Hodge-Riemann property of Griffiths positive matrices of (1, 1) forms

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Abstract

The classical Hard Lefschetz theorem (HLT), Hodge-Riemann bilinear relation theorem (HRR) and Lefschetz decomposition theorem (LD) are stated for a power of a Kähler class on a compact Kähler manifold. These theorems are not true for an arbitrary class, even if it contains a smooth strictly positive representative. Explicit counterexamples of bidegree (2, 2) classes in dimension 4 can be found in Timorin (1998) and Berndtsson-Sibony (2002).

Gromov (1990), Timorin (1998), Dinh-Nguyễn (2006, 2013) proved the mixed HLT, HRR, LD for a product of arbitrary Kähler classes. Instead of products, Dinh-Nguyễn (2013) conjectured that determinants of Griffiths positive $k \times k$ matrices with (1, 1) form entries in $C^n$ satisfies these theorems in the linear case.

This paper proves that Dinh-Nguyễn’s conjecture holds for $k = 2$ and $n = 2, 3$. Moreover, assume that the matrix only has diagonalized entries, for $k = 2$ and $n \geq 4$, the determinant satisfies HLT for bidegrees $(n-2,0), (n-3,1), (1, n-3)$ and $(0, n-2)$. In particular, Dinh-Nguyễn’s conjecture is true for $k = 2$ and $n = 4, 5$ with this extra assumption.

Keywords: Griffiths positivity, Hard Lefschetz theorem, Lefschetz decomposition theorem, Hodge-Riemann bilinear relation theorem, Hodge index theorem, compact Kähler manifold.

1 Introduction

Let $X$ be a compact Kähler manifold of dimension $n$ and let $\omega$ be a Kähler form. The cohomology of $X$ satisfies the Hodge-decomposition:

$$H^d(X, C) = \bigoplus_{p+q = d} H^{p,q}(X, C), \quad (0 \leq d \leq n); \quad H^{p,q}(X, C) = H^{q,p}(X, C);$$

where $H^{p,q}(X, C)$ is the Hodge cohomology group of bidegree $(p, q)$ of $X$ with the convention that $H^{p,q}(X, C) = 0$ unless $0 \leq p, q \leq n$. When $p, q \geq 0$ and $p + q \leq n$, let $k = n - p - q$ and define $\Omega := \omega^{n-p-q}$ a $(k, k)$ form on $X$.

Recall the classical hard Lefschetz theorem (HLT), the Hodge-Riemann bilinear relation theorem (HRR) and the Lefschetz decomposition theorem (LD). One may refer to BDIP [2], Griffiths and Harris [10] and Voisin [22].

**Theorem 1.1 (HLT).** The linear map

$$H^{p,q}(X, C) \rightarrow H^{n-q,n-p}(X, C)$$

$$\{\alpha\} \rightarrow \{\alpha\} \sim \{\Omega\}$$

1
is an isomorphism, where \( \sim \) denotes the cup-product on the cohomology ring \( \oplus H^*(X, \mathbb{C}) \).

Define the primitive subspace \( P^{p,q}(X, \mathbb{C}) \) of \( H^{p,q}(X, \mathbb{C}) \) by

\[
P^{p,q}(X, \mathbb{C}) := \left\{ \{\alpha\} \in H^{p,q}(X, \mathbb{C}), \{\alpha\} \sim \{\Omega\} \sim \{\omega\} = 0 \right\}.
\]

Note that this primitive subspace depends on the class of \( \Omega \). Define \( Q = Q_\Omega \) a Hermitian form on \( H^{p,q}(X, \mathbb{C}) \) by

\[
Q(\{\alpha\}, \{\beta\}) := (\sqrt{-1})^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} \int_X \alpha \wedge \beta \wedge \Omega
\]

for smooth closed \( (p, q) \) forms \( \alpha \) and \( \beta \) on \( X \). The integral depends only on the cohomology classes \( \{\alpha\}, \{\beta\} \) of \( \alpha, \beta \) in \( H^{p,q}(X, \mathbb{C}) \).

**Theorem 1.2** (HRR: Hodge-Riemann biliener relations). The Hermitian form \( Q \) is positive-definite on \( P^{p,q}(X, \mathbb{C}) \).

**Theorem 1.3** (LD: Lefschetz decomposition theorem). The decomposition

\[
H^{p,q}(X, \mathbb{C}) = \left( \{\omega\} \sim H^{p-1,q-1}(X, \mathbb{C}) \right) \oplus P^{p,q}(X, \mathbb{C})
\]

is orthogonal with respect to the Hermitian form \( Q \).

Thus we get the signature of \( Q \) in terms of the Hodge numbers \( h^{p,q} := \dim H^{p,q}(X, \mathbb{C}) \). For example when \( p = q = 1 \) we obtain

**Corollary 1.4** (Hodge index theorem). The signature of \( Q \) on \( H^{1,1}(X, \mathbb{C}) \) is \( (h^{1,1} - 1, 1) \).

The above theorems are not true if we replace \( \{\Omega\} = \omega^{n-p-q} \) with an arbitrary class in \( H^{n-p,q,n,p-q}(X, \mathbb{R}) \), even when the class contains a strictly positive form, see for example Berndtsson and Sibony \[1\] Sect. 9.

The sufficient conditions on \( \{\Omega\} \) for these theorems has been studied by different groups of mathematicians. We recall some positive results. Let \( \omega_1, \ldots, \omega_{n-p-q+1} \) be arbitrary \( \mathbb{C} \) Kähler forms on \( X \). Let \( \Omega = \omega_1 \wedge \cdots \wedge \omega_{n-p-q} \). Define the primitive space \( P^{p,q}(X, \mathbb{C}) \) and the Hermitian form \( Q = Q_\Omega \) with respect to this \( \Omega \). More precisely

\[
P^{p,q}(X, \mathbb{C}) := \left\{ \{\alpha\} \in H^{p,q}(X, \mathbb{C}), \{\alpha\} \sim \{\Omega\} \sim \{\omega_{n-p-q+1}\} = 0 \right\}.
\]

and

\[
Q(\{\alpha\}, \{\beta\}) := (\sqrt{-1})^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} \int_X \alpha \wedge \beta \wedge \Omega.
\]

**Theorem 1.5** (mixed HLT, HRR, LD). For all \( p+q \leq n \), the class of \( \Omega = \omega_1 \wedge \cdots \wedge \omega_{n-p-q} \) satisfies the hard Lefschetz theorem, the Hodge-Riemann theorem and the Lefschetz decomposition theorem for the bidegree \( (p, q) \).

Gromov in \[11\] stated that \( Q \) is positive semi-definite when \( p = q \). He gave a complete proof for the case \( p = q = 1 \). Later, Timorin in \[21\] proved the mixed HRR in the linear case, i.e. when \( X \) is a complex torus of dimension \( n \), see also \[13, 19\]. Let \( V \) be a complex vector space of dimension \( n \) and \( \nabla \) be its complex conjugate. Let \( V^{p,q} = \Lambda^p V \otimes \Lambda^q \nabla \) with the convention that \( V^{p,q} = 0 \) unless \( 0 \leq p, q \leq n \). A form \( \omega \in V^{1,1} \) is a Kähler form if \( \omega = \sum_{l=1}^{n} \sqrt{-1} dz_l \wedge d\overline{z_l} \) in some complex coordinates \( (z_1, \ldots, z_n) \) where \( z_i \otimes \overline{z_j} \) is identified with \( dz_i \wedge d\overline{z_j} \). Timorin proved the following.
Theorem 1.6 (Linear mixed HLT, HRR, LD). Let $\omega_1, \ldots, \omega_{n-p-q+1} \in V^{1,1}$ be Kähler forms. Let $\Omega = \omega_1 \wedge \cdots \wedge \omega_{n-p-q}$. Then $\wedge: V^{p,q} \to V^{n-q,n-p}$ is an isomorphism. Define a Hermitian form on $V^{p,q}$

$$Q(\alpha, \beta) = (\sqrt{-1})^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}}(\alpha \wedge \beta \wedge \Omega),$$

where $\ast$ is the Hodge-star operator, and define the mixed primitive space

$$P^{p,q} = \{\alpha \in V^{p,q} \mid \alpha \wedge \Omega \wedge \omega_{n-p-q+1} = 0\}$$

Then $Q$ is positive definite on $P^{p,q}$. Moreover, the decomposition

$$V^{p,q} = (\omega_{n-p-q+1} \wedge V^{p-1,q-1}) \oplus P^{p,q}$$

is orthogonal with respect to $Q$.

Dinh-Nguyễn in [6] proved Theorem 1.5 for general compact Kähler manifolds, see also Cattani [3] for a proof using the theory of variations of Hodge structures. Later, in [7] Dinh-Nguyễn posed a point-wise condition and proved a stronger version.

We will introduce in the next section the notion of Hodge-Riemann cone in the exterior product $V^{k,k} := \Lambda^k V \otimes \Lambda^k \bar{V}$ with $0 \leq k \leq n$. In practice, $V$ is the complex cotangent space at an arbitrary point $x$ of $X$ and we define Hodge-Riemann cone point-wisely on $X$.

Theorem 1.7 (Dinh-Nguyễn 2013). Let $\Omega$ be a closed smooth form of bidegree $(n-p-q, n-p-q)$ on $X$. Assume that $\Omega$ takes values only in the Hodge-Riemann cone associated with $X$ point-wisely. Then $\{\Omega\}$ satisfies HLT, HRR and LD for all bidegrees $(p,q)$.

Roughly speaking, taking values in the Hodge-Riemann cone means $\Omega$ can be continuously deformed to $\omega^{n-p-q}$ in a nice way that some hard Lefschetz properties are preserved. Such deformation does not need to depend on $x$ continuously. Moreover, it does not need to preserve the closedness nor smoothness of the form. The key is to check hard Lefschetz properties point-wisely.

Dinh-Nguyễn asked in [7] whether the Griffiths cone is contained in the Hodge-Riemann cone. Let $M = (\alpha_{i,j})_1 \in M_{k,k}(\Lambda^1 \times V)$ be a $k \times k$ matrix of constant coefficient $(1,1)$ forms. We will define the Griffiths positivity of $M$ in the next section. The Griffiths cone is the collection of $(k,k)$ forms which are determinants of Griffiths positive matrices.

Fix complex coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n$. In this paper, to simplify computations, we treat matrices with diagonalized entries, i.e.

$$\alpha_{i,j} = \sum_{l=1}^n b^{(l)}_{i,j} \frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l.$$ 

That is to say, there are no terms like $dz_1 \wedge d\bar{z}_2$.

Theorem 1.8 (Main theorem 1). Let $M$ be a $2 \times 2$ matrix of constant coefficient $(1,1)$ forms on $\mathbb{C}^n$ with $n = 2, 3$. Suppose $M$ is Griffiths positive. Let $\Omega = \det(M)$. Then $\Omega$ satisfies HLT, HRR and LD for all bidegrees.

Theorem 1.9 (Main theorem 2). Let $M$ be a $k \times k$ matrix of constant coefficient $(1,1)$ forms. Suppose $M$ is Griffiths positive and has only diagonalized entries. Let $\Omega = \det(M)$. Then

1. when $n \leq 5$, $k = 2$, the form $\Omega$ satisfies HLT, HRR and LD for all bidegrees $(p,q)$ with $p + q = n - k$;
2. when \( n \geq 6 \), \( k = 2 \), the form \( \Omega \) satisfies HLT for bidegrees \((n-2,0)\), \((n-3,1)\), \((1,n-3)\) and \((0,n-2)\).

HRR has applications in mixed-volumes inequalities, see Khovanskii \[16,17\], Teissier \[19,20\] and Dinh-Nguyên \[6\]. The reader will find some related results and applications in Cattani \[3\], de Cataldo and Migliorini \[4\], Gromov \[11\], Dinh and Sibony \[5,8\] and Keum, Oguiso and Zhang \[15,24\]. A generalization of mixed HRR and \(m\)-positivity is proved by Xiao in \[23\].

This paper is organized as follows. In Section 2 we recall the notion of Hodge-Riemann forms and the Griffiths cone defined by Dinh-Nguyen. In Section 3 we establish a normal form of Griffiths positive matrices under two group actions: \(GL_k(\mathbb{C})\) action and \(GL_n(\mathbb{C})\) action. In Section 4 we prove Theorem 1.8. In Section 5, we add the assumption that all entries of \(M\) are diagonalized, and we prove Theorem 1.9.

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2 Hodge-Riemann forms

In this section we recall the Hodge-Riemann form in the linear setting, defined by Dinh-Nguyen in \[7\].

Let \( V \) be an \( n \)-dimensional complex vector space and \( \overline{V} \) be its conjugate space. Denote by \( V_{p,q} = \Lambda^p V \otimes \Lambda^q \overline{V} \) the space of constant coefficient \((p,q)\) forms with the convention that \( V_{p,q} = 0 \) unless \( 0 \leq p, q \leq n \). It is a complex vector space of dimension \( \binom{n}{p} \binom{n}{q} \). A form \( \omega \in V_{1,1} \) is a Kähler form if

\[
\omega = \sqrt{-1} \frac{d}{2} \left( dz_1 \wedge d\overline{z}_1 + \cdots + dz_n \wedge d\overline{z}_n \right)
\]

in certain complex coordinates \((z_1, \ldots, z_n)\), where \( z_i \otimes \overline{z}_j \) is identified with \( dz_i \wedge d\overline{z}_j \).

A form \( \Omega \in V_{k,k} \) with \( 0 \leq k \leq n \) is real if \( \Omega = \overline{\Omega} \). Let \( V_{k,k}^R \) be the space of all real \((k,k)\) forms. A form \( \Omega \) is positive if it is a combination with positive coefficients of forms of type \((\sqrt{-1})^k \alpha \wedge \overline{\alpha}\) with \( \alpha \in V_{k,0} \). A positive \((k,k)\) form is strictly positive if its restriction on any \( k \) dimensional subspace is non-zero. Fix a Kähler form \( \omega \) as above.

Definition 2.1 (Lefschetz forms). A \((k,k)\) form \( \Omega \in V_{k,k} \) is said to be a Lefschetz form for the bidegree \((p,q)\) if \( k = n - p - q \) and the map \( \alpha \mapsto \alpha \wedge \Omega \) is an isomorphism between \( V_{p,q} \) and \( V_{n-q,n-p} \).

Definition 2.2 (Hodge-Riemann forms). A real \((k,k)\) form \( \Omega \in V_{k,k}^R \) is said to be a Hodge-Riemann form for the bidegree \((p,q)\) if there is a continuous deformation \( \Omega_t \in V_{k,k}^R \) with \( 0 \leq t \leq 1 \), \( \Omega_0 = \Omega \) and \( \Omega_1 = \omega^k \) such that

\[
(*) \quad \Omega_t \wedge \omega^{2r} \text{ is a Lefschetz form for the bidegree } (p-r, q-r)
\]

for every \( 0 \leq r \leq \min\{p,q\} \) and \( 0 \leq t \leq 1 \). The cone of all such forms is called the Hodge-Riemann cone for bidegree \((p,q)\). We say \( \Omega \) is Hodge-Riemann if it is a Hodge-Riemann form for any bidegree \((p,q)\) with \( p + q = n - k \).

\[1\]In \[13\] this is called strongly positive.
Note that a priori the definition of Hodge-Riemann forms depends on the choice of $\omega$. Due to Dinh-Nguyen’s results in [7], a Hodge-Riemann $(k, k)$ form satisfies HLT, HRR and Lefschetz decomposition for all bidegree $(p, q)$ such that $p + q = n - k$. It is proper to use the name “Hodge-Riemann” here.

According to the classical HLT in the linear case, $(*)$ holds for $t = 1$. Moreover, due to Timorin’s results in [21], for any Kähler forms $\omega_1, \ldots, \omega_k$, the product $\Omega := \omega_1 \wedge \cdots \wedge \omega_k$ is a Hodge-Riemann form. In this paper we assume $2 \leq k \leq n$ to avoid the trivial case $k = 1$.

Let $M = (\alpha_{i,j})$ be a $k \times k$ matrix with entries in $V_{1,1}$. Assume that $M$ is Hermitian, i.e. $\alpha_{i,j} = \overline{\alpha_{j,i}}$ for all $i, j$. We say that $M$ is Griffiths positive if for all $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{C}^k \setminus \{0\}$, $\theta \cdot M \cdot \theta^t$ is a Kähler form. We call Griffiths cone the set of $(k, k)$ forms $\Omega := \det(M)$ with $M$ Griffiths positive.

Dinh-Nguyễn conjectured in [7]

**Conjecture 2.3.** Let $M$ be a Griffiths positive $k \times k$ matrix. Then $\det(M)$ is a Hodge-Riemann form.

They also explained the relation between this conjecture and

**Conjecture 2.4.** Let $M$ be a Griffiths positive $k \times k$ matrix. Then $\det(M)$ is a Lefschetz form for all bidegree $(p, q)$ such that $p + q = n - k$.

in the following sense:

Conjecture 2.3 holds for $k$ and for some $\omega \Rightarrow$ Conjecture 2.4 holds for $k$.

Conjecture 2.4 holds for $k + 2r, \forall 0 \leq r \leq (n-k)/2 \Rightarrow$ Conjecture 2.3 holds for $k$ and for any $\omega$.

**Proof.** (relation between the two conjectures) Conjecture 2.3 implies Conjecture 2.4 for the same $k$ due to Dinh-Nguyen’s Theorem 1.7. Suppose Conjecture 2.4 holds for all $k + 2r$, $\forall 0 \leq r \leq (n-k)/2$. Then for any Griffiths positive $k \times k$ matrix and for any $t \in [0, 1]$, $(1-t)M + twId_k$ is again a Griffiths positive $k \times k$ matrix. Let $\Omega_t := \det \left( (1-t)M + twId_k \right)$. Then $\Omega_t$ is a continuous family such that $\Omega_0 = \Omega = \det(M)$ and $\Omega_1 = \omega^k$. Moreover, the block matrix

$$
\begin{pmatrix}
(1-t)M + twId_k & 0 \\
0 & \omega Id_{2r}
\end{pmatrix}
$$

is a Griffiths positive $(k+2r) \times (k+2r)$ matrix. So its determinant, $\Omega_t \wedge \omega^{2r}$ is Lefschetz for all suitable $p, q, r$. By definition, $\Omega = \det(M)$ is Hodge-Riemann. 

In this paper we treat Conjecture 2.4 whose statement does not depend on the choice of $\omega$.

## 3 Griffiths positive matrices

In this section we prove some properties of Griffiths positive $k \times k$ matrices.

### 3.1 Two group actions on Griffiths positive matrices

**($GL_k(\mathbb{C})$ action)** Let $M$ be a Griffiths positive $k \times k$ matrices and let $C = (c_{i,j}) \in GL_k(\mathbb{C})$. Then $C \cdot M \cdot C^H$ is also Griffiths positive. Thus we define a $GL_k(\mathbb{C})$ action on Griffiths positive matrices. Moreover, the $(k, k)$ form $\det(C \cdot M \cdot C^H) = \left[ \det(C) \right]^2 \det(M)$, is Lefschetz for the
bidegree \((p, q)\) if and only if \(\det(M)\) is Lefschetz for the same bidegree. It suffices to study conjecture 2.4 for one representative in each \(GL_k(\mathbb{C})\)-orbit.

\((GL_n(\mathbb{C}) \text{ action})\) Write \(M = (\alpha_{i,j})\) where each entry \(\alpha_{i,j}\) is a \((1, 1)\) form. The Griffiths positivity of \(M\) implies that each \(\alpha_{i,i}\) is a Kähler form. In complex coordinates \((z_1, \ldots, z_n)\), each \(\alpha_{i,j}\) can be expressed by a \(n \times n\) matrix \((\alpha_{i,j}^{(u,v)})\) such that

\[
\alpha_{i,j} = \sum_{1 \leq u, v \leq n} \alpha_{i,j}^{(u,v)} \frac{\sqrt{-1}}{2} d z_u \wedge d \bar{z}_v.
\]

The \((1, 1)\) form \(\alpha_{i,j}\) is real (resp. Kähler) if and only if the matrix \((\alpha_{i,j}^{(u,v)})\) indexed by \((u, v)\) is Hermitian (resp. positive definite). Follows from the Griffiths positivity, \((\alpha_{i,i}^{(u,u)})\) is positive definite for each \(1 \leq i \leq k\).

Let \((w_1, \ldots, w_n)\) be another complex coordinates and let \(P \in GL_n(\mathbb{C})\) be the transition matrix between the two coordinates systems, i.e.

\[
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_n
\end{pmatrix} = P \cdot 
\begin{pmatrix}
  w_1 \\
  \vdots \\
  w_n
\end{pmatrix}
\]

Then each \((1, 1)\) form \(\alpha_{i,j}\), in new coordinates, can be expressed as

\[
\alpha_{i,j} = \sum_{1 \leq u, v \leq n} \tilde{\alpha}_{i,j}^{(u,v)} \frac{\sqrt{-1}}{2} dw_u \wedge d \bar{w}_v,
\]

where the matrix \((\tilde{\alpha}_{i,j}^{(u,v)})\) = \(P \cdot (\alpha_{i,j}^{(u,v)}) P^H\).

Clearly, neither the Griffiths positivity nor the Lefschetz condition depends on the choice of complex coordinates. So it suffices to study conjecture 2.4 for one representative in each \(GL_n(\mathbb{C})\)-orbit.

We will use these \(GL_k(\mathbb{C})\) and \(GL_n(\mathbb{C})\) actions to simplify the expression of \(M\). Thus we can compute \(\det(M)\) and verify its Lefschetz property efficiently.

### 3.2 \(GL_n(\mathbb{C})\) reduction: Diagonalizability of \((1, 1)\) forms

Now we distinguish three notions for a matrix \(M = (\alpha_{i,j})\) of \((1, 1)\) forms.

- **Diagonal entries** of \(M\) are the \((1, 1)\) forms \(\alpha_{i,i}\), \(1 \leq i \leq k\).
- **Diagonalizable entries** of \(M\) are the \((1, 1)\) forms \(\alpha_{i,j}\) such that in some complex coordinates \((z_1, \ldots, z_n)\),

\[
(*) \quad \alpha_{i,j} = \sum_{l=1}^{n} b_{i,j}^{(l)} \frac{\sqrt{-1}}{2} d z_l \wedge d \bar{z}_l.
\]

- Fix a complex coordinate system \((z_1, \ldots, z_n)\). **Diagonalized entries** of \(M\) are the \((1, 1)\) forms \(\alpha_{i,j}\) of the form \((*)\).

Recall that the Griffiths positivity of \(M\) implies that all diagonal entries \(a_{i,i}\) are Kähler. Thus there exists complex coordinates \((z_1, \ldots, z_n)\) such that

\[
\alpha_{1,1} = \omega = \sum_{l=1}^{n} \frac{\sqrt{-1}}{2} d z_l \wedge d \bar{z}_l.
\]
In new coordinates \((w_1, \ldots, w_n)\), \(a_{1,1}\) has the same expression

\[
\alpha_{1,1} = \omega = \sum_{l=1}^{n} \frac{\sqrt{-1}}{2} dw_l \wedge d\overline{w_l}
\]

if and only if the transition matrix \(P\) is unitary, i.e. \(P \cdot P^H = Id_n\).

**Theorem 3.1** (Schur’s unitary triangularization theorem). (See [14] p79 Thm 2.3.1) Given an arbitrary \(n \times n\) matrix \(A\) with complex coefficients. There is a unitary matrix \(U\) such that \(U \cdot A \cdot U^H\) is upper triangular.

**Corollary 3.2.** Given an arbitrary \((n,n)\) Hermitian positive definite matrix \(A\), there is a unitary matrix \(U\) such that \(U \cdot A \cdot U^H = \text{diag}\{\lambda^{(1)}, \ldots, \lambda^{(n)}\}\) with \(\lambda^{(1)}, \ldots, \lambda^{(n)} > 0\).

Thus in \(M\) we can diagonalize at least two diagonal entries after a \(GL_n(\mathbb{C})\) action. We conclude the following.

**Corollary 3.3.** Let \(M = (a_{i,j})\) be a Griffiths positive matrix of \((1,1)\) forms. Then there exists complex coordinates \((z_1, \ldots, z_n)\) such that

\[
\alpha_{1,1} = \omega = \sum_{l=1}^{n} \frac{\sqrt{-1}}{2} dz_l \wedge d\overline{z_l},
\]

\[
\alpha_{2,2} = \sum_{l=1}^{n} \lambda^{(l)} \frac{\sqrt{-1}}{2} dz_l \wedge d\overline{z_l},
\]

for some \(\lambda^{(1)}, \ldots, \lambda^{(n)} > 0\).

### 3.3 \(GL_k(\mathbb{C})\) reduction: Lefschetz decomposition

Now we assume \(\alpha_{1,1} = \omega = \sum_{l=1}^{n} \frac{\sqrt{-1}}{2} dz_l \wedge d\overline{z_l}\). According to the Lefschetz decomposition in the linear case

\[
V^{1,1} = \mathbb{C}\omega \oplus P^{1,1}
\]

where \(P^{1,1} = \{\alpha \in V^{1,1}| \alpha \wedge \omega^{n-1} = 0\}\). Thus we may write

\[
M = (a_{i,j}) = \begin{pmatrix}
\omega & \lambda_{1,2} \omega + \rho_{1,2} & \ldots & \lambda_{1,k} \omega + \rho_{1,k} \\
\lambda_{1,2} \omega + \rho_{1,2} & \lambda_{2,2} \omega + \rho_{2,2} & \ldots & \lambda_{2,k} \omega + \rho_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{1,k} \omega + \rho_{1,k} & \lambda_{2,k} \omega + \rho_{2,k} & \ldots & \lambda_{k,k} \omega + \rho_{k,k}
\end{pmatrix}
\]

where \(\lambda_{i,j} \in \mathbb{C}\), \(\lambda_{j,j} \in \mathbb{R}_{>0}\) and \(\rho_{i,j} \in P^{1,1}\). The fact \(\lambda_{j,j} \in \mathbb{R}_{>0}\) follows from the Griffiths positivity of \(M\).

After a \(GL_k(\mathbb{C})\) reduction we may assume that \(\lambda_{1,j} = 0\) for all \(2 \leq j \leq k\). More precisely, let

\[
C = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
-\lambda_{1,2} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\lambda_{1,k} & 0 & \cdots & 1
\end{pmatrix} \in GL_k(\mathbb{C})
\]
then
\[
C \cdot M \cdot C^H = \begin{pmatrix}
\omega & \rho_{1,2} & \cdots & \rho_{1,k} \\
\rho_{1,2} & \lambda_{2,2} & \cdots & \lambda_{2,k} \\
\cdots & \cdots & \cdots & \cdots \\
\rho_{1,k} & \lambda_{2,k} & \cdots & \lambda_{k,k}
\end{pmatrix}
\]

where \( \lambda_{i,j}' \in \mathbb{C} \), \( \rho_{i,j}' \in P^{1,1} \) is also Griffiths positive and shares the same determinant with \( M \). By the Griffiths positivity \( \lambda_{i,j}' > 0 \).

Furthermore, after a \( GL_k(\mathbb{C}) \) action with

\[
C = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & -\frac{\lambda_{2,2}}{\lambda_{2,2}^k} & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & -\frac{\lambda_{k,2}}{\lambda_{2,2}^k} & 0 & \cdots & 1
\end{pmatrix}
\in GL_k(\mathbb{C})
\]

we may assume that \( \lambda_{2,j}' = 0 \) for \( 3 \leq j \leq k \). After finitely many such \( GL_k(\mathbb{C}) \) reductions and a positive dilation on the diagonal we may assume that \( \lambda_{i,j}' = 0 \) for \( i \neq j \) and \( \lambda_{j,j}' = 1 \) for \( 2 \leq j \leq k \). To conclude

**Proposition 3.4.** Let \( M \) be a Griffiths positive matrix with \( \alpha_{1,1} = \omega \). Then there is \( C \in GL_k(\mathbb{C}) \) such that

\[
C \cdot M \cdot C^H = \begin{pmatrix}
\omega & \rho_{1,2} & \cdots & \rho_{1,k} \\
\rho_{1,2} & \omega + \rho_{2,2} & \cdots & \rho_{2,k} \\
\cdots & \cdots & \cdots & \cdots \\
\rho_{1,k} & \rho_{2,k} & \cdots & \omega + \rho_{k,k}
\end{pmatrix}
\]

where \( \rho_{i,j} \in P^{1,1} \). Moreover, by Corollary 3.3 we may assume that \( \rho_{2,2} \) is a diagonalized entry, i.e. \( \rho_{2,2} = \sum_{l=1}^{n} b_{2,2}^{(l)} \sqrt{-1} dz_l \wedge d\bar{z}_l \) with \( \sum_{l=1}^{n} b_{2,2}^{(l)} = 0 \) and \( b_{2,2}^{(l)} > -1 \) for each \( 1 \leq l \leq n \).

**Proof.** The fact \( \sum_{l=1}^{n} b_{2,2}^{(l)} = 0 \) follows from \( \rho_{2,2} \in P^{1,1} \). By Griffiths positivity, \( \omega + \rho_{2,2} \) is a Kähler form. Thus \( b_{2,2}^{(l)} > -1 \) for each \( 1 \leq l \leq n \).

Thus to treat Conjecture 2.4, it suffices to study Griffiths positive matrices of the normalized form in Proposition 3.4.

## 4 Proof for \( k = 2 \) and \( n = 2, 3 \)

According to the appendix of Griffiths [9], when \( k = 2 \) the determinat \( \Omega \) is a weakly positive \((2,2)\)-form. In particular when \( n \leq 3 \), \( \Omega \) is strictly positive, hence Lefschetz for suitable bidegrees and hence Hodge-Riemann. In this section we reproduce the proof in the linear case.

### 4.1 Trivial case \( 2 = k = n \)

**Theorem 4.1.** For \( n = 2 \), let \( M \) be a Griffiths positive \( 2 \times 2 \) matrix. Then \( \Omega = \det(M) \) is a Lefschetz form for the bidegree \((0,0)\).
Proof. In $\mathbb{C}^2$, let
$$M = \begin{pmatrix} \omega & \rho_{1,2} \\ \frac{\rho_{1,2}}{\rho_{1,2}} & \omega + \rho_{2,2} \end{pmatrix}$$
where $\rho_{1,2}, \rho_{2,2} \in P^{1,1}$. Then
$$\Omega = \text{det}(M) = \omega^2 + \omega \wedge \rho_{2,2} - \rho_{1,2} \wedge \frac{\rho_{1,2}}{\rho_{1,2}} = 0$$

By definition of the primitive space, $\omega \wedge \rho_{2,2} = 0$. By linear HRR, $-\rho_{1,2} \wedge \frac{\rho_{1,2}}{\rho_{1,2}} = \lambda \cdot \text{Vol}$ where $\lambda > 0$ and Vol = $\omega^2$ is the euclidean volume form on $\mathbb{C}^2$. Thus $\Omega = (1 + \lambda) \cdot \text{Vol}$ defines an isomorphism $V^{1,0} = \mathbb{C} \to V^{2,2} = \mathbb{C}\text{Vol}$.

Corollary 4.2. For $n = 2$, let $M$ be a Griffiths positive $2 \times 2$ matrix. Then det$(M)$ is a Hodge-Riemann form.

4.2 Case $k = 2, n = 3$

Theorem 4.3. For $n = 3$, let $M$ be a Griffiths positive $2 \times 2$ matrix. Let $\Omega = \text{det}(M)$ be a real $(2, 2)$ form. Define a Hermitian form on $V^{1,0}$ by
$$Q(\alpha, \beta) = \sqrt{-1} \ast (\alpha \wedge \overline{\beta} \wedge \Omega)$$

Then $Q$ is positive definite on $V^{1,0}$.

Proof. Let
$$M = \begin{pmatrix} \omega & \rho_{1,2} \\ \rho_{1,2} & \omega' \end{pmatrix}$$
where $\rho_{1,2} \in P^{1,1}$, i.e. $\rho_{1,2} \wedge \omega^2 = 0$, and $\omega'$ is a Kähler form. Then $\Omega = \omega \wedge \omega' - \rho_{1,2} \wedge \frac{\rho_{1,2}}{\rho_{1,2}}$.

For any $\alpha \in V^{1,0}$, $\alpha \neq 0$, after a $U_\alpha(\mathbb{C})$ action on coordinates we may assume that $\alpha = \lambda dz_1$ for some $\lambda \in \mathbb{C}, \lambda \neq 0$, while $\omega$ and the subspace $P^{1,1}$ stays unchanged. However, $\omega'$ may no longer be diagonalized. Under these coordinates, each $(1,1)$ form can be expressed by a $3 \times 3$ matrix:
$$\sum_{1 \leq i,j \leq 3} a_{i,j} \sqrt{-1} dz_i \wedge d\overline{z}_j \quad \longleftrightarrow \quad (a_{i,j})$$

In particular
$$\omega \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \omega' \longleftrightarrow \begin{pmatrix} * & * & * \\ * & a \cdot b & * \\ * & \frac{|b|^2 + t}{a} & * \end{pmatrix},$$
$$\rho_{1,2} \longleftrightarrow \begin{pmatrix} * & * & * \\ * & c_{2,2} & c_{2,3} \\ * & c_{3,2} & c_{3,3} \end{pmatrix}, \quad \rho_{1,2} \longleftrightarrow \begin{pmatrix} * & * & * \\ * & c_{2,2} & c_{2,3} \\ * & c_{3,2} & c_{3,3} \end{pmatrix},$$
where $a > 0, t > 0, b \in \mathbb{C}, c_{i,j} \in \mathbb{C}$.

We calculate
$$Q(\alpha, \alpha) = 2|\lambda|^2 \ast (V_1 \wedge \omega \wedge \omega') - 2|\lambda|^2 \ast (V_1 \wedge \rho_{1,2} \wedge \frac{\rho_{1,2}}{\rho_{1,2}})$$
$$= 2|\lambda|^2 (a + \frac{|b|^2 + t}{a}) - 2|\lambda|^2 (c_{2,2} c_{3,3} + c_{3,2} c_{2,3} - |c_{2,3}|^2 - |c_{3,2}|^2)$$
$$= 2|\lambda|^2 \left( a + \frac{|b|^2 + t}{a} - c_{2,2} c_{3,3} - c_{3,2} c_{2,3} + |c_{2,3}|^2 + |c_{3,2}|^2 \right).$$
It suffices to analyse that \( a + \frac{|b|^2 + t}{a} - c_{2,2}\overline{c_{3,3}} - c_{3,3}\overline{c_{2,2}} > 0 \), which is the job of Griffiths positivity. Let \( \theta = (1, z) \) for an arbitrary \( z \in \mathbb{C} \). By Griffiths positivity, the form

\[
\theta \cdot M \cdot \overline{\theta} = \omega + z\rho_{1,2} + \overline{z}\rho_{1,2} + |z|^2 \omega'
\]
is a Kähler form, i.e. the matrix

\[
\begin{pmatrix}
* & 1 + zc_{2,2} + |z|^2a & * \\
* & * & * \\
* & 1 + zc_{3,3} + \overline{z}c_{3,3} + |z|^2(\frac{|b|^2 + t}{a})
\end{pmatrix}
\]
is positive definite. So for any \( z \in \mathbb{C} \)

\[
\begin{align*}
1 + zc_{2,2} + \overline{zc_{2,2}} + |z|^2|a| & > 0 \\
1 + zc_{3,3} + \overline{zc_{3,3}} + |z|^2(\frac{|b|^2 + t}{a}) & > 0.
\end{align*}
\]

Let \( f(z, \overline{z}) := 1 + zc_{2,2} + \overline{zc_{2,2}} + |z|^2|a| \). The minimum of this quadratic is achieved when \( \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \overline{z}} = 0 \), i.e. when \( z = \frac{-c_{2,2}}{a} \). Since \( f(z, \overline{z}) > 0 \) for all \( z \in \mathbb{C} \), we conclude that

\[
f(z, \overline{z}) \bigg|_{z = \frac{-c_{2,2}}{a}} = 1 - \frac{|c_{2,2}|^2}{a} > 0 \quad \Rightarrow |c_{2,2}|^2 < a.
\]

By the same method

\[
|c_{3,3}|^2 < \frac{|b|^2 + t}{a}.
\]

So

\[
Q(\alpha, \alpha) = 2|\lambda|^2 \left( a + \frac{|b|^2 + t}{a} - c_{2,2}\overline{c_{3,3}} - c_{3,3}\overline{c_{2,2}} + |c_{2,3}|^2 + |c_{3,2}|^2 \right) > 2|\lambda|^2 \left( |c_{2,2}|^2 + |c_{3,3}|^2 - c_{2,2}\overline{c_{3,3}} - c_{3,3}\overline{c_{2,2}} + |c_{2,3}|^2 + |c_{3,2}|^2 \right)
\]

\[
= 2|\lambda|^2 \left( |c_{2,2} - c_{3,3}|^2 + |c_{2,3}|^2 + |c_{3,2}|^2 \right) \geq 0.
\]

**Corollary 4.4.** For \( n = 3 \), let \( M \) be a Griffiths positive \( 2 \times 2 \) matrix. Then \( \Omega = \det(M) \) is a Lefschetz form for bidegree \((1,0)\), hence a Hodge-Riemann form.

**Proof.** Since \( Q \) is non-degenerate on \( V^{1,0} \), we know \( \wedge \Omega : V^{1,0} \to V^{3,2} \) is an injective linear map between vector spaces of the same dimension, hence an isomorphism. \( \square \)

### 4.3 Difficulty of the case \( k = 2, n = 4 \)

First we explain why the proof above does not work for \( n = 4 \). Still let \( M \) be normalized and \( \Omega = \omega \wedge (\omega + \rho_{2,2}) - \rho_{1,2} \wedge \overline{\rho_{1,2}} \). Define the Hermitian form

\[
Q(\alpha, \beta) := -((\sqrt{-1})^{p-q} \star (\alpha \wedge \overline{\beta} \wedge \Omega))
\]
on \( V^{p,q} \) with \( p + q = n - k = 2 \). For any \( \alpha \in V^{2,0} = P^{2,0}, \alpha \neq 0 \), in the sum

\[
Q(\alpha, \alpha) = \star (\alpha \wedge \overline{\alpha} \wedge \omega \wedge (\omega + \rho_{2,2})) - \star (\alpha \wedge \overline{\alpha} \wedge \rho_{1,2} \wedge \overline{\rho_{1,2}})
\]
the first part

\[
\star (\alpha \wedge \overline{\alpha} \wedge \omega \wedge (\omega + \rho_{2,2})) > 0
\]
by the mixed linear HRR. However, the second part

\[- \star (\alpha \wedge \pi \wedge \rho_{1,2} \wedge \rho_{1,2})\]

is not always non-negative. Although \(\alpha \wedge \pi + \omega^2\) is strictly positive for each \(\epsilon > 0\), \(\alpha \wedge \pi + \epsilon \omega^2\) may not satisfy HRR. For example when \(\alpha = dz_1 \wedge dz_2\), the bilinear form

\[\tilde{Q}(\beta, \gamma) := - \star (\alpha \wedge \beta \wedge \gamma) = -4 \star (V_{1,2} \wedge \beta \wedge \gamma)\]

is no longer semi-positive definite on \(P^{1,1}\). For \(\beta = 2V_1 - V_3 - V_4 \in P^{1,1}\),

\[\tilde{Q}(\beta, \beta) = -4 \star (V_{1,2} \wedge 2V_{3,4}) < 0\]

At this moment we cannot conclude that \(Q\) is non-degenerate on \(P^{2,0}\).

5 Case all entries of \(M\) being diagonalized

In this paper, for simplifications, we write \(V_j = \sqrt{-1} dz_j \wedge d\bar{z}_j\) for the euclidean volume form on the complex line of \(z_j\), which is two times the euclidean volume form. We write \(V_{j_1, j_2, \ldots, j_s} = V_{j_1} \wedge \cdots \wedge V_{j_s}\) for the volume form on the \(s\)-dimensional subspace spanned by those complex lines. Without extra specifications, \(\omega = \sum_{l=1}^{n} V_l\) denotes the standard Kähler form in the linear case. We write \(\text{Vol} = V_{1,2,\ldots,n}\) for the volume form on \(\mathbb{C}^n\).

Now we assume that all entries of \(M = (\alpha_{i,j})\) are diagonalized, i.e. \(\alpha_{i,j} = \sum_{l=1}^{n} b_{i,j}^{(l)} V_l\). The following lemma holds for general \(k \geq 2\).

**Proposition 5.1.** Let \(M\) be a \(k \times k\) matrix with diagonalized entries. We can write \(M\) as a matrix valued \((1,1)\) form

\[M = (\alpha_{i,j}) = \left(\sum_{l=1}^{n} b_{i,j}^{(l)} V_l\right) = \sum_{l=1}^{n} (b_{i,j}^{(l)}) V_l\]

Then \(M\) is Griffiths positive if and only if the matrix \(B^{(l)} := (b_{i,j}^{(l)})\) is a positive definite \(k \times k\) matrix for \(1 \leq l \leq n\).

**Proof.** For any \(\theta \in \mathbb{C}^k\), \(\theta \neq 0\). The \((1,1)\) form

\[\theta \cdot M \cdot \bar{\theta} = \sum_{l=1}^{n} \theta \cdot B^{(l)} \cdot \bar{\theta} V_l.\]

Thus \(M\) is Griffiths positive if and only if \(\theta \cdot M \cdot \bar{\theta}\) is a Kähler form if and only if \(\theta \cdot B^{(l)} \cdot \bar{\theta} > 0\) for \(1 \leq l \leq n\). \(\Box\)

After a dilation we may assume that \(\alpha_{1,1} = \omega = \sum_{l=1}^{n} V_l\), i.e. \(b_{1,1}^{(l)} = 1\) for \(1 \leq l \leq n\).

To calculate \(\Omega = \det(M)\), we introduce the hyperdeterminant among \(B^{(l)}\).
Definition 5.2. Let $B^{(1)}, \ldots, B^{(k)}$ be $(k, k)$ complex valued matrices with $B^{(i)} = (b^{(i)}_{i,j})$. We define the $k \times k \times k$ hypermatrix $B = (B^{(1)}, \ldots, B^{(k)})$ a 3-dim array whose layers $B^{(i)}$ are matrices. We define its hyperdeterminant by

$$
\text{hdet}(B) := \sum_{\sigma, \tau \in S_k} \text{sgn}(\sigma) \prod_{j=1}^{k} b^{(\tau(j))}_{j,\sigma(j)}
$$

where $S_k$ is the permutation group of $k$ elements.

We remark that switching two layers does not change the hyperdeterminant. The determinant

$$
\Omega = \det(M) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \Omega_{i_1,\ldots, i_k} V_{i_1,\ldots, i_k}
$$

where $\Omega_{i_1,\ldots, i_k} = \text{hdet}((B^{(i_1)}, \ldots, B^{(i_k)}))$, whose positivity is proved by Zheng (MCM, AMSS).

Theorem 5.3 (Zheng 2021). Let $B^{(1)}, \ldots, B^{(k)}$ be positive semidefinite Hermitian matrices. Denote by $\mu^{(l)}$ and $\lambda^{(l)}$ the minimal and the maximal eigenvalues of $B^{(l)}$. Then

$$
k! \mu^{(1)} \cdots \mu^{(k)} \leq \text{hdet}(B) \leq k! \lambda^{(1)} \cdots \lambda^{(k)}.
$$

In particular, if $B^{(1)}, \ldots, B^{(k)}$ are positive definite, then $\text{hdet}(B) > 0$.

Proof. We proceed by induction on $k$. The case $k = 1$ is trivial.

For the general case, note that for any $k \times k$ matrix $U$, we have $\text{hdet}(BU) = \det(U) \text{hdet}(B) = \text{hdet}(UB)$, where $BU$ and $UB$ are the hypermatrices defined by layerwise multiplication $(BU)^{(l)} := B^{(l)}U$ and $(UB)^{(l)} := UB^{(l)}$. Thus, up to replacing $B$ by $UBU^H$ for a unitary matrix $U$, we may assume that one layer, say $B^{(1)}$, is diagonal.

Once $B^{(1)}$ is diagonal, in the definition of $\text{hdet}(B)$ above, if $\tau(j) = 1$, then $b^{(1)}_{j,\sigma(j)}$ is nonzero only if $\sigma(j) = j$. Thus

$$
\text{hdet}(B) = \sum_{j=1}^{k} b^{(1)}_{j,j} \text{hdet}(B_j)
$$

where $B_j$ is the $(k-1) \times (k-1) \times (k-1)$ hypermatrix obtained from $B$ by removing the layer $B^{(1)}$ and removing the $j$-th row together with the $j$-th column in each layer $B^{(i)}$. We conclude by the induction hypothesis. \qed

Corollary 5.4. Let $M$ be a $(k, k)$ Griffiths positive matrix with diagonalized entries. Then $\Omega = \det(M)$ is a strictly positive $(k,k)$ form, hence a Lefschetz form for bidegree $(n-k,0)$ and $(0,n-k)$.

Corollary 5.5. Let $M$ be a $(k,k)$ Griffiths positive matrix with diagonalized entries. If $k = n - 1$ or $k = n$, then $\det(M)$ satisfies HLT, HRR, LD for all suitable bidegrees.

5.1 Case $k = 2$, $n = 4$, $M$ with diagonalized entries

When $k = 2$, each positive definite matrix $B^{(l)}$ can be written as

$$
B^{(l)} = \left( \begin{array}{cc} 1 & b_l \\ b_l & |b_l|^2 + t_l \end{array} \right)
$$
for some $b_i \in \mathbb{C}$ and $t_i > 0$. The determinant

$$\Omega = \det(M) = \sum_{1 \leq i < j \leq 4} \Omega_{i,j} V_{i,j}$$

where

$$\Omega_{i,j} = \text{hdet}(B^{(i)}, B^{(j)}) = |b_i|^2 + t_i + |b_j|^2 + t_j - b_i \overline{b_j} - b_j \overline{b_i} = |b_i - b_j|^2 + t_i + t_j > 0.$$

For simplifications, we define $b_{i,j} := b_i - b_j \in \mathbb{C}$.

**Theorem 5.6.** Let $n = 4$. Let $M$ be a $(2,2)$ Griffiths positive matrix with diagonalized entries. Then $\Omega = \det(M)$ is a Lefschetz form for bidegree $(2,0)$, $(1,1)$ and $(0,2)$.

**Proof.** It suffices to check bidegree $(2,0)$ and $(1,1)$.

(Trivial part) For bidegree $(2,0)$, we take the standard basis of $V^{2,0}$ by the lexicographical order

$$\{dz_1 \wedge dz_2, dz_1 \wedge dz_3, \ldots, dz_3 \wedge dz_4\}$$

and we take a basis of $V^{4,2}$ by the Hodge-star of their conjugates

$$\{dz_1 \wedge dz_2 \wedge V_{3,4}, dz_1 \wedge dz_3 \wedge V_{2,4}, \ldots, dz_3 \wedge dz_4 \wedge V_{1,2}\}.$$

Under these two basis, the linear map $\land \Omega : V^{2,0} \to V^{4,2}$ can be expressed as a diagonal matrix $\text{diag}\{\Omega_{3,4}, \Omega_{2,4}, \ldots, \Omega_{1,2}\}$ with positive entries. This map is an isomorphism.

(Non-trivial part) For bidegree $(1,1)$, again, we take the standard basis of $V^{1,1}$ as follows

$$\{V_1, V_2, V_3, V_4, dz_1 \wedge dz_2, dz_1 \wedge dz_3, \ldots, dz_3 \wedge dz_4\}$$

and we take a basis of $V^{3,3}$ by the Hodge star of their conjugates

$$\{V_{2,3,4}, V_{1,3,4}, V_{1,2,3}, V_{1,2,4}, dz_1 \wedge dz_2 \wedge V_{3,4}, dz_1 \wedge dz_3 \wedge V_{2,4}, \ldots, dz_3 \wedge dz_4 \wedge V_{1,2}\}.$$

Under these two basis, the linear map $\land \Omega : V^{1,1} \to V^{3,3}$ can be expressed as a blocked matrix

$$\begin{pmatrix} G & 0 \\ 0 & \text{diag}\{\Omega_{3,4}, \Omega_{2,4}, \ldots, \Omega_{1,2}\} \end{pmatrix}$$

where

$$G = \begin{pmatrix} 0 & \Omega_{3,4} & \Omega_{2,4} & \Omega_{2,3} \\ \Omega_{3,4} & 0 & \Omega_{1,4} & \Omega_{1,3} \\ \Omega_{2,4} & \Omega_{1,4} & 0 & \Omega_{1,2} \\ \Omega_{2,3} & \Omega_{1,3} & \Omega_{1,2} & 0 \end{pmatrix}.$$  

It suffices to verify that $\det(G) \neq 0$. Let

$$A = \sqrt{\Omega_{3,4}\Omega_{1,2}}, \quad B = \sqrt{\Omega_{2,4}\Omega_{1,3}}, \quad C = \sqrt{\Omega_{2,3}\Omega_{1,4}}.$$  

Then $\det(G) = -(A + B + C)(A + B - C)(A - B + C)(-A + B + C)$ has the form of the Heron formula which calculates the area of a triangle with side length $(A, B, C)$. We are going
to show that the side lengths \((A, B, C)\) actually forms a triangle. After permutations among \(b_j\) and among \(t_j\), it suffices to verify that

\[
\sqrt{\Omega_{3,4}\Omega_{1,2}} + \sqrt{\Omega_{2,4}\Omega_{1,3}} > \sqrt{\Omega_{2,3}\Omega_{1,4}},
\]
i.e.

\[
\Omega_{3,4}\Omega_{1,2} + \Omega_{2,4}\Omega_{1,3} + 2\sqrt{\Omega_{3,4}\Omega_{1,2}\Omega_{2,4}\Omega_{1,3}} - \Omega_{2,3}\Omega_{1,4} > 0.
\]
The left hand side is

\[
LHS = (|b_{3,4}|^2 + t_3 + t_4)(|b_{1,2}|^2 + t_1 + t_2) + (|b_{1,3}|^2 + t_1 + t_3)
+ 2\sqrt{(|b_{3,4}|^2 + t_3 + t_4)(|b_{1,2}|^2 + t_1 + t_2)(|b_{1,3}|^2 + t_1 + t_3)}
- (|b_{2,3}|^2 + t_2 + t_4)(|b_{1,4}|^2 + t_1 + t_4)
\]
We expand the product in the square-root and sort each summand according to the order of \(t_j\):

\[
\Omega_{3,4}\Omega_{1,2}\Omega_{2,4}\Omega_{1,3} = (\cdots) + \sum_{j=1}^{4} (\cdots)t_j + \sum_{j=1}^{4} (\cdots)t_j^2 + \sum_{1<j<l} (\cdots)t_j t_l + O_{t_j}(3)
\]
where

\[
(b\text{ part}) := |b_{3,4}|^2|b_{1,2}|^2|b_{2,4}|^2|b_{1,3}|^2
\]
\[
(t^2\text{ part}) := (t_1|b_{3,4}||b_{2,4}|)^2
\]
\[
(t_1\text{ part}) := t_1|b_{3,4}|^2|b_{2,4}|^2(|b_{1,2}|^2 + |b_{1,3}|^2)
\]
\[
\geq 2t_1|b_{3,4}|^2|b_{2,4}|^2|b_{1,2}||b_{1,3}|
= 2\sqrt{(t^2_1\text{ part})}\sqrt{(b\text{ part})}
\]
\[
(t_1 t_2\text{ part}) := t_1 t_2|b_{3,4}|^2(|b_{1,2}|^2 + |b_{2,4}|^2 + |b_{1,3}|^2)
\]
\[
\geq 2t_1 t_2|b_{3,4}|^2|b_{2,4}||b_{1,3}|
= 2\sqrt{(t^2_1\text{ part})}\sqrt{(t^2_2\text{ part})}
\]
\[
(t_1 t_4\text{ part}) := t_1 t_4(|b_{2,4}|^2 + |b_{3,4}|^2)(|b_{1,2}|^2 + |b_{1,3}|^2)
\]
\[
\geq 2t_1 t_4|b_{1,2}||b_{1,3}||b_{2,4}||b_{3,4}|
= 2\sqrt{(t^2_1\text{ part})}\sqrt{(t^2_2\text{ part})}
\]
\[
2\sqrt{\Omega_{3,4}\Omega_{1,2}\Omega_{2,4}\Omega_{1,3}}
\]
\[
> 2\sqrt{(b\text{ part})} + \sum_{j=1}^{4} (t^2_j\text{ part}) + \sum_{j=1}^{4} 2\sqrt{(t^2_j\text{ part})}\sqrt{(b\text{ part})} + \sum_{1<j<l} 2\sqrt{(t^2_j\text{ part})}\sqrt{(t^2_l\text{ part})}
\]
\[
= 2\left(\sqrt{(b\text{ part})} + \sum_{j=1}^{4} \sqrt{(t^2_j\text{ part})}\right).
\]
Here the first inequality is strict because there is a term $t_1t_2t_3t_4 > 0$. Thus
\[
LHS > |b_{3,4}|^2 |b_{1,2}|^2 + |b_{2,4}|^2 |b_{1,3}|^2 + 2\sqrt{(b \text{ part}) - |b_{2,3}|^2 |b_{1,4}|^2} + t_1(|b_{3,4}|^2 + |b_{2,4}|^2 + \frac{2|b_{3,4}||b_{2,4}|}{2} - |b_{2,3}|^2) \\
+ t_2(|b_{3,4}|^2 + |b_{1,3}|^2 + 2|b_{3,4}||b_{1,3}| - |b_{1,4}|^2) \\
+ t_3(|b_{1,2}|^2 + |b_{2,4}|^2 + 2|b_{1,2}||b_{2,4}| - |b_{1,4}|^2) \\
+ t_4(|b_{1,2}|^2 + |b_{2,3}|^2 + 2|b_{1,2}||b_{1,3}| - |b_{2,3}|^2) \\
+ (t_3 + t_4)(t_1 + t_2) + (t_2 + t_4)(t_1 + t_3) - (t_2 + t_3)(t_1 + t_4).
\]

The first line is $(|b_{3,4}||b_{1,2}| + |b_{2,4}||b_{1,3}|)^2 - |b_{2,3}|^2 |b_{1,4}|^2$. In fact
\[
b_{2,3}b_{1,4} = (b_2 - b_3)(b_1 - b_4) \\
= (b_3 - b_4)(b_2 - b_1) + (b_2 - b_4)(b_1 - b_3) = -b_{3,4}b_{1,2} + b_{2,4}b_{1,3} \\
|b_{2,3}b_{1,4}| \leq |b_{3,4}b_{1,2}| + |b_{2,4}b_{1,3}|
\]

Indeed this is Ptolemy Theorem.

![Ptolemy Theorem Diagram](image)

**Figure 1:** Ptolemy Theorem: $|b_{2,3}b_{1,4}| \leq |b_{3,4}b_{1,2}| + |b_{2,4}b_{1,3}|$

In the second line
\[
t_1(|b_{3,4}|^2 + |b_{2,4}|^2 + 2|b_{3,4}||b_{2,4}| - |b_{2,3}|^2) = t_1 \left( (|b_{3,4}| + |b_{2,4}|)^2 - |b_{2,3}|^2 \right) \geq 0
\]
by the triangle inequality. So $LHS > 0$ which implies $A + B - C > 0$. After permutations among $b_j$ and among $t_j$, we conclude that $\det(G) < 0$ and $\wedge \Omega : V^{1,1} \rightarrow V^{3,3}$ is an isomorphism. \hfill \Box

### 5.2 Case $k = 2, n \geq 4$, $M$ with diagonalized entries

As before, write $M = \sum_{i=1}^{n} B^{(i)}V_i$ with
\[
B^{(i)} = \left( \frac{1}{b_i} \right) \begin{pmatrix} b_i & b_i \\ b_i & b_i + t_i \end{pmatrix}
\]
for some $b_i \in \mathbb{C}$ and $t_i > 0$. The determinant
\[
\Omega = \det(M) = \sum_{1 \leq i < j \leq n} \Omega_{i,j} V_{i,j}
\]
where  
\[ \Omega_{i,j} = |b_{i,j}|^2 + t_i + t_j > 0. \]

**Theorem 5.7.** Let \( n \geq 4 \). Let \( M \) be a \((2,2)\) Griffiths positive matrix with diagonalized entries. Then \( \Omega = \det(M) \) is a Lefschetz form for bidegree \((n-2,0)\), \((n-3,1)\), \((1,n-3)\) and \((0,n-2)\).

**Proof.** The technique is the same as in Theorem 5.6. We only need to choose basis carefully.

For bidegree \((n-2,0)\), take the lexicographical ordered basis

\[ \{dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n-2}, \ldots, dz_3 \wedge \cdots \wedge dz_n\} \]

of \( V^{n-2,0} \) and take the Hodge-star of their conjugates as basis of \( V^{n,2} \). Then the matrix form of the linear map \( \wedge \Omega : V^{n-2,0} \to V^{n,2} \) is \( \text{diag}(\Omega_{n-1,n}, \Omega_{n-2,n}, \ldots, \Omega_{1,2}) \) where the indices are in the reversed lexicographical order. Each \( \Omega_{i,j} > 0 \) implies that \( \Omega \) is a Lefschetz form for bidegree \((n-2,0)\).

For bidegree \((n-3,1)\), take basis of \( V^{n-3,1} \) as follows

\[
\begin{align*}
\{ &dz_1 \wedge \cdots \wedge dz_{n-4} \wedge V_{n-3}, \\
&dz_1 \wedge \cdots \wedge dz_{n-4} \wedge V_{n-1}, \\
&dz_1 \wedge \cdots \wedge dz_{n-3} \wedge dz_{n-2}, \\
&dz_1 \wedge \cdots \wedge dz_{n-5} \wedge dz_{n-3} \wedge V_{n-4}, \\
&dz_1 \wedge \cdots \wedge dz_{n-3} \wedge V_{n-1}, \\
&dz_1 \wedge \cdots \wedge dz_{n-3} \wedge dz_{n-2}, \\
&dz_5 \wedge \cdots \wedge dz_n \wedge V_1, \\
&dz_5 \wedge \cdots \wedge dz_n \wedge V_3, \\
&dz_5 \wedge \cdots \wedge dz_1 \wedge dz_2,
\end{align*}
\]

and take basis of \( V^{n-1,3} \) by the Hodge-star of their conjugates. Then the matrix form of the linear map \( \wedge \Omega : V^{n-3,1} \to V^{n-1,3} \) is

\[
\begin{pmatrix}
G_{n-3,n-2,n-1,n} & 0 & \cdots & 0 & 0 \\
0 & \text{diag}\{\Omega_{n-1,n}, \ldots, \Omega_{n-3,n-2}\} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & G_{1,2,3,4} & 0 \\
0 & 0 & \cdots & 0 & \text{diag}\{\Omega_{3,4}, \ldots, \Omega_{1,2}\}
\end{pmatrix}
\]

where each

\[
G_{i_1,i_2,i_3,i_4} = \begin{pmatrix}
0 & \Omega_{i_3,i_4} & \Omega_{i_2,i_4} & \Omega_{i_2,i_3} \\
\Omega_{i_3,i_4} & 0 & \Omega_{i_1,i_4} & \Omega_{i_1,i_3} \\
\Omega_{i_2,i_4} & \Omega_{i_1,i_4} & 0 & \Omega_{i_1,i_2} \\
\Omega_{i_2,i_3} & \Omega_{i_1,i_3} & \Omega_{i_1,i_2} & 0
\end{pmatrix}
\]

is invertible. So \( \Omega \) is a Lefschetz form for bidegree \((n-3,1)\).

**Corollary 5.8.** Let \( n = 4,5 \). Let \( M \) be a \((2,2)\) Griffiths positive matrix with diagonalized entries. Then \( \det(M) \) is a Hodge-Riemann form. \( \square \)
5.3 Difficulty of the case $k = 2$, $n = 6$ and $M$ with diagonalized entries

We already checked that $\Omega$ is a Lefschetz form for bidegree $(4, 0)$, $(3, 1)$, $(1, 3)$ and $(0, 4)$. The only thing left is the bidegree $(2, 2)$. It amounts to prove that the matrix

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \Omega_{4, 6} & \Omega_{4, 5} \\
0 & 0 & 0 & 0 & 0 & \Omega_{3, 6} & \Omega_{3, 5} & \Omega_{2, 6} \\
0 & 0 & 0 & 0 & \Omega_{5, 6} & \Omega_{4, 6} & \Omega_{4, 5} & 0 \\
0 & 0 & 0 & \Omega_{3, 6} & \Omega_{3, 5} & 0 & \Omega_{2, 6} & 0 \\
0 & 0 & \Omega_{4, 6} & \Omega_{3, 6} & \Omega_{3, 5} & 0 & 0 & 0 \\
0 & \Omega_{5, 6} & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

is invertible. However, unlike the case before, the determinant is irreducible in $\mathbb{C}[\sqrt{\Omega_{i,j}}]$. Indeed, by a Mathematica program, the determinant is irreducible on $\mathbb{C}[\Omega_{i,j}^{1/t}]$ for $t = 1, 2, \ldots, 15$. It is difficult to prove that this determinant is non-zero by the techniques, analogues of Heron formula, as before.

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