The Functional Calculus Approach to the Spectral Theorem

Markus Haase

Abstract. A consistent functional calculus approach to the spectral theorem for strongly commuting normal operators on Hilbert spaces is presented. In contrast to the common approaches using projection-valued measures or multiplication operators, here the functional calculus is not treated as a subordinate but as the central concept.

Based on five simple axioms for a “measurable functional calculus”, the theory of such calculi is developed in detail, including spectral theory, uniqueness results and construction principles. Finally, the functional calculus form of the spectral theorem is stated and proved, with some proof variants being discussed.

Mathematics Subject Classification (2000). Primary 47B15 Secondary 46A60.
Keywords. Spectral theorem, measurable functional calculus.

1. Introduction

The spectral theorem (for normal or self-adjoint operators on a Hilbert space) is certainly one of the most important results of 20th century mathematics. It comes in different forms, two of which are the most widely used: the multiplication operator (MO) form and the one using projection-valued measures (PVMs). Associated with this variety is a discussion about “What does the spectral theorem say?” (Halmos [9]), where the pro’s and con’s of the different approaches are compared.

In this article, we would like to add a slightly different stance to this debate by advocating a consistent functional calculus approach to the spectral theorem. Since in any exposition of the spectral theorem one also will find results about functional calculus, some words of explanation are in order.

Let us start with the observation that whereas multiplication operators and projection-valued measures are well-defined mathematical objects, the concept of
a functional calculus as used in the literature on the spectral theorem is usually
defined only implicitly. One speaks of the functional calculus of a normal operator
(that is, the mapping whose properties are listed in Theorem X.Y) rather than of
a functional calculus as an abstract concept. As a result, such a concept remains
a heuristic one, and the concrete calculus associated with the spectral theorem
acquires and retains a subordinate status, being merely a derivation of the “main”
formulations by multiplication operators or projection–valued measures. (At this
point, we should emphasize that we have the full functional calculus in mind, the
one which comprises all measurable functions and not just bounded ones.) For the
mathematical practice, this expositional dependence implies that when using the
functional calculus (and one wants to use it all the time) one always has to resort
to one of its constructions.

In this respect, the multiplication operator version appears to have a slight
advantage, since deriving functional calculus properties from facts about multipli-
cation operators is comparatively simple. (This is probably the reason why eminent
voices like Halmos [9] and Reed–Simon [11, VII] prefer multiplication operators.)
However, this advantage is only virtual, since the MO-version has two major draw-
backs. Firstly, a MO-representation is not canonical and hence leads to the problem
whether functional calculus constructions (square root, semigroup, logarithm etc)
are independent of the chosen MO-representation. Secondly (and somehow related
to the first), the MO-version is hardly useful for anything else than for deriving
functional calculus properties. (For example, it cannot be used in constructions,
like that of a joint (product) functional calculus.)

In contrast, an associated PVM is canonical and PVMs are very good for
constructions, but the description of the functional calculus, in particular for un-
bounded functions, is cumbersome. And since one needs the functional calculus
eventually, every construction based on PVMs has, in order to be useful, to be
backed up by results about the functional calculus associated with the new PVM.

With the present article we propose a “third way” of treating the spectral
theorem, avoiding the drawbacks of either one of the other approaches. Instead of
treating the functional calculus as a logically subordinate concept, we put it in the
spotlight and make it our main protagonist. Based on an axiomatic definition of a
“measurable functional calculus”, we shall present a thorough development of the
associated theory entailing, in particular:

• general properties, constructions like a pull-back and a push-forward calculus
  (Section 2);
• projection-valued measures, the role of null sets, the concepts of concentration
  and support (Section 3);
• spectral theory (Section 4);
• uniqueness (and commutation) properties (Section 5);
• construction principles (Section 6).

Finally, in Section 7 we state and prove “our” version of the spectral theorem,
which assumes the following simple form (see Theorem 7.6).
Spectral Theorem: Let $A_1, \ldots, A_d$ be pairwise strongly commuting normal operators on a Hilbert space $H$. Then there is a unique Borel calculus $(\Phi, H)$ on $\mathbb{C}^d$ such that $\Phi(z_j) = A_j$ for all $j = 1, \ldots, d$.

Here, we use a notion of strong commutativity which is formally different from that used by Schm"udgen in [15], but is more suitable for our approach. In a final section we then show that both notions are equivalent.

In order to advertise our approach, let us point out some "special features". Firstly, the axioms for a measurable calculus are few and simple, and hence easy to verify. Restricted to bounded functions they are just what one expects, but the main point is that these axioms work for all measurable functions.

Secondly, the mentioned axioms are complete in the sense that each functional calculus property which can be derived with the help of a MO-representation can also be derived directly, and practically with the same effort, from the axioms. This is of course not a rigorous (meta)mathematical theorem, but a heuristic statement stipulated by the exhaustive exposition we give. In particular, we demonstrate that many properties of multiplication operators (for example its spectral properties) are consequences of the general theory, simply because the multiplication operator calculus satisfies the axioms of a measurable calculus (Theorem 2.9 and Corollary 4.6).

Thirdly, the abstract functional calculus approach leads to a simple method for extending a calculus from bounded to unbounded measurable functions (Theorem 6.1). This method, known as "algebraic extension" or "extension by (multiplicative) regularization", is well-established in general functional calculus theory for unbounded operators like sectorial operators or semigroup generators. (See [6, Appendix A] and the references therein.) It has the enormous advantage that it is elegant and perspicuous, and that it avoids cumbersome arguments with domains of operators, which are omnipresent in the PVM-approach (cf. Rudin’s exposition in [14]).

Notation and Terminology
We shall work generically over the scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The letters $H, K$ usually denote Hilbert spaces, the space of bounded linear operators from $H$ to $K$ is denoted by $\mathcal{L}(H; K)$, and $\mathcal{L}(H)$ if $H = K$.

A (closed) linear subspace of $H \oplus K$ is called a (closed) linear relation. Linear relations are called multi-valued operators in [6, Appendix A], and we use freely the definitions and results from that reference. In particular, we say that a bounded operator $T \in \mathcal{L}(H)$ commutes with a linear relation $A$ if $TA \subseteq AT$, which is equivalent to

$$(x, y) \in A \Rightarrow (Tx, Ty) \in A.$$

\[1\] Actually, with less effort, since one saves the work for establishing the MO-representation, Theorem 4.7.
A linear relation is called an operator if it is functional, i.e., it satisfies
\[(x, y), (x, z) \in A \Rightarrow y = z.\]

The set of all closed linear operators is
\[C(H; K) := \{ A \subseteq H \oplus K \mid A \text{ is closed and an operator} \},\]
with \(C(H) := C(H; H)\).

For the spectral theory of linear relations, we refer to [6, Appendix A]. For a closed linear relation \(A\) in \(H\) we denote by \(\sigma(A), \sigma_p(A), \sigma_{ap}(A), \rho(A)\) the spectrum, point spectrum, approximate point spectrum and resolvent set, respectively. The resolvent of \(A\) at \(\lambda \in \rho(A)\) is
\[R(\lambda, A) := (\lambda I - A)^{-1}.\]

A measurable space is a pair \((X, \Sigma)\) where \(X\) is a set and \(\Sigma\) is a \(\sigma\)-algebra of subsets of \(X\). A function \(f : X \to K\) is measurable if it is \(\Sigma\)-to-Borel measurable in the sense of measure theory. We abbreviate
\[M(X, \Sigma) := \{ f : X \to K \mid f \text{ measurable} \}\]
and
\[M_b(X, \Sigma) := \{ f \in M(X, \Sigma) \mid f \text{ bounded} \} .\]

Then \(M_b(X, \Sigma)\) is closed under \(bp\)-convergence, by which is meant that if a sequence \((f_n)_n\) in \(M_b(X, \Sigma)\) converges boundedly (i.e., with sup \(n\|f_n\|_{\infty} < \infty\)) and pointwise to a function \(f\), then \(f \in M_b(X, \Sigma)\) as well.

If the \(\sigma\)-algebra \(\Sigma\) is understood, we simply write \(M(X)\) and \(M_b(X)\). If \(X\) is a separable metric space, then we take per default \(\Sigma = \text{Bo}(X)\), the Borel \(\sigma\)-algebra on \(X\) generated by the family of open subsets (equivalently: closed subsets, open/closed balls).

Let \((\Omega, \mathcal{F}, \mu)\) be a measure space. A null set is any subset \(A \subseteq \Omega\) such that there is \(N \in \mathcal{F}\) with \(A \subseteq N\) and \(\mu(N) = 0\). A mapping \(a : \text{dom}(a) \to X\), \((X, \Sigma)\) any measurable set, is called almost everywhere defined if \(\Omega \setminus \text{dom}(a)\) is a null set. And it is called essentially measurable, if it is almost everywhere defined and there is a measurable function \(b : \Omega \to X\) such that \(\{ x \in \text{dom}(a) \mid a(x) \neq b(x) \}\) is a null set.

### 2. Measurable Functional Calculus — Definition and Basic Properties

#### 2.1. Definition

A measurable (functional) calculus on a measurable space \((X, \Sigma)\) is a pair \((\Phi, H)\) where \(H\) is a Hilbert space and
\[\Phi : M(X, \Sigma) \to \mathcal{C}(H)\]
is a mapping with the following properties \((f, g \in M(X, \Sigma), \lambda \in \mathbb{K})\):

\[\text{(MFC1)} \quad \Phi(1) = 1;\]
The Functional Calculus Approach to the Spectral Theorem

(MFC2) \( \Phi(f) + \Phi(g) \subseteq \Phi(f + g) \) and \( \lambda \Phi(f) \subseteq \Phi(\lambda f) \);
(MFC3) \( \Phi(f) \Phi(g) \subseteq \Phi(fg) \) and \( \text{dom}(\Phi(f) \Phi(g)) = \text{dom}(\Phi(f)) \cap \text{dom}(\Phi(g)) \);
(MFC4) \( \Phi(f) \in \mathcal{L}(H) \) and \( \Phi(f)^* = \Phi(f^*) \) if \( f \) is bounded;
(MFC5) If \( f_n \to f \) pointwise and boundedly, then \( \Phi(f_n) \to \Phi(f) \) weakly.

Property (MFC5) is called the **weak bp-continuity** of the mapping \( \Phi \). We shall see below, that a measurable functional calculus is actually (strongly) bp-continuous, i.e., one can replace “weakly” by “strongly” in (MFC5). (See Theorem 2.1.f below.)

Given a measurable functional calculus \( (\Phi, H) \) we denote by
\[
\text{bdd}(\Phi) := \{ f \in \mathcal{M}(X, \Sigma) \mid \Phi(f) \in \mathcal{L}(H) \}
\]
the set of **\( \Phi \)-bounded elements**.

### 2.2. First Properties

In the following, we shall explore and comment on the axioms. First of all, (MFC1)–(MFC3) simply say that a measurable functional calculus is a proto-calculus in the terminology of [8]. As a consequence, a measurable functional calculus has the properties of every proto-calculus. These account for a)–c) of the following theorem.

**Theorem 2.1.** Let \( \Phi : \mathcal{M}(X, \Sigma) \to \mathcal{C}(H) \) be a measurable functional calculus. Then the following assertions hold (\( f_n, f, g \in \mathcal{M}(X, \Sigma), \lambda \in \mathbb{C} \)):

a) If \( \lambda \neq 0 \) or \( \Phi(f) \in \mathcal{L}(H) \) then \( \Phi(\lambda f) = \lambda \Phi(f) \).

b) If \( \Phi(g) \in \mathcal{L}(H) \) then
\[
\Phi(f) + \Phi(g) = \Phi(f + g) \text{ and } \Phi(f) \Phi(g) = \Phi(fg).
\]

Moreover, \( \Phi(g) \Phi(f) \subseteq \Phi(f) \Phi(g) \), i.e., \( \Phi(f) \) commutes with \( \Phi(g) \).

c) If \( f \neq 0 \) everywhere then \( \Phi(f) \) is injective and \( \Phi(f)^{-1} = \Phi(f^{-1}) \).

d) \( \Phi(f) \) is densely defined.

e) If \( f \) is bounded, then \( \| \Phi(f) \| \leq \| f \|_{\infty} \).

f) (MFC5') \( \Phi \) is bp-continuous, i.e.: if \( f_n \to f \) pointwise and boundedly, then \( \Phi(f_n) \to \Phi(f) \) strongly.

**Proof.** Assertion a) and the first part of b) are straightforward consequences of the axioms (MFC1)–(MFC3) for a proto-calculus, see [8] Thm. 2.1]. The second assertion of b) follows since \( fg = gf \), and hence
\[
\Phi(g) \Phi(f) \subseteq \Phi(gf) = \Phi(fg) = \Phi(f) \Phi(g).
\]

For c) note that if \( f \neq 0 \) everywhere then \( g := 1/f \) satisfies \( fg = 1 \), and hence also c) follows from general properties of proto-calculi, cf. [8] Thm. 2.1].

---

1If \( K = \mathbb{R} \) then \( f = f \) for all \( f \in \mathcal{M}(X, \Sigma) \).
d) Let \( e := (1 + |f|)^{-1} \). Then \( e \) is bounded and real-valued, and hence \( \Phi(e) \) is self-adjoint by (MFC4). By c), \( \Phi(e) \) is injective, and hence \( \Phi(e) \) has dense range. But \( ef \) is bounded and hence, by b), \( \Phi(f)\Phi(e) = \Phi(ef) \) is bounded. It follows that \( \Phi(e) \) maps \( H \) into \( \text{dom}(\Phi(f)) \).

e) This follows from d) by a standard argument, which we give for the convenience of the reader. Let \( f \in \mathcal{M}(X, \Sigma) \) with \( |f| \leq 1 \). Then, with \( g := (1 - |f|^2)^{\frac{1}{2}} \),
\[
\langle (1 - \Phi(|f|^2))x, x \rangle = \langle \Phi(g^2)x, x \rangle = \langle \Phi(g)^*\Phi(g)x, x \rangle = \|\Phi(g)x\|^2 \geq 0
\]
and hence
\[
\|\Phi(f)x\|^2 = \langle \Phi(f)^*\Phi(f)x, x \rangle = \langle \Phi(|f|^2)x, x \rangle \leq \langle x, x \rangle = \|x\|^2
\]
for each \( x \in H \).

f) Suppose that \( f_n \to f \) pointwise and boundedly and let \( x \in H \). Then, by (MFC5), \( \Phi(f_n)x \to \Phi(f)x \) weakly. Furthermore,
\[
\|\Phi(f_n)x\|^2 = \langle \Phi(f_n)^*\Phi(f_n)x, x \rangle = \langle \Phi(|f_n|^2)x, x \rangle \to \langle \Phi(|f|^2)x, x \rangle = \|\Phi(f)x\|^2,
\]
because also \( |f_n|^2 \to |f|^2 \) pointwise and boundedly. By a standard fact from Hilbert space theory [4, Lemma D.18], it follows that \( \Phi(f_n)x \to \Phi(f)x \) in norm. \( \square \)

**Remark 2.2.** Revisiting the previous proof we see that a)–c) rest exclusively on (MFC1)–(MFC3), and only f) rests on (MFC5).

Let us derive some immediate consequences.

**Corollary 2.3.** Let \( \Phi : \mathcal{M}(X, \Sigma) \to \mathcal{C}(H) \) be a measurable functional calculus. Then the following assertions hold for \( f, g, h \in \mathcal{M}(X, \Sigma) \):

a) If \( |f| \leq |g| \) then \( \text{dom}(\Phi(g)) \subseteq \text{dom}(\Phi(f)) \) and
\[
\|\Phi(f)x\| \leq \|\Phi(g)x\| \quad (x \in \text{dom}(\Phi(g))).
\]
b) \( \text{dom}(\Phi(f)) = \text{dom}(\Phi(|f|)) \) and \( \|\Phi(f)x\| = \|\Phi(|f|)x\| \) for all \( x \in \text{dom}(\Phi(f)) \).

c) Let \( p(z) = \sum_{j=0}^n a_jz^j \in \mathbb{K}[z] \) be a polynomial of degree \( n \in \mathbb{N} \). Then
\[
\Phi(p(f)) = p(\Phi(f)) = \sum_{j=0}^n a_j\Phi(f)^j.
\]
In particular, \( \text{dom}(\Phi(p(f))) = \text{dom}(\Phi(f)^n) \).

**Proof.** a) If \( |f| \leq |g| \) then we can write \( f = hg \), where \( h \) is the function
\[
h := \begin{cases} \frac{f}{g} & \text{on } |g| \neq 0 \\ 0 & \text{on } |g| = 0. \end{cases}
\]
Then \( |h| \leq 1 \) and hence \( \Phi(h) \) is bounded with \( \|\Phi(h)\| \leq 1 \). From (MFC3) it follows that
\[
\Phi(h)\Phi(g) \subseteq \Phi(f)
\]
with \( \text{dom}(\Phi(g)) \subseteq \text{dom}(\Phi(f)) \). Furthermore, if \( x \in \text{dom}(\Phi(g)) \) we obtain
\[
\|\Phi(f)x\| = \|\Phi(h)\Phi(g)x\| \leq \|\Phi(g)x\|
\]
as claimed.
b) follows from a).
c) For \( n \geq 2 \) write \( f^n = f^n1_{|f| \leq 1} + f^n1_{|f| > 1} \). Since the first summand is bounded, one has
\[
\text{dom}(\Phi(f^n)) = \text{dom}(\Phi(f^n1_{|f| > 1})) \subseteq \text{dom}(\Phi(f^{n-1}1_{|f| > 1})) = \text{dom}(\Phi(f^{n-1}))
\]
by a). It follows that \( \Phi(f)\Phi(f^{n-1}) = \Phi(f^n) \).
by (MFC3). By induction, we obtain
\[
\Phi(f^n) = \Phi(f)^n \quad (n \geq 1).
\]
Now take \( p \in \mathbb{K}[z] \) as in the hypothesis. Since \( \text{deg}(p) = n \), we have \( a_n \neq 0 \) and there are numbers \( 0 < a < b \) and \( c > 0 \) such that
\[
a|z|^n \leq |p(z)| \leq b|z|^n \quad (|z| \geq c).
\]
Similarly as above, multiplying with \( 1_{|f| \geq c} \) shows that
\[
\text{dom}(\Phi(p(f))) = \text{dom}(\Phi(f^n)) = \text{dom}(\Phi(f)^n).
\]
Since
\[
\sum_{j=0}^{n} a_j \Phi(f)^j \subseteq \Phi(p(f)).
\]
by (MFC2), the assertion is proved. \( \square \)

So far, we have used (MFC5) only to establish the strengthening (MFC5'). We shall explore further consequences of (MFC5) in the following section.

2.3. Approximations of the Identity and Further Properties
Let \( (\Phi, H) \) be a measurable functional calculus on the measurable space \( (X, \Sigma) \).
An approximate identity in \( \mathcal{M}(X, \Sigma) \) is a sequence \((e_n)_n\) of bounded measurable functions such that \( e_n \to 1 \) pointwise and boundedly. It then follows from (MFC5') (see Theorem 2.1.f) that \( \Phi(e_n) \to I \) strongly on \( H \).

Such approximate identities abound. For instance, given \( f \in \mathcal{M}(X, \Sigma) \) both sequences of functions
\[
e_n := \frac{n}{n + |f|} \quad \text{and} \quad \bar{e}_n := 1_{|f| \leq n} \quad (n \in \mathbb{N})
\]
are approximate identities. Furthermore, as \( e_n^{-1} = 1 + \frac{1}{n} |f| \), one has
\[
\Phi(e_n)^{-1} = \Phi(e_n^{-1}) = 1 + \frac{1}{n} \Phi(|f|).
\]
This yields
\[
\text{dom}(\Phi(f)) = \text{dom}(\Phi(|f|)) = \text{dom}(\Phi(e_n)^{-1}) = \text{ran}(\Phi(e_n))
\]
for each \( n \in \mathbb{N} \). It follows once more that \( \Phi(f) \) must be densely defined. But more is true.
**Theorem 2.4.** Let $(\Phi, H)$ be a measurable functional calculus on the measurable space $(X, \Sigma)$. Then the following assertions hold $(f, g \in \mathcal{M}(X, \Sigma))$: 

a) $\text{dom}(\Phi(f)) \cap \text{dom}(\Phi(g))$ is a core for $\Phi(f)$. 

b) $\Phi(f) + \Phi(g) = \Phi(f + g)$ and $\Phi(f)\Phi(g) = \Phi(fg)$. 

c) $\Phi(f)^* = \Phi(f^*).$ 

d) $\Phi(f)$ is normal and $\Phi(f)\Phi(f) = \Phi(|f|^2)$. 

e) If $f$ is real-valued then $\Phi(f)$ is self-adjoint. Moreover, if $f$ and $g$ are real-valued and $f \leq g$, then 

$$\langle \Phi(f)x, x \rangle \leq \langle \Phi(g)x, x \rangle$$ 

for all $x \in \text{dom}(\Phi(f)) \cap \text{dom}(\Phi(g))$. 

f) The set $\text{bdd}(\Phi)$ of $\Phi$-bounded elements is a unital $*$-subalgebra of $\mathcal{M}(X, \Sigma)$ and 

$$\Phi : \text{bdd}(\Phi) \to \mathcal{L}(H)$$ 

is a unital $*$-homomorphism. Moreover, the following generalization of (MFC5) holds: if $(f_n)_n$ is a sequence in $\text{bdd}(\Phi)$ such that $f_n \to f$ pointwise and $\sup_n \|\Phi(f_n)\| < \infty$, then $f \in \text{bdd}(\Phi)$ and $\Phi(f_n) \to \Phi(f)$ strongly on $H$.

**Proof.** a) Define $e_n := n(n + |f|)^{-1}$ as in the remark above. Suppose that $x, y \in H$ are such that $\Phi(g)x = y$. Let $x_n := \Phi(e_n)x$ and $y_n := \Phi(e_n)y$. Then $\Phi(g)x_n = y_n$ (by Theorem 2.4(b) and $x_n \in \text{dom}(\Phi(f))$ (by definition of $e_n$). Since $(x_n, y_n) \to (x, y)$ as $n \to \infty$, the claim follows. 

b) To prove the first identity, let $h := |f| + |g|$. Then $\text{dom}(\Phi(h)) \subseteq \text{dom}(\Phi(f)) \cap \text{dom}(\Phi(g))$ by Corollary 2.4. On the other hand, 

$$\text{dom}(\Phi(h)) = \text{dom}(\Phi(h)) \cap \text{dom}(\Phi(f + g))$$ 

is a core for $\Phi(f + g)$ by a). The first identity follows. 

For the proof of the second identity, let $h := |g| + |fg|$. Then (by (MFC3)) 

$$\text{dom}(\Phi(h)) \subseteq \text{dom}(\Phi(g)) \cap \text{dom}(\Phi(fg)) = \text{dom}(\Phi(f)\Phi(g)).$$ 

By a), $\text{dom}(\Phi(h))$ is a core for $\Phi(fg)$. Hence the second identity. 

c) Let $e \in \mathcal{M}_B(X, \Sigma)$ real-valued and such that $ef$ is bounded. Then we have $\Phi(e) = \Phi(e)^*$ and hence 

$$\Phi(e)\Phi(f)^* \subseteq \Phi(e)\Phi(f)^* = (\Phi(f)\Phi(e))^* = \Phi(f^*) = \Phi(f^*) = \Phi(f)\Phi(e)$$ 

by [6] Prop.C.2.1.k. By taking $e = e_n = n(n + |f|)^{-1}$ we conclude that $\Phi(f)^* \subseteq \Phi(f)$. On the other hand, since $\Phi(ef) = \Phi(e)\Phi(f)$ by b) and again by [6] Prop.C.2.1.k we obtain 

$$\Phi(e)\Phi(f) \subseteq \Phi(e\Phi(f)) = \Phi(ef)^* = (\Phi(e)\Phi(f))^* = \Phi(f)^*\Phi(e).$$ 

With the same argument as before, now employing that $\Phi(f)^*$ is closed, we obtain $\Phi(f)^* \subseteq \Phi(f)^*.$
d) It is clear by (MFC3) that \( \Phi(\overline{f}) \Phi(f) \subseteq \Phi(|f|^2) \). Hence, by c),
\[
I + \Phi(f)^* \Phi(f) \subseteq \Phi(1 + |f|^2).
\]

By Theorem 2.1, the operator on the right-hand side is injective while the operator on the left-hand side is surjective. (This is standard Hilbert space operator theory, see [14, Thm. 13.13].) Hence, these operators must coincide. Normality of \( \Phi(f) \) follows readily.

e) The first assertion follows readily from c). For the second, let it follow readily.

For calculus on the measurable space \((X, \Sigma)\) see [14, Thm. 13.13].) Hence, these operators must coincide. Normality of \( \Phi(f) \) follows readily.

e) The first assertion follows readily from c). For the second, let \( h := \sqrt{g - f} \) and \( x \in \text{dom}(\Phi(f)) \cap \text{dom}(\Phi(g)) \). Then \( x \in \text{dom}(\Phi(g - f)) = \text{dom}(\Phi(u)^2) \) and since \( \Phi(u) \) is self-adjoint,
\[
\langle \Phi(g)x, x \rangle - \langle \Phi(f)x, x \rangle = \langle \Phi(g - f)x, x \rangle = \langle \Phi(u)^2x, x \rangle = \|\Phi(u)x\|^2 \geq 0.
\]

f) The first assertion follows readily from c) above and from Theorem 2.1. For the second assertion, suppose that \((f_n)_n\) is a sequence in \( \mathcal{M}(X, \Sigma) \) such that \( f_n \to f \) pointwise and \( \sup_n \|\Phi(f_n)\| < \infty \). Write
\[
\frac{f_n}{1 + |f|} = f_n \left( \frac{1}{1 + |f|} - \frac{1}{1 + |f_n|} \right) + \frac{f_n}{1 + |f_n|}.
\]

By Theorem 2.1 it follows that
\[
\Phi(f_n) \Phi(\frac{1}{1 + |f|}) = \Phi \left( \frac{f_n}{1 + |f|} \right) = \Phi(f_n) \Phi \left( \frac{1}{1 + |f|} - \frac{1}{1 + |f_n|} \right) + \Phi \left( \frac{f_n}{1 + |f_n|} \right).
\]

By (MFC5') and the uniform boundedness of the operators \( \Phi(f_n) \), we obtain
\[
\Phi(f_n) \Phi \left( \frac{1}{1 + |f|} \right) \to \Phi \left( \frac{f}{1 + |f|} \right) = \Phi(f) \Phi \left( \frac{1}{1 + |f|} \right)
\]
strongly on \( H \). Hence, for all \( x \in \text{dom}(\Phi(f)) \) one has \( \Phi(f_n)x \to \Phi(f)x \). The uniform boundedness assumption implies that \( \Phi(f) \) is norm bounded on its domain, and since it is a closed operator, \( \Phi(f) \in \mathcal{L}(H) \). Again from the uniform boundedness it follows that \( \Phi(f_n)x \to \Phi(f)x \) for all \( x \in H \).

\( \square \)

Remark 2.5. By essentially the same arguments, one can prove the following generalization of a) and b) from Theorem 2.4. Let \( (\Phi, H) \) be a measurable functional calculus on the measurable space \((X, \Sigma)\) and let \( f_1, \ldots, f_d \in \mathcal{M}(X, \Sigma) \). Then
\begin{align*}
a) & \quad \text{the space } \bigcap_{j=1}^d \text{dom}(\Phi(f_j)) \text{ is a core for each operator } \Phi(f_j), \ j = 1, \ldots, d, \\
b) & \quad \Phi(f_1) + \cdots + \Phi(f_d) = \Phi(f_1 + \cdots + f_d), \text{ and} \\
c) & \quad \Phi(f_1) \cdots \Phi(f_d) = \Phi(f_1 \cdots f_d).
\end{align*}

From now on, we shall use the properties of measurable functional calculi expressed in Theorems 2.1 and 2.4 and Corollary 2.3 without explicit reference.
2.4. Determination by Bounded Functions
Let \((\Phi, H)\) be a measurable functional calculus on the measurable space \((X, \Sigma)\) and let \(f \in \mathcal{M}(X, \Sigma)\) be arbitrary. Define
\[
e := \frac{1}{1 + |f|}
\]
Then \(e, ef\) are bounded functions and hence \(\Phi(e), \Phi(ef)\) are bounded operators. As \(e\) is nowhere zero, \(\Phi(e)\) is injective. (In the terminology of [8], this means that \(e\) is a regularizer of \(f\).) As a consequence, we obtain
\[
\Phi(f) = \Phi(e^{-1}ef) = \Phi(e^{-1}\Phi(ef)) = \Phi(e)^{-1}\Phi(ef).
\]
The thus established identity \(\Phi(f) = \Phi(e)^{-1}\Phi(ef)\) can be rephrased through the equivalence
\[
\Phi(f)x = y \iff \Phi(ef)x = \Phi(e)y \quad (x, y \in H).
\]
We realize that \(\Phi\) is completely determined by its restriction to \(\mathcal{M}_b(X, \Sigma)\). In particular, a measurable functional calculus is a calculus in the sense of [8], and \(\mathcal{M}_b(X, \Sigma)\) is an “algebraic core” in the terminology introduced there.

2.5. Restriction to Subspaces
Suppose that \((\Phi, H)\) is a measurable calculus on \((X, \Sigma)\), and that \(K \subseteq H\) is a closed subspace of \(H\), with \(P \in \mathcal{L}(H)\) being the orthogonal projection onto \(K\). For each \(f \in \mathcal{M}(X, \Sigma)\) we let \(\Phi_K(f)\) be the part of \(\Phi(f)\) in \(K\), that is, the operator
\[
\Phi_K(f) := \Phi(f) \cap (K \oplus K).
\]
The mapping \(\Phi_K : \mathcal{M}(X, \Sigma) \to \mathcal{C}(H)\) is called the restriction of \(\Phi\) to \(K\).

Lemma 2.6. In the situation just described, the following assertions hold:
\begin{itemize}
\item[a)] \(\Phi_K\) satisfies (MFC1)—(MFC3).
\item[b)] The following are equivalent:
\begin{itemize}
\item[(i)] \(\Phi_K\) is a measurable functional calculus.
\item[(ii)] \(\Phi_K(f) \in \mathcal{L}(K)\) for each \(f \in \mathcal{M}_b(X, \Sigma)\).
\item[(iii)] \(K\) is invariant under each \(\Phi(f), f \in \mathcal{M}_b(X, \Sigma)\).
\item[(iv)] \(\Phi(f)P = P\Phi(f)\) for each \(f \in \mathcal{M}_b(X, \Sigma)\).
\end{itemize}
\end{itemize}

Proof. a) is straightforward to verify.
b) It is clear that (i) implies (ii). If (ii) holds and \(f \in \mathcal{M}_b(X, \Sigma)\), then \(\Phi_K(f) = \Phi(f) \cap (K \oplus K)\) is fully defined. But that means that \(\Phi(f)\) must map \(K\) into \(K\). This yields (iii). Suppose that (iii) holds and let \(f \in \mathcal{M}_b(X, \Sigma)\). Then \(\Phi(f)P = P\Phi(f)P\). Since \(K\) is also invariant under \(\Phi(f) = \Phi(f)^*\), one also has
\[
\Phi(f)(1 - P) = (1 - P)\Phi(f)(1 - P)
\]
and combining both identities yields (iv).
The implications (iv)⇒(iii)⇒(ii) are trivial. If (ii) holds, then one obviously has (MFC4) and (MFC5). Hence, by a), (i) follows. □

2.6. Pull-back and Push-Forward of a Measurable Calculus

Suppose that \((\Phi, H)\) is a measurable calculus on \((X, \Sigma)\), and \(U : H \to K\) is an isometric isomorphism of Hilbert spaces \(H\) and \(K\). Then by

\[\Psi(f) := UF(f)U^{-1} \quad (f \in \mathcal{M}(X, \Sigma))\]

a measurable calculus \((\Psi, K)\) is given. (This is easily checked.)

In contrast to the above situation, in which the measurable space is kept and the Hilbert space is changed, one can transfer a measurable calculus to a different measurable space in the following way.

**Proposition 2.7.** Let \((X, \Sigma_X)\) and \((Y, \Sigma_Y)\) be measurable spaces, let \((\Phi, H)\) be a measurable functional calculus on \((X, \Sigma_X)\) and let \(T : \mathcal{M}(Y, \Sigma_Y) \to \mathcal{M}(X, \Sigma_X)\) be a \(*\)-homomorphism with the property that if \(f_n \to f\) pointwise and boundedly, then \(T f_n \to T f\) pointwise and boundedly. Then the mapping

\[T^* \Phi : \mathcal{M}(Y, \Sigma_Y) \to \mathcal{C}(H), \quad (T^* \Phi)(g) := \Phi(Tg),\]

is a measurable functional calculus on \((Y, \Sigma_Y)\).

**Proof.** Straightforward. □

The new functional calculus \(T^* \Phi\) is called the pull-back of \(\Phi\) along \(T\).

A particular instance of a pull-back occurs in the case of a measurable mapping \(\varphi : X \to Y\). The induced “Koopman operator”

\[T_\varphi : \mathcal{M}(Y, \Sigma_Y) \to \mathcal{M}(X, \Sigma_X), \quad T_\varphi(g) := g \circ \varphi\]

satisfies the hypothesis of Proposition 2.7. Hence, its pull-back is

\[\Phi_\varphi := (T_\varphi^* \Phi) : \mathcal{M}(Y, \Sigma_Y) \to \mathcal{C}(H), \quad \Phi_\varphi(f) := \Phi(f \circ \varphi).\]

**Corollary 2.8.** In the situation described above, the mapping \(\Phi_\varphi : \mathcal{M}(Y, \Sigma_Y) \to \mathcal{C}(H)\) is a measurable functional calculus.

The calculus \((\Phi_\varphi, H)\) is called the push-forward of \(\Phi\) along \(\varphi\).

This construction applies in particular in the case that \((Y, \Sigma_Y) = (\mathbb{K}, Bo(\mathbb{K}))\) or, more generally, \((Y, \Sigma_Y) = (\mathbb{K}^d, Bo(\mathbb{K}^d))\). To wit, each tuple \(\varphi := (\varphi_1, \ldots, \varphi_d)\) of measurable scalar functions induces a measurable calculus on \(\mathcal{M}(\mathbb{K}^d, Bo(\mathbb{K}^d))\). We shall see below in Proposition 5.11 that this calculus does only depend on the tuple of operators \(\Phi(\varphi_1), \ldots, \Phi(\varphi_d)\).
2.7. Multiplication Operators

Let $\Omega = (\Omega, \mathcal{F}, \mu)$ be any measure space. For a measurable function $a : \Omega \to \mathbb{K}$ we define the corresponding multiplication operator $M_a$ on $H := L^2(\Omega)$ through

$$M_a x = y \iff ax = y,$$

where on the right-hand side we mean equality of equivalence classes, i.e., almost everywhere equality of representatives. In other words,

$$M_a x := af \quad \text{for} \quad f \in \text{dom}(M_a) := \{f \in L^2(\Omega) \mid af \in L^2(\Omega)\}.$$

It is obvious that $M_a$ depends only on the equivalence class of the function $a$ modulo equality almost everywhere. We shall freely make use of this observation in the following and form operators $M_a$ also in the case when $a$ is just essentially measurable.

**Theorem 2.9.** Let $\Omega = (\Omega, \mathcal{F}, \mu)$ be a measure space and define

$$\Phi(a) := M_a \quad (a \in \mathcal{M}(\Omega, \mathcal{F}))$$

Then $\Phi$ is a measurable functional calculus.

**Proof.** This is straightforward. \(\Box\)

As described in the previous section, the measurable calculus described above generates a wealth of related measurable calculi as push-forwards. Let, as above, $\Omega = (\Omega, \mathcal{F}, \mu)$ be a measure space, and let $(X, \Sigma)$ be any measurable space and $a : \Omega \to X$ a measurable (or just essentially measurable) function. For a measurable function $f \in \mathcal{M}(X, \Sigma)$ define

$$\Phi(f) := M_{f \circ a},$$

which is a closed operator on $H := L^2(\Omega)$. By Corollary 2.8 the mapping $\Phi : \mathcal{M}(X, \Sigma) \to \mathcal{L}(H)$ is a measurable functional calculus.

3. Projection-Valued Measures and Null Sets

If $(\Phi, H)$ is a measurable functional calculus on a measurable space $(X, \Sigma)$, then the mapping

$$E_\Phi : \Sigma \to \mathcal{L}(H), \quad E_\Phi(B) := \Phi(1_B) \in \mathcal{L}(H) \quad (B \in \Sigma)$$

is a **projection-valued measure**. That means, $E := E_\Phi$ has the following, easy-to-check properties:

- **(PVM1)** $E(B)$ is an orthogonal projection on $H$ for each $B \in \Sigma$.
- **(PVM2)** $E(\emptyset) = 0$.
- **(PVM3)** If $B = \bigcup_{n=1}^\infty B_n$ with all $B_n \in \Sigma$ then $\sum_{n=1}^\infty E(B_n) = E(B)$ in the strong (equivalently: weak) operator topology.
A resolution of the identity shown similarly as f) in Theorem 2.1.) A projection-valued measure is nothing but a resolution of the identity in the terminology of Rudin [14, 12.17].

Several concepts and results treated from now on actually depend only on the properties of the projection-valued measure. However, it is well-known that to each projection-valued measure \( E \) on \((X, \Sigma)\) there exists a (unique) measurable functional calculus \( \Phi_E \) such that \( E = E_{\Phi_E} \) (see Theorem 6.3 below). It is therefore no loss of generality when we treat the said concepts and results in the framework of measurable calculi.

### 3.1. Null Sets

Let \((\Phi, H)\) be a fixed measurable functional calculus on \((X, \Sigma)\). Then a set \( B \in \Sigma \) is called a **null set** if \( \Phi(1_B) = 0 \). The set

\[ N_\Phi := \{ B \in \Sigma \mid \Phi(1_B) = 0 \} \]

of \( \Phi \)-null sets is a \( \sigma \)-ideal of \( \Sigma \). (This is a simple exercise.) Similarly to usual measure theory, we say that something happens \( \Phi \)-almost everywhere if it doesn’t happen at most on a \( \Phi \)-null set. For instance, the assertion “\( f = g \) \( \Phi \)-almost everywhere” for two functions \( f, g \in \mathcal{M}(X, \Sigma) \) means just that \([ f \neq g ] \in N_\Phi \).

**Lemma 3.1.** Let \((\Phi, H)\) be a measurable functional calculus on \((X, \Sigma)\) and let \( f, g \in \mathcal{M}(X, \Sigma) \). Then the following assertions hold:

a) \( \ker(\Phi(f)) = \ker(\Phi(1_{f \neq 0})) \).

b) \( \Phi(f) = 0 \iff f = 0 \ \Phi \text{-almost everywhere.} \)

c) \( \Phi(f) = \Phi(g) \iff f = g \ \Phi \text{-almost everywhere.} \)

d) \( \Phi(f) \) is injective \( \iff f \neq 0 \ \Phi \text{-almost everywhere.} \)

**Proof.** a) Since \( f = f1_{f \neq 0} \) one has \( \Phi(f) = \Phi(f)\Phi(1_{f \neq 0}) \). This yields the inclusion “\( \supseteq \)”. Next, define \( g := f^{-1}1_{f \neq 0} \). Then \( gf = 1_{f \neq 0} \) and hence \( \Phi(g)\Phi(f) \subseteq \Phi(1_{f \neq 0}) \). This yields the inclusion “\( \subseteq \)”.

b) By a), \( \Phi(f) = 0 \) if and only if \( \Phi(1_{f \neq 0}) = 0 \), if and only if \( f = 0 \) \( \Phi \)-almost everywhere.

c) If \( f = g \) \( \Phi \)-almost everywhere, then \( f - g = 0 \) \( \Phi \)-almost everywhere and hence, by b), \( \Phi(f - g) = 0 \). Since \( f = g + (f - g) \), it follows by general functional calculus properties that \( \Phi(f) = \Phi(g) + \Phi(f - g) = \Phi(g) \).

Conversely, suppose that \( \Phi(f) = \Phi(g) \). Abbreviate \( A_n := \{ |f| + |g| \leq n \} \) for \( n \in \mathbb{N} \). Then

\[ \Phi((f - g)1_{A_n}) = \Phi(f1_{A_n} - g1_{A_n}) = \Phi(f)\Phi(1_{A_n}) - \Phi(g)\Phi(1_{A_n}) = 0 \]

and hence, by b), \([ f \neq g ] \cap A_n \in N_\Phi \). Since \( N_\Phi \) is a \( \sigma \)-ideal, \( f = g \) \( \Phi \)-almost everywhere.

d) By a), \( \Phi(f) \) is injective if and only if \( \Phi(1_{f \neq 0}) \) is injective, if and only if \( \Phi(1_{f \neq 0}) = I \) (since it is an orthogonal projection), if and only if \( \Phi(1_{f = 0}) = I - \Phi(1_{f \neq 0}) = 0 \). \( \square \)
3.2. Concentration

Let $(\Phi, H)$ be a measurable calculus on $(X, \Sigma)$. We say that $\Phi$ is concentrated on a set $Y \in \Sigma$ if $Y^c$ is a $\Phi$-null set. For $Y \subseteq X$ denote by $\Sigma_Y$ the trace $\sigma$-algebra

$$\Sigma_Y := \{ Y \cap B \mid B \in \Sigma \}.$$

If $\Phi$ is concentrated on $Y \in \Sigma$ then one can induce a measurable calculus on $(Y, \Sigma_Y)$ by defining

$$\Phi_Y(f) := \Phi(f^Y) \quad (f \in M(Y, \Sigma_Y)),$$

where

$$f^Y := \begin{cases} f & \text{on } Y \\ 0 & \text{on } Y^c. \end{cases}$$

Axioms (MFC2)–(MFC5) are immediate, and Axiom (MFC1) holds since $\Phi$ is concentrated on $Y$.

Conversely, if $(\Phi, H)$ is a measurable functional calculus on $(Y, \Sigma_Y)$ then by

$$\Phi_X(f) := \Phi(f|_Y) \quad (f \in M(X, \Sigma))$$

one obtains a measurable functional calculus $(\Phi_X, H)$ on $(X, \Sigma)$ concentrated on $Y$. (This calculus is nothing but the push-forward of $\Phi$ along the inclusion mapping.) In this way, for a measurable set $Y \subseteq X$ a one-to-one correspondence is established between measurable functional calculi on $(Y, \Sigma_Y)$ on one side and measurable functional calculi on $(X, \Sigma)$ concentrated on $Y$ on the other.

3.3. Support of a Borel Calculus

A measurable functional calculus $(\Phi, H)$ on $(X, \Sigma)$ is called a Borel calculus if $X$ carries a topology and $\Sigma = \text{Bo}(X)$ is the Borel $\sigma$-algebra, i.e., the smallest $\sigma$-algebra on $X$ that contains all open sets.

In this case we call the closed subset

$$\text{supp}(\Phi) := X \setminus \bigcup \{ U \mid U \in \mathcal{N}_\Phi \text{ and } U \text{ is open in } X \}$$

the support of $\Phi$. A point $x \in X$ is contained in $\text{supp}(\Phi)$ if and only if no open neighbourhood of $x$ is a $\Phi$-null set. The next result is obvious.

**Proposition 3.2.** Let $(\Phi, H)$ be a Borel calculus on a second countable topological space $X$. Then $\Phi$ is concentrated on $\text{supp}(\Phi)$.

4. Spectral Theory

In this section we shall see that a measurable calculus $(\Phi, H)$ contains the full information about the spectrum of each operator $\Phi(f)$. To this end, define the $\Phi$-essential range of $f \in M(X, \Sigma)$ by

$$\text{essran}_\Phi(f) := \{ \lambda \in \mathbb{K} \mid \forall \varepsilon > 0 : [ |f - \lambda| \leq \varepsilon ] \notin \mathcal{N}_\Phi \}.$$  \hspace{1cm} (4.1)

Then we have the following important result.
Theorem 4.1. Let \((\Phi, H)\) be a measurable functional calculus on \((X, \Sigma)\), let \(f \in \mathcal{M}(X, \Sigma)\), \(c \geq 0\) and \(\lambda \in \mathbb{K}\). Then the following assertions hold:

a) \(\sigma(\Phi(f)) = \sigma_{ap}(\Phi(f)) = \text{essran}_\Phi(f)\).

b) \(f \in \text{essran}_\Phi(f)\) \(\Phi\)-almost everywhere.

c) \(\Phi(f) \in \mathcal{L}(H), \|\Phi(f)\| \leq c \iff |f| \leq c\) \(\Phi\)-almost everywhere.

d) \(\Phi(f)\) is self-adjoint iff \(f \in \mathbb{R}\) \(\Phi\)-almost everywhere.

e) \(\lambda\) is an eigenvalue of \(\Phi(f)\) iff \(|f = \lambda\) is not a \(\Phi\)-null set. And \(\Phi(1_{f=\lambda})\) is the projection onto the eigenspace \(\ker(\lambda - \Phi(f))\).

Proof. a) Passing to \(\lambda - f\) if necessary we only need to show that

\[0 \in \sigma(\Phi(f)) \implies 0 \in \text{essran}_\Phi(f) \implies 0 \in \sigma_{ap}(\Phi(f))\].

If \(0 \notin \text{essran}(f)\) then there is \(\varepsilon > 0\) such that \([|f| \leq \varepsilon] \in \mathcal{N}_\Phi\). Define

\[g = \begin{cases} f & \text{on } [|f| \geq \varepsilon] \\ \varepsilon & \text{else.} \end{cases}\]

Then \(g \neq 0\) everywhere and \(g^{-1}\) is bounded, so \(\Phi(g)\) is invertible. Since \(g = f\) \(\Phi\)-almost everywhere and hence \(\Phi(f) = \Phi(g)\) by Lemma 3.1, we obtain \(0 \in \rho(\Phi(f))\).

Suppose now that \(0 \in \text{essran}_\Phi(f)\) and fix \(n \in \mathbb{N}\). Then \(A_n := [\{|f| \leq 1/n\}]\) is not \(\Phi\)-null, hence there is a unit vector \(x_n \in H\) with

\[x_n = \Phi(1_{A_n})x_n\]

Then

\[\|\Phi(f)x_n\| = \|\Phi(f)\Phi(1_{A_n})x_n\| = \|\Phi(f1_{A_n})x_n\| \leq \|f1_{A_n}\|_\infty \|x_n\| \leq \frac{1}{n}\]

It follows that \((x_n)_n\) is an approximate eigenvector for \(0\), and therefore \(0 \in \sigma_{ap}(\Phi(f))\) as claimed.

b) Abbreviate \(M := \text{essran}_\Phi(f)\). For each \(\lambda \in \mathbb{K} \setminus M\) there is \(\varepsilon_\lambda > 0\) such that \([|f| \in B(\lambda, \varepsilon_\lambda)]\) is a \(\Phi\)-null set. Since countably many \(\mathcal{B}(\lambda, \varepsilon_\lambda)\) suffice to cover \(\mathbb{K} \setminus M\) and \(\mathcal{N}_\Phi\) is a \(\sigma\)-ideal, it follows that \([|f| \notin M]\) is a \(\Phi\)-null set, and hence that \(f \in M\) \(\Phi\)-almost everywhere.

c) If \(|f| \leq c\) \(\Phi\)-almost everywhere then \(f = f1_{|f| \leq c}\) \(\Phi\)-almost everywhere. Hence by Lemma 3.1

\[\|\Phi(f)\| = \|\Phi(f1_{|f| \leq c})\| \leq \|f1_{|f| \leq c}\|_\infty \leq c\]

Conversely, suppose that \(\|\Phi(f)\| \leq c\). Then \(\text{essran}_\Phi(f) = \sigma(\Phi(f)) \subseteq B[0, c]\), and therefore \(|f| \leq c\) \(\Phi\)-almost everywhere, by b).

d) If \(f \in \mathbb{R}\) \(\Phi\)-almost everywhere, then \(f = \overline{f}\) \(\Phi\)-almost everywhere, which implies (by Lemma 3.1)

\[\Phi(f) = \Phi(\overline{f}) = \Phi(f)^*\]

Conversely, if \(\Phi(f)\) is self-adjoint, then \(\sigma(\Phi(f)) \subseteq \mathbb{R}\), and hence \(f \in \mathbb{R}\) \(\Phi\)-almost everywhere, by b).
e) Without loss of generality \( \lambda = 0 \). Let \( P := \Phi(1_{\{ f \neq 0 \}}) \). Then
\[
\text{ran}(P) = \text{ker}(I - P) = \ker(\Phi(1_{\{ f \neq 0 \}})) = \ker(\Phi(f))
\]
by a) of Lemma 3.1.

Corollary 4.2. Let \((\Phi, H)\) be a measurable functional calculus on \((X, \Sigma)\) and \( f \in \mathcal{M}(X, \Sigma) \). Then
\[
\sigma_p(\Phi(f)) \subseteq f(X) \quad \text{and} \quad \sigma(\Phi(f)) = \text{supp}(\Phi(f)) \subseteq f(X),
\]
where \( \Phi^f \) is the push-forward calculus on \( K \) defined by \( \Phi^f(g) = \Phi(g \circ f) \). A fortiori, \( \Phi^f \) is concentrated on \( \sigma(\Phi(f)) \).

Proof. If \( \lambda \in \sigma_p(\Phi(f)) \) then \( f = \lambda \) is not \( \Phi \)-null, and in particular \( f = \lambda \neq \emptyset \). Hence \( \sigma_p(\Phi(f)) \subseteq f(X) \).

Since \( \sigma(\Phi(f)) = \text{essran}_\Phi(f) \), we have \( \lambda \notin \sigma(\Phi(f)) \) iff there is \( \varepsilon > 0 \) such that \([|f - \lambda| < \varepsilon]\) is \( \Phi \)-null, which is equivalent to say that the ball \( B(\lambda, \varepsilon) \) is \( \Phi^f \)-null. This shows that \( \sigma(\Phi(f)) = \text{supp}(\Phi^f) \). The inclusion \( \text{essran}_\Phi(f) \subseteq f(X) \) is clear. The last assertion then follows from Proposition 3.2.

In the special case \( X = \mathbb{K} \) one can apply the previous result to the mapping \( f = z := (z \mapsto z) \).

Corollary 4.3. Let \((\Phi, H)\) be a Borel functional calculus on \( K \), and let \( A := \Phi(z) \). Then \( \Phi \) is concentrated on \( \sigma(A) \) but on no strictly smaller closed set.

For the next corollary we denote by \( z_j = (z \mapsto z_j) : \mathbb{C}^d \to \mathbb{C}, \ j = 1, \ldots, d \), the coordinate projections, and consider \( \mathbb{R}^d \subseteq \mathbb{C}^d \) canonically.

Corollary 4.4. Let \((\Phi, H)\) be a Borel functional calculus on \( \mathbb{C}^d \) such that the operator \( A_j := \Phi(z_j) \) is self-adjoint for each \( j = 1, \ldots, d \). Then \( \Phi \) is concentrated on \( \mathbb{R}^d \).

Proof. By Theorem 4.1 the set \( \{ z_j \notin \mathbb{R} \} \) is \( \Phi \)-null for each \( j = 1, \ldots, d \). Hence, the set \( \mathbb{C}^d \setminus \mathbb{R}^d = \bigcup_{j=1}^d [z_j \notin \mathbb{R}] \) is \( \Phi \)-null as well.

4.1. Multiplication Operators Revisited

Let \( \Omega = (\Omega, \Sigma, \mu) \) be a measure space with associated multiplication calculus
\[
\Phi(a) := M_a \quad (a \in \mathcal{M}(\Omega, \Sigma))
\]
as in Section 2. The measure space \( \Omega \) is called **semi-finite** if each set of infinite measure has a subset of finite but non-zero measure.

Lemma 4.5. Let \( \Omega \) be a semi-finite measure space. Then for a set \( A \in \Sigma \) the following assertions are equivalent:

(i) \( A \) is \( \mu \)-null.

(ii) \( A \) is \( \Phi \)-null.
Proof. Clearly (i) implies (ii), even if Ω is not semi-finite. For the converse suppose that \( \Phi(1_A) = 0 \). Then \( 1_A f = 0 \) for each \( f \in L_2(\Omega) \). Then \( 1_B = 0 \) for all \( B \subseteq A \) with \( \mu(B) < \infty \). By semi-finiteness, this implies that \( \mu(A) = 0 \). \( \square \)

The following is a standard result from elementary operator theory. Here we obtain it as a corollary of functional calculus theory.

**Corollary 4.6.** Let \( \Omega = (\Omega, \Sigma, \mu) \) be a semi-finite measure space. Then the following assertions hold for each \( a \in M(\Omega, \Sigma) \).

a) \( M_a \) is injective if and only if \( \mu[a = 0] = 0 \). In this case, \( M_a^{-1} = M_a^{-1} \).

b) \( M_a \) is bounded if and only if \( a \in L_\infty(\Omega) \). In this case \( \| M_a \| = \|a\|_{L_\infty} \).

c) \( \sigma(M_a) = \sigma_{ap}(M_a) = \text{essran}(a) \), the essential range of \( a \).

d) \( \lambda \in \rho(M_a) \implies R(\lambda, M_a) = M(\lambda^{-1}) \).

e) Up to equality \( \mu \)-almost everywhere, \( a \) is uniquely determined by \( M_a \).

f) \( M_a \) is symmetric iff it is self-adjoint iff \( a \in \mathbb{R} \) almost everywhere.

We have seen that the standard spectral properties of multiplication operators are just special cases of the spectral properties of measurable functional calculi. On the other hand, one can derive properties of measurable functional calculi from properties of multiplication operators. This is due to the following theorem, which is stated here for the sake of completeness, but will not be used at any point in following sections.

**Theorem 4.7.** Let \( (\Phi, H) \) be a measurable calculus on the measurable space \( (X, \Sigma) \). Then there exists a semi-finite measure space \( (\Omega, F, \mu) \), a unitary operator \( U : H \to L_2(\Omega, F, \mu) \) and an injective \( * \)-homomorphism \( T : M(X, \Sigma) \to M(\Omega, F) \) with

\[
\Phi(f) = UM_T U^{-1} \quad \text{for all } f \in M(X, \Sigma).
\]

Moreover, \( T \) is continuous with respect to pointwise convergence of sequences.

**Proof.** The proof follows well-known lines, so we only sketch it. Details can be found in many books, e.g. in [11 VII], or [6 Appendix D].

For each \( x \in H \) let \( \mu_x \) be the measure on \( (X, \Sigma) \) defined by

\[
\mu_x(A) = \langle \Phi(1_A)x, x \rangle \quad (A \in \Sigma).
\]

Then

\[
\langle \Phi(f)x, x \rangle = \int_X f \, d\mu_x \quad (f \in M_b(X, \Sigma)).
\]

Define

\[
Z_x := \{ \Phi(f)x \mid f \in M_b(X, \Sigma) \}.
\]

Then

\[
M_b(X, \Sigma) \to Z_x, \quad f \mapsto \Phi(f)x
\]

is isometric with respect to \( \| \cdot \|_{L_2(\mu_x)} \) and hence extends to a unitary operator

\[
V_x : L_2(X, \Sigma, \mu_x) \to Z_x.
\]
Both spaces $Z_x$ and $Z_x^\perp$ are $\Phi(\mathcal{M}_b(X, \Sigma))$-invariant. Employing Zorn’s lemma, one finds a maximal set $(x_\alpha)_\alpha$ of unit vectors $x_\alpha$ in $H$ such that the spaces $Z_{x_\alpha}$ are pairwise orthogonal and satisfy

$$H = \ell^2 \oplus \bigoplus_{\alpha} Z_{x_\alpha}. $$

For each $\alpha$ let $X_\alpha := X \times \{\alpha\}$ be a copy of $X$, so that the $X_\alpha$ are pairwise disjoint. Let $\Omega := \bigsqcup_\alpha X_\alpha$ be their disjoint union. Let $\mathcal{F}$ be the largest $\sigma$-algebra on $\Omega$ such that all inclusion maps $X_\alpha \to \Omega$ are measurable. Let $\mu$ be the measure on $\mathcal{F}$ defined by

$$\mu(B) := \sum_\alpha \mu_{x_\alpha}(B \cap X_\alpha) \quad (B \in \mathcal{F})$$

and let $K := L_2(\Omega, \mathcal{F}, \mu)$. Define the unitary operator

$$U : H \to K$$

such that $U^{-1} = V_{x_\alpha}$ on $L_2(X_\alpha, \Sigma, \mu_{x_\alpha}) \subseteq K$. Then define the mapping $T : \mathcal{M}(X, \Sigma) \to \mathcal{M}(\Omega, \mathcal{F})$ by

$$(Tf)(x, \alpha) := f(x) \quad (x \in X).$$

Then $T$ has the desired properties. Define the measurable calculus $(\Psi, K)$ by $\Psi(f) := M_{Tf}$. (That is, $\Psi$ is the pull-back of the multiplication operator calculus by $T$.) By construction,

$$M_{Tf} = U\Phi(f)U^{-1}$$

is true for all $f \in \mathcal{M}_b(X, \Sigma)$. By the remarks in Section 2.4 it must then be true for all $f \in \mathcal{M}(X, \Sigma)$. (Cf. also Lemma 5.1 below.)

5. Uniqueness

5.1. Uniqueness for Measurable Calculi

In this section we shall establish several properties that determine a measurable functional calculus uniquely. The first one has already been mentioned in Section 2.4.

**Lemma 5.1.** Let $(\Phi, H)$ and $(\Psi, H)$ be two measurable calculi on $(X, \Sigma)$ such that $\Phi(f) = \Psi(f)$ for all bounded functions $f$, then $\Phi = \Psi$.

**Lemma 5.1** can be easily refined.

**Proposition 5.2.** Let $(\Phi, H)$ and $(\Psi, H)$ be two measurable calculi on $(X, \Sigma)$. If the corresponding projection-valued measures coincide, i.e., if $E_\Phi = E_\Psi$, then $\Phi = \Psi$.

**Proof.** It follows from the hypothesis that $\Phi$ and $\Psi$ agree on the linear span of characteristic functions. By (MFC5), they agree on all bounded measurable functions, hence by Lemma 5.1 they must be equal. □
For more refined uniqueness statements we need more information about the set of functions on which two measurable calculi agree. We prepare this by looking at a slightly more general result about intertwining operators.

**Theorem 5.3.** Let $H, K$ be Hilbert spaces, let $(\Phi, H)$ and $(\Psi, K)$ be two measurable calculi on $(X, \Sigma)$ and let $T : H \to K$ be a bounded operator. Then the “intertwining set”

$$E := E(\Phi, \Psi; T) := \{ f \in \mathcal{M}(X, \Sigma) \mid T\Phi(f) \subseteq \Psi(f)T\}$$

has the following properties:

1) $E$ is a unital $*$-subalgebra of $\mathcal{M}(X, \Sigma)$.

2) If $f \in E$ and $f \neq 0$ everywhere, then $f^{-1} \in E$.

3) $f \in E \iff \frac{f}{1+|f|^2}, \frac{1}{1+|f|^2} \in E$.

4) $E$ is closed under pointwise convergence of sequences.

5) If $f \in E$, then $|f| \in E$.

6) If $f, g \in E$ are real-valued, then $f \vee g, f \wedge g \in E$.

7) The set $\{ A \in \Sigma \mid 1_A \in E \}$ is a sub-$\sigma$-algebra of $\Sigma$.

8) If $f \in E$ then $1_{[f \in B]} \in E$ for each Borel set $B \subseteq \mathbb{C}$.

In order to streamline the proof, we single out a lemma first.

**Lemma 5.4.** In the situation of Theorem 5.3, let $f \in \mathcal{M}(X, \Sigma)$ and $A \in \Sigma$, and define $T_A := \Psi(1_A)T\Phi(1_A) \in \mathcal{L}(H, K)$. Then

$$f \in E(\Phi, \Psi; T) \Rightarrow f1_A \in E(\Phi, \Psi, T_A) \Rightarrow f \in E(\Phi, \Psi, T_A).$$

**Proof.** Abbreviate $P := \Phi(1_C)$ and $Q := \Psi(1_C)$. Suppose that $T\Phi(f) \subseteq \Phi(f)T$. Then

$$T_A\Phi(1_A)f = QT\Phi(1_A)\Phi(1_A)f \subseteq QT\Phi(1_A)f = QT\Phi(f)\Phi(1_A) \subseteq Q\Psi(f)TP \subseteq \Psi(f)QTP = \Psi(f1_A)T_A.$$

This proves the first implication. Next, suppose $T_A\Phi(1_A)f \subseteq \Psi(1_A)fT_A$. Then

$$T_A\Phi(f) = T_A\Phi(1_A)f \subseteq T_A\Phi(1_A)f \subseteq \Psi(1_A)fT_A = \Psi(f)\Psi(1_A)T_A = \Psi(f)T_A,$$

which proves the second implication.

We can now give the

**Proof of Theorem 5.3.** 1) We first note that the set $E \cap \mathcal{M}_b(X, \Sigma)$ is a unital $*$-subalgebra of $\mathcal{M}_b(X, \Sigma)$, closed under bp-convergence. (The closedness under conjugation follows from (the bounded operator version of) Fuglede’s theorem [13, 12.16].)
Now let $f, g \in E$ be arbitrary. For $n \in \mathbb{N}$ define $A_n := \{|f| + |g| \leq n\}$ and $T_n := T_{A_n} = \Psi(1_{A_n}T\Phi(1_{A_n}))$. Then by Lemma 5.4, $f1_{A_n}, g1_{A_n} \in E(\Phi, \Psi; T_n)$.

By what we have just observed, this implies that

$$
(f + g)1_{A_n}, (fg)1_{A_n} : T_{A_n} \in E(\Phi, \Psi; T_n).
$$

By another application of Lemma 5.4 we obtain $f + g, fg \in E(\Phi, \Psi; T_n)$.

Let $h$ be any one of the functions $f + g, fg, f$. Then, by what we have shown so far,

$$
T_{n}\Phi(h) = \Psi(h)T_{n}.
$$

This proves the claim.

3) follows from 1) and 2) since $|f|^2 = f\overline{f}$ and $f = \frac{f}{1 + |f|^2} \cdot (\frac{1}{1 + |f|^2})^{-1}$.

4) Suppose that $f_n \in E$ and $f_n \to f$ pointwise. If $(f_n)_n$ is uniformly bounded, then $f \in E$ by (MFC5). In the general case, note that

$$
\frac{f_n}{1 + |f_n|^2} \to \frac{f}{1 + |f|^2} \quad \text{and} \quad \frac{1}{1 + |f_n|^2} \to \frac{1}{1 + |f|^2}
$$

pointwise and boundedly. The claim now follows from 3).

5) Since, by 1), $E \cap M_b(X, \Sigma)$ is a norm-closed $\ast$-subalgebra of $M_b(X, \Sigma)$, it follows by standard arguments (as for instance in the proof of the Stone–Weierstraß theorem) that if $f \in E$ is bounded, then $|f| \in E$. For general $f \in E$ approximate $f_n := f1_{|f| \leq n} \to f$ pointwise.

6) This follows from 5).

7) Let $\mathcal{E} := \{A \in \Sigma \mid 1_A \in E\}$. It is clear that $X \in \mathcal{E}$ and $\mathcal{E}$ is closed under taking complements and disjoint countable unions (by (MFC5)). Also, $\mathcal{E}$ is stable under taking finite intersections and unions, by 6).

8) By 7) it suffices to prove the assertion for $B$ being any ball $B = B(\lambda, \varepsilon)$.

Replacing $f$ by $|f - \lambda 1|$ (which is possible by 1) and 5)) we may suppose that $f$ is real-valued and $B = (-\infty, \varepsilon)$. Now

$$
1_{|f| < \varepsilon} = \lim_n n(\varepsilon - f)^+ \wedge 1
$$

pointwise, and the claim follows from 4).

Remarks 5.5. 1) A more diligent reasoning would show that 7) is a more or less direct consequence solely of the axioms (PVM1)–(PVM3) for projection-valued measures.
2) Lemma 5.4 and its use in the proof of Theorem 5.3 are inspired by Fuglede’s original article [5], see also [13]. (Observe that Fuglede’s theorem is a corollary of Theorem 5.3.)

From Theorem 5.3 we obtain the following information about the coincidence set of two calculi.

**Corollary 5.6.** Let \((\Phi, H)\) and \((\Psi, H)\) be two measurable calculi on \((X, \Sigma)\). Then the coincidence set

\[ E := \{ f \in \mathcal{M}(X, \Sigma) \mid \Phi(f) = \Psi(f) \} \]

has the properties 1)–8) listed in Theorem 5.3.

**Proof.** Apply Theorem 5.3 with \(H = K\) and note that

\[ \Phi(f) = \Psi(f) \iff f \in E(\Phi, \Psi; 1) \cap E(\Psi, \Phi; 1). \] 

\[ \Box \]

**Remark 5.7.** Fuglede’s theorem, which was employed in the proof of Theorem 5.3, is not needed to establish Corollary 5.6. Indeed, the implication

\[ \Phi(f) = \Psi(f) \Rightarrow \Phi(f) = \Psi(f) \]

follows directly from (MFC4).

### 5.2. Uniqueness for Borel Calculi

Now we confine ourselves to Borel functional calculi, more precisely to calculi on subsets \(X\) of \(\mathbb{K}^d\) endowed with the trace of the Borel algebra. We denote by \(z_j = (z \mapsto z_j) : X \to \mathbb{K}, j = 1, \ldots, d\), the coordinate projections, and \(z := (z_1, \ldots, z_d) : X \to \mathbb{K}^d\) the inclusion mapping.

**Lemma 5.8.** Let \(E\) be a subset of \(\mathcal{M}(\mathbb{K}^d)\) that satisfies the properties 1)–8) listed in Theorem 5.3. Then the following assertions are equivalent.

(i) \(E = \mathcal{M}(X, \Sigma)\);

(ii) \(z_1, \ldots, z_d \in E\);

(iii) \(\frac{z_j}{1 + |z|^2} \in E \quad (j = 1, \ldots, d)\) and \(\frac{1}{1 + |z|^2} \in E\);

(iv) \(\frac{z_j}{(1 + |z|^2)^\frac{1}{2}} \in E \quad (j = 1, \ldots, d)\).

**Proof.** (ii) \(\Rightarrow\) (i): The coordinate projections generate the Borel algebra on \(\mathbb{K}^d\). By \(E\) having properties 7) and 8), it follows that \(1_A \in E\) for all \(A \in \Sigma\). By properties 1) and 4), \(E = \mathcal{M}(X, \Sigma)\) as desired.

(iii) \(\Rightarrow\) (ii): This follows from properties 1) and 2) and the representation

\[ z_j = \frac{z_j}{1 + |z|^2} \cdot \left(\frac{1}{1 + |z|^2}\right)^{-1}. \]
(iv)⇒(ii): By property 1) we obtain first
\[ \frac{|z_j|^2}{1 + |z|^2} \in E \quad (j = 1, \ldots, d); \]
From this, one concludes \( \frac{1}{1 + |z|^2} \in E \) and then proceeds as in the proof of the implication (ii)⇒(i).

**Theorem 5.9.** Let \( X \subseteq \mathbb{K}^d \), endowed with the trace \( \sigma \)-algebra. Let \((\Phi, H)\) and \((\Psi, H)\) be two measurable calculi on \( X \). Then each of the following conditions implies that \( \Phi = \Psi \).

1) \( \Phi \) and \( \Psi \) agree on the functions \( z_1, \ldots, z_d; \)
2) \( \Phi \) and \( \Psi \) agree on the functions
\[ \frac{z_j}{1 + |z|^2} \quad (j = 1, \ldots, d) \quad \text{and} \quad \frac{1}{1 + |z|^2}; \]
3) \( \Phi \) and \( \Psi \) agree on the functions
\[ \frac{z_j}{(1 + |z|^2)^{\frac{1}{2}}} \quad (j = 1, \ldots, d); \]

Let \( A \) be a normal (self-adjoint if \( \mathbb{K} = \mathbb{R} \)) operator on a Hilbert space \( H \) and let \( K \subseteq \mathbb{K} \) be a Borel subset of \( \mathbb{C} \). A Borel calculus \((\Phi, H)\) on \( K \) is called a Borel calculus for (the operator) \( A \), if \( \Phi(z) = A \). By Theorem 5.9 applied with \( d = 1 \), a Borel calculus for \( A \) is uniquely determined. We can even say a little more.

**Corollary 5.10.** Let \( K, L \) be Borel subsets of \( \mathbb{K} \) and let \((\Phi, H)\) and \((\Psi, H)\) be Borel functional calculi on \( K \) and \( L \), respectively, for the same operator \( A \) on \( H \). Then \( \Phi \) and \( \Psi \) are both concentrated on \( K \cap L \) and
\[ \Phi_{K \cap L} = \Psi_{K \cap L}. \]

**Proof.** By Theorem 5.9 applied with \( d = 1 \), one has \( \Phi_K = \Psi_K \). Hence
\[ \Phi(1_{K \setminus L}) = \Phi_C(1_{K \setminus L}) = \Psi_C(1_{K \setminus L}) = \Psi(0) = 0. \]
The rest is simple. \( \square \)

Theorem 5.9 has another consequence, already mentioned in Section 2.6.

**Proposition 5.11.** Let \((\Phi, H)\) and \((\Psi, H)\) be measurable functional calculi on the measurable spaces \((X, \Sigma_X)\) and \((Y, \Sigma_Y)\), respectively. Let \( f_1, \ldots, f_d \in \mathcal{M}(X, \Sigma_X) \) and \( g_1, \ldots, g_d \in \mathcal{M}(Y, \Sigma_Y) \) such that
\[ \Phi(f_j) = \Psi(g_j) \quad (j = 1, \ldots, d) \]
Then for each \( h \in \mathcal{M}(\mathbb{K}^d) \) one has
\[ \Phi(h \circ (f_1, \ldots, f_d)) = \Psi(h \circ (g_1, \ldots, g_d)). \]
6. Construction of Measurable Calculi

In this section we describe different steps that lead to the construction of a measurable functional calculus. In the results we have in mind one starts with a “partial calculus”, so to speak. That is, one is given a subset \( M \subseteq \mathcal{M}(X, \Sigma) \), in the following called our set of departure, and a mapping \( \Phi : M \to \mathcal{C}(H) \) that has the properties of a restriction of a measurable calculus. And one aims at asserting that this partial calculus is in fact such a restriction, that is, can be extended (uniquely, if possible) to a full measurable calculus.

In a sense, the spectral theorem itself is of this form. There, \( X = \mathbb{K} \), the set of departure is \( M = \{ z \} \) the coordinate mapping, and the only requirement is that the operator \( \Phi(z) \) is normal (self-adjoint if \( \mathbb{K} = \mathbb{R} \)).

In all what follows, \((X, \Sigma)\) is a measurable space and \(H\) a Hilbert space.

6.1. From Bounded to Unbounded Functions (Algebraic Extension)

Here we take \( M := \mathcal{M}_b(S, \Sigma) \), the bounded measurable functions, as our set of departure. We know already that each measurable functional calculus on \((X, \Sigma)\) is uniquely determined by its restriction to \(M\).

But more is true: each functional calculus defined originally on \(\mathcal{M}_b(X, \Sigma)\) can be uniquely extended to a full measurable functional calculus. The procedure for this is canonical and known as “algebraic extension” or “extension by (multiplicative) regularization”.

**Theorem 6.1.** Let \((X, \Sigma)\) be a measurable space, \(H\) a Hilbert space and

\[ \Phi : \mathcal{M}_b(X, \Sigma) \to \mathcal{L}(H) \]

a unital and (weakly) bp-continuous \(*\)-homomorphism. Then \(\Phi\) extends uniquely to a measurable functional calculus \(\mathcal{M}(X, \Sigma) \to \mathcal{C}(H)\).

**Proof.** Uniqueness is clear. For existence, let \( \tilde{\Phi} : \mathcal{M}_b(X, \Sigma) \to \mathcal{L}(H) \) be as stated in the theorem. If \( f \in \mathcal{M}(X, \Sigma) \) is arbitrary, we take any anchor element\(^3\) for \( f \) in \( \mathcal{M}_b(X, \Sigma) \), i.e., a function \( e \in \mathcal{M}_b(X, \Sigma) \) such that \( ef \) is bounded and \( \Phi(e) \) is injective. (The function \( e = (1 + |f|)^{-1} \) will do, see below.) Then define

\[ \tilde{\Phi}(f) := \Phi(e)^{-1}\Phi(ef). \]

It is easy to see that this definition does not depend on the choice of the function \( e \) and the so-defined mapping \( \tilde{\Phi} : \mathcal{M}(X, \Sigma) \to \mathcal{C}(H) \) extends \(\Phi\) and satisfies (MFC1)–(MFC3). As a matter of fact, it also satisfies (MFC4) and (MFC5), hence it is a measurable functional calculus.

It remains to show that an anchor element as above can always be found. To this end, for given \( f \in \mathcal{M}(X, \Sigma) \) define

\[ e := \frac{1}{1 + |f|} \quad \text{and} \quad e_n := \mathbf{1}_{|f| \leq n} \quad (n \in \mathbb{N}). \]

---

\(^3\)Such elements were called “regularizers” in [6], but the terminology has been modified in the meantime, cf. [8].
Obviously $e$ and $ef$ are both bounded functions. Also, $e_n|f|$ is bounded and
\[
e_n = ((1 + |f|)e_n)e,
\]
which leads to $\Phi(e_n) = \Phi((1 + |f|)e_n)\Phi(e)$. Since $\Phi(e_n) \to \Phi(1) = I$ strongly (by (MFC1) and (MFC5')), $\Phi(e)$ must be injective. $\square$

Theorem 6.1 tells that for establishing a measurable functional calculus it suffices to construct its restriction to bounded functions. The rest is canonical.

The following example shows that without the assumption of bp-continuity in Theorem 6.1 one can encounter quite degenerate situations. (I am indebted to Hendrik Vogt for providing the main idea.)

Example 6.2. Let $K = \mathbb{C}$, $X = \mathbb{N}$ and $\Sigma = \mathcal{P}(\mathbb{N})$, the whole power set. Then
\[
\mathcal{M}_b(X, \Sigma) = \ell^\infty \quad \text{and} \quad \mathcal{M}(X, \Sigma) = \mathbb{C}^\mathbb{N},
\]
the space of all sequences.

For each strictly increasing mapping (“subsequence”) $\pi : \mathbb{N} \to \mathbb{N}$ pick a non-zero multiplicative functional $\Phi_\pi : \ell^\infty \to \mathbb{C}$ which vanishes on the ideal of sequences $x = (x_n)_n \in \ell^\infty$ such that $\lim_{n \to \infty} x_{\pi(n)} = 0$. This exists: by Zorn’s lemma there is a maximal ideal $M_\pi$ containing this ideal and by the Gelfand–Mazur theorem $\ell^\infty/M_\pi \cong \mathbb{C}$ as Banach algebras. By the commutative Gelfand–Naimark theorem, $\Phi_\pi$ is a unital $*$-homomorphism. (Alternatively one can define $\Phi_\pi$ as the ultrafilter limit with respect to some ultrafilter that contains all the “tails” $\{\pi(k) | k \geq n\}$ for $n \in \mathbb{N}$.)

Now let $I$ be the set of all such subsequences $\pi$, let $H := \ell^2(I)$ and define
\[
\Phi : \ell^\infty \to \ell^\infty(I) \subseteq \mathcal{L}(H), \quad \Phi(x) := (\Phi_\pi(x))_\pi,
\]
where we identify a bounded function on $I$ with the associated multiplication operator on $H = \ell^2(I)$. Then $\Phi$ is a unital $*$-homomorphism.

If $f : \mathbb{N} \to \mathbb{C}$ is any unbounded sequence, then there is a subsequence $\pi \in I$ along which $|f|$ converges to $+\infty$. Hence, if $e \in \ell^\infty$ is such that $ef \in \ell^\infty$ as well, then $e(n)$ converges to zero along $\pi$. Consequently, $\Phi_\pi(e) = 0$. Let $\delta_\pi \in H$ be the canonical unit vector which is 1 at $\pi$ and 0 else. Then $\Phi(e)\delta_\pi = \Phi_\pi(e)\delta_\pi = 0$. This not only shows that $f$ does not admit any “anchor elements”, but even more: the set
\[
[f]_{\ell^\infty} := \{e \in \ell^\infty \mid ef \in \ell^\infty\}
\]
is not an “anchor set” (in the terminology of [8]). It follows that algebraic extension, even in its more general form discussed in [8], does not lead to a proper extension of the original calculus. Of course, $\Phi$ is not bp-continuous.
6.2. From Projection-Valued Measures to Measurable Functional Calculus

Next, we take \( M = \{1_A \mid A \in \Sigma\} \), the set of all characteristic functions, as our set of departure. In other words, we start with a projection-valued measure.

**Theorem 6.3.** Let \((X, \Sigma)\) be a measurable space, let \(H\) be a Hilbert space, and let \(E : \Sigma \to \mathcal{L}(H)\) be a projection-valued measure as defined in Section 3. Then there exists a unique measurable functional calculus \(\Phi : \mathcal{M}(X, \Sigma) \to \mathcal{C}(H)\) such that \(E(A) = \Phi(1_A)\) for each \(A \in \Sigma\).

**Proof.** It follows from the axioms that if \(A, B \in \Sigma\) are disjoint then the ranges of \(E(A)\) and \(E(B)\) are orthogonal. Define \(\Phi\) on simple functions \(f\) by

\[
f = \sum_{j=1}^{n} \alpha_j 1_{A_j} \Rightarrow \Phi(f) := \sum_{j=1}^{n} \alpha_j E(A_j)
\]

where \(A_1, \ldots, A_n\) is any finite measurable partition of \(X\) and \(\alpha_1, \ldots, \alpha_n\) are scalars. By standard arguments it is shown that \(\Phi\) is a (well-defined) contractive unital \(*\)-homomorphism. Hence \(\Phi\) extends continuously to \(\mathcal{M}_b(X, \Sigma)\), and this extension, again denoted by \(\Phi\), is still a contractive unital \(*\)-homomorphism.

In view of Theorem 6.1 it suffices to show that \(\Phi\) is weakly bp-continuous. For any pair \(x, y \in H\) the mapping

\[
\mu_{x,y} : \Sigma \to \mathbb{K}, \quad \mu_{x,y}(A) := \langle E(A)x, y \rangle
\]

is a \(\mathbb{K}\)-valued measure. Obviously,

\[
\langle \Phi(f)x, y \rangle = \int_X f \, d\mu_{x,y}
\]

for each \(f \in \mathcal{M}_b(X, \Sigma)\). Therefore, (MFC5) is a consequence of the dominated convergence theorem, and the proof is complete. \(\square\)

6.3. From Continuous to Measurable Functional Calculus

In this section we confine ourselves to a compact Hausdorff space \(X\) endowed with the Borel \(\sigma\)-algebra \(\Sigma = \text{Bo}(X)\). The set of departure is \(M = C(X)\), the space of continuous functions.

**Theorem 6.4.** Let \(X\) be a compact Hausdorff space and let \(\Phi : C(X) \to \mathcal{L}(H)\) be a unital \(*\)-homomorphism. Then \(\Phi\) extends uniquely to a measurable calculus \(\Phi\) on \((X, \text{Bo}(X))\) with the additional property:

\[
\langle \Phi(f)x, x \rangle = \sup\{ \langle \Phi(g)x, x \rangle \mid g \in C_c(X), 0 \leq g \leq f \} \quad (x \in H)
\]

whenever \(f \geq 0\) is a bounded and lower semi-continuous function on \(X\).

**Proof.** Existence: This is rather standard, so we just give a sketch. For more details cf. the proof of Theorem 12.2 in [14].
As in the proof of Theorem 2.1 one shows that $\Phi$ is contractive. It follows that for all $x, y \in H$ the linear functional $f \mapsto \langle \Phi(f)x, y \rangle$ is bounded. By the Riesz–Markov–Kakutani representation theorem, there is a unique regular $\mathbb{K}$-valued Borel measure $\mu_{x,y} \in M(X)$ such that

$$\langle \Phi(f)x, y \rangle = \int_X f \, d\mu_{x,y} \quad (f \in C(X), x, y, \in H).$$

One easily shows that the mapping $(x, y) \mapsto \mu_{x,y}$ is sesquilinear (bilinear if $\mathbb{K} = \mathbb{R}$). Given $g \in M_b(X, \text{Bo}(X))$, the sesquilinear/bilinear form

$$(x, y) \mapsto \int_X g \, d\mu_{x,y}$$

is bounded. By a standard result from Hilbert space theory, there is a unique operator $\Psi(g)$ such that

$$\langle \Psi(g)x, y \rangle = \int_X f \, d\mu_{x,y}$$

for all $x, y \in H$. It is then routine to show that $\Psi : M_b(X, \text{Bo}(X)) \to \mathcal{L}(H)$ is a weakly bp-continuous unital $*$-homomorphism that extends $\Phi$. Moreover, it follows from the regularity of the measures $\mu_{x,x}$ that $\Psi$ has the additional property asserted in the theorem. By Theorem 6.1, $\Psi$ extends to a full measurable calculus.

Uniqueness: Let $\Psi_1, \Psi_2$ be two extensions of $\Phi$ that both have the additional property. By Theorem 5.3, the set $\{ A \in \Sigma \mid \psi_1(1_A) = \psi_2(1_A) \}$ is a $\sigma$-algebra. By the additional property, this $\sigma$-algebra contains each open set, and hence coincides with $\text{Bo}(X)$. It follows that $\Psi_1 = \Psi_2$. \hfill \Box

If the compact space $X$ is metrizable, each open subset is $\sigma$-compact, and each bounded and positive lower semicontinuous function is the pointwise limit of a uniformly bounded sequence of continuous functions. It follows that in this case the additional property is automatic, and the uniqueness assertion holds without that requirement.

**Corollary 6.5.** Let $X$ be a compact and metrizable space and let $\Phi : C(X) \to \mathcal{L}(H)$ be a unital $*$-homomorphism. Then $\Phi$ extends uniquely to a measurable calculus $\Phi$ on $(X, \text{Bo}(X))$.

**Remark 6.6.** The Baire algebra $\text{Ba}(X)$ on a compact Hausdorff space $X$ is the smallest $\sigma$-algebra that renders each continuous function measurable. It coincides with the Borel algebra when $X$ is metrizable, but is generally different from it. A measure defined on the Baire algebra is called a **Baire measure**. Baire measures are automatically regular and uniquely determined by their associated linear functionals on $C(X)$. By using Baire measures instead of regular Borel measures, one sees that Corollary 6.5 stays true if one drops metrizability but replaces $\text{Bo}(X)$ by $\text{Ba}(X)$. 
6.4. Cartesian Products

In this last section we look at measurable functional calculi on Cartesian products, that is, tensor products of functional calculi. At least in a special case, we have a positive result.

**Theorem 6.7.** Let $(\Phi, H)$ and $(\Psi, H)$ be Borel functional calculi on the compact metric spaces $X$ and $Y$, respectively. Suppose that these calculi commute, in the sense that

$$
\Phi(f)\Psi(g) = \Psi(g)\Phi(f) \quad (f \in C(X)), \quad g \in C(Y)).
$$

Then there is a unique Borel calculus $(\Phi \otimes \Psi, H)$ on $X \times Y$ such that

$$
(\Phi \otimes \Psi)(f \otimes g) = \Phi(f)\Psi(g)
$$

for all $f \in \mathcal{M}_b(X)$ and $g \in \mathcal{M}_b(Y)$.

**Proof.** Uniqueness follows from Corollary 6.5. For existence, observe first that by Theorem 5.3 and the hypothesis on has

$$
\Phi(f)\Psi(g) = \Psi(g)\Phi(f) \quad (6.1)
$$

for all $f \in \mathcal{M}_b(X)$ and $g \in \mathcal{M}_b(Y)$. Now let

$$
\mathcal{E} := \text{span}\{1_{A \times B} \mid A \in \text{Bo}(X), \ B \in \text{Bo}(Y)\}
$$

and define a linear mapping

$$
\Lambda : \mathcal{E} \to \mathcal{L}(H), \quad \Lambda(1_{A \times B}) := \Phi(1_A)\Psi(1_B).
$$

(Of course, one has to show that this map is well defined.) From (6.1) it follows that $\Lambda$ is a unital $^*$-homomorphism, and since $\mathcal{E}$ is closed under taking square roots of positive functions, $\Lambda$ is contractive (cf. the proof of Theorem 2.1.e). Hence, $\Lambda$ extends uniquely to a bounded operator (again denoted by $\Lambda$ on the $\|\cdot\|_\infty$-closure $\overline{\mathcal{E}}$ of $\mathcal{E}$. As a matter of fact, this extension is still a unital $^*$-homomorphism.

By the Stone–Weierstraß theorem, $C(X \times Y)$ is the closure of $C(X) \otimes C(Y)$ and hence contained in $\overline{\mathcal{E}}$. So we may apply Corollary 6.5 to obtain an extension of $\Lambda$, denoted by $\Phi \otimes \Psi$, to a Borel functional calculus on $X \times Y$.

The mapping $f \mapsto (\Phi \otimes \Psi)(f \otimes 1)$ is a Borel calculus on $X$ (it is the pull-back with respect to the mapping $f \mapsto f \otimes 1$) and coincides with $\Phi$ on $C(X)$. It follows that

$$
(\Phi \otimes \Psi)(f \otimes 1) = \Phi(f)
$$

for all $f \in \mathcal{M}_b(X)$. Analogously, one obtains $(\Phi \otimes \Psi)(1 \otimes g) = \Psi(g)$ for all $g \in \mathcal{M}_b(Y)$. This implies

$$
(\Phi \otimes \Psi)(f \otimes g) = (\Phi \otimes \Psi)((f \otimes 1)(1 \otimes g)) = \Phi(f)\Psi(g)
$$

as desired. \qed
Remark 6.8. Recall from Remark 6.6 that one can allow for non-metrizable spaces in Corollary 6.5 when one uses the Baire instead of the Borel algebra. A similar remark applies to Theorem 6.7. Continuing in this line of thought, by adapting the proof of Theorem 6.7 one obtains the following generalization to arbitrary products:

Theorem: Let, for each \( \lambda \in \Lambda \), \((\Phi_\lambda, H)\) be a Baire functional calculus on the compact Hausdorff space \( X_\lambda \). Suppose that all these calculi commute, in the sense that

\[
\Phi_\lambda(f)\Phi_\mu(g) = \Phi_\mu(g)\Phi_\lambda(f)
\]

for all \( \lambda, \mu \in \Lambda \), \( f \in \mathcal{M}_b(X_\lambda, \text{Ba}(X_\lambda)) \) and \( g \in \mathcal{M}_b(X_\mu, \text{Ba}(X_\mu)) \).

Then there is a unique Baire calculus \((\Psi, H)\) on \( \prod_{\lambda \in \Lambda} X_\lambda \) such that

\[
\Psi(\otimes_{\lambda} f_\lambda) = \prod_{\lambda} \Phi_\lambda(f_\lambda)
\]

for all \( f_\lambda \in \mathcal{M}_b(X_\lambda) \) with \( f_\lambda = 1 \) for all but finitely many \( \lambda \in \Lambda \).

As a matter of fact, there is an analogue for arbitrary products of Borel calculi when one makes appropriate assumptions about positive lower semi-continuous functions as in Theorem 6.4.

7. The Spectral Theorem

Finally, we shall state and prove “our” version(s) of the spectral theorem.

7.1. Bounded Operators, Complex Case

We start with the bounded operator version in the case \( \mathbb{K} = \mathbb{C} \).

Theorem 7.1 (Spectral Theorem: Bounded Operators, \( \mathbb{K} = \mathbb{C} \)). Let \( A_1, \ldots, A_d \) be bounded normal and pairwise commuting operators on a complex Hilbert space \( H \). Then there is a unique Borel calculus \((\Phi, H)\) on \( \mathbb{C}^d \) such that \( \Phi(z_j) = A_j \) (\( j = 1, \ldots, d \)).

Proof. Uniqueness follows from 5.9 so we prove existence. By Fuglede’s theorem, the operators \( A_1, \ldots, A_d \) generate a commutative unital \( C^* \)-subalgebra \( \mathcal{A} \) of \( \mathcal{L}(H) \). By Gelfand’s theorem, there is a compact space \( X \) and an isometric isomorphism \( \Psi : C(X) \to \mathcal{A} \) of \( C^* \)-algebras. By Theorem 6.4 this map extends to a Borel calculus \((\Psi, H)\) on \( X \). Let \( f_j \in C(X) \) be such that \( \Phi(f_j) = A_j \) for \( j = 1, \ldots, d \). Let \((\Phi, H)\) be the push-forward of \( \Phi \) along the continuous mapping

\[
f = (f_1, \ldots, f_d) : X \to \mathbb{C}^d.
\]

Then \((\Phi, H)\) is a measurable calculus such that

\[
\Phi(z_j) = \Psi(z_j \circ f) = \Psi(f_j) = A_j \quad \text{for each } j = 1, \ldots, d,
\]

as desired. \( \square \)
The given proof rests on Gelfand’s theorem. If one wants to avoid that, one can proceed as follows. In a first step, the theorem is reduced to self-adjoint operators. Each normal operator $A_j$ can be written uniquely as

$$A_j = A_{j1} + iA_{j2}$$

where the operators $A_{j1}$ and $A_{j2}$ are self-adjoint. Also, the operators $A_{jk}$ ($k = 1, 2, j = 1, \ldots, d$) are pairwise commuting. Suppose that Theorem 7.1 is known provided all operators are self-adjoint. Then we obtain a Borel functional calculus $\Psi$ on $\mathbb{C}^{2d}$ such that $\Psi(z_{jk}) = A_{jk}$ for all $j = 1, \ldots, d$ and $k = 1, 2$. By Corollary 4.4 $\Psi$ is concentrated on $\mathbb{R}^{2d}$ and hence can be regarded as a Borel calculus on $\mathbb{R}^{2d}$. Write $x_j := z_{j1}$ and $y_j := z_{j2}$, as these coordinate functions are real-valued now. Identify $\mathbb{R}^{2d}$ with $\mathbb{C}^d$ via the mapping $\varphi := (x_1, y_1, \ldots, x_d, y_d) \mapsto (x_1 + iy_1, \ldots, x_d + iy_d)$ and let $\Phi$ be the push-forward of $\Psi$ along $\varphi$. Then $\Phi$ is a Borel calculus on $\mathbb{C}^d$ and

$$\Phi(z_j) = \Psi(x_j + iy_j) = \Psi(x_j) + i\Psi(y_j) = A_{j1} + iA_{j2} = A_j \quad (j = 1, \ldots, d).$$

Next, suppose that the theorem is true for $d = 1$, and let $\Phi_j$ be the Borel calculus on $\mathbb{C}$ (concentrated on $\mathbb{R}$) such that $\Phi(z) = A_j$. Since the $A_j$ are pairwise commuting, it follows from Theorem 6.3 that the associated functional calculi $\Phi_j$ are pairwise commuting, too. Therefore, one can apply Theorem 6.7 to find the “joint functional calculus” $\Phi$.

That leaves us to prove Theorem 7.1 for the case that $d = 1$ and $A_1 = A$ is self-adjoint. In that situation there is a remarkably elementary proof, which was already known to Halmos [9]. For convenience, we give the short argument.

**Proof of Theorem 7.1 for a single self-adjoint operator.** Let $A$ be a bounded, self-adjoint operator on $H$ and let $a, b \in \mathbb{R}$ such that $\sigma(A) \subseteq [a, b]$. For $p \in \mathbb{C}[z]$ denote by $p^*$ the polynomial $p^*(z) := p(\overline{z})$, and let $q := p^* p$. By the spectral inclusion theorem for polynomials,

$$\sigma(q(A)) \subseteq q(\sigma(A)).$$

Now observe that $p(A)^* p(A) = p^*(A)p(A) = q(A)$. Hence, $q(A)$ is self-adjoint and therefore its norm equals its spectral radius (see [7] Section 13.2) for an elementary proof). Since $q = |p|^2$ on $\mathbb{R}$,

$$\|p(A)\|^2 = \|p(A)^* p(A)\| = \|q(A)\| = r(q(A)) = \sup\{|\lambda| \mid \lambda \in \sigma(q(A))\} \leq \sup\{|q(\mu)| \mid \mu \in \sigma(A)\} \leq \|q\|_{\infty, \sigma(A)} \leq \|p\|_{\infty, [a,b]}^2.$$

It follows that the polynomial functional calculus for $A$ is contractive for the supremum-norm on $[a, b]$. By Weierstrass’ theorem, the polynomials are dense in $\mathbb{C}[a, b]$, and hence there is a contractive linear map

$$\Phi : \mathbb{C}[a, b] \to \mathcal{L}(H)$$

such that $\Phi(p) = p(A)$ for $p \in \mathbb{C}[z]$. It is easily seen that $\Phi$ is a unital $*$-homomorphism. By Corollary 5.3 $\Phi$ extends uniquely to a Borel functional calculus.
on \([a, b]\), and pushing that forward along the inclusion map, we obtain the desired Borel functional calculus on \(\mathbb{C}\).

\[\square\]

**Remark 7.2.** A likewise elementary proof, which even yields the better estimate \(\|p(A)\| \leq \|p\|_{\infty, \sigma(A)}\) for polynomials \(p\), is given in the lecture notes [2 Theorem E.3] by Arendt, Vogt and Voigt. It is inspired by the proof of Riesz and Sz.-Nagy from [12 VII, 106]. Compare this also with Lang’s approach in [10] XVIII, §4.

Applied to a single operator, Theorem 7.1 tells that each bounded normal operator \(A\) on a complex Hilbert space \(H\) comes with a unique Borel calculus \((\Phi, H)\) such that \(\Phi(z) = A\) for all \(z \in \sigma(A)\) (Corollary 4.4), but also on \(\sigma(A) \setminus \{\lambda\}\) whenever \(\lambda\) is not in the point spectrum of \(A\) (cf. Theorem 4.1). Since \(\sigma(A)\) is the smallest closed set on which \(\Phi\) is supported, we conclude that isolated points of the spectrum must be eigenvalues. (This can, of course, be proved more elementarily.)

Uniqueness of the calculus justifies the common habit to write

\[f(A) := \Phi_A(f) \quad (f \in \text{Bo}(\mathbb{C})).\]

Suppose that \(f(A)\) is again bounded. Then one has the **composition rule**

\[g(f(A)) = (g \circ f)(A) \quad (g \in \mathcal{M}(\mathbb{C}))\] (7.1)

just because, by uniqueness, the push-forward along \(f\) of \(\Phi_A\) must coincide with \(\Phi_{f(A)}\).

### 7.2. Bounded Operators, Real Case

We now consider the case that \(\mathbb{K} = \mathbb{R}\). As is well-known, normality is now not sufficient to imply the spectral theorem. We need to assume that all operators are self-adjoint.

**Theorem 7.3 (Spectral Theorem: Bounded Operators, \(\mathbb{K} = \mathbb{R}\)).** Let \(A_1, \ldots, A_d\) be bounded self-adjoint and pairwise commuting operators on a real Hilbert space \(H\). Then there is a unique Borel calculus \((\Phi, H)\) on \(\mathbb{R}^d\) such that \(\Phi(z_j) = A_j\) for all \(j = 1, \ldots, d\).

**Proof.** Complexify \(H\) to \(H^\mathbb{C} := H \oplus iH\) and let \(A_j^\mathbb{C}\) be the canonical \(\mathbb{C}\)-linear extension of \(A_j\) to \(H^\mathbb{C}\). Then the \(A_j^\mathbb{C}\) are bounded, pairwise commuting self-adjoint operators on \(H^\mathbb{C}\). Let \((\Psi, H^\mathbb{C})\) be the associated Borel calculus on \(\mathbb{C}^d\). By Corollary 4.4, \(\Psi\) is concentrated on \(\mathbb{R}^d\). So, effectively, \(\Psi\) is a Borel calculus on \(\mathbb{R}^d\).

Next, restrict \(\Psi\) to real-valued functions, view \(H^\mathbb{C}\) as a real Hilbert space and let \(\Phi\) be the part of \(\Psi\) in the real subspace \(H \oplus \{0\} \subseteq H^\mathbb{C}\). We claim that \(\Phi\) is a Borel functional calculus. To prove this, let \(P\) be the orthogonal projection with range \(H\) (i.e., projection onto the first component). By construction and the self-adjointness of the operators \(A_j^\mathbb{C}\), \(PA_j^\mathbb{C} = A_j^\mathbb{C}P = P\Psi(z_j)\) for all \(j = 1, \ldots, d\). By Lemma 5.8, \(P\Psi(f) = \Psi(f)P\) holds for all \(f \in \mathcal{M}(X, \Sigma; \mathbb{R})\). Hence, Lemma 2.6 tells that \(\Phi\) is a measurable functional calculus.
Finally, observe that
\[ \Phi(z_j) = A_j^C \cap (H \oplus \{0\}) \oplus (H \oplus \{0\}) = A_j \]
for each \( j = 1, \ldots, d \), and we are done. \( \square \)

An alternative to the given proof proceeds as follows. Let \( A \) be the real unital \( C^* \)-subalgebra of \( \mathcal{L}(H) \), generated by the operators \( A_1, \ldots, A_d \). We can view \( A \) as a subset of \( \mathcal{L}(H^C) \) (via the isometric embedding \( A \to A^C \) as in the proof above). By the following corollary of Gelfand’s theorem, communicated to us by Jürgen Voigt, there is a compact Hausdorff space \( K \) and an isometric isomorphism \( \Psi: C(K; \mathbb{R}) \to A \). Now proceed exactly as in the proof of Theorem 7.1.

**Proposition 7.4.** Let \( B \) be unital \( C^* \)-algebra and \( A \subseteq B \) a real, closed, unital and commutative \( * \)-subalgebra of \( B \) consisting entirely of self-adjoint elements. Then there is a compact Hausdorff space \( K \) and a unital and isometric \( * \)-isomorphism \( \Psi: C(K; \mathbb{R}) \to A \).

**Proof.** Let \( A^\wedge := A + iA \). Then \( A^\wedge \) is a unital, commutative, \( * \)-subalgebra of \( B \). Moreover, it is closed, since \( A \) is closed and the mapping
\[ c = a + ib \mapsto (a, b) = \left( \frac{1}{2}(c + c^*), \frac{1}{2i}(c - c^*) \right) \]
is bounded. Then, by Gelfand’s theorem, there is a compact Hausdorff space \( K \) and a unital and isometric \( C^* \)-isomorphism \( \Psi: C(K; \mathbb{C}) \to A^\wedge \). Obviously, \( \Psi(C(K; \mathbb{R})) = A \). \( \square \)

Actually, in order to arrive at a continuous calculus \( \Psi: C(K; \mathbb{R}) \to A \) passing to a complexification is not necessary. Instead, one can apply one of the existing purely real characterizations of real \( C(K) \)-spaces, see e.g., [1].

Finally, there is an alternative route to Theorem 7.3 avoiding both complexification and Gelfand-type theorems. Like in the complex case, one can reduce the theorem to the case \( d = 1 \) and the boundedness of the real polynomial calculus. The latter can be obtained, e.g., by the proofs given in [2, Theorem E.3] or [10, XVIII, §4], already mentioned in Remark 7.2 above.

### 7.3. Unbounded Operators

The spectral theorem for (in general) unbounded operators shall be reduced to the one for bounded operators. To this aim, we introduce for any densely-defined closed operator \( A \) on a \( K \)-Hilbert space \( H \) the bounded operators
\[ T_A := (1 + A^*A)^{-1} \quad \text{and} \quad S_A := AT_A = A(1 + A^*A)^{-1}. \]  
Note that if \( \Phi \) is a Borel calculus on \( K \) for \( A \), then
\[ T_A = \Phi\left((1 + |z|^2)^{-1}\right) \quad \text{and} \quad S_A = \Phi\left(z(1 + |z|^2)^{-1}\right). \]
The idea is, roughly, to apply Theorem 7.1 to the operators \( T_A \) and \( S_A \) and then construct a Borel calculus for \( A \) as a push-forward. In order to succeed with this idea, we need the following properties of the operators \( T_A \) and \( S_A \).
Lemma 7.5. Let $A$ be a densely defined and closed operator on a Hilbert space $H$. Then the operators $T_A, S_A$ have the following properties:

a) $T_A$ is an injective, bounded and positive self-adjoint operator with $\|T_A\| \leq 1$; $S_A$ is a bounded operator.

b) $A = S_A T_A^{-1}$.

c) If $A$ is normal then $T_A^* = T_A$ and $S_A^* = S_A^*$, and one has $A = T_A^{-1} S_A$. Moreover, $T_A S_A = S_A T_A$ in this case.

d) If $A$ is self-adjoint or normal, then so is $S_A$.

Proof. a) This is standard Hilbert space operator theory, see [14, Theorem 13.13]

b) $S_A T_A^{-1} = A T_A T_A^{-1} = A|_D$, where $D = \text{ran}(T_A) = \text{dom} A^* A$ is a core for $A$ [14, Theorem 13.13].

c) Suppose that $A$ is normal. Then $(A^*)^* = A$ since $A$ is closed, and hence $T_A^* = (1 + (A^*)^* A^*)^{-1} = (1 + A A^*)^{-1} = (1 + A^* A)^{-1} = T_A$. Next, we claim that

$$T_A A \subseteq A T_A = S_A.$$  \hfill (7.3)

Proof of claim: Let $x \in \text{dom}(A)$ and $y := T_A x$. Then $y + A^* A y = x$ and hence $A^* A y = x - y \in \text{dom}(A)$. Applying $A$ and using the normality we obtain

$$A x = A y + A A^* A y = (I + A A^*) A y = (I + A^* A) A y,$$

which results in $T_A A x = A y = A T_A x = S_A x$.

A consequence of (7.3) is that

$$T_A S_A = T_A T_A = A T_A T_A = S_A T_A$$

since $\text{ran}(T_A) \subseteq \text{dom}(A)$. Next, as $\text{dom}(A)$ is dense,

$$S_A^* = (T_A A)^* = A^* T_A = A^* T_A = S_A^*.$$  

Also, we obtain $A \subseteq T_A^{-1} S_A$ from (7.3). In order to establish equality here, let $x \in \text{dom}(T_A^{-1} S_A)$, i.e., $S_A x = A T_A x \in \text{ran}(T_A) = \text{dom}(A^* A) = \text{dom}(A^* A)$. Then

$$x = (I + A^* A) T_A x = T_A x + A^* (A T_A x) \in \text{dom}(A)$$

as desired.

d) If $A$ is self-adjoint, then $S_A^* = S_A^* = S_A$ by c). If $A$ is normal then

$$S_A^* S_A = S_A^* S_A = A^* T_A A T_A = A^* T_A A T_A = A^* A T_A^2 = A A^* T_A^2 = (A A^*)^2 = S_A S_A^* = S_A S_A^*.$$  \hfill $\blacksquare$

We say that two normal operators $A$ and $B$ are strongly commuting, if the bounded operators $T_A, S_A, T_B, S_B$ are pairwise commuting.

Theorem 7.6 (Spectral Theorem: General Case). Let $A_1, \ldots, A_d$ be pairwise strongly commuting normal operators on a Hilbert space $H$, all self-adjoint if $\mathbb{K} = \mathbb{R}$. Then there is a unique Borel calculus $(\Phi, H)$ on $\mathbb{K}^d$ such that $\Phi(z_j) = A_j$ for all $j = 1, \ldots, d$.  

32 Markus Haase
Proof. Uniqueness is clear by [5.3]. For existence, we apply Theorem 7.1 to the tuple \((T_{A_1}, \ldots, T_{A_d}, S_{A_1}, \ldots, S_{A_d})\) to obtain a unique Borel functional calculus \(\Psi\) on \(K^{2d}\) such that \(t_j = T_{A_j}, \ s_j = S_{A_j}\), where \(t_1, \ldots, t_d, s_1, \ldots, s_d\) are just the coordinate projections.

Since each \(T_{A_j}\) is self-adjoint, positive, contractive and injective, its associated Borel calculus is concentrated on \((0, 1]\). It follows that \(\Psi\) is concentrated on \((0, 1]\times K^d\). Define \(f : (0, 1]^d \times K^d \to K^d\), \(f(t_1, \ldots, t_d, s_1, \ldots, s_d) := (s_1/t_1, \ldots, s_d/t_d)\) and let \(\Phi\) be the push-forward functional calculus of \(\Psi\) along \(f\). Then \(\Phi(z_j) = \Psi(t_j^{-1}s_j) = \Psi(t_j)^{-1}\Psi(s_j) = T_{A_j}^{-1}S_{A_j} = A_j\).

The proof is complete. \(\Box\)

By Theorem 7.6, each normal (self-adjoint) operator, bounded or unbounded, on a complex (real) Hilbert space \(H\) comes with a unique Borel calculus \(\Phi_A\) on \(C(R)\) such that \(\Phi_A(z) = A\). We call this the Borel calculus for \(A\) or associated with \(A\). As before, one writes \(f(A) := \Phi_A(f)\) \((f \in M(K))\).

The Borel calculus for \(A\) is concentrated on \(\sigma(A)\) (Corollary 4.3) and, as in the bounded case, isolated spectral points must be eigenvalues. The composition rule \(g(f(A)) = (g \circ f)(A)\) now holds universally, for the same reason as in the bounded operator case.

7.4. Strong Commutativity

Formally, our notion of strong commutativity differs from that of Schmüdgen from [13]. Instead of the pair of operators \((T_A, S_A)\), Schmüdgen employs the notion of the bounded transform

\[Z_A := \left(\frac{z}{\sqrt{1 + |z|^2}}\right)(A)\]

of a normal operator \(A\). Alternatively, one can write

\[Z_A = A((1 + A^*A)^{-1})^{1/2},\]

where the square root is defined via the (continuous) functional calculus for the self-adjoint operator \((1 + A^*A)^{-1}\). The following proposition is the major step to showing that both notions of strong commutativity agree.

**Proposition 7.7.** Let \(A\) be a normal operator on \(H\), self-adjoint if \(K = \mathbb{R}\). Then for \(B \in \mathcal{L}(H)\) the following assertions are equivalent:

(i) \(BA \subseteq AB\), i.e., \(B\) commutes with \(A\).

(ii) \(Bf(A) \subseteq f(A)B\) for all \(f \in M(K)\).

(iii) \(B\) commutes with \(Z_A\).
(iv) \( B \) commutes with \( T_A \) and \( S_A \).

If \( B \) is also normal, then (i)-(iv) are also equivalent to the following assertions:

(v) \( A \) and \( B \) are strongly commuting.

(vi) \( Z_A \) and \( Z_B \) commute.

\textit{Proof.} The set \( E := E(\Phi_A, \Phi_A; B) = \{ f \in \mathcal{M}(\mathbb{K}) \mid Bf(A) \subseteq f(A)B \} \) has the properties 1)–8) of Theorem 5.3. Hence, by Lemma 5.8 (i)-(iv) are pairwise equivalent.

Suppose, in addition, that \( B \) is normal. Then we can apply the foregoing to \( B \) in place of \( A \) and \( T_A, S_A \) or \( Z_A \) in place of \( B \). This yields the equivalences (iv)⇔(v) and (iii)⇔(vi). \[\Box\]

As a corollary we obtain that two normal operators strongly commute in our sense if and only if they do in the sense of Schmüdgen from [15].

\textbf{Corollary 7.8.} Let \( A, B \) be normal operators on a Hilbert space \( H \), and self-adjoint if \( \mathbb{K} = \mathbb{R} \). Then the following assertions are equivalent:

(i) \( A \) and \( B \) are strongly commuting.

(ii) \( Z_A \) and \( Z_B \) commute.

(iii) \( f(A) \) commutes with \( g(B) \) whenever \( f, g \in \mathcal{M}(\mathbb{K}) \) and one of the operators is bounded.

(iv) \( f(A) \) commutes with \( g(B) \) whenever \( f, g \in \mathcal{M}_b(\mathbb{K}) \).

(v) The projection-valued measures associated with \( A \) and \( B \) commute.

\textit{Proof.} We note the trivial or close-to-trivial implications (iii)⇒(v)⇒(iv)⇒(ii), (i).

(i)⇒(iii): Suppose that (i) holds and \( f(A) \) is bounded. Then \( T_B \) commutes with \( T_A \) and \( S_A \). By Proposition 7.7, applied with \( T_B \) instead of \( B \), \( T_B \) commutes with each \( f(A) \). The same holds for \( S_B \). Hence, if \( f(A) \) is bounded, we can apply Proposition 7.7 again (now with \( B \) replaced by \( f(A) \) and \( A \) replaced by \( B \)) and conclude that \( f(A) \) commutes with \( g(B) \) whatever \( g \) is. This proves (iii).

(ii)⇒(iii): This is similar as before. \[\Box\]

\textbf{Remark 7.9.} The definition of the bounded transform goes back to [16]. Schmüdgen [15, Chapter 5] uses the bounded transform for passing from bounded to unbounded normal operators in the proof of Theorem 7.6, cf. also [3]. The advantage is that to cover the case of a single unbounded operator one only needs the result for a single bounded operator, and this may be helpful in a course situation. On the other hand, a nontrivial concept of functional calculus, the square root, is needed to define the bounded transform in the first place, whereas one has a direct access to the operators \( T_A \) and \( S_A \) used in our approach.
The Functional Calculus Approach to the Spectral Theorem

References

[1] Albiac, F. and Kalton, N. J. A characterization of real C(K)-spaces. Amer. Math. Monthly 114, 8 (2007), 737–743.

[2] Arendt, W., Vogt, H., and Voigt, J. Form Methods for Evolution Equations. Lecture Notes of the 18th International Internetseminar, version: 6 March 2019.

[3] Budde, C., and Landsman, K. A bounded transform approach to self-adjoint operators: functional calculus and affiliated von Neumann algebras. Ann. Funct. Anal. 7, 3 (2016), 411–420.

[4] Eisner, T., Farkas, B., Haase, M., and Nagel, R. Operator Theoretic Aspects of Ergodic Theory. Vol. 272 of Graduate Texts in Mathematics. Springer, Cham, 2015.

[5] Fuglede, B. A commutativity theorem for normal operators. Proc. Nat. Acad. Sci. U.S.A. 36 (1950), 35–40.

[6] Haase, M. The Functional Calculus for Sectorial Operators. Vol. 169 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 2006.

[7] Haase, M. Functional analysis. An Elementary Introduction. Vol. 156 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2014.

[8] Haase, M. On the fundamental principles of unbounded functional calculi. Preprint, 2020.

[9] Halmos, P. R. What does the spectral theorem say? Amer. Math. Monthly 70 (1963), 241–247.

[10] Lang, S. Real and Functional Analysis. Third edition, vol. 142 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1993.

[11] Reed, M. and Simon, B. Methods of Modern Mathematical Physics I. Functional analysis. Second edition. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980.

[12] Riesz, F., and Sz.-Nagy, B. Functional analysis. Translated by Leo F. Boron. Frederick Ungar Publishing Co., New York, 1955.

[13] Rosenblum, M. On a theorem of Fuglede and Putnam. J. London Math. Soc. 33 (1958), 376–377.

[14] Rudin, W. Functional Analysis. Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.

[15] Schmüdgen, K. Unbounded Self-Adjoint Operators on Hilbert Space. Vol. 265 of Graduate Texts in Mathematics. Springer, Dordrecht, 2012.

[16] Woronowicz, S. L. Unbounded elements affiliated with C*-algebras and noncompact quantum groups. Comm. Math. Phys. 136, 2 (1991), 399–432.

Acknowledgements

In preliminary form, parts of this work were included in the lecture notes to the 21st International Internet Seminar on “Functional Calculus” during the academic year 2017/2018. I am indebted to the participating students and colleagues, in particular to Jan van Neerven (Delft), Hendrik Vogt (Bremen) and, in particular, Jürgen Voigt (Dresden) for valuable remarks and discussions.
This work was completed while the author was spending a research sabbatical at UNSW in Sydney. The author is grateful to Fedor Sukochev for his kind invitation. Moreover, the author gratefully acknowledges the financial support from the DFG, project number 431663331.

Markus Haase  
Kiel University  
Mathematisches Seminar  
Ludewig-Meyn-Str.4  
42118 Kiel, Germany  
e-mail: haase@math.uni-kiel.de