The Hamiltonian formulation of N-bein, Einstein-Cartan, gravity in any dimension: the Progress Report
(Extended version of a talk given on CAIMS-2009, June 11-14, London, Canada)

N. Kiriushcheva
Department of Applied Mathematics,
University of Western Ontario, London, Canada

S.V. Kuzmin
Faculty of Arts and Social Science,
Huron University College and Department of Applied Mathematics,
University of Western Ontario, London, Canada

Abstract

The Hamiltonian formulation of N-bein, Einstein-Cartan, gravity, using its first order form in any dimension higher than two, is analyzed. This Hamiltonian formulation allows to explicitly show where peculiarities of three dimensional case (A.M.Frolov et al, 0902.0856 [gr-qc]) occur and make a conjecture, based on presented in this report results, that there is one general for all dimensions characteristic of N-bein formulation of gravity: after elimination of second class constraints the algebra of Poisson brackets among remaining first class secondary constraints is the Poincaré algebra and in all dimensions N-bein, Cartan-Einstein, gravity is the Poincaré gauge theory. The gauge symmetry corresponding to the algebra of first class constraints has two parameters- rotational (Lorentz) and translational. Translational invariance is common to all dimensions but some terms in general expressions for gauge transformations of N-beins and connections are zero in a particular, three dimensional, case.

The proof of our conjecture is outlined in detail. Some straightforward but tedious calculations remain to be completed to call our conjecture - a theorem and will be reported later.

*Electronic address: nkiriush@uwo.ca
†Electronic address: skuzmin@uwo.ca
I. INTRODUCTION

In this Report we continue the analysis of the Hamiltonian formulation of Einstein-Cartan, N-bein gravity (tetrads in four dimensional case), using its first-order form. This analysis was started in [1] where the complete treatment of three dimensional case (3D, \(D\) is the dimension of spacetime) was discussed. In the considering Hamiltonian formulation we will compare quite often higher dimensional cases with [1] because Dirac’s method of constraint dynamics [2] (and his more technical initial articles [3] based on the course of lectures given at Canadian Mathematical Seminar, Vancouver, August-September 1949) is perfectly suitable for such a task as this is the general method not related to a particular dimension. To be able to use advantages of this method, we do not specialize our formulation to any particular representation of variables which is valid only in a particular dimension and work with variables in which the Lagrangian of N-bein gravity is originally formulated. Note that this is a Progress Report, not a regular article or review, and our main goal here is to provide details of calculations (still in progress). Some related references can be found in [1].

The Lagrangian of Einstein-Cartan, N-bein, gravity written in its first order form is

\[ L \left( e_{\mu(\alpha)}, \omega_{\mu(\alpha\beta)} \right) = -e \left( e^{\mu(\alpha)} e^{\nu(\beta)} - e^{\nu(\alpha)} e^{\mu(\beta)} \right) \left( \omega_{\nu(\alpha\beta)},\mu + \omega_{\mu(\alpha\gamma)} \omega_{\nu(\gamma\beta)} \right), \quad (1) \]

where the covariant N-beins \( e_{\gamma(\rho)} \) and the connections \( \omega_{\nu(\alpha\beta)} \) (\( \omega_{\nu(\alpha\beta)} = -\omega_{\nu(\beta\alpha)} \)) are treated as independent fields, and \( e = \text{det} \left( e_{\gamma(\rho)} \right) \).\(^1\) Greek letters indicate covariant indices \( \alpha = 0, 1, 2, \ldots (D - 1) \). Indices in brackets (..) denote the internal (“Lorentz”) indices, whereas indices without brackets are external or “world” indices. Internal and external indices are raised and lowered by the Minkowski tensor \( \tilde{\eta}_{\alpha\beta} = (-,+,+,...) \) and the metric tensor \( g_{\mu\nu} = e_{\mu(\alpha)} e^{\nu(\alpha)} \), respectively (here and below we will use tilde for any combination with only internal indices and do not use brackets in these cases, except for antisymmetric indices). We assume that N-beins are invertible and \( e^{\mu(\alpha)} e_{\mu(\beta)} = \delta_{\beta}^\alpha \), \( e^{\mu(\alpha)} e_{\nu(\alpha)} = \delta_{\nu}^\mu \). For the Hamiltonian formulation where we have to separate space and time indices (not a spacetime itself on space and time) we are using 0 for an external time index ((0) for an

\(^1\) Usually variables \( e_{\gamma(\rho)} \) and \( \omega_{\nu(\alpha\beta)} \) are named tetrads and spin connection, but such names are specialized for \( D = 4 \). As we consider the Hamiltonian formulation in any dimension (\( D > 2 \), we will call \( e_{\gamma(\rho)} \) and \( \omega_{\nu(\alpha\beta)} \) N-beins and connections, respectively.
internal “time” index) and Latin letters for spatial external indices \( k = 1, 2, \ldots (D - 1) \) \((k)\) for internal “spatial” indices).

For the N-bein gravity, equation (1) represents the first order form because variation of this Lagrangian with respect to connections treated as independent variables gives an equation of motion that can be solved for connections and the solution of it gives exactly the definition of a connection

\[
\omega^{(\alpha\beta)}_\sigma = \frac{1}{2} e^{\sigma(\lambda)} \left( A^{\varepsilon(\mu)(\lambda)}_{\sigma(\mu)} e^{(\beta)}_{\varepsilon,\mu} + A^{\varepsilon(\mu)(\beta)}_{\sigma(\mu)} e^{(\alpha)}_{\varepsilon,\mu} - A^{\varepsilon(\beta)(\mu)}_{\sigma(\mu)} e^{(\lambda)}_{\varepsilon,\mu} \right)
\]  

(2)

where

\[
A^{\mu(\alpha)\nu(\beta)} = e^{\mu(\alpha)} e^{\nu(\beta)} - e^{\mu(\beta)} e^{\nu(\alpha)}.
\]  

(3)

The standard, second order form, is \((\text{II})\) with connections as short notation for the combination \((\text{II})\), not independent variables.

The proof that connections can be treated as independent fields, i.e. solving a corresponding variational equation for \(\omega_{\nu(\alpha\beta)}\) in terms of \(e_{\gamma(\rho)}\), closely resembles a solution of similar problem in formulation of metric gravity when possibility to treat affine connections as independent fields for the standard Einstein-Hilbert Lagrangian is discussed. This solution was first given by Einstein [4] and the similar problem for particular combinations of affine connections was considered in [5]. In case of N-bein formulation such a proof, to the best of our knowledge, was not published anywhere and for the pedagogical reason we provide it in next Section. This proof also allows us to establish notation that will be useful in considering the Hamiltonian formulation of N-bein gravity and to solve similar equation that arises in higher than three dimensions of N-bein formulation during elimination of second class constraints.

II. THE PROOF THAT CONNECTIONS CAN BE TREATED AS INDEPENDENT VARIABLES

First of all, to make the analysis more transparent, using integration by parts, we rewrite the Lagrangian \((\text{II})\) in the form

\[
L \left( e_{\mu(\alpha)}, \omega_{\mu(\alpha\beta)} \right) = e B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} e^{\gamma(\rho)\mu} e^{\nu(\alpha\beta)} - e A^{\mu(\alpha)\nu(\beta)} \omega_{\mu(\alpha\gamma)} \omega_{\nu(\beta)}
\]  

(4)
where
\[ B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} = e^{\gamma(\rho)} A^{\mu(\alpha)\nu(\beta)} + e^{\gamma(\alpha)} A^{\mu(\beta)\nu(\rho)} + e^{\gamma(\beta)} A^{\mu(\rho)\nu(\alpha)}. \] (5)

Of course, equations of motion for (1) and (4) remain the same and variation with respect to a connection is
\[ \frac{\delta L}{\delta \omega_{\sigma(\lambda \tau)}} = -e A^{\sigma(\lambda)\nu(\beta)} \omega_{\nu,\beta}^{-(\tau)} + e A^{\sigma(\tau)\nu(\beta)} \omega_{\nu,\beta}^{-(\lambda)} + e B^{\gamma(\rho)\mu(\lambda)\sigma(\tau)} e_{\gamma(\rho),\mu} = 0. \] (6)

In case of affine-metric gravity, to solve similar equations one has to have all free indices in one position (covariant or contravariant). For the N-bein Lagrangian it is more complicated because we have indices of different nature (internal and external) and cannot use permutation of indices immediately, like in Einstein’s solution [4]. So, the first step is to consider combinations with the same indices and try to solve for them. To accomplish this we introduce the following combinations
\[ \bar{\omega}^{\lambda(\tau)}_{\beta...} \equiv e^{\nu(\lambda)} \omega_{\nu,\beta}^{-(\tau)}. \] (7)

Using (7) the variation (6) can be rewritten as
\[ -e A^{\sigma(\lambda)\nu(\beta)} \bar{\omega}^{\beta(\tau)}_{\beta...} + e A^{\sigma(\tau)\nu(\beta)} \bar{\omega}^{\lambda(\tau)}_{\beta...} + e B^{\gamma(\rho)\mu(\lambda)\sigma(\tau)} e_{\gamma(\rho),\mu} = D^{\sigma(\lambda \tau)}, \] (8)

where
\[ D^{\sigma(\lambda \tau)} \equiv -B^{\gamma(\rho)\mu(\lambda)\sigma(\tau)} e_{\gamma(\rho),\mu}. \] (9)

Now, contracting with \( e^{(e)}_{\sigma} \) to eliminate only one external index that is left and to have all free indices of the same nature (internal only, in our case), we obtain
\[ -\bar{\eta}^{\epsilon \lambda} \bar{\omega}^{\beta(\tau)}_{\beta...} + \bar{\omega}^{\lambda(\tau)}_{\beta...} + \bar{\eta}^{\epsilon \tau} \bar{\omega}^{\beta(\lambda)}_{\beta...} - \bar{\omega}^{\tau(\lambda)}_{\beta...} = \tilde{D}^{\epsilon(\lambda \tau)}, \] (10)

where
\[ \tilde{D}^{\epsilon(\lambda \tau)} \equiv e^{(e)}_{\sigma} D^{\sigma(\lambda \tau)}. \] (11)

If we have only the second and fourth terms in equation (10) then it can be solved using Einstein’s permutation [4] (see below), however we have two terms of lower tensorial
dimensions with only one free index $\tilde{\omega}^{\beta(\tau)}_{\ldots \beta}$ (first and third terms). To find these “traces” we contract (10) with $\tilde{\eta}_{\varepsilon\beta}$

$$\tilde{\omega}^{\beta(\tau)}_{\ldots \beta} = \tilde{D}^{\beta(\tau\beta)}. \tag{12}$$

In two dimensions this equation cannot be solved and, as a result, connections can be treated as independent fields only for $D > 2$. In higher than two dimensions we have

$$\tilde{\omega}^{\beta(\tau)}_{\ldots \beta} = \frac{1}{D - 2} \tilde{D}^{\beta(\tau\beta)}. \tag{13}$$

Upon substitution of (13), equation (10) becomes

$$+ \tilde{\omega}^{\lambda(\tau\varepsilon)} - \tilde{\omega}^{\tau(\lambda\varepsilon)} = \tilde{D}^{\varepsilon(\lambda\tau)} \tag{14}$$

where

$$\tilde{D}^{\varepsilon(\lambda\tau)} = \tilde{D}^{\varepsilon(\lambda\tau)} + \tilde{\eta}^{\varepsilon\lambda} \frac{1}{D - 2} \tilde{D}^{\varepsilon(\tau\sigma)} - \tilde{\eta}^{\varepsilon\tau} \frac{1}{D - 2} \tilde{D}^{\varepsilon(\lambda\sigma)}. \tag{15}$$

Now, performing permutation of indices in (14) $(\lambda\tau\varepsilon) + (\varepsilon\lambda\tau) - (\tau\varepsilon\lambda)$ (in the way as was done by Einstein in [4], we obtain the solution

$$2\tilde{\omega}^{\lambda(\tau\varepsilon)} = \tilde{D}^{\varepsilon(\lambda\tau)} + \tilde{D}^{\varepsilon(\varepsilon\lambda)} - \tilde{D}^{\varepsilon(\lambda\tau)}. \tag{16}$$

To find an explicit form of this solution we, using properties (5) and definition (11), obtain

$$\tilde{D}^{\varepsilon(\lambda\tau)} = -\tilde{\eta}^{\varepsilon\tau} A^{(\lambda)(\gamma(\rho)_{,\mu})}_{\mu} - \tilde{\eta}^{\varepsilon\lambda} A^{(\rho)(\gamma(\tau))_{\mu}}_{\mu} - \tilde{\eta}^{\varepsilon\rho} A^{(\tau)(\gamma(\lambda))_{\mu}}_{\mu}. \tag{17}$$

The “trace” of this combination is

$$\tilde{D}^{\varepsilon(\lambda\varepsilon)} = - (D - 2) A^{\varepsilon(\lambda)(\gamma(\rho)_{,\mu})}_{\mu} \tag{18}$$

and, using definition of $\tilde{D}^{\varepsilon(\lambda\tau)}$ (15), it immediately follows

$$\tilde{D}^{\varepsilon(\lambda\tau)} = - A^{\varepsilon(\tau)(\gamma(\lambda)_{\mu})}_{\mu} \tag{19}$$

that upon substitution into (16) gives us the solution for $\tilde{\omega}^{\varepsilon\lambda(\tau\varepsilon)}$, or contracting with $e_{\nu(\varepsilon)}$ (see (7))
\[ e_{\nu(\mu)} \tilde{\omega}^{(\nu)} = \omega_{\nu(\mu)} = \frac{1}{2} e_{\nu(\mu)} \left( A^{\mu(\nu)} \right) \]

which is definition of a connection \( \omega_{\nu(\mu)} \). In all higher than two dimensions \( \omega_{\nu(\alpha\beta)} \) can be treated as an independent variable and equivalence of two Lagrangians, \( L(e_{\mu(\alpha)}, \omega_{\mu(\alpha\beta)}) \) and \( L(e_{\mu(\alpha)}) \), is established.

### III. THE N-BEIN HAMILTONIAN

As for any first order formulation (at most linear in “velocities” - time derivatives of fields), the first step of the Hamiltonian formulation is strikingly simple. Separating terms with “velocities” in our

\[ L = eB^{\gamma(\rho)(\alpha\beta)} e_{\gamma(\rho)} \omega_{\nu(\alpha\beta)} + eB^{\gamma(\rho)(k(\alpha))\nu(\beta)} e_{\gamma(\rho),k} \omega_{\nu(\alpha\beta)} - eA^{\mu(\alpha)(\nu(\beta))} \omega_{\mu(\alpha)(\gamma)} \omega_{\nu(\beta)()} \]

we can just read off the total Hamiltonian

\[ H_T = \pi_{\phi(\rho)}^0 \dot{e}_{0(\rho)} + \underbrace{\left( \pi_{k(\rho)} - eB^{k(\rho)(m(\alpha\beta))} \omega_{m(\alpha\beta)} \right)}_{\phi(k(\rho))} \dot{e}_{k(\rho)} + \underbrace{\Pi_{\mu(\alpha\beta)} \dot{\omega}_{\mu(\alpha\beta)}}_{\omega_{mu(\alpha\beta)}} - \underbrace{-eB^{\gamma(\rho)(k(\alpha))\nu(\beta)} e_{\gamma(\rho),k} \omega_{\nu(\alpha\beta)}}_{H_c = -L(\text{part without “velocities”)}} + eA^{\mu(\alpha)(\nu(\beta))} \omega_{\mu(\alpha)(\gamma)} \omega_{\nu(\beta)()} \]

where \( \pi_{\phi(\rho)} \) and \( \Pi_{\mu(\alpha\beta)} \) are momenta conjugate to N-beins and connections. As in any Hamiltonian formulation of a first order action, the number of primary constraints \( (\phi^{(\rho)}, \Phi^{(\alpha\beta)}) \) equals the number of independent variables, or canonical variables, with the fundamental Poisson brackets (PB)

\[ \{ e_{\mu(\alpha)}(x), \pi^{\gamma(\rho)}(y) \} = \delta_{\mu(\alpha)}^{\gamma(\rho)}(x - y) , \quad \{ \omega_{\lambda(\alpha\beta)}(x), \Pi^{(\mu(\rho))}(y) \} = \Delta_{(\alpha\beta)}^{(\mu(\rho))}(x - y) \]

where

\[ \Delta_{(\alpha\beta)}^{(\mu(\rho))} = \frac{1}{2} \left( \delta_{(\alpha\beta)}^{(\mu(\rho))} - \delta_{(\alpha\beta)}^{(\rho(\mu))} \right) . \]

The rest of PB are zero. (In the text we often write a PB without the factor \( \delta (x - y) \)).
Note that properties of antisymmetry (for any pair of internal and external indices) of coefficient functions $A, B$ are very helpful in calculations. For example, the absence of term with connections in primary constraint $\phi^{0(\rho)}$ is just a consequence of this property, because in this case we have $B^{0(\rho)0(\alpha)m(\beta)}$ with two equal indices, and so it is zero. Similarly, calculation of PBs among primary constraints $\phi^{\mu(\rho)}$ is almost obvious if similar properties of next generation of coefficient functions is used. We have already introduced $B$ which is the result of following variation
\[
\frac{\delta}{\delta e_{\gamma(\rho)}} \left( e A^{\mu(\alpha)\nu(\beta)} \right) = e B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} \quad (25)
\]
(it can be expressed in terms of $A$ (see (5))). Next generation of such functions is
\[
\frac{\delta}{\delta e_{\sigma(\tau)}} \left( e B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} \right) = e C^{\sigma(\tau)\gamma(\rho)\mu(\alpha)\nu(\beta)} = 0.
\]
These relations (25, 26) and antisymmetry of these functions will be very helpful for further calculations; we will call them $ABC$ properties. For calculation of PBs among primary constraints $\phi^{\mu(\rho)}$ the antisymmetry of $C$ is sufficient
\[
\left\{ \phi^{\mu(\alpha)}, \phi^{\nu(\beta)} \right\} = e C^{\mu(\alpha)\nu(\beta)0(\alpha')m'(\beta')} \omega_{m'(\alpha')\beta'} - e C^{\mu(\alpha)0(\alpha')m'(\beta')} \omega_{m'(\alpha')\beta'} = 0. \quad (28)
\]
The rest of PBs among primary constraints, $\{ \Pi^{\mu(\alpha\beta)}, \Pi^{\mu(\alpha\beta)} \} = \{ \Pi^{\mu(\alpha\beta)}, \pi^{0(\rho)} \} = \{ \pi^{0(\rho)}, \pi^{0(\rho)} \} = 0$, follows just from definition of the fundamental PB (23). The only non-zero PB among primary constraints is
\[
\left\{ \phi^{k(\rho)}, \Phi^{m(\alpha\beta)} \right\} = -e B^{k(\rho)0(\alpha)m(\beta)}. \quad (29)
\]
Based on this simple analysis, it is clear that only $\phi^{0(\rho)}$ and $\Phi^{0(\alpha\beta)}$ are candidates for first class constraints in any dimension higher than two (this is the case also in $3D$ [1]). The gauge
invariance is derivable if algebra of PBs among all first class constraints is known, an explicit form of transformations depends on this algebra and form of constraints. However, gauge parameters of transformations are defined by primary first class constraints only. It is clear from the first and very simple step of the Dirac procedure that for the primary first class constraints $\phi^0(\rho)$ and $\Phi^{0(\alpha \beta)}$ the only possible gauge parameters are $t(\rho)$ and $r_{(\alpha \beta)}$. There are no primary first class constraints that allow to have any parameter with an external index. As a result, diffeomorphism invariance that needs such a parameter, $\xi_\mu$, cannot be the *gauge symmetry* of N-bein gravity in any dimension. In works claiming that diffeomorphism is the gauge symmetry of N-bein gravity (even in a particular dimension) by referring to “results” of the Hamiltonian formulation with the “diffeomorphism constraint” (spatial or full, it does not matter) the non-canonical change of variables must be performed. So, any connection with the Einstein-Cartan formulation would be lost. Of course, such theories which differ from the original ones, despite that they are obtained by abandoning mathematical rules of ordinary mechanics, can be still considered as some toy models. We do not see any reason why the original formulation of Einstein should be abandoned and, in general, we doubt that any toy obtained by non-canonical transformations of a theory which has a mathematical beauty and experimental conformations has even small hope to produce any meaningful result. Contrary, a bad theory, in principle, can be “converted” into a good one by non-canonical transformations, because equivalence with bad one is lost, but such an approach is not scientific and chances to win such a “game” are infinitesimally small.

Our interest is to find the Hamiltonian formulation of the original Einstein-Cartan theory and we continue with the next step of the Dirac procedure.

### IV. TIME DEVELOPMENT OF PRIMARY CONSTRAINTS

The next step of the Dirac procedure is time development of primary constraints which is a PB of constraints with the total Hamiltonian

$$\dot{\phi}^\mu(\rho) = \{ \phi^\mu(\rho), H_T \},$$

and similarly for $\Phi^{\mu(\alpha \beta)}$. Note that PBs are calculated with the total Hamiltonian which includes all primary constraints. It is customary to call “velocities” in front of primary constraints as “undetermined multipliers” but they are rarely used. In case of N-bein gravity
the true “multiplier” nature of this coefficients (actually, they are the Lagrange multipliers for the second class constraints) becomes important which we clarify in Section VI. So, to make this more transparent, we rewrite the Hamiltonian \((22)\) in the following form

\[
H_T = H_c + \lambda_{\mu(\rho)}\phi^{\mu(\rho)} + \Lambda_{\mu(\alpha\beta)}\Phi^{\mu(\alpha\beta)}. \tag{31}
\]

Considering time development of primary constraints we classify them into three distinct groups.

**First group** consist of \(\phi^{k(m)}\) and \(\Phi^{p(k0)}\) (see \((22)\))

\[
\phi^{k(m)} = \pi^{k(m)} - 2eB^{k(m)0(q0)}\omega_{p(q0)} - eB^{k(m)0(p)0(q)}\omega_{p(q)p} = 0, \tag{32}
\]

\[
\Phi^{p(k0)} = \Pi^{p(k0)} = 0 \tag{33}
\]

and their time development

\[
\dot{\Pi}^{p(m0)} = -\frac{\delta H_c}{\delta \omega^{p(m0)}} + \lambda_{k(q)}eB^{k(q)0(m)p(0)}, \tag{34}
\]

\[
\dot{\phi}^{k(m)} = -\frac{\delta H_c}{\delta \epsilon^{k(m)}} - 2\Lambda_{p(q0)}eB^{k(m)0(q)p(0)} - \Lambda_{p(nq)}eB^{k(m)0(n)p(q)} = 0 \tag{35}
\]

In this group of equations there are extra terms compare with three dimensional case \([1]\). However, for this group such terms neither affect possibility to solve nor the way of solving these equations for \(\Pi^{p(k0)}\), \(\omega_{p(q0)}\), separately or together, with the corresponding to them multipliers \(\lambda_{k(q)}\), \(\Lambda_{p(q0)}\) (determined) in all dimensions. Of course, as in many known cases we can use a short cut and just solve pair of equations \((32, 33)\) for \(\Pi^{p(k0)}\), \(\omega_{p(q0)}\), and substitute the solution into \(H_c\) and the rest of constraints. This is the Hamiltonian reduction - elimination of a pair of canonical variables by solving a pair of second class constraints (this is what we did in \([1]\)). Of course, we have to be careful and calculate the Dirac brackets which after such eliminations quite often coincide with Poisson brackets. In the extended form (keeping multipliers and the corresponding equations), constraints are second class if we can solve these equations for the corresponding to them multipliers.

Note that coefficients in front of all fields that we want to find, \(\omega_{p(q0)}\), \(\lambda_{k(q)}\), and \(\Lambda_{p(q0)}\) \((\Pi^{p(k0)} = 0\) is trivial), are \(B\)–functions of the same structure:
To solve all above equations, we need the inverse for this particular combination. It was found in [1] that there is a combination (not a new variable), similar to those used by Dirac

\[
\gamma^{km} = g^{km} - \frac{g^{k0} g^{m0}}{g^{00}}, \quad \gamma^{km} g_{mn} = \delta^k_n
\]

when he considered Hamiltonian formulation of second order metric gravity [6]. For N-bein gravity it is

\[
\gamma^k(m) \equiv e^k(m) - \frac{e^k(0)e^{0(m)}}{e^{0(0)}}
\]

with properties

\[
\gamma^{m(p)} e_{m(q)} = \delta^p_q, \quad \gamma^{n(q)} e_{m(q)} = \delta^n_m.
\]

that allow to rewrite

\[
B^{0(0)k(q)p(m)} = e^{0(0)} E^{k(q)p(m)}
\]

where

\[
E^{k(m)p(q)} \equiv \gamma^{k(m)} \gamma^p(q) - \gamma^{k(q)} \gamma^p(m).
\]

Note that all B-coefficient in (32), (34) and (35) by permutations of indices can be converted in such a form. For any dimension \(D > 2\) (which is consistent with restriction on possibility to solve for connections (13)) we can find the inverse of \(E^{k(m)p(q)}\)

\[
I_{m(q)a(b)} \equiv \frac{1}{D-2} e^{m(q)} e_{a(b)} - e_{m(b)} e_{a(q)},
\]

\[
I_{m(q)a(b)} E^{a(b)m(p)} = E^{n(p)a(b)} I_{a(b)m(q)} = \delta^m_n \delta^p_q.
\]

So, for example, we have

\[
\omega_{k(0)} = -\frac{1}{2e^{0(0)}} I_{k(q)m(p)} \pi^{m(p)} + \frac{1}{2e^{0(0)}} I_{k(q)m(p)} B^{m(p)0(a)n(b)} \omega_{n(ab)}
\]

and

\[
\lambda_{a(b)} = -I_{a(b)p(m)} \frac{1}{e^{0(0)}} \frac{\delta H_c}{\delta \omega_{p(0)}}.
\]
These solutions suggest to rewrite all $B$ with different combinations of indices in such “$\gamma - E$-form”. Using the identity ((5), (37) and (40) has to be used to prove it)

$$B^0(p)m(a)n(b) = e^0(p)E^m(a)n(b) + e^0(a)E^m(b)n(p) + e^0(b)E^m(p)n(a)$$

we can present (43) in the form

$$\omega_k(q_0) = -\frac{1}{2e_0(0)}I_k(q)m(p)\pi^m(p) - \frac{e^0(p)}{2e_0(0)}I_k(q)m(p)E^m(a)n(b)\omega_n(ab) + \frac{e^0(a)}{e^0(0)}\omega_k(aq).$$

This is general solution valid in all dimensions and 3D form of it is just a limited case. In 3D all spatial components have only two values, 1, 2, for both internal and external indices and, in addition, $I_k(q)m(p)$ becomes antisymmetric as $A, B, C$ and $E$. Taking this into consideration, direct but simple calculations for all four possible components $\omega_1(10)$, $\omega_2(20)$, $\omega_1(20)$ and $\omega_2(10)$ lead to cancellation of two last terms. So, (46) gives the same $\omega_k(q_0)$ for $D = 3$ as was found in [1]. For the Hamiltonian method a separate treatment of 3D case is not needed and all results can be obtained from a general solution. We demonstrated this for (46) but it is also valid for other results, as it will be clear from our further calculations.

Now we consider second group of primary constraints $\phi^{k(0)}$ and $\Phi^{p(km)}$

$$\phi^{k(0)} = \pi^{k(0)} - eB^{k(0)0(p)m(q)}\omega_{m(pq)} = 0, \quad \Phi^{p(km)} = \Pi^{p(km)} = 0$$

and their time development

$$\dot{\phi}^{k(0)} = -\frac{\delta H_c}{\delta \epsilon^{k(0)}} - \Lambda_{m(pq)}eB^{k(0)0(p)m(q)}, \quad \dot{\Phi}^{p(mn)} = -\frac{\delta H_c}{\delta \omega^{p(mn)}} + \lambda_{k(q)}eB^{k(0)0(m)p(n)} + \lambda_{k(0)}eB^{k(0)0(m)p(n)}.$$

In general case neither a short cut (the system (47-48)) nor all equations including the corresponding multipliers (47-50) can be solved because the number of equations is smaller than the number of unknowns. The only exception here is 3D case where the number of equations and unknowns is the same because the number of independent components for all $\omega_{m(pq)}$, $\Pi^{p(km)}$, $\lambda_{k(0)}$, and $\Lambda_{m(pq)}$ is just two [1]. Note that in 3D case this group is
completely decoupled from the first one \((32-35)\). In higher than three dimensions we have next generation of constraints (secondary)

\[
\chi_{p(mn)} = \frac{\delta H_c}{\delta \omega_{p(mn)}}
\]

and only after calculation of its time development we have enough equations to find all fields and associated with them multipliers. We will solve these equations in Section VI and illustrate again that a separate treatment of 3D is not needed because 3D—limit of a general solution gives the same result as \([1]\), as was demonstrated for \((46)\).

Finally we consider the **third group** of primary constraints \(\Pi^0(\alpha \beta)\) and \(\pi^0(\rho)\). Time development of them is

\[
\dot{\Pi}^0(\alpha \beta) = eB^m(\rho)k(\alpha)0(\beta) e_{m(\rho),k} - eA^0(\alpha)k(\beta') \omega_k^{(\beta')} + eA^0(\beta)k(\alpha') \omega_k^{(\alpha')} = \chi^0(\alpha \beta),
\]

\[
\dot{\pi}^0(\rho) = \frac{\delta}{\delta e^{0(\rho)}} \left( eB_{\rho k(\alpha)0(\beta)} \omega_k^{(\beta')} \left( e^{0(\rho)k(\alpha)0(\beta)} \omega_{m(\alpha \beta)} \right)_{0} \right)_{k} = 0 \quad \text{(if } D=3 \text{ and } \rho, \alpha, \beta \neq 0) \]

\[
- \frac{\delta}{\delta e^{0(\rho)}} eA^k(\rho)k(m) \omega_k^{m(\rho)} \omega_m^{(n)} = - \frac{\delta}{\delta e^{0(\rho)}} 2eA^k(\rho)k(m) \omega_k^{m(\rho)} \omega_m^{(n)} = \chi^0(\rho).
\]

Note that both equations \((52)\) and \((53)\) do not have multipliers (the consequence of zero PBs of \(\Pi^0(\alpha \beta)\) and \(\pi^0(\rho)\) with the rest of primary constraints). So, time development of \(\Pi^0(\alpha \beta)\) and \(\pi^0(\rho)\) leads to secondary constraints (we cannot find the corresponding multipliers). Note also that the temporal connections \((\omega_{0(\alpha \beta)})\) are absent in both constraints, so time development of the secondary constraints \(\chi^0(\alpha \beta)\) and \(\chi^0(\rho)\) cannot give us an equation to find \(\Lambda_0(\alpha \beta)\) (PBs of primary constraint \(\Pi^0(\alpha \beta)\) are zero with both secondary constraints). PBs of both constraints with primary \(\pi^0(\rho)\) are also zero (almost manifestly) and cannot give us an equation to find \(\lambda_0(\rho)\). This is the result based on ABC properties. Second variation of each term in \((53)\) leads to antisymmetric combination with two external zeros, e.g.

\[
\frac{\delta}{\delta e^{0(\sigma)}} \left( eB^{m(\rho)k(\alpha)0(\beta)} \right) = eC^{0(\sigma)m(\rho)k(\alpha)0(\beta)} = 0.
\]
All these simple results are equivalent with three dimensional case [1] and provide strong indication that in all dimensions PBs among primary and secondary constraints are zero. We call this “indication” because after solving second class constraints we have to substitute solutions for spatial connections into (53) that will lead to it modification. However, as we will demonstrate later, such substitutions do not affect PBs and zero PBs among primary and secondary constraints are unaltered. So far we have the result that looks the same in all dimensions and equivalent with considered before 3D case [1]. What is the difference in higher dimensions? The difference is obviously in the secondary constraint $\chi^0(\rho)$ (we will call it “translational”, as in 3D case) where we have many “invisible” in three dimension terms (see (53)) that have to change drastically this constraint and make it much richer compare with 3D case. Secondary rotational constraint $\chi^0(\alpha\beta)$ is different compare to $\chi^0(\rho)$, it does not have any three dimensional peculiarities as $\chi^0(\rho)$ and looks absolutely the same in three [1] and all higher dimensions.

V. SOME SIMPLE PRELIMINARY RESULTS

If the Lagrangian with some variables is defined in all dimensions, then the Hamiltonian analysis based on such variables should also be independent on a particular dimension. Quite simple and straightforward first steps of the Dirac procedure performed in previous Sections allow to make some conclusions and perform quick calculations. It is clear that at least part of PB algebra of first class constraints should be the same as in three dimensions, as well as some parts of transformations found in 3D.

Let us briefly review the results of three dimensional case. The total Hamiltonian after elimination of second class constraints is [1] (disregarding a total derivative)

$$H_T = -e_{0(\rho)}\chi^{0(\rho)} - \omega_{0(\alpha\beta)}\chi^{0(\alpha\beta)} + \dot{\epsilon}_{0(\rho)}\pi^{0(\rho)} + \dot{\omega}_{0(\alpha\beta)}\Pi^{0(\alpha\beta)}. \quad (55)$$

Algebra of PB among constraints is the following: all PBs among primary and secondary constraints are zero and secondary constraints have ordinary (Poincaré) algebra as in ordinary field theories (no structure functions or non-localities - derivatives of delta functions)

$$\{\chi^{0(\rho)}, \chi^{0(\gamma)}\} = 0, \quad (56)$$
\[
\{ \chi^{0(\alpha\beta)}, \chi^{0(\rho)} \} = \frac{1}{2} \eta^{(\beta)(\rho)} \chi^{0(\alpha)} - \frac{1}{2} \eta^{(\alpha)(\rho)} \chi^{0(\beta)}, \tag{57}
\]

\[
\{ \chi^{0(\alpha\beta)}, \chi^{0(\mu\nu)} \} = \frac{1}{2} \eta^{(\beta)(\mu)} \chi^{0(\alpha\nu)} - \frac{1}{2} \eta^{(\alpha)(\mu)} \chi^{0(\beta\nu)} + \frac{1}{2} \eta^{(\beta)(\nu)} \chi^{0(\mu\alpha)} - \frac{1}{2} \eta^{(\alpha)(\nu)} \chi^{0(\mu\beta)}. \tag{58}
\]

The simplicity of the Hamiltonian and algebra of constraints (all of them are first class) makes derivation of generators straightforward and gauge invariance of all independent fields immediately follows [1]. The gauge transformations can be cast into a covariant form but we have to remember that this is the result of calculations in three dimensional case where many terms (see (32, 33, 50, 51, 53)) were not taken into account. Transformations for the first order formulation of N-bein gravity for \( D > 3 \) should be modified.

Transformations for 3D are [1]

\[
\delta e_{\gamma(\lambda)} = -t_{(\lambda),\gamma} - \omega_{\gamma(\lambda)}^{(\rho)} t_{(\rho)} - \frac{1}{2} \left( e_{\gamma}^{(\alpha)} \delta_{(\lambda)}^{(\beta)} - e_{\gamma}^{(\beta)} \delta_{(\lambda)}^{(\alpha)} \right) r_{(\alpha\beta)}, \tag{59}
\]

\[
\delta \omega_{\gamma(\sigma\lambda)} = -r_{(\sigma\lambda),\gamma} - \left( \omega_{\gamma}^{(\alpha)} \delta_{(\sigma)}^{(\beta)} - \omega_{\gamma}^{(\beta)} \delta_{(\sigma)}^{(\alpha)} \right) r_{(\alpha\beta)}. \tag{60}
\]

Here \( t_{(\rho)} \) and \( r_{(\alpha\beta)} \) are the translational and rotational gauge parameters, respectively.

Any conclusion about higher dimensions based on three dimensional case should be made with a great care. It is well known fact that the first order N-bein Lagrangian is not invariant under translational (proportional to \( t_{(\rho)} \)) part of (59-60) and only rotational part of these transformations can be promoted from three to any dimension higher than two. Translational part of three dimensional case is not gauge invariance in higher dimensions which is the expected result after neglecting so many terms in (32, 33, 50, 51, 53). However, such simple observation is not sufficient to make any general conclusion about gauge invariance in higher dimensions and especially to say, based only on three dimensional results and without calculations in higher dimensions, that N-bein gravity is not a Poincaré gauge theory. Such a conclusion is in contradiction with an ordinary logic. Much more reasonable expectation based on first steps of the Dirac procedure should be quite different: in higher dimensions a translational part of transformations should be different but algebra of constraints should be unchanged despite modifications of constraints themselves.
Gauge transformations (59, 60) were obtained in [1] using the Castellani procedure [7] and are the result of the PB algebra of constraints (56-58). We know that rotational invariance is the same in all dimensions \( D > 2 \) and at least part of algebra (56-58) must to be the same to preserve a corresponding part of generators. In [1] we built such generators using only this algebra. According to the Castellani procedure they are given by

\[
G = G^{(\rho)}(1) \dot{t}^{(\rho)} + G^{(\rho)}(0) t^{(\rho)} + G^{(\alpha\beta)}(1) \dot{t}^{(\alpha\beta)} + G^{(\alpha\beta)}(0) t^{(\alpha\beta)}. \tag{61}
\]

The functions \( G_{(1)}^{(\rho)} \) in (61) are the primary constraints

\[
G^{(\rho)}_{(1)} = \pi^{0(\rho)} \quad \text{and} \quad G^{(\alpha\beta)}_{(1)} = \Pi^{0(\alpha\beta)} \tag{62}
\]

and \( G_{(0)} \) are defined using the following relations [7]

\[
G^{(\rho)}_{(0)}(x) = - \left\{ \pi^{0(\rho)}(x), H_T \right\} + \int \left[ \tilde{\alpha}^{\rho}_\gamma(x, y) \pi^{0(\gamma)}(y) + \tilde{\alpha}^{\rho}_{(\alpha\beta)}(x, y) \Pi^{0(\alpha\beta)}(y) \right] d^{D-1}y, \tag{63}
\]

\[
G^{(\alpha\beta)}_{(0)}(x) = - \left\{ \Pi^{0(\alpha\beta)}(x), H_T \right\} + \int \left[ \tilde{\alpha}^{(\alpha\beta)}_\gamma(x, y) \pi^{0(\gamma)}(y) + \tilde{\alpha}^{(\alpha\beta)}_{(\nu\mu)}(x, y) \Pi^{0(\nu\mu)}(y) \right] d^{D-1}y, \tag{64}
\]

where the functions \( \tilde{\alpha}^{(\cdot)}_\gamma(x, y) \) have to be chosen in such a way that the chains end at primary constraints

\[
\left\{ G^{\sigma}_{(0)}, H_T \right\} = \text{primary.} \tag{65}
\]

To construct the generator (61), we have to find \( \tilde{\alpha}^{(\cdot)}_\gamma(x, y) \) using condition (65). This calculation, because of the simple PBs among the constraints, is straightforward:

\[
\left\{ G^{(\rho)}_{(0)}(x), H_T \right\} = \left\{ -\chi^{0(\rho)}(x) + \int \left[ \tilde{\alpha}^{\rho}_\gamma(x, y) \pi^{0(\gamma)}(y) + \tilde{\alpha}^{\rho}_{(\alpha\beta)}(x, y) \Pi^{0(\alpha\beta)}(y) \right] d^{D-1}y, H_T \right\} = 0, \tag{66}
\]

\[
\left\{ G^{(\alpha\beta)}_{(0)}(x), H_T \right\} = \left\{ -\chi^{0(\alpha\beta)}(x) + \int \left[ \tilde{\alpha}^{(\alpha\beta)}_\gamma(x, y) \pi^{0(\gamma)}(y) + \tilde{\alpha}^{(\alpha\beta)}_{(\nu\mu)}(x, y) \Pi^{0(\nu\mu)}(y) \right] d^{D-1}y, H_T \right\} = 0 \tag{67}
\]
where $H_T$ can be replaced by $H_c = -e_{0(\sigma)}\chi^{0(\sigma)} - \omega_{0(\sigma\lambda)}\chi^{0(\sigma\lambda)}$, because PBs among primary constraints themselves and among primary and secondary constraints are zero. (These calculations are simpler compared with the Hamiltonian formulation of metric gravity \[8\].)

From (66) and (67) and the PBs among first class constraints we found all the functions $\tilde{\alpha}^{(\cdot)}(x, y)$ in (63, 64):

\[
\tilde{\alpha}^{(\rho)}(x, y) = 0, \quad (68)
\]

\[
\tilde{\alpha}^{(\gamma)}(x, y) = \omega_{0(\gamma)} \delta(x - y), \quad (69)
\]

\[
\tilde{\alpha}^{(\gamma\beta)}(x, y) = \frac{1}{2} \left( e_{0(\alpha)} \delta_{(\gamma)}^{(\beta)} - e_{0(\beta)} \delta_{(\gamma)}^{(\alpha)} \right) \delta(x - y), \quad (70)
\]

\[
\tilde{\alpha}^{(\alpha\beta)}(x, y) = \left( \omega_{0(\alpha)} \delta_{(\beta)}^{(\gamma)} - \omega_{0(\beta)} \delta_{(\alpha)}^{(\gamma)} \right) \delta(x - y). \quad (71)
\]

This completes the derivation of the generator (61) as now

\[
G^{(\rho)}_0 = -\chi^{0(\rho)} + \omega_{0(\gamma)} \pi^{0(\gamma)}, \quad (72)
\]

and

\[
G^{(\alpha\beta)}_0 = -\chi^{0(\alpha\beta)} + \frac{1}{2} \left( e_{0(\alpha)} \delta_{(\gamma)}^{(\beta)} - e_{0(\beta)} \delta_{(\gamma)}^{(\alpha)} \right) \pi^{0(\gamma)} + \omega_{0(\mu)} \Pi^{0(\beta\mu)} - \omega_{0(\beta)} \Pi^{0(\alpha\mu)}. \quad (73)
\]

Using

\[
\delta(field) = \{G, field\} = \left\{ G^{(\rho)} \dot{t}(\rho) + G^{(\rho)}_0 t(\rho) + G^{(\alpha\beta)}_0 \dot{r}(\alpha\beta) + G^{(\alpha\beta)}_0 r(\alpha\beta), field \right\} \quad (74)
\]

we can find the gauge transformations of fields.

Because rotational invariance is the same in all dimensions, the absence of dimensional peculiarities in (52) is consistent with this fact. Part of the generator (61) responsible for this transformation must be the same in all dimensions ($D > 2$), so as the corresponding $\tilde{\alpha}^{(\cdot)}(x, y)$ (70, 71) that follow from (67). This makes even small deviations from the Poincaré algebra almost impossible. In addition, whatever modifications of a translational constraint
we have in higher dimensions they suppose to preserve the limit of three dimensional case where, as we know, we have the Poincaré algebra.

Let us provide some simple calculations that are based on and supported by our reasonings. The rotational constraint is the same in all dimensions, consequently, the solution for \( \omega_{m(pq)} \) (different in higher dimension, see (47-51)) should not affect \( \chi^{0(\alpha\beta)} \). We should be able to find this constraint in any dimension without solving for \( \omega_{m(pq)} \). In 3D we obtained

\[
\chi^{0(\alpha\beta)} = \frac{1}{2} e^{(\alpha)}_{k} \pi^{k(\beta)} - \frac{1}{2} e^{(\beta)}_{k} \pi^{k(\alpha)} - e B^{(\rho)k(\alpha)0(\beta)} e_{\gamma(\rho),k}
\]  

and demonstrated that such a constraint satisfies (58) without any reference to a particular dimension.

Without solving for \( \omega_{m(pq)} \) in higher dimensions using the secondary constraint, we have the relation from the primary constraint (22)

\[
\pi^{k(\rho)} = e B^{k(\rho)0(\alpha)m(\beta)} \omega_{m(\alpha\beta)}.
\]  

We substitute this into the first two terms of (75) and, after short rearrangements and using properties of \( B \) (5), we obtain

\[
\frac{1}{2} e^{(\alpha)}_{k} \pi^{k(\beta)} - \frac{1}{2} e^{(\beta)}_{k} \pi^{k(\alpha)} = -e A^{(0(\alpha)k(\gamma))} \omega_{k(\gamma)}^{(\beta)} + e A^{(0(\beta)k(\gamma))} \omega_{k(\gamma)}^{(\alpha)}
\]

which is exactly the expression that we have in a general case (52). We, almost at once, have the rotational constraint, moreover, as it was demonstrated (see discussion after (53)), before we used a solution of the secondary constraint, its PBs with primary constraints \( \Pi^{0(\alpha\beta)} \) and \( \pi^{0(\rho)} \) are zero and after substitution of (76) (expression of spatial connections in terms of momenta) they remain to be zero

\[
\{ \Pi^{0(\alpha\beta)}, \chi^{0(\alpha\beta)} \} = \{ \pi^{0(\rho)}, \chi^{0(\alpha\beta)} \} = 0.
\]  

So, as it was calculated in three dimensional case [1], PBs among (75) are (58). We demonstrated that this part of PB algebra is the same in all dimensions.

In a general expression of the translational constraint we also have one contribution that can be almost immediately obtained without solution for \( \omega_{m(pq)} \), just by using primary second class constraints (76). The second term of (53) consists of exactly the same combination (76)
and upon substitution gives a simple contribution into the secondary translational constraint in all dimensions and such a contribution supports three dimensional limit (see [1])

\[ \chi^0(\rho) = \pi^k(\rho) + \ldots \] (79)

This part by itself (as it was before substitution) has zero PBs with both primary constraints and supports all relations of the Poincaré algebra found in three dimensions (56, 57). To consider the rest of contributions in (53) the biggest part of which is zero in three dimensional case, we have to find the solution for \( \omega_{m(pq)} \). So, go back to the second group of equations (47-51).

VI. FINDING THE SOLUTION FOR \( \omega_{m(pq)} \)

Using the first group of equations (32-35), components of a spin connection \( \omega_{m(\rho0)} \) and the corresponding to them momenta can be solved as in 3D [1], of course, with some extra contributions. Now to find \( \omega_{m(pq)} \) we consider two equations (47) and (50) from the second group in \( D > 2 \):

\[ \pi^k(0) + 2e e^0(0) \gamma^k(p) \gamma^m(q) \omega_{m(pq)} = 0, \] (80)

\[ -\frac{\delta H_c}{\delta \omega_{p(mn)}} + \lambda_{k(q)} e B^{k(q)0(m)p(n)} - \lambda_{k(0)} e e^0(0) E^{k(m)p(n)} = 0. \] (81)

First equation is slightly modified using (39, 40) and antisymmetry of connections. In (81) the solutions for \( \omega_{k(\rho0)} \) (48) and the multiplier \( \lambda_{k(q)} \) (44) have to be substituted. We have two sets of equations to solve for two sets of variables \( \omega_{m(pq)} \) and \( \lambda_{k(0)} \). The number of unknowns and equations are the same in any dimension \( D > 2 \). This system of equations is a perfect illustration of the fact (which is often hidden in simple theories) that \( \lambda_{k(0)} \) are true Lagrange multipliers: we have to find a solution for \( \omega_{p(mn)} \) with extra condition (81) and the Dirac method automatically gives correct settings for the problem of finding conditional extremum.

To perform substitutions of (44) and (46), the explicit form of variation \( -\frac{\delta H_c}{\delta \omega_{p(mn)}} \) is needed

\[ -\frac{\delta H_c}{\delta \omega_{p(mn)}} = e B^\gamma(\rho) k(m)p(n)e_{\gamma(\rho),k} - e A_{p(m)} k(q) \omega^{(n)}_k \omega^{(m)}_q + e A_p(n) k(q) \omega^{(m)}_k \omega^{(n)}_q \]
as well as \( \frac{\delta H_c}{\delta \omega_{p(0)}} \):

\[
\frac{\delta H_c}{\delta \omega_{p(0)}} = -e B^\gamma(q)k(m)p(0)e_{\gamma(q),k}
\]

\[
+ e A^p(m)n(q)\omega_n^{(0)} - e A^p(0)n(q)\omega_n^{(m)} + e A^p(m)0(q)\omega_0^{(0)} - e A^p(0)0(q)\omega_0^{(m)}.
\]  

(83)

Solutions of previous equations suggest transition of all expressions into "\( \gamma - E \)"-form that after short rearrangements, using (15) and

\[
A^p(m)k(q) = E^p(m)k(q) + \frac{e^0(m)}{e^0(0)} A^p(0)k(q) + A^p(m)k(0) \frac{e^0(q)}{e^0(0)},
\]

(84)

converts (81) into

\[
-E^p(m)k(q)\omega_k^{(n)} + E^p(n)k(q)\omega_k^{(m)} + \frac{e^0(m)}{e^0(0)} E^p(n)n'(q') e^0(a) e^0(a') + \frac{e^0(n)}{e^0(0)} E^p(m)n'(q') e^0(a) e^0(a')
\]

\[
- \frac{e^0(q)}{e^0(0)} E^k(m)p(n)I_{k(q)p'(m')}A^p(0)n(q')\omega_n^{(m')} - \frac{e^0(q)}{e^0(0)} E^k(m)p(n)I_{k(q)p'(m')}A^p(0)n(q')\omega_n^{(m')}
\]

\[
+ D^{p(mn)}(e_{\mu(\nu),k}) + D^{p(mn)}(\omega_{0(\alpha,\beta)}) + D^{p(mn)}(\pi^{k(m)}) + D^{p(mn)}(\pi^{k(0)})
\]

\[
- \lambda_k(0)e^0(0)E^k(m)p(n) = 0
\]

(85)

where

\[
D^{p(mn)}(e_{\mu(\nu),k}) = B^{\gamma(\nu)k(m)p(n)}e_{\gamma(\rho),k} + B^{k(q)0(0)m(p(n))}I_{k(q)p'(m')} \frac{1}{e^0(0)} B^{\gamma(q')k'(m')}p'(0) e_{\gamma(q'),k'};
\]

(86)

\[
D^{p(mn)}(\omega_{0(\alpha,\beta)}) = -A^{p(m)0(\alpha)} \omega_0^{(n)} + A^{p(n)0(\alpha)} \omega_0^{(m)}
\]

\[
- B^{k(q)0(0)m(p(n))}I_{k(q)p'(m')} \frac{1}{e^0(0)} \left( A^{p'(m')0(q')} \omega_0^{(0)} - A^{p'(0)0(q')} \omega_0^{(m')} \right);
\]

(87)
\[- \frac{1}{2ee^{0(0)}} I_{k(\pi^0(p'))} = A^{(m)k(0)} \frac{1}{2ee^{0(0)}} I_{k(m')p'} \pi^{m'(p')} - A^{(n)k(0)} \frac{1}{2ee^{0(0)}} I_{k(m')p'} \pi^{m'(p')}
\]

\[- \frac{1}{2ee^{0(0)}} E^{k(m)p}(n(q')) \frac{1}{2ee^{0(0)}} I_{n(q')a(b)} \pi^{a(b)} \frac{1}{2ee^{0(0)}} I_{n(q')a(b)} \pi^{a(b)} \pi^{n'(p')}, \]  

\[D^{(p)(m)} \left( \pi^{k(0)} \right) = A^{(p)(m)k(0)} \frac{1}{2ee^{0(0)}} I_{k(m')p'} E^{m'(a)n(b)} \omega_{n(ab)}
\]

\[- A^{(n)k(0)} \frac{1}{2ee^{0(0)}} I_{k(m')p'} E^{m'(a)n(b)} \omega_{n(ab)}
\]

\[+ \frac{1}{2ee^{0(0)}} E^{k(m)p}(n(q')) \frac{1}{2ee^{0(0)}} I_{n(q')a(b)} \pi^{a(b)} \frac{1}{2ee^{0(0)}} I_{n(q')a(b)} \pi^{a(b)} \pi^{n'(p')}, \]  

\[D^{(p)(m)} \left( \pi^{k(0)} \right) = A^{(p)(m)k(0)} \frac{1}{2ee^{0(0)}} I_{k(m')p'} E^{m'(a)n(b)} \omega_{n(ab)}
\]

\[- A^{(n)k(0)} \frac{1}{2ee^{0(0)}} I_{k(m')p'} E^{m'(a)n(b)} \omega_{n(ab)}
\]

\[+ \frac{1}{2ee^{0(0)}} E^{k(m)p}(n(q')) \frac{1}{2ee^{0(0)}} I_{n(q')a(b)} \pi^{a(b)} \frac{1}{2ee^{0(0)}} I_{n(q')a(b)} \pi^{a(b)} \pi^{n'(p')}. \]  

All terms in equation (85) with a particular contraction \(E^{d(a)n(b)}\omega_{n(ab)}\) are in \(D^{(p)(m)} \left( \pi^{k(0)} \right)\) because they are expressible in terms of momenta

\[- \frac{1}{2ee^{0(0)}} \pi^{k(0)} = E^{k(p)m(q)} \omega_{m(pq)} = 2\gamma^{k(p)} \gamma^{m(q)} \omega_{m(pq)}. \]  

As in a covariant case considered in Introduction, to solve equation (85) (similar with (3)) we have to have combinations with internal indices only, which in this case are

\[\gamma^{k(m)} \omega_{k(n)} = \tilde{\omega}^{m(q)}, \]  


20
and contract (85) with $e_{p}^{(s)}$ (we just mimic a solution in a covariant case, see (18)). Note also that, compare with a covariant case (7), in (91) we have contraction in spatial indices only.

The resulting equation is (all free indices are internal)

$$\tilde{\omega}^{m(ns)} - \tilde{\omega}^{n(ms)} - \frac{\epsilon_{0}^{(m)}}{\epsilon_{0}^{(0)}} e_{(a)}^{0} \tilde{\omega}^{n(as)} + \frac{\epsilon_{0}^{(n)}}{\epsilon_{0}^{(0)}} e_{(a)}^{0} \tilde{\omega}^{m(as)}$$

$$- \tilde{\eta}^{sn} \tilde{\omega}^{q(n \ldots q)} + \tilde{\eta}^{sn} \tilde{\omega}^{q(n \ldots q)} + \tilde{\eta}^{sn} \frac{\epsilon_{0}^{(m)}}{\epsilon_{0}^{(0)}} \frac{\epsilon_{0}^{(a)}}{\epsilon_{0}^{(0)}} \tilde{\omega}^{q'(aq')} - \tilde{\eta}^{sm} \frac{\epsilon_{0}^{(n)}}{\epsilon_{0}^{(0)}} \frac{\epsilon_{0}^{(a)}}{\epsilon_{0}^{(0)}} \tilde{\omega}^{q'(aq')}$$

$$- \frac{\epsilon_{0}^{(q)}}{\epsilon_{0}^{(0)}} \left( \gamma^{k(m)} \tilde{\eta}^{sm} - \gamma^{k(n)} \tilde{\eta}^{sm} \right) I_{k(q)p'(m')} A^{p'(0)n(q')} \omega_{n}^{(m'q')}$$

$$- \frac{\epsilon_{0}^{(q)}}{\epsilon_{0}^{(0)}} \left( \gamma^{k(m)} \tilde{\eta}^{sm} - \gamma^{k(n)} \tilde{\eta}^{sm} \right) I_{k(q)p'(m')} \frac{\epsilon_{0}^{(m')}}{\epsilon_{0}^{(0)}} A^{p'(0)n(q')} \frac{\epsilon_{0}^{(a)}}{\epsilon_{0}^{(0)}} \omega_{n}^{(m'aq')}$$

$$+ \tilde{D}^{s(mn)} \left( \pi^{k(m)} \right) + \tilde{D}^{s(mn)} \left( \pi^{k(0)} \right) + \tilde{D}^{s(mn)} \left( \epsilon_{(\mu),k} \right) + \tilde{D}^{s(mn)} \left( \omega_{(\alpha\beta)} \right)$$

$$- e^{0(0)} \left( \tilde{\lambda}^{m} \tilde{\eta}^{sn} - \tilde{\lambda}^{n} \tilde{\eta}^{sm} \right) = 0 \quad (93)$$

where

$$\tilde{D}^{s(mn)} \equiv e_{p}^{(s)} D^{p(mn)} \quad (94)$$

and

$$\gamma^{k(n)} \lambda_{k(0)} \equiv \tilde{\lambda}^{n} \quad (95)$$

The solution for free part of (93), two first terms, is known, and we have to eliminate the contractions (as we did this in (12)). We have once contracted terms with two free indices in the first line of (93), $e_{(a)}^{0} \tilde{\omega}^{m(as)}$, and with one free index (third line), such as $\tilde{\eta}^{sn} \tilde{\lambda}^{m}$ (the explicit form of $\tilde{\lambda}^{m}$ can be read off from (93)). First, as elimination of “trace” in covariant case, we consider contraction of (93) with $\tilde{\eta}_{ms}$ that gives

$$\tilde{\omega}_{s}^{-(ns)} + \frac{\epsilon_{0}^{(n)}}{\epsilon_{0}^{(0)}} e_{(a)}^{0} \tilde{\omega}_{s}^{-(as)} \quad (96)$$
\[-(D - 2) \tilde{\omega}_{(n \omega)} - (D - 2) \frac{e^{0(q)}}{e^{0(0)}} \frac{e^{0(a)}}{e^{0(0)}} \tilde{\omega}_{(aq')}
\]

\[+ \frac{e^{0(q)}}{e^{0(0)}} \gamma^{k(n)} (D - 2) \tilde{\eta}^{sm} I_{k(q)p'(m')} A^{p'(0)n(q')} \omega_{(m' q')}^{(m')}
\]

\[+ \frac{e^{0(q)}}{e^{0(0)}} \gamma^{k(n)} (D - 2) I_{k(q)p'(m')} \frac{e^{0(m')}}{e^{0(0)}} A^{p'(0)n(q')} \frac{e^{0(a)}}{e^{0(0)}} \omega_{(aq')}
\]

\[+ \tilde{D}_{m}^{(mn)} (\pi^{k}(m)) + \tilde{D}_{m}^{(mn)} (\pi^{k(0)}) + \tilde{D}_{m}^{(mn)} (\eta^{\mu(\nu), k}) + \tilde{D}_{m}^{(mn)} (\omega_{0(\alpha\beta)}) + \tilde{\lambda}^{n} e^{0(0)} (D - 2) = 0.
\]

As before, \( \tilde{\omega}_{(ns)} \) is known and expressible in terms of \( \pi^{k(0)} \) using (91) and we can solve this equation for the multiplier \( \tilde{\lambda}^{n} \)

\[\tilde{\lambda}^{n} = \frac{D - 3}{D - 2} \frac{1}{e^{0(0)}} \left( \tilde{\omega}_{(ns)}^{(s \omega)} + \frac{e^{0(n)}}{e^{0(0)}} \frac{e^{0(a)}}{e^{0(0)}} \tilde{\omega}_{(as)}^{(s \omega)} \right)
\]

\[= \frac{1}{e^{0(0)}} \frac{e^{0(q)}}{e^{0(0)}} \gamma^{k(n)} \tilde{\eta}^{sm} I_{k(q)p'(m')} A^{p'(0)n(q')} \omega_{(m') q'}^{(m')}\]

\[- \frac{1}{e^{0(0)}} \frac{e^{0(q)}}{e^{0(0)}} \gamma^{k(n)} I_{k(q)p'(m')} \frac{e^{0(m')}}{e^{0(0)}} A^{p'(0)n(q')} \frac{e^{0(a)}}{e^{0(0)}} \omega_{(aq')}
\]

\[- \frac{1}{D - 2} \frac{1}{e^{0(0)}} \left( \tilde{D}_{s}^{(sn)} (\pi^{k}(m)) + \tilde{D}_{s}^{(sn)} (\pi^{k(0)}) + \tilde{D}_{s}^{(sn)} (\eta^{\mu(\nu), k}) + \tilde{D}_{s}^{(sn)} (\omega_{0(\alpha\beta)}) \right)
\]

and substitute its solution back into (93) (as elimination of “trace” in covariant case, see 13 14)

\[\tilde{\omega}^{m(ns)} - \tilde{\omega}^{n(ms)} = \frac{e^{0(m)}}{e^{0(0)}} \frac{e^{0(a)}}{e^{0(0)}} \tilde{\omega}^{n(as)} + \frac{e^{0(n)}}{e^{0(0)}} \frac{e^{0(a)}}{e^{0(0)}} \tilde{\omega}^{m(as)}
\]

\[+ \tilde{D}_{s}^{(sn)} (\pi^{k}(m)) + \tilde{D}_{s}^{(sn)} (\pi^{k(0)}) + \tilde{D}_{s}^{(sn)} (\eta^{\mu(\nu), k}) + \tilde{D}_{s}^{(sn)} (\omega_{0(\alpha\beta)}) = 0
\]

where

\[\tilde{D}_{s}^{(sn)} (\pi^{k}(m)) \equiv \tilde{D}_{s}^{(sn)} (\pi^{k(0)}) + \frac{\tilde{\eta}_{sm}^{(n)}}{(D - 2)} \left( \omega_{s}^{(ms)} + \frac{e^{0(m)}}{e^{0(0)}} \frac{e^{0(a)}}{e^{0(0)}} \omega_{s}^{(as)} + \tilde{D}_{s}^{(am)} (\pi^{k(0)}) \right)
\]
\[
- \frac{\tilde{\eta}^{sm}}{(D-2)} \left( \tilde{\omega}^{(ns)} + \frac{e^{0(n)}}{e^{0(0)}} \tilde{\omega}^{(as)} + D_a^{(an)} (\pi^{k(0)}) \right),
\]

(99)

\[
\tilde{D}^{s(mn)} (\pi^{k(m)}) = \tilde{D}^{s(mn)} (\pi^{k(m)}) + \frac{\tilde{\eta}^{sn}}{D-2} D_a^{(am)} (\pi^{k(m)}) - \frac{\tilde{\eta}^{sm}}{D-2} D_a^{(an)} (\pi^{k(m)})
\]

(100)

and similar expressions can be written for \( \tilde{D}^{s(mn)} (e_{(\nu),k}) \), \( \tilde{D}^{s(mn)} (\omega_{0(\alpha\beta)}) \). Note that the second line in (93) vanishes after substituting to (93) the multiplier \( \tilde{\lambda}^n \) from (??). Note that the same dimensional coefficient, \( \frac{1}{D-2} \), appears, as in a covariant case, reflecting the same fact: first order formulation (Lagrangian or Hamiltonian) is not valid in two dimensions. This coefficient was already uncounted in the definition of \( I_{k(m)n(p)} \) (11).

What is important that the direct calculations (using (100), (94), and (90)) shows that

\[
\tilde{D}^{s(mn)} (\omega_{0(\alpha\beta)}) = 0.
\]

(101)

This result is very significant: there are no temporal connections in a solution for spatial connections and so in the secondary translational constraint. We do not have reappearance of temporal connections that were absent in (53). This immediately allows to make a conclusion that the secondary translational constraint has zero PB with primary rotational one (again, as in 3D case [1]).

We present equation (98) using short notation

\[
\tilde{\omega}^{m(ns)} - \tilde{\omega}^{n(ms)} - \frac{e^{0(m)}}{e^{0(0)}} \tilde{\omega}^{n(as)} + \frac{e^{0(n)}}{e^{0(0)}} \tilde{\omega}^{m(as)} =
\]

(102)

where

\[
\tilde{D}^{\nu s(mn)} (\pi^{k(m)}, \pi^{k(0)}, e_{(\nu),k})
\]

(103)

and the part \( \tilde{D}^{s(mn)} (\omega_{0(\alpha\beta)}) \) is not here by virtue of (101). The solution of (102) is a little bit more involved compare with a covariant case (13). Part of terms with contractions were eliminated by solving for multipliers but we still have additional contraction in one index (third and fourth terms of (102)). We were not able to eliminate these terms by further
contractions, i.e. contraction with \( e^0_{(s)} \) leads only to a relation (that we will use), not to an elimination of it,

\[
e^0_{(s)} \tilde{\omega}^m_{(ns)} - e^0_{(s)} \tilde{\omega}^n_{(mn)} = e^0_{(s)} \tilde{D}^\mu s_{(mn)}. \tag{104}
\]

So, we have to perform Einstein’s permutation first (as in (15)): \((mns) + (smn) - (nsm)\), and try to eliminate a contraction after that. After permutation of (102) we obtain

\[
2\tilde{\omega}^m_{(ns)} + \frac{e^0_{(m)}}{e^0_{(0)} e^0_{(0)}} \left( e^0_{(a)} \tilde{\omega}^s_{(an)} - e^0_{(a)} \tilde{\omega}^z_{(an)} \right) - \frac{e^0_{(s)}}{e^0_{(0)} e^0_{(0)}} \left( e^0_{(a)} \tilde{\omega}^m_{(an)} + e^0_{(a)} \tilde{\omega}^m_{(an)} \right) + \frac{e^0_{(n)}}{e^0_{(0)} e^0_{(0)}} \left( e^0_{(a)} \tilde{\omega}^m_{(as)} + e^0_{(a)} \tilde{\omega}^s_{(am)} \right) = \tilde{D}^\mu s_{(mn)} + \tilde{D}^m_{(sm)} - \tilde{D}^mn_{(ns)}. \tag{105}
\]

The first bracket can be eliminated using (104) and for the last two we can eliminate half of contributions that gives

\[
2\tilde{\omega}^m_{(ns)} - 2 \frac{e^0_{(s)}}{e^0_{(0)} e^0_{(0)}} e^0_{(a)} \tilde{\omega}^m_{(an)} + 2 \frac{e^0_{(n)}}{e^0_{(0)} e^0_{(0)}} e^0_{(a)} \tilde{\omega}^m_{(as)} = \tilde{D}^\mu s_{(mn)} + \tilde{D}^m_{(sm)} - \tilde{D}^mn_{(ns)}. \tag{106}
\]

In this form terms \( e^0_{(a)} \tilde{\omega}^m_{(an)} \) will be found by contraction with \( e^0_{(n)} \)

\[
2e^0_{(n)} \tilde{\omega}^m_{(ns)} g^{00}_{e^0_{(0)} e^0_{(0)}} = e^0_{(n)} \tilde{D}^\mu s_{(mn)} + e^0_{(n)} \tilde{D}^m_{(sm)} - e^0_{(n)} \tilde{D}^mn_{(ns)}
\]

\[
- \frac{e^0_{(m)}}{e^0_{(0)} e^0_{(0)}} e^0_{(n)} e^0_{(a)} \tilde{D}^m_{(an)} + \frac{e^0_{(s)}}{e^0_{(0)} e^0_{(0)}} e^0_{(n)} e^0_{(a)} \tilde{D}^m_{(an)} - \frac{e^0_{(n)}}{e^0_{(0)} e^0_{(0)}} e^0_{(a)} \tilde{D}^m_{(as)} \tag{107}
\]

where \( g^{00} \) is just a short notation for \( e^0_{(0)} e^0_{(0)} \).

Equation (106) after expressing \( e^0_{(a)} \tilde{\omega}^m_{(an)} \) using (107) gives us the final solution:

\[
2\tilde{\omega}^m_{(ns)} = \tilde{D}^\mu s_{(mn)} + \tilde{D}^m_{(sm)} - \tilde{D}^mn_{(ns)}. \tag{108}
\]
According to the definition (99), we have for ˜

\[ D^{\mu_1 \nu_1} \pi^{\mu_2 \nu_2} = - \frac{e^{0(m)}}{e^{0(0)} e^{0(s)}} D_{(a)} \hat{D}^{\mu_2 \nu_2} + \frac{e^{0(s)}}{e^{0(0)} e^{0(0)}} e_{(a)} D^{\mu_2 \nu_2} + \frac{e^{0(n)}}{e^{0(0)} e^{0(0)}} e_{(a)} \hat{D}^{\mu_2 \nu_2} \]

+ \frac{e^{0(s)}}{g^{00}} \left[ e_{(a)} D^{\mu_2 \nu_2} + e_{(a)} \hat{D}^{\mu_2 \nu_2} - e_{(a)} \hat{D}^{\mu_2 \nu_2} \right]

\[ + \frac{e^{0(n)}}{g^{00}} \left[ e_{(a)} D^{\mu_2 \nu_2} + e_{(a)} \hat{D}^{\mu_2 \nu_2} - e_{(a)} \hat{D}^{\mu_2 \nu_2} \right] \]

\[ - \frac{e^{0(n)}}{g^{00}} \left[ e_{(a)} D^{\mu_2 \nu_2} + e_{(a)} \hat{D}^{\mu_2 \nu_2} - e_{(a)} \hat{D}^{\mu_2 \nu_2} \right]. \]

Note that the solution given by (108) is manifestly antisymmetric, as it should be. In addition, because of linearity of ˜\[ D^{\mu_2 \nu_2} \], which is, in turn linear in contribution with dependence on \[ \pi^{k(0)} \], \[ \pi^{k(m)} \], and \[ e_{\mu(k)} \] (see (103)), we can calculate these contributions separately.

Let us, as an example, consider the result for \[ \tilde{\omega}^{m(ns)} (\pi^{k(0)}) \].

Using explicit form of \[ D^{\mu_2 \nu_2} \] (89), after performing contractions and going to “tilde” notation (91), we obtain

\[ \tilde{D}^{s(mn)} (\pi^{k(0)}) = - \frac{e^{0(s)}}{e^{0(0)} e^{0(n)}} e_{(n)} \hat{V}^{m} - e^{0(n)} \hat{V}^{m} \]

(109)

where we introduce a short notation (solution of (91))

\[ \tilde{V}^{s} \equiv \tilde{\omega}^{s(n)} = - \frac{1}{2 e^{0(0)} e^{0(s)}} e^{0(s)} \pi^{k(0)}. \]

(110)

According to the definition (99), we have for \[ \tilde{D}^{s(mn)} (\pi^{k(0)}) \]

\[ \tilde{D}^{s(mn)} (\pi^{k(0)}) = - \frac{e^{0(s)}}{e^{0(0)} e^{0(n)}} e_{(n)} \hat{V}^{m} - e^{0(n)} \hat{V}^{m} \]

(111)

\[ + \frac{\tilde{\eta}^{sm}}{D - 2} \left( \tilde{V}^{m} + \frac{e^{0(m)}}{e^{0(0)} e^{0(s)}} e_{(n)} \tilde{V}^{a} - \tilde{V}^{m} + e^{0(0)} e^{0(n)} \tilde{V}^{m} + e^{0(m)} e^{0(n)} \tilde{V}^{a} \right) \]

\[ - \frac{\tilde{\eta}^{sm}}{D - 2} \left( \tilde{V}^{m} + \frac{e^{0(n)}}{e^{0(0)} e^{0(s)}} e_{(n)} \tilde{V}^{a} - \tilde{V}^{m} + e^{0(0)} e^{0(n)} \tilde{V}^{m} + e^{0(n)} e^{0(n)} \tilde{V}^{a} \right). \]
Substitution it into (108) after some simplifications gives

\[ 2 \tilde{\omega}^m(n) (\pi^{(k(0))}) = \frac{1}{D-2} \left( \eta^{sm} \tilde{V}^n - \tilde{\eta}^{nm} \tilde{V}^s \right) + e^{0(m)} \left( e^{(s)}_0 \tilde{V}^n - e^{(n)}_0 \tilde{V}^s \right) \]

\[ + \left( \tilde{V}^n e^{0(s)} - \tilde{V}^s e^{0(n)} \right) e^{0(a)}_0 e^{0(0)}_a g^{00} - e^{0(m)} \left( e^{0(n)}_0 \tilde{V}^s - e^{0(s)}_0 \tilde{V}^n \right) e^{0(0)}_0 e^{0(0)}_a \]

\[ + \left( e^{(n)}_0 e^{0(s)} - e^{(s)}_0 e^{0(n)} \right) \tilde{V}^m e^{0(0)}_m e^{0(0)}_n + e^{0(m)} \left( e^{(n)}_0 e^{0(s)} - e^{(s)}_0 e^{0(n)} \right) e^{0(0)}_n e^{0(0)}_a \]

\[ + \frac{2 \tilde{\eta}^{sm}}{D-2} \left[ \left( -e^{0(c)}_0 e^{0(c)}_n \tilde{V}^n + e^{0(n)}_0 e^{0(c)}_n \tilde{V}^c \right) e^{0(0)}_c e^{0(0)}_a - e^{0(n)}_0 e^{0(0)}_a \right] \]

\[ - \frac{2 \tilde{\eta}^{nm}}{D-2} \left[ \left( -e^{0(c)}_0 e^{0(c)}_n \tilde{V}^n + e^{0(n)}_0 e^{0(c)}_n \tilde{V}^c \right) e^{0(0)}_c e^{0(0)}_a - e^{0(n)}_0 e^{0(0)}_a \right]. \]

The solution for spatial connections (only dependence on \( \pi^{(k(0))} \)) is quite big and one possible way to check it is to consider a “trace” of it by contraction with \( \tilde{\eta}_{ms} \). After not long calculations it gives the correct result

\[ 2 \tilde{\omega}^{(nc)}_c (\pi^{(k(0))}) = -\frac{1}{e^{0(0)}_e} e^{(n)}_k \pi^{(k(0))}. \]  

(113)

Second consistency check is to consider three dimensional case which is also an illustration that the Dirac approach is valid in all dimensions. There are only two possible independent connections in three dimensional case, \( \tilde{\omega}^{(12)}_1 \) and \( \tilde{\omega}^{(12)}_2 \), and by direct calculation for both combinations we obtain

\[ 2 \tilde{\omega}^{(21)}_1 (\pi^{(k(0))}) = \frac{1}{e^{0(0)}_e} e^{(2)}_k \pi^{(k(0))}, \quad 2 \tilde{\omega}^{(12)}_2 (\pi^{(k(0))}) = \frac{1}{e^{0(0)}_e} e^{(1)}_k \pi^{(k(0))}. \]

Actually, direct calculations to find \( \omega^{(n)}_{p q} \) can be avoided, as it follows from the general “trace” relation (113), because in three dimensions (due to antisymmetry)

\[ \tilde{\omega}^{(1c)}_c = \tilde{\omega}^{(11)}_1 + \tilde{\omega}^{(12)}_2 = \tilde{\omega}^{(12)}_2 \]

and using the inverse to (92)
\[ \omega_p^{(n)} = e_{(m)p} \tilde{\omega}_m^{(n)} \]  

we can find three dimensional expressions \([1]\), e.g.

\[ \omega_1^{(1)} = e_{(m)} \tilde{\omega}_m^{(1)} = e_{(1)} \tilde{\omega}_m^{(1)} + e_{(2)} \tilde{\omega}_m^{(2)}. \]  

With the solution for \( \tilde{\omega}_m^{(n)} \) \([99]\) (or \( \omega_m^{(n)} \), using \([114]\)) and found before for \( \omega_m^{(p0)} \) \([16]\), all spatial connections are eliminated, as in 3D case, and we can find the reduced Hamiltonian with fewer variables (without all pairs of canonical variables: spatial spin connections and their momenta). The reduced Hamiltonian is the subject of next Section.

VII. THE REDUCED HAMILTONIAN

Now we have the solution for all spatial connections, \([99] \) and \([16]\), and can substitute them into the original Hamiltonian \([22]\) to obtain the reduced Hamiltonian, \( \hat{H}_T \), with fewer number of variables

\[ \hat{H}_T = \hat{H}_c + \pi^0(\rho) e_{0(\rho)} + \Pi^{0(\alpha\beta)} \tilde{\omega}_{0(\alpha\beta)}, \]  

\[ \hat{H}_c (\pi_{k(\rho)}, e_{\mu(\alpha)}, \omega_{0(\alpha\beta)}) = -eB^{\gamma(\rho)k(\alpha)\nu(\beta)} e_{\gamma(\rho),k} \omega_{\nu(\alpha\beta)} + eA^{\mu(\alpha)\nu(\beta)} \omega_{\mu(\alpha\gamma)} \omega_{\nu(\beta)}. \]  

Only two primary constraints survive the reduction and spatial connections \( \omega_{k(\alpha\beta)} \) in \([117]\) are just short notation; their expressions are given by \([99]\) and \([16]\). Let us analyze the canonical part of \( \hat{H}_T \). There is one, simple part of \( \hat{H}_c \), with contributions linear in temporal connections that we combine together keeping the rest of terms \( \hat{H}_c' \) separately (see Section V)

\[ \hat{H}_c = \left( \frac{1}{2} e_k^{(\alpha)} \pi_{k(\beta)} - e_k^{(\beta)} \pi_{k(\alpha)} - eB^{\gamma(\rho)k(\alpha)0(\beta)} e_{\gamma(\rho),k} \right) \omega_{0(\alpha\beta)} + \hat{H}_c', \]  

\[ \hat{H}_c' = -eB^{\gamma(\rho)k(\alpha)m(\beta)} e_{\gamma(\rho),k} \omega_{m(\alpha\beta)} + eA^{k(\alpha)m(\beta)} \omega_{k(\alpha\gamma)} \omega_{m(\beta)}. \]
The expression in brackets is the rotational constraint \( \chi^{0(\alpha\beta)} \) and, as expected, in all dimensions it is the same as in three dimensions. So, this part of the reduced Hamiltonian can be written as

\[
\hat{H}_c = -\omega_0^{0(\alpha\beta)} \chi^{0(\alpha\beta)} + \hat{H}_c'.
\]  

(120)

Moreover (it is not difficult to check and it was demonstrated in our [1], this secondary rotational constraint has zero PBs with both primary constraints and PB for two rotational constraints gives the Poincaré relation (see (58)) in all dimensions. Note that this is the only part of the Hamiltonian with temporal connections (solutions of second class constraints for spatial connections are independent on temporal one, as we demonstrated in previous Section).

Let us continue with (119). Here we have also one simple contribution which becomes transparent after separation of first term of (119)

\[
- eB^{\gamma(\rho)k(\alpha)m(\beta)} e_{\gamma(\rho),k} \omega_{m(\alpha\beta)} = - eB^{0(\rho)k(\alpha)m(\beta)} e_{0(\rho),k} \omega_{m(\alpha\beta)} - eB^{n(\rho)k(\alpha)m(\beta)} e_{n(\rho),k} \omega_{m(\alpha\beta)}.
\]  

(121)

First part, non-zero in three dimensions, was also discussed and is quite simple (see Section V)

\[
- eB^{0(\rho)k(\alpha)m(\beta)} e_{0(\rho),k} \omega_{m(\alpha\beta)} = - \pi^{k(\rho)} e_{0(\rho),k}.
\]  

(122)

It gives simple (first) contribution \( \chi^{0(\rho)}_1 \) into the secondary translational constraint in all dimensions, the same that was found in three dimensions [1],

\[
\chi^{0(\rho)}_1 = \pi^{k(\rho)}.
\]  

(123)

This contribution has zero PBs with all primary constraints and by itself gives all relations of the Poincaré algebra (56-57) and, after integrations by part, allows to write the corresponding part of the Hamiltonian as

\[
\hat{H}_c' = - e_{0(\rho)} \chi^{0(\rho)}_1.
\]  

(124)

The second term of (121) is manifestly zero in three dimensions (there are no three distinct values for spatial components of external indices in three spacetime dimensions -
only two spatial components are available) and the same happens with many terms quadratic in spatial connections (see (53)).

Let us discuss the effect of these (“invisible” in three dimensions) terms on the translational constraint and on PBs among constraints found in three dimensions [1].

First, we substitute our solution for \( \omega_{m(p0)} \) (46) into \( \hat{H}'_c \) (119) (remember that one small contribution is already found and only the second term of (121) is left), where we explicitly separate two kinds of spatial connections \( \omega_{m(p0)} \) and \( \omega_{m(pq)} \)

\[
\hat{H}'_c = -eB^{n(p)k(p)m(q)}\epsilon_{n(p),k}\omega_{m(pq)} - 2eB^{n(q)k(p)m(0)}\epsilon_{n(q),k}\omega_{m(p0)} \tag{125}
\]

\[
+eA^{k(p)m(q)}\omega_{k(pn)}\omega_{m}^{(n)} + eA^{k(p)m(q)}\omega_{k(p0)}\omega_{m}^{(0)} + 2eA^{k(0)m(q)}\omega_{k(0p)}\omega_{m}^{(p)}.
\]

The result of such a substitution converts \( \hat{H}'_c \) into the expression where we have three groups of terms

\[
\hat{H}'_c = \hat{H}'_c (0) + \hat{H}'_c (1) + \hat{H}'_c (2) \tag{126}
\]

classified by order of spatial connections \( \omega_{m(pq)} \) \( (\omega_{m(p0)} \) is substituted). For the first part, \( \hat{H}'_c (0) \), we have

\[
\hat{H}'_c (0) = eB^{n(q)k(p)m(0)}\epsilon_{n(q),k} \left[ \frac{1}{\epsilon\epsilon_0(0)} I_{m(p)a(b)}\pi_{a(b)} \right] \nonumber
\]

\[
-\frac{1}{4} eA^{k(p)m(q)} \left[ \frac{1}{\epsilon\epsilon_0(0)} I_{k(p)a(b)}\pi_{a(b)} \right] \left[ \frac{1}{\epsilon\epsilon_0(0)} I_{m(q)a(b)}\pi_{a(b)} \right]. \tag{127}
\]

Let us analyze contributions into the secondary constraint that are created but this part (we repeat, the first term of (127) is manifestly zero in three dimensions)

\[
\hat{\pi}^0(\sigma) = \left\{ \pi^0(\sigma), \hat{H}'_c (0) \right\} = -\frac{\delta \hat{H}'_c (0)}{\delta e_0(\sigma)}. \tag{128}
\]

We have

\[
-\frac{\delta \hat{H}'_c (0)}{\delta e_0(\sigma)} = -\frac{\delta}{\delta e_0(\sigma)} (eB^{n(q)k(p)m(0)}) \left[ \epsilon_{n(q),k} \right] \left[ \frac{1}{\epsilon\epsilon_0(0)} I_{m(p)a(b)}\pi_{a(b)} \right]. \tag{129}
\]
\[ + \frac{1}{4} \delta \frac{\delta}{\delta \varepsilon_0(\sigma)} \left( eA^{k(p)m(q)} \right) \left[ \frac{1}{e \varepsilon_0(0)} I_{k(p)a(b)} \pi^{a(b)} \right] \left[ \frac{1}{e \varepsilon_0(0)} I_{m(q)a(b)} \pi^{a(b)} \right] = \chi^{0(\sigma)}(0). \]

Note that variation of expressions in square brackets is zero. This part of the secondary constraint has obviously zero PBs with primary rotational and also with primary translational constraints just because of antisymmetry of \(ABC\) functions - the only part which is affected by a variation. For example,

\[ \frac{\delta}{\delta \varepsilon_0(\tau)} \frac{\delta}{\delta \varepsilon_0(\sigma)} \left( eA^{k(p)m(q)} \right) = \frac{\delta}{\delta \varepsilon_0(\tau)} \left( eB^{0(\sigma)k(p)m(q)} \right) = e C^{0(\tau)0(\sigma)k(p)m(q)} = 0. \] (130)

So, at least for this part, despite appearance of additional terms in higher dimensions, properties which were found in three dimensions \[1\] survive. Let us check possibility to present this part of the Hamiltonian as a linear combination of components of a translational constraint. Again, \(ABC\) properties make calculations simple. Let us illustrate this. In the second line we have

\[ \frac{\delta}{\delta \varepsilon_0(\sigma)} \left( eA^{k(p)m(q)} \right) = e B^{0(\sigma)k(p)m(q)} \] (131)

that we have to contract with \(e_0(\sigma)\) with a hope to have \(\hat{H}_c'(0) = -\varepsilon_0(\sigma) \chi^{0(\sigma)}(0)\). We used properties of \(B\) (see \[34\]) which is a kind of expansion in an external index but similar relation exists for the internal one

\[ B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} = e^{\gamma(\rho)A^{\mu(\alpha)\nu(\beta)}} + e^{\mu(\rho)A^{\nu(\alpha)\gamma(\beta)}} + e^{\nu(\rho)A^{\gamma(\alpha)\mu(\beta)}} \] (132)

In our case it gives

\[ \frac{\delta}{\delta \varepsilon_0(\sigma)} \left( eA^{k(p)m(q)} \right) = e \left( e^{0(\sigma)A^{k(p)m(q)}} + e^{k(\sigma)A^{m(p)0(q)}} + e^{m(\sigma)A^{0(p)k(q)}} \right) \] (133)

and after contraction we have

\[ e_0(\sigma) \frac{\delta}{\delta \varepsilon_0(\sigma)} \left( eA^{k(p)m(q)} \right) = eA^{k(p)m(q)}. \] (134)

A similar, external, expansion exists for \(C\)

\[ C^{\sigma(\tau)\gamma(\rho)\mu(\alpha)\nu(\beta)} = \]
\[ e^{\sigma(\tau) B^{\gamma(\rho) \mu(\alpha) \nu(\beta)}} - e^{\gamma(\tau) B^{\mu(\rho) \nu(\alpha) \sigma(\beta)}} + e^{\mu(\tau) B^{\nu(\rho) \sigma(\alpha) \gamma(\beta)}} - e^{\nu(\tau) B^{\sigma(\rho) \gamma(\alpha) \mu(\beta)}}. \] (135)

In short, the translational constraint has additional contributions in higher dimensions compare with 3D but properties of a constraint and possibility to present the Hamiltonian as a linear combination of it survives. For this part we demonstrated that

\[ \dot{H}'_c(0) = -\epsilon_{0(\sigma)} \chi^0(\sigma)(0) \] (136)

which is the same relation as in 3D case \[35\]. Of course, to make the final conclusion, all terms have to be considered, so we are looking now for contributions linear and quadratic in \( \omega_{m(pq)} \) which turned to be better to consider together \( \dot{H}'_c(1) + \dot{H}'_c(2) \).

Here we collect what is left after separating \( \dot{H}'_c(0) \) (we advise not to simplify expressions)

\[ \dot{H}'_c(1) + \dot{H}'_c(2) = \dot{H}'_c(1 + 2) = \]

\[ -e B^{(\rho)k(p)m(q)} \left[ e^{n(\rho),k} \right] \omega_{m(pq)} + 2e B^{(q)k(p)m(0)} \left[ e^{n(q),k} \right] \left[ \frac{\epsilon^0(d)}{2\epsilon^0(0)} I_{m(p)c(d)} E^{c(a)f(b)} \omega_{f(ab)} - \frac{\epsilon^0(a)}{\epsilon^0(0)} \omega_{m(ap)} \right] + e A^{k(p)m(q)} \omega_{k(pn)} \omega^{(n)}_{m(q)} \]

\[ + 2e A^{k(0)m(q)} \omega^{(p)}_{m(q)} \left[ \frac{1}{2e \epsilon^0(0)} I_{k(p)a(b)} \pi^{a(b)} + \frac{\epsilon^0(d)}{2\epsilon^0(0)} I_{k(p)c(d)} E^{c(a)f(b)} \omega_{f(ab)} - \frac{\epsilon^0(a)}{\epsilon^0(0)} \omega_{k(ap)} \right] - e A^{k(p)m(q)} \left[ \frac{\epsilon^0(d)}{2\epsilon^0(0)} I_{m(q)c(d)} E^{c(a)f(b)} \omega_{f(ab)} - \frac{\epsilon^0(a)}{\epsilon^0(0)} \omega_{m(ap)} \right] \]

\[ -e A^{k(p)m(q)} \left[ \frac{\epsilon^0(d)}{2\epsilon^0(0)} I_{k(p)c(d)} E^{c(a)f(b)} \omega_{f(ab)} - \frac{\epsilon^0(a)}{\epsilon^0(0)} \omega_{k(ap)} \right] \left[ \frac{\epsilon^0(d)}{2\epsilon^0(0)} I_{m(q)c(d)} E^{c(a)f(b)} \omega_{f(ab)} - \frac{\epsilon^0(a)}{\epsilon^0(0)} \omega_{m(ap)} \right]. \]

To find a contribution into the secondary constraint from this part we, as with part \( \dot{H}'_c(0) \), have to perform variation of this expression with respect to \( \epsilon_{0(\sigma)} \). Instead of explicit substitution of the solution for \( \omega_{m(pq)} \) (which would drug us into extremely cumbersome calculations), we use the following
\[
\frac{\delta \hat{H}'_c(1+2)}{\delta e_0(\sigma)} = \frac{\partial \hat{H}'_c(1+2)}{\partial e_0(\sigma)} + \frac{\partial \hat{H}'_c(1+2)}{\partial \omega_{x(yz)}} \frac{\partial \omega_{x(yz)}}{\partial e_0(\sigma)}. \tag{138}
\]

First variation is extremely simple and, actually, is exactly the same as what we did in \( \hat{H}'_c(0) \)-part. Variation of all expressions in square brackets are zero because of

\[
\frac{\partial}{\partial e_0(\sigma)} \left( \frac{1}{ee^{0(0)}} \right) = \frac{\partial}{\partial e_0(\sigma)} \left( \frac{e^{0(a)}}{e^{0(0)}} \right) = \frac{\partial}{\partial e_0(\sigma)} I_{k(p)a(b)} = \frac{\partial}{\partial e_0(\sigma)} \gamma^{k(m)} = \frac{\partial}{\partial e_0(\sigma)} E^{c(a)f(b)} = 0. \tag{139}
\]

As in previous case, we have to consider only variations of \( eA \) and \( eB \) which gives the same, as for \( \hat{H}'_c(0) \), result:

\[
\hat{H}'_c(1+2) = -e_0(\sigma) \chi^{0(\sigma)}(1+2). \tag{140}
\]

This is the final answer and the Hamiltonian is the linear combination of secondary constraints (up to a total derivative) as in three dimensional case if, of course, the second part of variation in (138) gives zero. Let us prove this. It is obvious from the solution for \( \omega_{x(yz)} \) (see (108)) that there are terms with non-zero variation in it but contraction \( \frac{\partial \hat{H}'(1+2)}{\partial \omega_{x(yz)}} \frac{\partial \omega_{x(yz)}}{\partial e_0(\sigma)} \) in (138) still can be zero. Again, the direct substitution here is too long and we can try to relate \( \frac{\partial \hat{H}'(1+2)}{\partial \omega_{x(yz)}} \) with the known equation for a connection (of course, with one where the multipliers were already eliminated).

So, as we did in obtaining (85), we perform variation of \( \frac{\partial \hat{H}'(1+2)}{\partial \omega_{x(yz)}} \) and present the result using “\( \gamma - E \)” notation

\[
\frac{\partial \hat{H}'(1+2)}{\partial \omega_{x(yz)}} = \]

\[
+ eE^{x(y)m(q)} \omega_{m}^{(z)} - eE^{x(z)m(q)} \omega_{m}^{(y)} q + eE^{k(p)x(z)} \frac{e^{0(y)}}{e^{0(0)}} \frac{e^{0(a)}}{e^{0(0)}} \omega_{k(a)} - eE^{k(p)x(y)} \frac{e^{0(a)}}{e^{0(0)}} \frac{e^{0(z)}}{e^{0(0)}} \omega_{k(a)}
\]

\[
+ eA^{k(0)m(q)} \omega^{(p)} \frac{e^{0(d)}}{e^{0(0)}} I_{k(p)c(d)} E^{c(y)x(z)} + eA^{k(p)m(0)} \frac{e^{0(q)}}{e^{0(0)}} \frac{e^{0(a)}}{e^{0(0)}} \omega_{k(a)} \frac{e^{0(d)}}{e^{0(0)}} I_{m(q)c(d)} E^{c(y)x(z)}
\]

\[
+ \hat{D}^{x(yz)}(e_{\mu(\nu),k}) + \hat{D}^{x(yz)}(\pi^{k(m)}) + \hat{D}^{x(yz)}(\pi^{k(0)})
\]
where, as in (85), we introduce $\hat{D}^{(yz)}$ for similar terms which are

$$
\hat{D}^{(yz)} \left( e_{\mu(\nu),k} \right) = -eB^{n(p)k(y)x(z)}e_{n(p),k} + eB^{m(q)k(p)m(0)}e_{n(q),k} \frac{\epsilon^0(y)}{\epsilon^0(0)} I_{m(p)c(d)} E^{c(y)x(z)}
$$

$$
- eB^{n(q)k(z)x(0)}e_{n(q),k} \frac{\epsilon^0(y)}{\epsilon^0(0)} + eB^{m(q)k(y)x(0)}e_{n(q),k} \frac{\epsilon^0(z)}{\epsilon^0(0)};
$$

$$
(142)
$$

$$
\hat{D}^{(yz)} \left( \pi^{k(m)} \right) = +eA^{k(0)x(z)} \left[ \frac{1}{2ee^0(0)} I_{k(y)\pi(a)b} \pi^{a(b)} \right] - eA^{k(0)x(y)} \left[ \frac{1}{2ee^0(0)} I_{k(z)\pi(a)b} \pi^{a(b)} \right]
$$

$$
- eA^{k(p)m(q)} \left[ \frac{1}{2ee^0(0)} I_{k(p)\pi(a)b} \pi^{a(b)} \right] \left[ \frac{\epsilon^0(d)}{\epsilon^0(0)} I_{m(q)c(d)} E^{c(y)x(z)} \right]
$$

$$
+ eA^{k(p)x(z)} \left[ \frac{1}{2ee^0(0)} I_{k(p)\pi(a)b} \pi^{a(b)} \right] \frac{\epsilon^0(y)}{\epsilon^0(0)} - eA^{k(p)x(y)} \left[ \frac{1}{2ee^0(0)} I_{k(p)\pi(a)b} \pi^{a(b)} \right] \frac{\epsilon^0(z)}{\epsilon^0(0)};
$$

$$
(143)
$$

$$
\hat{D}^{(yz)} \left( \pi^{k(0)} \right) = +eA^{k(0)x(z)} \frac{\epsilon^0(d)}{2ee^0(0)} I_{k(y)c(d)} E^{c(a)f(b)} \omega_{f(ab)} - eA^{k(0)x(y)} \frac{\epsilon^0(d)}{2ee^0(0)} I_{k(z)c(d)} E^{c(a)f(b)} \omega_{f(ab)}
$$

$$
- e \left( \frac{\epsilon^0(p)}{\epsilon^0(0)} A^{k(0)m(q)} + A^{k(p)m(0)} \frac{\epsilon^0(q)}{\epsilon^0(0)} \right) \frac{\epsilon^0(d)}{2ee^0(0)} I_{k(p)c(d)} E^{c(a)f(b)} \omega_{f(ab)} \frac{\epsilon^0(d)}{\epsilon^0(0)} I_{m(q)c(d)} E^{c(y)x(z)}
$$

$$
- e \left( \frac{\epsilon^0(q)}{2ee^0(0)} E^{m(a)f(b)} \omega_{f(ab)} \frac{\epsilon^0(d)}{\epsilon^0(0)} \right) I_{m(q)c(d)} E^{c(y)x(z)}
$$

$$
(144)
$$

Equation (144), or rather the result of variation $\frac{\partial \hat{I}^{(1+2)}}{\partial \omega_{x(y)}}$ (it is not an equation of motion), we compare with the similar equation (93), but not in “tilde” notation which is the result of further contractions that were needed for use of the Einstein permutation, and not one (85) which has the form of (141), but with the multiplier. We substitute the solution for the multiplier (97) into (85) and obtain the equation with which we compare (141). We use
the same letters \((x(yz))\) for free indices to make a comparison more transparent. Of course, terms with temporal connections are cancelled out, as it was found before, as well as some additional cancellation occurs, and the final result is

\[
-E^{x(y)k(q)}\omega_k^{(zq)} + E^{x(z)k(q)}\omega_k^{(yq)} + \frac{\epsilon^{0(y)}}{\epsilon^{0(0)}} E^{x(z)n'(q')}\epsilon^0(a) \omega_{n'(aq')} - \frac{\epsilon^{0(z)}}{\epsilon^{0(0)}} E^{x(y)n'(q')}\epsilon^0(a) \omega_{n'(aq')}
\]

\[
+ D^{x(yz)} (\pi^{k(m)}) + \gamma^{x(z)} \frac{1}{(D - 2)} \left( \tilde{\omega}_s^{(yz)} + \frac{\epsilon^{0(y)}}{\epsilon^{0(0)}} \tilde{\omega}_s^{(as)} + \tilde{D}_a^{(ay)} (\pi^{k(m)}) \right)
\]

\[
- \gamma^{x(y)} \frac{1}{(D - 2)} \left( \tilde{\omega}_s^{(zs)} + \frac{\epsilon^{0(z)}}{\epsilon^{0(0)}} \tilde{\omega}_s^{(as)} + \tilde{D}_s^{(zm)} (\pi^{k(m)}) \right)
\]

\[
+ D^{x(yz)} (\pi^{k(0)}) + \gamma^{x(z)} \frac{1}{D - 2} \tilde{D}_a^{(ay)} (\pi^{k(0)}) - \gamma^{x(y)} \frac{1}{D - 2} \tilde{D}_m^{(zm)} (\pi^{k(0)})
\]

\[
+ D^{x(yz)} (\epsilon_{\mu(v), k}) + \gamma^{x(z)} \frac{1}{D - 2} \tilde{D}_a^{(a(y)} (\epsilon_{\mu(v), k}) - \gamma^{x(y)} \frac{1}{D - 2} \tilde{D}_m^{(zm)} (\epsilon_{\mu(v), k}) = 0.
\]

Now, to simplify calculations, we can multiply this equality by \(\epsilon\) and add to RHS of (141). The most difficult for calculation part (first line of (141)) will disappear. The second line of (141) survives but the rest is already expressed in terms of momenta and N-beins, and we avoided a cumbersome substitution. Moreover, let us look at the second line of (141). Both contributions have \(x(yz)\) indices in the following form: \(E^{c(y)x(z)}\). We almost immediately have zero for the corresponding contributions

\[
\frac{\partial \tilde{H}^c_0 (1 + 2) \partial \omega_{x(yz)}}{\partial \omega_{x(yz)}} = X_c E^{c(y)x(z)} \frac{\partial \omega_{x(yz)}}{\partial \epsilon_{0(\sigma)}} = X_c \frac{\partial E^{c(y)x(z)} \omega_{x(yz)}}{\partial \epsilon_{0(\sigma)}} = 0.
\]

Here, because variation of \(E^{c(y)x(z)}\) is zero (139), we can interchange order of contraction and variation and obtain the combination

\[
E^{c(y)x(z)} \omega_{x(yz)} = 2 \gamma^{c(y)} \gamma^{x(z)} \omega_{x(yz)} = -\frac{1}{\epsilon \epsilon^{0(0)}} \pi^{c(0)},
\]

variation of which with respect to \(\epsilon_{0(\sigma)}\) is zero (139). Similarly, in the rest of expression we consider separately three different contributions which are proportional to \(\pi^{k(0)}, \pi^{k(m)}, and\)
terms without momenta (with derivatives of tetrads) and check this relation. Note that in (145) all terms proportional to $\frac{1}{D-2}$ have a factor $\gamma^{x(z)}$ or $\gamma^{x(y)}$. These factors, as in (146), can be moved under variation and contracted with $\omega_{x(yz)}$. This gives similar to (147) result

$$\gamma^{x(z)}\omega_{x(yz)} = -\frac{1}{ee^0(0)(D-2)}e_m(y)\pi^m(0)$$

which has zero variation. The rest of terms have the following structure

$$\hat{D}^{x(yz)}(\pi^{k(m)}) + eD^{x(yz)}(\pi^{k(m)})$$

and there are similar expressions for $\pi^{k(0)}$ and $\epsilon_{\mu(\nu),k}$. By performing the same as above operations, all of them give zero. So, the only non-zero contributions come from the first term in (138), and (140) is the final (complete) result. As in 3D, the canonical Hamiltonian is a linear combination of secondary translational and rotational constraints.

To prove that translational secondary and primary constraints have zero PBs (as in 3D), we have to find second variation of the secondary translational constraint, i.e. we have to calculate

$$\delta \frac{\delta \hat{H}'_{c}(1+2)}{\delta e_{0(\tau)}} = \frac{\partial}{\partial e_{0(\tau)}} \frac{\partial \hat{H}'_{c}(1+2)}{\partial e_{0(\sigma)}} + \frac{\partial}{\partial \omega_{x(yz)}} \frac{\partial \hat{H}'_{c}(1+2)}{\partial e_{0(\sigma)}} \cdot \frac{\partial \omega_{x(yz)}}{\partial e_{0(\tau)}}.$$  

The first term is zero, just based on ABC properties, as we have already discussed. The second term needs consideration. We demonstrated above that in all terms in $\frac{\partial \hat{H}'_{c}(1+2)}{\partial \omega_{x(yz)}}$ we have parts (as (147) or (148)) that are unaffected by variation with respect to components $e_{0(\sigma)}$, so we can freely move them trough both variations, e.g.

$$\frac{\partial}{\partial e_{0(\sigma)}} \frac{\partial \omega_{x(yz)}}{\partial e_{0(\sigma)}} = E^{c(y)x(z)} \frac{\partial}{\partial e_{0(\sigma)}} \frac{\partial e_{0(\sigma)}}{\partial e_{0(\sigma)}} \cdot \frac{\partial \omega_{x(yz)}}{\partial e_{0(\sigma)}} = 0.$$  

We almost at once obtained the expected result: in all dimensions PBs among primary and secondary translational constraints are zero. In this Section we demonstrated that, despite having so many additional terms that are present in higher than three dimensions and much richer structure of the secondary translational constraint, the reduced Hamiltonian has the same structure is in three dimensions: it is a linear combination of secondary
constraints and all secondary constraints have zero PBs with all primary. Next step of the
Dirac procedure is to prove its closure and consider PBs of secondary constraints with the
Hamiltonian. Because the Hamiltonian is a linear combination of constraints, these calcu-
lations are equivalent to calculations of PBs among all constraints. Complete calculations
of PBs among secondary constraints are in progress and only few contributions are checked
but they so far support the Poincaré algebra. Possible consequences of our calculations is
the subject of next Section.

VIII. DISCUSSION

We start our discussion by briefly summarizing the obtained so far results for the Hamil-
tonian formulation of N-bein gravity.

a) In any dimension after elimination of second class constraints and correponding to
them variables (spatial connections and conjugate to them momenta) the total Hamiltonian
of N-bein gravity is

\[
H_T \left( e_{\mu(\rho)}, \pi^{\mu(\rho)}, \omega_{0(\alpha\beta)}, \Pi^{0(\alpha\beta)} \right) =
\]

\[
- e_{0(\rho)} \chi^{0(\rho)} - \omega_{0(\alpha\beta)} \chi^{0(\alpha\beta)} + \dot{e}_{0(\rho)} \pi^{0(\rho)} + \dot{\omega}_{0(\alpha\beta)} \Pi^{0(\alpha\beta)}.
\]

(152)

b) In any dimension the canonical part of the Hamiltonian \((H_c)\) is a linear combination of
secondary constraints \((\chi^{0(\rho)}, \chi^{0(\alpha\beta)})\) which are time development of two primary constraints
\((\pi^{0(\rho)}, \Pi^{0(\alpha\beta)})\)

\[
\dot{\pi}^{0(\rho)} = \{ \pi^{0(\rho)}, H_T \} = \chi^{0(\rho)}, \quad \Pi^{0(\alpha\beta)} = \{ \Pi^{0(\alpha\beta)}, H_T \} = \chi^{0(\alpha\beta)}
\]

(153)

and all PBs among primary and secondary constraints are zero:

\[
\{ \pi^{0(\rho)}, \chi^{0(\sigma)} \} = \{ \pi^{0(\rho)}, \chi^{0(\alpha\beta)} \} = \{ \Pi^{0(\alpha\beta)}, \chi^{0(\sigma)} \} = \{ \Pi^{0(\alpha\beta)}, \chi^{0(\mu\nu)} \} = 0.
\]

(154)

c) In any dimension

\[
\{ \chi^{0(\alpha\beta)}, \chi^{0(\mu\nu)} \} = \frac{1}{2} \tilde{\eta}^{\beta\mu} \chi^{0(\alpha\nu)} - \frac{1}{2} \tilde{\eta}^{\alpha\mu} \chi^{0(\beta\nu)} + \frac{1}{2} \tilde{\eta}^{\beta\nu} \chi^{0(\alpha\mu)} - \frac{1}{2} \tilde{\eta}^{\alpha\nu} \chi^{0(\mu\beta)}
\]

(155)
and in three dimensional case also
\[
\{ \chi^{0(\alpha\beta)}, \chi^{0(\rho)} \} = \frac{1}{2} \tilde{\eta}^{\beta\rho} \chi^{0(\alpha)} - \frac{1}{2} \tilde{\eta}^{\alpha\rho} \chi^{0(\beta)}
\]
that must be true in any dimension to preserve a rotational invariance and

\[
\{ \chi^{0(\rho)}, \chi^{0(\gamma)} \} = 0.
\] (157)

The proof of (156) and (157) for any dimension is a quite involved calculation taking into account the complexity of \( \chi^{0(\rho)} \) in higher than three dimensions. We checked a few terms for the general \( \chi^{0(\rho)} \) and did not find contradictions with (156, 157). In addition, because rotational invariance is the same in all dimensions, PB (157) must be the same in all dimensions, which follows from the Castellani procedure. Moreover, general expressions for constraints and all calculated so far properties, in particular the secondary translational constraint \( \chi^{0(\rho)} \), satisfy 3D limit, i.e. equivalent with found before [1] where all calculations were performed using simplifications of 3D case right from the beginning.

The above results and observations seems to us sufficient to make the following conjecture:

in any dimension the algebra of secondary constraints is Poincaré (N-bein gravity is the Poincaré gauge theory), and consequently, N-bein gravity has rotational and translational gauge invariance.

For someone our conjecture can sound very reasonable, for others, maybe, even not reasonable at all. However, contrary to many well-known conjectures that no one knows how to prove or disprove, our conjecture is accompanied by the mathematically well-defined procedure of proving or disproving it: calculate PB among two translational constraints (157). Moreover, because it is always easier to disprove something: one counter-example is enough. We even can suggest, seems to us, a relatively simple calculation: consider PB, \( \{ \chi^{0(0)}, \chi^{0(k)} \} \), keeping only quadratic in \( \pi^{k(0)} \) contributions (note that there are no such contributions in 3D [1]). The result will be the third order in momenta \( \pi^{k(0)} \). If the result is not zero, our conjecture is wrong and N-bein gravity either has algebra of first class constraints different from Poincaré or, at least, third generation of constraints will appear that must be second class. In such a case the only gauge invariance would be rotational. There would be no translational or any other invariances.

Let us discuss consequences of our conjecture. With the same algebra of constraints
in all dimensions calculation of generators is independent on a dimension and we can just use results for the generator that was obtained for the Poincaré algebra in 3D case and recalculate transformations in parts where the full translational constraint is present. This simplicity is the reflection of the fact that PB algebra of first class constraints defines a generator and explicit form of constraints is irrelevant, especially when the Hamiltonian is a linear combination of constraints as (152).

We immediately have a gauge generator (using three dimensional result [1])

\[ G = G_t + G_r \]

where translational and rotational parts are

\[ G = \pi^0(\rho)\dot{t}(\rho) + \left( -\chi^0(\rho) + \omega^0(\rho)\pi^0(\gamma) \right) t(\rho), \]

\[ G_r = \Pi^{0(\alpha\beta)}\ddot{r}_{(\alpha\beta)} + \left( -\chi^{0(\alpha\beta)} + \frac{1}{2} \left( \epsilon_0^{(\alpha)} \pi^{0(\beta)} - \epsilon_0^{(\beta)} \pi^{0(\alpha)} \right) + \omega_0^{(\alpha)} \Pi^{0(\beta\mu)} - \omega_0^{(\beta)} \Pi^{0(\alpha\mu)} \right) r_{(\alpha\beta)}. \]

The only difference with 3D case is in translational part where \( \chi^0(\rho) \) is much richer. The knowledge of a generator allows us to find transformations of all fields and any combinations of them (in particular, transformations of secondary constraints)

\[ \delta(...) = \{ G, (...) \}. \]

The total Hamiltonian (152) is the result of the Hamiltonian reduction [5]: elimination of all spatial connections. This simple Hamiltonian can be converted by the inverse Legendre transformations into equivalent to it Lagrangian

\[ L \left( \pi^k(\rho), e_{\mu(\nu)}; \omega_0(\alpha\beta) \right) = \pi^k(\rho)\dot{e}_k(\rho) + e_0(\rho)\chi^0(\rho) + \omega_0(\alpha\beta)\chi^0(\alpha\beta). \]

This is a different first order formulation of the original Einstein-Cartan action obtained from the reduced Hamiltonian. As in the case of usual first order formulation discussed in Introduction, the correctness of (162) can be proven by elimination of auxiliary fields using a variational method. The result: the same equation of motion for N-beins, the only independent variables in the second order formulation.
The Hamiltonian and Lagrangian reductions must be equivalent and the Lagrangian obtained from the reduced Hamiltonian has to be derivable also in the Lagrangian approach. Here the method of Lagrange multipliers, used by Ostrogradsky for the Hamiltonian formulation of higher derivatives actions, should be implemented. We apply this method to the standard first order formulation but with different purpose (it is already linear in derivatives) - to simplify elimination of spatial connections. So, we define a coefficient in front of time derivatives of N-bein as

$$\pi^k(\rho) = e^{B_k(\rho)}\Omega^{(\alpha\beta)}_{\nu}(\alpha\beta)$$

is just an auxiliary field) and, to keep equivalence, we add this redefinition using the Lagrange multiplier $\Lambda^k(\rho)$ (this guaranties equivalence with the original Lagrangian)

$$L\left(\pi^k(\rho), e_{\mu(\nu)}, \omega_0(\alpha\beta), \omega_k(\alpha\beta), \Lambda_k(\rho)\right) = \pi^k(\rho)\dot{e}_k(\rho) + \Lambda_k(\rho)\left(\pi^k(\rho) - e^{B_k(\rho\alpha\beta)}\omega_0(\alpha\beta)\right)$$

$$+ eB^{(k\rho)}e_{\mu(\nu)}\omega_0(\alpha\beta) - eA^{(k\rho)}\omega_0(\alpha\beta)\omega_0^{(\gamma)}_{\nu}.$$  \hspace{1cm} (163)

Performing variation with respect to $\Lambda_k(\rho)$ and $\omega_m(\alpha\gamma)$, we obtain exactly the same equations as in the Hamiltonian approach (17) and (50) (or (80) and (81)) and after elimination of these fields we have as a result (162). This is exactly what we did in the Hamiltonian approach: they are equivalent, as it should be, and there is no advantage in amount of calculations. However, even performance of such an operation cannot be motivated by pure Lagrangian methods: why should one introduce even more fields to already a first order Lagrangian? Contrary, in the Hamiltonian approach the reason for such modifications is to simplify solutions of second class constraints and to have a formulation with only first class constraints that allows to find gauge transformations.

Let us discuss invariance of the Lagrangian (162) using the gauge generators (159), (160) which are built in the corresponding Hamiltonian formulation (152). First, we start from the rotational invariance of (162)

$$\delta_r L\left(\pi^k(\rho), e_{\mu(\nu)}, \omega_0(\alpha\beta)\right) =$$

$$\delta_r \pi^k(\rho)\dot{e}_k(\rho) + \pi^k(\rho)\delta_r \dot{e}_k(\rho) + \delta_r e_0(\rho)\dot{\chi}^{0}(\rho) + e_0(\rho)\delta_r \dot{\chi}^{0}(\rho) + \delta_r \omega_0(\alpha\beta)\dot{\chi}^{0}(\alpha\beta) + \omega_0(\alpha\beta)\delta_r \dot{\chi}^{0}(\alpha\beta).$$

All variations which are needed here are easy to find using the generator (160) and (161)
\begin{equation}
\delta_r e_{0(\rho)} = -\frac{1}{2} \left( e_{0}^{(\alpha)} r_{(\rho\beta)} - e_{0}^{(\beta)} r_{(\rho\alpha)} \right),
\end{equation}

\begin{equation}
\delta_r \omega_{0(\alpha\beta)} = -\dot{r}_{(\alpha\beta)} - \left( \omega_{0}^{(\gamma)} r_{(\gamma\alpha)} - \omega_{0}^{(\gamma)} r_{(\gamma\beta)} \right),
\end{equation}

\begin{equation}
\delta_r e_{k(\rho)} = r_{(\alpha\beta)} \frac{\delta}{\delta \pi^{k(\rho)}} \chi^{0(\alpha\beta)} = r_{(\alpha\rho)} e_{k}^{(\alpha)},
\end{equation}

\begin{equation}
\delta_r \pi^{k(\rho)} = -r_{(\alpha\beta)} \frac{\delta}{\delta e_{k(\rho)}} \chi^{0(\alpha\beta)} = -r_{(\rho\beta)} \pi^{k(\beta)} - e B^{k(\rho)m(\alpha)0(\beta)} \left( r_{(\alpha\beta)} \right)_m,
\end{equation}

\begin{equation}
\delta_r \chi^{0(\rho)} = -r_{(\rho\beta)} \chi^{0(\alpha)},
\end{equation}

\begin{equation}
\delta_r \chi^{0(\alpha\beta)} = -r_{(\gamma\alpha)} \chi^{0(\gamma\beta)} + r_{(\gamma\beta)} \chi^{0(\gamma\alpha)}.
\end{equation}

Note that only explicit form of the rotational constraint (75) was used (see (167), (168)) and this again explains why found in 3D rotational invariance remains to be the same in all dimensions despite modification of the translational constraint. Calculation of (169) and (170) are based on algebra of secondary constraints but not on explicit form of constraints. Note that our assumption \( \{ \chi^{0(\alpha)}, \chi^{0(\beta)} \} = 0 \) (157) was not applied here but (156) was essential for (169) and (170). This provides explicit illustration of the argument to support our conjecture that known rotational invariance of N-bein gravity imposes severe restrictions on possible modification of the Poincaré algebra in higher dimensions: (156) must be correct in any dimension.

Substitution of (165-170) into (164) gives

\begin{equation}
\delta_r L \left( \pi^{k(\rho)}, e_{\mu(\nu)}, \omega_{0(\alpha\beta)} \right) =
\end{equation}

\begin{equation}
\left( -r_{(\rho\beta)} \pi^{k(\beta)} - e B^{k(\rho)m(\alpha)0(\beta)} \left( r_{(\alpha\beta)} \right)_m \right) \dot{e}_{k(\rho)} + \pi^{k(\rho)} \left( r_{(\alpha\rho)} e_{k}^{(\alpha)} \right)_{,0} - \dot{r}_{(\alpha\beta)} \chi^{0(\alpha\beta)}.
\end{equation}

The contributions with the translational constraint automatically disappear and the only property that was used is its PB with the rotational one. Substitution of explicit form of the rotational constraint (75) gives
that complete the proof of invariance. We substituted the explicit form of the rotational constraint in (167), (168) and (171) but even this is not necessary for proof of invariance. Equation (171) can be written (see (167), (168)) as

$$\delta_r L \left( \pi^k(\rho), e^{\mu(\nu)}, \omega^0(\alpha\beta) \right) =$$

$$- r_{(\alpha\beta)} \frac{\delta \chi^{0(\alpha\beta)}}{\delta e_k(\rho)} \dot{e}_k(\rho) + \pi^k(\rho) \left( r_{(\alpha\beta)} \frac{\delta \chi^{0(\alpha\beta)}}{\delta \pi^k(\rho)} \dot{\pi}^k(\rho) \right)_{,0} - \dot{r}_{(\alpha\beta)} \chi^{0(\alpha\beta)}.$$  (173)

After integrations by parts in the second and third terms we obtain

$$\delta_r L \left( \pi^k(\rho), e^{\mu(\nu)}, \omega^0(\alpha\beta) \right) = r_{(\alpha\beta)} \left( \hat{\chi}^{0(\alpha\beta)} - \frac{\delta \chi^{0(\alpha\beta)}}{\delta e_k(\rho)} \dot{e}_k(\rho) - \hat{\pi}^k(\rho) \frac{\delta \chi^{0(\alpha\beta)}}{\delta \pi^k(\rho)} \right)_{,0}$$

$$- \left( r_{(\alpha\beta)} \left( \chi^{0(\alpha\beta)} - \hat{\pi}^k(\rho) \frac{\delta \chi^{0(\alpha\beta)}}{\delta \pi^k(\rho)} \right) \right)_{,0}.$$  (174)

where the first bracket is identically zero and the second term is the total temporal derivative.

The explicit form of the rotational constraint is not needed to prove rotational invariance of the Lagrangian. Zero value for the first bracket just follows from the definition of derivatives

$$\hat{\chi}^{0(\alpha\beta)} = \frac{\delta \chi^{0(\alpha\beta)}}{\delta e_k(\rho)} \dot{e}_k(\rho) + \frac{\delta \chi^{0(\alpha\beta)}}{\delta \pi^k(\rho)} \hat{\pi}^k(\rho) + \frac{\delta \chi^{0(\alpha\beta)}}{\delta e_0(\rho)} \dot{e}_0(\rho)$$  (175)

where the last term, as we demonstrated (see (154), \{\chi^{0(\alpha\beta)}, \pi^0(\rho)\} = 0 = \frac{\delta \chi^{0(\alpha\beta)}}{\delta e_0(\rho)}), is zero.

Of course, to find explicit form of the transformation in (167, 168) or total derivative (172), we need the exact expression for the rotational constraint. We will not give further detail here because they are the same in all dimensions and were discussed in [1].

Now we consider translational invariance of (162)

$$\delta_t L = \delta_t L \left( \pi^k(\rho), e^{\mu(\nu)}, \omega^0(\alpha\beta) \right)$$  (176)

$$\delta_t \pi^k(\rho) \dot{e}_k(\rho) + \pi^k(\rho) \delta_t \dot{e}_k(\rho) + \delta_t e_0(\rho) \chi^{0(\rho)} + e_0(\rho) \delta_t \chi^{0(\rho)} + \delta_t \omega_0(\alpha\beta) \chi^{0(\alpha\beta)} + \omega_0(\alpha\beta) \delta_t \chi^{0(\alpha\beta)}$$

where again we can find transformations of all presented fields and constraints.
\[ \delta_t e_0(\rho) = -i_{(\rho)} - \omega_{0(\rho)} t_{(\alpha)}, \]  

(177)

\[ \delta_t \alpha_{0(\alpha,\beta)} = 0, \]  

(178)

\[ \delta_t e_k(\rho) = t_{(\alpha)} \frac{\delta \chi^{0(\alpha)}}{\delta \pi^{k(\rho)}}, \]  

(179)

\[ \delta_t \pi^k(\rho) = -t_{(\alpha)} \frac{\delta \chi^{0(\alpha)}}{\delta e_k(\rho)}, \]  

(180)

\[ \delta_t \chi^{0(\rho)} = 0, \]  

(181)

\[ \delta_t \chi^{0(\alpha,\beta)} = \frac{1}{2} t_{(\alpha)} - \frac{1}{2} t_{(\beta)} \theta \]  

(182)

which are even simpler than for rotational invariance (of course, if explicit calculations of variations in (179) and (180) are not needed). Substitution of (177-182) into (176) gives

\[ \delta_t L = -t_{(\alpha)} \frac{\delta \chi^{0(\alpha)}}{\delta e_k(\rho)} e_k(\rho) + \pi^k(\rho) \left(t_{(\alpha)} \frac{\delta \chi^{0(\alpha)}}{\delta \pi^k(\rho)}\right)_{0} - i_{(\rho)} \theta. \]

(183)

As in the case of rotational invariance (173) (where the translational constraint automatically dropped out), here terms proportional to the rotational constraint cancel out without using its explicit form. Note that our assumption \( \{ \chi^{0(\alpha)}, \chi^{0(\beta)} \} = 0 \) was imposed here and triviality of (181) is the consequence of it.

The invariance of the Lagrangian (162) for 3D case can be easily verified because the form of constraints is quite simple due to absence of many terms presented in higher dimensions. We return to the expression for the translational constraint in any dimension and use it to illustrate one more time that there is nothing special about three dimensional case if the general Dirac procedure is used, all 3D results follow from general expressions for constraints.

The translational constraint was obtained in Section VII

\[ \chi^{0(\rho)} = \pi_{k(\rho)} - \frac{\delta}{\delta e_0(\rho)} \left( e B^{n(\rho)k(\alpha)m(\beta)} \right) e_{n(\rho),k} \omega_{m(\alpha,\beta)} - \frac{\delta}{\delta e_0(\rho)} \left( e A^{k(\rho)m(q)} \right) \omega_{k(p)n} \omega_{m q} \]
\[-\frac{\delta}{\delta e(\rho)} \left( e A^{k(p)m(q)} \right) \omega_{k(p)q} - 2 \frac{\delta}{\delta e_{0}(\rho)} \left( e A^{k(0)m(q)} \right) \omega_{k(0)p} \]  

\[(184)\]

where spatial connections are not independent fields but only a short notation for solutions given by (108) and (46).

In 3D case the term with \(B\) is manifestly zero (there are no three distinct external spatial indices to support its antisymmetry for all pairs of permutation \(nkm\)), as well as the last term in the first line (if \(n = 1\) (2) then \(p = q = 2\) (1) and \(A\) is zero). The solutions for spatial connections that we have to substitute into (184) are also considerably simplified in three dimensions.

The general solution for \(\omega_{k(p)n}\) (108) in 3D limit gives

\[\lim_{D \to 3} \omega_{k(p)} = -\frac{1}{2e e^{0}(0)} I^{k(p)}_{m(q)} \pi^{m(0)}(\rho)\]

The solution for \(\omega_{k(p)0}\) (46) in 3D case is

\[\lim_{D \to 3} \omega_{k(0)} = -\frac{1}{2e e^{0}(0)} I^{k(0)}_{m(p)} \pi^{m(p)}(\rho)\]

Performing variation in the second line of (184), we obtain \(B\) with one external zero which is expressible in terms of \(E\) that allows to perform contraction with one of \(I\) and, as a result, we obtain the secondary translational constrain in 3D case. Note that all the above 3D results are limits of general solutions and they are equivalent with [1].

As it was shown in [1] with the explicit form of constraints, we can derive transformations of \(\pi^{k(\rho)}\) and \(e_{k(\rho)}\). To find transformations of the original first order Lagrangian which has different set of variables, we have to find transformations of spatial connections first. Solutions to them (second class constraints) have to be used (for 3D see [1]). Finally, if we are interested in invariance of the original second order Lagrangian, we have to find a transformation of N-beins in terms of N-beins (the only independent variable) using the definition of a connection (20). Alternatively, we can obtain this transformation from equivalence of (162) with the original Lagrangian using its equations of motion and avoid intermediate calculations of transformations for spatial connections.

Now, in higher dimensions, we just demonstrate the complexity of transformations due to many additional contributions and essential modifications of the 3D result.
First, let us, as an example, consider only one (second term in (184)) which is zero in three dimensions

\[ \chi_2^{0(\rho)} = -\frac{\delta}{\delta e_0(\rho)} \left( eB^m(\rho)k(\alpha)m(\beta) \right) e_{n(\rho),k}\omega_{m(\alpha\beta)} \]  

and its contribution into the constraint \( \chi^{0(0)} \). Performing variation and using ABC properties, we obtain

\[ \chi_2^{0(0)} = e e_0^{0(0)} \left( \tilde{\omega}^\eta_{pq} E^m(b)p + \tilde{\omega}^b_{pq} E^m(p)k + \tilde{\omega}^\mu_{pq} E^m(q)k \right) e_{n(b),k} \]  

and even further restriction to a particular variation \( \frac{\delta \chi^{0(0)}}{\delta e_0^{0(0)}} \) (this choice is dictated by a solution for \( \omega_{m(pq)} \left( \pi^k(0) \right) \) given in Section VII). To find this single variation \( \frac{\delta \chi^{0(0)}}{\delta e_0^{0(0)}} \), we have to perform variation of (186) where there are two terms with “traces” \( \tilde{\omega}^\eta_{pq} \) which are simple, but in the second term we have three different indices and full solution is needed (see (108)). Of course, 3D limit is preserved on all stages of calculations, in particular, for (186). To prove this we have to consider all possible combinations of spatial indices, as 3D limit is not manifest in such a form. For \( n = k = 1(2) \) it is zero because of antisymmetry of \( E \), for non-equal indices \( n = 1(2) \ k = 2(1) \) we have to consider also particular values, (1) and (2), and find all contributions, sum of which is zero.

The appearance of additional contributions in the translational constraint and a comparison of them with derivation of transformations in 3D case allows to describe expected modifications (qualitatively) in transformations of different fields

\[ \delta \mu e_\mu(\alpha) \propto \ e_{\nu(\beta),\gamma} \text{ and } \omega_{\nu(\beta\sigma)}, \]  

\[ \delta \omega_{\mu(\alpha\beta)} \propto \omega_{\nu(\tau\sigma),\lambda} \text{ and } \omega_{\nu(\tau\sigma)} \times \omega_{\gamma(\rho\epsilon)}. \]

Of course, the completion of all, only partially described, calculations is needed to have the explicit form of transformations but to prove invariance of the Lagrangian (162) under translation in any dimension, we can repeat simple steps as it was done for rotational invariance (173)-(175), because the algebra of first class constraints (not an explicit form of constraints) defines invariance. Performing integration by parts in (183) we obtain

\[ \delta \mu L = t_0(\alpha) \left( \chi_2^{0(\alpha)} - \frac{\delta \chi^{0(\alpha)}}{\delta e_0(\rho)} \hat{e}_k(\rho) - \hat{\pi}^k(\rho) \frac{\delta \chi^{0(\alpha)}}{\delta \pi^k(\rho)} \right) + \]  

44
\[
-t_\alpha \left( \chi^{0,\alpha} - \pi^{k}(\rho \frac{\delta \chi^{0,\alpha}}{\delta \pi^{k}(\rho)}) \right), 0
\]

where the first bracket is zero because

\[
\dot{\chi}^{0,\alpha} = \frac{\delta \chi^{0,\alpha}}{\delta e^{k}(\rho)} \dot{e}^{k}(\rho) + \frac{\delta \chi^{0,\alpha}}{\delta \pi^{k}(\rho)} \dot{\pi}^{k}(\rho) + \frac{\delta \chi^{0,\alpha}}{\delta e^{0}(\rho)} \dot{e}^{0}(\rho)
\]

with the last term also zero which is again the consequence of the algebra of constraints \((\chi^{0,\rho}, \pi^{0,\rho}) = 0 = \frac{\delta \chi^{0,\rho}}{\delta e^{0}(\rho)})\). In all dimensions the Lagrangian is invariant under a translation, only algebra of constraints is needed to prove this, not the explicit form of constraints.

Of course, long calculations have to be performed to find the explicit form of transformations \((179, 180)\) and the total derivative in \((189)\). All qualitative changes that one can expect in translational invariance in higher dimensions were described in \((187, 188)\). We repeat that, based on \((174, 175)\) and \((189, 190)\), the invariance of the reduced Lagrangian in any dimension, as well as all equivalent to it (second or first order) formulations, is the consequence of \((152, 157)\). So, only our conjecture has to be proven. The complexity of general expression for the translational constraint makes this task very difficult if direct substitution is used. We are trying to find some short cuts using \(ABC\) properties and/or something similar with what was found for calculations of PB between primary and secondary translational constraints in Section VII. Whatever result of calculation for PB among translational constraints will produce we can make some conclusions.

**IX. CONCLUSION**

The Hamiltonian formulation of constraint systems developed by Dirac is indispensable in studies of complicated theories with unknown \(a\ priori\) gauge invariance. When original, not specialized to a particular dimension, variables are used it allows to consider all dimensions at once and see possible peculiarities and origin of them in particular dimensions.

The approach based on \(a\ priori\) assumptions about gauge invariance, attempts to build a gauge theory according to standard rules (as Yang-Mills) and comparison with known theories with a hope to obtain equivalence, is not productive. The transformations found by Witten \([11]\) recognizing relation of the Einstein-Cartan action in three dimensions to the Chern-Simons action is an interesting observation but not a mathematical method to
find out whether any field theoretical model in any dimension has a gauge symmetry or not and to obtain gauge transformations of fields. His conclusion: “we cannot hope that four-dimensional gravity would be a gauge theory in that sense” (as we understood, it means Einstein-Cartan is not Chern-Simons in four dimensions) is trivial but, of course, correct. According to Dirac [2], a gauge theory is a theory that has first class constraints which define the gauge invariance. And in this sense, using the well-defined and general procedure, we can always answer the question whether we have a gauge theory or not. However, Witten’s followers trying to apply gauge transformations found in 3D case to the Einstein-Cartan action in higher dimensions (obviously without success, after neglecting so many contributions as it is clear from the Hamiltonian approach presented here) made a non-trivial conclusion that N-bein gravity is not Poincaré gauge theory or that translational invariance is only property of 3D which, as we claim, is incorrect. Gauge invariance is defined by algebra of PBs among first class constraints and, as we illustrated in Discussion, the translational secondary constraint is much richer in higher dimensions but its PBs, e.g. with primary constraints, remains the same. Despite of different expression of the translational constraint, the algebra of first class constraints might be the same in all dimensions \( D > 2 \) although the form of transformations of fields might be different in higher dimensions. It is clear now that, for example, the gauge transformation of \( \omega_{\nu(\alpha\beta)} \) in dimensions higher than three should also have a dependence on a translational parameter.

The final answer on the question what gauge symmetry N-bein gravity has in dimensions higher than three depends on correctness of our conjecture. If \( \{ \chi^0(\alpha), \chi^0(\beta) \} = 0 \) then N-bein gravity is Poincaré gauge theory in all dimensions and 3D is not special at all. If \( \{ \chi^0(\alpha), \chi^0(\beta) \} \neq 0 \) but proportional to secondary constraints, i.e. \( \{ \chi^0(\alpha), \chi^0(\beta) \} = f^{0(\beta)} \chi^0(\alpha) - f^{0(\alpha)} \chi^0(\beta) \) (structure functions instead of structure constraints are unavoidable but in 3D this bracket must be zero) we still have closure of the Dirac procedure, all constraints are first class and new generators can be found easily. In this case N-bein gravity is the gauge theory but it is Poincaré only in 3D case. If \( \{ \chi^0(\alpha), \chi^0(\beta) \} \neq 0 \) and not proportional to secondary constraints, then we have next generation of constraints because multipliers would not be found on this stage (primary and secondary constraints have zero PBs). In this case, at least tertiary constraints would appear, all of them could not be first class, otherwise we would have a negative number of degrees of freedom. We reject such a nonphysical result. If tertiary, quarterly, etc. constraints appear for consistency they should be second class,
neither translational invariance nor diffeomorphism would be gauge symmetries and only rotational gauge invariance would survive.

The direct calculation of \( \{ \chi^{(\alpha)}, \chi^{(\beta)} \} \) is laborious because of complexity of constraints in higher dimensions and we are trying to find a way of dealing with such calculations. However, some results already allow to make the following conclusion, respectively what PBs among translational constraints are. It is clear that if all constraints (\( \pi^{0(\rho)}, \pi^{0(\alpha\beta)}, \chi^{0(\rho)}, \chi^{0(\alpha\beta)} \)) are first class we have two gauge parameters, \( t(\rho) \) and \( r(\alpha\beta) \), which correspond to two primary constraints, \( \pi^{0(\rho)} \) and \( \pi^{0(\alpha\beta)} \). Both parameters have internal indices, so there is no place for “diffeomorphism constraint” (spatial or full) and diffeomorphism is not a gauge invariance of N-bein gravity. All formulations that claim to have the “spatial diffeomorphism constraint” for tetrad gravity are the product of non-canonical change of variables. The similar loss of full diffeomorphism invariance in the metric gravity was discussed in [8], [12]. Actually, this non-canonical change of variables for tetrad gravity has the same origin as in the metric gravity [13]. Of course, after a non-canonical change of variables is performed, any connection with an original theory is lost. Loosely speaking (as a mathematical result cannot be more correct or more incorrect), a “deviation” from a correct formulation in case of tetrad gravity is more severe: the gauge parameter of diffeomorphism, \( \xi_{\mu} \), has an external index, whereas the gauge parameter of translation, \( t(\rho) \), has an internal index. The only possibility to reconcile translational invariance with diffeomorphism (of course, full, not spatial) in the Hamiltonian formulation, where these two symmetries cannot be present simultaneously as gauge symmetries (too many primary constraints are needed), is to find a canonical transformation that converts one into another. However, such a possibility seems to us quite bleak, in particular, because the nature of these two parameters is so different: translational invariance arises from the primary constraint \( \phi^{0(\rho)} \), whereas for diffeomorphism we need \( \phi^{\mu(0)} \).

Finally, if our conjecture is correct, the algebra of constraints is the Poincaré which is an ordinary Lie algebra: no structure functions, no derivatives of delta functions (non-locality). In contrast, formulations based on non-canonical changes of variables leading to the “spatial diffeomorphism” constraint for tetrad gravity have non-local algebra of constraints with structure functions. This algebra for a long time is the source of many troubles and numerous speculations.

Acknowledgements
We would like to thank D.G.C. McKeon for many helpful discussions and for invitation to give a talk on Theoretical Physics section at CAIMS. We are also thankful to A.M. Frolov for numerous discussions during preparation of our Report. The partial support of the Huron University College Faculty of Arts and Social Science Research Grant Fund is greatly acknowledged.

[1] A.M. Frolov, N. Kiriushcheva and S.V. Kuzmin, arXiv: 0902.0856 [gr-qc].
[2] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Sciences, Yeshiva University, New York, 1964).
[3] P.A.M. Dirac, Can. J. Math. **2** (1950) 129; *ibid.* **3** (1951) 1.
[4] A. Einstein, Sitzungsber. preuss. Akad. Wiss., Phys.-Math. K1 (1925) 414; *The Complete Collection of Scientific Papers* (Nauka, Moskva, 1966) vol. 2, p. 171; English translation of this and a few other articles is available from: <http://www.lrz-muenchen.de/~aunzicker/ae1930.html> and A.Unzicker, T. Case, arXiv: physics/0503046.
[5] N. Kiriushcheva and S.V. Kuzmin, Ann. Phys. **321** (2006) 958.
[6] P.A.M. Dirac, Proc. Roy. Soc. A **246** (1958) 333.
[7] L. Castellani, Ann. Phys. **143** (1982) 357.
[8] N. Kiriushcheva, S.V. Kuzmin, C. Racknor and S.R. Valluri, Phys. Lett. A **372** (2008) 5101.
[9] M. Ostrogradsky, Memoires l’Acad. Imperiale Sci. St.-Peterbourg, IV, 385 (1850); in *Variatsionnye printzipy mekhaniki*, ed. L.S. Polak (Fizmatgiz, Moskva, 1959) p. 315; E.T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, fourth ed. (Cambridge University Press, 1999).
[10] N. Kiriushcheva and S.V. Kuzmin, Class. Quant. Grav. **24** (2007) 1371.
[11] E. Witten, Nucl. Phys. B **311** (1988) 46.
[12] A.M. Frolov, N. Kiriushcheva and S.V. Kuzmin, arXiv: 0809.1198 [gr-qc].
[13] N. Kiriushcheva and S.V. Kuzmin, arXiv: 0809.0097 [gr-qc].