Negative Ricci curvature on some non-solvable Lie groups

Cynthia E. Will

Received: 29 December 2015 / Accepted: 21 July 2016 / Published online: 18 August 2016
© Springer Science+Business Media Dordrecht 2016

Abstract We show that for any non-trivial representation \((V, \pi)\) of \(u(2)\) with the center acting as multiples of the identity, the semidirect product \(u(2) \ltimes \pi V\) admits a metric with negative Ricci curvature that can be explicitly obtained. It is proved that \(u(2) \ltimes \pi V\) degenerates to a solvable Lie algebra that admits a metric with negative Ricci curvature. An \(n\)-dimensional Lie group with compact Levi factor \(SU(2)\) admitting a left invariant metric with negative Ricci is therefore obtained for any \(n \geq 7\).

Keywords Ricci curvature · Lie groups · Riemannian metrics

Mathematics Subject Classification 53C30 · 53C21

1 Introduction

The question of when a differentiable manifold admits a Riemannian metric with a particular sign of the curvature has been one of the foundational problems in geometry leading to a lot of fundamental results. For homogeneous Riemannian manifolds, the case of sectional curvature is quite restrictive and well-understood. For example, by [1, 8], any homogeneous manifold of negative sectional curvature is isometric to a left-invariant metric on a solvable Lie group whose Lie algebra \(g\) has a codimension one derived algebra \([g, g]\), and there exists an element \(Y \in g\) such that all the eigenvalues of \(\text{ad} Y|_{[g, g]}\) have positive real part.

We are interested in homogeneous negative Ricci curvature. Although a great progress has been made lately, specially in the solvable case, the general case seems to be far from being completely understood. We summarize the known results:

This research was partially supported by Grants from CONICET, FONCYT and SeCyT (Universidad Nacional de Córdoba).

Cynthia E. Will
cwill@famaf.unc.edu.ar

1 Universidad Nacional de Córdoba, FaMAF and CIEM, 5000 Córdoba, Argentina
• By Bochner’s Theorem the isometry group of a compact Riemannian manifold with negative Ricci curvature is discrete (see [2]). Consequently, a homogeneous space with negative Ricci curvature is necessarily non-compact.

• No solvable unimodular Lie group admits a left-invariant metric with $\text{Ric} < 0$ (see [4]).

• Any unimodular Lie group which admits a left-invariant metric with negative Ricci curvature is non-compact and semisimple (see [7]). Moreover, it has recently been proved in [9] that such semisimple Lie group can not have compact factors. In fact, [9, Proposition 1.2] asserts that if $G/K$ is a homogeneous space of negative Ricci curvature, where $G$ is a semisimple Lie group acting almost effectively on $G/K$, then $G$ has no compact normal subgroups.

• There is no any left-invariant metric on $\text{SL}(2, \mathbb{R})$ with negative Ricci curvature (see [13]), though there exist negative Ricci left-invariant metrics on $\text{SL}(n, \mathbb{R})$ for every $n \geq 3$ (see [6]) and on most non-compact simple groups (see [7]).

• In [5] it is studied, on unimodular Lie groups, the relationship between non-positive Ricci curvature and abelian ideals.

• Let $\mathfrak{g}$ be a solvable Lie algebra with nilradical $\mathfrak{n}$, and let $\mathfrak{z}$ denote the center of $\mathfrak{n}$. If $\mathfrak{g}$ admits an inner product with $\text{Ric} < 0$, then there exists an element $Y \in \mathfrak{g}$ such that all the eigenvalues of $\text{ad} Y|_{\mathfrak{z}}$ have positive real part. On the other hand, if there exists $Y \in \mathfrak{g}$ such that all the eigenvalues of $\text{ad} Y|_{\mathfrak{n}}$ have positive real part, then $\mathfrak{g}$ admits an inner product of negative Ricci curvature (see [14] or Theorem 2.2).

• Let $\mathfrak{g}$ be a solvable Lie algebra with nilradical $\mathfrak{n}$. In the case when $\mathfrak{n}$ is the Heisenberg Lie algebra or the standard filiform Lie algebra, there is a characterization in [14] of when they admit a metric with negative Ricci curvature. This was recently extended to the case when $\mathfrak{n}$ is any filiform Lie algebra in [15].

We should also mention that a non-flat Einstein solvmanifold is an example of a Riemannian homogeneous space with negative Ricci curvature and from there one obtains many examples. In [12], an example is given of a metric Lie algebra with negative Ricci curvature which is substantially different from any of the above ones. In fact, it is shown there that $u(2) \ltimes_\theta \mathbb{R}^4$ admits a metric with negative Ricci curvature, where $\theta$ is the standard representation (see [12, Examples 6.11, 6.14]). Note that $u(2) \ltimes_\theta \mathbb{R}^4$ is neither solvable nor unimodular and moreover, it has a compact Levi factor. A natural question is therefore if this representation is special in some sense or if this is a particular case of a more general fact.

Our main result is

**Theorem 1.1** Let $(V, \pi)$ be a non-trivial real representation of $\text{su}(2)$ extended to $u(2)$ by letting the center act as multiples of the identity, then the Lie algebra $u(2) \ltimes V$ admits an inner product with negative Ricci curvature.

Hence, this inner product defines a left-invariant metric on the corresponding simply connected Lie group $(\text{SU}(2) \times \mathbb{R}) \ltimes V$ with Lie algebra $u(2) \ltimes V$ (compare with [5, Corollary 3]). We start by showing that this holds for any irreducible representation of $\text{su}(2)$ given in terms of complex polynomials in two variables. In each case, we prove that $u(2) \ltimes_\pi V$ degenerates to a solvable Lie algebra that admits a metric with negative Ricci curvature. We note that even though this process is based on a continuous argument, we actually find explicitly the Ricci negative metric on the solvable Lie algebra, and from there it is not hard to get a metric on the starting algebra (see Lemma 3.4 and Remark 3.5). We note that the limit solvable Lie algebra does not satisfy any of the sufficient conditions for $\text{Ric} < 0$ given in [14,15].

The smallest example we obtain is 7-dimensional. Note that, according to [9, Corollary 1.3], this procedure can never give an Einstein metric.
2 Preliminaries and notation

2.1 Lie algebras

Let \( g = (\mathbb{R}^m, [\cdot, \cdot]) \) be a Lie algebra of dimension \( m \), that is, the underlying linear space of \( g \) is (identified with) \( \mathbb{R}^m \) and \( [\cdot, \cdot] \) belongs to the space of Lie brackets \( \mathcal{L}_m \subset \Lambda^2 (\mathbb{R}^m)^* \otimes \mathbb{R}^m \), defined as

\[
\mathcal{L}_m := \{ \mu : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m : \mu \text{ bilinear, skew-symmetric and satisfies Jacobi} \}.
\]

\( \mathcal{L}_m \) is also called the variety of Lie algebras of dimension \( m \). We consider the following action of \( \text{GL}_m(\mathbb{R}) \) on \( \mathcal{L}_m \):

\[
(g \cdot \mu)(X, Y) = g \mu(g^{-1}X, g^{-1}Y), \quad g \in \text{GL}_m(\mathbb{R}), \quad \mu \in \mathcal{L}_m, \quad X, Y \in g.
\]

Note that \( (\mathbb{R}^m, \mu) \) is isomorphic to \( (\mathbb{R}^m, g \cdot \mu) \) for any \( g \in \text{GL}_m(\mathbb{R}) \) though, \( (\mathbb{R}^m, \mu) \) is not isomorphic to \( (\mathbb{R}^m, \mu_o) \) for \( \mu_o \in \text{GL}_m(\mathbb{R}) \cdot \mu \setminus \text{GL}_m(\mathbb{R}) \cdot \mu \). Since \( \mathcal{L}_m \subset \Lambda^2 \mathbb{R}^m^* \otimes \mathbb{R}^m \) is defined by polynomials equations, any \( \mu_0 \) in the closure is also a Lie bracket. We will say that \( \mu_o \) is a degeneration of \( \mu \) or that \( \mu \) degenerates to \( \mu_o \) if \( \mu_o \in \text{GL}_m(\mathbb{R}) \cdot \mu \). Note that by continuity, many of the properties of \( \mu_o \) are shared by \( \mu \). In particular if \( (\mathbb{R}^m, \mu_o) \) admits a metric with negative (or positive) sectional or Ricci curvature, so does \( (\mathbb{R}^m, \mu) \) (see [16, Remark 6.2] or [14, Proposition 1]).

**Proposition 2.1** Suppose \( \mu, \lambda \in \mathcal{L}_m \) and that \( \lambda \) is in the closure of the orbit \( \text{GL}_m(\mathbb{R}) \cdot \mu \).

If the Lie algebra \( (\mathbb{R}^m, \lambda) \) admits an inner product of negative Ricci curvature, then so does the Lie algebra \( (\mathbb{R}^m, \mu) \).

Moreover, if we fix an inner product on \( g = (\mathbb{R}^m, \mu) \), or equivalently, an orthonormal basis, then the orbit \( \text{GL}(g) \cdot \mu \) parameterizes, from a different point of view, the set of all inner products on \( g \). Indeed,

\[
(g, g \cdot \mu, \langle \cdot, \cdot \rangle) \text{ is isometric to } (g, \mu, \langle g \cdot \cdot, g \cdot \rangle) \text{ for any } g \in \text{GL}(g).
\]  

(1)

Let \( (g, [\cdot, \cdot], \langle \cdot, \cdot \rangle) \) be a metric Lie algebra and \( H \in g \) the only element such that \( \langle H, X \rangle = \text{tr} \, \text{ad} \, X \) for any \( X \in g \), usually called the mean curvature vector, and let \( B \) denotes the symmetric map defined by the Killing form of \( (g, [\cdot, \cdot], \langle \cdot, \cdot \rangle) \) (i.e. \( \langle BX, X \rangle = \text{tr} (\text{ad} \, X)^2 \)). The Ricci operator of \( (g, [\cdot, \cdot], \langle \cdot, \cdot \rangle) \) is given by (see for instance [11, Appendix]):

\[
\text{Ric} = M - \frac{1}{2} B - S(\text{ad} \, H),
\]

(2)

where, \( S(\text{ad} \, H) = \frac{1}{2}(\text{ad} \, H + \text{ad} \, H^\dagger) \) is the symmetric part of \( \text{ad} \, H \) and \( M \) is the symmetric operator defined by

\[
\langle MX, X \rangle = -\frac{1}{2} \sum \langle [X, X_i], X_j \rangle^2 + \frac{1}{4} \sum \langle [X_i, X_j], X \rangle^2, \quad \forall X \in g.
\]

(3)

where \( \{X_i\} \) is any orthonormal basis of \( (g, \langle \cdot, \cdot \rangle) \). Note that if \( g \) is nilpotent, then \( \text{Ric} = M \).

A special case we will be interested in is when \( g \) is a solvable Lie algebra. If we consider an orthogonal decomposition

\[
g = a \oplus n,
\]

(4)
where \( n \) is the nilradical of \( \mathfrak{g} \) (i.e. maximal nilpotent ideal), there is a substantial simplification of the expression of \( \text{Ric} \) when \( \mathfrak{a} \) is abelian (see [10]). Indeed, we get

\[
\langle \text{Ric} \ A, \ A \rangle = - \text{tr} \, S(\text{ad} \ A|_n)^2, \\
\langle \text{Ric} \ A, \ X \rangle = - \frac{1}{2} \text{tr} \, (\text{ad} \ A|_n)^t \text{ad} \ X|_n \\
\langle \text{Ric} \ X, \ X \rangle = - \frac{1}{2} \sum_i \langle [X, X_i], X_j \rangle^2 + \frac{1}{4} \sum_i \langle [X_i, X_j], X \rangle^2 + \frac{1}{2} \sum_i \langle [\text{ad} \ A|_n, (\text{ad} \ A|_n)^t] \rangle - \langle [H, X], X \rangle,
\]

for all \( A \in \mathfrak{a} \) and \( X \in \mathfrak{n} \), where \( \{A_i\}, \{X_i\} \) are any orthonormal basis of \( \mathfrak{a} \) and \( \mathfrak{n} \), respectively. If in addition \( \text{ad} \ A \) are normal operators for all \( A \in \mathfrak{a} \), then we get that \( \text{tr} \, (\text{ad} \ A|_n)^t \text{ad} \ X|_n = 0 \) (see [10, (25) and Prop. 4.3]) and therefore

\[
\langle \text{Ric} \ A, \ A \rangle = - \text{tr} \, S(\text{ad} \ A|_n)^2, \\
\langle \text{Ric} \ A, \ X \rangle = 0 \\
\langle \text{Ric} \ X, \ X \rangle = - \frac{1}{2} \sum_i \langle [X, X_i], X_j \rangle^2 + \frac{1}{4} \sum_i \langle [X_i, X_j], X \rangle^2 - \langle [H, X], X \rangle.
\]

**Theorem 2.2** [14, Thm 2] Suppose \( \mathfrak{g} \) is a solvable Lie algebra. Let \( \mathfrak{n} \) be the nilradical of \( \mathfrak{g} \) and let \( \mathfrak{z} \) be the center of \( \mathfrak{n} \). Then

- If \( \mathfrak{g} \) admits an inner product of negative Ricci curvature, then there exists \( Y \in \mathfrak{g} \) such that \( \text{Tr} \, \text{ad} \ Y > 0 \) and all the eigenvalues of the restriction of the operator \( \text{ad} \ Y \) to \( \mathfrak{z} \) have a positive real part;
- If there exists \( Y \in \mathfrak{g} \) such that all the eigenvalues of the restriction of \( \text{ad} \ Y \) to \( \mathfrak{n} \) have positive real part, then \( \mathfrak{g} \) admits an inner product of negative Ricci curvature.

**2.2 Representations of \( su(2) \)**

It is well known (see for example [3, II 5]) that the (complex) irreducible representations of \( SU(2) \) are classified and there is one for each dimension. The first one, \( V_0 \) is the trivial representation on \( \mathbb{C} \) and \( V_1 \) is the standard representation on \( \mathbb{C}^2 \), explicitly given by

\[
g \cdot (z_1, z_2) = g \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},
\]

i.e. matrix multiplication by \( g \). For each \( n \geq 2 \), let \( V_n = P_{2,n}(\mathbb{C}) \) be the space of homogeneous polynomials in two variables of degree \( n \) on \( \mathbb{C} \). The action is given by

\[
(g \cdot P)(z_1, z_2) = P \left( g^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right).
\]

Concerning its Lie algebra, \( su(2) \), we set

\[
H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\]

where \( i^2 = -1 \). It is easy to see that \( \{H, X, Y\} \) is a basis of \( su(2) \) and

\[
[H, X] = 2Y, \quad [H, Y] = -2X, \quad [X, Y] = 2H.
\]

We have, as usual, the corresponding (complex) representation of \( su(2) \) on \( P_{2,n}(\mathbb{C}) \) by derivations. These are all the irreducible complex representations of \( su(2) \) and it is easy to see that \( \text{dim} \, V_n = n + 1 \).

To get real representations one can proceed in two ways. On the one hand we can see \( P_{2,n}(\mathbb{C}) \) as a real vector space. This gives a real representation of \( su(2) \) of real dimension twice the dimension of the original one and therefore this procedure always gives an even dimensional (real) representation, though not always irreducible. In fact they are irreducible.
only when \( n = 4k \). On the other hand, if \( J \) is a conjugate-linear \( \text{SU}(2) \)-map such that \( J^2 = 1d \), which is known as a real structure, then the eigenspace of \( J \) corresponding to the eigenvalue 1 is a real representation of \( g \) (see [3] II. 6). In this way one gets all the odd dimensional real irreducible representations of \( \text{su}(2) \).

From now on, \( V_n \) will be a real vector space and we fix the basis

\[
\left\{ z_1^n, i z_1^n, \ldots, z_1^{n-j} z_2, i z_1^{n-j} z_2, \ldots, z_2^n, i z_2^n \right\},
\]

that will be used to write elements in \( \text{gl}(V_n) \) as \( 2(n + 1) \times 2(n + 1) \) matrices.

A straightforward calculation shows that \( H \cdot z_1 = -iz_1 \) and \( H \cdot z_2 = iz_2 \). In fact,

\[
(H \cdot z_1)(u, v) = \frac{d}{dt} \bigg|_{t=0} z_1 \left( \exp(-tH) \begin{bmatrix} u \\ v \end{bmatrix} \right) = -iz_1(u, v).
\]

Therefore, by using that \( \text{su}(2) \) is acting by derivations on \( V_n \) we get that for \( n \geq 2 \),

\[
\pi_n(H) = \begin{bmatrix}
0 & n \\
-n & 0 \\
0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & 0 \\
-n+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

In the same way we obtain that

\[
X \cdot z_1 = -z_2, \quad X \cdot z_2 = z_1, \quad Y \cdot z_1 = -iz_2, \quad Y \cdot z_2 = -iz_1,
\]

and from this

\[
\pi_n(X) = \begin{bmatrix}
0 & 1 \\
-n & 0 \\
0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

or equivalently,

\[
\pi_n(X)(sz_1^k z_2^{n-k}) = -ks z_1^{k-1} z_2^{n-k+1} + (n - k) s z_1^{k+1} z_2^{n-k-1}, \quad s = 1, i,
\]

\[
\pi_n(Y)(z_1^k z_2^{n-k}) = -k i z_1^{k-1} z_2^{n-k+1} - (n - k) i z_1^{k+1} z_2^{n-k-1},
\]

\[
\pi_n(Y)(iz_1^k z_2^{n-k}) = k z_1^{k-1} z_2^{n-k+1} + (n - k) z_1^{k+1} z_2^{n-k-1}.
\]

It is easy to check that if \( n \) is even, then the conjugate-linear map \( J : \mathcal{P}_{2,n}(\mathbb{C}) \to \mathcal{P}_{2,n}(\mathbb{C}) \) such that

\[
J \left( z_1^{n-j} z_2^j \right) = (-1)^j z_1^{n-j} z_2^j
\]
is a real structure of \((V_n, \pi_n)\). Let us denote by \(V_n^o\) the eigenspace corresponding to the eigenvalue 1. One can check that if
\[
P_k^+ = z_1 z_2^{-n-k} + z_2^{-n-k} z_1^k, \quad \text{and} \quad P_k^- = z_1 z_2^{-n-k} - z_2^{-n-k} z_1^k,
\]
then a basis of \(V_n^o\) is given by
\[
\left\{ p_n^+, i p_{n-1}^-, p_{n-1}, i p_n^+, p_{n-2}^+, i p_{n-2}^-, \ldots, s p_s^+ \right\},
\]
where \(s = 1\) if \(n/2\) is even and \(s = i\) if \(n/2\) is odd. Straightforward calculation shows that in this basis,

\[
\pi_n(H) =
\begin{bmatrix}
0 & \pi_1 & \cdots & \pi_n & 0 \\
-\pi_1 & 0 & \cdots & 0 & \pi_n \\
\vdots & \ddots & \ddots & \ddots & \cdots \\
\cdots & \ddots & 0 & 0 & \pi_n \\
-\pi_n & \cdots & \cdots & 0 & 0
\end{bmatrix}
\]

\[
\pi_n(X) =
\begin{bmatrix}
0 & 1 & \cdots & \pi_n & 0 \\
-1 & 0 & \cdots & 0 & \pi_n \\
\vdots & \ddots & \ddots & \ddots & \cdots \\
\cdots & \ddots & 0 & 0 & \pi_n \\
-\pi_n & \cdots & \cdots & 0 & 0
\end{bmatrix}
\]

\[
\pi_n(Y) =
\begin{bmatrix}
0 & 0 & \cdots & \pi_n & 0 \\
0 & 0 & \cdots & 0 & \pi_n \\
\vdots & \ddots & \ddots & \ddots & \cdots \\
\cdots & \ddots & 0 & 0 & \pi_n \\
-\pi_n & \cdots & \cdots & 0 & 0
\end{bmatrix}
\]

where,

- if \(n/2\) is even: \(b = d = 0, a = n, c = -n/2, e = g = 0, f = -n, h = n/2 + 1\).
- If \(n/2\) is odd: \(a = c = 0, b = n, d = -n/2, e = h = 0, f = e = n, g = -n/2 - 1\).

\textbf{Remark 2.3} Let us consider \(u(2) = su(2) \oplus \mathbb{R}Z\), where \(Z = i Id\). It is easy to see that all of the above representations can be extended to \(u(2)\) by letting \(Z\) act as the identity on \(V_n\) or \(V_n^o\). We will also denote these representations of \(u(2)\) by \((V_n, \pi_n)\) and \((V_n^o, \pi_n)\), respectively.

\section{3 Construction}

Let \(g\) be a Lie algebra and \((V, \pi)\) be a real representation of \(g\) of dimension \(d\). By \(g \ltimes V\) we will denote the Lie algebra \((g \ltimes V, \mu)\) where
\[
\mu|_{g \times g} = [\cdot, \cdot], \quad \mu|_{g \times V} = \pi \quad \text{and} \quad \mu|_{V \times V} = 0,
\]
that is, the semidirect product of \(g\) with \(V\). Note that if \(g\) is solvable or nilpotent, so is \(g \ltimes V\) and in any case \(V\) is an abelian ideal of \(g \ltimes V\) and therefore this Lie algebra can never be semisimple.

\(\text{Springer}\)
In the following we will construct examples of Lie algebras with negative Ricci curvature as a semidirect product $u(2) \ltimes V$, where $V$ is a representation of $u(2)$ as in the previous section [see (2.2)]. According to that classification, we will split the construction in the even and odd dimensional cases.

### 3.1 Even dimensional representations

For $\mathfrak{g} = u(2)$ and for any $n \in \mathbb{N}$, $n > 1$, let $\mathfrak{h}_n = \mathfrak{g} \ltimes V_n$ be the semidirect product Lie algebra where $(V_n, \pi_n)$ is the representation introduced in Sect. 2.2. Thus $\mathfrak{h}_n$ is a $(2n + 6)$ non-solvable Lie algebra with Levi factor $su(2)$. Fix the basis of $\mathfrak{h}_n$ [see (9)],

$$\beta = \left\{ Z, H, X, Y, z_1^n, z_1, \ldots, z_1^{n-j}, i z_1^{n-j}, z_2^n, z_2, \ldots, z_2^{n-j}, i z_2^{n-j} \right\}. \quad (12)$$

For each $t > 0$ consider $\phi_t \in \text{GL}(\mathfrak{h}_n)$ such that

$$\phi_t|_g = \begin{bmatrix} 1 & 1 \\ t & t \end{bmatrix}, \quad \phi_t|_{V_n} = \begin{bmatrix} t & t^2 & \cdots \\ t^2 & \ddots & \vdots \\ \vdots & \ddots & t \\ t & t^2 & t \end{bmatrix}. \quad (13)$$

That is, if $p_k = z_1^{n-k}$, $k = 0, \ldots, n$ then $\phi_t(s p_k) = \begin{cases} t s p_k, & \text{if } k = 0, n, s = 1, i, \\ t^2 s p_k, & \text{if } k \neq 0, n, s = 1, i. \end{cases}$

It is straightforward to check that there exist

$$\lambda_n = \lim_{t \to \infty} \phi_t \cdot [\cdot, \cdot].$$

In fact, we have that for $s = 1, i$,

$$\lambda_n(H, X) = 2Y, \quad \lambda_n(H, Y) = -2X,$$

$$\lambda_n(Z, s p_k) = s p_k, \quad \lambda_n(H, s p_k) = s p_k, \quad \forall k,$$

$$\lambda_n(X, s p_n) = [X, s p_n] = -n s p_{n-1}, \quad \lambda_n(X, s p_0) = [X, s p_0] = n s p_1,$$

$$\lambda_n(Y, p_n) = [Y, p_n] = -n i p_{n-1}, \quad \lambda_n(Y, i p_n) = [Y, i p_n] = n p_{n-1},$$

$$\lambda_n(Y, p_0) = [Y, p_0] = -n i p_1, \quad \lambda_n(Y, i p_0) = [Y, i p_0] = n p_1. \quad (14)$$

From this we can see that $\mathfrak{h}_n^{\infty} = (\mathbb{R}^{2n+6}, \lambda_n)$ is a solvable Lie algebra. The nilradical of $\mathfrak{h}_n^{\infty}$ is $n = \text{Span}\{X, Y, V_n\}$ with center $\mathfrak{z} \subseteq V_n$. Note that $\text{ad}_{\lambda_n}(Z)|_{\mathfrak{z}} = 1d$ and therefore $\mathfrak{h}_n^{\infty}$ satisfy the first condition of Theorem 2.2 but not the second one.

**Remark 3.1** For $n = 1$, the above definition gives $\phi_t|_{V_1} = \text{Diag}(t, t, t, t)$, which leads to a solvable Lie algebra with an abelian nilradical. In fact, the non zero bracket of the limit Lie algebra would only be the first two rows of (14). In particular we could not get a negative defined Ricci operator by Theorem 2.2. Hence we will study this case separately and take a different degeneration (see Lemma 3.4).

**Lemma 3.2** For $n \neq 1$, the algebra $\mathfrak{h}_n^{\infty} = (\mathbb{R}^{2n+6}, \lambda_n)$ admits an inner product with negative Ricci curvature.

**Proof** Let $\mathfrak{h}_n^{\infty} = (\mathbb{R}^{2n+6}, \lambda_n, \langle \cdot, \cdot \rangle)$ where $\lambda_n$ is given by (14) and $\langle \cdot, \cdot \rangle$ be the inner product that makes $\beta$ an orthonormal basis [see (12)]. As we have noticed before, it is a solvable
Lie algebra with nilradical \( n = \text{Span}\{X, Y, V_n\} \). Note that the mean curvature vector is \( H = 2(n + 1)Z \) and

\[
\text{ad}_{\lambda_n}(Z)|_n = \begin{bmatrix}
0 \\
-\frac{1}{n^2} & - & - \\
0 & \text{Id}_{V_n}
\end{bmatrix}, \quad \text{ad}_{\lambda_n}(H)|_n = \begin{bmatrix}
-\frac{2}{n^2} \\
\pi_n(H)
\end{bmatrix}
\]

\[
\text{ad}_{\lambda_n}(X)|_n = \begin{bmatrix}
0 \\
- & - & - \\
0 & 0 & 0
\end{bmatrix}, \quad \text{ad}_{\lambda_n}(Y)|_n = \begin{bmatrix}
0 \\
- & - & - \\
0 & 0 & 0
\end{bmatrix}, \quad \text{ad}_{\lambda_n}(sp_k)|_n = \begin{bmatrix}
0 \\
B
\end{bmatrix}, \quad k = 0, n, \quad \text{ad}_{\lambda_n}(sp_k)|_n = 0, \quad k \neq 0, n,
\]

for \( s = 1, i \), where \( B \) is a \( 2(n + 1) \times 2 \) matrix (that depends on \( s \) and \( k \)) with only 2 non-zero entries. Note that since \( \text{ad}_{\lambda_n}(H)|_n \) is skew-symmetric, \( \langle \text{Ric}_{\lambda_n}H, H \rangle = 0 \) and \( \text{tr}(\text{ad}_{\lambda_n}(H))^t \text{ad}_{\lambda_n}(\tilde{X}) = 0 \) for any \( \tilde{X} \in n \) [see (6)]. Moreover, \( \text{Ric}_{\lambda_n} \) is diagonal in our fixed basis and

\[
\langle \text{Ric}_{\lambda_n}Z, Z \rangle = -2(n + 1), \quad \langle \text{Ric}_{\lambda_n}H, H \rangle = 0,
\]

\[
\langle \text{Ric}_{\lambda_n}X, X \rangle = \langle \text{Ric}_{\lambda_n}Y, Y \rangle = -2n^2,
\]

\[
\langle \text{Ric}_{\lambda_n}sp_k, sp_k \rangle = -n^2 - 2(n + 1), \quad \text{for } k = 0, n \text{ and } s = 1, i,
\]

\[
\langle \text{Ric}_{\lambda_n}sp_k, sp_k \rangle = n^2 - 2(n + 1), \quad \text{for } k = n - 1, 1, \text{ and } s = 1, i,
\]

\[
\langle \text{Ric}_{\lambda_n}sp_k, sp_k \rangle = -2(n + 1), \quad \text{for } k \neq n, n - 1, 1, 0 \text{ and } s = 1, i,
\]

where if \( n = 2 \), a line where \( k \neq n, n - 1, 1, 0 \), as in the the last one, should be ignored. Also, it is easy to see that for \( n \neq 2 \) \( \{X, Y, p_n, \ldots, i_0p_0\} \) is a nice basis of \( n \) (see [11]) but for \( n = 2 \) it is not. Therefore, if \( n \geq 3 \) \( \text{Ric}_n = M|_{n \times n} \) is diagonal for any rescaling of this basis (see [11]).

For \( n \geq 3 \) we will get a negatively defined Ricci operator by acting on the bracket with \( f \in \text{GL}_{2n+6}(\mathbb{R}) \), \( f = \text{Diag}(1, 1, 1, 1, \frac{1}{a}, \frac{1}{b}, 1, \ldots, 1, \frac{1}{a}, \frac{1}{b}) \), or equivalently by changing...
the basis for

$$\{Z, H, X, Y, a p_n, b i p_n, p_{n-1}, \ldots, i p_1, a p_0, b i p_0\}.$$  

Note that for any $a, b \neq 0$, the zero and non-zero brackets are the same and hence it is not hard to check that $\text{Ric}_{f, \lambda_n}(a, n) = 0$. Also, $[\text{ad}_{f, \lambda_n} H|_n, (\text{ad}_{f, \lambda_n} H|_n)^\dagger]$ is diagonal in our basis and therefore $[\text{Ric}_{f, \lambda_n}]_\beta$ is diagonal for any $a, b$. Moreover, if $a \neq b$, $\text{ad}_{f, \lambda_n}(H)$ is no longer skew-symmetric and therefore $\langle \text{Ric}_{f, \lambda_n} H, H \rangle < 0$. Direct calculation shows that

$$\langle \text{Ric}_{f, \lambda_n} Z, Z \rangle = -2(n + 1), \quad \langle \text{Ric}_{f, \lambda_n} H, H \rangle = -n^2 \left(\frac{b}{a} - \frac{a}{b}\right)^2$$

$$\langle \text{Ric}_{f, \lambda_n} X, X \rangle = \langle \text{Ric}_{f, \lambda_n} Y, Y \rangle = -n^2 (a^2 + b^2),$$

$$\langle \text{Ric}_{f, \lambda_n} sp_k, sp_k \rangle = -2(n + 1), \quad \text{for } k \neq n, n - 1, 1, 0 \text{ and } s = 1, i,$n

$$\langle \text{Ric}_{f, \lambda_n} p_k, p_k \rangle = -n^2 a^2 + \frac{n^2}{2} \left(\left(\frac{b}{a}\right)^2 - \left(\frac{a}{b}\right)^2\right) - 2(n + 1), \quad k = 0, n,$n

$$\langle \text{Ric}_{f, \lambda_n} i p_k, i p_k \rangle = -n^2 b^2 + \frac{n^2}{2} \left(\left(\frac{a}{b}\right)^2 - \left(\frac{b}{a}\right)^2\right) - 2(n + 1), \quad k = 0, n. \quad (15)$$

Therefore, to get negative Ricci curvature it is enough to choose $a > b > 0$ such that

$$a^2 + b^2 < \frac{4(n+1)}{n^2}, \quad -n^2 b^2 + \frac{n^2}{2} \left(\left(\frac{a}{b}\right)^2 - \left(\frac{b}{a}\right)^2\right) < 2(n + 1). \quad (16)$$

Let $t > 1$ so that $t^2 - \frac{1}{t^2} < \frac{4(n+1)}{n^2}$, and $b > 0$ such that $b^2 < \frac{4(n+1)}{(1+t^2)n^2}$. It is easy to see that if $a = t b$, (16) holds.

The case when $n = 2$ follows the same lines. The rescaling is different and we can take for example

$$\{Z, H, X, Y, \frac{1}{2} z_1^2, \frac{1}{\sqrt{8}} i z_1^2, z_1 z_2, i z_1 z_2, \frac{1}{2} z_2^2, \frac{1}{\sqrt{8}} i z_2^2\}$$

that gives $\text{Ric}_{f, \lambda_2} = \frac{1}{2} \text{Diag}(-12, -4, -3, -3, -7, -9, -9, -20, -7). \quad \square$

Finally, by the continuous argument in Proposition 2.1, we get the following corollary.

**Corollary 3.3** For $n \neq 1$, $h_n$ admits an inner product with negative Ricci curvature.

**Lemma 3.4** $h_1$ admits an inner product with negative Ricci curvature.

**Proof** This can be considered as an example since all the techniques are the same, we will just change the degeneration. Hence we will give a lot of details to clarify the general construction.

As we said above, an explicit realization of the 4-dimensional representation $V_1$ can be obtained by taking the basis

$$v_1 = z_1, \quad v_2 = i z_1, \quad v_3 = z_2, \quad v_4 = i z_2,$$

and the action in this basis is given by

$$\pi_1(Z) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \pi_1(H) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \pi_1(X) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \pi_1(Y) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$
Therefore, we get the metric Lie algebra \( h_1 = u(2) \ltimes V_1 = (\mathbb{R}^8, \langle \cdot, \cdot \rangle, [\cdot, \cdot]) \) where \( \langle \cdot, \cdot \rangle \) is the inner product such that \( \beta = \{ Z, H, X, Y, v_1, v_2, v_3, v_4 \} \) is an orthonormal basis and

\[
\begin{align*}
[ H, X ] &= 2Y, & [ H, Y ] &= -2X, & [ X, Y ] &= 2H, \\
[ Z, v_i ] &= v_i, & i &= 1, 2, 3, 4, \\
[ H, v_1 ] &= v_2, & [ H, v_2 ] &= -v_1, & [ H, v_3 ] &= -v_4, & [ H, v_4 ] &= v_3, \\
[ X, v_1 ] &= -v_3, & [ X, v_2 ] &= -v_4, & [ X, v_3 ] &= v_1, & [ X, v_4 ] &= v_2, \\
[ Y, v_1 ] &= v_4, & [ Y, v_2 ] &= -v_3, & [ Y, v_3 ] &= v_2, & [ Y, v_4 ] &= -v_1.
\end{align*}
\]

Let us consider \( \phi_t \in GL(h_1) \) such that

\[
\phi_t|_g = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad \phi_t|_{V_1} = \begin{bmatrix} t & t^2 \\ 0 & t^2 \end{bmatrix}.
\] (17)

It is easy to see that the limit, \( h_1^\infty = (\mathbb{R}^8, \langle \cdot, \cdot \rangle, \lambda_1) \) is given by

\[
\begin{align*}
\lambda_1(H, X) &= 2Y, & \lambda_1(H, Y) &= -2X, & \lambda_1(Z, v_i) &= v_i, & i &= 1, 2, 3, 4, \\
\lambda_1(H, v_1) &= v_2, & \lambda_1(H, v_2) &= -v_1, & \lambda_1(H, v_3) &= -v_4, & \lambda_1(H, v_4) &= v_3, \\
\lambda_1(X, v_1) &= -v_3, & \lambda_1(X, v_2) &= -v_4, & \lambda_1(Y, v_1) &= v_4, & \lambda_1(Y, v_2) &= -v_3.
\end{align*}
\] (18)

It is a solvable Lie algebra with nilradical \( n = \text{Span}\{X, Y, V_1\} \). The center of \( n \) is \( Z = \text{Span}\{v_3, v_4\} \).

Direct calculation shows that if we change the basis to

\[
\{ Z, H, X, Y, \sqrt{2}v_1, v_2, \sqrt{2}v_3, v_4 \}
\]

or equivalently, if \( f \) is the corresponding element in \( gl(h_1^\infty) \) then

\[
\text{Ric}_f \lambda_1 = \text{Diag} \left( -4, -\frac{1}{2}, -1, -\frac{5}{4}, -\frac{25}{4}, -4, -4, -\frac{7}{4} \right).
\] (19)

Now that we have explicitly the bracket that gives us the inner product on \( h_1^\infty \) with the desired property we can get the one \( h_1 \). In fact, one can show that for any \( t, [\cdot, \cdot]_t = \phi_t \cdot [\cdot, [\cdot, \cdot]]_t \) is given by

\[
\begin{align*}
[ H, X ]_t &= 2Y, & [ H, Y ]_t &= -2x, & [ X, Y ]_t &= \frac{2}{t^2} H, & [ Z, v_i ]_t &= v_i, & \forall i, \\
[ H, v_1 ]_t &= \sqrt{2}v_2, & [ H, v_2 ]_t &= -\frac{1}{\sqrt{2}} v_1, & [ H, v_3 ]_t &= -\sqrt{2}v_4, & [ H, v_4 ]_3 &= \frac{1}{\sqrt{2}} v_3, \\
[ X, v_1 ]_t &= -v_3, & [ X, v_2 ]_t &= -v_4, & [ X, v_3 ]_t &= \frac{1}{t} v_1, & [ X, v_4 ]_t &= \frac{1}{t^2} v_2, \\
[ Y, v_1 ]_t &= \sqrt{2}v_4, & [ Y, v_2 ]_t &= -\frac{1}{\sqrt{2}} v_3, & [ Y, v_3 ]_t &= \frac{\sqrt{2}}{t} v_2, & [ Y, v_4 ]_t &= -\frac{1}{\sqrt{2}t^2} v_1,
\end{align*}
\]

with corresponding Ricci operator

\[
\text{Ric}_{[\cdot, \cdot]}_t = \text{Diag} \left( -4, \frac{4-t^2}{2t^2}, \frac{13}{4t^4}, \frac{3}{2t^4}, \frac{3}{2t^4}, \frac{3}{2t^4}, \frac{3}{2t^4}, \frac{3}{2t^4}, \frac{3}{2t^4}, \frac{3}{2t^4} \right).
\]

It is easy to check that for any \( t > \sqrt{3} + \sqrt{6} \) all these constants are negatives and therefore \( h_1 = (\mathbb{R}^8, \langle \cdot, \cdot \rangle, [\cdot, \cdot]_t) \) has negative Ricci curvature for any \( t > \sqrt{3} + \sqrt{6} \).
Remark 3.5 Recall that \((\mathfrak{g}, g \cdot [\cdot, \cdot], \langle \cdot, \cdot \rangle)\) is isometric to \((\mathfrak{g}, [\cdot, \cdot], \langle g \cdot \cdot, g \cdot \cdot \rangle)\) for any \(g \in \text{GL}(\mathfrak{g})\) and therefore in the previous example we get that, for example,
\[
(u(2) \ltimes V_1, [\cdot, \cdot], \langle \cdot, \cdot \rangle) \text{ is isometric to } (u(2) \ltimes V_1, [\cdot, \cdot], \langle \cdot, \cdot \rangle_3)
\]
where \(\langle \cdot, \cdot \rangle_3 = \langle \phi_3 f \cdot, \phi_3 f \cdot \rangle\). It is not hard to check that this is the inner product that makes
\[
\{Z, H, \frac{1}{3} X, \frac{1}{3} Y, \frac{\sqrt{2}}{3} v_1, \frac{1}{3} v_2, \frac{\sqrt{2}}{3} v_3, \frac{1}{5} v_4\}
\]
an orthonormal basis of \((u(2) \ltimes V_1)\).

With these two lemmas we get the general result.

**Theorem 3.6** Let \((V_n, \pi_n)\) be the real \(2(n + 1)\)-dimensional representation of \(\mathfrak{su}(2)\) on \(P_{2,n}\) extended to \(u(2)\) by letting the center act as multiples of the identity. Hence the Lie algebra \((u(2) \ltimes V_n)\) admits an inner product with negative Ricci curvature.

**Remark 3.7** It is easy to check that if \(Z\) acts as a non-trivial multiple of the identity we will get, in every step of the construction, a Lie algebra isomorphic to the one where \(Z\) acts as the identity. Henceforth, by changing the action of \(Z\) we do not get any new examples.

### 3.2 Odd dimensional representations

For any \(n\) even, let us consider \(l_n = \mathfrak{g} \ltimes V_n^{\alpha}\) and fix the basis [see (10)]
\[
\beta = \{Z, H, X, Y, p_n^+, ip_n^-, p_{n-1}^+, i p_{n-1}^-, \ldots, sp_k^\pm\},
\]
where \(s = 1\) if \(n\) is even and \(s = i\) if it is odd. For each \(t > 0\) define \(\phi_t \in \text{GL}(l_n)\) such that
\[
\left.\phi_t\right|_{\mathfrak{g}} = \begin{bmatrix} 1 & t \\ t & t^2 \end{bmatrix}, \quad \left.\phi_t\right|_{V_n^{\alpha}} = \begin{bmatrix} t \\ \frac{1}{t} \\ \ldots \end{bmatrix}
\]
That is,
\[
\phi_t(s p_k^\pm) = \begin{cases} t s p_k^\pm, & k = n, s = 1, i; \\ t^2 s p_k^\pm, & k \neq n, s = 1, i. \end{cases}
\]
As before, straightforward calculation shows that \(\mu_n = \lim_{t \to \infty} \phi_t[\cdot, \cdot]\) is given by
\[
\mu_n(H, X) = 2Y, \quad \mu_n(H, Y) = -2X,
\]
\[
\mu_n(Z, sp_k) = sp_k, \quad \mu_n(H, sp_k) = [H, sp_k], \quad \forall k, s = 1, i,
\]
\[
\mu_n(X, p_n^+) = [X, p_n^+] = -n p_{n-1}^-, \quad \mu_n(X, ip_n^-) = [X, ip_n^-] = -n ip_{n-1}^-,
\]
\[
\mu_n(Y, p_n^+) = [Y, p_n^+] = -n ip_{n-1}^+, \quad \mu_n(Y, ip_n^-) = [Y, ip_n^-] = n p_{n-1}^-.
\]

From this we can see that \(\bigoplus_n = (\mathbb{R}^{2n+5}, \mu_n)\) is a solvable Lie algebra with nitradical \(\mathfrak{n} = \text{Span}[X, Y, V_n^{\alpha}]\) whose center is \(3 \subset V_n^{\alpha}\). As in the even case, we fix an inner product that makes \(\beta\) an orthonormal basis and calculate the Ricci operator for the metric Lie algebra.
Proposition 2.1). The identity. Hence the Lie algebra $u_v^1$ admits an inner product with negative Ricci curvature. For any even natural number $n$, let $V_n^e, \pi_n$ be the real $(n+1)$-dimensional representation of $su(2)$ as in (10) extended to $u(2)$ by letting the center act as multiples of the identity. Hence the Lie algebra $u(2) \ltimes V_n^e$ admits an inner product with negative Ricci curvature.

Example 3.9 The first example one can construct in this way corresponds to the 3-dimensional representation. Note that it is a 7-dimensional Lie algebra and therefore the smallest dimensional one in any case. An explicit realization of this representation is given by $V_2^e = \text{Span}\{v_1, v_2, v_3\}$ where

$$v_1 = z_1^2 + z_2^2, \quad v_2 = i(z_1^2 - z_2^2), \quad v_3 = 2iz_1z_2.$$ 

Note that $v_1 = p_n^+, v_2 = ip_n^-, v_3 = i p_n^+$ for $n = 2$. In this case the Lie algebra $l_2 = u(2) \ltimes V_2^e = (\mathbb{R}^7, [\cdot, \cdot])$ where

$$[H, X] = 2Y, \quad [H, Y] = -2X, \quad [X, Y] = 2H, \quad [Z, v_i] = v_i \quad i = 1, 2, 3, \quad [H, v_1] = -2v_2, \quad [H, v_2] = 2v_1, \quad [X, v_2] = -2v_3, \quad [X, v_3] = 2v_2, \quad [Y, v_1] = -2v_3, \quad [Y, v_3] = 2v_1.$$ 

The degeneration is obtained by $\phi_t \in \text{GL}(h)$,

$$\phi_t|_0 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad \phi_t|_{V_2^e} = \begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix}.$$
and therefore the limit \( l_2^\infty = (\mathbb{R}^7, \langle \cdot, \cdot \rangle, \mu_2) \) where \( \mu_2 = \lim_{t \to \infty} \phi_t \cdot [\cdot, \cdot] \) is given by

\[
\begin{align*}
\mu_2(H, X) &= 2Y, \\
\mu_2(H, Y) &= -2X, \\
\mu_2(Z, v_i) &= v_i, & i &= 1, 2, 3, \\
\mu_2(H, v_1) &= -2v_2, \\
\mu_2(H, v_2) &= 2v_1, \\
\mu_2(X, v_2) &= -2v_3, \\
\mu_2(Y, v_1) &= -2v_3.
\end{align*}
\]

As we have asserted before, \( l_2 \) and then \( l_2^\infty \) both admit an inner product with negative Ricci curvature. Indeed, we can proceed as in Lemma 3.4 and get explicitly one of these inner products. A direct calculation shows that if \( \langle \cdot, \cdot \rangle \) is the inner product such that \( \{Z, H, X, Y, v_1, v_2, v_3\} \) is an orthonormal basis of \( l_2^\infty \) and \( f \in \text{GL}_7(\mathbb{R}) \) corresponds to the change of basis given by

\[
\left\{Z, H, X, Y, \frac{1}{\sqrt{2}}v_1, \frac{1}{\sqrt{3}}v_2, v_3\right\},
\]

then the Ricci operator of the metric Lie algebra \( l_2^\infty = (\mathbb{R}^7, f \cdot \mu_2, \langle \cdot, \cdot \rangle) \) is negative definite. Note that the formula for this Ricci operator is slightly different form (23). One can check in this case that \( l_2 = (\mathbb{R}^7, \phi_{\sqrt{12}} \cdot f \cdot \mu_2, \langle \cdot, \cdot \rangle) \) has negative Ricci curvature. In fact, in \( l_2^\infty \) we get

\[
\text{Ric}_{f \cdot \mu_2} = \frac{1}{3} \text{Diag}(-9, -1, -2, -3, -17, -6, -4),
\]

and in \( l_2 \),

\[
\text{Ric}_{\phi_{\sqrt{12}} \cdot f \cdot [\cdot, \cdot]} = \text{Diag}\left(-3, -\frac{23}{72}, -\frac{1}{18}, -\frac{3}{8}, -\frac{203}{36}, -\frac{47}{24}, -\frac{101}{72}\right),
\]

as wanted.

**Remark 3.10** Note that in every case the restriction to \( u(2) \) of the degeneration \( \phi_1|_\mathfrak{g} \) is the same and therefore, we can proceed enterally analogous with direct sum of these representations, \( \mathfrak{g} \cong (V_{n_1} \oplus \cdots \oplus V_{n_l} \oplus V_{n_{l+1}}^{\alpha} \oplus \cdots \oplus V_{n_k}^{\alpha}) \).

**Theorem 3.11** Let \((V, \pi)\) be a non-trivial real representation of \( \mathfrak{su}(2) \) extended to \( \mathfrak{u}(2) \) by letting the center act as multiples of the identity, then the Lie algebra \( \mathfrak{u}(2) \ltimes V \) admits an inner product with negative Ricci curvature.

**Proof** Let \( V = V_1 \oplus \cdots \oplus V_r \) be the decomposition of \( V \) in irreducible representations of \( \mathfrak{su}(2) \). If the trivial representation does not appear in the decomposition, we can reorder so that \( V_i = V_{n_i} \) for \( 1 \leq i \leq l \) and \( V_j = V_{n_j}^{\alpha} \) if \( l + 1 \leq j \leq r \), as defined in 2.2, for some even \( n_i \) and odd \( n_j \).

For each \( 1 \leq i \leq r \) let \( \beta_i \) be the ordered basis of \( V_i \) we defined in (9) and (10) respectively and let us denote its elements by

\[
\beta_i = \{p_{n_i, i}, i p_{n_i, i}, \ldots p_{0, i}, i p_{0, i}\}, \quad 1 \leq i \leq l,
\]

\[
\beta_j = \left\{ p_{n_j, j}^+, i p_{n_j, j}^-, \ldots, s p_{n_j, j}^\alpha \right\}, \quad j > l,
\]

and let \( \beta \) be the basis of \( V \) we obtain by taking the union of them; i.e. \( \beta = \beta_1 \cup \cdots \cup \beta_r \). Consider now the metric Lie algebra \( \mathfrak{u}(2) \ltimes V = (\mathbb{R}^{d + 4}, \langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle) \), where \( d = \dim V = (\dim V_1 + \cdots + \dim V_r), Z \) is the element in the center of \( \mathfrak{u}(2) \) acting as the identity and \( \langle \cdot, \cdot \rangle \) the inner product that makes \( \{Z, H, X, Y, \beta\} \) an orthonormal basis.
For each \( t > 0 \) define \( \phi_t \in \text{GL}(u(2) \times V) \) such that
\[
\phi_t|_{u(2)} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad \phi_t|_{V_{n_j}} = \begin{bmatrix} t \\ t^2 \\ \ddots \\ t^2 \end{bmatrix}, \quad \phi_t|_{V_{n_j}^0} = \begin{bmatrix} t \\ t^2 \\ \ddots \\ t^2 \end{bmatrix},
\]
where \( 1 \leq i \leq l, l + 1 \leq j \leq r \) as in (13) or (20) respectively.

As before, straightforward calculation shows that there exists \( v = \lim_{t \to \infty} \phi_t \cdot [\cdot, \cdot] \) and it gives a solvable Lie algebra that will be denoted by \((s, v, \langle \cdot, \cdot \rangle)\). The nilradical of \( s, n, \) is \( \text{Span}\{X, Y, V_1, \ldots, V_r\} \) and its center is contained in \( V \). It is easy to see that if \( \tilde{X} \in u(2) \), \( p \in V_{n_i} \) and \( q \in V_{n_j}^0 \) then
\[
v(\tilde{X}, p) = \lambda_{n_i}(\tilde{X}, p) \text{ [see (14)]} \quad \text{and} \quad v(\tilde{X}, q) = \mu_{n_j}(\tilde{X}, q) \text{ [see (22)]}
\]
and moreover
\[
ad_v(Z)|_V = Id, \quad \ad_v(X)|_V = \begin{bmatrix} \ad_{\lambda_{n_j}}(X_1) & & & & \\ & \ddots & & & \\ & & \ad_{\lambda_{n_j}}(X_l) & & \\ & & & \ddots & \\ & & & & \ad_{\lambda_{n_j}}(X_1) \end{bmatrix},
\]
\[
ad_v(H)|_V = \begin{bmatrix} \pi_{n_j}(H) \\ \ddots \\ \pi_{n_j}(H) \end{bmatrix}, \quad \ad_v(Y)|_V = \begin{bmatrix} \ad_{\mu_{n_j}}(Y_1) & & & & \\ & \ddots & & & \\ & & \ad_{\mu_{n_j}}(Y_l) & & \\ & & & \ddots & \\ & & & & \ad_{\mu_{n_j}}(Y_1) \end{bmatrix}.
\]

The mean curvature vector is \( \mathbf{H} = dZ \) where \( d = \dim V \) and since \( \pi_{n_j}(H) \) is skew symmetric for every \( i, \langle \text{Ric} H, H \rangle = 0 \).

To get a negatively defined Ricci operator, we will proceed as in the previous cases by changing the basis in each \( V_i \) as we did before with the same \( a \) and \( b \) in all of them [see (15) and (23)]. Let \( f \in \text{GL}(\mathbb{R}^{d+4}) \) such that
\[
f|_{u(2)} = \text{Diag}(1, 1, 1, 1), \quad f|_{V_i} = \text{Diag} \left( \frac{1}{a^2}, \frac{1}{b^2}, 1, \ldots, 1, \frac{1}{a^2}, \frac{1}{b^2} \right), \quad \text{if } i \leq l,
\]
\[
f|_{V_j} = \text{Diag} \left( \frac{1}{a^2}, \frac{1}{b^2}, 1, \ldots, 1 \right), \quad \text{if } j > l.
\]

Note that by (28) the formula of the Ricci operator is very similar to the one in the previous cases. Indeed, straightforward calculation shows that every element in our fixed basis is an eigenvector of the Ricci operator and
\[
\langle \text{Ric}_{f \cdot v} Z, Z \rangle = -d, \quad \langle \text{Ric}_{f \cdot v} H, H \rangle = -\left( \frac{b}{a} - \frac{a}{b} \right)^2 N
\]
\[
\langle \text{Ric}_{f \cdot v} X, X \rangle = \langle \text{Ric}_v Y, Y \rangle = -(a^2 + b^2)N,
\]
\[
\langle \text{Ric}_{f \cdot v} p_{k,i}^\epsilon, p_{k,i}^\epsilon \rangle = \frac{n^2}{2} (a^2 + b^2) - d, \quad k \neq n_i, n_i - 1, 0, \epsilon = \pm 0, s = 1, i,
\]
\[
\langle \text{Ric}_{f \cdot v} p_{k,i}^\epsilon, p_{k,i}^\epsilon \rangle = -a^2 n_i^2 + \frac{n^2}{2} \left( \left( \frac{b}{a} \right)^2 - \left( \frac{a}{b} \right)^2 \right), \quad k = 0, n_i, \epsilon = \pm 0, s = 1, i,
\]
\[
\langle \text{Ric}_{f \cdot v} p_{k,i}^\epsilon, p_{k,i}^\epsilon \rangle = -a^2 n_i^2 + \frac{n^2}{2} \left( \left( \frac{a}{b} \right)^2 - \left( \frac{b}{a} \right)^2 \right), \quad k = 0, n_i, \epsilon = \pm 0, s = 1, i,
\]
where \( N = \sum_{i \leq l} n_i^2 + \frac{1}{2} \sum_{i > l} n_i^2 \). To reduce the formula we are using here the notation \( p_{k,i}^\epsilon = p_{k,i}^\epsilon \) for \( \epsilon = 0 \).
It is easy to see that if $a > b$ satisfy

$$(a^2 + b^2) < \frac{2d}{n_0}, \quad \left(\frac{a}{b}\right)^2 - \left(\frac{b}{a}\right)^2 < \frac{2d}{n_0},$$

(30)

for $n_0 = \max\{n_i, \ 1 \leq i \leq k\}$, then all the eigenvalues of the Ricci operator are negative. To see that such $a$ and $b$ do exist we can proceed as in (16) with $n = n_0$.

Assume now that the decomposition of $V$ is $V = V_0 \oplus V_1 \oplus \cdots \oplus V_r$, where $su(2)$ is acting trivially on $V_0$, $V_i = V_{n_i}$ for $1 \leq i \leq l$ and $V_j = V_{n_j}^0$ if $l + 1 \leq j \leq r$, for some even $n_i$ and odd $n_j$. We can proceed as before with

$$\phi_t|V_0 = Id, \quad f|V_0 = Id.$$ 

It is easy to see that we obtain (29) and $\text{Ric}_{f^t \cdot \nu}|V_0 = -dId$. Therefore we get the same conditions for $a$, $b$ as before [see (30)].

Remark 3.12 In the proof, for simplicity, we have not made any distinction when $V_i = V_1$ or $V_i = V_2^o$. Recall that if $V_i = V_1$, one has to consider a different degeneration (see Lemma 3.4) and when $V_i = V_2^o$ the formula for the Ricci operator is a little different (see Example 3.9). It is not hard to check that in the case of $V_2^o$, condition (30) is sufficient but when $V_i = V_1$, condition $\frac{a}{b} < 3, 5$ should be added. In any case, suitable real numbers $a$, $b$ always exist.

Remark 3.13 A very similar argument can be applied when an element in the center acts as a positive multiple of the identity in each irreducible component.

References

1. Alekseevskii, D.: Homogeneous Riemannian spaces of negative curvature. Math. USSR Sb. 25, 87–109 (1976)
2. Bochner, S.: Vector fields and Ricci curvature. Bull. Am. Math. Soc. 52, 776–797 (1946)
3. Bröker, T., Dieck, T.: Representations of Compact Lie groups. Springer, New York (1985)
4. Dotti, I.: Ricci curvature of left invariant metrics on solvable unimodular Lie groups. Math. Z. 180, 257–263 (1982)
5. Dotti, I.: Metrics with non-positive Ricci curvatures on semidirect products. Q. J. Math. 37, 309–314 (1986)
6. Dotti, I., Leite, M.L.: Metrics of negative Ricci curvature on SL$(n, \mathbb{R})$, $n \geq 3$. J. Differ. Geom. 17(4), 635–641 (1982)
7. Dotti, I., Leite, M.L., Miatello, R.: Negative Ricci curvature on complex semisimple Lie group. Geom. Dedic. 17, 207–218 (1984)
8. Heintze, E.: On homogeneous manifolds of negative curvature. Math. Ann. 211, 23–34 (1974)
9. Jablonski, M., Petersen, P.: A step towards the Alekseevskii conjecture. Math Ann. (2016). doi:10.1007/s00208-016-1429-7
10. Lauret, J.: Ricci soliton solvmanifolds. J. Reine Angew. Math. 650, 1–21 (2011)
11. Lauret, J., Will, C.E.: On the diagonalization of the Ricci flow on Lie groups. Proc. AMS 141(10), 36513663 (2013)
12. Lauret, J., Will, C.E.: On the symplectic curvature flow for locally homogeneous manifolds. J. Symp. Geom. (in press) (arXiv)
13. Milnor, J.: Curvatures of left invariant metrics on Lie groups. Adv. Math. 21(3), 293–329 (1976)
14. Nikolayevsky, Y., Nikonorov, YuG: On solvable Lie groups of negative Ricci curvature. Math. Z. 280, 1–16 (2015)
15. Nikolayevsky, Y.: Solvable extensions of negative Ricci curvature of filliform Lie groups. Math. Nachr. 289, 321–331 (2016)
16. Will, C.E.: The space of solsosolitons in low dimensions. Ann. Global Anal. Geom. 40, 291–309 (2011)