ARITHMETIC STATISTICS OF GALOIS GROUPS

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Abstract. We develop a computational framework for the statistical characterization of Galois characters with finite image, with application to characterizing Galois groups and establishing equivalence of characters of finite images of \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \).

1. Introduction

The absolute Galois group \( G = \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) is a fundamental object of study in number theory. The objective of this work is to develop an explicit computational framework for the study of its finite quotients. We may replace \( G \) with the absolute Galois group of any global field, but restrict to that of \( \mathbb{Q} \) for simplicity of exposition.

As point of departure, we consider an irreducible polynomial \( f(x) \in \mathbb{Z}[x] \) of degree \( n \) as input. We set \( K = \mathbb{Q}[x]/(f(x)) \), denote by \( L \) its normal closure and by \( G(K) \) the Galois group \( \text{Gal}(L/\mathbb{Q}) \) equipped with a permutation representation in \( S_n \) determined by the action on the roots of \( f(x) \). Let \( \mathcal{P}_S(\mathbb{Z}) \) be the set of primes, coprime to the finite set \( S \) of primes ramified in \( \mathbb{Z}[x]/(f(x)) \).

The statistical perspective we develop expresses the map from \( \mathcal{P}_S(\mathbb{Z}) \) to factorization data as an equidistributed map to a finite set \( \mathcal{X}(K) \) equipped with a probability function induced from the Haar measure on \( G(K) \). A Frobenius lift at \( p \), defined up to conjugacy, acts on the roots of \( f(x) \). The permutation action on the roots of \( f(x) \) induces a representation in \( O(n) \), fixing the formal sum of the roots. The orthogonal complement gives the standard representation in \( O(n - 1) \), spanned by differences of basis elements. Let \( P(x) \) be the characteristic polynomial of Frobenius in the permutation representation and

\[
S(x) = P(x)/(x-1) = x^{n-1} - s_1x^{n-2} + \cdots + (-1)^{n-1}s_{n-1}.
\]

the characteristic polynomial in the standard representation. This polynomial is independent of choices of lift of Frobenius and choice of basis. As such, the coordinates \( (s_1,\ldots,s_{n-1}) \in \mathbb{Z}^{n-1} \) are invariants of the Frobenius conjugacy class \( \text{Frob}_p \) in the set \( \mathcal{O}(G(K)) \) of conjugacy classes of \( G(K) \). Denote the finite set of such class points by \( \mathcal{X}(K) \). We note that the class points are entirely determined by the factorization data of \( f(x) \mod p \), and \( \mathcal{X}(K) \subset \mathbb{Z}^{n-1} \) is equipped with the structure of a finite probability space, induced from the cover \( \mathcal{O}(G(K)) \to \mathcal{X}(K) \). The irreducible characters are known to form an orthogonal basis for the class functions on \( \mathcal{O}(G(K)) \), and the rational characters are integer-valued class functions on the class space \( \mathcal{X}(K) \).

In what follows we develop this approach by describing systems of rational characters on \( G(K) \) algebraically as a basis of polynomials in \( \mathbb{Z}[s_1,\ldots,s_{n-1}] \) modulo the defining ideal for \( \mathcal{X}(K) \), together with their associated inner product. As a consequence we develop algorithms for the characterization of Galois groups, and more generally, tools for determining equivalence of finite Galois representations.

2. Representations of orthogonal groups

Let \( G \) be a compact Lie group. In practice, \( G \) will be an orthogonal group

\[
G = O(n-1) \subset O(n) \text{ or } G = \text{SO}(n-1) \subset O(n-1),
\]

or a finite permutation group, equipped with the standard representation in \( O(n-1) \),

\[
G \subseteq S_n \subset O(n-1) \text{ or } G \subseteq \mathcal{A}_n \subset \text{SO}(n-1).
\]
The standard representation of $S_n$ provides the motivation for an algebraic presentation of the character ring of a permutation group. For the character theory of permutation groups, we appeal to known algorithms for its computation.

The symmetric group $S_n$ acts on a set of $n$ elements, and the linear extension to a basis of $\mathbb{Z}^n \subset \mathbb{R}^n$ gives the permutation representation of $S_n$. Denote a basis $\{e_1, \ldots, e_n\}$. Since $e_1 + \cdots + e_n$ is fixed by $S_n$, a line is fixed, and we consider the action on the hyperplane spanned by the orthogonal complement. In the basis $\{e_1-e_2, \ldots, e_{n-1}-e_n\}$, we obtain the standard representation of $S_n$ in $O(n-1)$. The choice of basis is noncanonical, but the character theory is independent of any such choice. The orthogonal group $O(n)$ and its subgroup $O(n-1)$ have two connected components, with principal component $SO(n-1) \subset SO(n)$, such that $A_n = S_n \cap SO(n-1).

**Representation ring.** For a compact Lie group $G$, we denote the set of conjugacy classes of $G$ by $\mathcal{G}(G)$. We define the representation ring of $G$,

$$\mathcal{R}(G) = \bigoplus \mathbb{Z}\chi,$$

as the free abelian group on irreducible characters $\chi : G \to \mathbb{C}$ of finite degree. We identify addition with direct sum, and thereby the abelian submonoid $\bigoplus \mathbb{N}\chi \subseteq \mathcal{R}(G)$ with characters, and define multiplication on $\mathcal{R}(G)$ by the linear extension of tensor product on $\bigoplus \mathbb{N}\chi$. We refer to elements of $\mathcal{R}(G)$ as virtual characters.

As class functions, $\mathcal{R}(G)$ can be identified with a subring of complex-valued functions on $\mathcal{G}(G)$. Indeed, when $G$ is finite, the number $h$ of conjugacy classes (and of irreducible characters) is finite, and the character table is defined as the evaluation vectors

$$(\chi_i(C_1), \ldots, \chi_i(C_h)).$$

in the ring $\mathbb{C}^h = \mathbb{C} \times \cdots \times \mathbb{C}$, for $\chi_i$ running over the irreducible characters, forming a generator set for the representation ring. For a subfield $F \subset \mathbb{C}$, we denote by $\mathcal{R}_F(G)$ the subring of $F$-valued virtual characters. While $\mathcal{R}(G) = \mathcal{R}_\mathbb{Q}(G)$ for $G = S_n$ or $G = O(n-1)$, for a general finite group that we may consider, the field of definition of an irreducible character may be a proper extension of $\mathbb{Q}$.

Considering the group $O(n)$ in $GL_n(\mathbb{R})$, an element $g$ satisfies a characteristic polynomial of the form

$$x^n - s_1x^{n-1} + \cdots + (-1)^n s_n.$$

The coefficient $s_1$ is the trace in its representation on $\mathbb{R}^n$, and $s_n$ is its determinant character. We note that $s_k$ is an invariant of the class of $g$, and we can identify $g \mapsto s_k$ as characters. Specifically, $s_k$ is the character on the $k$-th exterior power $\bigwedge^k \mathbb{R}^n$. We recall the structure of the character ring for $O(n)$ (cf. Takeuchi [20]).

**Lemma 1.** The virtual character ring $\mathcal{R}(O(n))$ is generated by $s_k$, $1 \leq k \leq n$, and

$$\mathcal{R}(O(n)) \cong \frac{\mathbb{Z}[s_1, \ldots, s_n]}{(s_general}.$$

The restriction $\text{Res} : \mathcal{R}(O(n)) \to \mathcal{R}(SO(n))$ surjects on

$$\mathcal{R}(SO(n)) \cong \frac{\mathbb{Z}[s_1, \ldots, s_n]}{(s_k - s_{n-k}, s_n - 1)}$$

with kernel ideal $(s_n - 1)$.

**Remark.** If $n = 2m$ or $n = 2m+1$, then $\mathcal{R}(SO(n)) = \mathbb{Z}[s_1, \ldots, s_m]$, and $\mathcal{R}(O(n))$ is an extension by the quadratic character $\xi = s_n$ such that $\xi|_{SO(n)} = 1.$
Algebraic parametrization. If $H$ is a subgroup of $G$, there is an induced map $\mathcal{A}(H) \to \mathcal{A}(G)$ on conjugacy classes and concomitant restriction homomorphism $\text{Res} : \mathcal{R}(G) \to \mathcal{R}(H)$ on representation rings. Applied to the standard representation of $S_n$ in $O(n-1)$, the restriction homomorphism equips the representation ring of $\mathcal{R}(S_n)$ with a surjective restriction map from $\mathcal{R}(O(n-1))$, giving an algebraic presentation of $\mathcal{R}(S_n)$ by polynomials in $\mathbb{Z}[s_1, \ldots, s_{n-1}]$ modulo the defining ideal $(s_k s_{n-1} - s_{n-k-1}, s_{n-1}^2 - 1)$. Given a permutation group $G \subseteq S_n$, the subsequent restriction captures a significant subring of $\mathcal{R}_G(G) \subseteq \mathcal{R}(G)$.

As a tool to characterize permutation groups in $S_n$, for subgroups $G$ and $H$, with $H \subseteq G \subseteq S_n$, we develop the branching rules — explicit forms for the decomposition

$$\text{Res}(\chi_i) = \sum_{j=1}^{n_i} a_{ij} \psi_j,$$

of irreducible characters $\{\chi_1, \ldots, \chi_r\}$ on $G$ in terms of the irreducible characters $\{\psi_1, \ldots, \psi_s\}$ on $H$. In light of the algebraic parametrization by $\mathbb{Z}[s_1, \ldots, s_{n-1}]$, we deduce the kernel ideals $I_G \subseteq I_H$ for each permutation group in the lattice (poset) of subgroups. A basis of generators provides test functions for membership in a given subgroup. We develop the algorithmic details later.

Using the Brauer-Klimyk formula (see Bump [4 Proposition 22.9]), it is possible to develop recursive formulas for the character theory of orthogonal groups, as done in Shieh [18, 19] for $\text{USp}(2m)$, and using the algebraic presentation, to deduce recursive branching rules for $\text{Res} : \mathcal{R}(O(n-1)) \to \mathcal{R}(G)$. Instead, we content ourselves with the algebraic parametrization from $\mathcal{R}(O(n-1))$ and exploit the well-established computational character theory of permutation groups to develop branching rules in the lattice of permutation subgroups of $S_n$.

3. Representations of permutation groups

Let $G$ be a permutation group — a finite group equipped with an embedding in $S_n$. The cycle type of $g \in G$ is the multiset of cardinalities of its orbits under the action of $S_n$ on $\{1, \ldots, n\}$. A multiset can be denoted by a tuple $(d_1, \ldots, d_t)$ or a formal product $m_1^{e_1} \cdots m_s^{e_s}$, where

$$d_1 \leq d_2 \leq \cdots \leq d_t \text{ or } m_1 < \cdots < m_s \text{ such that } \sum_{i=1}^{t} d_i = \sum_{i=1}^{s} e_i m_i = n.$$

The cycle type is invariant under conjugation in $S_n$, thus the cycle type is well-defined for the conjugacy class $C = C(g) \in \mathcal{A}(G)$, where $C(g) = \{xgx^{-1} : x \in G\}$.

Lemma 2. The map $\mathcal{A}(S_n) \to \{ (d_1, \ldots, d_t) : \sum_{i=1}^{t} d_i = n \}$ from conjugacy classes of $S_n$ to cycle types is a bijection.

Proof. Clearly, giving a cyclic ordering to any partition of $\{1, \ldots, n\}$ into orbits determines an element of $S_n$, hence the map is surjective. Moreover, by definition the symmetric group is $n$-transitive, conjugating any cyclically ordered orbit partition to any another of the same cycle type. Consequently the map is injective.

Remark. For a permutation group $G \subseteq S_n$ the induced map $\mathcal{A}(G) \to \mathcal{A}(S_n)$ in general is neither injective nor surjective. The failure of injectivity means that the cycle type fails to distinguish the conjugacy classes. We will later see this in the failure of $\mathcal{R}(S_n)$ to surject on $\mathcal{R}(G)$. In fact, the irreducible characters are known to form a basis of the class functions on $G$ (cf. Serre [17 Theorem 6]), hence the failure to separate conjugacy classes means that the restriction homomorphism from $\mathcal{R}(S_n)$ does not surject on $\mathcal{R}(G)$.

On the one hand, the cycle type of a conjugacy class characterizes the class. On the other hand, the characteristic polynomial (hence its coefficients) is class invariant of an orthogonal group element, and the permutation and standard representations thus provide other class invariants. We make this association explicit. Given $(d_1, \ldots, d_t)$ be the cycle type of an element $g \in S_n$. It is easy to see that the characteristic polynomial of the permutation representation of $g$ is

$$P(x) = (x^{d_1} - 1) \cdots (x^{d_t} - 1) = (x^{m_1} - 1)^{e_1} \cdots (x^{m_s} - 1)^{e_s}. $$
The eigenvalue on the trivial space is 1, so the characteristic polynomial in the standard representation is

\[ S(x) = \frac{P(x)}{x-1} = x^{n-1} - s_1x^{n-2} + \cdots + (-1)^{n-1}s_{n-1}, \]

and \((s_1, \ldots, s_{n-1})\) is the tuple of class invariants associated to the the conjugacy class \(C(g)\) under the standard representation in \(O(n-1)\). This gives the following lemma.

**Lemma 3.** The map \(\mathcal{O}(S_n) \to \mathbb{Z}^{n-1}\) from conjugacy classes to the \((n-1)\)-tuples \((s_1, \ldots, s_{n-1})\) of coefficients of the characteristic polynomial under the standard embedding is injective.

**Proof.** By Lemma 2 the map from conjugacy classes to cycle types is a bijection. However, by unique factorization in \(\mathbb{Q}[x]\), a polynomial of the form \(x^{d_1-1}\cdots x^{d_t-1}\) is uniquely determined by the cycle type \((d_1, \ldots, d_t)\), hence the map to its coefficients \((s_1, \ldots, s_{n-1})\) is injective. \(\square\)

**Representation rings and character tables.** Let \(G\) be a permutation group and let \(\mathcal{O}(G) = \{C_1, \ldots, C_h\}\), and \(\chi_1, \ldots, \chi_h\) be its irreducible characters. For a conjugacy class \(C\), define the ideal

\[ m_C = \{ f \in \mathcal{R}(G) : f(C) = 0 \}. \]

such that the value \(f(C)\) of a virtual character \(f\) at \(C\) is a well-defined class in the residue class ring \(\mathcal{R}(G)/m_C\). The character table of \(G\) is typically represented as a matrix whose \(i\)-th row is the evaluation vector \((\chi_1(C_1), \ldots, \chi_1(C_h))\). With this notation, we interpret as the embedding of the character \(\chi_i\) in the product ring, under the injection

\[ \mathcal{R}(G) \to \mathcal{R}(G)/m_{C_1} \times \cdots \times \mathcal{R}(G)/m_{C_h}. \]

**Lemma 4.** The image of the homomorphism \(\mathcal{R}(G) \to \mathcal{R}(G)/m_{C_1} \times \cdots \times \mathcal{R}(G)/m_{C_h}\) has finite index in its codomain.

**Proof.** Clearly \(\mathcal{R}(G)\) is torsion-free, since the image of a virtual character is a subring of \(\mathbb{C}\). Thus \(\mathcal{R}(G)\) embeds in \(\mathcal{R}(G) \otimes \mathbb{Q}\), which is an étale algebra, and \(\mathcal{R}(G)/m_{C_i} \otimes \mathbb{Q}\) its component fields (see Brakenhoff [3] for details). It follows that the index is finite. \(\square\)

More generally in the direction of the Lemma, Brakenhoff finds that the center of the group ring \(\mathbb{Q}[G]\) over \(\mathbb{Q}\) and the tensor product of the representation ring \(\mathcal{R}(G) \otimes \mathbb{Q}\) are related by Brauer equivalence. We give two examples below. In view of the restriction map from \(\mathcal{R}(S_n)\) to \(\mathcal{R}(G)\), and since all characters on \(S_n\) are rational, the image of \(\mathcal{R}(S_n) = \mathcal{R}_Q(S_n)\) lies in the subring \(\mathcal{R}_Q(G) \subset \mathcal{R}(G)\). In the examples below, we illustrate the role of nontrivial Galois action and of quadratic characters in failure of surjectivity of \(\mathcal{R}(S_n)\) on \(\mathcal{R}(G)\) and on \(\mathcal{R}_Q(G)\). In the next section we exploit the embedding by interpolating the character table values by the polynomial presentation \(\mathbb{Z}[s_1, \ldots, s_{n-1}] \to \mathcal{R}_Q(G)\).

**Orthogonality relations.** The role of arithmetic statistics of \(G\) comes from the orthogonality relations for the irreducible characters. Let \(\{\chi_1, \ldots, \chi_h\}\) be the irreducible characters for \(G\), and \(A(G)\) be the character matrix:

\[ A(G) = \begin{bmatrix} \chi_1(C_1) & \cdots & \chi_1(C_h) \\ \vdots & \ddots & \vdots \\ \chi_h(C_1) & \cdots & \chi_h(C_h) \end{bmatrix} \]

The orthogonality relations for characters (see Serre [17] Section 2.3)), expressed in terms of group elements, reformulated in terms of conjugacy classes, takes the form

\[ \delta_{ij} = \langle \chi_i, \chi_j \rangle_G := \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \sum_{k=1}^h |C_k| \chi_i(C_k) \chi_j(C_k). \]

Set \(D(G)\) to be the diagonal matrix with diagonal entries \((p_1, \ldots, p_h)\), where \(p_k = |C_k|/|G|\) is the weight of the conjugacy class \(C_k\). The orthogonality relations are then expressed by the equality

\[ I_h = A(G)D(G)A(G)^\dagger, \]
where $\dagger$ denotes the conjugate transpose. The matrix $D(G)$ can be viewed as the inner product matrix of the Haar measure induced by $G$ on $\mathcal{O}(G)$.

**Rational character table.** Let $\chi$ be a character on $G$, let $m$ be the exponent of $G$, and let $\mathcal{C} = \mathcal{C}(g)$ be a conjugacy class. As the trace of a representation of $g$, the value $\chi(C)$ lies in $\mathbb{Z}[\zeta_m]$, since each of its eigenvalues are in $\mu_m = \{\zeta_m\}$. We thus obtain two actions of the Galois group $\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*$. Denote $\sigma : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ the isomorphism such that $\zeta_m^\sigma = \zeta_m^k$. The first of the actions is on conjugacy classes, by $\mathcal{C}(g) \rightarrow \mathcal{C}(g^k)$, and the second on characters by $\chi^\sigma(\mathcal{C}(g)) = \chi(\mathcal{C}(g))^\sigma$. Considering the action on eigenvalues we see immediately that

$$\chi^\sigma(\mathcal{C}(g)) = \chi(\mathcal{C}(g^k)).$$

**Restriction from $\mathfrak{R}(S_n)$.** Only characters in the image of $\mathfrak{R}(S_n)$ can be parametrized by polynomials in $\mathbb{Z}[s_1, \ldots, s_{n-1}]$ from the standard representation. We note by example, that the pre-image of $\mathcal{C}$ in $\mathfrak{R}(S_n)$ under the induced map $\mathfrak{O}(G) \rightarrow \mathfrak{O}(S_n)$ can split into an even number of conjugacy class separated by a quadratic character not coming from $S_n$. We observe this phenomenon for $G = D_4$ and $G = Q_8$ in the examples section below.

### 4. Algorithms for Galois representations

In what follows we describe algorithms for testing equivalence of finite Galois characters. As principal application, we consider input $f(x)$ of degree $n$, determining a number field $K = \mathbb{Q}[x]/(f(x))$, and describe how to evaluate a sample set of primes $S$ at characters on the permutation group $G(K)$. The approach is completely general, allowing one to compare the set of characters on the absolute group $G$ mapping through permutation groups $G(K_1)$ and $G(K_2)$ determined by number fields $K_1$ and $K_2$.

**Factorization types of irreducible polynomials.** Consider an irreducible polynomial $f(x)$ in $\mathbb{Z}[x]$ of degree $n$, set $K = \mathbb{Q}[x]/(f(x))$ and let $L$ be its normal closure with maximal order $\mathcal{O}_L$.

For a rational prime $p$ and prime $\mathfrak{p}$ over $p$ in $\mathcal{O}_L$, the Frobenius lift $\text{Frob}_p$ is the unique element of the decomposition subgroup $D_{\mathfrak{p}} \subset G = G(K)$, such that

$$\text{Frob}_p(a) = a^p \mod \mathfrak{p}$$

for all $a$ in $\mathcal{O}_L$. Denote by $\text{Frob}_p$ the conjugacy class of $\text{Frob}_p$ in $\mathfrak{O}(G)$.

For $p$ not dividing $\text{disc}(f(x))$ we define the **factorization type** of $f(x)$ mod $p$ to be the multiset of degrees of the factorization of $f(x)$ in $\mathbb{F}_p[x]$, which we may denote $(d_1, \ldots, d_t)$, where $d_1 \leq \cdots \leq d_t$ and $d_1 + \cdots + d_t = n$. We can now identify the data of the factorization type with the cycle type of the Galois group $G = G(K)$ equipped with its embedding in $S_n$.

**Lemma 5.** The factorization type of $f(x)$ mod $p$ is the cycle type of $\text{Frob}_p \subset G(K)$.

**Proof.** The factorization $p\mathcal{O}_K = \mathfrak{p}_1 \cdots \mathfrak{p}_t$ is determined from $f(x) \equiv f_1(x) \cdots f_t(x) \mod p$, with $\mathfrak{p}_k = (p, f_k(x))$ a prime of degree $d_k = \deg(f_k)$. The Galois group acts transitively on primes of $\mathcal{O}_L$ over $p$, and there exist conjugates $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ over $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$, from which we see that $d_k$ divides $\deg(\mathfrak{p})$, and each $d_k$ is the cardinality of an orbit of roots modulo $p$ under the action of $\text{Frob}_p$. \hfill $\square$

**Character inner products as expectation.** The factorization type of a polynomial gives a means of taking random samples of character values $(s_1, \ldots, s_{n-1})$ at a set $S$ of primes mapping to the group $G$. Other data, for particular characters, may come from weight one modular eigenforms, character sums, or Kronecker symbols. Let $S$ be such a sample set of primes, and $\psi, \chi$ two characters which can be evaluated on $S$. We write $\langle \psi, \chi \rangle$ for the value of the character at a sample point. We obtain an approximation for the orthogonal product $\langle \psi, \chi \rangle$ as the expectation of $\psi \chi$.

$$\langle \psi, \chi \rangle \approx E(\psi \chi) = E_S(\psi \chi) = \frac{1}{|S|} \sum_{p \in S} \psi(p) \chi(p).$$
Assuming the multiplicity of each irreducible character in the support of \( \psi \) and \( \chi \) is one, then \( m = \langle \psi, \chi \rangle \) is an integer counting the number of irreducible characters in the support of both \( \psi \) and \( \chi \). When \( \psi \) and \( \chi \) are irreducible, to determine equality \( \psi = \chi \), one needs only sufficient precision to distinguish the one bit \( \langle \psi, \chi \rangle = 0 \) or \( \langle \psi, \chi \rangle = 1 \).

The interest in working with irreducible characters, or nearly irreducible characters as captured by the image of restriction from \( \mathcal{R}(S_n) \), is that the variance of the character products \( \psi \chi \) is minimized, and the number of primes needed to recognize convergence small, as observed by Shieh [18, 19] in the case of symplectic groups \( \text{USp}(2n) \) (see also Fité and Guitart [8]).

One should note that in view of classifying the Galois group, nonvanishing of an element of the kernel ideal of the restriction \( \mathcal{R}(S_n) \to \mathcal{R}(G) \) can be used to provably exclude \( G \) as a Galois group. This was already observed by Pohst [15], who proposed the use of factorization types as a lower bound for the Galois group, and that for \( n \geq 8 \) the factorization types, and their probabilities, fail to separate groups. This statement, however, concerns the data of the induced Haar measure on \( \mathcal{O}(G) \), and not that of the character table of \( G \). Precisely we have two data structures on \( \mathcal{O}(G) \) at our disposal, that of a probability space and of class functions (given by a character table):

- \( \mathcal{O}(G) \) with Haar measure \( p : \mathcal{O}(G) \to \mathbb{R} \), and
- \( \mathbb{C}^h = \text{Hom}(\mathcal{O}(G), \mathbb{C}) \) with orthonormal basis \( \{\chi_1, \ldots, \chi_h\} \).

Due to failure of surjectivity of the restriction homomorphism from \( \mathcal{R}(S_n) \), the subset of characters determined from the cycle types are unlikely to separate groups for sufficiently large \( n \). Nevertheless, the joint data of Haar measure and character table, plus the system of restriction maps coming from common embeddings in \( S_n \) gives more information than either the Haar measure or character table alone.

**Restriction kernel ideal.** To a conjugacy class \( C \) for \( S_n \) we associate an ideal \( m_C \) in \( \mathbb{Z}[s_1, \ldots, s_{n-1}] \) of the form

\[
m_C = (s_1 - s_1(C), \ldots, s_{n-1} - s_{n-1}(C)),
\]

where \( (s_1(C), \ldots, s_{n-1}(C)) \) are the values of \( s_i \) at \( C \). Then the kernel ideal for the restriction of \( \mathcal{R}(S_n) \) to \( \mathcal{R}(G) \) is the intersection ideal

\[
I(G) = \bigcap_{C \in \pi(\mathcal{O}(G))} m_C,
\]

where \( \pi : \mathcal{O}(G) \to \mathcal{O}(S_n) \).

**Example.** Consider the restriction from \( \mathcal{R}(O(3)) \) to \( \mathcal{R}(S_4) \). Since

\[
\mathcal{R}(O(3)) = \frac{\mathbb{Z}[s_1, s_2, s_3]}{(s_1s_3 - s_2, s_3^2 - 1)}
\]

and the values of \( (s_1, s_2, s_3) \) are in

\[
\{ (3, 3, 1), (-1, -1, 1), (0, 0, 1), (-1, 1, -1), (1, -1, -1) \},
\]

we obtain a defining ideal of \( S_4 \) given by the additional generators:

\[
s_1(s_1 + 1)(s_1 - s_3 - 2), s_1(s_1 + 1)(s_1 - 1)(s_1 - 3), (s_1 + 1)(s_1 - 1)(s_3 - 1).
\]

The map \( \mathcal{O}(D_4) \to \mathcal{O}(S_4) \) fails to surject on \( (0, 0, 1) \), hence there are only four maximal ideals in the intersection and the kernel ideal for \( \mathcal{R}(S_4) \to \mathcal{R}(D_4) \) is generated by:

\[
s_1^2 - s_1 - s_2 - s_3 - 2, s_2 - s_1s_3, s_3^2 - 1.
\]

The first polynomial is not in the kernel ideal for \( \mathcal{R}(S_4) \) and its vanishing provides a test for \( D_4 \). Geometrically, it means that the tensor square of the representation with trace \( s_1 \) decomposes into a direct sum of representations with trace \( s_1 + s_2 + s_3 + 2 \).
**Restriction homomorphism.** Let $H \subset G$ be permutation groups, and set $\ell = |\mathcal{O}(H)|$ and $h = |\mathcal{O}(G)|$ their cardinalities of their conjugacy class sets. Suppose that $\{\psi_1, \ldots, \psi_\ell\}$ and $\{\chi_1, \ldots, \chi_h\}$ are the irreducible characters, which are given by embeddings in $\mathbb{C}^\ell$ and $\mathbb{C}^h$, respectively. We thus have isomorphisms

$$\mathfrak{R}(H) = \bigoplus_{i=1}^\ell \mathbb{Z}\psi_i \rightarrow \Lambda(H) \subset \mathbb{C}^\ell,$$

and

$$\mathfrak{R}(G) = \bigoplus_{j=1}^h \mathbb{Z}\chi_j \rightarrow \Lambda(G) \subset \mathbb{C}^h,$$

where $\Lambda(H)$ and $\Lambda(G)$ are the lattices in $\mathbb{C}^\ell$ and $\mathbb{C}^h$ spanned by the rows of the character table.

The restriction homomorphism $\mathfrak{R}(G) \rightarrow \mathfrak{R}(H)$ is induced by the map $\pi : \mathcal{O}(H) \rightarrow \mathcal{O}(G)$, by

$$\chi \mapsto (\chi(\pi(C_1)), \ldots, \chi(\pi(C_\ell))) \in \Lambda(H) \subset \mathbb{C}^\ell.$$

The linear transformation $\Lambda(G) \rightarrow \Lambda(H)$ gives the restriction homomorphism as an integral $(h \times \ell)$-matrix with respect to the respective bases of irreducible characters. The rows of this matrix can be interpreted as *branching rules*, giving the decomposition of an irreducible character on $G$ as a sum of irreducible characters on $H$.

Inside each $\Lambda(G)$ we have a sublattice (generally of lower rank) $\Lambda_\mathbb{Q}(G) = \Lambda(G) \cap \mathbb{Q}^h$ of rational-valued characters. We recall that for a conjugacy class $C$ of group elements of order $m$, the value of $\chi(C)$ is a sum of eigenvalues in $\mathbb{Q}(\zeta_m)$. We thus obtain an action by the Galois group of a cyclotomic field on the irreducible characters. As a consequence, the lattice $\Lambda_\mathbb{Q}(G)$ is generated by the sums over Galois orbits of irreducible characters. Since these orbits are disjoint, this basis of rational characters remains orthogonal, but not orthonormal, since $\langle \chi, \chi \rangle$ measures the cardinality of the orbit (assuming $\chi$ is a sum of irreducible characters of multiplicity one). On the other hand, the restriction images $\text{Res}_G^S(\Lambda(G)) \subset \Lambda(H)$ and $\text{Res}_H^G(\Lambda_\mathbb{Q}(G)) \subset \Lambda_\mathbb{Q}(H)$ do not possess natural reduced orthogonal bases. In order to determine a generating set which is small with respect to the orthogonality relations on characters, we need to apply a constrained lattice reduction inside the submonoid of characters:

$$\bigoplus_{j=1}^\ell N\psi_j \subset \bigoplus_{j=1}^\ell \mathbb{Z}\psi_j = \mathfrak{R}(H).$$

Rather than a generic LLL algorithm, we need to carry out a structured lattice reduction in the character monoid order to be able to invoke the heuristic arguments for convergence of small characters.

**Algebraic parametrization.** In order to interpret factorization types of polynomials (or splitting types of primes) as conjugacy classes on which we can apply the class functions $s_1, \ldots, s_{n-1}$, we need to find an explicit algebraic parametrization

$$\frac{\mathbb{Z}[s_1, \ldots, s_{n-1}]}{I(S_n)} \rightarrow \mathfrak{R}(S_n) \rightarrow \text{Res}_G^S(\Lambda(S_n)) \subseteq \Lambda(G).$$

The presentation $\mathbb{Z}[s_1, \ldots, s_{n-1}]/I(S_n) \rightarrow \mathfrak{R}(S_n)$ comes from the standard representation of $S_n$, and its composition into $\Lambda(S_n)$ can be effectively computed. In order to lift characters in $\Lambda(S_n)$ back to representative polynomials in $(s_1, \ldots, s_{n-1})$, we must invert

$$\frac{\mathbb{Z}[s_1, \ldots, s_{n-1}]}{I(S_n)} \rightarrow \Lambda(S_n).$$

As noted above, the isomorphism $\mathfrak{R}(S_n) \rightarrow \Lambda(S_n)$ is obtained by the Chinese remainder theorem. More precisely, over $\mathbb{Q}$, we obtain a product decomposition of the étale algebra $\mathfrak{R}(S_n) \otimes \mathbb{Q}$:

$$\mathfrak{R}(S_n) \otimes \mathbb{Q} \rightarrow \mathfrak{R}(S_n)_{m_{c_1}} \otimes \mathbb{Q} \times \cdots \times \mathfrak{R}(S_n)_{m_{c_h}} \otimes \mathbb{Q} \cong \mathbb{Q}^h,$$

under which $\mathfrak{R}(S_n) \cong \Lambda(S_n) \subseteq \mathbb{Z}^h$. Since the generators $s_1, \ldots, s_{n-1}$ can be evaluated at conjugacy classes, we can evaluate a basis of monomials modulo $I(S_n)$ and invert a matrix to determine the pre-image of a basis of irreducible characters. The same applies to a basis of characters in $\text{Res}_G^S(\Lambda(S_n))$ modulo the restriction kernel $I(G)$. 

5. EXPLICIT COMPUTATIONS

We illustrate the approach through arithmetic statistics of character theory by applying the methods to groups of low degree. First we analyze the dihedral and quaternionic groups $D_4$ and $Q_8$ of order 8, the smallest groups sharing the same character table. Then we consider an example of a pair of permutation groups of degree 8 and order 16 whose cycle types and induced Haar measure on $S_8$-conjugacy classes are equal. We show how an auxiliary (sub)field suffices to distinguish the characters using joint Frobenius cycle data. In a final example, we treat different permutation representations of $A_5$, to show how this approach can be used to establish the equivalence of the absolute Galois representations determined by different fields.

Dihedral and quaternionic groups of order 8. The groups $D_4$ and $Q_8$, known to share the same character table, can nevertheless be separated by the restriction data coming from a permutation representation. We first recall that the common character table takes the form

\[
A(G) = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1-1 \\
1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
2 & -2 & 0 & 0
\end{bmatrix}
\]

with weights $(1/8, 1/8, 1/4, 1/4, 1/4)$ on the conjugacy classes. The semisimple group algebras $\mathbb{Q}[D_4]$ and $\mathbb{Q}[Q_8]$ have Wedderburn decompositions

\[
\mathbb{Q}[D_4] \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q}), \quad \text{and} \quad \mathbb{Q}[Q_8] \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{H},
\]

where $\mathbb{H}$ is the quaternion algebra over $\mathbb{Q}$ ramified at 2 and $\infty$. These decompositions correspond to the four linear characters and sole degree-2 irreducible representation.

Only the former group, $D_4$, embeds in $S_4$, which shows that the permutation embedding contains distinguishing information not in the character table. We make explicit the above approach through character theory for the degree-4 permutation representation. Let $\{1, \chi_1, \chi_2, \chi_3, \chi_4\}$ be a basis of characters, with $\chi_1, \chi_2$, and $\chi_3 = \chi_1 \chi_2$ quadratic linear characters, and $\chi_4$ of degree 2. The standard representation of $S_4$ in $O(3)$ provides irreducible characters

\[
\{1, s_1, s_2, s_3, s_1^2 - s_1 - s_2 - 1\}
\]

where $s_3$ is the quadratic determinant character, $s_1$ and $s_2 = s_1 s_3$ are degree-3 representations, and the last one is of degree 2. Computing the inner product matrices for these characters on $S_4$ and $D_4$, we obtain

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 2
\end{bmatrix}.
\]

For example, this was the output to the nearest integer for the expectation method on a sample size of 16 unramified primes, for the polynomials $x^4 + x + 1$ and $x^4 - 2x^2 + 2$ with respective Galois groups $S_4$ and $D_4$.

One identifies the polynomial expression $\chi = s_3^2 - s_1 - s_2 - 1$ for the irreducible degree-2 character $\chi$ on $S_4$, which decomposes into a direct sum $1 + s_3$ on $D_4$, from which we deduce that $s_3^2 - s_1 - s_2 - s_3 - 2$ is in the kernel ideal $I(D_4)$. Similarly, we read from the inner products $\langle s_1, s_1 \rangle = \langle s_2, s_2 \rangle = 2$ and $\langle s_1, s_2 \rangle = 1$ on $D_4$ that each of $s_1$ and $s_2$ decompose into two irreducible
characters, which share a common irreducible summand. The restriction homomorphism from $\mathcal{R}(S_4)$ thus captures

$$1, s_1 = \chi_1 + \chi_4, s_2 = \chi_2 + \chi_4, s_3 = \chi_3.$$  

The restriction fails to span all characters, because the conjugacy classes are not separated by characters on $S_4$. Indeed the cycle types of the five conjugacy classes in $\mathcal{O}(D_4)$ are $1^4$, $1^22^1$, $2^2$, $2^2$, and $4^1$, and hence the two classes of cycle type $2^2$ map to the same class in $\mathcal{O}(S_4)$.

The missing character $\chi_1$ is easily recovered. It arises from the quadratic subfield (here with defining polynomial $x^2 - 2x + 2$), which can be expressed as a Legendre symbol. In terms of the basis of characters $\{1, s_1, s_2, s_3, \chi_1\}$, we now obtain an inner product matrix:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 1 \\
0 & 0 & 1 & 2 & 0 \\
0 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

which can be reduced to an orthonormal basis for $\mathcal{R}(D_4)$.

Since both $D_4$ and $Q_8$ admit permutation representations of degree 8, we carry out a similar analysis of the permutation representations of degree 8 for $D_4$ and $Q_8$, given by

$$D_4 \cong \left\{ (1,8),(2,7),(3,4),(5,6), \right\} \text{ and } Q_8 \cong \left\{ (1,2,4,7)(3,6,8,5), (1,3,4,8)(2,5,7,6) \right\}.$$

The cycle types $1^8, 2^4, 4^2$ arise with probabilities $(1/8, 5/8, 1/4)$ in $D_4$ whereas in $Q_8$, these same types have probabilities $(1/8, 1/8, 3/4)$. Both groups embed in $A_8 \subset SO(7)$, hence the character rings are parametrized by $\mathcal{R}(SO(7)) \cong \mathbb{Z}[s_1, s_2, s_3]$ ($s_7 = 1$ and $s_4 = s_3, s_5 = s_2, s_6 = s_1$). Since the cycle types are the same, the kernel ideals agree, but the Haar measures differentiate the groups. However, a naive tabulation of the probabilities gives a poor empirical invariant. In fact, computing these probabilities is tantamount to evaluating the expectations of the idempotents $e_1, e_2, e_3$ under the isomorphism

$$\mathcal{R}(G) \otimes \mathbb{Q} = \frac{\mathbb{Q}[s_1, s_2, s_3]}{I(G) \otimes \mathbb{Q}} \to \mathcal{R}(G) \otimes \mathbb{Q} \times \mathcal{R}(G) \otimes \mathbb{Q} \times \mathcal{R}(G) \otimes \mathbb{Q} \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}.$$  

To express this computation in the character ring framework, we scale by the group order to have integer values. As a general strategy for a group $G \subset S_n$ this amounts to asking whether the scaled idempotents converge to

$$(\langle |G|e_1, 1 \rangle, \ldots, \langle |G|e_n, 1 \rangle) = (|C_{\ell_1}|, \ldots, |C_{\ell_s}|),$$

where $C_{\ell}$ are the $S_n$-conjugacy classes for $G$.

Let $\{1, \chi_1, \chi_2, \chi_3, \psi\}$ be a basis of irreducible characters for $D_4$, and $\{1, \chi'_1, \chi'_2, \chi'_3, \psi'\}$ be a basis of irreducible characters for $Q_8$. The parametrization gives a $\mathbb{Q}$-basis $\{1, s_1, s_2\}$ and an idempotent basis $\{e_1, e_2, e_3\}$ which are characteristic functions for the evaluations on conjugacy classes. A reduced basis for the image of $\mathcal{R}(S_4)$ in $\mathcal{R}(D_4)$ is $\{1, \sigma_1, \sigma_2\}$, described as follows in these respective bases:

$$
\begin{array}{c|cccc}
D_4 & 1 & s_1 & s_2 & \{e_1, e_2, e_3\} \\
1 & 1 & e_1 + e_2 + e_3 \\
\sigma_1 & -s_1 + s_2/2 - 1/2 & 2e_1 - e_2 + e_3 & \chi_1 + \psi \\
\sigma_2 & 2s_1 - s_2/2 + 1/2 & 4e_1 - 2e_3 & \chi_1 + \chi_2 + \chi_3 \\
\end{array}
$$

Similarly, a reduced basis for the image of $\mathcal{R}(S_8)$ in $\mathcal{R}(Q_8)$ is $\{1, \tau_1, \tau_2\}$, expressed in the respective bases as follows:

$$
\begin{array}{c|cccc}
Q_8 & 1 & s_1 & s_2 & \{e_1, e_2, e_3\} \\
1 & 1 & e_1 + e_2 + e_3 \\
\tau_1 & -s_1 + s_2/2 - 3/2 & 2e_1 - 2e_2 & \psi' \\
\tau_2 & 3s_1 - s_2 + 3 & 3e_1 + 3e_2 - e_3 & \chi'_1 + \chi'_2 + \chi'_3 \\
\end{array}
$$
Relative to the parametrizations from $\mathfrak{R}(\text{SO}(7))$, the bases $(\sigma_1, \sigma_2)$ and $(\tau_1, \tau_2)$ are related by $(\sigma_1, \sigma_2) = (\tau_1 + 1, \tau_1 + \tau_2 - 1)$, and inversely $(\tau_1, \tau_2) = (\sigma_1 - 1, \sigma_1 + \sigma_2 + 2)$. We thus express $(8e_1, 8e_2, 8e_3)$ in the respective bases:

| $8e_1$ | $s_1 + 1$ | $1 + \sigma_1 + \sigma_2$ | $1 + 2\tau_1 + \tau_2$ | 
| $8e_2$ | $5s_1 - 2s_2 + 7$ | $5 - 3\sigma_1 + \sigma_2$ | $1 - 2\tau_1 + \tau_2$ | 
| $8e_3$ | $-6s_1 + 2s_2$ | $2 + 2\sigma_1 - 2\sigma_2$ | $6 - 2\tau_2$ |

Giving inclusions of submodules $\langle 8e_1, 8e_2, 8e_3 \rangle \subset \langle 1, \sigma_2, \sigma_3 \rangle = \langle 1, \tau_2, \tau_3 \rangle \subset \langle e_1, e_2, e_3 \rangle$.

Computing the expectations of the test functions $\{1, \sigma_1, \sigma_2\}$, for $D_4$ on polynomials with Galois groups $G = D_4$ or $Q_8$, the Gram matrix $M(G) = (\langle \sigma_i, \sigma_j \rangle)$ $\langle \sigma_0 = 1 \rangle$ takes the form

$$M(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 1 & 2 & 0 & 1 \\ -1 & 0 & 5 & 1 \end{bmatrix} \quad \text{where } G = D_4 \text{ and otherwise }$$

$$M(G) = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ -1 & 2 & -3 & 7 \end{bmatrix} \quad \text{where } G = Q_8 \text{ and otherwise }$$

With respect to test functions $\{1, \tau_1, \tau_2\}$ for $Q_8$, the Gram matrices are:

It should be clear that the full Gram matrix gives a more complete picture of the orthogonality relations of characters than the triple of inner products $\langle (8e_1, 1) \rangle, \langle (8e_2, 1) \rangle, \langle (8e_3, 1) \rangle$, which is just one linear combination of the rows in the above Gram matrices.

In the next section, we show that the choice of reduced basis for the target group gives a better set of test functions, converging more rapidly to the asymptotic Gram matrix. With respect to the polynomials $x^8 + 6x^4 + 1$ of Galois group $D_4$ and $x^8 - 12x^6 + 36x^4 - 36x^2 + 9$ of Galois group $Q_8$, we obtain reasonably good convergence (to within a half integer) with the first 80 primes.

**Non distinguished representations of degree 8.** The first example of nonisomorphic permutation representations not distinguished by their cycle types and Haar measure are the degree-8 groups of order 16 denoted $8T10$ and $8T11$ (see the LMFDB [13] Galois groups database). Specifically we define the representative groups

$$G_0 = \langle (1, 2, 3, 8)(4, 5, 6, 7), (1, 5)(3, 7) \rangle \quad \text{and} \quad G_1 = \langle (1, 3, 5, 7)(2, 4, 6, 8), (1, 4, 5, 8)(2, 3, 6, 7), (1, 5)(3, 7) \rangle$$

whose character tables are given by

$$A(G_0) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

and

$$A(G_1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

with respective probabilities $1/16, 1/16, 1/16, 1/16, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8$. We note that the first 8 characters are linear, and the latter two are of degree 2. The linear characters admit a group structure, isomorphic to $C_2 \times C_4$ and $C_2^2$, respectively. We denote the characters by $\{1, \chi_1, \chi_2, \chi_3, \rho_1, \rho_2, \rho_2, \psi_1, \psi_2\}$ and $\{1, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \psi, \tilde{\psi}\}$. In the groups $G_0$ and $G_1$ the character of the standard representation (of degree 7) decomposes as

$$s_1 = \chi_1 + \rho_1 + \rho_1 + \psi_1 + \psi_2$$

and

$$s_1 = \xi_1 + \xi_2 + \xi_2 + \psi + \tilde{\psi},$$

respectively, but the individual characters in $s_1$ are not separated.
Given the obvious Galois action (on the codomain field \( \mathbb{Q}(i) \)), we see that the subrings \( \mathcal{R}_0(G_0) \) and \( \mathcal{R}_2(G_1) \) have different ranks, 8 and 9. On the other hand, the images of the restriction homomorphism from \( \mathcal{R}(S_4) \) have rank 4 in each of \( \mathcal{R}(G_0) \) and \( \mathcal{R}(G_1) \), generated for instance by \( \{1, s_1, s_2, s_3\} \). Moreover, since exactly the same four cycle types occur, with the same probabilities \((1/16, 5/16, 1/8, 1/2)\), the characters in the image of restriction from \( \mathcal{R}(S_8) \) to \( \mathcal{R}(G_0) \) and \( \mathcal{R}(G_1) \) cannot be differentiated.

Let \( K_0 \) and \( K_1 \) be number fields whose normal closures have respective Galois groups \( G_0 \) and \( G_1 \). In order to distinguish these fields, it suffices to construct missing characters from the linear character groups. In fact these number fields have nontrivial automorphism groups, isomorphic to \( V_4 \) and \( C_4 \), respectively. This induces respective subfield lattices of the forms

\[
\begin{array}{c}
K_0 & \xrightarrow{F'_0} & K_1 \\
& F_0 & \xrightarrow{G_0} & F_0' \xrightarrow{G_0''} & Q \\
& G_0 & \xrightarrow{Q} & G_0' \xrightarrow{G_0''} & Q \\
\end{array}
\]

For each field we recover a significant subgroup of the linear character groups from the quartic and quadratic characters. In fact there is a unique cyclic subfield \( F_0/\mathbb{Q} \) in \( K_0 \) which recovers the characters \( \rho_1, \rho_2 \), and \( \chi_1 = \rho_1^2 \). (The other fields \( F_0' \) and \( F_0'' \) are non-normal.) And there exists a unique biquadratic field \( F_1/\mathbb{Q} \) in \( K_1 \) which yields the quadratic characters \( \xi_1, \xi_2, \xi_3 \). The pairs \((K_0, F_0)\) and \((K_1, F_1)\) give characters on the pairs of permutation groups of degree 8 and 4, \((G_0, G_0/H_0 \cong C_4)\) and \((G_1, G_1/H_1 \cong V_4)\), such that the joint factorization types of Frobenius characters separate the Galois structures.

**Representations of \( A_5 \).** We denote the irreducible characters of the alternating groups \( A_5 \) by \( \{1, \chi_1, \chi_2, \chi_3, \chi_4\} \), where \( \chi_1 \) is the character of the degree-4 standard representation, \( \chi_2 \) is the character of a degree-5 representation, and \( \chi_3 \) and \( \chi_4 \) are the conjugate characters of degree-3 icosahedral representations over \( \mathbb{Q}(\sqrt{5}) \). The rational representations are thus spanned by the orthogonal characters \( \{1, \chi_1, \chi_2, \chi_3 + \chi_4\} \) of degrees 1, 4, 5, and 6.

On the other hand, the permutation representation of \( A_5 \) in \( S_5 \) gives a parametrization by

\[ \mathcal{R}(SO(4)) = \frac{\mathbb{Z}[s_1, s_2, s_3, s_4]}{(s_1 - s_3, s_1 - 1)} \cong \mathbb{Z}[s_1, s_2], \]

and while \( |\mathcal{O}(A_5)| = 5 \), there are two conjugacy classes which map to the same cycle type \( 5^1 \) in \( \mathcal{O}(S_5) \). Thus the restriction from \( \mathcal{R}(S_5) \) gives a basis of four independent characters, and we identify:

\[ (1, s_1, s_1^2 - s_2 - s_1 - 1, s_2) = (1, \chi_1, \chi_2, \chi_3 + \chi_4). \]

In addition to its degree-5 permutation representation, \( A_5 \) admits a faithful permutation representation in \( S_5 \). In the restriction of \( \mathbb{Z}[s_1, s_2] \cong \mathcal{R}(SO(5)) \) we recognize the same characters equipped with a different parametrization

\[ (1, s_1^2 - 2s_1 - s_2 - 1, s_1, s_2 - \chi_1) = (1, \chi_1, \chi_2, \chi_3 + \chi_4). \]

Consider the number fields, each with Galois group \( A_5 \), defined by polynomials

\[
\begin{align*}
f &= x^5 - 5x^4 + 48x^3 + 28x^2 + 5x - 1, \\
g &= x^6 + 4x^5 + 10x^4 - 10x^3 + 17x^2 + 10x + 1
\end{align*}
\]

constructed as subfields of the same normal closure. Although not isomorphic, we can construct the inner product matrix of the same characters set \( \{1, \chi_1, \chi_2, \chi_3 + \chi_4\} \) on \( A_5 \) with respect to its different embeddings in \( S_5 \) and \( S_0 \). Jointly evaluating the characters on factorization types of \( f \) or \( g \) with those of either \( f \) or \( g \), yields the same diagonal inner product matrix (= diag(1, 1, 1, 2) to nearest integer). This gives a means of recognizing the same character of the absolute Galois group via different presentations. The arithmetic statistic approach through character theory
gives a powerful tool to not only characterize Galois groups, but to recognize equivalence of finite representations of the absolute Galois group $\mathcal{G}$ which may arise in different contexts.

6. VARIANCE, COVARIANCE AND CONVERGENCE

The focus on irreducible characters provides, on the one hand, a theoretic framework for understanding the arithmetic statistics of Frobenius distributions. On the computational side, irreducible characters provide test functions with optimal convergence properties. Naively, the orthogonality relations for a system $\{\chi_1, \ldots, \chi_r\}$ of irreducible characters as test functions, it suffices to recognize the integer $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ to one bit of precision. Furthermore, $\chi_i \neq 1$ and $\chi_j \neq 1$ the inner products $\langle \chi_i, 1 \rangle = \langle \chi_j, 1 \rangle = 0$ imply that $\chi_i$ and $\chi_j$ have mean $0$, hence we can interpret

$$E_S(\chi_i, \chi_j) = \frac{1}{|S|} \sum_{p \in S} \chi_i(p)\chi_j(p)$$

as a (sample) variance ($i = j$) or covariance ($i \neq j$) of the sample $S$, we see that the use of irreducible characters (or of reduced characters in $\mathcal{R}(G)$) as the next best approximation when irreducible characters are not in the restriction image from $\mathcal{R}(S_n)$ minimizes the variance of the test functions, and orthogonality minimizes the covariance.

We can illustrate the convergence properties with the lattice of subgroups between the representation of $\text{PSL}_2(\mathbb{F}_7)$ on $\mathbb{F}^4(\mathbb{F}_7)$ and $S_8$:

$$\text{PGL}_2(\mathbb{F}_7) \cong G_1 \rightarrow S_8$$

with respective orders $|H_1| = 168$, $|G_1| = 336$, and $|H_2| = 1344$.

Let $h(G)$ be the number of conjugacy classes of $G$, equal to the number of irreducible characters and to the rank of $\mathcal{R}(G)$; let $r(G)$ be the number of characters irreducible over $\mathbb{Q}$, equal to the rank of $\mathcal{R}_Q(G)$; and let $s(G)$ the rank of the image of the restriction of $\mathcal{R}(S_n)$ to $\mathcal{R}(G)$. For each of the groups we give the respective numbers $h(G)$, $r(G)$ and $s(G)$, as well as a representative polynomial (from the LMFDB [13]) with Galois group $G$.

| $G$ | $h(G)$ | $r(G)$ | $s(G)$ | $f_{G}(x)$ |
|-----|--------|--------|--------|------------|
| $S_8$ | 22     | 22     | 22     | $x^8 - x - 1$ |
| $A_8$ | 14     | 12     | 12     | $x^8 - 2x^7 + 3x^5 - 5x^4 + 2x^3 + 2x^2 - x + 1$ |
| $G_1$ | 9      | 8      | 8      | $x^8 - x^7 + x^6 + 4x^5 - x^4 - 3x^3 + 5x^2 - 2x + 1$ |
| $H_2$ | 11     | 10     | 10     | $x^8 - 4x^7 + 8x^6 - 9x^5 + 7x^4 - 4x^3 + 2x^2 + 1$ |
| $H_1$ | 6      | 5      | 5      | $x^8 - 4x^7 + 7x^6 - 7x^5 + 7x^4 - 7x^3 + 7x^2 + 5x + 1$ |

For the generic group $S_n$ the characters $(1, s_1, \ldots, s_{n-1})$ are irreducible on $S_n$ and form a system of test functions for $S_n$. On $A_n$ and its subgroups the relations $s_{n-1-n} = s_i$ hold, and so the characters $(1, s_1, \ldots, s_m)$, where $n = 2m + 1$ or $2m + 2$, form a system of test functions for $A_n$.

The Gram matrices $M(G)$ with respect to the test characters $(1, s_1, \ldots, s_7)$ for $G = S_8$, $A_8$, and $G_1$, respectively are:

\[
M(S_8) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix},
M(A_8) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix},
M(G_1) = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 3 & 2 & 1 & 1 & 1 & 0 \\
0 & 1 & 2 & 6 & 4 & 1 & 1 & 1 \\
1 & 1 & 1 & 4 & 6 & 2 & 1 & 0 \\
0 & 1 & 1 & 1 & 2 & 3 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

For the indicated representative polynomials, characters $(\chi_1, \ldots, \chi_r)$ and set of non-ramified primes $S$, we define the error matrix: $Z_S(G) = E_S(\chi_i, \chi_j) - M(G)$ and for an $(r \times r)$-matrix
segments \( \ell \) functions give good convergence in the

Extending the computation further, we find that the apparent stable convergence fails when

which determine the diagonalized matrix \( M(G) = A(G_1)^t D(G_1) A(G_1) \) with respect to the rational irreducible characters. With respect to this basis, in increments of 128k primes, we find stable convergence after just 512 = 128 · 4 primes:
For the subgroup chain $H_1 \subset H_2 \subset A_8$, starting with the characters $(1, s_1, s_2, s_3)$, irreducible on $A_8$, we find a similar analysis. In particular, the Gram matrices with respect to this basis are

$$M(A_8) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M(H_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad M(H_1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \\ 3 & 1 & 2 \end{bmatrix}. $$

In the former two cases, the characters are orthogonal and irreducible or nearly so ($S_3$ decomposes as a sum of three distinct irreducibles on $H_2$), and convergence is relatively good. In contrast, the Gram matrix $M(H_1)$ has determinant 14, and far from being orthogonal or irreducible (except for 1 and $s_1$) on $H_1$. In increments of 1024, we find stable convergence only after $2^{15} = 1024 \cdot 32$ primes:

| $H_1$ | $||Z_S(G)||_2$ | $||Z_S(G)||_8$ | $||Z_S(G)||_{\infty}$ | $H_1$ | $||Z_S(G)||_2$ | $||Z_S(G)||_8$ | $||Z_S(G)||_{\infty}$ |
|-------|----------------|----------------|------------------------|-------|----------------|----------------|------------------------|
| 1:    | 1.300776 < 2.685076 < 3.787109 | | | 17:   | 0.162082 < 0.331846 < 0.467773 | | |
| 2:    | 0.457035 < 0.943691 < 1.331054 | | | 18:   | 0.249476 < 0.507743 < 0.715332 | | |
| 3:    | 0.316304 < 0.671333 < 0.948242 | | | 19:   | 0.260497 < 0.533048 < 0.753136 | | |
| 4:    | 0.149549 < 0.327977 < 0.463623 | | | 20:   | 0.250136 < 0.514311 < 0.725195 | | |
| 13:   | : : : | : : : | : : : | 29:   | 0.183952 < 0.364100 < 0.511112 | | |
| 14:   | 0.219831 < 0.449876 < 0.634137 | | | 30:   | 0.193960 < 0.384122 < 0.539257 | | |
| 15:   | 0.207462 < 0.427431 < 0.602799 | | | 31:   | 0.148770 < 0.290129 < 0.406060 | | |
| 16:   | 0.170705 < 0.352045 < 0.496520 | | | 32:   | 0.132390 < 0.258615 < 0.362091 | | |

Going further one finds that the $\ell_{\infty}$-norm gradually decreases and does indeed stay below 0.50 after this point. In contrast, in terms of the basis $(1, \chi_1 = \varphi + \overline{\varphi}, \chi_2, \chi_3, \chi_4)$ of irreducible characters over $\mathbb{Q}$, of degrees $(1, 6, 6, 7, 8)$ given by

$$\chi_1 = (4s_2 + 3s_3 - s_1s_2 - 4s_1 - 2)/2, \quad \chi_3 = s_1,$$
$$\chi_2 = (2s_2 + 5s_3 - s_1s_2 - 6s_1 - 4)/4, \quad \chi_4 = (s_1s_2 + 2s_1 + 2 - 2s_2 - 3s_3)/2,$$

the test characters stable converge to $M(H_1)$ after only 128 primes, with results here in increments of 128 primes:

| $H_1$ | $||Z_S(G)||_2$ | $||Z_S(G)||_8$ | $||Z_S(G)||_{\infty}$ | $H_1$ | $||Z_S(G)||_2$ | $||Z_S(G)||_8$ | $||Z_S(G)||_{\infty}$ |
|-------|----------------|----------------|------------------------|-------|----------------|----------------|------------------------|
| 1:    | 0.227868 < 0.301886 < 0.406250 | | | 9:    | 0.064413 < 0.088651 < 0.114583 | | |
| 2:    | 0.225747 < 0.296604 < 0.398437 | | | 10:   | 0.079219 < 0.104501 < 0.132812 | | |
| 3:    | 0.127307 < 0.165588 < 0.216145 | | | 11:   | 0.091419 < 0.119029 < 0.154829 | | |
| 4:    | 0.149822 < 0.191605 < 0.250000 | | | 12:   | 0.056475 < 0.076844 < 0.097656 | | |
| 5:    | 0.166819 < 0.214155 < 0.271875 | | | 13:   | 0.047871 < 0.066901 < 0.086538 | | |
| 6:    | 0.085019 < 0.114926 < 0.148437 | | | 14:   | 0.041653 < 0.062817 < 0.083705 | | |
| 7:    | 0.101179 < 0.132950 < 0.166294 | | | 15:   | 0.029993 < 0.041051 < 0.053125 | | |
| 8:    | 0.114860 < 0.148922 < 0.193359 | | | 16:   | 0.041465 < 0.054989 < 0.069335 | | |

These convergence results give empirical support to the principle of using irreducible characters as test functions, based on the theoretical interpretation of inner product relations on characters.
The quotient group $G$ the convergence to the Gram matrix $A$ absolutely irreducible rational characters in the image. In increments of 256 primes, we compute by its center, which is the Galois group of the degree 120 polynomial $g$. Conjugacy classes is the partition number $p(n)$, whose asymptotic growth is known by Hardy and Ramanujan \[10\] to be
\[p(n) \sim \frac{1}{4n\sqrt{3}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right).\]

In particular, we will treat a nontrivial example of degree 120 (and 240) despite the large size $p(120) = 1844349560$ (and $p(240) = 105882246722733$) of the corresponding partition numbers. Finally, computation of the kernel ideal of the restriction $\mathcal{R}(O(n - 1)) \to \mathcal{R}(G)$ by Groebner basis algorithms is prohibitively expensive, even if the $s(G)$ points in the kernel can be computed.

Polynomials with interesting Galois groups of large degree, outside the generic groups $S_n$ and $A_n$ and cyclic and dihedral groups $C_n$ and $D_n$ rely on specific constructions. We consider such an example of Jouve, Kowalski and Zywina \[12\], a polynomial $f(x)$ of degree 240 with Galois group the Weyl group $W(E_8)$ of the lattice $E_8$, of order 696729600. In contrast to the large number of conjugacy classes of $S_{240}$, the number of conjugacy classes of $W(E_8)$ is 112, and the restriction homomorphism from $\mathcal{R}(S_{240})$ has full rank. We take the quotient of order 348364800 by its center, which is the Galois group of the degree 120 polynomial $g(x)$ such that $f(x) = g(x^2)$. The quotient group $G = W(E_8)/Z(W(E_8))$ has 67 conjugacy classes, all characters are rational, and the restriction homomorphism from $\mathcal{R}(S_{240})$ is a subring of rank 65. We consider the 18 absolutely irreducible rational characters in the image. In increments of 256 primes, we compute the convergence to the Gram matrix $A(G)$ for these 18 characters to $2^{13} = 256 \cdot 32$ primes:

| $W(E_8)/Z(W(E_8))$ | $W(E_8)/Z(W(E_8))$ |
|-------------------|-------------------|
| $||Z_2(G)||_2$ | $||Z_2(G)||_8$ | $||Z_2(G)||_\infty$ | $||Z_8(G)||_2$ | $||Z_8(G)||_8$ | $||Z_8(G)||_\infty$ |
| 1 : 0.512041 | 0.947691 | 3.843750 | 17 : 0.122505 | 0.279019 | 0.473345 |
| 2 : 0.256200 | 0.683283 | 1.609375 | 18 : 0.118703 | 0.265146 | 0.452473 |
| 3 : 0.180987 | 0.525172 | 0.929687 | 19 : 0.114018 | 0.254474 | 0.432360 |
| 4 : 0.251753 | 0.848164 | 1.571289 | 20 : 0.110361 | 0.248728 | 0.442968 |
| ... | ... | ... | ... | ... | ... |
| 13 : 0.161766 | 0.376064 | 0.648137 | 29 : 0.110513 | 0.283699 | 0.530980 |
| 14 : 0.151537 | 0.350967 | 0.611049 | 30 : 0.100819 | 0.275711 | 0.514713 |
| 15 : 0.139688 | 0.325884 | 0.550000 | 31 : 0.105191 | 0.262830 | 0.491053 |
| 16 : 0.128557 | 0.298173 | 0.504638 | 32 : 0.102228 | 0.251891 | 0.468505 |

Extending the computation further suggests that the convergence to $M(G)$ is stable for $m > 2^{13}$.

8. Conclusion

A standard tool in Galois group computation is to recognize the probable group from an analysis of Frobenius cycle types. We use an explicit polynomial parametrization of the character ring to identify the irreducible characters in the restriction from orthogonal groups and subsequently from the symmetric group. As in the thesis work of Shieh \[18\] \[19\], with the view to classifying Sato-Tate groups, it is recognized that the irreducible characters on the target group provide optimal test functions for recognizing (or rejecting) a given group coming from a Galois representation. We develop this perspective in the application to the parametrized representation rings of finite groups, with associated lattice structure. Although we focus on Galois groups arising from splitting fields as variance and covariance. Moreover, when using irreducible characters, the number of primes necessary to recognize the Gram matrix associated to a Galois group is strikingly small.
of polynomials over \( \mathbb{Q} \), the same methods apply to Galois representations coming form \( L \)-series and modular forms, families of exponential sums, and global fields of any characteristic.

At a higher level, the approach through character theory and arithmetic statistics lets us identify when Frobenius distributions of different degrees admit a common Galois subrepresentation. Examples arise in the form of fields with isomorphic normal closures, as described in the above examples of \( A_5 \) representations, but more generally one can recognize whether two normal fields admit a common subfield. In this framework orthogonality relations of characters are measured by correlations of Frobenius distributions associated to different representations of the absolute Galois group. This perspective has promising potential for the computational investigation of Galois representations.

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