A nonparametric doubly robust test for a continuous treatment effect

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Abstract

The vast majority of literature on evaluating the significance of a treatment effect based on observational data has been confined to discrete treatments. These methods are not applicable to drawing inference for a continuous treatment, which arises in many important applications. To adjust for confounders when evaluating a continuous treatment, existing inference methods often rely on discretizing the treatment or using (possibly misspecified) parametric models for the effect curve. Recently, Kennedy et al. (2017) proposed nonparametric doubly robust estimation for a continuous treatment effect in observational studies. However, inference for the continuous treatment effect is a harder problem. To the best of our knowledge, a completely nonparametric doubly robust approach for inference in this setting is not yet available. We develop such a nonparametric doubly robust procedure in this paper for making inference on the continuous treatment effect curve. Using empirical process techniques for local U- and V-processes, we establish the test statistic’s asymptotic distribution. Furthermore, we propose a wild bootstrap procedure for implementing the test in practice. We illustrate the new method via simulations and a study of a constructed dataset relating the effect of nurse staffing hours on hospital performance. We implement our doubly robust dose response test in the R package DRDRtest on CRAN.

1 Introduction

We are interested in hypothesis testing for a continuous (causal) treatment effect based on observational data. The fundamental challenge of causal

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inference with observational data is to account for confounding variables, which are variables related to both the outcome and the treatment. In the presence of confounding variables, it is well known that naive regression modeling does not lead to an unbiased estimate for the causal effect curve. While continuous treatments are common in many important applications, much of the existing literature on inference for a treatment effect from observational data has been focused on discrete treatments. Relatively few methods are available for testing hypotheses about a continuous treatment effect curve.

Under the popular ‘no unmeasured confounders’ assumption, there are two broad directions to adjust for the confounding variables. A procedure can start by estimating the outcome regression function, a function that relates the outcome to the treatment and the confounders, and then this can be weighted appropriately to yield an estimate of the causal estimand. Alternatively, a procedure can start by estimating the propensity score function, a function that relates the treatment to confounders, and then allows a variety of methods to be implemented to estimate the causal estimand. For instance, Imbens [2004] and Hill [2011] model only the outcome regression function; while Hirano and Imbens [2004], Imai and van Dyk [2004], Galvão and Wang [2015] model only the propensity score function. Following the terminology of semiparametric statistics, the outcome regression function and the propensity score function are often referred to as nuisance parameters (possibly infinite-dimensional). In the aforementioned approaches, an incorrectly specified model for either nuisance parameter would lead to inconsistent estimates of the treatment effect curve; and, regardless, the conclusions are susceptible to the curse of dimensionality: the rate of convergence of the estimator of the treatment effect curve is the same as that of the estimator of the nuisance parameter, which may be high dimensional.

In the so-called doubly robust approach, widely used for estimating the average treatment effect in the discrete treatment setting, one estimates both nuisance parameters and then combines them. The term “doubly robust” means that only one of the two nuisance parameters needs to be estimated consistently to achieve consistent estimation of the causal treatment effect. Thus even if the model for one of the two nuisance parameters is misspecified, the causal estimand can still be estimated consistently if the other nuisance parameter model is correctly specified; alternatively/similarly, the rate of the leading error term in estimating the causal estimand is determined by the product of the error terms for estimating the two nuisance parameters, allowing for efficient estimation.

If one wishes to apply a doubly robust test in the continuous treatment
setting, the simplest and likely the standard approach would be to discretize
the treatment and use the methodology for discrete treatments [Robins et al.,
2007, Van Der Laan and Dudoit, 2003]. Unfortunately, this could result in
misleading estimates, and can lead to possibly massive loss of power. Also,
in many applications maintaining the treatment as a continuous variable
is important for post-analysis interpretation. (As mentioned above, Gal-
vao and Wang [2015] develop inference procedures for the dose response
curve but require good, possibly parametric, estimators for the propensity
score (see their assumptions N.1, G.IV).) If one wishes to use doubly robust
methods without discretization, then Robins [2000] and Neugebauer and
van der Laan [2007] allow this, but requires specifying a parametric model
for the unknown causal treatment effect curve. If the parametric model is
not plausible, then the results can be unreliable. Recently, a nonparamet-
ric doubly robust estimation method has been proposed (Kennedy et al.
[2017]), allowing for greater flexibility in modeling the nuisance parameters.
Although the rates of nonparametrically estimating each nuisance function
may be slower than $\sqrt{n}$, the rate of estimating the causal estimand may be
much faster than that for estimating either of the individual nuisance pa-
rameters (by virtue of the product rate discussed earlier), while alleviating
the difficulty of model specification for nuisance parameters.

To the best of our knowledge, a completely nonparametric doubly robust
approach for inference for a continuous treatment effect is not yet available.
It is worth noting that “double robustness” for estimation does not automatic-
ally warrant “double robustness” for inference. See, for instance, related
discussions in Van der Laan [2014] and Benkeser et al. [2017].

In this paper, we develop a doubly robust procedure for testing the
null hypothesis that the treatment effect curve is constant. To do so, we
introduce a test statistic based on comparing the integrated squared distance
from an estimate under the alternative to the null estimate. We derive
the limit distribution of the proposed test statistic. In order to implement
the hypothesis test, the unknown parameters in the limit distribution must
be estimated. A natural approach is the bootstrap [Efron and Tibshirani,
1993]. Unfortunately, the naive bootstrap turns out to be inconsistent. We
propose a wild bootstrap procedure, which provides provable guarantees for
estimating the limit distribution, and thus allows the test to be implemented.
Code that implements our doubly robust dose response test is available in
the R package ‘DRDRTest’ on CRAN.

Our main contribution is thus a new doubly robust test procedure which
is consistent (in level and against fixed alternatives) as long as at least one
of the two nuisance parameters is specified correctly. It requires only non-
parametric assumptions on the nuisance parameters unlike Robins [2000] and Neugebauer and van der Laan [2007]. The proposed test is doubly robust in the sense that the p-values we generate are reliable (uniformly distributed under the null hypothesis) even if one of the nuisance parameter models is misspecified. Some may argue that one may use machine learning to estimate the nuisance parameters to alleviate model misspecification. Although this is true to some degree, popular machine learning methods such as random forests and neural networks are not immune from model misspecification; without structural assumptions (e.g., sparsity, additive structure) on the underlying model, they may have poor estimation accuracy (very slow rates of convergence). Their practical implementations also often require multiple tuning parameters.

Our statistic is inspired by the test of Härdle and Mammen [1993] (also Dette et al. [2001]) which were developed in the non-causal setting. Comparing with the non-causal setting, our theory is significantly more complicated due to the two infinite-dimensional nuisance parameters that are present in the causal inference setting. We introduce new empirical processes techniques for local U- and V-processes to handle this complexity, which may be of independent interest for nonparametric causal inference.

The rest of the paper is organized as follows. In Subsections 1.1 and 1.2, immediately following this one, we provide further discussions on related literature on nonparametric hypothesis testing and on the continuous treatment effect setting, respectively. Section 2 introduces the setup, notation, the new testing procedure, and underlying assumptions. In Section 3, we present the main results. Section 4 presents simulation studies and in Section 5 we present analysis of a dataset relating nurse staffing to hospital effectiveness.

1.1 Literature on nonparametric hypothesis testing

The simpler, non-causal, problem of hypothesis testing about a (non-causal) regression function when the alternative is a large nonparametric class has a very large literature already. There are many different approaches to this general problem; to start, one must decide on the definition of the nonparametric alternative class. The full, unrestricted, nonparametric alternative class is generally tested against by using a test based on a primitive of the function of interest: if \( m(\cdot) \) is the regression function then \( M(a) := \int_{-\infty}^{a} m(x) dx \) is the primitive. This approach is possibly more familiar to readers in the density/distribution testing setting where \( m \) and \( M \) would be replaced by a density and cumulative distribution function, re-
pectively. Such tests based on $M$ are “omnibus” in the sense that in theory they have power approaching one against any fixed alternative. However, for that theory to be relevant with certain fixed alternatives, extremely large sample sizes may be needed; or put another way, there are many alternatives that such tests for practical sample sizes are not well powered.

Another set of procedures is based on taking the alternative class to be some sort of smoothness class (e.g., a Hölder, Sobolev, or Besov class [Giné and Nickl, 2016]). Confusion may arise because some tests, e.g., those based on primitive functions, may have power against local alternatives converging at rate $n^{-1/2}$ whereas tests based on smoothness assumptions often require local alternatives to converge at a slower rate. However, this is a case where the (local) rates of convergence can be misleading. Rather than local rates, one can use global minimax rates of convergence over a given class or classes to compare procedures. Ingster [1993a,b,c] studies minimax rates in nonparametric hypothesis testing problems (in a white noise model and in density estimation, both with simple null models). We do not recount all the results here, but note that in general tests based on primitives will not attain minimax optimal rates against smoothness-based alternatives ([Ingster, 1993a, Section 2.5], Pouet [2001]). The minimax results are not just theoretical: Eubank and LaRiccia [1992] provide both theory and simulation results demonstrating that (in the context of density estimation) for any fixed sample size there are alternative sequences such that smoothness-based methods are more powerful than primitive-based methods (the Cramér-von Mises statistic in this case) even though the latter has a local $n^{-1/2}$ rate.

We develop our test statistic based on that of Härdle and Mammen [1993] which is a smoothness-based test in the non-causal setting; this allows us to develop a test that has power in all directions, and to develop a test that is doubly robust.

One of the difficulties in smoothness-based testing is the issue of bias. Nonparametric smoothness-based estimates generally have nontrivial bias which must be accounted for and the estimation of which entails complications. In our particular testing setting, actually there is no bias under the null (since our null hypothesis is the class of constant functions), so bias is not a major issue in the usual way. The bias will affect the estimator under the alternative and so will affect the power. A related issue is that in nonparametric testing, the asymptotic distributions of many test statistics have the property that their bias (mean) is of a larger order of magnitude than their variance. One consequence of this for us is that it makes the asymptotically negligible error terms in the analysis of our test statistic (in the causal setting of the current paper) much more complicated than they
would otherwise be; it turns out that the large bias of the main term gets multiplied by other error terms (after expanding a square) and this requires extra mathematical analysis.

1.2 Literature on continuous treatment effects

In the last few years there has been significant and increasing interest in causal inference with continuous treatment effects; this includes interest in the setting of optimal treatment regimes [Kallus and Zhou, 2018], and in specific scientific areas (e.g., Kreif et al. [2015] in the health sciences).

We briefly discuss here several statistics papers that have built theory and/or methods related to causal effect estimation and/or inference in the presence of a continuous treatment (based on observational data). We start with Kennedy et al. [2017], on which other works, including the present paper, build. Kennedy et al. [2017] have developed a method for efficient doubly robust estimation of the treatment effect curve. Denote the outcome regression function by \( \mu \) or \( \mu_0 \), and denote the propensity score function by \( \pi \) or \( \pi_0 \). Their method is based on a pseudo-outcome \( \xi \equiv \xi(Z; \pi, \mu) \), which depends on the sample point \( Z \), and on the nuisance functions \( \pi, \mu \). The pseudo-outcome \( \xi \) has the key double robustness property that if either \( \pi = \pi_0 \) or \( \mu = \mu_0 \), then \( E(\xi(Z; \pi, \mu)|A = a) \) is equal to the treatment effect curve (at the treatment value \( a \)). The estimation procedure of Kennedy et al. [2017] is then a natural two-step procedure: (1) estimate the nuisance functions \( (\pi_0, \mu_0) \) by some estimators \( (\hat{\pi}, \hat{\mu}) \) which the user can choose as they wish and construct (observable) pseudo-outcomes \( \hat{\xi}_i \) (which approximate \( \xi_i \) and depend on \( \hat{\pi}, \hat{\mu} \)), and (2) regress the pseudo-outcomes on \( A \) using some nonparametric method (e.g., local linear regression). As we described above, the error term from the nuisance parameter estimation is given by the product of the error term for estimating \( \pi_0 \) and for estimating \( \mu_0 \), so is smaller than either, partially alleviating the curse of dimensionality.

Several works have now made use of the pseudo-outcome approach of Kennedy et al. [2017], or similar approaches. Westling et al. [2020], Semenova and Chernozhukov [2020] use the pseudo-outcomes of Kennedy et al. [2017] with alternative estimation techniques, and Colangelo and Lee [2020], Su et al. [2019] use similar pseudo-outcomes (and study particular nuisance estimators). Like Kennedy et al. [2017], Westling et al. [2020] also develop a doubly robust estimator of a continuous treatment effect curve; they develop a different procedure, based on the assumption that the true effect curve satisfies the shape constraint of monotonicity. Colangelo and Lee [2020] provide an alternative motivation for a related pseudo-outcome Kennedy et al.
study a sample-splitting variation of the estimation methodology of Kennedy et al. [2017], and also consider estimating the gradient of the treatment curve. These works do not consider the global inference problem that we address here.

Many works since Kennedy et al. [2017] have considered doubly robust estimation of structural/causal functions based on nonparametric models. Some include or focus on continuous treatment effects, while others focus more on the problem of conditional average treatment effect (CATE) (or “partially conditional average treatment effect” (PCATE) [Wang et al., 2021]) based on a binary treatment variable, or other related quantities. The (P)CATE setting is of course different than the continuous treatment setting we consider, but may share some features with our setting when the covariates on which the treatment is conditioned are continuous, so we discuss some of the recent literature briefly. Chernozhukov et al. [2018], Semenova and Chernozhukov [2020], Chernozhukov et al. [2022] develop general “double/debiased” machine learning approaches to estimating causal estimands that are continuous functions. Chernozhukov et al. [2022] develop a Dantzig-type estimator based on estimating equations for the nuisance parameters. They consider four running example stimands, as well as “local” and “perfectly localized” functionals, the latter including the continuous treatment effect at a fixed point. They develop Gaussian approximations for the distributions of their estimators at a fixed point. They do not further develop inference methods, so their focus is distinct from our focus on a global testing problem. Semenova and Chernozhukov [2020] develop a general theory for debiased machine learning and uniform confidence bands for inference for different causal or missing data estimands, such as conditional average treatment effects, regression functions with partially missing outcomes, and conditional average partial derivatives. They also consider the causal effect curve with continuous treatments. However, in the latter setting, their assumptions are slightly too strong to allow double robustness for (pointwise) consistency, and their confidence band is centered at an approximation of the true function, rather than at the true function itself (i.e., there is an error term that is ignored; see their Theorem 4.7). In summary, the

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\[\text{1In discussing works that focus on continuous treatment effect estimation, they say “These works develop inference on perfectly localized average potential outcomes with continuous treatment effects, using a different approach than what we develop here. Our development is complementary as it covers a much broader collection of functionals.”}\]

\[\text{2“The only requirement we impose on the estimation of [the nuisance parameters] is that [they converge] to the true nuisance parameter }\eta_0\text{ at a fast enough rate }[o_p(n^{-1/4-\delta})]\text{ for some }\delta \geq 0\text{” [Semenova and Chernozhukov, 2020, page 271]; see their Assumption 4.9.}\]
ory and methodology of Semenova and Chernozhukov [2020], Chernozhukov et al. [2022] is built for a variety of settings and does not focus exclusively on the setting of continuous treatments, which we do focus on, and derive a powerful procedure for, here.

Wang et al. [2021], Luedtke and van der Laan [2016], Lee et al. [2017], Zimmert and Lechner [2019] and Foster and Syrgkanis [2019], consider various doubly robust types of estimation procedures for (P)CATE estimation, generally based on pseudo-outcomes. Here “PCATE” means the effect of a binary/discrete treatment conditional on some covariates which may be a strict subset of the set of all confounding variables. Nie and Wager [2021] and Kennedy [2020] consider a different type of estimator (dubbed the “R-learner” and “lp-R-Learner”, respectively, after Robinson [1988]); they provide double-robust-type conditions under which these two-step estimators can attain the oracle rate of convergence, with Kennedy [2020] able to weaken the conditions on the nuisances given in Nie and Wager [2021] by a new cross-validation technique (inspired by Newey and Robins [2018]) and undersmoothing. Very recently Kennedy et al. [2022] (on the arxiv) find minimax lower and upper bounds for CATE estimation (showing that the estimator/rates of Kennedy [2020] were optimal in some smoothness regimes but not others). None of the above works consider hypothesis tests or questions of inference except for Lee et al. [2017]. We do not know of an analog of the (lp-)R-Learner that has been directly studied for the case of estimation of continuous treatments. An open question is whether the approach we develop here can be applied also in the setting of inference for a (P)CATE. In a different setup Luedtke et al. [2019] develop a hypothesis test in a causal inference setting which could allow for continuous treatments. However, their null hypothesis is different than ours and their conditions (Condition 3) rule out our setting.

Very recently, Westling [2021] considered a similar problem to the one we consider, but using a different test statistic, based on a “primitive” (or “anti-derivative”) of the treatment effect curve. The present paper and Westling [2021] were developed entirely separately. A strength of the primitive-based method of Westling [2021] is that it naturally handles mixed discrete-continuous exposures, whereas our method would require further modifications (e.g., locally chosen bandwidths) to do so. Westling [2021]’s method is not quite doubly robust (requiring an $o_p(n^{-1/2})$ rather than $O_p(n^{-1/2})$ product-nuisance-estimation-rate), whereas our method (which allows an $o_p((n\sqrt{h})^{-1/2})$ rate) is. (See Westling [2021]’s Assumption A4 and Theorems 3 and 4, as well as Figure 1.) Also, the two methods have noticeably different power in different scenarios. Westling [2021]’s test has a local $n^{-1/2}$
convergence rate, whereas our test statistic has a local convergence rate of \((n\sqrt{h})^{-1/2}\). The implications are as discussed above: Westling [2021]’s test will have power focused in “one direction” and decaying in directions away from that one direction, and our power will be more uniformly spread over the alternatives. For instance, we will have noticeably higher power when the true (alternative) effect curve has significant peaks or valleys.

2 Setup and method

2.1 Notation, data and setup

We use the following notation throughout. Let \((Z_1, \ldots, Z_n)\) be the observed sample where each observation is an independent copy of the tuple \(Z = (L, A, Y)\) with support \(Z = L \times A \times Y\). Here \(L \subseteq \mathbb{R}^d\) and \(L\) is the vector containing the \(d\) potential confounder variables (covariates); \(A \subseteq \mathbb{R}\) and \(A\) is the continuous treatment dosage received; \(Y \subseteq \mathbb{R}\) and \(Y\) is the observable outcome of interest. We let \(P\) denote the distribution of \(Z\) and \(p_0(z) = p_0(y|l, a)p_0(a|l)p_0(l)\) denote the corresponding density function with respect to some dominating measure \(\nu\). Let \(\mu_0(l, a) := E(Y|L = l, A = a)\) denote the outcome regression function. Similarly, let \(\pi_0(a|l) := \frac{\partial}{\partial a} P(A \leq a|L = l)\) denote the conditional density or propensity score function of \(A\) given \(L\) and \(\varpi_0(a) := \frac{\partial}{\partial a} P(A \leq a)\) denote the marginal density function of \(A\) (both of which densities are assumed to exist). For a function \(f\) on \(\mathbb{R}\), we let \(\mathbb{P}\{f(Z)\} := \int_Z f(z) dP(z)\). And for \(p \geq 1\), we use \(\|f\|_p := \{\int_Z f(z)^p dP(z)\}^{1/p}\) to denote the \(L_p(P)\) norm and use \(\|f\|_{\mathcal{X}} := \sup_{x \in \mathcal{X}} |f(x)|\) to denote the uniform norm over the range \(\mathcal{X}\).

We use \(P_n\) to denote the empirical distribution defined on the observed data so that \(\mathbb{P}_n \{f(Z)\} := \int f(z) dP_n(z) = n^{-1} \sum_{i=1}^n f(Z_i)\).

To characterize the problem, let \(Y^a\) be the potential outcome [Rubin, 1975] when treatment level \(a\) is applied. Then the causal estimand that we are interested in learning about (developing a hypothesis test for) is \(\theta_0(a) := E(Y^a)\), and we wish to test if this function is constant. Specifically, we want to test

\[ H_0: \theta_0 \equiv c \in \mathbb{R} \text{ versus } H_1: \theta_0 \text{ is nonconstant,} \quad (2.1) \]

where we will assume that \(\theta_0(\cdot)\) satisfies some smoothness assumptions if it is nonconstant.
2.2 Proposed method

In Kennedy et al. [2017], the authors derived a doubly robust mapping for estimating the continuous treatment effect curve. Like doubly robust estimators in binary treatment cases, the doubly robust mapping depends on both the outcome regression function and the propensity score function, and can be written as

$$\xi(Z; \pi, \mu) = \frac{Y - \mu(L, A)}{\pi(A|L)} \int_L \pi(A|l) dP(l) + \int_L \mu(l, A) dP(l).$$  (2.2)

where \(\pi(a|l)\) and \(\mu(l, a)\) are some propensity score and outcome regression functions, respectively. The above mapping has the desired property of double robustness in that

$$E\{\xi(Z; \pi, \mu)|A = a\} = \theta_0(a),$$  (2.3)

provided either \(\mu = \mu_0\) or \(\pi = \pi_0\), under Assumptions I below [Kennedy et al., 2017]. Thus \(\theta_0(\cdot)\) could be estimated using standard nonparametric smoothing techniques if either \(\mu_0\) or \(\pi_0\) were known. Since we do not actually know \(\mu_0, \pi_0\), we plug in estimators \(\hat{\mu}, \hat{\pi}\) for \(\mu_0, \pi_0\). To compute \(\xi\) we also need to know \(dP(l)\) in two places; since we do not, we plug in \(P_n(l)\) for \(P(l)\), and we denote this by \(\hat{\xi}\). Thus, our estimate of the pseudo-outcome \(\xi(Z; \pi_0, \mu_0)\) is

$$\hat{\xi}(Z; \hat{\pi}, \hat{\mu}) = \frac{Y - \hat{\mu}(L, A)}{\hat{\pi}(A|L)} \int_L \hat{\pi}(A|l) dP_n(l) + \int_L \hat{\mu}(l, A) dP_n(l).$$  (2.4)

We can then apply a nonparametric estimation procedure to the (observed) tuples \(\{(\xi(Z_i; \hat{\pi}, \hat{\mu}), A_i)\}_{i=1}^n\). Kennedy et al. [2017] show that when at least one of the estimators is consistent then, under some assumptions about complexity and boundedness conditions of \(\hat{\mu}\) and \(\hat{\pi}\) and the product of their convergence rates, the convergence rate of the nonparametric estimator is the same as if we know the true \(\mu_0\) or \(\pi_0\). Kennedy et al. [2017] apply a local linear estimator to the pseudo-outcome to estimate \(\theta_0(\cdot)\) and show the above-stated property on the pointwise convergence rate of the nonparametric estimator.

In this paper, we are interested in a different problem: testing if \(\theta_0(\cdot)\) is constant or not. As in the estimation problem, in the setting where we (unrealistically) know one of \(\mu_0\) or \(\pi_0\), testing whether \(\theta_0(\cdot)\) is constant becomes a standard regression problem, and we can consider many possible nonparametric tests to the tuples \(\{(\xi(Z_i; \hat{\pi}, \hat{\mu}), A_i)\}_{i=1}^n\). As described in Section 1, not all tests will be doubly robust for testing, though.
In Härdle and Mammen [1993], the authors consider the problem of testing parametric null linear models (not in a causal setting) and construct test statistics based on the integrated difference between the nonparametric model estimated using the Nadaraya-Watson estimator and the parametric null model. Alcalá et al. [1999] extended the test to allow using a local polynomial estimator [Fan and Gijbels, 1996] for the nonparametric model.

We propose to test our hypothesis (M.4) of a constant treatment effect curve (i.e., no treatment effect) using the following statistic

$$T_n = n\sqrt{h} \int_A \left( \hat{\theta}_h(a) - \mathbb{P}_n \xi(Z) \right)^2 w(a) \, da, \quad (2.5)$$

where $w(\cdot)$ is a user-specified weight function, $\hat{\theta}_h(a)$ is the local linear estimator applied to $\{ (\xi(Z_i; \hat{\pi}, \hat{\mu}), A_i) \}_{i=1}^n$. To define the local linear estimator, we let

$$\hat{\beta}_h(a) = \text{argmin}_{\beta \in \mathbb{R}^2} \mathbb{P}_n \left[ K_{ha}(A) \left\{ \hat{\xi}(Z; \hat{\pi}, \hat{\mu}) - g_{ha}(A) \beta \right\}^2 \right],$$

where $g_{ha}(t) = (1, \frac{t-a}{h})^T$, $K_{ha}(t) = h^{-1}K \left\{ (t-a)/h \right\}$, and $K(\cdot)$ is a kernel function, and then we let $\hat{\theta}_h(a) = g_{h,0}(0)^T \hat{\beta}_h(a)$. Note: we could define the test statistic $T_n$ by a summation over $A_i$ rather than as an integral against (Lebesgue measure) $da$; as mentioned in Horowitz and Spokoiny [2001], Dette et al. [2001], similar results as ours would hold although some constants would change. Note that under the null hypothesis of no treatment effect, $\mathbb{P}_n \{ \xi(Z; \pi_0, \mu_0) \}$ is an $\sqrt{n}$-consistent estimator of the null model and so is $\mathbb{P}_n \xi$ provided $\hat{\mu}, \hat{\tau}$ are not converging too slowly (see the later discussion). We will see under some mild conditions on the convergence rates of $\hat{\tau}$ and $\hat{\mu}$, that $T_n$ converges to a normal distribution similar to the one in Alcalá et al. [1999] under the null model (given in (M.4)). However, similar to Härdle and Mammen [1993] and Alcalá et al. [1999], due to the slow order of convergence of the asymptotically negligible terms that arise in the proof, we do not suggest using the target distribution given in Theorem 3.1 to directly calculate the critical values under the null hypothesis. Instead we advocate using the bootstrap [Efron and Tibshirani, 1993] to estimate the distribution of $T_n$ to improve the finite sample performance and, more specifically, we use the so-called wild bootstrap [Davidson and Flachaire, 2008] as used in Härdle and Mammen [1993], Alcalá et al. [1999] and Dette et al. [2001].

In Härdle and Mammen [1993], the authors show the theoretical properties of three different bootstrap methods: (1) the naive resampling method; (2) the adjusted residual bootstrap; (3) the wild bootstrap and showed only the wild bootstrap gives consistent estimation of the null distribution. These results are again true in our setting: the wild bootstrap is valid whereas the other two are not. Here we provide a brief outline of our proposed test.
1. Estimate \((\pi_0, \mu_0)\) by (black-box estimators) \((\hat{\pi}, \hat{\mu})\).

2. Calculate the pseudo-outcomes \(\hat{\xi}(Z; \hat{\pi}, \hat{\mu})\) by (J.2) and construct the local linear estimator \(\hat{\theta}_h(a)\) using \(\{\{(\hat{\xi}(Z_i; \hat{\pi}, \hat{\mu}), A_i)\}_{i=1}^n\}\).

3. To generate wild bootstrap samples to estimate the distribution of \(T_n\) under the null hypothesis,
   
   (a) Calculate \(\hat{\varepsilon}_i = \hat{\xi}(Z_i; \hat{\pi}, \hat{\mu}) - \hat{\theta}_h(A_i)\) (we can also use \(\hat{\varepsilon}_i = \hat{\xi}(Z_i; \hat{\pi}, \hat{\mu}) - \sum_{i=1}^n \hat{\xi}(Z_i; \hat{\pi}, \hat{\mu})/n\)),
   
   (b) Do the following \(B\) times, where \(B\) is the desired number of bootstrap resamplings: for each \(i \in \{1, \ldots, n\}\), generate \(\varepsilon^*_i \sim \hat{F}_i\) (defined just below, based on \(\{\hat{\varepsilon}_i\}\) and use \((\xi^*_i = \mathbb{P}_n \hat{\xi} + \varepsilon^*_i, A_i)\) as bootstrap observations,

4. Use the wild bootstrap samples to compute \(T^*_n, j = 1, \ldots, B\) (according to (J.1) but using the bootstrap samples) and use \(\{T^*_n, j\}_{j=1}^B\) to estimate the distribution of \(T_n\) under the null hypothesis. Let \(\hat{t}^*_{n,1-\alpha}\) denote the \(1 - \alpha\) quantile of the estimated distribution, where \(0 < \alpha < 1\) is the predetermined significance level. Reject the null hypothesis if \(T_n > \hat{t}^*_{n,1-\alpha}\).

When generating bootstrap samples, we use \(\hat{F}_i\) to estimate the conditional distribution of \(\xi(Z_i; \hat{\pi}, \hat{\mu})\) based on the single residual \(\hat{\varepsilon}_i\). Härdele and Mammen [1993] use a “two point distribution” which matches the first three moments of \(\hat{\varepsilon}_i\) and is defined as

\[
\varepsilon^*_i = \begin{cases} 
-\hat{\varepsilon}_i(\sqrt{5} - 1)/2 & \text{with probability } (\sqrt{5} + 1)/(2\sqrt{5}) \\
\hat{\varepsilon}_i(\sqrt{5} + 1)/2 & \text{with probability } (\sqrt{5} - 1)/(2\sqrt{5}),
\end{cases} \tag{2.6}
\]

We also consider another common choice, a Rademacher type distribution, where \(\varepsilon^*_i\) equals \(\hat{\varepsilon}_i\) or \(-\hat{\varepsilon}_i\) with probability 1/2 each. (Davidson and Flachaire [2008]). Unlike the two point distribution, the Rademacher distribution matches the first two and the fourth (and all even) moments of \(\hat{\varepsilon}_i\), but imposes symmetry on \(\hat{F}_i\).

Remark 2.1. In Section H we also present an extension of the doubly robust pseudo-outcome to allow for possible (discrete or continuous) effect modifiers, and present the natural extension of the test to the case where the effect modifier is discrete.
2.3 Assumptions

Here we introduce the assumptions needed for our theoretical results. Our parameter of interest, $\theta_0(a)$, is defined on the potential outcome $Y^a$ which is not observable. Thus, we need the following identifiability conditions on the observed data.

Assumption I.

1. Consistency: $A = a$ implies $Y = Y^a$.
2. Positivity: $\pi_0(a|l) \geq \pi_{\text{min}} > 0$ for all $l \in \mathcal{L}$ and all $a \in A$.
3. Ignorability: $\mathbb{E}(Y^a|L, A) = \mathbb{E}(Y^a|L)$.

We need some further assumptions to regulate the distribution of the observed data and the treatment effect curve $\theta_0(a)$.

Assumption D.

1. The support of $A$ (i.e., $\mathcal{A}$), is a compact subset of $\mathbb{R}$.
2. The treatment effect curve $\theta_0(a)$ and the marginal density function $\varpi_0(a)$ are twice continuously differentiable.
3. The conditional density $\pi_0(a|l)$ and the outcome regression function $\mu_0(l, a)$ are uniformly bounded.
4. Let $\tau(l, a) := \text{Var}(Y|L = l, A = a)$ be the conditional variance of $Y$ given covariates and treatment level. Assume there exist $\tau_{\text{max}} > 0$ such that $0 < \tau(l, a) \leq \tau_{\text{max}}$ for all $l \in \mathcal{L}$ and $a \in A$. Moreover, define

\[ S_\tau := \{ l \in \mathcal{L} : \tau(l, a) \text{ is a continuous function of } a \}, \]
\[ S_{\pi_0} := \{ l \in \mathcal{L} : \pi_0(a|l) \text{ is a continuous function of } a \}, \]
\[ S_{\mu_0} := \{ l \in \mathcal{L} : \mu_0(l, a) \text{ is a continuous function of } a \}; \]

assume we have $P(S_\tau \cup S_{\pi_0} \cup S_{\mu_0}) = 1$.

Statement 4 about the sets $S_\tau$, $S_{\pi_0}$, $S_{\mu_0}$ is just a slight relaxation of the requirement that the given functions all be simultaneously almost surely continuous everywhere. We also impose some conditions on the estimators of the nuisance parameters and the local linear estimator of the treatment effect curve. We assume the estimators for nuisance parameters fall in classes with finite uniform entropy integrals. For a generic class of functions $\mathcal{F}$, let
\(F\) denote an envelope function for \(\mathcal{F}\), i.e., sup\(_{f \in \mathcal{F}} |f| \leq F\). Let \(N(\varepsilon, \mathcal{F}, \|\cdot\|)\) denote the covering number, i.e., the minimal number of \(\varepsilon\)-balls (with distance defined on \(\|\cdot\|\)) needed to cover \(\mathcal{F}\). Let

\[ J_m(\delta, \mathcal{F}, L_2) := \int_0^\delta \sup_Q (1 + \log N(\varepsilon \|F\|_Q, \mathcal{F}, L_2(Q)))^{m/2} \, d\varepsilon, \tag{2.7} \]

where the sup is over all probability measures \(Q\) and \(L_2(Q) \equiv \|\cdot\|_{2,Q}\) is the \(L_2\) semimetric under the distribution \(Q\), i.e., \(\|f\|_{2,Q} = (\int f^2 \, dQ)^{1/2}\). If \(J_1(1, \mathcal{F}, L_2) < \infty\) we say \(\mathcal{F}\) has a finite uniform entropy integral. We will at times require \(J_m(1, \mathcal{F}, L_2) < \infty\) for differing values of \(m \in \{1, 2, 3, 4\}\). Following standard convention, we sometimes let \(J(\cdot, \cdot, \cdot)\) refer to \(J_1(\cdot, \cdot, \cdot)\). For the assumption on our kernel, we also need to define a Vapnik-Chervonenkis (VC) (Dudley, 1999) class. If a class of functions \(\mathcal{F}\) is a VC class, we have that

\[ \sup_Q N(\tau \|F\|_{2,Q}, \mathcal{F}, L_2(Q)) \leq \left( \frac{C}{\tau^v} \right)^v \]  

for some positive \(C, v\) and all \(\tau > 0\) (and again the sup is over all probability measures \(Q\)). The assumptions we make on our estimators are as follows.

**Assumption E(A).**

1. The bandwidth \(h \equiv h_n\) fulfills \(c_1^1 n^{-1/5} \leq \liminf h_n \leq \limsup h_n \leq c_2^2 n^{-1/5}\) for some constants \(0 < c_1^1 \leq c_2^2 < \infty\).

2. Let \(\bar{\pi}\) and \(\bar{\mu}\) denote the limits of the estimators \(\hat{\pi}\) and \(\hat{\mu}\) such that \(\|\hat{\pi} - \bar{\pi}\|_Z = o_p(\sqrt{h})\) and \(\|\hat{\mu} - \bar{\mu}\|_Z = o_p(\sqrt{h})\), where \(h\) is the bandwidth used in local linear estimator. And we have either \(\bar{\pi} = \pi_0\) or \(\bar{\mu} = \mu_0\).

3. The kernel function \(K\) for the local linear estimator is a continuous symmetric probability density function with support on \([-1, 1]\). Moreover, we assume the class of functions \(\{K((\cdot - a)/h) : a \in \mathbb{R}, h > 0\}\) satisfies condition (2.8).

4. Let \(r_n^\infty\) and \(s_n^\infty\) be such that

\[
\sup_{a \in A} \|\hat{\pi}(a|L) - \pi_0(a|L)\|_2 = O_p(r_n^\infty) \\
\sup_{a \in A} \|\hat{\mu}(L,a) - \mu_0(L,a)\|_2 = O_p(s_n^\infty).
\]

We assume \(s_n^\infty r_n^\infty = o\{(n\sqrt{h})^{-1/2}\}\).
Assumption E(A).1 requires that \( h \) is of the order of magnitude for optimal estimation; such \( h \) can be achieved a variety of ways (for instance, one can minimize a risk estimate, or perform cross validation [Fan and Gijbels, 1996]). Assumption E(A).2 is not a stringent assumption; by definition \( \pi, \bar{\mu} \) are the limits of the estimators \( \hat{\pi}, \hat{\mu} \); here we require the rate of convergence (in \( \| \cdot \|_Z \) ) to these limits (which are not necessarily the truth) to be order \( \sqrt{h} = o(n^{-1/10}) \) which is quite slow.

Assumption E(A).3 is a standard assumption on the user-chosen kernel. Assumption E(A).4 is a somewhat non-standard assumption, since it combines \( L_\infty \) and \( L_2 \) norms. The \( L_2 \) aspect arises in the local asymptotics of Kennedy et al. [2017], and the \( L_\infty \) aspect arises because we consider a global test. Note that in parametric settings, \( r_n^\infty \) or \( s_n^\infty \) may attain \( \sqrt{n} \) rates. For instance, in a linear regression, if the regression model is \( \mu(l, a) = (l^T, a)\beta \), and \( \hat{\beta} \) is an estimator converging to the true parameter \( \beta_0 \) at \( \sqrt{n} \) rate, then 
\[
P \left( (l^T, a)(\hat{\beta} - \beta_0) \right)^2 \leq 2O_p(n^{-1/2})(a^2 + \|l\|_2^2),
\]
which follows from using the inequality \((a + b)^2 \leq 2a^2 + 2b^2\) and the fact that if \( (\hat{\beta} - \beta_0) = O_p(n^{-1/2}) \) then \((\hat{\beta} - \beta_0)(\hat{\beta} - \beta_0)^T\) has eigenvalues of order \( O_p(n^{-1}) \). Taking a square root and a supremum over the bounded set \( a \in A \) shows that in this case \( r_n^\infty \) is order \( \sqrt{n} \). Similar results hold in other parametric models for \( \mu \) or for \( \pi \). Thus indeed, the test is “doubly robust”: if one parametric model is well-specified and attains root-\( n \) rates, the other may be misspecified.

Many nonparametric or semiparametric examples will also satisfy Assumption E(A).4; if \( r_n^\infty \) and \( s_n^\infty \) are both, say, \( n^{-2/5} \) up to polylogarithmic factors, which is the rate one expects from for instance a generalized additive model under twice differentiability, then the assumption is satisfied.

In addition to the above E(A) (‘Estimator assumption part A’) assumption, we make one more assumption (‘Estimator assumption part B’). We subscript this next assumption by \( m \), which corresponds to the requirement that \( J_m(1, F, L_2) < \infty \). Differing results will require differing values of the integer \( m \) in \{1, 2, 3, 4\}. We label the below assumption as “E(B)_m” and when we want to assume that, for example, \( J_3(1, F, L_2) < \infty \), we refer to the assumption as “E(B)_3”.

**Assumption E(B)_m.** The estimators \( \hat{\pi}, \hat{\mu} \) and their limits \( \bar{\pi}, \bar{\mu} \) are contained in uniformly bounded function classes \( F_\pi, F_\mu \), which satisfy that \( J_m(1, F, L_2) < \infty \) for \( F = F_\pi \) or \( F = F_\mu \), with \( 1/\bar{\pi} \) also uniformly bounded.
Moreover, we assume $P(S_{\bar{\pi}} \cup S_{\bar{\mu}}) = 1$, where we let

\[ S_{\bar{\pi}} := \{ l \in L : \bar{\pi}(a|l) \text{ is a continuous function of } a \} \]

\[ S_{\bar{\mu}} := \{ l \in L : \bar{\mu}(l, a) \text{ is a continuous function of } a \} \].

Remark 2.2. To control rates of convergence, we make assumptions on the complexity of the classes being considered in $E(B)$. For our first main theorem (limit distribution of the test statistic) we require $E(B)_m$ i.e. $m = 3$ and for our bootstrap theorem we require $E(B)_4$ i.e. $m = 4$. For instance, if $\mathcal{F}_\mu$ is a class of Hölder continuous functions with Hölder exponent $\beta > 0$ on $D = d + 1$ dimensional Euclidean space, and we require $E(B)_m$ to hold, then the $\varepsilon$-entropy is of order $\varepsilon^{-D/\beta}$ so we require $mD/2\beta < 1$ or $\beta > mD/2$. When $m = 1$ the condition is the standard one and when $m = 3$ or $4$ it is more restrictive.

Remark 2.3. It may be possible to weaken these assumptions. The assumption $E(B)_m$ with $m > 1$ arises from certain (degenerate) U- or V-process terms in the analysis. Analyzing such terms requires these more stringent entropy conditions. On the other hand, an $m$th order (degenerate) U-process comes with a faster decay to 0, of order $n^{-m/2}$. When the class $\mathcal{F}$ (i.e., $\mathcal{F}_\mu, \mathcal{F}_\pi$) does not depend on $n$ this does not help us. But if we allow a sieve-type approach where the class $\mathcal{F} \equiv \mathcal{F}_n$ depends on $n$, then we need $J_1(1, \mathcal{F}_n)$ to be $O(1)$ but $J_m(1, \mathcal{F}_n)$ for $m > 1$ can be allowed to grow with $n$; if we allow such sieve classes $\mathcal{F}_n$, then the only entropy required to stay finite/bounded in $n$ is $J_1$, and so in this sense we can recover/require the more classical condition. At present we have phrased the conditions only in terms of independent-of-$n$ classes.

3 Main results

Now we present the asymptotic distribution of our test statistic $T_n$ under the null hypothesis. To metrize weak convergence, we use the Dudley metric [Chapter 14, Section 2, Shorack, 2000] (although any topologically equivalent metric would work), which is defined as

\[
d(\mu, \nu) := \sup \left\{ \int g \, d\mu - \int g \, d\nu : \|g\|_{BL} \leq 1 \right\}, \tag{3.1}\]

where $X$ and $Y$ are random variables with probability distributions/laws $\mu$ and $\nu$, respectively, and where $\|g\|_{BL} := \sup_{x \in \mathbb{R}} |g(x)| + \sup_{x \neq y} |g(x) - g(y)|/|x - y|$. For a kernel function $K$, we use $K^{(s)}$ to denote the $s$-times
convolution product of $K$, that is $K^{(s)}(x) = \int K^{(s-1)}(y)K(x-y)\,dy$, with $K^{(1)} = K$. And we let $K_h^{(s)}(x) := K^{(s)}(x/h)$. Let
\[ \sigma^2(a) = \mathbb{E} \left[ \frac{\tau(L, a) + \{ \mu_0(L, a) - \hat{\mu}(L, a) \}^2}{\bar{\pi}(a|L)/\bar{\varpi}(a)} \right] - \{ \theta_0(a) - \bar{m}(a) \}^2, \]
(3.2)
where $\bar{\varpi}(a) := \int \bar{\pi}(a|l)\,dP(l)$ and $\bar{m}(a) := \int \bar{\mu}(l, a)\,dP(l)$. We can now state our main theorem, which gives the limit distribution of our test statistic under the null hypothesis. Let $L(X)$ denote the probability law of the random variable $X$.

**Theorem 3.1.** Let Assumptions I1–I3, D1–D4, E(A).1–E(A).4, and E(B) hold and let $w(\cdot)$ be a continuously differentiable weight function on $A$. Let $\sigma^2(\cdot)$ and $T_n$ be as given in (3.2) and (J.1). Then under $H_0$ (from (M.4)), we have
\[ d\{ L(T_n), L(N(b_{0h}, V)) \} \to 0 \]
(3.3)
as $n \to \infty$, where
\[ b_{0h} = h^{-1/2}K^{(2)}(0) \int_A \frac{\sigma^2(a)w(a)}{\bar{\varpi}(a)}\,da, \quad V = 2K^{(4)}(0) \int_A \left[ \frac{\sigma^2(a)w(a)}{\bar{\varpi}(a)} \right]^2\,da. \]
(3.4)
The full proof is given in the appendices. We provide an outline of the proof here.

**Proof outline for Theorem 3.1.** We can decompose the statistic $T_n$ as
\[ T_n = n\sqrt{h} \int_A (D_1(a) + D_2(a) + D_3)^2 w(a)\,da, \]
(3.5)
where $D_1(a) := \hat{\theta}_h(a) - \hat{\theta}_h(a)$, $D_2(a) := \hat{\theta}_h(a) - \mathbb{P}_n\check{\xi}(Z)$, $D_3 := \mathbb{P}_n\check{\xi}(Z) - \mathbb{P}_n\check{\xi}(Z)$, and where $\hat{\theta}_h(a)$ is the local linear estimator regressing the oracle doubly robust mappings $\check{\xi}_i := \xi(Z_i; \bar{\pi}, \bar{\mu})$ on $A_i$. Expanding the square leads to 6 terms to be analyzed, so we break the proof up into 6 main steps corresponding to each of those terms. In step 1, we verify that $n\sqrt{h} \int_A D_2(a)^2 w(a)\,da$ is distributed approximately as the given $N(b_{0h}, V)$ limit distribution in the theorem. This follows essentially from Alcalá et al. [1999], which extends
the results of Härdle and Mammen [1993] to allow local polynomial estimators. In steps 2–6, we show that the remainder terms (which are the terms $n \sqrt{h} \int_A D_3^2 w(a) \, da$, $n \sqrt{h} \int_A D_1(a)^2 w(a) \, da$, $n \sqrt{h} \int_A D_2(a) w(a) \, da$, $n \sqrt{h} \int_A D_1(a) D_3 w(a) \, da$, and $n \sqrt{h} \int_A D_1(a) D_2(a) w(a) \, da$) are all $o_p(1)$ as $n \to \infty$. We now will discuss those remainder terms in slightly more detail, basically in parallel with steps 2–6.

Let $\eta := (\pi, \mu)$ (and $\tilde{\eta} := (\tilde{\pi}, \tilde{\mu})$, $\hat{\eta} := (\hat{\pi}, \hat{\mu})$). Both $D_1(a)$ and $D_3$ involve summations over $\tilde{\xi}(Z_i; \tilde{\eta}) - \xi(Z_i; \tilde{\eta})$ and in analyzing the remainder terms, we break these summations into sums over $\tilde{\xi}(Z_i; \tilde{\eta}) - \xi(Z_i; \tilde{\eta})$ and $\xi(Z_i; \tilde{\eta}) - \xi(Z_i; \hat{\eta})$. (Recall the definition of $\tilde{\xi}$ given in (J.2), in which $dP$ is replaced by $d\bar{P}_n$.) The sums over $\tilde{\xi}(Z_i; \tilde{\eta}) - \xi(Z_i; \tilde{\eta})$ yield terms that can be written as degenerate $V$-statistics (or, rather, because of the presence of the random $\tilde{\eta}$, terms whose size is governed by degenerate $V$-processes). We are are able to conclude that these are of very small order of magnitude, but unfortunately the empirical process tools (that we are aware of) for analyzing them requires the imposition of stronger entropy conditions than the normal Donsker-type condition. (Because of the presence of up to order 3 $V$-processes, we require $J_3(1, \mathcal{F}, L_2) < \infty$ rather than the weaker, more standard Donsker condition, $J_1(1, \mathcal{F}, L_2) < \infty$ (for $\mathcal{F}$ equal to both $\mathcal{F}_\mu, \mathcal{F}_n$).)

For instance, in Step 2 we write the term $D_3$ as

$$
P_n \{\xi(Z; \tilde{\pi}, \tilde{\mu}) - \tilde{\xi}(Z; \hat{\pi}, \hat{\mu})\} = P_n \{\xi(Z; \tilde{\pi}, \tilde{\mu}) - \xi(Z; \hat{\pi}, \hat{\mu})\} + P_n \{\xi(Z; \hat{\pi}, \hat{\mu}) - \tilde{\xi}(Z; \hat{\pi}, \hat{\mu})\}.
$$

The first summand (of the right hand side above) is further decomposed into two terms of types that commonly arise in causal inference or semiparametric problems one an empirical process one and one a ‘second order remainder’; the former is shown to be negligible by an empirical process asymptotic equicontinuity argument and the latter is small by assumptions on $r_n \sim n^\infty$. The second summand in the display above can be written as a degenerate order 2 $V$-process. We provide an introduction to and discussion of $U$- and $V$-processes in Section K Under Assumption $E(B)_2$ (implied by Assumption $E(B)_3$) we show that the $V$-process is negligible. Note that a degenerate order $m$ $U$- or $V$-statistic is usually of order $n^{-m/2}$ [van der Vaart, 1998, Chapter 12].

In Step 3, we write $D_1(a)$ as $D_1(a) = d_{1,1}(a) + d_{1,2}(a)$, where

$$
d_{1,1}(a) := P_n \left[ W_{ha}(A) \left( \tilde{\xi}(Z; \tilde{\pi}, \tilde{\mu}) - \xi(Z; \hat{\pi}, \hat{\mu}) \right) \right],
$$

$$
d_{1,2}(a) := P_n \left[ W_{ha}(A) \left( \xi(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \tilde{\pi}, \tilde{\mu}) \right) \right],
$$

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and \(W_{ha}(\cdot)\) are “equivalent” kernels for the local polynomial estimator (see the proof for definitions and references or see Fan and Gijbels [1996], page 63). The added difficulty here from Step 2 is that these terms are local i.e. they depend on \(h\), but thematically the decomposition works the same as in Step 2. The term \(d_{1,2}(a)\) is handled by an asymptotic equicontinuity argument and assumptions on nuisance estimators, and the term \(d_{1,1}(a)\) by a (nonasymptotic) V-process maximal inequality (see Proposition K.2) for an order 2 V-process.

In Step 4, we consider \(\int_{A} D_2(a) D_3 w(a) da\); here \(D_3\) can be factored out and handled by the result of Step 2, and the remaining integral of \(D_2(a)\) can be handled in an elementary way by Taylor expansion and the Central Limit Theorem. In Step 5, we consider \(\int_{A} D_1(a) D_3 w(a) da\), whose negligibility follows immediately from the analysis in Steps 2 and 3 and the Cauchy-Schwarz inequality.

Finally, in Step 6 an order 3 V-processes arises (and so Assumption E(B)\(^3\) is needed) in the analysis of \(\int_{A} D_1(a) D_2(a) w(a) da\). This can be essentially simplified into (a sum of) two main terms,

\[
\int \frac{1}{\omega_0^2(a)} \mathbb{P}_n \left[ K_{ha}(A) \left\{ \xi(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \right\} \right] \frac{1}{n} \sum_{i=1}^{n} K_h (A_i - a) \bar{\varepsilon}_i w(a) da,
\]

\[
\int \frac{1}{\omega_0^2(a)} \mathbb{P}_n \left[ K_{ha}(A) \left\{ \hat{\xi}(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \right\} \right] \frac{1}{n} \sum_{i=1}^{n} K_h (A_i - a) \bar{\varepsilon}_i w(a) da,
\]

where \(\bar{\varepsilon}_i := \xi(Z; \bar{\pi}, \bar{\mu}) - \mathbb{P}\xi(Z; \bar{\pi}, \bar{\mu})\). The first term in (E.6) is further decomposed, with \(\mathbb{P}_n\) written as \((\mathbb{P}_n - \mathbb{P}) + \mathbb{P}\); usually, the \((\mathbb{P}_n - \mathbb{P})\) term yields an empirical process and the \(\mathbb{P}\) term yields a second order remainder. Here the empirical process piece can be handled by previous arguments; the \(\mathbb{P}\) term can not be treated as a second order remainder because taking absolute values (as would be commonly done) breaks the mean zero structure of the term and does not yield the correct size (because of the integral over \(A\), the \(n^{-1} \sum_{i=1}^{n} K_h (A_i - a) \bar{\varepsilon}_i w(a)\) term and \(\mathbb{P}\) term cannot be analyzed separately). Rather, the term is handled by an empirical process (maximal inequality) argument (in Lemma E.4). Finally, the second term in (E.6) is the order three V-process, requiring \(J_3(1, \cdot, L_2) < \infty\) in order to apply a maximal inequality.\(^3\) That completes our proof outline; full details of the proof are given in Section B.

\(^3\)Again, in the analysis of this [and other] term[s] we cannot analyze the multiplicands’ orders of magnitudes separately, because taking absolute values breaks the mean zero structure; i.e., to explain, let \(C_i, D_i\) be generic random variables and then although we
Next we state the consistency of our test under alternatives as follows. (The $\delta_n$ sequence can in particular be $(n\sqrt{h})^{1/2}$.)

**Theorem 3.2.** Let Assumptions I1–I3, D1–D4, E(A)1–E(A)4 hold and let $w(\cdot)$ be a continuously differentiable weight function on $A$. Let $\sigma^2(\cdot)$ and $T_n$ be as given in (3.2) and (J.1). Then under the alternative $\theta_0(a) = c_0 + \delta_n(n\sqrt{h} - 1/2)g(a)$, where $c_0 = \mathbb{P}\xi(Z; \pi_0, \mu_0)$, where $g(\cdot)$ is not the constant 0, and where $\int g(a)\varpi(a)w(a)\,da = 0$. Moreover, $\delta_n$ is a sequence converging to $\infty$ such that $\lim_{n \to \infty} n^{1/40}/\delta_n = 0$. Then we have

$$P(T_n > z_{n,1-\alpha}) \to 1$$

as $n \to \infty$, where we use $z_{n,1-\alpha}$ to denote the upper $\alpha$ quantile of the $N(b_n, V)$ distribution in (3.3).

The proof is given in the appendices. The condition that $\int g(a)\varpi(a)w(a)\,da = 0$ is not a substantive restriction, it is just so that $c_0$ is ‘identifiable’ in a sense.

Finally, we show that the bootstrap distribution of the statistic can be used to approximate $T_n$’s unknown distribution (under the null). For our proof to hold, we require an extra entropy condition to accommodate a fourth order V-process that appears in the analysis of the wild bootstrap. Recall the definition of $J_4$ from (2.7) and the definition of the Dudley metric for weak convergence in (3.1). Let $\mathcal{L}^*(X) := \mathcal{L}(X|Z_1, \ldots, Z_n)$ denote the conditional law of a random variable $X$.

A variety of bootstrap definitions are possible (and are included in the simulations) and discussed in Subsection 2.2. For the following theorem, we let $\hat{\xi}_i := \hat{\xi}(Z_i; \hat{\pi}, \hat{\mu}) - \sum_{i=1}^n \hat{\xi}(Z_i; \hat{\pi}, \hat{\mu})/n$ be centered at the null estimate, and we take $\varepsilon^*_i$ to be the Rademacher choice so equal to $\pm\hat{\varepsilon}_i$ with probability $1/2$ each. Then we proceed as discussed in Subsection 2.2, with $(\xi^*_i = \mathbb{P}_n\hat{\xi} + \varepsilon^*_i, A_i)$ as bootstrap observations and defining $T^*_n$ by (J.1) but using the bootstrap observations.

**Theorem 3.3.** Let the assumptions of Theorem 3.1 hold. Let $T^*_n$ be the bootstrap test statistic defined in the paragraph preceding this theorem. Then

$$d(\mathcal{L}^*(T^*_n), \mathcal{L}(N(b_n, V))) \to_p 0$$

as $n \to \infty$.

---

\textsuperscript{20} can bound $|\sum_i C_iD_i| \leq (\max_i |C_i|) \sum_i |D_i|$, unfortunately then $\sum_i |D_i|$ (with absolute values) is not a sum of mean zero variables, even if the $\{D_i\}$ are mean zero; getting the right order of magnitude requires treating the multiplicands simultaneously as a V-process.
Next we consider the bootstrap where \( \hat{\varepsilon}_i := \tilde{\xi}(Z_i; \hat{\pi}, \hat{\mu}) - \hat{\theta}_h(A_i) \) (and the rest of the procedure is as described in the paragraph preceding the previous theorem). We study this case in the next theorem, which requires an extra entropy condition \( J_4 < \infty \).

**Theorem 3.4.** Let the assumptions of Theorem 3.1 hold. Further, assume that \( J_4(1, \mathcal{F}, L_2) < \infty \) for \( \mathcal{F} = \mathcal{F}_\mu \) and for \( \mathcal{F} = \mathcal{F}_\pi \). Let \( T^*_n \) be the bootstrap test statistic defined in the paragraph preceding this theorem. Then

\[
d(L^*(T^*_n), \mathcal{L}^*(N(b_n, V))) \to_p 0
\]

as \( n \to \infty \).

The structure of the proofs is analogous to that of Theorem 3.1, although the calculations are somewhat more intricate, and lead to a fourth order \( V \)-process which requires the finiteness of \( J_4 \) as stated in the theorem. The proof details are given in the appendices. Both bootstraps are studied in our simulations and seem to perform similarly.

## 4 Simulation studies

### 4.1 Simulation for testing constant average treatment effect

We use simulation to assess the performance of our proposed test in terms of both type I error probability and power. We consider two data generating models, one with a binary response and which is defined similarly as in Kennedy et al. [2017], and another one with a continuous response. In more detail, they are specified as follows.

**Model 1:** we simulate the covariates from independent standard normal distributions, \( L = (L_1, \ldots, L_4)^T \sim N(0, I_4) \), and simulate the treatment level from a Beta distribution,

\[
\frac{A}{20} | L \sim \text{Beta}(\lambda(L), 1 - \lambda(L)),
\]

\[
\text{logit} \lambda(L) = -0.8 + 0.1L_1 + 0.1L_2 - 0.1L_3 + 0.2L_4,
\]

and finally the binary outcome is simulated as \( Y | L, A \sim \text{Bernoulli}(\mu(L, A)) \), where \( \text{logit} \mu(L, A) = 1 + (0.2, 0.2, 0.3, -0.1) L + \delta A (0.1 - 0.1L_1 + 0.1L_3 - 0.13^2 A^2) \).

**Model 2:** the covariates are simulated the same as in Model 1. We simulate the treatment level from a Beta distribution,

\[
\frac{A}{5} | L \sim \text{Beta}(\lambda(L), 1 - \lambda(L)),
\]

\[
\text{logit} \lambda(L) = 0.1L_1 + 0.1L_2 - 0.1L_3 + 0.2L_4,
\]
and we simulate the continuous response from conditional normal distributions as $Y|\mathbf{L}, A \sim N(\mu(\mathbf{L}, A), 0.5^2)$, where

$$\mu(\mathbf{L}, A) = (0.2, 0.2, 0.3, -0.1)\mathbf{L} + A(-0.1L_1 + 0.1L_3) + \delta \exp \left\{ -\frac{(A - 2.5)^2}{(1/2)^2} \right\}.$$ 

In both models, we have a parameter $\delta$ that controls the distance between the true treatment effect and the null hypothesis, i.e., treatment effect is constant, with $\delta = 0$ indicating no treatment effect in both models. In Model 1, $\delta = 0$ yields a constant average treatment effect and is a ‘strong null’ meaning all individuals have the same treated and untreated outcomes; in Model 2 when $\delta = 0$ the ‘weak null’ holds meaning conditional average treatment effects are nonconstant but the average treatment effect is constant. Specifically, for Model 1, we let $\delta \in \{0, 0.002, 0.004, 0.006, 0.008, 0.01\}$, and we let $\delta \in \{0, 0.1, 0.2, 0.3, 0, 4, 0.5\}$ for Model 2. We plot the treatment effect curve with $\delta = 0.01$ for Model 1 and the treatment effect curve with $\delta = 0.5$ for Model 2 in Figure 1.

For each data generating model, we test the performance of our method under 4 scenarios: (1) $\pi$ is correctly specified with a parametric model, $\mu$ is incorrectly specified with a parametric model; (2) $\pi$ is incorrectly specified with a parametric model, $\mu$ is correctly specified with a parametric model; (3) both $\pi$ and $\mu$ are correctly specified with a parametric model; (4) both $\pi$ and $\mu$ are estimated with Super Learners [Van der Laan et al., 2007]. In the first two scenarios, the incorrect parametric models are constructed in the same fashion as in Kang et al. [2007]. The first three scenarios are used to test double robustness of our method and last one to show the empirical
performance of our method when we use flexible machine learning models to estimate the nuisance functions. After we calculate the pseudo-outcomes, we use the rule of thumb for bandwidth selection [Fan and Gijbels, 1996] for the local linear estimator. We compare the performance of our method with Westling [2021] in the first three scenarios; with Westling [2021] and a discretized version of TMLE [Gruber and van der Laan, 2012] with treatment dichotomized at the middle point in the last scenario. In our method, we choose the weight function \( w(a) \equiv 1 \). We implemented all three versions \((L_1, L_2, \text{and } L_\infty)\) of the methods in Westling [2021] for comparison. Rejection probabilities are estimated with 1000 independent replications of simulation. Finally, we consider sample sizes in \( \{500, 1000, 2000\} \).

Figures 5 and 6 show the results for Model 1. We can see when at least one of the nuisance functions is correctly estimated, our method and Westling’s methods performed similarly in terms of both type I error probability and power. When both nuisance functions are estimated with Super Learners, Westling’s methods have slightly larger power than our method but also have slightly larger type I error probabilities. Note that in this case, the discretized version of TMLE outperformed both our method and Westling’s method in terms of type I error probability and power. The reason may be the shape of the treatment effect curve in Model 1 is somewhat simple and monotone, and there isn’t much information loss if we dichotomize the continuous treatment to form a simpler testing problem. We also note that apparently Westling [2021]’s method is doubly robust on the specific data generating model we use here.

Figures 2 and 3 show the results for Model 2 where we have a slightly complicated and non-monotone treatment effect curve as in Figure 1. We observe that in the first three scenarios our methods outperform Westling’s methods in terms of power in all cases. Our method does better even with a small sample size and a weak deviation from the null model, compared with Westling’s method. Similar observations hold when we use Super Learner to estimate both nuisance functions. Another observation worth noting is that in Model 2 the discretized TMLE fails to detect any deviation from the null model since the treatment effect curve is symmetric, which provides an example in which discretizing a continuous treatment and applying a binary test may lead to a completely incorrect conclusion.

### 4.2 Cross-fitted test procedure

We also consider simulations to analyze how dimensionality of the confounders \( L \) can affect the performance of our test and how cross-fitting
Figure 2: Simulation result for Model 2 with $\pi$ and $\mu$ estimated from parametric models.

Figure 3: Simulation result for Model 2 with $\pi$ and $\mu$ estimated from nonparametric models.
could be applied to improve finite sample performance under high dimensional settings. To save space, we defer this part of the content to Section J. The detailed description of the test procedure with cross-fitting can be found in Section J.1. We conduct simulation studies to compare our main proposed test (without cross-fitting) and the cross-fitted test under low dimensional data and under high dimensional data, respectively, in Section J.2. The following is a summary of the results from the simulations. When the dimensionality of the covariates is small, both non-cross-fitted and cross-fitted tests can achieve the desired type I error probability but the cross-fitted version tends to have lower power; when the dimensionality of the covariates is large, only the cross-fitted version maintains the desired type I error probability.

5 Analysis of data on nursing hours and hospital performance

In this section we apply our test to a real data problem. In Kennedy et al. [2017] and McHugh et al. [2013], the authors were interested in whether nurse staffing (measured in nurse hours per patient day) affected a hospital’s risk of excess readmission penalty after adjusting for hospital characteristics (for more detail of the data and related background of the problem, see McHugh et al. [2013]). Kennedy et al. [2017] proposed a doubly robust procedure to estimate the probability of readmission penalty against average adjusted nursing hours per patient day, and provided pointwise confidence intervals for the estimated treatment curve. However, their method and analysis did not answer the question of whether nurse staffing significantly affects the probability of excess readmission penalty after adjusting for hospital characteristics. We apply our method to test the null hypothesis: nurse staffing does not affect hospital’s risk of excess readmission penalty after adjusting for hospital characteristics, with updated data from the year 2018. As a brief summary of the data, the outcome \( Y \) indicates whether the hospital was penalized due to excess readmissions and are calculated by the Center for Medicare & Medicaid Services (https://www.cms.gov). The treatment \( A \) measures nurse staffing hours and we calculate it as the ratio of registered nurse hours to inpatient days, which is slightly different from Kennedy et al. [2017] and McHugh et al. [2013], because we don’t have access to the hospitals’ financial data and thus are not able to calculate adjusted inpatient days. The covariates \( L \) include the following nine variables: the number of beds, the teaching intensity, an indicator for not-for-profit status, an indi-
Figure 4: Estimated treatment effect of average nursing hours on probability of readmission penalty.

cator for whether the location is urban or rural, the proportion of patients on Medicaid, the average patient socioeconomic status, a measure of market competition, an indicator for whether the hospital has a skilled nursing facility (because our measure of nurse staffing hours \( A \) will unfortunately include hours worked in such a skilled nursing facility), and whether open heart or organ transplant surgery is performed (which serves as a measurement of whether the hospital is high technology). We omitted patient race proportions and operating margin from the analysis (present in Kennedy et al. [2017] and McHugh et al. [2013]) because we don’t have access to those features. Figure 9 shows an unadjusted loess fit of the readmission penalty as a function of the average nursing hours and the loess fits of the covariates against the average nursing hours. The curves are not identical to those in Kennedy et al. [2017] since we’ve used updated data from 2018, but we observe generally similar patterns and nurse staffing hours is correlated with many hospital characteristics. In the analysis, we use Super Learner [Van der Laan et al., 2007] with the same implementation as in Kennedy et al. [2017] to estimate \( \pi_0 \) and \( \mu_0 \). We truncate \( \hat{\pi} \) to be 0.01 if it fell below that value. The rule of thumb Fan and Gijbels [1996] is applied for bandwidth selection as in Section 4. Since our test statistic is based on the integrated distance between the nonparametric fit of the treatment effect curve and the parametric fit of the treatment effect curve under the null hypothesis, a byproduct of the test is the estimated treatment effect curve. We plot the estimated treatment effect curve of average nurse staffing in Figure 4 (the solid red curve).

We apply our test, Westling’s test, and the discretized version of TMLE
to this data set. All versions of our methods and Westling’s methods have p-values of 0. (Exact zeros are due to the fact that we use simulated reference distributions.) The discretized TMLE reports a p-value of 0.0017. So all the tests suggest strong statistical evidence against the null hypothesis of constant treatment effect, meaning average nursing hours does have a significant causal impact on a hospital’s chance of being penalized for excess readmissions. This interpretation requires that we have included all important confounders in our analysis. If we have not, our test result is interpreted as being based on a partially adjusted estimate of association (rather than the treatment effect curve).

Finally, we test whether an indicator for whether a hospital is in a rural or in an urban setting is a treatment effect modifier. We present the estimated conditional treatment effect curves for rural hospitals and for the urban hospitals in Figure 4 (dashed green for the rural hospitals and dotted blue for the urban hospitals). We observe that for the hospitals in urban areas, the pattern of the effect curve has a shape that is close to concave and is similar to the pattern in the overall average treatment effect curve: after average nursing hours exceeds 10, increasing the average nursing hour results in a decrease in the probability of the readmission penalty. The increasing trend of the curve up to 8 average nursing hours seems to be counter-intuitive, but it turns out that there are not many hospitals in that range of the data and thus the left tail behavior is likely an artifact due to low sample size in that region. On the other hand, the effect curve for rural hospitals is wavy and does not suggest a clear pattern.

We first apply our main test separately to each of the two groups of hospitals to see whether the two individual treatment effect curves are constant or not. We obtain a p-value of 0.28 for the group of rural hospitals and a p-value of approximately 0 for the urban hospitals. This analysis suggests that the conditional treatment curve for the rural hospitals is not significantly different from constant, so the wavy pattern we see in the estimated curve is likely due to randomness. Next we apply the extended test procedure and obtain a p-value of 0.007, which indicates a significant difference between the two conditional treatment curves. Again, these interpretations require that we have included all important confounders in our analysis. It is somewhat surprising that in this dataset the rural hospitals show no significant effect of nurse staffing on readmission penalty. It is possible that hospital occupancy rates, case mix, financial stability and differing abilities to recruit and retain nurses are important for understanding the effect of nurse staffing on hospital performance, either as confounders or as treatment effect modifiers.
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A Empirical process lemmas

In this section, we first discuss the concept of stochastic equicontinuity and provide two lemmas that will be used in the proof of our main result. Let $G_n = \sqrt{n}(P_n - P)$. Then a sequence of empirical processes \( \{G_nV_n(f) : f \in \mathcal{F}\} \) indexed by elements \( f \) from a metric space \( \mathcal{F} \) equipped with semimetric \( \rho \) is stochastically equicontinuous [van der Vaart and Wellner, 1996] if for every \( \varepsilon > 0 \) and \( \zeta > 0 \) there exists a \( \delta > 0 \) such that

$$\limsup_{n \to \infty} P\left( \sup_{\rho(f_1,f_2) < \delta} |G_nV_n(f_1) - G_nV_n(f_2)| > \varepsilon \right) < \zeta.$$ 

We also rely on a standard measurability condition, that of “\( P \)-measurability” [Definition 2.3.3, van der Vaart and Wellner, 1996], which is generally satisfied and so we do not give the definition here.

**Lemma A.1** (Theorem 2.5.2, van der Vaart and Wellner [1996]). Consider the sequence of processes \( \{G_nV_n(\cdot) : n \geq 1\} \) where \( V_n \) is the identity mapping, i.e.,

$$V_n(f) = f,$$

here \( f \in \mathcal{F} \) with envelope \( F(Z) = \sup_{f \in \mathcal{F}} |f(Z)| \). Assume \( F \) is uniformly bounded, i.e., \( \|F\|_Z \leq F_{\max} < \infty \), and \( \mathcal{F} \) has a finite uniform entropy integral, i.e., \( J_1(\delta,\mathcal{F},L_2) < \infty \), and \( J_1(\delta,\mathcal{F},L_2) \) is as defined in (2.7) Let the class \( \mathcal{F}_\delta \) and \( \mathcal{F}_{\infty}^2 \) be \( P \)-measurable for every \( \delta > 0 \), where \( \mathcal{F}_\delta = \{f_1 - f_2 : f_1, f_2 \in \mathcal{F}, \|f_1 - f_2\|_2 < \delta\} \) and \( \mathcal{F}_{\infty}^2 = \{(f_1 - f_2)^2 : f_1, f_2 \in \mathcal{F}, \|f_1 - f_2\|_2 < \delta\} \). Then \( \{G_nV_n(\cdot) : n \geq 1\} \) is stochastically equicontinuous.

**Lemma A.2.** Consider the sequence of processes \( \{G_nV_n(\cdot) : n \geq 1\} \) with

$$V_n(f) = \int_A f(L,t) dt,$$

where \( f \in \mathcal{F} \) with envelope \( F \) as in Lemma A.1. Assume \( F \) is uniformly bounded and \( \mathcal{F} \) has a finite uniform entropy integral. Then \( \{G_nV_n(\cdot) : n \geq 1\} \) is stochastically equicontinuous.
Proof. We check conditions (1)-(3) of Theorem 2.11.1 from van der Vaart and Wellner [1996]. For the Lindeberg condition (1), note
\[
\mathbb{E} \left\{ \| V_n \| F > \varepsilon \sqrt{n} \right\} \leq \mathbb{E} \left\{ \left( \int_A F(Z) \, dt \right)^2 I \left( \int_A F(Z) \, dt > \varepsilon \sqrt{n} \right) \right\} \\
\leq [\lambda(A) F_{\text{max}}]^2 I(\lambda(A) F_{\text{max}} > \varepsilon \sqrt{n}) \to 0,
\]
as \( n \to \infty \) for any \( \varepsilon > 0 \), since by Assumption D1 \( A \) is compact. Here \( \lambda(\cdot) \) is the Lebesgue measure on \( \mathbb{R} \).

For the Lindeberg condition (2),
\[
\sup_{\rho(f_1,f_2) < \delta_n} \mathbb{E} \left\{ V_n(f_1) - V_n(f_2) \right\}^2 = \sup_{\rho(f_1,f_2) < \delta_n} \mathbb{E} \left\{ \int_A f_1(L,t) - f_2(L,t) \, dt \right\}^2 \\
\leq (2\lambda(A) \delta_n)^2 \to 0,
\]
for any \( \delta \to 0 \). So condition (2) is also satisfied.

For the complexity condition (3), again we check that the process is measure-like. Note
\[
\frac{1}{n} \{ V_n(f_1) - V_n(f_2) \}^2 = \frac{1}{n} \left[ \int_A f_1(L,t) - f_2(L,t) \, dt \right]^2 \\
\leq \frac{1}{n} \int_A \{ f_1(L,t) - f_2(L,t) \}^2 \, dt,
\]
by Jensen’s inequality. So \( V_n(f) \) is measure-like with \( \nu_n = \frac{1}{n} \delta L_i \times \lambda \). That completes the proof. \( \square \)

Recall that \( N(\varepsilon, F, \| \cdot \|) \) denotes the \( \varepsilon \)-covering number of \( F \), i.e., the minimal number of \( \varepsilon \)-balls (with distance defined by \( \| \cdot \| \)) needed to cover \( F \).

Lemma A.3. Let \( F \) be a class of functions \( f(x,y) \) on \( X \times Y \) with finite uniform entropy. Let \( \mathbb{P} \) be a measure on \( X \times Y \). Then the class of functions \( G := \{ g(y) = \mathbb{P} f(X,y) : f \in F \} \) has \( \sup_Q N(\varepsilon, G, L_2(Q)) \leq \sup_Q N(\varepsilon, F, L_2(Q)) \), for all \( \varepsilon > 0 \).

Proof. By Jensen’s inequality, for a measure \( Q \) on \( Y \),
\[
\int g^2(y) \, dQ(y) = \int \left[ \int f(x,y) \, d\mathbb{P}(x) \right]^2 \, dQ(y) \leq \int \int f^2(x,y) \, d\mathbb{P}(x) \, dQ(y),
\]
and \( d\mathbb{P}(x) \, dQ(y) \) is a measure on \( X \times Y \). Thus, with \( g_i(y) := \mathbb{P} f_i(X,y), i = 1,2 \), we have for a measure \( Q \), \( \int (g_1 - g_2)(y)^2 \, dQ(y) \leq \int (f_1 - f_2)(x,y)^2 \, dQ(x,y) \)
(with \( \widetilde{Q}(x,y) := \mathbb{P}(x) Q(y) \)), and so the conclusion follows. \( \square \)
The following two results are slight modifications of Theorem 3 from Andrews [1994], so that we can apply them to $J_2$ as well as to $J \equiv J_1$ (as defined in (2.7)). The proof of that theorem gives the following statements.

**Lemma A.4.** For two classes of measurable functions $\mathcal{G}, \mathcal{H}$, with envelopes $G$ and $H$, respectively, for any $\varepsilon > 0$ and probability measure $Q$, we have

\[
N(L_2(Q), \mathcal{G} + \mathcal{H}, \varepsilon \| G + H \|_{Q,2}) \\
\leq N(L_2(Q), \mathcal{G}, 2^{-1} \varepsilon \| G \|_{Q,2}) N(L_2(Q), \mathcal{H}, 2^{-1} \varepsilon \| H \|_{Q,2}).
\]

**Lemma A.5.** For two classes of measurable functions $\mathcal{G}, \mathcal{H}$, with envelopes $G$ and $H$, respectively, and for any $\varepsilon > 0$, we have

\[
\sup_Q N(L_2(Q), \mathcal{G} \mathcal{H}, \varepsilon \|(G \lor 1)(H \lor 1)\|_{Q,2}) \\
\leq \sup_Q N(L_2(Q), \mathcal{G}, 2^{-1} \varepsilon \| G \|_{Q,2}) \sup_Q N(L_2(Q), \mathcal{H}, 2^{-1} \varepsilon \| H \|_{Q,2}).
\]

### B Proof of Theorem 3.1

Here we present the proof of Theorem 3.1. An outline of the proof is given in the main text. The proof is broken into 5 steps and a 6th concluding step. Step 1 is about the main term. Steps 2–5 are about remainder terms. The remainder terms are all (essentially, perhaps after some initial analysis) broken up into two main terms, one being a $V$-process and the other depending essentially on the size of $(\hat{\pi}, \hat{\mu}) - (\bar{\pi}, \bar{\mu})$.

**Proof of Theorem 3.1.** We write the test statistic as

\[
T_n = n \sqrt{\hat{h}} \int_A \{D_1(a) + D_2(a) + D_3\}^2 w(a) da,
\]

where

\[
D_1(a) := \hat{\theta}_h(a) - \tilde{\theta}_h(a) \\
D_2(a) := \hat{\theta}_h(a) - \mathbb{P}_n \xi \\
D_3 := \mathbb{P}_n \xi - \mathbb{P}_n \hat{\xi},
\]

and $\hat{\theta}_h(a)$ is the local linear estimator regressing the oracle doubly robust mappings $\hat{\xi}_i := \xi(Z_i; \bar{\pi}, \bar{\mu})$ on $A_i$.

We will show the dominating term $n \sqrt{\hat{h}} \int_A \{D_2(a)\}^2 w(a) da$ converges to the Normal distribution given in the conclusion of the theorem. For the rest
of the terms, we will see that \( n \sqrt{h} \int_{\mathcal{A}} \{D_1(a)\}^2 w(a) \, da \), \( n \sqrt{h} \int_{\mathcal{A}} D_3^2 w(a) \, da \) and all the cross-product terms are asymptotically negligible.

Note that the negligibility of the cross-product terms does not follow from negligibility of the main squared terms and the Cauchy-Schwarz inequality. The reason is that the approximate distribution of \( n \sqrt{h} \int_{\mathcal{A}} \{D_2(a)\}^2 w(a) \, da \) has a mean of order \( 1/\sqrt{h} \to \infty \). Thus, to apply a Cauchy-Schwarz argument to say, \( n \sqrt{h} \int_{\mathcal{A}} D_1(a)D_2(a)w(a)da \), one would need to not just show that \( nh^{1/2} \int_{\mathcal{A}} D_1(a)^2 w(a)da \) is negligible but that it is \( o_p(\sqrt{h}) \), which will not generally be true.

**Step 1.** We first show the asymptotic distribution of \( n \sqrt{h} \int_{\mathcal{A}} \{D_2(a)\}^2 w(a) \, da \) by applying Theorem 2.1 of Alcalá et al. [1999] (which extends the results of Härdle and Mammen [1993] to allow local polynomial estimators) to \( n \sqrt{h} \int_{\mathcal{A}} \{D_2(a)\}^2 w(a) \, da \). We can see assumptions (A1) and (A2) of Alcalá et al. [1999] are automatically satisfied by our Assumption I2, D1 and D2.

Next, we check Assumption (A4) of Alcalá et al. [1999]. By Assumption D3, we know \( \mu_0(l,a) \) is uniformly bounded. And since \( m_0(a) = \int \mu_0(l,a) \, dP(l) \), for any \( a_0 \in \mathcal{A} \) and \( a_n \to a_0 \), by Assumption D4 and the dominated convergence theorem, we have \( |m_0(a_0) - m_0(a_n)| = |\int \{\mu_0(l,a_0) - \mu_0(l,a_n)\} \, dP(l)| \to 0 \) as \( a_n \to a_0 \) and thus \( m_0(a) \) is continuous and bounded. Similarly, we can show \( m, 1/\varpi_0, 1/\varpi \) are continuous and uniformly bounded. Then applying the dominated convergence theorem to (3.2) with \( a_n \to a_0 \) gives the continuity of \( \sigma^2(a) \). Then by the total variance formula, we have

\[
\text{Var} \left( \xi(Z; \bar{\pi}, \bar{\mu}) | A = a \right)
= \mathbb{E} \left\{ \text{Var} \left( \xi(Z; \bar{\pi}, \bar{\mu}) | L, A = a \right) | A = a \right]\}
+ \text{Var} \left\{ \mathbb{E} \left( \xi(Z; \bar{\pi}, \bar{\mu}) | L, A = a \right) | A = a \right\}.
\]

From (2.2), we have

\[
\text{Var} \left( \xi(Z; \bar{\pi}, \bar{\mu}) | L, A = a \right)
= \text{Var} \left( Y - \mu(L, A) | L, A = a \right) \left\{ \frac{1}{\bar{\pi}(a| L)} \int \bar{\pi}(a| L) \, dP(L) \right\}^2
= \tau(L, a) \left\{ \frac{1}{\bar{\pi}(a| L)} \int \bar{\pi}(a| L) \, dP(L) \right\}^2
\]

and by Assumption D4, we have \( \tau(l, a) \) is positive everywhere, so we know the expectation \( \mathbb{E} \{ \text{Var} \left( \xi(Z; \bar{\pi}, \bar{\mu}) | L, A = a \right) | A = a \} \) is positive for all \( a \in \mathcal{A} \) and thus the conditional variance \( \sigma^2(a) \) is positive. Then since \( \mathcal{A} \) is compact,
by the continuity of \( \sigma^2(a) \), we have \( \sigma^2(a) \) is bounded below from 0 and bounded above.

Assumption (A5) of Alcalá et al. [1999] also holds since our null parametric model is a linear model with only intercept term which can be estimated with \( \sqrt{n} \)-consistency. Assumptions (K1) and (K2) of Alcalá et al. [1999] are also met by our Assumption E(A)1 and E(A)3. So we have

\[
\left\{ n\sqrt{h} \int_A D_2(a)^2 w(a) \, da, \, N(b_0h, V) \right\} \to 0, \tag{B.2}
\]
as \( n \to \infty \).

**Step 2** \((D_3^2)\). We now show that \( D_3 = o_p(1/\sqrt{nh^{1/2}}) \) and so in particular, the integral \( n\sqrt{h} \int_A D_3^2 w(a) \, da \) is \( o_p(1) \) as \( n \to \infty \). We can write \( D_3 \) as

\[
\mathbb{P}_n \{ \xi(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \}
= \mathbb{P}_n \{ \xi(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \} + \mathbb{P}_n \{ \xi(Z; \bar{\pi}, \bar{\mu}) - \hat{\xi}(Z; \bar{\pi}, \bar{\mu}) \}. \tag{B.3}
\]

This type of decomposition is used repeatedly below and is discussed/explained in the proof outline in the main document. The first term above (is split up into an empirical process and a ‘second order remainder’ term and) has order of magnitude given by Lemma E.1 as \( O_p(1/\sqrt{n} + s_n^\infty r_n^\infty) \). The second term on the right side of (B.3) is a V-process and is shown to be \( O_p(n^{-1/2}) \) by Lemma C.1. So \( D_3 = O_p(1/\sqrt{n} + s_n^\infty r_n^\infty) \) and thus we conclude that \( D_3 = o_p((nh^{1/2})^{-1/2}) \) by Assumption E(A)4.

**Step 3** \((D_3^2)\). Now we show the asymptotic negligibility of \( n\sqrt{h} \int_A D_1(a)^2 w(a) \, da \).

To use the standard representation of a local polynomial estimator (see Lemma G.2), we let \( \hat{D}_{ha} = \mathbb{P}_n \{ gh_{ha}(A) K_{ha}(A) g_{ha}^T(A) \} \) and \( W_{ha}(t) := g_{ha}^T(t) \hat{D}_{ha}^{-1} g_{ha}(t) K_{ha}(t) \). Then (by Lemma G.2) we write \( D_1(a) \) as

\[
D_1(a) = g_{ha}^T(a) \hat{D}_{ha}^{-1} \mathbb{P}_n \left[ gh_{ha}(A) K_{ha}(A) \left\{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \hat{\pi}, \hat{\mu}) \right\} \right] \tag{B.4}
\]

where

\[
d_{1,1}(a) := \mathbb{P}_n \left[ W_{ha}(A) \left\{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \hat{\pi}, \hat{\mu}) \right\} \right],
\]

\[
d_{1,2}(a) := \mathbb{P}_n \left[ W_{ha}(A) \left( \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \hat{\pi}, \hat{\mu}) \right) \right]. \tag{B.5}
\]

We have that \( \int d_{1,2}(a)^2 w(a) \, da = o_p(1/n + (s_n^\infty r_n^\infty)^2) \) by Lemma E.2. For the other term, we have \( \int \{d_{1,1}(a)\}^2 w(a) \, da = O_p(1/n) \) by Lemma C.2. Applying
the Cauchy-Schwarz inequality yields $\int D_1(a)^2 w(a) \, da = o_p(1/n + (s_{n^2}^\infty)^2)$ and thus $n\sqrt{h} \int D_1(a)^2 w(a) \, da = o_p(1)$ by Assumptions E(A)2 and E(A)4.

**Step 4** ($D_2 D_3$). Next we will consider the integrated crossproduct term $n\sqrt{h} \int_A D_2(a) D_3 w(a) \, da$. This term can be analyzed using the above results, since $D_3$ factors out of the integral. The analysis is given in Lemma E.3, which shows that $n\sqrt{h} \int_A D_2(a) D_3 w(a) \, da$ is $o_p(1)$ by showing that $\int_A D_2(a) w(a) \, da$ is $O_p(n^{-1/2}) = o_p(1/\sqrt{n\sqrt{h}})$ (and combining that with the result of Step 2).

**Step 5** ($D_1 D_3$). Here we show $\int_A D_1(a) D_3 w(a) \, da$ is $o_p((n\sqrt{h})^{-1})$. This follows directly by the results of Step 2 and of Step 3, and the Cauchy-Schwarz inequality.

**Step 6** ($D_1 D_2$). Now we show that $\int_A D_1(a) D_2(a) w(a) \, da$ is $o_p((n\sqrt{h})^{-1})$. Recall in Step 4, we write

$$D_2(a) = \hat{\theta}_h(a) - \mathbb{P} \tilde{\xi} + \mathbb{P} \tilde{\xi} - \mathbb{P} n \tilde{\xi},$$  

and thus there we can focus on $\int D_1(a) \{\hat{\theta}_h(a) - \mathbb{P} \tilde{\xi}\} w(a) \, da$, because the other term

$$\int D_1(a) (\mathbb{P} \tilde{\xi} - \mathbb{P} n \tilde{\xi}) w(a) \, da = (\mathbb{P} \tilde{\xi} - \mathbb{P} n \tilde{\xi}) \int D_1(a) w(a) \, da$$

$$\leq |\mathbb{P} \tilde{\xi} - \mathbb{P} n \tilde{\xi}| \int |D_1(a)| w(a) \, da$$

$$\leq |\mathbb{P} \tilde{\xi} - \mathbb{P} n \tilde{\xi}| \sqrt{\int \{D_1(a)\}^2 w(a) \, da}$$

$$= O_p(n^{-1/2}) o_p((n\sqrt{h})^{-1/2}).$$

where the last line above comes from the results in Step 3. So we can write

$$\int_A D_1(a) D_2(a) w(a) \, da$$

$$= \int_A D_1(a) \{\hat{\theta}_h(a) - \mathbb{P} \tilde{\xi}\} w(a) \, da + o_p((n\sqrt{h})^{-1}),$$

and further we write $\int_A D_1(a) \{\hat{\theta}_h(a) - \mathbb{P} \tilde{\xi}\} w(a) \, da$ as the sum

$$\int_A d_{1,1}(a) \{\hat{\theta}_h(a) - \mathbb{P} \tilde{\xi}\} w(a) \, da + \int_A d_{1,2}(a) \{\hat{\theta}_h(a) - \mathbb{P} \tilde{\xi}\} w(a) \, da,$$  

(B.7)
where \( d_{1,1} \) and \( d_{1,2} \) are defined in equation (B.5). From Lemma G.1 \( g_{ha}(a) \tilde{D}_{ha}^{-1}g_{ha}(A_1) \) converges to \( \varpi_0(a)^{-1} \) almost surely (so also in-probability) regardless of the value of \( A_1 \). Further, as in the proof of Lemma E.3, we write out \( \theta_h(a) - \mathbb{P}\xi \) as

\[
\tilde{\theta}(a) - \mathbb{P}\xi = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h \varpi_0(a)} K \left( \frac{A_i - a}{h} \right) (\tilde{\xi}_i - \mathbb{P}\xi) \{1 + o_p(1)\}. 
\]  
(B.8)

Let \( \varepsilon := \xi(Z; \pi, \varpi) - \mathbb{P}\xi(Z; \pi, \varpi) \), and specifically \( \varepsilon_i := \xi(Z_i; \pi, \varpi) - \mathbb{P}\xi(Z_i; \pi, \varpi) \) for \( i = 1, \ldots, n \). So the order of the first term on right side of (B.7) is dominated by

\[
\int \frac{1}{\varpi_0(a)} \mathbb{P}_n \left[ K_{ha}(A) \left\{ \tilde{\xi}(Z; \hat{\pi}, \hat{\varpi}) - \xi(Z; \hat{\pi}, \hat{\varpi}) \right\} \right] \frac{1}{n} \sum_{i=1}^{n} K_h(A_i - a) \varepsilon_i w(a) \, da.
\]  
(B.9)

Since we can write

\[
\tilde{\xi}(Z; \hat{\pi}, \hat{\varpi}) - \xi(Z; \hat{\pi}, \hat{\varpi}) = Y - \hat{\mu}(L|A) \left( \frac{1}{n} \sum_{j=1}^{n} \hat{\pi}(A|L_j) - \int \hat{\pi}(A|l) \, d\mathbb{P}(l) \right) + \frac{1}{n} \sum_{j=1}^{n} \hat{\varpi}(L_j, A) - \int \hat{\varpi}(l, A) \, d\mathbb{P}(l)
\]

\[
= Y - \hat{\mu}(L, A) \frac{1}{n} \sum_{j=1}^{n} \tilde{\pi}(A|L_j) + \frac{1}{n} \sum_{j=1}^{n} \tilde{\varpi}(L_j, A),
\]

where we let tilde \( \sim \) operate on any \( \mu, \pi \) to yield \( \tilde{\pi}(a_1|l_2) := \pi(a_1|l_2) - \mathbb{P}\pi(a_1|L), \) and similarly \( \tilde{\varpi}(l_2, a_1) := \mu(l_2, a_1) - \mathbb{P}\mu(L, a_1), \) we can further decompose (B.9) as

\[
\int \frac{1}{\varpi_0(a)} \mathbb{P}_n \left[ K_{ha}(A) \left\{ \frac{Y - \hat{\mu}(L, A)}{\hat{\pi}(A|L)} \frac{1}{n} \sum_{j=1}^{n} \tilde{\pi}(A|L_j) \right\} \right] \frac{1}{n} \sum_{i=1}^{n} K_h(A_i - a) \varepsilon_i w(a) \, da
\]

plus

\[
\int \frac{1}{\varpi_0(a)} \mathbb{P}_n \left[ K_{ha}(A) \frac{1}{n} \sum_{j=1}^{n} \tilde{\varpi}(L_j, A) \right] \frac{1}{n} \sum_{i=1}^{n} K_h(A_i - a) \varepsilon_i w(a) \, da,
\]

which are in the forms of the two terms studied in Lemma C.3 (\( V_{1,2,\pi} \) and \( V_{1,2,\mu} \), respectively.). So applying Lemma C.3 yields that (B.9) is \( o_p((n\sqrt{h})^{-1}). \)
For the second term on the right side of (B.7), we have

\[
\int d_{1,2}(a) \left\{ \tilde{\theta}_h(a) - \mathbb{P}\tilde{\xi} \right\} \ w(a) \ da
\]

\[
= \int R_{n,1,a} \left\{ \tilde{\theta}_h(a) - \mathbb{P}\tilde{\xi} \right\} \ w(a) \ da + \int R_{n,2,a} \left\{ \tilde{\theta}_h(a) - \mathbb{P}\tilde{\xi} \right\} \ w(a) \ da,
\]

where

\[
R_{n,1,a} := g^T_{ha} \hat{D}^{-1}_{ha} (\mathbb{P}_n - \mathbb{P}) \left[ g_{ha} (A) K_{ha} (A) \left\{ \xi (Z; \hat{\pi}, \hat{\mu}) - \xi (Z; \bar{\pi}, \bar{\mu}) \right\} \right]
\]

\[
R_{n,2,a} := g^T_{ha} \hat{D}^{-1}_{ha} \mathbb{P} \left[ g_{ha} (A) K_{ha} (A) \left\{ \xi (Z; \hat{\pi}, \hat{\mu}) - \xi (Z; \bar{\pi}, \bar{\mu}) \right\} \right].
\]

The first term \( \int R_{n,1,a} \left\{ \tilde{\theta}_h(a) - \mathbb{P}\tilde{\xi} \right\} w(a) \ da \) could be bounded by Cauchy-Schwarz as

\[
\int R_{n,1,a} \left\{ \tilde{\theta}_h(a) - \mathbb{P}\tilde{\xi} \right\} w(a) \ da \leq \sqrt{\int (R_{n,1,a})^2 w(a) \ da \int \left\{ \tilde{\theta}_h(a) - \mathbb{P}\tilde{\xi} \right\}^2 w(a) \ da}.
\]

From Lemma E.2, we know \( \int (R_{n,1,a})^2 w(a) \ da = o_p (1/n) \) and from Step 1, we have \( \int \left\{ \tilde{\theta}_h(a) - \mathbb{P}\tilde{\xi} \right\}^2 w(a) \ da = \{O(1/\sqrt{n}) + O_p(1)/n\sqrt{n}\}. \) Thus

\[
\int R_{n,1,a} \left\{ \tilde{\theta}_h(a) - \mathbb{P}\tilde{\xi} \right\} w(a) \ da
\]

\[
= \sqrt{o_p(1/n) \left\{ O \left( \frac{1}{\sqrt{n}} \right) + O_p(1) \right\}} / (n\sqrt{n})
\]

\[
= o \left( (n\sqrt{n})^{-1} \right).
\]

Next, with a similar argument as in (B.9), we see the order of the second term \( \int R_{n,2,a} \left\{ \tilde{\theta}_h(a) - \mathbb{P}\tilde{\xi} \right\} w(a) \ da \) is

\[
\int \frac{1}{\omega_0^2(a)} \mathbb{P} \left[ K_{ha} (A) \left\{ \xi (Z; \hat{\pi}, \hat{\mu}) - \xi (Z; \bar{\pi}, \bar{\mu}) \right\} \right] \frac{1}{n} \sum_{i=1}^{n} K_h (A_i - a) \bar{\varepsilon}_i w(a) \ da \left\{ 1 + o_p(1) \right\},
\]

(B.10)

and thus dominated by

\[
\int \frac{1}{\omega_0^2(a)} \mathbb{P} \left[ K_{ha} (A) \left\{ \xi (Z; \hat{\pi}, \hat{\mu}) - \xi (Z; \bar{\pi}, \bar{\mu}) \right\} \right] \frac{1}{n} \sum_{i=1}^{n} K_h (A_i - a) \bar{\varepsilon}_i w(a) \ da.
\]

(B.11)
An empirical process argument, given in Lemma E.4, shows that the above term is \( o_p((n\sqrt{h})^{-1}) \). Finally, combining the above, it is straightforward to see \( \int_A D_1(a)D_2(a)w(a)\,da = o_p((n\sqrt{h})^{-1}) \).

**Step 7 (conclusions).** We have shown that \( T_n = n\sqrt{h} \int_A \{D_2(a)\}^2w(a)\,da + o_p(1) \). Then by the triangle inequality, we have

\[
\begin{align*}
d\{T_n, N(b_0h, V)\} &\leq d\left\{T_n, n\sqrt{h} \int_A \{D_2(a)\}^2w(a)\,da\right\} \\
&+ d\left\{n\sqrt{h} \int_A \{D_2(a)\}^2w(a)\,da, N(b_0h, V)\right\}.
\end{align*}
\]

The first term on the right side of the inequality has been shown to go to 0 as \( n \to \infty \). For the second term, by the definition of Dudley metric, we have

\[
\begin{align*}
d\left\{T_n, n\sqrt{h} \int_A \{D_2(a)\}^2w(a)\,da\right\} &= \sup \left\{ E g(T_n) - E g \left(n\sqrt{h} \int_A \{D_2(a)\}^2w(a)\,da\right) : \|g\|_{BL} \leq 1 \right\} \\
&\leq E \left\{ \left| T_n - n\sqrt{h} \int_A \{D_2(a)\}^2w(a)\,da \right| \wedge 2 \right\}.
\end{align*}
\]

Since \( |T_n - n\sqrt{h} \int_A \{D_2(a)\}^2w(a)\,da| = o_p(1) \), by the dominated convergence theorem, the expectation in the last line above converge to 0 as \( n \to \infty \) and thus we have

\[
d\{T_n, N(b_0h, V)\} \to 0,
\]

as \( n \to \infty \) and that completes the proof.

\[\square\]

**C Applying U- or V-process results to remainder terms**

The following three lemmas provide the negligibility of remainder terms in the analysis of our test statistic’s limit distribution. They are all in V-statistic form (if one regards \( \hat{\pi}, \hat{\mu} \) as fixed) and thus their analysis (allowing \( \hat{\pi}, \hat{\mu} \) to vary) requires the theory of V-processes.
Lemma C.1. Let the assumptions of Theorem 3.1 hold. Then

\[ \mathbb{P}_n \left\{ \xi(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \hat{\pi}, \hat{\mu}) \right\} = O_p(n^{-1/2}) \text{ as } n \to \infty. \] (C.1)

Lemma C.2. Let the assumptions of Theorem 3.1 hold. Let

\[ d_{1,1}(a) := g_{ha}^T \hat{D}_{ha}^{-1} \mathbb{P}_n \left[ g_{ha}(A) K_{ha}(A) \left\{ \hat{\xi}(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \hat{\pi}, \hat{\mu}) \right\} \right]. \]

Then \( n \sqrt{h}(\sup_{a \in A} d_{1,1}(a)^2) = O_p(\sqrt{h}) \) and so \( n \sqrt{h} \int (d_{1,1}(a))^2 w(a) da = O_p(\sqrt{h}) = o_p(1) \) as \( n \to \infty. \)

Recall in the above statement, \( \hat{D}_{ha} = \mathbb{P}_n \{ g_{ha}(A) K_{ha}(A) g_{ha}^T(A) \}. \) The V-statistic terms analyzed in the next lemma arise from certain cross product error terms. We let tilde \( \tilde{\cdot} \) operate on any \( \mu, \pi \) to yield \( \tilde{\pi}(a_1 | l_2) := \pi(a_1 | l_2) - \mathbb{P} \pi(a_1 | L) \) and similarly \( \tilde{\mu}(l_2, a_1) := \mu(l_2, a_1) - \mathbb{P} \mu(L, a_1) \). Let \( \tilde{\xi} := \xi(Z; \tilde{\pi}, \tilde{\mu}) - \mathbb{P} \xi(Z; \tilde{\pi}, \tilde{\mu}) \), and specifically \( \tilde{\xi}_i := \xi(Z; \tilde{\pi}, \tilde{\mu}) - \mathbb{P} \xi(Z; \tilde{\pi}, \tilde{\mu}) \) for \( i = 1, \ldots, n. \)

Lemma C.3. Let the assumptions of Theorem 3.1 hold. Let

\[ V_{12, \hat{\mu}} := \frac{1}{n^2} \sum_{i,j,k=1}^n \int_A \varpi_0(a)^{-2} [K_{ha}(A_i) \tilde{\mu}(L_j, A_i) K_{ha}(A_k) \tilde{\pi}_k] w(a) da, \]

\[ V_{12, \hat{\pi}} := \frac{1}{n^2} \sum_{i,j,k=1}^n \int_A \varpi_0(a)^{-2} \times \]

\[ K_{ha}(A_i) \left( \frac{Y_i - \mu(L_i, A_i)}{\pi(A_i | L_i)} \tilde{\pi}(A_i | L_j) \right) K_{ha}(A_k) \tilde{\pi}_k \] \( w(a) da. \)

Then \( V_{12, \hat{\mu}} \) and \( V_{12, \hat{\pi}} \) are \( O_p(n^{-2}h^{-3/2}) \) as \( n \to \infty. \)

Proof of Lemma C.1. We first write

\[ \mathbb{P}_n \left\{ \tilde{\xi}(Z; \tilde{\pi}, \tilde{\mu}) - \xi(Z; \tilde{\pi}, \tilde{\mu}) \right\} = \frac{1}{n} \sum_{i=1}^n \left[ \frac{Y_i - \tilde{\mu}(L_i, A_i)}{\pi(A_i | L_i)} \left\{ \frac{1}{n} \sum_{j=1}^n \tilde{\pi}(A_i | L_j) - \int \tilde{\pi}(A_i | l) dP(l) \right\} \right. \]

\[ + \left\{ \frac{1}{n} \sum_{j=1}^n \tilde{\mu}(L_j, A_i) - \int \tilde{\mu}(l, A_i) dP(l) \right\} \].

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Thus, the mean of the given term can be bounded by the mean of a process where \( \pi \), \( \mu \), range over \( \mathcal{F}_\pi \), \( \mathcal{F}_\mu \). We define the V-processes of interest as follows. For \( w_i \in \mathcal{Z} := \mathcal{L} \times \mathcal{A} \times \mathcal{Y} \), \( i = 1, 2 \), let

\[
h_1(w_1, w_2) = h_{1, \pi, \mu}(w_1, w_2) := \frac{y_1 - \mu(l_1, a_1)}{\pi(a_1|l_1)} \tilde{\pi}(a_1|l_2) + \tilde{\mu}(l_2, a_1) \quad (C.3)
\]

be a (non-symmetric) U- or V-statistic “kernel”, indexed by \( \mu, \pi \), and where, recall we let tilde \( \tilde{\cdot} \) operate on any \( \mu, \pi \) to yield \( \tilde{\pi}(a_1|l_2) := \pi(a_1|l_2) - \mathbb{P}\pi(a_1|L) \), and similarly \( \tilde{\mu}(l_2, a_1) := \mu(l_2, a_1) - \mathbb{P}\mu(L, a_1) \). Then, recalling \( \{W_i\} = \{(L_i, A_i, Y_i)\} \) is our i.i.d. sample, (C.1) is of the form

\[
n^{-2} \left( \sum_{i=1}^{n} h_1(W_i, W_i) + \sum_{1 \leq i < j \leq n} h_1(W_i, W_j) + h_1(W_j, W_i) \right). \quad (C.4)
\]

Note that \( \mathbb{P}h_1(w_1, W_2) = 0 \), for almost any \( w_1 \), so \( \mathbb{P}h_1(W_1, W_2) = 0 \). Now by Assumption D3 and D4, using the total variance formula, we can see \( Y \) has a finite variance. In addition, by Assumption E(B) \( \pi \) and \( \mu \) are bounded above, and \( \pi \) is bounded below, it is immediate that \( \mathbb{P}|h_1(W_1, W_1)| < \infty \) and \( \mathbb{P}h_1(W_1, W_2)^2 < \infty \). The larger term will be the double summation(s) in (C.4), which is in U-statistic form so we now introduce some notation so we can then apply U-process results.

Let

\[
g_1(w_1, w_2) = g_{1, \mu, \pi}(w_1, w_2) := h_1(w_1, w_2) + h_1(w_2, w_1). \quad (C.5)
\]

To apply the maximal inequality in Proposition K.1 to our particular U-processes we thus need to bound the appropriate uniform entropy-type integrals and compute the envelope moments, for the classes of functions

\[
\mathcal{G}_1 := \{g_{1, \mu, \pi} : \mu \in \mathcal{F}_\mu, \pi \in \mathcal{F}_\pi\}.
\]

We start by considering the covering numbers. Take a generic class of functions \( \mathcal{F} \) on a space \( \mathcal{X} \) with finite covering number \( N(\mathcal{F}, \| \cdot \|_{2, Q}, \tau) \) and envelope \( \hat{F} \). Then it is easy to verify that the class \( \mathcal{F}_o \) of functions \( f^o(x, z) := f(x) \) defined on the extended space \( \mathcal{X} \times \hat{\mathcal{X}} \), some measurable space \( \hat{\mathcal{W}} \), has \( N(\mathcal{F}, \| \cdot \|_{2, Q}, \tau) = N(\mathcal{F}_o, \| \cdot \|_{2, \mathcal{Q}^o}, \tau) \) for any \( Q^o \) on \( \mathcal{X} \times \mathcal{X} \) that extends \( Q \) in the sense that \( Q \) is the marginal of \( Q^o \) on \( \mathcal{X} \). In particular, \( \sup_Q N(\mathcal{F}, \| \cdot \|_{2, Q}, \tau) = \sup_{Q^o} N(\mathcal{F}_o, \| \cdot \|_{2, \mathcal{Q}^o}, \tau) \). Thus, if we consider the class of functions \( \mathcal{F}_\mu \) given by \((l_1, a_1, l_2, a_2) \mapsto \mu(l_2, a_1) \) for \( \mu \in \mathcal{F}_\mu \), then this class has the same uniform covering number as the original class \( \mathcal{F}_\mu \). Similarly for \( \mathcal{F}_\pi \).
Bounding entropy. First we bound $J$ and $J_2$ for the appropriate classes of functions. Let $\mathcal{H}_1$ be the class of $h_1$ functions (defined in (C.3)). We can write the class $\mathcal{H}_1$ as

$$\mathcal{H}_1 = (Y_1 - \mathcal{F}_\mu)\mathcal{F}_\mu^{-1}\mathcal{F}_\mu^0 + \mathcal{F}_\mu^0,$$

where: we abuse notation to let, e.g., $\mathcal{F}_\mu$ refer to the class $(l_1, a_1, y_1, l_2, a_2, y_2) \mapsto \mu(l_1, a_1)$, and similarly for $\mathcal{F}_\pi$; we define operations on classes of functions so $\mathcal{F} \mathcal{G} := \{fg : f \in \mathcal{F}, g \in \mathcal{G}\}$, $\mathcal{F} + \mathcal{G} := \{f + g : f \in \mathcal{F}, g \in \mathcal{G}\}$, $\mathcal{F}^{-1} := \{f^{-1} : f \in \mathcal{F}\}$; we let $Y_1$ denote the function on $W^2$ that returns just $y$; and let $\mathcal{F}_\mu^0$ be the class of functions $(l_1, a_1, l_2, a_2) \mapsto \bar{\mu}(l_2, a_1)$ and similarly $\mathcal{F}_\pi^0$ is the class of functions $(l_1, a_1, l_2, a_2) \mapsto \bar{\pi}(a_1 | l_2)$. The class $\mathcal{F}_\mu^0$ can be written as $\mathcal{F}_\mu^0 - \{\mathcal{P}\mu(L, \cdot ; \mu) \in \mathcal{F}_\mu^0\}$. By the proof of Lemma 20 of Nolan and Pollard [1987], $\{\mathcal{P}\mu(L, \cdot ; \mu) \in \mathcal{F}_\mu^0\}$ has uniform entropy smaller than that of $\mathcal{F}_\mu$. A similar statement holds for $\mathcal{F}_\pi^0$. Thus, we can apply Lemmas A.4 and A.5 (see also Lemma 16 of Nolan and Pollard [1987]) to $\mathcal{H}_1$. This shows $J_2(1, \mathcal{H}_1) < \infty$. Let $G_1 := \{g_1, \mu, \pi : \mu \in \mathcal{F}_\mu, \pi \in \mathcal{F}_\pi\}$. Note that $N(G_1, L_2(Q), \varepsilon \sqrt{2}) = N(H_1, L_2(Q), \varepsilon)$ so we can conclude $J_2(1, G_1) < \infty$. By Lemma 20 of Nolan and Pollard [1987], we also conclude that $J(1, G_1) < \infty$. By Proposition K.1, we conclude that $\mathbb{P}||U_n||_{G_1}$ is thus bounded above by a constant, as desired.

**Envelopes.** Now we consider envelopes and their squared expectation for the appropriate classes. Let $G_1$ be the envelope for $\mathcal{G}_1$. By our assumptions D3, D4 and E(B) we see that $EG^2_1 < \infty$ (and is independent of $n$). It’s also easily seen that $\mathcal{H}_1$ has squared-integrable envelope $H_1$.

Next, for the minimal envelope $K_1$ for $\mathbb{P}G_1$, it is clear again that we have $\mathbb{P}K_1(W)^2 < \infty$. Finally, we apply Proposition K.1 to (C.1) and conclude (C.1) is of order $O_p(n^{-1/2})$. □

**Proof of Lemma C.2.** For $n\sqrt{n} \int \{d_{1,1}(a)\}^2 w(a) da$, we apply a V-process approach, like that in Lemma C.1, but now we must accommodate the kernel. For $w_i \in \mathcal{Z} = \mathcal{L} \times \mathcal{A} \times \mathcal{Y}$, $i = 1, 2$, let $h_2 \equiv h_{2,\mu,\pi}$ be

$$h_2(w_1, w_2) := g_{ha}(a_1)K_{ha}(a_1)h_1(w_1, w_2), \quad (C.6)$$

where $h_1 \equiv h_{1,\mu,\pi}$ is as defined in (C.3), and then we can write

$$d'_{1,1}(a) := \mathbb{P}_n \left[ g_{ha}(A)K_{ha}(A) \left\{ \hat{\xi}(Z; \bar{\pi}, \tilde{\mu}) - \xi(Z; \bar{\pi}, \tilde{\mu}) \right\} \right]$$

$$= \frac{1}{n^2} \left( \sum_{i=1}^{n} h_{2,\bar{\mu},\hat{\pi}}(W_i, W_i) + \sum_{1 \leq i < j \leq n} h_{2,\bar{\mu},\hat{\pi}}(W_i, W_j) + h_{2,\bar{\mu},\hat{\pi}}(W_j, W_i) \right). \quad (C.7)$$

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Note that $\mathbb{P}h_2(w_1, W_2) = 0$, for almost any $w_1$, so $\mathbb{P}h_2(W_1, W_2) = 0$. And we have that $\mathbb{P}h_2(W_1, W_1) = 0$. Thus

$$n^{-2} \sum_{i=1}^{n} h_2(W_i, W_i) = O_p(n^{-1}).$$  \hfill (C.9)

The larger term will be the double summation(s) in (C.7). Now let

$$g_2(w_1, w_2) \equiv g_{2,\mu,\pi}(w_1, w_2) := h_2(w_1, w_2) + h_2(w_2, w_1).$$ \hfill (C.10)

Since $h_2$ and $g_2$ are vector functions (recall $g_{ha}(a_1) = (1, (a_1 - a)/h)^T$), we refer to their components as $h_{2;i} \equiv h_{2,\mu,\pi;i}$ and $g_{2;i} \equiv g_{2,\mu,\pi;i}$, $i = 1, 2$. Similarly as in Step 2, to apply the above maximal inequality in Proposition K.1 to the particular $U$-processes we thus need to bound the appropriate uniform entropy-type integrals and compute the envelope moments, for the class of functions

$$G_{2;i} := \{g_{2,\mu,\pi;i} : \mu \in \mathcal{F}_\mu, \pi \in \mathcal{F}_\pi, i = 1, 2\}.$$ \hfill (C.11)
defined on \( Z^2 \), letting \((z_1, z_2) \mapsto f(z_1)\) without changing the entropy.) By Assumption E(B) this then allows us to conclude that \( J_2(1, h^i G_{2;i}) < \infty \) (since \( J_2(1, h^i G_{2;i}) < \infty \)) and further, by Lemma 20 of Nolan and Pollard [1987], that \( J(1, h^i \mathbb{P} G_{2;i}) < \infty, \ i = 1, 2. \)

**Envelopes.** From the proof of Lemma C.1, we know \( \mathcal{H}_1 \) has square-integrable envelope \( H_1 \) (by the boundedness assumptions on the function classes, and the assumption \( Y \) has a finite variance), so an envelope for \( hG_{2;1} \) is \( G_{2;1} := K_{\text{max}}^\infty H_1 \) and for \( h^2 G_{2;2} \) is \( G_{2;2} := a_{\text{width}} K_{\text{max}}^\infty H_1 \) where \( a_{\text{width}} := \max_{a_1, a_2 \in \mathcal{A}} |a_2 - a_1| \), and where \( K_{\text{max}} := \sup K < \infty \) by assumption. These two envelopes have second moment finite (and independent of \( n \)). Next consider minimal envelopes \( K_{2;i}(w) \) for \( \mathbb{P} G_{2;i}, \ i = 1, 2. \) Note that

\[
\mathbb{P} g_2(w, W) = \mathbb{P} h_2(W, w)
\]

since \( \mathbb{P} h_2(w, W) = 0 \) for almost every \( w \). (Recall \( h_2 \) is defined in (C.6) and \( g_i \) is defined in (C.10).) And we can see, by the change of variables \( u = (a_1 - a)/h \), that

\[
\begin{align*}
\mathbb{P} h_2(W, w_2) &= \int \left( 1, \frac{a_1 - a}{h} \right)^T h^{-1} K \left( \frac{a_1 - a}{h} \right) \left( \frac{y - \mu(l, a_1)}{\pi(a_1 | l)} \right) (\pi(l_2, a_1) + \tilde{\mu}(l_2, a_1)) \\
&\quad \times p(l, a_1, y) d\nu(l, y) da_1 \\
&= \int \left( 1, u \right)^T K(u) \left( \frac{y - \mu(l, a + hu)}{\pi(a + hu | l)} \right) (\pi(l_2, a + hu) + \tilde{\mu}(l_2, a + hu)) \\
&\quad \times p(l, a + hu, y) d\nu(l, y) du \\
&= (C.12)
\end{align*}
\]

which, by our Assumptions E(B) and E(A) (and \( Y \) having a finite mean), is bounded in absolute value above by a constant (independent of \( n \)). We can check that the class of functions \( w \mapsto \mathbb{P} h_2(W, w) \) (and so \( w \mapsto \mathbb{P} g_2(w, W) \)) has a (vector) envelope \( (K_{2;1}, K_{2;2}) \) satisfying \( \mathbb{P} K_{2;1}(W)^2 < \infty \) and \( \mathbb{P} K_{2;2}(W)^2 < \infty \) (independent of \( n \)). Next we will analyze the two components of (C.7) separately. The term with the \((a_1 - a)/h\) factor will be larger in our analysis so we focus on it. This term can be written, as described above, as

\[
\int_{\mathcal{A}} \left( h^{-2} n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{h}_{2;\tilde{\mu}, \tilde{x};2}(W_i, W_j) \right)^2 w(a) da, \quad (C.13)
\]

where \( \tilde{h}_{2;\tilde{\mu}, \tilde{x};2} \in h^2 \mathcal{H}_{2;2} \). (Recall from (C.6) that \( h_{2;i} \) depends on \( a \) but we
notationally suppressed this dependence for simplicity.) Considering

\[ n^{-3/2} \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{h}_{2,\mu,\pi;2}(W_i, W_j), \]

and then taking a sup over \( \mu \in \mathcal{F}_\mu, \pi \in \mathcal{F}_\pi, a \in \mathcal{A} \), and applying (C.9) and Proposition K.1 with the class \( h^2 G_{2:2} \) and envelopes \( G_{2:2} \) and \( h^2 K_{2:2} \), (recalling (C.7) to go from \( H_{2:2} \) to \( G_{2:2} \)), we see that (C.13) is bounded above by

\[ O_p(n^{-2} h^{-4}) + O_p(n^{-1}). \]  

(C.14)

Similarly, we write the other term in (C.7) as

\[ \int_{\mathcal{A}} \left( h^{-1} n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{h}_{2,\tilde{\mu},\tilde{\pi};1}(W_i, W_j) \right)^2 w(a) da, \]

where \( \tilde{h}_{2,\tilde{\mu},\tilde{\pi};1} \in h\mathcal{H}_{2:1} \), and (by the same argument) be seen to be of smaller order, \( O_p(n^{-2} h^{-2}) + O_p(n^{-1}) \).

Thus, multiplying \( O_p(n^{-2} h^{-4}) + O_p(n^{-1}) \) by \( n\sqrt{h} \) we see that

\[ \sup_{a \in \mathcal{A}}(n\sqrt{h})d'_{1,1}(a)^2 = O_p(n^{-1} h^{-7/2}) + O_p(h^{1/2}), \]

so that

\[ n\sqrt{h} \int_{\mathcal{A}} d'_{1,1}(a)^2 w(a) da = O_p(n^{-1} h^{-7/2}) + O_p(h^{1/2}), \]

and is thus \( O_p(\sqrt{h}) \) by Assumption E(A)1. Lastly, in order to derive the order of \( n\sqrt{h} \int \{d'_{1,1}(a)\}^2 w(a) da \), similar to the proof of Lemma E.2,

\[ n\sqrt{h} \int_{\mathcal{A}} d_{1,1}(a)^2 w(a) da \]

\[ = n\sqrt{h} \int \left\{ g_{ha}^T \tilde{D}_{ha}^{-1} d'_{1,1}(a) \right\}^2 w(a) da \]

\[ = n\sqrt{h} \int \left\{ \left[ g_{ha}^T \tilde{D}_{ha}^{-1} - (\varpi(a)^{-1}, 0) + (\varpi(a)^{-1}, 0) \right] d'_{1,1}(a) \right\}^2 w(a) da. \]

Then

\[ n\sqrt{h} \int d_{1,1}(a)^2 w(a) da \leq n\sqrt{h} \int 2 \left\{ \left| g_{ha}^T \tilde{D}_{ha}^{-1} - (\varpi(a)^{-1}, 0) \right| s'_{1,1}(a) \right\}^2 w(a) da \]

\[ + n\sqrt{h} \int 2 \left\{ \left| (\varpi(a)^{-1}, 0) \right| d'_{1,1}(a) \right\}^2 w(a) da. \]
The first term on the last two lines above can be bounded as

\[
n\sqrt{h} \int 2 \left\{ |g_{ha}^T \hat{D}_{ha}^{-1} - (\varpi(a)^{-1}, 0)| d_{1,1}(a) \right\}^2 w(a) \, da \\
\leq 2n\sqrt{h} \int \|g_{ha}^T \hat{D}_{ha}^{-1} - (\varpi(a)^{-1}, 0)\|_{l_2}^2 \|d_{1,1}^i\|_{l_2}^2 w(a) \, da;
\]

From the proof of Lemma E.2, we know \(|g_{ha}^T \hat{D}_{ha}^{-1} - (\varpi(a)^{-1}, 0)| \) is uniformly \(O(1)\) a.s., then we have the right hand side of the above inequality is \(O_p(\sqrt{h})\).

By Assumption I2, \(\varpi(a)\) is bounded below from 0; similarly it is easy to see

\[
n\sqrt{h} \int 2 \left\{ |(\varpi(a)^{-1}, 0)| d_{1,1}(a) \right\}^2 w(a) \, da \\
\leq 2n\sqrt{h} \int \|(\varpi(a)^{-1}, 0)\|_{l_2}^2 \|d_{1,1}^i\|_{l_2}^2 w(a) \, da \\
= O_p(\sqrt{h}).
\]

That completes the proof of \(n\sqrt{h} \int \{d_{1,1}(a)\}^2 w(a) \, da = O_p(\sqrt{h})\).

**Proof of Lemma C.3.** First we focus on the \(V_{12,\vec{a}}\) term. This is a third degree \(V\)-statistic. Start by defining the asymmetric kernel \(H\) for a generic \(\mu\) to be

\[
H(Z_1, Z_2, Z_3) := \varepsilon_3 \int_A \varpi^{-2}(a) [K_{ha}(A1)\hat{\mu}(L_2, A1)K_h(A3 - a)] w(a) da.
\]

(C.15)

Recall that \(\hat{\mu}(l_2, a_1) := \mu(l_2, a_1) - \mathbb{P}\mu(L, a_1).\) Then the \(\mu\) term \(V\)-statistic is \(V_{12,\mu} := n^{-3} \sum_{i,j,k} H(Z_i, Z_j, Z_k).\) We begin by analyzing the corresponding \(U\)-statistic

\[
U_{12,\mu} := n^{-3} \sum_{i,j,k} H(Z_i, Z_j, Z_k)
\]

where the sum is over \((i, j, k)\) with unique coordinates (i.e., over the \(n(n-1)(n-2)\) ordered choices of 3 distinct elements of \(\{1, \ldots, n\}\)). Let the symmetrized kernel be \(G(Z_1, Z_2, Z_3) := 6^{-1} \sum_\sigma H(Z_{\sigma_1}, Z_{\sigma_2}, Z_{\sigma_3}),\) where \(\sigma\) ranges over the 6 permutations of 3 elements.

One can decompose any \(U\)-statistic (kernel) into sums of \(U\)-statistics (kernels) which have certain “degenerate” structure. (See Serfling [1980, pages 177–178]). We define as follows. Let

\[
G_{1,\{1\}}(z_1) := \mathbb{P}^2 H(z_1, Z_2, Z_3),
G_{1,\{2\}}(z_2) := \mathbb{P}^2 H(Z_1, z_2, Z_3),
G_{1,\{3\}}(z_3) := \mathbb{P}^2 H(Z_1, Z_2, z_3),
\]

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where $\mathbb{P}^i$ denotes the product measure of $\mathbb{P}$ ($i$ copies). Note that these three functions are all identically zero. It is immediately seen that $\mathbb{P}H(Z_1, Z_2, z) = \mathbb{P}H(z, Z_2, Z_3) = 0$, since $\mathbb{P} \tilde{\mu}(a, L) = 0$ for any $a$. It is also immediately clear that $\mathbb{P}H(Z_1, z_2, Z_3) = 0$, since $\mathbb{E}(\bar{z}_3|A_3) = 0$ almost surely. Therefore, $G$ is degenerate, in that

$$\mathbb{P}G(z, Z_2, Z_3) = 0$$

for any $z$. Next, let

$$G_{2\{1,2\}}(z_1, z_2) := \mathbb{P}H(z_1, z_2, Z_3),$$

$$G_{2\{1,3\}}(z_1, z_3) := \mathbb{P}H(z_1, Z_2, z_3),$$

$$G_{2\{2,3\}}(z_2, z_3) := \mathbb{P}H(Z_1, z_2, z_3).$$

Of the above three functions, only $G_{2\{2,3\}}(z_2, z_3)$ is not identically zero (again, since $\mathbb{E}(\bar{z}_3|A_3) = 0$ and $\mathbb{P} \tilde{\mu}(a, L) = 0$). Finally, let

$$G_3(z_1, z_2, z_3) := H(z_1, z_2, z_3) - G_{2\{2,3\}}(z_2, z_3).$$

(C.16)

Note that both $G_3$ and $G_{2\{2,3\}}$ are maximally degenerate (i.e., averaging over any variable yields the zero function).

Now trivially by definition (C.16),

$$H(z_1, z_2, z_3) = G_3(z_1, z_2, z_3) + G_{2\{2,3\}}(z_2, z_3),$$

meaning that $H$ is decomposed into a degree 2 degenerate kernel of degree 3, and a degree 1 degenerate kernel of degree 2.

First we will consider the latter term, $G_{2\{2,3\}}(z_2, z_3)$. We wish to find an envelope for this class of functions, and then can apply a maximal inequality to the sum. Since $\varpi$, $\varpi^{-1}$, and $\tilde{\mu}$ are all uniformly bounded,

$$\varpi_0^{-2}(a)\mathbb{P}[K_{ha}(A_1)\tilde{\mu}(L_2, A_1)] = \varpi_0^{-2}(a) \int K(u)\tilde{\mu}(a + uh, l_1)\varpi(a + uh)du,$$

is $O(1)$ (independently of $a$). Then, plugging the above in to $G_{2\{2,3\}}(z_2, z_3)$, we have

$$\bar{z}_3 \int_{A} O(1)K_h(a_3 - a)w(a) da = \bar{z}_3 \int_{(a_3 - A)/h} O(1)K(\bar{w})w(a_3 - hu)du.$$ 

Thus $|\bar{z}_3| \int_{\mathbb{R}} O(1)K(u)(\max w)du$ is an envelope for the class (and is independent of $h$ and $n$).

Now we consider entropies. By Assumption E(B)_3, we have $J_3(1, F_\mu, L_2) < \infty$; the class of functions $H$ under consideration is

$$\mathcal{H} := \{(z_1, z_2, z_3) \mapsto \varepsilon_3 \varpi_0^{-2}hK_{ha}(a_1)\tilde{\mu}(l_2, a_1)K_{ha}(a_3) : \mu \in F_\mu, a \in A\},$$
which by an argument almost identical to that in the proof of Lemma C.2, has $J_3(1, \mathcal{H}, L_2) < \infty$. Furthermore, by Lemmas A.3, A.4, and A.5, we can see that the class of functions $\{G_{2,\{2,3\}}\}$ and then $G_3$ (using shorthand notation for these classes) also have their corresponding $J_3$ integral finite.

Now, since $J_2(1, \{G_{2,\{2,3\}}\}, L_2) < J_3(1, \{G_{2,\{2,3\}}\}, L_2) < \infty$, by Proposition K.2 (alternatively, see Nolan and Pollard (1987)), $n^{-2} \sum_{i\neq j} G_{2,\{2,3\}}(Z_i, Z_j) = O_p(n^{-1})$, and so $n^{-3} \sum_{i,j,k} G_{2,\{2,3\}}(Z_i, Z_j) = O_p(n^{-1})$ (summing over $(i, j, k)$ not equal to each other).

Now consider $G_3(z_1, z_2, z_3)$. We have just seen $J_3(1, \{G_3\}, L_2) < \infty$ so we only need to focus on finding the envelope. Since we have an envelope for $G_{2,\{2,3\}}(z_2, z_3)$ we just need an envelope for $H(z_1, z_2, z_3)$. By the change of variables $u = (a_1 - a)/h$

$$H(z_1, z_2, z_3) = \mathcal{E}_3 \int_{(a_1-A)/h} (a_1 - uh)^{-2} K(u) \tilde{\mu}(a_1, l_2) \frac{1}{h} K\left(\frac{a_3 - a_1}{h} + u\right) w(a_1 - uh) du.$$  

(C.17)

Now $K$ has support $[-1, 1]$ and so

$$K(u) \frac{1}{h} K\left(\frac{a_3 - a_1}{h} + u\right) \leq K_{\max}^2 \mathbb{1}_{[-1,1]}(u) \mathbb{1}_{[-2h,2h]}(a_3 - a_1)$$

since $-h \leq a_3 - a_1 + uh \leq h$ implies $-2h \leq a_3 - a_1 \leq 2h$ for $u \in [-1, 1]$.

Thus, (C.17) is bounded above in absolute value by

$$\mathcal{E}_3 h^{-1} \mathbb{1}_{[-2h,2h]}(a_3 - a_1) \int \mathcal{E}_0^{-2} (a_1 - uh) K(u) \tilde{\mu}(a_1 - l_2) w(a_1 - uh) du$$

$$\leq C_1 \mathcal{E}_3 h^{-1} \mathbb{1}_{[-2h,2h]}(a_3 - a_1) =: F_3(z_1, z_2, z_3),$$

for a constant $C_1$, where we take $F_3$ as our envelope. We see that

$$\mathbb{P}^3 F_3^2(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) \leq C_2 h^{-3/2}$$

for a constant $C_2$. Therefore, by Proposition K.2, we have $n^{-3} \sum_{i,j,k} G_3(\mathbf{Z}_i, \mathbf{Z}_j, \mathbf{Z}_k) = O_p(n^{-3/2} h^{-3/4})$ (sum over $i, j, k$ not ever equal to each other).

Finally, the above analysis of $U_{12,\mu}$ ignored the summands in $V_{n,\mu}$, where an argument is repeated (i.e., $H(\mathbf{Z}_1, \mathbf{Z}_2)$, etc., or $H(\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_1)$). The sums of such terms are very small since there are many fewer terms than $n^3$. A very simple analysis can show each such sum is not larger than $O_p(n^{-2} h^{-3/2}) = O_p((n \sqrt{h})^{-1}(nh)^{-1})$, which is much smaller than we need.

Analyzing the “π term”, $V_{12,\pi}$, is similar to analyzing the μ term and we do not present the details. \[ \square \]
D Lemma D.1

The quantity \( \mathbb{P}_n \left[ g_{ha}(A) K_{ha}(A) \left\{ \hat{\xi}(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \right\} \right] \), broken into three pieces, is analyzed in the following lemma; the result allows to conclude that \(|\theta_h(a) - \theta(a)| = O_p((nh)^{-1/2} + h^2 + r_n s_n)\).

**Lemma D.1.** Let \( \pi \) and \( \mu \) denote fixed functions to which \( \hat{\pi} \) and \( \hat{\mu} \) converge in the sense that \( \sup_x |\hat{\pi} - \pi| = o_p(1) \) and \( \sup_x |\hat{\mu} - \mu| = o_p(1) \). Let \( a \in A \) denote a point in the interior of the compact support \( A \) of \( A \). Let Assumption I hold, Assumption D parts 1, 2, 3 hold, and Assumption E(A)3 holds. Assume \( E(B) \) denotes fixed functions to which \( \hat{\pi} \) and \( \hat{\mu} \) converge in the sense that \( \sup_x |\hat{\pi} - \pi| = o_p(1) \) and \( \sup_x |\hat{\mu} - \mu| = o_p(1) \). Let \( a \in A \) denote a point in the interior of the compact support \( A \) of \( A \). Let Assumption I hold, Assumption D parts 1, 2, 3 hold, and Assumption E(A)3 holds. Assume \( E(B) \) holds. Assume \( Y \) has finite variance. Assume the conditional density of \( \xi(Z; \bar{\pi}, \bar{\mu}) \) given \( A = a \) is continuous in \( a \). Assume \( h \equiv h_n \to 0 \) and \( nh^3 \to \infty \) as \( n \to \infty \). Assume either \( \pi = \pi_0 \) or \( \pi = \mu_0 \). Let \( r_n \equiv r_n(a) \) and \( s_n \equiv s_n(a) \) be given by \( \sup_{t:|t| \leq h} \| \hat{\pi}(t/L) - \pi(t/L) \|_2 = O_p(r_n) \) and \( \sup_{t:|t| \leq h} \| \hat{\mu}(L,t) - \mu(L,t) \|_2 = O_p(s_n) \). Then

\[
\mathbb{P}_n \left[ g_{ha}(A) K_{ha}(A) \left\{ \hat{\xi}(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \right\} \right] = O_p(1/\sqrt{nh}), \quad (D.1)
\]

\[
(\mathbb{P}_n - \mathbb{P}) \left[ g_{ha}(A) K_{ha}(A) \left\{ \xi(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \right\} \right] = o_p(1/\sqrt{nh}), \quad \text{and} \quad (D.2)
\]

\[
\mathbb{P} \left[ g_{ha}(A) K_{ha}(A) \left\{ \xi(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \right\} \right] = O_p(r_n s_n) \quad (D.3)
\]

as \( n \to \infty \).

**Proof of Lemma D.1. Second term, (D.2):** Kennedy et al. [2017] show that

\[
(\mathbb{P}_n - \mathbb{P}) \left[ g_{ha}(A) K_{ha}(A) \left\{ \xi(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \right\} \right] = o_p(1/\sqrt{nh})
\]

(with no hat over the first \( \xi \)). (This follows from their proof/analysis of their \( R_{n,1} \) in the proof of their Theorem 2 (page 13 of the Web Appendix of Kennedy et al. [2017]).)

**First term, (E.6):** The order of the first term, (E.6), follows from the proof of Lemma C.2; note that the proof of Lemma C.2 proceeds (initially) for a fixed \( a \). We make a few comments here about the differing assumptions for Lemma D.1 and for Lemma C.2 (i.e., for Theorem 3.1) in the context of the proof of Lemma C.2.

Lemma C.2 relies on the assumption that \( \pi, \mu, 1/\pi \) are uniformly bounded above, which both Theorem 3.1 and Lemma D.1 make (Assumption E(B)2). Similarly, both theorems make the assumption that \( K \) is bounded above (Assumption E(A)3). Both theorems assume that \( J_m(1, \mathcal{F}, L_2) < \infty \) for some \( m > 1 \), so \( J(1, \mathcal{F}, L_2) < \infty \), for \( \mathcal{F} \) equal to \( \mathcal{F}_{\mu, \mathcal{F}_{\pi}} \); this latter assumption is
all that is needed for Lemma C.2 (which calls upon Proposition K.1). Both theorems assume that \( Y \) has a finite variance, as needed by Lemma C.2.

Finally, by the assumption that \( nh^3 \to \infty \), we see that multiplying (E.15) by \( nh \) yields an order of \( o_p(1) \), so the term on the left of (E.6) is \( o_p(1/\sqrt{nh}) \) (rather than \( O_p(1/\sqrt{nh}) \)) under the stronger assumption of Theorem 3.1 that \( h \) is of order \( n^{-1/5} \) as desired.

**Third term, (E.14):** Note \( \Pr[g_{ha}(A)K_{ha}(A)\{\xi(Z;\hat{\pi},\hat{\mu}) - \xi(Z;\pi,\mu)\}] \) is a vector with \( j \)th element (\( j = 1, 2 \)) equal to

\[
\int g_{ha,j}(t)K_{ha}(t)\Pr\{\xi(Z;\hat{\pi},\hat{\mu}) - \xi(Z;\pi,\mu) | A = t\} \varpi_0(t) dt,
\]

where \( g_{ha,j}(t) = \{(t - a)/h\}^{j-1} \). Note that

\[
\Pr\{\xi(Z;\hat{\pi},\hat{\mu}) - \xi(Z;\pi,\mu) | A = t\} = \Pr\left[ \{\mu_0(L, t) - \hat{\mu}(L, t)\} \left\{ \frac{\pi_0(t|L)/\varpi_0(t)}{\hat{\pi}(t|L)/\int \hat{\pi}(t|l) dP(l)} + \int \hat{\mu}(l, t) dP(l) - \int \mu_0(l, t) dP(l) \right\} \right].
\]

(D.4)

Then by further calculation, we see

\[
\Pr\left[ \{\mu_0(L, t) - \hat{\mu}(L, t)\} \left\{ \frac{\pi_0(t|L) \int \hat{\pi}(t|l) dP(l) - \hat{\pi}(t|L) \varpi_0(t)} {\hat{\pi}(t|L) \varpi_0(t)} \right\} \right] \]

\[
= \Pr\left[ \{\mu_0(L, t) - \hat{\mu}(L, t)\} \left\{ \frac{\pi_0(t|L) \int \hat{\pi}(t|l) dP(l) - \hat{\pi}(t|L) \varpi_0(t)} {\hat{\pi}(t|L) \varpi_0(t)} \right\} \right]
\]

\[
+ \Pr\left[ \{\mu_0(L, t) - \hat{\mu}(L, t)\} \left\{ \frac{\hat{\pi}(t|L) \int \hat{\pi}(t|l) dP(l) - \hat{\pi}(t|L) \varpi_0(t)} {\hat{\pi}(t|L) \varpi_0(t)} \right\} \right]
\]

\[
= \frac{\int \hat{\pi}(t|l) dP(l)}{\varpi_0(t)} \Pr\left[ \{\mu_0(L, t) - \hat{\mu}(L, t)\} \left\{ \frac{\pi_0(t|L) - \hat{\pi}(t|L)} {\hat{\pi}(t|L)} \right\} \right]
\]

\[
+ \frac{1}{\varpi_0(t)} \Pr\{\hat{\pi}(t|L) - \pi_0(t|L)\} \Pr\{\mu_0(L, t) - \hat{\mu}(L, t)\}.
\]

(D.5)
By Assumptions 12, E(B), and the Cauchy-Schwarz inequality, we see the conditional integral
\[ P \{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) | A = t \} \]
is bounded by
\[ O \left( \| \bar{\pi}(t|L) - \pi_0(a|L) \|_2 \| \bar{\mu}(L, t) - \mu_0(L, t) \|_2 \right), \]
and thus
\[ |P \{ g_{ha,j}(A) K_{ha}(A) \{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \} | \]
\[ = O \left( \int_A g_{ha,j}(A) \| \bar{\pi}(t|L) - \pi_0(a|L) \|_2 \| \bar{\mu}(L, t) - \mu_0(L, t) \|_2 \| \omega_0(t) \| dt \right) \]
\[ = O_P(r_n s_n). \]
\[ \square \]

E  Proof of main lemmas for Theorem 3.1

The lemmas proved in this section, combined with several of those from Section C, form the backbone of the proof of Theorem 3.1.

**Lemma E.1.** Let the assumptions of Theorem 3.1 hold. Then
\[ P_n \{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \} = O_p \left( \frac{1}{\sqrt{n}} + s_n^\infty r_n^\infty \right), \]
as \[ n \to \infty. \]

**Proof of Lemma E.1.** We write the quantity of interest as
\[ P_n \{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \} = (P_n - P) \{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \} + P \{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \}. \]
Now, we apply the concept of stochastic equicontinuity to deal with first term on the right side of the previous display, using an argument similar to the one used by Kennedy et al. [2017]. Let
\[ \Xi = (Y \oplus F_{\mu}) F_{\pi}^{-1} F_{\omega} \oplus F_{m}, \] (E.1)

where \( (F_{\mu}, F_{\pi}, F_{\omega}, F_{m}) \) are the classes of functions containing \( (\pi, \mu, \omega, m) \) (Recall \( F_{\mu} \) and \( F_{\pi} \) are defined in Assumption E(B)) and specifically, the latter two are defined as \( F_{\omega} := \{ \int \pi(\cdot|l) dP(l), \pi \in F_{\pi} \} \) and \( F_{m} := \{ \int \mu(l, \cdot) dP(l), \mu \in F_{\mu} \} \). With slight abuse of notation, \( Y \) is the single identity function that takes \( z = (l, a, y) \) as input and outputs \( y \). Moreover, we define the operators \( F_1 \oplus F_2 = \{ f_1 + f_2 : f_j \in F_j \} \), \( F^{-1} = \{ 1/f : f \in F \} \) and \( F_1 F_2 = \{ f_1 f_2 : f_j \in F_j \} \), for arbitrary function classes \( F \).
By construction of Ξ and Assumption E(B), ξ(z; π, µ) and ξ(z; π, µ) fall in the class Ξ. Moreover, after some rearranging, we can write

\[ \xi(\mathbf{Z}; \pi, \mu) - \xi(\mathbf{Z}; \hat{\pi}, \hat{\mu}) \]

\[ = \frac{Y - \hat{\mu}(L, A)}{\pi(A|L)} \int \hat{\pi}(A|l) dP(l) + \int \hat{\mu}(l, A) dP(l) \]

\[ - \frac{Y - \hat{\mu}(L, A)}{\pi(A|L)} \int \hat{\pi}(A|l) dP(l) - \int \hat{\mu}(l, A) dP(l) \]

\[ = \frac{Y - \hat{\mu}(L, A)}{\pi(A|L)} \int \hat{\pi}(A|l) dP(l) \{\hat{\pi}(A|L) - \hat{\pi}(A|L)\} \]

\[ + \frac{\int \hat{\pi}(A|l) dP(l)}{\pi(A|L)} \{\hat{\mu}(L, A) - \hat{\mu}(L, A)\} \]

\[ + \frac{Y - \hat{\mu}(L, A)}{\pi(A|L)} \left\{ \int \hat{\pi}(A|l) P(l) - \int \hat{\pi}(A|l) dP(l) \right\} \]

\[ + \left\{ \int \hat{\mu}(l, A) dP(l) - \int \hat{\mu}(l, A) P(l) \right\} \]

\[ = O_p(\|\pi - \hat{\pi}\|_Z + \|\hat{\mu} - \mu\|_Z). \]

Thus, by Assumption E(A)2, we have \( \|\xi(z; \pi, \mu) - \xi(z; \hat{\pi}, \hat{\mu})\|_Z = o_p(\sqrt{h}). \) Then by Lemma A.1 we have

\[ (P_n - P)\{\xi(\mathbf{Z}; \pi, \mu) - \xi(\mathbf{Z}; \hat{\pi}, \hat{\mu})\} = o_p(1/\sqrt{n}). \]

(E.3)

From the proof of Lemma D.1, we have \( \mathbb{P}\{\xi(\mathbf{Z}; \pi, \mu) - \xi(\mathbf{Z}; \hat{\pi}, \hat{\mu})|A = t\} \) is \( O(||\hat{\pi}(t|L) - \pi_0(a|L)||_2||\hat{\mu}(L, t) - \mu_0(L, t)||_2). \) Then by Assumption D3, we have the uniform boundedness of \( \varpi_0. \) Thus we have

\[ |\mathbb{P}\{\xi(\mathbf{Z}; \pi, \mu) - \xi(\mathbf{Z}; \hat{\pi}, \hat{\mu})\}| \]

\[ = O \left( \left| \int A ||\hat{\pi}(t|L) - \pi_0(t|L)||_2||\hat{\mu}(L, t) - \mu_0(L, t)||_2 dt \right| \right) \]

\[ = O \left( \sup_{t \in A} ||\hat{\pi}(t|L) - \pi_0(t|L)||_2 \sup_{t \in A} ||\hat{\mu}(L, t) - \mu_0(L, t)||_2 \right) \]

\[ = O_p(s_n^\infty r_n^\infty). \]

(E.4)

\[ \square \]

**Lemma E.2.** Let the assumptions of Theorem 3.1 hold. Then

\[ \int \left( g_n^T \tilde{D}_{ha}^{-1} P_n [g_n(A)K_{ha}(A)\{\xi(\mathbf{Z}; \pi, \mu) - \xi(\mathbf{Z}; \hat{\pi}, \hat{\mu})\}]^2 w(a) da \]

is \( o_p(1/n + (s_n^\infty r_n^\infty)^2) \) as \( n \to \infty. \)
Proof of Lemma E.2. Here we use a further decomposition \( d_{1,2}(a) = R_{n,1,a} + R_{n,2,a} \) (recall the definition of \( d_{1,2} \) in (B.5)) with

\[
R_{n,1,a} := g_{h(a)}^T \tilde{D}_{h(a)}^{-1}(\mathbb{P}_n - \mathbb{P}) \left[ g_{h(a)}(A)K_{h(a)}(A) \left\{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \right\} \right] \\
R_{n,2,a} := g_{h(a)}^T \tilde{D}_{h(a)}^{-1} \mathbb{P} \left[ g_{h(a)}(A)K_{h(a)}(A) \left\{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \right\} \right].
\]

(E.5)

We will bound \( \sup_{a \in A} R_{n,2,a} \), whereas for \( R_{n,1,a} \) we will show a moment bound for each \( a \) and then to control the order of magnitude of the integral we control its expectation, by interchanging the integral and the expectation.

Lemma G.1 yields that \( \sup_{a \in A} |g_{h(a)}^T \tilde{D}_{h(a)}^{-1} - (\varpi_0(a)^{-1}, 0)| = o(1) \) a.s. for each element. Now we let

\[
R_{n,1,a}' := (\mathbb{P}_n - \mathbb{P}) \left[ g_{h(a)}(A)K_{h(a)}(A) \left\{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \right\} \right] \\
R_{n,2,a}' := \mathbb{P} \left[ g_{h(a)}(A)K_{h(a)}(A) \left\{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \right\} \right].
\]

We first show \( n\sqrt{h} \int (R_{n,1,a}')^2 w(a) \, da \) is \( o_p(\sqrt{h}) = o_p(1) \). For a measurable class of functions \( F \), recall that

\[
J(\delta, F, L_2) := \int_0^\delta \sup_Q \sqrt{1 + \log N(\varepsilon \|F\|_{Q,2}, F, L_2(Q))} d\varepsilon
\]

where the supremum is taken over all discrete probability measures \( Q \) with \( \|F\|_{Q,2} > 0 \). Here \( N(\varepsilon, F, L_2(Q)) \) is the \( L_2(Q) \)-\( \varepsilon \) covering number of \( F \) [van der Vaart and Wellner, 1996]. Also, let \( \|G_n\|_F := \sup_{f \in F} |G_n(f)| \).

For \( \delta > 0, j = 1, 2 \), let \( G_{\delta, j, a, n} \equiv G_{\delta, j, a} \) be the class of functions

\[
\left\{ Z \mapsto (A-a)^{j-1} K\left( \frac{A-a}{h} \right) (\xi_1(Z) - \xi_2(Z)) : \|\xi_1 - \xi_2\|_Z \leq \sqrt{h}\delta \right\}
\]

for \( Z = (L, A, Y) \). Here \( K \) is a kernel satisfying Assumption E(A)3. For any \( a \), \( G_{\delta, j, a} \) has envelope given by

\[
G_n(Z) := \left( \frac{A-a}{h} \right)^{j-1} K\left( \frac{A-a}{h} \right) \delta.
\]

By Theorem 2.14.1 of van der Vaart and Wellner [1996], we have (suppressing outer expectations related to measurability concerns)

\[
\mathbb{E}\|G_n\|_{G_{\delta, j, a}}^2 \lesssim J(1, G_{\delta, j, a})^2 \mathbb{E} |G_n|^2 \quad (E.6)
\]

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Thus, we can conclude that \( (E.6) \) is bounded above by \( O(\delta) \) which goes to 0 by letting \( \delta \). The latter probability converges to 0 since \( \| a \|_{\infty} \) is bounded.

A standard kernel density calculation shows that \( \mathbb{E} G^2_n \leq O(h) \delta \) independent of \( a \) (by Assumption E(A)3 \( \int |u|^{j-1} K(u)du < \infty \) and \( K \) is bounded).

Thus, we can conclude that \( (E.6) \) is bounded above by \( O(h) \), independently of \( a \).

Then for any \( \delta, \varepsilon > 0 \), we have

\[
P\left( n \int (R'_{n,1,a})^2 w(a) da > \varepsilon \right) \leq P\left( n \int (R'_{n,1,a})^2 w(a) da > \varepsilon, \| \xi(z; \widehat{\pi}, \widehat{\mu}) - \xi(z; \widehat{\pi}, \widehat{\mu}) \|_z \leq \sqrt{h}\delta \right) + P\left( \| \xi(z; \widehat{\pi}, \widehat{\mu}) - \xi(z; \widehat{\pi}, \widehat{\mu}) \|_z > \sqrt{h}\delta \right),
\]

where the latter probability converges to 0 since \( \| \xi(z; \widehat{\pi}, \widehat{\mu}) - \xi(z; \widehat{\pi}, \widehat{\mu}) \|_z = o_p(\sqrt{h}) \) from Step 2. And for the other term, by Markov’s theorem,

\[
P\left( n \int (R'_{n,1,a})^2 w(a) da > \varepsilon, \| \xi(z; \widehat{\pi}, \widehat{\mu}) - \xi(z; \widehat{\pi}, \widehat{\mu}) \|_z \leq \sqrt{h}\delta \right) \leq \varepsilon^{-1} \mathbb{E} \left\{ \int n(R'_{n,1,a})^2 w(a) da \right\} \leq \varepsilon^{-1} \mathbb{E} n(R'_{n,1,a})^2 w(a) da \leq \frac{w_{\max}}{\varepsilon} \mathbb{E} \| G_n \|_{G_{i,j,a}}^2 \leq \frac{w_{\max} O(\delta)}{\varepsilon},
\]

which goes to 0 by letting \( \delta \searrow 0 \), and here \( w_{\max} := \sup A |w(a)| \) \( < \infty \). Thus \( n\sqrt{h} \int (R'_{n,1,a})^2 w(a) da \) is \( o_p(1) \).

To go from \( R'_{n,1,a} \) to \( R_{n,1,a} \), we use the argument (used above) on \( g_{ha}^{T} \hat{D}_{ha}^{-1} \) being \( O_p(1) \); we have

\[
n\sqrt{h} \int R_{n,1,a}^2 w(a) da = n\sqrt{h} \int \left\{ g_{ha}^{T} \hat{D}_{ha}^{-1} R'_{n,1,a} \right\}^2 w(a) da \leq n\sqrt{h} \int 2 \left\{ |g_{ha}^{T} \hat{D}_{ha}^{-1} (\varphi_0(a)^{-1}, 0) | R'_{n,1,a} \right\}^2 w(a) da + n\sqrt{h} \int 2 \left\{ |(\varphi_0(a)^{-1}, 0) | R'_{n,1,a} \right\}^2 w(a) da.
\]

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The first term on the last two lines above can be bounded as
\[
n\sqrt{h} \int 2 \left\{ \left| g_{\text{ha}}^T \hat{D}_{\text{ha}}^{-1} - (\varpi_0(a)^{-1}, 0) \right| R'_{n,1,a} \right\}^2 w(a) \, da
\leq 2n\sqrt{h} \int \left\| g_{\text{ha}}^T \hat{D}_{\text{ha}}^{-1} - (\varpi_0(a)^{-1}, 0) \right\|_{l_2}^2 \| R'_{n,1,a} \|_{l_2}^2 w(a) \, da,
\]
where \( \|x\|_{l_2} := \sqrt{x^T x} \) is the \( l_2 \) norm for vectors on \( \mathbb{R}^d \). Since we have 
\( |g_{\text{ha}}^T \hat{D}_{\text{ha}}^{-1} - (\varpi_0(a)^{-1}, 0)| \) is uniformly \( o(1) \) a.s. and each element of \( n\sqrt{h} \int (R'_{n,1,a})^2 w(a) \, da \) is \( o_p(\sqrt{h}) \), we have the right hand side of the above inequality is \( o_p(\sqrt{h}) \). By Assumption I2, \( \varpi_0(a) \) is bounded below from 0; similarly it is easy to see
\[
n\sqrt{h} \int 2 \left\{ \left| (\varpi_0(a)^{-1}, 0) R'_{n,1,a} \right| \right\}^2 w(a) \, da
\leq 2n\sqrt{h} \int \left\| (\varpi_0(a)^{-1}, 0) \right\|_{l_2}^2 \| R'_{n,1,a} \|_{l_2}^2 w(a) \, da
= o_p(\sqrt{h}).
\]
So
\[
n\sqrt{h} \int R_{n,1,a}^2 w(a) \, da = o_p(\sqrt{h}).
\]
Next we show \( \int (R'_{n,2,a})^2 w(a) \, da \) is \( o_p((s_n^{-\infty} r_n^{-\infty})^2) \). From (D.4), we have
\[
|R'_{n,2,a}| = O \left( \left| \int_A g_{\text{ha}}(t) K_{\text{ha}}(t) \| \hat{\pi}(t|L) - \pi(t|L) \|_2 \| \hat{\mu}(L, t) - \mu(L, t) \|_2 dt \right| \right)
\]
We let
\[
I_a := \left| \int_A g_{\text{ha}}(t) K_{\text{ha}}(t) \| \hat{\pi}(t|L) - \pi(t|L) \|_2 \| \hat{\mu}(L, t) - \mu(L, t) \|_2 dt \right|
\]
We show \( \int_A I_a^2 w(a) \, da \) is \( o_p((n\sqrt{h})^{-1}) \). By Assumption E(A)4, we know \( K \) has bounded support, and thus
\[
I_a = O \left( \left| \int_A K_{\text{ha}}(t) \| \hat{\pi}(t|L) - \pi(t|L) \|_2 \| \hat{\mu}(L, t) - \mu(L, t) \|_2 dt \right| \right)
= O \left( \sup_{t \in A} \| \hat{\pi}(t|L) - \pi(t|L) \|_2 \sup_{t \in A} \| \hat{\mu}(L, t) - \mu(L, t) \|_2 \right)
= O_p(s_n^{-\infty} r_n^{-\infty}).
\]
With a similar argument as for \( n\sqrt{h} \int R_{n,1,a}^2 w(a) \, da \), we see the integral \( \int R_{n,2,a}^2 w(a) \, da = o_p((s_n^{-\infty} r_n^{-\infty})^2) \). Then by the Cauchy-Schwarz inequality, it’s easily seen that \( \int (d_{1,2}(a))^2 w(a) \, da = o_p(1/n + (s_n^{-\infty} r_n^{-\infty})^2) \).
\[\square\]
Lemma E.3. Let the assumptions of Theorem 3.1 hold. Let $D_2(a), D_3$ be as defined in (B.1). Then $\int_A D_2(a)w(a) \, da = O_p(n^{-1/2})$, and so

$$n\sqrt{h} \int_A D_2(a)D_3w(a) \, da = o_p(1) \text{ as } n \to \infty.$$  \hfill (E.7)

Proof of Lemma E.3. In step 2 of the proof of Theorem 3.1, we have seen that $D_3 = o_p(1/\sqrt{nh})$. So (E.7) follows from $\int_A D_2(a)w(a) \, da = O_p(n^{-1/2})$. We first write $D_2(a)$ as

$$D_2(a) = \tilde{h}_a(a) - \mathbb{P}_n\tilde{\xi} - \tilde{h}_a(a) - \mathbb{P}\tilde{\xi} - \mathbb{P}_n\tilde{\xi}.$$ \hfill (E.8)

By the Central Limit Theorem, it is easily seen that $\mathbb{P}\tilde{\xi} - \mathbb{P}_n\tilde{\xi} = O_p(1/\sqrt{n})$. By Lemma G.1, we can write

$$\tilde{h}_a(a) - \mathbb{P}_n\tilde{\xi} = g_{ha}(A)K_{ha}(A) \{ \tilde{\xi}(Z_i; \bar{\pi}, \bar{\mu}) - \mathbb{P}\tilde{\xi} \}$$

as defined in (B.1). Then

$$\mathbb{P}_n\tilde{\xi} = \mathbb{P}_n\tilde{\xi}.$$ \hfill (E.8)

uniformly for all $a \in A$. So we can focus on integrating the dominating term

$$\frac{1}{n} \sum_{i=1}^{n} \frac{1}{h\varpi_0(a)} K \left( \frac{A_i - a}{h} \right) (\xi_i - \mathbb{P}\tilde{\xi}) ,$$

and we have

$$\int_A \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h\varpi_0(a)} K \left( \frac{A_i - a}{h} \right) (\xi_i - \mathbb{P}\tilde{\xi}) w(a) \, da$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_A \frac{w(a)}{h\varpi_0(a)} K \left( \frac{A_i - a}{h} \right) \, da (\xi_i - \mathbb{P}\tilde{\xi}) .$$

By a change of variables $(a = A_i + hu)$, we can evaluate the integral as

$$\int_A \frac{w(a)}{h\varpi_0(a)} K \left( \frac{A_i - a}{h} \right) \, da = \int \frac{w(A_i + hu)}{\varpi_0(A_i + hu)} K(u) \, du .$$

By Taylor expansion, we can write $w(A_i + hu)/\varpi_0(A_i + hu)$ as

$$\frac{w(A_i)}{\varpi_0(A_i)} + \frac{w'(A_i + \eta u)\varpi_0(A_i + \eta u) - w(A_i + \eta u)\varpi_0'(A_i + \eta u)}{(\varpi_0(A_i + \eta u))^2} u h ,$$
where $0 < \eta < 1$ is some constant depending on $u$. By assumption D2 and our assumption on $w(a)$, we know $\varpi_0(a)$ is bounded below from 0 and bounded above, and $\varpi'$ and $w'$ are both bounded.

\[ \int_{A} \frac{w(a)}{h\varpi_0(a)} K \left( \frac{A_i - a}{h} \right) da = \frac{w(A_i)}{\varpi_0(A_i)} + O(h), \]

uniformly for all $A_i$. Then by central limit theorem,

\[ \frac{1}{n} \sum_{i=1}^{n} \int_{A} \frac{w(a)}{h\varpi_0(a)} K \left( \frac{A_i - a}{h} \right) da \left( \bar{\xi}_i - \mathbb{P} \xi \right) = O_p(1/\sqrt{n}), \]

and thus $\int D_2(a)w(a) da = O_p(1/\sqrt{n})$. So we have $n\sqrt{h} \int D_2(a)D_3w(a) da = o_p(h^{1/4}) = o_p(1)$.

**Lemma E.4.** Let the assumptions of Theorem 3.1 hold. Then

\[ \int \frac{1}{\varpi_0^2(a)} \mathbb{P} \left[ K_{h\varpi_0}(A) \{ \xi(Z; \bar{\pi}, \bar{\mu}) - \xi(Z; \bar{\pi}, \bar{\mu}) \} \right] \frac{1}{n} \sum_{i=1}^{n} K_h (A_i - a) \bar{\varepsilon}_i w(a) da \]

is $o_p((n\sqrt{h})^{-1})$.

**Proof of Lemma E.4.** Recall the definition of the function class $\Xi$ of functions $\xi(\cdot; \pi, \mu)$:

\[ \Xi := (Y + F_\mu)F_\pi^{-1}F_\varpi + F_m, \tag{E.9} \]

where $Y$ is shorthand for the single function outputting $Y$ from $Z$. Thus $\Xi$ is indexed by the above function classes. Define also the shifted class $\Xi_1 := \{ \xi - \bar{\xi} : \xi \in \Xi \}$. Now for functions $\pi \in F_\pi, \mu \in F_\mu$, let

\[ \psi_{\mu, \pi}(t) := \frac{\mathbb{P}_{\nu(t) \mid \mathbf{L}}}{\mathbb{P}_{\pi_0(t) \mid \mathbf{L}}} \left[ \left\{ \mu_0(\mathbf{L}, t) - \mu(\mathbf{L}, t) \right\} \left\{ \frac{\pi_0(t \mid \mathbf{L}) - \pi(t \mid \mathbf{L})}{\pi(t \mid \mathbf{L})} \right\} \right] \]

\[ + \frac{1}{\mathbb{P}_{\pi_0(t) \mid \mathbf{L}}} \left\{ \pi(t \mid \mathbf{L}) - \pi_0(t \mid \mathbf{L}) \right\} \mathbb{P}_{\nu_0(\mathbf{L}, t)} - \mu(\mathbf{L}, t) \}, \tag{E.10} \]

(/regarding $\varpi$ as dependent on $\pi$). Note that $\mathcal{F}_\psi := \{ \psi_{\mu, \pi} : \pi \in F_\pi, \mu \in F_\mu \}$ is uniformly bounded, and also has finite (independent of $n$) uniform entropy integral. This latter statement follows because, first, from the properties discussed in the previous paragraph (i.e., by [van der Vaart and Wellner, 1996, Theorem 2.10.20]) the classes of functions $(l, t) \mapsto \frac{\varpi(t \mid l)}{\varpi_0(l)} \{ \mu_0(l, t) - \mu(l, t) \} \left\{ \frac{\pi_0(l \mid t) - \pi(l \mid t)}{\pi(l \mid t)} \right\}$,
as well as \((l, t) \mapsto \frac{1}{\varpi_0(t)} \{ \pi(t|l) - \pi_0(t|l) \}\), and \((l, t) \mapsto \{ \mu_0(l, t) - \mu(l, t) \}\), all have finite uniform entropy integral. And then, second, applying \(\mathbb{P}\)
(with \(t\) fixed and \(L\) random) cannot increase the uniform entropy integral by Lemma A.3. Thus, 
\(t \mapsto \mathbb{P} \left[ \frac{\varpi(t)}{\varpi_0(t)} \{ \mu_0(L, t) - \mu(L, t) \} \right] \), 
\(t \mapsto \frac{1}{\varpi_0(t)} \mathbb{P} \left\{ \pi(t|L) - \pi_0(t|L) \right\}\) and 
\(t \mapsto \mathbb{P} \left\{ \mu_0(L, t) - \mu(L, t) \right\}\), all have finite uniform entropy integral; and then by the preservation properties of the uniform entropy integral (i.e., by [van der Vaart and Wellner, 1996, Theorem 2.10.20]), the class of functions \(\{ \psi_{\mu, \pi}, \mu \in \mathcal{F}_\mu, \pi \in \mathcal{F}_\pi \}\), given in (E.10), has a finite uniform entropy integral. It can also be checked to be uniformly bounded.

For a sequence \(\delta_n > 0\), define another class

\[ \mathcal{K}_\Xi_n := \left\{ f(a_1) := \varpi_0(a_1)^{-1} \int K_h(a_2 - a_1) \tilde{f}(l, a_2, y) d\mathbb{P}(l, a_2, y); \tilde{f} \in \Xi_1, \right. \]

\[ f(a_1) = \varpi_0(a_1)^{-1} \int_{\mathcal{A}} K_h(a_2 - a_1) \psi_{\mu, \pi}(a_2) \varpi_0(a_2) da_2, ||f||_A \leq \delta_n \}

where recall that functions in \(\Xi\) (and so \(\Xi_1\)) are indexed by the classes given in (E.9). Define a third class

\[ \mathcal{KK}_\Xi_n := \left\{ f(A) = \int K_h(A - a_1) \tilde{f}(a_1) da_1; \tilde{f} \in \mathcal{K}_\Xi_n \right\}. \]

And, for a probability measure \(Q\) on \(\mathcal{A}\), let \(\tilde{Q}\) be a measure on \(\mathcal{A}\) given by

\[ \tilde{Q} \psi := Q \left( \int_{\mathcal{A}} K_h(A - a_1) K_h(a_2 - a_1) \tilde{\psi}(a_2) da_2 da_1 \right). \] (E.12)

(The properties of a measure, e.g. additivity, can be trivially verified given the definition as an integral.)

For a function \(f \in \mathcal{KK}_\Xi_n\), we would like to use Jensen’s inequality to show \(Q f^2\) is bounded above by (a constant times) \(\tilde{Q} \tilde{\psi}^2\) for a \(\tilde{\psi} \in \mathcal{F}_\psi\). Any \(Q f^2\) is of the form

\[ E_Q \left( \int_{\mathcal{A}} K_h(A - a_1) K_h(a_2 - a_1) \tilde{f}(l, a_2, y) \varpi_0^{-1}(a_1) d\mathbb{P}(l, a_2, y) da_1 \right)^2 \]

which equals

\[ E_Q \left( \int_{\mathcal{A}} \varpi^{-1}(a_1) K_h(A - a_1) K_h(a_2 - a_1) \psi_{\mu, \pi}(a_2) \varpi_0^{-1}(a_2) da_2 da_1 \right)^2 \]

\[ \leq \varpi_{\min}^{-2} E_Q \left( \int_{\mathcal{A}} K_h(A - a_1) K_h(a_2 - a_1) \psi_{\mu, \pi}^2(a_2) da_2 da_1 \right) = \varpi_{\min}^{-2} \tilde{Q} \psi_{\mu, \pi}^2. \]
where the inequality is by applying Jensen’s inequality twice. (Since both kernels integrate to 1, Jensen’s inequality applies.)

It is immediately clear that \( F \equiv \delta_n \) is a (constant) envelope for the class \( KK \Xi_n \). (Recall \( \varpi_{\text{max}} := \max \varpi \) and \( \varpi_{\text{min}} := \min \varpi \).) Then, with \( F \) being the envelope for \( F_\psi \), we have

\[
N\left( \varepsilon \delta_n / \varpi_{\text{min}}, KK \Xi_n, L_2(Q) \right) \leq N\left( \varepsilon \| F \|_{Q,2}, F_\psi, L_2(Q) \right).
\]

Thus we can take a sup on the left over all measures \( Q \) and this is upper bounded by a sup on the right over all measures \( \tilde{Q} \). (We take sups over all measures, not just all discrete measures.) Thus by change of variables

\[
J(\infty) := \int_0^\infty \sup_Q \sqrt{\log N(\varepsilon \| F \|_{Q,2}, KK \Xi_n, L_2(Q))} \, d\varepsilon
\]

\[
\leq \int_0^\infty \sup_Q \sqrt{\log N(\varepsilon \| \tilde{F} \|_{Q,2}, F_\psi, L_2(Q))} \, d\varepsilon < \infty
\]

where the second integral does not depend on \( n \) and is finite as discussed above.

Finally, note that the entropy for \( KK \Xi_n \) equals the entropy for \( e \times KK \Xi_n \) where \( e \) stands for the singleton function giving \( \varepsilon \); and this class has envelope \( |e| \delta_n \) with \( L_2(P) \)-integral of order \( \delta_n \). Thus, applying Theorem 2.14.1 of van der Vaart and Wellner [1996] (see also van der Vaart and Wellner [2011]) to the previous display (and the order \( \delta_n \) envelope) we have

\[
P\| G_n \|_{e \times KK \Xi_n} \leq C_1 J(\infty) \delta_n \leq C_2 \delta_n
\]

(E.13)

where \( G_n = \sqrt{n}(P_n - P) = \sqrt{n}P_n \) since \( E(\varepsilon | A) = 0 \) and \( C_i \) are constants (depending on \( P \) and the classes/methods for \( \pi, \mu \)).

Finally, we will apply (E.13) to see that

\[
n \sqrt{n} \int \frac{1}{n} \sum_{i=1}^n K_h(A_i - a) \tilde{\varepsilon}_i \mathbb{P} \{ K_h(A - a)(\xi(\cdot; \hat{\pi}, \hat{\mu}) - \tilde{\xi}) \} da
\]

(E.14)

is \( O_p(\delta_n \sqrt{n}) \).

This is because by (D.4) and (D.5) we have

\[
P \{ K_h(A - a)(\xi(\cdot; \hat{\pi}, \hat{\mu}) - \tilde{\xi}) \} = \int K_h(a_2 - a) \psi_{\hat{\pi}, \hat{\mu}}(a_2) \varpi_0(a_2) da_2
\]

(E.15)
\[
P \{ K_h(A - a)(\xi(\cdot; \hat{\pi}, \hat{\mu}) - \bar{\xi}) \}
\]
\[
= \int K_h(t - a)P \{ \xi(\cdot; \hat{\pi}, \hat{\mu}) - \bar{\xi} | A = t \} \varpi_0(t)dt
\]
\[
= \int K_h(t - a)\psi_{\hat{\pi}, \hat{\mu}}(t)\varpi_0(t) \, dt,
\] (E.16)

and so by applying the Cauchy Schwartz inequality, \( \| \psi_{\hat{\pi}, \hat{\mu}} \|_A \leq \delta_n \) and so the sup over \( a \in A \) of (E.15) is also bounded above by \( \delta_n \). Now by Assumptions I2 and E(B), we have

\[
\psi_{\hat{\mu}, \hat{\pi}}(t) = \frac{P \pi(t|L)}{\bar{\pi}(t|L)} \left[ \left\{ \bar{\mu}_0(L, t) - \hat{\mu}(L, t) \right\} \left\{ \frac{\pi_0(t|L) - \bar{\pi}(t|L)}{\pi(t|L)} \right\} \right] \] (E.17)

\[
+ \frac{1}{\bar{\pi}_0(t|L)} \left\{ \bar{\pi}(t|L) - \pi_0(t|L) \right\} \left\{ \mu_0(L, t) - \hat{\mu}(L, t) \right\} \] (E.18)

\[
= O \left( \| \mu_0(L, t) - \hat{\mu}(L, t) \|_2 \| \pi_0(t|L) - \bar{\pi}(t|L) \|_2 \right) \] (E.19)

\[
= O(r_{\infty} s_{\infty}^2). \] (E.20)

Substituting \( \delta_n \) with \( r_{\infty} s_{\infty}^2 \) yields (E.14) is \( o_p((n\sqrt{h})^{-1}) \) because \( s_{\infty} r_{\infty} = o((n\sqrt{h})^{-1/2}) \) from Assumption E(A) 3.

\[ \blacksquare \]

F Proof of Theorem 3.2

Proof of Theorem 3.2. Recall that we can write our test statistic as

\[
T_n = n\sqrt{h} \int_A \{ D_1(a) + D_2(a) + D_3 \}^2 w(a) \, da.
\]

Under the stated alternative, by slightly modifying the proof of Theorem 2.1 of Alcalá et al. [1999], we have

\[
n\sqrt{h} \int \{ D_2(a) \}^2 w(a) \, da = b_{0h} + \delta_n^2 b_{1h} + o_p(\delta_n^2),
\]

where

\[
b_{1h} = \int_A [K_h * g(a)]^2 w(a) \, da.
\]

From step 3 of the proof of Theorem 3.1 we know that
\[
\int \{D_1(a)\}^2 w(a) \, da = O\left(\frac{1}{n} + \{r_n^\infty s_n^\infty\}^2\right), \quad (F.1)
\]

regardless of the validity of null hypothesis. then by Cauchy-Schwart, we have

\[
n\sqrt{h} \int D_1(a)D_2(a)w(a) \, da \leq \sqrt{n^2h \int \{D_1(a)\}^2 w(a) \, da \int \{D_2(a)\}^2 w(a) \, da} \quad (F.2)
\]

\[
= \sqrt{n^2hO\left(\frac{1}{n} + \{r_n^\infty s_n^\infty\}^2\right) \{b_{0h} + \delta_n^2 b_{1h} + o_p(\delta_n^2)\} / (n\sqrt{h})} \quad (F.3)
\]

\[
= \sqrt{(O(1) + n\{r_n^\infty s_n^\infty\}^2) \{O(1) + \delta_n^2 \sqrt{h} b_{1h} + o_p(\sqrt{h}\delta_n^2)\}} \quad (F.4)
\]

It’s easily seen that

\[
O(1) \left\{O(1) + \delta_n^2 \sqrt{h} b_{1h} + o_p(\sqrt{h}\delta_n^2)\right\} = o_p(\delta_n^2) \quad (F.6)
\]

For the other term,

\[
n\{r_n^\infty s_n^\infty\}^2 \left\{O(1) + \delta_n^2 \sqrt{h} b_{1h} + o_p(\sqrt{h}\delta_n^2)\right\} \quad (F.7)
\]

\[
= O\left(n\{r_n^\infty s_n^\infty\}^2 + n\{r_n^\infty s_n^\infty\}^2 \delta_n^2 \sqrt{h} b_{1h}\right). \quad (F.8)
\]

From Assumption E(A)4, We know \(\{r_n^\infty s_n^\infty\}^2 = o(1/(n\sqrt{h}))\), so the second above \(O(n\{r_n^\infty s_n^\infty\}^2 \delta_n^2 \sqrt{h} b_{1h})\) is \(o(\delta_n^2)\). Moreover, \(O(n\{r_n^\infty s_n^\infty\}^2) = o(\sqrt{h}) = o(n^{1/10})\) by Assumption E(A)2, thus \(O(n\{r_n^\infty s_n^\infty\}^2) = o(\delta_n^4)\). Then we have the cross-product term \(n\sqrt{h} \int D_1(a)D_2(a)w(a) \, da = o(\delta_n^2)\). With a similar argument, from step 1 of the proof of Theorem 3.1, we have

\[
\int \{D_3(a)\}^2 w(a) \, da = O\left(\frac{1}{n} + \{r_n^\infty s_n^\infty\}^2\right) \quad (F.9)
\]

under the alternative model stated and then \(n\sqrt{h} \int D_2(a)D_3(a)w(a) \, da = o(\delta_n^2)\). Finally we have \(T_n = b_{0h} + \delta_n^2 b_{1h} + o_p(\delta_n^2)\) and it’s straightforward to see

\[
P(T_n > t_{n,1-\alpha}^*) \rightarrow 1,
\]

as \(n \rightarrow \infty\). □
G Basic local polynomial estimator lemmas

Here we provide lemmas that are standard in the analysis of local polynomial estimators.

Lemma G.1. Let the assumptions of Theorem 3.1 hold. Recall that $\hat{D}_{ha} = \mathbb{P}_n \{ g_{ha}(A)K_{ha}(A)g_{ha}^T(A) \}$ and $g_{ha}(A) = (1, (A-a)/h)$. Then $\sup_{a \in A} |g_{ha}(a)^T \hat{D}_{ha}^{-1} - (\varpi_0(a)^{-1}, 0) | = o(1)$ almost surely. And thus $g_{ha}(a)^T \hat{D}_{ha}^{-1} g_{ha}(t)$ converges to $\varpi_0^{-1}(a)$ almost surely for any $t$.

Proof. Using Assumptions E(A)1 and E(A)3, by Section 4.1 of Dony et al. [2006], we have

$$\sup_{a \in A} \left| \mathbb{P}_n \left\{ K_{ha}(A) \left( \frac{A-a}{h} \right)^j \right\} - \varpi_0(a) \int (-u)^j K(u) \, du \right| \to 0, \quad \text{a.s.,}$$

for $j = 0, 1, 2$. Thus $\hat{D}_{ha}$ converges to $\text{diag}\{ \varpi_0(a), \varpi_0(a) \int u^2 K(u) \, du \}$ uniformly almost surely over all elements. Here $\text{diag}(c_1, c_2)$ is a $2 \times 2$ diagonal matrix with elements $c_1$ and $c_2$ on the diagonal. Then by Assumption I2 and D2, we have that $\sup_{A} \varpi_0(a) < \infty$ and $\inf_{A} \varpi_0(a) \geq \pi_{\text{min}} > 0$, so $\sup_{a \in A} |\hat{D}_{ha}^{-1} - \varpi_0(a)^{-1} \text{diag}\{ 1, 1/ \int u^2 K(u) \, du \} | = o(1)$ a.s. for each element. Thus

$$\sup_{a \in A} |g_{ha}(a)^T \hat{D}_{ha}^{-1} - (\varpi_0(a)^{-1}, 0) | = o(1)$$

a.s. as $n \to \infty$ for each element. \hfill \Box

The following is a standard representation of a local polynomial estimator.

Lemma G.2 (Fan and Gijbels [1996], page 63). Consider the local linear estimator at a point $a$ based on data $(A_1, Y_1), \ldots, (A_n, Y_n)$ and kernel $K$, given as

$$\hat{\beta}_h(a) = \arg \min_{\beta \in \mathbb{R}^2} \mathbb{Q}_n \left[ K_{ha}(A) \{ Y - g_{ha}(A)^T \beta \}^2 \right],$$

with $K_{ha}(t) = h^{-1} K\{(t-a)/h\}$ and $g_{ha}(t) = (1, (t-a)/h)^T$ (and $\mathbb{Q}_n$ being the empirical distribution of the $(A_i, Y_i)$ observations). The first coordinate of $\hat{\beta}_h$ (the function estimator) can be written as

$$\hat{\beta}_h = n^{-1} \sum_{i=1}^n W_h(A_i-a)Y_i = \mathbb{Q}_nW_h(A-a)Y,$$

where

$$W_h(t) := g_{ha}^T(0) \hat{D}_{ha}^{-1} g_{ha}(t) K_h(t), \quad \text{(G.1)}$$

and where $\hat{D}_{ha} = \mathbb{Q}_n \{ g_{ha}(A)K_{ha}(A)g_{ha}^T(A) \}$. 

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H Extension for testing for a treatment effect modifier

In this section, we briefly introduce an extension of our proposed test to a scenario where we are interested in testing whether a covariate is a treatment effect modifier or not, meaning the causal treatment effect is different conditional on the covariate value than it is unconditionally. The following provides a mapping or pseudo-outcome that has the double robustness property for assessing treatment effect modifiers.

**Theorem H.1.** Let Assumption I hold. Then the following mapping

\[ \phi(Z; \pi, \mu) = Y - \mu(L, A) \pi(A|L) \int_A \pi(A|l) dP(l|L_P) + \int_A \mu(l, A) dP(l|L_P) \]  

satisfies \( E\{\phi(Z; \pi, \mu)|A = a, L_P = l_P\} = E\{Y^a|L_P = l_P\} \) if either \( \pi = \pi_0 \) or \( \mu = \mu_0 \).

In real data analysis, we need to estimate the unknown aspects of (H.1), so we arrive at

\[ \hat{\phi}(Z; \hat{\pi}, \hat{\mu}) = Y - \hat{\mu}(L, A) \hat{\pi}(A|L) \int_A \hat{\pi}(A|l) dP_n(l|L_P) + \int_A \hat{\mu}(l, A) dP_n(l|L_P) \]

Once we have estimated the pseudo-outcome, we can use methodology analogous to the methodology we developed for testing \( \theta_0(\cdot) \) above, based on \( \{(A_i, L_{i,p}, \hat{\phi}(Z_i, \hat{\pi}, \hat{\mu})\} \) for \( i = 1 \).

In this manuscript we focus on the case of discrete variables that are treatment effect modifiers, in which case it is straightforward to extend the methodology from Subsection 2.2. Dette et al. [2001] proposed three test statistics based on Nadaraya-Watson estimators of the regression functions to test equality of regression functions. Specifically, one of the three test statistics they proposed has a similar form as our statistic \( T_n \) above, and can be written as

\[ \sum_{m=2}^{k} \sum_{j=1}^{m-1} \int_A \{\hat{g}_m(a) - \hat{g}_j(a)\}^2 w_{m,j}(a) da, \]  

where \( w_{m,j} \) are (positive, user-chosen) weight functions satisfying \( w_{m,j} = w_{j,m} \), there are \( k \) groups (i.e., values of the discrete variable), and \( \hat{g}_j, j = 1, \ldots, k, \) is the Nadaraya-Watson estimator for the \( j \)-th group regression function. Combining the conditional doubly robust mapping with (H.3), we
consider the following test statistic for testing whether a discrete covariate
with a finite number, \( k \), of possible values is an effect modifier or not:

\[
T_p^n = \sum_{m=2}^{k} \sum_{j=1}^{m-1} \int_{A} \left\{ \hat{\theta}_{h,m}(a) - \hat{\theta}_{h,j}(a) \right\}^2 w_{ij}(a) da,
\]

where \( \hat{\theta}_{h,j}(a) \) is the local linear estimator applied to the subset of the tuples 
\( \{(\hat{\phi}(Z; \hat{\pi}, \hat{\mu}), A)\} \) with the \( p \)-th covariate \( L_{i,p} \) taking the \( j \)-th category. We
summarize this extended testing procedure as follows.

1. Estimate the nuisance functions \((\pi_0, \mu_0)\) with \((\hat{\pi}, \hat{\mu})\).
2. Calculate the pseudo-outcomes \( \hat{\phi}(Z; \hat{\pi}, \hat{\mu}) \) according to \((H.2)\) and con-
struct the local linear estimator \( \hat{\theta}_{h,j}(a) \) for each conditional treatment
effect curve with the corresponding subset of \( \{(\hat{\phi}(Z; \hat{\pi}, \hat{\mu}), A)\} \).
3. Calculate the test statistic \( T_p^n \) according to \((H.4)\).
4. To estimate the critical value of the asymptotic distribution under the
null,
   (1) Use the pooled data \( \{(\hat{\phi}(Z_i; \hat{\pi}, \hat{\mu}), A_i)\}_{i=1}^{n} \) to construct a single
local linear estimator \( \hat{\theta}_h(a) \)
   (2) Calculate \( \hat{\theta}_h(A_i) \) and \( \hat{\varepsilon}_i = \hat{\phi}(Z_i; \hat{\pi}, \hat{\mu}) - \hat{\theta}_h(A_i) \). Generate the
bootstrap sample of residuals \( \hat{\varepsilon}_i^* \sim \hat{F}_i \) and response \( \hat{\phi}_i^* = \hat{\theta}_h(A_i) + \hat{\varepsilon}_i^* \), where choices of \( \hat{F}_i \) are the same as discussed in Section 2.2.
   (3) Calculate \( T_{p,n}^{\ast} \) using the bootstrap sample \( \{(A_i, L_{i,p}, \hat{\phi}_i^*)\}_{i=1}^{n} \).
   (4) Repeat (2) and (3) for \( B \) times, where \( B \) is the desired num-
ber of bootstrap samples; We estimate critical value of rejection
by \( \hat{t}_{p,n,1-\alpha}^{\ast} \), the \( 1 - \alpha \) sample quantile of \( T_{p,n}^{\ast} \) from the bootstrap
samples.
5. Reject the null hypothesis if \( T_p^n > \hat{t}_{n,1-\alpha}^{\ast} \).

We use simulation to show the validity of this extended procedure in
Section 4.

\textit{Proof of Theorem H.1.} We use \( I_{-p} \) to denote of vector of covariates after
deleting the \( p \)-th component.
Let \( \bar{m}(a, l_p) := \mathbb{E}\{\bar{\mu}(L, a)|L_p = l_p\} \), \( \bar{\nu}(a, l_p) := \mathbb{E}\{\bar{\pi}(a|L)|L_p = l_p\} \), and \( \theta_0(a, l_p) := \mathbb{E}(Y^a|L_p = l_p) \). Then

\[
\mathbb{E}\{\phi(Z; \bar{\pi}, \bar{\mu})|A = a, L_p = l_p\} = \mathbb{E}\left\{ \frac{Y - \bar{\mu}(L, A)}{\bar{\pi}(A|L)/\bar{\nu}(A, L_p)} \bigg| A = a, L_p = l_p \right\} = \int \{\mu_0(l, a) - \bar{\mu}(l, a)\} \frac{\pi_0(\|a\|)/\bar{\nu}(a, l_p)}{\bar{\pi}(a|l)/\bar{\nu}(a, l_p)} dP(l|L_p = l_p) + \bar{m}(a, l_p) = \theta_0(a, l_p) + \int \{\mu_0(l, a) - \bar{\mu}(l, a)\} \left\{ \frac{\pi_0(\|a\|)/\bar{\nu}(a, l_p)}{\bar{\pi}(a|l)/\bar{\nu}(a, l_p)} - 1 \right\} dP(l|L_p = l_p),
\]

where the last line shows the double robustness of the proposed mapping.

Some details of the calculation are given as follows.

- The second equality above is calculated by iterated expectations:

\[
\mathbb{E}\left\{ \frac{Y - \bar{\mu}(L, A)}{\bar{\pi}(A|L)/\bar{\nu}(A, L_p)} \bigg| A = a, L_p = l_p \right\} = \mathbb{E}\left[ \mathbb{E}\left\{ \frac{Y - \bar{\mu}(L, A)}{\bar{\pi}(A|L)/\bar{\nu}(A, L_p)} \bigg| A, L \right\} \bigg| A = a, L_p = l_p \right] = \mathbb{E}\left\{ \mu_0(L, A) - \bar{\mu}(L, A) \bigg| \bar{\pi}(A|L)/\bar{\nu}(A, L_p) \right\} = \int \frac{\mu(l, a) - \bar{\mu}(l, a)}{\bar{\pi}(a|l)/\bar{\nu}(a, l_p)} dP(l|a, l_p).
\]

- The third equality comes from the calculation that

\[
dP(l|a, l_p) = \frac{p(a, l)}{p(a, l_p)} d\nu(l) = \frac{p(a|l)p(l)}{p(a|l_p)p(l_p)} d\nu(l) = \frac{\pi_0(\|a\|)}{\bar{\nu}_0(a, l_p)p(l_p|l_p)} d\nu(l) = \frac{\pi_0(\|a\|)}{\bar{\nu}_0(a, l_p)} dP(l|l_p).
\]
• For the last equality, note that
\[
\bar{m}(a, l_p) = \int \bar{\mu}(l, a) \, dP(l|l_p)
= \int \{\bar{\mu}(l, a) - \mu_0(l, a)\} \, dP(l|l_p) + \int \mu_0(l, a) \, dP(l|l_p)
= \int \{\bar{\mu}(l, a) - \mu_0(l, a)\} \, dP(l|l_p) + \theta_0(a, l_p).
\]

\[ \Box \]

I Plots from simulations and from data analysis

I.1 Description of simulation for testing for a treatment effect modifier

We consider the following data generating process with a continuous outcome, which is similar to Model 2 above, to perform the simulations for testing for a treatment effect modifier. First, we let
\[
\bar{L} = (L_1, L_2, L_3, \bar{L}_4)^T \sim N(0, I_4).
\]
And let \( L_4 = 1\{\bar{L}_4 > 1\} \) and suppose we observe
\[
L = (L_1, L_2, L_3, L_4)^T.
\]
Then we simulate the treatment level from Beta distributions,
\[
(A/5)|L \sim \text{Beta}(\lambda(L), 1 - \lambda(L)),
\]
\[
\text{logit} \, \lambda(L) = 0.1L_1 + 0.1L_2 - 0.1L_3 + 0.2L_4,
\]
and we simulate the continuous response from a normal distribution,
\[
Y|L, A \sim N(\mu(L, A), 0.5^2),
\]
\[
\mu(L, A) = 0.2L_1 + 0.2L_2 + 0.3L_3 - 0.1\delta L_4 - 0.1AL_1 + 0.1\delta AL_4
+ \exp \left\{ \frac{(A - 2.5)^2}{(1/2)^2} \right\}.
\]
So we see that the parameter \( \delta \) controls the distance between the conditional treatments when \( L_4 = 1 \) and when \( L_4 = 0 \), and when \( \delta = 0 \), there is no difference between the two conditional treatment effect curves, i.e., \( L_4 \) is not an effect modifier. In the simulation, we let \( \delta \) take values in \( \{0, 0.1, 0.2, 0.3, 0.4, 0.5\} \). And similar to Section 4, we test the performance of this treatment modifier test under 4 scenarios: (1) \( \pi \) is correctly specified
with a parametric model, \( \mu \) is incorrectly specified with a parametric model; (2) \( \pi \) is incorrectly specified with a parametric model, \( \mu \) is correctly specified with a parametric model; (3) both \( \pi \) and \( \mu \) are correctly specified with a parametric model; (4) both \( \pi \) and \( \mu \) are estimated with Super Learners [Van der Laan et al., 2007].

We display the simulation results in Figures 7 and 8. In Figures 7, we see that when at least one of the nuisance functions are correctly estimated, we have type I error probability converges to the nominal significance level \( \alpha = 0.05 \) and rejection probability converging to 1 as we increase the sample size. So this suggests the extended test maintains double robustness as we expected. In Figures 8, with the nonparametric Super Learner, we also see type I error probability converges to the nominal significance level \( \alpha = 0.05 \) and the rejection probability converges to 1 as we increase the sample size.
I.2 Plots

Figure 5: Simulation result for Model 1 with $\pi$ and $\mu$ estimated from parametric models.
Figure 6: Simulation result for Model 1 with $\pi$ and $\mu$ estimated from non-parametric models.
Figure 7: Simulation result for testing effect modifier with $\pi$ and $\mu$ estimated from parametric models.
Figure 8: Simulation result for testing effect modifier with $\pi$ and $\mu$ estimated from nonparametric models.

Figure 9: Left: Unadjusted loess fit of outcome against average nursing hours. Right: Average covariate values as a function of exposure, after transforming to percentiles to display on common scale.
Figure 10: Plot of estimated propensity scores $\hat{\pi}(a|l)$ (truncated below by 0.01) against average nursing hours

\section{J Cross-fitted Test}

In this section, we develop and study a cross-fitted version of the test we presented in the main paper, and use simulation studies to show how the dimensionality of the confounder vector $L$ affects the performance of the two tests.

\subsection*{J.1 Test procedure with cross-fitting}

Suppose we randomly partition the index set $\{1, \ldots, n\}$ into $V$ disjoint sets $\mathcal{V}_{n,1}, \ldots, \mathcal{V}_{n,V}$ with cardinalities $N_1, \ldots, N_V$, where $V \in \{2, \ldots, \lfloor n \rfloor\}$ is the number of partitions and without loss of generality, here we assume all the partitions have equal size, i.e., $N_1 = N_2 = \cdots = N_V$. For each $v \in \{1, \ldots, V\}$, we define $\mathcal{T}_{n,v} = \{Z_i : i \notin \mathcal{V}_{n,v}\}$ as the training set for fold $v$. 
Recall our original test statistics $T_n$ can be written as

$$T_n = n\sqrt{h} \int_A \left( \hat{\theta}_h(a) - \mathbb{P}_n \hat{\xi} \right)^2 w(a) da,$$  \hspace{1cm} (J.1)

where

$$\hat{\xi}(Z; \hat{\pi}, \hat{\mu}) = \frac{Y - \hat{\mu}(L, A)}{\hat{\pi}(A|L)} \int_L \hat{\pi}(A|l) d\mathbb{P}_n(l) + \int_L \hat{\mu}(l, A) d\mathbb{P}_n(l).$$  \hspace{1cm} (J.2)

and $\hat{\theta}_h(a)$ is the local linear estimator applied to $\{(\hat{\xi}(Z_i; \hat{\pi}, \hat{\mu}), A_i)\}_{i=1}^n$. Given one splitting of the data, i.e., $\mathcal{V}_{n,v}$ and $\mathcal{T}_{n,v}$, we can calculate the pseudo-comes as

$$\hat{\xi}_{n,v}(Z_i; \hat{\pi}_{n,v}, \hat{\mu}_{n,v}) = \frac{Y_i - \hat{\mu}_{n,v}(L_i, A_i)}{\hat{\pi}(A|L)} \int_L \hat{\pi}_{n,v}(A|l) d\mathbb{P}_{\mathcal{V}_{n,v}}(l) + \int_L \hat{\mu}_{n,v}(l, A) d\mathbb{P}_{\mathcal{V}_{n,v}}(l),$$  \hspace{1cm} (J.3)

where $\hat{\pi}_{n,v}$ and $\hat{\mu}_{n,v}$ are estimated only using the observations in $\mathcal{T}_{n,v}$, and $\mathbb{P}_{\mathcal{V}_{n,v}}$ is the empirical measure defined on $\{Z_i : i \in \mathcal{V}_{n,i}\}$. Similarly, we can calculate the test statistic restricted to this splitting as

$$T_{n,v} = N_v n^{\frac{1}{2}} \int_A \left( \hat{\theta}_h(a) - \mathbb{P}_{\mathcal{V}_{n,v}} \hat{\xi}_{n,v} \right)^2 w(a) da.$$  \hspace{1cm} (J.4)

Here $\hat{\theta}_h(a)$ is the local linear estimator applied to $\{(\hat{\xi}_{n,v}(Z_i; \hat{\pi}_{n,v}, \hat{\mu}_{n,v}), A_i)\}_{i \in \mathcal{V}_{n,v}}$, and with slight abuse of notation, we let $\mathbb{P}_{\mathcal{V}_{n,v}} \hat{\xi}_{n,v} = 1/N_v \sum_{i \in \mathcal{V}_{n,v}} \hat{\xi}_{n,v}(Z_i; \hat{\pi}_{n,v}, \hat{\mu}_{n,v})$. We can do this for each splitting and aggregate the results from all the splittings to get the cross-fitted test statistic as

$$T_n^\circ = \frac{1}{V} \sum_{v=1}^V T_{n,v}.$$  \hspace{1cm} (J.5)

We also use a modified wild bootstrap procedure to estimate the distribution of the cross-fitted test statistic. We perform wild bootstrap for each splitting separately. The detailed test procedure with cross-fitting is summarized as follows.

1. Randomly partition the index set $\{1, \ldots, n\}$ into $V$ disjoint set $\mathcal{V}_{n,1}, \ldots, \mathcal{V}_{n,V}$ with equal cardinalities $N_V$ (assume $n$ is a multiplier of $V$).

2. For each split $\mathcal{T}_{n,v}, \mathcal{V}_{n,v}$,
Estimate the nuisance functions \((\pi_0, \mu_0)\) with \((\hat{\pi}_{n,v}, \hat{\mu}_{n,v})\) using observations from \(T_{n,v}\).

Calculate the pseudo outcomes \(\hat{\xi}_v(Z; \hat{\pi}_{n,v}, \hat{\mu}_{n,v})\) by (J.3) and construct the local linear estimator \(\hat{\theta}_v\) using \(\{(\hat{\xi}_v(Z; \hat{\pi}_{n,v}, \hat{\mu}_{n,v}), A_i)\}_{i \in V_{n,v}}\).

Calculate \(T^n_v\) using (J.4).

To generate wild bootstrap samples for this splitting,

i. For each \(i \in V_{n,v}\), calculate \(\hat{\epsilon}_i = \hat{\xi}_i(Z; \hat{\pi}_{n,v}, \hat{\mu}_{n,v}) - \hat{\theta}_v(h(A_i))\).

Generate \(\epsilon_{i,v,b}^* \sim \hat{F}_i\) and use \(\{(\xi_{i,v,b}^*, A_i)\}_{i \in V_{n,v}}\) as bootstrap observations.

ii. Calculate \(T_{n,v}^{*,b}\) as

\[
T_{n,v}^{*,b} = N_v \sqrt{h} \int_A \left( \hat{\theta}_{v,b}^*(a) - \frac{1}{N_v} \sum_{i \in V_{n,v}} \xi_{i,v}^* \right)^2 w(a) da,
\]

where \(\hat{\theta}_{v,b}^*(a)\) is the local linear estimator applied to \(\{(\xi_{i,v,b}^*, A_i)\}_{i \in V_{n,v}}\).

iii. Repeat i. and ii. for \(B\) times where \(B\) is the desired number of bootstrap samples.

3. Calculate \(T_0^n = \sum_{v=1}^V T^n_v / V, T_{n,v}^{*,b} = \sum_{v=1}^V T_{n,v}^{*,b} / V\) for \(b = 1, \ldots B\).

Let \(t_{0,1-\alpha}^*\) denote the \(1 - \alpha\) quantile of \(\{T_{n,v}^{*,b}\}_{b=1}^B\). Reject the null hypothesis if \(T_0^n > t_{0,1-\alpha}^*\).

**J.2 Simulation studies for the effect of data dimensionality**

We first compare the performance of the test with cross-fitting with our original non-cross-fitted test procedure. We use the same data generating models, Model 1 and Model 2, to perform the simulation. Both \(\pi\) and \(\mu\) are estimated with Super Learners. The results are shown in Figures 11 and 12. We can see under both two data models, both two versions of tests have type I error probability converging to the desired level as we increase the sample size. However, the cross-fitted test shows uniformly lower power under alternatives.

Then we compare the two versions of tests with high dimensional data. The data generating model comes from Colangelo and Lee [2020] with some...
Figure 11: Simulation result for comparing non-cross-fitted test and cross-fitted test under model 1

Figure 12: Simulation result for comparing non-cross-fitted test and cross-fitted test under model 2
Figure 13: Simulation result for comparing p-values of non-cross-fitted test and cross-fitted test under Model (1.7). Type I error probabilities for $\alpha = 0.05$ are: No Cross-fitting, $n=500$ (0.448); Cross-fitting, $n=500$ (0.056); No Cross-fitting, $n=1000$ (0.535); Cross-fitting, $n=1000$ (0.047).
slight modifications, where we let

\[ L = (L_1, \ldots, L_{100})' \sim N(0, \Sigma), \]

\[ A = \Phi(3L'\beta) + 0.75\nu, \]

\[ Y = \gamma(1.2A + A^2 + AL_1) + 1.2L'\tilde{\beta} + \epsilon \quad (J.7) \]

where \( \beta_j = 1/j^2, \text{diag}(\Sigma) = 1, \) the \((i, j)\)-entry \( \Sigma_{ij} = 0.5 \) for \(|i - j| = 1\) and 0 for \(|i - j| > 1, \) for \( i, j = 1, \ldots, 100. \) \( \Phi \) is the CDF of \( N(0, 1) \) and \( \tilde{\beta} = (\beta_{100}, \beta_{99}, \ldots, \beta_1). \) We set \( \gamma = 0 \) to show the effect of high dimensional data on type I error probability of the test and when \( \gamma > 0, \) the treatment effect is nonconstant. We use sample sizes 500 and 1000, and use Random Forest to estimate the nuisance functions \( \pi \) and \( \mu. \) The results are displayed in Figure J.2, where we plot the histograms for p-values from each replication and calculate the type I error probabilities for \( \alpha = 0.05 \) in the caption. We can see, the distribution of p-values of non-cross-fitted test is very skewed and far away from uniform distribution. Moreover, increasing the sample size to 1000 does not show a great improvement. On the other side, the p-values from the cross-fitted test are quite uniformly distributed. The calculations also shows the non-cross-fitted test failed to maintain the desired level of type I error probability and the cross-fitted test achieved the desired level.

In summary, the cross-fitted test has lower power, especially with low dimensional data. But the non-cross-fitted test tends to fail to maintain the type I error probability under high dimensional data. So in practice, we need to decide which version to use according the dimensionality of our real world problem.

K U- and V-process results

In analyzing the remainder terms in Theorems 3.1 and 3.3, we use the theory of U- or V-processes, which we thus discuss in this section. Let \( f: \mathbb{Z}^r \to \mathbb{R} \) be permutation symmetric in its arguments. A U-statistic based on the ‘kernel’ \( f, \) and our i.i.d. sample \( W_1, \ldots, W_n \in \mathbb{Z}, \) is \( (\binom{n}{r})^{-1} \sum_{\beta} f(W_{\beta_1}, \ldots, W_{\beta_r}) \) where \( (\binom{n}{r}) = n!/(n - r)!r! \) is the binomial coefficient (and \( ! \) means the \( i \)th factorial) and the sum is over all \( \binom{n}{r} \) combinations of \( r \) non-repeated/distinct elements out of \( n \) data points [van der Vaart, 1998, Chapter 12]. The kernel and the U-statistic have degree (sometimes “order”) \( r. \) If instead we sum over all size-\( r \)-subsets of the \( n \) data points, \( (n-r)\sum_{i_1, \ldots, i_r} f(W_{i_1}, \ldots, W_{i_r}) \) we get a degree \( r \) V-statistic. U- and V-statistics often have the same asymptotic distributions. In our present context, we will need to allow \( f \) to range
over a function class, and so we arrive at so-called \( U \)- or \( V \)-processes. For us \( V \)-processes arise as remainder terms and we wish to show they are asymptotically negligible; we will apply maximal inequalities to do so.

To start, consider an order 2 \( U \)-process

\[
U_n(f) := n^{-3/2} \sum_{1 \leq i < j \leq n} f(W_i, W_j) \tag{K.1}
\]

(with slight laziness in the normalization since here we will only be interested in order of magnitude) for \( f \) varying over some class \( F \) of centered functions, meaning \( \mathbb{P} f(W_1, W_2) = 0 \). We assume \( f(w_1, w_2) = f(w_2, w_1) \). (Note: some of the functions \( f \) we need to consider will not be symmetric, but they can and will be symmetrized by considering \( f(w_1, w_2) + f(w_2, w_1) \).) A degenerate process is one such that \( \mathbb{P} f(w, W) = 0 \) for almost every \( w \) (and all \( f \in F \)). Otherwise the process is non-degenerate. Like U-statistics, U-processes can be decomposed via a Hájek or Hoeffding decomposition into an i.i.d. process and a degenerate U-process. For any \( f \), define \( (\Pi f)(w_1, w_2) := f(w_1, w_2) - \mathbb{P} f(W_1, W_2) + \mathbb{P} f(W_1, W_2) \) (which satisfies \( \mathbb{P} \Pi f(w, W) = 0 \)). Then \( U_n(f) = U_n(\Pi f) + (n-1)n^{-3/2}(\mathbb{P}_n \otimes \mathbb{P})(f) \) where \( \mathbb{P}_n \otimes \mathbb{P} \) is the product measure of \( \mathbb{P}_n \) (the empirical measure of the \( W_i \)'s) and \( \mathbb{P} \) on \( Z \times Z \). We can combine maximal inequalities for i.i.d. empirical processes with a maximal inequality for degenerate U-statistics from Nolan and Pollard [1987] to yield a maximal inequality for the entire process. This is done in Proposition K.1 below. The former term generally dominates. We modify the result so it fits the details of our setting. The following result is a form of Theorem 6 of Nolan and Pollard [1987] for (degenerate) U-processes (combined with Theorem 2.14.1 of van der Vaart and Wellner [1996]). We do not discuss measurability difficulties here. For a class \( F \) of symmetric functions (i.e., \( f(w_1, w_2) = f(w_2, w_1) \)), we let \( \mathbb{P} F \) denote \( \{ \mathbb{P} f(\cdot, W) : f \in F \} \). Recall the definitions of \( J(\cdot, \cdot, L_2) \) and \( J_2(\cdot, \cdot, L_2) \) given in (2.7).

**Proposition K.1.** Assume \( F \) is a class of (measurable) functions on a (measure) space \( W \times W \), with (measurable) envelope \( F \). Assume \( f \in F \) satisfies \( f(w_1, w_2) = f(w_2, w_1) \) and \( \mathbb{P} f(W_1, W_2) = 0 \). Assume \( W_1, \ldots, W_n \) are i.i.d. and \( U_n \) is defined by (K.1). Let \( F_1(w) \) be an envelope for \( \mathbb{P} F \). Then for a universal constant \( C > 0 \),

\[
\mathbb{P} ||U_n||_F \leq C J_2(1, F, L_2) \sqrt{\mathbb{P} F(W_1, W_2)^2 n^{-1/2}} + C J(1, \mathbb{P} F, L_2) \sqrt{\mathbb{P} F_1(W)^2}. \tag{K.2}
\]

**Proof of Proposition K.1.** For any \( f \) defined on \( W \times W \), define \( (\Pi f)(w_1, w_2) := f(w_1, w_2) - \mathbb{P} f(w_1, W_2) - \mathbb{P} f(W_1, w_2) + \mathbb{P} f(W_1, W_2) \) (which satisfies \( \mathbb{P} \Pi f(w, W) = 0 \)).

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0). The result follows from the decomposition $U_n(f) = U_n(\Pi f) + (n - 1)n^{-3/2}(\mathbb{P}_n \otimes \mathbb{P})(f)$ from above. We apply Theorem 6 of Nolan and Pollard [1987] to the first summand which is a degenerate U process. That theorem states that

$$EU_n(\Pi f) \leq n^{-3/2}CE(\tau_n \int_0^{\theta_n/\tau_n} 1 + \log N(\varepsilon \tau_n, \mathcal{F}, L_2(T_n))d\varepsilon) \quad \text{(K.3)}$$

where $\tau_n = (T_n F^2)^{1/2}$, $\theta_n = \sup_{f \in \mathcal{F}}(T_n f^2)^{1/2}/4$, and where $T_n$ is a measure defined on page 782 of Nolan and Pollard [1987] which places mass 1 at $4n(n - 1)$ pairs of data points $(W_i, W_j)$ where we always have $i \neq j$. (Note that the covering numbers are unchanged under a rescaling of the measure $T_n$.) If, in the integral on the right side of (K.3), we replace $T_n$ by a generic probability measure (since $N$ is invariant under rescaling of the measure) $Q$ and take a sup over $Q$, upper bound $\theta_n/\tau_n$ by 1, then the integral becomes $J_2(1, \mathcal{F})$, which can be factored out of the expectation. The expectation is then just $E\tau_n$; by Jensen’s inequality $E(T_n F^2)^{1/2} \leq (ET_n F^2)^{1/2} = (4n(n - 1))^{1/2}(\mathbb{P}F^2(W_1, W_2))^{1/2}$, so we see the degenerate U-process is bounded by $CJ_2(1, \mathcal{F})\sqrt{\mathbb{P}F(W_1, W_2)^2}n^{-1/2}$.

We apply Theorem 2.14.1 of van der Vaart and Wellner [1996] to see that $(n - 1)n^{-3/2}(\mathbb{P}_n \otimes \mathbb{P})(f)$ is upper bounded by $CJ(1, \mathbb{P}\mathcal{F})\sqrt{\mathbb{P}F_1(W)^2}$, which completes the proof.

In addition to the above result, which applies nicely to order 2 U-processes, we also rely on results from Arcones and Giné [1993] which apply to higher order U-processes (see also Sherman [1994]). We provide a maximal inequality derived from the results of Arcones and Giné [1993] in Proposition K.2. We consider U-processes of order 3 in Lemma C.3.

Let us start by considering the term $D_3$ from (E.15). We can write $D_3$ as

$$\mathbb{P}_n\{\xi(Z; \hat{\pi}, \hat{\mu}) - \hat{\xi}(Z; \hat{\pi}, \hat{\mu})\} = \mathbb{P}_n\{\xi(Z; \hat{\pi}, \hat{\mu}) - \xi(Z; \hat{\pi}, \hat{\mu})\} + \mathbb{P}_n\{\xi(Z; \hat{\pi}, \hat{\mu}) - \hat{\xi}(Z; \hat{\pi}, \hat{\mu})\}. \quad \text{(K.4)}$$

The second term on right side of (K.4) can be written as an order 2 V-process, as it equals

$$\frac{1}{n} \sum_{i=1}^{n} \left[ \frac{Y_i - \hat{\mu}(L_i, A_i)}{\hat{\pi}(A_i|L_i)} \left\{ \frac{1}{n} \sum_{j=1}^{n} \hat{\pi}(A_i|L_j) - \int \hat{\pi}(A_i|l) dP(l) \right\} \right] + \left\{ \frac{1}{n} \sum_{j=1}^{n} \hat{\mu}(L_j, A_i) - \int \hat{\mu}(l, A_i) dP(l) \right\}. \quad \text{76}$$
This yields a V-process (indexed by $\pi \in \mathcal{F}_\pi$, $\mu \in \mathcal{F}_\mu$) based on the non-symmetric kernel

$$h_1(w_1, w_2) \equiv h_{1,\mu}(w_1, w_2) := \frac{y_1 - \mu(l_1, a_1)}{\pi(a_1|l_1)} \pi(a_1|l_2) + \tilde{\mu}(l_2, a_1),$$

where we let tilde $\tilde{}$ operate on any $\mu, \pi$ to yield $\tilde{\pi}(a_1|l_2) := \pi(a_1|l_2) - \mathbb{P}\pi(a_1|L)$, and $\tilde{\mu}(l_2, a_1) := \mu(l_2, a_1) - \mathbb{P}\mu(L, a_1)$. In Lemma C.1 we show that this V-process (i.e., the second term on right side of (K.4)) is $O_p(n^{-1/2})$. (This is indeed the order that would arise for a degenerate V-statistic (with fixed kernel) of order 2.) Similarly, V-process terms arise from the other remainder terms. The outline and explanation of where they arise is given in the proof outline in the main document Doss et al. [2023], after the statement of Theorem 3.1. Terms with order $\geq 3$ are handled by the following proposition. It uses results from the proof of Theorem 5.2 of Arcones and Giné [1993]. Recall that an order $m$ kernel $H(z_1, \ldots, z_m)$ for a U-statistic is maximally degenerate (for measure $P$) if integrating over any one of the $m$ arguments yields an identically zero function. The definition of $P$-measurable is given in Definition 2.3.3, page 110, of van der Vaart and Wellner [1996].

Let $U_m^n f := \frac{(n-m)!}{n!} \sum_{i_1 \neq i_2 \cdots \neq i_m} f(X_{i_1}, \ldots, X_{i_m})$, where $n!$ is the factorial of $n$, based on a $P$-i.i.d. sample of $X_i \in \mathcal{Z}$. Let $\| f \|_n := (U_m^n f^2)^{1/2}$ and recall the definition of $J_m(\delta, \mathcal{F}, L_2) \equiv J_m(\delta, \mathcal{F})$ given in (2.7).

**Proposition K.2.** Let $\mathcal{F}$ be a $P$-measurable class of maximally degenerate functions on a measurable product space $\mathcal{Z}^m$ with envelope $F$ satisfying $P^m F^2 < \infty$. Then

$$E\|n^{m/2}U_m^n f\|_F \lesssim E(J_m(\theta_n, \mathcal{F})\|F\|_n) \lesssim J_m(1, \mathcal{F})(P^m F^2)^{1/2}$$

where $\theta_n := \| f \|_n\|F\|/\|F\|_n$.

**Proof.** By the first lines of the proof of Theorem 5.2 of Arcones and Giné [1993], we have

$$E\|n^{m/2}U_m^n f\|_F \lesssim E \int_0^{\infty} (\log N(\varepsilon, \mathcal{F}, L_2(U_m^n)))^{m/2} d\varepsilon$$

where for $\delta > 0$, $\mathcal{F}_\delta := \{f - g : f, g \in \mathcal{F}, e_{P,m}(f, g) \leq \delta n^{-m/2(m+1)}\}$ and $e_{P,m}(f, g) := \|f - g\|_{L_2(P^m)}$.

Now replace $\mathcal{F}$ by $\mathcal{F} \cup \{0\}$ and so

$$E\|n^{m/2}U_m^n f\|_F \lesssim \int_0^{\|f\|_F} (\log 1 + N(\varepsilon, \mathcal{F}, L_2(U_m^n)))^{m/2} d\varepsilon,$$
by Propositions 2.1 and 2.6 in Arcones and Giné [1993] and the fact that $0 \in \mathcal{F} \cup \{0\}$ to replace $\mathcal{F}_0$ by $\mathcal{F} \cup \{0\}$ (see e.g. Corollary 2.2.8 of van der Vaart and Wellner [1996]). Then do a change of variable to see the previous expression equals

$$E \left( \|F\|_n \int_0^{\theta_n} \log^{m/2} 1 + N(\varepsilon\|F\|_n, \mathcal{F}, L_2(U^m_n)) d\varepsilon \right)$$

where $\theta_n := \|f\|_n \|F\|/\|F\|_n$. This gives the first inequality of the proposition. And then the previous expression is bounded above by

$$E\|F\|_n \sup_Q \int_0^1 (\log^{m/2} 1 + N(\varepsilon\|Q\|_2, \mathcal{F}, L_2(Q))) d\varepsilon.$$ 

By Jensen’s inequality, $E\|F\|_n \leq (P^m F^2)^{1/2}$ which gives the second inequality of the proposition.
L Proof of Theorem 3.3

The rough idea of the proof is that the oracle bootstrap is consistent, and then the bootstrap remainder terms (which we show to be asymptotically negligible) can either be dominated by, or analyzed in somewhat analogous fashion to, various remainder terms that arose in the analysis of $T_n$ (not bootstrap); sometimes relying on the fact that symmetrized (multiplier) terms are of the same order of magnitude as the non-symmetrized terms. Each bootstrap error term has three components, from the three summands $\hat{\xi}_i^* = \delta_i(\hat{\xi}_i - \mathbb{P}_n\hat{\xi}) + \mathbb{P}_n\hat{\xi}$. So the details are somewhat lengthy; we break the argument up into 6 steps again. The lengthiest computation is (again) in Step 6.

Proof of Theorem 3.3. Without loss of generality we take $w(a) \equiv 1$, which does not substantively modify the proof. Recall the setup: we let

$$T_n^* := n\sqrt{n} \int_A \left( \hat{\theta}^*(a) - \mathbb{P}_n^*\hat{\xi}^* \right)^2 da,$$

where $\mathbb{P}_n^*$ is the ‘empirical’ measure of $\{(\delta_i, \mathbb{Z}_i)\}_{i=1}^n$, with $\delta_i$ being i.i.d. Rademacher variables independent of $\{\mathbb{Z}_i\}_{i=1}^n$. We let

$$\hat{\xi}_i := \hat{\xi}(\mathbb{Z}_i; \hat{\pi}, \hat{\mu}) - \sum_{j=1}^n \hat{\xi}(\mathbb{Z}_j; \hat{\pi}, \hat{\mu})/n = \hat{\xi}_i - \mathbb{P}_n\hat{\xi},$$

$$\tilde{\xi}_i := \xi(\mathbb{Z}_i; \tilde{\pi}, \tilde{\mu}) - \sum_{j=1}^n \xi(\mathbb{Z}_j; \tilde{\pi}, \tilde{\mu})/n = \tilde{\xi}_i - \mathbb{P}_n\tilde{\xi},$$

and $\hat{\xi}_i^* = \delta_i\hat{\xi}_i$, and we also let $\tilde{\xi}_i^* := \delta_i\tilde{\xi}_i$. We let

$$\hat{\xi}_i^* := \hat{\xi}_i^* + \mathbb{P}_n\hat{\xi}(\mathbb{Z}), \quad \text{and} \quad \tilde{\xi}_i^* := \tilde{\xi}_i^* + \mathbb{P}_n\tilde{\xi}(\mathbb{Z}).$$

For notational ease let $W_{ha}(A) \equiv W_h(A-a) := g_{ha}^T \hat{D}_{ha}^{-1} g_{ha}(A)K_{ha}(A)$. We let

$$\hat{\theta}^*(a) := \mathbb{P}_n^* W_h(A-a)(\delta(\hat{\xi} - \mathbb{P}_n\hat{\xi}) + \mathbb{P}_n\hat{\xi}) = \mathbb{P}_n^* W_h(A)\hat{\xi}_i,$$

$$\tilde{\theta}^*(a) := \mathbb{P}_n^* W_h(A-a)(\delta(\tilde{\xi} - \mathbb{P}_n\tilde{\xi}) + \mathbb{P}_n\tilde{\xi}) = \mathbb{P}_n^* W_h(A)\tilde{\xi}_i.$$

We can now decompose $T_n^*$ (as we decomposed $T_n$ in the proof of Theorem 3.1), writing

$$\hat{\theta}^*(a) - \mathbb{P}_n\hat{\xi}^* = D_1^*(a) + D_2^*(a) + D_3^*,$$
with
\[ D_1^*(a) := \tilde{\theta}^*(a) - \tilde{\theta}^*(a), \quad D_2^*(a) := \tilde{\theta}^*(a) - \mathbb{P}_n^* \tilde{\xi}^* , \quad \text{and} \quad D_3^* := \mathbb{P}_n^* \tilde{\xi}^* - \mathbb{P}_n^* \tilde{\xi}^* . \]

We consider the terms \( \int (D_1^*)^2(a)da, \int D_1^*(a)D_2^*(a)da, \int (D_2^*)^2(a)da, \int D_1^*(a)D_3^*(a)da, \int D_2^*(a)D_3^*(a)da, \) and \( (D_3^*)^2. \) Except for the \( (D_2^*)^2 \) term, we will show the rest are \( o_P(1/\sqrt{n\sqrt{h}}) \) (unconditionally, since no conditional argument is needed for negligible remainder terms by the definition of convergence in probability to 0 in the (Dudley) metric \( d(.,.) \)).

**Step 1** \( ((D_2^*)^2) \). By Theorem 2 of Härdle and Mammen [1993] (in combination with the proof of Theorem 2.1 of Alcalá et al. [1999]), we have \( d(\mathcal{L}^*(n\sqrt{h}\int_A (D_2^*(a))^2da), \mathcal{L}(N(b_h,V))) \rightarrow_p 0 \) as \( n \rightarrow \infty \).

**Step 2** \( ((D_3^*)^2) \). We have \( D_3^* = \mathbb{P}_n^*(\tilde{\xi}^* - \tilde{\xi}^*) \) which equals
\[
(\mathbb{P}_n^* \tilde{\xi} - \mathbb{P}_n^* \tilde{\xi}) + \mathbb{P}_n^* \delta(\tilde{\xi} - \tilde{\xi} + \mathbb{P}_n(\tilde{\xi} - \tilde{\xi})).
\]

Note that \( \mathbb{P}_n^* \tilde{\xi} - \mathbb{P}_n^* \tilde{\xi} = D_3 \), which was shown in the proof of Theorem 3.1 to be \( o_P(1/\sqrt{n\sqrt{h}}) \). This also shows that \( \mathbb{P}_n^* \delta(\mathbb{P}_n^* \tilde{\xi} - \mathbb{P}_n^* \tilde{\xi}) = o_P(1/\sqrt{n\sqrt{h}})O_p(n^{-1/2}) = o_P(1/\sqrt{n\sqrt{h}}). \)

It remains to analyze \( \mathbb{P}_n^* \delta(\tilde{\xi} - \tilde{\xi}) \), the symmetrized version of \( D_3 \). As in the analysis of \( D_3 \), we decompose this into an empirical process and a V-process. The symmetrized V-process that arises is seen to be \( O_p(n^{-1/2}) \) because, by Arcones and Giné [1993] (see page 1509 and the argument there), it is of the same order of magnitude as the non-symmetrized V-process studied (and seen to be \( O_p(n^{-1/2}) \)) in Lemma C.1.

The empirical process term is then broken into two terms, where we write
\[
\mathbb{P}_n^* \delta(\xi(Z;\tilde{\eta}) - \xi(Z;\tilde{\eta})) = \frac{1}{n} \sum_{i=1}^{n} \delta_i \left[ (\xi(Z;\tilde{\eta}) - \xi(Z;\tilde{\eta})) - \mathbb{P}(\xi(Z;\tilde{\eta}) - \xi(Z;\tilde{\eta})) \right]
+ \frac{1}{n} \sum_{i=1}^{n} \delta_i \mathbb{P}(\xi(Z;\tilde{\eta}) - \xi(Z;\tilde{\eta}))
\]
(recall \( \eta \) represents the two nuisance parameters). The first term is centered and so Lemma 2.3.6 of van der Vaart and Wellner [1996] (which gives moment bounds of a Rademacher symmetrized mean zero process in terms of the unsymmetrized process) applies, which, together with the analysis of \( D_3 \) in the proof of Theorem 3.1 (see Lemma E.1), shows that
the first sum on the right in the display above is $O_p(n^{-1/2} + s_n^\infty r_n^\infty)$. The remaining term, $\sum_{i=1}^n n^{-1} \delta_i P(\xi(Z; \eta) - \xi(Z; \eta))$, is $o_p(n^{-1/2}) = o_p((n^{1/2}r_n)^{-1/2})O_p(n^{-1/2})$, again by (the proof of) Lemma E.1.

**Step 3** ($D_1^2$). We can decompose $D_1^2(a) = \hat{\theta}^*(a) - \tilde{\theta}^*(a)$ into the sum of the following three summands:

\[
\begin{align*}
P_n^* W_{ha}(A) \delta(\tilde{\xi} - \xi), & \quad \text{(L.1)} \\
-P_n^* (W_{ha}(A) \delta \hat{P}_n(\tilde{\xi} - \xi)), & \quad \text{(L.2)} \\
P_n^* W_{ha}(A) (\hat{P}_n(\tilde{\xi} - \xi) - \hat{P}_n(\xi - \tilde{\xi})). & \quad \text{(L.3)}
\end{align*}
\]

The term in (M.4) is negligible from Step 2 of the proof of Theorem 3.1. The term (M.3) is similarly negligible.

Finally we consider (M.2). Like in the proof of Theorem 3.1 we break this into a (standard) empirical process term and a V-process term. For the former term, we apply Lemma 2.3.6 of van der Vaart and Wellner [1996], which bounds the expectation of the empirical process implied by (M.2) by the expectation of the empirical process implied by $P_n W_{ha}(A)(\tilde{\xi} - \xi)$, which was shown to be negligible in Step 3 of the proof of Theorem 3.1. Analogously, for the V-process term, we use results of Arcones and Giné [1993] (see page 1509 and the argument there), together with the argument in Step 3 of the proof of Theorem 3.1 for the analogous V-process (without the $\delta$ multiplier) to see that term is also negligible. Thus, we conclude that (M.2) is negligible.

**Step 4** ($D_2^* D_3^*$). From Part 2, we have that $D_3^* = o_p(1/\sqrt{n/v})$, so to analyze $\tilde{D}_3 \int_A D_2^*(a) da$ it remains to analyze $\int_A D_2^*(a) da$. This reduces to analyzing $\int_A (\hat{\theta}^*(a) - \hat{P}_n(a)) da$. By the proof of Lemma E.3, $\int (\hat{\theta}^*(a) - \hat{P}(\tilde{\xi})) da = n^{-1} \sum_{i=1}^n \delta_i (\omega_0^{-1}(A_i) + O(h)) (\tilde{\xi}_i - \hat{P}(\tilde{\xi}))$. Then by the (classical) Central Limit Theorem, we conclude that $\int_A D_2^*(a) da = O_p(n^{-1/2})$, and thus $D_3^* \int_A D_2^*(a) da = o_p(1/nh^{1/4})$.

**Step 5** ($D_1^* D_3^*$). We can see that $D_3^* \int_A D_1^*(a) da = o_p(1/\sqrt{n/v})$ by the results of Step 2 and Step 3.

**Step 6** ($D_1^2 D_2^2$). We now analyze $\int_A D_1^*(a) D_2^*(a) da$. Write $\tilde{\theta}^*_h(a) - P_n^* \tilde{\xi}^* = \tilde{\theta}^*_h - P(\xi) + P(\xi) - P_n^* \tilde{\xi}^*$. We have $P_n^* \tilde{\xi}^* = n^{-1} \sum_{i=1}^n (\delta_i (\tilde{\xi}_i - \theta_h(A_i)) + P_n(\xi))$ where $P_n \tilde{\xi} \to \hat{P} \xi$ almost surely. Let $E^*$ and $\text{Var}^*$ denote mean and variance.
conditional on the data (so, averaging over \( \{ \delta_i \}_{i} \)). Then
\[
E^* \left( n^{-1} \sum_{i=1}^{n} \delta_i (\tilde{\xi}_i - P_n \tilde{\xi}) \right) = 0,
\]
\[
\text{Var}^* \left( n^{-1} \sum_{i=1}^{n} \delta_i (\tilde{\xi}_i - P_n \tilde{\xi}) \right) = n^{-2} \sum_{i=1}^{n} (\tilde{\xi}_i - P_n \tilde{\xi})^2.
\]

We can see that \( n^{-2} \sum_{i=1}^{n} (\tilde{\xi}_i - P_n \tilde{\xi})^2 \to_p \text{Var}(\tilde{\xi}(Z)) \) as \( n \to \infty \) under \( H_0 \).

Thus, \( P^*_n \tilde{\xi} - P \tilde{\xi} \) reduces to \( P_n \tilde{\xi} - P \tilde{\xi} = D_3 \) which by the proof of Theorem 3.1 is \( o_p(1/\sqrt{n\sqrt{h}}) \).

Thus as in Step 5 we consider
\[
\int_{A} (\tilde{\theta}_h^*(a) - \widetilde{\theta}_h^*(a)) (\tilde{\theta}_h^*(a) - P \tilde{\xi}) \, da,
\]
which, after ignoring the \( \int_{A} (P_n (\tilde{\xi} - \tilde{\xi})) (\tilde{\theta}_h^*(a) - P \tilde{\xi}) \, da = D_3 \int_{A} D_2^*(a) \, da \)

summand which has already been shown to be negligible (in Step 4), equals
\[
\int_{A} n^{-1} \sum_{i=1}^{n} W_{ha}(A_i) \delta_i (\tilde{\xi}_i - \tilde{\xi}) + P_n (\tilde{\xi} - \tilde{\xi})) (\tilde{\theta}_h^*(a) - P \tilde{\xi}) \, da. \tag{L.5}
\]
The term
\[
\int_{A} n^{-1} \sum_{i=1}^{n} W_{ha}(A_i) \delta_i (\tilde{\xi}_i - \tilde{\xi}) (\tilde{\theta}_h^*(a) - P \tilde{\xi}) \, da
\]
is again decomposed (analogously to the decompositions in (B.3), (C.2)); we get two V-processes, of second and third order, as in Step 6 of the proof of Theorem 3.1. The two terms are handled in a similar fashion as previously, using Proposition K.2.

Then the remaining term,
\[
\int_{A} n^{-1} \sum_{i=1}^{n} W_{ha}(A_i) \delta_i (P_n (\tilde{\xi} - \tilde{\xi})) (\tilde{\theta}_h^*(a) - P \tilde{\xi}) \, da,
\]
can be studied in analogous fashion to the study of \( \int_{A} D_2(a) \, da \) in Lemma E.3, except that a V-statistic arises instead of an average, as follows. We write the term in the above display as \( D_3 \int_{A} n^{-1} \sum_{i=1}^{n} W_{ha}(A_i) \delta_i (\tilde{\theta}_h^*(a) - P \tilde{\xi}) \, da \), which is
\[
D_3 n^{-2} \sum_{i=1}^{n} \int_{A} W_{ha}(A_i) \delta_i \left( \sum_{j=1}^{n} W_{ha}(A_j) (\delta_j \tilde{x}_j + P \tilde{\xi} - P \tilde{\xi}) \right) \, da.
\]

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By the property of local polynomial equivalent kernels (see (3.12) page 63 Fan and Gijbels [1996]) that says that they preserve identically polynomials, we have \( \sum_{j=1}^{n} W_{ha}(A_j) \mathbb{P} \xi = \mathbb{P} \tilde{\xi} \) and so the above display reduces to

\[
D_3 \left( n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{i} \delta_{j} \tilde{\varepsilon}_{j} \int_{A} W_{ha}(A_i) W_{ha}(A_j) \, da \right).
\]

The integral \( \int_{A} W_{ha}(A_i) W_{ha}(A_j) \, da \) is of order 1/h (use the representation of the first order equivalent kernel, page 63 Fan and Gijbels [1996], and do a change of variables, say \( u = (A_i - a)/h \)). This is sufficient to conclude that the V-statistic (not process) term inside parentheses in the above display is order \( 1/nh \) [van der Vaart, 1998, Chapter 12]. Multiplying by \( D_3 = O_p(r_n^{\infty} s_n^{\infty}) = o_p((nh^{1/2})^{-1/2}) \), we see that the entire term is negligible. This was the last term in Step 6 and so we have completed the proof.

\[\Box\]

M Proof of Theorem 3.4

Here we provide (only) the modifications to the proof of Theorem 3.3 needed to prove Theorem 3.4. The difference is in the definition of the \( \varepsilon \) variables, which are now defined to be

\[
\hat{\varepsilon}_i := \hat{\xi}(Z_i; \hat{\pi}, \hat{\mu}) - \hat{\theta}(A_i),
\]
\[
\tilde{\varepsilon}_i := \xi(Z_i; \hat{\pi}, \hat{\mu}) - \tilde{\theta}(A_i).
\]

**Proof of Theorem 3.4.** Again the proof is divided into 6 steps.

**Step 1** \(((D_2^*)^2)\). As previously, by Theorem 2 of Härdle and Mammen [1993] (in combination with the proof of Theorem 2.1 of Alcalá et al. [1999]), we have \( d(L^*(n \sqrt{h} \int_{A}(D_2^*)^2) da), L(N(b_h, V)) \) \( \rightarrow_p 0 \) as \( n \rightarrow \infty \).

**Step 2** \(((D_3^*)^2)\). We have \( D_3^* = \mathbb{P}_n^*(\hat{\xi}^* - \tilde{\xi}^*) \) which equals

\[
(\mathbb{P}_n \hat{\xi}(Z) - \mathbb{P}_n \tilde{\xi}(Z)) + \mathbb{P}_n^* \delta(\hat{\xi}(Z) - \tilde{\xi}(Z) + \hat{\theta}(A) - \tilde{\theta}(A)).
\]

The term \( \mathbb{P}_n^* \delta(\hat{\theta}(A) - \tilde{\theta}(A)) \) is the only one that is not analyzed and shown to be negligible in the proof of Theorem 3.3. Recall that (by definition) \( D_1(A) = \hat{\theta}(A) - \tilde{\theta}(A) \) and we can use the decomposition we have used
previously for $D_1$, namely,

$$D_1(A_i) = \frac{1}{n} \sum_{j=1}^{n} W_h(A_i - A_j)(\tilde{\xi}_j - \hat{\xi}_j) = d_{1,1}(A_i) + d_{1,2}(A_i)$$

$$= d_{1,1}(A_i) + R_{n,1,A_i} + R_{n,2,A_i} \tag{M.1}$$

(recall the definitions of $d_{1,1}$, $d_{1,2}$ in (B.5) and the definitions of $R_{n,i,a}$ in (E.5)). (Again, because of the integral over $A$, we cannot analyze the terms separately.) In Lemmas C.2 and E.2 we have shown $\sup_{a \in A}$ bounds for $d_{1,1}(a)$ and $R_{n,2,a}$, and those thus show that the corresponding sums here, $P_n^* \delta d_{1,1}(A)$ and $P_n^* \delta R_{n,2,A}$, are asymptotically negligible. For $R_{n,1,A_i}$, the treatment now is slightly different than in Lemma E.2. Previously $R_{n,1,a}$ was treated as an empirical (order 1 V-) process whereas now we treat $P_n^* \delta R_{n,1,A}$ as an order 2 (degenerate) V-process. The envelope and entropy calculations from the proof of Lemma E.2 still apply almost verbatim (the presence of $\delta$ changes neither), and then we apply Proposition K.1 (or Proposition K.2).

We thus conclude that $D_3^* = o_p(1/\sqrt{n\sqrt{h}})$.

**Step 3** ($(D_1^*)^2$). We consider $\int_A D_1^*(a)^2 da$; we can decompose $D_1^*(a) = \hat{\theta}^*(a) - \tilde{\theta}^*(a)$ into the sum of the following three summands:

$$P_n^* W_h(A) \delta(\tilde{\xi} - \hat{\xi}), \tag{M.2}$$

$$-P_n^* (W_h(A) \delta(\hat{\theta}_h(A) - \tilde{\theta}_h(A))), \tag{M.3}$$

$$P_n^* W_h(A) (\hat{\theta}(A) - \tilde{\theta}(A)), \tag{M.4}$$

and need to analyze their squared integrals (which suffices by Cauchy-Schwarz). The term (M.3) is the one that changed from the previous proof. The integral of that term squared is

$$\int_A \left[ n^{-2} \sum_{i=1}^{n} (\sum_{j=1}^{n} W_h(A_i - A_j)(\tilde{\xi}_j - \hat{\xi}_j)) W_h(A_i) \delta_i \right]^2 da. \tag{M.5}$$

The term (M.5) is asymptotically negligible (is $o_p(1/\sqrt{n\sqrt{h}})$) which follows a somewhat similar recipe as used in the previous step. We again use a decomposition as in (M.1) for $D_1(A) = \hat{\theta}_h(A) - \tilde{\theta}_h(A)$. Since $(b + c)^2 \leq
max(4b^2, 4c^2), it suffices to bound each of

\[ \int_{\mathcal{A}} [P_n^*(W_{ha}(A)\delta d_{1,1}(A))]^2 \, da \]  
\[ \int_{\mathcal{A}} [P_n^*(W_{ha}(A)\delta R_{n,1,A})]^2 \, da, \]  
\[ \int_{\mathcal{A}} [P_n^*(W_{ha}(A)\delta R_{n,2,A})]^2 \, da. \]

The handling of these three terms is somewhat analogous to what was done in the previous step. Again, by the proofs of Lemmas C.2 and E.2, we have \( \sup_{a \in \mathcal{A}} \) bounds for \( |d_{1,1}(a)| \) and \( |R_{n,2,a}| \) of order \( o_P(1/\sqrt{n\sqrt{h}}) \) which allows us to see that (M.6) and (M.8) are also \( O_P(1/n) \). For the term (M.7), we slightly extend the proof of Lemma E.2 (to include the \( W_{ha}(A)\delta \) terms in the corresponding function class and treat the term as an order 2 (degenerate) V-process) to see that the term inside the integrand has finite second moment and thus (by interchanging the integral and expectation) that the entire term is negligible.

**Step 4 (\( D_2^* D_3^* \)).** No new argument is needed here (since we rely in part on the result from Part 2 of the current proof, and because \( D_2^* \) is defined via oracle values).

**Step 5 (\( D_1^* D_3^* \)).** Again, we can see that \( D_3^* \int_{\mathcal{A}} D_1^*(a) \, da = o_P(1/\sqrt{n\sqrt{h}}) \) by the results of Step 2 and Step 3.

**Step 6 (\( D_1^* D_2^* \)).** We now analyze \( \int_{\mathcal{A}} D_1^*(a) D_2^*(a) \, da \). The analysis introduces a fourth order V-process and so we rely on the finiteness of the \( J_4 \) entropy term. As in Step 6 of the proof of Theorem 3.3 we consider

\[ \int_{\mathcal{A}} (\tilde{\theta}_h^*(a) - \delta^*_h(a))(\tilde{\theta}_h^*(a) - \mathbb{P}\tilde{\xi}) \, da, \]

which is again reduced to

\[ \int_{\mathcal{A}} n^{-1} \sum_{i=1}^{n} W_{ha}(A_i) \delta_i(\tilde{\xi}_i - \xi_i - (\tilde{\theta}_h(A_i) - \tilde{\theta}_h(A_i)))(\tilde{\theta}_h^*(a) - \mathbb{P}\tilde{\xi}) \, da. \]

The only term not already handled in the proof of Theorem 3.3 is

\[ \int_{\mathcal{A}} n^{-1} \sum_{i=1}^{n} W_{ha}(A_i) \delta_i(\tilde{\theta}_h(A_i) - \tilde{\theta}_h(A_i)))(\tilde{\theta}_h^*(a) - \mathbb{P}\tilde{\xi}) \, da, \]
which (somewhat in parallel to (M.5)) is reduced to (after decomposing 
\( \tilde{\theta}_h(a) - P\tilde{\xi} = \mathbb{P}_n \left[ W_h(a)(\delta(\tilde{\xi} - \tilde{\theta}_h(a)) + \tilde{\theta}_h(a)) \right] - P\tilde{\xi} \) and ignoring the smaller \( \tilde{\theta}_h(a) - P\tilde{\xi} \) term) the expression

\[
\frac{1}{n} \sum_{i=1}^{n} \left[ (\sum_j W_h(A_i - A_j)(\tilde{\xi}_j - \tilde{\xi}_j)) \int_A W_h(A_i) \delta_i \left( \sum_k W_h(A_k) \delta_k \tilde{\varepsilon}_k \right) da \right].
\]

We again use the decomposition given in (M.1). Previously, \( d_{1,1} \) yielded V-processes (one for each of \( \pi, \mu \), \( R_{n,1,a} \) corresponded to an empirical process expression and \( R_{n,2,a} \) corresponded to a ‘second order remainder’ type of term (handled in different ways previously). From (M.1) we decompose (M.11) into a sum of three terms,

\[
\begin{align*}
&n^{-2} \sum_{i=1}^{n} \left[ d_{1,1}(A_i) \int_A W_h(A_i) \delta_i \left( \sum_k W_h(A_k) \delta_k \tilde{\varepsilon}_k \right) da \right], \quad (M.12) \\
&n^{-2} \sum_{i=1}^{n} \left[ R_{n,1,A_i} \int_A W_h(A_i) \delta_i \left( \sum_k W_h(A_k) \delta_k \tilde{\varepsilon}_k \right) da \right], \quad \text{and} \quad (M.13) \\
&n^{-2} \sum_{i=1}^{n} \left[ R_{n,2,A_i} \int_A W_h(A_i) \delta_i \left( \sum_k W_h(A_k) \delta_k \tilde{\varepsilon}_k \right) da \right]. \quad (M.14)
\end{align*}
\]

The terms (M.13) and (M.14) are handled in a fashion analogous to the handling of the “\( R_{n,1,a} \)” and “\( R_{n,2,a} \)” type terms previously (see Lemmas E.2 and E.4) although with one more summation so yielding V-processes of order 3 and 2 with modified/extended function classes from the ones considered previously.

The term (M.12) (as in previous analyses of the “\( d_{1,1} \)” term”), is a sum of V-processes for \( \mu \) and for \( \pi \) which are handled similarly. Consider the case of \( \mu \) (with the \( \pi \) term being analogous). Let \( W_i = (\delta_i, Z_i) \) and then define the V-process ‘kernel’ \( H \equiv H_{\mu} \) by

\[
H(W_1, \ldots, W_4) := \int_A \tilde{\mu}(L_4, A_2) \delta_1 W_h(A_1) W_h(A_1 - A_2) \delta_3 W_h(A_3) \tilde{\varepsilon}_3 da,
\]

where recall \( \tilde{\varepsilon} := \xi(Z; \tilde{\eta}) - P\xi(Z; \tilde{\eta}) \). (Recall we let tilde \( \tilde{\cdot} \) operate on any \( \mu, \pi \) to yield \( \tilde{\pi}(a_1|l_2) := \pi(a_1|l_2) - P\pi(a_1|L) \), and similarly \( \tilde{\mu}(l_2, a_1) := \mu(l_2, a_1) - P\mu(L, a_1) \).)

Recall the discussion of V- or U-statistics and -processes from Section K; We do not repeat that discussion here, except to recall the intuition that a
V-statistic of order $k$ that is maximally degenerate (i.e., degenerate of degree $k-1$ meaning that averaging over any 1 argument yields 0) and with a kernel not changing with $n$ can be expected to be of order $n^{-k/2}$ [van der Vaart, 1998, Chapter 12]. V-statistics or V-processes have a so-called Hoeffding decomposition, and the least degenerate term generally governs the size of a V-statistic. Now we let $G_{1,S} \equiv G_{1,S,\mu}$, for a set $S \subset \{1, 2, 3, 4\}$ be the function given by averaging over independent copies of the $\{1, 2, 3, 4\} \setminus S$ variables in $H$, where the (slightly redundant) 1 indicates $S$ has size 1. So, e.g., $G_{1,\{1\}}(w_1) := \mathbb{P}^3 H(w_1, W_2, W_3, W_4)$. Then it is quick to check that

$$G_{1,\{i\}} \equiv 0 \text{ for } i \in \{1, 2, 3, 4\}.$$ 

Similarly, because averaging over any of the $1, 3, 4$ coordinates yields 0 (because of say $\delta_1, \delta_3, \mu(L_4, A_2)$ (where $\mu$ is centered on its first argument), $G_{2,\{i_1, i_2\}} \equiv 0$ (for any $i_1 \neq i_2$). Then $G_{3,\{i_1, i_2, i_3\}}$ is only nonzero for $\{i_1, i_2, i_3\} = \{1, 3, 4\}$ in which case

$$G_{3,\{1,3,4\}}(w_1, w_3, w_4)
= \delta_3 \varepsilon_3 \delta_1 (\mathbb{P} \mu(l_4, A_2) W_{h_0}(A_2)) \int_{\mathcal{A}} W_h(a - a_1) W_h(a - a_3) da.$$

Thus, the only nondegenerate terms to consider arise from the (order 3, maximally degenerate) kernel $G_{3,\{1,3,4\}}(w_1, w_3, w_4)$ and the (order 4, maximally degenerate) kernel $(w_1, w_2, w_3, w_4) \rightarrow H(w_1, w_2, w_3, w_4) - G_{3,\{1,3,4\}}(w_1, w_3, w_4)$.

Both are maximally degenerate (averaging over any coordinate yields the zero function). (Typically, if the kernel $H$ did not depend on $n$ (via $h \equiv h_n$), the orders of magnitude would thus be $O_p(n^{-3/2})$ and $O_p(n^{-4/2})$; as we will see, indeed the first term is of larger size.)

We can see that (with a change of variables) $\mathbb{P} \mu(l_4, A_2) W_h(a_1 - A_2) \leq 2 \| \varpi \|_\infty \| \mu \|_\infty$ where $\| \mu \|_\infty$ is uniformly bounded by assumption, and where the right side does not depend on $h$. The term $\int_{\mathcal{A}} W_h(a - a_1) W_h(a - a_3) da$, after change of variables, is seen to be of order $O(1/h)$. Thus we have an envelope for the class of functions $G_{3,\{1,3,4\}}(w_1, w_3, w_4)$ of size $O(1/h)$. The class of functions $\{H\}$ has an envelope of size $O(1/h^2)$ (after just doing one change of variables).

We now note that $J_4(1, \mathcal{H}, L_2) < \infty$ where $\mathcal{H} := \{H_\mu : \mu \in \mathcal{F}_\mu\}$; by Lemma A.3 (and Lemma A.4) this will also show that $J_3(1, \{G_{3,\{1,3,4\},\mu} : \mu \in \mathcal{F}_\mu\}, L_2) < \infty$. That $J_4(1, \mathcal{H}, L_2) < \infty$ follows from the assumption that $J_4(1, \mathcal{F}_\mu, L_2) < \infty$ (i.e., by Assumption E(B)4) and from the preservation arguments we have made previously (see e.g., the proof of Lemma C.3). Thus the larger term is $O_p(1/hn^{-3/2})$ (and the smaller bounded by $O_p(1/h^2n^2)$).
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