Shear Waves, Sound Waves

On

A Shimmering Horizon

Omid Saremi

Ernest Rutherford Physic Building
McGill University
Montreal QC
Canada H3A 2T8

Abstract

In the context of the so called “membrane paradigm” of black holes/branes, it has been known for sometime that the dynamics of small fluctuations on the stretched horizon can be viewed as corresponding to diffusion of a conserved charge in simple fluids. To study shear waves in this context properly, one must define a conserved stress tensor living on the stretched horizon. Then one is required to show that such a stress tensor satisfies the corresponding constitutive relations. These steps are missing in a previous treatment of the shear perturbations by Kovtun, Starinets and Son. In this note, we fill the gap by prescribing the stress tensor on the stretched horizon to be the Brown and York (or Balasubramanian-Kraus (BK) in the AdS/CFT context) holographic stress tensor. We are then able to show that such a conserved stress tensor satisfies the required constitutive relation on the stretched horizon using Einstein equations. We read off the shear viscosity from the constitutive relations in two different channels, shear and sound. We find an expression for the shear viscosity in both channels which are equal, as expected. Our expression is in agreement with a previous membrane paradigm formula reported by Kovtun, Starinets and Son.

March 20th 2007.
1 Introduction

It has been known for a long time that numerous properties of black holes can be reproduced by assuming the existence of a “dynamical membrane” sitting just outside and in the immediate vicinity of the actual event horizon. In order for the membrane paradigm to work, the membrane must be endowed with certain mechanical, electrodynamic and thermodynamical properties [1]. It was uncovered that in this membrane picture, the fluid living on the membrane acts as a viscous medium held at temperature $T$, the black hole’s Hawking temperature. For a Schwarzschild black hole, the ratio of the shear viscosity of the membrane fluid to the entropy density (entropy over the area of the event horizon ratio) is equal to $\hbar/(4\pi)$ [1]. Although the membrane paradigm, at first glance appears to be a realization of the holographic principle [2], it nevertheless does not yield one with a concrete holographic recipe for mapping distinct theories into each other. A much better understood picture is the celebrated AdS/CFT, where there exists a prescription for how to access to the information carried by the dual field theory correlators via a gravitational dual. It is only in this new context that the membrane fluid could acquire a physical interpretation as a finite temperature dual field theory plasma in its hydrodynamic limit. Following a proposal for calculating Lorentzian signature correlators in AdS/CFT [3], it became feasible to compute various transport coefficients including shear viscosity in the hot dual field theory plasma [4], [5]. A general formula, based on the membrane paradigm ideas, for the shear viscosity associated with a given gravitational background was derived in [6]. The formula was derived through mapping the shear wave propagation to a charge diffusion problem [6]. Utilizing the membrane paradigm formula, the shear viscosity (and the ratio of the shear viscosity to entropy density) was computed for various black-branes in type II string theories as well as membranes and M5-branes in M-Theory [6]. In all cases value for the shear viscosity computed by the membrane paradigm formula agrees with the AdS/CFT prediction. The ratio of the shear viscosity to entropy density computed using the membrane paradigm formula was found to be $\hbar/(4\pi)$, in agreement with AdS/CFT results.

Motivated by these observations, Kovtun, Son and Starinets proposed that the ratio of shear viscosity to entropy density is bounded from below by $\hbar/(4\pi)$ for all forms of matter [6] [6]. In all cases where the dual field theory is infinitely strongly coupled, the bound was

*In it has been argued that the bound may be violated for metastable fluids [7].
discovered to saturate. A no-go theorem was proved in [8]. The no-go theorem implies the saturation of the bound for a large class of supergravity backgrounds. More interesting set ups with R-charge background turned on were studied in [9]. Although the set-up didn’t fulfill the conditions of the no-go theorem, nevertheless the bound was found to saturate. An extended version of the no-go theorem which included the cases with an R-charge background was subsequently proved in [10].

In the context of the membrane paradigm, it was found [6] that small fluctuations of the stretched horizon have properties which can be viewed as corresponding to diffusion of the conserved charge in simple fluids. Shear perturbations were treated indirectly, by mapping the shear perturbation diffusion problem into a charge diffusion problem.

As was mentioned in [6], to study shear waves properly, one needs to define a conserved stress tensor living on the stretched horizon. Then one must show that such a stress tensor satisfies the constitutive relations using Einstein equations. This step is missing in the analysis of [6].

In this paper, we fill the gap by prescribing the stress tensor for the stretched horizon to be the Balasubramanian-Kraus (BK) holographic stress tensor [11] (which is the Brown and York prescription [12] for the stress tensor used in the context of AdS/CFT). We are then able to show that such conserved stress tensor satisfies the required constitutive relation on the stretched horizon using Einstein equations. We read off the shear viscosity from the constitutive relations in two different channels: sound and shear. We find an expression for the shear viscosity in each channel (which are equal as expected). Our expression is in agreement with the general membrane paradigm formula for the shear viscosity reported in [6].

In section 2, the BK holographic stress tensor prescription is reviewed. The constitutive relations are the subject of the section 3. We continue with description of general properties of the background and its symmetries in section 4. Section 5 is where we write down our results in detail. The constitutive relations on the stretched horizon are studied in two different channels; sound and shear. We conclude with a discussion in section 6.

2 Balasubramanian And Kraus Holographic Stress Tensor Prescription

Defining tensorial observables measuring “local” gravitational energy and momentum density in general relativity is problematic. Assigning a non-vanishing local energy and mo-
mentum density to a gravitational system is impossible as one can always switch to a “local” free falling frame where all the first derivatives of the metric are zero and spacetime is locally flat. However, there have been attempts to associate a “quasi-local” stress tensor to a given gravitational system. This definition, due to Brown and York [12] uses the conventional notion of Hamiltonian in particle mechanics. In the Hamilton-Jacobi formalism, the action functional is a function of the proper time elapsed between the initial and final configurations. The Hamilton-Jacobi equation implies \( H = -\partial S_{cl}/\partial T \) where \( T \) is the proper time between the initial and final hypersurfaces. Suppose \( M \) is a \( D \)-dimensional spacetime with topology \( M^{D-1} \times R \). Take \( \partial M \) to denote the \((D-1)\)-dimensional boundary of \( M \). Let \( \Sigma_t \) be a family of spacelike hypersurfaces foliating \( M \). The spacelike part of the boundary is denoted by \( D^{−2}M \). Take \( n^\mu \) to be the outward spacelike normal vector to the boundary and \( U^\mu \) to be the future directed timelike vector orthonormal to the spacelike section of the boundary i.e., \( D^{−2}M \), such that \( n^\mu U_\mu = 0 \). The induced metric on \( \partial M \) is represented by \( \gamma_{\mu\nu} \). The embedding of the boundary in the \( D \)-dimensional spacetime is characterized by its extrinsic curvature, defined as

\[
\Theta_{\mu\nu} = -\frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu).
\]

(1)

The action for general relativity coupled to matter, evaluated on \( M \) (a solution to the equations of motion), will be a function of the metric induced on the boundary. This metric plays the role of proper time elapsed between the initial and final hypersurfaces in its particle mechanics analogue. The Hamilton-Jacobi equation then implies \( H = -\partial S_{cl}/\partial T \) where \( T \) is the proper time between the initial and final hypersurfaces. In this way one ends up with a Hamiltonian.

**Fig. 1** – Manifold \( M \) with \( n^\mu \) being the spacelike normal vector to the boundary. The timelike vector \( U \) is orthogonal to the spatial part of the boundary i.e., \( D^{−2}M \).
Note that the quasi-local energy defined in this way is equal to the Hamiltonian and generates time translations. Recall that the metric on the boundary not only measures the proper time elapsed between the two initial and final configurations, but also calculates the spatial separation between two given events on the boundary. Therefore the above procedure yields an energy-momentum tensor, rather than just a Hamiltonian. For general relativity coupled to matter consider the following action

$$S = \frac{1}{2\kappa^2} \int_M \mathcal{R} + \frac{1}{\kappa^2} \int_{\mathcal{C}} d^{D-1}x \sqrt{hK} - \frac{1}{\kappa^2} \int_{\partial M} d^{D-1}x \sqrt{-\gamma} \Theta + S_{\text{matter}},$$

(2)

where $\kappa^2 = 8\pi G_N$ and $h_{\mu\nu}$ is the induced metric on $\Sigma_t$. Variation with respect to the metric and matter degrees of freedom gives rise to

$$\delta S = \text{bulk terms} + \int_{\mathcal{C}} d^{D-1}x \ P^{\mu\nu} \delta h_{\mu\nu} + \int_{\partial M} d^{D-1}x \ \pi^{\mu\nu} \delta \gamma_{\mu\nu},$$

(3)

where $P^{\mu\nu}$ denotes the gravitational momentum conjugate to $h_{\mu\nu}$ whereas, $\pi^{\mu\nu}$ is the gravitational momentum conjugate to $\gamma_{\mu\nu}$. The gravitational momenta associated with $\gamma_{\mu\nu}$ is expressed in terms of the extrinsic curvature of the boundary as follows

$$\pi^{\mu\nu} = \frac{1}{8\pi G} \sqrt{-\gamma} (\Theta^{\mu\nu} - \gamma^{\mu\nu} \Theta),$$

(4)

where $\Theta = \Theta^\mu_\mu$. Following the analogy with the Hamilton-Jacobi approach, the stress energy tensor is identified as

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{cl}}}{\delta \gamma_{\mu\nu}},$$

(5)

which is

$$T_{\mu\nu} = \frac{1}{8\pi G} \left[ -\frac{1}{2} (\nabla_\mu n_\nu + \nabla_\nu n_\mu) + \gamma_{\mu\nu} \nabla_\rho n^\rho \right].$$

(6)

Note that for minimally coupled matter to gravity (no derivatives of the metric in the matter sector) the gravitational momenta are independent of matter degrees of freedom. This implies that the stress tensor is the total (quasi-local\(^\dagger\)) energy and momentum density associated with both “matter” and “gravitational” degrees of freedom in a region of spacetime bounded by $\partial M$.

An implementation of this definition in the context of AdS/CFT was consider by Balasubramanian and Kraus (BK holographic stress tensor) \(^\dagger\). In this way the authors were able to compute the expectation value of the stress tensor in the field theory dual to the corresponding asymptotically AdS spaces in diverse dimensions.

\(^\dagger\)Since it is defined on the boundary of $M$ rather than just a point in spacetime.
2.1 Conservation Of Energy and Momentum For The BK Stress Tensor

A natural question would be to inquire how conservation of energy and momentum is implemented in the context of the work of Brown and York. Although briefly flashed on in [12], we repeat the argument here in more detail and with complimentary comments. To answer the above posed question one needs to consider the Gauss-Codazzi equation corresponding to an ADM decomposition in the radial direction
\[ \nabla_\nu \Theta^\nu_{\mu} - \nabla_\mu \Theta^\nu_{\nu} = R_{\rho\sigma} n^\sigma \gamma^\rho_{\mu}, \] (7)
where \( n^\mu \) is a hypersurface orthonormal vector. Now note that in the absence of matter, the initial value constraints
\[ \nabla_\nu \Theta^\nu_{\mu} - \nabla_\mu \Theta^\nu_{\nu} = R_{\rho\sigma} n^\sigma \gamma^\rho_{\mu} = G_{\rho\sigma} n^\sigma \gamma^\rho_{\mu} = 0 \] (8)
are equal to conservation for energy momentum for the BK stress tensor. In the presence of matter, the right hand side of (8) gets modified. Using the Einstein equations
\[ \nabla^\mu (\Theta_{\mu\nu} - \gamma_{\mu\nu} \Theta) = -8\pi G T_{\nu\sigma} n^\sigma = -8\pi G J_\nu, \] (9)
which is again the conservation law for the BK stress tensor defined as \( T_{\mu\nu} = (\Theta_{\mu\nu} - \gamma_{\mu\nu} \Theta)/(8\pi G) \) in the presence of a matter source term \( J_\nu \).

3 Constitutive Relations For The Stress Tensor

Hydrodynamics is an effective classical field theory describing degrees of freedom relevant to the long range (compared to any other scale in the theory) dynamics of a system. A collection of energy-momentum conservation law and the constitutive relations is what we call a hydrodynamic description of the corresponding system.

Using spacetime transformation properties of the stress tensor and general arguments as given in [13], the spatial components of the stress tensor \( T^{ij} \) (which are present near the equilibrium state), with at most one derivative in the spatial coordinates and linearized to the first order, are written (in 3+1 dimensions) as the following
\[ T^{ij} = g^{ij}(P + v_s^2 \delta \epsilon) - \gamma_\zeta g^{ij} \nabla \cdot \pi - \gamma_\eta (\nabla^i \pi^j + \nabla^j \pi^i) - \frac{2}{3} g^{ij} \nabla \cdot \pi, \] (10)
where \( g_{ij} \) is the spacetime metric and \( \nabla_i \) is the covariant derivative compatible with the metric \( g_{ij} \). \( v_s \) is the speed of sound, \( P \) is the equilibrium pressure and \( \pi^i = T^{0i} \). \( \delta \epsilon \) is the
energy density perturbation around the equilibrium state, where \( \langle T^{00} \rangle = \epsilon \) is the equilibrium energy density. The transport coefficients \( \gamma_\zeta \) and \( \gamma_\eta \) are defined as follows

\[
\gamma_\zeta = \frac{\zeta}{\epsilon + P}, \\
\gamma_\eta = \frac{\eta}{\epsilon + P},
\]

where \( \eta \) is the shear viscosity and \( \zeta \) is the bulk viscosity. Terms linear in perturbations around equilibrium do not exist due to the charge conjugation symmetry of the reference equilibrium state. Higher derivative terms are ignored as only the longest space and time scales will be focused on. Note that any other term consistent with spacetime symmetries besides the ones included in (10) either involve higher powers of fluctuations or else more derivatives (which are ignored as stated).

4 Background And Notations

In this paper we work with a General Relativity in \((p+2)\)-dimensions coupled to a matter sector. The background we consider here possesses \( p + 1 \) Killing vectors. In a suitable coordinate system the Killing directions are represented as \( \partial_{x_\mu} \), where \( \mu = 0 \ldots p \) and “0” refers to the timelike direction. The radial coordinate is denoted by \( r \). All the functions describing the background only depend on \( r \). For simplicity we only consider the \( p = 3 \) case. The generalization to arbitrary \( p \) is straightforward. The prime on the functions will refer to \( \partial_r \).

The following two different ADM decompositions of the general background are used

\[
\begin{align*}
    ds^2 &= -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \\
    ds^2 &= N_{(i)}^2 dr^2 + \gamma_{\mu\nu}(dx^\mu + N^{\mu}_{(r)} dr)(dx^\nu + N^{\nu}_{(r)} dr).
\end{align*}
\]

If \( n^\mu \) is the unit outward spacelike vector orthogonal to the boundary, the metric induced on the boundary i.e., \( \gamma_{\mu\nu} \) can be expressed as

\[
\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu.
\]

All the equations will be linearized in the perturbations. It is fruitful to note that at the linearized level \( h^{xy} = 0 \). In the near horizon limit similar to [6], we will assume the following
expansions for the metric components

\[ g_{00}(r) = \gamma_0 (r - r_0) + \mathcal{O}((r - r_0)^2), \quad (14) \]
\[ g_{rr}(r) = \frac{\gamma_r}{r - r_0} + \mathcal{O}((r - r_0)^0). \]

The Einstein equation in \( D \)-dimensions is written as follows

\[ R_{\mu \nu} = 8\pi G_D \left( T_{\mu \nu} - \frac{2}{D - 2} \frac{\delta^\mu_{\nu}}{T} \right), \quad (15) \]

where \( T_{\mu \nu} \) is the matter stress tensor. For the purpose of illustration, consider the following multi-scalar field stress tensor

\[ T_{\mu \nu} = \sum_i \partial^\mu \Phi_i \partial_\nu \Phi_i - \delta_{\mu \nu} \mathcal{L}(\Phi_i), \quad (16) \]

where \( \mathcal{L} \) is the matter Lagrangian. Our later arguments about the stress tensor will turn out to be general and must hold in more general situations with matter content of different type. The perturbed Einstein equation reads

\[ \delta R_{\mu \nu} = 8\pi G_D \left( \delta T_{\mu \nu} - \frac{2}{D - 2} \delta^\mu_{\nu} \delta T \right), \quad (17) \]

where, using (16), one can write

\[ \delta T_{\mu \nu} = \sum_i \partial^\mu \Phi_i \partial_\nu \Phi_i + \sum_i \partial^\mu \Phi_0 i \partial_\nu \delta \Phi_i - \delta_{\mu \nu} \delta \mathcal{L}. \quad (18) \]

where \( \Phi_0 i = \Phi_0 i (r) \) is the background profile of the field \( \Phi_i \).

5 Constitutive Relations On The Stretched Horizon

5.1 Sound Channel

In the sound channel, the perturbations \( h_{tt}, h_{x1x2}, h_{x3x3}, h_{tx3} \) are turned on.

The spacetime coordinates are ordered as \((t, x_1, x_2, x_3, r)\). These perturbations are assumed to have the following space and time dependence

\[ \zeta(t, x_3, r) = \zeta(r) e^{-i\Omega t + iq x_3}, \quad (19) \]

where \( \zeta \) stands for a typical sound perturbation. The full perturbed background is then written as

\[ ds^2 = (-c_0(r)^2 + h_{tt})dt^2 + 2h_{tx3}dtdx_3 \]
\[ + (c_x(r)^2 + h_{x1x1})dx_1^2 + (c_x(r)^2 + h_{x2x2})dx_2^2 + (c_x(r)^2 + h_{x3x3})dx_3^2 + c_r(r)^2 dr^2. \quad (20) \]
In what follows, we define \( h_{xx} = c_x^2 H_{xx}, \) \( H_{aa} = H_{x1x1} + H_{x2x2} \) and \( H_{ii} = H_{aa} + H_{x3x3} \). The idea is to calculate the BK stress tensor \([6]\) for the perturbed background and show (using the Einstein equations) that it satisfies the corresponding constitutive relations \([10]\).

Let us first compute the momentum flux, \( \pi^i \). We note that

\[
\pi^\mu = \sigma^{\mu\nu} U_\gamma T^\nu_\gamma, \tag{21}
\]

where \( T_{\mu\nu} \) is the BK stress tensor, \( U^\mu \) is the unit timelike vector orthogonal to the spacelike component of the boundary. The metric induced on the spatial sector of the boundary is denoted by \( \sigma^{\mu\nu} \). The \( U_\mu \) and \( \sigma^{\mu\nu} \) are written explicitly (up to the first order in the perturbations) as follows

\[
U_\mu = \left( \frac{1}{c_0(r)^2 - h_{tt}} \right)^{1/2} (0, 0, h_{tx3} / [(c_0(r)^2 - h_{tt})^{1/2}(c_x(r)^2 + h_{x3x3})], 0), \tag{22}
\]

\[
\sigma^{\mu\nu} = g^{\mu\nu} - n_\mu n_\nu + U_\mu U_\nu, \tag{23}
\]

where \( n_\mu \) the unit spacelike vector orthogonal to the boundary. It is given by

\[
n^\mu = (0, 0, 0, 1/c_r(r)). \tag{24}
\]

It is easily confirmed that \( U^\mu U_\mu = -1 \), utilizing the perturbed background \([20]\). Using \([21]\), it turns out that all components of the momentum flux \( \pi^\mu \) are zero except for

\[
\pi^{x3} = -\frac{1}{2} \frac{c_x^2}{c_0 c_r} H'_{tx3}. \tag{25}
\]

Using this result, it is straightforward to see that

\[
\nabla^{x2}\pi_{x2} = 0, \tag{26}
\]

\[
\nabla \cdot \pi = \nabla^{x3}\pi_{x3} = -\frac{1}{2c_0 c_r} \partial^2_{x3r} H_{tx3}.
\]

One the other hand using the constitutive relations \([10]\) one can write down the following two components of the stress tensor

\[
T^{x2}_{x2} = \left( \frac{2}{3} \gamma_\eta - \gamma_\zeta \right) \nabla \cdot \pi + v^2 \delta \epsilon + P, \tag{27}
\]

\[
T^{x3}_{x3} = \left( -\frac{4}{3} \gamma_\eta - \gamma_\zeta \right) \nabla \cdot \pi + v^2 \delta \epsilon + P,
\]

where we have used our knowledge of the relations \([25]\). Subtracting the above two components of the stress tensor

\[
T^{x2}_{x2} - T^{x3}_{x3} = 2\gamma_\eta \nabla \cdot \pi. \tag{28}
\]
This will be the constitutive relation that we will prove to hold on the stretched horizon using Einstein equations and in the near horizon limit. Using the holographic stress tensor prescription \(\mathcal{G}\), one computes
\[
T_{x_2x_2} - T_{x_3x_3} = \frac{1}{2c_r} (H_{x_3x_3} - H_{x_2x_2})'.
\] (28)

Therefore, in order to show the constitutive relation (27) is satisfied, it suffices to show that the following equality is fulfilled in the near horizon limit
\[
\frac{1}{2c_r} (H_{x_3x_3} - H_{x_2x_2})' = 2\gamma \eta \nabla \cdot \pi = -\frac{\gamma \eta q}{c_0 c_r} H'_{tx3}.
\] (29)

In order to prove (29), let us begin with a component of the Einstein equations i.e., \(R^{tx}_x = 0\) (note that this equation is source free using the explicit form of the perturbed stress tensor in subsection (4)). One finds
\[
H''_{tx3} + \left[\ln \left(\frac{c}{c_0 c_r}\right)\right]' H'_{tx3} + \frac{c^2}{c_x} \partial^2_{tx3} H_{aa} = 0.
\] (30)

Ignoring the last term in the hydrodynamic limit
\[
\partial_r \left(\frac{c_5}{c_0 c_r} H'_{tx3}\right) = 0.
\] (31)

Solving the above differential equation, one obtains
\[
H_{tx3} = C_0 \int_{r_0}^{\infty} \frac{c_0(r) c_r(r)}{c_x(r)^5} dr,
\] (32)

therefore
\[
\Gamma_{\text{sound}} = \frac{H_{tx3}}{H'_{tx3}} \bigg|_{r=r_0} = \frac{c_x(r_0)^5}{c_0(r_0) c_r(r_0)} \int_{r_0}^{\infty} \frac{c_0(r) c_r(r)}{c_x(r)^5} dr.
\] (33)

Consider the following combination of Einstein equations
\[
R^{x_2x_2} - R^{x_3x_3} = 8\pi G(\delta T^{x_2x_2} - \delta T^{x_3x_3}).
\] (34)

It is clear from the perturbed stress tensor given in subsection (4) that the right hand side of the above equation is zero as background itself does not depend either on \(x_2\) or \(x_3\) and the terms in the two stress tensor components proportional to the Kronecker delta cancel each
other off.

\[ R_{x2}^2 - R_{x3}^2 = \frac{1}{2c_0^2} \partial_t^2 H_{x2x2} + \frac{1}{c_0^2} \partial_{tx3}^2 H_{tx3} - \frac{1}{2c_0^2} \partial_{tx3}^2 H_{tx3} - \frac{1}{2c_0^2} \partial_t^2 H_{x3x3} - \frac{c_0'}{2c_0^2 c_r^2} H'_{x2x2} \tag{35} \]

\[ + \frac{c_0'}{2c_0 c_r^2} H'_{x3x3} + \frac{c_r'}{2c_r^2} H'_{x2x2} - \frac{c_r'}{2c_r^2} H'_{x3x3} - \frac{3c_r'}{2c_r^2 c_r^2} H'_{x2x2} + \frac{3c_r'}{2c_r^2 c_r^2} H'_{x3x3} \]

\[ - \frac{1}{2c_r^2} H''_{x2x2} + \frac{1}{2c_r^2} H''_{x3x3} + \frac{1}{2c_r^2} \partial_{tx3}^2 H_{x1x1} = 0. \]

The above equation in the near horizon limit leads to

\[ \frac{1}{2} \partial_t^2 (H_{x2x2} - H_{x3x3}) + \partial_{tx3}^2 H_{tx3} = 0. \tag{36} \]

Also, from \( R^t_r = 8\pi G \delta T^t_r \)

\[ \frac{c_r'}{2c_0 c_r} \partial_t H_{ii} - \frac{c_0'}{2c_0^2} \partial_t H_{ii} + \frac{1}{2c_0^2} \partial_t H_{ii} + (\frac{c_0'}{c_0^3} - \frac{c_r'}{c_r^3 c_r}) \partial_{tx3} H_{tx3} - \frac{1}{2c_0^2} \partial_{tx3} H'_{tx3} = 8\pi G \delta T^t_r \tag{37} \]

which in the near horizon limit gives rise to

\[ \partial_{tx3} H_{tx3} - \frac{1}{2} \partial_t H_{ii} = 0. \tag{38} \]

Using (19) one ends up with

\[ qH_{tx3} + \frac{1}{2} \Omega H_{ii} = 0. \tag{39} \]

Here we are assuming that \( \delta T^t_r \) is a smooth function near the horizon such that \( c_0^3 \delta T^t_r \to 0 \) as \( r \to r_0 \). Now let us concentrate on another component of the Einstein equation. It is straightforward to see

\[ c_0^2 R_{x3}^2 = 8\pi G (\delta T_{x3}^3 - \frac{1}{2} \delta T^3) c_0^2 = \frac{1}{2} \partial_t^2 H_{x3x3} - \partial_{tx3}^2 H_{tx3} \to 0, \tag{40} \]

where \( 8\pi G (\delta T_{x3}^3 - \frac{1}{2} \delta T^3) c_0^2 \to 0 \) as the horizon at \( r = r_0 \) is approached. Using (19), the equation (40) reads

\[ \frac{1}{2} \Omega^2 H_{x3x3} + \Omega qH_{tx3} = 0, \tag{41} \]

in the near horizon limit. Comparing (39) and (41) one find that

\[ H_{x1x1} + H_{x2x2} \to 0, \tag{42} \]

as one approaches the event horizon. Our next step is to recast the equation (35) in a suggestive form. Call \( \chi = H_{x2x2} - H_{x3x3} \)

\[ R_{x2}^2 - R_{x3}^2 = \frac{1}{2c_r^2} (\chi'' + (\frac{c_r^2 c_0}{c_r})') \frac{c_r^3 c_0}{c_r} \chi' - \frac{c_r^2}{c_0} \partial_t^2 \chi + \frac{1}{3} \frac{c_r^2}{c_r^2} \partial_{tx3}^2 \chi + \frac{2c_r^2}{c_0} \partial_{tx3} H_{tx3} = 0. \tag{43} \]
In the hydrodynamic limit the last two terms can be dropped. One finds
\[ \frac{c_0}{c_r^3} \partial_t (c_r^2 \chi') - \partial_r^2 \chi = 0. \] (44)
Now one has to use the expansions of \( c_0 \) and \( c_r \) in the near horizon region (14) and solve for \( \chi \). The boundary condition prescription of [3], [6] singles out the incoming wave solution on the horizon. Using this solution, one obtains
\[ \partial_r \chi = \sqrt{\frac{\gamma_r}{\gamma_0}} \frac{\partial_t \chi}{r-r_0}. \] (45)
Using the definition of \( \chi \)
\[ H'_{x1x} - H'_{x3x} = \sqrt{\frac{\gamma_r}{\gamma_0}} \frac{-i \Omega}{r-r_0} (H_{x1x} - H_{x3x}), \] (46)
where on the first and second line equations (42) and (39) are used respectively. Using (33) one can write
\[ H'_{x1x} - H'_{x3x} = \sqrt{\frac{\gamma_r}{\gamma_0}} \frac{2i q}{r-r_0} \Gamma_{sound} H'_{tx3}. \] (47)
Comparing the above equation with what we need to prove equation (29), we can read off \( \gamma_\eta \)
\[ \gamma_\eta = \sqrt{\gamma_0 \gamma_r} \Gamma_{sound}, \] (48)
which is exactly the general expression for the shear viscosity reported by Kovtun, Starinets and Son in [6].

5.2 Shear Channel

In the shear channel, the \( h_{xt} \) and \( h_{xy} \) perturbations are turned on. The full perturbed background is given by
\[ ds^2 = g_{00}(r) dt^2 + 2g_{xx}(r) \omega dtdx + 2g_{xx}(r) Q dxdy + g_{xx}(r) (dx^2 + dy^2 + dz^2) + g_{rr}(r) dr^2, \] (49)
where $\omega = \omega(r,t,y)$ and $Q = Q(r,t,y)$ and where $x$, $y$ and $z$ denote the world-volume directions. The following space and time dependence for the perturbations is assumed

$$\omega = \omega(r)e^{-i\Omega t + iqy},$$

$$Q = Q(r)e^{-i\Omega t + iqy}.$$  \hfill (50)

Notice that in this section, the only (up, down) components of the perturbed Einstein equations that we will be interested in are $(x,r), (t,x), (x,y)$. These components are all off diagonal, which implies that the second piece in (18) vanishes. Noticing that the background matter fields $\Phi_{0i}$ are only functions of $r$, one concludes that the above mentioned components of the perturbed Einstein equations are source free.

As in the previous section, our aim will be to compute the stress tensor (6) for the perturbed background and show (using the Einstein equations) that it satisfies the constitutive relations as given in (10). We concentrate on the following constitutive relation coming from (10)

$$T_{x}^{y} = -\gamma_{y}(\nabla_{x}\pi^{y} + \nabla^{y}\pi_{x}).$$ \hfill (51)

This is the constitutive relation which needs to hold on the stretched horizon. It turns out that $\pi_{x}$ is the only non-vanishing component of the flux

$$8\pi G\pi_{x} = \sigma_{x}^{\nu}U^{\mu}T_{\mu\nu},$$

$$= -\frac{1}{2}(\delta_{x}^{\nu} + U_{x}U^{\nu} - n_{x}n^{\nu})U_{\mu}\nabla_{\mu}n_{\nu} - \frac{1}{2}(\delta_{x}^{\nu} + U_{x}U^{\nu} - n_{x}n^{\nu})U^{\mu}\nabla_{\nu}n_{\mu},$$ \hfill (52)

where

$$U_{\mu} = (-N, 0, 0, 0, 0),$$ \hfill (53)

$$n_{\mu} = (0, 0, 0, 0, N(r)),$$

$$U^{\mu} = (\frac{1}{N}, -\frac{N_{x}}{N}, 0, 0, 0),$$

$$n^{\mu} = (0, 0, 0, 0, \frac{1}{N(r)}),$$

and $U^{\mu}U_{\mu} = -1$, $n^{\mu}n_{\mu} = 1$, $N(r) = \sqrt{g_{rr}}$ and $N = \sqrt{-g_{00}}$. Using (52) and recalling that $\delta g_{0x} = g_{xx}N^{x}$, after doing some algebra, one ends up with

$$8\pi G\pi_{x} = -\frac{g^{rr}g_{xx}}{2N}N(r)(\frac{\delta g_{0x}}{g_{xx}}).$$ \hfill (54)
After performing some rather tedious algebra, one is able to further check that
\[ 8\pi G \pi^x = -\frac{g^{xx}g_{xx}}{2NN_{(r)}} \left( \frac{\delta g_{0x}}{g_{xx}} \right), \] (55)
as expected. It is easily seen that
\[ \nabla_y \pi_x = \partial_y \pi_x, \] (56)
\[ \nabla_x \pi_y = 0. \]

On the other hand, notice that
\[ 8\pi GT^y_x = -\frac{1}{2} \nabla_x n^y - \frac{1}{2} g^{yy} \nabla_y n_x \] (57)
\[ = -\frac{g^{yy}}{2N_{(r)}} \delta g_{xy,r}. \]

Using the equation (56) and (51), it is clear that what needs to be proven is
\[ T^y_x = -\gamma \eta \partial_y \pi_x. \] (58)

From \( R^x_r = 0 \), one obtains
\[ g_{00} \partial_{y \omega}^2 Q + g_{xx} \partial_{t \omega}^2 Q = 0. \] (59)

Also the Einstein equation \( R^x_r = 0 \) gives us
\[ (g_0^{-%1/2} g_{rr}^{-1/2} g_{xx}^{3/2} \partial \omega, r) - g_0^{-%1/2} g_{rr}^{-1/2} g_{xx}^{3/2} \partial_y (\partial_t Q - \partial_y \omega) = 0. \] (60)

And finally \( R^y_y = 0 \) leads to
\[ (g_0^{1/2} g_{rr}^{-1/2} g_{xx}^{3/2} \partial_t Q, r) - g_0^{1/2} g_{rr}^{-1/2} g_{xx}^{3/2} \partial_t (\partial_y \omega - \partial_t Q) = 0. \] (61)

There is also the following trivial identity
\[ \partial_t (\partial_t Q) + \partial_y (-\partial_r \omega) - \partial_r (\partial_t Q - \partial_y \omega) = 0. \] (62)

From the above set of equations we derive the following equations. From (59) we get
\[ \partial_{t \omega}^2 Q = -\frac{g^{xx}}{\Omega^2} g_{00} \partial_y (\partial_{t \omega}^2 Q), \] (63)
where \( \Omega \) is a typical inverse time scale. Following arguments given in \[6\], one can show that the right hand side of (63), with an appropriate choice of the location of the stretched horizon can be made arbitrary small. Combining equations (61) and (62), one finds
\[ -g_{00}^{-%1/2} g_{rr}^{1/2} g_{xx}^{3/2} \partial_t^2 (\partial_y \omega - \partial_t Q) + [g_0^{1/2} g_{rr}^{-%1/2} g_{xx}^{3/2} \partial_t (\partial_t Q - \partial_y \omega)],_r + (g_0^{1/2} g_{rr}^{-%1/2} g_{xx}^{3/2} \partial_y \omega),_r = 0. \] (64)
Using (63) the last term is negligible compared to the rest if the location of the stretched horizon is chosen appropriately. Call

\[ P = \partial_t Q - \partial_y \omega. \]

therefore, we have

\[
\partial_t^2 P + \frac{\gamma_0}{\gamma_r} (r - r_0) \partial_r [(r - r_0) \partial_r P] = 0.
\]

(65)

Following the arguments given in [6] one ends up with

\[ \partial_r Q = \sqrt{\frac{\gamma_r}{\gamma_0}} \frac{P}{r - r_0}. \]

(66)

Following arguments in [6] one can show \( \partial_t Q \ll \partial_y \omega \). Using this, the equation (66) reduces to

\[ \partial_r Q = -\sqrt{\frac{\gamma_r}{\gamma_0}} \frac{\partial_y \omega}{r - r_0}. \]

(67)

Near the horizon from equation (60) in the hydrodynamic limit one has (ignoring the last two terms)

\[
\omega = \int_{r_0}^{\infty} \frac{g^{1/2}_{00} g^{1/2}_{rr}}{g_{xx}^{1/2}} = \int_{r_0}^{\infty} \frac{g_{00} g_{rr}}{g_{xx} \sqrt{-g}},
\]

(68)

Therefore

\[
\Gamma = \omega \bigg|_{r = r_0} = \frac{g_{xx}(r_0) \sqrt{-g(r_0)}}{g_{00}(r_0) g_{rr}(r_0)} \int_{r_0}^{\infty} \frac{g_{00} g_{rr}}{g_{xx} \sqrt{-g}}.
\]

(69)

Now we are in a position to demonstrate that the constitutive relation (58) is indeed satisfied on the stretched horizon. Recall that

\[
T_{xy} = -\frac{g_{xx}}{2N(r)} \partial_r Q.
\]

(70)

Using (66), (70), (54) and (69) one finds

\[
\frac{T_{xy}}{\partial_y \pi_x / N} = \sqrt{\gamma_0} \gamma_r \Gamma,
\]

(71)

which is to say

\[
\frac{T_{xy}}{\partial_y \pi_x / N} = \mathcal{D} = \frac{\sqrt{-g(r_0)}}{\sqrt{g_{00}(r_0) g_{rr}(r_0)}} \int_{r_0}^{\infty} \frac{g_{00} g_{rr}}{g_{xx} \sqrt{-g}}.
\]

(72)

This is the same equation for \( \mathcal{D} \), the shear diffusion constant, as given in [6]. The Einstein equation \( R^r_r = 0 \) can be recasted into

\[
\partial_t \frac{\pi_x}{N} + \partial_y T^y_x = 0.
\]

(73)

which is the conservation of momentum. Using this equation and the constitutive relation (58), one finds that the momentum flux fluctuation satisfies a diffusion equation with a diffusion constant given by \( \mathcal{D} \).
6 Discussion

In this work we filled a gap which was left open in the analysis of [6]. In this paper, we identified the stress tensor on the stretched horizon with the Brown and York stress tensor (known as the Balasubramanian-Kraus stress tensor in the context of AdS/CFT). We then moved to demonstrate that such stress tensor satisfies the constitutive relations in the near horizon limit. Reading off various near equilibrium transport coefficients from the resulting constitutive relations, we were able to find an expression for the shear viscosity which turned out to agree with what was computed in [6]. We repeat this calculation in two different channels, sound and shear and find the same expression. It would be great to try find a general membrane paradigm expression for the other transport coefficient e.g., $\gamma_c$. If such a formula exists it would be nice to compare its value against the existing AdS/CFT calculations for various backgrounds [14]. It would be exciting to see an agreement.

Another outstanding question is to see whether one could make sense of the membrane paradigm prediction i.e., $\eta/s = 1/4\pi$ for a Schwarzschild black hole. Here $\eta$ is the shear viscosity and $s$ is the entropy over the area of the event horizon. This case is rather different from the hydrodynamic limit of theories with extended spatial dimensions as these objects are point like in the transverse space. A natural context to look for a possible explanation would be to consider the BFSS matrix theory description of a Schwarzschild black hole in M-theory. However, in the M-theory description one encounters a somewhat paradoxical situation. Inside the D0-brane gas, the distances over which transverse momentum transfer takes place is of the order of the size of the bound state i.e., the horizon radius. So, assigning a shear viscosity to the gas becomes ambiguous. Shear viscosity is best defined when interactions in the system are all short range.

7 Acknowledgement

I would like to thank the Institute for Theoretical physics and Mathematics of Iran (IPM) for their hospitality near the end stage of this work. I am grateful of Mohsen Alishahiha, Guy Moore, Amir E. Mosaffa, Erich Poppitz and Shahin Sheikh-Jabbari for fruitful discussions. This work was supported by a Tomlinson Postdoctoral Award at McGill University and by the Natural Science and Engineering Research Council of Canada (NSERC).
Références

[1] K. S. Thorne, R. H. Price and D. A. Macdonald, “Black holes : the membrane paradigm”, Yale University Press, New Haven 1986.

[2] G. ’t Hooft, “Dimensional reduction in quantum gravity,” arXiv :gr-qc/9310026.

[3] D. T. Son and A. O. Starinets, “Minkowski-space correlators in AdS/CFT correspondence : Recipe and applications,” JHEP 0209, 042 (2002) arXiv :hep-th/0205051.

[4] G. Policastro, D. T. Son and A. O. Starinets, “From AdS/CFT correspondence to hydrodynamics,” JHEP 0209, 043 (2002) arXiv :hep-th/0205052.

[5] G. Policastro, D. T. Son and A. O. Starinets, “From AdS/CFT correspondence to hydrodynamics. II : Sound waves,” JHEP 0212, 054 (2002) arXiv :hep-th/0210220.

[6] P. Kovtun, D. T. Son and A. O. Starinets, “Holography and hydrodynamics : Diffusion on stretched horizons,” JHEP 0310, 064 (2003) arXiv :hep-th/0309213.

[7] T. D. Cohen, “Is there a ’most perfect fluid’ consistent with quantum field theory ?,” arXiv :hep-th/0702136.

[8] A. Buchel, “On universality of stress-energy tensor correlation functions in supergravity,” Phys. Lett. B 609, 392 (2005) arXiv :hep-th/0408095.

[9] Omid Saremi, “The viscosity bound conjecture and hydrodynamics of M2-brane theory at finite chemical potential,” JHEP 0610, 083 (2006) arXiv :hep-th/0601159.

J. Mas, “Shear viscosity from R-charged AdS black holes,” JHEP 0603, 016 (2006) arXiv :hep-th/0601144.

D. T. Son and A. O. Starinets, “Hydrodynamics of R-charged black holes,” JHEP 0603, 052 (2006) arXiv :hep-th/0601157.

[10] P. Benincasa, A. Buchel and R. Naryshkin, “The shear viscosity of gauge theory plasma with chemical potentials,” Phys. Lett. B 645, 309 (2007) arXiv :hep-th/0610145.

[11] V. Balasubramanian and P. Kraus, “A stress tensor for anti-de Sitter gravity,” Commun. Math. Phys. 208, 413 (1999) arXiv :hep-th/9902121.

[12] J. D. Brown and J. W. . York, “Quasilocal energy and conserved charges derived from the gravitational action,” Phys. Rev. D 47, 1407 (1993).

[13] P. Kovtun and L. G. Yaffe, “Hydrodynamic fluctuations, long-time tails, and supersymmetry,” Phys. Rev. D 68, 025007 (2003) arXiv :hep-th/0303010.
[14] Omid Saremi, in preparation.