C-VECTORS AND DIMENSION VECTORS FOR CLUSTER-FINITE QUIVERS

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Abstract. Let \((Q, W)\) be a quiver with a non degenerate potential. We give a new description of the \(c\)-vectors of \(Q\). We use it to show that, if \(Q\) is mutation equivalent to a Dynkin quiver, then the set of positive \(c\)-vectors of the cluster algebra associated to \(Q^{\ac}\) coincides with the set of dimension vectors of the indecomposable modules over the Jacobian algebra of \((Q, W)\).

1. Introduction

Since the introduction of cluster categories in [2], cf. also [4], it has been an interesting problem to find representation theoretic lifts of the concepts and constructions surrounding cluster algebras. In this note, we focus our attention on the \(c\)-vectors associated to a quiver and continue the approach started in [16]. We recall a fundamental fact first proved in [7], namely, each \(c\)-vector is non-zero and all its entries are either nonpositive or nonnegative. From [15, Section 8] (see also [16, Theorem 4]) we know that positive \(c\)-vectors associated to a quiver \(Q\) are always dimension vectors of indecomposable rigid modules over an appropriate algebra, namely, the Jacobian algebra \(J(Q, W)\) associated to \(Q^{\ac}\) and a generic potential \(W^{\ac}\) on it. We denote by \(\Phi_{\text{re, Sch}}(Q, W)\) the set of these dimension vectors and call its elements the real Schur roots associated to \((Q, W)\).

Let \(\mathfrak{c}_+(Q)\) be the set of positive \(c\)-vectors associated to \(Q\). We are interested in determining when the following equality holds

\[
\mathfrak{c}_+(Q) = \Phi_{\text{re, Sch}}(Q, W).
\]

A first step in this direction was accomplished in [16] where we proved that (1.1) holds if \(Q\) is acyclic. In this note, we introduce the so-called \(c\)-modules (see Theorem 2) which allow us to deduce (1.1) for all cluster-finite (skew-symmetric) cluster algebras. This qualitative result complements the work of T. Nakanishi and S. Stella [18], who gave an explicit diagrammatic description of the \(c\)-vectors for cluster-finite (skew-symmetrizable) cluster algebras.

2. \(c\)-Modules

Throughout this note, we will freely use the concepts from the study of cluster categories. We refer the reader to [6] for details concerning quivers with potentials (QP’s for short), to [1, 2, 17] for the definition and further properties of cluster categories and to [14] [15] and section 7 of [12] for the construction of distinguished triangulated equivalences induced by QP mutation.

Notation 1. Let \((Q, W)\) be a QP with a non degenerate potential \(W\). Let \(n\) be the number of vertices of \(Q\). We denote by \(\Gamma\) (resp. \(\mathcal{C}\)) the associated Ginzburg dg-algebra (resp. cluster category). Whenever there are unspecified morphisms or indices, we assume
they are the “most natural” ones, for instance, in the expression
\[ \bigoplus_{i \rightarrow j} \Gamma_i \rightarrow \Gamma_j \]
the sum is taken over all arrows in \( Q \) ending at \( j \) and the morphism is given by left multiplication by the arrows. In particular, associated to every cluster tilting object \( T' = \bigoplus T_i' \) in \( C \), we have exchange triangles
\[
T_j'^* \rightarrow \bigoplus_{i \rightarrow j} T_i' \rightarrow T_j' \rightarrow \Sigma T_j'^* \quad \text{and} \quad T_j' \rightarrow \bigoplus_{j \rightarrow i} T_i' \rightarrow T_j'^* \rightarrow \Sigma T_j'.
\]
When there is no risk of confusion, we will abbreviate the expression \( \text{Hom}(X,Y) \) by \( (X,Y) \).

Finally, for each \( i \in \mathbb{Z} \), denote by \( \text{pr}_{\Sigma^i} \Gamma \) the full subcategory of \( D \Gamma \) whose objects are the cones of morphisms in \( \text{add} \Sigma^i \Gamma \).

**Theorem 2.** Let \( t \) be a vertex of the \( n \)-regular tree \( T_n \) and \( T' = \bigoplus T_i' \) be the corresponding cluster-tilting object in \( C \), cf. section 7.7 of [12]. If \( c_j(t) \) is positive, then it equals the dimension vector of the \( J(Q,W) \)-module

\[
(2.1) \quad \text{coker} \left( \bigoplus_{j \rightarrow i} \text{Hom}(T, \Sigma T_i') \rightarrow \text{Hom}(T, \Sigma T_j'^*) \right).
\]

If \( c_j(t) \) is negative, then its opposite equals the dimension vector of the \( J(Q,W) \)-module

\[
(2.2) \quad \text{coker} \left( \bigoplus_{i \rightarrow j} \text{Hom}(T, \Sigma T_i') \rightarrow \text{Hom}(T, \Sigma T_j') \right).
\]

Moreover, for each pair \((t,j)\) exactly one of the modules described above is nonzero.

For obvious reasons we call modules of this form \( c \)-modules. Theorem 2 is a direct consequence of the following proposition and the fact that \( H^*(S_j(t)) \) is concentrated either in degree 0 or degree 1. Let \( \langle Q', W' \rangle \) be a QP which is mutation equivalent to \( \langle Q, W \rangle \). Denote by \( \Gamma' \) the corresponding Ginzburg dg algebra and by \( C' \) the corresponding cluster category.

**Proposition 3.** Let \( F : D\Gamma' \rightarrow D\Gamma \) be a triangle equivalence such that \( D_{\leq 0} \Gamma \subseteq FD_{\leq 0} \Gamma' \subseteq D_{\leq 1} \Gamma \) and that \( F(\Gamma') \) lies in \( \text{pr} \Sigma^{-1} \Gamma \). Suppose that \( T' \), the image of \( F(\Gamma') \) in \( C \), is a cluster tilting object. Then there are short exact sequences

\[
\bigoplus_{j \rightarrow i} \text{Hom}_C(T, \Sigma T_i') \rightarrow \text{Hom}_C(T, \Sigma T_j'^*) \rightarrow H^0(FS_j') \rightarrow 0
\]

and

\[
\bigoplus_{i \rightarrow j} \text{Hom}_C(T, \Sigma T_i') \rightarrow \text{Hom}_C(T, \Sigma T_j') \rightarrow H^1(FS_j) \rightarrow 0.
\]

**Proof.** Consider the triangles

\[
(2.3) \quad \bigoplus_{i \rightarrow j} \Gamma_i' \rightarrow \Gamma_j' \rightarrow U_j \rightarrow \Sigma \bigoplus_{i \rightarrow j} \Gamma_i'
\]

and

\[
(2.4) \quad \Gamma_j' \rightarrow \bigoplus_{j \rightarrow i} \Gamma_i' \rightarrow V_j \rightarrow \Sigma \Gamma_j'.
\]
induced by the natural morphisms. Then, there is a triangle \( \Sigma V_j \to U_j \to S_j' \to \Sigma^2 V_j \)
ind in \( \mathcal{D}_\Gamma' \). If we consider the long exact sequence in homology induced by the image of this
last triangle, then, since \( F \mathcal{D}_{\leq 0} \subseteq \mathcal{D}_{\leq 1} \), we obtain
\[
(2.5) \quad H^1(FU_j) \cong H^1(FS_j)
\]
and the exact sequence
\[
(2.6) \quad H^1(FV_j) \to H^0(FU_j) \to H^0(FS_j) \to 0.
\]
Recall that there is an isomorphism \( \text{Hom}_{\mathcal{D}_\Gamma'}(X, Y) \to \text{Hom}_C(X, Y) \), whenever \( X \) and \( Y \) lie
in \( \text{pr} \Sigma i \Gamma \) (for all \( i \in \mathbb{Z} \)). Since \( FT' \) and \( \Gamma \) (resp. \( \Sigma FT' \) and \( \Gamma \)) belong to \( \text{pr} \Sigma^{-1} \Gamma \) (resp.
and \( \text{pr} \Gamma \)), we can apply this to the image in \( \mathcal{D}_\Gamma \) of the triangle (2.3) to construct the following
diagram
\[
\begin{array}{cccc}
\bigoplus_{i \to j} (\Gamma, FT_i') & \to & (\Gamma, FU_j) & \to \bigoplus_{i \to j} (\Gamma, \Sigma FT_i') \\
\bigoplus_{i \to j} (T, T_i') & \to & (T, \Sigma T_j') & \to \bigoplus_{i \to j} (T, \Sigma T_j').
\end{array}
\]
We obtain an isomorphism
\[
(2.7) \quad H^0(FU_j) \cong \text{Hom}_C(T, \Sigma T_j')
\]
and an exact sequence
\[
(2.8) \quad \bigoplus_{i \to j} \text{Hom}_C(T, \Sigma T_i') \to \text{Hom}_C(T, \Sigma T_j') \to H^1(FU_j) \to 0.
\]
Applying a similar argument to the triangle (2.4), we obtain an epimorphism
\[
(2.9) \quad \bigoplus_{j \to i} \text{Hom}_C(T, \Sigma T_j') \to H^1(FV_j) \to 0.
\]
Combining (2.6) with (2.7) and (2.9) we obtain the natural morphism \( \bigoplus_{j \to i} \text{Hom}_C(T, \Sigma T_j') \to \text{Hom}_C(T, \Sigma T_j') \) and thus the first exact sequence of the statement. We obtain the second
sequence by combining (2.5) with (2.8). □

**Definition 4.** We say that a module \( M \) over a finite dimensional algebra is \( \tau \)-rigid if \( \text{Hom}(M, \tau M) = 0 \). Notice that the images in \( \text{mod} J(Q, W) \) of the rigid indecomposable
objects in \( C \) under the canonical functor \( \text{Hom}(\Gamma, \cdot) \) are the \( \tau \)-rigid indecomposable modules, cf. section 3.5 of [13].

**Corollary 5.** Let \( (Q, W) \) be a Jacobi-finite quiver with potential. Suppose \( M \) is a \( \tau \)-rigid
indecomposable \( J(Q, W) \)-module and that \( M \cong \text{Hom}_C(T, M) \) for an object \( M \) of \( C \). If there
is a cluster tilting object \( T' \) in \( C \) such that \( M \cong \Sigma T_j' \) where \( T_j' \) is a source of \( T' \), then
\( \dim(M) \) is a \( c \)-vector.

**Theorem 6.** Let \( (Q, W) \) be a quiver with potential which is mutation equivalent to a Dynkin quiver \( (\Delta, 0) \) (i.e. \( Q \) is cluster-finite). Then the set of positive \( c \)-vectors equals the
set of dimension vectors of the indecomposable \( J(Q, W) \)-modules.
Proof. Let $M$ be an indecomposable $J(Q,W)$-module. We can find an indecomposable object $X$ in $C_{Q,W}$ such that $\text{Hom}_{C}(T, X) = M$. Since $C_{Q,W}$ is triangle equivalent to the usual cluster category $C_{\tilde{X}}$, the object $X$ must be rigid and thus $M$ is $\tau$-rigid (in particular rigid). It is easy to see that for Dynkin quivers we can obtain every rigid indecomposable object in the cluster category as source of a cluster tilting object (see the proof of Proposition 3 in [16]). Now, using Corollary 5 we obtain the result. \[ \square \]

Remark 7. It follows that the sets of $c$-vectors of $Q$ and $Q^{\text{op}}$ coincide. Indeed, the sets of dimension vectors of indecomposable modules over $J(Q, W)$ and $J(Q, W)^{\text{op}}$ coincide.

3. Examples

We analyze two examples, one of them beyond cluster-finite type, to show how Theorem 2 can be used to understand families of $c$-vectors.

Remark 8. ([2]) Let $Q$ be a quiver mutation equivalent to an acyclic quiver $Q'$ and $k$ be a field. Recall that $\text{Hom}_{Q}(X, Y) = \text{Hom}_{kQ'(X, Y)} \oplus \text{Ext}_{kQ'}^{1}(X, \tau^{-1}Y)$ for any pair of objects $X$ and $Y$ in $C$ and that only one of these summands is non-zero. We call the elements of $\text{Hom}_{Q}(X, Y)$ (resp. $\text{Ext}_{Q}^{1}(X, \tau^{-1}Y)$) $H$-morphisms (resp. $F$-morphisms).

Definition 9. Let $U$ be a rigid indecomposable object of $C$ and $T'$ a reachable cluster-tilting object containing $U$ as a direct summand. Let $t$ be the vertex of $T_{n}$ corresponding to $T'$ and $j$ the vertex of $Q$ corresponding to $U$. We write $c_{U,T'}$ for the $c$-vector $c_{j}(t)$. Denote by $c_{U}$ the set of vectors obtained in this way and by $c_{U}^{-}$ (resp. $c_{U}^{+}$) its subset of negative (resp. positive) vectors. Note that this notation depends on the orientation of $Q$.

Example 10. (Type $\tilde{A}_{1,1}$) Let $C$ be a cluster category of type $\tilde{A}_{1,1}$, i.e. the cluster category associated with the quiver

\[
Q : \begin{array}{c}
1 \\
\downarrow \downarrow \\
2 & 3
\end{array}
\]

Here, we let $T_{i} = P_{i}$, i.e. the image in $C$ of the indecomposable projective $kQ$-module associated to the vertex $i$. We take $T = T_{1} \oplus T_{2} \oplus T_{3}$ as the initial cluster tilting object and let $T' = \Sigma^{-1}T_{1} \oplus S_{2} \oplus \Sigma^{-1}T_{3}$. We can visualize $\text{End}_{C}(T')$ as follows

\[
\begin{array}{c}
S_{2} \\
\downarrow \\
\Sigma^{-1}T_{1} \longrightarrow \Sigma^{-1}T_{3}
\end{array}
\]

The cluster-tilting objects of the cluster category $C$ are in bijection with the vertices of the planar graph in Figure 1 and the indecomposable rigid objects with the connected components of its complement in the plane, cf. [8] [5]. Let us compute $c_{\Sigma^{-1}T_{3},T'}$ using the formulas in Theorem 2. Since $T$ is the image of a projective $kQ$-module, we have $\text{Hom}_{C}(T, X) \cong \text{Hom}_{kQ}(T, X)$ for every $kQ$-module $X$ (see Remark 8). Therefore, the morphism in the equation (2.2) becomes

\[ \text{Hom}_{kQ}(T_{1}, T') \oplus \text{Hom}_{kQ}(T_{1}, T_{1}) \longrightarrow \text{Hom}_{kQ}(T, T_{3}). \]

Its cokernel is the indecomposable $kQ$-module with dimension vector $(0, 1, 1)$, thus $c_{\Sigma^{-1}T_{3},T'} = (0, -1, -1)$.

Now suppose our initial seed is given by the cluster tilting object $T = P_{1} \oplus \tau S_{2} \oplus P_{3}$. We proceed to compute the family $c_{S_{2}}$. Notice that the complements of $S_{2}$ are of the form:
where $i$ is an odd integer. First suppose $T'$ is as in cases $(A_i)$, $(B_i)$ or $(C_i)$ for $i \geq 1$. Note that there are only $H$-morphisms from $T$ to $\Sigma S_2$. Suppose that the morphism in (2.2) is non-zero for any of the cases mentioned above. Then there exists a commutative diagram

$$
\begin{array}{c}
X \\
g \\
T \\
\downarrow{f} \\
\Sigma S_2,
\end{array}
$$

for every $X \in \{\Sigma^i P_1, \Sigma^{i-1} P_2, \Sigma^i P_3 : i \geq 0\}$. By Remark 8 one of $g$ or $h$ is an $F$-morphism. This implies that $f$ is an $F$-morphism which is a contradiction. Thus $c_{S_2,T'} = (-1, -1, -1)$.

It is easy to see that in the rest of the cases, a factorization will occur. We obtain in this way $c_{S_2} = \{(−1, −1, −1), (0, −1, −1), (0, −1, 0)\}$.

Let $\mathcal{C}$ be a cluster category with finitely many isomorphism classes of indecomposables. By Theorem 6 we know that the (positive) $c$-vectors are of the form $\dim \text{Hom}_\mathcal{C}(T, U)$ where $U$ runs through the indecomposable objects of $\mathcal{C}$. This already has some nice consequences.

For example, we claim that the components of the $c$-vectors of a cluster-finite quiver $Q$ are bounded by 6 and that this bound is reached only for quivers of cluster type $E_8$. Indeed, by Remark 8, it follows that the components are bounded by the maximum of the $\dim \text{Hom}_{\mathcal{D}^b(Q)}(L,M)$, where $L$ and $M$ run through the indecomposable objects of $\mathcal{D}^b(Q)$.

Now by Happel’s description of $\mathcal{D}^b(Q)$ [10], these dimensions equal the coefficients of the simple roots in the decomposition of the positive roots in the root system of type $\Delta$, where $\Delta$ is the cluster type of $Q$. It follows that 6 is indeed the upper bound and that it is reached only for type $\Delta = E_8$. Let us determine the structure of the quivers of cluster type $E_8$ admitting a $c$-vector with a component equal to 6.

Example 11. (Type $E_8$) Let $\mathcal{C}$ be a cluster category of type $E_8$. We know that the coefficients of the positive roots (expressed in terms of the simple roots) in an $E_8$ root system are bounded by 6. Moreover, 6 can only appear as a coefficient of the simple root associated to the vertex of valency 3 in $E_8$. We claim that the cluster-finite quivers admitting a $c$-vector with an entry equal to 6 are those obtained by gluing quivers of cluster
type $A_2$, $A_3$ and $A_5$ in a common vertex. To see this, we can use Figure 2, which depicts the AR-quiver of an $E_8$ quiver. The vertices represented by $\square$ are the modules $Y$ for which $\dim \text{Hom}_{E_8}(X,Y) = 6$. This implies that if a $c$-vector $\dim \text{Hom}(T,U)$ associated with the quiver $Q$ of $\text{End}(T)$ has a component equal to 6, then $T$ contains an indecomposable factor $T_1$ in the orbit of $X$. The other factors $T_j$ satisfy $\text{Ext}^1(T_j,T_j) = 0$. Thus, if $T_j$ corresponds to $\diamondsuit$ in Figure 2, the factors $T_j$ correspond to vertices $\diamondsuit$. We know from [11] that the possible complements of $T_1$ are in bijection with the cluster-tilting objects in the Calabi-Yau reduction $T_1^*/\langle T_1 \rangle$, where

$$T_1^* = \{ X \in \mathcal{C} | \text{Ext}^1(T,X) = 0 \}$$

and $\langle T_1 \rangle$ is the ideal of morphisms factoring through a sum of copies of $T_1$. We see that $T_1^*/\langle T_1 \rangle$ is equivalent to the product $\mathcal{C}_{A_1} \times \mathcal{C}_{A_2} \times \mathcal{C}_{A_4}$. The claim follows.

Figure 2. The AR-quiver of an $E_8$ quiver.

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