BRIEF COMMUNICATIONS

An Analogue of the Perelomov–Popov Formula for the Lie Superalgebra $q(N)$

T. A. Grigoryev and M. L. Nazarov

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ABSTRACT. We study the center of the universal enveloping algebra of the strange Lie superalgebra $q(N)$. We obtain an analogue of the well-known Perelomov–Popov formula [6] for the central elements of this algebra—an expression of the central characters through the highest weight parameters.

KEY WORDS: Lie superalgebras, central characters.

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1. Introduction. 1.1. The Lie superalgebra $q(N)$. The strange Lie superalgebra $q(N)$ can be realized as a subalgebra in the general linear Lie superalgebra $gl(N|N)$ over the complex field; see, e.g., [2]. If the indices $i$ and $j$ range over $-N,\ldots,-1,1,\ldots,N$, then the $4N^2$ elements $E_{ij}$ span the algebra $gl(N|N)$ as a vector space. The Lie superbracket on $gl(N|N)$ is defined by

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - (-1)^{(i+j)(k+l)}\delta_{il}E_{kj},$$

where

$$\overline{k} = \begin{cases} 0 & \text{if } k > 0, \\ 1 & \text{if } k < 0. \end{cases}$$

Then $q(N) \subset gl(N|N)$ is the subalgebra of fixed points of the involution

$$\eta: E_{ij} \mapsto E_{-i-j}.$$ Hence, as a vector space, $q(N)$ is spanned by the $2N^2$ elements

$$F_{ij} = E_{ij} + E_{-i-j}$$ with $i > 0$. The Lie superbracket on $q(N)$ is then described by

$$[F_{ij}, F_{kl}] = \delta_{kj}F_{il} - (-1)^{(i+j)(k+l)}\delta_{il}F_{kj} + \delta_{k-j}F_{-i-l} - (-1)^{(i+j)(k+l)}\delta_{-i}F_{k-j}.$$ (1)

1.2. Casimir elements in $U(q(N))$. Unless otherwise stated, we will assume that the indices in the expressions below range over $-N,\ldots,-1,1,\ldots,N$. For any $n = 1,2,\ldots$, consider the elements of the universal enveloping algebra $U(q(N))$ first proposed in [7]:

$$C_{ij}^{(n)} = \sum_{k_1,\ldots,k_n-1} F_{i k_1} (-1)^{\overline{k}} F_{k_1 k_2} (-1)^{\overline{k}_2} \cdots F_{k_{n-2} k_{n-1}} (-1)^{\overline{k}_{n-1}} F_{k_{n-1} j}.$$ (2)

Taking into account the recurrence relations

$$C_{ij}^{(n+1)} = \sum_{k} F_{ik}(-1)^{\overline{k}} C_{kj}^{(n)},$$ one can find the supercommutator

$$[F_{ij}, C_{kl}^{(n)}] = \delta_{kj}C_{il}^{(n)} - (-1)^{(i+j)(k+l)}\delta_{il}C_{kj}^{(n)} + \delta_{k-j}C_{-i-l}^{(n)} - (-1)^{(i+j)(k+l)}\delta_{-i}C_{k-j}^{(n)}.$$
which is similar to the superbracket (1) between the generators of \(q(N)\). It is then easy to see that the elements

\[ c_n = \sum_i C_{ii}^{(n)} \]

are central in \(U(q(N))\). Moreover, by using the relation

\[ C_{-i-j}^{(n)} = (-1)^{n-1} C_i^{(n)} \]

which follows from (2), we see that \(c_n = 0\) if \(n\) is even. This is why we will be only interested in the \(c_n\) with \(n\) odd. These are the Casimir elements for the Lie superalgebra \(q(N)\), as introduced in [3]. It was shown in [4] that these elements generate the center of \(U(q(N))\).

1.3. The Harish-Chandra homomorphism. Let \(v\) be a singular vector of any irreducible finite-dimensional representation of the Lie superalgebra \(q(N)\) relative to the natural triangular decomposition \(q(N) = n_+ \oplus h \oplus n_+\), where

\[ n_+ = \text{span}\{F_{ij} : |i| > |j|\}, \quad h = \text{span}\{F_{ij} : |i| = |j|\}, \quad n_+ = \text{span}\{F_{ij} : |i| < |j|\}. \]

Then the following equalities hold:

\[ F_{ij} \cdot v = 0 \quad \text{for } |i| < |j|, \]
\[ F_{ii} \cdot v = \lambda_i v \quad \text{for } i > 0. \]

Here the \(\lambda_i\) are the eigenvalues of the elements \(F_{ii} = F_{-i-i}\) of the even part of the Cartan subalgebra \(h_0 = \text{span}\{F_{ii} : i > 0\}\). They depend on the particular representation of \(q(N)\). Let \(\lambda \in h_0^*\) be the highest weight of the representation, so that \(\lambda(F_{ii}) = \lambda_i\) for \(i > 0\).

The generators of \(q(N)\) in the summands of (2) can always be rearranged in such a way that, in each monomial, first (left-to-right) go the lowering operators, then the operators from the Cartan subalgebra, and last the raising operators. Here the lowering operators are elements of \(n_+\) and the raising operators are elements of \(n_+\). The operators from the Cartan subalgebra can also be rearranged so that the elements from its even part \(h_0\) go after those from its odd part \(h_1 = \text{span}\{F_{-ii} : i > 0\}\). It suffices to use the supercommutation relations (1) to achieve this. The part of the resulting sum which belongs to \(U(h)\) is well defined. For the sum (2) with \(i = j\), this is its image under the Harish-Chandra homomorphism \(\chi\); see [2].

The subspace formed by the vectors of an irreducible representation of \(q(N)\) satisfying (3) and (4) is called the singular subspace. The strangeness of \(q(N)\) shows in the fact that the singular subspace of an irreducible representation is not one-dimensional but is irreducible over the Cartan subalgebra \(h = h_0 \oplus h_1\), for which we have \(U(h) = S(h_0) \otimes (h_1)\); see [5].

Due to (2), the Casimir elements are even and therefore commute with the whole algebra \(U(q(N))\) in the usual non-\(\mathbb{Z}_2\)-graded sense. This implies that they act as scalar operators on the irreducible representations. This allows us to consider their eigenvalues when their action is restricted to singular vectors, and the computations in this case are rather simple.

Let \(n\) be odd, and let \(v\) be a singular vector of an irreducible representation of \(q(N)\). Then

\[ c_n \cdot v = \chi(c_n) \cdot v = \chi(c_n)|_{F_{ii}=\lambda_i} v \quad \text{for } i > 0. \]

In what follows, we will assume that the application of the homomorphism \(\chi\) is always followed by the substitution \(F_{ii} \mapsto \lambda_i\), as in (5). Below we will explicitly describe this action on the singular vector.

2. Computations. 2.1. Recurrence relations. Here we will derive a recurrence relation for the images of the elements \(C_{ii}^{(n)}\) with \(n = 1, 3, \ldots\) under the Harish-Chandra homomorphism. For brevity, we will denote the element \(F_{-ii} = F_{i-1}\) of the odd part of the Cartan subalgebra by \(G_{i-1}\).

**Proposition 1.** For \(i > 0\), \(\chi(G_i^2) = \lambda_i\).

**Proof.** We have \(G_i^2 \cdot v = F_{-ii}^2 \cdot v = \frac{1}{2}[F_{-ii}, F_{-ii}] \cdot v = F_{ii} \cdot v = \lambda_i v\) for \(i > 0\).

**Proposition 2.** Whenever \(|i| < |j|\), \(C_{ij}^{(n)} \cdot v = 0\).
Proposition 5. For $i > 0$,

$$
\chi(C_{ii}^{(n+1)}) = \lambda_i \chi(C_{ii}^{(n)}) - G_i \chi(C_{-ii}^{(n)}) - \sum_{|k| > i} \chi(C_{kk}^{(n)})
$$

Proof. For $i > 0$, the vector $C_{ii}^{(n+1)} \cdot v$ equals

$$
\sum_{|k| > i} F_{ik}(-1)^{k} C_{ki}^{(n)} \cdot v = (F_{ii} C_{ii}^{(n)} - F_{ii} C_{-ii}^{(n)}) \cdot v + \sum_{|k| > i} (-1)^k [F_{ik}, C_{ki}^{(n)}] \cdot v
\]

$$
= (\lambda_i C_{ii}^{(n)} - G_i C_{-ii}^{(n)}) \cdot v + \sum_{|k| > i} (-1)^k (C_{ii}^{(n)} - (-1)^k C_{kk}^{(n)}) \cdot v
\]

$$
= (\lambda_i C_{ii}^{(n)} - G_i C_{-ii}^{(n)} - \sum C_{kk}^{(n)}) \cdot v.
\]

Corollary 1. For $i > 0$ and $m = 1, 2, \ldots$, $\chi(C_{ii}^{(2m+1)}) = \lambda_i \chi(C_{ii}^{(2m)}) - G_i \chi(C_{-ii}^{(2m)})$.

Proposition 4. For $i > 0$,

$$
\chi(C_{-ii}^{(n+1)}) = G_i \chi(C_{ii}^{(n)}) - \lambda_i \chi(C_{-ii}^{(n)}) - \sum_{|k| > i} (-1)^k \chi(C_{kk}^{(n)})
$$

Proof. For $i > 0$, the vector $C_{-ii}^{(n+1)} \cdot v$ equals

$$
\sum_{|k| > i} F_{-ik}(-1)^{k} C_{ki}^{(n)} \cdot v = (F_{-ii} C_{ii}^{(n)} - F_{-ii} C_{-ii}^{(n)}) \cdot v + \sum_{|k| > i} (-1)^k [F_{-ik}, C_{ki}^{(n)}] \cdot v
\]

$$
= (G_i C_{ii}^{(n)} - \lambda_i C_{-ii}^{(n)}) \cdot v + \sum_{|k| > i} (-1)^k (C_{-ii}^{(n)} - (-1)^k C_{k-k}^{(n)}) \cdot v
\]

$$
= (G_i C_{ii}^{(n)} - \lambda_i C_{-ii}^{(n)} - \sum C_{k-k}^{(n)}) \cdot v.
\]

Corollary 2. For $i > 0$ and $m = 1, 2, \ldots$, $\chi(C_{-ii}^{(2m)}) = G_i \chi(C_{ii}^{(2m-1)}) - \lambda_i \chi(C_{-ii}^{(2m-1)})$.

Proposition 5. For $i > 0$ and $m = 1, 2, \ldots$,

$$
\chi(C_{ii}^{(2m+1)}) = \chi(C_{-i,-i}^{(2m+1)}) = \lambda_i (\lambda_i - 1) \chi(C_{ii}^{(2m-1)}) - 2\lambda_i \sum_{j > i} \chi(C_{jj}^{(2m-1)})
$$

Proof. If $i > 0$, then the vector $C_{ii}^{(2m+1)} \cdot v$ equals

$$
(\lambda_i C_{ii}^{(2m)} - G_i C_{-ii}^{(2m-1)}) \cdot v = \lambda_i \left( \lambda_i C_{ii}^{(2m-1)} - G_i C_{-ii}^{(2m-1)} - \sum_{|j| > i} C_{jj}^{(2m-1)} \right) \cdot v
\]

$$
- G_i (G_i C_{ii}^{(2m-1)} - \lambda_i C_{-ii}^{(2m-1)}) \cdot v
\]

$$
= (\lambda_i^2 - G_i^2) C_{ii}^{(2m-1)} \cdot v - \lambda_i \sum_{|j| > i} C_{jj}^{(2m-1)} \cdot v
\]

$$
= \lambda_i (\lambda_i - 1) C_{ii}^{(2m-1)} \cdot v - 2\lambda_i \sum_{j > i} C_{jj}^{(2m-1)} \cdot v.
\]
Corollary 3. For $i > 0$ and $m = 0, 1, 2, \ldots$, \( \chi(C_{ii}^{(2m+1)}) = \sum_{j=1}^{N} (A^m)_{ij} \lambda_j \), where

\[
A = \begin{pmatrix}
\lambda_1(\lambda_1 - 1) & -2\lambda_1 & \cdots & -2\lambda_1 \\
0 & \lambda_2(\lambda_2 - 1) & \cdots & -2\lambda_2 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \lambda_N(\lambda_N - 1)
\end{pmatrix}.
\]

Proof. This follows from Proposition 5 with the relation \( \chi(C_{ii}^{(1)}) = \chi(F_{ii}) = \lambda_i \) taken into account.

2.2. Generating functions. In order to compute \( \chi(c_{2m+1}) \) more explicitly, for each \( i > 0 \), we consider the generating function

\[
\mu_i(u) = \sum_{m=0}^{\infty} \chi(C_{ii}^{(2m+1)}) u^{-2m-1} = u \sum_{j=1}^{N} ((u^2 - A)^{-1})_{ij} \lambda_j
\]

and write

\[
A = \Lambda(\Lambda - 1) - 2\Lambda A(1 - \Delta)^{-1},
\]

where \( \Delta_{ij} = \delta_{i,j-1} \) and \( \Lambda = \text{diag}(\lambda_i) \). We set

\[
\Pi = (u^2 - \Lambda(\Lambda + 1))(u^2 - \Lambda(\Lambda - 1))^{-1}.
\]

Then

\[
(u^2 - A)^{-1} = \frac{1}{2} (1 - \Pi)(1 - \Delta \Pi)^{-1}(1 - \Delta)^{-1}.
\]

The last two factors here cause all but the last summand in (6) to cancel, leaving

\[
\mu_i(u) = \frac{u}{2} (1 - \Pi_{ii})(1 - \Delta \Pi)^{-1}_{ii} = \frac{u}{2} (1 - \Pi_{ii}) \prod_{j>i} \Pi_{jj}.
\]

This can also be written in a more straightforward way:

\[
\mu_i(u) = \frac{u \lambda_i}{u^2 - \lambda_i(\lambda_i - 1)} \prod_{j>i} \frac{u^2 - \lambda_j(\lambda_j + 1)}{u^2 - \lambda_j(\lambda_j - 1)}.
\]

In order to find a generating function \( \mu(u) \) of the images of the central elements under \( \chi \), it suffices to sum all expressions for \( \mu_i(u) \) obtained above over the positive values of the index \( i \) and recall that \( C_{-i-i}^{(2m+1)} = C_{ii}^{(2m+1)} \). Making further cancellations, we obtain

\[
\mu(u) = \sum_{m=0}^{\infty} \chi(c_{2m+1}) u^{-2m-1} = 2 \sum_{i=1}^{N} \mu_i(u) = u \left( 1 - \prod_{j=1}^{N} \Pi_{jj} \right).
\]

Finally, we arrive at

\[
\mu(u) = u \left( 1 - \prod_{i=1}^{N} \frac{u^2 - \lambda_i(\lambda_i + 1)}{u^2 - \lambda_i(\lambda_i - 1)} \right).
\]

2.3. The images of the central elements. We introduce a new variable \( z = u^2 \) and set

\[
\tilde{\mu}(z) = u \mu(u) = \sum_{m=0}^{\infty} \chi(c_{2m+1}) z^m = \frac{1}{z} \left( 1 - \prod_{i=1}^{N} \frac{1 - z \lambda_i(\lambda_i + 1)}{1 - z \lambda_i(\lambda_i - 1)} \right).
\]
From the above definition it immediately follows that \( \chi(c_{2m+1}) = \text{Res}_0 \tilde{\mu}(z) z^{-m-1} dz \). Combining this with our explicit expression for \( \tilde{\mu}(z) \), we see that \( \chi(c_{2m+1}) \) equals

\[
- \sum_{i=1}^N \text{Res}_{(\lambda_i(\lambda_i-1))^{-1}} \tilde{\mu}(z) z^{-m-1} dz = \sum_{i=1}^N \text{Res}_{(\lambda_i(\lambda_i-1))^{-1}} \prod_{j=1}^N \frac{1-z\lambda_j(\lambda_j+1)}{1-z\lambda_j(\lambda_j-1)} z^{-m-2} dz,
\]

where the regularity of the form at infinity is taken into account. Finally, for \( m \geq 0 \),

\[
\chi(c_{2m+1}) = 2 \sum_{i=1}^N \lambda_i^{m+1} \frac{(\lambda_i - 1)^m}{\prod_{j \neq i} \lambda_i(\lambda_i - 1) - \lambda_j(\lambda_j - 1)}.
\]

Note that, despite the fact that the last expression is formally a rational function of \( \lambda \), its denominator always cancels, which allows us to regard \( \chi(c_{2m+1}) \) as a polynomial in \( \lambda \).

**Remarks.** The analogue of the Perelomov–Popov formula for the Lie superalgebra \( q(N) \) presented here was obtained by the second-named author about 30 years ago but left unpublished. Independently but by the same method, this analogue was obtained in [1] and, more recently, by the first-named author. Publishing this note gives us an opportunity to review the history of this analogue.

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Skolkovo Institute of Science and Technology, Moscow, Russia
National Research University Higher School of Economics, Moscow, Russia
e-mail: grigorev.ta@phystech.edu

Department of Mathematics, University of York, York, United Kingdom
e-mail: maxim.nazarov@york.ac.uk

Translated by T. A. Grigoryev and M. L. Nazarov