Entire functions of exponential type represented by pseudo-random and random Taylor series

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To Alex Eremenko on occasion of his birthday

Abstract

We study the influence of the multipliers $\xi(n)$ on the angular distribution of zeroes of the Taylor series

$$F_\xi(z) = \sum_{n \geq 0} \xi(n) \frac{z^n}{n!}.$$

We show that the distribution of zeroes of $F_\xi$ is governed by certain autocorrelations of the sequence $\xi$. Using this guiding principle, we consider several examples of random and pseudo-random sequences $\xi$ and, in particular, answer some questions posed by Chen and Littlewood in 1967.

As a by-product we show that if $\xi$ is a stationary random integer-valued sequence, then either it is periodic, or its spectral measure has no gaps in its support. The same conclusion is true if $\xi$ is a complex-valued stationary ergodic sequence that takes values from a uniformly discrete set.

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1 Introduction

In this work, we consider entire functions of exponential type represented by the Taylor series

\[ F_\xi(z) = \sum_{n \geq 0} \xi(n) \frac{z^n}{n!}, \quad \xi : \mathbb{Z}_+ \to \mathbb{C}. \]

We are interested in the influence of the multipliers \( \xi(n) \) on the angular distribution of zeroes of the function \( F_\xi \). This question belongs to a “terra incognita” in the theory of entire functions that contains no general results going in this direction but several interesting examples. These examples include:

(a) random independent identically distributed \( \xi(n) \) (Littlewood–Offord [15], Kabluchko–Zaporozhets [10]),

(b) \( \xi(n) = e(qn^2) \) with quadratic irrationality \( q \) (Nassif [16], Littlewood [14]) and, more generally, arbitrary irrational \( q \) (Eremenko–Ostrovskii [7]),

(c) \( \xi(n) = e(n(\log n)^\beta) \) with \( \beta > 1 \), and \( e(n^\beta) \) with \( 1 < \beta < \frac{3}{2} \) (Chen–Littlewood [5]),

(d) uniformly almost periodic \( \xi(n) \) (Levin [13, Chapter VI, §7]),

Here and elsewhere, \( e(t) = e^{2\pi ti} \).

In this work, we consider the following four sequences \( \xi \):

(i) \( \xi(n) = e(Q(n)) \), where \( Q(x) = \sum_{k \geq 2} q_k x^k \) is a polynomial with real coefficients \( q_k \), at least one of which is irrational.

(ii) \( \xi(n) = e(n^\beta) \), where \( \beta \geq \frac{3}{2} \) is non-integer.

(iii) \( \xi(n) \) is a stationary sequence with a mild decay of the maximal correlation coefficient.

(iv) \( \xi(n) \) is a stationary Gaussian sequence.

In the cases (i), (ii), and (iii), using some potential theory, we reduce the question on the asymptotic distribution of zeroes of \( F_\xi \) to certain lower bounds for the exponential sums

\[ W_R(\theta) = \sum_{|n| \leq N} \xi(n + R)e(n\theta)e^{-\frac{n^2}{R^2}}. \]
when $R \gg 1$ and $N$ has the size $R^{4+\varepsilon}$ (see Lemmas 4.2.1 and 4.3.1). These lower bounds, in turn, depend on the behaviour of the autocorrelations

$$m \mapsto \frac{1}{N} \sum_{n=1}^{N} \xi(n + R) \bar{\xi}(n + m + R) e(m\theta).$$

In the case (iv) (similarly to the almost-periodic case (d)), the zero set of $F_\xi$ has an angular density that, generally speaking, is not constant, as in the cases (i), (ii) and (iii). This density is determined by the spectrum of the sequence $\xi$, that is, after all, also by the autocorrelations between the elements of $\xi$.

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## 2 Main results

### 2.1

We start with the cases when the zeroes of $F_\xi$ have the uniform angular distribution.

**Definition.** We say that the sequence $\xi: \mathbb{Z}_+ \to \mathbb{C}$ is an $L$-sequence, if

$$\frac{\log |F_\xi(tz)|}{t} \xrightarrow{t \to \infty} |z|, \quad \text{as distributions}. \quad (2.1.1)$$

Since the Laplacian is continuous in the distributional topology, (2.1.1) yields

$$\frac{1}{t} \Delta \log |F_\xi(tz)| \xrightarrow{t \to \infty} \Delta|z| = dr \otimes d\theta, \quad z = re^{i\theta}. \quad (2.1.2)$$

Denoting by $n_F(r; \theta_1, \theta_2)$ the number of zeroes (counted with multiplicities) of the entire function $F$ in the sector $\{ z : 0 \leq |z| \leq r, \theta_1 \leq \arg(z) < \theta_2 \}$ and recalling that $\frac{1}{2\pi} \Delta \log |F|$ is the sum of point masses at zeroes of $F$, we can rewrite (2.1.2) in a more traditional form: for every $\theta_1 < \theta_2$,

$$n_{F_\xi}(r; \theta_1, \theta_2) = \frac{(\theta_2 - \theta_1 + o(1)) r}{2\pi} \quad \text{as } r \to \infty. \quad (2.1.3)$$
Theorem 1. Suppose that
\[ Q(x) = \sum_{k=2}^{d} q_k x^k \]
is a polynomial with real coefficients \( q_k \) and that at least one of the coefficients \( q_k \) is irrational. Then \( \xi(n) = e(Q(n)) \) is an \( L \)-sequence.

For \( Q(x) = qx^2 \), \( q \) being a quadratic irrationality, this is a result of Nassif [16] and Littlewood [14]. For arbitrary irrational \( q \)'s, this was proven by Eremenko and Ostrovskii [7]. It seems that the methods used in these works cannot be extended to polynomials \( Q \) of degree bigger than 2. Quoting Chen and Littlewood [5], “many lines of experience converge to show that there can be nothing doing if \( \Lambda(n) \gg n^2 \)” (in their notation, \( \xi(n) = e(\Lambda(n)) \), and \( \Lambda(n) \gg n^2 \) means that \( \Lambda(n)/n^2 \rightarrow \infty \)).

Theorem 2. For any non-integer \( \beta > 1 \), the sequence \( \xi(n) = e(n^\beta) \) is an \( L \)-sequence.

As we have already mentioned, the case \( 1 < \beta < \frac{3}{2} \) is due to Chen and Littlewood [5]. They used the Poisson summation combined with the saddle point approximation and obtained much more accurate information about the asymptotic location of zeroes of the function \( F_\xi \). They write: “The gap \( \frac{3}{2} \leq \beta < 2 \) presents a most interesting unsolved problem”.

2.2

Now, we turn to the case when \( \xi : \mathbb{Z} \rightarrow \mathbb{C} \) is a stationary sequence of random variables (formally, we need only the restriction of \( \xi \) on \( \mathbb{Z}_+ \), but due to stationarity, this restriction determines a unique extension of \( \xi \) onto \( \mathbb{Z} \)). As usual, the stationarity means that, for every positive integer \( k \), every integers \( n_1, \ldots, n_k \), and every integer \( m \), the \( k \)-tuples of random variables
\[ \langle \xi(n_1), \ldots, \xi(n_k) \rangle, \quad \langle \xi(n_1 + m), \ldots, \xi(n_k + m) \rangle \]
are equidistributed. In what follows, we deal only with stationary sequences having a finite second moment. Then the sequence
\[ m \mapsto \mathbb{E}\{\xi(0)\xi(m)\} \]
is positive-definite, and therefore, is the Fourier transform of a non-negative measure \( \rho \in M_+(\mathbb{T}) \). Here and elsewhere, \( \mathbb{T} = \{e^{i\theta} : |\theta| \leq \pi\} \) is the unit circle. We call \( \rho \) the spectral measure of \( \xi \). Then the spectrum \( \sigma(\xi) \) of \( \xi \) is the support of the measure \( \rho \). Note that we do not require that \( \mathbb{E}\xi(0) = 0 \). The definition of the spectral measure we use here differs from the one, which is more customary in the theory of stationary processes [8], by the atom at \( \theta = 0 \) with the weight \( |\mathbb{E}\xi(0)|^2 \).

We also need the maximal correlation coefficient of the sequence \( \xi \)

\[
r(m) \overset{\text{def}}{=} \sup \left\{ \frac{\left| \mathbb{E}\{ (x - \mathbb{E}x)(y - \mathbb{E}y) \} \right|}{\sqrt{\mathbb{E}|x - \mathbb{E}x|^2 \cdot \mathbb{E}|y - \mathbb{E}y|^2}} : x \in L^2_{(-\infty,0]}, y \in L^2_{[m,\infty)} \right\}
\]

where \( L^2_{(-\infty,0]} \) consists of the elements of the \( \sigma \)-algebra generated by the set \( \{ \xi(n) : -\infty < n \leq 0 \} \) with finite second moment, and \( L^2_{[m,\infty)} \) consists of the elements of of the \( \sigma \)-algebra generated by the set \( \{ \xi(n) : m \leq n < +\infty \} \) with finite second moment.

**Theorem 3.** Let \( \xi \) be a bounded stationary sequence of random variables, and let the maximal correlation coefficient of \( \xi \) satisfy

\[
r(m) = O\left( (\log m)^{-\kappa} \right), \quad m \to \infty,
\]

with some \( \kappa > 1 \). Then, almost surely, \( \xi \) is an \( L \)-sequence.

**2.3**

Now, we turn to the Gaussian stationary sequences \( \xi \). In this case, the leading term of the asymptotics of \( \log |F_\xi| \) is determined by the support of the spectral measure \( \rho \) of the sequence \( \xi \).

We start with some preliminaries. For any set \( \sigma \subset \mathbb{T} \), we denote by \( \text{ch}(\sigma) \) the closed convex hull of \( \sigma \), and by

\[
H_\sigma(z) \overset{\text{def}}{=} \max_{\lambda \in \text{ch}(\sigma)} \text{Re} \left( z\lambda \right) = \sup_{\lambda \in \sigma} \text{Re} \left( z\lambda \right)
\]

the Minkowski functional of \( \text{ch}(\sigma) \). This function is subharmonic in \( \mathbb{C} \) and homogeneous, that is, \( H_\sigma(re^{i\theta}) = h_\sigma(\theta)r \), where \( h_\sigma \) is the so called supporting function of \( \text{ch}(\sigma) \). The distributional Laplacian of the function \( H_\sigma \) is \( \Delta H_\sigma = \)
dr \otimes ds_\sigma(\theta)$, where $ds_\sigma(\theta) = (h''_\sigma + h_\sigma) \, d\theta$, the second derivative $h''_\sigma$ is also understood in the sense of distributions.

**Definition.** Let $\sigma \subset \mathbb{T}$. We say that the sequence $\xi$ is an $L(\sigma)$-sequence, if

$$\frac{\log |F_\xi(tz)|}{t} \xrightarrow{t \to \infty} H_\sigma(z), \quad \text{as distributions.} \quad (2.3.1)$$

Obviously, $L$-sequences are a special case of $L(\sigma)$-sequences that correspond to the case when the set $\sigma$ is dense in $\mathbb{T}$.

In the language of the entire function theory [13, Chapters II and III] (see also [11] for a modern treatment), this definition says that $F_\xi$ is an entire function of completely regular growth in the Levin–Pfluger sense with the Phragmén–Lindelöf indicator $h_\sigma$. Condition (2.3.1) yields the angular asymptotics of zeroes of $F_\xi$:

$$n_{F_\xi}(r; \theta_1, \theta_2) = \frac{(s_\sigma(\theta_2) - s_\sigma(\theta_1) + o(1)) r}{2\pi}, \quad r \to \infty, \quad (2.3.2)$$

where $-\pi \leq \theta_1 < \theta_2 \leq \pi$ with at most countable set of exceptional values of $\theta_1$ and $\theta_2$ that correspond to possible atoms of the measure $s_\sigma$, cf. (2.1.3). It also yields the Lindelöf-type symmetry condition, namely, the existence of the limit

$$\lim_{r \to \infty} \sum_{|z_n| \leq r} \frac{1}{z_n}, \quad (2.3.3)$$

where the sum is taken over zeroes of $F_\xi$. In the reverse direction, for functions of exponential type, conditions (2.3.2) and (2.3.3) yield (2.3.1).

We say that the stationary sequence $\xi$ is Gaussian, if $(\text{Re} \xi(n), \text{Im} \xi(n))$ are random normal vectors in $\mathbb{R}^2$ with non-zero covariance matrix (so that this definition includes also real-valued Gaussian stationary sequences).

**Theorem 4.** Suppose $\xi$ is a Gaussian stationary sequence with the spectrum $\sigma = \sigma(\xi)$. Then, almost surely, $\xi$ is an $L(\sigma^*)$-sequence, where $\sigma^*$ is the reflection of $\sigma$ in the real axis.

Comparing this result with Theorem 3 we note that, by condition (2.2.1), the spectral measure $\rho$ has a density $|f|^2$, where $f$ belongs to the Hardy space $H^2(\mathbb{T})$. This follows from a classical result that goes back to Kolmogorov, see [8, Chapter XVII, § 1]. Since no function in $H^2(\mathbb{T}) \setminus \{0\}$ vanishes on an arc, in the assumptions of Theorem 3 we have $\sigma(\xi) = \mathbb{T}$. 
2.4

Theorems 3 and 4 have a counterpart for uniformly almost-periodic sequences found by Levin [13, Chapter VI, § 7], which we will recall here.

Let \( \xi : \mathbb{Z} \to \mathbb{C} \) be a uniformly almost-periodic sequence. Then the limit

\[
\hat{\xi}(e^{i\lambda}) = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{|n| \leq N} \xi(n)e^{-i\lambda n}
\]

exists for every \( e^{i\lambda} \in \mathbb{T} \), and does not vanish for a non-empty at most countable set of \( e^{i\lambda} \). This set is called the spectrum of \( \xi \).

**Theorem 5** (B. Ya. Levin). Suppose \( \xi \) is a uniformly almost periodic sequence with the spectrum \( \sigma \). Then \( \xi \) is an \( L(\sigma^*) \)-sequence, with \( \sigma^* \) being the reflection of \( \sigma \) in the real axis.

The proof of this theorem given in [13] is based on deep results on the zero distribution of entire functions approximated by finite linear combinations of exponents. For the reader’s convenience, we include the proof of this theorem, which is based on the same ideas as Levin’s original proof, but can be read independently of the theory developed in [13, Chapter VI].

2.5

Here, we briefly explain how Theorems 1–5 are related to a wealth of results, which deal with the analytic continuation of the Taylor series

\[
f_\xi(s) \overset{\text{def}}{=} \sum_{n \geq 0} \xi(n)s^n
\]

through the boundary of the disk of convergence. A survey of these results obtained prior to 1955 can be found in [2]. First, observe that the function

\[
w^{-1}f_\xi(w^{-1}) = \sum_{n \geq 0} \frac{\xi(n)}{w^{n+1}}
\]

is the Laplace transform of \( F_\xi \). Then, by Pólya’s theorem (see [13, Theorem 33, Chapter I] or [2, Theorem 1.1.5]), the upper limit

\[
H_{F_\xi}(z) = \limsup_{t \to \infty} \frac{\log |F_\xi(tz)|}{t}
\]  \hspace{1cm} (2.5.1)
is Minkowski’s functional of the closed convex hull of the set of singularities of the function $w^{-1}f_ξ(w^{-1})$, reflected in the real axis. Hence, the results about analytic continuation of $f_ξ$ provide information about the upper limit in (2.5.1) but not about the existence of the limit in (2.3.1).

For instance, the property that the unit circle is a natural boundary for the Taylor series $f_ξ$ is equivalent to the property that the upper limit $H^F(z) \equiv 1$, but it cannot guarantee that $ξ$ is an $L$-sequence.

2.6

Here, we mention two curious results, which follow from Lemma 7.2.1 and which might be of an independent interest.

2.6.1

The first result sheds some light on the nature of very strong cancellations in Taylor series: Suppose $ξ: \mathbb{Z} \to \mathbb{C}$ is a stationary sequence with the spectral measure $ρ$. Then, almost surely,

$$\limsup_{r \to \infty} \frac{\log |F_ξ(re^{iθ})|}{r} \leq \max_{t \in \text{spt}(ρ)} \cos(θ + t).$$

In particular, $F_ξ(re^{iθ})$, almost surely, exponentially decays on some angle $θ_1 < \arg(z) < θ_2$ provided that the origin does not belong to the convex hull of the support of $ρ$.

Note that this result is useful when there are no restrictions on the support of the spectral measure.

2.6.2

The second result says that in some situations such restrictions do exist. We say that a set $A \subset \mathbb{C}$ is uniformly discrete if $\inf\{|z-w|: z, w \in A, z \neq w\} > 0$.

**Theorem 6.** Suppose that $ξ: \mathbb{Z} \to \mathbb{Z}$ is a stationary integer-valued sequence. Let $ρ$ be the spectral measure of $ξ$. Then either $\text{spt}(ρ) = \mathbb{T}$, or the sequence $ξ$ is periodic and $\text{spt}(ρ) \subset \{w : w^N = 1\}$ for some $N \geq 1$.

The same conclusion holds if $ξ: \mathbb{Z} \to A$ is a stationary ergodic sequence and the set $A$ is uniformly discrete.
3 Subharmonic preliminaries

3.1

First, we recall several basic facts from Azarin’s theory of limit sets of subharmonic functions \[1\]. In what follows, we deal only with entire functions \(F\) of exponential type. That is, \(|F(z)| \leq Ae^{\tau|z|}\), \(z \in \mathbb{C}\). Consider a family of subharmonic functions

\[ u_t(z) = \frac{1}{t} \log |F(tz)|, \quad t \geq 1. \]

This family is pre-compact in the topology of distributions \(\mathcal{D}' = (C_0^\infty(\mathbb{R}^2))'\). That is, each sequence \(t_j \to \infty\) has a subsequence \(t_{jk}\) so that \(u_{t_{jk}}\) converges in \(\mathcal{D}'\) to a subharmonic function \(v\). By \(\mathcal{L}(F)\) we denote the set of all limiting functions \(v\). The set \(\mathcal{L}(F)\) is called the limit set of \(\log |F|\). This set is invariant with respect to the multiplicative action of \(\mathbb{R}_+\), that is, if \(v \in \mathcal{L}(F)\), then for each \(t > 0\),

\[ \text{the function } \quad v_t(z) = t^{-1}v(tz) \quad \text{also belongs to } \mathcal{L}(F). \tag{3.1.1} \]

Since \(F\) is an entire function of exponential type, every function \(v \in \mathcal{L}(F)\) satisfies

\[ v(z) \leq \tau |z|, \quad z \in \mathbb{C}, \]

and \(v(0) = 0\).

The homogeneous indicator \(H^F\) of \(F\) is the upper envelope of functions in \(\mathcal{L}(F)\):

\[ H^F(z) \overset{\text{def}}{=} \sup_{v \in \mathcal{L}(F)} v(z), \quad z \in \mathbb{C}. \]

Then \(H^F(re^{i\theta}) = h^F(\theta) r\), where

\[ h^F(\theta) = \sup_{v \in \mathcal{L}(F)} v(e^{i\theta}), \quad -\pi \leq \theta \leq \pi \]

is the Phragmén-Lindelöf indicator of \(F\). An equivalent (more traditional) definition of \(h^F\) is

\[ h^F(\theta) = \limsup_{r \to \infty} \frac{\log |F(re^{i\theta})|}{r}. \]

The homogeneous indicator \(H^F\) is Minkowski’s functional of a convex compact set called the indicator diagram \(I^F\) of \(F\).
3 Subharmonic preliminaries

The ray \( \{\arg(z) = \theta\} \) is called a ray of completely regular growth of the function \( F \) if the set \( \mathcal{L}(F) \) restricted on that ray is singleton. Then

\[
v(r e^{i\theta}) = H^F(re^{i\theta}) = h^F(\theta) r, \quad v \in \mathcal{L}(F). \tag{3.1.2}
\]

By continuity of the Phragmén–Lindelöf indicators, the set of rays of completely regular growth is closed. Clearly, the function \( F \) has completely regular growth in \( \mathbb{C} \) if it has a completely regular growth on any ray. Hence, it suffices to verify condition (3.1.2) on a dense set of rays.

3.2

**Definition.** We say that a sequence \( R_j \uparrow \infty \) is thick if \( \lim_{j \to \infty} \frac{R_{j+1}}{R_j} = 1 \).

**Lemma 3.2.1.** Let \( F \) be an entire function of exponential type. Let \( h^F(\theta) \leq \kappa \) for some \( \theta \in [-\pi, \pi] \). Suppose that there exist a thick sequence \( R_j \uparrow \infty \) and a sequence \( \theta_j \to \theta \) so that

\[
\lim_{j \to \infty} \frac{1}{R_j} \log |F(R_j e^{i\theta_j})| \geq \kappa. \tag{3.2.1}
\]

Then \( h^F(\theta) = \kappa \) and \( F \) has completely regular growth on the ray \( \{\arg(z) = \theta\} \).

**Proof.** Suppose that there exists a function \( v \in \mathcal{L}(F) \) so that \( v(e^{i\theta}) < \kappa \). Then

\[
\frac{1}{t_k} \log |F(t_k z)| \to v(z) \quad \text{in} \ D'
\]

for some sequence \( t_k \uparrow \infty \). By the upper semi-continuity of subharmonic functions, both point-wise and with respect to the \( D' \)-convergence (see, for instance, [9, Theorem 4.1.9]) we have

\[
\limsup_{k \to \infty} \frac{1}{t_k} \log |F(t_k z_k)| \leq v(e^{i\theta}) < \kappa,
\]

provided that \( z_k \to e^{i\theta} \).

Now, we choose \( j_k \) so that \( R_{j_k} \leq t_k < R_{j_k+1} \), and put \( \tau_k = t_k^{-1} R_{j_k} \) and \( z_k = \tau_k e^{i\theta_k} \). Then \( \tau_k \to 1 \) (this is the place where we use thickness of the sequence \( R_j \)), and therefore, \( z_k \to e^{i\theta} \). Thus,

\[
\limsup_{k \to \infty} \frac{1}{R_{j_k}} \log |F(R_{j_k} e^{i\theta_k})| = \limsup_{k \to \infty} \frac{1}{\tau_k t_k} \log |F(t_k z_k)| < \kappa,
\]

arriving at a contradiction. \( \square \)
3.3

The following lemma is a variation on the theme of the maximum principle. It will be needed for the proof of Theorem 5.

**Lemma 3.3.1.** Let $F$ be an entire function of exponential type, and let $\sigma \subset \mathbb{T}$. Suppose that

(i) $h^F \leq h_\sigma$ everywhere on $[-\pi, \pi]$;
(ii) $h^F = h_\sigma = 1$ everywhere on $\sigma$;
(iii) $F$ has completely regular growth on the set of rays $\{z : \arg(z) \in \sigma\}$.

Then $h^F = h_\sigma$ everywhere, and $F$ has completely regular growth in $\mathbb{C}$.

**Proof.** If $\sigma$ is dense on $\mathbb{T}$, then the statement is obvious. So we will concentrate on the case when $\sigma$ is not dense in $\mathbb{T}$.

Let $I^F$ be the indicator diagram of $F$. By condition (i), $I^F \subseteq \text{ch}(\sigma)$. By the definition of the convex hull, $\text{ch}(\sigma)$ is the smallest convex compact that contains the set $\sigma$. By condition (ii), $\sigma \subseteq I^F$. Hence, $I^F = \text{ch}(\sigma)$, that is, $h^F = h_\sigma$ everywhere.

Let $S = \{\theta : h^F(\theta) < 1\}$. The set $S$ is a union of disjoint open intervals, let $J = (\alpha, \beta)$ be one of them. That is, $h_\sigma(\alpha) = h_\sigma(\beta) = 1$, while $h_\sigma < 1$ everywhere on $(\alpha, \beta)$. For $\theta \in \bar{J}$, we have

$$h^F(\theta) = \max\left(\cos(\theta - \alpha), \cos(\theta - \beta)\right)$$

$$= \begin{cases} 
\cos(\theta - \alpha), & \alpha \leq \theta \leq \frac{1}{2}(\alpha + \beta), \\
\cos(\theta - \beta), & \frac{1}{2}(\alpha + \beta) \leq \theta \leq \beta.
\end{cases}$$

Consider the angle $\alpha \leq \arg(z) \leq \frac{1}{2}(\alpha + \beta)$. In this angle the indicator $h^F$ is trigonometric, and $F$ has a completely regular growth on the boundary ray $\arg(z) = \alpha$. Moreover, $(h^F)'(\alpha + 0) = 0$ and $(h^F)'(\alpha - 0) = 0$. The first relation is obvious. To see that the second relation holds, we consider two cases: (i) $\alpha$ is not an isolated point of $[-\pi, \pi] \setminus S$, and (ii) $\alpha$ is an isolated point of $[-\pi, \pi] \setminus S$. In the first case, there is a sequence $\theta_\ell \uparrow \alpha$ such that $h^F(\theta_\ell) = h^F(\alpha) = 1$. On each interval $(\theta_\ell, \theta_{\ell+1})$, we have

$$0 \leq 1 - h^F(\theta) \leq O\left((\theta_{\ell+1} - \theta)^2\right) \leq O\left((\alpha - \theta)^2\right).$$

Hence, $(h^F)'(\alpha - 0) = 0$. In the second case, this relation is obvious, since $\alpha$ is a maximum point of a trigonometric function. Thus, the indicator $h^F$ is $C^1$-smooth at $\theta = \alpha$, and we are in the assumptions of Levin’s theorem on entire
functions with Phragmén–Lindelöf indicator [13, Theorem 7, Chapter III]. By this theorem, \( F \) has completely regular growth in the angle \( \{ \alpha \leq \theta \leq \frac{1}{2}(\alpha + \beta) \} \). Similarly, \( F \) has completely regular growth in the angle \( \{ \frac{1}{2}(\alpha + \beta) \leq \theta \leq \beta \} \). This proves Lemma 3.3.1.

It is worth mentioning that Levin’s theorem used in the proof of Lemma 3.3.1 can be deduced from Hopf’s boundary maximum principle for non-positive subharmonic functions vanishing on a part of the boundary.

## 4 Exponential sums

### 4.1

For a bounded sequence \( \xi : \mathbb{Z}_+ \to \mathbb{C} \), introduce the exponential sum

\[
W_R(\theta) = \sum_{|n| \leq N} \xi(n + R)e(n\theta)e^{-\frac{n^2}{R^2}},
\]

where \( R \) and \( N \) are large integer parameters such that \( N = R^{1/2} \log R + O(1) \) (in principle, any choice of \( N \) in the range \( R^{1/2} \log R \ll N \ll \varepsilon R^{1/2 + \varepsilon} \) would suffice for our purposes).

**Lemma 4.1.1.** Let

\[
F_\xi(z) = \sum_{n \geq 0} \xi(n) \frac{z^n}{n!}
\]

with a bounded sequence \( \xi : \mathbb{Z}_+ \to \mathbb{C} \). Then, for each \( \varepsilon > 0 \),

\[
|F_\xi(Re(\theta))| \geq \mu(R)\left|W_R(\theta) - C_\varepsilon R^\varepsilon\right|,
\]

where \( \mu(R) = \frac{e^R}{\sqrt{2\pi R}} \).

**Proof.** First, we estimate the tails

\[
\left( \sum_{0 \leq n < R - N} + \sum_{n > R + N} \right) |\xi(n)| \frac{R^n}{n!}.
\]

Put \( N_1 = R - N \), \( N_2 = R + N \). These sums are bounded by

\[
O(1) \sum_{0 \leq n \leq N_1 - 1} \frac{R^n}{n!} \quad \text{and} \quad O(1) \sum_{n \geq N_2 + 1} \frac{R^n}{n!}.
\]
correspondingly. Note that the sequence \( n \mapsto \frac{R^n}{n!} \) increases for \( 0 \leq n \leq N_1 - 1 \) and decreases for \( n \geq N_2 + 1 \). For \( 0 \leq n \leq N_1 - 1 \), we have

\[
\frac{R^n}{n!} : \frac{R^{n+1}}{(n+1)!} = \frac{n+1}{R} \leq 1 - \frac{N}{R},
\]

while, for \( n \geq N_2 + 1 \),

\[
\frac{R^{n+1}}{(n+1)!} : \frac{R^n}{n!} = \frac{R}{n+1} < \frac{R}{N_2} = \frac{1}{1 + \frac{N}{R}}.
\]

Whence,

\[
\sum_{0 \leq n \leq N_1 - 1} \frac{R^n}{n!} < \frac{R^{N_1}}{N_1!} \frac{1}{1 - (1 - \frac{N}{R})} = \frac{R}{N} \cdot \frac{R^{N_1}}{N_1!}
\]

and

\[
\sum_{n \geq N_2 + 1} \frac{R^n}{n!} < \frac{R^{N_2}}{N_2!} \frac{1}{1 - \frac{1}{1 + \frac{N}{R}}} < 2 \frac{R}{N} \cdot \frac{R^{N_2}}{N_2!}.
\]

It remains to observe that each of the quantities \( \frac{R^{N_1}}{N_1!} \) and \( \frac{R^{N_2}}{N_2!} \) does not exceed \( C e^{-c R^{-1} N^2} \mu(R) \), provided that \( \sqrt{R} \ll N \ll R \). Therefore,

\[
F_\xi(Re(\theta)) = \sum_{|n - R| \leq N} \xi(n)e(n\theta) \frac{R^n}{n!} + O(1)\mu(R),
\]

provided that \( \sqrt{R \log R} \ll N \ll R \).

Now, we turn to the central group of terms of the series. By Stirling’s formula, we have

\[
\sum_{|n - R| \leq N} \xi(n)e(n\theta) \frac{R^n}{n!} = \mu(R) \sum_{|n - R| \leq N} \xi(n)e(n\theta) \frac{R^n}{n!} \cdot \frac{\sqrt{2\pi R}}{e^R}
\]

\[
= \mu(R) \sum_{|n - R| \leq N} \xi(n)e(n\theta) (1 + O(R^{-1})) \left( \frac{R}{n} \right)^{n + \frac{1}{2}} e^{n-R}. \quad (4.1.1)
\]
Put $t = n - R$. Then $|t| \leq N$, and

$$
\left(\frac{R}{n}\right)^{n+\frac{1}{2}} e^{n-R} = \exp\left(\left(R + t + \frac{1}{2}\right) \log\left(1 - \frac{t}{R + t}\right) + t\right)
= \exp\left(-\frac{t^2}{2(R + t)} - \frac{t}{2(R + t)} + O\left(\frac{|t|^3}{R^2}\right)\right)
= \exp\left(-\frac{t^2}{2R} + O\left(\frac{|t|}{R}\right) + O\left(\frac{|t|^3}{R^2}\right)\right)
= \exp\left(-\frac{t^2}{2R} + O\left(\frac{N^3}{R^2}\right)\right)
= \left(1 + O(R^{-\frac{1}{2}+3\varepsilon})\right)e^{-\frac{t^2}{2R}}. 
$$

Hence, the sum on the RHS of (4.1.1) equals

$$
\mu(R) \sum_{|n-R| \leq N} \xi(n) e(n\theta) e^{-\frac{1}{2\pi}(n-R)^2} + \Omega \mu(R)
$$

with

$$
|\Omega| \leq O(1) N \cdot R^{-\frac{1}{2}+3\varepsilon} = O(R^{4\varepsilon}).
$$

This completes the proof of Lemma 4.1.1.

4.2

Combining Lemmas 3.2.1 and 4.1.1 we arrive at

Lemma 4.2.1. Let

$$
F_\xi(z) = \sum_{n \geq 0} \xi(n) \frac{z^n}{n!},
$$

where $\xi: \mathbb{Z}_+ \to \mathbb{C}$ is a bounded sequence. Suppose that for every $a \in [0,1]$ there exist a thick sequence $R_j \uparrow \infty$, a sequence $\theta_j \to a$, and $\delta > 0$ so that

$$
|W_{R_j}(\theta_j)| \geq R_j^\delta. \quad (4.2.1)
$$

Then $\xi$ is an $L$-sequence.
4.3

In many instances it is easier to produce a lower bound for an average of $|W_R|^2$ over short intervals of $\theta$. The following lemma is a straightforward corollary to the previous one.

From now on, we fix a non-negative even function $g \in C^2_0[-\frac{1}{2}, \frac{1}{2}]$ with \(\int g(\theta) \, d\theta = 1\).

**Lemma 4.3.1.** Let \(F_\xi(z) = \sum_{n \geq 0} \xi(n) \frac{z^n}{n!}\), where $\xi : \mathbb{Z}_+ \rightarrow \mathbb{C}$ is a bounded sequence. Suppose that, for every $a \in [0, 1]$ and for every $m \in \mathbb{N}$, there exist a thick sequence $R_j \uparrow \infty$ and $\delta > 0$ so that

\[
\int_{a-\frac{1}{2m}}^{a+\frac{1}{2m}} |W_{R_j}(\theta)|^2 g(m(\theta - a)) \, d\theta \geq R_j^\delta, \quad j \geq j_0(a, m). \quad (4.3.1)
\]

Then $\xi$ is an $L$-sequence.

Curiously enough, assumptions of Lemmas 4.2.1 and 4.3.1 impose restrictions only on relatively short blocks $\bigcup_j [R_j - R_j^{\frac{3}{2} + \varepsilon}, R_j + R_j^{\frac{3}{2} + \varepsilon}]$ of elements of the sequence $\xi$. The values attained by $\xi$ off these blocks do not matter.

5 Proof of Theorems 1 and 2 ($\beta > \frac{3}{2}$)

In this part, we put $\xi(n) = e(f(n))$ for some real-valued $f$. Then

\[W_R(\theta) = \sum_{|n| \leq N} e(f(n + R) + n\theta) e^{-\frac{n^2}{R}}, \quad \sqrt{R \log R} \ll N \ll R^{\frac{3}{2} + \varepsilon},\]

and we are looking for a lower bound for

\[X_R = \int_{a-\frac{1}{2m}}^{a+\frac{1}{2m}} |W_R(\theta)|^2 g(m(\theta - a)) \, d\theta, \quad a \in [0, 1], \ m \in \mathbb{N}.
\]

The upper bound $X_R \leq C\sqrt{R}$ as well as the matching lower bound in the case when $m = 1$ follow from Parseval’s theorem. There are some reasons to expect that if there are no unreasonable cancellations, then a similar lower bound holds in all scales, that is, $X_R \geq c(a, m)\sqrt{R}$ for every $m \in \mathbb{N}$ and every $a \in [0, 1]$. In the next sections, we justify these expectations.
5.1
The following lemma reduces the lower bound for $X_R$ to upper bounds for certain Weyl sums. Put

$$S_T(M_1, M_2) = \sum_{M_1 \leq n < M_2} e\left(f(n + R) - f(n + R - T)\right).$$

Lemma 5.1.1. There exist positive numerical constants $c$ and $C$ so that

$$X_R \geq \frac{c\sqrt{R}}{m} - Cm \sum_{T=1}^{2N} \frac{1}{T^2} \max_{0 \leq M_2 - M_1 \leq \sqrt{R}, \text{ and } |M_1|, |M_2| \leq N} \left| S_T(M_1, M_2) \right|.$$

Proof. We have

$$X_R = \frac{1}{m} \int_{-1/2}^{1/2} \left| \sum_{|n| \leq N} e\left(f(n + R) + na + \frac{n\theta}{m}\right) e^{-n^2/(2R)} \right|^2 g(\theta) \, d\theta$$

$$= \frac{1}{m} \sum_{|n|, |n'| \leq N} e\left(f(n + R) - f(n' + R) + (n - n')a\right) e^{-n'^2 + n^2}/(2R) \, \hat{g}\left(\frac{n' - n}{m}\right),$$

where $\hat{g}$ denotes the Fourier transform of $g$ extended by $0$ to $\mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]$. The diagonal sum ($n = n'$) contributes

$$\frac{1}{m} \sum_{|n| \leq N} e^{-n^2/R} \geq \frac{c\sqrt{R}}{m}.$$

We need to estimate from above the contribution of non-diagonal terms

$$\frac{2}{m} \left| \sum_{|n|, |n'| \leq N, n' < n} e(f(n + R) - f(n' + R) + (n - n')a) e^{-n'^2 + n^2}/(2R) \, \hat{g}\left(\frac{n' - n}{m}\right) \right|.$$

Letting $T = n - n'$ and using that

$$\hat{g}\left(\frac{n' - n}{m}\right) = O\left(\frac{m^2}{(n - n')^2}\right),$$

we see that the contribution of non-diagonal terms is

$$\leq Cm \sum_{T=1}^{2N} \frac{1}{T^2} \left| \sum_{-N + T \leq n \leq N} e\left(f(n + R) - f(n + R - T)\right) e^{-n^2 + (n - T)^2}/(2R) \right|.$$

(5.1.1)
The function \( n \mapsto e^{-(n^2 + (n-T)^2)/(2R)} \) increases for \(-\infty < n \leq \frac{1}{2} T \) and decreases for \( \frac{1}{2} T \leq n < +\infty \). We consider these two ranges separately. Then the expression in (5.1.1) is
\[
\leq Cm \sum_{T=1}^{2N} \frac{1}{T^2} \left[ \sum_{-N+T \leq n < \frac{1}{2} T} \ldots + \left| \sum_{\frac{1}{4} T \leq n \leq N} \ldots \right| \right].
\]

Next, we split the sums in \( n \) into blocks of length \( \sqrt{R} \) (and several blocks of smaller length that are treated similarly). We set \( J_k = \left[ \frac{T}{2} + (k-1)\sqrt{R}, \frac{T}{2} + k\sqrt{R} \right) \) and put
\[
Y_{k,T} = \sum_{n \in J_k \cap [-N+T,N]} e(f(n+R) - f(n+R-T))e^{-(n^2+(n-T)^2)/(2R)},
\]
with \( |k| \leq \frac{1}{\sqrt{R}} (N - \frac{T}{2}) + 1 \). Note that for \( n = \frac{T}{2} + \lambda \sqrt{R} \) with \( k-1 \leq \lambda < k \), we have
\[
-\frac{1}{2R} \left( n^2 + (n-T)^2 \right) = -\frac{1}{2R} \left( (\lambda \sqrt{R} + \frac{T}{2})^2 + (\lambda \sqrt{R} - \frac{T}{2})^2 \right)
= -\frac{1}{2R} \left( 2\lambda^2 R + \frac{1}{2} T^2 \right) < -\lambda^2 \leq -ck^2 + c_1.
\]

Then applying the Abel summation to the sum \( Y_{k,T} \), we see that
\[
|Y_{k,T}| \leq Ce^{-ck^2} \max_{0 < M_2 - M_1 \leq \sqrt{R}} \max_{|M_1|,|M_2| \leq N} |S_T(M_1, M_2)|,
\]
and the sum of non-diagonal terms we are estimating is
\[
\leq Cm \sum_{T=1}^{2N} \frac{1}{T^2} \sum_{|k| \leq \frac{1}{\sqrt{R}} (N - \frac{T}{2}) + 1} |Y_{k,T}|
\leq Cm \sum_{T=1}^{2N} \frac{1}{T^2} \max_{0 < M_2 - M_1 \leq \sqrt{R}} \max_{|M_1|,|M_2| \leq N} |S_T(M_1, M_2)|.
\]

This completes the proof of Lemma 5.1.1.

Now, Theorems 1 and 2 (for \( \beta > \frac{3}{2} \)) will readily follow from the classical Weyl and van der Corput estimates of exponential sums.
5.2 Proof of Theorem 1

First, we fix \( T_0 = T_0(m) \) so large that

\[
Cm \sum_{T>T_0} \frac{1}{T^2} < \frac{1}{2} m,
\]

where the positive numerical constants \( C \) and \( c \) are the same as in the assertion of Lemma 5.1. Then, using the trivial bound \( |S_T(M_1, M_2)| \leq \sqrt{R} \), we get

\[
X_R > \frac{1}{2} \frac{c}{m} \sqrt{R} - CmT_0 \max_{1 \leq T \leq T_0} \max_{0 < M_2 - M_1 \leq \sqrt{R}} |S_T(M_1, M_2)|.
\]

Put

\[
P(x) = Q(x) - Q(x - T) = \sum_{k=1}^{d-1} p_k x^k,
\]

and observe that at least one of the coefficients \( p_k \) is irrational (if \( \ell \) is the maximal index such that the coefficient \( q_\ell \) of \( Q \) is irrational, then \( p_{\ell-1} \) must be irrational too). Then, by Weyl’s theorem [18, Section 3],

\[
\max_{0 < M_2 - M_1 \leq M} \left| \sum_{M_1 \leq n < M_2} e(P(n)) \right| = o(M), \quad \text{as } M \to \infty.
\]

Hence, for each \( T \in \{1, ..., T_0\} \),

\[
\max_{0 < M_2 - M_1 \leq \sqrt{R}} |S_T(M_1, M_2)| = o(\sqrt{R}), \quad \text{as } R \to \infty,
\]

and, for \( R > R_0(m) \), we have \( X_R > c(m) \sqrt{R} \) with \( c(m) > 0 \). An application of Lemma 4.3.1 completes the proof of Theorem 1. \( \square \)

5.3 Proof of Theorem 2

Here, we prove Theorem 2 for \( \beta > \frac{3}{2} \). The case \( \beta = \frac{3}{2} \) will be treated in Section 6. Put \( f_{T,R}(x) = (R + x)^\beta - (R + x - T)^\beta \).
5.3.1 \( \frac{3}{2} < \beta < 2 \)

In this case, we apply the classical summation formula (see, for instance, [17, (2.1.2)])

\[
\sum_{M_1 \leq n < M_2} \varphi(n) = \int_{M_1}^{M_2} \varphi(x) \, dx + \int_{M_1}^{M_2} (x - \lfloor x \rfloor - \frac{1}{2}) \varphi'(x) \, dx
+ \frac{1}{2} \varphi(M_1) - \frac{1}{2} \varphi(M_2)
\]

with \( \varphi(x) = e(f_{T,R}(x)) \) and with integer \( M_1 \) and \( M_2, |M_1|, |M_2| \leq N \). We get

\[
S_T(M_1, M_2) = \int_{M_1}^{M_2} e(f_{T,R}(x)) \, dx
+ 2\pi i \int_{M_1}^{M_2} (x - \lfloor x \rfloor - \frac{1}{2}) f'_{T,R}(x) e(f_{T,R}(x)) \, dx + O(1).
\]

Since the function \( f'_{T,R} \) is monotonically decreasing, applying a classical estimate on integrals of oscillating functions (see, e.g., [17, Lemma 4.2]), and recalling the \( M_2 \leq N \), we get

\[
\left| \int_{M_1}^{M_2} e(f_{T,R}(x)) \, dx \right| \leq \frac{4}{f'_{T,R}(N)}. \quad (5.3.1)
\]

For \( R \geq R_0(\beta), |x| \leq N, \) and \( 1 \leq T \leq 2N \), we have

\[
f'_{T,R}(x) = \beta((R + x)^{\beta-1} - (R + x - T)^{\beta-1}) = (\beta(\beta-1) + o(1)) \frac{T}{R^{2-\beta}},
\]

uniformly in \( x \in [-N, N] \). Therefore, the LHS of (5.3.1) is \( \lesssim T^{-1} R^{2-\beta} \).

Next,

\[
\left| \int_{M_1}^{M_2} (x - \lfloor x \rfloor - \frac{1}{2}) f'_{T,R}(x) e(f_{T,R}(x)) \, dx \right|
\leq (M_2 - M_1) \max_{|x| \leq N} |f'_{T,R}(x)| \lesssim N \frac{T}{R^{2-\beta}},
\]

whence, by Lemma 5.1.1

\[
X_R \geq \frac{c\sqrt{R}}{m} - C(\beta)m(R^{2-\beta} + NR^{3-2} \log N) \geq c(m, \beta)\sqrt{R},
\]

provided that \( R \geq R_0(m, \beta) \). In view of Lemma 4.3.1 this proves Theorem 2 in the case \( \frac{3}{2} < \beta < 2 \). \( \Box \)
5.3.2 $\beta > 2$

Suppose that $k < \beta < k + 1$ with an integer $k \geq 2$. To estimate

$$\max_{M_2 - M_1 \leq N} \left| S_T(M_1, M_2) \right|,$$

we apply a van der Corput bound \cite[Theorem 5.13]{17}. Using that

$$f_{T,R}^{(k)}(x) \asymp_{\beta,k} \frac{T}{R^{k+1-\beta}}$$

uniformly in $|x| \leq N$, $1 \leq T \leq 2N$, we get

$$\left| S_T(M_1, M_2) \right| \lesssim_{\beta,k} (M_2 - M_1) \left( \frac{T}{R^{k+1-\beta}} \right)^{\frac{1}{2k-2}} + (M_2 - M_1)^{1-\frac{2}{K}} \left( \frac{R^{k+1-\beta}}{T} \right)^{\frac{1}{2k-2}}$$

with $K = 2k-1$. Since $M_2 - M_1 \leq \sqrt{R}$, the RHS is

$$\lesssim \sqrt{R} \left( T^{1/2} R^{-\delta} + R^{-\frac{k+1-\delta}{k-2}} \right).$$

with some $\delta > 0$. Since $K \geq 2$, we have

$$k + 1 - \beta < 1 \leq 2 - \frac{2}{K} = \frac{2K - 2}{K}.$$

Therefore,

$$\max_{M_2 - M_1 \leq N} \left| S_T(M_1, M_2) \right| \lesssim_{\beta,\delta} T^{1/2} R^{1/2-\delta},$$

and, by Lemma \[5.1.1\] $X_R \geq c(m) \sqrt{R}$, provided that $R \geq R_0(m, \beta)$. \qed

6 Proof of Theorem 2 ($\beta = \frac{3}{2}$)

In \cite{5}, Chen and Littlewood showed that the zeroes of the function $F_\xi$ with $\xi(n) = e(n^{\beta})$, $1 < \beta < \frac{3}{2}$, asymptotically are very close to a sequence of points that are regularly distributed on the spiral given in polar coordinates by $\theta = -\pi + C(\beta)r^{\beta-1}$. Their analysis yields that this sequence $\xi$ is an $L$-sequence. In fact, they gave a detailed proof for another sequence $\xi(n) = e(n(\log n)^{\beta})$ with $\beta > 1$, and mention that their arguments work with minor changes in the case we consider here. Apparently, it is an intriguing question which part of their analysis can be extended to the case $\frac{3}{2} \leq \beta \leq 2$ (or,
even to $\beta = \frac{3}{2}$). Nevertheless, as we will show in this section, a certain combination of their method with our techniques is strong enough to show that the sequence $\xi(n) = e(n^{3/2})$ is an $L$-sequence.

Everywhere in this part,

$$W_R(\theta) = \sum_{|n| \leq N} e\left((n + R)^{3/2} + (n + R)\theta\right) e^{-n^2/(2R)}$$

with $N = R^{1/2} \log R + O(1)$.

### 6.1

Here, we give an asymptotic estimate of $W_R$, which will yield Theorem 2 in the case $\beta = \frac{3}{2}$.

**Lemma 6.1.1.** For $R \to \infty$,

$$W_R(\theta) = \frac{2e(1/8 + MR)R^{1/4}}{\sqrt{3}} \sum_{|m| \leq \frac{1}{4} \log R} e\left(mR - \frac{4}{27}(M + m - \theta)^3\right) e^{-\frac{8}{9}(m-\theta)^2} + O((\log R)^3)$$

with $M = \frac{3}{2}R^{1/2}$, uniformly in $\theta$.

It is worth mentioning that, in the case $1 < \beta < \frac{3}{2}$ considered by Chen and Littlewood, at most two terms contribute to the corresponding sum on the RHS. This was crucial for finding the asymptotic locations of zeroes of $F$.

We split the proof of Lemma 6.1.1 into several parts.

### 6.1.1

Take $\chi \in C_0^\infty[0, +\infty)$ with $\chi \geq 0$,

$$\chi(t) = \begin{cases} 1, & 0 \leq x \leq N, \\ 0, & x \geq N + 1, \end{cases}$$

and set $\chi(z) = \chi(|z|)$, and

$$u(t) = \chi(t-R)e(t^{3/2} + t\theta)e^{-(t-R)^2/(2R)}, \quad t \in \mathbb{R}.$$
Then
\[ W_R(\theta) = \sum_{n \in \mathbb{Z}} u(n) = \sum_{m \in \mathbb{Z}} \hat{u}(m) \quad \text{(the Poisson summation)}, \]
where
\[ \hat{u}(m) = \int_{\mathbb{R}} u(t)e(-mt) \, dt \]
\[ = e(mR) \int_{\mathbb{R}} \chi(t)e((t + R)^{3/2} - (m - \theta)(t + R))e^{-t^2/(2R)} \, dt \]
\[ = e(mR) \int_{\mathbb{R}} \chi(t)e(\psi_m(t))e^{-t^2/(2R)} \, dt, \]
where \( \psi_m(t) = (t + R)^{3/2} - \mu(t + R) \) is “a phase function”, and \( \mu = m - \theta \) is “a distorted \( m \)”. Put
\[ I_m = \int_{\mathbb{R}} \chi(t)e(\psi_m(t))e^{-t^2/(2R)} \, dt. \]

Estimating the integrals \( I_m \), we set \( M = \frac{3}{2} R^{1/2} \) and consider separately three cases: \( |m - M| > \log R, \frac{1}{2} \log R < |m - M| \leq \log R, \text{ and } |m - M| \leq \frac{1}{2} \log R \). In what follows, we will be using the Taylor approximation in the disk \( |z| \leq 10N \):
\[ \psi_m(z) = -\frac{1}{2} R^{3/2} - \sigma R - \sigma z + \frac{3}{8} z^2 R^{-1/2} - \frac{1}{16} z^3 R^{-3/2} + O(R^{-1/2}(\log R)^4), \quad (6.1.1) \]
where \( \sigma = \mu - M \).

6.1.2

We start with the case \( |m - M| > \log R \). Then the derivative of the phase \( \psi_m \) is large on the support of \( \chi \), see (6.1.3) below. We show that for \( R \geq R_0 \),
\[ |I_m| \leq \frac{e^{-c(\log R)^2}}{(m - M)^2}. \quad (6.1.2) \]

Integrating twice by parts we obtain
\[ I_m = \frac{1}{(2\pi i)^2} \int_{\mathbb{R}} \left[ \frac{1}{\psi_m'(t)} \left( \chi e^{-t^2/(2R)} \right)'(t)e(\psi_m(t)) \right] \, dt \]
\[ = \frac{1}{(2\pi i)^2} \int_{\mathbb{R}} \frac{\chi(t)}{\psi_m'(t)} e(\psi_m(t))e^{-t^2/(2R)} \, dt, \]
Proof of Theorem 2 ($\beta = \frac{3}{2}$)

where

$$\lambda = \chi'' - \frac{3\chi'\psi''}{\psi'_m} - \frac{2\chi'}{R} - \frac{\chi}{R} + \frac{\chi t^2}{R^2} + \frac{3\chi\psi'' t}{\psi'_m R} - \frac{\chi\psi''}{\psi'_m} + 3\chi\left(\frac{\psi''}{\psi'_m}\right)^2.$$  

For $|z| \leq N + 1$ and $R > R_0$, we have

$$|\psi'_m(z)| = \left|\frac{3}{2}(R + z)^{1/2} - \frac{3}{2}R^{1/2} - \sigma\right| \geq \left|\sigma - \frac{3}{2}R^{1/2}\left[\sqrt{1 + \frac{N + 1}{R}} - 1\right]\right| \geq \left|\sigma - (\frac{3}{4} + o(1))\log R \geq \frac{1}{5}|\sigma|\right. \quad (6.1.3)$$

Since the functions $\psi'', \psi'''_m$ are bounded on the disk $\{z: |z| \leq N + 1\}$, we conclude that $\lambda$ is bounded on the same disk.

Next, we set

$$H(z) = 2\pi i\psi'_m(z) - \frac{z^2}{2R} = 2\pi i(R + z)^{3/2} - 2\pi i\left(\frac{3}{2}R^{1/2} + \sigma\right)(z + R) - \frac{z^2}{2R}.$$  

Then

$$I_m = -\frac{1}{4\pi^2} \int_R \frac{\lambda(t)}{\psi'_m(t)} e^{H(t)} \, dt.$$  

Using the Taylor expansion (6.1.1), we get

$$\left|\exp H(x + iy)\right| \leq C \exp\left(2\pi|\sigma|y - \frac{3\pi}{2}R^{-1/2} + \frac{3\pi}{8}x^2yR^{-3/2} - \frac{\pi}{8}y^3R^{-3/2} - \frac{x^2}{2R} + \frac{y^2}{2R}\right) = C \exp\left(2\pi|\sigma| - \frac{3\pi}{2}|xR^{-1/2} + o(1)|y - \frac{x^2}{2R} + \frac{y^2}{2R}\right), \quad |x + iy| \leq 3N. \quad (6.1.4)$$

Now,

$$4\pi^2|I_m| \leq \left|\int_{\frac{N}{2} \leq |x| \leq N+1}\right| + \left|\int_{|x| \leq N/2}\right|. \quad (6.1.4)$$

For $|x| \geq \frac{1}{2}N$, $\exp H(x) \leq C \exp[-x^2/(2R)] \leq \exp[-c(\log R)^2]$. Thus, the first integral does not exceed

$$CN|\sigma|^{-2}e^{-c(\log R)^2} \leq \frac{e^{-c_1(\log R)^2}}{(m - M)^2}. \quad (6.1.4)$$

In the second integral, instead of integrating over the interval $[-\frac{1}{2}N, \frac{1}{2}N]$, we integrate over the contour $\Gamma_\sigma$ as on Figure 1.
Proof of Theorem 2 ($\beta = \frac{3}{2}$)

Estimate (6.1.4) shows that

$$|e^{H(x+iy)}| \leq \begin{cases} Ce^{-cR^{1/2}\log R}, & z \in \Gamma_\sigma, |y| = R^{1/2}, \\ Ce^{-c(\log R)^2}, & z \in \Gamma_\sigma, |x| = N/2. \end{cases}$$

Therefore, the integral over the contour $\Gamma_\sigma$ is also bounded by

$$(m - M)^{-2} e^{-c(\log R)^2},$$

and estimate (6.1.2) follows.

6.1.3

Now, $\frac{1}{2}\log R < |m - M| \leq \log R$. This case is similar to the previous one but is somewhat shorter since there is no need to integrate by parts (instead of (6.1.2) we check a simpler estimate (6.1.5)). We again split the integral into two parts:

$$|I_m| = \left| \int_{\mathbb{R}} \chi(t)e(\psi_m(t))e^{-t^2/(2R)} \, dt \right| \leq \left| \int_{\mathbb{R}} \chi(t)e(\psi_m(t))e^{-t^2/(2R)} \, dt \right| + \left| \int_{|x| \leq N/2} e^{H(z)} \, dz \right|.$$

$$\leq Ce^{-c(\log R)^2} + \left| \int_{\Gamma_\sigma} e^{H(z)} \, dz \right|.$$
and, arguing as above, we obtain
\[ |I_m| \leq e^{-c(log R)^2}. \]  
(6.1.5)

6.1.4

At last, we deal with \( I_m \) such that \( |m - M| \leq \frac{1}{2} \log R \). This case requires a saddle point approximation. Set
\[ z_0 = \frac{4}{3} \sigma R^{1/2}, \quad A_0 = -\frac{8}{9} \sigma^2 - \frac{8\pi i}{27} \mu^3. \]

Then, using the Taylor approximations (6.1.1), we get
\[ H(z_0) = A_0 + O\left(R^{-1/2}(\log R)^4\right), \]
\[ H'(z_0) = O\left(R^{-1/2}(\log R)^2\right), \]
\[ H''(z) = \frac{3\pi i}{2} R^{-1/2} + O\left(R^{-1} \log R\right), \quad |z| < 5N. \]

Now,
\[ I_m = \int_{|x| \leq N} + \int_{N \leq |x| \leq N + 1} = \int_{\Lambda_{\sigma}} e^{H(z)} \, dz + O\left(e^{-c(log R)^2}\right), \]
where the Fresnel-type contour \( \Lambda_{\sigma} \) is as on Figure 2.

Let \( \Lambda_{\sigma}^0 \) be the vertical part of \( \Lambda_{\sigma} \), and \( \Lambda_{\sigma}^1 \) be the rest. Then by estimate (6.1.4) we have
\[ \left| \int_{\Lambda_{\sigma}^0} e^{H(z)} \, dz \right| \leq e^{-c(log R)^2}. \]

Hence,
\[ I_m = \int_{\Lambda_{\sigma}^1} e^{H(z)} \, dz + O\left(e^{-c(log R)^2}\right). \]

Furthermore,
\[ \int_{\Lambda_{\sigma}^1} e^{H(z)} \, dz = e^{i\pi/4} \int_{R^{1/2}(NR^{-1/2} - 4\sigma/3)}^{R^{1/2}(NR^{-1/2} - 4\sigma/3)} e^{H(t)e^{i\pi/4} + z_0} \, dt. \]
For $|t| \leq 2N$ we have

$$H(te^{i\pi/4} + z_0) = A_0 + O\left(\frac{(\log R)^4}{R^{1/2}}\right) + O\left(\frac{t(\log R)^2}{R^{1/2}}\right) - \frac{3\pi}{4} t^2 R^{-1/2} + O\left(\frac{t^2 \log R}{R}\right),$$
and hence,
\[
\int_{\Lambda_\sigma^1} e^{H(z)} \, dz \underbrace{\int_{R^{1/4}(- \log R - 4\sigma/3 + o(1))} \cdots}_{\text{as in (14)}} = e^{i\pi/4 + A_0} R^{1/4} \int_{R^{1/4}(- \log R - 4\sigma/3 + o(1))} \exp \left[ - \frac{3\pi}{4} t^2 \right] \left\{ \sum_{|m| \leq \frac{1}{2} \log R} + \sum_{\frac{1}{2} \log R < m < \log R} + \sum_{|m| > \log R} \right\} e^{(M + m)R} I_{m+M} dt 
\]
\[
eq e^{i\pi/4 + A_0} R^{1/4} \int_{R^{1/4}(- \log R - 4\sigma/3 + o(1))} \exp \left[ - \frac{3\pi}{4} t^2 \right] dt + O \left( \frac{(\log R)^3}{R^{1/4}} \right) 
\]
\[
eq \frac{2}{\sqrt{3}} R^{1/4} e \left( \frac{1}{8} - \frac{4}{27} \mu^3 \right) e^{-\frac{8}{9} \sigma^2} + O \left( (\log R)^3 \right).
\]

Finally, for \(|m - M| \leq \frac{1}{2} \log R\), we get
\[
I_m = \frac{2}{\sqrt{3}} R^{1/4} e \left( \frac{1}{8} - \frac{4}{27} \mu^3 \right) e^{-\frac{8}{9} \sigma^2} + O \left( (\log R)^3 \right) + O \left( e^{-(\log R)^2} \right) 
\]
\[
= \frac{2}{\sqrt{3}} R^{1/4} e \left( \frac{1}{8} - \frac{4}{27} (m - \theta)^3 \right) e^{-\frac{8}{9} (m-M-\theta)^2} + O \left( (\log R)^3 \right).
\]

6.1.5

Thus,
\[
W_R(\theta) = \sum_{m \in \Z} \hat{u}(m + M) 
\]
\[
= \left[ \sum_{|m| \leq \frac{1}{2} \log R} + \sum_{\frac{1}{2} \log R < m < \log R} + \sum_{|m| > \log R} \right] e((M + m)R) I_{m+M} 
\]
\[
= \frac{2e(1/8 + MR)R^{1/4}}{\sqrt{3}} \sum_{|m| \leq \frac{1}{2} \log R} e \left( mR - \frac{4}{27} (M + m - \theta)^3 \right) e^{-\frac{8}{9} (m-M-\theta)^2} + O \left( (\log R)^3 \right),
\]

proving Lemma 6.1.1. \(\square\)
6.2

At last, we are able to prove Theorem 2 for $\beta = \frac{3}{2}$. Consider the shifts $W_R(\theta + t)$ with $0 \leq t \leq \frac{3}{8} M^{-1}$. We have

$$W_R(\theta + t) = \frac{2e(1/8 + MR)R^{1/4}}{\sqrt{3}} \sum_{|m| \leq \frac{1}{2} \log R} e\left(mR - \frac{4}{27}(M + m - \theta - t)^3\right)e^{-\frac{5}{9}(m-\theta-t)^2} + O((\log R)^3).$$

Furthermore, since $|t| = O(M^{-1})$ with $M = \frac{3}{2}R^{1/2}$, we have

$$e\left(-\frac{4}{27}(M + m - \theta - t)^3\right)e^{-\frac{5}{9}(m-\theta-t)^2} = e\left(-\frac{4}{27}(M + m - \theta)^3 + \frac{4}{9}(M^2 - 2M \theta)t + \frac{8}{9}Mmt\right)e^{-\frac{5}{9}(m-\theta)^2} + O((\log R)^3).$$

Therefore,

$$W_R(\theta + t) = KR^{1/4} \sum_{|m| \leq \frac{1}{2} \log R} e\left(\frac{8}{9}Mtm + mR - \frac{4}{27}(M + m - \theta)^3\right)e^{-\frac{5}{9}(m-\theta)^2} + O((\log R)^3)$$

with

$$K = K(M, \theta, t) = \frac{2e\left(\frac{1}{8} + MR + \frac{4}{9}(M^2 - 2M \theta)t\right)}{\sqrt{3}}.$$

Now, notice that the sum on the RHS is a Fourier series in the variable $\frac{8}{9}Mt$. Hence, by the Parceval theorem, there exists $t \in [0, \frac{3}{8} M^{-1}]$ so that

$$|W_R(\theta + t)| \geq \frac{2R^{1/4}}{\sqrt{3}} \left( \sum_{|m| \leq \frac{1}{2} \log R} e^{-\frac{10}{9}(m-\theta)^2} \right)^{1/2} - O((\log R)^3) \geq CR^{1/4}$$

with a positive numerical constant $C$. Applying Lemma 4.2.1 we finish off the proof. \qed
7 Wide-sense stationary sequences

Here, we prove several simple lemmas pertaining to the case when \( \xi : \mathbb{Z}_+ \to \mathbb{C} \) is a wide-sense stationary sequence, that is, \( \mathbb{E}|\xi(n)|^2 < \infty \) for every \( n \), and \( \mathbb{E}\xi(n) \) and \( \mathbb{E}\{\xi(n)\xi(n+m)\} \) do not depend on \( n \). We also always assume that \( \xi \) is not the zero sequence. By \( \rho \) we denote the spectral measure of such a sequence \( \xi \). That is, \( \rho \) is a finite non-negative measure on the unit circle \( \mathbb{T} \) such that

\[
\mathbb{E}[\xi(n_1)\xi(n_2)] = \hat{\rho}(n_2 - n_1),
\]

and by \( \sigma(\xi) \) we denote the spectrum of \( \xi \), that is, the closed support of the spectral measure \( \rho \). In what follows, by \( \sigma^* \) we always denote the reflection of the spectrum \( \sigma \) in the real axis.

Observe that if \( \xi \) is a wide-sense stationary sequence then, almost surely, \( F_\xi \) is an entire function of exponential type at most one. Indeed, for every \( \varepsilon > 0 \),

\[
\mathbb{P}\{|\xi(n)| > (1 + \varepsilon)^n\} \leq (1 + \varepsilon)^{-2n} \mathbb{E}|\xi(n)|^2,
\]

whence, by the Borel–Cantelli lemma,

\[
\limsup_{n \to \infty} |\xi(n)|^{1/n} \leq 1, \quad \text{almost surely},
\]

which is equivalent to the asymptotic inequality \( |F_\xi(z)| \leq C(\varepsilon)e^{(1+\varepsilon)|z|} \) valid for every \( z \in \mathbb{C} \) and every \( \varepsilon > 0 \).

7.1

First, we compute the variance of \( F_\xi \) in terms of the spectral measure \( \rho \).

**Lemma 7.1.1.** Suppose \( \xi \) is a wide-sense stationary sequence. Then

\[
\mathbb{E}|F_\xi(re^{i\theta})|^2 = \int_{-\pi}^{\pi} e^{2r\cos(\theta+t)} \, d\rho(t), \quad (7.1.1)
\]

and

\[
\log\mathbb{E}|F_\xi(re^{i\theta})|^2 = 2rh_{\sigma^*}(\theta) + o(r), \quad r \to \infty. \quad (7.1.2)
\]
7 Wide-sense stationary sequences

Proof. We have

\[ \mathbb{E}|F_\xi(re^{i\theta})|^2 = \sum_{n_1,n_2 \geq 0} \mathbb{E}[\xi(n_1)\overline{\xi(n_2)}] e^{i(n_1-n_2)\theta} \frac{r^{n_1+n_2}}{n_1!n_2!} \]

\[ = \sum_{n_1,n_2 \geq 0} \left[ \int_{-\pi}^{\pi} e^{-i(n_2-n_1)t} \, d\rho(t) \right] e^{i(n_1-n_2)\theta} \frac{r^{n_1+n_2}}{n_1!n_2!} \]

\[ = \int_{-\pi}^{\pi} \left[ \sum_{n_1,n_2 \geq 0} e^{in_1(\theta+t)} \frac{r^{n_1}}{n_1!} \cdot e^{-in_2(\theta+t)} \frac{r^{n_2}}{n_2!} \right] \, d\rho(t) \]

\[ = \int_{-\pi}^{\pi} e^{r[\cos(\theta+t)+e^{-i(\theta+t)}]} \, d\rho(t) \]

\[ = \int_{-\pi}^{\pi} e^{2r\cos(\theta+t)} \, d\rho(t) , \]

proving (7.1.1). Now, recalling the definition of the supporting function

\[ h_{\sigma^*}(\theta) = \max_{t \in \text{spt}(\rho)} \cos(\theta + t) , \]

we readily get asymptotics (7.1.2).

7.2

As a straightforward consequence of the previous lemma, we get

Lemma 7.2.1. Suppose \( \xi \) is a wide-sense stationary sequence. Then, almost surely,

\[ h_{\sigma^*}(\theta) \leq h_{\sigma^*}(\theta) , \quad \theta \in [-\pi, \pi] . \]

In other words, the indicator diagram \( I_{\sigma^*}(\rho) \) of \( F_\xi \) is contained in the closed convex hull of the spectrum \( \sigma(\xi) \) reflected in the real axis.

Proof. Using (7.1.2) and Chebyshev’s inequality, we see that, for every \( \varepsilon > 0 \),

\[ \mathbb{P}\{\log |F_\xi(re^{i\theta})| > (h_{\sigma^*}(\theta) + \varepsilon)r\} = \mathbb{P}\{|F_\xi(re^{i\theta})|^2 > e^{2(h_{\sigma^*}(\theta)+\varepsilon)r}\} \]

\[ \leq \mathbb{E}\{|F_\xi(re^{i\theta})|^2\} e^{-2(h_{\sigma^*}(\theta)+\varepsilon)r} = e^{-2\varepsilon r + o(r)} , \quad r \to \infty . \]

Whence, by the Borel–Cantelli lemma, for every \( \kappa > 0 \) and every \( \theta \in [-\pi, \pi] \),

\[ \limsup_{n \to \infty} \frac{\log |F_\xi(\kappa ne^{i\theta})|}{\kappa n} \leq h_{\sigma^*}(\theta) , \quad \text{almost surely} . \]
Since the exponential type of the entire function $F_\xi$ does not exceed 1, for any $\kappa < \pi$, we have
\[
\limsup_{r \to \infty} \frac{\log |F_\xi(re^{i\theta})|}{r} = \limsup_{n \to \infty} \frac{\log |F_\xi(\kappa ne^{i\theta})|}{\kappa n}.
\]
This is a special instance of a classical result that goes back to Pólya and to Vl. Bernstein. For a simple proof of this result see, for instance, [2, Theorem 1.3.5]. Therefore, given $\theta \in [-\pi, \pi]$, almost surely, we have $h^{F_\xi}(\theta) \leq h_{\sigma^*}(\theta)$. Since both functions in this inequality are continuous on $[-\pi, \pi]$, we immediately conclude that, almost surely, the inequality holds for all $\theta \in [-\pi, \pi]$.

We will be using Lemmas 7.1.1 and 7.2.1 in the Gaussian case (Theorem 4).

7.3

The next lemma is needed for the mixing case (Theorem 3). As above, we use the notation
\[
W_R(\theta) = \sum_{|n| \leq N} \xi(n + R)e(n\theta) e^{-\frac{n^2}{2R}} , \quad N = R^{1/2} \log R + O(1),
\]
and
\[
X_R = \int_{a-\frac{1}{2m}}^{a+\frac{1}{2m}} |W_R(\theta)|^2 g(m(\theta - a)) \, d\theta
\]
\[
= \frac{1}{m} \sum_{|n_1|,|n_2| \leq N} \xi(n_1 + R)\xi(n_2 + R)e((n_1 - n_2)a)e^{-\frac{(n_1^2 + n_2^2)/(2R)}{m}} \widehat{g}\left(\frac{n_2 - n_1}{m}\right).
\]

Lemma 7.3.1. Suppose $\xi$ is a wide-sense stationary sequence whose spectral measure $\rho$ has no gaps in its support. Then for every $\alpha \in [0,1]$ and every $m \in \mathbb{N}$, there exists a positive limit
\[
\lim_{R \to \infty} R^{-1/2} \mathbb{E}X_R = c(\alpha, m) > 0 . \quad (7.3.1)
\]
Proof. We have
\[ \mathbb{E} X_R = \frac{1}{m} \sum_{|n_1||n_2| \leq N} \hat{\rho}(n_2 - n_1) \hat{g} \left( \frac{n_2 - n_1}{m} \right) e((n_1 - n_2)a) e^{- (n_1^2 + n_2^2) / 2R}. \]

Put \( k = n_2 - n_1, \ell = n_2 + n_1 \). Then
\[ |k| \leq 2N, \quad |\ell| \leq 2N - k, \quad \ell \equiv k \mod 2, \]
and \( n_1^2 + n_2^2 = \frac{1}{2}(k^2 + \ell^2) \). Hence,
\[ \mathbb{E} X_R = \frac{1}{m} \sum_{|k| \leq 2N} \hat{\rho}(k) \hat{g} \left( \frac{k}{m} \right) e(-ka) e^{-k^2/(4R)} \sum_{|\ell| \leq 2N-k, \ell \equiv k \mod 2} e^{-\ell^2/(4R)}. \]

Because of the cut-off \( e^{-k^2/(4R)} \), we discard the sum over \( N \leq |k| \leq 2N \) (recall that \( N = R^{1/2} \log R + O(1) \)) and consider only the range \( |k| \leq N \). Then the inner “\( \ell \)-sum” equals \( \sqrt{\pi R} + O(R^{-1/2}) \), and we get
\[ \mathbb{E} X_R = \frac{\sqrt{\pi R}}{m} \sum_{|k| \leq N} \hat{\rho}(k) \hat{g} \left( \frac{k}{m} \right) e(-ka) e^{-k^2/(4R)} + O(\log R). \]

By the dominated convergence,
\[ \lim_{R \to \infty} R^{-1/2} \mathbb{E} X_R = \frac{\sqrt{\pi}}{m} \sum_{k \in \mathbb{Z}} \hat{\rho}(k) \hat{g} \left( \frac{k}{m} \right) e(-ka). \]

The sum on the RHS is the density of the convolution \( \rho * g_m \) at the point \(-a\), where
\[ g_m(\theta) = \begin{cases} mg(m\theta), & |\theta| \leq 1/(2m) \\ 0, & \text{otherwise}. \end{cases} \]

Since the support of \( \rho \) is the whole circle \( \mathbb{T} \), and the function \( g \) is non-negative, this value is positive. This proves the lemma.

8 Proof of Theorem 6

8.1

First, we assume that \( \xi : \mathbb{Z} \to \mathbb{Z} \) is an integer-valued stationary sequence with the spectral measure \( \rho \). Let \( K_\xi \) be the convex hull of \( \text{spt}(\rho) \), and let \( K^*_\xi \)
be its reflection in the real axis. Suppose that spt(\rho) \neq \mathbb{T}, that is \( K^*_\xi \neq \overline{D} \). By Pólya’s theorem (see [13, Theorem 33, Chapter I] or [2, Theorem 1.1.5]), the series
\[
f_\xi(w) = \sum_{n \geq 0} \frac{\xi(n)}{w^{n+1}}
\]
is analytic on \( \hat{\mathbb{C}} \setminus K^*_\xi \). Since \( \xi \) attains only integer values, another theorem of Pólya [2, Theorem 6.2.1] yields that for every fixed \( \xi \), the function \( f_\xi \) is rational with poles at roots of 1, \( f_\xi = P/Q \), with mutually prime \( P, Q \in \mathbb{Z}[w] \) and monic \( Q \).

Next, we use simple algebra. Noting that \( P \) is a product of irreducible polynomials, and recalling that if a polynomial is irreducible in \( \mathbb{Z}[w] \) then it is also irreducible in \( \mathbb{Q}[w] \) (“Gauss lemma”), and that two different irreducible polynomials in \( \mathbb{Q}[w] \) are mutually prime, we conclude that \( P \) has no common zeroes with \( Q \).

Since any polynomial in \( \mathbb{Z}[w] \) is a product of irreducible ones, and since cyclotomic polynomials
\[
\Phi_n(w) = \prod_{\gcd(k,n)=1} (w - e(k/n))
\]
belong to \( \mathbb{Z}[w] \) and are irreducible therein, we see that
\[
Q = \prod_{1 \leq k \leq u} \Phi_n(k).
\]
Since \( f_\xi \) is analytic on a fixed arc of the unit circle, we obtain that \( n(k) \leq M \) for some \( M \) independent of \( \xi \). Thus, the set of poles of \( f_\xi \) is contained in \( \{ w : w^N = 1 \} \) for some \( N \geq 1 \) independent of \( \xi \).

Furthermore, since \( \mathbb{E}|\xi(n)|^2 \) is finite (and does not depend on \( n \)), applying Chebyshev’s inequality and the Borel-Cantelli Lemma, we see that, for any \( \lambda > \frac{1}{2} \), almost surely, \( |\xi(n)| = o(n^\lambda) \), whence,
\[
\max_{|w|=r} |f_\xi(w)| = o((r - 1)^{-2}), \quad r \downarrow 1.
\]
Therefore, all poles of \( f_\xi \) are simple. Thus, \( f_\xi \) can be written in the form
\[
f_\xi(w) = (w^N - 1)^{-1}S(w),
\]
where \( S \) is a polynomial (depending on \( \xi \)) and \( N \in \mathbb{N} \) does not depend on \( \xi \). Hence, the coefficients \( \xi(n) \) of \( f_\xi \) are eventually periodic with period \( N \).
Since the sequence $\xi$ is stationary, we conclude that it is periodic with period $N$. \hfill \Box

Note that we used stationarity of $\xi$ only on the last step of the proof. The rest is valid for wide-stationary integer-valued sequences.

8.2

To prove the second part, we use a result of Hausdorff [2, Theorem 4.2.4]. It says that if the set $A$ is uniformly discrete then there exist at most countably many sequences $\xi$ such that the series $f_\xi(w)$ can be analytically continued through an arc in $T$.

Let $\mu$ be a translation invariant probability measure in the space of sequences $A^\mathbb{Z}$ corresponding to the stationary sequence $\xi$. Suppose that there exists a lacuna in the support of the spectral measure $\rho$. Then, as above, by Lemma 7.2.1 combined with Pólya’s theorem, almost surely, the function $f_\xi$ has an analytic continuation through an arc in $T$; and by the theorem of Hausdorff, the measure $\mu$ has at most countable support. Since $\mu$ is translation invariant, we conclude that, almost surely, the sequence $\xi(n)$ is periodic. Since $\mu$ is ergodic, the sequence $\xi$ is periodic. \hfill \Box

9 Proof of Theorem 3

9.1

The proof of Theorem 3 also needs an estimate of the fourth order correlations:

**Lemma 9.1.1.** Let $\xi$ be a bounded stationary sequence of random variables, and let the maximal correlation coefficient of $\xi$ satisfy

$$r(m) = O\left((\log m)^{-\kappa}\right), \quad m \to \infty,$$

with some $\kappa > 1$. Then, for every $a \in [0, 1]$ and every $m \in \mathbb{N}$

$$\mathbb{E}(X_R - \mathbb{E}X_R)^2 = O\left(\frac{R}{(\log R)^{\kappa_1}}\right), \quad R \to \infty,$$

with some $1 < \kappa_1 < \kappa$. 
Proof. We have
\[ \mathbb{E}(X_R - \mathbb{E}X_R)^2 \]
\[ = \mathbb{E} \left[ \sum_{|n_1|,|n_2| \leq N} \left( \xi(n_1 + R)\xi(n_2 + R) - \mathbb{E}\{\xi(n_1 + R)\xi(n_2 + R)\} \right) \tilde{g}(\frac{n_2 - n_1}{m}) \times \right. \]
\[ \times e((n_1 - n_2)a) e^{(n_1^2 + n_2^2)/(2R)} \left. \right]^2 \]
\[ = \sum_{|n_1|,\ldots,|n_4| \leq N} C(n_1, n_2, n_3, n_4) \tilde{g}(\frac{n_2 - n_1}{m}) \tilde{g}(\frac{n_4 - n_3}{m}) \times \]
\[ \times e((n_1 - n_2 + n_3 - n_4)a) e^{(n_1^2 + n_2^2 + n_3^2 + n_4^2)/(2R)} , \]
where
\[ C(n_1, n_2, n_3, n_4) = \mathbb{E}\{\eta(n_1, n_2) \cdot \eta(n_3, n_4)\} \]
with
\[ \eta(n_i, n_j) = \xi(n_i + R)\xi(n_j + R) - \mathbb{E}\{\xi(n_i + R)\xi(n_j + R)\} . \]
Letting
\[ t = \min\{|n_i - n_j|: i \in \{1, 2\}, j \in \{3, 4\}\} \, , \]
we estimate \( C \) by the maximal correlation coefficient \( r(t) \):
\[ |C(n_1, n_2, n_3, n_4)| \leq r(t) \sqrt{\mathbb{E}|\eta(n_1, n_2)|^2 \cdot \mathbb{E}|\eta(n_3, n_4)|^2} \leq 4r(t) \|\xi\|_4^4 . \]
Therefore,
\[ \mathbb{E}(X_R - \mathbb{E}X_R)^2 \leq O(1)\|\xi\|_4^4 \times \]
\[ \times \sum_{|n_1|,\ldots,|n_4| \leq N} r(t) \frac{m^2}{1 + (n_1 - n_2)^2} \frac{m^2}{1 + (n_3 - n_4)^2} e^{-(n_1^2 + n_2^2 + n_3^2 + n_4^2)/(2R)} . \]
To estimate the sum on the RHS, we put
\[ k_1 = n_1 - n_2, \, \ell_1 = n_1 + n_2, \, k_2 = n_3 - n_4, \, \ell_2 = n_3 + n_4 . \]
Then \( t = \frac{1}{2} \min |\pm k_1 \pm k_2 + (\ell_1 - \ell_2)| \), where the minimum is taken over all combinations of signs. Hence, \( t \geq \frac{1}{2}(|\ell_1 - \ell_2| - (|k_1| + |k_2|)) \), and we need to estimate the sum
\[ \sum_{|k_1|,|k_2|,|\ell_1|,|\ell_2| \leq 2N} r(\frac{1}{2}(|\ell_1 - \ell_2| - (|k_1| + |k_2|))) \frac{e^{-(k_1^2 + k_2^2 + \ell_1^2 + \ell_2^2)/(4R)}}{(1 + k_1^2)(1 + k_2^2)} . \]
Here and later on, $r(t) = r(\max([t], 0))$, where $[t]$ is the maximal integer not exceeding $t$.

We split this sum into two parts. The first one is taken over $|\ell_1 - \ell_2| \leq 2(|k_1| + |k_2|)$, while the second one is taken over $|\ell_1 - \ell_2| > 2(|k_1| + |k_2|)$.

The first sum does not exceed

$$
\sum_{k_1, k_2 \geq 0} \frac{e^{-(k_1^2 + k_2^2)/(4R)}}{(1 + k_1^2)(1 + k_2^2)} \cdot O(1 + k_1 + k_2) \cdot O(N)
$$

$$
= O(R^{1/2} \log R) \sum_{k_1, k_2 \geq 0} \frac{1 + k_1 + k_2}{(1 + k_1^2)(1 + k_2^2)} e^{-(k_1^2 + k_2^2)/(4R)}
$$

$$
= O(R^{1/2} \log R) \left[ \sum_{k \geq 1} \frac{e^{-k^2/(4R)}}{k} + O(1) \right] = O(R^{1/2}(\log R)^2),
$$

while the second sum is bounded by

$$
O(1) \sum_{|\ell_1|, |\ell_2| \leq 2N} r\left(\frac{1}{4} \cdot |\ell_1 - \ell_2|\right) e^{-(\ell_1^2 + \ell_2^2)/(4R)} \leq O(\sqrt{R}) \sum_{\ell > 1} r\left(\frac{1}{4} \ell\right) e^{-\ell^2/(8R)}.
$$

Recall that $r(t) = O\left(\frac{1}{\log^c R}\right)$ and let $\kappa = 1 + 2\varepsilon$. Then

$$
\sum_{\ell > 1} r\left(\frac{1}{4} \ell\right) e^{-\ell^2/(8R)} \leq \sum_{1 \leq \ell \leq \sqrt{R} \log^{\kappa} R} r\left(\frac{1}{4} \ell\right) + \sum_{\ell > \sqrt{R} \log^{\kappa} R} e^{-\ell^2/(8R)}
$$

$$
\leq O(1) \left[ \frac{\sqrt{R} \log^{\kappa} R}{\log^{1+2\varepsilon} R} + \sqrt{R} e^{-c\log^{2\varepsilon} R} \right] \leq O(1) \frac{\sqrt{R}}{\log^{1+c} R}.
$$

This completes the proof of Lemma 9.1.1.

9.2

Now, the proof of Theorem 9 is straightforward. Since the maximal correlation coefficient $r(m)$ decays to zero as $m \to \infty$, the bounded stationary sequence $\xi$ is linearly regular, that is, $\bigcap_{m=1}^{\infty} L^2_{[-\infty, m]} = \{0\}$, where $L^2_{[-\infty, m]}$ is the Hilbert space, which consists of elements of the $\sigma$-algebra generated by $\{\xi(n): -\infty < n \leq m\}$ that have a finite second moment. Then the spectral measure $\rho$ has a density $|f|^2$, where $f$ belongs to the Hardy space $H^2(\mathbb{T})$, see [8, Chapter XVII, §1], and therefore, $\text{spt}(\rho) = \mathbb{T}$. Hence, we are in
10 Proof of Theorem 4

Given \( z = re^{i\theta} \), \( F_\xi(z) \) is a Gaussian random variable. As before, \( \sigma^* \) is the reflection of the spectrum \( \sigma(\xi) \) in the real axis. By Lemma 7.1.1

\[
E|F_\xi(re^{i\theta})|^2 = e^{2h_{\sigma^*}(\theta)r + o(r)}, \quad r \to \infty.
\]

Then, for every \( \epsilon > 0 \), every \( r > r_\epsilon \), and every \( \theta \in [-\pi, \pi] \), we have

\[
\mathbb{P}\{\log |F_\xi(re^{i\theta})| < (h_{\sigma^*}(\theta) - \epsilon)r\} = \mathbb{P}\{F_\xi(re^{i\theta}) < e^{-\epsilon r + o(r)} \sqrt{E|F_\xi(re^{i\theta})|^2} \} < e^{-\frac{1}{2}\epsilon r}
\]

(the last inequality is the place where we are using the Gaussianity of \( \xi \)). Applying this with \( R = j \) and using the Borel–Cantelli lemma, we see that, given \( \theta \in [-\pi, \pi] \), we have

\[
\liminf_{j \to \infty} \frac{1}{j} \log |F_\xi(je^{i\theta})| \geq h_{\sigma^*}(\theta), \quad \text{almost surely}.
\]

By Lemma 7.2.1, \( h_\xi \leq h_{\sigma^*} \) everywhere on \([-\pi, \pi]\). Therefore, applying Lemma 3.2.1, we conclude that, almost surely, \( F_\xi \) has completely regular growth on the ray \( \{\arg(z) = \theta\} \) with the indicator \( h_{\sigma^*}(\theta) \). To complete the proof, we apply this argument to a dense countable set of \( \theta \)'s. \( \square \)
11 Proof of Theorem 5

Now, $\xi$ is a uniformly almost periodic sequence. By $\hat{\xi}$ we denote the Fourier transform of $\xi$, $\xi : \mathbb{T} \rightarrow \mathbb{C}$. The spectrum of $\xi$ is $\sigma(\xi) = \{e^{i\lambda} \in \mathbb{T} : \hat{\xi}(e^{i\lambda}) \neq 0\}$, this is an at most countable subset of $\mathbb{T}$.

We will be using Bochner’s theorem that states that there exists an enumeration of the spectrum $\sigma(\xi) = \{e^{i\lambda_1}, e^{i\lambda_2}, \ldots\}$ and a sequence of multipliers $\beta_k^{(m)}$ ($k \in \{1, \ldots, m\}$) satisfying $0 \leq \beta_k^{(m)} \leq 1$ and $\beta_k^{(m)} \rightarrow 1$ as $m \rightarrow \infty$, $k$ stays fixed, such that the finite exponential sums

$$\sum_{k=1}^{m} \beta_k^{(m)} \hat{\xi}(e^{i\lambda_k}) e^{i\lambda_k n}$$

converge to $\xi(n)$ uniformly in $n \in \mathbb{Z}$ as $m \rightarrow \infty$. For the proof, see, for instance, [12, Chapter VI, § 5]. Therein, the proof is given for almost periodic functions, the proof for almost periodic sequences is almost the same.

As before, by $\sigma^*$ we denote the reflection of $\sigma(\xi)$ in the real axis. First, we show that $h^{F_\xi} \leq h_{\sigma^*}$ everywhere, and then that $|F(re^{i\theta})| \geq c(\theta)e^r$ with some $c(\theta) > 0$, whenever $\theta \in \sigma^*$ and $r \geq r_0(\theta)$. Then Lemma 3.3.1 does the job.

11.1

The following lemma is an old result of Bochner and Bonnenblust [3]. The proof given here follows the one in [13, Chapter VI].

Lemma 11.1.1. Everywhere, $h^{F_\xi} \leq h_{\sigma^*}$.

Proof. If the spectrum $\sigma(\xi)$ is dense on $\mathbb{T}$, then $h_{\sigma^*} \equiv 1$, and there is nothing to prove. So we assume that there is an open arc $J \subset \mathbb{T}$ such that $\sigma(\xi) \cap J = \emptyset$. Rotating the complex plane, $z \mapsto ze^{-it}$, we shift the spectrum $\sigma^*(\xi)$ and the indicator function $h^{F_\xi}$ by $t$. Therefore, without loss of generality, we may assume that $\sigma(\xi)$ is contained in the arc $\{e^{i\theta} : |\theta| \leq \pi - \delta\}$ for some $\delta > 0$. We need to show that the indicator diagram $I^{F_\xi}$ is contained in the closed convex hull of $\{e^{i\theta} : |\theta| \leq \pi - \delta\}$.

By our assumption, the functions

$$w \mapsto \Xi_m(w) = \sum_{k=1}^{m} \beta_k^{(m)} \hat{\xi}(e^{i\lambda_k}) e^{i\lambda_kw}$$
are entire functions of exponential type at most $\pi - \delta$. By Bochner’s theorem, given $\varepsilon > 0$, there exists $M_\varepsilon$ so that, for all $m_1, m_2 > M_\varepsilon$,

$$\|\Xi_{m_1} - \Xi_{m_2}\|_{\ell^\infty(Z)} < \varepsilon.$$ 

Then, by Cartwright’s theorem [13, Chapter IV, Theorem 15],

$$\|\Xi_{m_1} - \Xi_{m_2}\|_{L^\infty(\mathbb{R})} < C(\delta)\varepsilon,$$

and, invoking one of the Phragmén–Lindelöf theorems, we conclude that the sequence of entire functions $\Xi_m$ converges to an entire function $\Xi$ uniformly in any horizontal strip. Obviously, the entire function $\Xi$ interpolates the sequence $\xi$ at $Z$, the exponential type of $\Xi$ does not exceed $\pi - \delta$, and $\Xi$ is bounded on $\mathbb{R}$. Thus, the indicator diagram of $\Xi$ is contained in the interval $[(-\pi + \delta)i, (\pi - \delta)i]$ of the imaginary axis. It is worth noting that in what follows we use only that the exponential type of $\Xi$ does not exceed $\pi - \delta$.

Now, consider the Taylor series

$$f(s) = \sum_{n \geq 0} \xi(n)s^n$$

analytic in the unit disk. Since the coefficients $\xi(n)$ can be interpolated by an entire function of exponential type at most $\pi - \delta$, the function $f$ can be analytically continued through the arc $\{e^{i\theta} : |\theta - \pi| < \delta\}$ to $\mathbb{C} \setminus \overline{D}$. This is a classical result that goes back to Carlson and Pólya (see [13, Appendix 1, § 5] or [2, Theorem 1.3.1]). On the other hand, the function $w^{-1}f(w^{-1})$ is nothing but the Laplace transform of the entire function $F_\xi$, and, as we have seen, this function is analytic outside the closed convex hull of the arc $\{e^{i\theta} : |\theta| \leq \pi - \delta\}$. Then, by Pólya’s theorem (see [13, Chapter I, Theorem 33] or [2, Theorem 1.1.5]), the indicator diagram $I^{F_\xi}$ is contained in the closed convex hull of $\{e^{i\theta} : |\theta| \leq \pi - \delta\}$. This completes the proof of the lemma. \qed

11.2

Here, we show that $F_\xi$ grows as $e^{r}$ on the rays corresponding to the set $\sigma^*$. 

**Lemma 11.2.1.** For every $\theta \in \sigma^*$, there exists $c(\theta) > 0$ and $r(\theta) < \infty$ so that

$$|F_\xi(re^{i\theta})| \geq c(\theta)e^r, \quad r \geq r(\theta).$$
Proof. Once again, we will be using Bochner’s theorem. We fix \( e^{i\lambda_j} \in \sigma(\xi) \), take \( m \geq j \), and put

\[
\xi_m(n) = \sum_{k=1}^{m} \beta_k^{(m)} \hat{\xi}(e^{i\lambda_k}) e^{i\lambda_k n}.
\]

Then, uniformly in \( z \),

\[
|F_\xi(z) - F_{\xi_m}(z)| \leq \varepsilon_m e^{|z|}, \quad \text{with } \varepsilon_m \to 0. \quad (11.2.1)
\]

Furthermore, \( F_{\xi_m}(z) \) is a finite sum of exponential functions

\[
F_{\xi_m}(z) = \sum_{k=1}^{m} \beta_k^{(m)} \hat{\xi}(e^{i\lambda_k}) e^{r e^{i\lambda_k}},
\]

whence

\[
|F_{\xi_m}(re^{-i\lambda_j})| \geq \beta_j^{(m)} |\hat{\xi}(e^{i\lambda_j})| e^{|r|} - \sum_{k=1, k \neq j}^{m} \beta_k^{(m)} |\hat{\xi}(e^{i\lambda_k})| e^{|r| \cos(\lambda_k - \lambda_j)}
\]

\[
\geq \beta_j^{(m)} |\hat{\xi}(e^{i\lambda_j})| e^{|r|} - C_m e^{(1-\delta_m)r},
\]

with some \( \delta_m > 0 \). Therefore,

\[
\liminf_{r \to \infty} e^{-r} |F_{\xi_m}(re^{-i\lambda_j})| \geq \beta_j^{(m)} |\hat{\xi}(e^{i\lambda_j})| \geq \frac{1}{2} |\hat{\xi}(e^{i\lambda_j})|, \quad (11.2.2)
\]

provided that \( m \geq m_0(j) \). Juxtaposing (11.2.1) and (11.2.2), we get Lemma 11.2.1.

\[ \square \]

To finish off the proof of Theorem 5 we observe that, by Lemmas 11.1.1 and 11.2.1 the function \( F_\xi \) satisfies the assumptions of Lemma 3.3.1 and then Theorem 5 readily follows.

\[ \square \]

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