Lie algebra and invariant tensor technology for $g_2$

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Abstract

Proceeding in analogy with $su(n)$ work on $\lambda$ matrices and $f$- and $d$-tensors, this paper develops the technology of the Lie algebra $g_2$, its seven dimensional defining representation $\gamma$ and the full set of invariant tensors that arise in relation thereto. A comprehensive listing of identities involving these tensors is given. This includes identities that depend on use of characteristic equations, especially for $\gamma$, and a good body of results involving the quadratic, sextic and (the non-primitivity of) other Casimir operators of $g_2$. 

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Contents

1 Introduction .......................................................... 3

2 Cartan-Weyl form of the Lie algebras \( b_3 \) and \( g_2 \) ................. 5
   2.1 General relations ................................................. 5
   2.2 The Lie algebra \( b_3 \) ............................................. 6
   2.3 The Lie algebra \( g_2 \) ............................................. 8
   2.4 \( g_2 \) as a Lie subalgebra of \( b_3 \) ............................ 9
   2.5 The matrices \( y_\alpha \) .......................................... 10

3 Properties of the \( 7 \times 7 \) matrices of \( g_2, b_3 \) and \( a_6 \) .......... 11
   3.1 Identities for the octonionic tensor \( \psi_{\alpha \beta \gamma} \) ............ 12

4 An equivalent \( 7 \times 7 \) matrix representation .............................. 13
   4.1 The matrices \( H_i \) and \( C_a \) .................................... 13
   4.2 The matrices \( Y_\alpha \) .......................................... 14
   4.3 Completeness considerations .................................... 15

5 Bilinear tensor identities .................................................. 16
   5.1 Some lemmas involving the matrices \( H_i, C_a \) and \( Y_\alpha \) ....... 16
   5.2 Completion of the proofs of bilinear identities ...................... 17

6 Casimir operators, projectors and characteristic equations ............... 17
   6.1 The quadratic Casimir operator .................................. 17
   6.2 Projectors ....................................................... 18
   6.3 Characteristic equations ......................................... 19
   6.4 On the non-primitive quartic Casimir operator .................... 22

7 Trilinear tensor identities ................................................. 22

8 Adjoint vectors and invariants ........................................... 24
   8.1 Results ............................................................. 24
   8.2 On the sixth order invariant and non-primitive invariants ......... 25
   8.3 Further use of characteristic equations .......................... 26
1 Introduction

This paper is devoted to the detailed study of such aspects of
a) the Lie algebra $g_2$ with generators $X_i$,
b) the matrices $x_i$ of its defining $7 \times 7$ representation $\gamma: X_i \mapsto x_i$,
c) the $L$-operator $L = x_i X_i$,
d) the $g_2$ invariant tensors that arise in the product laws associated with the $x_i$,
as are expected to be useful in the study of integrable systems for which $g_2$ is an invariance algebra.

We begin by giving references that have been found useful for general background information on Lie algebras [1–5], and for detailed information [6–10] on the exceptional Lie algebra $g_2$. For some indication of the programme we are aiming to follow for $g_2$, we cite [11] a study of some integrable systems with the invariance algebra $c_n = sp(2n, \mathbb{R})$. A $g_2$ application resembling the work [12] might usefully be undertaken.

Our approach is based on the fact that $g_2$ is a non-symmetric subalgebra of $b_3 = so(7)$, which is a symmetric subalgebra of $a_6 = su(7)$, the defining representation of all three being of dimension seven. To explain the distinction made here, let $g$ be a Lie algebra and $h$ a Lie subalgebra so that as vector spaces $g = h + \mathfrak{k}$. Then in addition to

$$[h, h] \subset h, \quad [h, \mathfrak{k}] \subset \mathfrak{k} \quad (1)$$

we have closure of $[\mathfrak{k}, \mathfrak{k}]$ on $h$ iff $h$ is a symmetric subalgebra of $g$, but a more general result

$$[\mathfrak{k}, \mathfrak{k}] \subset h + \mathfrak{k} \quad (2)$$

for non-symmetric cases like $g_2 = h \subset g = b_3$.

Our notation uses indices

$$i, j, k \ldots \in \{1, 2 \ldots, 14\}$$
$$a, b, c \ldots \in \{1, 2 \ldots, 7\}$$
$$\lambda, \mu, \nu \ldots \in \{1, 2 \ldots, 21\}$$
$$\alpha, \beta, \gamma \ldots \in \{1, 2 \ldots, 27\}$$
$$A, B, C \ldots \in \{1, 2 \ldots, 48\} \quad (3)$$

Thus $X_i, X_a, X_\mu, X_\alpha, X_A$ respectively denote generators of $g_2$, generators of $b_3$ which lie outside its $g_2$ subalgebra, generators of $b_3$, generators of $a_6$ which lie outside its $b_3$ subalgebra, and generators of $a_6$. Occasionally also we use $A, B, C$ to denote the generators of an arbitrary simple Lie algebra.

We begin by presenting $b_3$ in Cartan-Weyl form, and $g_2$ as a subalgebra of it also in such a form. In the sprit of [11][13], we encapsulate the information this entails in terms of the $L$-operators of $b_3$ and exhibiting all the matrices of the defining representations $\Gamma: X_\mu \mapsto x_\mu$ of $b_3$, and $\gamma: X_i \mapsto x_i$ of $g_2$, simultaneously in each case, in a single $7 \times 7$ array. These arrays a) allow convenient checking of claimed properties like Lie algebra relations, and b) are valuable artifacts, well-suited for use, cf. [11], in the solution of Yang-Baxter equations of $b_3$ and $g_2$. Likewise matrices such as $A = A_i x_i$, where $A_i \in \mathbb{R}$ allow easy check of trace and other properties of the $x_i$.

To exploit the value of the $L$-operators fully, one needs not only the matrices $x_i$ of $g_2$, and the $7 \times 7$ matrices $z_a$ of the defining representation of $b_3$ which lie outside $g_2$, but also
the matrices \( y\alpha \), which along with the \( x_i \) and the \( z_a \) span the vector space in which the \( 7 \times 7 \) defining representation of \( a_6 \) acts. Detailed information about all these matrices and about the various \( g_2 \) invariant tensors that enter various product laws (see Sec. 3) must be assembled. To obtain it, it is worthwhile to pass by similarity transformation

\[
    x_i \equiv H_i, \quad z_a \equiv C_a, \quad y\alpha \equiv Y\alpha,
\]

\[
    (H_i)^T = -H_i, \quad (C_a)^T = -C_a, \quad (Y\alpha)^T = +Y\alpha,
\]

from the matrix representations that our Cartan-Weyl starting point naturally leads us to, to an equivalent representation in which the matrices enjoy the simple transposition properties

The invariant tensors that occur in the product laws are of course unaffected by a similarity transformation, and their properties are more easily derived. We should remark however that we keep the original matrices in practical uses of our \( L \)-operators. One feature of the discussion deserves emphasis. Our product laws in either of their equivalent forms

\[
\begin{align*}
    [z_a, z_b] &= \ i c_{abc} z_c + i h_{iab} x_i, \\
    [C_a, C_b] &= \ i c_{abc} C_c + i h_{iab} H_i,
\end{align*}
\]

reflect by their first non-trivial first terms the fact that \( g_2 \) is not a symmetric subalgebra of \( b_3 \), and feature the totally antisymmetric third rank quantity \( \psi_{abc} \) that occurs in the multiplication law of octonions in two roles. These correspond to

\[
    c_{abc} = \sqrt{\frac{1}{3}} \psi_{abc}, \quad (C_a)_{bc} = i c_{abc}.
\]

We discuss identities including trace and completeness properties of matrices, contraction formulas and identities of first class for the invariant tensors. By first class identities we mean those that stem from identities of Jacobi type, ones which apply uniformly throughout say the \( a_n \) family, as opposed to the second class identities that involve the use in some way of characteristic equations and are specific to each Lie algebra \([14, 15]\). To access identities of the second class, we consider characteristic polynomials, which also give us a lot of important information about Casimir operators. Our approach follows a general method described in generality elsewhere \([15]\) and depends on the algebra of the projectors for the reduction of the representation \( ad \otimes ad \), where \( ad \) stands for adjoint, of \( g_2 \). Many results emerge like the fact \([16, 17]\) that the quartic Casimir of \( g_2 \) is not primitive, and also formulas for the traces of symmetrised products of adjoint matrices of \( g_2 \). Similar formulas for the traces of symmetrised products of matrices \( x_i \) can likewise be derived from the study of the characteristic equation of \( A = A_i x_i \). Finally the construction, \cf. \([18]\), of \( g_2 \)-vectors and \( g_2 \)-invariants out of the components \( A_i \) of a single adjoint \( g_2 \) vector is undertaken. We thereby find formulas expressing naturally occurring non-primitive scalars in terms of suitably constructed primitive scalars, there being one such of each of the orders 2 and 6 and no others.

There is of course a very large body of work on \( g_2 \) already in the literature. Our citations have mainly concentrated on works that are directly to studies and purposes like our own. However some things may be given a brief mention to place our work in a wider context.

We point out that \( g_2 \) has received attention in theoretical physics in several apparently very different contexts. One was motivated by the search in progress some forty years ago for the flavour symmetry group of the hadrons. This search studied \( g_2 \) and its representations using
roots and weights within the Cartan-Weyl description of Lie algebras. The papers [8], [20] are still excellent accounts of this work. Many physicist at this time relied on [21] for their understanding of Lie algebra theory.

A second definition of $g_2$ is to be found in the work of Racah [22] (see also [21]) on nuclear spectroscopy. This is couched in terms of $su(2)$ unit tensors of ranks 1, 3 and 5 with 3, 7 and 11 components. Their commutation relations are determined by Racah coefficients (or Wigner $6-j$ symbols). Not only does the algebra of tensor components close on $b_7 = so(7)$, but also, because the Racah coefficient $W(3 5 3 5; 3 3)$ vanishes (accidentally?), the set of 14 components of the rank one and five unit tensors close on a subalgebra of $b_3$, namely $g_2$. The thesis [9] contains a clear account of this. The vanishing of the Racah coefficient is obviously not accidental, since an analogous view of other exceptional Lie algebras and superalgebras exists, accessible from [23]. See also [24].

A third definition of $g_2$ realises it as the subgroup of $b_3 = so(7)$ that leaves an eight-component $so(7)$ spinor invariant. This is explained in [8] and in many other places; the discussion in [25] contains relevant detail but does not stress it in the $g_2$ context.

A fourth approach to $g_2$ is that of [34], which expresses the Lie bracket relations elegantly in terms of generators which transform according to the octet, triplet and antitriplet representations of its $su(3)$ subalgebra.

Other matters not considered here (because we deal mainly with certain low dimensional ones) include the state labelling problem, important for discussing the general representation of $g_2$. See e.g. [20], where matrix representations are constructed. The problem is also fully analysed in [35].

While [19] discusses the construction of Casimir operators for Lie algebras in terms of Lie algebra generators, for $g_2$ an explicit formula for the sixth order Casimir is not displayed, although probably accessible. See however [27] which describes an explicit construction of the sixth order Casimir, and a formula for its eigenvalues in a given representation in terms of the Dynkin indices of that representation.

See also [28] (as well as [8]) for information about $g_2$ characters and branching rules.

Finally we cite some interesting work on non-compact realisations of $G_2$ [35–38].

2 Cartan-Weyl form of the Lie algebras $b_3$ and $g_2$

2.1 General relations

Let the compact real simple Lie algebra $g$ with generators $X_A$ and totally antisymmetric structure constants be defined by

$$[X_A, X_B] = i c_{ABC} X_C .$$

Let $V$ denote the defining representation of $g$ with matrices $x_A$ given by $X_A \mapsto \gamma x_A$, where $\gamma = 1$ for all series of simple Lie algebras except $a_n = su(n+1)$, for which $\gamma = \frac{1}{2}$ and $x_A = \lambda_A$ are a set of Gell-Mann matrices [14]. We chose the basis so that

$$x_A^\dagger = x_A, \quad \text{tr } x_A = 0, \quad \text{tr } x_A x_B = 2 \delta_{AB} .$$

We define also the $L$-matrix by

$$L = x_A \otimes X_A \equiv x_A X_A$$

(11)
acting on $V \otimes \mathcal{H}$ where $V$ is the defining representation of $\mathfrak{g}$ and $\mathcal{H}$ any other representation. It is to be noted that $L$ is not only a quantity of central importance in the study of integrable systems with invariance algebra $\mathfrak{g}$, but that it also encapsulates in a very concise and useful manner many properties of $\mathfrak{g}$ and its defining representation $V$.

We present the Lie algebra $\mathfrak{g}$ of rank $n$ and $\dim \mathfrak{g}$ in Cartan-Weyl form with generators $H = (H_1, \ldots, H_n)$ of its Cartan subalgebra, positive roots $r_\alpha$, and raising and lowering operators $E_{\pm \alpha}, \alpha \in \{1, 2, \ldots, \frac{1}{2}(\dim \mathfrak{g} - n)\}$. Then we have

$$[H, E_{\pm \alpha}] = \pm r_\alpha E_{\pm \alpha}, \quad [E_{\alpha}, E_{-\alpha}] = r_\alpha \cdot H,$$  \hspace{1cm} (12)

together with well-known non-trivial expressions for $[E_{+\alpha}, E_{\pm \beta}]$ whenever $r_\alpha \pm r_\beta$ is a non-zero root of $\mathfrak{g}$. The $X_A$ of (7) are related to the Cartan-Weyl generators according to

$$\begin{cases} X_A: A \in \{1, \ldots, \dim \mathfrak{g}\} = \{H_r: r \in \{1, \ldots, n\} \} \\ U_\alpha, V_\alpha: \alpha \in \{1, \ldots, \frac{1}{2}(\dim \mathfrak{g} - n)\} \end{cases},$$  \hspace{1cm} (13)

where $\sqrt{2}E_{\pm \alpha} = U_\alpha \pm iV_\alpha$. For the defining representation $V$ of $\mathfrak{g}$, we have $X_A \mapsto x_A$, except, as noted, in the case of $a_n$. Ignoring this case, we employ the notation

$$H \mapsto h, \quad E_{\pm \alpha} \mapsto e_{\pm \alpha}, \quad \sqrt{2}e_{\pm \alpha} = u_\alpha \pm iv_\alpha,$$  \hspace{1cm} (14)

so that also

$$\begin{cases} x_A: A \in \{1, \ldots, \dim \mathfrak{g}\} = \{h_r: r \in \{1, \ldots, n\} \} \\ u_\alpha, v_\alpha: \alpha \in \{1, \ldots, \frac{1}{2}(\dim \mathfrak{g} - n)\} \end{cases}.$$  \hspace{1cm} (15)

General references for background on Lie algebras and their representations have been noted [1–3, 5] as well as the source [4] of valuable information.

### 2.2 The Lie algebra $b_3$

We turn now to the cases on which the present work focusses. Thus we start with the Lie algebra of $b_3 \cong so(7)$, because we intend to realise the Lie algebra of $g_2$ as a subalgebra of $b_3$ that is not symmetric. To distinguish between quantities referring to $b_3$ and their counterparts for $g_2$, we shall, for $b_3$, use $R_\alpha$ for its roots, $\mathcal{H}, \mathcal{E}$ for its generators, and $\mathcal{H} \mapsto k$ and $\mathcal{E} \mapsto \epsilon$ for the matrices of the defining representation $\Gamma$.

For $b_3$, the simple roots are [1]

$$R_1 = (1, -1, 0), \quad R_2 = (0, 1, -1), \quad R_3 = (0, 0, 1),$$  \hspace{1cm} (16)

and the remaining positive roots are given by

$$\begin{align*}
R_{12} &= R_1 + R_2 = (1, 0, -1), \\
R_{23} &= R_2 + R_3 = (0, 1, 0), \\
R_{123} &= R_1 + R_2 + R_3 = (1, 0, 0), \\
R_{233} &= R_2 + R_3 = R_2 + 2R_3 = (0, 1, 1), \\
R_{1233} &= R_1 + R_3 = R_1 + R_2 + 2R_3 = (1, 0, 1), \\
R_{1223} &= R_1 + R_2 + R_3 = R_1 + 2R_2 + 2R_3 = (1, 1, 0). \quad (17)
\end{align*}$$
Of these, only $R_3$, $R_{23}$ and $R_{123}$ are short.

For $b_3$ and $g_2$ it is convenient to denote by $\Gamma$ and $\gamma$ their $7 \times 7$ defining representations $V$. We build the matrix

$$L = \mathcal{H}_1 k_1 + \mathcal{H}_2 k_2 + \mathcal{H}_3 k_3 + \sum_{\alpha \geq 0} (\mathcal{E}_\alpha \epsilon_{-\alpha} + \mathcal{E}_\alpha \epsilon_{\alpha}) ,$$

explicitly in the form

$$L = \begin{pmatrix}
\mathcal{H}_1 & \mathcal{E}_1 & \mathcal{E}_{12} & \mathcal{E}_{123} & \mathcal{E}_{1233} & 0 & 0 \\
\mathcal{E}_1 & \mathcal{H}_2 & \mathcal{E}_2 & \mathcal{E}_{23} & \mathcal{E}_{233} & 0 & 0 \\
\mathcal{E}_{12} & \mathcal{E}_2 & \mathcal{H}_3 & \mathcal{E}_3 & \mathcal{E}_{23} & \mathcal{E}_{233} & 0 \\
\mathcal{E}_{123} & \mathcal{E}_{23} & \mathcal{E}_3 & \mathcal{H}_3 & \mathcal{E}_{12} & \mathcal{E}_{1233} & 0 \\
\mathcal{E}_{1233} & \mathcal{E}_{233} & \mathcal{E}_{23} & \mathcal{E}_3 & \mathcal{H}_3 & \mathcal{E}_{12} & \mathcal{E}_{1233} \\
0 & 0 & 0 & \mathcal{E}_{123} & \mathcal{E}_{1233} & \mathcal{E}_3 & \mathcal{H}_3 \\
0 & 0 & 0 & 0 & \mathcal{E}_{1233} & \mathcal{E}_3 & \mathcal{H}_3 \\
\end{pmatrix} .$$

Here and in all such cases below, the upper triangular part of $L$ has been suppressed for ease of reading but it can easily be supplied with the aid of hermiticity properties. One may read off (14) the explicit forms of the matrices $k_1, k_2, k_3$ and $\epsilon_{+\alpha}$ of $\Gamma$, and verify that they obey the Cartan-Weyl Lie algebra relations of $b_3$ that follow from (12) when the choice (16) and (17) of roots is made. One may write $L = \mathcal{N}_+ + \mathcal{H} + \mathcal{N}_-$ to expose the parts of $L$ upper-triangular, diagonal and lower-triangular, which corresponds to writing $b_3 = n_+ + h + n_-$, where $n_\pm$ is the span of the $\epsilon_{\pm\alpha}$ and $h$ refers to the Cartan subalgebra. Then calculation of $[\mathcal{H}, \mathcal{N}_\pm]$ and $[\mathcal{N}_+, \mathcal{N}_-]$ is easily done in each case at a single (MAPLE) stroke.

The matrix $L$ has many features of interest. In the main diagonal, and the first super-diagonal above it, one sees the non-zero entries of the matrices of the Cartan subalgebra and of those of the lowering operators that belong to the simple roots of $b_3$. The matrices $\epsilon_{-\alpha}$ for roots of heights greater than one are then found by looking at the other superdiagonals successively, all the way up $\epsilon_{-12233}$ corresponding to the highest root $R_{12233}$ of height five. One sees how the expansion of any root of $b_3$ in terms of simple roots is reflected in $L$. For example, there is an entry to $\epsilon_{-12233}$ in position (15), since $\mathcal{E}_{1233}$ is associated with the root $R_{1233} = R_1 + R_2 + R_3 + R_3$ whose summands are similarly related to the entries in places (12), (23), (34), (45), and so on. Another point is worth emphasising because it facilitates the writing down of the matrix $L$ for $g_2$ below: all the entries of the $p$-th subdiagonal of (19) refer to roots of heights $p$, $p \in \{1, \cdots, 5\}$, with $p = 1$ for the simple roots.

It was emphasised in [13] that such a construction as we have just described is available not only for $b_n$ for all $n$, but also for the other classical families of Lie algebra. It was also mentioned there that an analogous result can be obtained for $g_2$ – a matter to which we turn below.

It may also be verified explicitly that the matrices $x_\mu$ of the defining representation $\Gamma = V$ of $b_3$, namely

$$x_\mu = \{k_1, k_2, k_3, u_\alpha, v_\alpha, \alpha = 1, \cdots, 9\} ,$$

where $\sqrt{2} \epsilon_{\pm\alpha} = u_\alpha \pm iv_\alpha$, and $\alpha = 1, \cdots, 9$ corresponds to the index set

$$\{1, 2, 3, 12, 23, 123, 233, 1233, 12233\} ,$$

possess the properties (10). In addition they satisfy the antisymmetry properties

$$x_\mu^T = -M x_\mu M^{-1} ,$$

where the matrix $M = M^T = M^{-1}$ has ones down its main anti-diagonal and zeros elsewhere.
2.3 The Lie algebra $g_2$

In discussing the Lie algebra of $g_2$, we employ the standard notation. We thus employ for $g_2$ the simple roots

$$r_1 = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}\right), \quad r_2 = (0, \sqrt{2}),$$

so that the other four positive roots may be taken to be

$$r_{12} = r_1 + r_2, \quad r_{112} = 2r_1 + r_2, \quad r_{1112} = 3r_1 + r_2, \quad r_{11112} = 3r_1 + 2r_2.$$  \hfill (24)

We see that $r_\alpha$ for $\alpha = 1, 12, 112$ are short roots with norm-squared $\frac{2}{3}$ while the others are long with norm-squared equal to two. Then the Lie algebra relations for $g_2$ follow from \eqref{eq:lie_group_relations} supplemented by the relations

$$[E_1, E_2] = E_{12},$$
$$[E_{12}, E_1] = \frac{2}{\sqrt{3}} E_{112},$$
$$[E_1, E_{12}] = E_{112},$$
$$[E_2, E_{112}] = E_{1112},$$
$$[E_{112}, E_{12}] = E_{11122}.$$ \hfill (25)

One can give the embedding of this realisation of $g_2$ in the realisation of $b_3$ presented in Sec. 2.2 explicitly:

$$H_1 = \sqrt{\frac{2}{3}}(2\mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3),$$
$$H_2 = \sqrt{\frac{2}{3}}(\mathcal{H}_2 - \mathcal{H}_3),$$
$$E_1 = \sqrt{\frac{2}{3}} \mathcal{E}_1 + \sqrt{\frac{2}{3}} \mathcal{E}_3,$$
$$E_2 = \mathcal{E}_2,$$
$$E_{12} = \sqrt{\frac{2}{3}} \mathcal{E}_{12} - \sqrt{\frac{2}{3}} \mathcal{E}_{23},$$
$$E_{112} = \sqrt{\frac{2}{3}} \mathcal{E}_{123} + \sqrt{\frac{1}{3}} \mathcal{E}_{233},$$
$$E_{1112} = \mathcal{E}_{1233},$$
$$E_{11112} = \mathcal{E}_{1223}.$$ \hfill (26)

Again we note the agreement of the heights of the roots on the two sides of each of last six equations.

Turning to the construction of the matrices $x_i$ of the defining representation $\gamma$ of $g_2$, we know that it corresponds to the span of a subset of 14 of the 21 matrices taken above for the defining representation $\Gamma$ of $b_3$. We also know the weight diagram of the $7 \times 7$ representation $\gamma$ of $g_2$, which gives us the matrices of the Cartan sub-algebra. We expect an expression like that of \eqref{eq:lie_group_relations} to have the same characteristics for $g_2$ as \eqref{eq:lie_group_relations} itself did for $b_3$. Thus one is lead
with very little difficulty to the result for $g_2$

$$L = \begin{pmatrix}
  cH_1 & J_1 & 0 \\
  sE_1 & E_2 & -cE_1 & J_2 \\
  sE_{12} & E_{112} & cE_1 & -J_2 \\
  cE_{112} & -cE_{112} & 0 & -cE_{112} \\
  E_{1112} & 0 & -sE_{1112} & cE_{12} \\
  0 & -E_{11122} & -E_{1112} & -sE_{1112} \\
  0 & -E_{11122} & E_{1112} & -sE_{1112} \\
  0 & 0 & 0 & -cH_1 \\
\end{pmatrix}.$$  \hspace{1cm} (27)

Here we have used the abbreviations

$$J_1 = \left( \frac{1}{\sqrt{6}}H_1 + \frac{1}{\sqrt{2}}H_2 \right), \quad J_2 = \left( \frac{1}{\sqrt{6}}H_1 - \frac{1}{\sqrt{2}}H_2 \right)$$

$$c \equiv \cos \theta = \sqrt{\frac{2}{3}}, \quad s \equiv \sin \theta = \sqrt{\frac{1}{3}}.$$  \hspace{1cm} (28)

There is little difficulty because $g_2$ has exactly one positive root of each height $p = 2, 3, 4, 5$. Thus working downwards from the highest root, we get the placing of $E_{11122}$ and $E_{1112}$ directly. For $E_{112}$ we place down the third sub-diagonal the entries $(c_1, s_1, -s_1, -c_1)$, with values of $c_1$ and $s_1$ to be fixed. Here $c_1 = \cos \theta_1$. The entries for $E_{12}$ and $E_1$ likewise involve further angles $\theta_2$ and $\theta_3$, but the entries for $E_2$ are written down directly. One reads the implied matrices for the $e_{\pm\alpha}$ off the display for $L$ partially thereby obtained. Then it is easy to see the assignments represent (25) correctly for the values presented in (27) and (28). The assignments could also have been inferred in agreement with this using the defining representation matrices of (26).

Some of the useful features of (27) can be seen more clearly when we present in more schematic form with signs and constants supressed

$$\begin{pmatrix}
  x & 1 & x \\
  12 & 2 & x \\
  112 & 12 & 1 & x \\
  1112 & 112 & 0 & 1 & x \\
  11122 & 0 & 112 & 12 & 2 & x \\
  0 & 11122 & 1112 & 112 & 12 & 1 & x \\
\end{pmatrix}.$$  \hspace{1cm} (29)

The pattern of the places associated with the non-simple roots relative to the simple ones here, despite its non-trivial nature, conforms very closely to that found for the $c_n$ and $b_n$ families. See (13) and cf. (19).

We may read explicit expressions for the matrices $h_1$, $h_2$ and $e_{\pm\alpha}$ from (27) and (28), and check that they obey exactly the same Lie algebra relations as do the abstract generators. Because the $x_i$ for $g_2$ are a subset of those given above for $b_3$, they share the same properties, namely (10) and (22).

### 2.4 $g_2$ as a Lie subalgebra of $b_3$

We have seen that the 14 matrices $x_i$ of the defining representation $\gamma$ of $g_2$ share the same properties as the 21 matrices of $\Gamma$ for $b_3$. The explicit definitions show the former to be simple linear combinations of the latter. It is natural to want to define the 7 linear combinations of
the $b_3$ matrices that lie outside $g_2$, but which share the properties (10) and (22). To do so at a single stroke we define

$$K = H_3 h_3 + \sum_{\beta=1}^{3} (E'_\beta e'_{-\beta} + E'_{-\beta} e'_\beta) \quad ,$$

explicitly in the form

$$K = \begin{pmatrix} sH_3 & -cE'_1 & -sH_3 \\
-cE'_2 & 0 & -sH_3 \\
sE'_3 & sE'_2 & sE'_1 & 0 \\
0 & -cE'_3 & 0 & -sE'_1 & sH_3 \\
0 & 0 & cE'_3 & -sE'_2 & 0 & sH_3 \\
0 & 0 & 0 & -sE'_3 & -cE'_2 & cE'_1 & -sH_3 \end{pmatrix} \quad ,$$

where $c$ and $s$ are as given in (28). This implies

$$z_4 = h_3 = s \text{diag} (1, -1, -1, 0, 1, 1, -1) \quad ,$$

and we define the other $z_a$, $a \in \{1, \cdots \}$ by means of

$$\sqrt{2}e_{\pm 1} = z_7 \pm iz_1$$

$$\sqrt{2}e_{\pm 2} = z_6 \pm iz_2$$

$$\sqrt{2}e_{\pm 3} = z_5 \pm iz_3 \quad .$$

The reason for the arrangement of detail here is provided in Sec. 3.

The display (31) thus yields explicitly the matrices $z_a, a \in \{1, \cdots , 7\}$ which span the $b_3 - g_2$ part of the vector space of the representation $\Gamma$ of $b_3$. The following properties may be checked

$$\text{tr} z_a = 0 \ , \ \text{tr} (z_az_b) = \delta_{ab} \ , \ \text{tr} (z_ax_i) = 0 \ , \ z_a^T = -Mz_aM^{-1} \quad .$$

It is to be noted that $K$ simply uses those linear combinations of $b_3$ matrices that are excluded from (27).

### 2.5 The matrices $y_\alpha$

To complete the basis we need for $7 \times 7$ traceless hermitian matrices, we define a set of 27 linearly independent matrices $y_\alpha$ which enjoy the properties

$$y_\alpha^\dagger = y_\alpha \ , \ \text{tr} y_\alpha = 0 \ , \ y_\alpha^T = +My_\alpha M^{-1} \ , \ \alpha \in \{1, \cdots , 27\} \quad ,$$

and

$$\text{tr} (y_\alpha y_\beta) = 2\delta_{\alpha\beta} \ , \ \text{tr} (x_i y_\alpha) = 0 \ , \ i \in \{1, \cdots , 14\} \ , \ \text{tr} (z_ay_\alpha) = 0 \ , \ a \in \{1, \cdots , 7\} \quad .$$

By referring to the diagonal matrices of $\gamma$ and to the diagonal matrix $z_4 = h_3$ given by (32) we chose the three diagonal matrices $y_\alpha$ to be

$$y_1 = \frac{1}{\sqrt{6}} \text{diag} (2, -1, -1, 0, -1, 1, 2)$$

$$y_2 = \frac{1}{\sqrt{2}} \text{diag} (0, 1, -1, 0, -1, 1, 0)$$

$$y_3 = \frac{1}{\sqrt{21}} \text{diag} (1, 1, 1, -6, 1, 1, 1) \quad .$$
This completes the choice of a set of six diagonal traceless hermitian matrices tracewise mutually orthogonal and also orthogonal to the matrix diag\((1, 1, 1, 1, 1, 1)\). The other 24 matrices \(y_\alpha\) are easy to specify using the operator matrix

\[
\sum_{n=1}^{12}(\rho_{-n}R_n + \rho_nR_{-n}) = \begin{pmatrix}
x & R_1 & x & R_4 & x & R_6 & R_5 & x & R_8 & \sqrt{2}R_{12} & R_3 & x \\
R_9 & \sqrt{2}R_{11} & R_7 & R_5 & R_2 & x & R_9 & R_8 & R_6 & R_4 & R_1 & x
\end{pmatrix} . \tag{38}
\]

The diagonal elements \(x\) here can be inferred from \((37)\). Then setting

\[
\sqrt{2}\rho_{\pm n} = y_{2n+2} \pm iy_{2n+3} \quad n = 1, \cdots, 12 \quad , \tag{39}
\]

completes the definition of the \(y_\alpha, \alpha \in \{1, \ldots, 27\}\). Our display of an explicit choice of matrices \(x_i, z_a, y_\alpha\) is motivated by the need to perform/check our matrix, and later, tensorial manipulations by MAPLE.

### 3 Properties of the \(7 \times 7\) matrices of \(g_2, b_3\) and \(a_6\)

We have defined the traceless hermitian \(7 \times 7\) matrices of \(x_i, i \in \{1, \ldots, 14\}\) of \(g_2\), \(z_a, a \in \{1, \ldots, 7\}\) of \(b_3 - g_2\), and \(y_\alpha, \alpha \in \{1, \ldots, 27\}\) of \(a_6 - b_3\). All are normalised so that the trace of their square is 2, and they are tracewise mutually orthogonal. They have the symmetry properties

\[
x_i^T = -Mx_iM^{-1} , \quad z_a^T = -Mz_aM^{-1} , \quad y_\alpha^T = My_\alphaM^{-1} \quad . \tag{40}
\]

It is possible to view the set of 48 matrices \(x_i, z_a, y_\alpha\) as a set of lambda matrices of \(a_6\). Their single multiplication rule here is replaced by the following

\[
\begin{align*}
x_i x_j & = \frac{2}{7}\delta_{ij} + \frac{1}{2}i\epsilon_{ijk}x_k + \frac{1}{7}d_{ija}y_a \\
z_a z_b & = \frac{2}{7}\delta_{ab} + \frac{1}{2}i\epsilon_{abc}z_c + \frac{1}{7}ih_{iab}x_i + \frac{1}{7}d_{ab\gamma}y_\gamma \\
y_\alpha y_\beta & = \frac{2}{7}\delta_{\alpha\beta} + \frac{1}{2}i\phi_{\alpha\beta\gamma}x_i + \frac{1}{7}ih_{a\alpha\beta}z_a + \frac{1}{7}d_{\alpha\beta\gamma}y_\gamma \\
x_i z_a & = \frac{1}{7}ih_{iab}z_b + \frac{1}{7}d_{iaa}y_a \\
x_i y_\alpha & = \frac{1}{7}i\phi_{\alpha\beta\gamma}y_\beta + \frac{1}{7}d_{ija}x_j + \frac{1}{7}d_{iaa}z_a \\
z_a y_\alpha & = \frac{1}{7}ih_{a\alpha\beta}y_\beta + \frac{1}{7}d_{aba}z_b + \frac{1}{7}d_{iaa}x_i \quad . \tag{46}
\end{align*}
\]

First we note that since all tensors here are real, hermitian conjugation of \((41) - (46)\) implies certain evident symmetry and antisymmetry properties. Second we note that behaviour under conjugation with \(M\) is what determines which terms are allowed on the right hand sides. Third we note the tracelessness of symmetric tensors occurring in \((41) - (46)\)

\[
d_{ii\alpha} = 0 , \quad d_{aba} = 0 , \quad d_{\alpha\alpha\gamma} = 0 \quad . \tag{47}
\]

Fourth we note the absence from \((41)\) of a term in \(z_a\). This simply reflects the closure property of the Lie algebra \(g_2\):

\[
[x_i, x_j] = ic_{ijk}x_k \quad . \tag{48}
\]
Fifth (41) and (43) imply
\[
[z_a, z_b] = i c_{abc} z_c + i h_{iab} x_i
\]
\[\text{Eq. (49)}\]
\[
[x_i, z_a] = i h_{iab} z_b
\]
\[\text{Eq. (50)}\]
Eqs. (48), (49) and (50) represent the Lie algebra relations of $b_3$ as generated by $x_i$ and $z_a$. The presence of the $x_i$ term in $[z_a, z_b]$ shows that $g_2$ is not a symmetric subalgebra of $b_3$ merely a reductive one. Sixth we note that, in the use of eqs. (41) to (46) to define the various $(g_2)$ invariant tensors there are repetitions. This follows from easy trace considerations. The latter also imply the total antisymmetry of $c_{ijk}$ and $c_{abc}$ and the total symmetry of $d_{\alpha \beta \gamma}$.

The totally antisymmetric $b_3$ structure constants $c_{abc}$ occurring in (49) are particularly interesting quantities, because, with due arrangement of the details, they are constant multiples of the totally antisymmetric tensors that enter the multiplication law of octonions. This should not be entirely surprising in view of the well-known occurrence of octonions in connection with $g_2$ featured in [6] and in the other main $g_2$ sources cited.

It is easy to see this explicitly, and appreciate the arrangement of detail in (33), by showing by direct calculation that
\[
c_{abc} = \frac{1}{\sqrt{3}} \psi_{abc} , a, b, c \in \{1, \ldots, 7\}
\]
\[\text{Eq. (51)}\]
where the totally antisymmetric octonionic tensor $\psi_{abc}$ has the value one for these ordered triples
\[
(123), (147), (165), (246), (257), (354), (367)
\]
\[\text{Eq. (52)}\]
and no other non-zero components except for those implied by antisymmetry. This is already a striking result, but it will be further embellished in Sec. 4.1 below. One consequence of (51) is that it greatly helps in the search for identities involving invariant tensors just defined, because the many known identities involving the $\psi_{abc}$, see especially [6], give us good control of the $c_{abc}$. Octonionic tensor identities are reviewed in Sec. 3.1.

### 3.1 Identities for the octonionic tensor $\psi_{abc}$

The triples for which the totally antisymmetric third rank tensor $\psi_{abc}$ takes non-zero values appear in (52). We note the following identities which it satisfies
\[
\psi_{abc} \psi_{abd} = 6 \delta_{cd}
\]
\[\text{Eq. (53)}\]
\[
\psi_{abc} = -\frac{1}{4!} \varepsilon_{abdefg} \psi_{dgh} \psi_{fgh}
\]
\[\text{Eq. (54)}\]
Here we see the totally antisymmetric seventh rank epsilon tensor. It is easy to confirm the truth of these results directly, and to see that they imply the key result
\[
\psi_{dch} \psi_{fgh} = \delta_{df} \delta_{eg} - \delta_{dg} \delta_{ef} - \frac{1}{6} \varepsilon_{defgabc} \psi_{abc}
\]
\[\text{Eq. (55)}\]
whence we can obtain the results
\[
\psi_{fag} \psi_{gbe} \psi_{ecf} = 3 \psi_{abc}
\]
\[\text{Eq. (56)}\]
\[
\psi_{h[de} \psi_{f]gh} = -\frac{1}{6} \varepsilon_{abdefg} \psi_{abc}
\]
\[\text{Eq. (57)}\]
\[
\psi_{dch} \psi_{fgh} + \psi_{feh} \psi_{dgh} = 2 \delta_{df} \delta_{eg} - \delta_{dg} \delta_{ef} - \delta_{fg} \delta_{ed}
\]
\[\text{Eq. (58)}\]
The identity (55) is a very simple and convenient product law, but it involves the seventh rank epsilon tensor. If one wants a product law that does not then one must have recourse to the remarkable result – derived without use of (55) – from [8]:

\[
\psi_{abj} \psi_{dek} \psi_{hek} = 3 \delta_{a[e} \psi_{f]g} b - 3 \delta_{b[e} \psi_{f]g} a - \delta_{ef} \psi_{gab} + \delta_{eg} \psi_{fab} .
\]

(59)

To establish this, here on the basis of (55), we multiply (55) by \( \psi_{abj} \), and use a simple argument to treat the \( \epsilon \delta \delta \) term that arises.

The result (51) relates the octonionic tensor \( \psi_{abc} \) to structure constants \( c_{abc} \) of the Lie algebra \( b_3 \). However (57) proves that the \( c_{abc} \) by themselves do not constitute the full set of structure constants of any Lie algebra, since they do not satisfy a Jacobi identity. The latter result itself is of course well-known.

We draw attention also to [25], which derives identities for the torsion tensor that parallels the sphere \( S^7 \), this tensor being a constant multiple of the one that specifies the octonionic product law. In particular, one of these identities is equivalent to (55).

4 An equivalent \( 7 \times 7 \) matrix representation

4.1 The matrices \( H_i \) and \( C_a \)

The representation of the matrices \( x_i, z_a \) and \( y_\alpha \) that we have been lead to above is the one that emerges naturally from the Cartan-Weyl representation of \( g_2 \) as a subalgebra of \( b_3 \). This enabled the definition (41) – (46) of \( \gamma \) of \( g_2 \) invariant tensors. In virtue of the conjugation properties (40) the representation is probably not the most convenient one to use in deriving the identities involving these tensors. Thus we wish to pass to an equivalent basis in which the \( x_i \) and \( z_a \) are replaced by antisymmetric matrices, and the \( y_\alpha \) by symmetric ones. In fact the required basis is already to hand.

To see this we note first that the Lie bracket relation of (50) provides us with a set of 14 hermitian antisymmetric matrices \( H_i \) for the \( 7 \times 7 \) representation \( \gamma \) of \( g_2 \) via the definition

\[
(H_i)_{ab} = i h_{iab} ,
\]

(60)

since the Jacobi identity for \( x_i, x_k, z_a \) translates directly into the result

\[
[H_i, H_k] = ic_{ikl} H_l .
\]

(61)

There is clearly an equivalence relation of the form

\[
x_i = S H_i S^{-1} , \quad \det S \neq 0 ,
\]

(62)

so that we get

\[
\text{tr} H_i = 0 \quad , \quad \text{tr} (H_i H_j) = \delta_{ij} .
\]

(63)

This last result translates into the identity

\[
h_{iab} h_{jab} = 2 \delta_{ab} .
\]

(64)

Next we examine the Jacobi identity for \( x_i, z_a, z_b \). Using the two relations (43) and (44), we are lead by linear dependence to two relations, one of which reproduces (61). The other gives

\[
[H_i, C_a] = i h_{iab} C_b .
\]

(65)
where we have defined a set of \( 7 \times 7 \) antisymmetric matrices \( C \) via

\[
(C_a)_{bc} = +\eta c_{abc} \ ,
\]

where \( \eta \) is a constant that remains to be determined. If we normalise \( C_a \) so that

\[
\text{tr} (C_a C_b) = 2 \delta_{ab} \ ,
\]

then (61) and (63) give \( \eta^2 = 1 \), and it turns out below that \( \eta = 1 \). Eq. (65) implies that the \( C_a \) transform according to the 7 dimensional representation of \( g_2 \). Further conjugation by \( S \) of the Lie algebra relation of (50) for gives rise to (65) with the identification

\[
z_a = S C_a S^{-1} \ .
\]

In other words conjugation with \( S \) carries the basis \( x_i, z_a \) of \( b_3 \) into equivalence with the basis of antisymmetric matrices \( H_i, C_a \).

Returning to a topic broached at the end of Sec. 3, we recall that the \( c_{abc} \) are proportional to the third rank antisymmetric tensor of the multiplication law of octonions. They have entered first as the structure constants that reflect the non-symmetric nature of \( g_2 \) as a subalgebra of \( b_3 \). Second they determine the elements of the matrices \( C \) which enter the version

\[
[C_a, C_b] = i h_{iab} H_i + i c_{abc} C_c \ ,
\]

of the previous statement, i.e. (13), which arises by conjugation of it by \( S \). Use of (69) and (74) shows that

\[
\text{tr}(C_a C_b C_c) = i c_{abc} \ ,
\]

and comparison of this with (63) requires the choice \( \eta = 1 \) noted above. Also (71) now translates into the identity

\[
c_{ae} c_{be} = 2 \delta_{ab} \ .
\]

4.2 The matrices \( Y_\alpha \)

To get at the matrices generated by conjugation with \( S \) of the matrices \( y_\alpha \) we set out from the consequence of (45)

\[
[x_i, y_\alpha] = i \phi_{ia\beta} y_{\alpha\beta} \ .
\]

obtaining

\[
[H_i, Y_\alpha] = i \phi_{ia\beta} Y_{\beta} \ ,
\]

where

\[
Y_\alpha = S(Y_\alpha) S^{-1} \ .
\]

Also the Jacobi identity containing \( \{ z_a, z_b \}, y_\alpha \) leads to two relations. The one we want, from the \( x_i \) term, is of the expected form (73) if we set

\[
(Y_\alpha)_{ab} = \epsilon d_{aba} \ ,
\]

where \( \epsilon \) is a constant. To determine \( \epsilon \), we use (12), the equivalence relations (62) and (68), and (51) to obtain

\[
d_{aba} = \text{tr} (z_a z_b y_\alpha) = \text{tr} (C_a C_b Y_\alpha) = -\frac{1}{3} \psi_{ae} \psi_{fb} \epsilon d_{ge\alpha} \ .
\]
Since $Y_\alpha$ is traceless and symmetric, only one term of (55) contributes to (76). Hence we find

$$(Y_\alpha)_{ab} = -3 d_{ab\alpha}.$$  (77)

We also, from equivalence, have

$$\text{tr} (Y_\alpha Y_\beta) = 2 \delta_{\alpha\beta},$$  (78)

and hence

$$d_{aba} d_{ab\beta} = \frac{2}{9} \delta_{\alpha\beta}.$$  (79)

### 4.3 Completeness considerations

We are in a position to use as a set of 48 lambda matrices of $a_6 = su(7)$ the matrices

$$\lambda_A = \{ H_i, i \in \{1, \cdots, 14\} ; C_a, a \in \{1, \cdots, 7\} ; Y_\alpha, \alpha \in \{1, \cdots, 27\} \}.$$  (80)

They are linearly independent, hermitian and traceless, and have been normalised in agreement with

$$\text{tr} (\lambda_A \lambda_B) = 2 \delta_{AB}.$$  (81)

In addition, we have the symmetry properties

$$H_i = -H_i^T, \quad C_a = -C_a^T, \quad Y_\alpha = +Y_\alpha^T.$$  (82)

The completeness identities for $b_3$ and $a_6$ can be written down directly. For $b_3$ we have

$$(H_i)_{ab} (H_i)_{cd} + (C_e)_{ab} (C_e)_{cd} = \delta_{ad} \delta_{bc} - \delta_{ac} \delta_{bd},$$  (83)

and, with the aid of the well-known completeness result for $a_6$, we obtain

$$(Y_\alpha)_{ab} (Y_\alpha)_{cd} = -\frac{2}{7} \delta_{ab} \delta_{cd} + \delta_{ad} \delta_{bc} + \delta_{ac} \delta_{bd}.$$  (84)

Also (83) may be arranged as a completeness result for $g_2$ because of (51). This gives

$$(H_i)_{ab} (H_i)_{cd} = \delta_{ad} \delta_{bc} - \delta_{ac} \delta_{bd} + \frac{1}{4} \psi_{eab} \psi_{ecd}.$$  (85)

Finally by conjugating with $S$ the completeness result for the basis $x_\mu = \{ x_i, z_a \}$ for $b_3$, namely

$$(x_i)_{ab} (x_i)_{cd} + (z_e)_{ab} (z_e)_{cd} = \delta_{ad} \delta_{bc} - M_{ac} M_{bd},$$  (86)

and comparing with (82), one may use Schur’s lemma to deduce

$$S^T MS = f I,$$  (87)

where $f$ is a constant that has not been determined.
5 Bilinear tensor identities

We begin with a listing of the most important of these ‘two-tensor’ identities

\[
\begin{align*}
\text{5.1 Some lemmas involving the matrices } H_i, C_a \text{ and } Y_\alpha \\
\text{We begin with a listing}
\end{align*}
\]

\[
\begin{align*}
H_i H_j H_i &= 0 \quad (110) \\
H_i C_a H_i &= 2 C_a \quad (111)
\end{align*}
\]
\[ H_i Y_\alpha H_i = -\frac{2}{3} Y_\alpha \] \hspace{1cm} (112)
\[ C_a H_i C_a = H_i \] \hspace{1cm} (113)
\[ C_a C_b C_a = -C_b \] \hspace{1cm} (114)
\[ C_a Y_\alpha C_a = -\frac{3}{2} Y_\alpha \] \hspace{1cm} (115)
\[ Y_\alpha Y_\beta Y_\alpha = \frac{3}{7} Y_\beta \] \hspace{1cm} (116)
\[ Y_\alpha H_i Y_\alpha = -\frac{9}{7} H_i \] \hspace{1cm} (117)
\[ Y_\alpha C_b Y_\alpha = -\frac{9}{7} C_b \] \hspace{1cm} (118)

To prove these we first write down these consequences of the completeness relation (83)

\[ H_i H_k H_i + C_e H_k C_e = H_k \] \hspace{1cm} (119)
\[ H_i C_a H_i + C_e C_a C_e = -C_a \] \hspace{1cm} (120)
\[ H_i Y_\alpha H_i + C_e Y_\alpha C_e = -Y_\alpha \] \hspace{1cm} (121)

Also the total antisymmetry of the \( c_{abc} \) allows us to make these rearrangements

\[ (C_e H_k C_e)_{cd} = \text{tr} (C_d C_c H_k) = (H_k)_{cd} \]
\[ (C_e C_a C_e)_{cd} = \text{tr} (C_d C_a C_c) = -(C_a)_{cd} \]
\[ (C_e Y_\alpha C_e)_{cd} = \text{tr} (C_d C_c Y_\alpha) = -\frac{1}{7} (Y_\alpha)_{cd} \] \hspace{1cm} (122)

Hence all the first six results listed follow. The last three are easy consequences of (84).

5.2 Completion of the proofs of bilinear identities

To prove (91), we set out from

\[ d_{ij\alpha} d_{ij\beta} = \text{tr} (H_i H_j Y_\alpha) \text{tr} (H_i H_j Y_\beta) \] \hspace{1cm} (123)

and use completeness relation (84), followed by the results (106) and (110). This gives (91). Eqs. (117) and (118) follow similarly, and indeed (115), and hence (114) can be confirmed.

Turning finally to (88), we use the completeness relation (83) to derive

\[ c_{ijk} c_{ijl} = -\text{tr} (H_i H_j H_k) \text{tr} (H_i H_j H_l) = -\text{tr} (H_j H_k H_l H_i + H_l H_k H_j H_i) \] \hspace{1cm} (124)

Here the CC term of (83) has not given any contribution because \( \text{tr} (C_e H_j H_k) = 0 \), a result which reflects the closure properties of the \( g_2 \) matrices \( H_i \). The results (106) and (110) now lead to (88).

6 Casimir operators, projectors and characteristic equations

6.1 The quadratic Casimir operator

The irreducible representation of \( g_2 \) of highest weight \((\lambda, \mu)\) is well known, e.g. [4], to have dimension given by

\[ 120 \dim (\lambda, \mu) = (\lambda + 1)(\mu + 1)(\lambda + \mu + 2)(2\lambda + \mu + 3)(3\lambda + \mu + 4)(3\lambda + 2\mu + 5) \] \hspace{1cm} (125)
If $X_i \mapsto D_i$, where these the matrices of $(\lambda, \mu)$, then the quadratic Casimir operator
\[ C^{(2)} = X_i X_i \mapsto D_i D_i, \] (126)
of $g_2$ has the eigenvalue
\[ c^{(2)}(\lambda, \mu) = \lambda^2 + \frac{1}{3} \mu^2 + \lambda \mu + 3\lambda + \frac{2}{3} \mu, \] (127)
for $(\lambda, \mu)$. To within an overall normalisation constant fixed by reference to the defining representation of $g_2$, this agrees with the result of [19]. See also [20], especially the striking and amusing section III.

In our work so far the representations
\[(0, 1) = 7, \quad X_i \mapsto x_i \text{ or } X_i \mapsto H_i \]
\[(1, 0) = 14 = \text{ad}, \quad X_i \mapsto (a_{di}) \text{, } (a_{di})_{jk} = -i \epsilon_{ijk} \]
\[(0, 2) = 27, \quad X_i \mapsto \Phi_i, \text{ } (\Phi_i)_{\alpha\beta} = -i \phi_{\alpha\beta} \]
(128)
have occurred. For these (127) requires the eigenvalues $4$, $8$ and $\frac{28}{3}$ respectively, and we can see that (105) – (108) are in agreement with this.

6.2 Projectors

We consider first the Clebsch-Gordan series
\[ 7 \otimes 7 \equiv (1 + 27) + (7 + 14) \]
\[ (0, 1) \otimes (0, 1) \equiv \frac{1}{\sqrt{2}} (0, 0) + (0, 2) + (0, 1) + (1, 0). \] (129)

Suppose $A_a, B_a, \quad a \in \{1, \cdots, 7\}$ transform according to the defining representation $\gamma = 7 = (0, 1)$ of $g_2$. The tensors which transform according to the symmetric part of $7 \otimes 7$ are given by
\[ T^{(1)}_{ab} = \frac{1}{7} \delta_{ab} A_c B_c \]
\[ = \frac{1}{4} \delta_{ab} \delta_{cd} A_c B_d \] (130)
\[ T^{(27)}_{ab} = \frac{1}{2} (A_a B_b + A_b B_a) - \frac{1}{4} \delta_{ab} A_c B_c \]
\[ = \frac{1}{2} (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) - \frac{1}{4} \delta_{ab} \delta_{cd} A_c B_d. \] (131)

Similarly for the antisymmetric part, we have
\[ T^{(7)}_{ab} = \frac{1}{2} c_{abc} c_{de} A_c B_d \] (132)
\[ T^{(14)}_{ab} = \frac{1}{2} (A_a B_b - A_b B_a) - \frac{1}{2} c_{abe} c_{cde} A_c B_d \] (133)
\[ = \frac{1}{2} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) - \frac{1}{2} c_{abc} c_{cde} A_c B_d. \] (134)

Now
\[ T^{(R)}_{ab} = P^{(R)}_{ab, cd} A_c B_d, \] (135)
defines a set of orthogonal projectors onto the representations of $g_2$ contained in the reduction $[129]$ of $7 \otimes 7$. 

18
The above used the result (138). Also we see that (84) implies
\[(Y_\alpha \otimes Y_\alpha)_{ac, bd} = (Y_\alpha)_{ab}(Y_\alpha)_{cd} = 2P^{(27)}_{ab, cd} \quad .\] (136)

Also
\[(H_i)_{ab}(H_i)_{cd} + (C_e)_{ab}(C_e)_{cd} = 2P^{(A)}_{ab, cd} = (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \quad ,\] (137)
where \(P^{(A)} = P^{(7)} + P^{(14)}\) is the projector onto the antisymmetric tensor subspace. In fact, more explicitly
\[(C_e)_{ab}(C_e)_{cd} = -c_{eab}c_{ecd} = -2P^{(7)}_{ab, cd} \quad (138)\]
\[(H_i)_{ab}(H_i)_{cd} = (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) - c_{eab}c_{ecd} = 2P^{(14)}_{ab, cd} \quad .\] (139)

We consider next the Clebsch-Gordan series
\[14 \otimes 14 \equiv (1 + 27 + 77) + (14 + 77^\prime)\]
\[= (1, 0) + (0, 2) + (2, 0) + (0, 0) + (2, 2) + (0, 3) \quad .\] (140)

Suppose \(A_i, B_i, \quad i \in \{1, \ldots, 14\}\) transform according to the adjoint representation \(ad = 14 = (1, 0)\) of \(g_2\). The tensors which transform according to the symmetric part of \(14 \otimes 14\) are given by
\[T^{(1)}_{ij} = \frac{1}{14}\delta_{ij}\delta_{kl}A_k B_l \quad (141)\]
\[T^{(27)}_{ij} = \frac{3}{27}d_{ij0}d_{klo}A_k B_l \quad (142)\]
\[T^{(77)}_{ij} = \left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{14}\delta_{ij}\delta_{kl} - \frac{9}{27}d_{ij0}d_{klo}\right]A_k B_l \quad .\] (143)

The result (141) has been used here. Similarly, with the aid of (88), we get, for the antisymmetric part,
\[T^{(14)}_{ij} = \frac{1}{8}c_{ijp}c_{klp}A_k B_l \quad (144)\]
\[T^{(77^\prime)}_{ij} = \left[\frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \frac{1}{8}c_{ijp}c_{klp}\right]A_k B_l \quad .\] (145)

Now
\[T^{(R)}_{ij} = P^{(R)}_{ij, kl}A_k B_l \quad ,\] (146)
defines a set of orthogonal projectors onto the representations of \(g_2\) contained in the reduction (140) of \(14 \otimes 14\).

One aspect of (140) is worth clarifying. How did we assign \(77 = (2, 0)\) to the symmetric part of \(14 \times 14\), and \(77^\prime = (0, 3)\) to the antisymmetric part? One way, which follows a general argument given on P3196 of [15], is indicated at the end of Sec. 6.3.

There is a very interesting discussion of projectors for \(g_2\) in [16].

### 6.3 Characteristic equations

As indicated a long time ago for \(su(n)\) [14], and emphasised in [15], there are two classes of basic identities for the invariant tensors of any Lie algebra – first class identities that stem directly from Jacobi identities, and those of the second class. The latter depend on the use of the characteristic equation of the Lie algebra usually for the defining representation \(\mathcal{V}\). The
difficulty of finding them increases with the dimension of the Lie algebra or of $V$. \cite{15} presents a systematic account with lots of results including many for $g_2$. It is useful to present a slightly modified version of the latter discussion. Some of the intermediate equations of the discussion may here appear slightly different from their counterparts in \cite{15}, because normalisations used there were fixed in a way that allowed uniform programming for all Lie algebras, whereas here we do what seems most convenient for $g_2$ itself.

Let $ad_1$ and $ad_2$ denote adjoint representations of $g_2$ acting in vector spaces $V_1$ and $V_2$, of dimension 14. The Clebsch-Gordan series for $14 \otimes 14$ is displayed as (140). Let $X_i = X_{i1} + X_{i2}$ denote the ‘total’ $g_2$ generators, acting in $V_1 \otimes V_2$. Then

$$X_i X_i = X_{i1} X_{i1} + X_{i2} X_{i2} + 2 \Lambda \quad , \quad \Lambda = X_{i1} X_{i2} \quad ,$$

implies that $\Lambda$ has the same eigenspaces as does $X_i X_i$, and eigenvalues $-8, -\frac{10}{3}, 2, -4, 0$ for the representations $1, 27, 77, 14, 77'$ respectively. To use this information, we begin by recalling a well-known formula

If a hermitian operator $A$ has eigenvalues and eigenvectors given by

$$A|a_i\rangle = a_i|a_i\rangle \quad , \quad a \in \{1, \ldots, p\} \quad ,$$

then the projector onto its $i$-th eigenspace is

$$P_i = \prod_{k \neq i} \frac{A - a_k I}{a_i - a_k} \quad ,$$

where $I$ is the unit operator in the vector space spanned by the $|a_i\rangle$. Clearly

$$(A - a_i I) P_i = 0 \quad ,$$

with no sum on $i$ implied, is, for each $i$, a possibly reduced, version of the characteristic equation sought. Now we know that $\Lambda$ is a linear combination of the projectors onto the eigenspaces of $X_i X_i$. Further $\Lambda I_S$, where $(I_S)_{ij,kl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ is the unit of the symmetric subspace, is a linear combination of the symmetric projectors for the representations $1, 27, 77$. For these, (148) implies

$$0 = (\Lambda + 8) I_S P^{(1)} \quad ,$$

$$0 = (\Lambda + \frac{10}{3}) I_S P^{(27)} \quad ,$$

$$0 = (\Lambda - 2) I_S P^{(77)} \quad .$$

We could well have left out $I_S$ from these equations. Since

$$P^{(1)} + P^{(27)} + P^{(77)} = I_S \quad ,$$

the last four equations lead easily to

$$0 = \Lambda I_S + 8 P^{(1)} + \frac{10}{3} P^{(27)} - 2 P^{(77)} \quad ,$$

$$0 = \Lambda I_S + 10 P^{(1)} + \frac{16}{3} P^{(27)} - 2 I_S \quad .$$

Since

$$(\Lambda)_{rs,ij} = -c_{pri}c_{psj} \quad ,$$

20
and since we get the explicit expressions for the projectors from (141) and (142), we are
directly lead to the important result

\[ 3 d_{rso} d_{kla} = c_{prk} c_{psl} + c_{prl} c_{psk} + 2(\delta_{rk}\delta_{sl} + \delta_{rl}\delta_{sk}) - \frac{10}{3}\delta_{rs}\delta_{kl} \]  
(157)

This is a second class result because its derivation employs a characteristic equation. It is
also an analogue of eq. (2.23) of [14], and is found in [15] as eq. (4.32b) of that paper. We
note some immediate consequences. With the aid of the ordinary Jacobi identity, applied
twice, once to treat each of the first two terms of the right side of (157), we deduce a further
important result

\[ d_{(ij)} d_{k)l} = \frac{6}{7}\delta_{(ij}\delta_{k)l} \]  
(158)

Here the round brackets indicate symmetrisation of unit weight over the indices enclosed;
sometimes indices are raised in a manner with no metric significance in order to exempt them
from symmetrisation. The same comment will later apply to antisymmetrisation and square
brackets. Also, we may apply \( d_{is\beta} \) to (157) and deduce the identity

\[ c_{prt} c_{psj} d_{ij\alpha} = \frac{10}{3} d_{rs\alpha} \]  
(159)

Because it leads us to important results, we consider the what the above analysis can tell
us if we wish to avoid any mention of the tensors \( d_{ij\alpha} \). As in [15], we apply \( \Lambda \) to (155), and
then use this equation again to eliminate \( P(27) \). Hence we find

\[ 0 = \Lambda^2 I_S + \frac{4}{3}\Lambda I_S - \frac{140}{3} P^{(1)} - \frac{20}{3} I_S \]  
(160)

It is convenient to add to this the equation

\[ \Lambda^2 I_A + 4\Lambda I_A = 0 \]  
(161)

To establish (161), we make use of

\[ (\Lambda + 4) I_A P^{(14)} = 0 \]  
(162)

which follows from (150), and

\[ \Lambda I_A + 4P^{(14)} = 0 \]  
(163)

which coincides with the Jacobi identity of \( g_2 \). This leads us to (161) and to the desired
formula

\[ 0 = \Lambda^2 + \frac{4}{3}\Lambda I_S + 4\Lambda I_A - \frac{140}{3} P^{(1)} - \frac{20}{3} I_S \]  
(164)

and hence to

\[ 3c_{jmr} c_{knr} c_{mps} c_{nqs} = 10(\delta_{jp}\delta_{kq} + \delta_{jq}\delta_{kp} + \delta_{jk}\delta_{pq}) + 8c_{jpr} c_{kqr} - 4c_{jqr} c_{kpr} \]  
(165)

This can be written out as a formula for \( \text{tr} ad_j ad_k ad_p ad_q \), and implies

\[ \text{tr} ad_j ad_k ad_p ad_q = 10\delta_{(jk}\delta_{pq)} \]  
(166)

As a further consequence of (161), we note \( \Lambda I_A \) has eigenvalues \(-4\) and \(0\), corresponding
to the representations \( 14 = (1, 0) \) and \( 77' = (0, 3) \), thereby confirming that \((0, 3)\) rather than
\( 77 = (2, 0) \) is a constituent of the antisymmetric part of \( 14 \otimes 14 \).
6.4 On the non-primitive quartic Casimir operator

First we quote the evaluation of a trace

\[ \text{tr} \, x_{ij} x_{k} x_{l} \equiv \text{tr} \, H_{ij} H_{k} H_{l} = \delta_{ij} \delta_{kl} \]  

(167)

Proof: A routine calculation based on two uses of (41) and evaluation of elementary traces yields the answer

\[ \frac{4}{7} \delta_{ij} \delta_{kl} + \frac{1}{2} d_{ij \alpha} d_{k \alpha \lambda} \]  

(168)

and then (158) is used.

Second we define \( A = A_{i} x_{i}, A_{i} \in \mathbb{R} \), and show that

\[ \text{tr} \, A^{4} = 4 \left( \frac{1}{2} \text{tr} \, A^{2} \right)^{2} \]  

(169)

This trivial consequence of (167) indicates the absence (see [19] for an early proof) of a primitive quartic Casimir operator for \( g_{2} \). It reflects the well-known fact that there are just two primitive Casimir operators of \( g_{2} \), which have orders 2 and 6.

Third, we shall prove a stronger result regarding the non-primitive nature of the quartic Casimir of \( g_{2} \). We introduce \( X = x_{i} X_{i} \) where the \( x_{i} \) are the matrices of the \( 7 \times 7 \) defining representation, and the \( X_{i} \) are the abstract generators. Then define the quartic Casimir operator of \( g_{2} \) by means of

\[ C^{(4)} = \text{tr} \, X^{4} = \text{tr} \, (x_{i} x_{j} x_{k} x_{l}) X_{i} X_{j} X_{k} X_{l} \]  

(170)

To evaluate this correctly requires taking full account of the Lie algebra relations

\[ [X_{i}, X_{j}] = ic_{ijk} X_{k} \]  

(171)

It is routine to obtain

\[ \text{tr} \, (x_{i} x_{j} x_{k} x_{l}) = \text{tr} \, (x_{i} x_{j} x_{k} x_{l}) - \frac{1}{3} (c_{klt} c_{ijl} - c_{ikt} c_{jkl}) \]  

(172)

Using this result and (167) enables the contribution from the first term of (172) to (170) to be evaluated. The contribution from the other terms depends only on the use of (171), and of the identity (88). One is lead then to the answer

\[ C^{(4)} = (C^{(2)})^{2} + \frac{28}{3} C^{(2)} \]  

(173)

Again this result agrees with that given in [19]. Okubo did not give details of his proof, indicating just that it was ‘involved’. The present proof depends on the introduction of a full range of invariant tensors and on gaining full control of their properties.

7 Trilinear tensor identities

We first remark that the product laws (41) – (46) give rise easily to one family of ‘three-tensor’ identities. For example

\[ c_{ijk} = -i \text{tr} \, x_{i} x_{j} x_{k} = -i \text{tr} \, H_{i} H_{j} H_{k} = h_{iab} h_{jbc} h_{kca} \]  

(174)
upon use also of (72) and 60). Such expressions also, as noted, account for the appearance in later product laws of tensors defined in earlier ones. We quote two more results from this family

\[ c_{abc} = c_{eaf} c_{bgc} g_{ce} \]  
(175)

\[ d_{\alpha\beta\gamma} = -27 d_{abc} d_{bc\beta} d_{ca\gamma} \]  
(176)

Eq. (175) coincides with (114).

Next we give a listing of three tensor identities that do not arise in the same way as those just noted.

\[ c_{piq} c_{qjr} c_{rkp} = -4 c_{ijk} \]  
(177)

\[ d_{jka} d_{lia} c_{jq} = \frac{20}{7} c_{kiq} \]  
(178)

\[ c_{pri} c_{psj} d_{ijo} = \frac{10}{7} d_{rsa} \]  
(179)

\[ d_{i\alpha\beta} d_{\alpha\beta\gamma} = \frac{22}{21} d_{ik\gamma} \]  
(180)

\[ d_{pqa} d_{pib} d_{qj} = -\frac{58}{63} d_{ija} \]  
(181)

\[ d_{\lambda\mu\alpha} d_{\mu\beta\alpha} d_{\nu\lambda\gamma} = \frac{53}{63} d_{\alpha\beta\gamma} \]  
(182)

We turn to the proof of these results. All have been confirmed using MAPLE. By a method that applies to any Lie algebra, we see that (177) follows directly from the Jacobi identity for \( g_2 \), upon use of (88). To prove (178), we reduce

\[ [(x_i, x_j), x_k] = \{\{x_j, x_k\}, x_i\} - \{\{x_k, x_i\}, x_j\} \]  
(183)

and reach an identity of the first class

\[ c_{ijp} c_{klp} = \frac{8}{7} (\delta_{ik} \delta_{jl} - \delta_{jk} \delta_{il}) + (d_{i\alpha\beta} d_{j\alpha\gamma} - d_{j\alpha\beta} d_{i\alpha\gamma}) \]  
(184)

Now apply \( c_{jlq} \) to (184). The result simplifies, with the aid of (177) and (88) to yield (178).

We have obtained (179) already as (159). Next we apply \( d_{ik\beta} \) to (184) getting

\[ c_{pri} c_{psj} d_{ij\beta} + d_{pq\beta} d_{pr\alpha} d_{qsa} = \frac{152}{63} d_{rs\beta} \]  
(185)

Now (173) implies (181).

The results (180) and (182) are easy to prove because the simple completeness relation (84) can be used. For example

\[ d_{ij\alpha} d_{jk\beta} d_{\alpha\beta\gamma} = (H_i H_j)_{ha} (H_j H_k)_{dc} (Y_{\alpha})_{ab} (Y_{\beta})_{cd} (Y_{\gamma})_{ef} (Y_{\alpha})_{cf} (Y_{\beta})_{fg} (Y_{\gamma})_{ge} \]  
(186)

Two applications of (84) now allow the proof of (180) to be completed. The results (63), (106), (88), (84) and \( d_{ij\alpha} = \text{tr} H_i H_j Y_{\alpha} \) are all used in the process.

Finally we note the absence of any simple result like those just proved for the quantity

\[ d_{ij\alpha} d_{jk\beta} d_{ki\gamma} \]  
(187)

It would be wrong to suppose this is a constant multiple of \( d_{i\alpha\beta} \). No such result exists, and MAPLE rejects such a conjecture. The vector space of third rank tensors totally symmetric in \( \alpha, \beta, \gamma \) has been analysed fully but we have no good reason to record the details.
8 Adjoint vectors and invariants

8.1 Results

Let the vector \( A_i, \ i \in \{1, \cdots, 14\} \) transform under the action of \( g_2 \) according to its adjoint representation. Then the vector

\[
d_{ij}^{\alpha} d_{k\alpha} A_j A_k A_i = \frac{6}{7} (A_p A_p) A_i ,
\]

is seen not to be linearly independent of \( A_i \). Eq. (188) holds because we can freely put round symmetrisation brackets round the set of indices \( j, k, l \), and employ the identity (155). The result is tantamount to the fact that the quartic Casimir of \( g_2 \) is not primitive. It suggests [19] that there should be a second adjoint vector, whose components are quintic in those of \( A_i \), since we know a primitive sixth order invariant exists. In fact, the required vector \( C_i \) is given by

\[
C_i = d_{\alpha\beta\gamma} d_{ij}^{\alpha} d_{kl}^{\beta} d_{pq}^{\gamma} A_j A_k A_l A_p A_q .
\]

Here we can use the \( A_r \) factors to justify putting round brackets around the index set \( jklpq \), and then see that \( i \) can also be accommodated correctly within them because of the symmetry of the tensorial factors. Thus we write

\[
C_i = T_{ijklpq} A_j A_k A_l A_p A_q ,
\]

where the totally symmetric sixth rank tensor is given by

\[
T_{ijklpq} = d_{\alpha\beta\gamma} d_{ij}^{\alpha} d_{kl}^{\beta} d_{pq}^{\gamma} .
\]

Turning next to the construction of invariants or scalars out of the components of \( A_i \), we see that there are quadratic and sixth order invariants

\[
A_i A_i , A_i C_i ,
\]

as expected. However there are others. One is \( C_i C_i \). Others use the 27-component quantities

\[
B_\alpha = d_{ij\alpha} A_i A_j , \quad D_\alpha = d_{\alpha\beta\gamma} B_\beta B_\gamma ,
\]

to build other scalars

\[
B_\alpha B_\alpha , \quad B_\alpha D_\alpha , \quad D_\alpha D_\alpha .
\]

None of the additional scalars are primitive. Indeed easily we can see

\[
B_\alpha D_\alpha = A_i C_i , \quad B_\alpha B_\alpha = \frac{6}{7} A_i A_j A_j ,
\]

leaving \( C_i C_i \) and \( D_\alpha D_\alpha \) to be given explicitly in terms of primitive invariants.

Also, the tensor \( T_{ijklpq} \), while totally symmetric, is not traceless. For the optimal construction a sixth order scalar, we should construct [29] [31] [32] the traceless totally symmetric tensor

\[
S_{ijklpq} = T_{ijklpq} - \frac{88}{441} \delta_{(ij} \delta_{kl} \delta_{pq)} .
\]

such that

\[
S_{ijklpq} = 0 .
\]
To establish (197), we need to open out the round brackets of the definition (191). Upon putting \( j = i \), some terms vanish because of \( d_{ii\delta} = 0 \). In fact this applies to 3 out of 15 distinct terms. The remainder are all equivalent, and such that (180) can be applied. This means that

\[
T_{ijklpq} = \frac{4}{5} \frac{22}{21} \frac{6}{7} \delta_{(kj\delta pq)} \quad (198)
\]

in which, at the very end, (158) has been used. Since putting \( j = i \) in the \( \delta \delta \delta \) term gives the same result, the proof is done.

### 8.2 On the sixth order invariant and non-primitive invariants

To get some measure of control of the sixth order Casimir operator, and to treat explicitly the non-primitivity of scalars such as \( C_iC_i \) and \( D_\alpha D_\alpha \) of orders 10 and 8, we use a basis in which \( A_i \) has the components \( (a, b, 0, \cdots, 0) \) so that \( A = A_i x_i \) is diagonal

\[
A = \text{diag} \left( \sqrt{\frac{2}{3}} a, \sqrt{\frac{2}{3}} a + \sqrt{\frac{1}{3}} b, \sqrt{\frac{2}{3}} a - \sqrt{\frac{1}{3}} b, 0, -\sqrt{\frac{1}{3}} a + \sqrt{\frac{1}{3}} b, -\sqrt{\frac{1}{3}} a - \sqrt{\frac{1}{3}} b, -\sqrt{\frac{2}{3}} a \right) \quad (199)
\]

From this we find

\[
C^{(2)} = A_i A_i = \frac{1}{2} \text{tr} A^2 = (a^2 + b^2) \quad (200)
\]

It is easy to evaluate explicitly the few tensor components needed to show that

\[
B_\alpha = (\sqrt{\frac{2}{3}} (a^2 - b^2), \sqrt{\frac{2}{3}} 2ab, \sqrt{\frac{2}{7}} (a^2 + b^2), 0, \cdots, 0) \quad (201)
\]

\[
D_\alpha = (\sqrt{\frac{2}{3}} \frac{22}{21} a^4 - 4a^2 b^2 + \frac{2}{7} b^4, \sqrt{\frac{2}{3}} (-\frac{40}{21} a^3 b + \frac{24}{7} ab^3), -\frac{4}{7} (\sqrt{\frac{2}{21}} (a^2 + b^2)^2, 0, \cdots, 0),
\]

where the components \( \alpha = 1, 2, 3 \) correspond to the ordering of (37) and (39). This enables us to verify easily the second result in (195), and to use the first one to deduce

\[
C^{(6)} = A_i C_i = \frac{88}{63} C^{(2)} + \frac{4}{9} (a^2 - b^2)(a^4 - 14a^2 b^2 + b^4) \quad (202)
\]

The fraction in the first term is the same one as we found by independent calculations in (196). Thus if we define the optimal sixth order invariant \( \tilde{C}^{(6)} \) that can be built using six copies of the vector \( A_i \) as

\[
\tilde{C}^{(6)} = S_{ijklpq} A_i A_j A_k A_l A_p A_q \quad (203)
\]

then its value in terms of \( a, b \) is

\[
\tilde{C}^{(6)} = \frac{4}{9} (a^2 - b^2)(a^2 - 4ab + b^2)(a^2 + 4ab + b^2) \quad (204)
\]

The data accumulated in this section enables us to show

\[
D_\alpha D_\alpha = \frac{16}{21} C^{(2)} C^{(6)} + \frac{88}{343} C^{(2)}^4 \quad (205)
\]

To treat \( C_i C_i \), we note that

\[
C_i = d_{ij\alpha} a_j D_\alpha \quad (206)
\]

All the information is therefore at hand to allow us to obtain the result

\[
C_i C_i = \frac{16}{147} C^{(2)}^2 \left( \frac{11}{3} C^{(6)} + \frac{71}{39} C^{(2)} \right) \quad (207)
\]

it being easy to see how the factor \( C^{(2)}^2 \) arises. MAPLE confirms the above results.
8.3 Further use of characteristic equations

Setting \( A = A_i x_i \), we use the (easily programmable) methods of Sec. 8.2 to find
\[
\text{tr} A^4 = \frac{1}{4} (\text{tr} A^2)^2 ,
\]
the characteristic polynomial of \( A \)
\[
\chi_A(t) = t^7 - \frac{1}{2} (\text{tr} A^2) t^5 + \frac{1}{16} (\text{tr} A^2)^2 t^3 + \left( \frac{1}{96} (\text{tr} A^2)^3 - \frac{1}{6} (\text{tr} A^6) \right) t ,
\]
and hence
\[
\begin{align*}
\text{tr} A^8 &= -\frac{5}{192} (\text{tr} A^2)^4 + \frac{2}{3} (\text{tr} A^2) (\text{tr} A^6) , \\
\text{tr} A^{10} &= -\frac{1}{64} (\text{tr} A^2)^5 + \frac{3}{16} (\text{tr} A^2)^2 (\text{tr} A^6) ,
\end{align*}
\]
which agrees with results established in [33] using a different method. From these results we can deduce the trace formulas
\[
\begin{align*}
\text{tr} (x_{(i_1} x_{i_2} \cdots x_{i_8)}) &= \delta_{(i_1i_2} \delta_{i_3i_4)} , \\
\text{tr} (x_{(i_1} x_{i_2} \cdots x_{i_8)}) &= -\frac{5}{16} \delta_{(i_1i_2} \cdots \delta_{i_7i_8)} + \frac{4}{3} \delta_{i_1i_2} \text{tr} (x_{i_3} x_{i_4} \cdots x_{i_8}) , \\
\text{tr} (x_{(i_1} x_{i_2} \cdots x_{i_{10}}) &= -\frac{1}{2} \delta_{(i_1i_2} \cdots \delta_{i_9i_{10})} + \frac{5}{4} \delta_{i_1i_2} \delta_{i_3i_4} \text{tr} (x_{i_5} x_{i_6} \cdots x_{i_{10})} ,
\end{align*}
\]
given in terms of the two independent primitive traces
\[
\text{tr} (x_{i_1} x_{i_2}) = 2 \delta_{i_1i_2} , \quad \text{tr} (x_{i_1} x_{i_2} \cdots x_{i_6}) .
\]

We can relate the latter trace to the tensors used in Sec. 8.1 to define the sixth order Casimir operator of \( g_2 \). With the help of (41) and (191), we find
\[
\text{tr} (x_{i_1} x_{i_2} \cdots x_{i_{16}}) = \text{tr} (x_{(i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6)}) = \frac{260}{192} \delta_{(i_1i_2} \delta_{i_3i_4} \delta_{i_5i_6)} + \frac{1}{8} T_{i_1 \cdots i_6} .
\]

Turning next to the characteristic equation of the adjoint matrix \( B = A_i ad_i \), where \((ad_i)_{jk} = -ic_{ijk}\), we quote from [15] the results
\[
\begin{align*}
\text{tr} B^2 &= 4 (\text{tr} A^2) = 8 A_i A_i , \\
\text{tr} B^4 &= \frac{5}{2} (\text{tr} A^2)^2 , \\
\text{tr} B^6 &= \frac{15}{4} (\text{tr} A^2)^3 - 26 (\text{tr} A^6) , \\
\text{tr} B^8 &= \frac{515}{96} (\text{tr} A^2)^4 - \frac{160}{3} (\text{tr} A^2) (\text{tr} A^6) , \\
\text{tr} B^{10} &= \frac{431}{64} (\text{tr} A^2)^5 - \frac{905}{8} (\text{tr} A^2)^2 (\text{tr} A^6) ,
\end{align*}
\]
noting that the characteristic equation, [15] eq. (A44), allows higher traces to be computed. Eq. (218) is equivalent to (166). From (218), we get
\[
\text{tr} (ad_{(i_1} \cdots ad_{i_6)}) = \frac{794}{49} \delta_{(i_1i_2} \delta_{i_3i_4} \delta_{i_5i_6)} - \frac{13}{4} T_{i_1 \cdots i_6} ,
\]
and so on.
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