An estimate of the $B \to K^*\gamma$ decay form factor

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Abstract

We present the results of a lattice QCD calculation of the form factor relevant to $B \to K^*\gamma$ decay. Our final value, $T(0) = 0.24 \pm 0.03^{+0.04}_{-0.01}$, is obtained in the quenched approximation, and by extrapolating $M_H^{3/2} \times T_H^{H \to K^*}(q^2 = 0)$ from the directly accessed $H$-heavy mesons to the meson $B$. We also show that the extrapolation from $B \to K^*\gamma^*$ ($q^2 \neq 0$) to $B \to K^*\gamma$ ($q^2 = 0$), leads to a result consistent with the one quoted above. On the other hand, our results are not accurate enough to solve the $SU(3)$ flavor breaking effects in the form factor and we quote $T_{B \to K^*}(0)/T_{B \to \rho}(0) = 1.2(1)$, as our best estimate.

PACS: 12.38.Gc, 13.25.Hw, 13.25.Jx, 13.30.Ce, 13.75Lb
1 Introduction

The flavour changing neutral decays, $B \rightarrow V\gamma$ ($V = K^{*}, \rho, \omega$), are induced by penguin diagrams. Their accurate experimental measurement gives us information about the heavy particle content in the loops, and thus might be a window to the physics beyond the Standard Model (SM). This is why a huge amount of both experimental and theoretical research has been invested in studying these modes over the past decade.

The experimenters at CLEO were first to observe and measure the $B \rightarrow K^{*}\gamma$ decay rate [1]. Averaging over the neutral and charged $B$-mesons they reported

$$B(B \rightarrow K^{*}\gamma) = (4.5 \pm 1.5 \pm 0.9) \times 10^{-5}. \quad (1)$$

Today, the unprecedented statistical quality of the data collected at the $B$-factories made it possible to do precision measurements separately for $B^{0}$ and $B^{\pm}$ decays, namely

$$B(B^{0} \rightarrow K^{*0}\gamma) = \begin{cases} (3.92 \pm 0.20 \pm 0.24) \times 10^{-5} & \text{BaBar [2]} \\ (4.01 \pm 0.21 \pm 0.17) \times 10^{-5} & \text{Belle [3]} \end{cases}, \quad (2)$$

$$B(B^{+} \rightarrow K^{*+}\gamma) = \begin{cases} (3.87 \pm 0.28 \pm 0.26) \times 10^{-5} & \text{BaBar [2]} \\ (4.25 \pm 0.31 \pm 0.24) \times 10^{-5} & \text{Belle [3]} \end{cases}.$$  

Besides, the first significant measurements of $B \rightarrow \rho(\omega)\gamma$ [4], opened a discussion on the possibility of constraining $|V_{td}/V_{ts}|$, thus providing an alternative to the constraint arising from the ratio of the oscillation frequencies in the $B_{s}^{0} - B_{s}^{0}$ and $B_{d}^{0} - B_{d}^{0}$ systems, $\Delta m_{B_{s}}/\Delta m_{B_{d}}$ [5].

However, when looking for non-SM effects in these decays, one should be able to confront the above experimental results to the corresponding theoretical estimates within the SM. As usual, the main obstacle is a lack of good theoretical control over the hadronic uncertainties. The hadronic matrix element entering the analysis of these, electromagnetic penguin induced, decays is

$$\langle V(p'; e_{\lambda})|T^{\mu\nu}(0)|B(p)\rangle = e^{*}_{\alpha}(p', \lambda) \times T^{\alpha\mu\nu},$$

where

$$T^{\alpha\mu\nu} = e^{\alpha\mu\nu\beta} \left[ p_{\beta} + \frac{m_{B}^{2} - m_{V}^{2}}{q^{2}} q_{\beta} \right] T_{1}(q^{2}) + \frac{m_{B}^{2} - m_{V}^{2}}{q^{2}} q_{\beta} T_{2}(q^{2})$$

$$+ \frac{2p^{\alpha}}{q^{2}} e^{\mu\nu\sigma\lambda} p_{\sigma} p'_{\lambda} \left( T_{2}(q^{2}) - T_{1}(q^{2}) + \frac{q^{2}}{M_{B}^{2} - m_{V}^{2}} T_{3}(q^{2}) \right), \quad (3)$$

where $q = p - p'$, the tensor current $T^{\mu\nu} = i\bar{\sigma}^{\mu\nu} b$ for $V = K^{*}$, and $T^{\mu\nu} = i\bar{d} \sigma^{\mu\nu} b$ for $V = \rho$, with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu}, \gamma_{\nu}]$. The above definition is suitable for the extraction of the form factors from the correlation functions computed on the lattice. The form factors $T_{1,2,3}(q^{2})$ are the

\footnote{A review on rare $B$-decays, containing an extensive list of references can be found in [6].}
same as those computed by the QCD sum rules \[7, 8\]. For the physical photon \((q^2 = 0)\),
the form factors \(T_1(0) = T_2(0)\), while the coefficient multiplying \(T_3(0)\) is zero.

In this paper we show that the strategy which we previously employed to compute
the \(B \to \pi\) semileptonic form factors \[9\] can be used to compute the radiative decays as
well. Although the attainable accuracy is quite limited, we believe the values we get are
still phenomenologically useful. In what follows we will show how we obtain
\(T^{B \to K^*}(0) = 0.24(3)(1^\pi)\), from our quenched QCD calculations with \(O(a)\) improved Wilson quarks at
two lattice spacings. We also obtain \(T^{B \to K^*}(0)/T^{B \to \rho}(0) = 1.2(1)\) although that result is
unstable when applying different strategies and using different lattice spacings.

2 Methods to approaching \(q^2 = 0\) and \(M_B\)

Even though we work with ever smaller lattice spacings (“\(a\)”), we are still not able to work
directly with the heavy \(b\)-quark. Instead of simulating a meson with \(M_B = 5.28\) GeV, we compute
the form factors with fictitious heavy-light mesons \((H)\) of masses \(M_B > M_H \geq M_D\), and then extrapolate them in \(1/M_H\) to \(1/M_B\), guided by the heavy quark scaling
laws. Alternatively one can discretise the nonrelativistic QCD (NRQCD) which basically
means the inclusion of \(1/(am_b)\)-corrections to the static limit. This, in the lattice QCD
community, is known as the “NRQCD approach”. Finally, one can build an effective theory
that combines the above two, which is known as the “Fermilab approach”. Each of the
mentioned approaches has its advantages and drawbacks. They were all used in computing
the \(B \to \pi\) semileptonic form factors and the results show a pleasant overall agreement
(see e.g. fig.3 in the first ref. \[10\]).

Concerning the methodology employed while working with propagating heavy quarks,
one should keep in mind that the form factors are accessed for all \(q^2 \in [0, (M_H - m_V)^2]\).
Only after extrapolating to \(M_B\), at fixed values of \(v \cdot p'\), the \(q^2\)-region becomes large and
the form factors are shifted to large \(q^2\)’s. \(^2\) The assumption underlying this extrapolation
is that the HQET scaling laws \[11\] remain valid when \(E = v \cdot p' > m_V\). In the case of
\(B \to \pi\ell\nu\), it appears that this assumption is not particularly worrisome as the form factors,
after extrapolating to \(M_B\) \[9, 12\], are consistent with those obtained by the effective heavy
quark approaches \[13, 14\], at large \(q^2\)’s. If one is interested in the form factor at \(q^2 = 0\),
then in order to extrapolate from large \(q^2\)’s one has to make some physically motivated
assumption about the \(q^2\)-shapes of the form factors.

Otherwise, when working with propagating heavy quarks, one can extrapolate the form
factors directly computed on the lattice at \(q^2 = 0\) in \(1/M_H\) to \(1/M_B\). The useful scaling
law relevant to this situation was first noted in the framework of the light cone QCD sum
rules (LCSR) \[15\], then generalised to the large energy effective theory in ref. \[16\], and
finally confirmed in the soft collinear effective theory \[17\]. The underlying assumption in
this extrapolation is that the scaling law would remain valid even when the light meson
is not very energetic (in the rest frame of the heavy, the \(q^2 = 0\) point corresponds to
\(E = (M_H^2 + m_V^2)/2M_H\)).

\(^2\)\(v\) stands for the heavy quark (meson) four-velocity, so that \(q^2 = M_B^2 + m_V^2 - 2M_B v \cdot p'\). In the rest
frame of the heavy meson, \(E = v \cdot p'\) is the energy of the light meson emerging from the decay.
In ref. [9] we showed that the results for the $B \to \pi \ell \nu$ form factor at $q^2 = 0$, obtained by employing either of these two different ways of extrapolating to $M_B$, are fully compatible. The method of extrapolating in $1/M_B$ at $q^2 = 0$ fixed is particularly useful for $B \to K^*(\rho)\gamma$, where the main goal is to compute the form factor when the photon is on-shell, $T(q^2 = 0)$. This is what we do in this paper. As a cross-check of our result we also use the standard method (extrapolating in $1/M_B$ prior to the extrapolation in $q^2$, down to $q^2 = 0$).

3 Raw Lattice Results

The form factors are extracted from the study of suitable ratios of three- and two-point correlation functions, namely

$$ R_{\zeta \mu \nu}(t_y) = \frac{C_{\zeta \mu \nu}^{(3)}(t_x, t_y; \vec{q}, \vec{p}_H)}{\frac{1}{3} \sum_{\alpha=0}^{3} C_{\alpha \alpha}^{(2)}(t_y, \vec{p}')} C_{\alpha H}^{(2)}(t_x - t_y, \vec{p}_H) \times \sqrt{Z_H} \sqrt{Z_V}, \quad (4) $$

where $\vec{p}' = \vec{p}_H - \vec{q}$. The correlation functions and their asymptotic behavior are given by

$$ C_{\alpha H}^{(2)}(t, \vec{p}) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \langle P_5(\vec{x}, t) P_5^\dagger(0) \rangle \to \frac{Z_V}{2E_H} e^{-E_H t}, $$

$$ C_{\alpha \beta}^{(2)}(t, \vec{p}') = \sum_{\vec{x}} e^{i\vec{p}'\cdot\vec{x}} \langle V_{\alpha}(\vec{x}, t) V_{\beta}^\dagger(0) \rangle \to \frac{Z_V}{2E_V} e^{\ast_\alpha (p', \lambda)} e_{\beta}(p', \lambda) e^{-E_V t}, $$

$$ C_{\zeta \mu \nu}^{(3)}(t_x, t_y, \vec{q}, \vec{p}_H) = \sum_{\vec{x}, \vec{y}} e^{i(\vec{q}\cdot\vec{y} - \vec{p}_H\cdot\vec{x})} \langle V_{\zeta}(0) \hat{T}_{\mu \nu}(y) P_5^\dagger(x) \rangle \to \frac{\sqrt{Z_V}}{2E_V} e^{-E_V t_y} \times (V(\vec{p'}, \lambda)|\hat{T}_{\mu \nu}(0)|H(\vec{p}_H)) \times \frac{\sqrt{Z_H}}{2E_H} e^{-E_H (t_x - t_y)} $$

$$ = \frac{\sqrt{Z_V} Z_H}{4E_V E_H} \sum_{\lambda} e^{\ast_\xi(p', \lambda)} e_{\alpha}(p', \lambda) T_{\alpha \mu \nu} e^{-E_H t_x + (E_H - E_V) t_y}. \quad (5) $$

For the interpolating field we choose $P_5 = \overline{q} \gamma_5 Q$ and $V_\mu = \overline{q} \gamma_\mu q$, with $q$ and $Q$ being light and heavy quark respectively. We also defined $\sqrt{Z_H} = |\langle 0| P_5 |H \rangle|$, and $\sqrt{Z_V} e^{\ast_\mu (p', \lambda)} = |\langle 0| V_\mu |V (p', \lambda) \rangle|$. The hat over the tensor current indicates that it is $O(a)$ improved and renormalised at some scale $\mu$, i.e.,

$$ \hat{T}_{\mu \nu}(\mu) = Z_{T}^{(0)}(g_0^2, \mu) \left[ 1 + b_T(g_0^2) \frac{am_q}{m_\pi} \right] \left[ i \overline{Q} \sigma_{\mu \nu} q + ac_T(g_0^2) \left( \partial_\mu \overline{Q} \gamma_\nu q - \partial_\nu \overline{Q} \gamma_\mu q \right) \right], \quad (6) $$

where $am_q(\mu) = (1/\kappa_q(\mu)) - 1/\kappa_T(\mu)$, and $m_\pi = (m_q + m_Q)/2$. The renormalisation constant, $Z_{T}^{(0)}(g_0^2, \mu)$ and the operator improvement coefficients, $b_T(g_0^2)$ and $c_T(g_0^2)$, are specified in table 1, where we also give the basic information about our lattices. By using the standard
| Set 1       | $24^3 \times 64$, $\beta = 6.2$, $c_{SW} = 1.614$ |
|-------------|--------------------------------------------------|
|             | 200 configs; $a^{-1} = 2.7(1)$ GeV; $t_x = 27$  |
| $\kappa_Q$ | 0.125; 0.122; 0.119; 0.115                       |
| $\kappa_q$ | 0.1344; 0.1349; 0.1352                           |
| $Z_T^{(0)}$ | 0.876(2); $b_T^{bpt} = 1.22$; $c_T = 0.06$       |

| Set 2       | $24^3 \times 64$, $\beta = 6.2$, $c_{SW} = 1.614$ |
|-------------|--------------------------------------------------|
|             | 200 configs; $a^{-1} = 2.8(1)$ GeV; $t_x = 31$  |
| $\kappa_Q$ | 0.128; 0.125; 0.122; 0.119; 0.116                 |
| $\kappa_q$ | 0.1344; 0.1346; 0.1348; 0.1350; 0.1352            |
| $Z_T^{(0)}$ | 0.876(2); $b_T^{bpt} = 1.22$; $c_T = 0.06$       |

| Set 3       | $32^3 \times 70$, $\beta = 6.45$, $c_{SW} = 1.509$ |
|-------------|--------------------------------------------------|
|             | 100 configs; $a^{-1} = 3.8(1)$ GeV; $t_x = 34$  |
| $\kappa_Q$ | 0.1285; 0.125; 0.122; 0.119; 0.116; 0.114        |
| $\kappa_q$ | 0.1349; 0.1351; 0.1352; 0.1353                    |
| $Z_T^{(0)}$ | 0.883(2); $b_T^{bpt} = 1.20$; $c_T^{bpt} = 0.02$ |

Table 1: Details on the lattices used in this work including the values of the Wilson heavy ($\kappa_Q$) and light ($\kappa_q$) mass parameters; $O(a)$ improvement coefficient of the action $c_{SW}$ [18] and of the tensor current $c_T$ [20], $b_T$; renormalisation constant $Z_T^{(0)}$ (1/a) [19]. When a nonperturbative value is not available, we take its estimate in boosted perturbation theory (“bpt”).

The definition of the vector current form factor

$$
\langle V(p', \lambda) | V^\mu(0) | B(p) \rangle = \epsilon^{\mu \nu \alpha \beta} e_\nu^*(p', \lambda) p_\alpha p'_\beta \frac{2 V(q^2)}{M_B + m_V},
$$

one can easily see that the improvement of the bare tensor current leaves the form factor $T_2(q^2)$ unchanged, whereas the form factor $T_1(q^2)$ is modified as

$$
T_1^{\text{impr.}}(q^2) = T_1(q^2) - a c_T \frac{q^2}{M_B + m_V} V(q^2).
$$
To study the form factors’ \( q^2 \)-dependence we considered the following 12 combinations of \( \vec{p}_H \), and \( \vec{q} \):

\[
\begin{align*}
\vec{p}_H &= (0, 0, 0) \quad \& \quad \vec{q} \in \{ (0, 0, 0)_1; (1, 0, 0)_4; (1, 1, 0)_6; (1, 1, 1)_4; (2, 0, 0)_4 \}, \\
\vec{p}_H &= (1, 0, 0) \quad \& \quad \vec{q} \in \{ (0, 0, 0)_4; (0, 1, 0)_{12}; (0, 1, 1)_6; (1, 0, 0)_4; (1, 1, 0)_{12}; (1, 1, 1)_{12}; (2, 0, 0)_4 \},
\end{align*}
\]

in units of \((2\pi/La)\), the elementary momentum on the lattice with periodic boundary conditions. The index after each parenthesis in (9) denotes the number of independent correlation functions \( C^{(3)}_{\zeta\mu\nu}(t_x, t_y; \vec{q}, \vec{p}_H) \), for a given combination of \( \vec{p}_H \) and \( \vec{q} \). Those are deduced after applying the symmetries: parity, charge conjugation, and the discrete cubic rotations. The plateaux of the ratios (4) are typically found for deduced after applying the symmetries: parity, charge conjugation, and the discrete cubic rotations. The plateaux of the ratios (4) are typically found for \( t_y \in [10, 15] \). The form factors \( T_{1,2,3}(q^2) \) are then extracted by minimising the \( \chi^2 \) on the corresponding set of plateaux of (4). When both mesons are at rest only the form factor \( T_2 \) can be computed, whereas in other kinematical situations we obtain all 3 form factors. In the following we will focus on \( T_1 \) and \( T_2 \).

4 \( T^{\pi \rightarrow K^*}(0) \) and \( T^{\pi \rightarrow K^*}(0)/T^{\pi \rightarrow \rho}(0) \)

4.1 Extrapolating to \( B \) at \( q^2 = 0 \)

As we already mentioned, in our lattice study we can extract the form factors at \( q^2 = 0 \), for each combination of \( \kappa_Q-\kappa_q \), in all three of our datasets (see table 1). The form factors that we directly compute on the lattice cover a range of \( q^2 \)’s straddling around zero, so that either one of the kinematical configurations (9) coincides with \( q^2 = 0 \), or we have to interpolate the form factors calculated in the vicinity of \( q^2 = 0 \) to \( q^2 = 0 \). In the latter case the results are insensitive to the interpolation formula used. \(^4\)

A smooth linear mass interpolation (extrapolation) is needed to reach the \( H \rightarrow K^* \) (\( H \rightarrow \rho \)) form factor, where \( H \) is our fictitious heavy-light meson that is accessible from our lattice. This is done by fitting to

\[
T_1^{H \rightarrow V}(q^2 = 0) = T_2^{H \rightarrow V}(q^2 = 0) \equiv T^{H \rightarrow V}(0) = \alpha_H + \beta_H m_P^2,
\]

where \( m_P \) is the light pseudoscalar meson, while \( \alpha_H \) and \( \beta_H \) are the fit parameters. \( T^{H \rightarrow K^*}(0) (T^{H \rightarrow \rho}(0)) \) is then obtained after choosing \( m_P = m_{K^{\text{phys}}}, (m_{\pi^{\text{phys}}}) \), where \( m_{K(\pi)^{\text{phys}}} \)

\(^3\)Even after applying the available symmetries to the problem in hands, for each combination of \( \kappa_Q-\kappa_q \) we still have 73 correlation functions \( C^{(3)}_{\zeta\mu\nu}(t_x, t_y; \vec{q}, \vec{p}_H) \) when running over the ensemble of momenta (9). That means inspecting 4453 ratios (4) and from the corresponding plateaux we extracted 671 values for \( T_{1,3}(q^2) \), and 732 values of \( T_2(q^2) \) form factor. We decided not to insert such formidable tables of numbers in this paper. A reader interested in those numbers can obtain them upon request from the authors.

\(^4\)To check the insensitivity to the interpolation formula we used the forms discussed in eqs. (18,19) of the present paper, in addition to the pole/dipole form, i.e., \( T_1(q^2) = T(0)/(1 - q^2/m_1^2)^2 \), \( T_2(q^2) = T(0)/(1 - q^2/m_2^2) \).
obtained values for $T(0)$, together with the masses in physical units, are given in table 2. We also list our results for $T^{H \to K^*}(0)/T^{H \to \rho}(0)$, which are simply obtained as $(1 + m^\text{phys}_K \beta_H/\alpha_H)/(1 + m^\text{phys}_\pi \beta_H/\alpha_H)$.

| $\kappa_Q$ | $M_{H_s}$ [GeV] | $T^{H \to K^*}(0)$ | $T^{H \to \rho}(0)$ | $T^{H \to K^*}(0)/T^{H \to \rho}(0)$ |
|-----------|---------------|-----------------|-----------------|------------------|
| Set 1     |               |                 |                 |                  |
| 0.125     | 1.79(5)       | 0.74(5)         | 0.70(6)         | 1.06(3)          |
| 0.122     | 2.05(5)       | 0.70(5)         | 0.66(7)         | 1.07(3)          |
| 0.119     | 2.29(6)       | 0.65(6)         | 0.60(8)         | 1.09(4)          |
| 0.115     | 2.59(7)       | 0.60(7)         | 0.55(9)         | 1.10(6)          |
| Set 2     |               |                 |                 |                  |
| 0.128     | 1.57(4)       | 0.80(11)        | 0.75(16)        | 1.05(5)          |
| 0.125     | 1.87(6)       | 0.77(8)         | 0.73(12)        | 1.06(4)          |
| 0.122     | 2.13(7)       | 0.72(7)         | 0.68(10)        | 1.05(5)          |
| 0.119     | 2.39(7)       | 0.67(7)         | 0.63(10)        | 1.06(7)          |
| 0.116     | 2.62(8)       | 0.62(8)         | 0.57(11)        | 1.09(9)          |
| Set 3     |               |                 |                 |                  |
| 0.1285    | 1.80(6)       | 0.75(7)         | 0.72(10)        | 1.02(3)          |
| 0.125     | 2.26(7)       | 0.65(6)         | 0.61(9)         | 1.04(3)          |
| 0.122     | 2.62(9)       | 0.57(5)         | 0.51(7)         | 1.07(4)          |
| 0.119     | 2.96(10)      | 0.50(5)         | 0.43(7)         | 1.09(5)          |
| 0.116     | 3.28(11)      | 0.44(5)         | 0.38(7)         | 1.11(6)          |
| 0.114     | 3.48(12)      | 0.42(4)         | 0.35(7)         | 1.13(8)          |

Table 2: The form factors at $q^2 = 0$ computed directly on the lattice at a fixed value of the heavy quark and for all of our datasets.

To extrapolate in the heavy quark mass we then use the heavy quark scaling law which tells us that $T^{H \to V}(0) \times M^{3/2}_{H_s}$ should scale as a constant, up to corrections proportional to $1/M^2_{Q}$. Instead of the heavy quark mass, we may take the mass of the corresponding heavy-light meson consisting of a heavy $Q$-quark and the light $s$-quark. The reason for using the strange light quark is that it is directly accessible on our lattices whereas for the light $u/d$-quark one needs to make an extrapolation which increases the error on the heavy-light meson mass. In other words we fit our data to

$$T^{H \to V}(0) \times M^{3/2}_{H_s} = c_0 + c_1 M_{H_s} + c_2 M_{H_s}^{-2},$$

(11)

where $c_{0,1,2}$ are the fit parameters. From the plot in fig. 1 we see a pronounced linear behavior in $1/M_{H_s}$, which is why we will take the result of the linear extrapolation ($c_2 = 0$)
Figure 1: In the upper plot we show the extrapolation of the $H \rightarrow K^* \gamma$ form factor (multiplied by $M_{H^s}^{3/2}$) from $1/M_{H^s}$, directly accessible on the lattice at $\beta = 6.45$, to $1/M_{B^s}$. Linear and quadratic fit to the data are denoted by the full and dotted lines respectively. The result of the quadratic extrapolation (empty square) is slightly shifted to left to make it discernible from the linear extrapolation result (filled square). The equivalent situation for the SU(3) breaking effect is shown in the lower plot.
as our main result. As it could be guessed from fig. 1, at $\beta = 6.45$, the extrapolated value does not change if we leave out from the fit the point corresponding to the lightest of our heavy quarks. Since we have more (and heavier) masses at $\beta = 6.45$, we prefer to quote the results obtained from that dataset (Set 3), namely,

$$T_{\text{lin.}}^{B \rightarrow K^*}(0) = 0.24(3), \quad T_{\text{quad.}}^{B \rightarrow K^*}(0) = 0.23(4),$$

where “lin.” and “quad.” stand for the linearly and quadratically extrapolated form factors to $1/M_{B_s}$. The results of the strategy discussed in this subsection for all our lattices are listed in table 3.

| Set     | $T_{\text{lin.}}^{B \rightarrow K^*}(0)$ | $T_{\text{quad.}}^{B \rightarrow K^*}(0)$ | $T_{\text{lin.}}^{B \rightarrow K^*}(0)/T_{\text{lin.}}^{B \rightarrow \rho}(0)$ | $T_{\text{quad.}}^{B \rightarrow K^*}(0)/T_{\text{quad.}}^{B \rightarrow \rho}(0)$ |
|---------|----------------------------------------|-----------------------------------------|-----------------------------------------------------------------|-----------------------------------------------------------------|
| Set 1   | 0.25(3)                                | 0.28(7)                                 | 1.14(11)                                                        | 1.24(22)                                                        |
| Set 2   | 0.28(6)                                | 0.29(9)                                 | 1.08(12)                                                        | 1.17(31)                                                        |
| Set 3   | 0.24(3)                                | 0.23(4)                                 | 1.15(7)                                                         | 1.25(14)                                                        |

Table 3: The form factors resulting from the linear and quadratic extrapolation (11) as obtained from all of our 3 datasets specified in table 1.

As an illustration, the linear fit with our data at $\beta = 6.45$ gives

$$T^{B \rightarrow K^*}(0) = \frac{3.6(7) \text{ GeV}^{3/2}}{M_{B_s}^{3/2}} \times \left[ 1 - \frac{0.9(1) \text{ GeV}}{M_{B_s}} \right].$$

The slope in $1/M_{B_s}$ is very close to what has been observed in the lattice studies of the heavy-light decay constants [21], and of the $B \rightarrow \pi$ semileptonic decay form factor (see eq. (19) in ref. [9]).

From table 3 we see that the ratio of $B \rightarrow K^*$ and $B \rightarrow \rho$ form factors has a large error. This error comes from $T^{B \rightarrow \rho}(0)$, and in particular from the light mass extrapolation of the form factors to reach $T^{H \rightarrow \rho}(0)$. That error is larger for larger $m_H$, which gets further inflated after extrapolating to $B$. In contrast, the extrapolation to reach $T^{H \rightarrow K^*}(0)$ is not needed as the $K^*$ mass falls in the range of the vector meson masses that are directly simulated on our lattices. When extrapolated to $B$-meson, the SU(3) breaking ratio of the form factor has a large error and is very sensitive to the inclusion of the quadratic term in the extrapolation.

Before closing this subsection, let us also mention that, contrary to HQET, one cannot match the short distance behaviour of our QCD results to the soft collinear effective theory (SCET) in which the $T^{H \rightarrow V}(0) \times m_Q^{3/2}$ scaling law is manifest. This is still an unsolved theoretical problem and hopefully a recent development based on ref. [23] will help solving it. We note however that the inclusion of the matching of the tensor current anomalous dimension of QCD with HQET produces a numerically marginal effect (1 $\div$ 2% on the central values). We hope that the similar will hold once such a matching of QCD with SCET becomes possible.
4.2 Extrapolating to $B$ at $q^2 \neq 0$ and then to $q^2 = 0$

To check on the results obtained in the previous subsection we now also employ the standard method and extrapolate our results at fixed $v \cdot p'$ to $M_B$. The main assumption here is that the HQET scaling laws are valid for all our $v \cdot p'$, i.e., not only for those that are very small compared to the heavy meson mass.

- From our directly accessed masses and $q^2$’s, one identifies $v p' = (M_H^2 + m_V^2 - q^2)/2M_H$, where $H$ is the heavy-light meson and $V$ stands for either $K^*$ or $\rho$. In physical units, the range of available $v \cdot p'$ is nearly equal for all of our lattices, namely $0.9 \text{ GeV} \lesssim v \cdot p' \lesssim 1.8 \text{ GeV}$. We emphasize that the kinematical configurations which we were able to explore are those listed in eq. (9). Proceeding like in ref. [9], we chose 5, 6, 7 equidistant $v \cdot p'$ points for our dataset 1, 2, 3, respectively. The form factors, $T_{1,2}(v \cdot p')$, are then linearly interpolated (extrapolated) to $m_{K^*}$ ($m_\rho$) for each of our heavy quarks.

- We construct
\[
\Phi_1(M_H, v \cdot p') = w(M_H) \frac{T_1(v \cdot p')}{\sqrt{M_H}}, \quad \Phi_2(M_H, v \cdot p') = w(M_H)T_2(v \cdot p') \times \sqrt{M_H},
\]
which, in HQET, are expected to scale as constants, up to corrections $\propto 1/M_H^n$. Fitting our data to
\[
\Phi_{1,2}(M_H, v \cdot p') = d_0(v \cdot p') + \frac{d_1(v \cdot p')}{M_H} + \frac{d_2(v \cdot p')}{M_H^2},
\]
either linearly ($d_2 = 0$) or quadratically ($d_2 \neq 0$), we can extrapolate to the $B$-meson mass. Since we use the scaling law which is manifest in HQET the factor $w(M_H)$ in eq. (14) accounts for the mismatch of the leading order anomalous dimensions in QCD ($\gamma_T = 8/3$) and in HQET ($\tilde{\gamma}_T = -4$) for the tensor density, namely,
\[
w(M_H) = \left( \frac{\alpha_s(M_H)}{\alpha_s(M_B)} \right)^{-\frac{\gamma_T - \gamma_0}{2\gamma_0}} \times \left( \frac{\alpha_s(1/a)}{\alpha_s(M_H)} \right)^{-\frac{\gamma_T}{2\gamma_0}} \left[ 1 - J_T \frac{\alpha_s(1/a) - \alpha_s(M_H)}{4\pi} \right].
\]
The numerator in the first factor match our QCD form factors with their HQET counterparts, while the denominator does the opposite to the result of the extrapolation to $M_B$. The second factor, instead, provides the NLO evolution from $\mu = 1/a$ to $\mu = M_H$. For $N_f = 0$, $\beta_0 = 11$, and $J_T = 2.53$.

- The extrapolation (15) is made both linearly and quadratically. The differences between the corresponding results are essentially indistinguishable, as it can be seen in fig. 2 where we plot the results for $B \to K^{*+} \gamma^*$ form factors ($\gamma^*$ stands for the off-shell photon) obtained at $\beta = 6.45$ (Set 3). These and the similar results we obtained at $\beta = 6.2$ are collected in table 4.

To reach the physically interesting case of the photon on-shell one needs to assume some functional dependence of the form factors and extrapolate the results of table 4 down to
Figure 2: The form factors $T_{1,2}(q^2)$ relevant for $B \to K^*\gamma^*$ decay, obtained after extrapolating (linearly and quadratically) our data at $\beta = 6.45$ in inverse heavy meson mass. Also shown are the curves fitting the $q^2$ dependence to the expressions given in eqs. (18,19).
Table 4: The values of the $B \to K^*\gamma^*$ form factors at several values of $q^2$. The first error in each result is statistical and the second is the difference between the results of linear and quadratic extrapolation in $1/M_H$ to $1/M_B$.  

| $q^2$ [GeV$^2$] | $T_1^{B\to K^*}(q^2)$ | $T_2^{B\to K^*}(q^2)$ | $q^2$ [GeV$^2$] | $T_1^{B\to K^*}(q^2)$ | $T_2^{B\to K^*}(q^2)$ |
|-----------------|------------------------|------------------------|-----------------|------------------------|------------------------|
| 12.2            | 0.76(24)$^{+0.00}_{-0.02}$ | 0.50(13)$^{+0.00}_{-0.07}$ | 13.5            | 0.70(17)$^{+0.13}_{-0.00}$ | 0.52(9)$^{+0.00}_{-0.14}$ |
| 13.6            | 0.94(22)$^{+0.00}_{-0.06}$ | 0.54(12)$^{+0.00}_{-0.07}$ | 14.4            | 0.81(18)$^{+0.11}_{-0.00}$ | 0.56(9)$^{+0.00}_{-0.15}$ |
| 15.1            | 1.19(19)$^{+0.00}_{-0.12}$ | 0.60(10)$^{+0.00}_{-0.07}$ | 15.3            | 0.95(19)$^{+0.07}_{-0.00}$ | 0.61(9)$^{+0.00}_{-0.16}$ |
| 16.5            | 1.53(17)$^{+0.00}_{-0.21}$ | 0.64(6)$^{+0.00}_{-0.07}$  | 16.1            | 1.11(21)$^{+0.03}_{-0.00}$ | 0.66(9)$^{+0.00}_{-0.15}$ |
| 17.9            | 2.03(21)$^{+0.00}_{-0.37}$ | 0.72(2)$^{+0.00}_{-0.07}$  | 17.0            | 1.32(26)$^{+0.00}_{-0.04}$ | 0.72(9)$^{+0.00}_{-0.15}$ |
|                |                        |                         | 17.8            | 1.56(36)$^{+0.00}_{-0.12}$ | 0.80(9)$^{+0.00}_{-0.15}$ |

| $q^2$ [GeV$^2$] | $T_1^{B\to K^*}(q^2)$ | $T_2^{B\to K^*}(q^2)$ |
|-----------------|------------------------|------------------------|
| 11.2            | 0.73(10)$^{+0.00}_{-0.02}$ | 0.39(6)$^{+0.00}_{-0.02}$ |
| 12.4            | 0.86(11)$^{+0.00}_{-0.03}$ | 0.43(6)$^{+0.00}_{-0.03}$ |
| 13.6            | 1.02(13)$^{+0.00}_{-0.04}$ | 0.47(6)$^{+0.00}_{-0.03}$ |
| 14.7            | 1.23(16)$^{+0.00}_{-0.05}$ | 0.52(6)$^{+0.00}_{-0.03}$ |
| 16.0            | 1.52(23)$^{+0.00}_{-0.07}$ | 0.58(7)$^{+0.00}_{-0.03}$ |
| 17.2            | 1.93(34)$^{+0.00}_{-0.10}$ | 0.66(8)$^{+0.00}_{-0.03}$ |
| 18.3            | 2.54(55)$^{+0.00}_{-0.16}$ | 0.76(10)$^{+0.00}_{-0.03}$ |
\( q^2 = 0 \). It is very easy to convince oneself that the form factors \( T_1 \) and \( T_2 \) satisfy the constraints very similar to those that govern the shapes of \( F_+ \) and \( F_0 \) semileptonic heavy to light pseudoscalar form factors. More specifically:

- The nearest pole in the crossed channel, \( J^P = 1^- \), which contributes to the form factor \( T_1 \), is \( M_{B^*} = 5.42 \) GeV in \( B \to K^* \) transition. The form factor \( T_2 \), instead, receives the contribution from heavy \( J^P = 1^+ \) resonances and multiparticle states both below and above the cut \( (M_B + m_V)^2 \).

- HQET, which is relevant to the region of large \( q^2 \)'s, suggests that the form factors scale with heavy quark/meson mass as 

\[
T_1 \sim \sqrt{M} \quad \text{and} \quad T_2 \sim \frac{1}{\sqrt{M}} \quad [11],
\]

and therefore both form factors cannot be fit to the pole-like shapes.

- For the high energy region of the light meson in the rest frame of the heavy \( (q^2 \to 0) \), we also have the scaling laws 

\[
T_1(E) \sim \sqrt{M/E^2} \sim M^{-3/2}.
\]

Similar holds for the \( T_2(E) \) form factor, i.e., both form factors scale as \( M^{-3/2} \) \([16, 17]\). Moreover, the two are related via

\[
T_1(E) = \frac{M}{2E} T_2(E). \tag{17}
\]

Thus the situation is analogous to the one in \( B \to \pi \ell \nu \) decay, and we can use the simple parameterisation of ref. [24],

\[
T_1(q^2) = \frac{T(0)}{(1 - \tilde{q}^2)(1 - \alpha \tilde{q}^2)}, \quad T_2(q^2) = \frac{T(0)}{1 - \tilde{q}^2/\beta}, \tag{18}
\]

where \( \tilde{q}^2 = q^2/M_{B^*}^2 \). If we relax the \( T_1/T_2 = M/2E \) constraint, then a simple form (18) becomes

\[
T_1(q^2) = \frac{C_1}{1 - \tilde{q}^2} + \frac{C_2}{1 - C_3 \tilde{q}^2}, \quad T_2(q^2) = \frac{C_1 + C_2}{1 - C_4 \tilde{q}^2}. \tag{19}
\]

Our data from table 4 cannot distinguish between the two of the above parameterisations and in both cases we end up with the same value for \( T(0) \). As in the previous subsection, as our final results we will quote those obtained at \( \beta = 6.45 \), for which more and heavier mesons were accessible,

\[
T_{\text{lin.}}^{B \to K^*}(0) = 0.22(3), \quad T_{\text{quad.}}^{B \to K^*}(0) = 0.24(4). \tag{20}
\]

The results for all three datasets are collected in table 5.

## 5 Final results and conclusion

As our final results we will quote those obtained at \( \beta = 6.45 \), since they have smaller discretisation errors. As central value we take the results obtained from the first method (subsec. 4.1) which are given in table 3 at \( \mu = 1/a \), in the (Landau) RI/MOM scheme.
\[ T^{B\to K^*}(0; \mu) = T(0; 1/a) \left( \frac{\alpha_s(1/a)}{\alpha_s(m_b)} \right)^{-\gamma_T/2\beta_0} \left[ 1 - J_T \frac{\alpha_s(1/a) - \alpha_s(m_b)}{4\pi} \right]. \]

For \( N_f = 0 \) and \( m_b = 4.6(1) \) GeV, the running factor is very close to 1, i.e., 0.99 (0.97) for the form factors computed at \( \beta = 6.45 \) (\( \beta = 6.2 \)). Our final result is

\[ T^{B\to K^*}(q^2 = 0; \mu = m_b) = 0.24 \pm 0.03^{+0.04}_{-0.01}. \]

The spread of central values presented in Table 3 at \( \beta = 6.2 \) (when multiplied by 0.97) are used to attribute a systematic error to our final result (22). The values in Table 5 are already at the scale \( \mu = M_B \approx m_b \), and those given in eq. (20) are fully consistent with (22). Our number is smaller when compared to the QCD sum rule results, \( T^{B\to K^*}(0) = 0.33(3) \) [7], 0.38(6) [8],

although the recent progress show the tendency of lowering the central value obtained by using LCSR, i.e., \( T^{B\to K^*}(0) = 0.31(4) \) [25]. Concerning the previous lattice studies [26], most of them were made at the time before the \( T(0)M_H^{3/2} \) -scaling law was known or before a significant statistical quality of the lattice data was feasible.

Our values for the ratio \( T^{B\to K^*}(0) / T^{B\to \rho}(0) \), on the other hand, are unstable, and we will only quote the average of the values obtained by using two methods discussed in this paper at \( \beta = 6.45 \),

\[ T^{B\to K^*}(0) / T^{B\to \rho}(0) = 1.2(1), \]

as our best estimate. This result agrees with the most recent LCSR estimate, \( T^{B\to K^*}(0) / T^{B\to \rho}(0) = 1.17(9) \) [27].

The method employed in this work relies on the use of a propagating heavy quark. In order to reach smaller values of \( q^2 \)'s in an effective theory of heavy quark the so called moving NRQCD has been developed, but the numerical quality of the signal does not appear very encouraging so far [28]. Clearly, our method cannot be used to obtain a very

---

| Set       | \( T^{B\to K^*}_{\text{lin.}}(0) \) | \( T^{B\to K^*}_{\text{quad.}}(0) \) | \( T^{B\to K^*}_{\text{lin.}}(0)/T^{B\to \rho}_{\text{lin.}}(0) \) | \( T^{B\to K^*}_{\text{quad.}}(0)/T^{B\to \rho}_{\text{quad.}}(0) \) |
|-----------|-----------------|-----------------|-----------------|-----------------|
| Set 1     | 0.25(6)         | 0.28(10)        | 1.2(3)          | 1.3(7)          |
| Set 2     | 0.23(6)         | 0.23(6)         | 1.1(3)          | 1.1(2)          |
| Set 3     | 0.22(3)         | 0.24(4)         | 1.3(2)          | 1.3(3)          |

Table 5: The results of the extrapolation of the form factors from Table 4 assuming the \( q^2 \)-dependence given in eqs. (18,19).
accurate value of $T(0)$, mainly because the heavy quark extrapolations are involved. If we are to work with the physical $b$-quark mass on the lattice, we would need a very small lattice spacing, i.e., at least $a^{-1} \approx 10$ GeV. For the volume corresponding to the lattice size $La = 2$ fm that would require the simulations on the lattice with $100^3$ spatial points. Moreover, on the lattice with the periodic boundary conditions, the $q^2 = 0$ point is reached when the energy of the vector meson (in the rest frame of the heavy) is

$$E^2 = m^2_{K^*} + \left(\frac{2\pi}{La}\right)^2 |\vec{n}|^2 \left(\frac{M_B^2 + m^2_{K^*}}{2M_B}\right)^2$$

$$\Rightarrow |\vec{n}| = \frac{M_B^2 - m^2_{K^*}}{4\pi M_B} \times La = (2.07 \text{ fm}^{-1}) \times La.$$  (25)

Therefore on a lattice of the size $La = 2$ fm, one needs to give the kaon a momentum $|\vec{n}| \approx 4$, for which it is highly unlikely to observe any signal of the correlation functions (5). Even if one organises the kinematics so that the momenta are shared between $B$- and $K^*$-mesons, the required spatial momenta are still too large for the reasonably accurate computation of the correlation functions (5). Therefore, a progress in improving the quality of signals of correlation functions when the spatial momenta $|\vec{n}| > 1$ would be highly welcome. Summarising, as of now, a significant improvement on the precision of the $B \to K^*$ form factors does not look promising even in quenched approximation. The methodology of the extraction of the form factors can, however, be improved by combining the correlation functions (5) in double ratios, in a way similar to what has recently been implemented in the lattice computation of heavy→heavy [29] and light→light pseudoscalar meson decay form factors [30].

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