The distribution of extremal points of Gaussian scalar fields

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Abstract. We consider the signed density of the extremal points of (two-
dimensional) scalar fields with a Gaussian distribution. We assign a positive
unit charge to the maxima and minima of the function and a negative one to its
saddles. At first, we compute the average density for a field in half-space with
Dirichlet boundary conditions. Then we calculate the charge-charge correlation
function (without boundary). We apply the general results to random waves and
random surfaces. Furthermore, we find a generating functional for the two-point
function. Its Legendre transform is the integral over the scalar curvature of a
two-dimensional Riemannian manifold.

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1. Introduction

In two-dimensional systems with orientational degrees of freedom, it is often fruitful to
focus on the nodal points (zeros, defects), and to ignore other degrees of freedom. The
distribution of the nodal points keeps information about the degree of (orientational)
order within the system, and about the topology of the ‘substrate’. Well known
examples are defects in liquid crystal films, vortices in thin superconductors and
superfluid films (for an overview, see [1]), or chaotic wave functions in quantum
mechanics [2], microwave billiards [3, 4], and optical speckle pattern [5].

Here we take a closer look at the distribution of the nodal points of a gradient
field or likewise the extremal points (maxima, minima, saddles) of a scalar field \( \phi \) with
a Gaussian distribution.‡ The locations of the extrema of \( \phi \) are the points, where \( \phi \)
is stationary, i.e. the gradient \( \nabla \phi \) vanishes. The distribution of the stationary points
was discussed earlier in the context of random surfaces [6, 7, 8] and of excursion sets
[9, 10].

The extremal points carry a topological charge \( q \), which is the sign of the Jacobian
right at the extremum \( q = \text{sign}(\det(\partial_i \partial_j \phi)) = \pm 1 \). In two dimensions, the positive
extrema are the maxima and minima of the function \( \phi \), the negative extrema are the
saddle points. We will compute moments of the signed density

\[
\rho(r) = \sum_{\alpha} q_{\alpha} \delta^2(r - r_{\alpha})
\]

where \( r_{\alpha} \) are the positions of the extrema with sign (charge) \( q_{\alpha} \).

‡ The distribution of \( \phi \) is solely characterized by the correlation function \( \langle \phi(r)\phi'(r') \rangle \). Higher
moments are calculated with the help of the Wick theorem.
From the experimental point of view, measuring the absolute mean values of the density, \(|\rho|\), is much easier than measuring the usually small density difference between the density of the positive points minus the density of the negative ones. The signed density, however, is the much more appealing object from the mathematical point of view. The signed density is subject to a topological constraint, and its averages are differential geometrical objects, as shown in [11].

We represent the signed density of the extremal points of a \(d\)-dimensional scalar field \(\phi\) through

\[
\rho(r) = \det (\partial_i \partial_j \phi(r)) s^d (\nabla \phi(r)).
\]

In order to calculate the average of the absolute density \(\langle |\rho| \rangle\), one needs the simultaneous distribution of the \(d(d + 1)/2\) random variables \(\partial_i \partial_j \phi\) and of the \(d\) variables \(\nabla \phi\), where both sets are Gaussian random variables. Therefore, we need to know all moments \(\langle \partial_i \partial_j \phi \partial_k \partial_l \phi \rangle\), \(\langle \partial_i \partial_j \phi \partial_k \phi \rangle\), and \(\langle \partial_i \phi \partial_j \phi \rangle\) at coinciding points. For the averaged signed density, however, we need less information. According to [11], it is sufficient to know the moments

\[
g_{ij}(r) = \langle \partial_i \phi(r) \partial_j \phi(r) \rangle
\]

(which usually depend on the position \(r\)). The mean density \(\langle \rho \rangle\) turns out to be proportional to the (total) curvature of the \(d\)-dimensional Riemannian manifold which is described by the above metric tensor \(g_{ij}\). In two dimensions, the mean density is proportional to the Gaussian curvature \(K\) of that manifold (‘surface’) (times the invariant area element \(\sqrt{\det g_{ij}}\))

\[
2\pi \langle \rho(r) \rangle = K \sqrt{\det g_{ij}}.
\]

Also, higher-order correlation functions fit into this scheme. The correlation function \(\langle \rho(r_1) \ldots \rho(r_f) \rangle\) is the total curvature of a \(f \times d\) dimensional manifold with metric tensor§

\[
g_{i\alpha,j\beta} = \langle \partial_i \phi(r_\alpha) \partial_j \phi(r_\beta) \rangle, \quad i, j = 1 \ldots d, \quad \alpha, \beta = 1 \ldots f
\]

where a pair of a Greek and a Roman index is seen as a single composite index. The charge-charge correlation function for a Gaussian field in two dimensions is therefore the curvature of a particular four-dimensional Riemann manifold. We shall see, that the ‘Einstein’-action, i.e. the covariant integral over the scalar curvature \(R\) of the four-dimensional manifold, plays an important role as a generating functional, relating the two-point function and the correlation function of the scalar field \(\phi\). Before we discuss the two-point function, we present a simple example of a field \(\phi\) with a nonzero mean density \(\langle \rho(r) \rangle\), namely a system with a straight boundary. The plots of the charge-charge density and of the charge density near a wall are somewhat similar, since the latter quantity might be seen as the correlation of a charge with its mirror charge.

The general results are applied to random waves, and to thermally fluctuating surfaces (fluid membranes).

§ We define a \(f \times d\) dimensional Gaussian scalar function \(\Phi(r_1, \ldots, r_f) = \phi(r_1) + \ldots + \phi(r_f)\). It is obviously \(\rho_{\Phi}(r_1, \ldots, r_f) = \rho(r_1) \times \ldots \times \rho(r_f)\), where \(\rho_{\Phi}\) is the density of extrema of the field \(\Phi\). We obtain the metric tensor \(\langle \partial_{\alpha} \Phi \partial_{\beta} \Phi \rangle = \langle \partial_{\alpha} \phi(r_\alpha) \partial_{\beta} \phi(r_\beta) \rangle\).
2. Density of extremal points near a boundary

We calculate the signed density of extremal points of a Gaussian distributed random function \( \phi(x, y) \) in the upper half space \( y > 0 \) and assume Dirichlet boundary conditions \( \phi(x, y) = 0 \) (see [12]). We begin with a field \( \phi(r) \) in Euclidean, two-dimensional space with an isotropic and translationally invariant correlation function

\[
\langle \phi(x, y) \phi(x', y') \rangle = G \left( \sqrt{(x-x')^2 + (y-y')^2} \right). \tag{6}
\]

The correlation function for the upper half space is obtained by subtracting the appropriate mirror image

\[
G_+(x, y|x', y') \equiv \langle \phi(x, y) \phi(x', y') \rangle
= G \left( \sqrt{(x-x')^2 + (y-y')^2} \right) - G \left( \sqrt{(x-x')^2 + (y+y')^2} \right). \tag{7}
\]

The Green function obeys the boundary condition \( G_+(x, y|x', y') = 0 \). The components of the metric tensor (8) are easily obtained

\[
g_{ij}(x, y) = \langle \partial_i \phi(x, y) \partial_j \phi(x, y) \rangle:
\begin{align*}
g_{xx} &= -G''(0) + \frac{G''(2y)}{2y}, \\
g_{yy} &= -G''(0) - G''(2y), \\
g_{xy} = g_{yx} &= 0. \tag{8}
\end{align*}
\]

The signed density of extremal points is the scalar curvature of a two-dimensional manifold with metric tensor (8)

\[
4\pi \rho(y) = \sqrt{\det g} R \tag{9}
\]

where the scalar curvature is twice the Gaussian curvature \( R = 2K \) in two dimensions. The metric is orthogonal and depends on the distance to the wall \( y \) only. The non-vanishing affine connections are

\[
\begin{align*}
\langle \partial_x \partial_y \partial_x \phi \rangle &= \partial_y g_{xx}/2, \\
\langle \partial_y \partial_y \partial_y \phi \rangle &= \partial_y g_{yy}/2, \\
\langle \partial_x \partial_x \partial_y \phi \rangle &= -\partial_y g_{xx}/2. \tag{10}
\end{align*}
\]

since \( \langle \partial_x \phi \partial_y \phi \rangle = 0 \) and, therefore, \( \langle \partial_x \partial_x \phi \partial_y \phi \rangle + \langle \partial_y \partial_x \phi \partial_x \phi \rangle = 0 \). The scalar curvature is according to [11]

\[
R = (\det g)^{-1} e_{ij} e_{kl} \left( \langle \partial_i \partial_k \phi \partial_j \partial_l \phi \rangle - \langle \partial_i \partial_k \phi \partial_m \phi \rangle g^{mn} \langle \partial_n \phi \partial_j \partial_l \phi \rangle \right)
= \frac{2}{g_{xx}g_{yy}} \left( \frac{1}{2} \partial_y^2 g_{xx} + \frac{1}{4} \partial_y g_{xx} \left( (g_{yy})^{-1} \partial_y g_{yy} + (g_{xx})^{-1} \partial_y g_{xx} \right) \right) \tag{11}
\]

where \( e_{12} = 1, e_{21} = -1, e_{11} = e_{22} = 0 \). A straightforward calculation yields

\[
4\pi \rho(y) = -\partial_y \left( (g_{xx}g_{yy})^{-1/2} \partial_y g_{xx} \right) = \partial_y f(y) \tag{12}
\]

where \( f(y) = -(g_{xx}g_{yy})^{-1/2} \partial_y g_{xx} \) is the integrated charge density.
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2.1. Random waves in half space

Chaotic wave functions are well modelled by a superposition of partial waves with Gaussian distributed amplitudes and fixed energy \([13]\). The partial waves \(\phi_k\) must obey the Helmholtz equation \(-\nabla^2 \phi_k = k^2 \phi_k\) with \(k\) being fixed. Without lost of generality we can set \(k^2 = 1\). For time reversal symmetry, i.e. in the absence of a magnetic field, we can choose real amplitudes. The corresponding free-space correlation function of the \(\phi\) reads \([12, 13, 14, 15, 16, 17]\)

\[
G(r) = \frac{1}{(2\pi)^2} \int d^2p \delta(p^2 - 1) \exp(i pr) \propto J_0(r),
\]

where \(J_0\) is the zeroth-order Bessel function. \(G(r)\) also models the correlation of the undulations \(\phi(r)\) of capillary waves (with fixed energy) at the surface of a liquid. We might therefore also apply the following results to the extremal points of random capillary waves or even to ocean waves \([7, 8, 18, 19, 20]\).

We compute the density of extrema of a random wave near a straight wall with Dirichlet boundary conditions. We expect it to be a model for the distribution of extremal points of a chaotic wave function near the boundary \([12]\). According to section 2 we have to compute the nonvanishing components of the metric tensor \([6]\)

\[
\begin{align*}
g_{xx} &= 1/2 - \frac{J_1(2y)}{2y} = (1 - J_0(2y) - J_2(2y))/2, \\
g_{yy} &= 1/2 + J'_1(2y) = (1 + J_0(2y) - J_2(2y))/2
\end{align*}
\]

which can be found in \([12]\). We obtain \(\partial_y g_{xx} = 2J_2(2y)/(2y)\) and the integrated charge density \([12]\)

\[
f(y) = \frac{-\partial_y g_{xx}}{\sqrt{g_{xx}g_{yy}}} = \frac{-4J_2(2y)}{2y \left( (1 - J_2(2y))^2 - (J_0(2y))^2 \right)^{1/2}} = -(1 + y^2/8 + \ldots).
\]

Figure 1 shows \(f(y)\) and its derivative \(2\pi \rho = \partial_y f(y)/2\). Next to the wall one observes a negative density, i.e. an excess of saddle points, followed by a sharp positive peak. The system has, contrary to the free one, a net excess charge \(4\pi \int dy \rho = f(0) = 1/2\) per unit length.

2.2. A surface of revolution

There is a good way to visualize the charge density \([12]\). It is possible to embed the manifold with metric tensor \([8]\) in three-dimensional space. This allows us to literally see the defect density, which is the curvature of the embedded surface times the element of area \(\sqrt{\text{det} g}\). There is, however, a price to be paid – we must compactify the dimension \(x\) by making it \(2\pi\)-periodic. Then one can construct a surface of revolution with metric tensor \([8]\). The surface of revolution is parametrized through a coordinate \(y\) along the principal axis and the angle \(0 < x \leq 2\pi\)

\[
\vec{X}(x, y) = \begin{pmatrix} A(y) \cos x \\ A(y) \sin x \\ B(y) \end{pmatrix}
\]

with \(A > 0\) and a monotonically growing \(B(y)\). \(\vec{X}\) is a vector in three-dimensional, Euclidean space. For a compact overview of the differential geometry of surfaces,
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Figure 1. Integrated charge density \( f(y) \) (solid) and charge density \( 2\pi\rho \) (dashed) of a chaotic wave function next to a wall.

especially of embedded surfaces, see e.g. [21, 22]. The (induced) metric tensor of the surface of revolution reads

\[
g_{xx} = \partial_x \vec{X} \cdot \partial_x \vec{X} = A^2(y),
\]

\[
g_{xy} = 0,
\]

\[
g_{yy} = \partial_y \vec{X} \cdot \partial_y \vec{X} = (\partial_y A)^2 + (\partial_y B)^2.
\]

Thus, the above metric is equal to the metric (8) provided we identify

\[
A(y) = \sqrt{g_{xx}},
\]

\[
\partial_y B(y) = (g_{yy} - (\partial_y A)^2)^{1/2} = \sqrt{g_{yy}} \sqrt{1 - f^2(y)/4}
\]

\[
B(y) = \int_0^y d\bar{y} \sqrt{g_{yy}(\bar{y})(1 - f^2(\bar{y})/4)}.
\]

It is important to keep the information about the coordinate \( y \), e.g. by displaying a grid with lines at equidistant intervals in \( y \)-space to be able to reconstruct the element of area \( \sqrt{\det g} \). The embedding \( \{16\} \) of the metric tensor \( \{3\} \) is restricted to regions, where the integrated charge density obeys \( |f(y)| < 2 \). The contour of the surface of revolution for the case of random waves is shown in figure 2, where the function \( A(B) \) for \( 0 \leq y < 7 \) is displayed. The crosses display the \( y \)-coordinate—two consecutive crosses represent an interval \( \Delta y = 1/8 \). Figures 3 and 4 show two projections of the three-dimensional surface. The meridians are at values of \( y = k/4, k = 0, 1, 2, \ldots, 0 \leq y < 7 \). To obtain \( 4\pi\rho \) for a given \( y \), one has to find the meridian of number \( 4 \times y \) (counted from the cone), the local Gaussian curvature there and multiply it with \( \sqrt{\det g} \), which is proportional to the area of the corresponding grid element. The surface of revolution has a conical form for \( y \to 0 \), which is an artefact of the embedding \( \{16\} \). In fact, expanding the surface for small \( y \) we obtain a conical geometry: \( A(y) = y/2 + \ldots \) and \( B(y) = \sqrt{3}y/2 + \ldots \). The curvature itself is negative at the origin and has the limiting value \( R = f'(y)(g_{xx}g_{yy})^{-1/2} \to -1/2 \).
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Figure 2. Contour of the surface of revolution. Shows $A(B)$ for $0 \leq y < 7$. Two consecutive crosses correspond to an interval of $\Delta y = 1/8$.

Figure 3. Surface of revolution. Shown for $0 \leq y < 7$. Two consecutive meridian correspond to an interval of $\Delta y = 1/4$. 
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Figure 4. Different view on the surface of revolution. Shown for $0 \leq y < 7$. Two consecutive meridian correspond to an interval of $\Delta y = 1/4$.

Figure 5. Scalar curvature $R(y)$ for random waves.

Figure 5 displays the curvature as a function of the coordinate $R = R(y)$. Note the pronounced positive peak of the scalar curvature at $y \approx 2$. 

"sor3d.dat"
2.3. Pinned fluctuating surfaces

As a further application, we calculate the density of extrema for a nearly planar, fluctuating fluid membrane, attached to a straight line \( y = 0 \) (imagine a membrane suspended in a large frame). We represent the shape of the membrane by a height variable \( \phi(r) \). The domain of \( r \) is restricted to the upper half space \( y > 0 \), the boundary condition is \( \phi(x, y = 0) = 0 \). The thermal shape fluctuations of the fluid membrane are described by the Helfrich-Hamiltonian

\[
H/T = \int d^2 r \left( \kappa \langle \nabla^2 \phi \rangle^2 + \sigma \langle \nabla \phi \rangle^2 \right)
\]

(19)

plus higher-order terms. \( \kappa \) is the so-called bending rigidity, \( \sigma \) is the effective surface tension. For almost planar surfaces the fluctuations of the height variable \( \phi \) are Gaussian to lowest order. We rescale the height \( \sqrt{\kappa} \rightarrow \phi \) and set \( \tau = \sigma/\kappa \). The correlation function for the unconstrained \( \phi \) is now

\[
G(r) = \frac{1}{(2\pi)^2} \int d^2 p \frac{\exp(ip x)}{p^2 (p^2 + \tau)} = \frac{1}{2\pi} \left( C + \log(1/r) - K_0(\tau r) \right),
\]

(20)

where \( C \) is an irrelevant constant. We rescale the length scale that \( \tau = 1 \) and regularize \( G \) at distances \( r < a \). With the help of \( K'_0(\tau r) = -K_1(\tau r) \) and \( K_1(\tau r) = -(1/2)(K_0(\tau r) + K_2(\tau r)) \) and \( K_1(r)/r = -(1/2)(K_0(r) - K_2(r)) \) we obtain the components of the metric tensor

\[
g_{xx} = B - (2y)^{-2} + K_2(2y)/2 - K_0(2y)/2
\]

\[
g_{yy} = B - (2y)^{-2} + K_2(2y)/2 + K_0(2y)/2,
\]

(21)

where \( B = -G''(0) \). For distances \( y \) much larger than the correlation length (which is 1 in our units) we have \( g_{xx} = g_{yy} \approx B - (2y)^{-2} \). The corresponding charge density

\[
4\pi \rho \sim \frac{3}{2B} y^{-4}
\]

(22)

for \( r \gg 1 \), a reminiscence of the long ranged interactions (from the \( 1/p^2 \)-term in the Green function). The integrated charge density \( f(y) \) is

\[
f(y) = \frac{-1/(2y^3) + K_2(2y)/y}{\sqrt{g_{xx}g_{yy}}} = \frac{-1/(2y^3) + K_0(2y)/y + K_1(2y)/y^2}{\sqrt{g_{xx}g_{yy}}},
\]

(23)

For small arguments \( a < y \ll 1 \) the integrated charge density behaves as \( f(y) \sim y^{-1} \tilde{F}(\log y) + O(y^{-2}) \). Figure 6 shows \( yf(y) \). Figure 7 displays the function \( 4\pi y^2 \rho \), where again \( y^{-2} \) is the leading singular behaviour of the charge density for small arguments. The cutoff region \( 0 < y \lesssim a \) is not shown in both figures.

3. The two-point function

We present a convenient way of calculating the two-point function (charge-charge correlation function) \( C(r) = \langle \rho(0) \rho(r) \rangle \). We have mentioned in the introduction that the two-point function can be expressed through the Riemannian curvature tensor \( R_{\alpha\beta,\lambda,\mu} \) of a four-dimensional Riemann manifold with metric tensor \( g_{\alpha\beta} \), where \( \alpha = 1, 2, \beta = 1, 2 \) enumerate the two points. The Riemann curvature tensor is evaluated in Appendix B. The derivation of the correlation function with the help of differential geometry, however, is rather involved. Instead we utilize
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Figure 6. The function $y f(y)$ for membranes.

Figure 7. The function $4\pi y^2 \rho$ for membranes.

a less tedious approach, which is even valid for more general Gaussian vector fields $u_i$, $i = 1, 2$ with an isotropic, translationally invariant correlation function

$$\langle u_i(r) u_j(r') \rangle = \chi_{ij}(r - r').$$

In Appendix A we obtain for the charge-charge correlation function

$$(2\pi)^2 \langle \rho(r) \rho(0) \rangle = (2\pi)^2 C(r) = -\partial_i \partial_j \Omega_{ij}$$

where the potential $\Omega_{im}$ reads

$$\Omega_{im}(r) = e_{ij} e_{mn} \frac{1}{2} (\det h)^{-1/2} e_{ab} e_{cd} \partial_j \chi_{ac} \partial_n \chi_{bd}$$
with \( h_{ij} = \langle u_i u_m \rangle \langle u_j u_m \rangle - \chi_{im} \chi_{jm}. \) Summation over double indices is implied. Next, we restrict the vector field \( u_i \) to gradient fields \( u_i = \partial_i \phi. \) The nodal points of \( u_i, i.e. \) the points where \( u_i \) vanishes, are the extremal point of the scalar field \( \phi. \) The potential becomes

\[
\Omega_{im}(r) = e_{ij}e_{mn} \frac{1}{2} (\det h)^{-1/2} e_{ahl} e_{cd} \partial_j \partial_a \partial_c \partial_d \partial_0 \partial_a \partial_a \partial_0 \partial_0
\]

and \( h_{ij} = \delta_{ij} - \partial_i \partial_m \partial_m \partial_j G. \) For convenience, we have chosen the amplitude of \( \phi \) so that \(-G''(0) = 1.\) Therefore, it is \(-\partial_i \partial_j G(0) = \delta_{ij}.\) As a consequence of the polar invariance, matrices (such as \( \partial \partial G, \Omega \) and \( h \)) have the general form

\[
A_{ij} = A_{11} \frac{r_i r_j}{r^2} + A_{22} \left( \delta_{ij} - \frac{r_i r_j}{r^2} \right). \tag{28}
\]

For the particular choice \( r = (r, 0) \) the only remaining components are \( A_{11} \) and \( A_{22}. \) The non-vanishing derivatives of order 2, 3, 4 of \( G(r) \) and components of \( h \) and \( \xi \) for \( r = (r, 0) \) are given in equation (29) (the prime ' denotes the derivative with respect to \( r \))

\[
\begin{align*}
\partial_1 \partial_1 G &= Z_1 = G''(r) \\
\partial_2 \partial_2 G &= Z_2 = G'(r)/r \\
\partial_1 \partial_1 \partial_1 G &= Z_1' \\
\partial_1 \partial_2 \partial_2 G &= Z_2' = (Z_1 - Z_2)/r \\
\partial_1 \partial_1 \partial_2 \partial_2 G &= Z_1'' \\
\partial_1 \partial_2 \partial_2 \partial_2 G &= Z_2'' \\
\partial_2 \partial_2 \partial_2 \partial_2 G &= 3Z_2'/r \\
h_{11} &= D_1 = 1 - Z_1 \\
h_{22} &= D_2 = 1 - Z_2 \\
\xi_{11} &= Z_1/D_1 \\
\xi_{22} &= Z_2/D_2
\end{align*} \tag{29}
\]

A short calculation yields

\[
\Omega_{11} = -(Z_1')^2(D_1 D_2)^{-1/2}, \quad \Omega_{22} = Z_1' Z_2'(D_1 D_2)^{-1/2} \tag{30}
\]

where \( Z_1 = G''(r), Z_2 = G'/r, D_1 = 1 - Z_1', D_2 = 1 - Z_2'. \) We find for a matrix of the form (28)

\[
\partial_1 \partial_1 A_{ij} = \nabla^2 A_{11} + \frac{1}{r} \partial_r (A_{11} - A_{22}), \tag{31}
\]

where \( \nabla^2 A_{11} = (rA_1')'/r. \) The correlation function becomes

\[
(2\pi)^2 C(r) = -\partial_i \partial_j \Omega_{ij} = -\nabla^2 \Omega_{11} - (\Omega_{11} - \Omega_{22})' /r \equiv \psi'/r, \tag{32}
\]

with the ‘potential’

\[
\psi(r) = \Omega_{22} - \Omega_{11} - r\Omega_{11}' = \Omega_{22} - (r\Omega_{11})' = \frac{3(Z_1' - Z_2')(Z_1 - Z_2)}{r(D_1 D_2)^{1/2}} + \frac{(Z_1 - Z_2)^2}{r} \left( (D_1 D_2)^{-1/2} \right)',
\]

\[
= \frac{1}{r(Z_1 - Z_2)} \left( \frac{(Z_1 - Z_2)^3}{(D_1 D_2)^{1/2}} \right)' \tag{33}
\]
This result was also obtained by [26] using a direct approach. The correlation function of the Gaussian field $\phi$ can now be written as

$$G(r) = \int dp_1 dp_2 G\left(\sqrt{p_1^2 + p_2^2}\right) \cos(p_1 r) = \int dp \tilde{G}(p) \cos(pr)$$  \hspace{1cm} (34)

with $G(p) > 0$ and also $\tilde{G}(p) > 0$. It is

$$1 = -G''(r = 0) = \int dp \tilde{G}(p) p^2,$$  \hspace{1cm} (35)

i.e. $w(p) = p^2 \tilde{G}(p)$ has the properties of a probability density. The $r$-expansion of $G(r)$ reads

$$G(r) = -\frac{r^2}{2} + \frac{b r^4}{4!} - \frac{c r^6}{6!} + \frac{d r^8}{8!} - \frac{e r^{10}}{10!} + O(r^{12}),$$  \hspace{1cm} (36)

where $b = G''''(r = 0) = \int dp w(p) p^2$ and $c = -G^{(6)}(r = 0) = \int dp w(p) p^4$ (as well as $d$ and $e$) are positive constants. With the help of Mathematica we find the following series expansion of $\psi(r)$:

$$\psi(r) = \frac{4 b}{3 \sqrt{3}} + \frac{(b^2 - c) r^2}{3 \sqrt{3}} + \frac{(45 b^4 - 56 b^2 c + 3 c^2 + 10 b d) r^4}{540 \sqrt{3} b} + O(r^6).$$  \hspace{1cm} (37)

It is shown in equation (40) that $-\psi(0)$ is related to the absolute density of extrema $n_0 = \langle |\rho| \rangle$. The next term of the $\psi$ expansion determines the charge-charge correlation function at almost coinciding points $\parallel$

$$2\pi^2 \lim_{r \to 0} C(r) = \frac{-2(c - b^2)}{3 \sqrt{3}}$$

$$= -\frac{2}{3 \sqrt{3}} \left( \int dp w(p) p^4 - \left( \int dp w(p) p^2 \right)^2 \right).$$  \hspace{1cm} (38)

The latter expression is obviously negative–the vicinity of an extremum is populated by extrema of the opposite type [27, 28], which screen the central charge.

3.1. Perfect screening of gradient fields

We show that the charge-charge correlation function $C(r)$ obeys the first Stillinger-Lovett sum rule

$$\lim_{\epsilon \to 0} \int_{|r| > \epsilon} d^2 r C(r) = -n_0.$$  \hspace{1cm} (39)

It expresses the fact that a particular extremum is completely screened by extrema (charges) of the opposite sign [15, 29, 30, 31, 32, 33, 34]. Applying the results of section 3 we find for the left hand side

$$-n_0 = 2\pi \int_0^\infty rdr C(r) = \frac{1}{2\pi} \int_0^\infty dr r^3 \psi'(r) = -\frac{-2 G''''(r = 0)}{3 \pi \sqrt{3}}$$  \hspace{1cm} (40)

$\parallel C(0)$ is not well defined, since $C(r)$ has a $\delta$-like singularity at the origin, see [11]. The $\delta$-function, however, is lost ‘somewhere’ during the course of section 3. A careful analysis at the origin similar to the one presented in [11] is needed to recover the $\delta$-function.
where $G(r)$ is the correlation function of the field $\phi$. On the other hand, the absolute density of extremal points of the Gaussian field $\phi$ reads \cite{[7,8]} (for an arbitrary point $r$)

$$n_0 = \langle |\partial_x \partial_y \phi \partial_x \partial_y \phi - (\partial_x \partial_y \phi)^2 | \delta(\partial_x \partial_y \phi) \rangle .$$

(41)

The Gaussian variables $\partial_i \phi$ are independent of the second derivatives $\partial_i \partial_j \phi$ (evaluated at the same point). It is therefore $n_0 = \langle \ldots \rangle \langle \delta(\partial_x \partial_y \phi) \rangle$ with

$$\langle \delta(\partial_x \partial_y \phi) \delta(\partial_y \phi) \rangle = \int d^2 \! E \frac{1}{\pi \langle (\nabla \phi)^2 \rangle} \exp \left( \frac{E^2}{\langle (\nabla \phi)^2 \rangle} \right) \delta(E_x) \delta(E_y)$$

$$= \frac{1}{\pi \langle (\nabla \phi)^2 \rangle} = 1/(2\pi),$$

(42)

where we have chosen $G''(0) = 1$. The second moments of $\partial_i \partial_j \phi$ at coinciding points are

$$\langle \partial_x \partial_y \phi \partial_x \partial_y \phi \rangle = \langle \partial_x \partial_y \phi \partial_y \partial_x \phi \rangle = 0$$

$$\langle \partial_x \partial_y \phi \partial_x \partial_x \partial_y \phi \rangle = \langle \partial_x \partial_y \phi \partial_y \partial_y \partial_x \phi \rangle = G''''(0)$$

$$\langle \partial_x \partial_y \phi \partial_y \partial_x \partial_y \phi \rangle = \langle \partial_x \partial_y \phi \partial_y \partial_y \partial_y \phi \rangle = G''''(0)/3.$$  

(43)

Next, we define the mutually independent Gaussian variables $X = (\partial_x \partial_y \phi + \partial_y \partial_y \phi)\sqrt{2}$, $Y = (\partial_x \partial_x \phi - \partial_y \partial_y \phi)/\sqrt{2}$, and $Z = \partial_x \partial_y \phi$ with $\langle X^2 \rangle = (4/3)G''''(0)$, $\langle Y^2 \rangle = (2/3)G''''(0)$, and $\langle Z^2 \rangle = (1/3)G''''(0)$. Then

$$\langle \ldots \rangle = \langle \big| X^2/2 - Y^2/2 - Z^2 \big| \rangle$$

$$= \frac{1}{(2\pi)^{3/2}} \int dx dy dz \exp \left( -(x^2 + y^2 + z^2)/2 \right)$$

$$\times \left| \frac{2G''''(0)}{3} x^2 - \frac{G''''(0)}{3} (y^2 + z^2) \right| .$$

(44)

We introduce spherical coordinates $y = r \sin \theta \cos \varphi$, $z = r \sin \theta \sin \varphi$, and $x = r \cos \theta$, yielding

$$\langle \ldots \rangle = G''''(0) \frac{2}{3 \sqrt{2}} \int_0^\infty dr \int_0^{\pi/2} \int_0^{\pi/2} d\theta \sin \theta |3(\cos \theta)^2 - 1| \exp(-r^2/2) \int_0^{\pi/2} d\varphi \sin \varphi$$

$$= \frac{4G''''(0)}{3 \sqrt{3}} .$$

(45)

Finally, we obtain the absolute density of extremal points of $\phi$,

$$n_0 = \frac{2G''''(0)}{3 \pi \sqrt{3}}$$

(46)

in agreement with equation \cite{40}.

3.2. Generating functionals for the two-point function

We have found a representation of the two-point function $C(r)$ as a functional derivative. Although this section is less useful from the practical point of view, it highlights an important mathematical aspect of the theory and demonstrates once
more the applicability of the differential geometrical approach. We begin with the integral over the two-point function times the correlation function of the field $\phi$,

$$-2\pi \int d^2 r C(r) G(r) = (2\pi)^{-1} \int d^2 r \partial_i \partial_j \Omega_{ij} G$$

$$= \text{b.c.} + (2\pi)^{-1} \int d^2 r \Omega_{ij} \partial_i \partial_j G$$

$$= \text{b.c.} + \int r dr (\Omega_{11} Z_1 + \Omega_{22} Z_2)$$

$$= \text{b.c.} + \int r dr \frac{1}{(D_1 D_2)^{1/2}} (Z_1' Z_2 - Z_2' Z_1)$$

$$= \text{b.c.} + \int dr \frac{Z_1 - Z_2}{(D_1 D_2)^{1/2}} (Z_1' Z_2 - Z_2' Z_1)$$

$$= -\int dr \frac{D}{(D_1 D_2)^{1/2}} (Z_1' + Z_2')$$

$$+ \int dr \left( \frac{(D_2/D_1)^{1/2} Z_1'}{(D_1/D_2)^{1/2}} + \frac{(D_1/D_2)^{1/2} Z_2'}{D} \right)$$

$$= H - L$$

(47)

where b.c. represents contributions from the boundaries $r = 0, r \to \infty$, and

$$\mathcal{H} = -\int dr \frac{D}{(D_1 D_2)^{1/2}} (Z_1' + Z_2'),$$

$$\mathcal{L} = -\int dr \left( \frac{(D_2/D_1)^{1/2} Z_1'}{(D_1/D_2)^{1/2}} + \frac{(D_1/D_2)^{1/2} Z_2'}{D} \right)$$

(48)

with $D = 1 - Z_1 Z_2$. The action $\mathcal{H}$ is identified in Appendix B as the integral over the scalar curvature $R = g^{\kappa \gamma, i \alpha} g^{(A \beta)} R_{\gamma \lambda, j \delta, i \alpha, j \beta}$

$$\mathcal{H} = (4\pi)^{-1} \int d^2 r \sqrt{\det g} R.$$

(49)

This functional is well known from general relativity. The curved spacetime manifolds which obey the least-action principle $\delta \mathcal{H} = 0$ are the solutions of the Einstein field equations (without matter). We show below that action (49) is also meaningful for our manifold.

The action $\mathcal{L}$ is the generating functional of $C(r)$ as computed in Appendix C

$$\frac{1}{r} \frac{\delta \mathcal{L}}{\delta G} = \psi' / r = (2\pi)^2 C(r).$$

(50)

The Legendre transform of $\mathcal{L}$ reads, using (47) and the above equation

$$\mathcal{L} - \int d^2 r \frac{\delta \mathcal{L}}{\delta G(r)} G(r) = \mathcal{L} - \int dr \frac{\delta \mathcal{L}}{\delta G(r)} G(r) = \mathcal{H}.$$

(51)

The ‘Einstein’-action $\mathcal{H}$, expressed as a functional of the charge-charge correlation function $C(r)$ is therefore the generating functional of the conjugated field-field correlation function $G(r)$,

$$\frac{1}{(2\pi)^2 r} \frac{\delta \mathcal{H}}{\delta C(r)} = G(r).$$

(52)

\* In principle, one has to integrate over four coordinates. The metric, however, depends on the distance $|r^A - r^B|$ only. The integration over the centre of mass is therefore trivial.
3.3. Two point function for random waves

We now compute as an application the two-point function \(C(r)\) for Gaussian random waves \(\phi\) (without boundary). We rescale the correlation function \(\langle \phi(0) \phi(0) \rangle\) of the field \(\phi\) for practical reasons and use \(G(r) = \langle \phi(0) \phi(0) \rangle = 2J_0(r)\). It is \(-G''(0) = 1, Z_1 = G''(r) = -J_0(r) + J_2(r), Z_2 = -2J_0(r) + J_2(r)\), \(D_1 = 1 - (J_0(r) - J_2(r))^2, D_2 = 1 - (J_0(r) + J_2(r))^2\), yielding for the potential \(\psi(r)\)

\[
\psi(r) = \frac{4}{rJ_2(r)} \frac{d}{dr} \left( \frac{(J_2)^3}{(1 - (J_0 - J_2)^2)^{1/2}} \frac{1}{(1 - (J_0 + J_2)^2)^{1/2}} \right). \tag{53}
\]

With the help of the relations \(J_0 = -J_1, J_2 = J_1 - 2J_2/r\) and \(2J_1/r = J_0 + J_2\) we obtain the result

\[
\psi(r) = \frac{2(J_2)^3}{(1 - (J_0 - J_2)^2)^{1/2}} \frac{1}{(1 - (J_0 + J_2)^2)^{1/2}} \times \left( \frac{1 + J_0}{J_2} \left( \frac{3}{J_2} + \frac{2(J_2 - J_0)}{1 - (J_2 - J_0)^2} \right) \right.
\]

\[
- \frac{4}{r^2} \left( \frac{J_2 - J_0}{1 - (J_2 - J_0)^2} + \frac{J_2 + J_0}{1 - (J_2 + J_0)^2} \right) \right). \tag{54}
\]

The small argument expansion of \(\psi\) reads

\[
\psi(r) = \frac{1}{\sqrt{3}} \left( 1 - \frac{r^2}{48} - \frac{r^4}{2304} - \frac{139r^6}{29859840} \right) + O(r^8). \tag{55}
\]

The small \(r\) behaviour of the charge-charge correlation function \(\langle \phi(0) \phi(0) \rangle\) is therefore

\[
(2\pi)^2 C(r) = -\frac{1}{24\sqrt{3}} \left( 1 + \frac{r^2}{24} + \frac{139r^4}{207360} \right) + O(r^6). \tag{56}
\]

Figure 8 shows \(\psi(r)/2\) and \((2\pi)^2 C(r)\) for \(0 < r < 10\). The plateau of the correlation function for small radii is almost perfect due to the smallness of the higher order expansion coefficients. Note the pronounced, negative peak at \(r \approx 3.4\).

4. Conclusion

We have studied the signed density of extremal points of scalar field with a Gaussian distribution. We have calculated the average signed density in half space with Dirichlet boundary conditions. The density is expressed as the Gaussian curvature of an associated (abstract) manifold, depending on the correlations of the scalar field. The abstract manifold could be visualized by embedding it into three-dimensional space. Then we computed the two-point correlation function of the above density, which is in principle the total curvature of a four-dimensional manifold. We identified the ‘Einstein’-action as the generating functional, which relates the correlation function of the scalar field and the correlation function of the density of extrema. Currently, no application of this equation is known. Nevertheless, we hope that its further study will ultimately lead to a deeper understanding of the differential geometry of the abstract manifold, which governs the distribution of the extremal points.

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Appendix A. Computation of the general charge-charge correlation function

We calculate the charge-charge correlation function for Gaussian vector fields $u_i$, $i = 1, 2$ with an isotropic, translationally invariant correlation function

$\langle u_i(r) u_j(r') \rangle = \chi_{ij}(r - r')$. \hspace{1cm} (A.1)

The charge density $\rho$ is represented as

$\rho = \frac{1}{2\pi} \partial_i \left( \frac{e_{ij} e_{kl} u_k \partial_j u_l}{u^2 + \epsilon} \right) = \frac{1}{2} e_{ij} e_{kl} \partial_i u_k \partial_j u_l \frac{\epsilon}{\pi (u^2 + \epsilon)^2}$

$\rightarrow \det(\partial_i u_j) \delta^2(u) \text{ for } \epsilon \rightarrow 0$ \hspace{1cm} (A.2)

where $e_{12} = 1, e_{21} = -1, e_{11} = e_{22} = 0$. The charge-charge correlation function $C(r) \equiv \langle \rho(r) \rho(r') \rangle$ is

$4\pi^2 C(r) = \frac{\partial}{\partial r_i} \frac{\partial}{\partial r'_m} \left\langle e_{ij} e_{kl} e_{mn} e_{st} \frac{u_k(r) \partial_j u_l(r) u_s(r') \partial_n u_t(r')}{(u^2(r) + \epsilon)(u^2(r') + \epsilon)} \right\rangle$

$\equiv - \partial_i \partial_m \Omega_{im}(r - r')$ \hspace{1cm} (A.3)

utilizing the translational invariance of the correlation function \hspace{1cm} (A.1). With the help of the Fourier-transformation

$$(u^2 + \epsilon)^{-1} = \frac{1}{2\pi} \int d^2 p \exp(ip \mathbf{u}) K_0(p \sqrt{\epsilon})$$ \hspace{1cm} (A.4)
The term \( (A.7) \) does not yield a contribution to the charge-charge correlation function in the limit \( \epsilon \to 0 \). The latter expression was obtained with the help of two partial integrations and \( \partial_0 K_0(\sqrt{\epsilon}p) = -\sqrt{\epsilon} K_1(\sqrt{\epsilon}p)p_0 \to -p_0/p^2 \). The Gaussian average can now be done, yielding

\[
\Omega_{im}(r) = \frac{1}{4\pi^2} \int d^2 p d^2 q K_0(\sqrt{\epsilon}q) e_{ij} e_{kl} e_{mn} e_{st} \frac{\partial}{\partial p_k} \frac{\partial}{\partial q_s} \frac{\partial}{\partial A_{jl}} \frac{\partial}{\partial B_{nt}} |_{A=B=0} \\
\times \left( \exp \left( A_{ij} \partial_q u_j(0) + B_{ij} \partial_q u_j(r) + i g_{j}(0) + i g_q(r) \right) \right) \\
= \frac{1}{4\pi^2} \int d^2 p d^2 q e_{ij} e_{kl} e_{mn} e_{st} \frac{p_k q_s}{p^2 q^2} \frac{\partial}{\partial A_{jl}} \frac{\partial}{\partial B_{nt}} |_{A=B=0} \\
\times \left( \exp \left( A_{ij} \partial_q u_j(0) + B_{ij} \partial_q u_j(r) + i g_{j}(0) + i g_q(r) \right) \right)
\]  

(A.5)

where \( \langle u_i u_j \rangle = \langle u_i(0) u_j(0) \rangle = \langle u_i(r) u_j(r) \rangle \). We eliminate the second derivates \( \partial_0 \partial_n \chi_{lt} \) by a partial ‘integration’

\[
\Omega_{im}(r) = e_{mn} \partial_n \left( \frac{1}{4\pi^2} \int d^2 p d^2 q e_{ij} e_{kl} e_{mn} e_{st} \frac{p_k q_s}{p^2 q^2} \partial_j \chi_{lt} \right) \exp(\ldots) \\
+ \frac{1}{4\pi^2} \int d^2 p d^2 q e_{ij} e_{kl} e_{mn} e_{st} \frac{p_k p_s q_t q_w}{p^2 q^2} \left( \partial_j \chi_{lt} \partial_n \chi_{vw} - \partial_n \chi_{st} \partial_j \chi_{lw} \right) \\
\times \exp \left( - (p_i p_j + q_i q_j) \langle u_i u_j \rangle / 2 - p_i q_j \chi_{ij} \right).
\]  

(A.7)

The term \( (A.7) \) does not yield a contribution to the charge-charge correlation function \( \propto -\partial_0 \partial_n \Omega_{im} \) since it is a pure curl. The term \( \partial_1 \chi_{lt} \partial_n \chi_{vw} - \partial_1 \chi_{lw} \partial_n \chi_{vt} \) is antisymmetric with respect to the indices \( l, w \):

\[
\partial_1 \chi_{lt} \partial_n \chi_{vw} - \partial_1 \chi_{lw} \partial_n \chi_{vt} = \epsilon_{tw} e_{ab} \partial_0 \partial_0 \chi_{la} \partial_0 \chi_{vb}.
\]  

(A.9)

The \( q^2 \)-denominator is cancelled, since \( \epsilon_{tw} e_{st} q_t q_w = -q^2 \), yielding

\[
\Omega_{im}(r) \equiv - \frac{1}{4\pi^2} \int d^2 p d^2 q e_{ij} e_{kl} e_{mn} e_{st} \frac{p_k p_s q_t q_w}{p^2 q^2} e_{ab} \partial_0 \partial_0 \partial_n \chi_{vb} \\
\times \exp \left( -(p_i p_j + q_i q_j) \langle u_i u_j \rangle / 2 - p_i q_j \chi_{ij} \right),
\]  

(A.10)

where we have omitted the curl \( \chi_{lt} \) once more. Now \( \Omega_{im} \) is no longer symmetric with respect to \( i, m \); nevertheless only the symmetric part of \( \Omega_{im} \) contributes to the
charge-charge correlation function. Equivalently, we might replace \( e_{ab} \partial_j \chi_{la} \partial_n \chi_{lb} \) by the symmetrized version \( e_{ab} (\partial_j \chi_{la} \partial_n \chi_{lb} + \partial_n \chi_{la} \partial_j \chi_{lb})/2 \). The latter expression is antisymmetric in \( l, v \) and can therefore be written as \( e_{ce} e_{cd} \partial_j \chi_{ca} \partial_n \chi_{db}/2 \). Up to an antisymmetric component, \( \Omega_{im} \) reads

\[
\Omega_{im}(r) \equiv e_{ij} e_{mn} \frac{1}{8\pi^2} \int d^2 p d^2 q e_{ab} e_{cd} \partial_j \chi_{ca} \partial_n \chi_{db} \\
\quad \times \exp \left(-\left(p_i p_j + q_i q_j\right) / 2 - p_i q_j \chi_{ij}\right) \quad (A.11)
\]

Next we perform the \( p, q \)-integral, which is Gaussian now, and obtain finally

\[
\Omega_{im}(r) = e_{ij} e_{mn} \frac{1}{2} (\det h)^{-1/2} e_{ab} e_{cd} \partial_j \chi_{ac} \partial_n \chi_{bd},
\]

(A.12)

where \( h_{ij} = \langle u_i u_m \rangle \langle u_j u_m \rangle - \chi_{im} \chi_{jm} \). As an examination, we use equations \((A.3)\) and \((A.12)\) to rederive the well known two-point function for the special case of vector fields with independent components \( \chi_{ij}(r) = \delta_{ij} G(r) \) in two dimensions (see \([11, 15, 31, 33]\)). Then

\[
\Omega_{im} = e_{ij} e_{mn} \frac{1}{G(0)^2 - (G(r))^2} \partial_j G \partial_n G = e_{ij} e_{mn} \partial_j K(r) \partial_n K(r) \quad (A.13)
\]

where \( K(r) = \arcsin(G(r)/G(0)) \). Plugged into \((A.1)\) we get

\[
4\pi^2 C(r) = -e_{ij} e_{mn} \partial_j K \partial_n K = 2 \det(\partial_i \partial_j K) = 2K''(r)K'(r)/r \quad (A.14)
\]

in agreement with the known result.

Appendix B. The ‘Einstein’-action

We will now calculate the scalar curvature \( R \) and the ‘Einstein’-action \( \mathcal{H} = (4\pi)^{-1} \int d^2 r \sqrt{\det g} R \) for a four-dimensional Riemannian manifold with metric \( g_{\alpha\beta} = -\partial_i \partial_j G(|r^\alpha - r^\beta|) \). In principle, one has to integrate over 4 coordinates. The metric, however, depends on the distance \(|r^A - r^B|\) only. The integration over the centre of mass is therefore trivial. We represent the correlation function of the field \( \phi \) through its Fourier transform

\[
\langle \phi(r^A) \phi(r^B) \rangle = G(|r^A - r^B|) = \int d^2 p \tilde{G}(p) \exp \left(i p \cdot (r^A - r^B)\right) \quad (B.1)
\]

The \( 4 \times 4 \) metric tensor reads \( (i, j \in \{1, 2\}; \alpha, \beta \in \{A, B\}) \)

\[
g_{\alpha\beta, ij} = \langle \partial_i \phi(r^\alpha) \partial_j \phi(r^\beta) \rangle = \int d^2 p \frac{\partial \tilde{\psi}^\alpha}{\partial r^\alpha} \frac{\partial \tilde{\psi}^\beta}{\partial r^\beta}.
\]

(B.2)

where

\[
\psi(r^A, r^B; p) = \sqrt{\tilde{G}(p)} \sum_{\alpha = A, B} \exp(i p \cdot r^\alpha)
\]

is an immersion of the four-dimensional manifold in function space. Explicitly, we have

\[
g_{\alpha\beta, ij} = \delta_{\alpha\beta} \delta_{ij} - c_{\alpha\beta} \partial_i \partial_j G(r)
\]

(B.4)

where \( c_{\alpha\beta} = 1 - \delta_{\alpha\beta} \). The inverse metric tensor is easily calculated with the help of \( c_{\alpha\beta} c_{\beta\gamma} = \delta_{\alpha\gamma} \):

\[
g^{\alpha\beta, ij} = (\delta_{\alpha\beta} h_{ij} + c_{\alpha\beta} \xi_{ij})
\]

(B.5)
where \( h_{ij} = \delta_{ij} - \partial_i \partial_j G \partial_m \partial_n G \), \( h^{ij} = h^{-1}{}_{ij} \) and \( \xi_{ik} = \partial_i \partial_j G h^{jk} \). The determinant of the metric tensor is \( \det g = \det h \). The components of the affine connection are

\[
\Gamma_{k\gamma,ia,j\beta} = \int d^2 p \partial_{ia} \partial_{j\beta} \psi^* \partial_{k\gamma} \psi = \delta_{\alpha\beta} e_{\alpha\gamma} \partial_i \partial_j \partial_k G(r),
\]

where the index \( \alpha \) is not summed. The components of the Riemannian curvature tensor are calculated from

\[
R_{k\gamma,ia,j\beta} = \int d^2 p \left( \partial_{ia} \partial_{j\beta} \psi^* \partial_{k\gamma} \partial_\lambda \psi \right)
- \int d^2 p \left( \partial_{ia} \partial_{j\beta} \partial_{mn} \psi \right) g^{mn,uv} \int d^2 p' \left( \partial_{nu} \psi^* \partial_{j\beta} \partial_{k\gamma} \partial_\lambda \psi \right)
- (i\alpha \leftrightarrow j\beta).
\]

Explicitly we have

\[
\int d^2 p \left( \partial_{ia} \partial_{j\beta} \psi^* \partial_{k\gamma} \partial_\lambda \psi \right) = \delta_{\alpha\gamma} \delta_{\beta\lambda} c_{\alpha\beta} \partial_i \partial_j \partial_k G(r) + \delta_{\alpha\beta} \delta_{\gamma\delta} Z_{ijkl}
\]

and

\[
R_{k\gamma,ia,j\beta} = \left( \delta_{\alpha\gamma} \delta_{\beta\lambda} \delta_{\alpha\lambda} \delta_{\beta\gamma} \right) c_{\alpha\beta} \partial_i \partial_j \partial_k \partial_l G
- \delta_{\alpha\gamma} \delta_{\alpha\beta} e_{\alpha\mu} \partial_i \partial_k \partial_m G g^{\mu\nu,rs} \delta_{\beta\gamma} e_{\beta\lambda} \partial_j \partial_l \partial_n G
+ \delta_{\beta\gamma} \delta_{\beta\lambda} e_{\beta\mu} \partial_j \partial_k \partial_m G g^{\mu\nu,rs} \delta_{\alpha\gamma} e_{\alpha\lambda} \partial_i \partial_n G,
\]

where \( \alpha \) and \( \beta \) are not summed, \( Z_{ijkl} = \partial_i \partial_j \partial_k \partial_l G|_{r=0} \) and \( \delta_{\alpha\beta} = 1 \) if \( \alpha = \beta = \gamma = \delta \) and zero otherwise. The Riemannian curvature tensor can be decomposed into

\[
R_{k\gamma,ia,j\beta} = c_{\alpha\beta} \delta_{\gamma\lambda} c_{\alpha\lambda} S_{ijkl}\delta_{\beta\delta} c_{\beta\delta} \partial_i \partial_j \partial_k \partial_l G - \delta_{\beta\gamma} \delta_{\gamma\lambda} \delta_{\alpha\delta} c_{\alpha\delta} c_{\beta\lambda} e_{\alpha\beta} e_{\beta\lambda} \partial_i \partial_j \partial_k \partial_l G \Theta/2,
\]

where we have used \( \delta_{\alpha\gamma} \delta_{\beta\lambda} + \delta_{\beta\gamma} \delta_{\alpha\lambda} c_{\alpha\beta} = c_{\alpha\beta} c_{\gamma\lambda} \) and \( \delta_{\alpha\gamma} \delta_{\beta\lambda} - \delta_{\beta\gamma} \delta_{\alpha\lambda} c_{\alpha\beta} = c_{\alpha\beta} e_{\gamma\lambda} \) (no summation over \( \alpha, \beta \)) and the definitions

\[
S_{ijkl} = \partial_i \partial_j \partial_k \partial_l G + (\partial_i \partial_k \partial_p G \partial_j \partial_q G + \partial_i \partial_l \partial_p G \partial_j \partial_q G - \partial_j \partial_k \partial_p G \partial_i \partial_q G + \partial_j \partial_l \partial_p G \partial_i \partial_q G) \xi_{pq}/2
\]

\[
\Theta = e_{ij} e_{kl} \partial_i \partial_k \partial_m G \partial_j \partial_n G h^{mn}/2.
\]

We obtain easily

\[
g^{k\gamma,ia} R_{k\gamma,ia,j\beta} = \left( \delta_{\beta\lambda} h^{ik} - c_{\beta\lambda} \xi_{ik} \right) S_{ijkl} + \delta_{\beta\lambda} c_{\beta\lambda} e_{\gamma\lambda} e_{\alpha\beta} \xi_{ik} (-\omega + \Theta/2)
+ c_{\beta\lambda} e_{ij} e_{kl} \xi_{ik} \Theta/2.
\]

and

\[
R = g^{k\gamma,ia} g^{l\lambda,ib} R_{k\gamma,ia,j\beta}
= 2 \left( (h^{ik} h^{jl} - \xi_{ik} \xi_{jl}) S_{ijkl} - \frac{2\omega}{\det h} + \frac{\Theta}{\det \xi} + \Theta \det \xi \right).
\]

To simplify this expression, we define

\[
S_{ijkl} = T_{ijkl} - (2\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \Theta/2,
\]

where \( T_{ijkl} \) can be written as

\[
T_{ijkl} = \partial_i \partial_j \partial_k \partial_l G + \partial_i \partial_j \partial_k \partial_l G \xi_{pq}.
\]

The scalar curvature is now

\[
R = 2 \left( (h^{ik} h^{jl} - \xi_{ik} \xi_{jl}) T_{ijkl} - \frac{2\omega}{\det h} + \frac{2\Theta}{\det \xi} \right).
\]
Furthermore
\[ h^{ij}h^{kl}\partial_i\partial_j\partial_mG\partial_k\partial_l\partial_nG\xi_{mn} + 2\Theta/\det h = h^{ij}\partial_i\partial_j\partial_mGh^{kl}\partial_k\partial_l\partial_nG\xi_{mn}. \] (B.17)

For convenience we choose \( r^A - r^B = (r, 0) \). The non-vanishing partial derivatives of \( G(r) \) for that particular choice of \( r \) are listed in equation (B.19). We have \( \omega = Z_1'Z_2'/D_1 - (Z_2')^2/D_2 \)

\[ \frac{r\omega}{\sqrt{\det h}} = \frac{1}{\sqrt{D_1D_2}} \left( \frac{Z_1'(Z_1 - Z_2)}{D_1} - \frac{Z_2'(Z_1 - Z_2)}{D_2} \right) = \left( \frac{D}{\sqrt{D_1D_2}} \right)' \] (B.18)

where \( D = 1 - Z_1Z_2 \). The integral \( \int d^2r (\det h)^{-1/2}\omega \) is therefore zero up to boundary contributions from \( r = 0 \) and \( r \to \infty \). The integral over the scalar curvature \( \mathcal{H} = (4\pi)^{-1} \int d^2r \sqrt{\det gR} \) becomes (up to boundary contributions)

\[ \mathcal{H} = (2\pi)^{-1} \int d^2r \sqrt{\det h} \left( (h^{ij}h^{kl} - \xi_{ik}\xi_{jl}) T_{ijkl} \right) \]

\[ = \int_0^{\infty} r dr (D_1D_2)^{1/2} \left( \frac{Z_1''}{D_1} + \frac{2Z_2' D}{D_1D_2} + \frac{3Z_2'}{rD_2} + \left( \frac{Z_1'}{D_1} + \frac{Z_2'}{D_2} \right)^2 \frac{Z_1}{D_1} \right. \]

\[ - (Z_1')^2 \left( \frac{Z_1}{D_1} \right)^3 - 3(Z_2')^2 \frac{Z_1Z_2}{D_1D_2} \right) \]

\[ = \int_0^{\infty} r dr (D_1D_2)^{1/2} \left( \frac{Z_1''}{D_1} + \frac{2Z_2' D}{D_1D_2} + \frac{3Z_2'}{rD_2} + \left( \frac{Z_1'}{D_1} + \frac{Z_2'}{D_2} \right)^2 \frac{Z_1}{D_1D_2} \right. \]

\[ + (Z_1')^2 \frac{Z_1}{D_1D_2} - 2(Z_2')^2 \frac{Z_1Z_2}{D_1D_2} \right). \] (B.19)

\( Z_1'' \) and \( Z_2'' \) are eliminated with the help of two partial integrations

\[ \int_0^{\infty} r dr (D_1D_2)^{1/2} \frac{Z_1''}{D_1} \]

\[ = \text{b.c.} + \int_0^{\infty} r dr (D_1D_2)^{1/2} \left( - \frac{Z_1'}{rD_1} + \frac{Z_1'Z_2}{D_1D_2} - \frac{(Z_1')^2Z_1}{D_1^2} \right), \] (B.20)

\[ 2 \int_0^{\infty} r dr (D_1D_2)^{1/2} \frac{Z_2'D}{D_1D_2} \]

\[ = \text{b.c.} + 2 \int_0^{\infty} r dr (D_1D_2)^{1/2} \left( - \frac{Z_2'D}{rD_1D_2} + \frac{Z_1'Z_2}{D_1D_2} + \frac{(Z_2')^2Z_1}{D_1D_2} \right. \]

\[ \left. - DZ_2' \left( \frac{Z_1'Z_1}{D_1^2D_2} + \frac{Z_2'Z_2}{D_1D_2^2} \right) \right) \] (B.21)

yielding

\[ \mathcal{H} = \text{b.c.} + \int_0^{\infty} r dr (D_1D_2)^{1/2} \left( \frac{-Z_1'}{rD_1} - \frac{2Z_2'D}{rD_1D_2} + \frac{3Z_2'}{rD_2} \right. \]

\[ + 3Z_1'Z_2'Z_2 + \frac{3(Z_2')^2Z_1}{D_1D_2} - \frac{2(Z_2')^2Z_2^2}{D_1D_2^2} + \frac{2Z_1'Z_2'Z_2^2Z_2}{D_1D_2^3} \right) \]

\[ = \text{b.c.} + \int_0^{\infty} r dr (D_1D_2)^{1/2} \left( \frac{-Z_1'}{rD_1} - \frac{2Z_2'D}{rD_1D_2} + \frac{3Z_2'}{rD_2} \right. \]

\[ + \frac{Z_1'Z_2}{D_1D_2} + \frac{3(Z_2')^2Z_1}{D_1D_2} - \frac{2(Z_2')^2Z_2^2}{D_1D_2^2} + \frac{2Z_1'Z_2^2Z_2}{D_1D_2^3} \right). \] (B.22)
We gather terms \( H = b.c. + \int dr (D_1^2 + D_2^2 + D_1D_2) \) in the last four terms of the integral

\[
\mathcal{H} = b.c. + \int dr(D_1D_2)^{1/2} \left( \frac{Z_1'}{D_1} - \frac{2Z_1'D_1}{D_1D_2} + \frac{3Z_1'(Z_1 - Z_2)Z_2}{D_1D_2} \right) + \frac{3Z_2'(Z_1 - Z_2)Z_1}{D_1D_2} - \frac{2Z_2'(Z_1 - Z_2)Z_2}{D_1D_2} + \frac{2Z_1'(Z_1 - Z_2)Z_2}{D_1D_2} \] (B.23)

Identity (B.18) allows us to simplify the last two terms of the above equation

\[
2 \int dr (Z_1 - Z_2) \left( \frac{Z_1'}{D_1} - \frac{Z_2'}{D_2} \right) = 2 \int dr \left( \frac{D}{\sqrt{D_1D_2}} \right)' = b.c. - \int dr \frac{2Z_1'D_1}{\sqrt{D_1D_2}} \] (B.24)

We gather terms \( \propto Z_1' \) and \( Z_2' \) and gain the remarkably simple result

\[
\mathcal{H} = - \int dr \frac{D}{\sqrt{D_1D_2}} (Z_1 + Z_2)' \] (B.25)

up to contributions from the boundary.

**Appendix C. Functional derivative of the Lagrangian**

We calculate the variation of

\[
\mathcal{L} = - \int dr \left( (D_2/D_1)^{1/2}Z_1' + (D_1/D_2)^{1/2}Z_2' \right) \] (C.1)

under an infinitesimal variation of the correlation function \( G(r) \to G(r) + \eta(r) \). We introduce the shorthand notations \( \partial_1 = \partial/\partial Z_1 \) and \( \partial_2 = \partial/\partial Z_2 \) and obtain

\[
\delta \mathcal{L} = - \int dr \left[ \left( (D_2/D_1)^{1/2}\eta'' + (D_1/D_2)^{1/2}\eta'/r \right) \right] + \left( \partial_1(D_2/D_1)^{1/2}\eta'' + \partial_2(D_2/D_1)^{1/2}\eta'/r \right) Z_1' + \left( \partial_1(D_1/D_2)^{1/2}\eta'' + \partial_2(D_1/D_2)^{1/2}\eta'/r \right) Z_2' \] (C.2)

\[
= b.c. + \int dr \left[ \left( \partial_1(D_2/D_1)^{1/2}Z_1' + \partial_2(D_2/D_1)^{1/2}Z_2' \right) \eta'' + \left( \partial_1(D_1/D_2)^{1/2}Z_1' + \partial_2(D_1/D_2)^{1/2}Z_2' \right) \eta'/r \right] - \int dr \left[ \left( \partial_1(D_2/D_1)^{1/2}\eta'' + \partial_2(D_2/D_1)^{1/2}\eta'/r \right) Z_1' + \left( \partial_1(D_1/D_2)^{1/2}\eta'' + \partial_2(D_1/D_2)^{1/2}\eta'/r \right) Z_2' \right] \] (C.3)

\[
= b.c. - \int dr \left[ \left( \partial_1(D_2/D_1)^{1/2} - \partial_2(D_2/D_1)^{1/2} \right) (Z_2'\eta'' - Z_1'\eta'/r) \right] = b.c. + \int dr Y (Z_2'\eta'' - Z_1'\eta'/r) \] (C.4)

where

\[
Y = \frac{Z_1 - Z_2}{(D_1D_2)^{1/2}}. \] (C.5)
The variation becomes

$$\delta L = \text{b.c.} - \int dr \eta' \left( (YZ_2')' + YZ_1'/r \right)$$

$$= \text{b.c.} + \int dr \eta' \left( 2Y(Z_1' - Z_2') + Y'(Z_1 - Z_2) \right)$$ \hspace{1cm} (C.6)

or

$$\frac{1}{r} \frac{\delta L}{\delta G} = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{Y'(Z_1 - Z_2)^2}{r(Z_1 - Z_2)} \right) = \frac{\psi'}{r} = (2\pi)^2 C(r).$$ \hspace{1cm} (C.7)

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