1. Introduction

Frustrated quantum antiferromagnets have been at the center of intense experimental and theoretical investigations for many years. These relentless efforts have very recently resulted in a number of theoretical and experimental breakthroughs: quantum entanglement [1–3], density matrix renormalization group (DMRG) revolution and \( Z_2 \) liquids in Kagome and \( J_1-J_2 \) square lattice models [4–6], spin-liquid-like behavior in organic Mott insulators [7, 8] and Kagome lattice antiferromagnet herbertsmithite [9].

Along the way, a large number of frustrated insulating magnetic materials featuring rather unusual ordered phases, such as magnetization plateaux, longitudinal spin-density waves and spin nematics, has been discovered and studied. It is these ordered, yet sufficiently unconventional, states of magnetic matter and theoretical models motivated by them that are the subject of this Key Issue article.

This review focuses on materials and models based on simple triangular lattice, which, despite many years of fruitful research, continue to supply us with novel quantum states and phenomena. Triangular lattice represents, perhaps, the most widely studied frustrated geometry [10–12]. Indeed, the Ising antiferromagnet on the triangular lattice was the first spin model found to possess a disordered ground state and extensive residual entropy [13] at zero temperature. While the classical Heisenberg model on the triangular lattice does order at \( T = 0 \) into a well-known 120° commensurate spiral pattern (also known as a three-sublattice or \( \sqrt{3} \times \sqrt{3} \) state), the fate of the quantum spin-1/2 Heisenberg Hamiltonian has been the subject of a long and fruitful debate spanning over 30 years of research. Eventually it was firmly established that the quantum spin-1/2 model remains ordered in the classical 120° pattern [14–16]. Although the originally proposed resonating valence bond liquid [17, 18] did not emerge as the ground state of the spin-1/2 Heisenberg Hamiltonian, such a phase was later found in a related quantum dimer model on the triangular lattice [19].

It turns out that a simple generalization of the triangular lattice Heisenberg model whereby exchange interactions on the nearest-neighbor bonds of the triangular lattice take two different values—\( J \) on the horizontal bonds and \( J' \) on the diagonal bonds, as shown in figure 1—leads to a very rich and not yet fully understood phase diagram which sensitively depends on the magnitude of the site spin \( S \) and magnetic field \( h \). Such
2. Classical model in a magnetic field

Triangular antiferromagnets in an external magnetic field have been extensively studied for decades and found to possess unusual magnetization physics that remains only partially understood. Underlying much of this interesting behavior is the discovery, made long ago, \cite{26} that in a magnetic field, Heisenberg spins with isotropic exchange interactions exhibit a large \textit{accidental} classical ground-state degeneracy. That is, at finite magnetic fields, there exists an infinite number of continuously deformable classical spin configurations that constitute minimum energy states, but are \textit{not} symmetry related.

This degeneracy is understood by the observation that the Hamiltonian of the isotropic triangular lattice antiferromagnet in magnetic field $\mathbf{h}$ can be written, up to an unessential constant, as

$$H_0 = \frac{J}{2} \sum_r \left[ S_r + S_r + \mathbf{h} \cdot \mathbf{S}_r - \frac{\mathbf{h}^2}{3J} \right]. \quad (1)$$

The sum is over all sites $r$ of the lattice and nearest-neighbor vectors $\delta_{1,2}$ are indicated in the figure 1. Note that Zeeman terms $S_r \cdot \mathbf{h}$ appear three times for every spin in this sum, which explains the factor of 1/3 in the $\mathbf{h}$ term in (1). We immediately observe that every spin configuration which nullifies every term in the sum (1) belongs to the lowest energy manifold of the model. Given the \textit{side-sharing} property of the triangular lattice, so that fixing all spins in one elementary triangle fixes two spins in each of the adjacent triangles, sharing sides with the first one, this implies that all such states exhibit a three-sublattice structure and must satisfy

$$S_A + S_B + S_C = \frac{\mathbf{h}}{3J}. \quad (2)$$

This condition provides 3 equations for 6 angles needed to describe 3 classical unit vectors. In the absence of the field, the 3 undetermined angles can be thought of as Euler’s angles of the plane in which the spins spontaneously form a three-sublattice 120° structure. However, this remarkable feature persists for $\mathbf{h} \neq 0$ as well. There, the symmetry of the Hamiltonian (1) is reduced to $U(1)$ but the degeneracy persists: one of the free angles can be thought as gauge degree of freedom to rotate all spins about the axis of the field $\mathbf{h}$, while the remaining two constitute the phenomenon of \textit{accidental degeneracy}.

Remarkably, thermal (entropic) fluctuations lift this extensive degeneracy in favor of the two \textit{coplanar} (Y and V states) and one \textit{collinear} (UUD) spin configurations, shown in figure 2. Symmetry-wise, coplanar states break two different symmetries—a discrete $Z_3$ symmetry, which corresponds to the choice of sublattice on which the down spin (in the case of Y state) or the minority spin (in the case of V state) is located and a continuous $U(1)$ symmetry of spin rotations about the field axis. The collinear UUD state breaks only the discrete $Z_3$ symmetry (a choice of sublattice for the down spin). Selection of these simple states out of infinitely many configurations, which satisfy (2), by thermal fluctuations represents a textbook example of the ‘order-by-disorder’ phenomenon \cite{27}. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Deformed triangular lattice. Solid (thin) lines denote bonds with exchange constant $J(J’)$ correspondingly. Also indicated are nearest-neighbor vectors $\delta_j$ as well as A–C sublattice structure.}
\end{figure}
The resulting enigmatic phase diagram, first sketched by Kawamura and Miyashita in 1985, [26], continues to attract much attention—and in fact remains not fully understood. Figure 3 shows the result of recent simulations [28], which studied critical properties of various phase transitions in great detail. The obtained phase diagram differs in one important aspect from the original suggestion—it is established now that there is no direct transition between the Y state and the paramagnetic phase (very similar results were obtained in an extensive study [29]). The two phases are separated by the intervening UUD state which extends down to lowest accessible field values and before the transition to the paramagnetic state. Figure 3 also shows that of all entropically selected states, the UUD state is most stable—it extends to higher temperature T than either Y or V states.

The UUD is also the most ‘visible’ of the three selected states—it shows up as a plateau-like feature in the magnetization curve M(h), see figure 4 which shows experimental data for a $S = S/2$ triangular lattice antiferromagnet RbFe(MoO$_4$)$_2$ [30]. Notice that a strict magnetization plateau at 1/3 of the full (saturation) value $M_{sat}$, $M = M_{sat}/3$, is possible only in the quantum problem (i.e. the problem with finite spin $S$), when all spin-changing excitations with $S' \neq 0$ are characterized by gapped spectra and at absolute zero $T = 0$, when no excitations are present in the ground state (see next section 3 for complete discussion). At any finite $T$ thermally excited spin waves are present and lead to a finite, albeit different from the neighboring non-plateau states, slope of the magnetization $M(h)$. In the case of the classical problem we review here the gap in spin excitation spectra is itself $T$-dependent and disappears as $T \rightarrow 0$: as a result its magnetization ‘quasi-plateau’ too disappears in the $T \rightarrow 0$ limit. However, at any finite $T$, less than that of the transition to the paramagnetic state, the slope of the $M(h)$ for the UUD state is different from that of the Y and V states [31].

Figure 2. Various spin configurations from the classical ground state manifold. (a) coplanar Y state, (b) collinear UUD (up–up–down), (c) coplanar V state, (d) non-coplanar cone (umbrella) state, (e) coplanar inverted Y state. Also indicated (in blue) are the order parameter manifolds for each of the states. Magnetic field $h$ is directed vertically (along $z$ axis).

Figure 3. Magnetic field phase diagram of the classical triangular lattice antiferromagnet. (Adapted from Seabra et al [28]. Copyright 2011 by the American Physical Society.) Transition points determined by the Monte Carlo simulations are shown by filled symbols. Continuous phase transitions are drawn with a dashed line, while Berezinskii–Kosterlitz–Thouless phase transitions are drawn with a dotted line. For fields $h \leq 3$ a double transition is found upon cooling from the paramagnet, while for $h \geq 3$ only a single transition is found. Behavior of the phase transition lines in the low-field region $h \leq 0.2$, which is left unshaded in the diagram, is not settled at the present. See [32] for the recent study of $h = 0$ line.

2.1. Spin chirality and emergent Ising orders

Understanding the symmetries broken by a particular ordered state amounts to specifying an order parameter manifold associated with this state. Thus, order parameter manifold of the 3-fold degenerate UUD state is $Z_3$, while that for the coplanar Y, V and inverted Y states in figure 2 is $U(1) \times Z_2$. The non-coplanar cone state is characterized by $U(1) \times Z_2$ manifold where $Z_2$ is associated with spontaneous selection of the vector chirality $\kappa_v = \vec{S}_r \times \vec{S}_\ell$, which is the sense in which spins rotate—clockwise or counterclockwise—as one moves between the sites of the elementary triangle of the lattice. Closely related to it is the scalar chirality, $\chi = \vec{S}_r \cdot \vec{S}_{r+\delta} \times \vec{S}_{r+2\delta}$, which measures the volume enclosed by the three spins of the elementary triangle and, similarly to $\kappa_v$, is sensitive to the sense of spin’s rotation in the $x$–$y$ plane.

Composite nature of the order parameter manifold opens up a possibility of unusual sequence of phase transitions between the high temperature paramagnetic phase and the low temperature magnetically ordered phase. Namely, in the case of the non-coplanar cone state one can imagine existence of the intermediate chiral state with finite spin chirality $\langle \kappa \rangle \neq 0$ but no long-ranged magnetic order $\langle S_\ell \rangle = 0$.

Reference [33] was perhaps one of the first studies of this possibility. Interestingly, the system studied there was
antiferromagnetic XY model of spins on triangular lattice. Its magnetic field $h$—temperature $T$ phase diagram is composed of the same phases, Y, UUD and V, that show up in the phase diagram of the Heisenberg model in the magnetic field, figure 3. (The counting of degeneracies is of course different.) Extensive later studies, which are reviewed in [34], have indeed identified classical chiral spin liquid state (rotations of a rigid object) and admits stable point defects. One of the first explicit calculations of this effect was carried out by Chubukov and Golosov [52], who used semiclassical large-$S$ spin wave expansion in order to systematically separate classical and quantum effects. This well-known technique relies on Holstein–Primakoff representation of spin operators in terms of bosons. The representation is nonlinear, $S^z_e = S - a_e^+ a_e$, $S^x_e = (2S - a_e^+ a_e) a_e^{1/2} a_e$ and leads, upon expansion of square roots in powers of small parameter $1/S$, to bosonic spin-wave Hamiltonian $H = E_{cl} + \sum_{k=2}^{\infty} H^{(k)}$. Here $E_{cl}$ stands for the classical energy of spin configuration, which scales as $S^2$, while each of the subsequent terms $H^{(k)}$ are of $k$-th order in operators $a_e$ and scale as $S^{2-k}$. Diagonalization of quadratic term $H^{(2)}$ provides one with the dispersion $\omega_k^{(m)}$ of spin wave excitations ($k$ is the wave vector and $m$ is the band index), in terms of which quantum zero-point energy is given by $\langle H^{(3)} \rangle = (1/2) \sum_{\mathbf{k}} \omega_k^{(m)}$. This energy scales as $S$ and

2.2. Spatially anisotropic triangular lattice antiferromagnet

A classical system with spatially anisotropic interactions offers an interesting generalization of the ‘order-by-disorder’ phenomenon. Consider slightly deformed triangular lattice, with $J' < J$ (see figure 1). An arbitrary weak deformation lifts, at $T = 0$, the accidental degeneracy in favor of the simple non-coplanar umbrella state (configuration ‘d’ in figure 2) in the whole range of $h$ below the saturation field. The energy gain of this well-known spin configuration is of the order $(J - J')/J$. One thus can expect that, for sufficiently small difference $R = J - J'$, entropic fluctuations, which favor coplanar and collinear spin states at a finite $T$, can still overcome this classical energy gain and stabilize collinear and coplanar states above some critical temperature which can be estimated as $T_{umb-uud} \sim R^2/J$. As a result, the UUD phase ‘decouples’ from the $T = 0$ axis. The leftmost point of the UUD phase in the $h - T$ phase diagram now occurs at a finite $T_{umb-uud}$ as was indeed observed in Monte Carlo simulation of the simple triangular lattice model in [31] (see figure 10 of that reference), as well as in a more complicated ones, for models defined on deformed pyrochlore [48] and Shastry–Sutherland [49] lattices. It is worth noting that the roots of this behavior can be traced to the famous Pomeranchuk effect in $^3$He, where the crystal phase has higher entropy than the normal Fermi-liquid phase. As a result, upon heating, the liquid phase freezes into a solid [50, 51]. In the present classical spin problem it is ‘superfluid’ umbrella phase which freezes into a ‘solid’ UUD phase upon heating [31] (see discussion of the relevant terminology at the end of section 3.1 below).

3. Quantum model in magnetic field

3.1. Isotropic triangular lattice

Much of the intuition about selection of coplanar and collinear spin states by thermal fluctuations applies to the most interesting case of quantum spin model on a (deformed) triangular lattice. Only now it is quantum fluctuations (zero-point motion) which differentiate between different classically-degenerate (or, nearly degenerate) states and lift the accidental degeneracy.

One of the first explicit calculations of this effect was carried out by Chubukov and Golosov [52], who used semiclassical large-$S$ spin wave expansion in order to systematically separate classical and quantum effects. This well-known technique relies on Holstein–Primakoff representation of spin operators in terms of bosons. The representation is nonlinear, $S^z_e = S - a_e^+ a_e$, $S^x_e = (2S - a_e^+ a_e) a_e^{1/2} a_e$ and leads, upon expansion of square roots in powers of small parameter $1/S$, to bosonic spin-wave Hamiltonian $H = E_{cl} + \sum_{k=2}^{\infty} H^{(k)}$. Here $E_{cl}$ stands for the classical energy of spin configuration, which scales as $S^2$, while each of the subsequent terms $H^{(k)}$ are of $k$-th order in operators $a_e$ and scale as $S^{2-k}$. Diagonalization of quadratic term $H^{(2)}$ provides one with the dispersion $\omega_k^{(m)}$ of spin wave excitations ($k$ is the wave vector and $m$ is the band index), in terms of which quantum zero-point energy is given by $\langle H^{(3)} \rangle = (1/2) \sum_{\mathbf{k}} \omega_k^{(m)}$. This energy scales as $S$ and

Figure 4. Magnetization curve $M(h)$ of a spin-5/2 antiferromagnet RbFe(MoO$_4$)$_2$ at $T = 1.3$ K (Adapted from Smirnov et al [30]. Copyright 2007 by the American Physical Society).
thus provides the leading quantum correction to the classical ($\alpha S^2$) result.

The main outcome of the calculation [52] is the finding that quantum fluctuations also selects coplanar Y and V and collinear UUD states (states (a)–(c) in figure 2) out of many other classically degenerate ones. The authors also recognized the key feature of the UUD state—being collinear, this state preserves $U(1)$ symmetry of rotations about the magnetic field axis. The absence of broken continuous symmetry implies that spin excitations have a finite energy gap in the dispersion. This expectation is fully confirmed by the explicit calculation [52] which finds that the gaps of the two low energy modes are given by $|h - h^0_{\nu}|$, where the lower/upper critical fields are given by $h^0_{\nu} = 3JS - 0.5JS / (2S)$ and $h^0_{\nu} = 3JS + 1.3JS / (2S)$, correspondingly. (The third mode describes a high-energy precession with energy $h S$.) The uniform magnetization $M$, being the integral of motion, remains at 1/3 of the maximum (saturation) value, $M = M_{sat}/3$, in the UUD stability interval $h^0_{\nu} < h < h^0_{\nu}$.

As a result, magnetization curve $M(h)$ of the quantum triangular lattice antiferromagnet is non-monotonic and exhibit striking 1/3 magnetization plateau in the finite field interval $\Delta h = h^0_1 - h^0_0 = 1.8JS / (2S)$. It is worth noting that this is not a narrow interval at all, $\Delta h/h_{sat} = 0.2/0.2S$ in terms of the saturation field $h_{sat} = 9JS$. Extending this large-$S$ result all way to the $S = 1/2$ implies that the magnetization plateau takes up 20% of the whole magnetic field interval $0 < h < h_{sat}$.

Numerical studies of the plateau focus mostly on the quantum spin-1/2 problem (numerical studies of higher spins are given by [53–58]) which finds that the gaps of the two low energy modes are given by $[h^0_{\nu}]$, as is $h^0_{\nu} = -h^0_{\nu}$. Extending this large-$S$ result all way to the $S = 1/2$ [53–58].

The pattern of symmetry breaking by the coplanar/collinear states is described by the following spin expectation values

\begin{align}
Y \text{ state, } & \quad 0 < h < h^0_1 : \langle S_\phi^x \rangle = a e^{i\phi} \sin \frac{Q \cdot r}{2} \cos^2 \frac{Q \cdot r}{2} \epsilon \psi \sin \frac{Q \cdot r}{2} \cos^2 \frac{Q \cdot r}{2} \epsilon \psi, \\
& \left. \langle S_\phi^z \rangle = b - c \cos^2 \frac{Q \cdot r}{2} \right) \quad (3) \\
UDU \text{ state, } & \quad h^0_0 \leq h < h^0_1 : \langle S_\phi^x \rangle = 0, \\
& \left. \langle S_\phi^z \rangle = M - c \cos \frac{Q \cdot r}{2} \right) \quad (4) \\
V \text{ state, } & \quad h^0_1 < h < h_{sat} : \langle S_\phi^x \rangle = a e^{i\phi} \cos \frac{Q \cdot r}{2} \epsilon \psi \cos \frac{Q \cdot r}{2} \epsilon \psi, \\
& \left. \langle S_\phi^z \rangle = b - c \cos^2 \frac{Q \cdot r}{2} \right) \quad (5)
\end{align}

Here the ordering wave vector $Q = (4\pi^2/3, 0)$ is commensurate with the lattice which results in only three possible values that the product $Q \cdot r = 2\pi n/3$ can take ($n = 0, 1, 2$), modulo $2\pi$. Angle $\phi$ specifies orientation of the ordering plane for coplanar spin configurations within the $x$–$y$ plane, $M$ is magnetization per site and parameters $a, b, c$ are constants dependent upon the field magnitude.

3.1.1. Connection with ‘super’ phases of bosons. Equation (4) identifies UUD state as a collinear ordered state which can be thought of as solid. Its ‘density’ $\langle S_\phi^z \rangle$ is modulated periodically, relative to the uniform value $M$, as $\cos \frac{Q \cdot r}{2} \epsilon \psi$, as is appropriate for the solid and as a result its local magnetization follows simple ‘up–up–down’ pattern within each elementary triangle. This state respects $U(1)$ symmetry of rotations about the $S^z$ axis, as follows from the very fact that it has no order in the transverse directions, $\langle S_\phi^y \rangle = 0$.

The coplanar Y and V states break this $U(1)$ symmetry by spontaneously selecting ordering direction in the spin $x$–$y$ plane, i.e. by selecting angle $\phi$ in (3),(5). Note that at the same time they are characterized by the modulated density $\langle S_\phi^y \rangle$, which makes them supersolids: the superfluid order (magnetic order in the $x$–$y$ plane as selected by $\phi$) co-exists with the solid one (modulated $z$ component, or density).

This useful connection is easiest to make precise [59] in the case of $S = 1/2$ when the following mapping between a hard-core lattice Bose gas and a spin-1/2 quantum magnet can easily be established:

$$a_t^x \leftrightarrow S_\phi^x, a_t \leftrightarrow S_\phi^x, n_t = a_t^x a_t \leftrightarrow S_\phi^x - 1/2.$$

The superfluid order is associated with finite $\langle a_t^x \rangle$ while the solid one with modulated (with momentum $2Q$ in our notations) boson density $\langle n_t \rangle$.

3.2. Spatially anisotropic triangular antiferromagnet with $J \neq J$

Consider now a simple deformation of the triangular lattice which makes exchange interaction on diagonal bonds, $J$, different from those on horizontal ones $J$, so that $R = J - J$ ≠ 0. This simple generalization of the Heisenberg model leads to surprisingly complicated and not yet fully understood phase diagram in the magnetic field ($h$)–deformation ($J$) plane.

Semiclassical ($S \gg 1$) analysis of this problem is complicated by the fact that arbitrary small $R \neq 0$ removes accidental degeneracy of the problem in favor of the unique non-coplanar and incommensurate cone (umbrella) state, state ‘d’ in figure 2, which was discussed previously in section 2. This simple state gains energy of the order $\delta E_\text{class} \sim S^2 R^2/2I$ per spin. Its structure is described by

$$\langle S_\phi^y \rangle = M^2 + c (\cos [Q \cdot r + \phi] + \sin [Q \cdot r + \phi] \hat{\delta})^2,$$

where classically the ordering wave vector $Q' = 2(\cos^{-1}|J/2I|, 0)$ is a continuous function of $J/I$. Spontaneous selection of the ordering phase $\phi$ makes the cone state a spin superfluid (note that here the density $\langle S_\phi^z \rangle = M$ is constant). In addition, this non-coplanar state is characterized by the finite scalar chirality, $\chi \sim \tilde{S}_r \cdot \tilde{S}_r + \tilde{S}_r + \tilde{S}_r$ $\tilde{S}_r + \tilde{S}_r$ $\tilde{S}_r + \tilde{S}_r$ $\sim M \cos [Q' \cdot 2I / 2]$. Observe that under $Q' \rightarrow Q'$ replacement chirality $\chi$ changes sign, while the ground state energy does not change. Hence, together with breaking spin-rotational $U(1)$ symmetry, the cone state also spontaneously breaks $Z_2$ symmetry (spatial inversion). This consideration allows us to identify the order parameter manifold for this state as $U(1) \times Z_2$. At the same time, for sufficiently small $R$, the quantum energy gain due to zero-point motion of spins, which is of the order $\delta E_{\text{eq}} \sim SJ$ per spin, should be able to overcome $\delta E_\text{class}$ and still stabilize one of the coplanar/collinear states considered in section 3.1 above. Comparing the two contributions, we conclude, following [60], that the classical-quantum
The competition can be parameterized by the dimensionless parameter $\delta = \frac{\delta E_{\text{class}}}{\delta E_{\text{q}}} \sim \frac{S(R/J)}{J^2}$. In the following, we will use more precise value $\delta = \frac{40/3}{S(R/J)}$, with numerical factor 40/3 as introduced in [60] for technical convenience.

Explicit consideration of this competition is rather difficult due to complicated dependence of the parameters of the collinear states on magnetic field $h$ and exchange deformation $R$. However, inside the $M = M_{\text{sat}}/3$ magnetization plateau phase, the spin structure is actually pretty simple, as equation (4) shows, which suggests that UUD state can be used as a convenient starting point for accessing more complicated states.

This approach, which amounts to the investigation of the local stability of the UUD phase, was carried out in [60]. Similar to [52], the calculation is based on the three-spin UUD unit cell, resulting in three spin wave branches. One of these, describing total spin precession, is a high-energy mode not essential for our analysis, while the two others, which describe relative fluctuations of spins, are the relevant low-energy modes. They will be called mode 1 and mode 2 below (creation operators $d_{1,k}^*$ and $d_{2,k}^*$ correspondingly).

Within the UUD plateau phase, which is bounded by FACBG lines in figure 5, both low energy spin wave modes remain gapped. As discussed above, the only symmetry this collinear state breaks is $Z_3$. The gaps of the low-energy spin wave modes are given by $|h - h_{1,2}|$, where now the lower and upper critical fields $h_{1,2}(\delta)$ depend on the dimensionless deformation parameter $\delta$. The closing of the gap at the lower $h_{1,2}$ (upper $h_{3,3}$) critical fields of the plateau implies Bose–Einstein condensation (BEC) of the appropriate magnon mode (mode 1 or mode 2, correspondingly) and the appearance of the transverse to the field spin component $(S_\phi^\perp) \neq 0$, see (3) and (5). This BEC transition breaks spin rotational $U(1)$ symmetry via spontaneous selection of the ‘superfluid’ phase $\varphi$. The spin structure of the resulting ‘condensed’ state sensitively depends on the wave vector $k_i(k_2)$ of the condensed magnon.

Key results of the large-$S$ calculation in [60–62] can now be summarized as follows:

(a) In the interval $0 < \delta \leq 1$ the lower critical field is actually independent of $\delta$, $h_{1,2} = h_{1,2}^0$ and the minimum of the spin wave mode 1 remains at $k_i = 0$. As a result, BEC condensation of mode 1 at $h = h_{1,2}$ signals the transition to the commensurate Y state, which lives in the region OAF in figure 5. In addition to breaking the continuous $U(1)$ symmetry, the Y state inherits broken $Z_3$ from the UUD phase (which corresponds to the selection of the sublattice for the down spins).

(b) For $1 < \delta \leq 4$, the low critical field increases and, simultaneously, the spin wave minima shift from zero to finite momenta $\pm k_1 = (\pm k_1, 0)$. Thus, at $h = h_{1,2}$, the spectrum softens at two different wave vectors at the same time. This opens an interesting possibility of the simultaneous coherent condensation of magnons with opposite momenta $+k_1$ and $-k_1$ [63]. It turns out, however, that repulsive interaction between condensate densities at $\pm k_1$ makes this energetically unfavorable [60]. Instead, at the transition the symmetry between the two possible condensates is broken spontaneously and magnons condense at a single momentum, $+k_1$ or $-k_1$. The resulting state, denoted as distorted umbrella in [60], is characterized by the broken $Z_3$, $U(1)$ and $Z_2$ symmetries—the latter corresponds to the choice $+k_1$ or $-k_1$. This non-coplanar state lives in the narrow region bounded by lines $AC_1$ (solid) and $AaC_1$ (dashed) in figure 5.

(c) Similar developments occur near the upper critical field $h_{3,3}$. In the interval $0 < \delta \leq 3$ the upper critical field is actually independent of $\delta$, $h_{3,3} = h_{3,3}^0$ and the minimum of the spin wave mode 2 remains at $k_i = 0$. The transition on the line $GB$ is towards the coplanar and commensurate V state, bounded by GBH in figure 5. This state is characterized by broken $Z_3 \times U(1)$ symmetries.

(d) In the interval $3 < \delta \leq 4$, the upper field $h_{3,3}$ diminishes and simultaneously the minimum of the mode 2 shifts from zero momentum to the two degenerate locations at $\pm k_0 = (\pm k_0, 0)$. Here, too, at the condensation transition (along the line BC) it is energetically preferable to break $Z_3$ symmetry between the two condensates and to spontaneously select just one momentum, $+k_0$ or $-k_0$. This leads to distorted umbrella with broken $Z_3 \times U(1) \times Z_2$ symmetries. This state lives between (solid) BC and (dashed) BBc lines in figure 5.

(e) Spin nematic region. The critical fields $h_{1,2}$ and $h_{3,3}$ merge at the plateau’s end-point $\delta = 4$ (point C). The minima
of the spin wave modes 1 and 2 coincide at this point \( k_1 = k_2 = (k_0, 0) \) with \( k_0 = \sqrt{3/(10 S)} \). The end-point of the UUD phase thus emerges as a point of an extended symmetry hosting four linearly-dispersing gapless spin wave modes [60] (two branches, each gapless at \((\pm k_0, 0)\)). Single-particle analysis of possible instabilities at the plateau’s end-point (point C, \( \delta = 4 \)) shows that in addition to the expected \( U(1) \times U(1) \) symmetry (the two \( U(1) \)'s represent phases of the single particle condensates), the state at \( \delta = 4 \) point possesses an unusual \( P_1 \) symmetry—the magnitude of the condensate at this point is not constrained [60]. The enhanced degeneracy of the plateau’s end point C can also be understood from the observation that the two chiral distorted umbrella states merging at the point C are characterized, quite generally, by the different chiralities: Condensation of magnons at \( h_{1,2} \), described in items (b) and (c) above, proceeds independently of each other—hence the chiralities of the two states are not related in any way. Thus the merging point of the two phases, point C, must possess enhanced symmetry. Unconstrained magnitude mode hints at a possibility of a two-particle condensation—and indeed recent work [61] has found that a single-particle instability at \( \delta = 4 \) is pre-empted by the two-particle one at a slightly smaller value of \( \delta = \delta_C = 4 - O \left(1/S^2\right) \). This is indicated by the (solid) line \( C_1-C_2 \) in figure 5. This happens via the development of the ‘superconducting’-like instability of the magnon pair fields \( \Psi_{1,p} = d_1 + k_0 + p d_2, -k_0 - p \) and \( \Psi_{2,p} = d_1, -k_0 + p d_2, -k_0 + p \) and consists in the appearance of two-particle condensates \( \langle \Psi_{1,p} \rangle = \langle \Psi_{2,p} \rangle = \partial \langle \Psi \rangle \). The sign of the real-valued Ising order parameter \( \Upsilon \) determines the sense of spin-current circulation on the links of the triangular lattice, as illustrated in figure 6.

The spin current is defined as the ground state expectation value of the vector product of neighboring spins. For example, spin current on the AC link of the elementary triangle is given by \( J_{\text{AC}} = \xi \cdot (S_A \times S_C) \sim \Upsilon \). In addition to spin currents, this novel state also supports finite scalar chirality, \( (S_A \cdot S_B \times S_C) \sim \Upsilon \), even though \( \langle S_{A,B,C} \rangle = 0 \) for each of the spins individually. At the same time, in the absence of single-particle condensation, \( \langle d_{1,2} \rangle = 0 \), the usual two-point spin correlation function \( \langle S_{[i,j]}(r) \rangle \) is not affected by the two-particle \( \langle \Psi_{1,2} \rangle \neq 0 \) condensate: its transverse \((a = b = x \text{ or } y)\) components continue to decay exponentially because of the finite energy gap in the single magnon spectra, while the longitudinal \((a = b = z)\) components continue to show a perfect UUD crystal order. Hence the resulting state, which lives inside triangle-shaped region \( C_1-C_2 \) in figure 5, is a spin-nematic state. It can also be called a spin-current state [61]. It is uniquely characterized by the sign of \( \Upsilon \) which determines the sense (clockwise or counterclockwise) of spin current circulation. The spin-current nematic is an Ising-like phase with massive excitations, which are domain walls separating domains of oppositely circling spin currents. It is characterized by broken \( Z_2 \times Z_2 \), where \( Z_2 \) is the sign of the two-magnon order parameter \( \Upsilon \) in the ground state. It is useful to note that spontaneous selection of the circulation direction can also be viewed as a spontaneous breaking of the spatial inversion symmetry \( \mathcal{I} : x \rightarrow -x \), which changes direction of spin currents on all bonds. (A different kind of nematic is discussed in section 4.2.)

(f) high-field region, \( h_{sd}(\delta) \ll h_{sat}(\delta) \). The high field region, \( h = h_{sat}(\delta) \), can be conveniently analyzed within powerful Bose–Einstein condensation (BEC) framework. This follows from the simple observation that the ground state of the spin-3 quantum model becomes a simple fully polarized state once the magnetic field is greater than the saturation field, \( h > h_{sat}(\delta) \) (which itself is a function of spin \( S \) and exchange deformation \( R \)). Excitations above this exact ground state are standard spin waves minimal energy for creating which is given by \( h - h_{sat} \). Similar to the situation near plateau’s critical field \( h_{1,2} \), these spin waves are characterized by non-trivial dispersion with two degenerate minima at momenta \( \pm Q \) (see expression below (7)). Spontaneous condensation in one of the minima, which constitutes breaking of \( Z_2 \) symmetry, results in the usual cone, or umbrella, state which is characterized by the spin pattern (7) which breaks spin-rotational \( U(1) \). This state is realized to the right of the D-C2 line in figure 5.

Instead, simultaneous condensation of the high-field magnons in both minima results in the coplanar state [63] (also known as fan state) of the \( V \) type. For any \( \delta \neq 0 \) the wave vector \( Q \) is not commensurate with the lattice, which makes the coplanar state to be incommensurate as well. The condensates \( \Psi_{1,2} = \sqrt{p} e^{i Q \cdot r} \) at wave vectors \( \pm Q \) have equal magnitude \( \sqrt{p} \) and each breaks \( U(1) \) symmetry. By choosing the phases \( \theta_{1,2} \), the resulting state breaks two \( U(1) \) symmetries. In total, the coplanar state is characterized by the broken \( U(1) \times U(1) \). It is instructive to think of these symmetries in a slightly different way—as of those associated with the total \( \theta = \theta_1 + \theta_2 \) and the relative \( \theta_1 = \theta_1 - \theta_2 \) phases. The spin structure of this state is described by the incommensurate version of (5) which can be obtained by identifying \( p = \theta \), and replacing \( \cos [Q \cdot r] \rightarrow \cos [Q \cdot r + \theta] \). The latter replacement shows that the relative \( U(1) \) symmetry, associated with the phase \( \theta_2 \), can also be thought of as a translational symmetry associated with the shift of the spin configuration by the vector \( r_0 \) such that \( Q \cdot r_0 = \theta_2 \), modulo \( 2\pi \).

Point H of the diagram, at \( \delta = 0 \) and \( h = h_{sat} \), is the special commensurate point. Here \( Q \rightarrow Q = (4\pi/3, 0) \), resulting in the commensurate coplanar V state. As noted right below equation (5), here \( Q \cdot r = 2\pi \nu/3 \), with \( \nu = 0 \),
the nonlinear $\cos \theta$ is described by the classical 2D sine-Gordon model with a commensurate-incommensurate transition (CI) type. It is described by the classical 2D sine-Gordon model with the nonlinear cos $[3]$, potential which locks the relative phase to the $Z_3$ set [57]. In the vicinity of point H in the diagram the line of the CI transition presents $h_{sat} - h \sim \sqrt{\delta}$ [62]. While the arguments presented here are valid in the immediate vicinity of $h_{sat}$, the identification of the whole line H-B as the CIT line between the commensurate and incommensurate V states is possible due to the additional evidence reported in item (c) above—commensurate V state is reached from the UUD state in the whole interval $0 < \delta \leq 3$. The next task is to connect the incommensurate coplanar V state, which occupies region HDB in figure 5, with the incommensurate cone state, to the right of D-C2 line. Since the V state has equal densities of bosons in the $\pm Q'$ points, while the cone has finite density only in one of them, continuous transition between these two states at finite condensate density (that is, at any $h < h_{sat}$) is not possible. At infinitesimally small condensate density, i.e. at $h = h_{sat}$, direct transition is possible—it occurs at point D, which is a point of extended $U(1) \times U(1) \times U(1)$ symmetry: the two $U(1)'s$ are phase symmetries while the third one is an emergent symmetry associated with the invariance of the potential energy at the constant total condensate density, $\rho = \rho_1 + \rho_2$, with respect to the distribution of condensate densities $\rho_{1,2} = \rho'_{j} \rho'||_{j}$ between the $\pm Q'$ momenta. Large-$S$ calculation of the ladder diagrams, which describe quantum corrections to the condensate energy, place point D at $\delta = 2.91$ [62]. Assuming that the first order transition does not realize, we are forced to conclude that V and cone states must be separated by the intermediate phase, occupying DBBC2 region. This phase breaks the $U(1)$ symmetry between $p_1$ and $p_2$ and interpolates smoothly between the symmetric situation $p_1 = p_2$ on the D-B line and the asymmetric one $p = p_1$ and $p_2 = 0$ (or vice versa) on the line D-C2. In doing so the momentum of the ‘minority’ condensate is found to evolve continuously from the initial $q_2 = Q'$ (which coincides with the momentum of the ‘majority’ condensate $p_1$) on the line D-C2 to the final $q_2 = -Q'$ on the D-B line [62]. The resulting phase is a non-coplanar one, with strongly pronounced asymmetry in the $x-y$ plane: $\langle S_x^z \rangle = \sqrt{2} \rho e^{i\delta Q' x} + \sqrt{2} \rho e^{i\delta Q' y}$. The state is characterized by the broken $U(1) \times U(1) \times Z_2$. For the lack of better term we call it double cone [62]. Going down along the field axis takes us toward the dashed B-b-C2 line below which, according to the analysis summarized in item (c) above, represents a phase with broken $Z_3 \times Z_2 \times U(1)$. Hence along this line $Z_3$ is replaced by $U(1)$, which makes it a continuation of the C-IC transitions line H-B.

(g) low-field region, $0 \leq h \ll h_c(\delta)$. Semiclassical analysis at zero field $h = 0$ is well established and predicts incommensurate spiral state with zero total magnetization $M = 0$ of course. Quantum fluctuations renormalize strongly parameters of the spin spiral [64]. The most quantum case of the spin $S = 1/2$ remains not fully understood even for the relatively weak deformation of exchanges $R = J - J' \leq J$ and is described in more details in section 3.3.

At $\delta = 0$ one again has commensurate three-sublattice antiferromagnetic state, widely known as a 120° structure, which evolves into commensurate Y state in external magnetic field $h \neq 0$. Phenomenological analysis of [57], section 3.5, shows Y state becomes incommensurate when the deformation $R$ exceeds $R_c \sim h^{3/2}$. Analysis near $h_{c1}$, reported in (b), tells that C-IC transition line must end up at point A. Comparing the energies of the incommensurate coplanar Y and the incommensurate umbrella state in the limit of vanishing magnetic field $h \to 0$, described in [60], identifies point E at $\delta = 1.1$ and $h = 0$ as the point of the transition between the $U(1) \times U(1)$ (the incommensurate coplanar V) and $U(1) \times Z_2$ (the incommensurate cone) breaking states. Thus point E is analogous to point D.

By the arguments similar to those in part (f) above, there must be an intermediate phase with broken $U(1) \times U(1) \times Z_2$. It occupies region E-C1-a-A in figure 5. The state between A-C1 and A-a-C1 lines is characterized by different broken symmetries (see item (b) above) which makes the line A-a-C1 to be the line of the $Z_1 \to U(1)$ transition. It thus has to be viewed as a continuation of the CIT line O-A.

To summarize, the large-$S$ phase diagram in figure 5 contains many different phases. It worth keeping in mind that it has been obtained under assumption of continuous phase transitions between states with different orders. Several of the phase boundaries shown in figure 5 are tentative—their existence is conjectured based on different symmetry properties of the states they are supposed to separate. To highlight their conjectured nature, such lines are drawn dashed in figure 5. These include line B-b-C2 which separates distorted umbrella (with broken $Z_3 \times U(1) \times Z_2$) and double spiral (with broken $U(1) \times U(1) \times Z_2$) and similar to it line A-a-C1 located right below the UUD phase. The end-points of these dashed lines are conjectured to be C2 and C1, correspondingly, which are the points of the two-magnon condensation (item (e)). Line D-C2, separating phases with broken $U(1) \times U(1) \times Z_2$ and $U(1) \times Z_2$, established via high-field analysis in item (f), is conjectured to end at the same C2. Since different behavior cannot be ruled out at the present, its extension to the near-plateau region is indicated by the dashed line as well. Similar arguments apply to the line E-C1. Finally, line C2-C-C1, covering the very tip of the UUD plateau phase, is made dashed because it is located past the two-magnon condensation transition (line C1-C2) into the spin-nematic state instabilities of which have not been explored in sufficient details yet.

One of the most unexpected and remarkable conclusions emerging from the analysis summarized here is the identification of the continuous line of C-IC transitions (line
H-B-b-C2-C-C1-a-A-O), separating phases with discrete $Z_3$ from those with continuous $U(1)$ symmetry. Its existence owes to the non-trivial interplay between geometric frustration and quantum spin fluctuations in the triangular antiferromagnet.

3.2.1. Spin excitation spectra. Many of the ordered non-collinear states described above harbor spin excitations with rather unusual characteristics. It has been pointed out some time ago [65, 66] that local non-collinearity of the magnetic order results in the strong renormalization of the spin wave spectra at 1/S order. (This should be contrasted with the case of the collinear magnetic order, where quantum corrections to the excitation spectrum appear only at 1/S$^2$ order.) This interesting effect, reviewed in [67], is responsible for dramatic flattening of spin wave dispersion, appearance of ‘roton-like’ minima and significant shortening of spin wave lifetime. It also causes pronounced thermodynamic anomalies at temperatures as low as $J/10$ [68, 69]. Several recent experiments on ‘large-spin’ antiferromagnets $\pi$-CaCr$_2$O$_4$($S=3/2$) [70, 71] and LuMnO$_3$(S = 2) [72] have observed these theoretically predicted features.

3.3. Spin 1/2 spatially anisotropic triangular antiferromagnet with $J \neq J$

We now turn to the case of most quantum system: a spin 1/2 antiferromagnet. Qualitatively, one expects quantum fluctuations to be most pronounced in this case, which suggests, in line with ‘order-by-disorder’ arguments of section 3.1, a selection of the ordered Y, UUD and V states at and near the isotropic limit $J = J$. Behavior away from this isotropic line represents a much more difficult problem, mainly due to the absence of physically motivated small parameter, which would allow for controlled analytical calculations. Aside from the two limits where small parameters do appear, namely the high field region near the saturation field and the limit of weakly coupled spin chains (see below), the only available approach is numerical.

Most of recent numerical studies of triangular lattice antiferromagnets focus on the zero field limit, $h = 0$ and on the phase diagram as a function of the ratio $J'/J$. These studies agree that a 2D magnetic spiral order, well established at the isotropic $J = J$ point, becomes incommensurate with the lattice when $J' < J$ and persists down to approximately $J' = 0.5 J$. The ordering wave vector of the spiral $\mathbf{Q}$ is strongly renormalized by quantum fluctuations [64, 73, 74] away from the semiclassical result. Below about $J' = 0.5 J$, strong finite size effects and limited numerical accuracy of the exact diagonalization [75] and DMRG [73, 76] methods do not allow one to obtain a definite answer about the ground state of the spin-1/2 $J = J$ Heisenberg model, although most recent exact diagonalization studies employing twisted boundary conditions [74] and quantum fidelity analysis [77] have managed to access $J' = 0.2 J$ region.

The apparent difficulty of reaching the true ground state of the model is a direct consequence of the strong frustration inherent in the triangular geometry. In the $J' \ll J$ limit the lattice decouples into a collection of linear spin chains weakly coupled by the frustrated interchain exchange $\mathcal{H} = J \sum_{x,y} (S_{x-1/2,y+1} + S_{x-1/2,y+1})$. Even classically, spin-spin correlations between spins from different chains are strongly suppressed as can be seen from the limit $J'/J \rightarrow 0$ when classical spiral wave vector $Q_x = 2 \cos^{-1} [-J'/2J] \rightarrow \pi + J'/J$. In this limit the relative angle between the spin at (integer-numbered) site $x$ of the $y$-th chain and its neighbor at (half-integer-numbered) $x + 1/2$ site of the $y + 1$-th chain approaches $\pi/2 + J'/2J$. Thus the scalar product of two classical spins at neighboring chains vanishes as $J'/2J \rightarrow 0$.

Quantum spins add strong quantum fluctuations to this behavior, resulting in the numerically observed near-exponential decay of the inter-chain spin correlations, even for intermediate value $J'/J \lesssim 0.5$ [73, 74, 76]. While some of the studies interpret such effective decoupling as the evidence of the spin-liquid ground state [75, 76], the others conclude that the coplanar spiral ground state persists all the way to $J' = 0$ [73, 78].

Analytical renormalization group approach [79], which utilizes symmetries and algebraic correlations of low-energy degrees of freedom of individual spin-1/2 chains, finds that the system experiences quantum phase transition from the ordered spiral state to the unexpected collinear antiferromagnetic (CoAF) ground state. This novel magnetically ordered ground state is stabilized by strong quantum fluctuations of critical spin chains and weak fluctuation-generated interactions between next-nearest chains. This finding is supported by the coupled-cluster study [80, 81], functional renormalization group [82] as well as by the combined DMRG and analytical RG study [83]. Recent exact diagonalization study with twisted boundary conditions [74] have presented convincing evidence in favor of the CoAF state, while also noting that the competition between predominantly antiferromagnetic next-nearest chains correlations, which is a ‘smoking gun’ feature of the CoAF state [79, 83] and predominantly ferromagnetic ones is ‘extremely close’, resulting in the $\sim 10^{-3}$ difference of the appropriate dimensionless susceptibilities for the system of 32 spins [74]. In addition, variational wave function study of the spatially anisotropic $t - t' - U$ Hubbard model [84] also finds that quasi-one-dimensional collinear spin state is stable in a remarkably wide region of parameters, $t/t' \lessgtr 0.8$ and $4 \lessgtr U/t \lessgtr 8$, see section 5.3 for more details.

Having described the limiting behavior along $h = 0$ and $J = J(R = 0)$ axes, we now discuss the full $h - R$ phase diagram of the spin-1/2 Heisenberg model shown in figure 7 ($R = J - J'$). The diagram is derived from extensive DMRG study of triangular cylinders (spin tubes) composed of 3, 6 and 9 chains and of lengths 120–180 sites (depending on $R$ and the magnetization value) as well as detailed analytical RG arguments applicable in the limit $J' \ll J$ [57]. It compares well, in the regions of small and intermediate $R$, with the only other comprehensive study of the full anisotropy-magnetic field phase diagram—the variational and exact diagonalization study by Tay and Motrunich [55]. The comparison is less conclusive in the regime of large anisotropy, $R \rightarrow 1$, which is most challenging for numerical techniques as already discussed above. (The biggest uncertainty of the diagram in
the main text CoAF phase) phases have no analogues. (Adapted from
and quasi-collinear (which is a finite-field version of the described in
magnetic field $h$
triangular antiferromagnet in magnetic field. Vertical axis—
non-coplanar cone (with $U$
parameters manifold
incommensurate (IC) planar Y and V phases, as well as their
incommensurate (IC) planar versions (these are phases with the order
parameter manifold $U(1) \times U(1)$ in figure 5), are the same in the two
diagrams. The narrow cone phase corresponds to incommensurate
non-coplanar cone (with $U(1) \times Z_2$ manifold) in figure 5. The SDW
and quasi-collinear (which is a finite-field version of the described in
the main text CoAF phase) phases have no analogues. (Adapted from
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figure 7 consists in so far undetermined region of stability of
the cone phase and, to a lesser degree, the phase boundaries
between incommensurate (IC) planar and SDW phases.)

The main features of the phase diagram of the quantum
spin-1/2 model are:

(a) High-field incommensurate coplanar (incommensurate V
or fan) phase, which is characterized by the broken $U(1) \times
U(1)$ symmetry, is stable for all values of the exchange
anisotropy $1 \geq R \geq 0$. This novel analytical finding,
confirmed in DMRG simulations, is described in [57].
This result is specific to the quantum $S = 1/2$ model—for
any other value of the site spin $S \geq 1$ there is a quantum
phase transition between the incommensurate planar and
the incommensurate cone phases at some $R_S$. The critical
value $R_S$ is spin-dependent and decreases monotonically
with $S$, from $R_S = 0.9$ for $S = 1$ to $R_S = 0.4$ for $S = 2$.
(b) 1/3 Magnetization plateau (UUD phase) is present for all
values of $R$ too: it extends from $R = 0$ all the way to $R = 1$.
This striking conclusion is based on analytical calculations
near the isotropic point [60], discussed in the previous sec-
tion, complementary field-theoretical calculations near the
decoupled chains limit of $R = 1$ [57, 85] and on extensive
DMRG studies of the UUD plateau in [57].
(c) A large portion of the diagram in figure 7, roughly to the
right of $R = 0.5$, is occupied by the novel incommensurate
collinear SDW phase. Physical properties of this mag-
netically ordered and yet intrinsically quantum state are
summarized in section 4.1 below.

Comparing the quantum phase diagram in figure 7 with
the previously described large-$S$ phase diagram in figure 5,
one notices that both the high-field incommensurate coplanar
and the UUD phases are present there too. The fact that both
of these states become more stable in the spin-1/2 case and
expand to the whole range of exchange anisotropy $0 < R < 1$,
represents a striking quantitative difference between the
large-$S$ and $S = (1/2)$ cases. It should also be noticed that both
phase diagrams demonstrate that the range of stability of the
incommensurate cone (umbrella) phase is greatly diminished
in comparison with that of the classical spin model.

In contrast, a collinear SDW phase, which occupies a good
portion of the quantum phase diagram in figure 7, is not pre-
sent in figure 5 at all—and this constitutes a major qualita-
tive distinction between the large-$S$ and the quantum $S = (1/2)$
cases.

4. SDW and nematic phases of spin-1/2 models

4.1. SDW

The collinear SDW phase is characterized by the modulated
expectation value of the local magnetization

$$\langle S_i^z \rangle = M + \text{Re}[\Phi e^{i k_{sdw} \cdot r}],$$

where $\Phi$ is the SDW order parameter and SDW wave vec-
tor $k_{sdw}$ is generally incommensurate with the lattice and,
moreover, is the function of the uniform magnetization $M$
and anisotropy $R$. Equation (8) is very unusual for a classical (or
semi-classical) spin system, where magnetic moments tend
to behave as vectors of fixed length. It is, however, a rela-
tively common phenomenon in itinerant electron systems with
nested Fermi surfaces [86–88]. The appearance of such a state
in a frustrated system of coupled spin-1/2 chains is rooted
in a deep similarity between Heisenberg spin chain and one-
dimensional spin-1/2 Dirac fermions [89–91]: thanks to the
well-known phenomenon of 1D spin-charge separation, the
spin sectors of these two models are identical. Ultimately, it
is this correspondence that is responsible for the ‘softness’ of

$\text{Figure 7.} \quad \text{Phase diagram of the spin } S = 1/2 \text{ spatially anisotropic triangular antiferromagnet in magnetic field. Vertical axis—magnetic field } h/J, \text{ horizontal—dimensionless degree of spatial anisotropy } R/J = 1 - J/J. \text{ Notation C/IC stands for commensurate/incommensurate phases, accordingly. To compare this diagram with the large-} S \text{ one in figure 5, we note that collinear UUD and commensurate (C) planar Y and V phases, as well as their incommensurate (IC) planar versions (these are phases with the order parameter manifold } U(1) \times U(1) \text{ in figure 5), are the same in the two diagrams. The narrow cone phase corresponds to incommensurate non-coplanar cone (with } U(1) \times Z_2 \text{ manifold) in figure 5. The SDW and quasi-collinear (which is a finite-field version of the described in the main text CoAF phase) phases have no analogues. (Adapted from [57]. Copyright 2013 by the American Physical Society.)}$

$\text{Figure 8.} \quad \text{Schematics of spinon dispersion for 1D spin chain in magnetic field. Dashed line shows dispersion for zero field. } k \text{ denote Fermi momentum of spin ‘up’ (‘down’) spinons, accordingly. Fermi momentum in the absence of the field is } k_F = \pi/2.$
the amplitude-like fluctuations underlying the collinear SDW state of figure 7.

Figure 8 schematically shows a dispersion of $S = 1/2$ electron in a magnetic field. Fermi-momenta of up- and down-spin electrons $k_{1/2}$ are shifted from the Fermi momentum of non magnetized chain, $k_F = \pi/2$, by $\pm \Delta k_F = \pm \pi M$, where $M$ is the magnetization. As a result, the momentum of the spin-flip scattering processes, which determine transverse spin correlation function ($S' S$), is given by $(k_1 = -k_2) = \pm 2 k_F = \pm \pi$ and remains commensurate with the lattice. At the same time, longitudinal spin excitations, which preserve $S$, now involve momenta $\pm 2k_{1/2} = \pi (1 \pm 2M)$ and become incommensurate with the lattice. To put it differently, in the magnetized chain with $M \neq 0$, low-energy longitudinal spin fluctuations can be parameterized as $S'(x) \sim S^\alpha e^{i(\pi x - 2\pi M x)} + S^\alpha e^{i(\pi x + 2\pi M x)}$, where (calligraphic) $S^\alpha(x)$ represents slow (low-energy) longitudinal mode. Similarly, transverse spin fluctuations are written $S^\alpha(x) \sim S e^{i\pi x}$, with $S^\alpha$ representing low-energy transverse mode.

This simple fact has dramatic consequences for frustrated interchain interaction $\mathcal{H}' = J' \sum_{\alpha xy} S^\alpha_{xy}$. (6) Longitudinal ($z$) component of the sum of two neighboring spins on chain ($y + 1$) adds up to $S^z_{1/2, 1/2, y+1} + S^z_{1/2, y, y+1} - \sin[\pi M] S^z_{1/2, y, y} e^{i(\pi x - 2\pi M x)} + h.c.$, while the sum of their transverse components becomes a derivative of the smooth component of the transverse field $S^\alpha_S^z, S^\alpha_S^z, S^\alpha_S^z + S^\alpha_S^z \rightarrow e^{i\pi x} \partial_x S^\alpha_S^z$. Hence the low-energy limit of the inter-chain interaction reduces to $\mathcal{H}' \rightarrow \sum_{\alpha} \int d\chi J' \sin[\pi M] S^\alpha_S^z \partial_x S^\alpha_S^z + J' S^\alpha_S^z \partial_x S^\alpha_S^z + h.c.$. The presence of the spatial derivative in the second term severely weakens it [85] and results in the domination of the density-density interaction (first term) over the transverse one (second term). The field-induced shift of the Fermi-momenta from its commensurate value, $k_F \rightarrow k_{1/2}$, together with frustrated geometry of inter-chain exchanges, are the key reasons for the field-induced stabilization of the 2D longitudinal SDW state.

Symmetry-wise, the SDW state breaks no global symmetries (time reversal symmetry is broken by the magnetic field, which also selects the $z$ axis) and, in particular, it preserves $U(1)$ symmetry of rotations about the field axis. This crucial feature implies the absence of the off-diagonal magnetic order $\langle S^x \gamma \rangle = 0$ and would be gapless (Goldstone) spin waves. Instead, SDW breaks lattice translational symmetry. Its order parameter $\Phi \sim (S') \neq 0$ is determined by inter-chain interactions [85, 92]. Provided that $k_{sdw} = \pi (1 - 2\xi) \hat{z}$ is incommensurate with the lattice, the only low energy mode is expected to be the pseudo-Goldstone acoustic mode of broken translations, known as a phason. (Inter-chain interactions do affect $k_{sdw}$ but in the limit of small $J'/J$ this can be neglected.) The phason is a purely longitudinal mode corresponding to the phase of the complex order parameter $\Phi$ and hence represents a modulation of $S$ only. This too is unusual in the context of insulating magnets, where, typically, the low energy collective modes are transverse spin waves, associated with small rotations of the spins away from their ordered axes. In the spin wave theory, longitudinal modes are typically expected to be highly damped [67, 93, 94] and hence hard to observe. (For a recent notable exception to this rule see a study of amplitude-modulated magnetic state of PrNi$_2$Si$_2$ [95].) In the SDW state, the longitudinal phason mode in the only low energy excitation. Transverse spin excitations, which are also present in SDW, have finite energy gap. This, in fact, is one of key experimentally identifiable features of the SDW phase. More detailed description of spin excitations of this novel phase, as well as of the spin-nematic state reviewed below, can be found in the recent study [92].

At present, there are three known routes to the field-induced longitudinal SDW phase for a quasi-1D system of weakly coupled spin chains. The first, reviewed above, relies on the geometry-driven frustration of the transverse interchain exchange, which disrupts usual transverse spin ordering and promotes incommensurate order of longitudinal $S$ components. The other route, described in the section 4.1.1 below, relies on Ising anisotropy of individual chains. Lastly, it turns out that a 2D SDW state may also emerge in a system of weakly coupled nematic spin chains—this unexpected possibility is reviewed in the section 4.2.

4.1. SDW in a system of Ising-like coupled chains. There is yet another surprisingly simple route to the 2D SDW phase. It consists in replacing Heisenberg chains with XXZ ones with pronounced Ising anisotropy. It turns out that a sufficiently strong magnetic field, applied along the $z$ (easy) axis, drives individual chains into a critical Luttinger liquid state with dominant longitudinal, $S' S'$, correlations [96]. This crucial property ensures that weak residual inter-chain interaction selects incommensurate longitudinal SDW state as the ground state of the anisotropic 2D system.

We note, for completeness, that not every field-induced gapless spin state is characterized by the dominant longitudinal spin correlations. For example, another well-known gapped system, spin-1 Haldane chain, can also be driven into a critical Luttinger phase by sufficiently strong magnetic field [97–100]. However, that critical phase is instead dominated by strong transverse spin correlations [101, 102]. As a result, a 2D ground state of weakly coupled spin-1 chains is a usual cone state [103].
spin system is in the 2D collinear SDW phase). This condition selects \( M = 1/3 \) \( M_{\text{sat}} \) plateau \((q = 1, p = 3)\) as the most stable one, in a sense of the biggest energy gap with respect to creation of spin-flip excitation (which changes total magnetization of the system by \pm 1\). The next possible plateau is at \( M = 3/5 \) \( M_{\text{sat}} \) \((q = 1, p = 5)\) [85]—however this one apparently does not realize in the phase diagram in figure 7, perhaps because it is too narrow and/or happen to lie inside the (yet not determined numerically) cone phase.

Applied to the 1D spin chain, the above condition can be re-written as a particular \( S = (1/2) \) version of the Oshikawa–Yamanaka–Affleck condition [104, 105] for the period-p magnetization plateau in a spin-S chain, \( p S (1 - M/M_{\text{sat}}) = \) integer. Interestingly, this shows that \( p = 3 \) plateau at \( M = 1/3 \) \( M_{\text{sat}} \) of the total magnetization \( M_{\text{sat}} \) is possible for all values of the spin \( S \): the quantization condition becomes simply \( 2S = \) integer. This rather non-obvious feature has in fact been numerically confirmed in several extensive studies [106–108].

4.2. Spin nematic

Spin nematic represents another long-sought type of exotic ordering in the system of localized lattice spins. It breaks \( SU(2) \) symmetry while preserving translational and time reversal symmetries. Out of many possible nematic states [109, 110], our focus here is on bond-nematic order associated with the two-magnon pairing [111] and the appearance of the non-local order parameter \( Q_{-} = S_{x}^{2} S_{y}^{2} \) defined on the \((r, r')\) bond connecting sites \( r \) and \( r' \). Such order parameter can be build from quadrupolar operators \( Q_{+} = S_{x}^{2} S_{y}^{2} - S_{x} S_{y} \) and \( Q_{0} = S_{x}^{2} S_{y}^{2} + S_{x} S_{y} \), as \( Q_{-} = -Q_{+} - Q_{0} \) [112]. This bond-nematic order is possible in both \( S \geq 1 \), where quadrupolar order was originally suggested [113] and \( S = 1/2 \) systems of localized spins, coupled by exchange interaction.

The magnon pairing viewpoint, explored in great length in [37, 112, 114], is extremely useful for understanding basic properties of the spin-nematic state: the nematic can be thought of as a ‘bosonic superconductor’ formed as a result of two-magnon condensation \((S_{x}^{2} S_{y}^{2}) \sim \langle Q_{-}\rangle \neq 0\). As in a superconductor, a two-magnon condensate breaks \( U(1) \) symmetry, which in this case is a breaking of the spin rotational symmetry with respect to magnetic field direction. It does not, however, break time-reversal symmetry (which requires a single-particle condensation). Just as in a superconductor, single-particle excitations of the nematic phase are gapped. This implies that transverse spin correlation function \((S_{x}^{2} S_{y}^{2}) \sim e^{-r/\xi}, \) which probes single magnon excitations, is short-ranged and decays exponentially. At the same time, fluctuations of magnon density, which are probed by longitudinal spin correlation function \((S_{x}^{2} S_{y}^{2})\), are sound-like acoustic (Bogoliubov) modes.

Before proceeding further, it is useful to distinguish spin nematic we are discussing here from other frequently discussed in the solid state literature nematic phases of itinerant electrons such as nematic Fermi-liquid [115] and (now experimentally confirmed) nematic phase of pnictide superconductors [116]. There, nematic correlations develop in the position space and correspond to a spontaneous selection of the preferred spatial direction (that is, spontaneous breaking of the lattice rotational symmetry) while still in the paramagnetic phase with no long-ranged magnetic order.

In contrast, the ‘magnon-paired’ state discussed here corresponds to a spontaneous selection of the direction in the internal spin space and is not sensitive to the often low, e.g. quasi-1D, lattice symmetries of the system.

4.2.1. Weakly coupled nematic chains. Basic ingredients of this ‘bosonic superconductor’ picture—gapped magnon excitations and attractive interaction between them—are nicely realized in the spin-1/2 quasi-1D material LiCuVO\(_4\), reviewed in section 5.2: the gap in the magnon spectrum is caused by the strong external magnetic field \( h \) which exceeds (single-particle) condensation field \( h_{c}^{(1)} \), while the attraction between magnons is caused by the ferromagnetic (negative) sign of exchange interaction \( J_{1} \) between the nearest spins of the chain (see figure 9).

Under these conditions, the two-magnon bound state, which lies below the gapped single particle states, condenses at \( h_{c}^{(2)} \). As a result, a spin nematic state is naturally realized in each individual chain in the intermediate field interval \( h_{c}^{(2)} < h < h_{c}^{(1)} \) [114].

Note, however, that a true \( U(1) \) symmetry breaking is not possible in a single chain, where instead a critical Luttinger state with algebraically decaying nematic correlations is established [112]. To obtain a true 2D nematic phase, one needs to establish a phase coherence between the phases of order parameters \( Q_{-}(y) \) of different chains. By our superconducting analogy, this requires an inter-chain Josephson coupling. It transfers bound two-magnon pairs between nematic chains. The corresponding ‘pair-hopping’ reads \( K \sum_{x, y} (Q_{x}(x, y) Q_{-}(x, y + 1) + h.c.) \). Microscopically, such an interaction represents a four-spin coupling, which is not expected to be particularly large in a good Mott insulator with a large charge gap, such as LiCuVO\(_4\). However, even if \( K \) is absent microscopically, it will be generated perturbatively from the usual inter-chain spin exchange \( J' \sum_{x, y} (S_{x}^{2} S_{y}^{2} + h.c.) \), which plays the role of a single-particle tunneling process in the superconducting analogy. (Observe that expectation value of this interaction in the chain nematic ground state is zero—adding or removing of a single magnon to the ‘superconducting magnon’ ground state is forbidden at energies below the single magnon gap.) The pair-tunneling generated by fluctuations is estimated to be of the order \( K \sim (J')^{2}J_{1} \ll J' \).

At the same time, \( S' \rightarrow S \) interaction between chains does not suffer from a similar ‘low-energy suppression’. This is because \( S' \) is simply proportional to a number of magnon pairs, \( \eta = 2n(x, y) \), which is just twice the magnon number \( n(x, y) \). Hence \( S'(x, y) = 1/2 - 2n(x, y) \) differs only a by coefficient 2 from its usual expression in terms of magnon density.

We thus have a situation where the strength of interchain density–density \((S' \rightarrow S)\) interaction, which is determined by the original interchain \( J' \), is much stronger than that for the fluctuation-generated Josephson interaction \( K \sim (J')^{2}J_{1} \). In addition, more technical analysis of the scaling dimensions of the corresponding operators shows [92] that the two competing
interactions are characterized by (almost) the same scaling dimension (approximately equal to 1 for $h = h_{\text{sat}}$) which makes them both strongly relevant in the renormalization group sense. Given an inequality $J < (J')^2/J_1$, which selects inter-chain $S^-S^-$ interaction as the strongest one, we end up with a 2D collinear SDW phase built out of nematic spin chains \[92, 114, 117\]. This conclusion holds for all $h$ except for the immediate vicinity of the saturation field $h_{\text{sat}}^{(2)}$. There a separate fully 2D BEC analysis is required, due to the vanishing of spin velocity at the saturation field and the result is a true 2D nematic phase in the narrow field range $h_{\text{sat}}^{(1)} \lesssim h < h_{\text{sat}}^{(2)}$ \[92, 114, 117\]. A useful analogy to this competition is provided by models of striped superconductors, where the competition is between the superconducting order (a magnetic analogue of which is the spin nematic) and the charge-density wave order (a magnetic analogue of which is the collinear SDW), see \[118\] and references therein.

To summarize, weak inter-chain interaction $J'$ between $J_1 - J_2$ spin chains with strong nematic spin correlations actually stabilizes a 2D SDW phase as the ground state in a wide range of magnetization. This state preserves $U(1)$ symmetry of spin rotations and is characterized by short-ranged transverse spin correlations, similar to a nematic state.

### 4.2.2. Spin-current nematic state at the 1/3-magnetization plateau

The discussion in the previous subsection was focused on the systems with ferromagnetic exchange ($J_1 < 0$) on some of the bonds—as described there, negative exchange needed in order to obtain an attractive interaction between magnons.

Extensively used above superconducting ‘intuition’ forces one to ask, by analogy with superconducting states of repulsive fermion systems (such as, for example, pnictide superconductors or high-temperature cuprate ones), if it is possible to realize a spin-nematic in a spin system with only antiferromagnetic (that is, repulsive) exchange interactions between magnons. To the best of our knowledge, the first example of such a state is provided by the spin-current state described in part (e) of the section \(3.2\). Being nematic, this state is characterized by the chiral (spin current) long-range order and the absence of the magnetic long-range order in the transverse to the magnetic field direction \[61\], as sketched in figure 6.

A very similar state, named chiral Mott insulator, was recently discovered in the variational wave function study of a 2D system of interacting bosons on frustrated triangular lattice \[119\] as well as in a 2D system of bosons on frustrated ladder \[120, 121\]. In both cases, a chiral Mott insulator is an intermediate phase, which separates the usual Mott insulator state (which is a boson’s analogue of the UUD state) from the superfluid one (which is an analogue of the cone state). As in figure 5, it intervenes between the states with distinct broken symmetries ($Z_3$ and $U(1) \times Z_2$ in our case) and gives rise to two continuous transitions instead of a single discontinuous one.

### 4.3. Magnetization plateaus in itinerant electron systems

Up–up–down magnetization plateaus, found in the triangular geometry, are of classical nature. Over the years, several interesting suggestions of non-classical (liquid-like) magnetization plateaux have been put forward \[122–127\], but so far not observed in experiments or numerical simulations. Very recently, however, two numerical studies \[128, 129\] of the spin-1/2 Kagome antiferromagnet have observed non-classical magnetization plateaux at $M/M_{\text{sat}} = 1/3$, 5/9, 7/9. These intriguing findings, taken together with earlier prediction of a collinear spin liquid at $h = h_{\text{sat}}/3$ in the classical Kagome antiferromagnet \[130\], hint at a very rich magnetization process of the quantum model, the ground state of which at $h = 0$ is a $Z_2$ spin liquid! \[4\].

A different point of view on the magnetization plateau was presented recently in \[131\]. The authors asked if the plateau is possible in an itinerant system of weakly-interacting electrons. The answer to this question is affirmative, as can be understood from the following consideration.

Let us start with a system of non-interacting electrons on a triangular lattice. A magnetic field, applied in-plane in order to avoid complications due to orbital effects, produces magnetization $M = (n_1 - n_2)/2$, where densities $n_j$ of electrons with spin $\sigma = \uparrow, \downarrow$ are constrained by the total density $n = n_1 + n_2$. Consider now special situation with $n_1 = 3/4$, at which the Fermi-surface of $\sigma = \uparrow$ electrons, by virtue of lattice geometry, acquires particularly symmetric shape: a hexagon inscribed inside the Brillouin zone hexagon, see figure 10.

Points where $\sigma = \uparrow$ Fermi-surface touches the Brillouin zone (denoted by vectors $\pm \mathbf{Q}_j$ with $j = 1, 2, 3$ in figure 10) are the van Hove points, at which Fermi-velocity vanishes and electron dispersion becomes quadratic. They are characterized by the logarithmically divergent density of states. In addition, being a hexagon, $\sigma = \uparrow$ Fermi-surface is perfectly nested. As a result, static susceptibility $\chi(k)$ of spin-up electrons is strongly divergent, as $\log^2(|\mathbf{k} - \mathbf{Q}_j|)$, for wave vectors $\mathbf{k} = \mathbf{Q}_j$. 

![Figure 9](image-url)
Given this highly susceptible spin–↑ Fermi surface, it is not surprising that a weak interaction between electrons, either in the form of a direct density–density interaction $V_{nn'}\delta_{n+n'}$ between electrons or, e.g., nearest sites, or in the form of a local Hubbard interaction $U n_{r\uparrow} n_{r\downarrow}$ between the majority and minority particles, drives spin–↑ electrons into a gapped correlated state—the charge density wave (CDW) state [131] with fully gapped Fermi surface. Moreover, CDW ordering wave vectors $Q_j$ are commensurate with the lattice, leading to a commensurate CDW for the spin–↑ electrons [132].

Minority spin–↓ electrons experience position-dependent effective field $U(n_{r\downarrow})$ and form CDW as well. Depending on whether or not vectors $Q_j$ span the spin-down Fermi-surface (this depends on $n_1$ density), the Fermi-surface of $\sigma = \downarrow$ electrons may or may not experience reconstruction. However, being not nested, it is guaranteed to retain at least some parts of the critical Fermi-surface.

The resulting state is a co-existence of a charge- and collinear spin-density waves, together with critical $\sigma = \downarrow$ Fermi surface. Since the energy cost of promoting a spin–↓ electron to a spin–↑ state is finite (and given by the gap on the spin–↑ Fermi surface), the resulting state realizes fractional magnetization plateau, the magnetization of which is determined by the total density $n$ via $M = (3/2 – n)/2$. At half-filling, $n = 1$, the plateau is at 1/2 of the total magnetization, but for $n \neq 1$ it takes a fractional value. Amazingly, the obtained state is also a half-metal [133]—the only conducting band is that of (not gapped) minority spin–↓ electrons.

Theoretical analysis sketched here bears strong similarities with recent proposals [132, 134–137] of collinear and chiral spin-density wave (SDW) and superconducting states of itinerant electrons on a honeycomb lattice in the vicinity of electron filling factors 3/8 and 5/8 in zero magnetic field. Our analysis shows that even simple square lattice may host similar half-metallic magnetization plateau state [131]. Similar to the case of a magnetic insulator, described in the previous sections, external magnetic field sets the direction of the collinear CDW/SDW state. The resulting half-metallic state only breaks the discrete translational symmetry of the lattice, resulting in fully gapped excitations and remains stable to fluctuations of the order parameter about its mean-field value. In addition to standard solid state settings, the described phenomenon may also be observed in experiments on cold atoms, where desired high degree of polarization can be easily achieved [138]. It appears that, in addition to the half-metallic state, the system may also support p-wave superconductivity—a competition between these phases may be efficiently studied with the help of functional renormalization group [139].

5. Experiments

Much of the current theoretical interest in quantum antiferromagnetism comes from the amazing experimental progress in this area during the last decade. The number of interesting materials is too large to review here and for this reason we focus on a smaller sub-set of recently synthesized quantum spin–1/2 antiferromagnets, which realize some of quantum states discussed above.

One of the best known among this new generation of materials is Cs$_2$CuCl$_4$, extensively studied by Coldea and collaborators in a series of neutron scattering experiments [20, 140, 141] and by others via NMR [142] and, more recently, ESR [143–147] experiments. This spin–1/2 material represents a realization of a deformed triangular lattice with $J'/J = 0.34$ [20] and significant DM interactions on chain and inter-chain (zig–zag) bonds, connecting neighboring spins [20, 85]. Inelastic neutron scattering experiments have revealed unusually strong multi-particle continuum, the origin of which has sparked intense theoretical debate [148–155]. The current consensus is that Cs$_2$CuCl$_4$ is best understood as a weakly-ordered quasi-1D antiferromagnet, whose spin excitations smoothly interpolate from fractionalized spin–1/2 spinons of 1D chain at high- and intermediate energies to spin waves at lowest energy ($\ll J$) [155]. Although weak, residual inter-plane and DM interactions play the dominant role in the magnetization process of this material. The resulting $B – T$ phase diagram is rather complex and highly anisotropic [156] and does not contain a magnetization plateau. However it is worth mentioning that this was perhaps the first spin–1/2 material, a magnetic response of which featured a SDW-like phase ordering wave vector, which scales linearly with magnetic field in an about 1 tesla wide interval (denoted as phase ’S’ in [140] and as phase ’E’ in [156]). While still not well understood, this experimental observation have provided valuable hint to quasi-1d approach based on viewing Cs$_2$CuCl$_4$ as a collection of weakly coupled spin chains [155].

5.1. Magnetization plateau

Robust 1/3 magnetization plateau—the first of its kind among triangular spin–1/2 antiferromagnets—is present in Cs$_2$CuBr$_4$, which has the same crystal structure as Cs$_2$CuCl$_4$, but is
believed to be more 2D-like in a sense of having higher $J'/J$ ratio. What that ratio is however a subject of current debates. By fitting the experimentally determined ordering wave vector $q_0$ of the incommensurate helical state in zero magnetic field to the predictions of the classical spin model one obtains $J'/J = 0.47$ [157]. When fit to the predictions of the $S = 1/2$ series expansion study [64], the same $q_0$ results in a much bigger estimate $J'/J = 0.74$ [157]. High-temperature series fit of the uniform susceptibility results in $J'/J = 0.5$ [158]. Recent high-field ESR experiment, which probes magnetic excitation spectra of the magnons of the fully polarized state, gives $J'/J = 0.41$ [147]. Importantly, the same ESR experiment, when done on Cs$_2$CuCl$_4$, predicts $J'/J = 0.3$ [147], in close agreement with the estimate $J'/J = 0.34$, already quoted above and derived from earlier neutron scattering experiment [20]. It thus appears that $J'/J = 0.5$ should be accepted as an estimate of the exchange of ratio for Cs$_2$CuBr$_4$.

The observed plateau, which is about 1 tesla wide ($h_{c1} = 13.1$ T and $h_{c2} = 14.4$ T) [157, 159, 160], is clearly visible in both magnetization and elastic neutron scattering measurements [157, 160], which determined the UUD spin structure on the plateau. The observation of the magnetization plateau has generated a lot of experimental activity. The quantum origin of the plateau visibly manifests itself via essentially temperature-independent plateau’s critical fields $h_{c1,2}(T = h_{c1,2}(T = 0)$, as found in the thermodynamic study [161]. This behavior should be contrasted with the phase diagram of the spin-5/2 antiferromagnet RbFe(MoO$_4$)$_2$ [30, 162, 163], where the critical field $h_{c1}(T)$ does show strong downward shift with $T$. (Recall that in the classical model, figure 3, the UUD phase collapses to a single point at $T = 0$.)

Commensurate up–up–down spin structure of Cs$_2$CuBr$_4$ is also supported by NMR measurements [164, 165] which in addition find that transitions between the commensurate plateau and adjacent to it incommensurate phases are discontinuous (first-order). Extensive magnetocaloric effect and magnetic-torque experiments [166] have uncovered surprising cascade of field-induced phase transitions in the interval 10–30 T. The most striking feature of the emerging complex phase diagram, shown in figure 11, is that it appears to contain up to 9 different magnetic phases—in stark contrast with the ‘minimal’ theoretical model diagram in figure 3 which contains just 3 phases! This, as well as strong sensitivity of the magnetization curve to the direction of the external magnetic field with respect to crystal axis, strongly suggest that the difference in the phase diagrams has to do with spatial ($J \neq J$) and spin-space (asymmetric DM interaction) anisotropies present in Cs$_2$CuBr$_4$. Large-$S$ and classical Monte Carlo studies [31] do find the appearance of new incommensurate phases in the phase diagram, in qualitative agreement with the large-$S$ diagram of figure 5 (note that the latter does not account for the DM interaction which significantly complicates the overall picture [31]).

Perhaps the most puzzling of the ‘six additional’ phases is a narrow region at about $B = 23$ T, where $dM/dB$ exhibits sharp double peak structure, interpreted in [157, 167] as a novel magnetization plateau at $M/M_{sat} = 2/3$. Such a new $2/3$-magnetization plateau was observed in an exact diagonalization study of spatially anisotropic spin-1/2 model [168] but was not seen in more recent variational wave function [55] and DMRG [57], as well as in analytical large-$S$ [31, 60] studies.

Nearly isotropic, $J' = 1$, antiferromagnet Ba$_2$CoSb$_2$O$_8$ is believed to provide an ‘ideal’ realization of the spin-1/2 antiferromagnet on a uniform triangular lattice [21, 169]. And, indeed, its experimental phase diagram is in close correspondence with $J = J$ cut in figure 5 (along $\delta = 0$ line) and figure 7 (along $R = 0$ line): it has 120° spin structure at zero field, coplanar state at low fields, the 1/3 magnetization plateau in the $h_{c1}/h_{sat} = 0.3 \leq h_{c1}/h_{sat} \leq h_{c2}/h_{sat} = 0.47$ interval and coplanar state (denoted as 2:1 state in [169]) at higher fields.

Consider the lattice of the study [21, 169], the appearance of weak anomaly in $dM/dB$ at about $M/M_{sat} = 3/5$, which was interpreted as a quantum phase transition from the V phase to another coplanar phase—inverted V state (state ‘e’ in figure 2). Near the saturation field these two phases are very close in energy, the difference appears only in the 6th order in condensate amplitude [63]. Such a transition can be driven by sufficiently strong easy-plane anisotropy [170, 62] as well as by anisotropic DM interaction [31]. (Note, however, that recent DMRG study [171] claims that the inverted V state is a finite-size effect which is not present in the thermodynamic limit of the spin-1/2 problem.)

However, perhaps a more likely explanation is that the transition is due to the weak interlayer exchange interaction. Reference [172] has shown that weak inter-plane antiferromagnetic exchange interaction causes transition from the uniform V phase to the staggered V phase. The latter is described by the same equation (5) but with a $\phi$-dependent phase, $\phi \sim \phi + \pi z$ (here, $z$ is the integer coordinate of the triangular layer and $\phi$ is an overall constant phase), leading to the doubling of the period of the magnetic structure along the direction normal to the layer. It is easy to see that such a state actually gains energy from the antiferromagnetic interlayer exchange $J'$, while preserving the optimal in-plane configuration in every layer. Such a transition, denoted as HFC1–HFC2 transition, was also observed in recent semi-classical Monte Carlo simulations [173]. This development suggest that a mysterious ‘2/3-plaquette’ of Cs$_2$CuBr$_4$, mentioned above, may also be related to a transition between the lower-field uniform and higher-field staggered versions of the commensurate V phase.

### 5.2. SDW and spin nematic phases

An ‘Ising’ route to the collinear SDW order, described in section 4.1.1, has been realized in spin-1/2 Ising-like antiferromagnet BaCo$_2$V$_2$O$_8$. Experimental confirmations of this comes from specific heat [174] and neutron diffraction [175] measurements. The latter one is particularly important as it proves the linear scaling of the SDW ordering wave vector with the magnetization, $k_{sat} = \pi (1 - 2M)$, predicted in [176]. Subsequent NMR [177], ultrasound [178] and neutron scattering [179] experiments have refined the phase diagram and even proposed the existence of two different SDW phases [177] stabilized by competing interchain interactions.

Another Co-based quasi-1D material, Ca$_3$Co$_2$O$_6$, represents an outstanding puzzle. This strong $S = 2$ Ising
ferromagnet shows both a 1/3 magnetization plateau and an incommensurate longitudinal SDW ordering below it [180–183], which, however, do not seem to represent its true equilibrium low-temperature phases: the material exhibits strong history dependence [183] and associated with it extremely slow, on a scale of several hours, temporal evolution [184] towards an ultimate equilibrium ground state. This, in the case of no external magnetic field, appears to be a commensurate antiferromagnetic phase. References [183, 184] have argued that strong Ising character of spin-2 chains, as estimated by antiferromagnetic phase. References [183, 184] have argued that strong Ising character of spin-2 chains, as estimated by the large ratio of the easy-axis anisotropy $D = 110$ K to the in-chain ferromagnetic exchange constant $J_1 = 6$ K, is the reason for this extremely slow dynamics—very weak quantum fluctuations lead to a very small relaxation rate.

Most recently, spin-1/2 magnetic insulator LiCuVO$_3$ has emerged [185, 186] as a promising candidate to realize both a high-field spin nematic phase, right below the two-magnon saturation field, which is at about 45 tesla high and an incommensurate collinear SDW phase at lower fields, extending from about 40 T down to about 10 T. At yet lower magnetic field, the material realizes more conventional vector chiral (umbrella) state which can be stabilized by a moderate easy-plane anisotropy of exchange interactions [187] (which does not affect the high field physics discussed here).

This material seems to nicely realize theoretical scenario outlined in section 4.2.1: spin-nematic chains [188, 189] form a 2D nematic phase only in the immediate vicinity of the saturation field [190, 191]. (Apparently, more detailed analysis of this near-saturation region also requires disentangling contribution of the nonmagnetic vacancies from that of the ‘ideal’ spin system, which is not an easy task at all [191, 192].) At fields below that rather narrow interval, the ground state is an incommensurate longitudinal SDW state. Evidence for the latter includes detailed studies of NMR line shape [193–196] and neutron scattering [197, 198]. It is worth adding here that quasi-1D nature of this material is evident from the very pronounced multi-spinon continuum, observed at $h = 0$ in inelastic neutron scattering studies [199].

### 5.3. Weak Mott insulators: Hubbard model on anisotropic triangular lattice

Given that, quite generally, Heisenberg Hamiltonian can be viewed as a strong-coupling (large $U/t$) limit of the Hubbard model, it is natural to consider the fate of the Hubbard $t - t' - U$ model on (spatially anisotropic, in general) triangular lattice.

As a matter of fact, this very problem is of immediate relevance to intriguing experiments on organic Mott insulators of X[Pd(dmit)$_2$]$_2$ and κ-(ET)$_2$Z families. Recent experimental [8, 22] and theoretical [12, 200–202] reviews describe key relevant to these materials issues and we direct interested readers to these publications.

One of the main unresolved issues in this field is that of a proper minimal model that captures all relevant degrees of freedom. Highly successful initial proposal [205] models the system as a simple half-filled Hubbard model on a spatially anisotropic triangular lattice, as sketched in figure 12. Every ‘site’ of this lattice in fact represents closely bound dimer, made of two ET molecules [205] and occupied by single electron (or hole). Spatial anisotropy shows up via different hopping integrals $t$, $t'$. Within the standard large-$U$ description, where $J = 2t^2/U$ for the given bond, anisotropy of hopping $t/t'$ directly translates into that of exchange interactions on different bonds, $J'/J \sim (tt')^2$. Ironically, most of the studied materials fall onto $t' < t$ side [8, 203] of the diagram, which happens to be opposite to $J > J'$ limit of spatially anisotropic Heisenberg model, to which this review is devoted.

This description has generated a large number of interesting studies, the full list of which is beyond the scope of this review and most of which are again on the ‘wrong’ side of the $tt'$ ratio. One notable exception is provided by [84], which represents an extensive study of the $tt' - U$ phase diagram of the Hubbard model on spatially anisotropic triangular lattice, see figure 2 of that paper. Interestingly, this phase diagram contains an extended region of the large-$U$ spin-liquid phase (which occupies squarish region of $0 < tt' < 0.8$ and $U/t > 10$), which is separated from the metallic low-$U$ phase (which occupies a wedge of $0 < tt' < 0.8$ and $0 < U/t < 5$) by the extended region of magnetically ordered collinear phase. This unexpected phase appears to be similar to the CoAF state of the $J - U$ model described in section 3.3. Its static spin structure factor peaks at $k_s = \pi$ and is very much insensitive of the value of $k_s$, suggesting strong 1D character of the state. This is further confirmed by the analysis of renormalized ‘Fermi surfaces’ (see [84] for explanations) which also show very pronounced 1D feature of being very weakly modulated as a function of transverse component of the momentum $k_x$. Taken together with the fact that the spin liquid state is strongly 1D as well [84], these findings suggest that the whole region of $0 < tt' < 0.8$ and $U/t > 5$ is adequately described by the model of weakly coupled Hubbard chains—and that, similar to what happens in the spatially anisotropic triangular antiferromagnet with $J < J$, section 3.3, it is the frustrated (zig–zag) geometry of the interchain hopping $t$ which is responsible for the strong ‘dimensional reduction’ [12, 79], observed numerically in [84].

Let us now turn to a more experimentally relevant regime of $tt' > 1$. We start by focusing on two materials from...
Figure 12. (a) Spatially anisotropic single-band Hubbard model for organic Mott insulators, after [8, 201]. Note that assignment of the primed and non-primed bonds of the lattice, within the Hubbard model nomenclature which is used in this figure and the whole of section 5.3, is opposite to that in the Heisenberg model used in figure 1 and all other sections of the review. Thus, $t' t < 1$ here implies $J J < 1$ in figure 1: $J$ of figure 1 actually lives on $t$ bonds of the Hubbard model here. (b) Geometry of an extended two-band model, after [202]. The ‘sites’, shown by filled dots inside ovals representing dimers, are now two-molecule dimers. Two sites from the same dimer are connected by the intra-dimer tunneling amplitude $t_d$ and in the limit $t_d \gg \text{‘inter-dimer tunnelings’}$ the model reduces to that in (a). Note also the appearance of additional hopping amplitudes (dotted line) connecting ‘more distant’ sites of neighboring dimers.

$\kappa-(ET)_2Z$ family: $\kappa-(ET)_2Cu[N(CN)_2]Cl$ (abbreviated as $\kappa-$Cl in the following) and $\kappa-(ET)_2Cu_2(N(CN)_3)$ (abbreviated as $\kappa-$CN). At low temperature and ambient pressure, the first of these, $\kappa-$Cl, is an ordered antiferromagnetic insulator, while $\kappa-$CN is a Mott insulator showing no signs of magnetic order down to lowest temperature [7, 206]. Band structure calculations, based on extended Hückel method, have predicted [207] $t'/t = 0.75$ for $\kappa-$Cl and $t'/t = 1.06$ for $\kappa-$CN. Modern density functional calculations, however, predict [208, 209] quite different ratios: $t'/t = 0.4$ for $\kappa-$Cl and $t'/t = 0.83$ for $\kappa-$CN. Strong downward revision of the $t'/t$ ratio has equally strong affect on the ratio of the exchanges: $J'/J = 6$ for $\kappa-$Cl and $J'/J = 1.5$ for $\kappa-$CN, suggesting that both materials can be viewed as weakly distorted square lattice antiferromagnets (the square lattice is formed by $J'$ bonds). But this is cannot be the complete picture as we do know that $\kappa-$CN does not magnetically order. At the same time, high-temperature series expansion for the uniform susceptibility [158] is very much consistent with that of the uniform triangular lattice antiferromagnet with exchange constant of about 250 K—this however cannot be a full story either as the triangular lattice antiferromagnet has an ordered ground state as well!

Clearly, Heisenberg-type spin-only description of these interesting materials is not sufficient for explaining their puzzling magnetically-disordered Mott insulator behavior. It is important to realize that for not too large $U/t$, the standard Heisenberg model must be amended with ring-exchange terms involving four (or longer) loops of spin operators [210, 211]. This addition dramatically affects the regime of intermediate $U/t$ by stabilizing an insulating spin-liquid ground state [211, 212]. The nature of the emerging spin-liquid is subject of intense on-going investigations, with proposals ranging from $Z_2$ liquid [213, 214] to spin Bose-metal [215], to spin-liquid with quadratic band touching [216].

Recently, however, this appealing spin-only picture of the organic Mott insulators has been challenged by the experimental discovery of anomalous response of dielectric constant [217, 218] and lattice expansion coefficient [219] at low temperature. This finding imply that charge degrees of freedom, assumed frozen in the spin-only description, are actually present in the material and have to be accounted for in the theoretical modeling. Several subsequent papers [204, 220–222] have identified dimer units of the triangular lattice, viewed as sites in figure 12, as the most likely place where charge dynamics persists down to lowest temperatures. To describe these internal states of the two-molecule dimers, one need to go back to a two-band extended Hubbard model description [223]. (Note that magneto-electric coupling is in fact also present in a single-band Hubbard model, where it appears in certain frustrated geometries in the third order of $tU$ expansion [224].) Taking the strong-coupling limit of such a model, one derives [204] a coupled dynamics of interacting spins and electric dipoles on the triangular lattice. In turns out that sufficiently strong inter-dimer Coulomb interaction stabilizes charge-ordered state (dipolar solid) and suppresses spin ordering via non-trivial modification of exchange interactions $J, J'$. Clearly many more studies, both experimental and theoretical, are required in order to elucidate the physics behind apparent spin-liquid behavior of organic Mott insulators.

In place of a conclusion we just state the obvious: despite many years of investigations, quantum magnets on triangular lattices continue to surprise us. There are no doubts that future studies of new materials and models, inspired by them, will bring out new quantum states and phenomena.

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References

[1] Zhang Y, Grover T, Turner A, Oshikawa M and Vishwanath A 2012 Quasiparticle statistics and braiding from ground-state entanglement Phys. Rev. B 85 235151

[2] Jiang H-C, Wang Z and Balents L 2012 Identifying topological order by entanglement entropy Nat. Phys. 8 902–5
[3] Grover T, Zhang Y and Vishwanath A 2013 Entanglement entropy as a portal to the physics of quantum spin liquids New J. Phys. 15 025002

[4] Yan S, Huse D A and White S R 2011 Spin–liquid ground state of the s = 1/2 Kagome Heisenberg antiferromagnet Science 332 1173–6

[5] Jiang H-C, Yao H and Balents L 2012 Spin liquid ground state of the spin-1/2 square J1–J2 Heisenberg model Phys. Rev. B 86 024424

[6] Depenbrock S, McCulloch I P and Schollwöck U 2012 Nature of the spin–liquid ground state of the s = 1/2 Heisenberg model on the Kagome lattice Phys. Rev. Lett. 109 067201

[7] Shimizu Y, Miyagawa K, Kanoda K, Maesato M and Saito G 2003 Spin liquid state in an organic Mott insulator with a triangular lattice Phys. Rev. Lett. 91 107001

[8] Kanoda K and Kato R 2011 Mott physics in organic conductors with triangular lattices Ann. Rev. Condens. Matter Phys. 2 167–88

[9] Han T-H, Helton J S, Chu S, Nocera D G, Rodriguez-Rivera J A, Broholm C and Lee Y S 2012 Fractionalized excitations in the spin–liquid phase of a Kagome–lattice antiferromagnet Nature 492 406–10

[10] Collins M F and Petrenko O A 1997 Triangular antiferromagnets Can. J. Phys. 75 605–55

[11] Ramirez A P 2001 Handbook of Magnetic Materials vol 13 (New York: Elsevier) chapter 4 pp 423–520

[12] Balents L 2010 Spin liquids in frustrated magnets Phys. Rev. Lett. 105 2590–3

[13] Wannier G H 1950 Antiferromagnetism. The triangular Ising net Phys. Rev. 79 357–64

[14] Huse D A and Elser V 1988 Simple variational wave functions for 2D Heisenberg spin-1/2 antiferromagnets Phys. Rev. Lett. 60 2531–4

[15] Bernu B, Lhuillier C and Pierre L 1992 Signature of Néel order in exact spectra of quantum antiferromagnets on finite lattices Phys. Rev. Lett. 69 2590–3

[16] White S R and Chernyshev A L 2007 Neel order in square and triangular lattice Heisenberg models Phys. Rev. Lett. 99 127004

[17] Anderson P W 1973 Resonating valence bonds: a new kind of anisotropic Heisenberg model Phys. Rev. B 9 153–60

[18] Fazekas P and Anderson P W 1974 On the ground state properties of the anisotropic triangular quantum antiferromagnet Phil. Mag. 30 423–40

[19] Moessner R and Sandhii S L 2001 Resonating valence bond phase in the triangular lattice quantum dimer model Phys. Rev. Lett. 86 1881–4

[20] Coldea R, Tennant D A, Habicht K, Smeibidl P, Wolters C Fazekas P and Anderson P W 2011 On the ground state of the triangular lattice quantum antiferromagnet Cs2CuCl4 Phys. Rev. Lett. 107 137203

[21] Shirata Y, Tanaka H, Matsuo A and Kindo K 2012 Experimental realization of a spin-1/2 triangular-lattice Heisenberg antiferromagnet Phys. Rev. Lett. 108 057205

[22] Yamashita M, Shibouchi T and Matsuda Y 2012 Thermal-transport studies on two-dimensional quantum spin liquids ChemPhysChem 13 74–8

[23] Mendels P and Wills A S 2011 Kagome antiferromagnets: materials versus spin liquid behaviors Introduction to Frustrated Magnetism Lacroix C et al (Springer Series in Solid-State Sciences vol 164) (Berlin: Springer) pp 207–238

[24] Nussinov Z and van den Brink J 2015 Compass models: theory and physical motivations Rev. Mod. Phys. 87 1–59

[25] Witzczak-Krempa W, Chen G, Kim Y B and Balents L 2014 Correlated quantum phenomena in the strong spin–orbit regime Ann. Rev. Condens. Matter Phys. 5 57–82

[26] Kawamura H and Miyashita S 1985 Phase transition of the Heisenberg antiferromagnet on the triangular lattice in a magnetic field J. Phys. Soc. Japan 54 4530–8

[27] Chalker J T 2011 Geometrically frustrated antiferromagnets: statistical mechanics, dynamics Introduction to Frustrated Magnetism (Springer Series in Solid-State Sciences vol 164) Lacroix C et al (Berlin: Springer) pp 3–22

[28] Seabra L, Momoi T, Sindzinger P and Shannon N 2011 Phase diagram of the classical Heisenberg antiferromagnet on a triangular lattice in an applied magnetic field Phys. Rev. B 84 214418

[29] Gvozdikova M V, Melchy P-E and Zhitomirsky M E 2011 Magnetic phase diagrams of classical triangular and Kagome antiferromagnets J. Phys.: Condens. Matter 23 164209

[30] Smirnov A I, Yashiro H, Kimura M, Nishimoto Y, Kindo K, Kikkawa A, Katsumata K, Shapiro A Y and Demianets L N 2007 Triangular lattice antiferromagnet RbFe(MoO4)2 in high magnetic fields Phys. Rev. B 75 134412

[31] Griset C, Head S, Alicea J and Starykh O A 2011 Deformed triangular lattice antiferromagnets in a magnetic field: role of spatial anisotropy and Dzyaloshinskii-Moriya interactions Phys. Rev. B 84 245108

[32] Kawamura H, Yamamoto A and Okubo T 2010 z-vortex ordering of the triangular-lattice Heisenberg antiferromagnet J. Phys. Soc. Japan 79 023701

[33] Lee D H, Joannopoulos J D, Negele J W and Landau D P 1984 Discrete-symmetry breaking and novel critical phenomena in an antiferromagnetic planar (XY) model in two dimensions Phys. Rev. Lett. 52 433–6

[34] Korshunov S E 2006 Phase transitions in two-dimensional systems with continuous degeneracy Phys.—Usp. 49 225

[35] Hasenbusch M, Pelissetto A and Vicari E 2005 Multicritical behaviour in the fully frustrated xy model and related systems J. Stat. Mech. P12002

[36] Sorokin A O and Syromyatnikov A V 2012 Chiral spin liquid in two-dimensional xy helimagnets Phys. Rev. B 85 174404

[37] Chandra P, Coleman P and Larkin A I 1990 A quantum fluids approach to frustrated Heisenberg models J. Phys.: Condens. Matter 2 7933

[38] Chandra P, Coleman P and Larkin A I 1990 Ising transition in frustrated Heisenberg models Phys. Rev. Lett. 64 88–91

[39] Weber C, Capriotti L, Misguich G, Becca F, Elhajal M and Mila F 2005 Ising transition driven by frustration in a 2d classical model with continuous symmetry Phys. Rev. Lett. 91 177202

[40] Capriotti L and Sachdev S 2004 Low-temperature broken-symmetry phases of spiral antiferromagnets Phys. Rev. Lett. 93 257206

[41] Kawamura H and Miyashita S 1984 Phase transition of the two-dimensional Heisenberg antiferromagnet on the triangular lattice J. Phys. Soc. Japan 53 4138–54

[42] Wintel M, Everts H U and Apel W 1995 Monte Carlo simulation of the Heisenberg antiferromagnet on a triangular lattice: topological excitations Phys. Rev. B 52 13480–6

[43] Southern B W and Xu H-J 1995 Monte Carlo study of the Heisenberg antiferromagnet on the triangular lattice Phys. Rev. B 52 R3836–9

[44] Azaria P, Delamotte B and Mouhanna D 1992 Low-temperature properties of two-dimensional frustrated quantum antiferromagnets Phys. Rev. Lett. 68 1762–5

[45] Mouhanna D, Delamotte B, Kowacki J-P and Tissier M 2011 Non-perturbative renormalization group: basic principles and some applications Mod. Phys. Lett. B 25 873–89

[46] Hasselmann N and Sinner A 2014 Interplay of topology and geometry in frustrated two-dimensional Heisenberg magnets Phys. Rev. B 90 094404
Weihong Z, McKenzie R H and Singh R R P 1999 Phase
Nikuni T and Shiba H 1995 Hexagonal antiferromagnets in
Zheng W, Fjærestad J O, Singh R R P, McKenzie R H
and Coldea R 2006 Excitation spectra of the spin-1/2
triangular-lattice Heisenberg antiferromagnet Phys. Rev. B
74 224420

[69] Mezio A, Manuel L O, Singh R R P and Trumper A E 2012
Low temperature properties of the triangular-lattice
antiferromagnet: a bosonic spinon theory New J. Phys. 14 123033
[70] Toth S et al 2011 120° helical magnetic order in the distorted
triangular antiferromagnet α-CaCr₂O₄ Phys. Rev. B 84 054452
[71] Toth S, Lake B, Hradil K, Guidi T, Rule K C, Stone M B and
Islam A T M N 2012 Magnetic soft modes in the distorted
triangular antiferromagnet α-CaCr₂O₄ Phys. Rev. Lett. 109 127203
[72] Oh J, Le M D, Jeong J, Lee J-h, Woo H, Song W-Y, Perrig T G,
Buyers W J L, Cheong S-W and Park J-G 2013
Magnon breakdown in a two dimensional triangular lattice
Heisenberg antiferromagnet of multiferroic lumno₃ Phys. Rev. Lett. 111 257202
[73] Weichselbaum A and White S R 2011 Incommensurate
correlations in the anisotropic triangular Heisenberg lattice
Phys. Rev. B 84 245130
[74] Thesberg M and Sørensen E S 2014 Exact diagonalization
study of the anisotropic triangular lattice Heisenberg model using
twisted boundary conditions Phys. Rev. B 90 115117
[75] Heidarian D, Sorella S and Becca F 2009 Spin-1/2 Heisenberg
model on the anisotropic triangular lattice: from magnetism to
a one-dimensional spin liquid Phys. Rev. B 80 012404
[76] Weng M Q, Sheng D N, Weng Z Y and Bursill R J 2006
Spin-liquid phase in an anisotropic triangular-lattice Heisenberg model: exact diagonalization and density-matrix
renormalization group calculations Phys. Rev. B 74 012407
[77] Thesberg M and Sørensen E S 2014 A quantum fidelity study
of the anisotropic next-nearest-neighbour triangular lattice
Heisenberg model J. Phys.: Condens. Matter 26 425602
[78] Pardini T and Singh R R P 2008 Magnetic order in coupled
spin-half and spin-one Heisenberg chains in an anisotropic
triangular-lattice geometry Phys. Rev. B 77 214433
[79] Starykh O A and Balents L 2007 Ordering in spatially
anisotropic triangular antiferromagnets Phys. Rev. Lett. 98 77205
[80] Bishop R F, Li P H Y, Farnell D J J and Campbell C E 2009
Magnetic order in a spin-1/2 interpolating square-triangle
Heisenberg antiferromagnet Phys. Rev. B 79 174405
[81] Bishop R F, Li P H Y, Farnell D J J and Campbell C E 2010
Magnetic ordering of antiferromagnets on a spatially
anisotropic triangular lattice Int. J. Mod. Phys. B 24 5011–26
[82] Reuther J and Thomale R 2011 Functional renormalization
group for the anisotropic triangular antiferromagnet Phys. Rev. B 83 024402
[83] Ghamari S, Kallin C, Lee S-S and Sørensen E S 2011 Order in a
spatially anisotropic triangular antiferromagnet Phys. Rev. B 84 174415
[84] Tocchio L F, Gros C, Valentì R and Becca F 2014 One-
dimensional spin liquid, collinear, and spiral phases from
uncoupled chains to the triangular lattice Phys. Rev. B 89 235107
[85] Starykh O A, Katsura H and Balents L 2010 Extreme
sensitivity of a frustrated quantum magnet: Cs₅CuCl₃ Phys. Rev. B 82 014421
[86] Grüner G 1988 The dynamics of charge-density waves Rev.
Mod. Phys. 60 1129–81
[87] Grüner G 1994 The dynamics of spin-density waves Rev. Mod. Phys. 66 1–24
[88] Monceau P 2012 Electronic crystals: an experimental
overview Adv. Phys. 61 325–581
[89] Affleck I 1988 Quantum critical phenomena Fields, Strings, Critical Phenomena Brézin E and Zinn-Justin J (Amsterdam: North-Holland) pp 563–640
[90] Gogolin A O, Nersesyan A A and Tsvelik A M 2004 Strongly
Correlated Systems (Cambridge: Cambridge University Press)
[91] Starykh O A, Furusaki A and Balents L 2005 Anisotropic pyrochlores and the global phase diagram of the checkerboard antiferromagnet Phys. Rev. B 72 094416
[92] Starykh O A and Balents L 2014 Excitations and quasi-one-dimensionality in field-induced nematic and spin density wave states Phys. Rev. B 89 104407
[93] Affleck I and Wellman G F 1992 Longitudinal modes in quasi-one-dimensional antiferromagnets Phys. Rev. B 46 8934–53
[94] Schulz H J 1996 Dynamics of coupled quantum spin chains Phys. Rev. Lett. 77 2790–3
[95] Blanco J A, Fäk B, Jensen J, Rotter M, Hiess A, Schmitt D, Hikihara T and Momoi T 2010 Quantum spin nematics: magnetic field in one and two dimensions Phys. Rev. B 87 104411
[96] Okunishi K and Suzuki I 2007 Field-induced incommensurate order for the quasi-one-dimensional $x_\perp$ model in a magnetic field Phys. Rev. B 76 224411
[97] Affleck I 1990 Theory of Haldane-gap antiferromagnets in applied fields Phys. Rev. B 41 6697–702
[98] Sachdev S, Senthil T and Shankar R 1994 Finite-temperature properties of quantum antiferromagnets in a uniform magnetic field in one and two dimensions Phys. Rev. B 50 258–72
[99] Zhitomirsky M E and Tsunetsugu H 2010 Magnon pairing in quantum spin nematic Europhys. Lett. 92 37001
[100] Oshikawa M, Yamanaka M and Affleck I 2004 Haldane-gap chains in a magnetic field J. Stat. Mech. P12006
[101] Konik R M and Fendley P 2002 Haldane-gapped spin chains as luttinger liquids: correlation functions at finite field Phys. Rev. B 66 144416
[102] Fäth G 2003 Luttinger liquid behavior in spin chains with a magnetic field Phys. Rev. B 68 134445
[103] McCulloch I P, Kube R, Kurz M, Kleine A, Schollwöck U and Kolezhuk A K 2008 Vector chiral order in frustrated spin chains Phys. Rev. B 77 094404
[104] Oshikawa M, Yamanaka M and Affleck I 1997 Magnetization plateaus in spin chains: ‘Haldane gap’ for half-integer spins Phys. Rev. Lett. 78 1984–7
[105] Oshikawa M 2003 Insulator, conductor, and commensurability: a topological approach Phys. Rev. Lett. 90 236401
[106] Okunishi K and Tonegawa T 2003 Magnetic phase diagram of the $s=1/2$ antiferromagnetic zigzag spin chain in the strongly frustrated region: Cusp and plateau J. Phys. Soc. Japan 72 479–82
[107] Heidrich-Meisner F, Sergienko I A, Feiguin A E and Dagotto E 2007 Universal emergence of the one-third plateau in the magnetization process of frustrated quantum spin chains Phys. Rev. B 75 064413
[108] Hikihara T, Momoi T, Furusaki A and Kawamura H 2010 Magnetic phase diagram of the spin-1/2 antiferromagnetic zigzag ladder Phys. Rev. B 82 224433
[109] Andreev A F and Grishchuk I A 1984 Spin nematics J. Exp. Theor. Phys. 60 267
[110] Chandra P and Coleman P 1991 Quantum spin nematics: moment-free magnetism Phys. Rev. Lett. 66 100–3
[111] Chubukov A V 1991 Chiral, nematic, and dimer states in quantum spin chains Phys. Rev. B 44 4493–6
[112] Hikihara T, Kecke L, Momoi T and Furusaki A 2008 Vector chiral and multipolar orders in the spin-1/2 frustrated ferromagnetic chain in magnetic field Phys. Rev. B 78 144404
[113] Blume M and Hsieh Y Y 1969 Biquadratic exchange and quadrupolar ordering J. Appl. Phys. 40 1249
[114] Zhitomirsky M E and Tsunetsugu H 2010 Magnon pairing in quantum spin nematic Europhys. Lett. 92 37001
[115] Oganesyan V, Kivelson S A and Fradkin E 2001 Quantum theory of a nematic Fermi liquid Phys. Rev. B 64 195109
[116] Fernandes R M, Chubukov A V and Schmalian J 2014 What drives nematic order in iron-based superconductors? Nat. Phys. 10 97104
[117] Sato M, Hikihara T and Momoi T 2013 Spin-nematic and spin-density-wave orders in spatially anisotropic frustrated magnets in a magnetic field Phys. Rev. Lett. 110 077206
[118] Jaefari A, Lai S and Fradkin E 2010 Charge-density wave and superconductor competition in stripe phases of high-temperature superconductors Phys. Rev. B 82 144531
[119] Zaletel M P, Parameswaran S A, Rüegg A and Altman E 2014 Chiral bosonic nont magnon insulator on the frustrated triangular lattice Phys. Rev. B 89 155142
[120] Dhar A, Maji M, Mishra T, Pai R V, Mukerjee S and Paramekanti A 2012 Bose–Hubbard Model in strong effective magnetic field: emergence of a chiral Mott insulator ground state Phys. Rev. A 85 041602
[121] Dhar A, Mishra T, Maji M, Pai R V, Mukerjee S and Paramekanti A 2013 Chiral Mott insulator with staggered loop currents in the fully frustrated Bose–Hubbard model Phys. Rev. B 87 174401
[122] Misguich G, Jolicoeur T and Girvin S M 2001 Magnetization plateaus of SrCu$_2$(BO$_3$)$_2$ from a Chern–Simons theory Phys. Rev. Lett. 87 097203
[123] Hida K and Affleck I 2005 Quantum versus classical magnetization plateaus of $s=1/2$ frustrated Heisenberg chains J. Phys. Soc. Japan 74 1849–57
[124] Aliche J and Fisher M P A 2007 Critical spin liquid at 1/3 magnetization in a spin-1/2 triangular antiferromagnet Phys. Rev. B 75 144411
[125] Tanaka A, Totsuka K and Hu X 2009 Geometric phases and the magnetization process in quantum antiferromagnets Phys. Rev. B 79 064412
[126] Takagawa M and Mila F 2011 Magnetization plateaus Introduction to Frustrated Magnetism (Springer Series in Solid-State Sciences vol 164) Lacroix C et al (Berlin: Springer) pp 241–67
[127] Parameswaran S A, Kimchi I, Turner A M, Stamper-Kurn D M and Vishwanath A 2013 Wannier permanent wave functions for featureless bosonic Mott insulators on the 1/3-filled Kagome lattice Phys. Rev. Lett. 110 125301
[128] Nishimoto S, Shibata N and Hotta C 2013 Controlling frustrated liquids and solids with an applied field in a Kagome Heisenberg antiferromagnet Nat. Commun. 4 2287
[129] Capponi S, Derzhko O, Honecker A, Läuchli A M and Richter J 2013 Numerical study of magnetization plateaus in the spin-1/2 Kagome Heisenberg antiferromagnet Phys. Rev. B 88 144416
[130] Zhitomirsky M E 2002 Field-induced transitions in a Kagome antiferromagnet Phys. Rev. Lett. 88 057204
[131] Hao Z and Starykh O A 2013 Half-metallic magnetization plateaus Phys. Rev. B 87 161109
[132] Martin I and Batista C D 2008 Itinerant electron-driven chiral magnetic ordering and spontaneous quantum hall effect in triangular lattice models Phys. Rev. Lett. 101 156402
[133] Katsnelson M I, Irkhin V Y, Chioncel L, Lichtenstein A I and de Groot R A 2008 Hall-magnetic ferromagnets: from band structure to many-body effects Rev. Mod. Phys. 80 315–78
[134] Nandkishore R, Levitov L S and Chubukov A V 2012 Chiral superconductivity from repulsive interactions in doped graphene Nat. Phys. 8 158–63
[135] Nandkishore R, Chern G-W and Chubukov A V 2012 Itinerant half-metal spin-density-wave state on the hexagonal lattice Phys. Rev. Lett. 108 227204
[136] Kiesel M L, Platt C, Hanke W, Abanin D A and Thomale R 2012 Competing many-body instabilities and
unconventional superconductivity in graphene Phys. Rev. B 86 020507

[137] Chern G-W and Batista C D 2012 Spontaneous quantum hall effect via a thermally induced quadratic Fermi point Phys. Rev. Lett. 109 156801

[138] Zwierlein M W, Schirotzek A, Schunck C H and Ketterle W 2006 Fermionic superfluidity with imbalanced spin populations Science 311 492–6

[139] Platt C, Hanke W and Thomale R 2013 Functional renormalization group for multi-orbital Fermi surface instabilities Adv. Phys. 62 453–562

[140] Coldea R, Tennant D A, Tsvetlik A M and Tylnczynski Z 2001 Experimental realization of a 2d fractional quantum spin liquid Phys. Rev. Lett. 86 1335–8

[141] Coldea R, Tennant D A and Tylnczynski Z 2003 Extended scattering continua characteristic of spin fractionalization in the two-dimensional frustrated quantum magnet Cs2CuCl4 observed by neutron scattering Phys. Rev. B 68 134424

[142] Vachon M-A, Koutroulakis G, Mitrovi V F, Ma O, Marston J B, Reyes A P, Kuhns P, Coldea R and Tylnczynski Z 2011 The nature of the low-energy excitations in the short-range-ordered region of Cs2CuCl4 as revealed by 133 Cs nuclear magnetic resonance New J. Phys. 13 093029

[143] Povarov K Y, Smirnov A I, Starykh O A, Petrov S V and Shapiro A Y 2011 Modes of magnetic resonance in the spin-liquid phase of Cs2CuCl4 Phys. Rev. Lett. 107 037204

[144] Smirnov A I, Povarov K Y, Petrov S V and Shapiro A Y 2012 Magnetic resonance in the ordered phases of the two-dimensional frustrated quantum magnet Cs2CuCl4 Phys. Rev. B 85 184423

[145] Zvyagin S A et al 2013 Unconventional spin dynamics in the spin-1/2 triangular-lattice antiferromagnet Cs2CuBr4 arXiv:1306.3887

[146] Fayzullin M A, Eremina R M, Eremin M V, Dittl A, van Well N, Ritter F, Assmus W, Deisenhofer J, Krug von Nidda H-A and Loidl A 2013 Spin correlations and Dzyaloshinskii-Moriya interaction in Cs2CuCl4 Phys. Rev. B 88 174421

[147] Zvyagin S A et al 2014 Direct determination of exchange parameters in Cs2CuBr4 and Cs2CuCl4: high-field electron-spin-resonance studies Phys. Rev. Lett. 112 077206

[148] Bocquet M, Essler F H L, Tsvetlik A M and Gogolin A O 2001 Finite-temperature dynamical magnetic susceptibility of quasi-one-dimensional frustrated spin-1/2 Heisenberg antiferromagnets Phys. Rev. B 64 094425

[149] Alicea J, Motrunich O I and Fisher M P A 2005 Algebraic vortex liquid in spin-1/2 triangular antiferromagnets: scenario for Cs2CuCl4 Phys. Rev. Lett. 95 247203

[150] Isakov S V, Senthil T and Kim Y B 2005 Ordering in Cs2CuCl4: possibility of a proximate spin liquid Phys. Rev. B 72 174417

[151] Veillette M Y, James A J A and Essler F H L 2005 Spin dynamics of the quasi-two-dimensional spin-1/2 quantum magnet Cs2CuCl4 Phys. Rev. B 72 134429

[152] Veillette M Y, Chalker J T and Coldea R 2005 Ground states of a frustrated spin-1/2 antiferromagnet: Cs2CuCl4 in a magnetic field Phys. Rev. B 71 214426

[153] Zheng W, Figarestad J O, Singh R R P, McKenzie R H and Coldea R 2006 Anomalous excitation spectra of frustrated quantum antiferromagnets Phys. Rev. Lett. 96 057201

[154] Dalidovich D, Sknepnek R, John Berlinsky A, Zhang J and Kallin C 2006 Spin structure factor of the frustrated quantum magnet Cs2CuCl4 Phys. Rev. B 73 184403

[155] Kohno M, Starykh O A and Balents L 2007 Spinons and triplons in spatially anisotropic frustrated antiferromagnets Nat. Phys. 3 790–5

[156] Tokiwa Y, Radu T, Coldea R, Wilhelm H, Tylnczynski Z and Steglich F 2006 Magnetic phase transitions in the two-dimensional frustrated quantum antiferromagnet Cs2CuCl4 Phys. Rev. B 73 134414

[157] Ono T et al 2005 Field-induced phase transitions driven by quantum fluctuation in s = 1/2 anisotropic triangular antiferromagnet Cs2CuBr4 Prog. Theor. Phys. Suppl. 159 217–21

[158] Zheng W, Singh R R P, McKenzie R H and Coldea R 2005 Temperature dependence of the magnetic susceptibility for triangular-lattice antiferromagnets with spatially anisotropic exchange constants Phys. Rev. B 71 134422

[159] Ono T, Tanaka H, Aruga Katori H, Ishikawa F, Mitamura H and Goto T 2003 Magnetization plateau in the frustrated quantum spin system Cs2CuCl4 Phys. Rev. B 67 104431

[160] Ono T et al 2004 Magnetization plateaux of the s = 1/2 two-dimensional frustrated antiferromagnet Cs2CuBr4 J. Phys.: Condens. Matter 16 S773

[161] Tsujii H, Rotundu C R, Ono T, Tanaka H, Andraka B, Ingersent K and Takano Y 2007 Thermodynamics of the up–down phase of the s = 1/2 triangular-lattice antiferromagnet Cs2CuBr4 Phys. Rev. B 76 060406

[162] Sivost L E, Smirnov A I, Prozorova L A, Petenko O A, Micheler A, Büttgen N, Shapiro A Y and Demianets L N 2006 Magnetic phase diagram, critical behavior, and two-dimensional to three-dimensional crossover in the triangular lattice antiferromagnet RbFe(MoO4)2 Phys. Rev. B 74 024412

[163] White J S, Niedermyer C, Gasparovic G, Broholm C, Fujii Y, Hashimoto H, Yasuda Y, Kikuchi H, Chiba M, Matsubara S and Takigawa M 2007 Magnetic-field induced quantum phase transitions in spin-1/2 triangular antiferromagnet Cs2CuBr4 Phys. Rev. B 88 060409

[164] Fujiy Y, Nakamura T, Kikuchi H, Chiba M, Goto T, Matsubara S, Kodama K and Takigawa M 2004 [NMR] study of s = 12 quasi-two-dimensional antiferromagnet Cs2CuBr4: Phys. B: Condens. Matter 346–7 45–9 (Proc. of the 7th Int. Symp. on Research in High Magnetic Fields)

[165] Fujiy Y, Hashimoto H, Yasuda Y, Kikuchi H, Chiba M, Matsubara S and Takigawa M 2007 Commensurate and incommensurate phases of the distorted triangular antiferromagnet Cs2CuBr4 studied using 133 Cs nmr J. Phys.: Condens. Matter 19 145237

[166] Fortune N A, Hannaby S T, Yoshida Y, Sherline T E, Ono T, Tanaka H and Takano Y 2009 Cascade of magnetic-field-induced quantum phase transitions in a spin-1/2 triangular-lattice antiferromagnet Phys. Rev. Lett. 102 257201

[167] Ono T et al 2011 Magnetic-field induced quantum phase transitions in triangular-lattice antiferromagnets J. Phys.: Conf. Ser. 302 012003

[168] Miyahara S, Ogino K and Furukawa N 2006 Magnetization plateaux of Cs2CuBr4 Phys. B: Condens. Matter 378–80 587–8 (Proc. of the Int. Conf. on Strongly Correlated Electron Systems 2005)

[169] Susuki T, Kurita N, Tanaka T, Nojiri H, Matsuo A, Kindo K and Tanaka H 2013 Magnetization process and collective excitations in the s = 1/2 triangular-lattice Heisenberg antiferromagnet Ba4CoSb2O9 Phys. Rev. Lett. 110 267201

[170] Yamamoto D, Marmorini G and Dashti I 2014 Quantum phase diagram of the triangular-lattice XXZ model in a magnetic field Phys. Rev. Lett. 112 127203

[171] Sellmann D, Zhang X-F, and Eggert S 2015 The phase diagram of the antiferromagnetic XXZ model on the triangular lattice Phys. Rev. B 91 081104

[172] Gekht R S and Bondarenko I N 1997 Triangular antiferromagnets with a layered structure in a uniform field J. Exp. Theor. Phys. 83 345

[173] Koutroulakis G, Zhou T, Kiami Y, Thompson J D, Zhou H D, Batista C D and Brown S E 2015 Quantum phase
diagram of the $s = 1/2$ triangular-lattice antiferromagnet Ba$_2$Co$_3$Sb$_2$O$_9$. Phys. Rev. B 91 024410

[174] Kimura S, Takeuchi T, Okumishi K, Hagiwara M, He Z, Kindo K, Taniyama T and Itoh M 2008 Novel ordering of an $s = 1/2$ quasi-1d Ising-like antiferromagnet in magnetic field Phys. Rev. Lett. 100 057202

[175] Kimura S et al 2008 Longitudinal spin density wave order in a quasi-1d Ising-like quantum antiferromagnet Phys. Rev. Lett. 101 207201

[176] Suzuki T, Kawashima N and Okunishi K 2007 Exotic finite-temperature phase diagram for weakly coupled $s = 1/2$ xxz chain in a magnetic field J. Phys. Soc. Japan 76 123707

[177] Klanjesk M, Horvatic M, Berthier C, Mayaffre H, Canevet E, Grenier B, Lejay P and Orignac E 2012 Spin-chain system as a tunable simulator of frustrated planar magnetism arXiv:1202.6374

[178] Yamaguchi H et al 2011 Novel phase transition probed by sound velocity in quasi-one-dimensional Ising-like antiferromagnet BaCo$_2$V$_2$O$_7$. J. Phys. Soc. Japan 80 033701

[179] Canevet E, Grenier B, Klanjesk M, Berthier C, Horvatic M, Simonet V and Lejay P 2013 Field-induced magnetic behavior in quasi-one-dimensional Ising-like antiferromagnet BaCo$_2$V$_2$O$_7$: a single-crystal neutron diffraction study Phys. Rev. B 87 054408

[180] Haury V, Lees M R, Petrenko O A, McK D, Flahaut D, Hébert S and Maignan A 2014 Temperature and time dependence of the field-driven magnetization steps in Ca$_2$Co$_2$O$_7$ single crystals Phys. Rev. B 70 064424

[181] Takeshita S, Goko T, Araj I and Nishiyama K 2007 Magnetic phase diagram of frustrated triangular-lattice system J. Phys. Chem. Solids 68 2174–7 (QuB2S2006 (ICM2006 Satellite Conf.) Advances in Neutron, Synchrotron Radiation, SR and (NMR) Researches—Complementary Probes for Magnetism)

[182] Shimizu Y, Horibe M, Namba H, Takami T and Itoh M 2010 Anisotropic spin dynamics in the frustrated Ca$_2$Co$_2$O$_7$ detected by single-crystal $^{59}$Co nmr Phys. Rev. B 82 094430

[183] Paddison J A M, Agrestini S, Lees M R, Fleck C L, Deen P, Schneidewind A, Hiess A and Prokofiev A 2012 Evidence of a bond-nematic phase in LiCuVO$_4$. Phys. Rev. Lett. 109 027203

[184] Endler M, Fák B, Mikeska H-J, Kremer R K, Prokofiev A and Assmus W 2010 Two-spinon and four-spinon continuum in a frustrated ferromagnetic spin-1/2 chain Phys. Rev. Lett. 104 237207

[185] Lee P A, Nagaosa N and Wen X-G 2006 Doping a Mott insulator: physics of high-temperature superconductivity Rev. Mod. Phys. 78 17–85

[186] Sachdev S 2009 Exotic phases and quantum phase transitions: model systems and experiments arXiv:0901.4103

[187] Powell B J and McKenzie R H 2011 Quantum frustration in organic Mott insulators: from spin liquids to unconventional superconductors Rep. Prog. Phys. 74 056501

[188] Piraz L and Ogata M 2011 Metallic and superconducting materials with frustrated lattices Lacroix C et al Introduction to Frustrated Magnetism (Springer Series in Solid-State Sciences vol 164) (Berlin: Springer) pp 587–627

[189] Hotta C 2010 Quantum electric dipoles in spin-liquid dimer mott insulator $\kappa$-(bedt-ttf)$_2$Cu$_2$(CN)$_3$: Phys. Rev. B 82 241104

[190] Kino H and Fukuyama H 1995 Electronic states of conducting organic $\kappa$-(bedt-ttf)$_2$X J. Phys. Soc. Japan 64 2726–9

[191] Kurosaki Y, Shimizu Y, Miyagawa K, Kanoda K and Saito G 2005 Mott transition from a spin liquid to a Fermi liquid in the spin-frustrated organic conductor Phys. Rev. Lett. 95 177001

[192] Komatsu T, Matsukawa N, Inoue T and Saito G 1996 Realization of superconductivity at ambient pressure by band-filling control in (bedt-ttf)$_2$Cu$_2$(CN)$_3$. J. Phys. Soc. Japan 65 1340–54

[193] Kandpal H C, Ophale I, Zhang Y-Z, Jeschke H O and Valent R 2009 Revision of model parameters for kappa-type charge transfer salts: an ab initio study Phys. Rev. Lett. 103 067004

[194] Nakamura K, Yoshimoto Y, Kosugi T, Arita R and Imada M 2009 Ab initio derivation of low-energy model for kappa-type organic conductors J. Phys. Soc. Japan 78 083710
[210] Misguich G, Bernu B, Lhuillier C and Waldtmann C 1998 Spin liquid in the multiple-spin exchange model on the triangular lattice: $^3$He on graphite Phys. Rev. Lett. 81 1098–101

[211] Motrunich O I 2005 Variational study of triangular lattice spin-12 model with ring exchanges and spin liquid state in $\kappa$-(et)$_2$Cu$_2$(CN)$_3$ Phys. Rev. B 72 045105

[212] Yang H-Y, Läuchli A M, Mila F and Schmidt K P 2010 Effective spin model for the spin-liquid phase of the hubbard model on the triangular lattice Phys. Rev. Lett. 105 267204

[213] Xu C and Sachdev S 2009 Global phase diagrams of frustrated quantum antiferromagnets in two dimensions: doubled Chern–Simons theory Phys. Rev. B 79 064405

[214] Barkeshli M, Yao H and Kivelson S A 2013 Gapless spin liquids: stability and possible experimental relevance Phys. Rev. B 87 140402

[215] Sheng D N, Motrunich O I and Fisher M P A 2009 Spin Bose-metal phase in a spin-1/2 model with ring exchange on a two-leg triangular strip Phys. Rev. B 79 205112

[216] Mishmash R V, Garrison J R, Bieri S and Xu C 2013 Theory of a competitive spin liquid state for weak Mott insulators on the triangular lattice Phys. Rev. Lett. 111 157203

[217] Abdel-Jawad M, Terasaki I, Sasaki T, Yoneyama N, Kobayashi N, Uesu Y and Hotta C 2010 Anomalous dielectric response in the dimer Mott insulator $\kappa$-(bedtt-tf)$_2$Cu$_2$(CN)$_3$ Phys. Rev. B 82 125119

[218] Poirier M, Parent S, Côté A, Miyagawa K, Kanoda K and Shimizu Y 2012 Magnetodielectric effects and spin-charge coupling in the spin-liquid candidate $\kappa$-(bedtt-tf)$_2$Cu$_2$(CN)$_3$ Phys. Rev. B 85 134444

[219] Manna R S, de Souza M, Brühl A, Schluefer J A and Lang M 2010 Lattice effects and entropy release at the low-temperature phase transition in the spin-liquid candidate $\kappa$-(bedtt-tf)$_2$Cu$_2$(CN)$_3$ Phys. Rev. Lett. 104 016403

[220] Naka M and Ishihara S 2010 Electronic ferroelectricity in a dimer Mott insulator J. Phys. Soc. Japan 79 063707

[221] Dayal S, Clay R T, Li H and Mazumdar S 2011 Paired electron crystal: order from frustration in the quarter-filled band Phys. Rev. B 83 245106

[222] Gomes N, Clay R T and Mazumdar S 2013 Absence of superconductivity and valence bond order in the Hubbard–Heisenberg model for organic charge-transfer solids J. Phys.: Condens. Matter 25 385603

[223] Seo H, Hotta C and Fukuyama H 2004 Toward systematic understanding of diversity of electronic properties in low-dimensional molecular solids Chem. Rev. 104 5005–36

[224] Bulaevskii L N, Batista C D, Mostovoy M V and Khomskii D I 2008 Electronic orbital currents and polarization in Mott insulators Phys. Rev. B 78 024402