PULLBACK FORMULAS FOR NEARLY HOLOMORPHIC SAIITO-KUROKAWA LIFTS

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Abstract. We give explicit pullback formulas for nearly holomorphic Saito-Kurokawa lifts restrict to product of upper half-plane against with product of elliptic modular forms. We generalize the formula of Ichino to modular forms of higher level and free the restriction on weights. The explicit formulas provide non-trivial examples for the refined Gross-Prasad conjecture for \((\mathrm{SO}_5,\mathrm{SO}_4)\) in the non-tempered cases. As an application, we obtain new cases for Deligne’s conjecture for central critical values of certain automorphic \(L\)-functions for \(\mathrm{GL}_3 \times \mathrm{GL}_2\).

Contents

1. Introduction 1
2. Notation and conventions 6
3. Whittaker functions 8
4. Weil representations and theta correspondence 16
5. Shimura-Shintani-Waldspurger correspondence 18
6. Saito-Kurokawa lifts 26
7. Jacquet-Langlands-Shimizu correspondence 34
8. Base change for \(\mathrm{GL}_2\) 37
9. Triple product \(L\)-functions 40
10. Pullback formula 46
11. Local trilinear period integral in the \(\mathbb{C} \times \mathbb{R}\) case 52
References 60

1. Introduction

Period integrals of automorphic forms are often related to critical values of \(L\)-functions. It has important applications to the analytic and algebraic theory of \(L\)-functions. The main theme of this article concerns with a special case, namely the pullback of Saito-Kurokawa lifts. This is a special case for the Gross-Prasad conjecture (cf. [GP92] and [GGP12]) for \((\mathrm{SO}_5,\mathrm{SO}_4)\) in the non-tempered case. The purpose here is to give explicit pullback formulas for nearly holomorphic Saito-Kurokawa lifts. A Saito-Kurokawa lift of an elliptic newform \(f\) of weight \(2\kappa'\) is a Siegel modular form \(F\) (not necessarily holomorphic) on the Siegel upper half space of degree 2. The precise definition for Saito-Kurokawa lifts are given in §1.4. In particular, it has the property that

\[ L^S_{\text{spin}}(s,F) = \zeta^S(s-\kappa')\zeta^S(s-\kappa'+1)L^S(s,f) \]

for some finite set of primes \(S\) depending on \(F\). Let \(F\) be a Saito-Kurokawa lift associated to \(f\). We consider its decomposition when restricted to \(\mathfrak{H} \times \mathfrak{H}\). Let \(g\) be another elliptic newform of weight \(\kappa + 1\). The period integral we concerned is given by

\[ \langle F|_{\mathfrak{H} \times \mathfrak{H}},g \times g \rangle = \frac{1}{(\mathrm{SL}_2(\mathbb{Z}) : \Gamma)^2} \int_{\Gamma \backslash \mathfrak{H}} \int_{\Gamma \backslash \mathfrak{H}} F \left( \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \right) g(\tau_1)g(\tau_2)y_1^{\kappa-1}y_2^{\kappa-1}d\tau_1d\tau_2, \]

here \(\Gamma\) is any sufficiently small congruence subgroup. The vanishing of this period integral is related to the vanishing of the central value of the \(L\)-function \(L(s,\mathrm{Sym}^2(g) \otimes f)\) by the results of Gan-Gurevich and Qiu in [GG09] and [Qiu]. Ichino in [Ich05] gives an explicit formula relating these two values in the case when \(\kappa = \kappa'\), both \(f\) and \(g\) are of full level, and \(F\) is holomorphic. The purpose of this article is to generalize...
1.1. Main results. To state our results, we begin with some notations. Let \( f \in S_{2\kappa'}(\Gamma_0(N_1)) \) and \( g \in S_{\kappa+1}(\Gamma_0(N_2)) \) be normalized newforms. We impose the following hypothesis on the levels and weights:

**Hypothesis (H).**

(i) \( \kappa \geq \kappa' \).

(ii) \( N_1 \) is odd square-free.

(iii) \( N_2 \) is odd cubic-free.

(iv) \( N_1 | N_2 \).

(v) \( (N_1, N_2/N_1) = 1 \).

Let \( h(\tau) = \sum_{n \geq 0} c_h(n)q^n \in S_{\kappa'+1/2}(\Gamma_0(4N_1)) \) be a newform in the sense of \([\text{Koh82}]\) associated to \( f \) by the Shintani lift. Here \( S_{\kappa'+1/2}(\Gamma_0(4N_1)) \) is the Kohnen’s plus space, consisting of cusp forms \( h' \) of level \( \Gamma_0(4N_1) \) and weight \( \kappa' + 1/2 \) on \( \mathcal{H} \) such that

\[
c_h(n) = 0 \text{ for all } (-1)^\kappa' n \equiv 2, 3 \mod 4.
\]

Define the Petersson norms of \( f, g, \) and \( h \) by

\[
\langle f, f \rangle = \frac{1}{(|\text{SL}_2(\mathbb{Z}) : \Gamma_0(N_1)|)} \int_{\Gamma_0(N_1) \backslash \mathcal{H}} |f(\tau)|^2 y^{2\kappa' - 2} d\tau,
\]

\[
\langle g, g \rangle = \frac{1}{(|\text{SL}_2(\mathbb{Z}) : \Gamma_0(N_2)|)} \int_{\Gamma_0(N_2) \backslash \mathcal{H}} |g(\tau)|^2 y^{\kappa - 1} d\tau,
\]

\[
\langle h, h \rangle = \frac{1}{6(|\text{SL}_2(\mathbb{Z}) : \Gamma_0(N_1)|)} \int_{\Gamma_0(N_1) \backslash \mathcal{H}} |h(\tau)|^2 y^{2\kappa' - 3/2} d\tau.
\]

Assume further that \( \kappa' \) is odd and \( \kappa - \kappa' = 2m \). To obtain a non-trivial period integral \((1.1)\), it is necessary to consider a variant of the classical Saito-Kurokawa lift. Let

\[
F_{N_2}(Z) = \sum_{B > 0} A_{N_2}(B) e^{2\pi \sqrt{-1} \text{Tr}(BZ)} \in S_{\kappa'+1}(\Gamma_0^2(N_2))
\]

be a Saito-Kurokawa lift of \( f \) whose precise definition is given in \((6.2)\). In particular, if \( (\det(B), N_2) = 1 \), then

\[
A_{N_2}(B) = \sum_{d \mid (b_1, b_2, b_3), (d, N_2) = 1} d^{\kappa'} c_h \left( \frac{4b_1b_3 - b_2^2}{d^2} \right)
\]

for \( B = \left( \begin{array}{cc} b_1 & b_2/2 \\ b_2/2 & b_3 \end{array} \right) \). For \( r \in \mathbb{N} \), let \( \Delta_r \) be the Maass differential operator defined in \((6.3)\). Then

\[
\Delta_{\kappa'+1} F
\]

is a nearly holomorphic Siegel modular form of level \( \Gamma_0^2(N_2) \) and weight \( \kappa + 1 \) on \( \mathcal{H}_2 \). Our main theorem is the following pullback formula which relates the central value \( \Lambda(\kappa + \kappa', \text{Sym}^2(g) \otimes f) \) with the pullback of \( \Delta_{\kappa'+1} F \) on \( \mathcal{H} \times \mathcal{H} \) against \( g \times g \), here \( \Lambda(s, \text{Sym}^2(g) \otimes f) \) is the completed \( L \)-function of \( \text{Sym}^2(g) \otimes f \) in \((10.2)\). A consequence of the pullback formulas is the Galois equivariance of the central value \( \Lambda(\kappa + \kappa', \text{Sym}^2(g) \otimes f) \).

**Theorem 1.1.** *(Theorem [10.1])* Assume \( \kappa' \) is odd and \( \kappa - \kappa' = 2m \), and Hypothesis (H) holds. We have

\[
\frac{|\langle \Delta_{\kappa'+1} F_{N_2}(\mathcal{H} \times \mathcal{H}, g \times g) \rangle|^2}{\langle f, g \rangle^2} = 2^{-\kappa - 6m - 1} C(f, g) \langle h, h \rangle \Lambda(\kappa + \kappa', \text{Sym}^2(g) \otimes f).
\]

Here \( C(f, g) \) is a non-zero number in the Hecke field of \( f \) defined in Definition 4 of [10.7].

**Remark 1.** When \( N_1 = N_2 = 1 \), and \( \kappa = \kappa' \), we have \( C(f, g) = 1 \). **Theorem 1.1** in these cases were proved by Ichino in [Ich05, Theorem 2.1].
Example 1. Let $f \in S_{18}(\Gamma_0(1))$ and $g \in S_2(\Gamma_0(1))$ be the normalized newforms. Then $\kappa = 11$, $\kappa' = 9$, and $N_1 = N_2 = 1$. Let $h \in S_{18/2}(\Gamma_0(4))$ be the newform associated to $f$ normalized so that

$$h(\tau) = q^3 - 2q^4 - 16q^7 + 36q^8 + 99q^{11} + \cdots.$$ 

By computer calculation, we have

$${C(f, g) = 1,}$$

$${\langle g, g \rangle = 0.00000011353620568043209223478168122251645 \cdots,}$$

$${\langle f, f \rangle (h, h)^{-1} = 75633.94212156018996880460854760845132468 \cdots,}$$

$$\Lambda(20, \text{Sym}^2(g) \otimes f) = 0.0053135057875930754652200977341472154100 \cdots.$$ 

On the other hand, by the formula for $\Delta_{10}$ in \([6,3]\), we have

$$\frac{(\Delta_{10}F_1|_{5\times 5}, g \times g)}{(g, g)^2} = \sum_{b \in \mathbb{Z}, b^2 < 4} (1 - b^2/4) \cdot A_1 \left( \begin{pmatrix} 1 & b/2 \\ b/2 & 1 \end{pmatrix} \right) = \frac{3}{2} c_h(3) + c_h(4) = 2^{-1}. $$

Therefore the pullback formula in Theorem 1.1 holds numerically.

Example 2. Let $f = g \in S_2(\Gamma_0(15))$ be the normalized newform. Then $\kappa = \kappa' = 1$, and $N_1 = N_2 = 15$. Let $h \in S_{3/2}(\Gamma_0(60))$ be the newform associated to $f$ normalized so that

$$h(\tau) = q^3 - 2q^8 - q^{15} + 2q^{20} + 2q^{23} + \cdots.$$ 

By computer calculation, we have

$${C(f, g) = 2^{-6} \cdot 3^{-1} \cdot 5,}$$

$${\langle g, g \rangle = 0.0023596244145167680294160631624014882733 \cdots,}$$

$${\langle f, f \rangle (h, h)^{-1} = 1.0161993600970582320694739236097011625363 \cdots,}$$

$$\Lambda(2, \text{Sym}^2(g) \otimes f) = 0.0034762890966413331251690052554140352448 \cdots.$$ 

On the other hand, we have

$$\frac{(F_{15}|_{5\times 5}, g \times g)}{(g, g)^2} = \sum_{b \in \mathbb{Z}, b^2 < 4} A_{15} \left( \begin{pmatrix} 1 & b/2 \\ b/2 & 1 \end{pmatrix} \right) = 2c_h(3) + c_h(4) = 2.$$ 

Therefore the pullback formula in Theorem 1.1 holds numerically.

Corollary 1.2. (Corollary \([10,3]\)) Assume Hypothesis \((H)\) holds. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$\left( \frac{\Lambda(\kappa + \kappa', \text{Sym}^2(g) \otimes f)}{(g, g)^2\Omega_f^+} \right)^\sigma = \frac{\Lambda(\kappa + \kappa', \text{Sym}^2(g^\sigma) \otimes f^\sigma)}{(g^\sigma, g^\sigma)^2\Omega_f^+}. $$

Here $\Omega_f^+$ is the plus period of $f$ defined in \([\text{Sh17}]\).

Remark 2.

(i) The period in the corollary coincides with Deligne’s period (cf. \([\text{CC}]\) Proposition A]).

(ii) In \([\text{CC}]\) Theorem A1, following different approach, we prove the analogue results for $\kappa' > \kappa$, or $\kappa \geq \kappa'$ and $N_1 > 1$, regardless the parities of $\kappa$ and $\kappa'$.

(iii) In \([\text{Xue}]\) Theorem 1.1, by considering the period integral obtained from the restriction of nearly holomorphic Jacobi form associated to $h$ on $\mathfrak{H}$ against $g$, Xue gives an elegant proof of the same result in the case $N_1 = N_2 = 1$. From the representation-theoretic point of view, period integrals of type \([11]\) and type in \([\text{Xue}]\) Proposition 3.1 are essentially the same by a seesaw identity (cf. \([\text{Ich05}]\) Proposition 5.1 and Step 1 of \([\text{Qiu}]\) Theorem 5.3].) For instance, the corresponding SL$_2$-period considered here is \([10,4]\). However, it seems that \([10,4]\) is different from the SL$_2$-period considered in \([\text{Xue}]\) Proposition 2.1].
1.2. **An outline of the proof.** We sketch the proof of Theorem 1.1 following the idea of Ichino in [Ich05]. The main difficulties here are the calculations at the archimedean place and the places \( p \) with \( p^2 \mid N_2 \). The calculations occupy the main body of this article. The proof can be divided into three main parts.

(i) **Arithmeticity of theta lifts:** Propositions 5.10, 6.3, 7.8 and 8.7

(ii) **Algebraicity of the Fourier coefficients of Saito-Kurokawa lifts:** Propositions 6.2, 6.4

(iii) **Algebraicity of the local trilinear period integrals:** Propositions 9.1, 9.3, 9.7, 9.9

(iv) **Archimedean calculation of certain local trilinear period integral:** §11

Let \( f^2 \) and \( g^2 \) be modular forms defined by \( f^2(\tau) = f(N\tau) \) and \( g^2(\tau) = g(4\tau) \), here \( N = \prod p^2 | N_2 \). Let \( h_{N_2} \in \Gamma_0(N_2) \) be a modular form of half-integral weight obtained from \( h \) by a Hecke action. The definition of \( h_{N_2} \) is more subtle and is given in §5.1. In particular, \( h_{N_2} = h \) if \( N_1 = N_2 \). For \( 2r \in \mathbb{N} \), let \( \delta_r \) be the Maass-Shimura differential operator defined in (5.2). We choose an auxiliary fundamental discriminant \(-D < 0\) satisfies certain conditions (cf. §10.2) such that \( c_h(D) \neq 0 \). Put \( \mathcal{K} = \mathbb{Q}(\sqrt{-D}) \). Let \( g^2_\mathcal{K} \) be a modular form over \( \mathcal{K} \) associated to \( g \) by the base change lift defined in (8.4). Let \( \theta \) be the elementary theta function defined by \( \theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2} \). We have two seesaw diagrams:

\[
\begin{array}{c|c|c|c|c}
\text{SO}(3, 2) & \text{SL}_2 \times \text{SL}_2 & \Delta_{\kappa+1}^m F_{N_2} & \tau^g \times \theta \\
\hline
\text{SO}(2, 2) \times \text{SO}(1) & \text{SL}_2 & (g \times g) \times 1 & \delta_{\kappa+1/2}^m h_{N_2} \\
\hline
\widetilde{\text{SL}}_2 \times \widetilde{\text{SL}}_2 & \text{SO}(3, 1) & \delta_{\kappa+1/2}^m h_{N_2} \times \theta & g^2_\mathcal{K} \otimes \chi_D \\
\hline
\text{SL}_2 & \text{SO}(2, 1) \times \text{SO}(1) & g^2 & (\delta_{2\kappa}^m f^2 \otimes \chi_D) \times 1
\end{array}
\]

We solve two explicit seesaw identities in Propositions 10.4 and 10.5 by explicit calculation of four global \( \mathcal{K} \)-functions (cf. §10.2). Then the \( \text{SO}(2, 2) \)-period \( \langle \Delta_{\kappa+1}^m F_{N_2} \rangle_{0 \times 0} \) is equal, up to an explicit non-zero number, to the \( \text{SO}(2, 1) \)-period \( \mathcal{I}(\delta_{2\kappa}^m f^2 \otimes g^2_\mathcal{K}) \) defined by

\[
\mathcal{I}(\delta_{2\kappa}^m f^2 \otimes g^2_\mathcal{K}) = \frac{1}{(\text{SL}_2(\mathbb{Z}) : \Gamma_0(N_2))} \int_{\text{SL}_2(\mathbb{Z}) \backslash \Gamma_0(N_2)} \delta_{2\kappa}^m f^2(\tau) g^2_\mathcal{K}(\tau) y^{2\kappa - 2} \, d\tau.
\]

By Ichino’s solution to the Gross-Prasad conjecture for \( \text{SO}_4, \text{SO}_3 \) in [Ich08], the square of \( \mathcal{I}(\delta_{2\kappa}^m f^2 \otimes g^2_\mathcal{K}) \) is equal to the product of certain local trilinear period integrals times the central value of the triple product \( L \)-function \( L(s, f \otimes g_\mathcal{K}) \) (cf. §9.1). The explicit value of the local trilinear period integrals are given in §9.2 and we obtain an explicit central value formula for \( L(\kappa + \kappa', f \otimes g_\mathcal{K}) \) in Proposition 10.6. Note that we have a decomposition of \( L \)-functions:

\[
L(s, f \otimes g_\mathcal{K}) = L(s - \kappa, f \otimes \chi_{-D}) L(s, \text{Sym}^2(g) \otimes f).
\]

The theorem then follows from the explicit central value formula and the Kohnen-Zagier formula [Koh85] which relates the central value \( L(\kappa', f \otimes \chi_{-D}) \) with \( c_h(D) \).

1.3. **Structure of the article.** In §2 we fix some notations and conventions. In §3.1 and §3.2 we calculate some Whittaker functions on the metaplectic groups \( \text{SL}_2 \). The results in these two subsection are used in §3.3 and §3.4 especially in the proofs of Propositions 5.7 and 6.3. In §3.5 we recall some Whittaker functions on \( \text{GL}_2 \). The results in this subsection are used in §4.2 primarily for the calculations in Proposition 9.9. In §4 we recall Kudla’s splitting of the pullback of Weil representations to \( \text{Sp}_n \times O(V) \). In §5 and §6 we prove explicit formulas for the Shimura-Shintani-Waldspurger correspondence and Saito-Kurokawa lift, respectively. In §7 and §8 we prove explicit formulas for the Jacquet-Langlands-Shimizu correspondence and the base change lift, respectively. In §9 we prove explicit central value formulas, which might be of independent interest, of triple product \( L \)-functions of \( \text{GL}_2(\mathbb{Q}) \times \text{GL}_2(\mathcal{K}) \) for imaginary quadratic fields \( \mathcal{K} \). We postpone the proof of Proposition 9.10 to §11. The contents in §§5-9 are logically independent of each other. Readers can read these sections separately. In §10 we prove Theorem 1.1 and Corollary 1.2 by specializing the results in §10.2. In §11 we calculate certain archimedean local trilinear period integral and prove the nonvanishing of this integral.
1.4. Saito-Kurokawa packets. We explain the relation of Theorem 1.1 with the general result of Gan and Gurevich in [GG09], and point out the reasons for the assumptions on the levels and weights, and some difficulties one might encountered when trying to remove them. In the following discussion, we drop the assumptions on the levels and weights. Let $\pi$ and $\sigma$ be the cuspidal automorphic representations of $\text{PGL}_2(\mathbb{A})$ and $\text{GL}_2(\mathbb{A})$ generated by (the adelic lifts of) $f$ and $g$, respectively. Fix a non-trivial additive character $\psi$ of $\mathbb{A}_/\mathbb{Q}$.

We recall the definition of the Waldspurger packet and the Saito-Kurokawa packet associated to $\pi$. Let $(V, Q)$ be the quadratic space over $\mathbb{Q}$ defined by $V = M_2^{\mathbb{A}}$ and $Q(x) = -\det(x)$. Note that $\text{PGL}_2 \simeq \text{SO}(V)$.

Consider the theta correspondence for $(\text{SL}_2, \text{SO}(V))$. For each place $v$ of $\mathbb{Q}$, let $W_{d_v}(\pi_v)$ be a Waldspurger packet of $\text{SL}_2(\mathbb{Q}_v)$ defined by

$$W_{d_v}(\pi_v) = \left\{ \theta_{\psi_v}(\pi_v \otimes \chi_a) \mid a \in \mathbb{Q}_v^x / \mathbb{Q}_v^{x,2} \right\}.$$ 

By the results of Waldspurger in [Wal90] and [Wal91], if $\pi_v$ is not a discrete series, then $W_{d_v}(\pi_v)$ consisting of a single element denoted by $\tilde{\pi}_v^+$; if $\pi_v$ is a discrete series, then $W_{d_v}(\pi_v)$ consisting of two elements denoted by $\tilde{\pi}_v^+$ and $\tilde{\pi}_v^-$. Moreover, $\theta_{\psi_v}(\pi_v \otimes \chi_a) = \pi_v^\epsilon$ with

$$\epsilon = \chi_a(-1)\epsilon \left( \frac{1}{2}, \pi_v \right) \epsilon \left( \frac{1}{2}, \pi_v \otimes \chi_a \right)^{-1}.$$ 

The global Waldspurger packet $W_{d}(\pi)$ of $\text{SL}_2(\mathbb{A})$ is defined by

$$W_{d}(\pi) = \left\{ \tilde{\pi}^e = \otimes_v \tilde{\pi}_v^e \mid \epsilon = (\epsilon_v) \in \oplus_v \{ \pm 1 \} \right\}.$$ 

By the results of Waldspurger, the multiplicity of $\tilde{\pi}^e$ in the discrete spectrum of $\text{SL}_2(\mathbb{A})$ is equal to

$$\frac{1}{2} \prod_v \epsilon_v + \epsilon (1/2, \pi).$$ 

Let $(V', Q')$ be the quadratic space over $\mathbb{Q}$ defined in [6.1]. Note that $\text{PGSp}_2 \simeq \text{SO}(V')$. Consider the theta correspondence for $(\text{SL}_2, \text{SO}(V'))$. For each place $v$ of $\mathbb{Q}$, let $SK(\pi_v)$ be a Saito-Kurokawa packet of $\text{PGSp}_2(\mathbb{Q}_v)$ defined by

$$SK(\pi_v) = \left\{ \theta_{\psi_v}(\tilde{\pi}_v^+) \right\} \quad \text{if} \ \pi_v \text{ is not a discrete series},$$

$$\left\{ \theta_{\psi_v}(\tilde{\pi}_v^+), \theta_{\psi_v}(\tilde{\pi}_v^-) \right\} \quad \text{if} \ \pi_v \text{ is a discrete series}.$$ 

We denote $\eta_v^e = \theta_{\psi_v}(\tilde{\pi}_v^e)$. Note that $\eta_v^e$ is independent of the choice of $\psi_v$ (cf. [CPSS88 Lemma 2.1]). The global Saito-Kurokawa packet $SK(\pi)$ of $\text{PGSp}_2(\mathbb{A})$ is defined by

$$SK(\pi) = \left\{ \eta^e = \otimes_v \eta_v^e \mid \epsilon = (\epsilon_v) \in \oplus_v \{ \pm 1 \} \right\}.$$ 

By the results of Piatetski-Shapiro in [PS83] and Gan in [Gan07], the multiplicity of $\eta^e$ in the discrete spectrum of $\text{PGSp}_2(\mathbb{A})$ is equal to

$$\frac{1}{2} \prod_v \epsilon_v + \epsilon (1/2, \pi).$$ 

A Saito-Kurokawa lift of $f$ is a non-zero automorphic form in the automorphic realization of $\eta^e$ for some $\eta^e \in SK(\pi)$. The main theorem in [GG09] asserts that among the elements in the global Saito-Kurokawa packet $SK(\pi)$, there exist at most one $\eta^e \in SK(\pi)$ such that the period integral (1.1) might be non-zero for some Saito-Kurokawa lift in $\eta^e$, and the corresponding $\epsilon$ is given by

$$\epsilon_v = \epsilon \left( \frac{1}{2}, \pi_v \boxtimes \sigma_v \boxtimes \sigma_v^\vee \right).$$ 

From now on, let $\eta^e$ be this element. Let $S = S(\pi, \sigma)$ be the set of prime divisors $p$ of $N_1$ such that $\epsilon \left( \frac{1}{2}, \pi_p \right) = -\epsilon_p$. Let $s = \sigma S$. Note that $\pi_\infty$ and $\sigma_\infty$ are discrete series representations of weights $2\kappa'$ and $\kappa + 1$, respectively. Thus

$$\epsilon \left( \frac{1}{2}, \pi_\infty \boxtimes \sigma_\infty \boxtimes \sigma_\infty^\vee \right) = \begin{cases} 1 & \text{if } \kappa < \kappa', \\ -1 & \text{if } \kappa \geq \kappa'. \end{cases}$$

Therefore, $\eta^e$ occurs in the discrete spectrum if and only if

$$(-1)^{\kappa'+s} = \begin{cases} 1 & \text{if } \kappa < \kappa', \\ -1 & \text{if } \kappa \geq \kappa'. \end{cases}$$
Assume this condition is satisfied. There exist a fundamental discriminant $\Delta$ such that
\[ \tilde{\pi}' = \theta_{\psi,\Delta}(\pi \otimes \chi_{\Delta}). \]
Let $\psi_0$ be the standard additive character of $\mathbb{A}/\mathbb{Q}$ defined in [21]. Take $\psi$ to be $\psi_0^{\infty}$. Then $\tilde{\pi}_r^{\infty}$ is the holomorphic discrete series representation of weight $\kappa' + 1/2$. Let $\mathcal{K} = \mathbb{Q}(\sqrt{\Delta})$ and $\sigma_{\mathcal{K}}$ be the base change lift of $\sigma$ to $\text{GL}_2(\mathbb{A}_{\mathcal{K}})$. The explicit calculation of the pullback period [11] contains three parts:

- Four explicit theta lifts.
- Explicit central value formula for $L\left(\frac{1}{2}, \pi \otimes \sigma_{\mathcal{K}}\right)$.
- Generalized Kohnen-Zagier formula.

The second part is settled (cf. [3], [11], [CC], and [Hsi17]) except for the case $\kappa - \kappa' \geq 0$ is odd, in which case the explicit calculation of local trilinear period for $\pi_{\infty} \otimes \sigma_{\mathcal{K},\infty}$ remains unsolved. The third part is settled in [BM07] for $N_1$ is odd square-free. The main difficulties in the first part would be explicit calculation of local Shimura-Shintani-Waldspurger correspondence at $p \in S$ and $p = 2$ (cf. [5]). For this reason we assume $N_1$ is odd square-free, $N_2$ is odd cubic-free, $N_1 | N_2$, and $(N_1, N_2/N_1) = 1$ in Theorem [11]. In this case, $S$ is the empty set, and $\tilde{\pi}_r^{\infty}$ is a special representation for $p | N_1$. Note that the standard local Shintani lifts at $p \in S$ might lead to a trivial seesaw identity. The local Shimura-Shintani-Waldspurger correspondence at $p \in S$ that is suitable for our purpose are given in Propositions [5.4] and [5.7].

Note that the analogue results for Corollary [1.2] are proved in [CC] except when $N_1 = 1$ and $\kappa \geq \kappa'$. Therefore, we assume $\kappa \geq \kappa'$ in (i) of Hypothesis (H). Note that when $\kappa \geq \kappa'$, we have $\epsilon_{\infty} = -1$. Thus $\eta_{\kappa,\infty} = \eta_{\kappa,-}$ is a holomorphic discrete series representation (cf. [Sch05] §4 and [Can07] §6).

If one can explicitly realized the Kohnen’s plus space over finite extensions of $\mathbb{Q}_2$ as theta lifts, then the results could be generalized to Hilbert modular forms.

2. NOTATION AND CONVENTIONS

2.1. Notation. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$, $\mathbb{Q}$ be the finite part of $\mathbb{A}$, and $\mathbb{Z}$ be the closure of $\mathbb{Z}$ in $\mathbb{Q}$. If $R$ is a ring, we denote $R^\times$ to be the group of units of $R$ and $R^{\times,2} = \{a^2 \mid a \in R^\times\}$. If $S$ is a set, let $\mathbb{I}_S$ and $\mathbb{Z}_S$ be the characteristic function of $S$, respectively. The letter $\epsilon$ will be used to denote a place of $\mathbb{Q}$, unless otherwise specified. Let $| \ |_Q$ be the absolute value of $Q_v$ determined by $|p|_{Q_v} = p^{-1}$ and $|1|_R = 1$. Denote $(\cdot, Q_v)$ the Hilbert symbol of $Q_v$. If $v \in Q_v^\times$ (resp. $v \in Q_v^*$), let $\chi_v$ be the quadratic character of $\mathbb{A}^\times/\mathbb{Q}^\times$ (resp. $\mathbb{Q}_v^*$) defined by $\chi_v(x) = \prod_v(a)_{Q_v}$ (resp. $\chi_v(x) = (a, x)_{Q_v}$). Let $\psi_v$ be the standard additive character of $Q_v$, defined by
\[ \psi_v(x) = e^{2\pi i \chi(x)} \quad x \in \mathbb{Z}[p^{-1}], \]
\[ \psi_v(x) = e^{2\pi i \chi(x)} \quad x \in \mathbb{R}. \]

Then $\otimes_v \psi_v$ is a non-trivial additive character of $\mathbb{A}/\mathbb{Q}$. We call it the standard additive character of $\mathbb{A}/\mathbb{Q}$. If $\psi$ is a non-trivial additive character of $Q_v$ and $v \in Q_v^*$, we denote $\psi^v$ be the additive character of $Q_v$ defined by $\psi^v(a) = \psi(ax)$. Let $\gamma_{Q_v}(a, \psi)$ be the Weil index defined in [Rao93] Appendix, and put $\gamma_{Q_v}(a, \psi) = \gamma_{Q_v}^v(\psi^v)/\gamma_{Q_v}(\psi)$. The formulas for $\gamma_{Q_v}(\psi_v)$ and $\gamma_{Q_v}(a, \psi_v)$ are given in [Ich05] §A.1.1.

For $n \in \mathbb{N}$, let $\text{GSp}_n$ be the linear algebraic group over $\mathbb{Q}$ defined by
\[ \text{GSp}_n = \left\{ g \in \text{GL}_{2n} \mid g \begin{pmatrix} 0 & 1_n \\ -1_n & 0_n \end{pmatrix} g = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}, \nu(g) \in \mathbb{Z}_m \right\}. \]

The map $\nu : \text{GSp}_n \to \mathbb{Z}_m$ is called the scale map. The kernel of the scale map is denoted by $\text{Sp}_n$. Then $\text{GSp}_n$ (resp. $\text{Sp}_n$) is a symplectic similitude group (resp. symplectic group) of rank $n + 1$ (resp. $n$). For $c \in \mathbb{Z}_{\geq 0}$, let $K_0^{(n)}(p^c)$ be the compact subgroup of $\text{GSp}_n(\mathbb{Q}_p)$ defined by
\[ K_0^{(n)}(p^c) = \left\{ k \in \text{GSp}_n(\mathbb{Z}_p) \mid k \equiv \begin{pmatrix} * & * \\ 0_n & * \end{pmatrix} \pmod{p^c} \right\}. \]

Put $K_0^{(n)}(p^c) = \text{Sp}_n(\mathbb{Z}_p) \cap K_0^{(n)}(p^c)$. When $n = 1$, we denote $K_0^{(n)}(p^c) = K_0^{(1)}(p^c)$ and $K_0^{(n)}(p^c) = K_0^{(1)}(p^c)$. For $N = \prod_p p^{r_p} \in \mathbb{N}$, let $\text{GSp}_n(N\mathbb{Z}) = \prod_p K_0^{(n)}(p^{r_p})$. Let $\Gamma_0^{(n)}(N)$ be the congruence subgroup of $\text{Sp}_n(\mathbb{Z})$ defined by $\Gamma_0^{(n)}(N) = \text{Sp}_n(\mathbb{Z}) \cap \text{GSp}_n(N\mathbb{Z})$. When $n = 1$, we denote $\Gamma_0(N) = \Gamma_0^{(1)}(N)$.

For $n \in \mathbb{N}$, let $\mathcal{S}_n$ be the Siegel upper half space of degree $n$ defined by
\[ \mathcal{S}_n = \left\{ Z \in M_n(\mathbb{C}) \mid \text{Im}(Z) > 0 \right\}. \]
When \( n = 1 \), we write \( \mathcal{H} = \mathcal{H}_1 \).

Let \( B \) (resp. \( D \)) be the standard Borel subgroup (resp. maximal torus) of \( GL_2 \) consisting of upper triangular matrices (resp. diagonal matrices), and \( B = B \cap SL_2 \) be the standard Borel subgroup of \( SL_2 \). Let \( U \) be the unipotent radical of \( B \). We write

\[
\begin{align*}
   u(x) &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \\
   a(\nu) &= \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}, \\
   d(\nu) &= \begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix}, \\
   t(a) &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \\
   w &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{align*}
\]

Let

\[
\begin{align*}
   SO(2) &= \left\{ k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \middle| \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}, \\
   SU(2) &= \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.
\end{align*}
\]

Let \( G \) be a locally compact totally disconnected group. Denote \( \mathcal{H}(G) \) the Hecke algebra of \( G \) consisting of locally constant and compactly supported \( \mathbb{C} \)-valued functions on \( G \). If \( \pi \) is a representation of \( G \) on \( V \) and \( K \) is a subgroup of \( G \), denote \( \mathcal{V}^K \) the subspace of \( \mathcal{V} \) consisting of \( K \)-invariant vectors.

When we consider an induced representation, we always assume the representation is unitarily induced.

2.2. Measures. Let \( G \) be a connected linear algebraic group over \( \mathbb{Q} \). We take the Tamagawa measure on \( \mathbb{A}_f \). In particular, if \( G = SO(V) \) for some quadratic space \( V \) over \( \mathbb{Q} \), then

\[
\text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})) = 2.
\]

Define \( \xi_v(p)(s) = (1 - p^{-s})^{-1}, \xi_v(s) = \pi^{-s/2} \Gamma(s/2) \), and \( \xi_v(\mathbb{C})(s) = 2(2\pi)^{-s}\Gamma(s) \). Put \( \xi_v(s) = \prod_v \xi_v(s) \). Let \( dx_v \) be the Haar measure on \( \mathbb{Q}_v \) such that vol(\( \mathbb{Q}_p, dx_p \)) = 1 if \( p = p \), and \( dx_\infty \) is the Lebesgue measure if \( v = \infty \). Let \( d^\times a_v \) be the Haar measure on \( \mathbb{Q}_v^\times \) defined by

\[
d^\times a_v = \xi_v(1)|a_v|_\mathbb{Q}_v^{-1}da_v.
\]

Let \( dg_v \) be the Haar measure on \( SL_2(\mathbb{Q}_v) \) defined by

\[
dg_v = |a_v|_\mathbb{Q}_v^{-2}dx_x d^\times a_v dk_v
\]

for \( g_v = u(x_v)t(a_v)k_v \) with \( x_v \in \mathbb{Q}_v, a_v \in \mathbb{Q}_v^\times \), and

\[
k_v \in \begin{cases} 
   SL_2(\mathbb{Z}_p) & \text{if } v = p, \\
   SO(2) & \text{if } v = \infty.
\end{cases}
\]

Here \( dk_p \) (resp. \( dk_\infty \)) is the Haar measure on \( SL_2(\mathbb{Z}_p) \) (resp. \( SO(2) \)) so that \( \text{vol}(SL_2(\mathbb{Z}_p), dk_p) = 1 \) (resp. \( \text{vol}(SO(2), dk_\infty) = 1 \)). Then the Tamagawa measure on \( SL_2(\mathbb{A}) \) is given by

\[
dg = \xi(2)^{-1} \prod_v dg_v.
\]

Let \( dg_p \) be the Haar measure on \( GL_2(\mathbb{Q}_p) \) defined by

\[
dg_p = |t_v|_\mathbb{Q}_p^{-1}d^\times z_p dx_p d^\times t_p dk_p
\]

for \( g_p = z_p u(x_p)t_p k_p \) with \( z_p, t_p \in \mathbb{Q}_p^\times, x_p \in \mathbb{Q}_p, \) and \( k_p \in GL_2(\mathbb{Z}_p) \). Here \( dk_p \) is the Haar measure on \( GL_2(\mathbb{Z}_p) \) so that \( \text{vol}(GL_2(\mathbb{Z}_p), dk_p) = 1 \). Let \( dg_\infty \) be the Haar measure on \( GL_2(\mathbb{R}) \) defined by

\[
dg_\infty = z_\infty^{-1} |t_\infty|_\mathbb{R}^{-1}d^\times z_\infty dx_\infty d^\times t_\infty dk_\infty
\]

for \( g_\infty = \xi_\infty(u(x_\infty)t_\infty)k_\infty \) with \( z_\infty \in \mathbb{R}_+^\times, t_\infty \in \mathbb{R}_+^\times, x_\in \mathbb{R}, \) and \( k_\infty \in SO(2) \). Here \( dk_\infty \) is the Haar measure on \( SO(2) \) so that \( \text{vol}(SO(2), dk_\infty) = 1 \). By abuse of notation, we write \( dg_\infty \) for the Haar measure on \( \mathbb{A}_f^\times \backslash GL_2(\mathbb{A}) \) defined by the quotient of \( dg_\infty \) by \( d^\times z_\infty \). Then the Tamagawa measure on \( \mathbb{A}_f^\times \backslash GL_2(\mathbb{A}) \) is given by

\[
dg = 2\xi(2)^{-1} \prod_v dg_v.
\]

For \( z \in \mathbb{C} \), let \( |z| = \sqrt{z\overline{z}} \). Let \( dz \) be the Haar measure on \( \mathbb{C} \) defined to be twice the Lebesgue measure on \( \mathbb{C} \). Let \( d^\times z = |z|^{-2}dz \) be the Haar measure on \( \mathbb{C}^\times \). Let \( d_{\infty} \) be the Haar measure on \( SL_2(\mathbb{C}) \) defined by

\[
d_{\infty} = r^{-4} dx_\infty d^\times r_\infty d\theta dk_\infty
\]

for \( r_\in \mathbb{R}_+^\times, \theta \in \mathbb{R} \).
for \( g_\infty = u(x_\infty) t(r_\infty e^{-\theta}) k_\infty \) with \( x_\infty \in \mathbb{C} \), \( r_\infty \in \mathbb{R}_+^* \), \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \), and \( k_\infty \in \text{SU}(2) \). Here \( \text{vol}(\mathbb{R}/2\pi \mathbb{Z}, d\theta) = \text{vol}(	ext{SU}(2), dk_\infty) = 1 \).

2.3. Special functions. For \( p, q \in \mathbb{Z}_0 \), let \( {}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z) \) be the generalized hypergeometric function defined by

\[
{}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z) = \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_q)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_p)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + n) \Gamma(\alpha_p + n)}{n! \Gamma(\beta_1 + n) \Gamma(\beta_q + n)} z^n
\]

whenever the series is converge. When \( p = q + 1 \), the series converges for \( |z| < 1 \).

For \( n \in \mathbb{Z}_0 \), let \( H_n(x) \) be the Hermite polynomial defined by

\[
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}).
\]

Let \( K_\nu(z) \) be the modified Bessel function defined by

\[
K_\nu(z) = \frac{1}{2} \int_0^\infty e^{-z(t+1)} / (t^{\nu+1}) dt
\]

if \( \text{Re}(z) > 0 \).

3. Whittaker functions

3.1. The metaplectic groups. Let \( v \) be a place of \( \mathbb{Q} \). Let \( \widetilde{\text{SL}_2(\mathbb{Q}_v)} \) be the 2-fold metaplectic cover of \( \text{SL}_2(\mathbb{Q}_v) \). Recall that \( \text{SL}_2(\mathbb{Q}_v) \) is the unique non-trivial 2-fold central extension of \( \text{SL}_2(\mathbb{Q}_v) \). Let \( c_v : \text{SL}_2(\mathbb{Q}_v) \times \text{SL}_2(\mathbb{Q}_v) \to \{\pm 1\} \) be Kubota’s 2-cocycle defined by

\[
c_v(g_1, g_2) = (x(g_1 g_2) x(g_1)^{-1}, x(g_1 g_2) x(g_2)^{-1})_{\mathbb{Q}_v},
\]

here

\[
x \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{otherwise.} \end{cases}
\]

We identify \( \widetilde{\text{SL}_2(\mathbb{Q}_v)} \) with \( \text{SL}_2(\mathbb{Q}_v) \times \{\pm 1\} \) as sets, and the multiplication is given by

\[
(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \cdot c_v(g_1, g_2)).
\]

By abuse of notation, we write \( g \) for the element \((g, 1) \in \widetilde{\text{SL}_2(\mathbb{Q}_v)} \). If \( H \) is a subgroup of \( \text{SL}_2(\mathbb{Q}_v) \), denote \( \hat{H} \) the inverse image of \( H \) in \( \text{SL}_2(\mathbb{Q}_v) \). Note that we have a homomorphism

\[
U(\mathbb{Q}_v) \to \widetilde{\text{SL}_2(\mathbb{Q}_v)} \quad u \mapsto (u, 1).
\]

For \( v = p \), define \( s_p : \text{SL}_2(\mathbb{Q}_p) \to \{\pm 1\} \) by

\[
s_p \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} (c, d)_{\mathbb{Q}_p} & \text{if } cd \neq 0, \text{ord}_{\mathbb{Q}_p}(c) \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}
\]

For \( v = 2 \), define \( \epsilon_2 : \text{SL}_2(\mathbb{Q}_2) \to \{\pm 1\} \) by

\[
\epsilon_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \gamma_{\mathbb{Q}_2}(d, \psi_2)(c, d)_{\mathbb{Q}_2} & \text{if } c \neq 0, \\ \gamma_{\mathbb{Q}_2}(d, \psi_2)^{-1} & \text{otherwise.} \end{cases}
\]

Here \( \psi_2 \) is the standard additive character of \( \mathbb{Q}_2 \). For \( v = p \neq 2 \), we have a homomorphism

\[
\text{SL}_2(\mathbb{Z}_p) \to \widetilde{\text{SL}_2(\mathbb{Q}_p)} \quad k \mapsto (k, s_p(k)).
\]
For \( v = 2 \), we have a homomorphism
\[
K_1(4) \to \text{SL}_2(\mathbb{Q}_2)
\]
\[
k \mapsto (k, s_2(k)),
\]
here \( K_1(4) \) is the compact subgroup of \( \text{SL}_2(\mathbb{Q}_2) \) defined by
\[
K_1(4) = \left\{ k \in \text{SL}_2(\mathbb{Z}_p) \mid k \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{4} \right\}.
\]
For \( v = \infty \), we have an isomorphism
\[
\mathbb{R}/4\pi \mathbb{Z} \to \text{SO}(2)
\]
\[
\theta \mapsto \tilde{k}_\theta,
\]
where
\[
\tilde{k}_\theta = \begin{cases} (k_\theta, 1) & \text{if } -\pi < \theta \leq \pi \\ (k_\theta, -1) & \text{if } \pi < \theta \leq 3\pi. \end{cases}
\]
Let \( \prod_v \text{SL}_2(\mathbb{Q}_v) \) be the restricted direct product with respect to \( \text{SL}_2(\mathbb{Z}_p) \) for \( p \neq 2 \). The adelic metaplectic group \( \text{SL}_2(\mathbb{A}) \) is defined by
\[
\text{SL}_2(\mathbb{A}) = \prod_v \text{SL}_2(\mathbb{Q}_v) / \left\{ (1, \epsilon_v) \in \bigoplus_v \{ \pm 1 \} \mid \prod_v \epsilon_v = 1 \right\}.
\]
By abuse of notation, we write \( \prod_v (g_v, \epsilon_v) \) for its image in \( \text{SL}_2(\mathbb{A}) \) under the natural quotient map. We identify \( \text{SL}_2(\mathbb{Q}_v) \) with subgroup of \( \text{SL}_2(\mathbb{A}) \) via the natural quotient map. We also identify \( \text{SL}_2(\mathbb{Q}) \) with subgroup of \( \text{SL}_2(\mathbb{A}) \) via the homomorphism
\[
\text{SL}_2(\mathbb{Q}) \to \text{SL}_2(\mathbb{A})
\]
\[
\gamma \mapsto \prod_v (\gamma, 1).
\]
Note that we have an exact sequence
\[
1 \to \{ \pm 1 \} \to \text{SL}_2(\mathbb{A}) \to \text{SL}_2(\mathbb{A}) \to 1.
\]
Therefore, \( \text{SL}_2(\mathbb{A}) \) is a 2-fold central extension of \( \text{SL}_2(\mathbb{A}) \).

3.2. **Whittaker functions on \( \text{SL}_2 \).**

3.2.1. **Whittaker functions for induced representations.** In this subsection, we calculate some Whittaker functions associated to genuine representations of \( \text{SL}_2(\mathbb{Q}_p) \) induced from unramified characters.

Let \( \chi \) be a character of \( \mathbb{Q}_p^\times \) such that \(|\chi(p)| = p^{-\text{wt}(\chi)} \) with \( \text{wt}(\chi) \geq 0 \). Let \( \psi \) be a nontrivial additive character of \( \mathbb{Q}_p \). Let
\[
\tilde{\rho} = \tilde{\rho}_{\chi, \psi} = \text{Ind}_{B(\mathbb{Q}_p)}^{\text{SL}_2(\mathbb{Q}_p)}(\chi)^\psi
\]
be the induced representation of \( \text{SL}_2(\mathbb{Q}_p) \). Recall that \( \tilde{\rho} \) acts via right translation on \( \mathcal{V} = \mathcal{V}_{\chi, \psi} \), the space consisting of locally constant functions \( f : \text{SL}_2(\mathbb{Q}_p) \to \mathbb{C} \) such that
\[
f((u(x)t(a), \epsilon)g) = \epsilon \gamma_p(a, \psi)\chi(a)|a|_{\mathbb{Q}_p} f(g)
\]
for \( x \in \mathbb{Q}_p, a \in \mathbb{Q}_p^\times, \epsilon \in \{ \pm 1 \}, \) and \( g \in \text{SL}_2(\mathbb{Q}_p) \).

**Remark 3.** In the notation of [Los05 A.3] (resp. [Wal80 II.2]), we have \( \tilde{\rho}_{\chi, \psi} = \text{Ind}_{B(\mathbb{Q}_p)}^{\text{SL}_2(\mathbb{Q}_p)}((\chi_{-1, p})^\psi \chi) \) (resp. \( \tilde{\rho}_{\chi, \psi} = \tilde{\rho}_{\chi, \psi^{-1}} \)).
Let \( \xi \in \Q_p^\times \). For \( f \in \mathcal{V} \), let \( W_{f, \xi} \) be the Whittaker function on \( \SL_2(\Q_p) \) with respect to \( \psi^\xi \) defined by

\[
W_{f, \xi}(g) = \lim_{k \to \infty} \int_{\mathbb{P}^{1}(\Q_p)} f(w^{-1} u(x) g) \psi(-\xi x) dx
\]

for \( g \in \SL_2(\Q_p) \). Since \( \text{wt}(\chi) \geq 0 \), the integral defining \( W_{f, \xi} \) converge for all \( g \in \SL_2(\Q_p) \). Put

\[
W(\tilde{\rho}, \psi^\xi) = \{ W_{f, \xi} \mid f \in \mathcal{V} \}
\]

be the space of Whittaker functions of \( \tilde{\rho} \) with respect to \( \psi^\xi \). Let \( \rho \) be the representation of \( \SL_2(\Q_p) \) on \( \mathcal{W}(\tilde{\rho}, \psi^\xi) \) defined by right translation. We recall the results of Waldspurger in [Wal80] Propositions 1-3.

**Proposition 3.1.** Let \( \xi \in \Q_p^\times \).

(i) \( \tilde{\rho} \) is irreducible if and only if \( \chi^2 \not| \Q_p \), and \( \tilde{\rho} \) has a unique irreducible sub-representation \( \tilde{\sigma} = \tilde{\sigma}_\chi, \psi \) if \( \chi^2 \not| \Q_p \).

(ii) If \( \chi^2 \not| \Q_p \), then the map \( \mathcal{V} \to \mathcal{W}(\tilde{\rho}, \psi^\xi) \), \( f \mapsto W_{f, \xi} \) is an \( \SL_2(\Q_p) \)-equivariant isomorphism.

(iii) If \( \chi = \chi_\tau \mid_{\Q_p}^{1/2} \) for some \( \tau \in \Q_p^\times \), then the map the map \( \mathcal{V} \to \mathcal{W}(\tilde{\rho}, \psi^\xi) \), \( f \mapsto W_{f, \xi} \) has a non-trivial kernel if and only if \( \xi \in -\tau \Q_p^\times, 2 \). In the case \( \xi \in -\tau \Q_p^\times, 2 \), the kernel is equal to the unique irreducible invariant subspace of \( \mathcal{V} \).

(iv) The Whittaker model for the irreducible sub-representation of \( \tilde{\rho} \) with respect to \( \psi^\xi \), if exist, is unique.

If \( \chi = \chi_\tau \mid_{\Q_p}^{1/2} \), an additive character \( \psi^\xi \) is called a missing character of \( \tilde{\sigma}_\chi, \psi \) if \( \xi \in -\tau \Q_p^\times, 2 \).

From now on, until the end of [3,21], we assume \( p \not= 2 \), \( \psi = \psi_p \) is the standard additive character of \( \Q_p \), and \( \chi \) is an unramified character with \( \alpha = \chi(p) \). The subspace \( \mathcal{V}_{\SL_2(\Q_p)} \) consisting of \( \SL_2(\Q_p) \)-invariant vectors in \( \mathcal{V} \) is one dimensional. Let \( f_0 \in \mathcal{V}_{\SL_2(\Q_p)} \) be normalized so that \( f_0(1) = 1 \). For each \( c \geq 1 \), we have the double coset decomposition

\[
\overline{B(\Q_p) \SL_2(\Q_p)} / K_0(p^c) = \mathbb{T} \sqcup \mathbb{T} \sqcup \bigcup_{1 \leq i \leq c - 1, u \in \Z_p^\times / \Z_p^\times, 2} \begin{pmatrix} 1 & 0 \\ p^i u & 1 \end{pmatrix}.
\]

By [Wal81] Proposition 9, the subspace \( \mathcal{V}_{K_0(p^c)} \) consisting of \( K_0(p^c) \)-invariant vectors in \( \mathcal{V} \) is \( 2c \)-dimensional with a basis

\[
\{ F[c, 1], F[c, p^c], F[c, p^c u] \mid 1 \leq i \leq c - 1, u \in \Z_p^\times / \Z_p^\times, 2 \}.
\]

Here

\[
F[c, 1](1) = 0, \quad F[c, 1](w) = 1, \quad F[c, 1]\begin{pmatrix} 1 & 0 \\ p^i u & 1 \end{pmatrix} = 0, 
\]

\[
F[c, p^c](1) = 1, \quad F[c, p^c](w) = 0, \quad F[c, p^c]\begin{pmatrix} 1 & 0 \\ p^i u & 1 \end{pmatrix} = 0, 
\]

for \( 1 \leq i \leq c - 1, u \in \Z_p^\times / \Z_p^\times, 2 \), and

\[
F[c, p^i u](1) = F[c, p^i u](w) = F[c, p^i u]\begin{pmatrix} 1 & 0 \\ p^i u & 1 \end{pmatrix} = 0, \quad F[c, p^i u]\begin{pmatrix} 1 & 0 \\ p^i u & 1 \end{pmatrix} = 1,
\]

for \( 1 \leq i, j \leq c - 1, u, u' \in \Z_p^\times / \Z_p^\times, 2 \) with \( i \not= j \) and \( u \not= u' \). Note that

\[
f_0 = F[c, 1] + F[c, p^c] + \sum_{1 \leq i \leq c - 1, u \in \Z_p^\times / \Z_p^\times, 2} F[c, p^i u].
\]

In the rest of this subsection, let \( \chi = \chi_\tau \mid_{\Q_p}^{1/2} \) for some \( \tau \in \Z_p^\times \). In this case, \( \alpha = (p, \tau) \Q_p p^{-1/2} \). Let \( \tilde{\sigma} \) be the unique irreducible sub-representation of \( \tilde{\rho} \). Let \( f \in \mathcal{V}_{K_0(p^c)} \) defined by

\[
f = F[1, 1] - pF[1, p].
\]

Let \( \xi \in \Q_p^\times \) with \( m = \text{ord}_{\Q_p}(\xi) \). Put

\[
e_p(\xi; \alpha) = \begin{cases} 2^{-1} \alpha^{-m/2} & \text{if } m \text{ is even,} \\ (1 + p)^{-1} \alpha^{-(m+3)/2} & \text{if } m \text{ is odd.}
\end{cases}
\]
Define $W_{ξ,p} ∈ W(\tilde{ρ}, ψ^ξ)$ by

\[ W_{ξ,p} = c_p(ξ; α)W_{f,ξ} \]

Define $Ψ_p(ξ; X) ∈ \mathbb{C}[X]$ by

\[ Ψ_p(ξ; X) = \begin{cases} X^{m/2} & \text{if } m ≥ 0 \text{ is even}, \text{ and } ξ ≠ -τQ_p^{x,2} \\ X^{(m-1)/2} & \text{if } m ≥ 0 \text{ is odd}, \text{ and } ξ \neq -τQ_p^{x,2} \\ 0 & \text{otherwise}. \end{cases} \]

**Lemma 3.2.** Let $ξ ∈ Q_p^x$ with $m = \text{ord}_{Q_p}(ξ)$. We have

\[ W_{ξ,p}(1) = Ψ_p(ξ; α). \]

**Lemma 3.3.** Let $ξ ∈ Q_p^x$ with $m = \text{ord}_{Q_p}(ξ)$. If $m ≥ 0$, then

\[ W_{ξ,p}(w) = -p^{-1}Ψ_p(ξ; α). \]

If $m = -1$, then

\[ W_{ξ,p}(w) = -p^{-1}α^{-1}. \]

If $m ≤ -2$, then

\[ W_{ξ,p}(w) = 0. \]

Note that $W_{f_o,ξ}(1) ≠ 0$ for all $ξ ∈ Z_p^x$. Therefore, by Lemmas 3.2 and 3.3, the subspace consisting of $K_0(p)$-fixed elements in $\tilde{σ}$ is one dimensional and containing $f$. In particular, $W_{ξ,p} ∈ W(\tilde{σ}, ψ^ξ)$ for $ξ ≠ -τQ_p^{x,2}$.

**3.2.2. The space of $K_0(p^2)$-fixed vectors.** We keep the notation of 3.2.1. In this subsection, we give a specific basis for $V_{X,ψ}^{K_0(p^2)}$ and calculate the Whittaker functions corresponding to this basis.

Let $\tilde{π}$ be an admissible representation of $\text{SL}_2(Q_p)$ on $V_\tilde{π}$. For $f ∈ V_{X,ψ}^{K_0(p^2)}$, define $\tilde{U}(p)(f)$ by

\[ \tilde{U}(p)(f) = \sum_{x ∈ \mathbb{Z}/p^2\mathbb{Z}} \tilde{π}(u(x)t(p))f. \]

Put

\[ f_0 = F[2,1] + F[2,p^2] + \sum_{u ∈ Z_p^x/Z_p^{x,2}} F[2, pu], \]

\[ f_1 = γ_{Q_p}(p, ψ)^{-1} \tilde{U}(p)(f_0), \]

\[ f_2 = γ_{Q_p}(p, ψ)^{-1} ρ(t(p^{-1}))f_0, \]

\[ f_3 = -(1 - p^{-1})F[2,1] + \sum_{u ∈ Z_p^x/Z_p^{x,2}} α^{-1}(p^{1/2} + (p, u)_{Q_p}(p, ψ))F[2, pu]. \]

Let $ξ ∈ Q_p^x$ with $m = \text{ord}_{Q_p}(ξ)$. Put

\[ c_p(ξ; α) = \begin{cases} α^{-m/2}(1 + p^{-1/2}(p, -ξ)_{Q_p}α)^{-1} & \text{if } m \text{ is even}, \\ α^{-(m-1)/2}(1 - p^{-1}α^2)^{-1} & \text{if } m \text{ is odd}. \end{cases} \]

Note that $\{f_0, f_1, f_2, f_3\}$ is a basis for $V_{X,ψ}^{K_0(p^2)}$. For $1 ≤ i ≤ 4$, define $W_{ξ,p}^{(i)} ∈ W(\tilde{ρ}, ψ^ξ)$ by

\[ W_{ξ,p}^{(i)} = c_p(ξ; α)W_{f_i,ξ}. \]

We also put $W_{ξ,p} = W_{ξ,p}^{(0)}$. Define $Ψ_p(ξ; X) ∈ \mathbb{C}[X + X^{-1}]$ by

\[ Ψ_p(ξ; X) = \begin{cases} \frac{X^{m/2+1} - X^{-m/2-1}}{X - X^{-1}} - p^{-1/2}(p, -ξ)_{Q_p} X^{m/2} - X^{-m/2} & \text{if } m ≥ 0 \text{ is even}, \\ \frac{X^{(m-1)/2+1} - X^{-(m-1)/2-1}}{X - X^{-1}} & \text{if } m ≥ 0 \text{ is odd}, \\ 0 & \text{if } m < 0. \end{cases} \]

Note that if $m ≥ 0$, then

\[ c_p(ξ; α)(α^{-1} - p^{-1}α) + αΨ_p(ξ; α) = Ψ_p(p^2ξ; α). \]
Lemma 3.4. Let $\xi \in \mathbb{Q}_p^\times$ with $m = \text{ord}_{\mathbb{Q}_p}(\xi)$.

(i) $W^{(0)}_{\xi,p}(1) = \Psi_p(\xi; \alpha)$.

(ii) If $m \geq 0$, then $W^{(1)}_{\xi,p}(1) = p\Psi_p(p^2\xi; \alpha)$.

If $m < 0$, then $W^{(1)}_{\xi,p}(1) = 0$.

(iii) $W^{(2)}_{\xi,p}(1) = p\Psi_p(p^{-2}\xi; \alpha)$.

(iv) If $m = 0$, then $W^{(3)}_{\xi,p}(1) = ((p,-\xi)_{\mathbb{Q}_p} - 1)(1 - p^{-1/2}\alpha)^{-1}$.

If $m \neq 0$, then $W^{(3)}_{\xi,p}(1) = 0$.

Lemma 3.5. Let $\xi \in \mathbb{Q}_p^\times$ with $m = \text{ord}_{\mathbb{Q}_p}(\xi)$.

(i) $W^{(0)}_{\xi,p}(w) = \Psi_p(\xi; \alpha)$.

(ii) If $m \geq -1$, then $W^{(1)}_{\xi,p}(w) = p(\alpha + \alpha^{-1})\Psi_p(\xi; \alpha) - \Psi_p(p^2\xi; \alpha)$.

If $m < -1$, then $W^{(1)}_{\xi,p}(w) = 0$.

(iii) $W^{(2)}_{\xi,p}(w) = p^{-1}\Psi_p(p^2\xi; \alpha)$.

(iv) If $m \geq -1$, then $W^{(3)}_{\xi,p}(w) = -(1 - p^{-1})\Psi_p(\xi; \alpha) + c_p(\xi; \alpha)(1 - p^{-1})(1 - p^{-1/2}\alpha)^{-1}$.

If $m = -2$, then $W^{(3)}_{\xi,p}(w) = p^{-3/2}\alpha^{-1}((p,-\xi)_{\mathbb{Q}_p} - 1)$.

If $m < -2$, then $W^{(3)}_{\xi,p}(w) = 0$.

Lemma 3.6. Let $\xi \in \mathbb{Q}_p^\times$ with $m = \text{ord}_{\mathbb{Q}_p}(\xi)$.

(i) \[
\sum_{u \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times} W^{(2)}_{\xi,p} \begin{pmatrix} 1 & 0 \\ pu & 1 \end{pmatrix} = p\gamma_{\mathbb{Q}_p}(p, \psi)^{-1} \mathfrak{G}_{p^{-1}\xi}(p)\Psi_p(\xi; \alpha). \]

(ii) If $m \geq 0$, then
\[
\sum_{u \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times} W^{(3)}_{\xi,p} \begin{pmatrix} 1 & 0 \\ pu & 1 \end{pmatrix} = p^{1/2}(p-1)\alpha^{-1}\Psi_p(\xi; \alpha) + c_p(\xi; \alpha) \left[ -(p-1)(1 + p^{1/2}\alpha^{-1}) + p\mathfrak{G}_{p^{-1}\xi}(p)\mathfrak{G}_{p^{-1}\xi}(p) - p^{1/2}\gamma_{\mathbb{Q}_p}(p, \psi)^{-1}\mathfrak{G}_{p^{-1}\xi}(p) \right].
\]

If $m < 0$, then
\[
\sum_{u \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times} W^{(3)}_{\xi,p} \begin{pmatrix} 1 & 0 \\ pu & 1 \end{pmatrix} = 0.
\]

Here \[
\mathfrak{G}_\eta(p) = \int_{\mathbb{Z}_p^\times} (p,x)_{\mathbb{Q}_p} \psi(-\eta x)dx.
\]
3.2.3. The 2-adic case. In this section, we recall the representation-theoretic interpretation of Kohnen’s plus space, and the formulas for the corresponding Whittaker functions following [Ich05 §3.2 and §A.5].

Let \( \tilde{\pi} \) be an admissible representation of \( \text{SL}_2(\mathbb{Q}_2) \) on \( \mathcal{V}_{\tilde{\pi}}\). For \( f \in \mathcal{V}_{\tilde{\pi}} \), define \( U(f), W(f) \in \mathcal{V}_{\tilde{\pi}} \) by
\[
U(f) = \int_{\mathbb{Z}_2^2} \tilde{\pi}(u(x)t(2))f dx,
\]
\[
W(f) = \tilde{\pi}(w^{-1}t(2))f.
\]
Put
\[
\mathcal{V}^\kappa_{\tilde{\pi}}(\epsilon_2^{-1}) = \{ f \in \mathcal{V}_{\tilde{\pi}} \mid \tilde{\pi}(k)f = \epsilon_2(k)^{-1}f \text{ for } k \in K_0(4) \}.
\]
By [Ich05 Lemmas 3.4 and 3.5], the space \( \mathcal{V}^\kappa_{\tilde{\pi}}(\epsilon_2^{-1}) \) is invariant under \( U \) and \( W \). Let \( \mathcal{V}^+ \) be the subspace of \( \mathcal{V}^\kappa_{\tilde{\pi}}(\epsilon_2^{-1}) \) defined by
\[
\mathcal{V}^+ = \{ f \in \mathcal{V}^\kappa_{\tilde{\pi}}(\epsilon_2^{-1}) \mid W(U(f)) = 2^{-1/2}\epsilon_8^{-1}f \}.
\]
By [Ich05 Lemma 3.6], the space \( \mathcal{V}^+ \) is one-dimensional. Take \( \mathcal{V} \) to be the Whittaker model \( \mathcal{W}(\tilde{\rho}, \psi^\xi) \).

Let \( \psi = \psi_2 \) be the standard additive character of \( \mathbb{Q}_2 \). Let \( \chi \) be an unramified character of \( \mathbb{Q}_2^\times \) with \( \alpha = \chi(2) \). We assume \( \alpha \equiv 1 \) and \( \alpha^2 \neq 2^{-1} \). Let \( \tilde{\rho} = \tilde{\rho}_\chi,\psi \) be the irreducible induced representation of \( \text{SL}_2(\mathbb{Q}_2) \) defined in §3.2.1. For \( \xi \in \mathbb{Q}_2^\times \), we denote \( W_{\xi,2} \in \mathcal{W}(\tilde{\rho}, \psi^\xi)^+ \) be the Whittaker function normalized so that
\[
W_{\xi,2}(1) = 1.
\]
Define \( \Psi_2(\xi; X) \in \mathbb{C}[X + X^{-1}] \) by
\[
\Psi_2(\xi; X) = \begin{cases} 
\frac{X^{m/2+1} - X^{-m/2-1}}{X^{m/2} - X^{-m/2}} & \text{if } m \geq 0 \text{ is even, and } u \equiv -1 \text{ mod } 4, \\
\frac{X^{m/2} - X^{-m/2}}{X^{m/2} - X^{-m/2}} & \text{if } m \geq 0 \text{ is even, and } u \equiv 1 \text{ mod } 4, \\
\frac{X^{(m-1)/2} - X^{-(m-1)/2}}{X^{(m-1)/2} - X^{-(m-1)/2}} & \text{if } m \geq 0 \text{ is odd,} \\
0 & \text{if } m < 0.
\end{cases}
\]
Here \( \xi = 2^m u \) with \( u \in \mathbb{Z}_2^\times \). By [Ich05 Lemma A.8],
\[
W_{\xi,2}(1) = \Psi_2(\xi; \alpha).
\]

3.2.4. The archimedean case. In this section, we recall the definition for holomorphic discrete series representation of \( \text{SL}_2(\mathbb{R}) \), and the formulas for some Whittaker functions.

Let \( \kappa \) be a positive integer. Let \( \psi = \psi_\infty \) be the standard additive character of \( \mathbb{R} \). Let
\[
\tilde{\rho} = \tilde{\rho}(\kappa) = \text{Ind}_{\text{SL}_2(\mathbb{R})}^{\text{SL}_2(\mathbb{R})} (\text{sgn}^{\kappa-1}(|_\mathbb{R})^{1/2}) \psi
\]
be the induced representation of \( \text{SL}_2(\mathbb{R}) \). Recall that \( \tilde{\rho} \) acts via right translation on \( \mathcal{V} = \mathcal{V}(\kappa) \), the space consisting of smooth and \( \text{SO}(2) \)-finite functions \( f : \text{SL}_2(\mathbb{R}) \to \mathbb{C} \) such that
\[
f((u(x)t(\alpha), e)g) = e_{\gamma}(\alpha, \psi)\text{sgn}(\alpha)^{\kappa-1} |_{\mathbb{R}}^{\kappa+1/2} f(g)
\]
for \( x \in \mathbb{R}, \alpha \in \mathbb{R}^\times, e \in \{ \pm 1 \}, \) and \( g \in \text{SL}_2(\mathbb{R}) \). By [Wal80 Proposition 6], \( \tilde{\rho} \) has a unique irreducible sub-representation \( \tilde{\rho}^+ \) with minimal \( \text{SO}(2) \)-type \( \kappa + 1/2 \). We say \( \tilde{\rho}^+ \) the holomorphic discrete series of weight \( \kappa + 1/2 \). For \( \xi \in \mathbb{R}^\times \), let \( \mathcal{W}(\tilde{\rho}, \psi^\xi) \) be the space of Whittaker functions of \( \tilde{\rho} \) with respect to \( \psi^\xi \) defined as in §3.2.1. By [Wal80 Proposition 7], the map \( \mathcal{V} \to \mathcal{W}(\tilde{\rho}, \psi^\xi), f \to W_{f,\xi} \) has a non-trivial kernel if and only if \( \xi < 0 \), and \( \tilde{\rho}^+ \) is equal to the kernel if \( \xi < 0 \). If \( \xi > 0 \), we denote \( \mathcal{W}(\tilde{\rho}^+, \psi^\xi) \) the image of \( \tilde{\rho}^+ \). If \( \xi < 0 \), we say \( \psi^\xi \) is a missing character of \( \tilde{\rho}^+ \). For \( \xi > 0 \), let \( W_{\xi,\infty} \) be the Whittaker function of \( \text{SO}(2) \)-type \( \kappa + 1/2 \) in \( \mathcal{W}(\tilde{\rho}^+, \psi^\xi) \) such that \( W_{\xi,\infty}(1) = e^{-2\pi\xi} \). Then
\[
W_{\xi,\infty}(t(\alpha)k_\theta) = a^{\kappa+1/2} e^{-2\pi\xi a^2 e^{\sqrt{-1}(\kappa+1/2)\theta}}
\]
for \( x \in \mathbb{R}, \ a \in \mathbb{R}_+^\times, \) and \( \theta \in \mathbb{R}/4\pi\mathbb{Z} \). The formula is given in the proof of proposition 7 in [Wa80]. Let \( \mathfrak{g}_C \) be the complexified Lie algebra of \( \text{SL}_2(\mathbb{R}) \). Let

\[
(3.10) \quad \tilde{V}_+ = -\frac{1}{8\pi} \mathcal{V}_+ \in \mathfrak{g}_C
\]

be the normalized weight raising element, where

\[
\mathcal{V}_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sqrt{-1}.
\]

Let \( m \in \mathbb{Z}_{\geq 0} \). Then \( \tilde{V}_+^m W_{\xi,\infty} \) is a Whittaker function of \( SO(2) \)-type \( \kappa + 2m + 1/2 \) in \( \mathcal{W}(\tilde{\rho}^+, \psi^\xi) \). A simple induction shows that (cf. [CC, Lemma 3.1])

\[
(3.11) \quad \tilde{V}_+^m W_{\xi,\infty}(t(a)) = a^{\kappa+1/2} e^{-2\pi \xi a^2} \sum_{j=0}^{m} (-4\pi)^j -m a^2 \frac{\Gamma(\kappa + 1/2 + m)}{\Gamma(\kappa + 1/2 + j)} \binom{m}{j}
\]

for \( a \in \mathbb{R}_+^\times \).

3.3. Whittaker functions on \( \text{GL}_2 \).

3.3.1. Whittaker functions on \( \text{PGL}_2(\mathbb{Q}_p) \). Let \( \psi = \psi_p \) be the standard additive character of \( \mathbb{Q}_p \). Let \( \chi \) be a character of \( \mathbb{Q}_p^\times \). Denote \( a(\chi) \in \mathbb{Z}_{\geq 0} \) the conductor of \( \chi \). Let \( L(s, \chi) \) and \( \epsilon(s, \chi, \psi) \) be the \( L \)-factor and \( \epsilon \)-factor of \( \chi \), respectively. Then

\[
\epsilon(s, \chi, \psi) = \begin{cases} p^{a(\chi)(1-s)} \int_{\mathbb{Z}_p} \chi^{-1}(p^{-a(\chi)}t) \psi(p^{-a(\chi)}t) dt & \text{if } a(\chi) > 0, \\ 1 & \text{otherwise}. \end{cases}
\]

Let \( \pi \) be an irreducible admissible representation of \( \text{GL}_2(\mathbb{Q}_p) \) with trivial central character on \( \mathcal{V} \). Then \( \pi \) is one of the following representations:

(i) Principal series representation:

\[
\pi = \text{Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}(\chi \boxtimes \chi^{-1})
\]

for some character \( \chi \) of \( \mathbb{Q}_p^\times \) with \( \chi^2 \neq 1 \).

(ii) Twisted Steinberg representation (special representation):

\[
\pi = \text{St}_{\mathbb{Q}_p} \otimes \chi
\]

for some quadratic character \( \chi \) of \( \mathbb{Q}_p^\times \).

(iii) Supercuspidal representation.

As in [JL70], let \( L(s, \pi) \) and \( \epsilon(s, \pi, \psi) \) be the \( L \)-factor and \( \epsilon \)-factor of \( \pi \), respectively. By [Cas73], there exist a unique \( c(\pi) \in \mathbb{Z}_{\geq 0} \) such that \( \dim_{\mathbb{C}} \mathcal{V}_{\pi}^{K_0(\rho(\pi))} = 1 \) and \( \epsilon(s, \pi, \psi) p^{c(\pi) s} \in \mathbb{C}^\times \). We call \( p^{c(\pi) \mathbb{Z}_p} \) the conductor of \( \pi \). By [JL70], the space of Whittaker functional has dimension equal to one. Let \( \mathcal{W}(\pi, \psi) \) be the Whittaker model of \( \pi \) with respect to \( \psi \), and \( \rho \) be the right translation acting on it. Let \( \mathcal{W}_\pi \in \mathcal{W}(\pi, \psi) K_0(\rho(p^{c(\pi)})) \) be normalized so that \( W_\pi(1) = 1 \). We call \( W_\pi \) the Whittaker new form of \( \pi \). For \( i \in \mathbb{Z}_{\geq 0} \), let

\[
W^{(i)} = \rho \left( \begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix} \right) W_\pi.
\]

By the Iwasawa decomposition, the Whittaker function \( W_\pi \) is determined by \( W_\pi(a(t)), \rho(t)W_\pi(a(t)) \), and \( W^{(i)}_\pi(a(t)) \) for \( t \in \mathbb{Q}_p^\times, \ i \in \mathbb{Z}_{\geq 0} \).

We can solve the Whittaker function \( W_\pi \) for arbitrary \( \pi \). We only list the result when \( \pi \) is a supercuspidal representation. For a complete list, see [Che18 Lemmas 2.3.1-2.3.6].

For \( \chi \in \mathbb{Z}_p^\times \), we extend \( \chi \) to a character of \( \mathbb{Q}_p^\times \) so that \( \chi(p) = 1 \).

**Lemma 3.7.** Assume \( p \neq 2 \). Let \( \pi \) be a supercuspidal representation. Let \( c = c(\pi) \).
On the other hand, by the functional equation (cf. [JL70])
\[ W_\pi(a(p^{i-c}t)) = p^{(i-c)/2} \zeta_{\pi}(1) \epsilon \left( \frac{1}{2}, \pi \right) \sum_{\mu \in \hat{Z}_p^\times, a(\mu) = c-i} \epsilon \left( \frac{1}{2}, \mu^{-1}, \psi \right) \epsilon \left( \frac{1}{2}, \pi \otimes \mu^{-1}, \psi \right)^{-1} \mu(-t). \]

(i) For \( 0 \leq i < c/2 \), \( W_\pi^{(i)}(a(t)) \) is supported in \( p^{2i-c}Z_p^\times \). For \( t \in Z_p^\times \) and \( 0 \leq i < c/2 \), we have
\[ W_\pi^{(i)}(a(p^{i-c}t)) = p^{(i-c)/2} \zeta_{\pi}(1) \epsilon \left( \frac{1}{2}, \pi \right) \sum_{\mu \in \hat{Z}_p^\times, a(\mu) = c-i} \epsilon \left( \frac{1}{2}, \mu^{-1}, \psi \right) \epsilon \left( \frac{1}{2}, \pi \otimes \mu^{-1}, \psi \right)^{-1} \mu(-t). \]

(ii) For \( c/2 \leq i \leq c \), \( W_\pi^{(i)}(a(t)) \) is supported in \( Z_p^\times \). For \( t \in Z_p^\times \) and \( c/2 \leq i < c - 1 \), we have
\[ W_\pi^{(i-1)}(a(t)) = -p^{-1} \zeta_{\pi}(1) + p^{-1/2} \zeta_{\pi}(1) \epsilon \left( \frac{1}{2}, \pi \right) \sum_{\mu \in \hat{Z}_p^\times, a(\mu) = 1} \epsilon \left( \frac{1}{2}, \mu^{-1}, \psi \right) \epsilon \left( \frac{1}{2}, \pi \otimes \mu^{-1}, \psi \right)^{-1} \mu(-t), \]
\[ W_\pi^{(c)}(a(t)) = 1. \]

Proof. Note that (cf. [Sch02, §2.4])
\[ W_\pi(a(t)) = L_{Z_p^\times}(t), \quad \rho(w)W_\pi(a(t)) = \epsilon \left( \frac{1}{2}, \pi \right) L_{p^{-c}Z_p^\times}(t). \]

For \( i \in Z_{\geq 0}, n \in Z \), and \( \mu \in \hat{Z}_p^\times \), let
\[ C_n^{(i)}(\mu) = \int_{Z_p^\times} W_\pi^{(i)}(a(p^n t)) \mu^{-1}(t) dt. \]
Then
\[ W_\pi^{(i)}(a(p^n t)) = \sum_{\mu \in \hat{Z}_p^\times} C_n^{(i)}(\mu) \mu(t), \]
for \( t \in Z_p^\times \). Let \( i \in Z_{\geq 0} \) and \( \mu \in \hat{Z}_p^\times \). Then
\[ \int_{Q_p^\times} \rho(w)W_\pi^{(i)}(a(t)) \mu(a) |t|^{1/2-s} d^\times a \]

\[
\begin{aligned}
(3.12) & = \begin{cases} 
  p^{1/2-cs} \zeta_{\pi}(1) \mu(-1) & \text{if } 0 \leq i < c \text{ and } a(\mu) = c-i, \\
  -p^{-1+\epsilon(1/2-s)} \zeta_{\pi}(1) & \text{if } i = c-1 \text{ and } \mu = 1, \\
  p^{-1/2-s} & \text{if } i = c \text{ and } \mu = 1, \\
  0 & \text{otherwise.}
\end{cases}
\end{aligned}
\]

By [Hum16, Proposition A.1],
\[ \epsilon \left( s, \pi \otimes \mu^{-1}, \psi \right) = \epsilon \left( \frac{1}{2}, \pi \otimes \mu^{-1}, \psi \right), \quad \left\{ \begin{array}{ll} 
  p^{\epsilon(1/2-s)} & \text{if } 2a(\mu) \leq c, \\
  p^{2a(\mu)(1/2-s)} & \text{if } 2a(\mu) > c.
\end{array} \right. \]

On the other hand, by the functional equation (cf. [JL70])
\[ \sum_{n \in Z} C_n^{(i)}(\mu) p^{n(1/2-s)} = \epsilon \left( s, \pi \otimes \mu^{-1}, \psi \right)^{-1} \int_{Q_p^\times} \rho(w)W_\pi^{(i)}(a(t)) \mu(a) |t|^{1/2-s} d^\times a. \]

The assertion follows from (3.12), (3.13), and (3.14). \( \square \)

3.3.2. The real case. Let \( \psi \) be the standard additive character of \( \mathbb{R} \). Let \( \mathfrak{gl}(2, \mathbb{R}) \) be the Lie algebra of \( \text{GL}_2(\mathbb{R}) \) and \( \mathfrak{gl}(2, \mathbb{R})_\mathbb{C} \) be its complexification.

Let \( l \) be a positive integer. An irreducible admissible \( (\mathfrak{gl}(2, \mathbb{R}), O(2)) \)-module \( \pi \) is called a (limit of) discrete series representation of weight \( l \) if \( \pi \) is the unique subrepresentation of
\[ \text{Ind}^{\text{GL}_2(\mathbb{R})}(\mathbb{R}^+) |_{\mathfrak{gl}(2, \mathbb{R})} \otimes \text{sgn}^l |_{\mathbb{R}^+} |_{\mathbb{R}^+}. \]
Let \( W_\pi \in \mathcal{W}(\pi, \psi) \) be the Whittaker of SO(2)-type \( l \) such that \( W_\pi(1) = e^{-2\pi} \). Then
\[
W_\pi(a(y)) = y^{l/2}e^{-2\pi y}R^+_{\mathbb{R}}(y).
\]
Let
\[
(3.15) \quad \tilde{V}_+ = -\frac{1}{8\pi} V_+ \in \mathfrak{gl}(2, \mathbb{R})_C
\]
be the normalized weight raising element, where
\[
V_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sqrt{-1}.
\]

Let \( m \in \mathbb{Z}_{\geq 0} \). Then \( \tilde{V}_+^m W_\pi \) is a Whittaker function of SO(2)-type \( l + 2m \) in \( \mathcal{W}(\pi, \psi) \). A simple induction shows that (cf. [CC, Lemma 3.1])
\[
(3.17) \quad \tilde{V}_+^m W_\pi(a(y)) = y^{l/2}e^{-2\pi y} \sum_{j=0}^m (-4\pi)^{j-m}y^j \Gamma(l+m) \Gamma(l+j) \left( \begin{array}{c} m \\ j \end{array} \right) \tilde{W}^{\pi}_\mathbb{R}^+ (y).
\]

3.3.3. The complex case. Let \( \psi \) be the additive character of \( \mathbb{C} \) defined by \( \psi(z) = e^{2\pi \sqrt{-1} \text{tr}(z)} \). For \( n \in \mathbb{Z}_{\geq 0} \), let \( \rho_n \) be the irreducible representation of SU(2) with dimension \( n + 1 \). Let \( \mathfrak{gl}(2, \mathbb{C}) \) be the Lie algebra of the real Lie group GL_2(\mathbb{C}) and \( \mathfrak{gl}(2, \mathbb{C})_C \) be its complexification.

Let \( l \) be a positive integer. Let \( \pi \) be the principal series representation
\[
\text{Ind}_{B(\mathbb{C})}^{G(\mathbb{C})} (\mu^l \boxtimes \mu^{-l}).
\]
Here \( \mu(z) = z|z|^{-1} \) for \( z \in \mathbb{C}^\times \). Note that \( \pi \) is irreducible admissible with minimal SU(2)-type \( \rho_{2l} \). Since \( \pi|_{\text{SU}(2)} \) is of multiplicity free, up to scalars, there exist a unique element \( v \) in the space of \( \pi \) such that \( v \) is pure of SU(2)-type \( \rho_{2l} \) and \( \pi(t(e^{\sqrt{-1} \theta}))v = e^{2\sqrt{-1} \theta}v \) for all \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \). Equivalently, such elements are characterized by \( H \cdot v = 2l \cdot v \) and \( X \cdot v = 0 \). Here \( H, X \in \mathfrak{gl}(2, \mathbb{C})_C \) are defined by
\[
H = \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \otimes \sqrt{-1}, \quad X = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes 1 + \frac{1}{2} \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \otimes \sqrt{-1}.
\]

Let \( W_\pi \in \mathcal{W}(\pi, \psi) \) be the Whittaker function of \( \pi \) satisfying the above conditions such that \( W_\pi(1) = K_l(4\pi) \). Then the formula for \( W_\pi \) is give by
\[
(3.18) \quad W_\pi(zu(x)t(a)k) = e^{2\pi \sqrt{-1} \text{tr}(z/a^{2l+2}) + 2l} \sum_{n=0}^{2l} \left( \begin{array}{c} 2l \\ n \end{array} \right) \alpha^{2l-n} \beta^n K_{l-n}(4\pi a^2),
\]
for \( z \in \mathbb{C}^\times, x \in \mathbb{C}, a \in \mathbb{R}_{\mathbb{R}}^+ \) and \( k = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \text{SU}(2) \).

4. Weil representations and theta correspondence

Let \( F \) be a field of characteristic not 2. Let \( (V, Q) \) be a non-degenerate quadratic space of dimension \( m \) over \( F \). The associated non-degenerate symmetric bilinear form \( (, ) \) is defined by
\[
(x, y) = Q[x + y] - Q[x] - Q[y]
\]
for \( x, y \in V \). Define the orthogonal similitude group GO(V) by
\[
\text{GO}(V) = \{ h \in \text{GL}(V) \mid (hx, hy) = \nu(h)(x, y) \text{ for } x, y \in V \},
\]
here \( \nu : \text{GO}(V) \rightarrow \mathbb{G}_m \) is the scale map. When \( m \) is even, let
\[
\text{GSO}(V) = \{ h \in \text{GO}(V) \mid \det(h) = \nu(h)^{m/2} \}.
\]

Let O(V) and SO(V) be the orthogonal group and special orthogonal group defined by
\[
\text{O}(V) = \{ h \in \text{GO}(V) \mid \nu(h) = 1 \},
\]
\[
\text{SO}(V) = \{ h \in \text{GO}(V) \mid \det(h) = \nu(h) = 1 \}.
\]
4.1. **Local Weil representations.** Assume \( F \) is a local field. Denote \(( \ , \hbar )_F\) the Hilbert symbol of \( F \). Fix a non-trivial additive character \( \psi \) of \( F \). For \( a \in F^\times \), let \( \gamma_F(\psi^a) \) be the Weil index defined in [Rao93] Appendix. Let \( \{v_1, \ldots, v_m\} \) be an orthonormal basis for \((V, Q)\). Then

\[
Q[x_1v_1 + \cdots + x_mv_m] = a_1x_1^2 + \cdots + a_mx_m^2
\]

for some \( a_1, \ldots, a_m \in F^\times \). Let \( \det(V) \in F^\times / F^\times .2 \) and \( h(V) \in \{\pm 1\} \) be the discriminant and the Hasse invariant of \((V, Q)\) defined by

\[
\det(V) = a_1 \cdots a_m F^\times .2,
\]

\[
h(V) = \prod_{i < j}(a_i, a_j)_F.
\]

Note that the definition of \( \det(V) \) and \( h(V) \) dose not depend on the choice of orthonormal basis. Let \( \gamma_V(\psi) \) be the Weil index associated to \((V, Q)\) defined by

\[
\gamma_V(\psi) = \gamma_F(\psi^{\det(V)})\gamma_F(\psi)^{m-1}h(V).
\]

For \( n \in \mathbb{N} \), let \( \widehat{\text{Sp}_n(F)} \) be the 2-fold metaplectic cover of \( \text{Sp}_n(F) \). We identify \( \widehat{\text{Sp}_n(F)} \) with \( \text{Sp}_n(F) \times \{\pm 1\} \) as sets, and the multiplication is given by

\[
(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2 \cdot c_{\text{Rao}, n}(g_1, g_2)).
\]

Here \( c_{\text{Rao}, n} : \text{Sp}_n(F) \times \text{Sp}_n(F) \rightarrow \{\pm 1\} \) is Rao’s 2-cocycle defined in [Rao93] Theorem 5.3. Note that \( c_{\text{Rao}, 1} \) is equal to Kubota’s 2-cocycle defined in \([\text{Kud}94]\). The action is given as follows:

\[
\omega_{\psi, V,n}(1, h) \varphi(x) = \varphi(h^{-1}x),
\]

\[
\omega_{\psi, V,n} \left( \begin{pmatrix} a & 0 \\ 0 & t a^{-1} \end{pmatrix}, 1 \right) \varphi(x) = \frac{\gamma_V(\psi)}{\gamma_V(\psi^{\det(a)})} |\det(a)|_F^{m/2} \varphi(\frac{x}{a}),
\]

\[
\omega_{\psi, V,n} \left( \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix}, 1 \right) \varphi(x) = \psi(\text{tr}(bQ[x]))\varphi(x),
\]

\[
\omega_{\psi, V,n} \left( \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, 1 \right) \varphi(x) = \gamma_V(\psi) \int_{V^n(F)} \varphi(y)\psi(\text{tr}(x, y))dy,
\]

\[
\omega_{\psi, V,n}(1, \epsilon, 1) \varphi(x) = \epsilon^m \varphi(x),
\]

for \( h \in \text{O}(V)(F) \), \( a \in \text{GL}_n(F) \), \( b \in \text{Sym}_n(F) \), and \( \epsilon \in \{\pm 1\} \). Here \((x, y) = ((x_i, y_j))_{1 \leq i, j \leq n} \in M_n(F) \), \( Q[x] = \frac{1}{2}(x, x) \), and \( dy \) is the self-dual Haar measure on \( V^n(F) \) with respect to the pairing \( \psi(\text{tr}(x, y)) \). When \( F \) is archimedean, let \( S_{\psi}(V^n(F)) \) be the Harish-Chandra module of \( \omega_{\psi, V,n} \) on \( S(V^n(F)) \) with respect to the standard maximal compact subgroup of \( \widehat{\text{Sp}_n(F)} \). When \( F \) is non-archimedean, let \( S_{\psi}(V^n(F)) = S(V^n(F)) \).

When \( m \) is even, the Weil representation \( \omega_{\psi, V,n} \) can be extend to \( R(F) \). Here \( R = G(\text{GSp}_n \times \text{GO}(V)) \), and

\[
G(\text{GSp}_n \times \text{GO}(V)) = \{(g, h) \in \text{GSp}_n \times \text{GO}(V) \mid \nu(g) = \nu(h)\}.
\]

The action is given by

\[
\omega_{\psi, V,n}(g, h) \varphi = \omega_{\psi, V,n} \left( g \begin{pmatrix} 1_n & 0 \\ 0 & \nu(g)^{-1}1_n \end{pmatrix}, 1 \right) L(h) \varphi
\]

for \( (g, h) \in R(F) \) and \( \varphi \in S(V^n(F)) \). Here

\[
L(h) \varphi(x) = |\nu(h)|_F^{-mn/4} \varphi(h^{-1}x).
\]
4.2. Global Weil representations and theta functions. Assume $F$ is a number field. Fix a non-trivial additive character $\psi$ of $A_F/F$. Let $(V,Q)$ be a non-degenerate quadratic space over $F$. Let $\omega_{\psi,V,n} = \otimes_v \omega_{\psi,V,n}$ be the Weil representation of $\text{Sp}_n(A_F) \times \text{O}(V)(A_F)$ on $S(V^n)(A_F) = \otimes_v S(V^n(F_v))$. Here $v$ runs through the place of $F$. Let $S_\varphi(V^n(A_F)) = \otimes_v S_\varphi(V^n(F_v))$. For $\varphi \in S(V^n(A_F))$, let $\theta_{\psi,V,n}(\cdot,\cdot;\varphi)$ be the theta function defined by

$$
\theta_{\psi,V,n}(g,h;\varphi) = \sum_{x \in V^n(F)} \omega_{\psi,V,n}(g,h)\varphi(x)
$$

for $(g,h) \in \text{Sp}_n(A_F) \times \text{O}(V)(A_F)$. This is a slowly increasing function on $\text{Sp}_n(A_F) \times \text{O}(V)(A_F)$, and define an automorphic form if $\varphi \in S_\varphi(V^n(A_F))$. When $m$ is even, we regard it as a function on $\text{Sp}_n(A_F) \times \text{O}(V)(A_F)$.

5. Shimura-Shintani-Waldspurger correspondence

5.1. Setting. Let $\kappa'$ be an odd positive integer and $N_1$ (resp. $N_2$) be an odd square-free (resp. odd cubic-free) positive integer. We assume $N_1 | N_2$. Let $h \in S^+_{N_1+2}((\Gamma_0(4N_1)))$ be a newform in the sense of [Koh82] and $f \in S_{2\kappa'}((\Gamma_0(N_1)))$ be the normalized newform associated to $h$ by the Shimura lift (cf. [Shi73] and [Niw75]). For $p \nmid N_1$, let $(\alpha_p, \alpha_p^{-1})$ be the Satake parameter of $f$ at $p$, put $\lambda_p(f) = p^{1/2}(\alpha_p + \alpha_p^{-1})$ and fix $s_p \in \mathbb{C}$ such that $\alpha_p = p^{-s_p}$. Note that $\text{Re}(s_p) = 0$ by the Ramanujan conjecture. For $p | N_1$, fix an element $\tau_p \in \mathbb{Z}_p^*$ such that $(p, \tau_p)\mathbb{Q}_p = p^{-\kappa' + 1}a_f(p)$, and put $\alpha_p = p^{-1/2}(p, \tau_p)\mathbb{Q}_p$. Let $h$ (resp. $f$) be the cusp form on $\text{SL}_2(\mathbb{A})$ (resp. $\text{GL}_2(\mathbb{A})$) associated to $h$ (resp. $f$) as defined in [Ich03] (resp. [Ich05]). Let $\tau = \otimes_v \tau_v$ (resp. $\pi = \otimes_v \pi_v$) be the irreducible genuine cuspidal automorphic representation of $\text{SL}_2(\mathbb{A})$ (resp. $\text{GL}_2(\mathbb{A})$) generated by $h$ (resp. $f$). By the results in [Wal80], $\pi_p$ (resp. $\pi_p$) is the principal series representation

$$
\pi_p | \mathbb{Q}_p \otimes \mathbb{Q}_p
$$

for $p | N_1$, $\pi_p$ (resp. $\pi_p$) is the representation $\pi_{\mathbb{Q}_p} \otimes 1_{\mathbb{Q}_p} \otimes \mathbb{Q}_p$ defined in [32.1] (resp. the special representation $\text{St}_{\mathbb{Q}_p} \otimes \chi_{\tau_p}$) for $p | N_1$, and $\pi_{\mathbb{Q}_p}$ (resp. $\pi_{\mathbb{Q}_p}$) is the holomorphic discrete series representation of weight $2\kappa'$. Note that $h$ satisfies the following conditions:

- $\tilde{\pi}((k, s_p(k)))h = h$ for $k \in \text{SL}_2(\mathbb{Z}_p)$, if $p \nmid 2N_1$.
- $\tilde{\pi}((k, s_p(k)))h = h$ for $k \in K_0(p)$, if $p | N_1$.
- $\tilde{\pi}(k)e_2(k) = e_2(k)h$ for $k \in K_0(4)$.
- $\tilde{\pi}(W)h = 2^{-1/2}e_2^{-1}h$, where $W, U$ are defined in [32.3].
- $\tilde{\pi}(k_0)h = e^{\sqrt{-1}(\kappa'+1/2)^2}h$, for $k_0 \in S(2)$.

Similarly, $f$ satisfies the following conditions:

- $\pi(k)f = f$ for $k \in \text{GL}_2(\mathbb{Z}_p)$, if $p \nmid N_1$.
- $\pi(k)f = f$ for $k \in K_0(p)$, if $p | N_1$.
- $\pi(kf_0) = e^{2\sqrt{-1}\kappa'}f$ for $k_0 \in S(2)$.

The conditions characterize $h$ (resp. $f$) in the space of $\tilde{\pi}$ (resp. $\pi$) up to scalars.

Since $\kappa'$ is assumed to be odd, by the nonvanishing theorem in [BH95] and [Wal91], there exist a fundamental discriminant $-D < 0$ such that:

- $D$ is prime to $N_2$.
- $-D \equiv 1 \mod 8$.
- $(p, -\tau_p)\mathbb{Q}_p = -1$ for $p | N_1$.
- $(p, -D)\mathbb{Q}_p = 1$ for $p | N_2/N_1$.
- $L\left(\frac{1}{2}, \pi \otimes \chi_D\right) \neq 0$.

Here $\chi_D$ is the quadratic Hecke character of $\mathbb{A}^\times/\mathbb{Q}^\times$ associated to $\mathbb{Q}(/\sqrt{-D})$ by class field theory, and $L(s, \pi \otimes \chi_D)$ is the automorphic $L$-function for the automorphic representation $\pi \otimes \chi_D$ of $\text{GL}_2(\mathbb{A})$. We fix
such discriminant $-D$ throughout this section. Let $(V, Q)$ be the quadratic space over $\mathbb{Q}$ defined as follows:

$$V = \{ x \in M_2(\mathbb{Q}) \mid \text{tr}(x) = 0 \}$$

$$= \left\{ \begin{array}{c} x_1 \ x_2 \\ x_3 \ x_4 \end{array} \right| x_1, x_2, x_3, x_4 \in \mathbb{Q} \right\},$$

$$Q[x] = -D \text{det}(x).$$

Recall we have an exact sequence

$$1 \rightarrow G_m \rightarrow \text{GL}_2 \rightarrow \text{SO}(V) \rightarrow 1,$$

here $\iota(a) = a1_2$ and $\rho(h)x = h x h^{-1}$ for $a \in G_m$, $h \in \text{GL}_2$, and $x \in V$. Via the exact sequence, we identify automorphic forms on $\text{SO}(V(\mathbb{A}))$ with automorphic forms on $\text{PGL}_2(\mathbb{A})$. Let $\psi$ (resp. $\psi_v$) be the standard additive character of $\mathbb{A}$ (resp. $\mathbb{Q}_v$). Let $\omega_\psi = \omega_{\psi, V, 1}$ (resp. $\omega_\psi = \omega_{\psi, V, 1}$) denote the Weil representation of $\text{SL}_2(\mathbb{A}) \times O(V)(\mathbb{A})$ (resp. $\text{SL}_2(\mathbb{Q}_v) \times O(V)(\mathbb{Q}_v)$) on $S(V(\mathbb{A}))$ (resp. $S(V(\mathbb{Q}_v))$) defined in (4.1).

Denote $Q(\hat{\pi})$ the set consisting of $\xi \in \mathbb{Q}^\times$ so that $\psi_\xi$ is not a missing character of $\hat{\pi}_v$ for all place $v$. In particular, $\hat{\pi}$ does not have Whittaker model with respect to $\psi_\xi$ for $\xi \notin Q(\hat{\pi})$. Note that $D \in Q(\hat{\pi})$ by our assumption on $D$, and the definition of a missing character in (3.2.3) and (3.2.4). Let $W_{D,v}, W_{D,v}^D \in \mathcal{W}(\hat{\pi}_v, \psi_v^D)$ (resp. $W_{\pi_v} \in \mathcal{W}(\pi_v, \psi_v)$) be the Whittaker functions of $\hat{\pi}_v$ (resp. $\pi_v$) with respect to $\psi_v^D$ (resp. $\psi_v$) defined as in (3.3.1), (3.4.1), (3.7), and (3.9) (resp. (3.3.1) and (3.3.2)). Put $W_{f,v} = W_{\pi_v}$, and $W_{f \otimes \chi_{-D}, v} = (\chi_{-D,v} \circ \det) W_{f,v} \in \mathcal{W}(\pi_v \otimes \chi_{-D,v}, \psi_v)$. Let $W_{h,D}$ (resp. $W_{f,1}$) be the $D$-th (resp. the first) Fourier coefficient of $h$ (resp. $f$) defined as in (Ich05) §3.2 (resp. (Ich05) §3.1)), similar for $W_{f \otimes \chi_{-D}, 1}$. Then

$$W_{h,D} = c_h(D) \prod_v W_{D,v}, \quad W_{f,1} = \prod_v W_{f,v}, \quad W_{f \otimes \chi_{-D}, 1} = \prod_v W_{f \otimes \chi_{-D,v}}.$$

Let $\xi_{N_2} \in \text{GL}_2(\mathbb{Q})$ defined by

$$\xi_{N_2,p} = \begin{cases} d(p) & \text{if } p^2 \mid N_2, \\ 1 & \text{otherwise.} \end{cases}$$

Fix a function $\tilde{\mathcal{F}}_{N_2} = \otimes_p \tilde{\mathcal{F}}_{N_2,p} \in \mathcal{H}(\text{SL}_2(\mathbb{Q}))$ and a function $\mathcal{F}_{N_2} = \otimes_p \mathcal{F}_{N_2,p} \in \mathcal{H}(\text{GL}_2(\mathbb{Q}))$ such that:

- If $p \nmid 2N_2$, then
  $$\tilde{\mathcal{F}}_{N_2,p} = \text{vol}(\text{SL}_2(\mathbb{Z}_p))^{-1} \mathcal{F}_{\text{SL}_2(\mathbb{Z}_p)},$$
  $$\mathcal{F}_{N_2,p} = \text{vol}(\text{GL}_2(\mathbb{Z}_p))^{-1} \mathcal{F}_{\text{GL}_2(\mathbb{Z}_p)}.$$

- If $p \mid 2N_1$, then
  $$\rho(\tilde{\mathcal{F}}_{N_2,p}) W_{D,p} = W_{D,p},$$
  $$\rho(\mathcal{F}_{N_2,p}) W_{f,p} = W_{f,p}.$$

- If $p \mid N_2$ with $p \nmid N_1$, then
  $$\rho(\tilde{\mathcal{F}}_{N_2,p}) W_{D,p} = (1 + \lambda_p(f)) W_{D,p}^0 - p^{-1/2} W_{D,p}^{(1)},$$
  $$\rho(\mathcal{F}_{N_2,p}) W_{f,p} = W_{f,p} + \rho(d(p)) W_{f,p}.$$

- If $p^2 \mid N_2$ and $p \nmid N_1$, then
  $$\rho(\tilde{\mathcal{F}}_{N_2,p}) W_{D,p} = 2\lambda_p(f) W_{D,p}^0 - 2p^{-1/2} W_{D,p}^{(1)} + (p^{1/2} - p^{-1/2}) W_{D,p}^{(2)} + (-1 + p^{-1/2} \alpha_p) W_{D,p}^{(3)}$$
  $$\rho(\mathcal{F}_{N_2,p}) W_{f,p} = W_{f,p} + (1 - p^{-1}) \rho(d(p)) W_{f,p} + \rho(d(p^2)) W_{f,p}.$$

Put $h_{N_2} = \pi(\tilde{\mathcal{F}}_{N_2}) h$ and $f_{N_2} = \pi(\mathcal{F}_{N_2}) f$. Let $h_{N_2}$ (resp. $f_{N_2}$) be the cusp form associated to $h_{N_2}$ (resp. $f_{N_2}$). Recall the Maass-Shimura differential operators $\delta_r$ and $\delta^m_r$ defined in [Shi87] §5

$$\delta_r = \frac{1}{2\pi \sqrt{-1}} \left( \frac{\partial}{\partial \tau} + \frac{r}{2\sqrt{-1} \eta} \right), \tau = x + \sqrt{-1}y, y > 0, 2r \in \mathbb{N},$$

$$\delta^m_r = \delta_{r+2m-2} \circ \cdots \circ \delta_r, m \in \mathbb{Z}_{\geq 0}.$$
From now on, we fix a positive integer $m$ and put $\kappa = \kappa' + 2m$. Then $\delta_{\kappa'+1/2}^m h_{N_2}$ (resp. $\delta_{2\kappa}^m f_{N_2}$) is a nearly holomorphic modular form of weight $\kappa + 1/2$ (resp. $2\kappa$). Note that $\hat{V}_+^m h_{N_2}$ (resp. $\hat{V}_+^{2m} f$) is the cusp form on $\text{SL}_2(\mathbb{A})$ (resp. $\text{GL}_2(\mathbb{A})$) associated to $\delta_{\kappa'+1/2}^m h_{N_2}$ (resp. $\delta_{2\kappa}^m f_{N_2}$).

The main result of this section are Propositions 5.8 and 5.10 which assert that, in the special case $N_1 = N_2$, $\delta_{\kappa'+1/2}^m h$ and $\delta_{2\kappa}^m f$ correspond to each other via the global theta.

5.2. Local theta lifts. Define $\varphi = \otimes_v \varphi_v \in S_v(V(A))$ as follows:

- If $v = p$ with $p \nmid N_2$, then $\varphi_v$ is the function defined in [Ich05] §9.2.
- If $v = p$ with $p \parallel N_2$, then
  $$\varphi_p(x) = \mathbb{I}_p(x_1) \mathbb{I}_p(x_2) \mathbb{I}_p(x_3).$$
- If $v = \infty$, then
  $$\varphi_\infty(x) = (2x_1 - \sqrt{-1}x_2 - \sqrt{-1}x_3)^{\kappa} e^{-\pi Dtr(x^t x)}.$$ 

Note that if $v = p$ with $p \parallel N_2$, then

$$\omega_{\varphi_v}((k, s_p(k)), k') \varphi_p = \varphi_p$$

for $k \in K_0(p)$, $k' \in K_0(p)$. If $v = p$ with $p^2 \mid N_2$, then

$$\omega_{\varphi_v}((k, s_p(k)), k') \varphi_p = \varphi_p$$

for $k \in K_0(p^2)$, $k' \in K_0(p^2)$. If $v = \infty$, then

$$\omega_{\varphi_\infty}(\hat{k}_\varphi, k') \varphi_\infty = e^{(\kappa'+1/2)\sqrt{-1}x} e^{-2\sqrt{-1}x} \varphi_\infty$$

for $\hat{k}_\varphi \in SO(2)$, $k' \in SO(2)$.

In §5.2.1 (resp. §5.2.2), we determine the local components of $\theta_\psi(\hat{V}_+^m h, \varphi)$ (resp. $\theta_\psi(\hat{V}_+^{2m} f \otimes \chi_D, \varphi)$). As a consequence of the local calculation, we show in §5.3 that $\theta_\psi(\hat{V}_+^m h, \varphi)$ and $\theta_\psi(\hat{V}_+^{2m} f \otimes \chi_D, \varphi)$ are proportional to $\hat{V}_+^{2m} f_{N_2} \otimes \chi_D$ and $\hat{V}_+^m h_{N_2}$, respectively. Moreover, we determine the ratio explicitly.

5.2.1. Shimura lifts. Let $V_1 = \{ x \in V \mid x_2 = x_3 = 0 \}$ be a quadratic subspace of $V$. For each place $v$, let

$$S(V(Q_v)) \longrightarrow S(V_1(Q_v)) \otimes S(Q_v^2)$$

be the partial Fourier transform defined by

$$\hat{\varphi}(x_1; y) = |D|_{Q_v}^{1/2} \int_{Q_v} \varphi \left( \begin{array}{c} x_1 \\ y_1 \\ -x_1 \end{array} \right) \psi_v(Dy_2 z) dz$$

for $x_1 \in V_1(Q_v)$, $y = (y_1, y_2) \in Q_v^2$. Let $\omega_{\psi_v}$ be the representation of $\text{SL}_2(Q_v) \times \text{O}(V)(Q_v)$ on $S(V_1(Q_v)) \otimes S(Q_v^2)$ defined by

$$\omega_{\psi_v}(g, h) \varphi = (\omega_{\psi_v}(g, h) \varphi).$$

Note that

$$\omega_{\psi_v}(\hat{t}(a), 1) \varphi(x_1; y) = \gamma_{Q_v}(a, \psi_v)(-D, a)_{Q_v} |a|^{1/2} \varphi(ax_1; ay_1, a^{-1} y_2)$$

for $a \in Q_v^\times$. If $\hat{\varphi} = \varphi_1 \otimes \varphi_2$ with $\varphi_1 \in S(V_1(Q_v))$ and $\varphi_2 \in S(Q_v^2)$, then

$$\omega_{\psi_v}((g, \epsilon), 1) \varphi(x_1; y) = \omega_{\psi_v}((g, \epsilon), 1) \varphi_1(x_1) \cdot \varphi_2(yg)$$

for $(g, \epsilon) \in \text{SL}_2(Q_v)$. We have an equivariant map

$$S_{\psi_v}(V(Q_v)) \otimes W(\hat{\psi}^v, \psi_v^{-D}) \longrightarrow W(\pi_v \otimes \chi_D, \psi_v^{-1})$$

$$\varphi \otimes W \longmapsto W^{(1)}(\varphi, W),$$

20
\[ W_{\psi}^{(1)}(\varphi, W)(h) = \int_{U(\mathbb{Q}_p) \cap SL_2(\mathbb{Q}_p)} \hat{\omega}_{\varphi}(g, h) \hat{\varphi}(2^{-1}D^{-1}; 0, 1)W(t(2^{-1}D^{-1})g)dg. \]

**Proposition 5.1.** Let \( v = p \nmid 2DN_2 \). We have
\[ W_{\psi}^{(1)}(\varphi_p, W_{D,p}) = \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})}. \]

**Proof.** [Ich05] §9.3.

**Proposition 5.2.** Let \( v = p \mid D \). We have
\[ W_{\psi}^{(1)}(\varphi_p, W_{D,p}) = p^{3/2}(2, -D)_{Q_p} \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})}. \]

**Proof.** [Ich05] Lemma 9.3.

**Proposition 5.3.** Let \( v = 2 \). We have
\[ W_{\psi}^{(1)}(\varphi_2, W_{D,2}) = -2^{-3/2} \sqrt{-1} \cdot \overline{W_{\psi}^{(1)}(\varphi_2, W_{D,2})}. \]

**Proof.** [Ich05] Lemma 9.6.

**Proposition 5.4.** If \( v = p \mid N_1 \), then we have
\[ W_{\psi}^{(1)}(\varphi_p, W_{D,p}) = (1 + p)^{-1} \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})}. \]

If \( v = p \) with \( p \nmid N_2 \), \( p \nmid N_1 \), and \( vl \), then we have
\[ W_{\psi}^{(1)}(\varphi_p, W_{D,p}) = (1 + p)^{-1} \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})} + (1 + p)^{-1} \rho(d(p)) \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})}. \]

If \( v = p \mid p^2 \), \( N_2 \) and \( p \nmid N_1 \), then we have
\[ W_{\psi}^{(1)}(\varphi_p, W_{D,p}) = p^{1/2}(1 + p)^{-1} \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})} + p^{-1}(1 + p)^{-1}(1 - p^{-1}) \rho(d(p)) \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})} + p^{-1}(1 + p)^{-1} \rho(d(p^2)) \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})}. \]

**Proof.** We only consider the case when \( p^2 \mid N_2 \) and \( p \nmid N_1 \). The other cases are similar.

By (5.3)
\[ W_{\psi}^{(1)}(\varphi_p, W_{D,p}) \in W_0(p \varprojlim_{D,p, \psi^{-1}} K_0(p^2) = C \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})} \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})} + C \rho(d(p)) \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})} \overline{W_{\psi}^{(1)}(\varphi_p, W_{D,p})}. \]

Note that
\[ W_{\psi}^{(1)}(\varphi_p, W_{D,p})(h) = p^{-1}(1 + p)^{-1} \sum_{k \in SL_2(\mathbb{Z}_p) \cap K_0(p^2)} \int_{\mathbb{Q}_p^x} \hat{\omega}_{\varphi}(k, h) \hat{\varphi}(2^{-1}D^{-1}; 0, 1) \overline{W_{D,p}(t(2^{-1}D^{-1})t(a)k)}|a|_{Q_p}^{-2}d^x a \]
\[ = p^{-1}(1 + p)^{-1} \sum_{k \in SL_2(\mathbb{Z}_p) \cap K_0(p^2)} \sum_{n \in \mathbb{Z}} p^{3n/2}(p^n, -D)_{Q_p} \gamma_{Q_p}(p^n, \psi_p) \]
\[ \times \hat{\psi}_a(k, h) \hat{\psi}_p(2^{-1}D^{-1}p^n; 0, 0) \overline{W_{D,p}(t(2^{-1}D^{-1})t(p^n)k)}. \]

By direct calculations, we have
\[ W_{\psi}^{(1)}(\varphi_p, W_{D,p})(1) = p^{-1}(1 + p)^{-1}, \]
\[ W_{\psi}^{(1)}(\varphi_p, W_{D,p})(a(p)) = p^{-1}(1 + p)^{-1} \left[ (1 - p^{-1}) + p^{-1/2}(a_p + a_p^{-1})(p, -D)_{Q_p} \right], \]
\[ W_{\psi}^{(1)}(\varphi_p, W_{D,p})(a(p^2)) = p^{-1}(1 + p)^{-1} \left[ 1 + (1 - p^{-1})p^{-1/2}(a_p + a_p^{-1})(p, -D)_{Q_p} + p^{-1}(a_p^2 + a_p^{-2} + 1) \right]. \]

This completes the proof.

**Lemma 5.5.** Let \( r \in \mathbb{R}^, m, n \in \mathbb{Z}_{\geq 0} \). Put
\[ J(m, n; r) = \int_0^\infty a^{-2m-2}H_n(\sqrt{a}ra + a^{-1})e^{-\pi(ra^2 + a^{-2})}da, \]
\[ I(m, n; r) = \sum_{j=0}^m \binom{2m-j}{j}(4\pi)^{-m+j} \sum_{i=0}^j \frac{n!(-4\pi)^{-i}}{(n-i)!} \binom{j}{i} r^{j-i}. \]
If \( r > 0 \), then
\[
J(m, n; r) = 2^{2n-1} \pi n^2 e^{-4\pi r} I(m, n; r).
\]

If \( r < 0 \) and \( n > m \), then
\[
J(m, n; r) = 0.
\]

**Proof.** In a small enough neighborhood of \( x = 0 \),
\[
\sum_{n=0}^{\infty} \frac{1}{n!} (-\sqrt{x})^n J(m, n; r)
= \int_0^\infty a^{-2m-2} e^{-\pi (ax + x^2)^2} \, da
= e^{-2\pi (r+x)} |r + x|^{m+1/2} K_{m+1/2}(2\pi |r + x|)
= 2^{-1} e^{-2\pi (r+x)-2\pi |r+x|} \sum_{j=0}^{m} \frac{(m+j)!}{j!(m-j)!} (4\pi)^{-j} |r + x|^{-j},
\]
where the last equality follows from [GHO7, 8.468].

If \( r < 0 \), then \( r + x < 0 \) in a neighborhood of \( x = 0 \) and we have
\[
\sum_{n=0}^{\infty} \frac{1}{n!} (-\sqrt{x})^n J(m, n; r) = 2^{-1} \sum_{j=0}^{m} \frac{(m+j)!}{j!(m-j)!} (4\pi)^{-j} (-1)^{m-j} (r + x)^{m-j}.
\]

If \( r > 0 \), then \( r + x > 0 \) in a neighborhood of \( x = 0 \) and we have
\[
\sum_{n=0}^{\infty} \frac{1}{n!} (-\sqrt{x})^n J(m, n; r) = 2^{-1} e^{-4\pi (r+x)} \sum_{j=0}^{m} \frac{(2m-j)!}{j!(m-j)!} (4\pi)^{m+j} (r + x)^j.
\]

This completes the proof. \( \square \)

**Proposition 5.6.** Let \( v = \infty \). We have
\[
W_\infty^{(1)}(\varphi_\infty, V^m W_{D, \infty}) = 2^{-1/2} D^{-\kappa+m+1} e^{-2\pi D} V^m W_{f \otimes \chi_{-D, \infty}}.
\]

**Proof.** Note that ([Ich05, §9.6])
\[
\hat{\varphi}_\infty(x_1; y_1, y_2) = (2\sqrt{\pi D})^{-\kappa} H_{\kappa}(\sqrt{\pi D}(2x_1 - \sqrt{1} y_1 + y_2)) e^{-\pi D(2x_1^2 + y_1^2 + y_2^2)}.
\]

Let \( y > 0 \). By [4.11], [6.5], and Lemma 5.6
\[
W_\infty^{(1)}(\varphi_\infty, V^m W_{D, \infty})(a(y))
= \text{vol}(\text{SO}_2(\mathbb{R}))^{-1} y \int_{U(\mathbb{R}) \setminus SL_2(\mathbb{R})} \tilde{\omega}_{\varphi_\infty}(g, 1) \hat{\varphi}_\infty(2^{-1} D^{-1}; 0, y) V^m W_{D, \infty}(t(2^{-1} D^{-1}) y) \, dg
= 2y \int_0^\infty a^{-5/2} \hat{\varphi}_\infty(2^{-1} D^{-1}; a; 0, ya^{-1}) V^m W_{D, \infty}(t(2^{-1} D^{-1}) a) \, da
= 2y (2\sqrt{\pi D})^{-\kappa} \sum_{j=0}^{m} (-4\pi)^{-m} \frac{\Gamma(\kappa - m + 1/2)}{\Gamma(\kappa - 2m + 1/2 + j)} \left( \frac{m}{j} \right) (2^{-1} D^{-1})^{\kappa-2m+2j+1/2} D^j
\times \int_0^\infty a^{\kappa-2m+2j-2} H_{\kappa}(\sqrt{\pi D}(D^{-1} a + ya^{-1})) e^{-\pi D^{-1/2} a^2 + D y_1 a^{-2}} \, da
= 2^{-1/2} D^{-\kappa+m+1} e^{-2\pi y} \sum_{j=0}^{m} (-\pi)^{-m} y^{\kappa-2m+2j} \frac{\Gamma(\kappa - m + 1/2)}{\Gamma(\kappa - 2m + 1/2 + j)} \left( \frac{m}{j} \right) I(m - j, \kappa, y)
= 2^{-1/2} D^{-\kappa+m+1} e^{-2\pi y} \sum_{j=0}^{m} (-\pi)^{-m} y^{\kappa-2m+2j} \frac{\Gamma(\kappa - m + 1/2)}{\Gamma(\kappa - 2m + 1/2 + j)} \left( \frac{m}{j} \right) \times \sum_{i=0}^{m-j} \frac{(2m-2j-i)!}{i!(m-j-i)!} (4\pi)^{i+j-m} \sum_{l=0}^{i} \frac{\kappa!(-4\pi)^{-l}}{(\kappa-l)!} \left( \frac{i}{l} \right) y^{i-l}.
\]
By (5.3), there exist a constant $C$ such that
\[
W^{(1)}_{\infty}(\varphi, V^m_{\infty}W_{D,\infty})(g) = CV^{2m}_{\infty}W_{f \otimes \chi_{-D,\infty}}.
\]
Both $W^{(1)}_{\infty}(\varphi, V^m_{\infty}W_{D,\infty})(a(y))$ and $V^{2m}_{\infty}W_{f \otimes \chi_{-D,\infty}}(a(y))$ are product of $e^{-2\pi y}$ and polynomials in $y$. Comparing the coefficients for $y^k$, we conclude that $C = 2^{-1/2}D^{-k+m-1}e^{-2\pi}$. This completes the proof.

\[\]

5.2.2. Shintani lifts. For each place $v$, we have an equivariant map (cf. corollary to proposition 12 in [Wal80])
\[
S_{\psi_v}(V(\mathbb{Q}_v)) \otimes W(\pi_v \otimes \chi_{-D,v}, \psi_v) \rightarrow W(\tilde{\pi}_v, \psi_v^{(2)})
\]
\[
\varphi \otimes W \mapsto W^{(2)}_v(\varphi, W),
\]
\[
W^{(2)}_v(\varphi, W)(g) = \int_{D(\mathbb{Q}_v) \backslash GL_2(\mathbb{Q}_v)} \omega_{\psi_v}(g, h)\varphi(d(-1)) \int_{\mathbb{Q}_v^{\times}} W(a(t)h) d^x t dh.
\]

**Proposition 5.7.** If $v = p$ with $p \parallel N_2, p \nmid N_1$, then we have
\[
W^{(2)}_p(\varphi_p, W_{f \otimes \chi_{-D,p}}) = (1 + p)^{-1}L\left(1, \frac{1}{2}, \pi_p \otimes \chi_{-D,p}\right)
\]
\[
\times \left((1 + (p, -D)\mathbb{Q}_p, \lambda_p(f))W^{(0)}_{D,p} - p^{-1/2}(p, -D)\mathbb{Q}_p, W^{(1)}_{D,p}\right)
\]
\[\]
If $v = p$ with $p^2 \parallel N_2$ and $p \nmid N_1$, then we have
\[
W^{(2)}_p(\varphi_p, \rho(d(p))W_{f \otimes \chi_{-D,p}}) = p^{-1}(1 + p)^{-1}L\left(1, \frac{1}{2}, \pi_p\right)
\]
\[
\times \left(2\lambda_p(f)W^{(0)}_{D,p} - 2p^{-1/2}W^{(1)}_{D,p} + (p^{1/2} - p^{-1/2})W^{(2)}_{D,p} + (-1 + p^{-1/2}\alpha_p)W^{(3)}_{D,p}\right).
\]

**Proof.** We only consider the case when $p^2 \parallel N_2$ and $p \nmid N_1$. The other cases are similar.

Recall that in this case, we assume $-D \in \mathbb{Z}_p^{\times, 2}$. Thus $\chi_{-D,p} = 1$ and $W_{f \otimes \chi_{-D,p}} = W_{f,p}$. For $k \in GL_2(\mathbb{Z}_p)$ and $n \in \mathbb{Z}_{\geq 0}$, put
\[
I(k; n) = \int_{\mathbb{Q}_p} dx \int_{\mathbb{Q}_p^{\times}} d^x t \omega_{\psi_p}(t(p^n), u(x)k)\varphi_p(d(-1))\psi_p(tx)W_{f,p}(a(t)kd(p)).
\]

By direct calculations, we have
\[
W^{(2)}_p(\varphi_p, \rho(d(p))W_{f \otimes \chi_{-D,p}})(t(p^n)) = p^{-1}(1 + p)^{-1} \sum_{k \in GL_2(\mathbb{Z}_p)/K_0(p^2)} I(k; n)
\]
\[
= p^{-n-1}(1 + p)^{-1} \gamma_{\mathbb{Q}_p}(p^n, \psi_p)L\left(1, \frac{1}{2}, \pi_p\right)
\]
\[
\times \left\{ \begin{array}{cl}
2 & \text{if } n = 0, \\
(p^{1/2} + p^{3/2})\psi_p(p^{2n-2}D; \alpha_p) & \text{if } n \geq 1.
\end{array} \right.
\]
(5.7)

\[
W^{(2)}_p(\varphi_p, \rho(d(p))W_{f \otimes \chi_{-D,p}})(w) = p^{-1}(1 + p)^{-1} \left[(3 - p^{-1})L\left(1, \frac{1}{2}, \pi_p\right) - 2\right].
\]

On the other hand, by (5.4) and Lemma 8.3
\[
W^{(2)}_p(\varphi_p, \rho(d(p))W_{f \otimes \chi_{-D,p}}) \in W(\tilde{\pi}_p, \psi_p^{(2)}D, K_0(p)) = CW^{(0)}_{D,p} + CW^{(1)}_{D,p} + CW^{(2)}_{D,p} + CW^{(3)}_{D,p}.
\]

Let $A_0, A_1, A_2, A_3 \in \mathbb{C}$ such that
\[
W^{(2)}_p(\varphi_p, \rho(d(p))W_{f \otimes \chi_{-D,p}}) = A_0W^{(0)}_{D,p} + A_1W^{(1)}_{D,p} + A_2W^{(2)}_{D,p} + A_3W^{(3)}_{D,p}.
\]
By Lemmas 3.4 and 3.5

\[ W_p^{(2)}(\varphi_p, p(d(p))) W_{f, D, p}(t(p^n)) \]
\[ = p^{-n} q_{p^n}(p^n, \Psi_p) \left[ A_0 \Psi_p(p^{2n} D; \alpha_p) + p A_1 \Psi_p(p^{2n+2} D; \alpha_p) + p A_2 \Psi_p(p^{2n-2} D; \alpha_p) \right]. \]

The assertion follows from \([5.7]\) and \([5.8]\). This completes the proof. □

5.3. Global theta lifts. For \( f \in \hat{\pi}^\vee \) and \( \phi \in S_\psi(V(\hat{A})) \), let \( \theta_\psi(f, \phi) \) be the automorphic form on \( SO(V)(\hat{A}) \) defined by

\[ \theta_\psi(f, \phi)(h) = \int_{SL_2(\hat{A})} \theta_\psi(g, h; \phi) f(g) dg. \]

Let \( \theta_\psi(\hat{\pi}^\vee) \) be the global theta lift of \( \hat{\pi}^\vee \) with respect to \( \psi \) defined by

\[ \theta_\psi(\hat{\pi}^\vee) = \{ \theta_\psi(f, \phi) \mid f \in \hat{\pi}^\vee, \phi \in S_\psi(V(\hat{A})) \}. \]

For \( f \in \pi \otimes \chi_{-D} \) and \( \phi \in S_\psi(V(\hat{A})) \), let \( \theta_\psi(f, \phi) \) be the genuine automorphic form on \( SL_2(\hat{A}) \) defined by

\[ \theta_\psi(f, \phi)(g) = \int_{SO(V)(\hat{A})} \theta_\psi(g, h; \phi) f(h) dh. \]

Let \( \theta_\psi(\pi \otimes \chi_{-D}) \) be the global theta lift of \( \pi \otimes \chi_{-D} \) with respect to \( \psi \) defined by

\[ \theta_\psi(\pi \otimes \chi_{-D}) = \{ \theta_\psi(f, \phi) \mid f \in \pi \otimes \chi_{-D}, \phi \in S_\psi(V(\hat{A})) \}. \]

5.3.1. Shimura lifts. Let \( W_p^{(1)} \) be the Fourier coefficient of \( \theta_\psi(V_+^m h, \varphi) \) defined by

\[ W_p^{(1)}(h) = \int_{\hat{A}/\hat{Q}} \theta_\psi(V_+^m h, \varphi)(u(x) h) \psi(x) dx. \]

Since

\[ W_{V_+^m h, D} = c_h(D) V_+^m W_{D, \infty} \prod_p W_{D, p}, \]

by \([1ch05]\) Lemma 4.2

\[ W_p^{(1)} = c_h(D) \xi_{Q}(2)^{-1} W_p^{(1)}(\varphi, V_+^m W_{D, \infty}) \prod_p W_p^{(1)}(\varphi, W_{D, p}). \]

Proposition 5.8. We have

\[ \theta_\psi(V_+^m h, \varphi) = -2^{-2} (\sqrt{-1}) D^{-\kappa+m+1/2} c_h(D) \xi_{Q}(2)^{-1} (\text{GL}_2(\hat{Z}) : K_0(\hat{N}_2 \hat{Z}))^{-1} V_+^{2m} f_{N_2} \otimes \chi_{-D}. \]

Proof. By Propositions 5.1, 5.4 and 5.6 there exists a constant \( C \) such that

\[ \theta_\psi(V_+^m h, \varphi) = CV_+^{2m} f_{N_2} \otimes \chi_{-D}, \]

and

\[ C = e^{2\pi c_h(D) \xi_{Q}(2)^{-1} W_\infty^{(1)}(\varphi, V_+^m W_{D, \infty})(1) \prod_p W_p^{(1)}(\varphi, W_{D, p})(1)} \]
\[ = -2^{-2} (\sqrt{-1}) D^{-\kappa+m-1} c_h(D) \xi_{Q}(2)^{-1} \prod_{p \mid D} p^{3/2} (\sqrt{-1}) q_p \prod_{p \| N_2} (1 + p)^{-1} \prod_{p^2 | N_2} p^{-1}(1 + p)^{-1} \]
\[ = -2^{-2} (\sqrt{-1}) D^{-\kappa+m+1/2} c_h(D) \xi_{Q}(2)^{-1} (\text{GL}_2(\hat{Z}) : K_0(\hat{N}_2 \hat{Z}))^{-1}. \]

This completes the proof. □
5.3.2. Shintani lifts.

**Lemma 5.9.** For $p | 2N_2$, we have
\[
\frac{\langle \rho(\xi_{N_2,p})W_{f,p}, \rho(\mathcal{F}_{N_2,p})W_{f,p} \rangle_{1,p}}{\langle W_{f,p}, W_{f,p} \rangle_{1,p}} = \frac{\langle \rho(\mathcal{F}_{N_2,p})W_{D,p}^{(0)}, W_{D,p}^{(0)} \rangle_{2,p}}{\langle W_{D,p}^{(0)}, W_{D,p}^{(0)} \rangle_{2,p}}.
\]

Here $\langle \cdot, \cdot \rangle_{1,v}$ (resp. $\langle \cdot, \cdot \rangle_{2,v}$) is a Hermitian equivariant pairing on $W(\pi_v, \psi_v)$ (resp. $W(\tilde{\pi}_v, \psi_v^D)$).

**Proof.** We have two equivariant maps
\[
S(V(Q_v)) \otimes_{\mathbb{C}} W(\pi_v, \psi_v) \otimes_{\mathbb{C}} W(\tilde{\pi}_v, \psi_v^D) \rightarrow \mathbb{C}
\]
\[
\varphi \otimes W_1 \otimes W_2 \mapsto \langle W_1, W_0^{(1)}(\varphi, W_2) \rangle_{1,v},
\]
\[
\varphi \otimes W_1 \otimes W_2 \mapsto \langle W_0^{(2)}(\varphi, W_1), W_2 \rangle_{2,v}.
\]

By the Howe duality for $(\widetilde{\text{SL}}_2(Q_v), \text{SO}(V)(Q_v))$, the two equivariant maps are equal up to a constant. Let $W_1 = \rho(\mathcal{F}_{N_2,p})W_{f,p}$ and $W_2 = W_{D,p}^{(0)}$. The assertion follows from Lemmas 5.3 and 5.7. \hfill \Box

Define the Petersson norms of $f$ and $h$ by
\[
\langle f, f \rangle = \frac{1}{(\text{SL}_2(\mathbb{Z}) : \Gamma_0(N_1))} \int_{\Gamma_0(N_1) \backslash \text{SL}_2(\mathbb{Z})} |f(\tau)|^2 y^{2\kappa'-2} d\tau,
\]
\[
\langle h, h \rangle = \frac{1}{6 (\text{SL}_2(\mathbb{Z}) : \Gamma_0(N_1))} \int_{\Gamma_0(4N_1) \backslash \text{SL}_2(\mathbb{Z})} |h(\tau)|^2 y^{2\kappa'-3/2} d\tau.
\]

Let $\langle \cdot, \cdot \rangle_{\text{SO}(V)}$ and $\langle \cdot, \cdot \rangle_{\text{SL}_2}$ be the Petersson pairings on $\pi$ and $\tilde{\pi}$ defined by
\[
\langle f_1, f_2 \rangle_{\text{SO}(V)} = \int_{\text{SO}(V)(Q) \backslash \text{SO}(V)(A)} f_1(h) \overline{f_2(h)} \, dh,
\]
\[
\langle f_3, f_4 \rangle_{\text{SL}_2} = \int_{\text{SL}_2(Q) \backslash \text{SL}_2(A)} f_3(g) \overline{f_4(g)} \, dg,
\]
for $f_1, f_2 \in \pi$ and $f_3, f_4 \in \tilde{\pi}$.

**Proposition 5.10.** We have
\[
\theta_v(\tilde{V}_+^{2m} \pi(\xi_{N_2}) \mathbf{f} \otimes \chi_{-D}, \varphi) = -2^{-2m-1} \pi^{-2m}(\sqrt{-1})^{D-\kappa+m+1/2} c_h(D) \xi_{Q}(2)^{-1}
\times \left( \text{GL}_2(\mathbb{Z}) : \text{K}_0(N_2 \mathbb{Z}) \right)^{-1} \left( \frac{\Gamma(\kappa - m) \Gamma(2m + 1)}{\Gamma(\kappa - 2m) \Gamma(m + 1)} \right) \langle f, f \rangle_{\text{SO}(V)} h^{-1} \tilde{V}_+^{m} \mathbf{h}_{N_2}.
\]

**Proof.** By [HL70] Lemma 5.6 and its analogue for $\widetilde{\text{SL}}_d(\mathbb{R})$, we have
\[
\langle \tilde{V}_+^{2m} \pi(\xi_{N_2}) \mathbf{f}, \tilde{V}_+^{2m} \mathbf{f}_{N_2} \rangle_{\text{SO}(V)} = (4\pi)^{-4m} \frac{\Gamma(2\kappa' + 2m) \Gamma(2m + 1)}{\Gamma(2\kappa')} \langle \mathbf{f}, \mathbf{f} \rangle_{\text{SO}(V)} \prod_{\rho | N_2 / N_1} \langle \rho(\xi_{N_2,p})W_{f,p}, \rho(\mathcal{F}_{N_2,p})W_{f,p} \rangle_{1,p}
\]
\[
\langle \tilde{V}_+^{m} \mathbf{h}_{N_2}, \tilde{V}_+^{m} \mathbf{h} \rangle_{\text{SL}_2} = (4\pi)^{-2m} \frac{\Gamma(\kappa' + m + 1/2) \Gamma(m + 1)}{\Gamma(\kappa' + 1/2)} \langle \mathbf{h}_{N_2}, \mathbf{h} \rangle_{\text{SL}_2} \prod_{\rho | N_2 / N_1} \langle \rho(\mathcal{F}_{N_2,p})W_{D,p}^{(0)}, W_{D,p}^{(0)} \rangle_{2,p}
\]
\[
= (4\pi)^{-2m} \frac{\Gamma(\kappa' + m + 1/2) \Gamma(m + 1)}{\Gamma(\kappa' + 1/2)} \langle \mathbf{h}_{N_2}, \mathbf{h} \rangle_{\text{SL}_2} \prod_{\rho | N_2 / N_1} \langle \rho(\mathcal{F}_{N_2,p})W_{D,p}^{(0)}, W_{D,p}^{(0)} \rangle_{2,p}.
\]

25
Note that \((f, f)_{SO(V)} = \xi_0(2)^{-1} \langle f, f \rangle\) and \((h, h)_{SL_2} = 2^{-1} \xi_0(2)^{-1} \langle h, h \rangle\). We conclude from the identities above and Lemma 3.4 that

\[
\langle \tilde{V}_+^{2m} \pi(\xi_{N_2}) f, \tilde{V}_+^{2m} f_{N_2} \rangle_{SO(V)} \left( \tilde{V}_+^{m} h_{N_2}, \tilde{V}_+^{m} h \right)_{SL_2}^{-1} = 2(2\pi)^{-2m} \left( \frac{\Gamma(k-m)\Gamma(2m+1)}{\Gamma(k-2m)\Gamma(m+1)} \right) \langle f, f \rangle \langle h, h \rangle^{-1}.
\]

By [LCh05] (9.1)-(9.3), (5.3), (5.6), and Propositions 5.4, there exists a constant \(C\) such that

\[
\theta_{\psi}(\tilde{V}_+^{2m} \pi(\xi_{N_2}) f \otimes \chi_{-D}, \varphi) = C' \langle \tilde{V}_+^{m} h_{N_2}, \tilde{V}_+^{m} h \rangle_{SL_2}^{-1}.
\]

Hence

\[
\langle \theta_{\psi}(\tilde{V}_+^{2m} \pi(\xi_{N_2}) f \otimes \chi_{-D}, \varphi), \tilde{V}_+^{m} h \rangle_{SL_2} = \langle \theta_{\psi}(\tilde{V}_+^{2m} \pi(\xi_{N_2}) f, \tilde{V}_+^{2m} f_{N_2}) \rangle_{SO(V)}.
\]

On the other hand, by Proposition 5.8,

\[
\langle \theta_{\psi}(\tilde{V}_+^{2m} \pi(\xi_{N_2}) f \otimes \chi_{-D}, \varphi), \tilde{V}_+^{m} h \rangle_{SL_2} = \langle \tilde{V}_+^{2m} \pi(\xi_{N_2}) f, \tilde{V}_+^{m} f_{N_2} \rangle_{SO(V)}
\]

\[
= -2^{-2} (\sqrt{-1}) D^{-\kappa + m + 1/2} \langle \chi h(D) \xi_0(2) \rangle^{-1} \left( \text{GL}_2(\hat{\mathbb{Z}}) : K_0(N_2 \hat{\mathbb{Z}}) \right)^{-1} \langle \tilde{V}_+^{2m} \pi(\xi_{N_2}) f, \tilde{V}_+^{2m} f_{N_2} \rangle_{SO(V)}.
\]

Therefore, by (5.9)

\[
C' = -2^{-2m-1} (\sqrt{-1}) D^{-\kappa + m + 1/2} \langle \chi h(D) \xi_0(2) \rangle^{-1}
\]

\[
\times \left( \text{GL}_2(\hat{\mathbb{Z}}) : K_0(N_2 \hat{\mathbb{Z}}) \right)^{-1} \left( \frac{\Gamma(k-m)\Gamma(2m+1)}{\Gamma(k-2m)\Gamma(m+1)} \right) \langle f, f \rangle \langle h, h \rangle^{-1}.
\]

This completes the proof.

6. Saito-Kurokawa lifts

6.1. Setting. We keep the notation of §5.1 except for the definition of the quadratic space \((V, Q)\).

Let \((V, Q)\) be a quadratic space over \(\mathbb{Q}\) defined as follows:

\[
V = \left\{ x \in M_4(\mathbb{Q}) \mid x \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^t x \right\}
\]

\[
Q[x] = 4^{-1} \text{tr}(x^2).
\]

We identify \(V\) with the space of column vectors \(\mathbb{Q}^5\) via

\[
\mathbb{Q}^5 \rightarrow V
\]

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \rightarrow \begin{pmatrix} x_3 & x_2 & 0 & x_1 \\ x_4 & -x_3 & -x_1 & 0 \\ 0 & x_5 & x_3 & x_4 \\ -x_5 & 0 & x_2 & -x_3 \end{pmatrix}.
\]

Recall we have an exact sequence

\[
1 \rightarrow \mathbb{G}_m \rightarrow \text{GSp}_2 \xrightarrow{\rho} \text{SO}(V) \rightarrow 1,
\]

here \(\iota(a) = a1_4\) and \(\rho(h)x = h x h^{-1}\) for \(a \in \mathbb{G}_m, h \in \text{GSp}_2, \) and \(x \in V\). Via the exact sequence, we identify automorphic forms on \(\text{SO}(V(\mathbb{A}))\) with automorphic forms on \(\text{PGSp}_2(\mathbb{A})\). Let \(\psi\) (resp. \(\psi_v\)) be the standard additive character of \(\mathbb{A}\) (resp. \(\mathbb{Q}_v\)). Let \(\omega_{\psi_v} = \omega_{\psi, v, \psi, v, 1}\) (resp. \(\omega_{\psi_v} = \omega_{\psi, v, 1}\)) denote the Weil representation of \(\text{SL}_2(\mathbb{A}) \times O(V)(\mathbb{A})\) (resp. \(\text{SL}_2(\mathbb{Q}_v) \times O(V)(\mathbb{Q}_v)\)) on \(S(V(\mathbb{A}))\) (resp. \(S(V(\mathbb{Q}_v))\)) defined in (1.1).

Let \(\xi \in \mathbb{Q}^X\). Let \(W_{\xi, v} \) be the \(\xi\)-th Fourier coefficient of \(h\) defined as in [LCh05] §3.2. If \(\xi \in \mathbb{Q}(\pi)\), let \(W_{\xi, v}, W_{\xi, v}^{(i)} \in \mathcal{W}(\pi_v, \psi_v^i)\) be the Whittaker functions of \(\pi_v\) with respect to \(\psi_v^i\) defined as in (3.1), (3.4), (3.7), and (3.9). Let \(\Psi_p(X; \alpha_p)\) be the rational functions defined in (3.2), (3.5), and (3.8). In this case,
write $\xi = \partial \xi^2$ with $\partial \xi \in \mathbb{N}$, $\xi \in \mathbb{Q}_+^*$ so that $-\partial \xi$ is the fundamental discriminant of $\mathbb{Q}(\sqrt{-\xi})/\mathbb{Q}$. By our normalizations on Whittaker functions, we have

$$W_{\xi,p}(t(\xi^{-1})) = |\xi|_{\mathbb{Q}_p}^{-1/2} \mathfrak{f}_p(\xi, \psi_p)$$

for all $p$. Therefore, for all $\xi \in \mathbb{Q}_+^*$, we have

$$c_h(\xi) = c_h(\partial \xi)^{\xi_0^{-1/2}} \prod_p \Psi_p(\xi; \alpha_p)$$

and

$$(6.1) \quad W_{h,\xi} = \begin{cases} c_h(\partial \xi)^{\xi_0^{-1/2}} \prod_v W_{\xi,v} & \text{if } \xi \in \mathbb{Q}(\pi), \\ 0 & \text{if } \xi \notin \mathbb{Q}(\pi). \end{cases}$$

Let $F \in S_{N+1}(\Gamma_0^{(2)}(N_1))$ be the classical Saito-Kurokawa lift defined by

$$F(Z) = \sum_{B>0} A(B)e^{2\pi \sqrt{-\text{Tr}(BZ)}},$$

for $Z \in \mathcal{O}_2$, where $B$ runs over all positive definite half-integral symmetric matrices of size 2, and

$$A(B) = \sum_{d \mid (b_1, b_2, b_3, d, N_1) = 1} d^{n^2} c_h \left( \frac{4b_1b_3 - b_2^2}{d^2} \right)$$

for $B = \begin{pmatrix} b_1 & b_2/2 \\ b_2/2 & b_3 \end{pmatrix}$ with $b_1, b_2, b_3 \in \mathbb{Z}$, $b_1 > 0$, and $4b_1b_3 - b_2^2 > 0$.

We define a variant of the classical Saito-Kurokawa lift $F$ as follows: Let $B$ be a half-integral symmetric matrix of size 2 and put $\xi = \det(B)$. Recall $\lambda_p(f) = p^{1/2}(\alpha_p + \alpha_p^{-1})$ for $p \nmid N_1$. For $p \mid N_2/N_1$, define $A_f(p; B) \in \mathbb{Q}(\lambda_p(f))$ by

(i) If $p \nmid N_2$ and $p \nmid N_1$, then

$$\Psi_p(\xi; \alpha_p) A_f(p; B) = (1 + p + \lambda_p(f)) \sum_{n=0}^{\min(\text{ord}_{\mathbb{Q}_p}(b_1))} p^{n/2} \Psi_p(p^{-2n}\xi; \alpha_p) - p^{1/2} \Psi_p(p^2\xi; \alpha_p) - p \Psi_p(\xi; \alpha_p).$$

(ii) If $p^2 \nmid N_2$ and $\min(\text{ord}_{\mathbb{Q}_p}(b_1)) = 0$, then

$$\Psi_p(\xi; \alpha_p) A_f(p; B) = -2p^{1/2} \Psi_p(p^2\xi; \alpha_p) + 2\lambda_p(f) \Psi_p(\xi; \alpha_p) + p^{1/2}(p-1)\Psi_p(p^{-2}\xi; \alpha_p)$$

$$+ \begin{cases} (1 - (p, -\xi)_{\mathbb{Q}_p}) & \text{if ord}_{\mathbb{Q}_p}(\xi) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) If $p^2 \nmid N_2$ and $\min(\text{ord}_{\mathbb{Q}_p}(b_1)) \geq 1$, then

$$\Psi_p(\xi; \alpha_p) A_f(p; B) = (2p\lambda_p(f) + p^2 - 1) \sum_{n=0}^{\min(\text{ord}_{\mathbb{Q}_p}(b_1))} p^{n/2} \Psi_p(p^{-2n}\xi; \alpha_p)$$

$$- p^{1/2}(p + 1) \Psi_p(p^2\xi; \alpha_p) - (p - 1)\lambda_p(f) + 3p^2 - 1)\Psi_p(\xi; \alpha_p) - p^{3/2}(p-1)\Psi_p(p^{-2}\xi; \alpha_p)$$

$$+ \begin{cases} p^{\min(\text{ord}_{\mathbb{Q}_p}(b_1))}/2(p-1)(p, -\xi)_{\mathbb{Q}_p} & \text{if ord}_{\mathbb{Q}_p}(\xi) = 2\min(\text{ord}_{\mathbb{Q}_p}(b_1)), \\ 0 & \text{otherwise.} \end{cases}$$

Let $F_{N_2} \in S_{N+1}(\Gamma_0^{(2)}(N_2))$ defined by

$$F_{N_2}(Z) = \sum_{B>0} A_{N_2}(B)e^{2\pi \sqrt{-\text{Tr}(BZ)}},$$

for $Z \in \mathcal{O}_2$, where $B$ runs over all positive definite half-integral symmetric matrices of size 2, and

$$(6.2) \quad A_{N_2}(B) = \prod_{p \mid N_2/N_1} A_f(p; B) \sum_{d \mid (b_1, b_2, b_3, d, N_2) = 1} d^{n^2} c_h \left( \frac{4b_1b_3 - b_2^2}{d^2} \right)$$

for $B = \begin{pmatrix} b_1 & b_2/2 \\ b_2/2 & b_3 \end{pmatrix}$ with $b_1, b_2, b_3 \in \mathbb{Z}$, $b_1 > 0$, and $4b_1b_3 - b_2^2 > 0$. Let $F_{N_2}$ be the cusp form on $\text{GSp}_2(A)$ associated to $F_{N_2}$ defined as in [Tol05 §7.2].
Let $\tilde{sp}$ be the imaginary part of $Z$. For $r \in \mathbb{N}$, let $\Delta_r$ be the Maass differential operator defined by

$$
\Delta_r = \frac{1}{32\pi^2} \left[ r(2r-1) \det(Y)^{-1} - 8 \frac{\partial^2}{\partial \tau_1 \partial \tau_2} + 2 \frac{\partial^2}{\partial z^2} + (4r-2) \sqrt{-1} \det(Y)^{-1} \left( y_1 \frac{\partial}{\partial \tau_1} + y_2 \frac{\partial}{\partial \tau_2} + v \frac{\partial}{\partial z} \right) \right].
$$

Note that $\Delta_r$ sends a nearly holomorphic Siegel modular form $\tilde{F}$ of degree 2 and weight $r$ to a nearly holomorphic Siegel modular form of degree 2 and weight $r + 2$. Moreover, if $\tilde{F}$ is holomorphic with Fourier expansion

$$
\tilde{F}(Z) = \sum_{B \geq 0} A_\tilde{F}(B) e^{2\pi i \text{tr}(BZ)},
$$

then a routine induction argument shows that the Fourier expansion of $\Delta_r^m \tilde{F}$ is given by

$$
\Delta_r^m \tilde{F}(Z) = \sum_{B \geq 0} A_\tilde{F}(B) e^{2\pi i \text{tr}(BZ)} \times \sum_{j=0}^{m} (-4\pi)^{-m-j} \Gamma(r + m - 1/2) \Gamma(r + j - 1/2) \left( \frac{m!}{j!} \right) \det(B)^j \det(Y)^{j-m} \times \sum_{i=0}^{m-j} \frac{(2m - 2j - i)!}{i!(m - j)!} (4\pi)^{i+j-m} \sum_{l=0}^{i} \frac{(r + 2m)!(-4\pi)^{-l}}{(r + 2m - l)!} \left( \frac{r}{l} \right) \text{tr}(BY)^{i-l},
$$

for $Z = X + \sqrt{-1}Y \in \mathfrak{h}_2$, here $B$ runs over all positive definite half-integral symmetric matrices of size 2.

Let $\mathfrak{sp}(2, \mathbb{R})_\mathbb{C}$ be the complexified Lie algebra of $\text{Sp}_2(\mathbb{R})$ and $\mathcal{U}(\mathfrak{sp}(2, \mathbb{R})_\mathbb{C})$ be its universal enveloping algebra. As in [PSS16], define $X_+, P_{0+}, P_{1+}, D_+ \in \mathcal{U}(\mathfrak{sp}(2, \mathbb{R})_\mathbb{C})$ by

$$
X_+ = \frac{1}{2} \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) \otimes \frac{1}{2} \left( \begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) \otimes \sqrt{-1},
$$

$$
P_{0+} = \frac{1}{2} \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{array} \right) \otimes \frac{1}{2} \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array} \right) \otimes \sqrt{-1},
$$

$$
P_{1+} = \frac{1}{2} \left( \begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{array} \right) \otimes \frac{1}{2} \left( \begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array} \right) \otimes \sqrt{-1},
$$

$$
D_+ = P_{1+}^2 - 4X_+ P_{0+}.
$$

Let $\tilde{D}_+ \in \mathcal{U}(\mathfrak{sp}(2, \mathbb{R})_\mathbb{C})$ defined by

$$
\tilde{D}_+ = -\frac{1}{64\pi^2} D_+.
$$

Note that $\tilde{D}_+^m \tilde{F}$ is the adelic lift of $\Delta_r^m \tilde{F}$ (cf. [PSS16] Remark 3.12]), here $\tilde{F}$ is the automorphic form on $\text{GSp}_2(\mathbb{A})$ associated to $\tilde{F}$.

The main result of this section is Proposition 6.6 which asserts that $\Delta_r^m \delta_{\kappa+1/2}^N F_{N_2}$ is a global theta lift of $\delta_{\kappa+1/2}^N N_2$.

### 6.2. Local theta lifts

Define $\varphi = \otimes_v \varphi_v \in S_\varphi(V(\mathbb{A}))$ as follows:

- If $v = p$ with $p \nmid N_2$, then $\varphi_v$ is the function defined in [Ich05 §7.2].
- If $v = p$ with $p \mid N_2$, then

$$
\varphi_p(x) = I_{\mathbb{Z}_p}(x_1)I_{\mathbb{Z}_p}(x_2)I_{\mathbb{Z}_p}(x_3)I_{\mathbb{Z}_p}(x_4)I_{\mathbb{Z}_p}(x_5).
$$

- If $v = p$ with $p^2 \mid N_2$, then

$$
\varphi_p(x) = I_{\mathbb{Z}_p}(x_1)I_{\mathbb{Z}_p}(x_2)I_{\mathbb{Z}_p}(x_3)I_{\mathbb{Z}_p}(x_4)I_{\mathbb{Z}_p}(x_5).
$$
\documentclass{article}
\usepackage{amsmath}
\begin{document}
\begin{itemize}
\item If $v = \infty$, then
\begin{equation*}
\varphi_{\infty}(x) = (x_2 + \sqrt{-1}x_1 + \sqrt{-1}x_5 - x_4)^{\kappa+1}e^{-\pi^2tr(x^t x)}.
\end{equation*}
Note that if $v = p$ with $p \parallel N_2$, then
\begin{equation}
\omega_{\psi_x}((k, s_p(k)), k') \varphi_p = \varphi_p
\end{equation}
for $k \in K_0(p)$, $k' \in K_0^{(2)}(p)$.
If $v = p$ with $p^2 \mid N_2$, then
\begin{equation}
\omega_{\psi_x}((k, s_p(k)), k') \varphi_p = \varphi_p
\end{equation}
for $k \in K_0(p^2)$, $k' \in K_0^{(2)}(p^2)$.
If $v = \infty$, then
\begin{equation}
\omega_{\psi_{\infty}}(\tilde{k}_\theta, k') \varphi_{\infty} = e^{-\sqrt{-1}(\kappa+1/2)\theta} \det(k)^{\kappa+1} \varphi_{\infty}
\end{equation}
for $\tilde{k}_\theta \in \text{SO}(2)$,
\begin{equation*}
k' = \begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix} \in \text{Sp}_2(\mathbb{R})
\end{equation*}
with $k = \alpha + \sqrt{-1}\beta \in U(2)$.
Let $V_1 = \{x \in V \mid x_1 = x_5 = 0\}$ be a quadratic subspace of $V$. For each place $v$, let
\begin{equation*}
S(V(Q_v)) \to S(V_1(Q_v)) \otimes S(Q_v^2)
\end{equation*}
\begin{equation*}
\varphi \mapsto \tilde{\varphi}
\end{equation*}
be the partial Fourier transform defined by
\begin{equation*}
\tilde{\varphi}(x; y) = \int_{Q_v} \varphi \begin{pmatrix}
z \\
x \\
y_1
\end{pmatrix} \psi_v(-y_2z) dx
\end{equation*}
for $x \in V_1(Q_v), y = (y_1, y_2) \in Q_v^2$. Let $\tilde{\omega}_{\psi_v}$ be the representation of $\widetilde{\text{SL}_2(Q_v)} \times \text{O}(V)(Q_v)$ on $S(V_1(Q_v)) \otimes S(Q_v^2)$ defined by
\begin{equation*}
\omega_{\psi_v}(g, h) \tilde{\varphi} = (\omega_{\psi_v}(g, h) \varphi).
\end{equation*}
Note that
\begin{equation*}
\omega_{\psi_v}(t(a), 1) \varphi(x_1; y) = \gamma_{Q_v}(a, \psi_v)^{-1}a^{3/2} \varphi(ax_1; ay_1, a^{-1}y_2)
\end{equation*}
for $a \in Q_v^\times$. If $\tilde{\varphi} = \varphi_1 \otimes \varphi_2$ with $\varphi_1 \in S(V_1(Q_v))$ and $\varphi_2 \in S(Q_v^2)$, then
\begin{equation}
\omega_{\psi_v}((g, \epsilon), 1) \tilde{\varphi}(x_1; y) = \omega_{\psi_v}((g, \epsilon), 1) \varphi_1(x_1) \cdot \varphi_2(yg)
\end{equation}
for $(g, \epsilon) \in \widetilde{\text{SL}_2(Q_p)}$.
Let
\begin{equation*}
B = \begin{pmatrix}
b_1 & b_2/2 \\
b_2/2 & b_3
\end{pmatrix} \in \text{Sym}_2(Q_v)
\end{equation*}
and $\xi = \det(B)$. Put
\begin{equation*}
\beta = \begin{pmatrix}
b_1 \\
b_2/2 \\
-b_3
\end{pmatrix} \in V_1(Q_v).
\end{equation*}
From now on until the end of this subsection, we assume $\xi \neq 0$ and $\psi_v^\xi$ is not a missing character of $\tilde{\pi}_v$.
We have an equivariant map
\begin{equation*}
S_{\psi_v}(V(Q_v)) \otimes W(\tilde{\pi}_v, \psi_v^\xi) \to W_{\psi_v^B}(\varphi) \otimes W \mapsto W_{B, \psi}(\varphi, W),
\end{equation*}
\end{itemize}
\end{document}
\[ W_{B,v}(\varphi, W)(h) = \int_{U(Q_v)\backslash SL_2(Q_v)} \hat{\omega}_{\psi_v}(g, h) \hat{\varphi}(\beta; 0, 1) W(g) dg. \]

Here \( W_v(\psi_v) \) is the space consisting of smooth functions \( W : \text{GSp}_2(Q_v) \to \mathbb{C} \) such that
\[
W \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} X \begin{pmatrix} 12 \\ 0 \end{pmatrix} \right) = \psi_v(\text{tr}(BX))W(h)
\]
for \( h \in \text{GSp}_2(Q_v) \) and \( X \in \text{Sym}_2(Q_v) \).

**Remark 4.** If \( W_{B,v} \neq 0 \), then the image of the equivariant map is irreducible and is a Bessel model for the local component at \( v \) of the automorphic representation generated by \( F \).

**Lemma 6.1.** Let \( v = p \). Let \( c \in \mathbb{Z}_{>0} \), \( a \in \mathbb{Z}_p \), and \( \varphi^{(c)}(x) = \mathbb{I}_p(x)_{\mathbb{I}_p}(x_1)_{\mathbb{I}_p}(x_2)_{\mathbb{I}_p}(x_3)_{\mathbb{I}_p}(x_4)_{\mathbb{I}_p}(x_5) \).
We have
\[
\hat{\omega}_{\psi_p}(w, 1) \hat{\varphi}_p(x; y_1, y_2) = \mathbb{I}_p(x_1)_{\mathbb{I}_p}(x_2)_{\mathbb{I}_p}(x_3)_{\mathbb{I}_p}(y_1)_{\mathbb{I}_p}(y_2),
\hat{\omega}_{\psi_p}(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, 1) \hat{\varphi}_p(x; y_1, y_2) = \mathbb{I}_p(x_1)_{\mathbb{I}_p}(x_2)_{\mathbb{I}_p}(x_3)_{\mathbb{I}_p}(y_1 + ay_2)_{\mathbb{I}_p}(y_2).
\]

**Proof.** Note that
\[
\hat{\varphi}_p(x; y_1, y_2) = \mathbb{I}_p(x_1)_{\mathbb{I}_p}(x_2)_{\mathbb{I}_p}(x_3)_{\mathbb{I}_p}(y_1)_{\mathbb{I}_p}(y_2).
\]
The assertion follows from (6.9) and the formula above. \( \square \)

**Proposition 6.2.** Let \( v = p \mid 2N_2 \). We have
\[
W_{B,p}(\varphi_p, W_{\xi,p})(1) = \begin{cases} \sum_{n=0}^{\min(\text{ord}_{\mathbb{Q}_p}(b_i))} p^{n/2} \Psi_p(p^{-2n} \xi; \alpha_p) & \text{if } b_1, b_2, b_3 \in \mathbb{Z}_p, \\
0 & \text{otherwise}. \end{cases}
\]

**Proof.** \([\text{Ich05}, \S 7.3]\). \( \square \)

**Proposition 6.3.** Let \( v = 2 \). We have
\[
W_{B,2}(\varphi_2, W_{\xi,2})(1) = \begin{cases} 2^{-7/2} \sum_{n=0}^{\min(\text{ord}_{\mathbb{Q}_p}(b_i))} 2^{n/2} \Psi(2^{-2n+2} \xi; \alpha_2) & \text{if } b_1, b_2, b_3 \in \mathbb{Z}_2, \\
0 & \text{otherwise}. \end{cases}
\]

**Proof.** \([\text{Ich05}, \text{Lemma 7.3}]\). \( \square \)

**Proposition 6.4.** If \( v = p \mid N_1 \), then we have
\[
W_{B,p}(1) = \begin{cases} (1 + p)^{-1} \Psi_p(\xi; \alpha_p) & \text{if } b_1, b_2, b_3 \in \mathbb{Z}_p, \\
0 & \text{otherwise}. \end{cases}
\]
If \( v = p \mid p \nmid 2N_2 \), then we have
\[
W_{B,p}(\varphi_p, \rho(\mathring{F}_{N_2,p})W_{\xi,p})(1) = \begin{cases} (1 + p)^{-1} \Psi_p(\xi; \alpha_p)A_f(p; B) & \text{if } b_1, b_2, b_3 \in \mathbb{Z}_p, \\
0 & \text{otherwise}. \end{cases}
\]
If \( v = p \mid p^2 \mid N_2 \), then we have
\[
W_{B,p}(\varphi_p, \rho(\mathring{F}_{N_2,p})W_{\xi,p})(1) = \begin{cases} p^{-1}(1 + p)^{-1} \Psi_p(\xi; \alpha_p)A_f(p; B) & \text{if } b_1, b_2, b_3 \in \mathbb{Z}_p, \\
0 & \text{otherwise}. \end{cases}
\]

**Proof.** We only consider the case when \( p^2 \mid N_2 \) and \( p \mid N_1 \). The other cases are similar.

For \( k \in \text{SL}_2(\mathbb{Z}_p) \) and \( W \in W(\mathring{\pi}_p, \psi_p^\xi) \), let
\[
J(k, W) = \sum_{n \in \mathbb{Z}} p^{n/2} \gamma_{\mathbb{Q}_p}(p^n, \psi_p^{-1}) \hat{\omega}_{\psi_p}(k, 1) \hat{\varphi}_p(p^n \beta; 0, p^{-n}) W(t(p^n)k).
\]
Since
\[
\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = n(u^{-1}) \begin{pmatrix} -u & -1 \\ 0 & -u^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = n(-y)w
\]
for $u \in \mathbb{Z}_p^\times$ and $y \in \mathbb{Z}_p$, by Lemma 6.1

$$J(w, W) = J \left( \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, W \right) = J \left( w \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, W \right)$$

for $u \in \mathbb{Z}_p^\times$, $y \in \mathbb{Z}_p$, and $W \in \mathcal{W}(\tilde{\pi}_p, \psi_p)^{K_0(p^2)}$. Therefore,

$$\mathcal{W}_{B,p}(\varphi_p, W)(1) = p^{-1}(1 + p)^{-1} \sum_{SL_2(\mathbb{Z}_p) / K_0(p^2)} J(k, W)$$

(6.10)

$$= p^{-1}(1 + p)^{-1} \left[ J(1, W) + p^2 J(w, W) + \sum_{u \in (\mathbb{Z}_p/p^2) \times} J \left( \begin{pmatrix} 1 & 0 \\ pu & 1 \end{pmatrix}, W \right) \right]$$

for $W \in \mathcal{W}(\tilde{\pi}_p, \psi_p)^{K_0(p^2)}$. Furthermore, if $b_i \notin \mathbb{Z}_p$ for some $i$, by Lemma 6.1

$$\mathcal{W}_{B,p}(\varphi_p, W)(1) = 0$$

for $W \in \mathcal{W}(\tilde{\pi}_p, \psi_p)^{K_0(p^2)}$.

Assume $b_1, b_2, b_3 \in \mathbb{Z}_p$. By Lemmas 3.4 and 6.1

$$J(1, W_{\xi, p}^{(0)})(1) = \sum_{n=0}^{\min(\text{ord}_{\psi_p}(b_i))} p^{n/2} \Psi_p(p^{-2n} \xi; \alpha_p),$$

$$J(1, W_{\xi, p}^{(1)})(1) = p \sum_{n=0}^{\min(\text{ord}_{\psi_p}(b_i))} p^{n/2} \Psi_p(p^{-2n+2} \xi; \alpha_p),$$

$$J(1, W_{\xi, p}^{(2)})(1) = p \sum_{n=0}^{\min(\text{ord}_{\psi_p}(b_i))} p^{n/2} \Psi_p(p^{-2n-2} \xi; \alpha_p),$$

$$J(1, W_{\xi, p}^{(3)})(1) = \begin{cases} (p_\alpha - \xi)_p - 1(1 + p^{-1/2} p^{-1})^{-1} p^{\min(\text{ord}_{\psi_p}(b_i))}/2 & \text{if } \text{ord}_{\psi_p}(\xi) = 2 \min(\text{ord}_{\psi_p}(b_i)), \\
0 & \text{otherwise}. \end{cases}$$

By Lemmas 3.5 and 6.1

$$J(w, W_{\xi, p}^{(0)})(1) = \sum_{n=2}^{\min(\text{ord}_{\psi_p}(b_i))} p^{n/2} \Psi_p(p^{-2n} \xi; \alpha_p),$$

$$J(w, W_{\xi, p}^{(1)})(1) = p(\alpha_p + \alpha_p^{-1}) \sum_{n=2}^{\min(\text{ord}_{\psi_p}(b_i))} p^{n/2} \Psi_p(p^{-2n} \xi; \alpha_p) - \sum_{n=2}^{\min(\text{ord}_{\psi_p}(b_i))} p^{n/2} \Psi_p(p^{-2n+2} \xi; \alpha_p),$$

$$J(w, W_{\xi, p}^{(2)})(1) = p^{-1} \sum_{n=2}^{\min(\text{ord}_{\psi_p}(b_i))} p^{n/2} \Psi_p(p^{-2n+2} \xi; \alpha_p),$$

$$J(w, W_{\xi, p}^{(3)})(1) = -(1 - p^{-1}) \sum_{n=2}^{\min(\text{ord}_{\psi_p}(b_i))} p^{n/2} \Psi_p(p^{-2n} \xi; \alpha_p)$$

$$+ (1 - p^{-1}) (p^{-1} + p^{-1/2} \alpha_p^{-1}) \sum_{n=2}^{\min(\text{ord}_{\psi_p}(b_i))} p^{n/2} \Psi_p(p^{-2n} \xi; \alpha_p).$$
By Lemmas 3.6 and 6.1,
\[
\sum_{u \in (\mathbb{Z}_p/p\mathbb{Z})^\times} \mathcal{J} \left( \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \right), W^{(2)}_{\xi,p} \right) = p^{n/2} \mathcal{G}_{p^{-1} - 2n \xi}(p) \Psi_p(p^{-2n \xi}; \alpha_p)
\]
\[
\sum_{u \in (\mathbb{Z}_p/p\mathbb{Z})^\times} \mathcal{J} \left( \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \right), W^{(3)}_{\xi,p} \right) = p^{1/2}(p - 1) \alpha^{-1} \sum_{n=1}^{\min(\text{ord}_p(b_i))} p^{n/2} \Psi_p(p^{-2n \xi}; \alpha_p)
\]
\[
+ \sum_{n=1}^{\min(\text{ord}_p(b_i))} p^{n/2} \mathcal{G}_{p^{-1} - 2n \xi}(p) \Psi_p(p^{-2n \xi}; \alpha_p)
\]
\[
\times \left[ -(p - 1)(1 + p^{1/2} \alpha^{-1}) + p \mathcal{G}_{p^{-1} - 2n \xi}(p) \mathcal{G}_{p^{-1} - 2n \xi}(p) - p^{1/2} \mathcal{G}_{p^{-1} - 2n \xi}(p) \psi^{-1} \mathcal{G}_{p^{-1} - 2n \xi}(p) \right].
\]
Since
\[
\rho(\tilde{\mathcal{F}}_{N_2} p) W^{(0)}_{\xi,p} - 2 \lambda_p(f) W^{(0)}_{\xi,p} + (p^{1/2} - p^{-1/2}) W^{(2)}_{\xi,p} + (-1 + p^{-1/2} \alpha_p) W^{(3)}_{\xi,p}
\]
the assertion follows from (3.10) and the above formulas. This completes the proof. □

Proposition 6.5. Let \( v = \infty \). Let \( A \in GL_2^{\pm}(\mathbb{R}) \) and \( X \in \text{Sym}_2(\mathbb{R}) \). Put \( Y = A^t A \) and \( Z = X + \sqrt{-1} Y \).

If \( B > 0 \), then
\[
\mathcal{W}_{B,\infty}(\varphi_\infty, \tilde{V}_+^m W_{\xi,\infty}) \left( \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \right) = 2^{\kappa+1} \det(Y)^{\kappa+1/2} e^{2\pi \sqrt{-1} \text{tr}(BZ)}
\]
\[
\times \sum_{j=0}^{m} (-4\pi)^{j-m} \frac{\Gamma(\kappa - m + 1/2)}{\Gamma(\kappa - 2m + 1/2 + j)} (m)_j! (j)_j! \xi_j \det(Y)^{-j} \frac{1}{j!} \text{tr}(BY)^{-j}. \]

If \( B < 0 \), then
\[
\mathcal{W}_{B,\infty}(\varphi_\infty, \tilde{V}_+^m W_{\xi,\infty}) \left( \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \right) = 0.
\]

Proof. As in [IaOt 76] Lemma 7.6], we have
\[
\mathcal{W}_{B,\infty}(\varphi_\infty, \tilde{V}_+^m W_{\xi,\infty}) \left( \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \right) \varphi_\infty(\beta; 0, 1)
\]
\[
eq e^{-\sqrt{-1}(\kappa+1/2)\theta} (2\sqrt{-1})^{-\kappa} \det(A)_{\infty}^{-1} \left[ a^3/\sqrt{-1} H_{\kappa+1} (\sqrt{-1} \det(A)^{-1} \text{tr}(BY) a + \det(A) a^{-1}) \right]
\]
\[
\times e^{2\pi \sqrt{-1} \text{tr}(BZ)} e^{2\pi \text{tr}(BY)} e^{-\pi (\det(A)^{-1} \text{tr}(BY) a + \det(A) a^{-1})^2 e^{2\pi \xi \beta} \beta},
\]
for \( a \in \mathbb{R}_+^n \), and \( \theta \in \mathbb{R}/4\pi \mathbb{Z} \). Therefore, by (3.11) and Lemma 5.5,
\[
\mathcal{W}_{B,\infty}(\varphi_\infty, \tilde{V}_+^m W_{\xi,\infty}) \left( \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \right) \varphi_\infty(\beta; 0, 1)
\]
\[
eq 2^{\kappa+1} \det(Y)^{\kappa+1/2} e^{2\pi \sqrt{-1} \text{tr}(BZ)} e^{2\pi \text{tr}(BY)}
\]
\[
\times \sum_{j=0}^{m} (-4\pi)^{j-m} \frac{\Gamma(\kappa' + 1/2 + m)}{\Gamma(\kappa' + 1/2 + j)} (m)_j! (j)_j! \xi_j \det(Y)^{-j} \frac{1}{j!} \text{tr}(BY)^{-j}. \]

This completes the proof. □
6.3. Global theta lifts. For $f \in \tilde{\pi}$ and $\phi \in S_{\psi}(V(\mathbb{A}))$, let $\theta_{\psi}(f, \phi)$ be the automorphic form on $SO(V(\mathbb{A}))$ defined by

$$\theta_{\psi}(f, \phi)(h) = \int_{SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A})} \theta_{\psi}(g, h; \phi) f(g) dg.$$ 

Let $\theta_{\psi}(\tilde{\pi})$ be the global theta lift of $\tilde{\pi}$ with respect to $\psi$ defined by

$$\theta_{\psi}(\tilde{\pi}) = \{ \theta_{\psi}(f, \phi) \mid f \in \tilde{\pi}, \phi \in S_{\psi}(V(\mathbb{A})) \}.$$ 

Let $\mathcal{B} = \begin{pmatrix} b_1 & b_2/2 \\ b_2/2 & b_3 \end{pmatrix} \in \text{Sym}_2(\mathbb{Q})$ and $\xi = \det(B)$. Let $W_B$ be the $B$-th Fourier coefficient of $\theta_{\psi}(\tilde{\psi}_m h_{N_2}, \varphi)$ defined by

$$W_B(h) = \int_{\text{Sym}_2(\mathbb{Q}) \backslash \text{Sym}_2(\mathbb{A})} \theta_{\psi}(\tilde{\psi}_m h_{N_2}, \varphi) \left( \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} h \right) \psi(-\text{tr}(BX)) dX.$$ 

By [Ich05, Lemma 4.2] and note that $-1 \notin \mathbb{Q}(\tilde{\pi})$, we have

$$W_B = \begin{cases} c_h(0, \xi) \xi^{k-1/2} \xi_2^{-1} W_{B, \infty}(\varphi_\infty, \tilde{\psi}_m \xi, \infty) \prod_p \mathcal{W}_{B, p}(\varphi_p, \rho(\tilde{F}_{N_2, p}) W_{\xi, p}) & \text{if } \xi \in \mathbb{Q}(\pi), \\
0 & \text{otherwise.} \end{cases}$$

**Proposition 6.6.** We have

$$\theta_{\psi}(\tilde{\psi}_m h_{N_2}, \varphi) = 2^{m-2} \xi_2^{-1} \left( \text{GL}_2(\tilde{\mathbb{Z}}) : K_0(N_2 \tilde{\mathbb{Z}}) \right)^{-1} D_+^m F_{N_2}.$$ 

**Proof.** Since

$$\theta_{\psi}(\tilde{\psi}_m h_{N_2}, \varphi) = \sum_{B \in \text{Sym}_2(\mathbb{Q})} W_B,$$ 

it suffices to show that for $B \in \text{Sym}_2(\mathbb{Q})$ with $\xi = \det(B) \in \mathbb{Q}(\tilde{\pi})$, we have

$$W_B(h_\infty) = 2^{m-2} \xi_2^{-1} \left( \text{GL}_2(\tilde{\mathbb{Z}}) : K_0(N_2 \tilde{\mathbb{Z}}) \right)^{-1} \det(Y)^{(k+1)/2} A_{N_2}(B) e^{2\pi \sqrt{-\text{tr}(BZ)}}$$

$$\times \sum_{j=0}^{m} (-4\pi)^{j-m} \frac{\Gamma(k-m+1/2)}{\Gamma(k-2m+1/2+j)} \frac{m!}{j!} \det(B)^j \det(Y)^{j-m}$$

$$\times \sum_{i=0}^{m-j} \frac{(2m-2j-i)!}{i!(m-j-i)!} (4\pi)^{i+j-m} \sum_{l=0}^{\kappa+1} \frac{(k+1)!(-4\pi)^{-l}}{(k+1-l)!} \left( \frac{i}{l} \right) \text{tr}(BY)^{i-l}.$$ 

for $h_\infty = \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & t^{-1} A^{-1} \end{pmatrix} \in \text{Sp}_2(\mathbb{R})$ with $X \in \text{Sym}_2(\mathbb{R})$ and $A \in \text{GL}_2^+(\mathbb{R})$. By Propositions 6.2, 6.3 both sides are equal to 0 if $B$ is not positive definite or $b_i \notin \mathbb{Z}$ for some $i$. Therefore we may assume $B > 0$ and $b_i \in \mathbb{Z}$ for $i = 1, 2, 3$. By Propositions 6, 7.5

$$W_B(h_\infty) = 2^{-7/2} \xi_2^{-1} W_{B, \infty}(\varphi_\infty, \tilde{\psi}_m \xi, \infty)(h_\infty) \left( \text{GL}_2(\tilde{\mathbb{Z}}) : K_0(N_2 \tilde{\mathbb{Z}}) \right)^{-1}$$

$$\times c_h(0, \xi) \xi^{k-1/2} \prod_{p \mid N_2} A_f(p; B) \prod_{p \mid N_1} \Psi_p(4\xi; \alpha_p) \prod_{p \mid N_2} \sum_{n=0}^{\min(\text{ord}_{\xi}(b_i))} p^{n/2} \Psi_p \left( \frac{4\xi}{p^{2n}} ; \alpha_p \right)$$

$$= 2^{-\kappa-1/2} \xi_2^{-1} W_{B, \infty}(\varphi_\infty, \tilde{\psi}_m \xi, \infty)(h_\infty) \left( \text{GL}_2(\tilde{\mathbb{Z}}) : K_0(N_2 \tilde{\mathbb{Z}}) \right)^{-1} A_{N_2}(B)$$

$$= 2^{m-2} \xi_2^{-1} \left( \text{GL}_2(\tilde{\mathbb{Z}}) : K_0(N_2 \tilde{\mathbb{Z}}) \right)^{-1} \det(Y)^{(k+1)/2} A_{N_2}(B) e^{2\pi \sqrt{-\text{tr}(BZ)}}$$

$$\times \sum_{j=0}^{m} (-4\pi)^{j-m} \frac{\Gamma(k-m+1/2)}{\Gamma(k-2m+1/2+j)} \frac{m!}{j!} \det(B)^j \det(Y)^{j-m}$$

$$\times \sum_{i=0}^{m-j} \frac{(2m-2j-i)!}{i!(m-j-i)!} (4\pi)^{i+j-m} \sum_{l=0}^{\kappa+1} \frac{(k+1)!(-4\pi)^{-l}}{(k+1-l)!} \left( \frac{i}{l} \right) \text{tr}(BY)^{i-l}.$$
This completes the proof.

\[\square\]

7. Jacquet-Langlands-Shimizu Correspondence

7.1. Setting. Let \((V, Q)\) be a quadratic space over \(\mathbb{Q}\) defined as follows:
\[
V = M_2(\mathbb{Q}) = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right| x_1, x_2, x_3, x_4 \in \mathbb{Q} \},
\]
\[
Q[x] = \det(x).
\]
Recall we have an exact sequence
\[
1 \rightarrow \mathbb{G}_m \rightarrow \text{GL}_2 \times \text{GL}_2 \rightarrow \text{GSO}(V) \rightarrow 1,
\]
here \(\iota(a) = (a1_2, a1_2)\) and \(\rho(h_1, h_2)x = h_1xh_2^{-1}\) for \(a \in \mathbb{G}_m, h_1, h_2 \in \text{GL}_2\), and \(x \in V\). Note that \(\nu(\rho(h_1, h_2)) = \det(h_1h_2^{-1})\). Via the exact sequence, we identify automorphic forms on \(\text{GSO}(V(\mathbb{A}))\) with automorphic forms on \(\text{GL}_2(\mathbb{A}) \times \text{GL}_2(\mathbb{A})\) that factors over \(\iota(\mathbb{G}_m(\mathbb{A}))\). Let \(\psi\) (resp. \(\psi_v\)) be the standard additive character of \(\mathbb{A}\) (resp. \(\mathbb{Q}_v\)). Let \(\omega = \omega_{\psi, V, 1}\) (resp. \(\omega_{\psi_v} = \omega_{\psi_v, V, 1}\)) denote the Weil representation of \(\text{SL}_2(\mathbb{A}) \times \text{O}(V(\mathbb{A}))\) (resp. \(\text{SL}_2(\mathbb{Q}_v) \times \text{O}(V(\mathbb{Q}_v))\)) on \(S(V(\mathbb{A}))\) (resp. \(S(V(\mathbb{Q}_v))\)) defined in (4.1). We extend \(\omega_{\psi_v}\) to a representation of \(R(\mathbb{Q}_v)\) on \(S(V(\mathbb{Q}_v))\) as in (4.2).

Let \(N_2 = \prod p \rho_p\) be a positive integer and \(l\) be an even positive integer. Let \(g \in S_i(\Gamma_0(N_2))\) be a normalized newform and \(\hat{\varphi}\) be the cusp form on \(\text{GL}_2(\mathbb{A})\) associated to \(g\) as defined in [Ih03 §3.2]. Let \(\sigma = \otimes_v \sigma_v\) be the cuspidal automorphic representation of \(\text{GL}_2(\mathbb{A})\) generated by \(g\). For \(v = \infty\), let \(c_p \in \mathbb{N}\) be the exponent of the conductor of \(\sigma_p\). For \(v = \infty\), \(\sigma\) is the discrete series representation of weight \(l\). Let \(W_{\sigma_v} \subset W(\sigma_v, \psi_v)\) be the Whittaker functions of \(\sigma_v\) defined in [3.3]. Put \(W_{g,v} = W_{\sigma_v}\).

The main result of this section is Propositions [6.3] and [7.4] which assert that \(g\) and \(g \otimes \hat{\varphi}\) correspond to each other via the global theta lift.

7.2. Local theta lifts. Define \(\varphi = \otimes_v \varphi_v \in S_{\varphi}(V(\mathbb{A}))\) as follows:
- If \(v = p\), then
  \[
  \varphi_p(x) = \mathbb{I}_{Z_p}(x_1)\mathbb{I}_{Z_p}(x_2)\mathbb{I}_{Z_p}(x_3)\mathbb{I}_{Z_p}(x_4).
  \]
- If \(v = \infty\), then
  \[
  \varphi_\infty(x) = (x_1 + \sqrt{-1}x_2 + \sqrt{-1}x_3 - x_4)e^{-\pi \text{tr}(x^tx)}.
  \]
Note that if \(v = p\), then
\[
(7.1) \quad \omega_{\psi_p}(k, (k_1, k_2))\varphi_p = \varphi_p
\]
for \(k, k_1, k_2 \in \mathbb{K}_0(p^{c_p})\) such that \(\det(k) = \det(k_1k_2^{-1})\).

If \(v = \infty\), then
\[
(7.2) \quad \omega_{\psi_\infty}(k_\theta, (k_\theta_1, k_\theta_2))\varphi_\infty = e^{\sqrt{-1}(-\theta + \theta_1 + \theta_2)}\varphi_\infty
\]
for \(k_\theta, k_\theta_1, k_\theta_2 \in \text{SO}(2)\).

For each place \(v\), let
\[
S(V(\mathbb{Q}_v)) \rightarrow S(V(\mathbb{Q}_v))
\]
\[
\varphi \mapsto \hat{\varphi}
\]
be the partial Fourier transform defined by
\[
\hat{\varphi}\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right) = \int_{\mathbb{Q}_v} \varphi\left(\begin{pmatrix} x_1 & y_2 \\ x_3 & y_4 \end{pmatrix}\right) \psi_v(-x_1y_2 + x_2y_4)dy_2dy_4.
\]
Let \(\hat{\omega}_{\psi_v}\) be the representation of \(R(\mathbb{Q}_v)\) on \(S(V(\mathbb{Q}_v))\) defined by
\[
\hat{\omega}_{\psi_v}(g, h)\hat{\varphi} = (\omega_{\psi_v}(g, h)\varphi).
\]
Note that
\[
(7.3) \quad \hat{\omega}_{\psi_v}(g, 1)\hat{\varphi}(x) = \hat{\varphi}(xg)
\]
\[34\]}
for \( g \in \text{SL}_2(\mathbb{Q}_v) \). We have an equivariant map

\[
S_{\psi_v}(V(\mathbb{Q}_v)) \otimes W(\sigma_v, \psi_v) \longrightarrow W(\sigma_v, \psi_v) \otimes W(\sigma_v, \psi_v)
\]

\[\varphi \otimes W \longrightarrow W_v(\varphi, W),\]

\[
W_v(\varphi, W)(h) = \int_{\text{SL}_2(\mathbb{Q}_v)} \omega_{\psi_v}(g', h) \hat{\varphi}(g) W(gg') dg,
\]

with \( g' \in \text{GL}_2(\mathbb{Q}_v) \) such that \( \det(g') = \nu(h) \).

**Proposition 7.1.** Let \( v = p \). We have

\[
W_p(\varphi_p, W_{v,p}) = (\text{GL}_2(\mathbb{Z}_p) : K_0(p^{c_2}))^{-1} W_{v,p} \otimes W_{v,p}.
\]

**Proof.** By (7.1),

\[
W_p(\varphi_p, W_{v,p}) \in W(\sigma_v, \psi_v)^{K_0(p^{c_2})} \otimes W(\sigma_v, \psi_v)^{K_0(p^{c_2})} = CW_{v,p} \otimes W_{v,p}.
\]

It suffices to show that

\[
W_p(\varphi_p, W_{v,p})(1) = (\text{GL}_2(\mathbb{Z}_p) : K_0(p^{c_2}))^{-1}.
\]

Note that \( \hat{\varphi}_p = \varphi_p \). For \( k \in \text{SL}_2(\mathbb{Z}_p) \), let

\[
\mathcal{J}(k) = \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \hat{\varphi}_p(u(x)t(a)k) W_{v,p}(u(x)t(a)k) |a|_p^{-2} dxd^\alpha a.
\]

It is easy to verify that

\[ u(x)t(a)k \in \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^{c_2} \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \]

if and only if \( x \in \mathbb{Z}_p, a \in \mathbb{Z}_p^{c_2}, \) and \( k \in K_0(p^{c_2}) \). Therefore,

\[
\mathcal{J}(k) = \begin{cases} 1 & \text{if } k \in K_0(p^{c_2}), \\ 0 & \text{otherwise}. \end{cases}
\]

We conclude from the above calculation and (7.3) that

\[
W_p(\varphi_p, W_{v,p})(1) = \text{vol}(K_0(p^{c_2}), dk) \sum_{k \in \text{SL}_2(\mathbb{Z}_p)/K_0(p^{c_2})} \mathcal{J}(k)
\]

\[= \text{vol}(K_0(p^{c_2}), dk). \]

This completes the proof. \( \square \)

**Proposition 7.2.** Let \( v = \infty \). We have

\[
W_{\infty}(\varphi_{\infty}, W_{v,\infty}) = 2^l W_{v,\infty} \otimes W_{v,\infty}.
\]

**Proof.** [H0N, p.96]. \( \square \)

### 7.3. Global theta lifts.

For \( f \in \sigma \) and \( \phi \in S_\psi(V(\mathbb{A})) \), let \( \theta_\psi(f, \phi) \) be the automorphic form on \( \text{GSO}(V)(\mathbb{A}) \) defined by

\[
\theta_\psi(f, \phi)(h) = \int_{\text{SL}_2(\mathbb{Q}) \backslash \text{SL}_2(\mathbb{A})} \theta_\psi(gg', h; \phi) f(gg') dg
\]

where \( g' \in \text{GL}_2(\mathbb{A}) \) such that \( \det(g') = \nu(h) \). Let \( \theta_\psi(\sigma) \) be the global theta lift of \( \sigma \) with respect to \( \psi \) defined by

\[
\theta_\psi(\sigma) = \{ \theta_\psi(f, \phi) \mid f \in \sigma, \phi \in S_\psi(V(\mathbb{A})) \}.
\]

For \( f \in \sigma \boxtimes \sigma \) and \( \phi \in S_\psi(V(\mathbb{A})) \), let \( \theta_\psi(f, \phi) \) be the automorphic form on \( \text{GL}_2(\mathbb{A}) \) defined by

\[
\theta_\psi(f, \phi)(g) = \int_{\text{SO}(V)(\mathbb{Q}) \backslash \text{SO}(V)(\mathbb{A})} \theta_\psi(g, hh'; \phi) f(hh') dh
\]

where \( h' \in \text{GSO}(V)(\mathbb{A}) \) such that \( \det(g) = \nu(h') \). Let \( \theta_\psi(\sigma \boxtimes \sigma) \) be the global theta lift of \( \sigma \boxtimes \sigma \) with respect to \( \psi \) defined by

\[
\theta_\psi(\sigma \boxtimes \sigma) = \{ \theta_\psi(f, \phi) \mid f \in \sigma \boxtimes \sigma, \phi \in S_\psi(V(\mathbb{A})) \}.
\]
By the results in [Shi72],
\[ \theta_\psi(\sigma) = \sigma \boxtimes \sigma, \quad \theta_\psi(\sigma \boxtimes \sigma) = \sigma. \]

Let \( W \) be the \((1,1)\)-th Fourier coefficient of \( \theta_\psi(g, \varphi) \) defined by
\[
W(h) = \int_{A/\mathfrak{q}} \int_{A/\mathfrak{q}} \theta_\psi(g, \varphi)((u(x), u(y))h)\psi(-x)\psi(-y)dxdy.
\]
By \([108\text{ Lemma 5.1}]\),
\[
W = \xi_Q(2)^{-1} \prod_{\nu} W_\nu(\varphi_\nu, W_{g,\nu}).
\]

**Proposition 7.3.** We have
\[
\theta_\psi(g, \varphi) = 2^t \xi_Q(2)^{-1} \left( \text{GL}_2(\hat{\mathbb{Z}}) : K_0(N_2 \hat{\mathbb{Z}}) \right)^{-1} g \boxtimes g.
\]

*Proof.* By Propositions \([7.2]\) and \([7.2]\) there exists a constant \( C \) such that
\[
\theta_\psi(g, \varphi) = C g \boxtimes g
\]
and
\[
C = \xi_Q(2)^{-1} \prod_{\nu} W_\nu(\varphi_\nu, W_{g,\nu})(1) = 2^t \xi_Q(2)^{-1} \left( \text{GL}_2(\hat{\mathbb{Z}}) : K_0(N_2 \hat{\mathbb{Z}}) \right)^{-1}.
\]
This completes the proof. \(\square\)

Define the Petersson norm of \( g \) by
\[
\langle g, g \rangle = \frac{1}{\text{SL}_2(\mathbb{Z})} \int_{\Gamma_0(N_2) \backslash \text{SL}_2(\mathbb{Z})} |g(\tau)|^2 y^{c-1} d\tau.
\]
Let \( \langle \cdot, \cdot \rangle_{\text{SO}(V)} \) and \( \langle \cdot, \cdot \rangle_{\text{SL}_2} \) be the Petersson pairings on \( (\sigma \boxtimes \sigma)|_{\text{SO}(V)} \) and \( \sigma|_{\text{SL}_2} \) defined by
\[
\langle f_1, f_2 \rangle_{\text{SO}(V)} = \int_{\text{SO}(V)(\mathbb{Q}) \backslash \text{SO}(V)(\mathbb{A})} f_1(h)\overline{f_2(h)} dh,
\]
\[
\langle f_3, f_4 \rangle_{\text{SL}_2} = \int_{\text{SL}_2(\mathbb{Q}) \backslash \text{SL}_2(\mathbb{A})} f_3(g)\overline{f_4(g)} dg,
\]
for \( f_1, f_2 \in (\sigma \boxtimes \sigma)|_{\text{SO}(V)} \) and \( f_3, f_4 \in \sigma|_{\text{SL}_2} \).

**Proposition 7.4.** We have
\[
\theta_\psi(g \boxtimes g, \varphi) = 2^t \xi_Q(2)^{-2} \left( \text{GL}_2(\hat{\mathbb{Z}}) : K_0(N_2 \hat{\mathbb{Z}}) \right)^{-1} \langle g, g \rangle g.
\]

*Proof.* By \([7.2]\) and \([7.2]\), there exists a constant \( C' \) such that
\[
\theta_\psi(g \boxtimes g, \varphi) = C' g.
\]
Hence
\[
\langle \theta_\psi(g \boxtimes g, \varphi), g \rangle_{\text{SL}_2} = 2^{-1} \xi_Q(2)^{-1} \langle g, g \rangle C'.
\]
On the other hand, by Proposition \([7.3]\)
\[
\langle \theta_\psi(g \boxtimes g, \varphi), g \rangle_{\text{SL}_2} = (g \boxtimes g, \theta_\psi(g, \varphi))_{\text{SO}(V)} = 2^t \xi_Q(2)^{-1} \left( \text{GL}_2(\hat{\mathbb{Z}}) : K_0(N_2 \hat{\mathbb{Z}}) \right)^{-1} (g \boxtimes g, g \boxtimes g)_{\text{SO}(V)}
\]
\[
= 2^t \xi_Q(2)^{-3} \left( \text{GL}_2(\hat{\mathbb{Z}}) : K_0(N_2 \hat{\mathbb{Z}}) \right)^{-1} \langle g, g \rangle^2.
\]
Therefore, \( C' = 2^t \xi_Q(2)^{-2} \left( \text{GL}_2(\hat{\mathbb{Z}}) : K_0(N_2 \hat{\mathbb{Z}}) \right)^{-1} \langle g, g \rangle. \) This completes the proof. \(\square\)

Let
\[
(7.4) \quad g^2 = \sigma(t(2^{-1})_2)g,
\]
here \( t(2^{-1})_2 = \begin{pmatrix} 2^{-1} & 0 \\ 0 & 2 \end{pmatrix} \in \text{GL}_2(\mathbb{Q}_2) \). Define \( \varphi^\tau = \otimes v \varphi^\tau_v \in S(V(\mathbb{A})) \) as follows:

- If \( v \neq 2 \), then \( \varphi^\tau_v = \varphi_v \).
• If \( v = 2 \), then \( \varphi_2^v(x) = \varphi_2(2^{-1} x) \).

Note that \( \varphi_2^v = 2^{-2} \omega_{\psi_2}(t(2^{-1}), 1) \varphi_2 \). Therefore, we have the following corollary.

**Corollary 7.5.** We have
\[
\theta_\psi(g_{\mathbb{G} \otimes \mathbb{G}}, \varphi^v) = 2^{l-2} \zeta(2)^{-2} \left( \text{GL}_2(\widehat{\mathbb{Z}}) : K_\mathfrak{m}(N_2 \widehat{\mathbb{Z}}) \right)^{-1} (g,g)_{\mathbb{G}^v}.
\]

**8. Base change for \( \text{GL}_2 \)**

**8.1. Setting.** We keep the notation of [7] except for the definition of the quadratic space \((V, Q)\).

Let \( \mathcal{K} \) be an imaginary quadratic field with ring of integers \( \mathcal{O} \) and discriminant \( -D < 0 \). Let \( \tau \) be the nontrivial automorphism of \( \mathcal{K} \) over \( \mathbb{Q} \), and \( \chi_D \) be the quadratic Hecke character of \( \mathbb{A}^\times / \mathbb{Q}^\times \) associated to \( \mathcal{K} \) by class field theory. Put \( \delta = \sqrt{-D} \). For \( x = \left( \begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \right) \in M_2(\mathcal{K}) \), let \( x^\tau = \left( \begin{array}{cc} x_1^\tau & -x_2^\tau \\ -x_3^\tau & x_4^\tau \end{array} \right) \). Let \((V, Q)\) be the quadratic space over \( \mathbb{Q} \) defined as follows:
\[
V = \left\{ \left( \begin{array}{c} x_1 \\ x_3 \end{array} \right) \in M_2(\mathcal{K}) : x_1 \in \mathcal{K}, x_2, x_3 \in \mathbb{Q} \right\},
\]
\[
Q[x] = \det(x).
\]

Recall we have an exact sequence
\[
1 \rightarrow \text{R}_{\mathcal{K}/\mathbb{Q}} \mathbb{G}_m \rightarrow \mathbb{G}_m \times \text{R}_{\mathcal{K}/\mathbb{Q}} \text{GL}_2 \rightarrow \text{SO}(V) \rightarrow 1,
\]
here \( \iota(a) = (N_{\mathcal{K}/\mathbb{Q}}(a)^{-1}, a\mathbf{1}_2) \) and \( \rho(z, h) = z^2 h z^* \) for \( a \in \text{R}_{\mathcal{K}/\mathbb{Q}} \mathbb{G}_m, z \in \mathbb{G}_m, h \in \text{R}_{\mathcal{K}/\mathbb{Q}} \text{GL}_2, \) and \( x \in V \). Note that \( \nu(\rho(z, h)) = z^2 N_{\mathcal{K}/\mathbb{Q}}(\det(h)) \). Via the exact sequence, we identify automorphic forms on \( \text{SO}(V(\mathcal{A})) \) with automorphic forms on \( \mathbb{A}^\times \times \text{GL}_2(\mathbb{A}_\mathcal{K}) \) that factors over \( \iota(\mathbb{A}^\times) \). Let \( \psi \) (resp. \( \psi_\tau \)) be the standard additive character of \( \mathbb{A} \) (resp. \( \mathcal{K} \)). Let \( \omega_\psi = \omega_{\psi, \psi_\tau} \) (resp. \( \omega_{\psi_\tau} = \omega_{\psi_{\psi_\tau}, \psi_\tau} \)) denote the Weil representation of \( \text{SL}_2(\mathbb{A}_\mathcal{K}) \times O(V(\mathcal{A})) \) (resp. \( \text{SL}_2(\mathbb{Q}_\mathcal{K}) \times O(V(\mathcal{K})) \) on \( S(V(\mathcal{A})) \) (resp. \( S(V(\mathbb{Q}_\mathcal{K}))) \) defined in [4.4]. We extend \( \omega_{\psi_\tau} \) to a representation of \( R(Q_{\mathcal{K}}) \) on \( S(V(Q_{\mathcal{K}})) \) as in [4.2]. Let \( \psi_{\mathcal{K}} \) (resp. \( \psi_{\mathcal{K}, \tau} \)) be the additive character of \( \mathbb{A}_\mathcal{K} \) (resp. \( \mathcal{K}_\tau \)) defined by \( \psi_{\mathcal{K}}(x) = \psi(\text{tr}_{\mathcal{K}/\mathbb{Q}}(\delta^{-1} x)) \) (resp. \( \psi_{\mathcal{K}, \tau}(x) = \psi_\tau(\text{tr}_{\mathcal{K}_\tau/\mathbb{Q}_\mathcal{K}}(\delta^{-1} x)) \)).

We impose the following conditions on \( D \):

- \( D \) is prime to \( N_2 \).
- \( -D \equiv 1 \mod 8 \).
- \( (p, -D)^{\mathcal{Q}_p} = 1 \) for \( p^2 \mid N_2 \).

For each place \( v \), let \( \sigma_{\mathcal{K}, v} \) be the local base change lift of \( \sigma_\mathcal{K} \) to \( \text{GL}_2(\mathcal{K}_v) \). Note that \( \sigma_{\mathcal{K}, \infty} \) is the principal series representation (cf. [3.33])
\[
\text{Ind}_{\mathbb{B}(\mathcal{C})}^{\text{GL}_2(\mathcal{C})}(\mu^{-1} \otimes \mu^{-l+1}).
\]

Let \( \sigma_{\mathcal{K}} = \otimes_v \sigma_{\mathcal{K}, v} \) be the global base change lift of \( \sigma \) to \( \text{GL}_2(\mathbb{A}_\mathcal{K}) \). Then \( \sigma \) is an automorphic representation of \( \text{GL}_2(\mathbb{A}_\mathcal{K}) \). Note that \( \sigma \) is not dihedral with respect to \( \mathcal{K} \) as \( D \) is prime to \( N_2 \). Therefore, \( \sigma \) is cuspidal. Let \( g_{\mathcal{K}} \) be a cuspidal form in the space of \( \sigma_{\mathcal{K}} \) satisfies the following conditions:

- \( \sigma_{\mathcal{K}}(k)^g_{\mathcal{K}} = g_{\mathcal{K}} \) for \( k \in \mathbb{K}_0(p^r \tau) \).
- \( H \cdot g_{\mathcal{K}} = (2l - 2) g_{\mathcal{K}} \) and \( X \cdot g_{\mathcal{K}} = 0 \).

Here
\[
\mathbb{K}_0(p^r \tau) = \left\{ k \in \text{GL}_2(O_p) : k \equiv \left( \begin{array}{cc} * & * \\ 0 & * \end{array} \right) (\mod p^r O_p) \right\}.
\]

By the assumptions on \( D \), the conditions characterize \( g_{\mathcal{K}} \) in the space of \( \sigma_{\mathcal{K}} \) up to scalars. We normalize \( g_{\mathcal{K}} \) such that
\[
\int_{\mathcal{K} \backslash \mathbb{A}_{\mathcal{K}}} g_{\mathcal{K}}(u(x)) \psi_{\mathcal{K}}(x^{-1}) dx = (- \sqrt{-1})^{l-1} D^{-1/2} K_{l-1}(4\pi D^{-1/2}).
\]

For \( v = p \), let \( W_{g_{\mathcal{K}, p}} \in W(\sigma_{\mathcal{K}, v}, \psi_{\mathcal{K}, v})^{\mathcal{K}_0(p^r \tau)} \) be the Whittaker function normalized such that \( W_{g_{\mathcal{K}, p}}(1) = 1 \). For \( v = \infty \), let \( W_{g_{\mathcal{K}, \infty}} \in W(\sigma_{\mathcal{K}, \infty}, \psi_{\mathcal{K}, \infty}) \) be the Whittaker function defined by \( W_{g_{\mathcal{K}, \infty}}(g) = W_{\sigma_{\mathcal{K}, \infty}}(a(\delta^{-1}) g) \). Here \( W_{\sigma_{\mathcal{K}, \infty}} \) is the Whittaker function of \( \sigma_{\mathcal{K}, \infty} \) defined in [3.18].

The main result of this section is Proposition [8.0] which asserts that \( g_{\mathcal{K}} \) is a global theta lift of \( \mathbf{g}^v \).
8.2. Local theta lifts. Define $\varphi = \otimes_v \varphi_v \in S_\varphi(V(\mathbb{A}))$ as follows:

- If $v = p$ with $p \nmid 2D$, then $\varphi_v$ is the function defined in \cite{Ich05} §10.2.
- If $v = p$ with $p \nmid 2D$, then
  \[
  \varphi_p(x) = \mathbb{I} \mathbb{C}_p(x_1) \mathbb{I} \mathbb{L}_{p^\prime}(x_2) \mathbb{I} \mathbb{L}_{p^\prime}(x_3).
  \]
- If $v = \infty$, then
  \[
  \varphi_{\infty}(x) = \frac{1}{\pi} e^{-\pi \text{tr}(x^2)}.
  \]

Note that if $v = p$, then
\begin{equation}
\omega_{\psi_p}(k, k') \varphi_p = \varphi_p
\end{equation}
for $k \in K_0(p^{r_2})$, $k' \in K_0(p^{r_2'})$ such that $\det(k) = N_{K_{p'/Q_p}}(\det(k'))$.

If $v = \infty$, then
\[
\omega_{\psi_{\infty}}(k_0, 1) \varphi_{\infty} = e^{\sqrt{-1} \theta} \varphi_{\infty}
\]
for $k_0 \in \text{SO}(2)$, and
\begin{equation}
H \cdot \varphi_{\infty} = 2(l - 1) \varphi_{\infty}, \quad X \cdot \varphi_{\infty} = 0.
\end{equation}

Let $V_1 = \{ x \in V \mid x_2 = x_3 = 0 \}$ be a quadratic subspace of $V$. For each place $v$, let
\[
S(V(Q_v)) \longrightarrow S(V_1(Q_v)) \otimes S(Q_v^2)
\]
be the partial Fourier transform defined by
\[
\hat{\varphi}(x_1; y) = |D|^{1/2} \int_{Q_v} \varphi \left( \begin{pmatrix} x_1 & \delta z \\ \delta y_1 & x_1^T \end{pmatrix} \right) \psi_v(Dy_2z) dz
\]
for $x_1 \in V_1(Q_v), y = (y_1, y_2) \in Q_v^2$. Let $\hat{\omega}_{\psi_v}(g, h) \hat{\varphi} = (\omega_{\psi_v}(g, h) \varphi)$.

Note that
\[
\hat{\omega}_{\psi_v}(a, 1) \hat{\varphi}(x_1; y) = (-D, a)_{Q_v} |a|_{Q_v} \hat{\varphi}(ax_1; ay_1, a^{-1}y_2)
\]
for $a \in Q_v$. If $\hat{\varphi} = \varphi_1 \otimes \varphi_2$ with $\varphi_1 \in S(V_1(Q_v))$ and $\varphi_2 \in S(Q_v^2)$, then
\begin{equation}
\hat{\omega}_{\psi_v}(g, 1) \hat{\varphi}(x_1; y) = \omega_{\psi_v}(g, 1) \varphi_1(x_1) \cdot \varphi_2(gy)
\end{equation}
for $g \in \text{SL}_2(Q_p)$. We have an equivariant map (cf. \cite{Cog85} and \cite{Cog86})
\[
S_{\psi_v}(V(Q_v)) \otimes W(\sigma_v, \psi_v^{-1}) \longrightarrow W(\sigma_{\varphi_v}, \psi_{\varphi_v})
\]
\[
\varphi \otimes W \mapsto W_v(\varphi, W),
\]
\[
W_v(\varphi, W)(h) = \int_{U(Q_v) \backslash \text{SL}_2(Q_v)} \hat{\omega}_{\psi_v}(gg', h) \hat{\varphi}(D^{-1}; 0, 1) W(a(D - 2)gg') dg,
\]
with $g' \in \text{GL}_2(Q_v)$ such that $\det(g') = N_{K_{p'/Q_p}}(\det(h))$.

**Proposition 8.1.** Let $v = p \mid D$. We have
\[
W_p(\varphi_p, W_{\varphi_p}) = p \gamma_{Q_p}(-D, \psi_p)^{-1} W_{g_{\varphi_v}}.
\]

**Proof.** \cite{Ich05} Lemma 10.3. \qed

**Proposition 8.2.** Let $v = 2$. We have
\[
W_2(\varphi_2, p(D^{-1})W_{g_2}) = 2^{-2} W_{g_{\varphi_v}}.
\]

**Proof.** \cite{Ich05} Lemmas 10.5 and 10.7. \qed

**Proposition 8.3.** Let $v = p \nmid 2D$. We have
\[
W_p(\varphi_p, W_{\varphi_p}) = (\text{GL}_2(Z_p) : K_0(p^{r_2}))^{-1} W_{g_{\varphi_v}}.
\]
Proposition 8.4. We conclude from the above calculation that

\[ W_p(\varphi_p, W_{g,p}) \in W(\sigma_{K,p}, \psi_{K,p})K_0(p^{r_p}) = CW_{g,K,p}. \]

It suffices to show that

\[ W_p(\varphi_p, W_{g,p})(1) = (\text{GL}_2(\mathbb{Z}_p) : K_0(p^{r_p}))^{-1}. \]

For \( k \in \text{SL}_2(\mathbb{Z}_p) \), let

\[ J(k) = \sum_{n \in \mathbb{Z}} p^n(-D, p^n)\omega_p(k, 1)\hat{\varphi}_p(D^{-1}p^n; 0, p^{-n})W_{g,p}(a(D^{-2})t(p^n)k). \]

Note that \( \hat{\varphi}_p(x_1; y) = I_{\mathcal{O}_p}(x_1)\|\mathbb{Z}_p\| \) and \( \omega_p(k, 1)\|\mathcal{O}_p \| = I_{\mathcal{O}_p} \) for \( k \in \text{SL}_2(\mathbb{Z}_p) \). Therefore, by (8.3)

\[ \hat{\omega}_p(k, 1)\hat{\varphi}_p(x_1; y) = I_{\mathcal{O}_p}(x_1)\mathbb{Z}_p \times \mathbb{Z}_p(gk). \]

We deduce that \( \hat{\omega}_p(k, 1)\hat{\varphi}_p(D^{-1}p^n; 0, p^{-n}) \neq 0 \) if and only if \( n = 0 \) and \( k \in K_0(p^{r_p}) \). Therefore,

\[ J(k) = \begin{cases} 1 & \text{if } k \in K_0(p^{r_p}), \\ 0 & \text{otherwise.} \end{cases} \]

We conclude from the above calculation that

\[ W_p(\varphi_p, W_{g,p})(1) = \text{vol}(K_0(p^{r_p}), dk) \sum_{k \in \text{SL}_2(\mathbb{Z}_p)/K_0(p^{r_p})} J(k) = \text{vol}(K_0(p^{r_p}), dk). \]

This completes the proof. \( \square \)

Proposition 8.5. We have

\[ W_{\infty}(\varphi_{\infty}, W_{g,\infty}) = 2^{2-l}(\sqrt{-1})^{l-1}D^{-1/2}W_{g,\infty}. \]

Proof. By (8.2),

\[ W_{\infty}(\varphi_{\infty}, W_{g,\infty}) = CW_{g,\infty}. \]

Note that

\[ W_{g,\infty}(1) = (-\sqrt{-1})^{l-1}D^{-1/2}K_{l-1}(4\pi D^{-1/2}). \]

A similar calculation as in [ICh05] Lemma 10.8] shows that

\[ W_{\infty}(\varphi_{\infty}, W_{g,\infty})(1) = 2^{2-l}D^{-1/2}K_{l-1}(4\pi D^{-1/2}). \]

This completes the proof. \( \square \)

Remark 5. If \( l \) is odd, then one can show that \( W_{\infty}(\varphi_{\infty}, W_{g,\infty}) = 0 \).

8.3. Global theta lifts. For \( f \in \sigma \) and \( \phi \in S_{\psi}(V(\mathbb{A})) \), let \( \theta_{\psi}(f, \phi) \) be the automorphic form on \( \text{GSO}_1(V(\mathbb{A})) \) defined by

\[ \theta_{\psi}(f, \phi)(h) = \int_{\text{SL}_2(\mathbb{Q})\backslash \text{SL}_2(\mathbb{A})} \theta_{\psi}(gg', h; \phi)f(gg')dg' \]

where \( g' \in \text{GL}_2(\mathbb{A}) \) such that \( \det(g') = \nu(h) \). Let \( \theta_{\psi}(\sigma) \) be the global theta lift of \( \sigma \) with respect to \( \psi \) defined by

\[ \theta_{\psi}(\sigma) = \{ \theta_{\psi}(f, \phi) \mid f \in \sigma, \phi \in S_{\psi}(V(\mathbb{A})) \}. \]

Recall \( g^s = \sigma(t(2^{-1})_2)g \). Let \( \mathcal{W} \) be the first Fourier coefficient of \( \theta_{\psi}(g^s, \varphi) \) defined by

\[ \mathcal{W}(h) = \int_{\mathbb{A}_K/K} \theta_{\psi}(g^s, \varphi)(u(x)h)\psi_{\mathbb{K}}(-x)dx. \]

By [ICh05] Lemma 4.2], we have

\[ \mathcal{W}(h) = \xi_q(2)^{-1}W_2(q, \rho(t(2^{-1}))W_{g,\mathbb{K}})(h_2) \prod_{v \neq 2} \mathcal{W}_v(\varphi_v, W_{g,v})(h_v), \]

for \( h \in \text{GL}_2(\mathbb{A}_K) \).

Proposition 8.5. We have

\[ \theta_{\psi}(\sigma) = \chi_D \otimes \sigma_\mathbb{K}. \]
This completes the proof. □

We have $pV\, 7.1.2$ and Corollary 7.1.3, together with the strong multiplicity one theorem for $GL_\theta$ correspondence. Since § explained in [JLZ06, here (8.4) This completes the proof

Setting.

9.1. A

Proof. By (8.1), (8.2), and Proposition 8.5, there exist a constant

By the result in [Rob96, §4.2, Proposition 8.1-8.4, the Howe duality also hold for $(GL_\theta)$ is non-zero and cuspidal, by the argument in the proof of [KR94, Proposition 8.2, let $GL_\theta$ is the principal series representation (cf [Cog85] shows that $\theta_\psi(\sigma _p^\vee \otimes \sigma _K)$. On the other hand, the result in [Cog85] shows that $\theta_\psi(\sigma _p^\vee) = \chi_{-D,p} \boxtimes \sigma _K,p$ for $p \nmid 2D$. By the strong multiplicity one theorem for $GL_2$,

$$\theta_\psi(\sigma ) = \chi_{-D} \boxtimes \sigma _K.$$ 

This completes the proof. □

Proposition 8.6. We have

$$\theta_\psi(\mathcal{G}, \varphi ) = -2^{-1}((\sqrt{-1})^AD^{1/2}2^{-1}(GL_2(\mathcal{Z} : K_0(N_2\mathcal{Z}))^{-1} \cdot (\chi_{-D} \otimes g\mathcal{K}).$$

Proof. By (8.1), (8.2), and Proposition 8.5, there exist a constant $C$ such that

$$\theta_\psi(\mathcal{G}, \varphi ) = C \cdot (\chi_{-D} \otimes g\mathcal{K}).$$

By Propositions 8.1-8.4

$$C = \xi _2^{-1}(\sqrt{-1})^{-1}D^{1/2}K_{1-1}(4\pi D^{1/2})^{-1}W_2(\varphi _2, \rho (t(2^{-1})))W_{\mathcal{K}}(1) \prod _{v \neq 2} W_2(\varphi _v, \mathcal{W}_v(1))$$

$$= -2^{-1}((\sqrt{-1})^AD^{1/2}\xi _2^{-1}(GL_2(\mathcal{Z} : K_0(N_2\mathcal{Z}))^{-1} \cdot (\chi_{-D} \otimes g\mathcal{K}).$$

This completes the proof. □

Let

$$g\mathcal{K} = \sigma _K(t_{\infty })g\mathcal{K},$$

here $t_{\infty } = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right) \in SU(2)$. Define $\varphi \mathcal{K} = \otimes _v \varphi _v^\mathcal{K}$ as follows:

- If $v \neq \infty$, then $\varphi _v^\mathcal{K} = \varphi _v$.
- If $v = \infty$, then $\varphi _\infty = (-2\delta _{-1}^{-1}1\omega _\psi _\infty (1, t_{\infty }))\varphi _\infty$.

The following corollary follows immediately.

Corollary 8.7. We have

$$\theta_\psi(\mathcal{G}, \varphi \mathcal{K}) = 2^{-1}((\sqrt{-1})D^{1/2+1}\xi _2^{-1}(GL_2(\mathcal{Z} : K_0(N_2\mathcal{Z}))^{-1} \cdot (\chi_{-D} \otimes \varphi \mathcal{K}).$$

9. Triple product L-functions

9.1. Setting. Let $E = \mathbb{Q} \times \mathcal{K}$ be a cubic algebra over $\mathbb{Q}$. Here $\mathcal{K}$ is an imaginary quadratic field with ring of integers $\mathcal{O}$ and discriminant $-D < 0$. Let $\tau$ be the nontrivial automorphism of $\mathcal{K}$ over $\mathbb{Q}$, and $\chi_{-D}$ be the quadratic Hecke character of $\mathbb{A}^\times /\mathbb{Q}^\times$ associated to $\mathcal{K}$ by class field theory. Put $\delta = \sqrt{-D}$. Let $\Pi$ be an irreducible unitary cuspidal automorphic representation of $GL_2(\mathbb{A}_E)$ such that the central character of $\Pi$ is trivial on $\mathbb{A}^\times$. Then $\Pi = \pi \boxtimes \sigma _\mathcal{K}$ for some irreducible unitary cuspidal automorphic representations $\pi$ and $\sigma _\mathcal{K}$ of $GL_2(\mathbb{A})$ and $GL_2(\mathcal{K})$, respectively. We assume that at the archimedean place, $\pi _\infty$ is the discrete series representation of weight $2\kappa _\mathcal{K}$ and $\sigma _{\mathcal{K}, \infty}$ is the principal series representation (cf (3.3.3))

$$Ind_{B(\mathcal{C})}^{GL_2(\mathcal{C})}(\mu ^\kappa \boxtimes \mu ^{-\kappa}).$$
for some positive integers \( \kappa \) and \( \kappa' \). Let \( N \mathbb{Z} \) and \( \mathfrak{M} \) be the conductors of \( \pi \) and \( \sigma_\mathcal{K} \), respectively. Put \( M \mathbb{Z} = N_{\mathcal{K}/\mathcal{Q}}(\mathfrak{M}) \). We assume both \( M \) and \( N \) are positive. Let \( f \) be a cusp form in the space of \( \pi \) satisfies the following conditions:

- \( \pi(k)f = f \) for \( k \in K_0(N\mathbb{Z}) \).
- \( \pi(k)g = e^{2\sqrt{-\pi}k'g} \) for \( k \in SO(2) \).

The conditions characterize \( f \) in the space of \( \pi \) up to scalars. Let \( g_{\mathcal{K}} \) be a cusp form in the space of \( \sigma_\mathcal{K} \) satisfies the following conditions:

- \( \sigma_\mathcal{K}(k)g_{\mathcal{K}} = g_{\mathcal{K}} \) for \( k \in K_0(\mathfrak{M}_1) \).
- \( H \cdot g_{\mathcal{K}} = 2\kappa \cdot g_{\mathcal{K}} \) and \( X \cdot g_{\mathcal{K}} = 0 \).

Here \( \mathfrak{M}_1 = \prod_p \mathfrak{M}_p \) and

\[
\mathfrak{M}_p = \left\{ k \in GL_2(\hat{O}) \left| k \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\mathfrak{M}\hat{O}} \right. \right\}.
\]

The conditions characterize \( g_{\mathcal{K}} \) in the space of \( \sigma_\mathcal{K} \) up to scalars. We normalize \( f \) and \( g_{\mathcal{K}} \) so that

\[
\int_{\mathbb{Q}\setminus\hat{A}} f(u(x))\psi(-x)dx = e^{-2\pi},
\]

\[
\int_{K'\setminus\hat{A}_\mathcal{K}} g_{\mathcal{K}}(u(x))\psi_{\mathcal{K}}(-x)dx = (-\sqrt{-1})^{\kappa}D^{(-\kappa-1)/2}K_\kappa(4\pi D^{-1/2}).
\]

Here \( \psi \) is the standard additive character of \( \hat{A} \), and \( \psi_{\mathcal{K}} \) is the additive character of \( \hat{A}_\mathcal{K} \) defined by \( \psi_{\mathcal{K}}(x) = \psi(tr_{\mathcal{K}/\mathcal{Q}}(x^2)) \).

For each place \( v \) of \( \mathbb{Q} \), let \( L(s, \Pi_v, r) \) and \( L(s, \Pi_v, Ad) \) be the triple product \( L \)-function and the adjoint \( L \)-function of \( \Pi_v \), respectively. Here \( r \) and \( Ad \) are the representations of the \( L \)-group of \( \mathbb{R}_{\mathcal{K}/\mathcal{Q}}, GL_2 \) defined in [Ich08]. Let \( \epsilon(s, \Pi_v, r, \psi_v) \) be the \( \epsilon \)-factor associated to \( \Pi_v \) and \( r \) with respect to the standard additive character \( \psi_v \) of \( \mathbb{Q}_v \). Let

\[
L(s, \Pi, r) = \prod_v L(s, \Pi_v, r), \quad L(s, \Pi, Ad) = \prod_v L(s, \Pi_v, Ad)
\]

be the triple product \( L \)-function and the adjoint \( L \)-function of \( \Pi \), respectively. The \( L \)-functions defined by the Euler products are absolutely convergent for \( \text{Re}(s) > 0 \), admit meromorphic continuations to \( s \in \mathbb{C} \), and satisfy the usual functional equations. Moreover, the assumption that \( \Pi \) is unitary implies that the \( L \)-functions are holomorphic at \( s = \frac{1}{2} \) and \( s = 1 \), respectively.

We impose the following assumptions:

- The central characters of \( \sigma_\mathcal{K} \) and \( \pi \) are trivial.
- \( \kappa - \kappa' \) is even.
- \( N \) is square-free and \( \mathfrak{M} \) is cubic-free.
- If \( p \) is a prime of \( \mathcal{O} \) such that \( p^2 \mid \mathfrak{M} \), then \( p \neq p' = N_{\mathcal{K}/\mathcal{Q}}(p) \uparrow 2N \), and

\[
\sigma_{\mathcal{K}, p} = \sigma_{\mathcal{K}, p'} \quad \text{as representations of } GL_2(\mathbb{Q}_p).
\]

- For all place \( v \), we have

\[
\epsilon \left( \frac{1}{2}, \Pi_v, r \right) = \chi_{-D,v}(1) - 1.
\]

Let \( \Sigma_1, \Sigma_2, \Sigma_3 \) be the sets of finite places \( p \) that are split, ramified, inert in \( \mathcal{K} \), respectively. Define \( \zeta_{\mathcal{K}, p}(s) \) by

\[
\zeta_{\mathcal{K}, p}(s) = \begin{cases} 
\zeta_{p}(s)^2 & \text{if } p \in \Sigma_1, \\
\zeta_{p}(s) & \text{if } p \in \Sigma_2, \\
\zeta_{p}(2s) & \text{if } p \in \Sigma_3.
\end{cases}
\]

Put \( \xi_{\mathcal{K}}(s) = \zeta_{\mathcal{C}}(s)\prod_p \zeta_{\mathcal{K}, p}(s) \). For \( p \in \Sigma_1 \), fix an isomorphism \( K_p \cong \mathbb{Q}_p \times \mathbb{Q}_p \) once and for all. Let \( p_1 \) and \( p_2 \) be the prime ideals of \( \mathcal{O} \) divide \( p \) that correspond to the first and second coordinate, respectively. Via this isomorphism, we identify \( GL_2(K_p) \) with \( GL_2(\mathbb{Q}_p) \times GL_2(\mathbb{Q}_p) \). For each finite place \( p \) of \( \mathcal{K} \), fix an uniformizer \( \varpi_p \) of \( K_p \). Let \( t \in GL_2(\mathcal{A}_{\mathcal{K}}) \) be defined as follows:
Let \( \tilde{V}_+ \in \mathfrak{gl}(2, \mathbb{R})_C \) be the normalized weight raising element defined in \[3.3.2\]. We put

\[
\mathcal{I}(\tilde{V}_+^{2m} \mathbf{f} \otimes \sigma_K(t)g_K) = \int_{A^+ \mathbb{GL}_2(\mathbb{Q}) \backslash \mathbb{GL}_2(\mathbb{A})} \tilde{V}_+^{2m} \mathbf{f}(h)g_K(h)dt.
\]

The main result of this section is Proposition \[9.12\] which asserts that \( \mathcal{I}(\tilde{V}_+^{2m} \mathbf{f} \otimes \sigma_K(t)g_K)^2 \) is equal to the central value \( L\left(\frac{1}{2}, \Pi, r\right) \) up to an explicit non-zero constant.

### 9.2. Local trilinear period integrals.

Let \( v \) be a place. Let \( V_v \) and \( V_v \) be models of \( \Pi_v \) and \( \Pi_v \), respectively. Let \( \langle \ , \rangle_v \) be a non-zero \( \mathbb{GL}_2(E_v) \)-invariant pairing on \( V_v \otimes V_v \). Let

\[
\mathcal{I}_v \in \text{Hom}_{\mathfrak{gl}_2(\mathbb{Q}_v) \times \mathfrak{gl}_2(\mathbb{Q}_v)}(\Pi_v \boxtimes \Pi_v^\vee, \mathbb{C})
\]

be the trilinear form defined by

\[
\mathcal{I}_v(\phi \otimes \phi') = \frac{1}{\zeta_K(2)} \cdot \frac{L(1, \Pi_v, \text{Ad})}{L(1/2, \Pi_v, r)} \cdot \int_{\mathbb{Q}_v^+ \mathfrak{gl}_2(\mathfrak{q}_v)} \langle \Pi_v(g)\phi, \phi' \rangle_v dg.
\]

Note that the integral is absolutely convergent by \[\text{Lemma 2.1}\]. The last assumption in \[9.1\] is equivalent to the non-vanishing of the trilinear form \( \mathcal{I}_v \) for all places \( v \) (cf. \[\text{Pra92}\]).

For \( v = \infty \), the complexified Lie algebra of \( \mathbb{GL}_2(E_\infty) \) is identified with \( \mathfrak{gl}(2, \mathbb{R})_C \oplus \mathfrak{gl}(2, \mathbb{C})_C \). Let \( \phi_v \in V_v \) be an element satisfies the following conditions:

- If \( v = p \), then \( \Pi_p((k_1, k_2))\phi_p = \phi_p \) for \( (k_1, k_2) \in K_0(\mathbb{N}\mathbb{Z}_p) \times K_0(2\mathbb{R}\mathbb{O}_p) \).
- If \( v = \infty \), then \( \Pi_\infty((k_0, 1)) = e^{2\sqrt{-1}v}\phi_\infty \) for \( k_0 \in SO(2) \), and \( (0, H) \cdot \phi_\infty = 2k \cdot \phi_\infty \cdot (0, X) \cdot \phi_\infty = 0 \).

The conditions characterize \( \phi_v \) in \( V_v \) up to scalars. Similarly, let \( \phi_v^\vee \in V_v^\vee \) be an element satisfies the same conditions. Let \( t_v \in \mathbb{GL}_2(K_v) \) be the element defined in \[9.1\]. Let \( \mathcal{I}_v^*(\Pi_v) \in \mathbb{C} \) be defined as follows:

- If \( v = p \), then
  \[
  \mathcal{I}_v^*(\Pi_p) = \frac{\mathcal{I}_p(\Pi_p((1, t_p))\phi_p \otimes \Pi_v^\vee((1, t_p))\phi_v^\vee)}{\langle \phi_p, \phi_v^\vee \rangle_p}.
  \]

- If \( v = \infty \), then
  \[
  \mathcal{I}_\infty^*(\Pi_{\infty}) = \frac{\mathcal{I}_\infty(\Pi_{\infty}((1, t_{\infty})))(\tilde{V}_+^{2m}, 0)\phi_\infty \otimes \Pi_v^\vee((1, t_{\infty})) (\tilde{V}_+^{2m}, 0)\phi_v^\vee}{\langle \Pi_{\infty}((a(-1), a(-1)t_{\infty}))\phi_\infty, \Pi_v((1, t_{\infty}))\phi_v^{\vee} \rangle_{\infty}}.
  \]

Note that \( \mathcal{I}_v^*(\Pi_v) \) dose not depend on the choices of the model \( V_v \), the pairing \( \langle \ , \rangle_v \), and the elements \( \phi_v, \phi_v^\vee \) satisfying the required conditions.

The assumption that

\[
\epsilon \left( \frac{1}{2}, \Pi_v, r \right) = \chi_{-D,v}(-1)
\]

puts some restrictions on \( v \). For instance, there are no place \( v = p \in \Sigma_3 \) with \( p \parallel NM \).

In the following propositions, we give the value \( \mathcal{I}_v^*(\Pi_v) \) in all possible circumstances. For \( v = p \), we divide the calculation into two cases according to whether \( p^4 \) divide \( M \) or not.
9.2.1. Case \( p^4 \mid M \). Let \( v = p \) with \( p^4 \mid M \). Let \( p \) be a prime of \( K \) that divides \( p \). By our assumptions made in \( 9.2.1 \) \( p^4 \mid M \) is equivalent to \( p^2 \mid \mathcal{M} \). Therefore, \( c(\sigma_{K,p}), c(\sigma_{K,p'}) , c(\pi_p) \leq 1 \) and in this case the calculation was carried out before by some authors.

**Proposition 9.1.** Let \( v = p \in \Sigma_1 \). If \( p \nmid NM \), then
\[
\mathcal{I}_p^v(\Pi_p) = 1.
\]
If \( p \| NM \), then
\[
\mathcal{I}_p^v(\Pi_p) = (1 + p)^{-1}.
\]
If \( p^2 \| NM \), then
\[
\mathcal{I}_p^v(\Pi_p) = p^{-1}.
\]
If \( p^3 \| NM \), then
\[
\mathcal{I}_p^v(\Pi_p) = 2p^{-2}(1 + p).
\]
**Proof.** [Ich08 Lemma 2.2, [III, §7], [Nel11 Lemma 4.4], and [CC §4].]

**Proposition 9.2.** Let \( v = p \in \Sigma_2 \). If \( p \nmid NM \), then
\[
\mathcal{I}_p^v(\Pi_p) = 1.
\]
If \( p \| NM \) and \( p \mid N \), then
\[
\mathcal{I}_p^v(\Pi_p) = 2(1 + p)^{-1}.
\]
If \( p \| NM \) and \( p \mid M \), then
\[
\mathcal{I}_p^v(\Pi_p) = (1 + p)^{-1}.
\]
If \( p^2 \| NM \), then
\[
\mathcal{I}_p^v(\Pi_p) = p^{-1}.
\]
**Proof.** [CC §4.2.4].

**Proposition 9.3.** Let \( v = p \in \Sigma_3 \). If \( p \nmid NM \), then
\[
\mathcal{I}_p^v(\Pi_p) = 1.
\]
If \( p^2 \| NM \), then
\[
\mathcal{I}_p^v(\Pi_p) = p^{-1}(1 + p)^{-2}(1 + p^2).
\]
If \( p^3 \| NM \), then
\[
\mathcal{I}_p^v(\Pi_p) = 2p^{-2}(1 + p)^{-1}(1 + p^2).
\]
**Proof.** [Ich08 Lemma 2.2] and [CC §4.2.4].

9.2.2. Case \( p^4 \mid M \). Let \( v = p \) with \( p^4 \mid M \). Let \( \psi = \psi_p \) be the standard additive character of \( \mathbb{Q}_p \). By our assumptions made in \( 9.1 \) we have \( p \in \Sigma_1 \), \( \sigma_{K,p} = \sigma_{K,p_2} \), \( c(\sigma_{K,p_1}) = 2 \), and \( c(\pi_p) = 0 \). Put \( \sigma_p = \sigma_{K,p_1} = \sigma_{K,p_2} \). Let \( W_{\sigma_p} \in \mathcal{W}(\sigma_p, \psi) \) be the Whittaker newform of \( \sigma_p \) defined in \( [3.3.1] \) respectively. Since \( c(\pi_p) = 0 \), we have
\[
\pi_p = \text{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\langle | \sigma_p, | \sigma_p^- \rangle)
\]
for some \( s_p \in \mathbb{C} \). Let \( \alpha_p = p^{-s_p} \). Note that \( \text{Re}(s_p) = 0 \) by the Ramanujan conjecture.

Let \( V_p \) and \( V_p' \) be the space of \( \text{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\langle | \sigma_p, | \sigma_p^- \rangle) \) and \( \text{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\langle | \sigma_p, | \sigma_p^- \rangle) \), respectively. Let \( \langle , \rangle_1 \) and \( \langle , \rangle_2 \) be the \( GL_2(\mathbb{Q}_p) \)-invariant pairings on \( \mathcal{W}(\sigma_p, \psi) \otimes \mathcal{W}(\sigma_p, \psi) \) and \( V_p \otimes V_p' \) defined by
\[
\langle W_1, W_2 \rangle_1 = \int_{\mathbb{Q}_p^*} W_1(a(t))W_2(a(-t))dt, \quad \langle f_1, f_2 \rangle_2 = \int_{GL_2(\mathbb{Z}_p)} f_1(k)f_2(k)dk.
\]
Let \( V_p \) and \( V_p' \) be models of \( \Pi_p \) and \( \Pi_p' \) defined by
\[
V_p = \mathcal{W}(\sigma_{K,p_1}, \psi_p) \otimes \mathcal{W}(\sigma_{K,p_2}, \psi_p) \otimes V_p, \quad V_p' = \mathcal{W}(\sigma_{K,p_1}, \psi_p) \otimes \mathcal{W}(\sigma_{K,p_2}, \psi_p) \otimes V_p' .
\]
Define \( \Psi_p \in \text{Hom}_{GL_2(\mathbb{Q}_p)}(\Pi_p, \mathbb{C}) \) (cf. [Jac72 (14.5)]) by the Rankin-Selberg local zeta integral
\[
\Psi_p(W_1 \otimes W_2 \otimes f) = \int_{\mathbb{Q}_p^* U(\mathbb{Q}_p) GL_2(\mathbb{Q}_p)} W_1(g)W_2(a(-1)g)f(g)dg.
\]
for $W_1 \otimes W_2 \otimes f \in V_p$. Note that the integral is absolutely convergent. Similarly we define $\Psi^V_p \in \text{Hom}_{\text{GL}_2(Q_p)}(\Pi^V_p, \mathbb{C})$. Then $\Psi_p \otimes \Psi^V_p \in \text{Hom}_{\text{GL}_2(Q_p) \times \text{GL}_2(Q_p)}(\Pi_p \otimes \Pi^V_p, \mathbb{C})$. Let

$$T_p \in \text{Hom}_{\text{GL}_2(Q_p) \times \text{GL}_2(Q_p)}(\Pi_p \otimes \Pi^V_p, \mathbb{C})$$

be the trilinear form defined in the beginning of this section with respect to the models $W(\sigma, \psi), V_p, V^V_p$ and the pairings $(\ , \ ), (\ , \ )_2$. We normalize the measures $d^\times t$, $dk$, $dg$ so that

$$\text{vol}(Z_p^+, d^\times t) = \text{vol}(\text{GL}_2(Z_p), dk) = \text{vol}(Z_p^+, \text{GL}_2(Z_p), dg) = 1.$$

**Lemma 9.4.** We have

$$T_p = \frac{\zeta_p(1)}{\zeta_p(2)} \cdot \frac{L(1, \Pi_p, \text{Ad})}{L(1/2, \Pi_p, r)} \cdot \Psi_p \otimes \Psi^V_p.$$

**Proof.** [MV10] Lemma 3.4.2] and [HS17] Proposition 5.1].

**Lemma 9.5.** We have

$$L(s, \Pi_p, r) = L(s + s_p, \sigma_p \boxtimes \sigma_p)L(s - s_p, \sigma_p \boxtimes \sigma_p).$$

Here $L(s, \sigma_p \boxtimes \sigma_p)$ is the Rankin-Selberg $L$-function of $\sigma_p \boxtimes \sigma_p$ defined as in [Jac72] and [GJ78].

**Proof.** [Rama00] Theorem 4.4.1].

Let $f_0 \in V_p^{\text{GL}_2(Z_p)}$ and $f'_0 \in (V_p^{\text{GL}_2(Z_p)})^\vee$ be the spherical sections normalized so that $f_0(1) = f'_0(1) = 1$. Note that $(W_{\sigma_p}, W_{\sigma_p}) = (f_0, f'_0) = 1$. By Lemma 6.4 and the definition of $t_p$, we have

$$T^*_p(\Pi_p) = \frac{\zeta_p(1)}{\zeta_p(2)} \cdot \frac{L(1, \Pi_p, \text{Ad})}{L(1/2, \Pi_p, r)} \cdot \Psi_p(W_{\sigma_p} \otimes W_{\sigma_p} \otimes \pi_p(a(p^{-1})f_0) \cdot \Psi^V_p(W_{\sigma_p} \otimes W_{\sigma_p} \otimes \pi^{V}_p(a(p^{-1})f'_0)).$$

The calculation of $T^*_p$ thus boils down to the calculation of the local zeta integrals $\Psi_p$ and $\Psi^V_p$. In the following propositions, we give the value $T^*_p(\Pi_p)$. The calculations are all similar, we omit it except the proof of Lemma 9.3: local period integral supercuspidal.

**Proposition 9.6.** If $\sigma_p$ is a ramified principal series representation with $L(s, \sigma_p, \text{Ad}) = \zeta_p(s)$, then

$$T^*_p(\Pi_p) = p^{-1}.$$

**Proposition 9.7.** If $\sigma_p$ is a ramified principal series representation with $L(s, \sigma_p, \text{Ad}) = \zeta_p(s)^2$ then

$$T^*_p(\Pi_p) = p^{-1} \zeta_p(1)^2 \left[ (1 + p^{-1}) - 4p^{-1/2} \alpha_p + (1 + p^{-1}) \alpha_p^2 \right] \left[ (1 + p^{-1}) - 4p^{-1/2} \alpha_p^{-1} + (1 + p^{-1}) \alpha_p^{-2} \right].$$

**Proposition 9.8.** If $\sigma_p$ is a ramified Steinberg representation, then

$$T^*_p(\Pi_p) = p^{-1}.$$

**Proposition 9.9.** If $\sigma_p$ is a supercuspidal representation, then

$$T^*_p(\Pi_p) = p^{-1}.$$

**Proof.** For $k \in \text{GL}_2(Z_p)$, let

$$J(k) = \int_{\mathbb{Q}_p^\times} W_{\sigma_p}(a(t)k) W_{\sigma_p}(a(-t)k) |t|_{\mathbb{Q}_p}^{s_p - 1/2} d^\times t.$$

Note that

$$J\left(\begin{pmatrix} 1 & 0 \\ p^iu & 1 \end{pmatrix}\right) = J\left(\begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix}\right), \quad J\left(\begin{pmatrix} w & 0 \\ x & 1 \end{pmatrix}\right) = J(w),$$

for $i \in \mathbb{Z}_{\geq 0}, u \in \mathbb{Z}_p^\times$, and $x \in \mathbb{Z}_p$. By Lemma 8.7 applied to the case $c(\sigma_p) = 2$, we have

$$J\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right) = p^{-1} \alpha_p^{-2}, \quad J\left(\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}\right) = 1.$$

Since $W_{\sigma_p}(a(t)) = \mathbb{1}_{\mathbb{Z}_p^\times}(t)$ and $W_{\sigma_p}(a(t)w) = \epsilon\left(\frac{1}{2}, \sigma_p\right) \mathbb{1}_{p^{-1}2^{\mathbb{Z}_p^\times}}(t)$ (cf. [Sch02] §2), we have

$$J(1) = 1, \quad J(w) = p^{-1} \alpha_p^{-2}.$$
Therefore,
\[ \Psi_p(W_{\sigma} \otimes W_{\sigma} \otimes \pi_p(a(p^{-1})))f_0 = \int_{GL_2(\mathbb{Z}_p)} \int_{Q_p^*} W_{\sigma}(a(t)k)W_{\sigma}(a(-t)k)f_0(ka(p^{-1}))|t|^\alpha_p^{-1/2}d^\kappa dk \]
\[ = p^{-1}(1+p)^{1-\alpha} \sum_{k \in GL_2(\mathbb{Z}_p)/K_0(p^2)} f_0(ka(p^{-1}))\mathcal{I}(k) \]
\[ = p^{-1/2} \alpha_p^{-1}. \]

Similarly, we have \( \Psi_q(W_{\sigma} \otimes W_{\sigma} \otimes \pi_q(a(p^{-1})))f_0 = p^{-1/2} \alpha_q^{-1}. \)

By Lemma 9.4, \([\text{GJ78}, \text{Corollary (1.3)}]\), and \([\text{Hsi17}, \text{Lemma 6.8-(2)}]\), we have
\[ L(s, \Pi_p, r) = (1 - p^{-2s} \alpha_p^2)^{-1}(1 - p^{-2s} \alpha_p^{-2})^{-1}, \]
\[ L(s, \Pi_p, \text{Ad}) = \zeta_{Q_p}(2s)^2 \zeta_{Q_p}(s)^{-1}(1 - p^{-s} \alpha_p^{-2})^{-1}(1 - p^{-s} \alpha_p^{-2})^{-1}. \]

This completes the proof. \( \square \)

9.2.3. Archimedean case. Let \( v = \infty \). Recall we have assume \( \kappa - \kappa' \) is even. Note that the assumption
\( \epsilon \left( \frac{1}{2}, \Pi_{\infty}, r \right) = -1 \)
is equivalent to \( \kappa - \kappa' \geq 0 \). Put \( 2m = \kappa - \kappa' \in 2\mathbb{Z}_{\geq 0}. \) Let \( C(\kappa, \kappa') \) be a rational number defined by
\[ C(\kappa, \kappa') = \frac{\Gamma(2\kappa)}{\Gamma(4\kappa)\Gamma(\kappa')^2\Gamma(2m+1)} \sum_{j=0}^{2m} \sum_{n=0}^{2m-j} (-1)^{i+j} \binom{2m}{j} \binom{2\kappa}{n} \Gamma(\kappa + 1 + n)\Gamma(3\kappa - 1 + n) \]
\[ \times \frac{\Gamma(\kappa' + j + n)\Gamma(\kappa + \kappa' + j)}{\Gamma(2\kappa' + j)(2\kappa + \kappa' + j - n)} \]
\[ \times \frac{\Gamma(2\kappa + \kappa' + j - n + 1)\Gamma(\kappa + \kappa' + j + 1/2 + i)\Gamma(2m - j + 1)}{\Gamma(2\kappa + \kappa' + j - n + 1)\Gamma(\kappa + \kappa' + j + 1/2)\Gamma(2m - j - i)\Gamma(\kappa' + j + n + 1 + i)}. \]

Proposition 9.10. We have
\[ \mathcal{I}_{\infty}^*(\Pi_{\infty}) = 2^{-6\kappa + 6\kappa' - 2} \pi^{-4m}(2\kappa + 1)C(\kappa, \kappa'). \]

Moreover, \( C(\kappa, \kappa') \neq 0. \)

The proof is given in \( \text{[\text{??}]} \).

Remark 6.

(1) We conjecture that
\[ C(\kappa, \kappa') = \left( \frac{\Gamma(\kappa - m)\Gamma(2m + 1)}{\Gamma(\kappa - 2m)\Gamma(m + 1)} \right)^2. \]

Verified by using mathematica, the equality holds for all \( m \leq 500. \)

(2) If \( \kappa - \kappa' \geq 0 \) is odd, one can show that \( \mathcal{I}_{\infty}^*(\Pi_{\infty}) = 0. \) On the other hand, the condition \( \epsilon \left( \frac{1}{2}, \Pi_{\infty}, r \right) = -1 \) still holds. In the proof of Corollary \( \text{[\text{11.6}]} \) we have show that the trilinear form \( \mathcal{I}_{\infty} \) is non-zero. It would be interesting if one can find \( \phi \in \Pi_{\infty}, \phi' \in \Pi_{\infty}' \) such that \( \mathcal{I}_{\infty}(\phi \otimes \phi') \neq 0 \) and explicitly compute it.

9.3. Explicit central value formula. Let \( \mathcal{A}(\Pi) \) be the model of \( \Pi \) realized in the space of cusp forms on \( GL_2(\mathbb{A}_E) \). Let \( \Phi = f \otimes g_\kappa \). Then \( \Phi \in \mathcal{A}(\Pi) \) and
\[ \tilde{V}_+^{2m} f \otimes \sigma_\kappa(t)g_\kappa = \Pi((1, t))(\tilde{V}_+^{2m}, 0)\Phi. \]

We put
\[ \langle \Pi_{\infty}((a(-1), a(-1)t_{\infty}))\Phi, \Pi_{\infty}((1, t_{\infty}))\Phi \rangle = \int_{\mathbb{A}_E^{\text{GL_2}(E)\backslash GL_2(\mathbb{A}_E)}} \Phi(h((a(-1), a(-1)t_{\infty})))\Phi(h(1, t_{\infty}))dh. \]
Lemma 9.11. We have
\[
\frac{\langle \Pi_\infty((a(-1), a(-1)t_\infty))\Phi, \Pi_\infty((1, t_\infty))\Phi \rangle}{L(1, \Pi, \text{Ad})} = 2^{-2\kappa'-3}2\xi_\infty(2)^{-1}\xi_\kappa(2)^{-1}D^{-1/2}(2\kappa + 1)^{-1}
\times \left( \frac{\text{GL}_2(\hat{\mathcal{O}}) : \mathbb{H}_0(\mathfrak{M}\hat{\mathcal{O}})}{\text{GL}_2(\hat{\mathcal{Z}}) : \mathbb{K}_0(N\hat{\mathcal{Z}})} \right)^{-1}
\times \prod_{p\mid M} L(1, \sigma_{K,p}, \text{Ad})^{-1}.
\]

Proof. The formula is obtained by specializing the formula in [Wal85 Proposition 6]. We leave the details to the readers. \(\square\)

Proposition 9.12. We have
\[
I(\hat{\mathcal{V}}^2f \otimes \sigma_\kappa(t)g_\kappa)^2 = 2^{-6\kappa+4\kappa'-6+2\nu(\Pi)}\pi^{-4m}D^{-1/2}\xi_\infty(2)^{-2}C(\kappa, \kappa')
\times \prod_{p\mid M} p(1+p)^{-2} \cdot \prod_{p\mid M} p^2(1+p)^{-2}L(1, \sigma_{K,p}, \text{Ad})^{-1}I_p(\Pi_p) \cdot L\left(\frac{1}{2}, \Pi, r\right).
\]

Here \(2m = \kappa - \kappa'\), and \(\nu(\Pi) = 2\{p \in \mathbb{Z} \mid p^3 \mid NM\} + 2\{p \in \mathbb{Z} \mid p \| NM, p \mid N\} + 2\{p \in \mathbb{Z} \mid p^3 \| NM\}.

The values \(I_p(\Pi_p)\) for \(p^4 \mid M\) are given in Propositions 9.6–9.10.

Proof. By [Ich08 Theorem 1.1 and Remark 1.3],
\[
I(\hat{\mathcal{V}}^2f \otimes \sigma_\kappa(t)g_\kappa)^2 = 2^{-1}\xi_\infty(2)^{-1}\xi_\kappa(2) \cdot \prod_{p\mid M} p^2(1+p)^{-2}
\times \langle \Pi_\infty((a(-1), a(-1)t_\infty))\Phi, \Pi_\infty((1, t_\infty))\Phi \rangle
\times \frac{\langle \Pi_\infty((a(-1), a(-1)t_\infty))\Phi, \Pi_\infty((1, t_\infty))\Phi \rangle}{L(1, \Pi, \text{Ad})} \cdot L\left(\frac{1}{2}, \Pi, r\right).
\]

By Propositions 9.6–9.10
\[
I_\infty(\Pi_\infty) \prod_{p\mid M} I_p(\Pi_p) = 2^{-6\kappa+6\kappa'-2+2\nu(\Pi)}\pi^{-4m}(2\kappa + 1)^{-1}C(\kappa, \kappa') \prod_{p\mid M} p^2 \prod_{p\mid M} (1+p)^{-2}
\times (NM)^{-1}\left( \text{GL}_2(\hat{\mathcal{O}}) : \mathbb{H}_0(\mathfrak{M}\hat{\mathcal{O}}) \right) \left( \text{GL}_2(\hat{\mathcal{Z}}) : \mathbb{K}_0(N\hat{\mathcal{Z}}) \right).
\]

The proposition then follows from Lemma 9.11 and Ichino’s central value formula (9.3). \(\square\)

10. Pullback formula

10.1. Setting and main theorem. Let \(f \in S_{2\kappa}(\Gamma_0(N_1))\) and \(g \in S_{\kappa+1}(\Gamma_0(N_2))\) be normalized newforms. For \(p \mid N_1\), let \(\{\alpha_p, \alpha_p^{-1}\}\) be the Satake parameter of \(f\) at \(p\). For \(p \mid N_1\), fix an element \(\tau_p \in \mathbb{Z}_{p'}^\times\) such that \((p, \tau_p)\mathcal{O}_{p} = p^{-\kappa'+1}a_f(p)\). Let \(h \in S_{\kappa+1,2}(\Gamma_0(4N_1))\) be a newform in the sense of [Koh82] associated to \(f\) by the Shintani lift (cf. [Shi76b]). We impose the following hypothesis on the levels and weights:

Hypothesis (H).
(i) \(\kappa \geq \kappa'\).
(ii) \(N_1\) is odd square-free.
(iii) \(N_2\) is odd cubic-free.
(iv) \(N_1 \mid N_2\).
(v) \(N_1, N_2 / N_1 = 1\).

Assume further \(\kappa'\) is odd and \(\kappa - \kappa' = 2m\). Let \(F_{N_2}\) be the Saito-Kurokawa lift of \(h\) with respect to \(N_2\) defined in (5.2). Let \(f, g, h, F_{N_2}\) be the cusp forms on \(\text{GL}_2(\hat{\mathfrak{A}}), \text{SL}_2(\hat{\mathfrak{A}}),\) and \(\text{GSp}_2(\hat{\mathfrak{A}})\) associated to \(f, g, h, \) and \(F_{N_2},\) respectively. For \(2r \in \mathbb{N}\) and \(r' \in \mathbb{N},\) let \(\delta_r\) and \(\Delta_{r'}\) denote the differential operators...
For each place $v$, note that algebras defined in (3.10), (3.16), and (6.5), respectively. Define the $SO(2, 2)$-period
\[
\langle \Delta_{\epsilon^t+1}^m F_N^2 |_{\mathcal{S} \times \mathcal{S}}, g \times g \rangle
\]
\[(10.1)\]
\[
= \frac{1}{(\text{SL}_2(\mathbb{Z}) : \Gamma_0(N_2))^2} \int_{\Gamma_0(N_2) \backslash \mathcal{S}} \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathcal{S}} \Delta_{\epsilon^t+1}^m F_N^2 \left( \begin{array}{cc} \tau_1 & 0 \\ 0 & \tau_2 \end{array} \right) g(\tau_1)g(\tau_2)y_1^{\epsilon_t-1}y_2^{\epsilon_t-1}d\tau_1d\tau_2.
\]

Let $\pi$ and $\sigma$ be the cuspidal automorphic representations of $GL_2(\mathbb{A})$ generated by $f$ and $g$, respectively. For each place $v$, let $L(s, \text{Sym}^2(\sigma_v) \otimes \pi_v)$ be the $L$-function for $\text{Sym}^2(\sigma_v) \otimes \pi_v$ defined in [CC] §2.5. Let $L(s, \text{Sym}^2(\sigma) \otimes \pi)$ be the automorphic $L$-function for $\text{Sym}^2(\sigma) \otimes \pi$ defined by
\[
L(s, \text{Sym}^2(\sigma) \otimes \pi) = \prod_v L(s, \text{Sym}^2(\sigma_v) \otimes \pi_v).
\]

Note that $L(s, \text{Sym}^2(\sigma) \otimes \pi)$ is holomorphic at $s = \frac{1}{2}$. Define the motivic $L$-function for $\text{Sym}^2(g) \otimes f$ and its associated completed $L$-function by
\[
L(s, \text{Sym}^2(g) \otimes f) = \prod_p L\left(s + \kappa + \kappa' + \frac{1}{2}, \text{Sym}^2(\sigma_p) \otimes \pi_p\right),
\]
\[(10.2)\]
\[
\Lambda(s, \text{Sym}^2(g) \otimes f) = L\left(s + \kappa + \kappa' + \frac{1}{2}, \text{Sym}^2(\sigma) \otimes \pi\right).
\]

**Definition 1.** Let $C(f, g) = \prod_v C_v(f, g)$ be a non-zero number in the Hecke field of $f$ defined as follows:
- If $v \nmid N_2$, then $C_p(f, g) = 1$.
- If $v \mid N_2$, then $C_p(f, g) = p(1+p)^{-2}$.
- If $v$ with $p^2 \mid N_2$, and $\sigma_p$ is a ramified principal series representation with $L(s, \sigma_p, \text{Ad}) = \zeta_{Q_p}(s)$, then $C_p(f, g) = p^{-1}(1+p)^{-2}(p-1)^2$.
- If $v$ with $p^2 \mid N_2$, and $\sigma_p$ is a ramified principal series representation with $L(s, \sigma_p, \text{Ad}) = \zeta_{Q_p}(s)^3$, then $C_p(f, g) = p^{-1}(1+p)^{-2}(p-1)^2 + (1+p)^{-1}-4p^{-1/2}a_p + (1+p^{-1})a_p^2$.
- If $v$ with $p^2 \mid N_2$, and $\sigma_p$ is a ramified Steinberg representation, then $C_p(f, g) = p^{-3}(p-1)^2$.
- If $v$ with $p^2 \mid N_2$, and $\sigma_p$ is a supercuspidal representation, then $C_p(f, g) = p^{-1}$.
- If $v = \infty$, then $C_\infty(f, g) = C(\kappa, \kappa') \left( \frac{\Gamma(\kappa - 2m)\Gamma(m+1)}{\Gamma(\kappa - m)\Gamma(2m+1)} \right)^2$.

Here $C(\kappa, \kappa')$ is a non-zero rational number defined in (12).

Following is our main theorem.

**Theorem 10.1.** Assume $\kappa'$ is odd and $\kappa - \kappa' = 2m$, and Hypothesis (H) holds. We have
\[
|\langle \Delta_{\epsilon^t+1}^m F_N^2 |_{\mathcal{S} \times \mathcal{S}}, g \times g \rangle|^2 = 2^{-\kappa-6m-1}C(f, g)\left\langle h, h \right\rangle \Lambda(\kappa + \kappa', \text{Sym}^2(g) \otimes f).
\]

The proof is given in [10.3] As a corollary of Theorem 10.1 we obtain new cases of Deligne’s conjecture for the central critical value $\Lambda(\kappa + \kappa', \text{Sym}^2(g) \otimes f)$. We begin with

**Lemma 10.2.** Assume $\kappa'$ is odd and $c_h(n) \in \mathbb{R}$ for all $n \in \mathbb{N}$. For $\sigma \in \text{Aut}(\mathbb{C})$,
\[
\left( \frac{\langle f, f \rangle}{\langle h, h \rangle \Omega_f^2} \right)^\sigma = \frac{\langle f^\sigma, f^\sigma \rangle}{\langle h^\sigma, h^\sigma \rangle \Omega_{f^\sigma}^2}.
\]
Proof. Since $\kappa'$ is odd and $N_1$ is odd square-free, by the nonvanishing theorem in [FH95], there exist a fundamental discriminant $-D < 0$ such that:
- $D$ is prime to $2N_1$.
- $(p, -pD)_{Q_p} = -1$ for $p \mid N_1$.
- $L \left( \frac{1}{2}, \pi \otimes \chi_D \right) \neq 0$.

Here $\chi_D$ is the quadratic Hecke character of $\mathbb{A}^\times / \mathbb{Q}^\times$ associated to $\mathbb{Q}(\sqrt{-D})$ by class field theory, and $L(s, \pi \otimes \chi_D)$ is the automorphic $L$-function for the automorphic representation $\pi \otimes \chi_D$ of $GL_2(\mathbb{A})$.

Let $\sigma \in \text{Aut}(\mathbb{C})$. By the Kohnen-Zagier formula [Koh85, Corollary 1],

$$
\frac{c_h(D)}{\langle h, h \rangle} = 2^{\kappa - 1 + \nu(N_1)} L(1/2, \pi \otimes \chi_D) \langle f, f \rangle,
$$

By [Shi77],

$$
\left( \frac{L(1/2, \pi \otimes \chi_D)}{D^{1/2} \Omega_f^+} \right)^\sigma = \frac{L(1/2, \pi^\sigma \otimes \chi_D)}{D^{1/2} \Omega_f^+ \sigma}.
$$

This completes the proof. \qed

Corollary 10.3. Assume Hypothesis (H) holds. For $\sigma \in \text{Aut}(\mathbb{C})$,

$$
\left( \frac{\Lambda(\kappa + \kappa', \text{Sym}^2(g) \otimes f)}{\langle g, g \rangle^{2\Omega_f^+}} \right)^\sigma = \frac{\Lambda(\kappa + \kappa', \text{Sym}^2(g^\sigma) \otimes f^\sigma)}{\langle g^\sigma, g^\sigma \rangle^{2\Omega_f^+ \sigma}}.
$$

Here $\Omega_f^+$ is the plus period of $f$ defined in [Shi77].

Proof. If $\kappa'$ is even, then

$$
\epsilon \left( \frac{1}{2}, \text{Sym}^2(\sigma) \otimes \pi \right) = -1.
$$

Therefore $\Lambda(\kappa + \kappa', \text{Sym}^2(g) \otimes f) = 0$ by functional equation and the conclusion holds.

Assume $\kappa'$ is odd. Put $m = \frac{\kappa - \kappa'}{2}$. Let $\mathbb{Q}(f)$ and $\mathbb{Q}(g)$ be the Hecke fields of $f$ and $g$, respectively. We may assume $c_h(n) \in \mathbb{Q}(f)$ for all $n \in \mathbb{N}$ (cf. [Shi82, Proposition 4.5] and [Pra09, Theorem 4.5]). Then,

$$
\frac{\langle \Delta_{\kappa'}^{m+1} F_{N_2} | \delta \times \delta, g \times g \rangle}{\langle g, g \rangle^{2\Omega_f^+}} \in \mathbb{Q}(f) \mathbb{Q}(g).
$$

In particular, $\langle \Delta_{\kappa'}^{m+1} F_{N_2} | \delta \times \delta, g \times g \rangle \in \mathbb{R}$.

Let $\sigma \in \text{Aut}(\mathbb{C})$, then $h^\sigma \in S_{\kappa, \kappa'}^{+1}(\Gamma_0(4N))$ is a newform associated to $f^\sigma$ by the Shimura-Shintani-Waldspurger correspondence (cf. [Pra09, Theorem 4.5]), and $(F_{N_2})^\sigma$ is the Saito-Kurokawa lift of $h^\sigma$ with respect to $N_2$ defined in (6.24). Recall that if $\phi : \mathbb{H} \to \mathbb{C}$ is a nearly holomorphic modular form with Fourier expansion

$$
\phi(\tau) = \sum_{i=0}^{k} \left( \frac{1}{4\pi i} \right)^{i} \sum_{n=0}^{\infty} a_{i,n} q^{n},
$$

then $\phi^\sigma$ is a nearly holomorphic modular form whose Fourier expansion is given by

$$
\phi^\sigma(\tau) = \sum_{i=0}^{k} \left( \frac{1}{4\pi i} \right)^{i} \sum_{n=0}^{\infty} a_{i,n}^\sigma q^{n}.
$$

Therefore, by (6.22) and (6.3)

$$
\Delta_{\kappa'}^{m+1} F_{N_2} | \delta \times \delta = (\Delta_{\kappa'}^{m+1} F_{N_2} | \delta \times \delta)^\sigma.
$$

Since $g$ is a newform and $\Delta_{\kappa'}^{m+1} F_{N_2} | \delta \times \delta$ is a nearly holomorphic modular form on $\mathbb{H}$, we have (cf. [Stu80, Theorem 4] and [Shi76a])

$$
\left( \frac{\Delta_{\kappa'}^{m+1} F_{N_2} | \delta \times \delta, g \times g}{\langle g, g \rangle^{2}} \right)^\sigma = \langle (\Delta_{\kappa'}^{m+1} F_{N_2} | \delta \times \delta)^\sigma, g^\sigma \times g^\sigma \rangle \langle g^\sigma, g^\sigma \rangle^{2} = (\Delta_{\kappa'}^{m+1} F_{N_2} | \delta \times \delta, g^\sigma \times g^\sigma \rangle \langle g^\sigma, g^\sigma \rangle^{2}
$$

(10.3)
For $p | N_2/N_1$, let $A_f(p; B)$ be the algebraic number in $\mathbb{Q}(a_f(p))$ defined in §6.1. By definition, we have

$$A_f(p; B)^\sigma = A_{f^*}(p; B).$$

Therefore, $(F_{N_2})^\sigma$ is the Saito-Kurokawa lift of $h^\sigma$ with respect to $N_2$ defined in §6.2. By the definition of the constant $C(f, g)$, we have

$$C(f, g)^\sigma = C(f^*, g^*).$$

The corollary then follows from Theorem 10.1, Lemma 10.2, and (10.3). This completes the proof. \hfill $\square$

10.2. Seesaw identities and explicit central value formula. Let $\psi$ be the standard additive character of $\mathbb{A}$. Let $(V^{(0)}, Q^{(0)})$ be the quadratic space over $\mathbb{Q}$ defined by $V^{(0)} = \mathbb{Q}$ and $Q^{(0)}(x) = x^2$. Let $\omega_{\psi, V^{(0)}, 1}$ be the Weil representation of $\text{SL}_2(\mathbb{A}) \times \text{O}(V^{(0)})(\mathbb{A})$ on $S(\mathbb{A})$ defined in §4.1. Define $\varphi^{(0)} = \otimes_{v} \varphi^{(0)}_v \in S_v(\mathbb{A})$ as follows:

- If $v = p$, then $\varphi^{(0)}_p = \mathbb{I}_{\mathbb{Z}_p}$.
- If $v = \infty$, then $\varphi^{(0)}_\infty(x) = e^{-2\pi x^2}$.

Let $\Theta$ be the automorphic form on $\text{SL}_2(\mathbb{A})$ defined by

$$\Theta(g) = \sum_{x \in \mathbb{Q}} \omega_{\psi, V^{(0)}, 1}(g, 1)\varphi^{(0)}(x).$$

Note that the modular forms of weight 1/2 corresponding to $\Theta$ is the elementary theta function $\theta(\tau) = \sum_{n \in \mathbb{Z}} \varphi^{\sigma}(n)$. Since $\kappa'$ is assumed to be odd, by the nonvanishing theorem in [FH95] and [Wal91], there exist a fundamental discriminant $-D < 0$ such that:

- $D$ is prime to $N_2$.
- $-D \equiv 1 \mod 8$.
- $(p, -\tau_D)_{\mathbb{Q}_p} = -1$ for $p | N_1$.
- $(p, -D)_{\mathbb{Q}_p} = 1$ for $p | N_2/N_1$.
- $\mathbb{L}(\frac{\sqrt{-D}}{2}, \pi \otimes \chi_D) \neq 0$.

Let $K = \mathbb{Q}(\sqrt{-D})$ be the imaginary quadratic field with discriminant $-D$. Let $\delta = \sqrt{-D}$. Following the notation in §7.1, let $l = \kappa + 1$. Let $g_{\delta}^{l}$ be the cusp form on $\text{GL}_2(\mathbb{A})$ defined by (7.4) and $g_{\delta}^{l}$ be the cusp form on $\text{GL}_2(\mathbb{A}_K)$ defined by (8.4) associated to $\mathbb{F}^l$ via the base change lift.

10.2.1. The first seesaw identity. Let $(V^{(1)}, Q^{(1)})$ and $(V^{(2)}, Q^{(2)})$ be the quadratic spaces over $\mathbb{Q}$ defined in §7.1 and §6.1 respectively. Let $\varphi^{(1)} \in S(V^{(1)}(\mathbb{A}))$ and $\varphi^{(2)} \in S(V^{(2)}(\mathbb{A}))$ be the test functions appeared in Corollary 7.5 and 6.1 respectively. We identify $V^{(0)} \oplus V^{(1)}$ with $V^{(2)}$ by the following isomorphism of quadratic spaces:

$$V^{(0)} \oplus V^{(1)} \longrightarrow V^{(2)}$$

$$\left( x, \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) \mapsto \begin{pmatrix} x & x_1 & 0 & x_2 \\ x_4 & -x & -x_2 & 0 \\ 0 & x_3 & x & x_4 \\ -x_3 & 0 & x_1 & -x \end{pmatrix}.$$
we have an embedding $SO(V^{(1)}) \subset SO(V^{(2)})$. Defined the $SL_2$-period

\[(10.4) \quad (\tilde{V}_+^m h_{N_2} \Theta, \mathbf{g}^2) = \int_{SL_2(\mathbb{Q})/SL_2(\mathbb{A})} \tilde{V}_+^m h_{N_2}(g) \Theta(g) \mathbf{g}^2(g) dg.\]

**Proposition 10.4.** We have

\[
(\Delta_{\kappa+1}^m F_{N_2}|_{g \times g}, g \times g) = 2^{\kappa-2m+2} \xi_Q(2) \langle g, g \rangle (\tilde{V}_+^m h_{N_2} \Theta, \mathbf{g}^2).
\]

**Proof.** We have a seesaw identity

\[
(\theta_\psi(\tilde{V}_+^m h_{N_2}, \varphi^{(2)}), \mathbf{g} \otimes \mathbf{g}) = (\tilde{V}_+^m h_{N_2}, \theta_\psi(\mathbf{g} \otimes \mathbf{g}, \varphi^{(1)}) \Theta).
\]

By Proposition 6.6 the left-hand side is equal to

\[
2^{2m-3} \xi_Q(2)^{-3} \left( \text{GL}_2(\tilde{\mathbb{Z}}) : K_0(N_2 \tilde{\mathbb{Z}}) \right)^{-1} (\Delta_{\kappa+1}^m F_{N_2}|_{g \times g}, g \times g).
\]

By Proposition 6.6 the right-hand side is equal to

\[
2^{\kappa-1} \xi_Q(2)^{-2} \left( \text{GL}_2(\tilde{\mathbb{Z}}) : K_0(N_2 \tilde{\mathbb{Z}}) \right)^{-1} \langle g, g \rangle (\tilde{V}_+^m h_{N_2} \Theta, \mathbf{g}^2).
\]

This completes the proof. \(\square\)

10.2.2. *The second seesaw identity.* Let $(V^{(3)}, Q^{(3)})$ and $(V^{(4)}, Q^{(4)})$ be the quadratic spaces over $\mathbb{Q}$ defined in [5.1] and [8.1] respectively. Let $\varphi^{(3)} \in S(V^{(3)}(\mathbb{A}))$ and $\varphi^{(4)} \in S(V^{(4)}(\mathbb{A}))$ be the test functions appeared in [5.1] and Corollary 8.7 respectively. We identify $V^{(0)} \oplus V^{(3)}$ with $V^{(4)}$ by the following isomorphism of quadratic spaces:

\[
V^{(0)} \oplus V^{(3)} \longrightarrow V^{(4)}
\]

\[
\begin{pmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 -x_1
\end{pmatrix} \mapsto \begin{pmatrix}
 x + \delta x_1 \\
 \delta x_3 \\
 x - \delta x_1
\end{pmatrix}.
\]

Then $\varphi^{(0)} \circ \varphi^{(3)} = \varphi^{(4)}$. Recall we have exact sequences

\[
1 \longrightarrow \mathbb{G}_m \overset{\iota^{(3)}}{\longrightarrow} \text{GL}_2 \overset{\rho^{(3)}}{\longrightarrow} SO(V^{(3)}) \longrightarrow 1
\]

\[
1 \longrightarrow R_{K/\mathbb{Q}} \mathbb{G}_m \overset{\iota^{(4)}}{\longrightarrow} G(\mathbb{G}_m \times R_{K/\mathbb{Q}} \text{GL}_2) \overset{\rho^{(4)}}{\longrightarrow} SO(V^{(4)}) \longrightarrow 1,
\]

defined in [5.1] and [8.1] respectively. Here

\[
G(\mathbb{G}_m \times R_{K/\mathbb{Q}} \text{GL}_2) = \{ (z, h) \in \mathbb{G}_m \times R_{K/\mathbb{Q}} \text{GL}_2 \mid z^2 = N_{K/\mathbb{Q}}(\det(h))^{-1} \}.
\]

Together with the embedding

\[
\text{GL}_2 \hookrightarrow G(\mathbb{G}_m \times R_{K/\mathbb{Q}} \text{GL}_2)
\]

\[
h \mapsto (\det(h)^{-1}, h),
\]

we have an embedding $SO(V^{(3)}) \subset SO(V^{(4)})$. Defined the $SO(2,1)$-period

\[(10.5) \quad \mathcal{I}(\tilde{V}_+^{2m} \pi(\xi_{N_2}) f \otimes \mathbf{g}^2_K) = \int_{\mathbb{A} \times \text{GL}_2(\mathbb{Q})/\text{GL}_2(\mathbb{A})} \tilde{V}_+^{2m} f(h \xi_{N_2}) \mathbf{g}^2_K(h) dh.
\]

Here $\xi_{N_2} \in \text{GL}_2(\hat{\mathbb{Q}})$ is defined by [5.1].

**Proposition 10.5.** We have

\[
(\tilde{V}_+^m h_{N_2} \Theta, \mathbf{g}^2) = -(2\pi)^{2m} D^{\kappa'/2} c_h(D)^{-1} \left( \frac{\Gamma(\kappa - 2m) \Gamma(m + 1)}{\Gamma(\kappa - m) \Gamma(2m + 1)} \right) \langle f, f \rangle^{-1} \langle h, h \rangle \mathcal{I}(\tilde{V}_+^{2m} \pi(\xi_{N_2}) f \otimes \mathbf{g}^2_K).
\]

Here $c_h(D)$ is the $D$-th Fourier coefficient of $h$. 

50
Proof. We have a seesaw identity
\[
(\theta_0(\tilde{V}_+^{2m} \pi(\xi_{N_2}) f \otimes \chi_{-D}, \varphi^{(3)} \Theta, g^t) = (\tilde{V}_+^{2m} \pi(\xi_{N_2}) f \otimes \chi_{-D}, \theta_0(\tilde{g}_t^\sharp, \varphi^{(4)})).
\]
By Proposition 5.10 the left-hand side of the seesaw identity is equal to
\[
-2^{-2m-1} \pi^{-2m}(\sqrt{-1})D^{-\kappa + m + 1/2}e_{h(D)}\xi(2)^{-1}
\times \left( GL_2(\tilde{Z}) : K_0(N_2\tilde{Z}) \right)^{-1} \langle \Gamma(k-m)\Gamma(2m+1) \rangle \langle f,f \rangle \langle h,h \rangle^{-1} (\tilde{V}_+^{2m} h_{N_2} \Theta, g_t^\sharp).
\]
By Proposition 5.7 the right-hand side of the seesaw identity is equal to
\[
2^{-1}(\sqrt{-1})D^{-\kappa + 1/2} \xi(2)^{-1} \left( GL_2(\tilde{Z}) : K_0(N_2\tilde{Z}) \right)^{-1} \mathcal{I}(\tilde{V}_+^{2m} \pi(\xi_{N_2}) f \otimes g_t^\sharp).
\]
This completes the proof. \(\square\)

10.2.3. Explicit central value formula. Let \(\sigma_K\) be the cuspidal automorphic representations of \(GL_2(\mathbb{A}_K)\) generated by \(g_{\mathbb{A}_K}\). Let \(\Pi = \pi \boxtimes \sigma_K\) be a cuspidal automorphic representation of \(GL_2(\mathbb{A}) \times GL_2(\mathbb{A}_K)\). Let \(L(s, \Pi, r)\) be the triple product \(L\)-function of \(\Pi\) defined in 9.1. By the assumption \(\kappa - \kappa' \geq 0\) and the properties of \(D\),
\[
\epsilon \left( \frac{1}{2}, \Pi_{e,r} \right) = \chi_{-D,e}(-1)
\]
for all place \(v\). Therefore, the assumptions in 9.1 are satisfied.

**Proposition 10.6.** We have
\[
\mathcal{I}(\tilde{V}_+^{2m} \pi(\xi_{N_2}) f \otimes g_t^\sharp)^2 = 2^{-6\kappa + 4\kappa' - 6 + \nu(N_1)} \pi^{-4m} D^{-1/2} \xi(2)^{-2} C(\kappa, \kappa')
\times \prod_p C_p(f,g) \cdot L \left( \frac{1}{2}, \Pi, r \right).
\]
Here \(\nu(N_1)\) is the number of prime divisors of \(N_1\).

**Proof.** In the notation of Proposition 9.12 we have \(\nu(\Pi) = \nu(N_1)\). For \(p^2 \mid N_2\), we have
\[
C_p(f,g) = p^2(1 + p)^{-2} L(1, \sigma_{K,p}, \text{Ad})^{-1} \mathcal{I}_p^\star(\Pi_p).
\]
Let \(t \in GL_2(\mathbb{A}_K)\) be the element defined in 9.1. Then
\[
\mathcal{I}(\tilde{V}_+^{2m} f \otimes \sigma_K(t) g_{\mathbb{A}_K}) = \mathcal{I}(\tilde{V}_+^{2m} \pi(\xi_{N_2}) f \otimes g_t^\sharp).
\]
The proposition then follows from Proposition 9.12. \(\square\)

10.3. Proof of the main theorem. We may assume \(c_h(n) \in \mathbb{R}\) for all \(n \in \mathbb{N}\). Then \(\langle \Delta_{\kappa'}^{m} F_{N_2}|_{\delta \times \delta}, g \times g \rangle \in \mathbb{R}\). Note that
\[
L(s, \Pi, r) = L(s, \text{Sym}^2(\sigma) \boxtimes \pi) L(s, \pi \otimes \chi_{-D}).
\]
By Propositions 10.4, 10.6,
\[
\frac{|\Delta_{\kappa'+1}^{m} F_{N_2}|_{\delta \times \delta}, g \times g \rangle)}{\langle g, g \rangle^2} = 2^{-\kappa - 6m - 1} C(f,g) \langle h,h \rangle \Lambda (\kappa + \kappa', \text{Sym}^2(g) \otimes f)
\times 2^{\kappa' - 1 + \nu(N_1)} c_h(D) D^{\kappa' - 1/2} \langle f,f \rangle^{-1} \langle h,h \rangle L \left( \frac{1}{2}, \pi \otimes \chi_{-D} \right).
\]
By the Kohnen-Zagier formula [Kohh85 Corollary 1],
\[
c_h(D)^2 \langle h,h \rangle = 2^{\kappa' - 1 + \nu(N_1)} D^{\kappa' - 1/2} \langle f,f \rangle \langle 1/2, \pi \otimes \chi_{-D} \rangle.
\]
This completes the proof of Theorem 10.1.
11. LOCAL TRILINEAR PERIOD INTEGRAL IN THE $\mathbb{C} \times \mathbb{R}$ CASE

11.1. Setting. The aim of this section is to give a proof of Proposition 11.10. The main results of this section are Proposition 11.4 and Corollary 11.6. We follow the normalization of measures as in [11.2]. Let $E = \mathbb{R} \times \mathbb{C}$. Let $\psi_1$ be the standard additive character of $\mathbb{R}$, and $\psi_2 = \psi_1 \circ \text{tr}_{\mathbb{C}/\mathbb{R}}$ be an additive character of $\mathbb{C}$. Let $\pi_1$ be the discrete series representation of $\text{GL}_2(\mathbb{R})$ of weight $2\kappa'$, and $\pi_2$ be the principal series representation (cf 11.2.3.3).

$$\text{Ind}_{\mathbb{B}(\mathbb{C})}^{\text{GL}_2(\mathbb{C})}(\mu^\kappa \boxtimes \mu^{-\kappa})$$

for some positive integers $\kappa$ and $\kappa'$. Let $\Pi_\infty = \pi_1 \boxtimes \pi_2$ be an irreducible admissible representation of $\text{GL}_2(E)$. We assume $\kappa - \kappa' \geq 0$ is even. Put $2m = \kappa - \kappa'$. Let $W_\mathbb{R} \in \mathcal{W}(\pi_1, \psi_1)$ and $W_\mathbb{C} \in \mathcal{W}(\pi_2, \psi_2)$ be the Whittaker functions defined in [11.15] and [11.18], respectively. Let $(\ , \ )_1$ and $(\ , \ )_2$ be invariant pairings on $\mathcal{W}(\pi_1, \psi_1) \otimes \mathcal{W}(\pi_1, \psi_1)$ and $\mathcal{W}(\pi_2, \psi_2) \otimes \mathcal{W}(\pi_2, \psi_2)$ defined by

$$\langle W_1, W_2 \rangle_1 = \int_{\mathbb{R}^\times} W_1(a(t)) W_2(a(-t)) d^\times t,$$

$$\langle W_1, W_2 \rangle_2 = \int_{\mathbb{C}^\times} W_1(a(t)) W_2(a(-t)) d^\times t.$$

Define $I_\infty(\Pi_\infty)^* \in \mathbb{C}$ by

$$I_\infty(\Pi_\infty)^* = \frac{1}{\zeta(2)} \cdot \frac{L(1, \Pi_\infty, \text{Ad})}{L(1/2, \Pi_\infty, r)} \cdot \int_{\mathbb{R}^\times \setminus \text{GL}_2(\mathbb{R})} \frac{(\pi_1(g) \check{V}^2 g W_\mathbb{R}, V^2 g W_\mathbb{R})_1}{(\pi_1(a(-1)) W_\mathbb{R}, W_\mathbb{R})_1 (\pi_2(a(-1)) W_\mathbb{C}, W_\mathbb{C})_2} dg.$$

Here $t_\infty = \frac{1}{2} \left( \frac{1}{-\sqrt{-1}} - \sqrt{-1} \right) \in SU(2)$.

Let $(V, Q)$ be the quadratic space over $\mathbb{R}$ defined by $V = M_2$ and $Q(x) = \det(x)$. Note that $G_m \backslash (\text{GL}_2 \times \text{GL}_2) \cong \text{GO}(V)$ as described in [7.1]. Let $\omega_{\psi_1} = \omega_{\psi_1, V, 1}$ and $\omega_{\psi_2} = \omega_{\psi_2, V, 1}$ be the Weil representations of $\text{SL}_2(\mathbb{R}) \times \text{O}(V)(\mathbb{R})$ and $\text{SL}_2(\mathbb{C}) \times \text{O}(V)(\mathbb{C})$ on $S(V(\mathbb{R}))$ and $S(V(\mathbb{C}))$ defined in [11.1], respectively. We extend the Weil representations to representations of $R(\mathbb{R})$ and $R(\mathbb{C})$ as in [11.2]. Here $R = G(\text{SL}_2 \times \text{O}(V))$. Let $\varphi \in S_{\psi_1}(V(\mathbb{R}))$ and $\varphi \in S_{\psi_2}(V(\mathbb{C}))$ be defined by

$$\varphi_R(x) = (x_1 + \sqrt{-1} x_2 + \sqrt{-1} x_3 - x_4)^{2\kappa} e^{-\pi \text{tr}(x' x)},$$

$$\varphi_C(x) = (2\kappa + 1) x_3^{2\kappa} e^{-2\pi \text{tr}(x' x)}.$$

11.2. Comparison of invariant pairings. Following [Ich08] and [Wal85], we define invariant pairings on $\mathcal{W}(\pi_1, \psi_1) \otimes \mathcal{W}(\pi_1, \psi_1)$ and $\mathcal{W}(\pi_2, \psi_2) \otimes \mathcal{W}(\pi_2, \psi_2)$ by realize they as quotients of Weil representations via the Jacquet-Langlands-Shimizu lifts (cf. 7.7). Let

$$\theta_1 : S_{\psi_1}(V(\mathbb{R})) \otimes \mathcal{W}(\pi_1, \psi_1) \longrightarrow \mathcal{W}(\pi_1, \psi_1) \otimes \mathcal{W}(\pi_1, \psi_1),$$

$$\theta_2 : S_{\psi_2}(V(\mathbb{C})) \otimes \mathcal{W}(\pi_2, \psi_2) \longrightarrow \mathcal{W}(\pi_2, \psi_2) \otimes \mathcal{W}(\pi_2, \psi_2)$$

be the equivariant maps defined as in 7.7.2. Define a map $B_1 : S_{\psi_1}(V(\mathbb{R})) \otimes \mathcal{W}(\pi_1, \psi_1) \longrightarrow \mathbb{C}$ by

$$B_1(\varphi, W) = \int_{U(\mathbb{R}) \setminus \text{SL}_2(\mathbb{R})} \omega_{\psi_1}(g, 1) \varphi(1) W(a(-1) g) dg.$$

Similarly, define a map $B_2 : S_{\psi_2}(V(\mathbb{C})) \otimes \mathcal{W}(\pi_2, \psi_2) \longrightarrow \mathbb{C}$ by

$$B_2(\varphi, W) = \int_{U(\mathbb{C}) \setminus \text{SL}_2(\mathbb{C})} \omega_{\psi_2}(g, 1) \varphi(1) W(a(-1) g) dg.$$

By [Ich08] Lemma 3.2, there exists invariant pairings

$$B_1 : \mathcal{W}(\pi_1, \psi_1) \otimes \mathcal{W}(\pi_1, \psi_1) \longrightarrow \mathbb{C},$$

$$B_2 : \mathcal{W}(\pi_2, \psi_2) \otimes \mathcal{W}(\pi_2, \psi_2) \longrightarrow \mathbb{C}$$

such that

$$B_i = B_{i1} \circ \theta_i.$$
for $i = 1, 2$. In particular, for $\varphi \otimes W \in S_{\psi_1}(V(\mathbb{R})) \otimes \mathcal{W}(\pi_1, \psi_1)$, the map
$$\Psi_1(\cdot, \varphi, W) : \text{GL}_2(\mathbb{R}) \to \mathbb{C}$$
$$g \mapsto \tilde{B}_1(\omega_{\psi_1}(1, (g, 1))\varphi, W)$$
is a matrix coefficient of $\pi_1$. Similar for $\sigma_2$.

**Lemma 11.1.** We have
$$\theta_1(\varphi_R, \tilde{V}^{2m}_{+}W_R) = 2^{2k}\tilde{V}^{2m}_{+}W_R \otimes \tilde{V}^{2m}_{+}W_R,$$
$$\theta_2(\varphi_C, \sigma_2(w)W_C) = W_C \otimes W_C$$
and
$$(\cdot, )_i = B_i$$
for $i = 1, 2$.

**Proof.** For $l \in \mathbb{Z}_{\geq 0}$, let $\varphi_l \in S_{\psi_1}(V(\mathbb{R}))$ be defined by
$$\varphi_l(x) = (x_1 + \sqrt{-1}x_2 + \sqrt{-1}x_3 - x_4)^l e^{-\pi \text{tr}(x^2)}.$$ 
Note that $\omega_{\psi_1}(k, (k_\theta, k_\theta))\varphi_l = e^{\sqrt{-1}l(-\theta + \theta_1 + \theta_2)}\varphi_l$ for $k, k_\theta, k_\theta \in SO(2)$, and
$$\varphi_l(x)(x_1 - \sqrt{-1}x_2 + \sqrt{-1}x_3 + x_4)^l e^{-\pi \text{tr}(x^2)}.$$ 
Therefore, for each $l \in \mathbb{Z}_{\geq 0}$, there exists a constant $C_l$ such that
$$\theta_1(\varphi_{2k}W_R, \tilde{V}^{l}_{+}W_R) = C_l \tilde{V}^{l}_{+}W_R \otimes \tilde{V}^{l}_{+}W_R.$$ 
Then,
$$\theta_1(\omega_{\psi_1}(a(-1), (a(-1), 1)) \varphi_{2k}W_R, \pi_1(a(-1))\tilde{V}^{k}_{+}W_R) = C_l \pi_1(a(-1))\tilde{V}^{k}_{+}W_R \otimes \tilde{V}^{k}_{+}W_R.$$ 
By [3.15] and Lemma 5.5,
$$\theta_1(\varphi_{2k}, W_R)(1) = 2^{1-2k}l^{-k}e^{-4l} \int_{\mathbb{R} > 0} a^{2k-2} H_{2k}(\sqrt{|a|}a^{-1}) e^{-\pi(2a+a^{-1})^2} da$$
(1.1)
$$= 2^{2k}l^{-k}e^{-4l}.$$ 
A simple calculation shows that

\begin{align}
(\pi_1(a(-1))\tilde{V}^{l}_{+}W_R \otimes \tilde{V}^{l}_{+}W_R) & = (4\pi)^{-2} \frac{\Gamma(l+1)\Gamma(2k'+l)}{\Gamma(2k') \Gamma(l+1)} (\pi_1(a(-1))W_R \otimes W_R)_1 \\
& = (4\pi)^{-2} \frac{\Gamma(l+1)\Gamma(2k'+l)}{\Gamma(2k')} \end{align}
(1.2)

Note that
$$\omega_{\psi_1}(t(a)k_\theta a(a(-1), (a(-1), 1)) \varphi_l(1) = 2^{l} \sqrt{-1} \theta a l^2 e^{-2\pi a^2}.$$ 
By [3.17] and [Ike98] Lemma 2.1,
$$\tilde{B}_1(\omega_{\psi_1}(a(-1), (a(-1), 1)) \varphi_{2k}W_R, \pi_1(a(-1))\tilde{V}^{k}_{+}W_R)$$
$$= 2^{2k+2l} \int_{\mathbb{R}^4} |a|^{-2} d^4 a \int_{SO(2)} db^l d^4 a^{4k+2l+2j} e^{-4\pi a^2} (-4\pi)^{j-l} \frac{\Gamma(2k'+l)}{\Gamma(2k'+j)} \frac{l(j)}{
(1.3)
$$
$$= (-1)^j 2^{2k+2l} (4\pi)^{-2} \frac{\Gamma(l+1)\Gamma(2k'+l)}{\Gamma(2k')} \Gamma(l+1) \Gamma(2k'+l).$$
It follows from (1.1), (1.2) and (1.3) that $C_l = (-1)^j 2^{2k+2l}$ and $B_1$.

Note that $\varphi_C = \varphi_C$ and
$$\begin{align}
(H, 0) \cdot \varphi_C & = 2k \varphi_C, \\
(0, H) \cdot \varphi_C & = 2k \varphi_C, \\
(X, 0) \cdot \varphi_C & = 0, \\
(0, X) \cdot \varphi_C & = 0. 
\end{align}$$
Therefore, there exits a constant $C$ such that
\[
\theta_2(\varphi_C, \sigma_2(p)W_C) = CW_C \otimes W_C.
\]

By [3.18], [Ich05] Lemma 6.6, and [GI07] 6.653.2,
\[
\theta_2(\varphi_C, \sigma_2(w)W_C)(1)
\]
\[
= (2\kappa + 1) \sum_{n=0}^{2\kappa} \left( \frac{2\kappa}{n} \right) (-\sqrt{-1})^n \int_{SO(2)} e^{-2n\sqrt{-1}\theta} d\theta \int_{SU(2)} \alpha \beta^{2n-m-n} \beta^m dk
\]
\[
\times \int_{\mathbb{R}_+^\times} \int_{\mathbb{R}_+^\times} r^{-2} e^{-2\pi(r^2 + r^2 + r^2 + r^2) + 2\pi \sqrt{\pi r^2(4\pi r^2)}} K_{\kappa-n}(4\pi r^2) d^r r dx
\]
\[
= \int_{\mathbb{R}_+^\times} e^{-4\pi r^2 - 2\pi r^2} K_\kappa(4\pi r^2) d^r r
\]
\[
= K_{\kappa}(4\pi)^2.
\]

We conclude that $C = 1$ and
\[
\theta_2 \left( \omega_p \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (a(-1)\mathbf{t}_\infty, \mathbf{t}_\infty) \right) \varphi_C, \sigma_2(a(-1))W_C \right) = \sigma_2(a(-1)\mathbf{t}_\infty)W_C \otimes \sigma_2(\mathbf{t}_\infty)W_C.
\]

By [3.18] and [GI07] 6.576.4,
\[
\langle \sigma_2(a(-1)\mathbf{t}_\infty)W_C \otimes \sigma_2(\mathbf{t}_\infty)W_C \rangle_2
\]
\[
= 2^{-2\kappa+1} a^{2\kappa+2} \sum_{n=0}^{2\kappa} \sum_{m=0}^{2\kappa} \left( \frac{2\kappa}{n} \right) \left( \frac{2\kappa}{m} \right) \int_{\mathbb{R}/2\pi Z} e^{(2n-m-n)\sqrt{-1}\theta} d\theta \int_{\mathbb{R}_+^\times} r^{2\kappa+2} K_{\kappa-n}(4\pi r) K_{\kappa-m}(4\pi r) d^r r
\]
\[
(11.4)
\]
\[
= 2^{-2\kappa+2} \pi a^{2\kappa+2} \sum_{n=0}^{2\kappa} \left( \frac{2\kappa}{n} \right) 2 \int_{\mathbb{R}_+^\times} r^{2\kappa+2} K_{\kappa-n}(4\pi r) d^r r
\]
\[
= 2^{-2\kappa+3} \pi^{-2\kappa-1}(2\kappa+1)^{-1} \Gamma(\kappa+1)^2.
\]

Note that
\[
W_C(a(-1)t(r)k(-1)) = r^{2\kappa+2} \sum_{n=0}^{2\kappa} \left( \frac{2\kappa}{n} \right) (-\sqrt{-1})^n \alpha^{2n-\kappa} \beta^{2n} K_{\kappa-n}(4\pi r^2),
\]
\[
\omega_{\psi_2}(d(-1)k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1) \varphi_C(x) = (2\kappa + 1) \sum_{n=0}^{2\kappa} \left( \frac{2\kappa}{n} \right) (-\sqrt{-1})^{2n+2n} \alpha^{2n-\kappa} \beta^{2n} \beta^{2n} x_3^{-2n} e^{-2\pi t(x)}.
\]

for $r \in \mathbb{R}_+^\times$ and $k = \begin{pmatrix} \alpha & \beta \\ -\alpha & -\beta \end{pmatrix} \in SU(2)$. Therefore, by [Ich05] Lemma 6.6 and [GI07] 6.621.3
\[
\tilde{B}_2 \left( \omega_{\psi_2} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (a(-1)\mathbf{t}_\infty, \mathbf{t}_\infty) \right) \varphi_C, \sigma_2(a(-1))W_C \right)
\]
\[
= (2\kappa + 1) \sum_{n=0}^{2\kappa} \sum_{m=0}^{2\kappa} \left( \frac{2\kappa}{n} \right) \left( \frac{2\kappa}{m} \right) \int_{\mathbb{R}/2\pi Z} e^{(2n-m-n)\sqrt{-1}\theta} d\theta \int_{SU(2)} \pi^{2n-\kappa} \alpha^{2n-\kappa} \beta^{2n} m \beta^m dk
\]
\[
(11.5)
\]
\[
\times \int_{\mathbb{R}_+^\times} r^{4\kappa+2} e^{-4\pi r^2} K_{\kappa-m}(4\pi r^2) d^r r
\]
\[
= 2^{-1} \sum_{n=0}^{2\kappa} \left( \frac{2\kappa}{n} \right) \int_{\mathbb{R}_+^\times} r^{2n+1} e^{-4\pi r^2} K_{\kappa-n}(4\pi r) d^r r
\]
\[
= 2^{-2n-3} \pi^{-2n-1}(2\kappa+1)^{-1} \Gamma(n+1)^2.
\]

It follows from (11.9) and (11.5) that $(\ , \ )_2 = B_2$. This completes the proof. \[\Box\]
11.3. Local trilinear period integral and local zeta integral. In [Ich08 Proposition 5.1], Ichino established an elegant equality between the local trilinear period integral and the local zeta integral of Piatetski-Shapiro and Rallis [PSR87]. In this section, specializing to our case, we calculate the corresponding local zeta integral explicitly and deduce the value $I_{\infty}(\Pi_{\infty})^*$ from it.

Let

$$G = \{ g \in \operatorname{Res}_{E/R} \operatorname{GL}_2 \mid \nu(g) \in \mathbb{G}_m \}. $$

We regard the space $E^2$ of row vectors as a symplectic space over $R$ with nondegenerate antisymmetric bilinear form

$$\langle x, y \rangle = \operatorname{tr}_{E/R}(x_1y_2 - x_2y_1),$$

for $x = (x_1, x_2), y = (y_1, y_2) \in E^2$. We choose a basis $\{e_1, e_2, e_3, e'_1, e'_2, e'_3\}$ of $E^2$ over $R$ as follows:

$$e_1 = ((0, 1), (0, 0)), e_2 = ((0, \sqrt{-1}), (0, 0)), e_3 = ((1, 0), (0, 0)), e'_1 = (0, 0), \left(0, \frac{1}{2}\right), e'_2 = (0, 0), \left(0, \frac{-\sqrt{-1}}{2}\right), e'_3 = ((0, 0), (1, 0)).$$

With respect to this basis, we have an embedding

$$G(R) \rightarrow \operatorname{GSp}_3(R)$$

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \mapsto \begin{pmatrix} a & a_2 & 0 & 2b_1 & -2b_2 & 0 \\ -a_2 & a_1 & 0 & -2b_2 & -2b_1 & 0 \\ 0 & 0 & a' & 0 & 0 & b' \\ c_1/2 & c_2/2 & d_1 & -d_2 & 0 \\ c_2/2 & -c_1/2 & 0 & d_2 & d_1 & 0 \\ 0 & 0 & c' & 0 & 0 & d' \end{pmatrix},$$

here $a = a_1 + \sqrt{-1}a_2, b = b_1 + \sqrt{-1}b_2, c = c_1 + \sqrt{-1}c_2, d = d_1 + \sqrt{-1}d_2$. Let $\omega = \omega_{\psi_1, V, 3}$ be the Weil representation of $G(\operatorname{Sp}_3 \times \operatorname{O}(V))(R)$ on $S(V^3(R))$ defined by (4.1) and (4.2). We have an isomorphism

$$\mathbb{R}^3 \rightarrow E$$

$$(x_1, x_2, x_3) \mapsto (x_1, x_2 + \sqrt{-1}x_3).$$

We identify $S(V^3(R))$ with $S(V(E))$ by this isomorphism. Then,

$$\omega((g_1, g_2), h)(\varphi_1 \otimes \varphi_2) = \omega_{\psi_1}(g_1, h)\varphi_1 \otimes \omega_{\psi_2}(g_2, h)\varphi_2$$

for $(g_1, g_2) \in G(R)$ and $h \in \operatorname{GO}(V)(R)$ such that $\det(g_1) = \det(g_2) = \nu(h)$. Define $\varphi_E \in S_{\psi_1}(V^3(R))$ by

$$\varphi_E = \varphi_{\mathbb{R}} \otimes \omega_{\psi_2}(1, (t_{\infty}, t_{\infty}))\varphi_{\mathbb{C}}.$$

Let

$$P = \left\{ \begin{pmatrix} A & 0 \\ 0 & \nu^t A^{-1} \end{pmatrix} \in \operatorname{GSp}_3 \mid A \in \operatorname{GL}_3, \nu \in \mathbb{G}_m \right\}$$

be the Siegel parabolic subgroup of $\operatorname{GSp}_3$ and

$$\rho_P \begin{pmatrix} A & 0 \\ 0 & \nu^t A^{-1} \end{pmatrix} = |\det(A)|_{R}^2 |\nu|_{R}^{-3}.$$

For $s \in \mathbb{C}$, let $I(s) = \operatorname{Ind}_{P(R)}^{G_{\operatorname{Sp}_3}(R)}(\rho_P^s)$ be a degenerate principal series representation of $G_{\operatorname{Sp}_3}(R)$. Define

$$f_{\varphi_E}^{(0)}(g) = |\nu(g)|_{R}^{-3} \omega \left( \begin{pmatrix} 1_3 & 0 \\ 0 & \nu(g)^{-1}1_3 \end{pmatrix} g, 1 \right) \varphi_E(0)$$

for $g \in G_{\operatorname{Sp}_3}(R)$. Then $f_{\varphi_E}^{(0)}$ belongs to the space of $I(0)$. Let

$$K = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mid \alpha + \sqrt{-1}\beta \in \mathbb{U}(3) \right\}$$

55
be a maximal compact subgroup of $\text{GSp}_3(\mathbb{R})$ and $K' = \gamma K \gamma^{-1}$. Here
\[
\gamma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\
-1/2 & 0 & 0 & 0 & 0 & 1/\sqrt{2} \\
0 & -1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/\sqrt{2} & 0 & 0 & 0 \end{pmatrix} \in \text{GSp}_3(\mathbb{R}).
\]

Then $\text{SO}(2) \times \text{SU}(2) \subset K'$. For $s \in \mathbb{C}$, we extend $f_{\varphi_E}^{(0)}$ to a holomorphic section $f_{\varphi_E}^{(s)}$ in the space of $I(s)$ so that its restriction to $K'$ is equal to $f_{\varphi_E}^{(0)}$. Let
\[
U_0 = \{u(x) \mid x \in R_E/R, tr_{E/R}(x) = 0 \}
\]
and
\[
\eta = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \in \text{Sp}_3(\mathbb{Z}).
\]

Define the local zeta integral (cf. [PSR87])
\[
Z_\infty(s, W_E, f_{\varphi_E}) = \int_{R \times U_0(\mathbb{R}) \times G(\mathbb{R})} f_{\varphi_E}(\eta g) W_E(a(-1)g) dg.
\]
Here $W_E \in \mathcal{W}(\Pi_\infty, \psi_1 \circ tr_{E/R})$ is a Whittaker function of $\Pi_\infty$ defined by
\[
W_E(g) = (\tilde{V}_E^{2m} W_\mathbb{R} \otimes \sigma_2(w) W_\mathbb{C})(g).
\]
Note that the integral is absolutely convergent for $\text{Re}(s) > -1/2$ (cf. [Ike92 Lemma 2.1]). The measure on $R^\times \setminus G(\mathbb{R})$ is defined by
\[
\int_{R^\times \setminus G(\mathbb{R})} f(g) dg = \int_{\text{SL}_2(E)} f(g) dg + \int_{\text{SL}_2(E)} f(d(-1)g) dg
\]
for $f \in L^1(R^\times \setminus G(\mathbb{R}))$, here the Haar measure $dg$ on $\text{SL}_2(E)$ is the product measure of $\text{SL}_2(\mathbb{R})$ and $\text{SL}_2(\mathbb{C})$. Then,
\[
\int_{R^\times U_0(\mathbb{R}) \setminus G(\mathbb{R})} f(g) dg = \int_{U_0(\mathbb{R}) \setminus \text{SL}_2(E)} f(g) dg + \int_{U_0(\mathbb{R}) \setminus \text{SL}_2(E)} f(d(-1)g) dg
\]
for $f \in L^1(R^\times U_0(\mathbb{R}) \setminus G(\mathbb{R}))$, here
\[
\int_{U_0(\mathbb{R}) \setminus \text{SL}_2(E)} f(g) dg = \int_{U(E) \setminus \text{SL}_2(E)} \int_{R} f((u(x), 1) g) dx dg.
\]

**Lemma 11.2.** For $x \in \mathbb{R}$, $a_1, a_2 \in \mathbb{R}_+$, $k_\theta \in \text{SO}(2)$, and $k = \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix} \in \text{SU}(2)$, we have
\[
f_{\varphi_E}^{(s)}(\eta(u(x) t(a_1) k_\theta, t(a_2) k))
\]
\[
= \pi^{-2s} \Gamma(2k+2)e^{-2s\sqrt{-1}x} \sum_{n=0}^{2k} \binom{2k}{n} (\sqrt{-1})^n \alpha^n \beta^{2k-n} \times a_1^{2s+2k+2} a_2^{-4s+2k+4} (a_1^2 + a_2^2 + \sqrt{-1}x)^{-s-2k-1} (a_1^2 + 2a_2^2 - \sqrt{-1}x)^{-s-1}.
\]

**Proof.** By the Iwasawa decomposition of $\eta(t(a_1), u(x)t(a_2))$ with respect to $K'$,
\[
f_{\varphi_E}^{(s)}(\eta(u(x) t(a_1) k_\theta, t(a_2) k)) = a_1^{2s} a_2^{4s} (x^2 + (a_1^2 + 2a_2^2)^2)^{-s} f_{\varphi_E}^{(0)}(\eta(t(a_1) k, u(x) t(a_2) k)).
\]
By the Bruhat decomposition of $\eta$ (cf. [Ich05 Lemma 12.1]),
\[
f_{\varphi_E}^{(0)}(\eta g) = \int_{V(\mathbb{R})} \omega(g, 1) \varphi_E(y) dy
\]
for \(g \in \text{Sp}_3(\mathbb{R})\). Here \(dy\) is the Haar measure self-dual with respect to the pairing \(\psi_1((x, y))\). Therefore, by \([11.6]\), \([11.7]\), and \([108\text{, Lemma 6.9}]\)

\[
f^{(s)}_x(q(u(x))t(a_1)k_0, t(a_2)k_0)
\]

\[
= a_1^{2s+2}a_2^{4s+4}(x^2 + (a_1^2 + 2a_2^2)^2)^{-s} \int_{V(\mathbb{R})} \omega_{\psi_1}(k_0, 1) \varphi_\mathbb{R}(a_1 y)\omega_{\psi_2}(k, 1)\varphi_\mathbb{C}(a_2 t_\infty^-y t_\infty) \psi_1(x \det(y)) dy
\]

\[
= 2^{-2s}(2\kappa + 1)e^{-2\kappa \sqrt{-1}\theta} a_1^{2s+2}a_2^{4s+4}(x^2 + (a_1^2 + 2a_2^2)^2)^{-s} \sum_{n=0}^{2\kappa} \binom{2\kappa}{n} (\sqrt{-1})^n a_1^{2n}a_2^{4n} \Phi_{\kappa-n}^{\kappa} dy_1 dy_2 dy_3 dy_4
\]

\[
= \pi^{-2s} \Gamma(2\kappa + 2)\Gamma(2\kappa + 2) e^{-2\kappa \sqrt{-1}\theta} \sum_{n=0}^{2\kappa} \binom{2\kappa}{n} (\sqrt{-1})^n a_1^{2n}a_2^{4n} \Phi_{\kappa-n}^{\kappa} dy_1 dy_2 dy_3 dy_4
\]

\[
\times a_1^{2s+2}a_2^{4s+4}(a_1^2 + 2a_2^2 + \sqrt{-1}x)^{-s-2\kappa-1}(a_1^2 + 2a_2^2 - \sqrt{-1}x)^{-s-1}.
\]

This completes the proof. \(\square\)

**Proposition 11.3.** We have

\[
Z_{\infty}(s, W_E; f_{x^E})
\]

\[
= \pi^{-s-4\kappa+3/2} \frac{\Gamma(2\kappa + 1)\Gamma(\kappa + \kappa')}{\Gamma(s + 2\kappa + 1)\Gamma(s + 2\kappa + 1/2)} \sum_{j=0}^{2\kappa} \sum_{n=0}^{2\kappa} (-1)^j \binom{2\kappa}{j} \binom{2\kappa}{n} \Gamma(2s + j \kappa + j \kappa' + j - n + 1) \Gamma(2s + j \kappa + j - n + 1) \Gamma(2s + j \kappa + j + 1)
\]

\[
\times 3 F_2(s + 1, j - 2m, 2s + j \kappa + j + 1/2; 2s + j \kappa + j + n + 1, 2s + 2\kappa + j + n + 1; 1).
\]

**Proof.** For \(\lambda, \mu, \nu \in \mathbb{C}\), and \(n_1, n_2 \in \mathbb{Z}\), put

\[
I(\lambda, \mu, \nu; n_1, n_2) = \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} a_1^{n_1+1}a_2^{n_2+1} (a_1 + a_2 + \sqrt{-1}x)^{-\lambda-\nu} (a_1 + a_2 - \sqrt{-1}x)^{-\mu} e^{-a_1 + a_2} dx_1 dx_2.
\]

By \([105\text{, Lemmas 12.8 and 12.9}]\),

\[
I(\lambda, \mu, \nu; n_1, n_2)
\]

\[
= 2^{-3s-n_1-n_2} \frac{\Gamma(2s + \nu + n_2)\Gamma(2s - \nu + n_2)\Gamma(s + n_1)\Gamma(s + n_2)\Gamma(s - \lambda - \mu + \nu + n_1 + n_2 + 1)}{\Gamma(2s + n_2 + 1/2)\Gamma(2s - \lambda + \nu + n_1 + n_2 + 1)\Gamma(s + \lambda)}
\]

\[
\times 3 F_2(s + 1, j - 2m, 2s + j \kappa + j + 1/2; 2s + j \kappa + j + n + 1, 2s + 2\kappa + j + n + 1; 1).
\]

By \([3.17], [3.18], [105\text{, Lemma 6.6}],\) and Lemma \([11.2]\)

\[
Z_{\infty}(s, W_E; f_{x^E})
\]

\[
= 2\pi^{-2s} \Gamma(2\kappa + 1) \sum_{j=0}^{2\kappa} \sum_{n=0}^{2\kappa} \binom{2m}{j} \binom{2\kappa}{n} \Gamma(\kappa + \kappa') \Gamma(2\kappa' + j)
\]

\[
\times \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} a_1^{2s+2}a_2^{4s+4} (a_1^2 + 2a_2^2 + \sqrt{-1}x)^{-s-2\kappa-1}(a_1^2 + 2a_2^2 - \sqrt{-1}x)^{-s-1}
\]

\[
\times e^{-a_1 + a_2} \sqrt{-1}x K_{\kappa-n}(4a_2^2)dx_1 dx_2
\]

\[
= \pi^{-s-4\kappa} \Gamma(2\kappa + 1) \sum_{j=0}^{2\kappa} \sum_{n=0}^{2\kappa} (-1)^j \binom{2m}{j} \binom{2\kappa}{n} \Gamma(\kappa + \kappa')
\]

\[
\times 2^{-3s-4\kappa-2m-j} I(2\kappa + 1, 1, \kappa - n; \kappa + \kappa' + j, 2\kappa + 1).
\]
By (11.8),
\[ I(2\kappa + 1, 1, \kappa - n; \kappa + \kappa' + j, 2\kappa + 1) = 2^{-3s-3\kappa-\kappa'} j! \pi^{3/2} \Gamma(2s + \kappa + n + 1)\Gamma(s + \kappa + \kappa' + j)\Gamma(s + 2\kappa + \kappa' + j - n)\Gamma(2s + 3\kappa - n + 1) \]
\[ \times \frac{\Gamma(s + 2\kappa + 1)\Gamma(2s + 2\kappa + 3/2)\Gamma(2s + 2\kappa + \kappa' + j - n + 1)}{\Gamma(s + 2\kappa + 1/2)\Gamma(2s + \kappa' + j + n + 1)} \times 3F_2(s + 1, 2s + \kappa - n + 1, \kappa - n + 1/2; 2s + 2\kappa + 3/2, 2s + 2\kappa + \kappa' + j - n + 1; 1). \]

By a two-term relation for $3F_2$ (cf. [Sha90 (4.3.1.3)]),
\[ 3F_2(s + 1, 2s + \kappa - n + 1, \kappa - n + 1/2; 2s + 2\kappa + 3/2, 2s + 2\kappa + \kappa' + j - n + 1; 1) = \frac{\Gamma(s + \kappa' + j + n)\Gamma(2s + 2\kappa + 3/2)}{\Gamma(s + 2\kappa + 1/2)\Gamma(2s + \kappa' + j + n + 1)} \times 3F_2(s + 1, j - 2m, 2s + \kappa + \kappa' + j + 1/2; 2s + \kappa' + j + n + 1, 2s + 2\kappa + \kappa' + j - n + 1; 1). \]

This completes the proof. \qed

**Proposition 11.4.** We have
\[ T^\infty_\infty(\Pi_\infty) = 2^{-6\kappa+6\kappa'}-2\pi^{-4m}(2\kappa + 1)C(\kappa, \kappa'). \]

Here $C(\kappa, \kappa')$ is the rational number defined in (7.3).

**Proof.** Note that
\[ L(s, \Pi_\infty, r) = \zeta_C(s + \kappa + \kappa' - 1/2)\zeta_C(s + 2\kappa + m + 1/2), \]
\[ L(s, \Pi_\infty, Ad) = \zeta_R(s)\zeta_R(1)^2\zeta_C(s + 2\kappa' - 1)\zeta_C(s + \kappa), \]
\[ \langle \sigma_1(a(-1))W_\infty, W_\infty \rangle_1 = (4\pi)^{-2\kappa'}\Gamma(2\kappa'), \]
\[ \langle \sigma_2(a(1)\tau_\infty)W_\infty, \sigma_2(\tau_\infty)W_\infty \rangle_2 = 2^{-2\kappa - 1}(2\kappa + 1)^{-1}\Gamma(\kappa + 1)^2. \]

It suffices to show that
\[ \int_{\mathbb{R}^\times \setminus GL_2(\mathbb{R})} \Phi(g; \varphi_E, W_E)dg = 2^{-6\kappa + 6\kappa'} \cdot Z_\infty(0, W_E, f_\phi) = 2^{-4\kappa - 4m - 1}(2\kappa + \kappa')\Gamma(\kappa + 1) \Gamma(\kappa')^2 (2m + 1)C(\kappa, \kappa'). \]

Let $\Phi(\cdot; \varphi_E, W_E)$ be a matrix coefficient of $\Pi_\infty$ defined by
\[ \Phi((g_1, g_2); \varphi, W_E) = \tilde{B}_1(\omega_\varphi, 1, (g_1, 1)\varphi_\tilde{r}, V_{\varphi}^{2m}\varphi_\tilde{r})\tilde{B}_2(\omega_{\varphi_2}, 1, (g_2, \tau_\infty, \tau_\infty))\varphi_\tilde{c}, \sigma_2(w)W_\tilde{c}). \]

By [Ich08 Proposition 5.1] and Proposition [11.3]
\[ \int_{\mathbb{R}^\times \setminus GL_2(\mathbb{R})} \Phi(g; \varphi_E, W_E)dg = 2^{-1}\zeta_\infty(2) \cdot Z_\infty(0, W_E, f_\phi) = 2^{-4\kappa - 4m - 1}(2\kappa + \kappa')\Gamma(\kappa + 1) \Gamma(\kappa')^2 (2m + 1)C(\kappa, \kappa'). \]

Note that there is an extra factor $2^{-1}$, which is due to the normalization of measures, comparing with [Ich08 Proposition 5.1]. On the other hand, by Lemma [11.1]
\[ \int_{\mathbb{R}^\times \setminus GL_2(\mathbb{R})} \langle \sigma_1(g)\tilde{V}_{\varphi_\tilde{r}}^{2m}W_\varphi, \tilde{V}_{\varphi_\tilde{r}}^{2m}W_\varphi \rangle_1 \langle \sigma_2(\gamma_\infty)W_\infty, \sigma_2(\gamma_\infty)W_\infty \rangle_2dg = 2^{-2\kappa} \int_{\mathbb{R}^\times \setminus GL_2(\mathbb{R})} \Phi(g; \varphi_E, W_E)dg. \]

This completes the proof. \qed

To complete the proof of Proposition 9.10 it remains to show that $C(\kappa, \kappa') \neq 0$. We introduce yet another local trilinear form of $\Pi_\infty$ by the Rankin-Selberg local zeta integral in the non-split $\mathbb{C}/\mathbb{R}$ case (cf. [Fla88], [Fla93], and [CC]). The corresponding Rankin-Selberg local zeta integral was considered by Ghate in a classical context in [Gha99 §6] with a conjectural formula, which is proved by Lanphier-Skogman and Ochiai in [LS14].

Let $\mathcal{V}$ be a model of $\pi_1$ realized as the irreducible subspace of
\[ \text{Ind}_{\mathbb{B}^{GL_2(\mathbb{R})}}^{GL_2(\mathbb{R})} \left( |, |_{\kappa'}^{1/2} \mathbb{R} \right) \left|_{\mathbb{R}^{-1/2}} \right). \]
Let \( V = \mathcal{V} \otimes W(\sigma_2, \psi_2) \) be a model of \( \Pi_\infty \). Define \( \Psi_\infty \in \text{Hom}_{\text{GL}_2(\mathbb{R})}(\Pi_\infty, \mathbb{C}) \) by
\[
\Psi_\infty(f \otimes W) = \int_{\mathbb{R}^n \setminus U(\mathbb{R})/\text{GL}_2(\mathbb{R})} W(a(\sqrt{-1})f(g)dg
\]
for \( f \otimes W \in V \). Note that the integral is absolutely convergent. Then
\[
\Psi_\infty \otimes \Psi_\infty \in \text{Hom}_{\text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R})}(\Pi_\infty \boxtimes \Pi_\infty, \mathbb{C}).
\]
Let \( f_\mathbb{R} \in \mathcal{V} \) be the section of \( \text{SO}(2) \)-type \( 2\kappa \) normalized so that \( f_\mathbb{R}(1) = 1 \).

**Proposition 11.5.** ([Gha99] [6] and [LS14]) We have
\[
\Psi_\infty(f_\mathbb{R} \otimes \sigma_2(t_\infty)W) = (-1)^{\kappa+m}2^{-2\kappa-\kappa'}\pi^{-\kappa-\kappa'} \frac{\Gamma(2m+1)\Gamma(\kappa-m)\Gamma(\kappa')+1}{\Gamma(m+1)}.
\]

**Proof.** By (3.18) and [GI07 6.561.16],
\[
\Psi_\infty(f_\mathbb{R} \otimes \sigma_2(t_\infty)W) = 2^{-\kappa+1}(-1)^\kappa \sum_{n=0}^{2\kappa} \binom{2\kappa}{n} \int_{\mathbb{R}_+^n} y^{\kappa+\kappa'}(4\pi y)^d y \frac{\Gamma(\kappa'-n)}{\Gamma(\kappa'+n)} \left( \frac{\kappa'+2\kappa-n}{2} \right)
\]
\[
= (-1)^{\kappa}2^{-2\kappa-\kappa'-1} \pi^{-\kappa-\kappa'} \sum_{n=0}^{2\kappa} \binom{2\kappa}{n} (-1)^{(n-\kappa)/2} \frac{\Gamma(\kappa'+n)}{\Gamma(\kappa'+n)} \left( \frac{\kappa'+2\kappa-n}{2} \right).
\]

In the notation of [LS14 Theorem 1.1], put \( n = \kappa - 1, m = 0 \), and \( s = \kappa' - \kappa \), we have
\[
\sum_{n=0}^{2\kappa} \binom{2\kappa}{n} \frac{(-1)^{(n-\kappa)/2}}{2} \frac{\Gamma(\kappa'+n)}{\Gamma(\kappa'+n)} \left( \frac{\kappa'+2\kappa-n}{2} \right) = (-1)^m \frac{\Gamma(2m+1)\Gamma(\kappa-m)\Gamma(\kappa')}{\Gamma(m+1)}.
\]

This completes the proof. \( \square \)

**Corollary 11.6.** We have
\[
C(\kappa, \kappa') \neq 0.
\]

**Proof.** Let \( D \) be the division quaternion algebra over \( \mathbb{R} \). Let \( \Pi_\infty^D \) be the irreducible admissible representation of \( D^\times(\mathbb{R}) \) associated to \( \Pi_\infty \) by the Jacquet-Langlands correspondence. By [Lok01],
\[
\dim \text{Hom}_{\mathbb{R}}(\Pi_\infty, \mathbb{C}) = 1, \quad \dim \text{Hom}_{D^\times(\mathbb{R})}(\Pi_\infty^D, \mathbb{C}) = 0.
\]

Let
\[
\mathcal{I}_\infty \in \text{Hom}_{\text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R})}(\Pi_\infty \boxtimes \Pi_\infty, \mathbb{C})
\]
be the trilinear form defined in [9.2] with respect to the models \( \mathcal{W}(\pi_1, \psi_1), \mathcal{W}(\sigma_2, \psi_2) \), and the invariant pairings \( (,)_1 \) and \( (,)_2 \). By (11.9) and Propositions [11.4] and [11.3], we have \( C(\kappa, \kappa') \neq 0 \) if and only if \( \mathcal{I}_\infty \neq 0 \).

Let \( (V', Q') \) be the quadratic space over \( \mathbb{R} \) defined by \( V' = D \) and \( Q'(x) = N_{D/\mathbb{R}}(x) \). Let \( \omega_{\psi_1, \psi_2, \psi_3} \) be the Weil representation of \( \text{Sp}_3(\mathbb{R}) \times \text{O}(V')(\mathbb{R}) \) on \( S_{\psi_1}(V^3(\mathbb{R})) \) defined by (11.1). Let \( \Theta_{\psi_1, V, 3}(1) \) (resp. \( \Theta_{\psi_1, V', 3}(1) \)) be the maximal quotient of \( S_{\psi_1}(V^3(\mathbb{R})) \) (resp. \( S_{\psi_1}(V'^3(\mathbb{R})) \)) which is trivial as Harish-Chandra module of \( \text{O}(V')(\mathbb{R}) \) (resp. \( \text{O}(V')(\mathbb{R}) \)) with respect to certain maximal compact subgroups. In order to prove \( \mathcal{I}_\infty \neq 0 \), by (11.9), [PSR87 Proposition 3.3], and [Ich08 Proposition 5.1], it suffices to show that
\[
\mathcal{I}(0) = \Theta_{\psi_1, V, 3}(1) \oplus \Theta_{\psi_1, V', 3}(1).
\]

The above assertion is already proved in [LZ97 Theorem 4.12]. This completes the proof. \( \square \)
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