Sharp bounds on $2m/r$ for static spherical objects

Paschalis Karageorgis and John G Stalker

School of Mathematics, Trinity College, Dublin 2, Ireland

E-mail: pete@maths.tcd.ie and stalker@maths.tcd.ie

Received 26 June 2008, in final form 27 June 2008
Published 16 September 2008
Online at stacks.iop.org/CQG/25/195021

Abstract

Sharp bounds are obtained, under a variety of assumptions on the eigenvalues of the Einstein tensor, for the ratio of the Hawking mass to the areal radius in static, spherically symmetric spacetimes.

PACS number: 04.40.Dg

(Some figures in this article are in colour only in the electronic version)

1. Introduction

All of the spacetimes considered in this paper are connected, four-dimensional and satisfy the following conditions:

- **Spherical Symmetry**: There is a timelike curve, called the *time axis*, with the property that at any point all normal directions are equivalent, i.e., for any two spacelike normal unit vectors there is an isometry of the spacetime, which fixes the point and takes the first vector to the second. This defines an action of $SO(3)$ on the spacetime, whose orbits are called *spheres*.
- **Staticity**: There is a 1-parameter group of isometries, called time translations, whose generating vector field is everywhere timelike.
- **Regularity**: The spacetime, together with its metric, is of class $C^3$, except possibly on 3-surfaces of discontinuity, where the second derivatives of the metric are allowed to have jump discontinuities.

The somewhat odd-looking regularity assumption is borrowed from Lichnerowicz [9]. It is meant to allow such discontinuities as one expects to find at the interface between two different materials, but nothing worse.

1 We follow Synge [10] in calling this assumption ‘spherical symmetry’ for brevity. This is a bit misleading, as we are assuming more than just spherical symmetry. Our assumption excludes, for example, Schwarzschild space, which lacks a time axis.
The areal radius $r$ is defined by the requirement that the area of a sphere be $4\pi r^2$. In terms of the radius $r$ and metric tensor $g$, we may then define the Hawking mass $m$ by the relation

$$g^{jk} \partial_j r \partial_k r = 1 - \frac{2m}{r}.$$  

(1.1)

The purpose of this paper is to prove sharp bounds on the ratio $2m/r$ under various hypotheses on the eigenvalues of the Einstein tensor. The particular hypotheses considered, their history and the resulting bounds are discussed in section 3.

Three general comments should be made at this stage. First, the method employed is quite general, and can be used to obtain sharp bounds on $2m/r$ for any matter model, not just those described below. Second, obtaining sharp bounds is, in each case, relatively easy. Proving sharpness, while not conceptually difficult, requires considerably more effort. Third, we carefully avoid the assumption, made tacitly by previous authors, that $2m < r$. This point is discussed in more detail in the following section.

Section 2 is devoted to a discussion of coordinates and the components of the Einstein tensor in our chosen coordinate system. Section 3 introduces the various assumptions on this tensor which are needed for the statement of our theorem. Our main result, theorem 4.1, appears in section 4, while its proof is given in section 5.

2. Geometry and coordinates

A spacetime of the class considered above has coordinates $r, \theta, \phi$ and $t$, known as curvature coordinates, in which the metric takes the form

$$g_{jk} \, dx^j \, dx^k = e^\alpha \, dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) - e^\gamma \, dt^2. $$

(2.1)

Here, $\alpha$ and $\gamma$ are functions of $r$. As shown in Synge [10], the Einstein tensor in curvature coordinates is of the form

$$G^\gamma_r = r^{-2} - r^{-2} e^{-\alpha} (1 + r \gamma'),$$

(2.2)

$$G^\theta_\theta = G^\phi_\phi = e^{-\alpha} \left( -\frac{1}{2} \nabla'' + \frac{1}{4} \gamma' \gamma' - \frac{1}{2r} \gamma' + \frac{1}{2r} \alpha' + \frac{1}{4} \alpha' \gamma' \right),$$

(2.3)

$$G^t_t = r^{-2} - r^{-2} e^{-\alpha} (1 - 2 \dot{\alpha}).$$

(2.4)

Here, primes denote derivatives with respect to $r$, while the off-diagonal entries are all zero. The formulae become a bit cleaner, when one uses derivatives with respect to $\beta = 2 \log r$,

instead. Denoting such derivatives by dots, one obtains the equivalent system

$$G^\gamma_r = r^{-2} - r^{-2} e^{-\alpha} (1 + 2 \dot{\gamma}),$$

(2.5)

$$G^\theta_\theta = G^\phi_\phi = r^{-2} - r^{-2} e^{-\alpha} (-2 \dot{\gamma} + \dot{\gamma}^2 + \dot{\alpha} + \dot{\alpha} \dot{\gamma}),$$

(2.6)

$$G^t_t = r^{-2} - r^{-2} e^{-\alpha} (1 - 2 \ddot{\alpha}).$$

(2.7)

The corresponding Einstein tensor, given by Einstein’s field equations, has diagonal entries

$$G^\gamma_r = -8\pi p_R, \quad G^\theta_\theta = G^\phi_\phi = -8\pi p_T, \quad G^t_t = 8\pi \mu,$$

(2.8)

and all off-diagonal entries equal to zero. Here, $p_R$ and $p_T$ are interpreted as the radial and tangential pressures, respectively, while $\mu$ is interpreted as the energy density.
There are two annoying points about curvature coordinates.

- The functions $\alpha$ and $\gamma$ may be of lower regularity than the metric, since $r$ itself is of lower regularity than the metric. This is discussed in more detail by Israel [8]. For our purposes it suffices to note that regularity is, in the presence of the other assumptions, equivalent to the statement that $\alpha$ and $\gamma$ are $C^3$ functions of $r$, except possibly at certain points, where the radial pressure $p_R$ is continuous and the tangential pressure $p_T$ and energy density $\mu$ are allowed to have jump discontinuities. At $r = 0$, the correct condition is that $\alpha'(0) = \gamma'(0) = 0$.

- The coordinates may fail to cover the whole spacetime. In fact, they cover the region from the time axis out to the first marginally trapped sphere, i.e., the first sphere where $r = 2m$. If we were to assume, as most authors do, that curvature coordinates cover the whole spacetime, then we would, in effect, be making the very strong additional assumption that $2m/r < 1$ everywhere. This we wish to avoid. For the classes of spacetimes we consider it is, in fact, the case that $2m/r < 1$ everywhere, but this belongs to the conclusion of our theorem, not to its hypotheses.

The simplest example of a spacetime that satisfies our spherical symmetry, staticity and regularity assumptions, but has a marginally trapped surface is de Sitter space, for which $e^\alpha = -e^{-\gamma} = 1 - r^2/R^2$. In this case, the coordinates cover a region where $r < R$, but break down at the boundary. Outside this region, there is another which is isometric to the first, and it is easy to check that $2m = r$ at $r = R$. However, de Sitter space does not satisfy the hypotheses of our theorem because it has negative pressures everywhere.

3. Matter models

Various conditions on the three functions $p_R$, $p_T$ and $\mu$ are of interest:

- **Non-negative isotropic pressure.** For fluids, one expects $p_R = p_T \geq 0$. The sharp bound
  
  \[ 2m/r \leq 12\sqrt{2} - 16 \approx 0.9706 \]

  under this assumption, and no others, was derived by Bondi [4]. His method of proof is closely related to ours, but is not rigorous.

- **Buchdahl assumption.** For static stars with constant density, one has the bound
  
  \[ 2m/r \leq 8/9, \]

  derived by Buchdahl [5]. More generally, this bound holds when $p_R = p_T \geq 0$, as long as $\mu \geq 0$ is decreasing; see [5]. The isotropy assumption was relaxed in [7], where the case $p_R \geq p_T \geq 0$ was treated, still the monotonicity assumption remains crucial.

- **Dominant energy condition.** For almost any reasonable matter model, one expects $|p_R| \leq \mu$ and $|p_T| \leq \mu$. In the special case that $p_R, p_T \geq 0$, the bound,

  \[ 2m/r \leq 48/49 \approx 0.9796, \]

  is provided by [3]. Our bound for this special case is roughly 0.963, which we show to be sharp.

- **Vlasov–Einstein.** For Vlasov–Einstein matter, the stress–energy tensor is an integral of those of individual particles, each of which has rank one and satisfies the dominant energy condition. This implies that $p_R \geq 0$, $p_T \geq 0$ and $p_R + 2p_T \leq \mu$. Under these assumptions, Andréasson [3] has recently shown that the sharp\(^2\) bound is

\[ m/r \leq 8/9 \]

A somewhat unfortunate feature of our argument is that the sharpness of the estimate $\frac{m}{r} \leq \frac{8}{9}$ is proved only within the class of spacetimes satisfying the pressure conditions above, without considering whether such spacetimes arise from solutions of the full Vlasov–Einstein system. Andréasson’s argument, on the other hand, does provide solutions to the full system.

\[ \]
2m/r ≤ 8/9.

Our method provides a new, and considerably shorter, proof of this result.

- Zero radial pressure. The case, \( p_T \geq p_R = 0 \), was studied by Florides [6] who obtained the sharp bound

\[
2m/r \leq 2/3.
\]

This can also be proved using our method, but the resulting proof is neither shorter nor clearer than the original, so we do not consider this case further.

4. Our main result

**Theorem 4.1.** Consider a spacetime satisfying the regularity, staticity and spherical symmetry conditions described in the introduction. Suppose that the corresponding Hawking mass (1.1) is finite, and that the pressures \( p_R, p_T \) and energy density \( \mu \) are all non-negative.

1. **Vlasov–Einstein case.** Assuming that \( p_R + 2p_T \leq \mu \), one has

\[
\left( 4 - \frac{6m}{r} + 8\pi r^2 p_R \right)^2 \leq 16 \left( 1 - \frac{2m}{r} \right), \quad \frac{2m}{r} \leq \frac{8}{9}. \tag{4.1}
\]

2. **Isotropic case.** Assuming that \( p_R = p_T \), one has

\[
\left( 2 - \frac{2m}{r} + 8\pi r^2 p_R \right)^2 \leq 36 \left( 1 - \frac{2m}{r} \right), \quad \frac{2m}{r} \leq 12\sqrt{2} - 16. \tag{4.2}
\]

3. **Isotropic case with dominant energy.** Assuming that \( p_R = p_T \leq \mu \), one has

\[
\left( 2 - \frac{2m}{r} + 8\pi r^2 p_R \right)^2 \leq 9.551 \left( 1 - \frac{2m}{r} \right), \quad \frac{2m}{r} \leq 0.865. \tag{4.3}
\]

4. **Dominant energy in tangential direction.** Assuming that \( p_T \leq \mu \), one has

\[
\left( 6 - \frac{10m}{r} + 8\pi r^2 p_R \right)^2 \leq 40 \left( 1 - \frac{2m}{r} \right), \quad \frac{2m}{r} \leq \frac{2\sqrt{2} + 2}{5}. \tag{4.4}
\]

5. **Dominant energy case.** Assuming that \( p_R, p_T \leq \mu \), one has

\[
\left( 6 - \frac{10m}{r} + 8\pi r^2 p_R \right)^2 \leq 37.924 \left( 1 - \frac{2m}{r} \right), \quad \frac{2m}{r} \leq 0.963. \tag{4.5}
\]

Moreover, these ten estimates are all sharp for the class of spacetimes considered in each case. As for the numerical values that appear in (4.3) and (4.5), these can be described in terms of a system of ODEs which arises in the course of the proof; see (5.24). The values given here are accurate up to three decimal places.

**Remark 4.2.** There is no assumption on the behaviour of the spacetime as \( r \) tends to infinity. In fact, we do not even assume that \( r \) is unbounded. This point is crucial. It allows us to apply the theorem to the interior of a finite sphere and, in particular, to the interior of the first marginally trapped surface, if there is such a surface.

More precisely, suppose we can prove the theorem in the region where the curvature coordinates are defined, namely in the region where \( 2m < r \). For each matter model, we may then deduce that \( 2m/r \leq c < 1 \) for some constant \( c \), which depends on the matter model considered. Since \( 2m/r \) is continuous and our spacetime is connected, this actually implies that \( 2m/r \leq c \) throughout the spacetime. In other words, the marginally trapped surface that we allowed is not in fact present, and the curvature coordinates, which might \textit{a priori} have
covered only part of the spacetime, cover the whole spacetime. We therefore obtain the full theorem from the special case where the whole spacetime is covered by curvature coordinates. In particular, we may, and do, use curvature coordinates throughout the proof without further comment.

**Remark 4.3.** Our proof for the isotropic case \( p_R = p_T \) applies verbatim in the more general case \( p_R \geq p_T \), while the estimates (4.2) are sharp for that case as well.

**Remark 4.4.** The assumptions of theorem 4.1 can be slightly improved in the sense that we do not use our hypothesis \( p_T \geq 0 \) to establish the given estimates. This hypothesis is merely included to improve the conclusions of theorem 4.1, as sharpness is now shown over a smaller class of spacetimes. In fact, the examples we construct in order to prove sharpness belong to the even smaller class of spacetimes which are vacuum outside a sphere.

The proof of theorem 4.1 is based on the following elementary fact, which is essentially due to Bondi [4].

**Lemma 4.5.** Let the assumptions of theorem 4.1 hold. Then the variables,

\[
x \equiv 1 - e^{-\alpha} = \frac{2m}{r}, \quad y \equiv -r^2 G' = 8\pi r^2 p_R, \tag{4.6}
\]

give rise to a parametric curve, which lies in \([0, 1) \times [0, \infty)\) and satisfies the equations

\[
8\pi r^2 p_R = y, \tag{4.7}
\]

\[
8\pi r^2 p_T = \frac{x + y}{2(1 - x)} \dot{x} + \frac{(x + y)^2}{4(1 - x)}, \tag{4.8}
\]

\[
8\pi r^2 \mu = 2\dot{x} + x, \tag{4.9}
\]

where the dots denote derivatives with respect to \( \beta = 2 \log r \).

**Proof.** First of all, we combine equations (2.4) and (2.8) to write

\[
8\pi r^2 \mu = 1 - \alpha (r e^{-\alpha}).
\]

Integrating over \([0, r]\) and using the definition of the Hawking mass (1.1), we then get

\[
\frac{2m}{r} = 1 - e^{-\alpha} = x. \tag{4.10}
\]

This implies \( x \geq 0 \) because \( m \geq 0 \) whenever \( \mu \geq 0 \). Next, we use (2.8) to get

\[
y \equiv -r^2 G' = 8\pi r^2 p_R \geq 0.
\]

To establish our assertion (4.9), we combine (2.8), (2.7) and (4.10) to find that

\[
8\pi r^2 \mu = r^2 G' = 1 - e^{-\alpha} = 1 - \alpha = 1 - (1 - x) \left(1 - \frac{2\dot{x}}{1 - x}\right) = x + 2\dot{x}.
\]

To establish our remaining assertion (4.8), we first use (2.5) and (4.10) to get

\[
y \equiv -r^2 G' = -1 + (1 - x)(1 + 2\dot{\gamma}), \quad \dot{\alpha} = \frac{\dot{x}}{1 - x}.
\]

Solving the leftmost equation for \( \dot{\gamma} \) and differentiating, we conclude that

\[
2\dot{\gamma} = \frac{x + y}{1 - x}, \quad 2\ddot{\gamma} = \frac{(1 + y)\dot{x} + (1 - x)\dot{y}}{(1 - x)^2}.
\]

On the other hand, equations (2.8), (2.6) and (4.10) combine to give

\[
8\pi r^2 p_T = (1 - x)(2\dot{\gamma} + \dot{\gamma}^2 - \dot{\alpha} - \dot{\alpha} \dot{\gamma}).
\]

Using these facts and a simple computation, one may thus easily deduce (4.8). \( \square \)
5. Proof of theorem 4.1

To prove the desired estimates, we study the curve (4.6) provided by lemma 4.5. In each case, we are seeking an upper bound for $x = 2m/r$ and also an upper bound for $w_n(x, y) = \left(n(1 - x) + 1 + y\right)^2 / (1 - x)$, where the exact value of $n$ varies from case to case. Differentiating (5.1), we get

$$\dot{w}_n = \frac{n(1 - x) + 1 + y}{(1 - x)^2} \cdot \left[(1 + y - n(1 - x))\dot{x} + 2(1 - x)\dot{y}\right]$$

throughout the curve (4.6), where dots denote derivatives with respect to $\beta = 2 \log r$. In the special case that $n = 1$, this formula reads

$$\dot{w}_1 = \frac{2 - x + y}{(1 - x)^2} \cdot \left[(x + y)\dot{x} + 2(1 - x)\dot{y}\right],$$

and it is closely related to the tangential pressure $p_T$; see (4.8). Let us also recall that $0 \leq x < 1$, $y \geq 0$, throughout the curve (4.6), a fact we shall frequently need to use in what follows.

5.1. Vlasov–Einstein case

In this case, we are assuming that $p_R + 2p_T \leq \mu$. According to lemma 4.5, the corresponding curve (4.6) must thus satisfy

$$(3x + y - 2)\dot{x} + 2(1 - x)\dot{y} \leq -\frac{z_3(x, y)}{2}, \quad z_3 = 3x^2 - 2x^2 + 2y^2.$$ (5.4)

Combining the last equation with our computation (5.2), we now find

$$\dot{w}_3 = \frac{4 - 3x + y}{(1 - x)^2} \cdot \left[(3x + y - 2)\dot{x} + 2(1 - x)\dot{y}\right] \leq -\frac{4 - 3x + y}{2(1 - x)^2} \cdot z_3(x, y).$$

In particular, $w_3$ is decreasing whenever $z_3 > 0$, so it must be the case that

$$w_3 \leq \max_{0 \leq x < 1} w_3(x, y) = w_3(0, 0) = 16$$

throughout the curve. This proves the first inequality in (4.1), which also implies the second inequality because the maximum value of $x$ over the region $0 \leq x \leq 1$, $y \geq 0$, $w_3 \leq 16$ is attained at $(8/9, 0)$, namely at the point at which the curve $w_3 = 16$ intersects the $x$-axis. We refer the reader to figure 1 for a sketch of the curves $z_3 = 0$ and $w_3 = 16$.

To show that the estimates in (4.1) are sharp, we need to construct a spacetime such that the corresponding curve of lemma 4.5 intersects a small neighbourhood of $(8/9, 0)$. Let us now temporarily assume that we have a parametric curve,

$$x = x(\tau), \quad y = y(\tau), \quad \tau \in (0, \infty),$$

which passes near the point $(8/9, 0)$ and also satisfies the following properties:

(A1) $\frac{1}{2} \cdot \frac{d^2x}{d\tau^2}$ is both negative and integrable;

(A2) $0 \leq x(\tau) < 1$ and $y(\tau) \geq 0$ for each $\tau > 0$;

(A3) $y(\tau) = 0$ for all large enough $\tau$ and $x(\tau) \to 0$ as $\tau \to \infty$;

(A4) the curve is $C^1$ except for finitely many points.
Given such a curve, we can easily construct a spacetime as follows. First, we define
\[ \kappa(\tau) = -\frac{dw_3/d\tau}{z_3(x, y)} \cdot \frac{2(1-x)^2}{4-3x+y}, \] (5.5)
and we note that \( \kappa \) is both positive and integrable by (A1)–(A2). Next, we define
\[ \beta = \int \kappa \, d\tau, \quad r = \exp\left(\frac{\beta}{2}\right), \] (5.6)
and finally, we define the metric coefficients in (2.1) by
\[ \alpha(r) = -\log(1-x), \quad \gamma(r) = \int \frac{x+y}{2(1-x)} \cdot \kappa \, d\tau. \] (5.7)
Letting dots denote derivatives with respect to \( \beta = 2 \log r \), as usual, we then get
\[ \dot{w}_3 = \frac{1}{\kappa} \cdot \frac{dw_3}{d\tau} = -\frac{4-3x+y}{2(1-x)^2} \cdot z_3(x, y) \]
using our definitions (5.6) and (5.5). In view of our computation (5.2), this gives
\[ (3x+y-2)x + 2(1-x)y = -\frac{z_3(x, y)}{2}, \] (5.8)
which is equivalent to the equation \( p_R + 2p_T = \mu \) because of lemma 4.5.

To finish the proof for this case, it thus remains to construct the curve, whose existence we assumed in the previous paragraph. We have to ensure that the curve satisfies (A1)–(A4), that it passes arbitrarily close to \((8/9, 0)\) and that the corresponding quantities \( p_R, p_T, \mu \) provided by lemma 4.5 are non-negative. Let us then fix some small \( \varepsilon > 0 \) and consider the curve
\[ w_{3-\varepsilon}(x, y) = [\varepsilon \sqrt{1+3x+4(1-\varepsilon)}]^2. \] (5.9)
When \( \varepsilon = 0 \), this reduces to the curve \( w_3 = 16 \), which passes through the origin and \((8/9, 0)\). When \( \varepsilon = 1 \), it reduces to the curve \( z_3 = 0 \), which passes through the origin and \((2/3, 0)\). In the more general case \( 0 < \varepsilon < 1 \), it describes a curve that lies between these two curves. We start out at the origin and follow this curve until we hit the \( x \)-axis, and then we return to the origin along the \( x \)-axis. Let us henceforth denote by \( C_1 \) the curve obtained in this manner; we refer the reader to figure 1 for a typical sketch of this curve.

![Figure 1. The curve \( C_1 \) for the Vlasov–Einstein case.](image-url)
The fact that \( C_1 \) satisfies (A2)–(A4) is trivial. To check that it satisfies (A1) along the part defined by (5.9), we recall that this part lies between the curves \( w_3 = 16 \) and \( z_3 = 0 \). Thus, it is easy to see that \( z_3 > 0 \) along this part, and we need only check that
\[
\frac{dw_3}{d\tau} < 0, \tag{5.10}
\]
as one follows the curve (5.9) in the positive \( x \)-direction. Differentiation of (5.9) gives
\[
\frac{dw_3}{d\tau} = \frac{3\varepsilon \sqrt{w_3 - 3\alpha}}{\sqrt{1 + 3\alpha}} \cdot \frac{dx}{d\tau} = \frac{3\varepsilon}{\sqrt{1 + 3\alpha}} \cdot \frac{3(1 - \varepsilon)(1 - x) + 1 + y}{\sqrt{1 - x}} \frac{dx}{d\tau},
\]
along the curve (5.9), and we may compare this equation with (5.2) to find that
\[
2(1 - x) \cdot \frac{dy}{d\tau} = \left[ \frac{3\varepsilon(1 - x)^{3/2}}{\sqrt{1 + 3\alpha}} + 3(1 - \varepsilon)(1 - x) - 1 - y \right] \frac{dx}{d\tau}, \tag{5.11}
\]
along the curve (5.9). Employing our computation (5.2) once again, we deduce that
\[
\frac{dw_3}{d\tau} = \frac{3\varepsilon(4 - 3x + y)}{1 - x} \cdot \frac{\sqrt{1 - x} - \sqrt{1 + 3\alpha}}{\sqrt{1 + 3\alpha}} \cdot \frac{dx}{d\tau}.
\]
Since \( \frac{dy}{d\tau} > 0 \) here, the desired (5.10) follows. To show that (A1) also holds for the remaining part of the curve \( C_1 \), we need only note that
\[
\frac{1}{z_3} \cdot \frac{dw_3}{d\tau} = \frac{1}{x \cdot (1 - x)} \cdot \frac{4 - 3x}{d\tau} < 0
\]
along the line \( y = 0 \), because this line is traversed in the direction of decreasing \( x \).

Finally, we check that \( p_R, p_T, \mu \geq 0 \) throughout the curve \( C_1 \). The fact that \( p_R \geq 0 \) follows by (A2) because \( 8\pi r^2 p_R = y \) by definition. Since (5.8) ensures that \( \mu = p_R + 2p_T \), we need only check that \( p_T \geq 0 \) as well. Let us now write
\[
8\pi r^2 p_T = \frac{x + y}{2(1 - x)} \cdot \frac{x + y}{4(1 - x)} = \frac{1}{2(2 - x + y)} \cdot \frac{(x + y)^2}{4(1 - x)} \tag{5.12}
\]
using equations (4.8) and (5.3). Along the part of \( C_1 \) defined by (5.9), we have
\[
\dot{w}_1 = \frac{2 - x + y}{1 - x} \cdot \left[ \frac{3\varepsilon \sqrt{1 - x}}{\sqrt{1 + 3\alpha}} + 2 - 3\varepsilon \right] \cdot \dot{x}
\]
by (5.3) and (5.11). In view of our definition (5.6), we thus have
\[
\dot{w}_1 = \frac{2 - x + y}{1 - x} \cdot \left[ \frac{3\varepsilon \sqrt{1 - x}}{\sqrt{1 + 3\alpha}} + 2 - 3\varepsilon \right] \cdot \frac{dx/d\tau}{\kappa}.
\]
Since \( \varepsilon > 0 \) is small, and \( \kappa \) is positive by above, this implies \( \dot{w}_1 > 0 \), hence \( p_T > 0 \) by (5.12). For the remaining part of \( C_1 \) along the \( x \)-axis, lemma 4.5 and (5.8) give
\[
p_R = 0, \quad p_T = \frac{x \mu}{4(1 - x)}, \quad \mu = 2p_T,
\]
so it easily follows that \( p_R = p_T = \mu = 0 \) throughout this part of the curve.
5.2. Isotropic case

In this case, our assumption that \( p_R = p_T \) is equivalent to
\[
(x + y)x + 2(1 - x)y = \frac{-z_1(x, y)}{2}, \quad z_1 = (x + y)^2 - 4y(1 - x). \tag{5.13}
\]

Proceeding as before, we use our computation (5.3) to find that
\[
w_1 = \frac{2 - x + y}{(1 - x)^2} \cdot [(x + y)x + 2(1 - x)y] = \frac{-2 - x + y}{2(1 - x)^2} \cdot z_1(x, y), \tag{5.14}
\]

Once again, \( w_1 \) is decreasing as soon as \( z_1 > 0 \), so it must be the case that
\[
w_1 \leq \max_{0 \leq x, y \leq 1} w_1(x, y) = w_1(0, 4) = 36
\]
throughout the curve. This proves the first inequality in (4.2), while the second inequality follows because the maximum value of \( x \) over the region \( 0 \leq x \leq 1, y \geq 0, w_1 \leq 36 \) is attained at \((12\sqrt{2} - 16, 0)\).

To show that the estimates in (4.2) are sharp, we argue as in the previous case. Suppose we have a curve which passes near the point \((12\sqrt{2} - 16, 0)\), and satisfies
\[
(B1) \quad \frac{1}{z_1} \cdot \frac{dw_1}{d\tau} \text{ is both negative and integrable}
\]
as well as \((A2)\)–\((A4)\). Then we can follow our previous approach with
\[
\kappa(\tau) = -\frac{dw_1/d\tau}{z_1(x, y)} \cdot \frac{2(1 - x)^2}{2 - x + y} > 0, \tag{5.15}
\]

instead of (5.5). Our definitions (5.6) and (5.7) are still applicable, however they now imply
\[
w_1 = \frac{1}{\kappa} \cdot \frac{dw_1}{d\tau} = \frac{2 - x + y}{2(1 - x)^2} \cdot z_1(x, y). \tag{5.16}
\]

In view of our computation (5.3), they thus imply
\[
(x + y)x + 2(1 - x)y = \frac{-z_1(x, y)}{2}, \tag{5.17}
\]

which is equivalent to the equation \( p_R = p_T \) because of lemma 4.5.

To finish the proof for this case, it thus remains to construct the curve whose existence we assumed in the previous paragraph. Fix some small \( \varepsilon > 0 \) and set
\[
x_\varepsilon = \varepsilon, \quad y_\varepsilon = 2 - 6\varepsilon + 2\sqrt{(1 - \varepsilon)(1 - 2\varepsilon)}, \tag{5.18}
\]
for convenience. Then \((x_\varepsilon, y_\varepsilon)\) is a point on the curve \( z_1 = 0 \) which is close to \((0, 4)\). To define the first part of the desired curve, we use the equation
\[
\sqrt{w_1(x, y)} = \sqrt{w_1(0, 0)} + 2A_\varepsilon x_\varepsilon x - A_\varepsilon x^2, \tag{5.19}
\]

where \( A_\varepsilon \) is determined by requiring that the curve passes through \((x_\varepsilon, y_\varepsilon)\), namely
\[
A_\varepsilon = \frac{\sqrt{w_1(x_\varepsilon, y_\varepsilon)} - 2}{x_\varepsilon^2}. \tag{5.20}
\]

We start out at the origin and follow the curve (5.19) until we reach the point \((x_\varepsilon, y_\varepsilon)\), then we follow the curve
\[
\sqrt{w_1(x, y)} = \sqrt{w_1(x_\varepsilon, y_\varepsilon)} - \frac{\varepsilon(x - x_\varepsilon)^2}{\sqrt{1 - x}}, \quad x \geq x_\varepsilon, \tag{5.21}
\]
until we hit the \( x \)-axis, and finally we return to the origin along the \( x \)-axis. Let \( C_2 \) denote the curve obtained in this manner; a typical sketch of this curve appears in figure 2.
The fact that $C_2$ satisfies (A2)–(A4) is trivial; we now check that it satisfies (B1). When it comes to the part of $C_2$ defined by (5.19), we have $z_1 \leq 0$, $x \leq x_c$ and
\[
\frac{dw_1}{d\tau} = 2A_\varepsilon(x_c - x) \cdot \frac{dx}{d\tau}.
\] (5.22)
Since $x$ is increasing along this part of $C_2$, it thus suffices to check that $A_\varepsilon$ is positive. In view of (5.20), this is certainly the case for all small enough $\varepsilon > 0$ because
\[
\lim_{\varepsilon \to 0} \varepsilon^2 A_\varepsilon = \sqrt{w_1(0, 4)} - 2 = 4.
\]
When it comes to the part of $C_2$ defined by (5.21), we have $z_1 \geq 0$, $x \geq x_c$ and
\[
\frac{dw_1}{d\tau} = -\varepsilon(x - x_c) \cdot \frac{4(1 - x) + x - x_c}{2(1 - x)^{3/2}} \cdot \frac{dx}{d\tau} \leq 0,
\]
as needed. When it comes to the remaining part of $C_2$ along the $x$-axis, we have
\[
\frac{dw_1}{d\tau} = \frac{2 - x}{x(1 - x)^2} \cdot \frac{dx}{d\tau} < 0,
\]
and this shows that the desired property (B1) holds throughout the curve $C_2$.

Finally, we check that $p_R, p_T, \mu \geq 0$ throughout the curve $C_2$. The fact that $p_R \geq 0$ follows trivially as before, hence $p_T = p_R \geq 0$ by (5.17), and we need only worry about $\mu$. Since
\[
8\pi r^2 \mu = 2\dot{x} + x
\]
by (4.9), we have $\mu \geq 0$ as long as $x$ is increasing along the curve, so we need only check the part of $C_2$ along the $x$-axis. As in the previous case, however, lemma 4.5 and (5.17) combine to give $p_R = p_T = \mu = 0$ throughout this part, so the proof for this case is complete.

### 5.3. Isotropic case with dominant energy

Our assumption that $p_R = p_T \leq \mu$ gives
\[
(x + y)\dot{x} + 2(1 - x)\dot{y} = -\frac{z_1(x, y)}{2}, \quad 2\dot{x} \geq y - x,
\] (5.23)
with $z_1$ as in (5.13). Due to the isotropy condition, (5.14) remains valid, so $w_1$ is increasing if and only if $z_1 \leq 0$. Since the curve of lemma 4.5 starts out at the origin, where $z_1 = 0$, it may only attain the largest possible value of $w_1$ at a point along the curve $z_1 = 0$. It is easy to check that higher values of $w_1$ occur at higher points on this curve. To attain the largest possible value of $w_1$, the curve of lemma 4.5 must thus ascend as fast as possible within the region $z_1 \leq 0$. Since it starts out at the origin, it must satisfy

\[(x + y)\dot{x} + 2(1 - x)\dot{y} = -\frac{z_1(x, y)}{2}, \quad 2\dot{x} = y - x,\]

until it exits the region $z_1 \leq 0$. This gives rise to the system of ODEs

\[2\dot{x} = y - x, \quad 2\dot{y} = \frac{y(2 - 3x - y)}{1 - x}, \quad (5.24)\]

which has a saddle point at the origin. The solution of interest is the one corresponding to the unstable manifold associated with the origin. Using numerical integration, we find that it intersects the curve $z_1 = 0$ at the point $(x_1, y_1) = (0.4927, 0.6939)$; see figure 3. This makes $w_1(x_1, y_1) \approx 9.551$

the largest possible value of $w_1$, and then we can use the fact that $w_1 \leq 9.551$ to deduce that the largest possible value of $x$ is attained at $(0.865, 0)$.

To show that our results for this case are sharp, we need to find a curve which passes near the point $(0.865, 0)$ and satisfies (B1) as well as (A2)–(A4). Given such a curve, one can use our approach in the previous case to obtain a spacetime for which $p_R = p_T$. We start out at the origin and follow the solution to the ODE,

\[\frac{dy}{dx} = \frac{y(2 - 3x - y)}{(1 - x)(y - x)}, \quad (5.25)\]
corresponding to the associated unstable manifold; we do so until we reach the point \((x_1, y_1)\) that lies on the curve \(z_1 = 0\), then we follow the curve

\[
\sqrt{w_1(x, y)} = \sqrt{w_1(x_1, y_1)} - \frac{\varepsilon(x - x_1)^2}{\sqrt{1 - x}}, \quad x \geq x_1, \tag{5.26}
\]

until we hit the \(x\)-axis, and finally we return to the origin along the \(x\)-axis. We refer the reader to figure 3 for a typical sketch of the curve \(C_3\) obtained in this manner.

The only nontrivial properties we need to verify are (B1) and the fact that \(p_R \leq \mu\). When it comes to the part of \(C_3\) defined by (5.25), we have \(p_R = p_T = \mu\) and also

\[
\frac{1}{z_1} \frac{d w_1}{d \tau} = -\frac{2 - x + y}{(1 - x)^2(y - x)} \frac{dx}{d \tau} \leq 0, \tag{5.27}
\]

so the desired properties are easily seen to hold. The same is true for the part of \(C_3\) along the \(x\)-axis, because \(p_R = p_T = \mu = 0\) and since

\[
\frac{1}{z_1} \frac{d w_1}{d \tau} = 1 \frac{2 - x}{x(1 - x)^2} \frac{dx}{d \tau} < 0
\]

along this part. For the remaining part defined by (5.26), we have

\[
\frac{dw_1/d\tau}{2\sqrt{w_1}} = -\varepsilon(x - x_1) \frac{4(1 - x) + x - x_1}{2(1 - x)^{3/2}} \frac{dx}{d \tau} \leq 0, \tag{5.28}
\]

which implies property (B1), because \(z_1 \geq 0\) for this part. Writing (5.16) in the form

\[
\dot{w}_1 = -\frac{2 - x + y}{2(1 - x)^2} z_1(x, y) = -\frac{\sqrt{w_1(x, y)}}{2(1 - x)^{3/2}} z_1(x, y),
\]

we now combine the last two equations to deduce that

\[
2\dot{x} = -\frac{z_1(x, y)}{\varepsilon(x - x_1)(4 - 3x - x_1)}
\]

throughout the curve (5.26). According to lemma 4.5, the condition \(p_R \leq \mu\) we need to verify is equivalent to the condition \(2\dot{x} \geq y - x\), so we need to check that

\[
\frac{z_1(x, y)}{x - x_1} \geq \varepsilon(4 - 3x - x_1)(y - x) \tag{5.29}
\]

throughout the curve (5.26). Write equation (5.26) in the equivalent form:

\[
y = f(x) = \sqrt{w_1(x_1, y_1)}\sqrt{1 - x} + x - 2 - \varepsilon(x - x_1)^2.
\]

Then, \(z_1(x_1, f(x_1)) = z_1(x_1, y_1) = 0\) by construction, so one easily finds

\[
\lim_{x \to x_1} \frac{z_1(x, f(x))}{x - x_1} = 8(x_1 + y_1) - 4 - (3x_1 + y_1 - 2) \cdot \frac{\sqrt{w_1(x_1, y_1)}}{\sqrt{1 - x_1}} \approx 4.746,
\]

using (5.13). Thus, the left-hand side of (5.29) is bounded away from zero near \(x = x_1\). Since the same is true away from \(x = x_1\), where \(z_1\) itself is bounded away from zero, we can always find a small enough \(\varepsilon > 0\), so that (5.29) holds throughout the curve (5.26).

### 5.4. Dominant energy in tangential direction

Our assumption that \(p_T \leq \mu\) gives

\[
(5x + y - 4)x + 2(1 - x)y \leq \frac{-z_5(x, y)}{2}, \quad z_5 = (x + y)^2 - 4x(1 - x). \tag{5.30}
\]

Proceeding as before, we use our computation (5.2) to find that

\[
\dot{w}_5 = \frac{6 - 5x + y}{(1 - x)^2} \cdot [(5x + y - 4)x + 2(1 - x)y] \leq \frac{6 - 5x + y}{2(1 - x)^2} \cdot z_5(x, y). \tag{5.31}
\]
Once again, $w_5$ is decreasing as soon as $z_5 > 0$, so it must be the case that

$$w_5 \leq \max_{0 \leq \tau \leq 1} w_5(x, y) = w_5(1/10, 1/2) = 40$$

throughout the curve. This proves the first inequality in (4.4), and the second inequality follows as before.

To show that the estimates in (4.4) are sharp, we need to find a curve which satisfies

(C1) $\frac{1}{z_5} \cdot \frac{d}{d\tau} w_5$ is both negative and integrable

as well as (A2)–(A4). Given such a curve, one can use our previous approach to obtain a spacetime for which $p_R = \mu$. To define the first part of the curve, we use the equation

$$\sqrt{w_5(x, y)} = \sqrt{w_5(0, 0)} + \frac{Ax}{5} - Ax^2,$$

where $A$ is chosen so that the curve passes through $(1/10, 1/2)$, namely

$$A = 100(\sqrt{40} - 6) > 0.$$ 

We start out at the origin and follow the curve (5.32) until we reach the point $(1/10, 1/2)$, then we follow the curve

$$\sqrt{w_5(x, y)} = \sqrt{w_5(1/10, 1/2)} - \frac{\varepsilon(x - 1/10)^2}{\sqrt{1-x}}, \quad x \geq 1/10,$$

until we hit the $x$-axis, and finally we return to the origin along the $x$-axis. Since this curve is almost identical with that for the isotropic case, our previous approach applies with minor changes; we shall not bother to include the details here.

### 5.5. Dominant energy case

In this case, our assumption that $p_R, p_T \leq \mu$ gives

$$(5x + y - 4)x + 2(1-x)y \leq -\frac{z_5(x, y)}{2}, \quad 2x \geq y - x,$$

with $z_5$ as in (5.30). Since (5.31) remains valid, $w_5$ is decreasing when $z_5 > 0$, so its maximum value is attained in the region $z_5 \leq 0$. To obtain the largest possible value of $w_5$, we need to ensure that $w_5$ is as large as possible in this region. In view of (5.31), this simply means that equality must hold in the first inequality in (5.33). We are thus faced with a situation which is almost identical with (5.23). Arguing as before, we find that the curve must satisfy

$$2x = y - x, \quad (5x + y - 4)x + 2(1-x)y = -\frac{z_5(x, y)}{2},$$

until it exits the region $z_5 \leq 0$. This is the same system of ODEs that we had in (5.24), and the rest of our argument applies almost verbatim. The solution associated with the unstable manifold at the origin intersects the curve $z_5 = 0$ at the point $(0.2746, 0.6180)$, and so

$$w_5(0.2746, 0.6180) \approx 37.924$$

is the largest possible value of $w_5$. Using this fact, we get the upper bound on $x$ which is stated in the theorem. To show that our results for this case are sharp, we follow our approach in the isotropic case with dominant energy. As there are only minor changes that need to be made, we are going to omit the details.
Acknowledgments

We would like to thank Aurélien Decelle to whom we are indebted for both the numerical analysis and the figures which appear in this paper. We would also like to thank Petros Florides for his encouragement and Håkan Andréasson whose recent papers [1–3] have revived interest in this important problem.

References

[1] Andréasson H 2007 Sharp bounds on $2m/r$ of general spherically symmetric static objects Preprint gr-qc/0702137 (to appear in J. Diff. Eqns)
[2] Bondi H 1964 Massive spheres in general relativity Proc. R. Soc. A 282 303–17
[3] Buchdahl H A 1959 General relativistic fluid spheres Phys. Rev. 116 1027–34
[4] Florides P S 1974 A new interior Schwarzschild solution Proc. R. Soc. A 337 529–35
[5] Guven J and O Murchadha N 1999 Bounds on $2m/R$ for static spherical objects Phys. Rev. D 60 084020
[6] Israel W 1958 Discontinuities in spherically symmetric gravitational fields and shells of radiation Proc. R. Soc. A 248 404–14
[7] Lichnerowicz A 1955 Théories Relativistes de la Gravitation et de l’Électromagnétisme. Relativité Générale et Théories Unitaires (Paris: Masson et Cie)
[8] Synge J L 1960 Relativity: The General Theory (Series in Physics) (Amsterdam: North-Holland)