I. INTRODUCTION

When quantum computers are realized in the style of ‘cloud’, only a few companies can possess them because of expensive prices. Many clients will have requirements of quantum computation but they have limited quantum technologies. They outsource their quantum computation to servers, but how to ensure that clients’ secrets are not leaked? Fortunately, Blind quantum computation (BQC) is proposed to solve this problem securely [1][2]. In BQC, a client who has few quantum technologies at her disposal delegates her quantum computation to servers who have full-advanced quantum computers without sacrificing the privacy of her quantum inputs, outputs and algorithms. In 2005, Childs [1] first presented blind quantum computation based on circuits, where the client Alice has the abilities to store quantum states and route her qubits, and the server Bob can perform universal quantum computation. Fisher et al. [2] realized quantum computation (X, Z, H, S, R, CNOT) on encrypted quantum states similar to homomorphic encryption [13]. They use linear optics to implement a proof-of-principle of the protocol. Broadbent [3] introduced an entanglement-based protocol such that it only needs multiple auxiliary qubits or two-way quantum communication.

In 2009, Broadbent et al. [5] first implemented an universal BQC protocol by measuring on blind graph states, i.e. brickwork states, where the client has the abilities to prepare single qubits randomly chosen from a finite set \( \{ |0\rangle, |1\rangle, e^{i\theta}|1\rangle \} \). \( \theta = 0, \frac{\pi}{4}, \frac{2\pi}{4}, \ldots, \frac{N\pi}{4} \). The brickwork state is composed of many unit cluster entangled states presented in [3]. They exploited the conceptual framework of measurement-based quantum computation to implement an experimental demonstration ensuring the privacy of quantum inputs, computations, and outputs. After that, double-server and triple-server BQC protocols were proposed in Refs. [7][9].

Recently, verifiable BQC protocols widely attract a lot of attentions [14][21]. Morimae [14] proposed two verifiable BQC protocols. In the first BQC protocol, the server Bob sends resource states \( |\Phi\rangle = |R\rangle \otimes |\psi^+\rangle \otimes |0\rangle \otimes |N/3\rangle \), which is a N-qubit state and \( |R\rangle \) is an N/3-qubit universal resource state, to the client Alice. In this protocol, if all measurements on traps show the correct results, the probability for changes in a logical state of Alice’s computation is exponentially small. In the second protocol, it did not use any traps but the properties of the topological code. The no-signaling principle guarantees the device-independent security, that is, the second BQC protocol is also verifiable. The stabilizer testing [15] is used to verify the correctness of quantum computation, where Alice can obtain the correct computation results if Bob is honest to generate the correct graph state. But if Bob is malicious to prepare a fake graph state, Alice can directly examine the stabilizers of these graph states to verify Bob’s honesty. Since the quantum channel noises are unavoidable in practical quantum communication, numerous anti-noise BQC protocols [7][9][22][23] are proposed to solve them. Such as Takeuchi et al. [23] used decoherence-free subspace (DFS) to resist a collective-noise of quantum channels. There are also some other interesting BQC protocols to improve the functions further [24][25]. For example, in [33], Huang et al. implemented a proof-of-principle experiment to complete the factorization of the number 15 in which the client is classical.

In this paper, we study the quantum fourier transform (QFT) inspired by these works [35][36]. In [35], Marquezino et al. used QR decomposition to convert the classical fourier transform algorithm into the quantum fourier transform. Motivated by these works in [7][8], we consider Bell states as carrier in quantum channel to realize QFT. Bell states have already been prepared...
in experiment [39][41]. To be specific, in our proposed BQC protocols, the trusted center bears the task to prepare enough initial Bell states \(\{|\phi^+\rangle, |\phi^\rangle, |\psi^+\rangle, |\psi^-\rangle\}\), server Bob performs those operations in circuits, and the client Alice is almost classical. In these BQC protocols, the QFT can be replaced by other operators to get the target quantum states and Alice needs to communicate with Bob multiple rounds. Our main contributions are as follows.

(1) We analyze and present these primary BQC protocols with the equivalent quantum circuits of QFT performed on qubits 12 of Bell states \(\{|\phi^+\rangle_{12}, |\psi^\rangle_{12}\}\);

(2) We discuss and show these enhanced BQC protocols with the equivalent quantum circuits of QFT performed on qubits 13 of any two Bell states \(|\xi\rangle_{12} \otimes |\theta\rangle_{34}\) where \(|\xi\rangle_{12}\) and \(|\theta\rangle_{34}\) are Bell states;

(3) We give these generalized BQC protocols with the equivalent quantum circuits of QFT performed respectively on qubits 13, 24 of any two Bell states;

(4) In the end, we prove the blindness and correctness for every BQC protocol, where the proof of the correctness is attached in Appendix I, II and III.

The rest of this paper is organized as follows. The basic knowledges about QFT are introduced in Sec. II. We propose the BQC protocols in Sec. III. The analysis of blindness and correctness is presented in Sec. IV. At last, the conclusions are drawn in Sec. V.

II. PRILIMENARIES

The quantum Fourier transform is introduced in Ref. [42]. We review the principle of quantum Fourier transform in this section for obtaining our BQC protocols.

We use the mathematical notation to describe the discrete Fourier transform as follows

\[
y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{\frac{2\pi i j k}{N}} x_j,
\]

where \(i^2 = -1, k, j = 0, \ldots, N-1\), the input is denoted as a vector of complex numbers \(\{x_0, x_1, \ldots, x_{N-1}\}\) and the output is a vector of complex numbers \(\{y_0, y_1, \ldots, y_{N-1}\}\) (\(N\) is the length of the vector). Identically, in quantum mechanics system, the quantum Fourier transform on orthonormal basis \(|0\rangle, \ldots, |N-1\rangle\) is defined as

\[
QFT_{N}|j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i j k}{N}} |k\rangle,
\]

and the unitary matrix \(QFT_{N}\) is given by

\[
QFT_{N} = \frac{1}{\sqrt{N}} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 \\
1 & \omega^2 & \omega^4 & \omega^6 \\
\vdots & \vdots & \vdots & \vdots \\
1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{(N-1)(N-1)}
\end{pmatrix},
\]

where \(N = 2^n\) and \(\omega = e^{\frac{2\pi i}{2^n}}\). When \(N = 4 = 2^2\) and phase \(\omega = i\), the transformation matrix

\[
QFT_4 = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & i & -1 & -i \\
1 & -1 & 1 & -i \\
1 & -i & -1 & i
\end{pmatrix}.
\]

By performing quantum Fourier transform, an arbitrary state \(\sum_{j=0}^{N-1} x_j |j\rangle\) will be changed into

\[
\sum_{j=0}^{N-1} x_j |j\rangle \xrightarrow{QFT} \sum_{k=0}^{N-1} y_k |k\rangle,
\]

where the amplitudes \(y_k\) are the discrete Fourier transform values of amplitudes \(x_j\). Suppose the state \(|j\rangle = |j_1, j_2, \ldots, j_n\rangle\) represents the binary \(j = \sum_{j=1}^{n} j_2^{n-i} \). The notation \(0, j_1j_2, \ldots, j_m\) represent the binary \(\sum_{i=0}^{m-1} j_{l+i}/2^l+1\).

The quantum circuit for the quantum Fourier transform is shown in FIG. 1. Hadamard gate operated on the \(j\)-th qubit is denoted as \(H_j : |j\rangle \rightarrow 1/\sqrt{2}(|0\rangle + (-1)^j|1\rangle)\). The \(G_k\) denotes the unitary transformation, which is expressed as

\[
G_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}.
\]

![FIG. 1. The quantum circuit for quantum Fourier transform.](image)

In the circuit, when the state \(|j\rangle = |j_1 \ldots j_n\rangle\) is input, we get

\[
|j\rangle \xrightarrow{QFT} \frac{1}{2^n/2} \sum_{k=0}^{2^n-1} e^{2\pi i j/k/2^n} |k\rangle
\]

\[
= \frac{1}{2^n/2} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi i j \sum_{j=1}^{n} k_j/2^{j-1}} |k_1 \ldots k_n\rangle
\]

\[
= \frac{1}{2^n/2} \left( |0\rangle + e^{2\pi i j_1 1} |1\rangle \right) \left( |0\rangle + e^{2\pi i j_n 1} |1\rangle \right) \cdots \left( |0\rangle + e^{2\pi i j_1 j_2 \ldots j_n} |1\rangle \right).
\]

where \(N = 2^n\) and the computational basis \(|0\rangle, \ldots, |2^n-1\rangle\) is given. This construction also proves that the quantum Fourier transform is unitary, since each gate in the
circuit is unitary. In FIG. 2, For double-qubit, 

\[
\begin{align*}
|00\rangle & \xrightarrow{QFT} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle), \\
|01\rangle & \xrightarrow{QFT} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)(|0\rangle + i|1\rangle), \\
|10\rangle & \xrightarrow{QFT} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle), \\
|11\rangle & \xrightarrow{QFT} \frac{1}{2}(|0\rangle - |1\rangle)(|0\rangle - i|1\rangle).
\end{align*}
\]

(1)

FIG. 2. Quantum Fourier transform is performed on two qubits of Bell states \{\(|\phi^+\rangle_{12}, |\psi^+\rangle_{12}\}\).

We introduce the characteristics of some common gates since they have been used in our BQC protocols. For single-qubit gates, we have

\[
\begin{align*}
X : |j\rangle & \rightarrow |j \oplus 1\rangle, \\
S : |j\rangle & \rightarrow i^j |j\rangle, \\
Z : |j\rangle & \rightarrow (-1)^j |j\rangle, \\
T : |j\rangle & \rightarrow (e^{\frac{i \pi}{4}})^j |j\rangle.
\end{align*}
\]

(2)

For the double-qubit gates, we have

\[
\begin{align*}
\text{CNOT} : |j\rangle|k\rangle & \rightarrow |j\rangle|j \oplus k\rangle, \\
\text{CS} : |j\rangle|k\rangle & \rightarrow i^k |j\rangle|k\rangle, \\
\text{CZ} : |j\rangle|k\rangle & \rightarrow (-1)^j |j\rangle|k\rangle.
\end{align*}
\]

(3)

where C is controlled.

The relationship between SWAP and CNOT, CS, CZ and CT, S, Z and T.

\[
\begin{align*}
\text{SWAP}_{12} & = \text{CNOT}_{12}\text{CNOT}_{21}\text{CNOT}_{12}, \\
\text{CZ}_{12} & = \text{CS}_{12}^2 = \text{CT}_{12}^4, \\
Z & = S^2 = T^4.
\end{align*}
\]

(4)

III. BQC PROTOCOLS FOR QUANTUM FOURIER TRANSFORM BASED ON BELL STATES

A. Architecture of BQC protocols

The process of BQC protocols for QFT is described as follows. It contains five steps:

1) A trusted center prepares enough initial \(|\phi^+\rangle\) states and sends to Alice.
2) Alice disturbs the positions and sends qubits to Bob.
3) Bob performs the relative operations and returns results to Alice.
4) They repeat steps 2) and 3) until the computation halts, but Bob does not know which qubit he receives during the whole period.

Alice designs every quantum circuit which is equivalent to QFT performed on Bell states (See FIG. 3, FIG. 4 and FIG. 5). To improved the blindness, we adds some auxiliary gates in these BQC protocols. If a circuit lacks one gate of \{CT, T, X, H, CNOT\}, the gate should be added as a auxiliary gate. That is, it must ensure that every circuit is composed of CT, T, X, H and CNOT. For example, in FIG. 4, gates X, CNOT and T should be added as auxiliary gates. However, in order that auxiliary gates do not affect the original circuits, Bob needs to perform eight rounds CT and T gates, where \(CT^8 = T^8 = I\), and two rounds gates X, H, CNOT, where \(X^2 = H^2 = CNOT^2 = I\). Therefore, we suppose that some auxiliary gates are performed randomly but not shown on the FIG. 4, FIG. 5 and FIG. 6.

All BQC protocols are started in an intersecting way but the order of operations for every gate is not changed. To be precise, the order of execution must be from the first round to the last round. But for different circuits, they can be implemented in an intersectional way.

B. Primary BQC protocols of two-qubit QFT on one Bell state

To show our method clearly, we first design the a primary BQC model, that is, the QFT is performed on two qubits of one Bell state with the equivalent quantum circuits in FIG. 3.

\[
|\chi\rangle_{12} = |\rangle_{\theta_{12}}
\]

FIG. 3. The equivalent quantum circuits of QFT performed on 12 of \(|\chi\rangle_{12}\) (\(|\chi\rangle_{12}\) belongs to \{\(|\phi^\pm\rangle_{12}, |\psi^\pm\rangle_{12}\}\).

\[
\text{BQC protocol 1.} \quad \text{For Bell states } |\chi\rangle_{12} \text{ belonging to } \{\phi^\pm\rangle_{12}, |\psi^\pm\rangle_{12}\}, \text{Bob performs one round } H_2, \text{two rounds } CT_{12}, \text{and one round } H_1. \text{Bob returns results to Alice and Alice obtains } \text{QFT}_{12}|\chi\rangle_{12}.
\]

Notation: In FIG. 3, the subscripts 1 and 2 of \(H_1\) and \(H_2\) represent which qubit is performed on the gate. In FIG. 4 and FIG. 5, the meanings of the expression are similar.

C. Enhanced BQC protocols of two-qubit QFT on two Bell states

We have considered two Bell states in section B but its security is not strong enough. What’s more, in some cases, Alice wants to obtain the results that QFT is performed on two qubits belonged to two Bell states respectively. So we propose the enhanced BQC protocols QFT on 13 of \(|\xi\rangle_{12} \otimes |\theta\rangle_{34}\). The equivalent quantum circuits is given in FIG. 4. For \(|\xi\rangle_{12} \otimes |\theta\rangle_{34}\) (\(|\xi\rangle_{12}, |\theta\rangle_{34} \in \{|\phi^+, |\phi^-\rangle, |\psi^+, |\psi^-\rangle\}\), it should be sixteen cases. Based on these above cases, we design these corresponding BQC protocols as follows.
**BQC protocol 2.** For $|\psi^+\rangle_{12}|\phi^-\rangle_{34}$, $|\psi^-\rangle_{12}|\psi^+\rangle_{34}$ and $|\psi^-\rangle_{12}|\phi^-\rangle_{34}$, we first describe the same operations (FIG. 4). That is, Bob performs three rounds CNOT (i.e., CNOT, CNOT, and CNOT), two rounds $H$ ($H_2$ and $H_4$) and two rounds CT. However, the slight different operations are implemented. After the same operations, a CNOT is carried out in the circuit of $|\psi^+\rangle_{12}|\phi^-\rangle_{34}$. A $X_4$ needs to be implemented in the circuit of $|\psi^-\rangle_{12}|\phi^-\rangle_{34}$ before the same operations. Bob returns these operated qubits to Alice and thus Alice will obtain $\text{QFT}_{13}|\psi^+\rangle_{12}|\phi^-\rangle_{34}$, $\text{QFT}_{13}|\psi^-\rangle_{12}|\psi^+\rangle_{34}$ and $\text{QFT}_{13}|\psi^-\rangle_{12}|\phi^-\rangle_{34}$.

**BQC protocol 3.** For $|\phi^+\rangle_{12}|\phi^+\rangle_{34}$, $|\psi^-\rangle_{12}|\phi^+\rangle_{34}$, $|\phi^+\rangle_{12}|\phi^+\rangle_{34}$, $|\phi^+\rangle_{12}|\psi^-\rangle_{34}$ and $|\phi^+\rangle_{12}|\psi^+\rangle_{34}$, the same operations are showed. Bob performs three rounds CNOT (i.e., CNOT, CNOT, and CNOT), two rounds $H$ ($H_2$ and $H_4$) and two rounds CT. Some different operations are as follows after above same operations. 1) For $|\psi^-\rangle_{12}|\psi^-\rangle_{34}$, Bob performs two rounds $T_1$ and four rounds $CT_{14}$. 2) For $|\phi^+\rangle_{12}|\phi^-\rangle_{34}$, four rounds $T_1$ and two rounds $CT_{34}$ are carried out by Bob. 3) For $|\phi^+\rangle_{12}|\phi^-\rangle_{34}$, Bob performs two rounds $T_1$ and four rounds $CT_{14}$. 4) For $|\phi^+\rangle_{12}|\phi^+\rangle_{34}$, Bob performs four rounds $T_1$, two rounds $T_3$, four rounds $T_4$ and four rounds $CT_{14}$. Particularly, $X_4$ needs to be performed before implementing $H_4$ for $|\phi^+\rangle_{12}|\phi^+\rangle_{34}$. Bob returns those operated qubits to Alice and Alice gets $\text{QFT}_{13}|\phi^+\rangle_{12}|\phi^+\rangle_{34}$, $\text{QFT}_{13}|\psi^-\rangle_{12}|\psi^+\rangle_{34}$, $\text{QFT}_{13}|\phi^+\rangle_{12}|\phi^-\rangle_{34}$, $\text{QFT}_{13}|\phi^+\rangle_{12}|\phi^+\rangle_{34}$ and $\text{QFT}_{13}|\phi^+\rangle_{12}|\psi^-\rangle_{34}$.

**BQC protocol 4.** For $|\phi^-\rangle_{12}|\phi^-\rangle_{34}$, $|\phi^-\rangle_{12}|\psi^-\rangle_{34}$, $|\phi^-\rangle_{12}|\phi^-\rangle_{34}$ and $|\phi^-\rangle_{12}|\psi^+\rangle_{34}$, the same operations are that Bob implements $X_2$, three rounds CNOT (i.e., CNOT, CNOT, and CNOT), and two rounds $H$ ($H_2$
and $H_4$). After that, we discuss the different operations in different circuits. 1) Bob executes four rounds $T_1$ and two rounds $CT_{34}$ for $|\phi^-(12)\phi^+34\rangle$. 2) Bob performs six rounds $T_1$ and six rounds $CT_{14}$ for $|\phi^-(12)\psi^-34\rangle$. 3) Bob performs two rounds $T_1$, six rounds $CT_{14}$ and four rounds $T_1$ for $|\phi^-12\psi^-34\rangle$. 4) Bob executes four rounds $T_3$ and two rounds $CT_{32}$. Bob returns them to Alice and $\phi^+$. For quantum inputs, algorithms and outputs. For blindness, we regard blindness as the privacy of clients’ quantum inputs, algorithms and outputs. For blindness, we prove that they are correct and blindness in this section. Here we regard blindness as the privacy of clients’ quantum inputs, algorithms and outputs. For blindness, we give the unified proof because all BQC protocols are performed crossways. While for correctness, we give the proof respectively according to their own properties.

### IV. ANALYSIS OF BLINDNESS AND CORRECTNESS

Since all BQC protocols have been described in detail, we prove that they are correct and blindness in this section. Here we regard blindness as the privacy of clients’ quantum inputs, algorithms and outputs. For blindness, we give the unified proof because all BQC protocols are performed crossways. While for correctness, we give the proof respectively according to their own properties.

### V. CORRECTNESS

For blindness, this initial Bell states $\{|\phi^+,\psi^+\rangle\}$ are prepared by the trusted center only known buy Alice. Alice also adds some trap operations and disturbs the order. When Alice sends one qubit or two qubits to Bob, he gets nothing about Alice’s inputs because he dare not measure in ease. If Bob receives one qubit, it belongs to one standard Bell state or one evolutive Bell state, where standard Bell states are the...
original Bell states (i.e. \{|\phi^{\pm}\rangle, |\psi^{\pm}\rangle\}), the evolutive Bell states represent some operations performed on original Bell states. If Bob receives two qubits, they respectively belong to one Bell state or two Bell states which is standard or evolutive. If Bob measures the qubits he received, it will result in destroying of entanglement or collapse of quantum states. Therefore, it is impossible for Bob to get anything, keeping the Alice’s inputs private in these BQC protocols.

Now, we consider respectively quantum algorithm and outputs, where quantum algorithm is the quantum Fourier transform in our protocols. For quantum Fourier transform, it is decomposed into several independent unitary operations. In every round, Bob only performs partial unitary operators because all BQC protocols are running crossways. Therefore, Alice can successfully hide quantum algorithms and thus guarantees the blindness of QFT. Then we give respectively the definitions of blindness for the algorithm and outputs as follows.

**Definition** A single-server BQC protocol is blind if given all the classical information Bob can obtain during the protocol.

1. The blindness of algorithm: the conditional probability distribution of Bob’s operators is equal to the priori probability distribution of Bob’s operators;
2. The blindness of outputs: the conditional probability distribution of the output quantum states that Alice wants to get is equal to the priori probability distribution of the output quantum states.

**Proof:** In our proposed primary, enhanced and generalized BQC protocols, Bob’s knowledge from Alice’s classical information contains Bell states and operations. Such as the conditional probability distribution of SWAP \(j\) is given by CNOT \(j\) and \(\Gamma_j\), where \(\Gamma_j\) represents rotations. For the reason that SWAP \(j\), CNOT \(j\) and \(\Gamma_j\) are completely independent, SWAP \(j\) is unknown to Bob in these BQC protocols. Based on Bayes’ theorem, we have

\[
p(SWAP_j \mid \text{CNOT}_j, \Gamma_j)
\]
So these protocols satisfy the definition (1). Therefore, the algorithm and outputs on Bell states. We divide into three cases to discuss: I, II, and III. We prove that these BQC protocols are gates $T$, $CNOT$, $H$, we have the same conclusions by similar proof processes. Therefore, the algorithm and outputs in our proof processes. Hence, we can show that $QFT_j$ is unknown to Bob as follows.

$$p(QFT_j | X_j, \Gamma_j) = \frac{p(X_j | QFT_j, \Gamma_j)p(QFT_j, X_j)}{p(X_j | \Gamma_j)p(QFT_j)p(\Gamma_j)} = p(QFT_j).$$

So these protocols also satisfy the definition (2). For gates $T$, $CT$, $H$, we have the same conclusions by similar proof processes. Therefore, the algorithm and outputs in all BQC protocols are blind. So far, we have proved blindness for inputs, algorithm and outputs in our proposed BQC protocols.

In the end, we analyze the correctness in Appendix I, II, and III. We prove that these BQC protocols are correct.

V. CONCLUSIONS

In this paper, we propose a new blind quantum computation protocol of quantum Fourier transform performed on Bell states. We divide into three cases to discuss: primary BQC protocols for QFT on one Bell state, enhanced BQC protocols for QFT on two Bell states and generalized BQC protocols for QFT on two Bell states. At last, we give the proof of blindness and correctness for all BQC protocols. In a word, our works give a better understanding for quantum Fourier transform. For further work, we will try to construct BQC protocols on multi-qubit for QFT.

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Appendix I: Proof of the Correctness of primary BQC protocols

First, we consider that QFT is performed on qubits 12 of Bell states.

1) For $|\phi^+\rangle_{12}$, this equation

$$|\phi^+\rangle_{12} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}(|+\rangle|+\rangle + \frac{1}{\sqrt{2}}|0\rangle - |i\rangle)_{12}$$

is equivalent to

$$|\phi^+\rangle_{12} \xrightarrow{\text{H}_2} \frac{1}{2}[|0\rangle(|0\rangle + |1\rangle) + |1\rangle(|0\rangle - |1\rangle)]_{12}$$

$$\xrightarrow{\text{CT}_{12}} \frac{1}{2}[|0\rangle(|0\rangle + |1\rangle) + |1\rangle(|0\rangle - |i\rangle)]_{12}$$

$$\xrightarrow{\text{H}_1} \frac{1}{\sqrt{2}}(|+\rangle|+\rangle) + \frac{1}{\sqrt{2}}(|0\rangle - |i\rangle)]_{12}.$$  

2) For $|\phi^-\rangle_{12}$, this equation

$$|\phi^-\rangle_{12} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}(|+\rangle|+\rangle) - \frac{1}{\sqrt{2}}|0\rangle - |i\rangle)]_{12}.$$  

is equivalent to

$$|\phi^-\rangle_{12} \xrightarrow{\text{H}_2} \frac{1}{2}[|0\rangle(|0\rangle + |1\rangle) - |1\rangle(|0\rangle - |1\rangle)]_{12}$$

$$\xrightarrow{\text{CT}_{12}} \frac{1}{2}[|0\rangle(|0\rangle + |1\rangle) - |1\rangle(|0\rangle - |i\rangle)]_{12}$$

$$\xrightarrow{\text{H}_1} \frac{1}{\sqrt{2}}(|+\rangle|+\rangle) - \frac{1}{\sqrt{2}}(|0\rangle - |i\rangle)]_{12}.$$  

3) For $|\psi^+\rangle_{12}$, this equation

$$|\psi^+\rangle_{12} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}|\rangle(|0\rangle + |1\rangle) + |+\rangle]|-\rangle]_{12}.$$  

is equivalent to

$$|\psi^+\rangle_{12} \xrightarrow{\text{H}_2} \frac{1}{2\sqrt{2}}[|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle + |1\rangle)]_{12}$$

$$\xrightarrow{\text{CT}_{12}} \frac{1}{2\sqrt{2}}[|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle + |i\rangle)]_{12}$$

$$\xrightarrow{\text{H}_1} \frac{1}{\sqrt{2}}[|+\rangle|-\rangle + |\rangle(|0\rangle + |i\rangle)]_{12}.$$  

4) For $|\psi^-\rangle_{12}$, this equation

$$|\psi^-\rangle_{12} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}|\rangle(|0\rangle + |i\rangle) - |\rangle)]_{12}.$$  

is equivalent to

$$|\psi^-\rangle_{12} \xrightarrow{\text{H}_2} \frac{1}{2\sqrt{2}}[|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle + |i\rangle)]_{12}$$

$$\xrightarrow{\text{CT}_{12}} \frac{1}{2\sqrt{2}}[|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle + i\rangle)]_{12}$$

$$\xrightarrow{\text{H}_1} \frac{1}{\sqrt{2}}[|+\rangle|-\rangle + |\rangle(|0\rangle + |i\rangle)]_{12}.$$  


is equivalent to

\[ |\psi^+\rangle_{12} \xrightarrow{H_2} \frac{1}{2\sqrt{2}} [0)(0| - |1)(0| + i1)|12] \]
\[ \xrightarrow{C^{T_2}_{12}} \frac{1}{2\sqrt{2}} [0)(0| - |1)(0| + i1)|12] \]
\[ \xrightarrow{H_1} \frac{1}{2\sqrt{2}} [0)(0| - |1)(0| + i1)|12] \]

Appendix II: Proof of the Correctness of enhanced BQC protocols

In the following, we consider that QFT is performed on 13 of \(|\xi\rangle_{12}[\phi]\rangle_{34}.

1) For \(|\phi^+\rangle_{12}[\phi^+\rangle_{34}, this equation is equivalent to

\[ |\phi^+\rangle_{12}[\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}[0|0+ + \frac{1}{\sqrt{2}}[0-1](0| + i1)|12] \]
\[ + |1 + 0- + \frac{1}{\sqrt{2}}[1-1](0| - i1)|12] \]

2) For \(|\phi^+\rangle_{12}[\phi^-\rangle_{34}, this equation is equivalent to

\[ |\phi^+\rangle_{12}[\phi^-\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}[0|0+ + \frac{1}{\sqrt{2}}[0-1](0| - i1)|12] \]
\[ + |1 + 0- + \frac{1}{\sqrt{2}}[1-1](0| + i1)|12] \]

3) For \(|\phi^+\rangle_{12}[\phi^+\rangle_{34}, this equation is equivalent to

\[ |\phi^+\rangle_{12}[\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}[0|0+ + [0-1](0| - i1)|12] \]
\[ + |1 + 0- + \frac{1}{\sqrt{2}}[1-1](0| + i1)|12] \]

4) For \(|\phi^+\rangle_{12}[\phi^-\rangle_{34}, this equation is equivalent to

\[ |\phi^+\rangle_{12}[\phi^-\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}[0|0+ + \frac{1}{\sqrt{2}}[0-1](0| + i1)|12] \]
\[ - |1 + 0- + \frac{1}{\sqrt{2}}[1-1](0| - i1)|12] \]

5) For \(|\phi^-\rangle_{12}[\phi^+\rangle_{34}, this equation is equivalent to

\[ |\phi^-\rangle_{12}[\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}[0|0+ + \frac{1}{\sqrt{2}}[0-1](0| + i1)|12] \]
\[ + |1 + 0- + \frac{1}{\sqrt{2}}[1-1](0| + i1)|12] \]

6) For \(|\phi^-\rangle_{12}[\phi^-\rangle_{34}, this equation is equivalent to

\[ |\phi^-\rangle_{12}[\phi^-\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}[0|0+ + \frac{1}{\sqrt{2}}[0-1](0| - i1)|12] \]
\[ + |1 + 0- + \frac{1}{\sqrt{2}}[1-1](0| + i1)|12] \]
is equivalent to
\[ |\phi^-\rangle_{12} |\phi^-\rangle_{34} \rightarrow \frac{1}{2} |(0101) - [0110] - [1001] + [1010]\rangle_{1234} \]
\[ \text{SWAP}_{13} \rightarrow \frac{1}{2} |(0101) - [1001] - [0110] + [1010]\rangle_{1234} \]
\[ H_2 H_4 \rightarrow \frac{1}{2} |(00 - 0) - (1 - 0) + (0 + 1 - 1 + 1 + 1)\rangle_{1234} \]
\[ \text{T}_{14}\text{T}_{13} \rightarrow \frac{1}{2} |(00 - 0) + (1 - 0) + (0 + 1 - 1 + 1 + 1)\rangle_{1234} \]
\[ \text{CT}_{12} \rightarrow \frac{1}{2} |(00 - 0) + \frac{1}{\sqrt{2}} |0 + 1\rangle |(0) - i(1)\rangle \]
\[ + |1 - 0 - 0 + \frac{1}{\sqrt{2}} |1 + 1\rangle |(0) + i(1)\rangle\rangle_{1234}. \]

7) For \(|\phi^-\rangle_{12} |\psi^+\rangle_{34}\), this equation
\[ |\phi^-\rangle_{12} |\psi^+\rangle_{34} \rightarrow \frac{1}{2} |(0000) - |0011\rangle + |1100\rangle + |1111\rangle\rangle_{1234} \]
\[ \text{SWAP}_{24} \rightarrow \frac{1}{2} |(0000) - [0011] + [1100] + [1111]\rangle_{1234} \]
\[ H_2 H_4 \rightarrow \frac{1}{2} |(00 + 0) + [1 + 0] + [0 - 1 + 1 - 1]\rangle_{1234} \]
\[ \text{CT}_{12}\text{T}_{13} \rightarrow \frac{1}{2} |(00 + 0) - \frac{1}{\sqrt{2}} |0 - 1\rangle |(0) + i(1)\rangle \]
\[ + |1 + 0 + 0 + \frac{1}{\sqrt{2}} |1 - 1\rangle |(0) + i(1)\rangle\rangle_{1234}. \]

8) For \(|\phi^-\rangle_{12} |\psi^-\rangle_{34}\), this equation
\[ |\phi^-\rangle_{12} |\psi^-\rangle_{34} \rightarrow \frac{1}{2} |(0000) - [0011] + [1100] + [1111]\rangle_{1234} \]
\[ \text{SWAP}_{24} \rightarrow \frac{1}{2} |(0000) - [0011] + [1100] + [1111]\rangle_{1234} \]
\[ H_2 H_4 \rightarrow \frac{1}{2} |(00 + 0) + [1 + 0] + [0 - 1 + 1 - 1]\rangle_{1234} \]
\[ \text{CT}_{12}\text{T}_{13} \rightarrow \frac{1}{2} |(00 + 0) - \frac{1}{\sqrt{2}} |0 - 1\rangle |(0) + i(1)\rangle \]
\[ + |1 + 0 + 0 + \frac{1}{\sqrt{2}} |1 - 1\rangle |(0) + i(1)\rangle\rangle_{1234}. \]

11) For \(|\psi^+\rangle_{12} |\psi^+\rangle_{34}\), this equation
\[ |\psi^+\rangle_{12} |\psi^-\rangle_{34} \rightarrow \frac{1}{2} |(00 + 0) - \frac{i}{\sqrt{2}} |0 - 1\rangle |(0) - i(1)\rangle \]
\[ - |1 + 0 - 0 + \frac{i}{\sqrt{2}} |1 + 1\rangle |(0) + i(1)\rangle\rangle_{1234}. \]

12) For \(|\psi^+\rangle_{12} |\psi^-\rangle_{34}\), this equation
\[ |\psi^+\rangle_{12} |\psi^-\rangle_{34} \rightarrow \frac{1}{2} |(00 + 0) - \frac{i}{\sqrt{2}} |0 - 1\rangle |(0) + i(1)\rangle \]
\[ + |1 + 0 + 0 + \frac{i}{\sqrt{2}} |1 - 1\rangle |(0) - i(1)\rangle\rangle_{1234}. \]
is equivalent to
\[ |\psi^+\rangle_{12}|\psi^+\rangle_{34} \xrightarrow{\text{SWAP}_{13}} \frac{1}{2}((0001) - (0010) + (1001) - (1100))_{3214} \]
\[ \xrightarrow{H_x H_1} \frac{1}{2}((0 + 0) - (1 + 0 + 0) + (0 - 1 + 0) + (1 - 1 + 1))_{3214} \]
\[ \xrightarrow{T_z^2} \frac{1}{2}((0 + 0) - (1 + 0 + 0) + i(0 - 1 + 0) + i(1 - 1 + 1))_{3214} \]
\[ \xrightarrow{\text{CNOT}_{12}} \frac{1}{2}[i(0 + 0) - i^2(0 + 1)(0) + i(1)) - (1 + 0 + 0) + i^2(1 - 1)(0) - i(1))_{3214} \]

13) For \(|\psi^-\rangle_{12}|\phi^+\rangle_{34}\), this equation
\[ |\psi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{2}(((0 + 0) - \frac{1}{\sqrt{2}}(0 + 1)(0) + i(1)) + (1 - 0 - 0) - \frac{1}{\sqrt{2}}(1 + 1)(0) - i(1))_{1234} \]

is equivalent to
\[ |\psi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{SWAP}_{13}} \frac{1}{2}((0011) - (0010) + (1101) - (1100))_{3214} \]
\[ \xrightarrow{H_x H_1} \frac{1}{2}((0011) - (0010) + (1101) - (1100))_{3214} \]
\[ \xrightarrow{\text{SWAP}_{13}} \frac{1}{2}((0101) - (1001) + (0111) - (1100))_{1432} \]
\[ \xrightarrow{H_x H_1} \frac{1}{2}((0001) - (0010) + (1101) - (1100))_{1432} \]
\[ \xrightarrow{\text{CNOT}_{12}} \frac{1}{2}((0001) - (0010) + (1111) - (1110))_{1432} \]
\[ \xrightarrow{\text{CNOT}_{12}} \frac{1}{2}((0100) - (0110) + (1010) - (1101))_{1432} \]

14) For \(|\psi^-\rangle_{12}|\phi^+\rangle_{34}\), this equation
\[ |\psi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{2}(((0 + 0) - \frac{1}{\sqrt{2}}(0 + 1)(0) - i(1)) + (1 - 0 - 0) - \frac{1}{\sqrt{2}}(1 + 1)(0) + i(1))_{1234} \]

is equivalent to
\[ |\psi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{SWAP}_{13}} \frac{1}{2}((0011) - (0010) + (1101) - (1100))_{1432} \]
\[ \xrightarrow{T_z^2} \frac{1}{2}((0011) - (0111) + (1101) - (1100))_{1432} \]
\[ \xrightarrow{H_x H_1} \frac{1}{2}((0000) - (0100) + (1010) - (1110))_{1432} \]
\[ \xrightarrow{\text{CNOT}_{12}} \frac{1}{2}((0101) - (0111) + (1011) - (1111))_{1432} \]
\[ \xrightarrow{\text{CNOT}_{12}} \frac{1}{2}((0100) - (0110) + (1010) - (1110))_{1432} \]

Appendix III: Proof of the Correctness of generalized BQC protocols

At last, we consider that QFT are performed on 13 and 24 of \(|\xi\rangle_{12}|\theta\rangle_{34}\).

1) For \(|\phi^+\rangle_{12}|\phi^+\rangle_{34}\), this equation
\[ |\phi^+\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}(|\phi^+\rangle|00\rangle + |\psi^+\rangle|11\rangle)_{1234} \]

is equivalent to
\[ |\phi^+\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{CNOT}_{12}} \frac{1}{2}((\psi^+)|00\rangle + |\psi^+\rangle|11\rangle)_{1234} \]

2) For \(|\phi^+\rangle_{12}|\phi^-\rangle_{34}\), this equation
\[ |\phi^+\rangle_{12}|\phi^-\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{2}(|\phi^+\rangle|00\rangle + |\phi^+\rangle|11\rangle)_{1234} \]

is equivalent to
\[ |\phi^+\rangle_{12}|\phi^-\rangle_{34} \xrightarrow{\text{CNOT}_{12}} \frac{1}{2}(|\phi^+\rangle|11\rangle + |\phi^+\rangle|00\rangle)_{1234} \]
\[ \xrightarrow{H_x H_1} \frac{1}{2}((0000) - (0010) + (1101) - (1100))_{1432} \]
\[ \xrightarrow{\text{SWAP}_{13}} \frac{1}{2}((0011) - (0010) + (1101) - (1100))_{1432} \]
\[ \xrightarrow{\text{CNOT}_{12}} \frac{1}{2}((0101) - (0111) + (1011) - (1110))_{1432} \]
\[ \xrightarrow{\text{CNOT}_{12}} \frac{1}{2}((0100) - (0110) + (1010) - (1111))_{1432} \]
\[ \xrightarrow{\text{CNOT}_{12}} \frac{1}{2}((0100) - (0110) + (1010) - (1110))_{1432} \]
3) For $|\phi^+\rangle_{12}|\psi^+\rangle_{34}$, this equation

$$|\phi^+\rangle_{12}|\psi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}|\phi^+\rangle_{12}(|00\rangle + i|11\rangle)_{34}.$$ is equivalent to

$$|\phi^+\rangle_{12}|\psi^+\rangle_{34} \xrightarrow{T_4^1\sqrt{T_3^1X_4^1}} \frac{1}{\sqrt{2}}|\phi^-\rangle_{12}(|01\rangle + |10\rangle)_{34}.$$ 

4) For $|\phi^+\rangle_{12}|\psi^+\rangle_{34}$, this equation

$$|\phi^+\rangle_{12}|\psi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2}}|\phi^-\rangle_{12}(|00\rangle + i|11\rangle)_{34}.$$ is equivalent to

$$|\phi^+\rangle_{12}|\psi^+\rangle_{34} \xrightarrow{T_3^1X_4^1} \frac{1}{\sqrt{2}}|\phi^-\rangle_{12}(|00\rangle + i|11\rangle)_{34}.$$ 

5) For $|\phi^-\rangle_{12}|\phi^+\rangle_{34}$, this equation

$$|\phi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{2}(i|\phi^+\rangle + |\phi^+\rangle)_{12}|\psi^+\rangle_{34}.$$ is equivalent to

$$|\phi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{X_4^1T_4^1} \frac{1}{2}(|\phi^+\rangle + |\phi^+\rangle)_{12}|\psi^+\rangle_{34}.$$ 

6) For $|\phi^-\rangle_{12}|\phi^+\rangle_{34}$, this equation

$$|\phi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{2}(i|\phi^+\rangle + |\phi^+\rangle)_{12}|\psi^+\rangle_{34}.$$ is equivalent to

$$|\phi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{X_4^1T_4^1} \frac{1}{2}(|\phi^+\rangle + |\phi^+\rangle)_{12}|\psi^+\rangle_{34}.$$ 

7) For $|\phi^-\rangle_{12}|\psi^+\rangle_{34}$, this equation

$$|\phi^-\rangle_{12}|\psi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{2}((|\psi^-\rangle|\psi^-\rangle + i|\phi^-\rangle|\psi^+\rangle)_{1234}.$$ is equivalent to

$$|\phi^-\rangle_{12}|\psi^+\rangle_{34} \xrightarrow{H_2\sqrt{X_4^1T_3^1}} \frac{1}{2}((|\phi^+\rangle + |\psi^+\rangle)_{12}|\psi^+\rangle_{34}.$$ 

8) For $|\phi^-\rangle_{12}|\psi^+\rangle_{34}$, this equation

$$|\phi^-\rangle_{12}|\psi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{2}((-|\phi^-\rangle|\psi^-\rangle + i|\psi^-\rangle|\psi^+\rangle)_{1234}.$$ is equivalent to

$$|\phi^-\rangle_{12}|\psi^+\rangle_{34} \xrightarrow{T_3^1T_4^1} \frac{1}{2}((|\phi^+\rangle|\psi^-\rangle + i|\psi^-\rangle|\psi^+\rangle)_{1234}.$$ 

9) For $|\psi^+\rangle_{12}|\phi^+\rangle_{34}$, this equation

$$|\psi^+\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{QFT} \frac{1}{2}(|\phi^+\rangle|00\rangle - |\psi^+\rangle|11\rangle)_{1234}.$$ is equivalent to

$$|\psi^+\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{T_4^1X_4^1T_3^1X_4^1} \frac{1}{2}(|\phi^+\rangle|00\rangle - |\psi^+\rangle|11\rangle)_{1234}.$$ 

10) For $|\psi^+\rangle_{12}|\phi^+\rangle_{34}$, this equation

$$|\psi^+\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{2}(|\phi^+\rangle|00\rangle - |\psi^+\rangle|11\rangle)_{1234}.$$ is equivalent to

$$|\psi^+\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{CNOT}_{32}} \frac{1}{2}(|\phi^+\rangle|00\rangle - |\psi^+\rangle|11\rangle)_{1234}.$$ 

11) For $|\psi^+\rangle_{12}|\phi^+\rangle_{34}$, this equation

$$|\psi^+\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{QFT} \frac{1}{2}((-|\psi^-\rangle|\psi^-\rangle + i|\phi^-\rangle|\psi^+\rangle)_{1234}.$$ is equivalent to

$$|\psi^+\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{T_4^1X_4^1} \frac{1}{2}((-|\psi^-\rangle|\psi^-\rangle + i|\phi^-\rangle|\psi^+\rangle)_{1234}.$$ 

12) For $|\psi^+\rangle_{12}|\psi^+\rangle_{34}$, this equation

$$|\psi^+\rangle_{12}|\psi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{2}((|\psi^+\rangle|\psi^+\rangle + i|\phi^+\rangle|\psi^+\rangle)_{1234}.$$ is equivalent to

$$|\psi^+\rangle_{12}|\psi^+\rangle_{34} \xrightarrow{T_4^1} \frac{1}{2}((|\psi^+\rangle|\psi^+\rangle + i|\phi^+\rangle|\psi^+\rangle)_{1234}.$$ 

13) For $|\psi^-\rangle_{12}|\phi^+\rangle_{34}$, this equation

$$|\psi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1}{2}((-|\phi^+\rangle + |\psi^+\rangle|\psi^+\rangle)_{1234}.$$
13

is equivalent to

\[ |\psi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1-i}{\sqrt{2}} |(i|\phi^+\rangle + |\psi^+\rangle)|\psi^+\rangle\]_{1234}.  

14) For \(|\psi^-\rangle_{12}|\phi^+\rangle_{34}\), this equation

\[ |\psi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1-i}{\sqrt{2}} [(i|\phi^+\rangle + |\psi^+\rangle)|\psi^+\rangle]\]_{1234}.

is equivalent to

\[ |\psi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{T_{34}^1 H_1} \frac{1}{\sqrt{2}} [(i|\phi^+\rangle + |\psi^+\rangle)|\psi^+\rangle]\]_{1234}.

15) For \(|\psi^-\rangle_{12}|\phi^+\rangle_{34}\), this equation

\[ |\psi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{QFT}} \frac{1-i}{\sqrt{2}} [(i|\phi^+\rangle + |\psi^+\rangle)|\psi^+\rangle]\]_{1234}.

is equivalent to

\[ |\psi^-\rangle_{12}|\phi^+\rangle_{34} \xrightarrow{\text{QFT} T_{34}^1 T_{23}^2 H_1} \frac{i}{\sqrt{2}} [(i|\phi^+\rangle + |\psi^+\rangle)|\psi^+\rangle]\]_{1234}.

16) For \(|\psi^-\rangle_{12}|\psi^-\rangle_{34}\), this equation

\[ |\psi^-\rangle_{12}|\psi^-\rangle_{34} \xrightarrow{\text{QFT}} \frac{1-i}{\sqrt{2}} [(i|\phi^+\rangle + |\psi^+\rangle)|\psi^+\rangle]\]_{1234}.

is equivalent to

\[ |\psi^-\rangle_{12}|\psi^-\rangle_{34} \xrightarrow{T_{34}^2 H_1} \frac{1}{\sqrt{2}} [(i|\phi^+\rangle + |\psi^+\rangle)|\psi^+\rangle]\]_{1234}.

\[ \xrightarrow{T_{34}^2 H_1} \frac{1}{\sqrt{2}} [(i|\phi^+\rangle + |\psi^+\rangle)|\psi^+\rangle]\]_{1234}.