On a function introduced by Erdős and Nicolas

José Manuel Rodríguez Caballero
Département de Mathématiques
UQÀM
Case Postale 8888, Succ. Centre-ville
Montréal, Québec H3C 3P8 Canada
rodriguez_caballero.jose_manuel@uqam.ca

Abstract

Erdős and Nicolas [1] introduced an arithmetical function $F(n)$ related to divisors of $n$ in short intervals $]\frac{t}{2}, t]$. The aim of this note is to prove that $F(n)$ is the largest coefficient of polynomial $P_n(q)$ introduced by Kassel and Reutenauer [2]. We deduce that $P_n(q)$ has a coefficient larger than 1 if and only if $2n$ is the perimeter of a Pythagorean triangle. We improve a result due to Vatne [6] concerning the coefficients of $P_n(q)$.

1 Introduction

Erdős and Nicolas introduced in [1] the function

$$F(n) = \max \{ q_t(n) : \ t \in \mathbb{R}^*_+ \},$$

(1)

where $q_t(n) = \# \{ d : \ d|n \ \text{and} \ \frac{1}{2} t < d \leq t \}$, and they proved that

$$\lim_{x \to +\infty} \frac{1}{x} \sum_{n \leq x} F(n) = +\infty.$$ 

(2)

Kassel and Reutenauer introduced in [2] a $q$-analog of the sum of divisors, denoted $P_n(q)$, by means of the generating function

$$\prod_{m \geq 1} \frac{(1 - t^m)^2}{(1 - q t^m)(1 - q^{-1} t^m)} = 1 + (q + q^{-1} - 2) \sum_{n=1}^{\infty} \frac{P_n(q)}{q^{n-1}} t^n$$

(3)

and they proved that, for $q = \exp \left( \frac{2\pi k}{\sqrt{-1}} \right)$, with $k \in \{2, 3, 4, 6\}$, this infinite product can be expressed by means of the Dedekind $\eta$-function (see [4]). A consequence of this coincidence is that the corresponding arithmetic functions $n \mapsto P_n(q)$, for each of the above-mentioned
values of \( q \), are related to the number of ways to express a given integer by means of a quadratic form (see [2] and [3]).

The aim of this paper is to prove the following theorem.

**Theorem 1.** For each integer \( n \geq 1 \), the largest coefficient of \( P_n(q) \) is \( F(n) \).

Using this result, we will derive that \( P_n(q) \) has a coefficient larger than 1 if and only if \( 2n \) is the perimeter of a Pythagorean triangle. Also, we will prove that each nonnegative integer \( m \) is the coefficient of \( P_n(q) \) for infinitely many positive integers \( n \).

## 2 Proof of the main result

In order to simplify the notation in the proofs, we will consider two functions\(^1\) \( f : \mathbb{R} \rightarrow \mathbb{R}^*_+ \) and \( g : \mathbb{R}^*_+ \rightarrow \mathbb{R} \), defined by

\[
\begin{align*}
f(x) &= \frac{1}{2} \left( x + \sqrt{8n + x^2} \right), \\
g(y) &= y - \frac{2n}{y}.
\end{align*}
\]

**Lemma 2.** The functions \( f(x) \) and \( g(y) \) are well-defined, strictly increasing and mutually inverse. Furthermore, \( g(y) \) satisfies the identity

\[
g(y) = -g \left( \frac{2n}{y} \right). \tag{6}
\]

**Proof.** It follows in a straightforward way from the explicit expressions (4) and (5) that \( f(x) \) and \( g(y) \) are well-defined and strictly increasing. In particular, the inequality \( |x| < \sqrt{2n + x^2} \) guarantees that \( f(x) \in \mathbb{R}^*_+ \) for all \( x \in \mathbb{R} \).

On the one hand, for all \( x \in \mathbb{R} \), we have

\[
g(f(x)) = \frac{(f(x) - x - \sqrt{2n + x^2}) (f(x) - x + \sqrt{2n + x^2})}{2 f(x)} + x = x.
\]

On the other hand, for all \( y \in \mathbb{R}^*_+ \), we have

\[
f(g(y)) = \frac{y}{2} - \frac{n}{y} + \sqrt{\left( \frac{y}{2} + \frac{n}{y} \right)^2} = \frac{y}{2} - \frac{n}{y} + \frac{y}{2} + \frac{n}{y} = y,
\]

where we used the inequality \( \frac{y}{2} + \frac{n}{y} > 0 \), provided that \( y > 0 \), for the elimination of the square root.

Hence, \( f(x) \) and \( g(y) \) are mutually inverses. Furthermore, using the identity

\[
-g \left( \frac{2n}{y} \right) = - \left( \frac{2n}{y} - \frac{n}{2} \right) = - \left( \frac{n}{y} - \frac{y}{2} \right) = \frac{y}{2} - \frac{n}{y} = g(y),
\]

we conclude that (6) holds for all \( y \in \mathbb{R}^*_+ \). \(\square\)

---

\(^1\)The function \( g(y) \) was implicitly used in Proposition 2.2. in [4].

2
Lemma 3. For each integer $n \geq 1$,
\[
\frac{P_n(q)}{q^{n-1}} = \sum_{i \in \mathbb{Z}} a_{n,i} q^i,
\]  
where
\[
a_{n,i} = \# \left\{ d : d | n \quad \text{and} \quad \frac{1}{2} g(d) \leq i < \frac{1}{2} g(2d) \right\}.
\]

Proof. By Theorem 1.2 in [3],
\[
P_n(q) = a_{n,0} q^{n-1} + \sum_{i=1}^{n-1} a_{n,i} (q^{n-1+i} + q^{n-1-i}),
\]
where
\[
a_{n,i} = \# \left\{ d : d | n \quad \text{and} \quad \frac{f(2i)}{2} < d \leq f(2i) \right\}.
\]

The condition $\frac{f(2i)}{2} < d \leq f(2i)$ is equivalent to $d \leq f(2i) < 2d$. So, since $g(y)$ is strictly increasing by Lemma 2, the expression (8) follows for all $0 \leq i \leq n - 1$.

We will extend $a_{n,i}$ to any $i \in \mathbb{Z}$ using the expression (8) as the definition of $a_{n,i}$ for $i < 0$. Applying the identity (6) to (8),
\[
a_{n,i} = \# \left\{ d : d | n \quad \text{and} \quad \frac{1}{2} g(d) < -i \leq \frac{1}{2} g(2d) \right\}.
\]

Substituting $i$ by $-i$ in (8),
\[
a_{n,-i} = \# \left\{ d : d | n \quad \text{and} \quad \frac{1}{2} g(d) \leq -i < \frac{1}{2} g(2d) \right\}.
\]

Now, we will prove that
\[
\# \left\{ d : d | n \quad \text{and} \quad \frac{1}{2} g(d) < -i \leq \frac{1}{2} g(2d) \right\}
= \# \left\{ d : d | n \quad \text{and} \quad \frac{1}{2} g(d) \leq -i < \frac{1}{2} g(2d) \right\}.
\]
Suppose that
\[
\frac{1}{2} g(d) = -i,
\]
for some $d | n$. Transforming (14) into $d = 2 \left( \frac{n}{d} - i \right)$, it follows that $d$ is even. So, $-i = \frac{1}{2} g(2d')$, where $d' = \frac{d}{2}$ is a divisor of $n$.

Conversely, suppose that
\[
-i = \frac{1}{2} g(2d),
\]
for some $d | n$. Transforming (15) into $\frac{d}{n} = 2 (d + i)$, it follows that $\frac{d}{n}$ is even. Furthermore, $2d$ divides $n$, because $2d \frac{d}{2} = n$ and $\frac{d}{2} \in \mathbb{Z}$. So, $\frac{1}{2} g (d') = -i$, where $d' = 2d$ is a divisor of $n$. Hence, (14) holds.

Combining (14), (11) and (12), we obtain that

$$a_{n,i} = a_{n,-i}$$

holds for all $0 \leq i \leq n - 1$.

Furthermore, the bound $-(2n - 1) \leq g(y) \leq 2n - 1$ for all $1 \leq y \leq 2n$ and the equality (8) imply that

$$a_{n,i} = 0$$

for all $i \in \mathbb{Z}$ such that $|i| \geq n$.

Using that (16) holds for all $0 \leq i \leq n - 1$ and that (17) holds for all $i \in \mathbb{Z}$, with $|i| \geq n$, we conclude that the expression (9) can be transformed into (8), where $a_{n,i}$ is given by (8) for all $i \in \mathbb{Z}$. 

\[\square\]

Lemma 4. Let $y_1$ and $y_2$ be two divisors of $2n$. If $y_1 < y_2$ then

$$g(y_1) + 2 \leq g(y_2).$$

Proof. Using the expression (5) we obtain that, for any real number $y > 0$,

$$g(y + 1) - g(y) > 1,$$

because $g(y + 1) - g(y) = 1 + \frac{2n}{y + 1}$.

Let $y_1$ and $y_2$ be two positive real numbers satisfying $y_2 - y_1 \geq 1$. By Lemma 2, the function $g(y)$ is strictly increasing. So, (19) implies that

$$g(y_2) - g(y_1) > 1.$$  

Furthermore, suppose that $y_1$ and $y_2$ are divisors of $2n$. It follows that $g(y_2) - g(y_1)$ is an integer, because of (5). In this case, the inequality (20) becomes

$$g(y_2) - g(y_1) \geq 2.$$  

Therefore, (18) holds. \[\square\]

Now, we can prove our main result.

Proof of Theorem 1. By Lemma 3, the coefficient $a_{n,i}$ is defined for all $i \in \mathbb{Z}$ by the expression (8).

First, we will prove that the largest coefficient of $P_n(q)$ is at most $F(n)$. Take some $j \in \mathbb{Z}$ satisfying $a_{n,j} = \max \{a_{n,i} : i \in \mathbb{Z}\}$. By (8), there are $h = a_{n,j}$ divisors of $n$, denoted $d_1, d_2, ..., d_h$ satisfying

$$g(d_1) < g(d_2) < ... < g(d_h) \leq 2j < g(2d_1) < g(2d_2) < ... < g(2d_h).$$  


In particular,

\[ g(d_1) < g(d_2) < \cdots < g(d_h) < g(2d_1) < g(2d_2) < \cdots < g(2d_h). \]  

(23)

Applying \( f(x) \) to the inequalities (23) we obtain

\[ d_1 < d_2 < \cdots < d_h < 2d_1 < 2d_2 < \cdots < 2d_h, \]

(24)

because \( f(x) \) and \( g(y) \) are mutually inverses in virtue of Lemma 2. So, we guarantee that

\[ \frac{1}{2} t < d_1 < d_2 < \cdots < d_h \leq t, \]

(25)

where \( t = 2d_1 - \varepsilon \) for all \( \varepsilon > 0 \) small enough. Hence, \( a_{n,j} \leq F(n) \), because of (1).

Now, we will prove that there is at least one coefficient of \( P_n(q) \) which reaches the value \( F(n) \). Setting \( h = F(n) \) and applying (1), it follows that there are \( h \) divisors of \( n \) satisfying (25) for some \( t \in \mathbb{R}_+^* \). The inequalities (24) follow. Applying \( g(y) \) to (24) we obtain (23).

Setting

\[ j := \left\lceil \frac{g(d_h)}{2} \right\rceil, \]

(26)

we have the inequalities

\[ g(d_h) \leq 2j, \]

(27)

\[ 2j \leq g(d_h) + 1, \]

(28)

\[ g(d_h) + 1 < g(d_h) + 2, \]

(29)

\[ g(d_h) + 2 \leq g(2d_1). \]

(30)

The inequality (27) follows from (26). The inequality \( 2j < g(d_h) + 2 \) follows from (26) and the stronger inequality (28) is obtained using the fact \( g(d_h) \in \mathbb{Z} \), derived from (5). The inequality (29) is trivial. Finally, the inequality (30) follows by Lemma 4, because \( d_h \) and \( 2d_1 \) are divisors of \( 2n \) satisfying \( d_h < 2d_1 \).

Combining (27), (28), (29) and (30) we obtain that

\[ g(d_h) \leq 2j < g(2d_1). \]

(31)

The inequalities (22) holds, because of (31) and (23). Hence, \( a_{n,j} = F(n) \). Therefore, the largest coefficient of \( P_n(q) \) is \( F(n) \).

\[ \square \]

**Corollary 5.** The largest coefficient of \( P_n(q) \) is the largest value of \( h \) for which (24) holds.

**Example 6.** The polynomial \( P_{12}(q) \) was computed in [2],

\[ P_{12}(q) = q^{22} + q^{21} + q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15} + q^{14} + 2q^{13} + 2q^{12} + 2q^{11} + 2q^{10} + 2q^9 + q^8 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1. \]
Let us compute $j$ such that $a_{12,j} = a_{12,-j}$ are equal to the largest coefficient of $P_{12}(q)$.

| $d$ | 1  | 2  | 3  | 4  | 6  | 12 |
|-----|----|----|----|----|----|----|
| $g(d)$ | -23 | -10 | -5 | -2 | 2  | 10 |
| $g(2d)$ | -10 | -2  | 2  | 5  | 10 | 23 |

The equality $F(12) = 2$ implies the existence of 2 divisors of 12, for example $d_1 = 2$ and $d_2 = 3$, satisfying (24). In our case,

$$2 < 3 < 2 \cdot 2 < 2 \cdot 3.$$ 

Applying $g(y)$ to the above inequalities, we obtain a particular case of (23),

$$-10 < -5 < -2 < 2.$$ 

So, taking $2j = g(3) + 1 = -5 + 1 = -4$, we obtain $a_{12,-2} = a_{12,2} = 2$, which are the coefficients of $q^9$ and $q^{13}$.

### 3 Some consequences of the main result

Kassel and Reutenauer observed in [2] that $P_n(q)$ has a coefficient larger than 1 provided that $n$ is a perfect number or an abundant number. The corresponding necessary and sufficient condition is given in the following result.

**Corollary 7.** The polynomial $P_n(q)$ has a coefficient larger than 1 if and only if $2n$ is the perimeter of a Pythagorean triangle.

**Proof.** From the explicit formula for Pythagorean triples (see [5]), it follows in a straightforward way that $2n$ is the perimeter of a Pythagorean triangle if and only if $n$ has a pair of divisors $d$ and $d'$ satisfying the inequality $d < d' < 2d$. So, the result follows from Corollary 5. \hfill $\square$

Vatne proved in [6] that the set of coefficients of $P_n(q)$ is unbounded. The following result is a stronger version of this property.

**Corollary 8.** Let $a_{n,i}$ be the coefficients of $P_n(q)$ as shown in (7) and (8). For any integer $m \geq 0$, the equality $a_{n,i} = m$ holds for infinitely many $(n, i) \in \mathbb{Z}^2$, with $n \geq 1$.

In the proof of Corollary 8 we will use the following auxiliary result.

**Lemma 9.** Let $h \geq 1$ be an integer. If $h$ is a coefficient of the polynomial $P_n(q)$, then $h - 1$ is also a coefficient of the same polynomial.

**Proof.** Consider two fixed integers $n \geq 1$ and $h \geq 1$. Let $j$ be the largest integer such that $a_{n,j} \geq h$, where $a_{n,j}$ is given by (8). The inequalities (22) hold for $h$ divisors of $n$, denoted $d_1, d_2, \ldots, d_h$. Setting

$$i := \left\lfloor \frac{g(2d_j)}{2} \right\rfloor,$$

(32)
we have the inequalities

\[ g(d_h) < g(2d_1), \quad (33) \]
\[ g(2d_1) \leq 2i, \quad (34) \]
\[ 2i \leq g(2d_1) + 1, \quad (35) \]
\[ g(2d_1) + 1 < g(2d_1) + 2, \quad (36) \]
\[ g(2d_1) + 2 \leq g(2d_2). \quad (37) \]

The inequality (33) follows by (22). The inequality (34) follows from (32). The inequality \(2i < g(2d_1) + 2\) follows from (32) and the stronger inequality (35) is obtained using the fact \(g(2d_1) \in \mathbb{Z}\), derived from (5). The inequality (36) is trivial. Finally, the inequality (37) follows by Lemma 4, because \(2d_1 < 2d_2\) and both are divisors of \(2n\).

Combining (33), (34), (35), (36) and (37) we obtain that

\[ g(d_h) \leq 2i < g(2d_2). \quad (38) \]

Combining (38) with (24), it follows that

\[ d_2 < d_3 < \ldots < d_h \leq 2i < 2d_2 < 2d_3 < \ldots < 2d_h. \quad (39) \]

Notice that (22) and (34) imply

\[ j < i. \quad (40) \]

In virtue of the expression (8), the inequalities (39) imply that

\[ a_{n,i} \geq h - 1. \quad (41) \]

The inequalities (40) and (41) imply that \(a_{n,i} = h - 1\), because \(j\) is the largest integer satisfying \(a_{n,j} \geq h\).

**Proof of Corollary 8.** Using (2), it follows that the range of \(F(n)\) is unbounded. By Theorem 1, the set of coefficients of \(P_n(q)\), for all \(n \geq 1\), is unbounded.

Take an integer \(m \geq 0\). Consider a polynomial \(P_n(q)\) whose largest coefficient is \(h > m\). Applying Lemma 9 several times, we will obtain that \(m\) is a coefficient of \(P_n(q)\). As there are infinitely many values of \(n\) such that \(P_n(q)\) has a coefficient larger than \(m\), the equality \(a_{n,i} = m\) holds for infinitely many \((n, i) \in \mathbb{Z}^2\), with \(n \geq 1\).

**Acknowledge**

The author thanks S. Brlek and C. Reutenauer for their valuable comments and suggestions concerning this research.
References

[1] Paul Erdős, Jean-Louis Nicolas. Méthodes probabilistes et combinatoires en théorie des nombres. Bull. SC. Math 2 (1976): 301–320.

[2] Christian Kassel and Christophe Reutenauer, Counting the ideals of given codimension of the algebra of Laurent polynomials in two variables, https://arxiv.org/abs/1505.07229, 2015.

[3] Christian Kassel and Christophe Reutenauer, Complete determination of the zeta function of the Hilbert scheme of $n$ points on a two-dimensional torus, https://arxiv.org/abs/1610.07793, 2016.

[4] Christian Kassel and Christophe Reutenauer, The Fourier expansion of $\eta(z)\eta(2z)\eta(3z) / \eta(6z)$, Archiv der Mathematik 108.5 (2017): 453-463.

[5] Waclaw Sierpinski, Pythagorean triangles, Courier Corporation, 2003.

[6] Jon Eivind Vatne, The sequence of middle divisors is unbounded. Journal of Number Theory 172 (2017): 413–415.