CHARACTERS OF INTEGRABLE HIGHEST WEIGHT MODULES OVER A QUANTUM GROUP

TOSHIYUKI TANISAKI

ABSTRACT. We show that the Weyl-Kac type character formula holds for the integrable highest weight modules over the quantized enveloping algebra of any symmetrizable Kac-Moody Lie algebra, when the parameter $q$ is not a root of unity.

1. Introduction

It is well-known that the character of an integrable highest weight module over a symmetrizable Kac-Moody algebra $\mathfrak{g}$ is given by the Weyl-Kac character formula (see Kac [6]). In this paper we consider the corresponding problem for a quantized enveloping algebra (see Kashiwara [7]).

For a field $K$ and $z \in K^\times$ which is not a root of 1, we denote by $U_{K,z}(\mathfrak{g})$ the quantized enveloping algebra of $\mathfrak{g}$ over $K$ at $q = z$, namely the specialization of Lusztig’s $\mathbb{Z}[q, q^{-1}]$-form via $q \mapsto z$. It is already known that the Weyl-Kac type character formula holds for $U_{K,z}(\mathfrak{g})$ in some cases. When $K$ is of characteristic 0 and $z$ is transcendental, this is due to Lusztig [10]. When $\mathfrak{g}$ is finite-dimensional, this is shown in Andersen, Polo and Wen [1]. When $\mathfrak{g}$ is affine, this is known in certain specific cases (see Chari and Jing [2], Tsuchioka [15]).

We first point out that the problem is closely related to the non-degeneracy of the Drinfeld pairing for $U_{K,z}(\mathfrak{g})$. In fact, assume we could show that the Drinfeld pairing for $U_{K,z}(\mathfrak{g})$ is non-degenerate. Then we can define the quantum Casimir operator. It allows us to apply Kac’s argument for Lie algebras in [6] to $U_{K,z}(\mathfrak{g})$, and we obtain the Weyl-Kac type character formula for integrable highest weight modules over $U_{K,z}(\mathfrak{g})$. In particular, we can deduce the Weyl-Kac type character formula in the affine case from the case-by-case calculation of the Drinfeld pairing due to Damiani [3], [4].

The aim of this paper is to give a simple unified proof of the non-degeneracy of the Drinfeld pairing and the Weyl-Kac type character formula for $U_{K,z}(\mathfrak{g})$, where $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra, $K$ is a field not necessarily of characteristic zero, and $z \in K^\times$ is not a root of 1. Our argument is as follows. We consider the (possibly)
modified algebra $\overline{U}_{K,z}(\mathfrak{g})$, which is the quotient of $U_{K,z}(\mathfrak{g})$ by the ideal generated by the radical of the Drinfeld pairing. Then the Drinfeld pairing for $U_{K,z}(\mathfrak{g})$ induces a non-degenerate pairing for $\overline{U}_{K,z}(\mathfrak{g})$, by which we can define the quantum Casimir operator for $\overline{U}_{K,z}(\mathfrak{g})$. It allows us to apply Kac’s argument for Lie algebras to $\overline{U}_{K,z}(\mathfrak{g})$, and we obtain the Weyl-Kac type character formula for $\overline{U}_{K,z}(\mathfrak{g})$ with modified denominator. In the special case where the highest weight is zero, this gives a formula for the modified denominator. Comparing this with the ordinary denominator formula for Lie algebras, we conclude that the modified denominator coincides with the original denominator for the Lie algebra $\mathfrak{g}$. It implies that the Drinfeld pairing for $\overline{U}_{K,z}(\mathfrak{g})$ was already non-degenerate. This is the outline of our argument. In applying Kac’s argument to the modified algebra, we need to show that the modified denominator is skew invariant with respect to a twisted action of the Weyl group. This is accomplished using certain standard properties of the Drinfeld pairing.

The first draft of this paper contained only results when $K$ is of characteristic zero. Then Masaki Kashiwara pointed out to me that the arguments work for positive characteristic case as well. I would like to thank Masaki Kashiwara for this crucial remark.

2. QUANTIZED ENVELOPING ALGEBRAS

Let $\mathfrak{h}$ be a finite-dimensional vector space over $\mathbb{Q}$, and let $\{h_i\}_{i \in I}$ and $\{\alpha_i\}_{i \in I}$ be linearly independent subsets of $\mathfrak{h}$ and $\mathfrak{h}^*$, respectively such that $(\langle \alpha_j, h_i \rangle)_{i,j \in I}$ is a symmetrizable generalized Cartan matrix. We denote by $W$ the associated Weyl group. It is a subgroup of $GL(\mathfrak{h})$ generated by the involutions $s_i$ ($i \in I$) defined by $s_i(h) = h - \langle \alpha_i, h \rangle h_i$ for $h \in \mathfrak{h}$. The contragredient action of $W$ on $\mathfrak{h}^*$ is given by $s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i$ for $i \in I$, $\lambda \in \mathfrak{h}^*$. Set

$$E = \sum_{i \in I} \mathbb{Q}\alpha_i, \quad Q = \sum_{i \in I} \mathbb{Z}\alpha_i, \quad Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i.$$  

We can take a symmetric $W$-invariant bilinear form $(\ , \ ) : E \times E \to \mathbb{Q}$ such that

$$(2.1) \quad \frac{(\alpha_i, \alpha_i)}{2} \in \mathbb{Z}_{>0} \quad (i \in I).$$

For $\lambda \in E$ and $i \in I$ we obtain from $(\lambda, \alpha_i) = (s_i\lambda, s_i\alpha_i)$ that

$$(2.2) \quad \langle \lambda, h_i \rangle = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}.$$

In particular we have

$$(\alpha_i, \alpha_j) = \langle \alpha_j, h_i \rangle \frac{(\alpha_i, \alpha_i)}{2} \in \mathbb{Z},$$
and hence \((Q, Q) \subset \mathbb{Z}\). For \(i \in I\) set \(t_i = \frac{(\alpha_i, \alpha_i)}{2} h_i\), and for \(\gamma = \sum_i n_i \alpha_i \in Q\) set \(t_\gamma = \sum_i n_i t_i\). By (2.2) we have \((\lambda, \gamma) = (\lambda, t_\gamma)\) for \(\lambda \in E\), \(\gamma \in Q\). We fix a \(\mathbb{Z}\)-form \(h_\mathbb{Z}\) of \(h\) such that

\[
(\alpha_i, h_\mathbb{Z}) \subset \mathbb{Z}, \quad t_i \in h_\mathbb{Z} \quad (i \in I).
\]

We set

\[
P = \{ \lambda \in h^* \mid (\lambda, h_\mathbb{Z}) \subset \mathbb{Z} \}, \quad P^+ = \{ \lambda \in P \mid (\lambda, h_i) \in \mathbb{Z}_{\geq 0} \}.
\]

We fix \(\rho \in h^*\) such that \((\rho, h_i) = 1\) for any \(i \in I\), and define a twisted action of \(W\) on \(h^*\) by

\[
w \circ \lambda = w(\lambda + \rho) - \rho \quad (w \in W, \lambda \in h^*).
\]

This action does not depend on the choice of \(\rho\), and we have \(w \circ P = P\) for any \(w \in W\).

Denote by \(E\) the set of formal sums \(\sum_{\lambda \in P} c_{\lambda} e(\lambda) \in E\) such that there exist finitely many \(\lambda_1, \ldots, \lambda_r \in P\) such that

\[
\{ \lambda \in P \mid c_{\lambda} \neq 0 \} \subset \bigcup_{k=1}^r (\lambda_k - Q^+).
\]

Note that \(E\) is naturally a commutative ring by the multiplication \(e(\lambda)e(\mu) = e(\lambda + \mu)\).

Denote by \(\Delta^+\) the set of positive roots for the Kac-Moody Lie algebra \(g\) associated to the generalized Cartan matrix \((\langle \alpha_j, h_i \rangle)_{i,j \in I}\). For \(\alpha \in \Delta^+\) let \(m_\alpha\) be the dimension of the root space of \(g\) with weight \(\alpha\). We define an invertible element \(D\) of \(E\) by

\[
D = \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{m_\alpha}.
\]

For \(n \in \mathbb{Z}_{\geq 0}\) set

\[
[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}} \in \mathbb{Z}[x, x^{-1}], \quad [n]!_x = [n]_x[n - 1]_x \cdots [1]_x \in \mathbb{Z}[x, x^{-1}].
\]

We denote by \(\mathbb{F} = \mathbb{Q}(q)\) the field of rational functions in the variable \(q\) with coefficients in \(\mathbb{Q}\).

The quantized enveloping algebra \(U\) associated to \(h\), \(\{h_i\}_{i \in I}, \{\alpha_i\}_{i \in I},\)

\(h_\mathbb{Z}, (\ , \ )\) is the associative algebra over \(\mathbb{F}\) generated by the elements \(k_h,\)
$e_i, f_i \ (h \in \mathfrak{h}_Z, i \in I)$ satisfying the relations

\begin{align}
(2.4) & \quad k_0 = 1, \quad k_hk_{h'} = k_{h+h'} \quad (h, h' \in \mathfrak{h}_Z), \\
(2.5) & \quad k_he_i k_{-h} = q_i^{(\alpha_i,h)}e_i \quad (h \in \mathfrak{h}_Z, i \in I), \\
(2.6) & \quad k_hf_i k_{-h} = q_i^{-(\alpha_i,h)}f_i \quad (h \in \mathfrak{h}_Z, i \in I), \\
(2.7) & \quad e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \quad (i, j \in I), \\
(2.8) & \quad \sum_{r+s=1-\langle \alpha_j, h_i \rangle} (-1)^r e_i^{(r)} f_j e_i^{(s)} = 0 \quad (i, j \in I, i \neq j), \\
(2.9) & \quad \sum_{r+s=1-\langle \alpha_j, h_i \rangle} (-1)^r f_i^{(r)} f_j f_i^{(s)} = 0 \quad (i, j \in I, i \neq j),
\end{align}

where $k_i = k_{h_i}$, $q_i = q^{(\alpha_i,\alpha_i)/2} \in I$, and $e_i^{(r)} = \frac{1}{[r]_{q_i}} e_i^r$, $f_i^{(r)} = \frac{1}{[r]_{q_i}} f_i^r$ for $i \in I, r \in \mathbb{Z}_{\geq 0}$. For $\gamma \in Q$ we set $k_\gamma = k_{h_\gamma}$.

We have a Hopf algebra structure of $U$ given by

\begin{align}
(2.10) & \quad \Delta(k_h) = k_h \otimes k_h, \\
& \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \\
(2.11) & \quad \varepsilon(h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0, \\
(2.12) & \quad S(k_h) = k_h^{-1}, \quad S(e_i) = -k_i^{-1}e_i, \quad S(f_i) = -f_ik_i
\end{align}

for $h \in \mathfrak{h}_Z, i \in I$. We will sometimes use Sweedler’s notation for the coproduct;

$$\Delta(u) = \sum_{(u)} u_{(0)} \otimes u_{(1)} \quad (u \in U),$$

and the iterated coproduct;

$$\Delta_m(u) = \sum_{(u)_m} u_{(0)} \otimes \cdots \otimes u_{(m)} \quad (u \in U).$$

We define $\mathbb{F}$-subalgebras $U^0, U^+, U^-, U^{\geq 0}, U^{\leq 0}$ of $U$ by

$$U^0 = \langle k_h \mid h \in \mathfrak{h}_Z \rangle, \quad U^+ = \langle e_i \mid i \in I \rangle, \quad U^- = \langle f_i \mid i \in I \rangle, \quad U^{\geq 0} = \langle k_h, e_i \mid h \in \mathfrak{h}_Z, i \in I \rangle, \quad U^{\leq 0} = \langle k_h, f_i \mid h \in \mathfrak{h}_Z, i \in I \rangle.$$

For $\gamma \in Q$ set

$$U_\gamma = \{u \in U \mid k_hu k_h^{-1} = q^{(\gamma, h)}u \ (h \in \mathfrak{h}_Z)\}, \quad U^+_\gamma = U_\gamma \cap U^+.$$

Then we have

$$U^0 = \bigoplus_{h \in \mathfrak{h}_Z} \mathbb{F}k_h, \quad U^\pm = \bigoplus_{\gamma \in Q^\pm} U^\pm_\gamma.$$

It is known that the multiplication of $U$ induces isomorphisms

$$U \cong U^+ \otimes U^0 \otimes U^- \cong U^- \otimes U^0 \otimes U^+, \quad U^{\geq 0} \cong U^+ \otimes U^0 \cong U^0 \otimes U^+, \quad U^{\leq 0} \cong U^- \otimes U^0 \cong U^0 \otimes U^-$$
of vector spaces. It is also known that

\[(2.13) \quad \sum_{\gamma \in Q^+} \dim U_{-\gamma} e(-\gamma) = D^{-1}.\]

For a $U$-module $V$ and $\lambda \in P$ we set

\[V_\lambda = \{ v \in V \mid k_h v = q^{(h, h)} v \ (h \in \mathfrak{h}_\mathbb{C}) \}. \]

We say that a $U$-module $V$ is integrable if $V = \bigoplus_{\lambda \in P} V_\lambda$ and for any $v \in V$ and $i \in I$ there exists some $N > 0$ such that $e_i^{(n)} v = f_i^{(n)} v = 0$ for $n \geq N$.

For $i \in I$ and an integrable $U$-module $V$ define an operator $T_i : V \to V$ by

\[T_i v = \sum_{-a+b-c=0} (-1)^b q_i^{-ac+b} f_i^{(a)} e_i^{(c)} v \quad (v \in V_\lambda).\]

It is invertible, and satisfies $T_i V_\lambda = V_{\lambda \lambda}$ for $\lambda \in P$. There exists a unique algebra automorphism $T_i : U \to U$ such that for any integrable $U$-module $V$ we have $T_i uv = T_i(u)T_i v$ ($u \in U, v \in V$). Then we have $T_i(U_{\gamma}) = U_{\gamma}U_{\gamma}$ for $\gamma \in Q$. The action of $T_i$ on $U$ is given by

\[T_i(k_h) = k_{sh}, \quad T_i(e_i) = -f_i k_i, \quad T_i(f_i) = -k_i^{-1} e_i \quad (h \in \mathfrak{h}_\mathbb{C}),\]

\[T_i(e_j) = \sum_{r+s=-(a_j, h_i)} (-1)^r q_i^{-r} e_j e_i^{(r)} \quad (j \in I, i \neq j),\]

\[T_i(f_j) = \sum_{r+s=-(a_j, h_i)} (-1)^r q_i^{-r} f_j f_i^{(r)} \quad (j \in I, i \neq j).\]

(see [11, Section 37.1]).

The multiplication of $U$ induces

\[(2.14) \quad U^+ \cong (U^+ \cap T_i(U^+)) \otimes \mathbb{F}[e_i] \cong \mathbb{F}[e_i] \otimes (U^+ \cap T_i^{-1}(U^+)),\]

\[(2.15) \quad U^- \cong (U^- \cap T_i(U^-)) \otimes \mathbb{F}[f_i] \cong \mathbb{F}[f_i] \otimes (U^- \cap T_i^{-1}(U^-))\]

(see [11, Lemma 38.1.2]). Moreover,

\[(2.16) \quad \Delta(U^+ \cap T_i(U^+)) \subset U^\leq \otimes (U^+ \cap T_i(U^+)),\]

\[(2.17) \quad \Delta(U^+ \cap T_i^{-1}(U^+)) \subset U^0(U^+ \cap T_i^{-1}(U^+)) \otimes U^+,\]

\[(2.18) \quad \Delta(U^- \cap T_i(U^-)) \subset (U^- \cap T_i(U^-)) \otimes U^\leq,\]

\[(2.19) \quad \Delta(U^- \cap T_i^{-1}(U^-)) \subset U^- \otimes U^0(U^- \cap T_i^{-1}(U^-))\]

(see [14, Lemma 2.8]).

Set

\[z \mathbb{U}^0 = \bigoplus_{\gamma \in Q} Fk_{\gamma} \subset U^0, \quad z \mathbb{U}^{\geq 0} = z \mathbb{U}^0 U^+, \quad z \mathbb{U}^{\leq 0} = z \mathbb{U}^0 U^-\]

They are Hopf subalgebras of $U$. The Drinfeld pairing is the bilinear form

\[\tau : z \mathbb{U}^{\geq 0} \times z \mathbb{U}^{\leq 0} \to \mathbb{F}\]
characterized by the following properties:

\begin{align}
(2.20) & \quad \tau(x, y_1 y_2) = (\tau \otimes \tau)(\Delta(x), y_1 \otimes y_2) \quad (x \in U^\geq, y_1, y_2 \in U^\leq), \\
(2.21) & \quad \tau(x_1 x_2, y) = (\tau \otimes \tau)(x_2 \otimes x_1, \Delta(y)) \quad (x_1, x_2 \in U^\geq, y \in U^\leq), \\
(2.22) & \quad \tau(k_\gamma, k_\delta) = q^{-(\gamma, \delta)} \quad (\gamma, \delta \in Q), \\
(2.23) & \quad \tau(e_i, f_j) = -\delta_{ij}(q_i - q_i^{-1})^{-1} \quad (i, j \in I), \\
(2.24) & \quad \tau(e_i, k_\gamma) = \tau(k_\gamma, f_i) = 0 \quad (i \in I, \gamma \in Q).
\end{align}

It satisfies the following properties:

\begin{align}
(2.25) & \quad \tau(xk_\gamma, yk_\delta) = \tau(x, y)q^{-(\gamma, \delta)} \quad (x \in U^+, y \in U^-, \gamma, \delta \in Q), \\
(2.26) & \quad \tau(U^+_{\gamma}, U^-_{\delta}) = \{0\} \quad (\gamma, \delta \in Q^+, \gamma \neq \delta), \\
(2.27) & \quad \tau|_{U^+_{\gamma} \times U^-_{\delta}} \text{ is non-degenerate} \quad (\gamma \in Q^+), \\
(2.28) & \quad \tau(Sx, Sy) = \tau(x, y) \quad (x \in U^\geq, y \in U^\leq).
\end{align}

Moreover, for \(x \in U^\geq, y \in U^\leq\) we have

\begin{align}
(2.29) & \quad xy = \sum_{(x_1)_2, (y_2)_2} \tau(x_1(0), y_2(0))\tau(x_2(1), Sy_2(1)y_1(1)x_1(1)), \\
(2.30) & \quad yx = \sum_{(x_2)_2, (y_2)_2} \tau(Sx_2(0), y_2(0))\tau(x_2(1), y_2(1)x_1(1)y_1(1)).
\end{align}

(see [12, Lemma 2.1.2]).

For \(i \in I\) we define linear maps \(r_{i, \pm} : U^\pm \to U^\pm\) by

\begin{align}
\Delta(x) \in r_{i, +}(x)k_i \otimes e_i + \sum_{\delta \in Q^+ \setminus \{\alpha_i\}} U^\geq \otimes U^+_\delta \quad (x \in U^+), \\
\Delta(x) \in e_i k_{-\alpha_i} \otimes r'_{i, +}(x) + \sum_{\delta \in Q^+ \setminus \{\alpha_i\}} U^+_\delta U^0 \otimes U^+ \quad (x \in U^+_\gamma), \\
\Delta(y) \in r_{i, -}(y) \otimes f_i k_{-\alpha_i} + \sum_{\delta \in Q^+ \setminus \{\alpha_i\}} U^- \otimes U^- U^0 \quad (y \in U^-), \\
\Delta(y) \in f_i \otimes r'_{i, -}(y)k_i^{-1} + \sum_{\delta \in Q^+ \setminus \{\alpha_i\}} U^- U^0 \otimes U^\leq \quad (y \in U^-).
\end{align}
We have
\[
U^+ \cap T_i(U^+) = \{ u \in U^+ \mid \tau(u, U^- f_i) = \{0\}\} \\
= \{ u \in U^+ \mid r_{i+}(u) = 0 \},
\]
\[
U^+ \cap T_i^{-1}(U^+) = \{ u \in U^+ \mid \tau(u, f_i U^-) = \{0\}\} \\
= \{ u \in U^+ \mid r_{i+}'(u) = 0 \},
\]
\[
U^- \cap T_i(U^-) = \{ u \in U^- \mid \tau(U^ e_i, u) = \{0\}\} \\
= \{ u \in U^- \mid r_{i-}'(u) = 0 \},
\]
\[
U^- \cap T_i^{-1}(U^-) = \{ u \in U^- \mid \tau(e_i U^+, u) = \{0\}\} \\
= \{ u \in U^- \mid r_{i-}(u) = 0 \}
\]
(see [13, Proposition 38.1.6]).

By (2.16), (2.17), (2.18), (2.19), (2.31), (2.32), (2.33), (2.34) we easily obtain
\[
\tau(x e_i^m, y f_i^n) = \delta_{mn} \tau(x, y) \cdot \frac{q_i^{n(1-n)/2}}{(q_i^{-1} - q_i)^n} [n]_q,
\]
\[(x \in U^+ \cap T_i(U^+), y \in U^- \cap T_i(U^-)),
\]
\[
\tau(e_i^m x', f_i^n y') = \delta_{mn} \tau(x', y') \cdot \frac{q_i^{n(1-n)/2}}{(q_i^{-1} - q_i)^n} [n]_q,
\]
\[(x' \in U^+ \cap T_i^{-1}(U^+), y' \in U^- \cap T_i^{-1}(U^-)).
\]

We have also
\[
\tau(x, y) = \tau(T_i^{-1}(x), T_i^{-1}(y))
\]
\[(x \in U^+ \cap T_i(U^+), y \in U^- \cap T_i(U^-))
\]
(see [13, Proposition 38.2.1], [14, Theorem 5.1]).

3. Specialization

Let $R$ be a subring of $\mathbb{F} = \mathbb{Q}(q)$ containing $A = \mathbb{Z}[q, q^{-1}]$. We denote by $U_R$ the $R$-subalgebra of $U$ generated by $k_h, e_i^{(n)}, f_i^{(n)}$ ($h \in h, i \in I, n \geq 0$). It is a Hopf algebra over $R$.

We define subalgebras $U_R^0, U_R^+, U_R^-, U_R^{\geq 0}, U_R^{\leq 0}$ of $U_R$ by
\[
U_R^0 = U^0 \cap U_R, 
U_R^+ = U^+ \cap U_R, 
U_R^{-} = U^- \cap U_R, 
U_R^{\geq 0} = U^{\geq 0} \cap U_R, 
U_R^{\leq 0} = U^{\leq 0} \cap U_R.
\]

Setting $U_{R, \pm \gamma} = U_{\pm \gamma} \cap U_R$ for $\gamma \in \mathbb{Q}^+$ we have
\[
U_R^\pm = \bigoplus_{\gamma \in \mathbb{Q}^+} U_{R, \pm \gamma}.
\]
It is known that $U_{R,\pm \gamma}$ is a free $R$-module of rank $\dim U_{R,\pm \gamma}$ (see [11, Section 14.2]). Hence we have
\begin{equation}
\sum_{\gamma \in Q^+} \text{rank}_R(U_{R,-\gamma})e(-\gamma) = D^{-1}
\end{equation}
by (2.13).

The multiplication of $U_R$ induces isomorphisms
\begin{align*}
U_R &\cong U_R^+ \otimes U_R^0 \otimes U_R^- \cong U_R^0 \otimes U_R^- \cong U_R^+ \otimes U_R^-,
U_R^0 &\cong U_R^+ \otimes U_R^0 \cong U_R^0 \otimes U_R^+, 
U_R^\mp &\cong U_R^+ \otimes U_R^0 \cong U_R^0 \otimes U_R^+ \cong U_R^0 \otimes U_R^-
\end{align*}
of $R$-modules.

For $i \in I$ the algebra automorphisms $T_{i}^\pm : U \to U$ preserve $U_R$.

**Lemma 3.1.** The multiplication of $U_R$ induces isomorphisms
\begin{align}
\label{3.2}
U_R^+ &\cong (U_R^+ \cap T_i(U_R^+)) \otimes_R \left( \bigoplus_{n=0}^{\infty} Re_i^{(n)} \right), \\
\label{3.3}
U_R^+ &\cong \left( \bigoplus_{n=0}^{\infty} Re_i^{(n)} \right) \otimes_R (U_R^+ \cap T_{i}^{-1}(U_R^+)), \\
\label{3.4}
U_R^- &\cong (U_R^- \cap T_i(U_R^-)) \otimes_R \left( \bigoplus_{n=0}^{\infty} Rf_i^{(n)} \right), \\
\label{3.5}
U_R^- &\cong \left( \bigoplus_{n=0}^{\infty} Rf_i^{(n)} \right) \otimes_R (U_R^- \cap T_{i}^{-1}(U_R^-)).
\end{align}

**Proof.** We only show (3.2). The injectivity of the canonical homomorphism
\begin{equation}
(U_R^+ \cap T_i(U_R^+)) \otimes_R \left( \bigoplus_{n=0}^{\infty} Re_i^{(n)} \right) \to U_R^+
\end{equation}
is clear. To show the surjectivity it is sufficient to verify that its image is stable under the left multiplication by $e_j^{(n)}$ for any $j \in I$ and $n \geq 0$. If $j \neq i$, this is clear since $e_j^{(n)} \in U_R^+ \cap T_i(U_R^+)$. Consider the case $j = i$. By (2.31) and the general formula
\begin{equation}
r_{i,+}(x') = q_i^{(\gamma,\alpha_i)} r_{i,+}(x')x + x r_{i,+}(x') (x \in U^+, x' \in U_i^-)
\end{equation}
we easily obtain
\begin{equation}
x \in U^+ \cap T_i(U^+) \implies e_i x - q_i^{(\gamma,\alpha_i)} x e_i \in U^+_{\gamma+\alpha_i} \cap T_i(U^+).
\end{equation}
Now let $x \in U_{R,\gamma}^+ \cap T_i(U_R^+)$. Define $x_k \in U_{R,\gamma+\alpha_i}^+ \cap T_i(U_R^+)$ inductively by $x_0 = x$, $x_{k+1} = \frac{1}{[k+1]q_i} (e_i x_k - q_i^{(\gamma,\alpha_i)+2k} x_k e_i)$. Then we see by induction on $n$ that
\begin{equation}
e_i^{(n)} x = \sum_{k=0}^{n} q_i^{(n-k)(\gamma,\alpha_i)+k} x_k e_i^{(n-k)},
\end{equation}
or equivalently,

\[ x_n = e_i^{(n)} x - \sum_{k=0}^{n-1} q_i^{(n-k)((\gamma, \alpha_i^\vee) + k)} x_k e_i^{(n-k)}. \]

We obtain from (3.7) that \( x_n \in U_+^R \) by induction on \( n \). By \( T_i(U_R) = U_R \) we have \( x_n \in U_+^R \cap T_i(U^+) = U_R^+ \cap T_i(U^+_R) \). It follows that \( e_i^{(n)} (U_+^R \cap T_i(U^+_R)) \in \sum_{k=0}^n (U_+^R \cap T_i(U^+_R)) e_i^{(k)} \) by (3.6).

We set

\[
\mathcal{Z} = \bigoplus_{\gamma \in Q} Rk_{\gamma} \subset U_R^0, \quad \mathcal{Z}^+ = \mathcal{Z} U_R^+, \quad \mathcal{Z}^0 = \mathcal{Z} U_R^-.
\]

Define a subring \( \widehat{A} \) of \( F \) by

\[
\widehat{A} = \mathbb{Z}[q, q^{-1}, (q - q^{-1})^{-1}, [n]_q^{-1} | n > 0] = \mathbb{Z}[q, q^{-1}, (q^n - 1)^{-1} | n > 0].
\]

Then the Drinfeld pairing induces a bilinear form

\[
\tau_{\mathcal{Z}} : \mathcal{Z}^+ \times \mathcal{Z}^0 \rightarrow \widehat{A}.
\]

For \( \gamma \in Q^+ \) we denote its restriction to \( U_{\mathcal{Z}, \gamma}^+ \times U_{\mathcal{Z}, -\gamma}^- \) by

\[
\tau_{\mathcal{Z}, \gamma} : U_{\mathcal{Z}, \gamma}^+ \times U_{\mathcal{Z}, -\gamma}^- \rightarrow \widehat{A}.
\]

In the rest of this paper we fix a field \( K \) and \( z \in K^\times \) which is not a root of 1, and consider the Hopf algebra

\[
U_z = K \otimes_{\widehat{A}} U_{\mathcal{Z}},
\]

where \( \widehat{A} \rightarrow K \) is given by \( q \mapsto z \). We define subalgebras \( U_z^0, U_z^+, U_z^- \), \( U_z^{\geq 0}, U_z^{\leq 0} \) of \( U_z \) by

\[
U_z^{\geq 0} = K \otimes_{\mathcal{Z}} U_{\mathcal{Z}}^{\geq 0}, \quad U_z^\pm = K \otimes_{\mathcal{Z}} U_{\mathcal{Z}}^\pm, \quad U_z^{\leq 0} = K \otimes_{\mathcal{Z}} U_{\mathcal{Z}}^{\leq 0}.
\]

For \( \gamma \in Q^+ \) we set \( U_{z, \pm, \gamma}^\pm = K \otimes_{\mathcal{Z}} U_{\mathcal{Z}, \pm, \gamma}^\pm \). Then we have

\[
U_z^0 = \bigoplus_{h \in \mathfrak{h}_z} Kk_h, \quad U_z^\pm = \bigoplus_{\gamma \in Q^+} U_{z, \pm, \gamma}^\pm.
\]

By (3.4) we have

\[
\sum_{\gamma \in Q^+} \dim U_{z, -\gamma}^- e(-\gamma) = D^{-1}.
\]

Moreover, setting

\[
U_{z, \gamma} = \{ u \in U_z | k_h u k_h^{-1} = z^{(\gamma, h)} u \ (h \in \mathfrak{h}_z) \} \quad (\gamma \in Q),
\]
we have $U_{z,±;γ}^± = U_z^± ∩ U_{z,γ}^±$ since $z$ is not a root of 1. The multiplication of $U_z$ induces isomorphisms

\[(3.11) \quad U_z ≅ U_z^+ ⊗ U_z^0 ⊗ U_z^− ≅ U_z^− ⊗ U_z^0 ⊗ U_z^+ ,\]

\[(3.12) \quad U_z^{≥0} ≅ U_z^+ ⊗ U_z^0 ≅ U_z^0 ⊗ U_z^+, \quad U_z^{≤0} ≅ U_z^− ⊗ U_z^0 ≅ U_z^0 ⊗ U_z^− \]

of $K$-modules. Here, $\otimes$ denotes $⊗_K$.

For a $U_z$-module $V$ and $λ ∈ P$ we set

\[V_λ = \{ v ∈ V \mid k_h v = z^{(λ,h)} v (h ∈ h_z) \}.\]

We say that a $U_z$-module $V$ is integrable if $V = \bigoplus_{λ ∈ P} V_λ$ and for any $v ∈ V$ and $i ∈ I$ there exists some $N > 0$ such that $e_i^{(n)} v = f_i^{(n)} v = 0$ for $n ≥ N$.

For $i ∈ I$ and an integrable $U_z$-module $V$ define an operator $T_i : V → V$ by

\[T_i v = \sum_{-a+b=0} (-1)^{-b} z_i^{−ac+b} e_i^{(a)} f_i^{(b)} e_i^{(c)} v \quad (v ∈ V_λ),\]

where $z_i = z^{(α_i,α_i)/2}$. It is invertible, and satisfies $T_i V_λ = V_{λ,λ}$ for $λ ∈ P$.

We denote by $T_i : U_z → U_z$ the algebra automorphism of $U_z$ induced from $T_i : U_̂ → U_̂$. Then we have $T_i(U_{z,γ}) = U_{z,γ}$ for $γ ∈ Q$.

**Lemma 3.2.** The multiplication of $U_z$ induces isomorphisms

\[(3.13) \quad U_z^+ ≅ (U_z^+ ∩ T_i(U_z^+)) \otimes \left( \bigoplus_{n=0}^{∞} Ke_i^{(n)} \right) ,\]

\[(3.14) \quad U_z^+ ≅ \left( \bigoplus_{n=0}^{∞} Ke_i^{(n)} \right) \otimes (U_z^+ ∩ T_i^{-1}(U_z^+) ),\]

\[(3.15) \quad U_z^- ≅ (U_z^- ∩ T_i(U_z^-)) \otimes \left( \bigoplus_{n=0}^{∞} Kf_i^{(n)} \right) ,\]

\[(3.16) \quad U_z^- ≅ \left( \bigoplus_{n=0}^{∞} Kf_i^{(n)} \right) \otimes (U_z^- ∩ T_i^{-1}(U_z^-) ).\]

**Proof.** We only show (3.13). By Lemma 3.1 we have

\[U_z^+ ≅ \left( K ⊗ ̂_K (U_̂^+ ∩ T_i(U_̂^+)) \right) \otimes \left( \bigoplus_{n=0}^{∞} Ke_i^{(n)} \right) .\]

By $U_̂^+ ∩ T_i(U_̂^+) = U_̂^+ ∩ T_i(U^+)$ the canonical map $K ⊗ ̂_K (U_̂^+ ∩ T_i(U_̂^+)) → U_z^+ ∩ T_i(U_z^+)$ is injective. Hence we have a sequence of
canonical maps
\[ U_+ \cong \left( K \otimes_{\mathcal{A}} (U_+ \cap T_i(U_+)) \right) \otimes \left( \bigoplus_{n=0}^{\infty} K e_i^{(n)} \right) \]
\[ \hookrightarrow (U_+ \cap T_i(U_+)) \otimes \left( \bigoplus_{n=0}^{\infty} K e_i^{(n)} \right) \to U_+ \]
Therefore, it is sufficient to show that
\[ (U_+ \cap T_i(U_+)) \otimes \left( \bigoplus_{n=0}^{\infty} K e_i^{(n)} \right) \to U_+ \]
is injective. This follows by applying \( T_i \) to \( U_+ \otimes U_0 \cong U_+ \).
□

We set
\[ \sharp U_0 = K \otimes_{\mathcal{A}} \sharp U_0, \quad \sharp U_0 = K \otimes_{\mathcal{A}} \sharp U_0, \quad \sharp U_0 = K \otimes_{\mathcal{A}} \sharp U_0. \]
They are Hopf subalgebras of \( U_+ \). The Drinfeld pairing induces a bi-linear form
\[ \tau: \sharp U_0 \times \sharp U_0 \to K. \]
For \( \gamma \in Q^+ \) we denote its restriction to \( U_+ \times U_{-\gamma} \) by
\[ \tau_{+,\gamma}: U_+ \times U_{-\gamma} \to K. \]

4. The modified algebra

Set
\[ J_+ = \{ x \in U_+ | \tau(x, U_-) = \{ 0 \} \}, \]
\[ J_- = \{ y \in U_- | \tau(U_+, y) = \{ 0 \} \}. \]
For \( \gamma \in Q^+ \) we set
\[ J_{+,\pm\gamma} = J_+ \cap U_{\pm\gamma}. \]
By \( \text{[2.20]} \) we have
\[ J_+ = \bigoplus_{\gamma \in Q^+ \setminus \{ 0 \}} J_{+,\pm\gamma}. \]
Define a two-sided ideal \( J_+ \) of \( U_+ \) by
\[ J_+ = U_+ J_+ U_+ + U_+ J_- U_. \]

**Proposition 4.1.**

(i) We have
\[ \Delta(J_+) \subset U_+ \otimes J_+ + J_+ \otimes U_+, \quad \varepsilon(J_+) = \{ 0 \}, \quad S(J_+) \subset J_. \]
(ii) Under the isomorphism \( U_+ \cong U_+ \otimes U_0 \otimes U_- \) (resp. \( U_+ \cong U_- \otimes U_0 \otimes U_+ \)) induced by the multiplication of \( U_+ \) we have
\[ J_+ \cong J_+ \otimes U_0 \otimes U_+ + U_+ \otimes U_0 \otimes J_-, \]
(resp. \( J_- \cong J_- \otimes U_0 \otimes U_+ + U_- \otimes U_0 \otimes J_+ \)).
Proof. (i) It is sufficient to show

\[
\Delta(J_z^+) \subset J_z^+ U_z^0 \otimes U_z^+ + U_z^{\geq 0} \otimes J_z^+,
\]

\[
\Delta(J_z^-) \subset J_z^- \otimes U_z^{\geq 0} + U_z^- \otimes J_z^- U_z^0,
\]

\[
\varepsilon(J_z^+) = \{0\},
\]

\[
S(J_z^+) \subset J_z^+ U_z^0.
\]

By (2.25) we have

\[
J_z^+ U_z^0 = \{ x \in U_z^{\geq 0} \mid \tau_z(x, U_z^-) = \{0\} \}.
\]

Hence in order to verify (1.2) it is sufficient to show

\[
\tau_z(\Delta(J_z^+), U_z^- \otimes U_z^-) = \{0\}.
\]

This follows from (2.20). The proof of (1.3) is similar. The assertions (4.4) and (1.5) follow from (1.1) and (2.28), respectively.

(ii) It is sufficient to show

\[
J_z^+ U_z^\pm = U_z^\pm J_z^+ = J_z^+,
\]

\[
J_z^+ U_z^{\leq 0} = U_z^{\leq 0} J_z^+,
\]

\[
J_z^- U_z^{\geq 0} = U_z^{\geq 0} J_z^-.
\]

The assertion (4.6) follows from (2.20), (2.21), (2.25). By (4.1) we have

\[
J_z^+ U_z^{\geq 0} = U_z^{\geq 0} J_z^+.
\]

Hence in order to show (4.7) it is sufficient to show

\[
J_z^+ U_z^{\geq 0} = \tau U_z^{\geq 0} J_z^+ \quad \text{and} \quad J_z^- U_z^{\geq 0} = \tau U_z^{\geq 0} J_z^-.
\]

Let \(x \in J_z^+, y \in \tau U_z^{\geq 0}\). By (4.2) we have

\[
\Delta_2(x)
\]

\[
\varepsilon \in U_z^{\geq 0} \otimes U_z^{\geq 0} \otimes J_z^+ + U_z^{\geq 0} \otimes J_z^+ U_z^{\leq 0} \otimes U_z^+ + J_z^+ U_z^{\geq 0} \otimes U_z^+.
\]

Hence we have \(xy \in U_z^{\leq 0} J_z^+ \) and \(yx \in J_z^+ U_z^{\leq 0}\) by (2.29), (2.30). It follows that \(J_z^+ U_z^{\leq 0} = \tau U_z^{\geq 0} J_z^+\). The proof of \(J_z^- U_z^{\geq 0} = \tau U_z^{\geq 0} J_z^-\) is similar.

By (2.33), (2.34), (2.37) we see easily the following.

Lemma 4.2. For \(i \in I\) we have

\[
J_z^- \cong (J_z^- \cap T_i(U_z^-)) \otimes \left( \bigoplus_{n=0}^{\infty} K f_i^{(n)} \right),
\]

\[
J_z^- \cong \left( \bigoplus_{n=0}^{\infty} K f_i^{(n)} \right) \otimes (J_z^- \cap T_i^{-1}(U_z^-)).
\]

Moreover, we have

\[
T_i^{-1}(J_z^- \cap T_i(U_z^-)) = J_z^- \cap T_i^{-1}(U_z^-).
\]

We set

\[
\overline{U}_z = U_z / J_z.
\]
It is a Hopf algebra by Proposition 4.1. Denote by $\mathcal{U}_z^0, \mathcal{U}_z^\pm, \mathcal{U}_z^{\geq 0}, \mathcal{U}_z^{\leq 0}$, $\mathcal{U}_z^{\geq 0}, \mathcal{U}_z^{\leq 0}, \mathcal{U}_z^{\geq 0, \pm \gamma}$ ($\gamma \in Q^+$) the images of $U_z^0, U_z^\pm, U_z^{\geq 0}, U_z^{\leq 0}, U_z^0, U_z^0, U_z^0, U_z^{\geq 0, \pm \gamma}$ under $U_z \to \mathcal{U}_z$ respectively. By the above argument we have

$$\mathcal{U}_z \simeq \mathcal{U}_z^+ \otimes \mathcal{U}_z^0 \otimes \mathcal{U}_z^- \simeq \mathcal{U}_z^+ \otimes \mathcal{U}_z^0 \otimes \mathcal{U}_z^-, \quad \mathcal{U}_z^{\geq 0} \simeq \mathcal{U}_z^+ \otimes \mathcal{U}_z^0 \simeq \mathcal{U}_z^0 \otimes \mathcal{U}_z^-,$$

$$\mathcal{U}_z^{\leq 0} \simeq \mathcal{U}_z^+ \otimes \mathcal{U}_z^0 \simeq \mathcal{U}_z^0 \otimes \mathcal{U}_z^-, \quad \mathcal{U}_z^{\geq 0} \simeq \mathcal{U}_z^+ \otimes \mathcal{U}_z^0 \simeq \mathcal{U}_z^0 \otimes \mathcal{U}_z^-,$$

and

$$\mathcal{U}_z^0 = \bigoplus_{h \in h_\mathfrak{Z}} K k_h, \quad \mathcal{U}_z^0 = \bigoplus_{\gamma \in Q} K k_\gamma,$$

By definition $\tau_z$ induces a bilinear form

$$\tau_z: \mathcal{U}_z^{\geq 0} \times \mathcal{U}_z^{\leq 0} \to K$$

such that for any $\gamma \in Q^+$ its restriction

$$\tau_{z, \gamma}: \mathcal{U}_z^{\pm} \times \mathcal{U}_z^{\pm} \to K$$

is non-degenerate.

For $\lambda \in P$ and a $\mathcal{U}_z$-module $V$ we set

$$V_\lambda = \{v \in V \mid k_h v = z^{(h, \lambda)} v \ (h \in h_\mathfrak{Z})\}.$$ 

We define a category $\mathcal{O}(\mathcal{U}_z)$ as follows. Its objects are $\mathcal{U}_z$-modules $V$ which satisfy

$$V = \bigoplus_{\lambda \in P} V_\lambda, \quad \dim V_\lambda < \infty \ (\lambda \in P),$$

and such that there exist finitely many $\lambda_1, \ldots, \lambda_r \in P$ such that

$$\{\lambda \in P \mid V_\lambda \neq \{0\}\} \subset \bigcup_{k=1}^r (\lambda_k - Q^+).$$

The morphisms are homomorphisms of $\mathcal{U}_z$-modules.

We say that a $\mathcal{U}_z$-module $V$ is integrable if $V = \bigoplus_{\lambda \in P} V_\lambda$ and for any $v \in V$ there exists $N > 0$ such that for $i \in I$ and $n \geq N$ we have $e_i^{(n)} v = f_i^{(n)} v = 0$. We denote by $\mathcal{O}^{\text{int}}(\mathcal{U}_z)$ the full subcategory of $\mathcal{O}(\mathcal{U}_z)$ consisting of integrable $\mathcal{U}_z$-modules belonging to $\mathcal{O}(\mathcal{U}_z)$.

For each coset $C = \mu + Q \in P/Q$ we denote by $\mathcal{O}_C(\mathcal{U}_z)$ the full subcategory of $\mathcal{O}(\mathcal{U}_z)$ consisting of $V \in \mathcal{O}(\mathcal{U}_z)$ such that $V = \bigoplus_{\lambda \in C} V_\lambda$. We also set $\mathcal{O}^{\text{int}}_C(\mathcal{U}_z) = \mathcal{O}_C(\mathcal{U}_z) \cap \mathcal{O}^{\text{int}}(\mathcal{U}_z)$. Then we have

$$\mathcal{O}(\mathcal{U}_z) = \bigoplus_{C \in P/Q} \mathcal{O}_C(\mathcal{U}_z), \quad \mathcal{O}^{\text{int}}(\mathcal{U}_z) = \bigoplus_{C \in P/Q} \mathcal{O}^{\text{int}}_C(\mathcal{U}_z).$$
For \( \lambda \in P \) we define \( M_z(\lambda) \in \mathcal{O}_{\lambda+Q}(\overline{U}_z) \) by

\[
M_z(\lambda) = \overline{U}_z / \left( \sum_{h \in \mathfrak{h}_\mathbb{Z}} \overline{U}_z(h) - z^{(\lambda,h)} + \sum_{i \in I} \overline{U}_z e_i \right),
\]

and for \( \lambda \in P^+ \) we define \( V_z(\lambda) \in \mathcal{O}^{\text{int}}_{\lambda+Q}(\overline{U}_z) \) by

\[
V_z(\lambda) = \overline{U}_z / \left( \sum_{h \in \mathfrak{h}_\mathbb{Z}} \overline{U}_z(h) - z^{(\lambda,h)} + \sum_{i \in I} \overline{U}_z e_i + \sum_{i \in I} \overline{U}_z f_i^{(\lambda,h_i)+1} \right).
\]

Let \( \lambda \in P \). A \( \overline{U}_z \)-module \( V \) is called a highest weight module with highest weight \( \lambda \) if there exists \( v \in V_\lambda \setminus \{0\} \) such that \( V = \overline{U}_z v \) and \( xv = \varepsilon(x)v \) (\( x \in \overline{U}_z^+ \)). Then we have \( V \in \mathcal{O}_{\lambda+Q}(\overline{U}_z) \). A \( \overline{U}_z \)-module is a highest weight module with highest weight \( \lambda \) if and only if it is a non-zero quotient of \( M_z(\lambda) \). If there exists an integrable highest weight module with highest weight \( \lambda \), then we have \( \lambda \in P^+ \). For \( \lambda \in P^+ \) a \( \overline{U}_z \)-module is an integrable highest weight module with highest weight \( \lambda \) if and only if it is a non-zero quotient of \( V_z(\lambda) \).

For \( V \in \mathcal{O}(\overline{U}_z) \) we define its formal character by

\[
\text{ch}(V) = \sum_{\lambda \in P} \dim V_\lambda e(\lambda) \in \mathcal{E}.
\]

We have

\[
\text{ch}(M_z(\lambda)) = e(\lambda) \overline{D}^{-1} \quad (\lambda \in P),
\]

where

\[
\overline{D}^{-1} = \sum_{\gamma \in Q^+} \dim \overline{U}_{z,-\gamma} e(-\gamma) \quad (\lambda \in P).
\]

For each coset \( C = \mu + Q \in P/Q \) we fix a function \( f_C : C \to \mathbb{Z} \) such that

\[
f_C(\lambda) - f_C(\lambda - \alpha_i) = 2\langle \lambda, t_i \rangle \quad (\lambda \in C, i \in I).
\]

**Remark 4.3.** The function \( f_C \) is unique up to addition of a constant function. If we extend \( (\, , \lambda) : E 	imes E \to \mathbb{Q} \) to a \( W \)-invariant symmetric bilinear form on \( \mathfrak{h}^* \), then \( f_C \) is given by

\[
f_C(\lambda) = (\lambda + \rho, \lambda + \rho) + a \quad (\lambda \in C)
\]

for some \( a \in \mathbb{Q} \).

For \( \gamma \in Q^+ \) let \( \overline{C}_\gamma \in \overline{U}_{z,\gamma} \otimes \overline{U}_{z,-\gamma} \) be the canonical element of the non-degenerate bilinear form \( \overline{\tau}_{z,\gamma} \). Following Drinfeld we set

\[
\Omega_\gamma = (m \circ (S \otimes 1) \circ P)(\overline{C}_\gamma) \in \overline{U}_{z,-\gamma} \Delta \overline{U}_z \overline{U}_{z,\gamma},
\]

where \( m : \overline{U}_z \otimes \overline{U}_z \to \overline{U}_z \) and \( P : \overline{U}_z \otimes \overline{U}_z \to \overline{U}_z \otimes \overline{U}_z \) are given by \( m(a,b) = ab, P(a \otimes b) = b \otimes a \) (see [12, Section 3.2], [11, Section 6.1]). Let \( C \in P/Q \). For \( V \in \mathcal{O}_C(\overline{U}_z) \) we define a linear map

\[
(4.12) \quad \Omega : V \to V
\]
by
\[
\Omega(v) = z^{f_C(\lambda)} \sum_{\gamma \in Q^+} \Omega_{\gamma} v \quad (v \in V_{\lambda}).
\]

This operator is called the quantum Casimir operator. As in [12, Section 3.2] we have the following.

**Proposition 4.4.** Let \( C \in P/Q \). For \( \lambda \in C \) the operator \( \Omega \) acts on \( M_z(\lambda) \) as \( z^{f_C(\lambda)} \text{id} \).

Since \( z \) is not a root of 1, we have
\[
z^{f_C(\lambda)} = z^{f_C(\mu)} \implies f_C(\lambda) = f_C(\mu).
\]

5. **Main results**

For \( w \in W \) and \( x = \sum_{\lambda \in P} c_\lambda e(\lambda) \in \mathcal{E} \) we set
\[
wx = \sum_{\lambda \in P} c_\lambda e(w(\lambda)), \quad w \circ x = \sum_{\lambda \in P} c_\lambda e(w \circ \lambda).
\]

The elements \( wx, w \circ x \) may not belong to \( \mathcal{E} \); however, we will only consider the case where \( wx, w \circ x \in \mathcal{E} \).

We denote by \( \text{sgn} : W \to \{\pm 1\} \) the character given by \( \text{sgn}(s_i) = -1 \) for \( i \in I \).

**Proposition 5.1.** For any \( w \in W \) we have \( w \circ D = \text{sgn}(w)D \).

**Proof.** We may assume that \( w = s_i \) for \( i \in I \). Define \( D_i, \overline{D}_i \in \mathcal{E} \) by
\[
D = (1 - e(-\alpha_i))D_i, \quad \overline{D} = (1 - e(-\alpha_i))\overline{D}_i.
\]

Then we have \( D_i = \prod_{\alpha \in \Delta^+ \setminus \{\alpha_i\}} (1 - e(-\alpha))^m_\alpha \). Moreover, by Lemma 3.2 and 4.2 and (4.9) we have
\[
D_i^{-1} = \sum_{\gamma \in Q^+} \dim(U_{\gamma}^{-} \cap T_i(U_{\gamma})) e(-\gamma)
\]
\[
= \sum_{\gamma \in Q^+} \dim(U_{\gamma}^{-} \cap T_i^{-1}(U_{\gamma})) e(-\gamma),
\]
\[
\overline{D}_i^{-1} = D_i^{-1} - \sum_{\gamma \in Q^+} \dim(J_{\gamma}^{-} \cap T_i(U_{\gamma})) e(-\gamma)
\]
\[
= D_i^{-1} - \sum_{\gamma \in Q^+} \dim(J_{\gamma}^{-} \cap T_i^{-1}(U_{\gamma})) e(-\gamma).
\]
By \( s_i \circ D = -(1 - e(-\alpha_i))s_i D_i \) we have only to show \( s_i D_i = D_i \). By Lemma 4.2 we have
\[
\sum_{\gamma \in \mathcal{Q}^+} \dim(J_{z, -\gamma} \cap T_i(U_z^-))^e(-\gamma))
= \sum_{\gamma \in \mathcal{Q}^+} \dim(J_z^- \cap T_i(U_z^-))^e(-\gamma)
= \sum_{\gamma \in \mathcal{Q}^+} \dim((J_{z, -\gamma} \cap T_i^{-1}(U_z^-))^e(-\gamma),
\]
and hence the assertion follows from \( s_i D_i = D_i \).

\[\square\]

**Proposition 5.2.** Let \( \lambda \in \mathcal{P}^+ \). Assume that \( V \) is an integrable highest weight \( U_z \)-module with highest weight \( \lambda \). Then we have
\[
\text{ch}(V) = \sum_{w \in \mathcal{W}} \text{sgn}(w) \text{ch}(M_z(w \circ \lambda)).
\]

**Proof.** The proof below is the same as the one for Lie algebras in Kac [6, Theorem 10.4].

Set \( C = \lambda + Q \in \mathcal{P}/\mathcal{Q} \). Similarly to [6, Proposition 9.8] we have
\[
\text{ch}(V) = \sum_{\mu \in \lambda - Q^+, f_C(\mu) = f_C(\lambda)} c_\mu \text{ch}(M_z(\mu)) \quad (c_\mu \in \mathbb{Z}, c_\lambda = 1).
\]

Multiplying (5.1) by \( D \) we obtain
\[
D \text{ch}(V) = \sum_{\mu \in \lambda - Q^+, f_C(\mu) = f_C(\lambda)} c_\mu e(\mu).
\]

Using the action of \( T_i \) \((i \in I)\) on \( V \) we see that \( w \text{ch}(V) = \text{ch}(V) \) for \( w \in \mathcal{W} \), and hence \( w \circ ((D \text{ch}(V)) = \text{sgn}(w)D \text{ch}(V) \) for any \( w \in \mathcal{W} \). It follows that
\[
c_\mu = \text{sgn}(w)c_{w_0 \mu} \quad (\mu \in \lambda - Q^+, w \in \mathcal{W}).
\]

Assume that \( \mu \in \lambda - Q^+ \) satisfies \( c_\mu \neq 0 \). By (5.2) \( W \circ \mu \subseteq \lambda - Q^+ \), and hence we can take \( \mu' \in W \circ \mu \) such that \( \text{ht}(\lambda - \mu') \) is minimal, where \( \text{ht}(\sum_i m_i \alpha_i) = \sum_i m_i \). Then we have \( \langle \mu', h_i \rangle \geq 0 \) for any \( i \in I \) by \( s_i \circ \mu' = \mu' - (\langle \mu', h_i \rangle + 1)\alpha_i \) and (5.2). Namely, we have \( \mu' \in P^+ \). Then by [6, Lemma 10.3] we obtain \( \mu' = \lambda \). \[\square\]

**Remark 5.3.** I. Heckenberger pointed out to me that Proposition 5.2 also follows from the existence of the BGG resolution of integrable highest weight modules of quantized enveloping algebras given in [5].

Recall that any integrable highest weight module \( V \) with highest weight \( \lambda \) is a quotient of \( V_z(\lambda) \). Proposition 5.2 tells us that its character \( \text{ch}(V) \) only depends on \( \lambda \). It follows that any integrable highest weight module with highest weight \( \lambda \) is isomorphic to \( V_z(\lambda) \).
Consider the case $\lambda = 0$. Since $V_z(0)$ is the trivial one-dimensional module, we obtain the identity
\[
1 = \left( \sum_{w \in W} \text{sgn}(w)e(w \circ 0) \right) \left( \sum_{\gamma \in Q^+} \dim U_{z,-\gamma}^- e(\gamma) \right)
\]
in $E$ by Proposition 5.2. On the other hand by the corresponding result for the Kac-Moody Lie algebra we have
\[
1 = \left( \sum_{w \in W} \text{sgn}(w)e(w \circ 0) \right) \left( \sum_{\gamma \in Q^+} \dim U_{z,-\gamma}^- e(\gamma) \right).
\]
It follows that $U_{z,-\gamma}^- \cong U_{z,-\gamma}^-$ for any $\gamma \in Q^+$. By $\dim U_{z,-\gamma}^- = \dim U_{z,\gamma}^+$ and the non-degeneracy of $\tau_{z,\gamma}$ we also have $U_{z,\gamma}^+ \cong U_{z,\gamma}^+$ for any $\gamma \in Q^+$. We have obtained the following results.

**Theorem 5.4.** The Drinfeld pairing
\[
\tau_{z,\gamma} : U_{z,\gamma}^+ \times U_{z,-\gamma}^- \to K
\]
is non-degenerate for any $\gamma \in Q^+$.

**Theorem 5.5.** Let $\lambda \in P^+$. Assume that $V$ is an integrable highest weight $U_z$-module with highest weight $\lambda$. Then we have
\[
\text{ch}(V) = D^{-1} \sum_{w \in W} \text{sgn}(w)e(w \circ \lambda).
\]

By Theorem 5.6 we can define the quantum Casimir operator $\Omega$ for $U_z$. As in [11, Section 6.2] we have the following.

**Theorem 5.6.** Any object of $O^{\text{int}}(U_z)$ is a direct sum of $V_z(\lambda)$’s for $\lambda \in P^+$.

By Theorem 5.4 we have the following.

**Theorem 5.7.** Let $\gamma \in Q^+$. Take bases $\{x_r\}$ and $\{y_s\}$ of $U_{\hat{h},\gamma}^+$ and $U_{\hat{h},-\gamma}^-$ respectively, and set $f_\gamma = \det(\tau_{h,\gamma}(x_r,y_s))_{r,s}$. Then we have $f_\gamma \in \hat{A}^\times$. Namely, we have
\[
f_\gamma = \pm q^a f_1^{\pm 1} \cdots f_N^{\pm 1},
\]
where $a \in \mathbb{Z}$, and $f_1, \ldots, f_N \in \mathbb{Z}[q]$ are cyclotomic polynomials.

**Proof.** We can write $f_\gamma = mgh$, where $m \in \mathbb{Z}_{>0}$, $g \in \mathbb{Z}[q]$ is a primitive polynomial with $g(0) > 0$ whose irreducible factor is not cyclotomic, and $h \in \hat{A}^\times$. Note that for any field $K$ and $z \in K^\times$ which is not a root of 1, the specialization of $f_\gamma$ with respect to the ring homomorphism $\hat{A} \to K (q \mapsto z)$ is non-zero by Theorem 5.4. Hence we see easily that $m = 1$ and $g = 1$. \qed
In the finite case Theorem 5.7 is well-known (see [8], [9], [11]). In the affine case this is a consequence of Damiani [3], [4], where $\det(\tau_{h,\gamma}(x_r, y_s))_{r,s}$ is determined explicitly by a case-by-case calculation.

References

[1] H. Andersen, P. Polo, K. Wen, Representations of quantum algebras. Invent. Math. 104 (1991), 1–59.
[2] V. Chari and N. Jing, Realization of level one representations of $U_q(\hat{g})$ at a root of unity. Duke Math. J. 108 (2001), 183–197.
[3] I. Damiani, The highest coefficient of $\det H_q$ and the center of the specialization at odd roots of unity for untwisted affine quantum algebras. J. Algebra 186 (1996), no. 3, 736–780.
[4] I. Damiani, The R-matrix for (twisted) affine quantum algebras. Representations and quantizations (Shanghai, 1998), 89–144, China High. Educ. Press, Beijing, 2000.
[5] I. Heckenberger, S. Kolb, On the Bernstein-Gelfand-Gelfand resolution for Kac-Moody algebras and quantized enveloping algebras. Transform. Groups 12 (2007), no. 4, 647–655.
[6] V. Kac, Infinite dimensional Lie algebras. Third edition. Cambridge University Press, Cambridge, 1990.
[7] M. Kashiwara, On crystal bases. Representations of groups (Banff, AB, 1994), 155–197, CMS Conf. Proc., 16, Amer. Math. Soc., Providence, RI, 1995.
[8] A. N. Kirillov, N. Reshetikhin, $q$-Weyl group and a multiplicative formula for universal $R$-matrices. Comm. Math. Phys. 134 (1990), no. 2, 421–431.
[9] S. Z. Levendorskii, Ya. S. Soibelman, Some applications of the quantum Weyl groups. J. Geom. Phys. 7 (1990), no. 2, 241–254.
[10] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras. Adv. in Math. 70 (1988), 237–249.
[11] G. Lusztig, Introduction to quantum groups. Progr. Math., 110, Boston etc. Birkhäuser, 1993.
[12] T. Tanisaki, Killing forms, Harish-Chandra isomorphisms, and universal $R$-matrices for quantum algebras. Inter. J. Mod. Phys. A7, Suppl. 1B (1992), 941–961.
[13] T. Tanisaki, Modules over quantized coordinate algebras and PBW-bases. to appear in J. Math. Soc. Japan, arXiv:1409.7973
[14] T. Tanisaki, Invariance of the Drinfeld pairing of a quantum group. to appear in Tokyo J. Math., arXiv:1503.04573
[15] S. Tsuchioka, Graded Cartan determinants of the symmetric groups. Trans. Amer. Math. Soc. 366 (2014), 2019–2040.

Department of Mathematics, Osaka City University, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585 Japan
E-mail address: tanisaki@sci.osaka-cu.ac.jp