Disturbance Observer-Based Boundary Control for an Antistable Stochastic Heat Equation With Unknown Disturbance

Ze-Hao Wu, Member, IEEE, Hua-Cheng Zhou, Member, IEEE, Feiqi Deng, and Bao-Zhu Guo, Senior Member, IEEE

Abstract—In this article, a novel control strategy namely disturbance observer-based control is first applied to stabilization and disturbance rejection for an antistable stochastic heat equation with Neumann boundary actuation and unknown boundary external disturbance generated by an exogenous system. A disturbance observer-based boundary control is designed based on the backstepping approach and estimation/cancellation strategy, where the unknown disturbance is estimated in real time by a disturbance observer and rejected in the closed loop, while the in-domain multiplicative noise whose intensity is within a known finite interval is attenuated. It is shown that the resulting closed-loop system is exponentially stable in the sense of both mean square and almost surely. A numerical example is demonstrated to validate the effectiveness of the proposed control approach.

Index Terms—Backstepping approach, boundary control, disturbance rejection, stabilization, stochastic heat equation.

I. INTRODUCTION

Disturbances are ubiquitous in many practical control systems, which often cause negative effects on the performance of the control plant. For the sake of control precision, many control approaches have been developed since 1970s to cope with disturbances in term of disturbance attenuation or disturbance rejection. The stochastic control and robust control are two of representative disturbance attenuation approaches, where the former is often used for attenuating noises with known statistical characteristics while the latter can deal with more general disturbances. However, most of the robust control approaches are on the worst case scenario, which may lead to control design rather conservative. Based on estimation/cancellation strategy, some novel active antidisturbance control approaches such as disturbance observer-based control (DOBC) [1], [2] have been proposed for disturbance rejection for control systems over the past two decades. The core idea of these active antidisturbance control approaches is that the disturbances affecting system performance can be estimated by a disturbance observer and then be compensated in the closed loop. Owing to the estimation/cancellation characteristics, the active antidisturbance control is capable of eliminating the disturbances before negative effects are caused and at the same time, the control energy can be reduced significantly in engineering applications.

The disturbance rejection for distributed parameter systems driven by white noise, see, for instance, [8], [9] and the references therein. The DOBC approach to the stabilization of non-linear parabolic partial differential equation (PDE) systems subject to external disturbances has been investigated in [3]. The uncertainty and disturbance estimator-based robust control approach to the stabilization of an unstable parabolic PDE with a Dirichlet type boundary actuator and an unknown time-varying input disturbance has been addressed in [4]. An infinite-dimensional observer-based output feedback boundary control has been designed for a multidimensional heat equation subject to boundary unmatched disturbance in [5]. The output regulation has been developed for linear distributed-parameter systems by finite-dimensional dual observers [6] and parabolic PDEs by a backstepping approach [7], respectively.

Nevertheless, the external disturbance appears most often in random way in practice, which is neglected in literatures aforementioned. Actually, there have been many control designs for finite-dimensional stochastic systems driven by white noise, see, for instance, [8], [9] and the references therein. Specially, some active antidisturbance control methods to disturbance rejection for finite-dimensional stochastic systems have been proposed. For example, problems of the composite DOBC and $H_{\infty}$ control for Markovian jump systems and the DOBC for a class of stochastic systems with multiple disturbances have been studied in [10] and [11], respectively. An extended state observer-based output feedback stabilizing control has been designed for a class of stochastic systems subject to bounded stochastic noise [12].

As one of the active antidisturbance control approaches, the DOBC has been widely applied in engineering applications with good disturbance rejection performance and robustness, see, for instance, [13], [14] and the references therein. However, there is still no relevant study from theoretical perspective on DOBC for stochastic distributed parameter systems. In this article, we demonstrate for the first time, through an antistable stochastic heat equation with unknown boundary external disturbance, the DOBC approach to stabilization and disturbance rejection for stochastic distributed parameter systems. The main contributions and novelty of this article can be summarized as follows.

1) From a theoretical perspective, the applicability of the powerful DOBC control technology is first expanded to a class of stochastic distributed parameter systems with unknown boundary external disturbance.

2) The unknown boundary external disturbance is rejected completely by virtue of estimation/cancellation strategy of the DOBC approach, while the in-domain multiplicative noise with bounded intensity is attenuated.
3) Not only the mean square exponential stability but also the almost surely exponential stability are obtained for the resulting closed-loop system.

The rest of this article is organized as follows. In Section II, some problem formulation and preliminaries are presented. In Section III, both design of the DOBC boundary control and stability of the closed-loop system are discussed and stated. A numerical example is presented in Section IV. Finally, Section V concludes this article. The proofs of the main results are arranged in the Appendix.

II. PROBLEM FORMULATION AND PRELIMINARIES

We first introduce some notations. The $\mathcal{F}_t$ denotes the $n$-dimensional identity matrix and $L^2(0, 1)$ is the space of all real-valued functions that are square Lebesgue integrable over $(0, 1)$. Let $(\Omega, \mathcal{F}, \mathbb{P}, P)$ be a complete filtered probability space with a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ on which a one-dimensional (1-D) standard Brownian motion $B(t)$ is defined. A stochastic process $f(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}^n$ is called $\mathcal{F}$-adapted if for every $t \geq 0$, the function $\omega \to f(t, \omega)$ is $\mathcal{F}_t$-measurable. For notation simplicity, we use $f(t)$ to denote a stochastic process $f(t, \omega)$. Let $\mathcal{V}$ be a Banach space. A sub-$\sigma$-algebra $\mathcal{M}$ of $\mathcal{F}$, denoted by $L^2_{\mathcal{M}}(\Omega, \mathbb{N})$, is the set of all $\mathcal{M}$-measurable ($\mathbb{N}$-valued) random variables $f : \Omega \to \mathbb{N}$ such that $\mathbb{E}|f|^2 < \infty$. Let $H^2(0, 1)$ and $H^2(0, 1)$ be the Sobolev spaces. Set $L^2(T; \mathbb{R}^n)$ is the $\mathcal{F}$-adapted and $\int_0^T (\mathbb{E}[f(t)]^2 + \sigma^2) dt < \infty$, $C^t(0, T; L^2(\Omega; \mathbb{R}^n)) = \{ f : [0, T] \times \Omega \to V(f) \}$ is $\mathcal{F}$-adapted and $\int_0^T (\mathbb{E}[f(t)]^2 + \sigma^2) dt < \infty$, $C(0, \infty; L^2(\Omega; \mathbb{R}^n)) = \{ f : (0, \infty) \times \Omega \to V(f) \}$ is $\mathcal{F}$-adapted and $\mathbb{E}[f(t)]^2$ is continuous and $C(0, \infty; L^2(\Omega; \mathbb{R}^n)) = \{ f : (0, \infty) \times \Omega \to V(f) \}$ is $\mathcal{F}$-adapted and $\mathbb{E}[f(t)]^2$ is continuously differentiable. All the abovementioned spaces are endowed with the usual canonical norms.

In this article, we consider stabilization and disturbance rejection for a multi-dimensional (1-D) antistable stochastic heat equation driven by multiplicative white noise with unknown boundary external disturbance as follows:

$$
\begin{align*}
\frac{dy(x, t)}{dt} &= y_{xx}(x, t)dt + a(x)y(x, t)dt + \sigma y(x, t)dB(t) \\
y(x, 0) &= 0, \quad t \geq 0 \\
y(x, t) &= u(t) + w(x), \quad t \geq 0 \\
y(x, 0) &= y_0(x), \quad 0 \leq x \leq 1
\end{align*}
$$

where $y(x, t)$ represents the function $y(x, t)$ at the spatial position $x \in [0, 1]$ and the time $t \in [0, \infty)$. $a(\cdot) \in L^2(0, 1)$ is a constant representing the intensity of the multiplicative white noise with a known upper bound for its absolute value, $y_0(\cdot) \in L^2_{\mathcal{M}}(0, 1)$ is the initial value, and $u(t)$ is the boundary control input, $w(t)$ is the random disturbance, which could be the temperature perturbation generated from an exogenous system as follows:

$$
\begin{align*}
\xi(t) &= A\xi(t) \\
w(t) &= C\xi(t)
\end{align*}
$$

where $\xi(t) \in \mathbb{R}^n$ is an unknown exogenous signal, and $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{1 \times n}$ are known matrices. Throughout this article, the stochastic differentials including $dy(x, t)$ are with respect to the time $t$.

The following Assumption (A1) is for estimation of the unknown disturbance $w(t)$.

Assumption (A1). For the exogenous system (2), the pair $(A, C)$ is supposed to be observable.

Remark 1: The exogenous system (2) is a typical form for disturbance in output regulation, which covers all finite harmonic disturbance with known frequency but unknown amplitude and phase. The harmonic disturbance can be considered as an approximation of periodic disturbance (see [1], [10], [18]) and has been discussed intensively by means of the DOBC approach (see, e.g., [2, p.47]).

Similar to both mean square exponential stability and almost surely exponential stability of stochastic differential equations (see, e.g., [22]), we introduce the following stability for system (1):

III. DOBC BOUNDARY CONTROL DESIGN AND MAIN RESULTS

The framework of the DOBC boundary control design and theoretical approach can be simply explained in Fig. 1. Motivated by [19], we introduce an invertible transformation $\Lambda : y \in L^2_{\mathcal{M}}(\Omega; L^2(0, 1)) \to z \in L^2_{\mathcal{M}}(\Omega; L^2(0, 1))$ as follows:

$$
z(x, t) = y(x, t) - \int_0^x k(x, \zeta) \eta\zeta d\zeta \in \mathcal{L}\Lambda y(x, t)
$$

where $k(\cdot, \cdot)$ defined on $C := \{x, \zeta \in \mathbb{R}^2 : 0 < \zeta < x \leq 1\}$ is the kernel function specified in (5) later. Let $c$ be a positive constant. By Itô’s differentiation rule, a direct computation shows that

$$
\int dz(x, t) - \int_0^x k(x, \zeta) \eta\zeta d\zeta dt - \eta x \sigma \zeta(x) + \eta \xi(t) dB(t)
$$

$$
= \int_0^x [k_x(x, \zeta) - k_x(x, \zeta) - (c + a(x))k(x, \zeta)] d\zeta dt + \int_0^x \frac{dk(x, \zeta)}{dx} d\zeta + k_x(x, \zeta) + a(x) + c \int_0^x \eta(x, \zeta) d\zeta - k_x(x, 0) \eta(0, t) dt.
$$

The kernel function $k(\cdot)$ is chosen to satisfy the following PDE:

$$
\begin{align*}
k_x(x, \zeta) - k_x(x, \zeta) &= (c + a(x))k(x, \zeta) \\
k_x(x, 0) &= 0 \\
\frac{dk(x, \zeta)}{dx} + k_x(x, x) + a(x) + c &= -a(x) - c
\end{align*}
$$

It can be seen that the right side of (4) becomes zero. Then, the boundary condition in (1) yields $y_x(0, t) = y_x(0, t) - \int_0^x k(x, \zeta) \eta\zeta d\zeta.$
\( k(0,0)y(0,t) = -k(0,0)y(0,t) \), which implies \( k(0,0) = 0 \). This, together with the third equation in (5), indicates that the PDE (5) becomes

\[
\begin{align*}
\frac{\partial z}{\partial t} = (A + LC)z + (A + LC)Lz(t) + L[c(\pi^2)Z(t) + u_0(t)] \\
\xi(t) = \hat{\theta}(t) + LZ(t) \\
\hat{\omega}(t) = C\hat{\xi}(t)
\end{align*}
\]

Remark 3: By [19, Th. 2.1], (6) admits a unique solution which is twice continuously differentiable in the domain \( G = \{x, \xi \in \mathbb{R}^2 : 0 \leq \xi \leq \pi \leq 1\} \). In particular, if \( a(\xi) \) is a constant function, for example, \( a(\xi) \equiv 0 \), then \( k(x,\xi) \) can be found analytically as \( k(x,\xi) = -c z_2 I_1(\sqrt{c^2 \xi^2 - c^2}) \sqrt{c^2 \xi^2 - c^2} \), where \( I_1(\cdot) \) is a first-order modified Bessel function given by

\[
I_1(\pi) = \sum_{i=0}^{\infty} \frac{(-1)^i (\pi^2)^{i+1}}{2^{2i+1} i!(i+1)!}.
\]

The following Lemma 3.1 is brought from [19, Th. 2.2] that demonstrates the transformation operator \( A \) defined in (3) is invertible.

Lemma 3.1 [see [19]]: The linear operator \( A : L_2^2(\Omega; L^2(0,1)) \rightarrow L_2^2(\Omega; L^2(0,1)) \) defined in (3) is bounded invertible, and

\[
y(x,t) = z(x,t) + \int_0^x l(x,\xi)\xi z(\xi,t)d\xi
\]

where \( l \in C^2(\xi) \).

Under the invertible transformation (3), system (1) is transformed into the following equivalent one:

\[
\begin{align*}
z(x,t) &= \int_0^{\pi} l(x,\xi)\xi z(\xi,t)d\xi \quad \text{and} \quad \sigma(x,t) = (A + LC)z(t) + u_0(t) + \int_0^1 k_1(\xi)z(\xi,t)d\xi.
\end{align*}
\]

Remark 4: It is noteworthy that for any \( T > 0 \)

\[
Z(t) \equiv 0, u_0(t) \equiv 0, \forall t \in [0,T] \Rightarrow w(t) \equiv 0 \quad \forall t \in [0,T]
\]

which means that \( w(t) \) is uniquely determined by the average-type output \( Z(t) \). This is similar to the exact observability of \( \xi(t) \) to stochastic systems without external disturbance that the state is uniquely determined by output, see, for instance, [20]. Therefore, a natural way is to use the output \( Z(t) \) of system (11) to design a disturbance observer to estimate \( w(t) \), and hence, the idea of finite-dimensional disturbance observer can be adopted to design the observer (12).

Since \( A + LC \) is Hurwitz, the mean square exponential stability for the error system (13) can be expected if the intensity of the white noise is “small” to some extent, and then the almost surely exponential stability can also be concluded directly due to the common linear growth condition. Thus, we can design the control law \( u_0(t) = -\hat{\omega}(t) \) in (9) to cancel the unknown disturbance \( w(t) \) in real time. We first consider the mean square exponential stability for \( (Z(t), \eta(t)) \), which is governed by

\[
\begin{align*}
d\left[ Z(t)^T + \eta(t)^T \right] &= \left[ -(c + \pi^2) C \quad 0 \right] \left[ Z(t) \quad \eta(t) \right] dt + \left[ \sigma \quad \sigma L \right] \left[ Z(t) \quad \eta(t) \right] dB(t).
\end{align*}
\]

By Assumption (A1), \( -(c + \pi^2) C \) is Hurwitz, and hence, there exists a unique positive matrix \( Q_c := \frac{Q_{c_1} Q_{c_2}}{Q_{c_3}} \in \mathbb{R}^{(n+1) \times (n+1)} \) depending on \( c \) such that

\[
\begin{align*}
\left[ -(c + \pi^2) C \quad 0 \right]^T Q_c + Q_c \left[ -(c + \pi^2) C \quad 0 \right] &= -\mathbb{I}_{n+1}.
\end{align*}
\]

which where \( Q_{c_1} \in \mathbb{R}^{+}, Q_{c_2} = Q_{c_3}^T \in \mathbb{R}^{+}, \) and \( Q_{c_4} \in \mathbb{R}^{n+1} \). A direct computation can easily show that \( Q_{c_1} = \frac{1}{c^2 + \pi^2}, Q_{c_3} = \frac{1}{c^2 + \pi^2} [C(\xi) + (c + \pi^2) I_n]^{-1}, \) and \( Q_{c_4} \) satisfies (A1) \( Q_{c_1} + Q_{c_2}(A + LC) + C^T Q_{c_2} + CQ_{c_3} = -\mathbb{I}_n \). It can be further obtained that

\[
\mu_c := \max_{\alpha \in [0,1]} \left( \alpha \geq \min \left( \frac{Q_{c_1}}{2\lambda_{\max}(Q_{c_3})}, \frac{Q_{c_4}}{2\lambda_{\max}(Q_{c_3})} \right) \right)
\]

\[
\begin{align*}
u_c := \max \left\{ \text{sup}_{t \rightarrow \infty} \frac{1}{t} \log |Z(t)|, \text{sup}_{t \rightarrow \infty} \frac{1}{t} \log |\eta(t)| \right\}
\end{align*}
\]

(i) \(||Z(t)||^2 + |\eta(t)||^2| \leq \frac{1}{\mu_c \sigma^2} \left( \min \left( \frac{Q_{c_1}}{2\lambda_{\max}(Q_{c_3})}, \frac{Q_{c_4}}{2\lambda_{\max}(Q_{c_3})} \right) \right) \left( |Z(0)|^2 + |\eta(0)|^2 \right) e^{-\frac{\mu_c \sigma^2}{2\lambda_{\max}(Q_{c_3})} t} \quad \forall t \geq 0
\]

(ii) max \( \left\{ \text{lim sup}_{t \rightarrow \infty} \frac{1}{t} \log |Z(t)|, \text{lim sup}_{t \rightarrow \infty} \frac{1}{t} \log |\eta(t)| \right\} \leq - \left( 1 - \mu_c \sigma^2 \right) \frac{\mu_c}{2\lambda_{\max}(Q_{c_3})} \) almost surely.

Proof: See “Proof of Lemma 3.2” in Appendix A.

The DOB boundary control is

\[
u(t) = k(1,1)y(1,t) + \int_0^1 k_2(1,1)\xi(t)d\xi - \hat{\omega}(t)
\]

under which system (8) becomes

\[
\begin{align*}
z_2(t) &= -\hat{\omega}(t) + w(t) + \hat{\omega}(t) \\
z_2(0,t) &= 0, \quad t \geq 0 \\
z(0,t) &= 0, \quad 0 \leq x \leq 1
\end{align*}
\]

It is noted that in order to design disturbance observer (12) and DOB boundary control (20), we used actually three measured outputs \( Y(t) := \{y(1,t), Z(t), \int_0^1 k_2(1,1)\xi(t)d\xi \} \), where \( y(1,t) \) is a pointwise measurement output, and \( Z(t) \) defined in (10) and
\[
\int_{\mathbb{T}} k_c(t, \zeta) y(t, \zeta) d\zeta \text{ with } k(\ldots) \text{ specified in (5) are two average-type measurement outputs. Such kinds of measurement outputs can be found in [15, Chs. 4 or 5 of the monograph] where they are regarded as an average measurement of the temperature around a certain point. In a recent paper [16], these signals are regarded as the nonlocal ones. In addition, by choosing appropriately measurement distribution functions, they can be realized by a finite number of sensors in practice, see for instance, [6], [17]. Nevertheless, we consider article as state feedback. The output feedback stabilization is a separate issue based on state feedback stabilization. In the following, we first show the well-posedness of the closed-loop system (21).

\textbf{Lemma 3.3:} Suppose that Assumption (A1) and \(|\sigma| < \frac{1}{\sqrt{2\pi}}\) hold. Then, for any initial value \(z_0 \in L^2_T(\Omega; L^2(0,1))\), the closed-loop system (21) admits a unique weak solution \(z \in C_T(0, +\infty; L^2(\Omega; L^2(0,1)))\). Moreover, for any \(T > 0\), there exists a positive constant \(C(T)\) such that

\[
\|z\|_{C_T(0, T; L^2(0,t,L^2(0,1)))} + \|z\|_{L^2_T(0,T;L^2(0,L^2(0,1)))} \leq C(T) \left[\left\|z_0\right\|_{L^2_T(0,T;L^2(0,1))} + \left\|w\right\|_{L^2_T(0,T;L^2(0,R))}\right].
\]

\textbf{Proof:} See “Proof of Lemma 3.3” in Appendix B.

\textbf{Theorem III.2:} Suppose that Assumption (A1) and \(|\sigma| < \frac{1}{\sqrt{2\pi}}\) hold. Then, for any initial value \(y_0 \in L^2_T(\Omega; L^2(0,1))\), the closed-loop system (25) admits a unique weak solution \(y \in C_T(0, +\infty; L^2(\Omega; L^2(0,1)))\). Moreover, the solution of the closed-loop system (25) satisfies

\[
\limsup_{t \to \infty} \frac{1}{t} \log |y(t)|_{L^2(0,1)} \leq -\frac{\theta}{2}\]

\textbf{IV. Numerical Simulations}

In this section, we present some numerical simulations for the closed-loop system (25) for illustration of the effectiveness of the proposed DOBC control. As pointed out in Remark 2, we take \(a(x) = 4x^2 + 1.005\), \(\gamma = 0.1\), \(y_0(x) = \cos(2\pi x)\) in system (1). In this case, system
Estimation of boundary external disturbance \( \tilde{p} \); \( \tilde{p}(t) \), \( p(1 - \lambda \leq 0 \leq -\lambda) \) follows:

\[
C(x, t - 3.2. \lambda = 1.0 \in \eta = [1 - t 0] \geq \text{to guarantee that the closed-loop state } C(x, t) \text{ is defined as that in (17). The (18) can follow from (29) and } \mu = 3.3. \text{ This completes the proof of the lemma.}
\]

\[
\text{mean square and almost surely. Some numerical simulations validate the}
\text{theoretic results and effectiveness of the proposed DOBC approach. A}
\text{further development of this topic is to address the output feedback}
\text{boundary control for stochastic PDEs subject to unknown external}
\text{disturbances only by pointwise measurement.}
\]

**APPENDIX A**

**PROOF OF LEMMA 3.2.**

Let \( V(Z, \eta) = \begin{bmatrix} Z & \eta \end{bmatrix} \). By (15) and (16), and applying Itô’s formula to \( V(Z(t), \eta(t)) \) with respect to \( t \), we obtain

\[
\mathbb{E} V(Z(t), \eta(t)) = \mathbb{E} V(Z(0), \eta(0)) - \int_0^t \mathbb{E} \left[ |Z(s)|^2 + |\eta(s)|^2 \right] ds
\]

\[
+ \sigma^2 \mathbb{E} \int_0^t \begin{bmatrix} Z(s) \eta(s) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q_c & 0 \\ 0 & L_c \end{bmatrix} \begin{bmatrix} Z(s) \\ \eta(s) \end{bmatrix} ds.
\]

It is also easily obtained that

\[
\lambda_{\min}(Q_c) |Z|^2 + |\eta|^2 \leq V(Z, \eta) \leq \lambda_{\max}(Q_c) |Z|^2 + |\eta|^2.
\]

In this way

\[
\frac{d}{dt} \mathbb{E} V(Z(t), \eta(t)) \leq -\left( 1 - \mu c \sigma^2 \right) \mathbb{E} V(Z(t), \eta(t))
\]

where \( \mu c \) is defined as that in (17). The (18) can follow from (29) and (30) that

\[
\mathbb{E} \left[ |Z(t)|^2 + |\eta(t)|^2 \right]
\]

\[
\leq \mathbb{E} V(Z(t), \eta(t)) \leq \mathbb{E} V(Z(0), \eta(0)) e^{-\left( 1 - \mu c \sigma^2 \right) t}
\]

\[
\leq \frac{\lambda_{\max}(Q_c)}{\lambda_{\min}(Q_c)} \mathbb{E} \left[ |Z|^2 + |\eta|^2 \right] e^{-\left( 1 - \mu c \sigma^2 \right) t}.
\]

Since both the drift term and diffusion term of system (15) satisfy the linear growth condition, by [22, Th. 4.2, p. 128], (19) can be directly concluded from (18), where the relevant parameters are specified as

\[
\lambda = \frac{1 - \mu c \sigma^2}{\lambda_{\max}(Q_c)}, \quad \mu_c = 2.
\]

This completes the proof of the lemma. \( \square \)

**APPENDIX B**

**PROOF OF LEMMA 3.3**

Since the admissibility theory for stochastic setting is not available for our case, the well-posedness cannot be simply proved. To derive the well-posedness of (21), we split the proof into three steps.

**Step 1:** Suppose that \( \tilde{w} \in C^2_L(0, T; L^2(\Omega; \mathbb{R})) \). Consider the boundary value problem of a second order PDE subject to random boundary value \( \tilde{w}(t) \) as follows:

\[
\begin{cases}
p_x(x, t) - c(\rho, x) = 0, & x \in (0, 1) \\
p_x(0, t) = 0, & p_x(1, t) = \tilde{w}(t).
\end{cases}
\]

We first claim that \( p \in C^1_T(0, T; L^2(\Omega; H^1(0, 1))) \). Actually, by the classical theory of elliptic equations with Neumann boundary condition, for any \( T > 0 \), there exists a positive constant \( C(T) \) depending on \( T \) only, such that

\[
|p|^2_{H^2(0, 1)} \leq C(T) |\tilde{w}|_2 \quad \forall t \in [0, T] \quad \text{almost surely.}
\]

This, together with the fact \( \tilde{w}(t) \in L^2(\Omega; \mathbb{R}) \) for all \( t \geq 0 \), which is concluded from (18) and \( \tilde{w}(t) = -C\eta(t) \), implies that \( p(., t) \in L^2_{\mathbb{P}}(\Omega; H^1(0, 1)) \), and

\[
\mathbb{E} \left[ p(., t) - p(., s) \right]^2_{H^2(0, 1)} \leq C(T) \mathbb{E} |\tilde{w}(t) - \tilde{w}(s)|^2.
\]

Since \( \tilde{w} \in C^1_T(0, T; L^2(\Omega; \mathbb{R})) \), it follows that \( p \in C^1_T(0, T; L^2(\Omega; H^1(0, 1))) \). Next, we consider the following stochastic...
heat equation:
\[
\begin{align*}
\frac{dv(x, t)}{dt} &= [v_{xx}(x, t) - cv(x, t) - p(x, t)]dt \\
&+ \sigma(v(x, t) + p(x, t))dB(t) \\
v_y(0, t) &= 0, \quad v_y(1, t) = 0, \quad t \geq 0 \\
v(x, 0) &= z_0(x) - p(x, 0), \quad 0 \leq x \leq 1
\end{align*}
\]
which can be rewritten as
\[
\frac{dv}{dt}(t) = [σv' + p]dt + σ[v + p]dB(t)
\]
where the linear operator $σ$ is defined by
\[
\sigma(f)(x) = f''(x) - cf(x)
\]
and $D(σ) = \{ f \in H^2(0, 1) | f'(0) = f'(1) = 0 \}$. It is well known that $σ$ generates an exponentially stable $C_0$-semigroup on $L^2(0, 1)$ with the decay rate $-c$. This, together with [21, Th. 3.5], shows that system (35) admits a unique weak solution $v \in C_0(0, +∞; L^2(Ω; L^2(0, 1)))$. Define $z(x, t) = v(x, t) + p(x, t)$. It is easy to see that $z(x, t)$ is a unique weak solution to (21).

2) Let $\tau_n = T \land \inf \{ t \geq 0 : \| z(x, t) \|_{L^2(0, 1)} \geq n \}$ and set $\inf \emptyset = +∞$. Clearly, $\tau_n \uparrow \tau_∞$ almost surely for some random variable $τ_∞$. We will show that $τ_∞ = T$ almost surely. By the definition of the stopping time in (39), $\int_0^{τ_∞} 2σ|z(s, s)|^2_{L^2(0, 1)}dB(s)$ is a martingale for all $t \in [0, T]$. We then have
\[
E[z(\cdot, t \land τ_n)]_{L^2(0, 1)} = E[z(\cdot, 0)]_{L^2(0, 1)}
\]
\[
+ 2E \int_0^{τ_∞} z(s, 1)\bar{w}(s)ds
\]
\[
- 2E \int_0^{τ_∞} |z(s, -s)|_{L^2(0, 1)}ds + E \int_0^{τ_∞} (σ^2 - 2c)|z(s, s)|^2_{L^2(0, 1)}ds.
\]
By the Cauchy–Schwarz inequality, Young’s inequality and the classical embedding theorem in Sobolev spaces, we obtain
\[
E \int_0^{τ_∞} z(s, 1)\bar{w}(s)ds \leq \frac{1}{2\varepsilon} E \int_0^{τ_∞} \bar{w}(s)ds
\]
\[
+ \varepsilon E \int_0^{τ_∞} [\|z(s, s)\|^2_{L^2(0, 1)} + |z(s, s)|^2_{L^2(0, 1)}]ds
\]
where $0 < \varepsilon < 1$. The following proof is divided into two cases.

1. Case 1: $σ^2 - 2c + 2ε \leq 0$. In this case, by (40) and (41), it follows that
\[
E[z(\cdot, t \land τ_n)]_{L^2(0, 1)} \leq E[z(\cdot, 0)]_{L^2(0, 1)} + \frac{1}{\varepsilon} \int_0^{τ_∞} \bar{w}(s)ds.
\]

Thus, $nP(τ_n \leq t) \leq E[z(\cdot, 0)]_{L^2(0, 1)} + \frac{1}{\varepsilon} \int_0^t \bar{w}(s)ds$ for any $t \in [0, T]$. Letting $n \rightarrow ∞$, we have $P(τ_n \leq t) = 0$ or $P(τ_n \geq T = 1$ for any $t \in [0, T]$. It can be concluded that $τ_n = T$ almost surely and then $τ_n \uparrow T$ almost surely. By Fatou’s Lemma and passing to the limit as $n \rightarrow ∞$ for (42), we have, for all $t \in [0, T]$, that
\[
E[z(\cdot, t)]_{L^2(0, 1)} \leq E[z(\cdot, 0)]_{L^2(0, 1)} + \frac{1}{\varepsilon} \int_0^T \bar{w}(s)ds
\]
which implies (22).

2. Case 2: $σ^2 - 2c + 2ε > 0$. In this case, similarly to case 1, $τ_n \uparrow T$ almost surely, and by Fatou’s Lemma and passing to the limit as $n \rightarrow ∞$, we can also obtain for all $t \in [0, T]$ that
\[
E[z(\cdot, t)]_{L^2(0, 1)} \leq E[z(\cdot, 0)]_{L^2(0, 1)} + \frac{1}{\varepsilon} \int_0^T \bar{w}(s)ds
\]
\[
+ (σ^2 - 2c + 2ε) \int_0^T E[z(s, s)]^2_{L^2(0, 1)}ds.
\]
By Gronwall’s inequality, for all $t \in [0, T]$, we have
\[
E[z(\cdot, t)]^2_{L^2(0, 1)} \leq (E[z(\cdot, 0)]^2_{L^2(0, 1)} + \frac{1}{\varepsilon} \int_0^T \bar{w}(s)ds)e^{(σ^2 - 2c + 2ε)t}
\]
which implies (22).

3. Step 3: It follows from (18) in Lemma 3.2 that $\bar{w} \in C_0(0, T; L^2(Ω; L^2(0, 1)))$. We can then find a sequence $\{ \bar{w}_n \}_{n=1}^∞ \subset C_0(0, T; L^2(Ω; L^2(0, 1)))$ such that $\lim_{n \rightarrow ∞} \bar{w}_n = \bar{w}$ in $L^2(0, T; L^2(Ω; L^2(0, 1)))$. Denote by $z_n(x, t)$ the weak solution to (21) with the initial value $z_0(x)$ and random boundary value $w_n(t)$. Then, $\{z_n(x, t)\}_{n=1}^∞$ is a Cauchy sequence in $C_0(0, T; L^2(Ω; L^2(0, 1))) \cap L^2_0(0, T; L^2(Ω; H^1(0, 1)))$. Thus, there exists a unique $z \in C_0(0, T; L^2(Ω; L^2(0, 1))) \cap L^2_0(0, T; L^2(Ω; H^1(0, 1)))$ such that $\lim_{n \rightarrow ∞} z_n = z$ in $C_0(0, T; L^2(Ω; L^2(0, 1))) \cap L^2_0(0, T; L^2(Ω; H^1(0, 1)))$. From the definition of the stopping time $τ_n(x, t)$, we have, for all $t \in [0, T]$ and $ϕ \in H^1(0, 1)$ that
\[
\int_0^1 z_n(x, t)ϕ(x)dx - \int_0^1 z(x, 0)ϕ(x)dx
\]
\[
= \int_0^t \frac{∂z_n(1, s)}{∂x}ϕ(1)ds - \int_0^t \frac{∂z(x, s)}{∂x}ϕ(x)dxds - c \int_0^t \int_0^1 z_n(x, s)ϕ(x)dxds + \int_0^t \int_0^1 σz_n(x, s)ϕ(x)dxdB(s).
\]
This yields, for all $t \in [0, T]$ that
\[
\int_0^1 z(x, t)ϕ(x)dx - \int_0^1 z(x, 0)ϕ(x)dx
\]
\[
= \int_0^t z_0(x, 1)ϕ(x)ds - \int_0^t \int_0^1 z_n(x, s)ϕ(x)dxds - c \int_0^t \int_0^1 z_n(x, s)ϕ(x)dxds + \int_0^t \int_0^1 σz(x, s)ϕ(x)dxdB(s).
\]
Therefore, $z(x, t)$ is a weak solution to (21) and satisfies (22). □

APPENDIX C

PROOF OF LEmma 3.4

Let $β(x, t) = z(x, t) + \frac{x^2}{4} Cη(t)$. Clearly, $dβ(x, t) = dz(x, t) + \frac{x^2}{2} Cη(t)dt$ and $β(x, t) = z_{xx}(x, t) + Cη(t)$. By (13), a direct computation shows that $β(x, t)$ satisfies the following stochastic PDE:
\[
\begin{align*}
\frac{dβ(x, t)}{dt} &= [β_{xx}(x, t) - cβ(x, t) + \frac{x^2}{4} - 1]Cη(t) \\
&+ \frac{1}{2} C(A + LCη)dt \\
&+ σ[\frac{1}{2} CL(z, x, t) + β(x, t) - \frac{x^2}{2} Cη(t)]dB(t)
\end{align*}
\]
Thus implies (22).
By Itô’s formula, a direct computation shows that
\[
\begin{align*}
\frac{d\beta^2(x, t)}{dt} &= 2\beta(x, t)d\beta(x, t) + \sigma^2 \left[ \frac{x^2}{2} CLZ(t) + \beta(x, t) - \frac{x^2}{2} C(t) \right]^2 dt \\
&= 2\beta(x, t)[\beta_x(x, t) - c_x(x, t) + \left( \frac{c^2}{2} - 1 \right) C(t)] + \frac{x^2}{2} C(t),
\end{align*}
\]
(49)

Since from Lemmas 3.2 and 3.3, \(2\sigma \int_0^t \beta(x, s) \frac{x^2}{2} CLZ(s) + \beta(x, s) - \frac{x^2}{2} C(t) dB(s)\) is a martingale for all \(t \geq 0\). Integrating on both sides of (49) with respect to \(x\) and \(t\) and taking mathematical expectations, we obtain
\[
\begin{align*}
\mathbb{E} \int_0^t \beta^2(x, t) dt &= \mathbb{E} \int_0^t \beta^2(x, 0) dt - 2c \int_0^t \mathbb{E} \int_0^t \beta^2(x, s) ds dt \\
&+ \int_0^t \mathbb{E} \int_0^t \sigma^2 \left[ \frac{x^2}{2} CLZ(s) + \beta(x, s) - \frac{x^2}{2} C(t) \right]^2 ds dt
\end{align*}
\]
which implies that
\[
\begin{align*}
\frac{d\beta^2}{dt} &= \mathbb{E} \int_0^t \beta^2(x, t) dt \\
&\leq -2c \mathbb{E} \int_0^t \beta^2(x, t) dt + \mathbb{E} \int_0^t \beta^2(x, t) dt + \left( \frac{C}{2} \right)^2 \mathbb{E} \beta^2(t) \\
&+ \mathbb{E} \int_0^t \beta^2(x, t) dt + \mathbb{E} \int_0^t \sigma^2 \left[ \frac{x^2}{2} CLZ(s) + \beta(x, s) - \frac{x^2}{2} C(t) \right]^2 ds dt
\end{align*}
\]
(50)

By (18) in Lemma 3.2 and (51), we conclude that
\[
\begin{align*}
\mathbb{E} \int_0^t \beta^2(x, t) dt &\leq e^{-(2c-2.3\sigma^2)t} \mathbb{E} \int_0^t \beta^2(x, 0) dt \\
&+ \mathbb{E} \int_0^t e^{-(2c-2.3\sigma^2)(s-t)} \mathbb{E}[\beta^2(s)] ds + \mathbb{E} \int_0^t \sigma^2 \left[ \frac{x^2}{2} CLZ(s) + \beta(x, s) - \frac{x^2}{2} C(t) \right]^2 ds dt
\end{align*}
\]
(52)

where \(\theta^*\) is given in (23). 

\[\Gamma_1 = \max \left\{ \frac{c(1 + \sigma^2)C}{2c - 2.3\sigma^2}, \frac{C(1 + \sigma^2)}{4c} \right\},\]
\[\Gamma_2 = \frac{\lambda_{\max}(Q_c)}{\lambda_{\min}(Q_c)} \mathbb{E}[\beta^2(x, 0)] + \frac{1}{2c - 2.3\sigma^2} \max_{\theta^*} \mathbb{E}[\beta^2(x, 0)] + \mathbb{E} \int_0^t \sigma^2 \left[ \frac{x^2}{2} CLZ(s) + \beta(x, s) - \frac{x^2}{2} C(t) \right]^2 ds dt \]
where \(\theta^*\) is given in (23). 

\[\Gamma_3 = \frac{\lambda_{\max}(Q_c)}{\lambda_{\min}(Q_c)} \mathbb{E}[\beta^2(x, 0)] + \frac{1}{2c - 2.3\sigma^2} \max_{\theta^*} \mathbb{E}[\beta^2(x, 0)] + \mathbb{E} \int_0^t \sigma^2 \left[ \frac{x^2}{2} CLZ(s) + \beta(x, s) - \frac{x^2}{2} C(t) \right]^2 ds dt \]
(53)

where we set

\[\Gamma = 2\Gamma_2 + \frac{\lambda_{\max}(Q_c)}{\lambda_{\min}(Q_c)} \mathbb{E}[\beta^2(x, 0)] + \mathbb{E} \int_0^t \sigma^2 \left[ \frac{x^2}{2} CLZ(s) + \beta(x, s) - \frac{x^2}{2} C(t) \right]^2 ds dt \]
(54)

**APPENDIX D**

**PROOF OF THEOREM 3.1**

The existence of the solution \(y \in C_b([0, -\infty); L^2(\Omega; L^2(0, 1)))\) can be concluded directly from Lemma 3.3. In addition, it follows from (7), Lemma 3.4 and the Hölder’s inequality that

\[\mathbb{E} \int_0^t \beta^2(x, t) dt \leq 2\mathbb{E} \int_0^t \beta^2(x, t) dt + 2\mathbb{E} \int_0^t \left( \int_0^t l(x, \zeta) \zeta \left( \zeta, t \right) d\zeta \right)^2 dx \]

\[\leq 2\mathbb{E} \int_0^t \beta^2(x, t) dt \]

\[+ 2\mathbb{E} \int_0^t \left( \int_0^t l^2(x, \zeta) d\zeta \cdot \int_0^t \beta^2(x, t) dt \right) dx \]

\[\leq 2\mathbb{E} \int_0^t \beta^2(x, t) dt + \max_{0 \leq s \leq 1} \max_{0 \leq \xi \leq \zeta} l^2(x, t) \cdot \mathbb{E} \int_0^t \beta^2(x, t) dt \]

\[\leq \Gamma \Gamma^* \leq 2 \left( 1 + \max_{0 \leq s \leq 1} \max_{0 \leq \xi \leq \zeta} l^2(x, t) \right) \Gamma.\]

**APPENDIX E**

**PROOF OF THEOREM 3.2**

Let \(n = 1, 2, \ldots\). Similarly to the techniques in (38) and (41), it follows from Itô’s formula that for \(n - 1 \leq t \leq n\)

\[|z(t)|_{L^2(0,1)}^2 = |z(t)|_{L^2(0,1)}^2 + (\sigma^2 - 2c + 2\varepsilon) \int_{n-1}^t |z(s, \varepsilon)|_{L^2(0,1)}^2 ds \]

\[+ \int_{n-1}^t \tilde{w}^2(s) ds + \int_{n-1}^t 2\sigma |z(s, \varepsilon)|_{L^2(0,1)}^2 dB(s) \]

\[\leq |z(t)|_{L^2(0,1)}^2 + \frac{1}{\varepsilon} \int_{n-1}^t \tilde{w}^2(s) ds \]

\[+ \int_{n-1}^t 2\sigma |z(s, \varepsilon)|_{L^2(0,1)}^2 dB(s) \]

where \(0 < \varepsilon < 1.\)

\[\mathbb{E} \left( \sup_{n-1 \leq s \leq n} |z(t)|_{L^2(0,1)}^2 \right) \leq \mathbb{E} |z(t)|_{L^2(0,1)}^2 \]

\[+ \frac{1}{\varepsilon} \mathbb{E} \tilde{w}^2(s) \]

\[+ \mathbb{E} \left( \sup_{n-1 \leq s \leq n} \int_{n-1}^t 2\sigma |z(s, \varepsilon)|_{L^2(0,1)}^2 dB(s) \right).\]

By the Burkholder–Davis–Gundy inequality (see, e.g., [22, Th. 1.7.3, p. 40]).
\[
\leq 4\sqrt{2E} \left( \int_{n=1}^{\infty} 4\sigma^2 |z(s,t)|^2_{L^2(0,1)} ds \right)^{\frac{1}{2}}
\]

\[
\leq 4\sqrt{2E} \left( \sup_{n=1,2,3,\ldots} |z(s,t)|^2_{L^2(0,1)} \int_{n=1}^{\infty} 4\sigma^2 |z(s,t)|^2_{L^2(0,1)} ds \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} E \left( \sup_{n=1,2,3,\ldots} |z(s,t)|^2_{L^2(0,1)} \right) + 64\sigma^2 \int_{n=1}^{\infty} E|z(s,t)|^2_{L^2(0,1)} ds.
\]  

(58)

After substitution of (58) into (57), we obtain from Lemmas 3.2 and 3.4 that

\[
E \left( \sup_{n=1,2,3,\ldots} |z(s,t)|^2_{L^2(0,1)} \right) \leq 2E|z(s,n-1)|^2_{L^2(0,1)}
\]

\[
+ \frac{2}{\varepsilon} \int_{n=1}^{\infty} E\theta^2(s) ds + 128\sigma^2 \int_{n=1}^{\infty} E|z(s,t)|^2_{L^2(0,1)} ds \leq \Theta e^{-\theta(n-1)}
\]

where \( \Theta = (2 + 128\sigma^2) \Gamma + \frac{2}{\varepsilon} \sup_{n=1,2,3,\ldots} |z(n,0)|^2_{L^2(0,1)} E\|z(0)\|^2 + |\eta(0)|^2 \). Let \( \varepsilon \in (0, \theta^-) \) be arbitrary. By (59) and Chebyshev’s inequality, it follows that

\[
P \left( \sup_{n=1,2,3,\ldots} |z(s,t)|^2_{L^2(0,1)} > e^{-\theta(n-1)} \right)
\]

\[
\leq e^{\theta(n-1)} \mathbb{E} \left( \sup_{n=1,2,3,\ldots} |z(s,t)|^2_{L^2(0,1)} \right) \leq \Theta e^{-\theta(n-1)}
\]  

(59)

Applying the Borel–Cantelli lemma (22, Lemma 2.4, p. 7), we obtain for almost all \( \omega \in \Omega \), that

\[
\sup_{n=1,2,3,\ldots} |g(u,t)|_{L^2(0,1)} \leq \Gamma \sup_{n=1,2,3,\ldots} |z(s,t)|_{L^2(0,1)} 
\]

\[
\leq \Gamma e^{-\theta(n-1)}
\]  

(60)

which holds for all but finitely many \( n \) with \( \Gamma ^{-1} \) given in (56). Hence, there exists a random variable \( n_0 = n_0(\omega) \), such that for almost all \( \omega \in \Omega \), (60) holds whenever \( n \geq n_0 \). Hence, for almost all \( \omega \in \Omega \),

\[
\frac{1}{t} \log |g(u,t)|_{L^2(0,1)} = \frac{1}{2t} \log |g(u,t)|_{L^2(0,1)} \leq \frac{\log \Gamma^{-1}}{2(n-1)} - \frac{(\theta^- - \varepsilon)(n-1)}{2n}
\]

almost surely when \( n - 1 \leq t \leq n \). Therefore, \( \lim_{t \to \infty} \sup_{u \in [0,1]} \frac{1}{t} \log |g(u,t)|_{L^2(0,1)} \leq \frac{(\theta^- - \varepsilon)}{2} \) almost surely. Since \( \varepsilon > 0 \) is arbitrary, we then have

\[
\lim_{t \to \infty} \sup_{u \in [0,1]} \frac{1}{t} \log |g(u,t)|_{L^2(0,1)} \leq \frac{\theta^-}{2} \text{ almost surely.}
\]  

(62)

\[\square\]

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