Supplemental Material 2

Pochhammer–Chree equation solver for dispersion correction of elastic waves in a (split) Hopkinson bar

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1. Nomenclature

\( a \)  
bar radius

\( C \)  
normalised speed of sound (\( = c / c_o \))

\( C_a \)  
approximate \( C \) solution at current \( F \) before bisection method application

\( C_{dc} \)  
\( C \) solution at \( F_{dc} \) necessary for dispersion correction

\( C_{\text{max}} \)  
maximum \( C \) value in tracking process of dispersion curves

\( c \)  
sound speed

\( c_o \)  
sound speed in thin bar (\( = \sqrt{E/\rho} \))

\( dC \)  
interval in tracking process of dispersion curves (\( dC \) is noted as \( dC_{Xa}, dC_{Xb}, \) etc., depending on routines)

\( E \)  
elastic modulus

\( dF_{\text{dc}} \)  
required \( F \) interval for dispersion correction

\( dF_{\text{ref}} \)  
target \( F \) interval value at or near which \( dF \) value is to be determined (\( dF_{\text{ref}} \) is noted as \( dF_{\text{ref}_Xa}, dF_{\text{ref}_Xb}, \) etc., depending on routines)

\( dF \)  
actual \( F \) interval in tracking process of dispersion curves (\( dF \) is noted as \( dF_{Xa}, dF_{Xb}, \) etc., depending on routines)

\( F \)  
normalised frequency (\( = a f / c_o \))

\( F_a \)  
approximate \( F \) solution at current \( C \) before bisection method application

\( F_{dc} \)  
necessary \( F \) value for dispersion correction

\( f \)  
frequency

\( F_{ct} \)  
critical \( F \) above which no abrupt-change point appears in dispersion curve (\( n = 20 \))

\( F_{dc_{\text{max}}} \)  
maximum \( F \) value in dispersion correction

\( F_{\text{max}} \)  
maximum \( F \) value in tracking process of dispersion curves

\( F_{\text{ref}} \)  
target \( F \) value where \( F_{\text{start}(1)} \) is determined.

\( F_{\text{start}(1)} \)  
actual \( F \) value where order of solutions is counted (\( F_{\text{start}(2)} = F_{\text{start}(1)} - dF \))

\( \gamma \)  
wave number (\( = 2\pi / \Lambda \))

\( L \)  
normalised wavelength (\( = a / \Lambda \))

\( \Lambda \)  
wavelength

\( \lambda \)  
a Lamé constant

\( m \)  
order of dispersion curves (locus of solutions) when solutions are located between two PC function surfaces with (nearly) positive and negative infinity values
order of dispersion curves when solutions are located on PC function surface that relatively gradually vary in the vicinity of roots

number of time data within time window of elastic wave

number of frequencies in dispersion correction (Nyquist number)

Poisson’s ratio

abbreviation for Pochhammer–Chree

another Lamé constant

density

angular frequency (\(= 2\pi f\))

name of routine where \(F\) is independent variable and \(C\) is dependent variable

name of routine where \(C\) is independent variable and \(F\) is dependent variable

2. Supplemental introduction

2.1 Challenges in solving PC equation

The PC equation [S1–S3] cannot be solved analytically in a closed form. In general, numerical solutions were derived. The PC equation has real, imaginary, and complex roots [S4,S5]. Real roots correspond to propagating modes without attenuation, while imaginary and complex roots correspond to non-propagating modes with attenuation. This study considers the real solution for the propagating mode that necessitates dispersion correction.

As the PC equation was first reported in 1876 [S1], its first-order numerical solution seems to have become available only after 55 years, i.e., in 1931 (Ruedy [S6] and Field [S7]), most likely owing to the computational burden involved in the numerical solution process in early studies. The solutions to the PC equation were continually sought thereafter via various numerical methods including an analytical approach for an approximate solution [S8] and numerical approaches via the finite difference method [S9] and finite element method [S10]. Several studies [S11–S15] employed various mathematical techniques developed for nonlinear equations to solve the PC equation. However, studies that have applied the aforementioned methods to obtain necessary information for dispersion correction (\(c\) values in the range of \(f\) values) are rare. Most studies that solved the PC equation for dispersion correction used the root-finding process based on iterative techniques [S4,S5,S16–S46].
When a single root exists, iterative techniques such as linear or quadratic interpolation and extrapolation algorithms may be applied. However, in the PC equation, two roots are usually in close proximity to the bending or intersection points of the dispersion curves. In such a case, the function often changes sign twice [S44], rendering the above-mentioned solution schemes unstable. Hence, the PC equation should be solved by more robust iteration techniques such as the bisection method.

However, even when a robust method is used, a different type of problem arises: the initial guess solution selected must considerably approximate the solution. Furthermore, the PC function surfaces often vary extremely rapidly with practically infinite slopes near the root; they typically intersect with or are significantly close to other planes. Consequently, even the use of robust methods is never a simple task, especially near the bending or crossing points of dispersion curves (locus of solutions). Under such circumstances, a method of obtaining graphical solutions was reported [S44] that may be useful for quickly comprehending the overall shape of the dispersion curves (in $c-f$, $c-\Lambda$, and $\Lambda-f$ domains) while necessitating digitisation of the graph for use in dispersion correction; the accuracy of the solution is dependent on the graph resolution and digitisation.

2.2 State-of-the-art PC equation solutions

Despite the difficulty of solving the PC equation, obtaining a solver program or a detailed algorithm to solve the PC equation is challenging. Moreover, the studies reporting the numerical solutions did not reveal the details of their numerical schemes [S4,S5,S16–S43]. Without an available solver, handling the PC equation to obtain the necessary $c-f$ relation for a specific HB or SHB system is cumbersome.

Under such circumstances, making use of existing solutions such as those in tabular forms available in Bancroft [S17] and Davies [S20] may be more practical. In most dispersion correction studies in the literature [S45–S52], the $c-f$ relation necessary for dispersion correction was obtained by curve-fitting the existing $c-\Lambda$ solutions [S17,S20], followed by combining the fitted result with the relation, $c = f\Lambda$. There were also cases [S53–S59] where the $c-f$ relation for the considered bars or the method of obtaining it were undisclosed.
However, curve-fitting of existing solutions has the following three limitations. First, the range of $\Lambda$ in the available solutions ($c$–$\Lambda$ curve) themselves was relatively narrow; consequently, the $c$ values could not be obtained for a sufficiently wide range of frequencies. Second, the $c$ vs. $f$ solutions obtained via the curve-fitting approach are generally at frequencies unnecessary for dispersion correction, thereby necessitating the interpolation of the obtained $c$–$f$ solutions to those at exactly the necessary frequencies. Third, the existing solutions were available only for some limited values of Poisson’s ratios, while versatile material types are currently used as elastic bars for HB and SHB; the existing solutions need to be interpolated again to the ones at the Poisson’s ratio of the considered bar. Overall, unless the PC equation is directly solved at the required frequencies for the specific Poisson’s ratio of considered bar, complete dispersion correction is limited.

Despite the high necessity of solving the PC equation, only a few studies involving dispersion correction [S42,S43] recently solved the PC equation, probably owing to the cumbersome nature of handling the PC equation. In these studies, solutions for three considered Poisson’s ratios (0.2, 0.3, and 0.35) were obtained based on in-house schemes (undisclosed) and were provided in graphical forms within the limited ranges of frequencies and sound speeds.

3. Pochhammer–Chree equation

The PC equation was obtained by transforming the Lamé–Navier equations of motion in Cartesian coordinates into cylindrical coordinates and applying the boundary conditions for traction-free surfaces [S6,S60]. The solution of the PC equation is the relation between $c$ and $\Lambda$. Considering that the frequency ($f$) can be obtained from the $c$–$\Lambda$ relation via the equation, $c = f \Lambda$, the PC equation expressed in the $c$–$\Lambda$ domain is called the ‘frequency equation’. If the PC equation is established in the $c$–$f$ and $f$–$\Lambda$ domains, then the wavelength ($\Lambda$) and speed of sound ($c$) can be obtained, respectively. The PC equation in the $c$–$f$ and $f$–$\Lambda$ domains can be referred to as wavelength and sound speed equations, respectively.

There are many versions of the PC equation [S1–S3,S17,S35,S38,S39,S61–S63]. Among them, those by Love [S61], Redwood [S63], and Bancroft [S17] are introduced here.
3.1 Love version

Love [S61] was the first scholar who summarised and explained the PC equation in detail. Consequently, the Love version was generally introduced and utilised in the literature [S12–S17] (together with the Bancroft version) instead of the versions in original papers [S1–S3]. Bancroft [S17] extracted the PC frequency equation in the form of a 3×3 determinant from equation (54) of Love [S61] in p. 284 (4th edition). This determinant yields:

\[
\frac{4\pi^2}{\Lambda^2} \left( \frac{\rho c^2}{\mu} - 2 \right) J_1(k'a) \left[ 2\mu \frac{\partial^2 J_0(h'a)}{\partial a^2} - \frac{4\pi^2 \lambda \rho c^2}{\Lambda^2(\lambda + 2\mu)} J_0(h'a) \right] + \frac{16\pi^2 \mu}{\Lambda^2} \frac{\partial J_0(h'a)}{\partial a} \frac{\partial J_1(k'a)}{\partial a} = 0
\]

where \( c \) is the wave speed with wavelength \( \Lambda \); \( \lambda \) and \( \mu \) are Lamé constants; \( \rho \) is the density; \( a \) is the bar radius; \( h' \) and \( k' \) are given by

\[
h'^2 = \frac{\omega^2 \rho}{(\lambda + 2\mu)} - \gamma^2, \quad k'^2 = \frac{\omega^2 \rho}{\mu} - \gamma^2
\]

where \( \omega (= 2\pi f; f \) is frequency) is the angular frequency; \( \gamma (= 2\pi / \Lambda ) \) is the wave number\(^1\); \( J_0 \) and \( J_1 \) are the Bessel functions of the first kind of order zero and one, respectively, given by the following:

\[
J_0 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2}, \quad J_1 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} (m+1)!}
\]

The values of the Bessel function of the first kind were calculated using MATLAB® software for orders from –1 to 3; the results are illustrated in Figure S1. The Bessel function is the solution of Bessel’s ordinary differential equation. It arises when a number of physical phenomena, such as elastic vibration, fluid flow, thermal conduction, and particle scattering, are described in cylindrical coordinates.

Love approximated the Bessel function for the cylinder with a small radius using the following approximations.

\[
J_0(h'a) \approx 1 - \frac{1}{4} h'^2 a^2 + \frac{1}{64} h'^4 a^4
\]

\(^1\) In physics, this is called ‘wave vector’. The term ‘wave number’ is reserved in physics for the ‘number of wavelengths per unit length’.
Figure S1. Shape of the Bessel function of the first kind for orders −1, 0, 1, 2, and 3.

\[ J_1(k'a) \cong \frac{1}{2} \left( k'a - \frac{1}{8} k'^3 a^3 \right) \]

Finally, the PC frequency equation is obtained as follows:

\[ \left( \frac{\omega^2 \rho}{\mu} - 2\gamma^2 \right) k'a \left( 1 - \frac{k'^2 a^2}{8} \right) \left[ h'^2 \left( 1 - \frac{3}{8} h'^2 a^2 \right) + \frac{\lambda \omega^2 \rho}{\mu(\lambda + 2\mu)} \left( 1 - \frac{1}{4} h'^2 a^2 \right) \right] \]

\[ + 2\gamma^2 k' \left( 1 - \frac{3}{8} k'^2 a^2 \right) h'^2 a \left( 1 - \frac{1}{8} h'^2 a^2 \right) = 0 \]  

(S1)

Equation (S1) constitutes two variables (c and Λ) describing the characteristics of the elastic wave and four variables (a, ρ, λ, and μ) describing the characteristics of the medium; these are simply called ‘wave variables’ and ‘medium variables’, respectively, herein. Equation (S1) can be solved explicitly for any particular variable as a function of the other five variables. For instance, to obtain one of the wave variables, e.g., c, as a function of the other, say, Λ, four medium variables have to be known prior to solving equation (S1).

3.2 Redwood version

Redwood [S63] analysed the elastic wave propagation in rectangular coordinates by considering that the elastic wave constitutes two types of disturbance. One is the dilatational
wave\(^2\) that involves no rotation in the medium particles and travels at a speed \(c_d (= \sqrt{\lambda/\rho})\). The other type is the transverse wave that involves no volume change in the medium particles and travels at a speed \(c_t (= \sqrt{(\lambda + 2\mu)/\rho})\). The wave equation of motion, i.e., the Redwood version of the PC equation, was derived in terms of these two types of waves:

\[
\gamma^2 \frac{k_t J_0(k_t a)}{J_1(k_t a)} - \frac{1}{2} \left(\frac{\omega}{\rho c_t}\right)^2 \frac{1}{a} + \left[\frac{1}{2} \left(\frac{\omega}{\rho c_t}\right)^2 - \gamma^2\right]^2 \frac{j_0(k_d a)}{j_1(k_d a)} = 0
\]

(S2)

where \(k_d = \sqrt{(\omega/c_d)^2 - \gamma^2}\) and \(k_t = \sqrt{(\omega/c_t)^2 - \gamma^2}\). Similar to the Love version, the Redwood version (equation (S2)) is also composed of two wave variables (\(c\) and \(A\)) and four medium variables (\(a\), \(\rho\), \(\lambda\), and \(\mu\)). The latter four variables should be known prior to solving the Redwood version to obtain the wave variable relations (\(c\sim A\)).

The Redwood version was solved by Alterman and Karal Jr. [S9], and the derived solutions were regarded as the reference for their finite difference analysis results. The Laverty–Puckett [S64–S66] solver also solved the Redwood version including the calculation of the cut-off frequency.

### 3.3 Bancroft version

As mentioned, two wave variables and four medium variables are involved in equations (S1) and (S2). Bancroft [S17] simplified the PC equation in terms of five variables (i.e., \(c\), \(A\), \(\nu\), \(\rho\), and \(c_o\)), which, in turn, are merged into three non-dimensional variables (\(x\), \(\gamma a\), and \(\beta\)):

\[
(x - 1)^2 \Phi(\gamma a) - (\beta x - 1)[x - \Phi(\gamma a)] = 0
\]

(S3)

where

\[
\begin{align*}
x &= \frac{c}{c_o^2}(1 + \nu), & \beta &= \frac{(1 - 2\nu)}{(1 - \nu)} \\
h &= \gamma(\beta x - 1)^{1/2}, & k &= \gamma(2x - 1)^{1/2} \\
k &= \gamma(2x - 1)^{1/2}, & \Phi(y) &= j_0(y)/j_1(y)
\end{align*}
\]

\(^2\)The dilatational wave is also called longitudinal, compressional, or irrotational wave; the transverse wave is also named shear, equivoluminal, rotational, or distortional wave [S63].
The Bancroft version (equation (S3)) is in the form \( G(x, \gamma a, \beta) = 0 \). The simplification of the PC equation to the nominal three-variable version (equation (S3)) was possible because the two wave variables (\( c \) and \( \Lambda \)) and three medium variables (\( a, c_o, \) and \( \nu \)) were successfully embedded into \( x \) and \( \gamma a \). The first variable \( (x) \) is a function of \( c, c_o, \) and \( \nu \); the second variable \( (\gamma a) \) is a function of \( \Lambda, a, c, c_o, \) and \( \nu \); the last variable \( (\beta) \) is a function of only \( \nu \).

This nominal three-variable version (equation (S3)) affords a considerable advantage in the solution process compared with other existing versions with six variables. In the solution process of the Bancroft version, it is not necessary to identify what physically \( x \) and \( \gamma a \) are; they can be simply regarded as two arbitrary independent and dependent variables. Either \( x \) or \( \gamma a \) can be solved as a function of the other at a given value of \( \beta \) (a function of only \( \nu \)); this is a significant advantage of the nominal three-variable equation. No medium variables other than Poisson’s ratio (\( \nu \)) are necessary in the solution process itself of the Bancroft version.

Once the Bancroft version is solved (i.e., once the \( x - \gamma a \) relation is obtained at a given \( \beta \) (\( \nu \))), the \( c-\Lambda \) relation is extracted from the \( x-\gamma a \) relation in the post-processing stage using medium variable information; as mentioned, the wave variables (\( c \) and \( \Lambda \)) are embedded in \( x \) and \( \gamma a \), respectively. Another advantage afforded by the Bancroft version is found in the post-processing stage: only two more medium variables (\( a \) and \( c_o \)) in addition to the \( \nu \) information used in the solution process are required. In other words, only three medium variables (\( \nu, a, \) and \( c_o \)) in total are necessary in the Bancroft version compared with other versions, which require four medium variables (\( \nu, \rho, \mu, \) and \( \lambda \)).

### 3.4 Bancroft version with physics-friendly variables

As mentioned in the main text, the physics-friendly non-dimensional variables employed herein are \( C (=c/c_o) \), \( L (=a/\Lambda) \), and \( F (=af/c_o) \). Their relation, \( F = CL \), is illustrated in Figure S1. If \( F \) and \( C \) are explicitly expressed in equation (S3), then,

\[
G(C, F, \nu) = [C^2(1 + \nu) - 1]^2 \Phi \left[ 2\pi \sqrt{\beta F^2(1 + \nu) - (F/C)^2} \right] - [\beta C^2(1 + \nu) - 1] \\
\times \left\{ C^2(1 + \nu) - \Phi \left[ 2\pi \sqrt{2F^2(1 + \nu) - (F/C)^2} \right] \right\} = 0 \quad (S4)
\]
The Bancroft version can also be expressed in the $C$–$L$ or $L$–$F$ domain, which are shown in equations (S5) and (S6). If the $C$–$L$ or $L$–$F$ version is solved, then the solution should be converted to the $C$–$F$ domain via the relation $F = CL$. If $C$ and $L$ are explicitly expressed in equation (S3), then,

\[ G(L, C, \nu) = [C^2(1 + \nu) - 1]^2 \Phi \left[ 2\pi L \sqrt{\beta C^2(1 + \nu) - 1} \right] - [\beta C^2(1 + \nu) - 1] \]
\[
\times \left\{ C^2(1 + \nu) - \Phi \left[ 2\pi L \sqrt{2C^2(1 + \nu) - 1} \right] \right\} = 0. \quad (S5)
\]

It is convenient to use equation \((S5)\) for constructing the \(c/c_0 - a/A (C-L)\) diagram by considering either \(C\) or \(L\) as the independent variable.

If \(F\) and \(L\) are explicitly expressed in equation \((S3)\), then, its form is written as follows:

\[
G(F, L, \nu) = [(F/L)^2(1 + \nu) - 1]^2 \Phi \left[ 2\pi L \sqrt{\beta(F/L)^2(1 + \nu) - 1} \right] - [\beta(F/L)^2(1 + \nu) - 1]
\times \left\{ (F/L)^2(1 + \nu) - \Phi \left[ 2\pi L \sqrt{2(F/L)^2(1 + \nu) - 1} \right] \right\} = 0 \quad (S6)
\]

It is convenient to use equation \((S6)\) for constructing the \(a/f_0-a/A (F-L)\) diagram by considering either \(F\) or \(L\) as the independent variable.

4. PC function shape

4.1. Function shape in 2D

4.1.1 Stiffness of function curve around the solution

The PC function values in the \(C-F\) domain were calculated using equation \((S4)\). The calculated values are presented in Figure \(S2a\) and \(b\), where \(F\) and \(C\) are considered as independent variables, respectively. The PC function values in the \(L-F\) domain are calculated using equation \((S5)\); the results are shown in Figure \(S2c\) and \(d\). Finally, the PC function values in the \(C-L\) domain are calculated using equation \((S6)\); the results are presented in Figure \(S2e\) and \(f\). In each diagram in Figure \(S2\), only the blue curves are drawn using a ‘solid curve plus open circles’, whereas the green and red curves are drawn using ‘open circles’ only to avoid complexity.

In all the diagrams in Figure \(S2\), the dashed horizontal line is drawn at the PC function value of zero. For each curve, the abscissa values of the points of intersection with the dashed horizontal line are the solutions of the PC equation. For instance, in Figure \(S2a\), the solutions of the blue curve \((C = 1.0)\) are marked using black close squares (■). These solutions are located on the blue curves that vary with a gradual slope as they cross the dashed horizontal line. In the literature, the orders of the solutions on these gradually varying curves are traditionally noted as \(n = 1, 2, 3 \ldots\); this study follows the same convention.
Figure S2. PC function values: (a) and (b) $C–F$ domain calculated using equation (S4); (c) and (d) $L–F$ domain calculated using equation (7); (e) and (f) $C–L$ domain calculated using equation (8). Poisson’s ratio is 0.3 in each domain.

In Figure S2a, the points of intersection of the gradually varying green and red curves with the dashed horizontal line ($n$-series solutions) certainly exist; however, they are left
unmarked to avoid complexity. In the blue curves in Figure S2a, considerably stiff lines connect the positive and negative infinities of the PC function values. Such lines are also present in the L–F (Figure S2c and d; blue curves) and C–L (Figure S2e, and f; blue curves) domains whenever the positive and negative infinities of the PC function values are juxtaposed along the abscissa.

The presence of these stiff lines cannot be visualised if the PC function values are drawn using only open circles. For instance, the curves drawn using green and red open circles do not visually indicate the presence of the stiff lines. However, if these curves are drawn using a ‘solid curve plus open circles’ similar to the blue curves, such stiff lines actually exist. Some examples on the foregoing are provided in Figure S3.

**Figure S3.** PC function values in C–F domain calculated using equation (S4). (a) only green curve \( C = 0.8 \) is drawn in ‘solid curve plus open circles’, whereas two other curves in blue and red are plotted solely using open circles. (b) only red curve \( C = 0.7 \) is drawn in ‘solid curve plus open circles’, whereas two other curves solely using open circles.

Figures S3a and b illustrate the datasets for \( C = 0.8 \) (in green) and \( C = 0.7 \) (in red), respectively, using a ‘solid curve plus open circles’. It is observed that stiff lines actually exist between the two gradually varying curves in green (Figure S3a) and red (Figure S3b); these lines can also be observed when the curves are drawn using a ‘solid curve plus open circles’, such as the blue curves in Figure S2a.
4.1.2 Two types of solutions

In Figure S2a, it is important to note that the abscissa values of open squares (□) on the stiff lines cause the PC function value to become zero; these values are certainly *non-trivial solutions* of the PC equation. These solutions are marked as $m = 1, 2, 3\ldots$. Note that, the considerably stiff lines are also observed in the $L–F$ and $C–L$ domains; the $m$ series solutions are also present in those domains.

At present, it is unknown whether or not the state of elastic vibration in the circular bar can be excited to these $m$-series solutions ($m = 1, 2, 3\ldots$); this is beyond the scope of this study. Nevertheless, this study derives both $m$-series and $n$-series solutions because $m$-series solutions explain the nature of found dispersion curve when the tracking of $n$-series solutions fails; the $m$-series curves are generally found when the tracking process of $n$-series solutions fails. Further, there is no evidence that elastic vibration cannot be excited to the $m$-series solutions.

The Pochhammer–Chree modes of excitation were experimentally verified in a number of studies [S25–S28,S50,S62,S67–S70]. While the excitation to $n = 1$ is considered [S42,S43,S45–S52,S58] in usual SHB experiment, excitations up to $n = 2$ and $n = 3$ were also observed in the studies of Curtis [S67] and Lee et al. [S69], respectively. In a later study of Curtis [S68], an excitation of even as high as $n = 6$ was reported.

4.2 Function shape in 3D

4.2.1 Function shape in $C–F$ domain

To observe the PC function shape in three dimensions, function values were calculated for the ranges of $C$ and $F$ using equation (S4). The results are shown in Figure S4a, where the ordinate (PC function value) spans from –10 to +10. In this study, the horizontal plane with a zero ordinate value is called the ‘root plane’. In Figure S4a, the planes marked as ‘$p$’ gradually vary when they pass through the root plane. By contrast, the planes marked as ‘$q$’ are considerably stiff when they intersect the root plane. Some planes (both ‘$p$’ and ‘$q$’) intersect other ‘$p$’ and/or ‘$q$’ planes, making the overall shape of the PC function surfaces extremely complicated.

The shape of the PC function surfaces near the root plane is investigated by plotting the function values from 0 to 0.2 in Figure S4b. In this figure, the bottom contours of several PC
function planes, which are the graphical solutions of the PC equation, are on the root plane. Similar to that in Figure S4a, two types of planes are also shown in Figure S4b. The $p$-series planes are twisted in shape and vary gradually in the proximity of the root plane. The $q$-series planes are untwisted, and their slope is considerably stiff in the proximity of the root plane. The $p$-series planes appear in a darker colour than the $q$-series planes because more datasets exists in them at the given intervals of $F$ and $C$. 
Figure S4. 3D shapes (real part) of PC function in $F-C$ domain with Poisson’s ratio of 0.3. Function values range (a) from −10 to 10 and (b) from 0 to 0.2. (c) Graphical solutions in $F-C$ domain, i.e., 2D cross-sectional cut at zero-magnitude plane in (a) and (b).

Although the $p$-series planes gradually vary in the proximity of the root plane when both $C$ and $F$ values are small (e.g., $0 \leq C \leq 2$ and $0 \leq F \leq 2$), they can become considerably stiff if either the $C$ or $F$ value increases. This feature causes computational burden in finding the root especially when the $C$ value is extremely high (e.g., $C \geq 500$); this is discussed in a later section (routine Y).

Honarvar et al. [S44] obtained the graphical solution of the PC equation in the $c/c_0$ vs. $af$ domain ($af$ is the dimensional quantity with ms$^{-1}$ unit). The graphical solutions in the $c/c_0$ ($C$) vs. $af/c_0$ ($F$) domain, i.e., the bottom contours shown in Figure S4b, are obtained herein using the MATLAB® commands listed in the footnote$^3$. The results are presented in Figure S4c. In this figure, the solutions for $n = 1, 2, 3 \ldots$ result from planes $p = 1, 2, 3 \ldots$, respectively (Figure S4b), whereas the solutions for $m = 1, 2, 3 \ldots$ are derived from considerably stiff planes.

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3 figure(); contour(C, L, PC); view(−90,90); axis([0.55 2 0 2 0 10]). In these commands, C, L, and PC are the matrices of $C$, $L$ and PC function values, respectively.
As mentioned, this study aims to obtain numerical values of the solutions instead of graphical solutions for both $n$ series and $m$ series.

The 3D shapes of the PC function in the $L$–$F$ and $C$–$L$ domains were also calculated using equations (7) and (8), respectively. The results are shown in the next sections (Figures S5 and S6, respectively). Similar to the case of the $C$–$F$ domain (Figure S4), the planes in the $L$–$F$ (Figure S4) and $C$–$L$ (Figure S5) domains can also be categorised into $p$-series and $q$-series planes, leading to $n$-series and $m$-series solutions, respectively.

### 4.2.2 Function shape in $C$–$L$ domain

The PC function values were calculated for the ranges of $C$ and $L$ using equation (S5). The results are shown in Figure S5a, where the ordinate (PC function value) spans from $-10$ to $+10$. The shape of the PC function surfaces near the root plane is investigated by plotting the function values from 0 to 0.2 in Figure S5b. Figure S5c presents the graphical solution in the $C$–$L$ domain.
**Figure S5.** 3D shapes of PC function in $L$–$F$ domain when the Poisson’s ratio is 0.3. Function values are (a) from $-10$ to $10$ and (b) from $0$ to $0.2$; (c) Graphical solutions in the $L$–$F$ domain: 2D cross-sectional cut at the zero-magnitude plane in (a) and (b).
4.2.3 Function shape in $L$–$F$ domain

The PC function values were calculated for the ranges of $L$ and $F$ using equation (S6). The results are shown in Figure S6a, where the ordinate (PC function value) spans from $-10$ to $+10$. The shape of the PC function surfaces near the root plane is investigated by plotting the function values from 0 to 0.2 in Figure S6b. Figure S6c presents the graphical solution in the $L$–$F$ domain.
5. Solution process

5.1. Pre-processing

Before implementing routine $X (X_a$ and $X_b$) for a given dispersion curve, it is necessary to determine the order of the dispersion curve (either $n$ or $m$). The initial two solutions at $F_{\text{start}(1)}$ and $F_{\text{start}(2)}$ also have to be determined for the linear extrapolation of the $C$ solutions at other $F$ values along a given dispersion curve.

It is necessary to plot the diagram of the PC function value vs. $C$ (such as Figure S7a) at a given frequency to determine the order of the dispersion curves. To construct this diagram, any two frequencies, e.g., $F_{\text{max}} (= F_1)$ and $F_{\text{max}}-dF (= F_2)$, may be selected instead of selecting $F_{\text{start}(1)}$ and $F_{\text{start}(2)}$ at the intermediate $F$ values (Figure 1). Thereafter, routine $X_a$ is unnecessary (only $X_b$ is necessary). However, as the $F_{\text{start}(1)}$ value increases, the slope of the $p$-series planes rapidly changes (Figure S4a). If the $C$ value is allowed to vary at such a high $F$ value, the function plane exhibits a transcendental nature. Furthermore, the spacing between adjacent
planes of different orders becomes extremely narrow. Therefore, if the $F_{\text{start}(1)}$ and $F_{\text{start}(2)}$ values are overly high, then, some PC function planes may be missing during the process of determining the order unless the $C$ value is varied at an overly fine step, which will increase the computational burden.

If a smaller $F_{\text{start}(1)}$ value is selected, the number of dispersion curves decreases within a given $C$-search range. Consequently, a wider range of $C$ has to be scanned to determine the order because the dispersion curves are more widely spread in a low $F$ regime; this also imposes further computational burden. Therefore, intermediate values of $F_{\text{start}(1)}$ and $F_{\text{start}(2)}$ are preferable.

As described above, because the order-determining frequencies, $F_{\text{start}(1)}$ and $F_{\text{start}(2)}$, have to be intermediate, routine $X$ is divided into two parts: $X_a$ and $X_b$. The PC function values are calculated at $F_{\text{start}(1)}$ and $F_{\text{start}(2)} (= F_{\text{start}(1)} - dF)$ by varying the value of $C$ as the independent variable. As an example, the calculated PC function values at an $F_{\text{start}(1)}$ value of 7 as a function of $C$ for the case where $\nu = 0.3$ are shown in Figure S7a; the results calculated at $F_{\text{start}(2)}$ are considerably similar to those shown in the this figure.

In Figure S7a, the horizontal line is drawn at the zero magnitude of the ordinate; consequently, the cross sections between the blue curve and horizontal line are the roots of the PC function at $F_{\text{start}(1)}$. To determine the order of solutions, i.e., the order of dispersion curves, the PC function value ($G_j$) at $F_{\text{start}(1)}$ are monitored with the increase in the $C_j$ value, where subscript ‘$j$’ is the index of the current $C$ value. The $C_j$ value varies from 0.55 to 6, which may be modified by the user via variables $C_0\min$ and $C_0\max$. Once $G_{j-1} \times G_j < 0$, the $C_{j-1}$ and $C_j$ values are set as the lower and higher bound values ($C_{lb}$ and $C_{hb}$, respectively) for the given root to be found by the bisection method.

For the case where $G_{j-1} \times G_j < 0$, the value of $|G_j - G_{j-1}|$ is practically infinite for the $m$-series solutions, whereas the value is finite for the $n$-series solutions (Figure S7a). In this study, if the value of $|G_j - G_{j-1}|/dC$ is less than $10^7$, then, the order of the current root is determined as belonging to the $n$ series; otherwise, it belongs to the $m$ series. This algorithm for determining the order terminates when the user-specified maximum order is reached or $C_j$ attains $C_0\max$.

Examples of the lower and higher bound values obtained according to the foregoing procedure for ranges of $n$ and $m$ values are presented in Figures S7b and c, respectively.
the lower and higher bounds are determined for each order \((n \text{ or } m)\), the solutions \((C \text{ values})\) at \(F_{\text{start}(1)}\) and \(F_{\text{start}(2)}\) are determined up to the user-specified maximum order using the bisection method.

(a)

(b)

| Order \(n\) | Lower \(C\) bound | Higher \(C\) bound |
|-------------|---------------------|---------------------|
| 2           | 0.6                 | 0.7                 |
| 4           | 0.7                 | 0.8                 |
| 6           | 0.8                 | 0.9                 |
| 8           | 0.9                 | 1.0                 |
| 10          | 1.0                 | 1.1                 |
| 12          | 1.1                 |                      |
| 14          | 1.1                 |                      |
| 16          | 1.1                 |                      |
| 18          | 1.1                 |                      |
| 20          | 1.1                 |                      |
5.2. Order counting frequency

$F_{\text{ref}}$ is the target $F$ value at or near which it is aimed to determine $F_{\text{start}(1)}$ and $F_{\text{start}(2)}$. The order is actually counted at $F_{\text{start}(1)}$ and $F_{\text{start}(2)}$. The $F_{\text{ref}}$ value is determined based on the necessity of (1) suitably counting the order and (2) steadily tracking dispersion curves. The $F_{\text{ref}}$ value is pre-set as 7–11 in the proposed solver depending on the type of solution ($n$ or $m$) and Poisson’s ratio. The reason for setting varying $F_{\text{ref}}$ values is explained later in this Supplemental Material (Sections 6.1 and 6.2).

The following explain how $F_{\text{start}(1)}$ and $F_{\text{start}(2)}$ are determined from $F_{\text{ref}}$. The number of $F$ intervals ($Q$) is counted from $F_{\text{max}}$ to $F_{\text{ref}}$ at the $dF$ interval by calculating the quotient ($Q$) of $(F_{\text{max}} - F_{\text{ref}})/dF$. Then, $F_{\text{start}(1)}$ is obtained as $F_{\text{start}(1)} = F_{\text{max}} - Q \cdot dF$. The value of $F_{\text{start}(1)}$ is set to that of $F_{\text{ref}}$ if the division yields an integer; otherwise, it is set in the proximity of the $F_{\text{ref}}$ value. The users may specify the $F$ value where the order of the solution is counted by mandatorily setting the $F_{\text{ref}}$ value or by directly setting the $F_{\text{start}(1)}$ value.

Based on the above method of setting $F_{\text{start}(1)}$ and $F_{\text{start}(2)}$ ($=F_{\text{start}(1)} - dF$) from $F_{\text{ref}}$, orders of up to 20 may be suitably counted by varying the $C$ value from $C0_{\text{min}}$ to $C0_{\text{max}}$ (0.55 to 6.0).
at $F_{\text{start}}(1)$ and $F_{\text{start}}(2)$. The lower limit is set as 0.55 because there is no $C$ solution below this value (Figure S4(c)). The upper limit of 6 is arbitrarily set as the default value, which can be modified via the user variable named $C0_{\text{max}}$ in the program. Unless the counter for order determination reaches the user-specified maximum order value, the order determination continues up to the $C0_{\text{max}}$ value. The increase in the $C0_{\text{max}}$ value increases the maximum order number that can be counted by the program. If the $F_{\text{start}}(1)$ value increases, the maximum order number that can be counted up to a given $C0_{\text{max}}$ value increases because the spacing between adjacent dispersion curves decreases with $F$.

5.3. Routine X (F-driven C solutions)

Upon the completion of pre-processing, routine X is implemented to determine the $C$ solutions at a range of $F$ values (F-driven $C$ solutions). This routine initially uses the two $C$ solutions at $F_{\text{start}}(1)$ and $F_{\text{start}}(2)$ for a given order that are obtained in the pre-processing stage. Routine X is explained below using the case of routine $X_b$. For a given dispersion curve (either $n$ series or $m$ series), the approximate solution ($C_a$) at the $i$th frequency ($F_i$) is first guessed by linearly extrapolating the solutions ($C_{i-2}$ and $C_{i-1}$) at the two previous frequencies ($F_{i-2}$ and $F_{i-1}$, respectively). Then, the lower and higher bound values, $C_{lb} = C_a - r \cdot dC$ and $C_{hb} = C_a + r \cdot dC$, respectively, at a given $F_i$ are searched by increasing integer $r$ until $G(C_{lb})G(C_{hb}) < 0$, where $G$ is the PC function value and $dC$ is the $C$ search interval. Once the two bound values ($C_{lb}$ and $C_{hb}$) that cause the $G(C_{lb})G(C_{hb})$ value to become negative are found, these two values are used to determine solution $C_i$ at the current $F_i$ via the bisection method.

This study implements the frequency-driven routine X at a given frequency interval ($dF$). The relation between the necessary frequency interval for dispersion correction ($dF_{dc}$) and the actual frequency interval ($dF$) for implementing routine X is described in Supplemental Material 2 (Section 5.5).

5.4. Routine Y (C-driven F solutions)

As routine X proceeds for the cases where $n \geq 2$ and $m \geq 1$, a critical $F$ value is reached. Thereafter, the increase in $C$ with the change in $F$ ($dF$) becomes extremely high. In such a case, the condition that $G(C_{lb})G(C_{hb}) < 0$ cannot be satisfied within the given search range of $C$, i.e.,
$C_a \pm r \cdot dC$; consequently, $C_i$ cannot be obtained at $F_i$ via the bisection method. At this instance, routine $Y$ initiates.

Routine $Y$ determines the $F$ solutions within a range of $C$ values ($C$-driven $F$ solutions). The last two $C$ solutions in routine $X_b$ are used as the first two $C$ values (independent variables) in routine $Y$. In addition, the last two $F$ values in routine $X_b$ are used as the first two $F$ values (dependent variables; solutions) in routine $Y$. Then, an approximate solution ($F_a$) at the $j$th $C$ ($C_j$) is first guessed by linearly extrapolating the solutions ($F_{j-2}$ and $F_{j-1}$) at the two previous two speeds ($C_{j-2}$ and $C_{j-1}$, respectively). Next, the lower and higher bound values, $F_{lb} = F_a - s \cdot dF$ and $F_{hb} = F_a + s \cdot dF$, respectively, at the current $C_j$ value are searched by increasing integer $s$ until $G(F_{lb})G(F_{hb}) < 0$, where $dF$ is the $F$ search interval. Once the two bound values ($F_{lb}$ and $F_{hb}$) that cause $G(F_{lb})G(F_{hb})$ to become negative are found, these two values are used to find solution $F_j$ at the current $C_j$ via the bisection method.

When the change in the slope of the dispersion curve is overly high in routine $Y$, the program fails to find $F_{lb}$ and $F_{hb}$ within a given $s$ range. In such a case, the increment step of $C$ is reduced and the search step of $F$ ($dF$) is increased.

As routine $Y$ proceeds, the $C$ value increases towards infinity, whereas the $F$ value practically ceases to vary. In other words, a cut-off $F$ value ($F_{cut}$) has been reached; below this value, there is no $C$ solution, and the dispersion curve ($C$ value) becomes infinite. In the framework of numerical root-finding, it is undesirable to implement routine $Y$ until the $C$ value reaches an overly high value because such implementation will only increase the computational burden. Practically all of the existing studies reporting the solution of the PC equation considered a solution of up to a value of approximately two for both $C$ and $F$, i.e., $C_{max} = 2$ and $F_{max} = 2$. The proposed program calculates the solution up to 500 in $C$ and 30 in $F$, which can be controlled via the user variables, $C_{maxY}$ and $F_{dc_max}$, respectively.

To obtain the cut-off frequency using the Bancroft version of the PC equation, the program formulated in this study calculates the inverse of $C$ (denoted as $I$) at each step in routine $Y$. The cut-off frequency is predicted when the $C$ value in routine $Y$ reaches the arbitrarily selected value of 500 by linearly extrapolating the last two $F$ values to the $F$ value when the $I$ value is zero.
5.5. Determination of $dF$ in a routine

$dF_{\text{ref}}$ is the target $F$ interval at or near which it is aimed to determine the value of $dF$; $dF$ is the actual $F$ interval in the process of tracking a dispersion curve. The $dF_{\text{ref}}$ value is determined based on the necessity of steadily tracking the dispersion curves. In the proposed solver, the $dF_{\text{ref}}$ value is pre-set as 0.0005–0.01 depending on the routine, type of solution ($n$ or $m$), and Poisson’s ratio. The $dF_{\text{ref}}$ value is noted as $dF_{\text{ref}_X a}$, $dF_{\text{ref}_X b}$, etc., depending on the routines; similarly, $dF$ is noted as $dF_{X a}$, $dF_{X b}$, etc., depending on the routines.

To explain the $dF$ value determination process in a certain routine, it is assumed that $dF_{\text{ref}}$ is pre-set as 0.01. $dF_{dc}$ is assumed to be 0.253313 based on the necessity of dispersion correction. To determine the $dF$ value, the multiplication factor, $M$, is first calculated by obtaining the quotient of $dF_{dc}/dF_{\text{ref}} (M = 25)$. Then, the actual frequency interval, $dF$, is given by $dF = dF_{dc}/M = 0.010133$, which closely approaches the value of $dF_{\text{ref}} (= 0.01)$. If $dF_{dc}$ is in multiples of $dF_{\text{ref}}$, then, the $dF$ value is set as $dF_{\text{ref}}: dF = dF_{dc}/M = M dF_{\text{ref}}/M = dF_{\text{ref}}$.

In general, the $dF$ value obtained according to the foregoing process is finer than the frequency interval required for dispersion correction ($dF_{dc}$). The necessary frequency values for dispersion correction ($F_{dc}$) can be obtained by extracting every $M$th (25th) data point from the portion of the $F$ matrix with the aforementioned $dF$ interval. If the $dF_{dc}$ value is less than $dF_{\text{ref}}$ (which is pre-set by the solver), the solver sets the $dF$ value as $dF_{dc}$.

In case the $dF$ value needs to be refined (if any), it is advantageous to employ a finer $dF$ value in routine $X_b$ than $X_a$. There are two reasons for the advantage. First, there are ACPs in routine $X_b$. Accordingly, a finer $dF$ in routine $X_b$ prevents from loosing track of the original curve in the tracking process. Second, a refined $dF$ extends the routine $X_b$ towards a lower $F$ regime, which decreases the execution time of computationally demanding routine $Y$ (if an overly high computational burden occurs in routine $Y$ (if any), it may be solved by employing a finer $dF$ value in routine $X_b$).

5.6. Group velocity

A monochromatic wave cannot transmit information because it cannot create a distinctive (discernible) wave shape in time and space. However, a group of waves composed of varying frequencies form a distinctive wave shape called wave group, wave envelope, or wave packet.
The wave packet can convey information (e.g., energy); accordingly, the wave packet velocity, called group velocity \( c_g \), has received certain interest. In the \( C–F \) domain, the normalised group velocity \( C_g = c_g/c_o \) is given by

\[
C_g = C \left(1 - \frac{F}{C \, dC/dF}\right)^{-1}
\]

The \( C_g \) array was prepared from the arrays of \( F \) and \( C \) and the \( C_g \) vs. \( F \) plot is presented in Sections 6.4 and 6.5 of this Supplemental Material.

5.7. \( C_{dc} \) and \( F_{dc} \)

To determine \( C_{dc} \), the \( F_{dc} \) value is varied independently. Then, the approximate solution, \( C_a \), at a given \( F_{dc} \) value is obtained by first linearly interpolating the existing \((F, C)\) datasets. A search for the lower and higher bound values \((C_{lb} \text{ and } C_{hb}, \text{ respectively})\) at a given \( F_{dc} \) value proceeds until the \( G(C_{lb})G(C_{hb}) \) value becomes negative. The two bound values are used to finally determine \( C_{dc} \) at the current \( F_{dc} \) via the bisection method.

As presented above, this study determines the \((F_{dc}, C_{dc})\) dataset after first obtaining the \((F, C)\) dataset. There are two reasons for deriving the \((F_{dc}, C_{dc})\) dataset in two steps. First, in tracking the dispersion curves, it is straightforward to set up the values of user variables, such as \( dC \) and \( dF \), without regard to the \( F_{dc} \) and \( C_{dc} \) values. Second, once the \((F, C)\) dataset is available, the second step that obtains the \((F_{dc}, C_{dc})\) dataset from the former dataset imposes only a mild computational burden.

6. Solutions

6.1. \( n \)-series dispersion curves

This study aims at obtaining solutions up to order 20. In revisiting the dispersion curves for \( n = 20 \) in Figure 3, a critical \( F \) \((F_{cr})\) value is found above which there are no ACPs observed. In this research, the \( F \) coordinate of the last ACP of the curve for \( n = 20 \) is called critical \( F \) \((F_{cr})\). It is preferable to set up the \( dF \) and \( dC \) values differently at the left and right sides of \( F_{cr} \) because of the difference in the shapes of the dispersion curves on the two sides. In this regard, the \( F_{ref} \) value is set as an \( F \) with a slightly higher value than \( F_{cr} \), making the majority of the \( F \) values on the right side of \( F_{cr} \) to belong to routine \( X_a \) and those on the left side to routine \( X_b \).
Then, the $dF$ and $dC$ values are set differently in routines $X_a$ and $X_b$, allowing the successful tracking of the dispersion curves with different curve shapes across $F_{cr}$. This study employs more refined values of $dF$ and $dC$ in routine $X_b$ than in $X_a$ mainly because the ACPs belong to routine $X_b$ (left side of $F_{cr}$).

As shown in Figure 3, the $F_{cr}$ value for the $n$-series curves is dependent on Poisson’s ratio. Accordingly, a Poisson’s ratio-dependent $F_{ref}$ value is programmed in the proposed solver to set the $F_{ref}$ at an $F$ value slightly higher than the $F_{cr}$ at a given Poisson’s ratio. Although the Poisson’s ratio-dependent $F_{ref}$ scheme is employed in this manner, the order of the $n$-series curves is independent of the selected $F_{ref}$ value; the order is independent of where it is counted.

6.2. $m$-series dispersion curves

The $m$-series dispersion curves are presented in Figure S8. No distinct point, such as the $F_{cr}$ in the $n$-series curves (Figure 3), is observed. By this reason, the $F_{ref}$ value for order determination is set as an arbitrarily selected intermediate $F$ value (11.0) regardless of Poisson’s ratio. Although no distinct difference is found in curve shape across the $F_{ref}$ value, more refined $dF$ and $dC$ values are employed in routine $X_b$ (left side of $F_{ref}$) than in $X_a$ (right side of $F_{ref}$) because it is advantageous to widen the $F$ range of routine $X_b$ towards a lower $F$ value; it reduces the $F$ range of routine $Y$ which computationally demands more than routine $X_b$ does.
Solution, $C_c/c_o$ (b) $\nu = 0.15$ 

Solution, $C_c/c_o$ (c) $\nu = 0.30$ 

$m = 20$
6.3. Second group of \(m\)-series dispersion curves

If the order count reaches 20 at an \(F_{\text{ref}}\) value less than 11 or exceeds 20 at an \(F_{\text{ref}}\) value of 11, a different group of \(m\)-series dispersion curves can be observed; examples are illustrated in Figure S9. Knowledge on the presence of the second group \(m\)-series curves helps understanding the nature of the newly found dispersion curve when the original \(n\)-series dispersion curve is lost in the tracking process (Figure S9a).

If the \(F_{\text{ref}}\) value for the \(m\)-series is set as 7 (by modifying line 92 of the proposed solver) instead of the pre-set value of 11, then, the \(m\)-series dispersion curves (up to \(m = 20\)) for two different Poisson’s ratios (0.05 and 0.2) are obtained (Figure S9). As shown in Figure S9a, if the order of the curves is counted at the \(F_{\text{ref}}\) value of 7, then, the 17 first-group curves are counted first, followed by the three second-group curves. If the order is counted at the \(F_{\text{ref}}\) value of 11, then, more than 17 first-group curves are counted before the second-group curves are encountered. These findings explain why the second-group curves are only observed when the \(F_{\text{ref}}\) value is reduced from the pre-set value of 11 (within the order limit of 20). The question on how many groups different from the first group there are is beyond the scope of this study.
In Figure S9, if the Poisson’s ratio decreases from 0.2 to 0.05, then, the second group starts from a lower $C$ value. Consequently, when the $F_{\text{ref}}$ value is 7, more second-group curves are counted with the decrease in the number of first-group curves. In Figure S9b for $\nu = 0.05$ and under the condition of the present setting of the user variable values in the proposed solver, the second-group curves typically deviate from the first-group curves as the $F$ value increases.

**Figure S9.** $m$-series dispersion curves obtained up to $m = 20$ after setting $F_{\text{ref}}$ value as 7 instead of pre-set value of 11 for cases where Poisson’s ratios are (a) 0.20 and (b) 0.05.
To track the second group of the $m$-series dispersion curves up to the $F$ value of 30, which is the target value of the maximum $F$ in this study, more refined $dF$ and $dC$ values are required than the settings in the proposed solver; this is beyond the scope of the study. The second group curves may be worth tracking up to a higher $F$ value after their physical significance is uncovered; as mentioned, the physical meaning of even the first-group $m$-series curves are not yet revealed.

The reason for raising the issue regarding the second group $m$-series curves is that in tracking the $n$-series dispersion curves, they are frequently observed when the original $n$-series curve is lost track of and a second group $m$-series dispersion curve is tracked instead particularly around the ACPs; examples are shown in Figure S10. Figure S10a presents the $n$-series dispersion curves obtained using the proposed solver for the case where Poisson’s ratio is 0.211. Figure S10b shows the $n$-series dispersion curves obtained for the same Poisson’s ratio (0.211) after modifying the $dC$ value in routine $X_b$ to $1 \times 10^{-6}$ ($C_{\text{imax}} = 1 \times 10^4$) instead of the pre-set value of $5 \times 10^{-9}$ ($C_{\text{imax}} = 2 \times 10^6$) in lines 404 and 405 of the proposed solver. In Figure S10b, the curve with $n = 17$ deviates around the ACP from one of the second group $m$-series curves in routine $X_b$. Figures S9 and S10b illustrate the plurality of $m$-series curves. These provide insight on the nature of the found curve when the original curve is lost in the tracking process; the newly tracked curve does not belong to the $n$-series but $m$-series.
Figure S10. (a) $n$-series dispersion curves obtained using proposed solver for case where $\nu = 0.211$; (b) the curves when $n = 17$ curve is lost in the tracking process and a new curve is found.

6.4. $C_g$ vs. $F$ plot for $n$-series dispersion curves

The $C_g$ vs. $F$ plots for $n$-series solutions are presented in Figure S11.
Figure S11. \( C_g - F \) plot for \( n \)-series solutions from \( n = 1 \) to 20: (a) \( \nu = 0.05 \), (b) \( \nu = 0.15 \), (c) \( \nu = 0.30 \), and (d) \( \nu = 0.45 \). The spike marked as ‘sp’ in (a) is explained in the next section.

6.5. \( C_g \) vs. \( F \) plot for \( m \)-series dispersion curves

The \( C_g \) vs. \( F \) plots for \( m \)-series solutions are presented in Figure S12.
Figure S12. $C_g$–$F$ plot for $m$-series solutions from $m = 1$ to 20: (a) $\nu = 0.15$ and (b) $\nu = 0.30$.

6.6. $C_{dc}$–$F_{dc}$ plot

The $F_{dc}$ and $C_{dc}$ values were calculated for the case where $\nu = 0.3$ ($dF_{dc} = 0.02$) using the proposed solver. The results are presented in Figure S13 for $n$-series and $m$-series solutions up to order 20.
Figure S13. C_{dc} vs F_{dc} plots for the (a) n-series and (b) m-series solutions for the case where \( \nu = 0.3 \) and \( dF_{dc} = 0.02 \). Solid lines simply connect two data points (open circles) for visualisation of given dispersion curve.

The proposed program writes the \((F_{dc}, C_{dc})\) dataset of each order to an Excel® file (Cdc-Fdc.xlsx). All of the data presented in this study and the Supplemental Material may be similarly written to external files because the datasets of \((C, F)\) and \((C_{g}, F)\) are also calculated in the solver.

7. Supplemental discussion

The Newton–Raphson (NR) method (also called as Newton's method) is based on the idea that a continuous (and differentiable) function can be approximated by a line tangent to the function. It quickly finds the root of a real-valued function, \( f(x) = 0 \). In this regards, this study actually tried the NR method first in the stage where either \( C \) (routine X) or \( F \) (routine Y) solutions are determined from \( C_{a} \) or \( F_{a} \), respectively. When the NR algorithm was employed, as \( F \) and \( C \) varied independently in routines X and Y, respectively, the dispersion curve frequently deviated from the original curve to follow the curves of other orders or different types \((n \text{ and } m)\), especially around the ACPs. For the scheme shown in Figure 1 in the \( C–F \) domain, the NR algorithm may be applicable up to a limited order in limited ranges of \( F \) (e.g.,
0 \leq F \leq 2\), \(C\) (e.g., \(0 \leq C \leq 2\))\), and Poisson’s ratios; in the foregoing case, the dispersion curves are sufficiently separated and ACPs are scarce. For the purpose of establishing a robust solver up to higher orders of \(n\)-series and \(m\)-series solutions in wider ranges of Poisson’s ratios, higher \(F\) values, and higher \(C\) values, this study employs the bisection method. It is a much more robust method than the NR algorithm but it increases computational burden.

As mentioned, the dispersion curve for \(n = 1\) is usually considered for the dispersion correction of the split Hopkinson bar signals; this is one of the most extensive application areas of the PC equation solution. When the maximum order is one in \(n\)-series solutions, the current program only implements routine \(X\) (\(X_a\) and \(X_b\)), i.e., routine \(Y\) is excluded.

As mentioned in the main text (Overall discussion section), the validity of the pre-set user variable values in tracking the dispersion curves was checked up to the order of 20 (for both \(n\)- and \(m\)-series) for a wide range of Poisson’s ratios (\(0.02 \leq \nu \leq 0.48\)) at intervals of 0.001. If the order is less than or equal to 20 under unchecked Poisson’s ratio conditions (if any), a couple of curves or less may require modifications, whereas the rest of the curves can be successfully tracked. To reduce the computational burden in case modifications are required, the proposed solver allows for the tracking of any curve of interest of a particular order (e.g., \(n = 17\)) without tracking the curves of other orders. For this purpose, the same order number can be inputted to \(order\_max\) and \(order\_min\) variables in the input parameter section of the solver. A curve of interest of a particular order can be tracked again after modifying the user variable values, for example, \(dC\) and \(dF\), specific to the curve. Further discussion including the computation time of the proposed solver is available

If the target order of the dispersion curve to be solved is not high (e.g., \(n \leq 6\)) and/or if the range of Poisson’s ratio is not wide (e.g., \(0.25 \leq \nu \leq 0.35\)), coarser \(dF\) and \(dC\) values than the settings in the proposed solver can be employed; it decreases the computational burden significantly. However, the \(dF\) and \(dC\) values in the proposed solver were set finely to some extent with a view to tracking the dispersion curves up to the order of 20 in a wide range of Poisson’s ratios (\(0.02 \leq \nu \leq 0.48\)) to cope with scientific interest. However, the computation performance of these days allows the tracking process to be completed within a reasonable time. For instance, it took 68 s using a personal computer (4.1 GHz CPU) to track the dispersion curves up to the order \((n)\) of 20 \((F \leq 30\) and \(C \leq 500; F_{dc\_max} = 30, F_{dc} = 0.05\)) at the Poisson’s
ratio of 0.3. It is believed that employing the bisection method is worthwhile for solving the PC equation.

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