Stationary Scattering Theory for One-body Stark Operators, II

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Abstract. We study and develop the stationary scattering theory for a class of one-body Stark Hamiltonians with short-range potentials, including the Coulomb potential, continuing our study in Adachi et al. (JDE 268: 5179–5206, 2020; Stationary scattering theory for 1-body Stark operators). The classical scattering orbits are parabolas parametrized by asymptotic orthogonal momenta, and the kernel of the (quantum) scattering matrix at a fixed energy is defined in these momenta. We show that the scattering matrix is a classical type pseudodifferential operator and compute the leading order singularities at the diagonal of its kernel. Our approach can be viewed as an adaption of the method of Isozaki-Kitada (Tokyo Univ. 35: 81–107, 1985) used for studying the scattering matrix for one-body Schrödinger operators without an external potential. It is more flexible and more informative than the more standard method used previously by Kvitsinsky-Kostrykin (Teoret. Mat. Fiz. 75(3): 416-430, 1988) for computing the leading order singularities of the kernel of the scattering matrix in the case of a constant external field (the Stark case). Our approach relies on Sommerfeld’s uniqueness result in Besov spaces, microlocal analysis as well as on classical phase space constructions.
1. Introduction and Results

In this paper, we continue a study of the stationary scattering theory for a class of one-body Stark Hamiltonians with short-range potentials initiated in [4] (the conditions on the potentials will be stronger though). While the time-dependent scattering theory is well understood [2,8,23–25], the stationary scattering theory is in our opinion on a less complete form, even for short-range potentials. There are other papers in the literature on one-body Stark stationary scattering theory, see, for example, [16,17]; however, we found it useful and appealing for its intrinsic beauty to give a more systematic account done entirely within the framework of stationary theory. While the time-dependent framework was not discussed at all in [4], we make in the present continuation of [4] a link to the time-dependent framework, showing that the studied quantities are the same (in disguised form of course). Having this settled there is one remaining issue, which is a deeper study of the scattering matrix.

To obtain detailed properties of the scattering matrix is the main goal of the present paper. This goal is roughly the same as the one of [16] with which we have overlapping results. However, our approach is very different from Kvitsinsky-Kostrykin’s. It can be considered as an adaption of the method of Isozaki-Kitada [13] used for studying the scattering matrix for one-body Schrödinger operators without an external potential. It is more flexible and more informative than ‘the standard method’. While the paper by Kvitsinsky-Kostrykin can be seen as an application of the standard method in the case of a constant external potential, our scheme is closer to the one invented by Isozaki-Kitada. In particular, it relies on microlocal analysis and on classical phase space constructions. By using such notions, the accomplishment of [13] is a ‘trivialization’ of the detailed study of the scattering matrix. More precisely, Isozaki-Kitada realized the scattering matrix in the form of a pseudodifferential operator PsDO (this is for short-range potentials, however recently extended to a class of long-range potentials [19]) and extracted the singularities of its kernel from their representation.

In the present paper, we do the same in the Stark case with short-range potentials. (Note that the Coulomb potential is a ‘short-range’ potential in the Stark case.) In particular, and more precisely, we show how to isolate the local
singularities to be present only in a term expressed as an explicit oscillatory integral, which in turn can be realized as the kernel of a classical type PsDO. For example, this allows us to compute the leading order local singularities for the Coulomb potential, reproducing a result in [16].

Let us outline the relationship of our purely stationary setup to time-dependent scattering theory. We consider a $d$-dimensional particle (with $d \geq 2$) subject to a constant nonzero field pointing in the $x_1$-direction. For simplicity, we assume that its strength and the particle mass and charge are all taken to 1. We split the coordinates in $\mathbb{R}^d$ into $x_1$ and the coordinates for orthogonal directions, decomposing the configuration space variable as $(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}; \quad x = x_1, \ y = (x_2, \ldots, x_d)$.

Then, the classical free Stark Hamiltonian is given by

$$h_0(x, y, \eta, \zeta) = \frac{1}{2} (\eta^2 + \zeta^2) - x, \quad (x, y, \eta, \zeta) \in T^* \mathbb{R}^d \cong \mathbb{R}^{2d},$$

and hence, the associated Hamilton equations are

$$\dot{x} = \eta, \quad \dot{y} = \zeta, \quad \dot{\eta} = 1, \quad \dot{\zeta} = 0.$$  

The solution with the initial data $(x_0, y_0, \eta_0, \zeta_0) \in T^* \mathbb{R}^d$, defining the free classical flow (say denoted by $\Theta(t)$), is given by

$$x = \frac{1}{2} t^2 + t\eta_0 + x_0, \quad y = t\zeta_0 + y_0, \quad \eta = t + \eta_0, \quad \zeta = \zeta_0. \quad (1.1)$$

In particular, the classical orbits are parabolas of the form $x = \frac{1}{2} (\zeta \pm) y^2 + \mathcal{O}(t)$. The same asymptotics holds for $h = h_0 + q$, where $q$ is short-range, for example given as $q = q_1$ in the following condition which will be imposed throughout this paper.

**Condition 1.1.** The potential $q$ splits into real-valued functions as $q = q_1 + q_2$, where $q_2$ is compactly supported, $q_2(-\Delta + 1)^{-1}$ is compact, $q_1$ smooth and for some $\delta \in (0, 1/2]$

$$\partial^\beta q_1 = \mathcal{O}(r^{-(1/2+\delta+|\beta|)}); \quad r = (x^2 + y^2)^{1/2}. \quad (1.2)$$

We introduce for any scattering orbit (with potential $q = q_1$) the asymptotic orthogonal momenta $\zeta^\pm = \lim_{t \to \pm \infty} \zeta(t)$. The orbit is incoming and outgoing along parabolas given as sections of the paraboloids $x = \frac{1}{2(\zeta^+)^2} y^2$, respectively. It is part of the classical scattering problem to determine the transition from an incoming asymptotic parabola to an outgoing asymptotic parabola, or stated somewhat strongly, the transition from an incoming momentum $\zeta^-$ to an outgoing momentum $\zeta^+$. As we will outline below (with further details given in Sect. 4), this information is in quantum mechanics encoded in the subject of study in this paper, the scattering matrix.

Under Condition 1.1, the free and the perturbed Stark operators on $\mathcal{H} := L^2(\mathbb{R}^d)$ are given by $H_0 = p^2/2 - x$ and $H = p^2/2 - x + q$ (with $p = -i\nabla$), respectively. We shall throughout this paper impose Condition 1.1 as well as the unique continuation principle [3, Condition 2.4]. It is a well-established fact that asymptotic completeness holds, i.e., that the wave operators

$$W^\pm = s\lim_{t \to \pm \infty} e^{itH} e^{-itH_0}$$
exist and map onto $L^2(\mathbb{R}^d)$. The asymptotic orthogonal momenta read in this case

$$p_y^- = \lim_{t \to \pm \infty} e^{itH} p_y e^{-itH} = \lim_{t \to \pm \infty} e^{itH} y / e^{-itH},$$

where the limits are taken in the strong resolvent sense (cf. [1]).

In terms of the stationary wave operators $\mathcal{F}^-$ and $\mathcal{F}^+$, we can simultaneously diagonalize either $H$ and $p_y^-$ or $H$ and $p_y^+$, respectively. These wave operators have several representations, for example given in terms of asymptotic properties of the boundary values $\lim_{\epsilon \to 0} (H - \lambda \mp i\epsilon)^{-1}$ (taken in an appropriate space). However, they are also connected to the time-depending wave operators by the formulas $\mathcal{F}_0(W^\pm)^* = \mathcal{F}^\pm$, where $\mathcal{F}_0$ is given in terms of asymptotic properties of the boundary values $\lim_{\epsilon \to 0} (H_0 - \lambda \mp i\epsilon)^{-1}$, or alternatively and more useful, given by a Fourier–Airy transformation which in turn is defined by an explicit oscillatory integral. By joint diagonalization, we mean more precisely the assertions

$$H = (\mathcal{F}^\pm)^* M_\lambda \mathcal{F}^\pm \quad \text{and} \quad p_y^\pm = (\mathcal{F}^\pm)^* \left( \int_\mathbb{R}^{\oplus} M_\lambda \, d\lambda \right) \mathcal{F}^\pm,$$

where $M_\lambda$ refers to multiplication in $L^2(\mathbb{R}, d\lambda; \Sigma)$ or in $\Sigma := L^2(\mathbb{R}^{d-1}, d\zeta)$, respectively.

The scattering operator $S = (W^+)^* W^-$ is represented as

$$\mathcal{F}_0 S \mathcal{F}_0^{-1} = \int_\mathbb{R}^{\oplus} S(\lambda) \, d\lambda,$$

where $S(\lambda)$ is a unitary operator on $\Sigma$ called the scattering matrix at energy $\lambda$. Its Schwartz kernel $S(\lambda)(\zeta, \zeta')$ is defined in terms of variables $\zeta$ and $\zeta'$ which by the above formulas may be interpreted as outgoing and incoming asymptotic orthogonal momenta, respectively. This explains the physical relevance of detailed information on $S(\lambda)$ and its kernel.

In the main part of the paper, Sects. 5–8, we study somewhat refined representations of $S(\lambda)$ based on Sommerfeld’s uniqueness result in Besov spaces proven in [3]. We show mapping properties and we show that the principal symbol of $T(\lambda) := S(\lambda) - I$ viewed of as a PsDO is given by

$$t_{\text{psym}}(\zeta, \zeta', y) = t_{\text{psym}}(y) = -2i \int_0^\infty \frac{q_1(x, y)}{\sqrt{2x}} \, dx.$$

In general, a linear operator $T : C_c^\infty(\mathbb{R}^{d-1}) \to C_c^\infty(\mathbb{R}^{d-1})$ is called a pseudodifferential operator on $\Sigma$ of order $k \in \mathbb{R}$ if there exists a smooth function $t = t(\zeta, \zeta', y)$ such that the kernel of $T$ is given by

$$T(\zeta, \zeta') = (2\pi)^{1-d} \int e^{i(\zeta - \zeta') \cdot y} t(\zeta, \zeta', y) \, dy;$$

$$\forall \alpha, \alpha', \beta \in \mathbb{N}_0^{d-1} : |\partial_\zeta^\alpha \partial_{\zeta'}^{\alpha'} \partial_y^\beta t| \leq C_{\alpha, \alpha', \beta} \langle y \rangle^{k-|\beta|}$$

for all $y$ and locally uniformly in $\zeta, \zeta'$. Here, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and we call $t$ a symbol of $T$. If $k$ can be taken arbitrarily, $T$ is a smoothing operator. We show that the above operator $T(\lambda)$ has order
−δ, like the quantization $T_{\text{psym}}$ of the symbol $t_{\text{psym}}$ has (see (6.3)), while $T(\lambda) - T_{\text{psym}}$ has order $-2\delta$.

Applied to $q = \kappa r^{-1}$ for $d \geq 3$ the singularity structure of the kernel of the scattering matrix at the diagonal is

$$S(\lambda)(\zeta, \zeta') - \delta(\zeta, \zeta') = \kappa C_d |\zeta - \zeta'|^{3/2-d} + \mathcal{O}(|\zeta - \zeta'|^{2-d}),$$

locally uniformly in $\zeta, \zeta'$ and $\lambda$. This result conforms with [16] for $d = 3$ in which case $C_d = -i(2\pi)^{-1/2}$, and the order of the error term is optimal (by an assertion of [16]). Although we only compute the top order asymptotics of the singularity, our method would allow a further expansion (as in [16]), in fact in principle a complete expansion.

In the bulk of the paper, we impose Condition 1.1 with the extra condition $q_2 = 0$. It is a minor technical issue to treat the general case by using the arguments of the paper and the second resolvent equation (not to be elaborated on). The unique continuation principle [3, Condition 2.4] is fulfilled for $q_2 = 0$ as well as for the example $q = \kappa r^{-1}$, $d \geq 3$.

It is convenient in this paper to reserve the notation $\tilde{f}$ to mean any smooth convex function $\tilde{f}$ on $\mathbb{R}$ such that $\tilde{f}(t) = 1$ for $t \leq 1/2$ and $\tilde{f}(t) = t$ for $t \geq 2$. This quantity is fixed throughout the paper and will be used without reference.

We use the standard notation $\langle z \rangle = (1 + |z|^2)^{1/2}$ for $z$ in a normed space, while for any $m \in \mathbb{N}$ we denote $\langle z \rangle_m = (m^2 + |z|^2)^{1/2}$ and $\hat{z}_m = z/\langle z \rangle_m$. We use the (standard) notation $F(x \in M) = 1_M$ for the characteristic function of a set $M$. For any $\kappa \in \mathbb{R}$, the notation $\chi(\cdot < \kappa)$ stands for any smooth real function $\chi$ on $\mathbb{R}$ with $\chi' \in C_0^\infty(\mathbb{R})$ and $\text{supp} \chi \subseteq (-\infty, \kappa)$, and for $\kappa > 0$ we require $\chi(t) = 1$ for $t \leq 3\kappa/4$, while for $\kappa < 0$ we require $\chi(t) = 1$ for $t \geq 4\kappa/3$. Let $\chi(\cdot > \kappa) = \chi(-\cdot < -\kappa)$, $\chi^\perp(\cdot < \kappa) = 1 - \chi(\cdot < \kappa)$ and $\chi^\perp(\cdot > \kappa) = 1 - \chi(\cdot > \kappa)$.

Let $L_s^2 = L_s^2(\mathbb{R}^d) = \langle (x, y) \rangle^{-s} \mathcal{H}$ for $s \in \mathbb{R}$ and $L_\infty^2 = \cap_s L_s^2$. Let $H_\infty^2 = H_\infty^2(\mathbb{R}^d) = (p^2 + 1)^{-1} L_\infty^2$. For $\varphi \in \mathcal{H}$ and an operator $T$ on $\mathcal{H}$, the notation $\langle T \varphi \rangle$ means $\langle \varphi, T \varphi \rangle$. The notation $\mathcal{L}(\mathcal{H})$ refers to the set of bounded operators on $\mathcal{H}$. More generally for given normed spaces $X$ and $Y$, the set of bounded operators $T : X \to Y$ is denoted by $\mathcal{L}(X, Y)$.

2. Fourier-Airy Transformation and the Stationary Phase Method

One can explicitly specify a diagonalizing transform $\int_\mathbb{R} \mathcal{F}_0(\lambda) d\lambda$ such that $\delta(H_0 - \lambda) = \mathcal{F}_0(\lambda)^* \mathcal{F}_0(\lambda)$, cf. [8,24]. Writing the Airy function by its Fourier transform, the expression for the kernel of $\mathcal{F}_0(\lambda)^*$ (it is not unique) is an oscillatory integral. One can then look at stationary points when applying the operator to any $\xi \in C_0^\infty(\mathbb{R}^{d-1}) \subseteq \Sigma$ and conclude by stationary phase method considerations, cf. [9] (or, for example, [11,21]), what the asymptotics should be. Below we state the results without giving details of proof, deferring a more
precise treatment to Subsect. 2.1. For simplicity we take below $\lambda = 0$ and consider
\[
(F_0(0)\star \xi)(x, y) = c \int d\zeta \xi(\zeta) \int e^{i\theta} \, d\eta;
\]
(2.1)
\[c = (2\pi)^{-d+1/2}, \quad \theta = y \cdot \zeta - \eta^3/6 + (x - \zeta^2/2) \eta.
\]
For the general case, we would replace $x$ by $x + \lambda$ in this expression. We look for critical points
\[0 = \partial_\eta \theta = -\eta^2/2 - \zeta^2/2 + x \quad \text{(energy relation)},
\]
\[0 = \partial_\zeta \theta = y - \eta \zeta \quad \text{(velocity relation)}.
\]
Note that considering the momentum $\eta$ as an effective time indeed the last equation written as $\zeta = y/\eta$ is a ‘velocity relation’. If we substitute $\zeta = y/\eta$ in the argument of the function $\xi$, interchange order of integration and do the $\zeta$-integration we end up with
\[c \int e^{i\theta_1}(y/\eta)(i\eta/(2\pi))^{1-d/2} \, d\eta;
\]
\[\theta_1(\eta) = y^2/(2\eta) - \eta^3/6 + x\eta.
\]
The critical points of $\theta_1$ fulfill
\[0 = \partial_\eta \theta_1 = -\frac{1}{2} y^2/\eta^2 - \frac{1}{2} \eta^2 + x,
\]
which in turn fulfill
\[\eta^2 = x \pm (x^2 - y^2)^{1/2}.
\]
We choose ‘+’ (the case of ‘-’ does not contribute to the asymptotics) leading to the two critical points
\[\eta = \pm \sqrt{x + (x^2 - y^2)^{1/2}} \approx \pm (2x)^{1/2}.
\]
The second-order derivative is $-\eta(1 + \mathcal{O}(y^2/x^2))$, whence the asymptotics of the integral is given as the sum of the following two terms (to be justified in Subsect. 2.1 and Appendix A):
\[ce^{\pm i\theta_{ex}}\xi(y/\eta)(i\eta/(2\pi))^{-d/2}
\approx \frac{e^{\mp i\pi d/4}}{\sqrt{2\pi}} (2x)^{-d/4} e^{\pm i\theta_{ex}} \xi(\pm \omega);
\]
\[\theta_{ex} = \sqrt{x + (x^2 - y^2)^{1/2}} \left(\frac{1}{2} y^2/\eta^2 - \eta^2/6 + x\right),
\]
\[\eta^2 = x + (x^2 - y^2)^{1/2}, \quad \omega = (2x)^{-1/2} y.
\]
We may write the function $\theta_{ex}$ as
\[\theta_{ex} = f_{ex}^3/3 := \frac{4}{3} \sqrt{x + (x^2 - y^2)^{1/2}} \left(x - \frac{1}{2} (x^2 - y^2)^{1/2}\right).
\]
It is a solution to the eikonal equation on the domain of its definition, as demonstrated in [4], that is $\frac{1}{2} |\nabla \theta_{ex}|^2 - x = 0.$
2.1. The Stationary Phase Method

As in (2.1), we consider in this subsection double integrals of the form:

\[ \phi_{\lambda, \tilde{a}}[\xi](x, y) = c \int d\zeta \frac{\xi(\zeta)}{\xi(\zeta)} e^{i\theta_{\lambda} \tilde{a} d\eta}; \]

\[ c = (2\pi)^{-\frac{d+1}{2}}, \quad \theta_{\lambda} = y \cdot \zeta - \eta^3/6 + (x + \lambda - \zeta^2/2)\eta. \]  

(2.2a)

We call \( \tilde{a} = \tilde{a}(x, y; \eta, \zeta) \) a symbol. By definition \( \tilde{a} \) is smooth in \((\eta, \zeta)\) and for some \( n \in \mathbb{N}_0 \), possibly depending on \( \tilde{a} \),

\[ |\partial^{\alpha}_{\eta, \zeta} \tilde{a}| \leq C_\alpha(\zeta)\left(1 + \frac{n^2}{f(x)}\right)^n; \quad C_\alpha(\cdot) \text{ locally bounded.} \]  

(2.2b)

(We shall use more refined symbol classes in Sects. 7 and 8.) The function \( \xi \in C^\infty_c(\mathbb{R}^{d-1}) \), and \( \lambda \in \mathbb{R} \). Here, we derive the asymptotics of such integrals as \(|(x, y)| \to \infty\).

The critical points of \( \theta_{\lambda} \) are given by

\[ 0 = \partial_\eta \theta_{\lambda} = -\eta^2/2 - \zeta^2/2 + x + \lambda, \]

\[ 0 = \partial_\zeta \theta_{\lambda} = y - \eta \zeta. \]

These equations suggest two ways of integrating by parts based on the formulas

\[ e^{i\theta_{\lambda}} = \left(1 + (x + \lambda - \eta^2/2 - \zeta^2/2)^2\right)^{-1} \]

\[ \cdot (1 - i(x + \lambda - \eta^2/2 - \zeta^2/2)\partial_\eta)e^{i\theta_{\lambda}}, \]  

(2.3a)

\[ e^{i\theta_{\lambda}} = \left(1 + (y - \eta \zeta)^2\right)^{-1} (1 - i(y - \eta \zeta) \cdot \nabla_\zeta)e^{i\theta_{\lambda}}. \]  

(2.3b)

First we note that for \( x \leq 2R \) for any \( R > 2 \) (large), the function \( \phi_{\lambda, \tilde{a}}[\xi] \) has arbitrary polynomial decay in \((x, y)\). This follows readily by repeated integration by parts using (2.3a) and (2.3b).

Pick for any sufficiently small \( \epsilon > 0 \) a function \( \chi_{\epsilon} \in C^\infty_c((-2\epsilon, 2\epsilon)) \) such that \( \chi_{\epsilon} = 1 \) on \((-\epsilon, \epsilon)\). Let \( \chi_{\epsilon}^\perp = 1 - \chi_{\epsilon} \), and let \( \chi_R = \chi(\cdot > R) \) for \( R > 2 \). Then, the functions

\[ c \int d\zeta \xi(\zeta) e^{i\theta_{\lambda}} \tilde{a} \chi_{\epsilon}^\perp(|\eta| - \sqrt{2x})\chi_R(x) d\eta, \]

\[ c \int d\zeta \xi(\zeta) e^{i\theta_{\lambda}} \tilde{a} \chi_{\epsilon}(\pm \eta - \sqrt{2x})\chi_{\epsilon}^\perp(|\zeta + y/\sqrt{2x}|)\chi_R(x) d\eta, \]

also have arbitrary polynomial decay in \((x, y)\). This is seen by the same argument as for \( x \leq 2R \).

We conclude that the leading asymptotics of \( \phi_{\lambda, \tilde{a}}[\xi] \) at infinity is the same as that of \( \phi_{\lambda, \tilde{a}}^+[\xi] + \phi_{\lambda, \tilde{a}}^-[\xi] \), where

\[ \phi_{\lambda, \tilde{a}}^\pm[\xi] = c \int d\zeta \xi(\zeta) e^{i\theta_{\lambda}} \tilde{a} \chi_{\epsilon}(\pm \eta - \sqrt{2x})\chi_{\epsilon}(\pm \zeta + y/\sqrt{2x})\chi_R(x) d\eta. \]  

(2.4)
If \( C > 1 \) and \( \text{supp} \xi \subset \{|\xi| < C/2\} \), then \( \phi_{\lambda, \tilde{a}}^\pm[\xi] \) vanish unless \(|y| < C\sqrt{2x}\).

Moreover, in the support of the product of the \( \chi \)-factors in the integrands of (2.4), the stationary point is uniquely given for \( R = R(\epsilon) \) big enough as

\[
\eta^\pm = \pm \sqrt{x + \lambda + ((x + \lambda)^2 - y^2)^{1/2}} \quad \text{and} \quad \zeta^\pm = \frac{y}{\eta^\pm},
\]

respectively.

Note that uniformly in \(|y| < C\sqrt{2x}\),

\[
\eta^\pm = O\left(\frac{1}{\sqrt{2x}}\right) \quad \text{and} \quad \zeta^\pm = \pm \frac{y}{\sqrt{2x}} \left(1 + O\left(\frac{1}{x}\right)\right).
\]

Clearly we are left with the asymptotics in the set \( \{|y| < C\sqrt{2x}\} \) only.

We consider \( \sqrt{2x} = h^{-1} \) as a large parameter, write \( \theta_\lambda = h^{-1} \tilde{\theta}_\lambda \) and then use the stationary phase method. We can in this way obtain the following leading order asymptotics as \( h \to 0 \).

\[
\phi_{\lambda, \tilde{a}}^\pm[\xi](x, y) = \frac{e^{i\pi d/4}}{\sqrt{2\pi}} h^{d/2} e^{i\theta_\lambda(x + \lambda, y)} \xi(\pm hy) \tilde{a}(x, y; \pm h^{-1}, \pm hy) + O\left(h^{(d+2)/2}\right), \quad \text{as} \quad h = (2x)^{-1/2}
\]

\[
\to 0 \text{ uniformly in } y \in \{|y| < C h^{-1}\}.
\]

The present stationary phase problem has an additional parameter dependence (that is a dependence on the variable \( y \)). Such problem is rather standard, and we show the result (2.6) in Appendix A by mimicking the proof of [11, Theorem 4.3] in the presence of the additional parameter. For very similar results, we refer to [9, Theorems 7.7.5-6].

In particular, in terms of Besov spaces \( B = B(f), B^* = B^*(f) \) and \( B_0^* = B_0^*(f) \) to be introduced in the beginning of Subsection 3.1, we conclude from (2.6) that

\[
\phi_{\lambda, \tilde{a}}^\pm[\xi], \phi_{\lambda, \tilde{a}}^\pm[\xi], \phi_{\lambda, \tilde{a}}[\xi] \in B^*,
\]

\[
\phi_{\lambda, \tilde{a}}[\xi](x, y) - \sum_{\pm} \frac{e^{i\pi d/4}}{\sqrt{2\pi}} h^{d/2} e^{i\theta_\lambda(x + \lambda, y)} \xi(\pm hy) \tilde{a}(x, y; \pm h^{-1}, \pm hy) F(x > 1) \in B_0^*.
\]

3. Stationary Scattering Theory

We recall various results of [4], proven under weaker conditions than Condition 1.1.

3.1. Parabolic Coordinates and the Phase Function \( \theta^\lambda = f^3/3 + \lambda f \)

It is known since a long time ago that parabolic coordinates are useful for studying problems for Stark Hamiltonians, see [22]. Here, we recall the version of these coordinates used in [4], which in turn is a slight modification of the one of [3]. We introduce \( f \in C^\infty(\mathbb{R}^d) \) by the recipe

\[
f(x, y) = \sqrt{\tilde{f}(r + x)}; \quad r = (x^2 + y^2)^{1/2}.
\]
Note that this function obeys
\[ \nabla^2 f^2 \geq 0. \]  
(3.1b)

Let \( \mathcal{B} = \mathcal{B}(f) \), \( \mathcal{B}^* = \mathcal{B}^*(f) \) and \( \mathcal{B}_0^* = \mathcal{B}_0^*(f) \) be the corresponding Besov spaces with respect to the multiplication operator by the function \( f \), i.e.,
\[ \mathcal{B} = \left\{ \psi \in L_{\text{loc}}^2(\mathbb{R}^d) \mid \sum_{n \in \mathbb{N}_0} 2^{n/2} \| F_n \psi \|_{L^2} < \infty \right\}, \]
\[ \mathcal{B}^* = \left\{ \psi \in L_{\text{loc}}^2(\mathbb{R}^d) \mid \sup_{n \in \mathbb{N}_0} 2^{-n/2} \| F_n \psi \|_{L^2} < \infty \right\}, \]
\[ \mathcal{B}_0^* = \left\{ \psi \in \mathcal{B}^* \mid \lim_{n \to \infty} 2^{-n/2} \| F_n \psi \|_{L^2} = 0 \right\}, \]
where \( F_n = 1_{\{2^n \leq f < 2^{n+1}\}} \) is a characteristic function of the set specified by \( 2^n \leq f < 2^{n+1} \) for \( n \in \mathbb{N}_0 \). These are Banach spaces with respect to the norms
\[ \| \psi \|_{\mathcal{B}} = \sum_{n \in \mathbb{N}_0} 2^{n/2} \| F_n \psi \|_{L^2}, \quad \| \psi \|_{\mathcal{B}^*} = \| \psi \|_{\mathcal{B}_0^*} = \sup_{n \in \mathbb{N}_0} 2^{-n/2} \| F_n \psi \|_{L^2}, \]
respectively.

Of course a classical scattering orbit will have \( x > 1 \) eventually so that for large time \( f = (r + x)^{1/2} \). Since we will use the parabolic variable \( (r + x)^{1/2} \) in quantum mechanics it is convenient to introduce the above regularization \( f \), say also named a parabolic variable. We introduce other parabolic variables,
\[ g = y/f, \quad g_i = y_i/f; \quad i = 2, \ldots, d. \]
(3.1c)

We recall a few calculus formulas in parabolic coordinates, cf. [4, Lemma 3.2]. It is below tacitly assumed that \( r + x > 2 \).
\[ f^2 + g^2 = 2r, \quad f^2 - g^2 = 2x, \quad f |g| = |y|, \]
\[ 2r |\nabla f|^2 = 1, \quad \nabla f \cdot \nabla g_i = 0; \quad i = 2, \ldots, d. \]
(3.2)

Introducing \( \theta^\lambda = f^3/3 + \lambda f \) for any \( \lambda \in \mathbb{R} \) (note the superscript convention to distinguish this function and the phase \( \theta_\lambda \) of (2.2a)), we compute (taking here \( \lambda = 0 \) and denoting the \( ij \)'th entry of the matrix \( \nabla^2 \theta^0 \) by \( \nabla^2 \theta^0_{ij} \))
\[ \nabla \theta^0 = \frac{1}{2r} (f^3, fy), \]
\[ \nabla^2 \theta^0_{11} = -\frac{1}{2} \frac{x f^3}{r^3} + \frac{3}{4} \frac{f^3}{r^2}, \]
\[ \nabla^2 \theta^0_{1\alpha} = -\frac{1}{2} \frac{y^\alpha f^3}{r^3} + \frac{3}{4} \frac{y^\alpha f}{r^2}, \]
\[ \nabla^2 \theta^0_{\alpha\beta} = -\frac{1}{2} \frac{y^\alpha y^\beta f}{r^3} + \frac{1}{4} \frac{y^\alpha y^\beta}{r^2 f} + \frac{1}{2} \frac{f}{r} \delta^\alpha_\beta, \]
\[ \Delta \theta^0 = \frac{d f}{2r}. \]
(3.3)

Moreover
\[ \nabla \left( \frac{2r}{f^2} \right) = \frac{2}{f^2} (-y^2, xy). \]
(3.4)

Letting \( T \) denote the change to parabolic coordinates, \( T(x, y) = (f, g) \), then a computation using (3.2) shows that
\[ J := |\det T'| = \frac{r^{2-d}}{f^2 + y^2}. \]
(3.5)
Using again (3.2), we easily compute the partial derivative with respect to $f$
\[ \partial_f = |\nabla f|^{-2} \nabla f \cdot \nabla = F \cdot \nabla; \quad F := \frac{2r}{f^2} \nabla \theta^0 = 2r \nabla f, \tag{3.6} \]
and by using (3.3)–(3.5) we compute
\[ \partial_f \ln \left( J^{-1/2} \right) = \frac{1}{2} \text{div} F. \tag{3.7} \]

In the more restricted region $\{ x > 1, x > 2|y| \}$ (for example), the following uniform bounds hold, cf. [4, Lemma 3.4],
\[ \theta_0 = \left( \frac{2x}{3} \right)^{3/2} \left( 1 + \frac{3}{8} |y/x|^2 + O(|y/x|^4) \right), \]
\[ \theta_{ex} - \theta_0 = f^3 O(|y/x|^4), \]
\[ \nabla \theta_{ex} - \nabla \theta_0 = f O(|y/x|^3), \]
\[ \nabla^2 \theta_{ex} - \nabla^2 \theta_0 = f^{-1} O(|y/x|^2), \]
\[ f - f_{ex} = f O(|y/x|^4) . \tag{3.8} \]

We can obtain similar formulas for $\lambda \neq 0$ by using a Taylor expansion in combination with (3.3) and (3.8). Thus, in particular in the same restricted region
\[ \theta_{ex}(x + \lambda, y) - \theta_{ex}(x, y) = f^3 O(|y/x|^4) + f O(|y/x|^2) + f^{-1} O(|y/x|^0). \tag{3.9} \]

3.2. Stationary Wave Operators and the Scattering Matrix

We introduce the radiation operators
\[ \gamma^{\lambda \pm} = p \mp \nabla \theta^{\lambda} \quad \text{and} \quad \gamma_{l}^{\lambda \pm} = \text{Re} (2r \nabla f \cdot \gamma^{\lambda \pm}). \tag{3.10} \]

Recall from (3.6) that the vector field $2r \nabla f$ is the partial derivative with respect to $f$. Similarly, we denote $\Gamma^{\lambda \pm} = (\gamma^{\lambda \pm}, y/f^2)$ and use the notation $\Gamma_{k}^{\lambda \pm}$ for its components, although there is a $\lambda$-dependence only for $k \leq d$.

In terms of the Besov spaces introduced in Subsect. 3.1 for any $\lambda \in \mathbb{R}$ the limiting resolvent $R(\lambda \pm i0) \in \mathcal{L}(\mathcal{B}, \mathcal{B}^{*})$, cf. [3, Corollary 2.13], for which the following radiation condition bounds hold, cf. [4, Proposition 2.7]. We recall that $\delta \leq 1/2$.

**Proposition 3.1.** Let $\epsilon > 0$ and $\psi \in L^{2}_{\infty} (\subseteq \mathcal{B})$ be given. Let $\phi = R(\lambda \pm i0)\psi$ for $\lambda \in \mathbb{R}$. Then, for all $k, l = 1, \ldots, 2d - 1$ and locally uniformly in $\lambda$
\[ ||f^{(1+2\delta-\epsilon)/2} \Gamma_{k}^{\lambda \pm} \phi ||_{\mathcal{B}^{*}} < \infty, \tag{3.11a} \]
\[ ||f^{1+2\delta-\epsilon} \Gamma_{l}^{\lambda \pm} \Gamma_{k}^{\lambda \pm} \phi ||_{\mathcal{B}^{*}} < \infty, \tag{3.11b} \]
\[ ||f^{1+2\delta-\epsilon} \gamma_{l}^{\lambda \pm} \phi ||_{\mathcal{B}^{*}} < \infty. \tag{3.11c} \]

The stationary wave operators, the scattering matrix and the generalized eigenfunctions from [4] are constructed in terms of the phase $\theta^{\lambda}$ as follows.

We introduce, using parabolic coordinates,
\[ R_{\psi, f}^{\lambda \pm} (\zeta) := (J^{-1/2} e^{\mp i0^{\lambda}} R(\lambda \pm i0)\psi)(f, \pm \zeta); \quad \psi \in \mathcal{B}. \]
It is an almost trivial consequence of (3.6), (3.7) and Proposition 3.1, that for any \( \psi \in L^2_\infty \subseteq B \) there exist \( \Sigma - \lim_{f \to \infty} R^\lambda_{\psi,f} \). Indeed, the computation

\[
\partial_f R^\lambda_{\psi,f} = i \left( J^{-1/2} e^{\mp i0} \gamma^\lambda R(\lambda \pm i0) \right) (f, \pm \cdot),
\]

the Cauchy-Schwarz inequality and (3.11c) show that the functions \([0, \infty) \ni f \to R^\lambda_{\psi,f} \) have integrable derivatives. Consequently, there exist wave operators at fixed energy,

\[
F^\pm(\lambda) \psi := e^{\pm i\pi(d-2)/4} \sqrt{2\pi} \lim_{f \to \infty} R^\lambda_{\psi,f}; \quad \psi \in L^2_\infty.
\]

These obey the formulas, cf. [4, Theorem 2.2],

\[
||F^\pm(\lambda)\psi||^2 = \langle \psi, \delta(H - \lambda)\psi \rangle; \quad \delta(H - \lambda) = \pi^{-1} \text{Im} R(\lambda + i0).
\] (3.12)

Recalling that \( R(\lambda \pm i0) \in \mathcal{L}(B, B^*) \), it follows from (3.12) that \( ||F^+(\lambda)\psi|| = ||F^-(\lambda)\psi|| \) and that \( F^\pm(\lambda)\psi \in \Sigma \) are defined for \( \psi \in B \). This extension of \( F^\pm(\lambda) \) is given explicitly as follows. For vector-valued functions \( \xi \) on \( \mathbb{R} \) (or on \( \mathbb{R}_+ \)), we use the notation \( \int_\rho \xi(r) \, dr = \rho^{-1} \int_0^{\rho} \xi(r) \, dr, \rho > 0 \). Then, for any \( \psi \in B \) the vectors \( F^\pm(\lambda)\psi \) are given as the averaged limits

\[
F^\pm(\lambda)\psi = \Sigma - \lim_{\rho \to \infty} \int_\rho e^{\pm i\pi(d-2)/4} \sqrt{2\pi} R^\lambda_{\psi,f} \, df.
\] (3.13)

The maps \( B \ni \psi \to F^\pm(\lambda)\psi \in \Sigma \) are surjective (cf. Theorem 3.3 4) stated below), and it is also known, cf. [4, Theorem 2.2 (2)], that for any \( \psi \in B \) the maps \( \mathbb{R} \ni \lambda \to F^\pm(\cdot)\psi \in \Sigma \) are continuous.

Consequently, we can define the scattering matrix as the unique unitary operator \( S(\lambda) \) on \( \Sigma \) obeying

\[
F^+(\lambda)\psi = S(\lambda)F^-(\lambda)\psi,
\] (3.14)

and deduce that the map \( \mathbb{R} \ni \lambda \to S(\lambda) \in \mathcal{L}(\Sigma) \) is strongly continuous.

Along with \( \mathcal{H} = L^2(\mathbb{R}^d) \), we introduce the space

\[
\tilde{\mathcal{H}} = L^2(\mathbb{R}, d\lambda; \Sigma),
\]

and let \( M_\lambda \) be the operator of multiplication by \( \lambda \) on \( \tilde{\mathcal{H}} \). We introduce the operators

\[
\mathcal{F}^\pm = \int_\mathbb{R} \mathcal{F}^\pm(\lambda) \, d\lambda: \mathcal{B} \to C(\mathbb{R}; \Sigma),
\]

and recall [4, Theorem 2.4]:

**Theorem 3.2.** The operators \( \mathcal{F}^\pm \) extend uniquely as to become unitary operators \( \mathcal{F}^\pm: \mathcal{H} \to \tilde{\mathcal{H}} \). These extensions satisfy \( \mathcal{F}^\pm H = M_\lambda \mathcal{F}^\pm \).

The extensions \( \mathcal{F}^\pm \) in Theorem 3.2 are called stationary wave operators, and the first assertion of the theorem may be referred to as stationary completeness. On the other hand, the adjoint of the operators \( \mathcal{F}^\pm(\lambda) \), i.e., \( \mathcal{F}^\pm(\lambda)^* \in \mathcal{L}(\Sigma, B^*) \), are called stationary wave matrices.
3.3. Minimal Generalized Eigenfunctions

For any $\xi \in \Sigma$, we introduce purely outgoing/incoming approximate generalized eigenfunctions $\phi_{\lambda \pm}[\xi] \in B^*$ by, using the parabolic coordinates,

$$
\phi_{\lambda \pm}[\xi](f, g) = e^{\pi i \frac{d}{4}} \chi_{\perp}(f < 2) J^{1/2}(f, g) e^{\pm i \theta_{\lambda}}(f) \xi(\pm g).
$$

(3.15)

These functions may be seen as purely outgoing/incoming (zeroth order) WKB-approximations of generalized eigenfunctions. In fact, for $\xi \in C_c^\infty(\mathbb{R}^{d-1})$ we can compute $\psi_{\lambda \pm}[\xi] := (H - \lambda) \phi_{\lambda \pm}[\xi] \in B$, which allows us to consider the exact solutions

$$
\phi_{\text{ex}}^{\lambda \pm}[\xi] := \phi_{\lambda \pm}[\xi] - R(\lambda \mp i0) \psi_{\lambda \pm}[\xi]; \quad \xi \in C_c^\infty(\mathbb{R}^{d-1}).
$$

(3.16a)

Furthermore, for $\xi \in C_c^\infty(\mathbb{R}^{d-1})$ we have the formulas, cf. [4, Proposition 2.6],

$$
\phi_{\text{ex}}^{\lambda \pm}[\xi] = \mathcal{F}^{\pm}(\lambda)^* \xi,
$$

(3.16b)

$$
0 = \phi_{\lambda \pm}[\xi] - R(\lambda \pm i0) \psi_{\lambda \pm}[\xi],
$$

(3.16c)

$$
\xi = \pm i2 \pi \mathcal{F}^{\pm}(\lambda) \psi_{\lambda \pm}[\xi].
$$

(3.16d)

Here, (3.16c) is a consequence of the Sommerfeld uniqueness result of [3], and obviously in turn (3.16d) is a consequence of (3.16c).

The elements of the space

$$
\mathcal{E}_\lambda := \{ \phi \in B^* | (H - \lambda) \phi = 0 \}
$$

are called minimal generalized eigenfunctions. They are all of the form specified to the right in (3.16b) with $\xi \in \Sigma$, as stated in the following theorem from [4].

**Theorem 3.3.**

1) For any one of $\xi_\pm \in \Sigma$ or $\phi \in \mathcal{E}_\lambda$, the two other quantities in the triple $(\xi_-, \xi_+, \phi)$ uniquely exist such that

$$
\phi - \phi_{\lambda \pm}[\xi_+] - \phi_{\lambda \pm}[\xi_-] \in B^*_0.
$$

(3.17a)

2) The correspondences in (1) are given by the formulas

$$
\phi = \mathcal{F}^\pm(\lambda)^* \xi_\pm, \quad \xi_+ = S(\lambda) \xi_-,
$$

(3.17b)

$$
\xi_\mp = \mp \frac{1}{2} \frac{\sqrt{2\pi}}{e^{\pm i \pi / 4}} \Sigma \lim_{\rho \to \infty} \int_\rho \left( J^{-1/2}(f \sqrt{2r})^{-1} e^{\pm i \theta_{\lambda}} \gamma_\parallel^{\lambda \pm} \phi \right)(f, \mp \cdot) df.
$$

(3.17c)

In particular, the wave matrices $\mathcal{F}^\pm(\lambda)^* : \Sigma \to \mathcal{E}_\lambda$ are linear isomorphisms.

3) The wave matrices $\mathcal{F}^\pm(\lambda)^* : \Sigma \to \mathcal{E}_\lambda (\subseteq B^*)$ are bi-continuous. In fact,

$$
\|\xi_\parallel\|^2 = \pi \lim_{m \to \infty} 2^{-m} \|F(2^m \leq f < 2^{m+1}) \phi\|^2.
$$

(3.17d)

4) The operators $\mathcal{F}^\pm(\lambda) : B \to \Sigma$ and $\delta(H - \lambda) : B \to \mathcal{E}_\lambda$ map onto.
The assertions (2) and (3.17c) provide a general formula for \( S(\lambda)\xi \) (take \( \xi = \xi_- \) and \( \phi = \mathcal{F}^- (\lambda) \ast \xi \)). However, for smooth compactly supported \( \xi \), there are the following alternative recipes for calculating the scattering matrix.

**Corollary 3.4.** For any \( \xi \in C_c^\infty (\mathbb{R}^{d-1}) \)

\[
S(\lambda)\xi = -i2\pi \mathcal{F}^+ (\lambda) \psi^\lambda - [\mathcal{F}] = -\frac{\sqrt{2\pi}}{e^{-i\pi d/4}} \Sigma_\rho \lim_{\rho \to \infty} \int_\rho \mathcal{R}^\lambda_{\psi^\lambda-,f} \, df. \tag{3.18}
\]

For any \( \xi, \xi' \in C_c^\infty (\mathbb{R}^{d-1}) \)

\[
\frac{1}{2\pi} \langle \xi', S(\lambda)\xi \rangle = \langle \psi^\lambda [\xi'] R(\lambda + i0) \psi^\lambda - [\xi] \rangle - \langle \phi^\lambda [\xi'], \psi^\lambda - [\xi] \rangle. \tag{3.19}
\]

There are other representations like (3.19) of the scattering matrix, which we will state and examine in this paper (see Sects. 7 and 8).

We also note that our notation appears consistent in the case \( q = 0 \) in the following sense. Recall that \( \mathcal{F}_0(\lambda) \) is given by (2.1) (with \( x \to x + \lambda \)).

**Corollary 3.5.** For \( q = 0 \)

\[
\mathcal{F}_0(\lambda) = \mathcal{F}^+(\lambda) = \mathcal{F}^-(\lambda).
\]

In particular \( S(\lambda) = I \) in this case.

**Proof.** We prove the first identity only. The proof of the identity \( \mathcal{F}_0(\lambda) = \mathcal{F}^-(\lambda) \) is similar, so this suffices. In turn it suffices to show that \( \mathcal{F}_0(\lambda) \ast \xi = \mathcal{F}^+(\lambda) \ast \xi \) for any \( \xi \in C_c^\infty (\mathbb{R}^{d-1}) \). We write \( \mathcal{F}_0(\lambda) \ast \xi = \phi_{\lambda,1} [\xi] (\in \mathcal{E}_\lambda) \) in agreement with (2.2a). By (2.7) and Theorem 3.3 1)–2) we may write \( \phi_{\lambda,1} [\xi] = \mathcal{F}^+(\lambda) \ast \xi \) for a unique \( \xi \in \Sigma \), which may be computed by (3.17c). For this end we first compute

\[
\frac{1}{2} \frac{\sqrt{2\pi}}{e^{-i\pi d/4}} J^{-1/2} \left( f \sqrt{2r} \right)^{-1} e^{-i\theta^\lambda} \gamma_\parallel \lambda^\parallel - \phi_{\lambda,1} [\xi] = \frac{\sqrt{2\pi}}{e^{-i\pi d/4}} J^{-1/2} e^{-i\theta^\lambda} \phi_{\lambda,\tilde{a}} [\xi];
\]

\[
\tilde{a} = \frac{1}{2} \left( f \sqrt{2r} \right)^{-1} \left( 2r \nabla f \cdot (\eta, \zeta) + \nabla \theta^\lambda \right) - \frac{i}{2} \Delta (2r \nabla f).
\]

By using (2.7) to this \( \tilde{a} \), (3.9) and (3.17c) it follows that indeed \( \tilde{\xi} = \xi \). \( \square \)

### 4. Identification of Wave Operators and Scattering Matrices

The **time-dependent wave operators** are given by

\[
W^\pm := s\text{-}\lim_{t \to \pm \infty} e^{itH} e^{-itH_0},
\]

and the corresponding **scattering operator** \( S = (W^+) \ast W^- \) commutes with \( H_0 \), yielding the representation

\[
\mathcal{F}_0 \mathcal{S} \mathcal{F}_0^{-1} = \int_{\mathbb{R}} S(\lambda) \, d\lambda; \quad \mathcal{F}_0 = \int_{\mathbb{R}} \mathcal{F}_0(\lambda) \, d\lambda. \tag{4.1}
\]

In this section, we show that

\[
\mathcal{F}_0(W^\pm)^* = \mathcal{F}^\pm, \tag{4.2}
\]
which implies that the (almost everywhere defined) operator $S(\lambda)$ in (4.1) is equal to the (everywhere defined strongly continuous) scattering matrix of Subsect. 3.2.

It follows from the Avron–Herbst formula [2] that for any $\varphi \in \mathcal{H}$

\[
(e^{-itH_0}\varphi)(x, y) = e^{-it\varphi_d/4}t^{-d/2} e^{i\{(-t^3/6 + tx + [(x-t^2/2+y^2)]/(2t))\}} \\
\times \varphi((x-t^2/2)/t, y/t) + o_{\mathcal{H}}(|t|^0) \text{ as } |t| \to \infty;
\]

here by definition $||o_{\mathcal{H}}(|t|^0)||_{\mathcal{H}} \to 0$ for $|t| \to \infty$.

To identify the wave operators, it would be tempting, based on Sect. 3, to try to compute directly the $L^2$-asymptotics of integrals of the form

\[
\int e^{-it\lambda} f_1^{1/2} e^{i\theta(\lambda)} \xi(\pm y/f) h(\lambda) \, d\lambda,
\]

where $\xi \in C_c^\infty(\mathbb{R}^{d-1})$ and $h \in C_c^\infty(\mathbb{R})$ and compare with the right-hand side of the above formula, cf. [14]. However, such computation does not seem doable. We proceed differently introducing, cf. (2.2a) and (2.2b),

\[
(F_{\pm}^{0}(\lambda)^* \xi)(x, y) = c \int d\zeta \xi(\zeta) \int e^{i\theta(\lambda)} \chi_{\pm}(\eta) \, d\eta;
\]

in terms of the partition $\chi_{+} + \chi_{-} = 1$, $\chi_{+} = \chi(\cdot > 1)$ and $\chi_{-} = \chi(\cdot > 1)$.

Using the formula

\[
(H_0 - \lambda)e^{i\theta(\lambda)} = -i\partial_\eta e^{i\theta(\lambda)}
\]

we can integrate by parts and deduce that

\[
g_1(\lambda) := (H_0 - \lambda)F_{\pm}^{0}(\lambda)^* \xi = -ic \int d\zeta \xi(\zeta) \int e^{i\theta(\lambda)} \chi'_{\pm}(\eta) \, d\eta.
\]

By the method of non-stationary phase, cf. the first part of Subsect. 2.1, this integral has polynomial decay as $r = |(x, y)| \to \infty$. In particular $g_1$ is an $L^2_1$-valued function, in fact (by the same argument)

\[
g_1 \text{ is } C^1 \text{ as an } L^2_1-\text{valued function. (4.5a)}
\]

In particular for any $\xi \in C^\infty_c(\mathbb{R}^{d-1})$ and $h \in C^\infty_c(\mathbb{R})$ the product $hg_1 \in \widetilde{\mathcal{H}}$ and

\[
\int_0^\infty ||\hat{hg}_1(s)||_{\mathcal{H}} \, ds < \infty. \quad (4.5b)
\]

Here and below, the notation $\hat{\cdot}$ refers to the one-dimensional Fourier transform. Note that (4.5b) is a consequence of (4.5a) due to the Cauchy-Schwarz inequality and the Plancherel theorem (cf. a standard one-dimensional Sobolev space bound). Letting

\[
g_2(\lambda) := qF_{\pm}^{0}(\lambda)^* \xi,
\]
it is easy to check using (2.6) that

\[ g_2 \text{ is continuous as an } L^2_{1/4+\delta'} \text{-valued function for any } \delta' < \delta. \]  

(4.6a)

We claim that similarly the product \( hg_2 \in \mathcal{H} \) and

\[ \int_0^\infty ||\widehat{hg_2}(s)||_{\mathcal{H}} \, ds < \infty. \]  

(4.6b)

We compute using (2.6)

\[ \partial_\lambda (hg_2) \text{ is continuous as an } L^2_{1/4+\delta'} \text{-valued function for any } \delta' < \delta. \]  

(4.6c)

Then, (4.6b) follows from (4.6a) and (4.6c) by interpolation (more precisely by an application of the Hadamard three-lines theorem, here omitted).

The combination (4.5b) and (4.6b) allows us to record

\[ \int_0^\infty ||\widehat{hg}(s)||_{\mathcal{H}} \, ds < \infty; \quad \quad g := g_1 + g_2 = (H - \lambda) F_{0}^{\pm}(\lambda)^* \xi. \]  

(4.7)

(These assertions will later serve as a justification of a use of [11, Lemma 5.1] adapted to our setting.)

Next by using (2.7), (3.9) and Theorem 3.3 1), we deduce (cf. the proof of [4, Lemma 4.4]) that the generalized eigenfunctions in the formula (3.16a) are given by

\[ \phi_{\text{ex}}^{\lambda \pm}[\xi] = F_{0}^{\pm}(\lambda)^* \xi - R(\lambda \mp i0)(H - \lambda) F_{0}^{\pm}(\lambda)^* \xi. \]  

(4.8)

It is easy to analyze the integral (focusing below on the case \( t \to +\infty \))

\[ I^+(t) := \int e^{-it\lambda} (F_{0}^{\pm}(\lambda)^* \xi) h(\lambda) \, d\lambda; \quad t > 0, \xi \in C_c^\infty(\mathbb{R}^{d-1}), \h \in C_c^\infty(\mathbb{R}). \]

In fact by the method of non-stationary phase

\[ I^+(t) + o_H(t^0) = \int e^{-it\lambda} (F_{0}(\lambda)^* \xi) h(\lambda) \, d\lambda = e^{-itH_0} \varphi, \]  

(4.9)

where \( \varphi \in \mathcal{H} \) is fixed by \( \mathcal{F}_{0} \varphi = h \otimes \xi \in \mathcal{H}. \)

In the paper [11], completeness for Schrödinger operators is considered /proven from the stationary point of view. In our setting, one would look at the exact solution to the Schrödinger equation

\[ \int e^{-it\lambda} (\mathcal{F}_{0}^{\pm}(\lambda)^* \xi) h(\lambda) \, d\lambda = \int e^{-it\lambda} \phi_{\text{ex}}^{\lambda \pm}[\xi] h(\lambda) \, d\lambda. \]  

(4.10)

Now by using (4.5a), (4.6a), (4.7), (4.8), (4.9) and the proof of [11, Lemma 5.1], we obtain that the wave packet (4.10) is of the form:

\[ e^{-itH_0} \varphi + o_H(t^0) = e^{-itH} W^+ \varphi + o_H(t^0) \]

with the above \( \varphi \in \mathcal{H} \). Since \( e^{-itH} W^+ \varphi \) is also an exact solution to the Schrödinger equation, it follows from the unitary property of the Schrödinger propagator that \( \mathcal{F}^{++}(h \otimes \xi) = W^+ \varphi \). Consequently (by density) \( \mathcal{F}^{++} \mathcal{F}_{0} = W^+ \), which is the ‘plus case’ of (4.2). The ‘minus case’ of (4.2) can be derived similarly.
We learn from (4.2) that there exist the asymptotic orthogonal momenta
\[ p_y^\pm = \lim_{t \to \pm \infty} e^{itH} p_y e^{-itH} = (\mathcal{F}^\pm)^* \left( \int_{\mathbb{R}}^\oplus M_\zeta \, d\lambda \right) \mathcal{F}^\pm; \]
here the limit is taken in the strong resolvent sense and \( M_\zeta \) denotes multiplication by (the components of) \( \zeta \) on \( \Sigma = L^2(\mathbb{R}^{d-1}) \), whence formally the (Schwartz) kernel \( S(\lambda)(\zeta, \zeta') \) of the scattering matrix is defined in terms of incoming and outgoing asymptotic orthogonal momenta, cf. the discussion in Sect. 1.

5. Resolvent Bounds

We recall the following elementary result from [3].

\[ \forall \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq 1, \forall k \in \mathbb{N}, \forall h \in C_c^\infty(\mathbb{R}) : p^\alpha |x|^{k/2} \chi(x < 0) h(H) \in \mathcal{L}(\mathcal{H}). \]

(5.1)

Let \( A_m = \text{Re} (\nabla f_m \cdot p) \) where \( f_m(x, y) = \sqrt{f(2x + 2 \langle y \rangle_m)} \) with \( f \) given as in (3.1a) and with \( m \in \mathbb{N} \); recall the notation \( \langle y \rangle_m = (m^2 + |y|^2)^{1/2} \). Let
\[ \tilde{A}_m = \frac{1}{2} \text{Re} (\nabla f_m^2 \cdot p) = f_m^{1/2} A_m f_m^{1/2}. \]

We compute
\[ i[H, 2\tilde{A}_m] = p \cdot (\nabla^2 f_m^2) + \partial_x f_m^2 - (\nabla f_m^2) \cdot \nabla q - \frac{1}{4} \Delta^2 f_m^2, \]

(5.2)

which leads to
\[ i[H, 2\tilde{A}_m] \geq 2 - (\nabla f_m^2) \cdot \nabla q - \frac{1}{4} (\Delta^2 f_m^2) - C_1 F(2x + 2 \langle y \rangle_m \leq 2) \]
\[ \geq 2 - C_2 \frac{1}{x} - C_3 m^{-3} + C_4 \frac{x-1}{m} F(x - 1 + m \leq 0). \]

In combination with (5.1), we conclude that for any given energy the Mourre estimate (see [18]) holds for \( \tilde{A}_m \) with a constant as close to 1 as wished provided we take \( m \) large enough.

For convenience we abbreviate \( A_m = A \) and \( f_m = f \), noting that this \( f \) is different from (3.1a) used previously. Now the following estimates hold locally uniformly in \( \lambda \in \mathbb{R} \), cf. the multiple commutator methods of [5,7]. The parameter \( m \) may depend on \( \lambda \); however, it can be taken independently of \( \lambda \), and it may depend on the parameters \( t, t' \) appearing in the estimates (however, the dependence is only on \( \kappa \in (0, 1) \) provided \( t, t' \in [\kappa - 1, 1 - \kappa] \); in our application we consider fixed parameters only).

\[ f^{-s} R(\lambda \pm i0) f^{-s} \in \mathcal{L}(\mathcal{H}); \quad s > 1/2. \]

(5.3a)

\[ \chi(\pm A < t) f^s R(\lambda \pm i0) f^{-1-s} \in \mathcal{L}(\mathcal{H}); \]
\[ s > -1/2, \; t < 1. \]

(5.3b)

\[ \chi(\pm A < t) f^s R(\lambda \pm i0) f^s \chi(\pm A > t) \in \mathcal{L}(\mathcal{H}); \]
\[ s \in \mathbb{R}, \; -1 < t < t' < 1. \]

(5.3c)
∀ \texttt{k} \in \mathbb{N} : \; f^{-s} R(\lambda \pm i0)^k f^{-s} \in \mathcal{L}(\mathcal{H}); \; s > k - 1/2. \quad (5.4)

The last estimate is a consequence of (5.3a)–(5.3c) and an algebraic argument (cf. [12,15]), in fact there are ‘microlocal bounds’ in the spirit of (5.3a)–(5.3c) for powers R(\lambda \pm i0)^k also (deducible from the same argument). Such estimates would be useful for obtaining regularity of the S-matrix in the spectral parameter; however, this topic will not be studied in the paper.

6. Classical Mechanics Bounds and Transport Equations

We may associate with the operator \( A = A_m \) of the previous section the ‘symbol’

\[
a = a_m = \frac{n+\hat{y}_m \cdot \zeta}{f_m};
\]

\[
\langle y \rangle_m = (m^2 + |y|^2)^{1/2}, \; \hat{y}_m = y/\langle y \rangle_m, \; f_m(x,y) = \sqrt{\hat{f}(2x + 2 \langle y \rangle_m)}.
\]

Here, \( m \) is a fixed large positive integer, and by definition \( f_m = \sqrt{2x + 2 \langle y \rangle_m} \) for \( x + \langle y \rangle_m > 1 \). Let \( \tilde{a} = \frac{n+\hat{y}_m \cdot \zeta}{\sqrt{2x+2(\langle y \rangle_m^2}} \) for \( x + \langle y \rangle_m > 0 \).

We consider for any such \( m \) and for any \( \varepsilon \in (0,1) \)

\[
\mathcal{X}_\varepsilon^\pm = \mathcal{X}_{\varepsilon,m}^\pm := \{ x + \langle y \rangle_m > 0, \; \pm \tilde{a} > -\varepsilon \}.
\]

Lemma 6.1. The sets \( \mathcal{X}_\varepsilon^+ \) and \( \mathcal{X}_\varepsilon^- \) are preserved by the free classical forward and backward flow \( \Theta \) given by (1.1) with \( t \geq 0 \) and \( t \leq 0 \), respectively.

Proof. We estimate as follows on \( \mathcal{X}_\varepsilon^\pm \) for \( \pm t \geq 0 \),

\[
2x(t) + 2 \langle y(t) \rangle_m \geq t^2 + 2(t \eta + \hat{y}_m \cdot \zeta) + (2x + 2 \langle y \rangle_m) + 2(\sqrt{2ty \cdot \zeta + \langle y \rangle_m^2} - t\hat{y}_m \cdot \zeta - \langle y \rangle_m) \geq t^2 - 2|t| \varepsilon \sqrt{2x + 2 \langle y \rangle_m + (2x + 2 \langle y \rangle_m)} \geq (1 - \varepsilon)(t^2 + 2x + 2 \langle y \rangle_m).
\]

In particular, the left-hand side stays positive. The ‘symbol’ \( \tilde{a} \) is well defined on \( \mathcal{X}_\varepsilon^+ \cup \mathcal{X}_\varepsilon^- \), and by a free classical Mourre estimate, cf. the calculation (5.2),

\[
\frac{d}{dt} \tilde{a}(t) \geq (1 - \tilde{a}(t)^2) / \sqrt{2x(t) + 2 \langle y(t) \rangle_m} \quad \text{on} \; \mathcal{X}_\varepsilon^\pm \quad \text{for} \; \pm t \geq 0.
\]

In particular, since \( \pm \tilde{a}(0) > -\varepsilon \) on \( \mathcal{X}_\varepsilon^\pm \), also \( \pm \tilde{a}(t) > -\varepsilon \) on \( \mathcal{X}_\varepsilon^\pm \) for \( \pm t \geq 0 \).

For any \( n \in \mathbb{N} \) we construct smooth functions \( a_n^\pm = \sum_{k=0}^n b_k^\pm \) in the variables \( (x,y;\eta,\zeta) \) as follows (omitting superscripts). Let \( b_0 = 1 \) and \( q_0 = q \).
Suppose that $b_k$ and $q_k$ are constructed for a given $k \in \{0, \ldots, n - 1\}$, then these quantities with $k$ replaced by $k + 1$ are given by

$$b_{k+1} = i \int_0^{\pm \infty} q_k(\Theta(t)) \, dt,$$

$$q_{k+1} = q b_{k+1} - \frac{1}{2} (\Delta(x,y) b_{k+1}).$$

These functions solve transport equations, more precisely

$$i(\partial_\eta + (\eta, \zeta) \cdot \nabla_{(x,y)}) b_{k+1} = q_k = q b_k - \frac{1}{2} (\Delta(x,y) b_k). \quad (6.2)$$

For the sake of justification of the above recursion scheme, we note the elementary computation

$$\forall f > 0 : \quad \int_0^\infty (t^2 + f^2)^{-s_1} t^{s_2} \, dt = C_{s_1,s_2} f^{s_2+1-2s_1};$$

$$s_2 + 1 - 2s_1 < 0, \quad -1 < s_2. \quad (6.3)$$

It follows from Lemma 6.1, (6.1), (6.3), the Faà di Bruno formula and induction that for any $0 \leq k \leq n$ and any $\varepsilon \in (0,1)$

$$|\partial^\alpha_{\eta,\zeta} \partial^\beta_{x,y} b_k^\pm| \leq C_{\alpha,\beta} (1 + x + \langle y \rangle_m)^{-(k\delta + |\alpha|/2 + |\beta|)},$$

$$|\partial^\alpha_{\eta,\zeta} \partial^\beta_{x,y} q_k^\pm| \leq C_{\alpha,\beta} (1 + x + \langle y \rangle_m)^{-(1/2 + (k+1)\delta + |\alpha|/2 + |\beta|)}; \quad (x, y; \eta, \zeta) \in \mathcal{X}_\varepsilon^\pm. \quad (6.4)$$

In particular,

$$|\partial^\alpha_{\eta,\zeta} \partial^\beta_{x,y} a_n^\pm| \leq C_{\alpha,\beta} (1 + x + \langle y \rangle_m)^{-(|\alpha|/2 + |\beta|)},$$

$$|\partial^\alpha_{\eta,\zeta} \partial^\beta_{x,y} a_n^\pm| \leq C_{\alpha,\beta} (1 + x + \langle y \rangle_m)^{-(1/2 + (n+1)\delta + |\alpha|/2 + |\beta|)}; \quad (x, y; \eta, \zeta) \in \mathcal{X}_\varepsilon^\pm.$$

Although we could work with $a_n^\pm$ for a fixed large $n$, it is convenient to repeat the construction of the $b_k$’s without limit and then invoke the Borel construction to regularize the sum $\sum_0^\infty b_k^\pm$.

Whence we introduce for any $\varepsilon \in (0,1)$, the symbol

$$a_B^\pm = \chi_\varepsilon^\pm \sum_0^\infty \chi_k b_k^\pm, \quad \chi_\varepsilon^\pm = \chi(\pm a > -\varepsilon), \quad \chi_k = \chi(f > C_k), \quad (6.5)$$

for a suitable sequence $\sqrt{2} < C_0 < C_1 < \cdots < \infty$. Here and henceforth, we use the abbreviated notation $f = f_m$ and $a = a_m = \frac{\eta + y_m - \zeta}{f}$. As noted before $a = \bar{a}$ for $f > \sqrt{2}$. The relevant choice of $m$ depends on a bounding constant of the energy $\lambda$, cf. a discussion in Sect. 5, but for convenience we prefer to suppress the dependence on $m$ in our notation. The construction of such sequence ($C_k$) is standard (see, for example, the proof of [9, Theorem 1.2.6]). We provide the details for our setting in Appendix B. Due to the fact that (6.4) are uniform bounds, it is not important for the construction that the variable $\zeta$ is considered as a bounded variable. However, some derivatives of the factor $\chi_\varepsilon^\pm$ in (6.5) are only bounded locally uniformly in $\zeta$. Thus, the notation $\mathcal{O}(\cdot)$ below refers to a symbol obeying the indicated bound, however this only being locally uniform in $\zeta$. In addition, we use the notation $\mathcal{O}(f^{-\infty})$
to mean a smooth function (a symbol) with all derivatives being of the form $O(f^{-k})$ locally uniform in $\zeta$ (and uniform in the other variables) for any $k \in \mathbb{N}$.

In conclusion, thanks to (6.2), there exists a suitable sequence $(C_k)$ such that for arbitrarily localized $\zeta$ (and for any fixed $m$)

$$\partial_{\eta,\zeta}^{\alpha} \partial_{x}^{\beta} \partial_{y}^{\gamma} a_B^\pm = O\left(f^{-(|\alpha|+2|\beta|+|\gamma|)} \min\left(f^{2}, (y)_m\right)^{-|\gamma|}\right),$$

$$i(\partial_{\eta} + (\eta, \zeta) \cdot \nabla_{(x,y)}) a_B^\pm = qa_B^\pm - \frac{1}{2}(\Delta_{(x,y)} a_B^\pm) + \sum_0^\infty r_k^\pm + O(f^{-\infty});$$

$$r_k^\pm = ib_k^\pm \left(\chi_k \partial_{\eta} \chi^\pm_e + (\eta, \zeta) \cdot \nabla_{(x,y)}(\chi_k \chi^\pm_e)\right) + (\nabla_{(x,y)} b_k^\pm) \cdot \nabla_{(x,y)}(\chi_k \chi^\pm_e) + \frac{b^\pm}{2} \Delta_{(x,y)}(\chi_k \chi^\pm_e).$$

(6.6)

(Here $O(f^{-\infty})$ comes from harmless changes $\chi_{k+1} \rightarrow \chi_k$ when applying (6.2).) These bounds in combination with (4.4) will play a basic role in Sects. 7 and 8.

7. Analysis of the Scattering Matrix

We consider in this section double integrals of the form:

$$c \int d\zeta \xi(\zeta) \int e^{i\theta_{\lambda}} \tilde{a} \, d\eta;$$

$$c = (2\pi)^{-\frac{d+1}{2}}, \quad \theta_{\lambda} = y \cdot \zeta - \frac{\eta^2}{2} + (x + \lambda - \zeta^2/2)\eta.$$

Such integrals are studied in Subsect. 2.1 with symbols $\tilde{a} = \tilde{a}(x, y; \eta, \zeta)$ obeying (2.2b) (see also the examples (2.1) and (4.3)). The function $\xi$ can in some cases be considered as any compactly supported distribution, but of course the double integral has nicest properties for $\xi \in C_0^\infty(\mathbb{R}^{d-1})$ (as in Subsect. 2.1). Let $m \in \mathbb{N}$ (it is considered as a large fixed auxiliary parameter), let $\varepsilon \in (0, 1/2)$ (conveniently taken small) and let $a_B^\pm$ be the associated symbol given by (6.5). We introduce then the explicit example

$$F_{m, \varepsilon}^\pm(\lambda)^* \xi := c \int d\zeta \xi(\zeta) \int e^{i\theta_{\lambda}} a_B^\pm \, d\eta.$$  (7.1a)

We calculate using (4.4), (6.6) and an integration by parts

$$(H - \lambda) F_{m, \varepsilon}^\pm(\lambda)^* \xi = -c \int d\zeta \xi(\zeta) \int e^{i\theta_{\lambda}} \left(\mathcal{O}(f^{-\infty}) + \sum_0^\infty r_k^\pm\right) \, d\eta;$$

$$r_k^\pm = ib_k^\pm \left(\chi_k \partial_{\eta} \chi^\pm_e + (\eta, \zeta) \cdot \nabla_{(x,y)}(\chi_k \chi^\pm_e)\right) + (\nabla_{(x,y)} b_k^\pm) \cdot \nabla_{(x,y)}(\chi_k \chi^\pm_e) + \frac{b^\pm}{2} \Delta_{(x,y)}(\chi_k \chi^\pm_e).$$  (7.1b)

We can use (6.4) to estimate the quantities of (7.1) and (7.1), that is

$$\phi_{m, \varepsilon}^\pm[\xi] := F_{m, \varepsilon}^\pm(\lambda)^* \xi \quad \text{and} \quad \psi_{m, \varepsilon}^\pm[\xi] := (H - \lambda) \phi_{m, \varepsilon}^\pm[\xi].$$
We obtain for the corresponding symbols, say denoted by \( \tilde{a}_1 = a_{\pm}^1 \) and \( \tilde{a}_2 \), respectively, the bounds

\[
|\partial_{\eta,\zeta}^\alpha \partial_{x,y}^\beta \tilde{a}_1| \leq C_1 f^{-(|\alpha|+|2\beta|)} \min \left( f^2, \langle y \rangle \right)^{-|\gamma|}; \quad C_1 = C_1(\zeta),
\]

\[
|\partial_{\eta,\zeta}^\alpha \partial_{x,y}^\beta \tilde{a}_2| \leq C_2 \left( 1 + \frac{|n|}{f} \right) \min \left( f, \sqrt{\langle y \rangle} \right)^{-(|\alpha|+|2\beta|+1)}; \quad C_2 = C_2(\zeta).
\]

(7.1c)

Here and henceforth, we use the function \( f = f_m \) of Sections 5 and 6, and we use, slightly abusively, the notation \( \langle y \rangle \) for \( \langle y \rangle_m \). Note that the constants depend on \( \zeta \), although if \( \zeta \in B_R = \{|\zeta| < R\} \) for any given \( R > 0 \), then the dependence is via \( R \) only. Also there is a dependence on the multiindices, however for convenience not indicated. Note that the first bound of (7.1) corresponds to the first assertion of (6.6). The second bound of (7.1) follows readily from an examination of the expressions \( r_k^\pm \) and the concrete construction given in Appendix B.

We claim (to be justified in the proof of Lemma 7.1) the following formula for the generalized eigenfunction of (3.16a)

\[
\phi_{\text{ex}}^{\lambda \pm}[\xi] = \phi_{m,\varepsilon}^{\lambda \pm}[\xi] - R(\lambda \mp i0) \psi_{m,\varepsilon}^{\lambda \pm}[\xi]; \quad \xi \in C_c^\infty(\mathbb{R}^{d-1}),
\]

(7.2a)

and the related formulas, cf. (3.16c) and (3.16d),

\[
0 = \phi_{m,\varepsilon}^{\lambda \pm}[\xi] - R(\lambda \pm i0) \psi_{m,\varepsilon}^{\lambda \pm}[\xi],
\]

(7.2b)

leading to

\[
\xi = \pm i2\pi \mathcal{F}^\pm(\lambda) \psi_{m,\varepsilon}^{\lambda \pm}[\xi]; \quad \xi \in C_c^\infty(\mathbb{R}^{d-1}).
\]

(7.2c)

The analogue of (3.19) reads, thanks to (3.16b), (7.2a) and (7.2c),

\[
\frac{1}{2\pi i} \langle \xi, S(\lambda) \xi' \rangle = \langle \psi_{m,\varepsilon}^{\lambda \pm}[\xi], R(\lambda + i0) \psi_{m,\varepsilon}^{\lambda -}[\xi'] \rangle - \langle \phi_{m,\varepsilon}^{\lambda \pm}[\xi], \psi_{m,\varepsilon}^{\lambda -}[\xi'] \rangle.
\]

(7.3)

Note that the leading order asymptotics of \( \phi_{m,\varepsilon}^{\lambda \pm}[\xi] \) for \( \xi \in C_c^\infty(\mathbb{R}^{d-1}) \) follows from (2.7). The expansion terms from stationary phase analysis, cf. Appendix A, all vanish for \( \psi_{m,\varepsilon}^{\lambda \pm}[\xi] \). The following result is a manifestation of this fact.

**Lemma 7.1.** For all \( \xi \in C_c^\infty(\mathbb{R}^{d-1}) \), the functions \( \psi_{m,\varepsilon}^{\lambda \pm}[\xi] \in L^2_{\infty} \), and the formulas (7.2a) and (7.2b) are valid.

**Proof.** Write

\[
\psi_{m,\varepsilon}^{\lambda \pm}[\xi] = \psi_{m,\varepsilon,1}^{\lambda \pm}[\xi] + \psi_{m,\varepsilon,2}^{\lambda \pm}[\xi],
\]

corresponding to the splitting

\[
\int e^{i\theta_{\lambda}} \left( \mathcal{O}(f^{-\infty}) + \sum_0^{\infty} r_k^\pm \right) d\eta = \int e^{i\theta_{\lambda}} \mathcal{O}(f^{-\infty}) d\eta + \int e^{i\theta_{\lambda}} \sum_0^{\infty} r_k^\pm d\eta
\]

in (7.1). There are two ways of integrating by parts using (2.3a) and (2.3b), respectively.
I. The contribution from $O(f^{-\infty})$ takes the desired form since we have any (high) power $(x + \langle y \rangle)^{-t} = 2^{t}f^{-2t}$ at our disposal. We can use a part of this factor to obtain a high power of $(\langle \eta, \zeta \rangle)^{-1}$ as well as a high power of $\langle x \rangle^{-1}$ by integrating by parts using (2.3a) repeatedly. We can then use the decay in $x$ (needed for $x < 0$ only) and another part of the factor $(x + \langle y \rangle)^{-t}$ (for $x \in \mathbb{R}$ arbitrary) to obtain a high power of $\langle y \rangle^{-1}$ as well. Altogether we obtain a desirable factor $\langle x \rangle^{-s} \langle y \rangle^{-s}$ with $s > 1$ arbitrarily big for the contribution from $O(f^{-\infty})$.

II. As for the contribution from the terms $r_{k}^{\pm}$ of $\sum_{0}^{\infty} r_{k}^{\pm}$, we observe that terms for which the factor $\chi_{k}$ is differentiated can be treated exactly as above.

III. It remains to consider the contributions from terms where at least one derivative falls on the factor $\chi_{\pm}$. By definition any such term is supported in $\{x + \langle y \rangle > 1, -\varepsilon/2 \geq \pm a \geq -\varepsilon\}$. We mimic Subsect. 2.1 and Appendix A.

Since $\xi$ is compactly supported, the variable $\zeta$ is localized and we may for any such term consider

$$\frac{|\eta|}{\sqrt{2x+2\langle y \rangle}} \approx |a| \in [\varepsilon/2, \varepsilon] \text{ effectively.} \quad (7.4)$$

(In the complement of this localized region Step I applies.)

Since $\varepsilon \in (0, 1/2)$, the stationary points (2.5) do not conform with (7.4). This means that Remark A.1 applies, proving the first assertion of the lemma.

IV. As for the second assertion, the difference of the left- and right-hand sides in (7.2a) is a purely incoming or outgoing generalized eigenfunction in $B^{*}$, respectively, cf. (2.7) and the proof of [4, Lemma 4.4]. Hence, by Theorem 3.3 1) it vanishes. For (7.2b) we can argue similarly.

Let $E_{d-1}'$ denote the space of compactly supported distributions on $\mathbb{R}^{d-1}$. Any $\xi \in E_{d-1}'$ can be written as $\xi = Q_\varepsilon \xi'$, where $Q = Q(\zeta, p_\varepsilon)$ is a differential operator on $\mathbb{R}^{d-1}$ with smooth coefficients and $\xi' \in C_{c}(\mathbb{R}^{d-1})$. We shall use the quantity $A = A_{m}$ of Sect. 5. Let for any $\varepsilon \in (0, 1/2)$

$$\chi_{-\varepsilon}(t) = \chi(t < -\varepsilon/4)\chi(t > -2\varepsilon); \quad t \in \mathbb{R}. \quad (7.5)$$

Lemma 7.2. Let $n \in \mathbb{N}_{0}$ and consider a fixed $\xi \in E_{d-1}'$ of the form $\xi = \langle p_\varepsilon \rangle^{2n} \xi'$, $\xi' \in C_{c}(\mathbb{R}^{d-1})$. Then, there exists $s' = s'(n) \in \mathbb{R}$ such that for any $s \in \mathbb{R}$, the quantities $\psi_{m, \varepsilon}^{\pm}[\xi]$ are represented

$$\psi_{m, \varepsilon}^{\pm}[\xi] = f^{s'} \chi_{-\varepsilon}(\pm A) \varphi_{1}^{\pm} + f^{-s} \varphi_{2}^{\pm} \text{ for some } \varphi_{1}^{\pm}, \varphi_{2}^{\pm} \in \mathcal{H}. \quad (7.6)$$

The $\mathcal{H}$-norm of $\varphi_{1}^{\pm}$ and $\varphi_{2}^{\pm}$ can be estimated by $C R^{(d-1)/2}||\xi'||_{\mathcal{H}}$ provided $\xi' \subseteq B_{R}$, and $\varphi_{1}^{\pm}$ (and the corresponding constant $C$) can be chosen independent of $s$. 


Proof. I. The powers of $p_{\zeta}$ in the definition of the $\zeta$ can be moved to other factors of the $\zeta$-integral thereby producing additional factors of monomials in $y - \eta \zeta$. By the proof of Lemma 7.1, the contribution from the term $O(f^{-\infty})$ did not use integration by parts in the other direction, i.e., (2.3b) was not used. In fact an arbitrarily large negative power $\langle x \rangle^{-s} \langle y \rangle^{-s}$ was produced. This can bound a factor $\langle y \rangle^{2n}$, and we conclude that the contribution from the term $O(f^{-\infty})$ takes the form of the second term on the right-hand side of (7.6).

II. As for the contribution from the $r_{k}^{\pm}$’s, we observe that terms for which the factor $\chi_k$ is differentiated offer a factor $f^{-s}$ right away (in fact for any $s$) and we can also bound an additional factor $\langle y \rangle^{2n}$, so again there is agreement with the form of the second term on the right-hand side of (7.6).

III. As for the contribution from the terms of $r_{k}^{\pm}$ for which the factor $\chi_k^{\pm}$ is differentiated is more complicated. We cannot proceed as in Step III of the proof of Lemma 7.1, since now integration by parts using (2.3b) is not legitimate (since $\xi$ is presently less regular). In a region of the form

\[ \{x > 1, |\eta| - \sqrt{2x} > \varepsilon \} \cup \{x < 4\}; \quad \varepsilon > 0, \tag{7.7} \]

we obtain a high power of $\langle (x, \eta) \rangle^{-1}$ by the $\eta$-integration by parts. This power in combination with the growing factor $\langle y \rangle^{s'/2}$, $s'/2 = 2n + d$, can be bounded by $f^{s'}$. This leads us to writing the contribution from any term given by first localizing to (7.7) as $f^{s'} \varphi^{\pm}$ with $\varphi^{\pm} \in \mathcal{H}$, and therefore in turn as

\[ f^{s'} \varphi^{\pm} = f^{s'} \chi_{-\varepsilon}(\pm a) \varphi_{1}^{\pm} + f^{s'} (1 - \chi_{-\varepsilon}(\pm A)) \varphi_{1}^{\pm} \quad \text{with} \quad \varphi_{1}^{\pm} = \varphi^{\pm} \in \mathcal{H}. \tag{7.8} \]

The first term agrees with the first term on the right-hand side of (7.6), so it remains to show that the second term agrees with the second term on the right-hand side of (7.6). For the latter task, we observe that if we replace $A$ by its Weyl symbol $a_{W}$ ($= a = \frac{\eta + \mu \cdot \xi}{2}$ for $x + \langle y \rangle > 1$), then at this rough symbolic level obviously

\[ (1 - \chi_{-\varepsilon}(\pm a_{W})) f^{-s'} \chi_{k} \chi'(\pm a > -\varepsilon) = 0. \tag{7.9} \]

We are discussing the case where $\chi_{\xi}^{\pm}$ is differentiated and the prime for the third factor $\chi$ denotes the derivative of the function. Terms with the double derivative $\chi''(\pm a > -\varepsilon)$ can be treated similarly as below. We may move the factor $1 - \chi_{\varepsilon}(\pm A)$ inside the integrals pass the exponential $e^{i\theta \lambda}$ and then replace the operator by its symbol (which should be legitimate to leading order) and finally conclude by (7.9). However, there are ‘errors’ due to $(x, y)$-dependence of the given symbols. We implement a version of this scheme below.

Pick $\chi \in C_{c}^{\infty}(\mathbb{R})$ with $\chi(t) = 1$ on $\text{supp} \chi'(\cdot > -\varepsilon)$ but $\chi(t) = 0$ on $\text{supp}(1 - \chi_{-\varepsilon})$. Take an almost analytic extension $\tilde{\chi} \in C_{c}^{\infty}(\mathbb{C})$ of $\chi$, and set

\[ d\mu_{\chi}(z) = \pi^{-1}(|\partial \tilde{\chi}|)(z) du dv; \quad z = u + iv. \]

Then,

\[ \chi(t) = \int_{\mathbb{C}} (t - z)^{-1} d\mu_{\chi}(z); \quad t \in \mathbb{R}. \]
In particular
\[(1 - \chi_{-\varepsilon}(\pm A)) f^{-s'} \chi_k \chi'(\pm a > -\varepsilon)\]
\[= (1 - \chi_{-\varepsilon}(\pm A))(\chi(\pm a) - \chi(\pm A)) f^{-s'} \chi_k \chi'(\pm a > -\varepsilon)\]
\[= \pm \int_C (1 - \chi_{-\varepsilon}(\pm A))(\pm A - z)^{-1}((A - a)f^{-s'} \chi_k)(\pm a - z)^{-1} \chi'(\pm a > -\varepsilon)\vphantom{f^{-s'}}\]
\[\times d\mu_{\chi}(z).\vphantom{f^{-s'}}\]

We insert this formula in the expression for \(\varphi^\pm\) for those terms with a single derivative of \(\chi(\pm \cdot > -\varepsilon)\) (the one with a double derivative can be treated similarly). Then, we move the middle factor \((A - a)f^{-s'} \chi_k\) to the far right, in particular pass the exponential \(e^{\theta_k}\). This produces altogether an extra factor \(f^{-1}\) since all derivatives (i.e., components of \(p\) applied to functions) are bounded except when passing through the exponential where a cancellation occurs. Repeating this procedure, we gain a large power of \(f^{-1}\), in particular a factor \(f^{-s'-s'}\), which allows us to conclude that the second term of (7.8) agrees with the second term on the right-hand side of (7.6).

IV. For a localized term in the region of the form \(\{x > 2, ||\eta| - \sqrt{2x}| < 2\varepsilon\}\), which remains to be treated, the \(\eta\)-integration by parts in the beginning of Step III does not work. To get a weight like \(\langle x \rangle^{-j}\) (and therefore \(\langle (x, \eta) \rangle^{-j}\)) to insure the Hilbert space bound, we simply bound the \(j\)-th power of \(\langle x \rangle\) by the same power of \(f^2\), yielding the desired inverse power of \(\langle x \rangle\). Here \(j = 3 + n\) suffices, and with the next argument of Step III we conclude (7.8) with \(s' = 2j + 4n + 2d\). Then, we mimic the last part of Step III.

V. Clearly the \(\varphi^\pm_1\) resulting from the combination of Steps III and IV above is independent of \(s\), and our arguments lead in all cases to \(H\)-norm bounds with a dependence on \(\xi'\) only through a factor of \(\int |\xi'| d\zeta\), and therefore by the Cauchy-Schwarz inequality in turn through a factor of \(R^{(d-1)/2}||\xi'||\). \(\square\)

**Corollary 7.3.** For all \(\xi \in C^\infty_c(\mathbb{R}^d-1)\) the vector \(S(\lambda)\xi \in C^\infty(\mathbb{R}^d-1)\).

**Proof.** Let \(\xi = \xi_+ \in C^\infty_c(\mathbb{R}^d-1)\) be given. Let \(\xi_+ = (p\xi)^{2n} \xi'_+\) be given as in Lemma 7.2.

I. We look at the first term \(\langle \psi_{m,\varepsilon}^\lambda [\xi_+], R(\lambda + i0)\psi_{m,\varepsilon}^\lambda [\xi_-]\rangle\) of (7.3). By combining the representations of \(\psi_{m,\varepsilon}^\lambda [\xi_+]\) from Lemma 7.2 in combination with (5.3a)–(5.3c) it follows by the Sobolev embedding theorem [9, 4.5.13] that indeed the contribution to \(S(\lambda)\xi_-\) from the first term of (7.3) is smooth. This argument only uses the weak input \(\xi_- \in C^\infty_c(\mathbb{R}^d-1)\). More generally it works with also \(\xi_- = (p\xi)^{2n} \xi'_-\) given as in Lemma 7.2 for an arbitrary \(n \in \mathbb{N}_0\).

II. The arbitrary power decay of \(\psi_{m,\varepsilon}^\lambda [\xi_-]\) from Lemma 7.1 and repeated integration by parts (using (2.3a)) in the integral \(\phi_{m,\varepsilon}^\lambda [\xi_+]\) yield that also the second term of (7.3) contributes by a smooth term to \(S(\lambda)\xi_-\). Note that we can estimate the term by \(C ||\xi'_+||_\Sigma\) and consequently again invoke the Sobolev embedding theorem. \(\square\)
We noted in Step I in the above proof that the smoothness of $\xi_-(\in C^\infty_c(\mathbb{R}^{d-1})$ was not used for that part of the proof. We could have assumed $\xi_-(\in \mathcal{E}'_{d-1}$ only and concluded that the corresponding contribution to $S(\lambda)\xi_- \in C^\infty(\mathbb{R}^{d-1})$. In fact it follows readily (for example, by using the left Kohn–Nirenberg quantization discussed below) that the first term of (7.3) is represented by a smoothing operator in the sense of (1.3), whence the local singularities of the kernel $S(\lambda)(\zeta, \zeta')$ appear in the second term of (7.3) only. This term is an explicit oscillatory integral. We have

\[-2\pi i \langle \phi_{m,\epsilon}^{\lambda} [\xi], \psi_{m,\epsilon}^{\lambda-} [\xi'] \rangle = \frac{-i}{(2\pi)^d} \int dx dy \int d\zeta \bar{\xi}(\zeta) \int e^{-i\theta_\lambda a_B^+ d\eta} \int d\zeta' \xi'(\zeta') \times \int e^{i\theta'_\lambda} \left( \mathcal{O}(f^{-\infty}) + \sum_{0}^{\infty} r_k^- \right) d\eta',\]

where the first exponential $e^{-i\theta_\lambda}$ is considered as a function of $(\eta, \zeta)$ (and of $(x, y)$ as well), while the second exponential $e^{i\theta'_\lambda} = e^{i\theta_\lambda}$ is considered as a function of $(\eta', \zeta')$. Of course, the symbols $\mathcal{O}(f^{-\infty})$ and the $r_k^-$’s also depend on the variables $(\eta', \zeta')$, while $a_B^+$ rather depends on $(\eta, \zeta)$. Up to a convergence factor $\chi(||y||/R < 1)$ (with $R \to \infty$), we write the right-hand side as:

\[\int \int \bar{\xi}(\zeta) \bar{S}(\zeta, \zeta') \xi'(\zeta') d\zeta d\zeta',\]

and then in turn $\bar{S}$ as a pseudodifferential operator with corresponding symbol $\check{s}$

\[\check{S}(\zeta, \zeta') = (2\pi)^{-d} \int e^{i(\zeta-\zeta') \cdot y} \check{s}(\zeta, \zeta', -y) dy;\]

\[\check{s}(\zeta, \zeta', y) = (2\pi i)^{-1} \int dx \int e^{-i\varphi_\lambda(x, \eta, \zeta) a_B^+ d\eta} \int e^{i\varphi_\lambda(x, \eta', \zeta')} \left( \mathcal{O}(f^{-\infty}) + \sum_{0}^{\infty} r_k^- \right) d\eta';\]

\[\varphi_\lambda(x, \eta, \zeta) = -\eta^3/6 + (x + \lambda - \zeta^2/2)\eta.\]

**Theorem 7.4.** The scattering operator $S(\lambda)$ is a PsDO of order 0 in the sense of (1.3).

We are going to use Corollary 7.3 (and Step I in its proof) to establish the theorem. For a suitable realization of a symbol $s$ of $S(\lambda)$ (recall that symbols are not uniquely defined), we need to verify the following bounds:

\[\forall \alpha, \alpha', \beta \in \mathbb{N}_0^{d-1} : \]

\[|\partial_\zeta^\alpha \partial_{\zeta'}^\beta \partial_y^\beta s| \leq C_{\alpha, \alpha', \beta} \langle y \rangle^{-|\beta|} \text{ for all } y \text{ and locally uniformly in } \zeta, \zeta'.\]

Due to Step I of the proof of Corollary 7.3 and the subsequent discussion, only $\check{S}$ and a corresponding symbol $\check{s}$ need elaboration. Our realization of $\check{s}$ is...
conveniently given by the left Kohn–Nirenberg symbol $s_{KN}$, implicitly given by

$$
\tilde{S}(\zeta, \zeta') = (2\pi)^{1-d} \int e^{i(\zeta - \zeta') \cdot y} \tilde{s}_{KN}(\zeta, y) \, dy.
$$

(7.11)

**Proof of Theorem 7.4.** I. The contribution from $O(f^{-\infty})$, say with PsDO symbol $s^-$, is a smoothing operator, since we can use the bound $(x + \langle y \rangle)^{-s}$, $s$ large, and $\eta$- and $\eta'$-integration by parts to obtain the following bound of any derivative:

$$
\partial_{\zeta, \zeta'}^\alpha y^s (\zeta, \zeta', y) = O(\langle (\zeta, \zeta', y) \rangle^{-\infty}).
$$

II. We are left with examining the symbol

$$
t = (2\pi)^{-1} \int dx \int e^{-i\varphi_\lambda(x, \eta, \zeta)} \bar{a}_B \chi_\epsilon(\eta - \sqrt{2x}) \, d\eta \int e^{i\varphi_\lambda(x, \eta', \zeta')} \sum_0^\infty r_k^- \, d\eta'.
$$

(7.12)

We shall here bound this expression locally uniformly in $(\zeta, \zeta')$ and uniformly in $y$.

By the non-stationary phase argument (in the variable $(x, \eta, \eta')$), we obtain, computing up to a smoothing operator, that only a localization to \(\{x > R, \eta \approx \eta', \eta \approx \sqrt{2x}\}\) with $R > 1$ big matters. In turn with such localization, we can use the method of stationary phase, cf. Subsect. 2.1 and Appendix A. With reference to the notation of (2.4) and (7.5), it suffices to bound the following expression:

$$(2\pi)^{-1} \int dx \chi_R(x) \int e^{-i\varphi_\lambda(x, \eta, \zeta)} \bar{a}_B \chi_\epsilon(\eta - \sqrt{2x}) \, d\eta \int e^{i\varphi_\lambda(x, \eta', \zeta')} \chi_\epsilon(\eta' - \sqrt{2x}) \chi_{-\epsilon}(\eta' - \sqrt{2x}) \chi_{-\epsilon}(\eta' - \sqrt{2x}) \sum_0^\infty r_k^- \, d\eta'.
$$

Note that the functions $\chi_\epsilon$ and $\chi_{-\epsilon}$ are very different: By definition, $\chi_\epsilon$ is supported in $(-2\epsilon, 2\epsilon)$, while $\chi_{-\epsilon}$ is supported in $(-2\epsilon, -\epsilon/4)$. In particular we can for free insert the localization factor

$$
\chi(x, y; \epsilon) := \chi(\sqrt{2x}/\sqrt{2 \langle y \rangle} < 5\epsilon) \chi(\sqrt{2x}/\sqrt{2 \langle y \rangle} > \epsilon/9)
$$

provided $\epsilon, \epsilon > 0$ are chosen sufficiently small (which we can assume).

We introduce the ‘large parameter’ $h^{-1} = \sqrt{2 \langle y \rangle}$, make the change of variable $x \rightarrow h^{-2}x$ and write

$$
\varphi_\lambda(= \varphi_\lambda(x, \eta, \zeta)) = h^{-1} \bar{\varphi}_\lambda \quad \text{and} \quad \varphi'_\lambda(= \varphi_\lambda(x, \eta', \zeta')) = h^{-1} \bar{\varphi}'_\lambda.
$$

The integration in the new $x$ is over a compact interval due to the introduced factor $\chi(\sqrt{2x} < 5\epsilon) \chi(\sqrt{2x} > \epsilon/9)$ and the double $(\eta, \eta')$-integral can be estimated by the stationary phase method, cf. Appendix A. We skip the details of proof at this point noting that one may mimic Appendix A interchanging the roles of $x$ and $y$. This leads to the bound $h^{-2}O(h)O\left(\sum_0^\infty r_k^-=O(h^0),\right.$ i.e., uniform boundedness holds (locally only in $(\zeta, \zeta')$) as desired.
It remains to bound derivatives of the above symbol \( t \) of (7.12). Here we show the following weaker bounds.

\[
\forall \alpha, \alpha', \beta \in \mathbb{N}_0^{d-1} : \\
\left| \partial_\zeta^\alpha \partial_{\zeta'}^\alpha' \partial_y^\beta t \right| \leq C_{\alpha, \alpha', \beta} \langle y \rangle^{\left| \alpha \right|/2 + 1/2 - \left| \beta \right|} \quad \text{for all } y \text{ and locally uniformly in } \zeta, \zeta'.
\]

(7.13)

For that we first compute the derivatives by differentiating inside the integrals, and then, we mimic Step II invoking the bounds (7.1). Since the phases \( \varphi_\lambda \) and \( \varphi_\lambda' \) are independent of \( y \), derivatives in \( y \) conform with (7.13). However, derivatives in \( \zeta \) and \( \zeta' \) are not so good as (7.1) indicates. The reason is that the phases \( \varphi_\lambda \) and \( \varphi_\lambda' \) have a dependence on these variables, and we can only bound like

\[
\partial_\zeta^\alpha \varphi_\lambda = O(\langle \eta \rangle) \quad \text{and} \quad \partial_\zeta' \varphi_\lambda' = O(\langle \eta' \rangle) \quad \text{for } \left| \alpha \right| \geq 1.
\]

Effectively, cf. Step II, \( O(\langle \eta \rangle) = O(\langle y \rangle^{1/2}) \) and \( O(\langle \eta' \rangle) = O(\langle y \rangle^{1/2}) \), whence (7.13) follows.

IV. The left Kohn–Nirenberg symbol \( t_{\text{KN}} \), cf. (7.11), is obtained from \( t \) by the formula

\[
t_{\text{KN}}(\zeta, y) = e^{i p_{\zeta'} \cdot p_y t}(\zeta, \zeta', y) |_{\zeta' = \zeta},
\]

cf. [10, Theorems 18.4.10 and 18.5.10] (note that this symbol is conveniently represented in momentum space, cf. Appendix A), whence (formally)

\[
t_{\text{KN}}(\zeta, y) = (2\pi)^{-d} \int dx \int d\eta e^{-i \varphi_\lambda(x, \eta, \zeta)} e^{i p_{\zeta'} \cdot p_y \left( \frac{a_B}{a} \int e^{i \varphi_\lambda(x, \eta', \zeta')} \left( O(f^{-\infty}) + \sum_{0}^{\infty} r_{k}^{-} \right) d\eta' \right) |_{\zeta' = \zeta}}.
\]

However, we only need the formula with the cutoffs as in Step II. The point is that when expanding the exponential, although when \( p_{\zeta'} \) hits the factor \( e^{i \varphi_\lambda(x, \eta', \zeta')} \) a ‘growing’ factor \( \eta' \) is introduced which effectively counts for a factor \( f \), but the accompanying factor \( p_y \) effectively counts for a factor \( f^{-2} \).

This means that the terms in the expansion of \( e^{i p_{\zeta'} \cdot p_y} \) effectively have decreasing order. Truncating the series leads to a more complicated integrand; however, the variable \( \zeta' \) has disappeared and now the phase factors enter as

\[
e^{-i \varphi_\lambda(x, \eta, \zeta)} e^{i \varphi_\lambda(x, \eta', \zeta')}.
\]

The \( \zeta \)-derivatives applied to this product are accompanied by factors of powers of \( \eta - \eta' \) which can be written as powers of \( p_x \) applied to the same product. Integrating by parts in \( x \) then effectively gives inverse powers of \( f \). In particular, the symbol does not become any worse when differentiating with respect to \( \zeta \), as we wish. We can now improve (7.13), using the stationary phase method of Step II to bound derivatives as in (7.10).

\[
\square
\]

Remark. A closer examination of Step IV above shows the slightly stronger assertion on the symbol \( s = s_{\text{KN}} \):

\[
\forall \alpha, \beta \in \mathbb{N}_0^{d-1} : \\
\left| \partial_\zeta^\alpha \partial_y^\beta s \right| \leq C_{\alpha, \beta} \langle y \rangle^{-\left| \alpha \right|/2 - \left| \beta \right|} \quad \text{for all } y \text{ and locally uniformly in } \zeta.
\]

(7.14)
From the fact that \( S(\lambda) \) is a PsDO we deduce the following result (by a general argument).

**Corollary 7.5.** The kernel \( S(\lambda)(\zeta, \zeta') \) is smooth away from the diagonal \( \{\zeta = \zeta'\} \).

### 8. The Kernel of the Scattering Matrix at the Diagonal

We will derive yet another representation of the scattering matrix. This will be more suitable for computing singularities at the diagonal of its kernel. We do an analysis of the latter problem in Subsect. 8.3.

#### 8.1. Subtracting the \( \delta \)-singularity at the Diagonal

Let \( m \) and \( \varepsilon \) be as in Sect. 7. We shall use notation of (4.3), (6.4), (6.5) and the quantity

\[
F_{m,\varepsilon}^\pm(\lambda)^* \xi := c \int d\zeta \xi(\zeta) \int e^{i\theta_\lambda} (a_B^\pm + \chi_1 \chi_\varepsilon^{\perp \pm}) \, d\eta;
\]

\[\chi_\varepsilon^\pm = \chi(\pm a > -\varepsilon),\]
\[\chi_\varepsilon^{\perp \pm} = \chi^{\perp}(\pm a > -\varepsilon); \quad a = \frac{\eta + \hat{b}_m \zeta}{{\ell}}.\]  \hspace{1cm} (8.1)

Recall that \( \chi_1 = \chi(x + \langle y \rangle > C_1) \) and \( \chi_\varepsilon^\pm = \chi(\pm a > -\varepsilon) \) enter in the definition (6.5). Note in comparison with (7.1) the appearance of the term \( \chi_1 \chi_\varepsilon^{\perp \pm} \) (although the left-hand side notation for convenience is the same).

We calculate

\[
(H - \lambda)F_{m,\varepsilon}^\pm(\lambda)^* \xi = -c \int d\zeta \xi(\zeta) \int e^{i\theta_\lambda} \left( \mathcal{O}(f^{-\infty}) + \sum_0^{\infty} r_k^\pm + r_\varepsilon^\perp \pm \right) \, d\eta;
\]

\[r_k^\pm = i b_k^\pm (\chi_k \partial_\eta \chi_\varepsilon^\pm + (\eta, \zeta) \cdot \nabla_{(x,y)}(\chi_k \chi_\varepsilon^\pm))
+ (\nabla_{(x,y)}b_k^\pm) \cdot \nabla_{(x,y)}(\chi_k \chi_\varepsilon^\pm) + \frac{b_k^\pm}{2} \Delta_{(x,y)}(\chi_k \chi_\varepsilon^\pm),
\]

\[r_\varepsilon^{\perp \pm} = i \left( \chi_1 \partial_\eta \chi_\varepsilon^{\perp \pm} + (\eta, \zeta) \cdot \nabla_{(x,y)}(\chi_1 \chi_\varepsilon^{\perp \pm}) + \frac{1}{2} \Delta_{(x,y)}(\chi_1 \chi_\varepsilon^{\perp \pm}) - q \chi_1 \chi_\varepsilon^{\perp \pm} \right).\]  \hspace{1cm} (8.2)

These formulas simplify as

\[
a_B^\pm + \chi_1 \chi_\varepsilon^{\perp \pm} = 1 + \sum_1^{\infty} \chi_k \chi_\varepsilon^{\perp \pm} b_k^\pm + \mathcal{O}(f^{-\infty}),
\]

\[r_B^\pm := \sum_0^{\infty} r_k^\pm + r_\varepsilon^{\perp \pm} = -q \chi_1 \chi_\varepsilon^{\perp \pm} + \sum_1^{\infty} r_k^\pm + \left( 1 + \frac{|\eta|}{{\ell}} \right) \mathcal{O}(f^{-\infty}).\]
We compute, using (6.4) and (B.1)–(B.3),
\[
a_B^\pm = 1 - \chi_1 \lambda_\pm + \mathcal{O}(f^{-2\delta}) = \mathcal{O}(f^0),
\]
\[
r_B^\pm = \left(1 + \frac{|\eta|}{T}\right)\mathcal{O}(f^{-2\delta} \min (f^2, \langle y \rangle)^{-1/2}),
\]
\[
r_B^\pm := r_B^\pm - i\tilde{B} = \left(\chi_1 \partial_\eta \lambda_\pm + (\eta, \zeta) \cdot \nabla_{(x,y)}(\chi_1 \lambda_\pm)\right) + q \chi_1 \lambda_\pm
\]
\[
= \left(1 + \frac{|\eta|}{T}\right)\mathcal{O}(f^{-4\delta} \min (f^2, \langle y \rangle)^{-1/2}) + \mathcal{O}(f^{-2\delta} \min (f^2, \langle y \rangle)^{-2}).
\]

We may also compute derivatives and conclude that \(r_B^\pm\) fulfill the second bound of (7.1) with an additional factor \(f^{-2\delta}\). We may also derive similar (slightly stronger) bounds for derivatives of \(r_B^\pm\).

Parallel to Sect. 7, we introduce
\[
\phi^{\lambda,\pm}_{m,\varepsilon}[\xi] = F_{m,\varepsilon}^\pm(\lambda)^* \xi \quad \text{and} \quad \psi^{\lambda,\pm}_{m,\varepsilon}[\xi] = (H - \lambda) \phi^{\lambda,\pm}_{m,\varepsilon}[\xi],
\]
and note the following formulas for the generalized eigenfunctions of (3.16a), cf. (2.7),
\[
\phi^{\lambda,\pm}_{\text{ex}}[\xi] = \phi^{\lambda,\pm}_{m,\varepsilon}[\xi] - R(\lambda \mp i0)\psi^{\lambda,\pm}_{m,\varepsilon}[\xi]; \quad \xi \in C_c^\infty(\mathbb{R}^{d-1}).
\]

Since \(F^-(\lambda)^* \xi = \phi^{\lambda,\pm}_{\text{ex}}[\xi]\) (cf. (3.16b)), Theorem 3.3 then leads to
\[
T(\lambda)\xi := S(\lambda)\xi - \xi = -i2\pi F^+(\lambda) \psi^{\lambda,\pm}_{m,\varepsilon}[\xi].
\]

Using again (3.16b) and (8.4), we conclude the following analogue of (3.19) and (7.3),
\[
\frac{1}{2\pi^1} \langle \xi, T(\lambda)\xi \rangle = -\langle F^+(\lambda)^* \xi, \psi^{\lambda,\pm}_{m,\varepsilon}[\xi'] \rangle
\]
\[
= \langle \psi^{\lambda,\pm}_{m,\varepsilon}[\xi], R(\lambda + i0) \psi^{\lambda,\pm}_{m,\varepsilon}[\xi'] \rangle - \langle \phi^{\lambda,\pm}_{m,\varepsilon}[\xi], \psi^{\lambda,\pm}_{m,\varepsilon}[\xi'] \rangle.
\]

8.2. The Leading Order Symbol of \(T(\lambda)\)

The operator \(T(\lambda)\) is a pseudodifferential operator of order \(-\delta\), meaning that its kernel can be written as
\[
T(\zeta, \zeta') = (2\pi)^{1-d} \int e^{i(\zeta - \zeta') \cdot y} t(\zeta, y) dy,
\]
where locally uniformly in \(\zeta\) (possibly locally uniformly \(\lambda\) as well)
\[
\partial_\zeta^\alpha \partial_y^\beta t = \mathcal{O}(\langle y \rangle^{-\delta-|\beta|}).
\]

This definition is consistent with (1.3) by the theory of PsDOs, cf. Step IV of the proof of Theorem 7.4.

**Theorem 8.1.** The operator \(T(\lambda)\) has order \(-\delta\). The principal symbol of \(T(\lambda)\) is given by \(t_{\text{psym}}\) in the sense that \(T(\lambda) - T_{\text{psym}}\) has order \(-2\delta\); here \(T_{\text{psym}}\) denotes the quantization of \(t_{\text{psym}} = t_{\text{psym}}(y): = -2i\int_0^\infty q_1(x-y) \frac{1}{\sqrt{2\pi}} dx\).

**Proof.** I. Due to Step I of the proof of Corollary 7.3 (including a trivial modification of Lemma 7.2) and the subsequent discussion, the first term on the right-hand side of (8.5) is represented by a smoothing operator.
II. As for the second term we write
\[ -2\pi i \langle \phi_{m,\epsilon}^{\lambda+}[\xi], \psi_{m,\epsilon}^{\lambda-}[\xi'] \rangle = \frac{-i}{(2\pi)^d} \int \int d\xi \int \int d\xi' \]
\[ \int e^{-i\theta \lambda}(a_B^+ + \chi_1 \lambda^{\frac{1}{+}}) d\eta \int d\xi' \xi' \xi(\eta') \int e^{i\theta \lambda}(\Omega(f^{-\infty}) + r_B^-) d\eta', \]

The contribution from the term $\Omega(f^{-\infty})$ is represented by a smoothing operator, since we can use the bound $(x + \langle y \rangle)^{-s}$, $s$ large, and $\eta$- and $\eta'$- integration by parts to bound any derivative,
\[ \partial_{\xi,\xi'}^\alpha, y^-(\xi, \xi', y) = \Omega((\xi, \xi', y))^{-\infty} \]
for the corresponding symbol $t^-$, cf. Step I of the proof Theorem 7.4.

III. The ‘leading order’ contribution is by (8.3) given by
\[ -2\pi i \langle \phi_{m,\epsilon}^{\lambda+}[\xi], \psi_{m,\epsilon}^{\lambda-}[\xi'] \rangle \approx \frac{-i}{(2\pi)^d} \int \int d\xi \int \int d\xi' \xi(\eta') \int e^{i\theta \lambda}(a_B^+ + \chi_1 \lambda^{\frac{1}{+}}) d\eta \int d\xi' \xi' \xi(\eta') \int e^{i\theta \lambda}(\Omega(f^{-\infty}) + r_B^-) d\eta'. \]

We may treat the contribution from the error from the above approximation by mimicking Steps II–IV of the proof Theorem 7.4. It is represented by a PsDO of order $-2\delta$, as wanted. Note that indeed the same cut-off functions as in Step II in the proof of Theorem 7.4 apply.

However, there is a somewhat alternative treatment based on the van der Corput lemma, cf. [21, p. 332], and more related to Subsect 2.1. Since we need it below, this method is explained here. We consider the PsDO
\[ \tilde{R}(\xi, \xi') = (2\pi)^{-d} \int e^{i(\xi - \xi') \cdot y} \tilde{r}(\xi, \xi', -y) d\eta; \]
\[ \tilde{r}(\xi, \xi', y) = (2\pi)^{-1} \int dx \int e^{-i\varphi_{\lambda}(x, \eta, \xi)}(a_B^+ + \chi_1 \lambda^{\frac{1}{+}}) d\eta \int e^{i\varphi_{\lambda}(x, \eta', \xi')} r_B^- d\eta'; \]
\[ \varphi_{\lambda}(x, \eta, \xi) = -\eta^3/6 + (x + \lambda - \xi^2/2)\eta. \]
Here $\tilde{r}_B^-$ is the exact error from (8.3). Proceeding as in Step II of the proof Theorem 7.4 we can freely (i.e., up to a term representing a smoothing operator) insert the factors
\[ \chi_{\epsilon}(\cdot) := \chi_\epsilon(\eta' - \sqrt{2}x) \quad \text{and} \quad \chi_{-\epsilon}(\cdot) := \chi_{-\epsilon}(\cdot - \frac{\eta'}{\sqrt{2}x + 2\langle y \rangle}) \]
in the $\eta'$-integral, $\chi_{\epsilon}(\eta - \sqrt{2}x)$ in the $\eta$-integral and finally $\chi_{R}(x)$ and $\chi(x, y; \epsilon)$ in the $x$-integral. As before $R > 2$ is large and $\epsilon > 0$ small. Next we introduce $h^{-1} = \sqrt{2}x$ and write
\[ \varphi_{\lambda} = \varphi_{\lambda}(x, \eta, \xi) = h^{-1} \varphi_{\lambda} \quad \text{and} \quad \varphi'_{\lambda} = \varphi_{\lambda}(x, \eta', \xi') = h^{-1} \varphi'_{\lambda}. \]
To treat the $\eta'$-integral we rewrite it as
\[ \int e^{i\varphi_{\lambda}(x, \eta', \xi')} \tilde{r}_B^- \chi_{\epsilon}(\cdot) \chi_{-\epsilon}(\cdot) d\eta' = \int \left( \frac{d}{d\eta} \int_{\eta'}^{\eta} e^{i\varphi_{\lambda}(x, s, \xi')} ds \right) \tilde{r}_B^- \chi_{\epsilon}(\cdot) \chi_{-\epsilon}(\cdot) d\eta', \]
and integrate by parts. By the van der Corput lemma
\[ \left| \int_{\sqrt{2x}}^{\eta'} e^{ih^{-1} \phi} \, ds \right| \leq C \sqrt{h} \]
with a universal constant $C$. We proceed similarly for the $\eta$-integral. With the bounds on $\tilde{r}^{-}_{B}$ and $a^{+}_{B}$ from the remark below (8.3) we then obtain, thanks to the factor $\chi(x, y; \epsilon)$ and (6.3),
\[ |\tilde{r}(\zeta, \zeta', y)| \leq C_{1} \int \chi_{R}(x) f^{-(4\delta + 1)} x^{-1/2} \, dx \leq C_{2} \langle y \rangle^{-2\delta} \quad (8.7) \]
with locally bounded constants (in $(\zeta, \zeta')$). Derivatives can be treated as in Step IV of the proof Theorem 7.4. So in conclusion, $\tilde{R}$ is a PsDO of order $-2\delta$.

IV. We show that the ‘leading order’ contribution in Step III given by the symbol $r^{-}_{B} - \tilde{r}^{-}_{B}$ is represented by a PsDO of order $-\delta$.

First we compute, cf. (6.2),
\[ r^{-}_{B} - \tilde{r}^{-}_{B} = -ib_{1} \left( \chi_{1} \partial_{\eta} \chi_{\epsilon} + (\eta', \zeta') \cdot \nabla_{(x, y)} (\chi_{1} \chi_{\epsilon}) \right) + q \chi_{1} \chi_{\epsilon} = q \chi_{1} - i(\partial_{\eta'} + (\eta', \zeta') \cdot \nabla_{(x, y)}) (b_{1} \chi_{1} \chi_{\epsilon}) . \]
The second term contributes only by a term of order $-2\delta$, which may be seen as follows. By an $\eta'$-integration by part using (4.4) we may effectively replace
\[ -ie^{i\theta} (\partial_{\eta'} + (\eta', \zeta') \cdot \nabla_{(x, y)}) (b_{1} \chi_{1} \chi_{\epsilon}) \approx (H_{0} - \lambda) (e^{i\theta} b_{1} \chi_{1} \chi_{\epsilon}) \]
with error
\[ \frac{1}{2} e^{i\theta} p^{2} (b_{1} \chi_{1} \chi_{\epsilon}) = \mathcal{O} \left( f^{-2\delta} \min \left( \frac{f^{2}}{4}, \langle y \rangle \right)^{-\frac{1}{2}} \right) , \]
cf. (8.3), and for this error term we can proceed as in Step III. Now we take the factor $(H_{0} - \lambda)$ to the left in the integral (by integration by parts) thereby producing the factor $(H_{0} - \lambda) (e^{i\theta} (a^{+}_{B} + \chi_{1} \chi_{\epsilon}))$. By the same argument, now for the $\eta$-integration, we have
\[ (H_{0} - \lambda) (e^{i\theta} (a^{+}_{B} + \chi_{1} \chi_{\epsilon})) \approx -q e^{i\theta} a^{+}_{B} + \left( 1 + \frac{|\eta|}{T} \right) \times \mathcal{O} \left( f^{-2\delta} \min \left( \frac{f^{2}}{4}, \langle y \rangle \right)^{-1/2} \right) , \]
and we can proceed as in Step III for the second term obtaining again a term of order $-2\delta$. However, the first term $-q e^{i\theta} a^{+}_{B}$ is missing a support property (used for the second one), preventing us from using the factor $\chi(x, y; \epsilon)$, whence we need to consider the localization factors $\chi_{\epsilon}(-\eta - \sqrt{2x})$ and $\chi_{\epsilon}(-\eta' - \sqrt{2x})$, as well as $\chi_{\epsilon}(\eta - \sqrt{2x})$ and $\chi_{\epsilon}(\eta' - \sqrt{2x})$ used before. Since $b_{1} \chi_{1} \chi_{\epsilon} = \mathcal{O} (f^{-2\delta})$ and $q a^{+}_{B} = \mathcal{O} (f^{-1-2\delta})$ we may bound the symbol corresponding to the term $-q e^{i\theta} a^{+}_{B}$, with the localizations factors in place, exactly as in (8.7).

For the first term $q \chi_{1}$, we can invoke the first identity of (8.3) and replace the factor $(a^{+}_{B} + \chi_{1} \chi_{\epsilon})$ by 1 with an error corresponding to a PsDO of order
\(-2\delta\). This is justified as we treated \(-q\epsilon^{i\lambda}a_{R}^{\pm}\) above, whence we end up with the ‘Born term’
\[
\tilde{T}(\zeta, \zeta') = (2\pi)^{1-d} \int e^{i(\zeta - \zeta') \cdot y} \tilde{t}(\zeta, \zeta', -y) \, dy;
\]
\[
\tilde{t}(\zeta, \zeta', y) = (2\pi)^{-1} \sum_{\pm} \int dx \chi_R(x)q(x, y)
\int e^{-i\varphi(x, \eta, \zeta)} \chi_{\epsilon}(\pm \eta - \sqrt{2}x) \, d\eta \int e^{i\varphi(x, \eta', \zeta')} \chi_{\epsilon}(\pm \eta' - \sqrt{2}x) \, d\eta'.
\]
The \(\eta\)- and \(\eta'\)-integrals essentially represent Airy functions, and their asymptotics can be obtained by the stationary phase method, for example by a simplified version of Appendix A. Alternatively we may invoke [9, (7.6.21)]. Anyhow we conclude that \(\tilde{T}\) (and therefore also \(T(\lambda)\)) is a PsDO of order \(-\delta\) with leading order symbol given in terms of any big \(R > 1\) (making \(\eta = \sqrt{2}x + 2\lambda - \zeta^2\) well defined) by
\[
\tilde{t}_{KN}(\zeta, y) \approx -4i \int_{R}^{\infty} \frac{q(x, -y)}{\sqrt{2x + 2\lambda - \zeta^2}} \sin^2(\cdot) \, dx \approx -2i \int_{0}^{\infty} \frac{q_1(x, -y)}{\sqrt{2x}} \, dx = t_{psym}(y).
\]
The quantization of the error is of order \(-2\delta\).

By the definition (1.3) and according to Theorem 8.1, the corresponding symbol \(t - t_{psym}\) fulfills bounds (with derivatives) that are locally uniform in \(\zeta\). We remark that from the above proof one easily deduce that these bounds are also (in fact simultaneously) locally uniform in \(\lambda\). The operator \(T_{psym}\) has order \(-\delta\) and in general no better, see Subsect. 8.3.

### 8.3. Analysis of the Kernel of \(T(\lambda)\) at the Diagonal

We compute the top order singularity of the kernel of \(T(\lambda)\) at the diagonal for a class of slowly decaying potentials. This is done by combining Theorem 8.1 with [13, Lemma 4.1].

For a homogeneous potential \(q \approx \kappa r^{-\alpha},\ 1/2 < \alpha < d - 1/2\), we can compute using (6.3)
\[
\int_{0}^{\infty} \frac{q(x, -y)}{\sqrt{2x}} \, dx \approx \kappa c_1 |y|^{\frac{1}{2} - \alpha} \text{ for } |y| \to \infty,
\]
with
\[
c_1 = 2^{-3/2} \int_{0}^{\infty} (t + 1)^{-\alpha/2} t^{-3/4} \, dt = 2^{-3/2} \Gamma(1/4) \Gamma(\alpha/2 - 1/4) / \Gamma(\alpha/2),
\]
cf. [6, (13.2.5), (13.5.10)]. The Fourier transform of \(|y|^{\frac{1}{2} - \alpha}\) is known, whence, cf. [13, Lemma 4.1], in this case the top order singularity of the kernel of \(T(\lambda)\) is given by
\[
T(\zeta, \zeta') \approx \kappa c_2 |\zeta - \zeta'|^{\frac{1}{2} + \alpha - d};
\]
\[
c_2 = c_1 (2\pi)^{1-d} (-2i)(2\pi)^{(d-1)/2} 2^{d/2 - \alpha} \Gamma(\alpha - 1/2) / \Gamma(\alpha/2)
\]
\[
= -i(2\pi)^{(1-d)/2} 2^{(d-1)/2 - \alpha} \Gamma(1/4) \Gamma(\alpha - 1/2) / \Gamma(\alpha/2).
\]
For the Coulomb potential $q = \kappa r^{-1}$ with $d \geq 3$ the order of $T_{\text{psym}}$ is $-1/2$, and the singularity (at the diagonal) is the form

$$T(\zeta, \zeta') \approx \kappa c_2 |\zeta - \zeta'|^{3/2-d}.$$  \hspace{1cm} (8.8)

The error is in this case of order $O(|\zeta - \zeta'|^{2-d})$, cf. [13, Lemma 4.1], whence we can summarize as follows.

**Corollary 8.2.** For $q = \kappa r^{-1}$ and $d \geq 3$ the kernel

$$S(\lambda)(\zeta, \zeta') - \delta(\zeta, \zeta') = \kappa c_2 |\zeta - \zeta'|^{3/2-d} + O(|\zeta - \zeta'|^{2-d})$$

at the diagonal, locally uniformly in $\zeta$, $\zeta'$ and $\lambda$.

For $q = \kappa r^{-1}$ and $d = 3$, we can use that

$$\Gamma(1/4)\Gamma(d/2 - 1/4 - \alpha/2) = \sqrt{2\pi} \quad \text{and} \quad \Gamma(\alpha/2) = \sqrt{\pi},$$

yielding in that case $c_2 = -i(2\pi)^{-1/2}$ and $T(\zeta, \zeta') \approx -i\kappa(2\pi)^{-1/2}|\zeta - \zeta'|^{-3/2}$. This result agrees with [16], where the exact asymptotics at the diagonal is derived by a different method relying on explicit calculations of integrals involving powers of the potential and the free Stark resolvent kernel. This method seems restricted to homogeneous potentials. In our approach (which is valid for a wider class of potentials), the singularities are ‘sitting’ in an explicit oscillatory integral, which is amenable to analysis. In particular we extracted the top order singularity for homogeneous potentials from the oscillatory integral.

For a partial decomposition of the kernel of $S(\lambda)$ for the Coulomb potential, see [17], although this paper does not study the singularity problem.

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**Appendix A. Proof of (2.6)**

For convenience we only consider $\phi^+_{\lambda, a}[\xi]$ and $\lambda = 0$, whence we need to consider the asymptotics of

$$c \int d\zeta \xi(\zeta) \int e^{ih^{-1} \tilde{q} \tilde{y}} \tilde{a} \chi_\epsilon(\eta - h^{-1}) \chi_\epsilon(|\zeta - hy|) \, d\eta.$$
of a symbol $\tilde{a} = \tilde{a}(x,y;\eta,\zeta)$ obeying (2.2b). Recall that there is exactly one relevant stationary point for $x > R$ and $|y| < C\sqrt{2x}$, cf. (2.5), say denoted $z(h,y) = (\eta^+,\zeta^+)$. This is given by

$$\eta^+ = \sqrt{x + (x^2 - y^2)^{1/2}}$$

and $\zeta^+ = y/\eta^+$.

Let similarly $z = (\eta,\zeta)$. To obtain the asymptotics as $h \to 0$ uniformly in $y$ we write the phase $\theta = h\tilde{\theta}$ as

$$\tilde{\theta} = \tilde{\theta}_{|z=z(h,y)} + \frac{1}{2} \langle \phi, A\phi \rangle,$$

where $A = A(h,y) = \nabla^2 z \tilde{\theta}_{|z=z(h,y)}$ (A.1)

and $\phi$ is a diffeomorphism in $z$ from an open neighborhood $U$ of $z(h,y)$ onto an open neighborhood $V$ of $0$ with $\phi_{|z=z(h,y)} = 0$ and derivative $\nabla z \phi_{|z=z(h,y)} = I$.

The existence of such map $\phi$ follows as in the proof of [11, Lemma 4.2] and the computation

$$\nabla^2 z \tilde{\theta} = -\eta(I + O(\zeta/\eta)),$$

yielding roughly $\nabla^2 z \tilde{\theta} \approx -I$.

Indeed we can introduce $\tilde{\theta}(\tilde{z}) = \tilde{\theta}(z)$ by substituting $z = \tilde{z} + z(h,y)$, write

$$\tilde{\theta}(\tilde{z}) = \tilde{\theta}_{|z=z(h,y)} + \frac{1}{2} \langle \tilde{z}, B(\tilde{z}) \rangle; \quad B = B(\tilde{z}) = 2\int_0^1 (1 - \tau)\nabla^2 z \tilde{\theta}(\tau \tilde{z} + z(h,y)) \, d\tau,$$

and use the inverse function theorem to solve

$$\Phi(\Gamma) := \Gamma A^{-1} \Gamma = B$$

for a unique real symmetric $d \times d$ matrix $\Gamma = \Gamma(\tilde{z}) = \Gamma(z,h,y)$ near $A = A(h,y)$. Note that $\Phi(A) = A = B(0)$. Then, $\phi(z) := A^{-1} \Gamma(z) \tilde{z}$ works in (A.1), in fact with $U = z(h,y) + B_r(0)$ where $B_r(0)$ is the open ball centered at $0$ with radius $r > 0$ being independent of $(h,y)$ (seen conveniently by using [20, Lemma 1.18]). Fix such $r$. One easily checks that $\phi$ has bounded derivatives with bounds being independent of $(h,y)$ (using the same property of $\Gamma$).

The inverse map $\psi: V \to U$ has derivatives which similarly are bounded uniformly in $(h,y)$ (seen inductively by the Faà di Bruno formula). We change variable $z \to \phi$ and write, possibly at this point taking $\epsilon > 0$ smaller and $R = R(\epsilon) > 2$ larger,

$$\int d\zeta \xi(\zeta) \int e^{ih^{-1}\tilde{\theta}} \tilde{a}_\epsilon(\eta - h^{-1}) \chi_\epsilon(|\zeta - hy|) \, d\eta$$

$$= e^{i\tilde{\theta}_{|z=z(h,y)}} \int_V e^{ih^{-1}2^{-1}\langle \phi, A\phi \rangle} f(\phi) \, d\phi;$$

$$f(\phi) = \left((\xi(\zeta)\tilde{a}(\cdot)(\eta - h^{-1})\chi_\epsilon(|\zeta - hy|)) (\psi(\phi)) \right) |\det(\psi')(\phi)|,$$

and compute

$$\forall \alpha: \quad \partial^{\alpha}_\phi f = O((x,y)^0),$$

$$\forall \alpha: \quad ||\partial^{\alpha}_\phi f||_2 = O((x,y)^0),$$
(Note for the latter bounds that $\int 1_V \, d\phi < \infty$.) By the Plancherel theorem and [9, Theorem 7.6.1]
\[
\int e^{ih^{-1}2^{-1} \langle \phi, A \phi \rangle} f(\phi) \, d\phi = h^{d/2} e^{i\pi \text{sgn}(A)/4} |\text{det}(A)|^{-1/2} \times \int_{\mathbb{R}^d} e^{-ih^{-1}(\zeta, A^{-1} \zeta)} \hat{f}(\zeta) \, d\zeta.
\]
By the inversion formula
\[
\int \hat{f}(\zeta) \, d\zeta = (2\pi)^{d/2} f(0).
\]
On the other hand, by using the bound
\[
|e^{-ih^{-1} \langle \zeta, A^{-1} \zeta \rangle} - 1| \leq h^{-1} |\langle \zeta, A^{-1} \zeta \rangle|,
\]
we can estimate for any integer $n > 2 + d/2$
\[
\left| \int_{\mathbb{R}^d} (e^{-ih^{-1} \langle \zeta, A^{-1} \zeta \rangle} - 1) \hat{f}(\zeta) \, d\zeta \right| 
\leq h C_1 \max_{|\alpha| \leq n} ||\partial^\alpha \hat{f}||_2 
\leq C_2 h.
\]
Finally, by invoking $A = -I + \mathcal{O}(h)$ (uniformly in $y$) and (2.5), the asymptotics (2.6) follows. \qed

**Remark A.1.** We used above only the zeroth order Taylor expansion of the Gaussian function of $\zeta$ at zero. If the symbol $\tilde{a}$ vanishes to any order at the stationary point $(\eta^+, \zeta^+)$, then higher order Taylor expansion yields that the integral is $\mathcal{O}(h^\infty)$ rather than $\mathcal{O}(h^{d/2})$ as proved above.

### Appendix B: Borel Construction for (6.5)

We consider $c^\pm := \sum_0^\infty \chi_k b^\pm_k$, where $\chi_k = \chi(f/C_k > 1)$ needs to be determined. Fix $\epsilon > 0$ (for example take $\epsilon = 1$). Thanks to (6.4), we can for any $k \in \mathbb{N}_0$ find a sufficiently big $C_k \geq 2$ such that
\[
|\partial^\alpha \eta, \xi \partial^\beta \partial^\gamma y b^\pm_k| \leq 2^{-k} f^{\epsilon - (2k\delta + |\alpha| + 2|\beta| + 2|\gamma|)} 
\quad \text{for } |\alpha| + |\beta| + |\gamma| \leq k, \quad \pm a > -\epsilon \quad \text{and } f > C_k,
\]
and clearly we can take $C_0 = 2$ and assume that $C_k > 1 + C_{k-1}$ for $k \geq 1$.

Note that for all $l \in \mathbb{N}_0$ there exists $C(l) > 0$ such for all $k \in \mathbb{N}_0$
\[
|\partial^\beta \partial^\gamma_l \chi_k| \leq C(l) f^{-2|\beta|} \min (f^2, \langle y \rangle_m)^{-|\gamma|} \text{ for } |\beta| + |\gamma| \leq l. \tag{B.2}
\]

By combining (B.1) and (B.2) with the product rule we conclude that for all $l \in \mathbb{N}_0$, there exists $\tilde{C}(l) > 0$ such for all $k \geq l$
\[
|\partial^\alpha_{\eta, \xi} \partial^\beta \partial^\gamma_y (\chi_k b^\pm_k)| \leq \tilde{C}(l) 2^{-k} f^{\epsilon - (2k\delta + |\alpha| + 2|\beta|)} \min (f^2, \langle y \rangle_m)^{-|\gamma|} 
\quad \text{for } |\alpha| + |\beta| + |\gamma| \leq l \text{ and for } \pm a > -\epsilon.
\]
By summing up we conclude that $c^\pm$ are well-defined smooth functions in the regions $\{\pm a > -\varepsilon\}$, respectively, with bounds
\[
|\partial_\eta^\alpha \partial_\zeta^\beta \partial_y^\gamma c^\pm| \leq C_{\alpha,\beta,\gamma} f^{-|\alpha|+2\beta} \min \left( f^2, \langle y \rangle_m \right)^{-|\gamma|}; \quad \pm a > -\varepsilon.
\]

Since $a_\pm^B = \chi_\varepsilon^\pm c^\pm$, these bounds and the product rule yields the first bound of (6.6) (however being only locally uniform in $\zeta$). Note at this point that
\[
|\partial_\eta^\alpha \partial_x^\beta \partial_y^\gamma \chi_\varepsilon^\pm| \leq C_{\alpha,\beta,\gamma} f^{-|\alpha|+2\beta} \min \left( f^2, \langle y \rangle_m \right)^{-|\gamma|}; \quad f > C_0. \quad (B.3)
\]

For the second assertion of (6.6), the term $O(f^{-\infty})$ is given explicitly as
\[
O(f^{-\infty}) = \sum_{k=0}^{\infty} \chi_\varepsilon^\pm (\chi_{k+1} - \chi_k) (q b_k^\pm - \frac{1}{2} (\Delta(x,y)b_k^\pm)).
\]

By using (B.1)–(B.3) one easily checks that indeed the right-hand side is bounded along with all derivatives by any inverse power of $f$ with a bounding constant being locally uniform in $\zeta$. This completes the proof of (6.6). The related bounds (7.1) easily follow too.

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