Marcinkiewicz–Zygmund Strong Law of Large Numbers
for Pairwise i.i.d. Random Variables

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Abstract

It is shown that the Marcinkiewicz–Zygmund strong law of large numbers holds for pairwise independent identically distributed random variables. It is proved that if $X_1, X_2, \ldots$ are pairwise independent identically distributed random variables such that $E|X_1|^p < \infty$ for some $1 < p < 2$, then $(S_n - ES_n)/n^{1/p} \to 0$ a.s. where $S_n = \sum_{k=1}^n X_k$.

Keywords: strong law of large numbers, pairwise independent random variables, identically distributed random variables.

1. Introduction.

Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent identically distributed random variables. There are two famous theorems on the strong law of large numbers for such a sequence: The Kolmogorov theorem and the Marcinkiewicz–Zygmund theorem (see e.g. Loève [4]). Let $S_n = \sum_{k=1}^n X_k$. By Kolmogorov’s theorem, there exists a constant $b$ such that $S_n/n \to b$ a.s. if and only if $E|X_1| < \infty$; if the latter condition is satisfied then $b = EX_1$.

Now we state the Marcinkiewicz–Zygmund theorem:

Theorem A. Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent identically distributed random variables. If $0 < p < 2$ then the relation $E|X_1|^p < \infty$ is equivalent to the relation

$$\frac{S_n - nb}{n^{1/p}} \to 0 \quad a.s.$$  \hspace{1cm} (1)

Here $b = 0$ if $0 < p < 1$, and $b = EX_1$ if $1 \leq p < 2$.

The aim of this work is to show that Theorem A remains true if we replace the independence condition by the condition of pairwise independence of random variables $X_1, X_2, \ldots$.

Etemadi [2] proved the Kolmogorov theorem under the pairwise independence assumption instead of the independence condition. Sawyer [7] showed that if $0 < p < 1$ then the condition $E|X_1|^p < \infty$ implies $S_n/n^{1/p} \to 0$ a.s. without any independence condition. Petrov [6] proved that if $0 < p < 2$ then relation (1) (with $b = 0$ or $EX_1$ according as $p < 1$ or $p \geq 1$) implies that $E|X_1|^p < \infty$ assuming pairwise independence.

In the present work we shall prove that if $1 < p < 2$ then the condition $E|X_1|^p < \infty$ implies $(S_n - ES_n)/n^{1/p} \to 0$ a.s. under the pairwise independence assumption. There are

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a number of papers that contain results on the strong law of large numbers for sequences of pairwise independent identically distributed random variables. See Choi and Sung [1], Li [3], Martikainen [5], Sung [8, 9] (recent work [9] contains more detailed review). However, results in these papers do not generalize Theorem A to sequences of pairwise independent random variables.

2. Main results.

The aim of this paper is to prove the following result:

**Theorem 1.** Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of pairwise independent identically distributed random variables. If \( E|X_1|^p < \infty \) where \( 1 < p < 2 \), then

\[
\frac{S_n - ES_n}{n^{1/p}} \to 0 \quad \text{a.s.} \quad (2)
\]

If we combine Etemadi’s, Sawyer’s, and Petrov’s results mentioned in the previous section with Theorem A we get a generalization of the Marcinkiewicz–Zygmund theorem (Theorem A):

**Theorem 2.** Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of pairwise independent identically distributed random variables. If \( 0 < p < 2 \) then the relation \( E|X_1|^p < \infty \) is equivalent to relation (1).

3. Proof of Theorem 1

To prove our main result we need the following lemmas.

**Lemma 1.** Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of identically distributed random variables. If \( E|X_1|^p < \infty \) where \( 1 < p < 2 \), then

\[
\sum_{i=1}^{n} |X_i| I_{\{|X_i| > n^{1/p}\}} \to 0 \quad \text{a.s.} \quad (3)
\]

**Proof.** Let \( U_n = |X_n|^p I_{\{|X_n| > n^{1/p}\}} \), \( n \geq 1 \). Note that condition \( E|X_1|^p < \infty \) is equivalent to the relation

\[
\sum_{n=1}^{\infty} P(|X_1| > n^{1/p}) < \infty. \quad (4)
\]

Thus, we have

\[
\sum_{n=1}^{\infty} P(U_n \neq 0) = \sum_{n=1}^{\infty} P(|X_n| > n^{1/p}) = \sum_{n=1}^{\infty} P(|X_1| > n^{1/p}) < \infty.
\]

Therefore, by Borel–Cantelli lemma,

\[
U_n \to 0 \quad \text{a.s.} \quad (5)
\]

Moreover

\[
\frac{\sum_{i=1}^{n} |X_i|^p I_{\{|X_i| > n^{1/p}\}}}{n^{1/p}} \leq \frac{\sum_{i=1}^{n} |X_i|^p I_{\{|X_i| > n^{1/p}\}}}{n} \leq \frac{\sum_{i=1}^{n} |X_i|^p I_{\{|X_i| > n^{1/p}\}}}{n}.
\] (6)

By (5) the right-hand side of (6) converges to zero almost sure and relation (3) follows. \[\square\]
Lemma 2. Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of identically distributed random variables. If \( E|X_1|^p < \infty \) where \( 1 < p < 2 \), then

\[
\sum_{i=1}^{n} E(|X_i|I_{|X_i| > n^{1/p}}) \to 0 \quad (n \to \infty).
\] (7)

Proof. Note that for any non-negative random variable \( \xi \) and \( a > 0 \),

\[
E(\xi I_{\xi > a}) = aP(\xi > a) + \int_{a}^{\infty} P(\xi > x) \, dx.
\]

Hence

\[
\sum_{i=1}^{n} E(|X_i|I_{|X_i| > n^{1/p}}) = \sum_{i=1}^{n} \left( n^{1/p} P(|X_i| > n^{1/p}) + \int_{n^{1/p}}^{\infty} P(|X_i| > x) \, dx \right) = nP(|X_1| > n^{1/p}) + n^{1/p} \int_{n^{1/p}}^{\infty} P(|X_1| > x) \, dx = I_{1n} + I_{2n}.
\] (8)

Using (4), we get

\[
I_{1n} = nP(|X_1| > n^{1/p}) \to 0 \quad (n \to \infty).
\] (9)

From obvious equality

\[
E|X_1|^p = p \int_{0}^{\infty} x^{p-1} P(|X_1| > x) \, dx
\] (10)

it follows that

\[
I_{2n} = n^{p-1} \int_{n^{1/p}}^{\infty} P(|X_1| > x) \, dx \leq \int_{n^{1/p}}^{\infty} x^{p-1} P(|X_1| > x) \, dx \to 0 \quad (n \to \infty),
\]

which, in conjunction with (8) and (9), proves (7).

Lemma 3. Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of identically distributed random variables. If \( E|X_1|^p < \infty \) where \( 1 < p < 2 \), then

\[
\sum_{n=1}^{\infty} \frac{1}{2^{n/p}} \sum_{k=1}^{2^n} E(|X_k|^2 I_{|X_k| \leq 2^{k/p}}) < \infty.
\] (11)

Proof. Note that for any non-negative random variable \( \xi \) and \( a > 0 \),

\[
E(\xi I_{\xi \leq a}) \leq \int_{0}^{a} P(\xi > x) \, dx.
\]

Hence, using (10), for some positive constants \( C \) and \( C_1 \), we obtain
\[ \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{2^n} P(|X_k| > x^{1/2}) dx \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{2^n} y P(|X_k| > y) dy \leq C \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{i=1}^{2^n} y P(|X_1| > y) dy \]

\[ \leq C_1 + C \sum_{n=1}^{\infty} 2^{\frac{n(n-2)}{p}} \sum_{i=1}^{2^n} y P(|X_1| > y) dy \leq C_1 + C \sum_{n=1}^{\infty} \int_{2^{n-1}}^{2^n} y P(|X_1| > y) dy \sum_{i=1}^{2^n} 2^{\frac{n(n-2)}{p}} \]

\[ \leq C_1 + C \sum_{n=1}^{\infty} \int_{2^{n-1}}^{2^n} y^{p-1} P(|X_1| > y) dy \cdot 2^{\frac{n(n-2)}{p}} \leq C_1 + C \sum_{n=1}^{\infty} \int_{n-1}^{n} y^{p-1} P(|X_1| > y) dy \leq C_1 + C \int_{0}^{\infty} y^{p-1} P(|X_1| > y) dy \leq C_1 + CE|X_1|^p < \infty, \]

and (11) follows.

**Lemma 4.** Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of pairwise independent identically distributed random variables. If \( E|X_1|^p < \infty \) where \( 1 < p < 2 \), then

\[ \sum_{i=1}^{2^n} |X_i| \mathbb{1}_{\{|X_i| \leq 2^n\}} \to 0 \quad \text{a.s.} \]  

(12)

**Proof.** By Chebyshev’s inequality, using Lemma 3 for \( \varepsilon > 0 \), we get
$$\sum_{n=1}^{\infty} P \left( \left| \sum_{i=1}^{2^n} |X_i| \mathbb{1}_{\{|X_i| \leq 2^{2^n} p\}} \right| > \varepsilon \right) \leq$$

$$\leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{\text{Var} \left( \sum_{i=1}^{2^n} |X_i| \mathbb{1}_{\{|X_i| \leq 2^{2^n} p\}} \right)}{2^{2^n} p} \leq$$

$$\leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{\text{Var}(|X_i| \mathbb{1}_{\{|X_i| \leq 2^{2^n} p\}})}{2^{2^n} p} \leq$$

$$\leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{2^{2^n} p} \sum_{i=1}^{2^n} E(|X_i|^2 \mathbb{1}_{\{|X_i| \leq 2^{2^n} p\}}) < \infty,$$

and so the desired conclusion (12) follows from Borel–Cantelli lemma. □

**Proof of Theorem 1.** Without loss of generality it can be assumed that $EX_1 = 0$. Let

$$X_i^{(n)} = X_i \mathbb{1}_{\{|X_i| \leq n^{1/p}\}}, \quad i \geq 1, \ n \geq 1,$$

$$S_j^{(n)} = \sum_{i=1}^{j} X_i^{(n)}; \quad j \geq 1, \ n \geq 1.$$

**Step 1.** Let us prove that

$$\frac{S_n - S_n^{(n)}}{n^{1/p}} \rightarrow 0 \quad \text{a.s.} \quad (13)$$

We have

$$|S_n - S_n^{(n)}| \leq \frac{\sum_{i=1}^{n} X_i \mathbb{1}_{\{|X_i| > n^{1/p}\}}}{n^{1/p}} \leq \frac{\sum_{i=1}^{n} |X_i| \mathbb{1}_{\{|X_i| > n^{1/p}\}}}{n^{1/p}}.$$

Application of Lemma 1 yields to (13).

**Step 2.** Let us show that

$$\frac{ES_n^{(n)}}{n^{1/p}} \rightarrow 0 \quad (n \rightarrow \infty). \quad (14)$$

We have

$$\frac{|ES_n^{(n)}|}{n^{1/p}} = \frac{\sum_{i=1}^{n} EX_i^{(n)}}{n^{1/p}} \leq \frac{\sum_{i=1}^{n} |EX_i^{(n)}|}{n^{1/p}} =$$

$$= \frac{\sum_{i=1}^{n} |E(X_i - X_i^{(n)})|}{n^{1/p}} \leq \frac{\sum_{i=1}^{n} E(|X_i|^2 \mathbb{1}_{\{|X_i| > n^{1/p}\}})}{n^{1/p}}.$$
The application of Lemma 2 yields to (14).

Now we note that to conclude the proof of the theorem, it is sufficiently to show that
\[
\frac{S_n^{(n)} - ES_n^{(n)}}{n^{1/p}} \to 0 \quad \text{a.s.} \quad (15)
\]

Step 3. Let us prove that
\[
\frac{S_{2^n}^{(2^n)} - ES_{2^n}^{(2^n)}}{2^{p^n}} \to 0 \quad \text{a.s.} \quad (16)
\]

Using Lemma 3 by Chebyshev’s inequality, for any \( \varepsilon > 0 \), we obtain
\[
\sum_{n=1}^\infty P \left( \left| \frac{S_{2^n}^{(2^n)} - ES_{2^n}^{(2^n)}}{2^{p^n}} \right| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \sum_{n=1}^\infty Var(S_{2^n}^{(2^n)}) = \frac{1}{\varepsilon^2} \sum_{n=1}^\infty \sum_{k=1}^{2^n} Var(X_k^{(2^n)}) \leq \frac{1}{\varepsilon^2} \sum_{n=1}^\infty \frac{n^{2/p}}{2^{p^n}} \sum_{k=1}^{2^n} E(|X_k|^{2/p}_{\{X_k \leq 2^{p^n}\}}) < \infty.
\]

Thus, by Borel-Cantelli lemma, we have that relation (16) is proved.

Step 4. Let us prove that
\[
\lim_{n \to \infty} \max_{2^n < k < 2^{n+1}} \left| \frac{\sum_{i=2^n+1}^{k} (X_i^{(i)} - EX_i^{(i)})}{2^{p^{n+1}}} \right| = 0 \quad \text{a.s.} \quad (17)
\]

For \( n \geq 1 \) and \( k \) such that \( 2^n < k \leq 2^{n+1} \) we have
\[
\left| \sum_{i=2^n+1}^{k} (X_i^{(i)} - EX_i^{(i)}) \right| = \left| \sum_{i=2^n+1}^{k} X_i^{(i)} - \sum_{i=2^n+1}^{k} EX_i^{(i)} \right| \leq \sum_{i=2^n+1}^{k} |X_i^{(i)}|_{\{X_i \leq 1\}} + \sum_{i=2^n+1}^{k} |EX_i^{(i)}|_{\{X_i \leq 1\}} \leq \sum_{i=2^n+1}^{k} |X_i^{(i)}|_{\{X_i \leq 1\}} + \sum_{i=2^n+1}^{k} |EX_i^{(i)}|_{\{X_i \leq 1\}} = \sum_{i=2^n+1}^{2^{n+1}} |X_i^{(i)}|_{\{X_i \leq i^{1/p}\}} + \sum_{i=2^n+1}^{2^{n+1}} |EX_i^{(i)}|_{\{X_i \leq i^{1/p}\}} \leq \sum_{i=2^n+1}^{2^{n+1}} |X_i^{(i)}|_{\{X_i \leq i^{1/p}\}} + \sum_{i=2^{n+1}}^{2^{n+1}} E(|X_i|^{p}_{\{X_i > 2^{p^n}\}}) \leq \sum_{i=1}^{2^{n+1}} |X_i^{(i)}|_{\{X_i \leq 2^{p^n}\}} + \sum_{i=1}^{2^{n+1}} E(|X_i|^{p}_{\{X_i > 2^{p^n}\}}).
\]
It follows that
\[
\max_{2^n < k \leq 2^{n+1}} \left| \sum_{i=2^n+1}^{k} (X^{(i)}_i - E[X^{(i)}_i]) \right| \leq \sum_{i=1}^{2^{n+1}} |X_i| I_{\{|X_i| \leq 2^{2^n} \}} + \sum_{i=1}^{2^n} E(|X_i| I_{\{|X_i| > 2^{2^n} \}}).
\]

The application of Lemmas 2 and 4 yields to (17).

**Step 5.** We shall prove that
\[
\lim_{n \to \infty} \max_{2^n < k \leq 2^{n+1}} \left| S_k^{(2^n)} - E[S_k^{(2^n)}] \right| = 0 \quad \text{a.s.} \quad (18)
\]
For \( n \geq 1 \) and \( k \) such that \( 2^n < k \leq 2^{n+1} \) we have
\[
\begin{align*}
|S_k^{(2^n)} - ES_k^{(2^n)}| &= |S_k^{(2^n)} - ES_k^{(2^n)} + S_k^{(2^n)} - S_k^{(2^n)} + ES_k^{(2^n)} - ES_k^{(2^n)}| = \\
&= |S_k^{(2^n)} - S_k^{(2^n)} - E(S_k^{(2^n)} - S_k^{(2^n)}) + (S_k^{(2^n)} - ES_k^{(2^n)})| \\
&\leq |(S_k^{(2^n)} - S_k^{(2^n)}) - E(S_k^{(2^n)} - S_k^{(2^n)})| + |(S_k^{(2^n)} - ES_k^{(2^n)})| = \\
&= \left| \sum_{i=2^{n+1}}^{k} X_i \mathbb{1}_{\{|X_i| \leq 2^{n}\}} - E\left( \sum_{i=2^{n+1}}^{k} X_i \mathbb{1}_{\{|X_i| \leq 2^{n}\}} \right) \right| + \left| (S_k^{(2^n)} - ES_k^{(2^n)}) \right| = \\
&= | \sum_{i=2^{n+1}}^{k} (X_i \mathbb{1}_{\{|X_i| \leq 2^{n}\}} - X_i \mathbb{1}_{\{2^{n} < |X_i| \leq 2^{n}\}}) - E\left( \sum_{i=2^{n+1}}^{k} (X_i \mathbb{1}_{\{|X_i| \leq 2^{n}\}} - X_i \mathbb{1}_{\{2^{n} < |X_i| \leq 2^{n}\}}) \right) | + \left| (S_k^{(2^n)} - ES_k^{(2^n)}) \right| = \\
&= | \sum_{i=2^{n+1}}^{k} (X_i \mathbb{1}_{\{|X_i| \leq 2^{n}\}} - E(X_i \mathbb{1}_{\{|X_i| \leq 2^{n}\}}) | + \left| (S_k^{(2^n)} - ES_k^{(2^n)}) \right| = \\
&\leq \left| \sum_{i=2^{n+1}}^{k} (X_i^{(i)} - EX_i^{(i)}) \right| + \left| \sum_{i=2^{n+1}}^{k} X_i \mathbb{1}_{\{2^{n} < |X_i| \leq 2^{n}\}} + E\left( X_i \mathbb{1}_{\{2^{n} < |X_i| \leq 2^{n}\}} \right) \right| + \\
&+ \left| (S_k^{(2^n)} - ES_k^{(2^n)}) \right| = \\
&\leq \left| \sum_{i=2^{n+1}}^{k} (X_i^{(i)} - EX_i^{(i)}) \right| + \left| \sum_{i=2^{n+1}}^{k} X_i \mathbb{1}_{\{2^{n} < |X_i| \leq 2^{n}\}} \right| + \\
&+ \left| (S_k^{(2^n)} - ES_k^{(2^n)}) \right| = \\
&\leq \left| \sum_{i=2^{n+1}}^{k} (X_i^{(i)} - EX_i^{(i)}) \right| + \left| \sum_{i=2^{n+1}}^{k} X_i \mathbb{1}_{\{|X_i| > 2^{n}\}} \right| + \\
&+ \left| (S_k^{(2^n)} - ES_k^{(2^n)}) \right| = \\
&\leq \left| \sum_{i=2^{n+1}}^{k} (X_i^{(i)} - EX_i^{(i)}) \right| + \left| \sum_{i=2^{n+1}}^{k} X_i \mathbb{1}_{\{|X_i| > 2^{n}\}} \right| + \\
&+ \left| (S_k^{(2^n)} - ES_k^{(2^n)}) \right| = \\
&\leq \left| \sum_{i=2^{n+1}}^{k} (X_i^{(i)} - EX_i^{(i)}) \right| + \left| \sum_{i=2^{n+1}}^{2^{n+1}} X_i \mathbb{1}_{\{|X_i| \leq 2^{n+1}\}} \right| + \\
&+ \left| (S_k^{(2^n)} - ES_k^{(2^n)}) \right| = \\
&\leq \left| \sum_{i=1}^{k} (X_i^{(i)} - EX_i^{(i)}) \right| + \left| \sum_{i=1}^{2^{n+1}} X_i \mathbb{1}_{\{|X_i| \leq 2^{n+1}\}} \right| + \\
&+ \left| (S_k^{(2^n)} - ES_k^{(2^n)}) \right|.
\end{align*}
\]
The application of Lemmas 2 and 4 and relations (16) and (17) yields to (18).

Therefore

\[
\max_{2^n < k \leq 2^{n+1}} \left| S_k^{(2^n)} - ES_k^{(2^n)} \right| \leq \max_{2^n < k \leq 2^{n+1}} \left( \sum_{i=2^n+1}^{2^{n+1}} (X_i^{(i)} - EX_i^{(i)}) + \right.
\]

\[
+ \sum_{i=1}^{2^{n+1}} |X_i| \mathbb{I}_{\{|X_i| \leq \frac{k}{2^n}\}} + \sum_{i=1}^{2^n} E(|X_i| \mathbb{I}_{\{|X_i| > \frac{k}{2^n}\}}) + \left| S_{2^n}^{(2^n)} - ES_{2^n}^{(2^n)} \right|.
\]

The application of Lemmas 2 and 4 and relations (16) and (17) yields to (18).

Step 6. We shall prove that

\[
\lim_{n \to \infty} \max_{2^n < k \leq 2^{n+1}} \frac{|S_k^{(2^n)} - ES_k^{(2^n)}|}{2^{k/n}} = 0 \quad \text{a.s.} \quad (19)
\]

For \( n \geq 1 \) and \( k \) such that \( 2^n < k \leq 2^{n+1} \) we have

\[
\left| S_k^{(2^n)} - ES_k^{(2^n)} \right| = \left| S_{2^n}^{(2^n)} - ES_{2^n}^{(2^n)} + S_{2^n}^{(2^n)} - S_{2^n}^{(2^n)} + ES_{2^n}^{(2^n)} - ES_{2^n}^{(2^n)} \right| =
\]

\[
\leq \left| (S_{2^n}^{(2^n)} - S_{2^n}^{(2^n)}) - E(S_{2^n}^{(2^n)} - S_{2^n}^{(2^n)}) \right| + \left| (S_{2^n}^{(2^n)} - S_{2^n}^{(2^n)}) - E(S_{2^n}^{(2^n)} - S_{2^n}^{(2^n)}) \right|
\]

\[
= \sum_{i=1}^{2^n} \left( \mathbb{I}_{\{|X_i| \leq \frac{k}{2^n}\}} - \mathbb{I}_{\{|X_i| \leq \frac{k}{2^n}\}} \right) - E(\sum_{i=1}^{2^n} \left( \mathbb{I}_{\{|X_i| \leq \frac{k}{2^n}\}} - \mathbb{I}_{\{|X_i| \leq \frac{k}{2^n}\}} \right)) +
\]

\[
+ \left| (S_{2^n}^{(2^n)} - ES_{2^n}^{(2^n)}) \right| =
\]

\[
\leq \sum_{i=1}^{2^n} \left( \mathbb{I}_{\{|X_i| \leq \frac{k}{2^n}\}} - E(\mathbb{I}_{\{|X_i| \leq \frac{k}{2^n}\}}) \right) + \left| (S_{2^n}^{(2^n)} - ES_{2^n}^{(2^n)}) \right|
\]

\[
\leq \sum_{i=1}^{2^n} \left( \mathbb{I}_{\{|X_i| \leq \frac{k}{2^n}\}} + \sum_{i=1}^{2^n} E(\mathbb{I}_{\{|X_i| \leq \frac{k}{2^n}\}}) + \left| (S_{2^n}^{(2^n)} - ES_{2^n}^{(2^n)}) \right| \right)
\]

\[
\leq \sum_{i=1}^{2^n+1} \left| X_i \right| \mathbb{I}_{\{|X_i| \leq \frac{k}{2^n}\}} + \sum_{i=1}^{2^n} E(|X_i| \mathbb{I}_{\{|X_i| > \frac{k}{2^n}\}}) + \left| (S_{2^n}^{(2^n)} - ES_{2^n}^{(2^n)}) \right|.
\]

Therefore

\[
\max_{2^n < k \leq 2^{n+1}} \left| S_k^{(2^n)} - ES_k^{(2^n)} \right| \leq \sum_{i=1}^{2^n+1} \left| X_i \right| \mathbb{I}_{\{|X_i| \leq \frac{k}{2^n}\}} +
\]

\[
+ \sum_{i=1}^{2^n} E(|X_i| \mathbb{I}_{\{|X_i| > \frac{k}{2^n}\}}) + \left| S_{2^n}^{(2^n)} - ES_{2^n}^{(2^n)} \right|.
\]
The application of Lemmas 2 and 4 and relation (10) yields to (19).

*Proof.* We shall prove that for $n \geq 1$ and $k$ such that $2^n < k \leq 2^{n+1}$ we have

\[
|S_k^{(k)} - ES_k^{(k)}| = \left| \left( S_k^{(k)} - S_2^{(k)} + (S_2^{(k)} - S_2^{(2n)}) \right) - E \left( S_k^{(k)} - S_2^{(k)} + (S_2^{(k)} - S_2^{(2n)}) \right) \right| +
\left| S_2^{(k)} - ES_2^{(k)} \right| + \left| S_2^{(2n)} - ES_2^{(2n)} \right| + \left| S_2^{(2n)} - ES_2^{(2n)} \right| \leq \sum_{i=2^n+1}^{2^{n+1}} |X_i| 2^{\frac{n}{2}} + \sum_{i=2^n+1}^{2^{n+1}} E(|X_i| > 2^{\frac{n}{2}}) +
\left| S_2^{(2n)} - ES_2^{(2n)} \right| + \left| S_2^{(2n)} - ES_2^{(2n)} \right| \leq \sum_{i=1}^{2^{n+1}} |X_i| 2^{\frac{n}{2}-\frac{3}{2}} + \sum_{i=1}^{2^n} E(|X_i| > 2^{\frac{n}{2}}) +
\left| S_2^{(2n)} - ES_2^{(2n)} \right| + \left| S_2^{(2n)} - ES_2^{(2n)} \right| .
\]

Therefore

\[
\max_{2^n < k \leq 2^{n+1}} \left| S_k^{(k)} - ES_k^{(k)} \right| \leq \sum_{i=1}^{2^{n+1}} |X_i| 2^{\frac{n}{2}-\frac{3}{2}} +
\sum_{i=1}^{2^n} E(|X_i| > 2^{\frac{n}{2}}) + \max_{2^n < k \leq 2^{n+1}} \left| S_k^{(k)} - ES_k^{(k)} \right| +
\max_{2^n < k \leq 2^{n+1}} \left| S_2^{(2n)} - ES_2^{(2n)} \right| + \left| S_2^{(2n)} - ES_2^{(2n)} \right| .
\]

The application of Lemmas 2 and 4 and relations (10), (11) and (19) yields to (20). Relation (20) implies (19). Relations (13), (14) and (15) imply (2). Theorem 1 is proved. \(\square\)
References

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