§0 Introduction

Holomorphic torsion, also called $\bar{\partial}$-torsion, is a positive real number $\tau(X, V)$ defined by Ray and Singer [R-S II] for a compact complex Hermitian manifold $X$ endowed with a Hermitian complex vector bundle $V$, in terms of zeta-regularized determinants of Hodge-Laplace operators on $(0, q)$-forms on $X$ with values in $V$. It was shown by Quillen [Q] in the case of complex curves, and by Bismut, Gillet and Soulé [BGS I, II, III] in general, that the $\bar{\partial}$-torsion provides “the” natural smooth metric on the determinant line bundle for direct images over complex algebraic manifolds. Given the perceived pattern of formal analogies between the Arakelov theory in arithmetic geometry and dynamical systems, it appears desirable to have a dynamical description of the $\bar{\partial}$-torsion. Such an interpretation was obtained by Fried [F2] in the case of complex hyperbolic manifolds. These are compact quotients $X = \Gamma \backslash \mathfrak{H}^n_C$ of complex hyperbolic spaces modulo discrete subgroups of holomorphic isometries $\Gamma \subset SU(n, 1)$. In this paper we extend the dynamical description of the $\bar{\partial}$-torsion to the case of compact Hermitian locally symmetric manifolds $X = \Gamma \backslash G/K$ whose connected group $G$ of isometries for the universal cover has only one class of cuspidal maximal parabolic subgroups or, equivalently, whose Satake diagram has no double lines.

To formulate our main result, let us fix such a compact complex manifold $X$; thus, its universal cover is a Hermitian globally symmetric space $\tilde{X} \simeq G/K$, with $G$ semisimple.
non-compact, $K$ a maximal compact subgroup, and $\Gamma$ a discrete, co-compact, torsion-free subgroup of $G$. We endow $X$ with the locally symmetric metric $g$ induced by the standard Hermitian metric on $\tilde{X}$, and also fix a holomorphic, Hermitian, homogeneous vector bundle $V$ over $X$. Recall that the periodic set of the geodesic flow is a disjoint union of connected components $\bigsqcup_{[\gamma]} X_\gamma$, parametrized by the conjugacy classes in $\Gamma \backslash \{e\}$. The components $X_\gamma$ are themselves locally symmetric manifolds $X_\gamma \cong \Gamma_\gamma \backslash G_\gamma / K_\gamma$. All geodesics in $X_\gamma$ have common length $\ell_\gamma$, and we let $\mu_\gamma$ denote the generic multiplicity. Out of these data, to which we add the choice of a standard compactification of $\tilde{X}$, namely the closure $\overline{X} = G \cdot K \subseteq \mathbb{P}$ of the Borel embedding of $\tilde{X}$ into $G \mathbb{C} / K \mathbb{C} P_-$, we construct a Weil-type zeta function of the form

$$Z_V(z) = \exp \left( - \sum_{[\gamma] \neq e} \frac{\chi_\gamma(V) e^{-z\ell_\gamma}}{\mu_\gamma} \right), \quad \Re z^2 >> 0,$$

whose Lefschetz coefficients $\chi_\gamma(V)$ can be viewed as “mixed” Euler characteristics of $X_\gamma$ with local coefficients in normal directions. We prove that $Z_V(z)$ is analytic for $\Re z^2 >> 0$, has meromorphic continuation to $\mathbb{C}$, and its leading term in the Laurent expansion at $z = 0$ captures the $\bar{\partial}$-torsion $\tau(X, V)$. More precisely,

$$Z_V(z) \sim c \frac{\tau(X, V)^2 \tau(X^d, V)^{2\chi_a(X)} z^\nu}{\tau(X^d, V)^{2\chi_a(X)}}, \quad \text{for } z \text{ near } 0,$$

where $c$ is a universal constant, $X^d$ denotes the compact dual symmetric space, whose holomorphic torsion $\tau(X^d, V)$ has been computed by Köhler [K], and $\chi_a(X)$ is the arithmetic genus of $X$. Quite remarkably, the order $\nu$ of the zero is a characteristic number

$$\nu = 2 \int_X \left( \mathcal{T}d'(R_g) - n \mathcal{T}d(R_g) \right) ch(L_V),$$

where $\mathcal{T}d$ denotes the Todd polynomial, $\mathcal{T}d'$ is a derived version of it, while $R_g$ and $L_V$ stand for the curvature of the canonical Hermitian connection of $X$, resp. for that of the holomorphic bundle $V$.

The above zeta function is described in great detail in §2, while the representation-theoretic expression of its topological coefficients is worked out in §3. The analysis necessary to develop its properties, which is based on the Selberg trace formula applied to the heat kernel of the Hodge-Laplace operators, is carried out in §§4–8 under two basic assumptions,
viz. that there is only one class of cuspidal parabolic subgroup, and the validity of an Ansatz. Evaluating explicitly the orbital integrals involved in this approach is usually a formidable task, and especially so in the higher rank case. However, in this instance, the underlying geometric nature of the integral operator introduces enough cancellations to allow a tractable calculation. More precisely, these simplifications arise from the algebraic results concerning derived Euler numbers established in §1. Finally, in §9 we rely on the Bismut-Gillet-Soulé anomaly formula [B-G-S, III] for the Quillen metric, applied to the scaled family of metrics $g_\epsilon = \epsilon g$, to unravel the topological meaning of order of the zero of the zeta function $Z_V(z)$ at $z = 0$ as a characteristic number.

This work was essentially completed 20 years ago but for the determination of the validity of the Ansatz. Lacking a satisfactory understanding of its scope, we postponed its formal submission for publication. The efforts of our junior colleague, Jan Frahm, to clarify the problem and to resolve several cases of it are presented in an Appendix. We are extremely grateful to him for this. From his work one can see that for $\mathbb{R}$-rank one groups our construction works with completely general vector bundle coefficients, for some rank two groups with specific vector bundle coefficients, and also that there are groups which do not satisfy the Ansatz.

The above mentioned postponement of the present work for publication had the unfortunate effect of delaying the rectification of a gap in the justification of the convergence of certain orbital integral expansions in the paper [M-S;II]. Instead of the flawed presentation therein, one should use exactly the same reasoning as in §5 below, which essentially reproduces Fried’s method. In the case of trivial coefficients a different correct justification was supplied by Shu Shen [Sh]. His proof of Fried’s conjecture is based on Bismut’s orbital integral formula [Bi]. It would be interesting to investigate if Bismut’s method of hypoelliptic Laplacians gives rise to a canonical class of zeta function encoding the holomorphic torsion with coefficients, independent of the choice of compactification for the globally symmetric space $\tilde{X} = G/K$, and of the validity of the Ansatz.

§1 Derived Euler Characteristics

Let $\tilde{X}$ be a $2n$ dimensional globally symmetric space, Hermitian and of non-compact type. The connected group of isometries, $G$, is a semisimple Lie group. If we fix a basepoint in $\tilde{X}$ and let $K$ be the isotropy subgroup at this point, then $K$ is a maximal compact subgroup
of $G$ and $\tilde{X}$ is diffeomorphic with $G/K$. Let the Lie algebra of $G$ (resp. $K$) be denoted by $\mathfrak{g}$ (resp. $\mathfrak{k}$). The tangent space to $\tilde{X}$ at $eK$ can be identified with a subspace $\mathfrak{p} \subseteq \mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition. A complex structure as well as a Hermitian metric on $G/K$ arise naturally from $\mathfrak{g}$. Indeed, there is an element $H_0$ from the center of $\mathfrak{k}$ such that $ad H_0$ on $\mathfrak{p}$ defines an almost complex structure which we assume agrees with that on $\tilde{X}$. The usual metric on $G/K$ is obtained simply from the restriction of the Killing form to $\mathfrak{p}$ (then translated by $G$ around $G/K$). We shall also need metrics obtained by scaling the usual metric. The effect of scaling in the formulas will be indicated later, for now we assume that $\tilde{X}$ is isometric to $G/K$ with the usual metric.

One has the decomposition $\mathfrak{p}_\mathbb{C} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, with $\mathfrak{p}_\pm$ the $\pm i$ eigenspaces of $ad H_0$. As $H_0$ is central in $\mathfrak{k}$, each of $\mathfrak{p}_\pm$ is invariant by $Ad K$. We shall denote by $\Lambda^p_+$ (resp. $\Lambda^q_-$) the representation of $K$ on $\Lambda^p \mathfrak{p}_+$ (resp. $\Lambda^q \mathfrak{p}_-$). As $\mathfrak{p}_+$ is $K$-equivariantly isomorphic to $(\mathfrak{p}_-)^*$, we have $\Lambda^p_+$ is $K$-equivariantly isomorphic to $(\Lambda^p_-)^*$.

For later purposes let us fix now a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$ containing $H_0$. The usual accompanying baggage of structure shall be with respect to $\mathfrak{t}$ and used freely. In particular, we will use any order on $\Delta(\mathfrak{g})$ such that $\mathfrak{p}_+$ is the span of the positive non-compact root vectors. As $H_0$ is central in $\mathfrak{t}$ this does not select an order on $\Delta(\mathfrak{t})$.

We fix $(\tau, V)$ an irreducible, unitary representation of $K$ subject to the following restriction: if $\lambda$ is the highest weight of $V$ relative to any choice of order on the compact roots, then we require:

(i) $\lambda$ is $G$-integral

(ii) $\lambda + \rho$ is $G$-regular.

These restrictions will be needed for our proof of Theorem 1.12. That they do not seriously limit geometric applications is pointed out at the end of §1.

If $(\pi, H_\pi)$ is an admissible, quasi-simple representation of $G$ then the space of smooth vectors, $H_\pi^\infty$, is a module for $\mathfrak{g}$ (and $\mathfrak{g}_\mathbb{C}$) with the property that $K$ isotypic spaces have finite multiplicity. Usually $(\pi, H_\pi)$ will be unitary, but when it is finite-dimensional we assume it has been endowed with an admissible inner product. We recall the $\mathcal{J}$ complex
associated to the datum \((\pi, V, p_-)\). Let \(\{Z_i\}\) be any orthonormal basis of \(p_+\) and let \(e(\cdot)\) denote exterior multiplication on \(\Lambda^* p_C\). The operator defined by

\[
\overline{\partial}_\pi = \sum \pi(Z_i) \otimes e(Z_i) \otimes I
\]

is a linear map

\[
\overline{\partial}_\pi : H^\infty_\pi \otimes \Lambda^q p_+ \otimes V \to H^\infty_\pi \otimes \Lambda^{q+1} p_+ \otimes V.
\]

It can easily be seen to satisfy \((\overline{\partial}_\pi)^2 = 0\) and to commute with the action of \(K\) via \(\pi \otimes \Lambda^*_+ \otimes \tau\). Thus it restricts to a map on the spaces of \(K\)-invariant vectors, and hence one obtains the \(\overline{\partial}\) complex of finite dimensional vector spaces associated to \((\pi, V, p_-)\):

\[
0 \to [H^\infty_\pi \otimes \Lambda^0 p_+ \otimes V]^K \xrightarrow{\overline{\partial}_\pi} [H^\infty_\pi \otimes \Lambda^1 p_+ \otimes V]^K \to \cdots \rightarrow [H^\infty_\pi \otimes \Lambda^n p_+ \otimes V]^K \to 0.
\]

The homology groups of this complex are denoted by \(\{H^{0,q}(g, K; \pi \otimes \tau)\}\).

Relative to the Killing form on \(p\) and the (admissible) inner product on \(H_\pi\) the formal adjoint, \(\overline{\partial}_\pi^*\), is given by

\[
\overline{\partial}_\pi^* = \sum_{i=1}^n \pi(Z_i)^* \otimes e(Z_i)^* \otimes I,
\]

where \(\pi(Z_i)^* = -\pi(Z_i)\) and \(e(Z_i)^* = i(Z_i)\) with \(i(\cdot)\) denoting interior multiplication. The formal Laplacian

\[
\Box^{0,q}_\pi = \overline{\partial}_\pi \overline{\partial}_\pi^* + \overline{\partial}_\pi^* \overline{\partial}_\pi
\]

acts on \(H^\infty_\pi \otimes \Lambda^q p_+ \otimes V\) and restricts to an operator on the \(K\)-invariant vectors. A useful formula for these operators is given in

Lemma 1.1.

(i) On \(H^\infty_\pi \otimes \Lambda^q p_+ \otimes V\),

\[
\Box^{0,q}_\pi = \frac{1}{2} \left\{ -\pi(\Omega_G) \otimes I \otimes I + (\pi \otimes \Lambda^q_+)(H_{2\rho_n}) \otimes I + (\pi \otimes \Lambda^q_+)(\Omega_K) \otimes I \right\}.
\]

(ii) On \([H^\infty_\pi \otimes \Lambda^q p_+ \otimes V]^K\),

\[
\Box^{0,q}_\pi = -\frac{1}{2} \pi(\Omega_G) \otimes I \otimes I + \frac{1}{2} (\lambda^* + 2\rho, \lambda^*) I \otimes I \otimes I,
\]
where \( \lambda^* \) is any highest weight of \( V^* \).

Proof. (i) The proof in [0-0] of Proposition 5.1 carries over to \( \square_\pi^{0,q} \) mutatis mutandis.

(ii) On invariant vectors

\[
(\pi \otimes \Lambda^q_+)(H_{2\rho_n}) \otimes I = -I \otimes \tau(H_{2\rho_n}) = -\lambda(H_{2\rho_n})I
\]

and

\[
(\pi \otimes \Lambda^q_+)(\Omega_K) \otimes I = I \otimes \tau(\Omega_K) = (\|\lambda + \rho_c\|^2 - \|\rho_c\|^2)I.
\]

Now \( H_{2\rho_n} \) is central in \( \mathfrak{g} \), so

\[
-\lambda(H_{2\rho_n}) = \langle \lambda^*, 2\rho_n \rangle \quad \text{and} \quad \|\lambda + \rho_c\|^2 = \|\lambda^* + \rho_c\|^2.
\]

From these and the independence of \( \langle \lambda^* + 2\rho, \lambda^* \rangle \) on the choice of order for the compact roots, the result follows.

\[\blacksquare\]

Remark. Since \( G/K \) is Kähler one can show that \( 2\square_\pi^{0,q} = \Delta^q_\pi \), the usual Laplacian of the \( d \)-complex. Statement (ii) is then also obtainable from Kuga’s Lemma.

For any complex of finite dimensional vector spaces and linear maps

\[
0 \to E_0 \xrightarrow{\partial_0} E_1 \xrightarrow{\partial_1} E_2 \to \cdots \to E_r \xrightarrow{\partial_r} 0
\]

one has the Poincaré polynomial

\[
P(t) = \sum_{i=0}^r \dim E_i \, t^i
\]

and the Euler number

\[
e(\{E_i, \partial_i\}) = \sum (-1)^i \dim E_i = p(-1).
\]

As the vector spaces are finite dimensional and they form a complex (\( \partial_i+1 \partial_i = 0 \)), one also has

\[
e(\{E_i, \partial_i\}) = \sum (-1)^i \dim H_i,
\]

where \( H_i \) is the \( i \)th homology group of the complex.

One can define the derived Euler number \( e'(\{E_i, \partial_i\}) \) by

\[
e'(\{E_i, \partial_i\}) = \sum (-1)^i i \dim E_i = P'(-1).
\]

We impose this definition on the \( \bar{\mathfrak{g}}_\pi \) complex to obtain
Definition. Set \( e'(\pi, V, p_-) = \sum (-1)^q \dim [H^\infty \otimes \Lambda^q p_+ \otimes V]^K \) and call \( e'(\pi, V, p_-) \) a spectral derived Euler number.

To evaluate these derived Euler numbers one can use various methods, adjusted to the type of the representation \((\pi, H_{\pi})\). In [K] Köhler evaluates them for finite-dimensional representations \((\pi, H_{\pi})\); however, his parametrization of those with \( e'(\pi, V, p_-) \neq 0 \) is not suitable for our purpose, so we shall give an alternative parametrization. For \((\pi, H_{\pi})\) a discrete series representation, one can determine those satisfying \( e'(\pi, V, p_-) \neq 0 \) by decomposing \( \Lambda^q p_+ \otimes V \) and using Blattner’s formula, similar to the proof of II.5.3 in [B-W]. However, for us it will suffice to observe that only finitely many (unitary equivalence classes of) discrete series have \( e'(\pi, V, p_-) \neq 0 \). By contrast, the parabolically induced representations play a more important role in our context, and treating them will require some additional structure theory.

We take \( a \subseteq p \) a maximal abelian subalgebra constructed, for example, from a maximal set of strongly orthogonal non-compact roots. Let \( q \) be a parabolic subalgebra with Levi decomposition \( q = m_q^1 \oplus n_q \) and \( m_q^1 = m_q \oplus a_q, a_q \subseteq a \). Let \( Q \) be the normalizer of \( q \) in \( G \) and let \( Q = M_Q A_Q N_Q \) be the corresponding Langlands decomposition. We recall the construction of principal series representations. Let \((\xi, W_{\xi})\) be a unitary representation of \( M_Q \) and \( e^\nu \) a character of \( A_Q \). Form \( \pi_{\xi, \nu} = \text{Ind}_{G \times M_Q} \xi \otimes e^\nu \otimes 1 \), acting via the left regular representation on

\[
H_{\xi, \nu} = \{ f : G \rightarrow W_{\xi} | f(g) = e^{-(\nu + \rho_Q) \log a_{\xi}(m)^{-1} f(g)} \},
\]

with \( \|f\|^2 = \int_K \|f(k)\|^2 dk \). For reasons arising from the disconnectness of \( M_Q \) we shall use the subgroup commonly denoted \( M_Q^+, M_Q^+ = \{ m \in M_Q | \text{Ad}(m) : M_q \rightarrow M_q \text{ is inner} \} \). One knows that \( M_Q^+ = M_Q^0 F, F \) central in \( M_Q^+ \) and generated by elements of order 2 in \( \exp i a_q \) (e.g. [Sc] p. 67). Henceforth \( \pi_{\xi, \nu} \) shall refer to \( \text{Ind}_{M_Q^+ A_Q N_Q} G \xi \otimes e^\nu \otimes 1 \). The proof of the following result is from [M-S;II].

**Lemma 1.2.** Let \( Y \in p_+ \) be a non-zero vector fixed by \( K \cap M_Q^+ \) and set \( p_Y^+ = (\mathbb{C} Y)^{+} \cap p_+ \). Then

\[
e'(\pi_{\xi, \nu}, V, p_-) = \sum_{\ell=0}^{n-1} (-1)^{\ell+1} \dim [W_{\xi} \otimes \Lambda^\ell p_Y^+ \otimes V]^K \cap M_Q^+.
\]
Proof. From Frobenius reciprocity one obtains

\[ e'(\pi_{\xi,\nu}, V, p_-) = \sum (-1)^q q \dim [W_\xi \otimes \Lambda^q p_+ \otimes V]_{K \cap M^+_Q}. \]

Since \( p_+ = p_Y^+ \oplus \mathbb{C}Y \) as \( K \cap M^+_Q \) modules, we have in the Grothendieck ring of \( K \cap M^+_Q \)

\[ \sum_{q=0}^{n} (-1)^q q \Lambda^q p_+ = \sum_{q=0}^{n} (-1)^q [\Lambda^q p_Y^+ \oplus \Lambda^{q-1} p_Y^+] \]

\[ = \sum_{q=0}^{n-1} (-1)^q q \Lambda^q p_Y^+ \oplus \sum_{q=0}^{n-1} (-1)^{q+1} (q + 1) \Lambda^q p_Y^+ \]

\[ = \sum_{q=0}^{n-1} (-1)^{q+1} \Lambda^q p_Y^+. \]

If one tensors this with \( W_\xi \otimes V \) and takes \( K \cap M^+_Q \) invariants, one obtains the result. \( \square \)

Lemma 1.3. If \( q \) is a parabolic subalgebra with \( \dim a_q \geq 2 \) then \( e'(\pi_{\xi,\nu}, V, p_-) = 0. \)

Proof. We show there is a two dimensional subspace of \( p_+ \) on which \( K \cap M^+_Q \) acts trivially. Now \( a_q \) is contained in \( a \) and is spanned by \( X_\alpha + X_{-\alpha}, \alpha \in S \) a set of strongly orthogonal non-compact roots. \( M^+_Q \) centralizes \( a_q \) and \( K \) stabilizes \( p_\pm \). Hence \( K \cap M^+_Q \) must fix \( X_\pm \). If \( \dim a_q \geq 2 \) then \( \sum_{\alpha \in S} \oplus \mathbb{R} X_{\pm \alpha} \) furnishes the desired subspace of \( p_+ \). If \( \dim a_q \geq 2 \) then we may assume some \( X_{\pm \alpha}, \alpha \in S \), is in \( p_Y^+ \). But then \( I \otimes [e(X_{\pm \alpha}) + i(X_{\pm \alpha})] \otimes I \) is a \( K \cap M^+_Q \) intertwining operator between \( \Lambda^{ev} p_+^Y \) and \( \Lambda^{odd} p_+^Y \), and non-trivial since \( [e(X_{\pm \alpha}) + i(X_{\pm \alpha})]^2 = \pm ||X_{\pm \alpha}||^2 \neq 0 \). Hence \( \Lambda^{ev} p_+^Y \) and \( \Lambda^{odd} p_+^Y \) are equivalent \( K \cap M^+_Q \) modules and the result follows from Lemma 1.2. \( \square \)

Consequently for parabolically induced representations, it suffices to restrict our attention to maximal, proper, parabolic subgroups, that is \( \dim a_q = 1 \). For these there is a refined structure theory due to a large extent to Koranyi and Wolf, and for which an excellent presentation of most of what we shall need can be found in Chapter III in [Sa], to which we enthusiastically refer the reader. We shall summarize below the salient facts for our purposes.

To each such \( q \) there is a homomorphism

\[ \kappa : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{g} \]
so that $a_q = \mathbb{R}X_\kappa$ where

$$X_\kappa = \kappa \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and such $\kappa$ intertwines the almost complex structures induced by $ad \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}$ and $ad H_0$. If we set

$$H_\kappa = \kappa \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } c_\kappa = \exp \frac{\pi i}{4} \kappa \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then $H_\kappa$ is in $\mathfrak{t}$, and $C_\kappa = Ad c_\kappa$ is an automorphism of $g_\mathbb{C}$, the $\kappa$-Cayley transform,

$$C_\kappa(H_\kappa) = -iX_\kappa \text{ and } C_\kappa(X_\kappa) = -iH_\kappa.$$

To a maximal set of strongly orthogonal non-compact roots one can associate a maximal set of mutually commuting homomorphisms $\kappa_j$’s. For such a set, $a = \text{span} \{X_{\kappa_j}\}$ is a maximal abelian subalgebra of $\mathfrak{p}$. Then we may arrange a choice of representative, $q_{\kappa_j}$, of each maximal, proper parabolic corresponding to the so-called canonical homomorphisms $\kappa_j$. If $\dim a = r$, there are $r$ canonical $\kappa_j$’s. We will use $\kappa$ to denote a typical one of these. To avoid even more burdensome notation we will use subscripts involving $\kappa$ only when it seems useful.

We fix a canonical $\kappa$ and let $q_\kappa = m_{q_\kappa} \oplus n_{q_\kappa}$ be the corresponding maximal parabolic subalgebra.

The tds $\kappa(\mathfrak{sl}(2, \mathbb{R}))$ acts on $g$ giving the decomposition $g = g^{[0]} \oplus g^{[1]} \oplus g^{[2]}$ according to the $\mathfrak{sl}(2, \mathbb{R})$ isotypic components, here $0, 1, 2$ refer to the highest weight of the isotypic component. These subspaces are invariant under the Cartan involution as well as the complex structure on $\mathfrak{p}$, thus (after complexifying) we have

$$(1.4) \quad \mathfrak{p}_\pm = \mathfrak{p}_\pm^{[0]} \oplus \mathfrak{p}_\pm^{[1]} \oplus \mathfrak{p}_\pm^{[2]}.$$

The algebra $\mathfrak{t} \cap m_{q_\kappa}$ is equal to $\mathfrak{t}^{[0]}$, that is, it centralizes $\kappa(\mathfrak{sl}(2, \mathbb{R}))$. Thus $(K \cap M_{Q_\kappa})^0$ (which is $K \cap M_{Q_\kappa}^0$) respects the decomposition (1.4). The subgroup $F_\kappa$ is contained in $K \cap \exp i a_{q_\kappa}$, in fact is generated by $\gamma_\kappa = \exp \pi i X_\kappa$. Since $g^{[i]}$ is a $\kappa(\mathfrak{sl}(2, \mathbb{R}))$ isotypic component, $Ad F_\kappa$ preserves each $g^{[i]}$ and, since $H_0$ is central in $\mathfrak{t}$, $p^{[i]}$. Moreover, it is easy
to see that on $g^{[0]}$ and $g^{[2]}$, $Ad \exp \pi i X_\kappa = I$, while on $g^{[1]}$, $Ad \exp \pi i X_\kappa = -I$. Thus each $p_\pm^{[i]}$ is a module for $K \cap M_{Q_\kappa}^\perp$. With the use of additional structure theory each of these modules will be identified with ones having more intrinsic geometric nature.

We make a choice of $q_\kappa$ so that in terms of $ad X_\kappa$ the nilradical $n_{q_\kappa} = v \oplus u$ where $v$ (resp. $u$) is the +1 (resp. +2) eigenspace of $ad X_\kappa$. There is also a decomposition of the algebra $m_{q_\kappa}$ into $m_{q_\kappa}^{(1)} \oplus m_{q_\kappa}^{(2)}$ where

$$m_{q_\kappa}^{(1)} = l_2 \oplus g_\kappa^{(1)},$$

$l_2$ a compact ideal in $m_{q_\kappa}$ and $g_\kappa^{(1)}$ of Hermitian non-compact type having

$$H_0^{(1)} = H_0 - \frac{1}{2} H_\kappa$$

defining the almost complex structure. Let $M_{Q_\kappa}^{(i)}$ be the connected subgroup with Lie algebra $m_{q_\kappa}^{(i)}$. Then $M_{Q_\kappa}^0 = M_{Q_\kappa}^{(1)} M_{Q_\kappa}^{(2)}$ and $M_{Q_\kappa}^\perp = M_{Q_\kappa}^{(1)} M_{Q_\kappa}^{(2)} F_\kappa$.

For the restriction of the Cartan decomposition one gets $g_\kappa^{(1)} = \mathfrak{t}^{(1)} \oplus \mathfrak{p}^{(1)}$, and $p_\kappa^{(1)} = p_\kappa^{(1)} \oplus p_\kappa^{(1)}$, with $p_\kappa^{(1)}$ simultaneous $\pm i$ eigenspaces of $ad H_0^{(1)}$ and $ad H_0$. Moreover we have the identification $p_\kappa^{[0]} = p_\kappa^{(1)}$.

In order to identify $p_\kappa^{[2]}$ one considers $m_{q_\kappa}^{(2)}$, the other summand of $m_{q_\kappa}$, with Cartan decomposition $\mathfrak{g} \cap m_{q_\kappa}^{(2)} \oplus \mathfrak{p}^{(2)}$, together with the subalgebra of $\mathfrak{g}$, so that in the notation of [Sa]

$$\mathfrak{t}_\kappa^* = \mathfrak{g} \cap m_{q_\kappa}^{(2)} \oplus \mathfrak{t}^{[2]}.$$

For our purposes the useful facts about $\mathfrak{t}_\kappa^*$ are:

$$H_\kappa \in \text{Center} (\mathfrak{t}_\kappa^*),$$

$$C_\kappa^{-1} : (\mathfrak{t}_\kappa^*)_C \cong (m_{q_\kappa}^{(2)} \oplus a_{q_\kappa})_C.$$

In other words, $\mathfrak{t}_\kappa^*$ and $m_{q_\kappa}^{(2)} \oplus a_{q_\kappa}$ are $\mathbb{R}$-forms of isomorphic complex algebras.

Now $p_\kappa^{[2]} = C_\kappa (u_C)$ ([Sa] III 2.7), and $\mathfrak{g} \cap m_{q_\kappa} \subseteq g^{[0]}$ commutes with $C_\kappa$. So as $\mathfrak{g} \cap m_{q_\kappa}$ module, $p_\kappa^{[2]} \simeq u_C$. Furthermore, on $u_C$ the algebra $\mathfrak{g} \cap m_{q_\kappa}^{(1)}$ acts trivially, while $m_{q_\kappa}^{(2)} \oplus a_{q_\kappa}$ acts faithfully via the restriction of $Ad$ ([Sa] Th. III.2.3). By means of the isomorphism of
vector spaces \( u_C \cong p_C^{(2)} \oplus a_{q, \mathbb{C}} \) ([Sa] p.98) we transfer the action. Hence as \( \mathfrak{t} \cap m_{q, \mathbb{C}} \)-modules we have \( u_C \cong p_C^{(2)} \oplus a_{q, \mathbb{C}} \) with \( a_{q, \mathbb{C}} \) an invariant subspace.

To identify \( p_{\pm}^{[1]} \), as well as later use, we consider

\[
L_\kappa = \text{Centralizer}_K(H_\kappa) = \text{Centralizer}_K(H_0^{(1)}),
\]

with Lie algebra \( l_\kappa \) the sum of the ideals

\[
l_\kappa = \mathfrak{t} \cap m_{\kappa}^{(1)} \oplus \mathfrak{t}_\kappa^*.
\]

In fact \( H_\kappa \) defines an almost complex structure on \( K/L_\kappa \) and we take \( n_{\kappa}^c \) (resp. \( \overline{n}_{\kappa}^c \)) to be the \( +i \) (resp. \( -i \)) eigenspace in \( \mathfrak{t}_C \) of \( H_\kappa \). With this choice of \( n_{\kappa}^c \) one has from [Sa] 3.7, and the table on p.101,

\[
(1.10) \quad C_\kappa \circ \theta \circ C_\kappa^{-1}(p_{\pm}^{[1]}) = \overline{n}_{\kappa}^c,
\]

as \( K \cap M_{{Q_\kappa}^+} \)-modules.

For convenience we summarize these various identifications in the next statement.

**Lemma 1.4.** As \( K \cap M_{{Q_\kappa}^+} \) modules we have

(i) \( p_{\pm}^{[0]} = p_{\pm}^{(1)} \)  
(ii) \( p_{\pm}^{[1]} \cong \overline{n}_{\kappa}^c \)  
(iii) \( p_{\pm}^{[2]} \cong p_C^{(2)} \oplus a_{q, \mathbb{C}} \).

**Corollary 1.5.** As \( K \cap M_{{Q_\kappa}^+} \) virtual module

\[
\sum_{q=0}^{n}(\Lambda^q p_+ \simeq \sum_{\ell=0}^{\dim n_{\kappa}^c} (-1)^{\ell+1}(\Lambda^{ev} p_+^{(1)} - \Lambda^{odd} p_+^{(1)}) \otimes (\Lambda^{ev} p_C^{(2)} - \Lambda^{odd} p_C^{(2)}) \otimes \Lambda^\ell \overline{n}_{\kappa}^c .
\]

**Proof.** The result follows from Lemma 1.4 and the proof of Lemma 1.2 choosing for \( Y \) the projection of \( X_\kappa \) into \( p_+ \), namely \( X_\kappa + \overline{X}_\kappa \).

**Remark.** The difference between this virtual module for \( K \cap M_{{Q_\kappa}^+} \) versus \( K \cap M_0^{Q_\kappa} \) is slight. Let \( \varepsilon \) be the non-trivial character of \( F_\kappa, \varepsilon(\gamma_\kappa) = -1 \). Then using (1.10) and the earlier
observation about $Ad \exp \pi i X_\kappa$ we see that $F_\kappa$ acts on $\Lambda^\ell \pi_\kappa$ by $\varepsilon^\ell$ and on the other factors above trivially.

To obtain a further simplification we shall need a theorem of Kostant, the statement of which requires a subset of the Weyl group of $K$. Associated to $H_\kappa$ is a parabolic subalgebra $q^c_\kappa \subseteq \mathfrak{k}_C$, namely $q^c_\kappa = l_\kappa, C \oplus n^c_\kappa$. Recall that an arbitrary choice of order on $\Delta(\mathfrak{k})$ was made. Since $t \subseteq l_\kappa$ we may choose $\Delta^+(l_\kappa, C)$ compatibly with the positive $K$-roots. Set

$$W_\kappa = \{ w \in W(\mathfrak{k}_C) | w^{-1} \alpha > 0, \alpha \in \Delta^+(l_\kappa, C) \}. \tag{1.11}$$

Here, $K$ is reductive not semisimple, nevertheless we still have the familiar statement:

**Theorem.** *(Kostant)* Let $V$ be an irreducible unitary representation of $K$ with highest weight $\lambda$. Then as $L_\kappa$ virtual modules

$$\sum (-1)^\ell \Lambda^\ell \pi_\kappa \otimes V = \sum (-1)^\ell(w)W_{w(\lambda + \rho_c) - \rho_c},$$

where $W_\mu$ is the irreducible $L_\kappa$ module with highest weight $\mu$.

For convenience we denote the highest weight $w(\lambda + \rho_c) - \rho_c$ by $\lambda_w$, and let $\lambda_w^\vee$ be the highest weight of the contragredient representation of $L_\kappa$. As $l_\kappa = \mathfrak{t} \cap m^{(1)}_{q_\kappa} \oplus \mathfrak{k}_C^*$ a sum of ideals, we may, using obvious notation, express

$$W_{\lambda_w} = W_{\lambda^{(1)}_w} \otimes W_{\lambda^{(2)}_w},$$

where $W_{\lambda^{(1)}_w}$ is an irreducible module for $\mathfrak{t} \cap m^{(1)}_{q_\kappa}$ with highest weight $\lambda^{(1)}_w$, the restriction of $\lambda_w$ to $\mathfrak{t} \cap \mathfrak{t} \cap m^{(1)}_{q_\kappa}$, and $W_{\lambda^{(2)}_w}$ is one for $\mathfrak{k}_C^*$ with $\lambda^{(2)}_w$ defined similarly. The isomorphism $C_\kappa$ (cf. (1.9)) makes $W_{\lambda^{(2)}_w}$ a module for $m^{(2)}_{q_\kappa, C} \oplus a_{q_\kappa, C}$ and fixes an order on the roots of $m^{(2)}_{q_\kappa, C}$ for which $W_{\lambda^{(2)}_w}$ has highest weight $\lambda^{(2)}_w$, the restriction of $\lambda_w$ to $\mathfrak{t} \cap \mathfrak{t} \cap m^{(2)}_{q_\kappa}$. Furthermore, $X_\kappa \in a_{q_\kappa}$ then acts by the scalar

$$X_\kappa \mapsto \lambda_w(-iH_\kappa). \tag{1.12}$$
Since $F_\kappa$ is generated by elements in $\exp i a_{q_\kappa}$, $\gamma_\kappa$ acts only on the second factor and the module for $M_{Q_\kappa}^{(2)} F_\kappa$ is determined. As module for $K \cap M_{Q_\kappa}^{(1)} \times M_{Q_\kappa}^{(2)} F_\kappa$ we shall denote this by

$$W_{\lambda_w} = W_{\lambda_w}^{(1)} \otimes W_{\lambda_w}^{(2)}.$$  

The character through which $\exp a_{q_\kappa}, C$ acts ultimately will be related to principal series data.

For similar reasons an irreducible unitary representation $(\xi, W_\xi)$ of $M_{Q_\kappa}^+$ will be written $(\xi^{(1)}, W_\xi^{(1)}) \otimes (\xi^{(2)}, W_\xi^{(2)})$, here $\xi^{(2)}$ is a representation of $M_{Q_\kappa}^{(2)} F_\kappa$. As $W_{\lambda_w}^{(1)}$ is an irreducible module for $K \cap M_{Q_\kappa}^{(1)}$ we have $e(\xi^{(1)}, W_{\lambda_w}^{(1)}, p_-^{(1)})$ the Euler number for the $\overline{\partial}$ complex associated to this datum for $M_{Q_\kappa}^{(1)}$. In an entirely similar way, one has for $M_{Q_\kappa}^{(2)}$ and the datum $(\xi^{(2)}, W_{\lambda_w}^{(2)}, p_-^{(2)})$ the $d$ complex and the Euler number $e(\xi^{(2)}, W_{\lambda_w}^{(2)}, p_-^{(2)})$. Of course it is crucial here that $W_{\lambda_w}^{(2)}$ is a representation of $M_{Q_\kappa}^{(2)}$, not just $K \cap M_{Q_\kappa}^{(2)}$.

**Proposition 1.7.** Let $Q_\kappa$ be a maximal, proper, parabolic subgroup of $G$ and let $\pi_{\xi, \nu} = \text{Ind}_{M_{Q_\kappa}^+ A_{Q_\kappa} N_{Q_\kappa}}^G (\xi \otimes \nu^\vee \otimes 1)$. Then

$$e'(\pi_{\xi, \nu}, V, p_-) = \sum_{W_\kappa} (-1)^{\ell(w)} + e(\xi^{(1)}, W_{\lambda_w}^{(1)}, p_-^{(1)}) e(\xi^{(2)}, W_{\lambda_w}^{(2)}, p_-^{(2)}).$$

**Proof.** One uses in turn Lemma 1.2, Corollary 1.5, Kostant’s Theorem and the previous discussion. \qed

The non-vanishing of $e'(\pi_{\xi, \nu}, V, p_-)$ imposes strong restrictions on the infinitesimal character of $\xi$. To derive these restrictions we shall treat $\xi^{(1)}$ and $\xi^{(2)}$ separately.

Recall that $M_{Q_\kappa}^{(1)}$ is of Hermitian type, so for $M_{Q_\kappa}^{(1)}$ there are $\rho^{(1)}_h = \rho(p_+^{(1)})$, $\rho^{(1)} = \rho(\mathfrak{t} \cap \mathfrak{m}_{Q_\kappa}^{(1)})$ and $\rho^{(1)} = \rho^{(1)}_h + \rho_c^{(1)}$. Here $\rho_c^{(1)}$ is determined by the order on $\Delta(I_{\kappa, C})$ which was previously specified. Let $w^{(1)}$ denote the opposition element in $W(K \cap M_{Q_\kappa}^{(1)})$.

**Lemma 1.8.** If $e(\xi^{(1)}, W_{\lambda_w}^{(1)}, p_-^{(1)}) \neq 0$ then

$$\xi^{(1)}(\Omega_{M_{Q_\kappa}^{(1)}}) = \|\lambda_w^{(1)} + \rho^{(1)}\|^2 - \|\rho^{(1)}\|^2$$
and the infinitesimal character of $\xi^{(1)}, \Lambda_{\xi^{(1)}}$, is given by

\begin{align*}
\Lambda_{\xi^{(1)}} &= -w^{(1)}(\lambda^{(1)}_w + \rho^{(1)}) \\
&= \lambda^{(1)\vee} + \rho^{(1)}_c - \rho^{(1)}_n.
\end{align*}

**Proof.** This is essentially Theorem 5.1 in [De-W]. Indeed, as observed before

\[
e(\xi^{(1)}, W_{\lambda^{(1)}_w}, p^{-1}) = \sum (-1)^{\ell} \dim [\xi^{(1)} \otimes \Lambda^{\ell} p^{(1)}_+ \otimes W^{(1)}_{\lambda^{(1)}_w}]_{K \cap M^{(1)}_{Q^{(1)}_\kappa}}
\]

\[
= \sum (-1)^{\ell} H^{0,\ell}(m^{(1)}_{q^{(1)}}, K \cap M^{(1)}_{Q^{(1)}_\kappa}; \xi^{(1)} \otimes W^{(1)}_{\lambda^{(1)}_w}).
\]

If $e(\xi^{(1)}, W_{\lambda^{(1)}_w}, p^{-1}) \neq 0$ then one of the $H^{0,\ell}$ is non-zero, so by Lemma 1.1

\[
\xi^{(1)}(\Omega^{(1)}_{M^{(1)}_{Q^{(1)}_\kappa}}) = (\lambda^{(1)\vee}_w + 2\rho^{(1)}, \lambda^{(1)\vee}_w).
\]

If we denote the usual half-spin representations of $K \cap M^{(1)}_{Q^{(1)}_\kappa}$ arising from $p^{(1)}_C$ by $\sigma^{(1)}_{\pm}$ and $\dim p^{(1)}_+$ by $q$, then in the representation ring of $K \cap M^{(1)}_{Q^{(1)}_\kappa}$

\[
(1.13) \\
\Lambda^{\text{ev}} p^{(1)}_+ - \Lambda^{\text{odd}} p^{(1)}_+ = (-1)^q (\sigma^{(1)}_+ - \sigma^{(1)}_-) \otimes \xi^{(1)}_{\rho^{(1)}_n}.
\]

But if

\[
0 \neq e(\xi^{(1)}, W_{\lambda^{(1)}_w}, p^{-1}) = (-1)^q \dim [\xi^{(1)} \otimes (\sigma^{(1)}_+ - \sigma^{(1)}_-) \otimes \xi^{(1)}_{\rho^{(1)}_n} \otimes W^{(1)}_{\lambda^{(1)}_w}]_{K \cap M^{(1)}_{Q^{(1)}_\kappa}}
\]

then for the full spin representation $L$,

\[
0 \neq \dim [\xi^{(1)} \otimes L \otimes \xi^{(1)}_{\rho^{(1)}_n} \otimes W^{(1)}_{\lambda^{(1)}_w}]_{K \cap M^{(1)}_{Q^{(1)}_\kappa}},
\]

or

\[
\text{Hom}_{K \cap M^{(1)}_{Q^{(1)}_\kappa}} [W^{(1)}_{\lambda^{(1)}_w} \otimes \xi^{(1)}_{\rho^{(1)}_n} \otimes L, \xi^{(1)}] \neq 0.
\]

DeGeorge-Wallach’s result then says that if $\lambda^{(1)\vee}_w - \rho^{(1)}_n + \rho^{(1)}_c = -w^{(1)}(\lambda^{(1)}_w + \rho^{(1)})$ is $m^{(1)}_{q^{(1)}_n}$ integral and regular, then the infinitesimal character of $\xi^{(1)}$ is $-w^{(1)}(\lambda^{(1)}_w + \rho^{(1)})$. Thus the following Lemma will complete the proof and justify the restrictions on $\lambda$ originally imposed. ■
Lemma 1.9. If $\lambda$ is $K$ dominant, $\lambda + \rho$ is $G$ integral, and $\lambda + \rho$ is $G$ regular, then the same is true for $\lambda^{(1)}_w$ relative to $M^{(1)}_{Q_{\kappa}}$.

Proof. A useful observation is that $m^{(1)}_{q_{\kappa}}$ roots are in fact $g$ roots, something not true about $m^{(2)}_{q_{\kappa}}$. To see this it suffices to notice from (1.10) and Lemma 1.4 (i) that $m^{(1)}_{q_{\kappa}} \subseteq g^{[0]}$ and that $t \subseteq g^{[0]} \oplus \mathbb{R}H_{\kappa}$. Also $t^\ast$ is an ideal in $l_{\kappa}$. The significance of this is that to see $\lambda^{(1)}_w$ satisfies the conditions we may use $\lambda_w$ and do calculations with $g$-roots.

As $\lambda_w$ is $l_{\kappa}$ dominant and $t \cap m^{(1)}_{q_{\kappa}}$ is an ideal in $l_{\kappa}$, $\lambda^{(1)}_w$ is $t \cap m^{(1)}_{q_{\kappa}}$ dominant.

Now to show $m^{(1)}_{q_{\kappa}}$ regular and integral, we consider $\alpha$ a root of $m^{(1)}_{q_{\kappa}}$.

\[
\frac{2\langle \alpha, \lambda^{(1)}_w + \rho \rangle}{\langle \alpha, \alpha \rangle} = \frac{2\langle \alpha, w(\lambda + \rho_c) - \rho_c + \rho^{(1)} \rangle}{\langle \alpha, \alpha \rangle} = \frac{2\langle \alpha, w(\lambda + \rho) - \rho + \rho^{(1)} \rangle}{\langle \alpha, \alpha \rangle} = \frac{2\langle w^{-1}\alpha, \lambda + \rho \rangle}{\langle w^{-1}\alpha, w^{-1}\alpha \rangle} - \frac{2\langle \alpha, \rho - \rho^{(1)} \rangle}{\langle \alpha, \alpha \rangle}.
\]

Since $w^{-1}\alpha$ is a root for $g$ and $\lambda + \rho$ is $G$ regular and integral, the first term is a non-zero integer. On $m^{(1)}_{q_{\kappa}}$, $\rho = \rho^{(1)}$ so the second term is zero. \hfill \Box

Remark. Recall from (1.5) that $m^{(1)}_{q_{\kappa}} = l_2 \oplus g^{(1)}_{\kappa}$ and need not be of non-compact type. But since $l_2$ is an ideal in $m^{(1)}_{q_{\kappa}}$, one easily sees that Lemma 1.8 is valid in this generality.

Remark. The group $M^{(2)}_{Q_{\kappa}}$ can be handled somewhat analogously to deduce that $\xi^{(2)}$ is a fundamental series representation. But at this time we have no application of this complication so we have imposed the assumption that $q_{\kappa}$ is cuspidal. Let $\xi^{(2)} = \xi_0^{(2)} \otimes \chi$ as a representation of $M^{(2)}_{Q_{\kappa}}F_{\kappa}$ where $\xi_0^{(2)} = \chi$ on $M^{(2)}_{Q_{\kappa}} \cap F_{\kappa}$, and let $w^{(2)}$ be the opposition element for $M^{(2)}_{Q_{\kappa}}$.

Lemma 1.10. If $e(\xi^{(2)}, W^{(2)}_{\lambda^{(2)}_w}, p^{(2)}_c) \neq 0$ then

\[
\xi_0^{(2)}(\Omega_M^{(2)}) = \|\lambda^{(2)}_w + \rho^{(2)}\|^2 - \|\rho^{(2)}\|^2
\]
and the infinitesimal character of $\xi_0^{(2)}$, $\Lambda_{\xi_0^{(2)}}$, is given by

$$\Lambda_{\xi_0^{(2)}} = \lambda_w^{(2)} + \rho^{(2)} = -w^{(2)}(\lambda_w^{(2)} + \rho^{(2)}).$$

**Proof.** One observes that the non-vanishing of $H^\ell(m_{Q_0}^{(2)}, K \cap M_{Q_0}^{(2)}; \xi_0^{(2)} \otimes W_{\Lambda_{\xi_0^{(2)}}}^{(2)})$ allows one to invoke Proposition 2.4 in [B] (or I.5.3 in [B-W] for disconnected $K$) directly to obtain the result. □

For $\xi_0^{(2)}$ discrete series, it follows from [B-W, p. 60] that if $e(\xi_0^{(2)}, W_{\Lambda_{\xi_0^{(2)}}}^{(2)}, p^{(2)}_C) \neq 0$, then it is equal to $(-1)^{\frac{1}{2}\dim p^{(2)}_C}$.

We shall need to be more precise about the various lexicographic orders. Set $t_\kappa = t \cap \mathfrak{k} \cap m_{q_\kappa}$ and $\mathfrak{h}_\kappa = t_\kappa \oplus \mathbb{R}X_\kappa$. Essentially the order $\Delta^+(\mathfrak{g}_C, \mathfrak{h}_{\kappa,C})$ is chosen according to the positive eigenvalues of $X_\kappa$, and, since $C_\kappa : \mathfrak{h}_{\kappa,C} \rightarrow t_\kappa$, in such a way that $\Delta^+(\mathfrak{g}_C, \mathfrak{h}_{\kappa,C}) = C_\kappa^* \Delta^+(\mathfrak{g}_C, t_\kappa)$. Then relative to $t_\kappa$ we have $\Delta^+(\mathfrak{g}_C, t_\kappa) = \Delta^+_c \cup \Delta^+_n$, where $p_+ \leftrightarrow \Delta^+_n$, and $\Delta^+_c$ has $n^c_\kappa \leftrightarrow$ positive compact roots, and some order has been chosen for $l_\kappa, C$. Since $X_\kappa = C_\kappa(iH_\kappa)$, and $\text{ad}H_\kappa$ on $p_-$ has values $0, -i, -2i$, one can easily check, [Sa] §2 (2.1) and p. 143, that

$$C_\kappa(\mathfrak{k}_C \oplus p_-) \cap \mathfrak{g} \supset \mathfrak{k} \cap m_{q_\kappa} \oplus \mathfrak{a}_{q_\kappa} \oplus \mathfrak{n}_{q_\kappa}.$$

Relative to the Killing form on $\mathfrak{g}$ one has $H_\kappa$ orthogonal to $t_\kappa$, and similarly $X_\kappa$ orthogonal to $t_\kappa$. We set (recall (1.3))

$$\tilde{X}_\kappa = \frac{X_\kappa}{[-\|H_\kappa\|^2]^{1/2}}.$$

**Lemma 1.11.**

(i) $\langle H_0, H_0 \rangle = -2\dim p_-$

(ii) $\langle H_\kappa, H_\kappa \rangle = -4\dim p_+^{[1]} - 8\dim p_+^{[2]}$

(iii) $\langle H_0, H_\kappa \rangle = -2\dim p_+^{[1]} - 4\dim p_+^{[2]}$

(iv) $\langle H_0^{(1)}, H_0^{(1)} \rangle = -2\dim p_-^{(1)} - \dim p_+^{[1]}$

(v) $\langle H_0^{(1)}, H_\kappa \rangle = 0$
Proof. For any $H_1, H_2$ in $t$

\[
\langle H_1, H_2 \rangle = tr(adH_1 adH_2) = \sum_{\alpha \in \Delta} \alpha(H_1) \alpha(H_2)
\]

\[
= \sum_{\Delta(t)} \alpha(H_1) \alpha(H_2) + \sum_{\Delta(p)} \alpha(H_1) \alpha(H_2).
\]

(i) As $H_0$ is central in $t$, for all $\alpha \in \Delta(t)$, $\alpha(H_0) = 0$; while for $\alpha \in \Delta(p)$, $\alpha(H_0) = \pm i$ according as $\alpha \in \Delta(p^{\pm})$.

(ii) As $H_\kappa$ is central in $t_\kappa$, for all $\alpha \in \Delta(t_\kappa)$, $\alpha(H_\kappa) = 0$; while for $\alpha \in \Delta(n_\kappa^c) \alpha(H_\kappa) = i$. From [Sa] one finds for $\alpha \in \Delta(p_\kappa^{[2]})$ that $\alpha(H_\kappa) = \pm 2i$, and since $n_\kappa^c \cong p_\kappa^{[1]}$, if $\alpha \in \Delta(p_\kappa^{[1]})$, $\alpha(H_\kappa) = \pm i$. Then

\[
\langle H_\kappa, H_\kappa \rangle = -2\dim n_\kappa^c - 2\dim p_\kappa^{[1]} - 8\dim p_\kappa^{[2]}
\]

\[
= -4\dim p_\kappa^{[1]} - 8\dim p_\kappa^{[2]}. 
\]

(iii) This follows from the proofs of (i) and (ii).

(iv)

\[
\langle H_0^{(1)}, H_0^{(1)} \rangle = \langle H_0 - H_\kappa/2, H_0 - H_\kappa/2 \rangle
\]

\[
= \langle H_0, H_0 \rangle - \langle H_0, H_\kappa \rangle + \frac{1}{4} \langle H_\kappa, H_\kappa \rangle
\]

\[
= \langle H_0, H_0 \rangle - \langle H_0, H_\kappa \rangle + \frac{1}{2} \langle H_0, H_\kappa \rangle
\]

\[
= \langle H_0, H_0 \rangle - \frac{1}{2} \langle H_0, H_\kappa \rangle.
\]

The result then follows from (i), (iii) and (1.4).

(v)

\[
\langle H_0^{(1)}, H_\kappa \rangle = \langle H_0 - H_\kappa/2, H_\kappa \rangle
\]

\[
= \langle H_0, H_\kappa \rangle - \frac{1}{2} \langle H_\kappa, H_\kappa \rangle = 0.
\]

\[\blacksquare\]
Theorem 1.12. Let $q_\kappa = m_{q_\kappa} \oplus a_{q_\kappa} \oplus n_{q_\kappa}$ be a cuspidal maximal parabolic subalgebra in $g$. Let $\xi$ be a discrete series representation of $M_{Q_\kappa}^+$. If

$$e(\xi^{(1)}, W_{\lambda^{(1)}}, p_{\xi}^{(1)})e(\xi^{(2)}, W_{\lambda^{(2)}}, p_{\xi}^{(2)}) \neq 0$$

then

$$(1.14) \quad \Lambda_\xi = (w^* (\lambda^* + \rho - 2\rho_n)) \circ C_\kappa + (\lambda_w \mid_{C H_\kappa}) \circ C_\kappa + \rho_{Q_\kappa}$$

and

$$(1.15) \quad \|\Lambda_\xi\|^2 - \|\lambda^* + \rho\|^2 = -\{(\lambda_w \circ C_\kappa + \rho_{Q_\kappa})(\hat{X}_\kappa)\}^2 + 4(\lambda, \rho_n).$$

Proof. From Lemma 1.8 and Lemma 1.10 we have

$$\Lambda_\xi = \Lambda_{\xi^{(1)}} + \Lambda_{\xi^{(2)}} = -w^{(1)}(\lambda_w^{(1)} + \rho^{(1)}) - w^{(2)}(\lambda_w^{(2)} + \rho^{(2)}).$$

As $l_\kappa = t \cap m_{q_\kappa} \oplus t^*_\kappa$ with these ideals, and given (1.9) with the resulting orders, the pair $(w^{(1)}, w^{(2)})$ is the pull-back of $w^{L_\kappa}$, where $w^{L_\kappa}$ is the opposition element for $L_\kappa$, i.e.

$$\Lambda_\xi = (-w^{L_\kappa}(\lambda_w)) \circ C_\kappa \mid_{t_\kappa} + (\rho_c^{(1)} - \rho_n^{(1)} + \rho^{(2)})$$

$$= (-w^{L_\kappa}(\lambda_w - \lambda_w \mid_{C H_\kappa})) \circ C_\kappa + (\rho_c^{(1)} - \rho_n^{(1)} + \rho^{(2)}).$$

Now

$$-w^{L_\kappa}(\lambda_w) = (-w^{L_\kappa})(w(\lambda + \rho_c) - \rho_c)$$

$$= (-w^{L_\kappa})(w(-w^K)(\lambda^* + \rho_c) - \rho_c)$$

$$= w^*(\lambda^* + \rho_c) - \rho_c + 2\rho(n^c_\kappa).$$

It is easy to see that $w^*$ is in $W_\kappa$, so one could write $-w^{L_\kappa}(\lambda_w) = \lambda_w^* + 2\rho(n^c_\kappa)$. Substituting the above and using $\rho^{(2)} \circ C_\kappa = \rho(t^*_\kappa)$ one obtains

$$\Lambda_\xi = (w^*(\lambda^* + \rho)) \circ C_\kappa - \rho \circ C_\kappa + 2\rho(n^c_\kappa) \circ C_\kappa + (\lambda_w \mid_{C H_\kappa}) \circ C_\kappa + (\rho_c^{(1)} - \rho_n^{(1)} + \rho^{(2)})$$

$$= (w^*(\lambda^* + \rho)) \circ C_\kappa - 2\rho_n \circ C_\kappa + (\lambda_w \mid_{C H_\kappa}) \circ C_\kappa + (\rho_n - \rho_n^{(1)} + \rho(n^c_\kappa)) \circ C_\kappa.$$

Recall that $q_\kappa = m_{q_\kappa} \oplus n_{q_\kappa}$, and that $n_{q_\kappa} = v \oplus u$. Writing ([Sa] p.100) $v_C = v_+ \oplus v_-$, and using ([Sa] p.101) $C_\kappa v_+ = p^{[1]}_+, C_\kappa u_C = p^{[2]}_+$, and $C_\kappa v_- = n^c_\kappa$, it follows that $(\rho_n - \rho_n^{(1)} + \rho(n^c_\kappa)) \circ C_\kappa = \rho_{Q_\kappa}$. Thus

$$\Lambda_\xi = (w^*(\lambda^* + \rho)) \circ C_\kappa - 2\rho_n \circ C_\kappa + (\lambda_w \mid_{C H_\kappa}) \circ C_\kappa + \rho_{Q_\kappa}.$$
and (1.14) follows.

For (1.15) one uses (1.14) and that $H_\kappa$ is orthogonal to $t_\kappa$:

$$
\|\Lambda_\xi\|^2 = \| (w^*(\lambda^* + \rho - 2\rho_n)) \circ C_\kappa \|^2 + \| (\lambda_w|_{CH_\kappa}) \circ C_\kappa + \rho_Q \|^2
+ 2\langle (w^*(\lambda^* + \rho - 2\rho_n)) \circ C_\kappa, (\lambda_w|_{CH_\kappa}) \circ C_\kappa + \rho_Q \rangle
= \|\lambda^* + \rho\|^2 + 4\langle\lambda, \rho_n\rangle + \| (\lambda_w|_{CH_\kappa}) \circ C_\kappa + \rho_Q \|^2
+ 2\langle (-w^L_\kappa(\lambda_w + \rho_c + \rho_n)) \circ C_\kappa, (\lambda_w|_{CH_\kappa}) \circ C_\kappa + \rho_Q \rangle
= \|\lambda^* + \rho\|^2 + 4\langle\lambda, \rho_n\rangle + \| (\lambda_w|_{CH_\kappa}) \circ C_\kappa + \rho_Q \|^2
- 2\langle (\lambda_w + \rho_c + \rho_n) \circ C_\kappa, (\lambda_w|_{CH_\kappa}) \circ C_\kappa + \rho_Q \rangle
= \|\lambda^* + \rho\|^2 + 4\langle\lambda, \rho_n\rangle + \| (\lambda_w|_{CH_\kappa}) \circ C_\kappa + \rho_Q \|^2 - 2\| (\lambda_w|_{CH_\kappa}) \circ C_\kappa + \rho_Q \|^2
$$

where in the last line we used that $H_\kappa$ is orthogonal to $t_\kappa$ and that $(\rho_c + \rho_n)\circ C_\kappa|_{CH_\kappa} = \rho_Q$.

\[ \blacksquare \]

**Corollary 1.13.** Let $\xi$ be a discrete series representation of $M^{+}_{Q_\kappa}$ and $e^\nu$ a character of $A_{Q_\kappa}$. Set $\pi_{\xi,\nu} = Ind^G_{M^{+}_{Q_\kappa}A_{Q_\kappa}N_{Q_\kappa}} (\xi \otimes e^\nu \otimes 1)$. If

$$
eq 0,$$

then for some $w \in W_\kappa$

$$
\Lambda_\xi = (w^*(\lambda^* + \rho)) \circ C_\kappa - 2\rho_n \circ C_\kappa + (\lambda_w|_{CH_\kappa}) \circ C_\kappa + \rho_Q
$$

and

$$
\|\Lambda_\xi\|^2 - \|\lambda^* + \rho\|^2 = -\{(\lambda_w \circ C_\kappa + \rho_Q)(\hat{X}_\kappa)\}^2 + 4\langle\lambda, \rho_n\rangle.
$$

**Remarks.** (1) If $\lambda = 0$, then $\|\Lambda_\xi\|^2 - \|\lambda^* + \rho\|^2 = -(n - k)^2$ for some $k$, a fact proved in [M-S;II] Lemma 2.5 for non-equal rank groups $G$.

(2) The significance of (1.16) is that it will provide passage from the harmonic analysis (i.e., the infinitesimal characters of unitary representations) to geometry (i.e. the holonomy
action of geodesics on the coefficient bundle). Said plainly, the transfer from spectral information to geodesic properties makes possible the construction of a geometric zeta function.

(3) The term $4\langle \lambda, \rho \rangle$ seems a little strange especially because using $\|\lambda^* + \rho\|^2 + 4\langle \lambda, \rho_n \rangle = \|\lambda + \rho\|^2$ would yield a cleaner formula. However, as $4\langle \lambda, \rho_n \rangle$ is independent of $w$ it will introduce only a scalar into the trace of the heat operators. Also it follows easily from Lemma 1.11 that $4\langle \lambda, \rho_n \rangle = -i\lambda(H_0)$.

(4) Notice that

\[
\{(\lambda_w \circ C_\kappa + \rho_{Q_\kappa})(\hat{X}_\kappa)\}^2 = \{- (\lambda_w \circ C_\kappa + \rho_{Q_\kappa})(\hat{X}_\kappa)\}^2 = \{(\lambda_w \circ C_\kappa^{-1} - \rho_{Q_\kappa})(\hat{X}_\kappa)\}^2.
\]

Now $iH_\kappa \in \mathfrak{k}_C$ acts on $V$, as $\mathfrak{k}_C$ is part of the Lie algebra of the isotropy at $eK_CP_-$. As $X_\kappa = C_\kappa(iH_\kappa)$, it is in the Lie algebra of the isotropy at $c_\kappa K_CP_-,$ and the transferred action of $X_\kappa$ on $V$ is then via $C_\kappa^{-1}(X_\kappa).$ Thus $(\lambda_w \circ C_\kappa^{-1})(X_\kappa)$ is the action of $X_\kappa$ on $W_{\lambda_w}$. This will be used in later sections.

Before moving on to the analysis and topology sections of the paper we want to point out that the conditions imposed on $\lambda$ at the beginning of §1 do not exclude the case of holomorphic $p$-torsion as originally defined by Ray and Singer [R-S]. Indeed, since $\mathfrak{p}_- \cong \mathfrak{p}_+^*$, it is enough to consider $\Lambda^p \mathfrak{p}_+$ and take any irreducible $K$ submodule, say $V_\lambda$, and show that $\lambda$ is of the required form. Now any highest weight of $\Lambda^p \mathfrak{p}_+$ is of the form $\lambda = \alpha_{k_1} + \cdots + \alpha_{k_p}$ where $\alpha_{k_i}$ are roots of $\mathfrak{p}_+$. This is clearly $G$-integral. Notice that if $\alpha$ is a positive root of $K$

\[
\langle \alpha, \alpha_{k_1} + \cdots + \alpha_{k_p} + \rho \rangle = \langle \alpha, \alpha_{k_1} + \cdots + \alpha_{k_p} + \rho_c + \rho_n \rangle = \langle \alpha, \alpha_{k_1} + \cdots + \alpha_{k_p} + \rho_c \rangle > 0.
\]

So it is enough to show $\lambda + \rho$ is regular, and from the above we need consider only non-compact roots, and here for $\mathfrak{g}$ simple and of Hermitian type, any non-compact root $\alpha = \pm(\alpha_0 + \alpha_c)$ where $\alpha_0$ is a simple non-compact root and $\alpha_c$ a sum of compact roots (or zero). Then, say $\alpha > 0,$

\[
\langle \alpha, \lambda + \rho \rangle = \langle \alpha_0 + \alpha_c, \alpha_{k_1} + \cdots + \alpha_{k_p} + \rho \rangle = \langle \alpha, \alpha_{k_1} + \cdots + \alpha_{k_p} \rangle + \frac{1}{2}(\alpha_c, 2\rho_c) + \langle \alpha_0, \rho \rangle.
\]
Now the first is non-negative because $\alpha, \alpha_k$ are non-compact roots and the sum is never (Hermitian) a root. The second is non-negative always and positive if $\alpha_c \neq 0$. The third, $\langle \alpha_0, \rho \rangle$, is positive because $\alpha_0$ is simple and, well, $\rho$ is $\rho$.

The last business for this section is to consider a finite-dimensional irreducible representation $(\pi, H_\pi)$. A parametrization of those with $e'(\pi, V, p_\pi) \neq 0$ follows from [K]; however, a little structure theory gives a parametrization that makes transparent the proportionality between $L^2$-torsion and torsion on the compact dual. Indeed, it appears that the proportionality principle is a consequence of a more basic relationship. We begin with the following observation.

**Lemma 1.14.** Let $ch$ denote the virtual character of an element of the representation ring $R(K)$. Then

$$ch\left(\sum (-1)^q \Lambda^q_+\right) = ch\left(\sum (-1)^q \Lambda^q_+ \left(\frac{n}{2} + \frac{1}{2} \sum_{\Delta_n^+} \coth \frac{\alpha}{2}\right)\right).$$

**Proof.** This follows from the already mentioned fact that

$$\sum (-1)^q \Lambda^q_+ = (-1)^n \xi_{\rho_n} \otimes (\sigma^+ - \sigma^-)$$

and that the almost complex structure determines a number operator on $\Lambda^* p_+$. Thus,

$$ch\left(\sum (-1)^q \Lambda^q_+\right) = -i H_0 \cdot ch\left(\sum (-1)^q \Lambda^q_+\right)$$

$$= -i(-1)^n [\rho_n(H_0)\xi_{\rho_n} ch((\sigma^+ - \sigma^-) + \xi_{\rho_n} H_0 \cdot \prod_{\Delta_n^+} (e^\frac{\alpha}{2} - e^{-\frac{\alpha}{2}})]$$

$$= -i[i\frac{n}{2} ch\left(\sum (-1)^q \Lambda^q_+\right) + \frac{i}{2} ch\left(\sum (-1)^q \Lambda^q_+ \cdot \sum_{\Delta_n^+} \coth \frac{\alpha}{2}\right)]$$

$$= ch\left(\sum (-1)^q \Lambda^q_+\right) \cdot \left(\frac{n}{2} + \frac{1}{2} \sum_{\Delta_n^+} \coth \frac{\alpha}{2}\right).$$

Since $W_K = W(\mathfrak{k}_C)$ permutes $\Delta_n^+$, the function $\frac{n}{2} + \frac{1}{2} \sum_{\Delta_n^+} \coth \frac{\alpha}{2}$ has a Fourier series in characters of $K$. The elements of $\hat{K}$ which occur we shall identify in terms of holomorphically induced representations from the $L_\kappa$'s. For $(\xi, \nu) \in \hat{L}_\kappa$ where $e^\nu$ is a character of the
circle generated by $H_\kappa$ and $\xi$ is compatible with $e^\nu$ we let $\pi_{\xi,\nu} \in \hat{K}$ be the representation given by the theorem of Bott-Kostant.

For notational convenience we suppose $G$ is simple. From [Kn, p. 479-480] or [Sc] one finds that there are at most two $W_K$-orbits in $\Delta^+_n$. If $\alpha_{\kappa_1}$ is the unique simple non-compact root, it defines one orbit; the other, if at all, is defined by $\alpha_{\kappa_2} := \alpha_{\kappa_1} + \delta$, where $\delta$ denotes a sum of compact roots. As the notation suggests, $\alpha_{\kappa_i}$ is associated to a tds $\kappa_i$.

**Lemma 1.15.**

$$\frac{n}{2} + \frac{1}{2} \sum_{\Delta^+_n} \text{coth} \frac{\alpha}{2} = \sum_{\kappa_i} \sum_{k \geq 0} \sum_{W_{\kappa_i}} \varepsilon(w) \text{ch}(\pi_{0_w, -k\alpha_{\kappa_i}}).$$

**Proof.** As before, the notation $\lambda_w$ denotes $w(\lambda + \rho_c) - \rho_c$ and is the highest weight of the representation of $L_\kappa$. Also it is standard that $W_K = W(l_\kappa)W_\kappa$ and since $L_\kappa$ centralizes $H_\kappa$, the $W_\kappa$ orbit of $\alpha_\kappa$ is $\{w^{-1} \cdot \alpha_\kappa \mid w \in W_\kappa\}$. Now

$$\frac{n}{2} + \frac{1}{2} \sum_{\Delta^+_n} \text{coth} \frac{\alpha}{2} = \sum_{\Delta^+_n} \frac{1}{1 - e^{-\alpha}} = \sum_{\kappa_i} \sum_{W_{\kappa_i}} \frac{1}{1 - e^{-w^{-1}_i \alpha_{\kappa_i}}}.$$

Fix a $\kappa_i$ and denote it by $\kappa$. The following calculation involves only elementary manipulations.

$$\left(\sum_{W_\kappa} \frac{1}{1 - e^{-w^{-1}_i \alpha_\kappa}}\right) \left(\sum_{W_\kappa} \varepsilon(\sigma) e^{\sigma \rho_c}\right) = \sum_{k \geq 0} \sum_{W_\kappa} e^{-kw^{-1}_i \alpha_\kappa} \sum_{W_\kappa} \varepsilon(\sigma) e^{\sigma \rho_c}$$

$$= \sum_{k \geq 0} \frac{1}{|W(l_\kappa)|} \sum_{W_\kappa} e^{-kw^{-1}_i \alpha_\kappa} \sum_{W_\kappa} \varepsilon(\sigma) e^{\sigma \rho_c}$$

$$= \sum_{k \geq 0} \frac{1}{|W(l_\kappa)|} \sum_{\sigma \in W_\kappa} \varepsilon(\sigma) \sum_{t \in W_\kappa} e^{\sigma(-kt^{-1}_i \alpha_\kappa + \rho_c)}$$

$$= \sum_{k \geq 0} \sum_{\sigma \in W_\kappa} \varepsilon(\sigma) \sum_{w \in W_\kappa} e^{\sigma(-kw^{-1}_i \alpha_\kappa + \rho_c)}$$

$$= \sum_{k \geq 0} \sum_{w \in W_\kappa} \varepsilon(w) \sum_{\sigma \in W_\kappa} \varepsilon(\sigma) e^{\sigma(-k\alpha_\kappa + w \rho_c)}.$$
Next notice that \(-k\alpha + w\rho_c\) is \(\Delta^+(I_\kappa)\) dominant and regular. Indeed, for \(\beta \in \Delta^+(I_\kappa)\)

\[
\langle -k\alpha + w\rho_c, \beta \rangle = \langle w\rho_c, \beta \rangle = \langle \rho_c, w^{-1}\beta \rangle > 0
\]

since \(L_\kappa\) centralizes \(H_\kappa\) and \(w \in W_\kappa\). It follows that \(-k\alpha + w\rho_c - \rho_c\) corresponds to the representation, unfortunately denoted \(0_{w,-k\alpha_\kappa}\).

Repeating the argument for the other \(\kappa\)'s and using the Weyl character formula concludes the proof. 

\[\boxed{}\]

**Remark.** One can easily determine from the structure of \(K/L_\kappa\) at which threshold the weight \(-k\alpha + \sigma\rho_c\) is \(\Delta^+_i\) singular or not dominant. In the latter case the necessary \(W_K\) element to establish dominance is easily determined. In this way, in specific cases such as rank 1, the Fourier series in Lemma 1.15 becomes simpler.

**Proposition 1.16.** Let \((\pi, H_\pi)\) be a finite dimensional irreducible representation of \(G\). If

\[
e'(\pi, V, p_-) \neq 0,
\]

then for some \(\sigma \in W(g_C, t_C)\) and \(w^* \in W_{\kappa_i}\) and \(k \geq 1\)

\[
\sigma(\Lambda_\pi + \rho) = w^*(\lambda^* + \rho) - 2\rho_n + k\alpha_{\kappa_i}.
\]

**Proof.**

\[
e'(\pi, V, p_-) = \sum (-1)^q q \dim[H_\pi \otimes \Lambda^q p_+ \otimes V]^K
\]

\[
= \int_K ch\pi ch \left( \sum (-1)^q q\Lambda^q_+ \right) ch\tau
\]

\[
= \int_K ch\pi ch \left( \sum (-1)^q q\Lambda^q_+ \right) \left( \frac{n}{2} + \frac{1}{2} \sum_{\Delta^+_i} \coth \frac{\alpha}{2} \right) ch\tau
\]

\[
= (-1)^n \int_K ch\pi ch \left( \sum (-1)^q \Lambda^q_+ \right) ch2\rho_n \left( \frac{n}{2} + \frac{1}{2} \sum_{\Delta^+_i} \coth \frac{\alpha}{2} \right) ch\tau.
\]
Now we apply Kostant’s Theorem twice, obtaining (with $W^p = W(g_C, t_C)/W(t_C, t_C)$)

$$ch\pi ch \left( \sum (-1)^q \Lambda^q \right) = \sum_{W^p} (-1)^{\ell(\sigma)} chV_{\sigma \pi}$$

and

$$V_\lambda \otimes \sum (-1)^\ell \Lambda^\ell \pi^\ell_K = \sum (-1)^{\ell(w)} W_{\lambda w},$$

and then using the Weyl formulas to get

$$\sum_{\kappa_i} \frac{(-1)^{n+m_i}}{|W_K|} \int_{T} \sum_{W^p} (-1)^{\ell(\sigma)} chV_{\sigma \pi} \Delta_K ch2\rho_n \sum_{W_{\kappa_i}} \frac{1}{1 - e^{-s^{-1}\alpha_{\kappa_i}}} ch\rho(n^\ell_K) \sum_{W_{\kappa_i}} (-1)^{\ell(w)} \Delta_{L\kappa} chW_{\lambda w}.$$

Since after expanding $(1 - e^{-s^{-1}\alpha_{\kappa_i}})^{-1}$ the integrand will consist of products of exponentials, in order for $e'(\pi, V, p_-)$ to be non-zero we need

$$0 = \sigma(\Lambda_{\pi} + \rho) - \rho + \rho_c + 2\rho_n + \rho(n^\ell_K) + \lambda_w + \rho(\ell) - ks^{-1}\alpha_{\kappa_i}$$

$$= \sigma(\Lambda_{\pi} + \rho) + \lambda_w + \rho - ks^{-1}\alpha_{\kappa_i}$$

$$= \sigma(\Lambda_{\pi} + \rho) + w(\lambda + \rho_c) + \rho_n - ks^{-1}\alpha_{\kappa_i}.$$ 

Then

$$s\sigma(\Lambda_{\pi} + \rho) = -sw(\lambda + \rho_c) - \rho_n + k\alpha_{\kappa_i}$$

$$= -sw(-w^K)(\lambda^* + \rho_c) - \rho_n + k\alpha_{\kappa_i}$$

and since $w^{L\kappa}$ fixes $\alpha_{\kappa_i}$,

$$(w^{L\kappa}s\sigma)(\Lambda_{\pi} + \rho) = (-w^{L\kappa})(sw)(-w^K)(\lambda^* + \rho_c) - \rho_n + k\alpha_{\kappa_i}.$$ 

Relabeling gives the result. 

\[\square\]

Remark. A comparison of Theorem 1.12 and Proposition 1.16 shows that the unitary principal series and the finite dimensional representations that have non-zero derived Euler number are in the same parametrized family, with the unitary character $e^\nu$ on exp $tX_{\kappa}$ being replaced with the countable collection of characters on exp $tH_{\kappa}$. This, together with some analysis, eventually leads to a sharper result than the usual one relating the $L^2$-torsion to holomorphic torsion on the compact dual.
\section{Geometric Zeta Function}

Let $\Gamma$ be a discrete, co-compact, torsion-free subgroup of $G$. The quotient of $\tilde{X} \simeq G/K$ by the action of $\Gamma$, $X = \Gamma \backslash \tilde{X}$, is a complex Hermitian manifold having $\tilde{X}$ as its universal cover. For the unitary representation $(\tau, V)$ of $K$ on the complex vector space $V$ we let $\tilde{V}$, resp. $V$, denote the associated holomorphic, Hermitian, homogeneous vector bundle over $\tilde{X}$, resp. $X$. Without loss of generality, we can assume that $\tau$ is irreducible, in which case we shall denote by $\lambda$ a highest weight as before and by $V_\lambda$ its representation space.

The zeta function we will define in this section encodes geometric data from the Hermitian metric and coefficient bundle on $X$ onto the periodic set of the geodesic flow $\Phi$ on the tangent bundle $TX$. The periodic set of the geodesic flow is the disjoint union, $\bigsqcup_{[\gamma] \in [\Gamma \backslash \{e\}]} X_\gamma$, of its connected components. Since $X_\gamma \simeq \Gamma_\gamma \backslash G_\gamma/K_\gamma$, we denote by $\tilde{X}_\gamma \simeq G_\gamma/K_\gamma$ its simply connected cover, and by $\tilde{X}_\gamma = X_\gamma/\Phi$ the orbit space under the (circle) action of the flow. From Selberg we know that $\gamma$ is semisimple, so we may assume that $\gamma$ has been conjugated to a standard Cartan subgroup with split factor contained in the exp defined in §1. Then $\gamma = \gamma_I \exp \ell_\gamma Y$ with $Y$ in the closure of $A_+$ and of norm one. Each geodesic in $X_\gamma$ has the same length $\ell_\gamma$; we let $\mu_\gamma$ denote the generic multiplicity.

The construction of the zeta function also involves the choice of a compactification of $\tilde{X}$. We shall use one of the standard compactifications of $\tilde{X}$, namely the closure $\overline{X} = \overline{G \cdot K_\mathbb{C} P_-}$ of the Borel embedding of $\tilde{X}$ into $G_\mathbb{C}/K_\mathbb{C}P_-$. 

We recall from [Sa] that $\partial \overline{X} = \overline{X} - \tilde{X}$ is stratified, with each strata an orbit of $G$, and that the following are in $1-1$ correspondence: $G$-orbits $Y_\kappa$ in $\partial \overline{X} = \overline{X} - \tilde{X}$, \{maximal parabolic subalgebras\} and \{$\kappa$-Cayley transforms $C_\kappa = \text{Ad} c_\kappa$, $c_\kappa \in G_\mathbb{C}$, associated to canonical $\kappa$\}.

By sending each point in $\tilde{X}_\gamma$ to the point at infinity of the (unique) axis to which it belongs (see [Sa] p. 141), we get a map $\tilde{\nu}_\gamma$ from $\tilde{X}_\gamma$ to a unique $G$-orbit $Y_\kappa \subset \partial \overline{X}$. This map descends to $\nu_\gamma : \tilde{X}_\gamma/\Phi \to Y_\kappa$ which, in turn, factors as a fibration

$$\tilde{X}_\gamma^s \to \tilde{X}_\gamma/\Phi \subset \tilde{X}_\gamma^{hs} \subset Y_\kappa \simeq G \cdot c_\kappa K_\mathbb{C} P_-,$$

with base $\tilde{X}_\gamma^{hs}$ Hermitian symmetric and fiber $\tilde{X}_\gamma^s$ Riemannian symmetric. By means of this we associate the extrinsic geometric information to $\tilde{X}_\gamma$. We denote by $T^{us}(\tilde{X}_\gamma/\Phi)$
the sub-bundle tangent to the fibers in the tangent bundle and by \(T^h s(\tilde{X}_\gamma/\Phi)\) its orthogonal complement with respect to the \(G_\gamma\)-invariant metric on \(\tilde{X}_\gamma/\Phi \cong G_\gamma/A_\Phi K_\gamma\), where \(A_\Phi = \exp(\mathbb{R} Y)\). Note that \(T^h s(\tilde{X}_\gamma/\Phi) \cong \nu^*_\gamma(T\tilde{X}^{h s}_\gamma)\) and thus comes equipped with a \(G_\gamma\)-invariant complex structure. The appropriate normal bundle for our purposes is the pull-back \(N(\tilde{X}_\gamma/\Phi)\) of the orthocomplement (with respect to the \(G_\gamma\)-invariant metric) of \(T\tilde{X}^{h s}_\gamma\) in \(TY_\kappa\mid \tilde{X}^{h s}_\gamma\). As all the above defined bundles are \(G_\gamma\)-invariant, they descend to the orbifold \(\tilde{X}_\gamma \cong \Gamma_\gamma \backslash \tilde{X}_\gamma/\Phi\) where they shall be denoted, respectively, by \(T^u s(\tilde{X}_\gamma), T^h s(\tilde{X}_\gamma)\) and \(N(\tilde{X}_\gamma)\).

The complexified normal bundle \(N_C(\tilde{X}_\gamma/\Phi)\), resp. \(N_C(\tilde{X}_\gamma)\), has in turn some extra structure, inherited from the geometry of the \(G\)-orbit \(Y_\kappa\). (We refer the reader to [Wo, p. 299] for more details concerning that geometry.) Namely, \(Y_\kappa \cong G \cdot c_\kappa K C P_-\) fibers over a compact flag manifold \(Y^s_\kappa \cong G/Q\), with typical fiber a Hermitian symmetric space \(Y^{h s}_\kappa \cong M^{(1)}_Q/K \cup M^{(1)}_Q\). The latter contains \(\tilde{X}^{h s}_\gamma\) as a complex submanifold. Now \(Y_\kappa \cong G \cdot c_\kappa K C P_-\) is naturally embedded in \(U/K \cong G/K C P_-\), and as such it inherits a left-invariant Riemannian metric. The pull-back by \(\nu_\gamma\) of the orthocomplement of \(T_C \tilde{X}^{h s}_\gamma\) in \(T_C Y^{h s}_\kappa\mid \tilde{X}^{h s}_\gamma\) with respect to the latter metric will be denoted \(N_C(\tilde{X}_\gamma/\Phi)\). Now \(Y^s_\kappa \cong G/Q\) itself fibers over a compact Hermitian symmetric space.

By lifting the tangent bundle to the fibers, resp. to the base, we get a vector bundle, \(U^\theta(\tilde{X}_\gamma/\Phi)\), resp. \(V^\theta(\tilde{X}_\gamma/\Phi)\), over \(\tilde{X}_\gamma/\Phi\). Thus,

\[
N_C(\tilde{X}_\gamma/\Phi) = N_C(\tilde{X}_\gamma/\Phi) \oplus V^\theta_C(\tilde{X}_\gamma/\Phi) \oplus U^\theta_C(\tilde{X}_\gamma/\Phi)
\]

and therefore

\[
N_C(\tilde{X}_\gamma) = N_C(\tilde{X}_\gamma) \oplus V^\theta_C(\tilde{X}_\gamma) \oplus U^\theta_C(\tilde{X}_\gamma).
\]

Next we describe the localization to \(\tilde{X}_\gamma\) of the coefficient bundle \(V\) over \(X\). Its lift \(\tilde{V}\) to \(\tilde{X} = G/K\) has a natural holomorphic extension \(\nabla\) to the compactification \(\overline{X}\) in \(G_C/K C P_-\), obtained by extending the isotropy representation trivially on \(P_-\). In particular, \(\nabla\) is defined over \(\partial \overline{X}\) and restricts to a vector bundle \(V_\kappa\) over \(Y_\kappa\). This, in turn, has a lift \(\tilde{V}_{\gamma,\kappa} = \nu^*_\gamma(\nabla_\kappa)\) over \(\tilde{X}_\gamma/\Phi\), which descends to give a vector bundle \(V_{\gamma,\kappa}\) over the orbifold \(\tilde{X}_\gamma \cong \Gamma_\gamma \backslash \tilde{X}_\gamma/\Phi\).

All the vector bundles over \(\tilde{X}_\gamma/\Phi \cong G_\gamma/A_\Phi K_\gamma\) (resp. \(\tilde{X}_\gamma\)) constructed above are homogeneous (resp. locally homogeneous) bundles, arising from representations of the isotropy
subgroup $\tilde{K}_\gamma = A_\Phi K_\gamma$. As a central element in $\tilde{K}_\gamma$, $\gamma$ determines an automorphism of each such bundle; moreover, by construction (cf. the Cayley transform in §1) this automorphism is necessarily semisimple. In particular, with each of the above vector bundles over $\hat{X}_\gamma$ we can associate not only an element in the $K$-theory group $K^0(\hat{X}_\gamma)$ but also an element in the representation ring $\mathbb{Z}[\mathbb{C}^*]$ generated by all semisimple representations of the cyclic group $\{\gamma\}$, hence an element in $K^0(\hat{X}_\gamma) \otimes \mathbb{Z}[\mathbb{C}^*]$.

Essentially from (1.9) it follows that $V_{\gamma,\kappa}$ is actually holomorphic “in the direction” of the tangent sub-bundle $T^{hs}(\hat{X}_\gamma)$ and flat “in the direction” of $T^{vs}(\hat{X}_\gamma)$. This suggests we consider the mixed de Rham-Dolbeault complex on $\hat{X}_\gamma$ obtained by tensoring the de Rham complex associated to $T^{vs}(\hat{X}_\gamma)$ with the Dolbeault complex associated to the holomorphic part $T^{hs}_+(\hat{X}_\gamma)$ of $T^{hs}(\hat{X}_\gamma)$. Using the Levi-Civita connection, one can construct out of the two first-order operators naturally associated to the sub-bundles an elliptic operator $D_\gamma$ acting as a self-adjoint (unbounded) operator on the $L^2$-sections of $\wedge T^{us}(\hat{X}_\gamma) \otimes \wedge T^{hs}(\hat{X}_\gamma)$ and anti-commutes with the grading into even and odd forms. Thus, $D_\gamma$ defines a $K$-homology class $[D_\gamma] \in K_0(\hat{X}_\gamma)$, which pairs with any $K$-theory class $[E] \in K^0(\hat{X}_\gamma)$,

$$\langle [D_\gamma], [E] \rangle = \text{Index } D_{\gamma,E}.$$  

Being integer-valued, the pairing with $[D_\gamma]$ extends by linearity to $K^0(\hat{X}_\gamma) \otimes \mathbb{Z}[\mathbb{C}^*]$.

If $\xi \in K^0(\hat{X}_\gamma) \otimes \mathbb{Z}[\mathbb{C}^*]$, we shall denote by $\xi(\gamma)$ the image of $\xi$ under the map $I \otimes \varepsilon : K^0(\hat{X}_\gamma) \otimes \mathbb{Z}[\mathbb{C}^*] \to K^0(\hat{X}_\gamma) \otimes \mathbb{C}$, where $\varepsilon : \mathbb{Z}[\mathbb{C}^*] \to \mathbb{C}$ stands for the tautological evaluation map. With this convention, and using the standard notation $\wedge_{-1} = \wedge^{ev} - \wedge^{odd}$, we introduce the following characteristic number of $X_\gamma$:

$$\chi_\gamma(\mathcal{V}) = \left\langle [D_\gamma], \frac{[V_{\gamma,\kappa}]|^{(\gamma)}_{\wedge_{-1}(\mathcal{N}_+^{*}(\hat{X}_\gamma)^* \oplus V^\theta_+^{*}(\hat{X}_\gamma)^* \oplus U^\theta_+^{*}(\hat{X}_\gamma)^*)]}{[\wedge_{-1}(\mathcal{N}_+^{*}(\hat{X}_\gamma)^* \oplus V^\theta_+^{*}(\hat{X}_\gamma)^* \oplus U^\theta_+^{*}(\hat{X}_\gamma)^*)]} \right\rangle. $$

The denominator of the right-hand side of the pairing is invertible, since by restriction to a point it becomes

$$\det(I - \text{Ad}(\gamma))|_{\wedge_{-1}(\mathcal{N}_+^{*} \oplus V^\theta_+^{*} \oplus U^\theta_+^{*})},$$

which involves only eigenvalues $\neq 1$ of $\text{Ad}(\gamma)$.

In the course of the explicit calculation of this characteristic number, we shall see that the bundles occurring in the above formula are holomorphic in the direction of $T^{hs}(\hat{X}_\gamma)$.
and flat in the direction of $T^{vs}(\hat{X}_\gamma)$, i.e., admit connections with this property. For this reason, $\chi_\gamma(\mathbb{V})$ can be viewed as a “mixed Euler characteristic” and we shall use the more suggestive notation

\begin{equation}
\chi_\gamma(\mathbb{V}) = \chi_{\text{mix}}(\hat{X}_\gamma, \frac{[\mathbb{V}_\gamma, \kappa]}{[\wedge^{-1}(\mathcal{N}_+(\hat{X}_\gamma)^* + \mathcal{V}_+^0(\hat{X}_\gamma)^* + \mathcal{U}_0^0(\hat{X}_\gamma)^*)](\gamma)}) \right).
\end{equation}

Recall that the coefficient bundle is associated to $\tau : K \to U(V_\lambda)$ an irreducible, unitary representation of $K$. In addition to $(\tau, V_\lambda)$ we take $\varphi : \Gamma \to U(F)$ a unitary representation on a finite dimensional complex vector space. With this geometric data we make the following definition of a Weil-type zeta function.

\begin{equation}
Z_{\mathbb{V}, \varphi}(z) = \exp - \sum_{[\gamma] \neq e} \text{Tr} \varphi(\gamma) \chi_{\text{mix}}(\hat{X}_\gamma, \frac{[\mathbb{V}_\gamma](\gamma)}{[\wedge^{-1}(\mathcal{N}_+(\hat{X}_\gamma)^* + \mathcal{V}_+^0(\hat{X}_\gamma)^* + \mathcal{U}_0^0(\hat{X}_\gamma)^*)](\gamma)}) \frac{e^{-2\ell_\gamma}}{\mu_\gamma}.
\end{equation}
§3 Topological Calculations

First, we shall prove that $\chi_{\gamma}(V)$ vanishes, unless $\gamma \in \mathcal{E}_1(\Gamma)$, i.e., $\gamma$ belongs to a cuspidal maximal parabolic subgroup.

**Lemma 3.1.** If $\gamma \notin \mathcal{E}_1(\Gamma)$, then $[D_{\gamma}] = 0$ in $K_0(\hat{\mathcal{X}}_{\gamma})$, hence $\chi_{\gamma} = 0$.

**Proof.** Identify the tangent space to $\hat{\mathcal{X}}_{\gamma}$ at the base point $x_0 = \Gamma_{\gamma} \cdot e \cdot \tilde{K}_{\gamma}$ with $p_{\gamma}^Y$ as the orthocomplement of $\mathbb{R}Y$ in $p_{\gamma}$. With $\wedge^*\tau_{\gamma}$ denoting the fiber of $\wedge T^g_{\gamma}(\hat{\mathcal{X}}_{\gamma}) \otimes \wedge T^h_{\gamma}(\hat{\mathcal{X}}_{\gamma})$ at $x_0$, let

$$\sigma(\xi) := \sigma(D_{\gamma}^+)_(x_0, \xi) : E^+ = \wedge^*\tau_{\gamma} \to E^- = \wedge^*\tau_{\gamma} \quad \forall \xi \in p_{\gamma}^Y,$$

be the principal symbol of $D_{\gamma}^+$. Then

$$\sigma(\text{Ad}(k)\xi) = \rho^-(k)\sigma(\xi)\rho^+(k)^{-1}, \quad \forall k \in \tilde{K}_{\gamma},$$

where $\rho^\pm$ denotes the representation of $\tilde{K}_{\gamma}$ on $E^\pm$.

The hypothesis $\gamma \notin \mathcal{E}_1(\gamma)$ implies the existence of a non-zero $\xi_0 \in p_{\gamma}^Y$ such that

$$\text{Ad}(T_{\gamma})\xi_0 = \xi_0,$$

where $T_{\gamma}$ denotes the maximal torus of $K_{\gamma}$. By (3.3) and the ellipticity of $D_{\gamma}^+$, $\sigma(\xi_0)$ is an isomorphism of $T_{\gamma}$-modules. Then $E^+ = \wedge^*\tau_{\gamma}$ and $E^- = \wedge^*\tau_{\gamma}$ must be isomorphic as $\tilde{K}_{\gamma}$-modules too. Let $\sigma_0 : E^+ \to E^-$ be such an isomorphism. Then

$$\sigma_t(\xi) = (1 - t) \cdot \sigma_0 + t \cdot \sigma(\xi), \quad \forall \xi \in p_{\gamma}^Y, \ t \in [0, 1],$$

defines a homotopy between $\sigma$ and the constant symbol $\sigma_0(\xi) \equiv \sigma_0$, which gives rise to a homotopy between $\sigma(D_{\gamma}^+)$ and a constant bundle isomorphism. Therefore, the symbol class $[\sigma(D_{\gamma}^+)] = 0$ in $K^0_c(T\hat{\mathcal{X}}_{\gamma})$, which is equivalent with the vanishing of $[D_{\gamma}]$ in $K_0(\hat{\mathcal{X}}_{\gamma})$.

$\blacksquare$

We shall also need the following observation.
Lemma 3.2. Let \((F, \rho_F)\) be a \(G_\gamma\)-module. The locally homogeneous vector bundle \(\mathbb{F}\) over \(\tilde{\mathcal{X}}_\gamma\) associated to \(\rho_F \mid \tilde{K}_\gamma\) is flat.

Proof. Define \(\psi : G_\gamma \times F \to G_\gamma \times F\) by \(\psi(g, v) = (g, \rho_F(g)v), \forall (g, v) \in G_\gamma \times F.\) Then

\[
\psi(g \cdot g \cdot k^{-1}, \rho_F(k)v) = (g \cdot g \cdot k^{-1}, \rho_F(gg)v), \quad \forall \gamma \in \Gamma_\gamma \text{ and } k \in \tilde{K}_\gamma.
\]

This shows that \(\psi\) induces an isomorphism of \(\mathbb{F} = \Gamma_\gamma \backslash G_\gamma \times \tilde{K}_\gamma F\) with the flat bundle \(G_\gamma/\tilde{K}_\gamma \times \Gamma_\gamma F\) over \(\tilde{\mathcal{X}}_\gamma\) induced by \(\rho_F \mid \Gamma_\gamma.\)

In view of Lemma 3.1, we may (and do) assume from now on that \(\gamma \in \mathcal{E}_1(\Gamma)\) and hence \(A_\Phi = \exp(\mathbb{R}X_\kappa).\) The technical lemmas provided in [M-S II, §3] under this hypothesis remain valid, with the proviso that the connected component \(G^0_\gamma\) employed in loc. cit. has to be replaced by the Harish-Chandra class subgroup \(G^+_\gamma, G^0_\gamma \subseteq G^+_\gamma \subseteq G_\gamma;\) in particular, \(C_\gamma\) will denote the center of \(G^+_\gamma.\) Modulo dealing with the non-orientable case as indicated in [M-S I, §5], we can also make the simplifying assumption that \(T^{us}(\tilde{\mathcal{X}}_\gamma)\) is orientable.

Using the Atiyah-Singer cohomological index formula [A-S III, Thm. (2.12)], the pairing (2.1) can be expressed in cohomological terms as follows:

\[
\chi_\gamma(V) = \left\{ \frac{\mathcal{T}d(T^{hs}(\tilde{\mathcal{X}}_\gamma)) \cup e(T^{us}(\tilde{\mathcal{X}}_\gamma)) \cup (\text{ch} \{V, \kappa\}(\gamma))}{\text{ch} [\wedge^{-1}(\mathcal{N}_+ \tilde{\mathcal{X}}_\gamma)^* \oplus V_+^{(\tilde{\mathcal{X}}_\gamma)^*} \oplus \mathcal{U}_C^{(\tilde{\mathcal{X}}_\gamma)^*}](\gamma)]} \right\} [\tilde{\mathcal{X}}_\gamma];
\]

here \(\mathcal{T}d\) stands for the Todd class, \(e\) for the Euler class, and here \(\text{ch}\) the Chern Character

\[
\text{ch} \xi(\gamma) = \sum \varepsilon(\chi_i) \text{ch} a_i, \quad \forall \xi = \sum a_i \otimes \chi_i \in K^0(\tilde{\mathcal{X}}_\gamma) \otimes \mathbb{Z}[\mathbb{C}^*].
\]

The remainder of the section is devoted to the task of expressing \(\chi_\gamma\) in group-theoretical terms. To begin with, we note that \(\mathcal{U}_C^{(\tilde{\mathcal{X}}_\gamma)^*}\) is already flat, by Lemma 3.2, since the \(\tilde{K}_\gamma\)-module \(u_C^\theta\) is the restriction of a \(G_\gamma\)-module (in fact, of an \(M_Q\)-module). Its equivariant Chern character is 0-dimensional and thus, using \(u_C^\theta \sim u_C\), easily computable:

\[
\text{ch} [\wedge^{-1} u_C^\theta(\tilde{\mathcal{X}}_\gamma)^*](\gamma)^{-1} = \text{Tr} \left( \gamma \mid \wedge^{-1} u_C^\theta\right)^{-1} = \text{det}(I - \gamma \mid u_C)^{-1}.
\]

Next, we replace \(\text{ch} [\wedge^{-1} V_+^\theta(\tilde{\mathcal{X}}_\gamma)^*](\gamma)^{-1}\) by the equivalent expression

\[
\frac{\text{ch} [\wedge^{-1} V_+^\theta(\tilde{\mathcal{X}}_\gamma)^*](\gamma)}{\text{ch} [\wedge^{-1} V_+^\theta(\tilde{\mathcal{X}}_\gamma)^*](\gamma)}.
\]
Since \( v_\gamma^\theta \) is also a \( G_\gamma \)-module, and \( v_\gamma^{\theta, *}_C \sim v_\gamma^\theta \), one has
\[
\text{ch} [\wedge_1 v_\gamma^\theta (\hat{X}_\gamma)^*](\gamma)^{-1} = \text{Tr}(\gamma | \wedge_1 v_\gamma^{\theta, *})^{-1} = \det(I - \gamma | \gamma)C^{-1}.
\]
Therefore,
\[
\chi_{\gamma}(\gamma) = \det(I - \gamma | \gamma C^{-1})^{-1} \cdot \left\{ \tau d(T^{hs}(\hat{X}_\gamma)) \cup e(T^{vs}(\hat{X}_\gamma)) \cup \frac{\text{ch}[V_{\gamma, \kappa}](\gamma) \cup \text{ch}[\wedge_1 v_\gamma^\theta(\hat{X}_\gamma)^*](\gamma)}{\text{ch}[\wedge_1 N_{\gamma}(\hat{X}_\gamma)^*](\gamma)} \right\} [\hat{X}_\gamma].
\]
(3.3)

Recall now that the fiber of \( \mathcal{N}(\hat{X}_\gamma) \) is, by the very definition, the orthocomplement of \( p_\gamma^{(1)} \) in \( p^{(1)} \cong m_{\gamma}^{(1)}/\ell \cap m_{\gamma}^{(1)} \), which will be denoted \( p_\gamma^{(1)} \). In making this identification, we regard \( p_\gamma^{(1)} \) as a \( \widetilde{K}_\gamma \)-module as follows. First, \( K_\gamma^{(1)} = K_\gamma \cap M_\kappa^{(1)} \subset K_\gamma \cap M_\kappa^{(1)} \) acts on \( p^{(1)} \) via the adjoint representation \( \text{Ad} \). Now \( K_\gamma = K_\gamma^{(1)} \cdot K_\gamma^{(2)} \) (almost direct product, with central intersection), with \( K_\gamma^{(2)} = K_\gamma \cap M_\kappa^{(2)} \) acting trivially by \( \text{Ad} \). The remaining part, \( A_\Phi \), also acts trivially by \( \text{Ad} \). Thus, if we let \( S_{\gamma, \perp}^{(1)} \) denote the standard spin module of \( \text{Spin} p_\gamma^{(1)} \), regarded as a \( \widetilde{K}_\gamma \)-module, one has (see also (1.13)),
\[
(\wedge_1 p_{\gamma, \perp}^{(1)})^* = (\wedge_1 p_{\gamma, \perp}^{(1)})^* = (-1)^{q_{\gamma}} (S_{\gamma, \perp}^{(1)} - S_{\gamma, \perp}^{(1)}) \otimes \xi_{\rho_n, \gamma, \perp}^{(1)}.
\]
Consequently,
\[
\text{ch} [\wedge_1 N_{\gamma}(\hat{X}_\gamma^*)](\gamma) = (-1)^{q_{\gamma}} [\text{ch}(S_{\gamma, \perp}^{(1)} + - S_{\gamma, \perp}^{(1)}) \otimes \mathbb{L}_{\rho_n, \gamma, \perp}^{(1)}](\gamma),
\]
where \( S_{\gamma, \perp}^{(1)} \) stands for the vector bundle over \( \hat{X}_\gamma \) associated to \( S_{\gamma, \perp}^{(1)} \) and \( \mathbb{L}_{\rho_n, \gamma, \perp}^{(1)} \) is the line bundle associated to the character \( \xi_{\rho_n, \gamma, \ell, \perp}^{(1)} \).

The contribution of the tangent sub-bundle \( T^{hs}(\hat{X}_\gamma) \), accounted for by the Todd class \( \tau d(T^{hs}(\hat{X}_\gamma)) \), can also be expressed in terms of characters of spin bundles. Indeed,
\[
\tau d(T^{hs}(\hat{X}_\gamma)) = \hat{A}(T^{hs}(\hat{X}_\gamma)) \text{ch} \mathbb{L}_{\rho_n, \gamma}^{(1)},
\]
where \( \hat{A} \) denotes the stable Hirzebruch \( \hat{A} \)-class and \( \mathbb{L}_{\rho_n, \gamma}^{(1)} \) is the canonical bundle, associated to the character \( \xi_{\rho_n, \gamma}^{(1)} \). In turn,
\[
\hat{A}(T^{hs}(\hat{X}_\gamma)) = e(T^{hs}(\hat{X}_\gamma)) \text{ch} [S_{\gamma, \perp}^{(1)} + - S_{\gamma, \perp}^{(1)}]^{-1},
\]
where $S_{\gamma}^{(1)}$ is the spin bundle associated to $T^{hs}(\tilde{X}_\gamma)$. Taking into account that $\{\gamma\}$ acts trivially on $T(\tilde{X}_\gamma)$, we conclude that

$$T d(T^{hs}(\tilde{X}_\gamma)) = e(T^{hs}(\tilde{X}_\gamma)) \cup \text{ch}[(S_{\gamma}^{(1)} + - S_{\gamma}^{(1)}) \otimes \mathbb{L}_{\rho_n}^{(1)}(\gamma)]^{-1}.$$ 

Therefore, (3.3) becomes

$$\chi_{\gamma}(\mathbb{V}) = (-1)^{q_{\gamma}} \det(I - \gamma | v_C \oplus u_C)^{-1},$$

(3.4) $$\left\{ \frac{e(T(\tilde{X}_\gamma)) \cup \text{ch} [\mathbb{V}_{\gamma,\kappa} \otimes \wedge_{-1} \mathbb{V}^\theta(\tilde{X}_\gamma)^*](\gamma) \cup \text{ch} [(S_{\gamma}^{(1)} - S_{\gamma}^{(1)}) \otimes \mathbb{L}_{\rho_n}^{(1)}(\gamma)]}{\text{ch} [(S_+ - S_-)](\gamma)} \right\} [\tilde{X}_\gamma],$$

where $S_{\gamma}^{(1)}$ is the bundle associated to the spin module of Spin $p(1)$ and $\mathbb{L}_{\rho_n}^{(1)}$ corresponds to $\xi_{\rho_n}^{(1)}$.

There is a conspicuous lack of symmetry in the previous formula, with respect to the real tangent sub-bundle $T^{us}(\tilde{X}_\gamma)$. To remedy this, we shall complete the bundle $S_{\gamma}^{(1)}$ to a larger bundle of spinors, $S = S_{\gamma}^{(1)} \oplus S^{(2)}$. Here $S^{(2)}$ is associated to the standard spin module of Spin $p^{(2)}$, viewed as a $\tilde{K}_\gamma$-module via the adjoint representation (and therefore with $K_{\gamma}^{(1)} = K_{\gamma} \cap M_Q^{(1)}$ and $A_{\Phi}$ acting trivially). Amplifying by the corresponding class in the right-hand side of (3.4), one gets:

$$\chi_{\gamma}(\mathbb{V}) = (-1)^{q_{\gamma}} \det(I - \gamma | v_C \oplus u_C)^{-1},$$

(3.5) $$\left\{ \frac{e(T(\tilde{X}_\gamma)) \cup \text{ch} [\mathbb{V}_{\gamma,\kappa} \otimes \wedge_{-1} \mathbb{V}^\theta(\tilde{X}_\gamma)^*](\gamma) \cup \text{ch} [(S_{\gamma}^{(1)} - S_{\gamma}^{(1)}) \otimes \mathbb{L}_{\rho_n}^{(1)}(\gamma)]}{\text{ch} [(S_+ - S_-)](\gamma)} \right\} [\tilde{X}_\gamma].$$

By means of Kostant’s formula (§1) and the decomposition into $K \cap M_Q^+$ isotypic components, the virtual bundle in the numerator can be expressed as follows:

$$\mathbb{V}_{\gamma,\kappa} \otimes \wedge_{-1} \mathbb{V}^\theta(\tilde{X}_\gamma)^* \otimes (S_{\gamma}^{(2)} - S_{\gamma}^{(2)}) \otimes \mathbb{L}_{\rho_n}^{(1)} = \sum_{w \in W_{\kappa}} (-1)^{\ell(w)} \sum_{\mu} a_{\mu}^{w} \mathbb{W}_{\mu};$$

(3.6) $$\text{ch} [(S_+ - S_-)](\gamma)$$

Here $\mathbb{W}_{\mu}$ is the vector bundle associated to $W_{\mu}$ regarded as $\tilde{K}_\gamma$-module by restriction from $K \cap M_Q^+ \cdot A_{\Phi}$ (see §2). Substituting this expression into the formula (3.5), we obtain:
Lemma 3.3. With the above notation, one has for $\gamma \in \mathcal{E}_1(\Gamma)$,

$$
\chi_{\gamma}(\mathcal{V}) = (-1)^{q_\gamma} \det(I - \gamma | v_{\mathcal{C}})^{-1} \det(I - \gamma | u_{\mathcal{C}})^{-1} \sum_{w \in W_\kappa} (-1)^{\ell(w)} \sum_{\mu} a^{w}_{\mu} \chi_{\gamma, \mu},
$$

where

$$
\chi_{\gamma, \mu} = \left\{ \frac{e(T(\hat{X}_\gamma)) \cup \text{ch} [W_{\mu}](\gamma)}{\text{ch} [S^+_+ - S^-_-](\gamma)} \right\} [\hat{X}_\gamma].
$$

By means of a slight extension of the formalism carefully explained in [H-P, §2], the $\{\gamma\}$-equivariant characteristic classes in the above formula can be expressed in group-theoretical terms. Thus, noting that $\gamma$ belongs to the center of $\tilde{K}_\gamma$ and adopting the notation of loc. cit., one has

$$
\frac{\text{ch} [W_{\mu}](\gamma)}{\text{ch} ([S^+_+ - S^-_-](\gamma))} = \left( \frac{\text{ch} W_{\mu}(\gamma)}{\text{ch} ([S^+_+ - S^-_-](\gamma))} \right) (P(\gamma)),
$$

where $P(\gamma) = \Gamma_\gamma \setminus G_\gamma$. Furthermore, on applying Weyl’s character formula to the group $K \cap M^+_Q \cdot A_{\Phi}$ one obtains

$$
\frac{\text{ch} [W_{\mu}](\gamma)}{\text{ch} ([S^+_+ - S^-_-](\gamma))} = \sum_{w \in W(K \cap M^+_Q)} \frac{\varepsilon(w)e^{w(\mu + \rho_{M, c})(\gamma_I)}e^{\lambda_w \circ C_{\kappa}^{-1}(l_\gamma \hat{X}_\kappa)}e^{w(\mu + \rho_{M, c})}}{\prod_{\alpha \in P(K \cap M^+_Q)} (e^{\alpha/2(\gamma_I)e^{\alpha/2} - e^{-\alpha/2(\gamma_I)e^{-\alpha/2}}})}.
$$

It follows that

$$
(3.7) \quad \chi_{\gamma, \mu} = e^{\lambda_w \circ C_{\kappa}^{-1}(l_\gamma \hat{X}_\kappa)} \tilde{L}(\gamma; \mu),
$$

where

$$
(3.8) \quad \tilde{L}(\gamma; \mu) = \left\{ \frac{\prod_{\alpha \in P_{\mu, \gamma}} \alpha \sum_{w \in W(K \cap M^+_Q)} \varepsilon(w)e^{w(\mu + \rho_{M, c})(\gamma_I)}e^{w(\mu + \rho_{M, c})}}{\prod_{\alpha \in P(K \cap M^+_Q)} (e^{\alpha/2(\gamma_I)e^{\alpha/2} - e^{-\alpha/2(\gamma_I)e^{-\alpha/2}}}) (P(\gamma))} \right\} [\hat{X}_\gamma]
$$

is the analogue of the Lefschetz numbers computed in [H-P].

In fact, by appealing to the results in [M-S;I, §5; M-S;II, §3], we can reduce its computation to that already performed in [H-P]. To this end, recall that $X_\gamma$, admits a finite normal
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cover \( ^0X_\gamma = \Gamma_\gamma \setminus G_\gamma / K_\gamma \), equipped with a free circle action and with quotient space \( ^0X'_\gamma = \Gamma_\gamma' \setminus G'_\gamma / K'_\gamma \) (see Lemma 5.4 [M-S;I]). The commutativity of the diagram

\[
\begin{array}{ccc}
^0X_\gamma & \rightarrow & ^0X'_\gamma = ^0X_\gamma / S^1 \\
\downarrow & & \downarrow \\
X_\gamma & \rightarrow & \hat{X}_\gamma = X_\gamma / S^1
\end{array}
\]

implies the following “proportionality principle” for characteristic numbers:

**Lemma 3.4.** For any characteristic class \( \Xi \), one has

\[
\frac{\Xi(\hat{X}_\gamma)}{\chi(\hat{X}_\gamma)} = \frac{\Xi(^0X'_\gamma)}{\chi(^0X'_\gamma)}.
\]

The Euler numbers in the denominators have been computed in §3 [M-S;II]. In particular, the calculations in loc. cit. imply

\[
\frac{\chi(\hat{X}_\gamma)}{\chi(^0X'_\gamma)} = \frac{\text{vol}(\Gamma_\gamma \setminus G_\gamma)}{\text{vol}(\Gamma'_\gamma \setminus G'_\gamma) \cdot \lambda_\gamma}
\]

where \( \lambda_\gamma \) is the length of a generic orbit of \( \Phi \) in \( X_\gamma \). Thus, if we set

\[
0L'(\gamma; \mu) = \begin{cases}
\left( \prod_{\alpha \in P_{n_\gamma}} \alpha \sum_{w \in W(K \cap M^+_Q)} \varepsilon(w)e^{w(\mu + \rho_{M,c})}\langle \gamma_I \rangle e^{w(\mu + \rho_{M,c})} \right) \\
\prod_{\alpha \in P(K \cap M^+_Q)} (e^{\alpha/2} - e^{-\alpha/2})(\langle \gamma_I \rangle e^{\alpha/2} - e^{-\alpha/2})(P(\gamma))
\end{cases}
\]

(3.9)

one has

\[
\hat{L}(\gamma; \mu) = 0L'(\gamma; \mu) \frac{\text{vol}(\Gamma_\gamma \setminus G_\gamma)}{\text{vol}(\Gamma'_\gamma \setminus G'_\gamma) \lambda_\gamma}.
\]

(3.10)

Now \( 0L'(\gamma; \mu) \) has been computed in [H-P, §2]; in our (slightly different) notation, the result can be stated as follows:

\[
0L'(\gamma; \mu) = (-1)^{n(\gamma)}v(\Gamma_\gamma \setminus G_\gamma) |W_\gamma^c|^{-1} \prod_{\alpha \in P_{\gamma_I}} \langle \rho_{\gamma_I}, \alpha \rangle^{-1}
\]

\[
\sum_{w \in W(K \cap M^+_Q)/W_\gamma} \varepsilon(w)e^{w(\mu + \rho_{M,c}) - \rho_{M,c} - \rho_{M,c}}(\langle \gamma_I \rangle \prod_{\alpha \in P_{\gamma}} \langle w(\mu + \rho_{M,c}), \alpha \rangle)
\]

\[
\prod_{\alpha \in P(K \cap M^+_Q) \setminus P_{\gamma}} (1 - e^{-\alpha}(\langle \gamma_I \rangle))
\]

(3.11)

where \( v(\Gamma_\gamma \setminus G_\gamma) \) denotes the volume normalized according to [H-P].
Proposition 3.5. For any \( \gamma \in \mathcal{E}_1(\Gamma) \), one has

\[
\chi(\gamma)(\mathcal{V}) = (-1)^{q_\gamma}(2\pi)^{-\# P_{\gamma}} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \cdot \lambda_{\gamma}^{-1}|W_\gamma|^{-1}2^{-\nu_{\gamma}/2}v(K'_\gamma)v(A_{\gamma})^{-1} \\
\cdot \det(I - \gamma | v_C)^{-1} \det(I - \gamma | u_C)^{-1} \sum_{w \in W_\kappa} (-1)^{\ell(w)} \sum_{\mu} a_{\mu}^w e^\lambda_\gamma w G_\gamma^{-1}(\hat{l}_\gamma, \widehat{X}_\kappa) \\
\cdot \sum_{w \in W(K \cap M_Q^+)/W_\gamma} \varepsilon(w)e^{w(\mu + \rho_{M,c}) - \rho_{M,c}}(\gamma I) \prod_{\alpha \in P_{\gamma}} \langle w(\mu + \rho_{M,c}), \alpha \rangle \prod_{\alpha \in P(K \cap M_Q^+) \backslash P_{\gamma}} (1 - e^{-\alpha}(\gamma I)).
\]

Proof. This follows from the above relations after reconciling the different volume normalizations (cf. [M-S;I], p. 659 and [M-S;II], p. 203). \(\square\)

For future reference, for \( w \in W_\kappa \) we set

\[
(3.12) \quad \chi(\widetilde{X}_\gamma, \frac{[W^\Lambda w](\gamma)}{[\Lambda_1(\mathcal{N}_+(\widetilde{X}_\gamma)^* \oplus \mathcal{V}_C^0(\widetilde{X}_\gamma)^* \oplus \mathcal{U}_C^0((\widetilde{X}_\gamma)^*))](\gamma)} = \\
(-1)^{q_\gamma}(2\pi)^{-\# P_{\gamma}} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \cdot \lambda_{\gamma}^{-1}|W_\gamma|^{-1}2^{-\nu_{\gamma}/2}v(K'_\gamma)v(A_{\gamma})^{-1} \\
\cdot \det(I - \gamma | v_C)^{-1} \det(I - \gamma | u_C)^{-1} \sum_{\mu} a_{\mu}^w e^\lambda_\gamma w G_\gamma^{-1}(\hat{l}_\gamma, \widehat{X}_\kappa) \\
\cdot \sum_{w \in W(K \cap M_Q^+)/W_\gamma} \varepsilon(w)e^{w(\mu + \rho_{M,c}) - \rho_{M,c}}(\gamma I) \prod_{\alpha \in P_{\gamma}} \langle w(\mu + \rho_{M,c}), \alpha \rangle \prod_{\alpha \in P(K \cap M_Q^+) \backslash P_{\gamma}} (1 - e^{-\alpha}(\gamma I)).
\]
§4 Orbital Integral Expansion for $\theta_{\lambda,\phi}(t)$

We begin the analysis necessary to develop the properties of the geometric zeta function (2.3). In this section we use the procedure pioneered by Selberg to study the trace of an integral operator on a locally symmetric space through its expansion into orbital integrals. The evaluation of orbital integrals in simple terms, especially in the higher rank case, is usually a formidable task. In this instance, the underlying geometric nature of the integral operator introduces enough cancellations to lead to a tractable calculation.

Recall from §2 that $\Gamma$ is a discrete, torsion-free co-compact subgroup of $G$. The quotient of $\tilde{X}$ by the action of $\Gamma$, $X = \Gamma \backslash \tilde{X}$, is a compact complex manifold with Hermitian metric, having $\tilde{X}$ as the simply connected covering space.

Recall that $\tau : K \to U(V_\lambda)$ is an irreducible, unitary representation of $K$ ($\lambda$ the highest weight) on a complex vector space. Associated to $(\tau, V_\lambda)$ are holomorphic, Hermitian vector bundles $\tilde{V} \to \tilde{X}$ and $V \to X$. In addition to $(\tau, V_\lambda)$ we have taken $\varphi : \Gamma \to U(F)$ a unitary representation on a finite dimensional complex vector space. Associated to $(\varphi, F)$ is a Hermitian bundle $F$ with a flat connection. The sections of $F$ and the Kodaira Laplacian can be easily identified in this locally homogeneous setting. The group $\Gamma$ acts on $F \otimes C^\infty(G)$ via $\varphi \otimes L$, $L$ the left regular representation. The $\Gamma$-invariants, $(F \otimes C^\infty(G))^\Gamma$, can be identified with

$$C^\infty(\Gamma \backslash G; \varphi) = \{ f : G \to F | f \text{ is } C^\infty \text{ and } f(\gamma g) = \varphi(\gamma)f(g) \}.$$  

A similar statement holds for the $L^2$ sections. The group $G$ acts unitarily on $L^2(\Gamma \backslash G; \varphi)$ via the right regular representation $R_{\Gamma,\varphi}$. As in the first section, one has an operator between the spaces $C^\infty(\Gamma \backslash G; \varphi) \otimes \Lambda^q p_+ \otimes V_\lambda$ given by

$$\overline{\partial} = \sum R_{\Gamma,\varphi}(Z_i) \otimes e(Z_i) \otimes I$$

that commutes with the action $R_{\Gamma,\varphi} \otimes \Lambda^q \otimes \tau$, of $K$ to give the $\overline{\partial}$ complex (see, for example, [0-0]). Indeed the $K$ invariant vectors, $[C^\infty(\Gamma \backslash G; \varphi) \otimes \Lambda^q p_+ \otimes V_\lambda]^K$, can be identified with smooth $(0, q)$ forms. The Cauchy-Riemann operator $\overline{\partial}$ is a closed, densely defined operator. The formal Laplacian $\Box^{0,q}$ is defined as usual and its restriction to smooth forms of type $(0, q)$ gives an elliptic operator with a unique self-adjoint extension to $L^2$ forms.
The heat operator $e^{-t\Box^0,q}$ has Schwartz kernel $h^q_t(\cdot,\cdot)$ obtained by the familiar procedure of averaging over $\Gamma$ a heat kernel on $\tilde{X}$. For this purpose let $\Box^0,q$ be the operator in §1 acting on $[C^\infty(G) \otimes \Lambda^q p_+ \otimes V_\lambda]^K$. The Schwartz kernel $\tilde{h}^q_t$ of the heat operator $e^{-t\Box^0,q}$ is known to be in

$$[S(G) \otimes \text{End} (\Lambda^q p_+ \otimes V_\lambda)]^{K \times K},$$

where $S(G) = \bigcap_{p>0} C^p(G), C^p(G)$ the Harish-Chandra space of $p$-integrable functions, is the nec plus ultra Schwartz space. Then the heat kernel $h^q_t(\cdot,\cdot)$ for $e^{-t\Box^0,q}$ is given, in this identification, by

$$h^q_t(x,y) = \sum_{\gamma \in \Gamma} \varphi(\gamma) \otimes \tilde{h}^q_t(y^{-1}\gamma x).$$

$$\text{(4.1)}$$

We define the holomorphic torsion theta function by

$$\theta_{\lambda,\phi}(t) = \sum_{q=0}^n (-1)^q q \text{Tr} e^{-t\Box^0,q}. \quad \text{(4.2)}$$

In order to evaluate $\theta_{\lambda,\phi}(t)$ using Selberg’s technique, we set

$$\tilde{k}^\lambda_t = \sum_{q=0}^n (-1)^q q \text{tr} \tilde{h}^q_t, \quad \text{(4.3)}$$

$\text{tr}$ denoting a local trace. As $\tilde{k}^\lambda_t$ is in $S(G)$, it is admissible in the sense that one has an absolutely convergent orbital integral expansion

$$\theta_{\lambda,\phi}(t) = \sum_{\{\gamma\}} \text{Tr} \varphi(\gamma) \text{vol}(\Gamma \setminus G) \mathcal{O}_{\tilde{k}^\lambda_t}(\gamma). \quad \text{(4.4)}$$

As $\Gamma$ is torsion free, there are no non-trivial elliptic conjugacy classes. Also, as $\Gamma$ is co-compact, all elements are semisimple. For semisimple elements one knows from Harish-Chandra that $f \mapsto \mathcal{O}_f(\gamma)$ is an invariant, tempered distribution. As such, the inversion of the invariant Fourier transform will involve only $\text{Tr} \pi(f)$.

Now $\tilde{h}^q_t$ is in $[S(G) \otimes \text{End} (\Lambda^q p_+ \otimes V_\lambda)]^{K \times K}$, so if $(\pi, H_\pi)$ is an irreducible unitary representation of $G$, $\pi(\tilde{h}^q_t)$ is defined, trace-class, and in $[\text{End} (H_\pi^\infty) \otimes \text{End} (\Lambda^q p_+ \otimes V_\lambda)]^{K \times K}$. In fact one easily sees that it is in $\text{End} [(H_\pi^\infty \otimes \Lambda^q p_+ \otimes V_\lambda)^K]$ where it agrees with $e^{-t\Box^0,q}$. Then $\text{Tr} \pi(\text{tr} \tilde{h}^q_t) = \text{Tr} e^{-t\Box_\pi^0,q}$, in particular

$$\text{Tr} \pi(\tilde{k}^\lambda_t) = \sum_{q=0}^n (-1)^q q \text{Tr} \pi(\text{tr} \tilde{h}^q_t) = \sum_{q=0}^n (-1)^q q \text{Tr} e^{-t\Box_\pi^0,q}. \quad \text{(4.5)}$$

We are now in a position to use the results of §1.
Proposition 4.1. Let \( Q = M_Q A_Q N_Q \) be a parabolic subgroup of \( G \) and \( \pi_{\xi,\nu} = \text{Ind}_{M_Q^+ A_Q N_Q}^G (\xi \otimes e^\nu \otimes 1) \).

(i) If \( \dim a_q \geq 2 \) then \( \text{Tr} \pi_{\xi,\nu}(k_{\ell}^\lambda) = 0 \).

(ii) If \( \dim a_q = 1 \) then

\[
\text{Tr} \pi_{\xi,\nu}(k_{\ell}^\lambda) = e^{\frac{1}{2}[\chi_{\xi,\nu}(\Omega_G) - (\lambda^*, \lambda^* + 2\rho)]} \sum_{W} (-1)^{\ell(w)+1} e(\xi^{(1)}, W_{\lambda^{(1)}}, p^{(1)}) e(\xi^{(2)}, W_{\lambda^{(2)}}, p^{(2)}).
\]

(iii) If \( Q = G \) and \( \xi \in E_2(G) \) then

\[
\text{Tr} \pi_{\xi}(k_{\ell}^\lambda) = e^{\frac{1}{2}(\chi_{\xi}(\Omega_G) - (\lambda^*, \lambda^* + 2\rho))} e'(\xi, V, p_-).
\]

Proof. (i) and (ii) follow from Lemma 1.1, Lemma 1.3 and Proposition 1.7.

Remark. An examination of the classification of the relevant simple \( G \) shows that each has at least one and at most two cuspidal maximal parabolic subgroups. So unlike the cases of eta invariants ([M-S;I]) or \( R \)-torsion ([M-S;II]) one cannot conclude solely by this Fourier transform argument the triviality of holomorphic torsion for any class of compact Hermitian locally symmetric spaces.

Before actually evaluating the orbital integrals we must comment on the normalization of Haar measures, etc. We try to be consistent with that used by Harish-Chandra, as in say [H-C; I] and [H-C; S]. In [M-S;I] we gave a summary of these conventions and so we shall omit repeating it here. As is customary we shall work with the invariant integrals \( \mathcal{O}_{k_{\ell}^\lambda} \), which are related to the orbital integrals by

\[
(4.6) \quad \mathcal{O}_{k_{\ell}^\lambda}(h) = \frac{\mathcal{O}_{k_{\ell}^\lambda}(h)}{\Delta_+(h) \Delta_I(h)}, \quad h \in H'.
\]

The basic result we use to evaluate \( \mathcal{O}_{k_{\ell}^\lambda} \) is due to Harish-Chandra, although [Hb] could also be used. To state this result and because we shall use it in several proofs we recall the relevant notation.
Let $B \subseteq M_Q A_Q$ be a Cartan subgroup, say $B = A_I A_Q$, with $A_I$ fundamental in $M_Q$. As the dimension of $a_q$ is one, either $B$ is connected or $B = A_I^0 A_Q F$, $F$ as before. Let $\hat{a}^*$ denote the set of unitary characters of $B$. A character $\hat{a}^*_0 \in A_I^0 \ast$ is called regular if

\[
\prod_{\alpha \in \Delta_{a_q}} (\log a^*_0, \alpha) \neq 0 \quad \text{and to the } W(M_Q^0, A_I^0) = W(M_Q^+, A_I) \text{ orbit of } a^*_0 \text{ is associated a discrete series representation of } M_Q^0.
\]

For $\chi$ a character of $F$, we set $a^* = (a^*_0, \chi)$ and we let $\xi(a^*)$ denote the discrete series representation of $M_Q^+$ with global character $\Theta_{\xi(a^*)}$. For any $\hat{a}^* = (\hat{a}^*_0, \chi), \hat{a}^*_0 \in W(M_Q^+, A_I) \cdot a^*_0$, let $\Theta_{\hat{a}^*}$ be the invariant eigendistribution associated by Harish-Chandra to $\hat{a}^*$. One has $\Theta_{\hat{a}^*} = (-1)^q \Theta_{\xi(a^*)}$ for appropriate $q$. As before we let $\pi_{\xi, \nu}$ denote the induced representation of $G$ and $\Theta_{\pi_{\xi, \nu}}$ the character. Following the notation in [H-C; S] we set $\hat{f}_B(b^*) = \hat{f}_B(a^*, \nu) = \Theta_{\pi_{\xi, \nu}}(f)$ and extend $\hat{f}_B$ to $B^*$ as a $W(G/B)$ skew function.

For $a^*_0 \in A_I^0 \ast$ singular, the situation is more complicated but analogous notation will be used. Let $W(a^*_0) \subseteq W(M_Q^+, A_I)$ be the isotropy subgroup and let $\Theta_{a^*_0}$ be the invariant eigendistribution associated by Harish-Chandra to $a^*_0$. One has ([Hb-S])

\[
\Theta_{a^*_0} = |W(a^*_0)|^{-1} (-1)^q \sum_{W(a^*_0)} \varepsilon(w) \Theta^{w^*}_{a^*_0}
\]

where $\Theta^{w^*}_{a^*_0}$ is the character of an irreducible unitary representation of $M_Q^0$ occurring in a unitary principal series. As before we set $a^* = (a^*_0, \chi)$ and let $\xi^w(a^*)$ be the representation of $M_Q^+$ with character $\Theta^{w^*}_{a^*_0} = \Theta^{w^*}_{a^*_0} \otimes \chi$. Then we set $\hat{f}_B(a^*, \nu) = |W(a^*_0)|^{-1} \sum \varepsilon(w) \Theta_{\pi_{\xi^w, \nu}}(f)$ and extend $\hat{f}_B$ to the $W(G/B)$ orbit of $b^* = (a^*, \nu)$ by skew symmetry. We shall let $E_2(M_Q^+)$ denote the set of $W(M_Q^+, A_I)$ of regular $a^*$ and $E_2^s(M_Q^+)$ the $W(M_Q^+, A_I)$ orbits of singular $a^*$.

**Theorem.** (Harish-Chandra) Let $H$ be a Cartan subgroup, $h \in H'$ and $f \in C^2(G)$. Assume

\[
\hat{f}_A \equiv 0 \quad \text{if } A \triangleright H.
\]

Then

\[
(4.7) \quad \tilde{F}_f^H(h) = \int_{H^*} [W(G/H)]^{-1} \sum_{s \in W(G/H)} \varepsilon_I(s)(s \cdot h^*, h) \hat{f}_H(h^*) dh^*.
\]
From (4.7) it is clear that we will need to sum certain Fourier series. We will do this separately for \(M_1^{(1)}\) and \(M_2^{(2)}\). Since \(M_1^{(1)}, M_2^{(2)}F\) and their maximal compact subgroups are in the Harish-Chandra class, a glance at (1.13) and the material just before it together with the proof given in [M-S;I] of Lemma 4.2 gives the next two results.

**Lemma 4.2.** Let \(W^{(1)}\) be an irreducible unitary representation of \(K \cap M_1^{(1)}\). Then on \(A_1^{(1)'}\)
\[
\overline{\Delta}_{M_1^{(1)},c} \chi_{W^{(1)}}(\xi, W^{(1)}, p^{(1)}_\omega) = \sum_{\xi \in \mathcal{E}_2(M_1^{(1)})} e(\xi, W^{(1)}, p^{(1)}_\omega) \overline{\Phi}_\xi + \sum_{\xi \in \mathcal{E}_2(M_1^{(1)}) |W(\xi)|^{-1} \sum_{W(\xi)} \varepsilon(w) e(\xi, W^{(1)}, p^{(1)}_\omega) \overline{\Delta}_{M_1^{(1)}} \overline{\Theta}_{w, \xi}.
\]

**Lemma 4.3.** Let \(W^{(2)}\) be an irreducible finite dimensional representation of \(M_2^{(2)}F\). Then on \((A_2^{(2)}F)'
\[
\overline{\Delta}_{M_2^{(2)},c} \chi_{W^{(2)}}(\xi, W^{(2)}, p^{(2)}_\omega) = \sum_{\xi \in \mathcal{E}_2(M_2^{(2)}C)} e(\xi, W^{(2)}, p^{(2)}_\omega) \overline{\Phi}_\xi + \sum_{\xi \in \mathcal{E}_2(M_2^{(2)}C) |W(\xi)|^{-1} \sum_{W(\xi)} \varepsilon(w) e(\xi, W^{(2)}, p^{(2)}_\omega) \overline{\Delta}_{M_2^{(2)}} \overline{\Theta}_{w, \xi}.
\]

**Proposition 4.4.** Let \(B = A_I A_Q\) be a split rank one Cartan subgroup and \(h \in B'\). Then
\[
(4.8) \quad \mathcal{O}_{\hat{k}_\lambda}(h) = \frac{e^{-\|\log h_\lambda\|^2/2t}}{(2\pi t)^{1/2}} \frac{e^{2t(\lambda, \rho_n)}}{\Delta_+(h) \Delta_{I,n}^{(1)}(h_I)} \sum_{W_\kappa} (-1)^{\ell(w)+1} e^{-i \hat{c}_\lambda^2 X_{\lambda w}(h_I)},
\]
where
\[
\hat{c}_\lambda^2 = \{((\lambda_w \circ C_\kappa) + \rho_Q)(\hat{X}_\kappa)\}^2.
\]

**Proof.** The holomorphic torsion \(\theta\)-function \(\hat{k}_\lambda^A\) is in \(\mathcal{S}(G) \subseteq C^2(G)\). Also \((\hat{k}_\lambda^A)_A \equiv 0\) for \(A \gg B, A \neq B\), as follows from (4.5). Then \(F_B^B(h)\) can be evaluated from (4.7). Now for \(a^* = (a_0^*, \chi)\) regular
\[
(\hat{k}_\lambda^A)_{B}(b^*) = (\hat{k}_\lambda^A)_B(a^*, \nu) = \text{Tr} \pi_{\xi, \nu}(\hat{k}_\lambda^A)
\]
\[
= e^{i \pi \xi, \nu(\Omega) - (\lambda^*, \lambda^* + 2\rho)} \sum_{W_\kappa} (-1)^{\ell(w)+1} e(\xi^{(1)}, W_\lambda^{(1)}, p^{(1)}) e(\xi^{(2)}, W_\lambda^{(2)}, p^{(2)}).
\]

Since
\[ \pi_{\xi,\nu}(\Omega) = \| \Lambda_\xi \|^2 - \| \nu \|^2 - \| \rho \|^2, \]
and setting
\[ (4.9) \quad \hat{c}_{\lambda w} = ((\lambda_w \circ C_\kappa) + \rho_Q)(\hat{X}_\kappa), \]
we obtain from (1.16)
\[ (\hat{k}_t^\lambda)_B(b^*) = e^{-\frac{t}{2} \| \nu \|^2} e^{2t\langle \lambda, \rho_n \rangle} \sum_{W_\kappa} (-1)^{\ell(w)} e^{-\frac{t}{2} \hat{c}_{\lambda w}^2} e(\xi^{(1)}, W_{\lambda_w}^{(1)}, p_{\lambda_w}^{(1)}) e(\xi^{(2)}, W_{\lambda_w}^{(2)}, p_{\lambda_w}^{(2)}). \]

From the singular $a^*$ one obtains a similar expression but involving an additional sum over $W(a_0^*)$. We substitute this into (4.7) and carry out the calculation
\[ 'F^B_{k_t^\lambda}(h) = \sum_{E_2(M_Q^+, A_I)} \sum_{W(G, B)} \varepsilon(s) \langle s \cdot b^*, h \rangle (\hat{k}_t^\lambda)_B(a_I^*, i\nu) d\nu \]
\[ = \sum_{A_I^*/W(M_Q^+, A_I)} [W(G, B)]^{-1} \sum_{W(M_Q^+, A_I)} \sum_{W(G, B)} \varepsilon(s) \langle s \cdot w \cdot a_I^*, h_I \rangle \]
\[ \int_{a_q^*} e^{i(\log h_\Xi)} (\hat{k}_t^\lambda)_B(w \cdot \xi(a_I^*), i\nu) d\nu. \]

Any $w$ in $W(M_Q^+, A_I)$ satisfies $w = I$ on $a_q^*$, so
\[ (\hat{k}_t^\lambda)_B(w \cdot \xi(a_I^*), i\nu) = \varepsilon(w)(\hat{k}_t^\lambda)_B(\xi(a_I^*), w^{-1} \cdot i\nu) = \varepsilon(w)(\hat{k}_t^\lambda)_B(\xi(a_I^*), i\nu). \]

The integration over $a_q^*$ is just the Fourier transform of $e^{-\frac{t}{2} \| \nu \|^2}$, so we consider the contribution from $A_I^*$. For the regular elements we get
\[ \sum_{E_2(M_Q^+, W(M_Q^+, A_I))} \varepsilon(w)[W(G, B)]^{-1} \sum_{W(G, B)} \varepsilon(s) \langle s \cdot w \cdot \xi(a_I^*), h_I \rangle \]
\[ e^{2t\langle \lambda, \rho_n \rangle} \sum_{W_\kappa} (-1)^{\ell(w)} e^{-\frac{t}{2} \hat{c}_{\lambda w}^2} e(\xi^{(1)}, W_{\lambda_w}^{(1)}, p_{\lambda_w}^{(1)}) e(\xi^{(2)}, W_{\lambda_w}^{(2)}, p_{\lambda_w}^{(2)}). \]
We move the $s$ action onto $h_I$. Then the sum over $W(M_Q^+, A_I)$ rearranges to

$$
\sum_{W(M_Q^+, A_I)} \varepsilon(w) (w \cdot \xi(a_t^r), s^{-1} \cdot h_I) \xi_{s \cdot \rho_M - \rho_M}(h_I) = \xi_{s \cdot \rho_M - \rho_M}(h_I) \Phi_{\xi(a_t^r)}(s^{-1} \cdot h_I).
$$

The singular elements are done identically giving a similar expression. The summations over $\mathcal{E}_2(M_Q^+)$ and $\mathcal{E}_3^2(M_Q^+)$ give, from Lemma 4.2 and Lemma 4.3, the following expression for $'F_{k,l}^{B}(h)$

$$
n_{W_{n}} (-1)^{\ell(w)} e^{-\frac{t}{2} \epsilon^2} \chi_{\omega_{\lambda}(s^{-1} \cdot h_I)'} \Delta_{M_Q^{(1)}}(s^{-1} \cdot h_I) \Delta_{M_Q^{(2)}}(s^{-1} \cdot h_I).
$$

We write

$$
n'_{\Delta_{M_Q^{(1)}}} / \Delta_{M_Q^{(2)}} = \frac{\Delta_{M_Q}}{[\Delta_{M_Q^{(1)}}]} = \frac{\Delta_{M_Q}}{[\Delta_{M_Q^{(1)}}]^2} \sum_{\Delta_{M_Q^{(1)}}}.
$$

From the invariance of $[\Delta_{M_Q^{(1)}}]^2$ by $s$, together with Lemma 27.2 in [H-C; I] we get

$$
n'_{F_{k,l}^{B}(h)} = e^{-\| \log h \|^2/2t} e^{2t(\lambda, \rho_n)} \frac{[\Delta_{M_Q^{(1)}}]}{[\Delta_{M_Q^{(1)}}]^2} \sum_{W_{n}} (-1)^{\ell(w)} e^{-\frac{t}{2} \epsilon^2} \chi_{\omega_{\lambda}(s^{-1} \cdot h_I)'} \Delta_{M_Q^{(1)}}(s^{-1} \cdot h_I) \Delta_{M_Q^{(2)}}(s^{-1} \cdot h_I).
$$

For $s \in W(M_Q^+, A_I) = W(K \cap M_Q^+, A_I)$, $\chi_{\omega_{\lambda}(s^{-1} \cdot h_I)} = \chi_{\omega_{\lambda}(h_I)}$ because $\chi_{\omega_{\lambda}}$ is a class function; similarly $'\Delta_{M_Q^{(1)}}$ as the class function $(\chi_{\sigma_1^{(1)}} - \chi_{\sigma_-^{(1)}}) \chi_{\rho_1^{(1)}}$ is invariant under $W(K \cap M_Q^+, A_I)$. One knows that

$$
W(M_Q, A_I)/W(M_Q^+, A_I) \simeq M_Q/M_Q^+ \simeq K \cap M_Q/K \cap M_Q^+.
$$

But $G$ is connected and so acts holomorphically on $\tilde{X}$; then $(K \cap M_Q)(H_0) = H_0$. When $g$ is simple [Sa] III.8.3 shows that $K \cap M_Q$ then also fixes $H_0^{(1)}$. But $H_0 = 2H_0 - 2H_0^{(1)}$ and so $K \cap M_Q \subseteq L_K$. Since the $W_{\lambda}$ are $L_K$ modules it follows that $\chi_{\omega_{\lambda}(s^{-1} \cdot h_I)} = \chi_{\omega_{\lambda}(h_I)}$. Also since $H_0^{(1)}$ defines the complex structure on $p^{(1)}$ and $K \cap M_Q$ commutes with $H_0^{(1)}$, 

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then $K \cap M_Q$ acts on $\mathfrak{p}_t$. But $\Delta_{M_Q}(\cdot) = \det(I - Ad|_{\mathfrak{p}_t}(\cdot))$ and thus is also invariant by $K \cap M_Q$. More generally any $s \in W(G, B)$ may be represented by an element in $K$ and $s|_{a_q} = \pm I$. Thus it suffices to consider $s$ whose restriction is $-I$. But since $K$ is connected, $s \cdot H_0 = H_0$. An examination of the proof of [Sa] III.8.3 shows that $s \cdot H_\kappa = H_\kappa$ hence $s \in L_\kappa$. As above, the invariance follows. We then have for the invariant integral

\[
(4.10) \quad F_{k_\ell}^B(h) = \frac{e^{-\|h\|_2^2/2t}}{(2\pi t)^{1/2}} \frac{\Delta_{M_Q}(h)}{\Delta_{M_Q}^{\prime}(h)} e^{2t(h,\rho_n)} \sum_{W_\kappa} (-1)^{\ell(w)+1} e^{-\frac{1}{2} \xi_2 w} \chi_{\lambda_w}(h).
\]

For future use, we remark that an equivalent form of the above identity is

\[
(4.11) \quad F_{k_\ell}^B(h) = \frac{e^{-\|h\|_2^2/2t}}{(2\pi t)^{1/2}} e^{2t(h,\rho_n)} \sum_{W_\kappa} (-1)^{\ell(w)+1} e^{-\frac{1}{2} \xi_2 w} \sum_{W(l_w)} \epsilon(s) \xi(s(\lambda_w + \rho(n)))(h).
\]

The formula for $O_{k_\ell}^B(h)$ then follows from (4.10) and (4.6).

Suppose $h \in B$ is singular but $h \neq e$. There is a well-established procedure to obtain $O_{k_\ell}^B(h)$ from (4.10). If one multiplies (4.10) by $\xi_{\rho_M Q}$ to obtain $F_{k_\ell}^B$ and applies the differential operator $\tilde{\omega}_h$ to $F_{k_\ell}^B$, then from [H-C; DSII] one gets

\[
O_{k_\ell}^B(h) = \frac{N_{k_\ell}^{-1} c_{h}^{-1}}{\xi(h) \prod_{\alpha \in P^c} \xi^{-\alpha}(h)} (\tilde{\omega}_h F_{k_\ell}^B)(h),
\]

where $N_{h} = [G_h : G_h^0]$ and

\[
c_{h} = (-1)^{q_{h}} (2\pi)^{q_{h}} 2^{q_{h}/2} \frac{v(A_{L})}{v(K \cap G_{h})} |W(G_{h}/A_{L})|
\]

with $q_{h} = \# P_{h}$, $\nu_{h} = \dim G_{h}/K_{h} - \text{rank } G_{h}/K_{h}$.

Since $B$ is $R$–rank one and $\gamma$ is torsion-free, $\xi_{\alpha}(h) \neq 1$ for $\alpha \in \Delta R$ and $\alpha \in \Delta C$, provided $h \neq e$. Then $P_{h} = \{ \alpha > 0 | \xi_{\alpha}(h) = 1 \} \subseteq \Delta_{M_Q}^{\pm} = \Delta_{L}^{\pm}$, and so $\tilde{\omega}_h$ is independent of the $h_{R}$ variable.

The next result follows from (4.11) by differentiation.
Lemma 4.5. Let $h \in B$, $h \neq e$. Then
\[
\mathcal{O}_{\tilde{k}_t^\lambda}(h) = \frac{e^{-\| \log h \|_2^2 / 2t}}{(2\pi t)^{1/2}} N_h^{-1} c_h^{-1} \xi_{\rho}(h) \prod_{\alpha \in P_c^+} \left[ 1 - \xi_{-\alpha}(h) \right] \sum_{\kappa} (-1)^{\ell(w)+1} e^{-\frac{t}{2} \xi_{\lambda w}^2} \sum_{W(\iota_{\alpha})} \epsilon(s) \tilde{\omega}_h(s \cdot (\lambda_w + \rho(\ell_{\kappa}))) \xi_s(\lambda_w + \rho(\ell_{\kappa}))(h_I).
\]

Remark. The calculation of this orbital integral was facilitated by two facts. First, the invariant Fourier transform of $\tilde{k}_t^\lambda$ factors into a product, one for the Levi and the other for the split part reducing the calculation to the evaluation of a Fourier series and an integral. Second, Theorem 1.12 makes possible a clean and useful closed form of the Fourier series. To obtain the orbital integral expansion of $\theta_{\lambda,\phi}(t)$ we have as our final task to evaluate the contribution of the identity element, i.e. to determine $\tilde{k}_t^\lambda(e)$. For this we use Harish-Chandra’s inversion formula. Let $d_{\pi,\omega}$ denote (in his normalization) the formal degree of the discrete series representation $\pi_{\omega}$.

We recall that the form of the Plancherel density (see e.g. [Hb]) for an $\mathbb{R}$-rank one Cartan subgroup is
\[
(4.12) \quad \frac{d\mu}{dv}(a_I^*, \nu) = c_A \prod_{\alpha \in \Delta_c^+ \cup \Delta_R^+} \langle \log a_I^* + iv, \alpha \rangle \prod_{\alpha \in \Delta_R^+} \langle iv, \alpha \rangle \left[ \frac{\cosh \pi \langle \nu, \alpha^\vee \rangle - a_I^*(\gamma_{\alpha})}{\sinh \pi \langle \nu, \alpha^\vee \rangle} \right],
\]
where $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$. With $a_I^* = (a_0^*, \chi)$, according as $\chi(\gamma_{\alpha}) = \pm 1$ the term in brackets is $\tanh \frac{2}{\pi} \langle \nu, \alpha^\vee \rangle$ or $\coth \frac{2}{\pi} \langle \nu, \alpha^\vee \rangle$. One knows that in the cases at hand there is a unique positive real root, say having root vector in $p_{++}^1$ the image under $\kappa$ of the one in $\mathfrak{sl}(2, \mathbb{R})$. Denote it $\alpha_R \in \Delta_R^+$ and let $H_{\alpha_R}$ satisfy $\nu(H_{\alpha_R}) = \langle \nu, \alpha_R \rangle$. Then $X_{\kappa} = H_{\alpha_R}/\langle \alpha_R, \alpha_R \rangle$ and $\gamma_{\alpha_R} = \exp \pi i X_{\kappa}$.

Since $|\Delta_R^+| = 1$, we may identify $a_q^*$ with $\mathbb{R}$ so that $\nu \in a_q^*$ corresponds to $\frac{2}{\pi} \alpha_R^*, \nu \in \mathbb{R}$. Clearly an imaginary root $\alpha$ has $\langle \nu, \alpha \rangle = 0$, while for the real root we have $\langle \alpha_R, \log a_I^* + iv \rangle = \langle \alpha_R, iv \rangle$. Thus the polynomial factor of the Plancherel density can be written as $\omega(\log a_I^* + iv)$. Since the complex roots occur in conjugate pairs, notice that this polynomial factor is $\nu$ times a polynomial in $\nu^2$. 

Also, for any \( w \in W_\kappa \) in order that \( e(\xi^{(1)}, W_{\lambda^{(1)}_w}, p_{(1)}^w)e(\xi^{(2)}, W_{\lambda^{(2)}_w}p_{(2)}^w) \neq 0 \), we see from Corollary 1.13 that this can occur only for a unique discrete series representation of \( M_Q^+ \), say having infinitesimal character \( \Lambda_{\xi(w)} \) with parameter, say, \( a^*_I = (a^*_0, \chi) \). Thus for each \( w \in W_\kappa \) there is at most one possibility for \( \chi(\gamma_\alpha) \), i.e. appearance of \( \tanh \frac{\pi}{2} \langle \nu, \alpha^\vee \rangle \) or \( \coth \frac{\pi}{2} \langle \nu, \alpha^\vee \rangle \). As this integer will occur frequently in formulas we will abbreviate it with the notation

\[
e(\xi, \lambda_w) := e(\xi^{(1)}_w, W_{\lambda^{(1)}_w}, p_{(1)}^w)e(\xi^{(2)}_w, W_{\lambda^{(2)}_w}, p_{(2)}^w).
\]

**Lemma 4.6.**

\[
\tilde{k}^I_\xi(e) = \sum_{\omega \in E_2(G)} e^I_\omega(\Omega(I) - \langle \lambda^*, \lambda^* + 2\rho \rangle)d\pi_\omega e'(\omega, V_\lambda, p_-) + e^{2t(\lambda, \rho_n)} \sum_{\kappa_i} \sum_{W_{\kappa_i}} (-1)^{\ell(w)} \int_{a^*_\kappa_i} e(\xi, \lambda_w)e^{-\frac{t}{2} \varepsilon^2_{\lambda_w}} e^{-\frac{t}{2} \|\nu\|^2} \frac{d\mu_i}{d\nu} \left( w^*(\lambda^* + \rho - 2\rho_n)|_{t_{\kappa_i}}, \nu \right) d\nu.
\]

**Proof.** The inversion formula gives

\[
\tilde{k}^I_\xi(e) = \sum_{E_2(G)} d\pi_\omega \Theta_\omega(\tilde{k}^I_\xi) + \sum_{\{A\} A^*_{\xi}' \int_{a^*_\kappa_i} \Theta(\xi, \nu)(\tilde{k}^I_\xi) d\mu_I(\xi, \nu).
\]

In (4.5) (iii) the discrete series terms are found. Then from Proposition 4.1, Lemmas 4.2, 4.3 and Kostant’s Theorem we get

\[
\sum_{\{A\}} \sum_{A^*_{\xi} \int_{a^*_\kappa_i} \Theta(\xi, \nu)(\tilde{k}^I_\xi) d\mu_I(\xi, \nu) =
\sum_{\{A\}} \sum_{A^*_{\xi} \int_{a^*_\kappa_i} e^{-\frac{t}{2} \|\nu\|^2} \sum_{W_{\kappa}} (-1)^{\ell(w)} e^I_\xi(\|\Lambda_{\xi} - \|\lambda^* + \rho\|^2) e(\xi^{(1)}, W_{\lambda^{(1)}_w}, p_{(1)}^w)e(\xi^{(2)}, W_{\lambda^{(2)}_w}, p_{(2)}^w) d\mu_I(\xi, \nu).
\]

From Theorem 1.12 and (4.9), but omitting the evaluation of the integral, we obtain the above

\[
e^{2t(\lambda, \rho_n)} \sum_{\kappa_i} \sum_{W_{\kappa_i}} (-1)^{\ell(w)} \int_{a^*_\kappa_i} e(\xi, \lambda_w)e^{-\frac{t}{2} \varepsilon^2_{\lambda_w}} e^{-\frac{t}{2} \|\nu\|^2} \frac{d\mu_i}{d\nu} \left( w^*(\lambda^* + \rho - 2\rho_n)|_{t_{\kappa_i}}, \nu \right) d\nu.
\]

■
Remark. The asymptotic behavior of $k_t^\lambda(e)$ as $t \to +\infty$ or $t \to 0^+$ can be determined from Lemma 4.6. This will be done in Section 6 in conjunction with the asymptotics of other heat operators.

Collecting the various terms we get the orbital integral expansion of the holomorphic torsion $\theta$-function.

**Theorem 4.7.** Let $X$ be a compact, locally symmetric Hermitian manifold and $F$ a flat Hermitian bundle associated to the unitary representation $(\varphi, F)$ of $\pi_1(X)$. Let $(\tau, V_\lambda)$ be an irreducible, unitary representation of $K$ with $\lambda + \rho G$-integral and regular and $\nabla$ the associated holomorphic Hermitian vector bundle. Set

$$\theta_{\lambda, \varphi}(t) = \sum_{q=0}^n (-1)^q Tr e^{-t\Delta_{n,q}}.$$

Let $E_1(\Gamma)$ denote the set of nontrivial $\gamma$-conjugacy classes of elements that can be conjugated into a maximal parabolic subgroup. Then

$$\theta_{\lambda, \varphi}(t) = \sum_{[\gamma] \in E_1(\Gamma)} \text{Tr} \varphi(\gamma) \text{vol}(\Gamma \gamma \backslash G \gamma) \frac{e^{-\|\log \gamma_\kappa\|^2/2t}}{(2\pi t)^{1/2}} \frac{N^{-1}c^{-1}_\gamma}{\xi_\rho(\gamma) \prod_{\alpha \in P_\gamma^+} [1 - \xi_{-\alpha}(\gamma)]}$$

$$\times e^{2t(\lambda, \rho_n)} \sum_{W_\kappa} (-1)^{\ell(w)+1} e^{-\frac{1}{2}||w||^2} \sum_{W(1_\kappa)} e(s) \tilde{\omega}(s \cdot (\lambda_n + \rho(1_\kappa))) \tilde{s}(\lambda_n + \rho(1_\kappa))(\gamma_1)$$

$$+ \dim F \text{vol } X \sum_{\omega \in E_2(G)} d_{\pi_\omega} e'(\omega, V_\lambda, p_-) e^{\frac{1}{2}(\chi_\omega(\Omega_G) - (\lambda^*, \lambda^* + 2\rho))}$$

$$+ \dim F \text{vol } X e^{2t(\lambda, \rho_n)} \sum_{\kappa_i} \sum_{W_{\kappa_i}} (-1)^{\ell(w)+1} \int_{\alpha_i^*} e(\xi, \lambda_n) e^{-\frac{1}{2}||\nu||^2} d\mu_i \frac{d\nu}{d\nu} (w^*(\lambda + \rho - 2\rho_n)|_{\Omega_G})$$
§5 Heat operators: Ansatz and Expansion

Two analytical technicalities present themselves as obstacles to the construction of geometric zeta functions in the generality of this paper. The first technicality is the commonplace one of using admissible functions for convergence of the orbital integral expansion. This we handle by using heat kernels. The second is a consequence of the scalars $\hat{c}_\lambda^2 w$ which appear geometrically through the action of the geodesic flow on the bundles constructed in §2. Their effect is to prevent a uniform meromorphic continuation of the intermediate (Selberg-like) zeta functions. Hence the need to treat these functions individually. These complications were not present in [M-S;I] in which all the $\hat{c}_\lambda^2 w$ happened to be zero, but did occur in [M-S;II]. The first approach to handle them, to our knowledge, was introduced by Fried [F1], who used certain combinations of heat kernels on bundles to handle orbital integral expansions related to closed and co-closed forms on hyperbolic space. While aware that we could generalize this to the setting considered in [M-S;II], instead we chose there to introduce an alternative approach. However, the presentation of it is flawed, so we shall use Fried’s technique, in the process simultaneously justifying the results in [M-S;II] and herein.

Proceeding with this section, we fix $Q$ a cuspidal maximal parabolic subgroup, say corresponding to the homomorphism $\kappa$ of $\mathfrak{sl}(2,\mathbb{R})$, and, for convenience, set $K^M = K \cap M^+_Q$. We recall the decomposition (1.4) $p_+^+ = p_+^0 \oplus p_+^1 \oplus p_+^2$, and set $p^{[\text{ev}]}_+ = p_+^0 \oplus p_+^2$. Then as $K^M$-modules one has the equivalence

$$\Lambda^q p_+ \simeq \sum \oplus \Lambda^j p^{[\text{ev}]}_+ \otimes \Lambda^\ell p^{[1]}_+.$$  

Set $e_\kappa^+ = \kappa\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in p_+^2$ and $e_\kappa^- = \kappa\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in p_-^2$. We fix a $K^M$ complement to $\mathbb{C}e_\kappa^+$ in $p^{[\text{ev}]}_+$

$$p^{[\text{ev}]}_+ \simeq p^{[\text{ev}]}_+ \oplus \mathbb{C}e_\kappa^+,$$

so that

$$\Lambda^q p_+ \simeq \sum \oplus \Lambda^j (p^{[\text{ev}]}_+ \oplus \mathbb{C}e_\kappa^+) \otimes \Lambda^\ell p^{[1]}_+.$$  

As in Corollary 1.5 we have as $K^M$-modules

$$\sum (-1)^q \Lambda^q p_+ \simeq \sum (-1)^{j+\ell+1} \Lambda^j p^{[\text{ev}]}_+ \otimes \Lambda^\ell p^{[1]}_+,$$
\[ \sum_{q=0}^{n}(-1)^{q}\Lambda^q p_+ \simeq \sum_{\ell=0}^{\dim \pi^c_{\kappa}} (-1)^{\ell+1} (\Lambda^{ev} p^{(1)}_+ - \Lambda^{odd} p^{(1)}_+) \otimes (\Lambda^{ev} p^{(2)}_C - \Lambda^{odd} p^{(2)}_C) \otimes \Lambda^\ell \pi^c_{\kappa}, \]

and so as before for \( \sum_{q\in\mathbb{Z}}(-1)^{q}\Lambda^q p_+ \otimes \mathcal{V}_\lambda \) we get

\[ \simeq \sum_{\ell=0}^{\dim \pi^c_{\kappa}} (-1)^{\ell+1} (\Lambda^{ev} p^{(1)}_+ - \Lambda^{odd} p^{(1)}_+) \otimes (\Lambda^{ev} p^{(2)}_C - \Lambda^{odd} p^{(2)}_C) \otimes W_{\lambda_w}. \] (5.1)

In order to construct the virtual heat kernels, initially we appealed to the following Ansatz:

**Ansatz.** The restriction map \( \text{Res} \) induces a surjection of Grothendieck groups

\[ \text{Res}: K^0(K) \to K^0(K^M) \to 0. \] (5.2)

In fact, this statement is too strong for our purposes and certainly false. A revised version suitable for our purposes would have for each \( j \) and \( w \), for some integers and \( K \)-modules one has in \( K^0(K^M) \),

\[ \Lambda^j p^{[ev]}_+ \otimes W_{\lambda_w} = \text{Res} \sum_{\mu \in \hat{K}} m^{j,\lambda_w}_\mu V_\mu. \]

Even this is stronger than needed as one can group all terms in the sum over \( j \). However, there remains one more simplification, viz. as the scalars \( c^2_{\lambda_w} \) are the source of the problem, one can group terms in the Weyl sum that have the same scalars \( c^2_{\lambda_w} \). The following is the form that is actually used and the current status of the validity of this version of the **Ansatz** is contained in the Appendix done by Jan Frahm.

\[ (\sum_{j}(-1)^{j}\Lambda^j p^{[ev]}_+) \otimes (\sum_{w\in\mathcal{W}_\kappa,c^2_{\lambda_w}=c^2} (-1)^{\ell(w)} W_{\lambda_w}) = \text{Res} \sum_{\mu \in \hat{K}} m^{\ell,\lambda_w}_\mu V_\mu. \] (5.3)

Take the regular representation \((\mathcal{R}, L^2(G))\) and for each irreducible \( K \)-module \( V_\mu \) which occurs above we consider the operator \( e^{\frac{1}{2}(\Omega_G-(\lambda^*,\lambda^*+2\rho)I)} \) acting on the Hilbert space \([L^2(G)\otimes V_\mu]^K\). Recall that \( \|\lambda^*+\rho\|^2 + 4\langle \lambda, \rho_n \rangle = \|\lambda + \rho\|^2 \). From Proposition 5.1 in [O-O]
(or cf. Lemma 1.1) up to a scalar operator depending on $\mu$, $\Omega_G$ agrees on the Hilbert space with $-2\Box^{0,0} = -\Delta^0$. Since $e^{-\frac{1}{2}((\lambda^\ast,\lambda^\ast+2\rho)I}$ is a scalar operator, $e^{\frac{1}{2}(\Omega_G-(\lambda^\ast,\lambda^\ast+2\rho)I)}$ viewed as an operator on the space of sections of the associated homogeneous bundle over the symmetric space $G/K$ is $G$-equivariant and smoothing. As such, it must be of the form $R(\tilde{h}_{\mu,t})$. Moreover, since this operator differs by a scalar factor from the heat operator associated to the corresponding connection Laplacian, its invariant Schwartz kernel $\tilde{h}_{\mu,t}$ belongs to $[S(G) \otimes \text{End}(V_\mu)]^{K\times K}$. We set

\begin{equation}
(5.4) \quad f_{\mu,t} = \text{tr} \tilde{h}_{\mu,t} .
\end{equation}

The effect of the integers $m_{\mu}^2_{\lambda w}$ is handled in the usual way. For positive integers we take a direct sum of bundles corresponding to $V_\mu$, while for negative integers we take a direct sum of bundles and use a $\mathbb{Z}_2$ grading to compute the trace. With this convention we then define the virtual heat kernel associated to $\tilde{c}_{\lambda w}^2$ as:

\begin{equation}
(5.5) \quad f_{t}^{\tilde{c}_{\lambda w}^2} := \sum_{\mu} m_{\mu}_{\lambda w}^2 f_{\mu,t} .
\end{equation}

Remark. Due to the use of the Ansatz, we shall systematically attach $\tilde{c}_{\lambda w}^2$ to the notation (often as superscript) to objects for which we take the $\sum_{w \in W, c_{\lambda w}^2 = c^2(-1)\ell(w)}$. For example, instead of

\[ e(\xi, \lambda w) := e(\xi_{w}^{(1)}, W_{\lambda_w}^{(1)}, p_{-}^{(1)})e(\xi^{(2)}, W_{\lambda_w}^{(2)}, p_{C}^{(2)}) \]

we will write

\[ e(\xi, \tilde{c}_{\lambda w}^2) := \sum_{w \in W, c_{\lambda w}^2 = c^2} (-1)^{\ell(w)} e(\xi_{w}^{(1)}, W_{\lambda_w}^{(1)}, p_{-}^{(1)})e(\xi^{(2)}, W_{\lambda_w}^{(2)}, p_{C}^{(2)}). \]

To construct the zeta functions, in analogy with the above we consider the quasi-regular representation $(R_{\Gamma,\varphi}, L^2(\Gamma\backslash G; \varphi))$ and we define the operator, $H_{\varphi,t}^{\tilde{c}_{\lambda w}^2}$, acting on

\[ \sum_{\mu} \otimes |m_{\mu}_{\lambda w}^2| [L^2(\Gamma\backslash G; \varphi) \otimes V_\mu]^K \]
as the operator with Schwartz kernel

\[ h_{c^2_{\lambda w}}^{\hat{} \phi_t}(x, y) = \sum_{\gamma \in \Gamma} \varphi(\gamma) \otimes \tilde{h}_{c^2_{\lambda w}}(y^{-1} x), \quad \dot{x} = \Gamma x, \dot{y} = \Gamma y, \]

where

\[ \tilde{h}_{c^2_{\lambda w}} := \sum_{\mu} \bigoplus \dim[H_{\pi} \otimes V_{\mu}^K], \]

Equivalently

\[ H_{c^2_{\lambda w}}^{\hat{} \phi_t} = R_{\Gamma, \phi}(\tilde{h}_{c^2_{\lambda w}}). \]

Relative to the \( \mathbb{Z}_2 \) gradings one has a super-trace, denoted \( \text{Tr}_s \) and we set

\[ (5.7) \quad \theta_{c^2_{\lambda w}}(t) = \text{Tr}_s(H_{c^2_{\lambda w}}^{\hat{} \phi_t}). \]

The following lemma gathers the information available to us regarding the invariant Fourier transform of \( \hat{f}_{c^2_{\lambda w}}, \)

\[ \hat{f}_{c^2_{\lambda w}}(\pi) = \text{Tr} \pi(f_{c^2_{\lambda w}}), \pi \in \hat{G}. \]

**Lemma 5.1.** (a) For any \( \pi \in \hat{G}, \) one has

\[ \hat{f}_{\mu, t}(\pi) = e^{\frac{t}{2}(\chi_{\pi}(\Omega_G) - (\lambda^* + 2\rho))} \dim[H_{\pi} \otimes V_{\mu}^K], \]

and so

\[ \hat{f}_{c^2_{\lambda w}}(\pi) = e^{\frac{t}{2}(\chi_{\pi}(\Omega_G) - (\lambda^* + 2\rho))} \sum_{\mu} \dim[H_{\pi} \otimes V_{\mu}^K]. \]

(b) Let \( A \) be the standard Cartan subgroup corresponding to the fixed cuspidal maximal parabolic \( Q. \) If \( \pi_{A, \xi, \nu} = \text{Ind}^Q_G(\xi \otimes e^\nu \otimes 1) \) with \( e^\nu \) unitary then

\[ \text{Tr} \pi_{A, \xi, \nu}(\hat{f}_{c^2_{\lambda w}}) = e^{\frac{t}{2}(\langle \xi, \lambda^* \rangle - 4\rho_n + \|\nu\|^2)} e(\xi, c^2_{\lambda w}). \]

(c) Let \( B \neq A \) be a standard Cartan subgroup such that \( B \succ A \) and let \( Q_B \) denote the associated parabolic. If \( \pi_{B, \xi, \nu} = \text{Ind}^Q_G(\xi \otimes e^\nu \otimes 1) \) then

\[ \hat{f}_{c^2_{\lambda w}}(\pi_{B, \xi, \nu}) = 0. \]
Proof. Statement \((a)\) follows as \(\pi\) is in \(\hat{G}\). For \((b)\) the formula follows from \((a), (1.16), (4.9)\) and Frobenius reciprocity.

To prove \((c)\), we may assume that \(B \subseteq M_QA_R\), say \(B = H_IH_R\) with \(H_R \supseteq A_R\). Let \(Q_B = M_BH_RN_B\) with \(M_B^+ \subseteq M_Q^+\). Take any \(Y \in H_R \setminus A_R\) with \(|Y| = 1\). Let \(a^* = (\xi, \mu, \nu) \in B^* = H_I^* \times H_R^*\). By double induction

\[
\pi := \text{Ind}_{Q}^{G}(\xi \otimes e^\mu \otimes e^{\nu} \otimes 1) = \text{Ind}_{Q}^{G}(\text{Ind}(\xi \otimes e^{\mu}) \otimes e^{\nu} \otimes 1).
\]

Using this, and arguing as above, we get

\[
\hat{f}_{\ell}^{2} \hat{c}_{\lambda w}(\pi) = e^{\frac{1}{2}((\chi_\ast(\Omega_G) - (\lambda^\ast;\lambda^\ast + 2\rho))}
\times \sum_{w \in W, \ell^2 = c^2} (-1)^{\ell(w)} \sum_{j} \dim(\text{Ind}(\xi \otimes e^{\mu}) \otimes \Lambda \tilde{\mathfrak{p}}_{+}^{[\text{ev}]} \otimes W_{\lambda w})^{K^M}
\]

\[
= e^{\frac{1}{2}((\chi_\ast(\Omega_G) - (\lambda^\ast;\lambda^\ast + 2\rho))}
\times \sum_{w \in W, \ell^2 = c^2} (-1)^{\ell(w)} \sum_{j} \dim(W_{\xi} \otimes \Lambda \tilde{\mathfrak{p}}_{+}^{[\text{ev}]} \otimes W_{\lambda w})^{K^{MB}}.
\]

As in the proof of Lemma 1.3, the \(\mathfrak{p}_{+}^{[\text{ev}]}\) component of \(Y\) will provide a non-trivial \(K^{MB}\) intertwining operator between \(\Lambda^{[\text{ev}]}\tilde{\mathfrak{p}}_{+}^{[\text{ev}]}\) and \(\Lambda^{\text{odd}}\tilde{\mathfrak{p}}_{+}^{[\text{ev}]}\); hence \(\hat{f}_{\ell}^{2} \hat{c}_{\lambda w}(\pi) = 0\). \(\blacksquare\)

Remark. An examination of the simple \(\mathfrak{g}\) of Hermitian type shows that only \(\mathfrak{sp}(n, \mathbb{R})\) and \(\mathfrak{so}(2n+1, 2)\), the non-simply laced Dynkin diagrams, have more than one class of cuspidal maximal parabolic subgroup – they have two. Hence in the case of simply-laced diagrams the preceding Lemma computes the invariant Fourier transform on all principal series induced from cuspidal parabolic subgroups. However in the non-simply laced case, at this time we are unable to give a useful expression for the invariant Fourier transform when \(A\) and \(B\) are standard Cartan subgroups of split rank one and \(A \sim B\). Thus for notational simplicity we assume that \(\mathfrak{g}\) is simple, while for technical reasons we make the assumption that \(G\) has only one class of cuspidal maximal parabolic subgroup, i.e. the Dynkin diagram is simply-laced. In these cases one finds that \(M_Q^{(2)} = \{e\}\). Although using this observation from the classification would simplify the notation, we prefer to carry it along so that we can refer to the computations on a later occasion when we do the case of more than one cuspidal maximal parabolic.
Lemma 5.2. Assume that $G$ has only one class of cuspidal maximal parabolic subgroup. Let $h \in A$, $h \neq e$. Then

$$O_{f_{t}^{\lambda_w}}(h) = e^{-\frac{\|\log h\|}{2} t} \xi(h)^{-1} \prod_{\alpha \in P_h} \left(1 - \xi^{-\alpha}(h)\right) e^{-\frac{\hat{c}_2^{\lambda_w}}{t}} \sum_{w \in W_{\kappa}, \hat{c}_2^{\lambda_w} = e^2} (-1)^{\ell(w)} \sum_{\tilde{b}(h)} \epsilon(s) \omega_h(s \cdot (\lambda_w + \rho(1_\kappa))) \xi_s \cdot (\lambda_w + \rho(1_\kappa))(h I).$$

Proof. In light of Lemma 5.1 the invariant Fourier transform of $f_{t}^{\lambda_w}$ is a product of factors as for that of $k_{i}^{\lambda}$. The proof of the result then follows the same lines as that used for Proposition 4.4 and Lemma 4.5. Hence we omit the details. 

The contribution of the identity also is obtained as in §4. However first we introduce notation to express it more compactly. Recall that for any irreducible representation $\pi$ of $G$, we have introduced an associated Euler number

$$e(\pi, V, p_{-}) = \sum (-1)^{\ell} \dim[H_{\pi}^{\infty} \otimes \Lambda^{\ell} p_{+} \otimes V]^{K}.$$

With $M_{Q}$ and $\hat{c}_2^{\lambda_w}$ fixed, and with (5.1), (5.3) in mind, we set for $\pi \in \hat{G}$

$$e_{M_{Q}}^{\lambda_w}(\pi, \lambda) := \sum_{\mu} m_{\mu}^{\lambda_w} \dim[H_{\pi} \otimes V_{\mu}]^{K}.$$ (5.9)

For each $w \in W_{\kappa}$, $W_{\lambda_w}^{(2)}$ is an irreducible module for $m_{q, c}^{(2)} \oplus a_{q, c}$. Since $F_{\kappa}$ is generated by $\gamma_{\kappa} = \exp \pi i X_{\kappa}$, $\lambda_{w}^{(2)}(\gamma_{\kappa})$ acts by either $\pm I$ on $W_{\lambda_w}^{(2)}$. Since $\gamma_{\kappa}$ acts trivially on $p_{c}^{(2)}$, for $e(\xi^{(1)}, W_{\lambda_w}^{(2)}, p_{+}^{(2)}) e(\xi^{(2)}, W_{\lambda_w}^{(2)}, p_{c}^{(2)}) \neq 0$ one must have $a_{\pi}^{*}(\gamma_{\kappa}) = \lambda_{w}^{(2)}(\gamma_{\kappa})$. As remarked earlier, for each $w \in W_{\kappa}$ there is at most a unique discrete series representation of $M_{Q}^{T}$, say having infinitesimal character $\Lambda_{\xi(w)}$ with parameter, say, $a_{\pi}^{*} = (a_{\pi}^{0}, \chi)$ with $e(\xi^{(1)}, W_{\lambda_w}^{(2)}, p_{+}^{(2)}) e(\xi^{(2)}, W_{\lambda_w}^{(2)}, p_{c}^{(2)}) \neq 0$. Thus for each $w \in W_{\kappa}$ there is also at most one possibility for $\chi(\gamma_{\alpha})$, i.e., appearance of $\tanh \frac{\pi}{2} \langle \nu, \alpha^{\vee} \rangle$ or $\coth \frac{\pi}{2} \langle \nu, \alpha^{\vee} \rangle$. Unfortunately, it does happen that there are different $w$ with the same $\hat{c}_2^{\lambda_w}$ but different $\chi(\gamma_{\alpha})$. 
Lemma 5.3. Assume that $G$ has only one class of cuspidal maximal parabolic subgroups. Then

\begin{equation}
\hat{f}_t \hat{c}^2_{\lambda w}(e) = \sum_{\omega \in \mathcal{E}_2(G)} e^{t(\chi_\omega(\Omega_G) - \langle \lambda^*, \lambda^* + 2\rho \rangle)} d_{\pi_\omega} \hat{c}^2_{\lambda w}(\omega, V_\lambda) +
\end{equation}

\begin{equation}
e^{2t(\lambda, \rho_n)} \int_{\mathfrak{a}_q^c} e^{-\frac{1}{2}(\hat{c}^2_{\lambda w} + ||\nu||^2)} \sum_{A'_t \in \mathcal{W}, c^2_{\lambda w} = c^2} (-1)^{\ell(w)} e(\xi, \lambda_w) \frac{d\mu}{d\nu} (w^*(\lambda^* + \rho - 2\rho_n)|_{I_n} + iv) \, d\nu.
\end{equation}

Proof. As in the proof of Lemma 4.6,

\begin{equation}
\hat{f}_t \hat{c}^2_{\lambda w}(e) = \sum_{\mathcal{E}_2(G)} d_{\pi_\omega} \Theta_\omega(f_t \hat{c}^2_{\lambda w}) + \sum_{\{A\}} \int_{A'_t} \int_{\mathfrak{a}_q^c} \Theta_{\xi, \nu}(f_t \hat{c}^2_{\lambda w}) d\mu(\xi, \nu).
\end{equation}

Thus, the statement follows from Lemma 5.1 and the notation before Lemma 4.6. ■

Remark. The discrete series contribution for the expression similar to $f_t \hat{c}^2_{\lambda w}(e)$ appears to be overlooked in [F2]. However for the special case considered there (no coefficient bundle) it is conceivable that, perhaps using the Ansatz, one may find an element in $K_0(K)$ satisfying $\hat{c}^2_{\lambda w}(\pi, V_\lambda) = 0$ for $\pi$ any discrete series representation.

As we have made the ongoing assumption that $\Gamma$ is torsion free, the only elliptic conjugacy class is that of the identity element. The following orbital integral expansion of $\theta \hat{c}^2_{\lambda w}(t)$ should be considered as the sum of two terms - the elliptic contribution and the semisimple contribution.

Lemma 5.4.

\begin{equation}
\theta \hat{c}^2_{\lambda w}(t) = \dim F \, \text{vol} \, X \, f_t \hat{c}^2_{\lambda w}(e)
\end{equation}

\begin{equation}
+ e^{2t(\lambda, \rho_n)} e^{-\hat{c}^2_{\lambda w}} \sum_{[\gamma] \in \mathcal{A}_G \setminus \{e\}} \text{Tr} \varphi(\gamma) e^{-\frac{1}{2}||\gamma||^2/2t} N_{\gamma}^{-1} c_{\gamma}^{-1} \text{vol}(\Gamma_\gamma \setminus G_\gamma) \prod_{\alpha \in \mathcal{P}_\gamma} [1 - \xi - \alpha(\gamma)]
\end{equation}

\begin{equation}
\sum_{w \in \mathcal{W}, c^2_{\lambda w} = c^2} (-1)^{\ell(w)} \sum_{W(I_n)} \varepsilon(s) \tilde{\omega}_\gamma(s \cdot (\lambda_w + \rho(I_n))) \xi_{s \cdot (\lambda_w + \rho(I_n))}(\gamma_I).
\end{equation}

The intermediate zeta functions will require a finite part Laplace transform of $\theta \hat{c}^2_{\lambda w}(t)$. The behaviour of $\theta \hat{c}^2_{\lambda w}(t)$ for small (resp. large) time will be needed. We show first that the behavior of $\theta \hat{c}^2_{\lambda w}(t)$ as $t \to 0^+$ is determined by the elliptic contribution.
Lemma 5.5. Let \( C_\Gamma = \bigcup_{x \in G} x^{-1}(\Gamma \setminus \{e\})x \). Then there exists a neighborhood of \( e \), \( V \subseteq G \), such that \( V \cap KC_\Gamma K = \emptyset \).

Proof. First let us notice that \( KC_\Gamma K = C_\Gamma K \). We now argue by contradiction. Suppose there exists a sequence \( \{x_n^{-1}\gamma_n x_n k_n\} \to e \). As \( K \) is compact we may assume that \( \{k_n\} \to k \in K \) and thus \( \{x_n^{-1}\gamma_n x_n\} \to k^{-1} \).

Since \( \Gamma \) is co-compact, \( G = \Gamma Q \) with \( Q \) compact. Then there are \( q_n \in Q \) and \( \delta_n \) with \( x_n = \delta_n q_n \) and \( \{\delta_n^{-1}\gamma_n \delta_n q_n\} \to k^{-1} \). But again since \( Q \) is compact we may assume that \( \{q_n\} \to q \in Q \). Hence

\[
\{\delta_n^{-1}\gamma_n \delta_n\} \to qk^{-1}q^{-1} \in qKq^{-1}.
\]

But \( \delta_n^{-1}\gamma_n \delta_n \) is in \( \Gamma \), which is discrete, hence \( \{\delta_n^{-1}\gamma_n \delta_n\} \) is eventually constant. Then for \( n >> 0, \delta_n^{-1}\gamma_n \delta_n \in \Gamma \cap qKq^{-1} \). However \( \Gamma \) is also assumed torsion-free, hence \( \Gamma \cap qKq^{-1} = \{e\} \). Then \( \delta_n^{-1}\gamma_n \delta_n = e \) or \( \gamma_n = e \), a contradiction. \( \blacksquare \)

Lemma 5.6. There is a \( c > 0 \) so that

\[
\sum_{\gamma \neq e} |\text{Tr} \varphi(\gamma) \text{tr} \tilde{h}_{\mu,t}^w(x^{-1}\gamma x)| = O(e^{-c/t}), \quad 0 < t < T.
\]

Proof. Since \( \varphi \) is unitary we may ignore the \( \text{Tr} \varphi(\gamma) \). For small time, we want to use the uniform estimates in [D], valid for scalar heat kernels on manifolds admitting a proper discontinuous group of isometries with compact quotient. To this end, we take on \( G \) a left-invariant metric \( d(\cdot,\cdot) \) and let \( p_t(\cdot) \) be the heat kernel for the associated Laplacian \( \Delta \) on scalars. According to [D], the estimate for \( 0 < t < T \) is

\[
p_t(x) \leq ct^{-n/2} \exp\left(-\frac{d^2(x,e)}{4t}\right).
\]

Since \( \tilde{h}_{\mu,t}(x) \) is obtained, as in [B-M, (2.6)], by integrating \( p_t(\cdot) \) along \( K \times K \) against certain vector valued functions, one has for \( 0 < t < T \)

\[
\|	ilde{h}_{\mu,t}(x)\| \leq C t^{-n/2} \exp\left(-\frac{d(KxK,e)^2}{4t}\right)
\]
where \(d(\mathcal{K}xK, e)\) denotes the distance of the compact \(\mathcal{K}xK\) to the identity element.

According to Lemma 5.5 there is a \(c > 0\) such that
\[
d(\mathcal{K}x^{-1}x\mathcal{K}, e) > c.
\]

Then
\[
\sum_{\gamma \neq e} \|\bar{h}_{\mu,t}(x^{-1}x)\| \leq \frac{Ce^{-c^2/4t}}{t^{n/2}} \sum_{\gamma \neq e} e^{-\frac{d(\mathcal{K}x^{-1}x\mathcal{K}, e)^2 - c^2}{4t}}
\]
and the latter sum is bounded independently of \(x\). The Lemma follows easily from this and the finite sum
\[
\tilde{h}^2_{\mu} := \sum_{\mu} \oplus m_{\mu}^2 \tilde{h}_{\mu,t}.
\]

**Corollary 5.7.**
\[
\theta^2_{\lambda,w}(t) \sim \dim F \text{ vol } X f^2_{t\lambda,w}(e), \ t \to 0^+.
\]

**Proof.**

\[
|\theta^2_{\lambda,w}(t) - \dim F \text{ vol } X f^2_{t\lambda,w}(e)|
\]
\[
= \left| \sum_{\gamma \in \mathbb{A}\backslash\{e\}} Tr \varphi(\gamma) \mathcal{O} f^2_{t\lambda,w}(\gamma) vol(\Gamma_{\gamma}\backslash G_{\gamma}) \right|
\]
\[
\leq \sum_{\gamma \in \mathbb{A}\backslash\{e\}} |Tr \varphi(\gamma)| \int_{\Gamma_{\gamma}\backslash G} |f^2_{t\lambda,w}(x^{-1}\gamma x)| dx
\]
\[
\leq \sum_{\gamma \in \mathbb{A}\backslash\{e\}} \int_{\Gamma_{\gamma}\backslash G} |f^2_{t\lambda,w}(x^{-1}\gamma x)| dx
\]
\[
\leq \frac{Ce^{-c^2/4t}}{t^{n/2}} \sum_{\gamma \neq e} \int_{\Gamma_{\gamma}\backslash G} e^{-\frac{d(\mathcal{K}x^{-1}x\mathcal{K}, e)^2 - c^2}{4t}}
\]
\[
\leq Ce^{-c^2/4t}
\]
where we have used \(\varphi\) is unitary, \(\Gamma_{\gamma}\backslash G\) is compact, and the series converges.
§6 Analysis of the elliptic contribution

It is necessary to estimate \( f_t^{c_2 w} (e) \). The next several technical Lemmas contain these estimates as well as the important evaluation of the Pf Laplace transform of \( f_t^{c_2 w} (e) \). This latter result is the key to the functional equation of the intermediate zeta functions.

For clarity of exposition we present the Lemmas for a real variable. Later we make the transition to express their results in terms of Lie algebra variables. For background on pseudofunctions and Pf Laplace transforms we suggest [La].

The presentation could be streamlined were we to refer to only those cases in the Appendix for which the Ansatz has been proved, for it is observed there that \( K^M \) is connected in these cases, i.e. only the \( \tanh \) case occurs. But should other cases of the Ansatz be found, likely, the \( \coth \) case will arise. Thus we do both at the cost of some notational unpleasantness later.

Lemma 6.1. Let \( k \) be a non-negative integer.

\[
\begin{align*}
(i) & \quad \int_0^\infty e^{-\nu^2/2} \nu^{2k} \tanh \frac{\pi}{2} \nu \, d\nu \sim \frac{c}{t^{k+3/2}}, \quad t \to \infty; \\
(ii) & \quad \int_0^\infty e^{-\nu^2/2} \nu^{2k} \tanh \frac{\pi}{2} \nu \, d\nu \sim \sum_{j=0}^{\infty} a_k^{j} t^{-k+1-j}, \quad t \to 0^+.
\end{align*}
\]

Proof. For (i) observe that \( \nu \tanh \frac{\pi}{2} \nu \) is analytic at \( \nu = 0 \), with \( \nu \tanh \frac{\pi}{2} \nu \sim \frac{\pi}{2} \nu^2 \). Then (i) follows immediately from Watson’s Lemma.

For (ii) we consider first the case \( k = 0 \). Set

\[
I_0 = \int_0^\infty e^{-\nu^2/2} \tanh \frac{\pi}{2} \nu \, d\nu.
\]

Then

\[
I_0 = -\frac{1}{2t} \int_0^\infty \frac{d}{d\nu} e^{-\nu^2/2} \tanh \frac{\pi}{2} \nu \, d\nu
= \frac{\pi}{4t} \int_0^\infty e^{-\nu^2/2} \, \text{sech}^2 \frac{\pi}{2} \nu \, d\nu \quad \text{(Integration by parts)}
= \frac{1}{t} \int_0^\infty \frac{e^{-x^2/4t} x}{(4\pi t)^{1/2} \sinh x} \, dx \quad \text{(Parseval’s Identity)}
\]

(6.1)
We may now take the Taylor series of $x / \sinh x$ (in the variable $x^2$) and use Watson’s Lemma to get an asymptotic expansion as $\frac{1}{t} \to +\infty$

$$I_0 \sim \sum_{j=0}^{\infty} \frac{a_j}{t^{1-j}}, \quad t \to 0^+.$$ 

Next we consider the case of a monomial $\nu^{2k}$, i.e.

$$I_k = \int_0^\infty e^{-t\nu^2} \nu^{2k} \tanh \frac{\pi}{2} \nu d\nu.$$

Then

$$I_k = \left( -\frac{d}{dt} \right)^k \int_0^\infty e^{-t\nu^2} \nu^{2k} \tanh \frac{\pi}{2} \nu d\nu \tag{6.2}$$

$$= \left( -\frac{d}{dt} \right)^k \frac{\pi}{4t} \int_0^\infty \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \frac{x}{\sinh x} dx$$

$$= \left( -\frac{d}{dt} \right)^k \frac{\pi}{4t} \int_0^\infty -\frac{d}{dx} \left( \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \right) dx$$

$$= \pi \int_0^\infty \left( -\frac{d^2}{dx^2} \right)^k \left( -\frac{d}{dx} \left( \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \right) \right) \frac{dx}{\sinh x},$$

where we have used the heat equation on $\mathbb{R}^1$. It is straightforward to show

$$\frac{d^{2k+1}}{dx^{2k+1}} \left( \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \right) = \sum_{j=0}^{k} b_j \frac{x^{2j+1}}{t^{k+j+1}} \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}}.$$ 

Then

$$I_k = \pi \sum_{j=0}^{k} b_j \frac{x^{2j+1}}{t^{k+j+1}} \int_0^\infty \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \frac{x^{2j}}{x \sinh x} dx.$$

As before for $I_0$, one uses the Taylor expansion for $x$ and Watson’s Lemma to get

$$\frac{1}{t} \int_0^\infty \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \frac{x^{2j}}{x \sinh x} dx \sim \sum_{\ell=0}^{\infty} \frac{c_{\ell}^j}{t^{1-j-\ell}}.$$ 

Hence after grouping terms we obtain

$$I_k \sim \sum_{\ell=0}^{\infty} \frac{a_{\ell}^k}{t^{k+1-\ell}}.$$ 

(Notice that this can be seen formally by substituting the asymptotic expansion for $I_0$ into $I_k$ and differentiating in $t$.)
Lemma 6.2. Let $k$ be a non-negative integer.

(i) \[ \int_0^\infty e^{-tv^2}v^{2k}\nu \coth \frac{\pi}{2} \nu d\nu \sim \frac{c}{t^{k+\frac{1}{2}}}, \quad t \to +\infty; \]

(ii) \[ \int_0^\infty e^{-tv^2}v^{2k}\nu \coth \frac{\pi}{2} \nu d\nu \sim \sum_{j=0}^{\infty} \frac{b_j^k}{t^{k+1-j}}, \quad t \to 0^+. \]

Proof. The proof of (i) is identical to that used in the previous Lemma with the different exponent resulting from the constant in the Taylor series of $\nu \coth \frac{\pi}{2} \nu$.

The proof of (ii) begins by deriving a Parseval Identity. We start with formula 3.986 (4) in [G-R]

\[ \frac{\pi}{2} \nu \coth \frac{\pi}{2} \nu = 2\pi \int_0^\infty \frac{\sin^2 \frac{\pi}{2} \nu x}{\sinh^2 \pi x} dx + 1. \]

Then

\[
J_0 = \int_0^\infty e^{-tv^2}v \coth \frac{\pi}{2} \nu d\nu \\
= \int_0^\infty e^{-tv^2} \left[ 4 \int_0^\infty \frac{\sin^2 \frac{\pi}{2} \nu x}{\sinh^2 \pi x} dx + \frac{2}{\pi} \right] d\nu \\
= 4 \int_0^\infty \int_0^\infty e^{-tv^2} \sin^2 \frac{\pi}{2} \nu x d\nu \left( \frac{dx}{\sinh^2 \pi x} + \frac{2}{\pi} \right) \\
= 2 \int_0^\infty \left[ \left( \frac{\pi}{4t} \right)^{1/2} - \int_0^\infty e^{-tv^2} \cos \nu \nu x d\nu \right] \frac{dx}{\sinh^2 \pi x} + \left( \frac{1}{\pi t} \right)^{1/2} \\
= 2 \int_0^\infty \left[ \left( \frac{\pi}{4t} \right)^{1/2} - \left( \frac{\pi}{4t} \right)^{1/2} e^{-\pi^2 x^2/4t} \right] \frac{dx}{\sinh^2 \pi x} + \left( \frac{1}{\pi t} \right)^{1/2} \\
= 2 \int_0^\infty \left. \frac{1}{\pi t} - \frac{1}{\pi t} e^{-\pi^2 x^2/4t} \right|_{0}^{\infty} dx + \left( \frac{1}{\pi t} \right)^{1/2} \\
= -\frac{2}{\pi} \left( \frac{\pi}{4t} \right)^{1/2} + \frac{2^{3/2}}{t} \int_0^\infty x \coth \pi x e^{-\pi^2 x^2/4t} \left( 4t \right)^{1/2} dx + \left( \frac{1}{\pi t} \right)^{1/2} \\
= \frac{2^{3/2}}{t} \int_0^\infty x \coth \pi x e^{-\pi^2 x^2/4t} \left( 4t \right)^{1/2} dx.
\]

Hence we obtain the Parseval Identity

\[ \int_0^\infty e^{-tv^2} \nu \coth \frac{\pi}{2} \nu d\nu = \frac{1}{t} \int_0^\infty e^{-x^2/4t} \frac{1}{(4\pi t)^{1/2}} x \coth x dx. \]
Now the result for \( k = 0 \) follows directly from Watson’s Lemma, while the computation for \( k > 0 \) proceeds as before.

Since the Plancherel density is of the form \( \sum_{j=0}^{m} a_j \nu^{2m-2j} \left\{ \nu \tanh \frac{\pi}{2} \nu \right\} \) and only finitely many nonzero \( e(\xi, \lambda_w) \), putting together Lemma 6.1 and Lemma 6.2 with Lemma 5.3 we obtain

**Proposition 6.3.** Assume that \( G \) has only one class of cuspidal maximal parabolic subgroups. Then

\[
(i) f_t^{\hat{c}^2_{\lambda_w}} (e) - \sum_{\omega \in \mathcal{E}_2(G)} e^{\frac{\nu}{2} (\chi_\omega(\Omega_G) - (\lambda^*, \lambda^* + 2\rho))} d_{\nu\omega} e^{\hat{c}^2_{\lambda_w}}(\omega, V_\lambda) \sim e^{2t (\lambda, \rho)} e^{-\frac{t}{2} \hat{c}^2_{\lambda_w}} \left\{ \frac{c}{\nu \hat{c}^2_{\lambda_w}} \right\}, \ t \to +\infty;
\]

\[
(ii) f_t^{\hat{c}^2_{\lambda_w}} (e) - \sum_{\omega \in \mathcal{E}_2(G)} e^{\frac{\nu}{2} (\chi_\omega(\Omega_G) - (\lambda^*, \lambda^* + 2\rho))} d_{\nu\omega} e^{\hat{c}^2_{\lambda_w}}(\omega, V_\lambda) \sim \sum_{\omega \in \mathcal{W}, \hat{c}^2_{\lambda_w} = c^2} \sum_{j=0}^{\infty} \frac{\hat{c}^2_{\lambda_w}(c)}{\nu j! + 1} \ t \to 0^+;
\]

\[
(iii) f_t^{\hat{c}^2_{\lambda_w}} (e) - \sum_{\omega \in \mathcal{E}_2(G)} e^{\frac{\nu}{2} (\chi_\omega(\Omega_G) - (\lambda^*, \lambda^* + 2\rho))} d_{\nu\omega} e^{\hat{c}^2_{\lambda_w}}(\omega, V_\lambda) \sim \sum_{j=0}^{\infty} \frac{\hat{c}^2_{\lambda_w}(c)}{\nu j! + 1} \ t \to 0^+.
\]

**Remark.** In (5.10) there occurs \( \sum_{\omega \in \mathcal{W}, \hat{c}^2_{\lambda_w} = c^2} \). As remarked earlier, for each \( w \) there is at most one \( \xi \) for which this is nonzero. In (i) for the asymptotics we have grouped together the cases according to \( \chi(\gamma, \alpha) = \pm 1 \) and performed the sum to obtain one constant \( c \). In (ii) we keep the dependence of the constants on \( w \) as this will be needed later. In (iii) similarly we have grouped terms and performed the sum to obtain the coefficients \( \hat{c}^2_{\lambda_w} \). From Lemma 5.1 it follows that \( \hat{c}^2_{\lambda_w} \) is independent of the Ansatz. Finally, we caution the reader concerning a substitution of \( \frac{t}{2} \) for \( t \). The real variable integrals appear cleaner as presented, but in Lemma 5.3 the variable involves \( \frac{t}{2} \).

In order to compute \( \theta \)-regularized determinants we shall need the \( Pf \) Laplace transform of \( f_t^{\hat{c}^2_{\lambda_w}} (e) \) for each \( \hat{c}^2_{\lambda_w} \). Notice that due to Proposition 6.3, \( f_t^{\hat{c}^2_{\lambda_w}} (e) \) defines a pseudo-function, and so we may use the symbolic calculus of \( Pf \) Laplace transforms. Again the \( \tanh \) and \( \coth \) cases require very different formulas, so will be treated separately.
Lemma 6.4. Assume that $\Re z^2 > 0$ and that $\Re z \notin (-\infty, 0]$. Then

\[
\text{Pf} \int_0^\infty e^{-tz^2} \left[ \int_0^\infty e^{-tv^2} v^k \tanh \frac{\pi}{2} v \, dv \right] \, dt = -(iz)^k \sum_{j=0}^{k} c_j z^{2j}.
\]

Consequently it has a meromorphic continuation to $\mathbb{C}$.

Proof. First notice that from Lemma 6.1 it follows that the expression in brackets is a pseudofunction so that the $\text{Pf}$ Laplace transform is defined and holomorphic for $\Re z^2 > 0$. We will do the $k = 0$ case first, and then use the symbolic calculus to treat the general case. We begin with the Parseval Identity (6.1)

\[
\text{Pf} \int_0^\infty e^{-tz^2} \left[ \int_0^\infty e^{-tv^2} v \tanh \frac{\pi}{2} v \, dv \right] \, dt = \text{Pf} \int_0^\infty e^{-tx^2} \left[ \frac{1}{t} \int_0^\infty e^{-x^2/4t} \frac{x}{(4\pi t)^{1/2}} \sinh x \, dx \right] \, dt.
\]

From the symbolic calculus this Laplace transform is up to a constant an integral of

\[
\text{Pf} \int_0^\infty e^{-tx^2} \left[ \int_0^\infty e^{-x^2/4t} \frac{x}{(4\pi t)^{1/2}} \sinh x \, dx \right] \, dt.
\]

As $\Re z^2 > 0$, the integrand is not singular, so we may use the Fubini theorem to get

\[
\int_0^\infty \frac{x}{\sinh x} \int_0^\infty e^{-tx^2} \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \, dt \, dx = \int_0^\infty \frac{x}{\sinh x} \frac{e^{-zx}}{2z} \, dx \quad (\Re z \notin (-\infty, 0])
\]

\[
= \frac{1}{4z} \sum_{n \geq 0} \frac{1}{(n + \frac{z+1}{2})^2} \quad (3.552.1 \ [G-R]).
\]

Now

\[
\sum_{n \geq 0} \frac{1}{(n + w)^2} = \frac{d^2}{dw^2} \log \Gamma(w) = \frac{d}{dw} \frac{\Gamma'(w)}{\Gamma(w)},
\]
so with \( z^2 = p \) we obtain

\[
Pf \int_0^\infty e^{-tp} \int_0^\infty \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \frac{x}{\sinh x} dx \, dt = \frac{d}{dp} \left[ \frac{\Gamma'(\frac{p^{1/2}+1}{2})}{\Gamma\left(\frac{p^{1/2}+1}{2}\right)} \right], \quad (\Re(p^{1/2}) > 0).
\]

The symbolic calculus then gives

\[
Pf \int_0^\infty e^{-tp} \frac{1}{t} \int_0^\infty \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \frac{x}{\sinh x} dx \, dt = -\frac{\Gamma'(\frac{p^{1/2}+1}{2})}{\Gamma\left(\frac{p^{1/2}+1}{2}\right)} + c.
\]

To determine \( c \), notice that

\[
\frac{\Gamma'(\frac{p^{1/2}+1}{2})}{\Gamma\left(\frac{p^{1/2}+1}{2}\right)} \sim \log p^{1/2} - \log 2, \quad p \to +\infty.
\]

Then from the symbolic calculus we have

\[
c = -\frac{\gamma}{2} - \log 2
\]

where \( \gamma \) is Euler’s constant. So we obtain, for \( \Re z > 0 \) with \( \Re z^2 > 0 \),

\[
Pf \int_0^\infty e^{-tz^2} \left[ \int_0^\infty e^{-\nu^2} \nu \tanh \frac{\pi}{2} \nu \, d\nu \right] dt = \left( -\frac{\gamma}{2} - \log 2 \right) - \frac{\Gamma'(\frac{1+z}{2})}{\Gamma\left(\frac{1+z}{2}\right)}.
\]

Since \( \frac{\Gamma'}{\Gamma} \) is meromorphic on \( \mathbb{C} \), we get a meromorphic continuation of the \( Pf \) integral to \( \mathbb{C} \).

Next we consider the case \( k = 1 \).

\[
Pf \int_0^\infty e^{-tz^2} \left[ \int_0^\infty e^{-\nu^2} \nu^2 \nu \tanh \frac{\pi}{2} \nu \, d\nu \right] dt
= Pf \int_0^\infty e^{-tz^2} \left[ -\frac{d}{dt} \int_0^\infty e^{-\nu^2} \nu \tanh \frac{\pi}{2} \nu \, d\nu \right] dt
= Pf \int_0^\infty e^{-tz^2} \left[ -\frac{d}{dt} g(t) \right] dt.
\]

But from Lemma 6.1 (ii),

\[
g(t) \sim \frac{a_0}{t} + a_1 + O(t),
\]
so the symbolic calculus gives

\[ Pf \int_0^\infty e^{-tp} \left[ -\frac{dg}{dt}(t) \right] dt = -p Pf \int_0^\infty e^{-tp} g(t) dt + a_1 + a_0p. \]

Using the result already proved, \( k = 0 \), we get

\[
Pf \int_0^\infty e^{-tz^2} \left[ \int_0^\infty e^{-t\nu^2} \nu^2 \nu \tanh\frac{\pi}{2} \nu d\nu \right] dt
= z^2 \frac{\Gamma'(\frac{1+z}{2})}{\Gamma(\frac{1+z}{2})} + z^2 \left( -\frac{\gamma}{2} - \log 2 \right) + a_1 + a_0 z^2
= -(iz)^2 \frac{\Gamma'(\frac{1+z}{2})}{\Gamma(\frac{1+z}{2})} + \sum_{j=0}^1 c_j^1 z^{2j}.
\]

The result for general \( k \) then follows from induction.

\[ \square \]

**Lemma 6.5.** Assume that \( \Re z^2 > 0 \) and that \( \Re z \notin (-\infty, 0] \). Then

\[
Pf \int_0^\infty e^{-tz^2} \left[ \int_0^\infty e^{-t\nu^2} \nu^2 \nu \coth\frac{\pi}{2} \nu d\nu \right] dt
= (iz)^{2k} \left[ \frac{\Gamma'(\frac{z}{2} + 1)}{\Gamma(\frac{z}{2} + 1)} + \frac{1}{z} \right] + \sum_{j=0}^k d_j^k z^{2j},
\]

in particular has a meromorphic continuation to \( \mathbb{C} \).

**Proof.** As usual we will do the \( k = 0 \) case first and then the general case with the symbolic calculus. We begin with the Parseval Identity (6.3).

\[
Pf \int_0^\infty e^{-tz^2} \left[ \int_0^\infty e^{-t\nu^2} \nu^2 \nu \coth\frac{\pi}{2} \nu d\nu \right] dt
= Pf \int_0^\infty e^{-tz^2} \left[ \frac{1}{t} \int_0^\infty e^{-x^2/4t} \nu \coth x \nu \nu d\nu \right] dt
\]

From the symbolic calculus this Laplace transform is, up to a constant, an integral of

\[ Pf \int_0^\infty e^{-tz^2} \left[ \int_0^\infty \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \nu \coth x \nu \nu d\nu \right] dt. \]
From Lemma 6.2 we see that the integrand is not singular in $t$ so we may use the Fubini Theorem to get

$$\int_0^\infty x \coth x \left[ \int_0^\infty e^{-tx^2} e^{-x^2/4t} \frac{dt}{(4\pi)^{1/2}} \right] dx$$

$$= \int_0^\infty x \coth x \frac{e^{-x^2/2}}{2z} dx \quad (\Re z \notin (-\infty, 0])$$

$$= \frac{1}{2z} \left[ \frac{1}{2} \sum_{n \geq 0} \frac{1}{(n + \frac{z}{2})^2} - \frac{1}{z^2} \right] \quad (3.551.3 \text{ [G-R]})$$

As in Lemma 6.4, with $z^2 = p$ we get

$$Pf \int_0^\infty e^{-tp} \int_0^\infty e^{-x^2/4t} x \coth x \frac{dx}{(4\pi)^{1/2}} dt$$

$$= \left[ \frac{d}{dp} \frac{\Gamma'(\frac{p^{1/2}}{2})}{\Gamma'(\frac{p^{1/2}}{2})} + \frac{d}{dp} \frac{1}{p^{1/2}} \right].$$

The symbolic calculus then gives

$$Pf \int_0^\infty e^{-tp} \frac{1}{t} \int_0^\infty e^{-x^2/4t} x \coth x \frac{dx}{(4\pi)^{1/2}} dt$$

$$= \left[ - \frac{\Gamma'(\frac{p^{1/2}}{2})}{\Gamma(\frac{p^{1/2}}{2})} - \frac{1}{p^{1/2}} \right] + c$$

$$= \left[ - \frac{\Gamma'(\frac{p^{1/2}}{2})}{\Gamma(\frac{p^{1/2}}{2})} - \frac{2}{p^{1/2}} + \frac{1}{p^{1/2}} \right] + c$$

$$= \left[ - \frac{\Gamma'(\frac{p^{1/2}}{2} + 1)}{\Gamma(\frac{p^{1/2}}{2} + 1)} + \frac{1}{p^{1/2}} \right] + c.$$

To determine $c$ we, as before, use Stirling’s estimate

$$\frac{\Gamma'(\frac{p^{1/2}}{2} + 1)}{\Gamma(\frac{p^{1/2}}{2} + 1)} \sim \frac{1}{z} + \log z - \log 2, \quad z \to +\infty$$
and then the symbolic calculus to find \( c = -\frac{\gamma}{2} - \log 2 \). We complete the proof for \( k \geq 1 \) as in the previous Lemma.

\[ \blacksquare \]

**Remarks.** In order to put the \( Pf \) Laplace transform of \( \dim F \ \vol X \int_1^{\lambda w}(e) \) in manageable, geometric form, we make some observations and notational simplifications. It is useful to keep in mind (5.10).

Let \( \chi_a(X) \) denote the arithmetic genus of \( X \), i.e.

\[ \chi_a(X) = \sum_{q=0}^n (-1)^n \dim H^{0,q}(X). \]

We shall denote the integer \( \chi_a(X) \dim F \) by \( \chi_a(X,F) \), the arithmetic genus of \( X \) with coefficients in the flat bundle \( F \).

For \( \omega \in \mathcal{E}_2(G) \) with infinitesimal character \( \Lambda_{\omega} \), notice that

\[ d(\Lambda_{\omega}) := \frac{e(\Lambda_{\omega})\varpi(\Lambda_{\omega})}{\varpi(\rho)} \]

is integral, being, up to a sign, the dimension of a finite dimensional representation of \( G \).

In the discussion circa (4.12) we parametrized \( a^*_q \) as \( \frac{r}{2}\alpha_R \) where \( r \in \mathbb{R} \). We recall that \( \varpi(\log a^*_q + i\nu) \), the polynomial factor in the Plancherel density, is \( \nu \) times a polynomial in \( \nu^2 \).

Let \( c_j \) (resp. \( d_j \)), depending on \( w \), be the numbers obtained from summing the \( c^*_{j,k} \) in Lemma 6.4 (resp. \( d^*_{j,k} \) in Lemma 6.5) over the various powers \( \nu^{2k} \) from the polynomial factor of the Plancherel density. For each \( w \) we define a polynomial function on \( a^*_q, C \), \( \tilde{p}_{\lambda w}(\cdot) \), by the expression

\[ \frac{(-2i)(-1)^{g-M}}{\varpi(\rho)} \sum c_j z^{2j} \]  (resp. \( \sum d_j z^{2j} \)) when \( \lambda w(\gamma_\alpha) \) is 1 (resp. -1) and \( z \in \mathbb{C} \).

Again the term \( \sum_{w \in W_\nu} c^2_{\lambda w} = c^2(-1)^{\ell(w)} e(\xi, \lambda_w) \) introduces the, by now, familiar notational complications. As before we will have to group terms, but for clarity we first do the computations for one \( w \).

We come to a subtle observation concerning the variable \( z \) in the Laplace transform, and thus in the zeta function to be defined in the next section - it must belong to the same space.
as the variable \( \nu \). Hence the variable \( z \) of the zeta function is in \( a_q^* \mathbb{C} \). So we must choose an identification of \( \mathbb{C} \) with \( a_q^* \mathbb{C} \). For consistency with the previous parametrization for \( \nu \) we take 'z' \( \in a_q^* \mathbb{C} \) as \( \frac{z}{2} \alpha_R, z \in \mathbb{C} \). But we caution that this leads to annoying appearances of 'two' in several places. Also, with the parametrization of \( a_q^* \) as \( \frac{z}{2} \alpha_R \), the measure on \( a_q^* \) in these coordinates becomes \( \frac{2d\nu}{\|\alpha_R\|^2} \).

Ordinarily in \( Pf \) Laplace transforms, changes of variables by scaling are not valid. However, in the proofs of Lemma 6.4 and Lemma 6.5 the starting point is a Laplace transform of a non-singular function, followed by repeated application of the symbolic calculus. With a little care (as will be done in the proof of the next Proposition) one can easily verify that the Lie theoretic formulation of Lemma 6.4 becomes

\[
Pf \int_0^\infty e^{-t\|\frac{z}{2}\alpha_R\|^2} \int_{a_q^*} e^{-t\|\nu\|^2} \varpi \left( \Lambda_{\xi(w)} + i\nu \right) \tanh \left( \frac{\pi}{2} \langle \nu, \alpha_R^\vee \rangle \right) d\nu dt
\]

\[= \frac{(-8i)}{\|\alpha_R\|} \varpi \left( \Lambda_{\xi(w)} + i\left( \frac{z\alpha_R}{2} \right) \right) \frac{\Gamma'(\frac{1}{2}(\langle \alpha_R^\vee, \frac{z}{2}\alpha_R \rangle + 1))}{\Gamma\left(\frac{1}{2}(\langle \alpha_R^\vee, \frac{z}{2}\alpha_R \rangle + 1)\right)} + (-1)^{q-q_M} \varpi(\rho) \bar{p}_{\lambda_w}(\frac{z}{2}\alpha_R),
\]

and similarly for Lemma 6.5. Notice that this is consistent with a change of variables in an absolutely convergent integral.

Recall ((4.9)) that \( \hat{c}_{\lambda_w} = ((\lambda_w \circ C_{\kappa}) + \rho Q_{\kappa})(\hat{X}_\kappa) \), and that \( X_\kappa = H_{\alpha_R}/\langle \alpha_R, \alpha_R \rangle \). It is necessary to realize \( \hat{c}_{\lambda_w} \) also as an element of \( a_q^* \). Recalling that \( \hat{X}_\kappa \) is a unit vector it follows from the above that with

\[c_{\lambda_w} := ((\lambda_w \circ C_{\kappa}) + \rho Q_{\kappa})(X_\kappa)
\]

that \( \hat{c}_{\lambda_w} \) is replaced with

\[c_{\lambda_w} \alpha_R = ((\lambda_w \circ C_{\kappa}) + \rho Q_{\kappa})(X_\kappa)\alpha_R,
\]

so that \( \hat{c}_{\lambda_w}^2 = c_{\lambda_w}^2 \|\alpha_R\|^2 \).

Finally, from Lemma 1.1 we see that the use of the Kodaira Laplacian rather than the Hodge Laplacian introduces a numerical factor of \( \frac{1}{2} \), whose effect is seen clearly in, say, (5.10). As is seen in (6.4) the formulas are rather satisfactory with out the factor \( \frac{1}{2} \). There appear to be two methods to reconcile this: incorporate a factor of \( \frac{1}{2} \) in the definition of the
Laplace transform or change the ‘t’ parameter to ‘2t’. We have chosen the latter approach for what follows. Recall the factor $e^{2t \langle \lambda, \rho_n \rangle}$ appears independently of $w$. Also we have seen in §1 that $4 \langle \lambda, \rho_n \rangle = -i \lambda (H_0)$, i.e. $e^{2t \langle \lambda, \rho_n \rangle} = e^{\frac{2t}{2} (-i \lambda (H_0))}$. With the change ‘t’ parameter to ‘2t’ we obtain $e^{4t \langle \lambda, \rho_n \rangle} = e^{-it \langle \lambda (H_0) \rangle}$. While this displays a curious interplay between the choice of the complex structure and the coefficient bundle, the lack of dependence on $w$ makes it a good candidate for a uniform translation change of variable. We will highlight where this is done. Finally, for the convenience of the reader we recall the formula

$$∥\lambda^* + \rho∥^2 + 4 \langle \lambda, \rho \rangle = ∥\lambda + \rho∥^2,$$

in particular $\omega(\Omega_G) - \langle \lambda^*, \lambda^* + 2\rho \rangle = ∥\Lambda_\omega + \rho∥^2 - ∥\lambda^* + \rho∥^2 = ∥\Lambda_\omega + \rho∥^2 - ∥\lambda + \rho∥^2 + 4 \langle \lambda, \rho_n \rangle$.

**Proposition 6.6.**

$$2(\alpha^\vee_R, z\alpha_R) P f \int_0^\infty e^{-t(∥\frac{1}{2}\alpha_R∥^2 - ∥c_{\lambda_w} \alpha_R∥^2)} \text{dim } F \text{ vol } X e^{-4t\langle \lambda, \rho_n \rangle} f_{2t}^\omega(e) dt =$$

$$\chi_a(X, \mathbb{F}) \sum_{\omega \in \mathcal{E}_1(G)} \frac{2(\alpha^\vee_R, z\alpha_R) \tilde{c}_{\lambda_w}^2 (\omega, V_\lambda)}{∥\frac{1}{2}\alpha_R∥^2 - ∥c_{\lambda_w} \alpha_R∥^2 - ∥\Lambda_\omega∥^2 + ∥\lambda + \rho∥^2}$$

$$\chi_a(X, \mathbb{F}) \sum_{w \in W_\alpha, \tilde{c}_{\lambda_w}^2 = c^2} (-1)^{\ell(w)} e(\xi, \lambda_w) 4(-1)^{q - q_M} \frac{\varpi(\Lambda(\xi(w)) + i(\frac{i}{2}\alpha_R))}{\varpi(\rho)} \frac{1}{\Gamma(\frac{1 + (\alpha^\vee_R, z\alpha_R)}{2})}$$

$$\chi_a(X, \mathbb{F}) \sum_{w \in W_\alpha, \tilde{c}_{\lambda_w}^2 = c^2} (-1)^{\ell(w)} e(\xi, \lambda_w) 4(\alpha^\vee_R, z\alpha_R) p_{\lambda_w}^2 \left(\frac{\varpi}{2}\alpha_R\right)$$

for $\xi_w \leftrightarrow \Lambda_\xi(w)$, and accordingly as $\lambda^w(\gamma_\alpha) = ±1$. Moreover this is a meromorphic function on $\mathbb{C} \alpha_R$ with simple poles and integral residues.

**Proof.** We begin with Lemma 5.3,

$$\text{dim } F \text{ vol } X f_{2t}^\lambda \omega(e) = \sum_{\omega \in \mathcal{E}_1(G)} \text{dim } F \text{ vol } X e^{t(\omega(\Omega_G) - \langle \lambda^*, \lambda^* + 2\rho \rangle)} d_{\pi_w} \tilde{c}_{\lambda_w}^2 (\omega, V_\lambda) +$$

$$\sum_{A^I_1} \text{dim } F \text{ vol } X c_A \int_{\alpha^\vee} e^{-t(∥c_{\lambda_w} \alpha_R∥^2 + ∥\nu∥^2)} e^{4t(\lambda, \rho_n)} \times$$

$$\sum_{w \in W_\alpha, \tilde{c}_{\lambda_w}^2 = c^2} (-1)^{\ell(w)} e(\xi, \lambda_w) \varpi(\Lambda(\xi(w)) + i\nu) \left\{\tanh \left(\frac{\pi}{2}\nu, \alpha^\vee_R\right) \right\} du \cdot$$
Then multiplying the formula above by \( e^{-4t(\lambda, \rho_n)} \) gives
\[
\dim F \vol X e^{-4t(\lambda, \rho_n)} f^2_{2t}(e) = \sum_{\omega \in \mathcal{E}_2(G)} \dim F \vol X e^{-4t(\lambda, \rho_n)} e^{t(\omega(G) - (\lambda^*, \lambda^* + 2\rho))} d_{\pi, \omega} e^{\frac{c^2}{2}} (\omega, V_{\lambda})
\]
\[
\sum_{A_f \in \mathcal{A}} \int_{\mathfrak{a}_q} e^{-t(\|c_{\lambda, \omega} \alpha_{k}^2 + \|v^2\})} \times
\sum_{w \in W_\kappa, \epsilon_{\lambda, \omega} = c^2} (-1)^{\ell(w)} e(\xi, \lambda_w) \varpi \left( \Lambda \xi(w) + iv \right) \left\{ \tanh \left( \frac{\pi}{2} \nu, \alpha_{\kappa}^2 \right) \right\} d\nu.
\]
As we use Harish-Chandra’s normalization of measures, we have ([H-C;I]) for \( \omega \in \mathcal{E}_2(G) \)
(and summing over \( \mathcal{E}_2(G) \) not \( T^* \))
\[
d_{\pi, \omega} = c^{-1}_G \epsilon(b^*_\omega) \varpi(b^*_\omega)|W(G, T)|,
\]
where \( \varpi(b^*_\omega) = \varpi(\log b^*_\omega + \rho) = \varpi(\Lambda \omega) \), \( c_G = |W(G, T)|2^q \pi^q \varpi_k(k) \), \( q = \frac{1}{2} \dim G/K \), \( \nu = \dim G/K - \text{rank } G/K \). The arithmetic genus is the index of Dirac with coefficients in the line bundle associated to \( \xi_{\rho_n} \). A calculation (similar, though with minor corrections, to one in [B-M] for \( \mathbb{R} \)-rank one) gives this index to be
\[
\chi_\alpha(X) = c^{-1}_G \varpi(\rho)|W(G, T)| \vol X.
\]
Then the discrete series term becomes
\[
\chi_\alpha(X, F) \sum_{\omega \in \mathcal{E}_2(G)} e^{\frac{c^2}{2}} (\omega, V_{\lambda}) d(\Lambda \omega) e^{-4t(\lambda, \rho_n)} e^{t(\omega(G) - (\lambda^*, \lambda^* + 2\rho))}.
\]
The first term of (6.6) results from an elementary calculation. We note that, provided
\(-\|c_{\lambda, \omega} \alpha_{k}^2 - \|\Lambda \omega\|^2 + \|\lambda + \rho\|^2 \neq 0 \), the resulting function is meromorphic with simple poles and integer residues because \( \chi_\alpha(X, F) \), \( d(\Lambda \omega) \) and \( e^{\frac{c^2}{2}} (\omega, V_{\lambda}) \) are integers.

For the principal series contribution the first order of business is to identify \( c_A \) in (4.12). This can be done using Herb’s evaluation of the Plancherel measure and by comparing it on cusp forms to Harish-Chandra’s in order to reconcile the different choices of Haar measure. So from Herb and for the cases at hand,
\[
f(e) = (-1)^q \frac{1}{(2\pi)|\Delta||} \sum_{b^* \in \widehat{T}} \Theta(b^*)(f) \varpi(\log b^*) +
\]
\[
(-1)^q \frac{1}{(2\pi)|\Delta||} \frac{|W(G, T)|}{|W(G, H)|} \left( \frac{i}{2} \right) \left\{ \tanh \left( \frac{\pi}{2} \mu, \alpha_{\kappa}^2 \right) \right\} \coth \left( \frac{\pi}{2} \mu \right) \right\} d\mu,
\]

\[
\int_{\mathfrak{a}_q} e^{-t(\|c_{\lambda, \omega} \alpha_{k}^2 + \|v^2\})} \times
\sum_{w \in W_\kappa, \epsilon_{\lambda, \omega} = c^2} (-1)^{\ell(w)} e(\xi, \lambda_w) \varpi \left( \Lambda \xi(w) + iv \right) \left\{ \tanh \left( \frac{\pi}{2} \nu, \alpha_{\kappa}^2 \right) \right\} d\nu.
\]
where \( \mu_\alpha = \frac{2(\mu, \alpha)}{(\alpha, \alpha)} \), \( \alpha \in \Delta^+_R \). For cuspidal \( f \), in [H-C, I] (p.181) we find

\[
f(e) = c_G^{-1} \sum_{b^* \in T^*} (-1)^q \varpi (b^*) (\Theta_{b^*}, f).
\]

It follows that

\[
dx_{H^b} = (2\pi)^{|\Delta^+|} c_G^{-1} dx_{H-C},
\]

and hence that

\[
c_A = (-1)^q \frac{|W(G, T)|}{|W(G, H)|} \left( \frac{i}{2} \right) \frac{2}{\|\alpha_R\|} \int_{a^*_q} e^{-t\|c_{\lambda \omega}\|_2^2 + \|\nu\|_2^2} c(\xi, \lambda_w) \varpi (\Lambda_{\xi(w)} + i\nu) \left\{ \tan \frac{\pi}{2} \langle \nu, \alpha_R \rangle \right\} \coth \frac{\pi}{2} \langle \nu, \alpha_R \rangle d\nu.
\]

Rewriting the sum over \( A^* \) as one over \( E_2(M^+_Q) \) and, as noted before, the sum has at most one non-zero term associated to \( \xi(w) \); recognizing \( a^*_q \) in [Hb] represents \( H_R \) with normalized Haar measure, we get (6.8)

\[
\sum_{w \in W_A, c_{\lambda_w} = c^2} (-1)^{\xi(w)}
\]

\[
\chi_a(X, F)e(\xi, \lambda_w) (-1)^{q-M} \left( \frac{i}{2} \right) \frac{2}{\|\alpha_R\|} \int_{a^*_q} e^{-t\|c_{\lambda \omega}\|_2^2 + \|\nu\|_2^2} \varpi(\Lambda_{\xi(w)} + i\nu) \left\{ \tan \frac{\pi}{2} \langle \nu, \alpha_R \rangle \right\} \coth \frac{\pi}{2} \langle \nu, \alpha_R \rangle d\nu.
\]

Now when we coordinatise \( a^*_q \) by \( \nu = \frac{\mu}{2} \alpha_R, \mu \in \mathbb{R}, \alpha_R \in \Delta^+_R \) then the integral above becomes

\[
\frac{\|\alpha_R\|}{2} \int_0^\infty e^{-t\|c_{\lambda \omega}\|_2^2} e^{-t\|\alpha_R\|_2^2} \mu^2 \varpi (\Lambda_{\xi(w)} + i\frac{\mu \alpha_R}{2}) \left\{ \tan \frac{\pi}{2} \mu \right\} \coth \frac{\pi}{2} \mu d\mu.
\]

One can check that \((-1)^{q-M} i\varpi(\Lambda_{\xi} + i\frac{\mu \alpha_R}{2})\) is positive for \( \mu > 0 \), as is needed by the Plancherel density. The right side of (6.6) now follows from Lemma 6.4 and Lemma 6.5.

Some remarks on the constants would be helpful. For the third term, recall that the constant \( \frac{(-2i)(-1)^q}{\varpi(\rho)} \) was put into the definition of \( \rho'_{\lambda_w} \), in particular the minus sign was added to have a minus sign for this term in (6.6); also the scaling factor \( \frac{4}{\|\alpha_R\|^2} \) is incorporated into the factor \( \langle \alpha_R, \frac{\zeta}{2} \alpha_R \rangle \) producing the \( \alpha^V \) and the aforementioned 2 in
\[ \tilde{p}(\lambda w) . \] For the second term, the presence of the missing real root which resulted from the integration was added back into \( \varpi \) when one multiplies by \( \langle \alpha_R, \frac{1}{2} \alpha_R \rangle \) times \( \frac{1}{||\alpha_R||^2} \); the minus sign is clearly seen to result from (6.5).

The partial fraction decomposition of the derivative of \( \log \Gamma (W-W, \text{p. } 247) \) shows that its poles are simple, occur at negative integral values of the argument and are of constant sign. Thus there are poles possibly at \( z = -2m - 1 \) (resp. \( z = -2m - 2 \)). Consequently we consider \( \Lambda(\xi(w)) + \frac{1}{2} \alpha_R, m \geq 0 \) (resp. \( \Lambda(\xi(w)) + m\alpha_R, m \geq 1 \)). We claim that this is \( \Delta(g_C, \eta_C) \) integral. Now from Cor. 1.13

\[
\Lambda(\xi(w)) = (w^* (\lambda^* + \rho - 2\rho_n)) \circ C_\kappa + (\lambda w|_{\mathcal{C}H_\kappa}) \circ C_\kappa + \rho Q_\kappa,
\]

and from the standing hypothesis on \( \lambda \) the first term is \( G \)-integral. The remaining terms are supported on \( a_q \), and are an integral or half-integral multiple of \( \alpha_R \). Since the possible poles are similarly located it suffices to consider such multiples of \( \alpha_R \). But it follows easily from the structure of roots in the Hermitian case that \( \frac{1}{2} \alpha_R \) is integral for \( l \) an integer, and thus that

\[
2(-1)^{q-q_M} \frac{\varpi (\Lambda(\xi(w)) + \lambda^* \frac{1}{2} \alpha_R)}{\varpi(\rho)}
\]

is an even integer at each of these poles. Moreover, this is dominant for the given order on the roots. \[ \square \]

Remark 6.7. Each term on the right of (6.6) is of the form \( \chi_a(X, \mathbb{F}) \) times a function depending only on the simply connected cover \( \tilde{X} \). This is not surprising since \( \tilde{f}^{2t} \lambda w(e) \) is determined by \( \tilde{X} \) and the extension of the bundle to the compactification of \( \tilde{X} \).
§7 The Elliptic Factor

In Proposition 6.6 we evaluated the Pf Laplace transform of $\dim F \text{ vol } X \ e^{-4t\langle \lambda, \rho_n \rangle} f_{2t}^{\hat{\varepsilon}^2 w} (e)$ and we showed that it is a meromorphic function on $\mathbb{C}$ with simple poles and integral residues. We should regard the term $\dim F \text{ vol } X \ e^{-4t\langle \lambda, \rho_n \rangle} f_{2t}^{\hat{\varepsilon}^2 w} (e)$ as a trace, a $\Gamma$-trace. Then it is natural to apply to it the construction of the $\theta$-regularized determinant from [M-S;II]. The details of these steps are the content of this section.

To begin we recall the symbolic calculus used in the proof of Lemma 6.4. Suppose that

$$ Pf \int_0^\infty e^{-tp} f(t) \ dt = \frac{d}{dp} F(p), $$

and suppose that

$$ F(p) \sim A \log(p) + B, \quad p \to +\infty. $$

Then

$$ Pf \int_0^\infty e^{-tp} f(t) \frac{dt}{t} = -F(p) - \gamma A + B, \quad \gamma, \text{ again, Euler’s constant.} $$

By analogy with the discussion in [M-S;II], we could denote by $\log \det_\theta(I + pG_{e}^{\hat{\varepsilon}^2 w})$ the unique meromorphic function satisfying:

1. $Pf \int_0^\infty e^{-tp} \dim F \text{ vol } X e^{-4t\langle \lambda, \rho_n \rangle} f_{2t}^{\hat{\varepsilon}^2 w} (e) \ dt = \frac{d}{dp} \log \det_\theta(I + pG_{e}^{\hat{\varepsilon}^2 w}),$

2. $\log \det_\theta(I + pG_{e}^{\hat{\varepsilon}^2 w}) \mid_{p=0} = 0.$

Consequently,

$$ Pf \int_0^\infty e^{-tp} \dim F \text{ vol } X e^{-4t\langle \lambda, \rho_n \rangle} f_{2t}^{\hat{\varepsilon}^2 w} (e) \frac{dt}{t} = - \log \det_\theta(I + pG_{e}^{\hat{\varepsilon}^2 w}) + C. $$

To determine the constant $C$ either one uses the behaviour of $\log \det_\theta(I + pG_{e}^{\hat{\varepsilon}^2 w})$ for $p \to \infty$ as described above, or the behaviour of $\dim F \text{ vol } X e^{-4t\langle \lambda, \rho_n \rangle} f_{2t}^{\hat{\varepsilon}^2 w} (e)$ for $t \to 0^+$ in the following way. Set

$$ \zeta_{e}^{\hat{\varepsilon}^2 w} (s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \dim F \text{ vol } X e^{-4t\langle \lambda, \rho_n \rangle} f_{2t}^{\hat{\varepsilon}^2 w} (e) \ dt. $$
Then $\zeta e^{2\lambda w}(s)$ is regular at $s = 0$, and in [M-S;II] p.190 we show that

$$C = \zeta e^{2\lambda w}(0) + \gamma c_{e,0},$$

where $c_{e,0}$ is the coefficient of $t^0$ in the small time asymptotics of $\dim F \vol X e^{-4t(\lambda, \rho_n)} f_{2t}^2(\cdot)$. With obvious analogy with zeta regularized determinant we denote $\zeta e^{2\lambda w}'(0)$ by $-\log \det \zeta \Delta e^{2\lambda w}$.

A small technical correction is necessary in the preceding; namely, the procedure in [M-S;II] assumes that zero modes have been removed from the trace. To handle this complication, we separate the contributions from the discrete series and principal series; remove possible zero modes; then follow the above procedure.

To simplify the expressions in (6.6) we introduce some notation. Recall that $c_{\lambda w} = ((\lambda_w \circ C_\kappa) + \rho Q_\kappa)(X_\kappa)\alpha_R$

$$e(\xi, \lambda_w) = e(\xi^{(1)}_w, W^{(1)}_{\lambda_w}) e(\xi^{(2)}_w, W^{(2)}_{\lambda_w}, p^{(2)}_C);$$

then we set

$$p_{\lambda}^{\omega}(\frac{x_{\alpha R}}{2}) := 2(-1)^{q-M} \frac{\omega(\Lambda^{\xi}_{\omega}) - \frac{\omega(\rho)}{2}}{\omega(\rho)},$$

(7.1)

$$m_{\omega, \hat{c}_{\lambda w}^2} := d(\Lambda_{\omega}) e^{\hat{c}_{\lambda w}^2} e^{\hat{c}_{\lambda w}^2}$$

$$M_{\lambda_{\omega}}^2 := \sum_{\omega \in E_2(G)} m_{\omega, \hat{c}_{\lambda w}^2}.$$  

As in (6.6) we separate the contributions to $\dim F \vol X e^{-4t(\lambda, \rho_n)} f_{2t}^2(\cdot) (e)$ coming from the discrete series and the principal series. The discrete series contribution, (6.7), is

$$\chi_a(X, F) \sum_{\omega \in E_2(G)} m_{\omega, \hat{c}_{\lambda w}^2} e^{-4t(\lambda, \rho_n)} e^{t(\omega(\Omega_G) - (\lambda^*, \lambda^* + 2\rho))}.$$  

Evaluating an elementary integral and using the functional calculus it is straightforward to calculate

$$- Pf \int_0^\infty e^{-t(\|\frac{\omega R}{2}\|^2 - \|c_{\lambda_w} \alpha_R\|^2)} \sum_{\omega \in E_2(G)} m_{\omega, \hat{c}_{\lambda w}^2} e^{-4t(\lambda, \rho_n)} e^{t(\omega(\Omega_G) - (\lambda^*, \lambda^* + 2\rho))} \frac{dt}{t} =$$

$$\log \prod_{\omega \in E_2(G)} (\|\frac{\omega R}{2}\|^2 - \|c_{\lambda_w} \alpha_R\|^2 - \|\Lambda_\omega\|^2 + \|\lambda + \rho\|^2) m_{\omega, \hat{c}_{\lambda w}^2} + \gamma M_{\lambda_{\omega}}^2.$$
Notice that the second summand above contains $\gamma$ times the coefficient of $t^0$ in the small time asymptotics of the discrete series contribution to $e^{-4t(\lambda, \rho)} f^{\hat{\lambda}^2}_{2t}(e)$.

Taking the exponential gives

$$e^{\gamma M_{\lambda}^2} \prod_{\omega \in \mathcal{E}_2(G)} (\|\frac{z}{2}\alpha_\omega\|^2 - \|c_{\lambda_\omega} \alpha_\omega\|^2 - \|\Lambda_\omega\|^2 + \|\lambda + \rho\|^2)^{m_\omega, \hat{c}_\lambda^2}.$$ 

We rewrite this in a way suggestive of the spectral interpretations of various terms. Let $n_{\hat{c}_\lambda^2, \mathcal{E}_2}$ be the dimension of the “zero modes”, i.e. those $\omega$ with $-\|\Lambda_\omega\|^2 + \|\lambda + \rho\|^2 = 0$. Notationally, we set

$$\det_\theta \left( I + (\|\frac{z}{2}\alpha_\omega\|^2 - \|c_{\lambda_\omega} \alpha_\omega\|^2) G_{\mathcal{E}_2}^{\hat{c}_\lambda^2} \right) := \prod_{\omega \in \mathcal{E}_2(G)} \left( 1 + \frac{\|\frac{z}{2}\alpha_\omega\|^2 - \|c_{\lambda_\omega} \alpha_\omega\|^2}{-\|\Lambda_\omega\|^2 + \|\lambda + \rho\|^2} \right)^{m_\omega, \hat{c}_\lambda^2} \text{ for } -\|\Lambda_\omega\|^2 + \|\lambda + \rho\|^2 \neq 0,$$

$$\det_\zeta \Delta_{\hat{c}_\lambda^2, \mathcal{E}_2} := \prod_{\omega \in \mathcal{E}_2(G)} (-\|\Lambda_\omega\|^2 + \|\lambda + \rho\|^2)^{m_\omega, \hat{c}_\lambda^2}.$$ 

Both these are defined to be 1 if there is no discrete series contribution. Then the exponential which was

$$e^{\gamma M_{\lambda}^2} \prod_{\omega \in \mathcal{E}_2(G)} (-\|\Lambda_\omega\|^2 + \|\lambda + \rho\|^2)^{m_\omega, \hat{c}_\lambda^2} \cdot (\|\frac{z}{2}\alpha_\omega\|^2 - \|c_{\lambda_\omega} \alpha_\omega\|^2)^{n_{\hat{c}_\lambda^2, \mathcal{E}_2}} \prod_{\omega \in \mathcal{E}_2(G)} \left( I + \frac{\|\frac{z}{2}\alpha_\omega\|^2 - \|c_{\lambda_\omega} \alpha_\omega\|^2}{-\|\Lambda_\omega\|^2 + \|\lambda + \rho\|^2} \right)^{m_\omega, \hat{c}_\lambda^2},$$

is written as

$$\tilde{B}_{\mathcal{E}_2} \left( \frac{z}{2} \alpha_\omega \right) :=$$

$$\det_\zeta \Delta_{\hat{c}_\lambda^2, \mathcal{E}_2} e^{\gamma M_{\lambda}^2} \left( \|\frac{z}{2}\alpha_\omega\|^2 - \|c_{\lambda_\omega} \alpha_\omega\|^2 \right)^{\hat{c}_\lambda^2} \prod_{\omega \in \mathcal{E}_2(G)} \left( I + \frac{\|\frac{z}{2}\alpha_\omega\|^2 - \|c_{\lambda_\omega} \alpha_\omega\|^2}{-\|\Lambda_\omega\|^2 + \|\lambda + \rho\|^2} \right)^{m_\omega, \hat{c}_\lambda^2}.$$ 

(7.2) $$\tilde{B}_{\mathcal{E}_2} \left( \frac{z}{2} \alpha_\omega \right) = e^{-\gamma M_{\lambda}^2} \tilde{B}_{\mathcal{E}_2} \left( \frac{z}{2} \alpha_\omega \right).$$

$$\mathcal{B}_{\mathcal{E}_2} \left( \frac{z}{2} \alpha_\omega \right) := e^{-\gamma M_{\lambda}^2} \tilde{B}_{\mathcal{E}_2} \left( \frac{z}{2} \alpha_\omega \right).$$
Notice that
\[
\tilde{B}_{E_2}(-z\alpha_R) = \tilde{B}_{E_2}(z\alpha_R),
\]
(7.3)
\[
\lim_{z \to 0} \frac{\tilde{B}_{E_2}((z + 2c_{\lambda_w})\frac{\alpha_R}{2})}{\langle \alpha_R^\vee, \frac{1}{2}\alpha_R^\vee \rangle^{n_{c,E_2}}} = \det \zeta \Delta_{c_{\lambda_w}}^{\frac{c_{\lambda_w}^2}{\frac{1}{2}}} e^{\gamma M_2 c_{\lambda_w}^2}.
\]

We repeat this formalism for the principal series contribution to (6.6). As usual, there is the complication resulting from the use of the Ansatz in that we have a sum of terms
\[
\sum_{w \in W, c_{\lambda_w}^2 = c^2} (-1)^{\ell(w)}.
\]
First we construct the functions directly, then interpret them in the determinant formalism.

Consider a constituent of the third term in (6.6). As \(\tilde{p}_{\lambda_w}'\) is a polynomial in \(z^2\) we take \(\tilde{p}_{\lambda_w}\) to be an integral with respect to \(z^2\) of \(4\langle \alpha^\vee_R, z\alpha_R \rangle \tilde{p}_{\lambda_w}'(z\alpha_R)\), with constant so that \(\tilde{p}_{\lambda_w}(2c_{\lambda_w}\alpha_R) = 0\).

Recall from (6.8) the contribution of the principal series to \(\dim F \vol X e^{-4t\langle \lambda, \rho_n \rangle} f_{\frac{c_{\lambda_w}^2}{2}}(e)\), viz.
\[
\sum_{w \in W, c_{\lambda_w}^2 = c^2} (-1)^{\ell(w)}
\]

\[
\chi_{\alpha}(X, F)e(\xi, \lambda_w)(-1)^{q-q_M} \left(\frac{i}{2}\right) \left\{ \begin{array}{l}
\frac{2}{\|\alpha_R\|} \int_{a_q^*} e^{-t(\|\xi(w)\| + \|\nu\|^2)} \frac{\varpi(\xi(w) + i\nu)}{\varpi(\rho)} \left\{ \begin{array}{l}
\tanh \frac{t}{2} \langle \nu, \alpha^\vee_R \rangle \\
\coth \frac{t}{2} \langle \nu, \alpha^\vee_R \rangle
\end{array} \right. \\
d
\end{array} \right\} d\nu dt
\]

We have seen in Proposition (6.6) that
\[
2\langle \alpha^\vee_R, z\alpha_R \rangle Pf \int_{0}^{\infty} e^{-t(\|\xi(w)\|^2)} \int_{a_q^*} e^{-t(\|\nu\|^2)} \frac{\varpi(\xi(w) + i\nu)}{\varpi(\rho)} \left\{ \begin{array}{l}
\tanh \frac{t}{2} \langle \nu, \alpha^\vee_R \rangle \\
\coth \frac{t}{2} \langle \nu, \alpha^\vee_R \rangle
\end{array} \right. \}
\]

is meromorphic. Then
\[
2\langle \alpha^\vee_R, z\alpha_R \rangle Pf \int_{0}^{\infty} e^{-t(\|\xi(w)\|^2)} \int_{a_q^*} e^{-t(\|\nu\|^2)} \frac{\varpi(\xi(w) + i\nu)}{\varpi(\rho)} \left\{ \begin{array}{l}
\tanh \frac{t}{2} \langle \nu, \alpha^\vee_R \rangle \\
\coth \frac{t}{2} \langle \nu, \alpha^\vee_R \rangle
\end{array} \right. \frac{d\nu dt}{t}
\]

is meromorphic. This, together with those obtained from the other two terms in (6.6),
gives a meromorphic function $M_w(z_{\alpha_{\mathbb{R}}})$ satisfying

$$- Pf \int_0^\infty e^{-t(\|x_{\alpha_{\mathbb{R}}}^0\|^2 - \|c_{\lambda_w}\alpha_{\mathbb{R}}\|^2)} \dim F \ vol X \ e^{-4t(\lambda, \rho_n)} \int_2^{cz_w} (e) \ dt \ t =$$

$$- \chi_a(X,F) \log \tilde{B} \xi_x(\frac{z}{2}) \alpha_{\mathbb{R}} + \chi_a(X,F) \sum_{w \in W_{\kappa, c_{\lambda_w}^2}^2} (-1)^{\ell(w)} e(\xi, \lambda_w) \ log M_w(z_{\alpha_{\mathbb{R}}})$$

$$+ \chi_a(X,F) \sum_{w \in W_{\kappa, c_{\lambda_w}^2}^2} (-1)^{\ell(w)} e(\xi, \lambda_w) \tilde{p}_{\lambda_w}(\frac{z}{2}) \alpha_{\mathbb{R}} + \sum_{w \in W_{\kappa, c_{\lambda_w}^2}^2} (-1)^{\ell(w)} C'_w.$$

Now each constituent of the second term in (6.6) (without the factors $e(\xi, \lambda_w)$ and $\chi_a(X,F)$), as a function on $\mathbb{C} \alpha_{\mathbb{R}}$ has simple poles with integral residues of constant sign say $\epsilon$. The Weierstrass theorem gives an entire function $F_w$ on $\mathbb{C} \alpha_{\mathbb{R}}$ as an infinite product, with $\epsilon$ log-derivative with respect to $z$ having precisely these poles and residues. By construction $M_w$ has these same poles and residues, thus there is then an entire function $h^w$, unique up to a constant, with $M_w = F_w e^{h^w}$. We will choose the normalizing constant for $M_w$ so that the leading coefficient of $M_w(z_{\alpha_{\mathbb{R}}})$ at $z = 2c_{\lambda_w}$ is 1.

For notational convenience we denote the set $\{w \in W_{\kappa} : c_{\lambda_w}^2 = \epsilon^2\}$ by $W_{c^2}$.

For each $w \in W_{c^2}$ we have the constant $C'_w$ which arises from the log det$\theta$ procedure. To determine $C'_w$ we repeat the procedure used for the discrete series. Namely, we factor out the zero at $2c_{\lambda_w} \alpha_{\mathbb{R}}$; we construct a $\theta$-regularized determinant from the principal series contribution; we group it with a $\zeta$-regularized determinant; and then the coefficient of $t^0$ of the small time heat trace of the principal series contribution contributes a constant factor.

In Proposition 6.3 (ii) we find the coefficient of $t^0$, denoted $c_0^w$. From the proof of Proposition 6.6 we see that $\chi_a(X,F)/\dim F \ vol X = c_G^{-1} \varpi(\rho)|W(G,T)|$, which is independent of the Ansatz. So we set

$$\tilde{c}_0^w = c_0^w \dim F \ vol X/\chi_a(X,F).$$

Then because there is a common factor of $\chi_a(X,F)$ we get

$$\tilde{c}_{e,0}^2 = \chi_a(X,F) \left( \sum_{\omega \in \tilde{e}_{2}(G)} m_{\omega, e_{\lambda_w}^2} \right) + \chi_a(X,F) \sum_{W_{c^2}} (-1)^{\ell(w)} e(\xi, \lambda_w) \tilde{c}_0^w$$

$$= \chi_a(X,F) (M_2 e_{\lambda_w}^2 + \sum_{W_{c^2}} (-1)^{\ell(w)} e(\xi, \lambda_w) \tilde{c}_0^w).$$

(7.4)
We denote by \( n_{e,P}^{\lambda_w} \) the order of the zero of \( M_w(z\alpha_R)^e(\xi,\lambda_w) \) at \( 2c_{\lambda_w}\alpha_R \). Recall that \( M_w(z\alpha_R) \) is normalized to have leading coefficient 1 at \( 2c_{\lambda_w}\alpha_R \), and \( \tilde{p}_{\lambda_w} \) is normalized to be 0 at \( 2c_{\lambda_w}\alpha_R \). Define \( \det_\zeta \Delta_{w,e,P}^{\lambda_w} \) by

\[
-\log \det_\zeta \Delta_{w,e,P}^{\lambda_w} - \chi_a(X,\mathbb{F}) \gamma e(\xi,\lambda_w)\tilde{c}_0 = C'_w.
\]

Set

\[
\tilde{B}_P(\frac{z}{2}\alpha_R) := \prod_{W_{e2}} \{ \det_\zeta \Delta_{w,e,P}^{\lambda_w} e^{e(\xi,\lambda_w)\tilde{c}_0} e^{e(\xi,\lambda_w)\tilde{p}_{\lambda_w}(\frac{z}{2}\alpha_R)} M_w(z\alpha_R)^e(\xi,\lambda_w) \} (-1)^{\ell(w)} \;
\]

\[
B_P = \prod_{W_{e2}} \{ e^{-e(\xi,\lambda_w)\tilde{c}_0} \} (-1)^{\ell(w)} \tilde{B}_P
\]

With these normalizations we see that we might call

\[
\det_\theta(I + (\|z\alpha_R\|^2 - \|c_{\lambda_w}\alpha_R\|^2)G_{P}^{\lambda_w}) := \frac{e^{e(\xi,\lambda_w)\tilde{p}_{\lambda_w}(\frac{z}{2}\alpha_R)} M_w(z\alpha_R)^e(\xi,\lambda_w)}\langle \alpha_R^\vee, (z - 2c_{\lambda_w})\alpha_R \rangle^{-n_{e,P}^{\lambda_w}},
\]

so that analogous to (7.2) for \( \tilde{B}_{e_2} \) we would have

\[
\tilde{B}_P(z\alpha_R) = \prod_{W_{e2}} \{ \det_\zeta \Delta_{w,e,P}^{\lambda_w} e^{e(\xi,\lambda_w)\tilde{c}_0} \langle \alpha_R^\vee, (z - 2c_{\lambda_w})\alpha_R \rangle^{-n_{e,P}^{\lambda_w}} \} \det_\theta(I + (\|\frac{z}{2}\alpha_R\|^2 - \|c_{\lambda_w}\alpha_R\|^2)G_{P}^{\lambda_w}) \} (-1)^{\ell(w)}.
\]

However, this would be misleading because the notation for the \( \theta \)-determinant is an even function of \( z \). Rather, we must take into account the next result. The other possible zeros will be handled subsequently.

**Lemma 7.1.**

\[
M_w(-z\alpha_R) = M_w(z\alpha_R) e^{2\pi \int_0^z \tilde{p}_{\lambda_w}(\frac{z}{2}\alpha_R) \left\{ \frac{\tan \frac{\alpha^\vee}{2} (\alpha R)}{\cot \frac{\alpha^\vee}{2} (\alpha R)} \right\} dx}
\]

**Proof.** Since \( M_w \) has integral residues, we may use

\[
\log M_w(z\alpha_R) - \log M_w(-z\alpha_R) = \int_0^z \left[ \frac{M_w'(x\alpha_R)}{M_w(x\alpha_R)} + \frac{M_w'(-x\alpha_R)}{M_w(-x\alpha_R)} \right] dx.
\]
As

\[
\frac{M'_w(x\alpha_R)}{M_w(x\alpha_R)} = 2p^λ_w\left(\frac{x}{2}\alpha_R\right)\left\{ \frac{\Gamma'\left(\frac{1+(\alpha_k^\vee, \frac{x}{2}\alpha_R)}{2}\right)}{\Gamma\left(\frac{1+(\alpha_k^\vee, \frac{x}{2}\alpha_R)}{2}\right)} \right. \\
- \left. \frac{\Gamma'\left(1+\frac{(\alpha_k^\vee, \frac{x}{2}\alpha_R)}{2}\right)}{\Gamma\left(1+\frac{(\alpha_k^\vee, \frac{x}{2}\alpha_R)}{2}\right)} \frac{1}{\langle \alpha_k^\vee, \frac{x}{2}\alpha_R \rangle} \right\}
\]

and \(p^λ_w\) is an odd function, it is enough to compute

\[
\frac{\Gamma'\left(1+\frac{(\alpha_k^\vee, \frac{x}{2}\alpha_R)}{2}\right)}{\Gamma\left(1+\frac{(\alpha_k^\vee, \frac{x}{2}\alpha_R)}{2}\right)} - \frac{\Gamma'\left(1-\frac{(\alpha_k^\vee, \frac{x}{2}\alpha_R)}{2}\right)}{\Gamma\left(1-\frac{(\alpha_k^\vee, \frac{x}{2}\alpha_R)}{2}\right)}
\]

These are easily seen using [W-W] p. 247 and the partial fraction expansions of \(\tanh z\) and \(\coth z\) to give \(\pi \tan \frac{\pi}{2} \langle \alpha_k^\vee, \frac{x}{2}\alpha_R \rangle\) and \(\pi \cot \frac{\pi}{2} \langle \alpha_k^\vee, \frac{x}{2}\alpha_R \rangle\) respectively. The 2 in (7.7) comes from (6.6) and (7.1).

**Lemma 7.2.** (1) For each \(z_0\alpha_R\) define the integer \(n^λ_w(z_0\alpha_R)\) by

\[
n^λ_w(z_0\alpha_R) = \left\{ \begin{array}{ll}
0 & \text{if } z_0 \neq -(2n+1) \text{ (resp. } -2n) \\
2p^λ_w\left(\frac{z_0}{2}\alpha_R\right) & \text{if } z_0 = -(2n+1) \text{ (resp. } -2n)
\end{array} \right.
\]

according as \(\lambda^w(\gamma_0) = \pm 1\). Then the order of \(M_w\) at \(z_0\) is \(n^λ_w(z_0\alpha_R)\).

(2) A similar statement holds for

\[
e^{-4t\langle \lambda, \rho_\alpha \rangle} \int_{2t}^{2t} e^{\frac{z^2}{2}} p^λ_w\left(\frac{z}{2}\alpha_R\right) \left\{ \tan \frac{\pi}{2} \langle \alpha_k^\vee, \frac{z}{2}\alpha_R \rangle \right. \\
\left. \cot \frac{\pi}{2} \langle \alpha_k^\vee, \frac{z}{2}\alpha_R \rangle \right\} dz.
\]

**Proof.** (1) Since the principal series contribution to \(e^{-4t\langle \lambda, \rho_\alpha \rangle} \int_{2t}^{2t} e^{\frac{z^2}{2}} (e)\) is independent of \(\Gamma\) and \(M_w\) is meromorphic on \(\mathbb{C}\alpha_R\) a result of this type certainly holds. To determine the precise order at \(z = z_0\), notice that for \(z_0 \neq 0\)

\[
-\text{order } M_w |_{z=z_0} = \lim_{z \to z_0} z \frac{d}{dz} \log M_w((z + z_0)\alpha_R) \\
= 2p^λ_w\left(\frac{z_0}{2}\alpha_R\right) \lim_{z \to 0}(z) \left\{ \frac{\Gamma'\left(1+\frac{(\alpha_k^\vee, \frac{z+x_0}{2}\alpha_R)}{2}\right)}{\Gamma\left(1+\frac{(\alpha_k^\vee, \frac{z+x_0}{2}\alpha_R)}{2}\right)} \right. \\
\left. - \frac{1}{\langle \alpha_k^\vee, \frac{z+x_0}{2}\alpha_R \rangle} \right\}
\]

Proof (2)
From the partial fraction expansion of $\Gamma'/\Gamma$ one can evaluate this limit, obtaining the result for $z_0 \neq 0$. If $z_0 = 0$, then $p_e^\lambda w(0) = 0$ to order 1 so that the order of $M_w$ at zero is zero.

(2) is a consequence of (1) and the functional equation in Lemma 7.1.

**Remarks 7.3.**

(i) A different method to determine the order was used in Fried p. 48.

(ii) Since the poles and residues of the $\epsilon$-logderivative $F_w$ are easily determined, one can form the Weierstrass product representation of $F_w$ explicitly. This interesting expression shall be used elsewhere. However, we point out that for $SL(2, \mathbb{R})$ and the cot case it appears already in [W-W] p. 148, #20.

(iii) Following the convention introduced earlier, we set

$$n_{e,P}^\lambda w = \sum_{W,c} (-1)^{\ell(w)}n_{e,P}^\lambda w.$$ 

Thus, similar to (7.3), we have

$$
\lim_{z \to 0} \tilde{B}_P((z + 2c_{\lambda w})^\frac{\alpha_R}{2}) = \prod_{W,c} \{det\Delta_{e,P}^\lambda w e^{\gamma e(\xi,\lambda w)c_0^w}(-1)^{\ell(w)}\}.
$$

Putting together the various contributions to $\dim F \vol X e^{-4t(\lambda,\rho_n)} \int_{2t} e_2^\lambda w(e)$ we make for the bundle $\mathcal{V}_\lambda$ and for $\tilde{c}_w^\lambda$.

**Definition 7.4.** Define the $\tilde{B}$-factor associated to the elliptic elements by

$$\tilde{B}(\tilde{X}, \mathcal{W}_{\tilde{c}_w^\lambda}, \frac{z}{2}a) = \tilde{B}_{E_2}(\frac{z}{2}a)\tilde{B}_P(\frac{z}{2}a).$$ 

$$B(\tilde{X}, \mathcal{W}_{\tilde{c}_w^\lambda}, \frac{z}{2}a) = B_{E_2}(\frac{z}{2}a)B_P(\frac{z}{2}a).$$

Summarizing the preceding discussion we have the basic properties of $\tilde{B}(\tilde{X}, \mathcal{W}_{\tilde{c}_w^\lambda}, \frac{z}{2}a)$. 
Proposition 7.5.

(i) \( \tilde{B}(\tilde{X}, W_{\xi}^2; \frac{z}{2} \alpha) \chi_a(X,F) = e^{-Pf \int_0^\infty e^{-t}(\|z\|^2 - \|c\|^2) \dim F \ vol X} e^{-4t(\lambda, \rho n) \int_0^\infty e^{-4t} dt} \)

(ii) \( \tilde{B}(\tilde{X}, W_{\xi}^2; \cdot) \) is meromorphic on \( \mathbb{C} \alpha_R \);

(iii) \( \tilde{B}(\tilde{X}, W_{\xi}^2; \cdot) \) satisfies the functional equation

\[
\tilde{B}(\tilde{X}, W_{\xi}^2; -\frac{z}{2} \alpha) = \tilde{B}(\tilde{X}, W_{\xi}^2; \frac{z}{2} \alpha) e^{2\pi \sum_{W_c} \frac{1}{2} \frac{1}{\alpha} \sum_{n} \left\{ \tan \frac{1}{2} \langle \alpha, \alpha \rangle \ cot \frac{1}{2} \langle \alpha, \alpha \rangle \right\} dx}
\]

(iv) For the leading term near \( 2c_{\lambda} \alpha \) one has

\[
\lim_{z \to 0} \frac{\tilde{B}(\tilde{X}, W_{\xi}^2; ((z + 2c_{\lambda}) \alpha))}{\langle \alpha, \alpha \rangle} (\frac{\xi}{2} \alpha_{\xi}) = e^{\gamma \sum_{\xi} \Delta_{\xi}^2 \xi_{\xi}^2} \prod_{W_{\xi}^2} \left\{ \det \xi_{\xi}^2 \lambda_{\xi}^2 \ e^{\gamma \xi_{\xi}^2} \right\} (-1)^{\xi_{\xi}^2} \]
§8 The Semisimple Factor

We proceed to the construction of zeta functions specified on the (nontrivial) semisimple conjugacy classes in $\Gamma$, indeed supported on $E_1(\Gamma)$.

Recall from §5 the constructions obtained with the use of the Ansatz. For each $\hat{c}_w^2$ we have defined a “truncation” $H_{\varphi,t}^{\hat{c}_w^2}$ acting on

$$\sum_{\mu} \oplus |m_{\mu} \hat{c}_w^2|[L^2(\Gamma\backslash G; \varphi) \otimes V_\mu]^K$$

as the operator with Schwartz kernel

$$h_{\varphi,t}^{\hat{c}_w^2}(\hat{x}, \hat{y}) = \sum_{\gamma \in \Gamma} \varphi(\gamma) \otimes \tilde{h}_{\varphi,t}^{\hat{c}_w^2}(y^{-1} \gamma x), \quad \hat{x} = \Gamma x, \hat{y} = \Gamma y,$$

where

$$\tilde{h}_{\varphi,t}^{\hat{c}_w^2} = \sum_{\mu} \oplus |m_{\mu} \hat{c}_w^2| h_{\mu,t}$$

$$= \sum_{\mu} \oplus |m_{\mu} \hat{c}_w^2| e^{\frac{1}{2}(\Omega_G - (\lambda^*+,\lambda^*+2\rho)I)}$$

$$= \sum_{\mu} \oplus |m_{\mu} \hat{c}_w^2| e^{-t(\Delta_{\mu} + \frac{1}{2}(\lambda^*+,\lambda^*+2\rho)I - \frac{1}{2}(\mu^*+,\mu^*+2\rho)I)}.$$ 

Equivalently,

$$H_{\varphi,t}^{\hat{c}_w^2} = R_{\Gamma,\varphi}(\tilde{h}_{\varphi,t}^{\hat{c}_w^2})$$

acting on

$$\sum_{\mu} \oplus |m_{\mu} \hat{c}_w^2|[L^2(\Gamma\backslash G; \varphi) \otimes V_\mu]^K.$$ 

Also we recall the notation (5.7) for the super-trace

$$\theta^{\hat{c}_w^2}(t) = \text{Tr}_s(H_{\varphi,t}^{\hat{c}_w^2}).$$

The notation to represent the above direct sum of operators becomes cumbersome. So we choose to use the suggestive, albeit misleading, notation

$$e^{-\frac{1}{2}\Box^{\hat{c}_w^2}} := \sum_{\mu} \oplus |m_{\mu} \hat{c}_w^2| e^{-t(\Delta_{\mu} + \frac{1}{2}(\lambda^*+,\lambda^*+2\rho)I - \frac{1}{2}(\mu^*+,\mu^*+2\rho)I)},$$

where

- $\Gamma$ is a discrete subgroup of $\text{SL}(2, \mathbb{R})$.
- $\varphi$ is a theta function.
- $E_1(\Gamma)$ is the space of cusp forms.
- $\hat{c}_w^2$ are certain constants.
- $L^2(\Gamma\backslash G; \varphi)$ is the space of square-integrable functions on $\Gamma\backslash G$ with respect to the measure $\varphi$. 
- $V_\mu$ are representations of $\Gamma$. 
- $\text{Tr}_s$ denotes the supertrace.
and thus
\[ \theta^{c^2\lambda_w}(t) = Tr_s \exp(-t\Box^{c^2\lambda_w}) := \sum_{\mu} m_{\mu}^{c^2\lambda_w} Tr e^{-t(\Delta_{\mu} + \frac{1}{2}(\lambda^{*}, \lambda^{*} + 2\rho)I - \frac{1}{2}(\mu^{*}, \mu^{*} + 2\rho)I)} \].

In a similar way we denote the direct sum of Green's operators by
\[ (zI + \frac{1}{2}\Box^{c^2\lambda_w})^{-1} := \sum_{\mu} \bigoplus |m_{\mu}^{c^2\lambda_w}| (zI + \frac{1}{2}(\lambda^{*}, \lambda^{*} + 2\rho)I - \frac{1}{2}(\mu^{*}, \mu^{*} + 2\rho)I + \Delta_{\mu})^{-1}. \]

A minor complication that arises is that the operators in \( \Box^{c^2\lambda_w} \) need not be non-negative operators. However each of the operators \( \frac{1}{2}(\lambda^{*}, \lambda^{*} + 2\rho)I - \frac{1}{2}(\mu^{*}, \mu^{*} + 2\rho)I + \Delta_{\mu} \) has spectrum on \( L^2(\Gamma \setminus G; \varphi) \otimes V_{\mu} \) bounded below. Using operators for which \( m_{\mu}^{c^2\lambda_w} \neq 0 \) we define \( \mu_{b}^{c^2\lambda_w} \) by
\[ \mu_{b}^{c^2\lambda_w} = \min_{\mu} \min\{Spec(\Delta_{\mu} + \frac{1}{2}(\lambda^{*}, \lambda^{*} + 2\rho)I - \frac{1}{2}(\mu^{*}, \mu^{*} + 2\rho)I) \}. \]

Since the construction and analytic properties of \( \theta \)-regularized determinants in [M-S:II] applies to each of the operators \( e^{-t\Delta_{\mu}} \), we may apply the procedure to the finitely many involved in \( e^{-t\Box^{c^2\lambda_w}} \), and hence to \( \theta^{c^2\lambda_w}(t) \). For \( t \to 0^+ \), notice that \( \theta^{c^2\lambda_w}(t) = Tr_s H^{c^2\lambda_w}_{\varphi,t} \) has an asymptotic expansion. Indeed, from Corollary 5.7 and the fact that \( e^{-4t(\lambda,\rho_n)} \sim 1, t \to 0^+ \)
\[ \theta^{c^2\lambda_w}(2t) \sim \dim F \text{ vol } X \; e^{-4t(\lambda,\rho_n)} f_{2t}^{c^2\lambda_w}(e), \; t \to 0^+, \]
and from Proposition 6.3 we see that \( e^{-4t(\lambda,\rho_n)} f_{2t}^{c^2\lambda_w}(e) \) is a pseudofunction. Since
\[ \theta^{c^2\lambda_w}(t) = \sum_{\mu} m_{\mu}^{c^2\lambda_w} Tr e^{-t(\Delta_{\mu} + \frac{1}{2}(\lambda^{*}, \lambda^{*} + 2\rho)I - \frac{1}{2}(\mu^{*}, \mu^{*} + 2\rho)I)}, \]
writing
\[ L^2(\Gamma \setminus G; \varphi) = \sum m_i H_{\pi_i}, \]
and recalling (5.9), one has
\[ \theta^{c^2\lambda_w}(t) = \sum m_i \sum_{\mu} m_{\mu}^{c^2\lambda_w} \dim[H_{\pi_i} \otimes V_{\mu}] K e^{-t\lambda_i} \]
\[ = \sum_{i} e^{-t\lambda_i} m_i e_{M_Q}^{c^2\lambda_w}(\pi_i, V_{\lambda}). \]
Consequently the behavior at infinity is suitable to evaluate a Laplace transform, certainly for \( \Re z + \frac{\mu_t^2}{2} > 0 \). To handle zero modes notice that on \([L^2(\Gamma \backslash G; \varphi) \otimes V_\mu]^K\) the operator \( \Delta_\mu + \frac{1}{2} (\lambda^*, \lambda^* + 2\rho) I - \frac{1}{2} (\mu^*, \mu^* + 2\rho) I \) has finite dimensional kernel, with associated projection operator say \( P_\mu \). Set

\[
\frac{1}{2} \Box^2 \chi_w = \sum \oplus | m_{\mu}^2 \chi_w | (\Delta_\mu + \frac{1}{2} (\lambda^*, \lambda^* + 2\rho) I - \frac{1}{2} (\mu^*, \mu^* + 2\rho) I) (I - P_\mu),
\]

and let \( G^2 \chi_w \) denote the inverse of \( \frac{1}{2} \Box^2 \chi_w \) on \( \sum \oplus | m_{\mu}^2 \chi_w | (I - P_\mu) [L^2(\Gamma \backslash G; \varphi) \otimes V_\mu]^K \), i.e. the sum of operators \( \sum \oplus | m_{\mu}^2 \chi_w | (\Delta_\mu + \frac{1}{2} (\lambda^*, \lambda^* + 2\rho) I - \frac{1}{2} (\mu^*, \mu^* + 2\rho) I)^{-1} \). To subtract the zero modes we use

\[
\theta^2 \chi_w (t) := \text{Tr}_s e^{-\frac{1}{2} \Box^2 \chi_w} = \theta^2 \chi_w (t) - \sum_{\mu} m_{\mu}^2 \chi_w \text{Tr} P_\mu
\]

noting that each \( P_\mu \) has finite rank and only finitely many \( m_{\mu}^2 \chi_w \neq 0 \). Hence

\[
(8.1) \quad Pf \int_0^\infty e^{-tp} \theta^2 \chi_w (t) dt = Pf \int_0^1 e^{-tp} \theta^2 \chi_w (t) dt + \int_1^\infty e^{-tp} \theta^2 \chi_w (t) dt
\]

is defined and meromorphic on \( \mathbb{C} \). Indeed, notice that from the \( Pf \) Laplace transform theory the first integral on the right side gives an entire function in \( p \) and the second is

\[
\text{Tr}_s ((pI + \Box^2 \chi_w)^{-1} e^{-(\Box^2 \chi_w + pI)}) + \frac{N \chi_w^2 e^{-p}}{p}
\]

which is meromorphic in \( p \). Following [M-S;II] we can then introduce the heuristic notation

\[
\text{Tr}_\theta (pI + \Box^2 \chi_w)^{-1} := Pf \int_0^\infty e^{-tp} \text{Tr}_s e^{-t \Box^2 \chi_w} dt = Pf \int_0^\infty e^{-tp} \theta^2 \chi_w (2t) dt,
\]

here we replace the ‘\( t \)’ parameter with ‘\( 2t \)’ as done in §6.

**Lemma 8.1.** \( \text{Tr}_\theta (pI + \Box^2 \chi_w)^{-1} \) is analytic for \( \Re p + \mu_t^2 \chi_w > 0 \) and has meromorphic continuation to \( \mathbb{C} \). Moreover, with \( p = \| (s + c \lambda_w) \alpha_{\mathbb{R}} \|^2 - \| c \lambda_w \alpha_{\mathbb{R}} \|^2 \) we have the expansion
uniformly convergent for $\Re(s^2 + 2sc_{\lambda_w})\|\alpha_\mathbb{R}\|^2 + \frac{1}{2}\mu_b^2 > 0$

\begin{equation}
2\langle\alpha^\nu_{\mathbb{R}}, (s + c_{\lambda_w})\alpha_\mathbb{R}\rangle Pf \int_0^\infty e^{-t\|s + c_{\lambda_w}\alpha_\mathbb{R}\|^2 - \|c_{\lambda_w}\alpha_\mathbb{R}\|^2}\varphi_{\lambda_w}(2t) dt =
\end{equation}

$$2\langle\alpha^\nu_{\mathbb{R}}, (s + c_{\lambda_w})\alpha_\mathbb{R}\rangle Pf \int_0^\infty e^{-t\|s + c_{\lambda_w}\alpha_\mathbb{R}\|^2 - \|c_{\lambda_w}\alpha_\mathbb{R}\|^2} \dim F \ vol X e^{-4t(\lambda, \rho_n)} f_{2t}^{\lambda_w}(e) dt +$$

$$\sum_{[\gamma] \in \mathcal{E}_1(\Gamma)} \text{Tr} \varphi(\gamma) \sum_{W_{c,2}} (-1)^{\ell(w)} \chi \left( \widehat{X_\gamma}, \frac{[W_{\lambda_w}]}{[\wedge_1(N_+(X_\gamma) + \nu_{\mathbb{C}}(X_\gamma) + \nu_{\mathbb{C}}(X_\gamma)*)](\gamma)} \right) \frac{\lambda_\gamma}{\|\alpha_\mathbb{R}\|} e^{-s_\alpha_\mathbb{R}}$$

**Proof.** We begin with a number of remarks. Basic to the proof is the familiar integral

$$e^{-wx^2} = \int_0^\infty e^{-tx} e^{-x^2/4t} dt, \quad \Re w \not\in (-\infty, 0].$$

Notice that with $w^2 = \|s + c_{\lambda_w}\alpha_\mathbb{R}\|^2$ and $x^2 = \|\gamma_\lambda \widehat{X_\kappa}\|^2$ the integral has value

$$e^{-(s + c_{\lambda_w})\ell_\gamma \|\alpha_\mathbb{R}\|}$$

$$\frac{2(s + c_{\lambda_w})\|\alpha_\mathbb{R}\|}{(s + c_{\lambda_w})\|\alpha_\mathbb{R}\|}.$$

From the discussion circa Lemma 4.6 we see that $\|\alpha_\mathbb{R}\|\|X_\kappa\| = 1 = \alpha_\mathbb{R}(X_\kappa)$, so the value of the integral can be written as

$$e^{-(s + c_{\lambda_w})\alpha_\mathbb{R}(\ell_\gamma \widehat{X_\kappa})}$$

\begin{equation}
= \frac{(s + c_{\lambda_w})\alpha_\mathbb{R}(\ell_\gamma \widehat{X_\kappa})}{(s + c_{\lambda_w})\|\alpha_\mathbb{R}\|}.
\end{equation}

Next from (6.5)

$$c_{\lambda_w} \alpha_\mathbb{R} = ((\lambda_w \circ C_\kappa) + \rho_{Q_\kappa})(X_\kappa)\alpha_\mathbb{R}$$

and, as noted in the Remark following (1.16),

$$-c_{\lambda_w} \alpha_\mathbb{R} = -((\lambda_w \circ C_\kappa) + \rho_{Q_\kappa})(X_\kappa)\alpha_\mathbb{R} = ((\lambda_w \circ C_\kappa^{-1}) - \rho_{Q_\kappa})(X_\kappa)\alpha_\mathbb{R}$$

where $e^{(\lambda_w \circ C_\kappa^{-1})(\ell_\gamma \widehat{X_\kappa})}$ is the action on bundles described in (3.7).

Finally, notice that with $\gamma = \gamma_I e^{\ell_\gamma \widehat{X_\kappa}}$ (hopefully there is no confusion with Euler’s constant)

$$\frac{1}{\xi_\rho(\gamma) \prod_{\alpha \notin \Delta I} [1 - \xi - \alpha(\gamma)]} = \frac{(-1)^{q_\gamma} \xi_{\rho_{Q_\kappa}}(\gamma) \xi_{-\rho_{M_\kappa}}(\gamma_I)}{\det(I - \gamma | \nu_{\mathbb{C}}) \det(I - \gamma | u_{\mathbb{C}})} = \frac{(-1)^{q_\gamma} \xi_{-\rho_{M_\kappa}}(\gamma_I) e^{\rho_{Q_\kappa}(X_\kappa)\alpha_\mathbb{R}(\ell_\gamma \widehat{X_\kappa})}}{\det(I - \gamma | \nu_{\mathbb{C}}) \det(I - \gamma | u_{\mathbb{C}})}.$$
On the other hand, the term \( e^{-\rho\alpha_n(X_\gamma) \alpha_{\R}(\ell, \tilde{X}_\gamma)} \) coming from \( c_{\lambda_w} \alpha_{\R} \) will be in the numerator as a result of the integration. Given these remarks, the expansion (8.2) will follow then from (5.11), an interchange of integrals and the topological calculations in §3 resulting in (3.12). We caution the reader that the generic length \( \lambda_\gamma \) is not present in 5.11. So we have inserted \( \lambda_\gamma\lambda_\gamma^{-1} = 1 \) to match with 3.12. This accounts for the \( \lambda_\gamma \) term.

The technique to justify the interchange was introduced in [M-S; I], Proposition 6.1, and can be used here. So we just sketch the argument.

As in (8.1) for fixed \( T > 0 \), we set

\[
I_T = 2\langle \alpha_{\R}^\vee, (s + c_{\lambda_w})\alpha_{\R} \rangle \text{ Pf } \int_0^T e^{-t(\|s + c_{\lambda_w}\alpha_{\R}\|^2 - \|c_{\lambda_w}\alpha_{\R}\|^2)} \theta_{\lambda_w}^2 (2t) \, dt
\]

and

\[
I_\infty = 2\langle \alpha_{\R}^\vee, (s + c_{\lambda_w})\alpha_{\R} \rangle \int_T^\infty e^{-t(\|s + c_{\lambda_w}\alpha_{\R}\|^2 - \|c_{\lambda_w}\alpha_{\R}\|^2)} \theta_{\lambda_w}^2 (2t) \, dt.
\]

Then

\[
I_\infty = 2\langle \alpha_{\R}^\vee, (s + c_{\lambda_w})\alpha_{\R} \rangle \text{ Tr}_s ((s^2 + 2sc_{\lambda_w})\|\alpha_{\R}\|^2 I + \Box_{\lambda_w}^2) - \frac{1}{2} e^{-T(\Box_{\lambda_w}^2 + \|s + c_{\lambda_w}\alpha_{\R}\|^2 - \|c_{\lambda_w}\alpha_{\R}\|^2)}
\]

Since \( \Re(s^2 + 2sc_{\lambda_w})\|\alpha_{\R}\|^2 + \frac{1}{2} \mu_b^2_{\lambda_w} > 0 \), the Green’s operator \( G_{\lambda_w}^2 \) is smoothing. Also, as explained in §5, the operator \( H_{\varphi, t}^{\lambda_w} \) is a virtual direct sum of heat kernels \( h_{\mu, t} \) which at time \( T \), are certainly in \( [S(G) \otimes \text{End} (V_\mu)]^{K_M \times K_M} \) times a smoothing operator. Thus the local trace of the Schwartz kernel is in \( S(G) \), hence admissible. Finally, since trace class operators are an ideal and the Green’s operator is bounded, the operator is trace class. Since the Green’s operator acts by a scalar in any irreducible representation, the invariant Fourier transform of the operator will factor in such a way that we may use Lemma 5.1 and Lemma 5.2 to obtain that \( I_\infty \) will have the absolutely convergent orbital integral expansion

(8.3)

\[
I_\infty = \sum_{[\gamma] \in \xi_1(t)} \text{ Tr } \varphi(\gamma) N_{\gamma}^{\lambda - 1} c_{\gamma}^{-1} \frac{\text{ vol } (\Gamma_\gamma \backslash G_\gamma)}{\prod \left[ 1 - \xi_{\gamma}(\alpha_\gamma) \right]} \sum_{W_{\lambda, w}} (-1)^{\ell(w)} \sum_{W_{(t_n)}} \varepsilon(x) \tilde{\omega}_\gamma(x \cdot (\lambda_w + \rho(I_n)) \xi_{\gamma}(x \cdot (\lambda_w + \rho(I_n)) \gamma t)
\]

\[
\int_{\alpha_{\gamma}^\vee} 2\langle \alpha_{\R}^\vee, (s + c_{\lambda_w})\alpha_{\R} \rangle \frac{\text{ vol } (\Gamma_\gamma \backslash G_\gamma)}{\prod \left[ 1 - \xi_{\gamma}(\alpha_\gamma) \right]} \frac{1}{\| (s + c_{\lambda_w})\alpha_{\R} \|^2 + \|\nu\|^2} e^{-T(\|s + c_{\lambda_w}\alpha_{\R}\|^2 + \|\nu\|^2)} e^{i\nu(\gamma \cdot X_\nu)} \, d\nu
\]

+ identity term.
For the $I_T$ term, Lemma 5.6 shows that if the contribution of the identity term is omitted from $\theta^{c_2 \lambda_w}(2t)$, then the resulting series is uniformly, absolutely convergent. Hence the finite integral in $t$ may be interchanged with the summation giving

\begin{equation}
I_T = \sum_{[\gamma] \in E_1(\Gamma)} \text{Tr} \varphi(\gamma) \frac{N_{\gamma}^{-1} c_{\gamma}^{-1}}{\xi(\gamma)} \frac{\text{vol}(\Gamma \backslash G)}{\prod_{\alpha \in P_{\gamma}} [1 - \xi_{-\alpha}(\gamma)]}
\end{equation}

\begin{align*}
&+ \sum_{W(1_\mu)} (-1)^{\ell(w)} \sum_{W(1_\nu)} \varepsilon(x) \overline{\omega}_\gamma(x \cdot (\lambda_w + \rho(1_\nu))) \xi(x(\lambda_w + \rho(1_\nu))) (\gamma T) \\
&\int_0^T 2(\alpha_R, (s + c_{\lambda_w}) \alpha_R) e^{-\|\ell_{\gamma} \overline{X}_\kappa\|^2/4t} (4\pi t)^{1/2} e^{-t\|s+c_{\lambda_w} \alpha_R\|^2} dt \\
&+ \text{identity term}.
\end{align*}

The expansion in (8.2) results from interchanging the $t$ and $\nu$ in (8.3) and then adding the absolutely convergent expansions (8.3) and (8.4). Recalling the definition of $c_{\lambda_w}$ (6.5) and comparison with the geometric calculation gives the result. Linearity of the finite part gives the identity term.

It is time to nail down the various constants that have not yet been specified. This will be done by matching asymptotic expansions. Recall that

$$\frac{1}{2} \Box_{+}^{c_2 \lambda_w} = \sum_{\mu} \bigoplus m_{\mu}^{c_{\lambda_w}^2} |(\Delta_{\mu} + \frac{1}{2} \langle \lambda^*, \lambda^* + 2\rho \rangle - \frac{1}{2} \langle \mu^*, \mu^* + 2\rho \rangle) (I - P_{\mu})|,$$

and set

$$\log \text{det}_{\zeta} \frac{1}{2} \Box_{+}^{c_2 \lambda_w} = \sum_{\mu} \bigoplus m_{\mu}^{c_{\lambda_w}^2} \log \text{det}_{\zeta} [(\Delta_{\mu} + \frac{1}{2} \langle \lambda^*, \lambda^* + 2\rho \rangle - \frac{1}{2} \langle \mu^*, \mu^* + 2\rho \rangle) (I - P_{\mu})].$$

Recall that $G_{+}^{c_2 \lambda_w}$ is the inverse of $\frac{1}{2} \Box_{+}^{c_2 \lambda_w}$ on $\sum \bigoplus (I - P_{\mu})$ $[L^2(\Gamma \backslash G; \varphi) \otimes V_{\mu}]$. (The annoying $\frac{1}{2}$ will disappear when we use $2t$ as before.) We subtract the zero modes as before

$$\theta_{+}^{c_2 \lambda_w}(t) = \theta^{c_2 \lambda_w}(t) - \sum_{\mu} m_{\mu}^{c_{\lambda_w}^2} \text{Tr} P_{\mu}$$

$$= \theta_{+}^{c_2 \lambda_w}(t) - N^{c_2 \lambda_w}.$$

The symbolic calculus of $Pf$ Laplace transforms gives the relationship

$$\log \text{det}_{\theta}(I + zG_+^{c_2 \lambda_w}) = -Pf \int_0^\infty e^{-tz} \theta_{+}^{c_2 \lambda_w}(2t) \frac{dt}{t} + C.$$
To determine the constant $C$, notice that
\[
\theta^\ell_{+w}(2t) = \theta^\ell_{+w}(2t) - N\ell_{+w}^2
\]
\[
= \dim F \vol X \ e^{-4t(\lambda, \rho_n)} f_{2t}^{\ell_{+w}}(e) - N\ell_{+w}^2
\]
\[
+ [\theta^\ell_{+w}(2t) - \dim F \vol X \ e^{-4t(\lambda, \rho_n)} f_{2t}^{\ell_{+w}}(e)],
\]
and from Lemma 5.6 one finds the term in brackets is $O(e^{-c/t})$, $c > 0$, for $t \to 0^+$. Hence the asymptotic expansions, $t \to 0^+$, of $\theta^\ell_{+w}(2t)$ and $\dim F \vol X \ e^{-4t(\lambda, \rho_n)} f_{2t}^{\ell_{+w}}(e) - N\ell_{+w}^2$ agree. Then the asymptotic expansions, for large $z$, of $Pf \int_0^\infty e^{-tz} \frac{\theta^\ell_{+w}(2t)}{t} dt$ and $Pf \int_0^\infty e^{-tz} \frac{\dim F \vol X \ e^{-4t(\lambda, \rho_n)} f_{2t}^{\ell_{+w}}(e) - N\ell_{+w}^2}{t} dt$ must be equal, and also agree up to the constant with that of $-\log \det \theta(I + zG^2_{+w})$.

In Proposition 6.3 we see that only integral powers of $t$ occur in the asymptotics as $t \to 0^+$. From this and from (1.12) in [M-S;II] it follows that $z \to \infty$
\[
\log \det \theta(I + zG^2_{+w}) \sim \sum_{k=1}^{m+1} \sum_{W_{c,2}} (-1)^{\ell(w)} e(\xi, \lambda_w) c_{-k}^w \left( \gamma + \log z - \sum_{j=1}^{k} \frac{1}{j} \right) \left( -z \right)^k \frac{1}{k!}
\]
\[
+ \sum_{W_{c,2}} (-1)^{\ell(w)} e(\xi, \lambda_w) c_0^w \log z - \log \det \zeta \Box_{+w}^2 ;
\]
in particular, the only coefficient of a term of the form $\log^n z$ is $\sum_{W_{c,2}} (-1)^{\ell(w)} e(\xi, \lambda_w) c_0^w$.

Hence from the symbolic calculus for antiderivatives we get
\[
\text{(8.5)}
\]
\[
Pf \int_0^\infty e^{-tz} \frac{\theta^\ell_{+w}(2t)}{t} dt = -\log \det \theta(I + zG^2_{+w}) - \gamma \sum_{W_{c,2}} (-1)^{\ell(w)} e(\xi, \lambda_w) c_0^w - \log \det \zeta \Box_{+w}^2 .
\]

From Proposition 6.3 and (6.7) we see that the coefficient of $t^0$ of the small $t$ expansion of $\dim F \vol X \ e^{-4t(\lambda, \rho_n)} f_{2t}^{\ell_{+w}}(e) - N\ell_{+w}^2$ is
\[
\chi_a(X, \mathbb{F})(M_{2}^{\ell_{+w}} + \sum_{W_{c,2}} (-1)^{\ell(w)} e(\xi, \lambda_w) c_0^w) - N\ell_{+w}^2 .
\]

As the small $t$ expansion of $\theta^\ell_{+w}(2t)$ and $\dim F \vol X \ f_{2t}^{\ell_{+w}}(e) - N\ell_{+w}^2$ agree, then as explained in §7, we obtain
\[
\text{(8.6)}
\]
\[
\sum_{W_{c,2}} (-1)^{\ell(w)} e(\xi, \lambda_w) c_0^w = \chi_a(X, \mathbb{F})(M_{2}^{\ell_{+w}} + \sum_{W_{c,2}} (-1)^{\ell(w)} e(\xi, \lambda_w) c_0^w) - N\ell_{+w}^2 .
\]
Definition 8.3. For \( z \in \mathbb{C} \) with \( \Re(\|z\alpha_R\|^2) - \|c_{\lambda_w} \alpha_R\|^2 + \frac{1}{2} \mu_b > 0 \) and \( \Re z \not\in (-\infty, 0] \) define \( \log Z_{\varphi}^{c_{\lambda_w}}(z\alpha_R) \) by the absolutely convergent series

\[
\log Z_{\varphi}^{c_{\lambda_w}}(z\alpha_R) = - \sum_{[\gamma] \in \mathfrak{E}_1(\Gamma)} \text{Tr} \varphi(\gamma) \sum_{W_{c_{\lambda_w}}} (-1)^{\ell(w)} \chi(\tilde{X}_\gamma, -\frac{1}{2}(N_+(\tilde{X}_\gamma)^* \oplus \mathcal{V}_C(\tilde{X}_\gamma)^* \oplus \mathcal{U}_C(\tilde{X}_\gamma)^*))_{\gamma}(\gamma) e^{-(z-c_{\lambda_w})\alpha_R(2i, \tilde{X}_\alpha)}.
\]

Remark. With the substitution \( z \mapsto s + c_{\lambda_w} \), the convergence in this domain follows directly from Lemma 8.1. Also, as the conjugacy classes are bounded away from zero, \( \lim_{z \to +\infty} \log Z_{\varphi}^{c_{\lambda_w}}(z\alpha_R) = 0 \). Moreover, when \((\tau, V_\lambda)\) is the trivial representation of \( K \) and \( G = SL(2, \mathbb{R}) \), the function \( Z_{\varphi}^{c_{\lambda_w}} \) becomes Selberg’s zeta function.

Theorem 8.4. (i) \( Z_{\varphi}^{c_{\lambda_w}} \) has a meromorphic continuation to \( \mathbb{C} \alpha_R \) given by

\[
Z_{\varphi}^{c_{\lambda_w}}(z\alpha_R) = \mathcal{B}(\tilde{X}, \mathbb{W}_{c_{\lambda_w}}^{c_{\lambda_w}}, z\alpha_R)^{-X_a(X,F)} \det \chi(\tilde{X}_\gamma, -\frac{1}{2}(N_+(\tilde{X}_\gamma)^* \oplus \mathcal{V}_C(\tilde{X}_\gamma)^* \oplus \mathcal{U}_C(\tilde{X}_\gamma)^*))_{\gamma}(\gamma) e^{-\gamma(c_{\lambda_w}z + \alpha_R)} (I + (\|z\alpha_R\|^2 - \|c_{\lambda_w} \alpha_R\|^2) G_{+}^{c_{\lambda_w}} \alpha_R) \]

(ii) \( Z_{\varphi}^{c_{\lambda_w}} \) satisfies the functional equation

\[
Z_{\varphi}^{c_{\lambda_w}}(-z\alpha_R) = Z_{\varphi}^{c_{\lambda_w}}(z\alpha_R) e^{-2\pi X_a(X,F) \sum_{W_{c_{\lambda_w}}} (-1)^{\ell(w)} \int_0^\infty \rho_{c_{\lambda_w}}^\chi(x, \alpha_R^2) \left\{ \frac{\tan \pi x}{\cot \pi x} \right\} dx}.
\]

Proof. (i) From the definition of \( \log Z_{\varphi}^{c_{\lambda_w}} \) and Lemma 8.1 we get for \( \Re(\|z\alpha_R\|^2) - \|c_{\lambda_w} \alpha_R\|^2 + \frac{1}{2} \mu_b > 0 \)

\[
\frac{d}{dz} \log Z_{\varphi}^{c_{\lambda_w}}(z\alpha_R) = 2\langle \alpha_R^V, z\alpha_R \rangle \text{Tr} \theta((\|z\alpha_R\|^2 - \|c_{\lambda_w} \alpha_R\|^2) I + \Box_{c_{\lambda_w}}^{c_{\lambda_w}})^{-1}
\]

\[
-2\langle \alpha_R^V, z\alpha_R \rangle Pf \int_0^\infty e^{-t(\|z\alpha_R\|^2 - \|c_{\lambda_w} \alpha_R\|^2)} \dim F \text{vol} X e^{-4t(<\lambda, \rho_n)} f_{2t}^{c_{\lambda_w}}(e) dt
\]

\[
= 2\langle \alpha_R^V, z\alpha_R \rangle \text{Tr} \theta((\|z\alpha_R\|^2 - \|c_{\lambda_w} \alpha_R\|^2) I + \Box_{+}^{c_{\lambda_w}})^{-1}
\]

\[
-2\langle \alpha_R^V, z\alpha_R \rangle Pf \int_0^\infty e^{-t(\|z\alpha_R\|^2 - \|c_{\lambda_w} \alpha_R\|^2)} \left[ \dim F \text{vol} X e^{-4t(<\lambda, \rho_n)} f_{2t}^{c_{\lambda_w}}(e) - N^{c_{\lambda_w}} \right] dt
\]
The definition of \( \det_\theta \) and the symbolic calculus gives
\[
\frac{d}{dz} \log Z_{\varphi}^{\ell_+^2}(z \alpha_\mathbb{R}) = \frac{d}{dz} \log \det_\theta \left( I + (\|z \alpha_\mathbb{R}\|^2 - \|c_{\lambda_\omega} \alpha_\mathbb{R}\|^2)G_+^{\ell_+^2} \right) \\
+ \frac{d}{dz} Pf \int_0^\infty e^{-t(\|z \alpha_\mathbb{R}\|^2 - \|c_{\lambda_\omega} \alpha_\mathbb{R}\|^2)} \frac{\dim F \vol X e^{-4t(\lambda, \rho_w)} f_{2t}^{\ell_+^2}(\epsilon) - N_c^{\ell_+^2} \text{d}t}{t}
\]
Then
\[
\log Z_{\varphi}^{\ell_+^2}(z \alpha_\mathbb{R}) = \log \det_\theta \left( I + (\|z \alpha_\mathbb{R}\|^2 - \|c_{\lambda_\omega} \alpha_\mathbb{R}\|^2)G_+^{\ell_+^2} \right) \\
+ Pf \int_0^\infty e^{-t(\|z \alpha_\mathbb{R}\|^2 - \|c_{\lambda_\omega} \alpha_\mathbb{R}\|^2)} \frac{\dim F \vol X e^{-4t(\lambda, \rho_w)} f_{2t}^{\ell_+^2}(\epsilon) - N_c^{\ell_+^2} \text{d}t}{t} + C
\]
As \( \lim_{z \to +\infty} \log Z_{\varphi}^{\ell_+^2}(z \alpha_\mathbb{R}) = 0 \), using (8.5) and the fact that the large \( p \) asymptotics of
\[
Pf \int_0^\infty e^{-tp} \frac{\theta_+^{\ell_2}(2t)}{t} \text{d}t \quad \text{and} \quad Pf \int_0^\infty e^{-tp} \frac{\dim F \vol X e^{-4t(\lambda, \rho_w)} f_{2t}^{\ell_+^2}(\epsilon) - N_c^{\ell_+^2} \text{d}t}{t}
\]
are equal we find
\[
C = \gamma \sum_{W_{c_2}} (-1)^{\ell(w)} e(\xi, \lambda_w)c_{0}^{w} + \log \det_\zeta \Box_+^{\ell_+^2}
\]
Hence
\[
\log Z_{\varphi}^{\ell_+^2}(z \alpha_\mathbb{R}) = \log \det_\theta \left( I + (\|z \alpha_\mathbb{R}\|^2 - \|c_{\lambda_\omega} \alpha_\mathbb{R}\|^2)G_+^{\ell_+^2} \right) + \log \det_\zeta \Box_+^{\ell_+^2} \\
+ \gamma \sum_{W_{c_2}} (-1)^{\ell(w)} e(\xi, \lambda_w)c_{0}^{w} + Pf \int_0^\infty e^{-tp} \frac{\dim F \vol X e^{-4t(\lambda, \rho_w)} f_{2t}^{\ell_+^2}(\epsilon) - N_c^{\ell_+^2} \text{d}t}{t}
\]
Then from Proposition 7.5 and (8.6) we obtain
\[
\log Z_{\varphi}^{\ell_+^2}(z \alpha_\mathbb{R}) = \log \det_\theta \left( I + (\|z \alpha_\mathbb{R}\|^2 - \|c_{\lambda_\omega} \alpha_\mathbb{R}\|^2)G_+^{\ell_+^2} \right) + \log \det_\zeta \Box_+^{\ell_+^2} \\
+ \gamma \chi_\alpha(X, \mathbb{F})(M_2^{\ell_+^2} + \sum_{W_{c_2}} (-1)^{\ell(w)} e(\xi, \lambda_w)c_{0}^{w}) - \gamma N_c^{\ell_+^2} - \log \tilde{B}(\tilde{X}, \mathbb{W}_{c_2}^{\ell}, z \alpha_\mathbb{R}) \chi_\alpha(X, \mathbb{F}) \\
+ \log(\|z \alpha_\mathbb{R}\|^2 - \|c_{\lambda_\omega} \alpha_\mathbb{R}\|^2)N_c^{\ell_+^2} + \gamma N_c^{\ell_+^2}
\]
The result follows then from the formula relating \( \tilde{B} \) with \( B \).

(ii) This functional equation is an obvious consequence of the one for \( \tilde{B} \) given in Proposition 7.5.
§9 Geometric Zeta Function

We may now consider the Weil-type zeta function, \( Z_{V,F}(z^{\alpha_R}) \), and examine its behavior for small \( z \). We remind the reader that we have the ongoing assumption that \( G \) has only one class of cuspidal maximal parabolic subgroups and we have an appropriate form of the Ansatz.

Recall from §4 (4.2) the holomorphic torsion theta function given by

\[
\theta_{\lambda,\phi}(t) = \sum_{q=0}^{n} (-1)^q \text{Tr} e^{-t \Box^q_0},
\]

and the associated combination of heat kernels (4.3)

\[
\tilde{k}_t^\lambda = \sum_{q=0}^{n} (-1)^q \text{tr} \tilde{h}_t^q.
\]

For each \( q \) let \( \zeta_q(s) \) be the zeta function for \( \text{Tr} e^{-t \Box^q_0} \) which, of course, is computed from \( \text{tr} \tilde{h}_t^q \). The holomorphic torsion is defined by

\[
\log \tau(X,p-,V) = \frac{1}{2} \sum_{q \geq 0} (-1)^q \zeta_q'(0).
\]

In this section we shall prove

**Theorem 9.1.**

\[
Z_{V,F}(z^{\alpha_R}) = \exp - \sum_{[\gamma] \in \mathcal{E}_1(\Gamma)} \text{Tr} \varphi(\gamma) \chi_{\text{mix}} \left( \hat{X}_\gamma, \frac{[V_\gamma](\gamma)}{[\Lambda_-(N_+^+(\hat{X}_\gamma)^* \oplus V_+(\hat{X}_\gamma)^* \oplus U_c(\hat{X}_\gamma)^*)](\gamma)} \right) e^{-\frac{1}{2} \mu_\gamma}.
\]

is analytic for \( \Re z^2 > \max_{w \in W_\infty} \|c_{\lambda_w} \alpha_R\|^2 + \frac{1}{2} \mu_{\lambda_b}^2 \) and \( z \notin (-\infty,0] \). \( Z_{V,F} \) has meromorphic continuation to \( \mathbb{C} \) by

\[
Z_{V,F}(z^{\alpha_R}) = \prod_{\lambda} \frac{Z_{\phi,\nu}(z + c_{\lambda_w} \alpha_R)}{Z_{\phi,\nu}(c_{\lambda_w} \alpha_R)}.
\]

Also for \( z \) near zero

\[
Z_{V,F}(z^{\alpha_R}) \sim \tau(X,p-,V)^2 k z^\nu
\]
where
\[ \tau(X, p_-, V) \] is the holomorphic torsion of \( V \);
\[ k = \prod_{c^2_{\lambda_w}} \left\{ 2^{(n_{e,F} + n_{e,E_2})} \det_{e,E_2} \Delta_{e,F}^{\lambda_w} \det_{e,F} \Delta_{e,F}^{\lambda_w} \right\}^{(-1)^{\ell(w)+1}} \prod_{W_{c^2}} \{ \det_{e,F} \Delta_{e,F}^{\lambda_w} e^{\gamma e}(\xi, \lambda_w) \} (-1)^{\ell(w)} \]
and
\[ \nu = \sum_{c^2_{\lambda_w}} \left\{ (n_{e,F} + n_{e,E_2})(-\chi_a(X, F)) - N c^2_{\lambda_w} \right\} \]

Proof. For \( \Re z^2 \) suitably large, using Prop. 3.5, (3.12) and Def. 8.3 we have
\[
\log Z_{V,F}(z\alpha_R) = - \sum_{[\gamma] \in \mathcal{E}_1(\Gamma)} \text{Tr} \varphi(\gamma) \chi \left( X_\gamma, \frac{[V_\gamma](\gamma)}{[\Lambda_{-1}(N_+(X_\gamma) \oplus V_+(X_\gamma)^* \oplus U_\mathbb{C}(X_\gamma)^*)](\gamma)} \right) e^{-z\alpha_R} \frac{e^{-z\alpha_R}}{\mu} \\
= - \sum_{[\gamma] \in \mathcal{E}_1(\Gamma)} \text{Tr} \varphi(\gamma) \sum_{W_{c^2}} (-1)^{\ell(w)+1} \chi \left( X_\gamma, \frac{[W_{\lambda_w}](\gamma)}{[\Lambda_{-1}(N_+(X_\gamma) \oplus V_+(X_\gamma)^* \oplus U_\mathbb{C}(X_\gamma)^*)](\gamma)} \right) e^{-z\alpha_R} \frac{e^{-z\alpha_R}}{\mu} \\
= \sum_{c^2_{\lambda_w}} \log Z_{\lambda_w}^{c^2_{\lambda_w}} ((z + c_{\lambda_w})\alpha_R) .
\]
Hence
\[ Z_{V,F}(z\alpha_R) = \prod_{c^2} Z_{\varphi}^{c^2_{\lambda_w}} ((z + c_{\lambda_w})\alpha_R) , \]
and since each factor is meromorphic on \( \mathbb{C} \) by the previous theorem, we obtain the result for \( Z_{V,F}(z\alpha_R) \).

For the leading behaviour of \( Z_{V,F}(z\alpha_R) \) for \( z \to 0 \) we shall collect various facts from earlier sections as well as recall the technique from [M-S;II]. First we handle the scalar factor \( e^{2t(\lambda, \rho_n)} \). From Lemma 4.6 we have
\[
\tilde{k}_t^\lambda(e) = \sum_{\omega \in \mathcal{E}_2(G)} e^{\frac{1}{2}(\chi_\omega(\Omega_G) - \langle \lambda^*, \lambda^* + 2\rho \rangle)} d_{\pi, e^t} e'(\omega, V_\lambda, p_-) \\
+ e^{2t(\lambda, \rho_n)} \sum_{W_{c^2}} (-1)^{\ell(w)+1} \int_{\alpha^*_n} e(\xi, \lambda_w) e^{-\frac{e_{\lambda_w}^2}{2} w^*} e^{-\frac{1}{2} w^* \mu^2} d\mu (w^* (\lambda^* + \rho - 2\rho_n) |_{\alpha^*_n}) dw .
\]
Using the relationship $\|\lambda^* + \rho\|^2 + 4\langle\lambda, \rho_n\rangle = \|\lambda + \rho\|^2$ in the discrete series contribution above we obtain in each term for $\theta_{\lambda,\phi}(t)$ in Theorem 4.7 the factor $e^{2t\langle\lambda, \rho_n\rangle}$. Thus we have (9.1)

$$e^{-2t\langle\lambda, \rho_n\rangle} \theta_{\lambda,\phi}(t) = \sum_{q=0}^{n} (-1)^q q \operatorname{Tr} e^{-t\{\square^{0,q} + \langle\lambda, 2\rho_n\rangle I\}}$$

$$= \sum_{[\gamma] \in \mathcal{E}_1(\Gamma)} \operatorname{Tr} \varphi(\gamma) \operatorname{vol}(\Gamma \gamma \setminus G) \frac{e^{-\|\log \tau_\gamma\|^2/2t}}{(2\pi t)^{1/2}} \frac{N_\gamma^{-1} c^{-1}_\gamma}{\xi_\rho(\gamma) \prod_{\alpha \in P_c^\gamma} [1 - \xi - \alpha(\gamma)]} \sum_{W_{\lambda\omega}} (-1)^{\ell(w)} e^{-\frac{1}{2} \epsilon^2_{\lambda \omega}} \sum_{W(I_n)} \epsilon(s) \tilde{\omega}_\gamma(s \cdot (\lambda_\omega + \rho(I_n))) \tilde{\gamma}(\lambda_\omega + \rho(I_n))(\gamma I)$$

$$+ \operatorname{dim} F \operatorname{vol} X \sum_{\omega \in \mathcal{E}_2(G)} d_{\pi} e'(\omega, V_\chi, p_\omega) e^{\frac{1}{2} \epsilon^2} (\chi_{\omega}(\Omega_G) - \langle\lambda, \lambda + 2\rho\rangle)$$

$$+ \operatorname{dim} F \operatorname{vol} X \sum_{W_{\lambda\omega}} (-1)^{\ell(w)+1} \int_{a_{\lambda}} \epsilon(\xi, \lambda_\omega) e^{-\frac{1}{2} \epsilon^2_{\lambda \omega}} e^{-\frac{1}{2} \nu^2} \frac{d\mu}{d\nu} (w^*(\lambda^* + \rho - 2\rho_n)|_{I_n})$$

We compute $\sum_{W_{\lambda\omega}}$ by first doing $\sum_{W_c}$ and then $\sum_{c_2} \epsilon^2_{\lambda \omega}$.

(9.2)

$$e^{-2t\langle\lambda, \rho_n\rangle} \theta_{\lambda,\phi}(t) = \sum_{q=0}^{n} (-1)^q q \operatorname{Tr} e^{-t\{\square^{0,q} + \langle\lambda, 2\rho_n\rangle I\}}$$

$$= \sum_{[\gamma] \in \mathcal{E}_1(\Gamma)} \operatorname{Tr} \varphi(\gamma) \operatorname{vol}(\Gamma \gamma \setminus G) \frac{e^{-\|\log \tau_\gamma\|^2/2t}}{(2\pi t)^{1/2}} \frac{N_\gamma^{-1} c^{-1}_\gamma}{\xi_\rho(\gamma) \prod_{\alpha \in P_c^\gamma} [1 - \xi - \alpha(\gamma)]} \sum_{W_{\lambda\omega}} (-1)^{\ell(w)+1} e^{-\frac{1}{2} \epsilon^2_{\lambda \omega}} \sum_{W(I_n)} \epsilon(s) \tilde{\omega}_\gamma(s \cdot (\lambda_\omega + \rho(I_n))) \tilde{\gamma}(\lambda_\omega + \rho(I_n))(\gamma I)$$

$$+ \operatorname{dim} F \operatorname{vol} X \sum_{\omega \in \mathcal{E}_2(G)} d_{\pi} e'(\omega, V_\chi, p_\omega) e^{\frac{1}{2} \epsilon^2} (\chi_{\omega}(\Omega_G) - \langle\lambda, \lambda + 2\rho\rangle)$$

$$+ \operatorname{dim} F \operatorname{vol} X \sum_{c_2} \sum_{W_{c_2}} (-1)^{\ell(w)+1} \int_{a_{\lambda}} \epsilon(\xi, \lambda_\omega) e^{-\frac{1}{2} \epsilon^2_{\lambda \omega}} e^{-\frac{1}{2} \nu^2} \frac{d\mu}{d\nu} (w^*(\lambda^* + \rho - 2\rho_n)|_{I_n})$$

A comparison of $e^{-2t\langle\lambda, \rho_n\rangle} \theta_{\lambda,\phi}(t)$ in (9.2) with $\sum_{c_2} \epsilon^2_{\lambda \omega}(t)$ as given in (5.11) and (5.10)
shows that the two agree. From this it follows that we have
\[
\sum_{q \geq 0} (-1)^q q \zeta'_q(0) = \sum_{W_u} (-1)^{\ell(u)+1} \log \det \zeta \Box^\lambda_w.
\]
Returning to the derivation of the leading behaviour, we substitute from (8.7) then use Proposition 7.5

\[
Z_{\nu, \varphi}(z \omega) = \prod_{c^2_{\lambda u}} Z_{\varphi}^{c^2_{\lambda u}} ((z + c_{\lambda u}) \omega) = \prod_{c^2_{\lambda u}} Z_{\varphi}^{c^2_{\lambda u}} ((2z + 2c_{\lambda u}) \frac{\alpha \omega}{2})
\]

\[
= \prod_{c^2_{\lambda u}} \left\{ \mathcal{B} \left( \tilde{\mathcal{X}}, \mathcal{W}_{c^2_{\lambda u}}, (2z + 2c_{\lambda u}) \frac{\alpha \omega}{2} \right) \right\} - \chi_a(X, F) \det \zeta \Box^\lambda w
\]

\[
= \prod_{c^2_{\lambda u}} \left\{ \mathcal{B} \left( \tilde{\mathcal{X}}, \mathcal{W}_{c^2_{\lambda u}}, (2z + 2c_{\lambda u}) \frac{\alpha \omega}{2} \right) \right\} - \chi_a(X, F)
\]

\[
\det \zeta \Box^\lambda w \left\{ \mathcal{B} \left( \tilde{\mathcal{X}}, \mathcal{W}_{c^2_{\lambda u}}, (2z + 2c_{\lambda u}) \frac{\alpha \omega}{2} \right) \right\}
\]

\[
\sim \prod_{c^2_{\lambda u}} \left\{ \mathcal{B} \left( \tilde{\mathcal{X}}, \mathcal{W}_{c^2_{\lambda u}}, (2z + 2c_{\lambda u}) \frac{\alpha \omega}{2} \right) \right\} - \chi_a(X, F)
\]

\[
\sim \tau(X, p_{-}, \mathcal{V})^2 \prod_{c^2} \left\{ \mathcal{B} \left( \tilde{\mathcal{X}}, \mathcal{W}_{c^2_{\lambda u}}, (2z + 2c_{\lambda u}) \frac{\alpha \omega}{2} \right) \right\} - \chi_a(X, F)
\]

\[
\nu \prod_{c^2_{\lambda u}} \left\{ \mathcal{B} \left( \tilde{\mathcal{X}}, \mathcal{W}_{c^2_{\lambda u}}, (2z + 2c_{\lambda u}) \frac{\alpha \omega}{2} \right) \right\}
\]

Here \( \det \zeta \Box^\lambda_{w, P} = \prod_{W_{c^2_{\lambda u}}} \{ \det \zeta \Delta_{\lambda_{w, P}} e^{\gamma e(\xi, \lambda_w) \zeta_{w}} \} \) and

\[
\nu = \sum_{c^2_{\lambda u}} \left\{ (n_{c^2_{\lambda u}} + n_{c^2_{\lambda u}}) \right\} - \chi_a(X, F) - N c^2_{\lambda w},
\]
and we recall that $n_{e,\mathcal{E}_2}^\lambda$ is the dimension of the “zero modes” in the discrete series contribution to $f_t^{\mathcal{E}_2}(e)$, i.e. those $\omega$ with $-\|\Lambda \omega\|^2 + \|\lambda + \rho\|^2 = 0$; and $n_{e,\mathcal{P}}^\lambda$ is given by the Plancherel density in Lemma 7.2, and $N^{\mathcal{E}_2}$ are the “zero modes” coming from $\Box^\lambda$.

\textbf{Remarks.} (1) Using the results from \S\n1, e.g. Proposition 1.16, one can show that, up to a universal constant, $\prod_{e^2} \{\det \zeta \Delta_{e,\mathcal{E}_2}^{\mathcal{E}_2} \det \zeta \Delta_{e,\mathcal{P}}^{\mathcal{E}_2}\}$ is $\tau(X^d, \mathfrak{p}_-, \mathcal{V})^2$, the holomorphic torsion of the compact dual symmetric space.

(2) Concerning the exponent $\nu$, the discrete series is again somewhat problematic, in that $n_{e,\mathcal{E}_2}^\lambda$ might well be zero. Similarly the use of the Ansatz contributes the term $N^{\mathcal{E}_2}$ which might also be specious. Hence it is tempting to conjecture that the degree of $Z_{\mathcal{V},\mathcal{F}}(z\alpha_{\mathbb{R}})$ at zero is $\left\{\sum n_{e,\mathcal{P}}^\lambda\right\} \{\chi_a(X, \mathcal{F})\}$, and thus given by the Plancherel density. In particular $\nu$ would be an integral multiple of the arithmetic genus.

We shall now employ the Bismut-Gillet-Soulé anomaly formula for the Quillen metric [B-G-S, III], to express the order of the zero of the zeta function $Z_{\mathcal{V}}(z)$ at $z = 0$ in terms of characteristic numbers of $X$ with coefficients in $\mathcal{V}$. The local system $\mathcal{F}$ is unimportant here, and we remove it from consideration.

Let us recall some standard notation (comp. [B-G-S, III]). The Todd polynomial on the algebra of complex $(n, n)$ matrices is, by definition

$$Td(A) = det \frac{A}{I - e^{-A}}.$$  

Now if $R$ denotes the curvature of the Levi-Civita connection of $X$, we define the corresponding \textit{Todd form} by

$$\mathcal{T}d(R) = Td(-\frac{R}{2\pi i})$$

and the \textit{derived Todd form} by

$$\mathcal{T}d'(R) = \left. \frac{d}{db} \mathcal{T}d(R + bI) \right|_{b=0}.$$
Also, with $L_V$ denoting the curvature of the canonical connection on $V$, we set

$$\text{ch}(L_V) = \text{Tr} \left( e^{-\frac{1}{2\pi} L_V} \right).$$

Let now $\Lambda$ denote the determinant line associated to $X$ and $V$, more exactly

$$\Lambda = \prod_{0 \leq q \leq n} (\det H^{0,q}(X, V))^{-1}, \quad n = \dim_{\mathbb{C}} X$$

and let $| \cdot |_{\Lambda,g}$, resp. $\| \cdot \|_{\Lambda,g}$, denote the $L^2$-metric, resp. the Quillen metric on the determinant bundle corresponding to the fixed Riemannian metric $g$ on $X$. Thus,

$$\| \cdot \|_{\Lambda,g} = | \cdot |_{\Lambda,g} T_g(X, V)$$

where $\tau_g(X, V)$ is the holomorphic torsion relative to the metric $g$.

Scaling $g$ to $\epsilon g$, with $\epsilon > 0$, has the effect of multiplying the volume form by $\epsilon^n$. Therefore, the $L^2$-metric of $\Lambda$ scales as follows:

$$\text{Log} \| \cdot \|^2_{\Lambda,\epsilon g} = n \text{Log} \epsilon \sum_{0 \leq q \leq n} (-1)^q h^{0,q}(X, V) + \text{Log} | \cdot |_{\Lambda,g} = n \text{Log} \epsilon \chi_a(X, V) + \text{Log} | \cdot |^2_{\Lambda,g}.$$  

For the Quillen metric, the generalized anomaly formula [B-G-S III; Theorem 1.22] applied to the 1-parameter family of metrics $g_\epsilon = \epsilon g$ gives

$$\text{Log} \| \Lambda \|^2_{\epsilon g} = \text{Log} \epsilon \cdot \int_X \mathcal{T}d'(R) \text{ch}(L_V) + \text{Log} \| \Lambda \|^2_g.$$  

It follows that the torsion changes according to the formula

$$\text{Log} \tau_{\epsilon g}(X, V)^2 - \text{Log} \tau_g(X, V)^2 = \text{Log} \epsilon \left( \int_X \mathcal{T}d'(R) \text{ch}(L_V) - n \chi_a(X, V) \right)$$

$$= \text{Log} \epsilon \left( \int_X \mathcal{T}d'(R - n \mathcal{T}d(R)) \text{ch}(L_V) \right).$$

On the other hand, the results of the previous section give a different formula for the torsion anomaly. Namely, from the very definition of the geometric zeta function $Z^g_V$, where the superscript $g$ was added to keep track of the dependence on the metric, one clearly has

$$Z^{\epsilon g}_V(z) = Z^g_V(\epsilon^\frac{1}{2} z).$$
Thus, using Theorem 9.1 to compare the leading coefficients of the expansion around $z = 0$, one obtains

$$\log \tau_{eg}(X, \mathcal{V})^2 = \log \tau_g(X, \mathcal{V})^2 + \frac{\nu}{2} \log \epsilon.$$ 

Equating the two expressions for the torsion anomaly gives the result recorded below.

**Theorem 9.2.** The order $\nu$ of the zero of $Z_\mathcal{V}(z)$ at $z = 0$ is given by the characteristic number

$$\nu = 2 \int_X (T\!d'(R) - nT\!d(R)) \text{ch}(L_\mathcal{V}).$$
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Let $G$ be a connected simple Hermitian Lie group with finite center. Motivated by [4] we assume that $G$ has only one conjugacy class of cuspidal maximal parabolic subgroups and let $Q = MAN$ be the Langlands decomposition of such a parabolic. Choose a maximal compact subgroup $K \subseteq G$ such that $K_M = M \cap K$ is maximal compact in $M$. We write $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{m}$ and $\mathfrak{k}_m$ for the Lie algebras of $G$, $K$, $M$ and $K_M$. A classification of all possible $\mathfrak{g}$ is given in Table 1.

| $\mathfrak{g}$          | $\mathfrak{k}$          | $\mathfrak{m}$          | $\mathfrak{k}_m$          |
|------------------------|------------------------|------------------------|------------------------|
| $\mathfrak{su}(m,n)$   | $\mathfrak{su}(m) \oplus \mathfrak{u}(n)$ | $\mathfrak{u}(m-1,n-1)$ | $\mathfrak{u}(m-1) \oplus \mathfrak{u}(n-1)$ |
| $\mathfrak{so}^*(2n)$ | $\mathfrak{u}(n)$      | $\mathfrak{so}^*(2n-4) \oplus \mathfrak{su}(2)$ | $\mathfrak{u}(n-2) \oplus \mathfrak{su}(2)$ |
| $\mathfrak{so}(2,2n)$ | $\mathfrak{so}(2) \oplus \mathfrak{so}(2n)$ | $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(2n-2)$ | $\mathfrak{so}(2) \oplus \mathfrak{so}(2n-2)$ |
| $\mathfrak{e}_6(-14)$ | $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$ | $\mathfrak{su}(1,5)$ | $\mathfrak{u}(5)$ |
| $\mathfrak{e}_7(-25)$ | $\mathfrak{e}_6 \oplus \mathfrak{u}(1)$ | $\mathfrak{so}(2,10)$ | $\mathfrak{so}(2) \oplus \mathfrak{so}(10)$ |

Table 1. Simple Hermitian Lie algebras with a unique conjugacy class of maximal cuspidal parabolic subalgebras.

Write $K^0(K)$ resp. $K^0(K_M)$ for the Grothendieck group of representations of $K$ resp. $K_M$, and let

$$\text{Res} : K^0(K) \to K^0(K_M)$$

be the map restricting representations of $K$ to $K_M$. The image of the restriction map is denoted by $\text{Res}(K^0(K))$.

In what follows we use the notation introduced in [4]. In particular, we consider a holomorphic, Hermitian, homogeneous vector bundle $V_\lambda$ over a compact Hermitian locally symmetric manifold $X = \Gamma\backslash G/K$ associated with an irreducible, unitary representation $V_\lambda$ of $K$ of highest weight $\lambda$. The regularity assumptions on $\lambda$ stated in [4] at the beginning of §1 are irrelevant for our considerations, so we omit them.

In [4] page 46 the Ansatz is used that for every $c \in \mathbb{R}$ the virtual representation

$$(\sum_j (-1)^j \Lambda^j \mathfrak{p}_+^{[\nu]}) \otimes \left( \sum_{w \in W_n} \sum_{c^2=\Lambda^j \mathfrak{p}_+^{[\nu]}} (-1)^j \mathfrak{p}_-^{[\nu]} W_\Lambda^w \right)$$

of $K_M$ is contained in $\text{Res}(K^0(K))$. We say that $\lambda$ is a good highest weight if this is true.

**Theorem A.**

1. For $G = \text{SU}(1,n)$, $n \geq 1$, the restriction map $\text{Res} : K^0(K) \to K^0(K_M)$ is surjective. In particular, every highest weight is good.

2. For $G = \text{SO}_0(2,2n)$, $n \geq 2$, a highest weight $\lambda$ is good if and only if either $\lambda(H_0) = n\sqrt{-1}$ or $V_\lambda|_{\text{SO}(2n)}$ is self-dual.
(3) For $G = SU(2,3)$ no highest weight is good.

In the formulation of Theorem A (2) we have used the (unique) element $H_0 \in \mathfrak{h}(\mathfrak{g})$ with $\text{ad}(H_0)|_{p_{\pm}} = \pm i \cdot \text{id}_{p_{\pm}}$ (see [4, page 3]).

Remark B. (1) Part (1) of Theorem A shows that the Ansatz works without additional assumptions on $V_\lambda$ if $G/K$ is of rank one.

(2) Part (2) of Theorem A shows that for $G = SO_0(2,2n)$ the Ansatz only works if $V_\lambda$ is a vector bundle associated to a representation of $K = SO(2) \times SO(2n)$ for which either the $SO(2)$-factor acts by a fixed character depending on $n$, or the $SO(2n)$-factor acts in a self-dual way. In particular, the Ansatz works for all line bundles since in this case $SO(2n)$ acts by the trivial representation which is self-dual.

(3) The Hermitian symmetric spaces $G/K$ of rank two in the above list belong to the following cases:

1. $g = su(2, n)$ ($n \geq 2$),
2. $g = so^*(8), so^*(10)$,
3. $g = so(2,2n)$ ($n \geq 2$),
4. $g = e_6(-14)$,

with the low-dimensional isomorphisms $su(2, 2) \simeq so(2, 4)$, and $so^*(8) \simeq so(2, 6)$.

By part (2) of the previous theorem, the Ansatz works with additional restrictions for $g = so(2,2n)$, so the same is true for $g = su(2, 2)$ and $g = so^*(8)$. Already for $g = su(2, 3)$ the Ansatz fails as stated in part (3) of the theorem, and it is likely that the same is true for $g = su(2, n)$, $n \geq 3$. For a complete treatment of all rank two cases the algebras $g = so^*(10)$ and $g = e_6(-14)$ are missing.

(4) In Section 3 we provide for the case $G = SO^*(2n)$ an explicit description of both the restriction map $\text{Res} : K^0(K) \to K^0(K_M)$ and all quantities occurring in (\textcircled{2}). However, we did not succeed in obtaining a characterization of the good highest weights for $n \geq 5$. We conjecture that for $n \geq 5$ no highest weight is good, and some computations for $n = 5$ indicate that this is indeed the case.

Notation. $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z}_+^n = \{\lambda \in \mathbb{Z}^n : \lambda_1 \geq \ldots \geq \lambda_n\}$, $\mathbb{Z}_{++}^n = \{\lambda \in \mathbb{Z}^n : \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq |\lambda_n|\}$.

1. $G = SU(m,n)$

In this section we verify Theorem A (1) and (3).

1.1. Some subgroups of $SU(m,n)$. Let $G = SU(m,n)$ and choose the maximal compact subgroup

$$K = \left\{ \begin{pmatrix} g & \cr h & \end{pmatrix} : g \in U(m), h \in U(n), \text{det}(g) \text{det}(h) = 1 \right\} \simeq S(U(m) \times U(n)).$$

Put

$$X_\kappa = \begin{pmatrix} 0 & 1 \\ 0_{m+n-2} & 1 \\ 1 & 0 \end{pmatrix},$$

then $\text{ad}(X_\kappa)$ acts on $\mathfrak{g}$ with eigenvalues $0, \pm 1$ and $\pm 2$. Write $\mathfrak{m}^1$ for the 0-eigenspace and $\mathfrak{n}$ for the direct sum of the positive eigenspaces, then $\mathfrak{q} = \mathfrak{m}^1 \oplus \mathfrak{n}$ is a cuspidal maximal parabolic
subalgebra of $\mathfrak{g}$. We further decompose $\mathfrak{m}^1 = \mathfrak{m} \oplus \mathfrak{a}$ where $\mathfrak{a} = \mathbb{R}X_n$ and $\mathfrak{m}$ is a direct sum of semisimple and compact abelian ideals. On the group level, $Q = N_G(\mathfrak{q})$ is a cuspidal maximal parabolic subgroup of $G$ with Langlands decomposition $Q = MAN$, where $MA = Z_G(\mathfrak{a})$, $A = \exp(\mathfrak{a})$ and $N = \exp(\mathfrak{n})$. The intersection $K_M = K \cap M$ is maximal compact in $M$ and given by

$$K_M = \left\{ \begin{pmatrix} z & g \\ h & z \end{pmatrix} : z \in U(1), g \in U(m-1), h \in U(n-1), z^2 \det(g) \det(h) = 1 \right\}.$$ 

1.2. The branching law. Both $K$ and $K_M$ are connected, so that we can describe irreducible representations in terms of their highest weights. Let

$$t = \{ \sqrt{-1} \text{diag}(t_1, \ldots, t_{m+n}) : t_i \in \mathbb{R}, t_1 + \cdots + t_{m+n} = 0 \} \subseteq \mathfrak{t},$$

then $t$ is a maximal torus of $\mathfrak{k}$ and $\mathfrak{g}$. The root system $\Delta(\mathfrak{t}_C, \mathfrak{t}_C)$ is given by $\{ \pm(\varepsilon_i - \varepsilon_j) : 1 \leq i < j \leq m$ or $m + 1 \leq i < j \leq m + n \}$, where

$$\varepsilon_i(\sqrt{-1} \text{diag}(t_1, \ldots, t_{m+n})) = \sqrt{-1} t_i.$$

We choose the positive system $\Delta^+(\mathfrak{t}_C, \mathfrak{t}_C) = \{ \varepsilon_i - \varepsilon_j : 1 \leq i < j \leq m$ or $m + 1 \leq i < j \leq m + n \}$. With this notation, irreducible representations of $K$ are parametrized by their highest weights $\lambda = \lambda^1 \varepsilon_1 + \cdots + \lambda'_m \varepsilon_m + \lambda^1_1 \varepsilon_{m+1} + \cdots + \lambda''_{m+n} \varepsilon_{m+n}$, where $\lambda^1 = (\lambda^1_1, \ldots, \lambda^1_m) \in \mathbb{Z}_m^+$ and $\lambda'' = (\lambda''_1, \ldots, \lambda''_{m+n}) \in \mathbb{Z}_{m+n}^+$. We write $\lambda = (\lambda', \lambda'')$ for convenience and denote by $\pi_{\lambda'} = \pi_{\lambda', \lambda''}$ the corresponding equivalence class of representations. Note that $\pi_{\lambda'+k, \lambda''+k} \simeq \pi_{\lambda', \lambda''}$ for $k \in \mathbb{Z}$, where $\lambda' + k = (\lambda'_1 + k, \ldots, \lambda'_m + k)$ and similar for $\lambda''$.

The intersection $t_M = t \cap t_M$ is a maximal torus of $\mathfrak{k}_M$ and we write $\tau_i = \varepsilon_i|_{t_M}$. Then $\pi_1 = \tau_{m+n}$ and $\Delta^+(\mathfrak{t}_M, \mathfrak{t}_M, \mathfrak{t}_M, \mathfrak{C}) = \{ \tau_i - \tau_j : 2 \leq i < j \leq m$ or $m + 1 \leq i < j \leq m + n - 1 \}$ is a positive system in $\Delta(\mathfrak{t}_M, \mathfrak{t}_M, \mathfrak{t}_M, \mathfrak{C})$. With this notation, irreducible representations of $K_M$ are parametrized by highest weights $p \pi_1 + \mu_1 \tau_2 + \cdots + \mu_{m-1} \tau_m + \mu'_1 \tau_{m+1} + \cdots + \mu''_{m+n-1} \tau_{m+n}$, where $\mu^1 = (\mu'_1, \ldots, \mu'_{m-1}) \in \mathbb{Z}^{m-1}$, $\mu'' = (\mu''_1, \ldots, \mu''_{m+n-1}) \in \mathbb{Z}^{m+n-1}$, $p \in \mathbb{Z}$ and we write $\tau_{\mu''} \tau_{\mu''} \tau_{\mu''} \tau_{\mu''}$ for the corresponding equivalence class of representations. Note that $\tau_{\mu''} \tau_{\mu''} \tau_{\mu''} \tau_{\mu''} \simeq \tau_{\mu''} \tau_{\mu''} \tau_{\mu''} \tau_{\mu''}$.

For tuples $\lambda \in \mathbb{Z}_+^k$ and $\mu \in \mathbb{Z}_+^{k-1}$ we write

$$\mu \subseteq \lambda : \iff \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k-1} \geq \mu_{k-1} \geq \lambda_k$$

and denote $|\lambda| = \lambda_1 + \cdots + \lambda_k$.

Lemma 1.1. For $(\lambda', \lambda'') \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n$ the following branching law holds:

$$\pi_{\lambda', \lambda''}|_{K_M} \simeq \bigoplus_{\mu'' \subseteq \lambda''} \tau_{\mu''} |_{\lambda' + |\lambda'| - |\mu'|-|\mu''|}.$$

Proof. The classical branching law for the restriction $U(k) \setminus U(k-1)$ states that the irreducible representation of $U(k-1)$ of parameter $\mu \in \mathbb{Z}_+^{k-1}$ occurs in the restriction of the irreducible representation of $U(k)$ of parameter $\lambda \in \mathbb{Z}_+^k$ if and only if $\mu \subseteq \lambda$, and in this case it occurs with multiplicity one (see e.g. [2] Theorem 9.14]). Keeping track of the action of the $U(1)$-factor in $K_M$ gives the claimed branching formula. □
Proof of Theorem \[3\] (I). Assume \( n = 1 \), then \( \pi_{\lambda', \lambda''} \simeq \pi_{\lambda' - \lambda''} \), \( \lambda' \in \mathbb{Z}^n_+ \), \( \lambda'' \in \mathbb{Z}^n_+ \), and we abbreviate \( \pi_{\lambda} = \pi_{\lambda, 0} \), \( \lambda' \in \mathbb{Z}^n_+ \). We further write \( \tau_{\mu, p} = \tau_{\mu, 0, p} \) since \( \mathbb{Z}^{n-1}_+ = \mathbb{Z}^0 \). In this simplified notation the branching law reads

\[
\pi_{\lambda}|_{K_M} = \bigoplus_{\mu \leq \lambda} \tau_{\mu, |\lambda| - |\mu|}.
\]

The representations \( \pi_{\lambda} \) are pairwise inequivalent and \( \tau_{\mu, p} \simeq \tau_{\mu, k, p} + 2k \), \( k \in \mathbb{Z} \). We now show by induction on \( \ell(\mu) = (\mu_1 - \mu_{m-1}) + \cdots + (\mu_{m-2} - \mu_{m-1}) \) that every \( \tau_{\mu, p} \) is in the image of the restriction map. For \( \ell(\mu) = 0 \) we have \( \mu = (q, \ldots, q) \), \( q \in \mathbb{Z} \), and

\[
\tau_{\mu, p} \simeq \tau_{(2q - p, \ldots, 2q - p)}(2q - p) \simeq \pi_{(2q - p, \ldots, 2q - p)}|_{K_M}.
\]

Hence \( \tau_{\mu, p} \) is in the image of the restriction map. Next assume that \( \tau_{\nu, p} \) is in the image of \( \text{Res} \) for all \( \nu \in \mathbb{Z}^{m-1}_+ \) with \( \ell(\nu) \leq k \). Let \( \mu \in \mathbb{Z}^{m-1}_+ \) with \( \ell(\mu) = k + 1 \) and \( p \in \mathbb{Z} \). Since

\[
\tau_{\mu, p} \simeq \tau_{(\mu_1 + \mu_{m-1} - p, \ldots, \mu_{m-2} + \mu_{m-1} - p, 2\mu_{m-1} - p)}(2\mu_{m-1} - p)
\]

we may assume that \( \mu_{m-1} = p \). Put \( \lambda = (\mu_1, \ldots, \mu_{m-1}, \mu_{m-1}) \), then \( \nu \leq \lambda \) implies \( \nu_i \leq \mu_i \) for \( i = 1, \ldots, m - 2 \) and \( \nu_{m-1} = \mu_{m-1} \) so that

\[
\ell(\nu) = (\nu_1 - \nu_{m-1}) + \cdots + (\nu_{m-2} - \nu_{m-1}) \leq (\mu_1 - \mu_{m-1}) + \cdots + (\mu_{m-2} - \mu_{m-1}) = \ell(\mu).
\]

Moreover, the only \( \nu \leq \lambda \) with \( \ell(\nu) = \ell(\mu) = k + 1 \) is \( \nu = \mu \). Applying the induction hypothesis to every \( \tau_{\nu, |\lambda| - |\nu|} \) with \( \nu \leq \lambda \) and \( \nu \neq \mu \), we find that

\[
\pi_{\lambda}|_{K_M} \simeq \tau_{\mu, |\lambda| - |\mu|} \oplus \text{terms in Res}(K^0(K)).
\]

Since \( |\lambda| - |\mu| = \mu_{m-1} = p \) this shows that \( \tau_{\mu, p} \) is contained in \( \text{Res}(K^0(K)) \), and the proof is complete. \( \square \)

1.3. The space \( p^{[\text{ev}]}_+ \). Define a homomorphism \( \kappa : \mathfrak{su}(1, 1) \to \mathfrak{su}(m, n) \) as the composition of the homomorphism

\[
\tilde{\kappa} : \mathfrak{su}(1, 1) \to \mathfrak{su}(m, n), \quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} x & 0_{m+n-2} & y \\ z & & w \end{pmatrix}
\]

and the Lie algebra isomorphism

\[
\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{su}(1, 1), \quad \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} \frac{b - c \overline{b}}{2} & a - \frac{b + c \overline{b}}{2} \\ \frac{a + b \overline{c}}{2} & -\frac{b - c \overline{c}}{2} \end{pmatrix}.
\]

Then the decomposition \( \mathfrak{g} = \mathfrak{g}^{[0]} \oplus \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \) into \( \kappa(\mathfrak{sl}(2, \mathbb{R})) \)-isotypic components is given by

\[
\mathfrak{g}^{[0]} = \left\{ \begin{pmatrix} a & X \\ X & a \end{pmatrix} \middle| a \in i\mathbb{R}, X \in \mathfrak{u}(m - 1, n - 1), 2a + \text{tr}(X) = 0 \right\},
\]

\[
\mathfrak{g}^{[1]} = \begin{pmatrix} 0 & * & 0 \\ * & 0_{m+n-2} & * \\ 0 & * & 0 \end{pmatrix},
\]

\[
\mathfrak{g}^{[2]} = \kappa(\mathfrak{sl}(2, \mathbb{R})).
\]
We identify \( g_C \) with \( \mathfrak{sl}(m+n, C) \), then the choice of \( \kappa \) determines
\[
P_+ = \left\{ \begin{pmatrix} 0_m & X \\ 0_n & \end{pmatrix} : X \in M(m \times n, C) \right\},
\]
so that
\[
\tilde{p}_+^{[ev]} = p_+^{[0]} = \begin{pmatrix} 0 & 0_m & \ast \\ 0 & 0_n & \end{pmatrix}.
\]
As representation of \( K_M \) we have \( \tilde{p}_+^{[ev]} \simeq \tau(1,0,\ldots,0,0,\ldots,0,0) \). Using Exercise 6.11] we obtain the following decomposition for its exterior powers of:
\[
\Lambda^j \tilde{p}_+^{[ev]} \simeq \bigoplus_{\mu', \mu''} \tau_{\mu', \mu'', 0},
\]
where the summation is over all \( (\mu', \mu'') \in \mathbb{Z}_m \times \mathbb{Z}_n \) with \( \mu'_1 \geq \ldots \geq \mu'_{m-1} \geq 0 \geq \mu''_1 \geq \ldots \geq \mu''_{n-1} \), \( |\mu'| = j = -|\mu''| \) and such that the partition \( (\mu''_{n-2}, \ldots, -\mu''_1) \) of \( j \) is conjugate to \( \mu' \), i.e.
\[
-\mu''_j = \# \{ i : \mu'_i \geq n - j \}.
\]
Lemma 1.2. For \( n = 2 \) we have
\[
\tau_{0,-1,1} \otimes \left( \sum_j (1) \Lambda^j \tilde{p}_+^{[ev]} \right) \in \text{Res}(K^0(K)).
\]
Proof. We write
\[
(1)_j = (1, \ldots, 1, 0, \ldots, 0),
\]
the length of the vector being clear from the context. In the Grothendieck group \( K^0(K_M) \) we form the alternating sum
\[
\left( \sum_{j=0}^{m} (-1)^{j-1}(1)_{(0,-j)} \right) \Bigg|_{K_M} = \sum_{j=1}^{m-1} (-1)^{j-1} \sum_{k=0}^{j} \left( \tau_{1,j-k,j-k-1} + \tau_{1,j-k-1,-k,j+k+1} \right)
\]
\[
- \tau_{0,0,0} + (-1)^{m-1} \tau_{1,j-1,-j,1}
\]
\[
= \sum_{j=1}^{m} (-1)^{j-1} \tau_{1,j-1,-j,1}
\]
\[
= \sum_{j=0}^{m-1} (-1)^{j} \tau_{1,j,-j,0} \otimes \tau_{0,-1,1}
\]
\[
= \left( \sum_{j=0}^{m-1} (-1)^{j} \Lambda^j \tilde{p}_+^{[ev]} \right) \otimes \tau_{0,-1,1},
\]
where we have used Lemma 1.1 in the first step and a telescoping argument in the second step.

Remark 1.3. It is likely that a similar statement holds in general, but since the Ansatz already fails in the case \( n = 2 \), we did not attempt to prove such a generalization.
1.4. The numbers $\hat{c}_\lambda$. We now compute the numbers $\hat{c}_\lambda$. The element $H_\kappa \in \mathfrak{su}(m, n)$ is given by

$$H_\kappa = \kappa \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \text{diag}(i, 0, \ldots, 0, -i).$$

This implies that

$$L_\kappa = Z_K(H_\kappa) = S(U(1) \times U(m - 1) \times U(n - 1) \times U(1)).$$

The torus $t$ is contained in $l_\kappa$ and $\Delta(l_\kappa, t_\kappa) = \{ \pm(\varepsilon_i - \varepsilon_j) : 2 \leq i < j \leq m + 1 \leq i < j \leq m + n - 1 \}$. We choose the positive roots $\Delta^+(l_\kappa, t_\kappa) = \{ \varepsilon_i - \varepsilon_j : 2 \leq i < j \leq m + 1 \leq i < j \leq m + n - 1 \} \subseteq \Delta^+(t_\kappa, t_\kappa)$.

The Weyl group $W(t_\kappa)$ of $\Delta(t_\kappa, t_\kappa)$ is naturally isomorphic to $S_m \times S_n$, where $S_k$ denotes the symmetric group in $k$ letters. Then

$$W_\kappa = \{ w \in W(t_\kappa) : w^{-1} \alpha > 0 \forall \alpha \in \Delta^+(l_\kappa, t_\kappa) \}$$

$$= \{(w_1, w_2) \in S_m \times S_n : w_1^{-1}(2) < \ldots < w_1^{-1}(m) \text{ and } w_2^{-1}(1) < \ldots < w_2^{-1}(n - 1) \}$$

$$= \{(w_1^{(i)}, w_2^{(j)}) : i = 1, \ldots, m, j = 1, \ldots, n \},$$

where

$$w_1^{(i)}(k) = \begin{cases} k + 1 & \text{for } 1 \leq k < i, \\ 1 & \text{for } k = i, \\ k & \text{for } i < k \leq m, \end{cases}$$

$$w_2^{(j)}(k) = \begin{cases} k & \text{for } 1 \leq k < j, \\ n & \text{for } k = j, \\ k - 1 & \text{for } j < k \leq n. \end{cases}$$

Note that $\ell(w_1^{(i)}) = i - 1$ and $\ell(w_2^{(j)}) = j - 1$, and since $\ell(w_1^{(i)}, w_2^{(j)}) = \ell(w_1^{(i)}) + \ell(w_2^{(j)})$ we obtain

$$(-1)\ell(w_1^{(i)}, w_2^{(j)}) = (-1)^{i+j-1}.$$

We further have

$$\rho_c = (\rho_{c,1}, \rho_{c,2})$$

with

$$\rho_1 = \frac{m-1}{2}, \frac{m-3}{2}, \ldots, \frac{1-m}{2} = \left( \frac{m-2k+1}{2} \right)_{k=1, \ldots, m}, \quad \rho_{c,2} = \left( \frac{n-2k+1}{2} \right)_{k=1, \ldots, n}.$$

Now let $\lambda = (\lambda', \lambda'') \in \mathbb{Z}^m_+ \times \mathbb{Z}^n_+$ be a highest weight of an irreducible $K$-representation. Then

$$w_1^{(i)}(\lambda' + \rho_{c,1}) - \rho_{c,1} = w_1^{(i)}(\lambda'_1 + m-1, \ldots, \lambda'_m + \frac{1-m}{2}) - \left( \frac{m-1}{2}, \ldots, \frac{1-m}{2} \right)$$

$$= (\lambda'_1 + m-2+1, \lambda'_1 + m-1, \ldots, \lambda'_i + m-2+1, \ldots, \lambda'_m + \frac{1-m}{2}) - \left( \frac{m-1}{2}, \frac{m-3}{2}, \ldots, \frac{1-m}{2} \right)$$

$$= (\lambda'_1 + 1 - i, \lambda'_1 + 1, \ldots, \lambda'_{i-1} + 1, \lambda'_{i+1}, \ldots, \lambda'_m)$$

and similarly

$$w_2^{(j)}(\lambda'' + \rho_{c,2}) - \rho_{c,2} = (\lambda''_1, \ldots, \lambda''_{j-1}, \lambda''_{j+1} - 1, \ldots, \lambda''_m - 1, \lambda''_m + n - j).$$

Together this gives for $w = (w_1^{(i)}, w_2^{(j)})$:

$$\lambda_w = (w_1^{(i)}, w_2^{(j)})(\lambda', \lambda'') + \rho_c - \rho_c = \left( (\lambda'_1 + 1 - i, \lambda'_1 + 1, \ldots, \lambda'_{i-1} + 1, \lambda'_{i+1}, \ldots, \lambda'_m), \right.$$

$$\left. (\lambda''_1, \ldots, \lambda''_{j-1}, \lambda''_{j+1} - 1, \ldots, \lambda''_m - 1, \lambda''_m + n - j) \right).$$
Restricting this weight to $t_M$ we obtain a dominant integral weight for $\mathfrak{p}_M$ which belongs to the representation

$$\tau(\lambda_1 + 1, \ldots, \lambda_{r-1} + 1, \lambda_{r+1}, \ldots, \lambda_m), (\lambda_1', \ldots, \lambda_{r-1}', \lambda_{r+1}', \ldots, \lambda_m', 1, \lambda_1 + \lambda_{r+1} + n + 1 - i - j).$$

We now compute $c_\lambda = ((\lambda_w \circ C_\kappa) + \rho_Q)(X_\kappa)$:

$$(\lambda_w \circ C_\kappa) + \rho_Q)(X_\kappa) = -\sqrt{-1} \lambda_w(H_\kappa) + \rho_Q(X_\kappa) = (\lambda_i' - \lambda_j'' - n - i + j + 1) + (m + n - 1)$$
$$= \lambda_i' - \lambda_j'' + (m - i) + j.$$

1.5. **Invariants for** $(m, n) = (3, 2)$. For $(\mu', \mu'', p) \in \mathbb{Z}_+^2 \times \mathbb{Z} \times \mathbb{Z}$ with $\mu' = (\mu_1', \mu_2')$ we put

$$I(\tau_{\mu', \mu'', p}) := (\mu_1' + \mu'' - p) + (\mu_2' + \mu'' - p) + (\mu_1' + \mu'' - p)^2 - (\mu_2' + \mu'' - p)^2$$

and extend this $\mathbb{Z}$-linearly to the Grothendieck group $K^0(K_M)$ which is the free $\mathbb{Z}$-module generated by all $\tau_{\mu', \mu'', p}$. Then $I$ is an invariant for the image $\text{Res}(K^0(K))$ of the restriction map:

**Lemma 1.4.** For all $(\lambda', \lambda'') \in \mathbb{Z}_+^3 \times \mathbb{Z}_+^2$ we have $I(\pi_{\lambda', \lambda''} \mid K_M) = 0$. In particular, $I$ vanishes on the image $\text{Res}(K^0(K))$ of the restriction map.

**Proof.** Write $\lambda' = (a + K + L, a + L, a)$ and $\lambda'' = (b + M, b)$ with $a, b \in \mathbb{Z}$ and $K, L, M \geq 0$, then

$$\pi_{\lambda', \lambda''} \mid K_M \simeq \bigoplus_{k=0}^K \bigoplus_{\ell=0}^L \bigoplus_{m=0}^M \tau(a + L + k, a + \ell, b + m, a + b + (K - k) + (L - \ell) + (M - m)).$$

According to the four summands in (1.1) we split $I(\pi_{\lambda', \lambda''} \mid K_M)$ into four parts which are straightforward to compute:

$$\sum_{k, \ell, m} (\mu_1' + \mu'' - p) = \sum_{k, \ell, m} \left(2k + \ell + 2m - K - M\right) = \frac{L}{2}(K + 1)(L + 1)(M + 1),$$

$$\sum_{k, \ell, m} (\mu_2' + \mu'' - p) = \sum_{k, \ell, m} \left(k + 2\ell + 2m - K - L - M\right) = -\frac{K}{2}(K + 1)(L + 1)(M + 1),$$

$$\sum_{k, \ell, m} (\mu_1' + \mu'' - p)^2 = \sum_{k, \ell, m} \left(2k + \ell + 2m - K - M\right)^2$$
$$= (K + 1)(L + 1)(M + 1) \left[\frac{1}{3}K(K + 2) + \frac{1}{6}L(2L + 1) + \frac{1}{3}M(M + 2)\right],$$

$$\sum_{k, \ell, m} (\mu_2' + \mu'' - p)^2 = \sum_{k, \ell, m} \left(k + 2\ell + 2m - K - L - M\right)^2$$
$$= (K + 1)(L + 1)(M + 1) \left[\frac{1}{6}K(2K + 1) + \frac{1}{3}L(L + 2) + \frac{1}{3}M(M + 2)\right].$$

Subtracting the last expression from the sum of the first three expressions gives 0. \qed

We use the invariant $I$ to prove that for $(m, n) = (3, 2)$ no highest weight is good.
Proof of Theorem A. In view of (3) and Lemma 1.2, it suffices to show that for any λ there exists $c \in \mathbb{R}$ such that the sum

$$\tau_{0,1,-1} \otimes \left( \sum_{w \in W_\lambda} (-1)^{\ell(w)} W_{\lambda_w} \right) = \sum_{w \in W_\lambda} (-1)^{\ell(w)} \tau_{0,1,-1} \otimes W_{\lambda_w}$$

(1.2)

is not contained in Res($K^0(K)$). For $\lambda = (\lambda', \lambda'')$ we list all summands with corresponding $\hat{c}_{\lambda_w}$ and invariant $I$ in Table 2. The inequalities $\lambda'_1 \geq \lambda'_2 \geq \lambda'_3$ and $\lambda''_1 \geq \lambda''_2$ imply the following relations between $A, B, C, D, E, F$:

| $w$ | $(-1)^{\ell(w)} \tau_{0,1,-1} \otimes W_{\lambda_w}$ | $\hat{c}_{\lambda_w}$ | $I(\tau_{0,1,-1} \otimes W_{\lambda_w}) = 0$ |
|-----|-------------------------------------------------|----------------|---------------------------------|
| $a = (w_{1}^{(1)}, w_{1}^{(1)})$ | $\tau(\lambda'_2, \lambda'_3, \lambda''_1 + \lambda''_2)$ | $A = \lambda'_1 - \lambda''_2 + 3 \leq 2 \lambda'_1 - \lambda'_2 - \lambda'_3 + 2 \lambda''_1 - 2 \lambda''_2 = 0$ |
| $b = (w_{2}^{(1)}, w_{2}^{(1)})$ | $-\tau(\lambda'_3 + 1, \lambda'_3, \lambda''_1 + \lambda''_2 - 1)$ | $B = \lambda'_2 - \lambda'_1 + 2 \leq \lambda'_3 - 2 \lambda'_2 + \lambda'_3 - 2 \lambda'_1 + 2 \lambda''_2 = -3$ |
| $c = (w_{3}^{(1)}, w_{3}^{(1)})$ | $\tau(\lambda'_1 + 1, \lambda'_2 + 1, \lambda''_1 + \lambda''_2 - 2)$ | $C = \lambda'_3 - \lambda'_1 + 1 \leq \lambda'_1 + \lambda'_2 - 2 \lambda'_3 - 2 \lambda'_2 + 2 \lambda''_1 + 2 \lambda''_2 = -6$ |
| $d = (w_{1}^{(2)}, w_{1}^{(2)})$ | $-\tau(\lambda'_2, \lambda'_3, \lambda''_1 + 1, \lambda''_2 - 1)$ | $D = \lambda'_1 - \lambda'_2 + 4 \leq 2 \lambda'_1 - \lambda'_2 - \lambda'_3 - 2 \lambda''_1 + 2 \lambda''_2 = 4$ |
| $e = (w_{2}^{(2)}, w_{2}^{(2)})$ | $\tau(\lambda'_1 + 1, \lambda'_2 + 1, \lambda''_1 + \lambda''_2 - 2)$ | $E = \lambda'_2 - \lambda''_2 + 3 \leq \lambda'_1 - 2 \lambda'_2 + \lambda'_3 + 2 \lambda''_1 - 2 \lambda''_2 = -7$ |
| $f = (w_{3}^{(2)}, w_{3}^{(2)})$ | $-\tau(\lambda'_1 + 1, \lambda'_2 + 1, \lambda''_3 + \lambda''_3 - 3)$ | $F = \lambda'_3 - \lambda''_2 + 2 \leq \lambda'_1 + \lambda'_2 - 2 \lambda'_3 + 2 \lambda''_1 - 2 \lambda''_2 = -10$ |

Using these inequalities one can systematically consider all possible groupings of representations with the same $c_{\lambda_w}^2$ to show that there is always $c \in \mathbb{R}$ such that the sum (1.2) does not belong to Res($K^0(K)$). We illustrate this for the case $C^2 = D^2$ and $E^2 = F^2$. Since $D > C$ this implies $D = -C$ and we get a first identity

$$\lambda'_1 + \lambda''_2 - \lambda''_1 - \lambda'_2 = -5.$$ (1.3)

Further, since $E > F$ we have $E = -F$ which implies the additional identity

$$\lambda''_2 - 2 \lambda''_2 = -5.$$ (1.4)

We now have the following possibilities for $B^2$:

- $B^2 \not\in \{A^2, E^2 = F^2\}$. Since $D > B > C$ and $D^2 = C^2$ we further have $B^2 \notin \{C^2 = D^2\}$, so that for $c = B$ the sum (1.2) consists of only one representation. By Lemma 1.2 this representation needs to have vanishing invariant $I$ which implies

$$\lambda'_1 - 2 \lambda''_2 + \lambda'_3 - 2 \lambda''_1 + 2 \lambda''_2 = -3.$$ (1.5)

Solving (1.3), (1.4) and (1.5) yields

$$(\lambda'_1, \lambda'_2, \lambda'_3) = (-1 + \frac{1}{2} \lambda''_2, -1 + \frac{1}{2} \lambda''_2, -1 + \frac{1}{2} \lambda''_2, -1 + \frac{1}{2} \lambda''_2, -4 + \frac{1}{2} \lambda''_2, -4 + \frac{1}{2} \lambda''_2).$$

Moreover, since $D > A, B, E, F > C$ we have $C^2 = D^2 \notin \{A^2, B^2, E^2, F^2\}$, so that for $c = D = -C$ the sum (1.2) consists of only two representations, and the invariant $I$ of the sum evaluates to

$$-2(\lambda''_1 - \lambda''_2 + 1)(\lambda'_1 - \lambda''_2 - 4).$$
Again by Lemma 1.4 this invariant has to vanish, which implies $\lambda''_1 = \lambda''_2 + 4$ and hence

$$ (\lambda'_1, \lambda'_2, \lambda'_3) = (1 + \lambda''_2, -3 + \lambda''_2, -2 + \lambda''_2), $$

a contradiction to $\lambda'_2 \geq \lambda'_3$.

- $B^2 = A^2$. Since $A > B$ we have $A = -B$ which implies the additional identity

$$ \lambda'_1 + \lambda'_2 - 2\lambda''_1 = -5. \quad (1.6) $$

Solving (1.3), (1.4) and (1.6) yields

$$ (\lambda'_1, \lambda'_2, \lambda'_3) = (-\frac{5}{2} + \frac{3}{2}\lambda''_1 - \frac{1}{2}\lambda''_2, -\frac{5}{2} + \frac{1}{2}\lambda''_1 + \frac{1}{2}\lambda''_2, -\frac{5}{2} - \frac{1}{2}\lambda''_1 + \frac{3}{2}\lambda''_2). \quad (1.7) $$

Then for $c = D - C$ the sum (1.2) consists of only two representations, and the invariant $I$ of the sum evaluates to

$$ 2(\lambda''_1 - \lambda''_2 + 1)^2. $$

This is $\neq 0$ since $\lambda''_1 \geq \lambda''_2$, a contradiction to Lemma 1.4.

- $B^2 = E^2 = F^2$. Since $E > B$ we have $E = -B$ which implies the additional identity

$$ 2\lambda''_2 - \lambda''_1 - \lambda''_2 = -5. \quad (1.8) $$

Solving (1.3), (1.4) and (1.8) yields (1.7), and by the same argument as above this shows that for $c = D$ the sum (1.2) is not contained in $\text{Res}(K^0(K))$. \hfill $\square$

2. $G = \text{SO}_0(2, 2n)$

In this section we verify Theorem A [2].

2.1. Some subgroups of $\text{SO}_0(2, 2n)$. Let $G = \text{SO}_0(2, 2n)$, $n \geq 2$, and choose the maximal compact subgroup $K = \text{SO}(2) \times \text{SO}(2n) \subseteq G$. Put

$$ X_\kappa = \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \\ 0_{2n-2} \end{pmatrix}, $$

then $\text{ad}(X_\kappa)$ acts on $\mathfrak{g}$ with eigenvalues $0, \pm 1$ and $\pm 2$. Write $\mathfrak{m}^i$ for the 0-eigenspace and $\mathfrak{n}$ for the direct sum of the positive eigenspaces, then $\mathfrak{q} = \mathfrak{m}^1 \oplus \mathfrak{n}$ is a cuspidal maximal parabolic subalgebra of $\mathfrak{g}$. We further decompose $\mathfrak{m}^1 = \mathfrak{m} \oplus \mathfrak{a}$ where $\mathfrak{a} = \mathbb{R}X_\kappa$ and $\mathfrak{m}$ is a direct sum of semisimple and compact abelian ideals. On the group level, $Q = N_G(\mathfrak{q})$ is a cuspidal maximal parabolic subgroup of $G$ with Langlands decomposition $Q = MAN$, where $MA = Z_G(\mathfrak{a})$, $A = \text{exp}(\mathfrak{a})$ and $N = \text{exp}(\mathfrak{n})$. The intersection $K_M = K \cap M$ is maximal compact in $M$ and given by

$$ K_M = \left\{ \begin{pmatrix} g & \cdot \\ \cdot & g \end{pmatrix} : g \in \text{SO}(2), h \in \text{SO}(2n - 2) \right\} \simeq \text{SO}(2) \times \text{SO}(2n - 2). $$
2.2. The branching law. Both $K$ and $K_M$ are connected, so that we can describe irreducible representations in terms of their highest weights. Let
\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
and
\[ t = \{ \text{diag}(t_0 J, t_1 J, \ldots, t_n J) : t_i \in \mathbb{R} \} \subseteq \mathfrak{t}, \]
then $t$ is a maximal torus of $\mathfrak{t}$ and $\mathfrak{g}$. The root system $\Delta(\mathfrak{t}_C, \mathfrak{t}_C)$ is given by $\{ \pm \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n, \text{ where } \varepsilon_i(\text{diag}(t_0 J, \ldots, t_n J)) = \sqrt{-1} t_i \}$. We choose the positive system $\Delta^+(\mathfrak{t}_C, \mathfrak{t}_C) = \{ \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n \}$. With this notation, irreducible representations of $K$ are parametrized by their highest weights $\lambda = \lambda_0 \varepsilon_0 + \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n$, where $p = \lambda_0 \in \mathbb{Z}$ and $\lambda' = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{++}$. We denote by $\pi_\lambda = \pi_{\lambda', p}$ the corresponding equivalence class of representations.

The intersection $t_M = t \cap t_M$ is a maximal torus of $\mathfrak{t}_M$ and we write $\overline{\pi}_i = \varepsilon_i|_{t_M}$. Then $\overline{\pi}_0 = \varepsilon_1$ and $\Delta^+(\mathfrak{t}_M, \mathfrak{t}_M, t_M) = \{ \overline{\pi}_i - \overline{\pi}_j : 2 \leq i < j \leq n \}$ is a positive system in $\Delta(\mathfrak{t}_M, \mathfrak{t}_M, \mathfrak{t}_M)$. With this notation, irreducible representations of $K_M$ are parametrized by highest weights $\mu = q \varepsilon_1 + p_1 \varepsilon_2 + \cdots + p_{n-1} \varepsilon_n$, where $q \in \mathbb{Z}$ and $\mu = (\mu_1, \ldots, \mu_{n-1}) \in \mathbb{Z}^{n-1}_{++}$, and we write $\tau_{\mu, q}$ for the corresponding equivalence class of representations.

Lemma 2.1. For $(\lambda', p) \in \mathbb{Z}^n_{++} \times \mathbb{Z}$ the following branching law holds:
\[ \pi_{\lambda', p}|_{K_M} \cong \bigoplus_{\sum_{i=1}^{n-1} \ell_i = \ell} \bigoplus_{k=0}^{\ell} \oplus_{\mu_1, \ldots, \mu_{n-1}}^{m(k)} \]
where
\[ \ell_1 = \lambda_1 - \max(\lambda_2, \mu_1), \quad \ell_i = \min(\lambda_i, \mu_{i-1}) - \max(\lambda_{i+1}, \mu_i) \quad (2 \leq i \leq n - 2), \]
\[ \ell_{n-1} = \min(\lambda_{n-1}, \mu_{n-2}) - \max(|\lambda_n|, |\mu_{n-1}|), \quad \ell_n = \text{sgn}(\lambda_n) \text{sgn}(\mu_{n-1}) \min(|\lambda_n|, |\mu_{n-1}|), \]
and $\ell = \sum_{i=1}^{n-1} \ell_i$, and the multiplicities $m(k)$ are given by
\[ m(k) = \# \left\{ (k_1, \ldots, k_{n-1}) \in \mathbb{Z}^{n-1} : 0 \leq k_i \leq \ell_i, \sum_{i=1}^{n-1} k_i = k \right\}. \]

Note that $m(0) = m(\ell) = 1$.

Proof. By [5, Theorem 1.1] the restriction $\pi_{\lambda', p}|_{K_M}$ is the claimed direct sum with multiplicities $m(k)$ given by the coefficient of 1 in the Laurent series expansion of
\[ - \prod_{i=1}^{n-1} \frac{X^{\ell_{i+1}} - X^{-\ell_{i-1}}}{X - X^{-1}}. \]
But since
\[ \frac{X^{\ell_{i+1}} - X^{-\ell_{i-1}}}{X - X^{-1}} = \sum_{k_i=0}^{\ell_i} X^{2k_i-\ell_i}, \]
the claimed formula for the multiplicities follows.

\begin{proposition}
The image \( \text{Res}(K^0(K)) \) of the restriction map is spanned over \( \mathbb{Z} \) by
\[
\begin{align*}
\tau_{\mu,q} &\quad \text{for } \mu_{n-1} = 0, \text{ and} \\
\tau_{\mu,q+s} + \tau_{\mu^*,q-s} &\quad \text{for } \mu_{n-1} \neq 0, \text{ s } \in \mathbb{Z} \text{ and } \mu^* = (\mu_1, \ldots, \mu_{n-2}, -\mu_{n-1}),
\end{align*}
\]
with \((\mu, q) \in \mathbb{Z}^{n-1} \times \mathbb{Z}.
\end{proposition}

\begin{proof}
That the image is actually contained in this span follows immediately from the fact that in the branching law the representations appear in pairs. More precisely, for each \( \tau_{\mu,p+\ell_n+2k-\ell} \) that occurs in the restriction of \( \pi_{\lambda',p} \) with multiplicity \( m(k) \), the representation \( \tau_{\mu,p-\ell_n+2k-\ell} \), \( \mu = (\mu_1, \ldots, \mu_{n-2}, -\mu_{n-1}) \) occurs with the same multiplicity. (For this note that if \( \mu_{n-1} \) is changed to \(-\mu_{n-1} \) only the number \( \ell_n \) switches sign, everything else stays the same.) For \( \mu_{n-1} = 0 \) these are the same representations, and for \( \mu_{n-1} \neq 0 \) they form a pair
\[
\tau_{\mu,q+s} \oplus \tau_{\mu,q-s},
\]
where \( q = p + 2k - \ell \) and \( s = \ell_n = \text{sgn}(\lambda_n) \text{sgn}(\mu_{n-1}) \text{ min}(|\lambda_n|, |\mu_{n-1}|). \)
To show that the image is actually equal to the span, we use induction on \( \ell(\mu) = \mu_1 + \cdots + \mu_{n-2} + |\mu_{n-1}| \). For \( \ell(\mu) = 0 \) we have \( \mu = (0, \ldots, 0) \) and for \( \lambda' = (0, \ldots, 0) \):
\[
\pi_{\lambda',q}|K_M \simeq \tau_{(0,\ldots,0),q} = \tau_{\mu,q} \quad \forall q \in \mathbb{Z}.
\]
For \( \ell(\mu) > 0 \) and \( |s| \leq |\mu_{n-1}| \) let \( \lambda' = (\mu_1, \ldots, \mu_{n-2}, |\mu_{n-1}|, \text{sgn}(\mu_{n-1})s) \), then
\[
\pi_{\lambda',q}|K_M = \tau_{\mu,q+s} \oplus \tau_{\mu^*,q-s} \oplus \text{lower order terms} \quad \forall q \in \mathbb{Z}.
\]
(In fact, for \( \mu \) and \( \mu^* \) we have \( \ell_1 = \ldots = \ell_{n-1} = 0 \) in the branching law and hence the multiplicity is 1.) By applying the induction hypothesis it follows that \( \tau_{\mu,q+s} \oplus \tau_{\mu^*,q-s} \) is contained in the image of the restriction map. Now, let \( s \in \mathbb{Z} \) be arbitrary and write \( s = s_1 + \cdots + s_{2k+1} \) with \( |s_i| \leq |\mu_{n-1}| \). Then
\[
\begin{align*}
\tau_{\mu,q+s} + \tau_{\mu^*,q-s} &= \left(\tau_{\mu,q+s} + \tau_{\mu^*,q+s-2s_1}\right) - \left(\tau_{\mu^*,q+s-2s_1} + \tau_{\mu,q+s-2(s_1+s_2)}\right) \\
&\quad + \left(\tau_{\mu,q+s-2(s_1+s_2)} + \tau_{\mu^*,q+s-2(s_1+s_2+s_3)}\right) - \left(\tau_{\mu^*,q+s-2(s_1+s_2+s_3)} + \tau_{\mu,q+s-2(s_1+s_2+s_3+s_4)}\right) \\
&\quad + \cdots \\
&\quad + \left(\tau_{\mu,q+s-(s_1+\cdots+s_{2k})} + \tau_{\mu^*,q+s-2(s_1+\cdots+s_{2k+1})}\right)
\end{align*}
\]
where each sum in parenthesis is contained in the image of the restriction map. Hence also \( \tau_{\mu,q+s} + \tau_{\mu^*,q-s} \) is contained in the image and the proof is complete.
\end{proof}

\section*{2.3. The space \( p_{[\text{ev}]}^\mu \).}
Define a homomorphism \( \kappa : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{so}(2n, \mathbb{R}) \) on the standard basis \( E, F, H \in \mathfrak{sl}(2, \mathbb{R}) \) by
\[
\kappa(E) = \frac{1}{2} \begin{pmatrix} -J & J \\ -J & 0_{2n-2} \end{pmatrix}, \kappa(F) = \frac{1}{2} \begin{pmatrix} J & J \\ J & 0_{2n-2} \end{pmatrix}, \kappa(H) = X_\kappa.
\]
Then the decomposition $\mathfrak{g} = \mathfrak{g}^{[0]} \oplus \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]}$ into $\kappa(\mathfrak{sl}(2, \mathbb{R}))$-isotypic components is given by

$$\mathfrak{g}^{[0]} = \left\{\begin{pmatrix} A & B \\ B & A \\ Z \end{pmatrix} : A \in \mathfrak{so}(2), B \in \text{Sym}(2, \mathbb{R}), \text{tr}(B) = 0, Z \in \mathfrak{so}(2n-2) \right\},$$

$$\mathfrak{g}^{[1]} = \left\{\begin{pmatrix} 0_2 & 0_2 & X^T \\ 0_2 & 0_2 & -Y^T \\ X & Y & 0_{2n-2} \end{pmatrix} : X, Y \in M((2n-2) \times 2, \mathbb{R}) \right\},$$

$$\mathfrak{g}^{[2]} = \kappa(\mathfrak{sl}(2, \mathbb{R})).$$

The choice of $\kappa$ determines

$$p_+ = \left\{\begin{pmatrix} 0 & 0 & z^T \\ 0 & 0 & -iz^T \\ z & -iz & 0_{2n} \end{pmatrix} : z \in \mathbb{C}^{2n} \right\} \subseteq \mathfrak{so}(2, 2n)_C,$$

so that

$$p^{[ev]}_+ = p^{[0]}_+ = \left\{\begin{pmatrix} 0_2 & Z \\ Z & 0_2 \\ 0_{2n-2} \end{pmatrix} : Z = \begin{pmatrix} z & -iz \\ -iz & -z \end{pmatrix}, z \in \mathbb{C} \right\}.$$ 

As representation of $K_M$ we have $p^{[ev]}_+ \simeq \tau(0, \ldots, 0) \simeq \pi(0, \ldots, 0) \times K_M$. Hence

$$\sum J(\mu \mu^{[ev]}_+) \in \text{Res}(K^0(K)).$$

### 2.4. The numbers $\hat{c}_\lambda$.

We now compute the numbers $\hat{c}_\lambda$. The element $H_\kappa \in \mathfrak{g}$ is given by

$$H_\kappa = \kappa(E - F) = \begin{pmatrix} -J & J \\ 0 & 0_{2n-2} \end{pmatrix},$$

which implies that

$$L_\kappa = Z_K(H_\kappa) = \text{SO}(2) \times \text{SO}(2) \times \text{SO}(2n-2).$$

Note that $t \subseteq t_\kappa$ and that the choice of positive roots $\Delta^+(t_\kappa, t_\kappa)$ is compatible with $t_\kappa$ in the sense that $\Delta^+(t_\kappa, t_\kappa) = \Delta(t_\kappa, t_\kappa) \cap \Delta^+(t_\kappa, t_\kappa)$ is a positive system of roots for $t_\kappa$. The Weyl group $W(t_\kappa)$ of $\Delta(t_\kappa)$ is naturally isomorphic to $\{\pm 1\}^n_{\text{even}} \rtimes S_n$, where $S_n$ denotes the symmetric group in $n$ letters and $\{\pm 1\}^n_{\text{even}}$ is the kernel of the homomorphism $\{\pm 1\}^n \to \{\pm 1\}$, $(\delta_1, \ldots, \delta_n) \mapsto \delta_1 \cdots \delta_n$. Then

$$W_\kappa = \{w \in W(t_\kappa) : w^{-1} \alpha > 0 \forall \alpha \in \Delta^+(t_\kappa, t_\kappa)\} = \{(\delta_\pm, w^{(i)}_i) : i = 1, \ldots, n\},$$

where

$$w^{(i)}_i(k) = \begin{cases} k + 1 & \text{for } 1 \leq k < i, \\ 1 & \text{for } k = i, \\ k & \text{for } i < k \leq n, \end{cases} \quad \delta_\pm = (\pm 1, 1, \ldots, 1, \pm 1).$$

Note that $\ell(w^{(i)}) = i - 1$ and $\ell(\delta_\pm) = 1$, so that

$$(-1)^{\ell(\delta_\pm, w^{(i)})} = (-1)^{i-1}.$$

We further have

$$\rho_c = (n - 1, n - 2, \ldots, 1, 0) = (n - j)_{j=1,\ldots,n}.$$
Now let $\lambda = (\lambda', p) \in \mathbb{Z}_{n+}^n \times \mathbb{Z}$ be a highest weight of an irreducible $K$-representation. Then for $1 \leq i < n$ the Weyl group element $w = (\delta_1, w(i))$ only acts on $\lambda'$ and we have

$$\lambda'_w = (\delta_1, w(i))'(\lambda + \rho_c) - \rho_c = (\delta_1, w(i))(\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n) - (n - 1, n - 2, \ldots, 0) \quad (\text{for } +)$$

$$= \delta_\pm (\lambda_1 + n - i, \lambda_1 + n - 1, \ldots, \lambda_1 + n - i, \ldots, \lambda_n) - (n - 1, n - 2, \ldots, 0) \quad (\text{for } -)$$

$$= (\pm(\lambda_i + n - i), \lambda_1 + n - 1, \ldots, \lambda_i + n - i, \lambda_{n-1} + 1, \pm \lambda_n)$$

$$= (\lambda_i - i + 1, \lambda_1 + 1, \ldots, \lambda_i - i + 1, 1, \lambda_{n-1} + 1, \lambda_n) \quad \text{for } +,$$

$$= (-\lambda_i - 2n + i + 1, \lambda_1 + 1, \ldots, \lambda_i - 1 + 1, \lambda_{n-1} + 1, -\lambda_n) \quad \text{for } -.$$

Restricting the weight $\lambda_w = (\lambda'_w, p)$ to $\mathfrak{f}_M$ we obtain a dominant integral weight for $\mathfrak{f}_M$ which belongs to the representation

$$W_{\lambda_w} = \begin{cases} 
\tau(\lambda_1 + 1, \ldots, \lambda_i - 1, 1, \lambda_{i+1}, \ldots, \lambda_n), p + \lambda_i - i + 1 & \text{for } w = (\delta_+, w(i)), \\
\tau(\lambda_1 + 1, \ldots, \lambda_i - 1, 1, \lambda_{i+1}, \ldots, \lambda_n), p - \lambda_i - 2n + i + 1 & \text{for } w = (\delta_-, w(i)).
\end{cases}$$

For $i = n$ a similar computation yields

$$\lambda'_w = \begin{cases} 
(\lambda_n - n + 1, \lambda_1 + 1, \ldots, \lambda_n - 2 + 1, \lambda_n - 1 + 1) & \text{for } w = (\delta_+, w(n)), \\
(-\lambda_n - n + 1, \lambda_1 + 1, \ldots, \lambda_n - 2 + 1, -\lambda_n - 1 - 1) & \text{for } w = (\delta_+, w(n)),
\end{cases}$$

and

$$W_{\lambda_w} = \begin{cases} 
\tau(\lambda_1 + 1, \ldots, \lambda_n - 2 + 1, \lambda_n - 1 + 1), p + \lambda_n - n + 1 & \text{for } w = (\delta_+, w(n)), \\
\tau(\lambda_1 + 1, \ldots, \lambda_n - 2 + 1, \lambda_n - 1 + 1), p - \lambda_n - n + 1 & \text{for } w = (\delta_-, w(i)).
\end{cases}$$

We now compute $\hat{c}_{\lambda_w} = ((\lambda_w \circ C_r) + \rho_Q)(\hat{X}_r)$. Since $\hat{X}_r = (-\|H_r\|^2)^{1/2}X_r$ and $C_r(X_r) = -\sqrt{-1}H_r$, we have

$$\hat{c}_{\lambda_w} = (-\|H_r\|^2)^{-1/2}(\rho_Q(X_r) - \sqrt{-1}\lambda_w(H_r)).$$

Here $\rho_Q(X_r) = \frac{1}{2}(2(2n - 2) + 2) = 2n - 1$ and

$$\lambda_w(H_r) = \begin{cases} 
\sqrt{-1}(p + \lambda_i - i + 1) & \text{for } w = (\delta_+, w(i)), \\
\sqrt{-1}(p - \lambda_i - 2n + i + 1) & \text{for } w = (\delta_-, w(i)),
\end{cases}$$

so that

$$\hat{c}_{\lambda_w} = (-\|H_r\|^2)^{-1/2} \cdot \begin{cases} 
-p + \lambda_i - i + 2n & \text{for } w = (\delta_+, w(i)), \\
-p - \lambda_i + i & \text{for } w = (\delta_-, w(i)).
\end{cases}$$

Proof of Theorem 24 [24]. In view of (24) and (24.1), a highest weight $\lambda = (\lambda', p) \in \mathbb{Z}_{n+}^n \times \mathbb{Z}$ is good if and only if for all $c \in \mathbb{R}$ the sum

$$\sum_{w \in W_{\lambda}} (-1)^{\ell(w)} W_{\lambda_w}$$

(2.2)
is contained in $\text{Res}(K^0(K))$. Since $(-1)^{(\delta_{\pm})} = 1$ and $(-1)^{(w^{(i)})} = (-1)^{i-1}$ the sum (2.2) without the restriction $c^2_{\lambda_{\omega}} = c^2$ takes the form

$$
\sum_{i=1}^{n-1} (-1)^{i-1} \left( \tau(\lambda_1+1, \ldots, \lambda_{i-1}+1, \lambda_{i+1}, \ldots, \lambda_n, \lambda_n), p+\lambda_i-i+1 + \tau(\lambda_1+1, \ldots, \lambda_{i-1}+1, \lambda_{i+1}, \ldots, \lambda_n, -\lambda_n), p-\lambda_i-2n+i+1 \right) \\
+ (-1)^{n-1} \left( \tau(\lambda_1+1, \ldots, \lambda_{n-2}+1, \lambda_{n-1}+1), p+\lambda_{n-1}+n+1 + \tau(\lambda_1+1, \ldots, \lambda_{n-2}+1, -\lambda_{n-1}-1), p-\lambda_{n-1}-n+1 \right).
$$

Each expression in parentheses in (2.3) is of the form $\tau_{\mu, q+s} + \tau_{\mu^*, q-s}$ with $q = p+n+1$ and $s = \lambda_i + n - i$ and therefore contained in $\text{Res}(K^0(K))$, thanks to Proposition 2.2. Moreover, this is the only possible way of combining two representations in (2.3) to a sum of the form $\tau_{\mu, q+s} + \tau_{\mu^*, q-s}$. Therefore, a highest weight $\lambda = (\lambda', p)$ is good if and only if for each expression in parentheses in (2.3) either both representations are separately contained in $\text{Res}(K^0(K))$ or both representations have the same value of $c^2_{\lambda_{\omega}}$. Note that in the $i$-th expression in parentheses $(1 \leq i \leq n)$ the two representations correspond to the Weyl group elements $w = (\delta_{\pm}, w^{(i)})$ and have therefore the same value of $c^2_{\lambda_{\omega}}$ if and only if either $\lambda_i = -(n-i)$ (in which case the $\hat{c}_{\lambda_{\omega}}$’s agree) or $p = n$ (in which case the $\hat{c}_{\lambda_{\omega}}$’s are opposite numbers).

Assume first that $\lambda_n = 0$, i.e. $V_{\lambda|\text{SO}(2n)}$ is self-dual. Then in the first $n-1$ expressions in parentheses in (2.3) every single representation is contained in $\text{Res}(K^0(K))$ by Proposition 2.2. Moreover, for $i = n$ we have $\lambda_i = 0 = -(n-i)$ and hence the values of $c^2_{\lambda_{\omega}}$ of the two representations in the last expression in parentheses agree. This shows that $\lambda$ is a good highest weight.

Now assume that $\lambda_n \neq 0$. Then none of the representations in (2.3) is separately contained in $\text{Res}(K^0(K))$, whence the highest weight $\lambda$ is good if and only if for each $i \in \{1, \ldots, n\}$ the values of $c^2_{\lambda_{\omega}}$ agree for the two Weyl group elements $w = (\delta_{\pm}, w^{(i)})$. As remarked above, this is only the case if for every $i \in \{1, \ldots, n\}$ either $\lambda_i = -(n-i)$ or $p = n$. For $i = 1$ we have $\lambda_1 \geq 0$, but $-(n-i) = -(n-1) < 0$ since $n \geq 2$ and therefore the highest weight $\lambda$ is good if and only if $p = n$. This finishes the proof, since $p = -\sqrt{-1} \lambda(H_0)$ for

$$
H_0 = \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
0_{2n}
\end{pmatrix}.
$$

\[\square\]

3. $G = \text{SO}^*(2n)$

In this section we obtain for $G = \text{SO}^*(2n)$ the full branching law from $K$ to $K_M$, i.e. the explicit description of the restriction map $\text{Res} : K^0(K) \to K^0(K_M)$. Moreover, we compute $P^*[\text{ev}], W_{\lambda_{\omega}}$, and $c^2_{\lambda_{\omega}}$.

3.1. Some subgroups of $\text{SO}^*(2n)$. Let $G = \text{SO}^*(2n)$, realized as

$$
\text{SO}^*(2n) = \left\{ g \in \text{GL}(2n, \mathbb{C}) : g^\top \begin{pmatrix} 1_n \\ 1_n \end{pmatrix} g = \begin{pmatrix} 1_n & 1_n \\ 1_n & -1_n \end{pmatrix}, g^* \begin{pmatrix} 1_n & -1_n \\ 1_n & 1_n \end{pmatrix} g = \begin{pmatrix} 1_n & -1_n \\ 1_n & 1_n \end{pmatrix} \right\},
$$

and choose the maximal compact subgroup

$$
K = \text{SO}^*(2n) \cap U(2n) = \left\{ \text{diag}(k, k) : k \in U(n) \right\} \simeq U(n).
$$
The Lie algebra \( g \) of \( G \) is given by
\[
    g = \left\{ \begin{pmatrix} A & B \\ B^* & -A^* \end{pmatrix} : A \in \mathfrak{u}(n), B \in \text{Skew}(n, \mathbb{C}) \right\}
\]
Put
\[
    X_k = \begin{pmatrix} 0_2 & 0_{n-2} & J \\ -J & 0_2 & 0_{n-2} \\ 0_{n-2} & 0_2 & 0_{n-2} \end{pmatrix}, \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
then \( \text{ad}(X_k) \) acts on \( g \) with eigenvalues 0, \( \pm 1 \), and \( \pm 2 \). Write \( \mathfrak{m}^1 \) for the 0-eigenspace and \( \mathfrak{n} \) for the direct sum of the positive eigenspaces, then \( \mathfrak{q} = \mathfrak{m}^1 \oplus \mathfrak{n} \) is a cuspidal maximal parabolic subalgebra of \( g \). We further decompose \( \mathfrak{m}^1 = \mathfrak{m} \oplus \mathfrak{a} \) where \( \mathfrak{a} = \mathbb{R} X_n \) and \( \mathfrak{m} \) is a direct sum of semisimple and compact abelian ideals. On the group level, \( Q = N_G(\mathfrak{q}) \) is a cuspidal maximal parabolic subgroup of \( G \) with Langlands decomposition \( Q = MAN \), where \( MA = Z_G(\mathfrak{a}) \), \( A = \exp(\mathfrak{a}) \) and \( N = \exp(\mathfrak{n}) \). The intersection \( K_M = K \cap M \) is maximal compact in \( M \) and given by
\[
    K_M = \{ \text{diag}(k, \tilde{k}) : k = \text{diag}(k_1, k_2), k_1 \in \text{SU}(2), k_2 \in \text{U}(n-2) \} \simeq \text{SU}(2) \times \text{U}(n-2).
\]

3.2. The branching law. Both \( K \) and \( K_M \) are connected, so we can describe irreducible representations in terms of their highest weights. Let \( t = \{ \text{diag}(k, \tilde{k}) : k = \sqrt{-1} \text{diag}(t_1, \ldots, t_n), t_i \in \mathbb{R} \} \), then \( t \) is a maximal torus in \( t \) and \( g \). The root system \( \Delta(t_G, t_C) \) is given by \( \{ \pm(\epsilon_i - \epsilon_j) : 1 \leq i < j \leq n \} \), where
\[
    \epsilon_i(\text{diag}(k, \tilde{k})) = \sqrt{-1} t_i
\]
if \( k \) is of the above form. We choose the positive system \( \Delta^+(t_G, t_C) = \{ \epsilon_i - \epsilon_j : 1 \leq i < j \leq n \} \).

With this notation, irreducible representations of \( K \) are parametrized by their highest weights \( \lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_n \epsilon_n \), where \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_+ \). Denote by \( \pi_\lambda \) the corresponding equivalence class of representations of \( K \).

The intersection \( t_M = t \cap t_M \) is a maximal torus in \( t_M \) and we write \( \bar{\epsilon}_i = \epsilon_i|_{t_M} \). Then
\[
    \bar{\epsilon}_1 = -\bar{\epsilon}_2 \quad \text{and} \quad \Delta^+(t_M, t_{C,M}) = \{ 2\bar{\epsilon}_1 \} \cup \{ \bar{\epsilon}_i - \bar{\epsilon}_j : 3 \leq i < j \leq n \}
\]
is a positive system in \( \Delta(t_M, t_{C,M}) \).

With this notation, irreducible representations of \( K_M \) are parametrized by highest weights \( p\bar{\epsilon}_1 + \nu_1 \bar{\epsilon}_2 + \cdots + \nu_{n-2} \bar{\epsilon}_n \), where \( p \geq 0 \) and \( \nu = (\nu_1, \ldots, \nu_{n-2}) \in \mathbb{Z}^{n-2}_+ \). Write \( \tau_{\nu,p} \) for the corresponding equivalence class of representations.

Lemma 3.1.
\[
    \pi_\lambda|_{K_M} \simeq \bigoplus_{\nu \in \mathbb{Z}^{n-2}_+} \tau_{\nu,p(\lambda,\nu)},
\]
where
\[
    p(\lambda, \nu) = \lambda_1 - \sum_{i=2}^{n-1} |\lambda_i - \nu_{i-1}| - \lambda_n \geq 0.
\]

Proof. We use the Gel'fand–Tsetlin basis for the representation \( \pi_\lambda \) of \( \text{U}(n) \subseteq \text{GL}(n, \mathbb{C}) \) (see e.g. [3]). Let \( \Lambda = (\Lambda_{ij})_{1 \leq j \leq i \leq n} \) be a Gel'fand–Tsetlin pattern with \( \Lambda_{n,j} = \lambda_j \) and \( \xi_\lambda \) be the corresponding weight vector in \( \pi_\lambda \). Then \( \xi_\lambda \) is a highest weight vector in an irreducible \( \text{GL}(n-2, \mathbb{C}) \)-representation if and only if \( \Lambda_{i,j} = \nu_j \) for all \( 1 \leq j \leq i \leq n-2 \) with some
\( \nu \in \mathbb{Z}^{n-2}_+ \) with \( \lambda_j \geq \nu_j \geq \lambda_{j+2} \) for \( 1 \leq j \leq n - 2 \). This explains the direct sum in the decomposition. Further, such a vector \( \xi_\Lambda \) is also a highest weight vector for \( \text{GL}(2, \mathbb{C}) \) (i.e. \( \pi_\lambda(E_{n-1,n}) \xi_\Lambda = 0 \)) if and only if for the second row \( \mu_j = \Lambda_{n-1,j} \), \( 1 \leq j \leq n - 1 \), we have

\[
\mu_j = \lambda_j \quad \text{or} \quad \mu_j = \nu_{j-1} \quad \forall \ j = 1, \ldots, n - 1.
\]

It is easy to see that \( \nu \subseteq \mu \subseteq \lambda \) (i.e. \( \Lambda \) is in fact a Gelfand–Tsetlin pattern) only if \( \mu_1 = \lambda_1 \) and \( \mu_i = \min(\lambda_i, \nu_{i-1}) \), \( 2 \leq i \leq n - 1 \). In this case the vector \( \xi_\Lambda \) satisfies

\[
\pi_\lambda(E_{n-1,n-1}) \xi_\Lambda = (|\mu| - |\nu|) \xi_\Lambda \quad \text{and} \quad \pi_\lambda(E_{n,n}) \xi_\Lambda = (|\lambda| - |\mu|) \xi_\Lambda,
\]

and hence is the highest weight vector in an irreducible \( \text{GL}(2, \mathbb{C}) \)-representation of dimension \( (|\mu| - |\nu|) - (|\lambda| - |\mu|) + 1 = 2|\mu| - |\lambda| - |\nu| + 1 \). Inserting the explicit form of \( \mu \) then shows that

\[
2|\mu| - |\lambda| - |\nu| = \lambda_1 - \sum_{i=2}^{n-1} |\lambda_i - \nu_{i-1}| - \lambda_n.
\]

### 3.3. The space \( p_{+}^{[\text{ev}]} \)
Define a homomorphism \( \kappa : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{so}^*(2n) \) on the standard basis \( E, F, H \in \mathfrak{sl}(2, \mathbb{R}) \) by

\[
\kappa(E) = \frac{\sqrt{-1}}{2} \begin{pmatrix} 1_2 & 0_{n-2} & -J \\ -J & -1_2 & 0_{n-2} \end{pmatrix}, \quad \kappa(F) = \frac{\sqrt{-1}}{2} \begin{pmatrix} -1_2 & 0_{n-2} & -J \\ -J & 1_2 & 0_{n-2} \end{pmatrix}
\]

and \( \kappa(H) = X_\kappa \). Then the decomposition \( \mathfrak{g} = \mathfrak{g}[0] \oplus \mathfrak{g}[1] \oplus \mathfrak{g}[2] \) into \( \kappa(\mathfrak{sl}(2, \mathbb{R})) \)-isotypic components is given by

\[
\mathfrak{g}[0] = \left\{ \begin{pmatrix} a & A & 0_2 \\ 0_2 & -a^\top \ B \end{pmatrix} : a \in \mathfrak{u}(2), \ tr(a) = 0, A \in \mathfrak{u}(n-2), B \in \text{Skew}(n-2, \mathbb{C}) \right\},
\]

\[
\mathfrak{g}[1] = \left\{ \begin{pmatrix} z & 0_{n-2} & w \\ 0_2 & -w^\top & 0_{n-2} \end{pmatrix} : z \in M(2 \times (n-2), \mathbb{C}) \right\},
\]

\[
\mathfrak{g}[2] = \kappa(\mathfrak{sl}(2, \mathbb{R})).
\]

The choice of \( \kappa \) determines

\[
p_+ = \left\{ \begin{pmatrix} 0_n & B \\ 0_n & 0_n \end{pmatrix} : B \in \text{Skew}(n, \mathbb{C}) \right\} \subseteq \left\{ \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} : A \in \mathfrak{gl}(n, \mathbb{C}), B, C \in \text{Skew}(n, \mathbb{C}) \right\} = \mathfrak{so}^*(2n)_{\mathbb{C}},
\]

so that

\[
p_{+}^{[\text{ev}]} = \mathfrak{p}_+^{[0]} = \left\{ \begin{pmatrix} 0_n & B \\ 0_n & 0_n \end{pmatrix} : B \in \begin{pmatrix} 0_2 & \text{Skew}(n-2, \mathbb{C}) \end{pmatrix} \right\}.
\]

As representation of \( K_M \) we have \( p_{+}^{[\text{ev}]} \simeq \tau_{(1,1,0,...,0),0} \).
3.4. The numbers $\hat{c}_{\lambda_w}$. We now compute the numbers $\hat{c}_{\lambda_w}$. The element $H_\kappa \in \mathfrak{g}$ is given by

\begin{equation}
H_\kappa = \kappa(E - F) = \sqrt{-1} \text{diag}(1, 1, 0, \ldots, 0, -1, -1, 0, \ldots, 0)
\end{equation}

which implies that

\begin{equation}
L_\kappa = \{ \text{diag}(k, k) : k = \text{diag}(k_1, k_2), k_1 \in U(2), k_2 \in U(n - 2) \} \simeq U(2) \times U(n - 2).
\end{equation}

Note that $t \subseteq l_\kappa$ and that the choice of positive roots $\Delta^+(t_\mathfrak{C}, t_\mathfrak{C})$ is compatible with $l_\kappa$ in the sense that $\Delta^+(l_\kappa, t_\mathfrak{C}) = \Delta(t_\mathfrak{C}, t_\mathfrak{C}) \cap \Delta^+(t_\mathfrak{C}, t_\mathfrak{C})$ is a positive system of roots for $l_\kappa$. The Weyl group $W(t_\mathfrak{C})$ of $\Delta(t_\mathfrak{C})$ is naturally isomorphic to the symmetric group $S_n$ in $n$ letters. Then

\begin{equation}
W_\kappa = \{ w \in W(t_\mathfrak{C}) : w^{-1} \alpha > 0 \forall \alpha \in \Delta^+(l_\kappa, t_\mathfrak{C}) \} = \{ w_{ij} : 1 \leq i < j \leq n \},
\end{equation}

where

\begin{equation}
w_{ij}(k) = \begin{cases} k + 2 & \text{for } 1 \leq k < i, \\ 1 & \text{for } k = i, \\ k + 1 & \text{for } i < k < j, \\ 2 & \text{for } k = j, \\ k & \text{for } j < k \leq n. \end{cases}
\end{equation}

Note that

\begin{equation}
(-1)^{\ell(w_{ij})} = (-1)^{i+j+1}.
\end{equation}

We further have

\begin{equation}
\rho_c = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}) = (\frac{n-2i+1}{2})_{i=1,\ldots,n}.
\end{equation}

Now let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_+^n$ be a highest weight of an irreducible $K$-representation. Then for $w = w_{ij}$ we have

\begin{equation}
\lambda_w = w_{ij}(\lambda + \rho_c) - \rho_c
= w_{ij}(\lambda_1 + \frac{n-1}{2}, \ldots, \lambda_n + \frac{1-n}{2}) - (\frac{n-1}{2}, \ldots, \frac{1-n}{2})
= (\lambda_i + \frac{n-2i+1}{2}, \lambda_j + \frac{n-2j+1}{2}, \lambda_1 + \frac{n-1}{2}, \ldots, \lambda_i + \frac{n-2i+1}{2}, \ldots, \lambda_j + \frac{n-2j+1}{2}, \ldots, \lambda_n + \frac{1-n}{2})
- (\frac{n-1}{2}, \ldots, \frac{1-n}{2})
= (\lambda_i - i + 1, \lambda_j - j + 2, \lambda_1 + 2, \ldots, \lambda_{i-1} + 2, \lambda_{i+1} + 1, \ldots, \lambda_{j-1} + 1, \lambda_{j+1}, \ldots, \lambda_n).
\end{equation}

Restricting the weight $\lambda_w$ to $t_M$ we obtain a dominant integral weight for $K_M$ which belongs to the representation

\begin{equation}
W\lambda_w = \tau(\lambda_1 + 2, \ldots, \lambda_{i-1} + 2, \lambda_{i+1} + 1, \ldots, \lambda_{j-1} + 1, \lambda_{j+1}, \ldots, \lambda_n), \lambda_i - i - j - 1.
\end{equation}

We now compute $\hat{c}_{\lambda_w} = ((\lambda_w \circ C_\kappa + \rho Q)(\hat{X}_\kappa))$. Since $\hat{X}_\kappa = (-\|H_\kappa\|^2)^{-1/2}X_\kappa$ and $C_\kappa(X_\kappa) = -\sqrt{-1}H_\kappa$ we have

\begin{equation}
\hat{c}_{\lambda_w} = ((\lambda_w \circ C_\kappa + \rho Q)(\hat{X}_\kappa)) = C_\kappa((\lambda_w \circ C_\kappa + \rho Q)(\hat{X}_\kappa))
= (\lambda_i + \lambda_j - i - j + 3) + (2n - 3) = \lambda_i + \lambda_j + 2n - i - j.
\end{equation}
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