ON CONVEX POLYTOPES IN $\mathbb{R}^d$
CONTAINING AND AVOIDING ZERO

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Abstract

The goal of this paper is to establish certain inequalities between the numbers of convex polytopes in $\mathbb{R}^d$ “containing” and “avoiding” zero provided that their vertex sets are subsets of a given finite set $S$ of points in $\mathbb{R}^d$. This paper is motivated by a question about these quantities raised by E. Boros and V. Gurvich in 2002.

Keywords: $d$-dimensional space, convex polytopes, zero-containing polytopes, zero-avoiding polytopes.

1 Introduction

The notions and facts used but not described here can be found in [1].

Let $S$ be a finite set of points in $\mathbb{R}^d$, $X \subseteq S$, and $z \in \mathbb{R}^d \setminus S$.

A set $X$ is called a $z$-containing set (a $z$-avoiding set) if $z$ is in the interior of the convex hull of $X$ (respectively, $z$ is not in the interior of the convex hull of $X$).

A $z$-containing set $X$ is minimal if $X$ has no proper $z$-containing subset.

A $z$-avoiding set $X$ is maximal in $S$ if $S$ has no $z$-avoiding subset containing $X$ properly.

Let $\mathcal{C}(S)$ and $\mathcal{A}(S)$ denote the sets of minimal $z$-containing and maximal $z$-avoiding subsets of $S$, respectively.

In 2002 E. Boros and V. Gurvich raised the following interesting question.

Question 1.1 Suppose that $S$ is a finite set of points in $\mathbb{R}^d$, $z \in \mathbb{R}^d \setminus S$, and $S$ is a $z$-containing set. Is it true that $|\mathcal{A}(S)| \leq 2d |\mathcal{C}(S)|$?

Questions of this type arise naturally in the algorithmic theory of a so-called efficient enumeration of different type of geometric or combinatorial objects (see, for example, [24]).

For an affine subspace $F$ of $\mathbb{R}^d$, let $\dim(F)$ denote the affine dimension of $F$ and $R(X)$

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denote the minimal affine subspace in $\mathbb{R}^d$ containing $X$.

A finite set $S$ of points in $\mathbb{R}^d$ is said to be in a general position if for every $X \subseteq S$, $|X| - 1 \leq d \Rightarrow \dim(R(X)) = |X| - 1$.

We say that $z$ is in a general position with respect to $S$ if $\dim(R(X)) < d \Rightarrow z \notin R(X)$ for every $X \subseteq S$.

One of our main results is the following theorem.

**Theorem 1.1** Let $S$ be a finite set of points in the $d$-dimensional space $\mathbb{R}^d$, $z \in \mathbb{R}^d \setminus S$. Suppose that $z$ is in a general position with respect to $S$. Then $|\mathcal{A}(S)| \leq d |\mathcal{C}(S)| + 1$.

This theorem was announced in [3] and its proof was presented at the RUTCOR seminar directed by E. Boros and V. Gurvich in June 2002.

Later L. Khachian gave a construction providing for every $d \geq 4$ a counterexample $(S, z)$ in $\mathbb{R}^d$ to the inequality in Question 1.1 such that $S$ is not in a general position (see [2]). From Theorem 1.1 it follows that in all these counterexamples $z$ is not in a general position with respect to $S$.

In [3] it is shown that if $S$ is a finite set of points in the plane $\mathbb{R}^2$, $z \in \mathbb{R}^2 \setminus S$, and $S$ is a $z$-containing set, then $|\mathcal{A}(S)| \leq 3|\mathcal{C}(S)| + 1$, and so the inequality in Question 1.1 is true for the plane.

In this paper we also give some strengthenings of Theorem 1.1.

## 2 Some notions, notation, and auxiliary facts

Given a convex polytope $P$ in $\mathbb{R}^d$, a face $F$ of a polytope $P$ is called a facet of $P$ if $\dim(R(F)) = d - 1$. Obviously, $R(F)$ is a hyperplane; we call $R(F)$ a facet hyperplane of $P$.

Let, as above, $S$ be a finite set of points in $\mathbb{R}^d$, $X \subseteq S$, and $z \in \mathbb{R}^d \setminus S$. Let $s \in S$.

We will use the following notation:

- $\text{conv}(X)$ is the convex hull of $X$,
- if $P$ is a convex polytope, then $V(P)$ is the set of vertices of $P$ and $v(P) = |V(P)|$,
- as above, $\mathcal{C}(S)$ is the set of minimal $z$-containing subsets of $S$; also $\mathcal{C}_s(S)$ is the set of members of $\mathcal{C}(S)$ containing $s$,
- as above, $\mathcal{A}(S)$ is the set of maximal $z$-avoiding subsets of $S$,
- $\mathcal{A}^*(S)$ is the set of subsets $A$ in $S$ such that $A$ is maximal $z$ avoiding in $S$ and $A \setminus s$ is maximal $z$-avoiding in $S \setminus s$, and so $A \in \mathcal{A}(S)$ and $A \setminus s \in \mathcal{A}(S \setminus s)$,
- $\mathcal{A}_s(S) = \mathcal{A}(S) \setminus \mathcal{A}^*(S) = \{X \in \mathcal{A}(S) : s \in X \text{ and } X \setminus s \notin \mathcal{A}(S \setminus s)\}$,
- $\text{Smpl}(S)$ is the set of simplexes $C$ such that $z$ is an interior point of $C$ and $V(C) \subseteq S$ and $\text{Smpl}_s(S)$ is the set of simplexes $C$ in $\text{Smpl}(S)$ such that $s \in V(C)$,
Lemma 2.1 Let \( \text{Conv}(S) = \{\text{conv}(X) : X \in \mathcal{C}(S)\} \) and \( \text{Conv}_s(S) \) is the set of members of \( \text{Conv}(S) \) containing \( s \),

- \( \mathcal{H}(S) \) is the set of hyperplanes \( H \) such that \( H \) is a facet hyperplane of a simplex in \( \text{Smpl}(S) \) and \( \mathcal{H}_a(S) \) is the set of hyperplanes in \( \mathcal{H}(S) \) containing \( s \), and

- \( \mathcal{F}(S) \) is the set of subsets \( T \) of \( S \) such that \( |T| = d \) and \( \text{conv}(T) \) is a face of a simplex in \( \text{Smpl}(S) \) and \( \mathcal{F}_a(S) \) is the set of subsets of \( S \) in \( \mathcal{F}(S) \) containing \( s \).

Obviously, we have the following.

**Lemma 2.1** Let \( S \) be a finite set of points in \( \mathbb{R}^d \). Then

(a1) \( |\text{Conv}(S)| = |\mathcal{C}(S)| \),

(a2) \( |\mathcal{H}(S)| \leq |\mathcal{F}(S)| \leq (d + 1)|\text{Smpl}(S)| \), and \( |\text{Conv}_s(S)| = |\mathcal{C}_s(S)| \),

(a3) \( |\mathcal{H}_a(S)| \leq |\mathcal{F}_a(S)| \leq d |\text{Smpl}_a(S)| \), and

(a4) \( |\mathcal{A}^s(S)| = |\mathcal{A}(S \setminus s)| \), and so \( |\mathcal{A}(S) - |\mathcal{A}(S \setminus s)| = |\mathcal{A}_s(S)| \).

We recall that \( z \) is in a general position with respect to \( S \) if \( \dim(R(X)) < d \Rightarrow z \notin R(X) \) for every \( X \subseteq S \).

We will use the following well known and intuitively obvious fact.

**Lemma 2.2** Let \( P \) be a convex polytope in \( \mathbb{R}^d \) and \( z \) a point in the interior of \( P \). Suppose that \( z \) is in general position with respect to \( V(P) \). Then there exists \( X \subseteq V(P) \) such that \( \text{conv}(X) \) is a simplex of dimension \( d \) and \( z \) is in the interior of \( \text{conv}(X) \).

**Proof** We prove our claim by induction on the dimension \( d \). The claim is obviously true for \( d = 1 \). We assume that the claim is true for \( d = n - 1 \) and will prove that the claim is also true for \( d = n \), where \( n \geq 2 \). Thus, \( P \) is a convex polytope in \( \mathbb{R}^n \). Since \( z \) is in a general position with respect to \( V(P) \), clearly \( z \notin V(P) \). Let \( p \in V(P) \) and \( L \) the line containing \( p \) and \( z \). Then there exists the point \( t \) in \( L \cap P \) such that the closed interval \( pL \) contains \( z \) as an interior point and \( t \) is not an interior point of \( P \). In a plain language, \( pL \) is the ray going from point \( p \) through point \( z \) and \( t \) is the first point of the ray which is not the interior point of \( P \). Then \( t \) belongs to a face \( F \) of \( P \) which is a convex polytope of dimension at most \( n - 1 \). Since \( z \) is in a general position with respect to \( V(P) \), the dimension of \( F \) is \( n - 1 \) and \( t \) is in a general position with respect to \( V(F) \). By the induction hypothesis, there exists \( T \subseteq V(F) \) such that \( \text{conv}(T) \) is a simplex of dimension \( n - 1 \) and \( t \) is in the interior of \( \text{conv}(T) \). Put \( X = T \cup p \). Then \( X \subseteq V(P) \), \( \text{conv}(X) \) of dimension \( n \) is a simplex and \( z \) is in the interior of \( \text{conv}(X) \). \( \square \)

**Lemma 2.2** also follows from the Caratheodory Theorem (see [1]) and **Lemma 2.4** below.

Given \( X, Y \subset \mathbb{R}^d \) and a hyperplane \( H \), we say that \( H \) separates \( X \) and \( Y \) (or separates \( X \) from \( Y \)) if \( X \setminus H \) and \( Y \setminus H \) belong to different half-spaces of \( \mathbb{R}^d \setminus H \).

It is easy to see the following.
Lemma 2.3 Let $A$ and $A'$ be z-avoiding subsets of $S$. Then

(a1) there exists a hyperplane separating $A$ from $z$,

(a2) if there exists a hyperplane separating both $A$ and $A'$ from $z$, then $A \cup A'$ is also a z-avoiding subsets of $S$, and therefore

(a3) if $A$ and $A'$ are maximal z-avoiding subsets of $S$ and there exists a hyperplane separating both $A$ and $A'$ from $z$, then $A = A'$.

We also need the following simple facts.

Lemma 2.4 Let $z$ be in a general position with respect to $S$ and $X \subseteq S$. Then the following are equivalent:

(a1) $z \in \text{conv}(X)$ and

(a2) $z$ is in the interior of $\text{conv}(X)$.

Proof Obviously, (a2) implies (a1). Suppose, on the contrary, (a1) does not imply (a2). Then $z$ belongs to $\text{conv}(X')$ for some $X' \subseteq X$ with $\text{dim}(R(X')) < d$. Therefore $z$ is not in a general position with respect to $S$, a contradiction. \qed

Lemma 2.5 Let $z$ be in a general position with respect to $S$. If $C$ is a minimal z-containing subset of $S$, then $\text{conv}(C)$ is a simplex, and so $\text{Smpl}(S) = \text{Conv}(S)$.

Proof (uses Lemmas 2.2 and 2.4). Since $C$ is z-containing, there exists $X \subseteq C$ such that $z \in \text{conv}(X)$ and $\text{conv}(X)$ is a simplex. By Lemma 2.4, $z$ is in the interior of $\text{conv}(X)$. Since $C$ is minimal z-containing, clearly $C = X$. Now by Lemma 2.2, $\text{conv}(C)$ is a simplex. \qed

Lemma 2.6 Let $z$ be in a general position with respect to $S$. Suppose that $X \subseteq S$ and $s \in S \setminus X$. Let $L$ be the line in $\mathbb{R}^d$ containing $s$ and $z$. If $\text{dim}(R(X)) \leq d - 2$, then $L \cap R(X) = \emptyset$.

Proof Suppose, on the contrary, $L \cap R(X) \neq \emptyset$. Then $z \in R(X \cup s)$ and $\text{dim}(R(X \cup s)) < d$. Then $z$ is not in a general position with respect to $S$, a contradiction. \qed

Lemma 2.7 Let $S$ be a z-containing set and $z$ in a general position with respect to $S$. Let $A$ be a maximal z-avoiding set in $S$ and $s \in S \setminus A$. Then $A$ has a subset $T$ such that

(a1) $|T| = d$,

(a2) $T$ belongs to a facet of $\text{conv}(A)$,

(a3) $\text{conv}(T \cup s)$ is a simplex of dimension $d$ containing $z$ as an interior point, and

(a4) $R(T)$ is a facet hyperplane of $A$ separating $A$ from $z$.

Proof (uses Lemmas 2.2 and 2.6). Let $L$ be the line in $\mathbb{R}^d$ containing $s$ and $z$. Since $s \not\in A$ and $A$ is a maximal z-avoiding subset in $S$, clearly $z$ is in the interior of $\text{conv}(A \cup s)$.

Let $I = L \cap \text{conv}(A \cup s)$. Since $\text{conv}(A \cup s)$ is a convex set, clearly $I$ is a line segment $sLr$, where $r$ is not an interior point of $\text{conv}(A \cup s)$. We claim that $r \in \text{conv}(A)$. Indeed, if not, then $r$ belongs to a facet $R$ of $\text{conv}(A \cup s)$ containing $s$. Since $sLr$ is a convex set and
s, r ∈ R, we have: \( z ∈ sLr ⊆ R \), and so \( z ∈ R \). It follows that \( z \) is not an interior point of \( \text{conv}(A ∪ s) \), a contradiction. Thus, \( \text{conv}(A) ∩ L ≠ ∅ \).

Since \( s ∉ A \), there exists a unique point \( t \) in \( L \) such that the closed interval \( sLt \) in \( L \) with the end-points \( s \) and \( t \) has the properties: \( z \) is in the interior of \( sLt \) and \( \text{conv}(A) ∩ sLt = t \). In a plane language, \( sL \) is the ray going from point \( s \) through point \( z \) and \( t \) is the first point of the ray belonging to \( \text{conv}(A) \). By Lemma 2.6 \( t \) belongs to the interior of a facet \( F \) of \( \text{conv}(A) \). Hence, by Lemma 2.2 \( F \) has a set \( T \) of \( d \) vertices such that \( T ⊂ A \) and \( \text{conv}(T) \) is a simplex of dimension \( d − 1 \) containing \( t \) as an interior point. Then \( \text{conv}(T ∪ s) \) is a simplex of dimension \( d \) containing \( z \) as an interior point, and so \( t \) is an interior point of \( \text{conv}(T) \). □

# 3 Main results

First we will prove a weaker version of our main result to demonstrate the key idea concerning the relation between the minimal \( z \) containing and maximal \( z \)-avoiding subsets of \( S \).

**Theorem 3.1** Suppose that \( S \) is a \( z \)-containing set and \( z \) is in a general position with respect to \( S \). Then \( |A(S)| ≤ (d + 1)|C(S)| \).

**Proof** (uses Lemmas 2.3(a4), 2.5 and 2.7). Let \( A \) be a maximal \( z \)-avoiding subset of \( S \). Since \( S \) is a \( z \)-containing set, there exists \( s ∈ S \setminus A \). Since \( A \) is a maximal \( z \)-avoiding subset of \( S \), \( A ∪ s \) is not a \( z \)-avoiding subset of \( S \). Hence \( A ∪ s \) is a \( z \)-containing set, and so \( z \) is in the interior of \( A ∪ s \). By Lemma 2.7 \( A \) has a subset \( T \) such that \( |T| = d \), \( T \) belongs to a face of \( \text{conv}(A) \), and \( \text{conv}(T ∪ s) \) is a simplex containing \( z \) as an interior point.

Let, as above, \( \mathcal{H}(S) \) denote the set of hyperplanes \( H \) such that \( H \) is a facet hyperplane of a simplex in \( \text{Smpl}(S) \) and \( \mathcal{F}(S) \) denote the set of subsets \( T \) of \( S \) such that \( |T| = d \) and \( \text{conv}(T) \) is a facet of a simplex in \( \text{Smpl}(S) \). Obviously, \( |\mathcal{H}(S)| ≤ |\mathcal{F}(S)| = (d + 1)|\text{Smpl}(S)| \).

By Lemmas 2.3(a4) and 2.7, \( |A(S)| ≤ |\mathcal{H}(S)| \). Clearly, \( |\text{conv}(S)| = |C(S)| \) and, by Lemma 2.5 \( \text{Smpl}(S) = \text{Conv}(S) \). Thus, \( |A(S)| ≤ (d + 1)|C(S)| \). □

One of the referees informed us that Theorem 3.1 was formulated in terms of minimal infeasible subsystems and proved in a different way in [4].

A hyperplane \( H \) in \( \mathcal{H}(S) \) is said to be **essential** if \( H \) is a facet hyperplane of a maximal \( z \)-avoiding subset \( A \) in \( S \) separating \( A \) from \( z \), and **non-essential**, otherwise. Let \( \mathcal{H}(S) \) and \( \mathcal{H}_e(S) \) denote the sets of essential hyperplanes in \( \mathcal{H}(S) \) and \( \mathcal{H}_s(S) \), respectively.

**Lemma 3.1** Let \( S \) be a finite set of points in \( \mathbb{R}^d \), \( z ∈ \mathbb{R}^d \setminus S \), and \( s ∈ S \). Suppose that \( S \) is a \( z \)-containing set and \( z \) is in a general position with respect to \( S \). Then

\[
|A(S)| - |A(S \setminus s)| = |A_s(S)| ≤ |A_e(S)| ≤ |\mathcal{H}_s(S)| ≤ |\mathcal{F}_s(S)| ≤ |δ_1| \text{Smpl}_s(S) = d |C_s(S)|.
\]

**Proof** (uses Lemmas 2.3(a4), 2.5 and 2.7). We prove that \( |A_s(S)| ≤ |\mathcal{H}_e(S)| \). Let \( A ∈ A_s(S) \). Then \( s ∈ A \) and \( A' = A \setminus s \) is a \( z \)-avoiding but not maximal \( z \)-avoiding set in \( S' = S \setminus s \). Therefore there exists \( s' ∈ S' \setminus A' \) such that \( A' ∪ s' \) is also a \( z \)-avoiding set. Obviously, \( A' ∪ \{s, s'\} \) is a \( z \)-containing set in \( S \). By Lemma 2.7 \( A \) has a subset \( T \) such that \( |T| = d \), \( T \) belongs to a face of \( \text{conv}(A) \), and \( \text{conv}(T ∪ s) \) is a simplex containing
z as an interior point. Since $A' \cup s'$ is a z-avoiding set, clearly $s \in T$. Now by Lemmas 2.3(a4) and 2.7, $|A_s(S)| \leq |H_s(S)|$. By Lemma 2.5, $\text{Smpl}_s(S) = \text{Conv}_s(S)$, and clearly, $|\text{Conv}_s(S)| = |C_s(S)|$. All the other inequalities in our claim are obvious. □

Now we are ready to prove the following strengthening of Theorem 3.1 which is also an extension of Theorem 3.1.

**Theorem 3.2** Let $S$ be a finite set of points in the $d$-dimensional space $\mathbb{R}^d$. Suppose that $z$ is in a general position with respect to $S$ (and so $z \in \mathbb{R}^d \setminus S$). Then

(a1) if either $S$ is z-avoiding or $S$ is z-containing and $|S| = d + 1$ (and so $\text{conv}(S)$ is a $d$-dimensional simplex), then $|A(S)| = d |C(S)| + 1$,

(a2) if $S$ is z-avoiding or $S$ is z-containing and $|S| = d + 1$, then $|A(S)| = d |C(S)| + 1$,

(a3) if $S$ is z-containing and $|S| = d + 2$, then $|A(S)| = d |C(S)| - d + 1$, and

(a4) if $S$ is z-containing and $|S| \geq d + 3$, then $|A(S)| \leq d |C(S)| - d$.

**Proof** (uses Lemmas 2.5 and 3.1).

(p1) First we prove (a1). If $S$ is z-avoiding, then $|A(S)| = 1$ and $|C(S)| = 0$, and so $|A(S)| = d |C(S)| + 1$. If $S$ is z-containing and $|S| = d + 1$, then $\text{conv}(S)$ is a $d$-dimensional simplex, $z$ is in the interior of $\text{conv}(S)$, $|A(S)| = d + 1$, and $|C(S)| = 1$, and therefore $|A(S)| = d |C(S)| + 1$.

(p2) We prove (a2). Our claim is obviously true if $S$ is a z-avoiding set. Therefore we assume that $S$ is a z-containing set. We prove our claim by induction on $|S|$. By Lemma 2.5, $|S| \geq d + 1$. If $|S| = d + 1$, then $\text{conv}(S)$ is a simplex, and our claim is obviously true. Thus, we assume that our claim is true for every z-avoiding set $S$ with $|S| = k \geq d + 1$ and will prove that the claim is also true if $|S| = k + 1$. Since $S$ is z-containing, by Lemma 2.5 there exists $X \subseteq S$ such that $\text{conv}(X)$ is a simplex and $z$ is the interior point of $\text{conv}(X)$. Since $|X| = d + 1 < |S|$, there exists $s \in S \setminus X$. Obviously, $S' = S \setminus s$ is a z-containing set. Since $k = |S'| < |S| = k + 1$, by the induction hypothesis, our claim is true for $S' = S \setminus s$, i.e. $|A(S')| \leq d |C(S')| + 1$. By Lemma 3.1, $|A(S)| - |A(S')| = |A_s(S)| \leq d |C_s(S)|$. Now since $|C(S)| = |C(S')| + |C_s(S)|$, our inductive step follows.

(p3) We prove (a3). Since $S$ is z-containing, there exists $S' \subset S$ such that $\Delta = \text{conv}(S')$ is a simplex and $z$ is in the interior of $\text{conv}(S')$. We can assume that $\text{conv}(S')$ is a minimal (by inclusion) simplex such that $S' \subset S$ and $z$ is in the interior of $\text{conv}(S')$. Then the interior of $\Delta$ does not contain points from $S$. Clearly, there is $s \in S$ such that $S' = S \setminus s$. Let $L$ be the line containing $s$ and $z$ and $sLt$ be the closed interval in $L$ such that $z$ is in the interior of $sLt$ and $t$ belongs to a face of $\Delta$. Let $sLt'$ be the maximal closed interval in $L$ that has no interior point of $\Delta$. Obviously, there exist faces $F$ and $F'$ of $\Delta$ containing $t$ and $t'$, respectively. In plane language, $t'$ and $t$ are the first and the last common points of the ray $sLt$ with $\Delta$. Since $z$ is in a general position with respect to $S$, clearly $t$ and $t'$ are the interior points of $F$ and $F'$, respectively, and $\text{dim}(R(F)) = d - 1$, and so $\nu(F) = d$. Let $s'' = S' \setminus V(F)$. Then $\Delta' = \text{conv}(F \cup s) = \text{conv}(S \setminus s')$ is a z-containing simplex, and so $V(F) \cup s$ is a minimal z-containing subset of $S$. By the above definition, $\Delta = \text{conv}(S \setminus s) = \text{conv}(F \cup s')$ is another z-containing simplex, and so $V(F) \cup s' = S \setminus s$ is another minimal z-containing subset of $S$. 

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We claim that $S \setminus x$ is a $z$-avoiding subset of $S$ for every $x \in V(F)$. To prove this claim we consider two cases. Suppose first that $x \in V(F) \setminus V(F')$. Then $S \setminus x = V(F') \cup s$ and by the definition of $F'$, $z$ is not in the interior of $conv(V(F') \cup s)$. Therefore $S \setminus x$ is $z$-avoiding. Now suppose that $x \in V(F) \cap V(F')$. Then $S \setminus x = V(F'') \cup s$, where $F''$ is a face of $\Delta$ distinct from $F$ and $F'$. Obviously, $L \cap F'' = \emptyset$. Therefore $z$ is not an interior point of $Conv(V(F'') \cup s)$, and so again $S \setminus x$ is $z$-avoiding.

Obviously, if $S \setminus x$ is $z$-avoiding, then $S \setminus x$ is also maximal $z$-avoiding. Therefore $S \setminus x$ is a maximal $z$-avoiding subset of $S$ for every $x \in V(F)$. Also $V(F)$ is a $z$-avoiding subset of $S$ and since both $V(F) \cup s$ and $V(F') \cup s'$ are $z$-containing, clearly $V(F)$ is a maximal $z$-avoiding subset of $S$. Thus, $\mathcal{A}(S) = \{S \setminus x : x \in V(F)\}$. Also both $\Delta = conv(F \cup s')$ and $\Delta' = conv(F \cup s)$ are $z$-containing simplexes, and so $\mathcal{C}(S) = \{S \setminus s, S \setminus s'\}$. Therefore, $d + 1 = \vert \mathcal{A}(S) \vert = d \vert \mathcal{C}(S) \vert = d + 1$. It follows that (a3) holds.

(p4) The proof of (a4) is similar to that in (p3) because it can be checked that the inequality holds if $\vert S \vert = d + 3$. Claim (a4) is also a particular case of Theorem 3.2 below. □

Our next goal is to prove a strengthening of Theorem 3.2 that takes into consideration the size of $S$.

Lemma 3.2 Let $S$ be a finite set of points in $\mathbb{R}^d$, and $z$ in a general position with respect to $S$ (and so $z \in \mathbb{R}^d \setminus S$). Let $P = conv(S)$ and $V = V(P)$. Suppose that $\vert S \vert \geq d + 2$ and $S$ is a $z$-containing set. Then there exist a simplex $\Delta$ and $u \in S$ such that

(a1) $V(\Delta) \subseteq V$,

(a2) $z$ is an interior point of $\Delta$,

(a3) $\Delta$ and $P$ have a common facet hyperplane $H$, and

(a4) $u \in H$ and $S \setminus u$ is a $z$-containing set.

Proof (uses Lemma 2.2). We will consider two cases: $P$ is a simplex or $P$ is not a simplex.

(p1) Suppose that $P$ is a simplex. Let $\Delta = P$. Since $\vert S \vert \geq d + 2$, there exists $v \in S \setminus V(P)$. Suppose that $v$ is in the interior of $P$. Then obviously, there exists a facet $F$ of $P$ such that $z$ is in the interior of $conv(F \cup s)$. Let $u$ be a unique vertex of $V(P)$ not belonging to $F$. Then $(\Delta, u)$ satisfies (a1) - (a4), where $H$ is the hyperplane containing $F$.

Now suppose that $v$ is not in the interior of $P$. Then there exists a facet $T$ of $P$ containing $v$. Put $u = v$. Then again $(\Delta, u)$ satisfies (a1) - (a4), where $H$ is the hyperplane containing $T$.

(p2) Finally, suppose that $P$ is not a simplex. By Lemmas 2.2 there exists a simplex $\Delta_z$ such that $V(\Delta_z) \subseteq V$ and $z$ is in the interior of $\Delta_z$. Since $P$ is not a simplex, $\Delta_z \neq P$. Therefore $\Delta_z$ has a facet $F_z$ which is not a facet of $P$. Let $v$ be a unique vertex of $\Delta_z$ such that $v \notin F_z$. Let $L$ be the line containing $v$ and $z$. Let $vLz'$ be a maximal segment in $L$ such that $z \in vLz'$ and $vLz' \subseteq P$. Obviously, such segment exists (and is unique) and $z'$ belongs to a face $F$ of $P$. Moreover, since $z$ is in a general position with respect to $S$, clearly $F$ is a facet of $P$ and point $z'$ is in a general position with respect to $S' = V(F)$ (and so $S'$ is a $z'$-containing set) in $\mathbb{R}^{d-1}$. By Lemma 2.2 $F$ contains an $(d - 1)$-dimensional simplex $\Delta'$ such that $z'$ is in the interior of $\Delta'$. Then $\Delta = conv(\Delta' \cup v)$ is a $d$-dimensional simplex
satisfying \((a1), (a2),\) and \((a3)\) with \(H = R(F) = R(\Delta')\). Since \(F\) is a facet of \(P\), clearly \(V(F) \subset V(P) \subseteq S\) and \(V(F) \setminus V(F_z) \neq \emptyset\). Let \(u\) be an arbitrary point of \(V(F) \setminus V(F_z)\). Obviously, \(u \neq v\), for otherwise, \(z \in vLz' \subseteq F\), and so \(z \in F \subset H\). This is impossible because \(z\) is in the interior of \(P\). Thus, \(\Delta \subset \text{conv}(S \setminus u)\), and so \(S \setminus u\) is \(z\)-containing. \(\square\)

Now we are ready to prove the following strengthening of Theorem 3.2.

**Theorem 3.3** Let \(S\) be a finite set of points in the \(d\)-dimensional space \(\mathbb{R}^d\), \(z \in \mathbb{R}^d \setminus S\). Suppose that \(S\) is \(z\)-containing, \(z\) is in a general position with respect to \(S\), and \(|S| \geq d + 2\). Then \(|A(S)| \leq d \cdot |C(S)| - |S| + 3\).

**Proof** (uses Lemmas 3.1, 3.2 and Theorem 3.2). We prove our claim by induction on \(|S|\). If \(|S| = d + 2\), then by Theorem 3.2 \((a3)\), the claim is true. We assume that our claim is true for \(|S| = k \geq d + 2\) and prove that it is also true if \(|S| = k + 1\). By Lemma 3.1

\[|A(S)| - |A(S \setminus s)| = |A_s(S)| \leq |H^c_s(S)| \leq |H_s(S)| \leq d \cdot |S_{\text{mpl}}(S)| = d \cdot |C_s(S)|\]

for every \(s \in S\). By Lemma 3.2 there exist a point \(u\) in \(S\) and a simplex \(\Delta\) in \(S_{\text{mpl}}(S)\) such that \(S \setminus u\) is \(z\)-containing and \(R(T) = R(F)\) for some facets \(T\) and \(F\) of \(\Delta\) and \(\text{conv}(S)\), respectively. Then obviously, \(R(T)\) is a non-essential hyperplane in \(H_u(S)\), and therefore \(|H^c_u(S)| \leq |H_u(S)| - 1\). Therefore

\[|A(S)| - |A(S \setminus u)| = |A_u(S)| \leq |H^c_u(S)| < |H_u(S)| \leq d \cdot |C_u(S)|\]

By the induction hypothesis, we have: \(|A(S \setminus u)| \leq d \cdot |C(S \setminus u)| - |S \setminus u| + 3\).

Thus, our inductive step follows from the last two inequalities. \(\square\)

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