Large deviations for stochastic models of two-dimensional second grade fluids

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Abstract: In this paper, we established a large deviation principle for stochastic models of incompressible second grade fluids. The weak convergence method introduced by [2] plays an important role.

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1 Introduction

In this paper, we are concerned with large deviation principles for stochastic models for the incompressible second grade fluid which is a particular class of Non-Newtonian fluid. Let be a connected, bounded open subset of with boundary of class . We consider

\begin{align}
\begin{split}
&d(u^\varepsilon - \alpha \Delta u^\varepsilon) + \left( -\nu \Delta u^\varepsilon + \text{curl}(u^\varepsilon - \alpha \Delta u^\varepsilon) \times u^\varepsilon + \nabla \mathfrak{P}^\varepsilon \right) dt \\
&= F(u^\varepsilon, t) dt + \sqrt{\varepsilon}G(u^\varepsilon, t) dW, \quad \text{in } \mathcal{O} \times (0, T),
\end{split}
\end{align}

where \( u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2) \) and \( \mathfrak{P}^\varepsilon \) represent the random velocity and modified pressure, respectively. \( W \) is an \( m \)-dimensional standard Brownian motion defined on a complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P) \).

The interest in the investigation of the second grade fluids arises from the fact that it is an admissible model of slow flow fluids, which contains a large class Non-Newtonian fluids such as...
as industrial fluids, slurries, polymer melts, etc.. Furthermore, “the second grade fluid has general and pleasant properties such as boundedness, stability, and exponential decay” (see [15]). It also has interesting connections with many other fluid models, see [6, 7, 19, 20, 21, 28, 29] and references therein. For example, it can be taken as a generalization of the Navier-Stokes Equation. Indeed, they reduce to Navier-Stokes Equation when \( \alpha = 0 \). Furthermore, it was shown in [21] that the second grade fluids models are good approximations of the Navier-Stokes Equation. We refer to [15, 16, 17, 24] for a comprehensive theory of the second grade fluids.

Recently, the stochastic models of two-dimensional second grade fluids (1.1) have been studied in [25], [26] and [27], where the authors obtained the existence and uniqueness of solutions and investigated the behavior of the solution as \( \alpha \to 0 \). The martingale solution of the system (1.1) driven by Lévy noise is studied in [18].

In the present work we are concerned with large deviation principles of the solutions of the system (1.1). Large deviations have applications in many areas, such as in thermodynamics, statistical mechanics, information theory and risk management, etc., see [14], [34] and reference therein. Large deviations for stochastic evolution equations and stochastic partial differential equations driven by Gaussian processes have been investigated in many papers, see e.g. [3], [4], [5], [8], [9], [10], [22], [32], [35]. In this paper, we will apply the weak convergence approach introduced in [2]. This approach is mainly based on a variational representation formula for certain functionals of infinite dimensional Brownian Motion. Technical difficulties arise when implementing weak convergence approach to the system (1.1). One of them is to deal with the nonlinear term \( \text{curl}(u^\varepsilon - \alpha \Delta u^\varepsilon) \).

The organization of this paper is as follows. In Section 2, we introduce some functional spaces and preliminary facts needed later. Section 3 is to formulate the hypotheses and to recall the theorem of existence of solutions for system (1.1) obtained in [25]. The entire Section 4 is devoted to establishing the large deviation principle for system (1.1).

2 Preliminaries

In this section, we will introduce functional spaces and preliminary facts needed later.

Let \( 1 \leq p < \infty \), and \( k \) a nonnegative integer. We denote by \( L^p(\mathcal{O}) \) and \( W^{k,p}(\mathcal{O}) \) the usual \( L^p \) and Sobolev spaces, and write \( W^{k,2}(\mathcal{O}) = H^k(\mathcal{O}) \). Let \( W_0^{k,p}(\mathcal{O}) \) be the closure in \( W^{k,p}(\mathcal{O}) \) of \( C^\infty_c(\mathcal{O}) \) the space of infinitely differentiable functions with compact supports in \( \mathcal{O} \). We denote \( W_0^{k,2}(\mathcal{O}) \) by \( H^k_0(\mathcal{O}) \). We endow the Hilbert space \( H^1_0(\mathcal{O}) \) with the scalar product

\[
((u, v)) = \int_{\mathcal{O}} \nabla u \cdot \nabla v dx = \sum_{i=1}^{2} \int_{\mathcal{O}} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \tag{2.3}
\]

where \( \nabla \) is the gradient operator. The norm \( \| \cdot \| \) generated by this scalar product is equivalent to the usual norm of \( W^{1,2}(\mathcal{O}) \) in \( H^1_0(\mathcal{O}) \).

In what follows, we denote by \( X \) the space of \( \mathbb{R}^2 \)-valued functions such that each component belongs to \( X \). We introduce the spaces

\[
\mathcal{C} = \left\{ u \in [C^\infty_c(\mathcal{O})]^2 \text{ such that } \text{div } u = 0 \right\},
\]

\[
\mathbb{V} = \text{ closure of } \mathcal{C} \text{ in } H^1(\mathcal{O}),
\]

\[
\mathbb{H} = \text{ closure of } \mathcal{C} \text{ in } L^2(\mathcal{O}).
\]
We denote by $(\cdot, \cdot)$ and $|\cdot|$ the inner product and the norm induced by the inner product and the norm in $L^2(O)$ on $\mathbb{H}$, respectively. The inner product and the norm of $\mathbb{H}^1_0(O)$ are denoted respectively by $(\cdot, \cdot)$ and $\|\cdot\|$. We endow the space $\mathbb{V}$ with the norm generated by the following scalar product

$$(u, v)_V = (u, v) + \alpha((u, v)),$$ for any $v \in \mathbb{V};$$

which is equivalent to $\|\cdot\|$, more precisely, we have

$$(P^2 + \alpha)^{-1}\|v\|^2_V \leq \|v\|^2 \leq \alpha^{-1}\|v\|^2_V,$$

for any $v \in \mathbb{V}$,

where $P$ is the constant from Poincaré’s inequality.

We also introduce the following space

$\tilde{\mathbb{W}} = \{u \in \mathbb{V} \text{ such that } \text{curl}(u - \alpha \Delta u) \in L^2(O)\},$

and endow it with the norm generated by the scalar product

$$(u, v)_W = (u, v)_V + (\text{curl}(u - \alpha \Delta u), \text{curl}(v - \alpha \Delta v)).$$ (2.5)

The following result states that $(\cdot, \cdot)_W$ is equivalent to the usual $H^3(O)$-norm on $\tilde{\mathbb{W}}$, and can be found in [11] [12] and Lemma 2.1 in [25].

**Lemma 2.1** Set $\tilde{\mathbb{W}} = \{v \in H^3(O) \text{ such that } \text{div} v = 0 \text{ and } v|_{\partial O} = 0\}$. Then the following (algebraic and topological) identity holds:

$$\mathbb{W} = \tilde{\mathbb{W}}.$$ (2.6)

Moreover, there is a positive constant $C$ such that

$$\|v\|^2_{H^3(O)} \leq C\left(\|v\|^2_V + |\text{curl}(v - \alpha \Delta v)|^2\right),$$ (2.7)

for any $v \in \tilde{\mathbb{W}}$.

From now on, we identify the space $\mathbb{V}$ with its dual space $\mathbb{V}^*$ via the Riesz representation, and we have the Gelfand triple

$$\mathbb{W} \subset \mathbb{V} \subset \mathbb{W}^*.$$ (2.8)

We denote by $\langle f, v \rangle$ the action of the element $f$ of $\mathbb{W}^*$ on an element $v \in \mathbb{W}$. It is easy to see

$$\langle v, w \rangle_V = (v, w), \quad \forall v \in \mathbb{V}, \quad \forall w \in \mathbb{W}.$$ (2.9)

Note that the injection of $\mathbb{W}$ into $\mathbb{V}$ is compact. Thus, there exists a sequence $\{e_i : i = 1, 2, 3, \cdots \}$ of elements of $\mathbb{W}$ which forms an orthonormal basis in $\mathbb{W}$, and an orthogonal basis in $\mathbb{V}$. The elements of this sequence are the solutions of the eigenvalue problem

$$(v, e_i)_W = \lambda_i(v, e_i)_V, \text{ for any } v \in \mathbb{W}.$$ (2.10)

Here $\{\lambda_i : i = 1, 2, 3, \cdots \}$ is an increasing sequence of positive eigenvalues. We have the following important result from [12] about the regularity of the functions $e_i$, $i = 1, 2, 3, \cdots$. 

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Lemma 2.2 Let $\mathcal{O}$ be a bounded, simply-connected open subset of $\mathbb{R}^2$ with a boundary of class $C^3$, then the eigenfunctions of (2.9) belong to $H^4(\mathcal{O})$.

Consider the following “generalized Stokes equations”:

$$
\begin{align*}
\nu - \alpha \Delta \nu + \nabla q &= f \text{ in } \mathcal{O}, \\
\text{div } \nu &= 0 \text{ in } \mathcal{O}, \\
\nu &= 0 \text{ on } \partial \mathcal{O}.
\end{align*}
$$

(2.10)

The following result can be derived from [30], [31] and also can be found in [27] and [25].

Lemma 2.3 Let $\mathcal{O}$ be a connected, bounded open subset of $\mathbb{R}^2$ with boundary $\partial \mathcal{O}$ of class $C_l$ and let $f$ be a function in $H^l$, $l \geq 1$. Then the system (2.10) admits a solution $\nu \in H^{l+2} \cap \mathcal{V}$.

Moreover if $f$ is an element of $H^l$, then $\nu$ is unique and the following relations hold

$$
\begin{align*}
(v, g)_\mathcal{V} &= (f, g), \quad \forall g \in \mathcal{V}, \\
\|v\|_{H^{l+2}} &\leq C\|f\|_H.
\end{align*}
$$

(2.11)

(2.12)

Define the Stokes operator by

$$
Au = -P \Delta u, \quad \forall u \in D(A) = H^2(\mathcal{O}) \cap \mathcal{V},
$$

(2.13)

here we denote by $P : \mathbb{L}^2(\mathcal{O}) \to \mathbb{H}$ the usual Helmholtz-Leray projector. It follows from Lemma 2.3 that the operator $(I + \alpha A)^{-1}$ defines an isomorphism form $H^l(\mathcal{O}) \cap \mathcal{H}$ into $H^{l+2}(\mathcal{O}) \cap \mathcal{V}$ provided that $\mathcal{O}$ is of class $C^l$, $l \geq 1$. Moreover, the following properties hold

$$
\begin{align*}
((I + \alpha A)^{-1} f, v)_\mathcal{V} &= (f, v), \\
\| (I + \alpha A)^{-1} f \|_\mathcal{V} &\leq C|f|,
\end{align*}
$$

for any $f \in H^l(\mathcal{O}) \cap \mathcal{V}$ and any $v \in \mathcal{V}$. From these facts, $\hat{A} = (I + \alpha A)^{-1} A$ defines a continuous linear operator from $H^l(\mathcal{O}) \cap \mathcal{V}$ onto itself for $l \geq 2$, and satisfies

$$(\hat{A} u, v)_\mathcal{V} = (Au, v) = ((u, v)),
$$

for any $u \in \mathcal{W}$ and $v \in \mathcal{V}$. Hence

$$(\hat{A} u, u)_\mathcal{V} = \| u \|,
$$

for any $u \in \mathcal{W}$.

Let

$$
b(u, v, w) = \sum_{i,j=1}^2 \int_\mathcal{O} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,
$$

for any $u, v, w \in \mathcal{C}$. Then the following identity holds (see for instance [13]):

$$
((\text{curl} \Phi) \times v, w) = b(v, \Phi, w) - b(w, \Phi, v),
$$

(2.14)

for any smooth function $\Phi$, $v$ and $w$. Now we recall the following two lemmas which can be found in [25] (Lemma 2.3 and Lemma 2.4), and also in [13].
Lemma 2.4 For any \( u, v, w \in \mathbb{W} \), we have
\[
|\text{curl}(u - \alpha \Delta u) \times v, w| \leq C \|u\|_{H^3} \|v\|_V \|w\|_W, \tag{2.15}
\]
and
\[
|\text{curl}(u - \alpha \Delta u) \times u, w| \leq C \|u\|_V^2 \|w\|_W. \tag{2.16}
\]
Define the bilinear operator \( \hat{B}(\cdot, \cdot) : \mathbb{W} \times \mathbb{V} \to \mathbb{W}^* \) as
\[
\hat{B}(u, v) = (I + \alpha A)^{-1} \left( \text{curl}(u - \alpha \Delta u) \times v \right). \tag{2.17}
\]

Lemma 2.5 For any \( u \in \mathbb{W} \) and \( v \in \mathbb{V} \) there holds
\[
\|\hat{B}(u, v)\|_{\mathbb{W}^*} \leq C \|u\|_W \|v\|_V, \tag{2.18}
\]
and
\[
\|\hat{B}(u, u)\|_{\mathbb{W}^*} \leq C_B \|u\|_V^2. \tag{2.19}
\]
In addition
\[
\langle \hat{B}(u, v), v \rangle = 0, \tag{2.20}
\]
which implies
\[
\langle \hat{B}(u, v), w \rangle = -\langle \hat{B}(u, w), v \rangle, \tag{2.21}
\]
for any \( u, v, w \in \mathbb{W} \).

3 Hypotheses

In this section, we will state the precise assumptions on the coefficients and collect some preliminary results from \cite{27} and \cite{25}, which will be used in the later sections.

We endow the complete probability space \((\Omega, \mathcal{F}, P)\) with the filtration \(\mathcal{F}_t, t \in [0, T]\). Let \( F : \mathbb{V} \times [0, T] \to \mathbb{V} \) and \( G : \mathbb{V} \times [0, T] \to \mathbb{V}^\otimes m \) be given measurable maps. We introduce the following conditions:

(F) For any \( t \in [0, T] \) and for any \( u_1, u_2 \in \mathbb{V} \),
\[
F(0, t) = 0, \tag{3.22}
\]
and
\[
\|F(u_1, t) - F(u_2, t)\|_V \leq C \|u_1 - u_2\|_V. \tag{3.23}
\]

(G) For any \( t \in [0, T] \) and for any \( u_1, u_2 \in \mathbb{V} \),
\[
G(0, t) = 0, \tag{3.24}
\]
and
\[\|G(u_1, t) - G(u_2, t)\|_{\mathcal{V}^m} \leq C\|u_1 - u_2\|_{\mathcal{V}}. \tag{3.25}\]

We now define two operators \(\hat{F}\) and \(\hat{G}\) which map \(\mathbb{V} \times [0, T]\) into \(\mathcal{W}\) and \(\mathcal{W}^m\), respectively, by
\[\hat{F}(u, t) = (I + \alpha A)^{-1} F(u, t), \quad \hat{G}(u, t) = (I + \alpha A)^{-1} G(u, t).\]

**Condition (F) and Condition (G)** implies that there exist \(C_F, C_G\) such that
\[\|\hat{F}(u_1, t) - \hat{F}(u_2, t)\|_{\mathcal{V}} \leq C_F\|u_1 - u_2\|_{\mathcal{V}}, \tag{3.26}\]
\[\|\hat{G}(u_1, t) - \hat{G}(u_2, t)\|_{\mathcal{V}^m} \leq C_G\|u_1 - u_2\|_{\mathcal{V}}. \tag{3.27}\]

Alongside (1.1), we consider the abstract stochastic evolution equations
\[d\nu'(t) + \nu\Delta u'(t)dt + B(u'(t), u'(t))dt = \hat{F}(u'(t), t) + \sqrt{\nu}\hat{G}(u'(t), t)dW(t), \tag{3.28}\]
with initial value \(u_0 = u(0)\), which holds in \(\mathcal{W}^*\). It can be proved that a stochastic process \(\nu'\) satisfies (3.28) if and only if it verifies (1.1) in the weak sense of partial differential equations. Indeed, (3.28) is obtained by applying \((I + \alpha A)^{-1}\) to the equation (1.1).

Now we recall the concept of solution of the problem (1.1) in [25].

**Definition 3.1** A stochastic process \(\nu'\) is called a solution of the system (1.1), if the following three conditions hold

1. \(\nu' \in L^p(\Omega, \mathcal{F}, P; L^\infty([0, T], \mathcal{W}))\), \(2 \leq p < \infty\).
2. For all \(t\), \(\nu'(t)\) is \(\mathcal{F}_t\)-measurable.
3. For any \(t \in (0, T]\) and \(v \in \mathcal{W}\), the following identity holds almost surely
\[\int_0^t (u'(t) - u'(0), v)_{\mathcal{V}} + \int_0^t [\nu((u'(s), v)) + (\text{curl}(u'(s)) - \alpha \Delta u'(s)) \times u'(s, v)]ds\]
\[= \int_0^t (F(u'(s), s), v)ds + \sqrt{\nu}\int_0^t (G(u'(s), s), v)dW(s).\]

Or equivalently, for any \(t \in (0, T]\), the following equation
\[u'(t) + \int_0^t \left(\nu\tilde{A}u'(s) + \tilde{B}(u'(s), u'(s))\right)ds = u_0 + \int_0^t \hat{F}(u'(s), s)ds + \sqrt{\nu}\int_0^t \hat{G}(u'(s), s)dW(s),\]
holds in \(\mathcal{W}^*\) P-a.s.

Using Galerkin approximation scheme for the system (1.1), Razafimandimby and Sango [25] obtained the following theorem (see Theorem 3.4 and Theorem 4.1 in [25]).

**Theorem 3.2** Let \(u_0 \in \mathcal{W}\). Assume conditions (F) and (G) hold. Then the system (1.1) or the problem (3.28) has a unique solution. Moreover, the solution \(\nu'\) admits a version which is continuous in \(\mathcal{V}\) with respect to the strong topology and continuous in \(\mathcal{W}\) with respect to the weak topology.
4 Large Deviation Principle

In this section, we will establish a large deviation principle for system (1.1). We first recall the general criteria obtained in [2].

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with an increasing family \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) of the sub-\(\sigma\)-fields of \(\mathcal{F}\) satisfying the usual conditions. Let \(\mathcal{E}\) be a Polish space with the Borel \(\sigma\)-field \(\mathcal{B}(\mathcal{E})\).

**Definition 4.1 (Rate function)** A function \(I : \mathcal{E} \to [0, \infty]\) is called a rate function on \(\mathcal{E}\), if for each \(M < \infty\), the level set \(\{x \in \mathcal{E} : I(x) \leq M\}\) is a compact subset of \(\mathcal{E}\).

**Definition 4.2 (Large deviation principle)** Let \(I\) be a rate function on \(\mathcal{E}\). A family \(\{X_\varepsilon\}\) of \(\mathcal{E}\)-valued random elements is said to satisfy the large deviation principle on \(\mathcal{E}\) with rate function \(I\), if the following two conditions hold.

\[(a) \text{ (Upper bound)} \quad \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X_\varepsilon \in F) \leq -\inf_{x \in F} I(x).\]

\[(b) \text{ (Lower bound)} \quad \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X_\varepsilon \in G) \geq -\inf_{x \in G} I(x).\]

The Cameron-Martin space associated with the Wiener process \(\{W(t), t \in [0, T]\}\) is given by

\[\mathcal{H}_0 := \left\{ h : [0, T] \to \mathbb{R}^m; h \text{ is absolutely continuous and } \int_0^T \|\dot{h}(s)\|_{\mathbb{R}^m}^2 ds < +\infty \right\}. \quad (4.29)\]

The space \(\mathcal{H}_0\) is a Hilbert space with inner product

\[\langle h_1, h_2 \rangle_{\mathcal{H}_0} := \int_0^T \langle \dot{h}_1(s), \dot{h}_2(s) \rangle_{\mathbb{R}^m} ds.\]

Let \(\mathcal{A}\) denote the class of \(\mathbb{R}^m\)-valued \(\mathcal{F}_t\)-predictable processes \(\phi\) belonging to \(\mathcal{H}_0\) a.s.. Let \(S_N = \{h \in \mathcal{H}_0; \int_0^T \|\dot{h}(s)\|_{\mathbb{R}^m}^2 ds \leq N\}\). The set \(S_N\) endowed with the weak topology is a Polish space. Define \(\mathcal{A}_N = \{\phi \in \mathcal{A}; \phi(\omega) \in S_N, \mathbb{P}\text{-a.s.}\}\).

Recall the following result from Budhiraja and Dupuis [2].

**Theorem 4.3** (\([2]\)) For \(\varepsilon > 0\), let \(\Gamma_\varepsilon\) be a measurable mapping from \(C([0, T]; \mathbb{R}^m)\) into \(\mathcal{E}\). Let \(X_\varepsilon := \Gamma_\varepsilon(W(\cdot))\). Suppose that there exists a measurable map \(\Gamma^0 : C([0, T]; \mathbb{R}^m) \to \mathcal{E}\) such that

\[(a) \text{ for every } N < +\infty \text{ and any family } \{h_\varepsilon; \varepsilon > 0\} \subset \mathcal{A}_N \text{ satisfying that } h_\varepsilon \text{ converge in distribution as } S_N\text{-valued random elements to } h \text{ as } \varepsilon \to 0, \Gamma^0\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^T \dot{h}_\varepsilon(s) ds\right) \text{ converges in distribution to } \Gamma^0(\int_0^T \dot{h}(s) ds) \text{ as } \varepsilon \to 0;\]
(b) for every $N < +\infty$, the set
\[
\left\{ \Gamma^0 \left( \int_0^T \dot{h}(s) ds \right) ; h \in S_N \right\}
\]
is a compact subset of $E$.

Then the family $\{X^\varepsilon\}_{\varepsilon > 0}$ satisfies a large deviation principle in $E$ with the rate function $I$ given by
\[
I(g) := \inf_{\{h \in H_0: g = \Gamma^0(\int_0^T \dot{h}(s) ds)\}} \left\{ \frac{1}{2} \int_0^T \|\dot{h}(s)\|_{\mathbb{R}^m}^2 ds \right\}, \quad g \in E,
\]
with the convention $\inf\emptyset = \infty$.

### 4.1 Main Results

The strong solutions of equation (1.1) determine a measurable mapping $\Gamma^\varepsilon(\cdot)$ from $C([0, T]; \mathbb{R}^m)$ into $C([0, T], V)$ so that $\Gamma^\varepsilon(W) = u^\varepsilon$.

Let $N$ be any fixed positive number. Fixed $g \in S_N$, consider the following deterministic PDE:
\[
\frac{d}{dt} u^g(t) + \int_0^t \left( \nu \hat{A} u^g(s) + \hat{B}(u^g(s), u^g(s)) \right) ds = u_0 + \int_0^t \hat{F}(u^g(s), s) ds + \int_0^T \hat{G}(u^g(s), s) dW(s) + \int_0^T \hat{G}(u^g(s), s) \dot{g}(s) ds.
\]

For any family $\{h^\varepsilon; \varepsilon > 0\} \subset A_N$, let $u^{h^\varepsilon}$ be the solution of the following SPDE
\[
\frac{d}{dt} u^{h^\varepsilon}(t) + \int_0^t \left( \nu \hat{A} u^{h^\varepsilon}(s) + \hat{B}(u^{h^\varepsilon}(s), u^{h^\varepsilon}(s)) \right) ds
\]
\[
= u_0 + \int_0^T \hat{F}(u^{h^\varepsilon}(s), s) ds + \sqrt{\varepsilon} \int_0^T \hat{G}(u^{h^\varepsilon}(s), s) dW(s) + \int_0^T \hat{G}(u^{h^\varepsilon}(s), s) \dot{h}^\varepsilon(s) ds.
\]

Then it is easy to see that $\Gamma^\varepsilon \left( W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^T \dot{h}^\varepsilon(s) ds \right) = u^{h^\varepsilon}$. Define $\Gamma^0(\int_0^T \dot{g}(s) ds) = u^g$. Let $I : C([0, T], V) \to [0, \infty]$ be defined as in (4.30).

**Theorem 4.4** Assume that the Lipschitz conditions (F) and (G) hold. Then the solution family $\{u^\varepsilon\}_{\varepsilon > 0}$ of system (1.1) satisfies a large deviation principle on $C([0, T], V)$ with the good rate function $I$ with respect to the topology of uniform convergence.

**Proof of Theorem 4.4.**

According to Theorem 4.3, we need to prove that Condition (a), (b) are fulfilled. The verification of Condition (a) will be given by Theorem 4.7 below. Condition (b) will be established in Theorem 4.8 below.

### 4.2 Proof of Theorem 4.4

From now on, we denote by $C$ any generic constant which may change from one line to another.

First we will establish the following a priori estimate.
Lemma 4.1 There exists $\varepsilon_0 > 0$ such that
\[
\sup_{\varepsilon \in (0, \varepsilon_0)} E(\sup_{s \in [0, T]} \|u^{h^\varepsilon}(s)\|_W^p) \leq C_{p,N}, \text{ for any } 2 \leq p < \infty, \tag{4.33}
\]
here $C_{p,N}$ is independent of $\varepsilon$.

Set $\mathbb{W}_M = \text{Span}(e_1, \ldots, e_M)$. Let $u^M \in \mathbb{W}_M$ be the Galerkin approximations of $u^{h^\varepsilon}$ satisfying
\[
d(u^M, e_i)_V + \nu((u^M, e_i))dt + b(u^M, u^M, e_i)dt - \alpha b(u^M, \Delta u^M, e_i)dt + \alpha b(e_i, \Delta u^M, u^M)dt
= (F(u^M, t), e_i)dt + \sqrt{\varepsilon}(G(u^M, t), e_i)dW(t) + (G(u^M, t)\dot{h}^\varepsilon(t), e_i)dt \tag{4.34}
\]
for any $i \in \{1, 2, \ldots, M\}$.

As in the proof of Theorem 3.4 in [25], one can show that $u^M \to u^{h^\varepsilon}$ weakly-* in $L^p(\Omega, F, P, L^\infty([0, T], \mathbb{W}))$ for any $p \geq 2$. Hence Lemma 4.1 will follow from the following Lemma 4.2.

Lemma 4.2 For any $4 \leq p < \infty$, we have, for any $\varepsilon \in (0, 1)$
\[
E\left(\sup_{s \in [0, T]} \|u^M(s)\|_V^p\right) \leq C_{p,N}, \tag{4.35}
\]
and
\[
E\left(\sup_{s \in [0, T]} \|u^M(s)\|_W^p\right) \leq C_{p,N}. \tag{4.36}
\]

Proof: Set $\|v\|_* = |\text{curl}(v - \alpha \Delta v)|$ for any $v \in \mathbb{W}$. Define
\[
\tau_J = \inf\{t \geq 0, \|u^M(s)\|_V + \|u^M(s)\|_* \geq J\}.
\]

Applying Itô’s formula, we have
\[
d(u^M, e_i)_V^2 + 2(u^M, e_i)_V\left[\nu((u^M, e_i)) + b(u^M, u^M, e_i) - \alpha b(u^M, \Delta u^M, e_i) + \alpha b(e_i, \Delta u^M, u^M)\right]dt
= 2(u^M, e_i)_V\left[(F(u^M, t), e_i)dt + \sqrt{\varepsilon}(G(u^M, t), e_i)dW(t) + (G(u^M, t)\dot{h}^\varepsilon(t), e_i)dt\right]
+ \varepsilon(G(u^M, t), e_i)(G(u^M, t), e_i)'dt.
\]

Noting that $\|u^M\|_V^2 = \sum_{i=1}^M \lambda_i(u^M, e_i)_V^2$,
\[
d\|u^M\|_V^2 + 2\nu\|u^M\|_V^2dt
= 2(F(u^M, t), u^M)dt + 2\varepsilon(G(u^M, t), u^M)dW(t) + 2(G(u^M, t)\dot{h}^\varepsilon(t), u^M)dt
+ \varepsilon \sum_{i=1}^M \lambda_i(G(u^M, t), e_i)(G(u^M, t), e_i)'+dt, \tag{4.37}
\]
here we have used the fact that $b(u^M, u^M, u^M) = 0$. Applying Itô’s formula to $\|u^M\|_V^p$, we have
\[
d\|u^M\|_V^p = d(\|u^M\|_V^2)^{p/2}dt = \frac{p}{2}(\|u^M\|_V^2)^{p/2-1}d\|u^M\|_V^2 + \frac{p}{4}(\|u^M\|_V^2)^{p/2-2}d\|u^M\|_V^2
\]

9
By Lemma 2.3, there exists unique solution \( \tilde{G}(u^M, t) \in \mathcal{W} \) satisfying

\[
\tilde{G} - \alpha \Delta \tilde{G} = G(u^M, t) \text{ in } \mathcal{O},
\]

Recall that \( \mathcal{P} \) is the Poincaré’s constant. We have

\[
|\langle F(u^M(s), s), u^M(s) \rangle| \leq C \mathcal{P}^2 \|u^M(s)\|^2 \leq \frac{C \mathcal{P}^2}{\alpha} \|u^M(s)\|_V^2,
\]

and

\[
|\langle G(u^M(t), u^M(t), u^M) \rangle| \leq C \|u^M\|^4.
\]

By Burkholder-Davis-Gundy inequalities,

\[
E \left( \sup_{t \in [0,T]} \left| \int_0^{T \land \tau_J} \|u^M(s)\|_V^{p-2} |\langle G(u^M(s), s)\dot{h}^\varepsilon(s), u^M(s) \rangle| \, ds \right| \right)
\]

\[
\leq C E \left( \int_0^{T \land \tau_J} \|u^M(s)\|_V^{2p} \, ds \right)^{1/2}
\]

\[
\leq C E \left[ \sup_{t \in [0,T \land \tau_J]} \|u^M(t)\|_V^{p/2} \left( \int_0^{T \land \tau_J} \|u^M(s)\|_V^{p} \, ds \right)^{1/2} \right]
\]

\[
\leq \delta E \left( \sup_{t \in [0,T \land \tau_J]} \|u^M(t)\|_V \right) + C_\delta E \left( \int_0^{T \land \tau_J} \|u^M(s)\|_V^{p} \, ds \right).
\]

By Hölder’s inequality and Young’s inequality, for any \( \eta > 0 \)

\[
\int_0^{T \land \tau_J} \|u^M(s)\|_V^{p-2} |\langle G(u^M(s), s)\dot{h}^\varepsilon(s), u^M(s) \rangle| \, ds
\]

\[
\leq C \int_0^{T \land \tau_J} \|u^M(s)\|_V^{p} \|\dot{h}^\varepsilon(s)\|_\mathcal{W} \, ds
\]

\[
\leq C \sup_{s \in [0,T \land \tau_J]} \|u^M(s)\|_V^{p-1} \int_0^{T \land \tau_J} \|u^M(s)\|_V \|\dot{h}^\varepsilon(s)\|_\mathcal{W} \, ds
\]

\[
\leq \eta \sup_{s \in [0,T \land \tau_J]} \|u^M(s)\|_V^{p} + C_\eta \left( \int_0^{T \land \tau_J} \|u^M(s)\|_V \|\dot{h}^\varepsilon(s)\|_\mathcal{W} \, ds \right)^p
\]

\[
\leq \eta \sup_{s \in [0,T \land \tau_J]} \|u^M(s)\|_V^{p} + C_\eta \left( \int_0^{T \land \tau_J} \|u^M(s)\|_V^2 \, ds \int_0^{T \land \tau_J} \|\dot{h}^\varepsilon(s)\|_\mathcal{W}^2 \, ds \right)^{p/2}
\]

\[
\leq \eta \sup_{s \in [0,T \land \tau_J]} \|u^M(s)\|_V^{p} + C_\eta N^{p/2} \left( \int_0^{T \land \tau_J} \|u^M(s)\|_V^2 \, ds \right)^{p/2}
\]

\[
\leq \eta \sup_{s \in [0,T \land \tau_J]} \|u^M(s)\|_V^{p} + C_\eta N^{p/2} T^{-\frac{p}{2}} \int_0^{T \land \tau_J} \|u^M(s)\|_V^p \, ds.
\]
\[ \text{div } \tilde{G} = 0 \text{ in } \mathcal{O}, \]
\[ \tilde{G} = 0 \text{ on } \partial \mathcal{O}. \]

Moreover,
\[ (\tilde{G}(u^M, t), e_i)_V = (G(u^M, t), e_i), \quad \forall i \in \{1, 2, \cdots, M\}, \]
and there exists a positive constant \( C_0 \) such that
\[ \|\tilde{G}(u^M, t)\|_{W^m} \leq C_0 \|G(u^M, t)\|_{V^m}. \]

Hence by (2.9),
\[ \sum_{i=1}^{M} \lambda_i (G(u^M(s), s), e_i)(G(u^M(s), s), e_i)' = \sum_{i=1}^{M} \frac{1}{\lambda_i} (\tilde{G}(u^M(s), s), e_i)_V (\tilde{G}(u^M(s), s), e_i)'_V \]
\[ \leq \frac{1}{\lambda_1} \|\tilde{G}(u^M(s), s)\|^2_{W^m} \leq \frac{C_0}{\lambda_1} \|G(u^M(s), s)\|^2_{V^m} \leq C \|u^M(s)\|^2_V. \]

Combining (4.38)–(4.43), we have
\[ (1 - p\eta - \sqrt{\varepsilon}p\delta)E \left( \sup_{t \in [0,T \wedge \tau_j]} \|u^M(t)\|_V^p \right) + 2\nu E \int_0^{T \wedge \tau_j} \frac{p}{2} \|u^M\|_V^{p-2} \|u^M\|^2 ds \]
\[ \leq \|u(0)\|_V^p + C_{\eta,N,p,\delta} E \int_0^{T \wedge \tau_j} \|u^M(s)\|_V^p ds. \]

Choosing \( \eta = \delta = \frac{1}{4p} \), then for any \( \varepsilon \in (0,1) \)
\[ E \left( \sup_{t \in [0,T \wedge \tau_j]} \|u^M(t)\|_V^p \right) \leq C_{p,N}. \]

Let \( J \to \infty \) to obtain (4.35).

Now we prove (4.36).

Setting \( \phi(u^M) = -\nu \Delta u^M + \text{curl}(u^M - \alpha \Delta u^M) \times u^M - F(u^M, t) - G(u^M, t)\dot{h}^\varepsilon(t) \),
we have
\[ d(u^M, e_i)_V + (\phi(u^M), e_i)dt = \sqrt{\varepsilon}(G(u^M, t), e_i)dW(t). \]

Note that \( \phi(u^M) \in \mathbb{H}^1(\mathcal{O}) \). By Lemma 2.3, there exists a unique solution \( v^M \in \mathcal{W} \) satisfying
\[ v^M - \alpha \Delta v^M = \phi(u^M) \text{ in } \mathcal{O}, \]
\[
\text{div } v^M = 0 \text{ in } \mathcal{O},
\]
\[
v^M = 0 \text{ on } \partial \mathcal{O}.
\]

Moreover,
\[
(v^M, e_i)_V = (\phi(u^M), e_i), \quad \forall i \in \{1, 2, \cdots, M\}.
\]

Thus
\[
d(u^M, e_i)_V + (v^M, e_i)_V dt = \sqrt{\varepsilon}(G(u^M, t), e_i) dW(t).
\]

We introduce \(\tilde{G}\) as in the proof of (4.43) to get
\[
\lambda_i(G(u^M, t), e_i) = (\tilde{G}(u^M, t), e_i)_W.
\]

By (2.9),
\[
d(u^M, e_i)_W + (v^M, e_i)_W dt = \sqrt{\varepsilon}(\tilde{G}(u^M, t), e_i)_W dW(t).
\]

Applying Itô’s formula, we have
\[
d(u^M, e_i)^2_W + 2(u^M, e_i)_W(v^M, e_i)_W dt = 2\sqrt{\varepsilon}(u^M, e_i)_W(\tilde{G}(u^M, t), e_i)_W dW(t) + \varepsilon(\tilde{G}(u^M, t), e_i)_W(\tilde{G}(u^M, t), e_i)'_W dt,
\]
and
\[
d\|u^M\|^2_W + 2(v^M, u^M)'_W dt = 2\sqrt{\varepsilon}(\tilde{G}(u^M, t), u^M)_W dW(t) + \varepsilon \sum_{i=1}^M (\tilde{G}(u^M, t), e_i)_W(\tilde{G}(u^M, t), e_i)'_W dt.
\]

By (2.5) we rewrite the above equation as follows
\[
d\|u^M\|^2_W + \|u^M\|^2_W + 2\left[\varepsilon(\tilde{G}(u^M, t), u^M)_W dW(t) + \varepsilon \sum_{i=1}^M \lambda_i^2(G(u^M, t), e_i)(G(u^M, t), e_i)'_W dtight.
\]
\[
+ 2\sqrt{\varepsilon}\left(\text{curl}(u^M - \alpha \Delta u^M), \text{curl}(\tilde{G}(u^M, t) - \alpha \Delta \tilde{G}(u^M, t))\right) dW(t).
\]

By the definition of \(v^M\) and \(\tilde{G}\), we obtain
\[
d\|u^M\|^2_W + \|u^M\|^2_W + 2\left[\varepsilon(\tilde{G}(u^M, t), u^M)_W dW(t) + \varepsilon \sum_{i=1}^M \lambda_i^2(G(u^M, t), e_i)(G(u^M, t), e_i)'_W dtight.
\]
\[
+ 2\sqrt{\varepsilon}\left(\text{curl}(u^M - \alpha \Delta u^M), \text{curl}(G(u^M, t))\right) dW(t).
\]

Subtracting (4.37) from the above equation, we obtain
\[
d\|u^M\|^2_W + 2\left(\text{curl}(u^M - \alpha \Delta u^M), \text{curl}(\phi(u^M))\right) dt
\]
\[
= \varepsilon \sum_{i=1}^M (\lambda_i^2 - \lambda_i)(G(u^M, t), e_i)(G(u^M, t), e_i)'_W dt.
\]
Applying Itô's formula, we have
\[
Hence
\]
\[
= \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl}(\phi(u^M)) \right)
\]
\[
= \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl}(-\nu \Delta u^M) \right) - \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl}(F(u^M, t) + G(u^M)\dot{h}(t)) \right)
\]
\[
+ \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl} \left( \text{curl}(u^M - \alpha \Delta u^M) \times u^M \right) \right)
\]
\[
= \frac{\nu}{\alpha} \|u^M\|_{*}^{2} - \frac{\nu}{\alpha} \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl} u^M \right)
\]
\[
- \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl}(F(u^M, t) + G(u^M)\dot{h}(t)) \right).
\]

Hence
\[
d\|u^M\|_{*}^{p} = d\left(\|u^M\|_{*}^{2}\right)^{p/2} = \frac{P}{2} \left(\|u^M\|_{*}^{2}\right)^{p/2 - 1} d\|u^M\|^{2}_{*} + \frac{P}{4} (\frac{p}{2} - 1) (\|u^M\|_{*}^{2})^{p/2 - 2} d(\|u^M\|_{*}^{2})
\]
\[
= \frac{P}{2} \|u^M\|_{*}^{p - 2} \left(-\frac{2\nu}{\alpha} \|u^M\|_{*}^{2} dt + \frac{2\nu}{\alpha} \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl} u^M \right) dt \right.
\]
\[
+ 2 \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl}(F(u^M, t) + G(u^M)\dot{h}(t)) \right) dt
\]
\[
+ \varepsilon \sum_{i=1}^{M} (\lambda_{i}^{2} - \lambda_{i})(G(u^M, t), e_{i}) (G(u^M, t), e_{i})' dt
\]
\[
+ 2\sqrt{\varepsilon} \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl}(G(u^M, t)) \right) dW(t)
\] + \varepsilon (\frac{P}{2} - 1) \|u^M\|_{*}^{p - 4}
\]
\[
\left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl} G(u^M, t) \right) \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl} G(u^M, t) \right)' dt.
\]

Applying Itô's formula, we have
\[
|\text{curl}(\phi)|^{2} \leq \frac{2}{\alpha} \|\phi\|_{V}^{2}
\] for any \( \phi \in \mathbb{W} \),

we have
\[
\|u^M\|_{*}^{p-2} |(\text{curl}(u^M - \alpha \Delta u^M), \text{curl} u^M)| + \|u^M\|_{*}^{p-2} |(\text{curl}(u^M - \alpha \Delta u^M), \text{curl}(F(u^M, t)))|
\]
\[
\leq C \|u^M\|_{*}^{p-1} \|u^M\|_{V}
\]
\[ \leq \| u^M \|_p^p + C \| u^M \|_\nu^p, \quad (4.49) \]

and
\[
\| u^M \|_{p-1} \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl } G(u^M, t) \right) \left( \text{curl}(u^M - \alpha \Delta u^M), \text{curl } G(u^M, t) \right)' \\
\leq C \| u^M \|_{p-2} \| u^M \|_\nu^2 \\
\leq \| u^M \|_p^p + C \| u^M \|_\nu^p. \quad (4.50) \]

By the similar arguments as for the proof of (4.43), we obtain
\[
\sum_{i=1}^M (\lambda_i^2 + \lambda_i) (G(u^M(s), s), e_i')(G(u^M(s), s), e_i) \leq C \| u^M(s) \|_\nu^2.
\]

Thus
\[
\| u^M \|_{p-2}^2 \sum_{i=1}^M (\lambda_i^2 - \lambda_i) (G(u^M, t), e_i')(G(u^M, t), e_i) \\
\leq C \| u^M \|_{p-2} \| u^M \|_\nu^2 \\
\leq \| u^M \|_p^p + C \| u^M \|_\nu^p. \quad (4.51) \]

By Hölder’s inequality and Young’s inequality, for any \( \eta > 0 \)
\[
\int_0^{T \wedge \tau_J} \| u^M \|_{p-2}^2 |(\text{curl}(u^M - \alpha \Delta u^M), \text{curl } G(u^M, s) \dot{h}^\nu(s))| ds \\
\leq C \int_0^{T \wedge \tau_J} \| u^M \|_{p-1} \| u^M \|_\nu \| \dot{h}^\nu(s) \| ds \leq \eta \sup_{s \in [0, T \wedge \tau_J]} \| u^M \|_p^p + C_{\eta,N,p} \int_0^{T \wedge \tau_J} \| u^M \|_\nu^p ds. \quad (4.52) \]

Applying Burkholder-Davis-Gundy inequalities,
\[
E \left( \sup_{t \in [0,T]} \left| \int_0^{T \wedge \tau_J} \| u^M(s) \|_{p-2} \left( \text{curl}(u^M(s) - \alpha \Delta u^M(s)), \text{curl } G(u^M(s), s) \right) dW(s) \right| \right) \\
\leq C E \left( \int_0^{T \wedge \tau_J} \| u^M(s) \|_{2p-2}^2 \| u^M(s) \|_\nu^2 ds \right)^{1/2} \\
\leq C T^{1/2} E \left( \sup_{t \in [0,T \wedge \tau_J]} \| u^M(s) \|_{p}^p \sup_{t \in [0,T \wedge \tau_J]} \| u^M(s) \|_\nu \right) \\
\leq \delta E \left( \sup_{t \in [0,T \wedge \tau_J]} \| u^M(s) \|_{p}^p \right) + C_{\delta} E \left( \sup_{t \in [0,T \wedge \tau_J]} \| u^M(s) \|_\nu^p \right). \quad (4.53) \]

Combining (4.47)-(4.53), for every \( \varepsilon \in (0, 1) \)
\[
(1 - \eta - \sqrt{\varepsilon p} \delta) E \left( \sup_{t \in [0,T \wedge \tau_J]} \| u^M(t) \|_p^p \right) \\
\leq \| u(0) \|_p^p + C_p E \int_0^{T \wedge \tau_J} \| u^M(s) \|_p^p ds + C_{\eta,N,p,\delta} E \left( \sup_{t \in [0,T]} \| u^M(t) \|_\nu^p \right).
Let $\eta = \delta = \frac{1}{4^p}$, for every $\varepsilon \in (0, 1)$ to obtain

$$E\left( \sup_{s \in [0,T]} \|u^M(s)\|_{W}^p \right) \leq C_{p,N}.$$ \hfill (4.54)

By Fatou’s lemma, (4.54) implies (4.36). ■

Let $\mathbb{H}$ be a separable Hilbert space. Given $p > 1$, $\beta \in (0, 1)$, let $W^{\beta,p}([0,T]; \mathbb{H})$ be the space of all $u \in L^p([0,T]; \mathbb{H})$ such that

$$\int_0^T \int_0^T \frac{\|u(t) - u(s)\|_H^p}{|t - s|^{1+\beta p}} dt ds < \infty$$

endowed with the norm

$$\|u\|_{W^{\beta,p}(0,T; \mathbb{H})} := \int_0^T \|u(t)\|_H^p dt + \int_0^T \int_0^T \frac{\|u(t) - u(s)\|_H^p}{|t - s|^{1+\beta p}} dt ds.$$

The following result represents a variant of the criteria for compactness proved in [23] (Sect. 5, Ch. I) and [33] (Sect. 13.3).

**Lemma 4.3** Let $\mathbb{H}_0 \subset \mathbb{H} \subset \mathbb{H}_1$ be Banach spaces, $\mathbb{H}_0$ and $\mathbb{H}_1$ reflexive, with compact embedding of $\mathbb{H}_0$ into $\mathbb{H}$. For $p \in (1, \infty)$ and $\beta \in (0, 1)$, let $\Lambda$ be the space

$$\Lambda = L^p([0,T]; \mathbb{H}_0) \cap W^{\beta,p}([0,T]; \mathbb{H}_1)$$

endowed with the natural norm. Then the embedding of $\Lambda$ into $L^p([0,T]; \mathbb{H})$ is compact.

**Proposition 4.5** \{u^{h\varepsilon}\} is tight in $L^2([0,T]; \mathcal{V})$.

**Proof:**

Note that

$$u^{h\varepsilon}(t) = u_0 - \int_0^t \nu \hat{A} u^{h\varepsilon}(s) ds - \int_0^t \hat{B}(u^{h\varepsilon}(s), u^{h\varepsilon}(s)) ds \hfill (4.55)$$

$$+ \int_0^t \hat{F}(u^{h\varepsilon}(s), s) ds + \sqrt{\varepsilon} \int_0^t \hat{G}(u^{h\varepsilon}(s), s) dW(s) + \int_0^t \hat{G}(u^{h\varepsilon}(s), s) h^{\varepsilon}(s) ds, \hfill$$

$$= J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) + J_6(t).$$

Using Lemma 4.1, it is easy to show that

$$E(\int_0^T \|u^{h\varepsilon}(s)\|_{W}^2 ds) \leq C_{2,N},$$ \hfill (4.56)

where $C_{2,N}$ is a constant independent of $\varepsilon$. We next prove

$$\sup_{\varepsilon \in (0,\varepsilon_0)} E(\|u^{h\varepsilon}\|_{W^{\beta,2}(0,T); \mathcal{V}^*}) \leq C < \infty,$$ \hfill (4.57)

here $\varepsilon_0$ is the constant stated in Lemma 4.1. Noting that, for any $u \in \mathcal{W}$ and $v \in \mathcal{V},$

$$(\hat{A}u, v)_\mathcal{V} = ((u, v)),$$
we have
\[ \| \hat{Au} \|_V = \sup_{\|v\|_V \leq 1} |(\hat{Au}, v)_V| = \sup_{\|v\|_V \leq 1} |(u, v)| \leq \|u\| \sup_{\|v\|_V \leq 1} \|v\| \leq \alpha^{-1/2} \|u\|. \] (4.58)

Then
\[ \|J_2(t) - J_2(s)\|_V^2 = \| \int_s^t \nu \hat{Au}^{-\nu}(l)dl \|_V^2 \leq \int_s^t \| \nu \hat{Au}^{-\nu}(l) \|_V^2 \| \nu \hat{Au}^{-\nu}(l) \|_V \| \nu \hat{Au}^{-\nu}(l) \|_V \| \nu \hat{Au}^{-\nu}(l) \|_V \int_s^t \| \nu \hat{Au}^{-\nu}(l) \|_V^2 dl(t - s) \]
\[ \leq \nu^2 \alpha^{-1} \int_s^t \| \nu \hat{Au}^{-\nu}(l) \|_V^2 dl(t - s) \leq \nu^2 \alpha^{-2} \| \nu \hat{Au}^{-\nu}(l) \|_V^2 \| \nu \hat{Au}^{-\nu}(l) \|_V \| \nu \hat{Au}^{-\nu}(l) \|_V \int_s^t \| \nu \hat{Au}^{-\nu}(l) \|_V^2 dl(t - s)^2. \] (4.59)

For any \( \beta \in (0, 1/2) \), we have
\[ E(\|J_2\|_{W^{\beta, 2}(\Omega; V)}^2) = E(\int_0^T \|J_2(s)\|_V^2 ds + \int_0^T \int_0^T \frac{\|J_2(t) - J_2(s)\|_V^2}{|t - s|^{1+2\beta}} ds dt) \leq TE(\sup_{s \in [0, T]} \|J_2(s)\|_V^2) + E(\int_0^T \int_0^T \|J_2(s)\|_V^2(t - s)^{1-2\beta} ds dt) \leq C_{\beta, T} E(\sup_{t \in [0, T]} \|\nu \hat{Au}^{-\nu}(l)\|_V^2). \] (4.60)

By (2.19), we have
\[ \|J_3(t) - J_3(s)\|_{V^2}^2 = \| \int_s^t \hat{B}(u \hat{h}^{-\nu}(l), u \hat{h}^{-\nu}(l))dl \|_{V^2}^2 \]
\[ \leq \int_s^t \| \hat{B}(u \hat{h}^{-\nu}(l), u \hat{h}^{-\nu}(l)) \|_{V^2}^2 dl(t - s) \leq C \sup_{t \in [0, T]} \|u \hat{h}^{-\nu}(l)\|_{V^2}^2 \|t - s\|^2, \]
which yields
\[ E(\|J_3\|_{W^{\beta, 2}(\Omega; V)}^2) \leq C_{\beta, T} E(\sup_{t \in [0, T]} \|u \hat{h}^{-\nu}(l)\|_{V^2}^2). \] (4.61)

For \( J_6 \), we have
\[ \|J_6(t) - J_6(s)\|_V^2 \leq \left\| \int_s^t \hat{G}(u \hat{h}^{-\nu}(l)) \hat{h}^{-\nu}(l)dl \right\|_V^2 \leq \int_s^t \| \hat{G}(u \hat{h}^{-\nu}(l)) \|_{V^2}^2 \| \hat{h}^{-\nu}(l) \|_{V^2}^2 dl \]
\[ \leq CN \left( \sup_{t \in [0, T]} \|u \hat{h}^{-\nu}(l)\|_V^2 \right)(t - s), \]
which implies that
\[ E(\|J_6\|_{W^{\beta, 2}(\Omega; V)}^2) \leq C_{\beta, T, N} E(\sup_{t \in [0, T]} \|u \hat{h}^{-\nu}(l)\|_V^2). \] (4.62)

By similar arguments, we also have
\[ E(\|J_4\|_{W^{\beta, 2}(\Omega; V)}^2) + E(\|J_5\|_{W^{\beta, 2}(\Omega; V)}^2) \leq C_{\beta} E(\sup_{t \in [0, T]} \|u \hat{h}^{-\nu}(l)\|_V^2). \] (4.63)
Combining (4.56), (4.60), (4.62) and (4.63) and (4.33) in Lemma 4.1, we obtain (4.57).

Since the embedding \( \mathcal{W} \subset \mathcal{V} \) is compact, by Lemma 4.3
\[
\Lambda = L^2([0, T], \mathcal{W}) \cap \mathcal{W}^{\beta, 2}([0, T], \mathcal{W}^*)
\]
is compactly imbedded in \( L^2([0, T], \mathcal{V}) \). Denote \( \| \cdot \|_\Lambda := \| \cdot \|_{L^2([0, T], \mathcal{W})} + \| \cdot \|_{\mathcal{W}^{\beta, 2}([0, T], \mathcal{W}^*)} \).
Thus for any \( L > 0 \),
\[
K_L = \{ u \in L^2([0, T], \mathcal{V}), \| u \|_\Lambda \leq L \}
\]
is relatively compact in \( L^2([0, T], \mathcal{V}) \).

We have
\[
P(u^{h\varepsilon} \notin K_L) \leq P(\| u^{h\varepsilon} \|_\Lambda \geq L) \leq \frac{1}{L} E(\| u^{h\varepsilon} \|_\Lambda) \leq \frac{C}{L}.
\]
Choosing sufficiently large constant \( L \), we see that \( \{ u^{h\varepsilon}, \varepsilon > 0 \} \) is tight in \( L^2([0, T], \mathcal{V}) \). ■

Since the imbedding \( \mathcal{W}^{\beta, 2}([0, T], \mathcal{W}^*) \subset C([0, T]; \mathcal{W}^*) \) is compact, the following result is a consequence of (4.57).

**Proposition 4.6** \( \{ u^{h\varepsilon} \} \) is tight in \( C([0, T]; \mathcal{W}^*) \).

**Theorem 4.7** For every fixed \( N \in \mathbb{N} \), let \( h^{\varepsilon} \), \( h \in \mathcal{A}_N \) be such that \( h^{\varepsilon} \) converges in distribution to \( h \) as \( \varepsilon \to 0 \). Then
\[
\Gamma^\varepsilon \left( W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \dot{h}^{\varepsilon}(s) ds \right)
\]
converges in distribution to \( \Gamma^0(\int_0^\cdot \dot{h}(s) ds) \)
in \( C([0, T]; \mathcal{V}) \) as \( \varepsilon \to 0 \).

**Proof:** Note that \( u^{h^{\varepsilon}} = \Gamma^\varepsilon \left( W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \dot{h}^{\varepsilon}(s) ds \right) \). By Proposition 4.5 and Proposition 4.6 we know that \( \{ u^{h^{\varepsilon}} \} \) is tight in \( L^2([0, T], \mathcal{V}) \cap C([0, T], \mathcal{W}^*) \).

Let \((u, h, W)\) be any limit point of the tight family \( \{ (u^{h^{\varepsilon}}, h^{\varepsilon}, W), \varepsilon \in (0, \varepsilon_0) \} \). We must show that \( u \) has the same law as \( \Gamma^0(\int_0^\cdot \dot{h}(s) ds) \), and actually \( u^{h^{\varepsilon}} \Rightarrow u \) in the smaller space \( C([0, T]; \mathcal{V}) \).

Set
\[
\Pi = \left( L^2([0, T], \mathcal{V}) \cap C([0, T], \mathcal{W}^*), S_N, C([0, T], \mathbb{R}^m) \right).
\]

By the Skorokhod representation theorem, there exit a stochastic basis \( (\Omega^1, \mathcal{F}^1, \{ \mathcal{F}^1_t \}_{t \in [0, T]}, \mathbb{P}^1) \) and, on this basis, \( \Pi \)-valued random variables \((\hat{X}^{\varepsilon}, \tilde{h}^{\varepsilon}, \hat{W}^{\varepsilon}) \) \((\hat{X}, \tilde{h}, \hat{W})\) such that \((\hat{X}^{\varepsilon}, h^{\varepsilon}, W^{\varepsilon})\) (respectively \((\hat{X}, \tilde{h}, \hat{W})\)) has the same law as \( \{ (u^{h^{\varepsilon}}, h^{\varepsilon}, W), \varepsilon \in (0, \varepsilon_0) \} \) (respectively \((u, h, W)\)) and \((\hat{X}^{\varepsilon}, \tilde{h}^{\varepsilon}, W^{\varepsilon}) \to (\hat{X}, \tilde{h}, \hat{W})\) \(\mathbb{P}^1\) a.s. in \( \Pi \).

From the equation satisfied by \((u^{h^{\varepsilon}}, h^{\varepsilon}, W)\), we see that \((\hat{X}^{\varepsilon}, \tilde{h}^{\varepsilon}, W^{\varepsilon})\) satisfies the following integral equation
\[
\hat{X}^{\varepsilon}(t) = u_0 - \int_0^t \nu \hat{A} \hat{X}^{\varepsilon}(s) ds - \int_0^t \hat{B}(\hat{X}^{\varepsilon}(s), \hat{X}^{\varepsilon}(s)) ds
\]
\[
+ \int_0^t \hat{F}(\hat{X}^{\varepsilon}(s), s) ds + \sqrt{\varepsilon} \int_0^t \hat{G}(\hat{X}^{\varepsilon}(s), s) d\hat{W}^{\varepsilon}(s) + \int_0^t \hat{G}(\hat{X}^{\varepsilon}(s), s) \gamma^{\varepsilon}(s) ds,
\]
and (see (4.33))
\[
\sup_{\varepsilon \in (0, \varepsilon_0)} E \left( \sup_{s \in [0, T]} \| \hat{X}^{\varepsilon}(s) \|_{\hat{W}^*}^p \right) \leq C_p, \text{ for any } 2 \leq p < \infty.
\]
Using similar arguments as in the proof of Theorem 3.4 and Theorem 4.1 in [25], we can show that \( \tilde{X} \) is the unique solution of the following equation

\[
\tilde{X}(t) = u_0 - \int_0^t \nu \tilde{X}(s)ds - \int_0^t \tilde{B}(\tilde{X}(s), \tilde{X}(s))ds + \int_0^t \tilde{F}(\tilde{X}(s), s)ds + \int_0^t \tilde{G}(\tilde{X}(s), s)\tilde{h}(s)ds.
\]

Finally, we will prove that

\[
\lim_{\epsilon \to 0} \sup_{t \in [0,T]} \|\tilde{X}(t) - \tilde{X}^\epsilon(t)\|_V = 0, \quad \mathbb{P}^1 - a.s..
\]

Let \( v^\epsilon(t) = \tilde{X}^\epsilon(t) - \tilde{X}(t) \). Using Itô formula, we have

\[
\|v^\epsilon(t)\|^2_V + 2 \int_0^t \left( \|v^\epsilon(s)\|^2 + \langle \tilde{B}(\tilde{X}^\epsilon(s), \tilde{X}(s)) - \tilde{B}(\tilde{X}(s), \tilde{X}(s)), v^\epsilon(s) \rangle \right) ds
\]

\[
= \int_0^t 2 \left( \langle \tilde{F}(\tilde{X}^\epsilon(s), s) - \tilde{F}(\tilde{X}(s), s), v^\epsilon(s) \rangle \|G(\tilde{X}^\epsilon(s), s)\|^2_{V \otimes M} \right) ds
\]

\[
+ 2 \int_0^t \sqrt{\epsilon} \|G(\tilde{X}^\epsilon(s), s, v^\epsilon(s))\|_V d\tilde{W}^\epsilon(s)
\]

\[
+ 2 \int_0^t \langle \tilde{G}(\tilde{X}^\epsilon(s), s)\tilde{h}(s) - \tilde{G}(\tilde{X}(s))\tilde{h}(s), v^\epsilon(s) \rangle ds.
\]

Since

\[
\langle \tilde{B}(\tilde{X}^\epsilon(s), \tilde{X}(s)) - \tilde{B}(\tilde{X}(s), \tilde{X}(s)), v^\epsilon(s) \rangle = -\langle \tilde{B}(v^\epsilon(s), v^\epsilon(s)), \tilde{X}(s) \rangle,
\]

by (2.19) and (3.23), it follows that

\[
\|v^\epsilon(t)\|^2_V + 2 \nu \int_0^t \|v^\epsilon(s)\|^2 ds
\]

\[
\leq C \int_0^t \|v^\epsilon(s)\|^2_V \|\tilde{X}(s)\|_{W^2} ds + C \int_0^t \|v^\epsilon(s)\|^2_V ds + C \epsilon \int_0^t \|\tilde{X}^\epsilon(s)\|^2_V ds
\]

\[
+ 2 \int_0^t \sqrt{\epsilon} \|G(\tilde{X}^\epsilon(s), s, v^\epsilon(s))\|_V d\tilde{W}^\epsilon(s)
\]

\[
+ 2 \int_0^t \langle \tilde{G}(\tilde{X}^\epsilon(s), s)\tilde{h}(s) - \tilde{G}(\tilde{X}(s))\tilde{h}(s), v^\epsilon(s) \rangle ds.
\]

Since

\[
\int_0^t \langle \tilde{G}(\tilde{X}(s), s)\tilde{h}(s) - \tilde{G}(\tilde{X}(s), s)\tilde{h}(s), v^\epsilon(s) \rangle ds
\]

\[
\leq \int_0^t \|\tilde{G}(\tilde{X}(s), s)\tilde{h}(s) - \tilde{G}(\tilde{X}(s), s)\tilde{h}(s), v^\epsilon(s) \rangle ds
\]

\[
+ \int_0^t \|\tilde{G}(\tilde{X}(s), s)\tilde{h}(s) - \tilde{G}(\tilde{X}(s), s)\tilde{h}(s), v^\epsilon(s) \rangle ds
\]

\[
\leq C \int_0^t \|v^\epsilon(s)\|^2_V \|\tilde{h}(s)\|_{W^m} ds + C \int_0^t \|\tilde{X}(s)\|_V \|\tilde{h}(s)\|_{W^m} \|v^\epsilon(s)\|_V ds
\]

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Theorem 4.8

\[ \Gamma^0(\int_0^T g(s)ds) \]

is a continuous mapping from \( g \in S_N \) into \( C([0, T]; \mathbb{V}) \), in particular, \( \{ \Gamma^0(\int_0^T g(s)ds); \, g \in S_N \} \) is a compact subset of \( C([0, T]; \mathbb{V}) \).
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