ON THE UPPER SEMI-CONTINUITY OF HSL NUMBERS

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Abstract. Let $B$ be an affine Cohen-Macaulay algebra over a field of characteristic $p$. For every prime ideal $p \subset B$, let $H_p$ denote $H_{\dim B}(\hat{B}_p)$. Each such $H_p$ is an Artinian module endowed with a natural Frobenius map $\Theta$ and if $\text{Nil}(H_p)$ denotes the set of all elements in $H_p$ killed by some power of $\Theta$ then a theorem by Hartshorne-Speiser and Lyubeznik shows that there exists an $e \geq 0$ such that $\Theta^e \text{Nil}(H_p) = 0$. The smallest such $e$ is the HSL-number of $H_p$ which we denote $\text{HSL}(H_p)$.

The main theorem in this paper shows that for all $e > 0$, the sets $\{p \in \text{Spec } B \mid \text{HSL}(H_p) < e\}$ are Zariski open, hence HSL is upper semi-continuous.

This extends [B, Prop. 4.8] where M. Hashimoto proves that if $B$ is $F$-finite Cohen-Macaulay then the $F$-injective locus of $B$ is a Zariski open set.

1. Introduction

Throughout this paper every ring is assumed to be Noetherian, commutative, associative, with identity, and of prime characteristic $p$.

If $R$ is such a ring then for any positive integer $e$ we define the $e$th-iterated Frobenius endomorphism $T^e : R \to R$ to be the map $r \mapsto r^{p^e}$. For any $R$-module $M$, we can define $F^e_* M$ to be the Abelian group $M$ with $R$-module structure given by $r \cdot m = T^e(r)m = r^{p^e}m$ for all $r \in R$ and $m \in M$. We can also define a functor, the $e$th-Frobenius functor from $R$-modules to $R$-modules as follows. For any $R$-module $M$, we consider the $F^e_* R$-module $F^e_* R \otimes_R M$ and after identifying the rings $R$ and $F^e_* R$, we may regard $F^e_* R \otimes_R M$ as an $R$-module and denote it $F^e_R(M)$ or just $F^e(M)$ when $R$ is understood. The functor $F^e_R(\_)$ is exact when $R$ is regular, cf. [B-H, Corollary 8.2.8], and for any matrix $C$ with entries in $R$, $F^e_R(\text{Coker } C)$ is the cokernel of the matrix $C^{[p^e]}$ obtained from $C$ by raising its entries to the $p^e$th power.

The main result of this paper concerns the study of HSL numbers which we now define. For any $R$-module $M$ an additive map $\Theta : M \to
$M$ is an $e^{th}$-Frobenius map if it satisfies $\Theta(rm) = r^e \Theta(m)$ for all $r \in R$ and $m \in M$. Given such $\Theta$ we can define for $i \geq 0$ the $R$-submodule $M_i = \{m \in M \mid \Theta^i m = 0\}$. We define the submodule of nilpotent elements in $M$, denoted $\text{Nil}(M)$, to be $\cup_{i \geq 0} M_i$.

We have the following.

**Theorem 1** (cf. Proposition 1.11 in [H-S] and Proposition 4.4 in [L]). Assume $(R, m)$ is a complete regular ring, $M$ is an Artinian $R$-module and $\Theta : M \to M$ is a Frobenius map then the ascending sequence $\{M_i\}_{i \geq 0}$ above stabilises, i.e., there exists an $e \geq 0$ such that $\Theta^e \text{Nil}(M) = 0$.

**Definition 2.** We define the $HSL$ number or index of nilpotency of $\Theta$ on $M$, denoted $\text{HSL}(M)$, to be the smallest integer $e$ at which $\Theta^e \text{Nil}(M) = 0$, or $\infty$ if no such $e$ exists.

We can rephrase Theorem 1 by saying that under the hypothesis of the theorem, $\text{HSL}(M) < \infty$.

Our results are a generalisation of [H Prop. 4.8] where it is proven that under certain hypothesis, the $F$-injective locus of a ring, which we introduce below, is open.

Recall that a natural Frobenius map acting on any $R$-module $M$ induces a natural Frobenius map on $H^i_m(M)$, cf [K1 Sect.2].

**Definition 3.** A local ring $(R, m)$ is $F$-injective if the natural Frobenius map $T : H^i_m(R) \to H^i_m(R)$ is injective for all $i$.

**Remark 4.** If $(R, m)$ is a Cohen-Macaulay ring of dimension $d$ and $M$ is an $R$-module then the only non-trivial local cohomology module is the top local cohomology $H^d_m(M)$. Therefore, for such a module $M$ to be $F$-injective is equivalent to $\text{HSL}(H^d_m(M)) = 0$.

We shall say that $(R, m)$ is CMFI if it is Cohen-Macaulay and $F$-injective. A non-local ring $R$ is CMFI if for each maximal ideal $m \subset R$ the localisation $R_m$ is CMFI. We define the $F$-injective locus of $R$ to be:

$$\text{CMFI}(R) = \{p \in \text{Spec}(R) \mid R_p \text{ is CMFI}\}.$$

The structure of this paper is the following; in Section 2 we define the operator $I_e(-)$ and in the case of a polynomial ring $A$ we show that it commutes with completions and localisations with respect to any multiplicatively closed subset of $A$. In Section 3 we are given a quotient $S$ of a local ring $(R, m)$ and we give an explicit description of the $R$-module $F_e(H^d_{ms}(S))$ consisting of all $e^{th}$-Frobenius maps acting on the
top local cohomology module $H^d_{m_S}(S)$. Using the fact that $\mathcal{F}_e(H^d_{m_S}(S))$ is generated by one element which is the natural Frobenius map acting on $H^d_{m_S}(S)$ (cf. [Ly-Sm, Example 3.7]) along with our result we obtain an explicit description of any Frobenius map acting on $H^d_{m_S}(S)$. Once we know this, we prove the main theorem of this section whose aim is to give a formula to compute $\text{HSL}(H^d_{m_S}(S))$; as a corollary we get a characterisation for $S$ to be $F$-injective. The goal of Section 4 is to prove that the set $B_e = \{ p \in \text{Spec}(A) | \text{HSL} \left( H^d_{pB_p}(\hat{B}_p) \right) < e \}$, where $B$ is a quotient of a polynomial ring $A$, is a Zariski open set. Note that it follows that the $F$-injective locus of a quotient of a polynomial ring $B$, namely $B_1$, is Zariski open.

2. The $I_e(-)$ Operator

In this section we show that the $I_e(-)$ operator defined below commutes with localisations and completions. Note that the result was proven in [B-M-S] with the additional assumption of $F$-finitess.

For any ideal $L$ of a ring $R$, we shall denote by $L^{[p^e]}$ the $e^{th}$-Frobenius power of $L$, i.e. the ideal generated by $\{ a^{p^e} | a \in L \}$.

**Definition 5.** If $R$ is a ring and $J \subseteq R$ an ideal of $R$ we define $I_e(J)$ to be the smallest ideal $L$ of $R$ such that its $e^{th}$-Frobenius power $L^{[p^e]}$ contains $J$.

**Remark 6.** In general, such an ideal may not exist; however it does exist in polynomial rings and power series rings, cf [K1, Proposition 5.3].

Throughout Section 2 let $A$ be a polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ and $W$ be a multiplicatively closed subset of $A$ and $J \subseteq A$ an ideal.

**Lemma 7.** If $L \subseteq W^{-1}A$ is any ideal then $L^{[p^e]} \cap A = (L \cap A)^{[p^e]}$.

**Proof.** $L^{[p^e]} \cap A$ and $(L \cap A)^{[p^e]}$ have the same generators. In fact, let $g_1^{p^e}, \ldots, g_s^{p^e}$ generate $L$. Then $L^{[p^e]}$ is generated by $g_1^{p^e}, \ldots, g_s^{p^e}$ and $L^{[p^e]} \cap A$ is consequently generated by $g_1^{p^e}, \ldots, g_s^{p^e}$; on the other hand $L \cap A$ is generated by $g_1, \ldots, g_s$ therefore $(L \cap A)^{[p^e]}$ is generated by $g_1^{p^e}, \ldots, g_s^{p^e}$. 

**Lemma 8.** If $J$ is any ideal of $A$ then $I_e(W^{-1}J)$ exists for any integer $e$ and equals $W^{-1}I_e(W^{-1}J \cap A)$.

**Proof.** Given any ideal $L \subseteq W^{-1}A$ such that $W^{-1}J \subseteq L^{[p^e]}$ we have that $W^{-1}I_e(W^{-1}J \cap A) \subseteq L$; in fact, $W^{-1}J \cap A \subseteq L^{[p^e]} \cap A = (L \cap A)^{[p^e]}$. 

and consequently $I_e(W^{-1}J \cap A) \subseteq L \cap A$ so $W^{-1}I_e(W^{-1}J \cap A) \subseteq W^{-1}(L \cap A) = L$. Hence $W^{-1}I_e(W^{-1}J \cap A)$ is contained in all the ideals $L$ such that $W^{-1}J \subseteq L^{[p^r]}$. If we show that $W^{-1}J \subseteq (W^{-1}I_e(W^{-1}J \cap A))^{[p^r]}$ then $I_e(W^{-1}J)$ exists and equals $W^{-1}I_e(W^{-1}J \cap A)$. But since $W^{-1}J \cap A \subseteq I_e(W^{-1}J \cap A)^{[p^r]}$ then using Lemma 7 we obtain $W^{-1}J = W^{-1}(W^{-1}J \cap A) \subseteq W^{-1}(I_e(W^{-1}J \cap A))^{[p^r]} = (W^{-1}I_e(W^{-1}J \cap A))^{[p^r]}$.

\[\blacksquare\]

**Proposition 9.** Let $\hat{A}$ denote the completion of $A$ with respect to any prime ideal and $W$ any multiplicatively closed subset of $A$. Then the following hold:

1. $I_e(J \otimes_A \hat{A}) = I_e(J) \otimes_A \hat{A}$, for any ideal $J \subseteq A$;
2. $W^{-1}I_e(J) = I_e(W^{-1}J)$.

**Proof.** (1) Write $\tilde{J} = J \otimes_A \hat{A}$. Since $I_e(\tilde{J})^{[p^r]} \supseteq \tilde{J}$ using Lemma 6.6 we obtain

$$(I_e(\tilde{J}) \cap A)^{[p^r]} = I_e(\tilde{J})^{[p^r]} \cap A \supseteq \tilde{J} \cap A = J.$$  

But $I_e(J)$ is the smallest ideal such that $I_e(J)^{[p^r]} \supseteq J$, so $I_e(\tilde{J}) \cap A \supseteq I_e(J)$ and hence $I_e(\tilde{J}) = (I_e(\tilde{J}) \cap A) \otimes_A \hat{A} \supseteq I_e(J) \otimes_A \hat{A}$. On the other hand, $(I_e(J) \otimes_A \hat{A})^{[p^r]} = I_e(J)^{[p^r]} \otimes_A \hat{A} \supseteq J \otimes_A \hat{A}$ and so $I_e(J \otimes_A \hat{A}) \subseteq I_e(J) \otimes_A \hat{A}$.

(2) Since $J \subseteq W^{-1}J \cap A$ then $I_e(J) \subseteq I_e(W^{-1}J \cap A)$ and so $W^{-1}I_e(J) \subseteq W^{-1}I_e(W^{-1}J \cap A)$. By Lemma 8 $W^{-1}I_e(W^{-1}J \cap R) = I_e(W^{-1}J)$ hence $W^{-1}I_e(J) \subseteq I_e(W^{-1}J)$.

For the reverse inclusion it is enough to show that $W^{-1}J \subseteq (W^{-1}I_e(J))^{[p^r]}$ because from this it follows that $I_e(W^{-1}J) \subseteq W^{-1}I_e(J)$ which is what we require. Since $J \subseteq I_e(J)^{[p^r]}$ then $W^{-1}J \subseteq W^{-1}(I_e(J)^{[p^r]}) = W^{-1}(W^{-1}I_e(J))^{[p^r]}$ where in the latter equality we have used Lemma 7.

\[\blacksquare\]

3. **The local case**

In this section we give an explicit formula for the HSL-numbers (cf. Definition 2) under some technical hypothesis.

For the moment, assume $(R, m)$ is complete and local and let $(-)\,^\vee$ denote the Matlis dual, i.e. the functor $\Hom_R(-, E_R)$, where $E_R = E_R(\mathbb{K})$ is the injective hull of the residue field $\mathbb{K}$ of $R$. We start by recalling the notions of $\Delta^e$-functor and $\Psi^e$-functor which have been defined in [K1 Section 3]; let $\mathcal{C}^e$ be the category of Artinian $R$-modules with Frobenius maps and $\mathcal{D}^e$ the category of $R$-linear maps $\alpha_M : M \to F_R^e(M)$ with $M$ a Noetherian $R$-module and where a morphism between
$M \xrightarrow{\alpha_M} F^e_R(M)$ and $N \xrightarrow{\alpha_N} F^e_R(N)$ is a commutative diagram of $R$-linear maps:

$$
\begin{array}{ccc}
M & \xrightarrow{h} & N \\
\downarrow{\alpha_M} & & \downarrow{\alpha_N} \\
F^e_R(M) & \xrightarrow{F^e_R(h)} & F^e_R(N).
\end{array}
$$

We define a functor $\Delta^e : \mathcal{C}^e \to \mathcal{D}^e$ as follows: given an $e^{th}$-Frobenius map $\Theta$ of the Artinian $R$-module $M$, we obtain an $R$-linear map $\phi : F^e_(R) \otimes_R M \to M$ which sends $F^e_r \otimes m$ to $r \Theta m$. Taking Matlis duals, we obtain the $R$-linear map

$$
M^\vee \to (F^e_(R) \otimes_R M)^\vee \cong F^e_(R) \otimes_R M^\vee
$$

where the last isomorphism is the functorial isomorphism described in [L, Lemma 4.1]. This construction can be reversed, yielding a functor $\Psi^e : \mathcal{D}^e \to \mathcal{C}^e$ such that $\Psi^e \circ \Delta^e$ and $\Delta^e \circ \Psi^e$ can naturally be identified with the identity functor. See [K1, Section 3] for the details of this construction.

From now on let $(R, m)$ be a complete, regular and local ring, $I$ an ideal of $R$ and write $S = R/I$. Let $d$ be the dimension of $S$ and suppose $S$ is Cohen-Macaulay with canonical module $\bar{\omega}$. For our purpose, we assume that $S$ is generically Gorenstein (i.e. each localisation of $S$ at a prime ideal is Gorenstein) so that $\bar{\omega} \subseteq S$ is an ideal of $S$, cf. [M1, Proposition 2.4]. In which case we can consider the following short exact sequence:

$$0 \to \bar{\omega} \to S \to S/\bar{\omega} \to 0$$

that induces the long exact sequence

$$
\cdots \to H_{mS}^{d-1}(S) \to H_{mS}^{d-1}(S/\bar{\omega}) \to H_{mS}^d(\bar{\omega}) \to H_{mS}^d(S) \to 0.
$$

Since $S$ is Cohen-Macaulay, the above reduces to

$$
(1) \quad 0 \to H_{mS}^{d-1}(S/\bar{\omega}) \to H_{mS}^d(\bar{\omega}) \to H_{mS}^d(S) \to 0.
$$

As we noticed in the introduction, a natural Frobenius map acting on $S$ induces a natural Frobenius map acting on $H_{mS}^d(S)$. The following result allows us to talk about the natural Frobenius map acting on $H_{mS}^d(S)$.

**Theorem 10** (cf. in Example 3.7 [Ly-Sm]). Let $\mathcal{F}_e := \mathcal{F}_e(H_{mS}^d(S))$ be the $R$-module consisting of all $e^{th}$-Frobenius maps acting on $\bar{E}_S$ which induce a Frobenius map on $H_{mS}^d(S)$. Then $\mathcal{F}_e$ is generated by one element and the generator corresponds, up to unit, to the natural Frobenius map.
Moreover, we can give an explicit description of the $R$-module $F_e$ and consequently of the natural Frobenius which generates it. Our strategy consists of checking that the exact sequence (1) is an exact sequence in $\mathcal{C}^e$ so that we can apply to it the $\Delta^e$-functor defined above.

Firstly, note that the natural Frobenius acting on $S$ induces (by restriction) the natural Frobenius map on $\bar{\omega}$. Hence we also have a natural Frobenius map $F$ on $H^d_{mS}(\bar{\omega})$ and the surjection $\alpha: H^d_{mS}(\bar{\omega}) \to H^d_{mS}(S)$ is compatible with the Frobenius maps (cf. [K1, Section 7]). A Frobenius map is also induced on $H^{d-1}_{mS}(S/\bar{\omega})$ as the restriction of the natural Frobenius map acting on $H^d_{mS}(\bar{\omega})$ to $H^{d-1}_{mS}(S/\bar{\omega})$ and we need to check that this restriction is an endomorphism so that we have that (1) is a short exact sequence in $\mathcal{C}^1$. Consider the following diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & H^{d-1}_{mS}(S/\bar{\omega}) & \rightarrow & H^d_{mS}(\bar{\omega}) & \rightarrow H^d_{mS}(S) & \rightarrow 0 \\
\downarrow & & F & & \alpha & & \downarrow T \\
0 & \rightarrow & H^{d-1}_{mS}(S/\bar{\omega}) & \rightarrow & H^d_{mS}(\bar{\omega}) & \rightarrow H^d_{mS}(S) & \rightarrow 0
\end{array}
\]

where $T$ is the natural Frobenius acting on $H^d_{mS}(S)$. To show that $F \text{Ker}(\alpha) \subseteq \text{Ker}(\alpha)$ we use that the surjection $\alpha: H^d_{mS}(\bar{\omega}) \to H^d_{mS}(S)$ is compatible with the Frobenius maps. Pick $a \in \text{Ker}(\alpha)$ then $F(a) \in \text{Ker}(\alpha)$ because $\alpha(F(a)) = T(\alpha(a)) = T(0) = 0$.

Once we have fixed an isomorphism between $\text{Ann}_{E_R}(I)$ and $H^d_{mS}(\bar{\omega})$ we can then consider a Frobenius map induced on $\text{Ann}_{E_R}(I)$. By [K1, Prop. 4.1] all Frobenius actions on $\text{Ann}_{E_R}(I)$ are given by the restriction of $uF: E_R \to E_R$ where $u \in (I^{[p]} : I)$ and $F$ is the natural Frobenius map acting on $E_R$. Consequently the $e$th-iterated Frobenius map acting on $\text{Ann}_{E_R}(I)$ is of the form

\[(uF)^e = uF \circ \cdots \circ uF = u^{\nu_e} F^e \]

where $\nu_e = 1 + p + \cdots + p^{e-1}$ when $e > 0$ and $\nu_0 = 0$. We can then apply the $\Delta^e$-functor to (1) but before we do that we write such a short exact sequence in a useful way for our computations.

First note that identifying $H^d_{mS}(\bar{\omega})$ with $\text{Ann}_{E_R}(I)$ we obtain that $H^{d-1}_{mS}(S/\bar{\omega})$ must be of the form $\text{Ann}_{E_S}(J)$ for a certain ideal $J \subseteq R$. More precisely we have the following:

**Theorem 11.** $H^{d-1}_{mS}(S/\bar{\omega})$ and $\text{Ann}_{E_S}(\bar{\omega})$ are isomorphic.
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Proof. If \( \omega \) is the preimage of \( \bar{\omega} \) in \( R \) then \( H_{\text{m}S}^d(S/\bar{\omega}) \cong H_{\text{m}S}^d(R/\omega) \). Since \( H_{\text{m}S}^d(R/\omega) \) is an \( R \)-submodule of \( E_S = \text{Ann}_{E_R} I \subseteq E_R \) then \( H_{\text{m}S}^d(R/\omega) = \text{Ann}_{E_R}(0: R H_{\text{m}S}^d(R/\omega)) \). The fact that \( (0: H_{\text{m}S}^d(R/\omega)) = \omega \) follows from [LL, Theorem 2.17] replacing \( R \) with \( R/\omega \) and \( (0) \) with \( \omega \) and using the fact that \( \omega \) is unmixed. □

Hence we can rewrite (1) as

(2) \[
0 \to \text{Ann}_{E_S}(\bar{\omega}) \to \text{Ann}_{E_R}(I) \to H_{\text{m}S}^d(S) \to 0
\]

and therefore we have \( H_{\text{m}S}^d(S) \cong \text{Ann}_{E_R}(I)/\text{Ann}_{E_S}(\bar{\omega}) \). An application of the \( \Delta e \)-functor to the latter short exact sequence yields the short exact sequence in \( D^e \):

\[
\begin{array}{cccccc}
0 & \to & \bar{\omega}/I & \to & R/I & \to R/\omega & \to 0 \\
& & \downarrow & & \downarrow & & \\
0 & \to & \bar{\omega}/[\nu^e]I & \to & R/[\nu^e]I & \to R/[\nu^e]\omega & \to 0 \\
\end{array}
\]

where the central vertical map is the multiplication by \( u^{\nu e} \). Note that we have used the fact that \( R \) is regular and so \( F_R(\cdot) \) is exact [B-H, Corollary 8.2.8]. In order to make the latter a commutative diagram the only possibility is that the other two vertical maps are also the multiplication by \( u^{\nu e} \). Moreover, such a diagram is well defined only if \( u \in (I/[\nu^e]: I) \cap (\omega/[\nu^e]: \omega) \) and since \( \bar{\omega} \) contains a nonzerodivisor then the kernel of the surjective map \( (I/[\nu^e]: I) \cap (\omega/[\nu^e]: \omega) \to F_e(H_{\text{m}S}^d(S)) \) which sends \( u \mapsto u^{\nu e} \nu^e \) is \( I/[\nu^e] \). Hence we have the following:

**Theorem 12.** The \( R \)-module consting of all \( e \)-th Frobenius maps acting on \( E_S \) which induce a Frobenius map on \( H_{\text{m}S}^d(S) \) is of the form

\[
\mathcal{F}_e = \frac{(I/[\nu^e]: I) \cap (\omega/[\nu^e]: \omega)}{I/[\nu^e]}
\]

where \( \omega \) is the preimage of \( \bar{\omega} \) in \( R \).

We prove now the main result of this section:

**Theorem 13.** \( \text{HSL}(H_{\text{m}S}^d(S)) \) is the smallest integer \( e \) for which

\[
\frac{I_e(u^{\nu e} \omega)}{I_{e+1}(u^{\nu e+1} \omega)} = 0
\]

where \( \omega \) is the preimage of \( \bar{\omega} \) in \( R \) and \( \nu_e = 1 + p + \cdots + p^{e-1} \) when \( e > 0 \) and \( \nu_0 = 0 \).

Proof. Let \( T \) be the natural Frobenius map acting on \( H_{\text{m}S}^d(S) \) which is induced by the natural Frobenius map acting on \( E_S \) through the surjection \( E_S \to H_{\text{m}S}^d(S) \); by Theorem 12 we know that any Frobenius
map acting on $H^d_{mS}(S)$ is of the form $\Theta = uT$ for some $u \in (I^p : I) \cap (\omega^p : \omega)$. For all $e \geq 0$ define $M_e = \{x \in H^d_{mS}(S) | T^e(x) = 0\}$ which yields a chain of inclusions of $M_e$ that stabilises by Theorem 1. Moreover, each $M_e$ is a submodule of $H^d_{mS}(S)$ and therefore it is of the form $\text{Ann}_{ES} L_e \bar{\omega}$ for some $L_e \subseteq R$ contained in $I$. Our goal is to find $L_e$ for all $e$ in such a way that $\frac{\text{Ann}_{ES} L_e}{\text{Ann}_{ES} \bar{\omega}}$ is the biggest submodule of $\frac{\text{Ann}_{ES} I}{\text{Ann}_{ES} \bar{\omega}}$ such that we have

$$\Theta^e \left( \frac{\text{Ann}_{ES} L_e}{\text{Ann}_{ES} \bar{\omega}} \right) \subseteq \frac{\text{Ann}_{ES} I}{\text{Ann}_{ES} \bar{\omega}}.$$ 

Applying $\Delta^e$ to the inclusion $\frac{\text{Ann}_{ES} L_e}{\text{Ann}_{ES} \bar{\omega}} \hookrightarrow \frac{\text{Ann}_{ES} I}{\text{Ann}_{ES} \bar{\omega}}$ we get

$$\omega/I \xrightarrow{\omega/L_e} \omega/L_e \xrightarrow{\omega^p/I^p} \omega^p/L_e^p \xrightarrow{\omega^p/I^p} \omega^p/L_e^p$$

where the map $\omega/L_e \rightarrow \omega^p/L_e^p$ must be the multiplication by $u^e$ because of the surjectivity of the horizontal maps; note that such a map is well defined because $u^e \omega \subseteq \omega^p$, and then $L_e \subseteq \omega$. Moreover $\omega/L_e \rightarrow \omega^p/L_e^p$ must be the zero-map by construction. Hence, $u^e \omega \subseteq L_e^p$, and in order to have the inclusion (3), $L_e$ must be the smallest ideal such that its $e$th-Frobenius power contains $u^e \omega$ so it must be that $L_e = I_e(u^e \omega)$. □

**Corollary 14.** $S$ is $F$-injective if and only if $\omega = I_1(u\omega)$.

**Proof.** $S$ is $F$-injective if and only if the index of nilpotency is zero i.e. if and only if $\omega = I_1(u\omega)$. □

### 4. The non-local case

In this section let $A$ be a polynomial ring $K[x_1, \ldots, x_n]$ with coefficients in a perfect field of positive characteristic $p$ and $J \subseteq A$ an ideal of $A$. Let $B$ be the quotient ring $A/J$; if $B$ is Cohen-Macaulay of dimension $d$ then $\Omega = \text{Ext}_A^{\dim A-d}(B, A)$ is a global canonical module for $B$. In Proposition 16 we show that if $B$ is a domain then $\Omega$ can be assumed to be an ideal of $B$.

**Lemma 15.** Let $B$ be a non-local Cohen-Macaulay domain and let $\bar{\Omega}$ be a global canonical module for $B$. Then every element $x \neq 0$ in $B$ is a nonzerodivisor on $\bar{\Omega}$.
Proof. The sequence \( \ker(x) \rightarrow \bar{\Omega} \xrightarrow{\cdot x} \bar{\Omega} \), where \( \bar{\Omega} \xrightarrow{\cdot x} \bar{\Omega} \) is the multiplication by \( x \in B/\{0\} \), induces the following exact sequence for each prime ideal \( p \subset B \):

\[
0 \rightarrow (\ker(x))_p \rightarrow \bar{\Omega}_p \xrightarrow{\cdot x} \bar{\Omega}_p.
\]

By [Mi, Lemma 2.2] if \( x_1 \) is a nonzerodivisor on \( B_p \) then \( x_1 \) is a nonzerodivisor on \( \bar{\Omega}_p \). Since \( \bar{\Omega}_p \) is a maximal Cohen-Macaulay module then \( \ker(x)_p = 0 \) for each \( p \) and consequently by [Ei, Lemma 2.8] \( \ker(x) = 0 \).

Proposition 16. Assume that \( B \) is a non-local Cohen-Macaulay domain and let \( \bar{\Omega} \) be a global canonical module for \( B \). Then \( \bar{\Omega} \) is isomorphic to an ideal of \( B \).

Proof. If \( W = B/\{0\} \) then \( W^{-1}B \) is a field hence Gorenstein and consequently \( W^{-1}B \cong W^{-1}\Omega \). Let \( \eta: W^{-1}\bar{\Omega} \rightarrow W^{-1}B \) an isomorphism and let \( \bar{\eta}: \bar{\Omega} \rightarrow W^{-1}B \) be the restriction of \( \eta \) to \( \bar{\Omega} \). We need to show that \( \bar{\eta} \) is injective. If \( x \in \ker(\bar{\eta}) \) then \( \bar{x} \sim 0 \) which happens if and only if there exists \( y \in W \) such that \( \bar{x}y = 0 \); by Lemma 15 it must be that \( x = 0 \). Therefore \( \bar{\eta} \) is injective. Let \( \frac{a_1}{w_1}, \ldots, \frac{a_d}{w_d} \) be the image of the generators of \( \bar{\Omega} \) in \( W^{-1}B \); if \( w = w_1 \cdots w_d \) then the composition map \( w\bar{\eta}: \bar{\Omega} \rightarrow B \) is injective and hence \( \bar{\Omega} \) is isomorphic to an ideal of \( B \). \( \square \)

From now on suppose \( B \) is a domain; so we can assume \( \bar{\Omega} \subseteq B \) and the following quantity is well defined:

\[
U_e = \frac{(J^{[p^e]} : J) \cap (\Omega^{[p^e]} : \Omega)}{J^{[p^e]}}
\]

where \( \Omega \) is the preimage of \( \bar{\Omega} \) in \( A \). Note that \( U_e \) is a finitely generated \( A \)-module since \( A \) is Noetherian.

Now, for any prime ideal \( p \subset A \), the \( A \)-module consting of the Frobenius maps on \( H^{\dim B_p}_{p\hat{B}_p}(\hat{B}_p) \) is of the form:

\[
\frac{(J^{[p^e]} \hat{A}_p : J \hat{A}_p) \cap (\Omega^{[p^e]} \hat{A}_p : \Omega \hat{A}_p)}{J^{[p^e]} \hat{A}_p}.
\]

Moreover

\[
\frac{(J^{[p^e]} \hat{A}_p : J \hat{A}_p) \cap (\Omega^{[p^e]} \hat{A}_p : \Omega \hat{A}_p)}{J^{[p^e]} \hat{A}_p} = U_e \hat{A}_p
\]

and \( F_e \cong U_e \hat{A}_p \); consequently \( U_e \hat{A}_p \) is generated by one element by Theorem 10.
We prove now the following result for a generic finitely generated $A$-module.

**Theorem 17.** Let $M$ be a finitely generated $A$-module and let $g_1, \cdots, g_s$ be a set of generators for $M$. If $M$ is locally principal then for each $i = 1, \cdots, n$

$$G_i = \{ p \in \text{Spec}(A) \mid M \hat{A}_p \text{ is generated by the image of } g_i \}$$

is a Zariski open set and $\bigcup_i G_i = \text{Spec}(A)$.

Before we proceed with the proof of Theorem 17 we need the following lemma. Let $M$ be an $A$-module generated by $g_1, \cdots, g_s$. Let $e_1, \cdots, e_s$ be the canonical basis for $A^s$ and define the map

$$A^s \xrightarrow{\varphi} M$$

$$e_i \mapsto g_i.$$ 

Such a surjective map extends naturally to an $A$-linear map $J : A^t \to A^s$ with $\ker \varphi = \text{Im} J$. Let $J_i$ be the matrix obtained from $J \in \text{Mat}_{s,t}(A)$ by erasing the $i^{th}$-row. With this notation we have the following:

**Lemma 18.** $M$ is generated by $g_i$ if and only if $\text{Im} J_i = A^{s-1}$.

**Proof.** Firstly suppose $\text{Im} J_i = A^{s-1}$. We can add to $J$, columns of $\text{Im} J$ without changing its image so we can assume that $J$ looks like

$$J = \begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} & 1 & 0 & \cdots & 0 \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_1 & b_2 & \cdots & b_s \\
\vdots & \vdots & \ddots & \vdots & 0 & \cdots & 1 \\
a_{m,1} & a_{m,2} & \cdots & a_{m,n} & 0 & \cdots & 0 & 1
\end{pmatrix}$$

where $a_{k,i}, b_j \in A$. In this way for every $j = 1, \cdots, s$ we have $g_j - b_j g_i = 0$ i.e. $g_i$ generates $M$. Viceversa if $M$ is generated by $g_i$ then for all $j \neq i$ we can write $g_j = r_j g_i$ i.e. $g_j - r_j g_i = 0$ and this happens if and only if $e_j - r_j e_i = 0$ if and only if $e_j - r_j e_i \in \ker \varphi = \text{Im} J$. Hence we can assume $J$ to contain the vector whose entries are all zeros but in the $i$-th and $j$-th positions where there is 1 and $r_j$ respectively. Consequently $J_i$ contains the $(s - 1) \times (s - 1)$ identity matrix. $\square$

Let $M$ be a $A$-module generated by $g_1, \cdots, g_s$ and let $W$ be any multiplicatively closed subset of $A$. Localise the exact sequence $A^t \to A^s \to M \to 0$ with respect to $W$ obtaining the exact sequence $W^{-1}A^t \to W^{-1}A^s \to W^{-1}M \to 0$. With this notation we have:

**Proposition 19.** $W^{-1}M$ is generated by $g_i$ if and only if $W^{-1}J_i = (W^{-1}A)^{s-1}$
Proof. Apply Lemma 18 to the localised sequence $W^{-1}A^t \to W^{-1}A^s \to W^{-1}M \to 0$. □

The proof of Theorem 17 follows immediately since the above Proposition is equivalent to saying that the intersection of $W$ with the ideal of $(s - 1) \times (s - 1)$ minors of $J_i$ is not trivial.

Proof of Theorem 17. $p \in \mathcal{G}_i$ if and only if $p \not\supseteq J_i$. □

From Theorem 17 it follows that if we choose $M = U(e)$ then for every prime ideal $p \in \text{Spec}(A) = \bigcup_i \mathcal{G}_i$ there exists an $i$ such that $p \in \mathcal{G}_i$ and the $A$-module $U(e)\hat{A}_p$ is generated by one element which is precisely the image of $g_i$. Hence, once we have localised and completed $B$ with respect to any prime ideal of $A$ we can use the local theory developed so far. In particular, from Corollary 14 it follows:

Theorem 20. For every prime ideal $p \subset A$, $\hat{B}_p$ is $F$-injective if and only if $I_1(u\Omega_p) = \Omega_p$.

Remark 21. Since $u \in (\Omega_p^{[p^r]} : \Omega_p)$ then $u\Omega_p \subseteq \Omega_p^{[p^r]}$ and consequently $I_1(u\Omega_p) \subseteq I_1(\Omega_p^{[p^r]}) = \Omega_p$.

Now we prove our main result.

Theorem 22.

$\mathcal{B}_e = \left\{ p \in \text{Spec}(A) \mid \text{HSL} \left( H^{d(p)}_{p\hat{B}_p}(\hat{B}_p) \right) < e \right\}$

is a Zariski open set.

Proof. For each $p \in \mathcal{G}_i$ consider

$$\frac{u^{r_e}}{1} T_{(p)} : H^{d(p)}_{p\hat{B}_p}(\hat{B}_p) \to H^{d(p)}_{p\hat{B}_p}(\hat{B}_p)$$
where \( T^e_p \) is the natural Frobenius acting on \( H^{d(p)}_{pB_p}(\hat{B}_p) \) and \( d(p) = \dim \hat{B}_p \). Then for all \( i = 1, \ldots, n \):

\[
\mathcal{B}_e \cap \mathcal{G}_i = \left\{ p \in \mathcal{G}_i \mid \text{HSL} \left( H^{d(p)}_{pB_p}(\hat{B}_p) \right) < e \right\} = \\
\left\{ p \in \mathcal{G}_i \mid \frac{I_e(u^{\nu_e}(\Omega))}{I_{e+1}(u^{\nu_{e+1}}(\Omega))} \neq 0 \right\} = \\
\left\{ p \in \mathcal{G}_i \mid \frac{I_e(u^{\nu_e}(\Omega)) \otimes \hat{B}_p}{I_{e+1}(u^{\nu_{e+1}}(\Omega)) \otimes \hat{B}_p} \neq 0 \right\} = \\
\left\{ p \in \mathcal{G}_i \mid \frac{I_e(u^{\nu_e}(\Omega))}{I_{e+1}(u^{\nu_{e+1}}(\Omega))} \otimes \hat{B}_p \neq 0 \right\} = \\
\text{Supp} \left( \frac{I_e(u^{\nu_e}(\Omega))}{I_{e+1}(u^{\nu_{e+1}}(\Omega))} \right) \cap \mathcal{G}_i.
\]

The latter is a finite set therefore \( \mathcal{B}_e \cap \mathcal{G}_i \) is Zariski open for all \( i \). Since \( \bigcup_i \mathcal{G}_i = \text{Spec}(A) \) then \( \mathcal{B}_e \) is Zariski open. \( \square \)

**Corollary 23.** The index of nilpotency is bounded.

**Corollary 24.** The \( F \)-injective locus of the top local cohomology of a quotient of a polynomial ring is open.

**Proof.** It is the special case of Theorem 22 when \( e = 0 \). \( \square \)

Note that this argument can be implemented as an algorithm which takes \( B = A/J \) as input and produces a set of ideals \( K_1, \ldots, K_t \) such that, for each \( i = 1, \ldots, t \), \( \mathbb{V}(K_i) \) consists of a prime for which the index of nilpotency is greater than \( i \) and \( \text{Spec}(B) = \bigcup \mathbb{V}(K_i) \).

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**References**

[B-M-S] M. Blickle, M. Mustata, K. E. Smith, *Discreteness and rationality of \( F \)-thresholds*, Michigan Math. J. **57** (2008), pp.43–61.

[B-H] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press (1993).

[B-S] M. P. Brodmann, R. Y. Sharp, *Local Cohomology*. Cambridge Studies in Advanced Mathematics **60**, Cambridge University Press (1998).

[Ei] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*. Springer **150** (1991).
[H-S] R. Hartshorne, R. Speiser, Local cohomology dimension in characteristic \( p \). Ann. of Math. 105 (1977), pp.45–79.

[H] M. Hashimoto, \( F \)-pure homomorphisms, strong \( F \)-regularity, and \( F \)-injectivity, (2010), eprint arXiv:0908.2703.

[Hoch] M. Hochster, Local Cohomology. Notes for the module Mathematics 615, (Winter 2011).

[K1] M. Katzman, Parameter test ideals of Cohen Macaulay rings. Compos. Math. 144 (2008), 933–948.

[L] G. Lyubeznik, \( F \)-modules: applications to local cohomology and \( D \)-modules in characteristic \( p > 0 \). J.f.d. reine u. angew. Math. 491 (1997), 65–130. MR 99c:13005.

[LL] L. Lynch, Annihilators of local cohomology modules (2011). Dissertations, Theses, and Student Research Papers in Mathematics.

[Ly-Smith] G. Lyubeznik, K. E. Smith, On the commutation of the test ideal with localization and completion, Transactions of the AMS 353 (2001), no. 8, 3149–3180.

[M1] Linquan Ma Finiteness property of local cohomology for \( F \)-pure local rings, (2012) arXiv:1204.1539 [math.AC]

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