Ubiquitous finite-size scaling features in $I$-$V$ characteristics of various dynamic $XY$ models in two dimensions

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Two-dimensional (2D) $XY$ model subject to three different types of dynamics, namely Monte Carlo, resistivity shunted junction (RSJ), and relaxational dynamics, is numerically simulated. From the comparisons of the current-voltage ($I$-$V$) characteristics, it is found that up to some constants $I$-$V$ curves at a given temperature are identical to each other in a broad range of external currents. Simulations of the Villain model and the modified $2D$ $XY$ model allowing stronger thermal vortex fluctuations are also performed with RSJ type of dynamics. The finite-size scaling suggested in Medvedyeva et al. [Phys. Rev. B (in press)] is confirmed for all dynamic models used, implying that this finite-size scaling behaviors in the vicinity of the Kosterlitz-Thouless transition are quite robust.

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I. INTRODUCTION

The phase transition between the superconducting and normal states in many two-dimensional (2D) systems is of Kosterlitz-Thouless (KT) type [1]. The thermally excited vortices in a large enough sample, interacting via a logarithmic potential, are bound in neutral pairs below the KT transition temperature $T_{KT}$ [1], and as the temperature $T$ is increased across $T_{KT}$ from below these pairs start to unbind. The KT transition has been observed in experiments on superconducting films [3,4], 2D Josephson junction arrays [5], and cuprate superconductors [6]. In these experiments, the current-voltage ($I$-$V$) characteristics have been commonly measured to detect the transition. For small enough currents it has a power law form $V \propto I^a$ (or equivalently, $E \propto J^a$ with the electric field $E$ and the current density $J$), where the $I$-$V$ exponent $a$ is known to have a universal value 3 precisely at the transition [3]; For $T < T_{KT}$ one has $a > 3$, whereas $a = 1$ for $T > T_{KT}$ [7]. The dynamic critical exponent $z$ which relates the relaxation time $\tau$ to the vortex correlation length $\xi$ via $\tau \sim \xi^z$, is connected with $a$ through the relation $a = 1 + z/6$. Consequently, $z$ has the value 2 at $T_{KT}$.

Indeed, the values $z = 2$ and $a = 3$ at the KT transition have been confirmed in many numerical simulations, e.g., the lattice Coulomb gas with Monte Carlo dynamics [5], the Langevin-type molecular dynamics of Coulomb gas particles [10], and the 2D $XY$ model both with the resistively shunted junction dynamics (RSJD) and the relaxational dynamics (RD) [11,12]. That is why $z \approx 6$ obtained by Pierson et al. in Ref. [13] at the resistive transition in many 2D systems is very intriguing. We in Ref. [14] have re-analyzed the experimental $I$-$V$ characteristics for an ultra thin YBCO sample in Ref. [10] and compared with the 2D RSJD model. This led to the suggestion that a novel finite-size type scaling effect of the $I$-$V$ characteristics (which can possibly be caused not only by the actual finite size of the sample but also by a finite perpendicular penetration depth or a residual weak magnetic field [14]) is responsible for the large value of $z \approx 6$ in Ref. [13], but at the same time it was concluded that this exponent has nothing to do with the true dynamic critical exponent which at $T_{KT}$ has value two [14]. Since similar conclusions were also reached in Ref. [15], we believe that there now seems to emerge some consensus [14,15], although the physical mechanism causing this finite-size scaling behavior is still unclear.

In the present paper we search for a possible reason of this finite-size scaling behavior. We use the RSJD [11,12,16], the RD (often referred to as time-dependent Ginzburg-Landau dynamics) [11,12], and the Monte Carlo dynamics (MCD) simulations [7,18] to study the $I$-$V$ characteristics of three different models of 2D superconductors: the Villain model [19], the $XY$ model in its original form, and the $XY$ model modified with $p$-type of potential (see Ref. [20] for details). These three models differ from each other by the density of the thermally created vortices. We come to the conclusion that the qualitative features of the scaling suggested in Ref. [14] are ubiquitous and independent of both the vortex density and the type of dynamics. We also demonstrate that the $I$-$V$ curves obtained with different types of dynamics, up to constant scale factors, coincide very well in a broad range of the external current density. This opens a possibility to use MCD simulation, which from a simulation point of view is more efficient than RSJD or RD, to examine long-time dynamic properties of the models.

The layout of the paper is as follows: In Sec. II we recapitulate the Hamiltonians for the generalized 2D $XY$ model with the $p$-type of potential (the usual $XY$ model is recovered when $p = 1$) and the Villain model, and describe details of the dynamics used (RSJD, RD, and MCD). The results from the usual $XY$ model with $p = 1$ subject to different dynamics (RSJD, RD, and MCD) are presented in Sec. III, while the results from RSJD simulations applied to the different types of models, i.e., the usual $XY$ model with $p = 1$, the $XY$ model with $p = 2$ and the Villain model, are described and analyzed in Sec. IV. We summarize and make final remarks in Sec. V.
A. Model Hamiltonians

The 2D XY model defined on a square lattice, where each lattice point \( i \) is associated with the phase \( \theta_i \) of the superconducting order parameter, is often used for studies of the KT transition. The phase variables in this model interact via the Hamiltonian with nearest neighbor coupling, which in the absence of frustration is given by

\[
H = \sum_{\langle ij \rangle} U(\phi_{ij} = \theta_i - \theta_j),
\]

where \( \langle ij \rangle \) denotes sum over nearest neighbor pairs, \( \phi \) is the angular difference between nearest neighbors, and the interaction potential \( U(\phi) \) is written as

\[
U(\phi) \equiv E_J(1 - \cos \phi),
\]

with the Josephson coupling strength \( E_J \).

The dominant characteristic physical features close to the KT transition are associated with vortex pair fluctuations. One interesting aspect is then how the density of the thermally excited vortex fluctuation effects the critical properties. To study this we generalize the interaction potential by using a parameter \( p \) \[20\]:

\[
U^p(\phi) \equiv 2E_J \left[1 - \cos^{2p^2}\left(\frac{\phi}{2}\right)\right],
\]

where \( U^{p=1}(\phi) \) corresponds to the potential of the usual XY model \[see Eq. (2)\]. The practical point with such generalization is that the vortex density increases with increasing \( p \) \[20\]. The variation of the parameter \( p \) can also change the nature of the transition: for \( p \) exceeding some maximum value \( (p > p_{\text{max}} \approx 5) \) the type of the phase transition changes from KT to the first order \[20,21\]. In the present paper we choose \( p = 2 \) which is well inside the KT transition region, yet is large enough to ensure substantially more vortex fluctuation over a temperature region around the phase transition in comparison with the usual \( p = 1 \) XY model given by Eq. (3). While the XY model with the \( p \)-type potential with \( p > 1 \) has more vortices than the usual \( p = 1 \) XY model, we also study the Villain model \[22\] which has less vortex-antivortex pairs \[22\]. The interaction potential \( U(\phi) \) in the Villain model is given by

\[
e^{-U(\phi)/T} = \sum_{n=-\infty}^{\infty} \exp \left[-\frac{E_J}{2T} (2\pi n - \phi)^2\right].
\]

To simulate the dynamic behaviors of these models we use several types of dynamics: RSJD, RD, and MCD. All these dynamics should result in the same equilibrium static behaviors if we apply them to the models with the same interaction potential. However, dynamic properties of the systems can be different. Of course, different types of dynamics have their own advantages and disadvantages. The RSJD is constructed from the elementary Josephson relations for single Josephson junction that forms the array units, plus Kirchhoff’s current conservation condition at each lattice site\[14,23\]. Therefore, this type of dynamics has a firm physical realization. On the other hand, RSJD is quite slow which leads to the limitation in the time scale one can probe in simulations. Although the RD \[14,21\] is much easier to implement than RSJD, it does not converge much faster than RSJD and it does not have a similar direct physical realization as RSJD. However, a superconductor has been argued to have a RD type of dynamics rather than a RSJD \[22,23\]. The MCD simulations \[18\] are much faster than RSJD or RD, which allows one to investigate dynamic behaviors in much longer time scale (one can also study dynamic behaviors at much lower temperatures with MCD). However, since there is no direct physical realization of the MCD in practice the applicability of this dynamics to a specific physical system must then be explicitly demonstrated. In the following discussions on the details of the different dynamics used, we focus on the original 2D XY model with the interaction potential in Eq. (3) since the extensions to a modified 2D XY model \[6\] and Villain model \[4\] are straightforward.

B. Dynamic models

In this section we briefly review the dynamical equations of motion for RSJD, RD, and MCD, in the presence of the fluctuating twist boundary condition (FTBC) \[11,18\]. We perform simulations of unfrustrated square \( L \times L \) lattices with \( L = 6, 8, \) and \( 10 \) at various temperatures to measure the voltage across the lattice as a function of the external current. Although the system sizes are relatively small, which is inevitable because of the low temperatures and the small external currents used here, the FTBC has been shown to be very efficient in reducing the artifact due to small system sizes \[11,18\], and reliable results can be established \[11,13,18\].

In the FTBC, the twist variable \( \Delta = (\Delta_x, \Delta_y) \) is introduced and the phase difference \( \phi_{ij} \) on the bond \( (i, j) \) is changed into \( \theta_i - \theta_j - \mathbf{r}_{ij} \cdot \Delta \), with the unit vector \( \mathbf{r}_{ij} \) from site \( i \) to site \( j \), while the periodicity on \( \theta_i \) is imposed: \( \theta_i = \theta_{i+Lx} = \theta_{i+Ly} \). The Hamiltonian of 2D \( L \times L \) XY model under FTBC without external current has been introduced in Ref. \[25\], and is written as [compare with Eq. (1)]

\[
H = -E_J \sum_{\langle ij \rangle} \cos(\phi_{ij} \equiv \theta_i - \theta_j - \mathbf{r}_{ij} \cdot \Delta),
\]

which later in Ref. \[18\] has been extended to the system in the presence of an external current and written as

\[
H = -E_J \sum_{\langle ij \rangle} \cos \phi_{ij} + \frac{\hbar}{2e} L^2 J \Delta_x,
\]

where \( J \) the current density in the \( x \) direction.
1. RSJD and RD

We introduce first the RSJD equations of motion for phase variables and twist variables, which are generated from the local (global) current conservation for the phase (twist) variables (see Ref. [11] for details and discussions). The net current $I_{ij}$ from site $i$ to site $j$ is the sum of the supercurrent $I^s_{ij}$, the normal resistive current $I^r_{ij}$, and the thermal noise current $I^t_{ij}$: $I_{ij} = I^s_{ij} + I^r_{ij} + I^t_{ij}$. The supercurrent is given by the Josephson current-phase relation, $I^s_{ij} = I_c \sin \phi_{ij}$, where $I_c = 2eE_J/\hbar$ is the critical current of the single junction. The normal resistive current is given by $I^r_{ij} = V_{ij}/r$, where $V_{ij}$ is the potential difference across the junction, and $r$ is the shunt resistance. Finally, the thermal noise current $I^t_{ij}$ in the shunt at temperature $T$ satisfies $\langle I^t_{ij} \rangle = 0$ and $\langle I^t_{ij}(t)I^t_{ij}(0) \rangle = (2k_B T/r) \delta(t) \delta_{ij} \delta_{ij} = \delta(t) \delta_{ij}$, where $\langle \cdots \rangle$ is thermal average, and $\delta(t)$ and $\delta_{ij}$ are Dirac and Kronecker delta, respectively. Using the current conservation law at each site of the lattice together with the Josephson relation $\phi_{ij} = d\phi_{ij}/dt = 2eV_{ij}/\hbar$ one can derive the RSJD equations of motion for phase variables:

$$\dot{\phi}_i = -\sum_j G_{ij} \sum_k \langle \sin \phi_{jk} + \eta_{jk} \rangle,$$

where the primed summation is over the four nearest neighbors of $j$, $G_{ij}$ is the lattice Green function for 2D square lattice, and $\eta_{jk}$ is the dimensionless thermal noise current defined by $\eta_{jk} = \frac{I^t_{jk}}{I_c} = \frac{V_{ij}/r}{2eV_{ij}/\hbar}$. The time, the current, the distance, the energy, and the temperature are normalized in units of $\hbar/2eI_c$, $I_c$, the lattice spacing $a$, the Josephson coupling strength $E_J$, and $E_J/k_B$, respectively. In order to get a closed set of equations we further specify the dynamics of the twist variable $\Delta$ from the condition of the global current conservation that the summation of the all currents through the system in each direction should vanish [11]:

$$\Delta_x = \frac{1}{L^2} \sum_{(ij)_x} \sin \phi_{ij} + \eta_{\Delta_x} = J,$$

$$\Delta_y = \frac{1}{L^2} \sum_{(ij)_y} \sin \phi_{ij} + \eta_{\Delta_y},$$

where $\sum_{(ij)_x}$ denotes the summation over all nearest neighbor links in the $x$ direction, and we apply the external dc current with the current density $J$ in the $x$ direction. Here, the thermal noise terms $\Delta_x$ and $\Delta_y$ obey the conditions $\langle \eta_{\Delta_x} \rangle = \langle \eta_{\Delta_y} \rangle = 0$, and $\langle \eta_{\Delta_x}(t)\eta_{\Delta_y}(0) \rangle = (2T/L^2) \delta(t)$.

In the RD, the equations of motion for the phase variables are written as [11,12,26]

$$\dot{\phi}_i = -\Gamma \frac{\partial H}{\partial \phi_i} + \eta_i = -\sum_j \langle \sin \phi_{ij} + \eta_i \rangle,$$

where $\Gamma$ is a dimensionless constant (we set $\Gamma = 1$ from now one), $H$ is in Eq. (8), $t$ is in units of $\hbar/\Gamma E_J$, and the thermal noise $\eta_i(t)$ at site $i$ satisfies $\langle \eta_i(t) \rangle = 0$ and $\langle \eta_i(t)\eta_i(0) \rangle = 2T \delta(t)$.

The equation of motion for the twist variables in the absence of an external current is of the form (see Ref. [11] for more details)

$$\Delta = -\frac{1}{L^2} \frac{\partial H}{\partial \Delta} + \eta\Delta,$$

which is the same as Eqs. (8) and (9) for RSJD. Accordingly, to some extent the RD may be viewed as a simplified version of the RSJD where the global current conservation is kept but the local current conservation is relaxed.

Consequently, the equations for the phase variables are different for RSJD and RD [Eqs. (8) and (10) respectively] while the same equations (8) and (9) apply to the twist variables for both dynamics. These coupled equations of motion are discretized in time with the time step $\Delta t = 0.05$ and $\Delta t = 0.01$ for RSJD and RD, respectively, and numerically integrated using the second order Runge-Kutta-Helfand-Greenside algorithm [27]. The voltage drop $V$ across the system in the $x$ direction is written as $V = -\Delta \Delta_x$ (see Ref. [11]) in units of $I_c r$ for RSJD and in units of $\Gamma E_J/2e$ for RD, respectively. We measure the electric field $E = \langle V/l \rangle$, to obtain I-V characteristics, where $\langle \cdots \rangle$ denotes the time average performed over $O(10^6)$ time steps for large currents for both RSJD and RD, and $O(5 \times 10^7)$ and $O(10^9)$ steps for small currents for RSJD and RD, respectively.

2. MCD

The technique to simulate 2D XY model with MCD is based on the Hamiltonian [13] and the standard Metropolis algorithm [58]. The one MC step, which we identify as a time unit, is composed as follows [18]:

1. Pick one lattice site and try to rotate the phase angle at the site by an amount randomly chosen in $[-\delta \theta, \delta \theta]$ (we call $\delta \theta$ the trial angle range). The twist variable $\Delta$ is kept constant during the update of the phase variables.

2. Compute the energy difference $\Delta H$ before and after the above try; If $\Delta H < 0$ or if $e^{-\Delta H/T}$ is greater than a random number chosen on the interval $[0,1)$, accept the trial move.

3. Repeat steps 1 and 2 for all the lattice sites to update the phase variables.

4. Update the fluctuating twist variables $\Delta_x$ in the similar way that $L \Delta_x$ is tried to rotate within the angle range $\delta \Delta_x$ with $\theta$, and $\Delta_x$ kept unchanged. (For convenience, we use $\delta \Delta = \delta \theta$).

5. Compute the energy difference $\Delta H$ before and after the trial step 1 for $\Delta_x$. Accept the step 1 if $\Delta H < 0$ otherwise accept it with probability $e^{-\Delta H/T}$ like in step 2; then repeat for $\Delta_y$.

6. Repeat steps 4 and 5 to update $\Delta_y$. 

3
In the MCD simulation the trial angle $\delta \theta = \pi/6$ has been chosen since it is sufficiently small in order to obtain the correct $I$-$V$ characteristics while it is big enough to make MCD much faster than the other dynamic methods \cite{18}. The time-averaged electric field is obtained after equilibration from the averages over $O(10^9)$ (at large currents) to $O(10^{10})$ (at small currents) MC steps.

III. 2D $XY$ Model Subject to Different Types of Dynamics

In this section we use three different types of dynamics, the RSJD, the RD, and the MCD to study dynamic behavior of the 2D $XY$ model with $p = 1$ under the FTBC. The dynamic behavior of the system can be obtained from the complex conductivity, the flux noise spectrum, as well as the $I$-$V$ characteristics which is commonly measured in experiments. We will focus on the $I$-$V$ characteristics in the present paper. As pointed out in Sec. I the RD to some extent may be considered as a simplified version of the RSJD. Thus from this point of view it is perhaps not surprising that these two models (as we will see) contain similar features of the vortex dynamics. In Ref. \cite{29} from the study of a simple dynamic model of isolated magnetic particles in a uniform field it has been shown that the actual dynamics of the model and MCD are in a good agreement when the acceptance ratio of the Metropolis step is low enough. This implies that the MCD should give the same $I$-$V$ characteristics as the RSJD after an appropriately chosen normalization of time, when the trial angle $\delta \theta$ is sufficiently small (it was shown in Ref. \cite{18} that $\delta \theta = \pi/6$ is sufficiently small). We will confirm this further in the present simulations.

In Fig. 1 we compare $I$-$V$ characteristics in the form of the electric field $E$ versus the current density $J$ obtained from RSJD, RD, and MCD simulations in the temperature range $0.70 \leq T \leq 1.50$ (the temperature interval is 0.05 if $0.70 \leq T \leq 1.00$ and 0.10 otherwise) in logs scales for the system size $L = 8$. In order to make $I$-$V$ data of RSJD and RD simulations coincide, $E$ obtained with RD ($E_{RD}$) is multiplied by a temperature-independent factor represented by the horizontal line in the inset of Fig. 1. Since the measured time-averaged electric field is inversely proportional to the time scale for a given dynamics, one can from the ratio $E_{RSJD}/E_{RD}$ infer the correspondence between times of RD and RSJD. From this comparison, we find for $0.70 \leq T \leq 1.00$ that one unit of time in RD approximately corresponds to 0.526 time unit in RSJD, independent of the temperature. Note that the $I$-$V$ curves corresponding to the temperatures exceeding 1.00 are almost a straight lines. Therefore the collapse of data between the different dynamics is trivial for $T > 1.00$. Also the $I$-$V$ characteristics obtained from MCD can be made to collapse on top of the corresponding curves for the RSJD and RD, as shown in Fig. 1. However, in this case the time scale factor, describing how many RSJD time steps one MCD step corresponds to, depends on the temperature, as shown in the inset of Fig. 1, where this factor is shown to be a linear function of temperature for system size $L = 6$, 8, and 10 up to $T = 1.00$. This is in accordance with the model studied in Ref. \cite{29}, where the same linear behavior in terms of the temperature has been found. Thus for the $I$-$V$ curves we have a precise relation between RSJD and MCD: For example, at $T = 0.80$ we get 1 MCS $= 10.2 \times$ RSJD time unit. This opens a practical possibility to study some aspects of dynamic behavior of the $XY$ model by using MCD simulation, which is usually much more efficient than RSJD and RD.

We have shown in this section that the RSJD, the RD, and the MCD applied to the usual $XY$ model gives basically the identical $I$-$V$ characteristics up to some constant factors. From this observation, one can conclude that the dynamic critical behaviors of the $XY$ model inferred from the $I$-$V$ characteristics should be identical for all these dynamic models. We in next section use the RSJD to study the Villain model and $XY$ models with $p = 2$ and presume from the observation in the present section that the conclusion drawn in Sec. IV for the RSJD case should be also valid in the other dynamics (RD and MCD).

IV. Various $XY$ Models Subject to RSJD

To study the critical behavior of the system in the vicinity of the transition one can use scaling relations. Fisher, Fisher, and Huse (FFH) in Ref. \cite{8} proposed that the nonlinear $I$-$V$ characteristics in a $D$-dimensional superconductor scales as

$$E = J\xi^{D-2-z_{\pm}}(J\xi^{D-1}/T),$$

where $\xi$ and $z$ are the correlation length and the dynamic critical exponent, respectively, and $\chi_{\pm}$ is the scaling function above (+) and below (−) the transition. Pierson et al. in Ref. \cite{13} have applied a variant of this FFH scaling approach to the $I$-$V$ data for thin ($D = 2$) superconductors and superfluids and suggested a phase transition with $z \approx 6$. In Ref. \cite{14} another scaling relation has been introduced and it has been shown that a certain finite-size effect which is not included in the FFH scaling may have caused the large $z$ in spite of the fact that the finite size effect precludes the possibility of a real phase transition. This finite-size scaling around and below the KT transition is given by the form (see Ref. \cite{14} for the details)

$$\frac{E}{JR} = h_T(JLg_L(T)),$$

where $R = \lim_{J \to 0}(E/J) \propto L^{-z(T)}$ is a finite-size induced resistance without external current, $h_T(0) = 1$, $h(x) \propto x^{z(T)}$ for large $x$, and $g_L(T)$ is a function of at most $T$ and $L$ such that a finite limit function $g_L(T)$ exists in the large-$L$ limit. For small values of the variable $x = JLg_L(T)$ the $T$-dependence of the scaling is absorbed in a function $g_L(T)$ for each fixed size $L$ giving rise to the scaling form

$$\frac{E}{JR} = h(JLg_L(T)).$$

In Fig. 8 we demonstrate the existence of the finite-size scaling given by Eq. (14) for $L = 8$ within the temperature
intervals $0.70 \leq T \leq 1.10$. The data are obtained for MCD and scaled by the appropriate factor so as to correspond to RSJD and RD (compare Fig. 3). Because of the correspondence between the three types of dynamics (see Sec. I), this also means that the existence of the finite-size scaling given by Eq. (4) is insensitive to the choice of dynamics.

The FFH scaling given by Eq. (12) is correct only in the thermodynamic limit $\xi/L \to 0$. However, from a practical point of view there is a connection between Pierson method, which is based on FFH scaling, and the finite-size scaling introduced by Eq. (14). If we assume that $\xi$ in Eq. (12) is proportional to $R^{-\alpha}$, where $\alpha$ is $T$-independent constant, the connection between the two different scaling approach becomes: $g_L(T) = A_L R^{-\alpha}$ with $A_L$ being a constant which may depend on $L$. In Fig. 3 there are presented three different functions $g_L(T)$ corresponding to $L = 6, 8,$ and 10. These functions are determined from the condition that curves corresponding to the different temperatures should collapse when plotted as $E/JR$ vs $Lg_L(T)$. Since it is well established that the 2D XY model on the square lattice has the KT transition at $T_{KT} \approx 0.892$ (Ref. [10]). Fig. 3 shows that $g_L(T)$ over a limited region in the vicinity of the KT transition is very well represented by the $R^{-\alpha}$ with $\alpha = 1/6$ for all investigated system sizes.

Next we demonstrate how the finite-size scaling given by Eq. (14) works for the $I-V$ data obtained by simulations of the modified XY model. These simulations are done with RSJD. Fig. 4(a) verifies that this scaling indeed exists for the XY model modified with the Villain type of potential introduced by Eq. (3). The inset in Fig. 4(a) shows the scaling function $g_L(T)$ determined by finding the best data collapse for small values of $Lg_L(T)$. One can see that in the vicinity of $T_{KT}$, which is approximately equal to 1.4 for this model, $g_L(T)$ can be fitted by $A_L R^{-\alpha}$ with $\alpha = 1/6$. The data collapse in Fig. 4(b) shows that $E/JR$ is only a function of the scaling variable $Lg_L(T)$ (when $Lg_L(T)$ is small enough) for the XY model modified with $p$-type of potential (Eq. (3)), where $p = 2$. The inset shows the function $g_L(T)$ together with $A_L R^{-\alpha}$. Since $T_{KT} \approx 1.15$ for this model one can again see that the scaling function $g_L(T)$ in the vicinity of KT transition is proportional to $R^{-\alpha}$ with the same exponent $\alpha = 1/6$ as for the original XY and Villain models.

The crucial difference between the Villain model, the usual XY model and the $p = 2$ XY model in the present context is the vortex density. The KT-transitions for these models occur at the Coulomb gas temperatures $T^{CG}_{c} = 0.23, 0.2,$ and 0.1, respectively ($T^{CG}_{c} = \frac{1}{2\pi} (U''(\nu))$ (see Ref. [2])). Lower $T^{CG}_{c}$ means higher vortex density. Thus the finite-size scaling property given by Eq. (4) appears to be independent of vortex density.

V. DISCUSSION AND CONCLUSIONS

We have simulated 2D XY model with three types of dynamics: RSJD, RD, and MCD. The main conclusion of the paper is that the qualitative features of the finite-size scaling given by Eq. (4) are independent of both the vortex density and the type of dynamics. Therefore the finite-size scaling behavior given by Eq. (4) of the finite-size induced tails of the $I-V$ characteristics appears to be a robust feature. From the comparisons of the current-voltage characteristics obtained for each type of dynamics we found that, up to some scale factor, $I-V$ curves at a given temperature are identical over a broad range of external currents. This makes it possible to use MCD simulations, which are more computer efficient than RSJD and RD simulations, to obtain $I-V$ curves corresponding to both RSJD and RD dynamics.

The phase transition for the 2D XY-type models are of KT type with $z = 2$. This raises the intriguing question of the origin of the large $z$ obtained by Pierson et al. [13]. In Ref. [14] it was argued that the Pierson scaling in relation to the finite-size scaling given by Eq. (4) corresponds to the proportionality $g_L(T) \sim R^{-\alpha}$ where $1/\alpha$ is the exponent which corresponds to the "$z\" obtained by the Pierson scaling. The reason for the existence of this scaling like behavior is still unclear.

In the present paper we have shown that, within the class of 2D XY-type models studied, a value $1/\alpha \approx 6$ is obtained independently of the type of dynamics, as well as, of the vortex density. The origin of this seemingly robust behavior calls for further investigations.

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FIG. 1. Comparisons of the $I$-$V$ characteristics for the 2D $XY$ model with MC, RSJ, and RD dynamics on a square lattice with the finite size $L = 8$ plotted as $E = V/L$ against $J = I/L$ in log scales at temperatures $T = 1.5, 1.3, 1.2, 1.1, 1.0, 0.95, 0.90, 0.85, 0.80, 0.75, \text{ and } 0.70$ (from top to bottom). Multiplication by a factor (presented in the inset), which in the case of MCD depends on temperature, makes the curves coincide in a broad range of external current density. The straight line from the origin in the inset shows that the factor in the MCD case is a linear function of temperature for system sizes, $L = 6, 8, 10$ in accordance with expectation [29]. However, for higher $T$ there is a deviation. The horizontal full line in the inset shows that the factor between the RSJD and RD $I$-$V$ curves is independent of $T$ (the factor is multiplied by 10 in the inset).

FIG. 2. The finite-size scaling of the current-voltage characteristics, $E/JR$ vs $LJg_L(T)$, given by Eq. (14). The function $g_L(T)$ is determined from data collapse for small values of $LJg_L(T)$. The data was obtained from MCD and was scaled with a $T$-dependent factor so as to correspond to RSJD and RD; system size $L = 8$ and $1.10 \leq T \leq 0.70$ where each curve corresponds to a fixed $T$.

FIG. 3. The function $g_L(T)$ determined for $I$-$V$ characteristics with size $L = 6$ (open squares), $L = 8$ (open circles), $L = 10$ (open triangles) obtained as in Fig. 2. One notes that the function $g_L$ for these sizes over a limited $T$ interval around KT-transition (at $T \approx 0.90$) is well approximated by $g_L \propto R^{-\alpha}$ with $\alpha \approx 1/6$. The data for $L = 10$ function $gL(T)$ as well as $A_L R^{-\alpha}$ are multiplied by 1.5 for convenience.

FIG. 4. Existence of the scaling in the form Eq. (14) for modified 2D $XY$ models corresponding to different vortex density. Systems with size $L = 8$ have been simulated with RSJD. The finite-size scaling of the $I$-$V$ characteristics obtained for 2D $XY$ model with the Villain potential in the temperature range $1.70 \leq T \leq 1.15$ is shown in (a) whereas (b) shows the same thing for the $p = 2$-potential for $1.30 \leq T \leq 0.95$. The inset on both (a) and (b) shows the function $g_L(T)$ determined from the condition of the best data collapse. It is also shown there that the function $g_L$ for both cases over a limited $T$ interval is well approximated by $g_L \propto R^{-\alpha}$ with $\alpha \approx 1/6$. In (a) $g_L \propto 0.76 R^{-1/6}$ and in (b) $g_L \propto 1.04 R^{-1/6}$.