On Grids in Point-Line Arrangements in the Plane

Mozhgan Mirzaei $^1$ · Andrew Suk $^1$

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Abstract
The famous Szemerédi–Trotter theorem states that any arrangement of $n$ points and $n$ lines in the plane determines $O(n^4/3)$ incidences, and this bound is tight. In this paper, we prove the following Turán-type result for point-line incidence. Let $L_a$ and $L_b$ be two sets of $t$ lines in the plane and let $P = \{\ell_a \cap \ell_b : \ell_a \in L_a, \ell_b \in L_b\}$ be the set of intersection points between $L_a$ and $L_b$. We say that $(P, L_a \cup L_b)$ forms a natural $t \times t$ grid if $|P| = t^2$, and conv $P$ does not contain the intersection point of some two lines in $L_a$ and does not contain the intersection point of some two lines in $L_b$. For fixed $t > 1$, we show that any arrangement of $n$ points and $n$ lines in the plane that does not contain a natural $t \times t$ grid determines $O(n^4/3 - \varepsilon)$ incidences, where $\varepsilon = \varepsilon(t) > 0$. We also provide a construction of $n$ points and $n$ lines in the plane that does not contain a natural $2 \times 2$ grid and determines at least $\Omega(n^{1+1/14})$ incidences.

1 Introduction
Given a finite set $P$ of points in the plane and a finite set $L$ of lines in the plane, let $I(P, L) = \{(p, \ell) \in P \times L : p \in \ell\}$ be the set of incidences between $P$ and $L$. The incidence graph of $(P, L)$ is the bipartite graph $G = (P \cup L, I)$, with bipartition classes $P$ and $L$, and $E(G) = I(P, L)$. If $|P| = m$ and $|L| = n$, then the celebrated theorem of Szemerédi and Trotter [17] states that

$$|I(P, L)| \leq O(m^{2/3}n^{2/3} + m + n).$$

(1.1)
Moreover, this bound is tight which can be seen by taking the \( \sqrt{m} \times \sqrt{m} \) integer lattice and bundles of parallel “rich” lines (see [13]). It is widely believed that the extremal configurations maximizing the number of incidences between \( m \) points and \( n \) lines in the plane exhibit some kind of lattice structure. The main goal of this paper is to show that such extremal configurations must contain large natural grids.

Let \( P \) and \( P_0 \) (respectively, \( L \) and \( L_0 \)) be two sets of points (respectively, lines) in the plane. We say that the pairs \((P, L)\) and \((P_0, L_0)\) are isomorphic if their incidence graphs are isomorphic. Solymosi made the following conjecture (see [2, p. 291]).

**Conjecture 1.1** For any set of points \( P_0 \) and for any set of lines \( L_0 \) in the plane, the maximum number of incidences between \( n \) points and \( n \) lines in the plane containing no subconfiguration isomorphic to \((P_0, L_0)\) is \( o(n^{4/3}) \).

In [15], Solymosi proved this conjecture in the special case that \( P_0 \) is a fixed set of points in the plane, no three of which are on a line, and \( L_0 \) consists of all of their connecting lines. However, it is not known if such configurations satisfy the following stronger conjecture.

**Conjecture 1.2** For any set of points \( P_0 \) and for any set of lines \( L_0 \) in the plane, there is a constant \( \varepsilon = \varepsilon(P_0, L_0) \), such that the maximum number of incidences between \( n \) points and \( n \) lines in the plane containing no subconfiguration isomorphic to \((P_0, L_0)\) is \( O(n^{4/3-\varepsilon}) \).

Our first theorem is the following.

**Theorem 1.3** For fixed \( t > 1 \), let \( L_a \) and \( L_b \) be two sets of \( t \) lines in the plane, and let \( P_0 = \{ \ell_a \cap \ell_b : \ell_a \in L_a, \ell_b \in L_b \} \) be such that \(|P_0| = t^2\). Then there is a constant \( c = c(t) \) such that any arrangement of \( m \) points and \( n \) lines in the plane that does not contain a subconfiguration isomorphic to \((P_0, L_a \cup L_b)\) determines at most \( c(m^{(2r-2)/(3r-2)}n^{(2r-1)/(3r-2)} + m^{1+1/6r-3} + n) \) incidences.

See Fig. 1. As an immediate corollary, we prove Conjecture 1.2 in the following special case.

**Corollary 1.4** For fixed \( t > 1 \), let \( L_a \) and \( L_b \) be two sets of \( t \) lines in the plane, and let \( P_0 = \{ \ell_a \cap \ell_b : \ell_a \in L_a, \ell_b \in L_b \}\). If \(|P_0| = t^2\), then any arrangement of \( n \) points and \( n \) lines in the plane that does not contain a subconfiguration isomorphic to \((P_0, L_a \cup L_b)\) determines at most \( O(n^{4/3-1/(9r-6)}) \) incidences.

In the other direction, we prove the following.

**Theorem 1.5** Let \( L_a \) and \( L_b \) be two sets of \( t \) lines in the plane, and let \( P_0 = \{ \ell_a \cap \ell_b : \ell_a \in L_a, \ell_b \in L_b \} \) be such that \(|P_0| = 4\). For \( n > 1 \), there exists an arrangement of \( n \) points and \( n \) lines in the plane that does not contain a subconfiguration isomorphic to \((P_0, L_a \cup L_b)\), and determines at least \( \Omega(n^{1+1/14}) \) incidences.

Given two sets \( L_a \) and \( L_b \) of \( t \) lines in the plane, and the point set \( P_0 = \{ \ell_a \cap \ell_b : \ell_a \in L_a, \ell_b \in L_b \} \), we say that \((P_0, L_a \cup L_b)\) forms a natural \( t \times t \) grid if \(|P_0| = t^2\), and the convex hull of \( P_0 \), conv \( P_0 \), does not contain the intersection point of any two lines in \( L_a \) and does not contain the intersection point of any two lines in \( L_b \). See Fig. 2.
Fig. 1 An example with $|L_a| = |L_b| = 3$ and $|P| = 9$ in Theorem 1.3

Fig. 2 An example of a natural $3 \times 3$ grid

**Theorem 1.6** For fixed $t > 1$, there is a constant $\varepsilon = \varepsilon(t)$, such that any arrangement of $n$ points and $n$ lines in the plane that does not contain a natural $t \times t$ grid determines at most $O(n^{4/3-\varepsilon})$ incidences.

Let us remark that $\varepsilon = \Omega(1/t^2)$ in Theorem 1.6, and it can be easily generalized to the off-balanced setting of $m$ points and $n$ lines. We systemically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of our presentation. All logarithms are assumed to be base 2. For $N > 0$, we let $[N] = \{1, \ldots, N\}$.

**2 Proof of Theorem 1.3**

In this section we will prove Theorem 1.3. We first list several results that we will use. The first lemma is a classic result in graph theory.

**Lemma 2.1** (Kővari–Sós–Turán [10]) Let $G = (V, E)$ be a graph that does not contain a complete bipartite graph $K_{r,s}$, $1 \leq r \leq s$, as a subgraph. Then $|E| \leq c_s |V|^{2-1/r}$, where $c_s > 0$ is a constant which only depends on $s$. 

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The next lemma we will use is a partitioning tool in discrete geometry known as simplicial partitions. We will use the dual version which requires the following definition. Let $\mathcal{L}$ be a set of lines in the plane. We say that a point $p$ crosses $\mathcal{L}$ if it is incident to at least one member of $\mathcal{L}$, but not incident to all members in $\mathcal{L}$.

**Lemma 2.2** (Matoušek [12]) Let $\mathcal{L}$ be a set of $n$ lines in the plane and let $r$ be a parameter such that $1 < r < n$. Then there is a partition of $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_r$ into $r$ parts, where $n/(2r) \leq |\mathcal{L}_i| \leq 2n/r$, such that any point $p \in \mathbb{R}^2$ crosses at most $O(\sqrt{r})$ parts $\mathcal{L}_i$.

**Proof of Theorem 1.3** Set $t \geq 2$. Let $P$ be a set of $m$ points in the plane and let $\mathcal{L}$ be a set of $n$ lines in the plane such that $(P, \mathcal{L})$ does not contain a subconfiguration isomorphic to $(P_0, \mathcal{L}_a \cup \mathcal{L}_b)$.

If $n \geq m^2/100$, then (1.1) implies that $|I(P, \mathcal{L})| = O(n)$ and we are done. Likewise, if $n \leq m^{t/(2r-1)}$, then (1.1) implies that $|I(P, \mathcal{L})| = O(m^{1+1/(6r-3)})$ and we are done. Therefore, let us assume $m^{t/(2r-1)} < n < m^2/100$. In what follows, we will show that $|I(P, \mathcal{L})| = O(m^{(2t-2)/(3r-2)}n^{(2r-1)/(3r-2)})$. For the sake of contradiction, suppose that $|I(P, \mathcal{L})| \geq cm^{(2t-2)/(3r-2)}n^{(2r-1)/(3r-2)}$, where $c$ is a large constant depending on $t$ that will be determined later.

Set $r = \lceil 10n^{(4t-2)/(3r-2)}/m^{2t/(3r-2)} \rceil$. Let us remark that $1 < r < n/10$ since we are assuming $m^{t/(2r-1)} < n < m^2/100$. We apply Lemma 2.2 with parameter $r$ to $\mathcal{L}$, and obtain a partition $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_r$ with the properties described above. Note that $|\mathcal{L}_i| > 1$. Let $G$ be the incidence graph of $(P, \mathcal{L})$. For $p \in P$, consider the set of lines in $\mathcal{L}_i$. If $p$ is incident to exactly one line in $\mathcal{L}_i$, then delete the corresponding edge in the incidence graph $G$. After performing this operation between each point $p \in P$ and each class $\mathcal{L}_i$, by Lemma 2.2, we have deleted at most $c_1m\sqrt{r}$ edges in $G$, where $c_1$ is an absolute constant. By setting $c$ sufficiently large, we have

$$c_1m\sqrt{r} = \sqrt{10}c_1m^{(2t-2)/(3r-2)}n^{(2r-1)/(3r-2)} \leq \frac{c}{2}m^{(2t-2)/(3r-2)}n^{(2r-1)/(3r-2)}.$$  

Therefore, there are at least $cm^{(2t-2)/(3r-2)}n^{(2r-1)/(3r-2)}/2$ edges remaining in $G$. By the pigeonhole principle, there is an $i$ such that the number of edges between $P$ and $\mathcal{L}_i$ in $G$ is at least

$$\frac{cm^{(2t-2)/(3r-2)}n^{(2r-1)/(3r-2)}}{2r} = \frac{cm^{(4t-2)/(3r-2)}}{20n^{(2r-1)/(3r-2)}}.$$  

Hence, every point $p \in P$ has either 0 or at least 2 neighbors in $\mathcal{L}_i$ in $G$. We claim that $(P, \mathcal{L}_i)$ contains a subconfiguration isomorphic to $(P_0, \mathcal{L}_a \cup \mathcal{L}_b)$. To see this, let us construct a graph $H = (\mathcal{L}_i, E)$ as follows. Set $V(H) = \mathcal{L}_i$. Let $Q = \{q_1, \ldots, q_w\} \subset P$ be the set of points in $P$ that have at least two neighbors in $\mathcal{L}_i$ in the graph $G$. For $q_j \in Q$, consider the set of lines $\{\ell_1, \ldots, \ell_s\}$ from $\mathcal{L}_i$ incident to $q_j$, such that $\{\ell_1, \ldots, \ell_s\}$ appear in clockwise order. Then we define $E_j \subset \binom{s}{2}$ to be a matching on $\{\ell_1, \ldots, \ell_s\}$, where
\[ E_j = \begin{cases} 
\{(\ell_1, \ell_2), (\ell_3, \ell_4), \ldots, (\ell_{s-1}, \ell_s)\} & \text{if } s \text{ is even}, \\
\{(\ell_1, \ell_2), (\ell_3, \ell_4), \ldots, (\ell_{s-2}, \ell_{s-1})\} & \text{if } s \text{ is odd}.
\end{cases} \]

Set \( E(H) = E_1 \cup E_2 \cup \cdots \cup E_w \). Note that \( E_j \) and \( E_k \) are disjoint, since no two points are contained in two lines. Since \(|E_j| \geq 1\), we have

\[ |E(H)| \geq \frac{cm^{(4t-2)/(3t-2)}}{60n^{(2t-1)/(3t-2)}}. \]

Since

\[ |V(H)| = |\mathcal{L}_i| \leq \frac{m^{2t/(3t-2)}}{5n^{t/(3t-2)}}, \]

this implies

\[ |E(H)| \geq \frac{c|V(H)|^{2-1/t}}{60 \cdot 25}. \]

By setting \( c = c(t) \) to be sufficiently large, Lemma 2.1 implies that \( H \) contains a copy of \( K_{t,t} \). Let \( \mathcal{L}_1', \mathcal{L}_2' \subset \mathcal{L}_i \) correspond to the vertices of this \( K_{t,t} \) in \( H \), and let \( P' = \{\ell_1 \cap \ell_2 \in P : \ell_1 \in \mathcal{L}_1', \ell_2 \in \mathcal{L}_2'\} \). We claim that \( (P', \mathcal{L}_1' \cup \mathcal{L}_2') \) is isomorphic to \((P_0, \mathcal{L}_a \cup \mathcal{L}_b)\). It suffices to show that \( |P'| = t^2 \). For the sake of contradiction, suppose \( p \in \ell_1 \cap \ell_2 \cap \ell_3 \), where \( \ell_1, \ell_2 \in \mathcal{L}_1' \) and \( \ell_3 \in \mathcal{L}_2' \). This would imply \((\ell_1, \ell_3), (\ell_2, \ell_3) \in E_j\) for some \( j \) which contradicts the fact that \( E_j \subset (\mathcal{L}_2')^2 \) is a matching. Same argument follows if \( \ell_1 \in \mathcal{L}_1' \) and \( \ell_2, \ell_3 \in \mathcal{L}_2' \). This completes the proof of Theorem 1.3.

\[ \square \]

### 3 Natural Grids

Given a set of \( n \) points \( P \) and a set of \( n \) lines \( \mathcal{L} \) in the plane, if \(|I(P, \mathcal{L})| \geq cn^{4/3-1/(9k-6)}\), where \( c \) is a sufficiently large constant depending on \( k \), Corollary 1.4 implies that there are two sets of \( k \) lines such that each pair of them from different sets intersects at a unique point in \( P \). Therefore, Theorem 1.6 follows by combining Theorem 1.3 with the following lemma.

**Lemma 3.1** There is a natural number \( c \) such that the following holds. Let \( \mathcal{B} \) be a set of \( ct^2 \) blue lines in the plane, and let \( \mathcal{R} \) be a set of \( ct^2 \) red lines in the plane such that for \( P = \{\ell_1 \cap \ell_2 : \ell_1 \in \mathcal{B}, \ell_2 \in \mathcal{R}\} \) we have \(|P| = ct^4\). Then \((P, \mathcal{B} \cup \mathcal{R})\) contains a natural \( t \times t \) grid.

To prove Lemma 3.1, we will need the following lemma which is an immediate consequence of Dilworth’s Theorem.

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Lemma 3.2 For \( n > 0 \), let \( \mathcal{L} \) be a set of \( n^2 \) lines in the plane, such that no two members intersect the same point on the y-axis. Then there is a subset \( \mathcal{L}' \subset \mathcal{L} \) of size \( n \) such that the intersection point of any two members in \( \mathcal{L}' \) lies to the left of the y-axis, or the intersection point of any two members in \( \mathcal{L}' \) lies to the right of the y-axis.

Proof Let us order the elements in \( \mathcal{L} = \{ \ell_1, \ldots, \ell_{n^2} \} \) from bottom to top according to their y-intercept. By Dilworth’s Theorem [5], \( \mathcal{L} \) contains a subsequence of \( n \) lines whose slopes are either increasing or decreasing. In the first case, all intersection points are to the left of the y-axis, and in the latter case, all intersection points are to the right of the y-axis. \( \square \)

Proof of Lemma 3.1 Let \((P, B \cup R)\) be as described above, and let \( \ell_y \) be the y-axis. Without loss of generality, we can assume that all lines in \( B \cup R \) are not vertical, and the intersection point of any two lines in \( B \cup R \) lies to the right of \( \ell_y \). Moreover, we can assume that no two lines intersect at the same point on \( \ell_y \).

We start by finding a point \( y_1 \in \ell_y \) such that at least \( |B|/2 \) blue lines in \( B \) intersect \( \ell_y \) on one side of the point \( y_1 \) (along \( \ell_y \)) and at least \( |R|/2 \) red lines in \( R \) intersect \( \ell_y \) on the other side. This can be done by sweeping the point \( y_1 \) along \( \ell_y \) from bottom to top until \( ct^2/2 \) lines of the first color, say red, intersect \( \ell_y \) below \( y_1 \). We then have at least \( ct^2/2 \) blue lines intersecting \( \ell_y \) above \( y_1 \). Discard all red lines in \( R \) that intersect \( \ell_y \) above \( y_1 \), and discard all blue lines in \( B \) that intersect \( \ell_y \) below \( y_1 \). Hence, \( |B| \geq ct^2/2 \).

Set \( s = \lfloor ct^2/4 \rfloor \). For the remaining lines in \( B \), let \( B = \{ b_1, \ldots, b_{2s} \} \), where the elements of \( B \) are ordered in the order they cross \( \ell_y \), from bottom to top. We partition \( B = B_1 \cup B_2 \) into two parts, where \( B_1 = \{ b_1, \ldots, b_s \} \) and \( B_2 = \{ b_{s+1}, \ldots, b_{2s} \} \). By applying an affine transformation, we can assume all lines in \( R \) have positive slope and all lines in \( B_1 \cup B_2 \) have negative slope. See Fig. 3.

Let us define a 3-partite 3-uniform hypergraph \( H = (R \cup B_1 \cup B_2, E) \), whose partition classes are \( R, B_1, B_2 \), and \((r, b_i, b_j) \in R \times B_1 \times B_2 \) is an edge in \( H \) if
and only if the intersection point $p = b_i \cap b_j$ lies above the line $r$. Note, if $b_i$ and $b_j$ are parallel, then $(r, b_i, b_j) \notin E$. Then a result of Fox et al. on semi-algebraic hypergraphs implies the following (see also [3,9]).

\[ \square \]

**Lemma 3.3** (Fox et al. [8, Thm. 8.1]) There exists a positive constant $\alpha$ such that the following holds. In the hypergraph above, there are subsets $R' \subseteq R$, $B'_1 \subseteq B_1$, $B'_2 \subseteq B_2$, where $|R'| \geq \alpha |R|$, $|B'_1| \geq \alpha |B_1|$, $|B'_2| \geq \alpha |B_2|$, such that either $R' \times B'_1 \times B'_2 \subseteq E$, or $(R' \times B'_1 \times B'_2) \cap E = \emptyset$.

We apply Lemma 3.3 to $H$ and obtain subsets $R', B'_1, B'_2$ with the properties described above. Without loss of generality, we can assume that $R' \times B'_1 \times B'_2 \subseteq E$, since a symmetric argument would follow otherwise. Let $\ell_1$ be a line in the plane such that the following holds:

1. The slope of $\ell_1$ is negative.
2. All intersection points between $R'$ and $B'_1$ lie above $\ell_1$.
3. All intersection points between $R'$ and $B'_2$ lie below $\ell_1$.

See Fig. 4.

**Observation 3.4** Line $\ell_1$ defined above exists.

**Proof** Let $U$ be the upper envelope of the arrangement $\bigcup_{\ell \in R'} \ell$, that is, $U$ is the closure of all points that lie on exactly one line of $R'$ and strictly above exactly the $|R'| - 1$ lines in $R'$.

Let $P_1$ be the set of intersection points between the lines in $B'_1$ with $U$. Likewise, we define $P_2$ to be the set of intersection points between the lines in $B'_2$ with $U$. Since $U$ is $x$-monotone and convex the set $P_2$ lies to the left of the set $P_1$. Then the line $\ell_1$ that intersects $U$ between $P_1$ and $P_2$ and intersects $\ell_y$ between $B'_1$ and $B'_2$ satisfies the conditions above.
Now we apply Lemma 3.2 to \( R' \) with respect to the line \( \ell_1 \), to obtain \( t\sqrt{ac/2} \) members in \( R' \) such that every pair of them intersects on one side of \( \ell_1 \). Discard all other members in \( R' \). Without loss of generality, we can assume that all intersection points between any two members in \( R' \) lie below \( \ell_1 \), since a symmetric argument would follow otherwise. We now discard the set \( B'_2 \).

Notice that the order in which the lines in \( R' \) cross \( b \in B'_1 \) will be the same for any line \( b \in B'_1 \). Therefore, we order the elements in \( R' = \{r_1, \ldots, r_m\} \) with respect to this ordering, from left to right, where \( m = \lceil t\sqrt{ac/2} \rceil \). We define \( \ell_2 \) to be the line obtained by slightly perturbing the line \( r_{\lfloor m/2 \rfloor} \) so that:

1. The slope of \( \ell_2 \) is positive.
2. All intersection points between \( B'_1 \) and \( \{r_1, \ldots, r_{\lfloor m/2 \rfloor}\} \) lie above \( \ell_2 \).
3. All intersection points between \( B'_1 \) and \( \{r_{\lfloor m/2 \rfloor + 1}, \ldots, r_m\} \) lie below \( \ell_2 \).

See Fig. 5. Finally, we apply Lemma 3.2 to \( B'_1 \) with respect to the line \( \ell_2 \), to obtain at least \( t\sqrt{ac/2} \) members in \( B'_1 \) with the property that any two of them intersect on one side of \( \ell_2 \). Without loss of generality, we can assume that any two such lines intersect below \( \ell_2 \) since a symmetric argument would follow. Set \( B^* \subset B'_1 \) to be this set of lines. Then \( B^* \cup \{r_1, \ldots, r_{\lfloor m/2 \rfloor}\} \) and their intersection points form a natural grid. By setting \( c = c(t) \) to be sufficiently large, we obtain a natural \( t \times t \) grid.

\[ \square \]

### 4 Lower Bound Construction

In this section, we will prove Theorem 1.5. First, we recall the definitions of Sidon and \( k \)-fold Sidon sets. Let \( A \) be a finite set of positive integers. Then \( A \) is a Sidon set if the sums of all pairs are distinct, that is, the equation \( x + y = u + v \) has no solutions with \( x, y, u, v \in A \), except for trivial solutions given by \( u = x, y = v \) and \( x = v, y = u \). We define \( s(N) \) to be the size of the largest Sidon set \( A \subset \{1, \ldots, N\} \). Erdős and Turán proved the following.

**Lemma 4.1** ([7] and [14]) For \( N > 1 \) we have \( s(N) = \Theta(\sqrt{N}) \).
Let us now consider a more general equation. Let \( u_1, u_2, u_3, u_4 \) be integers such that \( u_1 + u_2 + u_3 + u_4 = 0 \), and consider the equation

\[
u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0. \tag{4.1}\]

We are interested in solutions to (4.1) with \( x_1, x_2, x_3, x_4 \in \mathbb{Z} \). Suppose \( (x_1, x_2, x_3, x_4) = (a_1, a_2, a_3, a_4) \) is an integer solution to (4.1). Let \( d \leq 4 \) be the number of distinct integers in the set \( \{a_1, a_2, a_3, a_4\} \). Then we have a partition of the indices, \( \{1, 2, 3, 4\} = T_1 \cup \cdots \cup T_d \), where \( i \) and \( j \) lie in the same part \( T_v \) if and only if \( x_i = x_j \). We call \( (a_1, a_2, a_3, a_4) \) a trivial solution to (4.1) if

\[
\sum_{i \in T_v} u_i = 0, \quad v = 1, \ldots, d.
\]

Otherwise, we will call \( (a_1, a_2, a_3, a_4) \) a nontrivial solution to (4.1).

In [11], Lazebnik and Verstraëte introduced \( k \)-fold Sidon sets which are defined as follows. Let \( k \) be a positive integer. A set \( A \subset \mathbb{N} \) is a \( k \)-fold Sidon set if each equation of the form (4.1) where \( |u_i| \leq k \) and \( u_1 + u_2 + u_3 + u_4 = 0 \), has no nontrivial solutions with \( x_1, x_2, x_3, x_4 \in A \). Let \( r(k, N) \) be the size of the largest \( k \)-fold Sidon set \( A \subset \{1, \ldots, N\} \).

**Lemma 4.2** There is an infinite sequence \( 1 = a_1 < a_2 < \cdots \) of integers such that \( a_m \leq 2^8k^4m^3 \), and the system of equations (4.1) has no nontrivial solutions in the set \( A = \{a_1, a_2, \ldots\} \). In particular, for integers \( N > k^4 \geq 1 \), we have \( r(k, N) \geq ck^{-4/3}N^{1/3} \), where \( c \) is a positive constant.

The proof of Lemma 4.2 is a slight modification of the proof of [14, Thm. 2.1]. For the sake of completeness, we include it here.

**Proof** We put \( a_1 = 1 \) and define \( a_m \) recursively. Given \( a_1, \ldots, a_{m-1} \), let \( a_m \) be the smallest positive integer satisfying

\[
-a_m \sum_{i \in S} u_i \neq \sum_{1 \leq i \leq 4, \ n \neq S} u_i x_i, \tag{4.2}
\]

for every choice \( u_i \) such that \( |u_i| \leq k \), for every set \( S \subset \{1, 2, 3, 4\} \) of subscripts such that \( \sum_{i \in S} u_i \neq 0 \), and for every choice of \( x_j \in \{a_1, \ldots, a_{m-1}\} \), where \( i \notin S \). For a fixed \( S \) with \( |S| = j \), this excludes \( (m - 1)^{4-j} \) numbers. Since \( |u_i| \leq k \), the total number of excluded integers is at most

\[
(2k + 1)^4 \sum_{j=1}^{3} \binom{4}{j} (m - 1)^{4-j} = (2k + 1)^4(m^4 - (m - 1)^4 - 1) < 2^8k^4m^3.
\]

Consequently, we can extend our set by an integer \( a_m \leq 2^8k^4m^3 \). This will automatically be different from \( a_1, \ldots, a_{m-1} \), since putting \( x_j = a_j \) for all \( i \notin S \) in (4.2) we get \( a_m \neq a_j \). It will also satisfy \( a_m > a_{m-1} \) by the minimal choice of \( a_{m-1} \).
We show that the system of equations (4.1) has no nontrivial solutions in the set \( \{a_1, \ldots, a_m\} \). We use induction on \( m \). The statement is obviously true for \( m = 1 \). We establish it for \( m \) assuming it holds for \( m - 1 \). Suppose that there is a nontrivial solution \((x_1, x_2, x_3, x_4)\) to (4.1) for some \( u_1, u_2, u_3, u_4 \) with the properties described above. Let \( S \) denote the set of those subscripts for which \( x_i = a_m \). If \( \sum_{i \in S} u_i \neq 0 \), then this contradicts (4.2). If \( \sum_{i \in S} u_i = 0 \), then by replacing each occurrence of \( a_m \) by \( a_1 \), we get another nontrivial solution, which contradicts the induction hypothesis. \(\Box\)

For more problems and results on Sidon sets and \( k \)-fold Sidon sets, we refer the interested reader to [4,11,14]. We are now ready to prove Theorem 1.5.

**Proof of Theorem 1.5** We start by applying Lemma 4.1 to obtain a Sidon set \( M \subset [n^{1/7}] \), such that \( |M| = \Theta(n^{1/4}) \). We then apply Lemma 4.2 with \( k = n^{1/7} \) and \( N = n^{11/4}/4 \), to obtain a \( k \)-fold Sidon set \( A \subset [N] \) such that

\[
|A| \geq cn^{1/14},
\]

where \( c \) is defined in Lemma 4.2. Without loss of generality, assume \( |A| = cn^{1/14} \).

Let \( P = \{(i, j) \in \mathbb{Z}^2 : i \in A, 1 \leq j \leq n^{13/4}\} \), and let \( L \) be the family of lines in the plane of the form \( y = mx + b \), where \( m \in M \) and \( b \) is an integer such that \( 1 \leq b \leq n^{13/4}/2 \). Hence, we have

\[
|P| = |A| \cdot n^{13/14} = \Theta(n), \quad |L| = |M| \cdot \frac{n^{13/4}}{2} = \Theta(n).
\]

Notice that each line in \( L \) has exactly \( |A| = cn^{1/14} \) points from \( P \) since \( 1 \leq b \leq n^{13/4}/2 \). Therefore,

\[
|I(P, L)| = |L| \cdot |A| = \Theta(n^{1+1/14}). \quad \square
\]

**Claim 4.3** There are no four distinct lines \( \ell_1, \ell_2, \ell_3, \ell_4 \in L \) and four distinct points \( p_1, p_2, p_3, p_4 \in P \) such that \( \ell_1 \cap \ell_2 = p_1, \ell_2 \cap \ell_3 = p_2, \ell_3 \cap \ell_4 = p_3, \ell_4 \cap \ell_1 = p_4 \).

**Proof** For the sake of contradiction, suppose there are four lines \( \ell_1, \ell_2, \ell_3, \ell_4 \) and four points \( p_1, p_2, p_3, p_4 \) with the properties described above. Let \( \ell_i = m_i x + b_i \) and \( p_i = (x_i, y_i) \). Therefore,

\[
\ell_1 \cap \ell_2 = p_1 = (x_1, y_1), \quad \ell_2 \cap \ell_3 = p_2 = (x_2, y_2),
\]

\[
\ell_3 \cap \ell_4 = p_3 = (x_3, y_3), \quad \ell_4 \cap \ell_1 = p_4 = (x_4, y_4).
\]

Hence,

\[
p_1 \in \ell_1, \ell_2 \implies (m_1 - m_2)x_1 + b_1 - b_2 = 0,
p_2 \in \ell_2, \ell_3 \implies (m_2 - m_3)x_2 + b_2 - b_3 = 0,
p_3 \in \ell_3, \ell_4 \implies (m_3 - m_4)x_3 + b_3 - b_4 = 0,
p_4 \in \ell_4, \ell_1 \implies (m_4 - m_1)x_4 + b_4 - b_1 = 0.
\]
By summing up the four equations above, we get

\[(m_1 - m_2)x_1 + (m_2 - m_3)x_2 + (m_3 - m_4)x_3 + (m_4 - m_1)x_4 = 0.\]

By setting \(u_1 = m_1 - m_2, u_2 = m_2 - m_3, u_3 = m_3 - m_4, u_4 = m_4 - m_1,\) we get

\[u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0, \tag{4.3}\]

where \(u_1 + u_2 + u_3 + u_4 = 0\) and \(|u_i| \leq n^{1/7}.\) Since \(x_1, x_2, x_3, x_4 \in A, (x_1, x_2, x_3, x_4)\) must be a trivial solution to (4.3). The proof now falls into the cases that follow. Note that no line in \(\mathcal{L}\) is vertical.

Case 1. Suppose \(x_1 = x_2 = x_3 = x_4.\) Then \(\ell_i\) is vertical and we have a contradiction.

Case 2. Suppose \(x_1 = x_2 = x_3 \neq x_4, u_1 + u_2 + u_3 = 0,\) and \(u_4 = 0.\) Then \(\ell_1\) and \(\ell_4\) have the same slope which is a contradiction. The same argument follows if \(x_1 = x_2 = x_4 \neq x_3, x_1 = x_3 = x_4 \neq x_2,\) or \(x_2 = x_3 = x_4 \neq x_1.\)

Case 3. Suppose \(x_1 = x_2 \neq x_3 = x_4, u_1 + u_2 = 0,\) and \(u_3 + u_4 = 0.\) Since \(p_1, p_2 \in \ell_2\) and \(x_1 = x_2,\) this implies that \(\ell_2\) is vertical which is a contradiction. A similar argument follows if \(x_1 = x_4 \neq x_2 = x_3, u_1 + u_4 = 0,\) and \(u_2 + u_3 = 0.\)

Case 4. Suppose \(x_1 = x_3 \neq x_2 = x_4, u_1 + u_3 = 0,\) and \(u_2 + u_4 = 0.\) Then \(u_1 + u_3 = 0\) implies that \(m_1 + m_3 = m_2 + m_4.\) Since \(M\) is a Sidon set, we have either \(m_1 = m_2\) and \(m_3 = m_4\) or \(m_1 = m_4\) and \(m_2 = m_3.\) The first case implies that \(\ell_1\) and \(\ell_2\) are parallel which is a contradiction, while the second case implies that \(\ell_2\) and \(\ell_3\) are parallel, which is again a contradiction. \(\square\)

This completes the proof of Theorem 1.5.

5 Concluding Remarks

An old result of Erdős states that every \(n\)-vertex graph that does not contain a cycle of length \(2k,\) has \(O_k(n^{1+1/k})\) edges. It is known that this bound is tight when \(k = 2, 3, 5,\) but it is a long standing open problem in extremal graph theory to decide whether or not this upper bound can be improved for other values of \(k.\) Hence, Erdős’s upper bound of \(O(n^{5/4})\) when \(k = 4\) implies Theorem 1.3 when \(t = 2\) and \(m = n.\) It would be interesting to see if one can improve the upper bound in Theorem 1.3 when \(t = 2.\) For more problems on cycles in graphs, see [18].

The proof of Lemma 3.1 is similar to the proof of the main result in [1]. The main difference is that we use the result of Fox et al. [8] instead of the Ham-Sandwich Theorem. We also note that a similar result was established by Dujmović and Langerman (see Theorem 6 in [6]).

Recently, Tomon and the second author [16] improved the lower bound in Theorem 1.5 to \(n^{9/8+o(1)}\), and more generally, gave a construction of \(n\) points and \(n\) lines in the plane with no \(k \times k\) grid and with at least \(n^{4/3-\Theta(1/k)}\) incidences.
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