Generalizing prime avoidance: the ideal avoidance property

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Abstract. Recently in [6, Theorem 2.2] we generalized the classical prime avoidance lemma to radical ideals. Motivated by this result, we introduce and study a general notion of avoidance for a commutative ring, without requiring any condition of primeness. We show that a finite product of rings has avoidance if and only if each factor does. It is proved that the avoidance property is preserved under flat ring epimorphisms. Dually, we also formulate a notion of strong avoidance, and show that it is reflected by pure morphisms (e.g. faithfully flat or split ring maps).

1. Introduction

One of the main and surprising results of [6, Theorem 2.2] asserts that the classical prime avoidance lemma can be generalized to radical ideals: if an ideal in a commutative ring is contained in the union of finitely many ideals, then it is contained in the radical of one of them. This leads to a natural question of how necessary the condition of being prime is for prime avoidance. In this article, we first apply the above theorem to formulate a general avoidance lemma, cf. Theorem 2.2 (although Theorem 2.2(iii) is well known, the remaining criteria are new). Next, we consider a general notion of avoidance (without requiring prime/radical ideals – cf. Definition 2.3), and investigate the rings which satisfy avoidance for any finite set of ideals. In particular, we study when the property of avoidance is preserved under various constructions and ring maps. For instance, we prove that the avoidance property is preserved by finite products of rings (see Theorem 2.11). Finally, we investigate the behavior of the avoidance property along ring maps in Theorem 3.1 and Lemma 3.6 and their consequences.

2. The ideal avoidance property

We begin with a general version of the avoidance lemma, encompassing all known conditions under which avoidance holds. In this article, all rings are commutative with 1 ≠ 0.

Definition 2.1. Let R be a ring. We say that an ideal I ⊆ R is immersive if whenever I ⊆ √J for some ideal J ⊆ R, one has I ⊆ J.

For example, every ideal generated by a set of idempotents is an immersive ideal.

Theorem 2.2. Let I, I₁, ..., Iₙ be finitely many ideals of a ring R such that I ⊆ ⋃ₖ=₁ⁿ Iₖ. If any of the following conditions holds, then I ⊆ Iₖ for some k.

(1) I is a principal ideal or an immersive ideal.

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(ii) All but two of the $I_k$ are radical ideals.

(iii) $R$ contains an infinite field as a subring.

Proof. (i): The assertion for principal ideals is clear. Suppose $I$ is immersive. Then by [6, Theorem 2.2], $I \subseteq \sqrt{I_k}$ for some $k$, hence $I \subseteq I_k$.

(ii): The assertion is clear for $n \leq 2$, so we may assume $n \geq 3$. Suppose $I \not\subseteq I_k$ for all $k$. After renumbering if necessary, we may assume $I_k$ is radical for all $k \geq 3$. So for each $k \geq 3$, there exists a prime ideal $p_k$ of $R$ containing $I_k$ such that $I \not\subseteq p_k$. Then $I \subseteq I_1 \cup I_2 \cup \left( \bigcup_{k=3}^n p_k \right)$ which is in contradiction with the classical prime avoidance lemma (see e.g. [9, Tag 00DS] or [3, Theorem 3.61]). Hence $I$ is contained in some $I_k$.

(iii): We have $I = \bigcup_{k=1}^n (I \cap I_k)$. If $K$ is a field contained in $R$, then every ideal of $R$ is naturally a $K$-vector space. It is well known that a vector space over an infinite field is not a finite union of proper subspaces. Thus $I = I \cap I_k$ for some $k$, hence $I \subseteq I_k$. □

Theorem 2.2 together with [6, Theorem 2.2], leads us to the following natural definition:

**Definition 2.3.** We say that a ring $R$ has the ideal avoidance property (or simply has avoidance) if whenever $I, I_1, \ldots, I_n$ are finitely many ideals of $R$ with $I \subseteq \bigcup_{k=1}^n I_k$, then $I \subseteq I_k$ for some $k$.

We can now globalize Theorem 2.2.

**Corollary 2.4.** Let $R$ be a ring. If any of the following conditions holds, then $R$ has avoidance.

(i) $R$ is a principal ideal ring.

(ii) $R$ is an absolutely flat (von Neumann regular) ring.

(iii) $R$ contains an infinite field as a subring.

(iv) $R$ is a valuation ring.

Proof. (i): Follows from Theorem 2.2(i).

(ii): If $R$ is absolutely flat, then every ideal is radical, hence the assertion follows from Theorem 2.2(ii). (Note that in this case every ideal is also immersive, so one could also appeal to Theorem 2.2(ii).)

(iii): Follows from Theorem 2.2(iii).

(iv): If $R$ is a valuation ring, then every set of ideals is linearly ordered, hence any finite family of ideals has a unique maximal element. □

**Remark 2.5.** Regarding Corollary 2.4(ii): note that a ring $R$ is absolutely flat if and only if every ideal of $R$ is a radical ideal, or equivalently, every ideal of $R$ is an immersive ideal, or equivalently, every ideal of $R$ is generated by idempotents. Every Boolean ring is absolutely flat, and every absolutely flat ring is a zero dimensional ring. Indeed, a ring is absolutely flat if and only if it is reduced and zero dimensional. In particular, any reduced Artinian ring is absolutely flat, so it has avoidance. One can also find many other characterizations for absolutely flat rings in the literature.

In Corollary 2.4 we observed that every principal ideal ring (PIR) has avoidance. In the presence of some finiteness assumptions, the converse can hold as well:
Proposition 2.6. Let $R$ be a finite ring. Then $R$ has avoidance if and only if $R$ is a PIR.

Proof. Suppose $R$ is not a PIR, and let $I \subseteq R$ be a nonprincipal ideal. Then $I = \bigcup_{x \in I} Rx$ is a finite union of proper principal ideals, so $R$ does not have avoidance. □

Example 2.7. It is useful to have a concrete example of a ring that does not have avoidance. Let $K$ be a finite field. Then the ring $K[x, y]/(x^2, xy, y^2)$ is a typical example of a finite ring with a nonprincipal ideal $I = (x, y)$, hence, does not have avoidance. More generally, let $X = \{x_i : i \in S\}$ be any set of indeterminates over $K$ with $|S| \geq 2$. Consider $R = K[X]/m^2$ (where $m = (x_i : i \in S)$ is the homogeneous maximal ideal of $K[X]$), and the nonprincipal ideal $(x_1, x_2) \subseteq R$, which can be written as a finite union of principal ideals

$$(x_1, x_2) = \bigcup_{a, b \in K} (ax_1 + bx_2).$$

Hence, $R$ does not have avoidance.

Our next goal is to study when the avoidance property is preserved by various ring-theoretic operations. Let $\phi : R \to S$ be a ring map. Recall that an ideal $J$ of $S$ is extended under $\phi$ if $J = I^\phi$ for some ideal $I$ of $R$, or equivalently $J^e = J$. Dually, an ideal $I$ of $R$ is contracted under $\phi$ if $I = J^c$ for some ideal $J$ of $S$, or equivalently $I = I^c$.

Lemma 2.8. Let $\phi : R \to S$ be a ring map such that every ideal of $S$ is extended under $\phi$. If $R$ has avoidance, then $S$ has avoidance.

Proof. If $J, J_1, \ldots, J_n$ are finitely many ideals of $S$ with $J \subseteq \bigcup_{k=1}^n J_k$, then $\phi^{-1}(J) \subseteq \phi^{-1}(\bigcup_{k=1}^n J_k) = \bigcup_{k=1}^n \phi^{-1}(J_k)$. Since $R$ has avoidance, $J^c = \phi^{-1}(J) \subseteq (J_k)^c = \phi^{-1}(J_k)$ for some $k$. Thus $J = J^c \subseteq (J_k)^c = J_k$. □

Corollary 2.9. If a ring $R$ has avoidance, then any quotient or localization of $R$ has avoidance.

Proof. It follows from Lemma 2.8. □

Remark 2.10. Let $K$ be a finite field. Then the polynomial ring $K[x_1, \ldots, x_n]$ does not have avoidance for all $n \geq 2$, because its quotient modulo $m^2$ fails to have this property where $m = (x_1, \ldots, x_n)$. Thus the avoidance property is not preserved by adjoining indeterminates. For example, $K[x]$ has avoidance, since it is a PID, but $K[x, y]$ does not have avoidance. Furthermore, the avoidance property of rings need not pass to subrings or extensions. As an example, for the ring $K[x, y]$, its subring $K$ and its field of fractions $K(x, y)$ both have avoidance. Finally, avoidance need not be preserved by tensor products: take e.g. $K[x, y] \cong K[x] \otimes_K K[y]$.

Next we consider finite products. In commutative algebra, checking a property for finite products of rings is often straightforward, but in the following result we will observe that establishing the avoidance property for finite products of rings is not as easy as one may initially think.

Theorem 2.11. If $R = \prod_{i=1}^n R_i$ is a finite product of rings, then $R$ has avoidance if and only if $R_i$ has avoidance for all $i$. 

Proof. By Corollary 2.9, if \( R \) has avoidance then each \( R_i \) has avoidance. Conversely, by induction it suffices to consider a product of two rings. Suppose the rings \( R_1, R_2 \) have avoidance. It is well known that every ideal of \( R_1 \times R_2 \) is precisely of the form \( I \times J \) where \( I \) (resp. \( J \)) is an ideal of \( R_1 \) (resp. \( R_2 \)). Let \( I \times J \) be an ideal in \( R_1 \times R_2 \) with \( I \times J \subseteq \bigcup_{i=1}^{n} (I_i \times J_i) \). It follows that \( I \subseteq \bigcup_{i=1}^{n} I_i \). Since \( R_1 \) has avoidance, there exists at least one \( i \) such that \( I \subseteq I_i \). By reordering we may assume there exists some \( d \) with \( 1 \leq d \leq n \) such that that \( I \subseteq I_i \) for all \( 1 \leq i \leq d \) and \( I \nsubseteq I_i \) for all \( d + 1 \leq i \leq n \). Then we claim that \( I \times J \subseteq \bigcup_{i=1}^{d} I_i \times J_i \). Indeed, suppose \( I \times J \nsubseteq \bigcup_{i=1}^{d} (I_i \times J_i) \), thus \( d < n \). Choose \((a, b) \in (I \times J) \setminus \bigcup_{i=1}^{d} (I_i \times J_i) \). Since each \( I_i \ldots I_d \) contains \( I \) (and thus \( a \)), it follows that \( b \) is not in any of \( J_1, \ldots, J_d \). Next, since \( I \) is not contained in any of \( I_{d+1}, \ldots, I_n \), there exists \( c \in I \setminus \bigcup_{i=d+1}^{n} I_i \) (since \( R_1 \) has avoidance). Then \((c, b) \in (I \times J) \setminus \bigcup_{i=1}^{n} (I_i \times J_i) \), contradiction. This establishes the claim. It follows that \( J \subseteq \bigcup_{i=1}^{d} J_i \). Since \( R_2 \) has avoidance, there exists \( 1 \leq k \leq d \) such that \( J \subseteq J_k \). So \( I \times J \subseteq I_k \times J_k \). \( \square \)

The avoidance property is also preserved by certain infinite products. In fact, every direct product of fields has avoidance. More generally, every direct product of absolutely flat rings is absolutely flat, and hence it has avoidance by Corollary 2.4(ii).

Remark 2.12. If a ring \( R \) has avoidance, then by Corollary 2.4 \( R_p \) has avoidance for all \( p \in \text{Spec}(R) \). But the converse does not seem to be true. In order to give a counterexample, it would suffice to find a Dedekind domain which does not have avoidance, because the localization of a Dedekind domain at any prime ideal is a PID, which has avoidance by Corollary 2.4(i).

Remark 2.13. Let \( R \) be a ring such that every ideal can be written as an intersection (possibly infinite) of primary ideals (e.g. a Noetherian or Laskerian ring). Then \( R \) has avoidance if and only if it has “primary avoidance”, i.e. if an ideal of \( R \) is contained in a finite union of primary ideals, then it is contained in one of them. Indeed, assume \( I, I_1, \ldots, I_n \) are finitely many ideals of \( R \) with \( I \subseteq \bigcup_{k=1}^{n} I_k \). Suppose \( I \) is not contained in any of the \( I_k \). Thus for each \( k \) there exists a primary ideal \( q_k \) of \( R \) containing \( I_k \) such that \( I \) is not contained in \( q_k \). But \( I \subseteq \bigcup_{k=1}^{n} q_k \) which is a contradiction.

Another setting in which the avoidance property is closely related to being a PIR is for graded algebras over finite fields:

Theorem 2.14. Let \( R \) be an \( \mathbb{N} \)-graded ring with \( R_0 = K \) a finite field. If \( R \) has avoidance, then it is a PIR.

Proof. Any \( \mathbb{N} \)-graded \( K \)-algebra has a presentation of the form \( K[X]/I \), where \( X \) is a set of indeterminates over \( K \) and \( I \) is a homogeneous ideal in \( K[X] \) contained in \( \mathfrak{m}^2 \), where \( \mathfrak{m} = (X) \) is the homogeneous maximal ideal of \( K[X] \). If \( R \) is not a PIR, then necessarily \(|X| \geq 2\) (any quotient of a univariate polynomial ring over \( K \))
is a PIR). Since there exists a surjection $R \rightarrow K[X]/m^2$, by Example 2.7, $R$ does not have avoidance.

In fact, one sees from this that the only $\mathbb{N}$-graded $K$-algebras which have avoidance (for $K$ a finite field) are $K[x]$ and $K[x]/(x^i)$ for some $i \geq 1$.

\section{The avoidance property along ring maps}

We next turn towards avoidance in the relative setting, i.e. its behavior along a ring map. Note that a ring map $\phi : R \rightarrow S$ is an epimorphism if and only if for each $S$-module $M$ the canonical morphism of $S$-modules $M \otimes_R S \rightarrow M$ given by $x \otimes s \mapsto sx$ is injective (in fact, an isomorphism).

\textbf{Theorem 3.1.} Let $\phi : R \rightarrow S$ be a flat ring epimorphism. If $R$ has avoidance, then $S$ has avoidance.

\textit{Proof.} It is well known that every ideal of $S$ is extended under $\phi$ (cf. [5] or [7, Theorem 2.5(ii)]). Here we provide a short proof for completeness. Let $J$ be an ideal of $S$, and set $I := \phi^{-1}(J)$. Clearly $IS \subseteq J$. For the reverse inclusion, note that the canonical ring map $\phi' : R/I \rightarrow S/J$ given by $r + I \mapsto \phi(r) + J$ is injective. This gives an injective map $\phi' \otimes 1_S : R/I \otimes_R S \rightarrow S/J \otimes_R S$, because $S$ is $R$-flat. Thus the canonical ring map $f : S/IS \rightarrow S/J$ given by $s + IS \mapsto s + J$ factors as $S/IS \xrightarrow{\sim} R/I \otimes_R S \xrightarrow{\phi' \otimes 1_S} S/J \otimes_R S \xrightarrow{\sim} S/J$. Hence $f$ is injective, so $J \subseteq IS$. Now the assertion follows from Lemma 2.8.

The above result, in particular, yields an alternative proof to the fact that the avoidance property is preserved by localizations (see Corollary 2.9). Indeed, for any multiplicative subset $S$ of a ring $R$, the canonical ring map $R \rightarrow S^{-1}R$ is a typical example of a flat epimorphism.

We now seek dual versions of Lemma 2.8 and Theorem 3.1. If $\phi : R \rightarrow S$ is a ring map and $I, I_1, \ldots, I_n$ are ideals of $R$ with $I \subseteq \bigcup_{k=1}^n I_k$, then $\phi(I) \subseteq \phi\left(\bigcup_{k=1}^n I_k\right) = \bigcup_{k=1}^n \phi(I_k)$. However, the extension ideal $I^c = IS$ is not necessarily contained in $\bigcup_{k=1}^n (I_k)^c$. This leads us to the following definition.

\textbf{Definition 3.2.} We say that a ring $S$ has \textit{strong avoidance} if for every ring map $\phi : R \rightarrow S$ and for any finitely many ideals $I, I_1, \ldots, I_n$ of $R$, if $I \subseteq \bigcup_{k=1}^n I_k$ then $IS \subseteq I_kS$ for some $k$.

Strong avoidance easily implies avoidance by considering the identity ring map. However, the converse does not hold: there are rings which have avoidance, but do not have strong avoidance (see Example 3.3).

\textbf{Proposition 3.3.} Every absolutely flat ring has strong avoidance.

\textit{Proof.} Consider a ring map $\phi : R \rightarrow S$ and finitely many ideals $I, I_1, \ldots, I_n$ of $R$ with $S$ an absolutely flat ring. If $I \subseteq \bigcup_{k=1}^n I_k$, then by [6, Theorem 2.2], $I \subseteq \sqrt{I_k}$ for some $k$. This yields that $I^c \subseteq (\sqrt{I_k})^c \subseteq \sqrt{(I_k)^c} = (I_k)^c$, the latter equality follows from the fact every ideal of an absolutely flat ring is a radical ideal. \hfill $\square$
Remark 3.4. The converse of the above result does not hold. For example, we show that the ring of integers \( \mathbb{Z} \) has strong avoidance. Let \( \phi : R \to \mathbb{Z} \) be a ring map. Since \( \mathbb{Z} \) is the initial object in the category of rings, there exists a unique ring map \( \psi : \mathbb{Z} \to R \) such that \( \phi \psi \) is the identity map. Thus \( \phi \) is surjective, so for any ideal \( I \) of \( R \), one has \( I^\phi = \phi(I) \). Now if \( I, I_1, \ldots, I_n \) are ideals of \( R \) with \( I \subseteq \bigcup_{k=1}^n I_k \), then \( I^\phi = \phi(I) \subseteq \phi\left( \bigcup_{k=1}^n I_k \right) = \bigcup_{k=1}^n \phi(I_k) = \bigcup_{k=1}^n (I_k)^\phi \). But \( (I_k)^\phi \) is a principal ideal since \( \mathbb{Z} \) is a PID, so \( I^\phi \subseteq (I_k)^\phi \) for some \( k \).

In order to correctly dualize Theorem 3.1, one must find a suitable class of monomorphisms. Recall that a ring map \( \phi : R \to S \) is called a pure morphism (or universally injective) if for every \( R \)-module \( M \), the induced map \( 1_M \otimes \phi : M \otimes_R R \to M \otimes_R S \), or equivalently, the map \( M \to M \otimes_R S \) given by \( x \mapsto x \otimes 1 \) is injective. Pure morphisms have very nice properties, and starting with Grothendieck, have been extensively studied in the literature (see e.g. \[3, \text{Tag \text{08WE}}\]).

By restricting to cyclic modules \( M = R/I \), we obtain a notion of semi-pure morphism: we say that a ring map \( \phi : R \to S \) is semi-pure if for each ideal \( I \) of \( R \) the induced ring map \( R/I \to S/IS \) given by \( r + I \mapsto \phi(r) + IS \) is injective. In particular, a semi-pure morphism is injective. Note that a ring map is semi-pure if and only if every ideal of the source ring is contracted under this map.

Some of the classes of rings in Corollary 2.4 are closed under reflection by semi-pure maps:

**Proposition 3.5.** Let \( \phi : R \to S \) be a semi-pure ring map. Then:

(i) If \( S \) is absolutely flat, then \( R \) is as well.

(ii) If \( S \) is a valuation ring, then \( R \) is as well.

In particular, in both cases above \( R \) has avoidance.

**Proof.** (i): If \( S \) is absolutely flat, then every ideal of \( S \) is radical. Since the contraction of a radical ideal is radical, this shows that every ideal of \( R \) is radical, so \( R \) is absolutely flat.

(ii): If \( S \) is a valuation ring, then \( S \) is a domain whose ideals are totally ordered under inclusion. Since \( \phi \) is injective, \( R \) is also a domain, and any two ideals \( I_1, I_2 \) of \( R \) are contractions of comparable ideals \( J_1, J_2 \) of \( S \), hence \( I_1, I_2 \) are comparable in \( R \), so \( R \) is a valuation ring. \( \square \)

Finally, we are ready to state the dual version of Lemma 2.8.

**Lemma 3.6.** Let \( \phi : R \to S \) be a semi-pure ring map. If \( S \) has strong avoidance, then \( R \) has strong avoidance.

**Proof.** Let \( h : R' \to R \) be any ring map and \( I, I_1, \ldots, I_n \) ideals of \( R' \) with \( I \subseteq \bigcup_{k=1}^n I_k \). Considering extensions of ideals under the ring map \( \phi h : R' \to S \), we have \( IS \subseteq I_k S \) for some \( k \), because \( S \) has strong avoidance. Since \( \phi \) is semi-pure, \( IR = \phi^{-1}((IR)S) = \phi^{-1}(IS) \), and similarly \( I_k R = \phi^{-1}(I_k S) \). Therefore \( IR \subseteq I_k R \), and hence \( R \) has strong avoidance. \( \square \)

**Corollary 3.7.** Let \( \phi : R \to S \) be a pure morphism. If \( S \) has strong avoidance, then \( R \) has strong avoidance.
Proof. Every pure morphism is semi-pure, so the assertion follows from Lemma 3.6.

Remark 3.8. As special cases of Corollary 3.7 we have the following examples of pure morphisms:

(i) Every faithfully flat ring map is pure.

(ii) If a ring map \( \phi: R \to S \) splits as an \( R \)-module (i.e. there exists a morphism of \( R \)-modules \( \psi: S \to R \) with \( \psi \phi = 1_R \)), then it is pure. For example, for any ring map \( \phi: R \to S \), the canonical ring map \( S \to S \otimes_R S \) given by \( s \mapsto s \otimes 1 \) splits as an \( S \)-module. As another example, by the direct summand theorem (see [1] or [2, Theorem 5.4]), any finite ring map \( R \to S \) with \( R \) a regular Noetherian ring splits as an \( R \)-module.

Example 3.9. The assumption of “strong avoidance” in Lemma 3.6 is crucial. In other words, the naive dual statement of Lemma 2.8 does not hold. For example, let \( K \) be a finite field, \( R \) a \( K \)-algebra which does not have avoidance (see e.g. Example 2.7) and \( F \) an infinite field containing \( K \) as a subfield (e.g. \( F = \bar{K} \), the algebraic closure of \( K \)). Then the canonical ring map \( R \to R \otimes_K F \) given by \( r \mapsto r \otimes 1_F \) is faithfully flat, since it makes \( R \otimes_K F \) a nonzero free \( R \)-module. Moreover, note that the ring \( R \otimes_K F \) has avoidance by Corollary 2.4(iii), but does not have strong avoidance by Lemma 3.6.

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