Filter Dimension

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1 Introduction

Throughout the paper, $K$ is a field, a module $M$ over an algebra $A$ means a left module denoted $AM$, $\otimes = \otimes_K$.

Intuitively, the filter dimension of an algebra or a module measures how ‘close’ standard filtrations of the algebra or the module are. In particular, for a simple algebra it also measures the growth of how ‘fast’ one can prove that the algebra is simple.

The filter dimension appears naturally when one wants to generalize the Bernstein’s inequality for the Weyl algebras to the class of simple finitely generated algebras.

The $n$’th Weyl algebra $A_n$ over the field $K$ has $2n$ generators $X_1, \ldots, X_n, \partial_1, \ldots, \partial_n$ that satisfy the defining relations

$$\partial_iX_j - X_j\partial_i = \delta_{ij}, \text{ the Kronecker delta, } X_iX_j - X_jX_i = \partial_i\partial_j - \partial_j\partial_i = 0,$$
for all $i, j = 1, \ldots, n$. When char $K = 0$ the Weyl algebra $A_n$ is a simple Noetherian finitely generated algebra canonically isomorphic to the ring of differential operators $\mathbb{K}[X_1, \ldots, X_n, \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}]$ with polynomial coefficients ($X_i \leftrightarrow X_i, \partial_i \leftrightarrow \frac{\partial}{\partial X_i}, i = 1, \ldots, n$).

Let $\text{K.dim}$ and $\text{GK}$ be the (left) Krull (in the sense of Rentschler and Gabriel, [22]) and the Gelfand-Kirillov dimension respectively.

**Theorem 1.1 (The Bernstein’s inequality, [8])** Let $A_n$ be the $n$’th Weyl algebra over a field of characteristic zero. Then $\text{GK}(M) \geq n$ for all nonzero finitely generated $A_n$-modules $M$.

Let $A$ be a simple finitely generated infinite dimensional $K$-algebra. Then $\dim_K(M) = \infty$ for all nonzero $A$-modules $M$ (the algebra $A$ is simple, so the $K$-linear map $A \to \text{Hom}_K(M, M), a \mapsto (m \mapsto am)$, is injective, and so $\infty = \dim_K(A) \leq \dim_K(\text{Hom}_K(M, M))$ hence $\dim_K(M) = \infty$). So, the Gelfand-Kirillov dimension (over $K$) $\text{GK}(M) \geq 1$ for all nonzero $A$-modules $M$.

**Definition.** $h_A := \inf\{\text{GK}(M) \mid M$ is a nonzero finitely generated $A$-module$\}$ is called the **holonomic number** for the algebra $A$.

**Problem.** For a simple finitely generated algebra find its holonomic number.

To find an approximation of the holonomic number for simple finitely generated algebras and to generalize the Bernstein’s inequality for these algebras was a main motivation for introducing the filter dimension, [4]. In this paper $d$ stands for the filter dimension $\text{fd}$ or the left filter dimension $\text{lfd}$. The following two inequalities are central for the proofs of almost all results in this paper.

- **The First Filter Inequality, [4].** Let $A$ be a simple finitely generated algebra. Then

$$\text{GK}(M) \geq \frac{\text{GK}(A)}{d(A) + \max\{d(A), 1\}}$$

for all nonzero finitely generated $A$-modules $M$.

- **The Second Filter Inequality, [5].** Under a certain mild conditions (Theorem 4.2) the (left) Krull dimension of the algebra $A$ satisfies the following inequality

$$\text{K.dim}(A) \leq \text{GK}(A)(1 - \frac{1}{d(A) + \max\{d(A), 1\}})$$

The paper is organized as follows. Both filter dimensions are introduced in Section 2. In Sections 3 and 4 the first and the second filter inequalities are proved respectively. In Section 4 we use both filter inequalities for giving short proofs of some classical results about the rings $\mathcal{D}(X)$ of differential operators on smooth irreducible affine algebraic varieties. The (left) filter dimension of $\mathcal{D}(X)$ is 1 (Section 5). A concept of multiplicity for the filter dimension and a concept of holonomic module for (simple) finitely generated algebras appear in Section 6. Every holonomic module has finite length (Theorem 6.8). In Section 7
an upper bound is given (i) for the Gelfand-Kirillov dimension of commutative subalgebras of simple finitely generated infinite dimensional algebras (Theorem 7.2), and (ii) for the transcendence degree of subfields of quotient rings of (certain) simple finitely generated infinite dimensional algebras (Theorems 7.4 and 7.5). In Section 8 a similar upper bound is obtained for the Gelfand-Kirillov dimension of \textit{isotropic} subalgebras of strongly simple Poisson algebras (Theorem 8.1).

2 Filter dimension of algebras and modules

In this section, the filter dimension of algebras and modules will be defined.

The Gelfand-Kirillov dimension. Let \( F \) be the set of all functions from the set of natural numbers \( \mathbb{N} = \{0, 1, \ldots \} \) to itself. For each function \( f \in F \), the non-negative real number or \( \infty \) defined as

\[
\gamma(f) := \inf\{ r \in \mathbb{R} | f(i) \leq i^r \text{ for } i \gg 0 \}
\]

is called the degree of \( f \). The function \( f \) has \textbf{polynomial growth} if \( \gamma(f) < \infty \). Let \( f, g, p \in F \), and \( p(i) = p^*(i) \) for \( i \gg 0 \) where \( p^*(t) \in \mathbb{Q}[t] \) (a polynomial algebra with coefficients from the field of rational numbers). Then

\[
\gamma(f + g) \leq \max\{\gamma(f), \gamma(g)\}, \quad \gamma(fg) \leq \gamma(f) + \gamma(g),
\]

\[
\gamma(p) = \deg_t(p^*(t)), \quad \gamma(pg) = \gamma(p) + \gamma(g).
\]

Let \( A = K\langle a_1, \ldots, a_s \rangle \) be a finitely generated \( K \)-algebra. The finite dimensional filtration \( F = \{A_i\} \) associated with algebra generators \( a_1, \ldots, a_s \):

\[
A_0 := K \subseteq A_1 := K + \sum_{i=1}^{s} Ka_i \subseteq \cdots \subseteq A_i := A_i^1 \subseteq \cdots
\]

is called the \textbf{standard filtration} for the algebra \( A \). Let \( M = AM_0 \) be a finitely generated \( A \)-module where \( M_0 \) is a finite dimensional generating subspace. The finite dimensional filtration \( \{M_i := A_iM_0\} \) is called the \textbf{standard filtration} for the \( A \)-module \( M \).

Definition. \( \text{GK}(A) := \gamma(i \mapsto \dim_K(A_i)) \) and \( \text{GK}(M) := \gamma(i \mapsto \dim_K(M_i)) \) are called the \textbf{Gelfand-Kirillov} dimensions of the algebra \( A \) and the \( A \)-module \( M \) respectively.

It is easy to prove that the Gelfand-Kirillov dimension of the algebra (resp. the module) does not depend on the choice of the standard filtration of the algebra (resp. and the choice of the generating subspace of the module).

The return functions and the (left) filter dimension.

Definition [1]. The function \( \nu_{F,M_0} : \mathbb{N} \to \mathbb{N} \cup \{\infty\} \),

\[
\nu_{F,M_0}(i) := \min\{ j \in \mathbb{N} \cup \{\infty\} : A_jM_{i,\text{gen}} \supseteq M_0 \text{ for all } M_{i,\text{gen}} \}
\]

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is called the return function of the $A$-module $M$ associated with the filtration $F = \{A_i\}$ of the algebra $A$ and the generating subspace $M_0$ of the $A$-module $M$ where $M_{i,\text{gen}}$ runs through all generating subspaces for the $A$-module $M$ such that $M_{i,\text{gen}} \subseteq M_i$.

Suppose, in addition, that the finitely generated algebra $A$ is a simple algebra. The return function $\nu_F \in F$ and the left return function $\lambda_F \in F$ for the algebra $A$ with respect to the standard filtration $F := \{A_i\}$ for the algebra $A$ are defined by the rules:

$$
\nu_F(i) := \min\{j \in \mathbb{N} \cup \{\infty\} \mid 1 \in A_j a A_j \text{ for all } 0 \neq a \in A_i\},
$$

$$
\lambda_F(i) := \min\{j \in \mathbb{N} \cup \{\infty\} \mid 1 \in A a A_j \text{ for all } 0 \neq a \in A_i\},
$$

where $A_j a A_j$ is the vector subspace of the algebra $A$ spanned over the field $K$ by the elements $x y$ for all $x, y \in A_j$; and $A a A_j$ is the left ideal of the algebra $A$ generated by the set $a A_j$. The next result shows that under a mild restriction the maps $\nu_F(i)$ and $\lambda_F(i)$ are finite.

Recall that the centre of a simple algebra is a field.

**Lemma 2.1** Let $A$ be a simple finitely generated algebra such that its centre $Z(A)$ is an algebraic field extension of $K$. Then $\lambda_F(i) \leq \nu_F(i) < \infty$ for all $i \geq 0$.

**Proof.** The first inequality is evident.

The centre $Z = Z(A)$ of the simple algebra $A$ is a field that contains $K$. Let $\{\omega_j \mid j \in J\}$ be a $K$-basis for the $K$-vector space $Z$. Since $\dim_K(A_i) < \infty$, one can find finitely many $Z$-linearly independent elements, say $a_1, \ldots, a_s$, of $A_i$ such that $A_i \subseteq Z a_1 + \cdots + Z a_s$. Next, one can find a finite subset, say $J'$, of $J$ such that $A_i \subseteq V a_1 + \cdots + V a_s$, where $V = \sum_{j \in J'} K \omega_j$. The field $K'$ generated over $K$ by the elements $\omega_j$, $j \in J'$, is a finite field extension of $K$ (i.e. $\dim_K(K') < \infty$) since $Z/K$ is algebraic, hence $K' \subseteq A_n$ for some $n \geq 0$. Clearly, $A_i \subseteq K' a_1 + \cdots + K' a_s$.

The $A$-bimodule $A A_j$ is simple with ring of endomorphisms $\text{End}(A A_j) \cong Z$. By the Density Theorem, [21], 12.2, for each integer $1 \leq j \leq s$, there exist elements of the algebra $A$, say $x_1^j, \ldots, x_m^j, y_1^j, \ldots, y_m^j$, $m = m(j)$, such that for all $1 \leq l \leq s$

$$
\sum_{k=1}^m x_k^j a_l y_k^j = \delta_{j,l}, \text{ the Kronecker delta.}
$$

Let us fix a natural number, say $d = d_i$, such that $A_d$ contains all the elements $x_k^j, y_k^j$, and the field $K'$. We claim that $\nu_F(i) \leq 2d$. Let $0 \neq a \in A_i$. Then $a = \lambda_1 a_1 + \cdots + \lambda_s a_s$ for some $\lambda_i \in K'$. There exists $\lambda_j \neq 0$. Then $\sum_{k=1}^m \lambda_j^{-1} x_k^j a_j y_k^j = 1$, and $\lambda_j^{-1} x_k^j, y_k^j \in A_{2d}$. This proves the claim and the lemma. □

**Remark.** If the field $K$ is uncountable then automatically the centre $Z(A)$ of a simple finitely generated algebra $A$ is algebraic over $K$ (since $A$ has a countable $K$-basis and the rational function field $K(x)$ has uncountable basis over $K$ since elements $\frac{1}{x+\lambda}, \lambda \in K$, are $K$-linearly independent).

It is easy to see that for a finitely generated algebra $A$ any two standard finite dimensional filtrations $F = \{A_i\}$ and $G = \{B_i\}$ are equivalent, $(F \sim G)$, that is, there exist
natural numbers \(a, b, c, d\) such that

\[ A_i \subseteq B_{ai+b} \text{ and } B_i \subseteq A_{ci+d} \text{ for } i \gg 0. \]

If one of the inclusions holds, say the first, we write \(F \leq G\).

**Lemma 2.2** Let \(A\) be a finitely generated algebra equipped with two standard finite dimensional filtrations \(F = \{A_i\}\) and \(G = \{B_i\}\).

1. Let \(M\) be a finitely generated \(A\)-module. Then \(\gamma(\nu_{F,M}) = \gamma(\nu_{G,M})\) for any finite dimensional generating subspaces \(M_0\) and \(N_0\) of the \(A\)-module \(M\).

2. If, in addition, \(A\) is a simple algebra then \(\gamma(\nu_F) = \gamma(\nu_G)\) and \(\gamma(\lambda_F) = \gamma(\lambda_G)\).

**Proof.** 1. The module \(M\) has two standard finite dimensional filtrations \(\{M_i = A_iM_0\}\) and \(\{N_i = B_iN_0\}\). Let \(\nu = \nu_{F,M_0}\) and \(\mu = \nu_{G,N_0}\).

Suppose that \(F = G\). Choose a natural number \(s\) such that \(M_0 \subseteq N_s\) and \(N_0 \subseteq M_s\), so \(N_i \subseteq M_{i+s}\) and \(M_i \subseteq N_{i+s}\) for all \(i \geq 0\). Let \(N_{i,gen}\) be any generating subspace for the \(A\)-module \(M\) such that \(N_{i,gen} \subseteq N_i\). Since \(M_0 \subseteq A_{\nu(i+s)}N_{i,gen}\) for all \(i \geq 0\) and \(N_0 \subseteq A_{\nu}M_0\), we have \(N_0 \subseteq A_{\nu(i+s)+s}N_{i,gen}\), hence, \(\mu(i) \leq \nu(i + s) + s\) and finally \(\gamma(\mu) \leq \gamma(\nu)\). By symmetry, the opposite inequality is true and so \(\gamma(\mu) = \gamma(\nu)\).

Suppose that \(M_0 = N_0\). The algebra \(A\) is a finitely generated algebra, so all standard finite dimensional filtrations of the algebra \(A\) are equivalent. In particular, \(F \sim G\) and so one can choose natural numbers \(a, b, c, d\) such that

\[ A_i \subseteq B_{ai+b} \text{ and } B_i \subseteq A_{ci+d} \text{ for } i \gg 0. \]

Then \(N_i = B_iN_0 \subseteq A_{ci+d}M_0 = M_{ci+d}\) for all \(i \geq 0\), hence \(N_0 = M_0 \subseteq A_{\nu(ci+d)}N_{i,gen} \subseteq B_{av(ci+d)+b}N_{i,gen}\), therefore \(\mu(i) \leq av(ci + d) + b\) for all \(i \geq 0\), hence \(\gamma(\mu) \leq \gamma(\nu)\). By symmetry, we get the opposite inequality which implies \(\gamma(\mu) = \gamma(\nu)\). Now, \(\gamma(\nu_{F,M_0}) = \gamma(\nu_{F,N_0}) = \gamma(\nu_{G,N_0})\).

2. The algebra \(A\) is simple, equivalently, it is a simple (left) \(A \otimes A^0\)-module where \(A^0\) is the opposite algebra to \(A\). The opposite algebra has the standard filtration \(F^0 = \{A_i^0\}\), opposite to the filtration \(F\). The tensor product of algebras \(A \otimes A^0\), so-called, the enveloping algebra of \(A\), has the standard filtration \(F \otimes F^0 = \{C_n\}\) which is the tensor product of the standard filtrations \(F\) and \(F^0\), that is, \(C_n = \sum\{A_i \otimes A_j^0, i + j \leq n\}\). Let \(\nu_{F \otimes F^0, K}\) be the return function of the \(A \otimes A^0\)-module \(A\) associated with the filtration \(F \otimes F^0\) and the generating subspace \(K\). Then

\[ \nu_F(i) \leq \nu_{F \otimes F^0, K}(i) \leq 2\nu_F(i) \text{ for all } i \geq 0, \]

and so

\[ \gamma(\nu_F) = \gamma(\nu_{F \otimes F^0, K}), \] (1)

and, by the first statement, we have \(\gamma(\nu_F) = \gamma(\nu_{F \otimes F^0, K}) = \gamma(\nu_{G \otimes G^0, K}) = \gamma(\nu_G)\), as required. Using a similar argument as in the proof of the first statement one can proof that \(\gamma(\lambda_F) = \gamma(\lambda_G)\). We leave this as an exercise. □
Definition 4.1. \( \text{fd}(M) = \gamma(\nu_{F,M_0}) \) is the \textit{filter dimension} of the \( A \)-module \( M \), and \( \text{fd}(A) := \text{fd}(A_{0\otimes A_0}A) \) is the \textit{filter dimension} of the algebra \( A \). If, in addition, the algebra \( A \) is simple, then \( \text{fd}(A) = \gamma(\nu_F) \), and \( \text{lfd}(A) := \gamma(\lambda_F) \) is called the \textit{left filter dimension} of the algebra \( A \).

By the previous lemma the definitions make sense (both filter dimensions do not depend on the choice of the standard filtration \( F \) for the algebra \( A \)).

By Lemma 2.1 \( \text{lfd}(A) \leq \text{fd}(A) \).

**Question.** What is the filter dimension of a polynomial algebra?

### 3 The first filter inequality

In this paper, \( d(A) \) means either the filter dimension \( \text{fd}(A) \) or the left filter dimension \( \text{lfd}(A) \) of a simple finitely generated algebra \( A \) (i.e. \( d = \text{fd, lfd} \)). Both filter dimensions appear naturally when one tries to find a lower bound for the holonomic number (Theorem 3.1) and an upper bound (Theorem 4.2) for the (left and right) Krull dimension (in the sense of Rentschler-Gabriel, [22]) of simple finitely generated algebras.

The next theorem is a generalization of the **Bernstein’s Inequality** (Theorem 1.1) to the class of simple finitely generated algebras.

**Theorem 3.1 (The First Filter Inequality, [4, 6])** Let \( A \) be a simple finitely generated algebra. Then

\[
\text{GK} (M) \geq \frac{\text{GK} (A)}{d(A) + \max\{d(A), 1\}}
\]

for all nonzero finitely generated \( A \)-modules \( M \) where \( d = \text{fd, lfd} \).

**Proof.** Let \( \lambda = \lambda_F \) be the left return function associated with a standard filtration \( F \) of the algebra \( A \) and let \( 0 \neq a \in A_i \). It suffices to prove the inequality for \( \lambda \) (since \( \text{fd}(A) \geq \text{lfd}(A) \)). It follows from the inclusion

\[
AaM_{\lambda(i)} = AaA_{\lambda(i)}M_0 \supseteq 1M_0 = M_0
\]

that the linear map

\[
A_i \to \text{Hom}(M_{\lambda(i)}, M_{\lambda(i)+i}), a \mapsto (m \mapsto am),
\]

is injective, so \( \dim A_i \leq \dim M_{\lambda(i)} \dim M_{\lambda(i)+i} \). Using the above elementary properties of the degree (see also [19], 8.1.7), we have

\[
\text{GK} (A) = \gamma(\dim A_i) \leq \gamma(\dim M_{\lambda(i)}) + \gamma(\dim M_{\lambda(i)+i})
\]

\[
\leq \gamma(\dim M_i)\gamma(\lambda) + \gamma(\dim M_i) \max\{\gamma(\lambda), 1\}
\]

\[
= \text{GK} (M)(\text{lfd}A + \max\{\text{lfd}A, 1\})
\]

\[
\leq \text{GK} (M)(\text{lfd}A + \max\{\text{lfd}A, 1\}). \quad \square
\]
The result above gives a lower bound for the holonomic number of a simple finitely generated algebra

\[ h_A \geq \frac{\GK(A)}{d(A) + \max\{d(A), 1\}}. \]

**Theorem 3.2** Let \( A \) be a finitely generated algebra. Then

\[ \GK(M) \leq \GK(A) \fd(M) \]

for any simple \( A \)-module \( M \).

**Proof.** Let \( \nu = \nu_{F,Km} \) be the return function of the module \( M \) associated with a standard finite dimensional filtration \( F = \{ A_i \} \) of the algebra \( A \) and a fixed nonzero element \( m \in M \). Let \( \pi : M \to K \) be a non-zero linear map satisfying \( \pi(m) = 1 \). Then, for any \( i \geq 0 \) and any \( 0 \neq u \in M_i : 1 = \pi(m) \in \pi(A_{\nu(i)}u) \), and so the linear map

\[ M_i \to \Hom(A_{\nu(i)}, K), \ u \mapsto (a \mapsto \pi(au)), \]

is an injective map hence \( \dim M_i \leq \dim A_{\nu(i)} \) and finally \( \GK(M) \leq \GK(A) \fd(M) \). \( \Box \)

**Corollary 3.3** Let \( A \) be a simple finitely generated infinite dimensional algebra. Then

\[ \fd(A) \geq \frac{1}{2}. \]

**Proof.** The algebra \( A \) is a finitely generated infinite dimensional algebra hence \( \GK(A) > 0 \). Clearly, \( \GK(A \otimes A^0) \leq \GK(A) + \GK(A^0) = 2\GK(A) \). Applying Theorem 3.2 to the simple \( A \otimes A^0 \)-module \( M = A \) we finish the proof:

\[ \GK(A) = \GK(A \otimes A^0) \leq \GK(A \otimes A^0) \fd(A \otimes A^0) \leq 2\GK(A) \fd(A) \]

hence \( \fd(A) \geq \frac{1}{2} \). \( \Box \)

**Question.** Is \( \fd(A) \geq 1 \) for all simple finitely generated infinite dimensional algebras \( A \)?

**Question.** For which numbers \( d \geq \frac{1}{2} \) there exists a simple finitely generated infinite dimensional algebra \( A \) with \( \fd(A) = d \)?

**Corollary 3.4** Let \( A \) be a simple finitely generated infinite dimensional algebra. Then

\[ \fd(M) \geq \frac{1}{\fd(A) + \max\{\fd(A), 1\}} \]

for all simple \( A \)-modules \( M \).

**Proof.** Applying Theorem 3.1 and Theorem 3.2 we have the result

\[ \fd(M) \geq \frac{\GK(M)}{\GK(A)} \geq \frac{\GK(A)}{\GK(A)(\fd(A) + \max\{\fd(A), 1\})} = \frac{1}{\fd(A) + \max\{\fd(A), 1\}}. \] \( \Box \)
4 Krull, Gelfand-Kirillov and filter dimensions of simple finitely generated algebras

In this section, we prove the second filter inequality (Theorem 4.2) and apply both filter inequalities for giving short proofs of some classical results about the rings of differential operators on a smooth irreducible affine algebraic varieties (Theorems 1.1, 4.4, 4.5, 4.7).

We say that an algebra $A$ is (left) finitely partitive \cite[8.3.17]{19} if, given any finitely generated $A$-module $M$, there is an integer $n = n(M) > 0$ such that for every strictly descending chain of $A$-submodules of $M$:

$$M = M_0 \supset M_1 \supset \cdots \supset M_m$$

with $\text{GK} (M_i/M_{i+1}) = \text{GK} (M)$, one has $m \leq n$. McConnell and Robson write in their book \cite[8.3.17]{19}, that "yet no examples are known which fail to have this property."

Recall that $\text{K.dim}$ denotes the (left) Krull dimension in the sense of Rentschler and Gabriel, \cite{22}.

**Lemma 4.1** Let $A$ be a finitely partitive algebra with $\text{GK} (A) < \infty$. Let $a \in \mathbb{N}$, $b \geq 0$ and suppose that $\text{GK} (M) \geq a + b$ for all finitely generated $A$-modules $M$ with $\text{K.dim} (M) = a$, and that $\text{GK} (N) \in \mathbb{N}$ for all finitely generated $A$-modules $N$ with $\text{K.dim} (N) \geq a$. Then $\text{GK} (M) \geq \text{K.dim} (M) + b$ for all finitely generated $A$-modules $M$ with $\text{K.dim} (M) \geq a$. In particular, $\text{GK} (A) \geq \text{K.dim} (A) + b$.

**Remark.** It is assumed that a module $M$ with $\text{K.dim} (M) = a$ exists.

**Proof.** We use induction on $n = \text{K.dim} (M)$. The base of induction, $n = a$, is true. Let $n > a$. There exists a descending chain of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_m$ with $\text{K.dim} (M_i/M_{i+1}) = n - 1$ for $i \geq 1$. By induction, $\text{GK} (M_i/M_{i+1}) \geq n - 1 + b$ for $i \geq 1$. The algebra $A$ is finitely partitive, so there exists $i$ such that $\text{GK} (M) > \text{GK} (M_i/M_{i+1})$, so $\text{GK} (M) - 1 \geq \text{GK} (M_i/M_{i+1}) \geq n - 1 + b$, since $\text{GK} (M) \in \mathbb{N}$, hence $\text{GK} (M) \geq \text{K.dim} (M) + b$. Since $\text{K.dim} (A) \geq \text{K.dim} (M)$ for all finitely generated $A$-modules $M$ we have $\text{GK} (A) \geq \text{K.dim} (A) + b$. $\square$

**Theorem 4.2** \cite{12} Let $A$ be a simple finitely generated finitely partitive algebra with $\text{GK} (A) < \infty$. Suppose that the Gelfand-Kirillov dimension of every finitely generated $A$-module is a natural number. Then

$$\text{K.dim} (M) \leq \text{GK} (M) - \frac{\text{GK} (A)}{d(A) + \max \{d(A), 1\}}$$

for any nonzero finitely generated $A$-module $M$. In particular,

$$\text{K.dim} (A) \leq \text{GK} (A)(1 - \frac{1}{d(A) + \max \{d(A), 1\}}).$$
Proof. Applying the lemma above to the family of finitely generated $A$-modules of Krull dimension $0$, by Theorem 3.1, we can put $a = 0$ and
\[ b = \frac{\text{GK}(A)}{\text{d}(A) + \max\{\text{d}(A), 1\}}, \]
and the result follows. □

Let $K$ be a field of characteristic zero and $B$ be a commutative $K$-algebra. The ring of $(K$-linear) differential operators $\mathcal{D}(B)$ on $B$ is defined as $\mathcal{D}(B) = \bigcup_{i=0}^{\infty} \mathcal{D}_i(B)$ where $\mathcal{D}_0(B) = \text{End}_R(B) \simeq B$, $((x \mapsto bx) \leftrightarrow b)$,
\[ \mathcal{D}_i(B) = \{ u \in \text{End}_K(B) : [u, r] \in \mathcal{D}_{i-1}(B) \text{ for each } r \in B \}. \]
Note that the $\{\mathcal{D}_i(B)\}$ is, so-called, the order filtration for the algebra $\mathcal{D}(B)$:
\[ \mathcal{D}_0(B) \subseteq \mathcal{D}_1(B) \subseteq \cdots \subseteq \mathcal{D}_i(B) \subseteq \cdots \text{ and } \mathcal{D}_i(B)\mathcal{D}_j(B) \subseteq \mathcal{D}_{i+j}(B), \ i, j \geq 0. \]
The subalgebra $\Delta(B)$ of $\text{End}_K(B)$ generated by $B \equiv \text{End}_R(B)$ and by the set $\text{Der}_K(B)$ of all $K$-derivations of $B$ is called the derivation ring of $B$. The derivation ring $\Delta(B)$ is a subring of $\mathcal{D}(B)$.

Let the finitely generated algebra $B$ be a regular commutative domain of Krull dimension $n < \infty$. In geometric terms, $B$ is the coordinate ring $\mathcal{O}(X)$ of a smooth irreducible affine algebraic variety $X$ of dimension $n$. Then

- Der$_K(B)$ is a finitely generated projective $B$-module of rank $n$;
- $\mathcal{D}(B) = \Delta(B)$;
- $\mathcal{D}(B)$ is a simple (left and right) Noetherian domain with $\text{GK} \mathcal{D}(B) = 2n$ ($n = \text{GK} (B) = K.\text{dim} (B)$);
- $\mathcal{D}(B) = \Delta(B)$ is an almost centralizing extension of $B$;
- the associated graded ring $\text{gr} \mathcal{D}(B) = \oplus \mathcal{D}_i(B)/\mathcal{D}_{i-1}(B)$ is a commutative domain;
- the Gelfand-Kirillov dimension of every finitely generated $\mathcal{D}(B)$-module is a natural number.

For the proofs of the statements above the reader is referred to [19], Chapter 15. So, the domain $\mathcal{D}(B)$ is a simple finitely generated infinite dimensional Noetherian algebra ([19], Chapter 15).

Example. Let $P_n = K[X_1, \ldots, X_n]$ be a polynomial algebra. Der$_K(P_n) = \bigoplus_{i=1}^{n} P_n \frac{\partial}{\partial X_i}$,
\[ \mathcal{D}(P_n) = \Delta(P_n) = K[X_1, \ldots, X_n, \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}] \]
is the ring of differential operators with polynomial coefficients, i.e. the $n$’th Weyl algebra $A_n$.

In Section 5, we prove the following result.
Theorem 4.3 ([15]) The filter dimension and the left filter dimension of the ring of differential operators $\mathcal{D}(B)$ are both equal to 1.

As an application we compute the Krull dimension of $\mathcal{D}(B)$.

Theorem 4.4 ([19], Ch. 15)

$$\text{K.dim } \mathcal{D}(B) = \frac{\text{GK}(\mathcal{D}(B))}{2} = \text{K.dim}(B).$$

Proof. The second equality is clear ($\text{GK}(\mathcal{D}(B)) = 2\text{GK}(B) = 2\text{K.dim}(B)$). It follows from Theorems 4.2 and 4.3 that

$$\text{K.dim } \mathcal{D}(B) \leq \frac{\text{GK}(\mathcal{D}(B))}{2} = \text{K.dim}(B).$$

The map $I \rightarrow \mathcal{D}(B)I$ from the set of left ideals of $B$ to the set of left ideals of $\mathcal{D}(B)$ is injective, thus $\text{K.dim}(B) \leq \text{K.dim } \mathcal{D}(B)$. □

This result shows that for the ring of differential operators on a smooth irreducible affine algebraic variety the inequality in Theorem 4.2 is the equality.

Theorem 4.5 ([19], 15.4.3) Let $M$ be a nonzero finitely generated $\mathcal{D}(B)$-module. Then

$$\text{GK}(M) \geq \frac{\text{GK}(\mathcal{D}(B))}{2} = \text{K.dim}(B).$$

Proof. By Theorems 4.1 and 4.3

$$\text{GK}(M) \geq \frac{\text{GK}(\mathcal{D}(B))}{1 + 1} = \frac{2\text{GK}(B)}{2} = \text{GK}(B) = \text{K.dim}(B).$$

So, for the ring of differential operators on a smooth affine algebraic variety the inequality in Theorem 4.1 is in fact the equality.

In general, it is difficult to find the exact value for the filter dimension but for the Weyl algebra $A_n$ it is easy and one can find it directly.

Theorem 4.6 Both the filter dimension and the left filter dimension of the Weyl algebra $A_n$ over a field of characteristic zero are equal to 1.

Proof. Denote by $a_1, \ldots, a_{2n}$ the canonical generators of the Weyl algebra $A_n$ and denote by $F = \{A_{n,i}\}_{i \geq 0}$ the standard filtration associated with the canonical generators. The associated graded algebra $\text{gr } A_n := \oplus_{i \geq 0} A_{n,i}/A_{n,i-1}$, $(A_{n,-1} = 0)$ is a polynomial algebra in $2n$ variables, so

$$\text{GK}(A_n) = \text{GK}(\text{gr } A_n) = 2n.$$

For every $i \geq 0$:

$$\text{ad } a_j : A_{n,i} \rightarrow A_{n,i-1}, \ x \mapsto \text{ad } a_j(x) := a_jx - xa_j.$$
The algebra $A_n$ is central ($Z(A_n) = K$), so
\[ \text{ad} a_j(x) = 0 \text{ for all } j = 1, \ldots, 2n \Leftrightarrow x \in Z(A_n) = K = A_{n,0}. \]

These two facts imply $\nu_F(i) \leq i$ for $i \geq 0$, and so $d(A_n) \leq 1$.

The $A_n$-module $P_n := K[X_1, \ldots, X_n] \simeq A_n/(A_n \partial_1 + \cdots + A_n \partial_n)$ has Gelfand-Kirillov dimension $n$. By Theorem 3.1 applied to the $A_n$-module $P_n$, we have
\[ 2n = \text{GK}(A_n) \leq n(d(A) + \max\{d(A), 1\}), \]
hence $d(A_n) \geq 1$, and so $d(A_n) = 1$. □

**Proof of the Bernstein’s inequality (Theorem 1.1).**

Since $\text{GK}(A_n) = 2n$ and $d(A_n) = 1$, Theorem 3.1 gives $\text{GK}(M) \geq \frac{2n}{2} = n$. □

One also gets a short proof of the following result of Rentschler and Gabriel.

**Theorem 4.7 ([22]).** If $\text{char } K = 0$ then the Krull dimension of the Weyl algebra $A_n$ is
\[ \text{K.dim } (A_n) = n. \]

**Proof.** Putting $\text{GK}(A_n) = 2n$ and $d(A_n) = 1$ into the second formula of Theorem 4.2 we have $\text{K.dim } (A_n) \leq \frac{2n}{2} = n$. The polynomial algebra $P_n = K[X_1, \ldots, X_n]$ is the subalgebra of $A_n$ such that $A_n$ is a free right $P_n$-module. The map $I \to A_n I$ from the set of left ideals of the polynomial algebra $P_n$ to the set of left ideals of the Weyl algebra $A_n$ is injective, thus $n = \text{K.dim } (P_n) \leq \text{K.dim } (A_n)$, and so $\text{K.dim } (A_n) = n$. □

5 **Filter dimension of the ring of differential operators on a smooth irreducible affine algebraic variety (proof of Theorem 4.3)**

Let $K$ be a field of characteristic 0 and let the algebra $B$ be as in the previous section, i.e. $B$ is a finitely generated regular commutative algebra which is a domain. We keep the notations of the previous section. Recall that the derivation ring $\Delta = \Delta(B)$ coincides with the ring of differential operators $\mathcal{D}(B)$ ([19], 15.5.6) and is a simple finitely generated finitely partitive $K$-algebra ([19], 15.3.8, 15.1.21). We refer the reader to [19], Chapter 15, for basic definitions. We aim to prove Theorem 4.3.

Let $\{B_i\}$ and $\{\Delta_i\}$ be standard finite dimensional filtrations on $B$ and $\Delta$ respectively such that $B_i \subseteq \Delta_i$ for all $i \geq 0$. Then the enveloping algebra $\Delta^i := \Delta \otimes \Delta^0$ can be equipped with the standard finite dimensional filtration $\{\Delta^i\}$ which is the tensor product of the filtrations $\{\Delta_i\}$ and $\{\Delta^0\}$ of the algebras $\Delta$ and $\Delta^0$ respectively.
Then $B \simeq \Delta/\Delta \text{Der}_K B$ is a simple left $\Delta$-module ([19], 15.3.8] with $\text{GK}(\Delta) = 2\text{GK}(B)$ ([19], 15.3.2). By Theorem 3.1,

$$d(\Delta) + \max\{d(\Delta), 1\} \geq \frac{\text{GK}(\Delta)}{\text{GK}(B)} = \frac{2\text{GK}(B)}{\text{GK}(B)} = 2,$$

hence $d(\Delta) \geq 1$. It remains to prove the opposite inequality. For, we recall some properties of $\Delta$ (see [19], Ch. 15, for details).

Given $0 \neq c \in B$, denote by $B_c$ the localization of the algebra $B$ at the powers of the element $c$, then $\Delta(B_c) \simeq \Delta(B)_c$ and the map $\Delta(B) \to \Delta(B)_c$, $d \to d/1$, is injective ([19], 5.1.25). There is a finite subset $\{c_1, \ldots, c_t\}$ of $B$ such that the algebra $\prod_{i=1}^{t} \Delta(B_{c_i})$ is left and right faithfully flat over its subalgebra $\Delta$.

$$\sum_{i=1}^{t} Bc_i = B \text{ (see the proof of 15.2.13, [19]). (2)}$$

For each $c = c_i$, $\text{Der}_K(B_c)$ is a free $B_c$-module with a basis $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$ for some $x_1, \ldots, x_n \in B$ ([19], 15.2.13). Note that the choice of the $x_j$'th depends on the choice of the $c_i$. Then

$$\Delta(B)_c \simeq \Delta(B_c) = B_c < \partial_1, \ldots, \partial_n > \supseteq K < x_1, \ldots, x_n, \partial_1, \ldots, \partial_n >$$

Fix $c = c_i$. We aim to prove the following statement:

- there exist natural numbers $a, b, \alpha, \text{ and } \beta$ such that for any $0 \neq d \in \Delta_k$ there exists

  $$w \in \Delta^e_{ak+b} : wd = c^{\alpha k + \beta}.$$

Suppose that we are done. Then one can choose the numbers $a, b, \alpha, \text{ and } \beta$ such that (*) holds for all $i = 1, \ldots, t$. It follows from [2] that

$$\sum_{i=1}^{t} f_i c_i = 1 \text{ for some } f_i \in A.$$

Choose $\nu \in \mathbb{N}$: all $f_i c_i \in \Delta_\nu$ and set $N(k) = \alpha k + \beta$, then

$$1 = (\sum_{i=1}^{t} f_i c_i)^t N(k) = \sum_{i=1}^{t} g_i c_i^{N(k)} = \sum_{i=1}^{t} g_i w_i d = wd,$$

where the $w_i$ are from (*), i.e. $w_i \in \Delta^e_{ak+b}$, $w_i d = c_i^{N(k)}$. So, $w = \sum_{i=1}^{t} g_i w_i \in \Delta^e_{\nu t N(k) + ak+b}$ and so $d(\Delta) \leq 1$, as required.

Fix $c = c_i$. By ([19], 15.1.24) $\text{Der}_K(B_c) \simeq \text{Der}_K(B)_c$ and $\text{Der}_K B$ can be seen as a finitely generated $B$-submodule of $\text{Der}_K(B_c)$ ([19], 15.1.7).
exists an element

Corollary 5.1

There exist natural numbers

α

where

α

polynomial

p

has coefficients in

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j

l

β

B

Then

P

algebra

Q

over

= Q

a natural number

l

has the same transcendence degree

B

L

is a finitely generated algebra, hence L is a finite field extension of Q of dimension, say m, over Q. Let e₁, . . . , eₘ ∈ B be a Q-basis for the vector space L over Q. Note that L = QB. One can find a natural number β ≥ 1 and a nonzero polynomial p ∈ Pₛ such that

\{B₁, e_j e_k | j, k = 1, . . . , m\} ⊆ \sum_{\alpha=1}^{m} p^{-1} P_{\beta} e_{\alpha}.

Then Bₘ ⊆ \bigoplus_{j=1}^{m} p^{-2k} P_{3\beta k} e_{j} and Bₘ e_{i} ⊆ \bigoplus_{j=1}^{m} p^{-3k} P_{3\beta k} e_{j} for all k ≥ 1 and i = 1, . . . , m. Let 0 ≠ d ∈ Bₘ. The m × m matrix of the bijective Q-linear map \( L \to L, x \mapsto dx \), with respect to the basis e₁, . . . , eₘ has entries from the set \( p^{-3k} P_{3\beta k} \). So, its characteristic polynomial

\[ \chi_d(t) = t^m + \alpha_{m-1} t^{m-1} + \cdots + \alpha_0 \]

has coefficients in \( p^{-3mk} P_{3\beta k} \), and \( \alpha_0 \neq 0 \) as \( x \mapsto dx \) is a bijection. Now,

\[ P_{6m\beta k} \ni p^{3mk} \alpha_0 = p^{3mk} (-d^{m-1} - \alpha_{m-1} d^{m-2} - \cdots - \alpha_1) d \in B_{4m\beta k} P_{3\beta k} \subseteq B_{m\beta k(4+3\ell)} d. \] (3)

Let \( \delta_1, \ldots, \delta_t \) be a set of generators for the left B-module \( \text{Der}_K(B) \). Then

\[ \partial_j \in \sum_{j=1}^{t} c^{-l_1} B_{l_1} \delta_j \] for \( i = 1, \ldots, n, \)

for some natural number \( l_1 \). Fix a natural number \( l_2 \) such that \( \delta_j(B_1) \subseteq B_{l_2} \) and \( \delta_j(c) \in B_{l_2} \) for \( j = 1, \ldots, t \). Then

\[ \partial^\alpha(B_k) \subseteq c^{-2|\alpha|(l_1+1)} B_{k+|\alpha|(l_1+l_2)} \] for all \( \alpha \in \mathbb{N}^n \), \( k \geq 1, \)

where \( \alpha = (\alpha_1, \ldots, \alpha_n), |\alpha| = \alpha_1 + \cdots + \alpha_n, \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \). It follows from (3) that one can find \( \alpha \in \mathbb{N}^n \) such that \( |\alpha| \leq 6m\beta k \) and

\[ 1 \in K^* \partial^\alpha(p^{3mk} \alpha_0) \subseteq \partial^\alpha(B_{m\beta k(4+3\ell)} d) \subseteq c^{-12m\beta(l_1+1)k} \Delta_{2(m\beta k(4+3\ell)+6m\beta k(l_1+l_2))} d \]

where \( K^* = K\setminus\{0\} \). Now (*) follows. □

In fact we have proved the following corollary.

**Corollary 5.1** There exist natural numbers a and b such that for any \( 0 \neq d \in \Delta_k \) there exists an element \( w \in \Delta_{ak+b} \) satisfying \( wd = 1 \).
6 Multiplicity for the filter dimension, holonomic modules over simple finitely generated algebras

In this section, we introduce a concept of multiplicity for the filter dimension and a concept of holonomic module for (some) finitely generated algebras. We will prove that a holonomic module has finite length (Theorem 6.8). The multiplicity for the filter dimension is a key ingredient in the proof.

First we recall the definition of multiplicity in the commutative situation and then for certain non-commutative algebras (somewhat commutative algebras).

Multiplicity in the commutative situation. Let $B$ be a commutative finitely generated $K$-algebra with a standard finite dimensional filtration $F = \{B_i\}$, and let $M$ be a finitely generated $B$-module with a finite dimensional generating subspace, say $M_0$, and with the standard filtration $\{M_i = B_iM_0\}$ attached to it. Then there exists a polynomial $p(t) = lt^d + \cdots \in \mathbb{Q}[t]$ with rational coefficients of degree $d = GK(M)$ such that $\dim_K(M_i) = p(i)$ for all $i \gg 0$.

The polynomial $p(t)$ is called the Hilbert polynomial of the $B$-module $M$. The Hilbert polynomial does depend on the filtration $\{M_i\}$ of the module $M$ but its leading coefficient $l$ does not. The number $e(M) = dl!$ is called the multiplicity of the $B$-module $M$. It is a natural number which does depend on the filtration $F$ of the algebra $B$.

In the case when $M = B$ is the homogeneous coordinate ring of a projective algebraic variety $X \subseteq \mathbb{P}^m$ equipped with the natural filtration that comes from the grading of the graded algebra $B$, the multiplicity is the degree of $X$, the number of points in which $X$ meets a general plane of complementary degree in $\mathbb{P}^m$ ($K$ is an algebraically closed field).

Somewhat commutative algebras. A $K$-algebra $R$ is called a somewhat commutative algebra if it has a finite dimensional filtration $R = \bigcup_{i \geq 0} R_i$ such that the associated graded algebra $\text{gr } R := \bigoplus_{i \geq 0} R_i/R_{i-1}$ is a commutative finitely generated $K$-algebra where $R_{-1} = 0$ and $R_0 = K$. Then the algebra $R$ is a Noetherian finitely generated algebra since $\text{gr } R$ is so. A finitely generated module over a somewhat commutative algebra has the Gelfand-Kirillov dimension which is a natural number. We refer the reader to the books [16, 19] for the properties of somewhat commutative algebras.

Definition. For a somewhat commutative algebra $R$ we define the holonomic number,

$$h_R := \min\{\text{GK}(M) \mid M \neq 0 \text{ is a finitely generated } R \text{-module}\}.$$ 

Definition. A finitely generated $R$-module $M$ is called a holonomic module if $\text{GK}(M) = h_R$. In other words, a nonzero finitely generated $R$-module is holonomic iff it has the least Gelfand-Kirillov dimension. If $h_R = 0$ then every holonomic $R$-module is finite dimensional and vice versa.

Examples. 1. The holonomic number of the Weyl algebra $A_n$ is $n$. The polynomial algebra $K[X_1, \ldots, X_n] \cong A_n/\sum_{\delta \in \mathbb{N}} A_n \partial_\delta$ with the natural action of the ring of differential operators $A_n = \mathbb{K}[X_1, \ldots, X_n, \partial_{X_1}, \ldots, \partial_{X_n}]$ is a simple holonomic $A_n$-module.
2. Let \( X \) be a smooth irreducible affine algebraic variety of dimension \( n \). The ring of differential operators \( \mathcal{D}(X) \) is a simple somewhat commutative algebra of Gelfand-Kirillov dimension \( 2n \) with holonomic number \( h_{\mathcal{D}(X)} = n \). The algebra \( \mathcal{O}(X) \) of regular functions of the variety \( X \) is a simple \( \mathcal{D}(X) \)-module with respect to the natural action of the algebra \( \mathcal{D}(X) \). In more detail, \( \mathcal{O}(X) \cong \mathcal{D}(X)/\mathcal{D}(X)\text{Der}_K(\mathcal{O}(X)) \) where \( \text{Der}_K(\mathcal{O}(X)) \) is the \( \mathcal{O}(X) \)-module of derivations of the algebra \( \mathcal{O}(X) \).

Let \( R = \bigcup_{i \geq 0} R_i \) be a somewhat commutative algebra. The associated graded algebra \( \text{gr} \, R \) is a commutative affine algebra. Let us choose homogeneous algebra generators of the algebra \( \text{gr} \, R \), say \( y_1, \ldots, y_s \), of graded degrees \( 1 \leq k_1, \ldots, k_s \) respectively (that is \( y_i \in R_{k_i}/R_{k_i-1} \)). A filtration \( \Gamma = \{ \Gamma_i, i \geq 0 \} \) of an \( R \)-module \( M = \bigcup_{i=0}^{\infty} \Gamma_i \) is called good if the associated graded \( \text{gr} \, R \)-module \( \text{gr}_R M := \bigoplus_{i \geq 0} \Gamma_i/\Gamma_{i-1} \) is finitely generated. An \( R \)-module \( M \) has a good filtration if it is finitely generated, and if \( \{ \Gamma_i \} \) and \( \{ \Omega_i \} \) are two good filtrations of \( M \), then there exists a natural number \( t \) such that \( \Gamma_i \subseteq \Omega_{i+t} \) and \( \Omega_i \subseteq \Gamma_{i+t} \) for all \( i \). If an \( R \)-module \( M \) is finitely generated and \( M_0 \) is a finite dimensional generating subspace of \( M \), then the standard filtration \( \{ \Gamma_i = R_i M_0 \} \) is good (see [9, 10, 11, 16, 19, 20] for details). The following two Lemmas are well-known by specialists (see their proofs, for example, in [3], Theorem 3.2 and Proposition 3.3 respectively).

**Lemma 6.1** Let \( R = \bigcup_{i \geq 0} R_i \) be a somewhat commutative algebra, \( k = \text{lcm}(k_1, \ldots, k_s) \), and let \( M \) be a finitely generated \( R \)-module with good filtration \( \Gamma = \{ \Gamma_i \} \).

1. There exist \( k \) polynomials \( \gamma_0, \ldots, \gamma_{k-1} \in \mathbb{Q}[t] \) with coefficients from \([k^{\text{GK}(M)} \text{GK}(M)!]^{-1} \mathbb{Z}\) such that

\[
\dim \Gamma_i = \gamma_j(i) \quad \text{for all } i \geq 0 \text{ and } j \equiv i \pmod{k}.
\]

2. The polynomials \( \gamma_j \) have the same degree \( \text{GK}(M) \) and the same leading coefficient \( e(M)/\text{GK}(M)! \) where \( e(M) \) is called the multiplicity of \( M \). The multiplicity \( e(M) \) does not depend on the choice of the good filtration \( \Gamma \). \( \square \)

**Remark.** A finitely generated \( R \)-module \( M \) has \( e(M) = 0 \) iff \( \dim_K(M) < \infty \).

**Lemma 6.2** Let \( 0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0 \) be an exact sequence of modules over a somewhat commutative algebra \( R \). Then \( \text{GK}(M) = \max\{ \text{GK}(N), \text{GK}(L) \} \), and if \( \text{GK}(N) = \text{GK}(M) = \text{GK}(L) \) then \( e(M) = e(N) + e(L) \). \( \square \)

**Corollary 6.3** Let the algebra \( R \) be as in Lemma 6.1 with holonomic number \( h > 0 \).

1. Let \( M \) be a holonomic \( R \)-module with multiplicity \( e(M) \). The \( R \)-module \( M \) has finite length \( \leq e(M)^h \).

2. Every nonzero submodule or factor module of a holonomic \( R \)-module is a holonomic module.
Proof. This follows directly from Lemma 6.2 □

Multiplicity. Let $f$ be a function from $\mathbb{N}$ to $\mathbb{R}_+ = \{ r \in \mathbb{R} : r \geq 0 \}$, the leading coefficient of $f$ is a non-zero limit (if it exists)

$$\text{lc}(f) = \lim_{i \to \infty} \frac{f(i)}{i^d} \neq 0,$$

where $d = \gamma(f)$. If $d \in \mathbb{N}$, we define the multiplicity $e(f)$ of $f$ by

$$e(f) = d! \text{lc}(f).$$

The factor $d!$ ensures that the multiplicity $e(f)$ is a positive integer in some important cases. If $f(t) = a_dt^d + a_{d-1}t^{d-1} + \cdots + a_0$ is a polynomial of degree $d$ with real coefficients then $\text{lc}(f) = a_d$ and $e(f) = d!a_d$.

**Lemma 6.4** Let $A$ be a finitely generated algebra equipped with a standard finite dimensional filtration $F = \{A_i\}$ and $M$ be a finitely generated $A$-module with generating finite dimensional subspaces $M_0$ and $N_0$.

1. If $\text{lc}(\nu_{F,M_0})$ exists then so does $\text{lc}(\nu_{F,N_0})$, and $\text{lc}(\nu_{F,M_0}) = \text{lc}(\nu_{F,N_0})$.
2. If $\text{lc}(\dim A_iM_0)$ exists then so does $\text{lc}(\dim A_iN_0)$, and $\text{lc}(\dim A_iM_0) = \text{lc}(\dim A_iN_0)$.

Proof. 1. The module $M$ has two filtrations $\{M_i = A_iM_0\}$ and $\{N_i = A_iN_0\}$. Let $\nu = \nu_{F,M_0}$ and $\mu = \nu_{F,N_0}$. Choose a natural number $s$ such that $M_0 \subseteq N_s$ and $N_0 \subseteq M_s$, so $N_i \subseteq M_{i+s}$ and $M_i \subseteq N_{i+s}$ for $i \geq 0$. Since $M_0 \subseteq A_{\nu(i+s)}N_{i,\text{gen}}$ for each $i$ and $N_0 \subseteq A_{\mu(i+s)}M_{i,\text{gen}}$, we have $N_0 \subseteq A_{\nu(i+s)}N_{i,\text{gen}}$, hence, $\mu(i) \leq \nu(i+s) + s$. By symmetry, $\nu(i) \leq \mu(i+s) + s$, so if $\text{lc}(\mu)$ exists then so does $\text{lc}(\nu)$ and $\text{lc}(\mu) = \text{lc}(\nu)$.

2. Since $\dim N_i \leq \dim M_{i+s}$ and $\dim M_i \leq \dim N_{i+s}$ for $i \geq 0$, the statement is clear. □

Lemma 6.3 shows that the leading coefficients of the functions $\dim A_iM_0$ and $\nu_{F,M_0}$ (if exist) do not depend on the choice of the generating subspace $M_0$. So, denote them by

$$l(M) = l_F(M) \quad \text{and} \quad L(M) = L_F(M)$$

respectively (if they exist). If $\text{GK}(M)$ (resp. $d(A)$) is a natural number, then we denote by $e(M) = e_F(M)$ (resp. $E(M) = E_F(M)$) the multiplicity of the function $\dim A_iM_0$ (resp. $\nu_{F,M_0}$).

We denote by $L(A) = L_F(A)$ the leading coefficient $L_F(A \otimes A^0)$ of the return function $\nu_{F@F^0, K}$ of the $A \otimes A^0$-module $A$.

**Holonomic modules.** Definition. Let $A$ be a finitely generated $K$-algebra, and $h_A$ be its holonomic number. A nonzero finitely generated $A$-module $M$ is called a **holonomic** $A$-module if $\text{GK}(M) = h_A$. We denote by $\text{hol}(A)$ the set of all the holonomic $A$-modules.

Since the holonomic number is an infimum it is not clear at the outset that there will be modules which achieve this dimension. Clearly, $\text{hol}(A) \neq \emptyset$ if the Gelfand-Kirillov dimension of every finitely generated $A$-module is a natural number.
A nonzero submodule or a factor module of a holonomic is a holonomic module (since the Gelfand-Kirillov dimension of a submodule or a factor module does not exceed the Gelfand-Kirillov of the module). If, in addition, the finitely generated algebra $A$ is left Noetherian and finitely partitive then each holonomic $A$-module $M$ has finite length and each simple sub-factor of $M$ is a holonomic module.

Let us consider algebras $A$ having the following properties:

- (S) $A$ is a simple finitely generated infinite dimensional algebra.
- (N) There exists a standard finite dimensional filtration $F = \{ A_i \}$ of the algebra $A$ such that the associated graded algebra $\text{gr} \ A := \oplus_{i \geq 0} A_i/A_{i-1}$, $A_{-1} = 0$, is left Noetherian.
- (D) $\text{GK} \ (A) < \infty$, $\text{fd} \ (A) < \infty$, both $l(A) = l_F(A)$ and $L(A) = L_F(A)$ exist.
- (H) For every holonomic $A$-module $M$ there exists $l(M) = l_F(M)$.

In many cases we use the weaker form of the condition (D).

- (D') $\text{GK} \ (A) < \infty$, $d = \text{fd} \ (A) < \infty$, there exist $l(A) = l_F(A)$ and a positive number $c > 0$ such that $\nu(i) \leq ci^d$ for $i \gg 0$ where $\nu$ is the return function $\nu_{F \otimes F^0, K}$ of the left $A \otimes A^0$-module $A$.

It follows from (N) that $A$ is a left Noetherian algebra.

**Lemma 6.5 (4)**

1. The Weyl algebra $A_n$ over a field of characteristic zero with the standard finite dimensional filtration $F = \{ A_{n,i} \}$ associated with the canonical generators satisfies the conditions (S), (N), (D), (H). The return function $\nu_F(i) = i$ for $i \geq 0$, and so the leading coefficient of $\nu_F$ is $L_F(A_n) = 1$.

2. $\nu_{G,K}(i) = i$ for $i \geq 0$ and $L_G(P_n) = 1$ where $\nu_{G,K}$ is the return function of the $A_n$-module $P_n = K[X_1, \ldots, X_n] = A_n/(A_n \partial_1 + \cdots + A_n \partial_n)$ with the usual filtration $G = \{ P_{n,i} \}$ of the polynomial algebra.

**Proof.** 1. The only fact that we need to prove is that $\nu_F(i) = i$ for $i \geq 0$. We keep the notation of Theorem 4.6. In the proof of Theorem 4.6 we have seen that $\nu_F(i) \leq i$ for $i \geq 0$. It remains to prove the reverse inequality.

Each element $u$ in $A_n$ can be written in a unique way as a finite sum $u = \sum \lambda_{\alpha \beta} X^\alpha \partial^\beta$ where $\lambda_{\alpha \beta} \in K$ and $X^\alpha$ denotes the monomial $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and similarly $\partial^\beta$ denotes the monomial $\partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$. The element $u$ belongs to $A_{n,m}$ iff $|\alpha| + |\beta| \leq m$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $\alpha \in K[X_1, \ldots, X_n]$, then

$$\partial_i^m \alpha = \sum_{j=0}^m \binom{m}{j} \frac{\partial^j \alpha}{\partial X_i^j} \partial_i^{m-j}, \ m \in \mathbb{N}.$$
It follows that for any \( v \in \sum A_{n,i} \otimes A_{n,j}^0, \) \( i + j < m, \) the element \( vX_1^m = \sum \lambda_{\alpha\beta}X_1^\alpha \partial_\beta \) has the coefficient \( \lambda_{0,0} = 0, \) hence it could not be a non-zero scalar, and so \( \nu(i) \geq i \) for all \( i \geq 0. \) Hence \( \nu(i) = i \) all \( i \geq 0 \) and then \( L_F(A_n) = 1. \)

2. The standard filtration of the \( A_n \)-module \( P_n \) associated with the generating subspace \( K \) coincides with the usual filtration of the polynomial algebra \( P_n. \) Since \( \partial_j(P_{n,i}) \subseteq P_{n,i-1} \) for all \( i \geq 0 \) and \( j, \nu_{G,K}(i) \leq i \) for \( i \geq 0. \) Using the same arguments as above we see that for any \( u \in \sum_{j=0}^{l} A_{n,j} \otimes A_{n,i-j}^0 \) the element \( vX_1^i \) belongs to the ideal of \( P_n \) generated by \( X_1, \) hence, \( \nu_{G,K}(i) \geq i, \) and so \( \nu_{G,K}(i) = i \) for all \( i \geq 0 \) and \( L_G(P_n) = 1. \)

**Theorem 6.6** Assume that an algebra \( A \) satisfies the conditions \( (S), (H), (D) \) resp. \( (D') \) for some standard finite dimensional filtration \( F = \{A_i\} \) of \( A. \) Then for every holonomic \( A \)-module \( M \) its leading coefficient is bounded from below by a nonzero constant:

\[
\ell(M) \geq \sqrt{\frac{\ell(A)}{(L(A)L'(A))^{h_A}}},
\]

where

\[
L'(A) = \begin{cases} L(A), & \text{if } \ell(A) > 1, \\ L(A) + 1, & \text{if } \ell(A) = 1, \\ 1, & \text{if } \ell(A) < 1. \end{cases}
\]

resp.

\[
\ell(M) \geq \sqrt{\frac{\ell(A)}{(c(c + 1))^{h_A}}},
\]

**Proof.** Let \( M_0 \) be a generating finite dimensional subspace of \( M \) and \( \{M_i = A_iM_0\} \) be the standard finite dimensional filtration of \( M. \) In the proof of Theorem 3.1 we proved that \( \dim A_i \leq \dim M_\lambda(i) \dim M_{\lambda(i)+i} \) for \( i \geq 0 \) where \( \lambda \) is the left return function of the algebra \( A \) associated with the filtration \( F. \) Since \( \lambda(i) \leq \nu(i) \) for \( i \geq 0 \) we have \( \dim A_i \leq \dim M_{\nu(i)} \dim M_{\nu(i)+i}, \) hence, if \( (D) \) holds then

\[
\ell(A)^{\ell GK(A)} + \cdots \leq \ell^2(M)(L(A)L'(A))^{\ell GK(M)}\ell^{\ell GK(M)(\ell(A)+\max\{\ell(A),1\})} + \cdots,
\]

where three dots denote smaller terms.

If \( (D') \) holds then

\[
\ell(A)^{\ell GK(A)} + \cdots \leq \ell^2(M)(c(c + 1))^{\ell GK(M)}\ell^{\ell GK(M)(\ell(A)+\max\{\ell(A),1\})} + \cdots.
\]

The module \( M \) is holonomic, i.e. \( GK(A) = GK(M)(\ell(A) + \max\{\ell(A),1\}). \) Now, comparing the “leading” coefficients in the inequalities above we finish the proof. \( \square \)

Let \( A \) be as in Theorem 6.6. We attach to the algebra \( A \) two positive numbers \( c_A \) and \( c'_A \) in the cases \( (D) \) and \( (D') \) respectively:

\[
c_A = \sqrt{\frac{\ell(A)}{(L(A)L'(A))^{h_A}}} \quad \text{and} \quad c'_A = \sqrt{\frac{\ell(A)}{(c(c + 1))^{h_A}}}.
\]

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Corollary 6.7 Assume that an algebra $A$ satisfies the conditions $(S)$, $(N)$, $(H)$, $(D)$ or $(D')$. Let $0 \to N \to M \to L \to 0$ be an exact sequence of nonzero finitely generated $A$-modules. Then $M$ is holonomic if and only if $N$ and $L$ are holonomic, in that case $l(M) = l(N) + l(L)$.

Proof. The algebra $A$ is left Noetherian, so the module $M$ is finitely generated iff both $N$ and $L$ are so. The proof of Proposition 3.11 ([19], p. 295) shows that we can choose finite dimensional generating subspaces $N_0, M_0, L_0$ of the modules $N, M, L$ respectively such that the sequences

$$0 \to N_i = A_i N_0 \to M_i = A_i M_0 \to L_i = A_i L_0 \to 0$$

are exact for all $i$, hence, $\dim M_i = \dim N_i + \dim L_i$ and the results follow. □

Theorem 6.8 ([4]) Suppose that the conditions $(S)$, $(N)$, $(H)$, $(D)$ (resp. $(D')$) hold. Then each holonomic $A$-module $M$ has finite length which is less or equal to $l(M)/c_A$ (resp. $l(M)/c'_A$).

Proof. If $M = M_1 \supset M_2 \supset \ldots \supset M_m \supset M_{m+1} = 0$ is a chain of distinct submodules, then by Corollary 6.7 and Theorem 6.6

$$l(M) = \sum_{i=1}^{m} l(M_i/M_{i+1}) \geq mc_A, \quad (\text{resp. } l(M) \geq mc'_A),$$

thus $m \leq l(M)/c_A$ (resp. $m \leq l(M)/c'_A$). □

7 Filter dimension and commutative subalgebras of simple finitely generated algebras and their division algebras

In this section, using the first and the second filter inequalities, we obtain (i) an upper bound for the Gelfand-Kirillov dimension of commutative subalgebras of simple finitely generated infinite dimensional algebras (Theorem 7.2), and (ii) an upper bound for the transcendence degree of subfields of quotient division rings of (certain) simple finitely generated infinite dimensional algebras (Theorems 7.4 and 7.5).

For certain classes of algebras and their division algebras the maximum Gelfand-Kirillov dimension/transcendence degree over the commutative subalgebras/subfields were found in [1], [12], [13], [14], [15], [2], and [23].

Recall that

the Gelfand – Kirillov dimension $\text{GK}(C) = \text{the Krull dimension K.dim } (C)$

$= \text{the transcendence degree tr.deg}_R(C)$

for every commutative finitely generated algebra $C$ which is a domain.

An upper bound for the Gelfand-Kirillov dimensions of commutative subalgebras of simple finitely generated algebras.
Proposition 7.1 Let $A$ and $C$ be finitely generated algebras such that $C$ is a commutative domain with field of fractions $Q$, $B := C \otimes A$, and $B := Q \otimes A$. Let $M$ be a finitely generated $B$-module such that $\mathcal{M} := B \otimes_B M \neq 0$. Then $\text{GK}_B(M) \geq \text{GK}_Q(B \mathcal{M}) + \text{GK}_C(C)$.

Remark. $\text{GK}_Q$ stands for the Gelfand-Kirillov dimension over the field $Q$.

Proof. Let us fix standard filtrations $\{A_i\}$ and $\{C_i\}$ for the algebras $A$ and $C$ respectively. Let $h(t) \in Q[t]$ be the Hilbert polynomial for the algebra $C$, i.e. $\text{dim}_k(C_i) = h(i)$ for $i \gg 0$. Recall that $\text{GK}_C(C) = \text{deg}_t(h(t))$. The algebra $B$ has a standard filtration $\{B_i\}$ which is the tensor product of the standard filtrations $\{C_i\}$ and $\{A_i\}$ of the algebras $C$ and $A$, i.e. $B_i := \sum_{j=0}^i C_j \otimes A_{i-j}$. By the assumption, the $B$-module $M$ is finitely generated, so $M = B M_0$ where $M_0$ is a finite dimensional generating subspace for $M$. Then the $B$-module $M$ has a standard filtration $\{M_i := B_i M_0\}$. The $Q$-algebra $B$ has a standard (finite dimensional over $Q$) filtration $\{B_i := Q \otimes A_i\}$, and the $B$-module $\mathcal{M}$ has a standard (finite dimensional over $Q$) filtration $\{\mathcal{M}_i := B_i M_0' = QA_i M_0'\}$ where $M_0'$ is the image of the vector space $M_0$ under the $B$-module homomorphism $M \to \mathcal{M}, m \mapsto m' := 1 \otimes_B m$.

For each $i \geq 0$, one can fix a $K$-subspace, say $L_i$, of $A_i M_0'$ such that $\text{dim}_Q(Q A_i M_0') = \text{dim}_K(L_i)$. Now, $B_{2i} \supseteq C_i \otimes A_i$ implies $\text{dim}_K(B_{2i} M_0) \geq \text{dim}_K((C_i \otimes A_i) M_0)$, and $((C_i \otimes A_i) M_0)' \supseteq C_i L_i$ implies $\text{dim}_K((C_i \otimes A_i) M_0)' \supseteq \text{dim}_K(C_i L_i) = \text{dim}_K(C_i) \text{dim}_K(L_i) = \text{dim}_K(C_i) \text{dim}_Q(\mathcal{M}_i)$. It follows that

\[
\text{GK}_B(M) = \gamma(\text{dim}_K(M_i)) \geq \gamma(\text{dim}_K(M_{2i})) = \gamma(\text{dim}_K(B_{2i} M_0)) \geq \gamma(\text{dim}_K((C_i \otimes A_i) M_0)) \\
\quad \geq \gamma(\text{dim}_K((C_i \otimes A_i) M_0)') \geq \gamma(\text{dim}_K(C_i) \text{dim}_K(\mathcal{M}_i)) \\
\quad = \gamma(\text{dim}_K(C_i)) + \gamma(\text{dim}_Q(\mathcal{M}_i)) \quad \text{(since } \gamma(\text{dim}_K(C_i)) = h(i), \text{ for } i \gg 0) \\
\quad = \text{GK}_C(C) + \text{GK}_Q(\mathcal{M}). \quad \square
\]

Recall that $d = \text{fd}$, $\text{lfd}$. A $K$-algebra $A$ is called central if its centre $Z(A) = K$.

Theorem 7.2 Let $A$ be a central simple finitely generated $K$-algebra of Gelfand-Kirillov dimension $0 < n < \infty$ (over $K$). Let $C$ be a commutative subalgebra of $A$. Then $\text{GK}_C(C) \leq \text{GK}_A(A) \left(1 - \frac{1}{f_A + \max\{f_A, 1\}}\right)$

where $f_A := \max\{d_{Q_m}(Q_m \otimes A) \mid 0 \leq m \leq n\}$, $Q_0 := K$, and $Q_m := K(x_1, \ldots, x_m)$ is a rational function field in indeterminates $x_1, \ldots, x_m$.

Proof. Let $P_m = K[x_1, \ldots, x_m]$ be a polynomial algebra over the field $K$. Then $Q_m$ is its field of fractions and $\text{GK}(P_m) = m$. Suppose that $P_m$ is a subalgebra of $A$. Then $m = \text{GK}(P_m) \leq \text{GK}(A) = n$. For each $m \geq 0$, $Q_m \otimes A$ is a central simple $Q_m$-algebra ([19], 9.6.9) of Gelfand-Kirillov dimension (over $Q_m$) $\text{GK}_{Q_m}(Q_m \otimes A) = \text{GK}_A(A) > 0$, hence

\[
\text{GK}_A = \text{GK}_A(A A A) \geq \text{GK}_A(A A P_m) = \text{GK}(P_m \otimes A) \quad (P_m \text{ is commutative}) \\
\quad \geq \text{GK}_{Q_m}(Q_m \otimes A \otimes P_m) + \text{GK}(P_m) \quad (\text{Lemma 7.1}) \\
\quad \geq \frac{\text{GK}_A}{d_{Q_m}(Q_m \otimes A) + \max\{d_{Q_m}(Q_m \otimes A), 1\}} + m \quad (\text{Theorem 5.1}).
\]

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Hence,

\[ m \leq \text{GK}(A) \left( 1 - \frac{1}{d_{Q_m}(Q_m \otimes A)} + \max\{d_{Q_m}(Q_m \otimes A), 1\} \right) \leq \text{GK}(A), \]

and so

\[ \text{GK}(C) \leq \text{GK}(A) \left( 1 - \frac{1}{f_A + \max\{f_A, 1\}} \right). \]

As a consequence we have a short proof of the following well-known result.

**Corollary 7.3** Let \( K \) be an algebraically closed field of characteristic zero, \( X \) be a smooth irreducible affine algebraic variety of dimension \( n := \dim(X) > 0 \), and \( C \) be a commutative subalgebra of the ring of differential operators \( D(X) \). Then \( \text{GK}(C) \leq n \).

**Proof.** The algebra \( D(X) \) is central since \( K \) is an algebraically closed field of characteristic zero [19], Ch. 15. By Theorem 4.3, \( f_{D(X)} = 1 \), and then, by Theorem 7.2,

\[ \text{GK}(C) \leq 2n(1 - \frac{1}{1 + 1}) = n. \]

**Remark.** For the ring of differential operators \( D(X) \) the upper bound in Theorem 7.2 for the Gelfand-Kirillov dimension of commutative subalgebras of \( D(X) \) is an exact upper bound since as we mentioned above the algebra \( \mathcal{O}(X) \) of regular functions on \( X \) is a commutative subalgebra of \( D(X) \) of Gelfand-Kirillov dimension \( n \).

**An upper bound for the transcendence degree of subfields of quotient division algebras of simple finitely generated algebras.**

Recall that the transcendence degree \( \text{tr.deg}_{K}(L) \) of a field extension \( L \) of a field \( K \) coincides with the Gelfand-Kirillov dimension \( \text{GK}_K(L) \), and, by the Goldie’s Theorem, a left Noetherian algebra \( A \) has a quotient algebra \( D = D_A \) (i.e. \( D = S^{-1}A \) where \( S \) is the set of regular elements = the set of non-zerodivisors of \( A \)). As a rule, the quotient algebra \( D \) has infinite Gelfand-Kirillov dimension and is not a finitely generated algebra (e.g., the quotient division algebra \( D(X) \) of the ring of differential operators \( D(X) \) on each smooth irreducible affine algebraic variety \( X \) of dimension \( n > 0 \) over a field \( K \) of characteristic zero contains a noncommutative free subalgebra since \( D(X) \supseteq D(A^1) \) and the first Weyl division algebra \( D(A^1) \) has this property [17]). So, if we want to find an upper bound for the transcendence degree of subfields in the quotient algebra \( D \) we can not apply Theorem 7.2. Nevertheless, imposing some natural (mild) restrictions on the algebra \( A \) one can obtain exactly the same upper bound for the transcendence degree of subfields in the quotient algebra \( D_A \) as the upper bound for the Gelfand-Kirillov dimension of commutative subalgebras in \( A \).

**Theorem 7.4** ([7]) Let \( A \) be a simple finitely generated \( K \)-algebra such that \( 0 < n := \text{GK}(A) < \infty \), all the algebras \( Q_m \otimes A, m \geq 0 \), are simple finitely partitive algebras where \( Q_0 := K, Q_m := K(x_1, \ldots, x_m) \) is a rational function field and, for each \( m \geq 0 \),
the Gelfand-Kirillov dimension (over $Q_m$) of every finitely generated $Q_m \otimes A$-module is a natural number. Let $B = S^{-1}A$ be the localization of the algebra $A$ at a left Ore subset $S$ of $A$. Let $L$ be a (commutative) subfield of the algebra $B$ that contains $K$. Then

$$\text{tr.deg}_K(L) \leq \text{GK}(A) \left(1 - \frac{1}{f_A + \max\{f_A, 1\}}\right)$$

where $f_A := \max\{d_{Q_m}(Q_m \otimes A) \mid 0 \leq m \leq n\}$.

**Proof.** It follows immediately from a definition of the Gelfand-Kirillov dimension that $	ext{GK}_{K'}(K' \otimes C) = \text{GK}(C)$ for any $K$-algebra $C$ and any field extension $K'$ of $K$. In particular, $\text{GK}_{Q_m}(Q_m \otimes A) = \text{GK}(A)$ for all $m \geq 0$. By Theorem 4.2

$$\text{K.dim}(Q_m \otimes A) \leq \text{GK}(A) \left(1 - \frac{1}{d_{Q_m}(Q_m \otimes A) + \max\{d_{Q_m}(Q_m \otimes A), 1\}}\right).$$

Let $L$ be a subfield of the algebra $B$ that contains $K$. Suppose that $L$ contains a rational function field (isomorphic to) $Q_m$ for some $m \geq 0$.

$$m = \text{tr.deg}_K(Q_m) \leq \text{K.dim}(Q_m \otimes Q_m) \leq \text{K.dim}(Q_m \otimes B) \text{ (by Proposition 6.5.3 since } Q_m \otimes B \text{ is a free } Q_m \otimes Q_m \text{-module)} = \text{K.dim}(Q_m \otimes S^{-1}A) = \text{K.dim}(S^{-1}(Q_m \otimes A)) \leq \text{K.dim}(Q_m \otimes A) \text{ (by Proposition 6.5.3.(ii),(b))} \leq \text{GK}(A) \left(1 - \frac{1}{d_{Q_m}(Q_m \otimes A) + \max\{d_{Q_m}(Q_m \otimes A), 1\}}\right) \leq \text{GK}(A).$$

Hence

$$\text{tr.deg}_K(L) \leq \text{GK}(A) \left(1 - \frac{1}{f_A + \max\{f_A, 1\}}\right). \quad \square$$

Recall that every somewhat commutative algebra $A$ is a Noetherian finitely generated finitely partitive algebra of finite Gelfand-Kirillov dimension, the Gelfand-Kirillov dimension of every finitely generated $A$-modules is an integer, and (Quillen’s Lemma): the ring $\text{End}_A(M)$ is algebraic over $K$ (see [19], Ch. 8 or [16] for details).

**Theorem 7.5** ([7]) Let $A$ be a central simple somewhat commutative infinite dimensional $K$-algebra and let $D = D_A$ be its quotient algebra. Let $L$ be a subfield of $D$ that contains $K$. Then the transcendence degree of the field $L$ (over $K$)

$$\text{tr.deg}_K(L) \leq \text{GK}(A) \left(1 - \frac{1}{f_A + \max\{f_A, 1\}}\right)$$

where $f_A := \max\{d_{Q_m}(Q_m \otimes A) \mid 0 \leq m \leq \text{GK}(A)\}$. 

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Proof. The algebra $A$ is a somewhat commutative algebra, so it has a finite dimensional filtration $A = \bigcup_{i \geq 0} A_i$ such that the associated graded algebra is a commutative finitely generated algebra. For each integer $m \geq 0$, the $Q_m$-algebra $Q_m \otimes A = \bigcup_{i \geq 0} Q_m \otimes A_i$ has the finite dimensional filtration (over $Q_m$) such that the associated graded algebra $\text{gr}(Q_m \otimes A) = \oplus_{i \geq 0} Q_m \otimes A_i/Q_m \otimes A_{i-1} \simeq Q_m \otimes \text{gr}(A)$ is a commutative finitely generated $Q_m$-algebra. So, $Q_m \otimes A$ is a somewhat commutative $Q_m$-algebra.

By the assumption $\dim_K(A) = \infty$, hence $\dim_K(\text{gr}(A)) = \infty$ which implies $\text{GK}(\text{gr}(A)) > 0$, and so $\text{GK}(A) > 0$ (since $\text{GK}(A) = \text{GK}(\text{gr}(A))$). The algebra $A$ is a central simple $K$-algebra, so $Q_m \otimes A$ is a central simple $Q_m$-algebra ([19], 9.6.9). Now, Theorem 7.5 follows from Theorem 7.4 applied to $B = D$. \Box

**Theorem 7.6** Let $K$ be an algebraically closed field of characteristic zero, $D(X)$ be the ring of differential operators on a smooth irreducible affine algebraic variety $X$ of dimension $n > 0$, and $D(X)$ be the quotient division ring for $D(X)$. Let $L$ be a (commutative) subfield of $D(X)$ that contains $K$. Then $\text{tr.deg}_K(L) \leq n.$

**Remark.** This inequality is, in fact, an exact upper bound for the transcendence degree of subfields in $D(X)$ since the field of fractions $Q(X)$ for the algebra $\mathcal{O}(X)$ is a commutative subfield of the division ring $D(X)$ with $\text{tr.deg}_K(Q(X)) = n.$

**Proof.** Since $Q_m \otimes D_K(\mathcal{O}(X)) \simeq D(Q_m(\mathcal{O}(X)))$ and $d(D(Q_m(\mathcal{O}(X)))) = 1$ for all $m \geq 0$ we have $f_{D(X)} = 1.$ Now, Theorem 7.5 follows from Theorem 7.4.

$$\text{tr.deg}_K(L) \leq 2n(1 - \frac{1}{1 + 1}) = n.$$ \Box

Following [15] for a $K$-algebra $A$ define the **commutative dimension**

$$\text{Cdim}(A) := \max\{\text{GK}(C) \mid C \text{ is a commutative subalgebra of } A\}.$$ 

The commutative dimension $\text{Cdim}(A)$ (if finite) is the largest non-negative integer $m$ such that the algebra $A$ contains a polynomial algebra in $m$ variables ([15], 1.1, or [19], 8.2.14). So, $\text{Cdim}(A) = \mathbb{N} \cup \{\infty\}.$ If $A$ is a subalgebra of $B$ then $\text{Cdim}(A) \leq \text{Cdim}(B)$.

**Corollary 7.7** Let $X$ and $Y$ be smooth irreducible affine algebraic varieties of dimensions $n$ and $m$ respectively, let $D(X)$ and $D(Y)$ be quotient division rings for the rings of differential operators $D(X)$ and $D(Y)$. Then there is no $K$-algebra embedding $D(X) \to D(Y)$ if $n > m$.

**Proof.** By Theorem 7.6 $\text{Cdim}(D(X)) = n$ and $\text{Cdim}(D(Y)) = m.$ Suppose that there is a $K$-algebra embedding $D(X) \to D(Y)$. Then $n = \text{Cdim}(D(X)) \leq \text{Cdim}(D(Y)) = m.$ \Box

For the Weyl algebras $A_n = D(\mathbb{A}^n)$ and $A_m = D(\mathbb{A}^m)$ the result above was proved by Gelfand and Kirillov in [12]. They introduced a new invariant of an algebra $A$, so-called the
(Gelfand-Kirillov) transcendence degree \( \text{GKtr.deg}(A) \), and proved that \( \text{GKtr.deg}(D_n) = 2n \). Recall that
\[
\text{GKtr.deg}(A) := \sup \inf_V \text{GK}(K[bV])
\]
where \( V \) ranges over the finite dimensional subspaces of \( A \) and \( b \) ranges over the regular elements of \( A \). Another proofs of the corollary based on different ideas were given by A. Joseph [14] and R. Resco [23], see also [19], 6.6.19. Joseph’s proof is based on the fact that the centralizer of any isomorphic copy of the Weyl algebra \( A_n \) in its division algebra \( D_n := D(A^n) \) reduces to scalars ([15], 4.2), Resco proved that \( \text{Cdim}(D_n) = n \) ([23], 4.2) using the result of Rentschler and Gabriel [22] that \( \text{Kdim}(A_n) = n \) (over an arbitrary field of characteristic zero).

8 Filter Dimension and Isotropic Subalgebras of Poisson Algebras

In this section, we apply Theorem 7.2 to obtain an upper bound for the Gelfand-Kirillov dimension of isotropic subalgebras of certain Poisson algebras (Theorem 8.1).

Let \( (P, \{\cdot, \cdot\}) \) be a Poisson algebra over the field \( K \). Recall that \( P \) is an associative commutative \( K \)-algebra which is a Lie algebra with respect to the bracket \( \{\cdot, \cdot\} \) for which Leibniz’s rule holds:
\[
\{a, xy\} = \{a, x\}y + x\{a, y\} \quad \text{for all} \quad a, x, y \in P,
\]
which means that the inner derivation \( \text{ad}(a) : P \to P, x \mapsto \{a, x\} \), of the Lie algebra \( P \) is also a derivation of the associative algebra \( P \). Therefore, to each Poisson algebra \( P \) one can attach an associative subalgebra \( A(P) \) of the ring of differential operators \( D(P) \) with coefficients from the algebra \( P \) which is generated by \( P \) and \( \text{ad}(P) := \{\text{ad}(a) \mid a \in P\} \). If \( P \) is a finitely generated algebra then so is the algebra \( A(P) \) with \( \text{GK}(A(P)) \leq \text{GK}(D(P)) < \infty \).

Example. Let \( P_{2n} = K[x_1, \ldots, x_{2n}] \) be the Poisson polynomial algebra over a field \( K \) of characteristic zero equipped with the Poisson bracket
\[
\{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_{n+i}} - \frac{\partial f}{\partial x_{n+i}} \frac{\partial g}{\partial x_i} \right).
\]
The algebra \( A(P_{2n}) \) is generated by the elements
\[
x_1, \ldots, x_{2n}, \quad \text{ad}(x_i) = \frac{\partial}{\partial x_{n+i}}, \quad \text{ad}(x_{n+i}) = -\frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n.
\]
So, the algebra \( A(P_{2n}) \) is canonically isomorphic to the Weyl algebra \( A_{2n} \).

Definition. We say that a Poisson algebra \( P \) is a strongly simple Poisson algebra if
1. $P$ is a finitely generated (associative) algebra which is a domain,

2. the algebra $A(P)$ is central simple, and

3. for each set of algebraically independent elements $a_1, \ldots, a_m$ of the algebra $P$ such that $\{a_i, a_j\} = 0$ for all $i, j = 1, \ldots, m$ the (commuting) elements $a_1, \ldots, a_m, \text{ad}(a_1), \ldots, \text{ad}(a_m)$ of the algebra $A(P)$ are algebraically independent.

**Theorem 8.1** ([7]) Let $P$ be a strongly simple Poisson algebra, and $C$ be an isotropic subalgebra of $P$, i.e. $\{C, C\} = 0$. Then

$$\text{GK}(C) \leq \frac{\text{GK}(A(P))}{2} \left(1 - \frac{1}{f_{A(P)} + \max\{f_{A(P)}, 1\}}\right)$$

where $f_{A(P)} := \max\{d_{Q_m}(Q_m \otimes A(P)) | 0 \leq m \leq \text{GK}(A(P))\}$.

**Proof.** By the assumption the finitely generated algebra $P$ is a domain, hence the finitely generated algebra $A(P)$ is a domain (as a subalgebra of the domain $D(Q(P))$, the ring of differential operators with coefficients from the field of fractions $Q(P)$ for the algebra $P$). It suffices to prove the inequality for isotropic subalgebras of the Poisson algebra $P$ that are polynomial algebras. So, let $C$ be an isotropic polynomial subalgebra of $P$ in $m$ variables, say $a_1, \ldots, a_m$. By the assumption, the commuting elements $a_1, \ldots, a_m, \text{ad}(a_1), \ldots, \text{ad}(a_m)$ of the algebra $A(P)$ are algebraically independent. So, the Gelfand-Kirillov dimension of the subalgebra $C'$ of $A(P)$ generated by these elements is equal to $2m$. By Theorem 7.2,

$$2\text{GK}(C) = 2m = \text{GK}(C') \leq \text{GK}(A(P)) \left(1 - \frac{1}{f_{A(P)} + \max\{f_{A(P)}, 1\}}\right),$$

and this proves the inequality. □

**Corollary 8.2**

1. The Poisson polynomial algebra $P_{2n} = K[x_1, \ldots, x_{2n}]$ (with the Poisson bracket) over a field $K$ of characteristic zero is a strongly simple Poisson algebra, the algebra $A(P_{2n})$ is canonically isomorphic to the Weyl algebra $A_{2n}$.

2. The Gelfand-Kirillov dimension of every isotropic subalgebra of the polynomial Poisson algebra $P_{2n}$ is $\leq n$.

**Proof.** 1. The third condition in the definition of strongly simple Poisson algebra is the only statement we have to prove. So, let $a_1, \ldots, a_m$ be algebraically independent elements of the algebra $P_{2n}$ such that $\{a_i, a_j\} = 0$ for all $i, j = 1, \ldots, m$. One can find polynomials, say $a_{m+1}, \ldots, a_{2n}$, in $P_{2n}$ such that the elements $a_1, \ldots, a_{2n}$ are algebraically independent, hence the determinant $d$ of the Jacobian matrix $J := \left(\frac{\partial a_i}{\partial x_j}\right)$ is a nonzero polynomial. Let $X = (\{x_i, x_j\})$ and $Y = (\{a_i, a_j\})$ be, so-called, the Poisson matrices associated with the
elements \( \{x_i\} \) and \( \{a_i\} \). It follows from \( Y = J^T X J \) that \( \det(Y) = d^2 \det(X) \neq 0 \) since \( \det(X) \neq 0 \). The derivations

\[
\delta_i := d^{-1} \det \begin{pmatrix}
\{a_1, a_1\} & \ldots & \{a_1, a_{i-1}\} & \{a_1, a_{i+1}\} & \ldots & \{a_1, a_{2n}\} \\
\{a_2, a_1\} & \ldots & \{a_2, a_{i-1}\} & \{a_2, a_{i+1}\} & \ldots & \{a_2, a_{2n}\} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\{a_{2n}, a_1\} & \ldots & \{a_{2n}, a_{i-1}\} & \{a_{2n}, a_{i+1}\} & \ldots & \{a_{2n}, a_{2n}\}
\end{pmatrix},
\]

\( i = 1, \ldots, 2n \), of the rational function field \( Q_{2n} = K(x_1, \ldots, x_{2n}) \) satisfy the following properties: \( \delta_i(a_j) = \delta_{i,j} \), the Kronecker delta. For each \( i \) and \( j \), the kernel of the derivation \( \Delta_{ij} := \delta_i \delta_j - \delta_j \delta_i \in \text{Der}_K(Q_{2n}) \) contains \( 2n \) algebraically independent elements \( a_1, \ldots, a_{2n} \). Hence \( \Delta_{ij} = 0 \) since the field \( Q_{2n} \) is algebraic over its subfield \( K(a_1, \ldots, a_{2n}) \) and \( \text{char}(K) = 0 \). So, the subalgebra, say \( W \), of the ring of differential operators \( D(Q_{2n}) \) generated by the elements \( a_1, \ldots, a_{2n}, \delta_1, \ldots, \delta_{2n} \) is isomorphic to the Weyl algebra \( A_{2n} \), and so \( \text{GK}(W) = \text{GK}(A_{2n}) = 4n \).

Let \( U \) be the \( K \)-subalgebra of \( D(Q_{2n}) \) generated by the elements \( x_1, \ldots, x_{2n}, \delta_1, \ldots, \delta_{2n} \), and \( d^{-1} \). Let \( P' \) be the localization of the polynomial algebra \( P_{2n} \) at the powers of the element \( d \). Then \( \delta_1, \ldots, \delta_{2n} \in \sum_{i=1}^{2n} P' \text{ad}(a_i) \) and \( \text{ad}(a_1), \ldots, \text{ad}(a_{2n}) \in \sum_{i=1}^{2n} P' \delta_i \), hence the algebra \( U \) is generated (over \( K \)) by \( P' \) and \( \text{ad}(a_1), \ldots, \text{ad}(a_{2n}) \). The algebra \( U \) can be viewed as a subalgebra of the ring of differential operators \( D(P') \). Now, the inclusions, \( W \subset U \subset D(P') \) imply \( 4n = \text{GK}(W) \leq \text{GK}(U) \leq \text{GK}(D(P')) = 2\text{GK}(P') = 4n \), therefore \( \text{GK}(U) = 4n \). The algebra \( U \) is a factor algebra of an iterated Ore extension \( V = P'[t_1; \text{ad}(a_1)] \cdots [t_{2n}; \text{ad}(a_{2n})] \). Since \( P' \) is a domain, so is the algebra \( V \). The algebra \( P' \) is a finitely generated algebra of Gelfand-Kirillov dimension \( 2n \), hence \( \text{GK}(V) = \text{GK}(P') + 2n = 4n \) (by [19], 8.2.11). Since \( \text{GK}(V) = \text{GK}(U) \) and any proper factor algebra of \( V \) has Gelfand-Kirillov dimension strictly less than \( \text{GK}(V) \) (by [19], 8.3.5, since \( V \) is a domain), the algebras \( V \) and \( U \) must be isomorphic. Therefore, the (commuting) elements \( a_1, \ldots, a_m, \text{ad}(a_1), \ldots, \text{ad}(a_m) \) of the algebra \( U \) (and \( A(P) \)) must be algebraically independent.

2. Let \( C \) be an isotropic subalgebra of the Poisson algebra \( P_{2n} \). Note that \( f_{A(P_{2n})} = f_{A_{2n}} = 1 \) and \( \text{GK}(A_{2n}) = 4n \). By Theorem 8.1

\[
\text{GK}(C) \leq \frac{4n}{2} \left( 1 - \frac{1}{1+1} \right) = n. \quad \Box
\]

Remark. This result means that for the Poisson polynomial algebra \( P_{2n} \) the right hand side of the inequality of Theorem 8.1 is the exact upper bound for the Gelfand-Kirillov dimension of isotropic subalgebras in \( P_{2n} \) since the polynomial subalgebra \( K[x_1, \ldots, x_n] \) of \( P_{2n} \) is isotropic.

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