Propagation Stability Concepts for Network Synchronization Processes

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Abstract—A notion of disturbance propagation stability is defined for dynamical network processes, in terms of decrescence of an input-output energy metric along cutsets away from the disturbance source. A characterization of the disturbance propagation notion is developed for a canonical model for synchronization of linearly-coupled homogeneous subsystems. Specifically, propagation stability is equivalent with the frequency response of a certain local closed-loop model, which is defined from the subsystem model and local network connections, being sub-unity gain. For the case where the subsystem is single-input single-output (SISO), a further simplification in terms of the subsystem’s open loop Nyquist plot is obtained. An extension of the disturbance propagation stability concept toward imperviousness of subnetworks to disturbances is briefly developed, and an example focused on networks with planar subsystems is considered.

I. INTRODUCTION

There has been a substantial cross-disciplinary research effort to define and characterize synchronization phenomena in networks of coupled systems [1]–[3]. These studies define synchronization in terms of the internal asymptotic stability of an equilibrium manifold, on which each coupled system has an identical state or output. However, synchronization in a practical sense also requires that this equilibrium is impervious to external disturbances. Based on this recognition, a body of recent work has considered the disturbance responses of network synchronization processes [4]–[6]. Broadly, these efforts model disturbances as impinging on all or a subset of subsystems within a network, and evaluate their potential impacts on network-wide synchrony according to a performance metric (typically, a $H_2$ or $H_\infty$ gain).

Engineers working with large-scale built networks often do not think about coordination in terms of either internal stability characteristics or global (network-wide) disturbance response metrics. Rather, they are concerned with the extent of propagation of local disturbances, whether arising from exogenous inputs or state deviations. For instance, bulk power grid operators often distinguish well-damped networks where oscillatory disturbance responses remain localized from poorly-damped ones where such disturbances have network-wide impact [7]. Similar assessments of network performance in terms of disturbance propagation are of interest in disciplines ranging from air traffic management to infectious-disease epidemiology and cyber-security [8].

The spatial propagation of disturbances among interconnected systems in cascade or line-topology configurations has been extensively researched under the heading of string stability [9], [10]. The string stability concept has also been extended to directionally-connected mesh networks [11]. The recent study [12] has recognized the need for disturbance-propagation stability notions for general (bi-directionally connected) dynamical networks, and therefore has proposed a definition for network stability in terms of boundedness of input-to-state or input-to-output gains which parallels the basic string stability definition. It has also introduced an alternate network stability definition for tree-like graphs which captures monotonic decrescence of the propagative response away from the disturbance source; this definition is a generalization of the strong string stability concept [10]. Other recent studies have also examined disturbance propagation (e.g., [13]) and other local-input-to-output properties of dynamical networks (e.g., gains, zeros) [14], [15]. However, definitions for propagation stability are not yet mature, and the formal analysis of propagation is incomplete even for networks of coupled linear systems.

The purpose of this article is to define and characterize a notion of disturbance propagation stability in the context of a canonical network model for coupled linear systems. The main contributions are two-fold:

1) A general definition is introduced for (strict) propagation stability, based on decrescence of response norms across cutsets in the network’s digraph, or equivalently along paths away from the disturbance source.

2) Propagation stability is characterized in terms of the closed-loop frequency responses of the subsystem with a proportional feedback applied. Further simplifications in terms of the subsystem’s open-loop frequency response are obtained, when the subsystems are single-input single-output.

The rest of the article is organized as follows. The network model is described in Section II, and propagation stability notions are defined in Section III. Main results are given in Section IV. Finally, in Section V, an example is given which focuses on the impact of damping on propagation stability when the subsystems are planar devices. Proofs are omitted to save space, see the extended version [18].

II. MODEL

A network with $N$ identical interconnected devices or nodes or subsystems, labeled $1, \ldots, N$, is considered. Each subsystem $i \in 1, \ldots, N$ has a state $x_i \in \mathbb{R}^n$ and output $y_i \in \mathbb{R}^m$ which are governed by the following linear or linearized state-space equations:

$$\dot{x}_i = Ax_i + B \left( \alpha \sum_{j \neq i} g_{ij} (y_j - y_i) + \gamma_i w_i \right)$$

$$y_i = Cx_i,$$

where $x_i$ is the state of the $i$th subsystem, $y_i$ is its output, and $w_i$ is an exogenous disturbance. The matrix $A$ is the state transition matrix, $B$ is the input matrix, $C$ is the output matrix, $g_{ij}$ are the coupling coefficients, and $\gamma_i$ are the sub-unity gains.

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Here, $A$, $B$, and $C$ are a subsystem’s state, input, and output matrices, respectively; the scalars $g_{ij} \geq 0$ are coupling weights; the vector $w_i \in \mathbb{R}^m$ represents an external disturbance input at subsystem $i$; and $\alpha$ is a global coupling-strength parameter which allows tuning of the network connectivity (see [1]). Our focus here is on an external disturbance impinging on a single source node $s \in 1, \ldots, N$, which is modeled by setting $\gamma_s = 1$ and $\gamma_i = 0$ for $i \neq s$.

The model (1) is a standard representation for the small-signal dynamics of synchronizing coupled oscillators [1], [2], with two distinctions. First, a disturbance is applied at a single subsystem, to allow for analysis of propagative impacts. Second, the subsystems are modeled as being interconnected through commensurately-dimensioned inputs and outputs, rather than through an explicit inner-coupling term or alternately through a designable protocol. This format is used to stress that the network is made up of input-output devices with fixed connections, but the formulation encompasses the scenarios with an inner coupling or a designable protocol.

Analyses of (1) often are phrased in terms of a graph that represents the network interconnections. For our development, a weighted digraph $\Gamma$ is defined with $N$ vertices corresponding to the $N$ subsystems. A directed edge is drawn from vertex $j$ to vertex $i$ if $g_{ij} > 0$, reflecting a direct influence of the output of subsystem $j$ on the state evolution of subsystem $i$. The edge is assigned a weight of $g_{ij}$. We use the notation $\mathcal{V}$ for the set of vertices, and $\mathcal{E}$ for the set of edges. Additionally, it is convenient to define an (asymmetric) Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$. Each off-diagonal entry $l_{ij}$ is given by $-g_{ij}$, while the diagonal entries are selected so that each row sums to 0 (i.e., $l_{ii} = \sum_{j \neq i} g_{ij}$).

For the model (1), the synchronization manifold where the states $x_1, \ldots, x_n$ are identical is known to be asymptotically stable under broad conditions on the network graph, the subsystem model, and the coupling-strength parameter. More precisely, stability can be related to Hurwitz stability of the $N$ complex matrices $A - \lambda_i \alpha BC$, where $0 = \lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of the Laplacian matrix $L$. From this analysis, stability can be distilled to a simple test on the Laplacian matrix’s spectrum via the master stability function construct, see e.g. [1] for details. A number of other characteristics of (1), including global disturbance stability and controllability via external stimulation, can also be related to the matrices $A - \lambda_i \alpha BC$ [4], [16].

From here on we refer to the model (1) as the network synchronization model. The model is approximate of a number of synchronization phenomena, including the swing dynamics of the bulk power grid, multi-vehicle formation flight, and the nonlinear dynamics of electrical oscillator networks.

III. PROPAGATION STABILITY DEFINITION

A notion of propagation stability is defined based on the spatial patterns of output energies (squared two norms) at network subsystems over a time interval $[0, T]$, when an exogenous disturbance input $w_s(t)$ is applied at a single node. In our development, the disturbance is assumed to satisfy the Dirichlet conditions (absolute integrability over any period, finite number of discontinuities and minima/maxima, bounded over any interval), but otherwise may be arbitrary. In defining stability, the squared two-norm metric $E_i(T) = \int_{t=0}^{T} y_i^2(t) dt$ is considered for each network subsystem $i$. Conceptually, the network can be viewed as propagation stable, if these energies are attenuated away from the disturbance source with respect to the network graph. However, since the network’s graph in general has a spatially inhomogeneous structure, defining attenuation requires some care.

One natural way to assess disturbance propagation is to consider the energy metric $E_b(T)$ for vertex-cutsets in the network’s graph (i.e., sets of vertices whose removal partition the graph). If the metric value for at least one cutset vertex is larger than the metric values for vertices that are separated from the source by the cutset, then the response can be viewed as being attenuated away from the source (see Figure 1). To formalize this notion, let us consider a set of vertices $\mathcal{V}_C \subseteq \mathcal{V}$. We refer to $\mathcal{V}_C$ as a separating cutset for the source $s$, if the remaining vertices $\mathcal{V} \setminus \mathcal{V}_C$ can be partitioned into two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that: 1) there are no edges from vertices in $\mathcal{V}_1$ to vertices in $\mathcal{V}_2$ and 2) $\mathcal{V}_A = \mathcal{V}_1 \cup \mathcal{V}_C$ contains the source vertex $s$. Using this notation, the following definition for propagation stability is proposed:

**Definition 1:** The network synchronization model is **propagation stable** if the following two conditions hold.

1. The synchronization manifold is asymptotically stable in the sense of Lyapunov.
2. For every source location $s$, disturbance signal $w_s(t)$, separating cutset $\mathcal{V}_C$ for $s$, time horizon $T > 0$, and vertex $b \in \mathcal{V}_B$, the following majorization holds: $E_b(T) \leq \max_{c \in \mathcal{V}_C} E_c(T)$.

The definition captures that the output signal energy for a subsystem associated with a graph cutset upper bounds the signal energy for all subsystems beyond the cutset, and hence the output signal energy is attenuated at cutsets away from the source. An immediate consequence of the definition is that the maximum response energy among subsystems at a distance $r$ from the source location on the network graph, say $E(r)$, is a non-increasing function of $r$. 

![Fig. 1. Illustration of the propagation stability concept.](image-url)
With some further thought, one also sees that propagation stability can be equivalently expressed in terms of paths in the network graph. In particular, the network model is propagation stable if and only if there is at least one path in the network graph from the disturbance source to each other vertex such that the response energy is non-increasing along the path. From this equivalent form, it is evident that propagation stability is a generalization of the classical strict string-stability definition and a recent definition for (strict) network stability in tree-like networks [10], [12], in that decrescence or attenuation along paths away from the disturbance source is enforced. We stress that our definition only requires attenuation along one path for each pair of vertices, so as to encompass the varying attenuation patterns that may arise in inhomogeneously-structured networks.

In some circumstances, the propagation stability definition may be too rigid to capture attenuative dynamics in a network. In particular, it is possible that small regions in a network may be susceptible to disturbance amplification, but a substantial portion of the network nevertheless attenuates disturbances. This partial notion of propagation attenuation is captured in the following definition:

**Definition 2:** Consider a set \( \mathcal{V}_D \) of vertices in the network graph (respectively, subsystems in the network model). Assume that the induced subgraph of \( \Gamma \) defined by \( \mathcal{V}_D \) is strongly connected. The subnetwork defined by \( \mathcal{V}_D \) is said to be propagation impervious if the following two conditions hold.

1. The synchronization manifold for the network synchronization model is asymptotically stable in the sense of Lyapunov.
2. For every source location \( s \), disturbance signal \( w_s(t) \), separating cutset \( \mathcal{V}_C \) for \( s \) contained within \( \mathcal{V}_D \), time horizon \( T > 0 \), and vertex \( b \in \mathcal{V}_B \cap \mathcal{V}_D \), the following majorization holds: \( E_b \leq \max_{s \in \mathcal{V}_C} E_c \).

The definition asserts that disturbances that enter the propagation impervious subnetwork (whether from an outside or an inside source) then exhibit a spatial attenuation within that subnetwork. The definition for propagation imperviousness also aligns with concepts in the strict string stability literature [12], which allow for amplification in a radius around the disturbance source provided that attenuation is guaranteed elsewhere.

### IV. Propagation Stability Analysis

Propagation stability is concerned with the ability of network subsystems to attenuate impinging disturbances, such that a disturbance response falls off spatially in the network regardless of the source. Thus, one might expect propagation stability to be related to local characteristics of the network synchronization model, specifically the structure of each subsystem and its interconnections with its neighbors. In the following development, we present several conditions for propagation stability and imperviousness, which are phrased in terms of local (subsystem-level) frequency response characteristics. Formally, these conditions are expressed in terms of the following Laplace-domain local feedback transfer matrices \( H_i(s) \), defined for each subsystem \( i = 1, \ldots, N \):

\[
H_i(s) = (\alpha \sum_{j \in \mathcal{N}(i)} g_{ij}C(sI - A + \alpha \sum_{j \in \mathcal{N}(i)} g_{ij}BC)^{-1}B. \tag{2}
\]

Here, the notation \( \mathcal{N}_i \) refers to the set of incoming neighbors of vertex \( i \) in the graph, i.e. the set containing vertices \( j \) such that \( g_{ij} > 0 \).

The first main result of our development is a sufficient condition for propagation stability:

**Theorem 1:** The network synchronization is propagation stable if: 1) the synchronization manifold is asymptotically stable in the sense of Lyapunov and 2) \( \sup_{\omega} \sigma_{\max}(H_i(j\omega)) \leq 1 \) for all \( i = 1, \ldots, N \). Further, if the network graph has any vertex that has an incoming edge from only one other vertex, the condition is necessary and sufficient.

Theorem 1 is tight in a certain sense, for the case that the subsystem model is single-input single-output (SISO). Specifically, for this case, consider that the condition of the theorem is not met, i.e. \( \max_{\omega} \sigma_{\max}(H_i(j\omega)) = \max_{\omega} |H_i(j\omega)| > 1 \) for some vertex \( i \). Then for some network graphs, the network synchronization model will not be propagation stable. To see why, consider a network graph for which the vertex \( i \) has only a single incoming edge, say from vertex \( j \). It is easy to check in this case that \( Y_i(s) = H_i(s)Y_j(s) \), i.e. the disturbance response at vertex \( i \) is the filtration of the disturbance response at vertex \( j \) by \( H_i(s) \). Then consider a single tone disturbance \( w_s(t) \) (where \( s \neq i \)) at a frequency \( \omega \) such that \( |H_i(j\omega)| > 1 \). For this disturbance, the response at each subsystem is also asymptotically a single-tone frequency at the same frequency \( \omega \). Since \( |H_i(j\omega)| > 1 \), it thus follows that the amplitude of the sinusoidal response at vertex \( i \) is larger than that at vertex \( j \). Thus, for a sufficiently long time horizon \( T \), \( E_i(T) > E_j(T) \). Therefore, since vertex \( j \) is a cutset that separates the source vertex \( s \) from vertex \( i \), the network model is not propagation stable. If the subsystem model is multi-input multi-output (MIMO), then characterization of response amplitudes for a single tone input is more complex, because the response vector at a subsystem may not coincide with the maximum amplification direction. However, the network certainly may be susceptible to disturbance amplification if \( |H_i(j\omega)| > 1 \).

Conditions for propagation imperviousness in a region of the network graph can also be developed using a parallel argument to the proof of Theorem 1. Here is the result:

**Theorem 2:** Consider a network synchronization model. For this model, consider a set \( \mathcal{V}_D \) of the vertices in the network graph (respectively, subsystems in the network model), such that the induced subgraph defined by \( \mathcal{V}_D \) is strongly connected. The subnetwork defined by \( \mathcal{V}_D \) is propagation impervious if the following two conditions hold: 1) the synchronization manifold is asymptotically stable in the sense of Lyapunov and 2) \( \sup_{\omega} \sigma_{\max}(H_i(j\omega)) \leq 1 \) for all \( i \in \mathcal{V}_D \).

Since propagation stability depends critically on the gains of the local transfer matrices \( H_i(s) \), it is useful to further interpret these matrices from a system-theoretic standpoint. Of
interest, the transfer matrices can be interpreted as the closed-loop reference-signal-to-output transfer function when a certain static feedback controller is applied to the subsystem model. Specifically, the local transfer matrix $H_i(s)$ is the transfer function from $r(t)$ to $y(t)$ of the following closed-loop system model:

$$
\dot{x} = Ax + Bu \\
y = Cx \\
u = (\alpha \sum_{j \neq i} g_{ij})(r - y).
$$

The local transfer matrices are thus seen to capture the closed-loop dynamics of the subsystem model, when an identical proportional feedback controller with gain $k_i = \alpha \sum_{j \neq i} g_{ij}$ is applied at each channel. This feedback control interpretation is useful for characterizing propagation stability in terms of only the subsystem model, as we will do for the SISO case in the following section.

A. SISO Case: Subsystem-Based Characterization

The interpretation of the local transfer matrix $H_i(s)$ as a closed-loop model allows easy development of conditions for propagation stability phrased in terms of either the frequency response or the transfer function of the subsystem model, when the subsystem is SISO. To develop this analysis, let us denote the transfer function for the subsystem model as $T(s) = C(sI - A)^{-1}B$. Then, for the SISO case, the local transfer matrix can be written as

$$H_i(s) = \frac{k_i T(s)}{1 + k_i T(s)},$$

where $k_i = \alpha \sum_{j \neq i} g_{ij}$.

We now develop a check for propagation stability in terms of the subsystem frequency response $T(j\omega)$. To do so, recall from Theorem 1 that propagation stability requires $|H_i(j\omega)| \leq 1$ for all $\omega$ and $i = 1, \ldots, N$, which we call the local requirement, in addition to stability of the synchronization manifold. With some algebra (see extended version [18]), one can show that the local requirement is met if and only if the following condition on the subsystem model $T(s)$ holds:

$$Re(T(j\omega)) \geq -\frac{1}{2\alpha \max_i \sum_{j \neq i} g_{ij}}.$$

Equivalently, the local requirement is met if and only if a trace of $T(j\omega)$ in the complex plane lies entirely to the right of $-\frac{1}{2\alpha \max_i \sum_{j \neq i} g_{ij}}$.

For the SISO subsystem case, the characterization of the local requirement provides a means to verify propagation stability entirely in terms of the frequency response of the subsystem model. Specifically, both the local requirement and the standard condition for asymptotic stability of the synchronization manifold can be determined from the Nyquist plot of the subsystem model. For the local requirement to be met, the Nyquist plot must lie entirely to the right of the vertical line with intercept $-\frac{1}{2\alpha \max_i \sum_{j \neq i} g_{ij}}$. Meanwhile, the characterization of asymptotic stability of the synchronization manifold in terms of the Nyquist plot is well known [17]. Briefly, asymptotic stability of the manifold can be equivalenced with simultaneous Hurwitz stability of matrices $A + \lambda_i BC$, $i = 2, \ldots, N$, where the $\lambda_i$ $(i = 2, \ldots, N)$ are the non-zero eigenvalues of the Laplacian matrix $L$.

For the SISO subsystem case, the matrices $A + \lambda_i BC$ are the state matrices of the closed-loop system, when a proportional feedback controller with gain $\lambda_i$ is applied to the subsystem. Thus, it is seen that Hurwitz stability of each matrix $A + \lambda_i BC$ can be determined through the standard Nyquist criterion, i.e. by comparing encirclements of the point $\frac{-1}{\lambda_i}$ by the Nyquist plot with the number of open-loop right-half-plane poles. Since this stability analysis for the synchronization manifold has been developed extensively in previous work, details are omitted.

The propagation stability analysis for the SISO subsystem case gives insight into the role of the coupling constant $\alpha$. As the coupling is weakened, the permissible region for the Nyquist plot such that the local requirement holds is widened, including more of the left half of the complex plane. In fact, by decreasing the coupling, the local requirement can be met for any subsystem model unless the Nyquist plot diverges in the left-half-plane (which corresponds to subsystem models with repeated poles on the $j\omega$ axis). This ability to meet the local requirement by reducing the coupling is logical, since reduced couplings should attenuate propagation of a disturbance through the network. However, scaling down the coupling may also influence the stability of the synchronization manifold. Analyses using the master stability function have shown the intermediate couplings are needed for stability of the synchronization manifold, for many subsystem models (e.g., several chaotic oscillators like the Rossler oscillator). These characterizations, which can also be verified through the Nyquist analyses presented above, indicate that the local requirement and manifold stability may conflict for some subsystem models: a small gain may be needed for the local requirement to hold, while a larger gain is needed for stability of the synchronization manifold.

The frequency-domain analysis developed above immediately yields conditions for propagation stability in terms of the subsystem transfer function. First, general characterizations of propagation stability can be obtained in the cases where the subsystem model is either strictly unstable (has open right-half-plane poles) or strictly stable:

**Corollary 1:** Consider the network synchronization model, and assume that the subsystem model is SISO.

- If the subsystem model has an open right-half-plane pole, then the network synchronization model is not propagation stable for any network graph $\Gamma$ and coupling constant $\alpha$.
- If the subsystem model is strictly stable (has all poles strictly in the open left-half-plane), then there exists a positive constant $\bar{\alpha}$ such that the network synchronization model is propagation stable for $\alpha \leq \bar{\alpha}$.

Keener characterizations of propagation stability can be
developed by considering what transfer functions meet the local requirement for propagation stability. The following is a main corollary, which uses the fact that the Nyquist plot for a passive system lies in the closed right half plane:

**Corollary 2:** Consider the network synchronization model, and assume that the subsystem model is SISO and passive. Also, assume that the synchronization manifold is asymptotically stable. Then the network synchronization model is necessarily propagation stable.

In the case where the subsystem model is not passive, propagation stability depends on the network graph and the coupling strength, as well as the extent to which the subsystem’s Nyquist plot encroaches on the left half of the complex plane. This is formalized in the following theorem:

**Theorem 3:** Consider the network synchronization model, and assume that the subsystem model is SISO. Also, assume that the synchronization manifold is asymptotically stable. If

\[
Re(T(j\omega)) \geq -\frac{1}{2\alpha \max_i \sum_{j \neq i} g_{ij}}
\]

for all \( \omega \), then the network synchronization model is propagation stable.

**V. Example**

The propagation stability analysis is illustrated for an example model with planar subsystems. Specifically, let us consider a synchronization network process whose subsystem model \((C, A, B)\) is defined as follows:

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & -d \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},
\]

where we refer to the positive scalar \( d \) as a damping constant. Several common network models are well-approximated by this form, including the classical model for the bulk power grid’s swing dynamics, mass-spring-damper network models, and models for vehicle teams engaged in formation flight. These networks are known to exhibit a dichotomy of responses to sinusoidal or periodic disturbances, depending on the damping constant \( d \). When the system is sufficiently damped, periodic/sinusoidal disturbances cause only localized responses; on the other hand, if the damping is low, disturbances at certain frequencies incur network-wide responses.

The propagation stability concept provides a means for assessing how the damping influences the disturbance response pattern for the defined class of synchronization network processes. In particular, for this example, the frequency-domain criterion for propagation stability can readily be phrased in terms of the damping constant \( d \). If the criterion is met, then disturbance responses are necessarily localized. If not, the network model potentially may be susceptible to network-wide responses for disturbances over certain frequency ranges.

The criterion for propagation stability includes a standard requirement for stability of the synchronization manifold, and an additional local requirement. For the network model considered in this example, stability of the synchronization manifold has been precisely characterized in prior work. Provided that the eigenvalues of the Laplacian are real (which encompasses the symmetric and diagonally symmetrizable cases), stability of the synchronization manifold holds for any damping. If the Laplacian has complex eigenvalues, then a sufficiently large damping is needed for stability of the synchronization manifold.

In our development here, we assume that the criterion for stability of the synchronization manifold is met, and focus on relating the local requirement with the damping ratio. The local requirement is met if the subsystem model \( Re(T(j\omega)) \geq -\frac{1}{2\alpha \max_i \sum_{j \neq i} g_{ij}} \) for all \( \omega \). The transfer function for the planar subsystem model is \( T(s) = \frac{1}{s^2 + ds} \), and hence the frequency response is \( T(j\omega) = \frac{1}{\omega^2 + j\omega d} = \frac{-\omega^2 - j\omega d}{\omega^2 + d^2 \omega^2} \). It immediately follows that

\[
Re(T(j\omega)) = \left( \frac{d^2}{\omega^2} \right) - \frac{1}{\omega^2 + d^2 \omega^2},
\]

Thus, we find that \( Re(T(j\omega)) \) is bounded by the interval \([\frac{1}{\omega^2}, 0]\), with the lower bound achieved asymptotically as \( \omega \to 0 \). The local requirement is therefore met if

\[
\frac{1}{\omega^2} \leq \frac{1}{\omega^2 + d^2 \omega^2},
\]

or equivalently \( d \geq \sqrt{2\alpha \max_i \sum_{j \neq i} g_{ij}} \). Thus, a sufficiently large damping relative to the largest (weighted) in-degree in the network graph guarantees propagation stability. If the damping ratio is not sufficiently large, the network is potentially susceptible to wide-area responses, for low-frequency disturbances.

Thus, we have shown that an additional requirement of sufficient damping is needed to ensure propagation stability in addition to stability of the synchronization manifold. In the case where the criterion is not met, disturbances with certain frequency components have the potential for amplification across the network. On the other hand, when the damping requirement is met, disturbances are restricted to have local spheres of influence. We note that the condition for propagation stability is phrased entirely in terms of the subsystem model and local graph properties, which then allows the development of a condition on the damping for propagation stability.

![Fig. 2. The dolphin network, with the source vertex highlighted in red and each undirected edge corresponding to two oppositely pointing directed edges.](image)

**REFERENCES**

[1] Pecora, Louis M., and Thomas L. Carroll. “Master stability functions for synchronized coupled systems.” Physical review letters 80, no. 10 (1998): 2109.
Fig. 3. Zero-state response of the dolphin network, with each vertex representing the planar subsystem model.

Fig. 4. The disturbance response of the dolphin network at two output location is illustrated, when the model is propagation stable with damping $d = 4$. In this instance, the maximum response at the target (dashed) at a distance of 2 from the source does not surpass the maximum responses of some vertices at the separating cutset at distance 1 from the source.

Fig. 5. The disturbance response of the dolphin network at two output location is illustrated, when the model is not propagation stable with damping $d = 0.08$. In this instance, the maximum response at the target (dashed) at a distance of 2 from the source surpasses the maximum responses of all vertices at the separating cutset at distance 1 from the source.

[2] Wu, Chai Wah, and Leon O. Chua. “Synchronization in an array of linearly coupled dynamical systems.” IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications 42, no. 8 (1995): 430-447.
[3] Fax, J. Alexander, and Richard M. Murray. "Information flow and cooperative control of vehicle formations." IEEE transactions on automatic control 49, no. 9 (2004): 1465-1476.
[4] Wen, Guanghui, Guoqiang Hu, Wenwu Yu, and Guanrong Chen. “Distributed $H_{\infty}$ Consensus of Higher Order Multiagent Systems With Switching Topologies.” IEEE Transactions on Circuits and Systems II: Express Briefs 61, no. 5 (2014): 359-363.
[5] Saberi, Ali, Anton A. Stoerlovogel, Meirong Zhang, and Peddapullaiah Sannuti. Synchronization of Multi-Agent Systems in the Presence of Disturbances and Delays. Springer Nature, 2022.
[6] Tegling, Emma, and Henrik Sandberg. "On the coherence of large-scale networks with distributed PI and PD control." IEEE control systems letters 1, no. 1 (2017): 170-175.
[7] Chevalier, Samuel, Petr Vorobev, and Konstantin Turitsyn. "A passivity interpretation of energy-based forced oscillation source location methods." IEEE Transactions on Power Systems 35, no. 5 (2020): 3588-3602.
[8] Li, Max Z., Karthik Gopalakrishnan, Kristyn Pantoja, and Hamsa Balakrishnan. "Graph signal processing techniques for analyzing aviation disruptions." Transportation Science 55, no. 3 (2021): 553-573.
[9] Swaroop, Darbha, and J. Karl Hedrick. "String stability of interconnected systems." IEEE transactions on automatic control 41, no. 3 (1996): 349-357.
[10] Ploeg, Jeroen, Nathan Van De Wouw, and Henk Nijmeijer. "Lp string stability of cascaded systems: Application to vehicle platooning." IEEE Transactions on Control Systems Technology 22, no. 2 (2013): 786-793.
[11] Pant, Aniruddha, Pete Seiler, and Karl Hedrick. "Mesh stability of look-ahead interconnected systems." IEEE Transactions on Automatic Control 47, no. 2 (2002): 403-407.
[12] Stüdli, Sonja, María M. Seron, and Richard H. Middleton. "From vehicular platoons to general networked systems: String stability and related concepts." Annual Reviews in Control 44 (2017): 157-172.
[13] Mirabiilo, Marco, Alessio Iovine, Elena De Santis, Maria Domenica Di Benedetto, and Giordano Pola. "Scalable Mesh Stability of Nonlinear Interconnected Systems." IEEE Control Systems Letters 6 (2021): 968-973.
[14] Pirani, Mohammad, John W. Simpson-Porco, and Baris Fidan. "System-theoretic performance metrics for low-inertia stability of power networks." In 2017 IEEE 56th Annual Conference on Decision and Control (CDC), pp. 5106-5111. IEEE, 2017.
[15] Koorehdavoudi, Kasra, Mohammadreza Hatami, Sandip Roy, Vaihtianathan Venkatasubramanian, Patrick Panciatici, Florent Xavier, and Jackeline Abad Torres. “Input-output characteristics of the power transmission network’s swing dynamics.” In 2016 IEEE 55th Conference on Decision and Control (CDC), pp. 1846-1852. IEEE, 2016.
[16] Hao, Yuqing, Zhisheng Duan, Guanrong Chen, and Fen Wu. “New controllability conditions for networked, identical LTI systems.” IEEE Transactions on Automatic Control 64, no. 10 (2019): 4223-4228.
[17] Li, Zhongkui, Zhisheng Duan, and Guanrong Chen. “Global synchronised regions of linearly coupled Lur’e systems.” International Journal of Control 84, no. 2 (2011): 216-227.
[18] Roy, Sandip, Subir Sarker, and Sandip Roy. “Propagation stability concepts for network synchronization processes (extended version).” Submitted to Arxiv.