Instability of a free boundary in a Hele-Shaw cell with sink/source and its parameter dependence

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Abstract

We carry out a linear stability analysis for a free boundary between two fluids in a Hele-Shaw cell, which is driven by the Darcy’s law and an injection or suction at the center. In general, stability of an interface is determined by the well-known Saffman–Taylor instability condition. The linear growth rate of the perturbation depends on two parameters; the rate of the injection/suction and the viscosity contrast of the two fluids. In this paper, we numerically find a parameter region, in which the interface can be stable (resp. unstable) even though it has been considered to be unstable (resp. stable) due to the Saffman–Taylor instability. For the case of the injection, it is suggested that such parameter region vanishes as time increases to infinity. However, the destabilization can be retarded for a sufficiently long time, as one tunes the viscosity and the injection rate of the injecting fluid.

Keywords

Hele-Shaw cell with sink/source, Saffman–Taylor instability, linear stability analysis, two phase Hele-Shaw problem, free boundary

Research Activity Group

Mathematical Aspects of Continuum Mechanics

1. Introduction

Hele-Shaw problem, to find a free boundary and the velocity field in a Hele-Shaw cell, has been investigated from not only the mathematical and the physical interests but also importance from engineering request. Here, Hele-Shaw cell is an apparatus which consists of two parallel transparent plates with a narrow gap compared with the total length of the interface and the size of the cell, as shown in Fig. 1. It is known that an interface between two fluids in a Hele-Shaw cell becomes unstable due to the viscosity difference of the fluids; if the less viscous fluid displaces the more viscous one, then the interface becomes unstable, otherwise remains stable [1]. This instability is well known as the Saffman–Taylor instability, and examined experimentally in previous studies [2–4].

As shown in [1], once the interface becomes unstable, then its perturbation grows to be finger-like ramified patterns, referred to as the viscous fingering. Previous studies about the viscous fingering in Hele-Shaw cell can be divided into two categories, one is the rectangular geometry, and the other is the radial geometry. In the former case, the initially planer interface moves from one end to another and then becomes unstable [1–4], and in the latter case, the circular interface grows up radially because of the point sink or source at the center [5–10].

For the rectangular geometry, the linear stability analysis has been carried out in [5,6]. Afterward, the critical and the minimum wavelength are examined by experiments and also with an approximate form of the growth rate [7]. The weakly nonlinear stability analysis was first applied to Hele-Shaw problem in [8], and have been investigated, for instance, for the nonlinear effects on Hele-Shaw problem such as the thin wetting-layer in the cell [9], and the viscous stress [10]. However, in these previous studies including the ones in the rectangular geometry, the dependence of the stability on the parameters such as the injection rate or the viscosity difference has not been sufficiently clarified yet. Under these backgrounds, we numerically investigate the parameter dependence of the instability of the radial Hele-Shaw problem with a sink or source.

In the next section, we formulate the Hele-Shaw problem with a sink or source. In Section 3, the linear stability analysis is introduced, and the dependence of the stability on the injection rate and viscosity different is investigated.

2. Formulation

Fig. 1 depicts the schematic configuration of our system. Let the free boundary \( \Gamma(t) \) be a closed Jordan curve between two regions \( \Omega_1(t) \) (inner) and \( \Omega_2(t) \) (outer), which are filled with fluid 1 and 2, respectively. As shown in Fig. 1, we assume that the interface can be divided into the sum of an unperturbed circular interface with the radius \( R(t) \) and a deviation \( \zeta(\theta,t) \) from it, which
obeys the Darcy’s law

\[ Q = \frac{R(t)}{R_0} \]

while \( R \) changes the area inside the unperturbed interface.

The thickness of the gap \( b \) is narrow compared with the total length of \( \Gamma(t) \) and the size of the cell.

\[ \nabla \cdot (\sigma \kappa) \bigg|_{\Gamma(t)} \]

depends on the angle \( \theta \) from prescribed axis and the time \( t \).

The perturbed interface is sometimes denoted by \( R = R(\theta, t) = R(t) + \zeta(\theta, t) \). In this paper, we consider the case where the inner fluid (fluid 1) is injected or sucked from the center \( O \) with a constant volume flow rate \( Q \), which satisfies

\[ R(t) = \sqrt{R_0^2 + \frac{QT}{\pi}}, \quad (1) \]

because of the volume conservation of the fluid 1. It is to be noted that the case of \( Q > 0 \) represents the injection, while \( Q < 0 \) represents the suction. In addition, we also assume a constraint that the deviation \( \zeta(\theta, t) \) does not change the area inside the unperturbed interface \( R(t) \).

In general, the flow in a Hele-Shaw cell is assumed to obey the Darcy’s law

\[ \mathbf{v}_i = -\frac{b^2}{12\mu_i} \nabla p_i \quad (i = 1, 2), \quad (2) \]

where \( \mathbf{v}_i, p_i, \) and \( \mu_i \) are the velocity vector, the pressure, and the viscosity of fluid \( i \) (\( i = 1, 2 \)), respectively, and \( b \) is the thickness of the cell. Then, the motion of the boundary \( \Gamma(t) \) is determined by the Hele-Shaw flow (2), with the aid of the incompressibility condition \( \nabla \cdot \mathbf{v}_1 = 0 \), and boundary conditions. Thus the Hele-Shaw problem is formulated as

\[
\begin{aligned}
\Delta p_1 &= 0 \quad \text{in } \Omega_1(t), \\
\Delta p_2 &= 0 \quad \text{in } \Omega_2(t), \\
\mathbf{v}_1 \cdot \mathbf{n} &= \mathbf{v}_2 \cdot \mathbf{n} = V_n \quad \text{on } \Gamma(t), \\
p_1 - p_2 &= \sigma \kappa \quad \text{on } \Gamma(t), 
\end{aligned}
\]

where \( \sigma > 0 \) is the surface tension coefficient, \( \kappa \) is the curvature (it takes positive sign for the convex interface from fluid 1 to fluid 2), and \( \mathbf{n} \) is the unit outward normal. The third equation in (HS) can be rewritten as

\[
\begin{aligned}
\frac{dR}{dt} &= \frac{-\partial \phi_1}{\partial \theta} + \frac{1}{R^2} \frac{\partial R}{\partial \theta} \frac{\partial \phi_1}{\partial \theta} \bigg|_{\Gamma(t)} \quad (i = 1, 2), 
\end{aligned}
\]

where \( \phi_i \) is the velocity potential defined by \( \phi_i = (b^2/12\mu_i)p_i \). The effect of sink or source is taken into account by (1) with (3).

Note that the Saffman–Taylor instability gives the relation such that the interface becomes unstable if \( \mu_1 \leq \mu_2 \) (resp. \( \mu_1 \geq \mu_2 \)) in the case of source \( Q > 0 \) (resp. sink \( Q < 0 \)).

3. Results and Discussions

For the free boundary in a Hele-Shaw cell, the linear stability analysis was carried out in the previous studies [7, 8]. We assume that the solution for (HS) can be written explicitly in terms of \( \phi_i \) as

\[
\begin{aligned}
\phi_1 &= \phi_1^{(0)} + \sum_{n \neq 0} \phi_1^{(n)}(t) \left( \frac{R}{R_0} \right)^{|n|} e^{in\theta}, \\
\phi_2 &= \phi_2^{(0)} + \sum_{n \neq 0} \phi_2^{(n)}(t) \left( \frac{R}{R_0} \right)^{|n|} e^{in\theta},
\end{aligned}
\]

where \( \phi_i^{(0)} \) is the potential for stationary flow at \( r = 0 \) (\( i = 1 \)) and \( r \to \infty \) (\( i = 2 \)). In addition, the perturbation \( \zeta(\theta, t) \) is also assumed to be written as

\[ \zeta(\theta, t) = \sum_{n = -\infty}^{\infty} \zeta_n(t) e^{in\theta}. \]

Note that the perturbation \( \zeta_n \) vanishes for \( n = 0, 1 \), since \( \zeta_0 \) is equivalent to the change of the unperturbed circular interface, and \( \zeta_1 \) is excluded by an assumption of the rotational symmetry of the system. Within the linear stability theory, substituting (4), (5) and (6) into (HS), one can obtain the following time evolution equation for perturbation up to the linear terms with respect to \( \zeta_n \)

\[
\frac{d}{dt} \zeta_n(t) = \lambda(n, t) \zeta_n(t),
\]

where \( \lambda(n, t) \) is the linear growth rate defined as

\[ \lambda(n, t) = Q \frac{2\pi R(t)^3}{n |\sigma|} (\Delta \phi_1 - 1) - \frac{\sigma}{R(t)^3} |n| (n^2 - 1) \]

for all integer \( n \).

Here, \( A \) is a viscosity contrast

\[ A = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}, \quad (9) \]

and \( \sigma \) is defined as

\[ \sigma = \frac{b^2 \sigma}{12(\mu_1 + \mu_2)}. \quad (10) \]

It seems that for a positive constant \( c \), \( |\zeta_n(t)| \) decays exponentially if \( \lambda(n, t) \leq -c \) holds, and \( |\zeta_n(t)| \) grows up exponentially if \( \lambda(n, t) \geq c \) holds. However, the criterion for the sign of \( \lambda \) is a bit complicated, since \( \lambda(n, t) \) depends not only on \( n \) and \( t \) but also \( A, Q \). In the injection case \( Q > 0 \), \( R(t) \) grows up to infinity as \( t \) tends to infinity, and in the suction case \( Q < 0 \), \( R(t) \) vanishes within a finite time \( T \), defined as

\[ T = \frac{\pi R_0^3}{|Q|}. \quad (11) \]

According to the criterion of the Saffman–Taylor instability [1], the interface becomes unstable if \( \mu_1 \leq \mu_2 \), i.e., \( 0 \leq A \leq 1 \) holds for the case of a source.
3.1 The case of the injection

From (8), one can solve \( \lambda(n, t) = 0 \) with respect to \( R(t) \), and obtain

\[
R(t) = R_n := \frac{2\pi |n|(n^2 - 1)}{Q} \left| \frac{A}{|n| - 1} \right|.
\]

Then, in the injection case \( Q > 0 \), the condition \( \lambda(n, t) > 0 \) is equivalent to the two conditions: \( |n| > 1 \) and \( t \geq \max(0, t_n) \), where \( t_n \) is the critical time satisfying \( R(t_n) = R_n \), i.e.

\[
t_n = \frac{\pi}{Q} (R_n^2 - R_0^2).
\]

Note that \( \hat{s} \) also depends on \( A \), and can be rewritten as

\[
\hat{s} = \frac{b^2 \sigma}{12(\mu_1 + \mu_2)} = \frac{b^2 \sigma (1 + A)}{24\mu_2}.
\]

From (12) it is shown that there exists \( n_* \) such that \( t_n \) satisfies the relation \( t_n < t_{n+1} < t_{n+2} \ldots \) for \( n \geq n_* \). \( n_* \) is dependent on \( A \), and determined as \( n_* = 2 \) for \( 11/16 \leq A \leq 1 \), and \( n_* > 2 \) for \( 0 \leq A < 11/16 \). Thus the mode \( n_* \) is the minimum of \( n \) such that \( \lambda(n, t) > 0 \) holds for fixed values of \( (A, Q) \), for each \( t \). In other words, \( n_* \) is the first mode which becomes unstable, and then other modes \( n \) become unstable as the increasing time \( t \) passes \( t_n \), for each \( (A, Q) \). It is also found that if \( A \) and \( n \) do not satisfy \( |A| > 1 \) then \( \lambda < 0 \) for any \( Q \) and \( t \), which gives the stable interface with respect to such \( A \) and \( n \).

Figs. 2–5 show the smallest mode \( n (2 \leq n \leq N) \) represented by the colored region which satisfies \( \lambda(n, t) > 0 \), with respect to the set of the parameter values \( (A, Q) \), and \( t \). In the black region, \( \lambda(n, t) < 0 \) holds for all \( n (2 \leq n \leq N) \). Here we take \( N = 10 \), since it is numerically examined that the colored region does not change for \( N \geq 10 \) in each figure. Following the experimental values in [7], we chose that \( 0 \leq Q < 10 \) cm²/s, \( b = 0.1 \) cm, and \( \sigma = 63 \) dyne/cm, and \( R_0 = 0.01 \). We also take \( \mu_2 = 5.21 \) P, and change \( \mu_1 \) as \( 0 \leq \mu_1 \leq \mu_2 \) so that \( A \) varies as \( 0 \leq A \leq 1 \). Fig. 2 shows the early stage of the injection at \( t = 1.0 \), which is after the minimum of \( t_{10} \). At this stage, the condition \( \lambda(n, t) > 0 \) can hold for all \( n \in [2, 10] \) with sufficiently large values of \( Q \) and \( A \), which are represented by the colored region in Fig. 2. Then, comparing Fig. 3 with Fig. 2, we see that the colored region expands, which means that the condition \( \lambda(n, t) > 0 \) can hold for wider ranges of the values of \( (A, Q) \) than those for Fig. 2. For instance, at \( t = 1.0 \) the interface is stable for \( (A, Q) = (0.2, 6) \). However, at \( t = 6.0 \) it becomes unstable for the mode around \( n = 5 \). As shown in Fig. 4, the ranges of \( (A, Q) \) such that \( \lambda(n, t) > 0 \) holds become much wider at \( t = 6.0 \), while there remains the black region for small values of \( (A, Q) \). This black region becomes narrower but survives for a long time, as shown in Fig. 5. In the discussion stated above we make use of the condition \( \lambda(n, t) > 0 \) instead of \( \lambda(n, t) \geq c \) where \( c \) is a positive constant. From our numerical results, the existence of such a \( c \) is directly confirmed, and hence the latter condition is satisfied.

It should be noted that for a larger value of \( R_0 \), the colored region appears immediately at the beginning of the injection as shown in Fig. 6. Actually, Fig. 6 and Fig. 7 depict the smallest mode \( n_* \) for \( R_0 = 10.0 \) cm. As shown in these figures, the colored region does not change significantly, except for the appearance of the narrow higher-mode region on the top left. Note that Fig. 7 and Fig. 5 are very similar with each other, though \( R(t) \) at \( t = 2.59 \times 10^7 \) in Fig. 7 with \( R_0 = 10 \) is different from that in Fig. 5 with \( R_0 = 0.01 \), which suggest that the dependence on \( R_0 \) is negligible at a sufficiently large time \( t \).

3.2 The case of the suction

We can treat the case of the suction by the same formulation with \( Q < 0 \). Similarly to the case of the injection, the condition \( \lambda(n, t) > 0 \) is equivalent to two conditions: \( |A| < 1 \) and \( t \geq \max(0, t_n) \), where \( t_n \) is defined as (11) and satisfies that \( t_n > t_{n+1} > t_{n+2} \ldots \) for \( n_* \leq n \), where \( n_* = 2 \) for \( 11/16 < A \leq 1 \) and \( n_* = 2 \) for \( 0 \leq A < 11/16 \). It should be noted in this case there exists \( T \) defined as (11) and \( t \) is bounded as \( 0 \leq t < T \), since the fluid is totally sucked at \( t = T \). Then, for fixed \( A \) and \( Q \), the modes \( n+2, n+1, n \ldots \) drop out of unstable modes as the increasing time \( t \) passes \( t_{n+2}, t_{n+1}, t_n, \ldots \), respectively. It is also found that if \( |A| < 1 \) is satisfied then \( \lambda(n, t) < 0 \) for any \( Q < 0 \) and \( t \in [0, T] \), which gives the stable interface.

Thus, we can find the largest mode \( n \) such that \( \lambda(n, t) > 0 \) for given \( (A, Q) \), as shown in Figs. 8–11. Each figure represents \( t = 1.0, 10, 30, \) and 31.4 sec, after the beginning of the suction with \( R_0 = 10 \) cm, respectively. Here, for \( Q = -10 \), \( T \) is calculated as \( T = 31.4 \). As shown in these figures, the colored region does not change significantly with respect to \( t \) until \( t = 10 \) in
Fig. 9. However, as shown in Fig. 10 and Fig. 11, the mode $n = 10$ in the bottom left of the figures gradually disappears as $t$ approaches $T$. This is because $t$ approaches $T$ and turned to be larger than $t_{10}$ around $t = 30$, which means that the mode $n = 10$ is not an unstable mode any more. At $t = 31.4$, which is very close to $T$, the interface almost vanishes.

From (8), it is also found that if $A|n| > 1$ then \( \lambda(n, t) < 0 \) for all $t$, and $A|n| > 1$ holds only if $A > 1/2$, as the black regions in the figures show. These figures suggest that the interface can be unstable for $A < 1/2$ although there is a suction. It is also examined that the colored region does not change significantly for the different values of $R_0$.

4. Conclusion and future works

We carry out the linear stability analysis for a free boundary in a Hele-Shaw cell. If the linear growth rate for a mode $n$, which depends on $R_0$, $t$, $A$ and $Q$, is positive, then the mode $n$ becomes unstable. The previous criterion given by the Saffman–Taylor instability tells that for a positive $A$ the interface becomes unstable for an injection, while it remains stable for a suction. On the other hand, even though the case of the injection, Figs. 2–7 numerically show that for sufficiently small values of $(A, Q)$ the interface can be stable for extremely long time, compared with the typical experiment (usually, at most several tens of seconds) as in [7]. Since such a parameter region of $(A, Q)$ which satisfies $\lambda(n, t) < 0$ vanishes as $t \to \infty$, our results do not contradict with the Saffman–Taylor instability. However, our results suggest that one can delay the destabilization of an interface by tuning $A$ and $Q$, i.e., by choosing the viscosity and the volume flow rate of the injecting fluid. In fact, as shown in Fig. 4, the interface becomes unstable for the parameter values $(A, Q) = (0.2, 2)$, while it remains stable for $(A, Q) = (0.1, 2)$ until $t = 3.6 \times 10^3$, without changing the volume flow rate $Q$.

For the case of $Q < 0$, the interface remains stable for $1/2 \leq A \leq 1$, which is given by the Saffman–Taylor instability. However, for $0 \leq A < 1/2$, the interface can be unstable even though there is the suction, as shown in Figs. 8–11. This result is not indicated by the work by Saffman and Taylor [1]. Thus our results give a complement to the criterion by the Saffman–Taylor instability.

Our results are based on the linear stability analysis. As a future work, these results should be examined by numerical calculations. It is also interesting to carry out the weakly nonlinear stability analysis, by taking the quadratic terms for $\zeta$ into consideration. This is also to be investigated as a future work.

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