On geometric properties of the functors of positively homogenous and semiadditive functionals

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Abstract

In this paper we investigate the functors of $OH$ of positively homogenous functionals and $OS$ of semiadditive functionals. We show that $OH(X) \in AR$ if and only if $X$ is openly generated, and $OS(X) \in AR$ if and only if $X$ is an openly generated compactum of weight $\leq \omega_1$. In section 3 we investigate the multiplication maps of monads generated by the abovementioned functors and consider when these mappings are soft.

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0. Introduction. V.Fedorchuk posed a general problem concerning geometric properties of functors, that is, how functors affect certain geometric properties of spaces and mappings between them [11]. Under geometric properties we understand the property of being an $AR$ for a space, the properties of being soft or a Tychonov fibering for a mapping etc.

There were many investigations in this direction involving such functors as the hyperspace functor $exp$, the probability measures functor $P$, the superextension functor $\lambda$, the inclusion hyperspace functor $G$ and others (see, e.g. [10] or [11]).

Let us now consider as an example the functors of probability measures $P$ and superextension $\lambda$. There is a natural structure of linear convexity on $P(X)$. As for $\lambda$, de Groot constructed some abstract convexity (not linear) on any space of the form $\lambda(X)$ (see [12]), and this convexity is binary, whereas the linear convexity on $P(X)$ is not.
The functors $\lambda$ and $P$ differ in their geometric properties as well. Consider the property of being an AR, for instance. In the metrizable case, $\lambda(X) \in AR$ if and only if $X$ is a continuum, and $P(X)$ is an absolute retract for each compactum $X$. When $X$ is not metrizable, the space $P(X)$ can be AR only in case $X$ is openly generated and of weight $\leq \omega_1$. As for the superextension functor, $\lambda(X) \in AR$ whenever $X$ is an openly generated continuum, without limitations on weight.

The algebraic aspects of functors are formalized by the notion of a monad in the sense of Eilenberg and Moore [13].

The notion of convexity considered in this paper is considerably broader than the classic one: specifically, it is not restricted to the context of linear spaces. Such convexities appeared in the process of studying different structures like partially ordered sets, semilattices, lattices, superextensions etc. We base our approach on the notion of topological convexity from [14] where the general convexity theory is covered from axioms to application in different areas. T.Radul assigned to each monad $F$ some abstract convexity structure on every space $FX$, where $F$ is the functorial part of the monad $F$. Some additional conditions on these monads (that they are $L$-monads which weakly preserve preimages) guarantee that the considered convexities generate the topology of the space $FX$ for the functor $F$ included in an $L$-monad. It was shown that $L$-monads which weakly preserve preimages and with binary convexities can give absolute retracts in all weights [3]. Also, the morphisms of their algebras can be soft in nonmetrizable case under certain conditions. Note that the property of binarity of the convexity generated by monad $F$ is equivalent to the superextension monad being the submonad of $F$ (again [3]).

In this article we consider functors $OS$ and $OH$ (introduced in [5], [6]), which both generate $L$-monads. The monad $\mathcal{OS}$ does not generate binary convexities, in turn $\mathcal{OH}$ does, and this as well appears to be the reason for the difference in their geometric properties: the properties of $OS$ are close to that of $P$, and $OH$ is closer to $\lambda$.

1. Definitions and facts. In the present paper we shall deal with objects and morphisms of the category $\text{Comp}$, that is, with compact Hausdorff spaces and continuous mappings.

By $C(X)$, where $X \in \text{Comp}$, we denote the Banach space of all continuous real-valued functions on $X$ with the sup-norm $\|\varphi\| = \sup\{|\varphi(x)| \mid x \in X\}$. By $c_X$, where $c \in \mathbb{R}$, we denote the constant function: $c_X(x) = c$ for all $x \in X$.

Let $X \subset Y$. We say that a space $X$ is a retract of $Y$ if there exists a map $r : Y \to X$ such that $r|_X = \text{id}_X$. The space $X$ is an absolute retract (shortly $X \in AR$), if for any embedding
$i : X \hookrightarrow Y$ the subspace $i(X)$ is a retract of $Y$.

Recall that a τ-system, where $\tau$ is any cardinal number, is a continuous inverse system consisting of compacta of weight $\leq \tau$ and epimorphisms over a $\tau$-complete indexing set. As usual, $\omega$ stands for the countable cardinal number. A compactum $X$ is called openly generated, if it can be represented as the limit of some $\omega$-system with open bonding mappings [1].

The mapping $f : X \to Y$ is called soft if for any space $Z$ and its closed subset $A$, any functions $\psi : A \to X$, $\Psi : Z \to Y$ with $\Psi |_A = f \circ \psi$ there is a mapping $G : Z \to X$ such that $G |_A = \psi$ and $\Psi = f \circ G$ [1].

We say that a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{p} & Y \\
\downarrow{q} & & \downarrow{f} \\
Z & \xrightarrow{g} & T
\end{array}
$$

is soft, if its characteristic map $\chi : X \to Y \times_T Z = \{(y, z) \in Y \times Z \mid f(y) = g(z)\}$ defined by $\chi(x) = (p(x), q(x))$ is soft.

A triple $(F, \eta, \mu)$, where $F$ is an endofunctor in category $\text{Comp}$, $\eta : \text{Id}_{\text{Comp}} \to F$ and $\mu : F^2 \to F$ are natural transformations, is called monad (in sense of Eilenberg and Moore), if

1) $\mu X \circ \eta F(X) = \mu X \circ F(\eta X) = \text{id}_{F(X)}$; 2) $\mu X \circ \mu F(X) = \mu F(\mu X)$ [13].

Suppose that $\mathbb{F} = (F, \eta, \mu)$ is a monad. A pair $(X, \xi)$, where $\xi : F(X) \to X$, is called an $\mathbb{F}$-algebra, if $\xi \circ \eta X = \text{id}_X$ and $\xi \circ \mu X = \xi \circ F(\xi)$.

Let $\nu : C(X) \to \mathbb{R}$ be a functional. We say that $\nu$ is: 1) normed, if $\nu(1_X) = 1$; 2) weakly additive, if for any $\phi \in C(X)$ and $c \in \mathbb{R}$ we have $\nu(\phi + c_X) = \nu(\phi) + c$; 3) order-preserving, whenever for any $\varphi, \psi \in C(X)$ such that $\varphi(x) \leq \psi(x)$ for all $x \in X$ (i.e. $\varphi \leq \psi$) the inequality $\nu(\varphi) \leq \nu(\psi)$ holds; 4) positively homogeneous, if for any $\varphi \in C(X)$ and any real $t \geq 0$ we have $\nu(t\varphi) = t\nu(\varphi)$; 5) semiadditive, if $\nu(\varphi + \psi) \leq \nu(\varphi) + \nu(\psi)$.

Now for any space $X$ denote $V(X) = \prod_{\varphi \in C(X)}[\min \varphi, \max \varphi]$. For any mapping $f : X \to Y$ let $V(f)$ be a mapping such that $V(f)(\nu)(\varphi) = \nu(\varphi \circ f)$ for any $\nu \in V(X)$, $\varphi \in C(Y)$. Defined in that way, $V$ forms a covariant functor in the category $\text{Comp}$.

For any space $X$ by $O(X)$ denote the set of functionals satisfying 1–3) (order-preserving functionals), by $OH(X)$ the set of all functionals on $C(X)$ which satisfy properties 1–4) (positively homogenous functionals), and by $OS(X)$ we denote the set of functionals on $C(X)$ which satisfy properties 1–5) (semiadditive functionals). Also recall that $P(X)$ stands for the set of all functionals on $C(X)$ which are normed ($\|\mu\| = 1$), positive $\langle \mu(\varphi) \geq 0 \text{ for all } \varphi \geq 0 \rangle$ and linear. Let $F$ stand for one of $O, OH, OS, P$. The space $F(X)$ is considered as the subspace
of $V(X)$. For any function $f : X \to Y$, the map $F(f) : F(X) \to F(Y)$ is the restriction of $V(f)$ on the corresponding space $F(X)$. Then $F$ forms a covariant functor in $\text{Comp}$, which is a subfunctor of $V$.

It was shown in [5] and [6] that the functor $OS$ is normal, and $OH$ is weakly normal, both $OH(X)$ and $OS(X)$ being convex compacta for any space $X$.

Each of the abovementioned functors generates a monad. If $F$ is one of $V, O, OH, OS, P$, the identity and multiplication maps are defined as follows. The natural transformation $\eta : \text{Id}_{\text{Comp}} \to F$ is given by $\eta_X(x)(\varphi) = \varphi(x)$ for any $x \in X$ and $\varphi \in C(X)$, and the natural transformation $\mu : F^2 \to F$ given by $\mu_X(\nu)(\varphi) = \nu(\pi_\varphi)$, where $\pi_\varphi : F(X) \to \mathbb{R}$, $\pi_\varphi(\lambda) = \lambda(\varphi)$. Later by $\mu_F X$ we shall denote the multiplication map for the corresponding functor $F$.

According to the characterization given in [15], by $L$-monad we mean any submonad of $V$. Hence, $\mathcal{O}H$ and $\mathcal{O}S$, being submonads of $V$ are both $L$-monads.

We say that an $L$-monad $F = (F, \eta, \mu)$ weakly preserves preimages ([3]) if for any mapping $f : X \to Y$ and any closed subset $A \subset Y$ we have $\nu(\varphi) \in [\min \varphi(f^{-1}(A)), \max \varphi(f^{-1}(A))]$ for all $\nu \in (Ff)^{-1}(F(A))$ and all $\varphi \in C(X)$.

Let us recall the notion of convexities introduced in [3]. Let $(F, \eta, \mu)$ be a monad, and $(X, \xi)$ be an $F$-algebra. Let $A$ be a closed subset of $X$. By $f_A$ denote the quotient map $f_A : X \to X/A$, $a = f_A(A)$. We say that $C_F(A) = \xi(\langle Ff_A^{-1}(\eta(X/A)(a)) \rangle)$ is the $F$-convex hull of $A$. Also put $C_F(\emptyset) = \emptyset$. The set $A$ is called $F$-convex if $A = C_F(A)$. Define $C_F(X, \xi) = \{A \subset X \mid A$ is closed and $A = C_F(A)\}$. The family $C_F(X, \xi)$ forms a convexity on $X$. Also, any $F$-algebras morphism preserves convexities defined above [3]. Later we’ll restrict ourselves with the binary monads. A monad $F$ is binary if $C_F(X, \xi)$ is binary, i.e. the intersection of each linked subsystem of $C_F(X, \xi)$ is not empty (we call a family of subsets of a space linked if the intersection of the finite number of any of its elements is not empty).

**Theorem A.** ([3, Theorem 3.3]) Let $F$ be a binary $L$-monad which weakly preserves preimages, and let $X$ be such that $FX$ is an openly generated (connected) compactum. Then each map $f : FX \to Y$ with $F$-convex fibers is $0$-soft (soft) provided $f$ is open.

By $\exp X$, for any compact $X$, we denote the space of all nonempty closed subsets of $X$ equipped with the Vietoris topology (see, e.g., [10]).

In what follows we shall need the characterization of $OS(X)$, given in [5]. In particular, the following facts take place:

- For any $A \in \exp P(X)$ the functional $\nu_A$ given by $\nu_A(\varphi) = \sup\{\mu(\varphi) \mid \mu \in A\}$, where
\( \varphi \in C(X) \), exists and belongs to \( OS(X) \). Also \( \nu_A = \nu_{\text{conv}(A)} \) for any \( A \in \exp P(X) \) (Proposition 3.2);

- Any \( \nu \in OS(X) \) coincides with a functional of the form \( \nu_A \), where \( A = \{ \mu \in P(X) | \mu(\varphi) \leq \nu(\varphi) \ \forall \varphi \in C(X) \} \) is a convex compactum in \( P(X) \), in addition, for each \( \varphi \in C(X) \) there is \( \mu \in A \) such that \( \mu(\varphi) = \nu(\varphi) \) (Theorem 3.3);

- The correspondence between functionals from \( OS(X) \) and closed convex subsets of \( P(X) \) is one-to-one (Theorem 3.4);

- For any \( f : X \to Y \) and \( \nu_A \in OS(X) \) we have \( OS(f)(\nu_A) = \nu_{P(f)(A)} \).

2. When \( OS(X) \) and \( OH(X) \) are absolute retracts?

For any subset \( A \subset OH(X) \), we see that sup \( A \), \( \inf A \) also belong to \( OH(X) \). Thus, \( OH(X) \) is a compact sublattice of \( \prod_{\varphi \in C(X)} [\min \varphi, \max \varphi] \).

The following statement can be obtained by applying the same arguments as in [4, Theorem 1].

**Proposition 1.** For any surjective function \( f : X \to Y \) the mapping \( OH(f) \) is open if and only if \( f \) is open.

From the remarks on \( OS \) made in the first section one can see that \( OS \) is in fact isomorphic to the composition of the functors \( cc \) and \( P \). Some properties of the functor \( cc \) were studied in [8]. For any convex compact \( X \), \( ccX \) is defined to be the set of all nonempty closed convex subsets of \( X \), \( ccX \) is considered as the subspace of \( \exp X \). For any affine mapping \( f : X \to Y \) function \( cc(f) \) is given by \( cc(f)(A) = f(A) \) where \( A \in ccX \). From [8, Proposition 3.1] and openness of the functor of probability measures follows

**Proposition 2.** The functor \( OS \) is open, i.e. for any open mapping \( f : X \to Y \) the map \( OS(f) \) is open.

It was shown in [3] that the monad \( \mathcal{O} \) generated by the functor of weakly additive functionals weakly preserves preimages (Theorem 4.2). Since \( \mathcal{O} \mathcal{H} \) and \( OS \) are submonads of \( \mathcal{O} \), they weakly preserve preimages as well.
Recall that the notation \( \mathbb{L} \) stands for the superextension monad generated by the superextension functor \( \lambda \) (see [10] for details). For any compact \( X \), the space \( \lambda X \) has a functional representation which can be defined by the embedding \( i_X : \lambda X \to \prod_{\varphi \in \mathcal{C}X} [\min \varphi, \max \varphi] \) such that \( i_X(A)(\varphi) = \sup\{\inf \varphi(A) | A \in \mathcal{A}\} \), where \( \mathcal{A} \) is from \( \lambda X \) and \( \varphi \in \mathcal{C}(X) \). It is easy to see that the image \( i_X(\lambda X) \) lies in \( OH(X) \). Actually, the natural transformation \( i = \{i_X\} \) is a monad morphism which embeds the superextension monad in \( \mathcal{O}H \). Therefore, by [3, Theorem 3.2], \( \mathcal{O}H \) is binary.

Now take any openly generated compactum \( X \). Whereas the functor \( OH \) is open and the space \( OH(X) \) is convex, \( OH(X) \) is an openly generated continuum. From Theorem A we see that whenever \( \mathbb{F} \) is a binary \( L \)-monad that weakly preserves preimages, then \( F(X) \in AR \) for some compact \( X \) provided \( F(X) \) is an openly generated connected compactum. Applying this fact in our case we see that \( OH(X) \in AR \).

Conversely, if we suppose that \( OH(X) \in AR \) for some compact \( X \), then an argumentation similar to that of [4, Theorem 2] provides that \( X \) is an openly generated compactum.

We therefore obtain the following fact:

**Theorem 1.** \( OH(X) \) is an absolute retract if and only if \( X \) is an openly generated compactum.

So what we get is that \( OH(X) \) can be an \( AR \) even when the weight of \( X \) exceeds \( \omega_1 \). The same could be said on some other functors which generate \( L \)-monads and contain \( \mathbb{L} \) as submonad, for instance \( G, O, \lambda \) by itself. The functor \( OS \) seems to be closer to \( P \). It does not give an \( AR \) in weights higher than \( \omega_1 \):

**Proposition 3.** \( OS(X) \) is an absolute retract if and only if \( X \) is openly generated with \( w(X) \leq \omega_1 \).

**Proof.** Follows from the results of [8], namely [8, Theorem 4.1] combined with results of [7] providing that a statement analogous to that of the proposition holds for the functors \( cc \) and \( P \).

**Corollary 1.** There is no monad embedding \( i : \mathbb{L} \rightarrow OS \).

Indeed, assuming the contrary, we would obtain that \( OS \) is binary. Therefore, according to
[3, Theorem 3.3], the space $OS(D^w)$, for example, must be an absolute retract, a contradiction.

3. The softness of multiplication maps for $OH$ and $OS$.

**Theorem 2.** If the multiplication map $\mu_{OS}X$ for $OS$ is soft then $X$ is metrizable.

**Proof.** Suppose that $X$ is not metrizable and $\mu_{OS}X$ is soft. Use [9, Theorem 3] to obtain that $X$ is openly generated.

Represent $X$ as the limit of an $\omega$-system $S = \{X_\alpha, p_\alpha, A\}$ with open bonding maps. Whereas $\mu X$ is soft, we can assume that all limit diagrams

$$
\begin{array}{ccc}
OS^2(X) & & OS(X) \\
\downarrow \mu X & & \downarrow \mu X_\alpha \\
OS(X) & & OS(X_\alpha)
\end{array}
$$

are soft [9, Theorem 2], hence open.

Now our aim is to obtain $\alpha_0 \in A$ and an accumulation point $x \in X_{\alpha_0}$ such that $p_{\alpha_0}^{-1}(x)$ contains more than one point. The weight of $X$ is uncountable, so its character is uncountable too, since $w(X) = \chi(X)$ for any openly generated compactum [4]. Choose $x_0 \in X$ with $\chi(x_0, X) > \omega$ and some $\alpha \in A$, put $x_\alpha = p_\alpha(x_0)$. Then $p_{\alpha}^{-1}(x_\alpha)$ contains more than one point, otherwise $x_0$ would have the countable character. If $x_\alpha$ is not isolated, then $x_\alpha$ is the required point. Suppose that $x_\alpha$ is isolated. Consider $x_1 \in p_{\alpha}^{-1}(x_\alpha)$ distinct from $x_0$. We can choose $\alpha_1 > \alpha$ with $p_{\alpha_1}(x_1) \neq p_{\alpha_1}(x_0)$. Again $p_{\alpha_1}^{-1}(x_0)$ is not a singleton, and if $p_{\alpha_1}(x_0)$ is an accumulation point, we are done. Assume the opposite. Take any $x_2 \in p_{\alpha_1}^{-1}(p_{\alpha_1}(x_0))$ with $x_2 \neq x_0$ and $\alpha_2 > \alpha_1$ such that $p_{\alpha_2}(x_2) \neq p_{\alpha_2}(x_0)$ and continue the process as described above.

If on any step $i$ the point $p_{\alpha_i}(x_0)$ is not an accumulation point, we obtain the sequence $\{x_i\}_{i \in \mathbb{N}}$ of points in $X$ and the up-directed chain of elements $\{\alpha_i\}_{i \in \mathbb{N}}$ of $A$ which has the least upper bound $\alpha_0 \in A$. Then the space $X_{\alpha_0}$ is the limit of the inverse system $\{X_{\alpha_i}, p_{\alpha_i}, i \leq j\}$ and $\lim_{i \to \infty} p_{\alpha_0}(x_i) = p_{\alpha_0}(x_0)$. Indeed, the family $\{(p_{\alpha_0}^{-1})^{-1}(p_{\alpha_i}(x_0))\}$ forms a base of neighborhoods at $p_\alpha(x_0)$, and for any such $(p_{\alpha_0}^{-1})^{-1}(p_{\alpha_i}(x_0))$ we see that $p_{\alpha_0}(x_j)$ is contained in it for all $j \geq i$. Therefore, $\alpha_0 \in A$ and $x = p_{\alpha_0}(x_0)$ chosen above are as required.

According to our assumption, the diagram
Consider the accumulation point \( x \in X_{\alpha_0} \) chosen above and distinct \( y_1, y_2 \in p_{\alpha_0}^{-1}(x) \). Let \( \{x_i\}_{i \in I} \) be the net converging to \( x \). Choose \( y_i \in p_{\alpha_0}^{-1}(x_i) \) for every \( i \in I \) the way that \( \{y_i\}_{i \in I} \) would converge to \( x_1 \).

Denote \( \nu = (\delta_{y_1} + \delta_{y_2})/2 \) and \( \mathcal{V} = \Delta_{\delta_x} \). Then the net \( (\nu_i, \mathcal{V}_i) = ((\delta_{y_i} + \delta_{y_2})/2, \Delta_{(\delta_x + \delta_x_i)/2}) \) converges to \( (\nu, \mathcal{V}) \). To obtain a contradiction with openness of \( \chi \), and therefore softness of \( \mu_{OS}X \), show that the inverse of the characteristic map of the considered diagram is not continuous. Indeed, let us consider \( \chi^{-1}((\delta_{y_i} + \delta_{y_2})/2, \Delta_{(\delta_x + \delta_x_i)/2}) = (OS^2(p_{\alpha_0}))^{-1}(\nu_i) \cap (\mu_{OS}X)^{-1}(\nu_i) \). Suppose that \( \Theta \in (OS^2(p_{\alpha_0}))^{-1}(\nu_i) \). Then \( \text{supp} \Theta \) is in \( OS((p_{\alpha_0})^{-1}(x_i)) \). Now take any \( \Theta = \Theta_A \in (\mu_{OS}X)^{-1}(\nu_i) \) (recall that any functional \( \eta \in OS(Y) \) is of the form \( \eta = \eta_B \), where \( B \in ccP(Y) \)). We want to show that \( \text{supp} \Theta_A \) is in \( OS(\{y_i, y_2\}) \). Indeed, assuming the contrary, we obtain that there is a measure \( M \in A \) that is not supported on \( OS(\{y_i, y_2\}) \), therefore there exists \( \theta \notin OS(\{y_i, y_2\}) \) from the support of \( M \). So we can choose a function \( \varphi \in C(X) \) which is zero at \( \{y_2, y_i\} \), \( \theta(\varphi) > 0 \) and \( 0_X \leq \varphi \). Since \( \pi_\varphi \) is continuous, there exists a closed neighborhood \( V \) of \( \theta \) on which \( \pi_\varphi \) is strictly greater than zero. Also, \( \text{supp} M \cap V \neq \emptyset \), so \( M(V) > 0 \). This implies \( M(\pi_\varphi) > 0 \), and hence \( \mu_{OS}X(\Theta_A)(\varphi) = \Theta_A(\pi_\varphi) = \sup \{M(\pi_\varphi) \mid M \in A\} > 0 \), whereas \( \nu_i(\varphi) = 0 \) which gives us \( \mu_{OS}X(\Theta_A) \neq \nu_i \). That’s why any \( \Theta \in (OS^2(p_{\alpha_0}))^{-1}(\mathcal{V}_i) \cap (\mu_{OS}X)^{-1}(\nu_i) \) must be supported on \( OS(\{y_i, y_2\}) \). The only such functional \( \Theta \) which also satisfies the condition \( \chi(\Theta) = ((\delta_{y_i} + \delta_{y_2})/2, \Delta_{(\delta_x + \delta_x_i)/2}) \) is the measure \( \Delta(\delta_{y_1} + \delta_{y_2})/2 \). Therefore, there is some neighborhood \( V_1 \) of the functional \( (\Delta_{\delta_{y_1}} + \Delta_{\delta_{y_2}})/2 \in \chi^{-1}(\nu, \mathcal{V}) \) that contains no elements of the form \( \Delta(\delta_{y_1} + \delta_{y_2})/2 \) starting from some \( i_0 \in I \), hence \( \chi^{-1} \) is not continuous, and the diagram is not open, a contradiction with the initial assumption. Theorem is proved.

The following are the results for \( \mathbb{O} \mathbb{H} \) which show that it behaves the same way as the monad \( \mathbb{O} \).

**Theorem 3.** \( \mu_{OH}X \) is open for any compactum \( X \).

Proof of Theorem 3 is the same as that for \( \mathbb{O} \) [3].

**Theorem 4.** \( \mu_{OH}X \) is soft if and only if \( X \) is an openly generated compactum.
Proof. Necessity. Let \( X = \lim_{\alpha} \mathcal{S} \), where \( \mathcal{S} = \{X_\alpha, p_\alpha, A\} \) is an \( \omega \)-system consisting of metrizable compacta and epimorphisms. The mapping \( \mu_{OH} X \) is soft, hence we can assume that all the limit diagrams of the form

\[
\begin{array}{ccc}
OH^2(X) & \overset{\mu_{OH} X}{\longrightarrow} & OH(X) \\
\downarrow & & \downarrow \\
OH(X) & \overset{\mu_{OH} X_\alpha}{\longrightarrow} & OH(X_\alpha)
\end{array}
\]

are open. Assume that \( X \) is not openly generated, so that there exists \( \alpha \in A \) such that \( p_\alpha \) is not open. Then by Proposition 2 the mapping \( OH(p_\alpha) \) is not open. Therefore, there is a functional \( \nu \in OH(X_\alpha) \) and a net \( \{\nu_i\}_{i \in I} \) converging to \( \nu \) such that the net \( OH(p_\alpha)^{-1}(\nu_i) \) converges to some \( A \neq OH(p_\alpha)^{-1}(\nu) \). We have that \( A \subset OH(p_\alpha)^{-1}(\nu) \). Choose two comparable elements \( \theta_1 \in A \) and \( \theta_2 \in OH(p_\alpha)^{-1}(\nu) \). Let, for example, \( \theta_1 \leq \theta_2 \). Let \( \{\theta_i\} \) be a net converging to \( \theta_2 \) such that \( \theta_i \in OH(p_\alpha)^{-1}(\nu_i) \) for all \( i \in I \). We see that the net \( \{(\theta_i, \eta OH(X_\alpha)(\nu_i))\} \) converges to \( (\theta_2, \eta OH(X_\alpha)(\nu)) \). Now let \( \nu \in OH^2(X) \) be a functional such that \( \nu(\Phi) = \max\{\Phi(\theta_1), \Phi(\theta_2)\} \).

Then \( \chi(\nu) = (\theta_2, \eta OH(X_\alpha)(\nu)) \), where \( \chi \) is the characteristic map of the diagram. Choose \( \Phi \in C(OH(X)) \) with \( \Phi(\theta_2) = 1 \) and \( \Phi(\theta) = 0 \) for any \( \theta \in A \). Then we may take that \( \Phi(\theta) \leq \text{frm}[\alpha] - 1/2 \) for any \( OH(p_\alpha)^{-1}(\nu_i) \), hence, using [4, Lemma 2], we get that \( \Theta(\Phi) \leq 1/2 \) for all \( \Theta \in (OH^2(p_\alpha))^{-1}(\eta OH(X_\alpha)(\nu_i)) \). Thus we obtained an open neighborhood of \( \nu \) of the form \( V = \{\Theta \in OH^2(X) \mid \Theta(\Phi) > 1/2\} \) with \( V \cap \chi^{-1}(\theta_i, \eta OH(X_\alpha)(\nu_i)) = \emptyset \), a contradiction which shows that \( X \) must be openly generated.

Sufficiency. The monad \( \mathcal{OH} \) is a binary monad which weakly preserves preimages. Since \( \mu_{OH} X : OH^2(X) \to OH(X) \) is an open \( \mathcal{OH} \)-algebras morphism and \( OH^2(X) \) is openly generated (by Theorem 3), the softness of \( \mu_{OH} X \) follows from Theorem A. The statement is proved.

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