THE RIGIDITY THEOREMS FOR LAGRANGIAN SELF SHRINKERS

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ABSTRACT. By the integral method we prove that any space-like entire graphic self-shrinking solution to Lagrangian mean curvature flow in $\mathbb{R}^{2n}$ with the indefinite metric $\sum dx_i dy_i$ is flat. This result improves the previous ones in [9] and [1] by removing the additional assumption in their results. In a similar manner, we reprove its Euclidean counterpart which is established in [1].

1. Introduction

Let $M$ be a submanifold in $\mathbb{R}^{m+n}$. Mean curvature flow is a one-parameter family $X_t = X(\cdot, t)$ of immersions $X_t : M \to \mathbb{R}^{m+n}$ with corresponding images $M_t = X_t(M)$ such that

$$\begin{aligned}
\frac{d}{dt} X(x, t) &= H(x, t), \quad x \in M \\
X(x, 0) &= X(x)
\end{aligned}$$

is satisfied, where $H(x, t)$ is the mean curvature vector of $M_t$ at $X(x, t)$ in $\mathbb{R}^{m+n}$.

An important class of solutions to the above mean curvature flow equations are self-similar shrinkers, whose profiles, self-shrinkers, satisfy a system of quasi-linear elliptic PDE of the second order

$$(1.1) \quad H = -\frac{X^N}{2},$$

where $(\cdots)^N$ stands for the orthogonal projection into the normal bundle $NM$.

In the ambient pseudo-Euclidean space we can also study the mean curvature flow (see [5] [6] [7] [11] and [8], for example). And self-shrinking graphs with high codimensions in pseudo-Euclidean space has been studied in [3]. Let $\mathbb{R}^{2n}_n$ be Euclidean space with null coordinates $(x, y) = (x_1, \ldots, x_n; y_1, \ldots, y_n)$, which means that the indefinite metric is defined by $ds^2 = \sum dx_i dy_i$. If $M = \{(x, Du(x)) | x \in \mathbb{R}^n\}$ is a space-like submanifold in $\mathbb{R}^{2n}_n$, then $u$ is convex (In this paper, we say that a smooth function $f$ is convex, if

The research was partially supported by NSFC.
\( D^2 f > 0 \), i.e., hessian of \( f \) is positive definite in \( \mathbb{R}^n \). The underlying Euclidean space \( \mathbb{R}^{2n} = \mathbb{C}^n \) of \( \mathbb{R}_n^{2n} \) has the usual complex structure. It is easily seen that \( M \) is a Lagrangian submanifold in \( \mathbb{R}^{2n} \) ([10], Lemma 5.2.11), as well as in \( \mathbb{R}_n^{2n} \). Moreover, if \( M \) is also a self-shrinker, namely, the convex function \( u \) satisfies (1.1). It has been shown that up to an additive constant \( u \) satisfies the elliptic equation (see [1][8][9])

\[
\log \det D^2 u(x) = \frac{1}{2} x \cdot Du(x) - u(x). \tag{1.2}
\]

Huang-Wang [9] and Chau-Chen-Yuan [1] have investigated the entire solutions to the above equation and showed that an entire smooth convex solution to (1.2) in \( \mathbb{R}^n \) is the quadratic polynomial under the decay condition on Hessian of \( u \).

In [1], Chau-Chen-Yuan introduce a natural geometric quantity \( \phi = \log \det D^2 u \) which obeys a second order elliptic equation with an “amplifying force”. Based on it, we consider an important operator: the drift Laplacian operator \( \mathcal{L} \), which was introduced by Colding-Minicozzi [2], and we can also write the second order equation for \( \phi \) in [1] as \( \mathcal{L} \phi = 0 \). This enables us to apply integral method to prove any entire smooth proper convex solution to (1.2) in \( \mathbb{R}^n \) is the quadratic polynomial, Theorem 2.3, where the case \( n = 1 \) is simple.

It is worth to note that when \( \phi \) is constant the mean curvature of \( M \) vanishes (see (8.5.7) of Chap. VIII in [10]), namely, the gradient graph of a solution \( u \) to (1.2) defines a space-like minimal Lagrangian submanifold in \( \mathbb{R}_n^{2n} \).

By thoroughly analysing the convexity of \( u \), we could prove that any solution of (1.2) is proper, which is showed in Theorem 2.6. Thus, we remove the additional condition of the corresponding results in [9] and [1]. Precisely, we obtain

**Theorem 1.1.** *Any entire smooth convex solution \( u(x) \) to (1.2) in \( \mathbb{R}^n \) is the quadratic polynomial \( u(0) + \frac{1}{2}(D^2 u(0)x, x) \).*

We also consider the corresponding problem in ambient Euclidean space: a Lagrangian graph \( \{(x, Du(x)) \mid x \in \mathbb{R}^n \} \) in \( \mathbb{R}^{2n} \) satisfying (1.1). Now, \( u \) is an entire solution to the following equation:

\[
\arctan \lambda_1(x) + \cdots + \arctan \lambda_n(x) = \frac{1}{2} x \cdot Du(x) - u(x) \tag{1.3}
\]
where $\lambda_1(x), \cdots, \lambda_n(x)$ are the eigenvalues of the Hessian $D^2u$ of $u$ at $x \in \mathbb{R}^n$. Chau-Chen-Yuan [1] constructed a barrier function to show that the phase function

$$\Theta = \arctan \lambda_1(x) + \cdots + \arctan \lambda_n(x)$$

on this Lagrangian graph is a constant via the maximum principles. A geometric meaning of the phase function is the summation of the all Jordan angles of the Gauss map $\gamma : M \to \mathbb{G}_{n,n}$ (see [10] Chap 7, for example). They proved the following theorem.

**Theorem 1.2.** If $u(x)$ is an entire smooth solution to (1.3) in $\mathbb{R}^n$, then $u(x)$ is the quadratic polynomial $u(0) + \frac{1}{2} \langle D^2u(0)x, x \rangle$.

We could also derive the phase function satisfies: $\mathcal{L}\Theta = 0$. This enables us to use the integral method to reprove the above rigidity theorem.

**Acknowledgement** The authors would like to express their sincere thanks to Jingyi Chen for his valuable comments on the first draft of this paper.

2. **Space-like Lagrangian self-shrinkers in pseudo-Euclidean space**

Let $M = \{(x, Du(x)) | x \in \mathbb{R}^n\}$ be a space-like submanifold satisfying (1.2) in ambient space $\mathbb{R}^{2n}$ with the induced metric $g_{ij}dx_idx_j$, where $Du = (u_1, u_2, \cdots, u_n)$. Then $g_{ij} = \partial_i \partial_j u = u_{ij}$, and let $(g^{ij})$ denote the inverse matrix $(g_{ij})$. We write $g = \det g_{ij}$ for simplicity and $\xi \cdot \eta = \langle \xi, \eta \rangle$ for any vectors $\xi, \eta \in \mathbb{R}^n$. By (1.2), we have

$$\partial_j (\log g) = \frac{1}{2} u_j + \frac{1}{2} x_i u_{ij} - u_j = \frac{1}{2} x_i u_{ij} - \frac{1}{2} u_j, \quad (2.1)$$

and

$$\partial_i (\sqrt{g} g^{ij}) = \frac{1}{2} \sqrt{g} g^{kl} \partial_i g_{kl} g^{ij} - \sqrt{g} g^{ki} \partial_i g_{kl} g^{lj}$$

$$= - \frac{1}{2} \sqrt{g} g^{kl} u_{kl} g^{ij} = - \frac{1}{2} \sqrt{g} g^{ij} \partial_i (\log g). \quad (2.2)$$

Let $\mathcal{L}$ be a differential operator defined by

$$\mathcal{L} \phi = \frac{1}{\sqrt{g}} e^{\frac{1}{4} x \cdot Du} \frac{\partial}{\partial x_i} \left( g^{ij} \sqrt{g} e^{-\frac{1}{4} x \cdot Du} \frac{\partial}{\partial x_j} \phi \right),$$
for any function $\phi \in C^2(\mathbb{R}^n)$. Combining (2.1) and (2.2), we have

$$L\phi = \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \phi_j) + \epsilon^i \partial_i (e^{-\frac{1}{4}x \cdot Du}) g^{ij} \phi_j$$

$$= g^{ij} \phi_{ij} + \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \phi_j) - \frac{1}{4} (u_i + x_k u_{ki}) g^{ij} \phi_j$$

$$= g^{ij} \phi_{ij} - \frac{1}{4} (x_k u_{ki} - u_i) g^{ij} \phi_j - \frac{1}{4} (u_i + x_k u_{ki}) g^{ij} \phi_j$$

$$= g^{ij} \phi_{ij} - \frac{1}{2} x_k u_{ki} g^{ij} \phi_j$$

$$= g^{ij} \phi_{ij} - \frac{1}{2} x_j \phi_{ij}.$$  

(2.3)

**Remark 2.1.** The submanifold $M$ in $\mathbb{R}^{2n}$ is defined by $(\mathbb{R}^n, ds^2 = u_{ij} dx_i dx_j)$. The operator $L$ is also defined on $M$. It is precisely the drift Laplacian $L$ in the version of space-like self-shrinkers in pseudo-Euclidean space, which was introduced by Colding-Minicozzi [2] in the ambient Euclidean space.

**Lemma 2.2.** Let $\Omega$ be a convex domain in $\mathbb{R}^n (n \geq 2)$ and $u$ be a smooth proper convex function in $\Omega$, then for any $\alpha > 0$

$$\int_{\Omega} |x \cdot Du| e^{-\alpha u} dx < +\infty.$$  

Proof. Let $\Gamma_t = \{ x \in \Omega \mid u(x) = t \}$ and $\Omega_t = \{ x \in \Omega \mid u(x) < t \}$ for each $t \in \mathbb{R}$. By the convexity of $u$, we know that $\Gamma_t \cap L$ contains two point at most, where $L$ is any line in $\mathbb{R}^n$. Since $u$ is proper, then $\Gamma_t$ is homotopic to $(n-1)$-sphere in $\mathbb{R}^n$, which implies $\Omega_t$ is a bounded domain enclosed by $\Gamma_t$. Thus, $\inf_{x \in \Omega} u(x) > -\infty$ and $\lim_{x \to \partial \Omega} u(x) = +\infty$. By translating $\Omega$ in the plane $\mathbb{R}^n$, we can assume $0 \in \Omega$ and $u(0) = \inf_{x \in \Omega} u(x)$. Moreover, by the convexity of $u$, there exist constants $C, \delta > 0$ such that for any $x \in \Omega$

$$u(x) + C \geq \delta |x|.$$  

(2.4)

It suffices to show

$$\int_{\Omega} |Du| e^{-\beta u} dx < +\infty$$  

holds for some $0 < \beta < \alpha$.

Set $x' = (x_1, \cdots, x_{n-1})$. Let

$$\Omega' = \{ x' \in \mathbb{R}^{n-1} \mid \exists x_n \text{ s.t.} (x', x_n) \in \Omega \}.$$  

For every fixed $x' \in \Omega'$, $u_{nn}(x', x_n) = \partial_{x_n} \partial_{x_n} u(x', x_n)$ is positive, and $u_n(x', x_n)$ is monotonic increasing in $x_n$. Since $\lim_{x \to \partial \Omega} u(x) = +\infty$, then there is $x_n^* \in \Omega$
and \( u_n(x', x^*_n) = 0 \). Furthermore, we have \( x^1_n, x^2_n \in [-\infty, +\infty] \) satisfying \( x^1_n < x^2_n \) and \( (x', x^i_n) \in \partial \Omega \) for \( i = 1, 2 \).

For each fixed \( x' \in \Omega' \), we have

\[
\int_{(x', x_n) \in \Omega} |u_n|e^{-\beta u}dx_n = -\int_{x^1_n}^{x^2_n} u_n(x', x_n)e^{-\beta u(x', x_n)}dx_n + \int_{x^1_n}^{x^2_n} u_n(x', x_n)e^{-\beta u(x', x_n)}dx_n
\]

\[
= \frac{1}{\beta} \int_{x^1_n}^{x^2_n} de^{-\beta u(x', x_n)} - \frac{1}{\beta} \int_{x^1_n}^{x^2_n} de^{-\beta u(x', x_n)}
\]

\[
= \frac{2}{\beta} e^{-\beta u(x', x^*_n)}.
\]

Since \( u(x', x^*_n) + C \geq \delta \sqrt{|x'|^2 + (x^*_n)^2} \geq \delta |x'| \), then by (2.6),

\[
\int_{\Omega} |u_n|e^{-\beta u}dx = \int_{x' \in \Omega} \int_{(x', x_n) \in \Omega} |u_n|e^{-\beta u}dx_n dx' = \frac{2}{\beta} e^{-\beta u(x', x^*_n)}dx' \leq \frac{2}{\beta} e^{\beta C - \beta \delta |x'|}dx' < \infty.
\]

By the same way to \( \{u_i\} \) for \( i = 1, \cdots, n - 1 \), we know (2.5) holds. This shows the Lemma. \( \square \)

**Theorem 2.3.** If \( \Omega \) is a convex domain containing the origin in \( \mathbb{R}^n (n \geq 2) \) and \( u(x) \) is a smooth proper convex solution to (1.2) in \( \Omega \), then \( \Omega \) is \( \mathbb{R}^n \) and \( u(x) \) is the quadratic polynomial \( u(0) + \frac{1}{2} \langle D^2u(0)x, x \rangle \).

**Proof.** Let \( \phi = \log g \), then by (2.1), \( \phi_{ij} = \frac{1}{2} x_k u_{ij,k} \) and

\[
g_{ij} \phi_{ij} = \frac{1}{2} g_{ij} x_k u_{ij,k} = \frac{1}{2} x_k \phi_k.
\]

(2.8) was found by Chau-Chen-Yuan [1]. Combining (2.3) and (2.8), we have

\[
\mathcal{L} \phi = g_{ij} \phi_{ij} - \frac{1}{2} x_j \phi_j = 0.
\]

Let \( F \) be a positive monotonic increasing \( C^1 \)-function on \( \mathbb{R} \), and \( \eta \) be a nonnegative Lipschitz function in \( \Omega \) with compact support, both to be defined later. Using (1.2) and (2.9) and integrating by parts, we have

\[
0 = -\int_{\Omega} F(\phi) \eta^2 \mathcal{L} \phi e^{-\frac{1}{2} x^D u} \sqrt{g} dx
\]

\[
= \int_{\Omega} g^{ij} \partial_i \left( F(\phi) \eta^2 \right) \phi_j e^{-\frac{1}{2} x^D u} \sqrt{g} dx
\]

\[
= \int_{\Omega} g^{ij} \phi_i \phi_j F' \eta^2 e^{-\frac{1}{2} u} dx + 2 \int_{\Omega} F(\phi) \eta g^{ij} \phi_i \phi_j e^{-\frac{1}{2} u} dx.
\]
Since $u$ is proper convex, then $\lim_{x \to \partial \Omega} u(x) = +\infty$ and we define the set $\Omega_t = \{x \in \Omega \mid u(x) < t\}$ as Lemma 2.2, which is an exhaustion of the domain $\Omega$. Let

$$
\eta(x) \triangleq \begin{cases} 1 & \text{if } x \in \Omega_t \\
\quad t + 1 - u(x) & \text{if } x \in \Omega_{t+1} \setminus \Omega_t \\
\quad 0 & \text{if } x \in \Omega \setminus \Omega_{t+1},
\end{cases}
$$

and

$$
F(s) \triangleq \begin{cases} e^s & \text{if } s < 0 \\
\quad 1 & \text{if } s = 0 \\
\quad 1 + \arctan s & \text{if } s > 0.
\end{cases}
$$

By (2.1) and (2.10), we have

$$
\int_{\Omega_t} g^{ij} \phi_i \phi_j F' e^{-\frac{u}{2}} dx \leq \int_{\Omega} g^{ij} \phi_i \phi_j F' e^{-\frac{u}{2}} dx = -2 \int_{\Omega} F(\phi) \eta g^{ij} \phi_i \phi_j e^{-\frac{u}{2}} dx
$$

$$
= 2 \int_{\Omega_{t+1} \setminus \Omega_t} F(\phi) \eta g^{ij} u_i \left( \frac{1}{2} x_k u_{jk} - \frac{1}{2} u_j e^{-\frac{u}{2}} dx 
\right)
$$

$$
\leq \int_{\Omega_{t+1} \setminus \Omega_t} F(\phi) \eta x_i u_i e^{-\frac{u}{2}} dx
$$

$$
\leq (1 + \pi/2) \int_{\Omega_{t+1} \setminus \Omega_t} |x_i u_i| e^{-\frac{u}{2}} dx.
$$

(2.11)

By Lemma 2.2, let $t$ go to infinity in (2.11), we know $D\phi = 0$ and $\phi = \log g$ is a constant in $\Omega$. By the equation (1.2) and $0 \in \Omega$, as shown in [1], we know $u(x)$ is the quadratic polynomial $u(0) + \frac{1}{2}(D^2 u(0)x, x)$. Since $\lim_{x \to \partial \Omega} u(x) = +\infty$, then $\Omega = \mathbb{R}^n$.  

\[\Box\]

As for $n = 1$, the equation (1.2) gives the equation

$$
u'' = e^{\frac{1}{2}xu'} - u.
$$

Since $(xu' - u)' = xu''$, we have $xu' - u \geq -u(0)$ and

$$
u''(x) \geq e^{-\frac{u(0) + u(x)}{2}}.
$$

If $\lim_{x \to +\infty} u(x) = C_0 \in [-\infty, +\infty)$, then $u(x) \leq \max\{u(0), C_0\}$ on $[0, +\infty)$. Then

$$
u''(x) \geq e^{-\frac{u(0) + \max\{u(0), C_0\}}{2}}.
$$

This means that

$$
\lim_{x \to +\infty} u'(x) = +\infty,
$$

Since $u$ is proper convex, then $\lim_{x \to \partial \Omega} u(x) = +\infty$ and we define the set $\Omega_t = \{x \in \Omega \mid u(x) < t\}$ as Lemma 2.2, which is an exhaustion of the domain $\Omega$. Let
which contradicts with \( \lim_{x \to +\infty} u(x) < +\infty \). A similar argument concludes that
\( \lim_{x \to -\infty} u(x) = +\infty \). Thus,
\[
\lim_{|x| \to \infty} u(x) = +\infty.
\]
Combining (2.4), we have
\[
\int_{\mathbb{R}} |xu'|e^{-\frac{u^2}{2}}dx < \infty.
\]
Following the argument of Theorem 2.3, we could prove Theorem 1.1 for the case 
\( n = 1 \).

For proving Theorem 1.1 completely, it suffices to remove the proper condition of \( u(x) \)
in Theorem 2.3 when \( \Omega = \mathbb{R}^n \). Now we give two lemmas on convex functions which will
be used in Theorem 2.6 in the case \( n \geq 2 \). One is an algebraic property for the Hessian of
convex functions, the other is on the size of Lebesgue measure of a set which arises from
the equation (1.2).

**Lemma 2.4.** Let \( u \) be a smooth convex function in a domain of \( \mathbb{R}^n \). If \( \xi_1, \ldots, \xi_n \) is an
arbitrary orthonormal basis of \( \mathbb{R}^n \), then
\[
\det D^2 u \leq u_{\xi_1} u_{\xi_2} \cdots u_{\xi_n},
\]
where \( u_{\xi_i \xi_j} = \text{Hessian}(u)(\xi_i, \xi_j) \) in \( \mathbb{R}^n \) for \( 1 \leq i, j \leq n \).

**Proof.** By an orthogonal transformation, we have
\[
(2.12) \quad \det D^2 u = \det u_{\xi_i \xi_j}.
\]
Let \( \alpha \) be a \((n-1)\)-dimensional vector defined by \( (u_{\xi_1 \xi_2}, u_{\xi_1 \xi_3}, \ldots, u_{\xi_1 \xi_n}) \) and \( A \) be a
\((n-1) \times (n-1)\) matrix \((u_{\xi_i \xi_j})_{2 \leq i, j \leq n}\). Since
\[
\begin{pmatrix}
1 & 0 \\
\frac{1}{u_{\xi_1 \xi_1}} \alpha^T & I_{n-1}\end{pmatrix}
\begin{pmatrix}
u_{\xi_1 \xi_1} & \alpha \\
\alpha^T & A\end{pmatrix}
\begin{pmatrix}
1 & -\frac{1}{u_{\xi_1 \xi_1}} \alpha \\
0 & I_{n-1}\end{pmatrix}
= \begin{pmatrix}
u_{\xi_1 \xi_1} & 0 \\
0 & A - \frac{\alpha^T \alpha}{u_{\xi_1 \xi_1}}\end{pmatrix},
\]
then \( A - \frac{\alpha^T \alpha}{u_{\xi_1 \xi_1}} \) is a positive definite matrix and
\[
(2.13) \quad \det D^2 u = \det u_{\xi_i \xi_j} = u_{\xi_1 \xi_1} \det \left(A - \frac{\alpha^T \alpha}{u_{\xi_1 \xi_1}}\right) \leq u_{\xi_1 \xi_1} \det(A).
\]
By induction,
\[
(2.14) \quad \det D^2 u \leq u_{\xi_1 \xi_1} u_{\xi_2 \xi_2} \cdots u_{\xi_n \xi_n}.
\]
\[
\square
\]
Lemma 2.5. Let $B_δ$ be an open ball with radius $δ$ and centered at the origin in $\mathbb{R}^m$, $v$ be a smooth convex function in $\overline{B_δ}$ with $v|_{\partial B_δ} \leq C_1$, then there is a constant $C_2 > 0$ depending only on $m, δ, C_1$ such that the set

$$E = \{x \in B_δ \mid e^{\frac{v(x)}{2}} \det D^2 v > C_2^m\}$$

has the measure $|E| < \frac{1}{2}|B_δ|$.

Proof. Suppose that the measure $|E| \geq \frac{1}{2}|B_δ|$ for some sufficiently large $C_2$, and we will deduce the contradiction. Denote the open sets

$$E_i = \{x \in B_δ \mid D_{ii}v(x) > C_2 e^{-\frac{v(x)}{2m}}\}$$

for $i = 1, \cdots, m$. By Lemma 2.4,

$$D_{11}vD_{22}v \cdots D_{mm}v \geq \det D^2 v,$$

then

$$E \subset \bigcup_{1 \leq i \leq m} E_i.$$

Thus,

$$\frac{1}{2}|B_δ| \leq |E| \leq \bigcup_{1 \leq i \leq m} |E_i| \leq \sum_{i=1}^{m} |E_i|,$$

which implies there is a $E_i$ with

$$|E_i| \geq \frac{1}{2m}|B_δ|.$$

Without loss of generality set $E_1 = E_i$, then there is

$$L = \{x = (x_1, \cdots, x_m) \in B_δ \mid x_2 = y_2, \cdots, x_m = y_m\}$$

such that the measure of $L \cap E_1$ is no less than $C_3δ$ for some constant $0 < C_3 < 1$ depending only on $m$.

Let $f(s) = v(s, y_2, y_3, \cdots, y_m)$, $I = \{s \in \mathbb{R} \mid (s, y_2, y_3, \cdots, y_m) \in L\}$, then $I = (-s_0, s_0)$ with $\frac{C_2δ}{2} \leq s_0 \leq δ$ and

$$E_1 = \{|s| < s_0 \mid f''(s) > C_2 e^{-\frac{f(s)}{2m}}\}.$$

Without loss of generality, we select $0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_N < b_N$ for some $N < \infty$ such that $L \cap E_1 \supset \bigcup_{i=1}^{N} (a_i, b_i) \times (y_2, \cdots, y_m)$ and $\sum_{i=1}^{N} (b_i - a_i) \geq \frac{C_2δ}{3}$. 
For deducing the contradiction, we need prove \( f(s_0) \) is sufficiently large as \( C_2 \) is sufficiently large, which violates our assumption \( v|_{\partial B_1} \leq C_1 \). Since \( v \) is convex, then \( \sup_{x \in \overline{B_1}} v(x) \leq C_1 \) and \( f'' = D_{11}v > 0 \). By Newton-Leibnitz formula, we get

\[
f(0) - C_1 \leq f(0) - f(-s_0) = \int_{-s_0}^{0} f'(s) ds \leq f'(0)s_0,
\]

which implies

\[
f'(0) \geq \frac{f(0) - C_1}{s_0}.
\]

Let \( C_4 = \frac{1}{s_0} (C_1 e^{\frac{C_1}{2m}} - f(0) e^{\frac{f(0)}{2m}}) \), which depends only on \( m, \delta \) and \( C_1 \). Since \( C_2 \) is sufficiently large, then there is a \( c \in (a_j, b_j) \) such that

\[
\sum_{i=1}^{j-1} (b_i - a_i) + c - a_j \in \left( \frac{C_4}{C_2}, \frac{2C_4}{C_2} \right).
\]

If \( f'(c) < 0 \), then \( f(0) \geq f(s) \) for \( s \in [0, c] \). Combining (2.15), (2.16) and the definition of \( E_1 \) and \( C_4 \), we have

\[
f'(c) = f'(0) + \int_{0}^{c} f''(s) ds \geq f'(0) + \sum_{i=1}^{j-1} \int_{a_i}^{b_i} f''(s) ds + \int_{a_j}^{c} f''(s) ds
\]

\[
\geq f'(0) + \sum_{i=1}^{j-1} \int_{a_i}^{b_i} C_2 e^{-\frac{f(s)}{2m}} ds + \int_{a_j}^{c} C_2 e^{-\frac{f(s)}{2m}} ds
\]

\[
\geq f'(0) + \sum_{i=1}^{j-1} \int_{a_i}^{b_i} C_2 e^{-\frac{f(0)}{2m}} ds + \int_{a_j}^{c} C_2 e^{-\frac{f(0)}{2m}} ds
\]

\[
\geq \frac{f(0) - C_1}{s_0} + C_4 e^{-\frac{f(0)}{2m}} = e^{-\frac{f(0)}{2m}} \left( C_4 + \frac{1}{s_0} (f(0) e^{\frac{f(0)}{2m}} - C_1 e^{\frac{f(0)}{2m}}) \right)
\]

\[
\geq e^{-\frac{f(0)}{2m}} \left( C_4 + \frac{1}{s_0} (f(0) e^{\frac{f(0)}{2m}} - C_1 e^{\frac{f(0)}{2m}}) \right) = 0.
\]

Thus, \( f'(c) \geq 0 \). Together with \( f'' > 0 \), we have

\[
0 \leq f'(s_1) \leq f'(s_2) \text{ and } f(s_1) \leq f(s_2) \text{ for } c \leq s_1 \leq s_2 \leq s_0.
\]

Denote \( \delta_j = b_j - c \) and \( \delta_k = b_k - a_k \) for \( k = j + 1, \ldots, N \). By (2.18) and the definition of \( E_1 \), for \( t \in (c, b_j) \) we obtain

\[
f'(t) = f'(c) + \int_{c}^{t} f''(s) ds \geq C_2 \int_{c}^{t} e^{-\frac{f(s)}{2m}} ds \geq C_2 (t - c) e^{-\frac{f(0)}{2m}},
\]

then

\[
e^{\frac{f(t)}{2m}} = e^{\frac{f(c)}{2m}} + \int_{c}^{t} \frac{f''(s)}{2m} e^{\frac{f(s)}{2m}} ds \geq \int_{c}^{t} \frac{C_2}{2m} (s - c) ds = \frac{C_2}{4m} (t - c)^2.
\]
So we claim
\[(2.19) \quad f'(b_k) \geq C_2 \sum_{i=j}^k \delta_i e^{-f(b_k)/2m}, \quad \text{and} \quad e^{f(b_k)/2m} \geq \frac{C_2}{4m} \left( \sum_{i=j}^k \delta_i \right)^2 \] for \( k = j, \ldots, N \).

If \((2.19)\) holds for some \( k < N \), then \( f'(a_{k+1}) \geq f'(b_k) \) and \( f(a_{k+1}) \geq f(b_k) \) by \((2.18)\).

For any \( t \in (a_{k+1}, b_{k+1}) \), we get
\[(2.20) \quad f'(t) = f'(a_{k+1}) + \int_{a_{k+1}}^t f''(s) ds \geq C_2 \sum_{i=j}^k \delta_i e^{-f(b_k)/2m} + C_2 \int_{a_{k+1}}^t e^{-f(s)/2m} ds \]
\[\geq C_2 \left( t - a_{k+1} + \sum_{i=j}^k \delta_i \right) e^{-f(b_k)/2m}, \]
and
\[(2.21) \quad e^{f(t)/2m} = e^{f(a_{k+1})/2m} + \int_{a_{k+1}}^t e^{f(s)/2m} ds \]
\[\geq \frac{C_2}{4m} \left( \sum_{i=j}^k \delta_i \right)^2 + \int_{a_{k+1}}^t \frac{C_2}{2m} \left( s - a_{k+1} + \sum_{i=j}^k \delta_i \right) ds \]
\[= \frac{C_2}{4m} \left( t - a_{k+1} + \sum_{i=j}^k \delta_i \right)^2. \]

By induction, we complete this claim. Combining the selection of \( a_i, b_i \) and \((2.16)(2.18)(2.19)\), we conclude
\[C_1 \geq f(s_0) \geq f(b_N) \geq 2m \log \frac{C_2}{4m} + 4m \log \sum_{i=j}^N \delta_i \geq 2m \log \frac{C_2}{4m} + 4m \log \left( \frac{C_3 \delta}{3} - \frac{2C_4}{C_2} \right), \]
which is impossible for sufficiently large \( C_2 \).

\[\square\]

**Theorem 2.6.** Any entire smooth convex solution \( u \) to \((1.2)\) in \( \mathbb{R}^n \) is proper.

**Proof.** To prove the result, it suffices to show \( \lim_{|x| \to \infty} u(x) = +\infty \) for \( n \geq 2 \). Let \( B^n_r \) be an open ball in \( \mathbb{R}^n \) with radius \( r \) and centered at the origin. Suppose that
\[(2.22) \quad \liminf_{|x| \to \infty} u(x) < +\infty. \]

Since \( \frac{\partial}{\partial r} \left( r \langle \beta, Du(r\beta) \rangle - u(r\beta) \right) = ru_{ij} \beta_i \beta_j > 0 \) for every \( \beta = (\beta_1, \ldots, \beta_n) \in S^{n-1}(1) \), then
\[r \partial_r u(r\beta) - u(r\beta) \geq -u(0), \]
and
\[\left( \frac{u(r\beta) - u(0)}{r} \right)' = \frac{r \partial_r u(r\beta) - u(r\beta) + u(0)}{r^2} \geq 0. \]
So \( \lim_{r \to \infty} \frac{u(r\beta)}{r} \) always exists (may be infinity) and is denoted by \( \kappa_{\beta} \). Let \( \Lambda = \{ \beta \in S^{n-1}(1) \mid \kappa_{\beta} \leq 0 \} \). If \( \Lambda = \emptyset \), then for any \( \beta \in S^{n-1}(1) \), there is a \( r_{\beta} > 0 \) such that \( u(r_{\beta}\beta) - u(0) \geq \frac{1}{2}\tilde{\kappa}_{\beta} r_{\beta} \), where \( \tilde{\kappa}_{\beta} = \min\{\kappa_{\beta}, 1\} > 0 \). By the continuity of \( u \), there is an open domain \( S_{\beta} \subset S^{n-1}(1) \) containing \( \beta \) such that \( u(r_{\beta}\gamma) - u(0) \geq \frac{1}{4}\tilde{\kappa}_{\beta} r_{\beta} \) for each \( \gamma \in S_{\beta} \).

Since \( u \) is convex, then \( \partial_{r} u(r\gamma) \geq \frac{1}{4}\tilde{\kappa}_{\beta} \) for \( r \geq r_{\beta} \), which implies

\[
    u(r\gamma) - u(0) = \int_{r_{\beta}}^{r} \partial_{r} u(s\gamma) ds + u(r\beta) - u(0) \geq \frac{1}{4}\tilde{\kappa}_{\beta}(r - r_{\beta}) + \frac{1}{4}\tilde{\kappa}_{\beta} r_{\beta} = \frac{1}{4}\tilde{\kappa}_{\beta} r
\]

for each \( \gamma \in S_{\beta} \) and \( r \geq r_{\beta} \). By the finite cover property, there is a sequence \( \{\tilde{\beta}_{i}\}_{i=1}^{N} \) such that \( S^{n-1}(1) = \bigcup_{1 \leq i \leq N} S_{\tilde{\beta}_{i}} \). Let \( r^{*} = \max_{1 \leq i \leq N} r_{\tilde{\beta}_{i}} \) and \( \kappa^{*} = \min_{1 \leq i \leq N} \tilde{\kappa}_{\tilde{\beta}_{i}} > 0 \), then for any \( \beta \in S^{n-1}(1) \) and \( r \geq r^{*} \), we have \( u(r\beta) - u(0) \geq \frac{1}{4}\kappa^{*}r \). This contradicts with (2.22).

Therefore, \( \Lambda \) is nonempty.

There is a sequence \( \{\tilde{\beta}_{i}\} \subset \Lambda \) such that

\[
    \lim_{i \to \infty} \kappa_{\tilde{\beta}_{i}} = \inf_{\beta \in \Lambda} \kappa_{\beta}.
\]

And we can assume \( \lim_{i \to \infty} \tilde{\beta}_{i} = \theta \) for some \( \theta = (\theta_{1}, \cdots, \theta_{n}) \in S^{n-1}(1) \). For every fixed \( r > 0 \), there is a \( i_{0} > 0 \) such that for all \( i \geq i_{0} \), \( u(r\tilde{\beta}_{i}) \geq u(r\theta) - 1 \). Then

\[
    0 \geq \kappa_{\tilde{\beta}_{i}} \geq \frac{u(r\tilde{\beta}_{i}) - u(0)}{r} \geq \frac{u(r\theta) - u(0) - 1}{r}.
\]

Hence \( u(r\theta) \leq u(0) + 1 \), and

\[
    \lim_{i \to \infty} \kappa_{\tilde{\beta}_{i}} \leq \lim_{r \to \infty} \frac{u(r\theta) - u(0) - 1}{r} = \kappa_{\theta}.
\]

Therefore \( \kappa_{\theta} = \inf_{\beta \in \Lambda} \kappa_{\beta} \leq 0 \). Let \( \kappa = \kappa_{\theta} \) for simplicity. For each \( \beta \in S^{n-1}(1) \), we obtain

\[
    \kappa = \lim_{r \to \infty} \frac{u(r\theta)}{r} = \lim_{r \to \infty} \langle \theta, Du(r\theta) \rangle \leq \lim_{r \to \infty} \frac{u(r\beta)}{r}.
\]

Let

\[
    U = \{ x \in \mathbb{R}^{n} \mid u(x) < \kappa(\theta, x) + u(0) \}.
\]

Since \( u \) is an entire convex function in \( \mathbb{R}^{n} \), then \( U \) is a convex domain in \( \mathbb{R}^{n} \). The definition of \( \kappa \) implies \( r\theta \in U \) for any \( r > 0 \). We then can find a slim column region around the ray \( r\theta \) inside the convex domain \( U \). Precisely, there exist \( r_{0}, \delta > 0 \) such that

\[
    U_{\theta} = \{ r\theta + \alpha \in \mathbb{R}^{n} \mid r \geq r_{0}, \alpha \perp \theta \text{ and } |\alpha| < \delta \} \subset U.
\]

Let

\[
    \mathcal{X}_{r} = \{ r\theta + \alpha \mid \alpha \perp \theta \text{ and } |\alpha| < \delta \}
\]
be a slice of $C$. Let $u_\theta(r\theta + \alpha) = \partial_\theta u(r\theta + \alpha) = \langle \theta, Du(r\theta + \alpha) \rangle$ denote the $\theta$-directional derivative of $u$ and 

$$u_\theta(r\theta + \alpha) = \frac{\partial^2}{\partial r^2} u(r\theta + \alpha) = \sum_{i,j} u_{ij}(r\theta + \alpha) \theta_i \theta_j.$$ 

By $C \subset U$ and (2.23), we conclude $\lim_{r \to \infty} u_\theta(r\theta + \alpha) = \kappa$ for any $\alpha \perp \theta, |\alpha| < \delta$. We don’t have the pointwise estimate for $u_\theta$ in $C$, but have the following integral estimate

\begin{equation}
\int_r^\infty \int_{S^s} u_{\theta\theta} dV_{S^s} ds = \int_r^\infty \int_{\alpha \perp \theta, |\alpha| < \delta} u_{\theta\theta}(s\theta + \alpha) dV_{\alpha} ds
= \int_{\alpha \perp \theta, |\alpha| < \delta} \int_r^\infty u_{\theta\theta}(s\theta + \alpha) ds dV_{\alpha}
= \int_{\alpha \perp \theta, |\alpha| < \delta} (\kappa - u_\theta(r\theta + \alpha)) dV_{\alpha} < \infty.
\end{equation}

Let $\omega_{n-1}$ be the standard volume of $(n-1)$-dimensional unit balls. From (2.24), we can find a sequence $\{r_i\}_{i=1}^{\infty} \subset \mathbb{R}$ with $\lim_{i \to \infty} r_i = +\infty$ such that the open set

$$\tilde{S}_{r_i} = \{x \in S_{r_i} \mid u_{\theta\theta}(x) < \frac{1}{r_i}\}$$

has measure

$$|\tilde{S}_{r_i}| \geq \frac{1}{2} |S_{r_i}| = \frac{1}{2} \omega_{n-1} \delta^{n-1}.$$ 

Here, the factor $\frac{1}{2}$ is not essential and could be replaced by any positive constant which is less than 1.

Since \( \frac{\partial}{\partial r} \left( r \langle \beta, Du(r\beta) \rangle - u(r\beta) \right) = ru_{ij} \beta_i \beta_j > 0 \) for every $\beta \in S^{n-1}(1)$, then

$$x \cdot Du(x) - u(x) \geq -u(0),$$

and

\begin{equation}
\det D^2 u(x) = e^{\frac{1}{2} x \cdot Du(x) - u(x)} \geq e^{-\frac{u(0) + u(x)}{2}}.
\end{equation}

Let $D^2 u$ be the $(n-1)$-Hessian matrix of $u$ in $S_r$ for each $r \geq r_0$, then $D^2 u > 0$. By (2.13) and (2.25), we get

$$\det D^2 u \geq u_{\theta\theta}^{-1} \det D^2 u \geq u_{\theta\theta}^{-1} e^{-\frac{u(0) + u(x)}{2}}.$$ 

The definition of $U$ implies that $u(x) \leq u(0)$ for any $x \in S_{r_i} \subset U$. Combining the measure $|\tilde{S}_{r_i}| = \{|x \in S_{r_i} \mid u_{\theta\theta}^{-1}(x) > r_i| \geq \frac{1}{2} |S_{r_i}|$ and Lemma 2.5, we arrive at a contradiction if $i$ goes to infinity. Therefore, $\lim_{|x| \to \infty} u(x) = +\infty$ when $n \geq 2$. We complete the proof. □
Proof of Theorem 1.1. Noting the case \( n = 1 \) and Combining Theorem 2.3 and Theorem 2.6, we finish the proof.

Let \( a > 0 \), \( c \) be constant numbers and \( b \in \mathbb{R}^n \) be a constant vector. The entire solution to the following general type equation

\[
\log \det D^2 u(x) = a\left(\frac{1}{2} x \cdot Du(x) - u(x)\right) + b \cdot x + c
\]

is a quadratic polynomial. In fact, let \( w(x) = au(x) - 2b \cdot x - c - n \log a \), then \( w \) satisfies the equation (1.2).

3. Application to other equations

Let’s prove Theorem 1.2 by the integral method, which is similar to the previous section.

Proof of Theorem 1.2. Let \( M \) be a Lagrangian submanifold satisfying (1.3) in \( \mathbb{R}^{2n} \) with induced metric \( g_{ij} dx_i dx_j \). Then \( g_{ij} = \delta_{ij} + \sum_k u_{ik} u_{jk} \), and denote \( g = \det g_{ij} \) for short.

\[
\partial_i (g^{ij} \sqrt{g}) = \frac{1}{2} \sqrt{g} g^{kl} \partial_i g_{kl} g^{ij} - \sqrt{g} g^{ki} \partial_i g_{kj} g^{ij}
\]

\[
= \frac{1}{2} \sqrt{g} g^{kl} g^{ij} (u_{ks} u_{ls} + u_{ks} u_{ls}) - \sqrt{g} g^{ki} g^{lj} (u_{ks} u_{ls} + u_{ks} u_{ls})
\]

\[= - \sqrt{g} g^{kl} g^{ij} u_{kl} u_{is}. \tag{3.1}\]

Define the differential operator \( \mathcal{L} \) on \( C^2(\mathbb{R}^n) \) by

\[
\mathcal{L} \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( g^{ij} \sqrt{g} e^{-\frac{|x|^2}{4} |D u|^2} \frac{\partial}{\partial x_j} \phi \right)
\]

which is the same as the drift Laplacian in [2].

Let \( \Theta = \arctan \lambda_1(x) + \cdots + \arctan \lambda_n(x) \), which is the phase function on Lagrangian submanifold \( M \in \mathbb{R}^{2n} \). By [1],

\[
\Theta_k = g^{ij} u_{ijk} = -\frac{1}{2} u_k + \frac{1}{2} x \cdot Du_k \tag{3.2}
\]

and we have

\[
\mathcal{L} \Theta = g^{ij} \Theta_k = \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g}) \Theta_j - \frac{1}{2} g^{ij} (x_i + u_k u_{ki}) \Theta_j
\]

\[
= g^{ij} \Theta_j - g^{kl} g^{ij} u_{kl} u_{is} \Theta_j - \frac{1}{2} g^{ij} (x_k \delta_{ki} + u_k u_{ki}) \Theta_j
\]

\[
= g^{ij} \Theta_j + g^{ij} \left( \frac{1}{2} u_s - \frac{1}{2} x_k u_{ks} \right) u_{is} \Theta_j - \frac{1}{2} g^{ij} (x_k (g_{ki} - u_{ks} u_{ki}) + u_k u_{ki}) \Theta_j
\]

\[
= g^{ij} \Theta_j - \frac{1}{2} g^{ij} x_k g_{ki} \Theta_j = g^{ij} \Theta_j - \frac{1}{2} x^j \Theta_j. \tag{3.3}\]
By (3.2), $\Theta_{kl} = \frac{1}{2} x_s u_{skl}$. Then $g^{kl} \Theta_{kl} = g^{kl} \frac{1}{2} x_s u_{skl} = \frac{1}{2} x_j \Theta_j$ (see also [1]), which implies

\begin{equation}
L \Theta = 0.
\end{equation}

Let $\nabla$ and $d\mu$ be Levi-Civita connection and volume element of $M$ with respect to the metric $g_{ij} dx_i dx_j$, and $\rho = e^{-\frac{|x|^2}{4}+|Du|^2}$. If $\eta$ is a smooth function in $M$ with compact support, then by integral by parts we have

\begin{equation}
0 = -\int_M \eta^2 \Theta L \Theta \rho d\mu = 2 \int_M \eta \Theta \nabla \eta \cdot \nabla \Theta \rho d\mu + \int_M |\nabla \Theta|^2 \eta^2 \rho d\mu
\geq -2 \int_M |\nabla \eta|^2 \Theta^2 \rho d\mu - \frac{1}{2} \int_M |\nabla \Theta|^2 \eta^2 \rho d\mu + \int_M |\nabla \Theta|^2 \eta^2 \rho d\mu,
\end{equation}

which implies

\begin{equation}
\int_M |\nabla \Theta|^2 \eta^2 \rho d\mu \leq 4 \int_M |\nabla \eta|^2 \Theta^2 \rho d\mu.
\end{equation}

Since $\Theta$ is a bounded function and $M$ has Euclidean volume growth [4], then we obtain $\Theta$ is a constant. Then, as shown in [1], we obtain Theorem 1.2.

Now, let’s consider another equation. If $v$ is a smooth subharmonic function on $\mathbb{R}^n$ satisfying

\begin{equation}
\log \Delta v = \frac{1}{2} x \cdot Dv - v.
\end{equation}

Let $\phi = \log \Delta v$, then $\phi_i = -\frac{1}{2} v_i + \frac{1}{2} x_j v_{ij}$ and $\phi_{ii} = \frac{1}{2} x_j v_{iij}$. We have

\begin{equation}
\Delta \phi = \frac{1}{2} x_j \partial_j (\Delta v) = \frac{1}{2} e^\phi x \cdot D\phi.
\end{equation}

**Theorem 3.1.** Let $\phi(x)$ be an entire smooth solution to (3.8) in $\mathbb{R}^n$ and $\eta$ be a Lipschitz function in $\mathbb{R}^n$ with compact support and $\eta|_{B_r} \equiv 1$. If

\begin{equation}
\lim_{r \to +\infty} \int_{\mathbb{R}^n \setminus B_r} \frac{|D\eta|^2}{|x|^2} e^{-\phi} e^{-\frac{|x|^2}{4} e^\phi} = 0,
\end{equation}

then $\phi$ is a constant.

**Proof.** Let $\eta$ be a Lipschitz function on $\mathbb{R}^n$ with compact support and $\eta|_{B_r} \equiv 1$, then we multiply $\eta^2 e^{-\frac{|x|^2}{4} e^\phi}$ on both sides of (3.8) and integral by parts,

\begin{equation}
\int_{\mathbb{R}^n} \frac{1}{2} x \cdot D\phi e^\phi \eta^2 e^{-\frac{|x|^2}{4} e^\phi} = -\int_{\mathbb{R}^n} D\phi \cdot D(\eta^2 e^{-\frac{|x|^2}{4} e^\phi})
= \int_{\mathbb{R}^n} \eta D\eta \cdot D\phi e^{-\frac{|x|^2}{4} e^\phi} - 2 \int_{\mathbb{R}^n} \eta D\eta \cdot D\phi e^{-\frac{|x|^2}{4} e^\phi}.
\end{equation}
Hence we have
\[
\frac{1}{4} \int_{\mathbb{R}^n} |x|^2 |D\phi|^2 e^\phi \eta^2 e^{-\frac{|x|^2}{4} e^\phi} = 2 \int_{\mathbb{R}^n} \eta D\eta \cdot D\phi e^{-\frac{|x|^2}{4} e^\phi}
\]
(3.10)
\[
\leq \frac{1}{4} \int_{\mathbb{R}^n \setminus B_r} |x|^2 |D\phi|^2 e^\phi \eta^2 e^{-\frac{|x|^2}{4} e^\phi} + 4 \int_{\mathbb{R}^n \setminus B_r} \frac{|D\eta|^2}{|x|^2} e^{-\phi} e^{-\frac{|x|^2}{4} e^\phi},
\]
then
\[
\int_{B_r} |x|^2 |D\phi|^2 e^\phi e^{-\frac{|x|^2}{4} e^\phi} \leq 16 \int_{\mathbb{R}^n \setminus B_r} \frac{|D\eta|^2}{|x|^2} e^{-\phi} e^{-\frac{|x|^2}{4} e^\phi}. \tag{3.11}
\]
Let \( r \to \infty \), then (3.11) implies \( \phi \) is a constant.

For the case \( n \geq 3 \), let
\[
\eta(x) \triangleq \begin{cases} 
1 & \text{if } x \in B_r \\
2 - \frac{|x|}{r} & \text{if } x \in B_{2r} \setminus B_r \\
0 & \text{if } x \in \mathbb{R}^n \setminus B_{2r}.
\end{cases}
\]
If \( e^\phi(x) \geq 4(n - 2) \frac{\log |x|}{|x|^2} \) for \( |x| \geq r \), then
\[
\int_{\mathbb{R}^n \setminus B_r} \frac{|D\eta|^2}{|x|^2} e^{-\phi} e^{-\frac{|x|^2}{4} e^\phi} \leq \frac{1}{4(n - 2)} \int_{B_{2r} \setminus B_r} \frac{1}{r^2 |x|^{n-2} \log |x|} \leq \frac{C_n}{\log r}.
\]
Here, \( C_n \) is a positive constant depending only on \( n \).

For the case \( n = 2 \), let
\[
\eta(x) \triangleq \begin{cases} 
1 & \text{if } x \in B_r \\
2 - \frac{\log \log |x|}{\log \log r} & \text{if } x \in B_{r \log r} \setminus B_r \\
0 & \text{if } x \in \mathbb{R}^2 \setminus B_{r \log r}.
\end{cases}
\]
If \( |x|^2 \log |x| e^\phi \geq C > 0 \) for \( |x| \geq r \geq e \), then
\[
\int_{\mathbb{R}^2 \setminus B_r} \frac{|D\eta|^2}{|x|^2} e^{-\phi} e^{-\frac{|x|^2}{4} e^\phi} \leq \frac{1}{C} \int_{B_{r \log r} \setminus B_r} \frac{\log |x|}{|x|^2 (\log |x|)^2 (\log \log r)^2} = \frac{2\pi}{C \log \log r}.
\]
Hence, \( \phi \) is a constant. By (3.7), as shown in [1], \( v(x) \) is the quadratic polynomial \( v(0) + \frac{1}{2} (D^2 v(0)) x, x \). For \( n = 2 \), up to an additive constant (3.7) is equivalent to
\[
\log \det \partial \tilde{\Omega} v(x) = \frac{1}{2} x \cdot Dv(x) - v(x).
\]
Thus, our condition \( \Delta v \geq \frac{C}{|x|^2 \log |x|} \) for any \( C > 0 \) as \( |x| \to \infty \) is weaker than \( \partial \tilde{\Omega} v(x) \geq \frac{1 + \delta}{2 |x|} \) for any \( \delta > 0 \) as \( |x| \to \infty \) in [1].
Remark 3.2. We don’t know whether every entire smooth subharmonic solution to (3.7) is the quadratic polynomial $v(0) + \frac{1}{2}(D^2v(0)x,x)$. Here, we provide a function $\phi(x) = \log(2n - 4) - 2 \log |x|$ for $n \geq 3$ and $x \in \mathbb{R}^n \setminus \{0\}$, which satisfies (3.8) in $\mathbb{R}^n \setminus \{0\}$.

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