Space-Time Compactification, Non-Singular Black Holes, Wormholes and Braneworlds via Lightlike Branes

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Abstract

We describe a concise general scheme for constructing solutions of Einstein-Maxwell-Kalb-Ramond gravity-matter system in bulk space-time interacting self-consistently with one or more (widely separated) codimension-one electrically charged lightlike branes. The lightlike brane dynamics is explicitly given by manifestly reparametrization invariant world-volume actions. We present several explicit classes of solutions with different physical interpretation as wormhole-like space-times with one, two or more “throats”, singularity-free black holes, brane worlds and space-times undergoing a sequence of spontaneous compactification-decompactification transitions.

1. Introduction

Lightlike branes (LL-branes for short) are singular null (lightlike) hypersurfaces in Riemannian space-time which provide dynamical description of various physically important phenomena in cosmology and astrophysics such as: (i) impulsive lightlike signals arising in cataclysmic astrophysical events (supernovae, neutron star collisions) [1]; (ii) dynamics of horizons in black hole physics – the so called “membrane paradigm” [2]; (iii) the thin-wall approach to domain walls coupled to gravity [3, 4, 5].

More recently, the relevance of LL-branes in the context of non-perturbative string theory has also been recognized [6].

Starting with the pioneering papers [3, 4, 5] the LL-branes have been exclusively treated in a “phenomenological” manner, i.e., without specifying an underlying Lagrangian dynamics from which they may originate. On the other hand, in the last few years we have proposed in a series of papers [7, 8,
9, 10] a new class of concise manifestly reparametrization invariant world-volume Lagrangian actions, providing a derivation from first principles of the \textit{LL-brane} dynamics. The following characteristic features of the new \textit{LL-branes} drastically distinguish them from ordinary Nambu-Goto branes:

(a) They describe intrinsically lightlike modes, whereas Nambu-Goto branes describe massive ones.

(b) The tension of the \textit{LL-brane} arises as an additional degree of freedom, whereas Nambu-Goto brane tension is a given \textit{ad hoc} constant. The latter characteristic feature significantly distinguishes our \textit{LL-brane} models from the previously proposed \textit{tensionless} \textit{p-branes} (for a review, see Ref.[11]). The latter rather resemble \textit{p}-dimensional continuous distributions of independent massless point-particles without cohesion among the latter.

(c) Consistency of \textit{LL-brane} dynamics in a spherically or axially symmetric gravitational background of codimension one requires the presence of an event horizon which is automatically occupied by the \textit{LL-brane} ("horizon straddling" according to the terminology of Ref.[4]).

(d) When the \textit{LL-brane} moves as a test brane in spherically or axially symmetric gravitational backgrounds its dynamical tension exhibits exponential “inflation/deflation” time behavior [8] – an effect similar to the “mass inflation” effect around black hole horizons [12].

An intriguing novel application of \textit{LL-branes} as natural self-consistent gravitational sources for \textit{wormhole} space-times has been developed in a series of recent papers [9, 10, 13, 14]. In what follows, when discussing wormholes we will have in mind precisely this physically important class of “thin-shell” traversable Lorentzian wormholes first introduced by Visser [15, 16]. For a comprehensive general review of wormhole space-times, we refer to [16, 17].

In the present work we describe a concise systematic scheme for constructing solutions of Einstein-Maxwell-Kalb-Ramond gravity-matter system in bulk space-time coupled self-consistently to one or more (widely separated) codimension-one electrically charged \textit{LL-branes}. The solutions describe bulk space-time manifolds consisting of several space-time regions (“universes”) with different (in general) geometries such that: (i) each separate “universe” is a “vacuum” solution of Einstein-Maxwell-Kalb-Ramond equations (i.e., without the presence of \textit{LL-branes}); (ii) the separate “universes” are pairwise matched (glued together) along some of their common horizons; (iii) each of these common matching horizons is automatically occupied by one \textit{LL-brane} (“horizon straddling”) which generates space-time varying cosmological constants in the various matching “universes”.

We present several explicit types of solutions with different physical interpretation such as: (a) wormhole-like space-times with one, two or more “throats”; (b) non-singular black holes; (c) brane worlds; (d) space-times undergoing a sequence of spontaneous compactification/decompactification transitions triggered by \textit{LL-branes}. 
2. Lagrangian Formulation of Lightlike Brane Dynamics

In a series of previous papers [7, 8, 9, 10, 13, 18] we have proposed manifestly reparametrization invariant world-volume Lagrangian formulation of LL-branes in several dynamically equivalent forms. Here we will use the Nambu-Goto-type formulation given by the world-volume action:

\[ S_{\text{LL}} = - \int dp^1 \sigma T \sqrt{\det |g_{ab} - \epsilon \frac{1}{T^2} \partial_a u \partial_b u|} , \quad \epsilon = \pm 1 . \]  

(1)

Here and below the following notations are used:

- \( g_{ab} \) is the induced metric on the world-volume:
  \[ g_{ab} \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) , \]
  which becomes singular on-shell (manifestation of the lightlike nature, cf. Eq.(6) below).

- \( X^\mu(\sigma) \) are the \( p \)-brane embedding coordinates in the bulk \( D \)-dimensional space-time with Riemannian metric \( G_{\mu\nu}(X) \) (\( \mu, \nu = 0, 1, \ldots, D-1 \));
  \( (\sigma) \equiv (\sigma^0 = \tau, \sigma^i) \) with \( i = 1, \ldots, p \);
  \( \partial_a \equiv \frac{\partial}{\partial \sigma^a} \).

- \( u \) is auxiliary world-volume scalar field defining the lightlike direction (see Eq.(6) below); the choice of the sign of \( \epsilon \) in (1) does not have physical effect because of the non-propagating nature of the \( u \)-field (see Appendix).

- \( T \) is dynamical (variable) brane tension (also a non-propagating degree of freedom, cf. Appendix).

The corresponding equations of motion w.r.t. \( X^\mu, u \) and \( T \) read accordingly (with \( \Gamma^\mu_{\lambda\nu} \) – Christoffel connection for the bulk metric):

\[ \partial_a \left( T \sqrt{|g|} \tilde{g}^{ab} \partial_b X^\mu \right) + T \sqrt{|g|} \tilde{g}^{ab} \partial_b X^\lambda \partial_b X^\nu \Gamma^\mu_{\lambda\nu} = 0 \]  

(3)

\[ \partial_a \left( \frac{1}{T} \sqrt{|g|} \tilde{g}^{ab} \partial_b u \right) = 0 \]  

(4)

where we have introduced the convenient notations:

\[ \tilde{g}_{ab} = g_{ab} - \epsilon \frac{1}{T^2} \partial_a u \partial_b u \]  

and \( \tilde{g}^{ab} \) is the inverse matrix w.r.t. \( \tilde{g}_{ab} \).

From the definition (5) and second Eq.(4) one easily finds that the induced metric on the world-volume is singular on-shell:

\[ g_{ab} \left( g^{bc} \partial_c u \right) = 0 \]  

(6)

exhibiting the lightlike nature of the \( p \)-brane described by (1).
Similarly to the ordinary bosonic \( p \)-brane we can rewrite the Nambu-Goto-type action for the \( LL \)-brane (1) in a Polyakov-like form by employing an \emph{intrinsic} Riemannian world-volume metric \( \gamma_{ab} \):

\[
S_{LL-Pol} = -\frac{1}{2} \int d^{p+1}\sigma \left( \sqrt{-\gamma} \left[ \gamma_{ab} \left( g_{ab} - \frac{1}{T^2} \partial_a u \partial_b u - \epsilon b_0 (p-1) \right) \right] \right) ,
\]

where \( b_0 \) is a positive constant. The world-volume action (7) produces the same equations of motion (3)–(4) together with the relation:

\[
\gamma_{ab} = \frac{\epsilon}{b_0} \bar{g}_{ab} .
\]

In particular, relation (8) reveals the meaning of \( b_0 \) as (inverse) proportionality factor between the intrinsic world-volume metric and the “extended” induced metric (5).

\textbf{Remark.} Let us note that consistency between the Lorentz nature of the intrinsic world-volume metric \( \gamma_{ab} \) and the Lorentz nature of the embedding space-time metric \( G_{\mu \nu} \), taking into account (8), requires to set \( \epsilon = 1 \) in the Polyakov-type action (7).

As shown in our previous papers [7, 8, 9], using the above world-volume Lagrangian framework one can add in a natural way couplings of the \( LL \)-brane to bulk space-time Maxwell \( A_\mu \) and Kalb-Ramond \( A_{\mu_1 \ldots \mu_{D-1}} \) gauge fields (the latter – in the case of codimension one \( LL \)-branes, \( i.e. \), for \( D = (p+1) + 1 \)). For the Nambu-Goto-type action (1) these couplings read (second ref.[19]):

\[
\bar{S}_{LL}[q, \beta] = -\int d^{p+1}\sigma T \sqrt{\det ||g_{ab} - \frac{1}{T^2} (\partial_a u + q A_a)(\partial_b u + q A_b)||} - \frac{\beta}{(p+1)!} \int d^{p+1}\sigma \epsilon^{a_1 \ldots a_{p+1}} \partial_a X^{\mu_1} \ldots \partial_{a_{p+1}} X^{\mu_{p+1}} A_{\mu_1 \ldots \mu_{p+1}} \]

with \( g_{ab} \) denoting the induced metric on the world-volume (2) and \( A_a \equiv \partial_a X^\mu A_\mu \). Using the short-hand notation generalizing (5):

\[
\bar{g}_{ab} \equiv g_{ab} - \epsilon \frac{1}{T^2} (\partial_a u + q A_a)(\partial_b u + q A_b) , \quad A_a \equiv \partial_a X^\mu A_\mu .
\]

the equations of motion w.r.t. \( X^\mu \), \( u \) and \( T \) acquire the form:

\[
\partial_a \left( T \sqrt{|\bar{g}|} \bar{g}^{ab} \partial_b X^\mu \right) + T \sqrt{|\bar{g}|} \bar{g}^{ab} \partial_b X^\lambda \partial_a X^\nu \Gamma^\mu_{\lambda \nu} + \epsilon \frac{q}{T} \sqrt{|\bar{g}|} \bar{g}^{ab} \partial_a X^\nu (\partial_b u + q A_b) F^{\lambda \nu} G^{\mu \lambda} - \frac{\beta}{(p+1)!} \epsilon^{a_1 \ldots a_{p+1}} \partial_{a_1} X_{\mu_1} \ldots \partial_{a_{p+1}} X_{\mu_{p+1}} F_{\lambda_1 \ldots \lambda_{p+1}} G^{\lambda_{p+1}} = 0 ,
\]
with

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \quad F_{\mu_1...\mu_D} = D \partial_{[\mu_1} A_{\mu_2...\mu_D]} = F \sqrt{-G} \varepsilon_{\mu_1...\mu_D} \]  \hspace{1cm} (12)

being the field-strengths of the electromagnetic \( A_\mu \) and Kalb-Ramond \( A_{\mu_1...\mu_{D-1}} \) gauge potentials [20], and

\[ \partial_a \left( \frac{1}{T} \sqrt{|g|} \tilde{g}^{ab}(\partial_b u + q A_b) \right) = 0 , \quad T^2 + \varepsilon g^{ab}(\partial_a u + q A_a)(\partial_b u + q A_b) = 0 . \]  \hspace{1cm} (13)

The on-shell singularity of the induced metric \( g_{ab} \) (2), i.e., the lightlike property, now reads (using notation (10), cf. Eq.(6)):

\[ g_{ab} \left( \tilde{g}^{bc}(\partial_c u + q A_c) \right) = 0 . \]  \hspace{1cm} (14)

The Polyakov-type form of the world-volume action (9) becomes (using short-hand notation (10)):

\[ \bar{S}_{LL-\text{Pol}}[q, \beta] = -\frac{1}{2} \int d^{p+1} \sigma T b_0^{p+1} \sqrt{-\gamma} \left[ \gamma^{ab} \tilde{g}_{ab} - \epsilon b_0 (p - 1) \right] \]

\[ -\frac{\beta}{(p + 1)!} \int d^{p+1} \sigma \varepsilon a_1...a_{p+1} \partial a_1 X^{\mu_1} ... \partial a_{p+1} X^{\mu_{p+1}} A_{\mu_1...\mu_{p+1}} , \]  \hspace{1cm} (15)

yielding the same set of equations of motion (11)–(13) plus the counterpart of (8):

\[ \gamma_{ab} = \frac{\epsilon}{b_0} \tilde{g}_{ab} \]  \hspace{1cm} (16)

with \( \tilde{g}_{ab} \) as in (10). Here again the above remark after Eq.(8) applies, i.e., that for consistency we must set \( \epsilon = 1 \) within the Polyakov-type action (15).

3. Bulk Gravity/Gauge-Field System Self-Consistently Interacting With Lightlike Branes

3.1. Lagrangian Formulation

Let us now consider self-consistent bulk Einstein-Maxwell-Kalb-Ramond system coupled to \( N \geq 1 \) distantly separated charged codimension-one lightlike \( p \)-branes (in this case \( D = (p + 1) + 1 \)). The pertinent Lagrangian action reads:

\[ S = \int d^D x \sqrt{-G} \left[ \frac{R(G)}{16\pi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{D!^2} F_{\mu_1...\mu_D} F^{\mu_1...\mu_D} \right] \]

\[ + \sum_{k=1}^{N} \tilde{S}_{LL}[q^{(k)}, \beta^{(k)}] , \]  \hspace{1cm} (17)
where again $\mathcal{F}_{\mu\nu}$ and $\mathcal{F}_{\mu_1...\mu_D}$ are the Maxwell and Kalb-Ramond field-strengths (12) and $\bar{S}_{LL}[q^{(k)},\beta^{(k)}]$ indicates the world-volume action of the $k$-th $LL$-brane of the form (9) (or (15)).

The corresponding equations of motion are as follows:

(a) Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R = 8\pi \left( T_{\mu\nu}^{(EM)} + T_{\mu\nu}^{(KR)} + \sum_{k=1}^{N} T_{\mu\nu}^{(brane-k)} \right).$$

The energy-momentum tensors of bulk gauge fields are given by:

$$T_{\mu\nu}^{(EM)} = \mathcal{F}_{\mu\kappa}\mathcal{F}^{\kappa\nu} - G_{\mu\nu} \frac{1}{4} \mathcal{F}_{\kappa\lambda} \mathcal{F}^{\kappa\lambda}, \quad T_{\mu\nu}^{(KR)} = -\frac{1}{2} F^2 G_{\mu\nu},$$

where the last relation indicates that $\Lambda \equiv 4\pi F^2$ can be interpreted as dynamically generated cosmological “constant”. The energy-momentum (stress-energy) tensor of $k$-th $LL$-brane is straightforwardly derived from the pertinent $LL$-brane action (9):

$$T_{\mu\nu}^{(brane-k)} = -\int d^{p+1}\sigma \frac{\delta^{(D)}(x - X_{(k)}(\sigma))}{\sqrt{-G}} T^{(k)} \sqrt{|\bar{g}_{(k)}|} \bar{g}_{ab}^{(k)} \partial_a X_{(k)}^\mu \partial_b X_{(k)}^\nu,$$

(b) Maxwell equations:

$$\partial_\nu \left( \sqrt{-G} \mathcal{F}^{\mu\nu} \right) - \sum_{k=1}^{N} q^{(k)} \int d^{p+1}\sigma \frac{\delta^{(D)}(x - X_{(k)}(\sigma))}{\sqrt{-G}} \times \sqrt{|\bar{g}_{(k)}|} \bar{g}_{ab}^{(k)} \partial_a X_{(k)}^\mu \partial_b u_{(k)}^{(k)} + q^{(k)} A_{a}^{(k)} = 0,$$

(c) Kalb-Ramond equations of motion (recall definition of $\mathcal{F}$ in (12)):

$$\varepsilon^{\mu_1...\mu_{p+1}} \partial_\nu \mathcal{F} - \sum_{k=1}^{N} \beta^{(k)} \int d^{p+1}\sigma \frac{\delta^{(D)}(x - X_{(k)}(\sigma))}{\sqrt{-G}} \times \varepsilon^{a_1...a_{p+1}} \partial_{a_1} X_{(k)}^{\mu_1} \ldots \partial_{a_{p+1}} X_{(k)}^{\mu_{p+1}} = 0.$$ 

(d) The $LL$-brane equations of motion have already been written down in (11)–(13) above.
3.2. LL-Brane Dynamics in Static “Spherically Symmetric” Backgrounds

We will be interested in static “spherically-symmetric”-type solutions of Einstein-Maxwell-Kalb-Ramond equations with the following generic form of the bulk Riemannian metric:

\[
    ds^2 = -A(\eta)dt^2 + \frac{d\eta^2}{A(\eta)} + C(\eta)h_{ij}(\theta)d\theta^i d\theta^j ,
\]

or, in Eddington-Finkelstein coordinates (\(dt = dv - \frac{d\eta}{A(\eta)}\)):

\[
    ds^2 = -A(\eta)dv^2 + 2dv d\eta + C(\eta)h_{ij}(\theta)d\theta^i d\theta^j .
\]

Here \(h_{ij}\) indicates the standard metric on \(p\)-dimensional sphere, cylinder, torus or flat Euclidean section. The “radial-like” coordinate \(\eta\) will vary in general from \(-\infty\) to \(+\infty\).

We will consider the simplest ansatz for the LL-brane embedding coordinates:

\[
    X^0 \equiv v = \tau \ , \quad X^1 \equiv \eta = \eta(\tau) \ , \quad X^i \equiv \theta^i = \sigma^i \ (i = 1, \ldots, p) .
\]

Furthermore, we will use explicit world-volume reparametrization invariance of the LL-brane actions ((7) and (15)) to introduce the standard synchronous gauge-fixing conditions for the intrinsic world-volume metric:

\[
    \gamma^{00} = -1 \ , \quad \gamma^{0i} = 0 \ (i = 1, \ldots, p) .
\]

The latter together with second Eq.(13) and (16) (and accounting for the definition (10)) implies for the 00-component of the induced metric (2) on the LL-brane world-volume:

\[
    g_{00} \equiv \dot{X}^\mu G_{\mu\nu} \dot{X}^\nu = \frac{b_0}{T^2} g^{ij} (\partial_i u + A_i) (\partial_j u + A_j) \geq 0 \quad (28)
\]

which must match the condition \(g_{00} \leq 0\) required by consistency between the Lorentz form of the bulk space-time metric and the Lorentz form of the LL-brane world-volume. Hence we are led to impose the ansatz:

\[
    \partial_i u + A_i = 0 \quad (29)
\]

which is consistent for static spherically symmetric bulk space-time Maxwell field \(A_\mu\) and whose physical meaning is that the lightlike direction for the induced metric in Eq.(14) (or Eq.(6) for electrically neutral LL-brane) coincides with the brane proper-time \(\tau\)-direction on the world-volume.

Thus, taking into account (27) and (29), the LL-brane equations of motion (13) (or, equivalently, (14)) reduce to:

\[
    g_{00} \equiv \dot{X}^\mu G_{\mu\nu} \dot{X}^\nu = 0 \ , \quad g_{0i} \equiv \dot{X}^\mu G_{\mu\nu} \partial_i X^\nu = 0 \ , \quad T^2 = \frac{1}{b_0} (\partial_0 u + A_0)^2 , \quad \partial_i T = 0 , \quad \partial_0 g^{(p)} = 0 \quad (g^{(p)} \equiv \det \|g_{ij}\|) \quad (30)
\]
with $g_{ij}$ being the spacelike part of the induced metric (2). Eqs. (30)–(31) with LL-brane embedding (26) and metric of the form (25) imply:

$$-A(\eta) + 2 \dot{\eta} = 0 \quad , \quad \partial_\tau C = \dot{\eta} \quad . \quad \partial_\eta C \big|_{\eta=\eta(\tau)} = 0 .$$

(32)

Here we will distinguish two cases. First, let us consider the case of $C(\eta)$ as non-trivial function of $\eta$ (i.e., proper spherically-symmetric-type space-time). In this case Eqs. (32) imply:

$$\dot{\eta} = 0 \quad \Rightarrow \quad \eta(\tau) = \eta_0 = \text{const} , \quad A(\eta_0) = 0 .$$

(33)

Eq. (33) tells us that consistency of LL-brane dynamics in a proper spherically-symmetric-type gravitational background of codimension one requires the latter to possess a horizon (at some $\eta = \eta_0$), which is automatically occupied by the LL-brane (“horizon straddling” according to the terminology of Ref. [4]). Similar property – “horizon straddling”, has been found also for LL-branes moving in rotating axially symmetric (Kerr or Kerr-Newman) and rotating cylindrically symmetric black hole backgrounds [9, 10].

Next, consider the case $C(\eta) = \text{const}$ in (25), i.e., the corresponding space-time manifold is of product type $\Sigma_2 \times S^p$. A physically relevant example is the Bertotti-Robinson [21, 22] space-time in $D = 4$ (i.e., $p = 2$) describing Anti-de-Sitter$_2 \times S^2$ with metric (in Eddington-Finkelstein coordinates):

$$ds^2 = -\frac{r^2}{r_0^2}d\tau^2 + 2dv d\eta + r_0^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right] .$$

(34)

At $\eta = 0$ the Bertotti-Robinson metric (34) possesses a horizon. Further, we will consider the case of Bertotti-Robinson universe with constant electric field $F_{\nu\eta} = \pm \frac{\mp}{2r_0\sqrt{\pi}}$. In the present case the second Eq. (32) is trivially satisfied whereas the first one yields: $\eta(\tau) = \eta(0) \left( 1 - \tau \frac{\eta(0)}{2r_0} \right)^{-1}$. In particular, if the LL-brane is initially (at $\tau = 0$) located on the Bertotti-Robinson horizon $\eta = 0$, it will stay there permanently. It is this particular solution which we will consider in what follows.

4. Self-Consistent Wormhole-Like Solutions with LL-Branes – General Scheme

We will construct self-consistent static “spherically symmetric” solutions of the system of Einstein-Maxwell-Kalb-Ramond equations (18)–(23) and LL-brane Eqs. (11)–(13) following the steps:

(i) The bulk space-time metric will be of the form:

$$ds^2 = -A(\eta)dv^2 + 2dv d\eta + C(\eta)h_{ij}(\theta) d\theta^i d\theta^j ,$$

$$A(\eta_0^{(k)}) = 0 \quad (k = 1, \ldots, N) , \quad A(\eta) > 0 \quad \text{for all } \eta \neq \eta_0^{(k)} .$$

(35)
Each horizon at $\eta = \eta_0^{(k)}$ is automatically occupied by (one of the) LL-brane(s) according to the LL-brane dynamics (“horizon straddling”, cf. (32)–(33)).

(ii) Choose “vacuum” solutions of Einstein-Maxwell-Kalb-Ramond equations (18)–(23) (i.e., without the delta-function terms due to the LL-branes) in each region $-\infty < \eta < \eta_0^{(1)}$, $\eta_0^{(1)} < \eta < \eta_0^{(2)}$, $\ldots$, $\eta_0^{(N)} < \eta < \infty$.

(iii) Match the discontinuities across each horizon at $\eta = \eta_0^{(k)}$ of the derivatives of the bulk metric, Maxwell and Kalb-Ramond field strengths using the explicit expressions for the LL-brane stress-energy tensors, electric and Kalb-Ramond currents systematically derived from the underlying LL-brane world-volume actions (15).

In particular, for the stress-energy tensor of each $k$-th LL-brane we obtain (here we suppress the index $(k)$):

$$T_{(brane)}^{\mu\nu} = S_{\mu\nu} \delta(\eta - \eta_0)$$

with surface energy-momentum tensor:

$$S_{\mu\nu} = \frac{T}{eb_0^{1/2}} \left( \partial_{\tau}X^\mu \partial_{\tau}X^\nu - e b_0 G_{ij} \partial_i X^\mu \partial_j X^\nu \right)_{v=\tau, \eta=\eta_0, \theta'=\sigma'}$$

where $G_{ij} = C(\eta) h_{ij}(\theta)$ (cf. (25)). For the non-zero components of (37) (with lower indices) and its trace we find:

$$S_{\eta\eta} = \frac{T}{b_0^{1/2}} , \quad S_{ij} = -T b_0^{1/2} G_{ij} , \quad S_{\lambda} = -p T b_0^{1/2}$$

Taking into account (36)–(38) Einstein equations (18) yield:

$$[\partial_\eta A]_{\eta_0^{(k)}} = -16\pi T(k) \sqrt{b_0^{(k)}} \ , \quad [\partial_\eta \ln C]_{\eta_0^{(k)}} = - \frac{16\pi}{p} T(k)$$

with notation $[Y]_{\eta_0} \equiv Y |_{\eta \rightarrow \eta_0 + 0} - Y |_{\eta \rightarrow \eta_0 - 0}$ for any quantity $Y$.

Maxwell and Kalb-Ramond equations yield:

$$[F_{\eta\eta}]_{\eta_0^{(k)}} = q^{(k)} , \quad [F_\eta]_{\eta_0^{(k)}} = -\beta^{(k)}$$

In Eqs.(39)–(40) $(T(k), b_0^{(k)})$ indicate the dynamical tension and $b_0$ parameter of the $k$-th LL-brane occupying horizon $\eta_0^{(k)}$, with electric charge surface density $q^{(k)}$ and Kalb-Ramond coupling $\beta^{(k)}$. The second relation
in (40) gives the jump of the dynamically generated cosmological constant \( \Lambda \equiv 4\pi F^2 \) across the \( k \)-th \( LL\)-brane.

The only non-trivial contribution of \( LL\)-brane equations of motion comes from the \( X^0 \)-equation which yields:

\[
\partial_0 T^{(k)} + T^{(k)} \frac{1}{2} \left( (\partial_0 A)_{\eta_0^{(k)}} + p b_0^{(k)} (\partial_0 \ln C)_{\eta_0^{(k)}} \right)
\]
\[
- \sqrt{b_0^{(k)}} \left( q^{(k)} (F_{\eta_0})_{\eta_0^{(k)}} - \beta^{(k)} (F)_{\eta=\eta_0^{(k)}} \right) = 0
\]

(41)

with notation \( \langle Y \rangle_{\eta_0} \equiv \frac{1}{2} \left( Y \bigg|_{\eta\to\eta_0+0} + Y \bigg|_{\eta\to\eta_0-0} \right) \).

In what follows we will take time-independent dynamical \( LL\)-brane tension(s) \( (\partial_0 T^{(k)} = 0) \) because of matching static bulk space-time geometries. Let us also note that the appearance of mean values of the corresponding quantities with discontinuities across the horizons follows the resolution of the discontinuity problem given in [3] (see also [23]).

The wormhole-like solutions presented in the next Section share the following important properties:

- (a) The \( LL\)-branes at the wormhole “throats” represent “exotic” matter \( - T \leq 0 \), i.e., negative or zero brane tension implying violation of null-energy conditions as predicted by general wormhole arguments [16] (although the latter could be remedied via quantum fluctuations).

- (b) The wormhole-like space-times constructed via \( LL\)-branes at their “throats” are not traversable w.r.t. the “laboratory” time of a static observer in either of the different “universes” comprising the pertinent wormhole space-time manifold. On the other hand, they are traversable w.r.t. the proper time of a traveling observer.

Proper-time traversability can be easily seen by considering dynamics of test particle of mass \( m_0 \) (“traveling observer”) in a wormhole background, which is described by the world-line action:

\[
S_{\text{particle}} = \frac{1}{2} \int d\lambda \left[ \frac{1}{e} \dot{x}^\mu \dot{x}^\nu G_{\mu\nu} - em_0^2 \right].
\]

(42)

Using energy \( \mathcal{E} \) and orbital momentum \( \mathcal{J} \) conservation and introducing the proper world-line time \( s \left( \frac{ds}{d\lambda} = em_0 \right) \), the “mass-shell” equation (the equation w.r.t. the “einbein” \( e \) produced by the action (42)) yields:

\[
\left( \frac{d\eta}{ds} \right)^2 + \mathcal{V}_{\text{eff}}(\eta) = \frac{\mathcal{E}^2}{m_0^2} , \quad \mathcal{V}_{\text{eff}}(\eta) \equiv A(\eta) \left( 1 + \frac{\mathcal{J}^2}{m_0^2 C(\eta)} \right)
\]

(43)

where the metric coefficients \( A(\eta), C(\eta) \) are those in (35). Irrespective of the specific form of the “effective potential” in (43), a “radially” moving (with zero “impact” parameter \( \mathcal{J} = 0 \)) traveling observer (and with sufficiently large energy \( \mathcal{E} \)) will always cross within finite amount of proper-time through any “throat” \( (\eta = \eta_0^{(k)} \) from one “universe” to another and possibly even shuttle between them (cf. Subsection 5.4 below).
5. Examples

Henceforth we will use the following acronyms for brevity: “BR” = “Bertotti-Robinson”, “Schw” = “Schwarzschild”, “RN” = “Reissner-Nordström”, “(A)dS” = “(Anti-)de-Sitter”, “SdS” = “Schwarzschild-de-Sitter”, and LL-brane matching will be denoted by “|”.

5.1. Symmetric Wormhole with Reissner-Nordström Geometry

It consists of two identical copies of exterior RN region ($r > r_0$, $r_0$ denoting the outer RN horizon) – “left” RN “universe” ($\eta < 0$) and “right” RN “universe” ($\eta > 0$) glued together via a LL-brane sitting on $r = r_0$ ($\eta = 0$):

$$ds^2 = -A(\eta)dv^2 + 2dv d\eta + C(\eta) \left[ d\theta^2 + \sin^2 \theta d\varphi^2 \right] ,$$

$$A(\eta) = 1 - \frac{2m}{r_0 + |\eta|} + \frac{Q^2}{(r_0 + |\eta|)^2} , \quad C(\eta) = (r_0 + |\eta|)^2 , \quad A(0) = 0 , \quad A(\eta) > 0 \text{ for } \eta \neq 0 .$$

RN mass is determined by the dynamical LL-brane tension $T$:

$$(16\pi |T|\sqrt{b_0} m - 1) (m^2 - Q^2) + 16\pi^2 T^2 b_0 Q^4 = 0 .$$

In the particular case of Schwarzschild wormhole (Einstein-Rosen “bridge”, $Q = 0$): $m = 1/8\pi |T|$.

5.2. Non-singular Black Hole

It is described by the metric:

$$ds^2 = -A(r)dv^2 + 2dv dr + r^2 \left[ d\theta^2 + \sin^2 \theta d\varphi^2 \right] ;$$

$$A(r) \equiv A(\pm)(r) = 1 - K r^2 , \quad \text{for } r < r_0 \quad (\text{de Sitter}) ,$$

$$A(r) \equiv A(\pm)(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} , \quad \text{for } r > r_0 \quad (\text{RN}) ,$$

where $r_0$ is the common horizon $A(\pm)(r_0) = 0$ , $r_0 = m - \sqrt{m^2 - Q^2}$ (internal RN).

An electrically charged LL-brane occupies the horizon $r = r_0$ and uniquely determines all parameters $r_0 = \frac{1}{\sqrt{K}} , \quad m = \frac{2}{\sqrt{K}} , \quad Q^2 = \frac{3}{K} , \quad \text{with} \quad \Lambda = 3K = \frac{4\pi}{3} \beta^2 - \text{dynamically generated cosmological cont} \text{in the interior de-Sitter region through the Kalb-Ramond LL-charge } \beta$. Apparently there is no black hole singularity at $r = 0$.

5.3. Asymmetric Wormhole – Schw-dS | RN

The overall metric is $ds^2 = -A(\eta)dv^2 + 2dv d\eta + (r_0 + |\eta|)^2 \left[ d\theta^2 + \sin^2 \theta d\varphi^2 \right]$ with $A(0) = 0$. Here we have:
(i) “left universe” – exterior region of Schwarzschild-de-Sitter space-time above the inner (Schwarzschild-type) horizon $r_0$:

$$A(\eta) = 1 - \frac{2m_1}{r_0 - \eta} - K(r_0 - \eta)^2 \quad \text{for} \; \eta < 0 ; \quad (51)$$

(ii) “right universe” – exterior Reissner-Nordström region beyond the outer RN horizon $r_0$:

$$A(\eta) = 1 - \frac{2m_2}{r_0 + \eta} + \frac{Q^2}{(r_0 + \eta)^2} \quad \text{for} \; \eta > 0 . \quad (52)$$

Charged LL-brane occupies the common horizon (wormhole “throat”) and determines all wormhole parameters via its charges $(q, \beta)$:

$$m_1 = \sqrt{b_0} \left(1 - b_0 \beta^2 \right) / (3 \pi T^2) , \quad m_2 = \sqrt{b_0} \left(1 + \frac{4q^2}{\pi T^2} \right) , \quad (53)$$

$$r_0 = \frac{\sqrt{b_0}}{2 \pi |T|} , \quad T^2 = \frac{\beta^2 + 4q^2}{2 \pi (1 - 4b_0)} , \quad Q^2 = \frac{16 \pi b_0 q^2 r_0^4}{b_0} . \quad (54)$$

including the dynamically generated cosmological const $\Lambda = 3K = 4\pi \beta^2$ in the “left” universe.

5.4. Compactification/Decompactification Transitions

These are wormhole-like solution with two widely separated LL-branes sitting at horizons $\eta = \eta_0 \equiv 0$ and $\eta = \bar{\eta}_0$, with metric:

$$ds^2 = -A(\eta) dv^2 + 2 dv d\eta + C(\eta) \left[ d\theta^2 + \sin^2 \theta d\varphi^2 \right] , \quad (55)$$

$$A(0) = 0 , \quad A(\bar{\eta}_0) = 0 , \quad \bar{\eta}_0 \equiv r_0 - r_0 > 0 , \quad A(\eta) > 0 \; \text{for} \; \eta \neq 0 , \; \bar{\eta}_0 , \quad (56)$$

describing three pairwise matched space-time regions:

(i) “left” Bertotti-Robinson “universe” ($AdS_2 \times S^2$) for $\eta < 0$ where:

$$A(\eta) = \frac{\eta^2}{r_0^2} , \quad C(\eta) = r_0^2 , \quad F_\eta = \pm \frac{1}{2 \sqrt{\pi} r_0} ; \quad (57)$$

(ii) “middle” Reissner-Nordström-de-Sitter “universe” for $0 < \eta < r_0 - r_0$ with:

$$A(\eta) = 1 - \frac{2m}{r_0 + \eta} + \frac{Q^2}{(r_0 + \eta)^2} - \frac{4\pi \beta^2}{3 (r_0 + \eta)^2} , \quad (58)$$

$$C(\eta) = (r_0 + \eta)^2 , \quad F_\eta = \frac{Q}{\sqrt{4\pi (r_0 + \eta)^2}} , \quad (59)$$

including the dynamically generated cosmological const $\Lambda = 3K = 4\pi \beta^2$ in the “left” universe.
where \( r_0 \) and \( \tilde{r}_0 \) (\( \tilde{r}_0 > r_0 \)) are the intermediate (outer RN) and the outermost (de-Sitter) horizons of the standard RN-de-Sitter space-time (note the dynamically generated cosmological const \( \Lambda = 4\pi\beta^2 \) in (58));

(iii) another “right” Bertotti-Robinson “universe” \( (AdS_2 \times S^2) \) for \( \eta > \tilde{r}_0 - r_0 \):

\[
A(\eta) = \frac{(\eta - \tilde{r}_0 + r_0)^2}{\tilde{r}_0^2}, \quad C(\eta) = \tilde{r}_0^2, \quad F_{\eta\eta} = \pm \frac{1}{2\sqrt{\pi}\tilde{r}_0}.
\]

Traveling observer along \( \eta \)-direction will “shuttle” between the three “universes” crossing consecutively both LL-branes at the “throats” within finite intervals of his/her proper time.

5.5. Multi-“throat” wormhole Schw | SdS | SdS | Schw

This is a wormhole-like solution with metric:

\[
ds^2 = -A(\eta)d\sigma^2 + 2d\nu d\eta + (r_0 + \eta)^2 [d\theta^2 + \sin^2 \theta d\varphi^2]
\]

\[
A(0) = 0, \quad A(\pm(\tilde{r}_0 - r_0)) = 0
\]

describing four pairwise matched space-time regions via 3 widely separated LL-branes located at \( \eta = 0 \) and \( \eta = \pm(\tilde{r}_0 - r_0) \):

(i) “left-most” \( (\eta < -(\tilde{r}_0 - r_0)) \) and “right-most” \( (\eta > \tilde{r}_0 - r_0) \) “universes” comprising the exterior Schwarzschild region beyond the Schwarzschild horizon at \( \tilde{r}_0 \):

\[
A(\eta) = 1 - \frac{\tilde{r}_0}{r_0 + |\eta|} \quad \text{for } |\eta| > \tilde{r}_0 - r_0 ,
\]

(ii) two “middle” “universes”, for \( -(\tilde{r}_0 - r_0) < \eta < 0 \) and for \( 0 < \eta < \tilde{r}_0 - r_0 \) — two identical copies of the intermediate region of Schwarzschild-de-Sitter space-time between the inner (Schwarzschild) horizon at \( r_0 \) and the outer (de-Sitter) horizon at \( \tilde{r}_0 \):

\[
A(\eta) = 1 - \frac{2m}{r_0 + |\eta|} - \frac{4\pi\beta^2}{3}(r_0 + |\eta|)^2 \quad \text{for } |\eta| < \tilde{r}_0 - r_0 ,
\]

where \( A(0) = 0 \) (inner SdS horizon) and \( A(\pm(\tilde{r}_0 - r_0)) = 0 \) (outer SdS horizon) and with dynamically generated (by the LL-branes) cosmological const \( \Lambda = 4\pi\beta^2 \).

5.6. Lightlike Braneworld

This is a solution with a bulk \( D = 5 \) space-time consisting of two identical copies of the exterior region of \( D = 5 \) AdS-Schwarzschild black hole beyond the horizon \( r_0 \) (“left” universe for \( \eta < 0 \) and “right” universe for \( \eta > 0 \))
glued together by a lightlike 3-brane with flat 4-dim world-volume located at the horizon \((\eta = 0)\):

\[
ds^2 = -A(\eta)d\tau^2 + 2dv d\eta + K(r_0 + |\eta|)^2 d\vec{x}^2,
\]

\[A(\eta) = K(r_0 + |\eta|)^2 - \frac{m}{(r_0 + |\eta|)^2}\]

with \(A(0) = 0\) and \(A(\eta) > 0\) for \(\eta \neq 0\), where \(\Lambda = -6K\) is the bare \(D=5\) cosmological constant.

The bulk space-time parameters \((K, m)\) are related to the LL-brane parameters \((T, b_0)\) as: \(T^2 = 3K/8\pi^2\) and \(b_0 = \frac{2}{3}\sqrt{Km}\).

Because of the shape of the “effective potential” \(A(\eta)\) a traveling observer along the extra 5-th dimension will “shuttle” between the two “universes” crossing in either direction the \(D=4\) braneworld within finite intervals of his/her proper time.

6. Conclusions

To conclude let us recapitulate the crucial properties of the dynamics of \(LL\)-branes interacting with gravity and bulk space-time gauge fields:

(i) “Horizon straddling” – automatic positioning of \(LL\)-branes on (one of) the horizon(s) of the bulk space-time geometry.

(ii) Intrinsic nature of the \(LL\)-brane tension as an additional degree of freedom unlike the case of standard Nambu-Goto \(p\)-branes (where it is a given \(ad\ hoc\) constant), and which might in particular acquire zero or negative values.

(iii) The stress-energy tensors of the \(LL\)-branes are systematically derived from the underlying \(LL\)-brane world-volume Lagrangian actions and provide the appropriate source terms on the r.h.s. of Einstein equations to enable the existence of consistent non-trivial wormhole-like solutions.

(iv) \(LL\)-branes naturally couple to Kalb-Ramond bulk space-time gauge fields which results in dynamical generation of space-time varying cosmological constant. In particular, the latter is responsible for creation of a non-singular black hole with de Sitter interior region below the horizon.

(v) The above properties of \(LL\)-branes trigger spontaneous compactification/decompactification transitions in the bulk space-time manifold.

Further explicit solutions describing multi-“throat” wormhole-like space-times of the form “\(BR\) | \(SdS\) | \(SdS\) | \(BR\)” , “\(BR\) | \(SdS\) | \(Schw\)” , “Cyclic” \(SdS\), as well as “flat Minkowski | \(AdS-RN\)” will appear in a subsequent paper.

Appendix

Let us consider for simplicity the \(LL\)-brane Polyakov-type action (7) for \(p = 0\), i.e., the case of lightlike \((LL-\) particle:

\[
S_{LL-\text{particle}} = \frac{1}{2} \int d\tau T b_0^{\frac{1}{2}} \left[ \frac{1}{\epsilon} \left( X^2 - \frac{\dot{u}^2}{T^2} \right) - \epsilon b_0 \right], \quad (65)
\]
where $\dot{X}^2 \equiv \dot{X}^\mu G_{\mu\nu} \dot{X}^\nu$ and $e$ is the einbein ($\gamma_{00} = -e^2$, $\sqrt{-\gamma} = e$). We will show that the LL-particle (65) is dynamically equivalent to the standard massless particle described by the action (42) with $m_0 = 0$.

Indeed, the action (65) produces the following equations of motion w.r.t. $e$, $T$, $u$ and $X^\mu$:

$$\dot{X}^2 + \epsilon \left( b_0 e^2 - \frac{\dot{u}^2}{T^2} \right) = 0 \quad \dot{X}^2 - \epsilon \left( b_0 e^2 - \frac{\dot{u}^2}{T^2} \right) = 0 ,$$

$$\partial_\tau \left( \frac{\dot{u}}{eT} \right) = 0 \quad \partial_\tau \left( \frac{T}{e} \dot{X}^\mu \right) + \frac{T}{e} \dot{X}^\nu \dot{X}^\lambda \Gamma^\mu_{\nu\lambda} = 0 .$$

Eqs.(66) imply $\ddot{X}^2 = 0$ and $e^2 b_0 = \dot{u}^2 / T^2$, where the first expression is the standard massless constraint following from the standard action (42) (with $m_0 = 0$) upon varying w.r.t. $e$, whereas the second relation makes the first Eq.(67) an identity. The last Eq.(67) is obviously equivalent to the standard geodesic equation up to a world-line $\tau$-reparametrization.

Within the canonical Hamiltonian approach, introducing the canonical momenta (using the short-hand notation $\epsilon \equiv e b_0^{1/2}$) $P_\mu = \frac{T}{e} G_{\mu\nu} \dot{X}^\nu$ and $p_u \equiv -\frac{T}{e} \dot{u}$ we obtain the canonical Hamiltonian:

$$H_c = \frac{\bar{e}}{2T} P^2 - \frac{\bar{e}T}{2} \left( p_u^2 - 1 \right) , \quad P^2 \equiv P_\mu G^{\mu\nu} P_\nu .$$

Preservation of the primary constraints $p_e = 0$ and $p_T = 0$ (vanishing canonical momenta of $e$ and $T$) by (68) yields the secondary first-class constraints:

$$P^2 = 0 \quad p_u^2 - 1 = 0 .$$

Thus, we deduce that $e, T, u$ are non-propagating “pure-gauge” degrees of freedom and we are left with the first relation (69) which is the standard canonical massless constraint resulting from the standard action (42) (with $m_0 = 0$) within the Hamiltonian formalism.

Acknowledgments

E.N. is sincerely grateful to Prof. Branko Dragovich and the organizers of the Sixth Meeting in Modern Mathematical Physics (Belgrade, Sept 2010) for cordial hospitality. E.N. and S.P. are supported by Bulgarian NSF grant DO 02-257. Also, all of us acknowledge support of our collaboration through the exchange agreement between the Ben-Gurion University of the Negev and the Bulgarian Academy of Sciences.

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