A detailed analysis for the fundamental solution of fractional vibration equation

1 Introduction

The fractional calculus has been extensively applied to various science and engineering problems to describe the memory and hereditary properties of various materials and processes [1–9]. In particular, the fractional calculus has been applied to the mathematical modelling of viscoelastic materials. For some viscoelastic materials the stress-strain constitutive relation can be more accurately described by introducing the fractional derivatives [9–13].

Scott-Blair [14, 15] introduced the fractional calculus to characterize a viscoelastic material whose mechanical properties are intermediate between those of a pure elastic solid (Hooke model) and a pure viscous fluid (Newton model) [6, 16]. In [11], a fractional calculus element whose constitutive law obeys stress is proportional to a fractional derivative of strain is said to be a spring-pot.

Fractional oscillation or vibration was discussed by Gorenflo and Mainardi [2], Bagley and Torvik [17], Beyer and Kempfle [18], and others [19–28]. In [2], the fractional relaxation and oscillation equations with the Caputo derivative were considered by means of the Laplace transform and Mittag-Leffler function. In [17], the damping term was described by using the Riemann-Liouville fractional derivative, and the Laplace transform was applied to analyze the behavior of the oscillator. In [18], the causal condition for solution of the fractional vibration equation was investigated with the help of the Fourier transformation, operator theory and complex analysis.

Afterwards, Achar et al. [19] studied the response characteristics of the fractional oscillator by extending the classic integral equation to fractional case and applying the Laplace transform and Mittag-Leffler function. Li et al. [20] considered the impulse response and the stability behavior of a class of fractional oscillators with multi-term
Caputo fractional derivatives by using the Laplace transform and Bessel functions. Lim et al. [21] established the relationship between fractional oscillator processes and the corresponding fractional Brownian motion processes.

Shen et al. [22, 23] analyzed the dynamical behavior and resonance for linear and Duffing-type nonlinear fractional oscillators in Caputo sense, respectively, using the averaging method.

Furthermore, the theory of fractional dynamic system has been founded and developed in [29–34]. In [29], Chaos synchronization of the Chua system with Caputo fractional derivative was considered by using numeric method. In [30], the nonlinear dynamic behaviors of oscillators described by fractional derivative were studied, the numerical scheme was developed, and the bifurcation and chaos of the oscillator in forced vibration were shown. In [31], stability of the vibration system with the Caputo fractional derivative was investigated based on stability switch. In [32], the linearization and stability theorems of the nonlinear fractional system were presented. In [33–37], the local fractional calculus and its fractal geometrical explanation and applications were considered.

Next, we recall the definitions of the related fractional derivatives. For additional details we refer the readers to references as [1–9]. Let $f(t)$ be piecewise continuous on $(t_0, +\infty)$ and integrable on any finite subinterval of $(t_0, +\infty)$. Then for $t > t_0$, the Riemann-Liouville fractional integral of $f(t)$ of order $\beta$ is defined as

$$
\int_{t_0}^{t} J_{t_0}^\beta f(\tau) d\tau,
$$

where $\beta$ is a positive real number, and $\Gamma(\cdot)$ is Euler’s gamma function.

The Riemann-Liouville fractional derivative of $f(t)$ of order $\alpha$ is defined as (when it exists)

$$
t_0 D_t^\alpha f(t) = \frac{d^m}{dt^m} \left( t_0 J_t^{m-\alpha} f(t) \right), \quad t > t_0, \quad m - 1 < \alpha < m, \quad m \in \mathbb{N}^+.
$$

Let $f^{(m)}(t)$ exist and be integrable on any finite subinterval of $(t_0, +\infty)$. Then the Caputo fractional derivative of $f(t)$ of order $\alpha$ is defined as

$$
t_0 D_t^\alpha f(t) = t_0 J_t^{m-\alpha} f^{(m)}(t), \quad t > t_0, \quad m - 1 < \alpha < m, \quad m \in \mathbb{N}^+.
$$

For a linear fractional system, the Laplace transform is frequently used for analytical research [9, 37–39]. Unlike the Riemann-Liouville fractional derivative, the Laplace transform of the Caputo fractional derivatives only involves the initial values of the integer-order derivatives [9]. We model the fractional vibration equation with the Caputo fractional derivatives associated with the initial conditions in the traditional form. In the sequel, we denote the operator $t_0 D_t^\alpha$ as $D_t^\alpha$ for short. The following formula of the Laplace transform will be used,

$$
\mathcal{L} \left[ D_t^\alpha f(t) \right] = s^\alpha F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad m - 1 < \alpha < m, \quad m \in \mathbb{N}^+,
$$

where

$$
F(s) = \int_0^{+\infty} f(t) e^{-st} dt,
$$

is the Laplace transform of the function $f(t)$.

In this paper we investigate the solution of the fractional vibration equation, where the damping term is characterized by means of the Caputo fractional derivative with the order $\alpha$ satisfying $0 < \alpha < 1$ or $1 < \alpha < 2$. We give a detailed analysis for the fundamental solution $y(t)$ through the complex inversion integral of the Laplace transform.

The text is organized as follows. In the next section, we recall the classical vibration equation and its resolution by the Laplace transform. In Section 3, we consider the fractional vibration equation for the two cases of the order $0 < \alpha < 1$ and $1 < \alpha < 2$ simultaneously. Section 4 summarizes our conclusions.
2 Classical vibration equation

First we consider the classic vibration equation

\[ x''(t) + b x'(t) + c x(t) = q(t), \quad b > 0, c > 0, \]
\[ x(0) = x_0, \quad x'(0) = x_1. \]  

By using the Laplace transform we have

\[ s^2 X(s) - sx_0 - x_1 + b(sX(s) - x_0) + cX(s) = Q(s). \]

The transform function is solved to be

\[ X(s) = \frac{s + b}{s^2 + bs + c} x_0 + \frac{1}{s^2 + bs + c} x_1 + \frac{Q(s)}{s^2 + bs + c}. \]  

Introducing function \( y(t) \), such that

\[ \mathcal{L}[y(t)] = Y(s) := \frac{s + b}{s^2 + bs + c}, \]

we have

\[ \mathcal{L}[y'(t)] = sY(s) - y(0) = s \frac{s + b}{s^2 + bs + c} - 1 = \frac{-c}{s^2 + bs + c}. \]

where we have used the initial-value theorem \( y(0) = \lim_{s \to \infty} sY(s) = 1 \). Applying the inverse Laplace transform to Eq. (7) we obtain

\[ x(t) = y(t)x_0 - \frac{1}{c} y'(t)x_1 - \frac{1}{c} y'(t) * q(t). \]  

Thus \( y(t) \) and \( -\frac{1}{c} y'(t) \) are the fundamental solution and the impulse-response solution, respectively, for Eq. (5).

In order to compare with the fractional case, we prefer to express the fundamental solution \( y(t) \) by using the complex inversion integral formula

\[ y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2\pi i} \int_{Br} Y(s)e^{st} ds = \frac{1}{2\pi i} \int_{Br} s + b e^{st} ds, \]

where \( Br \) denotes the Bromwich path, i.e. the straight line from \( s = \gamma - i \infty \) to \( s = \gamma + i \infty \), where \( \gamma \) is chosen so that all the singularities of the integrand lie to the left of the line.

In the integer-order case, \( Y(s) \) in Eq. (8) is a meromorphic function having two poles at most depending on the cases of the roots of the characteristic equation

\[ w(s) := s^2 + bs + c = 0. \]

Hence we have

\[ y(t) = \sum_i \text{Res}[Y(s)e^{st}, s_i], \]

where \( s_i \) are the poles of \( Y(s) \). There are the following three cases.

Case 1. Overdamped case \( b^2 - 4c > 0 \).

The characteristic equation has two different negative real roots \( s_{1,2} = -\frac{b \pm \sqrt{b^2 - 4c}}{2} \), which are two simple poles of \( Y(s) \). Calculating the residues we have

\[ y(t) = \frac{s_1 + b}{s_1 - s_2} e^{s_1 t} + \frac{s_2 + b}{s_2 - s_1} e^{s_2 t}. \]  

Case 2. Critically damped case \( b^2 - 4c = 0 \).
The characteristic equation has two identical negative real roots \( s_3 = -\frac{b}{2} \), which is a double pole of \( Y(s) \). Calculating the residues we have

\[
y(t) = e^{s_3 t} + (s_3 + b)t e^{s_3 t}.
\] (14)

**Case 3. Underdamped case** \( b^2 - 4c < 0 \).

The characteristic equation has a couple of conjugate complex roots with negative real parts, \( s_{4,5} = -\frac{b}{2} \pm \frac{\sqrt{4c-b^2}}{2}i \), which are two simple poles of \( Y(s) \). Calculating the residues we have

\[
y(t) = e^{-\frac{b}{2}t} \cos\left(\frac{\sqrt{4c-b^2}}{2}t\right) + \frac{b}{\sqrt{4c-b^2}} e^{-\frac{b}{2}t} \sin\left(\frac{\sqrt{4c-b^2}}{2}t\right).
\] (15)

### 3 Fractional vibration equation

In this section, we consider the fractional vibration equation

\[
x''(t) + b D^\alpha x(t) + c x(t) = q(t), \quad b > 0, c > 0,
\] (16)

\[
x(0) = x_0, \quad x'(0) = x_1,
\] (17)

where \( \alpha \) satisfies \( 0 < \alpha < 1 \) or \( 1 < \alpha < 2 \). We consider the two fractional cases simultaneously, and express their results separately as Case i and Case ii, if necessary.

#### 3.1 General form of solutions

Applying the Laplace transform to Eq. (16) leads to

\[
s^2 X(s) - sx_0 - x_1 + b(s^\alpha X(s) - s^{\alpha-1}x_0 - [\alpha]s^{\alpha-2}x_1) + c X(s) = Q(s),
\]

where we use the round-off notation

\[
[\alpha] = \begin{cases} 0, & 0 < \alpha < 1, \\ 1, & 1 < \alpha < 2, \end{cases}
\]

to combine the two fractional cases. Solving the transform function we have

\[
X(s) = \frac{s + bs^\alpha - 1}{s^2 + bs^\alpha + c} x_0 + \frac{1 + b[\alpha]s^{\alpha-2}}{s^2 + bs^\alpha + c} x_1 + \frac{Q(s)}{s^2 + bs^\alpha + c}.
\] (18)

We express the fundamental solution as \( y(t) \), i.e. it satisfies

\[
\mathcal{L}[y(t)] = Y(s) := \frac{s + bs^\alpha - 1}{s^2 + bs^\alpha + c}.
\] (19)

Since \( y(0) = \lim_{s \to \infty} s Y(s) = 1 \), we have the following two formulas

\[
\mathcal{L}[y'(t)] = sY(s) - y(0) = \frac{-c}{s^2 + bs^\alpha + c},
\] (20)

and

\[
\mathcal{L}\left[ \int_0^t y(t) \, dt \right] = \frac{1}{s} Y(s) = \frac{1 + bs^\alpha - 2}{s^2 + bs^\alpha + c}.
\] (21)

Therefore, the inverse Laplace transform of Eq. (18) yields the solutions for the two cases,

**Case i.** \( 0 < \alpha < 1 \):

\[
x(t) = y(t)x_0 - \frac{1}{c} y'(t)x_1 - \frac{1}{c} y(t) * q(t),
\] (22)
We note that the impulse-response solution is $-\frac{1}{c}y'(t)$. The fundamental solution $y(t)$ satisfies the corresponding homogeneous equation $y''(t) + b\, D^\alpha_t x(t) + c\, x(t) = 0$ subject to the initial conditions $x_0 = 1$ and $x_1 = 0$, and plays an important role for construction of the solution. We focus our attention to the fundamental solution $y(t)$ in the sequel.

### 3.2 Representation of fundamental solution

We investigate the fundamental solution $y(t)$ by the complex inversion formula

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2\pi i} \int_{\text{Br}} Y(s) e^{st} \, ds,$$

where $\text{Br}$ denotes the Bromwich path, i.e. the straight line from $s = \gamma - i\infty$ to $s = \gamma + i\infty$, where $\gamma$ is chosen so that all the singularities of the integrand lie to the left of the line.

For the fractional cases, the original point $s = 0$ is a branch point of the integrand. We make a branch cut along the negative real axis and consider the problem on the principal Riemann surface. Due to Cauchy’s theorem and the residue theorem we can rewrite the right hand side of Eq. (24) as the sum of residues plus a Hankel contour integral, i.e.

$$y(t) = f_1(t) + f_2(t),$$

where

$$f_1(t) = \sum_i \text{Res}[Y(s)e^{st}, s_i],$$

and

$$f_2(t) = \frac{1}{2\pi i} \int_{\text{Ha}} Y(s) e^{st} \, ds,$$

where $s_i$ are the relevant singularities of $Y(s)$, and $\text{Ha}$ denotes the Hankel path, a loop which starts from $-\infty$ along the lower side of the negative real axis, en circules the origin counter clockwise, and ends at $-\infty$ along the upper side of the negative real axis. We display the Bromwich path and the Hankel path in Fig. 1.

**Fig. 1.** The Bromwich path, Hankel path and the simple poles $s_6$ and $s_7$ for the fractional cases.
3.3 Calculation of residues for $f_1(t)$

To calculate $f_1(t)$, we need to find out the relevant singularities of $Y(s)$. First we consider the roots of the characteristic equation

$$w(s) := s^2 + b s^\alpha + c = 0,$$  

(28)

for the two cases.

**Case i.** $0 < \alpha < 1$.

In [24], it was proved in detail that the characteristic equation (28) has a couple of conjugate complex roots with negative real parts for arbitrary real coefficients $b, c > 0$.

**Case ii.** $1 < \alpha < 2$.

Eq. (28) may be rewritten as

$$\left(\frac{1}{s}\right)^2 + \frac{b}{c} \left(\frac{1}{s}\right)^{2-\alpha} + \frac{1}{c} = 0,$$  

(29)

by dividing both sides of Eq. (28) by $s^2c$. For Eq. (29), we can use the result in Case i for $0 < \alpha < 1$ due to $0 < 2 - \alpha < 1$. If we denote the two roots of Eq. (29) in $\frac{1}{s}$ as

$$s_6, 7 = \beta \pm i \sigma = \frac{1}{r_1} e^{\pm i \theta_1}, \quad (\beta < 0, \sigma > 0, r_1 > 0, \frac{\pi}{2} < \theta_1 < \pi),$$

then the roots of the characteristic equation (28) are

$$s_{6,7} = \beta \pm i \sigma = \frac{1}{r_1} e^{\pm i \theta_1}, \quad (\beta < 0, \sigma > 0, r > 0, \frac{\pi}{2} < \theta < \pi),$$

(30)

where $r = \sqrt{\beta^2 + \sigma^2}$ and $\theta = \pi + \arctan(\sigma/\beta)$. Since the numerator and the denominator of the right hand side of Eq. (19) cannot equal zero simultaneously, so the roots $s_6$ and $s_7$ of the characteristic equation are the simple poles of $Y(s)e^{st}$. See Fig. 1 for a graphical representation of the locations of the two simple poles.

For specified $b, c$ and $\alpha$, we can calculate the two roots $s_6$ and $s_7$ of the characteristic equation by a software such as MATHEMATICA. In Table 1, we list the values of $\beta$, $\sigma$, $r$, $\theta$ for $b = c = 1$ and different $\alpha$.

**Table 1.** The data for the roots of Eq. (28) for $b = c = 1$.

| $\alpha$ | $\beta$ | $\sigma$ | $r$ | $\theta$ |
|----------|---------|----------|-----|---------|
| 0.25     | -0.156825 | 1.42239  | 1.43101 | 1.68061 |
| 0.50     | -0.343815 | 1.35843  | 1.40127 | 1.81869 |
| 0.75     | -0.512231 | 1.16666  | 1.27416 | 1.98451 |
| 1.25     | -0.315514 | 0.718617 | 0.784831 | 1.98451 |
| 1.50     | -0.175098 | 0.691825 | 0.713639 | 1.81869 |
| 1.75     | -0.076582 | 0.694597 | 0.698806 | 1.68061 |

Thus $f_1(t)$ is expressed as the sum of the residues

$$f_1(t) = \text{Res}[Y(s)e^{st}, s_6] + \text{Res}[Y(s)e^{st}, s_7] = \frac{s_6 + b s_6^{\alpha-1}}{2s_6 + ab s_6^{\alpha-1}} e^{s_6 t} + \frac{s_7 + b s_7^{\alpha-1}}{2s_7 + ab s_7^{\alpha-1}} e^{s_7 t}.$$  

(31)

Note that $s_7 = \bar{s}_6$, the two terms on the right hand side of Eq. (31) are conjugate. We obtain $f_1(t)$ by calculating the real part as

$$f_1(t) = 2\text{Re}\left[ \frac{s_6 + b s_6^{\alpha-1}}{2s_6 + ab s_6^{\alpha-1}} e^{s_6 t} \right] = 2e^{\beta t} \cos(\sigma t) \frac{2r^2 + (2 + \alpha)br^\alpha \cos(2\theta - \alpha \theta) + ab^2 r^{2(\alpha-1)}}{4r^2 + 4abr^\alpha \cos(2\theta - \alpha \theta) + \alpha^2 b^2 r^{2(\alpha-1)}}$$


\[ +2e^{\beta t}\sin(\sigma t) \frac{(2 - \alpha)be^\alpha \sin(2\theta - \alpha \theta)}{4r^2 + 4abr^2 \cos(2\theta - \alpha \theta)} + \alpha^2 b^2 r^2(\alpha - 1). \]  

(32)

Utilizing the results in (32) and the data in Table 1, we plot the curves of \( f_1(t) \) in Fig. 2 for \( \alpha = 0.25, 0.5, 0.75 \) and in Fig. 3 for \( \alpha = 1.25, 1.5, 1.75 \). The behavior of \( f_1(t) \) is similar to the classical underdamped vibration. As \( \alpha \to 1 \), \( f_1(t) \) becomes the result of the classical underdamped vibration as in Eq. (15).

### 3.4 Simplification of integrals for \( f_2(t) \)

We look into the complex contour integral in Eq. (27). Since \( sY(s) \to 0 \) as \( s \to 0 \), the integration on the small circle satisfies

\[ \lim_{\epsilon \to 0^+} \int_{C(\epsilon)} Y(s)e^{st} ds = i \lim_{\epsilon \to 0^+} \int_{-\pi}^{\pi} Y(e^{i\theta})e^{\epsilon \theta} e^{i\theta} d\theta = 0. \]  

(33)

On the upper side and lower side of the negative real axis, we express \( s = ue^{\pm i\pi} \), and simplify \( f_2(t) \) as

\[ f_2(t) = \frac{1}{2\pi i} \int_{0}^{+\infty} [Y(u e^{-i\pi}) - Y(u e^{i\pi})]e^{-ut} du = i \int_{0}^{+\infty} \text{Im}[Y(u e^{i\pi})]e^{-ut} du = \int_{0}^{+\infty} K_\alpha(u)e^{-ut} du, \]  

(34)

where \( \text{Im}[\cdot] \) denote the imaginary part and

\[ K_\alpha(u) = \frac{1}{\pi} \text{Im}[Y(u e^{i\pi})] = \frac{bc \sin(\alpha \pi)u^{\alpha-1}}{\pi[(u^2 + c)^2 + 2b(u^2 + c)u^\alpha \cos(\alpha \pi) + b^2 u^{2\alpha}]} \]  

(35)

Note that

\[ (u^2 + c)^2 + 2b(u^2 + c)u^\alpha \cos(\alpha \pi) + b^2 u^{2\alpha} > (u^2 + c - bu^\alpha)^2, \]  

(36)

i.e. the denominator of Eq. (35) is always positive. Then we have \( K_\alpha(u) > 0 \) as \( 0 < \alpha < 1 \) while \( K_\alpha(u) < 0 \) as \( 1 < \alpha < 2 \). We draw the following conclusions about \( f_2(t) \) from Eqs. (34)–(36).

**Case i.** \( 0 < \alpha < 1 \): \( f_2(t) > 0 \), \( f_2(t) \) decreases monotonically and approaches zero.

**Case ii.** \( 1 < \alpha < 2 \): \( f_2(t) < 0 \), \( f_2(t) \) increases monotonically and approaches zero.

In Figs. 4–6, we display the curves of \( f_2(t) \) for \( b = c = 1 \) and specified values of \( \alpha \). From Eqs. (34) and (35), we observe that \( f_2(t) \to 0 \) as \( \alpha \to 1 \).
3.5 The fundamental solution and asymptotic behavior

From the above three subsections, the fundamental solution $y(t)$ can be expressed in the form

$$y(t) = f_1(t) + f_2(t) = C e^{\beta t} \cos(\sigma t) + D e^{\beta t} \sin(\sigma t) + \int_0^{+\infty} K_\alpha(u) e^{-ut} du,$$  \hspace{1cm} (37)

where

$$C = 2 \left( \frac{2r^2 + (2 + \alpha)br^\alpha \cos(2\theta - \alpha \theta) + ab^2 r^{2(\alpha-1)}}{4r^2 + 4abr^\alpha \cos(2\theta - \alpha \theta) + \alpha^2 b^2 r^{2(\alpha-1)}} \right),$$  \hspace{1cm} (38)

$$D = 2 \left( \frac{(2 - \alpha)br^\alpha \sin(2\theta - \alpha \theta)}{4r^2 + 4abr^\alpha \cos(2\theta - \alpha \theta) + \alpha^2 b^2 r^{2(\alpha-1)}} \right).$$  \hspace{1cm} (39)

Here, $f_1(t)$ represents a decaying oscillation along the $t$ axis, where the amplitude decays exponentially. For $f_2(t)$, we consider its asymptotic behavior using the Hankel integral representation in Eq. (27) by means of the Watson’s lemma for loop integrals [40]; see Appendix.

Expanding $Y(s)$ in Eq. (27) for small $s$, we have

$$Y(s) = \frac{s + bs^{\alpha-1}}{s^2 + bs^{\alpha} + c} = \frac{1}{s} \left( 1 - \frac{1}{1 + \frac{bs^{\alpha} + s^2}{c}} \right) = \frac{1}{s} \left( \frac{bs^{\alpha} + s^2}{c} - \left( \frac{bs^{\alpha} + s^2}{c} \right)^2 + \ldots \right).$$

Reserving the principal term in the asymptotic series we obtain

$$Y(s) \sim \frac{b}{c} s^{\alpha-1}, \quad s \to 0.$$  \hspace{1cm} (40)
Hence the monotone part $f_2(t)$ has the asymptotic representation

$$f_2(t) \sim \frac{b}{c} \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}, \quad t \to +\infty,$$

which exhibits an algebraic decay in the form of power of negative exponent. Therefore, the fundamental solution $y(t)$ is dominated by $f_2(t)$ as $t \to +\infty$, and we have

$$y(t) \sim \frac{b}{c} \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}, \quad t \to +\infty.$$  

The derivative of the fundamental solution has the similar expression and asymptotic behavior,

$$y'(t) = (C\beta + D\sigma)e^{\beta t} \cos(\sigma t) + (D\beta - C\sigma)e^{\beta t} \sin(\sigma t) - \int_0^{+\infty} uK_\alpha(u)e^{-ut} du,$$

$$y'(t) \sim \frac{b}{c} \frac{(-\alpha)t^{-\alpha-1}}{\Gamma(1 - \alpha)}, \quad t \to +\infty.$$  

If $0 < \alpha < 1$, $y'(t)$ is always negative for large enough $t$. If $1 < \alpha < 2$, $y'(t)$ is invariably positive for large enough $t$. We conclude the behaviour of $y(t)$ as follows.

**Case i.** $0 < \alpha < 1$: $y(t)$ is ultimately positive, and ultimately decreases monotonically and approaches zero.

**Case ii.** $1 < \alpha < 2$: $y(t)$ is ultimately negative, and ultimately increases monotonically and approaches zero.

**Fig. 7.** Curves of $y(t)$ for $b = c = 1$ and for $\alpha = 0.25$ (solid line), $\alpha = 0.5$ (dash line) and $\alpha = 0.75$ (dot-dash line).

**Fig. 8.** Curves of $y(t)$ for $b = c = 1$ and for $\alpha = 1.25$ (solid line), $\alpha = 1.5$ (dash line) and $\alpha = 1.75$ (dot-dash line).
In Fig. 7 and Fig. 8, we plot the curves of the fundamental solution \( y(t) \) for \( b = c = 1 \) and for different values of \( \alpha \). In order to clarify the variation near the \( t \)-axis, in Fig. 9 we display the curves in Fig. 8 with different vertical scales.

### 3.6 The zeros and the maximum extreme point

In [41] and [42], the zeros of solutions and eigenvalue problems for a class of simple fractional differential equations were considered. The two issues were found out to be closely related.

Due to \( y(0) = 1 \), we deduce that

**Case i.** \( 0 < \alpha < 1 \): If \( y(t) \) has zeros, then the number of zeros is an even number, unless the critical case where the maximum zero satisfies \( y'(t) = 0 \).

**Case ii.** \( 1 < \alpha < 2 \): \( y(t) \) has one zero at least, and the number of zeros is an odd number, unless the critical case where the maximum zero satisfies \( y'(t) = 0 \).

Let \( \tau \) be the maximum zero of \( y(t) \), then \( \tau \) is the least number such that \( y(t) \) is constantly positive on \((\tau, +\infty)\) for the case \( 0 < \alpha < 1 \) and \( y(t) \) is constantly negative on \((\tau, +\infty)\) for the case \( 1 < \alpha < 2 \). The zeros of \( y(t) \) can be calculated with the help of MATHEMATICA.

Let \( T \) be the maximum extreme point of \( y(t) \), then \( T \) is the least number such that \( y(t) \) is monotonically decreasing on \((T, +\infty)\) for the case \( 0 < \alpha < 1 \) and \( y(t) \) is monotonically increasing on \((T, +\infty)\) for the case \( 1 < \alpha < 2 \). The maximum extreme point \( T \) of \( y(t) \) is the maximum value among all the solutions of \( y'(t) = 0 \) and \( y''(t) \neq 0 \).

For example, we consider the case for \( b = 1, c = 1 \) and \( \alpha = 1.25 \) as follows. The number of zeros of function \( y(t) \) is 3 and the maximum zero is \( \tau = 10.850127 \ldots \). On the interval \((\tau, +\infty)\), \( y(t) \) remains as negative; see Fig. 10.
We calculate the maximum zero of $y'(t)$ such that $y''(t) \neq 0$ to be $T = 21.255755\ldots$. On the interval $(T, +\infty)$, $y(t)$ increases monotonically and approaches zero; see Figs. 10 and 11.

**Fig. 11.** The function $y'(t)$ for $b = 1, c = 1$ and $\alpha = 1.25$.

In Table 2, we list the number of zeros, the maximum zero and the maximum extreme point of $y(t)$ for $b = c = 1$ and ten different values of $\alpha$. It shows that the number of zeros and the value of the maximum zero increase with increasing fractional order $\alpha$ for both cases $0 < \alpha < 1$ and $1 < \alpha < 2$. We note that the maximum extreme points firstly decrease and then increase with increasing fractional order $\alpha$ for the case of $0 < \alpha < 1$.

| the fractional order $\alpha$ | the number of zeros $N$ | the maximum zero $\tau$ | the maximum extreme point $T$ |
|-----------------------------|-------------------------|-------------------------|-----------------------------|
| 0.25                        | 0                       | none                    | 39.206049\ldots            |
| 0.50                        | 0                       | none                    | 13.505562\ldots            |
| 0.75                        | 2                       | 3.429712\ldots          | 10.109469\ldots            |
| 0.8                         | 2                       | 3.86771\ldots           | 10.5277\ldots              |
| 0.9                         | 2                       | 4.90762\ldots           | 12.1037\ldots              |
| 0.95                        | 2                       | 5.49164\ldots           | 13.2575\ldots              |
| 0.99                        | 4                       | 12.5758\ldots           | 20.4578\ldots              |
| 1.25                        | 3                       | 10.850127\ldots         | 21.255755\ldots            |
| 1.50                        | 9                       | 37.478529\ldots         | 57.999544\ldots            |
| 1.75                        | 29                      | 127.792254\ldots        | 193.290362\ldots           |

**4 Conclusions**

We have investigated the solution of the fractional vibration equation, where the damping term is characterized by means of the Caputo fractional derivative with the order $\alpha$ satisfying $0 < \alpha < 1$ or $1 < \alpha < 2$. A detailed analysis for the fundamental solution $y(t)$ is carried out through the Laplace transform and its complex inversion integral formula. Unlike the integer-order case, the Laplace phase function has a branch point at the original point.

The fundamental solution $y(t)$ is expressed as a sum of the oscillative part $f_1(t)$ and the monotone part $f_2(t)$. We deduce that $f_1(t)$ represents a decaying oscillation whose amplitude decays exponentially, while $f_2(t)$ exhibits an algebraic decay in the form of power of negative exponent.

We have concluded that $y(t)$ is ultimately positive, and ultimately decreases monotonically and approaches zero for the case of $0 < \alpha < 1$, while $y(t)$ is ultimately negative, and ultimately increases monotonically and approaches zero for the case of $1 < \alpha < 2$. We have also considered the number of zeros, the maximum zero and the maximum extreme point of the fundamental solution $y(t)$ for specified values of the coefficients and fractional order.
Appendix: Watson’s lemma for loop integrals [40]

If $F(s)$ has the asymptotic expansion

$$
F(s) \sim \sum_{\nu=1}^{\infty} a_{\nu} s^{\lambda_{\nu}}, \quad s \to 0, \quad -\pi < \arg(s) < \pi,
$$

where $\text{Re}(\lambda_1) < \text{Re}(\lambda_2) < \cdots$, and $\text{Re}(\lambda_{\nu})$ increases without bound as $\nu \to \infty$, then the loop integral

$$
f(t) = \frac{1}{2\pi i} \int_{\gamma} F(s) e^{st} ds,
$$

has the asymptotic expansion

$$
f(t) \sim \sum_{\nu=1}^{\infty} \frac{a_{\nu}}{\Gamma(-\lambda_{\nu}) t^{\lambda_{\nu}+1}}, \quad t \to \infty, \quad -\pi/2 < \arg(t) < \pi/2.
$$

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