A note on module-composed graphs

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Abstract

In this paper we consider module-composed graphs, i.e. graphs which can be defined by a sequence of one-vertex insertions \( v_1, \ldots, v_n \), such that the neighbourhood of vertex \( v_i \), \( 2 \leq i \leq n \), forms a module (a homogeneous set) of the graph defined by vertices \( v_1, \ldots, v_{i-1} \).

We show that module-composed graphs are HHDS-free and thus homogeneously orderable, weakly chordal, and perfect. Every bipartite distance hereditary graph, every (co-2C_4, P_4)-free graph and thus every trivially perfect graph is module-composed. We give an \( O(|V_G| \cdot (|V_G| + |E_G|)) \) time algorithm to decide whether a given graph \( G \) is module-composed and construct a corresponding module-sequence.

For the case of bipartite graphs, module-composed graphs are exactly distance hereditary graphs, which implies simple linear time algorithms for their recognition and construction of a corresponding module-sequence.

Keywords: graph algorithms, homogeneous sets, HHDS-free graphs, distance hereditary graphs, bipartite graphs

1 Preliminaries

Let \( G = (V_G, E_G) \) be a graph. For some vertex \( v \in V_G \) we denote the *neighbourhood* of \( v \) by \( N(v) = \{ w \in V_G \mid \{v, w\} \in E_G \} \). \( M \subseteq V_G \) is called a *module* (homogeneous set) of \( G \), if and only if for all \( (v_1, v_2) \in M^2 \): \( N(v_1) - M = N(v_2) - M \), i.e. \( v_1 \) and \( v_2 \) have identical neighbourhoods outside \( M \). \( M \subseteq V_G \) is called a *trivial module*, if \( |M| = 0 \), \( |M| = 1 \), or \( M = V_G \), see [CH94]. A graph \( G \) is called *prime* if every module of \( G \) is trivial. A module \( M \) is *maximal* if there is no non-trivial module \( N \) such that \( M \subseteq N \). A module is called *strong* if it does not overlap with any other module.

While the set of modules of a graph \( G \) can be exponentially large, the set of strong modules is linear in the number of vertices. The inclusion order of the set of all strong modules defines a tree-structure which is denoted as *modular decomposition* \( T_G \), see [MR84]. The root of \( T_G \) represents the graph \( G \) and the leaves of \( T_G \) correspond to the vertices of \( G \). Every inner node, i.e. non-leaf node, \( w \) of \( T_G \) corresponds to an induced subgraph of \( G \) consisting of the leaves of \( T_G \) in subtree with root \( w \), which is called the *representative graph* of \( w \) and is denoted by \( G(w) \). Vertex set \( V_{G(w)} \) is a strong module of \( G \). For some inner node \( v \) of \( T_G \), the *quotient graph* \( G[v] \) is obtained by substituting in \( G(v) \) every strong module, represented

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by some child of \( v \) in \( T_G \), by a single vertex. For some inner node \( v \) of \( T_G \), quotient graph \( G[v] \) is either an independent set (\( v \) is denoted as \textit{co-join node}), a clique (\( v \) is denoted as \textit{join node}), or a prime graph (\( v \) is denoted as \textit{prime node}).

For \( U \subseteq V_G \), we define by \( G[U] \) the subgraph of \( G \) induced by the vertices of \( U \). For some graph \( G \), we denote its edge complement by co-\( G \). For a set of graphs \( \mathcal{F} \), we denote by \( \mathcal{F} \)-free graphs the set of all graphs that do not contain a graph of \( \mathcal{F} \) as an induced subgraph.

In Table 1 we show some special graphs to which we refer during the paper. A \textit{hole} is a chordless cycle with at least five vertices. A \textit{k-sun} is a chordal graph \( G \) on \( 2k \) vertices for some \( k \geq 3 \) whose vertex set can be partitioned into \( V_G = U \cup W \) such that \( U = \{u_0, \ldots, u_{k-1}\} \) and \( W = \{w_0, \ldots, w_{k-1}\} \) is an independent set. Additionally vertex \( u_i \) is adjacent to vertex \( w_j \) if and only if \( i = j \) or \( i = j + 1 \mod k \). \( G \) is called a \textit{sun} if it is a \( k \)-sun for some \( k \geq 3 \). If graph \( G[U] \) is a clique, then \( G \) is called a \textit{complete \( k \)-sun}.

\begin{table}[h]
\centering
\begin{tabular}{cccc}
\hline
\( C_5 \) & \text{hole} & \text{house} & \text{gem} \\
\hline
\end{tabular}
\caption{Special graphs}
\end{table}

2 Module-composed graphs

There are several graph classes which are defined by a sequence of one-vertex extensions of restricted form. Some well known examples are trees, co-graphs, and distance hereditary graphs, see [Rao07] for a survey. We next analyze a closely related but new concept.

Graph \( G \) is \textit{module-composed}, if and only if there exists a linear ordering \( \varphi : V_G \rightarrow [|V_G|] \), such that for every \( 2 \leq i \leq |V_G| \) the neighbourhood of vertex \( \varphi^{-1}(i) \) in graph \( G[\{\varphi^{-1}(1), \ldots, \varphi^{-1}(i-1)\}] \) forms a module. For some module-composed graph \( G \), \( \varphi \) is called a \textit{module-sequence} for \( G \).

The definition of module-composed graphs was introduced [AGK+06] for computing connectivity ratings for vertices in special graph classes, see also [AKKW06]. We first recall the following easy but important lemma from [AGK+06].

\textbf{Lemma 2.1 (Induced subgraph)} If a graph \( G \) is module-composed, then every induced subgraph of \( G \) is also module-composed.

Given two module-sequences \( \varphi_1, \varphi_2 \) for two graphs \( G_1 \) and \( G_2 \), sequence \( \varphi(v) = \varphi_1(v), v \in V_{G_1} \) and \( \varphi(v) = \varphi_2(v) + |V_{G_1}|, v \in V_{G_2} \) is a possible module-sequence for the disjoint union of these two graphs.
Lemma 2.2 (Disjoint union)  For two module-composed graphs $G_1, G_2$, the disjoint union $G_1 \cup G_2$ is also module-composed.

The following observation follows from Lemma 2.1 and the definition of module-composed graphs.

Lemma 2.3 A graph $G$ is module-composed, if and only if there exists a vertex $v \in V_G$ such that $N(v)$ is a module in graph $G[V_G - \{v\}]$ and graph $G[V_G - \{v\}]$ is module-composed.

By Lemma 2.3 the following graphs (see Table 1) are not module-composed, since none of them contains a vertex $v$ such that $N(v)$ is a module in graph $G[V_G - \{v\}]$:

- $C_n$, $n \geq 5$ (i.e. holes), co-$C_n$, $n \geq 5$ (i.e. anti-holes), house, domino, co-$(K_{3,3} - e)$, 3-sun, co-$2C_4$.

The example of graph co-$2C_4$ shows that not every co-graph is module-composed. Graph co-$2C_4$ can even be used to characterize those co-graphs which are module-composed.

Lemma 2.4 Let $G$ be a co-graph. The following conditions are equivalent.

1. $G$ is module-composed.
2. $G$ is (co-$2C_4$)-free.

Proof If $G$ is module-composed, then by Lemma 2.1 it obviously contains no co-$2C_4$ as induced subgraph.

Let $G$ be (co-$2C_4$)-free co-graph. Then there exists a co-graph expression $X$ defined by the three co-graph operations (single vertex $\bullet$, disjoint union $G_1 \cup G_2$ of two co-graphs $G_1, G_2$, join $G_1 \times G_2$ of two co-graphs $G_1, G_2$) for $G$. Any subexpression $\bullet$ and $G_1 \cup G_2$ are also feasible for a module-sequence.

Let $X' = X_1 \times X_2$ be a subexpression of $X$. Since the graph defined by $X'$ contains no co-$2C_4$ as an induced subgraph either graph defined by $X_1$ or that by $X_2$ defines a subgraph of $K_1 \cup K_2$, i.e. the disjoint union of a clique on two vertices and a clique on one vertex. Let us assume that $X_2$ does so. This allows us to define a module decomposition for $X$ as follows. We start with a module-sequence for $X_1$, which exists by induction, proceed with the vertices of $K_2$ and finish with vertex of graph $K_1$, which leads a module-sequence for graph defined by $X$.

Co-graphs are exactly $P_4$-free graphs which implies our next corollary.

Corollary 2.5 (co-$2C_4$, $P_4$)-free graphs are module-composed.

Further it is known that trivially perfect graphs are exactly $(C_4, P_4)$-free graphs [Gol78], which obviously form a subclass of (co-$2C_4$, $P_4$)-free graphs.

Corollary 2.6 Trivially perfect graphs are module-composed.

1 A co-graph is either a single vertex $\bullet$, the disjoint union $G_1 \cup G_2$ of two co-graphs $G_1, G_2$, or the join $G_1 \times G_2$ of two co-graphs $G_1, G_2$, which connects every vertex of $G_1$ with every vertex of $G_2$.

2 A graph is trivially perfect if for every induced subgraph $H$ of $G$, the size of the largest independent set in $H$ equals the number of all maximal cliques in $H$. 

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1

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2

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Next we conclude results on super classes of module-composed graphs.

It is easy to see that the house, every hole and the domino are not module-composed. By a result shown in Far83 each sun contains a complete sun as induced subgraph, which is obviously not module-composed. By Lemma 2.1 the next result follows.

**Lemma 2.7** Module-composed graphs are HHDS-free.

Since HHDS-free graphs are perfect, the same holds true for module-composed graphs.

**Corollary 2.8** Module-composed graphs are perfect.

Further, HHDS-free graphs are homogeneously orderable by the results shown in BDN97, which implies the same for module-composed graphs.

**Corollary 2.9** Module-composed graphs are homogeneously orderable.

Since the graph $C_4$ is module-composed but not chordal, we conclude that module-composed graphs are not chordal, but they are weakly chordal, since they are HHD-free and HHD-free graphs are weakly chordal.

**Corollary 2.10** Module-composed graphs are weakly chordal.

### 3 Algorithms for module-composed graphs

Next we give a polynomial time algorithm to recognize module-composed graphs. Our algorithm is based on Lemma 2.3. In order to find some vertex $v$ that satisfies the conditions of Lemma 2.3 we use a modular decomposition in our following Algorithm 3.1. A basic observation is that for every connected module-composed graph $G$ vertex $v$ is either a child or a grandchild of the root of $T_G$.

**Algorithm 3.1**

*Input:* Graph $G$

*Output:* Module-sequence $\varphi : V_G \to \|V_G\|$ or the answer NO

1. $\text{mod-com}(G)$
2. if ($G$ disconnected)
3. for every connected component $H$ of $G$: $\text{mod-com}(H)$;
4. else {
5. construct $T_G$ with root $r$;
6. if ($r$ is join node) {
7. if ($\exists$ child $v_l$ of $r$ which is a leaf in $T_G$) {
8. for every such child $v_l$ of $r$ $\{\varphi(v_l) = i + +; G = G - \{v_l\}\}$
9. }
10. }
11. }
12. }
13. }

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3 (house,hole,domino,sun)-free
4 A graph $G$ is perfect if, for every induced subgraph $H$ of $G$, the chromatic number of $H$ is equal to the size of a maximum clique of $H$.
5 A graph is weakly chordal if it does not contain any induced cycles of length greater than four or their complements.
6 (house,hole,domino)-free
The construction of the modular decomposition $T_G$ in Line (5) of Algorithm 3.1 can be realized in time $O(|V_G| + |E_G|)$ by CH94, MS99.

**Theorem 3.2** Given a graph $G$, one can decide in time $O(|V_G| \cdot (|V_G| + |E_G|))$ whether $G$ is module-composed, and in the case of a positive answer, constructs a module-sequence.

Since module-composed graphs are HHD-free, we conclude by the results shown in JOSS the following theorem.

**Theorem 3.3** For every module-composed graph which is given together with a module-sequence the size of a largest independent set, the size of a largest clique, the chromatic number and the minimum number of cliques covering the graph can be computed in linear time.

## 4 Independent module-composed graphs

Next we want to characterize module-composed graphs for a restricted case.

A graph $G$ is **independent module-composed**, if and only if there exists a linear ordering $\varphi : V_G \rightarrow [|V_G|]$, such that for every $2 \leq i \leq |V_G|$ the neighbourhood of vertex $\varphi^{-1}(i)$ in graph $G[\{\varphi^{-1}(1), \ldots, \varphi^{-1}(i-1)\}]$ forms a module which is an independent set.

It is easy to see that independent module-composed graphs do not contain any of the graphs of Table 1 as induced subgraph.

**Lemma 4.1** Independent module-composed graphs are HHDG-free.\footnote{(house,hole,domeino,gem)-free}
HHDG-free are also known as distance hereditary graphs \cite{HM90, BM86}. Examples for distance hereditary graphs are co-graphs and trees. For the case of bipartite graphs, the notion module-composed even is equivalent to the notion of distance hereditary.

**Theorem 4.2** (\cite{AGK+06}) Let \(G\) a bipartite graph. The following conditions are equivalent.

1. \(G\) is module-composed.
2. \(G\) is domino and hole free.
3. \(G\) is distance hereditary.
4. \(G\) is \((6,2)\)-chordal.

For general graphs Theorem 4.2 does not hold true, since there are module-composed graphs which are not distance hereditary, e.g. the gem and there are distance hereditary graph which are not module-composed, e.g. the co\-(K\(_{3,3} - e\)).

The problem to decide whether a given graph is bipartite distance hereditary and to construct a corresponding pruning sequence can be done in linear time by the well known characterization for bipartite graphs as 2-colorable graphs and the linear time recognition algorithms for distance hereditary graphs shown in \cite{HM90, BM86}. By Theorem 4.2 this immediately implies a linear time algorithms for recognizing independent module-composed graphs. A corresponding module-sequence can be constructed in linear time from a pruning sequence as shown in \cite{AGK+06}. Since both known linear time recognition algorithms for distance hereditary graphs shown in \cite{HM90, BM86} are based on the fact that the neighbourhood of every vertex in a distance hereditary graph is a co-graph and additional conditions, both algorithms are not simple.

In \cite{JO88} it is shown that for HHD-free graphs every Lex-BFS (Lexicographic Breadth First Search) ordering is a semi perfect elimination ordering, i.e. every vertex \(\varphi^{-1}(i)\) is no midpoint of an induced \(P_4\) in graph \(G[[\varphi^{-1}(1), \ldots, \varphi^{-1}(i - 1)]]\). In the case of bipartite graphs this ordering obviously is even an independent module-sequence.

**Theorem 4.3** Given an independent module-composed graph \(G\), every Lex-BFS ordering constructs in time \(O(|V_G| + |E_G|)\) an independent module-sequence for \(G\).

To decide whether a given graph is bipartite distance hereditary can be done by Corollary 5 shown in \cite{BM86} using the fundamental search strategy of BFS (Breadth First Search) which produces a classification of the vertices into levels, with respect to a start vertex \(u\). Level \(i\) is the set of vertices with distance \(i\) to vertex \(u\) and is denoted by \(N_i(u)\).

**Theorem 4.4 (Corollary 5 of BM86)** Let \(G\) be a connected graph and let \(u\) be a vertex of \(G\). Then \(G\) is bipartite distance hereditary if and only if all levels \(N_k(u)\) are edgeless, and for every vertices \(v, w\) in \(N_k(u)\) and neighbours \(x\) and \(y\) of \(v\) in \(N_{k-1}(u)\), we have \(N(x) \cap N_{k-2}(u) = N(y) \cap N_{k-2}(u)\), and further \(N(v) \cap N_{k-1}(u)\) and \(N(w) \cap N_{k-1}(u)\) are either disjoint or one is contained in the other.

\(^8\) A graph is bipartite if it is \(C_{2n+1}\)-free, for \(n \geq 1\).

\(^9\) A graph is \((k, l)\)-chordal if each cycle of length at least \(k\) has at least \(l\) chords.
A BFS starting at a vertex $u$ can compute the level sets $N_k(u)$ in time $O(|V_G| + |E_G|)$ and using these levels, the conditions of Corollary 5 of [BM86] can be verified in the same time.

A BFS numbering $\varphi$ of the vertices with respect to some vertex $u$ can be used to obtain a module-sequence $\varphi_1$ as follows. We start with $\varphi_1(v) = \varphi(v)$, $\forall v \in V_G$. For the first $|N_0(u)| + |N_1(u)|$ vertices we obviously can choose $\varphi_1(v) = \varphi(v)$. For the vertices of $w \in N_k(u)$, $k \geq 2$, we know that their neighbours in set $N_{k-1}(u)$ are modules which can be ordered by a series of inclusions $N^1 \subseteq N^2 \subseteq \ldots \subseteq N^3$. We rearrange the order of the vertices in $N_k(u)$ with respect to $\varphi_1$ such that for every such series of inclusions $\varphi_1(w_1) < \varphi_1(w_2)$ if and only if $N_{k-1}(u) \cap N(w_1) \supseteq N_{k-1}(u) \cap N(w_2)$. This obviously leads a module-sequence for graph $G$ if $G$ is bipartite distance hereditary.

**Theorem 4.5** Given a graph $G$, one can decide using BFS in time $O(|V_G| + |E_G|)$ whether $G$ is independent module-composed, and in the case of a positive answer, construct a module-sequence.

On bipartite distance hereditary graphs, and so on independent module-composed graphs, the path-partition problem [YC98], hamiltonian circuit and path problem [MN93], and the computation of shapley value ratings [AGK++06] can be solved in polynomial time.

It is well known that distance hereditary graphs and thus independent module-composed graphs have clique-width at most 3 [GR00]. This implies that all graph properties which are expressible in monadic second order logic with quantifications over vertices and vertex sets (MSO$_1$-logic) are decidable in linear time on independent module-composed graphs [CMR00]. Some of these problems are partition into $k$ independent sets or cliques, $k$-dominating set, $k$-achromatic number, for every fixed integer $k$.

Furthermore, there are a lot of NP-complete graph problems which are not expressible in MSO$_1$-logic like chromatic number, partition problems, vertex disjoint paths, and bounded degree subgraph problems but which can also be solved in polynomial time on clique-width bounded graphs and thus on bipartite distance hereditary graphs [EGW01, GW06].

Note that general module-composed graphs are of unbounded clique-width. For example every graph which can be constructed from a single vertex by a sequence of one vertex extensions by a domination vertex\(^{10}\) or a pendant vertex\(^{11}\) is obviously module-composed. But the set of all such defined graphs have unbounded clique-width [Rao07].

### 5 Graph class inclusions

In Table 2 we summarize the relation of module-composed graphs and related graph classes. For the definition and relations of special graph classes we refer to the survey of Brandstädt et al. [BLS99].

### References

[AGK++06] M. Abraham, F. Gurski, A. Krumnack, R. Kotter, and E. Wanke. A connectivity rating for vertices in networks. submitted, 2006.

\(^{10}\) A vertex $v \in V_G$ is a dominating vertex of $G$, if it is adjacent to all other vertices in $G$.

\(^{11}\) A vertex $v \in V_G$ of degree one is called a pendant vertex of $G$. 
Table 2: Inclusion of special graph classes
[AKKW06] M. Abraham, R. Kötter, A. Krumnack, and E. Wanke. A connectivity rating for vertices in networks. In Proceedings of the 4th IFIP International Conference on Theoretical Computer Science-TCS, pages 283–298. Springer, 2006.

[BDN97] A. Brandstädt, F. F. Dragan, and F. Nicolai. Homogeneously orderable graphs. Theoretical Computer Science, 172:209–232, 1997.

[BLS99] A. Brandstädt, V.B. Le, and J.P. Spinrad. Graph Classes: A Survey. SIAM Monographs on Discrete Mathematics and Applications. SIAM, Philadelphia, 1999.

[BM86] H.-J. Bandelt and H.M. Mulder. Distance-hereditary graphs. Journal of Combinatorial Theory, Series B, 41:182–208, 1986.

[CH94] A. Cournier and M. Habib. A new linear time algorithm for modular decomposition. In Proceedings of CAAP, volume 787 of LNCS, pages 68–84. Springer, 1994.

[CMR00] B. Courcelle, J.A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. Theory of Computing Systems, 33(2):125–150, 2000.

[EGW01] W. Espelage, F. Gurski, and E. Wanke. How to solve NP-hard graph problems on clique-width bounded graphs in polynomial time. In Proceedings of Graph-Theoretical Concepts in Computer Science, volume 2204 of LNCS, pages 117–128. Springer, 2001.

[Far83] M. Farber. Characterizations of strongly chordal graphs. Discrete Mathematics, 43:173–189, 1983.

[Gol78] M.C. Golumbic. Trivially perfect graphs. Discrete Mathematics, 24:105–107, 1978.

[GR00] M.C. Golumbic and U. Rotics. On the clique-width of some perfect graph classes. International Journal of Foundations of Computer Science, 11(3):423–443, 2000.

[GW06] F. Gurski and E. Wanke. Vertex disjoint paths on clique-width bounded graphs. Theoretical Computer Science, 359(1-3):188–199, 2006.

[HM90] P.L. Hammer and F. Maffray. Completely separable graphs. Discrete Applied Mathematics, 27:85–99, 1990.

[JO88] B. Jamison and S. Olariu. On the semi-perfect elimination. Advances in applied mathematics, 9:364–376, 1988.

[MN93] H. Müller and F. Nicolai. Polynomial time algorithms for hamiltonian problems on bipartite distance-hereditary graphs. Information Processing Letters, 46(5):225–230, 1993.

[MR84] R.H. Möhring and F.J. Radermacher. Substitution decomposition for discrete structures and connections with combinatorial optimization. Annals of Discrete Mathematics, 19:257–365, 1984.
[MS99] R.M. McConnell and J. Spinrad. Modular decomposition and transitive orientation. *Discrete Mathematics*, 201(1-3):189–241, 1999.

[Rao07] M. Rao. Clique-width of graphs defined by one-vertex extensions. Manuscript, 2007.

[YC98] H.-G. Yeh and G.J. Chang. The path-partition problem in bipartite distance-hereditary graphs. *Taiwanese Journal of Mathematics*, 2(3):353–360, 1998.