Fast Low Rank column-wise Compressive Sensing

Seyedehsara (Sara) Nayer and Namrata Vaswani
Dept. of Electrical and Computer Engineering
Iowa State University, Ames, IA, USA.
Email: namrata@iastate.edu

Abstract

In this work, we study the following problem, that we refer to as Low Rank column-wise Compressive Sensing (LRcCS): how to recover an \( n \times q \) rank-\( r \) matrix, \( \mathbf{X}^* = [x_1^*, x_2^*, \ldots, x_q^*] \) from \( m \) independent linear projections of each of its \( q \) columns, i.e., from \( y_k := \mathbf{A}_k \mathbf{x}_k^*, k \in [q] \), when \( y_k \) is an \( m \)-length vector. The matrices \( \mathbf{A}_k \) are known and mutually independent for different \( k \). The regime of interest is low-rank, i.e., \( r \ll \min(n, q) \), and undersampled measurements, i.e., \( m < n \). Even though many LR recovery problems have been extensively studied in the last decade, this particular problem has received little attention so far in terms of methods with provable guarantees. We introduce a novel gradient descent (GD) based solution called altGDmin. We show that, if all entries of all \( \mathbf{A}_k \)s are i.i.d. Gaussian, and if the right singular vectors of \( \mathbf{X}^* \) satisfy the incoherence assumption, then \( \epsilon \)-accurate recovery of \( \mathbf{X}^* \) is possible with \( mq > C(n + q)r^2 \log(1/\epsilon) \) total samples and \( O(mqnr \log(1/\epsilon)) \) time. Compared to existing work, to our best knowledge, this is the fastest solution and, for \( \epsilon < 1/\sqrt{r} \), it also has the best sample complexity. Moreover, we show that a simple extension of our approach also solves LR Phase Retrieval (LRPR), which is the magnitude-only generalization of LRcCS. It involves recovering \( \mathbf{X}^* \) from the magnitudes of entries of \( y_k \). We show that altGDmin-LRPR has matching sample complexity and better time complexity when compared with the (best) existing solution for LRPR.

I. INTRODUCTION

Large scale usage of Internet-of-Things (IoT) devices, smartphones, and surveillance cameras have resulted in large amounts of geographically distributed data in current times. There is a need to compress/sketch this data before it can be stored, processed, or transmitted. The term sketch refers to a compression approach where the compression end is inexpensive (usually simple linear projections) [1], [2], [3], [4]. The data is often distributed on different devices and hence a federated sketching solution is often desirable. On the other extreme, in projection imaging settings, such as magnetic resonance imaging (MRI), computed tomography (CT), or Fourier ptychography, data acquisition is very slow. Exploiting the structural properties of the underlying image, or of a set of similar images, has proved quite fruitful in both accelerating the image acquisition (scanning) for projection imaging applications and in designing efficient sketching approaches. Since the first papers on compressed sensing (CS) in the 2000s [5], [6], sparsity and structured sparsity assumptions have been heavily exploited for both type of problems, e.g., [7], [2], [1]. For settings where joint reconstruction of a set of similar images is needed, a low rank (LR) assumption on the matrix formed by arranging the vectorized images as its columns is a more flexible model than sparsity since it does not require knowledge of the sparsifying basis or dictionary. This has been extensively used in practical algorithms for accelerating dynamic MRI [8], [9], [10], [11], [12], as well as for federated (distributed data) sketching [13], [3], [14], [4]. However, the underlying problem, which we henceforth refer to as LR column-wise Compressive Sensing (LRcCS), has received little attention so far in terms of approaches with provable guarantees. We are only aware of [4] which studies a convex relaxation based solution. Its magnitude-only generalization, LR phase retrieval (LRPR), is studied in our recent work [15], [16], [17]. This studies an alternating minimization (AltMin) solution to it.

A. Problem Setting and Notation

The goal is to recover an \( n \times q \) rank-\( r \) matrix \( \mathbf{X}^* = [x_1^*, x_2^*, \ldots, x_q^*] \) from \( m \) linear projections (sketches) of each of its \( q \) columns, i.e. from

\[
y_k := \mathbf{A}_k \mathbf{x}_k^*, \quad k \in [q]
\]

where each \( y_k \) is an \( m \)-length vector, \( [q] := \{1, 2, \ldots, q\} \), and the measurement/sketching matrices \( \mathbf{A}_k \) are mutually independent and known. The setting of interest is low-rank (LR), \( r \ll \min(n, q) \), and undersampled measurements, \( m < n \). Our guarantees assumes that each \( \mathbf{A}_k \) is random-Gaussian: each entry of it is independent and identically distributed (i.i.d.) standard Gaussian.
We also study the magnitude-only measurements’ setting, LR phase retrieval (LRPR), which involves recovering $X^*$ from

$$y_{(mag)}_k := |A_k x_k^*|, \ k \in [q].$$

Here $|z|$ takes the entry-wise absolute value of entries of the vector $z$. As we will explain below, LRcCS and LRPR are significantly different from the three other well-investigated LR recovery problems – LR matrix sensing (LRMS) [18], LR matrix completion (LRMC) [19], [18], and multivariate regression (MVR) [20].

Everywhere, $\|\cdot\|_F$ denotes the Frobenius norm, $\|\cdot\|$ without a subscript denotes the (induced) $l_2$ norm (often called the operator norm or spectral norm), $\|M\|_{\text{max}}$ is the maximum magnitude entry of the matrix $M$, $^\top$ denotes matrix or vector transpose, and $|z|$ for a vector $z$ denotes element-wise absolute values. $I_n$ (or sometimes just $I$) denotes the $n \times n$ identity matrix. We use $e_k$ to denote the $k$-th canonical basis vector, i.e., the $k$-th column of $I$. For a vector $w$, we sometimes use $w(k)$ to denote the $k$-th entry of $w$.

We say $U$ is a basis matrix if it contains orthonormal columns. For basis matrices $U_1, U_2$, we use

$$\text{SD}(U_1, U_2) := \|(I - U_1U_1^\top)U_2\|_F$$

as the Subspace Distance (SD) measure. For two $r$-dimensional subspaces, this is the $l_2$ norm of the sines of the $r$ principal angles between $\text{span}(U_1)$ and $\text{span}(U_2)$. SD($U_1, U_2$) is symmetric when $U_1, U_2$ are both $n \times \sigma$ basis matrices. Notice here we are using the Frobenius SD, unlike many recent works including our older work [16] that use the induced 2-norm one. This is done because it enables us to prove the desired guarantees easily. We reuse the letters $c, C$ is done because it enables us to prove the desired guarantees easily. We reuse the letters $c, C$ is done because it enables us to prove the desired guarantees easily. We reuse the letters $c, C$.

We let $X^* \overset{\text{SVD}}{=} U^* \Sigma^* V^* := U^* B^*$

descend its reduced (rank $r$) SVD, i.e., $U^*$ and $V^*\top$ are matrices with orthonormal columns (basis matrices), $U^*$ is $n \times r$ and $V^*$ is $r \times q$, and $\Sigma^*$ is an $r \times r$ diagonal matrix with non-negative entries. We use $\kappa := \sigma_{\text{max}}^*/\sigma_{\text{min}}^*$ to denote the condition number of $\Sigma^*$. This is not the condition number of $X^*$ (whose minimum singular value is zero). Observe the non-standard way of writing the SVD (use of $V^*$ instead of $V^*\top$ for right singular vectors). We do this because it makes it simpler to explain our recovery algorithms. We let $B^* := \Sigma^* V^*$.

We use the phrase $\epsilon$-accurate recovery to refer to $\text{SD}(U, U^*) \leq \epsilon$ or $\|X - X^*\|_F \leq \epsilon \|X^*\|_F$ or both.

Another way to understand $(1)$ is as follows: each scalar measurement $y_{ki}$ ($i$-th entry of $y_k$) satisfies

$$y_{ki} := \langle a_{ki}, x_k^* \rangle, \ i \in [m], \ k \in [q]$$

with $a_{ki}^\top$ being the $i$-th row of $A_k$. Observe that the measurements are not global, i.e., no $y_{ki}$ is a function of the entire matrix $X^*$. They are global for each column ($y_{ki}$ is a function of column $x_k^*$) but not across the different columns. We thus need an assumption that enables correct interpolation across the different columns. The following incoherence (w.r.t. the canonical basis) assumption on the right singular vectors suffices for this purpose [16].

**Assumption 1.1** (Right singular vectors’ incoherence). Assume that $\max_k \|v_k^*\| \leq \mu \sqrt{r/q}$ for a constant $\mu \geq 1 \ (\mu$ does not grow with $n, q, r$). This implies that

$$\max_k \|x_k^*\| = \max_k \|b_k^*\| \leq \sigma_{\text{max}}^* \mu \sqrt{r/q},$$

This further implies that $\max_k \|x_k^*\| \leq \kappa \mu \|X^*\|_F / \sqrt{q}$.

Such an assumption on both left and right singular vectors was first introduced for guaranteeing correct “interpolation” across rows and columns for LRMC [19]. In case of LRMC, measurements are both row-wise and column-wise local and hence incoherence of left and right singular vectors is required.

**B. Existing Work and Our Contributions**

Existing solutions for LRcCS, LRPR. The discussion below treats $\kappa, \mu$ as numerical constants. There is only one existing 2019 work that provably solves LRcCS [4]. The work of [4] introduced and studied a convex relaxation (mixed norm minimization) for the noisy version of LRcCS. In the noise-free case, this guaranteed $\epsilon$-accurate recovery if $mq \in \Omega((n + q)r/\epsilon^2)$ and if an
TABLE I: Existing work versus our work. All approaches also need $m \geq \max(\log q, \log n)$.

| Sample Comp. | Time Comp. |
|--------------|------------|
| Convex [4]   | $mq \geq (n+q)r^{1/2}$ | $mqnr^{1/2}$ |
| AltMin [16]  | $mq \geq (n+q)r^4 \log(1/\epsilon)$ | $mqnr \log^2(1/\epsilon)$ |
| AltMin [17]  | $mq \geq (n+q)r^2 (r + \log(1/\epsilon))$ | $mqnr \log^2(1/\epsilon)$ |
| altGDmin (proposed) | $mq \geq (n+q)r^2 \log(1/\epsilon)$ | $mqnr \log(1/\epsilon)$ |

assumption similar to (but stronger than) Assumption 1.1 holds. Its time complexity is not discussed in the paper, however, it is well known that solvers for convex programs are slow compared to direct iterative algorithms: they either require number of iterations proportional to $1/\sqrt{\kappa}$ or the per-iteration cost has cubic dependence on the problem size (here $((n+q)r)^3$) [18].

LRPR was studied in our recent work [15], [16] where we introduced an Alternating Minimization (AltMin) solution and showed that it converges geometrically to the true $X^*$ as long as Assumption 1.1 holds and $mq \in \Omega((n+q)r^4 \log(1/\epsilon))$. In [17], we improved the AltMin sample complexity guarantee to $mq \in \Omega((n+q)r^2 (r + \log(1/\epsilon)))$. AltMin is faster than the convex solution with a time complexity of $O(mqr \log^2(1/\epsilon))$. Since LRPR is a generalization of LRcCS, a solution for LRPR also solves LRcCS. However, as we show in this paper, since LRcCS is an easier problem, the sample complexity guarantee can be improved to just $mq \in \Omega((n+q)r^2 \log(1/\epsilon))$. Also, the squared dependence of the time complexity on $\log(1/\epsilon)$ makes AltMin slower than the GD-based solution we develop in this work. This discussion is summarized in Table I. While the speed difference is just one log factor, in practice it is often significant, e.g., see Fig. 1.

Other well-studied LR recovery (LRR) problems. The multivariate regression (MVR) problem, studied in [20], is our problem with $A_k = A$. This is a very different setting than ours because the different $y_k$'s are no longer mutually independent. As a result, one cannot exploit law of large numbers’ arguments over all $mq$ scalar measurements $y_{ki}$. Consequently, the required value of $m$ can never be less than $n$. The result of [20] shows that $m$ of order $(n+q)r$ is both necessary and sufficient.

LR matrix sensing (LRMS) involves recovering $X^*$ from $y_i = (A_i, X^*)$, $i = 1, 2, \ldots, m$ with $A_i$ being dense matrices, typically i.i.d. Gaussian [18]. Thus all measurements are i.i.d. and global: each contains information about the entire quantity-of-interest, here $X^*$. Because of this, for LRMS, one can prove a LR Restricted Isometry Property (RIP) that simplifies the rest of the analysis. This is what makes it very different from, and easier than, our problem.

LR matrix completion (LRMC), which involves recovering $X^*$ from a subset of its observed entries, is the most closely related problem to ours since it also involves recovery from non-global measurements. The typical model assumed is that each matrix entry is observed with probability $p$ independent of others [19], [18]. Setting unobserved entries to zero, this can be written as $y_{jk} = \delta_{jk} X^*_{jk}$ with $\delta_{jk} \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$. LRMC measurements are both row-wise and column-wise local. To allow correct interpolation across both rows and columns, it needs the incoherence assumption on both its left and right singular vectors. For our problem (LRcCS), the measurements are global for each column, but not across the different columns. For this reason, only right singular vectors’ incoherence is needed. In fact, because of the nature of our measurements, even if left incoherence were assumed, it would not help. This asymmetry in our measurement model and the fact that our measurements are unbounded are two key differences between LRMC and LRcCS that prevent us from borrowing LRMC proof techniques for our work. Here symmetric means: if we replace $X^*$ by its transpose, the probability distribution of the set of measurements does not change. Bounded means the measurements’ magnitude has a uniform bound. The bound is $\|X^*\|_\text{max}$ for LRMC measurements. A related point is, even though the LRMC measurements $y_{jk}$ are not identically distributed, $y_{jk}/X^*_{jk}$ are i.i.d. and in fact, are i.i.d. Bernoulli. This fact, along with a good bound on $\|X^*\|_\text{max}$ (obtained using the incoherence assumptions), is used in works that study projected-GD for LRMC [21], [22]. On the other hand, for LRcCS, $y_{ki}$ are identically distributed for the same $k$, but not for different $k$. There is no way to define a function of $y_{ki}$ and $X^*$ that is i.i.d. for all $i \in [m], k \in [q]$.

The fastest algorithm for LRMC that also has the best sample complexity guarantee is projected GD on $X$ (projGD-X) studied in [21]. This is shown to converge geometrically and with a constant GD step size, while needing only $\Omega((n+q)r^2 \log^2 n \log^2(1/\epsilon))$ samples on average. Similar sample complexity bounds were proved in [23], [24] for alternating GD (altGD) approaches. But these need a GD step size of order $1/r$ or smaller and hence their convergence is $r$-times slower than that of projGD-X.

Contributions. We treat $\kappa, \mu$ as numerical constants in the discussion below. (1) This work provides a fast and provably correct GD-based solution to LRcCS, called altGDmin. We show that, with high probability (w.h.p.), altGDmin obtains an $\epsilon$-accurate estimate in order $\log(1/\epsilon)$ iterations (geometric convergence), as long as Assumption 1.1 holds, the matrices $A_k$ are i.i.d., with
Algorithm 1 The altGDmin algorithm. Let $M^† := (M^TM)^{-1}M^T$.

1: **Input:** $y_k, A_k, k \in [q]$
2: **Parameters:** Multiplier in specifying $\alpha$ for init step, $\tilde{C}$; GD step size, $\eta$; Number of iterations, $T$
3: **Sample-split:** Partition the measurements and measurement matrices into $2T + 1$ equal-sized disjoint sets: one set for initialization and $2T$ sets for the iterations. Denote these by $y_k^{(\tau)}, A_k^{(\tau)}, \tau = 0, 1, \ldots 2T$.
4: **Initialization:**
5: Set $\alpha = \tilde{C} \frac{1}{mq} \sum_{k} |y_{ki}|^2$ and $X_0 := \frac{1}{m} \sum_{k} a_{ki} y_{ki} e_k^T 1\{y_{ki}^2 \leq \alpha\}$ with $y_{ki} \equiv y_{ki}^{(0)}, a_{ki} \equiv a_{ki}^{(0)}$.
6: Set $U_0 \leftarrow$ top-$r$-singular-vectors of $X_0$
7: **GDmin Iterations:**
8: for $t = 1$ to $T$
9:   Let $U \leftarrow U_{t-1}$
10:   **Update $b_k, x_k$:** For each $k \in [q]$, set $(b_k)_t \leftarrow (A_k^{(t)}U)^T y_k^{(t)}$ and set $(x_k)_t \leftarrow U(b_k)_t$
11:   **Gradient w.r.t. $U$:** Compute $\nabla_U f(U, B_t) = \sum_{k} (y_{ki} - a_{ki}^T (x_k)_t) a_{ki} (b_k)_t^T$ with $y_{ki} \equiv y_{ki}^{(T+t)}, a_{ki} \equiv a_{ki}^{(T+t)}$
12:   **GD step w.r.t. $U$:** Set $U^+ \leftarrow U - (\eta/m) \nabla_U f(U, B_t)$.
13:   **Orthonormalize:** Compute $U^+ \overset{QR}{=_{t}} U^+ R^+$. Set $U_t \leftarrow U^+$.
15: end for

each containing i.i.d. standard Gaussian entries, $mq \in \Omega((n + q) r^2 \log(1/\epsilon))$, and $m \in \Omega(\max(\log q, \log n) \log(1/\epsilon))$. Its time complexity is $O(mqr \log(1/\epsilon))$. To our best knowledge, this is the fastest existing solution to LRcCS and, for $\epsilon < 1/\sqrt{T}$, it also has the best sample complexity. See Table I. Ignoring log factors, our sample complexity guarantee matches that of the best results for LRMC solutions that are not convex relaxation based [21], [23], [24] (described above). The need for exploiting incoherence while obtaining the high probability bounds on the recovery error terms is what introduces the extra factor of $r$ for both LRMC and LRcCS. LRMC has been extensively studied for over a decade and there does not seem to be a way to obtain an order-optimal sample complexity guarantee for it except when studying convex relaxation solutions (which are much slower). Here and below optimal (or order-optimal) means sample complexity is of order log factors times the number of degrees of freedom; for an LR problem this number is $(n + q)r$. Moreover, the altGDmin solution can also be federated efficiently.

(2) We show that a simple extension of altGDmin also provably solves LRPR, which is the phaseless generalization of LRcCS. We prove that altGDmin-LRPR has the same sample complexity as that for the altMin solution [17]: both need $mq \in \Omega(nr^2 (r + \log(1/\epsilon)))$ along with $m \in \Omega(\max(r, \log q, \log n) \log(1/\epsilon))$. But its time complexity is $O(mqr \log(1/\epsilon))$. This makes it faster than the altMin solution. As already argued in [17], for solutions to the two related problems – sparse PR (phaseless but global measurements) and LRMC (linear but non-global measurements) – that have been extensively studied for nearly a decade, the best sample complexity guarantees are still sub-optimal. The best sparse PR guarantee [25], [26] requires $m$ to be of order $s^2$ for the initialization step. Here $s$ is the sparsity level. LRPR has both phaseless and non-global measurements. This is why its initialization step needs two extra factors of $r$ compared to the optimal. Once initialized close enough to the true solution, it is well known that a PR problem behaves like a linear one. This is true for altGDmin-LRPR too.

C. Organization

The rest of this paper is organized as follows. In Sec. II, we develop the altGDmin algorithm and its guarantee for solving LRcCS. A detailed discussion is provided in Sec. III. This section explains the technical details of the algorithm, the need for a different type of GD approach than what is used for LRMC (and other LR recovery (LRR) problems), and summarizes the novelty of both our algorithm design and our proof techniques. The extension of altGDmin for solving LRPR is developed in Sec. IV. This section also provides our guarantee for altGDmin-LRPR. We prove the results in Sec. V and Sec. VI: the latter proves the lemmas given in the former. Simulation experiments are provided in Sec. VII.

II. THE PROPOSED ALTGDMIN ALGORITHM AND GUARANTEE

A. The altGDmin algorithm

We would like to design a fast GD algorithm to find the matrix $X$ that minimizes the squared-loss cost function $f(X) := \sum_{k=1}^{n} \| y_k - A_k x_k \|^2$ over $X$, subject to the constraint that its rank is $r$ or less. There are two commonly used GD approaches
in the LR recovery (LRR) literature, and in particular for LRMC. The first is to use projGD-X (also referred to as Iterative Hard Thresholding): at each iteration, perform one step of GD for minimizing \( f(X) \) w.r.t. \( X \), followed by projecting the resulting matrix onto the space of rank \( r \) matrices (by SVD) [21]. The second is to let \( X = UB \) where \( U \) is \( n \times r \) and \( B \) is \( r \times q \) and perform alternating GD (altGD): update \( B \) with one step of GD for minimizing \( f(UB) \) while keeping \( U \) fixed at its previous value, and then do the same for \( U \) with \( B \) fixed, and repeat. AltGD for just \( f(UB) \) can result in the norm of one of \( U \) or \( B \) growing in an unbounded fashion. This issue is resolved either by including an orthonormalization (by QR decomposition) step after each GD step or by developing altGD for minimizing \( f(UB) + \lambda f_2(U, B) \) where \( f_2(U, B) := \|U\Sigma U^\top - BB^\top\|_F \) is a term that helps ensure that norms of \( U \) and \( B \) remain similar [23], [24]. As explained in detail in Sec. III-B, it is not clear how to analyze either of the above approaches for solving LRCsCS. The reason is, in both cases, the estimates of columns of \( X^* \) are coupled (the estimate \( x_k \) depends on all columns of \( X^* \) and not just on \( x_k^* \)). Consequently it is not possible to get a tight bound on \( \max_k \|x_k^* - x_k\| \). However this is needed to get a tight enough concentration bound on \( \|\nabla_X f(X)\| \) or on \( \|\nabla_U f(UB)\| \). Moreover, even for LRMC, for which both the altGD and projGD-X approaches do work, altGD needs a step size that is \( 1/r \) or smaller [23], [24], making its convergence \( r \)-times slower than geometric.

The following simple modification of altGD, that we dub \( altGDmin \), resolves the above issues. Let

\[
    f(U, B) := \sum_k \|y_k - A_k UB_k\|^2.
\]

It proceeds as follows. At each new iteration,

- use projected GD for updating \( U \): perform one GD step for \( U \) followed by projecting the result onto the space of orthonormal matrices (by QR decomposition);
- for each new estimate of \( U \), solve for \( B \) by minimizing over it while keeping \( U \) fixed at its current value. This becomes a column-wise LS step: \( b_k = (A_k U)^\top y_k \) for each \( k \); here \( M^\top := (M^\top M)^{-1} M^\top \).

Because of the asymmetric nature of our measurement model, the minimization step is decoupled for the different columns of \( B \) as shown above: it only involves solving \( q \times r \)-dimensional Least Squares (LS) problems, in addition to also first computing the matrices, \( A_k U \). Computing the matrices needs time of order \( mnr \), and solving one LS problem needs time of order \( rmr^2 \). Thus, the LS step needs time \( O(q \max(mn, mr^2)) = O(mqr) \) since \( r \leq n \). This is equal to the time needed to compute the gradient w.r.t. \( U \); and thus, the per-iteration cost of altGDmin is only \( O(mqr) \).

For \( m < n \), our cost function is not strongly convex (does not have a unique minimizer) and thus the algorithm needs a careful initialization. We initialize as follows. Compute \( U_0 \) as the top \( r \) left singular vectors of

\[
    X_0 := \frac{1}{m} \sum_{k=1}^q \sum_{i=1}^m a_{ki} y_{ki} e_k^\top 1 \{ y_{ki}^2 \leq \alpha \}
\]

where \( \alpha = \tilde{C}_\eta \sum_{i=1}^m (y_{ki})^2 \). Observe that we are summing \( a_{ki} y_{ki} e_k^\top \) only over those values \( i, k \) for which \( y_{ki}^2 \) is not too large (is not much larger than its empirically computed average value). This truncation approach filters out the too large (outlier-like) measurements and sums over the rest. Theoretically, this converts the summands into sub-Gaussian r.v.s which have lighter tails than the un-truncated ones which are sub-exponential. We summarize the complete algorithm in Algorithm 1. This uses sample-splitting which is a commonly used approach in LR recovery literature; we explain it in Sec. III-A below.

We should mention here that the above algorithm can also be federated easily. If a subset of \( q/L \) columns are measured at each of \( L \) distributed nodes, we can develop a federated implementation that has a communication cost of just order \( nr \) per node per iteration; and that guarantees privacy: it globally recovers only the column span \( \text{span}(U^*) \). The coefficients \( b_k^\star \) are recovered only locally.

1) Practical algorithm and setting algorithm parameters: First, when we implement the algorithm, we use Algorithm 1 with using the full set of measurements for all the steps (no sample-splitting). The algorithm has 3 parameters: \( \eta \), \( \tau \), and \( \tilde{C} \). We can set \( \eta \) as specified in the theorem below as \( \eta = c/\sigma^2_{\max} \) with \( c < 1 \). To set the total number of algorithm iterations \( T \), we can use a large maximum value along with breaking the loop if a stopping criterion is satisfied. A common stopping criterion for GD is to stop when the iterates do not change much. One way to do this is to stop when \( \text{SD}(U_t, U_{t-1}) \leq 0.01 \sqrt{r} \) for last few iterations. Consider \( \tilde{C} \). The theorem requires setting \( \tilde{C} = 9\kappa^2 \mu^2 \), however \( \kappa, \mu \) are functions of \( X^* \), and hence unknown. Recall that \( \mu = \sqrt{\max_k \|y_{ki}^2\|^2 / q} \). We can estimate it as \( \mu = \sqrt{\max_k \|X_{0,full,k}\| / q} \). There is no good way to estimate \( \kappa \). We can try to estimate it as \( \kappa = \max(X_{0,full}) / \min(X_{0,full}) \) where \( X_{0,full} = (1/m) \sum_{k=1}^1 a_{ki} y_{ki} e_k^\top \). Clearly \( \mathbb{E}[X_{0,full}] = X^* \). However, as explained later, we do not have a way to prove a concentration bound for \( X_{0,full} - X^* \) and hence, it is not clear...
if $X_{0,\text{full}}$ is a good estimate of $X^*$. Alternatively, a lower bound on $\kappa$ is given by $\max_k \|x_k^\perp\|/\min_k \|x_k^\perp\|^2$. We can use $(1/m) \sum_i y_{ki}^2$ as an estimate of $\|x_k^*\|^2$ to get estimate the lower bound on $\kappa$. This can provide a starting point for setting $C$.

B. Main Result

**Theorem 2.1.** Consider Algorithm 1. Set $\tilde{C} = 9\kappa^2\mu^2, \eta = c/\sigma_{\max}^2$ with a $c < 0.8$, and $T = Cn^2\log(1/\epsilon)$. Assume that Assumption 1.1 holds and that the $A_k$s are i.i.d. and each contains i.i.d. standard Gaussian entries. If $mq$ satisfies

$$mq \geq Cn^6\mu^2(n + q)r^2$$

for the initialization step, and

$$mq \geq C^4\mu^2(n + q)r^2\log \kappa$$

for each GD iteration, then, with probability (w.p.) at least $1 - n^{-10},$

$$\text{SD}(U_T, U^*) \leq \epsilon, \max_k \|x_k^T - x_k^*\| \leq \epsilon, \text{ and } \|X^T - X^*\|_F \leq 1.4\epsilon\|X^*\|$$

Thus, the total sample complexity is $m_{\text{tot}}q \geq Cn^6\mu^2(n + q)r^2\log(1/\epsilon)\log \kappa$ along with $m_{\text{tot}} \geq C\max(\log q, \log n)\log(1/\epsilon)$. The time complexity is $O(n^2mqnr\log(1/\epsilon)).$

Observe that the above result shows that after $T = Cn^2\log(1/\epsilon)$ iterations, $\text{SD}(U_T, U^*) \leq \epsilon$ and $\|X^T - X^*\|_F \leq 3\epsilon\|X^*\|$. The RHS in the second bound does indeed contain $\|X^*\|$ (the induced 2-norm). This is correct because, the subspace distance being used is a Frobenius norm distance. We show below (see Lemma 5.2) that, at any iteration, $\|X - X^*\|_F \leq 1.4\|I - UU^\perp\|X^*\|_F = 1.4\|I - UU^\perp\|U^*B^\perp\|_F \leq 1.4\|I - UU^\perp\|U^*\|B^\perp\| = 1.4\text{SD}(U, U^*) \|B^\perp\| = 1.4\text{SD}(U, U^*) \|X^*\|.$

The degrees of freedom (number of unknowns) of a rank-$r$ $n \times q$ matrix are $(n + q)r$. A sample complexity of $\Omega((n + q)r)$ samples (or, sometimes this times log factors) is called “optimal”. Thus, ignoring the log factors, our sample complexity of $m_{\text{tot}}q \geq (n + q)r^2$ is sub-optimal only by a factor of $r$. As noted earlier, when developing non-convex solutions (solutions that do not solve a convex relaxation) to problems involving recovery from non-global measurements such as LRMC, the best achievable sample complexity has this extra factor of $r$. This is a consequence of needing to exploit the incoherence assumption while applying concentration bounds on the various error terms.

In addition, we also need $m_{\text{tot}} \geq \max(r, \log q, \log n)$. This is usually redundant except when $q > nr$. It is needed to guarantee correct recovery of all the $q$ columns of $B^*$ w.p. at least $1 - n^{-10}$. Recall that the recovery of each column is a decoupled $r$-dimensional LS problem.

Our time complexity of $mqnr\log(1/\epsilon)$ is only $\log(1/\epsilon)$ times more than the time needed to compute one gradient of the cost function w.r.t. $X$ or $U$. A summary comparison of time and sample complexities with existing work is provided in Table I.

C. Time complexity derivation

This discussion treats $\kappa, \mu$ as constants. The initialization step needs time $mqn$ for computing $X_0$; and time of order $nqr$ times the number of iterations used in the $r$-SVD step. Since we only need a $\delta_0$-accurate initial estimate of $\text{span}(U^*)$, with $\delta_0$ being a constant $c < 1$, a constant number of iterations suffice for this SVD step. Thus the complexity is $O(mq(m + r)) = O(mqn)$ since $m \geq r$. One gradient computation needs time $O(mqnr)$. The QR decomposition needs time of order $nr^2$. The update of columns of $B$ by LS also needs time $O(mqnr)$ (explained earlier). As we prove above, we need to repeat these steps $T = O(\log(1/\epsilon))$ times. Thus the total time complexity is $O(mqn + \max(mqnr, nr^2, mqnr) \cdot T) = O(mqnr\log(1/\epsilon))$.

III. DISCUSSION

This section consists of three subsections. In Sec. III-A below, we explain why we use sample-splitting in our algorithm. We explain why LRMC-style GD approaches cannot be used to provably solve our problem in Sec. III-B. A summary of the novelty of our algorithm and proof techniques and is provided in Sec. III-C.
TABLE II: Why the LRMC initialization approach cannot be directly borrowed?

|                      | LRMC                                                                 | Our Problem, LReCS                          |
|----------------------|----------------------------------------------------------------------|-----------------------------------------------------------------------------------------|
|                      | \( \mathbf{X}_{0,full} = \sum_{k} \sum_{j} \frac{\delta_{jk}}{p} \mathbf{y}_{kj} \mathbf{e}_{j}^{\top} \mathbf{e}_{k}^{\top} \) | \( \frac{1}{m} \frac{1}{m} \sum_{k} \sum_{i} \mathbf{a}_{ki} \mathbf{y}_{ki} \mathbf{e}_{k}^{\top} \) |
|                      | \( \mathbf{H}_{0} = \mathbf{X}_{0,full} - \mathbf{X}^{*} \sum_{k=1}^{m} \sum_{j=1}^{n} (1 - \frac{\delta_{jk}}{p}) \mathbf{X}_{kj} \mathbf{e}_{j}^{\top} \mathbf{e}_{k}^{\top} \) | \( \frac{1}{m} \sum_{i} \sum_{k=1}^{m} (I - \mathbf{a}_{ki} \mathbf{a}_{ki}^{\top}) \mathbf{x}_{k}^{\top} \mathbf{e}_{k}^{\top} \) |

Each summand is nicely bounded by unbounded & sub-expo. norm
Concen. ineq. Matrix Bernstein [32] Sub-expo Bernstein [30]

**:** "max sub-expo. norm": max sub-exponential norm of \((\mathbf{a}_{ki}^{\top} \mathbf{w})(\mathbf{a}_{ki}^{\top} \mathbf{x}_{k}^{\top})(\mathbf{e}_{k}^{\top} \mathbf{z})\) for any unit vectors \(\mathbf{w}, \mathbf{z}\).

A. Sample splitting

**Sample-splitting.** In Algorithm 1, sample-splitting (line 3) helps ensure that the measurements and measurement matrices in each iteration for updating each of \(\mathbf{U}\) and \(\mathbf{B}\) are independent of all previous iterates: we split our sample set into \(2T+1\) subsets, we use one subset for initialization of \(\mathbf{U}\) and one subset each for \(T\) iterations of updating \(\mathbf{B}\) and updating \(\mathbf{U}\). This helps prove the desired error decay bound. Since we prove convergence in order \(\log(1/\epsilon)\) iterations, this only adds a multiplicative factor of \(\log(1/\epsilon)\) in the sample complexity. Sample-splitting is a standard approach used in many older works; in fact it is assumed for most of the LRMC guarantees for solutions that do not solve a convex relaxation (are iterative algorithms) [18], [22], [21].

There are two commonly used approaches in the literature to avoid sample splitting. One is using the leave-one-out strategy as done in [27]. But this means that the sample complexity dependence on \(r\) worsens: the LRMC sample complexity with this approach is \((n + q)r^3\) times log factors. The second is to use the "epsilon-net" idea which has been successfully used for analyzing standard PR algorithms, e.g., see [28], [29] but does not always work for other problems. Developing the same idea does not seem possible for our setting because of the decoupled measurements of different columns; see footnote for details.  

**Sample-splitting to simplify initialization proof.** In order to show that \(\mathbf{U}_0\) is a good enough initialization, we need to show that \(\text{SD}(\mathbf{U}_0, \mathbf{U}^{*}) \leq \delta_0\) for a constant \(\delta_0 < 1\) that is small enough. This is typically done using a \(\sin \Theta\) theorem, e.g., Davis-Kahan or Wedin [31], which uses a bound on the error between \(\mathbf{X}_0\) and a matrix whose span of top \(r\) singular vectors equals that of \(\mathbf{U}^{*}\). Such a matrix may be \(\mathbf{E}[\mathbf{X}_0]\) or something else that can be shown to be close to \(\mathbf{X}_0\). For our approach, it is not easy to compute \(\mathbb{E}[\mathbf{X}_0]\) because the threshold, \(\alpha\), used in the indicator function depends on all the \(\mathbf{y}_{ki}^{2}\). There are two ways to resolve this. A novel, but somewhat complicated, argument can be developed, see Appendix A. Or, we can use sample-splitting here too: use a different independent set of measurements to compute \(\alpha\) than those used for the rest of \(\mathbf{X}_0\). With this, the proof becomes simple; see Sec. V-B. Since this is a one-time step, it does not change the sample complexity order.

B. Understanding why altGDmin works for our problem but LRMC-style approaches do not?

**Initialization.** The simplest initialization approach would be to borrow the approach of LRMC (and also other existing LRR) literature. This computes \(\mathbf{U}_0\) as the top \(r\) left singular vectors of \(\mathbf{X}_{0,full}\) defined in Table II for both LRMC and LReCS. It is not hard to see that, in both cases, \(\mathbb{E}[\mathbf{X}_{0,full}] = \mathbf{X}^{*}\). To show that this approach works (i.e. that \(\mathbf{U}_0\) is a good estimate of the true column span, \(\text{span}(\mathbf{U}^{*})\)), one typically uses Davis-Kahan or Wedin’s \(\sin \Theta\) theorems to bound \(\text{SD}(\mathbf{U}^{*}, \mathbf{U}_0)\) as functions of terms that depend on \(\mathbf{H}_0 := \mathbf{X}_{0,full} - \mathbf{X}^{*}\). Thus a first requirement is bounding \(\|\mathbf{H}_0\|\). Expressions for \(\mathbf{H}_0\) are also given in the table. For LRMC, this can be done easily since \(\mathbf{H}_0\) is a sum of independent one-sparse random matrices shown in the table with each containing an i.i.d. Bernoulli r.v. times \(\mathbf{X}_{kj}^{*}\) as its nonzero entry. Using the left and right singular vectors’ incoherence (assumed in all LRMC guarantees), one can argue that, for unit vectors \(\mathbf{w}, \mathbf{z}\), each summand of \(\mathbf{w}^{\top} \mathbf{H}_0 \mathbf{z}\) is of order at most

\[1\] At iteration \(t\), suppose that the estimate \(\mathbf{U}\) satisfies \(\text{SD}(\mathbf{U}, \mathbf{U}^{*}) \leq \delta_t\). Consider the next iterate \(\mathbf{U}_{t+1} := q(\mathbf{U}, \{\mathbf{A}_{k}, \mathbf{y}_k, k \in [q]\})\), this is a function of \(\mathbf{U}\) and the current measurement matrices and measurements. We need to try to show that, for all \(\mathbf{U} \in \mathcal{T} := \{\mathbf{U} \in \mathbb{R}^{n \times r} : \mathbf{U}^{\top} \mathbf{U} = \mathbf{I}, \text{SD}(\mathbf{U}, \mathbf{U}^{*}) \leq \delta_t\}, \text{SD}(\mathbf{U}_{t+1}, \mathbf{U}^{*}) \leq \delta_t\) for a \(c < 1\) showing this for all \(\mathbf{U} \in \mathcal{T}\) requires covering \(\mathcal{T}\) by a net containing a finite number of points, and first proving that this bound holds for all \(\mathbf{U}\) in the net. The size of the smallest net to cover \(\mathcal{T}\) with accuracy \(c < 1\) will be \(C^\text{nr} \leq 30\). The first step for proving such a bound is to bound the error in the estimates \(b_{kj}(U) := (\mathbf{A}_{kj})^{\top} \mathbf{y}_k\) for all \(\mathbf{U}\) in the net. Consider a fixed \(\mathbf{U}\) in the net. Since this step is decoupled for the different columns of \(\mathbf{B}\), we use only \(m\) measurements for updating each column, followed by a union bound over all \(q\) columns. We can show that the error bound on \(b_{kj}(U)\) holds w.p. \(\geq 1 - \exp(-q + r - cm)\) for one fixed \(\mathbf{U}\). This is done in Lemma 5.2. If we want this bound to hold for all \(\mathbf{U}\)’s in the net covering \(\mathcal{T}\), we will need another union bound over all points in the net, which has at most \(C^\text{nr}\) points. Since union bound is used, one can only show that this holds w.p. \(\geq 1 - \exp(nr + q + r - cm)\). For this probability to be non-negative, we will then need \(m > C^nr\) which makes the result useless.
\( \sigma_{\text{max}}^* r / \sqrt{nq} \). This bound, along with a bound on the “variance parameter” needed for applying matrix Bernstein helps show that \( \| H_0 \| \leq c \sigma_{\text{max}}^* \) w.h.p., under the desired sample complexity bound. The proof uses the matrix Bernstein inequality \([32],[30, \text{Chap 5}]\), followed by an epsilon-net argument. For LReCS, the summands of \( X_{0, \text{full}} \) are sub-exponential \([30]\). These can be bounded using the sub-exponential Bernstein inequality \([30, \text{Chap 2}]\). This requires a bound on the maximum sub-exponential norm of any summand. Denote this by \( K_c \). In order to show that \( \| H_0 \| \leq c \sigma_{\text{max}}^* \) w.h.p, under the desired sample complexity, we need \( K_c \) to be of order \( (r/q) \) or smaller. However, we can only guarantee \( K_c \leq \max_k \| x_k^* \| \leq \mu_\sqrt{r/q} \sigma_{\text{max}}^* \). This is not small enough: it will require \( mq \gtrsim (n+q)r \cdot \sqrt{q} \) which is too large.

To resolve this issue, we borrow the truncation idea from \([28],[29],[16]\). We compute \( U_0 \) as the top \( r \) left singular vectors of \( X_0 \) defined in (2). This truncation converts the summands into sub-Gaussian r.v.s for which we can use the sub-Gaussian Hoeffding inequality \([30, \text{Chap 2}]\). This needs a small enough bound on only the squared sum of the sub-Gaussian norms of the \( mjq \) summands. This is an easier requirement that gets satisfied for our problem. Of course the truncation also means that \( \mathbb{E}[X_0] \) is no longer equal to \( X^* \). We actually only need it to be such that its span of top \( r \) left singular vectors equals that of \( U^* \). To show this, we can either use sample-splitting for computing the truncation threshold followed by a simple proof, or a more interesting (but somewhat complicated) argument can be developed to analyze it without sample-splitting. This latter argument replaces the sandwiching arguments from earlier works that used truncation for symmetric matrices \([28],[29],[16]\).

Briefly, we define a matrix \( X^+ \) whose expected value has the desired property and which is close to \( X_0 \). We bound \( \| X_0 - \mathbb{E}[X_+] \| \) by bounding \( \| X_0 - X_+ \| \) and \( \| X_+ - \mathbb{E}[X_+] \| \). The approach for bounding \( \| X_0 - X_+ \| \) given in the proof of Lemma A.1 in Appendix A is different from that used in related works in LRR or PR literature. Similar ideas are used to also bound the other terms in the SD bound obtained by applying Wedin’s \( \sin \theta \) result.

**Gradient Descent.** To prove geometric convergence of a GD algorithm, we need to show that, w.h.p., the norm of the gradient decays exponentially at each iteration, under the desired sample complexity lower bound.

Consider projected GD on \( X \) which was studied in \([21]\) for LRMC. Its iterations involve computing \( X^+ := \mathcal{P}_r(X - \nabla f(X)) \). Here \( \mathcal{P}_r(M) \) projects its argument onto the space of rank-\( r \) matrices. We show the squared loss cost function and the gradient of it w.r.t. \( X \) for both LRMC and LReCS in Table III. We need to bound \( \| w^\top \nabla_X f(X) z \| \) for any unit norm vectors \( w, z \). For LReCS, this is a sum of sub-exponential r.v.s with subexponential norms bounded by \( \max_k \| x_k^* \| \cdot \| x_k^* - x_k \| \cdot \| z_k \| \leq \max_k \max \| x_k^* - x_k \| \). Thus, in order to show that, under the desired sample complexity bound, \( \| w^\top \nabla_X f(X) z \| \leq m \delta \sigma_{\text{max}}^* \) w.h.p. with \( \delta \) decaying exponentially, we need a small enough bound on \( \| x_k^* - x_k \| \) (column-wise error bound). The projection step introduces coupling between the different columns of \( X \) (the estimate \( x_k \) depends on all columns of \( X^* \) and not just on \( x_k^* \)), which makes it difficult to get a tight enough column-wise bound. We explain this point further in Appendix B. Moreover, even if we could somehow get such a bound, in the best case, it would be proportional to \( \max_\| x_k^* \| \delta \| x_k \| \) with \( \delta \) decaying exponentially. Using incoherence, this is bounded by \( \delta \sqrt{r/q} \sigma_{\text{max}}^* \) which is not small enough. We need a bound proportional to \( \delta \sqrt{r/q} \) to get the desired bound under the desired sample complexity. Consider altGD studied in \([24],[23]\) for LRMC. In this case again, for LReCS, the desired column-wise error bound cannot be obtained because the update step for \( B \) involves GD w.r.t. \( f(U, B) + f_2(U, B) \). The gradient w.r.t \( f_2 \) introduces coupling between the different columns of \( B \), and hence, also between columns of \( X = UB \). Thus, once again, it is not clear how to get a tight bound on \( \| x_k^* - x_k \| \).

On the other hand, for altGDmin, we can get the required bounds under the desired sample complexity. At iteration \( t \), suppose that \( \text{SD}(U_t, U^*) \leq \delta_t \). Let \( U = U_t \). The LS step to recover \( b_k^* \) guarantees that (see Lemma 5.2) \( \| b_k - U^\top U^* b_k^* \| \leq 0.4 \| (I - UU^\top) U^* b_k^* \| \) and consequently \( \| x_k - x_k^* \| \leq 1.4 \| (I - UU^\top) U^* b_k^* \| \). Using the error bound on \( U \), this implies that \( \max_k \| x_k - x_k^* \| \leq 1.4 \delta_t \max_k \| x_k^* \| \). This, along with Assumption 1.1, implies that (i) \( \| x_k - x_k^* \| \leq 1.5 \delta_t \sigma_{\text{max}}^* \sqrt{r/q} \); (ii) \( b_k \) is also incoherent, i.e., that \( \| b_k \| \leq 1.1 \mu \sigma_{\text{max}}^* \sqrt{r/q} \); and that (iii) \( \sigma_{\text{max}}(B) \leq 1.1 \sigma_{\text{max}}^* \). As a result, it is possible to show that, for unit vectors \( w, z \), the maximum sub-exponential norm of any summand of \( w^\top \nabla U f(U, B) z = \sum_k w_\top \delta t \mu \sigma_{\text{max}}^* \| x_k^* - x_k \| b_k^\top z \) is at most \( 1.1 \mu^2 \sigma_{\text{max}}^* \delta_t \sqrt{r/q} \). This is small enough. Thus, along with bounding \( \| \nabla f \| \) and using the sub-exponential Bernstein inequality, helps guarantee that \( \| \nabla U f \| \leq 3 m \delta_t \sigma_{\text{max}}^* \) w.h.p. as long as \( mq \geq C(n+q)^2 \).

To bound \( \| \nabla U f \| \), along with Lemma 5.9 and bounds on the two other terms in that lemma, helps guarantee that \( \text{SD}(U_{t+1}, U^*) \leq 0.7 \delta_t \) w.h.p., i.e., that the error decays exponentially, while only needing \( mq \geq C(n+q)^2 \).

Observe that the use of the sub-exponential Bernstein inequality is possible because the summands are mutually independent. This holds because the \( a_k \)’s are independent of the previous iterates \( U, B \) which is true because of sample-splitting.
When we studied the altMin approach for LRPR [16], [17], we could directly modify proof techniques from altMin for LRMC. This allows us to prove geometric convergence w.h.p. with the desired sample complexity.

The reason is it results in the error term containing a sum of $mq$ sub-exponential r.v.s that are not nice enough: the upper bound on their largest sub-exponential norm is not small enough. We address this issue by borrowing the truncation idea from the PR literature [28], [29], [16]. But, in our case, truncation is applied to a non-symmetric matrix. Thus the sandwiching arguments developed for symmetric matrices in [28], and modified in [29], [16], cannot be borrowed. We need a different argument which is given in the proof of Lemma A.1 in Appendix A.

(2) For designing the GD algorithm, the two standard GD approaches for LRMC (and other LR recovery work) cannot be used for LRcCS either. The reason is again that the terms that need to be bounded are sums of sub-exponential r.v.s that are not nice enough. On the other hand, an altMin approach, analogous to that of [16], [17] for LRPR, can be easily developed and analyzed (the resulting terms will be sums of nice sub-exponentials) and we can show that it converges geometrically. However, it is slow by an extra factor of $\log(1/\epsilon)$. The min step for updating $U$ is the bottleneck, its per iteration complexity is order $mqnr \log(1/\epsilon)$. Our proposed approach, altGDmin, removes both the above limitations: its per iteration cost is the same as that of GD (does not have the log factor); and the terms that need to be bounded are sums of nice enough sub-exponential r.v.s. This allows us to prove geometric convergence w.h.p. with the desired sample complexity.

In order to get the terms in the subspace error bound to be as desired above, we use the Frobenius norm subspace distance, $SD(U, U^*)$, and bound it using the fundamental theorem of calculus [33]. This is done in Lemma 5.9. The use of this result is motivated by its use in [27] where it was used in a standard way: to bound the Euclidean distance for standard GD to solve the PR problem for a single vector $x$; thus, at the true solution $x = x^*$, the gradient of the cost function was zero. In our work, we use it to bound $\| (I - U^*U^*)\vec{U}\|_F$ for altGDmin, followed by using $\vec{U}^+ = Q\vec{U}^+$ to get a bound on $SD(U^+, U^*)$. Recall that $\vec{U}^+ = \vec{U} - (\eta/m)\nabla_U f(U, B)$. We apply the result to $\nabla_U f(U, B) - \nabla_U f(U^*U^+U, B)$. This is a nonstandard application of the result for two reasons: (i) $f(U, B)$ is a function of two variables and thus $\nabla_U f(U^*U^+U, B) \neq 0$; and (ii) we want to bound the subspace distances. Because of this, we have to deal with extra terms than those in the standard application. Since we bound $SD(U^+, U^*)$ using $SD(U^+, U^*) \leq \| (I - U^*U^+)\vec{U}\|_F/\sigma_{\min}(\vec{R})$, the bound consists of a denominator term of the form $1 - (\eta/m)\|\nabla_U f(U, B)\|$. We also have an extra numerator term, $\|\text{Tern2}\|_F$ with $\text{Tern2} = (I - U^*U^+)\nabla_U f(U^*U^+U, B)$. This term nonzero but its expected value is zero. Neither of these terms appear in a standard application of this result, e.g., see [27, Lemma 2]. We needed the above different approach because existing literature on LRMC and other LR problems studies different GD approaches, which we cannot modify (explained in Sec. III-B).

When we studied the altMin approach for LRPR [16], [17], we could directly modify proof techniques from altMin for LRMC [18] for this first step to get a bound that contained sums of nice sub-exponentials.

### IV. Extension to Low Rank Phase Retrieval (LRPR)

In Low Rank Phase Retrieval (LRPR), recall that, we measure $y_{(mag)} = |A_kx_k|$ where $|.|$ denotes element-wise magnitudes. This problem commonly occurs in dynamic phaseless imaging applications such as Fourier psychography. Because of the magnitude-only measurements, we can recover each column only up to a global phase uncertainty. We use $\text{dist}(x^*, x) := \min_{\theta \in [-\pi, \pi]} \| x^* - e^{-j\theta} x \|$ to quantify this phase invariant distance. Also, for a complex number, $z$, we use $\bar{z}$ to denote its conjugate and $\text{phase}(z) := z/|z|$ [34], [29].

| TABLE III: Understanding why LRMC style projected-GD on $X$ does not work in our case. |
|----------------|----------------|
| $f(X)$ | $\sum_{k=1}^{q} \sum_{j=1}^{n} (y_{jk} - \delta_{jk}X_{jk})^2$ |
| $\delta_{jk} \sim \text{Bernoulli}(p)$ | $a_k \sim N(0, I_n)$ |
| $\nabla_X f(X)$ | $\sum_{k=1}^{q} \sum_{j=1}^{n} \delta_{jk}(y_{jk} - \delta_{jk}X_{jk})e_j e_k^\top$ |
| $\nabla_X f(X)$ | $\sum_{k=1}^{q} \sum_{j=1}^{m} (y_{kj} - a_k e_k e_k^\top)$ |
| $\nabla_X f(X)$ | $\sum_{k=1}^{q} \sum_{j=1}^{m} a_k (x_k^* - x_k) a_k e_k e_k^\top$ |
Algorithm 2 The altGDmin-LRPR algorithm.

1: **Input:** $y_{(\text{mag})k}, A_k, k \in [q]$
2: **Parameters:** GD step size, $\eta$; Number of iterations, $T$
3: **Sample-split:** Partition the measurements and measurement matrices into $2T + 1$ equal-sized disjoint sets: one set for initialization and $2T$ sets for the iterations. Denote these by $y_{(\text{mag})k}^{(\tau)}, A_k^{(\tau)}, \tau = 0, 1, \ldots, 2T$.
4: **Initialization:**
5: Compute $U_0$ as the top $r$ singular vectors of $Y_U := \frac{1}{mq} \sum_k (y_{(\text{mag})k} a_k^t a_k^\top \{ y_{(\text{mag})k} a_k \} \leq C_{\text{max}} \frac{1}{mq} \sum_k (y_{(\text{mag})k})^2 \}$ with $y_{(\text{mag})k} \equiv y_{(\text{mag})k}^{(0)}, a_k \equiv a_k^{(0)}$.
6: **GDmin Iterations:**
7: for $t = 1$ to $T$
8: \hspace{1em} Let $U \leftarrow U_{t-1}$.
9: \hspace{1em} Update $b_k, x_k$: For each $k \in [q]$, set $(b_k)_t \leftarrow \text{RWF}(y_{(\text{mag})k}^{(t)}, (U^\top A_k^{(t)}))$. Set $(x_k)_t \leftarrow U(b_k)_t$
10: \hspace{1em} Estimate gradient w.r.t. $U$: With $y_{(\text{mag})k} \equiv y_{(\text{mag})k}^{(t)}, a_k \equiv a_k^{(t+t)}$,
\hspace{1.5em} compute $\hat{y}_{kt} := y_{(\text{mag})k} c_{kt}$ with $c_{kt} = \text{phase}(a_k^t x_k)$ and
\hspace{1.5em} compute $\text{GradU} = \sum_k (\hat{y}_{kt} - a_k^t x_k)_t a_k b_k)_t^\top$.
11: \hspace{1em} Set $U^+ \leftarrow U - (\eta/m) \text{GradU}$
12: **Orthornormalize to get new $U$:** Compute $U^+ \leftarrow U^- R^\top$. Set $U_t \leftarrow U^+$.
13: end for

With three simple changes that we explain next, the altGDmin approach also solves LRPR and provides the fastest existing solution for it. First, observe that because of the magnitude-only measurements, we cannot use $X_0$ with $y_{kt}$ replaced by $y_{(\text{mag})k}$ for initialization. The reason is $\mathbb{E}[a_k y_{(\text{mag})k}] = 0$ and, with a little bit of work, one can also show that $\mathbb{E}[a_k y_{(\text{mag})k} y_{(\text{mag})k}] \leq \sqrt{\sigma}$ = 0 too. In fact, because of this, it is not even possible to define a matrix $X$ whose expected value can be shown to be close to $X^*$. Instead, we have to use the initialization approach of [16]; see line 5 of Algorithm 2. This matrix $Y_U$ is such that its expected value is close to $X^* X^\top + CT$ so that the matrix of top $r$ singular vectors is still close to span($U^*$).

Next, consider the GDmin iterations. We use the following idea to deal with the magnitude-only measurements: $y_{(\text{mag})k} := \|y_{kt}\|$. Let $c_{kt} := \text{phase}(a_k^t x_k)$. Then, clearly,

$$y_{kt} = c_{kt} y_{(\text{mag})k}$$

and $y_{(\text{mag})k} = \bar{c}_{kt} y_{kt}$. We do not observe $c_{kt}$, but we can estimate it using $y_k$ which is an estimate of $x_k^*$. Using the estimated phase, we can get an estimate $y_k$ of $y_{kt}$.

We replace the gradient of our earlier cost function, $f(U, B)$, w.r.t $U$, by its estimate which uses $y_{kt} = y_{(\text{mag})k} \bar{c}_{kt}$, with $c_{kt} = \text{phase}(a_k^t x_k)$, to replace $y_{kt}$. See line 10 of Algorithm 2.

Lastly, because of the magnitude-only measurements, the update step for updating $b_k$s is no longer an LS problem. We now need to solve an $r$-dimensional standard PR problem: $\min_b \|y_{(\text{mag})k} - |A_k b|\|^2$. This can be solved using any of the order-optimal algorithms for standard PR, e.g., Truncated Wirtinger Flow (TWF) or Reshaped WF (RWF) [28], [29]. For concreteness, we assume that RWF is used. The entire algorithm, altGDmin-LRPR, is summarized in Algorithm 2.

We can prove the following result with simple changes to the proof of Theorem 2.1.

**Theorem 4.1.** Consider Algorithm 2. Set $\eta = c/\sigma_{\max}^2$, $\bar{C} = 9 \sigma^2 \mu^2$, and $T = C \kappa^2 \log(1/\epsilon)$. Assume that Assumption 1.1 holds. If

$$mq \geq C \kappa^6 \mu^2(n + q)r^2(r + \log(1/\epsilon) \log \kappa)$$

and $m \geq C \max(\log q, \log n) \log(1/\epsilon)$, then, w.p. $1 - n^{-10}$, $\text{SD}(U^*, U_T) \leq \epsilon$, and $\sum_k \text{dist}^2((x_k)_T, x_k^*) \leq \epsilon^2 \sigma_{\max}^2$.

We prove this result in Sec. V-D. Notice the $\log(1/\epsilon)$ in the sample complexity of Theorem 2.1 is now replaced by $(r + \log(1/\epsilon))$. The reason is because of the different initialization approach which needs $n r^3$ samples instead of $n r^2$. This is needed because PR is a more difficult problem: we cannot define a matrix $X_0$ for it for which $\mathbb{E}[X_0]$ is close to $X^*$. This trend is also seen for sparse PR which is very well studied; it needs $s$-times more samples than its linear counterpart, sparse CS (what is just known is CS or sparse recovery), see [25], [26]. Here $s$ is the sparsity size.
V. PROOFS OF THEOREM 2.1 AND THEOREM 4.1

We provide the three main results that help prove Theorem 2.1 and its proof next. We prove the first result (initialization theorem) in Sec. V-B, while proving the third result (GDmin error descent theorem) in Sec V-C. The second result given below is a lemma that bounds the error in updating the columns of $B$ and its consequences. This has a simple proof that is provided in Sec. VI-F. We prove Theorem 4.1 in Sec. V-D. All the lemmas used for the proofs in this section are proved in Sec. VI.

A. Proving Theorem 2.1

The SD($U_T, U^*$) bound of Theorem 2.1 is an immediate consequence of the following three results. The theorem below analyzes the initialization step.

**Theorem 5.1** (Initialization). Pick a $\delta_0 < 0.1$. If $mq \geq C\kappa^2\mu^2(n + q)r^2/\delta_0^2$, then w.p. at least $1 - \exp(-c(n + q))$, SD($U^*, U_0$) $\leq \delta_0$.

Given an estimate $U$ satisfying SD($U, U^*$) $\leq \delta_i < 0.1$, the next lemma proves that the error in estimating $g_k := U^T x_k^*$ is bounded by a constant times $\| (I - UU^T) x_k^* \|$. This in turn implies that $b_k$ is also incoherent, that $\max_k \| x_k - x_k^* \|/\| x_k^* \| \leq 1.4\delta_i$ and $\| x^* - X \|_F \leq 1.5\delta_i\sigma_{\max}$, and that the singular values of $B$ lie between $0.9\sigma_{\min}^*$ and $1.1\sigma_{\max}^*$.

**Lemma 5.2.** Let $g_k := U^T x_k^*$. Assume that SD($U^*, U_t$) $\leq \delta_t$ with $\delta_t < \delta_0 = c/\kappa^2$ (this bound on $\delta_t$ is needed for the second part of this lemma). Then, w.p. $\geq 1 - \exp(q + r - cm)$,

1) $\| g_k - b_k \| \leq 0.4\| (I_n - UU^T) U^* b_k^* \|

2) This in turn implies all of the following.

   a) $\| g_k - b_k \| \leq 0.4\delta_t\| b_k^* \|

   b) $\| x_k - x_k^* \| \leq 1.4\| (I_n - UU^T) U^* b_k^* \| \leq 1.4\delta_t\| x_k^* \|

   c) $\| U^T U b_k - b_k^* \| \leq 2.4\delta_t\| b_k^* \|

   d) $\| G - B \|_F \leq 0.4\delta_t\| b_k^* \|

   e) $\| x^* - X \|_F \leq \sqrt{1.16}\delta_t\| x_k^* \|

   f) $\| b_k \| \leq 1.1\mu\sigma_{\max}\sqrt{r/q}.

   g) $\sigma_{\min}(B) \geq 0.9\sigma_{\min}^*$ and $\sigma_{\max}(B) \leq 1.1\sigma_{\max}^*.

The next theorem analyzes the GD steps and proves exponential error decay. It uses the above lemma in its proof.

**Theorem 5.3** (GD Descent). If, at each iteration, $mq \geq C\kappa^4\mu^2(n + q)r^2 \log \kappa$ and $m > C \max(\log q, \log n)$; if SD($U^*, U_0$) $\leq \delta_0 = c/\kappa^2$ for a $c$ small enough; and if $\eta = 0.5/\sigma_{\max}^*$, then w.p. at least $1 - n^{-10}$, SD($U^*, U_{t+1}$) $\leq (1 - \frac{c}{\kappa^2})^{t+1} \delta_0$.

**Proof of Theorem 2.1.** The SD(·) bound is an immediate consequence of Theorems 5.1 and 5.3. To apply Theorem 5.3, we need $\delta_0 = c/\kappa^2$. Using this value for $\delta_0$, by Theorem 5.1, if $mq \geq C\kappa^4\mu^2(n + q)r^2$, then SD($U^*, U_0$) $\leq \delta_0 = c/\kappa^2$. With this, if $\eta = 0.5/\sigma_{\max}^*$ and if, at each iteration, $mq \geq C\kappa^4\mu^2(n + q)r^2 \log \kappa$ and $m \geq C \max(\log q, \log n)$, then by Theorem 5.3, the stated bound on SD($U^*, U_{t+1}$) holds. By setting $T = C\kappa^2\log(1/\epsilon)$ in this bound, we can guarantee $(1 - \frac{c}{\kappa^2})^T \leq \epsilon$. This proves the SD($U_T, U^*$) bound. The bounds on $\| x_k - x_k^* \|/\| x_k^* \|$ and $\| X - X^* \|_F$ follow by Lemma 5.2.

1) **Proofs of above results:** We prove the above two theorems in the next two subsections. Lemma 5.2 has a simple proof that is proved later in Sec. VI-F. The first part uses standard concentration bounding ideas for random Gaussian matrices. The second part relies on triangle inequality and other simple linear algebra tracks. In particular the following idea is used for the Frobenius norm error bounds on the matrix estimates: $\| X - X^* \|_F \leq \sqrt{\sum_k 1.4\| (I - UU^T) U^* b_k^* \|^2} = 1.4\| (I - UU^T) U^* b_k^* \|_F \leq 1.4\delta_t\| B^* \| = 1.4 \cdot 1.1\delta_t\| x_k^* \|$. Observe that the RHS contains $\sigma_{\max}^*$ and not $\| X^* \|_F$. This type of a bound is possible because we use the Frobenius norm subspace distance for quantifying subspace recovery error.

B. Proving Initialization Theorem 5.1: simpler proof that assumes independent measurements used for computing $\alpha$

Recall that we initialize $U_0$ as the top $r$ left singular vectors of $X_0$ defined in (2). For the simpler proof given here, assume that we use a different independent set of measurements for computing $\alpha$ than those used for the rest of $X_0$, i.e., let

$$\alpha = \hat{C} \frac{\sum_k (y_{k1}^n x_1^m x_2^m)^2}{mq}.$$
Lemma 5.4. Conditioned on \( \alpha \), we have the following conclusions.

1. Let \( \zeta \) be a scalar standard Gaussian r.v. Define
   \[
   \beta_k(\alpha) := \mathbb{E}[\zeta^2 \mathbb{I}_{\{\|\mathbf{x}_k\|^2 \leq \alpha\}}].
   \]
   Then,
   \[
   \mathbb{E}[\mathbf{X}_0|\alpha] = \mathbf{X}^* \mathbf{D}(\alpha), \text{ where } \mathbf{D}(\alpha) := \text{diagonal}(\beta_k(\alpha), k \in [q])
   \]
   i.e. \( \mathbf{D}(\alpha) \) is a diagonal matrix of size \( q \times q \) with diagonal entries \( \beta_k(\alpha) \) defined above.

2. Let \( \mathbb{E}[\mathbf{X}_0|\alpha] = \mathbf{X}^* \mathbf{D}(\alpha) \overset{\text{SVD}}{=} \mathbf{U}^* \mathbf{\Sigma}^* \mathbf{V} \) be its r-SVD. Then,
   \[
   \text{SD}(\mathbf{U}_0, \mathbf{U}^*) \leq \sqrt{2} \max \left(\|\mathbf{X}^* - \mathbf{E}[\mathbf{X}_0|\alpha]\|_F, \|(\mathbf{X}_0 - \mathbf{E}[\mathbf{X}_0|\alpha])\mathbf{V}^\top\|_F\right) \frac{\sigma_{\min} \min_k \beta_k(\alpha) - \|\mathbf{X}_0 - \mathbf{E}[\mathbf{X}_0|\alpha]\|}{\sigma_{\min}^2}
   \]
   as long as the denominator is non-negative.

Proof. See Sec. VI-A

Define the event
\[
\mathcal{E} := \left\{ \hat{C}(1 - \epsilon_1) \frac{\|\mathbf{X}^*\|_F^2}{q} \leq \alpha \leq \hat{C}(1 + \epsilon_1) \frac{\|\mathbf{X}^*\|_F^2}{q} \right\}.
\]

The lemma below bounds the terms on the RHS of Lemma 5.4.

Lemma 5.5. Fix \( 0 < \epsilon_1 < 1 \). Then,

1. w.p. at least \( 1 - \exp\left(\left(n + q\right) - c\epsilon_1^2 m q / \mu^2 \kappa^2\right) \), conditioned on \( \alpha \), for an \( \alpha \in \mathcal{E} \),
   \[
   \|\mathbf{X}_0 - \mathbb{E}[\mathbf{X}_0|\alpha]\| \leq 1.1\epsilon_1 \|\mathbf{X}^*\|_F
   \]

2. w.p. at least \( 1 - \exp\left(\left(q r - c\epsilon_1^2 m q / \mu^2 \kappa^2\right)\right) \), conditioned on \( \alpha \), for an \( \alpha \in \mathcal{E} \),
   \[
   \|(\mathbf{X}_0 - \mathbb{E}[\mathbf{X}_0|\alpha])\mathbf{V}^\top\|_F \leq 1.1\epsilon_1 \|\mathbf{X}^*\|_F
   \]

3. w.p. at least \( 1 - \exp\left(\left(n r - c\epsilon_1^2 m q / \mu^2 \kappa^2\right)\right) \), conditioned on \( \alpha \), for an \( \alpha \in \mathcal{E} \),
   \[
   \|(\mathbf{X}_0 - \mathbb{E}[\mathbf{X}_0|\alpha])\mathbf{V}^\top\|_F \leq 1.1\epsilon_1 \|\mathbf{X}^*\|_F
   \]

Proof. See Sec. VI-B

We also need to the following two simple facts which are proved in Sec. VI-C.

Fact 5.6. W.p. at least \( 1 - \exp(-\tilde{c} m q \epsilon_1^2) \), \( \mathcal{E} \) holds. Here \( \tilde{c} = c / \hat{C} = c / \kappa^2 \mu^2 \).

Fact 5.7. For any \( \epsilon_1 \leq 0.1 \), \( \min_{\zeta} \mathbb{E}\left[\zeta^2 \mathbb{I}_{\left\{\zeta \leq \hat{C} \frac{\|\mathbf{x}_k\|_F}{\sqrt{\|\mathbf{X}^*\|_F}}\right\}}\right] \geq 0.92\).

Proof of Theorem 5.1. Set \( \epsilon_1 = 0.4\delta_0 / \sqrt{T\kappa} \). Define
\[
p_0 = 2 \exp((n + q) - c m q \delta_0^2 / r \kappa^2) + 2 \exp(n r - c m q \delta_0^2 / r \kappa^2) + 2 \exp(q r - c m q \delta_0^2 / r \kappa^2)
\]
and
\[
p_\alpha = \exp(-\tilde{c} m q \epsilon_1^2) = \exp(-c m q \delta_0^2 / r \mu^2 \kappa^2).
\]
so that \( \text{Pr}(\alpha \in \mathcal{E}) \geq 1 - p_\alpha \).

Using Lemma 5.5, conditioned on \( \alpha \), for an \( \alpha \in \mathcal{E} \),

- w.p. at least \( 1 - p_0 \)
  \[
  \|\mathbf{X}_0 - \mathbb{E}[\mathbf{X}_0|\alpha]\| \leq 1.1\epsilon_1 \|\mathbf{X}^*\|_F = 0.4\delta_0 \sigma_{\min} \text{ and }
  \max \left(\|\mathbf{X}_0 - \mathbb{E}[\mathbf{X}_0|\alpha]\|_F, \|(\mathbf{X}_0 - \mathbb{E}[\mathbf{X}_0|\alpha])\mathbf{V}^\top\|_F\right) \leq 0.4\delta_0 \sigma_{\min}^2
  \]

• For any $\alpha \in \mathcal{E}$,

$$\min_k \beta_k(\alpha) \geq \min_k \mathbb{E} \left[ \zeta^2 1 \right]_{\{||z|| \leq C \sqrt{\frac{q \alpha}{\sqrt{\delta} \kappa_n}}\}} \geq 0.9$$

The first inequality is an immediate consequence of $\alpha \in \mathcal{E}$. The second follows using Fact 5.7.

Plugging the above bounds into (4) of Lemma 5.4, conditioned on $\alpha$, for any $\alpha \in \mathcal{E}$, w.p. at least $1 - p_0$, $\text{SD}(U_0, U^*) \leq 0.44\delta_0 < \delta_0$ since $\delta_0 < 0.1$. In other words,

$$\Pr (\text{SD}(U_0, U^*) \geq \delta_0|\alpha) \leq p_0 \text{ for any } \alpha \in \mathcal{E}. \quad (5)$$

Since (i) $\Pr(\text{SD}(U_0, U^*) \geq \delta_0) \leq \Pr(\text{SD}(U_0, U^*) \geq \delta_0$ and $\alpha \in \mathcal{E}) + \Pr(\alpha \notin \mathcal{E})$, and (ii) $\Pr(\text{SD}(U_0, U^*) \geq \delta_0$ and $\alpha \in \mathcal{E}) \leq \Pr(\alpha \in \mathcal{E}) \max_{\alpha \in \mathcal{E}} \Pr(\text{SD}(U_0, U^*) \leq \delta_0|\alpha)$, thus, using Fact 5.6 and (5), we can conclude that

$$\Pr(\text{SD}(U_0, U^*) \geq \delta_0) \leq p_0(1 - p_0) + p_0 \leq p_0 + p_0$$

Thus, for a $\delta_0 < 0.1$, $\text{SD}(U_0, U^*) < \delta_0$ w.p. at least $1 - p_0 - p_\alpha = 1 - 2 \exp(-n + q - cmq\delta_0^2/r\kappa^2) - 2 \exp(nq - cmq\delta_0^2/r\kappa^2) - \exp(-cmq\delta_0^2/r\mu_k^4)$. This is $\geq 1 - 5 \exp(-c(n + q))$ if $mq > C\kappa^2\mu(n + q)^2/\delta_0^2$. This finishes our proof. \hfill \Box

1) Proof outline for the lemmas: Lemma 5.4 is an almost immediate consequence of Wedin’s sin $\Theta$ theorem for Frobenius norm subspace distance [35], [31][Theorem 2.3.1, second row] summarized next. We apply it conditioned on $\alpha$. We use $M = X_0$ and $M^* = \mathbb{E}[X_0|\alpha]$. Since we show in the first part that $\mathbb{E}[X_0|\alpha] = X^* D(\alpha)$, its top $r$ singular vectors span $\text{span}(U^*)$.

**Theorem 5.8** (Wedin sin $\Theta$ theorem for Frobenius norm subspace distance [35], [31][Theorem 2.3.1].) For two $n_1 \times n_2$ matrices $M^*$, $M$, let $U^*, U$ denote the matrices containing their top $r$ singular vectors and let $V^*^\top, V^\top$ be the matrices of their right singular vectors (recall from problem definition that we defined SVD with the right matrix transposed). Let $\sigma_r^*, \sigma_{r+1}^*$ denote the $r$-th and $(r + 1)$-th singular values of $M^*$. If $\|M - M^*\| \leq \sigma_r^* - \sigma_{r+1}^*$, then

$$\text{SD}(U, U^*) \leq \sqrt{2} \max(\|M - M^*\|^2 \|U^*\|_F, \|M - M^*\|^2 V^* V^\top_\|F\|) / \sigma_r^* - \sigma_{r+1}^* - \|M - M^*\|$$

The proof of Lemma 5.5 uses the following basic idea which we explain by sketching the proof of its first part. Let $S_d$ denote the unit $l_2$-norm hypersphere in $\mathbb{R}^d$. We have that $\|X_0 - \mathbb{E}[X_0]\| = \max_{w \in S_d, z \in S_n} |z^\top (X_0 - \mathbb{E}[X_0]) w|$. We first bound this for a fixed $w, z$ using the sub-Gaussian Hoeffding inequality [30][Chapter 2] with $t = c_1 \|X^*\|_F$ and with sub-Gaussian norm of the $k$-th summand bounded by $K_{k} = C \|X^*\|_F w(k)/\sqrt{q}$. Thus, $\sum_k K_k \leq C m \|X^*\|_F^2 / m^2 q$. Here $w(k)$ is the $k$-th entry of $w$. Next, we extend the bound to all $w, z$ by using an epsilon-net based argument [30][Chapter 4].

The complete proofs are provided in Sec. VI.

C. Proving GD descent Theorem 5.3

This will be proved if we can show the following: Starting with $\text{SD}(U^*, U_t) \leq \delta_t$ with $\delta_t < \delta_0 = c/\kappa^2$, $\text{SD}(U^*, U_{t+1}) \leq (1 - c/\kappa^2) \delta_t = \delta_{t+1}$. Let $U \equiv U_t$, $B \equiv B_t$. The proof follows using the following two lemmas.

**Lemma 5.9.** Let $\otimes$ denote the Kronecker product. We have

$$\text{SD}(U_{t+1}, U^*) \leq \frac{\|I - (\eta/m) \text{Hess} \cdot \text{SD}(U^*, U_t) + (\eta/m) \|\text{Term2}\|_F}{1 - (\eta/m) \|\text{GradU}\|}$$

where

$$\text{GradU} := \nabla_U f(U, B) = \sum_{ki}(y_{ki} - a_{ki}^T U b_k) a_{ki} b_k^\top$$

$$\text{Term2} := (I - U^* U^*^\top) \nabla_U f((U^* U^*^\top), B) = (I - U^* U^*^\top) \sum_{ki}(y_{ki} - a_{ki}^T U^* U^*^\top U b_k) a_{ki} b_k^\top$$

$$\text{Hess} := \sum_{ki} (a_{ki} \otimes b_k)(a_{ki} \otimes b_k)^\top$$

Proof. See Sec. VI-D \hfill \Box
The next lemma bounds the terms from the above lemma.

**Lemma 5.10.** Assume $\text{SD}(U^*, U_i) \leq \delta_t < \delta_0 = c/\kappa^2$. Then,

1) w.p. at least $1 - \exp((n + r) - cmq_1^2/\mu^2) - \exp(\log q + r - cm)$,

$$\|\text{Grad}U\| \leq 1.5(1 + \epsilon_1)m\delta_1\sigma_{\max}^2;$$

2) w.p. at least $1 - \exp(nr - cmq_2^2/\mu^2) - \exp(\log q + r - cm)$,

$$\|\text{Term2}\|_F \leq 1.1m\epsilon_2\delta_0\sigma_{\max}^2;$$

3) w.p. at least $1 - \exp(nr \log \kappa - cmq_3^2/r\kappa^4\mu^2) - \exp(log q + r - cm)$,

$$m(0.65 - 1.2\epsilon_3)\sigma_{\min}^2 \leq \lambda_{\min}(\text{Hess}) \leq \lambda_{\max}(\text{Hess}) \leq m(1 + \epsilon_3)\sigma_{\max}^2$$

**Proof.** See Sec. VI-E.

**Proof of Theorem 5.3.** The upper bound on $\lambda_{\max}(\text{Hess})$ along with setting $\epsilon_3 = 0.01$ and $\eta = 0.5/\sigma_{\max}^2$ implies that

$$\lambda_{\min}(I_{nr} - (\eta/m)\text{Hess}) = 1 - (\eta/m)\lambda_{\max}(\text{Hess}) \geq 1 - \frac{0.1(1 + 0.01)m\sigma_{\max}^2}{m\sigma_{\max}^2} > 1 - 0.555 > 0$$

i.e. that $I_{nr} - (\eta/m)\text{Hess}$ is positive definite. Thus,

$$\|\text{Hess}I_{nr} - (\eta/m)\text{Hess}\| = \lambda_{\max}(I_{nr} - (\eta/m)\text{Hess}) = 1 - (\eta/m)\lambda_{\min}(\text{Hess}) \leq 1 - (\eta/m)m(0.65 - 1.2\epsilon_3)\sigma_{\min}^2 \leq 1 - 0.3/\kappa^2.$$  

Assume that $\text{SD}(U^*, U_i) \leq \delta_t$ with $\delta_t < \delta_0 = c_0/\kappa^2$ with $c_0 = 0.1$. Set $\epsilon_1 = 0.1$, $\epsilon_3 = 0.01$, $\epsilon_2 = 0.01/1.1\kappa^2$ and, $\delta_0 = 0.1/1.5(1 + 0.1)\kappa^2$.

By Lemma 5.9, Lemma 5.10, and the above,

$$\text{SD}(U^*, U_{i+1}) \leq \left(1 - 0.3/\kappa^2\right) \cdot \delta_t + \frac{\eta_1m\epsilon_2\sigma_{\max}^2\delta_t}{1 - (\eta/m)1.5(1 + \epsilon_1)m\delta_t\sigma_{\max}^2} \leq \left(1 - 0.29/\kappa^2\right) \cdot \delta_t \leq \left(1 - 0.26/\kappa^2\right) \cdot \delta_t$$

w.p. at least $1 - \exp((n + q) - cmq/\mu^2) - \exp(nr - cmq/\mu^4\mu^2) - \exp(log q + r - cm)$. The second inequality substituted the values of $\epsilon_1$'s and $\eta$ and used $\delta_t < \delta_0$ for its denominator term. The third inequality used $(1 - 0.01/\kappa^2)^{-1} \leq (1 + 0.02/\kappa^2)$ for $0 < x < 1, 1/(1 - x) \leq 1 + 2x$.

By plugging in the epsilon values in the probability, the above holds w.p. $\geq 1 - \exp((n + q) - cmq/\mu^2) - \exp(nr - cmq/\mu^4\mu^2) - \exp(log q + r - cm)$. If $m \geq C\kappa^4(n + q)^2/\log \kappa$ and $m \geq C\max(r, \log q, \log n)$ for a $C$ large enough, then, this probability is $\geq 1 - \exp(-c(n + q)) - 2\exp(-cnr) - n^{-10} > 1 - 5n^{-10}$.  

1) **Proof outline for the lemmas: Lemma 5.9 needs the following definition.**

**Definition 5.11.** For any $n \times r$ matrix $Z$, let $Z_{\text{vec}}$ denote the $nr$ length vector formed by arranging all $r$ columns of $Z$ one below the other.

Thus,

1. $(a_{ki}^T b_k^T)_{\text{vec}} = a_{ki} \otimes b_k$ with $\otimes$ being the Kronecker product.
2. $a_{ki}^T U b_k = \text{trace}(b_k a_{ki}^T U) = \langle a_{ki}^T b_k, U \rangle = \langle a_{ki} \otimes b_k, U_{\text{vec}} \rangle$ and hence $f(U_{\text{vec}}, B) = \sum_{ki}(a_{ki} \otimes b_k)^T U_{\text{vec}} - y_{ki})^2$
3. $(\nabla_j U_{f(U, B)})_{\text{vec}} = \nabla U_{f(U_{\text{vec}}, B)}$

**Lemma 5.9** bounds $\|[(I - U U^*) U^+]_F\|$ using the fundamental theorem of calculus [33, Chapter XIII, Theorem 4.2],[27] applied to $\nabla U_{f(U_{\text{vec}}, B)}$ and $\nabla U_{f(U^* U^+ U_{\text{vec}}, B)}$, followed by some linear algebra tricks. The bound on $\text{SD}(U^*, U)$ then follows by $\text{SD}(U^*, U) \leq \|[(I - U U^*) U^+]_F\|_F = \|(R^+)\|_F(\|(R^+)\|_F)^{-1} \leq \|\text{Grad}U\| = \sigma_{\min}(R^+) = \sigma_{\min}(U^+) = \sigma_{\min}(U - (\eta/m)\nabla U_{f(U, B)}) \geq 1 - (\eta/m)\|\nabla U_{f(U, B)}\|$.

**Lemma 5.10.** first part uses (i) $\text{sd}(\text{Grad} U) \leq n\|X^* - X\|B^T \leq n\|X^* - X\|B^T \leq n\|X^* - X\|B^T \leq m(1.4\delta_t\sigma_{\max}^* \cdot 1.1\sigma_{\max}^*)$ by Lemma 5.2; (ii) $\text{Grad} U - \text{E}(\text{Grad} U) = \max_{w \in \mathcal{S}_n, z \in S, w^T (\text{Grad} U - \text{E}(\text{Grad} U)) z$; (iii) bounds this for a fixed $w, z$ using sub-exponential Bernstein inequality [30, Chap 2] along with Lemma 5.2 to bound the terms in its probability; (iv) followed by a (mostly standard) epsilon-net argument.
Its second part use $\mathbb{E}[\text{Term}2] = 0$ and $\|\text{Term}2\|_F = \max_{W \in S_{nr}} \text{trace}(W^T \text{Term}2) = \max_{W \in S_{nr}} \sum_{k_i} (a_{ki}^T (U^*(U^*Ub_k - b_k^*)) (a_{ki}^T (I - U^*U^*)Wb_k))$ followed by similar ideas to first part.

Proof of the third part starts with a variational definition of the eigenvalues. It then uses the fact that, for a $u \in S_{nr}$ and its rearranged unit Frobenius norm matrix $W$, $\mathbb{E}[u^T \text{Hess} \, w] = \mathbb{E}[\sum_{k_i} (a_{ki}^T Wb_k)^2] = m \|WB\|_F^2$, along with using Lemma 5.2 to lower and upper bound this. The concentration bounding uses similar ideas to first two parts, except the epsilon-net argument for lower bounding the minimum eigenvalue is a little different from a standard one, and is provided in the proof.

All the proofs are in Sec. VI.

D. Proof of Theorem 4.1

For the initialization, we bound from [16].

Lemma 5.12 ([16]). Let $SD_2(U_0, U^*) = \|(I - U^*U^*)U_0\|$. Pick a $\delta_{init} < 0.1$. Then, w.p. at least $1 - 2\exp\left(n(\log 17) - c\frac{\delta_{init}^2 m q}{\mu^2 r^2}\right) - 2\exp\left(-c\frac{\delta_{init}^2 m q}{\mu^2 r^2}\right)$,

$$SD_2(U_0, U^*) \leq \delta_{init} \text{ and so } SD(U_0, U^*) \leq \sqrt{r}\delta_{init}.$$ 

For the iterations, without loss of generality, as also done in past works on PR, e.g., [34], [28], [29], [17], to make things simpler, we assume that, for each $k$, $x_k^*$ is replaced by $\hat{z} x_k^*$ where $\hat{z} = \text{phase}(x_k^*, x_k)$. With this, $\text{dist}(x_k^*, x_k) = \|x_k^* - x_k\|$

We modify Lemma 5.9 using the following idea. Let $U = U_t$ and $B = B_t$. For LRPR, the GD step uses an approximate gradient w.r.t. the old cost function $f(U, B)$. Let

$$\text{Err} := \text{Grad}U - \text{Grad}U.$$

Here $\text{Grad}U = \sum_{k_i} (\hat{y}_{ki} - a_{ki}^T x_k) a_{ki} b_k^T$ and $\text{Grad}U = \nabla f(U, B) = \sum_{k_i} (y_{ki} - a_{ki}^T x_k) a_{ki} b_k^T$ is the same as earlier. Thus,

$$\text{Err} = \sum_{k_i} (\hat{y}_{ki} - y_{ki}) a_{ki} b_k^T = \sum_{k_i} (\hat{c}_{ki} - c_{ki}) a_{ki}^T x_k^* \ | a_{ki} a_{ki}^T b_k^T = \sum_{k_i} (\hat{c}_{ki} - c_{ki} - 1) a_{ki}^T x_k^* a_{ki} b_k^T$$

Proceeding as in the proof of Lemma 5.9, and using $\|\text{Term}2\|_F \leq \|\text{Err}\|_F$ and $\|\text{Err}\|_F \leq \|\text{Err}\|_F$, we can conclude the following

$$SD(U^*, U^+) \leq \frac{(1 - \eta/m)\text{Hess} \cdot SD(U^*, U) + (\eta/m)\|\text{Term}2\|_F + (\eta/m)\|\text{Err}\|_F}{1 - (\eta/m) \text{Grad}U - (\eta/m)\|\text{Err}\|_F}$$

where the expressions for $\text{Grad}U, \text{Term}2, \text{Hess}$ are the same as before with one change: $b_k$ is now obtained by solving a noisy $r$-dimensional PR problem (instead of a LS problem) using RWF [29]. Thus, to complete the proof, (i) we need to bound

$$\|\text{Err}\|_F = \max_{W \in S_{nr}} \sum_{k_i} (\hat{c}_{ki} - c_{ki} - 1) (a_{ki}^T x_k^*) (a_{ki}^T Wb_k)$$

and (ii) we need bounds on the three other terms that were also bounded earlier for the linear case.

The term $\|\text{Err}\|_F$, is bounded in Lemma 4 of [17]. We repeat the lemma below.

Lemma 5.13. Assume that $SD(U_t, U^*) \leq \delta_t$ with $\delta_t < c/r^2$. Then, w.p. at least $1 - 2\exp\left(nr \log(17) - c\frac{m q \sigma_2^2}{\mu^2 r^2}\right) - \exp(\log q + r - cm)$,

$$\|\text{Err}\|_F \leq Cm (\epsilon_2 + \sqrt{\delta_t}) \delta_t \sigma_{\max}^2$$

Consider the other three terms: $\text{Grad}U, \text{Term}2, \text{Hess}$. These were bounded in Lemma 5.10 for the linear case. The statement and proof of this lemma remain the same as earlier because of the following reason. Its proof uses the bounds on $b_k, x_k$ from Lemma 5.2. The statement of this lemma also remains the same with one change: we replace $\|x^* - x\|$ by $\text{dist}(x^*, x)$ and $\|X^* - X\|^2_F$ by $\sum_{k_i=1} q_i \text{dist}^2(x_k^*, x_k)$, and the same for $b_k^*, \mu$. The first part of Lemma 5.2 now follows by the first bound of Lemma 5 of [17]: this lemma itself is an easy consequence of Theorem 1 of [29]. All the subparts of the second part of Lemma 5.2 follow exactly as given in its proof in Sec. VI-F.

VI. PROOFS OF ALL THE LEMMAS

Everywhere we use $S_{dr}$ to denote both $\{W \in \mathbb{R}^{d \times r} : \|W\|_F = 1\}$ and its vectorized version $\{w \in \mathbb{R}^{dr} : \|w\| = 1\}$
A. Proof of Initialization lemmas/ facts: Proof of Lemma 5.4

To see why (3) holds, it suffices to show that \( \mathbb{E}[\langle X_0 \rangle_k \mid \alpha] = x_k^\top \beta_k(\alpha) \) for each \( k \). The easiest way to see this is to express \( x_k^\top = \|x_k^\top \| Q_k e_1 \) where \( Q_k \) is an \( n \times n \) unitary matrix with first column \( x_k^\top / \|x_k^\top \| \); and to use the fact that \( \tilde{a}_{ki} := Q_k^\top a_{ki} \) has the same distribution as \( a_{ki} \), both are \( N(0, I_n) \). Using \( Q_k Q_k^\top = I \), \( \langle X_0 \rangle_k = (1/m) \sum_i Q_k Q_k^\top a_{ki} \tilde{a}_{ki}(1) \mathbb{1}_{\|a_{ki} \| \leq \sqrt{n} \|x_k^\top \|} \). Thus \( \mathbb{E}[\langle X_0 \rangle_k] = (1/m) m Q_k \|x_k^\top \| \|e_1 \| \mathbb{E}[\|Q_k^\top a_{ki} \| \|\tilde{a}_{ki}(1) \| \leq \sqrt{n} \|x_k^\top \|] \). This follows because \( \mathbb{E}[\tilde{a}(1) \mathbb{1}_{\|a \| < \beta}] = e \mathbb{E}[\tilde{a}(1)^2 \mathbb{1}_{\|a \| < \beta}] \).

Recall that \( \tilde{C} = \sqrt{n^2 \mu^2} \) and \( \bar{c} = c / \tilde{C} \) for a \( c < 1 \). Recall also that \( X^* \overset{\text{SVD}}{=} U^* \Sigma^* V^* \) and \( \mathbb{E}[X_0] = \overset{\text{SVD}}{=} U^* \Sigma^* V^* \). Thus, using (3), \( \Sigma^* = \Sigma^* V^* D \Sigma^* \). Hence,

\[
\sigma_r(\mathbb{E}[X_0]) = \sigma_{\min}(\Sigma^*) = \sigma_{\min}(\Sigma^* V^* D \Sigma^*) \geq \sigma_{\min}(\Sigma^*) \sigma_{\min}(V^*) \sigma_{\min}(D) = \sigma_{\min}(\Sigma^*) \cdot \min(\beta(\alpha)) \cdot 1
\]

Also, \( \sigma_{r+1}(\mathbb{E}[X_0]) = 0 \) since it is a rank \( r \) matrix. Thus, using Wedin’s \( \sin \theta \) theorem for the Frobenius norm subspace distance \( SD \) [35], [31][Theorem 2.3.1, second row] (specified in Theorem 5.8 above) applied with \( M = X_0, M^* = \mathbb{E}[X_0] \) we get (4).

B. Proof of Initialization lemmas and facts: Proof of Lemma 5.5

**Proof of first part of Lemma 5.5.** The proof involves an application of the sub-Gaussian Hoeffding inequality, Theorem 2.6.2 of [30], followed by an epsilon-net argument. The application of sub-Gaussian Hoeffding uses conditioning on \( \alpha \) for \( \alpha \in \mathcal{E} \). For \( \alpha \in \mathcal{E}, \alpha \leq \sqrt{\tilde{C}(1 + \epsilon_1)} \|X^*\|_F / \sqrt{\gamma} \) and this helps get a simple probability bound. Since \( \alpha \) is independent of all \( a_{ki}, y_{ki} \)’s used in defining \( X_0 \), the conditioning does not change anything else in our proof. For example, the different summands are mutually independent even conditioned on it.

We have,

\[
\|X_0 - \mathbb{E}[X_0] \| = \max_{z, w} \langle X_0 - \mathbb{E}[X_0], zw^\top \rangle.
\]

For a fixed \( z \in S_n, w \in S_q \), we have

\[
\langle X_0 - \mathbb{E}[X_0], zw^\top \rangle = \frac{1}{m} \sum_{k_i} \langle w(k) y_{ki}(a_{ki}^\top z) \mathbb{1}_{\|y_{ki}\|^2 \leq \alpha} - \mathbb{E}[w(k) y_{ki}(a_{ki}^\top z) \mathbb{1}_{\|y_{ki}\|^2 \leq \alpha}], zw^\top \rangle.
\]

The summands are mutually independent, zero mean sub-Gaussian r.v.s with sub-Gaussian norm \( K_{ki} \leq C \|w(k)\| \sqrt{n} / m \). For \( \alpha \in \mathcal{E}, \alpha \leq \sqrt{\tilde{C}(1 + \epsilon_1)} \|X^*\|_F / m \sqrt{n} \). Let \( t = \epsilon_1 \|X^*\|_F \). Then, for any \( \alpha \in \mathcal{E},
\]

\[
\frac{t^2}{\sum_k K_{ki}^2} \geq \frac{\epsilon_1^2 \|X^*\|_F^2}{C m \mu^2} \geq \frac{\epsilon_1^2 m q}{C m \mu^2 \kappa^2},
\]

since \( \sum_k k^2 w(k)^2 = \|w\|^2 = 1 \). Thus, for a fixed \( z \in S_n, w \in S_q \), by sub-Gaussian Hoeffding, we conclude that, conditioned on \( \alpha \), for any \( \alpha \in \mathcal{E} \), w.p. at least \( 1 - \exp \left[ \frac{\|X^*\|_F^2}{\epsilon_1^2 m q / (\mu^2 \kappa^2)} \right],
\]

\[
\langle X_0 - \mathbb{E}[X_0], zw^\top \rangle \leq C \epsilon_1 \|X^*\|_F.
\]

The rest of the proof follows by a standard epsilon net argument for any \( z \in S_n, w \in S_q \). The above bound then holds w.p. at least \( 1 - \exp \left[ \frac{(n + q) - \epsilon_1^2 m q / (\mu^2 \kappa^2)}{(n + q) - \epsilon_1^2 m q / (\mu^2 \kappa^2)} \right],
\]

We provide the epsilon net argument here for completeness. For let \( \tilde{S}_n, \tilde{S}_q \) be corresponding epsilon nets so that for any \( z \in S_n, w \in S_q \) there exists \( \tilde{z} \in \tilde{S}_n, \tilde{w} \in \tilde{S}_q \) so that \( \|z - \tilde{z}\|, \|w - \tilde{w}\| \leq \epsilon_0 = 1/4 \) and \( \|z\|, \|w\| \leq (1 + 2 \epsilon_0)^n \). We also let \( \gamma^* = \max_{z \in \mathcal{S}_n, w \in \mathcal{S}_q} \langle X_0 - \mathbb{E}[X_0], zw^\top \rangle \). Now, by applying union bound over the nets, we have that, w.p.

\[
1 - \exp \left[ \frac{(n + q) - \epsilon_1^2 m q / (\mu^2 \kappa^2)}{(n + q) - \epsilon_1^2 m q / (\mu^2 \kappa^2)} \right],
\]

\[
\langle X_0 - \mathbb{E}[X_0], zw^\top \rangle \leq C \epsilon_1 \|X^*\|_F \quad \text{for all } z \in \tilde{S}_n, w \in \tilde{S}_q.
\]

Now, for any \( z \in S_n, w \in S_q \)

\[
\langle X_0 - \mathbb{E}[X_0], zw^\top \rangle \leq \langle X_0 - \mathbb{E}[X_0], \tilde{z} \tilde{w}^\top \rangle + \|z - \tilde{z}\| \|X_0 - \mathbb{E}[X_0], \tilde{z} \tilde{w}^\top \rangle + \|w - \tilde{w}\| \|z - \tilde{z}\| \langle X_0 - \mathbb{E}[X_0], zw^\top \rangle,
\]

\[
\leq \epsilon_1 \|X^*\|_F + 2 \epsilon_0 \gamma^* + \epsilon_0^2 \gamma^*.
\]
Thus, we can conclude that $\gamma^* \leq \frac{\epsilon}{1 - 2\epsilon_0 - \epsilon_0} \|X^*\|_F$. Recall that $\epsilon_0 = 1/4$. This conclude the proof. \hfill \Box

**Proof of second part of Lemma 5.5.** We have

$$\| (X_0 - \mathbb{E}[X_0|\alpha])^T U^* \|_F = \max_{W \in S_{qr}} \langle W, (X_0 - \mathbb{E}[X_0|\alpha])^T U^* \rangle$$

For a fixed $W \in S_{qr}$,

$$\langle W, (X_0 - \mathbb{E}[X_0|\alpha])^T U^* \rangle = \text{trace} (W^T (X_0 - \mathbb{E}[X_0|\alpha])^T U^*) = \frac{1}{m} \sum_{k_i} (y_{ki}(a_{ki}^T U^* w_k) \mathbb{I}|y_{ki}|^2 \leq \alpha) - \mathbb{E}[:])$$

Conditioned on $\alpha$, for an $\alpha \in \mathcal{E}$, the summands are independent zero mean sub-Gaussian r.v.s with subGaussian norm $K_{ki} \leq \alpha \|w_k\| / m \leq \sqrt{C(1 + \epsilon_1)} \|X^*\|_F / w_k / \sqrt{q}$. Thus,

$$\sum_{ki} K_{ki}^2 \leq m\bar{C}(1 + \epsilon_1) \|W\|_F^2 ||X^*||_F^2 / m^2 q = \bar{C} ||X^*||_F^2 / \epsilon q$$

Applying the sub-Gaussian Hoeffding inequality Theorem 2.6.2 of [30], for a fixed $W \in S_{qr}$, conditioned on $\alpha$, for an $\alpha \in \mathcal{E}$, w.p. $1 - \exp[-\epsilon q^2 / \mu^2 \|W\|_F^2]$,

$$\text{trace} (W^T (X_0 - \mathbb{E}[X_0|\alpha])^T U^*) \leq \epsilon_1 \|X^*\|_F$$

The rest of proof follows by a (standard) epsilon net argument over all $W \in S_{qr}$ (union bound on the epsilon-net $\bar{S}_{qr}$ of $S_{qr}$ which is of size at most $Cq^r$, followed by extending the bound from the net to the entire sphere). Because of the union bound, the final result holds w.p. $1 - \exp[-\epsilon q^2 / \mu^2 \|W\|_F^2]$.

Epsilon-net argument: let $\gamma^* = \max_{W \in S_{qr}} \text{trace} (W^T (X_0 - \mathbb{E}[X_0|\alpha])^T U^*)$. We also let $\bar{S}_{qr} \in S_{qr}$ to be an epsilon net over $S_{qr}$ so that for $W \in S_{qr}$ there exists $\bar{W} \in \bar{S}_{qr}$ with $\|W - \bar{W}\|_F \leq \epsilon_{net}$ and that $|\bar{S}_{qr}| \leq (1 + 2/\epsilon_{net})^q$. We set $\epsilon_{net} = 1/8$. By applying union bound we can show that w.p. $1 - \exp[-\epsilon q^2 / \mu^2 \|W\|_F^2]$,

$$\text{trace} (W^T (X_0 - \mathbb{E}[X_0|\alpha])^T U^*) \leq \epsilon_1 \|X^*\|_F$$

Then, for any $W \in S_{qr}$

$$\text{trace} (W^T (X_0 - \mathbb{E}[X_0|\alpha])^T U^*) \leq \text{trace} (W^T (X_0 - \mathbb{E}[X_0|\alpha])^T U^*) + \|W - \bar{W}\|_F \text{trace} (\frac{\|W - \bar{W}\|_F}{\|W - \bar{W}\|_F} (X_0 - \mathbb{E}[X_0|\alpha])^T U^*) \leq \epsilon_1 \|X^*\|_F + \epsilon_{net} \gamma^*.$$

We can conclude from the above inequality that $\gamma^* \leq \epsilon q^2 / \mu^2 \|W\|_F + \epsilon_{net} \gamma^*$ and finally $\gamma^* \leq \epsilon q^2 / \mu^2 \|W\|_F / (1 - \epsilon_{net})$. This concludes the proof. \hfill \Box

**Proof of third part of Lemma 5.5.** We have

$$\| (X_0 - \mathbb{E}[X_0|\alpha]) \tilde{V}^T \|_F = \max_{W \in S_{nr}} \langle (X_0 - \mathbb{E}[X_0|\alpha]) \tilde{V}^T, W \rangle.$$ 

For a fixed $W \in S_{nr}$ we have,

$$\langle (X_0 - \mathbb{E}[X_0|\alpha]) \tilde{V}^T, W \rangle = \frac{1}{m} \sum_{ki} (y_{ki}(a_{ki}^T W \tilde{v}_k) \mathbb{I}|y_{ki}|^2 \leq \alpha) - \mathbb{E}[:])$$

where $\mathbb{E}[:]$ is the expected value of the first term. Conditioned on $\alpha$, for an $\alpha \in \mathcal{E}$, the summands are independent, zero mean, sub-Gaussian r.v.s with subGaussian norm $K_{ki} \leq C \alpha \|W \tilde{v}_k\| \leq C \sqrt{C(1 + \epsilon_1)} \|X^*\|_F \|W \tilde{v}_k\| / \sqrt{q}$. Thus, by applying the sub-Gaussian Hoeffding inequality Theorem 2.6.2 of [30], with $t = \epsilon q \|X^*\|_F$, and using $\|W \tilde{V}\|_F = 1$, we can conclude that, conditioned on $\alpha$, for an $\alpha \in \mathcal{E}$, w.p. $1 - \exp[-\epsilon q^2 / \mu^2 \|W\|_F^2]$,

$$\langle (X_0 - \mathbb{E}[X_0|\alpha]) \tilde{V}^T, W \rangle \leq C \epsilon q \|X^*\|_F.$$

By a standard epsilon net argument, the above bound holds for all $W \in S_{nr}$ w.p. $1 - \exp[-nr \epsilon q^2 / \mu^2 \|W\|_F^2]$. \hfill \Box
We can prove our final result by using the following linear algebra facts:

\[
\gamma_k = \frac{\sqrt{C(1+\epsilon)}}{\sqrt{q||x_k||}}. \quad \text{Since } C = 9\mu^2\kappa^2 \text{ and } ||x_k||^2 \leq \mu^2\kappa^2 ||x||^2 \text{ (Assumption 1.1) thus }
\]

\[
\gamma_k \geq 3.
\]

Now,

\[
E[\zeta^2 \mathbb{1}_{||\zeta|| \leq \gamma_k}] = 1 - E[\zeta^2 \mathbb{1}_{||\zeta|| \geq \gamma_k}] \\
\geq 1 - \frac{2}{\sqrt{2\pi}} \int_{\gamma_k}^\infty z^2 \exp(-z^2/2)dz \\
\geq 1 - \frac{2e^{-1/4}}{\sqrt{\pi}} \int_{\gamma_k}^\infty z \exp(-z^2/4)dz = 1 - \frac{2e^{-11/4}}{\sqrt{\pi}} \geq 0.92.
\]

The first inequality used \(\gamma_k \geq 3\). The second used the fact that \(z \exp(-z^2/4) \leq \sqrt{2e}\) for all \(z \in \mathbb{R}\).

D. Proving GD iterations’ lemmas: Proof of Lemma 5.9 (algebra lemma)

Everywhere below \(\nabla f(U, B)\) is short for \(\nabla f(U, B) = \sum_k a_k b_k^\top (a_k^\top U b_k - y_k)\) and similarly \(\nabla f(U_{vec}, B)\) is short for \(\nabla f(U_{vec}, B) = \sum_k (a_k^\top \otimes b_k)(a_k^\top b_k^\top U_{vec} - y_k)\).

We use the vectorized versions so that we can apply the simple vector version of the fundamental theorem of calculus [33, Chapter XIII, Theorem 4.2],[27, Lemma 2 proof] on the \(nr\) length vector \(\nabla f(U_{vec}, B)\), and so that the Hessian can be expressed as an \(nr \times nr\) matrix.

For any vector \(w\), we use \(w(k)\) to denote its \(k\)-th entry. For a differentiable vector function \(g(z)\), and two vectors \(z_0, z^*\), the fundamental theorem of calculus [33, Chapter XIII, Theorem 4.2] says the following

\[
g(z_0) - g(z^*) = \left(\int_{\tau=0}^1 \nabla g(z(\tau))d\tau\right)(z_0 - z^*), \quad \text{where } z(\tau) = z^* + \tau(z_0 - z^*)
\]

When \(z\) is a \(d\)-length vector and \(g(z)\) is a \(d_2\) length vector function, \(\nabla z g(z)\) is a \(d_2 \times d\) matrix.

We apply the above result with \(z_0 \equiv U_{vec}, z^* \equiv (U^*U^\top U)_{vec}\), and \(g(z) = \nabla f(z, B)\). Thus \(d = d_2 = nr\) and \(\nabla g(z)\) is the Hessian of \(f(z, B)\) computed at \(z\). Let \(U(\tau) := U^*U^\top U + \tau(U - U^*U^\top U)\). Applying the theorem,

\[
\nabla f(U_{vec}, B) - \nabla f((U^*U^\top U)_{vec}, B) = \left(\int_{\tau=0}^1 \nabla^2_{U_{vec}} f(U(\tau)_{vec}, B) d\tau\right)(U_{vec} - (U^*U^\top U)_{vec})
\]

where

\[
\nabla^2_{U_{vec}} f(U(\tau)_{vec}, B) = \sum_k (a_k^\top \otimes b_k)(a_k^\top b_k^\top) := \text{ Hess}
\]

This is an \(nr \times nr\) matrix. Because the cost function is quadratic, the Hessian is constant w.r.t. \(\tau\). Henceforth, we refer to it as Hess. With this, the above simplifies to

\[
\nabla f(U_{vec}, B) - \nabla f((U^*U^\top U)_{vec}, B) = \text{Hess } (U_{vec} - (U^*U^\top U)_{vec}) = \text{Hess } (P_{vec})
\]

with

\[
P := I - U^*U^\top
\]

is an \(n \times n\) matrix that projects orthogonal to \(U^*\).

This proof is motivated by a similar approach used in [27, Lemma 2 proof] to analyze GD for standard PR. However, there the application was much simpler because \(f(.)\) was a function of one variable and at the true solution the gradient was zero, i.e., \(\nabla f(x^*) = 0\). In our case \(\nabla f(U^*U^\top U, B) \neq 0\) because \(B \neq B^*\). But we can show that \(\mathbb{E}[(U - U^*U^\top) \nabla f(U^*U^\top U, B)] = 0\) and this helps us get the final desired result.

From Algorithm 1, recall that \(\hat{U}^+ = U - (\eta/m)\nabla f(U, B)\). Vectorizing this equation, and using (8), we get

\[
(\hat{U}^+)_{vec} = U_{vec} - (\eta/m)\nabla f(U_{vec}, B) \]

\[
= U_{vec} - (\eta/m) \text{ Hess } (P_{vec})_{vec} - (\eta/m)\nabla f((U^*U^\top U)_{vec}, B))
\]

We can prove our final result by using the following linear algebra facts:
1) \( \nabla f(U_{vec}, B) = (\nabla f(U, B))_{vec} \)
2) For an \( n \times n \) matrix \( M \), let \( \text{big}(M) := I_r \otimes M \) be an \( nr \times nr \) block diagonal matrix with \( M \) in the diagonal blocks. For any \( n \times r \) matrix \( Z \),

\[
\text{big}(M)Z_{vec} = (MZ)_{vec}
\]

(10)
3) Notice that \( P \) is idempotent so that \( P = P^2 \). Also, because of its block diagonal structure, \( \text{big}(M^2) = (\text{big}(M))^2 \).

Thus,

\[
\text{big}(P) = \text{big}(P^2) = (\text{big}(P))^2 = \text{big}(P)I_{nr}(\text{big}(P))
\]

(11)
Left multiplying both sides of (9) by \( \text{big}(P) \), and using (10) and (11) and \( \nabla f(U_{vec}, B) = (\nabla f(U, B))_{vec} \),

\[
\text{big}(P)(\hat{U}^+)_{vec} = \text{big}(P)U_{vec} - (\eta/m)\text{big}(P)\text{Hess}(PU)_{vec} - (\eta/m)\text{big}(P)\nabla f((U^*U^\top)_{vec}, B)
\]

\[
= \text{big}(P)I_{nr}\text{big}(P)U_{vec} - (\eta/m)\text{big}(P)\text{Hess}(\text{big}(P)U_{vec} - (\eta/m)\text{big}(P)\nabla f((U^*U^\top)_{vec}, B)
\]

\[
= \text{big}(P)(I_{nr} - (\eta/m)\text{Hess})\text{big}(P)U_{vec} - (\eta/m)\text{big}(P)\nabla f((U^*U^\top)_{vec}, B)
\]

Thus, using \( ||\text{big}(P)|| = ||P|| = 1 \), and (10),

\[
\|(P\hat{U}^+)_{vec}\| \leq ||I_{nr} - (\eta/m)\text{Hess}|| \|(P\nabla f((U^*U^\top)_{vec}, B))_{vec}\|
\]

(12)
Converting the vectors to matrices, vector \( l_2 \) norm to Frobenius norm and substituting for \( P \),

\[
||(I - U^*U^\top)\hat{U}^+||_{F} \leq ||I_{nr} - (\eta/m)\text{Hess}|| ||(I - U^*U^\top)U||_{F} + (\eta/m)||((I - U^*U^\top)\nabla f((U^*U^\top)_{vec}, B)||_{F}
\]

Since \( \hat{U}^+ \leq U^+R^+ \) and since \( ||M_1M_2||_{F} \leq ||M_1||_{F}||M_2|| \), this means that \( \text{SD}(U^*, U^+) \leq ||(I - U^*U^\top)\hat{U}^+||_{F}||(R^+)^{-1}||. \) Since \( ||(R^+)^{-1}|| = 1/\sigma_{\min}(R^+) = 1/\sigma_{\min}(U^+) \), using \( U^+ = U - (\eta/m)\nabla f(U, B) \),

\[
||(R^+)^{-1}|| = \frac{1}{\sigma_{\min}(U - (\eta/m)\nabla f(U, B))} \leq \frac{1}{1 - (\eta/m)||\nabla f(U, B)||}
\]

where we used \( \sigma_{\min}(U - (\eta/m)\nabla f(U, B)) \geq \sigma_{\min}(U) - (\eta/m)||\nabla f(U, B)|| = 1 - (\eta/m)||\nabla f(U, B)|| \) for the last inequality. Thus,

\[
\text{SD}(U^*, U^+) \leq \frac{||I_{nr} - (\eta/m)\text{Hess}|| \text{SD}(U^*, U) + (\eta/m)||((I - U^*U^\top)\nabla f((U^*U^\top)_{vec}, B))_{vec}||_{F}}{1 - (\eta/m)||\nabla f(U, B)||}
\]

E. Proof of GD iterations’ lemma: Proof of Lemma 5.10

1) Upper and Lower bounding the Hessian eigenvalues and hence HessTerm: Recall from (7) that \( \text{Hess} := \nabla^2_{U_{vec}} f(\hat{U}_{vec}; B) = \sum_{ki}(a_{ki} \otimes b_{ki})(a_{ki} \otimes b_{ki})^\top \). Since \( \text{Hess} \) is a positive semi-definite matrix, \( \lambda_{\min}(\text{Hess}) = \min_{w \in S_{nr}} w^\top \text{Hess} w \) and \( \lambda_{\max}(\text{Hess}) = \max_{w \in S_{nr}} w^\top \text{Hess} w \). For a fixed \( w \in S_{nr} \),

\[
w^\top \text{Hess} w = \sum_{ki} (a_{ki}^\top W b_{ki})^2
\]

where \( W \) is an \( n \times r \) matrix with \( ||W||_{F} = 1 \). Clearly \( (a_{ki}^\top W b_{ki})^2 \) are mutually independent sub-exponential random variables (r.v.) with sub-exponential norm \( K_{ki} \leq ||W b_{ki}||^2 \). Also, \( \mathbb{E}[(a_{ki}^\top W b_{ki})^2] = ||W b_{ki}||^2 \) and thus \( \mathbb{E}[(\sum_{ki} (a_{ki}^\top W b_{ki})^2)] = m||W B||^2 \). Applying the sub-exponential Bernstein inequality, Theorem 2.8.1 of [30], for a fixed \( W \in S_{nr} \) yields

\[
\Pr\left\{ \left| \sum_{ki} (a_{ki}^\top W b_{ki})^2 - m||W B||^2 \right| \geq t \right\} \leq \exp\left[ -c_{\min} \left( \frac{t^2}{\sum_{ki} K_{ki}^2}, \frac{t}{\max_{ki} K_{ki}} \right) \right].
\]

We set \( t = c_{\min} \sigma_{\max}^2 \). By Lemma 5.2, \( ||b_{ki}||^2 \leq 1.1\mu^2 \sigma_{\max}^2 (r/q) = 1.1\mu^2 \sigma_{\max}^2 (r/q) \). Thus,

\[
\frac{t^2}{\sum_{ki} K_{ki}^2} \geq \frac{c_{\min} \sigma_{\max}^4 \mu^2}{\max_{ki} ||W b_{ki}||^4} \geq \frac{c_{\min} \sigma_{\max}^4 \mu^2}{\max_{ki} ||W b_{ki}||^4} \sum_{ki} ||W b_{ki}||^2 \geq \frac{c_{\min} \sigma_{\max}^4 \mu^2}{\max_{ki} ||W b_{ki}||^4} \frac{1}{1.1 \sigma_{\max}^2} = \frac{c_{\min} \sigma_{\max}^4 \mu^2}{1.1 \sigma_{\max}^2} \frac{1}{1.1 \sigma_{\max}^2} = \frac{c_{\min} \sigma_{\max}^4 \mu^2}{1.1 \sigma_{\max}^2} \frac{1}{1.1 \sigma_{\max}^2}
\]

Here we used \( \sum_{ki} ||W b_{ki}||^2 = ||W B||_{F}^2 \leq ||W||_{F} ||B||_{2} \leq 1.1 \sigma_{\max} \) using the bound on \( ||B||_{2} \) from Lemma 5.2. Also,

\[
\frac{t}{\max_{ki} K_{ki}} \geq \frac{c_{\min} \sigma_{\max}^2 \mu^2}{\max_{ki} ||W b_{ki}||^2} \geq \frac{c_{\min} \sigma_{\max}^2 \mu^2}{1.1 \sigma_{\max}^2} = \frac{c_{\min} \sigma_{\max}^2 \mu^2}{1.1 \sigma_{\max}^2}.
\]
Therefore, for a fixed \( W \in S_{nr} \), w.p. \( 1 - \exp \left[ -\epsilon_3^2 m q / r \mu^2 \kappa^4 \right] - \exp(\log q + r - cm) \) we have

\[
\sum_{k_i} |a_{k_i}^\top W b_k|^2 - m\|WB\|^2_F \leq \epsilon_3 m \sigma_{\min}^2.
\]

and hence, by Lemma 5.2,

\[
\sum_{k_i} |a_{k_i}^\top W b_k|^2 \leq m\|WB\|^2_F + \epsilon_3 m \sigma_{\min}^2 \\
\leq m\|B\|^2 + \epsilon_3 m \sigma_{\min}^2 \leq m(1.1 + \epsilon_3 / \kappa^2) \sigma_{\max}^2.
\]

Next, one can use a standard epsilon net argument [30] over all \( W \in S_{nr} \) to show that w.p. \( 1 - \exp(nr - \epsilon_3^2 m q / r \mu^2 \kappa^4) - \exp(\log q + r - cm) \)

\[
\max_{W \in S_{nr}} \sum_{k_i} |a_{k_i}^\top W b_k|^2 \leq 1.5m(1.1 + \epsilon_3 / \kappa^2) \sigma_{\max}^2.
\]

The epsilon net argument is analogous to the one in [16] for their Term3.

Similarly, using (13) followed by the lower bound on \( \sigma_{\min}(B) \) from Lemma 5.2, we can show that, for a fixed \( W \in S_{nr} \), w.p. \( 1 - \exp(-\epsilon_3^2 m q / r \mu^2 \kappa^4) - \exp(\log q + r - cm) \)

\[
\sum_{k_i} |a_{k_i}^\top W b_k|^2 \geq m\|WB\|^2_F - \epsilon_3 m \sigma_{\min}^2 \geq m\|W\|^2_F \sigma_{\min}(B)^2 - \epsilon_3 m \sigma_{\min}^2 \geq (0.9 - \epsilon_3) m \sigma_{\min}^2
\]

To extend this lower bound to all \( W \in S_{nr} \), we use the following epsilon-net argument. Let \( \delta_{nr} \) be an \( \epsilon_{net} = 1 / 8 \kappa^2 \) net of \( S_{nr} \), i.e., for any \( W \in S_{nr} \), there exists a \( \tilde{W} \in \delta_{nr} \) for which \( \|W - \tilde{W}\|_F \leq \epsilon_{net} = 1 / 8 \kappa^2 \). By [30], we can find a net with \( |\delta_{nr}| \leq (1 + 2 / \epsilon_{net})^{nr} \leq (24 \kappa^2)^{nr} \). Thus, by union bound, w.p. \( 1 - \exp(nr \log \kappa - \epsilon_3^2 m q / r \mu^2 \kappa^4) \)

\[
\sum_{k_i} |a_{k_i}^\top W b_k|^2 \geq m(0.9 - \epsilon_3) \sigma_{\min}^2 \quad \text{for all} \quad \tilde{W} \in \delta_{nr}.
\]

For any \( W \in S_{nr} \) we have

\[
\sum_{k_i} |a_{k_i}^\top W b_k|^2 = \sum_{k_i} |a_{k_i}^\top W b_k|^2 + \sum_{k_i} |a_{k_i}^\top (W - \tilde{W}) b_k|^2 + 2 \sum_{k_i} a_{k_i}^\top W b_k \cdot a_{k_i}^\top (W - \tilde{W}) b_k \\
\geq \sum_{k_i} |a_{k_i}^\top W b_k|^2 - 2 \sum_{k_i} |a_{k_i}^\top W b_k|^2 \cdot |a_{k_i}^\top (W - \tilde{W}) b_k|^2 \\
\geq \sum_{k_i} |a_{k_i}^\top W b_k|^2 - 2 \sqrt{\sum_{k_i} |a_{k_i}^\top W b_k|^2 \cdot \sum_{k_i} |a_{k_i}^\top (W - \tilde{W}) b_k|^2} \cdot \|W - \tilde{W}\|_F^2.
\]

The last inequality follows by Cauchy-Schwarz. Since \( (W - \tilde{W})/(W - \tilde{W})_F \in S_{nr} \) and \( W \in S_{nr} \), thus, we can use (15) to upper bound both the factors in the second term above. Further, using (16) to lower bound the first term, and using \( \|W - \tilde{W}\|_F \leq \epsilon_{net} = 1 / 8 \kappa^2 \), we have that, w.p. \( 1 - 4 \exp(nr \log \kappa - \epsilon_3^2 m q / r \mu^2 \kappa^4) - 2 \exp(\log q + r - cm) \)

\[
\min_{W \in S_{nr}} \sum_{k_i} |a_{k_i}^\top W b_k|^2 \geq m(0.9 - \epsilon_3) \sigma_{\min}^2 - 1.5m(1.1 + \epsilon_3 / \kappa^2) \sigma_{\max}^2 \cdot \epsilon_{net} \\
\quad = m(0.9 - \epsilon_3 - 1.5 \epsilon_{net}(1.1 + \epsilon_3) \sigma_{\min}^2 \\
\quad = m(0.9 - \epsilon_3 - 1.5 \cdot (1/8)(1.1 + \epsilon_3) \sigma_{\min}^2 \geq m(0.7 - 1.2 \epsilon_3) \sigma_{\min}^2
\]

Thus, we have shown that, w.p. at least \( 1 - \exp(nr \log \kappa - cm \epsilon_3^2 q / r \mu^2 \kappa^4) - \exp(\log q + r - cm) \),

\[
m(0.7 - 1.2 \epsilon_3) \sigma_{\min}^2 \leq \lambda_{min}(\text{Hess}) \leq \lambda_{max}(\text{Hess}) \leq m(1 + \epsilon_3) \sigma_{\max}^2
\]

2) **Bounding the GradU Term:** We have \( \|\nabla f(U, B)\| = \max_{z \in S_{nr}, w \in S_r} z^\top \nabla f(U, B)w \). For a fixed \( z \in S_n, w \in S_r \) we have

\[
z^\top (\nabla f(U, B) - \mathbb{E}[\nabla f(U, B)]) w = \sum_{k_i} [(a_{k_i}^\top Ub_k - y_{k_i}) (a_{k_i}^\top z) (w^\top b_k) - \mathbb{E}[\cdot]]
\]

where \( \mathbb{E}[\cdot] \) is the expected value of the first term. Clearly, the summands are independent sub-exponential r.v.s with norm \( K_{ki} \leq C\|x_k - x_k^*\|_1 \|b_k\| \). We apply the sub-exponential Bernstein inequality, Theorem 2.8.1 of [30], with \( t = \epsilon_1 \delta t \sigma_{\max}^2 \). To apply this, we use bounds on \( \|b_k\|, \|X^* - X\|_F \) and \( \|x_k - x_k^*\|_F \) from Lemma 5.2 to show that

\[
\sum_{k_i} K_{ki}^2 \geq C m \max_k \|b_k\|^2 \sum_k \|x_k - x_k^*\|_2^2 \geq C m \sigma_{\max}^4 \|X - X^*\|_F^2 \geq C m \sigma_{\min}^4 (r/q) \|X - X^*\|_F^2 \geq C \mu^2 \sigma_{\max}^4 \epsilon_1^2 \sigma_{\max}^4 (r/q) \sigma_{\max}^4 \geq C \mu^2 \sigma_{\max}^4 \epsilon_1^2 \sigma_{\max}^4 / r \mu^2.
\]
We extend the proof for any \( z \in S_n, w \in S_r \) w.p. \( 1 - \exp(-c_1^2mq/r\mu^2) \),
\[
\mathbb{E}[\nabla f(U, B)] = m \sum_k (x_k - x_k^*) b_k^\top = m (X - X^*) B^\top.
\]
Using the bounds on \( \|X^* - X\|_F \) and \( \|B\| \) from Lemma 5.2,
\[
\|\mathbb{E}[\nabla f(U, B)]\| = m \|X - X^*\|_B^\top \leq m \|X - X^*\|_F \|B\| \leq 1.1m\delta_3\sigma_{\max}^*\gamma^2.
\]
Hence, for a fixed \( z \in S_n, w \in S_r \) w.p. \( 1 - \exp(-c_1^2mq/r\mu^2) \) we have
\[
|z^\top \nabla f(U, B)w| \leq (1 + \epsilon_1)m\delta_3\sigma_{\max}^*\gamma^2, \quad \text{for all } z \in S_n, \ w \in S_r.
\]
Define \( \gamma^* := \max_{z \in S_n, w \in S_r} z^\top \nabla f(U, B)w \). Then, for any \( z \in S_n, w \in S_r \),
\[
z^\top \nabla f(U, B)w \leq |z^\top \nabla f(U, B)w| + |(z - z)^\top \nabla f(U, B)w| + |(z - z)^\top \nabla f(U, B)(w - w)|
\leq (1 + \epsilon_1)m\delta_3\sigma_{\max}^*\gamma + \epsilon_1\gamma + \epsilon_3\gamma^2.
\]
Thus, \( \gamma^* \leq (1 + \epsilon_1)m\delta_3\sigma_{\max}^*\gamma/(1 - \epsilon_3\gamma + 2\epsilon_1\gamma) \) with \( \epsilon_3\gamma = 1/8 \). Thus, \( \gamma^* < 1.5(1 + \epsilon_1)m\delta_3\sigma_{\max}^*\gamma^2 \).

3) Bounding Term2: First, since Term2 = \( (I - U^{*\top}U^{*\top}) \sum_k a_k \sum_i U_i(U^{*\top}U b_k - b_k^*) b_k^\top \), and \( \mathbb{E}[a_k a_k^\top] = I \),
\[
\mathbb{E}[\text{Term}2] = 0
\]
We have
\[
\|((I - U^{*\top}U^{*\top}) \nabla f(U^{*\top}U), B)\|_F = \max_{W \in S_{nr}} \langle (I - U^{*\top}U^{*\top}) \nabla f(U^{*\top}U), W \rangle.
\]
For a fixed \( n \times r \) matrix \( W \) with unit Frobenius norm,
\[
\langle (I - U^{*\top}U^{*\top}) \nabla f(U^{*\top}U), W \rangle = \sum_{ki} (a_k U_i^{*\top}U b_k - b_k^*) (a_k^\top (I - U^{*\top}U^{*\top}) W b_k
\]
Observe that the summands are independent, zero mean, sub-exponential r.v.s with sub-exponential norm \( K_{ki} \leq C\|U^{*\top}U b_k - b_k^*\| \|I - U^{*\top}U^{*\top}\| \|W b_k\| \leq \|U^{*\top}U b_k - b_k^*\| \|W b_k\| \). We can now apply the sub-exponential Bernstein inequality Theorem 2.8.1 of [30]. Let \( t = c_2\delta_3m\sigma_{\max}^*\gamma^2 \). Using the bound on \( \|U^{*\top}U b_k - b_k^*\| \) from Lemma 5.2 followed by Assumption 1.1 (right incoherence), and also the bound on \( \|B\| \) from Lemma 5.2,
\[
\sum_{ki} K_{ki} \geq \frac{c_2^2\delta_3^2 m^2\sigma_{\max}^*\gamma^2}{\delta_3^2m^2\sigma_{\max}^*\gamma^2} \geq \frac{c_2^2m^2\sigma_{\max}^*\gamma^2}{\mu^2(r/q)m\|W B\|_F^2} \geq \frac{c_2^3m^2\sigma_{\max}^*\gamma^2}{\mu^2(r/q)m\|W B\|_F^2} \geq c_2^2mq/r\mu^2,
\]
and
\[
\frac{t^2}{\max_{ki} K_{ki}} \geq \frac{c_2\delta_3m\sigma_{\max}^*\gamma^2}{\delta_3\sigma_{\max}^*\gamma^2\mu^2(r/q)} \geq \frac{c_2\delta_3m\sigma_{\max}^*\gamma^2}{\sigma_{\max}^*\gamma^2\mu^2(r/q)} \geq c_2\delta_3m\sigma_{\max}^*\gamma^2.
\]
Thus, by the sub-exponential Bernstein inequality, for a fixed \( W \in S_{nr} \), w.p. \( 1 - \exp(-c_2^2mq/r\kappa^2\mu^2) \),
\[
\langle (I - U^{*\top}U^{*\top}) \nabla f(U^{*\top}U), W \rangle \leq c_2\delta_3m\sigma_{\max}^*\gamma^2.
\]
By a standard epsilon net argument, the above bound (with \( c_2 \) replaced by 1.1\( c_2 \)) holds for all \( W \in S_{nr} \) w.p. \( 1 - \exp(nr - c_2^2mq/r\kappa^2\mu^2) \).
F. Proof of Lemma 5.2

1) Proof of Lemma 5.2: first part: We bound $\|g_k - b_k\|$ here. Recall that $g_k = U^T \omega_k$. Since $y_k = A_k \omega_k = A_k U^T \omega_k + A_k (I - UU^T) \omega_k$, therefore

$$b_k = (U^T A_k^T A_k U)^{-1} (U^T A_k^T) A_k U U^T \omega_k + (U^T A_k^T A_k U)^{-1} (U^T A_k^T) A_k (I - UU^T) \omega_k,$$

$$= (U^T A_k^T A_k U)^{-1} (U^T A_k^T) A_k U \omega_k + (U^T A_k^T A_k U)^{-1} (U^T A_k^T) A_k (I - UU^T) \omega_k,$$

$$= g_k + (U^T A_k^T A_k U)^{-1} (U^T A_k^T) A_k (I - UU^T) \omega_k.$$

Thus,

$$\|b_k - g_k\| \leq \|(U^T A_k^T A_k U)^{-1} \| \|(U^T A_k^T A_k (I - UU^T) \omega_k)\|.$$

We have $\|(U^T A_k^T A_k U)^{-1} \| = \sigma_{\min}(U^T A_k A_k U)$ and since $U^T A_k A_k U$ is a positive semi-definite (PSD) matrix,

$$\sigma_{\min}(U^T A_k A_k U) = \min_{w \in S_r} w^T U^T A_k A_k U w = \sum_i |a_{ki} U w|^2.$$

Consider a fixed $w \in S_r$. Clearly, $E[\sum_i |a_{ki} U w|^2] = m$. Since we use different set of measurements in each update step, thus $U$ is independent of all $a_{ki}$. Hence, $\tilde{a}_{ki} := U^T a_{ki}$ is a standard Gaussian random vector of length $r$. By applying the sub-exponential Bernstein inequality Theorem 2.8.1 of [30] for the above term with $t = \epsilon_0 m$, we can show that, for a fixed $w \in S_r$, w.p. $1 - \exp(-c_2^2 m)$ we have

$$m(1 + \epsilon_0) \geq \sum_i |a_{ki} U w|^2 \geq m(1 - \epsilon_0).$$

By a standard epsilon net argument\(^2\), we can argue that, w.p. $\geq 1 - \exp(r - c_2^2 m)$,

$$\min_{w \in S_r} \sum_i |a_{ki} U w|^2 \geq (1 - \epsilon_0)m - \epsilon_{net} \frac{1}{(1 - \epsilon_{net})} (1 + \epsilon_0)m \geq (1 - \epsilon_0 - (1 + \epsilon_0)/T)m$$

where $\epsilon_{net} = 1/8$ is the value of epsilon used for defining the net. By setting $\epsilon_0 = 0.1$, followed by a union bound over all $q$ columns, we conclude that, w.p. $\geq 1 - q \exp(r - cm)$, for all $k \in [q]$, $\min_{w \in S_r} \sum_i |a_{ki} U w|^2 \geq 0.7m$ and so

$$\|(U^T A_k^T A_k U)^{-1} \| \leq 1.5/m.$$ 

Now, we proceed with bounding $\|(U^T A_k^T A_k (I - UU^T) \omega_k)^{\ast}\|$. We have

$$\|(U^T A_k^T A_k (I - UU^T) \omega_k)^{\ast}\| = \max_{w \in S_r} \sum_i |a_{ki} U w| \|(U^T A_k A_k (I - UU^T) \omega_k)^{\ast}\| = \max_{w \in S_r} \sum_i |a_{ki} U w| |a_{ki}^T (I - UU^T) \omega_k|.$$ 

We note that $E[U^T A_k^T A_k (I - UU^T) \omega_k] = U^T (I - UU^T) \omega_k = 0$. Also, the summands are thus mutually independent zero mean sub-exponential r.v.s. Applying sub-exponential Bernstein with $t = m\epsilon_0 \|(I - UU^T) \omega_k\| m$ for a fixed $w \in S_r$,

$$\sum_i |a_{ki} U w| |a_{ki}^T (I - UU^T) \omega_k| \leq m\epsilon_0 \|(I - UU^T) \omega_k\|$$

w.p. $1 - \exp(-c_2^2 m)$. Let $\epsilon_0 = 0.1$. Using another standard epsilon-net argument, followed by a union bound over all $q$ columns, we can argue that, w.p. at least $1 - q \exp(r - cm)$, for all $k \in [q]$,

$$\|(U^T A_k^T A_k (I - UU^T) \omega_k)^{\ast}\| \leq m(0.1/(1 - \epsilon_{net}))(I - UU^T) \omega_k \| \leq m0.15 \|(I - UU^T) \omega_k\|$$

Combining the bounds on the two terms in (18), w.p. at least $1 - \exp(\log q + r - cm)$, for all $k \in [q]$,

$$\|g_k - b_k\| \leq 0.4 \|(I_n - UU^T) U^* b_k\|$$

\(^2\)Define a net $\tilde{S}_r$ of $S_r$ using $\epsilon_{net} = 1/8$. The size of the smallest such net on $S_r$ is $C^r$ [30, Chap. 4]. Using a proof idea to that of Lemma 4.4.2 of [30], w.p. $\geq 1 - \exp(r - c_2^2 m)$, $\min_{w \in \tilde{S}_r} \sum_i |a_{ki} U w|^2 \geq \min_{w \in S_r} \sum_i |a_{ki} U w|^2 - \epsilon_{net} \max_{w \in \tilde{S}_r} \sum_i |a_{ki} U w|^2 \geq \min_{w \in S_r} \sum_i |a_{ki} U w|^2 - \epsilon_{net} \max_{w \in \tilde{S}_r} \sum_i |a_{ki} U w|^2/(1 - \epsilon_{net}) \geq m(1 - \epsilon_0) - \epsilon_{net} \frac{1}{1 - \epsilon_{net}} m(1 + \epsilon_0)$
2) Proof of Lemma 5.2: other parts: All these parts follow easily from the bound on \( \|g_k - b_k\| \) and the assumption that \( \text{SD}(U^*, U) = \text{SD}(U, U^*) \leq \delta_t < c/\kappa \).

Recall that \( g_k = U^T x_k^* = U^T U^* b_k^* \), and \( G = U^T U^* B^* \).

Part 2a: Using the SD bound and the first part, \( \|g_k - b_k\| \leq 0.4\delta_t\|b_k^*\| \).

Part 2b: Since \( x_k^* - x_k = U^T g_k + (I - UU^T) x_k^* - Ub_k = U(g_k - b_k) + (I - UU^T) x_k^* \), using (19),

\[
\|x_k^* - x_k\| \leq \|g_k - b_k\| + \|(I - UU^T)U^*b_k^*\| \leq 1.4\delta_t\|b_k^*\|.
\]

Part 2c: \( \|U^T Ub_k - b_k^*\| = \|U^T U^* U^* b_k - b_k^*\| = \|Ub_k - (I - U^* U^*)Ub_k - U^* b_k^*\| = \|x_k - (I - U^* U^*)Ub_k - x_k^*\| \leq \|x_k - x_k^*\| + \|(I - U^* U^*)Ub_k - b_k^*\| \leq 2.4\delta_t\|b_k^*\| \)

Parts 2d and 2e: Since \( \sum_k \|Mb_k\|^2 = \|MB^*\|_F^2 \leq \|M\|_F^2 \|B^*\|^2 = \|M\|_F^2 \sigma_{\max}^* \), we can use the first bound from (19) to conclude that

\[
\|G - B\|_F^2 = \sum_k \|g_k - b_k\|^2 \\
\leq 0.4^2 \sum_k \|(I - UU^T)U^*b_k^*\|^2 = 0.4^2 \|(I - UU^T)U^* B^*\|_F^2 \leq 0.4^2 \delta_t^2 \sigma_{\max}^* \]

and, similarly,

\[
\|X^* - X\|_F^2 \leq \sum_k \|g_k - b_k\|^2 + \sum_k \|(I - UU^T)U^*b_k^*\|^2 \leq (0.4^2 + 1^2) \delta_t^2 \sigma_{\max}^* \]

Part 2f (incoherence of \( b_k \)): Using the bound on \( \|b_k - g_k\| \), and using \( \|g_k\| \leq \|b_k^*\| \) and the right incoherence assumption,

\[
\|b_k\| = \|(b_k - g_k + g_k)\| \leq (1 + 0.4\delta_t)\|b_k^*\| \leq 1.04\sigma_{\max} \sqrt{r/q}.
\]

Part 2g (bounds on \( \sigma_i(B) \)): Using the bound on \( \|G - B\|_F \) and using \( \text{SD}(U, U^*) \leq \delta_t < c/\kappa \),

\[
\sigma_{\min}(B) \geq \sigma_{\min}(G) - \|G - B\| \\
\geq \sigma_{\min}(U^* U^*)\sigma_{\min}(B^*) - \|G - B\|_F \\
\geq \sqrt{1 - \|U^* U^*\|^2} \sigma_{\min} - 0.4\delta_t \sigma_{\max} \geq \sqrt{1 - \delta_t^2} \sigma_{\min} - 0.4\delta_t \sigma_{\max} \geq 0.9\sigma_{\min}
\]

since we assumed \( \delta_t \leq \delta_0 < 0.1/\kappa \). Similarly,

\[
\|B\| = \sigma_{\max}(B) \leq \sigma_{\max}(U^* U^*)\sigma_{\max}(B^*) + \|G - B\|_F \\
\leq \sigma_{\max}^* + 0.4\delta_t \sigma_{\max} \leq 1.1\sigma_{\max}
\]

VII. NUMERICAL EXPERIMENTS

We would like to compare our proposed altGDMin algorithm with both the AltMin solution from [16], [17] and the convex optimization one from [4]. A complete step-wise algorithm to solve their convex program is not provided in their paper or in their supplement. For example, the paper does not tell us how the optimization in step 4 of Sec 4.0.2 is solved. The code provided on the website of the first author of [4] does not generate the results provided in [4] and is much slower. Hence, we
only provide comparisons with the AltMin algorithm of [16], [17]. The approach was designed for the phaseless setting. We compare with its (obvious) modification for the linear setting. In Fig. 1, we compare two versions: AltMin initialized using the phaseless init (AltMin-YU-init) and AltMin with our truncated linear init (AltMin-our-init). We implemented altGDMin with \( \eta = 1/\sigma_{\text{max}}^2 \) and \( \tilde{C} = 9 \). This, as well as the AltMin algorithms, reused all the measurements in each iteration.

We generated the data as follows. We used \( n = 600, q = 600, r = 4 \) and \( m = 30 \) and \( m = 50 \) for Fig. 1a and 1b. For each of 100 Monte Carlo runs, we simulated \( U^* \) by orthogonalizing an \( n \times r \) standard Gaussian matrix; and \( b_k^* \)'s were generated i.i.d. from \( \mathcal{N}(0, I_r) \). We plot the empirical average of SD(\( U^*, U_t \)) at each iteration \( t \) on the y-axis and the time taken by the algorithm until iteration \( t \) on the x-axis. For Fig. 1c, we used \( n = 600, q = 1000, r = 4 \) and \( m = 250 \) and 90 Monte Carlo runs. For this figure, we used the magnitude-only version of the same data as \( y_{(\text{mag})_{ki}} \) to also compare GD-min-LRPR and the algorithm of [15] (labeled AltMin-LRPR).

Notice from Fig. 1 that GD-min is 10-times faster than the AltMin approaches. The speed gap increases as we increase \( n \) or \( q \). Moreover, for very large values of \( q \), the computation can be efficiently parallelized or federated in case of the GD solutions. For AltMin, this is not possible because it requires too many data exchanges. The second key observation is that GD-min and AltMin-our-init converge even for \( m = 30 \), but AltMin-YU-init does not. The reason is our initialization only needs \( m \geq (n/q)^2 \) instead of \( (n/q)^3 \). For \( m = 50 \), all the three linear approaches work, but GD-min is much faster than the other two. From Fig 1c, notice that GD-min-LRPR is also much faster than AltMin-LRPR. We should mention that both the LRPR approaches need a larger \( m \) to work. This is both because of the initialization step and because the constants are larger.

**Appendix A**

**Proof of initialization Theorem 5.1 without sample-splitting**

Consider the initialization using \( X_0 \) defined in (2). We want to bound the initialization error without sample-splitting. This means that the threshold \( \beta \) is not independent of the \( a_{ki}, y_{ki} \) used in the expression for \( X_0 \) and thus, it is not clear how to compute its expected value even if we condition on \( \alpha \). However, the following slightly more complicated approach can be used. Using Fact 5.6 and Assumption 1.1, it is possible to show that \( X_0 \) is close to a matrix, \( X_+(\epsilon_1) \) given next for which \( \mathbb{E}[X_+] \) is easily computed: Let

\[
\alpha_+ := \tilde{C}(1 + \epsilon_1) \frac{\|X^*\|_F^2}{q}
\]

and define

\[
X_+(\epsilon_1) := \frac{1}{m} \sum_{ki} a_{ki} y_{ki} e_k^T \mathbb{1}(y_{ki} \geq \alpha_+).
\]

Then,

\[
\mathbb{E}[X_+] = X^* D(\epsilon_1), \quad D := \text{diagonal}(\beta_k(\epsilon_1)), \quad \beta_k(\epsilon_1) := \mathbb{E} \left[ \zeta^2 \mathbb{1} \left\{ \zeta^2 \leq \frac{\alpha_+}{\|e_k\|^2} \right\} \right]
\]

with \( \zeta \) being a scalar standard Gaussian. Thus \( X_+ \) is \( X_0 \) with the threshold \( \alpha \) replaced by \( \alpha_+ \) which is deterministic. Consequently \( \mathbb{E}[X_+] \) has a similar form too and is obtained as explained in the proof of Lemma 5.4 given in Sec. VI-A.

Next, recall that

\[
X^* \overset{\text{SD}}{=} U^* \Sigma^* V^* \quad \text{and} \quad \tilde{C} = 9n^2 \mu^2.
\]

Let \( \tilde{c} = c/\tilde{C} \) for a \( c < 1 \). Clearly, the span of the top \( r \) singular vectors of \( \mathbb{E}[X_+] = X^* D \) equals \( \text{span}(U^*) \) and it is rank \( r \) matrix. Let

\[
\mathbb{E}[X_+] = X^* D \overset{\text{SD}}{=} U^* \Sigma^* V
\]

be its r-SVD (here \( V \) is an \( r \times q \) matrix with its rows containing the \( r \) right singular vectors). We thus have

\[
\sigma_r(\mathbb{E}[X_+]) = \sigma_{\min}(\Sigma^*) = \sigma_{\min}(\Sigma^* V^* D V^T) \geq \sigma_{\min}(\Sigma^*) \sigma_{\min}(V^*) \sigma_{\min}(D) = \sigma_{\min}^* \cdot 1 \cdot (\min_k \beta_k) \cdot 1
\]

Fact 5.7 given earlier shows that \( (\min_k \beta_k) \geq 0.9 \) and thus,

\[
\sigma_r(\mathbb{E}[X_+]) \geq 0.9 \sigma_{\min}^*
\]

Also, \( \sigma_{r+1}(\mathbb{E}[X_+]) = 0 \) since it is a rank \( r \) matrix. Thus, using Wedin’s \( \sin \Theta \) theorem for SD (summarized in Theorem 5.8) applied with \( M \equiv X_0, \ M^* \equiv \mathbb{E}[X_+] \) gives

\[
\text{SD}(U_0, U^*) \leq \frac{\sqrt{2} \max(\|X_0 - \mathbb{E}[X_+]\|_F, \|X_0 - \mathbb{E}[X_+]\|_F)}{0.9 \sigma_{\min}^* - \|X_0 - \mathbb{E}[X_+]\|_F}
\]

(21)
In the next three subsections, we prove a set of six lemmas that help bound the three terms in the expression above. The main new ideas over the proof given earlier in Sec V-B, are in the proof of the first lemma, Lemma A.1 given below, and in the proof of Claim A.7 that is used in this proof.

Combining Lemmas A.2 and A.1 and using Fact 5.6, and setting \( \epsilon_1 = c\delta_0/\sqrt{r\kappa} \), we conclude that, w.p. at least 
\[1 - 2 \exp((n + q - \tilde{c}_1^2mq) - \exp(-\tilde{c}_1^2mq\delta_0^2/(nr^2)) - \exp(-\tilde{c}_1^2mq\delta_0^2/(nr^2)),\]
\[||X_0 - E[X_+]|| \lesssim \epsilon_1 ||X^*||_F \lesssim c\delta_0 \sigma_{\min}^*\]

By combining Lemmas A.3, A.4, A.5, and A.6 and using Fact 5.6, and setting \( \epsilon_1 = c\delta_0/\sqrt{r\kappa} \), we conclude that, w.p. at least 
\[1 - 2 \exp(nr - \tilde{c}_1^2mq\delta_0^2/(nr^2)) - \exp(qr - \tilde{c}_1^2mq\delta_0^2/(nr^2)) - \exp(-\tilde{c}_1^2mq\delta_0^2/(nr^2)),\]
\[\max (||X_0 - E[X_+]||^T \tilde{U}^* ||_F, ||X_0 - E[X_+]|| \tilde{V}^T ||_F) \lesssim c\delta_0 \sigma_{\min}^*\]

Plugging these into (21) proves Theorem 5.1

A. Bounding the denominator term

By triangle inequality, \( ||X_0 - E[X_+]]|| \leq ||X_+ - E[X_+]]|| + ||X_0 - X_+|| \). The next two lemmas bound these two terms.

The lemmas assume the claim of Fact 5.6 holds, i.e., that \( \frac{1}{mq} \sum_{k_i} y_{k_i}^2 \in [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \) where \( \tilde{C} = 9\mu^2\kappa^2 \).

Lemma A.1. Assume that \( \frac{1}{mq} \sum_{k_i} y_{k_i}^2 \in [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \) (claim of Fact 5.6 holds). Then, w.p. \( 1 - \exp(C(n + q - \epsilon_1^2mq/\mu^2\kappa^2)) \),
\[||X_0 - X_+|| \leq C \epsilon_1 \mu \kappa ||X^*||_F\]

Proof of Lemma A.1. We have
\[||X_+ - X_0)|| = \max_{z \in S^n, w \in S^q} z^T (X_+ - X_0) w = \max_{z \in S^n, w \in S^q} \frac{1}{m} \sum_{k_i} w(k) y_{k_i} (a_{k_i}^T z) \mathbb{1}_{\left\{ \frac{1}{mq} \sum_{k_i} y_{k_i}^2 \leq [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \right\}}\]
\[= \frac{1}{m} \sum_{k_i} |w(k) y_{k_i} (a_{k_i}^T z)| \mathbb{1}_{\left\{ |y_{k_i}|^2 \leq [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \right\}}\]
\[\leq \frac{1}{m} \sum_{k_i} |w(k) y_{k_i} (a_{k_i}^T z)| \mathbb{1}_{\left\{ |y_{k_i}|^2 \leq [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \right\}}\]
\[= \max_{z \in S^n, w \in S^q} \frac{1}{m} \sum_{k_i} |w(k) y_{k_i} (a_{k_i}^T z)| \mathbb{1}_{\left\{ |y_{k_i}|^2 \leq [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \right\}}\]
\[\leq \epsilon_1 ||X^*||_F \]

For the last expression above, we have used the assumption \( \sum_{k_i} y_{k_i}^2/m \leq \tilde{C}(1 + \epsilon_1)||X^*||^2_F/q \). Consider the RHS for a fixed unit norm \( z \) and \( w \). The lower threshold of the indicator function is itself a r.v. To convert it into a deterministic bound, we need the following sequence of bounding steps: To use our assumption that \( \sum_{k_i} y_{k_i}^2/m \geq (1 - \epsilon_1)\tilde{C}||X^*||^2_F/q \), we first need to bound the summands by their absolute values. This is done as follows:
\[|z^T (X_+ - X_0) w| \leq \frac{1}{m} \sum_{k_i} |w(k) y_{k_i} (a_{k_i}^T z)| \mathbb{1}_{\left\{ \frac{1}{mq} \sum_{k_i} y_{k_i}^2 \leq [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \right\}}\]
\[\leq \frac{1}{m} \sum_{k_i} |w(k) y_{k_i} (a_{k_i}^T z)| \mathbb{1}_{\left\{ |y_{k_i}|^2 \leq [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \right\}}\]
\[\leq \epsilon_1 ||X^*||_F \]

where in the last line we used our assumption that \( \sum_{k_i} y_{k_i}^2/m \geq (1 - \epsilon_1)\tilde{C}||X^*||^2_F/q \). This final expression is a sum of mutually independent sub-Gaussian r.v.s with subGaussian norm \( K \leq (1 + \epsilon_1)||X^*||_F \). Thus, by applying the sub-Gaussian Hoeffding inequality, Theorem 2.6.2 of [30],
\[\Pr \left\{ \sum_{k_i} |w(k) y_{k_i} (a_{k_i}^T z)| \mathbb{1}_{\left\{ |y_{k_i}|^2 \leq [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \right\}} - \mathbb{E} \left( \sum_{k_i} |w(k) y_{k_i} (a_{k_i}^T z)| \mathbb{1}_{\left\{ |y_{k_i}|^2 \leq [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \right\}} \right) \geq t \right\} \leq 2 \exp \left[ - \frac{t^2}{\sum_{k_i} K_{k_i}^2} \right] \]

By setting \( t = \epsilon_1 m ||X^*||_F \),
\[\frac{t^2}{\sum_{k_i} K_{k_i}^2} \geq \frac{m q \epsilon_1^2 ||X^*||_F^2}{C ||X^*||_F} = \epsilon_1^2 mq \]
\[\frac{t^2}{\sum_{k_i} K_{k_i}^2} \geq \frac{m q \epsilon_1^2 ||X^*||_F^2}{C ||X^*||_F} = \epsilon_1^2 mq \]
\[\frac{t^2}{\sum_{k_i} K_{k_i}^2} \geq \frac{m q \epsilon_1^2 ||X^*||_F^2}{C ||X^*||_F} = \epsilon_1^2 mq \]

Since \( \tilde{C} = 9\mu^2\kappa^2 \), thus, w.p. \( 1 - \exp(-\epsilon^2 mq/\mu^2\kappa^2) \), for a fixed \( z \) and \( w \),
\[z^T (X_0 - X_+) w \leq \epsilon_1 ||X^*||_F + \mathbb{E} \left( \frac{1}{m} \sum_{k_i} |w(k) y_{k_i} (a_{k_i}^T z)| \mathbb{1}_{\left\{ |y_{k_i}|^2 \leq [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \right\}} \right)\]
\[\leq \sqrt{\tilde{C}(1 + \epsilon_1)} \epsilon_1 ||X^*||_F \]

By using Claim A.7 and \( |w(k)||z| = |w(k)| \) we have
\[\mathbb{E} \left( \frac{1}{m} \sum_{k_i} |y_{k_i} (a_{k_i}^T z)| w(k) \mathbb{1}_{\left\{ |y_{k_i}|^2 \leq [1 \pm \epsilon_1]\tilde{C}||X^*||^2_F/q \right\}} \right) \leq \sqrt{\tilde{C}(1 + \epsilon_1)} \epsilon_1 ||X^*||_F \frac{1}{m} \sum_k |w(k)| / \sqrt{q} \leq C \epsilon_1 \mu \kappa ||X^*||_F \]
where in the last inequality we used Cauchy-Schwarz to show that \(\sum_k |w(k)|/\sqrt{\eta} \leq \sqrt{\sum_k |w(k)|^2 \sum_k (1/q)} = 1\). Or this also follows by \(\|w\|_1/\sqrt{\eta} \leq \|w\|_2 = 1\). Also, we used \(\sqrt{C} = C\mu\).

Next, we need to show that the statement holds for any \(z \in S_n, w \in S_q\). We let \(S_n\) and \(S_q\) to be epsilon nets corresponding to \(S_n\) and \(S_q\) so that for any \(z \in S_n, w \in S_q\) there are \(\tilde{z} \in S_n, \tilde{w} \in S_q\) so that \(\|z - \tilde{z}\| \leq \epsilon_{net}, \|w - \tilde{w}\| \leq \epsilon_{net}\) where \(|S_n| \leq (1 + 2/\epsilon_{net})^n\) and \(|S_q| \leq (1 + 2/\epsilon_{net})^q\). Let \(\epsilon_{net} = 1/8\). By union bound, w.p. \(1 - \exp((n + q) \log(17) - \epsilon_{net}^2 m q/\mu^2 k^2)\)

\[
\tilde{z}^T (X_0 - X_+) w \leq C\epsilon_{net} \|X^+\|_F \quad \text{for all } \tilde{z} \in S_n, \tilde{w} \in S_q.
\]

Define \(\gamma^* := \max_{z \in S_n, w \in S_q} z^T (X_0 - X_+) w\). For any \(z \in S_n, w \in S_q\) we have,

\[
z^T (X_0 - X_+) w \leq z^T (X_0 - X_+) \bar{w} + \|z - \bar{z}\| \left( \frac{z - \bar{z}}{\|z - \bar{z}\|} \right)^T (X_0 - X_+) \bar{w} + \|w - \bar{w}\| z^T (X_0 - X_+) \left( \frac{w - \bar{w}}{\|w - \bar{w}\|} \right)
\]

\[
\leq C\epsilon_{net} \|X^+\|_F + \epsilon_{net} \gamma^* + \epsilon_{net} \gamma^* + \epsilon_{net}^2 \gamma^*
\]

Since the above holds for any \(z \in S_n, w \in S_q\), we thus can conclude that \(\gamma^* \leq C\epsilon_{net} \|X^+\|_F / (1 - 2\epsilon_{net} - \epsilon_{net}^2) = C\epsilon_{net} \|X^+\|_F\) w.p. at least \(1 - \exp((n + q) \log(17) - \epsilon_{net}^2 m q/\mu^2 k^2)\).

**Lemma A.2.** Consider \(X_+\). Fix \(1 < \epsilon_1 < 1\). Then, w.p. \(1 - \exp \left[ C(n + q) - c\epsilon_1^2 m q/\mu^2 k^2 \right]\)

\[
\|X_+ - \mathbb{E}[X_+]\| \leq C\epsilon_1 \|X^+\|_F.
\]

**Proof of Lemma A.2.** The proof involves an application of the sub-Gaussian Hoeffding inequality followed by an epsilon-net argument, both almost the same as those used in the proof of Lemma A.1 given above. We have,

\[
\|X_+ - \mathbb{E}[X_+]\| = \max_{z \in S_n, w \in S_q} \langle X_+ - \mathbb{E}[X_+], zw^T \rangle.
\]

For a fixed \(z \in S_n, w \in S_q\), we have,

\[
\langle X_+ - \mathbb{E}[X_+], zw^T \rangle = \frac{1}{m} \sum_{k} \left( w(k) y_{ki}(a_{ki}^T z) \mathbb{1}_{\|y_{ki}\|^2 \leq C(1 + \epsilon_1) \|X^+\|_F^2} - \mathbb{E} \left[ w(k) y_{ki}(a_{ki}^T z) \mathbb{1}_{\|y_{ki}\|^2 \leq C(1 + \epsilon_1) \|X^+\|_F^2} \right] \right).
\]

The summands are mutually independent, zero mean Gaussian r.v.s with norm \(K_{ki} \leq C \|w(k)\| \sqrt{C(1 + \epsilon_1)} \|X^+\|_F / \sqrt{q}\). We will again apply the sub-Gaussian Hoeffding inequality Theorem 2.6.2 of [30]. Let \(t = \epsilon_1 m\|X^+\|_F\). Then

\[
\sum_{k} K_{ki}^2 \geq \sum_{k} C(1 + \epsilon_1) \|X^+\|_F^2 / q \geq \frac{\epsilon_1^2 m q}{C \mu^2 k^2}
\]

Thus, for a fixed \(z \in S_n, w \in S_q\), by sub-Gaussian Hoeffding, we conclude that, w.p. at least \(1 - \exp \left[ -c\epsilon_1^2 m q/\mu^2 k^2 \right]\),

\[
\langle X_+ - \mathbb{E}[X_+], zw^T \rangle \leq C\epsilon_1 \|X^+\|_F.
\]

The rest of the proof follows by applying epsilon net argument for any \(z \in S_n, w \in S_q\). The above bound then holds w.p. at least \(1 - \exp \left[ (n + q) - c\epsilon_1^2 m q/\mu^2 k^2 \right]\).

**B. Bounding the \(\tilde{V}\) numerator term**

We bound \(||(X_0 - \mathbb{E}[X_+])\tilde{V}^T||_F\) in this section. By triangle inequality, it is bounded by \(||(X_0 - X_+)\tilde{V}^T||_F + ||(X_+ - \mathbb{E}[X_+])\tilde{V}^T||_F\).

**Lemma A.3.** Assume that \(\frac{1}{m} \sum_{k} y_{ki}^2 \in [1 + \epsilon_1] \|X^+\|_F^2\). Then, w.p. \(1 - \exp \left[ nr - c\epsilon_1^2 m q/\mu^2 k^2 \right]\),

\[
||X_0 - X_+|| \tilde{V}^T \|_F \leq C\epsilon_1 \mu k \|X^+\|_F.
\]

**Proof of Lemma A.3.** The initial part of the proof is very similar to the that of the proof of Lemma A.1. We have, \(||(X_0 - X_+)\tilde{V}^T||_F = \max_{W \in S_{nr}} (W, (X - X_+)\tilde{V}^T)\). For a fixed \(W \in S_{nr}\),

\[
\langle W, (X_0 - X_+)\tilde{V}^T \rangle = \frac{1}{m} \sum_{k} y_{ki}(a_{ki}^T \tilde{W} \tilde{v}_k) \mathbb{1}_{\|\tilde{W} \tilde{v}_k\|^2 \leq C(1 + \epsilon_1) \|X^+\|_F^2}.
\]
Proceeding as in the proof of Lemma A.1,
\[ \frac{1}{m} \sum_{k_i} y_{ki}(a_{ki}^\top W \hat{v}_k) \mathbb{I} \left\{ \frac{\epsilon^2}{m} \sum_{k_i} y_{ki}^2 \leq |y_{ki}|^2 \leq \frac{C(1+\epsilon_1)}{q} \|X^*\|^2 \right\} \leq \frac{1}{m} \sum_{k_i} \left| y_{ki}(a_{ki}^\top W \hat{v}_k) \right| \mathbb{I} \left\{ \frac{\epsilon^2}{m} \sum_{k_i} y_{ki}^2 \leq |y_{ki}|^2 \leq \frac{C(1+\epsilon_1)}{q} \|X^*\|^2 \right\}, \]
\[ \leq \frac{1}{m} \sum_{k_i} \left| y_{ki}(a_{ki}^\top W \hat{v}_k) \right| \mathbb{I} \left\{ |y_{ki}|^2 \in [1 \pm \epsilon_1] \frac{q}{C} \|X^*\|^2 \right\}. \]

The summands are mutually independent sub-Gaussian r.v.s with norm \( K_{ki} \leq C \sqrt{C(1+\epsilon_1)} \|W \hat{v}_k\|_F \|X^*\|_F/\sqrt{q} \). Thus, we can apply the sub-Gaussian Hoeffding inequality Theorem 2.6.2 of [30]. Set \( t = \epsilon_1 m \|X^*\|_F \). Then we have
\[ \sum_{k_i} K_{ki}^2 \geq \left( \sum_{k_i} \|W \hat{v}_k\|_F^2 \right) C(1+\epsilon_1) \|X^*\|_F^2 / q \geq \frac{\epsilon^2 m q}{C \mu^2 \kappa^2}, \]
where we used the fact that \( \hat{V} \hat{V}^\top = I \) (\( \hat{V}^\top \) contains right singular vectors of a matrix) and thus \( \|W \hat{V}\|_F = 1 \). Applying sub-Gaussian Hoeffding, we can conclude that, w.p. 1 – \( \exp \left[-\epsilon^2 m q / (C \mu^2 \kappa^2)\right] \)
\[ \frac{1}{m} \sum_{k_i} \left| y_{ki}(a_{ki}^\top W \hat{v}_k) \right| \mathbb{I} \left\{ |y_{ki}|^2 \in [1 \pm \epsilon_1] \frac{q}{C} \|X^*\|^2 \right\} \leq \epsilon_1 \|X^*\|_F \mbox{ and } \frac{1}{m} \sum_{k_i} \mathbb{E} \left[ \left| y_{ki}(a_{ki}^\top W \hat{v}_k) \right| \mathbb{I} \left\{ |y_{ki}|^2 \in [1 \pm \epsilon_1] \frac{q}{C} \|X^*\|^2 \right\} \right] \leq \frac{1}{m} \sum_{k_i} \sqrt{C(1+\epsilon_1)} \epsilon_1 \|X^*\|_F \|W \hat{v}_k\|_F / \sqrt{q} \leq C \epsilon_1 \mu \kappa \|X^*\|_F. \]

We use Claim A.7 to bound the expectation term. Using this lemma with \( \alpha^2 \equiv C(1+\epsilon_1) \|X^*\|_F^2 / q, z \equiv W \hat{v}_k \)
\[ \frac{1}{m} \sum_{k_i} \mathbb{E} \left[ \left| y_{ki}(a_{ki}^\top W \hat{v}_k) \right| \mathbb{I} \left\{ |y_{ki}|^2 \in [1 \pm \epsilon_1] \frac{q}{C} \|X^*\|^2 \right\} \right] \leq \frac{1}{m} \sum_{k_i} \sqrt{C(1+\epsilon_1)} \epsilon_1 \|X^*\|_F \|W \hat{v}_k\|_F / \sqrt{q} \leq C \epsilon_1 \mu \kappa \|X^*\|_F. \]
where the last inequality used Cauchy-Schwarz on \( \sum_k \|W \hat{v}_k\|_F / \sqrt{q} \) to conclude that \( \sum_k \|W \hat{v}_k\|_F (1/\sqrt{q}) \leq \sqrt{\sum_k \|W \hat{v}_k\|_F^2 / (1/q)} = \sqrt{\|W \hat{V}\|_F^2 \cdot 1} \) since \( \|W \hat{V}\|_F = 1 \).

By a standard epsilon net argument, the above bound holds for all \( W \in S_{nr} \), w.p. at least \( 1 - \exp \left[ n r \log(1+2/\epsilon_{net}) - \epsilon^2 m q / (C \mu^2 \kappa^2) \right] \).

**Lemma A.4.** Consider \( 0 < \epsilon_1 < 1 \). Then, w.p. 1 – \( \exp \left[ n r - \epsilon^2 m q / (C \mu^2 \kappa^2) \right] \)
\[ \| (X_+ - E[X_+]) \hat{V}^\top \|_F \leq C \epsilon_1 \|X^*\|_F. \]

**Proof of Lemma A.4.** The proof is quite similar to the previous one. For a fixed \( W \in S_{nr} \) we have,
\[ \langle (X_+ - E[X_+]) \hat{V}^\top, W \rangle = \frac{1}{m} \sum_{k_i} \left( y_{ki}(a_{ki}^\top W \hat{v}_k) \mathbb{I} \left\{ |y_{ki}|^2 \leq \frac{C(1+\epsilon_1)}{q} \|X^*\|^2 \right\} - \mathbb{E}[\cdot] \right) \]
where \( \mathbb{E}[\cdot] \) is the expected value of the first term. The summands are independent, zero mean, sub-Gaussian r.v.s with subGaussian norm less than \( K_{ki} \leq C \sqrt{C(1+\epsilon_1)} \|X^*\|_F \|W \hat{v}_k\|_F / \sqrt{q} \). Thus, by applying the sub-Gaussian Hoeffding inequality Theorem 2.6.2 of [30], with \( t = \epsilon_1 m \|X^*\|_F \), and using \( \|W \hat{V}\|_F = 1 \), we can conclude that, w.p. 1 – \( \exp \left[ -\epsilon^2 m q / (C \mu^2 \kappa^2) \right] \)
\[ \langle (X_+ - E[X_+]) \hat{V}^\top, W \rangle \leq C \epsilon_1 \|X^*\|_F. \]

By a standard epsilon net argument, the above bound holds for all \( W \in S_{nr} \) w.p. 1 – \( \exp \left[ n r - \epsilon^2 m q / (C \mu^2 \kappa^2) \right] \).

**C. Bounding the U^* numerator term**

We bound \( \| (X_0 - E[X_+])^\top U^* \|_F \) here. By triangle inequality, it is bounded by \( \| (X_0 - X_+)^\top U^* \|_F + \| (X_+ - E[X_+])^\top U^* \|_F \).

**Lemma A.5.** Assume that \( \frac{1}{mq} \sum_{k_i} y_{ki}^2 \in [1 \pm \epsilon_1] \|X^*\|^2 / q \). Then, w.p. 1 – \( \exp \left[ n r - \epsilon^2 m q / (C \mu^2 \kappa^2) \right] \)
\[ \| (X_0 - X_+)^\top U^* \|_F \leq C \epsilon_1 \mu \kappa \|X^*\|_F. \]

**Proof of Lemma A.5.** The proof is similar to that of Lemmas A.1 and A.3. We have, \( \| (X_0 - X_+)^\top U^* \|_F = \max_{W \in S_{qr}} \langle W, (X_+ - X_+)^\top U^* \rangle \). For a fixed \( W \in S_{qr} \), using the same approach as in Lemma A.1, and letting \( w_k \) be the \( k \)-th column of the \( r \times q \) matrix \( W \),
\[ \langle W, (X_0 - X_+)^\top U^* \rangle \leq \frac{1}{m} \sum_{k_i} \left| y_{ki}(a_{ki}^\top U^* w_k) \right| \mathbb{I} \left\{ \frac{\epsilon^2}{m} \sum_{k_i} |y_{ki}|^2 \leq |y_{ki}|^2 \leq \frac{C(1+\epsilon_1)}{q} \|X^*\|^2 \right\}, \]
\[ \leq \frac{1}{m} \sum_{k_i} \left| y_{ki}(a_{ki}^\top U^* w_k) \right| \mathbb{I} \left\{ |y_{ki}|^2 \in [1 \pm \epsilon_1] \frac{q}{C} \|X^*\|^2 \right\}. \]
The summands are now mutually independent sub-Gaussian r.v.s with norm \( K_{ki} \leq \sqrt{\bar{C}(1 + \epsilon_1)} \| w_k \| \| X^* \|_F / \sqrt{q} \). Thus, we can apply the sub-Gaussian Hoeffding inequality Theorem 2.6.2 of [30], to conclude that, for a fixed \( W \in S_{gr} \), w.p. \( 1 - \exp \left[ -c_1^2mq/\mu^2\kappa^2 \right] \),

\[
\frac{1}{m} \sum_{k_i} \left| y_{ki}(a_{ki}^\top U^*w_k) \right| \mathbb{I} \left\{ \| y_{ki} \|_F^2 \leq \frac{\bar{C}_1(1 + \epsilon_1)}{q} \| X^* \|_F^2 \right\} \leq \epsilon_1 \| X^* \|_F + \frac{1}{m} \sum_k \mathbb{E} \left| y_{ki}(a_{ki}^\top U^*w_k) \right| \mathbb{I} \left\{ \| y_{ki} \|_F^2 \leq \frac{\bar{C}_1(1 + \epsilon_1)}{q} \| X^* \|_F^2 \right\}
\]

By Claim A.7, and using \( \sum_k \| w_k \| / \sqrt{q} \leq \sqrt{\sum_k \| w_k \|^2} \sqrt{\sum_k 1/q} = 1 \),

\[
\frac{1}{m} \sum_k \mathbb{E} \left| y_{ki}(a_{ki}^\top U^*w_k) \right| \mathbb{I} \left\{ \| y_{ki} \|_F^2 \leq \frac{\bar{C}_1(1 + \epsilon_1)}{q} \| X^* \|_F^2 \right\} \leq \frac{1}{m} \sum_k \epsilon_1 \| w_k \| \sqrt{\bar{C}(1 + \epsilon_1)/q} \| X^* \|_F,
\]

\[
\leq C\epsilon_1 \mu \| X^* \|_F.
\]

The rest of proof follows by epsilon net argument over all \( W \in S_{gr} \)(union bound on the epsilon-net \( S_{gr} \) of \( S_{gr} \) which is of size at most \( C^\alpha \), followed by extending the bound from the net to the entire sphere). Because of the union bound, the final result holds w.p. \( 1 - \exp \left[ qr - c_1^2mq/\mu^2\kappa^2 \right] \).

**Lemma A.6.** Consider \( 0 < \epsilon_1 < 1 \). Then, w.p. \( 1 - \exp \left[ qr - c_1^2mq/\mu^2\kappa^2 \right] \)

\[
\| (X_+ - \mathbb{E}[X_+])^\top U^* \|_F \leq C\epsilon_1 \| X^* \|_F.
\]

**Proof of Lemma A.6.** For fixed \( W \in S_{gr} \),

\[
\text{trace} \left( W^\top (X_+ - \mathbb{E}[X_+])^\top U^* \right) = \frac{1}{m} \sum_{k_i} \left( y_{ki}(a_{ki}^\top U^*w_k) \mathbb{I} \left\{ \| y_{ki} \|_F^2 \leq \frac{\bar{C}_1(1 + \epsilon_1)}{q} \| X^* \|_F^2 \right\} - \mathbb{E} \left| y_{ki}(a_{ki}^\top U^*w_k) \right| \mathbb{I} \left\{ \| y_{ki} \|_F^2 \leq \frac{\bar{C}_1(1 + \epsilon_1)}{q} \| X^* \|_F^2 \right\} \right)
\]

The summands are independent zero mean sub-Gaussian r.v.s with norm less than \( K_{ki} \leq \sqrt{\bar{C}(1 + \epsilon_1)} \| X^* \|_F \| w_k \| / \sqrt{q} \). Thus, by applying the sub-Gaussian Hoeffding inequality Theorem 2.6.2 of [30], with \( t = \epsilon_1 m \| X^* \|_F \), we can conclude that, for a fixed \( W \in S_{gr} \), w.p. \( 1 - \exp \left[ -c_1^2mq/\mu^2\kappa^2 \right] \),

\[
\text{trace} \left( W^\top (X_+ - \mathbb{E}[X_+])^\top U^* \right) \leq \epsilon_1 \| X^* \|_F.
\]

The rest of proof follows by epsilon net argument over all \( W \in S_{gr} \)(union bound on the epsilon-net \( S_{gr} \) of \( S_{gr} \) which is of size at most \( C^\alpha \), followed by extending the bound from the net to the entire sphere). Because of the union bound, the final result holds w.p. \( 1 - \exp \left[ qr - c_1^2mq/\mu^2\kappa^2 \right] \).

\[D. \text{ A claim used above and its proof}\]

**Claim A.7.** Let \( x^* \in \mathbb{R}^n \), \( z \in \mathbb{R}^n \) be two deterministic vectors and let \( \alpha \) be a deterministic scalar. Let \( \alpha \sim \mathcal{N}(0, I_n) \) be a standard Gaussian vector and define \( y := \alpha^\top x^* \). For an \( 0 < \epsilon < 1 \),

\[
\mathbb{E} \left[ |y(\alpha^\top z)| \mathbb{I} \{ |y^2 \in [1 + \epsilon] \alpha \} \right] \leq C\epsilon \| z \| \sqrt{\alpha}.
\]

**Proof.** We can write \( x^* = \| x^* \| Qe_1 \) where \( Q \) is a unitary matrix with first column proportional to \( x^*_1 \). We need to bound

\[
\mathbb{E} \left[ |x^*| \cdot |(a^\top Qe_1)(a^\top QQ^\top z)| \mathbb{I} \{ |x^*| \in [1 + \epsilon] \alpha \} \right] = \| x^* \| \cdot |z| \cdot \mathbb{E} \left[ |\bar{a}_1^\top \bar{Q}z_1| \mathbb{I} \{ |\bar{a}_1| \in [1 + \epsilon] |\alpha| \right]
\]

where \( \bar{Q} := Q^\top z / |z| \), \( \bar{a} := Q^\top a \) and \( \beta := \sqrt{\alpha} / \| x^* \| \). Since \( Q \) is unitary and \( a \) Gaussian, thus \( \bar{a} \) has the same distribution as \( a \). Let \( \bar{a}(1) \) be its first entry and \( \bar{a} \) (rest) be the \( (n - 1) \)-length vector with the rest of the \( n - 1 \) entries and similarly for \( \bar{z}_Q \). Since \( \bar{a}(1) \) and \( \bar{a} \) (rest) are independent,

\[
\mathbb{E}\left| \bar{a}_1^\top \bar{Q}z_1 | \mathbb{I} |\bar{a}_1| \in [1 + \epsilon] |\alpha| \right| \leq |\bar{z}_Q(1)| \mathbb{E} \left| \bar{a}(1)^2 |\bar{a}(1)| \in [1 + \epsilon] |\alpha| \right| + \mathbb{E} \left| (\bar{a}(\text{rest})^\top \bar{Q}(\text{rest})) | \mathbb{I} |\bar{a}(\text{rest})| \in [1 + \epsilon] |\alpha| \right|
\]

\[
\leq \mathbb{E} \left| \bar{a}(1)^2 |\bar{a}(1)| \in [1 + \epsilon] |\alpha| \right| + 2\mathbb{E} \left| \bar{a}(1)^2 |\bar{a}(1)| \in [1 + \epsilon] |\alpha| \right| \leq \epsilon \beta + 2\epsilon \beta = 3\epsilon \beta = C\epsilon \| x^* \| \sqrt{\alpha}.
\]
The second inequality used the facts that (i) \(|z_Q(1)| \leq \|z_Q\| = 1\) by definition and (ii) \(\zeta := \tilde{a}(\text{rest})^T z_Q(\text{rest})\) is a scalar standard Gaussian r.v. and so \(\mathbb{E}[|\zeta|] \leq 2\). The third one relies on the following two bounds

\[
\mathbb{E} \left[ |a(1)|^2 \mathbb{I}_{\{|a(1)|^2 \in [1 \pm \epsilon/b^2]\}} \right] = \frac{2}{\sqrt{2\pi}} \int_{1-\epsilon/b^2}^{1+\epsilon/b^2} z^2 \exp(-z^2/2) dz,
\]

where we used the facts that \(z^2 \exp(-z^2/2) \leq \exp(-1/2)\) for all \(z \in \mathbb{R}; \sqrt{1-\epsilon} \geq 1 - \epsilon/2\) and \(\sqrt{1+\epsilon} \leq 1 + \epsilon/2\) for \(0 < \epsilon < 1\). Similarly, we can show that

\[
\mathbb{E} \left[ |a(1)| \mathbb{I}_{\{|a(1)| \in [1 \pm \epsilon/b]\}} \right] = \frac{2}{\sqrt{2\pi}} \int_{1-\epsilon/b}^{1+\epsilon/b} z \exp(-z^2/2) dz,
\]

\[
\leq \frac{2e^{-1/2}}{\sqrt{2\pi}} \int_{1-\epsilon/b}^{1+\epsilon/b} dz = \frac{2e^{-1/2}}{\sqrt{2\pi}} \epsilon \leq \epsilon/3
\]

The claim follows by combining the first two equations given above. \(\square\)

**Appendix B**

**Why Projected GD on \(X\) does not work for our problem but works for LRMC**

Recall that projected GD on \(X\) involves the following iteration: \(X^+ \leftarrow \mathcal{P}_r(X - \eta \nabla_X f(X))\). Here \(\mathcal{P}_r(.)\) projects onto the set of rank \(r\) matrices. The cost functions and their gradients for both LReCS and LRMC are given in Table III. Let \(H := X - X^+\), \(\hat{H} := (X - \eta \nabla f(X)) - X^+ = H - \eta \nabla f(X)\), and \(H^+ = X^+ - X^* = \mathcal{P}_r(\nabla f(X)) - X^* = \mathcal{P}_r(\nabla f(X) + \hat{H}) - X^*\).

To bound the LReCS gradient and show that it decays exponentially under the desired sample complexity, we need a tight column-wise bound of the form \(\max_k \|x_k^+ - x_k\| \leq \delta_k \|x_k^+\|\) with \(\delta_k\) decaying exponentially. To get a similar bound for LRMC we need an entry-wise bound of the form \(\|X^+ - X\|_{\text{max}} \leq \delta_k \|X^+\|_{\text{max}}\).

Consider LRMC. We will get the above bound if we can show \(\|H^+\|_{\text{max}} \leq c \|H\|_{\text{max}}\) for a \(c < 1\). Using the gradient expression, \(H = \sum_{k=1}^{q} \sum_{j=1}^{n} (1 - \delta_{jk}/p) H_{jk} e_j e_k^T\): different entries are mutually independent and each depends on only one entry of \(H\). This fact is carefully exploited in [21, Lemma 1] and [22, Lemma 1]. By borrowing ideas from the literature on spectral statistics of Erdos-Renyi graphs [36], the authors are able to obtain expressions for higher powers of \((H^T H)^T\). These expressions help them get the desired bound under the desired sample complexity. For LReCS, using the gradient expression, \(H = \sum_{k=1}^{q} \sum_{j=1}^{m} (I - a_k a_k^T) h_k e_k^T\). We need a bound on \(\max_k \|h_k^+\|\) in terms of \(\|h_k\|\) in order to show its exponential decay. Since the different entries of \(H\) are not mutually independent and not bounded, the LRMC proof approach cannot be borrowed. Moreover, as explained in Sec. III-B, even if we somehow obtain a different proof to get this bound, we will only be able to bound the sub-exponential norms of the summands by \(\delta_k \sqrt{r/q}\) with \(\delta_k\) decaying exponentially; but this is not small enough to obtain the result with just order \((n + q)r^2\) samples.

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