Integral Expressions for the Vassiliev Knot Invariants

Dylan Thurston

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Abstract

It has been folklore for several years in the knot theory community that certain integrals on configuration space, originally motivated by perturbation theory for the Chern-Simons field theory, converge and yield knot invariants. This was proposed independently by Gaudagnini, Martellini, and Mintchev [11] and Bar-Natan [4]. The analytic difficulties involved in proving convergence and invariance were reportedly worked out by Bar-Natan [4, 3], Kontsevich [13, 16], and Axelrod and Singer [1, 2]. But I know of no elementary exposition of this fact. Bar-Natan [3] only proves invariance for the degree 2 invariant. Kontsevich’s exposition is decidedly non-elementary and leaves many details implicit. Axelrod and Singer’s papers take a physics point of view (and are thus difficult for mathematicians to read) and only discuss the related invariants for 3-manifolds. More recently, Bott and Taubes [7] have explained the degree 2 invariant from a purely topological point of view, but again omit the higher degree cases.

This thesis is an attempt to remedy this lack. I adopt an almost exclusively topological point of view, rarely mentioning Chern-Simons theory. For an explanation of the physics involved, see, for instance, Bar-Natan [4].

There are also a few new results in this thesis. These include a new construction of the functorial compactification of configuration space (Section 3.2) as well as some variations on the integrals. For a suitable choice of this variation, the integral reduces to counting *tinkertoy diagrams* (Section 4.3). In particular, the invariants constructed take values in $\mathbb{Q}$. 


## Contents

1 Introduction  
1.1 Motivation .............................................. 2  
1.2 Linking Number ....................................... 3  
1.3 Back to Knots .......................................... 6  
1.4 Acknowledgements ...................................... 8  

2 Vassiliev Invariants and Weight Diagrams .......... 9  

3 Configuration space and its Compactification .... 14  
3.1 A description of the compactification ................. 14  
3.2 Precise definitions .................................... 16  
3.3 Further variations ..................................... 19  

4 Tying it Together ......................................... 21  
4.1 Integrals are Vassiliev .................................. 21  
4.2 On the Pushforward ..................................... 22  
4.3 The Form of the Integrals .............................. 24  
4.4 Principal Faces Cancel ................................. 25  
4.5 Tinkertoys ............................................. 27  

5 The Hidden Faces ......................................... 29  
5.1 Preliminaries ............................................ 29  
5.2 Conditions for Vanishing ............................... 31  
5.3 Most Hidden Faces Vanish ............................. 32  

6 Anomalous Faces ......................................... 34  
6.1 Preliminary Remarks .................................... 34  
6.2 Parity of Invariants .................................... 36  

A Some linear algebra ...................................... 40  
A.1 Unordered Collections of Vector Spaces .............. 41  
A.2 Exact Sequences ....................................... 43  

B Graphs ..................................................... 45
Chapter 1

Introduction

1.1 Motivation

First, I’ll present some motivation for introducing these invariants, primarily intended for topologists.

A (mathematical) knot is a loop in \( \mathbb{R}^3 \): an embedding \( K : S^1 \to \mathbb{R}^3 \). An embedding \( K \) is a smooth map with two conditions:

1. Immersion condition: \( \frac{dK}{d\theta}(\theta) \neq 0, \forall \theta \in S^1 \)
2. Non self-intersecting: \( K(\theta_0) \neq K(\theta_1), \forall \theta_0, \theta_1 \in S^1 \)

The first condition is purely local and not terribly interesting. (In fact, removing that condition yields an essentially equivalent space.) The second condition is much more interesting. It’s a global condition and is what makes the study of knots much harder than the study of loops. How can we reduce it to a finite condition? One answer comes from the tradition of raising maps to the product of the spaces involved. Observe that from the map \( K \), we can construct

\[
K^{\times n} : (S^1)^n \to (\mathbb{R}^3)^n
\]

\[
(\theta_1, \ldots, \theta_n) \mapsto (K(\theta_1), \ldots, K(\theta_n))
\] (1.1)

This may seem like a trivial remark, but it has far-reaching consequences. In particular, because of condition 2, \( K^{\times n} \) restricts to a map between the configuration spaces. The configuration space \( C^n_0(M) \) of nil points in a manifold \( M \) is the space of \( n \) distinct points in \( M \). Formally,

\[
C^n_0(M) = \{(x_1, x_2, \ldots, x_n) \in M^n : x_i \neq x_j \text{ for } i \neq j\}
\] (1.2)

and, if \( f : M \to N \) is an embedding, we can define

\[
C^n_0(f) : C^n_0(M) \to C^n_0(N)
\] (1.3)

\(^1\)As opposed to a physical knot, which has a finite thickness and has ends.
Figure 1.1: Sign convention for the linking number. The link is viewed from directly above.

\[ K_2 \rightarrow K_1 \quad -1 \quad K_2 \rightarrow K_1 \]

Figure 1.2: Computation of the linking number for the Hopf link

as the restriction of \( f^n \) to \( C^0_n(M) \). You can therefore try to gain information about the map \( f \) by looking at the map on the configuration spaces \( f: C^n(M) \rightarrow C^n(N) \).

In this thesis, I begin to carry out this program in the case of knots. \( C^0_n(\mathbb{R}^3) \) has a useful set of homology generators given by the pullbacks of a 2-form on \( S^2 \) via the direction map for each pair of points in \( C^0_n(\mathbb{R}^3) \). By pulling back these forms to \( C^0_n(S^1) \) and taking sums so that the boundary terms cancel, I will construct explicit formulas for the Vassiliev knot invariants.

### 1.2 Linking Number

To help explain these ideas, I’ll give a simple example. The linking number is a well-known invariant of two-component links. (An \( n \)-component link is an embedding of a disjoint union of \( n \) circles in \( \mathbb{R}^3 \).) Label the two circles \( K_1 \) and \( K_2 \) and consider the set of \( K_1K_2 \)-crossings—points of \( K_2 \) that lie directly above a point of \( K_1 \). If the link is in general position, there will only be a finite number of crossings, and they will project to distinct points in the \( xy \) plane. Count up the crossings with signs, as in Figure 1.1. A crossing is counted with a positive sign if \( K_2 \) runs from left to right as you look along the direction of \( K_1 \), negative otherwise. See Figure 1.2 for an example of this computation of the linking number.

Because of the sign convention, the linking number turns out to be an invariant of the link; it doesn’t depend on the position of the link in space. See Figure 1.3, which shows invariance under the one Reidemeister move (see Kauffman 12) in which the set of \( K_1K_2 \)-crossings changes.
Figure 1.3: The linking number is an invariant.

Figure 1.4: The sign of a crossing is the sign of the local map from the torus $A \times B$ to the sphere. On the left are the sections of a knot near this crossing. Note that the view is not from above, so it doesn’t look like a crossing. On the right are the corresponding directions on the sphere.

Configuration space provides a nice way to understand this combinatorial formula topologically. Because of the type of crossings that we count, it makes sense to consider the subset of $C^0_2(K_1 \sqcup K_2)$ in which one point lies on $K_1$ and the other on $K_2$. Since $K_1 \cap K_2 = \emptyset$, this is isomorphic to $C^0_1(K_1) \times C^0_1(K_2) \simeq K_1 \times K_2$, which is a torus.

As in Section 1.1, we get a map $C^2(L) : C^2(K_1 \sqcup K_2) \to C^2_2(\mathbb{R}^3)$. There is also a map $\phi_{12} : C^0_2(\mathbb{R}^3) \to S^2$, given by

$$\phi_{12}(x_1, x_2) = \frac{x_2 - x_1}{|x_2 - x_1|} \quad (1.4)$$

(the denominator does not vanish in $C^0_2(\mathbb{R}^3)$). Composing these two, we get a map $\phi_{12} \circ C^2(L)$ from the torus to the sphere. The “crossings” we counted above were just those points on the torus mapping to the vertical direction on the sphere. If we orient both the torus and the sphere, the sign of the crossing defined above turns out to be the sign of the local mapping from the torus to the sphere (i.e., whether it locally preserves or reverses orientation), which will be non-degenerate for knots in general position. See Figure 1.4.

In fact, this number is independent of the direction (= point on the sphere) that we choose, as long as it’s a regular value. See Figure 1.5. This type of counting is

\footnote{In fact, a homotopy equivalence.}
Figure 1.5: The degree is independent of the point on the sphere used to choose it. As the point on the sphere moves from left to right, it crosses a fold, introducing two new inverse images; but they’re counted with opposite signs.

well-known in topology; it is known as the degree, and can be defined for any map $f : M \to N$ between oriented compact manifolds of the same dimension $n$.

To define the degree rigourously, note that the highest cohomology of a compact oriented $n$-manifold $M$, $H^n(M)$, is always 1-dimensional and has a natural generator $\omega$ (so that the integral of $\omega$ over the manifold is 1); this is called the fundamental class of $M$. The map $f : M \to N$ induces a map $f^* : H^n(N) \to H^n(M)$, taking the fundamental class of $N$ to some multiple of the fundamental class of $M$, which can be computed by integrating $f^*\omega$ over $M$. This multiple turns out to be exactly the degree. By Sard’s theorem, we know that there is some regular point $x$ of $N$, i.e., points $x$ for which $f$ is a diffeomorphism in the neighborhood of each inverse image of $x$. Since $M$ is compact, there are only a finite number of inverse images of $x$. If we take a representative $\omega$ of the fundamental class of $N$ that is sufficiently localized around $x$, then the integral of $f^*\omega$ over $M$ is exactly the degree as defined above.

But you need not pick a localized $\omega$. In our case, of a map from $S^1 \times S^1 \to S^2$, it’s natural to choose $\omega$ to be the unique $SO(3)$ invariant volume form on $S^2$,

$$\omega = \frac{\epsilon_{\mu\nu\sigma}}{8\pi} \frac{x^\mu \, dx^\nu \, dx^\sigma}{|x|^3}$$

(using physicists’ conventions: the components of the 3-vector $x$ are $x^\mu$, $\mu = 1, 2, 3$ and repeated indices in an expression are to be summed over, while $\epsilon_{\mu\nu\sigma}$ is the basic totally anti-symmetric tensor), then the integral of the pullback of $\omega$ over the knot becomes becomes

$$\text{Linking number} = \int_{S^1} dx_1^\mu \int_{S^1} dx_2^\nu \frac{\epsilon_{\mu\nu\sigma}}{4\pi} \frac{(x_2 - x_1)^\sigma}{|x_2 - x_1|^3}$$

$$= \int_{S^1 \times S^1} \theta_{12}$$

The first expression is again written in physicists’ notation (the $x_1^\mu$ are the components of the embedding of the $i$’th circle into $\mathbb{R}^3$). In the second, I use notation from
Bott and Taubes \[7\]:

\[ \theta_{12} = C_2(K)^* \phi_{12}^* \omega. \]

This is the simplest case of the integrals considered in this thesis. (Although I won’t consider links any further, the generalization of these integrals to links is straightforward.)

1.3 Back to Knots

But this case is misleadingly simple. Consider trying to modify the linking invariant to get an invariant of a single knot. As before, there is a canonical map \( \phi_{12} : C_2(K) : C_2(S^1) \to S^2 \). But now \( C_2(S^1) \) is not compact (it is the torus \( S^1 \times S^1 \) minus the diagonal) so there is no well-defined degree. Alternatively, if you compactify \( C_2(S^1) \) by adding a boundary, the number of inverse images of a point in \( S^2 \) changes as you cross the image of the boundary. See Figure 1.6.

The resolution of this difficulty is to take linear combinations of maps from spaces associated with the knot, each with a boundary, in such a way that the boundaries exactly cancel out in the target manifold. The resulting “degree” will automatically be an invariant of knots.

In the case of \( C_2(S^1) \), the necessary correction term comes from a framing of the knot: a nowhere zero section of the normal bundle to the knot (as imbedded in \( \mathbb{R}^3 \)). (A framed knot can also be thought of as an imbedded (oriented or two-sided) ribbon in \( \mathbb{R}^3 \).) With this correction, you get the framing number of the knot: the linking number of the knot with a copy slightly displaced in the direction of the framing.

But it would take us too far afield to describe this in more detail. Instead, let’s go to level 4, where there is an honest knot invariant (rather than an invariant of framed knots). That is, consider \( C_4(S^1) \). Unlike \( C_2(S^1) \), this is a disconnected space. Pick the connected component in which the points 1, 2, 3, 4 appear in that (cyclic) order around the circle. There are now maps \( \phi_{ij} : C_4(S^1) \to S^2 \) (for \( i \neq j, 1 \leq i, j \leq 4 \)), given by normalizing the vector \( K(s_j) - K(S_i) \). To define a degree of any sort, we’ll need a map from the 4-dimensional space \( C_4(S^1) \) to another 4-dimensional manifold. We can find one such by considering, for instance, \( \phi_{13} \times \phi_{24} \). (This is really the only possible choice using our \( \phi_{ij} \). All of 1, 2, 3, 4 must be represented, or

Figure 1.6: Number of inverse images changes as you cross a boundary.
the map is degenerate, and if we have a term like $\phi_{12}$ (with two adjacent points on the circle) we'll get a boundary just like that from $\phi_{12}$ on $C^0_2(S^1)$, which requires a framing-dependent correction.

The term to counter the boundary of the map $\phi_{13} \times \phi_{24}$ turns out to be slightly more complicated. It involves a map from $C^0_{4;3}(\mathbb{R}^3; K)$, the space of 4 points in $\mathbb{R}^3$, with the first 3 restricted to lie on the knot $K$. From this 6-dimensional space we consider the map $\phi_{14} \times \phi_{24} \times \phi_{34} : C^0_{4;3}(\mathbb{R}^3; K) \rightarrow S^2 \times S^2 \times S^2$. Then the resulting integral

$$\frac{1}{4} \int_{C^0_{4;3}} \theta_{13} \theta_{24} - \frac{1}{3} \int_{C^0_{4;3}} \theta_{14} \theta_{24} \theta_{34}$$

(1.8)

converges and is a knot invariant. (As before, $\theta_{ij} = C_k(K)^* \circ \phi^*_{ij}(\omega)$, with $\omega$ the normalized volume form on the sphere.)

Raoul Bott and Cliff Taubes [7] proved this integral is invariant. (Earlier, Dror Bar-Natan [1] had proven invariance in another way.) Their proof relies crucially on the functorial compactification $C_4(\mathbb{R}^3)$ of the configuration space $C^0(\mathbb{R}^3)$, described in detail in Section 3. This compactification yields a manifold with corners, whose (codimension 1) faces correspond to the points in a subset $B \subset \{1, 2, 3, 4\}$ of size $\geq 2$ approaching each other. (Since the compactification is functorial, the same description holds for the other spaces involved, $C_4(S^1)$ and $C_{4;3}(\mathbb{R}^3; K)$.)

The proof is essentially Stokes’ theorem: the variation of the integral of a (closed) form is the integral of the form on the boundary. (This yields a 1-form on the space of knots $\mathcal{K}$.) See Proposition 4.2 for the precise statement. Bott and Taubes showed that (a) either integral restricted to any face in which more than two points come together (as a 1-form on the space of knots) vanishes and (b) the integrals restricted to the principal faces, those in which exactly two points come together, cancel each other out.

The principal faces are more interesting, so let’s look at them first. On the face of $C_4(S^1)$ in which points 1 and 2 come together, the form $\theta_{13} \theta_{24}$ becomes $\theta_{1,2,3} \theta_{12,4}$. (The notation $\theta_{1,2,3}$ indicates that $\theta_{13}$ is equal to $\theta_{23}$. It also has another meaning, relating to the coordinates on $C_4(S^1)$: on this face, coordinates are the positions $s_{\{1,2\}}, s_3,$ and $s_4$. Thus this form can be considered a form on $C_{\{1,2,3,4\}}(S^1)$. See Section 3.) Similarly for the other principal faces (with 2 and 3, 3 and 4, and 4 and 1 colliding) of $C_4(S^1)$. In $C_{4;3}(\mathbb{R}^3; K)$, consider the face in which points 1 and 4 come together. On this face, the form $\theta_{14} \theta_{24} \theta_{34}$ becomes $\theta_{14} \theta_{21,4} \theta_{31,4}$. But, on this face, the map $\phi_{14}$ varies over $S^2$ without changing the directions to any of the other points; so we can integrate over this $S^2$, yielding the form $\theta_{21,4} \theta_{31,4}$ on $C_{\{1,4\},2,3}(S^1)$. Similarly for the faces with 2 and 4 and 3 and 4 colliding. On the faces in which two points on the knot collide (for instance, 1 and 2), the restriction of this form vanishes.

Thus each of the four principal faces of $C_4(S^1)$ contributes a form like $\theta_{1,2,3} \theta_{1,2,4}$. All of these yield the same 1-form on $\mathcal{K}$. Each of the three interesting principal faces of $C_{4;3}(\mathbb{R}^3; K)$ yields another form like this, again yielding the same 1-form on $\mathcal{K}$. Because of the factors and signs in the expression (1.8), all these forms cancel.
This result was not news to the knot theory community. This compactification of configuration space had been known and used, for instance by Axelrod and Singer [1, 2], who proved existence and invariance of the similar integral invariants for homology 3-spheres, and it was common knowledge that the techniques worked for the knot invariants too. But I know of no elementary exposition. The Bott-Taubes paper [7] is a first step; unfortunately, it omits description of the higher invariants, which can be treated with the same methods. This thesis attempts to rectify the situation.

The invariants are all constructed from configuration space integrals of this basic type. Namely, given a graph $\Gamma$ with a cycle $c$ representing the knot, the vertices of $\Gamma$ define a configuration space $C_{V(\Gamma);V(c)}(\mathbb{R}^3;K)$, and the edges of $\Gamma$ define a map $\phi(\Gamma)$ to $(S^2)^{|e|}$ (after choosing an ordering of the edges $e$ and a direction for each edge). The corresponding integral $I(\Gamma)$ is the integral over $C_{V(\Gamma);V(c)}(\mathbb{R}^3;K)$ of the pullback of the canonical form on $(S^2)^{|e|}$ (which is just the product of the canonical forms on the $S^2$ factors). Note that for this to be well-defined, there must be an orientation on $C_{V(\Gamma);V(c)}(\mathbb{R}^3;K)$. An ordering of the vertices of $\Gamma$ gives such an orientation.

Chapter 2 defines the Vassiliev invariants and introduces weight diagrams, with a plausibility argument why they should yield invariant integrals. Chapter 3 introduces the functorial compactification of configuration space that is essential to removing the analytic difficulties involved. Chapter 4 ties up loose ends, completing the description of the invariants, showing that the principle faces cancel, and describing the tinkertoy diagram formula. Chapter 5 shows that the contributions to the variation of the integral from most hidden or non-principle faces vanish. The remaining faces (the anomalous faces) are the subject of Chapter 6.

There are two appendices, describing in more detail the signs involved in relating weight diagrams and configuration space integrals. Appendix A gives some linear algebra preliminaries for the main statements in Appendix B.

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Chapter 2

Vassiliev Invariants and Weight Diagrams

This chapter is a brief introduction to Vassiliev invariants and the combinatorics of weight diagrams, which are basic to the construction of configuration space integrals. The reader is referred to Bar-Natan’s excellent paper [6] for a more in-depth treatment.

Any knot invariant $V : \mathcal{K} \to \mathbb{R}$ can be extended to an invariant $V^{(k)}$ of immersed circles with exactly $k$ transversal self-intersections inductively, using

$$V^{(0)} = V$$

$$V^{(k)}(\begin{array}{c}
\includegraphics{crossing.png}
\end{array}) = V^{(k-1)}(\begin{array}{c}
\includegraphics{crossing.png}
\end{array}) - V^{(k-1)}(\begin{array}{c}
\includegraphics{crossing.png}
\end{array})$$

Equation 2.2 is to be interpreted a local statement. The loose ends are connected by some paths, possibly with crossings. Note that this definition depends on the orientation on $\mathbb{R}^3$, but not the plane projection or the orientation on $S^1$. This is a good definition, since (expanding out) it expresses $V^{(k)}$ on a “knot” with $k$ transversal self-intersections as a signed sum of the values of $V$ on the $2^k$ ways of resolving the singularity, which won’t depend on the order of expansion.

**Definition 2.1 (The Birman-Lin Condition).** A Vassiliev invariant \cite{18, 19, 6} of degree $k$ is a knot invariant that vanishes on any knot with $k + 1$ double points, when extended as above.

One way to think of these Vassiliev invariants is as invariants of polynomial type. Taking the difference of the value of the knot invariant between the two sides of a crossing can be seen as “differentiation” of a sort. A polynomial is a function that becomes 0 if you differentiate it enough, similar to the definition of a Vassiliev invariant above.

Directly from the definition, we can see that the invariants we’ll construct will be Vassiliev invariants.
Proposition 2.2. A sum of canonical configuration space integrals of degree $k$ (involving a configuration space with $2k$ points) that is a knot invariant is a Vassiliev invariant of degree $k$.

The proof is in Section 4.1. The essential idea is that there can only be a difference between an undercrossing and an overcrossing if there is at least one point on each strand very near the crossing and a propagator connecting them. If you differentiate a Vassiliev invariant of degree $k$ just $k$ times, you get the analog of a constant. This is an invariant of knots with $k$ double points that doesn’t change as the knot passes through itself: this is exactly the condition that it vanishes on knots with $k + 1$ double points. Thus the invariant evaluated on a knot with $k$ double points doesn’t depend on the embedding in space, but only on the topological structure of the image. This is completely determined by labelling the points of self-intersection of the knot and recording the labels of the points as you travel around the circle in the increasing coordinate. You can get a graphical representation (called a chord diagram) by drawing the circle as a circle and connecting points on the circle that get mapped to the same point in $\mathbb{R}^3$, as in Figure 2.1.

Each Vassiliev invariant of degree $k$ therefore determines a function on chord diagrams with $k$ chords. We can evaluate this function explicitly for configuration space integrals. By the proof of Proposition 2.2 the only configuration space integrals of degree $k$ that don’t vanish when evaluated on a knot with $k$ double points are those that involve $2k$ points, all on the knot, connected by propagators in pairs that are near the double points. In particular, the graph of the propagators must be isomorphic the the graph of the chord diagram. Since the integral of a basic propagator near a double point is $\pm 1$, the value of the configuration space integral on this knot with double points will be (up to sign) the number of such isomorphisms. See Section 4.1 for more details.

Not all functions on chord diagrams can be achieved as derivatives of knot invariants. In particular, imagine evaluating the invariant on a knot with $k - 1$ double points and moving one strand in a small circle around one of the other double points, as on the left in Figure 2.2. The difference as you pass through each strand is the value of the invariant on a certain chord diagram. Since you end up at the same knot you started at, the sum of the value on the four diagrams produced must be 0. Unfolding the relation, you get the sum of diagrams on the right in Figure 2.2, call the four term or $4T$ relation. Functions on chord diagrams satisfying the $4T$ relation are called weight diagrams. One of the consequences of this thesis is that any weight
Figure 2.2: The 4 term relation, imbedded in $\mathbb{R}^3$ (left) and unfolded into a chord diagram (right). The dotted sections of the circle are portions that may have other chords attached (possibly crossing those drawn).

diagram can be “integrated” to yield an invariant of knots. This was proved earlier by Kontsevich using a different type of integral. See [6] for a description.

The number of weight diagrams of a given degree is an important open question. Bar-Natan [6] has done extensive computer computations, computing all weight diagrams up to degree 9. There seem to be a fair number of them, probably growing exponentially, although the difficulty of computation grows as a considerably faster exponential.

It’s an amazing fact that any weight diagram can be extended to a function on diagrams with internal vertices of a certain type. Namely, a trivalent graph (termed a Chinese Character Diagram by Bar-Natan [6]) is a connected graph $\Gamma$ with a directed cycle $c$ (with distinct vertices) and all vertices trivalent. Vertices that do not meet $c$ are the internal (or free or interaction) vertices. The value of a weight diagram on a trivalent graph will depend on an orientation on the edges incident to each internal vertex. Since the vertices are trivalent, this is a cyclic order.

**Proposition 2.3 (Bar-Natan [6],Theorem 6).** A weight diagram $w$ extends uniquely to a function on trivalent graphs with an orientation on the edges incident to each vertex satisfying the STU relation

$$S = T - U$$

**(2.3)**

*Proof.* It’s clear that (since trivalent graphs are connected) the $STU$ relation defines inductively the value of $w$ on any trivalent graph, as long as the definition does not introduce inconsistency.

The $4T$ relation is a difference of two $STU$ relations involving graphs with zero or one internal vertices. Because of this, the value of $w$ on a graph with exactly
one internal vertex is well-defined. The proof then proceeds by induction. Suppose \( w \) has been defined on graphs with \( k \geq 1 \) or fewer internal vertices. Consider \( w(\Gamma_{k+1}) \) for a graph \( \Gamma_{k+1} \) with \( k + 1 \) vertices. This can be defined from the value of \( w \) on graphs with \( k \) internal vertices via \( STU \) on any of the edges that connect an internal vertex of \( \Gamma_{k+1} \) to the circle. If two of these edges connect to different internal vertices, then the corresponding two potential expressions for \( w(\Gamma_{k+1}) \) can be expanded into a sum of \( w(\Gamma'_{k-1}) \) for graphs \( \Gamma'_{k-1} \) with \( k - 1 \) vertices via \( STU \) at the next lower level on the other edge. The resulting graphs \( \Gamma' \) are the same, so the two potential expressions for \( w(\Gamma_{k+1}) \) are consistent. If two edges of \( \Gamma_{k+1} \) connect the same internal vertex to the circle but there is a third edge connecting a different internal vertex to the circle, we can apply this procedure in two steps to deduce consistency. There is one remaining case, as in Figure 2.3. Since the value of the weight diagram on trivalent diagrams like this will turn out to be irrelevant, I defer to Bar-Natan [6] for the completion of the proof.

Internal vertices arise naturally in the Feynmann diagrams for Chern-Simons perturbation theory, where they represent the interactions of the theory. But they are still a mystery from other points of view. \( w(\Gamma) \) has not yet been given a sensible interpretation in terms of knots when \( \Gamma \) has internal vertices. Bott and Taubes [7] have suggested that, from the differential point of view they are related to Sullivan’s minimal model for the rational homotopy of \( C^0_n(\mathbb{R}^3) \), but this has not yet been explained clearly. In any case, we have the following theorem. \( I(\Gamma) \) is, as in the introduction, the configuration space integral corresponding to the trivalent graph \( \Gamma \) (with a sign yet to be defined).

**Theorem.** For any weight diagram \( w \), the following sum of integrals is a knot invariant:

\[
I(w) = \sum_{\text{isomorphism classes of graphs } \Gamma} \frac{w(\Gamma)I(\Gamma)}{|\text{Aut}(\Gamma)|} + (\text{anomaly correction term}) \quad (2.4)
\]

The expression [2.4], without the correction term, is exactly the integral that would be written down in perturbation theory, but it has been decoupled from all physics considerations here.

The proof is spread out over the remainder of the paper. The basic idea is that the differential form extends smoothly to the compactification of configuration
space. In this context, the variation of the integral as the knot varies can be written (up to sign) as the sum of the restrictions of the integral to the faces of configuration space. (See Proposition 4.2.)

The most interesting faces are the principle faces, which correspond to exactly two points (connected by either a segment of the knot or a propagator) approaching each other. The STU relation expresses the cancellation of faces in which two vertices on the knot or a vertex on the knot and an internal vertex approach each other. The remaining principle faces (two internal vertices colliding) cancel by the following proposition:

**Proposition 2.4 (Bar-Natan [6], Theorem 6).** Every weight diagram $w$ extended to graphs satisfies the IHX relation:

$$I = H - X \quad \quad \quad \quad \quad (2.5)$$

I will not repeat the proof here. It reduces to the case when one of the edges is connected to the circle, and in this case you get a sum of terms very much like the Jacobi identity for matrices.

The factor $|\text{Aut}(\Gamma)|$, the size of the automorphism group of the graph $\Gamma$, arises for the same reason a similar factor appeared for the evaluation of an integral on a knot with double points above. A different explanation is that the number of edges like the any given edge in $\Gamma$, and hence the number of faces of configuration space whose variation is the contraction of that edge, is usually the number of automorphisms of $\Gamma$.

The remaining faces of configuration space correspond to some number $k \geq 3$ of vertices approaching each other. Most of these are shown to vanish in Chapter 5. The remaining faces are the *anomalous faces*: those faces in which the $k$ degenerating vertices are not connected to any other vertex by a propagator. (This includes the case in which all the vertices degenerate.) The variations resulting from these faces is described at length in Chapter 6, and is the source of the (potential) correction term in equation 2.4.
Chapter 3

Configuration space and its Compactification

Definition 3.1. The configuration space $C^0_A(M)$ of points indexed by the finite set $A$ in a smooth manifold $M$ is the subset of $M^A$ given by

$$C^0_A(M) = \{(x_a)_{a \in A} \in M^A : x_a \neq x_b \text{ for } a, b \in A, a \neq b\} \quad (3.1)$$

This is a smooth manifold, but it is not compact if the size of $A$, $|A|$, is greater than 1. The construction is functorial on the category of smooth manifolds and imbeddings, with, for $f : N \rightarrow M$ an imbedding and $(x_a)_{a \in A} \in L^A$,

$$C^0_A(f)(x_a)_{a \in A} = (f(x_a))_{a \in A} \quad (3.2)$$

which is in $C^0_A(M)$ since $f$ is an imbedding.

I sometimes use the notation $C^0_A(M)$ as a shorthand for $C^0_{\{1, \ldots, n\}}(M)$ when I need to refer to the points specifically by name.

The compactification of $C_A(M)$ of $C^0_A(M)$ appropriate for this context was first written down in an algebro-geometric context\footnote{Using the projective sphere bundle rather than the sphere bundle so the result is an honest manifold (or variety).} by Fulton and Macpherson \[10\], and in the case of manifolds by Axelrod and Singer \[2\].

Section 3.1 aims to give an intuitive explanation and complete description of this wonderful compactification. Many details of the construction are deferred until Section 3.2, which is somewhat tangential to the main thrust of the thesis. It contains a new definition of the compactification. Section 3.3 describes some variations on the compactification necessary in our context.

3.1 A description of the compactification

Our basic requirement on the compactification is that the maps $\phi_{ab} : C^0_A(\mathbb{R}^3) \rightarrow S^2$ must extend smoothly to the compactification for all $a, b \in A$. Since the compact-
ification should be compact, every path in $C_A^0(\mathbb{R}^3)$ must have a limit.\footnote{Actually, I won’t allow paths that approach spatial infinity for the moment, so “compactification” is a slight misnomer. See Section 3.3 for a description of the true compactification.} Every such path has a well-defined limit in $(x_a) \in (\mathbb{R}^3)^A$. Group $A$ into classes that map into the same point in $\mathbb{R}^3$ in the limit. Then the direction between elements $a, b$ in different classes is determined by the limit of $x_a$ and $x_b$; so to guarantee that the corresponding $\phi_{ab}$ extends continuously, the compactification will come with a map $C_A(\mathbb{R}^3) \rightarrow (\mathbb{R}^3)^A$.

Now consider a path in $C_A(\mathbb{R}^3)$ in which the points of just one subset $B$ of size greater than one approach the same point in $\mathbb{R}^3$. The $\phi_{ab}$ with $a, b \in B$ are completely determined by the relative configuration of the points in $B$: the positions of the points in $B$, modulo overall translation and scaling. This relative configuration comes with a natural map to the space of points $(\mathbb{R}^3)^B$ whose coordinates are not all identical, modulo translation and scaling; this latter space is compact (it is the sphere $(((\mathbb{R}^3)^B/\Delta) \setminus \{0\})/\mathbb{R}^+$, where $\Delta$ is the diagonal in $(\mathbb{R}^3)^B$), so every path has a limit point in it. So to help define the $\phi_{ab}$, those portions of $C_A(\mathbb{R}^3)$ in which the points of $B$ approach each other will come with a map to the space of relative configurations, which I’ll call $C_B(\mathbb{T}\mathbb{R}^3)$.

But the relative configuration might not be enough: the relative configuration might approach a degenerate relative configuration, in which some set $C \subset B$ come together. This corresponds to all the points in $B$ approaching each other, but the points in $C$ approaching each other faster. But all the points of $B$ cannot come together in the limit, so we always gain something: $C$ will be strictly smaller than $B$. So if the points in $C$ come together in $C_B(\mathbb{T}\mathbb{R}^3)$, new coordinates, the relative configuration of the points in $C$, will help define more of the $\phi_{ab}$.

In general, the coordinates of $C_A(\mathbb{R}^3)$ will depend on the particular point. The points in some subsets $B_i \subset A$ which approach each other at the top level (only counting maximal $B_i$), and the directions between points in different $B_i$ are determined by a macroscopic configuration, a point in $C_{A/B_i}(\mathbb{R}^3)$. $(A/B_i$ means $A$ with points in each $B_i$ identified.) Then, for each $B_i$, the microscopic configuration near the $B_i$, the limit point in $C_{B_i}(\mathbb{T}\mathbb{R}^3)$ determine some more directions. Possibly some $C_{ij} \subset B_i$ still approach each other; then the submicroscopic configuration at the $C_{ij}$, the limit point in $C_{C_{ij}}(\mathbb{T}\mathbb{R}^3)$ determines yet more directions, and so forth.

The set of subsets $\mathcal{S}$ that occur in this construction characterize the coordinates used. The valid $\mathcal{S}$ are those which consist of subsets $S_i$ of $A$ of size $\geq 2$ for which every distinct pair $S_i, S_j$ satisfy one of

- $S_i \cap S_j = \emptyset$
- $S_i \subset S_j$
- $S_j \subset S_i$

These $\mathcal{S}$ characterize the faces $C_{A,\mathcal{S}}(\mathbb{R}^3)$ of $C_A(\mathbb{R}^3)$, which is a manifold with corners, a generalization of a manifold with boundary. (See below for a precise definition.)
The reader should verify that the dimension of the face \( C_{A,\mathcal{S}}(\mathbb{R}^3) \) is \( 3|A| - |\mathcal{S}| \), or codimension \( |\mathcal{S}| \). (\(|\mathcal{S}|\) is the number of distinct subsets of \( A \) in \( \mathcal{S} \).) The stratum corresponding to \( \mathcal{S} \) is, intuitively, the set of limit points in the open configuration space in which points in each subset \( S_i \) approach each other at comparable speeds.

Although we won’t use the compactification for any spaces other than \( \mathbb{R}^3 \) or \( S^1 \), it’s worth pointing out that a similar compactification can be defined for any \( n \)-dimensional manifold \( M \). The one essential difference is that, without the Euclidean structure of \( \mathbb{R}^3 \), the relative configurations \( C_B(TM_x) \) are defined only locally. Specifically, a coordinate chart \((U, \psi)\) near \( x \) yields a homeomorphism \( \psi' \) from \( U \) to a neighborhood of 0 in the tangent space to \( M \) at \( x \), \( TM_x \), whose Jacobian at \( x \) is the identity. In \( TM_x \), the limiting configuration of the points in \( B \) is well defined (taking values in \( C_B(TM_x) = (((TM_x)^B/\Delta) \setminus \{0\})/\mathbb{R}^+ \): non-constant maps from \( B \) into the tangent space to \( M \) at \( x \), modulo overall translation and scaling). The limit point in \( C_B(TM_x) \) of a path in \( C^0_A(M) \) in which the points in \( B \) all approach the point \( x \) does not depend on the coordinate chart.

### 3.2 Precise definitions

It’s now time for some proper definitions. This section is somewhat off the main track of my thesis, and is somewhat independent of the other material. Statements are made in full generality that are not needed for these knot invariants. But the results are of some independent interest. In particular, I give a new and somewhat simpler definition of the compactification of configuration space.

As I mentioned, the compactification will be a manifold with corners.

**Definition 3.2.** A (smooth) \( n \)-dimensional manifold with corners \( M \) is a Hausdorff topological space covered by open sets, each homeomorphic to

\[
\{x_1, \ldots, x_n\} \in \mathbb{R}^n : x_1 \geq 0, \ldots, x_m \geq 0\} = (\mathbb{R}^+)^m \times \mathbb{R}^{(n-m)}
\]

for some integer \( 0 \leq m \leq n \). The transition functions between two such open sets must extend to a smooth function on an open subset of \( \mathbb{R}^n \).

This definition allows us to define smooth functions, smooth differential forms, the tangent space, etc. The general principle is that smooth functions (et al) are required to have a smooth extension to a neighborhood of the coordinate chart.

Any manifold with corners \( M \) can be written as the disjoint union of \( m \)-dimensional manifolds, for \( 0 \leq m \leq n \), consisting of those points \( p \in M \) which have a neighborhood isomorphic to \((\mathbb{R}^+)^m \times \mathbb{R}^{(n-m)}\), with \( p \) mapping to the origin. These spaces (or the connected components of them) are called the strata of \( M \).

The notion of blowup will be central in the construction of the space \( C_n(M) \). Intuitively, the blowup of a manifold \( M \) along a submanifold \( L \), remove \( L \) from \( M \) and glue in a replacement that records the directions points approach \( L \). The replacement is \( SNL \), meaning the sphere bundle of the normal bundle of \( L \). The normal bundle of \( L \) is \( TM/TL \)—that is, tangent vectors of paths approaching \( L \),
modulo translation along $L$ itself. And the sphere bundle $SV$ of a vector bundle $V$ is, over each point $x$, $SV_x = V_x \setminus \{0\}/\mathbb{R}^+$. Thus $SNL$ removes paths that sit on $L$ and takes the quotient by the length of the tangent vector in $NL$.

**Definition 3.3.** The blowup of a smooth manifold with corners $M$ along a closed imbedded submanifold with corners $L$ is the manifold with boundary $\text{Bl}(M, L)$ that is $M$ with $L$ replaced by those points of $S(N(L))$ that are actually the images of paths in $M$. In particular, there is a natural smooth map $\pi : \text{Bl}(M, L) \to M$ and a partial inverse $M \setminus L \to \text{Bl}(M, L) \setminus \pi^{-1}(L)$.

To construct a manifold diffeomorphic to this blowup explicitly, pick a Riemannian metric on $M$ and find an $\epsilon > 0$ so that $B_\epsilon(L) = \{x \in M : d(x, L) < \epsilon\}$ is a tubular neighborhood of $L$. Then $M - B_{\epsilon/2}(L)$ is diffeomorphic to our desired blowup.

If $M$ has a Riemannian structure, then $S(N(L))$ can be identified with the unit vectors in $T_M|_L$ that are perpendicular to $TL$.

Note the similarity with the coordinates on configuration space $CA(M)$ constructed above. What I called the “relative configuration” of $B$ points at $x \in M$ is an element on the boundary of the blowup of $MB$ along the main diagonal $\Delta = \{(p, \ldots, p) \in MB, p \in M\}$. The configuration space $CA(M)$ should therefore be obtained, in some sense, by blowing up $MA$ along each of its diagonals $\Delta_B$, $B \subset A, |B| \geq 2$. But it’s impossible to do all the blowups at once, and successive blowups are slightly tricky. If you blow up along $\Delta_B$ before $\Delta_C$, for $B$ a proper subset of $C$, then $\Delta_C$ is part of the boundary of the partial blowup. The blowup along $\Delta_C$ can be done, but it’s easier to just require that $\Delta_C$ be blown up before $\Delta_B$ is.

With this restriction, when we blow up a diagonal $\Delta_B$ it will always intersect the interior of the partial blowup, and we can define successive blowups using the proper transform: the proper transform of $\Delta_B$ will be the closure of $\Delta_B$ minus its subdiagonals within the partial blowup. Blowing up $\Delta_B$ means blowing up the proper transform of $\Delta_B$.

Another worry is that the resulting configuration space might depend on the order chosen for the blowups, which would mean that it wouldn’t be truly functorial. For an example of blowups that don’t commute, see Figure 3.1 for the blowup of two intersecting lines in $\mathbb{R}^3$.

Another example that’s more relevant to our situation but harder to visualize is the blowup of three 2-planes in $\mathbb{R}^4$ intersecting at 0. (In fact, this is exactly what the intersection of the diagonals $\Delta_{12}, \Delta_{23},$ and $\Delta_{13}$ in $(\mathbb{R}^2)^3$ looks like, after taking the quotient by overall translation.) The blowup along the first two (in either order) will be isomorphic $\text{Bl}(\mathbb{R}^2, 0) \times \text{Bl}(\mathbb{R}^2, 0)$. In particular, the stratum lying over 0 is isomorphic to $S^1 \times S^1$. But the third 2-plane will intersect the boundary of the partial blowup only in the stratum lying over 0, and in some diagonal on the torus $S^1 \times S^1$. Blowing up along this plane turns the stratum over 0 into a torus minus a diagonal, which isn’t very nice and won’t be symmetric.
With the restriction that all the $\Delta_C$ for $B \subset C$ and $B \neq C$ be blown up before $\Delta_B$, all diagonals become disjoint before you get around to blowing them up. Blowups along disjoint submanifolds clearly commute (and can in fact be done simultaneously), so any permissible order of doing the blowups would yield the same result. But this is not true. For instance, after blowing up $M^4$ along each of the diagonals $\Delta_S$, $|S| = 3, 4$, the diagonals $\Delta_{12}$ and $\Delta_{34}$ will still intersect. But this intersection seems somehow “nicer” than the nasty cases given above. This can be made precise.

**Definition 3.4.** A collection $N_i$, $1 \leq i \leq k$ of submanifolds of a manifold $M$ intersect generically at a point $p \in M^o$ if the map

\[
\bigoplus_{i=1}^{k} \mathcal{T}N_i|_p \to (\mathcal{T}M|_p)^k/\Delta
\]

is surjective, where $\Delta$ is the image of the diagonal imbedding of $\mathcal{T}M|_p$ in $(\mathcal{T}M|_p)^k$ and the map is the direct sum of the imbeddings of the $\mathcal{T}N_i|_p$ in $\mathcal{T}M|_p$.

(The motivation is that this map is surjective exactly when the appropriate linear equations can be solved to find a common intersection point for small displacements of the $N_i$).

The submanifolds $N_i$ intersect generically if, for every $p \in M$ in at least one of the $N_i$, the $N_i$ such that $p \in N_i$ intersect generically at $p$.

Equivalently, the manifolds $N_i$ intersect generically at the point $p \in M$ if and only if $p$ has a neighborhood $U$ such that

\[
U \simeq L_0 \times L_1 \times \cdots \times L_k
\]

\[
N_i \cap U = \pi^{-1}_i(p_i), \quad 1 \leq i \leq k
\]

for some manifolds $L_0, L_i$ and points $p_i \in L_i$, $0 \leq i \leq k$, so that $p = p_0 \times p_1 \times \cdots \times p_k$. ($\pi_i$ is the projection of $L \times L_1 \times \cdots \times L_k$ onto the factor $L_i$.)

18
Proposition 3.5. If the $N_i$, $1 \leq i \leq k$, are proper submanifolds of a manifold with corners $M$ that intersect generically, then successive blowups along the $N_i$ yield a result independent of the order.

Proof. It’s enough to proceed locally, in the neighborhood of a point $p \in M$. For ease of notation, assume that all the $N_i$ intersect at $p$. Choose a neighborhood $U$ of $p$ that can be written as in Definition 3.4 as $U \simeq L \times L_1 \times \cdots \times L_k$, with $N_i = \pi^{-1}_i(p_i)$. Then the blowup of $U$ along $N_1$ is $L \times \text{Bl}(L_1, p_1) \times L_2 \times \cdots \times L_k$, the blowup of this space along $L_2$ is $L \times \text{Bl}(L_1, p_1) \times \text{Bl}(L_2, p_2) \times \cdots \times L_k$, and so forth; after all the blowups, the result is $L \times \text{Bl}(L_1, p_1) \times \cdots \times \text{Bl}(L_k, p_k)$, independently of the order they were performed.

(The condition that the $N_i$ be proper submanifolds is only required because we can’t blow up a manifold $M$ along $M$ itself.)

So the notion of generic intersection seems to correspond to intuition about “niceness” of the intersection. We’re ready to define the space $C_A(M)$, in stages from $M^A$. At each stage, we’ve blown up some collection $\Delta_{B_i}$ of diagonals of $M^A$. The next stage is the blow up of any diagonal $\Delta_C$ of $M^A$ for which all proper supersets of $C$ are contained in the $B_i$. To prove the result doesn’t depend on the choice of which $\Delta_C$ to blow up at each stage, it suffices to check that adjacent steps commute.

Proposition 3.6. If the diagonals corresponding to all proper supersets of $C$ and $D$ have been blown up at some stage in the construction of $C_A(M)$, but not $\Delta_C$ and $\Delta_D$, then $\Delta_C$ and $\Delta_D$ intersect generically.

Proof. If $C \cap D \neq \emptyset$, then their intersection at this stage will be empty. In this case, $\Delta_C \cap \Delta_D = \Delta_{C \cup D}$, which, by hypothesis, we’ve already blown up; and in the coordinates over $\Delta_{C \cup D}$, there exist points with either all the points in $C$ identical, or all the points in $D$ identical, but not both.

On the other hand, if $C \cap D = \emptyset$, $\Delta_C$ and $\Delta_D$ will intersect, but the intersection will be generic.

at a given stratum, want to reduce to each subst. individually. Right: enough to show that each tangent factor is a product as we want.

For any embedding $f : N \to M$, the diagonals $\Delta_B$ of $N^A$ are the intersections of the diagonals $\Delta_B$ of $M^A$ with $N^A$, with a generic intersection. Since the sequence of blowups is the same in the two cases, the blowup will be functorial.

3.3 Further variations

The configuration spaces of interest to us differ from the spaces $C_A(\mathbb{R}^3)$ defined above in two ways. First, as previously mentioned, $C_A(\mathbb{R}^3)$ is not actually compact as $\mathbb{R}^3$ is not compact. Instead of $C_A(\mathbb{R}^3)$, the proper compactification is $C_{A \cup \{\infty\}}(S^3)$ for some (fixed) point $\infty \in S^3$. $\mathbb{R}^3 \simeq S^3 \setminus \infty$, so $C_A^0(\mathbb{R}^3) \simeq C_{A \cup \{\infty\}}^0(S^3)$, and $C_{A \cup \{\infty\}}(S^3)$
is a proper compactification of \( C_A^0(\mathbb{R}^3) \). The maps \( \phi_{ab} \) extend continuously to this compactification as well; the direction a point \( a \in A \) approaches \( \infty \) determines the direction from \( a \) to points that do not approach \( \infty \), and the relative configuration at \( \infty \) determines directions between points that approach \( \infty \).

Since points approaching \( \infty \) turn out to be mostly irrelevant for the purposes of this thesis, in the future I’ll write \( C_A(\mathbb{R}^3) \) to mean \( C_{A\cup(\infty)}(S^3) \), leaving the strata at \( \infty \) implicit.

The other difference is the presence of the knot. As explained above, this is a functorial construction, so a knot \( K : S^1 \to \mathbb{R}^3 \) yields an imbedding \( C_A(K) \) (of manifolds with corners) of \( C_A(S^1) \) in \( C_A(\mathbb{R}^3) \), which defines \( \phi_{ab} \) for points on the knot and use these to construct certain invariants. But as explained in Section 1, for any but the simplest integrals this is not enough to construct an invariant. Most invariants involve integration of certain points over \( \mathbb{R}^3 \) and other points over \( S^1 \). To do this properly, we need the notion of a restricted configuration space \( C_{A,B}^0(\mathbb{R}^3; K) \) of maps from \( A \) to distinct points in \( \mathbb{R}^3 \), with the points in \( B \) landing in the submanifold \( K(S^1) \). (Note that points in \( A \setminus B \) range over the complement of the \( x_b, b \in B \), not the complement of the knot \( K(S^1) \).) Its closure in \( C_A(\mathbb{R}^3) \), \( C_{A,B}(\mathbb{R}^3; K) \), is the desired compact space.

Since the knot \( K \) is one-dimensional, the spaces \( C_{A,B}(\mathbb{R}^3; K) \) as I’ve defined them are not connected for \( |B| > 2 \). The (cyclic) order the points in \( B \) occur around \( K \) is significant. In this notation, \( B \) will therefore be taken to include a cyclic order of its points.

It’s important to note that \( C_{A,B}(\mathbb{R}^3; K) \) is smoothly fibered over the space of knots \( \mathcal{K} \). \( C_{A,B}^0(\mathbb{R}^3; K) \) is a subset of \( (\mathbb{R}^3)^{A\setminus B} \times (S^1)^B \) defined by equations like \( x_a \neq x_b \) and \( x_a \neq K(s_b) \) and so is fibered over \( \mathcal{K} \). Since \( C_{A,B}(\mathbb{R}^3; K) \) is just the closure of \( C_{A,B}^0(\mathbb{R}^3; K) \) in \( C_A(\mathbb{R}^3) \), it too is fibered over \( \mathcal{K} \). The notation \( C_{A,B} \) without qualifications will mean this bundle over \( \mathcal{K} \).

Finally, since the codimension 1 strata of \( C_{A,B} \) become important in Stokes’ theorem, a brief description is appropriate. As usual, the strata are parametrized by subsets \( D \subset A \). If \( D \subset A \setminus B \) consists of only free vertices, then the stratum can be written \( C_{(A,B),D} = C_{(A/D),B} \times C_D(\mathbb{T} \mathbb{R}^3) \); that is, this stratum of configuration space is the product of the macroscopic configuration with the microscopic configuration.

On the other hand, if \( B \cap D \neq \emptyset \) (so \( D \) contains some points on the knot, necessarily a contiguous subset in the cyclic order on \( B \)), \( C_{(A,B),D}(\mathbb{R}^3; K) \) is fibered over \( C_{(A/D),B}(\mathbb{R}^3; K) \), with fiber \( C_D(\mathbb{T} \mathbb{R}^3; \hat{K}(x_D)) \). This is again the macroscopic configuration and the microscopic configuration, but now the microscopic configuration is relative the line \( \hat{K}(x_D) \), the derivative of \( K \) at the common limit of \( D \). This latter space is the intersection of the closure of \( C_{D,(B\cap D)}^0(\mathbb{R}^3; K) \) with the stratum \( C_{D,D}(\mathbb{R}^3) \simeq C_D(\mathbb{T} \mathbb{R}^3) \) near the point \( x_D \) (the common limit point of the points in \( D \)). Specifically, this is the space of non-constant maps \( D \to \mathbb{R}^3 \) with points in \( D \cap B \) in the subspace spanned by \( \hat{K}(x_D) \), modulo translations in the direction \( \hat{K}(x_D) \) (so that points in \( D \cap B \) remain in the right subspace) and overall scaling.

20
Chapter 4

Tying it Together

In this chapter I bring together a number of loose ends, completing the proof that the invariants I’ll construct will be Vassiliev (Section 4.1), explaining why the variation of an integral is the integral restricted to the boundary and how the same cancellations can prove that a number of different integrals are invariants and give the same value (Section 4.2), making some notes on the form of the integrals resulting (Section 4.3), completing the proof that the contributions from principal faces cancel for an integral $I(w)$ corresponding to a weight diagram $w$ (Section 4.4), and introducing tinkertoy diagrams (Section 4.5).

4.1 Integrals are Vassiliev

We can define the configuration space $C_{A,B}(\mathbb{R}^3; K)$ of points on an immersed curve $K$ with $k$ double points: it is the closure in $C_A(\mathbb{R}^3)$ of $C_{A,B}^0(\mathbb{R}^3; K)$, the space of $B$ distinct points on the immersed curve and $A \setminus B$ points in space. This is a perfectly fine compact manifold with corners, so configuration space integrals make sense for curves with double points as well.

Given any configuration space integral on a curve $K$ with $k$ double points, suppose we approach a curve $K_1$ with $k+1$ double points by varying $K$. As the strands near $s_1, s_2 \in S^1$ approach each other (with the tangents at $s_1$ and $s_2$, $\bar{K}(s_1)$ and $\bar{K}(s_2)$, remaining fixed), the submanifold $C_{A,B}(\mathbb{R}^3; K)$ of $C_A(\mathbb{R}^3)$ approaches a limiting position. The limiting position is not a smooth submanifold of $C_A(\mathbb{R}^3)$. But there is an open dense subset of the limiting position that is smooth, and so the limit of the integral will be equal to the integral over this subset of the limiting position.

There are a number of connected components to this smooth portion of the limiting position, indexed by the set $D \subset A$ of points that get very close to the crossing, together with the subsets $D_1 \subset D \cap B$ and $D_2 \subset D \cap B$ that approach the crossing along the strands near $s_1$ and $s_2$ respectively. (Of necessity, $D_1$ and $D_2$ will each be connected subsets of $B$ and $D \cap B = D_1 \cup D_2$.) This portion of the limiting position then lies in the stratum $C_{A,D}(\mathbb{R}^3)$ of $C_A(\mathbb{R}^3)$ in which the
points $D$ approach each other (at comparable speeds) and is the product of the macroscopic configuration space $C_{A/D;B/D}(\mathbb{R}^3; K^x)$ with a microscopic configuration space, whose details don’t matter much, but is the image of [configurations of points $D_1$ on a line in the direction $\dot{K}(s_1)$, points $D_2$ on a line in the direction $\dot{K}(s_2)$, and points $D \setminus (D_1 \cup D_2)$ in $\mathbb{R}^3$] in $C_D(\mathcal{T} \mathbb{R}^3)$.

See Figure 4.1 for a picture of the situation in the most relevant case, when $D$ has two points, one on each strand.

So the limiting integral can be broken into components with different subsets $D$ very near the crossing. But the integral over a component with 0 or 1 points near the crossing will clearly not depend on the direction of the crossing. Hence the $k + 1$st derivative of a knot invariant with $2k$ crossings will be 0. This completes the proof of Proposition 2.2.

### 4.2 On the Pushforward

Our integrals are of the general form

$$
\int_{C_{A;B}(\mathbb{R}^3; K)} \omega
$$

where $\omega$ is an $n$-form on $C_{A;B}$, with $n = 3|A| - 2|B| = \dim C_{A;B}(\mathbb{R}^3; K)$. (Recall that $C_{A;B}$ is the fiber bundle over the space $\mathcal{K}$ of knots whose fiber over $K$ is $C_{A;B}(\mathbb{R}^3; K)$. For us, $\omega$ will be the pullback of a top-dimensional form on some manifold $M$, and so will be closed.) This type of expression is called *integrating over the fiber* or the *pushforward*.

**Definition 4.1.** Given any $n+k$-form $\omega$ on a fiber bundle $\pi : B \rightarrow X$ whose fibers are oriented $n$-dimensional manifolds with corners, the *pushforward* of $\omega$ along $\pi$, denote $\pi_*\omega$, is the $k$-form on $X$ whose value on a $k$-chain $c$ is

$$
\int_c \pi_*\omega = \int_{\pi^{-1}(c)} \omega
$$

Figure 4.1: Local situation of two points near the limit of a crossing, drawn as a subset of $C_2(\mathbb{R}^3)$ modulo overall translation. The flat section represents points macroscopically separated and the hemisphere in the center is the portion of $C_2(\mathcal{T} \mathbb{R}^3)$ that is covered by the limit.
When $B$ is a smooth manifold (without boundary) the pushforward of a closed form will be closed. But when $B$ is a manifold with corners, this is not true; instead, we have

**Proposition 4.2.** For bundle $\pi : B \to X$ whose fibers are smooth oriented manifolds with corners with boundary $(\partial \pi) : \partial B \to X$ (with the orientation induced from $B$),

$$d\pi_* \omega = \pi_* d\omega - (\partial \pi)_* \omega$$

(4.3)

**Proof.** For any $k+1$-chain $c$ in $X$, we have

$$\int_c \pi_* d\omega = \int_{\pi^{-1}(c)} d\omega$$

$$= \int_{\partial(\pi^{-1}(c))} \omega$$

(4.4)

$$= (\int_{\pi^{-1}(\partial c)} + \int_{(\partial \pi)^{-1}(c)}) \omega$$

$$= d\pi_* \omega + (\partial \pi)_* \omega$$

using Stokes’ theorem and $\partial(\pi^{-1}(c)) = \pi^{-1}(\partial c)$ II $(\partial \pi)^{-1}(c)$.

This proposition justifies my claim that the variation of an integral is (up to sign) the integral restricted to the boundary: since the forms $\omega$ we integrate are closed (being the pullback of a closed form), $\pi_* d\omega$ will be 0, and $d\pi_* \omega$, the variation of the integral of $\omega$, will be $-(\partial \pi)_* \omega$.

This proposition also tells us what happens as we vary the $n$-form $\omega$. If we pick a form $\omega'$ in the same cohomology class as $\omega$, then $\omega' - \omega = d\alpha$ for some $n-1$-form $\alpha$. Applying the above proposition, $\pi_* \omega' - \pi_* \omega = \pi_* d\alpha = d\pi_* \alpha + (\partial \pi)_* \alpha$; but $\pi_* \alpha$ is a $-1$-form on $X$ and hence 0. Thus the difference between the integrals of $\omega$ and $\omega'$ will be $(\partial \pi)_* \alpha$ for some form $\alpha$.

Therefore an integral will both

- be an invariant under deformation of the knot and
- be independent of the form $\omega$ within a cohomology class

if the boundary of the bundle $B$ is 0 in some sense. More precisely, since the forms we use are all $\phi^* \psi$ for some $n$-form $\psi$ on a manifold $M$ (which will be a product of 2-spheres) the image of the boundary of $B$ in this product should be 0 (as an $n-1$-chain). But since there is not yet any sensible homology theory with general manifolds with corners, I’ll state this in a different form.

Precisely, the arguments I’ll use to show the vanishing or cancellation of the pushforward of $\phi^* \omega$ along a given non-anomalous face $\partial B$ will either be

---

1 Even though the dimension of the fiber is equal to the degree of $\omega$, this is not a trivial statement, since it requires $\omega$ to be closed as a form on the bundle.
• degeneration: a proof that the image of each fiber of \( \partial B \) in \( M \) lies in an \( n - 2 \) dimensional submanifold; or

• cancellation: a map \( \Phi \) from \( \partial B \) to some other boundary component \( \partial B' \) that is orientation reversing but commutes with the maps to \( M \) (for example, or else \( \Phi \) might be orientation preserving but the two faces appear with an opposite numerical factor, or 3 (or more) faces might cancel).

In either case, it follows that both desiderata above hold. If the face degenerates, then \( (\partial \pi)_* \phi^* \psi \) evaluated on any 1-chain is zero, as the image of the fiber will be a codimension 1 submanifold; and \( (\partial \pi)_* \phi^* \alpha \), for an \( n - 1 \) from \( \alpha \) on \( M \), will be zero as a function on \( K \) since the image of the fiber is of dimension \( n - 2 \). The arguments are even easier in the case of cancellation.

Separate arguments are needed for some of the anomalous faces. These are discussed in Chapter 6.

4.3 The Form of the Integrals

As mentioned in Chapter 1, the configuration space integrals are the integrals of the pullback of the canonical form on \( (S^2)^e \) via the map \( \phi(\Gamma) : C_{V,K} \rightarrow (S^2)^e \) defined by some oriented trivalent graph \( \Gamma \). Recall that the canonical form on \( S^2 \) is

\[
\omega = \frac{\epsilon_{\mu\nu\sigma}}{8\pi} \frac{x^\mu dx^\nu dx^\sigma}{|x|^3}
\]  

(4.5)

Pulling this back to \( C_{A,B} \) via the map \( \phi_{ab} : C_{A,B} \rightarrow S^2 \), the direction map between \( x_a \) and \( x_b \), yields

\[
\theta_{ab} = \phi_{ab}^* \theta = \frac{\epsilon_{\mu\nu\sigma}}{4\pi} \frac{x_b - x_a}{|x_b - x_a|^3} \left( \frac{1}{2} dx_a^\nu dx_a^\sigma - dx_a^\nu dx_b^\sigma + \frac{1}{2} dx_b^\nu dx_b^\sigma \right)
\]  

(4.6)

All the invariants considered in this thesis have trivalent vertices, both internal and on the knot (counting the knot as 2 edges incident to a vertex on the knot). A simple dimension check verifies that these yield functions on knots. (The dimension of the configuration space is \((\# \text{ vertices on knot}) + 3(\# \text{ vertices off knot})\), while the dimension of the product of spheres is \(2(\# \text{ propagators})\); since each propagator has two endpoints while each vertex is incident to 1 or 3 propagators, these match.) There are other graphs that yield non-zero functions on the space of knots. See figure 4.2 for an example. These graphs will not be considered in this thesis. But since every configuration space integral is a Vassiliev invariant (Proposition 2.2) and I give an explicit construction of an integral for every Vassiliev invariant, no new invariants can be constructed from them.

For a trivalent diagram and the canonical form on the product of \( S^2 \)'s, the integral can be written in another way. The product of the various \( \theta_{ab} \)'s as in 4.6 is a sum of terms with 0, 1, or 2 \( dx_a \)'s and a complementary number of \( dx_b \)'s. But note that, for the integral of this form over \( C_{A,B}(\mathbb{R}^3; K) \) not to vanish, exactly one
\[ dx_a \] must appear for each \( a \in B \) on the knot and exactly three \( dx_a \)'s must appear for each \( a \in A \setminus B \) in \( \mathbb{R}^3 \). A little combinatorial juggling shows that only the terms of the various \( \theta_{ab} \) with exactly one \( dx_a \) and one \( dx_b \) give a non-zero contribution to the integral. Note that this is only true for the integral over the fiber. In particular, the term you integrate here is not a closed form on the bundle, so computing the variation with this expression involves more than just restriction to the boundary.

For example, one of the integrals in the degree 2 Vassiliev invariant is (from the introduction)

\[ \int C_0^{4;3} \theta_{14} \theta_{24} \theta_{34}. \]

By the above remarks, this integral can be rewritten to look like

\[
\int ds_1 \frac{dx_1^{\mu_1}}{ds_1} \int ds_2 \frac{dx_2^{\mu_1}}{ds_2} \int ds_3 \frac{dx_3^{\mu_1}}{ds_3} \int d^3 x_4 \epsilon^{\nu_1 \nu_2 \nu_3} \Delta_{\mu_1 \nu_1}(x_4 - x_1) \Delta_{\mu_2 \nu_2}(x_4 - x_2) \Delta_{\mu_3 \nu_3}(x_4 - x_3) \quad (4.7)
\]

where \( \Delta_{\mu \nu}(x) \) is the propagator written as a tensor:

\[
\Delta_{\mu \nu}(x) = \frac{\epsilon_{\mu \nu \sigma}}{4\pi} \frac{x^\sigma}{|x|^3} \quad (4.8)
\]

But note what’s required for this type of expression. In general, assign a tensor index to each propagator leaving each vertex and write down \( \int ds_a \frac{ds_a^\mu}{ds_a} \) for each point \( a \) on the knot, \( \int d^3 x_b \) for each point \( b \) in space, \( \Delta_{\mu \nu}(x_b - x_a) \) for each propagator, and a term \( \epsilon^{\mu \nu \sigma} \) for each internal vertex, with the appropriate choices of tensor index. As each integral is paired with a corresponding differential form, the order of the integrals (which also determines the orientation on the space) is irrelevant. The only sign choice in an expression like this is the internal vertex \( \epsilon^{\mu \nu \sigma} \) tensors. But this is exactly an ordering on the edges meeting each internal vertex, which is \textit{exactly} what’s needed to define weight diagrams. This gives an algorithm for relating the

For a more complete discussion of these signs, see Appendix B.

\section*{4.4 Principal Faces Cancel}

There are several fixes that need to be applied to the naive definition of \( \phi(\Gamma) \) in the introduction in order to get a meaningful formula.
To apply the results of Section 4.2 on methods of proving invariance and non-dependence on the form $\omega$ on $M$, you need maps from the bundle $C_{A,B}$ to a target manifold that is the same for all maps. But the number of propagators in the graphs associated to a weight diagram varies. (The number of propagators is the degree plus the number of internal vertices.) To fix this, $\phi(\Gamma)$ will be modified slightly: it will be a map $C_{A,B} \times (S^2)^x \to (S^2)^{\alpha+x}$, the product of the naive $\phi(\Gamma)$ with the identity map on enough spheres so that all these maps have the same target space.

Equation 2.4, the definition of the integrals considered, was not actually well defined, since in the definition of the integral associated to a graph in the Introduction I needed an ordering of the vertices (to define the orientation on configuration space) and an ordering of the edges and a direction on each edge (to define the map to $(S^2)^e$).

The ordering on the edges is actually irrelevant for the canonical integrals, as the product of canonical forms on each $S^2$ is symmetric under permutation of the edges. But for general forms that are not products of the same form on each $S^2$ factor, this won’t be true. As explained in the next section, we’ll need these more general forms. To solve this, $I(\Gamma)$, the integral associated to the graph $\Gamma$, will include averaging over all orderings of the edges. The canonical form is also antisymmetric under the antipodal map $N$ on $S^2$; this will be needed below. Since not all forms on $(S^2)^{e+x}$ will have this property, $I(\Gamma)$ will also be antisymmetrized under the antipodal map on each edge.

(Alternatively, you could view this as a restriction to forms on $(S^2)^{e+x}$ that are symmetric under permutations of the factors and antisymmetric under the antipodal map on each sphere.)

To define $I(\Gamma)$ now, you need an ordering on the vertices and a choice of direction on each edge. A transposition of two vertices and changing the direction on an edge both change the sign of the resulting integral. This defines a notion of orientation on the graph. But in equation 2.4, the orientation on a graph used was an orientation on each trivalent edge. For trivalent diagrams, these two notions of orientation turn out to be equivalent. See Appendix B, or alternatively take the canonical integral and rewrite it as described in the last section to get it into a form as in equation 4.7, which only depends on an orientation at each vertex.

Although $I(\Gamma)$ as it stands now is an invariant in the desired sense, the automorphism factors $|\text{Aut}(\Gamma)|$ in equation 2.4 still appear a bit mysterious. To make them transparent, note that the number of ways of labelling the vertices of an $n$-vertex graph $\Gamma$ with $\{1, \ldots, n\}$ is $\frac{n!}{|\text{Aut}(\Gamma)|}$. So instead of the equation 2.4 as it stands (with its annoying reference to “isomorphism classes of graphs”), we can equivalently write

$$I(w) = \frac{1}{n!} \sum_{\text{labelled graphs } \Gamma} w(\Gamma) I(\Gamma) + \text{(anomaly correction term)} \quad (4.9)$$

With all these modifications, $I(\Gamma)$ is now properly defined. The proof that the principal faces cancel now falls into place easily.
First, the principal face corresponding to the collision of two points not connected by a propagator or a section of the knot is degenerate: the stratum of configuration space (for the collision of a and b) is isomorphic to \( C_{A/\{a,b\};B/\{a,b\}}(\mathbb{R}^3, K) \times C_{\{a,b\}}(T \mathbb{R}^3) \), and the map \( \phi(\Gamma) \) on this configuration space is independent of the second component. Therefore the image of this stratum must have codimension at least 3 in \((S^2)^{e+x}\).

The remaining cases will correspond to either the STU or IHX relation. When two points connected by a propagator collapse, the stratum of configuration space is again isomorphic to \( C_{A/\{a,b\};B/\{a,b\}}(\mathbb{R}^3, K) \times C_{\{a,b\}}(T \mathbb{R}^3) \); the map \( \phi(\Gamma) \) will be a product of \( \phi(\Gamma_e) \) of the map (corresponding to the labelled graph \( \Gamma_e \) with the edge connecting a and b contracted) with the identity map on the second component, which is isomorphic to \( S^2 \). When two points connected by a segment of the knot collapse, the stratum is \( C_{A/\{a,b\};B/\{a,b\}}(\mathbb{R}^3, K) \times C_{\{a,b\};\{a,b\}}(T \mathbb{R}^3; \dot{K}(s_{\{a,b\}})) \). This latter space is just a point and so can be omitted. \( \phi(\Gamma) \) on this stratum is in this case just \( \phi(\Gamma_e) \).

So in any case, \( \phi(\Gamma) \) restricted to this stratum is \( \phi(\Gamma_e) \) cross the identity on some number of spheres. Each labelled graph \( \Gamma_e \) that arises in this way comes from exactly six labelled trivalent graphs \( \Gamma \) by contraction. (The vertex resulting from the contraction will be labelled with the set of the two contracted vertices. There are 6 ways of partitioning the 4 edges incident to this vertex into 2 (labelled) partitions of two edges each.) Depending on whether the contracted vertex is internal or on the knot, these 6 diagrams correspond to the 3 diagrams in the IHX or STU relation in a natural way. A simple sign check then verifies that these diagrams cancel each other out, using the corresponding equation for the weight diagram.

### 4.5 Tinkertoy Diagrams

In this section I discuss briefly a combinatorial formula for these configuration space integrals. This will be expanded in a future paper. Right now the main consequence is

**Theorem.** For a weight diagram with rational coefficients, the corresponding configuration space integral invariant is rational.

(In fact, it’s easy to bound the demoninator by counting everything we averaged over.)

As I mentioned in Section 4.2, my proof of invariance of the configuration space integrals proves, at the same time, that any other form on \((S^2)^{e+x}\) yields the same invariant. A particular choice of the form on \((S^2)^{e+x}\) will yield a counting formula for this invariant by the following general remarks, which apply to any invariant for which “boundaries cancel” as in Section 4.2.

Suppose \( p \in M \) is a regular value of each of the maps \( f_i : B_i \to M \), so that each \( f_i \) is a diffeomorphism in a neighborhood of each inverse image of \( p \). (We can always find such a \( p \) by Sard’s theorem.) Then \( p \) will have only a finite number of inverse images (since each of the \( B_i \) is compact) and \( f_i \) will necessarily be non-degenerate.
(and hence a diffeomorphism) in a neighborhood $U_{ij}$ of each of the inverse images $x_{ij}$ of $p$ in $B_i$. Pick an $n$-form $\omega$ so that $\int_{U} \omega = 1$ and $\text{supp}\, \omega \subset \bigcap_i f_i(U_i)$. (\text{supp}\, \omega, the support of $\omega$, is the closure of the set of points in $M$ where $\omega$ is non-zero.) Then $\int f_i^* \omega$ is $\text{sgn} \, f_i(x_{ij})$ near $x_{ij}$ of $p$, and $\sum_i \int_{B_i} f_i^* \omega$ is the sum of $\text{sgn} \, f_i(x_{ij})$ over all inverse images of $p$. (\text{sgn} \, f_i(x_{ij})$ is $\pm 1$ depending on whether $f_i$ is orientation preserving or reversing near that inverse image.)

Thus counting the number of inverse images of a generic point $p \in (S^2)^{e+x}$ (with appropriate signs) yields a purely combinatorial formula for each invariant. These inverse images can be interpreted as \textit{tinkertoy diagrams}: some nodes, either anchored to the knot or free in space, and rods between certain pairs of nodes. The rods are constrained to lie in certain directions, each corresponding to the projection of $p$ onto the $S^2$ factor for that edge. Because of the symmetrization on the $S^2$ factors described in the previous section, only the set of allowable directions matters; each rod must lie in an allowable direction (or its negative), with no direction used more than once.

The sign associated with each tinkertoy diagram can be computed using the pull-back of the canonical form as in equation 4.7. By definition, the form $ds_1 ds_2 ds_3 d^3x_A$ (for equation 4.7, or in general the product of the $d^3x_a$ and $ds_b$ for $a \in A \setminus B$ and $b \in B$ in the proper order) is positive (has positive value when integrated over a small region in $C_{A,B}$), so the total form will be positive exactly when the numerical factor is positive. This numerical factor can be interpreted in various ways, using the interpretation of $\epsilon_{\mu\nu\sigma} x^\mu y^\nu y^\sigma$ as the cross product of the vectors $x$ and $y$.

For the degree 2 Vassiliev invariant and an almost planar knot (as in equation 1.8), this can be simplified somewhat. The two types of tinkertoy structures in this case are

- have two rods with both endpoints anchored to the knot (and a constraint on the cyclic order the endpoints occur around the knot) or
- have a tripod of rods anchored at one end and joined at an internal node.

For an almost planar knot, rods with both endpoints anchored to the knot can only occur near a crossing, while tripods occur exactly when there is a triangle of points on the knot similar to a given triangle. So this invariant can be reduced to a sum over certain pairs of crossings, certain crossings (depending on the directions of the strands at the crossing), and certain triangles of points on the knot.

By choosing the 3 directions all near a plane, two things happen: the contributions from single crossings drop out and the triangle degenerates into counting critical points and points in the same horizontal slice through the knot with a given ratio.

Dror Bar-Natan has suggested that this approach can be used to construct a quasi-triangular quasi-Hopf algebra \textit{à la} Drinfel’d and in particular to show that these integrals are the same as those arising from the Knizhnik-Zamolodchikov connection [8, Section 4]. This program will be carried out in a future paper.
Chapter 5

The Hidden Faces

In Chapters 2 and 4, I discussed in some detail how to form an integral with trivalent graphs in which the principal faces cancel. In this chapter I’ll show that, for almost all the remaining faces of the fibered chain for a trivalent graph \( \Gamma \), the restriction to that face is 0. The proofs are mostly after Bott and Taubes [7].

5.1 Preliminaries

A general principle that will be useful is the notion of pullbacks for bundles. Given three bundles \( B_1, B_2, B_3 \) over \( K \) and fibration preserving maps \( i_1 : B_1 \to B_3 \) and \( i_2 : B_2 \to B_3 \), we can “complete the square” in a canonical way. We construct a bundle \( B \) over \( K \) with fiber maps \( j_1 : B \to B_1 \) and \( j_2 : B \to B_2 \) so that the following square commutes:

\[
\begin{array}{ccc}
B & \xrightarrow{j_1} & B_1 \\
\downarrow{j_2} & & \downarrow{i_1} \\
B_2 & \xrightarrow{i_2} & B_3 \\
\end{array}
\]

and \( B \) is minimal: given any other \( B' \) and maps \( j'_1, j'_2 \), there is a unique map \( \kappa : B' \to B \) so that \( j_1 \circ \kappa = j'_1 \) and \( j_2 \circ \kappa = j'_2 \). Concretely, \( B \) is the bundle

\[
B = \{(x \in B_1, y \in B_2) : i_1(x) = i_2(y)\}
\]

(5.2)

For example, for any finite set \( A \) and \( B, C \subset A \) with \( B \cup C = A \), there is a commutative square

\[
\begin{array}{ccc}
C_A(\mathbb{R}^3) & \longrightarrow & C_B(\mathbb{R}^3) \\
\downarrow & & \downarrow \\
C_C(\mathbb{R}^3) & \longrightarrow & C_{B \cap C}(\mathbb{R}^3) \\
\end{array}
\]

(5.3)
and $C_A(\mathbb{R}^3)$ is a pullback in this diagram. (Each of the maps is a version of the natural project $C_A(\mathbb{R}^3) \to C_B(\mathbb{R}^3)$ for $B \subset A$, forgetting the points in $A \setminus B$.) The spaces $C_{A,B}$ can be constructed in this way: they are the pullbacks in

$$
\begin{array}{ccc}
C_{A;B} & \longrightarrow & C_B(S^1) \\
\downarrow & & \downarrow C_B(K) \\
C_A(\mathbb{R}^3) & \longrightarrow & C_B(\mathbb{R}^3)
\end{array}
$$

This is how $C_{A,B}$ was defined in [7], but it’s a little bit tricky to show that it is a manifold with corners in this way.

As an application, we have the following proposition.

**Proposition 5.1.** The fibered chain corresponding to any trivalent graph $\Gamma$ with a subset $A \subset V_i(\Gamma)$ of the internal vertices of $\Gamma$ such that

- $|A| \geq 2$
- The number of edges of $\Gamma$ connecting a vertex in $A$ to a vertex not in $A$ is $\leq 3$

is degenerate.

**Proof.** Let $B$ be the set of vertices not in $A$ that are joined to a vertex in $A$. By hypothesis, $|B| \leq 3$. Now, $C_{V;c}$ (the configuration space appropriate to this graph) is the pullback in

$$
\begin{array}{ccc}
C_{V;c} & \longrightarrow & C_{A\cup B}(\mathbb{R}^3) \\
\downarrow & & \downarrow \\
C_{(V\setminus A);c} & \longrightarrow & C_B(\mathbb{R}^3)
\end{array}
$$

and each summand in the fibered chain $\mathcal{F}(\Gamma)$ will be the restriction of a product of two maps, from $C_{A\cup B}(\mathbb{R}^3)$ and $C_{(V\setminus A);c}$, to $C_{V;c} \subset C_{A\cup B}(\mathbb{R}^3) \times C_{(V\setminus A);c}$. The two maps correspond, respectively, to the graph $\Gamma_A$ whose edges are all edges in $\Gamma$ meeting $A$ and the graph $\Gamma_A$ containing all other edges of $\Gamma$.

But the first map (corresponding to $\Gamma_A$) lands in a proper submanifold of $S^2E(\Gamma_A)$. The source, $C_{A\cup B}(\mathbb{R}^3)$, has dimension $3(|A| + |B|)$, while the target, $S^2E(\Gamma_A)$, has dimension $2|E(\Gamma_A)| \geq 3|A| + |B| + d$ ($d$ is the number of vertices of $B$ connected to more than one vertex of $A$), so this isn’t immediately obvious. But the directions between all these points is unchanged under

- overall translation of $A$ and $B$ in $\mathbb{R}^3$ (3 dimensions)
- moving a vertex in $B$ away from the vertex it’s connected to in $A$, for vertices in $B$ connected to exactly one vertex of $A$ ($|B| - d$ dimensions)
- overall scaling (1 dimension)
All of these transformations are independent because \(|A| \geq 2\). Taking the quotient by all these transformations, we see that the map from \(C_{A \cup B}(\mathbb{R}^3)\) factors through a space of dimension \(3(|A| + |B|) - 3 - |B| + d - 1 \leq 3|A| + |B| + d - 1 < \dim(S^2)^{E(\Gamma_A)}\), so indeed we do land in a proper submanifold.

The homology chain \(\mathcal{F}(\Gamma)\) will be inside the product of this submanifold with \((S^2)^{E(\Gamma_A)}\), and thus in a proper submanifold of \((S^2)^{E(\Gamma)}\).

Arguments along similar lines show that boundary components \(C_{(A; B), D}\) are degenerate if they have enough (or too few) external edges: edges connecting a point in \(D\) with a point not in \(D\). (For trivalent diagrams, boundary components with \(B \cap D = \emptyset\) are degenerate if there are more than 4 external edges, while boundary components with \(B \cap D \neq \emptyset\) are degenerate if there are more than 2 external edges.) See [7] for details. See below for a proof by global symmetry (or sometimes degeneracy) that these faces vanish as chains.

### 5.2 Conditions for Vanishing

There are three different arguments to show that a boundary face vanishes. Throughout, we’re considering the face of the configuration space \(C_{V,c}\) corresponding to the trivalent graph \(\Gamma\) in which points in the subset \(A\) collide. Since these are non-principal faces, by definition \(|A| \geq 3\). \(\Gamma_A\) is the subgraph of \(\Gamma\) whose edges all propagators that connect points of \(A\) (note that this differs from the definition in the proof of Proposition [5.1]), and \(\Gamma_A\) consists of all other edges of \(\Gamma\). \(M = (S^2)^{E(\Gamma)}\) is the target for the fibered homology chain \(\mathcal{F}(\Gamma)\).

**Case 1.** If \(\Gamma_A\) is not connected, the fibered chain \(\mathcal{F}(C_{(V; c), A})\) is degenerate.

**Proof.** In this case, we can independently translate the vertices of two disjoint subsets of \(\Gamma_A\) within \(C_{A \cap c}(\mathbb{R}^3; K(s_A))\) without changing the map to \(M\). The translation is in any direction in \(\mathbb{R}^3\) (if \(A \cap c = \emptyset\) or along the direction \(K(s_A)\) (otherwise), but in either case the map to \(M\) factors through a map to a bundle of strictly lower dimension. The quotient is non-trivial since \(|A| \geq 3\). (Recall that the local coordinates are already the quotient by translation and scaling of all the points in \(A\).)

Note that this same argument works even if \(|A| = 2\) but both vertices are free. This justifies ignoring certain principal faces.

**Case 2.** If \(\Gamma\) has a subgraph like

```
    d    not A
   / \
  a --
     /
    /   A
   /    \
 b     c
```

(5.6)
(with \(a, b, c \in A\) and \(d \notin A\)) then the fibered chain \(\mathcal{F}(C(V; c), A)\) vanishes by a global symmetry.

**Proof.** The map on \(C_{A \setminus (c \cap A)}(\mathcal{T} \mathbb{R}^3, \hat{K}(s_A))\) that takes \(x_a\) to \(x_b + x_c - x_a\) and leaves all other coordinates unchanged is an orientation-reversing automorphism on the stratum that takes the fibered chain to itself (swapping and negating the spheres corresponding to the edges \(a-b\) and \(a-c\)).

**Case 3.** If \(\Gamma\) has a subgraph like

\[
\begin{align*}
&\text{not } A \\
a & \\
b & \\
c & \text{not } A \\
d & \\
e & \\
f &
\end{align*}
\]

(with \(a, b, c, d \in A\) and \(e, f \notin A\)) then the fibered chain \(\mathcal{F}(C(V; c), A)\) vanishes by a global symmetry.

**Proof.** The map on \(C_{A \setminus (e \cap A)}(\mathcal{T} \mathbb{R}^3, \hat{K}(s_A))\) that takes \(x_b\) to \(x_c + x_d - x_b\), \(x_a\) to \(x_c + x_d - x_a\), and leaves all other coordinates fixed is an orientation-preserving automorphism on the stratum that takes the fibered chain to its negative (swapping and negating the spheres corresponding to the edges \(b-c\) and \(b-d\) and negating the sphere corresponding to \(a-b\)).

### 5.3 Most Hidden Faces Vanish

**Proposition 5.2.** Any non-principal face with external edges vanishes as a fibered chain.

**Proof.** Suppose the face corresponding to the subset \(A\) has external edges. If some vertex \(a \in A\) is connected to exactly one external vertex, either \(a\) is on the knot, in which case \(\Gamma_A\) (as in the previous section) cannot be connected and the face vanishes by Case \(\Box\); or \(a\) is not on the knot, in which case \(\Gamma\) has a subgraph as in Case \(\Box\). If some vertex \(a\) is connected to three external vertices, \(\Gamma_A\) cannot be connected and the face is degenerate by Case \(\Box\). So suppose boundary vertices (vertices meeting at least one external edge) are meet exactly two external edges. If some boundary vertex \(a\) is also connected to a vertex \(b \in A\) on the knot or another boundary vertex, the pedantic reader will note that this is not quite true, since \(x_b + x_c - x_a\) might happen to coincide with \(x_e\) for some other point \(e \in A\). For this reader, I will point out that we didn’t actually need to blow up on the diagonal \(\Delta_{ae}\), since \(a\) and \(e\) cannot be connected by an edge of \(\Gamma\), and if we don’t do this blowup we do get a true automorphism.\(^1\) See previous footnote \(^2\).
then (since there is at least one other vertex in $A$), $\Gamma_A$ is not connected and the face is degenerate. Otherwise, $a$ must be connected to another non-boundary vertex not on the knot and $\Gamma$ will have a subgraph as in Case 3.

There are two cases left: faces in which some points $A \subset V(\Gamma)$ approach $\infty$, and faces in which some points $A$, with no external edges, approach each other. The first case is easy (by Proposition 5.1, if $\mathcal{F}(\Gamma)$ is non-degenerate $A$ will have at least three external edges, all of which map to the same point in $S^2$, so the image of the map has codimension at least 4; since the face has codimension 1, the map is degenerate). The second is not so easy, and in fact, it’s not always true that these faces vanish. They are called anomalous faces, and are the subject of the next chapter.
Chapter 6

Anomalous Faces

In the previous section, I showed that most hidden faces (faces that aren’t just two vertices coming together) vanish. The remaining faces, dubbed “anomalous faces” by Bott and Taubes in [7], are much more interesting. They do not all vanish as homology chains. Fortunately, most of those anomalous faces that do not vanish can be corrected by an additional term.

6.1 Preliminary Remarks

First, some definitions and reductions of the problem.

Definition 6.1. A split graph $\Gamma$ is a graph whose vertices can be partitioned into two sets, $A$ and $B$, such that the only edges running between $A$ and $B$ are cycle (knot) edges.

The notion of a split graph is closely related to the notion of connected sum of knots.

Definition 6.2. The connected sum of two knots $K_1$ and $K_2$ is the knot formed by embedding $K_1$ and $K_2$ in $\mathbb{R}^3$ so that $K_1$ is contained in a ball that does not intersect $K_2$, removing a short section of $K_1$ and a short section of $K_2$ and connecting them with a strand from $K_1$ to $K_2$ and a strand from $K_2$ to $K_1$ so that the orientations match and the loop formed by the two sections removed and the two strands added is an unknot unlinked with $K_1$ and with $K_2$. See Figure 6.1 for an example. For a proof that the isotopy class of the resulting knot does not depend on the choices, see [12].

A split trivalent graph can be thought of as the connected sum of two trivalent graphs (although this operation depends on the base point chosen; but see Proposition 6.3 below). A weight system with no split graphs (called a primitive weight system) corresponds to an invariant (to be constructed) whose value on a connected sum of two knots is the sum of the values on the individual knots. For the corresponding homology semicycle, the same thing is true, at least if the two knots are widely separated and the strands connecting them are near each other.
Figure 6.1: Connected sum of two knots

Figure 6.2: The product of graphs

There is a natural product on the vector space spanned by trivalent graphs: the sum of all ways of interleaving the knot vertices of the two graphs, as in Figure 6.2. (The orientations carry over as well.) This product clearly corresponds to taking the product of the corresponding counting formulas or integrals. The product of two weight systems will again be a weight system. (The STU relation between two knot vertices from different original graphs is the only new thing to check, and it’s satisfied because of the sum over all interleavings.)

Note that the product of two non-trivial weight systems will not be primitive. Conversely, we have the following proposition, which corresponds nicely to the (unique) decomposition of a knot into pieces that cannot be expressed as a connected sum (“prime” knots).

**Proposition 6.3.** *Every weight system can be expressed uniquely as a sum of products of primitive weight systems.*

*Proof.* See [6], and especially Proposition 3.4.

Since it’s obviously desirable to have the product of weight systems correspond to the product of the (numerical) invariants, it suffices to construct invariants for only the primitive weight diagrams. For these diagrams, the anomalous faces must have the entire graph collapsing to a single point on the knot. It’s interesting to note that there are several connected components to this face for a given graph, corresponding to different ways to make the cyclic order of vertices into a linear order.

One note: the anomalous face is degenerate if the graph consisting of just the propagators is not connected, by Case 1 of the previous section. This means that any graph with a non-trivial anomalous face has a relatively large number of internal vertices.
For a given non-split, connected graph $\Gamma$, the restriction of $\mathcal{F}(\Gamma)$ to the (unique) anomalous stratum defines a map from the appropriate configuration space, $C_{(V,c)}V$, to $M = (S^2)^E(\Gamma)$. Since all the points collapse, the configuration space is fibered over $C_1(K) \simeq S^1$. Each component of the fiber is $C_{V;c}(T \mathbb{R}^3; K(s_V))$ (where $c(\Gamma)$ is given a linear order compatible with its cyclic order). This can be viewed as a bundle $B$ over $S^2$, with projection map $\pi: B \to S^2$; then $C_{V;c}(V)$ is the pullback to $C_1(K)$ of $B_{V;c}$ via the (normalized) tangent map. This is a different kind of pullback than we saw before. This kind of pullback can be constructed whenever you have a bundle $B$ over a manifold $Y$ (with projection map $\pi: B \to Y$) and a map $f: X \to Y$ from another manifold $X$; then the pullback of the bundle $B$ to $X$ is the universal $A$ in

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \pi \\
X & \longrightarrow & Y 
\end{array}
$$

or, concretely, the manifold $A = \{(b \in B, x \in X) : f(x) = \pi(b)\}$.

The sum of maps $\mathcal{F}(\Gamma): C_{V;c}(T \mathbb{R}^3; K(s_V)) \to M$ is the pullback of a sum of maps $\phi: B \to M$. $\phi$ gives a codimension 2 fibered chain on $S^2$ that captures many of the properties of $\mathcal{F}(\Gamma)$. For instance, if $B(\pi^{-1}(\xi)) = 0$ as a chain (modulo singular chains) in $M$ for each $\xi \in S^2$, then $\mathcal{F}(\Gamma)$ restricted to the anomalous face is, like $\phi$, a degenerate fibered chain. Or in the integral formulation: if the integral over the fibers of $\phi$ of the pullback of a volume form on $M$ is the 2-form $\omega$ on $S^2$, the corresponding integral over the fibers of $\mathcal{F}(\Gamma)$ is the integral of the pullback of $\omega$ via the tangent map. To show that the anomalous face vanishes as a fibered chain, it suffices to show that $\phi$ does (or equivalently, that the integral over the fibers of $\phi$ of an $n$-form $\omega$ on $M$ is 0 for every $\omega$). To show that the integral constructed with the canonical form on $M$ is an invariant, it suffices to show that the integral over the fibers of the canonical form $\theta$ on $M$ is the zero 2-form.

### 6.2 Parity of Invariants

There are two orientations involved in the definition of a knot: the orientation on $S^1$ and the orientation on $\mathbb{R}^3$. Reversing the orientation on $S^1$ yields the inverse of the knot; reversing the orientation on $\mathbb{R}^3$ yields its reverse. A (numerical) knot invariant that takes the same value on a knot and its inverse is called even under reversal of $S^1$; invariants that take the negative value on the inverse are called odd under reversal of $S^1$. Even and odd invariants under reversal of $\mathbb{R}^3$ are defined similarly. Every invariant can be uniquely expressed as a sum of invariants that are either even or odd under reversal of $S^1$ and even or odd under reversal of $\mathbb{R}^3$.

Knots that are not equal to their reverse are easy to come by. The first non-trivial knot, the trefoil, is one such. Invariants odd under reversal of $\mathbb{R}^3$ are correspondingly easy to find. Every weight diagram of odd degree yields an invariant odd under...
reversal of $\mathbb{R}^3$. We expect this to be true since the definition of derivative depends on the orientation of $\mathbb{R}^3$, so the derivative of an invariant (its evaluation on an immersed loop with one more crossing) changes its $\mathbb{R}^3$ parity, and a constant (an invariant whose derivative is 0) is necessarily even under reversal of $\mathbb{R}^3$. And in fact, by looking at the tinkertoy counting formula and noting that each $\epsilon$-tensor in the sign algorithm changes sign when you reverse the orientation of $\mathbb{R}^3$, one can show that the function on loops associated to a trivalent graph changes sign under reversal of $\mathbb{R}^3$ exactly when the graph is of odd degree (i.e., has $2k$ vertices, for $k$ odd). Our correction term below will not change this.

Knots that are not equal to their inverse are much rarer. The knot with the fewest number of crossings that is not equal to its inverse has eight crossings. Vassiliev invariants that are odd under reversal of $S^1$ are also rare, and may not exist at all. In [6], Bar-Natan presents the results of computer calculations enumerating the weight diagrams up to degree 9; so far, none that are odd under reversal of $S^1$ have been found. (The definition of derivative of a knot invariant does not depend on the orientation of the knot (check this!), so the $S^1$ parity of a knot invariant can be determined by looking at its largest non-zero derivative, or a weight diagram, which is what Bar-Natan computed.)

The parities of a (pure in parity, primitive) weight diagram $D$ give important information about the bundle map $\phi$ discussed above. In particular, they tell us the parity of $\phi$: whether $\phi \circ N$ (with $N : S^2 \to S^2$ the antipodal map) is $\pm \phi$. If $D$ is of degree $k$, then $\phi \circ N = (-1)^{k+1} \phi$, while if $D$ is of $S^2$ parity $\epsilon = \pm 1$, then $\phi \circ N = \epsilon \phi$. The easiest way to see this is to note that the variation of a loop function of a given parity will have the same parity (for either type of parity). The integral of a 2-form of a given $\mathbb{R}^3$ parity over the knot will have the opposite $\mathbb{R}^3$ parity (as a 1-form on $\mathfrak{K}$), since this integral depends on the orientation on $S^2$, which is inherited from the orientation of $\mathbb{R}^3$. The integral of a 2-form of a given $S^1$ parity over the knot has the same parity: the integral depends on the orientation on $S^1$, but it also depends on the tangent map $\dot{K}$, which is reversed (an orientation-reversing map on the sphere) when the orientation on $S^1$ is changed. (It’s possible to exhibit explicit automorphisms of the bundle $B$ showing both these facts.)

**Theorem.** There exists a canonically-defined fibered homology cycle for each weight diagram $D$, with a correction term if $D$ has odd $\mathbb{R}^3$ parity and even $S^1$ parity. In any case, no correction term is necessary for almost-planar knots or for the canonical integral.

**Proof.** It’s natural to proceed by cases, depending on the parities of the weight diagram $D$.

**Case 2.** $\mathbb{R}^3$ parity even, $S^1$ parity even or $\mathbb{R}^3$ parity odd, $S^1$ parity odd

**Proof.** In either of these two cases, the fibered chain $\phi$ must be both odd and even under the antipode map on $S^2$ and hence vanishes. \(\square\)

**Case 3.** $\mathbb{R}^3$ parity odd, $S^1$ parity even
Proof. This is the only case in which we’ll need a correction factor. The correction
will be given by \(-\frac{1}{2}\) the pullback of the bundle \(B\) and map \(\phi\) to the space \(C_2(K)\)
via the map \(\frac{x_2-x_1}{|x_2-x_1|}\). \(C_2(K)\) has a single boundary stratum with two connected
components (the points 1 and 2 coming together in either of the two possible orders).
The variations of these two faces are the pullbacks to \(C_1(K)\) via the tangent map
\(-\frac{1}{2}\phi\) and \(-\frac{1}{2}\phi \circ N\). Since \(\phi \circ N = \phi\), these two terms are the same and combine
to cancel the original variation, yielding a fibered homology cycle and hence a knot
invariant.

The tinkertoy structures for this correction term are a little more involved than
usual. They consist of two nodes on the knot and a “virtual” rod between them (a rod not
constrained to lie in one of the chosen directions), together with a tinkertoy
structure (for the graph whose anomaly we’re correcting) constructed on the virtual
rod; i.e., each node that would have gone on the knot in a regular tinkertoy diagram
must lie on the virtual rod. For general sets of chosen directions, this condition can
only be satisfied if the virtual rod is in a finite number of positions. Given any such
“virtual” tinkertoy diagram, its clear that translations (along the rod) and scalings
of it will also satisfy the conditions; these should only be counted once. The sign
will also be more involved. The only property of the sign I’ll need is that (like all
bundles over \(C_2(K)\) in this way) the sign of an overcrossing and an undercrossing
(with the same virtual diagram, relative to the corresponding virtual direction) are
opposite.

This correction term takes a special form for almost planar knots. An almost
planar knot is, as the name implies, very close to lying in a plane (over which the
knot should have generic crossings). Although only the unknot can exactly lie in a
plane, any knot can come arbitrarily close to a plane, with the only deviations from
true planarity near the crossings of the plane projection.

For a knot in such a position, the virtual directions (and hence a correction
term) can only occur at the crossings, since, relative to the almost planar knot, these
directions are almost vertical (for generic initially chosen directions). Furthermore,
at each crossing every virtual direction occurs (some with point 1 on the lower
strand and point 2 on the upper strand, some vice versa). The net contribution
from each overcrossing will therefore be the same, say \(\lambda\) and the net contribution
from each undercrossing will be \(-\lambda\). The correction term for almost planar knots is
then \(\lambda(\#\text{overcrossings} - \#\text{undercrossings})\). This difference between the number of
overcrossings and undercrossings of an almost planar knot is called the writhe; it is
the framing number of the framing given by choosing normal vectors parallel to the
plane. (See [12]).

One can show \(\lambda\) does not depend on the form \(\omega\) chosen on the product of \(S^2\):
this boils down to the familiar fact that the boundary of a boundary is 0.

From field theory arguments one expects this correction term \(\lambda\) to always be 0,
but I don’t know any proof. The only non-trivial case which I know is that for the
unique degree 3 Vassiliev invariant (shown in Figure 6.3) the only anomaly term
comes from the last factor and I have found a direct argument that \(\lambda\) is 0 for this
diagram, although the correction term does not vanish for all forms \(\omega\).
Since the canonical form on $(S^2)^e$ is SO(3) invariant, the correction form will also be SO(3) invariant, and hence will vanish if $\lambda$, its integral over the sphere, is 0.

\[ w\begin{pmatrix} \mathcal{O} \end{pmatrix} = 2 \quad w\begin{pmatrix} \mathcal{O} \end{pmatrix} = -1 \]

\[ w\begin{pmatrix} \mathcal{O} \end{pmatrix} = 1 \quad w\begin{pmatrix} \mathcal{O} \end{pmatrix} = 1 \]  

(6.2)

**Case 4.** $\mathbb{R}^3$ parity even, $S^1$ parity odd

**Proof.** In this case, the variation form $\psi$ is odd under the antipodal map $N$, and thus has total integral 0 over the sphere. This implies that $\psi$ is cohomologous to 0, or equivalently $\psi = d\alpha$ for some 1-form $\alpha$ on $S^2$.

**Proposition 6.4.** If $d\alpha = \psi$, then $I(\Gamma)(K) - \int_{S^1} K^*\alpha$ is a knot invariant that does not depend form $\alpha$ chosen, and is independent of $\omega$.

Now consider the integral $\int_{S^1} K^*\alpha$. By Proposition 4.2, the variation of this integral is $\int_{S^1} K^*d\alpha = \int_{S^1} K^*\psi$ is exactly the variation of our original form, so subtracting this term yields a knot invariant.

although not cancellation formula of original type, doesn’t change as vary the knot.

For plane curves, multiple of the winding number. (0 for canonical integral.)

Surprising if it vanishes for all $\omega$, because then it vanishes identically; but we’ve already looked at the obvious symmetries that would keep direction fixed.

This completes the proof of the proposition, and hence our main theorem.
Appendix A

Some linear algebra

In this section I will present some results in linear algebra, related to the determinant, that may not be familiar to all readers. I assume a basic knowledge of linear algebra, including tensor products and antisymmetric products and the exterior algebra.

The determinant of an $n$-dimensional vector space $V$ is the 1-dimensional vector space

$$\det V = \bigwedge^\text{top} V = \bigwedge^n V$$

This is a functor from the category $\text{Vect}^i$ of vector spaces and isomorphisms to itself. ($\bigwedge^n$ is a functor on the ordinary category of vector spaces and linear maps. The restriction to isomorphisms ensures that the domain and range have the same dimension.) It’s called the determinant because the ordinary determinant of a linear map $f : V \to V$ is the (unique) eigenvalue of the linear map $\bigwedge^n f : \bigwedge^n V \to \bigwedge^n V$.

**Lemma A.1.** $\det V \otimes \det W$ is canonically isomorphic to $\det(V \oplus W)$.

**Proof.** These are both 1-dimensional vector spaces, so a canonical non-zero map between them will suffice. In general, a linear map from $A \otimes B$ to $C$ is a bilinear map from $A$ and $B$ to $C$. So suppose $a \in \det V$ and $b \in \det W$. Since $V$ and $W$ are both subspaces of $V \oplus W$, $\det V$ is a subspace of $\bigwedge^\dim V (V \oplus W)$ and $\det W$ is a subspace of $\bigwedge^\dim W (V \oplus W)$. If we consider both $a$ and $b$ in $\bigwedge^n (V \oplus W)$, then $a \wedge b$ is the desired element of $\det(V \oplus W)$. This is a bilinear map and is clearly functorial.

Note that the isomorphism in the above lemma depends on the order $V$ and $W$ are specified. Specifically, we have the following:

**Lemma A.2.** Under the isomorphisms

$$\det V \otimes \det W \simeq \det(V \oplus W) \simeq \det(W \oplus V) \simeq \det W \otimes \det V \simeq \det V \otimes \det W$$

(A.2)
in which the first and third isomorphism are those given by Lemma A.1 (with \(V, W\) and \(W, V\) respectively), \(\det V \otimes \det W\) is mapped into itself by multiplication by \((-1)^{\dim V \cdot \dim W}\).

**Proof.** This follows immediately from the familiar equation in anti-commuting algebras (such as \(\wedge^* V\)),

\[
a \wedge b = (-1)^{\deg a \cdot \deg b} b \wedge a
\]  
(A.3)

which in turn follows from the equation for elements of degree 1

\[
v \wedge w = -w \wedge v
\]  
(A.4)

by induction. \(\square\)

### A.1 Unordered Collections of Vector Spaces

Now suppose we have an unordered collection of \(k\) vector spaces. Formally, this is an element of the category \(\text{Vect}^{i,u}_k\) whose objects are \(k\)-tuples of vector spaces \(\{V_i\}_{i=1}^k\) and in which a morphism from \(\{V_i\}_{i=1}^k\) to \(\{W_i\}_{i=1}^k\) is a permutation \(\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}\) and vector space isomorphisms \(l_i : V_i \to W_{\pi(i)}\). There is a functor from this category to \(\text{Vect}^1\), the direct sum, given by

\[
\{V_i\}_{i=1}^k \mapsto \bigoplus_{i=1}^k V_i
\]  
(A.5)

where \(\sigma_\pi : \bigoplus_{i=1}^k W_{\pi(i)} \to \bigoplus_{i=1}^k W_i\) is the canonical isomorphism rearranging the factors of \(\bigoplus_{i=1}^k W_i\). Similarly, there is another functor to \(\text{Vect}^1\), the tensor product, given by \(\{V_i\}_{i=1}^k \mapsto \bigotimes_{i=1}^k V_i\) and the corresponding map on morphisms.

Then, for any \(\{V_i\}_{i=1}^k\) we can define the two 1-dimensional vector spaces

\[
\det\left(\bigoplus_{i=1}^k V_i\right)
\]  
(A.6)

and

\[
\bigotimes_{i=1}^k \det V_i
\]  
(A.7)

by analogy with the construction above for \(A \oplus B\). But there is no canonical isomorphism between them in general. The story is more complicated and more interesting.
Lemma A.3. If all the \( \{V_i\}_{i=1}^k \) are even-dimensional, then there is a canonical isomorphism
\[
\det(\bigoplus_{i=1}^k V_i) \simeq \bigotimes_{i=1}^k \det V_i
\] (A.8)

Proof. This is just an iterated version of Lemma A.1 above, with the further note that the isomorphism is independent of the order of the \( V_i \) by Lemma A.2. \( \square \)

For odd-dimensional vector spaces, we find

Lemma A.4. If all the \( \{V_i\}_{i=1}^k \) are odd-dimensional, there is a canonical isomorphism
\[
\det(\bigoplus_{i=1}^k V_i) \simeq \bigotimes_{i=1}^k \det V_i \otimes \det(\bigoplus_{i=1}^k \mathbb{R}v_i^*)
\] (A.9)

where the \( v_i \) are arbitrary independent vectors (so \( \bigoplus_{i=1}^k \mathbb{R}v_i^* \) is the dual of the free vector space generated by \( \{1, \ldots, k\} \)).

Proof. Given an element \( a_1 \otimes \cdots \otimes a_k \otimes \lambda \) of the second vector space, the element
\[
\lambda(v_1, \ldots, v_k) \cdot a_1 \wedge \cdots \wedge a_k
\] (using the inclusions of \( \det V_i \) in \( \bigwedge^*(\bigoplus_{i=1}^k V_i) \)) is a non-zero element of \( \det(\bigoplus_{i=1}^k V_i) \) which doesn’t depend on the particular ordering. (By Lemma A.2 both \( \lambda(v_1, \ldots, v_k) \) and \( a_1 \wedge \cdots \wedge a_k \) change sign under a simple transposition.) \( \square \)

Since we only admit vector space isomorphisms, we can sort the vector spaces by dimension. In a somewhat weaker and more precise form, we have a functor from \( \text{Vect}^{i,u}_k \) to \( \text{Vect}^2 \) (the category of pairs of vector spaces with isomorphisms) given by
\[
\bigoplus V_i \mapsto \bigoplus_{\dim V_i \text{ even}} V_i \oplus \bigoplus_{\dim V_i \text{ odd}} V_i
\] (A.11)
(i.e., divide the vector spaces into even and odd dimensional pieces).

Proposition A.5. For arbitrary \( V_i \), there is a canonical isomorphism
\[
\det(\bigoplus_{i=1}^k V_i) \simeq \bigotimes_{i=1}^k \det V_i \otimes \det \left( \bigoplus_{\dim V_i \text{ odd}} \mathbb{R}v_i^* \right)
\] (A.12)

\(^1\)Of course, this vector space has a canonical basis and so is naturally isomorphic with its dual, but this formulation makes the result easier to state.
Proof. By equation A.11, Lemma A.1, and Lemmas A.3 and A.4, we have

\[
\det\left(\bigoplus_{i=1}^{k} V_i\right) \simeq \det\left(\bigoplus_{\text{dim } V_i \text{ even}} V_i \bigoplus \bigoplus_{\text{dim } V_i \text{ odd}} V_i\right)
\]

\[
\simeq \det\left(\bigoplus_{\text{dim } V_i \text{ even}} V_i\right) \otimes \det\left(\bigoplus_{\text{dim } V_i \text{ odd}} V_i\right)
\]

\[
\simeq \bigotimes_{\text{dim } V_i \text{ even}} \det V_i \otimes \bigotimes_{\text{dim } V_i \text{ odd}} \det V_i \otimes \det\left(\bigoplus_{\text{dim } V_i \text{ odd}} R_{v_i}^*\right)
\]

\[
\simeq \bigotimes_{i=1}^{k} \det V_i \otimes \det\left(\bigoplus_{\text{dim } V_i \text{ odd}} R_{v_i}^*\right)
\]

(A.13)

A.2 Exact Sequences

In addition to the isomorphisms above, I will need some more general isomorphisms involving \(\det\) associated with any exact sequence.

Lemma A.6. For any vector space \(V\) of dimension \(n\) and subspace \(W\) of dimension \(m\), there is a canonical isomorphism

\[
\det V \simeq \det W \otimes \det(V/W)
\]

(A.14)

Proof. As noted above, the natural injection \(W \to V\) induces an injection \(\det W \to \Lambda^m V\). Similarly, the natural surjective map \(V \to V/W\) induces a surjective map \(\Lambda^{n-m} V \to \det(V/W)\), whose kernel is generated by elements of the form \(w \wedge v_1 \wedge \cdots \wedge v_{n-m-1}, w \in W\). For any two elements \(a \in \det W, b \in \det(V/W)\), pick an arbitrary preimage \(b^* \in \Lambda^{n-m}\) of \(b\). Then \(a \wedge b^*\) is independent of the choice of \(b^*\), since \(a \wedge c\) is 0 for any \(c\) in the kernel of the map \(\Lambda^{n-m} V \to \det(V/W)\) and defines a canonical (non-zero) bilinear map from \(\det W\) and \(\det(V/W)\) to \(\det V\), as desired.

Proposition A.7. In any exact sequence

\[
0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{k-1}} V_k \xrightarrow{f_k} 0
\]

of vector spaces, there is a canonical isomorphism

\[
\bigotimes_{1 \leq i \leq k, k \text{ odd}} \det V_i \simeq \bigotimes_{1 \leq i \leq k, k \text{ even}} \det V_i
\]

(A.16)
Proof. For each $i$, we have, by Lemma 6, an isomorphism

$$\det V_i \cong \det(ker l_i) \otimes \det(V_i/ker l_i) \quad (A.17)$$

so, combining these (and using $ker l_i \cong V_{i-1}/ker l_{i-1}$),

$$\bigotimes_{1 \leq i \leq k} \det V_i \simeq \bigotimes_{1 \leq i \leq k} \det(ker l_i) \otimes \det(V_i/ker l_i)$$

$$\simeq \bigotimes_{1 \leq i \leq k} \det(V_{i-1}/ker l_{i-1}) \otimes \det ker l_{i+1} \quad (A.18)$$

$$\simeq \bigotimes_{1 \leq i \leq k} \det V_i$$

$k$ odd

$k$ even

\qed
Appendix B

Graphs

In this Appendix I formally prove the equivalence of three different notions of orientation on a graph, generalizing the comment in Section 4.3 about the two different ways of writing the integral invariants.

An (unoriented) graph based on a circle is an undirected graph $\Gamma$ with a choice of a (directed) cycle $c(\Gamma)$ with distinct edges and vertices. An orientation on a graph based on a circle $\Gamma$ is an orientation on the 1-dimensional vector space

$$\text{or}(\Gamma) = \det \mathbb{R}V(\Gamma) \otimes \bigotimes_{e \in E(\Gamma)} \mathbb{R}V(e) \quad (B.1)$$

where

- $\det W$, for $W$ an $n$-dimensional vector space, is $\Lambda^n W$.
- $V(\Gamma)$ is the set of vertices of $\Gamma$.
- $E(\Gamma)$ is the set of edges of $\Gamma$.
- $VE(\Gamma)$ is the set of incident vertex-edge pairs of $\Gamma$.
- $V(e)$ is the set of vertices that are endpoints of $e$ (a two element set).
- $E(v)$, which we will need later, is the set of edges that have the vertex $v$ as an endpoint.
- $V(c)$ and $E(c)$ are the vertices and edges, respectively, in $c(\Gamma)$.
- $V_i(\Gamma)$ and $E_i(\Gamma)$, the internal vertices and edges of $\Gamma$, are $V(\Gamma)$ $V(c)$ and $E(\Gamma)$ $E(c)$, respectively.

Two equivalent definitions are given in Proposition 3.3 below.

A labelling $l$ of the internal edges of $\Gamma$ is a bijective map

$$l : \{1, \ldots, |E_i(\Gamma)|\} \to E_i(\Gamma) \quad (B.2)$$

45
together with bijective maps
\[ l_e : \{ s, d \} \to V(e), \quad e \in E_i(\Gamma) \quad (B.3) \]

A labelling \( l \) and an orientation \( o \) together define a chain in \((S^2)^{E_i(\Gamma)}\) for any given knot \( K \). Pick an arbitrary bijection
\[ t : V(\Gamma) \to \{1, \ldots, |V(\Gamma)|\} \quad (B.4) \]

Then the chain in \((S^2)^{E_i(\Gamma)}\) is
\[ \phi_{\Gamma, o, l}(K) = \epsilon \phi_{l(1)(s), l(1)(d)}(C^t_{|V(e)|, |V_i(\Gamma)|}(K)) \times \phi_{l(2)(s), l(2)(d)}(C^t_{|V(e)|, |V_i(\Gamma)|}(K)) \times \cdots \times \phi_{l(|E_i(\Gamma)|)(s), l(|E_i(\Gamma)|)(d)}(C^t_{|V(e)|, |V_i(\Gamma)|}(K)) \quad (B.5) \]

where
\[ \cdot \]

- \( C^t_{|V(e)|, |V_i(\Gamma)|}(K) \) is the component of \( C_{|V(e)|, |V_i(\Gamma)|}(K) \) with corresponding labellings of the points and with the points on the circle occuring in the same cyclic order as they do in \( \Gamma \), imbued with the orientation the interior inherits as a subset of \((S^1)^{|V(e)|} \times (\mathbb{R}^3)^{|V_i(\Gamma)|}\) with the factors in the appropriate order.

- \( \epsilon \) is the appropriate sign: \( \pm 1 \), depending on whether or not the element
\[ (t(1) \land \cdots \land t(n)) \otimes \bigotimes_{e \in V_i(\Gamma)} (l_e(1) \land l_e(2)) \otimes \bigotimes_{e \in V(e)} (c_e(1) \land c_e(2)) \in \text{or}(\Gamma) \]

is positively oriented. \( (c_e : \{1, 2\} \to V(e) \) is the map determining the orientation of the circle.\)

Note, in the configuration space of the knot, that cycle vertices are constrained to lie on the knot, internal vertices are not so constrained, internal edges give rise to propogators, while cyelic edges are merely place-holders to indicate the order of points around the knot.

This is a good definition: it doesn’t depend on the labelling \( t \) of \( V(\Gamma) \) we chose, because of the choice of \( \epsilon \). (Note in particular that both \( S^1 \) and \( \mathbb{R}^3 \) are odd-dimensional, so the orientation of \( C^t_{|V(e)|, |V_i(\Gamma)|} \) depends on the sign of \( t \) (considered as a permutation), exactly as \( \epsilon \) does.)

In addition, if given an orientation \( o \) and a labelling \( t \) of the vertices as above, there is a well-defined form
\[ \theta_{\Gamma, o, t} = \phi_{\Gamma, o, t}^{\epsilon} \otimes |E_i(\Gamma)| \quad (B.7) \]
on \( C^t_{|V(e)|, |V_i(\Gamma)|} \) independent of the labelling of the edges \( l \). With the orientation defined by \( t \), the integral of this form is the volume of the chain \( \phi_{\Gamma, o, l} \) and is thus independent of \( t \) and \( l \). These integrals are exactly those defined in BT.

These integrals and chains are the basic building blocks to construct knot invariants. But there are two important equivalent ways of defining the orientation above.

46
**Proposition B.1.** Orientations on the following 1-dimensional vector spaces are equivalent:

1. \( \text{or}(\Gamma) = \det \mathbb{R}V(\Gamma) \otimes \bigotimes_{e \in E(\Gamma)} \mathbb{R}V(e) \)
2. \( \bigotimes_{v \in V(\Gamma)} \det \mathbb{R}E(v) \otimes \det(\bigoplus_{\deg v \ even} \mathbb{R}v) \)
3. \( \det \mathbb{R}E(\Gamma) \otimes \det H_1(\Gamma) \otimes \det H_0(\Gamma) \)

Note that we have the following equivalences:

\[
\bigotimes_{e \in E(\Gamma)} \det \mathbb{R}V(e) \simeq \det \mathbb{R}VE(\Gamma) \simeq \bigotimes_{v \in V(\Gamma)} \det \mathbb{R}E(v) \otimes \det(\bigoplus_{\deg v \ odd} \mathbb{R}v)
\]  

(B.8)

The first isomorphism arises from grouping \( VE(\Gamma) \) by vertex-edge pairs by those sharing an edge, while the second comes about by grouping by vertices. Note that when we group vertex-edge pairs in this way, we must be careful to observe the ordering of the groups. A group with an even number of elements will commute with all other groups, while a group with an odd number of elements will anticommute. Every edge has two endpoints, yielding the first isomorphism, while the second isomorphism is just the above observation. Tensoring equation (B.8) with \( \det V(\Gamma) \) yields \((1) \simeq (2)\).

To see \((1) \simeq (3)\), note that, since a graph has no two-cells, we have an exact sequence

\[
0 \to H_1(\Gamma) \to C_1(\Gamma) \to C_0(\Gamma) \to H_0(\Gamma) \to 0
\]

(B.9)
yielding an isomorphism

\[
\det H_1(\Gamma) \otimes \det H_0(\Gamma) \simeq \det C_1(\Gamma) \otimes \det C_0(\Gamma)
\]

(B.10)

(all these spaces have canonical (unordered) bases, so there’s no worry about dual spaces). \( C_0(\Gamma) \) is just \( \mathbb{R}V(\Gamma) \), but \( C_1(\Gamma) \) is the vector space spanned by oriented edges, so

\[
C_1(\Gamma) \simeq \bigoplus_{e \in E(\Gamma)} \det V(e)
\]

(B.11)

\[
\det C_1(\Gamma) \simeq \det \mathbb{R}E(\Gamma) \otimes \bigotimes_{e \in E(\Gamma)} \det V(e)
\]

(B.12)

Therefore, tensoring equation (B.10) above with \( \det \mathbb{R}E(\Gamma) \), we find

\[
\det \mathbb{R}E(\Gamma) \otimes \det H_1(\Gamma) \otimes \det H_0(\Gamma) \simeq \det \mathbb{R}V(\Gamma) \otimes \bigotimes_{e \in E(\Gamma)} \mathbb{R}V(e)
\]

(B.13)
as desired.

Note that orientations of type (2) in the proposition correspond to Bar-Natan’s trivalent diagrams, as in BN. Orientations of type (3) are mentioned in Kontsevich’s paper K. (He omits the $H_0$ term. In the case of knots, this is OK, as all integrals of interest are connected (to the knot). However, to get invariants of homology 3-spheres the $H_0$ term is necessary.)

Also note that in each of the cases above the cycle of the boundary could have been omitted from the graph $\Gamma$ without essentially changing the definition. For instance, for definition (1) of $or(\Gamma)$, if we omit the circle from the graph we omit no vertices and some edges, each with a canonical orientation. This would have made the definition of the chain $\phi_{r, o, l}^{\Gamma}$ slightly cleaner. However, the definition of the boundary operator is easier to state if the cycle is included in the graph.
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