Sobolev, Hardy, Gagliardo–Nirenberg, and Caffarelli–Kohn–Nirenberg-type inequalities for some fractional derivatives

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Abstract
In this paper, we show different inequalities for fractional-order differential operators. In particular, the Sobolev, Hardy, Gagliardo–Nirenberg, and Caffarelli–Kohn–Nirenberg-type inequalities for the Caputo, Riemann–Liouville, and Hadamard derivatives are obtained. In addition, we show some applications of these inequalities.

Keywords Sobolev inequality · Hardy inequality · Gagliardo–Nirenberg inequality · Caffarelli–Kohn–Nirenberg inequality · fractional-order differential operator · Caputo derivative · Riemann–Liouville derivative · Hadamard derivative

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1 Introduction

There is no doubt that the inequalities not depending on a type of operators are very powerful for integral and differential equations. Without them, the progress of integro-differential equations would not be at its present level. Fractional-order differential operators are not an exception.

Let us recall some classical results. Let \( \Omega \subset \mathbb{R}^N \) be a measurable set and let \( 1 < p < N \), and then, the classical Sobolev inequality is formulated as:

\[
\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad u \in C_0^\infty(\Omega),
\]

(1.1)

where \( C = C(N, p) > 0 \) is a positive constant, \( p^* = \frac{Np}{N-p} \) and \( \nabla \) is the standard gradient in \( \mathbb{R}^N \). The inequality (1.1) is one of the most important tools in PDEs and variational problems.

Further generalizations of the Sobolev inequality were obtained by Gagliardo and Nirenberg, independently. In [10, 17], they independently from each other proved the interpolation inequality:

\[
\|u\|_{L^p(\mathbb{R}^N)}^p \leq C \|\nabla u\|_{L^p(\mathbb{R}^N)}^{N(p-2)/2} \|u\|_{L^{p^*}(\mathbb{R}^N)}^{(2p-N(p-2))/2}, \quad u \in H^1(\mathbb{R}^N),
\]

(1.2)

where

\[
\left\{ \begin{array}{l}
2 \leq p \leq \infty \text{ for } N = 2, \\
2 \leq p \leq \frac{2N}{N-2} \text{ for } N > 2.
\end{array} \right.
\]

Now, it is called the Gagliardo–Nirenberg inequality.

The next important generalization of the Sobolev inequality is the Caffarelli–Kohn–Nirenberg inequality. In 1984, Caffarelli, Kohn, and Nirenberg [7] established the following result:

**Theorem 1.1** Let \( N \geq 1 \). Assume that \( l_1, l_2, l_3, a, b, d, \delta \in \mathbb{R} \) be such that \( l_1, l_2 \geq 1, l_3 > 0, 0 \leq \delta \leq 1 \), and:

\[
\frac{1}{l_1} + \frac{a}{N}, \quad \frac{1}{l_2} + \frac{b}{N}, \quad \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} > 0.
\]

Then:

\[
\|x^{\delta d + (1-\delta)b} u\|_{L^5(\mathbb{R}^N)} \leq C \|x^{a \nabla u}\|_{L^5(\mathbb{R}^N)}^{\delta} \|x^{b \nabla u}\|_{L^5(\mathbb{R}^N)}^{1-\delta}, \quad u \in C_c^\infty(\mathbb{R}^N),
\]

(1.3)

if and only if

\[
\frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \delta \left( \frac{1}{l_1} + \frac{a-1}{N} \right) + (1-\delta) \left( \frac{1}{l_2} + \frac{b}{N} \right),
\]

\[
a - d \geq 0, \quad \text{if } \delta > 0,
\]

\[
a - d \leq 1, \quad \text{if } \delta > 0 \text{ and } \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \frac{1}{l_1} + \frac{a-1}{N},
\]
where $C$ is a positive constant independent of $u$.

Recently, mathematicians started to develop the classical inequalities (1.1), (1.2), and (1.3) for the $p$-Laplacian operator. In [9], Nezza, Palatucci, and Valdinoci obtained the $p$-Laplacian version of the Sobolev inequality:

$$
\|u\|_{L^p(\mathbb{R}^N)} \leq C[u]_{s,p},
$$

(1.4)

for the parameters $N > sp$, $1 < p < \infty$, and $s \in (0, 1)$, for any measurable and compactly supported function $u$. Here, $C = C(N, p, s) > 0$ is a suitable constant, and $[u]_{s,p}$ defined by:

$$
[u]^p_{s,p} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy,
$$

is the Gagliardo seminorm and $p^* = \frac{Np}{N-sp}$.

Using different techniques, the authors of the papers [8, 18, 19] proved the Gagliardo–Nirenberg inequality for the $p$–Laplacian operator:

$$
\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C[u]^a_{s,p} \|u\|_{L^p(\mathbb{R}^N)}^{1-a} \quad \forall \, u \in C^1_c(\mathbb{R}^N),
$$

(1.5)

for $N \geq 1$, $s \in (0, 1)$, $p > 1$, $a \geq 1$, $\tau > 0$, and $a \in (0, 1]$, such that:

$$
\frac{1}{\tau} = a \left( \frac{1}{p} - \frac{s}{N} \right) + \frac{1-a}{\alpha}.
$$

In [13, 14], Hughes derived a Hardy–Landau–Littlewood inequality [12] for the Riemann–Liouville fractional integral, then for the Riemann–Liouville fractional derivatives in weighted $L^p$ spaces. For more information about inequalities related to the fractional-order operators, the reader is referred to [5] and references therein.

In this paper, we deal with new inequalities related to some fractional-order differential operators. Especially, the Caputo derivative analogues of the above inequalities are in the field of our interest. Here, we derive the generalizations of the classical Sobolev, Hardy, Gagliardo–Nirenberg, and Caffarelli–Kohn–Nirenberg inequalities. Note that, in this direction, systematic studies of different functional inequalities on general homogeneous (Lie) groups were initiated by the book [20].

Recently, more attention has been paid to the study of fractional analogues of known functional inequalities (see, e.g., [2–5, 15]). Also, we note that, in [2], the author considered Sobolev-type inequality for the Caputo and Riemann–Liouville derivatives of order $\alpha \geq 1$.

We start by compiling basic definitions of fractional differential operators.
2 Preliminaries

Let us recall the Riemann–Liouville fractional integrals and derivatives. Also, we give definitions of the Caputo fractional derivatives. In [16, p.394], the sequential differentiation was formulated in a way that we will use in the further investigations. We refer to [16, 21] and references therein for further properties.

**Definition 2.1** The left Riemann–Liouville fractional integral \( I_{a+}^\alpha \) of order \( \alpha > 0 \), and derivative \( D_{a+}^\alpha \) of order \( 0 < \alpha \leq 1 \) are given by:

\[
I_{a+}^\alpha [f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t \in (a, b],
\]

and

\[
D_{a+}^\alpha [f](t) = \frac{d}{dt} I_{a+}^{1-\alpha} [f](t), \quad t \in (a, b],
\]

respectively, and \( f \in AC[a, b] \). Here, \( \Gamma \) denotes the Euler gamma function.

Since \( I_{a+}^{\alpha} f(t) \to f(t) \) almost everywhere as \( \alpha \to 0 \), then, by definition, we suppose that \( I_{a+}^{0} f(t) = f(t) \). Hence, \( D_{a+}^{1} f(t) = f'(t) \).

**Definition 2.2** The left Caputo fractional derivative of order \( 0 < \alpha \leq 1 \) is given by:

\[
\partial_{a+}^\alpha [f](t) = D_{a+}^\alpha [f(t) - f(a)] = I_{a+}^{1-\alpha} f'(t), \quad t \in (a, b].
\]

**Proposition 2.1** In Definition 2.2, if \( f(a) = 0 \), then \( \partial_{a+}^\alpha = D_{a+}^\alpha \).

**Proposition 2.2** If \( f \in L^1([a, b]) \) and \( \alpha > 0, \beta > 0 \), then the following equality holds

\[
I_{a+}^\alpha p_{a+}^\beta f(t) = p_{a+}^{\alpha+\beta} f(t).
\]

**Proposition 2.3** ([16]) If \( f \in L^1([a, b]) \) and \( f' \in L^1([a, b]) \), then the equality

\[
I_{a}^{\alpha} \partial_{a+}^\alpha f(t) = f(t) - f(a), \quad 0 < \alpha \leq 1,
\]

holds almost everywhere on \([a, b] \).

3 The main results

In this section, we derive the main results of this paper.

**Remark 3.1** We note that, in all statements of this section, we will work with the Caputo fractional derivative \( \partial_{a+}^\alpha \). However, analogous results can be easily obtained...
for the Riemann–Liouville derivative $D_{a+}^\alpha$ with the same order $\alpha \leq 1$ by adopting the techniques in the proofs and taking into account Property 2.1.

### 3.1 Poincaré–Sobolev-type inequality

In this subsection, we show the Poincaré–Sobolev-type inequality for fractional-order operators.

**Theorem 3.1** Let $u \in L^p(a, b)$, $u(a) = 0$, $\partial_{a+}^\alpha u \in L^p(a, b)$ and $p > 1$. Then, for the Caputo fractional derivative $\partial_{a+}^\alpha$ of order $\alpha \in \left(\frac{1}{p}, 1\right]$, we have the inequality:

$$
\|u\|_{L^\infty(a, b)} \leq \frac{(b-a)^{\frac{1}{p}-\frac{1}{p}}}{(ap - \frac{1}{p-1})} \left\|\partial_{a+}^\alpha u\right\|_{L^p(a, b)}.
$$

**Proof** Let $u \in L^p(a, b)$, $u(a) = 0$, $\partial_{a+}^\alpha u \in L^p(a, b)$, and consider the function:

$$
u(t) = I_{a+}^\alpha \partial_{a+}^\alpha u(t).$$

Using the Hölder inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we obtain:

$$
\left|I_{a+}^\alpha \partial_{a+}^\alpha u(t)\right| \leq \frac{1}{\Gamma(\alpha)} \int_a^t \left|(t-s)^{\alpha-1} \partial_{a+}^\alpha u(s)\right| ds
$$

$$
\leq \frac{1}{\Gamma(\alpha)} \left\{ \int_a^t (t-s)^{\alpha q - q} ds \right\} \left( \int_a^t \left| \partial_{a+}^\alpha u(s)\right|^p ds \right)^{\frac{1}{p}}
$$

$$
= \frac{(b-a)^{-\frac{1}{p}}}{(\alpha q - q + 1)^{\frac{1}{q}} \Gamma(\alpha)} \left( \int_a^t \left| \partial_{a+}^\alpha u(s)\right|^p ds \right)^{\frac{1}{p}}
$$

$$
\leq \frac{(b-a)^{\alpha - \frac{1}{p}}}{(\alpha q - q + 1)^{\frac{1}{q}} \Gamma(\alpha)} \left\| \partial_{a+}^\alpha u\right\|_{L^p(a, b)}
$$

$$
= \frac{\left(\frac{ap}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}} \Gamma(\alpha)}{(\alpha q - q + 1)^{\frac{1}{q}} \Gamma(\alpha)} \left\| \partial_{a+}^\alpha u\right\|_{L^p(a, b)}.
$$

where $q = \frac{p}{p-1} > 1$.

Then:
\[
\|u\|_{L^\infty(a,b)} = \| I_{a+}^\alpha\, \partial_{a+}^\alpha u\|_{L^\infty(a,b)} \leq \frac{(b-a)^{\frac{1}{p}-1}}{\left(\frac{ap}{p-1} - \frac{1}{p}\right)^{\frac{p-1}{p}}} \| \partial_{a+}^\alpha u\|_{L^p(a,b)},
\]
(3.3)

showing (3.1).

\[\square\]

**Remark 3.2** In Theorem 3.1, by taking \(1 < q < \infty\), we obtain:

\[
\|u\|_{L^q(a,b)} \leq \frac{(b-a)^{a-\frac{1}{p}+\frac{1}{q}}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}} \| \partial_{a+}^\alpha u\|_{L^p(a,b)},
\]
(3.4)

Let us also present the following result.

**Theorem 3.2** Let \(\partial_{a+}^\alpha u \in L^p(a,b)\) with \(p > 1\) and let \(\beta \in [0,1)\) be such that \(\alpha \in \left(\beta + \frac{1}{p}, 1\right]\). Then, for the Caputo fractional derivative \(\partial_{a+}^\beta\), we have:

\[
\|\partial_{a+}^\beta u\|_{L^\infty(a,b)} \leq \frac{(b-a)^{\alpha-\beta-\frac{1}{p}+\frac{1}{q}}}{\left(\alpha q - \beta q - q + 1\right)^\frac{1}{q} \Gamma(\alpha - \beta)} \| \partial_{a+}^\alpha u\|_{L^p(a,b)},
\]
(3.5)

for all \(1 < p \leq q < \infty\), where \(\frac{1}{p} + \frac{1}{q} = 1\).

**Proof** Using Definition 2.2 and Properties 2.2 and 2.3, we introduce the function:

\[
\partial_{a+}^\beta u(t) = I_{a+}^{1-\beta} u'(t) = I_{a+}^{\alpha-\beta} I_{a+}^{1-a} u'(t) = I_{a+}^{\alpha-\beta} \partial_{a+}^\alpha u(t).
\]
(3.6)

Using the Hölder inequality with \(\frac{1}{p} + \frac{1}{q} = 1\), we get:

\[
\left| I_{a+}^{\alpha-\beta} \partial_{a+}^\alpha u(t) \right| \leq \frac{1}{\Gamma(\alpha - \beta)} \int_a^t (t-s)^{\alpha-\beta-1} \partial_{a+}^\alpha u(s) \, ds
\]

\[
\leq \frac{1}{\Gamma(\alpha - \beta)} \left( \int_a^t (t-s)^{\alpha q - \beta q - q} \, ds \right)^{\frac{1}{q}} \left( \int_a^t \left| \partial_{a+}^\alpha u(s) \right|^p \, ds \right)^{\frac{1}{p}}
\]

\[
= \frac{(t-a)^{\alpha-\beta-\frac{1}{q}+\frac{1}{q}}}{(\alpha q - \beta q - q + 1)^\frac{1}{q} \Gamma(\alpha - \beta)} \left( \int_a^t \left| \partial_{a+}^\alpha u(s) \right|^p \, ds \right)^{\frac{1}{p}}
\]

\[
= \frac{(t-a)^{\alpha-\beta-\frac{1}{p}}}{(\alpha q - \beta q - q + 1)^\frac{1}{q} \Gamma(\alpha - \beta)} \left( \int_a^t \left| \partial_{a+}^\alpha u(s) \right|^p \, ds \right)^{\frac{1}{p}}
\]

\[
\leq \frac{(b-a)^{\alpha-\beta-\frac{1}{p}}}{(\alpha q - \beta q - q + 1)^\frac{1}{q} \Gamma(\alpha - \beta)} \left( \int_a^t \left| \partial_{a+}^\alpha u(s) \right|^p \, ds \right)^{\frac{1}{p}}
\]

\[
\| \partial_{a+}^\alpha u\|_{L^p(a,b)},
\]
where by assumption $\alpha > \beta + \frac{1}{p}$, we have $\alpha q - \beta q - q + 1 > 0$. From this, we obtain:
\[
\|\partial_{a+}^{\beta} u\|_{L^q(a,b)} \leq \frac{(b-a)^{\alpha-\beta-q-1}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}} \|\partial_{a+}^\alpha u\|_{L^p(a,b)},
\]
showing (3.5).

**Remark 3.3** In (3.5), if $\beta = 0$, we obtain the Sobolev-type inequality.

**Remark 3.4** In Theorem 3.2, by taking $1 < q < \infty$, we get:
\[
\|\partial_{a+}^{\beta} u\|_{L^q(a,b)} \leq \frac{(b-a)^{\alpha-\beta-q-1}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}} \|\partial_{a+}^\alpha u\|_{L^p(a,b)},
\]

### 3.2 Hardy-type inequality

Let us show the Hardy inequality.

**Theorem 3.3** Let $\alpha > 0$, $u(a) = 0$ and $\partial_{a+}^\alpha u \in L^p(a,b)$ with $p > 1$ and $\alpha \in \left(\frac{1}{p}, 1\right]$. Then, for the Caputo fractional derivative $\partial_{a+}^\alpha$, we have the inequality:
\[
\left\|\frac{u}{x}\right\|_{L^p(a,b)} \leq \frac{a^{-1}(b-a)^\alpha}{\left(\frac{ap}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}} \|\partial_{a+}^\alpha u\|_{L^p(a,b)},
\]

**Proof** From $a < x < b$, we have $\frac{1}{b} < \frac{1}{x} < \frac{1}{a}$. By using Theorem 3.1, we calculate:
\[
\left(\int_a^b \frac{|u(x)|^p}{x^p} dx\right)^{\frac{1}{p}} \leq a^{-1}\|u\|_{L^p(a,b)} \leq \frac{a^{-1}(b-a)^\alpha}{\left(\frac{ap}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}} \|\partial_{a+}^\alpha u\|_{L^p(a,b)},
\]
showing (3.9).

Let us give the weighted one-dimensional Hardy-type inequality.

**Theorem 3.4** Let $a > 0$, $u \in L^p(a,b)$, $u(a) = 0$ and $\partial_{a+}^\alpha u \in L^p(a,b)$ with $p > 1$ and $\alpha \in \left(\frac{1}{p}, 1\right]$. Then, for the Caputo fractional derivative $\partial_{a+}^\alpha$ of order $\alpha$ and $\gamma \in \mathbb{R}$, we have:
\[ \left\| \frac{u}{x^{q+1}} \right\|_{L^p(a,b)} \leq \frac{a^{-|r| - 1} b^{|r|}(b - a)^{\frac{a}{p}}}{(aq - q + 1)^{\frac{1}{p}}} \left\| \frac{\partial_a^a u}{x^q} \right\|_{L^p(a,b)}, \tag{3.11} \]

where \( q = \frac{p}{p-1} \).

**Proof** We prove our statement in two stages, namely, when \( \gamma \geq 0 \) and \( \gamma < 0 \). First, let us study the case \( \gamma \geq 0 \). For \( a > 0 \), we have \( b^{-\gamma - 1} < x^{-\gamma - 1} < a^{-\gamma - 1} \), so that:

\[
\left( \int_a^b \frac{|u(x)|^p}{x^{(\gamma + 1)p}} dx \right)^{\frac{1}{p}} \leq a^{-\gamma - 1} \left( \int_a^b |u(x)|^p dx \right)^{\frac{1}{p}} \leq \left( \int_a^b \left| \frac{\partial_a^a u}{x^{\gamma + 1}} \right|^p dx \right)^{\frac{1}{p}}, \tag{3.12}
\]

\[
= \frac{a^{-\gamma - 1} (b - a)^{\frac{a}{p}}}{\frac{a p}{p-1} - 1} \left( \int_a^b \left| \frac{\partial_a^a u}{x^{\gamma + 1}} \right|^p dx \right)^{\frac{1}{p}}
\]

\[
= \frac{a^{-\gamma - 1} b^r (b - a)^{\frac{a}{p}}}{\frac{a p}{p-1} - 1} \left( \int_a^b \frac{x^{\gamma + 1}}{x^{\gamma + 1}} \left| \frac{\partial_a^a u}{x^{\gamma + 1}} \right|^p dx \right)^{\frac{1}{p}}
\]

To show the case \( \gamma < 0 \), one obtains:
\[
\left( \int_a^b \frac{|u(x)|^p}{x^{(r+1)p}} \, dx \right)^\frac{1}{p} = \left( \int_a^b \frac{|u(x)|^p}{x^{(r+p)p}} \, dx \right)^\frac{1}{p} \leq b^{-r} \left( \int_a^b \frac{|u(x)|^p}{x^p} \, dx \right)^\frac{1}{p} \\
\leq \frac{1}{\alpha} b^{-r} (b-a)^\alpha \left( \int_a^b \frac{\partial^\alpha u}{x^p} \, dx \right)^\frac{1}{p} = \frac{1}{\alpha} b^{-r} (b-a)^\alpha \left( \int_a^b \frac{\partial^\alpha u}{x^p} \, dx \right)^\frac{1}{p} (3.9)
\]

\[
= \frac{1}{\alpha} b^{-r} (b-a)^\alpha \left( \int_a^b \frac{\partial^\alpha u}{x^p} \, dx \right)^\frac{1}{p} (3.13)
\]

implying (3.11).

\[\square\]

### 3.3 Gagliardo–Nirenberg-type inequality

Now, we are on a way to establish the Gagliardo–Nirenberg inequality for differential operators of fractional orders. We show that the Sobolev-type inequality formulated in Theorem 3.2 implies a family of Gagliardo–Nirenberg inequalities.

**Theorem 3.5** Assume that \(1 \leq p, q < \infty, \alpha \in \left( \frac{1}{q}, 1 \right] \) and \(u(a) = 0\). Then, we have the following Gagliardo–Nirenberg-type inequality:

\[
\|u\|_{L^q(a,b)} \leq C \|\partial^\alpha_a u\|_{L^r(a,b)} \|u\|_{L^p(a,b)}^{1-s}, \quad (3.14)
\]

with

\[
\frac{\gamma s}{q} + \frac{\gamma (1-s)}{p} = 1, \quad (3.15)
\]

where \(s \in [0, 1]\).
Proof Using the Hölder inequality with $\frac{qs}{q} + \frac{(1-s)}{p} = 1$, we have:

$$
\int_a^b |u(x)|^q \, dx = \int_a^b |u(x)|^{qs} |u(x)|^{q(1-s)} \, dx \\
\leq \left( \int_a^b |u(x)|^{qs} \, dx \right)^{\frac{q}{qs}} \left( \int_a^b |u(x)|^p \, dx \right)^{\frac{q(1-s)}{p}}
$$

(3.16)

$$
(3.1) \quad \leq C\|\partial_a^s u\|_{L^s(a,b)} \|u\|_{L^p(a,b)}^{q(1-s)},
$$

showing (3.14).

Let us consider the space $H^s_+(a, b)$ with $\alpha \in \left(\frac{1}{2}, 1\right]$ of the following form:

$$
H^s_+(a, b) := \{ u \in L^2(a, b), \ \partial_a^s u \in L^2(a, b), \ u(a) = 0 \}.
$$

In particular case of Theorem 3.5, which is important for our further analysis, when $q = 2$ and $\alpha = 1$, one obtains the classical Gagliardo–Nirenberg inequality:

**Corollary 3.1** We have the following Gagliardo–Nirenberg-type inequality:

$$
\|u\|_{L^s(a,b)} \leq C\|u\|_{s}^{s} \|u\|_{L^p(a,b)}^{1-s},
$$

(3.17)

for $s \in [0, 1]$.

We also recall another more general special case of Theorem 3.5 with $q = 2$:

**Corollary 3.2** Let $\alpha \in \left(\frac{1}{2}, 1\right]$. Assume also that $1 \leq p < \infty$ and $s \in [0, 1]$. Then, we have the following Gagliardo–Nirenberg-type inequality:

$$
\|u\|_{L^s(a,b)} \leq \|u\|_{H^s_+(a,b)}^{s} \|u\|_{L^p(a,b)}^{1-s},
$$

(3.18)

for $\frac{1}{s} = \frac{s}{2} + \frac{1-s}{p}$.

### 3.4 Caffarelli–Kohn–Nirenberg-type inequality

Now, let us show the fractional Caffarelli–Kohn–Nirenberg-type inequality.

**Theorem 3.6** Assume that $a > 0$, $\alpha \in \left(1 - \frac{1}{q}, 1\right)$, $1 < p, q < \infty$, $0 < r < \infty$, and $p + q \geq r$. Let $\delta \in [0, 1] \cap \left[\frac{c-r}{r}, \frac{c}{r}\right]$ and $c, d, e \in \mathbb{R}$ with $\frac{\delta}{p} + \frac{1-\delta}{q} = \frac{1}{r}$, $c = \delta(d - 1) + e(1 - \delta)$ and $u(a) = 0$. If $1 + (d-1)p > 0$, then we have:

$$
\|x^\delta u\|_{L^r(a,b)} \leq C\|x^d \partial_a^s u\|_{L^p(a,b)} \|x^\delta u\|_{L^q(a,b)}^{1-\delta},
$$

(3.19)
**Proof** Case $\delta = 0$.

If $\delta = 0$, then $c = e$ and $q = r$. Then, (3.19) is the inequality:

$$\|x^\delta u\|_{L^r(a,b)} \leq \|x^\delta u\|_{L^r(a,b)}.$$

Case $\delta = 1$.

If $\delta = 1$, then we have $c = d - 1$ and $p = r$. Also, we have $1 + cp = 1 + (d - 1)p > 0$. Then, using weighted fractional Hardy inequality (Theorem 3.4), we obtain:

$$\|x^\delta u\|_{L^p(a,b)} \leq C \left( \|x^{\delta+1} \partial_a^\alpha u\|_{L^p(a,b)} \right) = C \left( \|x^{\delta} \partial_a^\alpha u\|_{L^p(a,b)} \right).$$

(3.20)

Case $\delta \in [0, 1] \cap \left[ \frac{q - r}{r}, \frac{p}{r} \right]$.

By assumption $c = \delta(d - 1) + e(1 - \delta)$ and using Hölder’s inequality with $\frac{\delta}{p} + \frac{1 - \delta}{q} = \frac{1}{r}$, we calculate:

$$\|x^\delta u\|_{L^r(a,b)} = \left( \int_a^b x^\delta |u(x)|^r \, dx \right)^{\frac{1}{r}} = \left( \int_a^b \left| \frac{u(x)}{x^\delta} \right|^r \left| \frac{u(x)}{x^\delta} \right|^{(1-\delta)r} \, dx \right)^{\frac{1}{r}} \leq \left\| \frac{u}{x^{1-d}} \right\|_{L^p(a,b)}^{\delta} \left\| \frac{u}{x^{e}} \right\|_{L^q(a,b)}^{1-\delta}. \quad (3.21)$$

Using weighted fractional Hardy inequality (Theorem 3.4) with $1 + (d - 1)p > 0$, we obtain:

$$\|x^\delta u\|_{L^p(a,b)} \leq \left\| \frac{u}{x^{1-d}} \right\|_{L^p(a,b)}^{\delta} \left\| \frac{u}{x^{e}} \right\|_{L^q(a,b)}^{1-\delta} \leq C \left\| x^d \partial_a^\alpha u \right\|_{L^p(a,b)}^{\delta} \|x^\delta u\|_{L^q(a,b)}^{1-\delta}, \quad (3.22)$$

completing the proof.

\[\square\]

### 4 Sequential derivation case

In this subsection, we collect results for the sequential derivatives. Indeed, these results are important due to the non–commutativity and the absence of the semigroup property of fractional differential operators.
4.1 Fractional Poincare–Sobolev-type inequality

**Theorem 4.1** Let $\partial_a^\beta u(a) = 0$, $\partial_a^\alpha \partial_a^\beta u \in L^p(a, b)$ with $\alpha \in \left(\frac{1}{q}, 1\right)$ and $\beta \in (0, 1)$. Then, the following inequality is true:

$$\|\partial_a^\beta u\|_{L^\infty(a, b)} \leq \frac{(b - a)^{\frac{1}{p} - \frac{1}{q}}}{(\alpha q - q + 1)^\frac{1}{q} \Gamma(\alpha)} \|\partial_a^\alpha \partial_a^\beta u\|_{L^p(a, b)},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof** Consider the function:

$$\partial_a^\beta u(t) = \Gamma_a^{\alpha} \partial_a^\alpha \partial_a^\beta u(t).$$

Using the Hölder inequality, one has:

$$\left|\Gamma_a^{\alpha} \partial_a^\alpha \partial_a^\beta u(t)\right| \leq \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} \partial_a^\alpha \partial_a^\beta u(s) ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^t (t - s)^{\alpha q - q} ds\right)^\frac{1}{q} \left(\int_a^t \|\partial_a^\alpha \partial_a^\beta u(s)\|^p ds\right)^\frac{1}{p}$$

$$= \frac{(t - a)^{\alpha - 1 + \frac{1}{q}}}{(\alpha q - q + 1)^\frac{1}{q} \Gamma(\alpha)} \left(\int_a^t \|\partial_a^\alpha \partial_a^\beta u(s)\|^p ds\right)^\frac{1}{p}$$

$$\leq \frac{(b - a)^{\alpha - 1 + \frac{1}{q}}}{(\alpha q - q + 1)^\frac{1}{q} \Gamma(\alpha)} \left\|\partial_a^\alpha \partial_a^\beta u\right\|_{L^p(a, b)}.$$}

Then, we obtain:

$$\|\partial_a^\beta u\|_{L^\infty(a, b)} \leq \frac{(b - a)^{\frac{1}{p} - \frac{1}{q}}}{(\alpha q - q + 1)^\frac{1}{q} \Gamma(\alpha)} \left\|\partial_a^\alpha \partial_a^\beta u\right\|_{L^p(a, b)},$$

completing proof. \(\square\)

**Remark 4.1** If $1 < \theta < \infty$ in Theorem 4.1, then we have:

$$\|\partial_a^\beta u\|_{L^\theta(a, b)} \leq \frac{(b - a)^{\frac{1}{p} - \frac{1}{q} - \frac{1}{\theta}}}{(\alpha q - q + 1)^\frac{1}{q} \Gamma(\alpha)} \left\|\partial_a^\alpha \partial_a^\beta u\right\|_{L^p(a, b)}.$$
4.2 Fractional Hardy-type inequality

Now, we show the following sequential fractional Hardy inequality.

**Theorem 4.2** Let \( a > 0, \gamma \in \mathbb{R} \) and \( \partial^\beta_{a+} u(a) = 0 \) and \( \partial^a_{a+} \partial^\beta_{a+} u \in L^p(a, b) \) with \( \alpha \in \left( \frac{1}{q}, 1 \right) \). Then, the following inequality is true:

\[
\left\| \partial^\beta_{a+} u \right\|_{L^p(a,b)} \leq C \left\| \partial^a_{a+} \partial^\beta_{a+} u \right\|_{L^p(a,b)},
\]

with \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof** From \( a < x < b \), we have \( \frac{1}{b} < \frac{1}{x} < \frac{1}{a} \). Using Theorem 4.1, we calculate:

\[
\left( \int_a^b \frac{\left| \partial^\beta_{a+} u(x) \right|^p}{x^{\frac{1}{p}}} \right)^{\frac{1}{p}} = \left( \int_a^b x^{-p} \left| \partial^\beta_{a+} u(x) \right|^p dx \right)^{\frac{1}{p}} \\
\leq a^{-1} \left\| \partial^a_{a+} u \right\|_{L^p(a,b)}^{\frac{1}{p}} \tag{4.4}
\]

\[
\leq \frac{a^{-1}(b - a)^{\frac{1}{p}}}{(\alpha q - q + 1)^{\frac{1}{p}} \Gamma(\alpha)} \left\| \partial^a_{a+} \partial^\beta_{a+} u \right\|_{L^p(a,b)},
\]

showing (4.3). \( \square \)

4.3 Fractional Gagliardo–Nirenberg-type inequality

In the same way as Theorem 3.5 is proved, we can prove the following statement.

**Theorem 4.3** Assume that \( 1 \leq p, q < \infty \), and let \( \alpha \in (0, 1) \) be such that \( \beta \in \left( \frac{1}{q}, 1 \right) \).

Suppose that \( \partial^a_{a+} \partial^\beta_{a+} u \in L^q(a, b) \) and \( \partial^a_{a+} u \in L^p(a, b) \). Then, we have the following Gagliardo–Nirenberg-type inequality:

\[
\int_a^b \left| \partial^a_{a+} u(x) \right|^q dx \leq \left( \int_a^b \left| \partial^\beta_{a+} \partial^a_{a+} u(x) \right|^q dx \right)^{\frac{1}{q}} \left( \int_a^b \left| \partial^a_{a+} u(x) \right|^p dx \right)^{\left( \frac{1}{p} - s \right)}, \tag{4.5}
\]

with

\[
\frac{s q}{p} + \left( 1 - s \right) \frac{q}{p} = 1, \tag{4.6}
\]

where \( s \in [0, 1] \).

**Proof** Let us calculate the following integral:
\[
\int_a^b |\partial_{a^+} u(x)|^\gamma \, dx = \int_a^b |\partial_{a^+} u(x)|^\gamma |\partial_{a+} u(x)|^{(1-\gamma)} \, dx \\
\leq \left( \int_a^b |\partial_{a+} u(x)|^q \, dx \right)^{\frac{\gamma}{q}} \left( \int_a^b |\partial_{a+} u(x)|^p \, dx \right)^{\frac{(1-\gamma)p}{p}},
\]
with
\[
\frac{sq}{q} + \frac{(1-s)p}{p} = 1.
\]

Then, using Theorem 4.1, we obtain:
\[
\int_a^b |\partial_{a^+} u(x)|^\gamma \, dx \leq \left( \int_a^b |\partial_{a^+} u(x)|^q \, dx \right)^{\frac{\gamma}{q}} \left( \int_a^b |\partial_{a+} u(x)|^p \, dx \right)^{\frac{(1-\gamma)p}{p}} \\
\leq C \left( \int_a^b |\partial_{a^+} \partial_{a+} u(x)|^q \, dx \right)^{\frac{\gamma}{q}} \left( \int_a^b |\partial_{a+} u(x)|^p \, dx \right)^{\frac{(1-\gamma)p}{p}}.
\]

The theorem is proved. \(\square\)

## 5 Hadamard fractional derivative

Let us give the definition of the Hadamard fractional derivative.

**Definition 5.1** The left Hadamard fractional integral \(\mathfrak{I}_{a+}^\alpha\) of order \(\alpha > 0\) and derivative \(\mathfrak{D}_{a+}^\alpha\) of order \(0 < \alpha < 1\) are given by:

\[
\mathfrak{I}_{a+}^\alpha [f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left( \frac{t}{s} \right)^{\alpha-1} f(s) \frac{ds}{s}, \quad t \in (a, b],
\]

and

\[
\mathfrak{D}_{a+}^\alpha [f](t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left( \frac{t}{s} \right)^{-\alpha} f'(s) \frac{ds}{s}. \quad t \in (a, b].
\]

Here, \(\Gamma\) denotes the Euler gamma function.

**Proposition 5.1** ([16]) If \(f \in L^1(a, b)\) and \(f' \in L^1_a(a, b)\), then the equality
\[
\mathfrak{I}_{a+}^\alpha \mathfrak{D}_{a+}^\alpha f(t) = f(t) - f(a), \quad 0 < \alpha < 1,
\]
holds almost everywhere on \([a, b]\).
Now, for $p \geq 1$, we define the weighted Lebesgue space $L^p_{\frac{1}{2}}(a, b)$ with the norm:

$$
\| u \|_{L^p_{\frac{1}{2}}(a, b)} := \left( \int_a^b |u(x)|^p \frac{dx}{x} \right)^{\frac{1}{p}}.
$$

(5.1)

For our further purpose, we will need the following property of the weighted space $L^p_{\frac{1}{2}}(a, b)$.

**Proposition 5.2** ([16]) Suppose that $f \in L^1_{\frac{1}{2}}(a, b)$. Then, for the parameters $\alpha > 0$ and $\beta > 0$, we have the following equality:

$$
\mathcal{F}^\alpha_a \mathcal{F}^\beta_a f(t) = \mathcal{F}^{\alpha+\beta}_{a+} f(t),
$$

for almost all $t \in (a, b)$.

### 5.1 Poincaré–Sobolev-type inequality

In this subsection, we show the fractional-order Poincaré–Sobolev-type inequality.

**Theorem 5.1** Let $a > 0$ and $p > 1$. Assume that $u \in L^p(a, b)$ and $\mathcal{D}^\alpha_{a+} u \in L^p_{\frac{1}{2}}(a, b)$ with $u(a) = 0$. Then, for the Hadamard fractional derivative $\mathcal{D}^\alpha_{a+}$ of order $\alpha \in \left( \frac{1}{p}, 1 \right]$, we have:

$$
\| u \|_{L^\infty(a, b)} \leq \frac{\log \left( \frac{b}{a} \right)^{\frac{a-1}{p}}}{\left( \frac{ap}{p-1} - \frac{1}{p-1} \right)^{\frac{p-1}{p}} \Gamma(\alpha)} \| \mathcal{D}^\alpha_{a+} u \|_{L^p_{\frac{1}{2}}(a, b)}.
$$

(5.2)

**Proof** Let $u \in L^p_{\frac{1}{2}}(a, b)$, $u(a) = 0$, $\mathcal{D}^\alpha_{a+} u \in L^p(a, b)$ and consider the function:

$$
u(t) = \mathcal{F}^\alpha_{a+} \mathcal{F}^\alpha_{a+} u(t).
$$

(5.3)

Using the Hölder inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we obtain:
\[
\left| \mathfrak{I}_{a+}^{\alpha} \mathfrak{D}_{a+}^{\alpha} u(t) \right| \leq \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left( \log \frac{t}{s} \right)^{\alpha-1} \left| \mathfrak{D}_{a+}^{\alpha} u(s) \right| \frac{ds}{s^{\frac{1}{p} + \frac{1}{q}}} \\
\leq \frac{1}{\Gamma(\alpha)} \left( \int_{a}^{t} \left| \log \frac{t}{s} \right|^{a_q - \frac{1}{q}} \frac{ds}{s} \right)^{\frac{1}{q}} \left( \int_{a}^{t} \left| \mathfrak{D}_{a+}^{\alpha} u(s) \right|^{p} \frac{ds}{s} \right)^{\frac{1}{p}} \\
= \frac{1}{\Gamma(\alpha)} \left( a_q - q + 1 \right)^{\frac{1}{q}} \int_{a}^{t} \left( a_q - q + 1 \right)^{\frac{1}{q}} \left( \int_{a}^{t} \left| \mathfrak{D}_{a+}^{\alpha} u(s) \right|^{p} \frac{ds}{s} \right)^{\frac{1}{p}} \\
= \frac{1}{\Gamma(\alpha)} \left( a_q - q + 1 \right)^{\frac{1}{q}} \left( \int_{a}^{t} \left| \mathfrak{D}_{a+}^{\alpha} u(s) \right|^{p} \frac{ds}{s} \right)^{\frac{1}{p}}.
\]

where \( q = \frac{p}{p-1} > 1 \), showing (5.2). \( \square \)

**Remark 5.1** In Theorem 5.1, by taking \( 1 < \theta < \infty \), we have:

\[
\| u \|_{L^p(a,b)} \leq \frac{b - a}{\left( \frac{ap}{p-1} - \frac{1}{p-1} \right)^{\frac{p-1}{p}}} \left( \int_{a}^{t} \left| \mathfrak{D}_{a+}^{\alpha} u(s) \right|^{p} \frac{ds}{s} \right)^{\frac{1}{p}} \left( \frac{b - a}{\left( \frac{ap}{p-1} - \frac{1}{p-1} \right)^{\frac{p-1}{p}}} \Gamma(\alpha) \right) \left( \int_{a}^{t} \left| \mathfrak{D}_{a+}^{\alpha} u(s) \right|^{p} \frac{ds}{s} \right)^{\frac{1}{p}}.
\] (5.4)

### 5.2 Hardy-type inequality

Here, we show the Hardy inequality for the Hadamard derivative.

**Theorem 5.2** Let \( a > 0 \) and \( p > 1 \). Assume that \( \mathfrak{D}_{a+}^{\alpha} u \in L^p_{\frac{1}{p}}(a,b) \) and \( u(a) = 0 \). Then, for the Hadamard fractional derivative \( \mathfrak{D}_{a+}^{\alpha} \) of order \( \alpha \in \left( \frac{1}{p}, 1 \right) \), we have:
\[
\left\| \frac{u}{x} \right\|_{L^p(a,b)} \leq \frac{a^{-1}(b-a)^{\frac{1}{p}}}{\left( \frac{ap}{p-1} - \frac{1}{p-1} \right) \Gamma(\alpha)} \left\| \mathfrak{D}^a_{a+} u \right\|_{L^p_{\frac{1}{2}}(a,b)}^q. \quad (5.5)
\]

**Proof** From \( a < x < b \), we have \( \frac{1}{b} < \frac{1}{x} < \frac{1}{a} \). Using Theorem 5.1, we calculate:

\[
\left( \int_a^b \frac{|u(\xi)|^p}{x^p} d\xi \right)^{\frac{1}{p}} \leq \left( \int_a^b x^{-p}|u(\xi)|^p d\xi \right)^{\frac{1}{p}} \leq a^{-1}\|u\|_{L^p(a,b)}
\]

\[
(5.6)
\]

\[
\leq \frac{a^{-1}(b-a)^{\frac{1}{p}}}{\left( \frac{ap}{p-1} - \frac{1}{p-1} \right) \Gamma(\alpha)} \left\| \mathfrak{D}^a_{a+} u \right\|_{L^p_{\frac{1}{2}}(a,b)},
\]

showing (5.5). \( \square \)

Let us show the weighted Hardy inequality with the Hadamard derivative.

**Theorem 5.3** Let \( a > 0 \), \( u(a) = 0 \) and \( \mathfrak{D}^a_{a+} u \in L^p_{\frac{1}{2}}(a, b) \) with \( p > 1 \). Then, for the Hadamard fractional derivative \( \mathfrak{D}^a_{a+} \) of order \( \alpha \in \left( \frac{1}{p}, 1 \right] \) and \( \gamma \in \mathbb{R} \), we have inequality:

\[
\left\| \frac{u}{x^{\gamma+1}} \right\|_{L^p(a,b)} \leq C \left\| \mathfrak{D}^a_{a+} u \right\|_{L^p_{\frac{1}{2}}(a,b)}. \quad (5.7)
\]

**Proof** We prove this result in two steps. Let us first show the case \( \gamma \geq 0 \). Since \( a > 0 \), we have \( b^{-\gamma-1} < x^{-\gamma-1} < a^{-\gamma-1} \), and by the direct calculations, one obtains:
\[
\left( \int_a^b \frac{|u(x)|^p}{x^{(\gamma+1)p}} \, dx \right)^{\frac{1}{p}} \leq a^{-\gamma-1} \left( \int_a^b |u(x)|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}} \log \frac{b}{a}^{\frac{\alpha-1}{p}}}{\left( \frac{ap}{p-1} - \frac{1}{p-1} \right)^{\frac{p-1}{p}} \Gamma(\alpha)} \left( \int_a^b |\mathcal{D}_a u|^p \, dx \right)^{\frac{1}{p}}
\]

\[
= \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}} \log \frac{b}{a}^{\frac{\alpha-1}{p}}}{\left( \frac{ap}{p-1} - \frac{1}{p-1} \right)^{\frac{p-1}{p}} \Gamma(\alpha)} \left( \int_a^b \frac{|\mathcal{D}_a u|^p}{x^{\gamma p}} \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}} b^{\gamma} \log \frac{b}{a}^{\frac{\alpha-1}{p}}}{\left( \frac{ap}{p-1} - \frac{1}{p-1} \right)^{\frac{p-1}{p}} \Gamma(\alpha)} \left( \int_a^b \frac{|\mathcal{D}_a u|^p}{x^{\gamma p}} \, dx \right)^{\frac{1}{p}}
\]

\[
= \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}} b^{\gamma} \log \frac{b}{a}^{\frac{\alpha-1}{p}}}{\left( \frac{ap}{p-1} - \frac{1}{p-1} \right)^{\frac{p-1}{p}} \Gamma(\alpha)} \left\| \mathcal{D}_a u \right\|_{L^p(\frac{\gamma}{p},(a,b))}.
\]

Now, to prove the case \( \gamma < 0 \), we arrive at:
\[
\left(\int_a^b \frac{|u(x)|^p}{x^{(r+1)p}} \, dx\right)^{\frac{1}{p}} = \left(\int_a^b \frac{|u(x)|^p}{x^{(r+p)p}} \, dx\right)^{\frac{1}{p}} \\
\leq b^{-\gamma} \left(\int_a^b \frac{|u(x)|^p}{x^{p}} \, dx\right)^{\frac{1}{p}} \\
\leq b^{-\gamma} a^{-1} (b-a)^{\frac{1}{p}} \log \left(\frac{b}{a}\right)^{\frac{a-1}{p}} \left(\int_a^b \frac{|\mathcal{D}_a^s u|^p}{x} \, dx\right)^{\frac{1}{p}} \\
\leq \frac{b^{-\gamma} a^{-1} (b-a)^{\frac{1}{p}} \log \left(\frac{b}{a}\right)^{\frac{a-1}{p}}}{\left(\frac{ap}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}} \Gamma(\alpha) \\
\leq \frac{b^{-\gamma} a^{-1} (b-a)^{\frac{1}{p}} \log \left(\frac{b}{a}\right)^{\frac{a-1}{p}}}{\left(\frac{ap}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}} \left(\int_a^b \frac{|\mathcal{D}_a^s u|^p}{x} \, dx\right)^{\frac{1}{p}} \\
\leq \frac{b^{-\gamma} a^{-1} (b-a)^{\frac{1}{p}} \log \left(\frac{b}{a}\right)^{\frac{a-1}{p}}}{\left(\frac{ap}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}} \left(\int_a^b \frac{|\mathcal{D}_a^s u|^p}{x^{\gamma p}} \, dx\right)^{\frac{1}{p}} \\
\leq \frac{b^{-\gamma} a^{-1} (b-a)^{\frac{1}{p}} \log \left(\frac{b}{a}\right)^{\frac{a-1}{p}}}{\left(\frac{ap}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}} \Gamma(\alpha) \\
\leq b^{-\gamma} a^{-1} (b-a)^{\frac{1}{p}} \log \left(\frac{b}{a}\right)^{\frac{a-1}{p}} \left(\int_a^b \frac{|\mathcal{D}_a^s u|^p}{x^{\gamma p}} \, dx\right)^{\frac{1}{p}} \\
\leq \frac{b^{-\gamma} a^{-1} (b-a)^{\frac{1}{p}} \log \left(\frac{b}{a}\right)^{\frac{a-1}{p}}}{\left(\frac{ap}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}} \Gamma(\alpha) \\
= \frac{b^{-\gamma} a^{-1} (b-a)^{\frac{1}{p}} \log \left(\frac{b}{a}\right)^{\frac{a-1}{p}}}{\left(\frac{ap}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}} \left\| \mathcal{D}_a^s u \right\|_{L^p(a,b)} \left\| u \right\|_{L^p(a,b)}^{\frac{1}{p}} 
\]
Proof Using the Hölder inequality with \( \frac{rs}{q} + \frac{(1-s)}{p} = 1 \), we get:

\[
\int_a^b |u(x)|^r \, dx = \int_a^b |u(x)|^{rs} |u|^{r(1-s)} \, dx
\leq \left( \int_a^b |u(x)|^q \, dx \right)^{\frac{r}{q}} \left( \int_a^b |u(x)|^p \, dx \right)^{\frac{r(1-s)}{p}}
\]

(5.12)

\[
\leq C \|x^d \mathcal{D}^\alpha_{a+}u\|_{L^q_{\frac{a}{2},(a,b)}}^{rs} \|u\|_{L^p(a,b)}^{r(1-s)},
\]

completing the proof. \(\square\)

5.4 Fractional Caffarelli–Kohn–Nirenberg-type inequality with Hadamard derivative

Now, we are in a position to show the fractional Caffarelli–Kohn–Nirenberg-type inequality.

Theorem 5.5 Let \( a > 0 \), \( 1 < p, q < \infty \), \( \alpha \in \left( 1 - \frac{1}{q}, 1 \right) \), and \( 0 < r < \infty \), such that \( p + q \geq r \). Suppose that \( \delta \in [0, 1] \cap \left[ \frac{r-q}{r}, \frac{p}{r} \right] \) and \( c, d, e \in \mathbb{R} \) with \( \frac{\delta}{p} + \frac{1-\delta}{q} = \frac{1}{r} \) and \( c = \delta(d-1) + e(1-\delta) \). Assume that \( x^d \mathcal{D}^\alpha_{a+}u \in L^q_{\frac{a}{2},(a,b)} \), \( x^e u \in L^q(a,b) \) and \( u(a) = 0 \).

Moreover, let \( 1 + (d-1)p > 0 \), and then, we have \( x^e u \in L^r(a,b) \) and:

\[
\|x^e u\|_{L^r(a,b)} \leq C \|x^d \mathcal{D}^\alpha_{a+}u\|_{L^q_{\frac{a}{2},(a,b)}}^{\delta} \|x^e u\|_{L^q(a,b)}^{1-\delta}. \tag{5.13}
\]

Proof Case \( \delta = 0 \).

If \( \delta = 0 \), then \( c = e \) and \( q = r \). Then, (3.19) is the inequality:

\[
\|x^e u\|_{L^r(a,b)} \leq \|x^e u\|_{L^r(a,b)}.
\]

Case \( \delta = 1 \).

If \( \delta = 1 \), then we have \( c = d-1 \) and \( p = r \). Also, we have \( 1 + cp = 1 + (d-1)p > 0 \). Then, using weighted fractional Hardy inequality (Theorem 5.3), we obtain:
\[ \|x^\delta u\|_{L^p(a,b)} \leq C\|x^{\delta+1}D_a^\alpha u\|_{L^{p_1}(a,b)} \]
\[ = C\|x^\delta D_a^\alpha u\|_{L^{p_1}(a,b)}. \]  

(5.14)

Case \( \delta \in [0, 1] \cap \left[ \frac{r-q}{r}, \frac{p}{r} \right] \).

By assuming \( c = \delta(d-1) + e(1-\delta) \) and using the Hölder’s inequality with
\[ \frac{\delta}{p} + \frac{1-\delta}{q} = \frac{1}{r}, \]
we calculate:

\[ \|x^\delta u\|_{L^p(a,b)} = \left( \int_a^b x^\delta |u(x)|^r \, dx \right)^{\frac{1}{r}} \]
\[ = \left( \int_a^b \frac{|u(x)|^\delta |u(x)|^{(1-\delta)r}}{x^{\delta r(1-d)} - \epsilon^{(1-\delta)d}} \, dx \right)^{\frac{1}{r}} \]  

(5.15)

Using the weighted fractional Hardy inequality (Theorem 5.3) with \( 1 + (d-1)p > 0 \), we obtain:

\[ \|x^\delta u\|_{L^p(a,b)} \leq \left\| \frac{u}{x^{1-d}} \right\|_{L^p(a,b)}^{\delta} \left\| \frac{u}{x^{-e}} \right\|_{L^q(a,b)}^{1-\delta} \]  

(5.16)

showing (5.13).

\[ \square \]

6 Applications

In this section, we show some applications of the obtained inequalities for the real-valued functions \( u \).

6.1 Uncertainty principle

The inequality (3.9) implies the following uncertainty principle:

**Corollary 6.1** Let \( a > 0 \), \( u(a) = 0 \) and \( D_a^\alpha u \in L^p(a,b) \) with \( p > 1 \). Then, for the Caputo fractional derivative \( D_a^\alpha \) of order \( \alpha \in \left( \frac{1}{p}, 1 \right] \), we have the following inequality:
where \( q = \frac{p}{p-1} \).

**Proof** Using (3.9), we obtain:

\[
\frac{a^{-1}(b-a)^a}{\left(\frac{ap}{p-1} - \frac{1}{p-1}\right)^\frac{p-1}{p}} \left\| \partial_{a+}^\alpha u \right\|_{L^p(a,b)} \left\| xu \right\|_{L^q(a,b)},
\]

completing the proof. \(\square\)

**Remark 6.1** Also, the uncertainly principle holds for the Riemann–Liouville derivative.

Let us show uncertainly principle for the Hadamard derivative.

**Corollary 6.2** Let \( a > 0 \) and \( p > 1 \). Assume that \( \mathfrak{D}_{a+}^\alpha u \in L^p \left( \frac{a}{2}, b \right) \) and \( u(a) = 0 \). Then, for the Hadamard fractional derivative \( \mathfrak{D}_{a+}^\alpha \) of order \( \alpha \in \left( \frac{1}{p}, 1 \right) \), we have:

\[
\left\| u \right\|_{L^2(a,b)}^2 \leq \frac{a^{-1}(b-a)^a}{\left(\frac{ap}{p-1} - \frac{1}{p-1}\right)^\frac{p-1}{p}} \log \frac{b}{a} \left\| \mathfrak{D}_{a+}^\alpha u \right\|_{L^p \left( \frac{a}{2}, b \right)} \left\| xu \right\|_{L^q(a,b)},
\]

where \( q = \frac{p}{p-1} \).

**Proof** Proof is similar to Corollary 6.1 using Theorem 5.2. \(\square\)

### 6.2 Embedding of spaces

Let us consider the space \( H^\alpha_+(a,b) \) with \( \alpha \in \left( \frac{1}{2}, 1 \right] \) introduced in [6, 11] in the following form:

\[
H^\alpha_+(a,b) := \left\{ u \in L^2(a,b), \ \partial_{a+}^\alpha u \in L^2(a,b), \ u(a) = 0 \right\}.
\]

If \( \alpha < \beta \), then by the Poincaré–Sobolev-type inequality (3.1), we have \( H^\beta_+(a,b) \hookrightarrow H^\alpha_+(a,b) \).

Let us introduce the space \( W^\alpha_+ \) in the following form:

\[
W^\alpha_+(a,b) := \left\{ u \in L^2(a,b), \ \mathfrak{D}_{a+}^\alpha u \in L^2(a,b), \ u(a) = 0 \right\}.
\]
where $\mathcal{D}^a_{a+}$ is the left Hadamard derivative. If $\alpha < \beta$, then by the Poincaré–Sobolev-type inequality (5.2), we have: $\mathcal{W}^\beta_+ (a, b) \hookrightarrow \mathcal{W}^\alpha_+ (a, b)$.

### 6.3 A-priori estimate

Here, we seek a real-valued solution to the following space-fractional diffusion problem:

$$
\begin{align*}
\begin{cases}
  u_t(x, t) + D^a_{b-} \partial^a_{a+} u(x, t) &= 0, \quad (x, t) \in (a, b) \times (0, T), \\
  u(x, 0) &= u_0(x), \quad \forall x \in (a, b),
\end{cases}
\end{align*}
$$

where $\alpha \in \left( \frac{1}{2}, 1 \right)$, $u \in L^\infty(0, T; H^a_+(a, b))$, $u_t \in L^2(0, T; \dot{H}^a_+(a, b))$, and $u_0 \in L^2(a, b)$.

Now, we show an a-priori estimate for this problem. Let us define:

$$I(t) = \| u(x, \cdot) \|_{L^2(a, b)}^2 = \int_a^b |u(x, t)|^2 \, dx.$$  

Then, by multiplying (6.4) by $u$, integrating over $(a, b)$, and using integration by parts, we compute:

$$
\int_a^b u_t(x, t) u(x, t) \, dx + \int_a^b u(x, t) D^a_{b-} \partial^a_{a+} u(x, t) \, dx \\
= \frac{1}{2} \frac{d}{dt} \int_a^b |u(x, t)|^2 \, dx + \int_a^b |\partial^a_{a+} u(x, t)|^2 \, dx
\tag{6.5}
$$

Using (3.1) with $p = 2$ in (6.5), we get:

$$0 = \frac{1}{2} \frac{dI(t)}{dt} + \int_a^b |\partial^a_{a+} u(x, t)|^2 \, dx \geq \frac{1}{2} \frac{dI(t)}{dt} + \frac{(2\alpha - 1) \Gamma^2(\alpha)}{(b - \alpha)^{2\alpha}} \int_a^b |u(x, t)|^2 \, dx.$$ 

Consequently, we arrive at $\frac{dI(t)}{dt} \leq 0$. This means that $I(t)$ is a non-decreasing function. Then, for all $t > 0$, we have $I(t) \leq I(0)$. Thus:

$$\| u(x, \cdot) \|_{L^2(a, b)} \leq \| u_0 \|_{L^2(a, b)}.$$

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