Recursion operators in the cotangent covering of the rdDym equation

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Received: 21 June 2021 / Revised: 6 October 2021 / Accepted: 9 October 2021 / Published online: 24 October 2021 © The Author(s), under exclusive licence to Springer Nature Switzerland AG 2021

Abstract
We describe a general method of constructing nonlocal recursion operators for symmetries of PDEs. As an example, the cotangent equation to the 3D rdDym equation $u_{yt} = u_x u_{xy} - u_y u_{xx}$ is considered for which two mutually inverse operators are found. The exposition includes a rigorous criterion to check the hereditary property of nonlocal recursion operators.

Keywords Partial differential equations · Integrable linearly degenerate equations · Nonlocal symmetries · Recursion operators

Mathematics Subject Classification 35B06

1 Introduction

The rdDym equation ($r$th dispersionless Dym equation)

$$u_{yt} = u_x u_{xy} - u_y u_{xx}$$

(1)

belongs to the class of linearly degenerate equations [8] and arose in the Martínez Alonso-Shabat hierarchy [16]. It is Lax integrable (i.e., admits a Lax pair with non-
removable parameter) and its symmetry properties were studied in [1,3]. Recursion operators for symmetries of Eq. (1) was found by O. Morozov in [19].

In [22], V. Ovsienko considered (in slightly different notation) the system

\[
\begin{align*}
v_{yt} &= 2(v_y u_{xx} - v_x u_{xy}) + u_x v_{xy} - u_y v_{xx} - 2(u_y u_{xx} + 2u_x u_{xy}), \\
u_{yt} &= u_x u_{xy} - u_y u_{xx},
\end{align*}
\]

which is a two-component extension of the rdDym equation (another two-component generalization of (1) was studied in [20]). Treating (2) as the Euler equation on the Virasoro algebra, its bi-Hamiltonian structure was established.

System (2) may be also considered in the following way. Recall (see [12,13]) that to any equation \( E = \{ F[u] = 0 \} \) in unknowns \( u = (u^1, \ldots, u^m) \) its cotangent equation \( \mathcal{T}^* E \)

\[
F[u] = 0, \quad \ell_F^*[u, p] = 0
\]

is naturally associated, where \( \ell_F^* \) is the operator adjoint to the linearisation of \( F \) (here and below \([u], [u, p], \text{etc.}\) denotes the variables together with their derivatives up to certain order). Then (2) may be understood as the cotangent equation of (1), with \( v \) instead of \( p \), ‘deformed’ by the Lagrangian term

\[
\frac{\delta L}{\delta u} \quad \text{with} \quad L = u_y u_x^2.
\]

System (2) is an Euler–Lagrange equations.

Lagrangian deformations of some linearly degenerate equations were studied in [2]. In particular, the system

\[
\begin{align*}
v_{yt} &= 2(v_y u_{xx} - v_x u_{xy}) + u_x v_{xy} - u_y v_{xx} - 2(2ku_{xy} u_x + (ku_y + l)u_{xx}), \\
u_{yt} &= u_x u_{xy} - u_y u_{xx},
\end{align*}
\]

which is considered below, is the deformation of the cotangent to rdDym equation with the deforming Lagrangian \( L = (ku_y + l)u_x^2, \ k, l \in \mathbb{R} \). System (2) is obtained from (3) when \( k = 1, l = 0 \). We find below recursion operators for symmetries of (3).

**Remark 1 (courtesy O.I. Morozov)** Actually, both parameters \( k \) and \( l \) are fake in a sense: the change of variables \( v \mapsto v - 2kxu_x - ku + ly \) kills them and transforms System (3) to the pure cotangent equation of (1). In particular, the Ovsienko system (2) also reduces to the cotangent equation. Though the computations in the general case are not much harder than in a particular one, the result looks more complicated. For that reason we set \( k = l = 0 \) everywhere below, i.e., the equation we study below will be of the form

\[
\begin{align*}
v_{yt} &= 2(v_y u_{xx} - v_x u_{xy}) + u_x v_{xy} - u_y v_{xx}, \\
u_{yt} &= u_x u_{xy} - u_y u_{xx}.
\end{align*}
\]

There exist several approaches to construct recursion operators (see, for example, [21,25]). Our method is based on the technology described in [13] and originating from
It is based on geometrical interpretation of recursion operators for symmetries as Bäcklund auto-transformations of the equation tangent to the given one (cf. [17]) and essentially uses the theory of differential coverings [14].

**Remark 2** Recursion operators as Bäcklund auto-transformations of the tangent equations initially aroused in M. Marvan’s paper [17], though implicitly the tangent equation as an instrument for computation of recursion operators appears in [9]. Informally, tangent and cotangent bundles to PDEs are discussed in the paper [15] by B. Kupershmidt. A detailed discussion of the cotangent covering and its role in the theory of Hamiltonian (Poisson) structures can be found in [12] and especially in [10].

The necessary theoretical matters needed for computations are shortly described in Sect. 2, while Sect. 3 contains the main results.

### 2 Preliminaries and notation

The exposition here is based on [5,13,14].

Let \( \pi : E \to M \) be a vector bundle over a smooth manifold \( M \), \( \text{dim } M = n \), rank \( \pi = m \), and \( \pi_\infty : J_\infty^\infty(\pi) \to M \) be the bundle of its infinite jets. Consider an infinitely prolonged differential equation \( \mathcal{E} \in J_\infty^\infty(\pi) \) and assume that \( \mathcal{E} = \{ F = 0 \} \), where \( F \in \Gamma(\pi_\infty(\xi)) \), where \( \xi : N \to M \) is another vector bundle. Any \( \mathcal{E} \) is naturally endowed with an \( n \)-dimensional Frobenius integrable distribution \( \mathcal{E} \) (the Cartan distribution) that defines its geometry. The Cartan distribution is \( \pi_\infty \)-horizontal and, consequently, defines a flat connection in \( \pi_\infty : \mathcal{E} \to M \).

A morphism of equations \( f : \tilde{\mathcal{E}} \to \mathcal{E} \) is a smooth map, such that \( f_*(\tilde{\mathcal{E}}_\theta) \subset \mathcal{E}_f(\theta) \) for any \( \theta \in \tilde{\mathcal{E}} \). A morphism is called a (differential) covering if it is a submersion and the restriction \( f_*|_{\mathcal{E}_\theta} \) of its differential to any Cartan plane is an isomorphism. Two coverings \( f_i : \mathcal{E}_i \to \mathcal{E}, i = 1, 2 \), are called equivalent if there exists a diffeomorphism \( g : \mathcal{E}_1 \to \mathcal{E}_2 \) that preserves the Cartan distributions and such that \( f_1 = f_2 \circ g \).

Denote by \( \mathcal{F} = \mathcal{F}(\mathcal{E}) \) the algebra of smooth functions on \( \mathcal{E} \). A (higher infinitesimal) symmetry of \( \mathcal{E} \) is a \( \pi_\infty \)-vertical derivation \( S : \mathcal{F} \to \mathcal{F} \) (i.e., a vector field on \( \mathcal{E} \)) that preserves the Cartan distribution. There exists a one-to-one correspondence between symmetries and solutions of the equation

\[
\ell_\mathcal{E} (\varphi) = 0,
\]

where \( \ell_\mathcal{E} \) is the restriction of the linearisation \( \ell_F : \varphi \to P \) to \( \mathcal{E} \) and \( \varphi \) denotes the module of sections \( \Gamma(\pi_\infty(\pi)) \). A conservation law of \( \mathcal{E} \) is a \( \pi_\infty \)-horizontal form \( \omega \in \Lambda^{n-1}_h(\mathcal{E}) \) closed with respect to the horizontal de Rham differential \( d_h \). Trivial conservation laws are those of the form \( \omega = d_h \rho, \rho \in \Lambda^{n-2}(\mathcal{E}) \).

Given a covering \( \tau : \tilde{\mathcal{E}} \to \mathcal{E} \), symmetries and conservation laws of \( \tilde{\mathcal{E}} \) are called nonlocal for \( \mathcal{E} \). Let \( S : \mathcal{F}(\mathcal{E}) \to \mathcal{F}(\tilde{\mathcal{E}}) \) be a \( \pi_\infty \)-vertical derivation. It is called a \( \tau \)-shadow if

\[
\tilde{\mathcal{E}}_X \circ S = S \circ \mathcal{E}_X
\]
for any vector field $X$ on $M$. We say that a symmetry $\tilde{S}: \mathcal{F}(\tilde{E}) \to \mathcal{F}(\tilde{E})$ is a lift of the shadow $S$ if $\tilde{S}|_{\mathcal{F}(E)} = S$.

Let an equation $\mathcal{E}$ be given by $\{F = 0\}$. Then the system $TE$

$$F[u] = 0, \quad \ell_F[u, q] = 0$$

is called the tangent equation of $\mathcal{E}$, while the projection $t: [u, q] \mapsto [u]$ is the tangent covering over $\mathcal{E}$. It must be stressed that $q$ is considered as an odd variable of degree 1. Sections of $t$ that preserve the Cartan distributions are identified with symmetries of $\mathcal{E}$ and consequently Bäcklund auto-transformations $R_{\tau, \tau'}$ of the form

\[
\begin{align*}
W & \xleftarrow{\tau} \mathcal{TE} & \xrightarrow{\tau'} & \mathcal{TE},
\end{align*}
\]

where $\tau, \tau'$ are coverings, are naturally interpreted as recursion operators for symmetries of $\mathcal{E}$.

To construct these operators, we use a scheme consisting of two steps. In the first step, we are looking for two-component conservation laws $\omega$ of $\mathcal{TE}$ linear on the fibers of the projection $t$. Provided such a law was found, we consider the covering $\tau = \tau_\omega$ associated with it and, as the second step, try to find $\tau_\omega$-shadows (also linear on fibers both of $t$ and $\tau_\omega$). If such a shadow $S$ exists, it delivers the covering $\tau' = \tau_S$ and the desired recursion operator $R_{\tau_\omega, \tau_S}$.

Finally, few words about the type of equations that are the subject of our study. Assume $\mathcal{E} = \{F = 0\}, F \in P$, and consider the operator $\ell^*_F: \hat{P} \to \hat{L}$ formally adjoint to $\ell_F$, where $\hat{\bullet} = \text{hom}(\bullet, \Lambda^*_n)$. Then the system $T^*E$

$$\ell^*_F[u, p] = 0, \quad F[u] = 0$$

is said to be the cotangent equation of $\mathcal{E}$, while the projection $t^*: T^*E \to \mathcal{E}, (u, p) \mapsto (u)$, is called the cotangent covering. Similar to the case of the tangent equation, the variable $p$ is odd. System (6) is always an Euler–Lagrange equation with the Lagrangian density $L = \sum p^j F^j$ and we are interested in equations

$$\ell^*_F[u, p] + G[u, p] = 0, \quad F[u] = 0,$$

(‘Lagrangian deformations’ of (6)), such that (a) they are Lagrangian as well and (b) they possess a nontrivial recursion operator. Equation (2) is exactly of this type. Note that for Euler–Lagrange equations $T\mathcal{E}$ and $T^*\mathcal{E}$ coincide, since $\ell_F = \ell^*_F$ in this case.

### 2.1 Coordinates

Let $\mathcal{U} \subset M$ be a coordinate neighborhood with local coordinates $x = (x^1, \ldots, x^n)$ and $E \supset \pi^{-1}(\mathcal{U}) \cong \mathcal{U} \times \mathbb{R}^m$ be a trivialization of $\pi$ with fiber-wise coordinates $u =$
$(u^1, \ldots, u^m)$. Then adapted coordinates $u^i_\sigma$ arise on $J^\infty(\pi)$, where $u^i_\sigma$ corresponds to the partial derivative $\partial^{|\sigma|} u^i / \partial x^\sigma$. The Cartan distribution is spanned by the total derivatives

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\sigma} u^i_\sigma \frac{\partial}{\partial u^i_\sigma},$$

while the Cartan connection takes $\partial / \partial x^i$ to $D_i$. Equation $\mathcal{E}$ is given by the system of relations $F^i(x, \ldots, u^j_\sigma, \ldots) = 0$, $l = 1, \ldots, r, |\sigma| \leq k$. To restrict necessary objects to $\mathcal{E}$, one needs to choose internal coordinates on $\mathcal{E}$ (cf. [18]) and express these objects in terms of internal coordinates. In particular, the Cartan distribution on $\mathcal{E}$ is got by rewriting the operators $D_i$ in internal coordinates.

The coordinate presentation of the linearisation operator acts on $\varphi = (\varphi^1, \ldots, \varphi^m)$ by

$$\ell_F(\varphi) = \sum_{\sigma, j} \partial F^i_\sigma \frac{\partial}{\partial u^i_\sigma} D^\sigma(\varphi^j),$$

where $D^\sigma$ is the composition of the total derivatives corresponding to the multi-index $\sigma$, while for the adjoint one we have

$$\ell^* F(\psi) = \sum_{\sigma, l} (-1)^{|\sigma|} D^\sigma \left( \frac{\partial F^i_\sigma}{\partial u^i_\sigma} \psi^l \right),$$

where $\psi = (\psi^1, \ldots, \psi^r)$. Thus, the tangent equation of $\mathcal{E}$ is

$$\sum_{\sigma, j} \frac{\partial F^i_\sigma}{\partial u^i_\sigma} q^j_\sigma = 0, \quad F(x, \ldots, u^j_\sigma, \ldots) = 0, \quad q = (q^1, \ldots, q^m),$$

while while the cotangent one is given by

$$\sum_{\sigma, l} (-1)^{|\sigma|} D^\sigma \left( \frac{\partial F^i_\sigma}{\partial u^i_\sigma} p^l \right) = 0, \quad F(x, \ldots, u^j_\sigma, \ldots) = 0, \quad p = (p^1, \ldots, p^r).$$

Without loss of generality, we can define two-component conservation laws as the forms

$$\omega = (X_1 dx^1 + X_2 dx^2) \wedge dx^3 \wedge \cdots \wedge dx^n, \quad D_1(X_2) = D_2(X_1),$$

where $X_1$ and $X_2$ are smooth functions on $\mathcal{E}$. Triviality of $\omega$ means existence of a potential $Y$, such that $D_i(Y) = X_i$, $i = 1, 2$.

A covering structure in a locally trivial vector bundle $\tau: W \to \mathcal{E}$ is determined by the system of pair-wise commuting vector fields

$$\bar{D}_i = D_i + X_i, \quad i = 1, \ldots, n,$$  

(7)
where \( X_i = \sum X_i^\alpha \partial / \partial w^\alpha \) are \( \tau \)-vertical fields and \( w^\alpha \) are coordinates in the fiber of \( \tau \) (nonlocal variables). The equalities \([\tilde{D}_i, \tilde{D}_j] = 0\) amount to the fact that the system

\[ w^\alpha_{x^i} = X_i^\alpha, \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, \text{rank} \, \tau, \]

is compatible modulo \( \mathcal{E} \). Note that (7) allows to lift any differential operator \( \Delta \) in total derivatives from \( \mathcal{E} \) to an operator \( \tilde{\Delta} \) on the covering equation.

Let \( \omega \) be a two-component conservation law like above. Then one can construct the covering \( \tau_\omega \) as follows. In the case \( \dim M = 2 \) we set

\[ w_{x^1} = X_1, \quad w_{x^2} = X_2, \]

and \( \text{rank} \, \tau_\omega = 1 \). When \( \dim M > 2 \), we introduce infinite number of nonlocal variables \( w^\sigma \), where \( \sigma \) is a multi-index consisting of integers \( 2, \ldots, n \), \( |\sigma| \geq 0 \), and set

\[ w^\sigma_{x^i} = \begin{cases} D_\sigma (X_i), & i = 1, 2, \\ w^{\sigma_i}, & i > 2. \end{cases} \]

Obviously, we obtain covering structures in both cases.

Consider an \( m \)-component function \( S = (S^1, \ldots, S^m) \). It defines a \( \tau \)-shadow if and only if

\[ \tilde{\ell}_\mathcal{E} (S) = 0, \]

where \( \tilde{\ell}_\mathcal{E} \) is the natural lift of the operator \( \ell_\mathcal{E} \) to the covering equation (see above).

Let now \( S \) be a \( \tau \)-shadow, where \( \tau = \tau_\omega \) in Diagram (5). Denote by \( p_\sigma \) the fiber coordinates in the left copy of \( \mathcal{T}\mathcal{E} \) and by \( \tilde{p}_\sigma \) the same coordinates in the right one and set \( \tilde{p}^j_\sigma = \tilde{D}_\sigma (S^j) \), where \( \tilde{D}_\sigma \) is the composition of total derivatives in \( W \). This gives the covering \( \tau' = \tau_\sigma \). The desired recursion operator is obtained, when we assume that \( S \) is linear in the odd variables, i.e.,

\[ S^j = \sum_{i, \sigma} S^{j, \sigma}_l p^l_\sigma + \sum_{\sigma} S^{j}_\sigma w^\sigma, \]

where \( S^{j, \sigma}_l, S^{j}_\sigma \) are smooth functions on \( \mathcal{E} \).

### 3 The main result

We now pass to the equation \( \mathcal{E} \) given by (4) and choose for internal coordinates on \( \mathcal{E} \) the functions \( x, y, t \) and

\[ u_{x^i} = u_{x^{i_1} \ldots x^{i_t}}, \quad u_{x^i, y^j} = u_{x^{i_1} \ldots x^{i_t} y^{j_1} \ldots y^{j_j}}, \quad u_{x^i, y^j} = u_{x^{i_1} \ldots x^{i_t} t^{j_1} \ldots t^{j_j}}, \quad i \geq 0, \; j > 0, \]

where \( i_1, \ldots, i_t \) and \( j_1, \ldots, j_j \) are integers.
and similar for \( v \). Then the total derivatives on \( \mathcal{E} \) acquire the form

\[
D_x = \frac{\partial}{\partial x} + \sum_{i,j} \left( u_{xi+1} \frac{\partial}{\partial u_{xi}} + u_{xi+1,yj} \frac{\partial}{\partial u_{xi,yj}} + u_{xi+1,tj} \frac{\partial}{\partial u_{xi,tj}} \right),
\]

\[
D_y = \frac{\partial}{\partial y} + \sum_{i,j} \left( u_{xi,y} \frac{\partial}{\partial u_{xi}} + u_{xi,yj+1} \frac{\partial}{\partial u_{xi,yj+1}} + D_x^i D_y^{i-1} (U) \frac{\partial}{\partial u_{xi,tj}} \right),
\]

\[
D_t = \frac{\partial}{\partial t} + \sum_{i,j} \left( u_{xi,i} \frac{\partial}{\partial u_{xi}} + D_x^i D_y^{j-1} (U) \frac{\partial}{\partial u_{xi,yj}} + u_{xi,tj+1} \frac{\partial}{\partial u_{xi,tj+1}} \right),
\]

where \( V = 2(v_y u_{xx} - v_x u_{xy}) + u_x v_{xy} - u_y v_{xx} \) and \( U = u_x u_{xy} - u_y u_{xx} \) are right-hand sides of the first and second equations in (4), respectively.

### 3.1 Symmetries

The defining equation \( \ell_{\mathcal{E}} (S) = 0 \) for symmetries of (4) is

\[
D_y D_t (\psi) = 2 v_y D_x^2 (\psi) - 2 v_x D_x D_y (\psi) + v_{xy} D_x (\psi) - v_{xx} D_y (\psi)
\]

\[
- 2 u_{xy} D_x (\psi) - u_y D_x^2 (\psi) + u_x D_x D_y (\psi) + 2 u_x D_y (\psi).
\]

\[
D_y D_t (\varphi) = u_{xy} D_x (\varphi) - u_y D_x^2 (\varphi) + u_x D_x D_y (\varphi) - u_{xx} D_y (\varphi).
\]  

Solving System (8) for \( \varphi \) and \( \psi \) of order \( \leq 3 \), we get the symmetries

\[ S_1 = (xu_x - 2u, xv_x + 2v), \]

\[ S_2 (T) = (T,0), \]

\[ S_3 (T) = (Tu_x + \ddot{T}x, Tv_x), \]

\[ S_4 (T) = \left( \frac{1}{2} \dddot{T}x^2 + \ddot{T} (xu_x - u) + Tu_t, (xv_x + 2v) \ddot{T} + Tv_t \right), \]

\[ S_5 (Y) = (Yu_y, Yv_y), \]

\[ S_6 = (0, u_{xxx}), \]

\[ S_7 = (0, \frac{1}{2} u_{xx}^2 + u_x u_{xxx} - u_{xxt}), \]
\( S_8 = \left( 0, u_x u_{xx}^2 - u_{xx} u_{xx} + u_{xx}^2 u_{xxx} - 2u_x u_{xxx} + u_{xxt} \right), \)
\( S_9 = \left( 0, \frac{3}{2} u_x u_{xx} - 3u_x u_{xx} + u_{xx}^2 - 3u_x u_{xx} + u_{xxt} + u_{xxx} - u_{xxt} \right), \)
\( S_{10} = \left( 0, \frac{u_{xyy} - 2u_y u_{xyy}}{u_y^2} \right), \)
\( S_{11} = \left( 0, \frac{u_{xyy} u_{yy} - u_y u_{xyy}}{u_y^3} \right), \)
\( S_{12} = \left( 0, v \right), \)
\( S_{13}(T) = \left( 0, Tu_x - \frac{1}{2} \dot{T}x \right), \)
\( S_{14}(T) = \left( 0, \frac{1}{6} \ddot{T}x^2 + T u_x^2 + \left( T - \frac{2}{3} \dot{T}x \right) u_x - \frac{1}{6} (2u + 3x) \ddot{T} - \frac{2}{3} Tu_t \right), \)
\( S_{15}(T) = \left( 0, T \right), \)
\( S_{16}(Y) = \left( 0, \frac{Y}{u_x^2} \right). \)

where \( Y = Y(y) \) and \( T = T(t) \) are arbitrary smooth functions and ‘dot’ denotes the \( t \)-derivative. These symmetries commute as follows\(^1\)

\[
[S_1, S_2(T)] = 2S_2(T), \quad [S_1, S_3(T)] = S_3(T),
[S_1, S_6] = -S_6, \quad [S_1, S_7] = -2S_7, \quad [S_1, S_8] = -3S_8, \quad [S_1, S_9] = -4S_9,
[S_1, S_{11}] = S_{11}, \quad [S_1, S_{13}(T)] = -3S_{13}(T),
[S_1, S_{14}(T)] = 4S_{14}(T) - S_{13}(T),
[S_1, S_{15}(T)] = -2S_{15}(T), \quad [S_1, S_{16}(Y)] = 2S_{16}(Y);
[S_2(T_1), S_4(T_2)] = -S_2(T_1 T_2 - T_1 T_2), \quad [S_2(T), S_9] = -S_{15}(T),
[S_2(T_1), S_{14}(T_2)] = -\frac{1}{3} S_{15}(T_1 T_2 + 2T_1 T_2);
[S_3(T_1), S_3(T_2)] = S_2(T_1 - T_1 T_2 - T_1 T_2),
[S_3(T_1), S_4(T_2)] = S_3(T_1 T_2 - T_2 T_1),
[S_3(T), S_8] = S_{15}(T T), \quad [S_3(T), S_9] = 2S_{13}(T T),
[S_3(T_1), S_{13}(T_2)] = S_{15}(T T_2 + \frac{1}{2} T_1 T_2),
[S_3(T_1), S_{14}(T_2)] = \frac{2}{3} S_{13}(2T_1 T_2 + T_1 T_2) - S_{15}(T_1 T_2 + \frac{1}{2} T_1 T_2);
[S_4(T_1), S_4(T_2)] = S_4(T_1 T_2 - T_1 T_2), \quad [S_4(T), S_7] = -S_{15}(T T),
[S_4(T), S_8] = -2S_{13}(T T), \quad [S_4(T), S_9] = 3S_{13}(T T) + 3S_{14}(T T),
[S_4(T_1), S_{13}(T_2)] = -S_{13}(2T_1 T_2 + T_1 T_2), \quad [S_4(T_1), S_{14}(T_2)] = -S_{14}(2T_1 T_2 + T_1 T_2);
[S_4(T_1), S_{15}(T_2)] = S_{15}(2T_1 T_2 + T_1 T_2);
\]

\(^1\) We omit zero commutators. ‘Prime’ denotes the \( y \)-derivative.
\[ [S_5(Y_1), S_5(Y_2)] = S_5(Y_1'Y_2 - Y_2'Y_1), \quad [S_5(Y_1), S_{16}(Y_2)] = -S_{16}(Y_1'Y_2 + 2Y_1'Y_2); \]
\[ [S_6, S_{12}] = S_6; \quad [S_7, S_{12}] = S_7; \]
\[ [S_8, S_{12}] = S_8; \quad [S_9, S_{12}] = S_9; \]
\[ [S_{10}, S_{12}] = S_{10}; \quad [S_{11}, S_{12}] = S_{11}; \]
\[ [S_{12}, S_{13}(T)] = -S_{13}(T), \quad [S_{12}, S_{15}(Y)] = -S_{15}(Y). \]

3.2 Tangent equation

Due to (8), the tangent equation is obtained by adding to System (4) the equations

\[ q_yt = 2v_y p_{xx} - 2v_x p_{xy} + v_{xy} p_x \]
\[ -v_{xx} p_y - 2u_{xy} q_x - u_y q_{xx} + u_x q_{xy} + 2u_x q_y, \]
\[ p_yt = u_{xy} p_x - u_y p_{xx} + u_x p_{xy} - u_{xx} p_y. \]  

3.3 Conservation laws

There exist four two-component conservation laws on \( \mathcal{T} E \) of order \( \leq 2 \) and linear with respect to the variables \( p_\sigma, q_\sigma \):

\[ \omega_1 = (X_1 \, dx + T_1 \, dt) \wedge dy, \quad \omega_2 = (X_2 \, dx + T_2 \, dt) \wedge dy, \]
\[ \omega_3 = (X_3 \, dx + Y_3 \, dy) \wedge dt, \quad \omega_4 = (X_4 \, dx + Y_4 \, dy) \wedge dt, \]

where

\[ X_1 = -\frac{Y p_y}{u_y^2}, \quad T_1 = \frac{Y(u_y p_x - u_x p_y)}{u_y^2}; \]
\[ X_2 = Y(v_y p_y + u_y q_y), \quad T_2 = Y(v_y u_y p_x - (2v_x u_y - u_x v_y) p_y - u_y^2 q_x + u_y u_x q_y); \]
\[ X_3 = T(2u_x p_x - p_t), \quad Y_3 = T(u_y p_x + u_x p_y); \]
\[ X_4 = T(v_x p_x + u_x q_x + q_t), \quad Y_4 = -T(v_x p_y + 2v_y p_x + u_y q_x - 2u_x q_y). \]

Setting \( Y = T = 1 \) and after slight relabeling, we obtain the following nonlocal variables associated with the above conservation laws:

\[ w_{1,x} = \frac{p_y}{u_y^2}, \quad w_{1,t} = \frac{u_x p_y - u_y p_x}{u_y^2}; \]
\[ w_{2,x} = v_y p_y + u_y q_y, \quad w_{2,t} = v_y u_y p_x + (u_x v_y - 2v_x u_y) p_y u_y^2 q_x + u_y u_x q_y; \]
\[ w_{3,x} = 2u_x p_x - p_t. \]
\[ w_{3,y} = u_y p_x + u_x p_y; \]
\[ w_{4,x} = v_x p_x + u_x q_x + q_t; \]
\[ w_{4,y} = 2v_x p_x - v_x p_y - u_y q_x + 2u_x q_y. \] (11)

All the nonlocal variables \( w_i \) are odd of degree 1.

### 3.4 Shadows

Direct computations reveal two shadows of symmetries that are linearly depend on the variables \( p_\sigma, q_\sigma, \) and \( w_{i,\sigma} \) up to second order:

- \( s_0 : \bar{p}_0 = p, \quad \bar{q}_0 = q; \)
- \( s_1 : \bar{p}_1 = u_x p - w_3, \quad \bar{q}_1 = v_x p - 2u_x q + w_4; \)
- \( s_2 : \bar{p}_2 = u_y w_1, \quad \bar{q}_2 = v_y w_1 + \frac{w_2}{u_y^2}. \)

The shadow \( s_0 \) is responsible for the identical operator and thus is of no interest, while the other two ones provide nontrivial results.

### 3.5 Recursion operators

Using the last formulas, we obtain the following expressions for the nonlocal variables \( w_i \):

\[ w_1 = \frac{\bar{p}_2}{u_y}, \quad w_2 = u_y (u_y \bar{q}_2 - v_y \bar{p}_2); \] (12)
\[ w_3 = u_x p - \bar{p}_1, \quad w_4 = -v_x p + 2u_x q + \bar{q}_1. \] (13)

Let us now substitute these expressions to the defining equations of the coverings (10), (11). As the result, we obtain two Bäcklund auto-transformations of \( T\mathcal{E} \):

\[ \bar{p}_{1,x} = u_{xx} p - u_x p_x + p_t, \]
\[ \bar{p}_{1,y} = u_{xy} p - u_y p_x, \]
\[ \bar{q}_{1,x} = 2v_x p_x - u_x q_x - 2u_{xx} q + v_{xx} p + q_t, \]
\[ \bar{q}_{1,y} = 2v_y p_x - u_y q_x - 2u_{xy} q + v_{xy} p; \] (14)

and

\[ \bar{p}_{2,x} = \frac{1}{u_y} (u_{xy} \bar{p}_2 + p_y), \]
\[ \bar{p}_{2,t} = \frac{1}{u_y} ((u_x u_{xy} - u_y u_{xx}) \bar{p}_2 - u_y p_x + u_x p_y), \]
\[ \ddot{q}_{2,x} = \frac{1}{u_y^2} ((2v_y u_{xy} + u_y v_{xy}) \ddot{p}_2 + 2v_y p_y) - \frac{1}{u_y} (2u_{xy} \ddot{q}_2 - q_y), \]
\[ \ddot{q}_{2,t} = \left( \frac{u_x v_{xy} - 2v_x u_{xy}}{u_y} \right) \dddot{p}_2 + \left( \frac{2v_y u_x}{u_y^2} - \frac{2v_x}{u_y} \right) \dddot{p}_y + \frac{2(u_x u_{xx} - u_x u_{xy})}{u_y} \ddot{q}_2 + \frac{u_x}{u_y} q_y - q_x. \] (15)

Thus, relations (14) and (15) define recursion operators for symmetries of Equation (4). The \( p \)-components of these operators give recursion operators for the rdDym equation (cf. [3]).

**Remark 3** Though formulas (14) and (15) look different, essentially they present the same object. Namely, if we resolve (15) with respect to the variables \( p_x, p_y, q_x, \) and \( q_y \), we shall obtain relations (14) up to some relabeling. Consequently, the above presented Bäcklund transformations are mutually inverse.

**Remark 4** Note also that (14), understood as a covering over \( TE \) with the nonlocal variables \( \ddot{p} \) and \( \ddot{q} \) by the very construction is equivalent to the one with the nonlocal variables \( w_3 \) and \( w_4 \). Explicitly, the corresponding gauge transformation is given by formulas (13). In a similar way, (12) delivers equivalence between (15) and the covering with the variables \( w_1 \) and \( w_2 \).

### 3.6 Actions

Let us indicate how the operator (14) acts on symmetries of the equation \( \mathcal{E} \); due to Remark 3, the action of (15) is the opposite. Note also that of (14) is defined up to the image of 0, which is \( S_2(T) \). Keeping in mind these remarks, we have:

\[
\begin{align*}
0 & \mapsto S_2(T) \mapsto S_3(T) \mapsto S_4(T) \mapsto * \\
S_1 & \mapsto * \\
S_5(Y) & \mapsto 0 \\
S_{11} & \mapsto -\frac{1}{2} S_{10} \mapsto 2S_6 \mapsto * \\
S_7 & \mapsto -S_8 \mapsto * \\
S_9 & \mapsto * \\
S_{12} & \mapsto * \\
S_{15}(T) & \mapsto -2S_{13}(T) \mapsto \frac{3}{2}(S_{13}(T) - S_{14}(T)) \\
S_{14}(T) & \mapsto * \\
S_{16}(Y) & \mapsto 0.
\end{align*}
\]

Here \( * \) denotes a nonlocal result.
3.7 Lifts of shadows

Note finally that any nonlocal symmetry in the coverings (10) and (11) is determined by the coefficients at $\partial/\partial u$, $\partial/\partial v$, $\partial/\partial p$, $\partial/\partial q$, $\partial/\partial w_1$, $\partial/\partial w_2$ (in the case of (10)), and $\partial/\partial w_3$, $\partial/\partial w_4$ (for (11)). Denote these coefficients by $U$, $V$, $P$, $Q$, $W_1$, $W_2$, $W_3$, $W_4$, respectively. We state that there exist nonlocal symmetries that are the lifts of the shadows $s_1$ and $s_2$ to the corresponding coverings. Namely, these symmetries are

$$\sigma_1: \quad U = u_y w_1,$$
$$V = \frac{w_2}{u_y} + v_y w_1,$$
$$P = w_1 p_y,$$
$$Q = w_1 q_y + 2 \frac{p_y w_2}{u_y^3},$$
$$W_1 = w_1 w_{1,y},$$
$$W_2 = 2w_{1,y} w_2 - w_{2,y} w_1$$

and

$$\sigma_2: \quad U = u_x p - w_3,$$
$$V = -2u_x q + v_x p - w_4,$$
$$P = p_x p,$$
$$Q = 2qp_x + q_x p,$$
$$W_3 = pp_t + 2p_t q - v_x pp_x + u_x p q_x + 2u_x p x q,$$
$$W_4 = pp_t - 2u_x pp_x.$$

Moreover, $\sigma_i$ are odd vector fields of degree 1. A direct computation shows that the super-commutators $[\sigma_1, \sigma_1]$ and $[\sigma_2, \sigma_2]$ of these fields vanish, which means that the constructed recursion operators are hereditary, [13].

4 Conclusions

The method used here to construct recursion operators for symmetries of Eq. (4) seems to be of a universal nature. In particular, it allows to test rigorously the hereditary property of nonlocal recursion operators. It would be interesting to apply it to other two-component extensions of linearly degenerate equations constructed in [2], as well as to the Dunajski equation [7], etc.
It would be also interesting to describe (similar to how it was done in [3]) the algebra of nonlocal symmetries (and the corresponding coverings) under the action of recursion operators described above.

**Acknowledgements** Computations were done using the JETS [4] and CADABRA [6,23,24] software. The authors are grateful to O.I. Morozov for discussion.

**Data availability** The authors confirm that the data supporting the findings of this study are available within the article [and/or] its supplementary materials.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**References**

1. Baran, H., Krasil’chik, I.S., Morozov, O.I., Vojčák, P.: Coverings over Lax integrable equations and their nonlocal symmetries. Theor. Math. Phys. 188 (3) 1273–1295 (2016). Russian version 188 (3) 361–385 (2016), https://doi.org/10.1134/S0040577916090014, arXiv:1507.00897
2. Baran, H., Krasil’schik, I.S., Morozov, O.I., Vojčák, P., Higher symmetries of cotangent coverings for Lax-integrable multi-dimensional partial differential equations and Lagrangian deformations. Journal of Physics: Conference Series, 482, 012002, Physics and Mathematics of Nonlinear Phenomena, (PMNP2013) 22–29 June 2013. Gallipoli, Italy (2013). arXiv:1309.7435
3. Baran, H., Krasil’schik, I.S., Morozov, O.I., Vojčák, P.: Nonlocal Symmetries of Integrable Linearly Degenerate Equations: A Comparative Study. Theor. Math. Phys. 196, 1089–1110 (2018). https://doi.org/10.1134/S0040577918080019, arXiv:1611.04938
4. Baran, H., Marvan, M.: Jets. A software for differential calculus on jet spaces and diffieties, https://doi.org/jets.math.slu.cz, http://jets.math.slu.cz
5. Bocharov, A.V., Symmetries of Differential Equations in Mathematical Physics and Natural Sciences, edited by A.M. Vinogradov and I.S. Krasil’schik). Factorial Publ. House, 2nd edition, et al.: in Russian, p. (1999). Amer. Math. Soc, English translation (2005)
6. Cadabra: a field-theory motivated approach to computer algebra, https://cadabra.science/
7. Dunajski, M.: A class of Einstein-Weyl spaces associated to an integrable system of hydrodynamic type. J. Geom. Phys. 51, 1, 126–137 (2004), https://doi.org/10.1016/j.geomphys.2004.01.004, arXiv:nlin:0311024
8. Ferapontov, E.V., Moss, J.: Linearly degenerate partial differential equations and quadratic line complexes. Commun. Anal. Geom. 23(1), 91–127 (2015). https://doi.org/10.4310/CAG.2015.v23.n1.a3. arXiv:1204.2777
9. Krasil’chik, I.S., Kersten, P.H.M.: Deformations and recursion operators for evolution equations, Memorandum of the Twente University (1992), no. 1104, Enschede, 47 pp. Also, In: Prastaro, A., Rassias, Th. M. (eds.) Geometry in Partial Differential Equations. World Scientific, Singapore (1994)
10. Krasil’chik, I.S., Verbeytovksy, A.: Geometry of jet spaces and integrable systems. J. Geom. Phys. 61(9), 1633–1674 (2011). arXiv:1002.0077
11. Kersten, P.H.M., Krasil’schik, I.S.: Symmetries and recursion operators for classical and supersymmetric differential equations. Kluwer Academic Publishers, Dordrecht (2000)
12. Kersten, P., Krasil’schik, I.S., Verbeytovsky, A.M.: Hamiltonian operators and ℓ∞-coverings, J. Geom. Phys. 50, 273–302 (2004). https://doi.org/10.1016/j.geomphys.2003.09.010. arXiv:math/0304245
13. Krasil’schik, I.S., Verbeytovsky, A.M., Vitolo, R.: The Symbolic Computation of Integrability Structures for Partial Differential Equations, Texts & Monographs in Symbolic Computation, Springer, Berlin (2017)
14. Krasil’schik, I.S., Vinogradov, A.M.: Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations. Acta Appl. Math. 15(1–2), 161–209 (1989). https://doi.org/10.1007/BF00131935
15. Kupershmidt, B.A.: Dark equations. J. Nonlin. Math. Phys. 8(3), 363–445 (2001). arXiv:nlin/0107076
16. Martínez Alonso, L., Shabat, A.B.: Hydrodynamic reductions and solutions of the universal hierarchy. Theor. Math. Phys. 140(2), 1073–1085 (2004). https://doi.org/10.1023/B:TAMP.0000036538.41884.57. arXiv:nlin/0312043
17. Marvan, M.: Another look on recursion operators. In: Differential Geometry and Applications, Proc. Conf. Brno, 1995 (Masaryk University, Brno, 1996) 393–402
18. Marvan, M.: Sufficient set of integrability conditions of an orthonomic system. Found. Comput. Math. 9(6), 651–674 (2009)
19. Morozov, O.I.: Recursion operators and nonlocal symmetries for integrable rmdKP and rdDym equations, arXiv:1202.2308
20. Morozov, O.I.: A two-component generalization of the integrable rdDym equation, SIGMA 8, 051, 5 pp (2012), https://doi.org/10.3842/SIGMA.2012.051, arXiv:1205.1149
21. Morozov, O.I.: A recursion operator for the universal hierarchy equation via Cartan’s method of equivalence. Cent. Eur. J. Math. 12(2), 271–283 (2014). https://doi.org/10.2478/s11533-013-0345-2. arXiv:1205.5748
22. Ovsienko, V.: Bi-Hamiltonian nature of the equation $u_{tx} = u_{xy} u_y - u_{yy} u_x$. Adv. Pure Appl. Math. 1, 7–17 (2010). https://doi.org/10.1515/apam.2010.002. arXiv:0802.1818
23. Peeters, K.: Cadabra2: computer algebra for field theory revisited. J. Open Source Softw. 3, 32, 1118 (2018). https://doi.org/10.21105/joss.01118
24. Peeters K., Introducing Cadabra: a symbolic computer algebra system for field theory problems, arXiv:hep-th/0701238
25. Sergiyev, A.: A simple construction of recursion operators for multidimensional dispersionless integrable systems. J. Math. Anal. Appl. 454(2), 468–480 (2017). arXiv:1501.01955

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