Isothermic Triangulated Surfaces

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Abstract

We define a simply connected triangulated surface in Euclidean space to be isothermic if it admits an infinitesimal rigid deformation preserving the mean curvature integrand. The same definition characterizes isothermic surfaces also in the smooth theory.

We show that the class of isothermic triangulated surfaces is Möbius invariant. As in the smooth case, closed triangulated surfaces are isothermic if and only if they are singular points in the space of all conformally equivalent surfaces. In addition, being an isothermic surface is equivalent to the existence of an infinitesimal deformation preserving the intersection angles of circumscribed circles and spheres.

We show that all infinitesimally flexible triangulated surfaces inscribed in a sphere are isothermic. Moreover, starting with an isothermic quadrilateral mesh in the sense of Bobenko and Pinkall, we obtain an isothermic triangulation by subdividing in an arbitrary fashion each face into two triangles.

Finally we construct discrete analogs of minimal surfaces based on Weierstrass data consisting of a triangulated plane domain equipped with a discrete harmonic function.

1 Introduction

A surface \( f : M \to \mathbb{R}^3 \) is called isothermic if there exists locally an \( \mathbb{R}^3 \)-valued closed 1-form \( \tau \) such that

\[
\text{df} \wedge \tau = 0.
\]

Here we have identified \( \mathbb{R}^3 \) with the space \( \text{Im} \mathbb{H} \) of purely imaginary quaternions \cite{23}. The class of isothermic surfaces includes all surfaces of revolution, quadrics, constant mean curvature surfaces and many other interesting surfaces \cite{14}.

It is known (although not well-known) that a surface is isothermic if and only if locally it admits an infinitesimal isometric deformation preserving the mean curvature. The only reference that we could find is from Ciesiński et al. \cite{10}, stating that this theorem was known in the 19th century.

Infinitesimal isometric deformations of triangulated surfaces have been extensively studied \cite{22, 29}. They are infinitesimal deformations of the vertex positions which preserve the edge lengths. The property of infinitesimal rigidity is projectively invariant \cite{17}.

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For a given infinitesimal rigid deformation $\dot{f} : V \rightarrow \mathbb{R}^3$, each face $\phi$ rotates according to an infinitesimal rotation given by a vector $Z_\phi \in \mathbb{R}^3$. These rotation vectors $Z_\phi$ satisfy a compatibility condition on each edge:

$$df(e_{ij}) \times Z_{\phi_{ijk}} = df(e_{ij}) \times Z_{\phi_{jil}} = d\dot{f}(e_{ij}),$$

where $\phi_{ijk} \in F$ is the left face of $e_{ij}$ and $\phi_{jil} \in F$ is the right face of $e_{ij}$. Note that the right-hand side is given by the discrete exact 1-form $d\dot{f}$.

It is well-known that the integral $\int H dA$ of the mean curvature has a very canonical discrete analogue $\sum H_e$ where for each edge $e$ we have defined the integrated mean curvature associated to $e$ as

$$H_e := \alpha_e |df(e)|.$$  

Here $\alpha_e$ is the dihedral angle $[27]$. Under the rigid deformation (1) given by $Z$ on faces, we have

$$\dot{H}_{e_{ij}} = \dot{\alpha}_{e_{ij}} |df(e_{ij})| = (df(e_{ij}), Z_{\phi_{ijk}} - Z_{\phi_{jil}}).$$

If we would demand $\dot{H}_e = 0$ there would be no non-trivial infinitesimal rigid deformations with this property. Hence we consider instead the change of the integrated mean curvature around vertices

$$\dot{H}_{v_i} := \sum_{e_{ij} \in E_{\text{or}}; i} \dot{\alpha}_{e_{ij}} |df(e_{ij})| = \sum_{e_{ij} \in E_{\text{or}}; i} \langle df(e_{ij}), Z_{\phi_{ijk}} - Z_{\phi_{jil}} \rangle.$$  

Combining the above notions, we proposed a discrete analogue of isothermic triangulated surfaces of arbitrary genus.

**Definition 1.1.** An immersed oriented triangulated surface (with boundary) $f : M \rightarrow \mathbb{R}^3$ is called isothermic if there exists a $\mathbb{R}^3$-valued discrete dual 1-form $\tau : E_{\text{or}} \rightarrow \mathbb{R}^3$, not identically zero, such that

$$df(e_{ij}) \times \tau(*e_{ij}) = 0 \quad \forall e_{ij} \in E_{\text{or}}$$

$$\sum_{e_{ij} \in E_{\text{or}}; i} \tau(*e_{ij}) = 0 \quad \forall v_i \in V_{\text{in}}$$

$$\sum_{e_{ij} \in E_{\text{or}}; i} \langle df(e_{ij}), \tau(*e_{ij}) \rangle = 0 \quad \forall v_i \in V_{\text{in}}$$

Here $*e$ denotes the combinatorial dual edge of $e$, while $E_{\text{or}}$ and $V_{\text{in}}$ are denoted as the set of oriented interior edges and the set of interior vertices of $M$.

Previous definitions of discrete isothermic surfaces were all based on quadrilateral meshes which are discrete versions of a conformal curvature line parametrization of a smooth surface. For example, Bobenko and Pinkall [2] considered surfaces consisting of quadrilaterals with factorized real cross ratios. Not all planar quadrilateral nets are isothermic in this sense, although in the smooth theory the plane is an isothermic surface. On the other hand, it is easy to see that under our definition all planar triangle meshes are isothermic. So one might say that our definition captures a property of isothermic surfaces as such, not of a particular class of parametrizations.

We will state several results about discrete isothermic triangle meshes that closely reflect known theorems from the smooth theory. In Section 3, 4 and 5, we prove

**Theorem 1.2.** The class of isothermic surfaces is M"obius invariant.
Theorem 1.3. For a closed triangulated surface \( f : M \to \mathbb{R}^3 \) of genus \( g \) the space of infinitesimal conformal deformations is of dimension greater or equal to \( |V| - 6g + 6 \). The inequality is strict if and only if the surface is isothermic.

Theorem 1.4. Suppose \( f : M \to \mathbb{R}^3 \) is a simply connected triangulated surface. Then \( f \) is isothermic if and only if there exists a non-trivial infinitesimal deformation of \( f \) preserving the intersection angles of neighboring circles and neighboring spheres.

We study special classes of isothermic triangle meshes in Section 6, 7 and 8. Subdividing any isothermic quadrilateral net (in the sense of Bobenko and Pinkall [2]) in an arbitrary way we obtain an isothermic triangle mesh. Moreover, triangulated cylinders generated by discrete groups and all triangulated surfaces with boundary inscribed in a round sphere are isothermic.

In Section 9 we define discrete minimal surfaces as the reciprocal-parallel meshes for triangulated surfaces (with boundary) inscribed in the unit sphere. We show pictures of discrete minimal surfaces, which are constructed by a discrete analogue of Weierstrass representation of minimal surfaces.

In Section 10, we review the smooth theory and also prove some new theorems that are similar to discrete results established in earlier sections.

Throughout we use the language of discrete differential forms and quaternionic analysis as introduced by Desbrun et al. [11] and Pedit and Pinkall [23].

2 Some notations

Definition 2.1. A triangulated surface \( M \) is a finite simplicial complex whose underlying topological space is a connected 2-manifold with boundary. The set of vertices (0-cells), edges (1-cells) and triangles (2-cells) are denoted as \( V \), \( E \) and \( F \). Without further notice we will assume that all triangulated surfaces under consideration are oriented.

Definition 2.2. An immersion of a triangulated surface \( M \) into \( \mathbb{R}^3 \) is a topological immersion \( f : M \to \mathbb{R}^3 \) which is linear on each triangle.

Note that an immersion \( f \) of a triangulated surface is determined by its vertex positions (i.e. the images of the vertices under \( f \)).

We consider oriented edges \( e_{ij} \) connecting the vertex \( v_i \) to the vertex \( v_j \) which are interior edges in the sense that at least one of the vertices is not a boundary vertex. We denote the set of such oriented edges by \( E_{or} \). Note that if \( e_{ij} \in E_{or} \), then

\[
e_{ij} \neq e_{ji} \in E_{or}.
\]

Given an immersion \( f : M \to \mathbb{R}^3 \), we write

\[
df(e_{ij}) := f_j - f_i
\]

which implies

\[
df(e_{ij}) = -df(e_{ji}).
\]

The set of non-oriented edges of \( M \) will be denoted by \( E \) and the set of non-oriented interior edges by \( E_{in} \). We abuse of notations. The unoriented edge connecting the vertices \( v_i \) and \( v_j \)
will be denoted by $e_{ij}$ as well. It will be clear from the context whether we mean oriented edges or non-oriented ones.

An immersion $f$ induces a discrete metric $l : E \to \mathbb{R}$ on $M$ via

$$\forall e_{ij} \in E, \quad l_{ij} := |f_j - f_i| = |df(e_{ij})| = |df(e_{ji})|.$$ 

### 3 Möbius invariance

In this section we prove that the class of isothermic triangulated surface is invariant under Möbius transformations.

Given a triangulated surface $f : M \to \mathbb{R}^3$ and a Möbius transformation $\sigma : \mathbb{R}^3 \cup \{\infty\} \to \mathbb{R}^3 \cup \{\infty\}$, we define $\sigma \circ f : M \to \mathbb{R}^3$ as the triangulated surface with vertices $(\sigma \circ f)_i := \sigma \circ f_i$. We consider only the Möbius transformations which do not map any vertex to the infinity.

Taking $\sigma$ to be minus the inversion in the unit sphere, we obtain a triangulated surface $f^{-1} : M \to \mathbb{R}^3$ with

$$f^{-1} := \sigma \circ f = -\frac{f}{||f||^2}.$$ 

We are going to show that $f$ is isothermic if and only if $f^{-1}$ is isothermic. We first rewrite the equations from Definition [1.1].

**Lemma 3.1.** Given a triangulated surface $f : M \to \mathbb{R}^3$ and a $\mathbb{R}^3$-valued dual 1-form $\tau : E_{or} \to \mathbb{R}^3$ satisfies

$$df(e_{ij}) \times \tau(*e_{ij}) = 0 \quad \forall e_{ij} \in E_{or}$$

$$\sum_{e_{ij} \in E_{or}} \tau(*e_{ij}) = 0 \quad \forall v_i \in V_{in}$$

$$\sum_{e_{ij} \in E_{or}} \langle df(e_{ij}), \tau(*e_{ij}) \rangle = 0 \quad \forall v_j \in V_{in}$$

if and only if there exists $k : E_{in} \to \mathbb{R}$ such that

$$\tau(*e_{ij}) = k_{ij} df(e_{ij}) \quad \forall e_{ij} \in E_{or}$$

$$\sum_{e_{ij} \in E_{or}} k_{ij} df(e_{ij}) = 0 \quad \forall v_i \in V_{in}$$

$$\sum_{e_{ij} \in E_{or}} k_{ij}(|f_j|^2 - |f_i|^2) = 0 \quad \forall v_i \in V_{in}.$$

**Proof.** Suppose $k : E_{in} \to \mathbb{R}$ satisfies

$$\sum_{e_{ij} \in E_{or}} k_{ij} df(e_{ij}) = 0 \quad \forall v_i \in V_{in}.$$ 

Then, we have the identity

$$\sum_{e_{ij} \in E_{or}} \langle df(e_{ij}), k_{ij} df(e_{ij}) \rangle$$
Hence, the surface

Second, for any vertex \( i \)

discrete dual 1-form

Proof of Theorem 1.2. It follows from the previous lemma and the fact that Möbius transformations are generated by inversions and Euclidean transformations.

Lemma 3.2. Suppose \( f : M \to \mathbb{R}^3 \) is an isothermic triangulated surface with a non-trivial discrete dual 1-form \( \tau \) satisfying Definition 1.1. We write

\[
\tau(\ast e_{ij}) = k_{ij} df(e_{ij})
\]

for some \( k : E_{in} \to \mathbb{R} \) that does not vanish identically. Then, the triangulated surface \( f^{-1} : M \to \mathbb{R}^3 \) is isothermic with corresponding dual 1-form

\[
\tilde{\tau}(\ast e_{ij}) := k_{ij} |f_i|^2 |f_j| df^{-1}(e_{ij}).
\]

Proof. We check that the 1-form \( \tilde{\tau} \) satisfies the three equations from Definition 1.1 by applying the previous lemma. First, for any vertex \( i \in V_{in} \), we have

\[
\sum_{e_{ij} \in E_{or^{-1}}} \tilde{\tau}(\ast e_{ij}) = \sum_{e_{ij} \in E_{or^{-1}}} k_{ij} |f_i|^2 |f_j| df^{-1}(e_{ij})
\]

\[
= \sum_{e_{ij} \in E_{or^{-1}}} (k_{ij} |f_i|^2 |f_j|^2 f_i - f_i [f_i] f_i^2 + k_{ij} |f_i|^2 f_i - k_{ij} |f_i|^2 f_j [f_j] f_j)
\]

\[
= f_i \sum_{e_{ij} \in E_{or^{-1}}} k_{ij} (|f_i|^2 - |f_j|^2) + |f_i|^2 \sum_{e_{ij} \in E_{or^{-1}}} k_{ij} (f_i - f_j)
\]

\[
= 0.
\]

Second, for any vertex \( i \in V_{in} \) we have

\[
\sum_{e_{ij} \in E_{or^{-1}}} k_{ij} |f_i|^2 |f_j|^2 (|f_i^{-1}|^2 - |f_j^{-1}|^2) = \sum_{e_{ij} \in E_{or^{-1}}} k_{ij} (|f_i|^2 - |f_j|^2)
\]

\[
= 0.
\]

Hence, the surface \( f^{-1} \) is isothermic with 1-form \( \tilde{\tau} \) in the sense of Definition 1.1.

Proof of Theorem 1.2. It follows from the previous lemma and the fact that Möbius transformations are generated by inversions and Euclidean transformations.
Remark 3.3. The above calculation can be simplified if written in terms of quaternions. If we identify the Euclidean 3-space with the space of purely imaginary quaternions we obtain
\[ df^{-1}(e_{ij}) = f_i^{-1}df(e_{ij})f_j^{-1} = f_j^{-1}df(e_{ij})f_i^{-1} \]
and
\[ \tilde{\tau}(*e_{ij}) = f_i\tau(*e_{ij})\tilde{f}_j = f_j\tau(*e_{ij})\tilde{f}_i. \]
These two formulas are similar to the smooth case (Theorem 10.5) and they are also proved in a similar way.

Lemma 3.1 provides another characterization of isothermic triangulated surfaces: We consider the light cone
\[ L := \{ x \in \mathbb{R}^5 | x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 = 0 \}. \]

Corollary 3.4. A triangulated surface \( f : M \to \mathbb{R}^3 \) is isothermic with corresponding dual 1-form \( \tau := kdf \) if and only if
\[ \sum_{e_{ij} \in E_{\text{cong}}} k_{ij}d\hat{f}(e_{ij}) = 0 \quad \forall v_i \in V_\text{in} \tag{5} \]
where \( \hat{f} : M \to L \subset \mathbb{R}^5 \) is the lift of \( f \) to \( \mathbb{R}^5 \) defined by
\[ \hat{f}_i := (f_i, 1 - |f_i|^2, 1 + |f_i|^2) \in L \subset \mathbb{R}^5. \]

A function \( k : E_{\text{in}} \to \mathbb{R} \) satisfying equation (5) is called a self-stress of \( \hat{f} \).

It is known that the Möbius geometry of \( \mathbb{R}^3 \cup \{ \infty \} \) is a subgeometry of the projective geometry of \( \mathbb{R}P^4 \). Möbius transformations of \( \mathbb{R}^3 \cup \{ \infty \} \) are represented as projective transformations of \( \mathbb{R}P^4 \) preserving the quadric defined by the light cone \( L \). If two immersions are related by a projective transformation, then the spaces of self-stress of the corresponding two surfaces are isomorphic \[17\]. Hence, we obtain another proof of Theorem 1.2.

4 Infinitesimal conformal deformations

We consider infinitesimal conformal deformations of a closed triangulated surface immersed in Euclidean space and show that isothermic surfaces \( f \) are the singular points in the space of all surfaces conformally equivalent to \( f \). In this section, surfaces are assumed to be closed.

4.1 Conformal Equivalence

Definition 4.1 (Luo \[21\]). Two discrete metrics \( l \) and \( \tilde{l} \) on \( M \) are conformally equivalent if there exists \( u : V \to \mathbb{R} \) such that
\[ \tilde{l}_{ij} = e^{u_i + u_j}l_{ij}. \]
Two immersions \( f, \tilde{f} : M \to \mathbb{R}^3 \) are conformally equivalent if their induced discrete metrics are conformally equivalent.

It leads naturally to the infinitesimal case.
Definition 4.2. An infinitesimal deformation of a triangulated surface \( f : M \to \mathbb{R}^3 \) is a map \( \dot{f} : V \to \mathbb{R}^3 \). It is called \textit{conformal} if there exists \( u : V \to \mathbb{R}^3 \) such that the change of the length function \( \dot{l} \) satisfies

\[
\frac{u_i + u_j}{2} l_{ij} = \dot{l}_{ij} = \frac{\langle \dot{f}_j - \dot{f}_i, f_j - f_i \rangle}{|f_j - f_i|}.
\]

We call \( \dot{f} \) an infinitesimal rigid deformation if \( u \equiv 0 \).

The conformal equivalence class of a triangulated surface in Euclidean space is Möbius invariant [3]. The conformal equivalence classes can be distinguished via the logarithmic length cross ratio.

Definition 4.3. Given an oriented triangulated surface \( M \), the \textit{logarithmic length cross ratio} is the operator on functions defined on edges, \( \log \text{lc} : \mathbb{R}|E| \to \mathbb{R}|E| \), by

\[
\log \text{lc}(l)_{ij} = \log l_{jk} - \log l_{ki} + \log l_{im} - \log l_{mj}.
\]

Theorem 4.4 (Bobenko et al. [3]). Two discrete metric \( l \) and \( \tilde{l} \) on \( M \) are conformally equivalent if and only if

\[
\log \text{lc}(l) = \log \text{lc}(\tilde{l}).
\]

Corollary 4.5 (Bobenko et al. [3]). The dimension of the space of the conformal equivalence classes of a closed triangulated surface is \(|E| - |V|\).

4.2 Infinitesimal Deformations

Assume \( f : M \to \mathbb{R}^3 \) is a closed triangulated surface. Suppose the change of length function \( \dot{l} \) is given by \( l = \sigma l \) for some infinitesimal scaling \( \sigma : E \to \mathbb{R} \). Then the corresponding change of logarithmic length cross ratio is

\[
(\log \text{lc}(l))_{e_{ij}} = \sigma_{jk} - \sigma_{ki} + \sigma_{im} - \sigma_{mj} =: L(\sigma)_{e_{ij}}.
\]

The image of the linear map \( L : \mathbb{R}|E| \to \mathbb{R}|E| \) is the tangent space of the space of conformal equivalence classes (which is the same space at all metrics \( l \)).

Lemma 4.6. The operator \( L \) is skew adjoint with respect to the standard product \( \langle , \rangle \) on \( \mathbb{R}|E| \) given by \( \langle a, b \rangle := \sum_{e \in E} a(e)b(e) \) \( \forall a, b \in \mathbb{R}|E| \).

Proof. Let \( \delta_{ij} : E \to \mathbb{R} \) be the function defined by \( \delta_{ij}(e_{ij}) = 1 \) and zero otherwise. Then for any \( b \in \mathbb{R}|E| \), we have

\[
L^*(b)_{ij} = \langle \delta_{ij}, L^*(b) \rangle = \langle L(\delta_{ij}), b \rangle = -b_{jk} + b_{ki} - b_{it} + b_{ij} = -L(b)_{ij}.
\]

Hence we have an orthogonal decomposition of \( \mathbb{R}|E| \) as

\[
\mathbb{R}|E| = \text{Ker}(L) \oplus \text{Im}(L^*) = \text{Ker}(L) \oplus \text{Im}(L).
\]
Lemma 4.7.

\[ \text{Ker}(L) = \{ a : E \to \mathbb{R} \mid \exists u \in \mathbb{R}^V \text{ s.t. } a(e_{ij}) = u_i + u_j \quad \forall e \in E \} \]

\[ \text{Im}(L) = \{ a : E \to \mathbb{R} \mid \sum_{e_{ij} \in E:i} a(e_{ij}) = 0 \quad \forall v_i \in V \} \]

Proof. It is obvious that

\[ \{ a : E \to \mathbb{R} \mid \exists u \in \mathbb{R}^V \text{ s.t. } a(e_{ij}) = u_i + u_j \quad \forall e_{ij} \in E \} \subset \text{Ker}(L). \]

Assume \( a \in \text{Ker}(L) \). For each triangle \( \triangle_{ijk} \) we define

\[ u_i := \frac{a_{ij} + a_{ki} - a_{jk}}{2}. \]

Suppose \( \tilde{\triangle}_{i\tilde{k}j} \) is the neighboring triangle sharing the edge \( e_{ij} \) with \( \triangle_{ijk} \). Because of \( L(a)_{ij} = 0 \) we have

\[ u_i = \frac{a_{ij} + a_{ki} - a_{jk}}{2} = \frac{a_{ij} + a_{il} - a_{lj}}{2} = \tilde{u}_i. \]

Since the link of each vertex is a disk this in fact defines a function \( u : V \to \mathbb{R} \) such that for any edge \( e_{ij} \) we have

\[ a_{ij} = u_i + u_j. \]

Hence

\[ \text{Ker}(L) = \{ a : E \to \mathbb{R} \mid \exists u \in \mathbb{R}^V \text{ s.t. } a(e_{ij}) = u_i + u_j \quad \forall e \in E \}. \]

On the other hand, it is obvious that

\[ \text{Im}(L) \subset \{ a : E \to \mathbb{R} \mid \sum_{e_{ij} \in E:i} a(e_{ij}) = 0 \quad \forall v_i \in V \}. \]

Since

\[ \text{rank}(L) = |E| - \dim \text{Ker}(L) = |E| - |V|, \]

the two vector spaces are indeed the same. \( \square \)

Given a dual 1-form \( \tau \), we denote \( \delta \tau : V \to \mathbb{R} \) by

\[ (\delta \tau)_i := \sum_{e_{ij} \in E:i} \tau(*e_{ij}). \]

Recall that conformal equivalence classes are parametrized by the logarithmic length cross ratio. By the inverse function theorem the theorem below implies that by deforming a non-isothermic surface in space we can reach all nearby conformal equivalence classes. It is precisely in the case of an isothermic surface that the hypothesis of the inverse function theorem fails to be satisfied. Thus the space of all immersions in a fixed equivalence class is a smooth manifold at all immersions \( f \) that are not isothermic.
Theorem 4.8. Suppose \( f : M \to \mathbb{R}^3 \) is a closed triangulated surface. Then \( f \) is isothermic if and only if there exists non-trivial \( a \in \text{Im}(L) \) such that

\[
(a, L(\sigma)) = 0
\]

for all infinitesimal scalings \( \sigma \in \mathbb{R}^{|E|} \) coming from extrinsic deformations in Euclidean space, i.e. for which there exists \( \hat{f} : V \to \mathbb{R}^3, W : E \to \mathbb{R}^3 \) such that \( df = \sigma df + df \times W \).

Proof. Suppose \( f \) is isothermic and \( \tau \) satisfies Definition 1.1. Let \( \hat{f} : V \to \mathbb{R}^3 \) be an arbitrary deformation of \( f \) in Euclidean space and write \( d\hat{f} = \sigma df + df \times W \). Since \( \delta \tau \equiv 0 \) we have

\[
0 = 2 \sum_{i \in V} \langle (\delta \tau)_i, \hat{f}_i \rangle = \sum_{e \in E_{or}} \langle \tau(*e), d\hat{f}(e) \rangle = \sum_{e \in E_{or}} \langle \tau(*e), \sigma_e df(e) + df(e) \times W_e \rangle.
\]

From \( df(e_{ij}) \times \tau(*e_{ij}) = 0 \) we obtain

\[
0 = \sum_{e \in E_{or}} \langle \tau(*e), \sigma_e df(e) + df(e) \times W_e \rangle = \sum_{e \in E_{or}} \langle \tau(*e), df(e) \rangle \sigma_e.
\]

Using

\[
\langle \tau(*e_{ij}), df(e_{ij}) \rangle = \langle \tau(*e_{ji}), df(e_{ji}) \rangle
\]

we see that \( \langle \tau, df \rangle : E \to \mathbb{R} \) is well defined. Since we know that

\[
\sum_{e_{ij} \in E_{or} : i} \langle df(e_{ij}), \tau(*e_{ij}) \rangle = 0
\]

on all vertices we must have \( \langle \tau, df \rangle \in \text{Im}(L) \). Hence there exists \( a \in \text{Im}(L) \) such that

\[
L(a)_e = -\langle \tau(*e), df(e) \rangle.
\]

Since \( \hat{f} \) is arbitrary we conclude that

\[
0 = \langle (\tau, df), \sigma \rangle = (L(a), \sigma) = (a, L(\sigma))
\]

for all infinitesimal scaling \( \sigma \in \mathbb{R}^{|E|} \) coming from extrinsic deformations.

On the other hand, suppose there exists a non-trivial \( a \in \text{Im}(L) \) such that

\[
(a, L(\sigma)) = 0
\]

for all infinitesimal scaling \( \sigma \in \mathbb{R}^{|E|} \) coming from extrinsic deformations. We then define an \( \mathbb{R}^3 \)-valued dual 1-form \( \tau \) by

\[
df(e) \times \tau(*e) = 0 \quad \forall e \in E_{or},
\]

\[
\langle df(e), \tau(*e) \rangle = -L(a)_e \quad \forall e \in E_{or}.
\]

Since \( \langle df(e), \tau(*e) \rangle \in \text{Im}(L) \), we have

\[
\sum_{e_{ij} \in E : i} \langle df(e_{ij}), \tau(*e_{ij}) \rangle = 0 \quad \forall i \in V.
\]

In addition, for any deformation \( \hat{f} : V \to \mathbb{R}^3 \) we write \( d\hat{f} = \sigma df + df \times W \) and obtain

\[
\sum_{i \in V} \langle (\delta \tau)_i, \hat{f}_i \rangle = \sum_{e \in E} \langle \tau(*e), d\hat{f}(e) \rangle
\]

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\[
\begin{align*}
= \sum_{e \in E} \langle \tau(e), \sigma_e df(e) + df(e) \times W_e \rangle \\
= \sum_{e \in E} \langle \tau(e), df(e) \rangle \sigma_e \\
= (-L(a), \sigma) \\
= (a, L(\sigma)) \\
= 0.
\end{align*}
\]

Since \( \dot{f} \) is arbitrary we conclude
\[
\sum_{e_{ij} \in E_{\alpha \gamma i}} \tau(*e_{ij}) = (\delta \tau)_i = 0 \quad \forall i \in V.
\]

Hence, \( f \) is isothermic with dual 1-form \( \tau \).

**Proof of Theorem 1.3.** Consider the composition of maps
\[
\{\text{infinitesimal deformations in } \mathbb{R}^3\} \xrightarrow{\sigma} \{\text{infinitesimal scalings}\} \xrightarrow{L} \{\text{change of lcrs}\}.
\]

The space of infinitesimal conformal deformations is exactly \( \text{Ker}(L \circ \sigma) \). Moreover, we know
\[
\dim(\text{Ker}(L \circ \sigma)) = 3|V| - \text{rank}(L \circ \sigma) \\
\geq 3|V| - (|E| - |V|) \\
= |V| - 6g + 6.
\]

Finally we conclude: The inequality is strict \( \iff \) \( L \circ \sigma \) is not surjective \( \iff \) \( f \) is isothermic.

Since the conformal equivalence classes are parametrized by length cross ratio, we can rephrase the previous theorems.

**Corollary 4.9.** Given a closed triangulated surface, isothermic immersions are precisely the points in the space of all immersions where the map that takes an immersion to the conformal equivalence class of its induced metric fails to be a submersion.

It is interesting to see how the combinatorics affects the geometry. For closed triangulated surfaces of genus \( g \) the number of vertices satisfies the Heawood bound [13]
\[
|V| \geq \frac{7 + \sqrt{1 + 48g}}{2}.
\]

This condition is sufficient except for \( g = 2 \). Comparing it with the inequality in the above theorem we obtain more examples of isothermic surfaces.

**Corollary 4.10.** If \( f : M \to \mathbb{R}^3 \) is a triangulated surface with \( |V| < 6g + 4 \) then \( f \) is isothermic.

**Proof.** The space of infinitesimal conformal deformations contains all deformations that come from infinitesimal Möbius transformations. Therefore this space has dimension at least 10 and therefore a surface must be isothermic if \( 10 > |V| - 6g + 6 \).

Some of these surfaces with small number of vertices can be realized in Euclidean space without self-intersection. For example, there are embedded surfaces with \( g = 2 \) and \( |V| = 10 \) [10].
5 Preserving intersection angles

Given a triangulated surface in Euclidean space, any triangle determines a circumscribed circle and two triangles sharing an edge determines a circumscribed sphere if the vertices are not concircular. Two circumscribed circles are called neighboring if their corresponding triangles share an edge. We will call two circumscribed spheres neighboring if they have a common vertex.

We are going to prove Theorem 1.4. Here deformations were called trivial if they are induced from infinitesimal Möbius transformations.

Proof of Theorem 1.4. Suppose we have a non-trivial infinitesimal deformation $\dot{f}$ that preserves the angles between circumcircles and circumspheres. Then it cannot be that $\dot{f}$ also preserves the length cross ratios (because it is not hard to see that in this case $\dot{f}$ would come from an infinitesimal Möbius transformation). We write $\ddot{f} = \sigma df + df \times W$ for some $\sigma : E \to \mathbb{R}$ and $W : E \to \mathbb{R}^3$. Then the change of conformal equivalence class is given by $L(\sigma)$ where

$$L(\sigma)_{ij} := \sigma_{ik} - \sigma_{kj} + \sigma_{j\tilde{k}} - \sigma_{\tilde{k}i}.$$ 

By our assumptions $L(\sigma)$ does not vanish identically.

We define a dual 1-form

$$\tau(e) := -L(\sigma)\frac{df(e)}{|df(e)|^2}.$$ 

Then we have

$$\sum_{e_{ij} \in E_{\text{or}}} \langle \tau(e_{ij}), df(e_{ij}) \rangle = -\sum_{e_{ij} \in E} L(\sigma)_{e_{ij}} = 0 \quad \forall v_i \in V_{\text{in}}.$$ 

In order to show that $f$ is isothermic it remains to prove

$$\sum_{e_{ij} \in E_{\text{or}}} \tau(e_{ij}) = 0$$

for any vertex $v_i \in V_{\text{in}}$.

We identify Euclidean space $\mathbb{R}^3$ with the space $\text{Im} \mathbb{H}$ of purely imaginary quaternions. Pick any vertex $v_0$ and denote its neighboring vertices by $v_1, v_2, \ldots, v_n$. Then we take an inversion with respect to the unit sphere centered at $f_0 := f(v_0)$ and denote the images of the neighboring vertices by $\tilde{f}_i$. We have the following relations:

$$\tilde{f}_j = (f_j - f_0)^{-1}$$

$$\tilde{f}_j = -(f_j - f_0)^{-1}(\tilde{f}_j - f_0)(f_j - f_0)^{-1}$$

$$\tilde{f}_{j+1} - \tilde{f}_j = -(f_j - f_0)^{-1}(f_{j+1} - f_j)(f_{j+1} - f_0)^{-1}.$$ 

We define the infinitesimal scaling

$$\tilde{\sigma}_{j,j+1} := \frac{|\tilde{f}_{j+1} - \tilde{f}_j|}{|\tilde{f}_{j+1} - \tilde{f}_j|}.$$ 

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Note that this is a logarithmic derivative and by writing
\[
\left| \tilde{f}_{j+1} - \tilde{f}_j \right| \left| \tilde{f}_j - \tilde{f}_j - 1 \right| = \left| f_{j+1} - f_j \right| \left| f_j - f_{j-1} \right|
\]
we obtain
\[
\tilde{\sigma}_{j+1} - \tilde{\sigma}_{j-1} = (L(\sigma))_{e_0},
\]
where \( \sigma_{e_j} := \left| f_j - f_i \right| / \left| f_j - f_i \right| \). On the other hand, the vertices \( \tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n, \tilde{f}_1 \) form a closed polygon in \( \mathbb{R}^3 \). Write
\[
\tilde{\ell}_{j,j+1} := \left| \tilde{f}_{j+1} - \tilde{f}_j \right|, \\
\tilde{T}_{j,j+1} := \frac{\tilde{f}_{j+1} - \tilde{f}_j}{\left| \tilde{f}_{j+1} - \tilde{f}_j \right|}.
\]
Since the polygon is closed, we have
\[
0 = \sum \tilde{\ell}_{j,j+1} \tilde{T}_{j,j+1}.
\]
The fact that the deformation \( \dot{f} \) preserves the intersection angles of neighboring circles and neighboring spheres implies that the angles between the neighboring segments and osculating planes of the closed curve remain constant. Thus there exists a constant vector \( c \in \mathbb{R}^3 \) such that
\[
0 = \sum \dot{\tilde{\ell}}_{j,j+1} \tilde{T}_{j,j+1} + \sum \tilde{\ell}_{j,j+1} \tilde{T}_{j,j+1} \times c
\]
\[
= \sum \tilde{\sigma}_{j+1} (\tilde{f}_{j+1} - \tilde{f}_j)
\]
\[
= \sum_{j=1}^n (-\tilde{\sigma}_{j+1} + \tilde{\sigma}_{j-1,j}) \tilde{f}_j
\]
\[
= -\sum_{c_{e_0} \in E_{ar}} L(\sigma)_{e_0} (f_j - f_0)^{-1}
\]
\[
= -\sum_{e_0 \in E_{ar}} \tau(*e_0).
\]
To show that the converse is true one only has to reverse the previous argument. \( \square \)

6 Isothermic quadrilateral meshes

We show that discrete isothermic nets as defined by Bobenko and Pinkall \[2\] are isothermic under our definition (after an arbitrary subdivision into triangles). Discrete isothermic nets are quadrilateral meshes known to be a discrete analogue of conformal curvature line parametrization. The class of such meshes is Möbius invariant and (like its smooth counterpart) can be treated using the theory of integrable systems. One can apply discrete Darboux transformations and Christoffel dual transformations to obtain new discrete isothermic nets from a given one \[15\]. Special discrete surfaces related to isothermic quadrilateral meshes were studied in \[4, 5, 8\].
Questions about infinitesimal rigidity of quadrilateral meshes often lead to the consideration of special types discrete surfaces. For example, Schief et al. [26] consider infinitesimal rigid deformations of quadrilateral meshes where the shape of each elementary quadrilateral is unchanged and Wallner and Pottmann [28] relate infinitesimal flexible conical meshes to discrete minimal surfaces.

We first review some results on the discrete isothermic nets from [6]. Then we construct for any given discrete isothermic net an infinitesimal rigid deformation and we show that the change of mean curvature around each vertex is zero. In this way we obtain the dual 1-form required in Definition 1.1.

### 6.1 Review

**Definition 6.1** (Bobenko and Pinkall [2]). A discrete isothermic net is a map \( F : \mathbb{Z}^2 \to \mathbb{R}^3 \), for which all elementary quadrilaterals have factorized real cross-ratios in the form

\[
q(F_{m,n}, F_{m+1,n}, F_{m+1,n+1}, F_{m,n+1}) = \frac{\alpha_m}{\beta_n} \quad \forall m, n \in \mathbb{Z},
\]

where \( \alpha_m \in \mathbb{R} \) does not depend on \( n \) and \( \beta_n \in \mathbb{R} \) not on \( m \).

![Figure 1: An Elementary Quadrilateral](image)

**Theorem 6.2** (Bobenko and Pinkall [2]). Let \( F : \mathbb{Z}^2 \to \mathbb{R}^3 \) be a discrete isothermic net. Then the discrete net \( F^* : \mathbb{Z}^2 \to \mathbb{R}^3 \) defined (up to translation) by the formulas

\[
F^*_{m+1,n} - F^*_{m,n} = \alpha_m \frac{F_{m+1,n} - F_{m,n}}{||F_{m+1,n} - F_{m,n}||^2},
\]

\[
F^*_{m,n+1} - F^*_{m,n} = \beta_n \frac{F_{m,n+1} - F_{m,n}}{||F_{m,n+1} - F_{m,n}||^2},
\]

is isothermic. \( F^* \) is called Christoffel dual of \( F \).

We need a formula for the diagonals of its Christoffel dual (Corollary 4.33 in [6]).

**Lemma 6.3.** Given a discrete isothermic net \( F \), the diagonals of any elementary quadrilateral of its Christoffel dual are given by

\[
F^*_{m+1,n} - F^*_{m,n+1} = (\alpha_m - \beta_n) \frac{F_{m+1,n+1} - F_{m,n}}{||F_{m+1,n+1} - F_{m,n}||^2},
\]

\[
F^*_{m+1,n+1} - F^*_{m,n} = (\alpha_m - \beta_n) \frac{F_{m+1,n} - F_{m,n+1}}{||F_{m+1,n} - F_{m,n+1}||^2}.
\]

### 6.2 Infinitesimal flexibility of discrete isothermic nets

Given a discrete isothermic net we first arbitrarily introduce a diagonal for each quadrilateral in order to get a triangulation. Then, we define infinitesimal rotations on faces as follows.
Rule: Suppose $ABCD$ is an elementary quadrilateral of a discrete isothermic net $F : \mathbb{Z} \to \mathbb{R}^3$ and the diagonal $AC$ is inserted. Then we get two triangles $ABC$ and $ACD$. We define infinitesimal rotations $Z_{ABC} := B^*$ and $Z_{ACD} := D^*$ where $B^*$ and $D^*$ are the corresponding vertices of the isothermic dual $F^* : \mathbb{Z} \to \mathbb{R}^3$.

Theorem 6.4. Suppose we are given a discrete isothermic net $F : \mathbb{Z}^2 \to \mathbb{R}^3$ and its dual net $F^* : \mathbb{Z}^2 \to \mathbb{R}^3$. We assume that the faces of the net have been subdivided into triangles in an arbitrary way. Then the infinitesimal rotations defined by the above rule for each triangle are induced by a non-trivial infinitesimal rigid deformation of the triangulated surface. Moreover, the integrated mean curvature is preserved under this deformation.

Proof. By Theorem 6.2 and Lemma 6.3, the infinitesimal rotations of two adjacent triangles are compatible on the common edge. Therefore they define an infinitesimal rigid deformation on the surface.

It remains to show that around an arbitrary vertex the change of mean curvature is zero. For every vertex there are $2^4 = 16$ ways of inserting diagonals on the four neighboring quadrilaterals. Using the symmetries of the problem we can reduce this to 6 cases. We enumerate these 6 cases and calculate the change of mean curvature on each edge in Table 1. It can be checked directly that in all cases the sum around the vertex is zero.

Remark 6.5. Although the infinitesimal rotations on faces depend on the triangulation, the deformations of the edges already present in the quadmesh do not. For example, the change of the edge $F_{m+1,n} - F_{m,n}$ is given by

$$
(F_{m+1,n} - F_{m,n}) = (F_{m+1,n} - F_{m,n}) \times F^*_{m+1,n} = (F_{m+1,n} - F_{m,n}) \times F^*_{m,n} = (F_{m+1,n} - F_{m,n}) \times \frac{F^*_{m+1,n} + F^*_{m,n}}{2}.
$$

Here we have used $(F^*_{m+1,n} - F^*_{m,n}) \parallel (F_{m+1,n} - F_{m,n})$. Moreover, the quadrilaterals do not stay con-circular under the infinitesimal deformation.

The infinitesimal rigid deformation defined above has an exact counterpart in the smooth theory: In the smooth case, the infinitesimal rigid deformation $f$ preserving the mean curvature is given by $df = df \times f^*$, where $f^*$ is the Christoffel dual of $f$. It does not preserve curvature lines. Under these infinitesimal rigid deformations the mean curvature $H$ is preserved. If in addition also the curvature lines were preserved, the whole shape operator would remain unchanged and the deformation would be an infinitesimal Euclidean transformation.

7 Homogeneous discrete cylinders

In this section we show that every homogeneous triangulation of a circular cylinder in $\mathbb{R}^3$ is isothermic. Here “homogeneous” means that the group of Euclidean symmetries acts transitively on vertices and respects the combinatorics. Note that in general none of the edges of such an isothermic discrete cylinder is aligned with the curvature directions of the underlying smooth cylinder.
Table 1: The six types of triangulations around a vertex $F_{m,n}$ and the corresponding change of mean curvature on the edges.

We consider the group $G$ of all Euclidean motions that fix the $z$-axis. Every element $g \in G$ is of the form that acts on points $p \in \mathbb{R}^3$ like

$$g(p) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} p + \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

where $\theta, h \in \mathbb{R}$.

We pick two elements $g_1, g_2$ of $G$ in general position and consider the group $H$ generated by $g_1, g_2$. For a generic choice of $g_1, g_2$ the group $H$ is isomorphic to $\mathbb{Z}^2$. An element $(s, t) \in \mathbb{Z}^2$ corresponds to the element $g_1^s g_2^t \in H$.

We also consider $\mathbb{Z}^2$ as the vertex set of a triangulated surface with faces of the form $\{(s, t), (s+1, t), (s, t+1)\}$ or $\{(s+1, t), (s+1, t+1), (s, t+1)\}$ (Figure 2).
We now define a map $f : \mathbb{Z}^2 \to \mathbb{R}^3$ by picking $r > 0$ and setting
\[ f(s, t) = g_1^s g_2^t (r, 0, 0). \]
For suitable $g_1, g_2 \in G$ this map $f$ will be an immersion. Figure 3 shows a piece of such a discrete surface.

We now prove that immersions $f : M \to \mathbb{R}^3$ constructed as above are isothermic. We will do this by showing that they admit a non-trivial infinitesimal rigid deformation preserving the integrated mean curvature. Note that up to symmetry there are only three types of edges, represented by $\{f(0, 0), f(1, 0)\}, \{f(1, 0), f(0, 1)\}$ and $\{f(0, 1), f(0, 0)\}$. We denote their lengths by
\[ \ell_a(r, \theta_1, h_1, \theta_2, h_2), \quad \ell_b(r, \theta_1, h_1, \theta_2, h_2), \quad \ell_c(r, \theta_1, h_1, \theta_2, h_2). \]
The integrated mean curvature function is the same at all vertices. We denote it by
\[ H(r, \theta_1, h_1, \theta_2, h_2). \]
Now the derivative of the map $\rho := (\ell_a, \ell_b, \ell_c, H) : \mathbb{R}^5 \to \mathbb{R}^4$ has a non-trivial kernel at every point $(r, \theta_1, h_1, \theta_2, h_2) \in \mathbb{R}^5$. Moreover, it is easy to see that any non-zero
\[ (\dot{r}, \dot{\theta}_1, \dot{h}_1, \dot{\theta}_2, \dot{h}_2) \in \ker d\rho \]
will correspond to a non-trivial infinitesimal deformation of $f$. This infinitesimal deformation preserves all edge lengths and the integrated mean curvature around vertices. Therefore the triangulated surface $f$ is isothermic.
8 Inscribed triangular meshes

In this section we will show that any triangulated disk with vertices on a sphere is isothermic. In the smooth theory this corresponds to the fact that any immersed disk on a sphere is isothermic.

**Lemma 8.1.** For any infinitesimal rigid deformation of a triangulated surfaces with vertices on a sphere the integrated mean curvature around each vertex remains unchanged.

**Proof.** Suppose we have an infinitesimal rigid deformation given by rotation vectors \( Z \) on faces that on each edge satisfy the compatible condition

\[
df(e_{ij}) \times (Z_{\phi_{ijk}} - Z_{\phi_{jil}}) = 0 \quad \forall e_{ij} \in E_{or}.
\]

Then we have

\[
Z_{\phi_{ijk}} - Z_{\phi_{jil}} = k_{ij} df(e_{ij})
\]

for some \( k : E_{in} \to \mathbb{R} \). Here \( \phi_{ijk} \) denotes the left face of \( e_{ij} \) while \( \phi_{jil} \) denotes the right face of \( e_{ij} \). For any vertex \( v_i \in V_{in} \),

\[
\sum_j k_{ij} df(e_{ij}) = 0.
\]

Then, since \( |f_i|^2 = |f_j|^2 = 1 \), around any vertex \( v_i \in V_{in} \), the change of integrated mean curvature vanishes:

\[
\dot{H}_i = \sum_j \langle df(e_{ij}), (Z_{\phi_{ijk}} - Z_{\phi_{jil}}) \rangle = \sum_j k_{ij} |df(e_{ij})|^2 = \sum_j k_{ij}(2|f_i|^2 - 2\langle f_i, f_j \rangle) = 2\langle f_i, \sum_j k_{ij}(f_i - f_j) \rangle = 0.
\]

**Remark 8.2.** For any triangulated mesh on a sphere, the above proof shows that a dual 1-form satisfying Equation (2) and (3) will satisfy Equation (4) automatically.

Combining the definition of isothermic triangulated surfaces, the previous lemma implies the following.

**Theorem 8.3.** For a triangulated surface with vertices on a sphere the following is true: If it is infinitesimal flexible, then it is isothermic. For a simply connected surface the converse also holds.

**Example 8.4.** Triangulated disks with vertices on a sphere for which the number of boundary vertices is larger than 4 are known to be infinitesimally flexible and hence are isothermic. We will construct the infinitesimal rigid deformations in Section 9.2.

**Example 8.5.** Jessen’s orthogonal icosahedron is obtained from a regular icosahedron by flipping 6 edges symmetrically without self intersection [12, 13]. Its vertices are the same as for the regular icosahedron and hence lie on a sphere. It is infinitesimal flexible and thus isothermic.
Since the property of being isothermic is Möbius invariant we obtain the following

**Corollary 8.6.** For simply connected triangulated surfaces with vertices on a sphere, the property of being infinitesimal rigid is invariant under Möbius transformations.

Since Möbius transformations of triangulated disks inscribed in a sphere can also be induced from projective transformations of the ambient space. Thus the above corollary is a special case of the projective invariance of infinitesimal rigidity [17].

## 9 Discrete minimal surfaces

In the smooth theory a minimal surface is the Christoffel dual of its Gauß map. We are going to give a definition of discrete minimal surfaces that is inspired by this fact.

**Definition 9.1.** Given a triangulated surface \( f : M \to \mathbb{R}^3 \), a non-constant map \( f^* : M^* \to \mathbb{R}^3 \) is called a Christoffel dual of \( f \) if

\[
\begin{align*}
\sum_{e_{ij} \in E_{or} \setminus \varnothing} \langle df(e_{ij}), df^*(\gamma e_{ij}) \rangle &= 0 \quad \forall \gamma \in V_{in}, \\
\end{align*}
\]

where \( M^* \) denotes the combinatorial dual of the triangulated surface \( M \).

Since we know from the previous section that triangulated disks inscribed in a sphere are isothermic, the following definition seems natural.

**Definition 9.2.** Given a triangulated surface \( f : M \to S^2 \) inscribed in a sphere, its Christoffel dual \( f^* : M^* \to \mathbb{R}^3 \) is called a discrete minimal surface.

Equivalently, discrete minimal surfaces are the reciprocal-parallel meshes of inscribed triangular surfaces. This means that the combinatorics of a discrete minimal surface is that of the dual cell complex and each dual edge is parallel to the corresponding primal edge. Figure 6 shows an inscribed triangulated surfaces together with a corresponding discrete minimal surface.

In Section 9.2 we will obtain a Weierstrass representation for discrete minimal surfaces. The only ingredient that will be needed to describe the surface will be a single harmonic function on a triangulated piece of the plane.
9.1 Smooth minimal surfaces from harmonic functions

In this section we develop a variant of the classical Weierstrass representation for smooth minimal surfaces. This variant needs only a single harmonic function on a planar domain as its input. In Section 9.2 we will give the discrete version of this Weierstrass representation.

We start with investigating infinitesimal normal deformations of surfaces immersed into the plane. Suppose we are given an immersion $f : M \to \text{span}\{j, k\} \subset \mathbb{R}^3$. Its normal $N := i$ is constant. Consider an infinitesimal normal deformation $\dot{f} = uN$ for some $u : M \to \mathbb{R}$. Then the metric is preserved and we can write the deformation as an infinitesimal spin transformation (see [19]). This means that there exists $f^* : M \to \text{Im} \mathbb{H}$ such that

$$duN = df = 2 \text{Im}(df^*).$$

Comparing the tangential components of both sides of this equation we see that $f^* \perp i$. Therefore

$$du = -N df f^* + N f^* df = -* df f^* - f^* * df$$

$$*du = 2 \text{Re}(df f^*)$$

$$d * du = -2 \text{Re}(df \wedge df^*).$$

Hence we obtain

$$\frac{1}{2}(*du + duN) = df f^*$$

(6)

and

$$df \wedge df^* = 0 \iff d * du = 0.$$

Recalling Definition 10.6 of Christoffel duals the above discussion shows the first part of the following lemma.

**Lemma 9.3.** Suppose we are given an immersion $f : M \to \text{span}\{j, k\} \subset \mathbb{H}$ of a disk $M$ with normal $N := i$ and a function $u : M \to \mathbb{R}$. Then the following is true:

1. The map $f^* : M \to \text{Im} \mathbb{H}$ defined by

$$\frac{1}{2}(*du + duN) = df f^*$$

is an immersion and a Christoffel dual of $f$ if and only if

$$d * du = 0$$

and $du$ vanishes nowhere.

2. If $f^*$ is a Christoffel dual of $f$ then $\tilde{f}^* : M \to \text{Im} \mathbb{H}$ defined by

$$d\tilde{f}^* = -(f - i) df^* (f - i).$$

is a minimal surface.
Proof. The first part follows from the previous discussion. For the second part we consider an inversion \( \sigma \) with respect to the sphere centered at \( i \) with radius \( \sqrt{2} \). Restricted to the \((j, k)\)-plane \( \sigma \) is just stereographic projection and we obtain an embedding \( \tilde{f} := \sigma \circ f : \mathbb{R}^2 \to S^2 \setminus \{i\} \). It satisfies

\[
d\tilde{f} = -(f - i) d\sigma(f - i)^{-1}.
\]

The proof of Theorem 10.5 shows that the Christoffel dual \( \tilde{f}^* \) of \( \tilde{f} \) is given by

\[
d\tilde{f}^* = -(f - i) df^*(f - i).
\]

Now our claim follows from the fact that Christoffel duals of conformal immersions into a round sphere are minimal surfaces.

We compare the above lemma with the Weierstrass representation of minimal surfaces. Suppose we have a holomorphic function \( f : \mathbb{C} \to \mathbb{C} \) with \( f_z(0) \neq 0 \). On a neighborhood \( U \subset \mathbb{C} \) of the origin, we can assume \( f(z) = z \) without loss of generality.

**Theorem 9.4.** Given an immersion \( f : U \subset \mathbb{C} \to \text{Im} \mathbb{H} \) defined by \( f(x, y) = x j + y k \) and a harmonic function \( u : U \to \mathbb{R} \). The induced minimal surface \( f^* : U \to \mathbb{R}^3 \cong \text{Im} \mathbb{H} \) from Lemma 9.3 satisfies

\[
df^* = -(xj + yk - i) df^*(xj + yk - i) = \text{Re} \left( \frac{z}{(1 - z^2)^{1/2}} \right) dz
\]

where \( f^* := \frac{1}{2}(u_yj + u_xk) \), \( z = x + iy \in \mathbb{C} \) and \( h := u_{yz} - iu_{yy} = 2iu_{z2} \). Here \( h \) is holomorphic, since \( u \) is harmonic.

**Proof.** Substituting \( f(x, y) = x j + y k \) into Equation (6), we have

\[
\frac{1}{2} (u_y + u_xi) dx + (-u_x + u_yi) dy = (jdx + kdy)f^*,
\]

which implies

\[
f^* = \frac{1}{2}(u_yj + u_xk).
\]

Lemma 9.3 implies the map \( \tilde{f}^* : U \to \text{Im} \mathbb{H} \) defined by

\[
d\tilde{f}^* = -(f - i) df^*(f - i)
\]

\[
= (u_{yz}(xi - x^2 + y^2 + 1) - xyk) + u_{yy}(yi - xyj + x^2 - y^2 + 1) + u_{xx}(yi - xyj + x^2 - y^2 + 1) + u_{yy}(yi - xyj + x^2 - y^2 + 1) dy
\]

is a minimal surface. The second equality in equation (7) follows by a straightforward computation. Here we only expand the first component of the right hand side.

\[
\text{Re}(h(z) zdz) = \text{Re}((u_{yz} - iu_{yy})(x + iy)(dx + idy)) = u_{yz}(xdx - ydy) + u_{yy}(ydx + xdy)
\]

Note that if we pick \( u(x, y) = xy \), we get Enneper’s minimal surface (See Figure 6 for its discrete analogue).
9.2 Harmonic functions on planar triangular meshes

In Section 9.2 we described a Weierstrass representation for smooth minimal surfaces in terms of a single harmonic function on a simply connected planar domain. In this section we will develop a discrete analog. We first show how all Christoffel duals of a planar triangle mesh can be obtained in terms of a single discrete harmonic function. The discrete analogue of the Laplace operator turns out to be the cotangent Laplacian [24].

Theorem 9.5. Let $f : M \to \mathbb{R}^2 \subset \mathbb{R}^3$ be a triangulated disk immersed in a plane. Then to each dual 1-form $\tau$ satisfying Definition 1.1 there corresponds a harmonic function $u : V \to \mathbb{R}$. $u$ is unique up to a linear function. Here $u$ being harmonic means that for any vertex $v_i \in V$ we have

$$\sum_{e_{ij} \in E_i} (\cot \beta_k + \cot \beta_l)(u_j - u_i) = 0.$$  

$\tau$ is obtained from $u$ via

$$\tau(\ast e_{ij}) = \frac{u_i df(e_{jk}) + u_j df(e_{ki}) + u_k df(e_{ij})}{2A_{ijk}} - \frac{u_j df(e_{ij}) + u_i df(e_{ij}) + u_k df(e_{il})}{2A_{jil}}.$$  

Figure 5: Two neighboring triangles sharing the edge $e_{ij}$

Proof. Since the surface is simply connected, all dual 1-forms satisfying Definition 1.1 come from infinitesimal rigid deformations preserving the integrated mean curvature.

Let $N$ denote the normal of the plane. It is easy to see that for any function $u : V \to \mathbb{R}$ the infinitesimal deformation $\dot{f} = uN$ preserves edge lengths. Up to global rotations and translation these are the only infinitesimal rigid deformations of $f$. The corresponding infinitesimal rotations of the faces $\phi_{ijk}$ are given by

$$Z_{\phi_{ijk}} = \frac{u_i df(e_{jk}) + u_j df(e_{ki}) + u_k df(e_{ij})}{2A_{\phi_{ijk}}}.$$  

where $A_{\phi_{ijk}}$ is the area of the triangle $\phi_{ijk}$. This follows from

$$(df(\dot{e}_{ij})) = u_j N - u_i N = df(e_{ij}) \times Z_{\phi_{ijk}}.$$  

In order for the change of mean curvature on vertices to vanish the map $Z : F \to \mathbb{R}^3$ has to satisfy

$$0 = \sum_{e_{ij} \in E_{\text{ver}, i}} \langle df(e_{ij}), Z_{\phi_{ijk}} - Z_{\phi_{jil}} \rangle.$$  

(9)
for all vertices \( v_i \in V_\alpha \). Expanding this expression we collect terms and obtain

\[
\text{the coefficient of } u_j = -\frac{\langle df(e_{ij}), df(e_{ik}) \rangle}{2A_{ijk}} - \frac{\langle df(e_{ij}), df(e_{ik}) \rangle}{2A_{ijk}} \\
+ \frac{\langle df(e_{il}), df(e_{ij}) \rangle}{2A_{ijk}} + \frac{\langle df(e_{il}), df(e_{ij}) \rangle}{2A_{ijk}}
\]

\[
= -\frac{\langle df(e_{ij}) - df(e_{il}), df(e_{ij}) \rangle}{2A_{ijl}} \\
+ \frac{\langle df(e_{ik}), df(e_{ij}) - df(e_{ik}) \rangle}{2A_{ijk}}
\]

\[
= -\frac{\langle df(e_{ij}), df(e_{il}) \rangle}{2A_{ijl}} - \frac{\langle df(e_{ik}), df(e_{lj}) \rangle}{2A_{ilj}}
\]

\[
= \cot \beta_l + \cot \beta_k,
\]

the coefficient of \( u_i = \sum_{ijk \in F:i} \frac{\langle df(e_{ij}), df(e_{jk}) \rangle - \langle df(e_{ik}), df(e_{jk}) \rangle}{2A_{ijk}}
\]

\[
= -\sum_{ijk \in F:i} (\cot \beta_j + \cot \beta_k)
\]

\[
= -\sum_{ij \in E:i} (\cot \beta_k + \cot \beta_l).
\]

Hence, equation \([9]\) is equivalent to the condition that around any vertex \( v_i \in V_\alpha \) we have

\[
0 = \sum_{e_{ij} \in E:i} (\cot \beta_k + \cot \beta_l)(u_j - u_i).
\]

We now show the uniqueness. Note that the dual 1-form \( \tau \) defined by

\[
\tau(e_{ij}) = Z_{\phi_{ijk}} - Z_{\phi_{jl}}
\]

vanishes identically if and only if for every edge \( e_{ij} \in E \)

\[
0 = Z_{\phi_{ijk}} - Z_{\phi_{jl}}
\]

\[
= u_idf(e_{jk}) + u_idf(e_{ki}) + u_kdf(e_{ij}) - u_idf(e_{lj}) + u_idf(e_{kj}) + u_kdf(e_{ij})
\]

\[
= -N \times (\text{grad } u_{ijk} - \text{grad } u_{jl}).
\]

Here the function \( \text{grad } u : F \to \mathbb{R}^2 \in \mathbb{R}^3 \) denotes the gradient of \( u \) given by

\[
\text{grad } u_{ijk} = N \times \frac{u_jdf(e_{jk}) + u_jdf(e_{ki}) + u_kdf(e_{ij})}{2A_{ijk}}
\]

which satisfies

\[
\langle \text{grad } u_{ijk}, df(e) \rangle = du(e) \forall e \subset \phi_{ijk} \in F.
\]

Hence, the dual 1-form vanishes identically if and only if there exists \( a \in \mathbb{R}^3 \) such that

\[
\text{grad } u \equiv a,
\]

or equivalently

\[
u = \langle a, f \rangle + c
\]

for some \( a \in \mathbb{R}^3 \) and \( c \in \mathbb{R}. \) \( \square \)
Starting with an immersed triangulated disk \( f \) in the plane together with a discrete harmonic function \( u \) we first apply a Möbius transformation to \( f \) and obtain a triangulated disk \( \tilde{f} \) on the unit sphere. The dual 1-form \( \tau \) corresponding to \( u \) is transformed via Lemma 3.2. Then we integrate the resulting \( \tilde{\tau} \) and obtain a discrete minimal surface.

Figure 6: Right: A discrete Enneper surface corresponding to the discrete harmonic function \( u|_{\partial M} = xy \). The edges are parallel to the primal edges of the Gauss image shown on the left.

As an example we take a triangulated square \( M \) in the \((x,y)\)-plane (Figure 6 left shows the stereographic projection of \( M \)). The harmonic function \( u \) can be defined by its boundary values. The choice \( u|_{\partial M} = xy \) leads to a discrete Enneper surface (Figure 6 right).

Figure 7: A discrete catenoid and a discrete helicoid.

As a second example, we consider a triangulated annulus centered at the origin with a cut along the positive x-axis. We solve for the discrete harmonic function \( u \) with boundary values either given by \( u|_{\partial M} = \log |z| \) or by \( u|_{\partial M} = \arg z \). We obtain a discrete helicoid and a discrete catenoid (Figure 7) respectively.
10 Smooth analogs

The main goal of this section is to prove the smooth analogue of Lemma 8.1:

**Theorem 10.1.** For every infinitesimal rigid deformation of an immersion \( f : M \to S^2 \) into a round sphere, the mean curvature is preserved.

Beyond this we also use the opportunity to review some known results on smooth isothermic surfaces that directly correspond to our discrete results. We rely on the treatment of smooth isothermic surfaces by means of quaternionic analysis as developed in [19, 20, 25].

**Theorem 10.2.** Suppose \( f : M \to \mathbb{R}^3 = \text{Im} \mathbb{H} \) is a simply connected smooth surface and \( \lambda : M \to \mathbb{H} \) is a function. Then there exists an infinitesimal conformal deformation \( \hat{f} : M \to \mathbb{R}^3 = \text{Im} \mathbb{H} \) given by

\[
\frac{d\hat{f}}{df} = 2 \text{Im}(df\lambda)
\]

if and only if there exists \( \hat{\rho} : M \to \mathbb{R} \) such that

\[
D\lambda := - \frac{df \wedge d\lambda}{|df|^2} = \hat{\rho}.
\]

In particular \( \hat{\rho} = (H|df|) \) is the change of mean curvature half density.

**Definition 10.3.** A smooth surface \( f : M \to \mathbb{R}^3 \cong \text{Im} \mathbb{H} \) is called isothermic if locally there exists a non-trivial \( \text{Im} \mathbb{H} \)-valued closed 1-form \( \tau \) such that

\[
df \wedge \tau = 0.
\]

**Theorem 10.4.** A smooth surface \( f : M \to \mathbb{R}^3 \cong \text{Im} \mathbb{H} \) is isothermic if and only if locally there exists a non-trivial infinitesimal rigid deformation such that the mean curvature is unchanged.

**Proof.** Let \( f \) be isothermic and \( \tau \) be the corresponding 1-form. The closedness of \( \tau \) implies that on any simply connected open subset \( U \subset M \) there exists \( \lambda : U \to \text{Im} \mathbb{H} \) such that

\[
\tau = d\lambda.
\]

Because of \( df \wedge d\lambda = 0 \) the 1-form \( \text{Im}(df\lambda) \) is closed. Thus there exists \( \hat{f} : U \to \text{Im} \mathbb{H} \) such that

\[
\frac{d\hat{f}}{df} = 2 \text{Im}(df\lambda)
\]

\[
(H|df|)' = \hat{\rho} = 0.
\]

Since \( \lambda \) is purely imaginary \( \hat{f} \) is a rigid deformation and \( \hat{H} = 0 \). The converse is proved similarly. \( \square \)

**Theorem 10.5.** The class of isothermic surfaces is Möbius invariant.

**Proof.** It suffices to consider the Möbius transformation as minus the inversion in the unit sphere:

\[
f^{-1} = -\frac{f}{|f|^2}.
\]

We have

\[
|df^{-1}| = |df| |f^{-1}|.
\]

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Suppose $\tau$ is a non-trivial $\text{Im} \mathbb{H}$-valued closed 1-form $\tau$ such that
\[ df \wedge \tau = 0. \]
Then the $\text{Im} \mathbb{H}$-valued 1-form $\tilde{\tau} := f\tau \bar{f}$ is closed:
\[ d\tilde{\tau} = df \wedge \tau \bar{f} - f\tau \wedge d\bar{f} = 0 \]
Moreover, it satisfies
\[ df^{-1} \wedge \tilde{\tau} = \bar{f}^{-1}df \wedge \tau \bar{f} = 0. \]

We need the global existence of $\tau$ in order to relate it to the space of immersions.

**Definition 10.6.** A smooth surface $f : M \to \mathbb{R}^3 \cong \text{Im} \mathbb{H}$ is called strongly isothermic if there exists a non-trivial $\text{Im} \mathbb{H}$-valued closed 1-form $\tau$ on $M$ such that
\[ df \wedge \tau = 0. \]
In addition, if $\tau$ is exact and $\tau = df^*$ for some $f^* : M \to \mathbb{R}^3 \cong \text{Im} \mathbb{H}$, then $f^*$ is called a Christoffel dual of $f$.

The smooth analog of Theorem 4.8 in Section 4 is known:

**Theorem 10.7** (Bohle et al. [7]). Strongly isothermic immersions of a compact surface are the points in the space of immersions where the map from the space of immersions to Teichmüller space (which assigns to each immersion the conformal class of its induced metric) fails to be a submersion.

A result related to Theorem 1.4 can be found in [9]:

**Theorem 10.8.** Let $f_1 : M \to S^3$ be two non-congruent conformal immersions inducing the same Hopf differentials. Then $f_1$ and $f_2$ are isothermic surfaces in the same associated family.

Finally, we establish a smooth counterpart of Theorem 8.1 in Section 8. We first need a lemma.

**Lemma 10.9.** Let $M$ be a surface with a Riemannian metric and $f : M \to \mathbb{R}^3$ be an isometric immersion. Let $\lambda = g + df(Y) + hN$ be an $\mathbb{H}$-valued function on $M$ where $g, df(Y)$ and $hN$ are its scalar, tangential and normal components. Then we can express the Dirac operator of $f$ in terms of standard operators from the vector calculus on $M$ as
\[ D\lambda = -\text{curl} Y + df(J \text{grad} g - AY + \text{grad} h) - (\text{div} Y + 2hH)N. \]

**Proof.** In the following $X \in T_p M$ will be some unit tangent vector. We first consider the action of the Dirac operator on the scalar component.

\[ -df \wedge dg(X, JX) = -df(X)dg(JX) + df(JX)dg(X) \]
\[ = Ndf(dg(X)X + dg(JX)JX) \]
\[ = Ndf(\text{grad} g). \]

Then we consider the normal component.

\[ -df \wedge d(hN)(X, JX) = ((-df \wedge dh)N - hdf \wedge dN)(X, JX) \]
\[ = Ndf(\text{grad} h)N - h(df(X)dN(JX) - df(JX)dN(X)) \]
Finally we look at the tangential component. Notice that for an immersed surface in Euclidean space the induced Levi-Civita connection is given as follows: for any tangent vector field \( Y \) and tangent vector \( Z \),

\[
df(\nabla_Z Y) = df(\nabla_Z)(Y) - \langle df(Y), df(AZ) \rangle N
\]

where \( A \) is the shape operator of the immersion \( f \). We recall the definition of curl and divergent operator of a tangent vector field \( Y \):

\[
div(Y) : = \langle X, \nabla_X Y \rangle + \langle X, \nabla Y \rangle
\]

\[
curl(Y) : = \langle JX, \nabla_X Y \rangle - \langle X, \nabla JX Y \rangle
\]

\[
= -\langle X, \nabla_X JY \rangle - \langle JX, \nabla JX Y \rangle
\]

Collecting the above information we now obtain

\[
- df \wedge d(df(Y))(X, JX)
= - df(X)(df(\nabla_J Y) - \langle Y, AJX \rangle) + df(JX)(df(\nabla X Y) - \langle Y, AX \rangle N)
= (X, \nabla_J X Y) - (JX, \nabla JX Y) N
+ (\langle -X, \nabla_J Y \rangle N - \langle AY, JX \rangle df(JX) - \langle AY, X \rangle df(X)

= - curl Y - (div Y) N - df(AY).
\]

Suppose now that \( f : M \to S^2 \) is an immersion and \( \dot{\lambda} : M \to \text{Im}(\text{H}) \) is an infinitesimal rigid deformation of \( f \). Then writing \( \dot{\lambda} = df(Y) + hN \), we have

\[
(H|df|) = D \dot{\lambda} = - \text{curl} Y + df(-Y + \text{grad} h) - ((\text{div} Y + 2h)) N.
\]

Comparing the imaginary part yields

\[
Y = \text{grad} h
\]

and thus

\[
(H|df|) = - \text{curl}(Y) = - \text{curl}(\text{grad} h) = 0.
\]

The above argument provides a proof for Theorem 10.1.

References

[1] A. Bobenko, U. Hertrich-Jeromin, and I. Lukyanenko, Discrete constant mean curvature nets in space forms: Steiners formula and christoffel duality, Discrete Comput. Geom., 52 (2014), pp. 612–629.

[2] A. Bobenko and U. Pinkall, Discrete isothermic surfaces, J. Reine Angew. Math., 475 (1996), pp. 187–208.
A. Bobenko, U. Pinkall, and B. Springborn, *Discrete Conformal Maps and Ideal Hyperbolic Polyhedra*, Preprint, (2010). [http://arxiv.org/abs/1005.2698](http://arxiv.org/abs/1005.2698).

A. I. Bobenko, T. Hoffmann, and B. A. Springborn, *Minimal surfaces from circle patterns: geometry from combinatorics*, Ann. of Math. (2), 164 (2006), pp. 231–264.

A. I. Bobenko, H. Pottmann, and J. Wallner, *A curvature theory for discrete surfaces based on mesh parallelity*, Math. Ann., 348 (2010), pp. 1–24.

A. I. Bobenko and Y. B. Suris, *Discrete differential geometry: Integrable structure*, vol. 98 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2008.

C. Bohle, G. P. Peters, and U. Pinkall, *Constrained Willmore surfaces*, Calc. Var. Partial Differential Equations, 32 (2008), pp. 263–277.

F. Burstall, U. Hertrich-Jeromin, and W. Rossman, *Discrete linear Weingarten surfaces*, Preprint, (2014). [http://arxiv.org/abs/1406.1293](http://arxiv.org/abs/1406.1293).

F. Burstall, F. Pedit, and U. Pinkall, *Schwarzian derivatives and flows of surfaces*, in *Discrete geometry and integrable systems (Tokyo, 2000)*, vol. 308 of *Contemp. Math.*, Amer. Math. Soc., Providence, RI, 2002, pp. 39–61.

J. Cieśliński, P. Goldstein, and A. Sym, *Isothermic surfaces in $\mathbb{E}^3$ as soliton surfaces*, Phys. Lett. A, 205 (1995), pp. 37–43.

M. Desbrun, E. Kanso, and Y. Tong, *Discrete differential forms for computational modeling*, in *Discrete differential geometry*, vol. 38 of Oberwolfach Semin., Birkhäuser, Basel, 2008, pp. 287–324.

M. Goldberg, *Unstable polyhedral structures*, Math. Mag., 51 (1978), pp. 165–170.

P. Heawood, *Map-color theorem*, Quart. J Pure Appl. Math, 24 (1890), pp. 332–338.

U. Hertrich-Jeromin, *Introduction to Möbius differential geometry*, vol. 300 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2003.

U. Hertrich-Jeromin, T. Hoffmann, and U. Pinkall, *A discrete version of the Darboux transform for isothermic surfaces*, in *Discrete integrable geometry and physics (Vienna, 1996)*, vol. 16 of *Oxford Lecture Ser. Math. Appl.*, Oxford Univ. Press, New York, 1999, pp. 59–81.

S. Hougardy, F. H. Lutz, and M. Zelke, *Polyhedral of genus 2 with 10 vertices and minimal coordinates*, Electronic Geometry Model No. 2005.08.001, (2007).

I. Izmestiev, *Projective background of the infinitesimal rigidity of frameworks*, Geom. Dedicata, 140 (2009), pp. 183–203.

B. Jessen, *Orthogonal icosahedra*, Nordisk Mat. Tidskr, 15 (1967), pp. 90–96.

G. Kamberov, P. Norman, F. Pedit, and U. Pinkall, *Quaternions, spinors, and surfaces*, vol. 299 of Contemporary Mathematics, American Mathematical Society, Providence, RI, 2002.

G. Kamberov, F. Pedit, and U. Pinkall, *Bonnet pairs and isothermic surfaces*, Duke Math. J., 92 (1998), pp. 637–644.
[21] F. Luo, *Combinatorial Yamabe flow on surfaces*, Commun. Contemp. Math., 6 (2004), pp. 765–780.

[22] I. Pak, *Lectures on Discrete and Polyhedral Geometry*, 2010.

[23] F. Pedit and U. Pinkall, *Quaternionic analysis on Riemann surfaces and differential geometry*, in Proceedings of the International Congress of Mathematicians, no. Vol. II, 1998, pp. 389–400.

[24] U. Pinkall and K. Polthier, *Computing discrete minimal surfaces and their conjugates*, Experiment. Math., 2 (1993), pp. 15–36.

[25] J. Richter, *Conformal maps of a Riemann surface into space of quaternions*, PhD thesis, TU Berlin, 1997.

[26] W. K. Schief, A. I. Bobenko, and T. Hoffmann, *On the integrability of infinitesimal and finite deformations of polyhedral surfaces*, in Discrete differential geometry, vol. 38 of Oberwolfach Semin., Birkhäuser, Basel, 2008, pp. 67–93.

[27] J. M. Sullivan, *Curvatures of smooth and discrete surfaces*, in Discrete differential geometry, vol. 38 of Oberwolfach Semin., Birkhäuser, Basel, 2008, pp. 175–188.

[28] J. Wallner and H. Pottmann, *Infinitesimally flexible meshes and discrete minimal surfaces*, Monatsh. Math., 153 (2008), pp. 347–365.

[29] W. Whiteley, *Rigidity and scene analysis*, in Handbook of discrete and computational geometry, CRC Press Ser. Discrete Math. Appl., CRC, Boca Raton, FL, 1997, pp. 893–916.

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