The Application of Simulation Method in Isothermal Elastic Natural Gas Pipeline

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Abstract. This Elastic pipeline mathematic model is of crucial importance in natural gas pipeline simulation because of its compliance with the practical industrial cases. The numerical model of elastic pipeline will bring non-linear complexity to the discretized equations. Hence the Newton-Raphson method cannot achieve fast convergence in this kind of problems. Therefore A new Newton Based method with Powell-Wolfe Condition to simulate the Isothermal elastic pipeline flow is presented. The results obtained by the new method are given based on the defined boundary conditions. It is shown that the method converges in all cases and reduces significant computational cost.

1. Introduction
Natural gas transmission pipelines occupy more than 90 percentages in terms of the total miles within China. In view of practice, most of natural gas pipelines are elastic pipeline, therefore the elastic model of gas pipeline application is of vital importance \textsuperscript{[1-2]}. However the research in the elastic pipeline area seems not very attractive and doesn’t receive a widespread focus. In summary, it is still in the initial stage.

As the natural gas flow through the elastic pipe, the structure and characteristic of the elastic pipe change and hence the flow characters inside the pipeline change distinctly. Therefore for elastic pipe, care must be taken to make sure that the mathematic flow model is precisely correct in side of the pipes.

Previous research mainly focus on the flow of rigid pipe, elasticity of wall, viscosity and non-constant flow were not mentioned in the research of pipe flow. However, in the practical natural gas transmission pipelines all of the characters are included. Kuchar and Ostrach gave\textsuperscript{[3-4]} a detailed research on the coupling problem of wall deformation of elastic pipe and gas flow. Flow distribution formula was given in the condition of constant flow of gas. So, non-constant flow of elastic pipe is still a problem to solve.

This paper presents a method dealing with the problem of variable diameter of elastic pipe. On the basis of flow conservation and governing equations for continuous medium flow, combined with
BWRS state equation, differential form of conservation equation of one-dimensional flow of compressible variable section pipe flow is brought out.

2. Mathematic model
Assume elastic pipe is an unlimited long cylindrical pipe with variable section. Diameter is $D(x)$, cross-sectional area is $A(x)$. And we propose the pipe is one-dimensional pipe, fluid flow direction is defined as $X$ axis, and flow velocity is replaced by sectional average velocity, defined as $u$. The elastic pipe flow model is built as follows: the velocity of inlet flow is uniform, which defined as $u(0)$, inlet pressure is defined as $P_s$. Due to the effect of viscosity, velocity and pressure reduce gradually. So the magnitude of elastic pipe expansion is also gradually reduced. At the outlet of the elastic pipe, pipe flow turns to a constant flow as in rigid pipe.

For this one dimension unsteady case, if the transaction surface in the pipeline is changeable, and assume the elastic pipe is straight, semi-infinite pipe, the governing equation for all the pipeline segment can defined as follows:

The continuity equation

$$A \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 A)}{\partial x} = -g \rho A \theta - \frac{\partial(PA)}{\partial x} - \frac{\lambda}{D} \frac{u^2}{2 \rho A}$$

(1)

The momentum equation

$$A \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u A)}{\partial x} = -A \frac{\partial P}{\partial x} - A \rho g \theta - \frac{A \rho A^2}{2D}$$

(2)

The energy equation

$$A \frac{\partial(\rho (u + u^2))}{\partial t} + \frac{\partial(\rho u (u + u^2) A)}{\partial x} = -A \frac{\partial(PA u)}{\partial x} - A \rho g \theta - \frac{\pi D h_1 (T - T_0)}{2}$$

(3)

And the frictional resistance coefficient is computed by F.Colebrook-White formular [3]:

$$\frac{1}{\sqrt{f}} = 1.7385 - 2 \ln(2 e / D + 18.574 / \sqrt{f} + R_e)$$

(4)

Where $Re$ refers to Reynolds number, $e/D$ denotes the ratio of roughness with respect to diameter [5].

$$Z = 1 + \frac{DB}{K^5} - \frac{D_r^{18}}{13} \sum_{n=13}^{18} C_n T^{-n} + \sum_{n=13}^{58} \sqrt{2} C_n T^{-n} \left( b_n - \frac{c_n D_r}{D} \right) D_r^{b_n} \exp \left( -c_n D_r^{b_n} \right)$$

(5)

3. Numerical model
Using $(\rho, u, T)$ as the basic variables, assume the pipeline a horizontal pipeline, the transaction surface is not changeable with position; the pipeline changed in spatial position can be obtained by processing the following contents.

3.1. Discretization governing equations
Discretization of continuity equation of one-dimension flow:

Take Infinitesimal pipe as the control body, the transaction surface acreage is specified as the average acreage of length and step of each point. The discretization form is as follows [6]:

$$A_{i+1}^{k+1} + A_i^{k+1} + A_{i+1}^{k+1} + A_i^{k+1} \times \rho_{i+1}^{k+1} - \rho_i^{k+1} + \rho_{i+1}^{k+1} - \rho_i^{k+1} + \rho_{i+1}^{k+1} u_{i+1}^{k+1} A_{i+1}^{k+1} - \rho_i^{k+1} u_i^{k+1} A_i^{k+1}$$

$$\frac{4}{2 \Delta t} \frac{2 \Delta x}{}$$

(6)

Discretization of energy equation of one-dimension flow:

$$A \frac{\partial(\rho (u + u^2 / 2))}{\partial t} + \frac{\partial(\rho u (u + u^2 / 2) A)}{\partial x} = -g \rho A \theta - \frac{\partial(PA u)}{\partial x} - \pi D h_1 (T - T_0)$$

(7)
Analogy of discretization of continuity equation, we can get the discretization of momentum equation as equation (8):
\[ \frac{A_{t+1}^{k+1} + A_t^k + A_{t+1}^k}{4} + \frac{\rho_{i+1}^{k+1}(u_{i+1}^{k+1})^2}{2\Delta t} \times \Delta t - \frac{\rho_{i+1}^k(u_{i+1}^k)^2}{2\Delta t} \times \Delta t + \frac{\rho_{i+1}^{k+1}(u_{i+1}^{k+1})^2}{2\Delta x} \times \Delta x - \frac{\rho_{i+1}^k(u_{i+1}^k)^2}{2\Delta x} \times \Delta x \]
\[ + \frac{P_{t+1}^{k+1} + P_t^k}{2\Delta x} \times \Delta x + \frac{\rho_{i+1}^{k+1}A_{i+1}^{k+1}u_{i+1}^{k+1}}{2\Delta x} + \frac{\rho_{i+1}^kA_i^ku_i^k}{2\Delta x} \]
\[ + \frac{\rho_{i+1}^{k+1}A_{i+1}^{k+1}u_{i+1}^{k+1}}{2\Delta x} + \frac{\rho_{i+1}^kA_i^ku_i^k}{2\Delta x} \times \Delta t - \frac{\rho_{i+1}^k(u_{i+1}^k)^2}{2\Delta t} \times \Delta t + \frac{\rho_{i+1}^{k+1}(u_{i+1}^{k+1})^2}{2\Delta x} \times \Delta x = 0 \] (8)

Analogy of discretization of continuity equation, we can also get the discretization of energy equation as equation (9).
\[ \frac{A_{t+1}^{k+1} + A_t^k + A_{t+1}^k}{4} \times \Delta t - \frac{\rho_{i+1}^{k+1}(u_{i+1}^{k+1})^2}{2\Delta t} \times \Delta t + \frac{\rho_{i+1}^k(u_{i+1}^k)^2}{2\Delta t} \times \Delta t + \frac{\rho_{i+1}^{k+1}A_{i+1}^{k+1}u_{i+1}^{k+1}}{2\Delta x} \times \Delta x + \frac{\rho_{i+1}^kA_i^ku_i^k}{2\Delta x} \]
\[ + \frac{\rho_{i+1}^{k+1}A_{i+1}^{k+1}u_{i+1}^{k+1}}{2\Delta x} + \frac{\rho_{i+1}^kA_i^ku_i^k}{2\Delta x} \times \Delta t - \frac{\rho_{i+1}^k(u_{i+1}^k)^2}{2\Delta t} \times \Delta t + \frac{\rho_{i+1}^{k+1}(u_{i+1}^{k+1})^2}{2\Delta x} \times \Delta x = 0 \] (9)

4. Iterative method
The discrete control equations with boundary conditions and initial conditions can be closed, then form the nonlinear equations after closed. If use matrix form can be expressed as \( C(x)x = b \), \( C(x) \) is the coefficient matrix, \( X = (x_1, x_2, x_3, ..., x_n) \) is solved variables, \( b = (b_1, b_2, b_3, ..., b_N) \) is the right side of the vector equation. Suppose that we use the basic NEWTON method to solve the nonlinear equations. First we introduce the function \( F = (F_1, F_2, F_3, ..., F_N) \).

\[ F_i = r_i = C_i(x) \cdot x - b_i \quad i = 1, 2, 3, ..., N \] (10)

Analogy of discretization of continuity equation, we can also get the discretization of energy equation as equation (9).

Among them, namely \( F_i \) is the residual vector. With the introduction of the function \( F \), we will solve the nonlinear equations into seeking the problems. The formula was established as follow:
\[ F = (F_1, F_2, F_3, ..., F_N)^T = 0 \] (11)

For arbitrary point \( X_0 \) and its adjacent points in the neighborhood \( X_0 + \sigma_x \). By TAYLOR expansion, we can achieve that:

\[ F_i(X_0 + \delta_x) = F_i(X_0) + \sum_{j=1}^{N} \frac{\partial F_j}{\partial x_j} \delta x_j + O(\delta x^2), i = 1, 2, 3, ..., N \] (12)

If we use the matrix form:

\[ F(X_0 + \delta x) = F(X_0) + J \delta x + O(\delta x^2) \] (13)

The J is N*N Jacobian matrix and \( J_{ij} = \frac{\partial F_i}{\partial x_j} \).

If we omit one of the higher order term, and demand \( F(X_0 + \delta x) \) as 0, we can obtain:

\[ \delta x = -J^{-1} \cdot F \] (14)

This condition is called Newton Condition. Who solving method based on this condition can be called Newton method, the basic Newton method (Newton-Raphson method) use the iterative method as follow:

\[ X_{k+1} = X_k + \delta x \] (15)

Obviously Newton-Raphson method is not a method of global convergence, and it is not even a drop algorithm. Overcoming the defect of Newton Raphson method has a lot of measures, we introduce the one below.

We first introduce the objective function: \( f = 0.5F \cdot F \).

So through simple mathematical operation, we can know the Newton iteration along the gradient direction of the objective function \( \nabla f \cdot \alpha \). Can always be found to make objective function decline.

\[ \nabla f \cdot \delta x = FJ \cdot (-J^{-1}F) = -F \cdot F < 0 \] (16)

So we can adopt the new iteration method:

\[ X_{k+1} = X_k + \alpha \delta x \] (17)

\( \alpha \) is Step length of the gradient direction. The initial value of \( \alpha \) must be 1.

By using Powell-Wolfe Condition, \( \alpha \) can be achieved. If the value cannot satisfy drop conditions, we can use the backtracking algorithm (backtracking) to realize Powell-Wolfe Condition.

When we are not close enough to the minimum of \( f \), taking the full Newton step \( p = \delta x \) need not decrease the function; we may move too far for the quadratic approximation to be valid. All we are guaranteed is that initially \( f \) decreases as we move in the Newton direction. So the goal is to move to a new point \( X_{k+1} \) along the direction of the Newton step \( p \), but not necessarily all the way:

\[ X_{k+1} = X_k + \alpha \delta x, \quad 0 < \alpha \leq 1 \] (18)

The aim is to find \( \alpha \) so that \( f(xk + \alpha p) \) has decreased sufficiently. Until the early 1970s, standard practice was to choose \( \alpha \) so that \( xk+1 \) exactly minimize \( f \) in the direction \( p \) [8]. However, we now know that it is extremely wasteful of function evaluations to do so. A better strategy is as follows: Since \( p \) is always the Newton direction in our algorithms, we first try \( \alpha = 1 \), the full Newton step. This will lead to quadratic convergence when \( x \) is sufficiently close to the solution. However, if \( f(xk+1) \) does not meet our acceptance criteria, we backtrack along the Newton direction, trying a smaller value of \( \alpha \), until we find a suitable point. Since the Newton direction is a descent direction, we are guaranteed to decrease for sufficiently small \( \alpha \).

A simple way to fix the problem of construct a sequence of steps satisfying this criterion with \( f \) decreasing too slowly relative to the step lengths is to require the average rate of decrease of \( f \) to be at least some fraction \( \beta \) of the initial rate of decrease \( \nabla f \cdot p \):
\[ f(x_{k+1}) \leq f(x_k) + \beta \nabla f \cdot (x_{k+1} - x_k) \quad (19) \]

Here the parameter \( \beta \) satisfies \( 0 < \beta < 1 \). We can get away with quite small values of \( \beta \); \( \beta = 10^{-4} \) is a good choice.

The problem of sequence where the step lengths are too small relative to the initial rate of decrease of \( f \) can be fixed by requiring the rate of decrease of \( f \) at \( x_k \) to be greater than some fraction \( \beta \) of the rate of decrease of \( f \) at \( x_k \)[7]. In practice, we will not need to impose this second constraint because our backtracking algorithm will have a built-in cutoff to avoid taking steps that are too small.

Here is the strategy for a practical backtracking routine: Define
\[
g(\alpha) \equiv f(x_k + \alpha p) , \quad (20) \]
\[
g'(\alpha) = \nabla f \cdot p \quad (21) \]

If we need to backtrack, then we model \( g \) with the most current information we have and choose \( \alpha \) to minimize the model. We start with \( g(0) \) and \( g'(0) \) available. The first step is always the Newton step, \( \alpha = 1 \). If this step is not acceptable, we have available \( g(1) \) as well. We can therefore model \( g(\lambda) \) as a quadratic:
\[
g(\lambda \alpha) \approx [g(1) - g(0) - g'(0)] \alpha^2 + g'(0) \alpha + g(0) \quad (22) \]
Taking the derivative of this quadratic, we find that it is a minimum when
\[
\alpha = \frac{-g'(0)}{2[g(1) - g(0) - g'(0)]} \quad (23) \]
Since the Newton step failed, we can show that \( \alpha \leq 1/2 \) for small \( \alpha \). We need to guard against too small a value of \( \alpha \), however. We set \( \alpha_{min} = 0.1 \).

On second and subsequent backtracks, we model \( g \) as a cubic in \( \alpha \), using the previous value \( g(\alpha_1) \) and the second most recent value \( g(\alpha_2) \):
\[
g(\alpha) = a \alpha^3 + b \alpha^2 + g'(0) \alpha + g(0) \quad (24) \]
Requiring this expression to give the correct values of \( g \) at \( \alpha_1 \) and \( \alpha_2 \) gives two equations that can be solved for the coefficients \( a \) and \( b \):
\[
\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\alpha_i - \alpha_j} \begin{bmatrix} 1/\alpha_i^2 & -1/\alpha_j^2 \\ -\alpha_j/\lambda_i^2 & \alpha_i/\alpha_j^2 \end{bmatrix} \begin{bmatrix} g(\alpha_i) - g(0) \alpha_i - g(0) \\ g(\alpha_j) - g(0) \alpha_j - g(0) \end{bmatrix} \quad (25) \]
The minimum of the cubic (9.7.12) is at
\[
\alpha = \frac{-b + \sqrt{b^2 - 3ag'(0)}}{3a} \quad (26) \]
We enforce that \( \alpha \) lie between \( \alpha_{max} = 0.5 \alpha_i \) and \( \alpha_{min} = 0.1 \alpha_i \).

5. Calculation result
In order to solve the above mathematical model, set the initial condition and boundary condition equations to close: a) The initial conditions: Inlet pressure: \( \text{PIN} \); Outlet pressure: \( \text{POUT} \); Mass flow rate: \( \text{M} \). b) Pipeline import and export of boundary conditions:

The pipe diameter at the end of pipe up hypothesis and cross-sectional area is kept constant, for the first kind of boundary condition, the expression is as follows:
\[
A_y = A_y = \frac{\pi D_o^2}{4}, D_o = D_n = D_o \quad (27) \]

Calculated by programming the following result:
1) Set the initial condition: \( \text{PIN} = 3 \text{MPa}, \text{POUT} = 0.5 \text{MPa}, \text{M} = 60.7 \text{kg/s} \)
2) Set the initial condition: PIN=5MPa, POUT=2MPa, M=94.9kg/s

6. Conclusion
In this article, the whole process is described in detail from the establishment of mathematical model to mathematic solving process. Numerical simulation research to the distribution of flow field, such as velocity field and pressure field can be obtained by using the above model and method.

To further study of the flow of the elastic pipeline, we must consider different factors of fluid flow, consider the higher precision of difference format and better algorithms, and all of these are the important research content of our the next phase, and from one-dimensional flow extends to the two dimensional flow are the focus of future research.

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