Model-based design of riblets for turbulent drag reduction

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Both experiments and direct numerical simulations have been used to demonstrate that riblets can reduce turbulent drag by as much as 10%, but their systematic design remains an open challenge. In this paper, we develop a model-based framework to quantify the effect of streamwise-aligned spanwise-periodic riblets on kinetic energy and skin-friction drag in turbulent channel flow. We model the effect of riblets as a volume penalization in the Navier-Stokes equations and use the statistical response of the eddy-viscosity enhanced linearized equations to quantify the effect of background turbulence on the mean velocity and skin-friction drag. For triangular riblets, our simulation-free approach reliably predicts drag-reducing trends as well as mechanisms that lead to performance deterioration for large riblets. We investigate the effect of height and spacing on drag reduction and demonstrate a correlation between energy suppression and drag-reduction for appropriately sized riblets. Our framework lays the ground for the optimal design of periodic surfaces for turbulent drag reduction using models of low complexity.

Key words: Drag reduction, turbulence control, turbulence modeling

1. Introduction

Surface roughness typically increases skin-friction drag and degrades performance of engineering systems that involve the motion of rigid bodies in turbulent flows, e.g., ships and submarines with biofouled hulls (Schultz et al. 2011). Using both experiments and numerical simulations, Yusim & Utama (2017) reported an increase in skin-friction drag by about 41% per year because of marine fouling growth. In contrast, carefully designed surface corrugations can reduce skin-friction drag by as much as 10% (Bechert et al. 1997; Gad-el Hak 2000). Patterned surface modifications have been used to reduce drag in a number of engineering applications (Coustols & Savill 1989). Success stories include the 2% drag reduction by spanwise-periodic riblets in commercial aircrafts (Szodruch 1991), and the 7% drag reduction by shark-skin-inspired design of swimsuits for olympic swimmers (Benjamuvatra et al. 2002; Mollendorf et al. 2004).

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1.1. Previous studies of drag reduction by riblets

Given the potential economic benefits of riblets, many experimental and numerical studies have been dedicated to examining the dependence of skin-friction drag on design parameters \cite{Bechert2000,Bechert1997,Choi1993,Garcia-Mayoral2011,Walsh1982,Walsh1984}. These efforts provided a broad range of guidelines for characterizing the drag-reducing trends of riblets based on their size and shape (blade-like, triangular, T-shaped, etc.). In particular, turbulent drag-reduction as a function of various metrics of size (e.g., rib spacing or groove area) appears to follow a consistent trend over a host of riblet shapes, e.g., see \cite{Bechert1997} (figure 15). For example, for small riblets (i.e., in the so-called “viscous” regime), the drag reduction is proportional to the riblet size. This linear trend gradually saturates at an optimal size before eventually degrading and leading to a drag increase for large riblets. Furthermore, for various riblet shapes, the optimal riblet spacing (in inner units) satisfies $s^+ \in [10, 20]$ \cite{Bechert1997,Garcia-Mayoral2011}. \cite{Garcia-Mayoral2011} discovered that the cross-sectional area $A_g$ of the grooves provides the best predictor of drag-reducing trends over various shapes and identified $l_g^+ = \sqrt{A_g^+} \approx 10.7 \pm 1$ as the optimal size of riblets.

Beyond parametric studies, a considerable effort was made to uncover mechanisms responsible for drag reduction. The presence of small-size riblets results in the suppression of the cross-flows introduced by near-wall turbulence, which weakens the near-wall quasi-streamwise vortices and pushes them away from the wall. This limits the transfer of mean momentum toward the wall and creates a zone of suppressed turbulence within the grooves, thereby reducing skin-friction drag \cite{Choi1993, LeeLee2001, Sirovich1997}. Various notions of protrusion height have been proposed to quantify the effect of riblets on near-wall turbulence. \cite{Bechert1989} defined the protrusion height as the offset between the virtual origin for the mean flow and a measure of the average wall location. In contrast, \cite{Luchini1991} proposed to use the difference between the virtual origin for the streamwise and spanwise flows. For blade-like and scalloped riblets, the latter approach provides a good indicator of the shift in mean velocity and it offers a better surrogate for predicting drag reduction in the viscous regime. \cite{Garcia-Mayoral2019, Ibrahim2019} proposed to quantify the shift in turbulence arising from quasi-streamwise vortices as a function of the wall-normal and spanwise slip lengths. This method can be used to predict the shift in the mean velocity, which is typically difficult to quantify in flows over complex surfaces. On the other hand, by examining 2D/3D roughness, \cite{Orlandi2006} demonstrated a linear relation between the roughness function, i.e., shift of mean velocity in the logarithmic region and the rms of wall-normal velocity at the tip of roughness elements.

Both experiments and simulations have been used to demonstrate that the drag-reducing performance of riblets eventually saturates and degrades with increase in their size. \cite{Goldstein1998} suggested that the creation of small secondary streamwise vortices around the tips of riblets by the unsteady crossflow degrades performance. \cite{Choi1993, LeeLee2001, SuzukiKasagi1994} related this phenomenon to the lodging of streamwise vortices into the grooves, which breaks down the viscous regime near the wall. More recently, the numerical study of \cite{Garcia-Mayoral2011} suggested that the breakdown of the viscous regime is accompanied by the emergence of spanwise rollers of typical streamwise length $\lambda_r^+ \sim 150$ that develop from a two-dimensional Kelvin-Helmholtz (K-H) instability. The emergence of these coherent flow
structures was also connected to an increase in the Reynolds shear stress in the vicinity of the corrugated surface.

While these studies offer valuable insights into drag reduction mechanisms, their reliance on costly experiments and simulations has hindered the model-based design of riblet-mounted surfaces. This motivates the development of low-complexity models that capture the essential physics of turbulent flows over riblets and are well-suited for analysis, design, and optimization. Previously proposed notions of protrusion height (Bechert & Bartenwerfer 1989), spanwise slip length (García-Mayoral et al. 2019), Ibrahim et al. (2019), and the roughness function (Orlandi & Leonard 2006) provide surrogate measures for the performance of specific riblet geometries, but are typically constrained to the viscous regime. The self-regular model for wall turbulence regeneration proposed by Bandyopadhyay & Hellum (2014) accounts for the spatio-temporal evolution of flow structures over patterned surfaces and matches experimental results for transitional and turbulent flows at low Reynolds numbers. More recently, the receptivity of channel flow over riblets was studied using the $H_2$ norm of the linearized dynamics (Kasliwal, Duncan & Papachristodoulou 2012) and the resolvent analysis (Chavarin & Luhar 2019). While the former study used a change of coordinates to translate spatially-periodic geometry into spatially-periodic differential operators, the latter utilized a volume penalization technique to represent the effect of riblets as a feedback term in the dynamics. Moreover, Chavarin & Luhar (2019) showed that the dependence of the resolvent gain on the spacing of riblets closely follows previously reported drag reducing trends in turbulent flows.

While prior model-based efforts can be used to predict the energetics of turbulent flows in the presence of riblets, they fail to account for the interactions among harmonics of flow fluctuations that are induced by spatially-periodic geometry. Furthermore, in the absence of a systematic framework to account for the influence of background turbulence on the mean velocity, such studies cannot provide accurate predictions of skin-friction drag in the presence of riblets. In this paper, we account for dynamical interactions and utilize turbulence modeling in conjunction with the eddy-viscosity-enhanced linearized NS equations to quantify the effect of background turbulence on skin-friction drag in turbulent channel flow over riblets.

1.2. Preview of modeling framework and main results

The linearized NS equations have been used to capture structural and statistical features of transitional (Bamieh & Dahleh 2001, Butler & Farrell 1992, Farrell & Ioannou 1993, Jovanović 2004, Jovanović & Bamieh 2005, Ran et al. 2019, Trefethen et al. 1993) and turbulent (Hwang & Cossu 2010, McKeon & Sharma 2010, Zare, Georgiou & Jovanović 2020, Zare, Jovanović & Georgiou 2017b) wall-bounded shear flows. In these studies, the effect of disturbances was modeled as an additive source of deterministic or stochastic excitation in the NS equations. This approach was also used for model-based design of sensor-free control strategies for suppressing turbulence via streamwise traveling waves (Lieu, Moarref & Jovanović 2010, Moarref & Jovanović 2010) and transverse wall oscillations (Jovanović 2008, Moarref & Jovanović 2012). To capture the influence of background turbulence, Moarref & Jovanović (2012) developed a framework to determine the turbulent viscosity of channel flow in the presence of control from the statistics of the eddy-viscosity enhanced linearized NS equations. This study showed that, by accounting for the influence of fluctuation dynamics on the turbulence model, reliable predictions of the mean velocity and the skin-friction drag can be obtained.

In this paper, we extend the framework developed in Moarref & Jovanović (2012) to quantify the effect of riblets on a turbulent channel flow. Following Chavarin & Luhar
we use a volume penalization technique \cite{Khadra2000} to approximate the effect of spatially-periodic surface on turbulent flow. This method introduces a static feedback term that captures the shape of riblets via a resistive function in the momentum equation. Additionally, we augment kinematic viscosity with turbulent eddy-viscosity and examine the dynamics of flow fluctuations around the steady-state solution of the modified governing equations. The spatially-periodic nature of the mean flow introduces interaction between different harmonics of the mean and fluctuating velocity fields, which complicates frequency response analysis relative to the flow over smooth walls. We utilize the second-order statistics of velocity fluctuations to determine the turbulent viscosity for the flow over riblets and compute their effect on the skin-friction drag.

We use our simulation-free approach to examine the effect of triangular riblets in turbulent channel flow. For various shapes and sizes of riblets, our results are in excellent agreement with experimental and numerical studies \cite{Bechert1997, Garcia-Mayoral2011}. We also study the kinetic energy of velocity fluctuations and observe a strong correlation between energy suppression and drag-reduction trends for certain sizes of riblets. In addition, we use our model to examine dominant flow structures and mechanisms for drag reduction. The close agreement between our predictions and prior experimental and DNS results demonstrates that our model-based approach can be used for systematic design of periodic drag-reducing surfaces.

1.3. Paper outline

The rest of our presentation is organized as follows. In §2 we formulate the problem and provide an overview of the volume penalization technique that is used to account for the presence of riblets. In §3 we utilize the linearized eddy-viscosity-enhanced NS equations to study the dynamics of velocity fluctuations around the turbulent base flow induced by riblets. The second-order statistics computed from this linearized model are used to modify turbulent viscosity and refine predictions of the mean velocity and skin-friction drag in turbulent channel flow over riblets. In §4 we demonstrate the merits of our framework and its ability to capture the drag-reducing trends of triangular riblets. In §5 we show that our framework uncovers mechanisms for drag reduction and, in §6, we provide summary of our results and outlook for future research directions.

2. Problem formulation

The pressure-driven channel flow of incompressible Newtonian fluid, with geometry shown in figure 1(a) is governed by the Navier-Stokes and continuity equations

\[
\begin{align*}
\partial_t \mathbf{u} &= - (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla P + \frac{1}{Re_\tau} \Delta \mathbf{u}, \\
0 &= \nabla \cdot \mathbf{u},
\end{align*}
\]

where \( \mathbf{u} \) is the velocity vector, \( P \) is the pressure, \( \nabla \) is the gradient operator, \( \Delta = \nabla \cdot \nabla \) is the Laplacian, \( (x, y, z) \) are the streamwise, wall-normal, and spanwise directions, and \( t \) is time. The friction Reynolds number \( Re_\tau = u_\tau \delta / \nu \) is defined in terms of the channel’s half-height \( \delta \) and the friction velocity \( u_\tau = \sqrt{\tau_w / \rho} \), where \( \tau_w \) is the wall-shear stress (averaged over horizontal directions and time), \( \rho \) is the fluid density, and \( \nu \) is the kinematic viscosity. In (2.1) and throughout this paper, spatial coordinates are nondimensionalized by \( \delta \), velocity by \( u_\tau \), time by \( \delta / u_\tau \), and pressure by \( \rho u_\tau^2 \). We also assume that the bulk flux, which is obtained by integrating the streamwise velocity over spatial dimensions and time, remains constant via adjustment of the uniform streamwise pressure gradient \( \partial_x P \).

When the lower channel wall is corrugated with a spanwise-periodic surface \( r(z) \) that is aligned with the flow, as shown in Fig. 1(b) boundary conditions on \( \mathbf{u} \) are given by
the no-slip and no penetration conditions,
\[ u(x, y = 1, z, t) = u(x, y = -1 + r(z), z, t) = 0. \] (2.2)

Solving the NS equations (2.1) subject to these boundary conditions requires a stretched mesh that conforms to the geometry dictated by \( r(z) \). This approach is computationally inefficient because it requires large number of discretization points to resolve the grid in the vicinity of the wall. This motivates the development of low-complexity models for analysis, optimization, and design. The key challenge is to capture the effect of riblets on the turbulent flow so that skin-friction drag is accurately predicted.

As skin-friction drag depends on the gradient of the turbulent mean velocity at the wall, a natural first step is to determine an approximation to the mean velocity in the presence of riblets. To this end, we adopt the Reynolds decomposition to split the velocity and pressure fields into their time-averaged mean and fluctuating parts as

\[
\begin{align*}
\mathbf{u} &= \bar{\mathbf{u}} + \mathbf{v}, \quad \langle \mathbf{u} \rangle = \bar{\mathbf{u}}, \quad \langle \mathbf{v} \rangle = 0, \\
P &= \bar{P} + p, \quad \langle P \rangle = \bar{P}, \quad \langle p \rangle = 0.
\end{align*}
\] (2.3)

Here, \( \bar{\mathbf{u}} = [U \ V \ W]^T \) is the vector of mean velocity components, \( \mathbf{v} = [u \ v \ w]^T \) is the vector of velocity fluctuations, \( p \) is the fluctuating pressure field around the mean \( \bar{P} \), and \( \langle \cdot \rangle \) denotes the expected value,

\[
\langle \mathbf{u}(x, y, z, t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \mathbf{u}(x, y, z, t + \tau) \, d\tau.
\] (2.4)

Substituting the decomposition (2.3) into the NS equations (2.1) and taking the expectation yields the Reynolds-averaged NS equations

\[
\begin{align*}
\partial_t \bar{\mathbf{u}} &= - (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} - \nabla \bar{P} + \frac{1}{Re} \Delta \bar{\mathbf{u}} - \nabla \cdot \langle \mathbf{vv}^T \rangle, \\
0 &= \nabla \cdot \bar{\mathbf{u}}.
\end{align*}
\] (2.5)

The Reynolds stress tensor \( \langle \mathbf{vv}^T \rangle \) quantifies the transport of momentum arising from turbulent fluctuations \( \text{McComb}\, [1991] \), and its value significantly affects the solution of equations (2.5). The difficulty in obtaining the fluctuation correlations stems from closure problem. We overcome this challenge by utilizing the turbulent viscosity hypothesis \( \text{McComb}\, [1991] \), which considers the turbulent momentum to be transported in the direction of the mean rate of strain

\[
\langle \mathbf{vv}^T \rangle - \frac{1}{3} \text{trace} \langle \mathbf{vv}^T \rangle I = - \frac{\nu_T}{Re} \left( \nabla \bar{\mathbf{u}} + (\nabla \bar{\mathbf{u}})^T \right),
\] (2.6)

where \( \nu_T(y) \) is the turbulent eddy viscosity normalized by kinetic energy, overline denotes averaging over homogenous dimensions \( (x \text{ and } z) \), and \( I \) is the identity operator. As we discuss what follows, our choice of turbulence model is motivated by \( \text{Moarrel & Jovanovic}\).
2.1. Modeling surface corrugation

To account for the effect of riblets we use the volume penalization technique proposed by Khadra et al. (2000). In this method, the solid obstruction of the flow is modeled as a spatially-varying permeability function $K$ that enters the governing equations through an additive body forcing term. This modulation brings the mean flow equations (2.5) into the following form,

$$
\begin{align*}
\partial_t \bar{u} &= - (\bar{u} \cdot \nabla) \bar{u} - \nabla P - K^{-1} \bar{u} + \frac{1}{Re_T} \nabla \cdot \left( (1 + \nu_T)(\nabla \bar{u} + (\nabla \bar{u})^T) \right), \\
0 &= \nabla \cdot \bar{u}.
\end{align*}
$$

The permeability function $K$ takes on two values: within the fluid, $K \rightarrow \infty$ yields back the original mean flow equations (2.5); and within the riblets, $K \rightarrow 0$ forces the velocity field to zero. Following Chavarin & Luhar (2019), we account for streamwise-constant, spanwise-periodic corrugation by considering the harmonic resistance

$$
K^{-1}(y, z) = \sum_{m \in \mathbb{Z}} a_m(y) \exp(i m \omega_z z). \tag{2.8}
$$

Here, $\omega_z$ is the fundamental spatial frequency of the riblets and the Fourier series coefficients $a_m(y)$ of $K^{-1}(y, z)$, which specify riblets’ height, are parameterized by the wall-normal coordinate. The base flow, i.e., the solution to the steady-state mean flow equations (2.7), can be also decomposed into the Fourier series, i.e.,

$$
\bar{u}(y, z) = \sum_{m \in \mathbb{Z}} \bar{u}_m(y) \exp(i m \omega_z z). \tag{2.9}
$$

The steady-state solution to the nonlinear mean flow equations (2.7) is obtained via Newton’s method and it only contains a streamwise velocity component, $\bar{u} = [U(y, z) \ 0 \ 0]^T$. Since the spanwise and wall-normal base flow components are zero, the nonlinear terms in mean flow equation (2.7) vanish and the equations for $U(y, z)$ is linear.

Ideally, at any spanwise location $z$, coefficients $a_m(y)$ in (2.8) should emulate a wall-normal step function at the interface of the solid surface and the fluid; see figure 2. However, in favor of wall-normal differentiability, we use the hyperbolic approximation

$$
K^{-1}(y, z) = \frac{R}{2} \left( 1 - \tanh(s_f(y + 1 - r(z))) \right). \tag{2.10}
$$

where $-1 + r(z)$ indicates the location of the lower corrugated wall (cf. Eq. (2.2)), $s_f$ is a smoothness factor that modifies the slope of the hyperbolic curve, and $R$ is a resistance rate that controls the accuracy of the solution in the solid region. While larger values of $s_f$ yield a better approximation of the step function, they require the use of a larger number of harmonics to maintain the smoothness of the resistance field. Herein, we choose $s_f$ to be inversely proportional to the height of the riblets $h$. On the other hand, while large values of the resistance rate $R$ induce a smaller velocity field within the riblets, they may trigger spurious negative solutions. In view of this fundamental trade-off, we relax the non-negativity constraint on $\bar{u}$ and choose $R$ to guarantee that the solution to (2.7) is larger than $-1 \times 10^{-5}$. In particular, for turbulent channel flow with $Re_T = 186$ over the triangular lower-wall riblets with frequency $\omega_z = 30$ and height to spacing ratio $h/s = 0.38$, our computational experiments show that $R = 1.5 \times 10^5$, $s_f = 3.6\pi/h$, and 25 spanwise harmonics ($m = -12, \ldots, 12$) yield small negative mean velocity while
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Figure 2. The wall-normal dependence of the resistance function $K^{-1}$ at the tip of a triangular riblet. The dashed curve results from equation (2.10) and it represents a smooth hyperbolic approximation to the step function (solid line). Here, $h = 0.075$, $R = 1.5 \times 10^5$, $s_f = 150$, $r_p = 0.73$, and the function $r(z)$ represents triangular riblets.

Figure 3. (a) Resistance function $K^{-1}$ given by equation (2.10) with $h = 0.075$, $R = 1.5 \times 10^5$, $s_f = 150$, $\omega_z = 30$, and $r_p = 0.73$. (b) The streamwise mean velocity for turbulent channel flow with $Re_\tau = 186$ over the triangular riblets shown in (a).

preserving the smoothness of the resistance field. For triangular riblets with $\omega_z = 30$ and height 0.075, figure 3(a) shows the resistance field $K^{-1}$ resulting from Eq. (2.10) with,

$$r(z) = -h r_p + \frac{h \omega_z}{\pi} \left| z - \frac{2\pi}{\omega_z} \left( 1 + \left\lfloor \frac{z \omega_z}{2\pi} - \frac{1}{2} \right\rfloor \right) \right|. \quad (2.11)$$

Here, $| \cdot |$ is the absolute value, $\lfloor \cdot \rfloor$ is the floor function, and $r_p$ denotes the proportion of the riblet height in the extended channel, i.e., below $y = -1$. In this study, we tune $r_p$, and thereby adjust the wall-normal position of riblets, so that the mean velocity profile resulting from (2.7) has the same bulk, $U_B$, as the channel flow with smooth walls, i.e.,

$$\frac{\omega_z}{2\pi} \int_{-1-hr_p}^{1-hr_p} U(y, z) \, dz \, dy = \int_{-1}^{1} U_0(y) \, dy = U_B. \quad (2.12)$$

Here, $U_0(y)$ is the turbulent mean velocity profile in the absence of riblets, and the mean velocity of the flow over spanwise-periodic surface corrugation, $U(y, z)$, is averaged over one period.

2.2. Prediction of drag reduction using turbulent eddy-viscosity $\nu_T$

We approach the problem of quantifying the influence of riblets on skin-friction drag by developing robust models that approximate the turbulent viscosity $\nu_T$ in equations (2.7). Several studies have offered expressions for $\nu_T$ that yield the turbulent mean velocity...
The following turbulent viscosity model for channel flow was developed by Reynolds & Tiederman (1967) as an extension of the model introduced by Cess (1958) for pipe flow:

$$\nu_{T0}(y) = \frac{1}{2} \left( \left( 1 + \left( \frac{c_2}{3} Re_r (1 - y^2)(1 + 2y^2)(1 - e^{-(1-|y|)Re_r/c_1}) \right)^2 \right)^{1/2} - 1 \right).$$

(2.13)

In this expression, parameters $c_1$ and $c_2$ are selected to minimize the least squares deviation between the mean streamwise velocity obtained in experiments and simulations and the steady-state solution to Eq. (2.7) without riblets using the averaged wall-shear stress $\tau_w = 1$ and $\nu_T$ given by Eq. (2.13). For turbulent channel flow with $Re_r = 186$, the optimal parameters are given by $c_1 = 46.2$ and $c_2 = 0.61$ (Del Álamo & Jiménez 2003). Even though the turbulent viscosity model given by Eq. (2.13) no longer holds in the presence of riblets, we use $\nu_{T0}$ for channel flow over smooth walls as a starting point for determining the mean flow in the presence of riblets. Furthermore, in the vicinity of the solid wall the flow is dominated by viscosity and, for small-size riblets, the flow in the grooved region can be assumed to be laminar. Thus, we consider small-size riblets and set $\nu_T = 0$ for $y \leq -1$.

The rate of drag reduction caused by riblets is given by

$$\Delta D := \frac{(D - D_0)}{D_0},$$

where $D_0$ denotes the slope of the mean velocity at the lower wall in a flow without riblets. In the presence of riblets, the skin-friction drag at the lower wall can be computed using the pressure gradient, which maintains a constant bulk, and the slope of the mean velocity at the upper wall using

$$D = \bar{P}_x - \frac{\omega_z}{2\pi} \int_0^{2\pi/\omega_z} \frac{\partial U}{\partial y} (y = 1, z) \, dz.$$

Since $\bar{P}_{x,0} = 2D_0$, $\Delta D$ can be determined from the difference between the pressure gradient adjustment and the drag reduction at the upper wall, i.e.,

$$\Delta D = \frac{1}{D_0} \left[ \bar{P}_{x,c} - \left( \frac{\omega_z}{2\pi} \int_0^{2\pi/\omega_z} \frac{\partial U}{\partial y} (y = 1, z) \, dz - D_0 \right) \right],$$

(2.14)

where $\bar{P}_{x,c}$ denotes the change in pressure gradient, which can be computed using the procedure outlined in Appendix C.

A pseudospectral scheme with Chebyshev polynomials (Weideman & Reddy 2000) is used to discretize the differential operators in the wall-normal dimension. To avoid numerical oscillations in the solution to equations (2.7), we divide the wall-normal extent of the computational domain into two parts using block operators (Aurentz & Trefethen 2017) and discretize the wall-normal dimension using $N_i = 179$ collocation points for $y \in [-1, 1]$ and $N_o = 20$ collocation points for $y \in [-1 - r_p h, 1]$. We impose no-slip boundary conditions (2.2) on the upper wall of the channel. The adopted volume penalization method automatically enforces immersed boundary conditions on the non-smooth lower wall without the need for additional boundary conditions. The boundary conditions at the intersection of the aforementioned wall-normal regimes ($y = -1$) enforce smoothness
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Figure 4. Prediction of drag reduction in a turbulent channel flow with \( \text{Re}_\tau = 186 \) resulting from the steady-state solution of equations (2.7) with \( \nu_T(y) \) given by (2.13). Triangular riblets, shown in figure 6, with different peak-to-peak spacing but a constant height to spacing ratio \( h/s = 0.38 \) are considered and the spacing is reported in inner viscous units, i.e., \( s^+ = \text{Re}_\tau s \).

on physical quantities, i.e.,

\[
\bar{u}(y = -1^+, z) = \bar{u}(y = -1^-, z) \]

\[
\frac{\partial \bar{u}}{\partial y}(y = -1^+, z) = \frac{\partial \bar{u}}{\partial y}(y = -1^-, z).
\]

For a turbulent channel flow with \( \text{Re}_\tau = 186 \) subject to a streamwise pressure gradient \( \bar{P}_x = -1 \) over triangular riblets with \( \omega_z = 30 \), \( h = 0.075 \), \( R = 1.5 \times 10^5 \), \( s_f = 150 \), and \( r_p = 0.73 \), figure 3(b) shows the steady-state solution \( \bar{u} \) to Eq. (2.7) with \( m = -12, \ldots, 12 \); see Eq. (2.9). Even though the mean velocity profile respects the shape of riblets and goes to zero within the solid region, the resulting drag reduction does not follow the trends reported in literature. As demonstrated in figure 4, the mean velocity profile resulting from the use of \( \nu_{T0} \) implies that the drag reduction is obtained regardless of the size of riblets and that there is no optimal spacing that maximizes drag reduction.

In §3, we extend the framework proposed in Moarref & Jovanović (2012) to account for the effect of velocity fluctuations in a flow over riblets on the turbulent viscosity \( \nu_T \) in order to improve predictions of the mean velocity and the resulting skin-friction drag.

3. Stochastically-forced dynamics of velocity fluctuations

In this section, we compute a correction to the turbulent viscosity and, subsequently, the mean velocity of a turbulent channel flow over riblets using second-order statistics of velocity fluctuations. To this end, we examine the dynamics of fluctuations around the mean velocity profile computed in §2.2. Our model-based framework for studying the effect of riblets involves the following steps; see figure 5 for an illustration.

(i) §2.2 The turbulent mean velocity \( \bar{u} \) is obtained from Eqs. (2.7), where closure is achieved using the turbulent viscosity \( \nu_{T0} \) for the channel flow with smooth walls.

(ii) §3.4 The stochastically forced linearized NS equations around the mean flow \( \bar{u} \) resulting from step (i) are used to compute the second-order statistics of the fluctuating velocity field and provide a correction to \( \nu_{T0} \).

(iii) [Appendix C] The modification to turbulent viscosity is used to correct the mean velocity and compute skin-friction drag.
Even though our discussion focuses on spanwise-periodic triangular riblets, the methodology and theoretical framework that we develop can be used to study turbulent flows over much broader class of periodic surface corrugations.

### 3.1. Model equation for $\nu_T$

As described in §2.2, $\nu_{T0}$ does not provide the proper eddy-viscosity model for the channel flow over corrugated walls. Establishing a relation between $\nu_T$ and the second-order statistics of velocity fluctuations represents the main challenge for identifying the appropriate eddy-viscosity model. With appropriate choices of velocity and length scales, turbulent viscosity can be expressed as (Pope 2000)

$$\nu_T(y) = cRe^2 \frac{k^2(y)}{\epsilon(y)}$$

where $c = 0.09$, $k$ is the turbulent kinetic energy, and $\epsilon$ is the rate of dissipation. The $k$-$\epsilon$ model (Jones & Launder 1972; Launder & Sharma 1974) provides two differential transport equations for $k$ and $\epsilon$, but is computationally demanding and does not offer insight into analysis, design, and optimization. On the other hand, wall-normal profiles for $k$ and $\epsilon$ can be obtained by averaging the second-order statistics of velocity fluctuations over the streamwise coordinate and one period of the spanwise surface corrugation:

$$k(y) = \frac{1}{2} \left( \overline{uu} + \overline{vv} + \overline{ww} \right)$$

$$\epsilon(y) = 2 \left( \overline{u_xu_x} + \overline{v_yv_y} + \overline{w_zw_z} + \overline{u_yv_x} + \overline{u_zw_x} + \overline{v_zw_y} \right) + \overline{u_yw_y} + \overline{w_yw_y} + \overline{v_xv_x} + \overline{w_zw_x} + \overline{u_zu_z} + \overline{v_zv_z}.$$

We next demonstrate how second-order statistics, e.g., $uu$, can be computed using the stochastically forced linearized NS equations.
3.2. Stochastically forced linearized Navier-Stokes equations

The dynamics of small velocity \( \mathbf{v} = [u \ v \ w]^T \) and pressure \( p \) fluctuations around \( \bar{u} = [U(y, z) \ 0 \ 0]^T \) and \( \bar{p} \) are governed by the linearized NS and continuity equations:

\[
\begin{align*}
\partial_t \mathbf{v} &= - (\nabla \cdot \bar{u}) \mathbf{v} - (\nabla \cdot \mathbf{v}) \bar{u} - \nabla p - K^{-1} \mathbf{v} + \frac{1}{Re} \nabla \cdot ((1 + \nu_T) (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)) + \mathbf{f}, \\
0 &= \nabla \cdot \mathbf{v},
\end{align*}
\]

(3.2)

where \( \mathbf{f} \) is a zero-mean white-in-time additive stochastic forcing. The normal modes in \( x \) are given by \( e^{ik_x x} \), where \( k_x \) is the streamwise wavenumber, and the normal modes in \( z \) are given by the Bloch waves \( \text{[Bensoussan et al. 1978] [Odeh & Keller 1964]} \), which are determined by the product of \( e^{i\theta} \) with \( \theta \in [0, \omega_z/2) \) and a \( 2\pi/\omega_z \) periodic function in \( z \). For example, the forcing field in (3.2) can be represented as

\[
\begin{align*}
\mathbf{f}(x, y, z, t) &= e^{ik_x x} e^{i\theta} \hat{f}(k_x, y, z, t) \\
\hat{f}(k_x, y, z, t) &= \hat{f}(k_x, y, z + 2\pi/\omega_z, t)
\end{align*}
\]

(3.3)

where real parts are used to represent physical quantities. The Fourier series expansion of the \( 2\pi/\omega_z \) periodic function \( \hat{f}(k_x, y, z, t) \) can be used to obtain,

\[
\mathbf{f}(x, y, z, t) = \sum_{n \in \mathbb{Z}} \hat{f}_n(k_x, y, \theta, t) e^{i(k_x x + \theta n z)}, \quad \theta_n = \theta + n\omega_z, \\
k_x \in \mathbb{R}, \quad \theta \in [0, \omega_z/2)
\]

(3.4)

Substituting (3.4) into the linearized equations (3.2) and eliminating pressure through a standard conversion \( \text{[Schmid & Henningson 2001]} \) yields the evolution form

\[
\begin{align*}
\partial_t \varphi_\theta(k_x, y, t) &= [A_\theta(k_x) \varphi_\theta(k_x, \cdot, t)](y) + d_\theta(k_x, y, t), \\
v_\theta(k_x, y, t) &= [C_\theta(k_x) \varphi_\theta(k_x, \cdot, t)](y)
\end{align*}
\]

(3.5)

with the state \( \varphi_\theta \) consisting of the wall-normal velocity \( u \) and vorticity \( \eta = \partial_x u - \partial_y w \). The state-space representation (3.5) is parameterized by the streamwise wavenumber \( k_x \) and the spanwise wavenumber offset \( \theta \): for each \( k_x \) and \( \theta \), \( \varphi_\theta, v_\theta \), and \( d_\theta := B_\theta f_\theta \) are \( \infty \times \infty \) column vectors, e.g., \( \varphi_\theta(k_x, y, t) = \text{col}\{\hat{\varphi}_n(k_x, y, \theta, t)\}_{n \in \mathbb{Z}} \), and \( A_\theta(k_x), B_\theta(k_x), \) and \( C_\theta(k_x) \) are \( \infty \times \infty \) matrices whose elements are operators in \( y \), e.g.,

\[
A_\theta := \begin{bmatrix}
\ddots & \cdots & \cdots & \cdots \\
\ddots & A_{n-1,0} & A_{n-1,1} & A_{n-1,2} \\
\ddots & A_{n,-1} & A_{n,0} & A_{n,1} \\
\ddots & A_{n+1,-2} & A_{n+1,-1} & A_{n+1,0} \\
& \ddots & \cdots & \cdots & \ddots
\end{bmatrix}
\]

(3.6)

where the off-diagonal term \( A_{n,m} \) denotes the influence of the \( (n+m) \)th harmonic \( \hat{\varphi}_{n+m} \) on the dynamics of the \( n \)th harmonic \( \hat{\varphi}_n \); see Appendix [A] for details. At the upper-wall of the channel, homogenous Dirichlet boundary conditions are imposed on \( \eta \), and homogeneous Dirichlet and Neumann boundary conditions are imposed on \( v \). Similar to the mean flow equations (2.7), the boundary conditions at the corrugated surface are automatically satisfied via volume penalization. Finally, smoothness of all physical quantities at the intersection of the inner and outer wall-normal regimes \( (y = -1) \) is
imposed by enforcing the following conditions:

\[
\begin{align*}
    v(y = -1^+, z) &= v(y = -1^-, z), \\
    \frac{\partial^2 v}{\partial y^2}(y = -1^+, z) &= \frac{\partial^2 v}{\partial y^2}(y = -1^-, z), \\
    \eta(y = -1^+, z) &= \eta(y = -1^-, z), \\
    \frac{\partial \eta}{\partial y}(y = -1^+, z) &= \frac{\partial \eta}{\partial y}(y = -1^-, z).
\end{align*}
\]

A pseudospectral scheme used for discretizing the mean flow equations (2.7) is utilized to discretize the wall-normal operators in (3.5). In addition, a change of variables is employed to obtain a state-space representation in which the kinetic energy is determined by the Euclidean norm of the state vector in a finite-dimensional approximation of the evolution model (Zare et al. 2017a, Appendix A),

\[
\begin{align*}
    \dot{\psi}_\theta(k_x, t) &= A_\theta(k_x) \psi_\theta(k_x, t) + d_\theta(k_x, t), \\
    \nu_\theta(k_x, t) &= C_\theta(k_x) \psi_\theta(k_x, t).
\end{align*}
\] (3.7)

For \(N_i\) and \(N_o\) collocation points in the inner and outer wall-normal regimes, respectively, and a Fourier series expansion \(2.8\) with \(M\) harmonics, \(\psi_\theta(k_x, t)\) and \(\nu_\theta(k_x, t)\) are vectors with \(2 \times M \times (N_i + N_o)\) and \(3 \times M \times (N_i + N_o)\) complex-valued entries, respectively. The state-space matrices \(A_\theta(k_x)\) and \(C_\theta(k_x)\) are discretized versions of the operators in (3.5) that incorporate the aforementioned change of coordinates.

### 3.3. Second-order statistics of velocity fluctuations and forcing

Let the linearized dynamics (3.7) be driven by zero-mean white-in-time stochastic forcing \(d_\theta(k_x, t)\) with the covariance matrix \(M_\theta(k_x) = M_\theta^*(k_x) \succeq 0\), i.e.,

\[
    \langle d_\theta(k_x, t_1) d_\theta^*(k_x, t_2) \rangle = M_\theta(k_x) \delta(t_1 - t_2), \quad (3.8)
\]

where \(\delta\) is the Dirac delta function. Following Moarref & Jovanović (2012), we select \(M_\theta(k_x)\) to guarantee equivalence between the two-dimensional energy spectrum of the turbulent channel flow with smooth walls and the flow governed by the stochastically forced NS equations linearized around \(\bar{u} = [U_0(y) 0 0]^T\). This is achieved via

\[
    M_\theta(k_x) = \frac{E_\theta(k_x)}{E_{\theta 0}(k_x)} M_{\theta 0}(k_x),
\]

where \(\bar{E}_\theta(k_x) = \int_{-1}^{1} E_\theta(y, k_x) \, dy\) is the two-dimensional energy spectrum of turbulent channel flow with smooth walls, which is obtained from the DNS-based energy spectrum \(E_\theta(y, k_x)\) (Del Álamo & Jiménez 2003; Del Álamo et al. 2004), and \(\bar{E}_{\theta 0}(k_x)\) is the energy spectrum resulting from the linearized NS equations subject to white-in-time stochastic forcing with the covariance matrix

\[
    M_{\theta 0}(k_x) = \begin{bmatrix}
    \sqrt{E_\theta(y, k_x)} I & 0 \\
    0 & \sqrt{E_\theta(y, k_x)} I
    \end{bmatrix} \begin{bmatrix}
    \sqrt{E_\theta(y, k_x)} I & 0 \\
    0 & \sqrt{E_\theta(y, k_x)} I
    \end{bmatrix}^*.
\]

The steady-state covariance of the state \(\psi_\theta\) in equations (3.7) can be determined from the solution \(X_\theta(k_x)\) to the Lyapunov equation (Fardad et al. 2008; Moarref & Jovanović 2010)

\[
    A_\theta(k_x) X_\theta(k_x) + X_\theta(k_x) A^*_\theta(k_x) = -M_\theta(k_x),
\] (3.9)

where the \((i, j)\)th block of \(X_\theta(k_x)\) determines the correlation matrix associated with the \(i\)th and \(j\)th harmonics of the state \(\psi_\theta\).
For the linearized NS equations, the covariance blocks $X_d(k_x, \theta_n)$ on the main diagonal of $X_\theta(k_x)$ can be decomposed as

$$X_d(k_x, \theta_n) = X_0(k_x, \theta_n) + X_c(k_x, \theta_n), \quad (3.10)$$

where $X_0(k_x, \theta_n)$ and $X_c(k_x, \theta_n)$ represent the steady-state covariance matrix in a flow over the smooth walls and the modification resulting from the inclusion of riblets, respectively. Finally, the energy spectrum of velocity fluctuations is determined by,

$$E(k_x, \theta) = \text{trace}(X_\theta(k_x)) = \sum_{n \in \mathbb{Z}} \text{trace}(X_d(k_x, \theta_n)).$$

3.4. Correction to turbulent viscosity

The turbulent viscosity $\nu_T(y)$ is determined by the second-order statistics of velocity fluctuations, i.e., the kinetic energy $k(y)$ and its rate of dissipation $\epsilon(y)$; see equation (3.11). The statistics can be computed using the covariance matrix $X_d(k_x, \theta_n)$ in equation (3.10) and $k(y)$, $\epsilon(y)$ can be decomposed as

$$k(y) = k_0(y) + k_c(y), \quad \epsilon(y) = \epsilon_0(y) + \epsilon_c(y), \quad (3.11)$$

where the subscript 0 signifies channel flow with smooth walls, and the subscript c quantifies the influence of fluctuations in the flow over riblets. The DNS results for turbulent channel flow yield $k_0$ and $\epsilon_0$ is computed using $\epsilon_0(y) = cRe^2_\tau k_0^2(y)/\nu_\tau(y)$. On the other hand, the corrections $k_c$ and $\epsilon_c$ can be determined from the second-order statistics in $X_c(k_x, \theta_n)$; see Appendix B for details. Substitution of $k(y)$ and $\epsilon(y)$ from (3.11) into equation (3.1) and application of the Neumann series expansion yields

$$\nu_T(y) = \nu_{T0}(y) + \nu_{Tc}(y), \quad (3.12)$$

where the correction $\nu_{Tc}(y)$ to turbulent viscosity $\nu_{T0}(y)$ is given by

$$\nu_{Tc}(y) = \nu_{T0}(y) \left( \left( 1 + \frac{k_c(y)}{k_0(y)} \right)^2 \left( 1 - \frac{\epsilon_c(y)}{\epsilon_0(y)} + \left( \frac{\epsilon_c(y)}{\epsilon_0(y)} \right)^2 \right) - 1 \right), \quad (3.13)$$

The influence of fluctuations on the turbulent mean velocity and, consequently, skin-friction drag can be evaluated by substituting $\nu_T(y)$ from (3.12) and solving equations (2.7); see Appendix C for details regarding the correction to the mean flow profile.

4. Turbulent drag reduction and energy suppression

In this section, we use the framework developed in §3 to examine the effect of triangular riblets shown in Figure 6 on the mean velocity, skin-friction drag, and kinetic energy in turbulent channel flow with $Re_\tau \approx 186$. We assume that the influence of small-size riblets on the channel height and shear velocity is negligible, thereby implying that the Reynolds number remains unchanged over various case studies. By letting the ratio between the height and spacing of riblets be fixed, the riblets of different sizes are obtained by modifying the frequency $\omega_z$; see Table 1 for a list of cases considered in our study. In the absence of riblets, DNS results (Del Álamo & Jiménez 2003, Del Álamo et al. 2004) provide second-order statistics which are used to determine the covariance of stochastic forcing $d_\theta(k_x, t)$ in equation (3.8) and to compute the kinetic energy $k_0(y)$; see §3.3.

We use a total of 199 Chebyshev collocation points to discretize the operators in the wall-normal direction ($N_t = 179$, $N_o = 20$). Furthermore, we parameterize the linearized equations (3.7) using 48 logarithmically spaced streamwise wavenumbers with $0.03 < k_x < 40$ and utilize 25 harmonics of $\omega_z$ ($n = -12, \ldots, 12$) with 50 equally spaced offset
Figure 6. Triangular riblets with height $h$, spacing $s = 2\pi/\omega_z$, and tip angle $\alpha$.

Table 1. Triangular riblets with different height to spacing ratios ($h/s$), tip angles $\alpha$, and spanwise frequencies $\omega_z$ that we examine in our study.

| $h/s$ | $\alpha$  | $\omega_z$ |
|-------|-----------|------------|
| 0.38  | 105°      | 30, 35, 40, 45, 50, 60, 80, 100, 160 |
| 0.5   | 90°       | 30, 35, 40, 45, 50, 60, 80, 100, 160 |
| 0.65  | 75°       | 35, 40, 45, 50, 60, 80, 100, 160 |
| 0.87  | 60°       | 50, 60, 70, 80, 100, 120, 160 |
| 1.2   | 45°       | 60, 80, 100, 120, 160, 210 |

Figure 7. Turbulent drag reduction for triangular riblets with different tip angles $\alpha$ as a function of (a) spacing $s^+$; and (b) height $h^+$ in a channel flow with $Re_\tau = 186$ and $\alpha = 105^\circ$ (▽); 90$^\circ$ (△); 75$^\circ$ (○); 60$^\circ$ (▽); and 45$^\circ$ (×).

4.1. Drag reduction

We first examine the effect of riblet size on turbulent drag. In our parametric study, we follow García-Mayoral & Jiménez (2011) and refer to the regime of vanishing riblet spacing, in which the drag reduction is proportional to the size of riblets, as the viscous regime. For a turbulent channel flow with $Re_\tau = 186$ subject to triangular riblets on the lower wall, figure 7 shows the influence of the height and peak-to-peak spacing of riblets on $\Delta D$ in Eq. (2.14). In this figure, the height and spacing are reported in inner viscous units, i.e. $h^+ = Re_\tau h$ and $s^+ = Re_\tau s$, and various curves represent different tip angles $\alpha$ as a measure of riblet geometry. As shown in figure 6, a particular tip angle $\alpha$ corresponds to a specific height to spacing ratio. Clearly, small-size riblets can indeed reduce skin-friction drag. Figure 7(a) demonstrates that, for $s^+ < 20$, the drag reduction first increases as $h^+$ increases, saturates, and then decreases. This trend, however, slows down for smaller values of $s^+$. For $\alpha = 90^\circ$ and $\alpha = 60^\circ$, the drag reduction trends...

points $\theta \in [0, \omega_z/2]$ to parameterize $\theta_n = \theta + n\omega_z$. Finally, to capture the triangular shape of riblets via (2.8), we use 25 harmonics in $z$ ($m = -12, \ldots, 12$).
and optimal $s^+$ values resulting from our method reliably capture the trends reported in experimental studies (Bechert et al. 1997). On the other hand, as shown in figure 7(b) for a fixed height, as the spacing $s^+$ of riblets decreases, drag reduction increases, saturates, and then decreases. Furthermore, figures 7(a) and 7(b) show that as the riblet tip angle $\alpha$ decreases, maximum drag reduction is achieved for less separated and taller riblets, respectively. Finally, as $\alpha$ decreases, the maximum value of drag reduction first increases and then decreases, which is also in agreement with experimental observations (Bechert et al. 1997). The trends predicted by our framework indicate an optimal height to spacing ratio of $h/s \approx 0.65$ ($\alpha = 75^\circ$) for triangular riblets, which over-predicts the previously reported optimal tip angle $\alpha = 54^\circ$ (Dean & Bhushan 2010).

As demonstrated in figure 7, the optimal height and spacing can be quite different for riblets of different shape (i.e., different values of $\alpha$), thereby indicating that the height and spacing may not be suitable metrics for characterizing the breakdown of the linear viscous regime. Instead, the groove cross-section area $l_g^+ := \sqrt{A^+}$ (for triangular riblets, $A^+ = h^+s^+/2$) provides the best collapse of the critical breakdown dimension across different riblet shapes (García-Mayoral & Jiménez 2011). Furthermore, to remove the effect of riblets’ shape on their slope in the viscous regime $m_l := \lim_{l_g^+ \to 0} \Delta D/l_g^+$, we normalize the drag reduction curves by $m_l$ (García-Mayoral & Jiménez 2011).

For turbulent channel flow over triangular riblets, figure 8 shows the $m_l$-normalized drag reduction as a function of $l_g^+$. The normalization factor $m_l$ is computed by averaging the slope obtained from the first two points on each curve, which are both in the viscous regime. The shaded region represents the envelope of normalized drag reduction values resulting from prior experimental and numerical studies (García-Mayoral & Jiménez 2011). For riblets with $\alpha = 105^\circ, 90^\circ, 75^\circ, 60^\circ$, and $45^\circ$, figure 8 shows collapse of drag reduction curves with the largest drag reduction occurring within a tight range of cross-section areas; $l_g^+ = 11.4, 11.7, 11.1, 11.0$, and $10.5$, for $\alpha = 105^\circ, 90^\circ, 75^\circ, 60^\circ$, and $45^\circ$, respectively. This prediction agrees well with the values reported by García-Mayoral & Jiménez (2011), $l_g^+ \in [9.7, 11.7]$. Moreover, the drag reduction curves resulting from our framework are primarily located within the shaded region and they reliably predict the overall trend.
4.2. Effect of riblets on turbulent viscosity and turbulent mean velocity

We next examine the effect of riblets on turbulent viscosity and mean velocity. Figure 9 shows the turbulent eddy viscosity $\nu_{T0}$ and mean velocity $U_0$ of channel flow over smooth walls with $Re_z = 186$ along with the corresponding corrections, $\nu_{Tc}$ and $U_c$, introduced by riblets with $\alpha = 90^\circ$ on the lower wall. Among the cases listed in table 1, two cases with maximum ($\omega_z = 50$) and minimum ($\omega_z = 30$) drag reduction are chosen. For $\omega_z = 30$, figure 9(c) shows that turbulence is slightly promoted at the beginning of the buffer layer, but is then suppressed in regions farther away from the wall. On the other hand, for $\omega_z = 50$, turbulence is always suppressed ($\nu_{Tc} \leq 0$) and the region of suppression shifts closer to the wall ($y^+ \gtrsim 3$). Figure 9(d) shows that, in both cases, riblets reduce the mean velocity gradient in the immediate vicinity of the wall ($y^+ \lesssim 6$). These results show that for the same shape of riblets (i.e., same tip angle $\alpha$), riblets of sizes that are larger than the optimal yield provide smaller amount of turbulence suppression and mean shear reduction. To illustrate the dependence of turbulent viscosity and mean streamwise velocity on the shape of riblets, figure 10 shows $\nu_{Tc}$ and $U_c$ for turbulent channel flow over riblets with different tip angles $\alpha$ and with spanwise frequencies $\omega_z$ that correspond to the maximum drag reduction. The largest turbulence suppression and mean velocity reduction is achieved for $\alpha = 75^\circ$, which is in agreement with the drag reduction trends observed in figure 7. Between $\alpha = 45^\circ$ and $\alpha = 105^\circ$, the reduction in turbulent viscosity is more pronounced for the latter, which again reflects the drag reduction trends reported in figure 7. For riblets with $\alpha = 60^\circ$ and $\omega_z = 60$ ($s^+ = 19.5$), figure 11 shows the variation of the mean velocity in the spanwise plane. No variation is found above $y > -0.9$, which is in agreement with the result of numerical simulations (Choi et al. 1993).

4.3. Effect of riblets on turbulent kinetic energy

We next examine the effect of triangular riblets on the fluctuations’ kinetic energy. Figure 12 compares the energy spectrum of turbulent channel flow with smooth walls to the changes in the energy spectrum caused by equally shaped ($\alpha = 90^\circ$) riblets of different sizes ($\omega_z = 160, 50$, and $30$). The energy spectra are premultiplied by the logarithmically scaled streamwise wavenumber $k_x$ so that the areas under the plots determine the total kinetic energy. Since the spanwise direction involves the parameterization $\theta_n = \theta + n\omega_z$, summation over $n$ is performed to integrate the energetic contribution of various harmonics in $\omega_z$ and identify the dependence of the energy spectrum on $\theta$.

For channel flow over smooth walls with $Re_z = 186$, figure 12(a) shows that the most energetic modes take place at $(k_x, \theta) = (2.5, 3.5)$. As blue regions in figures 12(b) and 12(c), and 12(d) illustrate, riblets reduce energy content of flow structures with smaller streamwise wavenumbers. Moreover, yellow and red regions in figures 12(c) and 12(b) demonstrate that larger riblets increase energy content of flow structures with larger streamwise wavenumbers. For these three cases, the largest energy amplification takes place around $(k_x, \theta) = (5.5, 2.4)$, $(6.4, 4.6)$, and $(6.4, 0.6)$, respectively. On the other hand, the maximum energy reduction occurs around $(k_x, \theta) = (4.9, 0.9)$, $(4.0, 0.5)$, and $(4.0, 0.5)$, respectively. Although the peak points are different, figures 12(b) and 12(c) and 12(d) demonstrate similar amplification/suppression trends: riblets suppress/increase energy content of long/short streamwise length scales. These results provide evidence that the analysis of spatially-periodic systems, e.g., the one considered in this paper, cannot be limited to a single horizontal wavenumber pair associated with the peak of the energy spectrum or the dominant near-wall cycle. We note that similar conclusions were reached in the analysis of turbulent channel flow subject to transverse wall oscillations (Moarref & Jovanović 2012). Finally, the dependence of correction $E_c(k_x, \theta)$ on the shape of riblets is shown in figure 13. For all cases shown in this figure, similar modes are affected by the
Model-based design of riblets for turbulent drag reduction

Figure 9. (a) The turbulent viscosity, \( \nu_T(0)(y^+) \); and (b) the turbulent mean velocity \( U_0(y^+) \), in uncontrolled channel flow with \( Re_\tau = 186 \). The correction to (c) turbulent viscosity, \( \nu_Tc(y^+) \); and (d) the mean velocity, \( U_c \), in the presence of riblets with \( \alpha = 90^\circ \). Red solid lines correspond to riblets with \( \omega_z = 30 \) (minimum drag reduction) and blue dashed lines correspond to riblets with \( \omega_z = 50 \) (maximum drag reduction).

Figure 10. Correction to (a) turbulent viscosity \( \nu_Tc(y^+) \); and (b) mean velocity \( U_c(y^+) \) in a turbulent channel flow with \( Re_\tau = 186 \) over triangular riblets on the lower wall. The spanwise frequency \( \omega_z \) associated with different shapes is selected to maximize drag reduction: \( \alpha = 105^\circ, \omega_z = 45 \) (solid black); \( \alpha = 75^\circ; \omega_z = 60 \) (dotted red); \( \alpha = 45^\circ, \omega_z = 80 \) (dot-dashed blue).

Presence of riblets and the suppression of kinetic energy is more pronounced for riblets with \( \alpha = 75^\circ \) and \( \omega_z = 60 \), which also yield the largest drag reduction (cf. figure 7). This suggests synchrony between the dependence of drag reduction and energy suppression on the geometry of triangular riblets.

Figure 14(b) shows the percentage of kinetic energy variation

\[
\Delta \bar{E} := (\bar{E} - \bar{E}_0) / \bar{E}_0,
\]


Figure 11. Mean velocity profiles $U(y, z)$ for $\alpha = 60^\circ$ and $\omega_z = 60$ (i.e., $s^+ = 19.5$): (a) One-dimensional view for different spanwise locations. The direction of the arrow points to velocity profiles corresponding to spanwise locations farther away from the tip of riblets. (b) Color-plot of the streamwise mean velocity in the cross-plane.

Figure 12. (a) Premultiplied energy spectrum $k_x E_0(k_x, \theta)$ resulting from DNS of a turbulent channel flow with $Re_\tau = 186$ (Del Álamo & Jiménez 2003); and correction to the premultiplied spectrum $k_x E_c(k_x, \theta)$ resulting from second-order statistics $X_c(k_x, \theta_n)$ of the linearized dynamics around the base flow induced by triangular riblets ($\alpha = 90^\circ$) with (b) $\omega_z = 160$ ($\ell_g^+ = 3.6$); (c) $\omega_z = 50$ ($\ell_g^+ = 11.7$, optimal); and (d) $\omega_z = 30$ ($\ell_g^+ = 19.5$).

for triangular riblets as a function of the riblet groove area $\ell_g^+$. Here, $\bar{E}$ and $\bar{E}_0$ denote the kinetic energy of velocity fluctuations in the presence and absence of riblets, respectively. These two quantities can be computed by integrating the energy spectrum $E(k_x, \theta)$ over all horizontal wavenumbers $k_x$ and $\theta$. On the other hand, figure 14(a) shows the
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Figure 13. Correction to the premultiplied energy spectrum \( k_x E_c(k_x, \theta) \) in a turbulent channel flow with \( Re_\tau = 186 \) triangular riblets of various sizes. The spanwise frequency \( \omega_z \) associated with different shapes of riblets corresponds to maximum drag reduction: (a) \( \alpha = 105^\circ, \omega_z = 45; \) (b) \( \alpha = 75^\circ, \omega_z = 60; \) and (c) \( \alpha = 45^\circ, \omega_z = 80. \)

Figure 14. (a) Kinetic energy suppression; and (b) turbulent drag reduction in a channel flow with \( Re_\tau = 186 \) over triangular riblets of various sizes. Symbols denote different shapes of triangular riblets with \( \alpha = 105^\circ(\triangledown); 90^\circ(\triangle); 75^\circ(\bigcirc); 60^\circ(\bigtriangleup); \) and \( 45^\circ(\times). \)

percentage of drag reduction for the same values of \( l_g^+ \). Our computations demonstrate similar trends in the dependence of \( \Delta E \) and \( \Delta D \) on \( l_g^+ \) (cf. figures 14(b) and 14(a)), especially for the riblets in viscous regime. Based on the various cases considered in figure 14, the linear regression model \( \Delta D = 1.6787 \Delta E + 0.9014 \) can be extracted with a coefficient of determination \( R^2 = 0.9937 \) for riblets with \( l_g^+ \leq 10 \). Strong correlation between changes in turbulent drag and kinetic energy suggests that energy can be used as a surrogate for predicting the effect of riblets on skin-friction drag; see figure 15(a).

As shown in figure 15(c), riblets can suppress or enhance turbulence near the wall. Small riblets can disturb the near-wall cycle in the turbulent flow by generating and preserving laminar regions within the grooves. On the other hand, for larger riblets, streamwise rollers penetrate into the grooves which enhances turbulence close to the wall. As a result, for larger riblets, nonlinear effects take over and the linear relation between drag/energy reduction and any metric of the riblet size (e.g., \( l_g \)) no longer holds. As illustrated in figure 15(b), the quality of linear regression models drops (i.e., \( R^2 \) becomes smaller) when data for riblets of larger size is taken into account.

Remark 1. The performance deterioration for large riblets is associated with the breakdown of the viscous regime within the riblet grooves that arises from the lodging of near-wall vortices (Lee & Lee 2001), the generation of secondary flow vortices (Goldstein & Tuan 1998), or the emergence of spanwise coherent rollers (García-Mayoral & Jiménez 2011). When turbulence moves into the grooves, a
A linear relation between the turbulent drag reduction $\Delta D$ and the kinetic energy suppression $\Delta \bar{E}$ for a channel flow with $Re_\tau = 186$ over triangular riblets of different sizes $l_g^+$. Circles mark cases listed in Table 1 and crosses mark data for riblets whose size is smaller than the optimal (i.e., $l_g^+ \leq 10$). The red line provides linear interpolation for crosses. (b) The coefficient of determination $R^2$ for linear regression models resulting from data with $l_g^+ \leq l_{gT}^+$. 

The turbulence model, which assumes that the wall-normal region with $y < -1$ is laminar (i.e., $\nu_T = 0$), loses its validity. To support an extended turbulent regime and go beyond the breakdown of the viscous regime, our model of surface corrugation assumes the tip of riblets to be located within the channel region (i.e., above $y = -1$); see §2.1. The parameter $r_p$ in (2.11), which controls the level of protrusion into the turbulent regime, is determined to satisfy a constant bulk assumption. Because of this, our model remains valid even for riblets that are larger than the optimal ($l_g^+ \lesssim 20$).

5. Turbulent flow structures

In this section, we use the stochastically forced linearized model (3.2) to examine the effect of riblets of different sizes and shapes on the dominant turbulent flow structures. We extract flow structures from our model using the eigenvalue decomposition of the velocity covariance matrix in statistical steady-state,

$$\Phi_\theta(k_x) = C_\theta(k_x) X_\theta(k_x) C_\theta^*(k_x)$$

where $X_\theta(k_x)$ represents the solution of Lyapunov equation (3.9). The eigenvectors associated with the principal pair of eigenvalues form the energetically dominant flow structures that are located in the vicinity of the upper and lower channel walls. The first pair of eigenvalues are usually one order of magnitude larger than the second pair. The velocity components are constructed by integrating over all spanwise harmonics and by accounting for the symmetry in the streamwise direction as

$$u(x, y, z) = \sum_{n \in \mathbb{Z}} \cos(\theta_n z) \text{Re} \left( \tilde{u}(k_x, \theta_n) e^{ik_x x} \right),$$

$$v(x, y, z) = \sum_{n \in \mathbb{Z}} \cos(\theta_n z) \text{Re} \left( \tilde{v}(k_x, \theta_n) e^{ik_x x} \right),$$

$$w(x, y, z) = -\sum_{n \in \mathbb{Z}} \sin(\theta_n z) \text{Im} \left( \tilde{w}(k_x, \theta_n) e^{ik_x x} \right).$$
Here, $\text{Re}$ and $\text{Im}$ denote real and imaginary parts, and $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ correspond to the streamwise, wall-normal, and spanwise components of the principal eigenvector of the matrix $\Phi_\theta(k_x)$, given in equation (5.1), associated with lower-wall flow structures.

**Remark 2.** In a turbulent channel flow with smooth walls, the dominant eigenmodes of the velocity covariance matrix appear in pairs and represent symmetric flow structures that reside in the vicinity of the upper and lower walls. Surface corrugation on the lower wall breaks this symmetry and can cause a suppression of near-wall structures in the lower half of the channel. In other words, the flow structures that dominate the flow close to the riblets can be less energetic than the flow structures close to the upper wall. As a result, physically relevant flow structures near the riblets, e.g., the dominant flow structures associated with the near-wall cycle over riblets of optimal size, are often associated with the second, less energetic, eigenmode of $\Phi_\theta(k_x)$; see figure 16(b) in §5.1.

### 5.1. Near-wall cycle

In the absence of riblets, the so-called near-wall cycle dominates the physics of the turbulent channel flow by generating streamwise streaks from the advection of the mean shear by streamwise vortices and the formation of streamwise vortices through streak instability and nonlinear interactions (Hamilton, Kim & Waleffe 1995; Jiménez & Pinelli 1999; Robinson 1991). Riblets can break this near-wall cycle and push the streamwise vortices and streaks away from the wall so that a laminar region is retained within the grooves, which ultimately reduces skin-friction drag. The typical wavelength of the flow structures in the near-wall cycle are reported as $(\lambda_x^+, \lambda_z^+) = (1000, 100)$, which corresponds to $(k_x, k_z) = (1.1687, 11.687)$ in a turbulent channel flow with $Re_\tau = 186$.

For different sizes of riblets, figure 16 compares the flow structures that correspond to the near-wall cycle in a turbulent channel flow with $Re_\tau = 186$. Flow patterns resulting from the combination of streaks and vortices can be clearly observed. In particular, it is evident that the quasi-streamwise vortices and regions of high and low streamwise velocity are pushed above the riblet tips creating a region of limited turbulence in the riblet grooves, and effectively impeding the transfer of mean momentum toward the lower wall. Figures 16(a) and 16(b) illustrate the flow structures over small- and optimal-sized riblets, respectively. The dominance of streamwise elongated structures that follow the length-scales of the near-wall cycle is evident in these two scenarios. As shown in figure 16(c), however, the flow over larger riblets is contaminated by multiple energetically relevant spanwise length-scales, which indicates a distribution of energy across multiple Fourier modes beyond the ones that are relevant in the near-wall cycle. This distribution of energy is caused by the interaction of near-wall turbulence with the spanwise-periodic surface, which leads to the generation of secondary flow structures that follow the spatial frequency of riblets close to the riblet tips (Goldstein & Tuan 1998); see $(y, z)$ slice on the right of figure 16(c). The cross-plane view also shows that for larger riblets, such secondary flow structures begin to penetrate into the grooves. This induces high-momentum flow into the viscous flow regime, which is reflected by an increase in the amplitude of velocity fluctuations at $y^+ \approx 6$ and the breakdown stage of the viscous regime which precedes the deterioration of drag reduction. We note that similar observations were recently made via a gain-based analysis of the resolvent modes (Chavarin & Luhar 2019).

### 5.2. Spanwise rollers resembling Kelvin-Helmholtz vortices

In addition to the influence of secondary flow structures around the tip of riblets, the breakdown of the viscous regime and decrease in drag reduction can also result from the amplification of spanwise rollers that are induced by a two-dimensional K-H
Figure 16. Dominant flow structures in the vicinity of the lower wall of a turbulent channel flow with $Re_\tau = 186$ over triangular riblets with $\alpha = 90^\circ$ and (a) $\omega_z = 160$; (b) $\omega_z = 50$; and (c) $\omega_z = 30$. These flow structures are extracted from the dominant eigenmode pair of the covariance matrix $\Phi_\theta(k_x)$ for $(k_x, \theta) = (1.1687, 11.687)$. Left column: $x-z$ slice of the streamwise velocity $u$ at $y^+ \approx 6$; Right column: $y-z$ slice of $u$ along with the vector field $(v, w)$ at $x^+ = 500$, which corresponds to the thick black lines in the left column. Color plots are used for the streamwise velocity fluctuation $u$ and vector fields identify streamwise vortices.

The amplifications of long spanwise scales for large riblets is evident from the energy spectra shown in figure 12. Figure 17 shows the DNS-based premultiplied streamwise co-spectra of various Reynolds stresses for infinitely wide scales ($\theta = 0$) in a turbulent channel flow with $Re_\tau = 186$ (Del Alamo & Jiménez 2003; Del Alamo et al. 2004), as well as the corresponding co-spectra resulting form our analysis of the flow over triangular riblets of different sizes. For larger riblets, the amplification of the co-spectra corresponding to streamwise and wall-normal intensities become stronger and occur closer to the wall. However, this trend is not observed for the co-spectrum corresponding to the spanwise turbulence intensity, which shows smaller amplification of channel-wide scales for riblets of larger size. Nevertheless, as the size of riblets increases, the co-spectrum corresponding to the wall-shear stress, $k_x E_{uv}$, starts to show signs of suppression in the vicinity of the wall; the penetration of negative shear stress into the riblet grooves is evident from the co-spectrum associated with spatial
Figure 17. Premultiplied streamwise co-spectra at $\theta = 0$ for a turbulent channel flow with $Re_\tau = 186$ over triangular riblets of tip angle $\alpha = 90^\circ$. The four rows correspond to $k_x E_{uu}^+$, $k_x E_{vv}^+$, $k_x E_{ww}^+$, and $k_x E_{uv}^+$; the four columns represent DNS data for the uncontrolled channel flow (Del Álamo & Jiménez 2003; Del Álamo et al. 2004) and data resulting from our computations for the flow with riblets of spatial frequency $\omega_z = 160, 50, \text{ and } 30$, respectively.

Frequency $\omega_z = 30$. The co-spectrum $k_x E_{uv}$ shows the largest suppression of shear stress for streamwise wavelengths $\lambda_x^+ \approx 200$. Our results demonstrate that large riblets result in the suppression of shear stress within the grooves, which is consistent with our earlier findings that showed a degradation of drag reduction for such sizes (cf. figure 8). We note that our computations illustrate that the trends observed for the energy spectra do not vary for different shapes of riblets (i.e., different values of $\alpha$).

For largest riblets ($\omega_z = 30$), figure 17 illustrates that the streamwise ($k_x E_{uu}$) and wall-normal ($k_x E_{vv}$) contributions to the energy spectra are significantly larger than the spanwise ($k_x E_{ww}$) contribution. Furthermore, the wall-normal spectrum shows significant amplification of wall-separated flow structures. Figure 18 shows the premultiplied streamwise energy spectrum, $k_x k_z E_{vv}$, at $y^+ = 5$ for different sizes of riblets. We note that energy amplification becomes larger as the size of riblets increases. For the riblets with $\omega_z = 30$, figure 18(c) shows that the band of streamwise and spanwise scales corresponding to $\lambda_x^+ \in [100, 300]$ and $\lambda_z^+ > 300$, is significantly more amplified than the other two cases, which is consistent with the DNS-result of García-Mayoral & Jiménez (2011).

Figure 19 shows the flow structures that are extracted from our model for $(k_x, \theta) = (5.76, 0)$ which correspond to spanwise-averaged infinitely long scales in $z$ and the peak in the shear stress co-spectrum for riblets with $\omega_z = 30$. These flow structures indicate that the dominant eigenmode of the covariance matrix (5.1) resembles a spanwise-constant
Figure 18. Premultiplied energy spectra of wall-normal velocity, $k_z k_x E_{vv}$, at $y^+ = 5$ in turbulent channel flow with $Re_{\tau} = 186$ over triangular riblets of tip angle $\alpha = 90^\circ$ and (a) $\omega_z = 160$; (b) $\omega_z = 50$; and (c) $\omega_z = 30$.

roller centered near $y^+ \approx 17.7$ which penetrates well into the grooves, thereby causing breakdown of the viscous regime (figure 19(a)). The wall-normal velocity $v$ at $y^+ = 5$ corresponding to the same horizontal wavenumbers is plotted in figure 19(b). These flow structures are in close agreement with the spanwise rollers identified using DNS which where centered around $y^+ \approx 15$; see figure 14 in García-Mayoral & Jiménez (2011). On the other hand, the streamwise wavelength of these structures, which corresponds to the highest suppression in the streamwise co-spectrum $k_x E_{uv}$ is slightly larger than the wavelength reported for spanwise rollers using DNS ($\lambda_x^+ \approx 200$ vs. $\lambda_x^+ \approx 150$).

Figure 20 shows the wall-normal location corresponding to the center of the spanwise rollers that appear above riblets with $\alpha = 90^\circ$ and larger size relative to the optimal value $l_{y}^+ = 11.7$. For these riblets, the spanwise rollers have similar dominant streamwise length scales ($\lambda_x^+ \approx 200$). For larger riblets, the core of the spanwise rollers moves down toward the riblets. Thus, as the size of riblets increases and the grooves become deeper, the dominant turbulent flow structures penetrate further down into the viscous region in the grooves.

6. Concluding remarks

We have developed a model-based framework for evaluating the effect of surface corrugation on skin-friction drag and kinetic energy in turbulent channel flows. The influence of the corrugated surface is captured via a volume penalization technique that enters as a feedback term into the governing equations. Our simulation-free approach utilizes eddy-viscosity enhanced NS equations and it consists of two steps: (i) we use the turbulent viscosity of the turbulent channel flow with smooth walls to capture the effect of the corrugated surface on the turbulent base velocity; and (ii) we use second-order statistics of stochastically forced equations linearized around this base velocity profile to assess the role of velocity fluctuations and correct the turbulent viscosity model. This correction perturbs the turbulent base velocity profile obtained in the first step and refines our prediction of skin-friction drag.

For a turbulent channel flow with streamwise-aligned spanwise-periodic triangular riblets on the lower-wall, we demonstrate that the base flow computed in the first step of our approach does not capture drag-reducing trends reported in experiments and simulations. Incorporating the influence of fluctuations on the turbulent viscosity significantly improves our predictions. Our results demonstrate excellent agreement with experimental and numerical results both in capturing drag-reducing trends and in identifying optimal shapes and sizes of riblets for largest drag reduction. We also investigate the dependence of the turbulent kinetic energy of fluctuations on the size of
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Figure 19. Turbulent flow structures corresponding to spanwise rollers extracted from the dominant eigenmode of the covariance matrix $\Phi_\theta(k_x)$ at $(k_x, \theta) = (5.76, 0)$ for a turbulent channel flow with $Re_\tau = 186$ over triangular riblets of $\alpha = 90^\circ$ and $\omega_z = 30$. (a) Vector field denotes the in-plane velocities $(u, v)$; and (b) $x-z$ slice of the wall-normal velocity $v$ at $y^+ \approx 5$.

Figure 20. Wall-normal locations of the core of spanwise rollers corresponding to $(k_x, \theta) = (5.76, 0)$ in a turbulent channel flow with $Re_\tau = 186$ over triangular riblets with $\alpha = 90^\circ$ and sizes specified by $l_g^+$. Riblets and demonstrate similar trends to what we observe for drag reduction. Building on this similarity and data obtained through a parametric study for riblets of various shapes and sizes, we extract a linear regression model and show that energy can be used as a surrogate for predicting the effect of riblets on skin-friction drag in the viscous regime.

The steady-state covariance matrices that we compute also allow us to examine
the impact of riblets on dominant turbulent flow structures. We show that small-size triangular riblets limit the wall-normal transfer of momentum associated with the near-wall cycle and the generation of secondary flow structures around the tips. Our model captures the penetration of secondary vortices into riblet grooves and predicts that drag reduction reduces for large-size riblets. We also investigate the amplification of spanwise rollers that resemble K-H vortices and show excellent agreement between our predictions of their streamwise length-scale and core location with previous numerical studies.

Our study paves the way for the optimal design of periodic surfaces using models of low complexity that can bypass the need for costly numerical simulations and experiments. We anticipate that further incorporation of data resulting from numerical simulations and experiments can further improve the predictive capability of our framework that utilizes stochastically forced linearized NS equations \cite{Zare et al. 2017a, b}.

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**Appendix A. Operators \(A_\theta, B_\theta,\) and \(C_\theta\) in equations (3.5)**

The dynamical generator \(A_\theta\) in equations (3.5) has a bi-infinite structure shown in equation (3.6), in which elements \(A_{n,m}\) contain four operators,

\[
A_{n,m} = \begin{bmatrix}
A_{n,m,1,1} & A_{n,m,1,2} \\
A_{n,m,2,1} & A_{n,m,2,2}
\end{bmatrix}.
\]

For the operators on the main diagonal, \(A_{n,0}\), we have

\[
A_{n,0,1,1} = \Delta^{-1}_n \left[ 2 (1 + \nu_T) \Delta_n^2 + \nu_T'' \left( \frac{\partial^2}{\partial y^2} + k_n^2 \right) + \frac{2\nu_T'}{Re_T} \right] + \Gamma_{n,0,1,1}
\]

\[
A_{n,0,1,2} = \Gamma_{n,0,1,2}
\]

\[
A_{n,0,2,1} = \Gamma_{n,0,2,1}
\]

\[
A_{n,0,2,2} = \Delta^{-1}_n \left[ 2 (1 + \nu_T) \Delta_n^2 + \nu_T' \frac{\partial^2}{\partial y^2} \right] + \Gamma_{n,0,2,2}
\]

and for the off-diagonal ones, \(A_{n,m}\) with \(m \neq 0\), we have

\[
A_{n,m,1,1} = \Gamma_{n,m,1,1}, \quad A_{n,m,1,2} = \Gamma_{n,m,1,2},
\]

\[
A_{n,m,2,1} = \Gamma_{n,m,2,1}, \quad A_{n,m,2,2} = \Gamma_{n,m,2,2}
\]
where

\[
\Gamma_{n,m,1,1} = \Delta_n^{-1} \left[ 2i m k_x \omega_z \frac{\theta_{n,m}}{k_{n+m}^2} (U'_{-m} \partial_y + U_{-m} \partial_{yy}) + ik_x (U''_{-m} - U_{-m} \Delta_{n+m}) + i k_x (m \omega_z)^2 U_{-m} - 2m k_x \omega_z \theta_{n,m} U_{-m} + m \omega_z (m \omega_z - 2 \theta_{n,m}) a_{-m} - a_{-m} \Delta_{n+m} - a'_{-m} \partial_y + m \omega_z \frac{\theta_{n,m}}{k_{n+m}^2} (a'_{-m} \partial_y + a_{-m} \partial_{yy}) \right]
\]

\[
\Gamma_{n,m,1,2} = \Delta_n^{-1} \left[ 2 \frac{i m k_x^2}{k_{n+m}^2} (U'_{-m} + U_{-m} \partial_y) + \frac{m k_x}{k_{n+m}^2} \omega_z (a'_{-m} + a_{-m} \partial_y) \right]
\]

\[
\Gamma_{n,m,2,1} = \frac{i m \omega_z}{k_{n+m}} (U'_{-m} - U_{-m} \partial_y) - i \theta_{n,m} U'_{-m} + \left[ (m \omega_z)^2 \theta_{n,m} U'_{-m} - \frac{m k_x \omega_z}{k_{n+m}^2} a_{-m} \right] \partial_y
\]

\[
\Gamma_{n,m,2,2} = -i k_x U_{-m} - a_{-m} + \frac{i}{k_{n+m}^2} \omega_z U_{-m} + m \omega_z \theta_{n,m} a_{-m}
\]

Here, \( \theta_{n,m} = (n + m) \omega_z + \theta, \ k_{n+m}^2 = k_x^2 + \theta^2_{n,m}, \) and \( \Delta_{n+m} = \partial_{yy} - k_{n+m}^2. \)

The input operator \( B_\theta \) takes the form \( B_\theta = \text{diag} \{ \ldots, B_{n-1}, B_n, B_{n+1}, \ldots \} \) where

\[
B_n = \begin{bmatrix} B_v \\ B_\eta \end{bmatrix} = \begin{bmatrix} -i k_x \Delta_n^{-1} \partial_y & -i k_n^2 \Delta_n^{-1} \partial_y \\ i \theta_n I & 0 \end{bmatrix}.
\]

Similarly, the output operator \( C_\theta \) is given by \( C_\theta = \text{diag} \{ \ldots, C_{n-1}, C_n, C_{n+1}, \ldots \} \) where

\[
C_n = \begin{bmatrix} C_u \\ C_v \\ C_w \end{bmatrix} = \begin{bmatrix} (i k_x / k_n^2) \partial_y & -(i \theta_n / k_n^2) I \\ I & 0 \\ (i \theta_n / k_n^2) \partial_y & (i k_x / k_n^2) I \end{bmatrix}.
\]

**Appendix B. Computing corrections \((k_c, \epsilon_c)\) to \((k, \epsilon)\)**

We show that the effect of fluctuations around the mean velocity on corrections \( k_c \) and \( \epsilon_c \) can be obtained from the correction \( X_c(k_x, \theta) \) to the steady-state covariance:

\[
k_c(y) = \int_0^\infty \int_0^{\omega_z/2} \sum_{n \in \mathbb{Z}} K_k(y, k_x, \theta_n) \, d\theta \, dk_x,
\]

\[
\epsilon_c(y) = \int_0^\infty \int_0^{\omega_z/2} \sum_{n \in \mathbb{Z}} K_\epsilon(y, k_x, \theta_n) \, d\theta \, dk_x
\]

where \( K_k(y, k_x, \theta_n) \) and \( K_\epsilon(y, k_x, \theta_n) \) are obtained by taking the diagonal components of matrices \( N_k \) and \( N_\epsilon \), respectively:

\[
N_k(k_x, \theta) = \frac{1}{2} \left( C_u X_c C_u^* + C_v X_c C_v^* + C_w X_c C_w^* \right),
\]

\[
N_\epsilon(k_x, \theta) = 2 \left( k_x^2 C_u X_c C_u^* + D_y C_v X_c C_v^* D_y^* + \theta_n^2 C_w X_c C_w^* - i k_x D_y C_u X_c C_v^* + k_x \theta_n C_u X_c C_w^* + i \theta_n D_y C_v X_c C_v^* \right) + \theta_n^2 C_u X_c C_w^* + \theta_n^2 C_v X_c C_v^* + k_x^2 C_w X_c C_w^*.
\]

Here, \( D_y \) denotes the finite-dimensional representation of \( \partial_y \) and \( C_u, C_v, \) and \( C_w \) are finite-dimensional approximations of the output operators in \((A2)\).
Appendix C. Computing the effect of fluctuations on the mean velocity

By substituting (3.12) into the mean flow equations (2.7) we arrive at the following equation for the corrected turbulent mean velocity $\bar{U}(y)$, which is of harmonic form (2.9):

$$
(1 + \nu T_0 + \nu T_c) \Delta \bar{U} + (\nu' T_0 + \nu' T_c) \bar{U}' - K^{-1} \bar{U} = Re_p P_{x,0} \quad (C1)
$$

Further inclusion of the harmonics of $K^{-1}$ yields the equation for $m$th harmonic of $\bar{U}(y)$

$$
\left[(1 + \nu T_0 + \nu T_c) \left(\partial_y^2 - m^2 \omega_s^2\right) + (\nu' T_0 + \nu' T_c) \partial_y - a_0\right] \bar{U}_m
$$

main-diagonal

$$
+ \sum_{n \in \mathbb{Z}\{0\}} a_n \bar{U}_{m-n} = \left\{ \begin{array}{ll}
Re_p P_{x,0}, & m = 0, \\
0, & m \neq 0
\end{array} \right.
$$

off-diagonal

The constant bulk flux requirement is enforced via a variation in the driving pressure gradient, which introduces a correction to zeroth harmonic of mean velocity as

$$
U_c(y) = \bar{U}_0(y) - (1 - P_{x,c}) U_0(y).
$$

Here, the correction $P_{x,c}$ is obtained from the change of bulk from the solution $\bar{U}(y)$ to equation (C1) as

$$
P_{x,c} = 1 - \frac{1}{U_B} \int_{-1-h_{rp}}^{1} \bar{U}_0(y) \, dy.
$$

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