Modified semiclassical approximation for trapped Bose gases

V.I. Yukalov

Institut f¨ur Theoretische Physik,
Freie Universit¨at Berlin, Arnimallee 14, D-14195 Berlin, Germany
and
Bogolubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, Dubna 141980, Russia

Abstract

A generalization of the semiclassical approximation is suggested allowing for an essential extension of its region of applicability. In particular, it becomes possible to describe Bose-Einstein condensation of a trapped gas in low-dimensional traps and in traps of low confining dimensions, for which the standard semiclassical approximation is not applicable. The results of the modified approach are shown to coincide with purely quantum-mechanical calculations for harmonic traps, including the one-dimensional harmonic trap. The advantage of the semiclassical approximation is in its simplicity and generality. Power-law potentials of arbitrary powers are considered. Effective thermodynamic limit is defined for any confining dimension. The behaviour of the specific heat, isothermal compressibility, and density fluctuations is analyzed, with an emphasis on low confining dimensions, where the usual semiclassical method fails. The peculiarities of the thermodynamic characteristics in the effective thermodynamic limit are discussed.

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I. INTRODUCTION

Physics of Bose gases, exhibiting Bose-Einstein condensation, is currently a subject of intensive investigations, both experimental and theoretical (see, e.g., the review works [1–5]). A very efficient method for describing Bose-Einstein condensation of trapped atoms is the semiclassical approximation, which has been employed for considering the properties of Bose-condensed ideal gases trapped in power-law potentials [2,6–11]. The advantage of the semiclassical method is its simplicity, as compared to quantum-mechanical calculations, and its generality allowing for the treatment of power-law potentials of arbitrary powers, except those cases, when the effective confining dimension becomes low. The latter happens for the gas of low spatial dimensionality and when the power of the confining potential is large. For instance, the semiclassical approximation predicts the absence of Bose-Einstein condensation at finite temperature in a one-dimensional harmonic trap [2,7,11], while this exists in a quantum-mechanical picture [12].

In the present paper, a modification of the semiclassical approximation is advanced, which makes it possible to essentially extend the region of its applicability. Thus, the low-dimensional gases can be successfully described, with confining potentials of arbitrary powers. The definition of an effective thermodynamic limit for trapped atoms, known for the case of a three-dimensional harmonic trap [3], can be generalized to arbitrary space dimensionality and any confining powers. The specific-heat discontinuity at the condensation point can now be studied for all those cases, for which the standard approach does not work. The behaviour of isothermal compressibility and the peculiarity of density fluctuations of a finite number of trapped atoms, not studied earlier, are investigated.

Throughout the paper, the system of units is employed, where the Planck constant $\hbar \equiv 1$ and the Boltzmann constant $k_B \equiv 1$ are set to unity.

II. SEMICLASSICAL DENSITY OF STATES

Let us start with very briefly recalling the basic notions and introducing the necessary notation that will be used in the following sections. We shall consider the ideal Bose gas confined by means of a trapping potential $U(r)$. The Cartesian vector $r = \{r_\alpha\}$, with $\alpha = 1, 2, \ldots, d$, is defined in a $d$-dimensional space, so that $d = 1, 2, \ldots$. The trapping potential is assumed to be slowly varying in space, such that its characteristic length $l_0$ be much larger than the thermal wavelength $\lambda_T$,

$$\frac{\lambda_T}{l_0} \ll 1, \quad \lambda_T \equiv \sqrt{\frac{2\pi}{mT}}, \quad (1)$$

where $m$ is atomic mass and $T$, temperature. Condition (1) can be rewritten as an inequality for the characteristic frequency of the trapping potential $\omega_0$,

$$\frac{\omega_0}{T} \ll 1, \quad \omega_0 \equiv \frac{1}{ml_0^2}. \quad (2)$$

Under condition (1) or (2), the semiclassical approach is applicable [2,6–10], described by the density of states
\[
\rho(\varepsilon) = \frac{(2m)^{d/2}}{(4\pi)^{d/2}\Gamma(d/2)} \int_{\mathbb{V}_\varepsilon} [\varepsilon - U(r)]^{d/2-1} \, dr ,
\]
where \(d \geq 1\) and the integration is over the available volume
\[
\mathbb{V}_\varepsilon \equiv \{ r | U(r) \leq \varepsilon \} .
\]
In particular, for a one-dimensional system, one has
\[
\rho(\varepsilon) = \frac{\sqrt{2m}}{2\pi} \int_{x_\varepsilon}^{x_\varepsilon} \frac{dx}{\sqrt{\varepsilon - U(x)}} ,
\]
with the turning points \(\pm x_\varepsilon\) given by the equality \(U(x_\varepsilon) = \varepsilon\). Note that Eq. (4) follows exactly from Eq. (3).

The general form of a power-law confining potential can be represented as
\[
U(r) = \sum_{\alpha=1}^{d} \frac{\omega_\alpha}{2} \left| \frac{r_\alpha}{l_\alpha} \right|^{n_\alpha} \left( l_\alpha \equiv \frac{1}{\sqrt{m\omega_\alpha}} \right) ,
\]
where \(\omega_\alpha, l_\alpha,\) and \(n_\alpha\) are positive parameters. The characteristic trapping frequency and potential length are given by the geometric averages
\[
\omega_0 \equiv \left( \prod_{\alpha=1}^{d} \omega_\alpha \right)^{1/d} , \quad l_0 \equiv \left( \prod_{\alpha=1}^{d} l_\alpha \right)^{1/d} .
\]
An important notion, arising for the confining potential (5), is the confining dimension
\[
s \equiv \frac{d}{2} + \sum_{\alpha=1}^{d} \frac{1}{n_\alpha} .
\]
The density of states (3), for potential (5), becomes
\[
\rho(\varepsilon) = \frac{\varepsilon^{s-1}}{\gamma_d \Gamma(s)} ,
\]
where \(\Gamma(s)\) is a gamma-function. Form (8) is valid for any \(d \geq 1\) and \(s \geq 1/2\). The parameter \(\gamma_d\) is
\[
\gamma_d \equiv \frac{\pi^{d/2}}{2^s} \prod_{\alpha=1}^{d} \frac{\omega_\alpha^{1/2+1/n_\alpha}}{\Gamma(1+1/n_\alpha)}
\]
for \(d \geq 1\). For instance, for \(d = 1\), it is
\[
\gamma_1 = \frac{\sqrt{\pi}}{\Gamma(1+1/n)} \left( \frac{\omega_0}{2} \right)^{s} \quad (d = 1) .
\]
In the case of harmonic traps, when \(n_\alpha = 2\), the confining dimension (7) coincides with the spatial dimension \(d = s\). Then \(\gamma_d = \omega_0^d\) for all \(d \geq 1\), and the density of states is
\[
\rho(\varepsilon) = \frac{\varepsilon^{d-1}}{\Gamma(d) \omega_0^d} \quad (d \geq 1) .
\]
III. MODIFIED SEMICLASSICAL APPROXIMATION

Observable quantities, calculated with the density of states (8), are expressed through the Bose-Einstein function \( g_s(z) \), in which \( s \) is the confining dimension (7) and \( z \equiv e^{\beta \mu} \) is fugacity, with \( \beta \equiv 1/T \) being inverse temperature and \( \mu \), chemical potential. Below the Bose condensation temperature \( T_c \), one has \( \mu \to 0 \) and \( z \to 1 \). The Bose-Einstein function \( g_s(z) \) diverges as \( z \to 1 \), if \( s \leq 1 \). Those observable quantities that contain \( g_s(1) \), with \( s \leq 1 \), cannot be defined, because of the divergence of \( g_s(1) \). This imposes the limits of applicability for the semiclassical approximation.

However the divergence of \( g_s(1) \), for \( s \leq 1 \), is related to the usage of the thermodynamic limit assuming an infinite system with an infinite number of particles. For this case, integrations over momenta start from zero, which implies that the minimal momentum is zero. But if the system is finite, containing a finite number of particles \( N \), though may be very large, and in addition is confined in space by a trapping potential with a characteristic length \( l_0 \), then the minimal momentum of a particle is not zero, but rather is a finite quantity \( k_{\text{min}} = 1/l_0 \). Respectively, the minimal energy is \( k_{\text{min}}^2/2m \), and the related dimensionless minimal energy is

\[
 u_{\text{min}} = \frac{\beta k_{\text{min}}^2}{2m} = \frac{\beta}{2ml_0^2} = \frac{\omega_0}{2T}. 
\]

The Bose-Einstein functions \( g_n(z) \), arising in the process of calculating physical quantities, are defined through integrals over the dimensionless energy variable \( u = \beta \varepsilon \). For a finite confined system, the integration should start from the minimal value (11), which yields

\[
 g_n(z) = \frac{1}{\Gamma(n)} \int_{u_{\text{min}}}^{\infty} zu^{n-1} e^{-u} du. 
\]

The lower integration limit, according to inequality (2) is small,

\[
 u_{\text{min}} = \frac{\omega_0}{2T} \ll 1, 
\]

though it is not strictly zero. For \( n > 1 \), when the integral (12) converges, the value (13) is negligible and can be replaced by zero. However, for \( n \leq 1 \), when the integral can diverge at \( z \to 1 \), the lower integration limit must be kept finite, being given by Eq. (11). In this way, we can define the integral (12) for all \( n \), which gives

\[
 g_1(1) \approx \ln \frac{2T}{\omega_0} \quad (n = 1),
\]

\[
 g_n(1) \approx \frac{1}{(1-n)\Gamma(n)} \left( \frac{2T}{\omega_0} \right)^{1-n} \quad (0 < n < 1),
\]

\[
 g_0(1) \approx \frac{2T}{\omega_0} \quad (n = 0).
\]

For the last equality in Eqs. (14), the relation

\[
 \frac{\partial}{\partial z} g_n(z) = \frac{1}{z} g_{n-1}(z)
\]
is used, being valid for all $n$. For negative values $n < 0$, the integral (12) has the form of the second of Eqs. (14). Note that for all $n \geq 0$, the values $g_n(1)$ are positive, while for $1/2 \leq n < 0$, these values become negative, since then $\Gamma(n) < 0$.

The total number of particles can be represented as

$$N = N_0 + \frac{T^s}{\gamma_d} g_s(z),$$

(15)

where $N_0$ is the number of condensed atoms. From here, the critical temperature $T_c$ follows as the temperature at which $\mu \to 0$, $z \to 1$, and $N_0 \to 0$, which results in

$$T_c = \left[ \frac{\gamma dN}{g_s(1)} \right]^{1/s}.$$  

(16)

For $s > 1$, one has $g_s(1) = \zeta(s)$, with $\zeta(s)$ being a Riemann zeta function. In the standard picture, $g_s(1)$ would diverge for all $s \leq 1$, which would lead to the conclusion that then $T_c \to 0$. That is, then a finite condensation temperature would not exist for one-dimensional systems trapped by a potential with the power $n \geq 2$, which includes the harmonic potential [2,7,11]. But, as is explained above, for a finite confined system, we have to take the value $g_n(1)$ given by Eqs. (14).

For $s = 1$, which happens for a one-dimensional harmonic trap ($d = 1, n = 2$), we find the critical temperature

$$T_c = \frac{N \omega_0}{\ln(2 T_c / \omega_0)} \quad (s = 1),$$

(17)

which is an immediate consequence of Eqs. (14) and (16). Equation (17) exactly coincides with the equation for $T_c$ obtained in a purely quantum-mechanical calculation [12]. Iterating Eq. (17), with taking account of the inequality

$$\frac{T_c}{\omega_0} \ll e^{2N} \quad (N \gg 1),$$

we get the condensation temperature

$$T_c = \frac{N \omega_0}{\ln(2N)} \quad (s = d = 1, n = 2)$$

(18)

for the one-dimensional harmonic trap.

In the case of one-dimensional traps, with confining potentials of powers $n > 2$, we can have the confining dimension in the interval $1/2 \leq s < 1$. In such a case, the condensation temperature is

$$T_c = \frac{\sqrt{\pi}(1-s)\Gamma(s)}{2\Gamma(1+1/n)} N \omega_0 \quad (s < 1).$$

(19)

Thus, we come to a conclusion that finite critical temperatures do exist for one-dimensional traps for all powers of the confining potential.

Dealing with harmonic traps, one has $s = d$, so that for two- and three-dimensional traps, the critical temperatures are

$$T_c = \omega_0 \left[ \frac{N}{\zeta(d)} \right]^{1/d} \quad (s = d = 2, 3).$$

(20)

And for a one-dimensional harmonic trap, $T_c$ is given by Eq. (18).
IV. EFFECTIVE THERMODYNAMIC LIMIT

Effective thermodynamic limit for three-dimensional harmonic traps has been defined earlier [3]. Here this notion is generalized for all spatial dimensions \( d \geq 1 \) and arbitrary confining dimensions \( s \geq 1/2 \).

The basic idea in defining an effective thermodynamic limit is the common agreement that, when the number of particles tends to infinity, \( N \to \infty \), the extensive observable quantities must vary proportionally to \( N \). One of the main extensive observables is the internal energy \( E \). Hence, the thermodynamic limit in the most general sense can be understood as the limit

\[
N \to \infty , \quad E \to \infty , \quad \frac{E}{N} \to \text{const} ,
\]

where a nonzero constant is assumed. For the internal energy, we have

\[
\frac{E}{N} = \frac{s g_{s+1}(z)}{N \gamma_d} T^{s+1}.
\]

Since \( s \geq 1/2 \), the value of \( g_{s+1}(z) \) is always finite. This means that the limit (21) can be rewritten as

\[
N \to \infty , \quad \gamma_d \to 0 , \quad N \gamma_d \to \text{const} .
\]

This is the most general form of the thermodynamic limit, valid for all power-law potentials in any spatial dimensionality.

Let us consider in more detail the case of unipower potentials, for which \( n_{\alpha} = n \). Then the confining dimension (7) becomes

\[
s = \left( \frac{1}{2} + \frac{1}{n} \right) d.
\]

The coefficient (9) takes the form

\[
\gamma_d = \frac{\pi^{d/2}}{\Gamma^d(1 + 1/n)} \left( \frac{\omega_0}{2} \right)^s
\]

where \( \omega_0 \) is defined in Eq. (6). This shows that \( \gamma_d \propto \omega_0^s \) for all \( d \geq 1 \). Thence, the limit (23) can be represented as the effective thermodynamic limit

\[
N \to \infty , \quad \omega_0 \to 0 , \quad N \omega_0^s \to \text{const} .
\]

The latter implies that

\[
\omega_0 \propto N^{-1/s} \quad (N \to \infty).
\]

In a particular case of a three-dimensional harmonic trap, when \( s = d = 3 \), the limit (26) reduces to the known result [3]. But, generally, the limit (26) is valid for all confining dimensions (24).
The condensation temperatures, found above, have sense for confined systems with a finite, though very large, number of particles $N$. The notion of the thermodynamic limit (26) allows one to analyze the behaviour of $T_c$ as $N \to \infty$. To this end, from Eq. (16) we have

$$T_c \propto \text{const} \quad (s > 1).$$  \hspace{1cm} (28)

For the one-dimensional harmonic trap, Eq. (18) gives

$$T_c \propto \frac{1}{\ln N} \to 0 \quad (s = 1).$$  \hspace{1cm} (29)

And for very low confining dimensions $s$, from Eq. (19) we find

$$T_c \propto N^{1-1/s} \to 0 \quad (s < 1).$$  \hspace{1cm} (30)

Equations (28) to (30) demonstrate the behaviour of $T_c$ as $N \to \infty$. But for finite $N$, the corresponding values in Eqs. (16), (18), and (19) can be finite and well defined.

V. SPECIFIC HEAT DISCONTINUITY

Specific heat, under a fixed number of particles, is given by the derivative

$$C_N \equiv \frac{1}{N} \frac{\partial E}{\partial T}. \hspace{1cm} (31)$$

With the internal energy (22), for the temperatures above $T_c$, one has

$$C_N = s(s + 1) \frac{g_{s+1}(z)}{g_s(z)} - s^2 \frac{g_s(z)}{g_{s-1}(z)} \quad (T > T_c).$$  \hspace{1cm} (32)

One may notice that for a three-dimensional uniform system, when $n \to \infty$ and $s \to 3/2$, Eq. (32) reduces to the known expression [13,14]. Below the condensation temperature $T_c$, we get

$$C_N = s(s + 1) \frac{g_{s+1}(1)}{g_s(1)} \left( \frac{T}{T_c} \right)^s \quad (T < T_c).$$  \hspace{1cm} (33)

Expression (33) is finite and positive for all $s \geq 1/2$. Defining the specific heat discontinuity at the critical point as

$$\Delta C_N \equiv C_N(T_c + 0) - C_N(T_c - 0),$$  \hspace{1cm} (34)

we find

$$\Delta C_N = -s^2 \frac{g_s(1)}{g_{s-1}(1)}. \hspace{1cm} (35)$$

In the standard approach, because of the divergence of $g_{s-1}(1)$ for $s \leq 2$, the specific heat discontinuity (35) could be considered [11] only for $s > 2$. Here we extend the analysis for all $s$. 7
First, for higher confining dimensions, we have

\[ \Delta C_N = -s^2 \frac{\zeta(s)}{\zeta(s-1)} \quad (s > 2), \]  

(36)

which agrees with the previous result [11]. For lower confining dimensions, which could not be treated earlier, we obtain the following expressions:

\[ \Delta C_N = -2 \frac{\pi^2}{3 \ln(2T_c/\omega_0)} \quad (s = 2), \]  

(37)

\[ \Delta C_N = -s^2 \zeta(s)(2-s)\Gamma(s-1)\left(\frac{\omega_0}{2T_c}\right)^{2-s} \quad (1 < s < 2), \]  

(38)

\[ \Delta C_N = -\frac{\omega_0}{2T_c} \ln\left(\frac{2T_c}{\omega_0}\right) \quad (s = 1), \]  

(39)

\[ \Delta C_N = -\frac{s^2(2-s)\Gamma(s-1)\omega_0}{2(1-s)\Gamma(s)T_c} \quad (s < 1). \]  

(40)

For all \( s \geq 1 \), the specific heat jump is negative, which tells that

\[ C_N(T_c - 0) > C_N(T_c + 0) \quad (s \geq 1). \]

But the sign of the discontinuity changes for \( s < 1 \), demonstrating that

\[ C_N(T_c - 0) < C_N(T_c + 0) \quad (s < 1). \]

The jump (40) becomes positive, since \( \Gamma(s-1) \) is negative for \( 1/2 \leq s < 1 \).

In the case of harmonic traps, the specific heat discontinuity \( \Delta C_N \) is given by Eq. (39) for \( d = 1 \), by Eq. (37) for \( d = 2 \), and for a three-dimensional harmonic trap, one has

\[ \Delta C_N = -\frac{54}{\pi^2} \zeta(3) \quad (s = d = 3). \]

For finite systems, the specific heat is always discontinuous at \( T_c \).

**VI. DENSITY FLUCTUATIONS AND COMPRESSIBILITY**

Density fluctuations in any statistical system can be quantified [13,15] by the isothermal compressibility

\[ \kappa_T = \frac{\Delta^2(\hat{N})}{\rho_T N} = \frac{1}{\rho^2} \left( \frac{\partial \rho}{\partial \mu} \right), \]  

(41)

where \( \rho \) is a mean particle density, and which is connected with the number-of-particle dispersion.
\[ \Delta^2(\hat{N}) \equiv <\hat{N}^2> - <\hat{N}>^2. \quad (42) \]

For the latter, one has
\[ \Delta^2(\hat{N}) = T \frac{\partial N}{\partial \mu} = \frac{T N}{\rho} \left( \frac{\partial \rho}{\partial \mu} \right). \quad (43) \]

The total number of particles \( N = N_0 + N_1 \) is a sum of the number of condensed atoms and the number of atoms outside the condensate. In a gauge-symmetric grand canonical ensemble, the fluctuations of condensate are known to be anomalous [16,17]. However, as is also perfectly known [16], this is just an artifact that can be easily removed by breaking the gauge symmetry. The most efficient way of gauge-symmetry breaking is by means of the so-called Bogolubov shift [18,19], when the field operators of condensed particles are replaced by the nonoperator quantities, representing the condensate wave functions. In that way, the operator of the number of condensed particles is replaced by its average, as a result of which only the noncondensed particles contribute to the dispersion
\[ \Delta^2(\hat{N}) = \Delta^2(\hat{N}_1) = T \frac{\partial N_1}{\partial \mu}. \quad (44) \]

The necessity of introducing broken gauge symmetry in grand canonical ensemble, in order to eliminate fictitious condensate fluctuations, was discussed in great detail by ter Haar [16] and Hohenberg and Martin [20], and carefully explained by Bogolubov [18,19]. The asymptotic exactness of the Bogolubov shift was proven by Ginibre [21].

The number of noncondensed particles, according to Eq. (15), is
\[ N_1 = \frac{T^s}{\gamma_d} g_s(z). \quad (45) \]

From here, the dispersion (44) writes as
\[ \Delta^2(\hat{N}_1) = \frac{T^s}{\gamma_d} g_{s-1}(z). \quad (46) \]

This, invoking relation (16), can be rewritten as
\[ \Delta^2(\hat{N}_1) = N \frac{g_{s-1}(z)}{g_s(1)} \left( \frac{T}{T_c} \right)^s. \quad (47) \]

Involving the notion of the thermodynamic limit, particle fluctuations can be classified into normal and anomalous. This is described more fully in the review article [22]. When \( \Delta^2(\hat{N}_1) \propto N \), fluctuations are called normal, while if \( \Delta^2(\hat{N}_1) \propto N^\alpha \), with \( \alpha > 1 \), they are termed anomalous.

At temperatures \( T > T_c \), above the condensation point, one has \( N_1 = N \) and \( z < 1 \). For the confining dimension \( s \geq 1 \), particle fluctuations are normal, since
\[ \Delta^2(\hat{N}_1) \propto N \quad (s \geq 1, T > T_c). \quad (48) \]

Respectively, the compressibility (41) is finite for all \( N \to \infty \). For lower confining dimensions \( s < 1 \), we find
\[ \Delta^2(\hat{N}_1) = \frac{zT}{(1-z)(1-s)\Gamma(s-1)\gamma_d} \left( \frac{2}{\omega_0} \right)^{1-s}. \]  

(49)

Then, because \( \Gamma(s-1) < 0 \) for \( 1/2 < s < 1 \), the compressibility (41) becomes divergent and negative,

\[ \kappa_T \propto -N^{-1+1/s} \quad (s < 1, \ T > T_c). \]  

(50)

Such a behaviour of the compressibility means that the system is unstable.

Above the condensation temperature \( T_c \), the trapped gas is stable only for the confining dimensions \( s \geq 1 \). For harmonic traps, for which \( s = d \), the gas is stable in all spatial dimensions \( d \geq 1 \).

The situation is more interesting for the temperatures \( T < T_c \) below the condensation point. Then the dispersion (46) becomes

\[ \Delta^2(\hat{N}_1) = \frac{T^s}{\gamma_d} g_{s-1}(1). \]  

(51)

There exists a rich variety of different cases:

\[ \Delta^2(\hat{N}_1) = N \frac{\zeta(s-1)}{\zeta(s)} \left( \frac{T}{T_c} \right)^s \quad (s > 2), \]  

(52)

\[ \Delta^2(\hat{N}_1) = \frac{N}{\zeta(2)} \left( \frac{T}{T_c} \right)^2 \ln \left( \frac{2T}{\omega_0} \right) \quad (s = 2), \]  

(53)

\[ \Delta^2(\hat{N}_1) = \frac{N}{(2-s)\zeta(s)\Gamma(s-1)} \left( \frac{2T_c}{\omega_0} \right)^{2-s} \left( \frac{T}{T_c} \right)^2 \quad (1 < s < 2), \]  

(54)

\[ \Delta^2(\hat{N}_1) = 2 \left( \frac{T}{\omega_0} \right)^2 \quad (s = 1). \]  

(55)

And for \( s < 1 \), the dispersion \( \Delta^2(\hat{N}_1) \) has the same form as in Eq. (54). In the thermodynamic limit, we find

\[ \Delta^2(\hat{N}_1) \propto N \quad (s > 2), \]

\[ \Delta^2(\hat{N}_1) \propto N \ln N \quad (s = 2), \]

\[ \Delta^2(\hat{N}_1) \propto N^{2/s} \quad (1 < s < 2), \]

\[ \Delta^2(\hat{N}_1) \propto N^2 \quad (s = 1), \]

\[ \Delta^2(\hat{N}_1) \propto -N^{2/s} \quad (s < 1). \]  

(56)

Respectively, the behaviour of the compressibility is

\[ \kappa_T \propto const \quad (s > 2), \]
\[ \kappa_T \propto \ln N \quad (s = 2), \]
\[ \kappa_T \propto N^{2/s-1} \quad (1 < s < 2), \]
\[ \kappa_T \propto N \quad (s = 1), \]
\[ \kappa_T \propto -N^{2/s-1} \quad (s < 1). \]

(57)

These equations demonstrate that the fluctuations are anomalous for all \( s \leq 2 \). A negative compressibility for \( s < 1 \) implies strong instability of the system. For the confining dimensions in the interval \( 1 \leq s \leq 2 \), the compressibility is positive but displays nonthermodynamic behaviour diverging in the thermodynamic limit. An actual divergence of \( \kappa_T \) happens only for infinite systems, when \( N \rightarrow \infty \), which would imply instability. When one deals with finite system, with a large, though finite, number of particles \( N \gg 1 \), then the compressibility does not really diverge, but becomes very large. This means that there exist very strong density fluctuations in the system. Such strong fluctuations arise in the case of the low confining dimension \( s \leq 2 \), which corresponds to large confining powers \( n_\alpha \) and low spatial dimensions \( d \). Fluctuations are known to be strong in low-dimensional systems \([23,24]\). For higher confining dimensions \( s > 2 \), fluctuations are always normal, so that for

\[ \frac{d}{2} + \sum_{\alpha=1}^{d} \frac{1}{n_\alpha} > 2 \]

(58)

the system is stable.

Since the standard traps are usually harmonic, let us pay a special attention to the harmonic confining potentials, when \( s = d \) and \( \gamma_d = \omega_0^d \). Then the dispersion (51) becomes

\[ \Delta^2(\hat{N}_1) = g_{d-1}(1) \left( \frac{T}{\omega_0} \right)^d. \]

(59)

For different spatial dimensions, we find

\[ \Delta^2(\hat{N}_1) = 2 \left( \frac{T}{\omega_0} \right)^2 \quad (d = 1), \]
\[ \Delta^2(\hat{N}_1) = \left( \frac{T}{\omega_0} \right)^2 \ln \left( \frac{2T}{\omega_0} \right) \quad (d = 2), \]
\[ \Delta^2(\hat{N}_1) = N \frac{\pi^2}{6\zeta(3)} \left( \frac{T}{T_c} \right)^3 \quad (d = 3). \]

(60)

The last expression agrees with the corresponding result by Politzer \([25]\) for a three-dimensional harmonic trap. All other formulas of this section are new.

Dispersions (60) in the thermodynamic limit behave as

\[ \Delta^2(\hat{N}_1) \propto N^2 \quad (d = 1), \]
\[ \Delta^2(\hat{N}_1) \propto N \ln N \quad (d = 2), \]
\[ \Delta^2(\hat{N}_1) \propto N \quad (d = 3). \]
So that the dispersions for $d = 1$ and $d = 2$ are anomalous. The related compressibilities possess the limits
\[
\begin{align*}
\kappa_T &\propto N \quad (d = 1), \\
\kappa_T &\propto \ln N \quad (d = 2), \\
\kappa_T &\propto \text{const} \quad (d = 3).
\end{align*}
\]
Anomalous values of the compressibilities for low-dimensional harmonic traps signify instability caused by the existence of very strong fluctuations in such traps.

**VII. CONCLUSION**

The semiclassical approximation is generalized, which makes it possible to essentially extend the region of its applicability. Bose-Einstein condensation in traps of low confining dimensions is described, for which the standard approach could not be used. Trapping potentials of arbitrary power laws are considered. Specific-heat discontinuities and isothermal compressibilities are analyzed. Effective thermodynamic limit is defined for any spatial dimension and for arbitrary powers of confining potentials. It is shown that the modified semiclassical method for harmonic traps yields the results coinciding with those obtained by means of quantum-mechanical calculations, when these are available.

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