A New Framework of Multistage Hypothesis Tests *

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In Memory of My Dear Father Hualong Chen (1933–1990)

Abstract

In this paper, we have established a general framework of multistage hypothesis tests which applies to arbitrarily many mutually exclusive and exhaustive composite hypotheses. Within the new framework, we have constructed specific multistage tests which rigorously control the risk of committing decision errors and are more efficient than previous tests in terms of average sample number and the number of sampling operations. Without truncation, the sample numbers of our testing plans are absolutely bounded.

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1 Introduction

Let $X$ be a random variable defined in a probability space $(\Omega, \mathcal{F}, \Pr)$. Suppose the distribution of $X$ is determined by an unknown parameter $\theta$ in a parameter space $\Theta$. In many applications, it is desirable to infer the true value of $\theta$ from random samples $X_1, X_2, \cdots$ of $X$. This topic can be formulated as a general problem of testing $m$ mutually exclusive and exhaustive composite hypotheses:

$$
\mathcal{H}_0 : \theta \in \Theta_0, \quad \mathcal{H}_1 : \theta \in \Theta_1, \quad \ldots, \quad \mathcal{H}_{m-1} : \theta \in \Theta_{m-1},
$$

where $\Theta_0 = \{ \theta \in \Theta : \theta \leq \theta_1 \}$, $\Theta_{m-1} = \{ \theta \in \Theta : \theta > \theta_{m-1} \}$ and $\Theta_i = \{ \theta \in \Theta : \theta_i < \theta \leq \theta_{i+1} \}$, $i = 1, \cdots, m-2$ with $\theta_1 < \theta_2 < \cdots < \theta_{m-1}$. To control the probabilities of making wrong decisions, it is typically required that, for pre-specified numbers $\delta_i \in (0,1)$,

$$
\Pr\{\text{Accept } \mathcal{H}_i \mid \theta \} \geq 1 - \delta_i, \quad \forall \theta \in \Theta_i, \quad i = 0, 1, \cdots, m-1
$$
with $\Theta_0 = \{ \theta \in \Theta_0 : \theta \leq \theta_1' \}$, $\Theta_{m-1} = \{ \theta \in \Theta_{m-1} : \theta \geq \theta_{m-1}' \}$ and $\Theta_i = \{ \theta \in \Theta_i : \theta_i'' \leq \theta \leq \theta_{i+1}' \}$, $i = 1, \ldots, m-2$, where $\theta_i', \theta_i''$ are parametric values in $\Theta$ such that $\theta_1' < \theta_1$, $\theta_{m-1}' > \theta_{m-1}$ and $\theta_{i-1}' < \theta_i'' < \theta_i < \theta_i' < \theta_{i+1}'$ for $i = 2, \ldots, m-2$. For $i = 0, 1, \ldots, m-1$, $\Pr\{\text{Accept } \mathcal{H}_i | \theta\}$ is referred to as an Operating Characteristic (OC) function. Since there is no requirement imposed for controlling the risk of making wrong decisions for $\theta$ in $\Theta \setminus \bigcup_{j=0}^{m-1} \Theta_j = \bigcup_{i=1}^{m-1} (\theta_i', \theta_i'')$, such a remainder set, $\bigcup_{i=1}^{m-1} (\theta_i', \theta_i'' \setminus \Theta_j)$, is referred to as an indifference zone. The concept of indifference zone was introduced by Wald [8] for two main reasons. First, when the parameter $\theta$ is close to $\theta_i$, the margin between adjacent parameter subsets $\Theta_{i-1}$ and $\Theta_i$, it is immaterial to decide whether $\mathcal{H}_{i-1}$ or $\mathcal{H}_i$ should be accepted. Second, the sample size required to make a reliable decision between consecutive hypotheses $\mathcal{H}_{i-1}$ and $\mathcal{H}_i$ becomes increasingly intolerable as $\theta$ tends to $\theta_i$. Undoubtedly, the indifference zone should be sufficiently “narrow” so that the consequence of making erroneous decision is practically unimportant when $\theta$ lies in it.

The general problem of hypothesis testing described above has been a fundamental issue of research for many decades. The well-known sequential probability ratio test (SPRT) has been developed by Wald [8] to address such testing problem in the special case of two hypotheses. In addition to the limitation associated with the number of hypotheses, the SPRT suffers from other drawbacks. First, the sample number of SPRT is a random number which is not bounded. However, to be useful, the sample number of any testing plan should be bounded by a deterministic number. Although this can be fixed by forced termination (see, e.g., [6] and the references therein), the prescribed level of power may not be ensured as a result of truncation. Second, the number of sampling operations of SPRT is as large as the number of samples. In practice, it is usually much more economical to take a batch of samples at a time instead of one by one. Third, the efficiency of SPRT is optimal only for the endpoints of the indifference zone. For other parametric values, the SPRT can be extremely inefficient. Needless to say, a truncated version of SPRT may suffer from the same problem due to the partial use of the boundary of SPRT.

In this paper, to overcome the limitations of SPRT and its variations, we have established a new framework of hypothesis testing which applies to arbitrary number of composite hypotheses. Our testing plans have the following features: i) The testing process has a finite number of stages and thus the cost of sampling operations is reduced as compared to SPRT; ii) The sample number is absolutely bounded without truncation; iii) The prescribed level of power is rigorously guaranteed; iv) The testing is not only efficient for the endpoints of indifference zone, but also efficient for other parametric values. The remainder of the paper is organized as follows. In Section 2, we present our general theory and computational mechanisms for the design and analysis of multistage testing plans. In Section 3, we first present more specific construction of testing procedures and then apply the general method to common important problems. Specially, we demonstrate that the principle can be used for testing a binomial proportion, the proportion of a finite population, a Poisson parameter, the mean of a normal distribution with known variance, the variance of a normal distribution, the parameter of an exponential distribution and life testing. Section 4 is dedicated to tests of the mean of a normal distribution with unknown variance. Section 5 is the
conclusion. All proofs of theorems are given in Appendices.

Throughout this paper, we shall use the following notations. The set of real numbers is denoted by \( \mathbb{R} \). The set of integers is denoted by \( \mathbb{Z} \). The set of positive integers is denoted by \( \mathbb{N} \). The notation \( \emptyset \) denotes an empty set. The ceiling function and floor function are denoted respectively by \( \lceil . \rceil \) and \( \lfloor . \rfloor \) (i.e., \( \lfloor x \rfloor \) represents the smallest integer no less than \( x \); \( \lceil x \rceil \) represents the largest integer no greater than \( x \)). The gamma function is denoted by \( \Gamma(.) \). For any integer \( i \), the combinatoric function \( \binom{i}{j} \) with respect to integer \( j \) takes value \( \frac{\Gamma(i+1)}{\Gamma(j+1)\Gamma(i-j+1)} \) for \( j \leq i \) and value 0 otherwise. The expectation of a random variable is denoted by \( \mathbb{E}[.] \). We use the notation \( \Pr\{. \ | \ \theta \} \) to denote the probability of an event which is defined in terms of random variables parameterized by \( \theta \). The parameter \( \theta \) in \( \Pr\{. \ | \ \theta \} \) may be dropped whenever this can be done without introducing confusion. If \( Z \) is parameterized by \( \theta \), we denote \( \Pr\{Z \leq z \ | \ \theta \} \) by \( F_Z(z,\theta) \) and \( \Pr\{Z \geq z \ | \ \theta \} \) by \( G_Z(z,\theta) \) respectively. The cumulative distribution function of a Gaussian random variable is denoted by \( \Phi(.) \). For \( \alpha \in (0,1) \), \( Z_\alpha \) denotes the critical value satisfying \( \Phi(Z_\alpha) = 1 - \alpha \). For \( \alpha \in (0,1) \), let \( \chi^2_{n,\alpha} \) denote the 100\( \alpha \)% percentile of a chi-square distribution of \( n \) degrees of freedom. For \( \alpha \in (0,1) \), let \( t_{n,\alpha} \) denote the 100\((1 - \alpha)\)% percentile of a Student \( t \)-distribution of \( n \) degrees of freedom. The support of a random variable \( Z \) is denoted by \( I_Z \), i.e., \( I_Z = \{Z(\omega) : \omega \in \Omega \} \). We write \( \delta = O(\zeta) \) if \( \delta \) is a function of \( \zeta > 0 \) such that there exist constants \( A \) and \( B \) such that \( A < \frac{\delta}{\zeta} < B \) provided that \( \zeta > 0 \) is sufficiently small. The other notations will be made clear as we proceed.

## 2 General Theory and Computational Machinery

In this section, we shall discuss a general theory of multistage hypothesis tests. A central theme of our theory is on the reduction of the computational complexity associated with the design and analysis of multistage testing plans.

### 2.1 Basic Structure

In general, a testing plan in our proposed framework consists of \( s \) stages. For \( \ell = 1, \ldots, s \), the number of available samples (i.e., sample size) of the \( \ell \)-th stage is denoted by \( n_\ell \). For the \( \ell \)-th stage, a decision variable \( D_\ell = \mathcal{D}_\ell(X_1, \ldots, X_{n_\ell}) \) is defined in terms of samples \( X_1, \ldots, X_{n_\ell} \) such that \( D_\ell \) assumes \( m + 1 \) possible values \( 0, 1, \ldots, m \) with the following notion:

(i) Sampling is continued until \( D_\ell \neq 0 \) for some \( \ell \in \{1, \ldots, s\} \).

(ii) The hypothesis \( \mathcal{H}_j \) is accepted at the \( \ell \)-th stage if \( D_\ell = j + 1 \) and \( D_i = 0 \) for \( 1 \leq i < \ell \).

For practical considerations, we shall only focus on sampling schemes which are closed in the sense that \( \Pr\{D_s = 0\} = 0 \). For efficiency, a sampling scheme should satisfy the condition that both \( \Pr\{D_1 \neq 0\} \) and \( \Pr\{D_{s-1} = 0\} \) are greater than zero.

Let \( l \) denote the index of stage when the sampling is terminated. Then, the sample number when the sampling is terminated, denoted by \( n_l \), is equal to \( n_l \). For the \( \ell \)-th stage, an estimator
\( \hat{\theta}_\ell \) for \( \theta \) can be defined based on samples \( X_1, \ldots, X_n \). Consequently, the overall estimator for \( \theta \), denoted by \( \hat{\theta} \), is equal to \( \hat{\theta}_\ell \). In many cases, decision variables \( D_\ell \) can be defined in terms of \( \hat{\theta}_\ell \). Specially, if \( \hat{\theta}_\ell \) is a Unimodal-Likelihood Estimator (ULE) of \( \theta \) for \( \ell = 1, \ldots, s \), the design and analysis of multistage sampling schemes can be significantly simplified. For a random tuple \( X_1, \ldots, X_r \) (of deterministic or random length \( r \)) parameterized by \( \theta \), we say that the estimator \( \varphi(X_1, \ldots, X_r) \) is a ULE of \( \theta \) if \( \varphi \) is a multivariate function such that, for any observation \((x_1, \ldots, x_r)\) of \((X_1, \ldots, X_r)\), the likelihood function is non-decreasing with respect to \( \theta \) no greater than \( \varphi(x_1, \ldots, x_r) \) and is non-increasing with respect to \( \theta \) no less than \( \varphi(x_1, \ldots, x_r) \). For discrete random variables \( X_1, \ldots, X_r \), the associated likelihood function is the joint probability mass function \( \Pr\{X_i = x_i, i = 1, \ldots, r \mid \theta\} \). For continuous random variables \( X_1, \ldots, X_r \), the corresponding likelihood function is, \( f_{X_1, \ldots, X_r}(x_1, \ldots, x_r, \theta) \), the joint probability density function of random variable \( X_1, \ldots, X_r \). It should be noted that a ULE may not be a maximum-likelihood estimator (MLE). On the other side, a MLE may not be a ULE.

In the sequel, we shall focus on multistage sampling schemes which can be defined in terms of estimator \( \varphi_n = \varphi(X_1, \ldots, X_n) \) such that \( \varphi_n \) is a ULE of \( \theta \) for every \( n \) and that \( \varphi_n \) converges in probability to \( \theta \) in the sense that, for any \( \varepsilon > 0 \) and \( \delta \in (0, 1) \), \( \Pr\{|\varphi_n - \theta| \geq \varepsilon\} < \delta \) provided that \( n \) is sufficiently large. Such estimator \( \varphi_n \) is referred to as a Unimodal-likelihood Consistent Estimator (ULCE) of \( \theta \). For the \( \ell \)-th stage, the estimator \( \hat{\theta}_\ell \) is defined as \( \varphi_{n_\ell} = \varphi(X_1, \ldots, X_{n_\ell}) \). Accordingly, the decision variables \( D_\ell \) can be defined in terms of estimator \( \hat{\theta}_\ell = \varphi_{n_\ell} \).

### 2.2 Bisection Risk Tuning

To avoid prohibitive burden of computational complexity in the design process, our global strategy is to construct multistage sampling schemes of certain structure such that the risks of erroneously accepting or rejecting a hypothesis can be adjusted by some parameter \( \zeta > 0 \). Such a parameter \( \zeta \) is referred to as a risk tuning parameter in this paper to convey the idea that \( \zeta \) is used to “tune” the risk of making a wrong decision to be acceptable. As will be seen in the sequel, by virtue of the concept of ULE, we are able to construct a class of multistage testing plans such that the risks can be “tuned” to be no greater than prescribed levels by making the risk tuning parameter \( \zeta \) sufficiently small. Moreover, the risk tuning can be accomplished by a bisection search method. Furthermore, the OC functions of these multistage testing plans possess some monotonicity which makes it possible to control the probabilities of committing decision errors by checking the endpoints of indifference zone.

For the ease of presentation of our sampling schemes, we need to introduce some multivariate functions regarding estimator \( \varphi_n = \varphi(X_1, \ldots, X_n) \) of \( \theta \). For \( n \in \mathbb{N}, \theta \in \Theta, \delta \in (0, 1) \), define

\[
\begin{align*}
  f(n, \theta, \delta) &= \begin{cases} 
    \max\{z \in I_{\varphi_n} : F_{\varphi_n}(z, \theta) \leq \delta, z \leq \theta\} & \text{if } \{F_{\varphi_n}(\varphi_n, \theta) \leq \delta, \varphi_n \leq \theta\} \neq \emptyset, \\
    -\infty & \text{otherwise}
  \end{cases} \\
  g(n, \theta, \delta) &= \begin{cases} 
    \min\{z \in I_{\varphi_n} : G_{\varphi_n}(z, \theta) \leq \delta, z \geq \theta\} & \text{if } \{G_{\varphi_n}(\varphi_n, \theta) \leq \delta, \varphi_n \geq \theta\} \neq \emptyset, \\
    \infty & \text{otherwise}
  \end{cases}
\end{align*}
\]
For $\theta' < \theta''$ contained in $\Theta$ and $\delta', \delta'' \in (0, 1)$, define
\[
\ell_i = f(n, \theta', \theta'', \delta', \delta'') = \begin{cases} f(n, \theta', \theta'') & \text{if } f(n, \theta'', \delta'') < g(n, \theta', \delta'), \\ \frac{1}{2}[f(n, \theta', \theta'') + g(n, \theta', \delta')] & \text{if } f(n, \theta'', \delta'') \geq g(n, \theta', \delta'). \end{cases}
\]
\[
\gamma(n, \theta', \theta'', \delta', \delta'') = \begin{cases} g(n, \theta', \delta') & \text{if } f(n, \theta'', \delta'') < g(n, \theta', \delta'), \\ \frac{1}{2}[f(n, \theta', \theta'') + g(n, \theta', \delta')] & \text{if } f(n, \theta'', \delta'') \geq g(n, \theta', \delta'). \end{cases}
\]

Our general principle for constructing multistage test plans and their properties can be described by Theorem 1 as follows.

**Theorem 1** Let $\alpha_i = O(\zeta) \in (0, 1), \beta_i = O(\zeta) \in (0, 1)$ for $i = 1, \ldots, m - 1$ and $\alpha_m = \beta_0 = 0$. Define $\bar{\alpha}_i = \max\{\alpha_j : i < j \leq m\}$ and $\bar{\beta}_i = \max\{\beta_j : 0 \leq j \leq i\}$ for $i = 0, 1, \ldots, m - 1$. Suppose that $\varphi_n$ is a ULCE of $\theta$. Suppose that the maximum sample size $n_s$ is no less than $\bar{n}$ which is the minimum integer $n$ such that $f(n, \theta'_i, \beta_i) \geq g(n, \theta'_i, \alpha_i)$ for $i = 1, \ldots, m - 1$. Define $f_{\ell,i} = f(n_{\ell,i}, \theta'_i, \theta''_i, \alpha_i, \beta_i)$ and $g_{\ell,i} = \gamma(n_{\ell,i}, \theta'_i, \theta''_i, \alpha_i, \beta_i)$ for $i = 1, \ldots, m - 1$. Define
\[
D_{\ell} = \begin{cases} 1 & \text{if } \hat{\theta}_{\ell} \leq f_{\ell,1}, \\ i & \text{if } g_{\ell,i-1} < \hat{\theta}_{\ell} \leq f_{\ell,i} \text{ where } 2 \leq i \leq m - 1, \\ m & \text{if } \hat{\theta}_{\ell} > g_{\ell,m-1}, \\ 0 & \text{else} \end{cases}
\]

for $\ell = 1, \ldots, s$. The following statements (I)-(VI) hold true for $m \geq 2$.

(I) $\Pr\{\text{Reject } \mathcal{H}_0 | \theta\}$ is non-decreasing with respect to $\theta \in \Theta_0$.

(II) $\Pr\{\text{Reject } \mathcal{H}_{m-1} | \theta\}$ is non-increasing with respect to $\theta \in \Theta_{m-1}$.

(III) $\Pr\{\text{Reject } \mathcal{H}_i | \theta\} \leq s(\alpha_i + \beta_i)$ for any $\theta \in \Theta_i$ and $i = 0, 1, \ldots, m - 1$.

(IV) For $0 < i \leq m - 1$, $\Pr\{\text{Accept } \mathcal{H}_i | \theta\}$ is no greater than $s\alpha_i$ and is non-decreasing with respect to $\theta \in \Theta$ no less than $\theta'_i$.

(V) For $0 \leq i \leq m - 2$, $\Pr\{\text{Accept } \mathcal{H}_i | \theta\}$ is no greater than $s\beta_{i+1}$ and is non-increasing with respect to $\theta \in \Theta$ no less than $\theta''_{i+1}$.

(VI) Assume that $\mathbb{E}[e^{\theta X}]$ exists for any $t \in \mathbb{R}$ and that $\varphi_n = \frac{\sum_{i=1}^{n} X_i}{n}$ is an unbiased and unimodal-likelihood estimator of $\theta$, where $X_1, X_2, \ldots$ are i.i.d. samples of $X$. Assume that $n_s$ is equal to $\bar{n}$. Then, $\lim_{\ell \to 0} \Pr\{\text{Reject } \mathcal{H}_i | \theta\} = 0$ for any $\theta \in \Theta_i$ and $i = 0, 1, \ldots, m - 1$.

Moreover, the following statements (VII), (VIII) and (IX) hold true for $m \geq 3$.

(VII)
\[
\Pr\{\text{Reject } \mathcal{H}_i | \theta\} \leq \Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \leq a | a\} + \Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \geq b | b\},
\]
\[
\Pr\{\text{Reject } \mathcal{H}_i | \theta\} \geq \Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \leq a | b\} + \Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \geq b | a\}
\]

for any $\theta \in [a, b] \subseteq \Theta_i$ and $1 \leq i \leq m - 2$.

(VIII) $\Pr\{\text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_{m-1} | \theta\}$ is non-decreasing with respect to $\theta \in \Theta_0$ and is non-increasing with respect to $\theta \in \Theta_{m-1}$.
(IX) \( \Pr\{\text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_{m-1} \mid \theta\} \) is no greater than \( s \times \max\{\alpha_i : 1 \leq i \leq m-2\} \) for \( \theta \in \Theta_0 \) and is no greater than \( s \times \max\{\beta_i : 2 \leq i \leq m-1\} \) for \( \theta \in \Theta_{m-1} \).

See Appendix for a proof.

In order to develop test plans with simple stopping boundary, we define multivariate functions

\[
\begin{align*}
    f_c(n, \theta, \delta) &= \left\{ \begin{array}{ll}
        \max\{z \in I_{\nu_n} : \mathcal{G}_n(z, \theta) \leq \delta, z \leq \theta\} & \text{if } \{\mathcal{G}_n(\varphi, \theta) \leq \delta, \varphi_n \leq \theta\} \neq \emptyset, \\
        -\infty & \text{otherwise}
    \end{array} \right.

    g_c(n, \theta, \delta) &= \left\{ \begin{array}{ll}
        \min\{z \in I_{\nu_n} : \mathcal{G}_n(z, \theta) \leq \delta, z \geq \theta\} & \text{if } \{\mathcal{G}_n(\varphi, \theta) \leq \delta, \varphi_n \geq \theta\} \neq \emptyset, \\
        +\infty & \text{otherwise}
    \end{array} \right.
\]

for \( n \in \mathbb{N}, \theta \in \Theta, \delta \in (0, 1) \), where \( \mathcal{G}_n(z, \theta) = \inf_{\rho \in \mathbb{R}} \mathbb{E}[e^{\rho(z-\theta)}] \). Moreover, define

\[
\begin{align*}
    f_c(n, \theta', \theta'', \delta', \delta'') &= \begin{cases} 
        f_c(n, \theta'', \delta'') & \text{if } f_c(n, \theta'', \delta'') < g_c(n, \theta', \delta'), \\
        \frac{1}{2}[f_c(n, \theta'', \delta'') + g_c(n, \theta', \delta')] & \text{if } f_c(n, \theta'', \delta'') \geq g_c(n, \theta', \delta')
    \end{cases} \\
    g_c(n, \theta', \theta'', \delta', \delta'') &= \begin{cases} 
        g_c(n, \theta', \delta') & \text{if } f_c(n, \theta'', \delta'') < g_c(n, \theta', \delta'), \\
        \frac{1}{2}[f_c(n, \theta'', \delta'') + g_c(n, \theta', \delta')] & \text{if } f_c(n, \theta'', \delta'') \geq g_c(n, \theta', \delta')
    \end{cases}
\]

for \( \theta' < \theta'' \) in \( \Theta, \delta', \delta'' \in (0, 1) \) and \( n \in \mathbb{N} \).

Our sampling schemes and their properties can be described by Theorem as follows.

**Theorem 2** Let \( \alpha_i = O(\zeta) \in (0, 1), \beta_i = O(\zeta) \in (0, 1) \) for \( i = 1, \cdots, m-1 \) and \( \alpha_m = \beta_0 = 0 \). Define \( \overline{\alpha}_i = \max\{\alpha_j : i < j \leq m\} \) and \( \overline{\beta}_i = \max\{\beta_j : 0 \leq j \leq i\} \) for \( i = 0, 1, \cdots, m-1 \). Suppose that \( \mathbb{E}[e^{\rho X}] \) exists for any \( \rho \in \mathbb{R} \) and \( \varphi_n = \frac{\sum_{i=1}^{\overline{\alpha}_m} X}{n} \) is an unbiased and unimodal-likelihood estimator of \( \theta \). Suppose that the maximum sample size \( n_s \) is no less than \( \overline{n} \) which is the minimum integer \( n \) such that \( f_c(n, \theta', \alpha_i) \geq g_c(n, \theta', \alpha_i) \) for \( i = 1, \cdots, m-1 \). Define decision variable \( D_\ell \) by \( \alpha_i \) for \( \ell = 1, \cdots, s \) with \( f_{i, \ell} = \int_{n_s} f_c(n, \theta', \theta''_i, \alpha_i) \) and \( g_{i, \ell} = \int_{n_s} g_c(n, \theta', \theta''_i, \alpha_i) \). Then, the same conclusion as described by statements (I)–(IX) of Theorem holds true.

See Appendix for a proof.

Theorem 1 and 2 establish the groundwork for bisection risk tuning. In the design of multistage test plans, for risk tuning purpose, we recommend choosing \( \alpha_i = \min\{\zeta \delta_{i-1}, 1\} \) and \( \beta_i = \min\{\zeta \delta_i, 1\} \) for \( i = 1, \cdots, m-1 \). According to statement (VI) of Theorem 1, \( \Pr\{\text{Reject } \mathcal{H}_i \mid \theta\} \) tends to 0 as \( \zeta \) tends to 0. This implies that we can ensure 2 by choosing a sufficiently small risk tuning parameter \( \zeta \). Clearly, every value of \( \zeta \) determines a test plan and consequently its performance specifications such as average sample number (ASN) and risks of making wrong decisions. Intuitively, under the constraint of risk requirements, the risk tuning parameter \( \zeta \) should be chosen as large as possible in order to reduce the sample number. To achieve such an objective, it is a critical subroutine to determine whether a given \( \zeta \) is sufficient to ensure the risk requirement 2. Since there may be an extremely large number or infinite parametric values in
$\cup_{i=0}^{m-1} \Theta_i$, it is essential to develop an efficient method to check the risk requirement (2) without exhaustive computation. For this purpose, statements (I), (II) and (VI) of Theorem 1 can be very useful. As a consequence of statement (I), to check if $\Pr\{\text{Reject } H_0 \mid \theta \} \leq \delta_0$ for any $\theta \in \Theta_0$, it suffices to check whether $\Pr\{\text{Reject } H_0 \mid \theta'_i \} \leq \delta_0$ is true. By virtue of statement (II), for purpose of determining whether $\Pr\{\text{Reject } H_{m-1} \mid \theta \} \leq \delta_{m-1}$ for any $\theta \in \Theta_{m-1}$, it is sufficient to check if $\Pr\{\text{Reject } H_{m-1} \mid \theta''_{m-1} \} \leq \delta_{m-1}$ is true. For $i \in \{1, \cdots, m-2\}$, to determine whether $\Pr\{\text{Reject } H_i \mid \theta \} \leq \delta_i$ for any $\theta \in \Theta_i$, we can apply the bounding results in statement (VI) of Theorem 1 and the Adaptive Maximum Checking Algorithm (AMCA) established in [1]. Therefore, it is clear that we can develop an efficient subroutine to determine whether a given $\zeta$ guarantees the risk requirement (2). Now, let $\zeta$ be the maximum number in the set $\{10 \times 2^{-i} : i \in \mathbb{N}\}$ such that the risk requirement (2) is satisfied when the risk tuning parameter $\zeta$ assumes value $\zeta$. Such number $\zeta$ can be obtained by using the subroutine to check the risk requirement (2). Once $\zeta$ is found, we can apply a bisection search to obtain a number $\zeta^*$ as large as possible from interval $[\zeta, 2\zeta]$ such that the risk requirement (2) is satisfied when the risk tuning parameter $\zeta$ assumes value $\zeta^*$.

The above bisection risk tuning technique can be straightforwardly extended to control the following error probabilities:

$$
\begin{align*}
\Pr\{\text{Accept } H_i \mid \theta \in \Theta_j\}, & \quad 0 \leq i < j \leq m - 1 \\
\Pr\{\text{Accept } H_i \mid \theta \in \Theta_j\}, & \quad 0 \leq j < i \leq m - 1 \\
\Pr\{\text{Accept } H_i \mid \theta \in \Theta_j\}, & \quad 0 \leq i \leq j - 2 < j \leq m - 1 \\
\Pr\{\text{Accept } H_i \mid \theta \in \Theta_j\}, & \quad 0 \leq j \leq i - 2 < i \leq m - 1
\end{align*}
$$

For this purpose, statements (IV) and (V) of Theorem 1 can be used to develop efficient method of checking the above risk requirements. In a similar spirit, by virtue of statements (VII) and (VIII) of Theorem 1 the control of $\Pr\{\text{Reject } H_0 \text{ and } H_{m-1} \mid \theta \in \Theta_0 \cup \Theta_{m-1}\}$ can be incorporated in the bisection risk tuning technique. As can be seen from above discussion, a critical idea in the tuning technique is to avoid exhaustive computation by making use of monotonicity of error probabilities with respect to $\theta$.

### 2.3 Recursive Computation

As will be seen in the sequel, for most multistage sampling schemes for testing parameters of discrete variables, the computation of the OC functions involve probabilistic terms like $\Pr\{K_i \in \mathcal{H}_i, i = 1, \cdots, \ell\}$, $\ell = 1, 2, \cdots$, where $K_{\ell} = \sum_{i=1}^{n_{\ell}} X_i$ and $\mathcal{H}_i$ is a subset of integers. The calculation of such terms can be performed by virtue of the following recursive relationship:

$$
\Pr\{K_{\ell+1} = k_{\ell+1}; \ K_i \in \mathcal{H}_i, i = 1, \cdots, \ell\}
= \sum_{k_i \in \mathcal{H}_i} \Pr\{K_{\ell} = k_{\ell}; \ K_i \in \mathcal{H}_i, i = 1, \cdots, \ell - 1\} \Pr\{K_{\ell+1} - K_{\ell} = k_{\ell+1} - k_{\ell}\},
$$

8
where the computation of probability \( \Pr\{K_{\ell+1} - K_\ell = k_{\ell+1} - k_\ell\} \) depends on specific problems.

In the context of testing a binomial parameter \( p \), we have
\[
\Pr\{K_{\ell+1} - K_\ell = k_{\ell+1} - k_\ell \mid p\} = \binom{n_{\ell+1} - n_\ell}{k_{\ell+1} - k_\ell} p^{k_{\ell+1} - k_\ell} (1 - p)^{n_{\ell+1} - n_\ell - k_{\ell+1} + k_\ell}.
\]

In the context of testing a Poisson parameter \( \lambda \), we have
\[
\Pr\{K_{\ell+1} - K_\ell = k_{\ell+1} - k_\ell \mid \lambda\} = \binom{n_{\ell+1} - n_\ell}{k_{\ell+1} - k_\ell} \lambda^{k_{\ell+1} - k_\ell} \exp\left( -(n_{\ell+1} - n_\ell)\lambda \right) \frac{(k_{\ell+1} - k_\ell)!}{(k_{\ell+1} - k_\ell)!}.
\]

In the context of testing the proportion, \( p \), of a finite population of size \( N \) using multistage sampling schemes to be described in Section 3.6.2, we have
\[
\Pr\{K_{\ell+1} - K_\ell = k_{\ell+1} - k_\ell \mid p\} = \binom{pN - k_\ell}{k_{\ell+1} - k_\ell} \left( \frac{N - n_\ell - pN + k_\ell}{n_{\ell+1} - n_\ell - k_{\ell+1} + k_\ell} \right) \frac{N - n_\ell}{n_{\ell+1} - n_\ell}.
\]

It should be noted that the domain truncation technique to be described in subsection 2.4 can be used to significantly reduce computation.

### 2.4 Domain Truncation

In the design and analysis of multistage sampling schemes, the associated computational complexity can be high because the domain of summation or integration is large. The truncation techniques recently established in [2] have the power to considerably simplify the computation by reducing the domain of summation or integration to a much smaller set. The following result, quoted from [2], shows that the truncation can be done with controllable error.

**Theorem 3** Let \( a_i, b_i, u_i, v_i, \eta_i, i = 1, \ldots, k \) be real numbers. Suppose that \( \Pr\{u_i \leq Z_i \leq v_i\} \geq 1 - \eta_i \) for \( i = 1, \ldots, k \). Then, \( P' \leq \Pr\{a_i \leq Z_i \leq b_i, i = 1, \ldots, k\} \leq P' + \sum_{i=1}^{k} \eta_i \), where \( P' = \Pr\{a'_i \leq Z_i \leq b'_i, i = 1, \ldots, k\} \) with \( a'_i = \max\{a_i, u_i\} \) and \( b'_i = \min\{b_i, v_i\} \) for \( i = 1, \ldots, k \).

### 3 Construction of Sampling Schemes

In this section, we shall discuss the applications of the fundamental principle described in the previous section to the design and analysis of multistage testing plans.

#### 3.1 One-sided Tests

In order to infer from random samples \( X_1, X_2, \ldots \) of \( X \) whether the true value of \( \theta \) is greater or less than a certain number \( \vartheta \in \Theta \), a classical problem is to test one-sided hypothesis \( \mathcal{H}_0 : \theta \leq \vartheta \) versus \( \mathcal{H}_1 : \theta > \vartheta \). This problem can be cast in the general formulation (1) with \( m = 2 \), \( \Theta_0 = \)
\{\theta \in \Theta : \theta \leq \vartheta\} \text{ and } \Theta_1 = \{\theta \in \Theta : \theta > \vartheta\}. \text{ To control the probabilities of making wrong decisions, it is typically required that, for a priori numbers } \alpha, \beta \in (0, 1),\]

\[
\Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} \leq \alpha \quad \text{for any } \theta \in \Theta_0, \quad (4)
\]

\[
\Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \leq \beta \quad \text{for any } \theta \in \Theta_1
\]

with \(\Theta_0 = \{\theta \in \Theta_0 : \theta \leq \theta_0\} \text{ and } \Theta_1 = \{\theta \in \Theta_1 : \theta \geq \theta_1\}, \) where \(\theta_0 \text{ and } \theta_1\) are numbers in \(\Theta\) such that \(\theta_0 < \vartheta < \theta_1\). The inequalities in (4) and (5) specify, respectively, the upper bounds for the probabilities of committing a Type I error and a Type II error. Clearly, the interval \((\theta_0, \theta_1)\) is an indifference zone, since there is no requirement imposed on probabilities of committing decision errors for \(\theta \in (\theta_0, \theta_1)\).

Applying Theorem 1 to the special case of \(m = 2\), we have the following results.

**Corollary 1** Let \(\alpha_0, \beta_1 \in (0, 1)\). Suppose that \(\varphi_n\) is a ULCE of \(\theta\) and that the maximum sample size \(n_s\) is no less than \(\bar{n}\) which is the minimum integer \(n\) such that \(f(n, \theta_1, \beta_1) \geq g(n, \theta_0, \alpha_0)\). Define

\[
D_\ell = \begin{cases} 
1 & \text{if } \hat{\theta}_\ell \leq f(n, \theta_0, \theta_1, \alpha_0, \beta_1), \\
2 & \text{if } \hat{\theta}_\ell > g(n, \theta_0, \theta_1, \alpha_0, \beta_1), \\
0 & \text{else}
\end{cases}
\]

for \(\ell = 1, \cdots, s\). Then, \(\Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \leq s\beta_1\) for \(\theta \in \Theta\) no less than \(\theta_1\), and \(\Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} \leq s\alpha_0\) for \(\theta \in \Theta\) no greater than \(\theta_0\). Moreover, \(\Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\}\) is non-increasing with respect to \(\theta \in \Theta\) such that \(\theta \notin (\theta_0, \theta_1)\).

Applying Theorem 2 to the special case of \(m = 2\), we have the following results.

**Corollary 2** Let \(\alpha_0, \beta_1 \in (0, 1)\). Suppose that \(f_c(n, \theta_1, \beta_1) \geq g_c(n, \theta_0, \alpha_0)\) if \(n\) is sufficiently large. Suppose that the maximum sample size \(n_s\) is no less than \(\bar{n}\) which is the minimum integer \(n\) such that \(f_c(n, \theta_1, \beta_1) \geq g_c(n, \theta_0, \alpha_0)\). Suppose that \(\hat{\theta}_\ell\) is an unbiased and unimodal-likelihood estimator of \(\theta\) for \(\ell = 1, \cdots, s\). Define

\[
D_\ell = \begin{cases} 
1 & \text{if } \hat{\theta}_\ell \leq f_c(n, \theta_0, \theta_1, \alpha_0, \beta_1), \\
2 & \text{if } \hat{\theta}_\ell > g_c(n, \theta_0, \theta_1, \alpha_0, \beta_1), \\
0 & \text{else}
\end{cases}
\]

for \(\ell = 1, \cdots, s\). Then, the same conclusion as that of Corollary 1 holds true.

In Corollaries 1 and 2 we can choose \(\alpha_0 = \min\{\zeta \alpha, 1\}, \beta_1 = \min\{\zeta \beta, 1\}\) for risk tuning purpose.
In order to develop a class of test plans with OC functions being monotone in the overall parameter space $\Theta$, we shall introduce multivariate functions

\[
\hat{F}(n, \theta, \delta) = \begin{cases} 
\max\{z \in I_{\varphi_n} : F_{\varphi_n}(z, \theta) \leq \delta\} & \text{if } \{F_{\varphi_n}(\varphi_n, \theta) \leq \delta\} \neq \emptyset, \\
-\infty & \text{otherwise}
\end{cases}
\]

\[
\hat{G}(n, \theta, \delta) = \begin{cases} 
\min\{z \in I_{\varphi_n} : G_{\varphi_n}(z, \theta) \leq \delta\} & \text{if } \{G_{\varphi_n}(\varphi_n, \theta) \leq \delta\} \neq \emptyset, \\
\infty & \text{otherwise}
\end{cases}
\]

for $n \in \mathbb{N}$, $\theta \in \Theta$, $\delta \in (0, 1)$ and

\[
\begin{align*}
\overline{F}(n, \theta', \theta'', \delta', \delta'') &= \begin{cases} 
\hat{F}(n, \theta'', \delta'') & \text{if } \hat{F}(n, \theta'', \delta'') < \hat{G}(n, \theta', \delta'), \\
\frac{1}{2}[\hat{F}(n, \theta'', \delta'') + \hat{G}(n, \theta', \delta')] & \text{if } \hat{F}(n, \theta'', \delta'') \geq \hat{G}(n, \theta', \delta')
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\overline{G}(n, \theta', \theta'', \delta', \delta'') &= \begin{cases} 
\hat{G}(n, \theta', \delta') & \text{if } \hat{G}(n, \theta', \delta') < \hat{G}(n, \theta', \delta'), \\
\frac{1}{2}[\hat{F}(n, \theta'', \delta'') + \hat{G}(n, \theta', \delta')] & \text{if } \hat{F}(n, \theta'', \delta'') \geq \hat{G}(n, \theta', \delta')
\end{cases}
\end{align*}
\]

for $\theta' < \theta''$ in $\Theta$ and $\delta', \delta'' \in (0, 1)$. Moreover, we need to make use of the concept of monotone likelihood ratio. The likelihood ratio is said to be monotonically increasing with respect to $\varphi_n$ if, for arbitrary $\theta' < \theta''$ in $\Theta$, the likelihood ratio $\frac{P_r(X_{i=1}, \ldots, n|\theta'')}{P_r(X_{i=1}, \ldots, n|\theta')}$ (or $\frac{f_{X_{i=1}, \ldots, n}(x_{i=1}, \ldots, x_n|\theta'')}{f_{X_{i=1}, \ldots, n}(x_{i=1}, \ldots, x_n|\theta')}$ for the continuous case) is monotonically increasing with respect to $\varphi_n$.

Now we are ready to describe a new class of test plans by Theorem 4 as follows.

**Theorem 4** Let $\alpha_0, \beta_1 \in (0, 1)$. Suppose that $\varphi_n$ is a ULCE of $\theta$ and that the likelihood ratio is monotonically increasing with respect to $\varphi_n$. Suppose that the maximum sample size $n_\alpha$ is no less than $\pi$ which is the minimum integer $n$ such that $\hat{F}(n, \theta_1, \beta_1) \geq \hat{G}(n, \theta_0, \alpha_0)$. Define

\[
D_\ell = \begin{cases} 
1 & \text{if } \hat{\theta}_\ell \leq \overline{F}(n_\ell, \theta_0, \theta_1, \alpha_0, \beta_1), \\
2 & \text{if } \hat{\theta}_\ell > \overline{G}(n_\ell, \theta_0, \theta_1, \alpha_0, \beta_1), \\
0 & \text{else}
\end{cases}
\]

for $\ell = 1, \ldots, s$. Then, $\Pr\{\text{Accept } \mathcal{H}_0 | \theta\} \leq s\beta_1$ for $\theta \in \Theta$ no less than $\theta_1$, and $\Pr\{\text{Reject } \mathcal{H}_0 | \theta\} \leq s\alpha_0$ for $\theta \in \Theta$ no greater than $\theta_0$. Moreover, $\Pr\{\text{Accept } \mathcal{H}_0 | \theta\}$ is non-increasing with respect to $\theta \in \Theta$.

See Appendix D for a proof.

### 3.2 Two-sided Tests

In order to infer from random samples $X_1, X_2, \ldots$ of $X$ whether the true value of $\theta$ is equal to a certain number $\theta_1 \in \Theta$, it is a frequent problem to test two-sided hypothesis $\mathcal{H}_0 : \theta = \theta_1$ versus $\mathcal{H}_1 : \theta \neq \theta_1$. To control the probabilities of making wrong decisions, it is typically required that, for a priori numbers $\alpha, \beta \in (0, 1),$

\[
\Pr\{\text{Reject } \mathcal{H}_0 | \theta_1\} \leq \alpha, \tag{6}
\]

\[
\Pr\{\text{Accept } \mathcal{H}_0 | \theta\} \leq \beta \quad \text{for } \theta \in \Theta \text{ such that } \theta \notin (\theta_0, \theta_2), \tag{7}
\]
where \( \theta_0 \) and \( \theta_2 \) are two numbers in \( \Theta \) such that \( \theta_0 < \theta_1 < \theta_2 \). The inequalities in (6) and (7) specify, respectively, the upper bounds for the probabilities of committing a Type I error and a Type II error. Since there is no requirement imposed on probabilities of committing errors for \( \theta \in (\theta_0, \theta_1) \cup (\theta_1, \theta_2) \), the union of intervals \((\theta_0, \theta_1) \cup (\theta_1, \theta_2)\) is referred to as an indifference zone.

Applying Theorem 2 to test hypotheses

\[
\mathcal{H}_0 : \theta \leq \frac{\theta_0 + \theta_1}{2}, \quad \mathcal{H}_1 : \frac{\theta_0 + \theta_1}{2} < \theta \leq \frac{\theta_1 + \theta_2}{2}, \quad \mathcal{H}_2 : \theta > \frac{\theta_1 + \theta_2}{2}
\]

with indifference zone \((\theta_0, \theta_1) \cup (\theta_1, \theta_2)\), we have \( \Pr\{\text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_2 \mid \theta\} = \Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \) and the following results follow immediately.

**Corollary 3** Let \( \alpha_0, \alpha_1, \beta_1, \beta_2 \in (0, 1) \). Suppose that \( \varphi_n \) is a ULCE of \( \theta \) and that the maximum sample size \( n_s \) is no less than \( \overline{n} \) which is the minimum integer \( n \) such that \( f(n, \theta_1, \beta_1) \geq g(n, \theta_0, \alpha_0) \) and \( f(n, \theta_2, \beta_2) \geq g(n, \theta_1, \alpha_1) \). Define

\[
D_\ell = \begin{cases} 1 & \text{if } \varphi(n_\ell, \theta_0, \theta_1, \alpha_0, \beta_1) < \hat{\theta}_\ell \leq f(n_\ell, \theta_1, \theta_2, \alpha_1, \beta_2), \\ 2 & \text{if } \hat{\theta}_\ell \leq \frac{f(n_\ell, \theta_0, \theta_1, \alpha_0, \beta_1)}{\varphi(n_\ell, \theta_1, \theta_2, \alpha_1, \beta_2)} \text{ or } \hat{\theta}_\ell > \frac{\varphi(n_\ell, \theta_0, \theta_2, \alpha_1, \beta_2)}{f(n_\ell, \theta_1, \theta_2, \alpha_1, \beta_2)}, \\ 0 & \text{else} \end{cases}
\]

for \( \ell = 1, \ldots, s \). Then, \( \Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \leq s \times \max\{\alpha_0, \beta_2\} \) for \( \theta \in \Theta \) such that \( \theta \notin (\theta_0, \theta_2) \), and \( \Pr\{\text{Reject } \mathcal{H}_0 \mid \theta_1\} \leq s(\alpha_1 + \beta_1) \). Moreover, \( \Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \) is non-decreasing with respect to \( \theta \in \Theta \) no greater than \( \theta_0 \) and is non-increasing with respect to \( \theta \in \Theta \) no less than \( \theta_2 \).

Applying Theorem 2 to test hypotheses \( \mathcal{H}_0, \mathcal{H}_1 \) and \( \mathcal{H}_2 \) with indifference zone \((\theta_0, \theta_1) \cup (\theta_1, \theta_2)\), we have the following results.

**Corollary 4** Let \( \alpha_0, \alpha_1, \beta_1, \beta_2 \in (0, 1) \). Suppose that

\[
f_\ell(n, \theta_1, \beta_1) \geq g_\ell(n, \theta_0, \alpha_0), \quad f_\ell(n, \theta_2, \beta_2) \geq g_\ell(n, \theta_1, \alpha_1) \tag{8}
\]

if \( n \) is sufficiently large. Suppose that the maximum sample size \( n_s \) is no less than \( \overline{n} \) which is the minimum integer \( n \) such that (8) is satisfied. Suppose that \( \hat{\theta}_\ell \) is an unbiased and unimodal-likelihood estimator of \( \theta \) for \( \ell = 1, \ldots, s \). Define

\[
D_\ell = \begin{cases} 1 & \text{if } \varphi_\ell(n_\ell, \theta_0, \theta_1, \alpha_0, \beta_1) < \hat{\theta}_\ell \leq f_\ell(n_\ell, \theta_1, \theta_2, \alpha_1, \beta_2), \\ 2 & \text{if } \hat{\theta}_\ell \leq \frac{f_\ell(n_\ell, \theta_0, \theta_1, \alpha_0, \beta_1)}{\varphi_\ell(n_\ell, \theta_1, \theta_2, \alpha_1, \beta_2)} \text{ or } \hat{\theta}_\ell > \frac{\varphi_\ell(n_\ell, \theta_0, \theta_2, \alpha_1, \beta_2)}{f_\ell(n_\ell, \theta_1, \theta_2, \alpha_1, \beta_2)}, \\ 0 & \text{else} \end{cases}
\]

for \( \ell = 1, \ldots, s \). Then, the same conclusion as that of Corollary 3 holds true.

In Corollaries 3 and 4 we can choose \( \alpha_0 = \beta_2 = \min\{\zeta \beta, 1\} \) and \( \alpha_1 = \beta_1 = \min\{\frac{\zeta \alpha}{2}, 1\} \) for risk tuning purpose.
3.3 Tests of Triple Hypotheses

As compared to two-sided tests, a more realistic formulation is to test three hypotheses \( \mathcal{H}_0 : \theta < \theta_1, \mathcal{H}_1 : \theta = \theta_1 \) and \( \mathcal{H}_2 : \theta > \theta_1 \), where \( \theta_1 \in \Theta \). To control the risks of committing decision errors, it is typically required that, for prescribed numbers \( \delta_0, \delta_1, \delta_2 \in (0,1) \),

\[
\Pr\{\text{Accept } \mathcal{H}_0 | \theta\} \geq 1 - \delta_0 \quad \text{for } \theta \in \Theta \text{ such that } \theta \leq \theta_0, \\
\Pr\{\text{Accept } \mathcal{H}_1 | \theta\} \geq 1 - \delta_1, \\
\Pr\{\text{Accept } \mathcal{H}_2 | \theta\} \geq 1 - \delta_2 \quad \text{for } \theta \in \Theta \text{ such that } \theta \geq \theta_2,
\]

where \( \theta_0 \) and \( \theta_2 \) are numbers in \( \Theta \) such that \( \theta_0 < \theta_1 < \theta_2 \). Clearly, \((\theta_0, \theta_1) \cup (\theta_1, \theta_2)\) is an indifference zone. Applying Theorem 1 to test hypotheses \( \mathcal{H}_0 : \theta \leq \frac{\theta_0 + \theta_1}{2}, \mathcal{H}_1 : \frac{\theta_0 + \theta_1}{2} < \theta \leq \frac{\theta_1 + \theta_2}{2} \) and \( \mathcal{H}_2 : \theta > \frac{\theta_1 + \theta_2}{2} \) with indifference zone \((\theta_0, \theta_1) \cup (\theta_1, \theta_2)\), we have the following results.

**Corollary 5** Let \( \alpha_0, \alpha_1, \beta_1, \beta_2 \in (0,1) \). Suppose that \( \varphi_n \) is a ULCE of \( \theta \). Suppose that the maximum sample size \( n_s \) is no less than \( \bar{n} \) which is the minimum integer \( n \) such that \( f(n, \theta_0, \alpha_0) \geq g(n, \theta_0, \alpha_0) \) and \( f(n, \theta_2, \beta_2) \geq g(n, \theta_1, \alpha_1) \). Define

\[
D_\ell = \begin{cases} 
1 & \text{if } \hat{\theta}_\ell \leq f(n_\ell, \theta_0, \theta_1, \alpha_0, \beta_1), \\
2 & \text{if } f(n_\ell, \theta_0, \theta_1, \alpha_0, \beta_1) < \hat{\theta}_\ell < f(n_\ell, \theta_1, \theta_2, \alpha_1, \beta_2), \\
3 & \text{if } \hat{\theta}_\ell > f(n_\ell, \theta_1, \theta_2, \alpha_1, \beta_2), \\
0 & \text{else}
\end{cases}
\]

for \( \ell = 1, \ldots, s \). Then, the following statements hold true.

(i) \( \Pr\{\text{Reject } \mathcal{H}_0 | \theta\} \leq s \times \max\{\alpha_0, \alpha_1\} \) for \( \theta \in \Theta \) no greater than \( \theta_0 \). Moreover, \( \Pr\{\text{Reject } \mathcal{H}_0 | \theta\} \) is non-decreasing with respect to \( \theta \in \Theta \) no greater than \( \theta_0 \).

(ii) \( \Pr\{\text{Reject } \mathcal{H}_2 | \theta\} \leq s \times \max\{\beta_1, \beta_2\} \) for \( \theta \in \Theta \) no less than \( \theta_2 \). Moreover, \( \Pr\{\text{Reject } \mathcal{H}_2 | \theta\} \) is non-increasing with respect to \( \theta \in \Theta \) no less than \( \theta_2 \).

(iii) \( \Pr\{\text{Reject } \mathcal{H}_1 | \theta_1\} \leq s(\alpha_1 + \beta_1) \).

Applying Theorem 2 to test hypotheses \( \mathcal{H}_0 : \theta \leq \frac{\theta_0 + \theta_1}{2}, \mathcal{H}_1 : \frac{\theta_0 + \theta_1}{2} < \theta \leq \frac{\theta_1 + \theta_2}{2} \) and \( \mathcal{H}_2 : \theta > \frac{\theta_1 + \theta_2}{2} \) with indifference zone \((\theta_0, \theta_1) \cup (\theta_1, \theta_2)\), we have the following results.

**Corollary 6** Let \( \alpha_0, \alpha_1, \beta_1, \beta_2 \in (0,1) \). Suppose that

\[
f_\ell(n, \theta_1, \beta_1) \geq g_\ell(n, \theta_0, \alpha_0), \quad f_\ell(n, \theta_2, \beta_2) \geq g_\ell(n, \theta_1, \alpha_1) \quad (9)
\]

if \( n \) is sufficiently large. Suppose that the maximum sample size \( n_s \) is no less than \( \bar{n} \) which is the minimum integer \( n \) such that \((9)\) is satisfied. Suppose that \( \hat{\theta}_\ell \) is an unbiased and unimodal-
likelihood estimator of $\theta$ for $\ell = 1, \cdots, s$. Define

$$D_\ell = \begin{cases} 
1 & \text{if } \hat{\theta}_\ell \leq f_c(n_\ell, \theta_0, \theta_1, \alpha_0, \beta_1), \\
2 & \text{if } \overline{f}_c(n_\ell, \theta_0, \theta_1, \alpha_0, \beta_1) < \hat{\theta}_\ell \leq f_c(n_\ell, \theta_1, \theta_2, \alpha_1, \beta_2), \\
3 & \text{if } \hat{\theta}_\ell > \overline{f}_c(n_\ell, \theta_1, \theta_2, \alpha_1, \beta_2), \\
0 & \text{else}
\end{cases}$$

for $\ell = 1, \cdots, s$. Then, the same conclusion as that of Corollary 5 holds true.

In Corollaries 5 and 6 we can choose $\alpha_0 = \min\{\zeta_0, 1\}$, $\alpha_1 = \beta_1 = \min\{\frac{\zeta_0}{\overline{f}}, 1\}$, $\beta_2 = \min\{\zeta_2, 1\}$ for risk tuning purpose.

### 3.4 Interval Tests

It is a frequent problem is to test hypothesis $H_0 : \theta \in [\theta_1, \theta_2]$ versus $H_1 : \theta \notin [\theta_1, \theta_2]$. For risk control purpose, it is typically required that, for two prescribed numbers $\alpha, \beta \in (0, 1),$

$$\Pr\{\text{Reject } H_0 \mid \theta\} \leq \alpha \quad \text{for } \theta \in \Theta \text{ such that } \theta \in [\theta'_1, \theta'_2],$$

$$\Pr\{\text{Accept } H_0 \mid \theta\} \leq \beta \quad \text{for } \theta \in \Theta \text{ such that } \theta \notin (\theta'_1, \theta'_2),$$

where $\theta'_1, \theta'_2$ are parametric values in $\Theta$ such that $\theta'_1 < \theta_1 < \theta'_2 < \theta_2 < \theta'_2$. Since there is no requirement imposed on probabilities of committing decision errors for $\theta \in (\theta'_1, \theta'_2) \cup (\theta'_2, \theta'_2)$, the union of intervals, $(\theta'_1, \theta'_2) \cup (\theta'_2, \theta'_2)$, is referred to as an indifference zone.

In view of the fact that the objective of the test is to decide whether the parameter $\theta$ falls into a specified interval, such a test is called an “interval test”.

Applying Theorem 1 to test hypotheses $H_0 : \theta \leq \theta_1$, $H_1 : \theta_1 < \theta \leq \theta_2$ and $H_2 : \theta > \theta_2$ with indifference zone $(\theta'_1, \theta'_2) \cup (\theta'_2, \theta'_2)$, we have $\Pr\{\text{Reject } H_0 \text{ and } H_2 \mid \theta\} = \Pr\{\text{Accept } H_0\}$ and the following result follows immediately.

**Corollary 7** Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, 1)$. Suppose that $\varphi_n$ is a ULCE of $\theta$ and that the maximum sample size $n_s$ is no less than $\overline{f}$ which is the minimum integer $n$ such that $f(n, \theta'_1, \beta_1) \geq g(n, \theta'_1, \alpha_1)$ and $f(n, \theta'_2, \beta_2) \geq g(n, \theta'_2, \alpha_2)$. Define

$$D_\ell = \begin{cases} 
1 & \text{if } \overline{f}(n_\ell, \theta'_1, \alpha_0, \beta_1) < \hat{\theta}_\ell \leq f(n_\ell, \theta'_2, \beta_2), \\
2 & \text{if } \hat{\theta}_\ell \leq f(n_\ell, \theta'_1, \alpha_0, \beta_1) \text{ or } \hat{\theta}_\ell > \overline{f}(n_\ell, \theta'_2, \beta_2), \\
0 & \text{else}
\end{cases}$$

for $\ell = 1, \cdots, s$. Then, the following statements hold true.

(i) $\Pr\{\text{Accept } H_0 \mid \theta\} \leq s \times \max\{\alpha_1, \beta_2\}$ for $\theta \in \Theta$ such that $\theta \notin (\theta'_1, \theta'_2)$.

(ii) $\Pr\{\text{Reject } H_0 \mid \theta\} \leq s(\alpha_2 + \beta_1)$ for $\theta \in \Theta$ such that $\theta \in [\theta'_1, \theta'_2]$. 
(iii) \( \Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \) is non-decreasing with respect to \( \theta \in \Theta \) no greater than \( \theta'_i \) and is non-increasing with respect to \( \theta \in \Theta \) no less than \( \theta''_i \). Moreover,

\[
\begin{align*}
\Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} & \leq \Pr\{\text{Reject } \mathcal{H}_0, \hat{\theta} \leq a \mid a\} + \Pr\{\text{Reject } \mathcal{H}_0, \hat{\theta} \geq b \mid b\}, \\
\Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} & \geq \Pr\{\text{Reject } \mathcal{H}_0, \hat{\theta} \leq a \mid b\} + \Pr\{\text{Reject } \mathcal{H}_0, \hat{\theta} \geq b \mid a\}
\end{align*}
\]

for any \( \theta \in [a, b] \subseteq [\theta''_1, \theta''_2] \cap \Theta \).

Applying Theorem 2 to test hypotheses \( \mathcal{H}_0 : \theta \leq \theta_1 \), \( \mathcal{H}_1 : \theta_1 < \theta \leq \theta_2 \) and \( \mathcal{H}_2 : \theta > \theta_2 \) with indifference zone \((\theta'_1, \theta''_1) \cup (\theta'_2, \theta''_2)\), we have the following results.

**Corollary 8** Let \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, 1) \). Suppose that

\[
f_c(n, \theta'_1, \beta_1) \geq g_c(n, \theta'_1, \alpha_1), \quad f_c(n, \theta''_2, \beta_2) \geq g_c(n, \theta''_2, \alpha_2) \tag{10}
\]

if \( n \) is sufficiently large. Suppose that the maximum sample size \( n_s \) is no less than \( \pi \) which is the minimum integer \( n \) such that \((10)\) is satisfied. Suppose that \( \hat{\theta}_\ell \) is an unbiased and unimodal-likelihood estimator of \( \theta \) for \( \ell = 1, \ldots, s \). Define

\[
D_\ell = \begin{cases} 
1 & \text{if } g_c(n, \theta'_1, \theta''_1, \alpha_1, \beta_1) < \hat{\theta}_\ell \leq f_c(n, \theta'_2, \theta''_2, \alpha_2, \beta_2), \\
2 & \text{if } \hat{\theta}_\ell \leq f_c(n, \theta'_1, \theta''_1, \alpha_1, \beta_1) \text{ or } \hat{\theta}_\ell > g_c(n, \theta'_2, \theta''_2, \alpha_2, \beta_2), \\
0 & \text{else}
\end{cases}
\]

for \( \ell = 1, \ldots, s \). Then, the same conclusion as that of Corollary 7 holds true.

In Corollaries 7 and 8 we can choose \( \alpha_2 = \beta_1 = \min\{\zeta \alpha, 1\} \) and \( \alpha_1 = \beta_2 = \min\{\zeta \beta, 1\} \) for risk tuning purpose.

### 3.5 Tests of Simple Hypotheses

In some situations, it may be interesting to test multiple simple hypotheses \( \mathcal{H}_i : \theta = \theta_i \) for \( i = 0, 1, \ldots, m - 1 \). For risk control purpose, it is typically required that, for prescribed numbers \( \delta_i \in (0, 1) \),

\[
\Pr\{\text{Accept } \mathcal{H}_i \mid \theta_i\} \geq 1 - \delta_i, \quad i = 0, 1, \ldots, m - 1.
\]

Applying Theorem 1 to the following hypotheses

\[
\mathcal{H}_0 : \theta \leq \vartheta_1, \quad \mathcal{H}_1 : \vartheta_1 < \theta \leq \vartheta_2, \quad \ldots, \quad \mathcal{H}_{m-2} : \vartheta_{m-2} < \theta \leq \vartheta_{m-1}, \quad \mathcal{H}_{m-1} : \theta > \vartheta_{m-1}
\]

with \( \vartheta_i = \frac{\vartheta_{i-1} + \vartheta_i}{2}, \ i = 1, \ldots, m - 1 \) and indifference zone \( \cup_{i=1}^{m-1}(\theta_{i-1}, \theta_i) \), we have the following results.
In this section, we shall demonstrate that the general principle proposed above can be applied to the following properties:

\( i < j \)

Suppose that

\[
\hat{\theta}_i = \max\{\alpha_j : i < j \leq m\} \quad \text{and} \quad \beta_i = \max\{\beta_j : 0 \leq j \leq i\} \quad \text{for} \quad i = 0, 1, \ldots, m - 1. \]

Suppose that \( \varphi_n \) is a ULCE of \( \theta \) and that the maximum sample size \( n_s \) is no less than \( \pi \) which is the minimum integer \( n \) such that \( f(n, \theta_i, \beta_i) \geq g(n, \theta_{i-1}, \alpha_i) \) for \( i = 1, \ldots, m - 1 \). Define \( f_{\ell,i} = f(n_\ell, \theta_{i-1}, \theta_i, \alpha_i, \beta_i) \) and \( g_{\ell,i} = g(n_\ell, \theta_{i-1}, \theta_i, \alpha_i, \beta_i) \) for \( i = 1, \ldots, m - 1 \). Define decision variable \( D_\ell \) by (3) for \( \ell = 1, \ldots, \). Then,

\[
\Pr\{\text{Reject } \mathcal{H}_i \mid \theta_i\} \leq s(\alpha_i + \beta_i) \quad \text{for} \quad i = 0, 1, \ldots, m - 1.
\]

Applying Theorem 2 to hypotheses \( \mathcal{H}_i, \ i = 0, 1, \ldots, m-1 \) with indifference zone \( \bigcup_{i=1}^{m-1} (\theta_{i-1}, \theta_i) \), we have the following results.

**Corollary 9** Let \( \alpha_i, \beta_i \in (0, 1) \) for \( i = 1, \ldots, m - 1 \) and \( \alpha_m = \beta_0 = 0 \). Define \( \alpha_i = \max\{\alpha_j : i < j \leq m\} \) and \( \beta_i = \max\{\beta_j : 0 \leq j \leq i\} \) for \( i = 0, 1, \ldots, m - 1 \). Suppose that

\[
f_c(n, \theta_i, \beta_i) \geq g_c(n, \theta_{i-1}, \alpha_i), \quad i = 1, \ldots, m - 1
\]

if \( n \) is sufficiently large. Suppose that the maximum sample size \( n_s \) is no less than \( \pi \) which is the minimum integer \( n \) such that (11) is satisfied. Define \( f_{\ell,i} = f(n_\ell, \theta_{i-1}, \theta_i, \alpha_i, \beta_i) \) and \( g_{\ell,i} = g(n_\ell, \theta_{i-1}, \theta_i, \alpha_i, \beta_i) \) for \( i = 1, \ldots, m - 1 \). Define decision variable \( D_\ell \) by (3) for \( \ell = 1, \ldots, \). Suppose that \( \hat{\theta}_i \) is an unbiased and unimodal-likelihood estimator of \( \theta \) for \( \ell = 1, \ldots, \). Then,

\[
\Pr\{\text{Reject } \mathcal{H}_i \mid \theta_i\} \leq s(\alpha_i + \beta_i) \quad \text{for} \quad i = 0, 1, \ldots, m - 1.
\]

In Corollaries 9 and 10 for risk tuning purpose, we recommend choosing \( \alpha_i = \min\{\zeta_{i-1}, 1\} \) and \( \beta_i = \min\{\zeta_i, 1\} \) for \( i = 1, \ldots, m - 1 \).

### 3.6 Applications

In this section, we shall demonstrate that the general principle proposed above can be applied to develop specific test plans for common important distributions. To apply our general method, we need to choose appropriate estimator \( \varphi_n = \varphi(X_1, \ldots, X_n) \) for \( \theta \) and investigate whether \( \varphi_n \) has the following properties:

(i) \( \varphi_n \) is a ULE of \( \theta \);  
(ii) \( \varphi_n \) converges in probability to \( \theta \);  
(iii) \( \varphi_n \) is an unbiased estimator of \( \theta \);  
(iv) The likelihood ratio is monotonically increasing with respect to \( \varphi_n \);  
(v) For \( \theta' < \theta'' \) in \( \Theta \) and \( \delta', \delta'' \in (0, 1) \), \( f_c(n, \theta'', \delta'') \) is no less than \( g_c(n, \theta', \delta') \) if \( n \) is sufficiently large.

#### 3.6.1 Testing a Binomial Proportion

Let \( X \) be a Bernoulli random variable with distribution \( \Pr\{X = 1\} = 1 - \Pr\{X = 0\} = p \in (0, 1) \). To test hypotheses regarding \( p \) based on i.i.d. samples \( X_1, X_2, \ldots \) of \( X \), we shall take
\[ \varphi_n = \varphi(X_1, \cdots, X_n) = \frac{1}{n} \sum_{i=1}^{n} X_i \] as an estimator of \( p \). With such a choice of estimator, it can be shown that, for \( n \in \mathbb{N}, p \in (0, 1), \delta \in (0, 1), \)

\[
\hat{F}(n, p, \delta) = \begin{cases} \frac{1}{n} \times \max \left\{ k \in \mathbb{Z} : \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \leq \delta, \ 0 \leq k \leq n \right\} & \text{for } n \geq \frac{\ln(\delta)}{\ln(1-p)}, \\ -\infty & \text{for } n < \frac{\ln(\delta)}{\ln(1-p)} \end{cases}
\]

\[
\hat{G}(n, p, \delta) = \begin{cases} \frac{1}{n} \times \min \left\{ k \in \mathbb{Z} : \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i} \leq \delta, \ 0 \leq k \leq n \right\} & \text{for } n \geq \frac{\ln(\delta)}{\ln(1-p)}, \\ \infty & \text{for } n < \frac{\ln(\delta)}{\ln(1-p)} \end{cases}
\]

\[
f(n, p, \delta) = \begin{cases} \frac{1}{n} \times \max \left\{ k \in \mathbb{Z} : \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \leq \delta, \ 0 \leq k \leq np \right\} & \text{for } n \geq \frac{\ln(\delta)}{\ln(1-p)}, \\ -\infty & \text{for } n < \frac{\ln(\delta)}{\ln(1-p)} \end{cases}
\]

\[
g(n, p, \delta) = \begin{cases} \frac{1}{n} \times \min \left\{ k \in \mathbb{Z} : \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i} \leq \delta, \ np \leq k \leq n \right\} & \text{for } n \geq \frac{\ln(\delta)}{\ln(1-p)}, \\ \infty & \text{for } n < \frac{\ln(\delta)}{\ln(1-p)} \end{cases}
\]

and

\[
f_c(n, p, \delta) = \begin{cases} \max \{ z \in [0, p] : \mathcal{M}_B(z, p) \leq \frac{\ln(\delta)}{n} \} & \text{for } n \geq \frac{\ln(\delta)}{\ln(1-p)}, \\ -\infty & \text{for } n < \frac{\ln(\delta)}{\ln(1-p)} \end{cases}
\]

\[
g_c(n, p, \delta) = \begin{cases} \min \{ z \in [p, 1] : \mathcal{M}_B(z, p) \leq \frac{\ln(\delta)}{n} \} & \text{for } n \geq \frac{\ln(\delta)}{\ln(1-p)}, \\ \infty & \text{for } n < \frac{\ln(\delta)}{\ln(1-p)} \end{cases}
\]

where

\[
\mathcal{M}_B(z, p) = \begin{cases} z \ln \frac{p}{z} + (1-z) \ln \frac{1-p}{1-z} & \text{for } z \in (0, 1), \\ \ln(1-p) & \text{for } z = 0, \\ \ln p & \text{for } z = 1. \end{cases}
\]

Moreover, it can be verified that the estimator \( \varphi_n \) possesses all properties described at the beginning of Section 3.6. This implies that all testing methods proposed in previous sections are applicable.

### 3.6.2 Testing the Proportion of a Finite Population

It is a frequent problem to test the proportion of a finite population. Consider a population of \( N \) units, among which there are \( Np \) units having a certain attribute, where \( p \in \Theta = \{ \frac{i}{N} : i = 0, 1, \cdots, N \} \). The procedure of sampling without replacement can be described as follows:

Each time a single unit is drawn without replacement from the remaining population so that every unit of the remaining population has equal chance of being selected.

Such a sampling process can be exactly characterized by random variables \( X_1, \cdots, X_N \) defined in a probability space \((\Omega, \mathcal{F}, \Pr)\) such that \( X_i \) denotes the characteristics of the \( i \)-th sample in the sense that \( X_i = 1 \) if the \( i \)-th sample has the attribute and \( X_i = 0 \) otherwise. By the nature of the sampling procedure, it can be shown that

\[
\Pr\{X_i = x_i, i = 1, \cdots, n \mid p\} = \left( \frac{Np}{\sum_{i=1}^{n} x_i} \right) \left( \frac{N - Np}{n - \sum_{i=1}^{n} x_i} \right) / \left[ \left( \sum_{i=1}^{n} x_i \right) \binom{N}{n} \right] \] (12)
for any $n \in \{1, \cdots, N\}$ and any $x_i \in \{0, 1\}, \ i = 1, \cdots, n$. By virtue of [12], it can be shown that $\Pr\{X_i = 1\} = 1 - \Pr\{X_i = 0\} = p \in \Theta$, which implies that $X_1, \cdots, X_N$ can be treated as identical but dependent samples of a Bernoulli random variable $X$ such that $\Pr\{X = 1\} = 1 - \Pr\{X = 0\} = p \in \Theta$. Recently, we have shown in [11] that, for any $n \in \{1, \cdots, N\}$, the sample mean $\varphi_n = \frac{\sum_{i=1}^n X_i}{n}$ is a ULE for $p \in \Theta$. Clearly, $\varphi_n$ is not a MLE for $p \in \Theta$. Hence, we can develop multistage testing plans in the framework outlined in Section 2.1. With the choice of $\varphi_n = \frac{\sum_{i=1}^n X_i}{n}$ as the estimator of $p$, it can be shown that

$$
\hat{F}(n, p, \delta) = \begin{cases} 
\frac{1}{n} \times \max \left\{ k \in \mathbb{Z} : \sum_{i=0}^{k} \left( \binom{pN}{i} - \binom{N-pN}{n-i} \right) / \binom{N}{n} \leq \delta, \ 0 \leq k < n \right\} & \text{for } \binom{N-pN}{n} \leq \delta \binom{N}{n}, \\
-\infty & \text{for } \binom{N-pN}{n} > \delta \binom{N}{n}
\end{cases}
$$

$$
\hat{G}(n, p, \delta) = \begin{cases} 
\frac{1}{n} \times \min \left\{ k \in \mathbb{Z} : \sum_{i=k}^{n} \left( \binom{pN}{i} - \binom{N-pN}{n-i} \right) / \binom{N}{n} \leq \delta, \ 0 < k \leq n \right\} & \text{for } \binom{N-pN}{n} \leq \delta \binom{N}{n}, \\
\infty & \text{for } \binom{N-pN}{n} > \delta \binom{N}{n}
\end{cases}
$$

$$
f(n, p, \delta) = \begin{cases} 
\frac{1}{n} \times \max \left\{ k \in \mathbb{Z} : \sum_{i=0}^{k} \left( \binom{pN}{i} - \binom{N-pN}{n-i} \right) / \binom{N}{n} \leq \delta, \ 0 \leq k \leq np \right\} & \text{for } \binom{N-pN}{n} \leq \delta \binom{N}{n}, \\
-\infty & \text{for } \binom{N-pN}{n} > \delta \binom{N}{n}
\end{cases}
$$

$$
g(n, p, \delta) = \begin{cases} 
\frac{1}{n} \times \min \left\{ k \in \mathbb{Z} : \sum_{i=k}^{n} \left( \binom{pN}{i} - \binom{N-pN}{n-i} \right) / \binom{N}{n} \leq \delta, \ np \leq k \leq n \right\} & \text{for } \binom{N-pN}{n} \leq \delta \binom{N}{n}, \\
\infty & \text{for } \binom{N-pN}{n} > \delta \binom{N}{n}
\end{cases}
$$

for $n \in \{1, \cdots, N\}, \ p \in \Theta$ and $\delta \in (0, 1)$. Clearly, $\varphi_n$ converges in probability to $p$ and thus is a UCE of $p$. Moreover, it can be verified that the likelihood ratio is monotonically increasing with respect to $\varphi_n$. This implies that the general results described in the previous sections can be useful.

In order to develop test plans with simple stopping boundary, we define multivariate functions

$$f_c(n, p, \delta) = \begin{cases} 
\max \{ z \in I_{\varphi_n} : C(n, z, p) \leq \delta, \ z \leq p \} & \text{if } \{C(n, \varphi_n, p) \leq \delta, \ \varphi_n \leq p \} \neq \emptyset, \\
-\infty & \text{otherwise}
\end{cases}
$$

$$g_c(n, p, \delta) = \begin{cases} 
\min \{ z \in I_{\varphi_n} : C(n, z, p) \leq \delta, \ z \geq p \} & \text{if } \{C(n, \varphi_n, p) \leq \delta, \ \varphi_n \geq p \} \neq \emptyset, \\
\infty & \text{otherwise}
\end{cases}
$$

for $n \in \mathbb{N}, \ p \in \Theta, \ \delta \in (0, 1)$, where

$$C(n, z, p) = \frac{\binom{Np}{n-z}}{\binom{N}{n}} \quad \text{for } z = 1,$$

$$C(n, z, p) = \frac{\binom{Np}{n-z} \binom{N-Np}{n-z}}{\binom{N}{n} \binom{N-Np}{n-z}} \quad \text{for } z \in \{ \frac{k}{n} : k \in \mathbb{Z}, \ 0 \leq k < n \}. \quad (13)$$

Moreover, define

$$f_c(n, p', p'', \delta', \delta'') = \begin{cases} 
f_c(n, p'', \delta'') & \text{if } f_c(n, p'', \delta'') < g_c(n, p', \delta'), \\
\frac{1}{2} [f_c(n, p'', \delta'') + g_c(n, p', \delta')] & \text{if } f_c(n, p'', \delta'') \geq g_c(n, p', \delta')
\end{cases}
$$

$$g_c(n, p', p'', \delta', \delta'') = \begin{cases} 
g_c(n, p', \delta') & \text{if } f_c(n, p'', \delta'') < g_c(n, p', \delta'), \\
\frac{1}{2} [f_c(n, p'', \delta'') + g_c(n, p', \delta')] & \text{if } f_c(n, p'', \delta'') \geq g_c(n, p', \delta')
\end{cases}
$$

for $p' < p''$ in $\Theta$, $\delta', \delta'' \in (0, 1)$ and $n \in \mathbb{N}$. 18
For the multi-hypothesis testing problem stated in the introduction with \( \theta \) replaced by \( p \), we have the following results.

**Theorem 5** Let \( \alpha_i, \beta_i \in (0, 1) \) for \( i = 1, \cdots, m-1 \) and \( \alpha_m = \beta_0 = 0 \). Define \( \pi_i = \max\{\alpha_j : i < j \leq m\} \) and \( \beta_i = \max\{\beta_j : 0 \leq j < i\} \) for \( i = 0, 1, \cdots, m-1 \). Suppose that the maximum sample size \( n_s \) is no less than \( n \) which is the minimum integer \( n \) such that

\[
  f_c(n, p_i', \alpha_i, \beta_i) \geq g_c(n, p_i', \alpha_i)
\]

for \( i = 1, \cdots, m-1 \). Define \( f_{\ell,i} = f_c(n_{\ell,i}, p_i', p_i'', \alpha_i, \beta_i) \) and \( g_{\ell,i} = g_c(n_{\ell,i}, p_i', p_i'', \alpha_i, \beta_i) \) for \( i = 1, \cdots, m-1 \). Define \( \hat{p}_{\ell} = \varphi_{n_{\ell}} = \frac{\sum_{i=1}^{s} X_i}{n} \) and

\[
  D_{\ell,i} = \begin{cases} 
  1 & \text{if } \hat{p}_{\ell} \leq f_{\ell,1}, \\
  i & \text{if } g_{\ell,i-1} \leq \hat{p}_{\ell} \leq f_{\ell,i} \text{ where } 2 \leq i \leq m-1, \\
  m & \text{if } \hat{p}_{\ell} > g_{\ell,m-1}, \\
  0 & \text{else}
  \end{cases}
\]  

for \( \ell = 1, \cdots, s \). The following statements (I)-(V) hold true for \( m \geq 2 \).

(I) \( \Pr\{\text{Reject } H_0 \mid p\} \) is non-decreasing with respect to \( p \in \Theta_0 \).

(II) \( \Pr\{\text{Reject } H_{m-1} \mid p\} \) is non-increasing with respect to \( p \in \Theta_{m-1} \).

(III) \( \Pr\{\text{Reject } H_i \mid p\} \leq s(\pi_i + \beta_i) \) for any \( p \in \Theta_i \) and \( i = 0, 1, \cdots, m-1 \).

(IV) For \( 0 < i \leq m-1 \), \( \Pr\{\text{Accept } H_i \mid p\} \) is no greater than \( s\alpha_i \) and is non-decreasing with respect to \( p \in \Theta \) no greater than \( p_i' \).

(V) For \( 0 \leq i \leq m-2 \), \( \Pr\{\text{Accept } H_i \mid p\} \) is no greater than \( s\beta_{i+1} \) and is non-increasing with respect to \( p \in \Theta \) no less than \( p_i'' \).

Moreover, the following statements (VI), (VII) and (VIII) hold true for \( m \geq 3 \).

(VI)

\[
  \Pr\{\text{Reject } H_i \mid p\} \leq \Pr\{\text{Reject } H_i, \hat{p} \leq a \mid a\} + \Pr\{\text{Reject } H_i, \hat{p} \geq b \mid b\},
\]

\[
  \Pr\{\text{Reject } H_i \mid p\} \geq \Pr\{\text{Reject } H_i, \hat{p} \leq a \mid a\} + \Pr\{\text{Reject } H_i, \hat{p} \geq b \mid a\}
\]

for any \( p \in [a, b] \subseteq \Theta_i \) and \( 1 \leq i \leq m-2 \).

(VII) \( \Pr\{\text{Reject } H_0 \text{ and } H_{m-1} \mid p\} \) is non-decreasing with respect to \( p \in \Theta_0 \) and is non-increasing with respect to \( p \in \Theta_{m-1} \).

(VIII) \( \Pr\{\text{Reject } H_0 \text{ and } H_{m-1} \mid p\} \) is no greater than \( s \times \max\{\alpha_i : 1 \leq i \leq m-2\} \) for \( p \in \Theta_0 \) and is no greater than \( s \times \max\{\beta_i : 2 \leq i \leq m-1\} \) for \( p \in \Theta_{m-1} \).

It should be noted that \( p_i', p_i'' \) in Theorem 5 play similar roles as \( \theta_i', \theta_i'' \) in the introduction in defining the requirement of risk control. Accordingly, \( \Theta_i \) in Theorem 5 has the same notion as \( \Theta_i \) in introduction with parameter \( \theta \) identified as \( p \).

Theorem 5 can be shown by using a similar argument as that for Theorem 1 and the following
results obtained by Chen [3],

\[
Pr\left\{ \frac{\sum_{i=1}^{n} X_i}{n} \leq z \mid p \right\} \leq C(n, z, p) \quad \text{for } z \in \left\{ \frac{k}{n} : k \in \mathbb{Z}, \, np \leq k \leq n \right\},
\]

(15)

\[
Pr\left\{ \frac{\sum_{i=1}^{n} X_i}{n} \geq z \mid p \right\} \leq C(n, z, p) \quad \text{for } z \in \left\{ \frac{k}{n} : k \in \mathbb{Z}, \, 0 \leq k \leq np \right\},
\]

(16)

where \( p \in \Theta \) and \( C(n, z, p) \) is defined by (13). Since \( \sum_{i=1}^{n} X_i \) has a hypergeometric distribution, the above inequalities (15) and (16) provide simple bounds for the tail probabilities of hypergeometric distribution, which are substantially less conservative than Hoeffding’s inequalities [7].

3.6.3 Testing the Parameter of a Poisson Distribution

Let \( X \) be a Poisson variable of mean \( \lambda > 0 \). We shall consider the test of hypotheses regarding \( \lambda \) based on i.i.d. random samples \( X_1, X_2, \cdots \) of \( X \). Choosing \( \varphi_n = \frac{\sum_{i=1}^{n} X_i}{n} \) as an estimator for \( \lambda \), we can show that, for \( n \in \mathbb{N}, \, \lambda \in (0, \infty), \, \delta \in (0, 1), \)

\[
\hat{F}(n, \lambda, \delta) = \begin{cases} \frac{1}{n} \times \max \left\{ k \in \mathbb{Z} : \sum_{i=0}^{k} (n\lambda)^{i} e^{-n\lambda} i! \leq \delta, \, k \geq 0 \right\} & \text{for } n \geq \frac{\ln(\delta)}{\lambda}, \\ -\infty & \text{for } n < \frac{\ln(\delta)}{\lambda} \end{cases}
\]

\[
\hat{G}(n, \lambda, \delta) = \frac{1}{n} \times \min \left\{ k \in \mathbb{Z} : \sum_{i=0}^{k-1} (n\lambda)^{i} e^{-n\lambda} i! \geq 1 - \delta, \, k \geq 1 \right\}
\]

\[
f(n, \lambda, \delta) = \begin{cases} \frac{1}{n} \times \max \left\{ k \in \mathbb{Z} : \sum_{i=0}^{k} (n\lambda)^{i} e^{-n\lambda} i! \leq \delta, \, 0 \leq k \leq n\lambda \right\} & \text{for } n \geq \frac{\ln(\delta)}{\lambda}, \\ -\infty & \text{for } n < \frac{\ln(\delta)}{\lambda} \end{cases}
\]

\[
g(n, \lambda, \delta) = \frac{1}{n} \times \min \left\{ k \in \mathbb{Z} : \sum_{i=0}^{k-1} (n\lambda)^{i} e^{-n\lambda} i! \geq 1 - \delta, \, k \geq n\lambda \right\}
\]

and

\[
f_e(n, \lambda, \delta) = \begin{cases} \max \{ z \in [0, \lambda] : M_P(z, \lambda) \leq \frac{\ln(\delta)}{n} \} & \text{for } n \geq \frac{\ln(\delta)}{\lambda}, \\ -\infty & \text{for } n < \frac{\ln(\delta)}{\lambda} \end{cases}
\]

\[
g_e(n, \lambda, \delta) = \min \left\{ z \in [\lambda, \infty) : M_P(z, \lambda) \leq \frac{\ln(\delta)}{n} \right\}
\]

where

\[
M_P(z, \lambda) = \begin{cases} z - \lambda + z \ln \left( \frac{z}{\lambda} \right) & \text{for } z > 0, \\ -\lambda & \text{for } z = 0. \end{cases}
\]

Moreover, it can be verified that the estimator \( \varphi_n \) possesses all properties described at the beginning of Section 3.6. This implies that all testing methods proposed in previous sections are applicable.

3.6.4 Testing the Mean of a Normal Distribution with Known Variance

It is an important problem to test the mean, \( \mu \), of a Gaussian random variable \( X \) with known variance \( \sigma^2 \) based on i.i.d. random samples \( X_1, X_2, \cdots \) of \( X \). Choosing \( \varphi_n = \frac{\sum_{i=1}^{n} X_i}{n} \) as an
estimator of \( \mu \), we have

\[
\hat{F}(n, \mu, \delta) = f(n, \mu, \delta) = \mu - \sigma \frac{Z_\delta}{\sqrt{n}}, \quad \hat{G}(n, \mu, \delta) = g(n, \mu, \delta) = \mu + \sigma \frac{Z_\delta}{\sqrt{n}}
\]

for \( n \in \mathbb{N}, \ \mu \in (-\infty, \infty), \ \delta \in (0, \frac{1}{2}) \). It can be shown that the estimator \( \varphi_n \) possesses all properties described at the beginning of Section 3.6 and consequently, all testing methods proposed in previous sections can be used.

### 3.6.5 Testing the Variance of a Normal Distribution

Let \( X \) be a Gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \). In many applications, it is important to test the variance based on i.i.d. random samples \( X_1, X_2, \ldots \) of \( X \).

In situations that the mean \( \mu \) of the Gaussian variable \( X \) is known, we shall use \( \varphi_n = \sqrt{\frac{1}{n} \sum_{i=1}^{n}(X_i - \mu)^2} \) as an estimator of \( \sigma \). It can be verified that

\[
\hat{F}(n, \sigma, \delta) = \sigma \sqrt{\frac{\chi^2_{n,\delta}}{n}}, \quad \hat{G}(n, \sigma, \delta) = \sigma \sqrt{\frac{\chi^2_{n,1-\delta}}{n}}.
\]

\[
f(n, \sigma, \delta) = \sigma \min \left\{ 1, \sqrt{\frac{\chi^2_{n,\delta}}{n}} \right\}, \quad g(n, \sigma, \delta) = \sigma \max \left\{ 1, \sqrt{\frac{\chi^2_{n,1-\delta}}{n}} \right\}
\]

for \( n \in \mathbb{N}, \ \sigma \in (0, \infty), \ \delta \in (0, 1) \). Moreover, it can be verified that the estimator \( \varphi_n \) possesses all properties described at the beginning of Section 3.6. This implies that all testing methods proposed in previous sections are applicable.

In situations that the mean \( \mu \) of the Gaussian variable \( X \) is unknown, we shall use \( \varphi_n = \sqrt{\frac{1}{n} \sum_{i=1}^{n}(X_i - \bar{X}_n)^2} \), where \( \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n}X_i \), as an estimator of \( \sigma \). To design multistage sampling schemes for testing \( \sigma \), we shall make use of the observation that \( \varphi_n \) is a ULCE of \( \sigma \) and relevant results described in previous sections. By the definition of \( \varphi_n \), it can be readily shown that

\[
f(n, \sigma, \delta) = \sigma \min \left\{ 1, \sqrt{\frac{\chi^2_{n-1,\delta}}{n}} \right\}, \quad g(n, \sigma, \delta) = \sigma \max \left\{ 1, \sqrt{\frac{\chi^2_{n-1,1-\delta}}{n}} \right\}
\]

for \( n \in \mathbb{N}, \ \sigma \in (0, \infty), \ \delta \in (0, 1) \). Let \( \alpha = O(\zeta) \in (0, 1), \ \beta = O(\zeta) \in (0, 1) \) and \( 0 < \sigma' < \sigma'' \). Let \( \pi(\zeta) \) be the minimum integer \( n \) such that \( f(n, \sigma'', \beta) \geq g(n, \sigma', \alpha) \). We can show that

\[
\pi(\zeta) \leq \max \left\{ \frac{2 \ln \alpha}{1 - \sigma''} + \ln \frac{\sigma''}{\sigma'}, \frac{2 \ln \beta}{1 - \sigma''} + \ln \frac{\sigma''}{\sigma'} + 1, \frac{1}{1 - \sigma''} \right\} = O \left( \ln \frac{1}{\zeta} \right).
\]

To show (17), note that \( f(n, \sigma'', \beta) \geq g(n, \sigma', \alpha) \) is equivalent to

\[
\max\{n, \chi^2_{n-1,1-\alpha}\} \leq \left( \frac{\sigma''}{\sigma'} \right)^2 \min\{n, \chi^2_{n-1,\beta}\}.
\]

Let \( Z \) be a chi-square variable of \( n - 1 \) degrees of freedom. Then, \( \Pr\{Z \geq \chi^2_{n-1,1-\alpha}\} = \alpha \) and \( \Pr\{Z \leq \chi^2_{n-1,\beta}\} = \beta \). By Lemma 7 in Appendix E we have

\[
\Pr\{Z \geq (n-1) \left( \frac{\sigma''}{\sigma'} \right)^2 \} \leq \left( \frac{\sigma''}{\sigma'} \right)^2 \exp \left( 1 - \frac{\sigma''}{\sigma'} \right) \leq \alpha
\]
and thus $\chi^2_{n-1, \alpha} \leq (n-1) \left( \frac{\sigma'}{\sigma''} \right)^2$ provided that

$$\frac{n-1}{2} \geq \frac{\ln \alpha}{1 - \frac{\sigma'}{\sigma''} + \ln \frac{\sigma'}{\sigma''}}.$$  

Similarly, by Lemma [1] in Appendix [2] we have

$$\Pr \left\{ Z \leq (n-1) \left( \frac{\sigma'}{\sigma''} \right) \right\} \leq \left[ \left( \frac{\sigma'}{\sigma''} \right) \exp \left( 1 - \frac{\sigma'}{\sigma''} \right) \right]^{(n-1)/2} \leq \beta$$

and thus $\chi^2_{n-1, \beta} \geq (n-1) \left( \frac{\sigma'}{\sigma''} \right)^2$ provided that

$$\frac{n-1}{2} \geq \frac{\ln \beta}{1 - \frac{\sigma'}{\sigma''} + \ln \frac{\sigma'}{\sigma''}}.$$  

It can be seen that a sufficient condition for (18) is

$$n \geq \max \left\{ \frac{2 \ln \alpha}{1 - \frac{\sigma'}{\sigma''} + \ln \frac{\sigma'}{\sigma''} + 1}, \frac{2 \ln \beta}{1 - \frac{\sigma'}{\sigma''} + \ln \frac{\sigma'}{\sigma''} + 1}, \frac{1}{1 - \frac{\sigma'}{\sigma''}} \right\}.$$  

It follows immediately that (17) is true. Making use of (17), we can show that, in the context of testing multiple hypotheses regarding $\sigma$ with our proposed multistage testing plan, the risk of making wrong decisions can be made arbitrarily small by choosing a sufficiently small $\zeta > 0$. Specifically, if we identify parameter $\theta$ in Theorem 1 as $\sigma$, using (17), we can show that

$$\lim_{\zeta \to 0} \Pr \left\{ \text{Reject } H_i \mid \theta \right\} = 0$$

for any $\theta \in \Theta_i$ and $i = 0, 1, \ldots, m-1$.

Our method for the exact computation of the OC function $\Pr \{ \text{Accept } H_0 \mid \sigma \}$ is described as follows. Since $\Pr \{ \text{Accept } H_0 \mid \sigma \} = 1 - \Pr \{ \text{Reject } H_0 \mid \sigma \}$, it suffices to compute $\Pr \{ \text{Reject } H_0 \mid \sigma \}$. By the definition of the testing plan, we have

$$\Pr \{ \text{Reject } H_0 \mid \sigma \} = \sum_{\ell=1}^{s} \Pr \left\{ \varphi_{n_{\ell}} > b_{\ell}, a_j \leq \varphi_{n_j} \leq b_j, 1 \leq j < \ell \mid \sigma \right\}. \quad (19)$$

If we choose the sample sizes to be even numbers $n_{\ell} = 2k_{\ell}$, $\ell = 1, \ldots, s$ for the case of known variance and odd numbers $n_{\ell} = 2k_{\ell} + 1$, $\ell = 1, \ldots, s$ for the case of unknown variance, we can rewrite (19) as

$$\Pr \{ \text{Reject } H_0 \mid \sigma \} = \sum_{\ell=1}^{s} \Pr \left\{ \sum_{q=1}^{k_{\ell}} Z_q \geq \frac{n_{\ell}}{2} \left( \frac{b_{\ell}}{\sigma} \right)^2, \frac{n_{j}}{2} \left( \frac{a_{j}}{\sigma} \right)^2 \leq \sum_{q=1}^{k_{j}} Z_q \leq \frac{n_{j}}{2} \left( \frac{b_{j}}{\sigma} \right)^2 \text{ for } 1 \leq j < \ell \mid \sigma \right\}, \quad (20)$$

where $Z_1, Z_2, \ldots$ are i.i.d. exponential random variables with common mean unity. To compute the probabilities in the right-hand side of (20), we can make use of the following results established by Chen [1].
Theorem 6 Let $1 = k_0 < k_1 < k_2 < \cdots$ be a sequence of positive integers. Let $0 = z_0 < z_1 < z_2 < \cdots$ be a sequence of positive numbers. Define $w(0, 1) = 1$ and

$$w(\ell, 1) = 1, \quad w(\ell, q) = \sum_{i=1}^{k_\ell} \frac{w(r, i) (z_{i - z_r})^{q-i}}{(q-i)!}, \quad k_r < q \leq k_{r+1}, \quad r = 0, 1, \ldots, \ell - 1$$

for $\ell = 1, 2, \ldots$. Let $Z_1, Z_2, \cdots$ be i.i.d. exponential random variables with common mean unity. Then,

$$\Pr\left\{\sum_{q=1}^{k_j} Z_q > z_j \text{ for } j = 1, \ldots, \ell\right\} = e^{-z\ell} \sum_{q=1}^{k_\ell} w(\ell, q)$$

for $\ell = 1, 2, \ldots$. Moreover, the following statements hold true.

(I)

$$\Pr\left\{a_j < \sum_{q=1}^{k_j} Z_q < b_j \text{ for } j = 1, \ldots, \ell\right\} = \left[\sum_{i=1}^{2^{\ell-1}} \Pr\left\{\sum_{q=1}^{k_j} Z_q > [A_i]_{i,j} \text{ for } j = 1, \ldots, \ell\right\}\right] - \left[\sum_{i=1}^{2^{\ell-1}} \Pr\left\{\sum_{q=1}^{k_j} Z_q > [B_i]_{i,j} \text{ for } j = 1, \ldots, \ell\right\}\right],$$

where $A_1 = [a_1], B_1 = [b_1]$ and

$$A_{r+1} = \begin{bmatrix} A_r & a_{r+1}I_{2^{r-1} \times 1} \\ B_r & b_{r+1}I_{2^{r-1} \times 1} \end{bmatrix}, \quad B_{r+1} = \begin{bmatrix} B_r & a_{r+1}I_{2^{r-1} \times 1} \\ A_r & b_{r+1}I_{2^{r-1} \times 1} \end{bmatrix}, \quad r = 1, 2, \ldots,$$

where $I_{2^{r-1} \times 1}$ represents a column matrix with all $2^{r-1}$ elements assuming value 1.

(II)

$$\Pr\left\{a_j < \sum_{q=1}^{k_j} Z_q < b_j \text{ for } j = 1, \ldots, \ell, \sum_{q=1}^{k_{\ell+1}} Z_q > b_{\ell+1}\right\} = \left[\sum_{i=1}^{2^{\ell-1}} \Pr\left\{\sum_{q=1}^{k_j} Z_q > [E]_{i,j} \text{ for } j = 1, \ldots, \ell + 1\right\}\right] - \left[\sum_{i=1}^{2^{\ell-1}} \Pr\left\{\sum_{q=1}^{k_j} Z_q > [F]_{i,j} \text{ for } j = 1, \ldots, \ell + 1\right\}\right],$$

where $E = \begin{bmatrix} A_\ell & b_{\ell+1}I_{2^{\ell-1} \times 1} \end{bmatrix}$ and $F = \begin{bmatrix} B_\ell & b_{\ell+1}I_{2^{\ell-1} \times 1} \end{bmatrix}.$

(III)

$$\Pr\left\{a_j < \sum_{q=1}^{k_j} Z_q < b_j \text{ for } j = 1, \ldots, \ell, \sum_{q=1}^{k_{\ell+1}} Z_q < b_{\ell+1}\right\} = \Pr\left\{a_j < \sum_{q=1}^{k_j} Z_q < b_j \text{ for } j = 1, \ldots, \ell, \sum_{q=1}^{k_{\ell+1}} Z_q < b_{\ell+1}\right\} - \Pr\left\{a_j < \sum_{q=1}^{k_j} Z_q < b_j \text{ for } j = 1, \ldots, \ell, \sum_{q=1}^{k_{\ell+1}} Z_q > b_{\ell+1}\right\}. $$

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3.6.6 Testing the Parameter of an Exponential Distribution

Let \( X \) be a random variable with density function \( f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \) for \( 0 < x < \infty \), where \( \theta \) is a parameter. In many applications, it is important to test the parameter \( \theta \) based on i.i.d. random samples \( X_1, X_2, \cdots \) of \( X \). We shall use \( \varphi_n = \frac{\sum_i x_i}{n} \) as an estimator for \( \theta \). Accordingly, for \( \ell = 1, \cdots, s \), the estimator of \( \theta \) at the \( \ell \)-th stage is \( \hat{\theta}_\ell = \varphi_{n_\ell} = \frac{\sum_i x_i}{n_\ell} \). It can be shown that
\[
\hat{F}(n, \theta, \delta) = \frac{\theta \chi^2_{2n, \delta}}{2n}, \quad \hat{G}(n, \theta, \delta) = \frac{\theta \chi^2_{2n, 1-\delta}}{2n}, \quad f(n, \theta, \delta) = \min \left\{ \frac{1}{\theta} \chi^2_{2n, \delta}, 1 \right\}, \quad g(n, \theta, \delta) = \max \left\{ \frac{1}{\theta} \chi^2_{2n, 1-\delta}, 1 \right\}
\]
for \( n \in \mathbb{N}, \theta \in (0, \infty), \delta \in (0, 1) \). Since the estimator \( \varphi_n \) possesses all properties described at the beginning of Section 3.6, all testing methods proposed in previous sections are applicable. Moreover, it is possible to exactly compute the OC function \( \Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \). Since \( \Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} = 1 - \Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} \), it suffices to compute \( \Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} \). By the definition of the stopping rule, we have
\[
\Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} = \sum_{\ell=1}^s \Pr\left\{ \hat{\theta}_\ell > b_\ell, a_j \leq \hat{\theta}_\ell \leq b_j, 1 \leq j < \ell \mid \theta \right\}. \tag{21}
\]
Let \( Z_1, Z_2, \cdots \) be i.i.d. exponential random variables with common mean unity. Then, we can rewrite (21) as
\[
\Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} = \sum_{\ell=1}^s \Pr\left\{ \sum_{q=1}^{n_\ell} Z_q \geq n_\ell \left( \frac{b_\ell}{\theta} \right), n_j \left( \frac{a_j}{\theta} \right) \leq \sum_{q=1}^{n_j} Z_q \leq n_j \left( \frac{b_j}{\theta} \right) \text{ for } 1 \leq j < \ell \mid \theta \right\}. \tag{22}
\]
To evaluate the probabilities in the right-hand side of (22), we can make use of the results in Theorem 6.

3.6.7 Life Testing

In this section, we shall consider the problem of life testing using the classical exponential model \[5\]. Suppose the lengths of life of all components to be tested can be modeled as i.i.d. random variables with common probability density function \( f_T(t) = \lambda \exp(-\lambda t) \), where the parameter \( \lambda > 0 \) is referred to as the failure rate and its reciprocal is referred to as the mean time between failures. In reliability engineering, it is a central issue to test the failure rate \( \lambda \) based on i.i.d. random samples \( X_1, X_2, \cdots \) of \( X \).

In practice, for purpose of efficiency, multiple components are initially placed on test. The test can be done with or without replacement whenever a component fails. The decision of rejecting, or accepting hypotheses or continuing test is based on the number of failures and the accumulated test time. Here it should be emphasized that the accumulated test time is referred to as the total running time of all components placed on test instead of the real time.

The main idea of existing life-testing plans is to check how much test time has been accumulated whenever a failure occurs. The test plans are designed by truncating the sequential
probability ratio tests (SPRT). There are several drawbacks with existing test plans. First, the existing test plans are limited by the number of hypotheses. Currently, there is no highly effect methods for testing more than two hypotheses. Second, when the indifference zone is narrow, the required accumulated test time may be very long. Third, the specified level of power may not be satisfied due to the truncation of SPRT. Four, the administrative cost may be very high in the situations of high failure rate, since it requires to check the status of test whenever a component fails. To overcome such drawbacks, we can apply the general principle described in previous sections to the life testing problems. We proceed as follows.

Let $\Delta > 0$. Let $Z$ be the number of failures in a time interval of length $\Delta$. Then, $Z$ is a Poisson variable of mean value $\lambda \Delta$. Define $X = \frac{Z}{\Delta}$. The distribution of $X$ is determined as

$$\Pr \left\{ X = \frac{k}{\Delta} \right\} = \frac{(\lambda \Delta)^k e^{-\lambda \Delta}}{k!}, \quad k = 0, 1, 2, \cdots .$$

Let $X_i = \frac{Z_i}{\Delta}$, where $Z_i$ is the number of failures in time interval $[(i-1)\Delta, i\Delta)$ for $i = 1, 2, \cdots$. Then, $X_1, X_2, \cdots$ are i.i.d. samples of $X$. Therefore, we can cast life testing problems in our general framework of multistage hypothesis tests with sample sizes $n_1, n_2, \cdots, n_s$. Accordingly, the testing time is $t_\ell = n_\ell \Delta$, $\ell = 1, \cdots, s$. We shall take $\varphi_n = \sum_{i=1}^{n} X_i$ as an estimator of $\lambda$. It follows that, for $\ell = 1, \cdots, s$, the estimator of $\lambda$ at the $\ell$-th stage is taken as

$$\hat{\lambda}_\ell = \varphi(X_1, \cdots, X_{n_\ell}) = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell} = \frac{\sum_{i=1}^{n_\ell} Z_i}{n_\ell \Delta} = \frac{\text{Number of failures in } [0, t_\ell]}{t_\ell}.$$

Clearly, $\varphi_n$ is a ULCE of $\lambda$; $\varphi_n$ is an unbiased estimator of $\lambda$; the likelihood ratio is monotonically increasing with respect to $\varphi_n$. Hence, the estimator $\varphi_n$ possesses all the properties described at the beginning of Section 3.6. This implies that all testing methods proposed in previous sections are applicable.

It can be seen that all tests described above depend on $\Delta$, the unit of time used to convert the continuous time process to a discrete time process. In applications, it may be preferred to use the test derived by letting $\Delta \to 0$. This can be accomplished as follows. Define

$$\varphi_i = \frac{\text{Number of failures in } [0, t]}{t},$$

$$\hat{F}^0(t, \lambda, \delta) = \begin{cases} \max \{ z \in I_{\varphi_i} : F_{\varphi_i}(z, \lambda) \leq \delta \} & \text{if } \{ F_{\varphi_i}(\varphi_i, \lambda) \leq \delta \} \neq \emptyset, \\ -\infty & \text{otherwise} \end{cases}$$

$$\hat{G}^0(t, \lambda, \delta) = \begin{cases} \min \{ z \in I_{\varphi_i} : G_{\varphi_i}(z, \lambda) \leq \delta \} & \text{if } \{ G_{\varphi_i}(\varphi_i, \lambda) \leq \delta \} \neq \emptyset, \\ \infty & \text{otherwise} \end{cases}$$

$$f^0(t, \lambda, \delta) = \begin{cases} \max \{ z \in I_{\varphi_i} : F_{\varphi_i}(z, \lambda) \leq \delta, z \leq \lambda \} & \text{if } \{ F_{\varphi_i}(\varphi_i, \lambda) \leq \delta, \varphi_i \leq \lambda \} \neq \emptyset, \\ -\infty & \text{otherwise} \end{cases}$$

$$g^0(t, \lambda, \delta) = \begin{cases} \min \{ z \in I_{\varphi_i} : G_{\varphi_i}(z, \lambda) \leq \delta, z \geq \lambda \} & \text{if } \{ G_{\varphi_i}(\varphi_i, \lambda) \leq \delta, \varphi_i \geq \lambda \} \neq \emptyset, \\ \infty & \text{otherwise} \end{cases}$$
Since \( \frac{\ell}{\delta} \rightarrow \Delta \), and find the corresponding limiting numbers defining indifference zones and \( \delta' \), \( \delta'' \) depend on the risk tuning parameter \( \zeta \). Hence, the limit of \( \pi \Delta \) can be determined by virtue of the following observations:

\[
\lim_{\Delta \to 0} \Delta \times \min \{ n \in \mathbb{N} : \hat{F}(n, \lambda'', \delta'') \geq \hat{G}(n, \lambda', \delta') \} = \min \{ t : \hat{F}^0(t, \lambda'', \delta'') \geq \hat{G}^0(t, \lambda', \delta') \},
\]

\[
\lim_{\Delta \to 0} \Delta \times \min \{ n \in \mathbb{N} : f(n, \lambda'', \delta'') \geq g(n, \lambda', \delta') \} = \min \{ t : f^0(t, \lambda'', \delta'') \geq g^0(t, \lambda', \delta') \},
\]

\[
\lim_{\Delta \to 0} \Delta \times \min \{ n \in \mathbb{N} : f_c(n, \lambda'', \delta'') \geq g_c(n, \lambda', \delta') \} = \min \{ t : f_c^0(t, \lambda'', \delta'') \geq g_c^0(t, \lambda', \delta') \}.
\]

If the sample sizes \( n_1, \cdots, n_s \) are chosen as the ascending arrangement of all distinct elements of \( \{ [C_{r-t} \pi] \} \), then we can determine the limit of \( t \) as \( \Delta \to 0 \) as follows.
4 Tests for the Mean of a Normal Distribution with Unknown Variance

In this section, we shall focus on tests for the mean, $\mu$, of a Gaussian variable $X$ with unknown variance $\sigma^2$ based on i.i.d. samples $X_1, X_2, \ldots$ of $X$. Our objective is to develop multistage sampling schemes for testing hypotheses regarding $\theta = \frac{\mu}{\sigma}$, which is the ratio of the mean to the standard deviation.

4.1 General Principle

A general problem regarding $\theta = \frac{\mu}{\sigma}$ is to test $m$ mutually exclusive and exhaustive composite hypotheses:

$$\mathcal{H}_0 : \theta \in \Theta_0, \quad \mathcal{H}_1 : \theta \in \Theta_1, \quad \ldots, \quad \mathcal{H}_{m-1} : \theta \in \Theta_{m-1},$$

where $\Theta_0 = (-\infty, \theta_1], \Theta_{m-1} = (\theta_{m-1}, \infty)$ and $\Theta_i = (\theta_i, \theta_{i+1}]$, $i = 1, \ldots, m-2$ with $\theta_1 < \theta_2 < \cdots < \theta_{m-1}$. To control the probabilities of making wrong decisions, it is typically required that, for pre-specified numbers $\delta_i \in (0, 1)$,

$$\Pr \{ \text{Accept } \mathcal{H}_i \mid \theta \} \geq 1 - \delta_i \quad \forall \theta \in \Theta_i, \quad i = 0, 1, \ldots, m-1$$

with $\Theta_0 = (-\infty, \theta_1], \Theta_{m-1} = (\theta_{m-1}, \infty)$ and $\Theta_i = [\theta_i, \theta_{i+1}]$ for $i = 1, \ldots, m-2$, where $\theta_i, \theta_i''$ satisfy $\theta_i' < \theta_1, \theta_i'' > \theta_{m-1}$ and $\theta_{i-1} < \theta_{i-1}' < \theta_i' < \theta_i'' < \theta_{i+1}' < \theta_{i+1}$ for $i = 2, \ldots, m-2$.

**Theorem 7** Suppose that $\alpha_i = O(\zeta) \in (0, 1)$ and $\beta_i = O(\zeta) \in (0, 1)$ for $i = 1, \ldots, m-1$. Let $n$ be the minimum integer $n$ such that $(\theta_i'' - \theta_i')\sqrt{n-1} \geq t_{n-1, \alpha_i} + t_{n-1, \beta_i}$ for $i = 1, \ldots, m-1$. Let $2 \leq n_1 < n_2 < \cdots < n_s = n$ be the sample sizes. Define

$$f_{\ell,i} = \begin{cases} \theta_i'' - \frac{t_{n_{\ell-1}, \beta_i}}{\sqrt{n_\ell}}, & \text{if } (\theta_i'' - \theta_i')\sqrt{n_{\ell-1}} - 1 < t_{n_{\ell-1}, \alpha_i} + t_{n_{\ell-1}, \beta_i}, \\ \theta_i' + \frac{t_{n_{\ell-1}, \alpha_i}}{2} + \frac{t_{n_{\ell-1}, \alpha_i} - t_{n_{\ell-1}, \beta_i}}{2\sqrt{n_{\ell-1}}}, & \text{if } (\theta_i'' - \theta_i')\sqrt{n_{\ell-1}} - 1 \geq t_{n_{\ell-1}, \alpha_i} + t_{n_{\ell-1}, \beta_i}, \end{cases}$$

$$g_{\ell,i} = \begin{cases} \theta_i' + \frac{t_{n_{\ell-1}, \alpha_i}}{2} + \frac{t_{n_{\ell-1}, \alpha_i} - t_{n_{\ell-1}, \beta_i}}{2\sqrt{n_{\ell-1}}}, & \text{if } (\theta_i'' - \theta_i')\sqrt{n_{\ell-1}} - 1 < t_{n_{\ell-1}, \alpha_i} + t_{n_{\ell-1}, \beta_i}, \\ \theta_i' + \frac{t_{n_{\ell-1}, \alpha_i}}{\sqrt{n_{\ell-1}}} - t_{n_{\ell-1}, \beta_i}, & \text{if } (\theta_i'' - \theta_i')\sqrt{n_{\ell-1}} - 1 \geq t_{n_{\ell-1}, \alpha_i} + t_{n_{\ell-1}, \beta_i}, \end{cases}$$

for $i = 1, \ldots, m-1$. Define

$$\overline{X}_{n_\ell} = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell}, \quad \overline{\sigma}_{n_\ell} = \sqrt{\frac{\sum_{i=1}^{n_\ell} (X_i - \overline{X}_{n_\ell})^2}{n_\ell}}, \quad \widehat{\theta}_\ell = \frac{\overline{X}_{n_\ell}}{\overline{\sigma}_{n_\ell}},$$

$$D_\ell = \begin{cases} 1 & \text{for } \widehat{\theta}_\ell \leq f_{\ell,1}, \\ i & \text{for } g_{\ell,i-1} < \widehat{\theta}_\ell < f_{\ell,i} \text{ where } 2 \leq i \leq m-1, \\ m & \text{for } \widehat{\theta}_\ell > g_{\ell,m-1}, \\ 0 & \text{else} \end{cases} \quad (23)$$

for $\ell = 1, \ldots, s$. Then, the following statements (I)-(V) hold true for $m \geq 2$. 

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(I) \( \Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} \) is non-decreasing with respect to \( \theta \in \Theta_0 \).

(II) \( \Pr\{\text{Reject } \mathcal{H}_{m-1} \mid \theta\} \) is non-increasing with respect to \( \theta \in \Theta_{m-1} \).

(III) \( \Pr\{\text{Reject } \mathcal{H}_i \mid \theta\} \) is no greater than \( \delta_i \) for any \( \theta \in \Theta_i \) and \( i = 0, 1, \cdots, m-1 \) provided that \( \zeta \) is sufficiently small.

(IV) For \( 0 < i \leq m-1 \), \( \Pr\{\text{Accept } \mathcal{H}_i \mid \theta\} \) is non-decreasing with respect to \( \theta \) no greater than \( \theta_i' \).

(V) For \( 0 \leq i \leq m-2 \), \( \Pr\{\text{Accept } \mathcal{H}_i \mid \theta\} \) is non-increasing with respect to \( \theta \) no less than \( \theta_{i+1}' \).

Moreover, the following statements (VI), (VII) and (VIII) hold true for \( m \geq 3 \).

(VI)
\[
\Pr\{\text{Reject } \mathcal{H}_i \mid \theta\} \leq \Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \leq a \mid a\} + \Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \geq b \mid b\},
\]
\[
\Pr\{\text{Reject } \mathcal{H}_i \mid \theta\} \geq \Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \leq a \mid b\} + \Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \geq b \mid a\}
\]
for any \( \theta \in [a, b] \subseteq \Theta_i \) and \( 1 \leq i \leq m-2 \).

(VII) \( \Pr\{\text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_{m-1} \mid \theta\} \) is non-decreasing with respect to \( \theta \in \Theta_0 \) and is non-increasing with respect to \( \theta \in \Theta_{m-1} \).

(VIII) For any pre-specified \( \delta \in (0, 1) \), \( \Pr\{\text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_{m-1} \mid \theta\} \) is no greater than \( \delta \) for \( \theta \in \Theta_0 \cup \Theta_{m-1} \) provided that \( \zeta \) is sufficiently small.

See Appendix E for a proof. By virtue of Theorem 7 and similar ideas as described after Theorem 2, we can develop bisection risk tuning techniques for designing multistage test plans. For risk tuning purpose, we can choose \( \alpha_i = \zeta \delta_{i-1} \) and \( \beta_i = \zeta \delta_i \) with \( \zeta \in (0, 1) \) for \( i = 1, \cdots, m-1 \).

4.2 Applications

In this section, we shall study the applications of Theorem 7 to specific testing problems. Specially, the following Sections 4.2.1, 4.2.2 and 4.2.3 are devoted to the discussion of hypotheses concerned with the comparison of the mean \( \mu \) of Gaussian variable \( X \) with a pre-specified number \( \gamma \). Such issues can be formulated as problems of testing hypotheses regarding \( \vartheta = \frac{\mu - \gamma}{\sigma} \). To develop concrete testing plans, we make use of the following statistics
\[
\bar{X}_{n_\ell} = \frac{1}{n_\ell} \sum_{i=1}^{n_\ell} X_i, \quad \hat{\sigma}_{n_\ell} = \sqrt{\frac{\sum_{i=1}^{n_\ell} (X_i - \bar{X}_{n_\ell})^2}{n_\ell - 1}}, \quad \hat{T}_\ell = \frac{\sqrt{n_\ell} (\bar{X}_{n_\ell} - \gamma)}{\hat{\sigma}_{n_\ell}}
\]
for \( \ell = 1, \cdots, s \).

4.2.1 One-sided Tests

In many situations, it is an important problem to test hypotheses \( \mathcal{H}_0 : \vartheta < 0 \) versus \( \mathcal{H}_1 : \vartheta > 0 \). To control the risks of committing decision errors, it is typically required that, for prescribed
numbers $\alpha, \beta \in (0,1)$,

\[
\Pr \{ \text{Accept } \mathcal{H}_0 \mid \vartheta \} > 1 - \alpha \quad \text{for } \vartheta \leq -\varepsilon, \\
\Pr \{ \text{Accept } \mathcal{H}_1 \mid \vartheta \} > 1 - \beta \quad \text{for } \vartheta \geq \varepsilon,
\]

where the indifference zone is $(-\varepsilon, \varepsilon)$. Applying Theorem 7 to the special case of $m = 2$, we have the following results.

**Corollary 11** Let $\alpha = O(\zeta) \in (0,1)$ and $\beta = O(\zeta) \in (0,1)$. Let $n$ be the minimum integer $n$ such that $t_{n-1,\alpha} + t_{n-1,\beta} \leq 2\varepsilon\sqrt{n-1}$. Let $2 \leq n_1 < n_2 < \cdots < n_s = n$ be the sample sizes. Define $a_\ell = \varepsilon\sqrt{n_\ell - 1} - t_{n_\ell-1,\alpha}$, $b_\ell = t_{n_\ell-1,\alpha} - \varepsilon\sqrt{n_\ell - 1}$ for $\ell = 1, \cdots, s-1$, and $a_s = b_s = \frac{t_{n_s-1,\alpha} - t_{n_s-1,\beta}}{2}$. Define

\[
D_\ell = \begin{cases} 
1 & \text{for } \hat{T}_\ell \leq a_\ell, \\
2 & \text{for } \hat{T}_\ell > b_\ell, \\
0 & \text{else}
\end{cases}
\]

for $\ell = 1, \cdots, s$. Then, the following statements hold true.

(i) $\Pr \{ \text{Accept } \mathcal{H}_0 \mid \vartheta \}$ is less than $\beta$ for $\vartheta$ no less than $\varepsilon$ if $\zeta > 0$ is sufficiently small.

(ii) $\Pr \{ \text{Reject } \mathcal{H}_0 \mid \vartheta \}$ is less than $\alpha$ for $\vartheta$ no greater than $-\varepsilon$ if $\zeta > 0$ is sufficiently small.

(iii) The OC function $\Pr \{ \text{Accept } \mathcal{H}_0 \mid \vartheta \}$ is monotonically decreasing with respect to $\vartheta \in (-\infty, -\varepsilon] \cup [\varepsilon, \infty)$.

For the sake of risk tuning, we recommend choosing $\alpha = \zeta\alpha$ and $\beta = \zeta\beta$, where $\zeta \in (0,1)$.

**4.2.2 Two-sided Tests**

It is a frequent problem to test hypotheses $\mathcal{H}_0 : \vartheta = 0$ versus $\mathcal{H}_1 : \vartheta \neq 0$. To control the risks of committing decision errors, it is typically required that, for prescribed numbers $\alpha, \beta \in (0,1)$,

\[
\Pr \{ \text{Accept } \mathcal{H}_0 \mid \vartheta \} > 1 - \alpha \quad \text{for } \vartheta = 0, \\
\Pr \{ \text{Accept } \mathcal{H}_1 \mid \vartheta \} > 1 - \beta \quad \text{for } |\vartheta| \geq \varepsilon,
\]

where the indifference zone is $(-\varepsilon, 0) \cup (0, \varepsilon)$. Applying Theorem 7 to test hypotheses $\mathcal{H}_0 : \vartheta \leq -\frac{\varepsilon}{2}$, $\mathcal{H}_1 : -\frac{\varepsilon}{2} < \vartheta \leq \frac{\varepsilon}{2}$ and $\mathcal{H}_2 : \vartheta > \frac{\varepsilon}{2}$ with indifference zone $(-\varepsilon, 0) \cup (0, \varepsilon)$, we have $\Pr \{ \text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_2 \mid \vartheta \} = \Pr \{ \text{Accept } \mathcal{H}_0 \mid \vartheta \}$ and the following results follow immediately.

**Corollary 12** Let $\alpha = O(\zeta) \in (0,1)$ and $\beta = O(\zeta) \in (0,1)$. Let $n$ be the minimum integer $n$ such that $t_{n-1,\alpha} + t_{n-1,\beta} \leq 2\varepsilon\sqrt{n-1}$. Let $2 \leq n_1 < n_2 < \cdots < n_s = n$ be the sample sizes. Define $a_\ell = \varepsilon\sqrt{n_\ell - 1} - t_{n_\ell-1,\alpha}$, $b_\ell = t_{n_\ell-1,\alpha} - \varepsilon\sqrt{n_\ell - 1}$ for $\ell = 1, \cdots, s-1$, and $a_s = b_s = \frac{t_{n_s-1,\alpha} - t_{n_s-1,\beta}}{2} + \varepsilon\sqrt{n_s - 1}$. Define

\[
D_\ell = \begin{cases} 
1 & \text{for } |\hat{T}_\ell| \leq a_\ell, \\
2 & \text{for } |\hat{T}_\ell| > b_\ell, \\
0 & \text{else}
\end{cases}
\]
for $\ell = 1, \ldots, s$. Then, the following statements hold true.

(i) $\Pr\{\text{Accept } \mathcal{H}_0 \mid \vartheta\} > 1 - \beta$ for any $\vartheta \in (\mathcal{A}, \mathcal{B})$ if $\zeta > 0$ is sufficiently small.

(ii) $\Pr\{\text{Reject } \mathcal{H}_0 \mid \vartheta\} > 1 - \alpha$ for $\vartheta = 0$ if $\zeta > 0$ is sufficiently small.

(iii) The OC function $\Pr\{\text{Accept } \mathcal{H}_0 \mid \vartheta\}$ is monotonically increasing with respect to $\vartheta \in (\mathcal{A}, \mathcal{B})$ and is monotonically decreasing with respect to $\vartheta \in [\mathcal{A}, \mathcal{B}]$.

For the purpose of risk tuning, we recommend choosing $\alpha = \frac{\zeta_\alpha}{2}$ and $\beta = \zeta \beta$, where $\zeta \in (0, 1)$.

### 4.2.3 Tests of Triple Hypotheses

In many applications, it is desirable to test three hypotheses

\[\mathcal{H}_0 : \vartheta < 0, \quad \mathcal{H}_1 : \vartheta = 0, \quad \mathcal{H}_2 : \vartheta > 0.\]

To control the risks of committing decision errors, it is typically required that, for prescribed numbers $\alpha$, $\beta \in (0, 1)$,

\[
\begin{align*}
&\Pr\{\text{Accept } \mathcal{H}_0 \mid \vartheta\} > 1 - \beta \quad \text{for } \vartheta \leq \mathcal{A}, \\
&\Pr\{\text{Accept } \mathcal{H}_1 \mid \vartheta\} > 1 - \alpha \quad \text{for } \vartheta = 0, \\
&\Pr\{\text{Accept } \mathcal{H}_2 \mid \vartheta\} > 1 - \beta \quad \text{for } \vartheta \geq \mathcal{B},
\end{align*}
\]

where the indifference zone is $(\mathcal{A},0) \cup (0,\mathcal{B})$. Applying Theorem 7 to test hypotheses $\mathcal{H}_0 : \vartheta \leq -\frac{\mathcal{B}}{2}$, $\mathcal{H}_1 : -\frac{\mathcal{B}}{2} < \vartheta \leq \frac{\mathcal{B}}{2}$ and $\mathcal{H}_2 : \vartheta > \frac{\mathcal{B}}{2}$ with indifference zone $(\mathcal{A},0) \cup (0,\mathcal{B})$, we have the following results.

**Corollary 13** Let $\alpha = O(\zeta) \in (0, 1)$ and $\beta = O(\zeta) \in (0, 1)$. Let $\overline{n}$ be the minimum integer $n$ such that $t_{n-1,\alpha} + t_{n-1,\beta} \leq \sqrt{n-1}$. Let $2 \leq n_1 < n_2 < \cdots < n_s = \overline{n}$ be the sample sizes. Define $a_\ell = \sqrt{n_\ell - 1} - t_{n_\ell - 1,\alpha}$ for $\ell = 1, \ldots, s-1$, and $a_s = t_{n_s - 1,\alpha}$. Define

\[
D_\ell = \begin{cases} 
1 & \text{for } \hat{T}_\ell < -b_\ell, \\
2 & \text{for } |\hat{T}_\ell| \leq a_\ell, \\
3 & \text{for } \hat{T}_\ell > b_\ell, \\
0 & \text{else}
\end{cases}
\]

for $\ell = 1, \ldots, s$. Then, the following statements hold true.

(i) $\Pr\{\text{Accept } \mathcal{H}_0 \mid \vartheta\} > 1 - \beta$ for any $\vartheta \in (\mathcal{A}, \mathcal{B})$ if $\zeta > 0$ is sufficiently small. Moreover, $\Pr\{\text{Accept } \mathcal{H}_0 \mid \vartheta\}$ is monotonically decreasing with respect to $\vartheta \in (\mathcal{A}, \mathcal{B})$.

(ii) $\Pr\{\text{Accept } \mathcal{H}_2 \mid \vartheta\} > 1 - \beta$ for any $\vartheta \in [\mathcal{B}, \mathcal{B}]$ if $\zeta > 0$ is sufficiently small. Moreover, $\Pr\{\text{Accept } \mathcal{H}_2 \mid \vartheta\}$ is monotonically increasing with respect to $\vartheta \in [\mathcal{B}, \mathcal{B}]$.

(iii) $\Pr\{\text{Accept } \mathcal{H}_1 \mid \vartheta\}$ for $\vartheta = 0$ is greater than $1 - \alpha$ if $\zeta > 0$ is sufficiently small.

For the purpose of risk tuning, we recommend choosing $\alpha = \frac{\zeta_\alpha}{2}$ and $\beta = \zeta \beta$, where $\zeta \in (0, 1)$. 30
4.2.4 Interval Tests

In some situations, it is desirable to test hypothesis \( \mathcal{H}_0 : \theta \in [\theta_1, \theta_2] \) versus \( \mathcal{H}_1 : \theta \notin [\theta_1, \theta_2] \). For risk control purpose, it is typically required that, for two prescribed numbers \( \alpha, \beta \in (0, 1) \),

\[
\begin{align*}
\Pr\{\text{Reject } \mathcal{H}_0 \mid \theta \} &\leq \alpha \quad \text{for } \theta \in \Theta \text{ such that } \theta \in [\theta_1'', \theta_2''], \\
\Pr\{\text{Accept } \mathcal{H}_0 \mid \theta \} &\leq \beta \quad \text{for } \theta \in \Theta \text{ such that } \theta \notin (\theta_1', \theta_2') ,
\end{align*}
\]

where \( \theta_1 < \theta_1' < \theta_1'' < \theta_2 < \theta_2' < \theta_2'' \). Since there is no requirement imposed on probabilities of committing decision errors for \( \theta \in (\theta_1', \theta_1'') \cup (\theta_2', \theta_2'') \), the union of intervals, \( (\theta_1', \theta_1'') \cup (\theta_2', \theta_2'') \), is referred to as an indifference zone.

Applying Theorem 7 to test hypotheses \( \mathcal{H}_0 : \theta \leq \theta_1 \), \( \mathcal{H}_1 : \theta_1 < \theta \leq \theta_2 \) and \( \mathcal{H}_2 : \theta > \theta_2 \) with indifference zone \( (\theta_1', \theta_1'') \cup (\theta_2', \theta_2'') \), we have \( \Pr\{\text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_2 \mid \theta\} = \Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \) and the following results.

**Corollary 14** Let \( \alpha_i = O(\zeta) \in (0, 1) \) and \( \beta_i = O(\zeta) \in (0, 1) \) for \( i = 1, 2 \). Let \( \bar{n} \) be the minimum integer \( n \) such that \( (\theta_1'' - \theta_1')\sqrt{n - 1} \geq t_{n-1, \alpha_i} + t_{n-1, \beta_i} \) for \( i = 1, 2 \). Let \( 2 \leq n_1 < n_2 < \cdots < n_s = \bar{n} \) be the sample sizes. Define

\[
\ell, i = \begin{cases} \theta_1'' - t_{n_{\ell-1, \alpha_i}} \sqrt{n_\ell - 1}, & \text{if } (\theta_1'' - \theta_1')\sqrt{n_\ell - 1} < t_{n_{\ell-1, \alpha_i}} + t_{n_{\ell-1, \beta_i}}, \\
\theta_1' + t_{n_{\ell-1, \alpha_i}} \sqrt{n_\ell - 1}, & \text{if } (\theta_1'' - \theta_1')\sqrt{n_\ell - 1} \geq t_{n_{\ell-1, \alpha_i}} + t_{n_{\ell-1, \beta_i}} \
\end{cases}
\]

\[
g_{\ell, i} = \begin{cases} \theta_2'' - \theta_1' + t_{n_{\ell-1, \alpha_i}} \sqrt{n_\ell - 1}, & \text{if } (\theta_2'' - \theta_1')\sqrt{n_\ell - 1} < t_{n_{\ell-1, \alpha_i}} + t_{n_{\ell-1, \beta_i}}, \\
\theta_2' + t_{n_{\ell-1, \alpha_i}} \sqrt{n_\ell - 1}, & \text{if } (\theta_2'' - \theta_1')\sqrt{n_\ell - 1} \geq t_{n_{\ell-1, \alpha_i}} + t_{n_{\ell-1, \beta_i}} \
\end{cases}
\]

for \( \ell = 1, 2 \). Define

\[
D_{\ell} = \begin{cases} 1 & \text{if } g_{\ell, 1} < \hat{\theta}_\ell \leq f_{\ell, 2}, \\
2 & \text{if } \hat{\theta}_\ell \leq f_{\ell, 1} \text{ or } \hat{\theta}_\ell > g_{\ell, 2}, \\
0 & \text{else}
\end{cases}
\]

for \( \ell = 1, \cdots, s \). Then, the following statements hold true.

(i) \( \Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \leq \beta \) for \( \theta \notin (\theta_1', \theta_2') \) if \( \zeta \) is sufficiently small.

(ii) \( \Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} \leq \alpha \) for \( \theta \in [\theta_1'', \theta_2''] \) if \( \zeta \) is sufficiently small.

(iii) \( \Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \) is non-decreasing with respect to \( \theta \) no greater than \( \theta_1' \) and is non-increasing with respect to \( \theta \) no less than \( \theta_2'' \). Moreover,

\[
\begin{align*}
\Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} &\leq \Pr\{\text{Reject } \mathcal{H}_0, \hat{\theta} \leq a \mid a\} + \Pr\{\text{Reject } \mathcal{H}_0, \hat{\theta} \geq b \mid b\}, \\
\Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} &\geq \Pr\{\text{Reject } \mathcal{H}_0, \hat{\theta} \leq a \mid b\} + \Pr\{\text{Reject } \mathcal{H}_0, \hat{\theta} \geq b \mid a\}
\end{align*}
\]

for any \( \theta \in [a, b] \subseteq [\theta_1'', \theta_2''] \).

We can choose \( \alpha_2 = \beta_1 = \zeta \alpha \) and \( \alpha_1 = \beta_2 = \zeta \beta \) with \( \zeta \in (0, 1) \) for risk tuning purpose.
4.2.5 Tests of “Simple” Hypotheses

In some situations, it may be interesting to test multiple simple hypotheses \( \mathcal{H}_i : \theta = \theta_i \) for \( i = 0, 1, \ldots, m - 1 \). For risk control purpose, it is typically required that, for prescribed numbers \( \delta_i \in (0, 1) \),

\[
\Pr \{ \text{Accept } \mathcal{H}_i \mid \theta_i \} \geq 1 - \delta_i, \quad i = 0, 1, \ldots, m - 1.
\]

Applying Theorem 7 to test the following hypotheses

\[
\mathcal{H}_0 : \theta \leq \vartheta_1, \quad \mathcal{H}_1 : \vartheta_1 < \theta \leq \vartheta_2, \quad \ldots, \quad \mathcal{H}_{m-2} : \vartheta_{m-2} < \theta \leq \vartheta_{m-1}, \quad \mathcal{H}_{m-1} : \theta > \vartheta_{m-1}
\]

with \( \vartheta_i = \frac{\theta_{i-1} + \theta_i}{2}, \quad i = 1, \ldots, m - 1 \) and indifference zone \( \bigcup_{i=1}^{m-1} (\theta_{i-1}, \theta_i) \), we have the following results.

**Corollary 15** Let \( \alpha_i = O(\zeta) \in (0, 1) \) and \( \beta_i = O(\zeta) \in (0, 1) \) for \( i = 1, \ldots, m - 1 \). Let \( \Pi \) be the minimum integer \( n \) such that \( (\theta_i - \theta_{i-1})\sqrt{n-1} \geq t_{n-1,\alpha_i} + t_{n-1,\beta_i} \) for \( i = 1, \ldots, m - 1 \). Let \( 2 \leq n_1 < n_2 < \cdots < n_s = \Pi \) be the sample sizes. Define

\[
f_{\ell,i} = \begin{cases} 
\theta_i - \frac{t_{n_{\ell-1},\beta_i}}{\sqrt{n_{\ell-1}} - 1} & \text{if } (\theta_i - \theta_{i-1})\sqrt{n_{\ell-1}} - 1 < t_{n_{\ell-1},\alpha_i} + t_{n_{\ell-1},\beta_i}, \\
\theta_i + \frac{t_{n_{\ell-1},\alpha_i} - t_{n_{\ell-1},\beta_i}}{2} & \text{if } (\theta_i - \theta_{i-1})\sqrt{n_{\ell-1}} - 1 \geq t_{n_{\ell-1},\alpha_i} + t_{n_{\ell-1},\beta_i}
\end{cases}
\]

\[
g_{\ell,i} = \begin{cases} 
\theta_{i-1} + \frac{t_{n_{\ell-1},\alpha_i}}{\sqrt{n_{\ell-1}} - 1} & \text{if } (\theta_i - \theta_{i-1})\sqrt{n_{\ell-1}} - 1 < t_{n_{\ell-1},\alpha_i} + t_{n_{\ell-1},\beta_i}, \\
\theta_i + \frac{t_{n_{\ell-1},\alpha_i} - t_{n_{\ell-1},\beta_i}}{2} & \text{if } (\theta_i - \theta_{i-1})\sqrt{n_{\ell-1}} - 1 \geq t_{n_{\ell-1},\alpha_i} + t_{n_{\ell-1},\beta_i}
\end{cases}
\]

for \( i = 1, \ldots, m - 1 \). Define \( f_{\ell,i} = f(n_{\ell}, \theta_{i-1}, \theta_i, \alpha_i, \beta_i) \) and \( g_{\ell,i} = g(n_{\ell}, \theta_{i-1}, \theta_i, \alpha_i, \beta_i) \) for \( i = 1, \ldots, m - 1 \). Define decision variable \( D_{\ell} \) by \((23)\) for \( \ell = 1, \ldots, s \). Then, \( \Pr\{\text{Reject } \mathcal{H}_i \mid \theta_i\} \leq \delta_i \) for \( i = 0, 1, \ldots, m - 1 \) if \( \zeta \) is sufficiently small.

For risk tuning purpose, we recommend choosing \( \alpha_i = \zeta \delta_{i-1} \) and \( \beta_i = \zeta \delta_i \) for \( i = 1, \ldots, m - 1 \), where \( \zeta \in (0, 1) \).

5 Conclusion

In this paper, we have established a new framework of multistage hypothesis tests which applies to arbitrary number of mutually exclusive and exhaustive composite hypotheses. Specific testing plans for common problems have also been developed. Our test plans have several important advantages upon existing tests. First, our tests are more efficient. Second, our tests always guarantee prescribed requirement of power. Third, the sample number or test time of our tests are absolutely bounded. Such advantages have been achieved by means of new structure of testing plans and powerful computational machinery.
A Preliminary Results

We need some preliminary results. The following Lemmas 1 and 2 have been established in [1].

**Lemma 1** $\Pr\{F_Z(Z) \leq \alpha\} \leq \alpha$ and $\Pr\{G_Z(Z) \leq \alpha\} \leq \alpha$ for any random variable $Z$ and positive number $\alpha$.

**Lemma 2** Let $\mathcal{E}$ be an event determined by random tuple $(X_1, \cdots, X_r)$. Let $\varphi(X_1, \cdots, X_r)$ be a ULE of $\theta$. Then,

(i) $\Pr\{\mathcal{E} \mid \theta\}$ is non-increasing with respect to $\theta \in \Theta$ no less than $z$ provided that $\mathcal{E} \subseteq \{\varphi(X_1, \cdots, X_r) \leq z\}$.

(ii) $\Pr\{\mathcal{E} \mid \theta\}$ is non-decreasing with respect to $\theta \in \Theta$ no greater than $z$ provided that $\mathcal{E} \subseteq \{\varphi(X_1, \cdots, X_r) \geq z\}$.

**Lemma 3** Let $X_1, X_2, \cdots, X_r$ be a sequence of samples of random variable $X$ parameterized by $\theta \in \Theta$. Let $Z = \varphi(X_1, \cdots, X_r)$ be an unbiased and unimodal-likelihood estimator of $\theta$. Suppose that the moment generating function $\mathcal{M}(\rho, \theta) = \mathbb{E}[e^{\rho Z}]$ of $Z$ exists for any $\rho \in \mathbb{R}$. Define $\mathcal{C}(z, \theta) = \inf_{\rho \in \mathbb{R}} e^{-\rho z} \mathbb{E}[e^{\rho Z}]$. Then,

$$\Pr\{Z \leq z\} \leq \mathcal{C}(z, \theta), \quad \forall z \leq \theta$$

$$\Pr\{Z \geq z\} \leq \mathcal{C}(z, \theta), \quad \forall z \geq \theta.$$

Moreover, $\mathcal{C}(z, \theta)$ is non-decreasing with respect to $\theta$ no greater than $z$ and is non-increasing with respect to $\theta$ no less than $z$. Similarly, $\mathcal{C}(z, \theta)$ is non-decreasing with respect to $z$ no greater than $\theta$ and is non-increasing with respect to $z$ no less than $\theta$.

**Proof.** By the convexity of function $e^x$ and Jensen’s inequality, we have $\inf_{\rho \geq 0} \mathbb{E}[e^{\rho(Z-z)}] = \mathbb{E}[e^{\rho(Z)}}] \geq e^{\rho\mathbb{E}[Z-Z]} \geq 1$ for $\theta \geq z$. In view of $\inf_{\rho \leq 0} \mathbb{E}[e^{\rho(Z-z)}] \leq 1$, we have $\mathcal{C}(z, \theta) = \inf_{\rho \leq 0} \mathbb{E}[e^{\rho(Z)}}]$ for $\theta \geq z$. Clearly, $\mathcal{C}(z, \theta) = \inf_{\rho \leq 0} e^{-\rho z} \mathbb{E}[e^{\rho Z}]$ is non-decreasing with respect to $z$ less than $\theta$. Since $Z$ is a ULE of $\theta$, we have that $\mathbb{E}[e^{\rho(Z-z)}] = e^{-\rho z} \mathbb{E}[e^{\rho Z}] = e^{-\rho z} \int_{u=0}^{\infty} \Pr\{e^{\rho Z} > u\} du$ is non-increasing with respect to $\theta$ for $\rho \leq 0$ and thus $\mathcal{C}(z, \theta)$ is non-increasing with respect to $\theta$ greater than $z$.

Observing that $\inf_{\rho \geq 0} \mathbb{E}[e^{\rho(Z-z)}] \leq 1$ and that $\inf_{\rho \leq 0} \mathbb{E}[e^{\rho(Z-z)}] \geq e^{-\rho u} \mathbb{E}[e^{\rho Z}] \geq 1$ for $\theta < z$, we have $\mathcal{C}(z, \theta) = \inf_{\rho \geq 0} \mathbb{E}[e^{\rho(Z)}}]$ for $\theta < z$. Clearly, $\mathcal{C}(z, \theta) = \inf_{\rho \geq 0} e^{-\rho z} \mathbb{E}[e^{\rho Z}]$ is non-increasing with respect to $z$ greater than $\theta$. Since $Z$ is a ULE of $\theta$, we have that $\mathbb{E}[e^{\rho(Z-z)}] = e^{-\rho z} \int_{u=0}^{\infty} \Pr\{e^{\rho Z} > u\} du$ is non-decreasing with respect to $\theta$ for $\rho > 0$ and consequently, $\mathcal{C}(z, \theta)$ is non-decreasing with respect to $\theta$ smaller than $z$.

Making use of the established fact $\inf_{\rho \leq 0} \mathbb{E}[e^{\rho(Z)}}] = \mathcal{C}(z, \theta)$ and the Chernoff bound $\Pr\{Z \leq z\} \leq \inf_{\rho \leq 0} \mathbb{E}[e^{\rho(Z)}}]$, we have $\Pr\{Z \leq z\} \leq \mathcal{C}(z, \theta)$ for $z \leq \theta$. Making use of the established fact $\inf_{\rho \geq 0} \mathbb{E}[e^{\rho(Z)}}] = \mathcal{C}(z, \theta)$ and the Chernoff bound $\Pr\{Z \geq z\} \leq \inf_{\rho \geq 0} \mathbb{E}[e^{\rho(Z)}}]$, we have $\Pr\{Z \geq z\} \leq \mathcal{C}(z, \theta)$ for $z \geq \theta$. This concludes the proof of Lemma 3. □
B Proof of Theorem 1

For arbitrary parametric values $\theta_0 < \theta_1$ in $\Theta$, by the assumption that $\varphi_n$ converges in probability to $\theta$, we have that $\Pr\{\varphi_n \geq \frac{\theta_0 + \theta_1}{2} \mid \theta_0\} \leq \Pr\{\varphi_n \geq \frac{\theta_1 - \theta_0}{2} \mid \theta_0\} \rightarrow 0$ and $\Pr\{\varphi_n \leq \frac{\theta_1 + \theta_0}{2} \mid \theta_1\} \leq \Pr\{\varphi_n \leq \frac{\theta_1 - \theta_0}{2} \mid \theta_1\} \rightarrow 0$ as $n \rightarrow \infty$. This shows that $\pi$ exists and is finite.

Since $F_{\widehat{\theta}_i}(z, \theta) = \Pr\{\widehat{\theta}_i \leq z \mid \theta\} = 1 - \Pr\{\widehat{\theta}_i > z \mid \theta\}$, making use of Lemma 2 and the assumption that $F_{\widehat{\theta}_i}$ is a ULE of $\theta$, we have that $F_{\widehat{\theta}_i}(z, \theta)$ is non-decreasing with respect to $\theta \in \Theta$. Similarly, since $G_{\widehat{\theta}_i}(z, \theta) = \Pr\{\widehat{\theta}_i \leq z \mid \theta\} = 1 - \Pr\{\widehat{\theta}_i < z \mid \theta\}$, making use of Lemma 2 and the assumption that $G_{\widehat{\theta}_i}$ is a ULE of $\theta$, we have that $G_{\widehat{\theta}_i}(z, \theta)$ is non-decreasing with respect to $\theta \in \Theta$.

To show statement (I), notice that $\{\text{Reject } \mathscr{H}_0\} \subseteq \{\widehat{\theta} \geq \theta_1'\}$ as a consequence of the definition of the test plan. Hence, statement (I) is proved by virtue of Lemma 2.

To show statement (II), notice that $\{\text{Reject } \mathscr{H}_{m-1}\} \subseteq \{\widehat{\theta} \leq \theta_m''\}$ as a consequence of the definition of the test plan. Hence, statement (II) is proved by virtue of Lemma 2.

To show statement (III), we first claim that $\Pr\{1 \leq D_{\ell} \leq i \mid \theta\} \geq \overline{\beta}_i$ for $0 \leq i \leq m - 1$ and $\theta \in \Theta_i$. Clearly, $\{\widehat{\theta}_i \leq f_{\ell,j}\} = \{\widehat{\theta}_i \leq f(n_{\ell}, \theta_j', \theta_j'', \alpha_j, \beta_j)\} \subseteq \{\widehat{\theta}_i \leq f(n_{\ell}, \theta_j'', \beta_j)\}$ for $1 \leq j \leq i$. Since $F_{\widehat{\theta}_i}(z, \theta)$ is non-decreasing with respect to $z$, we have $\{\widehat{\theta}_i \leq f(n_{\ell}, \theta_j', \beta_j)\} \subseteq \{\widehat{\theta}_i \leq \theta_j'\}$, $F_{\widehat{\theta}_i}(\theta_j, \theta_j') \leq \beta_j \} \subseteq \{F_{\widehat{\theta}_i}(\theta_j, \theta_j') \leq \beta_j\} \subseteq \{F_{\widehat{\theta}_i}(\theta_j, \theta_j'') \leq \beta_j\} \subseteq \{\widehat{\theta}_j \leq f_{\ell,j}\}$. Recalling that $F_{\widehat{\theta}_i}(z, \theta)$ is non-increasing with respect to $\theta \in \Theta$ and invoking Lemma 1, we have

$$\Pr\{\widehat{\theta}_i \leq f_{\ell,j} \mid \theta\} \leq \Pr\{F_{\widehat{\theta}_i}(\theta_j, \theta_j'') \leq \beta_j \mid \theta\} \leq \Pr\{F_{\widehat{\theta}_i}(\theta_j, \theta) \leq \beta_j \mid \theta\} \leq \beta_j \leq \overline{\beta}_i$$ (24)

for $1 \leq j \leq i$ and $\theta \in \Theta_i$. For $i = 0$, it is clear that $\Pr\{1 \leq D_{\ell} \leq i \mid \theta\} = 0 \leq \overline{\beta}_0$ for $\theta \in \Theta_0$. For $i = 1$, by virtue of (24), we have $\Pr\{1 \leq D_{\ell} \leq i \mid \theta\} = \Pr\{\widehat{\theta}_{\ell} \leq f_{\ell,1} \mid \theta\} \leq \overline{\beta}_1$ for $\theta \in \Theta_1$.

For $2 \leq i \leq m - 1$, define $S = \{j : g_{\ell,j-1} < f_{\ell,j}, 2 \leq j \leq i\}$ and let $r$ be an integer such that $r$ assumes value 1 if $S$ is empty and that $r \in S$, $f_{\ell,r} = \max\{f_{\ell,j} : j \in S\}$ if $S$ is not empty. It follows from (24) that $\Pr\{1 \leq D_{\ell} \leq i \mid \theta\} \leq \Pr\{\widehat{\theta}_{\ell} \leq f_{\ell,r} \mid \theta\} \leq \overline{\beta}_i$ for $2 \leq i \leq m - 1$ and $\theta \in \Theta_i$. This proves our first claim. Next, we claim that $\Pr\{i + 2 \leq D_{\ell} \leq m \mid \theta\} \leq \overline{\theta}_i$ for $0 \leq i \leq m - 1$ and $\theta \in \Theta_i$. Clearly, $\{\widehat{\theta}_{\ell} \leq g_{\ell,j}\} = \{\widehat{\theta}_{\ell} \leq g(n_{\ell}, \theta_j', \theta_j'', \alpha_j, \beta_j)\} \subseteq \{\widehat{\theta}_{\ell} \leq g(n_{\ell}, \theta_j'', \alpha_j)\}$ for $i < j \leq m - 1$. Since $G_{\widehat{\theta}_i}(z, \theta)$ is non-increasing with respect to $z$, we have $\{\widehat{\theta}_i \geq g(n_{\ell}, \theta_j'', \alpha_j)\} \subseteq \{\widehat{\theta}_i \geq \theta_j'\}$, $G_{\widehat{\theta}_i}(\theta_j', \theta_j'' \leq \alpha_j\} \subseteq \{G_{\widehat{\theta}_i}(\theta_j', \theta_j'') \leq \alpha_j\} \subseteq \{G_{\widehat{\theta}_i}(\theta_j, \theta_j'') \leq \alpha_j\} \subseteq \{\widehat{\theta}_j \leq f_{\ell,j}\}$. Recalling that $G_{\widehat{\theta}_i}(z, \theta)$ is non-decreasing with respect to $\theta \in \Theta$ and invoking Lemma 1, we have

$$\Pr\{\widehat{\theta}_i \geq g_{\ell,j} \mid \theta\} \leq \Pr\{G_{\widehat{\theta}_i}(\theta_j', \theta_j'') \leq \alpha_j \mid \theta\} \leq \Pr\{G_{\widehat{\theta}_i}(\theta_j, \theta_j'') \leq \alpha_j \mid \theta\} \leq \alpha_j \leq \overline{\theta}_i$$ (25)

for $i < j \leq m - 1$ and $\theta \in \Theta_i$. For $i = m - 1$, it is evident that $\Pr\{i + 2 \leq D_{\ell} \leq m \mid \theta\} = 0 \leq \overline{\theta}_{m-1}$ for $\theta \in \Theta_{m-1}$. For $i = m - 2$, making use of (25), we have $\Pr\{i + 2 \leq D_{\ell} \leq m \mid \theta\} = \Pr\{\widehat{\theta}_{\ell} \leq g_{\ell,m-1} \mid \theta\} \leq \overline{\theta}_{m-2}$ for $\theta \in \Theta_{m-2}$. For $0 \leq i \leq m - 3$, define $S = \{j : g_{\ell,j-1} < f_{\ell,j}, i + 2 \leq j \leq m - 1\}$, and let $r$ be an integer such that $r$ assumes value $m - 1$ if $S$ is empty and that $r \in S$, $g_{\ell,r} = \min\{g_{\ell,j-1} : j \in S\}$ if $S$ is not empty. It follows from (25) that $\Pr\{i + 2 \leq D_{\ell} \leq m \mid \theta\} \leq \Pr\{\widehat{\theta}_i \leq g_{\ell,r} \mid \theta\} \leq \overline{\theta}_i$ for $0 \leq i \leq m - 3$ and $\theta \in \Theta_i$. This proves our second claim. Making use of these two established claims, we have

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\(\Pr\{\text{Reject } \mathcal{H}_i \mid \theta\} \leq \sum_{i=1}^n \Pr\{1 \leq D_i \leq i \mid \theta\} + \Pr\{i + 2 \leq D_i \leq m \mid \theta\} \leq \sum_{i=1}^n (\alpha_i + \beta_i)\) for \(i = 0, 1, \ldots, m - 1\) and \(\theta \in \Theta_i\). This establishes statement (III).

Statements (IV) and (V) can be shown by virtue of Lemma 2 and the observation that \{Accept \( \mathcal{H}_i \)\} \( \subseteq \{\theta' \leq \hat{\theta} \leq \theta''_{i+1}\} \) and that \{Accept \( \mathcal{H}_i \)\} is determined by the random tuple \((X_1, \ldots, X_n)\) as a consequence of the definition of the test plan.

We now want to show statement (VI). Observing that \(G_{\varphi_n}(z, \theta)\) is non-increasing with respect to \(z\), we have that \(g(n, \theta'_i, \alpha_i) \leq \frac{\theta''_i + \theta'_i}{2}\) if \(G_{\varphi_n}\left(\frac{\theta''_i + \theta'_i}{2}, \theta'_i\right) \leq \alpha_i\). Since \(\varphi_n = \sum_{i=1}^n X_i\) is an unbiased ULE for \(\theta\), it follows from Lemma 3 that
\[
G_{\varphi_n}\left(\frac{\theta''_i + \theta'_i}{2}, \theta'_i\right) = \Pr\{\varphi_n \geq \frac{\theta''_i + \theta'_i}{2} \mid \theta'_i\} \leq \left[C\left(\frac{\theta''_i + \theta'_i}{2}, \theta'_i\right)\right]^n \leq \alpha_i
\]
if \(n \geq \frac{\ln(\alpha_i)}{C(\frac{\theta''_i + \theta'_i}{2}, \theta'_i)}\), where \(C(z, \theta) = \inf_{\rho \in \mathbb{R}} \mathbb{E}[e^{\rho(X-z)}]\). On the other hand, observing that \(F_{\varphi_n}(z, \theta)\) is non-decreasing with respect to \(z\), we have that \(f(n, \theta''_i, \beta_i) \geq \frac{\theta''_i + \theta'_i}{2}\) if \(F_{\varphi_n}\left(\frac{\theta''_i + \theta'_i}{2}, \theta''_i\right) \leq \beta_i\). Since \(\varphi_n\) is an unbiased ULE for \(\theta\), it follows from Lemma 3 that
\[
F_{\varphi_n}\left(\frac{\theta''_i + \theta'_i}{2}, \theta''_i\right) = \Pr\{\varphi_n \leq \frac{\theta''_i + \theta'_i}{2} \mid \theta''_i\} \leq \left[C\left(\frac{\theta''_i + \theta'_i}{2}, \theta''_i\right)\right]^n \leq \beta_i
\]
if \(n \geq \frac{\ln(\beta_i)}{C(\frac{\theta''_i + \theta'_i}{2}, \theta''_i)}\). Therefore, \(f(n, \theta''_i, \beta_i) \geq g(n, \theta'_i, \alpha_i)\) if
\[
n \geq \max\left\{\frac{\ln(\alpha_i)}{C(\frac{\theta''_i + \theta'_i}{2}, \theta'_i)}, \frac{\ln(\beta_i)}{C(\frac{\theta''_i + \theta'_i}{2}, \theta''_i)}\right\}.
\]
It follows from the definition of \(\pi\) that
\[
\pi \leq \max_{i \in \{1, \ldots, m-1\}} \max\left\{\frac{\ln(\alpha_i)}{C(\frac{\theta''_i + \theta'_i}{2}, \theta'_i)}, \frac{\ln(\beta_i)}{C(\frac{\theta''_i + \theta'_i}{2}, \theta''_i)}\right\} = O\left(\frac{1}{\zeta}\right).
\]
Invoking the established statement (III), we have that, as \(\zeta \rightarrow 0\),
\[
\Pr\{\text{Reject } \mathcal{H}_i \mid \theta\} \leq \pi (\alpha_i + \beta_i) \leq O\left(\frac{1}{\zeta}\right) O(\zeta) \rightarrow 0
\]
for any \(\theta \in \Theta_i\) and \(i = 0, 1, \ldots, m - 1\). This proves statement (VI).

To show statement (VII), by the definition of the test plan, we have that \{Reject \( \mathcal{H}_i \)\} is determined by the random tuple \((X_1, \ldots, X_n)\). Moreover, for any numbers \(a\) and \(b\) such that \(\theta''_i \leq a < b \leq \theta''_{i+1}\), we have that \{Reject \( \mathcal{H}_i \)\} \( = \{\text{Reject } \mathcal{H}_i, \ \hat{\theta} \leq a\} \cup \{\text{Reject } \mathcal{H}_i, \ \hat{\theta} \geq b\} \cap \{\text{Reject } \mathcal{H}_i, \ \hat{\theta} \leq a\} \cap \{\text{Reject } \mathcal{H}_i, \ \hat{\theta} \geq b\} = \emptyset\), which imply that \(\Pr\{\text{Reject } \mathcal{H}_i \mid \theta\} = \Pr\{\text{Reject } \mathcal{H}_i, \ \hat{\theta} \leq a \mid \theta\} + \Pr\{\text{Reject } \mathcal{H}_i, \ \hat{\theta} \geq b \mid \theta\}\). By Lemma 2, we have that \(\Pr\{\text{Reject } \mathcal{H}_i, \ \hat{\theta} \leq a \mid \theta\}\) is non-increasing with respect to \(\theta \in \Theta\) no less than \(a\) and that \(\Pr\{\text{Reject } \mathcal{H}_i, \ \hat{\theta} \geq b \mid \theta\}\) is non-decreasing with respect to \(\theta \in \Theta\) no greater than \(b\). This leads to the upper and lower bounds of \(\Pr\{\text{Reject } \mathcal{H}_i \mid \theta\}\) in statement (VII).

Statement (VIII) can be shown by virtue of Lemma 2 based on the observation that
{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_{m-1} \subseteq \{ \theta' \leq \hat{\theta} \leq \theta'' \}_{m-1} \text{ and that } \{ \text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_{m-1} \} \text{ is determined by the random tuple } (X_1, \cdots, X_n) \text{ as a consequence of the definition of the test plan.}

Finally, we shall show statement (IX). Note that \( \Pr(\text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_{m-1} \mid \theta) \leq \sum_{\ell=1}^s \Pr\{2 \leq D_{\ell} \leq m-1 \mid \theta\} \). Define \( S = \{ j : g_{\ell,j-1} < f_{\ell,j}, \ 2 \leq j \leq m-1 \} \). In the case that \( S \) is empty, \( \Pr\{2 \leq D_{\ell} \leq m-1 \mid \theta\} = 0 \). In the case that \( S \) is not empty, let \( r \in S \) be an integer such \( f_{\ell,r} = \max\{f_{\ell,j} : j \in S\} \). Then, \( \Pr\{2 \leq D_{\ell} \leq m-1 \mid \theta\} \leq \Pr\{\hat{\theta}_{\ell} \leq f_{\ell,r} \mid \theta\} \leq \max\{\beta_j : 2 \leq j \leq m-1\} \) for \( \theta \in \Theta_{m-1} \). On the other hand, if we let \( r \in S \) be an integer such that \( g_{\ell,r-1} = \min\{g_{\ell,j-1} : j \in S\} \), then \( \Pr\{2 \leq D_{\ell} \leq m-1 \mid \theta\} \leq \Pr\{\hat{\theta}_{\ell} \geq g_{\ell,r-1} \mid \theta\} \leq \max\{\alpha_j : 1 \leq j \leq m-2\} \) for \( \theta \in \Theta_0 \). This proves statement (IX) and concludes the proof of the theorem.

C Proof of Theorem 2

Theorem 2 can be established by making use of Lemmas 1, 2, and 3 and an argument similar to the proof of Theorem 1.

D Proof of Theorem 4

For arbitrary parametric values \( \theta_0 < \theta_1 \) in \( \Theta \), by the assumption that \( \varphi_n \) converges in probability to \( \theta \), we have that \( \Pr\{\varphi_n \geq \frac{\theta_0 + \theta_1}{2} \mid \theta_0\} \leq \Pr\{\|\varphi_n - \theta_0\| \geq \frac{\theta_0 + \theta_1}{2} \mid \theta_0\} \to 0 \) and \( \Pr\{\varphi_n \leq \frac{\theta_0 + \theta_1}{2} \mid \theta_1\} \leq \Pr\{\|\varphi_n - \theta_1\| \geq \frac{\theta_0 + \theta_1}{2} \mid \theta_1\} \to 0 \) as \( n \to \infty \). This shows that \( \varphi_n \) exists and is finite. By the definition of the testing plan, we have

\[
\Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} = \sum_{\ell=1}^s \Pr\{\text{Accept } \mathcal{H}_0, \ l = \ell \mid \theta\} \leq \sum_{\ell=1}^s \Pr\{D_{\ell} = 1 \mid \theta\} = \sum_{\ell=1}^s \Pr\{\hat{\theta}_{\ell} \leq \hat{F}(n_{\ell}, \theta_0, \theta_1, \alpha_0, \beta_1) \mid \theta\} \leq \sum_{\ell=1}^s \Pr\{\hat{\theta}_{\ell} \leq \hat{F}(n_{\ell}, \theta_1, \beta_1) \mid \theta\}.
\]

Since \( F_{\hat{\theta}_{\ell}}(z, \theta) \) is non-decreasing with respect to \( z \in I_{\hat{\theta}_{\ell}} \) for any given \( \theta \in \Theta \), we have \( \Pr\{\hat{\theta}_{\ell} \leq \hat{F}(n_{\ell}, \theta_1, \beta_1) \mid \theta\} \leq \Pr\{F_{\hat{\theta}_{\ell}}(\hat{\theta}_{\ell}, \theta_1) \leq \beta_1 \mid \theta\} \) for \( \ell = 1, \cdots, s \). Since \( \hat{\theta}_{\ell} \) is a ULE of \( \theta \), by Lemma 2 we have that \( F_{\hat{\theta}_{\ell}}(z, \theta) = \Pr\{\hat{\theta}_{\ell} \leq z \mid \theta\} \) is non-increasing with respect to \( \theta \) no less than \( z \). This implies that \( \Pr\{F_{\hat{\theta}_{\ell}}(\hat{\theta}_{\ell}, \theta_1) \leq \beta_1 \mid \theta\} \leq \Pr\{F_{\hat{\theta}_{\ell}}(\hat{\theta}_{\ell}, \theta) \leq \beta_1 \mid \theta\} \), \( \ell = 1, \cdots, s \) for \( \theta \in \Theta \) no less than \( \theta_1 \). Therefore, \( \Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \leq \sum_{\ell=1}^s \Pr\{D_{\ell} = 1 \mid \theta\} \leq \sum_{\ell=1}^s \Pr\{F_{\hat{\theta}_{\ell}}(\hat{\theta}_{\ell}, \theta) \leq \beta_1 \mid \theta\} \leq s\beta_1 \) for \( \theta \in \Theta \) no less than \( \theta_1 \), where the last inequality follows from Lemma 1. By a similar method, we can show that \( \Pr(\text{Reject } \mathcal{H}_0 \mid \theta) \leq \sum_{\ell=1}^s \Pr\{D_{\ell} = 2 \mid \theta\} \leq s\alpha_0 \) for \( \theta \in \Theta \) no greater than \( \theta_0 \). By the definition of the testing plan and the assumption that the likelihood ratio is monotonically increasing with respect to \( \hat{\theta}_{\ell} \), we have that the test procedure is a generalized SPRT. Hence, the monotonicity of \( \Pr\{\text{Accept } \mathcal{H}_0 \mid \theta\} \) with respect to \( \theta \) is established. This concludes the proof of the theorem.
Proof of Theorem 7

We need some preliminary results.

**Lemma 4** Let $X_1, \ldots, X_n$ be i.i.d. Gaussian random variables with mean $\mu$ and variance $\sigma^2$. Define $\tilde{T} = \frac{X_n}{\tilde{\sigma}_n}$ where $X_n = \frac{\sum_{i=1}^{n} X_i}{n}$ and $\tilde{\sigma}_n = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - X_n)^2}$. Then, $\tilde{T}$ is a ULE of $\frac{\mu}{\sigma}$.

**Proof.** Let $x_1, \ldots, x_n$ be observations of $X_1, \ldots, X_n$. Then, the logarithm of the corresponding likelihood function can be expressed as

$$h(x_1, \ldots, x_n, \mu, \theta) = \sum_{i=1}^{n} \ln \left[ \frac{1}{\sqrt{2\pi} \theta} \exp \left( -\frac{(x_i - \mu)^2}{2 \theta^2} \right) \right]$$

where $\theta = \frac{\mu}{\sigma}$.

Define $g(x_1, \ldots, x_n, \mu, \sigma) = \sum_{i=1}^{n} \left[ \ln \left( \frac{1}{\sqrt{2\pi} \sigma} \right) - \frac{(x_i - \mu)^2}{2 \sigma^2} \right]$. Then,

$$\frac{\partial h(x_1, \ldots, x_n, \mu, \theta)}{\partial \mu} = \frac{\partial g(x_1, \ldots, x_n, \mu, \sigma)}{\partial \mu} + \frac{\partial g(x_1, \ldots, x_n, \mu, \sigma)}{\partial \sigma} \frac{\partial \sigma}{\partial \mu} = 0,$$  \hspace{1cm} (26)

$$\frac{\partial h(x_1, \ldots, x_n, \mu, \theta)}{\partial \theta} = \frac{\partial g(x_1, \ldots, x_n, \mu, \sigma)}{\partial \sigma} \frac{\partial \sigma}{\partial \theta} = 0.$$  \hspace{1cm} (27)

Since $\sigma = \frac{\mu}{\theta}$ and $\frac{\partial \sigma}{\partial \theta} \neq 0$, equations (26) and (27) can be written as

$$\frac{\partial g(x_1, \ldots, x_n, \mu, \sigma)}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) = 0,$$  \hspace{1cm} (28)

$$\frac{\partial g(x_1, \ldots, x_n, \mu, \sigma)}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (x_i - \mu)^2 = 0.$$  \hspace{1cm} (29)

Define $\bar{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}$ and $\bar{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{\mu})^2}$. Then, $\mu = \bar{\mu}$, $\sigma = \bar{\sigma}$ is the solution of equations (28), (29) with respect to $\mu$ and $\sigma$. Hence, setting $\tilde{\theta} = \frac{\bar{\mu}}{\bar{\sigma}}$, we have that

$$\frac{\partial h(x_1, \ldots, x_n, \mu, \theta)}{\partial \mu} \bigg|_{\theta=\tilde{\theta}, \mu=\bar{\mu}} = \frac{\partial h(x_1, \ldots, x_n, \mu, \theta)}{\partial \theta} \bigg|_{\theta=\tilde{\theta}, \mu=\bar{\mu}} = 0$$

and that the likelihood function is monotonically increasing with respect to $\theta < \tilde{\theta}$ and is monotonically decreasing with respect to $\theta > \tilde{\theta}$. This implies that $\tilde{T}$ is a ULE of $\frac{\mu}{\sigma}$. The proof of the lemma is thus completed.

**Lemma 5** For any $\delta \in (0, 1)$, $\frac{t_{\text{ln}, \delta}}{\sqrt{n}}$ is monotonically decreasing to 0 as $n$ increases from 2 to $\infty$.  

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Proof. For simplicity of notations, let \( \psi(n) = \frac{\tan \delta}{\sqrt{n}} \). Then, \( \delta = \Pr\left\{ \frac{|U|}{\sqrt{Z/n}} > \frac{1}{\sqrt{n}} \right\} = \Pr\left\{ \frac{|U|}{\sqrt{Z}} > \psi(n) \right\} \), where \( U \) and \( Z \) are independent random variables such that \( U \) is a Gaussian variable with zero mean and unit variance and that \( Z \) is a chi-squared variable of \( n \) degrees of freedom. Since \( \frac{U}{\sqrt{Z/n}} \) possesses a Student’s \( t \)-distribution of \( n \) degrees of freedom, its mean and variance are, respectively, 0 and \( \frac{n}{n-2} \). Accordingly, the mean and variance of \( \frac{U}{\sqrt{Z}} \) are, respectively, 0 and \( \frac{1}{n-2} \). By Chebyshev’s inequality, \( \Pr\left\{ \frac{|U|}{\sqrt{Z}} > \psi \right\} \leq \frac{1}{(n-2)|\psi(n)|^2} \), leading to \( \delta < \frac{1}{(n-2)|\psi(n)|^2} \), i.e., \( \psi(n) < \frac{1}{\sqrt{(n-2)\delta}} \) → 0 as \( n \to \infty \). This proves \( \lim_{n \to \infty} \frac{t_n}{\sqrt{n}} = 0 \).

To show the monotonicity, it suffices to show that, for any fixed \( t > 0 \), \( \Pr\{|U|/\sqrt{Z} > t\} \) decreases monotonically with respect to \( n \). Let \( V_1, \cdots, V_n, V_{n+1} \) be i.i.d. Gaussian random variables which have zero mean, unity variance and are independent with \( U \). Then, \( \Pr\{|U|/\sqrt{Z} > t\} = \Pr\{|U|/\sqrt{\sum_{i=1}^n V_i^2} > t\} \). In view of \( \Pr\{|U|/\sqrt{\sum_{i=1}^{n+1} V_i^2} > t\} > \Pr\{|U|/\sqrt{\sum_{i=1}^n V_i^2} > t\} \) and \( \Pr\{|U|/\sqrt{\sum_{i=1}^n V_i^2} > \psi(n)\} = \Pr\{|U|/\sqrt{\sum_{i=1}^{n+1} V_i^2} > \psi(n+1)\} = \delta \), we have \( \Pr\{|U|/\sqrt{\sum_{i=1}^n V_i^2} > \psi(n)\} = \Pr\{|U|/\sqrt{\sum_{i=1}^{n+1} V_i^2} > \psi(n+1)\} = \delta \), which implies \( \psi(n+1) < \psi(n) \). This completes the proof of the lemma.

\[ \square \]

Lemma 6 \( \lim_{\delta \to 0} \frac{Z_{\delta}}{\sqrt{2 \ln \frac{\pi}{\delta}}} = 1 \).

Proof. For simplicity of notations, we abbreviate \( Z_{\delta} \) as \( z \) when this can be done without introducing confusion. By virtue of the well-known inequality \( 1 \leq \Phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \left(\frac{1}{\Gamma}\right) \), we have \( \delta < \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \left(\frac{1}{\Gamma}\right) \), or equivalently, \( 2\ln \frac{\pi}{\delta} > 2\ln(\sqrt{2\pi z}) + 1 \), which implies \( \liminf_{z \to \infty} \frac{2\ln \frac{\pi}{\delta}}{2 \ln \frac{\pi}{\delta}} \geq 1 \) and, consequently, \( \limsup_{\delta \to 0} \frac{Z_{\delta}}{\sqrt{2 \ln \frac{\pi}{\delta}}} \leq 1 \). On the other hand, making use of the well-known inequality \( \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \left(\frac{1}{\Gamma}\right) \leq 1 - \Phi(z) \), we have \( \delta > \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \left(\frac{1}{\Gamma}\right) \left(1 - \frac{1}{\Gamma}\right) \), which implies \( \frac{2\ln \frac{\pi}{\delta}}{2 \ln \frac{\pi}{\delta}} > 1 \) and thus \( \liminf_{\delta \to 0} \frac{Z_{\delta}}{\sqrt{2 \ln \frac{\pi}{\delta}}} \geq 1 \). This establishes \( \lim_{\delta \to 0} \frac{Z_{\delta}}{\sqrt{2 \ln \frac{\pi}{\delta}}} = 1 \). \[ \square \]

Lemma 7 Let \( X \) be a chi-squared random variable with \( n \) degrees of freedom. Then, \( \Pr\{X \geq n(1 + \kappa)\} \leq [(1 + \kappa)e^{-\kappa}]^{n\kappa} \) for any \( \kappa > 0 \) and \( \Pr\{X \leq n(1 - \kappa)\} \leq [(1 - \kappa)e^{\kappa}]^{n\kappa} \) for \( 0 < \kappa < 1 \).

Proof. For simplicity of notations, let \( c = n(1 + \kappa) \). Then,

\[
\Pr\{X \geq c\} \leq \inf_{\rho > 0} E\left[ e^{\rho (X-c)} \right] = \inf_{\rho > 0} \int_0^\infty \frac{1}{2^{\frac{\rho}{2}} \Gamma\left(\frac{\rho}{2}\right)} x^{\frac{n\rho}{2}-1} e^{-\frac{x}{2}} e^{\rho(x-c)} dx
\]

\[
= \inf_{\rho > 0} e^{-\rho c}(1 - 2\rho)^{-\frac{n}{2}} \int_0^\infty \frac{1}{2^{\frac{\rho}{2}} \Gamma\left(\frac{\rho}{2}\right)} y^{\frac{n}{2}-1} e^{-\frac{y}{2}} dy = \inf_{\rho > 0} e^{-\rho c}(1 - 2\rho)^{-\frac{n}{2}},
\]

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Lemma 8 Let \( F(t) \) be Student’s t-distribution of \( n \) degrees of freedom. Let \( x(t) \) be the root of equation \( \Phi(x) = F(t) \) with respect to \( x \). Then, \( \sqrt{(n-\frac{1}{2}) \ln (1+\frac{t^{2}}{n})} \leq x(t) \leq \sqrt{\ln (1+\frac{t^{2}}{n})} \) for any \( t > 0 \).

Lemma 9 For any \( \epsilon > 0 \), there exists a number \( \zeta^{*} > 0 \) such that \( \left| \frac{t \alpha - 1 - \beta}{\sqrt{n}} \right| < \epsilon \) for any \( \zeta \in (0, \zeta^{*}) \) and all \( n \geq \kappa(\zeta, \varrho) = \min \left\{ \left\lfloor \frac{1}{2} \right\rfloor, \left\lfloor 2 \rho \zeta \right\rfloor \right\} > 1 \), where \( \zeta^{*} \) is independent of \( n \) and \( \varrho > 0 \).

Proof. Define
\[
h(\zeta, n) = \left[ \ln \left( 1 + \frac{t_{n, \alpha}^{2}}{n} \right) \right] \left[ \ln \left( 1 + \frac{t_{n, \beta}^{2}}{n} \right) \right]^{-1}
\]
for \( n \geq \kappa(\zeta, \varrho) \). We shall first show that \( h(\zeta, n) \) tends to 1 uniformly for \( n \geq \kappa(\zeta, \varrho) \) as \( \zeta \to 0 \). Applying Lemma 8 we have
\[
\frac{Z_{\alpha}^{2}}{n} \leq \ln \left( 1 + \frac{t_{n, \alpha}^{2}}{n} \right) \leq \frac{Z_{\alpha}^{2}}{n-\frac{1}{2}}, \quad \frac{Z_{\beta}^{2}}{n} \leq \ln \left( 1 + \frac{t_{n, \beta}^{2}}{n} \right) \leq \frac{Z_{\beta}^{2}}{n-1}
\]
and thus
\[
\left( 1 - \frac{1}{2\kappa(\zeta, \varrho)} \right) \left( \frac{Z_{\alpha}}{Z_{\beta}} \right)^{2} < \frac{n-\frac{1}{2}}{n} \left( \frac{Z_{\alpha}}{Z_{\beta}} \right)^{2} \leq h(\zeta, n) \leq \frac{n}{n-\frac{1}{2}} \left( \frac{Z_{\alpha}}{Z_{\beta}} \right)^{2} < \left( 1 + \frac{1}{2\kappa(\zeta, \varrho) - 1} \right) \left( \frac{Z_{\alpha}}{Z_{\beta}} \right)^{2}
\]
for \( n \geq \kappa(\zeta, \varrho) \). By Lemma 8 we have
\[
\lim_{\zeta \to 0} \frac{Z_{\alpha}}{Z_{\beta}} = \lim_{\zeta \to 0} \left[ \frac{Z_{\alpha}}{\sqrt{2\ln \frac{1}{\alpha}}} \times \frac{\sqrt{2\ln \frac{1}{\beta}}}{\sqrt{2\ln \frac{1}{\beta}}} \right] = 1.
\]
It follows that \( h(\zeta, n) \) tends to 1 uniformly for \( n \geq \kappa(\zeta, \varrho) \) as \( \zeta \to 0 \). By virtue of (30), we have
\[
\ln \left( 1 + \frac{t_{n, \alpha}^{2}}{n} \right) \leq \frac{Z_{\alpha}^{2}}{n-\frac{1}{2}} \leq \frac{Z_{\alpha}^{2}}{\kappa(\zeta, \varrho) - \frac{1}{2}} \to \frac{2}{\varrho}
\]
and
\[
\ln \left( 1 + \frac{t_{n, \beta}^{2}}{n} \right) \leq \frac{Z_{\beta}^{2}}{n-\frac{1}{2}} \leq \frac{Z_{\beta}^{2}}{\kappa(\zeta, \varrho) - \frac{1}{2}} \to \frac{2}{\varrho}
\]
uniformly for \( n \geq \kappa(\zeta, \vartheta) \) as \( \zeta \to 0 \). Therefore, both \( \frac{t_{a,a}}{\sqrt{n}} \) and \( \frac{t_{b,b}}{\sqrt{n}} \) are bounded uniformly for all \( n \geq \kappa(\zeta, \vartheta) \) and any \( \zeta \in (0, 1) \). By virtue of this result and recalling that \( h(\zeta, n) \) tends to 1 uniformly for \( n \geq \kappa(\zeta, \vartheta) \) as \( \zeta \to 0 \), we have that \( \ln(1 + \frac{t_{a,a}}{\sqrt{n}}) - \ln(1 + \frac{t_{b,b}}{\sqrt{n}}) \) tends to 0 and thus \( \frac{t_{a,a} - t_{b,b}}{\sqrt{n}} \) tends to 0 uniformly for \( n \geq \kappa(\zeta, \vartheta) \) as \( \zeta \to 0 \). This completes the proof of the lemma.

\[ \square \]

**Lemma 10** For any \( \Delta > 0 \), 
\[ \sum_{n=\kappa(\zeta, \vartheta)+1}^{\infty} \Pr\{\frac{X_n}{\sigma_n} - \theta \geq \Delta \mid \theta\} \to 0 \]  
\( \) as \( \zeta \to 0 \), where \( \kappa(\zeta, \vartheta) = \min\left\{ \frac{\varrho}{\sqrt{n}}, \frac{\alpha}{\sqrt{n}} \right\} \).

**Proof.** We shall first show that 
\[ \sum_{n=\kappa(\zeta, \vartheta)+1}^{\infty} \Pr\{\frac{X_n}{\sigma_n} \leq \theta - \Delta \mid \theta\} \to 0 \]  
\( \) as \( \zeta \to 0 \) by considering two cases: (i) \( \theta \geq \Delta \); (ii) \( \theta < \Delta \).

In the case of \( \theta \geq \Delta \), let \( \eta \) be a positive number such that \( (1 + \eta)(\theta - \Delta) < \theta \). Then,

\[
\Pr\left\{ \frac{X_n}{\sigma_n} \leq \theta - \Delta \mid \theta \right\} \leq \Pr\left\{ \frac{X_n}{\sigma_n} \leq \theta - \Delta, \sigma_n \leq (1 + \eta)\sigma \mid \theta \right\} + \Pr\{\sigma_n > (1 + \eta)\sigma \mid \theta\} \\
\leq \Pr\{\sigma_n \leq (1 + \eta)\sigma(\theta - \Delta) \mid \theta\} + \Pr\{\sigma_n > (1 + \eta)\sigma \mid \theta\} \\
= \Pr\{U \geq \sqrt{n}(1 + \eta)\Delta - \eta\theta\} + \Pr\{\chi^2_{n-1} > (1 + \eta)^2\} \\
< \Pr\{U \geq \sqrt{n}(1 + \eta)\Delta - \eta\theta\} + \Pr\{\chi^2_{n-1} > (n - 1)(1 + \eta)\},
\]  

where \( U \) is a Gaussian random variable with zero mean and unit variance and \( \chi^2_{n-1} \) is a chi-square variable of \( n - 1 \) degrees of freedom. By the choice of \( \eta \), we have \( (1 + \eta)\Delta - \eta\theta > 0 \) as a consequence of \( (1 + \eta)(\theta - \Delta) < \theta \). Hence,

\[
\Pr\{U \geq \sqrt{n}(1 + \eta)\Delta - \eta\theta\} < \exp\left(-\frac{n}{2}(1 + \eta)\Delta - \eta\theta^2\right). \tag{32}
\]

On the other hand, by Lemma 7 we have

\[
\Pr\{\chi^2_{n-1} > (n - 1)(1 + \eta)\} \leq [(1 + \eta)e^{-\eta}]^{(n-1)/2}. \tag{33}
\]

Combining (31), (32) and (33) yields

\[
\sum_{n=\kappa(\zeta, \vartheta)+1}^{\infty} \Pr\left\{ \frac{X_n}{\sigma_n} \leq \theta - \Delta \mid \theta \right\} < \sum_{n=\kappa(\zeta, \vartheta)+1}^{\infty} \exp\left(-\frac{n}{2}(1 + \eta)\Delta - \eta\theta^2\right) + [(1 + \eta)e^{-\eta}]^{(n-1)/2},
\]

where the right side tends to 0 as \( \zeta \to 0 \) because \( \kappa(\zeta, \vartheta) \to \infty \) as \( \zeta \to 0 \).

In the case of \( \theta < \Delta \), let \( \eta \in (0, 1) \) be a number such that \( (1 - \eta)(\theta - \Delta) < \theta \). Then,

\[
\Pr\left\{ \frac{X_n}{\sigma_n} \leq \theta - \Delta \mid \theta \right\} \leq \Pr\left\{ \frac{X_n}{\sigma_n} \leq \theta - \Delta, \sigma_n \geq (1 - \eta)\sigma \mid \theta \right\} + \Pr\{\sigma_n < (1 - \eta)\sigma \mid \theta\} \\
\leq \Pr\{X_n \leq (1 - \eta)\sigma(\theta - \Delta) \mid \theta\} + \Pr\{\sigma_n < (1 - \eta)\sigma \mid \theta\} \\
= \Pr\{U \geq \sqrt{n}\eta\theta + (1 - \eta)\Delta\} + \Pr\{\chi^2_{n-1} < (1 - \eta)^2\}. \tag{34}
\]
By the choice of \( \eta \), we have \( \eta \theta + (1 - \eta) \Delta > 0 \) as a consequence of \( (1 - \eta)(\theta - \Delta) < \theta \). Hence,

\[
\Pr\{U \geq \sqrt{n[\eta \theta + (1 - \eta) \Delta]}\} < \exp\left(-\frac{n}{2}[\eta \theta + (1 - \eta) \Delta]^2\right). \tag{35}
\]

For small enough \( \zeta > 0 \), we have \( n > \kappa(\zeta, \varrho) > \frac{1}{n} \eta \) and thus

\[
\Pr\{\chi_{n-1}^2 < n(1 - \eta)^2\} < \Pr\{\chi_{n-1}^2 < (n - 1)(1 - \eta)\} \leq [(1 - \eta)e^{\eta}]^{(n-1)/2}, \tag{36}
\]

where the last inequality follows from Lemma 7. Combining (34), (35) and (36) yields

\[
\sum_{n=\kappa(\zeta, \varrho)+1}^{\infty} \Pr\left\{ \frac{X_n}{\sigma_n} \leq \theta - \Delta \mid \theta \right\} < \sum_{n=\kappa(\zeta, \varrho)+1}^{\infty} \left[ \exp\left(-\frac{n}{2}[\eta \theta + (1 - \eta) \Delta]^2\right) + [(1 - \eta)e^{\eta}]^{(n-1)/2} \right],
\]

where the right side tends to 0 as \( \zeta \to 0 \) because \( \kappa(\zeta, \varrho) \to \infty \) as \( \zeta \to 0 \). This proves that

\[
\sum_{n=\kappa(\zeta, \varrho)+1}^{\infty} \Pr\left\{ \frac{X_n}{\sigma_n} \geq \theta + \Delta \mid \theta \right\} \to 0 \text{ as } \zeta \to 0. \]

In a similar manner, we can show that

\[
\sum_{n=\kappa(\zeta, \varrho)+1}^{\infty} \Pr\left\{ \frac{X_n}{\sigma_n} \geq \theta + \Delta \mid \theta \right\} \to 0 \text{ as } \zeta \to 0. \]

This concludes the proof of the lemma. \( \square \)

**Lemma 11** Let \( \delta = O(\zeta) \in (0, 1) \). If \( \zeta > 0 \) is sufficiently small, then

\[
\frac{1}{|\theta|} \left( \frac{t_{n, \delta}}{\sqrt{n}} - \frac{t_{n, \sqrt{\delta}}}{\sqrt{n}} \right) > \exp\left( \frac{\ln 1}{4n} \right) > 1
\]

for \( 2 \leq n < \left| \mathbb{Z}^2_{\sqrt{\delta}} \right| \), where \( 0 < \varrho < \frac{1}{4(1+|\theta|)^2} \).

**Proof.** From Wallace’s inequality restated in Lemma 8, we have

\[
\sqrt{\exp\left( \frac{Z^2_{\delta}}{n} \right) - 1} \leq \frac{t_{n, \delta}}{\sqrt{n}} \leq \sqrt{\exp\left( \frac{Z^2_{\delta}}{n - \frac{1}{2}} \right) - 1}, \quad \forall \delta \in (0, 1)
\]

and thus

\[
\frac{1}{|\theta|} \left( \frac{t_{n, \delta}}{\sqrt{n}} - \frac{t_{n, \sqrt{\delta}}}{\sqrt{n}} \right) > \frac{1}{|\theta|} \left[ \sqrt{\exp\left( \frac{Z^2_{\delta}}{n} \right) - 1} - \sqrt{\exp\left( \frac{Z^2_{\sqrt{\delta}}}{n - \frac{1}{2}} \right) - 1} \right].
\]

Therefore, to show the lemma, it suffices to show that

\[
\frac{1}{|\theta|} \left[ \sqrt{\exp\left( \frac{Z^2_{\delta}}{n} \right) - 1} - \sqrt{\exp\left( \frac{Z^2_{\sqrt{\delta}}}{n - \frac{1}{2}} \right) - 1} \right] > \exp\left( \frac{\ln 1}{4n} \right) > 1 \tag{37}
\]

for \( 2 \leq n < \left| \mathbb{Z}^2_{\sqrt{\delta}} \right| \) if \( \zeta > 0 \) is small enough. By Lemma 6, for small enough \( \zeta > 0 \), we have

\[
\ln \frac{1}{|\beta|} < \frac{2}{3} \mathbb{Z}^2_{\sqrt{\delta}} \quad \text{and thus}
\]

\[
\frac{\exp\left( \frac{Z^2_{\delta}}{n} \right) - 1}{\exp\left( \frac{\ln 1}{3n} \right)} > \exp\left( \frac{Z^2_{\sqrt{\delta}}}{3n} \right) - 1 > \exp\left( \frac{Z^2_{\sqrt{\delta}}}{3(n+1)} \right) - 1
\]

\[
\geq \exp\left( \frac{1}{3\varrho} \right) - 1 > \frac{1}{3\varrho} > \frac{4(1 + |\theta|^2)}{3} > 1
\]

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for $2 \leq n < \left\lfloor \frac{Z^2}{\sqrt{\delta}} \right\rfloor$. Hence,

$$\sqrt{\exp\left(\frac{Z^2}{n - \frac{1}{2}\delta}\right) - 1} > \sqrt{\exp\left(\frac{Z^2}{n}\right) - 1} > \exp\left(\frac{\ln \frac{1}{\delta}}{4n}\right) > 1$$

for $2 \leq n < \left\lfloor \frac{Z^2}{\sqrt{\delta}} \right\rfloor$ if $\zeta$ is small enough. Therefore, to guarantee (37), it suffices to make $\zeta$ small enough and ensure that

$$\sqrt{\exp\left(\frac{Z^2}{n}\right) - 1} > (1 + |\theta|) \sqrt{\exp\left(\frac{Z^2}{n - \frac{1}{2}}\right) - 1}$$

By Lemma 6, we have $\lim_{\zeta \to 0} \frac{Z^2}{\sqrt{\theta}} = 2$. This implies that, if $\zeta > 0$ is sufficiently small, then

$$\frac{Z^2}{\sqrt{\delta}} > \frac{5}{3}, \text{ and consequently, }$$

$$\frac{Z^2}{n - \frac{1}{2}} > \frac{Z^2}{n - \frac{1}{2}} \left(\frac{n - \frac{1}{2}}{n - \frac{1}{2}} \cdot \frac{Z^2}{2\delta} - 1\right) > \frac{1}{4} \left(\frac{2 - \frac{1}{2}}{2} \times \frac{5}{3} - 1\right) = \frac{1}{4q}$$

for $2 \leq n < \left\lfloor \frac{Z^2}{\sqrt{\delta}} \right\rfloor$. Hence,\n
$$\frac{\exp\left(\frac{Z^2}{n}\right) - 1}{\exp\left(\frac{Z^2}{n - \frac{1}{2}}\right) - 1} > \frac{\exp\left(\frac{Z^2}{n}\right) - 1}{\exp\left(\frac{Z^2}{n - \frac{1}{2}}\right) - 1} > \frac{\exp\left(\frac{Z^2}{n}\right) - 1}{\exp\left(\frac{Z^2}{n - \frac{1}{2}}\right) - 1} > \frac{1}{4} > (1 + |\theta|)^2$$

for $2 \leq n < \left\lfloor \frac{Z^2}{\sqrt{\delta}} \right\rfloor$, and consequently (37) is ensured if $\zeta > 0$ is small enough. This completes the proof of the lemma.

\[\square\]

**Lemma 12** Let $\theta' < \theta''$ and $\kappa(\zeta, \theta) = \min\left\{\left\lfloor \frac{Z^2}{\sqrt{\alpha}} \right\rfloor, \left\lfloor \frac{Z^2}{\sqrt{\beta}} \right\rfloor\right\}$. Then,

$$\lim_{\zeta \to 0} \left[ \sum_{n=2}^{\kappa(\zeta, \theta)} \Pr\left\{ \frac{X_n}{\sigma_n} \leq \theta'' - \frac{t_{n-1, \alpha}}{\sqrt{n - 1}} \ | \ \theta \right\} + \sum_{n=\kappa(\zeta, \theta)}^{\infty} \Pr\left\{ \frac{X_n}{\sigma_n} \leq \frac{\theta'' + t_{n-1, \alpha} - t_{n-1, \beta}}{2\sqrt{n - 1}} \ | \ \theta \right\} \right] = 0 \tag{38}$$

for $\theta > \theta''$ provided that $0 < \varrho < \frac{1}{6(1 + |\theta|)^2}$. Similarly,

$$\lim_{\zeta \to 0} \left[ \sum_{n=2}^{\kappa(\zeta, \theta)} \Pr\left\{ \frac{X_n}{\sigma_n} \geq \theta' + \frac{t_{n-1, \alpha}}{\sqrt{n - 1}} \ | \ \theta \right\} + \sum_{n=\kappa(\zeta, \theta)}^{\infty} \Pr\left\{ \frac{X_n}{\sigma_n} \geq \frac{\theta' + t_{n-1, \alpha} - t_{n-1, \beta}}{2\sqrt{n - 1}} \ | \ \theta \right\} \right] = 0 \tag{39}$$

for $\theta \leq \theta'$ provided that $0 < \varrho < \frac{1}{6(1 + |\theta|)^2}$.
Proof. Without loss of generality, assume that $\zeta$ is sufficiently small so that $\kappa(\zeta, \varrho)$ is greater than 2. We shall first show that
\[
\lim_{\zeta \to 0} \sum_{n=2}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\bar{X}_n}{\sigma_n} \leq \theta'' - \frac{t_{n-1, \beta}}{\sqrt{n-1}} \mid \theta \right\} = 0 \tag{40}
\]
for $\theta \geq \theta''$. Obviously, $\lim_{\zeta \to 0} \Pr \left\{ \frac{\bar{X}_n}{\sigma_n} \leq \theta'' - \frac{t_{n-1, \beta}}{\sqrt{n-1}} \mid \theta \right\} = 0$ for $n = 2$ and $\theta \geq \theta''$. Hence, to show (40), it remains to show
\[
\lim_{\zeta \to 0} \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\bar{X}_n}{\sigma_n} \leq \theta'' - \frac{t_{n-1, \beta}}{\sqrt{n-1}} \mid \theta \right\} = 0 \tag{41}
\]
for $\theta \geq \theta''$. We shall show (41) by considering three cases: (i) $\theta = 0$; (ii) $\theta < 0$; (iii) $\theta > 0$.

In the case of $\theta = 0 \geq \theta''$, we have
\[
\sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\bar{X}_n}{\sigma_n} \leq \theta'' - \frac{t_{n-1, \beta}}{\sqrt{n-1}} \mid \theta \right\} \leq \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\sqrt{n} \bar{X}_n}{\sigma_n} \leq -t_{n-1, \beta} \mid \theta \right\} < \kappa(\zeta, \varrho) \beta.
\]
Noting that
\[
\kappa(\zeta, \varrho) \beta \leq \varrho \frac{Z^2}{\sqrt{\beta}} \times \beta = \varrho \times \frac{Z^2}{2 \ln \frac{1}{\sqrt{\beta}}} \times \beta \times 2 \ln \frac{1}{\sqrt{\beta}} \to 0
\]
as $\zeta \to 0$, we have that (41) is true for the case of $\theta = 0 \geq \theta''$. Hence, it remains to show that (41) is true for the cases of $\theta < 0$ and $\theta > 0$. Let
\[
\Delta_n = \theta \sqrt{n-1} \left( 1 - \frac{\sigma}{\sigma_n} \right) + t_{n-1, \sqrt{\beta}} - t_{n-1, \beta}, \quad n = 3, 4, \ldots.
\]
Note that
\[
\sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\bar{X}_n}{\sigma_n} \leq \theta'' - \frac{t_{n-1, \beta}}{\sqrt{n-1}} \mid \theta \right\} = \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\sqrt{n} \bar{X}_n}{\sigma_n} \leq -t_{n-1, \beta} + \theta'' \sqrt{n-1} \mid \theta \right\}
\leq \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\bar{X}_n - \sigma \theta}{\sigma_n} \leq -t_{n-1, \beta} - \frac{\sqrt{n-1} \sigma \theta}{\sigma_n} \mid \theta \right\}
= \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\bar{X}_n - \sigma \theta}{\sigma_n} \leq -t_{n-1, \sqrt{\beta}} + \Delta_n \mid \theta \right\}
\leq \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\bar{X}_n - \sigma \theta}{\sigma_n} \leq -t_{n-1, \sqrt{\beta}} \mid \theta \right\} + \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \{ \Delta_n \geq 0 \mid \theta \}
\leq \kappa(\zeta, \varrho) \sqrt{\beta} + \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \{ \Delta_n \geq 0 \mid \theta \}.
\]
Clearly,
\[
\kappa(\zeta, \varrho) \sqrt{\beta} \leq \varrho \frac{Z^2}{\sqrt{\beta}} \times \sqrt{\beta} = \varrho \times \frac{Z^2}{2 \ln \frac{1}{\sqrt{\beta}}} \times \beta \times 2 \ln \frac{1}{\sqrt{\beta}} \to 0
\]
as \( \zeta \to 0 \). Hence, to show (41), it suffices to show \( \lim_{\zeta \to 0} \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr\{\Delta_n \geq 0 \mid \theta \} = 0 \) for \( \theta < 0 \) and \( \theta > 0 \).

In the case of \( \theta < 0 \), we have

\[
\Pr\{\Delta_n \geq 0 \mid \theta\} = \Pr\left\{ \frac{\sigma}{\sigma_n} - 1 \geq \frac{1}{|\theta|} \left( \frac{t_{n-1, \beta}}{\sqrt{n-1}} - \frac{t_{n-1, \sqrt{\beta}}}{\sqrt{n-1}} \right) \right\} \leq \Pr\left\{ \frac{\sigma}{\sigma_n} \geq \frac{1}{|\theta|} \left( \frac{t_{n-1, \beta}}{\sqrt{n-1}} - \frac{t_{n-1, \sqrt{\beta}}}{\sqrt{n-1}} \right) \right\}.
\]

By Lemma 11 for small enough \( \zeta > 0 \), we have

\[
\Pr\{\Delta_n \geq 0 \mid \theta\} \leq \Pr\left\{ \frac{\sigma}{\sigma_n} > \exp\left( \frac{\ln \frac{1}{\beta}}{4(n-1)} \right) \right\} = \Pr\left\{ \frac{\sigma}{\sigma_n} > \sqrt{\frac{n-1}{n}} \exp\left( \frac{\ln \frac{1}{\beta}}{6(n-1)} \right) \right\}
\]

for \( 3 \leq n \leq \kappa(\zeta, \varrho) \). By Lemma 6, we have that \( \ln \frac{1}{\beta} > \frac{1}{3} 2^2 \sqrt{\beta} \) if \( \zeta \) is small enough. This implies that

\[
\exp\left( \frac{\ln \frac{1}{\beta}}{6(n-1)} \right) > \exp\left( \frac{2^2 \sqrt{\beta}}{18(n-1)} \right) > \exp\left( \frac{2^2 \sqrt{\beta}}{18n} \right) \geq \exp\left( \frac{1}{18\varrho} \right)
\]

and thus

\[
\sqrt{\frac{n-1}{n}} \exp\left( \frac{\ln \frac{1}{\beta}}{4(n-1)} \right) > \sqrt{\frac{2}{3}} \exp\left( \frac{\ln \frac{1}{\beta}}{6(n-1)} \right) = \sqrt{\frac{2}{3}} \exp\left( \frac{\ln \frac{1}{\beta}}{6(n-1)} \right) \exp\left( \frac{\ln \frac{1}{\beta}}{6(n-1)} \right)
\]

\[
> \sqrt{\frac{2}{3}} \exp\left( \frac{1}{18\varrho} \right) \exp\left( \frac{\ln \frac{1}{\beta}}{6(n-1)} \right)
\]

\[
> \sqrt{\frac{2}{3}} \exp\left( \frac{1}{18 \times \frac{1}{6}} \right) \exp\left( \frac{\ln \frac{1}{\beta}}{6(n-1)} \right) > \exp\left( \frac{\ln \frac{1}{\beta}}{6(n-1)} \right)
\]

for \( 3 \leq n \leq \kappa(\zeta, \varrho) \) if \( \zeta \) is small enough, where we have used the assumption that \( \varrho < \frac{1}{6(1+|\theta|)^2} < \frac{1}{6} \). Therefore, for small enough \( \zeta > 0 \), we have

\[
\Pr\{\Delta_n \geq 0 \mid \theta\} < \Pr\left\{ \frac{\sigma}{\sigma_n} > \exp\left( \frac{\ln \frac{1}{\beta}}{6(n-1)} \right) \right\}
\]

for \( 3 \leq n \leq \kappa(\zeta, \varrho) \) and it follows that

\[
\Pr\{\Delta_n \geq 0 \mid \theta\} < \Pr\left\{ \frac{\sigma}{\sigma_n} < \sqrt{\beta^{\frac{1}{n-1}}} \mid \theta \right\} = \Pr\left\{ \chi_{n-1}^2 < (n-1) \beta^{\frac{1}{n-1}} \right\}
\]

\[
\leq \left[ \beta^{\frac{1}{n-1}} \exp\left( 1 - \beta^{\frac{1}{n-1}} \right) \right]^{(n-1)/2} < \beta^\frac{1}{2} e^{(n-1)/2}
\]

(42)

for \( 3 \leq n \leq \kappa(\zeta, \varrho) \). Noting that \( \frac{1}{2} \kappa(\zeta, \varrho) < 2\varrho \ln \frac{1}{\sqrt{\beta}} \) for small enough \( \zeta \) and invoking the assumption that \( 0 < \varrho < \frac{1}{6(1+|\theta|)^2} < \frac{1}{6} \), we have

\[
\beta^\frac{1}{2} \exp\left( \frac{\kappa(\zeta, \varrho)}{2} \right) < \beta^\frac{1}{2} \exp\left( 2\varrho \ln \frac{1}{\sqrt{\beta}} \right) = \beta^\frac{1}{2} e^{-\varrho} \to 0
\]

(43)
as \( \zeta \to 0 \). It follows from (42) and (43) that, in the case of \( \theta < 0 \),

\[
\sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr\{\Delta_n \geq 0 \mid \theta\} < \beta^\frac{1}{2} \sum_{n=3}^{\kappa(\zeta, \varrho)} e^{(n-1)/2} = \beta^\frac{1}{2} \times \frac{\exp\left( \frac{\kappa(\zeta, \varrho)}{2} \right) - e}{e^{1/2} - 1} < 2 \beta^\frac{1}{2} \exp\left( \frac{\kappa(\zeta, \varrho)}{2} \right) \to 0
\]

(44)
as $\zeta \to 0$.

In the case of $\theta > 0$, by virtue of Lemma 11 we have

$$\Pr\{\Delta_n \geq 0 \mid \theta\} = \Pr\left\{1 - \frac{\sigma}{\sigma_n} \geq \frac{1}{\theta} \left(\frac{t_{n-1,\beta}}{\sqrt{n-1}} - \frac{t_{n-1,\sqrt{\beta}}}{\sqrt{n-1}}\right) \mid \theta\right\} \leq \Pr\left\{1 - \frac{\sigma}{\sigma_n} > 1\right\} = \Pr\{\sigma_n < 0\} = 0$$

for $3 \leq n \leq \kappa(\zeta, \varrho)$ provided that $\zeta$ is small enough. It follows that $\sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr\{\Delta_n \geq 0 \mid \theta\} = 0$ for $\theta > 0$ if $\zeta > 0$ is sufficiently small. Therefore, we have shown that (40) holds for $\theta \geq \theta''$.

Next, we shall show that

$$\lim_{\zeta \to 0} \sum_{n=\kappa(\zeta, \varrho)+1}^{\infty} \Pr\left\{\frac{X_n}{\sigma_n} \leq \frac{\theta' + \theta''}{2} + \frac{t_{n-1,\alpha} - t_{n-1,\beta}}{2\sqrt{n-1}} \mid \theta\right\} = 0$$

for $\theta \geq \theta''$. By Lemma 9, there exist a number $\Delta > 0$ and $\zeta^* \in (0, 1)$ such that

$$\frac{\theta' + \theta''}{2} + \frac{t_{n-1,\alpha} - t_{n-1,\beta}}{2\sqrt{n-1}} < \theta - \Delta, \quad \forall \theta \geq \theta''$$

for any $\zeta \in (0, \zeta^*)$. It follows from Lemma 10 that

$$\sum_{n=\kappa(\zeta, \varrho)+1}^{\infty} \Pr\left\{\frac{X_n}{\sigma_n} \leq \theta - \Delta \mid \theta\right\} < \sum_{n=\kappa(\zeta, \varrho)+1}^{\infty} \Pr\left\{\frac{X_n}{\sigma_n} - \theta \left| \theta\right. \geq \Delta \mid \theta\right\} \to 0 \quad (45)$$

as $\zeta \to 0$. Combining (40) and (45) leads to (38).

Now we want to show that (39) is true. It suffices to show that

$$\lim_{\zeta \to 0} \sum_{n=2}^{\kappa(\zeta, \varrho)} \Pr\left\{\frac{X_n}{\sigma_n} \geq \theta' + \frac{t_{n-1,\alpha}}{\sqrt{n-1}} \mid \theta\right\} = 0 \quad (46)$$

and

$$\lim_{\zeta \to 0} \sum_{n=\kappa(\zeta, \varrho)+1}^{\infty} \Pr\left\{\frac{X_n}{\sigma_n} \geq \theta' + \frac{t_{n-1,\alpha} - t_{n-1,\beta}}{2\sqrt{n-1}} \mid \theta\right\} = 0 \quad (47)$$

for $\theta \leq \theta''$ under the assumption that $0 < \varrho < \frac{1}{6(1+|\theta|)^2}$. Clearly, for $n = 2$ and $\theta \leq \theta''$, $\Pr\left\{\frac{X_n}{\sigma_n} \geq \theta' + \frac{t_{n-1,\alpha}}{\sqrt{n-1}} \mid \theta\right\} \to 0$ as $\zeta \to 0$. Hence, to show (46), it suffices to show that

$$\lim_{\zeta \to 0} \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr\left\{\frac{X_n}{\sigma_n} \geq \theta' + \frac{t_{n-1,\alpha}}{\sqrt{n-1}} \mid \theta\right\} = 0 \quad (48)$$

for $\theta \leq \theta'$. We can show (48) by considering three cases: (i) $\theta < 0$; (ii) $\theta > 0$; (iii) $\theta = 0$. 45
Note that, for \( \theta \leq \theta' \),

\[
\sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\overbar{X}_n}{\sigma_n} \geq \theta' + \frac{t_{n-1, \alpha}}{\sqrt{n-1}} \mid \theta \right\} = \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\sqrt{n} \overbar{X}_n}{\sigma_n} \geq t_{n-1, \alpha} + \theta' \sqrt{n-1} \mid \theta \right\}
\]

\[
\leq \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\sqrt{n}(\overbar{X}_n - \sigma \theta)}{\sigma_n} \geq t_{n-1, \alpha} - \frac{\sigma \theta}{\sqrt{n-1}} + \theta \sqrt{n-1} \mid \theta \right\}
\]

\[
= \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\sqrt{n}(\overbar{X}_n - \sigma \theta)}{\sigma_n} \geq t_{n-1, \alpha} + \Delta_n \mid \theta \right\}
\]

\[
\leq \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\sqrt{n}(\overbar{X}_n - \sigma \theta)}{\sigma_n} \geq t_{n-1, \alpha} \mid \theta \right\} + \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \{ \Delta_n \leq 0 \mid \theta \}
\]

\[
< \kappa(\zeta, \varrho) \sqrt{\alpha} + \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \{ \Delta_n \leq 0 \mid \theta \},
\]

where

\[
\Delta_n = \theta \sqrt{n-1} \left(1 - \frac{\sigma}{\sigma_n}\right) - t_{n-1, \alpha} \sqrt{\alpha} + t_{n-1, \alpha}, \quad n = 3, 4, \ldots
\]

and

\[
\kappa(\zeta, \varrho) \sqrt{\alpha} \leq g \cdot \frac{2^2}{\sqrt{\alpha}} \times \sqrt{\alpha} = g \times \frac{2^2}{\sqrt{\alpha}} \times \sqrt{\alpha} \times 2 \ln \frac{1}{\sqrt{\alpha}} \rightarrow 0
\]

as \( \zeta \rightarrow 0 \).

In the case of \( \theta > 0 \), by Lemma 11 we have

\[
\Pr \{ \Delta_n \leq 0 \mid \theta \} = \Pr \left\{ 1 - \frac{\sigma}{\sigma_n} \leq -\frac{1}{\theta} \left( t_{n-1, \alpha} - t_{n-1, \alpha} \right) \right\}
\]

\[
\leq \Pr \left\{ \frac{\sigma}{\sigma_n} \geq -\frac{1}{\theta} \left( t_{n-1, \alpha} - t_{n-1, \alpha} \right) \right\} \leq \Pr \left\{ \frac{\sigma}{\sigma_n} > \exp \left( \frac{\ln \frac{1}{\alpha}}{4(n-1)} \right) \right\}
\]

for \( 3 \leq n \leq \kappa(\zeta, \varrho) \) if \( \zeta \) is small enough. Hence, by a similar method as that for proving (43), we have \( \lim_{\zeta \rightarrow 0} \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \{ \Delta_n \leq 0 \mid \theta \} \rightarrow 0 \) as \( \zeta \rightarrow 0 \).

In the case of \( \theta < 0 \), by Lemma 11 we have

\[
\Pr \{ \Delta_n \leq 0 \mid \theta \} = \Pr \left\{ 1 - \frac{\sigma}{\sigma_n} \geq 1 \mid \theta \left( t_{n-1, \alpha} - t_{n-1, \alpha} \right) \right\} \leq \Pr \left\{ 1 - \frac{\sigma}{\sigma_n} > 1 \right\} = 0
\]

for \( 3 \leq n \leq \kappa(\zeta, \varrho) \) if \( \zeta \) is small enough. Hence, \( \lim_{\zeta \rightarrow 0} \sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \{ \Delta_n \leq 0 \mid \theta \} = 0 \) for \( \theta < 0 \) if \( \zeta > 0 \) is small enough.

In the case of \( \theta = 0 \leq \theta' \), we have

\[
\sum_{n=3}^{\kappa(\zeta, \varrho)} \Pr \left\{ \frac{\overbar{X}_n}{\sigma_n} \geq \theta' \mid \theta \right\} \leq \kappa(\zeta, \varrho) \alpha \rightarrow 0
\]

as \( \zeta \rightarrow 0 \). Therefore, 23 is true for all three cases. As a result, 15 is true for \( \theta \leq \theta' \).

By a similar method as that for 15, we can show that 17 is true. Finally, combining 16 and 17 leads to 19. This completes the proof of the lemma.
Now we are in a position to prove the theorem. Statement (I) can be shown by virtue of Lemma 2 and the observation that \( \{\text{Reject } \mathcal{H}_0\} \subseteq \{\hat{\theta} \geq \theta'_i\} \) and that \( \{\text{Reject } \mathcal{H}_0\} \) is determined by the random tuple \((X_1, \cdots, X_n)\) as a consequence of the definition of the test plan. Similarly, statement (II) can be shown by virtue of Lemma 2 and the observation that \( \{\text{Reject } \mathcal{H}_{m-1}\} \subseteq \{\hat{\theta} \leq \theta''_{m-1}\} \) and that \( \{\text{Reject } \mathcal{H}_{m-1}\} \) is determined by the random tuple \((X_1, \cdots, X_n)\) as a consequence of the definition of the test plan.

To show statement (III), note that

\[
\Pr\{\text{Reject } \mathcal{H}_j \mid \theta\} \leq \sum_{i=1}^{j} \Pr\{\text{Accept } \mathcal{H}_{i-1} \mid \theta\} + \sum_{i=j+1}^{m-1} \Pr\{\text{Accept } \mathcal{H}_i \mid \theta\}. \tag{49}
\]

By Lemma 5, we have

\[
f_{\ell,i} \leq \theta''_i, \quad g_{\ell,i} \geq \theta'_i, \quad f_{\ell,i} \leq \frac{\theta'_i + \theta''_i}{2} + \frac{t_{n-1,\alpha_i} - t_{n-1,\beta_i}}{2\sqrt{n-1}} \leq g_{\ell,i}
\]

for \( i = 1, \cdots, m - 1 \) and \( \ell = 1, \cdots, s \). Hence, by the definition of the testing plan, we have

\[
\Pr\{\text{Accept } \mathcal{H}_{i-1} \mid \theta\} < \sum_{n=2}^{\kappa} \Pr\left\{\frac{\overline{X}_n}{\sigma_n} \leq \theta''_i - \frac{t_{n-1,\beta_i}}{\sqrt{n-1}} \mid \theta\right\} + \sum_{n=\kappa+1}^{\infty} \Pr\left\{\frac{\overline{X}_n}{\sigma_n} \leq \frac{\theta'_i + \theta''_i}{2} + \frac{t_{n-1,\alpha_i} - t_{n-1,\beta_i}}{2\sqrt{n-1}} \mid \theta\right\}, \tag{50}
\]

for \( i = 1, \cdots, m \), where \( \kappa \) can be any integer greater than 2. Making use of (50) and applying Lemma 12 with \( \kappa = \kappa(\zeta, \varrho) \), we have that

\[
\lim_{\zeta \to 0} \Pr\{\text{Accept } \mathcal{H}_{i-1} \mid \theta\} = 0, \quad \forall \theta \geq \theta''_i, \quad i = 1, \cdots, m. \tag{51}
\]

Similarly, by the definition of the testing plan, we have

\[
\Pr\{\text{Accept } \mathcal{H}_i \mid \theta\} < \sum_{n=2}^{\kappa} \Pr\left\{\frac{\overline{X}_n}{\sigma_n} \geq \theta'_i + \frac{t_{n-1,\alpha_i}}{\sqrt{n-1}} \mid \theta\right\} + \sum_{n=\kappa+1}^{\infty} \Pr\left\{\frac{\overline{X}_n}{\sigma_n} \geq \frac{\theta'_i + \theta''_i}{2} + \frac{t_{n-1,\alpha_i} - t_{n-1,\beta_i}}{2\sqrt{n-1}} \mid \theta\right\}, \tag{52}
\]

for \( i = 1, \cdots, m - 1 \), where \( \kappa \) can be any integer greater than 2. Making use of (52) and applying Lemma 12 with \( \kappa = \kappa(\zeta, \varrho) \), we have that

\[
\lim_{\zeta \to 0} \Pr\{\text{Accept } \mathcal{H}_i \mid \theta\} = 0, \quad \forall \theta \leq \theta'_i, \quad i = 1, \cdots, m - 1. \tag{53}
\]

Therefore, statement (III) follows from (49), (51) and (53).
Statements (IV) and (V) can be shown by virtue of Lemma 2 and the observation that 
\{\text{Accept } \mathcal{H}_i\} \subseteq \{\theta'_i \leq \hat{\theta} \leq \theta''_{i+1}\} and that \{\text{Accept } \mathcal{H}_i\} is determined by the random tuple 
\((X_1, \cdots, X_n)\) as a consequence of the definition of the testing plan.

To show statement (VI), by the definition of the testing plan, we have that 
\{\text{Reject } \mathcal{H}_i\} is determined by the random tuple 
\((X_1, \cdots, X_n)\). Moreover, for any numbers 
\(a \leq b \leq \theta''_{i+1}\), we have that 
\{\text{Reject } \mathcal{H}_i, \hat{\theta} \leq a \} \cup \{\text{Reject } \mathcal{H}_i, \hat{\theta} \geq b\} = \emptyset, which imply that 
\(\Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \leq a \mid \theta\} + \Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \geq b \mid \theta\}. By Lemma 2 we have that 
\(\Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \leq a \mid \theta\} is non-increasing with respect to \(\theta\ no less than \(a\) and that 
\(\Pr\{\text{Reject } \mathcal{H}_i, \hat{\theta} \geq b \mid \theta\} is non-decreasing with respect to \(\theta\ no greater than \(b\). This leads to the 
upper and lower bounds of \(\Pr\{\text{Reject } \mathcal{H}_i \mid \theta\} in statement (VI).

Statement (VII) can be shown by virtue of Lemma 2 and the observation that 
\{\text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_{m-1}\} \subseteq \{\theta'_1 \leq \hat{\theta} \leq \theta''_{m-1}\} and that \{\text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_{m-1}\} is determined 
by the random tuple \((X_1, \cdots, X_n)\) as a consequence of the definition of the testing plan.

Finally, statement (VIII) can be shown by making use of statement (III) and the observation 
that \{\text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_{m-1}\} \subseteq \{\text{Reject } \mathcal{H}_0\} and \{\text{Reject } \mathcal{H}_0 \text{ and } \mathcal{H}_{m-1}\} \subseteq \{\text{Reject } \mathcal{H}_{m-1}\}. This concludes the proof of Theorem 7.

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