Semiclassical computation of quantum effects in multiparticle production at large $\lambda n$

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**ABSTRACT:** We use the semiclassical formalism based on singular solutions in complex time to compute scattering rates for multiparticle production at high energies. In a weakly coupled $\lambda \phi^4$ scalar field theory in four dimensions, we consider scattering processes where the number of particles $n$ in the final state approaches its maximal value $n \to E/m \gg 1$, where $m$ is the particle mass. Quantum corrections to the known tree-level amplitudes in this regime are characterised by the parameter $\lambda n$ and we show that they become large at sufficiently high multiplicities. We compute full amplitudes in the large $\lambda n$ limit on multiparticle mass thresholds using the thin-wall realisation of the singular solutions in the WKB approach. We show that the scalar theory with spontaneous symmetry breaking, used here as a simplified model for the Higgs sector, leads to exponentially growing multiparticle rates within our regime which is likely to realise the high-energy Higgsplosion phenomenon. We also comment on realisation of Higgsplosion in dimensions lower than four.
1 Introduction

The aim of this paper is to present and explain the semiclassical calculation of few \( \rightarrow \) \( n \) particle processes in the limit of ultra-high particle multiplicities \( n \). The underlying semiclassical formalism, was originally developed by Son in Ref. [1], while a first version of the calculation was presented in my earlier paper [2]. The present paper seeks to provide a more detailed justification of the main result and its derivation.

We are interested in \( 1^* \rightarrow n \) decay rates where \( 1^* \) is a virtual state created by a local operator \( O(x) \) at a point \( x = 0 \). In high-energy scattering processes the highly virtual states \( 1^* \) with \( Q^2 = s \) would correspond to the s-channel resonances created by the two incoming colliding particles. For example in the gluon fusion process, \( gg \rightarrow h^* \rightarrow n \times h \) the highly virtual Higgs boson \( h^* \) is created by the two initial gluons before decaying into \( n \) Higgs bosons in the final state. The \( 1^* \rightarrow n \) decay rates we are interested in, correspond in this example to the \( h^* \rightarrow n \times h \) part of the process.

As this paper is about proving a technical point by providing a non-perturbative calculation of the \( n \)-particle decay rates, we leave the discussion and interpretations of the resulting rates, which will turn out to be unsuppressed in the model we are considering, to other papers and future work. The calculation that we present is aimed towards developing a theoretical foundation for the phenomenon of Higgsplosion proposed in [3] and further investigated in the recent papers [4–7].
As in Refs. [2, 3] we are interested in the scalar sector of the theory which for simplicity we will take to be a quantum field theory of a single real degree of freedom \( h(x) \) described by the Lagrangian,

\[
\mathcal{L} = \frac{1}{2} \partial^\mu h \partial_\mu h - \frac{\lambda}{4} (h^2 - v^2)^2.
\] (1.1)

The theory has a non-zero vacuum expectation value \( \langle h \rangle = v \) which breaks spontaneously the \( \mathbb{Z}_2 \) symmetry, and gives the mass \( m = \sqrt{2\lambda} v \) to the elementary scalar particle described by the shifted field,

\[
\phi(x) = h(x) - v.
\] (1.2)

This model can be viewed as a reduction of the SM Higgs sector in the unitary gauge to a single scalar field. In this simplified model the scalar boson is all there is, and since all other SM-like degrees of freedom (vector bosons and fermions) are decoupled, the scalar \( h(x) \) is stable.

Our goal is to compute the multi-boson production rate in the large \( \lambda n \) limit, where \( \lambda \) is the coupling constant and \( n \) is the particle number in the final state. On the technical side, the idea which makes this calculation possible, is to combine the semiclassical formalism developed by Son in Ref. [1] based on singular classical solutions with the idea [2] to search for these solutions in the form of thin-walled singular bubbles. The thin-wall approximation has been already adopted to multiparticle production processes earlier in Ref. [8] in the case of standard non-singular smooth bubble configurations as in the false vacuum decay. We will instead tie the appearance of the semiclassical configurations with singular thin-wall surfaces to the requirements of the semiclassical approach Ref. [1].

In the scattering processes at very high energies, production of large numbers of particles in the final state becomes possible. These processes were studied in some detail in the literature and we refer the reader to papers [8–22] and references therein.

This paper is organised as follows. In section 2 we briefly recall the known results for the multiparticle scattering rates obtained in perturbation theory at tree-level, before proceeding with the non-perturbative calculation in the main body of the paper. In section 3 we will summarise the semiclassical approach of Son as a series of steps needed to identify the saddle-point solution in Minkowski space. In section 4, still following [1], we simplify and refine this prescription as the extremization over singular surfaces approach in complex time. The resulting set-up is ideal for using the thin-wall approach which we develop is sections 5 and 6. In particular, in section 5 we will recover tree-level results familiar from section 2 along with the prescription for computing the quantum corrections. These quantum contributions to the multi-article rates are computed in section 6 using the thin-walled singular classical solutions. In section 7 we consider multiparticle processes in 3 dimensions and provide a successful test for the semiclassical results. Finally, we present our conclusions in section 8.
2 Simple classical solutions and tree-level amplitudes at threshold

The purpose of this paper is to compute the amplitudes and the corresponding probabilistic rates for processes involving multiparticle final states in the large $\lambda n$ limit non-perturbatively – i.e. using a semiclassical approach with no reference to perturbation theory and without artificially separating the result into a tree-level and a ‘quantum corrections’ contributions. Their entire combined contribution should emerge from the unified semiclassical algorithm. But to first set the scene for such a computation we need to recall the known properties of the tree-level amplitudes and their relation with certain classical solutions. This is the aim of this section.

Thus, we start here with tree-level $n$-point scattering amplitudes computed on the $n$-particle mass thresholds. This is the kinematics regime where all $n$ final state particles are produced at rest. These amplitudes for all $n$ are conveniently assembled into a single object – the amplitude generating function – which at tree-level is described by a particular solution of the Euler-Lagrange equations. The classical solution which provides the generating function of tree-level amplitudes on multi-particle mass thresholds in the model (1.1) is given by [11],

$$h_0(z_0; t) = v \left( \frac{1 + z_0 e^{int} / (2v)}{1 - z_0 e^{int} / (2v)} \right)^{\frac{1}{2}} - \frac{1}{2} z_0 e^{int},$$  

and where $z_0$ is an auxiliary variable. It is easy to check with the direct substitution that the expression in (2.1) does indeed satisfy the Euler-Lagrange equation resulting from our theory Lagrangian (1.1) for any value of the $z_0$ parameter. It then follows that all $1^* \rightarrow n$ tree-level scattering amplitudes on the $n$-particle mass thresholds are given by the differentiation of $h_0(z_0; t)$ with respect to $z_0$,

$$A_{1\rightarrow n} = \langle n | S \phi(0) | 0 \rangle = \left( \frac{\partial}{\partial z_0} \right)^n h_0 \bigg|_{z_0=0}$$  

The classical solution in (2.1) is uniquely specified by requiring that it is a holomorphic function of the complex variable $z(t) = z_0 e^{int}$,

$$h_0(z) = v + 2v \sum_{n=1}^{\infty} \left( \frac{z}{2v} \right)^n, \quad z = z(t) = z_0 e^{int},$$  

so that the amplitudes in (2.2) are given by the coefficients of the Taylor expansion in (2.3) times $n!$ from differentiating $n$ times over $z$,

$$A_{1\rightarrow n} = \left( \frac{\partial}{\partial z} \right)^n h_0(z) \bigg|_{z=0} = n! \left( \frac{1}{2v} \right)^{n-1} = n! \left( \frac{\lambda}{2m^2} \right)^{n-1}.$$  

These formulae and the characteristic factorial growth of $n$-particle amplitudes, $A_n \sim \lambda^{n/2} n!$, form the essence of the elegant formalism pioneered by Brown in Ref. [11] that is based on solving classical equations of motion and bypasses the summation over individual Feynman diagrams. In the following sections we will see how these (and also more general
In terms of the Wick rotated time variable it is defined as

\[ t = -i(-t) \rightarrow \tau \]

In this limit the classical solution approaches the vacuum implies that the early time \( t \rightarrow -t \) maps to the contour in the complex plane \( t_C \) as in Fig. 1. This corresponds to the \((-t) \times e^{i\pi} = \tau\) rotation, at early times, \(-\infty < t < 0\):

\[ t \rightarrow i\tau \]  \hspace{1cm} \text{(2.6)}

We also note that \( \tau \) corresponds to minus the Euclidean time \( t_{\text{Eucl}} \) defined by the standard Wick rotation via \( t \rightarrow -it_{\text{Eucl}} \).

Expressed as the function of the complexified time variable \( t_C \), the classical solution

\[ t \rightarrow t_C = t + i\tau, \]  \hspace{1cm} \text{(2.5)}

where \( t \) and \( \tau \) are real valued. We will use the deformation the time-evolution contour from the real time axis \(-\infty < t < +\infty\) to the contour in the complex \( t_C \) plane depicted in Fig. 1 in such a way that the initial time \( t = -\infty \) maps on the imaginary time \( \text{Im} t_C = \tau = +\infty \).
Figure 2: Singular classical solution (2.9) uniform in space: flat domain wall located at $\tau_\infty$ in the imaginary time.

(2.1) reads,

$$h_0(t_\text{cl}) = v \left( \frac{1 + e^{im(t_\text{cl}-i\tau_\infty)}}{1 - e^{im(t_\text{cl}-i\tau_\infty)}} \right), \quad (2.7)$$

where $\tau_\infty$, a constant,

$$\tau_\infty := \frac{1}{m} \log \left( \frac{z_0}{2v} \right) \quad (2.8)$$

it parameterises the location (or the centre) of the solution in imaginary time. If the time-evolution contour of the solution in the $t_\text{cl}$ plane is along the the imaginary time with the real time $t = 0$, the field configuration (2.7) becomes real-valued,

$$h_0(\tau) = v \left( \frac{1 + e^{-m(\tau-\tau_\infty)}}{1 - e^{-m(\tau-\tau_\infty)}} \right), \quad (2.9)$$

and singular at $\tau = \tau_\infty$.

Having already noted that the solution is complex-valued we note another important feature of the solution (2.3) that is for the forthcoming semiclassical analysis, namely that the configuration $h_0$ is singular in imaginary time, in particular at $\tau = \tau_\infty$ when $t = 0$.

The expression on the right hand side of (2.9) has an obvious interpretation in terms of a singular domain wall located at $\tau = \tau_\infty$ that separates two domains of the field $h(\tau, \vec{x})$ as shown in Fig. 2. The domain on the right of the wall $\tau \gg \tau_\infty$ has $h = +v$, and the domain on the left of the wall, $\tau \ll \tau_\infty$, is characterised by $h = -v$. The field configuration is singular at the position of the wall, $\tau = \tau_\infty$, for all values of $\vec{x}$, i.e. the singularity surface is flat (or uniform in space). The thickness of the wall is set by the inverse mass $1/m$.

The field configuration (2.9) can be used to compute the surface tension of the domain wall. The surface tension is defined as the Euclidean action computed on (2.9) per unit area of the 3-dimensional surface $\tau_0(\vec{x}) = \tau_\infty$. Since the $\tau_\infty$ surface is uniform in space, the surface tension is given by the integral,

$$\mu = \int_{-\infty+\epsilon}^{+\infty+\epsilon} d\tau \left( \frac{1}{2} \left( \frac{dh}{d\tau} \right)^2 + \frac{\lambda}{4} \left( h^2 - v^2 \right)^2 \right) = \frac{m^3}{3\lambda}, \quad (2.10)$$
This integral is finite for the contour along $\tau$ shifted by $\pm i\epsilon$; the rational for this procedure will be explained in section 6 cf. Eq. (6.20).

In the following section we will summarise the results of the semiclassical formalism for computing probability rates of $1^* \rightarrow n$ processes in which the complex-valued singular configurations of the type (2.7) appear naturally as the solution of the boundary value problem.

3 The semiclassical formalism of Son

Motivated in part by the Landau formulation of the WKB approach in the non-relativistic quantum mechanics [23, 24], D. T. Son developed in Ref. [1] a semiclassical formalism for computing multi-particle cross-sections in a quantum field theory. It relies on functional integrals in the coherent state representation to specify the initial and final states as the boundary conditions at early and late times. The functional integrals are then evaluated using the steepest descent method with the dominant field configurations and other relevant parameters taking in general complex values. The complex-valued saddle points (local minima in our model) and the presence of singularities in the solutions of the boundary value problem are the essential characteristics of the Landau-WKB and the Son’s approach in quantum field theory.\footnote{Earlier work on generalisations of the Landau-WKB formalism to problems with many degrees of freedom includes Refs. [25–28] and in section 4 of the review [16].}

In this section we will list the main steps that specify the steepest descent solution of the boundary value problem in the formalism of Son. These steps follow directly from the construction in [1], and for the convenience of the reader in the Appendix A we provide additional comments on the algorithm. No prior familiarity with the formalism in [1] is required to follow the algorithm for finding the solution, however a pedagogical overview of Ref. [1] is beyond the scope of this paper; this task is postponed to a separate work [29].

The central quantity is the dimensionless probability rate $R_n(E)$ for a local operator $O(x)$ at a point $x = 0$ to create $n$ particles of total energy $E$ from the vacuum. It is given by [1],

$$ R_n(E) = \int d\Phi_n \langle 0| O^\dagger P_E|n\rangle \langle n| P_E S O |0\rangle, \quad (3.1) $$

where the matrix element involves the operator $O$ between the vacuum state $|0\rangle$ and the $n$-particle state of fixed energy $\langle n| P_E$ (here $P_E$ is the projection operator on states with fixed energy $E$), along with the $S$ matrix to evolve between the initial and final times. The matrix element is squared and integrated over the $n$-particle Lorentz-invariant phase space. The local operator $O$ appearing in the matrix elements in (3.1) is conventionally [1] in the form

$$ O = j^{-1} e^{j(h(0)-v)} = j^{-1} e^{j\phi(0)}, \quad (3.2) $$

where $j$ is a constant, and the limit $j \rightarrow 0$ is taken in the computation of the probability rates (3.1) to select the single particle initial state $|\phi(0)\rangle$.

The cross-sections for few to many particles, $\sigma_{\text{few} \rightarrow n}(E)$ as well as multi-particle partial decay rates $\Gamma_{\text{n}}(E)$ of a single particle state $X \rightarrow n \times h$, are determined by the exponential
factor for $R_n(E)$ in (3.1) times a non-exponential prefactor of appropriate dimensionality which is of no interest in a semiclassical approximation.

In the construction of [1] the expression on the right hand side of (3.1) is represented as a functional integral, which is subsequently computed in the steepest descent approximation for all integration variables. The steepest descent method relies on having a single large parameter in front of all terms in the exponent. This parameter is the inverse coupling constant $1/\lambda \gg 1$ in the weak-coupling limit of the theory. The final state particle number $n = \lambda n/\lambda$ is $\sim 1/\lambda$ for $\lambda n = \text{fixed}$. Thus the steepest descent method is justified in the double-scaling weak-coupling and large-$n$ semiclassical limit:

$$\lambda \to 0, \quad n \to \infty, \quad \text{with} \quad \lambda n = \text{fixed}, \quad \varepsilon = \text{fixed}.$$  

(3.3)

Here $\varepsilon$ denotes the average kinetic energy per particle per mass in the final state,

$$\varepsilon = (E - nm)/(nm).$$  

(3.4)

Holding $\varepsilon$ fixed implies that in the large-$n$ limit we are raising the total energy linearly with $n$. The semiclassical result for the rate has the characteristic exponential form [1],

$$R_n(E) \simeq \exp [W(E, n)],$$  

(3.5)

where

$$W(E, n) \equiv \frac{1}{\lambda} \mathcal{F}(\lambda n, \varepsilon) = ET - n\theta - 2\text{Im}[S].$$  

(3.6)

$S$ is the action on the complex-valued field solution and $T$ and $\theta$ are the auxiliary parameters that will be specified momentarily.

The algorithm [1] to find the saddle-point configuration on which to compute the semiclassical rate $R_n(E)$ is as follows:

1. Solve the classical equation without the source-term,

$$\frac{\delta S}{\delta h(x)} = 0,$$  

(3.7)

by finding a complex-valued solution $h(x)$ with a point-like singularity at the origin $x^\mu = 0$ and regular everywhere else in Minkowski space. The singularity at the origin is selected by the location of the operator $\mathcal{O}(x = 0)$.

2. Impose the initial and final-time boundary conditions,

$$\lim_{t \to -\infty} h(x) = v + \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} a_k^\dagger e^{ik_\mu x^\mu},$$  

(3.8)

$$\lim_{t \to +\infty} h(x) = v + \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left( b_k e^{i\omega_k T - \theta e^{-ik_\mu x^\mu}} + b_k^\dagger e^{ik_\mu x^\mu} \right).$$  

(3.9)
3. Compute the energy and the particle number using the $t \to +\infty$ asymptotics of $h(x)$,

$$
E = \int d^3k \, \omega_k b_k^\dagger b_k e^{\omega_k T - \theta}, \quad n = \int d^3k \, b_k^\dagger b_k e^{\omega_k T - \theta}.
$$

(3.10)

At $t \to -\infty$ the energy and the particle number are vanishing. The energy is conserved by regular solutions and changes discontinuously from 0 to $E$ at the singularity at $t = 0$.

4. Eliminate the $T$ and $\theta$ parameters in favour of $E$ and $n$ using the expressions above. Finally, compute the function $W(E, n)$

$$
W(E, n) = ET - n\theta - 2\text{Im}S[h]
$$

(3.11)

on the set $\{h(x), T, \theta\}$ and compute the semiclassical rate $\mathcal{R}_n(E) = \exp[W(E, n)]$.

To implement this programme one starts with the specified expressions (3.8) and (3.9) for $h(x)$ at the $t \to \pm\infty$ boundaries and classically evolves them by solving the equation of motion into the region of finite $t$. We thus have two trial functions, one at $t < 0$ and the second at $t > 0$ which we would like to match at $t = 0$. The field configuration at $t < 0$ is given by a regular classical solution $h_1(t, \vec{x})$ which satisfies the initial time boundary condition with the Fourier coefficient functions $a_{k}^\dagger$. The second trial function, $h_2(t, \vec{x})$, is a regular classical solution on the Minkowski half-plane $t > 0$ which is evolved from the final-time boundary condition with the coefficient functions $b_{k} e^{\omega_k T - \theta}$ and $b_{k}^\dagger$. One then contemplates scanning over the space of the functions $a_{k}$ and $b_{k}$ to achieve the matching at $t = 0$ between the two branches $h_1$ and $h_2$ of the solution, $h_1(\vec{x}) = h_2(\vec{x})$, and all of its time derivatives for all values of $\vec{x} \neq 0$. The only allowed singularity of the full solution is point-like, and located at the origin $t = 0 = \vec{x}$.

A practical difficulty in implementing the matching between $h_1$ and $h_2$ is that $h_1(x)$ should be equal to $h_2(x)$ on the entire hyperplane $(t = 0, \vec{x})$ with the exception of the single point $t = 0 = \vec{x}$. This technical difficulty can be bypassed following [1], by analytically continuing to complex time as we will explain in the following section.

4 Refining the method in complex time $t_C$

In Minkowski space-time $x^\mu = (t, \vec{x})$ the desired solution $h(x)$ should contain a point-like singularity at the origin $x = 0$, and be regular everywhere else. In the Euclidean space-time, $(\tau, \vec{x})$, however, such a solution will in general be singular on a 3-dimensional hypersurface $\tau = \tau_0(\vec{x})$ located at $t = 0$.

To illustrate this point consider the already familiar from section 2 classical solution (2.7). We now modify this field configuration by replacing the collective coordinate parameter $\tau_\infty$ by a function $\tau_0(\vec{x})$ that is no longer uniform in space, but interpolates between 0 at $\vec{x} = 0$ and a constant $\tau_\infty$ at $|\vec{x}| \to \infty$. The configuration

$$
h_0(t_C; \vec{x}) = v \left( \frac{1 + e^{im(t_C - i\tau_0(\vec{x}))}}{1 - e^{im(t_C - i\tau_0(\vec{x}))}} \right),
$$

(4.1)
deviates from an exact solution of equations of motion by terms involving derivatives of
\( \tau_0(\vec{x}) \) and requires additional corrections on the right hand side, but for a slowly varying
\( \tau_0 \) it is a good trial function to expand around and use in a variational principle. It
then immediately follows that in Minkowski spacetime where \( t_C = t \) is real, the field
configuration (4.1) is singular at the point \( t = 0 = \vec{x} \), while in complex time, it is singular
on the surface located at \( t_C = 0 + i\tau_0(\vec{x}) \) spanned by the 3-dimensional variable \( \vec{x} \).

We now describe the extremization procedure for finding the solution to the boundary
value problem in complexified time \( t_C = t + i\tau \), following [1]:

1. Select a trial singularity surface located at \( \tau = \tau_0(\vec{x}) \). The surface profile \( \tau_0(\vec{x}) \) is an
\( O(3) \) symmetric function of \( \vec{x} \) and is given by a local deformation of the flat singularity
domain wall at \( \tau_\infty \) with the single maximum touching the origin \( (\tau, \vec{x}) = 0 \) as shown
in Fig. 3 (a). In Minkowski space the singularity is point-like at \( t = 0 = \tau \) and \( \vec{x} = 0 \)
as required.

2. Deform the time evolution contour specifying the paths in the Feynman path integral
to follow the contour on the complex plane \((t, \tau)\),

\[
(0, \infty) \rightarrow (0, \tau_0(\vec{x})) \oplus [(0, \tau_0(\vec{x})) \rightarrow (0, 0)] \oplus [(0, 0) \rightarrow (\infty, 0)]
\]  (3.1)
as shown in Figs. 3 (b) and 1 (a). More precisely, in order to be able to linearise
the late time asymptotics of the solution, as in (4.5) below, we should make the final
third segment of the contour in (4.2) to have an infinitesimal positive angle w.r.t. the
real time axis, i.e. \( t(1 + \delta) \) for \( 0 \leq t < +\infty \) with \( \delta = 0_+ \).

3. Find a classical trajectory \( h_1(\tau, \vec{x}) \) on the first segment, \( +\infty > \tau > \tau_0(\vec{x}) \), of the
contour (4.2) that satisfies the initial time (vanishing) boundary condition (3.8),

\[
\lim_{\tau \to +\infty} h_1(\tau, \vec{x}) - v \to 0
\]  (3.3)
and becomes singular as \( \tau \to \tau_0(\vec{x}) \) so that\(^2 h_1(\tau, \vec{x})|_{\tau=\tau_0(\vec{x})} = \Phi_0(\vec{x}) \to \infty.\)

4. Find another classical solution \( h_2(\tau, \vec{x}) \) on the remaining part of the contour \((3.8),\) that at \( \tau \to \tau_0(\vec{x}) \) is singular and matches with \( h_1,\)

\[
h_2(\tau_0, \vec{x}) = h_1(\tau_0, \vec{x}) = \Phi_0(\vec{x}) \to \infty, \tag{4.4}
\]

and also satisfies the final time boundary condition \((3.9),\)

\[
\lim_{t \to +\infty} h_2(t, \vec{x}) - v = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left( b_k e^{i\omega_k T - \theta} e^{-ik\mu x^\mu} + b_k^\dagger e^{i\omega_k T + \theta} e^{ik\mu x^\mu} \right). \tag{4.5}
\]

The two functions \( h_1(\tau, \vec{x}) \) and \( h_2(\tau, \vec{x}) \) can be viewed as the two branches of a trial configuration \( h(x). \) The action of \( h(x) \) along our complex-time contour is the sum of the action integrals\(^3\) of \( h_1(\tau, \vec{x}) \) and \( h_2(t, \vec{x}) \) on the parts of the contour,

\[
iS[h] = \int d^3x \left( \int_{+\infty}^{\tau_0(\vec{x})} d\tau \mathcal{L}_{\text{Eucl}}(h_1) + \int_0^{\tau_0(\vec{x})} d\tau \mathcal{L}_{\text{Eucl}}(h_2) + i \int_0^\infty dt \mathcal{L}(h_2) \right) \tag{4.6}
\]

5. Up to this point we have not imposed the matching conditions on the derivatives of \( h_1(\tau, \vec{x}) \) and \( h_2(\tau, \vec{x}) \) at the singularity surface at \( \tau_0. \) A priori, the normal derivatives to the surface will be different, \( \partial_n(h_1 - h_2)|_{\tau=\tau_0(\vec{x})} \neq 0,\) and the equation of motion \((3.7)\) will not be satisfied at the matching surface \( \tau_0(\vec{x}).\) For the combined configuration \( h(x) \) to solve the classical equation \((3.7)\) everywhere, including the \( \tau_0 \) surface, one simply needs to extremize the action integral \((4.6)\) over all singularity surfaces \( \tau = \tau_0(\vec{x}) \) containing the point \( t = 0 = \vec{x}.\)

6. Finally, determine the semiclassical rate by evaluating

\[
W(E, n) = ET - n\theta - 2\text{Im}S[h] \tag{4.7}
\]

on the extremum, using \((4.6)\) for the action, and expressions for \( T \) and \( \theta \) in terms of \( E \) and \( n \) found from \((3.10)\) as before. The imaginary part of the Minkowski action in \((4.6),\) \((4.7)\) is the same as the real part of the Euclidean action, \( \text{iS} := -S_{\text{Eucl}}.\)

This is the general outcome of the semiclassical construction of Ref. [1]. One starts with the two individual solutions satisfying the boundary conditions \((4.3)-(4.5)\) and then varies over the profiles of the singular matching surface \( \tau_0(\vec{x}) \) to find an extremum of the imaginary part of the action \((4.6). \) On the extremal surface not only the field configurations, but also their normal derivatives match \( \partial_n(h_1 - h_2) = 0 \) at all \( \vec{x} \) except \( \vec{x} = 0. \) This implies that \( h_1 = h_2 \) on the entire slice of the spacetime where they are both defined, i.e. for \( \tau \) in the interval \([0, \tau_0],\) except at the point at the origin. Restricting to the Minkowski space slice, i.e. at \( \tau = 0, \) this implies \( h_1(0, \vec{x}) = h_2(0, \vec{x}), \) as it should be. It does not mean however

\(^2\)One can always assume a regularisation procedure that keeps \( \Phi_0 \) finite at intermediate stages of the calculation, i.e. before taking the limit of the operator source \( j \to 0.\)

\(^3\)As usual, \( \mathcal{L}_{\text{Eucl}}[h] := \text{kinetic} + \text{potential} = \frac{1}{2} (\partial_\nu h)^2 + \frac{1}{2} (h^2 - v^2)^2.\)
that the real part of the action in (4.6) vanishes, as the sum of the first two integrals can be viewed as encircling the singularity of the solution at $\tau_0$.

In summary, the highly non-trivial problem of searching for the appropriate singular field solutions $h(x)$ is reduced to a geometrical problem – extremization over the surface shapes $\tau_0(\vec{x})$ and accounting for the appropriate boundary conditions (4.3)-(4.5). This formulation of the problem is now well-suited for using the thin-wall approximation that will be described in section 6 and will allow us to address the previously unexplored in [1] regime at large values of $\lambda n$ where quantum non-perturbative effects are large.

We proceed with the practical implementation of the steps 1.-6. for the model (1.1) in the following two sections.

5 Computing the rate: setting the scene

In this section we will specify and solve the boundary conditions at the initial and final times deriving the coefficient functions in (4.3), (4.5), will determine the $T$ and $\theta$ parameters and compute the general expression for the exponent of the rate $W(E,n)$ in (3.11). This is the last section where we are still following Son (specifically section 4 of Ref. [1]) before we move on to the thin-wall bubble analysis of the expression for $W(E,n)$ in section 6.

In the limit $\varepsilon = 0$, the scattering amplitude is on the multiparticle threshold, the final state momenta are vanishing and one would naively assume that the classical solution describing this limit is uniform in space. This is correct for the tree-level solution but not for the solution incorporating quantum effects. In the latter case, the correct is the less restrictive assumption that the presence of the singularity at $x = 0$ deforms the flat surface of singularities near its location, as shown in Fig. 3. From now on we will concentrate on the physical case where $\varepsilon$ is non-vanishing and non-relativistic, $0 < \varepsilon \ll 1$. At the same time, the parameter $\lambda n$ is held fixed and arbitrary. It will ultimately be taken to be large.

The initial-time boundary condition (4.3) dictates that the solution $h_1(t_C = i\tau, \vec{x}) - v$ must vanish with exponential accuracy as $e^{-m\tau}$ in the limit $\tau \to \infty$. This

Next we investigate the final-time boundary condition (4.5) of the finite-energy solution $h_2(x)$. This solution should be singular on the singularity surface $\tau_0(\vec{x})$. Following Son, without loss of generality, we can search for $h_2$ near the final-time asymptotics in the form,

$$h_2(t_C, \vec{x}) = v \left(1 + \frac{e^{im(t_C - i\tau_\infty)}}{1 - e^{im(t_C - i\tau_\infty)}}\right) + \tilde{\phi}(t_C, \vec{x}).$$  (5.1)

The first term on the right hand side is the $\vec{x}$-independent field configuration. It is an exact classical solution (2.7) with the surface of singularities at $\tau = \tau_\infty$ which is a plane. The second term, $\tilde{\phi}(t_C, \vec{x})$, describes the deviation of the singular surface from the $\tau_\infty$-plane. It is non-trivial locally around $\vec{x} = 0$ and vanishes at $\vec{x} \to \infty$.

On the final segment of the time evolution contour $t(1 + i\delta)$ as $t \to +\infty$, the first term in (5.1) can be Taylor-expanded in powers of $e^{im(1+i\delta)}$ and linearised thanks to $\delta$ being positive, giving,

$$\lim_{t \to +\infty} h_0(x) - v = 2v e^{m\tau_\infty} e^{int},$$  (5.2)
while for the second term in (5.1) we will write the general expression involving the positive-frequency and the negative frequency components. Taking the 3-dimensional Fourier transform, \( \tilde{\phi}(t, k) = \int \frac{dk}{(2\pi)^3/2} \hat{\phi}(t, x) e^{-ikx} \), we have,

\[
\lim_{t \to +\infty} \tilde{\phi}(t, k) = \frac{1}{\sqrt{2\omega_k}} \left( f_k e^{-i\omega_k t} + g_{-k} e^{i\omega_k t} \right).
\]

(5.3)

Combining this with the Fourier transform of (5.2) we get the full solution in (5.1) in the form,

\[
\lim_{t \to +\infty} h_2(t, k) - v = \frac{1}{\sqrt{2\omega_k}} \left( f_k e^{-i\omega_k t} + \left\{ g_{-k} + 2v e^{i\pi\tau} (2\pi)^{3/2} \delta^{(3)}(k) \right\} e^{i\omega_k t} \right).
\]

(5.4)

Comparing with the the final-time boundary condition (4.5) we read off the expressions for the coefficient functions,

\[
b_k e^{i\omega_k T - \theta} = f_k
\]

(5.5)

\[
b_k^\dagger = g_k + 2v e^{i\pi\tau} (2\pi)^{3/2} \delta^{(3)}(k).
\]

(5.6)

We will now make an educated guess that the parameter \( T \) will be infinite in the limit \( \varepsilon \to 0 \). In fact we will soon derive that \( T = 3/(2m\varepsilon) \), so this assumption will be justified a posteriori. We can then re-write (5.5) as

\[
b_k = f_0 e^{-\omega_k T} e^{\theta}
\]

(5.7)

Essentially the factor \( e^{-\omega_k T} \) is thought as regularisation of the momentum-space delta-function by \( \varepsilon \). In the limit where \( \varepsilon \to 0 \), the factor is proportional to \( e^{-\omega_k T} \) as it cuts-off all non-vanishing values of \( k \) and reduces \( k \) to zero. Hence we set \( f_k \) to \( f_0 \) in the equation above. Furthermore, since the function \( b_k \) is proportional to the (regularised) delta-function, so must be its complex conjugate, \( b_k^\dagger \). This implies that the coefficient function \( g_k \) in (5.6) must be zero \([1]\), so that (5.3) becomes,

\[
\lim_{t \to +\infty} \tilde{\phi}(t, k) = \frac{1}{\sqrt{2\omega_k}} f_k e^{-i\omega_k t}.
\]

(5.8)

We have obtained the expression for the coefficient function \( b_k \) (and its complex conjugate) and also obtained a symbolic identity involving the parameters \( T \) and \( \theta \) and the delta-function,

\[
b_k = f_0 e^{-\omega_k T} e^{\theta} = 2v e^{i\pi\tau} (2\pi)^{3/2} \delta^{(3)}(k) = b_k^\dagger.
\]

(5.9)

This symbolic identity should be interpreted as follows. In the limit of strictly vanishing \( \varepsilon \), all these terms are proportional to the delta-function. Away from this limit, i.e. in the case of processes near the multiparticle threshold where \( 0 < \varepsilon \ll 1 \) the function \( \delta^{(3)}(k) \) appearing in the third term above, is not the strict delta-function, but a narrow peak with the singularity regulated by \( \varepsilon \). This can be derived by allowing the surface \( \tau_\infty \) in the first term in (5.1) to be not completely flat at small non-vanishing \( \varepsilon \), but to have a tiny curvature \( 2\varepsilon/3 \ll 1 \) \([1]\), thus leading to a regularised expression for \( \delta^{(3)}(k) \) in the final term in (5.4).
To proceed, we integrate the two middle terms in (5.9) over $d^3k$ to find at large $T$,

$$ f_0 = \frac{2\sqrt{2m}}{m^3} (mT)^{3/2} e^{mT - \theta + m\tau_\infty}. \quad (5.10) $$

We can now compute the particle number $n$ and the energy $E$ in the final state using equations (3.10) and the now known coefficient functions (5.9) along with (5.10). We find,

$$ n = \int d^3k \; b_k^\dagger b_k e^{\omega_k T - \theta} = \frac{8\nu^2}{m^2} (2\pi mT)^{3/2} e^{mT - \theta + 2m\tau_\infty} \quad (5.11) $$

$$ mn\varepsilon = E - mn = \int d^3k \; \frac{k^2}{2} b_k^\dagger b_k e^{\omega_k T - \theta} = \frac{8\nu^2}{m^2} (2\pi mT)^{3/2} e^{mT - \theta + 2m\tau_\infty} \frac{3}{2T} \quad (5.12) $$

Dividing the second expression by the first we find,

$$ T = \frac{1}{m} \frac{3}{2} \frac{1}{\varepsilon}, \quad (5.13) $$

and the second parameter $\theta$ is found to be,

$$ \theta = - \log \frac{\lambda n}{4} + \frac{3}{2} \log \frac{3\pi}{\varepsilon} + 2m\tau_\infty + \frac{3}{2} \frac{1}{\varepsilon}. \quad (5.14) $$

We now finally substitute these parameters into the equation (4.7) for the ‘holy grail’ function $W(E, n)$, and find,

$$ W(E, n) = ET - n\theta - 2\text{Re}S_{\text{Eucl}}[h] = mn(1 + \varepsilon)T - n\theta - 2\text{Re}S_{\text{Eucl}}[h] $$

$$ = n \log \frac{\lambda n}{4} + \frac{3n}{2} \left( \log \frac{\varepsilon}{3\pi} + 1 \right) - 2nm\tau_\infty - 2\text{Re}S_{\text{Eucl}}[h]. \quad (5.15) $$

Before interpreting this expression, we would like to separate the terms appearing on the right hand side into those that depend on the location and shape of the singularity surface $\tau_0(\vec{x})$, and those that do not. The first two terms in (5.15) have no dependence on the singularity surface; the third term, $2nm\tau_\infty$, depends on its location at $\tau_\infty$. The final term, $2\text{Re}S_{\text{Eucl}}$, is obtained by taking the real part of the three integrals appearing in (4.6).

The first two integrals are along the Euclidean time $\tau$ segments of the contour and are real-valued

$$ 2\text{Re}S_{\text{Eucl}}^{(1,2)} = 2 \int d^3x \left[ - \int_{\tau_0(\vec{x})}^{\infty} d\tau L_{\text{Eucl}}(h_1) - \int_{\tau_0(\vec{x})}^{0} d\tau L_{\text{Eucl}}(h_2) \right], \quad (5.16) $$

while the remaining integral along the third segment of the contour appears to be purely imaginary. This last statement is almost correct, as it applies to the bulk contribution of the Minkowski-time integral $\int_0^\infty dt L(h_2)$, but not to the boundary contribution at $t \to \infty$. The full contribution from the third segment of the contour is,\textsuperscript{4}

$$ 2\text{Re}S_{\text{Eucl}}^{(3)} = 2 \int d^3x \left[ - i \int_0^\infty dt L(h_2) \right] = - \int d^3k \; b_k^\dagger b_k e^{\omega_k T - \theta} = - n. \quad (5.17) $$

\textsuperscript{4}The expression (5.17) for the boundary contribution to the Minkowski action is also in agreement with the construction in [1] and [16].
Accounting for the effect of the boundary contribution (5.17) we can write the expression for the rate (5.15) in the form:

\[
W(E,n) = n \left( \log \frac{\lambda n}{4} + \frac{3}{2} \log \frac{\varepsilon}{3\pi} + \frac{1}{2} \right) - 2nm \tau_\infty - 2 \text{Re} S_E^{(1,2)}(\tau_0). \tag{5.18}
\]

This is a remarkable formula in the following sense. The expression on the right hand side of (5.18) cleanly separates into two parts: The first part is \(n \left( \log \frac{\lambda n}{4} + \frac{3}{2} \log \frac{\varepsilon}{3\pi} + \frac{1}{2} \right)\), it does not depend on the shape of the singularity surface \(\tau_0(\vec{x})\) and furthermore, it coincides with the known tree-level result for the scattering rate in the non-relativistic limit \(0 < \varepsilon \ll 1\), as we will demonstrate below. Then the entire dependence of \(W(E,n)\) on \(\tau_0(\vec{x})\) is contained in the last two terms in (5.18) which correspond to the purely quantum contribution in the \(\varepsilon \to 0\) limit.

The tree-level contribution to \(W\) is well-known, it was computed using the resummation of Feynman diagrams by solving solving the tree-level recursion relations \([15]\) and integrating over the phase-space. In the model (1.1) the tree-level result to the order \(\varepsilon^1\) was derived in \([18]\) and reads,

\[
W(E,n; \lambda)^\text{tree} = n \left( f_1(\lambda n) + f_2(\varepsilon) \right), \tag{5.19}
\]

where

\[
f_1(\lambda n) = \log \left( \frac{\lambda n}{4} \right) - 1, \tag{5.20}
\]

\[
f_2(\varepsilon)|_{\varepsilon \to 0} \to f_2(\varepsilon)^\text{asymp} = \frac{3}{2} \left( \log \left( \frac{\varepsilon}{3\pi} \right) + 1 \right) - \frac{25}{12} \varepsilon. \tag{5.21}
\]

First ignoring the order-\(\varepsilon^1\) terms in the tree-level contribution, we see that the perturbative result is correctly reproduced by the first two terms in the semiclassical expression on the right hand side of (5.18),

\[
W(E,n)^\text{tree} = n \left( \log \frac{\lambda n}{4} - 1 \right) + \frac{3n}{2} \left( \log \frac{\varepsilon}{3\pi} + 1 \right). \tag{5.22}
\]

Schematically, the contribution \(n \log \lambda n \in W^\text{tree}\) comes from squaring the tree-level amplitude on threshold and dividing by the Bose symmetry factor, \(\frac{1}{n!} (n!\lambda^{n/2})^2 \sim n!\lambda^n \sim e^{n \log \lambda n}\), while the contribution \(\frac{3}{2}n \log \varepsilon\) comes from the non-relativistic \(n\)-particle phase space volume factor \(\varepsilon^{\frac{3n}{2}} \sim e^{\frac{3}{2}n \log \varepsilon}\). \(^5\)

The apparent agreement between the first term in the expression on the right hand side of (5.18) and the result of an independent tree-level perturbative calculation (5.22), provides a non-trivial consistency check of the semiclassical formalism that led us to (5.18).

Furthermore, it was shown in [1] that the tree-level results are correctly reproduced by the semiclassical result also including the order-\(\varepsilon^1\) terms. It would be interesting to pursue such terms also at the quantum level, but this is beyond the scope of this paper and we will neglect all \(O(\varepsilon)\) terms as they are vanishing in the \(\varepsilon \to 0\) limit.

\(^5\)See Refs. [1, 15, 18] for more details on the derivation of \(W(E,n)^\text{tree}\).
We can finally re-write the expression (5.18) for the rate $W(E, n)$ in the form [1],

$$W(E, n) = W(E, n; \lambda)^{\text{free}} + \Delta W(E, n; \lambda)^{\text{quant}}, \quad (5.23)$$

where the quantum contribution is given by

$$\Delta W^{\text{quant}} = -2nm \tau_{\infty} - 2 \text{Re} S^{(1,2)}_{\text{Eucl}}$$

$$= 2nm |\tau_{\infty}| + 2 \int d^3x \left[ \int_{+\infty}^{\tau_0(\vec{x})} d\tau \mathcal{L}_{\text{Eucl}}(h_1) + \int_{\tau_0(\vec{x})}^{0} d\tau \mathcal{L}_{\text{Eucl}}(h_2) \right] \quad (5.24)$$

$$= 2nm |\tau_{\infty}| - 2 \int d^3x \left[ \int_{\tau_0(\vec{x})}^{+\infty} d\tau \mathcal{L}_{\text{Eucl}}(h_1) - \int_{\tau_0(\vec{x})}^{0} d\tau \mathcal{L}_{\text{Eucl}}(h_2) \right]$$

Here we have used the fact that $\tau_{\infty}$ is manifestly negative (as the singularity surface away at $\vec{x} \neq 0$ is by construction assumed to be located at negative $\tau$) to indicate that $-2nm \tau_{\infty}$ is a positive-valued contribution $+2nm |\tau_{\infty}|$.

The problem of finding the singularity surface $\tau_0(\vec{x})$ that extremizes the expression (5.24) thus has a simple physical interpretation [1, 2]: it is equivalent to finding the shape of the membrane $\tau_0(\vec{x})$ at equilibrium, that has the surface energy $\text{Re} S^{(1,2)}_{\text{Eucl}}$ and is pulled at the point $\vec{x} = 0$ by a constant force equal to $nm$. Note that even before the extremization of (5.24) with respect to $\tau_0(\vec{x})$, both configurations, $h_1(x)$ and $h_2(x)$ are tightly constrained. They are required to be solutions of the classical equations; they have to have satisfy the correct boundary conditions in time, and consequentially, the energy of these classical solutions is fixed: $h_1$ has $E = 0$ and $h_2$ has $E = nm$ (in the $\varepsilon \to 0$ limit). These conditions constrain the extremization of (5.24) with respect to $\tau_0(\vec{x})$.

6 Computing the rate: using thin-singular-wall bubbles

The main idea on which our calculation will be based is the geometrical interpretation of the saddle-point field configuration as a domain wall solution separating the vacua with different VEVs $h \to \pm v$ on the different sides of the wall. Our scalar theory with the spontaneous symmetry breaking in (1.1) clearly supports such field configurations. The solution is singular on the surface of the wall, and the wall thickness is $\sim 1/m$. The effect of the ‘force’ $nm$ applied to the domain wall locally pulls upwards the centre of the wall and gives it a profile $\tau_0(\vec{x})$ depicted in Fig. 3. When computing the Euclidean action on the solution characterised by the domain wall at $\tau_0(\vec{x})$, it will be represented by the action of a thin-wall bubble. The shape of the bubble will be straightforward to determine by extremizing the action in the thin-wall approximation, and the validity of this approximation will be be justified in the limit $\lambda n \to \infty$.6

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6The idea to use of the thin-wall approximation in the large $\lambda n$ limit was pursued earlier by Gorsky and Voloshin in Ref. [8] where it was applied to the standard regular bubbles of the false vacuum that were interpreted as intermediate physical bubble states in the process $1^* \to \text{Bubble} \to n$. Conceptually, this is different from our approach where the thin-wall solutions are singular points on the deformed contours of the path integral; they cannot be obviously interpreted as physical macroscopic states supposedly occurring as intermediate states in the $1^* \to n$ process.
Our first task is to implement the realisation of the singular field configuration \( h(x) \) in terms of bubbles with thin-wall singular surfaces. The \( h_1 \) branch of the solution is defined on the first part of the time-evolution contour, i.e. the imaginary time \( \tau \) in the interval \(+\infty > \tau \geq \tau_0(x)\). It is given by

\[
h_1(\tau, \vec{x}) = h_0(\tau - \tau_0(\vec{x})) + \delta h_1(\tau, \vec{x}). \tag{6.1}
\]

The first term on the right hand side of (6.1) is the familiar singular bubble (or a kink)

\[
h_0(\tau - \tau_0(\vec{x})) = v \left( \frac{1 + e^{-m(\tau - \tau_0(\vec{x}))}}{1 - e^{-m(\tau - \tau_0(\vec{x}))}} \right). \tag{6.2}
\]

depicted in Fig. 2 which interpolates between \( h = +v \) at \( \tau \gg \tau_0(\vec{x}) \) and \( h = -v \) at \( \tau \ll \tau_0(\vec{x}) \) and is singular on the 3-dimensional surface \( \tau = \tau_0(\vec{x}) \). Since \( \tau_0(\vec{x}) \) depends on the spacial variable, the correction \( \delta h_1(\tau, \vec{x}) \) is required in (6.1) to ensure that the entire field configuration \( h_1(x) \) satisfies the classical equations. On the singularity surface \( \delta h_1 \) vanishes, in fact it is straightforward to show that \( \delta h_1 \sim (\tau - \tau_0(\vec{x}))^2 \) near the singularity surface by solving the linearised classical equations for \( \delta h_1 \) in the background of the singular \( h_0 \) [1]. The initial time condition on \( h_1 \) is

\[
\lim_{\tau \to \infty} h_1(x) = v + \mathcal{O}(e^{-m\tau}), \tag{6.3}
\]

which also guarantees that \( \delta h_1(x) \to 0 \) exponentially fast at large \( \tau \). Hence in computing the action integral on \( h_1(x) \) in the thin-wall approximation, where the main contribution comes from \( \tau \) in the vicinity of \( \tau_0(\vec{x}) \), it will be a good approximation to use \( h_1(\tau, \vec{x}) = h_0(\tau - \tau_0(\vec{x})) \).

The second branch of the solution \( h_2(x) \) we will search in the form required by Eq. (5.1),

\[
h_2(t_C, \vec{x}) = h_0(\tau - \tau_\infty) + \tilde{\phi}(t_C, \vec{x}), \tag{6.4}
\]

The first term on the right hand side of (6.4) is the uniform in space solution singular on the plane \( \tau = \tau_\infty \),

\[
h_0(\tau - \tau_\infty) = v \left( \frac{1 + e^{-m(\tau - \tau_\infty)}}{1 - e^{-m(\tau - \tau_\infty)}} \right). \tag{6.5}
\]

This singular expression is accompanied in (6.4) by the second field, \( \tilde{\phi}(t_C, \vec{x}) \), whose purpose was to modify the singularity surface of \( h(t_C, \vec{x}) \) from the plane at \( \tau = \tau_\infty \) to the curved in space surface \( \tau_0(\vec{x}) \). We will now construct \( \tilde{\phi}(t_C, \vec{x}) \) using the kink-anti-kink (co-centric bubble–anti-bubble) configuration, as follows. At \( \tau > \tau_0(\vec{x}) \) the field is in the original vacuum \( h \to v \). Then at \( \tau_0(\vec{x}) \) it passes through the singularity and approaches value \( h = -v \) in the \( \tau \)-interval \( \tau_1(\vec{x}) < \tau < \tau_0(\vec{x}) \). Then the field passes through zero at \( \tau = \tau_1(\vec{x}) \), to then ascend to \( h = +v \) where it matches with the first term \( h_0(\tau - \tau_\infty) \) in the interval between \( \tau_1 \) and \( \tau_\infty \).

The \( \tau \)-profile of the entire configuration (6.4) is shown in Fig. 4 and can be visualised as follows,

\[
v \left( \frac{1 + e^{-m(\tau - \tau_0(\vec{x})) + imt}}{1 - e^{-m(\tau - \tau_0(\vec{x})) + imt}} \right) \left( \frac{1 - e^{-m(\tau - \tau_1(\vec{x})) + imt}}{1 + e^{-m(\tau - \tau_1(\vec{x})) + imt}} \right) \left( \frac{1 + e^{-m(\tau - \tau_\infty) + imt}}{1 - e^{-m(\tau - \tau_\infty) + imt}} \right) + \delta h(x) \tag{6.6}
\]
The two singularity surfaces \( \tau_0(\vec{x}) \) and \( \tau_{\infty} \) are fixed (they will be ultimately determined by extremization of the exponent) while the intermediate surface \( \tau_1(\vec{x}) \) where the field vanished will be sent to approach \( \tau(\vec{x}) \) from the left. In this way, the overall field configuration is still singular on \( \tau_0(\vec{x}) \), but at \( \tau \) away from \( \tau_0(\vec{x}) \), it is effectively indistinguishable from \( h_0(\tau - \tau_{\infty}) \) in (6.5).

The correction term \( \delta h(x) \) appearing on the right hand side of (6.6) is to ensure that the configuration \( h_2(x) \) will ultimately satisfy the classical equations and also to have the required large-\( t \) asymptotics. We will discuss the contribution \( \delta h(x) \) in a moment.

In Fig. 4 we set \( t = 0 \) and plot (6.6) as a function of \( \tau \), ignoring \( \delta h(x) \). The first singularity surface reached from the right is \( \tau_0(\vec{x}) \), this is followed by the vanishing field surface at \( \tau_1(\vec{x}) \) and finally one arrives at the second singularity surface at \( \tau_{\infty} \). For clarity, the position of the intermediate surface \( \tau_1(\vec{x}) \) was kept in Fig. 4 at a generic value between \( \tau_{\infty} \) and \( \tau_0(\vec{x}) \).

Next step is to take the limit \( \tau_1(\vec{x}) \to \tau_0(\vec{x}) \) which ensures that the first two terms in brackets on the right hand side of (6.6) cancel each other everywhere, except the single point at \( t = 0 = \tau \) where the first term is singular,

\[
\text{For } t \neq 0 : \quad \lim_{\tau_1(\vec{x}) \to \tau_0(\vec{x})} \left( \frac{1 + e^{-m(\tau - \tau_0(\vec{x})) + imt}}{1 - e^{-m(\tau - \tau_0(\vec{x})) + imt}} \right) \left( \frac{1 - e^{-m(\tau - \tau_1(\vec{x})) + imt}}{1 + e^{-m(\tau - \tau_1(\vec{x})) + imt}} \right) = 1. \quad (6.7)
\]

This corresponds to an annihilation of the kink and the anti-kink and is best expressed as the Taylor expansion of these two terms in powers of \( e^{imt} \) at \( \tau = 0 \),

\[
\left( \frac{1 + e^{-m|\tau_0| + imt}}{1 - e^{-m|\tau_0| + imt}} \right) \left( \frac{1 - e^{-m|\tau_1| + imt}}{1 + e^{-m|\tau_1| + imt}} \right)
= \left( 1 + 2 \sum_{n=1}^{\infty} e^{-nm|\tau_0|} e^{imnt} \right) \left( 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-nm|\tau_1|} e^{imnt} \right)
\approx 1 + 2 \left( e^{-m|\tau_0|} - e^{-nm|\tau_1|} \right) e^{imt} \to_{\tau_1 \to \tau_0} 1. \quad (6.8)
\]
where we have used the fact that \( \tau_0 \) and \( \tau_1 \) are negative semi-definite, which guarantees the convergence of the Taylor expansion in \( e^{\imath t} \).

After taking the \( \tau_1 \to \tau_0 \) limit of (6.6) the resulting field configuration becomes

\[
h_2(x) = h_{[\tau_0, \tau_1(\vec{x}) \to \tau_0(\vec{x})]} + \delta h(x),
\]

where the kink-antikink-kink configuration \( h_{[\tau_0, \tau_1(\vec{x}) \to \tau_0(\vec{x})]} \) is effectively identical\(^7\) to the single uniform in space solution \( h_0 \) in (6.5). The correction \( \delta h(x) \) is present in (6.9) to generate the \( e^{-\imath t} \) asymptotics at \( t \to +\infty \), as required by (5.8). All singularities are contained in the first term in (6.9), while the correction \( \delta h(x) \) is regular.

The final time asymptotics at \( t \to +\infty \) along the real axis of the solution \( h_2(x) \) in (6.9) are then given by,

\[
\lim_{t \to +\infty} h_{[\tau_0, \tau_1(\vec{x}) \to \tau_0(\vec{x})]} = v + 2v e^{m\tau_0} e^{\imath nt},
\]

\[
\lim_{t \to +\infty} \delta h(t, k) = \frac{1}{\sqrt{2\omega_k}} f_k e^{-\imath \omega_k t}.
\]

This is in agreement with Eqs. (5.2) and (5.8) derived in the previous section.

We now turn to the evaluation of the Euclidean action integrals appearing in (5.24) on the two branches \( h_1 \) and \( h_2 \) of the classical field configuration \( h(x) \). Each of the two individual integrals in

\[
- \text{Re} S_{\text{Eucl}}^{(1, 2)} = \int d^3 x \left[ \int_{+\infty}^{\tau_0(\vec{x})} d\tau \mathcal{L}_{\text{Eucl}}(h_1) + \int_{\tau_0(\vec{x})}^{0} d\tau \mathcal{L}_{\text{Eucl}}(h_2) \right],
\]

are singular at the integration limit \( \tau = \tau_0(\vec{x}) \), but their sum is expected to be finite, as it always turns out to be in the Landau-WKB approach in Quantum Mechanics. In our approach, instead of reaching the singularity and then cancelling the resulting infinite contributions at \( \tau \to \tau_0(\vec{x}) \) between the two integrals in (6.12) we will instead deform the integration contour over \( \tau \) to encircle the singularity, as shown in the transition from Fig. 1 (a) to Fig. 1 (b). This is achieved by shifting the contour infinitesimally by \( t = -\epsilon \) in the first integral and by \( t = +\epsilon \) in the second.

Since our integration contour in Fig. 1 (b) passes on the \( \pm \imath \epsilon \) sides of the singularity at \( \tau = \tau_0(\vec{x}) \), the action integrals and the solutions themselves are finite, and one can continue the integration contours down in \( \tau \) to \( -\infty \) or to any arbitrary values \( -A \). At \( \tau = -A \) where \( \tau \) is well below the final singularity surface \( \tau_\infty \), the two contours are joined, and as the result we have,

\[
- \text{Re} S_{\text{Eucl}}^{(1, 2)} = \int_{+\infty}^{-A-\imath \epsilon} d\tau \mathcal{L}_{\text{Eucl}}[h_1] + \int_{-A+\imath \epsilon}^{0+\imath \epsilon} d\tau \mathcal{L}_{\text{Eucl}}[h_2],
\]

\(^7\)By this we mean that contributions to the action and the energy of \( h_{[\tau_0, \tau_1(\vec{x}) \to \tau_0(\vec{x})]} \) from \( \tau \sim \tau_0 \) are negligible; the only non-trivial contributions come from \( \tau \sim \tau_\infty \) and are the same as for \( h_0(\tau - \tau_\infty) \). The configuration nevertheless is still formally singular when \( \tau \) approaches \( \tau_0(\vec{x}) \) from the right, as required by the semiclassical construction.
Figure 5: Singularity surfaces of the two branches of the solution. Plot (a) depicts the solution branch $h_1(x)$ with the singularity surface at $\tau_0(\vec{x})$. The energy of the solution $E(h_1) = 0$ at any value of $\tau$. The integration contour $\int d\tau$ is shown as the horizontal red arrow with intermediate points $\tau_a$, $\tau_b$ and $\tau_c$. Plot (b) shows $h_2(x)$ with the null surface at $\tau_1(\vec{x})$ annihilating the singularity surface $\tau_0(\vec{x})$. The remaining singular surface is flat $\tau_\infty$. The integration contour $\int d\tau$ is shown as the horizontal red arrow with intermediate points $\tau_c$, $\tau_d$ and $\tau_e$. The energy of the solution $E(h_2) = nm$ for $\tau > \tau_\infty$.

where $L_{Eucl} = \int d^3x \mathcal{L}_{Eucl}$, and each of the two integrals in (6.13) is finite. We make use of the constraint that the energies of the classical solutions to the right of the first singular surface on Fig. 4, i.e. for $\tau > \tau_0(\vec{x})$, are known:

$$\tau > \tau_0(\vec{x}) : \quad E[h_1] = 0, \quad E[h_2] = nm = E > 0. \quad (6.14)$$

The first integral on the right hand side of (6.13) is over the zero-energy Euclidean-time solution $h_1(x)$ in (6.1)-(6.2) with the singular surface $\tau_0(\vec{x})$ shown in Fig. 5 (a). Integration over $\tau$ proceeds from right to left through the sequence of intermediate values of $\tau$, with three snapshots at $\tau_a$, $\tau_b$ and $\tau_c$ shown in Fig. 5 (a). At $\tau = \tau_a$ the Lagrangian and the Euclidean energy of the solution,

$$L_{Eucl} = \int d^3x \left( \frac{1}{2} (\partial_\tau h)^2 + \frac{1}{2} (\partial_\vec{x} h)^2 + V(h) \right), \quad (6.15)$$

$$E = \int d^3x \left( -\frac{1}{2} (\partial_\tau h)^2 + \frac{1}{2} (\partial_\vec{x} h)^2 + V(h) \right), \quad (6.16)$$

are vanishing. At the lower fixed value $\tau = \tau_b$ the integrals over $d^3x$ cross the $\tau_0(\vec{x})$ surface passing between the $h_1 = +v$ and the $h_1 = -v$ domains along the dotted line at $\tau_b$ in Fig. 5 (a). The Lagrangian density $L_{Eucl}$ has a peak at the crossing of $\tau_0(\vec{x})$ and as the result the Lagrangian in (6.15) becomes positive at $\tau = \tau_b$, and the action increases from $\tau_a$ to $\tau_b$. The energy of the solution $h_1$, however, must remain vanishing at $\tau_b$, as it is conserved on a classical solution (recall that there are no infinities of the solution on our shifted contour). It is easy to understand why the energy in (6.16) can remain vanishing –
it is the consequence of the negative relative sign between the kinetic and potential energy density terms in (6.16); hence unlike the expression for the Lagrangian, the energy density has a positive and the negative energy peaks at the crossing that compensate one another on the classical solution.

In summary, the action integral \[ \int_{\tau^*_0(x)}^{\tau^*_\infty(x)} \] grows with \( \tau \) from \( \tau = 0 \) to \( \tau = \tau^*_\infty \) where it crosses the surface \( \tau_0(x) \) separating the \( \pm v \) domains of the solution. The action on \( h_1 \) has a non-trivial dependence on the shape of \( \tau_0(x) \). At the same time, the energy of the solution \( h_1 \) is vanishing at all values of \( \tau \), \( E = 0 \), as it should (6.14).

We now consider the second integral on the right hand side of (6.13) computed on the finite-energy solution \( h_2(x) \) defined in (6.9). As we already explained in the beginning of this section, the effect of the singularity surface at \( \tau = \tau_0(x) \) is cancelled by the null-surface \( \tau_1(x) \) on the solution (6.9). This is shown in the second plot, Fig. 5 (b). The action integral \[ \int d\tau L_{\text{Eucl}}(h_2) \] now starts at \( \tau_c \) and progresses through the snapshots at \( \tau_d \) and \( \tau_e \) as indicated in the Figure. Only the flat in \( x \) surface at \( \tau_\infty \) matters for the calculation of the action and the energy. They both jump by a constant amount between \( \tau < \tau_\infty \) and \( \tau < \tau_\infty \). The energy of \( h_2 \) \( \tau < \tau_\infty \) is \( E = mn \) and the action will be calculated momentarily. The action integral on \( h_2 \) is a constant and, importantly, it does not depend on the shape of \( \tau_0(x) \).

Following from the discussion in the previous section, the shape of the singular surface, \( \tau_0(x) \), should be determined by extremizing the function \( \Delta W^{\text{quant}} \) in the exponent of multiparticle probability rate. This is equivalent to searching for a stationary i.e. equilibrium surface configuration described by the ‘surface energy’ functional,

\[
\frac{1}{2} \Delta W^{\text{quant}} = nm \left| \tau_\infty \right| - \int_{A_{-i\epsilon}}^{A_{+i\epsilon}} d\tau L_{\text{Eucl}}(h; \tau_0(x)) S_{\text{Eucl}\left[ \tau_0(x) \right]} + \int_{A_{+i\epsilon}}^{A_{+i\epsilon}} d\tau L_{\text{Eucl}}(h; \tau_\infty),
\]

and we no longer need to distinguish between \( h_1 \) and \( h_2 \). The stationary point corresponds to balancing the action of the surface tension term given by the first integral in (6.17) against the force \( mn \) that stretches the surface \( \tau_0(x) \) by the amount \( \left| \tau_\infty \right| \). The second integral in (6.17) plays no role in the extremization procedure over \( \tau_0(x) \) and gives a positive-valued constant contribution to \( \frac{1}{2} \Delta W^{\text{quant}} \).

We now compute the constant contribution of the second integral in (6.17). In the thin wall approximation, the action on the solution with a domain wall separating the two domains is dominated by the wall whose width is taken to be small. In this case, the relevant solution \( h_2 \), is well approximated by the bubble \( h_0(\tau - \tau_\infty) \) in (6.5). Along the contour with \( \tau \) shifted by \( i\epsilon \) this configuration becomes,

\[
h_1(\tau + i\epsilon) = v \left( \frac{1 + e^{-m(\tau - \tau_\infty + i\epsilon)}}{1 - e^{-m(\tau - \tau_\infty + i\epsilon)}} \right),
\]

The action integral on this uniform in space solution (for simplicity we extend the integration limits to \( \pm \infty \) but it should not matter for a narrow wall) can be calculated exactly,
giving
\[ \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\tau \int d^3x L_{\text{Eucl}}(h) = \mu \int_0^R 4\pi r^2 dr = \mu \frac{4\pi}{3} R^3 , \]  \hspace{1cm} (6.19)

where \( R \) is the spatial radius; the limit \( R \to \infty \) will be taken in the infinite volume limit at the end of the calculation, after combining the two action integrals in (6.12). The parameter \( \mu \) appearing on the right hand side in (6.19) is the surface tension on the bubble solution (6.18)

\[ \mu = \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\tau \left( \frac{1}{2} \left( \frac{dh}{d\tau} \right)^2 + \frac{\lambda}{4} (h^2 - v^2)^2 \right) = \frac{m^3}{3\lambda} , \]  \hspace{1cm} (6.20)

and it is easily checked (e.g. by using the residue theorem) that the value of \( \mu \) does not depend on the numerical value of \( i\epsilon \) in the shift of the integration contour, any value of \( i\epsilon \) that shifts the contour that it does not pass directly through the singularity at \( \tau_\infty \) is fine.

Summarising our construction up to this point: we have derived the expression for the contribution of quantum effects (5.24) to the semiclassical rate \( W \) (5.23) in the form,

\[ \frac{1}{2} \Delta W_{\text{quant}} = nm |\tau_\infty| - \int_{-\infty-i\epsilon}^{+\infty+i\epsilon} d\tau L_{\text{Eucl}}(h; \tau_0(\vec{x})) + \frac{4\pi}{3} \mu R^3 . \]  \hspace{1cm} (6.21)

We note that no extremization of the rate with respect to the surface \( \tau = \tau_0(\vec{x}) \) has been carried so far. The expression in (6.21) is the general formula equivalent to the expression in (5.24). It will be now extremized with respect to the domain wall surface \( \tau_0(\vec{x}) \). The constant term \( \frac{4\pi}{3} \mu R^3 \) will be cancelled with its counterpart arising from the action integral in (6.21) before the infinite volume limit is taken.

The action \( S_{\text{Eucl}}[\tau_0(\vec{x})] \) can now be written as an integral over the domain wall surface \( \tau_0(\vec{x}) \) in the thin-wall approximation. This is equivalent to stating that the action is equal to the surface tension of the domain wall \( \mu \) already computed in (6.20) times the area. The infinitesimal element of the 3-dimensional area of a surface curved in 3+1 dimensions is \( 4\pi \mu r^2 \sqrt{(dr)^2 + (\dot{r})^2} \). Hence the action reads,

\[ S_{\text{Eucl}}[\tau_0(r)] = \int_{\tau_\infty}^0 d\tau 4\pi \mu r^2 \sqrt{1 + (\dot{r})^2} \equiv \int_{\tau_\infty}^0 d\tau L(r, \dot{r}) , \]  \hspace{1cm} (6.22)

where \( r = |\vec{x}| \) and \( \dot{r} = dr/d\tau \). The integral depends on the choice of the domain wall surface \( \tau_0(\vec{x}) \) implicitly via dependence on \( \tau \) of \( r(\tau) \) and \( \dot{r}(\tau) \) which are computed on the domain wall.

Since \( L(r, \dot{r}) \) has the meaning of the Lagrangian, we can introduce the Hamiltonian function defined in the standard way\(^8\) as the Legendre transformation,

\[ H(p, r) = L(r, \dot{r}) - p \dot{r} , \]  \hspace{1cm} (6.23)

\(^8\)In Euclidean space \( L = K + P \) and \( H = P - K \) where \( K \) and \( P \) are the kinetic and potential energies respectively.
where the momentum $p$, conjugate to the coordinate $r$, is

$$p = \frac{\partial L(r, \dot{r})}{\partial \dot{r}} = 4\pi \mu \frac{r^2 \dot{r}}{\sqrt{1 + \dot{r}^2}} \quad (6.24)$$

On a classical trajectory $r = r(\tau)$ that satisfies the Euler-Lagrange equations corresponding to $L(r, \dot{r})$, the Hamiltonian is time-independent, $dH/d\tau = 0$, and is given by the energy $E$ of the classical trajectory $r = r(\tau)$.

Hence, on a stationary point of $S_{\text{Eucl}}[\tau_0(r)]$ that has the energy $E$ we can rewrite the action as

$$S_{\text{Eucl}}[\tau_0(r)]_{\text{stationary}} = -\tau_\infty E + \int_{\tau_\infty}^{0} d\tau \left( L - H \right) = -E\tau_\infty + \int_{0}^{R} p(E) \, dr. \quad (6.25)$$

Here we added and subtracted the constant energy of the solution $E = H$ in the integral, used the fact that $L - H = p\dot{r}$ and have set the lower and upper integration limits at $r(\tau_\infty) = R$ and $r(0) = 0$. The expression above gives us $S_{\text{Eucl}}[\tau_0(r)]$ on a trajectory $r(\tau)$, or equivalently $\tau = \tau_0(r)$ which is a classical trajectory i.e. an extremum of the action for a fixed energy $E$. Equivalently, for the stationary point of the expression in (6.21) we have,

$$\frac{1}{2} \Delta W_{\text{quant}} = (E - nm)\tau_\infty - \int_{R}^{0} p(E) \, dr + \frac{4\pi}{3} \mu R^3. \quad (6.26)$$

Extremization of this expression with respect to the parameter $\tau_\infty$ gives $E = nm$ thus selecting the energy of the classical trajectory to be set at $nm$ as required,

$$\frac{1}{2} \Delta W_{\text{quant}} \text{ stationary} = - \int_{R}^{0} p(E) \, dr + \frac{4\pi}{3} \mu R^3, \quad E = nm. \quad (6.27)$$

To evaluate (6.27) we need to determine the dependence of the momentum of the classical trajectory on its energy. To find $p(E)$, we start by writing the expression for the energy, $E = L - p\dot{r}$, in the form

$$E = 4\pi \mu r^2 \sqrt{1 + \dot{r}^2} - 4\pi \mu \frac{r^2 \dot{r}}{\sqrt{1 + \dot{r}^2}} = 4\pi \mu \frac{r^2}{\sqrt{1 + \dot{r}^2}}, \quad (6.28)$$

and then compute the combination $E^2 + p^2$ using the above expression and (6.24),

$$E^2 + p^2 = \left(4\pi \mu r^2\right)^2 \left( \frac{1}{1 + \dot{r}^2} + \frac{\dot{r}^2}{1 + \dot{r}^2} \right) = \left(4\pi \mu r^2\right)^2. \quad (6.29)$$

This gives the desired expression for the momentum $p = p(E)$,

$$p(E, r) = -4\pi \mu \sqrt{r^4 - \left( \frac{E}{4\pi \mu} \right)^2}, \quad (6.30)$$

\footnote{It is important not to confuse the energy of the classical trajectory $r = r(\tau)$ – which is essentially the Euclidean surface energy of the domain wall – with the energy of the classical solutions $h_1$ and $h_2$. Both energy variables are denoted as $E$, but the energy of the domain wall at the stationary point will turn out to be $E = mn$ while the energy of the corresponding field configuration $h_1$ was $E = 0$.}
where have selected in (6.30) the negative root for the momentum in accordance with the fact that \( p(\tau) \propto \dot{r} \) (as follows from (6.24)) and that \( r(\tau) \) is a monotonically decreasing function.

Substituting this into the expression (6.27) we have,

\[
\frac{1}{2} \Delta W_{\text{quant}} = - \int_{r_0}^{R} p(E) \, dr + \frac{4\pi}{3} \mu R^3 = - \int_{r_0}^{R} 4\pi \mu \sqrt{r^4 - r_0^4} \, dr + \frac{4\pi}{3} \mu R^3 . \tag{6.31}
\]

The minimal value of the momentum (and the lower bound of the integral in (6.31)) is cut-off at the critical radius \( r_0 \),

\[
r_0^2 = \frac{E}{4\pi \mu} , \tag{6.32}
\]

Below we will also consider the contribution to the integral (6.31) on the interval \( 0 \leq r \leq r_0 \) but for now we will temporarily ignore it.

The integral on the right hand side of (6.31) is evaluated as follows,

\[
\int_1^{R/r_0} \sqrt{x^4 - 1} \, dx = \left[ \frac{1}{3} x \sqrt{x^4 - 1} - \frac{2}{3} i \text{EllipticF}[\text{ArcSin}(x), -1] \right]_{x=1}^{x=R/r_0}
\]

where the \textit{Mathematica} function \( \text{EllipticF}[z, m] \) is also known as the elliptic integral of the first kind \( F(z|m) \). The integral simplified in the \( R/r_0 \to \infty \) limit giving,

\[
(-4\pi \mu r_0^3) \int_1^{R/r_0} \sqrt{x^4 - 1} \, dx \to -\frac{4\pi}{3} \mu R^3 + \frac{4\pi}{3} \mu \sqrt{\frac{1}{3} \Gamma(5/4)} \Gamma(3/4) \left( \frac{E}{4\pi \mu} \right)^{1/2} = -\frac{4\pi}{3} \mu R^3 + \frac{E^{3/2}}{\sqrt{\mu}} \frac{1}{3} \left( \frac{E}{4\pi \mu} \right) \Gamma(5/4) \Gamma(3/4) . \tag{6.33}
\]

We see that the large volume constant term \( \frac{4\pi}{3} \mu R^3 \) cancels between the expressions in (6.33) and (6.31), as expected. The final result for the thin-wall trajectory contribution to the quantum rate is given by,

\[
\Delta W_{\text{quant}} = \frac{E^{3/2}}{\sqrt{\mu}} \frac{2}{3} \frac{\Gamma(5/4)}{\Gamma(3/4)} = \frac{1}{\lambda} \left( \lambda n \right)^{3/2} \frac{2}{\sqrt{3}} \frac{\Gamma(5/4)}{\Gamma(3/4)} \simeq 0.854 n \sqrt{\lambda n} . \tag{6.34}
\]

We note that this expression is positive-valued, that it grows in the limit of \( \lambda n \to \infty \), and that it has the correct scaling properties for the semiclassical result, i.e. it is of the form \( 1/\lambda \) times a function of \( \lambda n \).

Our result (6.34) reproduces the expression derived in our earlier paper [2] and is also in agreement with the expression derived even earlier in Ref. [8].

It also follows that the thin-wall approximation is fully justified in the \( \lambda n \gg 1 \) limit as originally noted in [2, 8]. The thin-wall regime corresponds to the radius of the bubble being much greater than the thickness of the wall, \( r \gg 1/m \). In our case the radius is always greater than the critical radius,

\[
r m \geq r_0 m = m \left( \frac{E}{4\pi \mu} \right)^{1/2} \propto \left( \frac{\lambda E}{m} \right)^{1/2} = \sqrt{\lambda n} \gg 1 , \tag{6.35}
\]
Figure 6: Extremal surface $\tau = \tau_0(r)$ of the thin wall bubble solution (6.38). Solid line denotes the bubble wall profile of the bubble radius $r$ above the critical radius $r_0$. The dashed line corresponds to the branch of the classical trajectory beyond the turning point at $r_0$.

where we have used the value for the energy $E = nm$ on our solution.

One can ask what is the actual classical trajectory $r(\tau)$ or equivalently the wall profile $\tau = \tau_0(r)$ of the classical bubble on which the rate $W$ was computed in (6.34). To find it we can integrate the equation for the conserved energy (6.28) on our classical solution,

$$E = 4\pi \mu \frac{r^2}{\sqrt{1 + r'^2}},$$

or, equivalently, the expression $(r/r_0)^4 = 1 + r'^2$. One finds,

$$\int_{\tau_\infty}^{\tau} d\tau = -\int_{r}^{r_\infty} \frac{dr}{\sqrt{\left( \frac{r}{r_0} \right)^4 - 1}},$$

which after integration can be expressed in the form,

$$\tau(r) = \tau_\infty + r_0 \left( \frac{\Gamma^2(1/4)}{4\sqrt{2\pi}} + \text{Im}(\text{EllipticF}[\text{ArcSin}(r/r_0), -1]) \right).$$

This classical trajectory gives the thin-wall bubble classical profile for $r_0 < r(\tau) < \infty$ which the result (6.34) for the quantum contribution to the rate $\Delta W^{\text{quant}}$. This trajectory is plotted in Fig. 6.

What happens when the radius of the bubble $r(\tau)$ approaches the critical radius $r_0$ (6.32) where the momentum (6.30) vanishes? Recall that in the language of a mechanical analogy we are searching for an equilibrium (i.e. the stationary point solution) where the surface $\tau_0(r)$ located at $\tau_\infty$ at large values of $r$ is pulled upwards (in the direction of $\tau$) by a constant force $E = nm$ acting at the point $r = 0$. This is what corresponds to finding an extremum – in our case the true minimum – of the expression in (6.21), which we rewrite now in the form,

$$\frac{1}{2} \Delta W^{\text{quant}} = \frac{E |\tau_\infty|}{\text{Force} \times \text{height}} - \mu \int \frac{d^{2+1} \text{Area}}{\text{surface Energy}}.$$
Figure 7: Stationary surface configuration obtained by gluing two branches. Plot (a) shows the surface in the thin-wall approximation which glues the original solution (6.38) to the infinitely stretchable cylinder solution of (6.40). Plot (b) depicts its more realistic implementation where the infinite cylinder is replaced by a cone as a consequence of allowing the surface tension $\mu$ to increase with $|\tau|$ in the regime where the highly stretched surface becomes effectively a 1-dimensional spring.

Sufficiently far away from the point at the origin where the force acts, the surface is nearly flat and does not extend in the $\tau$ direction. As the distance in the $r$-direction closer to the point where the force is applied, the surface is getting more and more stretched in the $\tau$ direction, until the critical radius $r_0$ is reached where the surface approaches the shape of a cylinder $R^1 \times S^2$ with $R^1$ along the $\tau$ direction.

Up to the critical point $\tau_c$ where $r = r_0$, the force and the surface tension have to balance each other in the expression,

$$E|\tau_\infty - \tau_c| - \left( \int_{\tau_\infty}^{\tau_c} 4\pi \mu r^2 \sqrt{1 + \frac{r^2}{\tau^2}} - \frac{4\pi}{3} \mu R^3 \right),$$

and this is what we have calculated in Eqs. (6.31) and (6.34). But when the critical point $r_0$ is reached at a certain $\tau_c$ the balance of forces becomes trivial,

$$E|\tau_c| - \mu \frac{4\pi r_0^2}{\tau_c} = 0.$$

Clearly, the branch of the classical trajectory shown as the dashed line in Fig. 6 is unphysical in the sense that it does not describe the membrane pulled upwards with the force $E = mn$. The vanishing of the expression (6.41) is the consequence of the definition of the critical radius in (6.32). As soon as the radius $r(\tau)$ approaches the critical radius $r_0$, the radius freezes at this value (since $p \propto d_r r = 0$), the two terms in (6.41) become equal, $E = \mu \frac{4\pi r_0^2}{\tau_c}$, and remain so at all times above the critical time $\tau_c$. The thin-wall profile becomes an infinitely stretchable cylinder, as shown in Fig. 7 (a), giving no additional contribution to $\Delta W_{\text{quant}}$ on top of (6.40).

The stationary solution in the form where it becomes at $r \to r_0$ a cylinder that can be freely stretched in the vertical (i.e. $\tau$) direction is an idealised approximation to the more realistic configuration that would be realised in our mechanical analogy of the surface.
stretched by the force in practice. It is easy to see how this realistic mechanical solution looks like. For the coordinate along the vertical axis, \( d := \tau + \tau_\infty \simeq 0 \), the bubble profile is nearly flat in the \( \tau \) direction. As \( d \) increases from 0, the radius \( r(\tau) \) grows smaller, following the profile of the thin-wall solution contour in the lower part of Fig. 7. As \( r \) approaches the critical radius \( r_0 \), the surface becomes almost entirely along the \( d \) (or \( \tau \)) direction. Such a surface looks more like a spring along the \( \tau \) coordinate. For the strict thin-wall approximation, the surface tension \( \mu \) is assumed to be a constant. But in the case of the spring, it should be the Young’s elastic modulus \( k_{\text{Young}} \) that takes a constant value. Hence for a highly stretched surface in the \( \tau \) direction we should introduce some dependence on \( d = \tau + \tau_\infty \) into the surface tension,

\[
\mu = \mu_0 \left( 1 + \hat{k} (\tau + \tau_\infty) \right),
\]

(6.42)

where \( \mu_0 = \frac{m^3}{4\lambda} \) is the same constant contribution to the surface tension as before in (6.20), and \( \hat{k} \ll 1 \) is a dimensionless constant. The corresponding Young’s modulus of the spring-shaped stretched surface would be \( k_{\text{Young}} = \mu_0 \hat{k} \). The equation (6.42) describes a small deviation from the standard thin wall approximation where the surface tension is now dependent on the stretching of the surface. This expression can be thought of as the zeroth and the first order terms in the Taylor expansion of the function \( \mu(\tau + \tau_\infty) \). The result of this improvement on \( \mu \) is that the balance between the two terms in (6.41) continues to hold, but now in the form,

\[
(E - \mu(d) \cdot 4\pi r(d)^2) d = 0, \quad \text{where} \ d \geq |\tau_c|.
\]

(6.43)

For every infinitesimal increase in the vertical coordinate \( d \) above \( |\tau_c| \), the radius \( r(d) \) gets a little smaller than its value \( r_0 \) at the base of the cylinder in Fig. 7 (a). As a result the cylinder gets narrower as \( d \) increases and turns into the cone-like shape shown in Fig. 7 (b). The actual choice of the modification of the surface tension expression, such as in (6.42), is of course determined by the field configurations themselves, so it can be seen as a part of the extremization procedure. One can always find an adiabatically slowly varying \( \mu \) such that the contribution from the cone to \( W \) is negligible, and the overall contribution is dominated by the surface at \( r > r_0 \) in the large \( \lambda n \) limit. Hence we conclude that

\[
\Delta W^{\text{quant}} = \frac{1}{\lambda} \left( \frac{\lambda n}{3} \right)^{3/2} \frac{2}{\sqrt{3}} \frac{\Gamma(5/4)}{\Gamma(3/4)} \simeq 0.854 \ n \sqrt{\lambda n}.
\]

(6.44)

7 Quantum rate in (2+1) dimensions

All our calculations can be straightforwardly generalised to any number of dimensions \( (d + 1) \) in the same as before scalar QFT model (1.1) with the VEV \( v \neq 0 \).

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\(^{10}\)Recall that the tip of the surface is at \( \tau = 0 \) where \( d = |\tau_\infty| \), and that the surface’s base is at a negative \( \tau = \tau_\infty = -|\tau_\infty| \) which corresponds to \( d = 0 \).
The expression $W(E, N)_d$ in the exponent of the multiparticle rate $R_n(E)$ has the same general decomposition into the tree-level and the quantum parts as before,

$$W(E, n)_d = W(E, n; \lambda)_d^{\text{tree}} + \Delta W(E, n; \lambda)_d^{\text{quant}},$$

(7.1)

where the tree-level expression in $(d + 1)$ dimensions reads (cf. (5.22)),

$$W(E, n)_d^{\text{tree}} = n \left( \log \frac{\lambda n}{4} - 1 \right) + \frac{dn}{2} \left( \log \frac{\varepsilon}{d\pi} + 1 \right),$$

(7.2)

and the quantum contribution is given by

$$\Delta W_d^{\text{quant}} = 2nm |\tau_\infty| + 2 \int d^d x \left[ \int_{\tau_0(x)}^{+\infty} d\tau L_{\text{Eucl}}(h_1) - \int_{\tau_0(x)}^{0} d\tau L_{\text{Eucl}}(h_2) \right]$$

(7.3)

being extremized over the singularity surfaces $\tau_0(x)$ in a complete analogy with (5.24).

For the rest of this section we will consider the case of $d = 2$ spacial dimensions and will concentrate on the contribution of the stationary surface to the quantity $\frac{1}{2} \Delta W_d^{\text{quant}}$, which we write as,

$$\frac{1}{2} \Delta W_d^{\text{quant}} = E |\tau_\infty| - 2\pi \mu \left( \int_{r_0}^R r\sqrt{1 + \dot{r}^2} dr - \int_{0}^R r dr \right),$$

(7.4)

where the surface tension is the same as before, $\mu = m^3/\lambda$, and the critical radius in $d = 2$ is given by $r_0 = E/(2\pi m)$. Proceeding with the evaluation of (7.4) on the classical trajectory $r(\tau)$ analogously to the calculation in the previous section we get,

$$\frac{1}{2} \Delta W^{\text{quant}} = - \int_{r_0}^R 2\pi \mu \sqrt{r^2 - r_0^2} dr + 2\pi \mu R^2,$$

(7.5)

which in the $Rm \to \infty$ limit becomes,

$$\simeq \frac{n^2 \lambda}{m} \frac{3}{4\pi} \left( \log(Rm) + \frac{1}{2} + \log \left( \frac{2\pi}{3} \frac{m}{\lambda n} \right) + \mathcal{O} \left( \frac{1}{Rm m} \right) \right).$$

(6.6)

Adopting the infinite volume limit where limit $Rm \to \infty$ is taken first, while the quantity $\frac{n^2 \lambda}{m}$ is held fixed, we can drop the $R$-independent and $1/R$-suppressed terms, leaving only the logarithmically divergent contribution,

$$\frac{1}{2} \Delta W^{\text{quant}} \simeq \frac{3}{4\pi} \frac{n^2 \lambda}{m} \log(Rm)$$

(7.7)

We see that all power-like divergent terms in $mR$ have cancelled in the expressions (7.5) and (7.7), but the logarithmic divergence $\log(Rm)$ remains. This result is not surprising in $d < 3$ dimensions and is the consequence of the infrared divergencies in the amplitudes at thresholds due to the rescattering effects of final particles. In fact, the appropriate coupling constant in the lower-dimensional theory is not the bare coupling $\lambda$ but the running quantity $\lambda t$ where $t$ is the logarithm of the characteristic momentum scale in the final state. In our case we can set,

$$t = \log(Rm)$$

(7.8)
and treat $R$ as one over the average momentum scale in the final state, i.e. $Rm = 1/\varepsilon^{1/2}$.

The semiclassical result obtained in (7.7) is the effect of taking into account quantum corrections to the scattering amplitudes into $n$-particle states near their threshold, and implies

$$A_n \simeq A_n^{\text{tree}} \exp\left(\frac{3n^2\lambda t}{4\pi m}\right).$$

(7.9)

It is important to recall the semiclassical limit assumed in the derivation of the above expression. It is as always the weak-coupling plus large multiplicity limit, such that

$$\text{dimensionless running coupling : } \frac{\lambda t}{m} \to 0 \quad \text{and multiplicity : } n \to \infty \quad (7.10)$$

with the quantity $n \frac{\lambda t}{m}$ held fixed (and ultimately large), and $t = -1/2 \log \varepsilon \to 0$ to ensure the non-relativistic limit which selects the amplitudes close to their multiparticle thresholds.

It is important that it is the running coupling $\lambda t$ that is required to be small in the semiclassical exponent\(^{12}\). This implies that the semiclassical expression would in general include unknown corrections in

$$A_n \simeq A_n^{\text{tree}} \exp\left(\frac{3n^2\lambda t}{4\pi m}\left(1 + \sum_{k=1}^{\infty} c_k \left(\frac{\lambda t}{m}\right)^k\right)\right),$$

(7.11)

parameterised by the sum $\sum_{k=1}^{\infty} c_k \left(\frac{\lambda t}{m}\right)^k$. Of course, there is a well-defined regime corresponding to the small values of the effective coupling $\lambda t$ where these corrections are negligible and the leading order semiclassical result in (7.9) is justified.

Remarkably, the semiclassical formula (7.9) can be tested against an independent computation of quantum effects in the $(2 + 1)$-dimensional theory obtained in [15, 30] using the RG resummation of perturbative diagrams. The result is,

$$A_n^{\text{RG}} = A_n^{\text{tree}} \left(1 - \frac{3\lambda t}{2\pi m}\right)^{-\frac{n(n-1)}{2}}.$$  

(7.12)

This expression is supposed to be valid for any values of $n$, and in the regime where the effective coupling $\lambda t$ is in the interval,

$$0 \leq \frac{\lambda t}{m} \lesssim 1.$$  

(7.13)

Now taking the large-$n$ limit the RG-technique based result of [15, 30] gives

$$A_n^{\text{RG}} = A_n^{\text{tree}} \exp\left(\frac{3n^2\lambda t}{4\pi m}\left(1 + \sum_{k=1}^{\infty} \frac{1}{k+1} \left(\frac{3\lambda t}{2\pi m}\right)^k\right)\right),$$

(7.14)

\(^{11}\)Recall that in $(2 + 1)$ dimensions, $\lambda$ has dimensions of mass.

\(^{12}\)For example it is completely analogous to the instanton action $S_{\text{inst}} = \frac{8\pi^2}{g^2(0)}$ in the Yang-Mills theory, where the inclusion of quantum corrections from the determinants into the instanton measure in the path integral ensures that $S_{\text{inst}}$ in the exponent depends on the correct RG coupling $g^2(t)$ and not the unphysical bare coupling $g^2_{\text{bare}}$. 

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It is a nice test of the semiclassical approach that the leading order terms in the exponent in both expressions, (7.11) and (7.14) are exactly the same and given by \( \frac{3n^2\lambda t}{4\pi m} \). An equally important observation is that the subleading terms are of the form \( \sum_{k=1} c_k \left( \frac{\lambda t}{m} \right)^k \) which is suppressed in the semiclassical limit \( \lambda t \to 0 \). There is no contradiction between the two expressions in the regime where the semiclassical approach is justified.

It thus follows that there is a regime in the \((2 + 1)\)-dimensional theory where the multiparticle amplitudes near their thresholds, and consequently the probabilistic rates \( R_n(E) \) become large. In the case of the RG expression (7.12), this is the consequence of taking a large negative power \(-n^2/2\) of the term that is smaller than 1. This implies that there is a room for realising Higgsplosion in this \((2 + 1)\)-dimensional model in the broken phase.

In the case of a much simpler model – the quantum mechanical anharmonic oscillator in the unbroken phase – it was recently shown in Ref. [22] that the rates remain exponentially suppressed in accordance with what would be expected from unitarity in QM.

8 Conclusions

In this paper, following the idea outlined in our earlier work [2] we computed the semiclassical exponent of the multi-particle production rate in the high-particle-number \( \lambda n \to \infty \) limit in the kinematical regime where the final state particles are produced near their mass thresholds. This corresponds to the limit

\[
\lambda \to 0, \quad n \to \infty, \quad \text{with} \quad \lambda n = \text{fixed} \gg 1, \quad \varepsilon = \text{fixed} \ll 1.
\]  

(8.1)

Combining the tree-level (5.22) and the quantum effects (6.44) contributions,

\[
W(E, n) = W(E, n; \lambda)^\text{tree} + \Delta W(E, n; \lambda)^\text{quant},
\]  

(8.2)

we can write down the full semiclassical rate,

\[
R_n(E) = e^{W(E, n)} = \exp \left[ n \left( \log \frac{\lambda n}{4} + 0.85 \sqrt{\lambda n} + \frac{3}{2} \log \frac{\varepsilon}{3\pi} + \frac{1}{2} \right) \right]
\]  

(8.3)

computed in the high-multiplicity non-relativistic limit (8.1). This expression for the multi-particle rates was first written down in the precursor of this work [2], and was used in Refs. [3, 4] and subsequent papers to introduce and motivate the Higgsplosion mechanism.

The energy in the initial state and the final state multiplicity are related linearly via

\[
E/m = (1 + \varepsilon) n,
\]  

(8.4)

and thus for any fixed non-vanishing value of \( \varepsilon \), one can raise the energy to achieve any desired large value of \( n \) and consequently a large \( \sqrt{\lambda n} \). Clearly, at the strictly vanishing value of \( \varepsilon \), the phase-space volume is zero and the entire rate (8.3) vanishes. Then by increasing \( \varepsilon \) to a positive but still small values, the rate increases. The competition is
between the negative log $\varepsilon$ term and the positive $\sqrt{\lambda n}$ term in (8.3), and there is always a range of sufficiently high multiplicities where $\sqrt{\lambda n}$ overtakes the logarithmic term $\log \varepsilon$ for any fixed (however small) value of $\varepsilon$. This leads to the exponentially growing multi-particle rates above a certain critical energy, which in the case described by the expression in (8.3) is in the regime of $E_c \sim 200m$. We refer the reader to Fig. 8 and to section 5 of Ref. [2] for a detailed discussion of the exponential rate (8.3) and its relevance for Higgsplosion [3].

Our discussion concentrated entirely on a simple scalar QFT model. If more degrees of freedom were included, for example the $W$ and $Z$ vector bosons and the SM fermions, new coupling parameters (such as the gauge coupling and the Yukawas) would appear in the expression for the rate along with the final state particle multiplicities. As there are more parameters, the simple scaling properties of $\mathcal{R}_n$ in the pure scalar theory will be modified. Understanding how this would work in practice and investigating the appropriate semiclassical limits is one (of the admittedly many) tasks for future work on exploring realisations of Higgsplosion in particle physics.

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Appendix: Comments on the semiclassical method

The aim of the semiclassical approach of \cite{1} is to compute the probability rate $\mathcal{R}_n(E)$ for a local operator $\mathcal{O}(x)$ at a point $x = 0$ to create $n$ particles of total energy $E$ from the vacuum,

$$\mathcal{R}_n(E) = \int d\Phi \langle 0 | \mathcal{O}^\dagger P_\mathcal{E} | n \rangle \langle n | P_\mathcal{E} S \mathcal{O} | 0 \rangle .$$  \hspace{1cm} (A.1)

The matrix element is squared and integrated over the $n$-particle Lorentz-invariant phase space $\Phi_n$

$$\int d\Phi = \frac{1}{n!} \left( \frac{2\pi}{3} \right)^{n/2} \delta^{(4)} \left( P_{\text{in}} - \sum_{j=1}^{n} p_j \right) \prod_{j=1}^{n} \int d^3 p_j \left( \frac{2\pi}{3} \right)^{3/2} p_0^{j/2} ,$$  \hspace{1cm} (A.2)

Note that in our conventions the bosonic phase-space volume element (A.2) includes the $1/n!$ symmetry factor for the production of the $n$ equivalent Higgs bosons.\footnote{Hence the $n$-particle cross-sections $\mathcal{R}_n(E)$ still retains a single factor of $n!$. Indeed, according to (2.4), the amplitude squared contributes the factor of $(n!)^2$, and combining with the symmetry factor from the bosonic $n$-particle phase space we have $\mathcal{R}_n(E) \sim \frac{n!}{n!} n! \sim n!$.}

The original Landau WKB method \cite{23} was setup for computing matrix elements of generic operators in Quantum Mechanics between the initial and final states with different energy eigenvalues. In the QFT settings, the initial state is a vacuum and the final state is the $n$-particle final state with $n \gg 1$. It is known that to the leading exponential accuracy the transition rates computed using the Landau WKB method do not depend on the specific form of the operator $\mathcal{O}$ used to deform the initial state, if this deformation is not exponential. It is then similarly expected that the choice of the operator in (3.2) does not affect the exponent in the transition rates in the QFT settings either.

The multiparticle rate (A.1) in question is represented as the double functional integral (one for each of the matrix elements) with additional integrations over the Lagrange multipliers implementing the projections onto final states with finite energy and particle number. All these integrals are subsequently computed in the steepest descent approximation for all integration variables which is justified in the double-scaling weak-coupling / large-$n$ semiclassical limit (3.3).

The semiclassical result for the rate then takes the form \cite{1},

$$\mathcal{R}_n(E) \simeq \exp \left[ W(E, n) \right]$$  \hspace{1cm} (A.3)

$$W(E, n) \equiv \frac{1}{\lambda} \mathcal{F}(\lambda n, \varepsilon) = ET - n\theta - 2\text{Im}[h] .$$  \hspace{1cm} (A.4)

Let us now examine the structure of this result. The function $\mathcal{F}(\lambda n, \varepsilon)$ appearing in (A.4), is a function of two arguments, $\lambda n$ and $\varepsilon$, characterising the final $n$-particle state with the average kinetic energy per particle per mass $\varepsilon$. All the integrations in the path integral representation of $\mathcal{R}_n(E)$ in (A.1) were carried out and saturated by their saddle-point values in the large-$n$, large-$1/\lambda$ limit (3.3). At negative values of $\mathcal{F}(\lambda n, \varepsilon)$ the multiparticle rate $\mathcal{R}_n(E)$ is exponentially suppressed, while if $\mathcal{F}(\lambda n, \varepsilon)$ crosses zero and becomes positive above some critical energy or multiplicity, the multi-particle processes enter the Higgslosion phase \cite{3}.
The function $W(E, n)$ is computed on the saddle-point value of the path integral. We now consider the terms appearing in the final expression in (A.4). First, the combination $-2\text{Im}S[h]$ follows from the $e^{-iS}e^{iS}$ factor (where $S$ is the action) in the product of the matrix elements in (3.1). The integration contours and the resulting saddle-points in the steepest descent integration are complex-valued, hence $iS[h] - iS[h]^* = -2\text{Im}S[h]$ or equivalently $-2S_{\text{Eucl}}[\hat{h}]$ using the Euclidean notation. The remaining parameters, $T$ and $\theta$, appearing on the right hand side of (A.4), are the Lagrange multipliers that emerged from the projection operators $P_E$ and $P_n$ onto the final states with defined values of the energy $E$ and the particle number $n$ in (A.1). The parameters $T$ and $\theta$ are some of the integration variables in the integral representation of (A.1); in the steepest descent approximation, they form a part of the saddle point parameter set and take the fixed value on a given saddle point solution.

Prior to taking the $j \to 0$ limit, the saddle-point field configuration $h(x)$ is given by a particular solution to the classical equation of motion with the singular source term $j(x) = j\delta^{(4)}(x)$ on the right hand side,

$$\frac{\delta S}{\delta h(x)} = i j \delta^{(4)}(x), \quad \text{(A.5)}$$

where $S = \int d^4x \mathcal{L}$ is the action of the theory and $j$ is a constant. After taking the limit $j \to 0$, the right hand side of the defining equation (A.5) vanishes but the required solution nevertheless remains singular at $x = 0$ in Minkowski space. The saddle-point solution also depends on the parameters $T$ and $\theta$, as will be explained below, while the overall expression $W(E, n)$ is independent of $T$ and $\theta$. Hence,

$$2 \frac{\partial \text{Im}S}{\partial T} = E, \quad 2 \frac{\partial \text{Im}S}{\partial \theta} = -n, \quad \text{(A.6)}$$

and $W(E, n)$ is the Legendre transformation of the action $2\text{Im}S$ with respect to $T$ and $\theta$.\footnote{Indeed, it follows from the definition of $W$ that $\frac{\partial W}{\partial T} = T$ and $\frac{\partial W}{\partial \theta} = -\theta$. The action $S[h]$ depends on the parameters $T$ and $\theta$ through the classical solution $h(x)$, but in the final expression for $W(E, n)$ these parameters are traded for $E$ and $n$.}

Next step is to specify the boundary conditions of the solution $h(x)$ at $t_{\text{fin}} \to -\infty$ and $t_{\text{fin}} \to +\infty$. At the initial and final time boundaries $h(x)$ satisfies the free Klein-Gordon equation, thus

$$h(\vec{x}, t)|_{t \to -\infty} \to v + \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} a_k^\dagger e^{ik\mu x^\mu} \quad \text{(A.7)}$$

$$h(\vec{x}, t)|_{t \to +\infty} \to v + \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left( c_k e^{-ik\nu x^\nu} + b_k^\dagger e^{ik\nu x^\nu} \right). \quad \text{(A.8)}$$

where we used the standard notation $k_0 = \omega_k = \sqrt{m^2 + \vec{k}^2}$ so that $e^{\pm ik\nu x^\nu} = e^{\pm i(\omega_k t - \vec{k} \cdot \vec{x})}$. The $t \to -\infty$ boundary condition in Eq. (A.7) contains only the positive frequency components $a_k^\dagger e^{-i\omega_k t}$ and no negative frequency ones $a_k e^{i\omega_k t}$. In the second quantisation operator formalism, this condition implements the requirement that there are no particles in the initial state, since the creation operator $a_k^\dagger$ annihilates the bra-state vacuum $\langle 0 |$. The
second boundary condition (A.8) at the final time $t \to +\infty$ contains both positive and negative frequency components. Following [1] we parameterise its $c_k$ coefficient in terms of the complex conjugate of its $b_k^\dagger$ coefficient,
\[
c_k = b_k e^{i\omega_k T - \theta}.
\]
(A.9)

The solution is complex-valued since $c_k \neq b_k$, and the corresponding parameters $T$ and $\theta$ are precisely those appearing in (A.6).

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