CR REGULAR EMBEDDINGS OF $S^{4n-1}$ IN $\mathbb{C}^{2n+1}$

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Abstract. Ahern and Rudin have given an explicit construction of a totally real embedding of $S^3$ in $\mathbb{C}^3$. As a generalization of their example, we give an explicit example of a CR regular embedding of $S^{4n-1}$ in $\mathbb{C}^{2n+1}$. Consequently, we show that the odd dimensional sphere $S^{2m-1}$ with $m > 1$ admits a CR regular embedding in $\mathbb{C}^{m+1}$ if and only if $m$ is even.

1. Introduction

Suppose $F : M^n \to \mathbb{C}^q$ is a smooth embedding of an $n$-manifold in $\mathbb{C}^q$. Then, for any point $x \in M^n$ and the standard complex structure $J$ on $\mathbb{C}^q$, the following inequality holds:

$$\dim_{\mathbb{C}}(dF_x(T_x M^n) \cap JdF_x(T_x M^n)) \geq n - q.$$ 

If the equality holds for each point $x \in M^n$, the embedding $F$ is called a CR regular embedding, and when $n = q$, we say that $F$ is a totally real embedding and $F(M^n)$ is a totally real submanifold.

Totally real submanifolds have been investigated by many geometers and topologists. Especially, the problem of determining which manifolds admit a totally real embedding has been widely studied from the viewpoint of the $h$-principle (Gromov [6, 7, 8], Lees [13], Forstnerič [5], Elgindi [4]). In [10, Section 5] and [11], the author points (Cartan [3], Tanaka [16, 17], Wells [20, 21], Lai [12], Jacobowitz-Landweber [9], Slapar [14, 15], Torres [18, 19], Audin [2]). On the other hand, Ahern and Rudin [1] have constructed an explicit example of a totally real embedding of $S^3$ in $\mathbb{C}^3$. In the following, let $z = (z_1, z_2, \ldots, z_m)$ be the coordinates on $\mathbb{C}^m$ and we regard $S^{2m-1}$ as the unit sphere in $\mathbb{C}^m$.

Theorem 1.1 (Ahern-Rudin [1]). Let $P(z_1, z_2) = z_2 \overline{z_1} z_2^2 + i z_1^2 \overline{z}_2$. Then, the embedding $F : S^3 \to \mathbb{C}^5$ defined by $F(z_1, z_2) = (z_1, z_2, P(z_1, z_2))$ is a totally real embedding.

CR regular embeddings also have been studied by many authors from various viewpoints (Cartan [3], Tanaka [16, 17], Wells [20, 21], Lai [12], Jacobowitz-Landweber [9], Slapar [14, 15], Torres [18, 19], Elgindi [4]). In [10, Section 5] and [11], the author and Takase have worked on the problem of determining when the $n$-sphere $S^n$ admits a CR regular embedding in $\mathbb{C}^q$ and have given some necessary conditions on $(n, q)$. In particular, we have proved that the $(4n + 1)$-dimensional sphere $S^{4n+1}$ does not admit a CR regular embedding in $\mathbb{C}^{2n+2}$ (Theorem 5.2 (c))). In this paper, we settle the remaining codimension three case by generalizing Theorem 1.1. The following is our main theorem.

Theorem 1.2. Let

$$Q(z_1, z_2, \ldots, z_{2n-1}, z_2) = \sum_{k=1}^{n} P(z_{2k-1}, \overline{z}_2),$$

where $P(x, y) = y \overline{x}^2 + ix^2 \overline{y}$. Then, the embedding $F : S^{4n-1} \to \mathbb{C}^{2n+1}$ defined by

$$F(z_1, z_2, \ldots, z_{2n-1}, z_2) = (z_1, z_2, \ldots, z_{2n-1}, z_2, Q(z_1, z_2, \ldots, z_{2n-1}, z_2))$$

is a CR regular embedding.
Corollary 1.3. Let m be an integer greater than 1. The odd dimensional sphere $S^{2m-1}$ admits a CR regular embedding in $\mathbb{C}^{m+1}$ if and only if m is even.

2. Proof of Main Theorem

For a smooth complex-valued function $f$ on $\mathbb{C}^n$, we use the following notations:

$$\frac{\partial f}{\partial z} = \sum_{j=1}^{m} \frac{\partial f}{\partial z_j} dz_j, \quad \bar{f} = \sum_{j=1}^{m} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$  

Lemma 2.1. Let $u$ and $v$ be the real part and the imaginary part of a smooth function $f : \mathbb{C}^n \to \mathbb{C}$, respectively. Then, $\partial u \wedge \partial v = \frac{1}{2} \partial f \wedge (\bar{\partial} f)$.

Proof. Since $f = u + iv$, we have $\partial f = \partial u + i\partial v$ and $\bar{\partial} f = \bar{\partial} u - i\partial v$. Hence,

$$\partial f \wedge (\bar{\partial} f) = (\partial u + i\partial v) \wedge (\partial u - i\partial v) = -2i\partial u \wedge \partial v.$$  

□

In [9], Jacobowitz and Landweber have given a necessary and sufficient condition for an embedding to be a CR regular embedding.

Proposition 2.2 (Jacobowitz-Landweber [9]). An embedding $F : M^{2n+k} \to \mathbb{C}^{m+k}$ is a CR regular embedding if and only if the submanifold $F(M^{2n+k})$ is given by simultaneous real equations

$$\rho_j(z_1, z_2, \ldots, z_{m+k}) = 0 \quad (j = 1, \ldots, k)$$

satisfying $\partial \rho_1 \wedge \cdots \wedge \partial \rho_k \neq 0$ at each point of $F(M^{2n+k})$.

Applying this proposition to the case where the submanifold is the graph of a function, we obtain the following.

Proposition 2.3. Let $f_j (j = 1, \ldots, q)$ be smooth complex-valued functions on $\mathbb{C}^n$ with $1 \leq q \leq m-1$. The embedding $F : S^{2m-1} \to \mathbb{C}^{m+q}$ defined by

$$F(z) = (z, f_1(z), \ldots, f_q(z))$$

is a CR regular embedding if and only if the $(q+1)$ complex vectors $z \frac{\partial f_j}{\partial z}(z) (j = 1, \ldots, q)$ are linearly independent over $\mathbb{C}$ for each $z \in S^{2m-1}$.

Proof. The submanifold $F(S^{2m-1})$ is described as

$$\{(z, z_{m+1}, \ldots, z_{m+q}) \in \mathbb{C}^{m+q} \mid \|z\|^2 = 1, z_{m+1} = f_1(z), \ldots, z_{m+q} = f_q(z)\}.$$

We define smooth real functions $\rho_1(z), \ldots, \rho_{2q+1}(z)$ by

$$\rho_1 = -1 + \sum_{k=1}^{m} z_k \bar{z}_k, \quad \rho_{2j}(z) = f_j(z) + i\rho_{2j+1}(z) = f_j(z) (j = 1, \ldots, q),$$

for which we have

$$F(S^{2m-1}) = \rho_1^{-1}(0) \cap \rho_2^{-1}(0) \cap \cdots \cap \rho_{2q+1}^{-1}(0).$$

By Lemma[2.1],

$$\partial \rho_{2j} \wedge \partial \rho_{2j+1} = \frac{i}{2} \partial(z_{m+j} - f_j(z)) \wedge \partial(z_{m+j} - f_j(z)) = \frac{i}{2} (\partial f_j) \wedge (dz_{m+j} - \partial f_j).$$

Therefore, $\partial \rho_1 \wedge \partial \rho_2 \wedge \cdots \wedge \partial \rho_{2q+1} \neq 0$ holds if and only if

$$(\bar{z}_1 dz_1 + \cdots + \bar{z}_m dz_m) \wedge \left( \frac{\partial f_1}{\partial z_1} dz_1 + \cdots + \frac{\partial f_1}{\partial z_m} dz_m \right) \wedge \cdots \wedge \left( \frac{\partial f_q}{\partial z_1} dz_1 + \cdots + \frac{\partial f_q}{\partial z_m} dz_m \right) \wedge \left( \frac{\partial f_q}{\partial z_1} dz_1 + \cdots + \frac{\partial f_q}{\partial z_m} dz_m \right) \neq 0$$

holds. This condition is equivalent to the complex vectors

$$z = (z_1, \ldots, z_m), \quad \frac{\partial f_1}{\partial z}, \ldots, \frac{\partial f_q}{\partial z} = (\frac{\partial f_1}{\partial z_1}, \ldots, \frac{\partial f_q}{\partial z_m})$$

being linearly independent over $\mathbb{C}$.

Now, we are ready to prove our main theorem. First we reprove Ahern-Rudin’s result from the viewpoint of Proposition 2.3 and then, prove Theorem 1.2.

**Proof of Theorem 1.2.**

When $(z_1, z_2) \neq (0, 0)$, the two vectors $(z_1, z_2)$ and $\left( \frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2} \right)$ are linearly independent over $\mathbb{C}$. Indeed, the function

$$z_2 \frac{\partial P}{\partial z_1} - z_1 \frac{\partial P}{\partial z_2} = |z_2|^2 (|z_1|^2 - 2|z_1|^2 - 2|z_2|^2)$$

vanishes only at the origin $(0, 0) \in \mathbb{C}^2$. Therefore, by Proposition 2.3, the embedding $F$ is a totally real embedding.

**Proof of Theorem 1.1.** Suppose $z = (z_1, z_2, \ldots, z_{2n-1}, z_{2n}) \neq (0, 0, \ldots, 0, 0)$. Then there exists $j$ such that $(z_{j-1}, z_j) \neq (0, 0)$. For such a $j$, the two vectors $(z_{j-1}, z_j)$ and $\frac{\partial Q}{\partial z} (z_{j-1}, z_j)$ are linearly independent over $\mathbb{C}$ by the proof of Theorem 1.1. Hence, the two vectors $z = (z_1, z_2, \ldots, z_{2n-1}, z_{2n})$ and

$$\frac{\partial Q}{\partial z} (z) = \left( \frac{\partial P}{\partial z} (z_1, z_2), \ldots, \frac{\partial P}{\partial z} (z_{2n-1}, z_{2n}) \right)$$

are linearly independent over $\mathbb{C}$. Then, by Proposition 2.3, the embedding $F$ is CR regular.

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**References**

1. Patrick Ahern and Walter Rudin, *Totally real embeddings of $S^3$ in $\mathbb{C}^3$*, Proc. Amer. Math. Soc. 94 (1985), no. 3, 460–462. MR 787894

2. Michèle Audin, *Fibrés normaux d’immersions en dimension double, points doubles d’immersions lagrangiennes et plongements totalement réels*, Comment. Math. Helv. 63 (1988), no. 4, 593–623.

3. Elie Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de l’espace de deux variables complexes*, Ann. Mat. Pura Appl. 11 (1933), no. 1, 17–90.

4. Ali M. Elgindi, *On the non-existence of CR-regular embeddings of $S^5$*, Complex Var. Elliptic Equ. 64 (2019), no. 9, 1564–1567.

5. Franc Forstnerič, *On totally real embeddings into $\mathbb{C}^n$*, Exposition. Math. 4 (1986), no. 3, 243–255.

6. M. L. Gromov, *A topological technique for the construction of solutions of differential equations and inequalities*, Proceedings ICM (Nice 1970), vol. 2 (1971), 221–225.

7. M. L. Gromov, *Convex integration of differential relations. I*, Izv. Akad. Nauk SSSR Ser. Mat. 37 (1973), 329–343.

8. Mikhail Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 9, Springer-Verlag, Berlin, 1986.

9. Howard Jacobowitz and Peter Landweber, *Manifolds admitting generic immersions into $\mathbb{C}^n$*, Asian J. Math. 11 (2007), no. 1, 151–165.

10. Naohiko Kasuya and Masamichi Takase, *Generic immersions and totally real embeddings*, Internat. J. Math. 29 (2018), no. 11, 1850073.

11. Naohiko Kasuya and Masamichi Takase, *Erratum to “Generic immersions and totally real embeddings”*, Int. J. Math. (2019), online. DOI: 10.1142/S0129167X19920034
[12] Hon Fei Lai, *Characteristic classes of real manifolds immersed in complex manifolds*, Trans. Amer. Math. Soc. **172** (1972), 1–33.

[13] J. Alexander Lees, *On the classification of Lagrange immersions*, Duke Math. J. **43** (1976), no. 2, 217–224.

[14] Marko Slapar, *Cancelling complex points in codimension two*, Bull. Aust. Math. Soc. **88** (2013), no. 1, 64–69.

[15] Marko Slapar, *CR regular embeddings and immersions of compact orientable 4-manifolds into $\mathbb{C}^3$*, Internat. J. Math. **26** (2015), no. 5, 1550033.

[16] Noboru Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of $n$ complex variables*, J. Math. Soc. Japan **14** (1962), 397–429.

[17] Noboru Tanaka, *On generalized graded Lie algebras and geometric structures. I*, J. Math. Soc. Japan **19** (1967), 215–254.

[18] Rafael Torres, *CR regular embeddings and immersions of 6-manifolds into complex 4-space*, Proc. Amer. Math. Soc. **144** (2016), no. 8, 3493–3498.

[19] Rafael Torres, *An equivalence between pseudo-holomorphic embeddings into almost-complex Euclidean space and CR regular embeddings into complex space*, Enseign. Math. **63** (2017), no. 1, 165–180.

[20] R. O. Wells, Jr., *Holomorphic hulls and holomorphic convexity*, Rice Univ. Studies **54** (1968), no. 4, 75–84.

[21] R. O. Wells, Jr., *Compact real submanifolds of a complex manifold with nondegenerate holomorphic tangent bundles*, Math. Ann. **179** (1969), 123–129.

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