Solvable Leibniz algebras with quasi-filiform Lie algebras of maximum length nilradicals

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\textbf{ABSTRACT}

In this article, solvable Leibniz algebras, whose nilradical is quasi-filiform Lie algebra of maximum length, are classified. The rigidity of such Leibniz algebras with two-dimensional complemented space to the nilradical is proved.

\textbf{1. Introduction}

Leibniz algebras are “non-commutative” analog of Lie algebras. They were introduced about twenty-five years ago by J.-L. Loday in his cyclic homology study [9, 19, 20]. Since the theory of Leibniz algebras has been actively studied and many results of the theory of Lie algebras have been extended to Leibniz algebras. Leibniz algebras inherit an important property of Lie algebras, the right multiplication operator of a Leibniz algebra is a derivation.

Thanks to Levi-Malcev’s theorem, the solvable Lie algebras [21, 22] have played an important role in the theory of Lie algebras over the last years, either in the classification theory or in geometrical and analytical applications. The investigation of solvable Lie algebras with special types of nilradicals was the subject of various paper [3, 6, 22, 24]. In Leibniz algebras, the analog of Levi-Malcev’s theorem was recently proved in [8], thus solvable Leibniz algebras also play a central role in the study of Leibniz algebras. In particular, the classifications of \( n \)-dimensional solvable Leibniz algebras with some restriction on their nilradicals have been obtained (see [1, 2, 11–13, 23]).

Since the description of finite-dimensional solvable Lie algebras is a boundless problem, lately geometric approaches have been developed. Relevant tools of geometric approaches are Zariski topology and natural action of the linear reductive group on varieties of algebras in such a way that the orbits under the action consist of isomorphic algebras. It is a well-known result of algebraic geometry that any algebraic variety (evidently, algebras defined via identities form an
algebraic variety) is a union of a finite number of irreducible components. The most important algebras are those whose orbits under the action are open sets in Zariski topology. The algebras of a variety with open orbits are important since the closures of the orbits of such algebras form irreducible components of the variety. At the same time, there are irreducible components which cannot be obtained via closure of the orbit of any algebra. This fact does not detract the importance of algebras with open orbits. This is a motivation of many works focused to discovering algebras with open orbits and to describe sufficient properties of such algebras [10, 16, 17].

The purpose of the present work is to continue the study of solvable Leibniz algebras with a given nilradical. Thanks to the work [15] the quasi-filiform Lie algebras of maximum length are known and we focus our attention to the description of the solvable Leibniz algebras with quasi-filiform Lie algebras of maximum length nilradicals. In order to achieve our description, we use the method which involve non-nilpotent outer derivations of the nilradicals.

Throughout this article vector spaces and algebras are finite-dimensional over the field of the complex numbers. Moreover, in the table of multiplication of an algebra the omitted products are assumed to be zero and, if it is not noted, we consider non-nilpotent solvable algebras.

2. Preliminaries

In this section we give necessary definitions and preliminary results.

**Definition 2.1** ([19]). A vector space $L$ over a field $\mathbb{F}$ equipped with bilinear bracket $[-,-]$ is called a *Leibniz algebra* if for any $x,y,z \in L$ the so-called Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [x,[z,y]]$$

holds.

For examples of Leibniz algebras we refer to the papers [2, 11, 19, 20] and references therein.

Further we use the following notation

$$\mathcal{L}(x,y,z) = [x,[y,z]] - [[x,y],z] + [[x,z],y].$$

It is obvious that Leibniz algebras are determined by the identity $\mathcal{L}(x,y,z) = 0$.

From the Leibniz identity we conclude that the elements $[x,x], [x,y] + [y,x]$ for any $x,y \in L$ lie in $\text{Ann}_r(L) = \{x \in L | [y,x] = 0, \forall y \in L\}$, the *right annihilator* of the Leibniz algebra $L$. Note that $\text{Ann}_r(L)$ is a two-sided ideal of $L$.

The set $\text{Center}(L) = \{x \in L | [x,y] = 0, \forall y \in L\}$ is said to be the *center* of $L$ and it forms a two-sided ideal of $L$.

The notion of derivation for the Leibniz algebras case is defined as usual, that is, a linear map $d : L \to L$ of a Leibniz algebra $L$ is called a *derivation* if it satisfies the condition

$$d([x,y]) = [d(x),y] + [x,d(y)], \forall x,y \in L. \quad (2.1)$$

The Leibniz identity implies that the right multiplication operator $R_x : L \to L, R_x(y) = [y,x], y \in L$, is a derivation.

For a given Leibniz algebra $L$ the *lower central* and *derived series* are defined as follows:

$$L^1 = L, L^{k+1} = [L^k, L], k \geq 1, \quad L^{[s]} = L, L^{[s+1]} = [L^{[s]}, L^{[s]}], s \geq 1,$$

respectively.

**Definition 2.2.** A Leibniz algebra $L$ is said to be *nilpotent* (respectively, *solvable*), if there exists $k \in \mathbb{N}$ ($s \in \mathbb{N}$) such that $L^k = \{0\}$ (respectively, $L^{[s]} = \{0\}$). The minimal number $k$ with such property is said to be the *index of nilpotency* of the algebra $L$.

Clearly, the index of nilpotency of an $n$-dimensional nilpotent Leibniz algebra is not greater than $n + 1$.

Recall, the maximal nilpotent ideal of a Leibniz algebra is said to be the *nilradical* of the algebra.

Let $R$ be a solvable Leibniz algebra with a nilradical $N$. We denote by $Q$ the complementary vector space of the nilradical $N$ to the algebra $R$. Let us consider the restrictions to $N$ of the right
multiplication operator on an element \( x \in Q \) (denoted by \( R_{x|N} \)). From [12] we know that for any \( x \in Q \), the operator \( R_{x|N} \) is a non-nilpotent derivation of \( N \).

Let \( \{x_1, \ldots, x_m\} \) be a basis of \( Q \), then for any scalars \( \{x_1, \ldots, x_m\} \in \mathbb{C} \setminus \{0\} \), the matrix \( x_1 R_{x_i|N} + \ldots + x_m R_{x_m|N} \) is non-nilpotent, which means that the operators \( \{R_{x_1|N}, \ldots, R_{x_m|N}\} \) are nil-independent [22]. Therefore, we have that the dimension of \( Q \) is bounded by the maximal number of nil-independent derivations of the nilradical \( N \) (see [12, Theorem 3.2]).

Below we define the notion of quasi-filiform Leibniz algebra.

**Definition 2.3.** An \( n \)-dimensional Leibniz algebra is called *quasi-filiform* if its index of nilpotency is equal to \( n - 1 \).

A Leibniz algebra \( L \) is \( \mathbb{Z} \)-graded, if \( L = \bigoplus_{i \in \mathbb{Z}} V_i \), where \( [V_i, V_j] \subseteq V_{i+j} \) for any \( i, j \in \mathbb{Z} \) with a finite number of non-null spaces \( V_i \).

We say that a nilpotent Leibniz algebra \( L \) admits the connected gradation \( L = V_{k_1} \oplus \ldots \oplus V_{k_t} \), if \( V_{k_i} \neq \{0\} \) for any \( 1 \leq i \leq t \).

**Definition 2.4.** The number \( l(\bigoplus L) = l(V_{k_1} \oplus \ldots \oplus V_{k_t}) = k_t - k_1 + 1 \) is called the *length of the gradation*. A gradation is called of maximum length, if \( l(\bigoplus L) = \dim(L) \).

We denote by \( l(L) = \max\{l(\bigoplus L) \text{ such that } L = V_{k_1} \oplus \ldots \oplus V_{k_t} \text{ is a connected gradation}\} \) the length of an algebra \( L \).

**Definition 2.5.** A Leibniz algebra \( L \) is called of maximum length if \( l(L) = \dim(L) \).

In the next theorem, we present the classification of quasi-filiform Lie algebras of maximum length given in [15].

**Theorem 2.6.** Let \( L \) be an \( n \)-dimensional quasi-filiform Lie algebra of maximum length. Then the algebra \( L \) is isomorphic to one of the following pairwise non-isomorphic algebras:

\[
g_{1(n,1)}^1 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq n - 2, \\ [e_i, e_n] = (-1)^i e_n, & 2 \leq i \leq \frac{n-1}{2}, n \geq 5 \text{ and } n \text{ is odd}; \end{cases}
\]

\[
g_{2(n,1)}^2 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq n - 2, \\ [e_i, e_n] = e_{i+2}, & 2 \leq i \leq n - 3, n \geq 5; \end{cases}
\]

\[
g_{2(n,1)}^3 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq 7, \\ [e_2, e_i] = e_{i+2}, & 3 \leq i \leq 4, \\ [e_3, e_i] = -2e_{i+3}, & 4 \leq i \leq 5, \\ [e_i, e_n] = (-1)^i e_n, & 2 \leq i \leq 4; \end{cases}
\]

\[
g_{11}^1 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq 9, \\ [e_2, e_i] = e_{i+2}, & 3 \leq i \leq 4, \\ [e_3, e_i] = -e_{i+2}, & 6 \leq i \leq 7, \\ [e_4, e_i] = e_{i+4}, & 5 \leq i \leq 6, \\ [e_i, e_{n-1}] = (-1)^i e_{n+1}, & 2 \leq i \leq 5, \\ \end{cases}
\]

where \( \{e_1, e_2, \ldots, e_n\} \) is a basis of the algebra.
We call a vector space $M$ a module over a Leibniz algebra $L$ if there are two bilinear maps:

$$[-,-]: L \times M \to M \quad \text{and} \quad [-,-]: M \times L \to M$$

satisfying the following three axioms

$$[m,[x,y]] = [[m,x],y] - [[m,y],x],$$

$$[x,[m,y]] = [[x,m],y] - [[x,y],m],$$

$$[x,y,m] = [[x,y],m] - [[x,m],y],$$

for any $m \in M, x, y \in L$.

For a Leibniz algebra $L$ and a module $M$ over $L$ we consider the spaces

$$CL^0(L,M) = M, \quad CL^n(L,M) = \text{Hom}(L\otimes^n, M), n > 0.$$

Let $d^n : CL^n(L,M) \to CL^{n+1}(L,M)$ be an $\mathbb{F}$-homomorphism defined by

$$(d^n \varphi)(x_1, \ldots, x_{n+1}) := [x_1, \varphi(x_2, \ldots, x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [\varphi(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}), x_i]$$

$$+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \varphi(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}),$$

where $\varphi \in CL^n(L,M)$ and $x_i \in L$. The property $d^{n+1}d^n = 0$ leads that the derivative operator $d = \sum_{i \geq 0} d^i$ satisfies the property $d \circ d = 0$. Therefore, the $n$-th cohomology group is well defined by

$$HL^n(L,M) := ZL^n(L,M)/BL^n(L,M),$$

where the elements $ZL^n(L,M) := \text{Ker} d^{n+1}$ and $BL^n(L,M) := \text{Im} d^n$ are called $n$-cocycles and $n$-coboundaries, respectively.

Although in general the computation of cohomology groups is complicated, the Hochschild-Serre factorization theorem simplifies its computation for semidirect sums of algebras [18]. If $R = NQ$ is a solvable Lie algebra such that $Q$ is abelian and the operators $R_x$ for all $x \in Q$ are diagonal, then the adjoint Lie $n$-th cohomology $H^n(R, R)$ satisfies the following isomorphism [4, 5]

$$H^n(R, R) \simeq \sum_{a+b=n} H^a(Q, F) \otimes H^b(N, R)^Q,$$

where

$$H^b(N, R)^Q = \{ \varphi \in H^b(N, R) | (d^b \varphi)(x, z_1, \ldots, z_b) = 0, x \in Q, z_i \in N \}.$$

The linear reductive group $GL_n(\mathbb{F})$ acts on $\text{Leib}_n$, the variety of $n$-dimensional Leibniz algebra structures, via change of basis, i.e.

$$(g \cdot \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y))), \quad g \in GL_n(\mathbb{F}), \lambda \in \text{Leib}_n.$$

The orbits $\text{Orb}(-)$ under this action are the isomorphism classes of algebras. Recall, Leibniz algebras with open orbits are called rigid. The importance of rigid algebras follows from the following fact: the closure of the orbit of a rigid algebra forms an irreducible component of the variety and the variety is the union of finite numbers of irreducible components.

Here we give a result, which asserts that the triviality of the second cohomology of a Leibniz algebra implies its rigidity.

**Theorem 2.7** ([7]). If the second cohomology $HL^2(L,L)$ of a Leibniz algebra $L$ is trivial, then it is a rigid algebra.
3. Solvable Leibniz algebras with quasi-filiform Lie algebras of maximum length nilradicals

In this section, we describe solvable Leibniz algebras whose nilradical is a quasi-filiform Lie algebra of maximum length. The classification of solvable Leibniz algebras with nilradical a five-dimensional quasi-filiform Lie algebras of maximal length was already obtained in [23]. Therefore, we focus on the case in which the nilradical is a quasi-filiform Lie algebra of maximum length of dimension greater than five.

For a solvable Leibniz algebra \( R = N \oplus Q \) with nilradical \( N \) and \( s \)-dimensional complemented space \( Q \) we use notation \( R(N, s) \).

In order to start the description we need to know the derivations of quasi-filiform Lie algebras of maximum length.

3.1. Derivations of algebras \( g^i, i=1, 2, 3 \)

In this subsection, we describe the spaces of derivations of quasi-filiform Lie algebras of maximum length.

Proposition 3.1. Any derivation of quasi-filiform Lie algebras of maximum length has the following form:

for the algebra \( g^1(n, 1) \),

\[
\begin{align*}
    d(e_1) &= \sum_{i=1}^{n} a_i e_i, \\
    d(e_2) &= \sum_{i=2}^{n} b_i e_i, \\
    d(e_i) &= ((i - 2)a_1 + b_2)e_i + \sum_{t=i+1}^{n-1} b_{t-i+2} e_t + (-1)^i a_{n-i+1} e_n, 3 \leq i \leq n - 2, \\
    d(e_{n-1}) &= ((n - 3)a_1 + b_2)e_{n-1} + a_2 e_n, \\
    d(e_n) &= ((n - 4)a_1 + 2b_2)e_n,
\end{align*}
\]

where \( b_{2k} = 0, 2 \leq k \leq \frac{n-3}{2} \);

for the algebra \( g^2(n, 1) \),

\[
\begin{align*}
    d(e_1) &= \sum_{i=1}^{n-1} a_i e_i, \\
    d(e_i) &= ((i - 2)a_1 + b_2)e_i + \sum_{t=i+1}^{n-1} b_{t-i+2} e_t, 2 \leq i \leq n - 2, \\
    d(e_{n-1}) &= ((n - 3)a_1 + b_2)e_{n-1}, \\
    d(e_n) &= -\sum_{t=3}^{n-2} a_{t-1} e_t + c_{n-1} e_{n-1} + 2a_1 e_n,
\end{align*}
\]

for the algebra \( g^3(n, 1) \).
\[
\begin{align*}
  d(e_1) &= a_1 e_1 + \sum_{i=3}^{n-1} a_i e_i, \\
  d(e_2) &= 3a_1 e_2 + \sum_{i=3}^{n-1} b_i e_i, \\
  d(e_i) &= (i + 1)a_1 e_i + b_3 e_{i+1} + b_4 e_{i+2} + \sum_{j=i+3}^{n-1} (b_{j-i+2} - a_{j-i})e_j, \quad 3 \leq i \leq n - 1, \\
  d(e_n) &= -\sum_{t=4}^{n-2} a_{t-1} e_t + c_{n-1} e_{n-1} + 2a_1 e_n,
\end{align*}
\]

for the algebra \(g_1^1\),
\[
\begin{align*}
  d(e_1) &= a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7, \\
  d(e_2) &= 2a_1 e_2 + b_3 e_3 + b_5 e_5 + b_6 e_6 + b_7 e_7, \\
  d(e_3) &= 3a_1 e_3 + b_3 e_4 - a_3 e_5 + (b_5 - a_4) e_6 - a_5 e_7, \\
  d(e_4) &= 4a_1 e_4 + b_5 e_5 - a_5 e_6 + a_4 e_7, \\
  d(e_5) &= 5a_1 e_5 + b_6 e_6 - a_6 e_7, \\
  d(e_6) &= 6a_1 e_6, \\
  d(e_7) &= 7a_1 e_7,
\end{align*}
\]

for the algebra \(g_2^2\),
\[
\begin{align*}
  d(e_1) &= a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7 + a_8 e_8 + a_9 e_9, \\
  d(e_2) &= 2a_1 e_2 + b_2 e_3 + a_4 e_4 + a_5 e_5 + b_7 e_7 + b_8 e_8 + b_9 e_9, \\
  d(e_3) &= 3a_1 e_3 + b_3 e_4 - a_3 e_5 - 2a_5 e_7 + (b_7 - 5a_6) e_8 - a_7 e_9, \\
  d(e_4) &= 4a_1 e_4 + b_5 e_5 - a_5 e_6 + 2a_4 e_7 + a_6 e_9, \\
  d(e_5) &= 5a_1 e_5 + b_6 e_6 - 3a_3 e_7 + 2a_4 e_8 - a_5 e_9, \\
  d(e_6) &= 6a_1 e_6 + b_7 e_7 - 5a_3 e_8 + a_4 e_9, \\
  d(e_7) &= 7a_1 e_7 + b_9 e_9 - a_5 e_9, \\
  d(e_8) &= 8a_1 e_8, \\
  d(e_9) &= 9a_1 e_9,
\end{align*}
\]

for the algebra \(g_{11}^3\),
\[
\begin{align*}
  d(e_1) &= a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7 + a_8 e_8 + a_9 e_9 + a_{10} e_{10} + a_{11} e_{11}, \\
  d(e_2) &= 2a_1 e_2 + b_3 e_3 + a_4 e_4 + a_5 e_5 + b_7 e_7 - a_7 e_8 + b_9 e_9 + b_{10} e_{10} + b_{11} e_{11}, \\
  d(e_3) &= 3a_1 e_3 + b_5 e_4 - a_3 e_5 + a_5 e_7 + (b_7 + a_6) e_8 - b_9 e_9 - a_{11} e_{11}, \\
  d(e_4) &= 4a_1 e_4 + b_7 e_5 - a_5 e_6 - a_4 e_7 + (a_6 + b_7) e_9 + a_7 e_{10} + a_8 e_{11}, \\
  d(e_5) &= 5a_1 e_5 + b_8 e_6 - a_6 e_8 - a_5 e_9 + b_{10} e_{10} - a_7 e_{11}, \\
  d(e_6) &= 6a_1 e_6 + b_9 e_7 + a_3 e_8 - a_5 e_{10} + a_6 e_{11}, \\
  d(e_7) &= 7a_1 e_7 + b_{10} e_8 + a_5 e_9 + a_{10} e_{10} - a_5 e_{11}, \\
  d(e_8) &= 8a_1 e_8 + b_{11} e_9 + a_{11} e_{11}, \\
  d(e_9) &= 9a_1 e_9 + b_{10} e_{10} - a_5 e_{11}, \\
  d(e_{10}) &= 10a_1 e_{10}, \\
  d(e_{11}) &= 11a_1 e_{11}.
\end{align*}
\]
Proof. From Theorem 2.6 we conclude that \(e_1\) and \(e_2\) are the generator basis elements of the algebra \(g_{(n,1)}^1\).

We put

\[
d(e_1) = \sum_{i=1}^{n} a_i e_i, \quad d(e_2) = \sum_{i=1}^{n} b_i e_i.
\]

From the derivation property (2.1) we have

\[
d(e_3) = d([e_1, e_2]) = [d(e_1), e_2] + [e_1, d(e_2)] = (a_1 + b_2) e_3 + \sum_{t=4}^{n-1} b_{t-1} e_t - a_{n-2} e_n.
\]

By induction and the derivation property, we derive

\[
d(e_i) = [(i - 2) a_1 + b_2] e_i + \sum_{t=i+1}^{n-1} b_{t-i+2} e_t + (-1)^i a_{n-i+1} e_n,
\]

where \(3 \leq i \leq n - 1\).

From \(0 = d([e_2, e_i]) = [d(e_2), e_i] + [e_2, d(e_i)], \quad 3 \leq i \leq n - 3, \) we conclude

\[
b_1 = 0, \quad (-1)^i b_{n-i} - b_{n-i} = 0, \quad 3 \leq i \leq n - 3.
\]

Consequently,

\[
b_{2k} = 0, \quad 2 \leq k \leq \frac{n-3}{2}.
\]

The equality \(d(e_n) = d([e_2, e_{n-2}])\) implies

\[
d(e_n) = d([e_2, e_{n-2}]) = ((n - 4) a_1 + 2 b_2) e_n.
\]

The descriptions of derivations for other algebras are obtained by applying similar calculations used for the algebra \(g_{(n,1)}^1\).

\[\square\]

3.2. Descriptions of algebras \(R(g^1, 1), i=1, 2, 3\)

In this subsection, we classify solvable Leibniz algebras with quasi-filiform Lie nilradicals of maximum length under the condition that the complemented space is one-dimensional.

Theorem 3.2. An arbitrary algebra of the family \(R(g_{(n,1)}^1, 1)\) admits a basis \(\{e_1, e_2, \ldots, e_n, x\}\) such that its multiplication table has one of the following types:
Proof. Let \( \{e_1, e_2, \ldots, e_n\} \) be a basis of the algebra \( G_{\mathcal{n}}^{(1)} \) such that the table of multiplication in this basis has the form of Theorem 2.6. We know that the operator \( R_{x_{\mathcal{n}}^{(1)}} \) for \( x \in Q \) is a non-nilpotent derivation of \( G_{\mathcal{n}}^{(1)} \). Taking into account the structure of the algebra \( G_{\mathcal{n}}^{(1)} \) we conclude \( \{e_1, \ldots, e_{n-2}\} \cap \text{Ann}_r(R_{x_{\mathcal{n}}^{(1)}}) = \emptyset \). Moreover, we have \([x, e_i] + [e_i, x], [x, x] \in \text{Ann}_r(R_{x_{\mathcal{n}}^{(1)}})\). Now using the description of derivations of the algebra \( G_{\mathcal{n}}^{(1)} \) from Proposition 3.1 we get the following products in \( R(G_{\mathcal{n}}^{(1)}, 1)\):

\[
R_1(g_{\mathcal{n}}^{(1)}, 1)(a_2, b_4, \ldots, b_{n-1}) : \begin{cases}
[e_1, x] = -[x, e_1] = a_2 e_2, \\
[e_i, x] = -[x, e_i] = e_i + \sum_{t=i+2}^{n-1} b_{t-i+2} e_t, 2 \leq i \leq n - 2, \\
[e_{n-1}, x] = -[x, e_{n-1}] = e_{n-1} + a_2 e_n, \\
[e_n, x] = -[x, e_n] = 2e_n, \quad \text{where } b_{2k} = 0, 2 \leq k \leq \frac{n-3}{2},
\end{cases}
\]

\[
R_2(g_{\mathcal{n}}^{(1)}, 1)(a_{n-1}) : \begin{cases}
[e_1, x] = -[x, e_1] = e_1 + a_{n-1} e_{n-1}, \\
[e_i, x] = -[x, e_i] = (i + 2 - n) e_i, 2 \leq i \leq n - 1, \\
[e_n, x] = -[x, e_n] = (4 - n) e_n,
\end{cases}
\]

\[
R_3(g_{\mathcal{n}}^{(1)}, 1)(\delta_{n-1}) : \begin{cases}
[e_1, x] = -[x, e_1] = e_1, \\
[e_i, x] = -[x, e_i] = (i + 1 - n) e_i, 2 \leq i \leq n - 2, \\
[e_n, x] = -[x, e_n] = (2 - n) e_n, \\
[x, x] = \delta_{n-1} e_{n-1},
\end{cases}
\]

\[
R_4(g_{\mathcal{n}}^{(1)}, 1)(\delta_n) : \begin{cases}
[e_1, x] = -[x, e_1] = e_1, \\
[e_i, x] = -[x, e_i] = \left(\frac{i - n}{2}\right) e_i, 2 \leq i \leq n - 1, \\
[x, x] = \delta_n e_n,
\end{cases}
\]

\[
R_5(g_{\mathcal{n}}^{(1)}, 1)(a_n) : \begin{cases}
[e_1, x] = -[x, e_1] = e_1 + a_n e_n, \\
[e_i, x] = -[x, e_i] = \left(\frac{i - n}{2}\right) e_i, 2 \leq i \leq n - 1, \\
[e_n, x] = -[x, e_n] = e_n,
\end{cases}
\]

\[
R_6(g_{\mathcal{n}}^{(1)}, 1)(a_2) : \begin{cases}
[e_1, x] = -[x, e_1] = e_1 + a_2 e_2, \\
[e_i, x] = -[x, e_i] = (i - 1) e_i, 2 \leq i \leq n - 2, \\
[e_{n-1}, x] = -[x, e_{n-1}] = (n - 2) e_{n-1} + a_2 e_n, \\
[e_n, x] = -[x, e_n] = (n - 2) e_n,
\end{cases}
\]

\[
R_7(g_{\mathcal{n}}^{(1)}, 1)(b_2) : \begin{cases}
[e_1, x] = -[x, e_1] = e_1, \\
[e_i, x] = -[x, e_i] = (i - 2 + b_2) e_i, 2 \leq i \leq n - 1, \\
[e_n, x] = -[x, e_n] = (n - 4 + 2b_2) e_n, b_2 \notin \left\{4 - n,\ 3 - n,\ \frac{4 - n}{2},\ \frac{5 - n}{2},\ 1\right\}.
\end{cases}
\]
The equalities

\[ [e_1, x] = \sum_{l=1}^{n} a_l e_l, \]
\[ [e_2, x] = \sum_{l=2}^{n} b_l e_l, \]
\[ [e_i, x] = ((i - 2)a_1 + b_2)e_i + \sum_{l=i+1}^{n-1} b_{l-i+2} e_l + (-1)^i a_{n-i+1} e_n, \text{ for } 3 \leq i \leq n - 1, \]
\[ [e_n, x] = ((n - 4)a_1 + 2b_2)e_n, \]
\[ [x, e_1] = -\sum_{l=1}^{n-2} a_l e_l + \mu_{1, n-1} e_{n-1} + \mu_{1, n} e_n, \]
\[ [x, e_i] = -((i - 2)a_1 + b_2)e_i - \sum_{l=i+1}^{n-2} b_{l-i+2} e_l + \mu_{i, n-1} e_{n-1} + \mu_{i, n} e_n, \text{ for } 2 \leq i \leq n - 2, \]
\[ [x, e_{n-1}] = \mu_{n-1, n-1} e_{n-1} + \mu_{n-1, n} e_n, \]
\[ [x, e_n] = \mu_{n-1, n} e_{n-1} + \mu_{n, n} e_n, \]
\[ [x, x] = \delta_{n-1} e_{n-1} + \delta_n e_n. \]

Consider the Leibniz identity for the following triples of elements to deduce

\[
\begin{align*}
\mathcal{L}(x, e_1, e_{n-2}) &= 0 \quad \Rightarrow \quad [x, e_{n-1}] = -((n - 3)a_1 + b_2)e_{n-1} - a_2 e_n, \\
\mathcal{L}(x, e_2, e_{n-2}) &= 0, \quad \Rightarrow \quad [x, e_n] = -((n - 4)a_1 + 2b_2)e_n, \\
\mathcal{L}(x, x, x) &= 0, \quad \Rightarrow \quad ((n - 3)a_1 + b_2)e_{n-1} = 0, a_2 e_{n-1} + ((n - 4)a_1 + 2b_2)\delta_n = 0.
\end{align*}
\]

The equalities \( \mathcal{L}(x, e_i, e_i) = \mathcal{L}(x, e_2, x) = \mathcal{L}(x, x, e_2) = \mathcal{L}(x, e_1, x) = \mathcal{L}(x, x, e_1) = 0 \) for \( 2 \leq i \leq n - 2 \) imply \( [e_i, x] = -[x, e_i], 1 \leq i \leq n. \)

Thus, we obtain the following table of multiplications of the algebra \( R(g^1_{(n,1)}, 1) \):

\[
\begin{align*}
[e_1, x] &= \sum_{l=1}^{n} a_l e_l, \\
[e_2, x] &= \sum_{l=2}^{n} b_l e_l, \\
[e_i, x] &= ((i - 2)a_1 + b_2)e_i + \sum_{l=i+1}^{n-1} b_{l-i+2} e_l + (-1)^i a_{n-i+1} e_n, \text{ for } 3 \leq i \leq n - 1, \\
[e_n, x] &= ((n - 4)a_1 + 2b_2)e_n, \\
[x, e_1] &= -\sum_{l=1}^{n} a_l e_l, \\
[x, e_2] &= -\sum_{l=2}^{n} b_l e_l, \\
[x, e_i] &= -((i - 2)a_1 + b_2)e_i - \sum_{l=i+1}^{n-1} b_{l-i+2} e_l - (-1)^i a_{n-i+1} e_n, \text{ for } 3 \leq i \leq n - 1, \\
[x, e_n] &= -((n - 4)a_1 + 2b_2)e_n, \\
[x, x] &= \delta_{n-1} e_{n-1} + \delta_n e_n.
\end{align*}
\]

where \( b_{2k} = 0, 2 \leq k \leq \frac{n-3}{2} \) and
\[(n - 3)a_1 + b_2) \delta_{n-1} = a_2 \delta_{n-1} + ((n - 4)a_1 + 2b_2) \delta_n = 0. \quad (3.1)\]

Let us take the general change of generators basis elements:
\[e'_1 = \sum_{i=1}^{n} A_i e_i, \quad e'_2 = \sum_{i=1}^{n} B_i e_i, \quad x' = Hx + \sum_{i=1}^{n} C_i e_i.\]

Consider the table of multiplications of the algebra \(R(g_{(n,1)}^{1},1)\) in the new basis and then express those products in terms of the old basis to obtain the relations:
\[a'_1 = Ha_1, b'_2 = Hb_2.\]

It is known that \((a_1, b_2) \neq (0,0)\), otherwise the algebra \(R(g_{(n,1)}^{1},1)\) is a nilpotent algebra.

Now we need to consider the following cases.

**Case 1.** Let \(a_1 = 0\). Then by choosing an appropriate value of \(H\) one can assume \(b_2 = 1\) and from (3.1) we have \(\delta_{n-1} = \delta_n = 0\).

Taking the change of generator basis elements in the following form:
\[e'_1 = e_1 - \frac{a_2(b_n - a_{n-1}) + a_n}{2} e_n, e'_2 = e_2 - (b_n - a_{n-1}) e_n, x' = x + b_3 e_1 - \sum_{i=3}^{n-1} a_i e_{i-1},\]
we can assume \(b_3 = b_n = a_t = 0, 3 \leq t \leq n\). Thus, the algebra \(R_1(g_{(n,1)}^{1},1)(a_2, b_4, ..., b_{n-1})\) is obtained.

**Case 2.** Let \(a_1 \neq 0\). Then for a suitable value of \(H\) one can assume \(a_1 = 1\) and from (3.1) we derive
\[(n - 3 + b_2) \delta_{n-1} = 0, a_2 \delta_{n-1} + (n - 4 + 2b_2) \delta_n = 0.\]

The change of generator basis elements of the form
\[e'_1 = e_1, e'_2 = e_i + \sum_{j=i+2}^{n-1} A_{j-i+2} e_j, 2 \leq i \leq n - 1,\]
with
\[A_3 = -b_3, A_i = -\frac{1}{i-2} \left(b_i + \sum_{j=3}^{i-1} b_{i-j+2} A_j\right), 4 \leq i \leq n - 1.\]

lead to \(b_t = 0, 3 \leq t \leq n - 1\).

Setting \(x' = x - \sum_{i=3}^{n-2} a_i e_{i-1} - b_n e_{n-2}\), we derive \(b_n = a_t = 0, 3 \leq t \leq n - 2\).

Thus, the multiplication table of the algebra \(R(g_{(n,1)}^{1},1)\) has the form:
\[
\begin{align*}
[e_1, x] &= e_1 + a_2 e_2 + a_{n-1} e_{n-1} + a_n e_n, \\
[e_i, x] &= (i - 2 + b_2) e_i, \quad 2 \leq i \leq n - 2, \\
[e_{n-1}, x] &= (n - 3 + b_2) e_{n-1} + a_2 e_n, \\
[e_n, x] &= (n - 4 + 2b_2) e_n, \\
[x, e_1] &= -e_1 - a_2 e_2 - a_{n-1} e_{n-1} - a_n e_n, \\
[x, e_i] &= -(i - 2 + b_2) e_i, \quad 2 \leq i \leq n - 2, \\
[x, e_{n-1}] &= -(n - 3 + b_2) e_{n-1} - a_2 e_n, \\
[x, e_n] &= -(n - 4 + 2b_2) e_n, \\
[x, x] &= \delta_{n-1} e_{n-1} + \delta_n e_n,
\end{align*}
\]

with
\[(n - 3 + b_2)\delta_{n-1} = 0, a_2\delta_{n-1} + (n - 4 + 2b_2)\delta_n = 0. \quad (3.3)\]

Applying the following change of generator basis elements \(e_1\) and \(e_{n-1}\) in (3.2):
\[e'_1 = e_1 + Ae_2 + Be_{n-1} + Ce_n, \quad e'_{n-1} = e_{n-1} + Ae_n,\]
we derive the restrictions
\[a'_2 = a_2 + A(b_2 - 1), a'_{n-1} = a_{n-1} + B(n - 4 + b_2), a'_n = a_n + Ba_2 + C(n - 5 + 2b_2) - a'_{n-1}A. \quad (3.4)\]

Now we study the following possible subcases.

**Subcase 2.1.** Let \(b_2 = 4 - n\). Then from (3.3) we get \(\delta_{n-1} = \delta_n = 0\) and choosing \(A = \frac{a_2}{n-3}, C = \frac{a_2 - a_{n-1}}{(n-3)^2}\) in (3.4) we can assume \(a_2 = a_n = 0\). Hence, the algebra \(R_2(g_{(n,1)}^1, 1)(a_{n-1})\) is obtained.

**Subcase 2.2.** Let \(b_2 = 3 - n\). Then setting \(A = \frac{a_2}{n-2}, B = a_{n-1}, C = \frac{a_2 + a_{n-1}}{n-1}\) in (3.4) we obtain \(a_2 = a_{n-1} = a_n = 0\). The restriction (3.3) leads to \(\delta_n = 0\). So, we get the algebra \(R_3(g_{(n,1)}^1, 1)(\delta_{n-1})\).

**Subcase 2.3.** Let \(b_2 = \frac{4-n}{2}\). Then from (3.3) we deduce \(\delta_{n-1} = \delta_n = 0\) and putting \(A = \frac{2a_2}{n-3}, B = -\frac{2a_2}{n-4}\) in (3.4) we have \(a_2 = a_{n-1} = a_n = 0\). Therefore, we get the algebra \(R_4(g_{(n,1)}^1, 1)(a_n)\).

**Subcase 2.4.** Let \(b_2 = \frac{5-n}{2}\). Then restrictions (3.3) imply \(\delta_{n-1} = \delta_n = 0\) and putting \(A = \frac{2a_2}{n-3}, B = -\frac{2a_2}{n-4}\) in (3.4) we have \(a_2 = a_{n-1} = a_n = 0\). Therefore, we get the algebra \(R_5(g_{(n,1)}^1, 1)(a_n)\).

**Subcase 2.5.** Let \(b_2 = 1\). Then we have \(\delta_{n-1} = \delta_n = 0\) and taking \(B = -\frac{a_2}{n-3}, C = -\frac{a_2}{n-3} + \frac{a_{n-1}a_2}{(n-3)^2}\) in (3.4) one can assume \(a_{n-1} = a_n = 0\). Hence, we derive the algebra \(R_6(g_{(n,1)}^1, 1)(a_2)\).

**Subcase 2.6.** Finally, let \(b_2 \neq 4 - n, 3 - n, \frac{4-n}{2}, \frac{5-n}{2}, 1\). Then from (3.3) we have \(\delta_{n-1} = \delta_n = 0\) and choosing \(A = -\frac{a_2}{b_2-1}, B = -\frac{a_2}{n-4+b_2}, C = -\frac{a_2}{n-5+2b_2} + \frac{a_{n-1}}{(n-5+2b_2)(n-4+b_2)}\) in (3.4) one can assume \(a_2 = a_{n-1} = a_n = 0\). Thus, we obtain the algebra \(R_7(g_{(n,1)}^1, 1)(b_2)\).

In the next theorem we investigate the isomorphisms inside each family of algebras of Theorem 3.2.

**Theorem 3.3.** An arbitrary algebra of the family \(R(g_{(n,1)}^1, 1)\) is isomorphic to one of the following pairwise non-isomorphic algebras:
\[
R_1(g_{(n,1)}^1, 1)(a_2, b_4, ..., b_{n-1}), R_2(g_{(n,1)}^1, 1)(1), R_3(g_{(n,1)}^1, 1)(1), R_4(g_{(n,1)}^1, 1)(1),
R_5(g_{(n,1)}^1, 1)(1), R_6(g_{(n,1)}^1, 1)(1), R_7(g_{(n,1)}^1, 1)(b_2).
\]

Note that the first non-vanishing parameters \(\{a_2, b_4, ..., b_{n-1}\}\) in the algebra \(R_1(g_{(n,1)}^1, 1)\) \((a_2, b_4, ..., b_{n-1})\) can be scaled to 1.

**Proof.** We start with the general change of generator basis elements of the algebra \(R(g_{(n,1)}^1, 1)\):
\[e'_1 = \sum_{i=1}^{n} A_i e_i, \quad e'_2 = \sum_{i=1}^{n} B_i e_i, \quad x' = Hx + \sum_{i=1}^{n} C_i e_i.\]

Write the multiplication table of an algebra from the considered family in the new basis \(\{e'_1, ..., e'_n, x'\}\) and express them via the basis \(\{e_1, ..., e_n, x\}\) to get invariant relations on the structure constants of the algebra.
For the algebra $R_1(g^1_{(n,1)}, 1)(a_2, b_4, \ldots, b_{n-1})$, we have the following invariant relations:

$$a'_2 = \frac{A_1 a_2}{B_2}, b'_t = \frac{b_t}{A_1^{-2}}, 4 \leq t \leq n - 1, \text{where } b_{2k} = 0, 2 \leq k \leq \frac{n-3}{2}.$$  

Note that the first non-vanishing parameter $\{a_2, b_4, \ldots, b_{n-1}\}$ in the algebra can be scaled to 1.

- Consider the family of algebras $R_2(g^1_{(n,1)}, 1)(a_{n-1})$. Then we have the following invariant relation:

$$a'_{n-1} = \frac{a_{n-1}}{A_1^{-4} B_2}.$$  

If $a_{n-1} \neq 0$, then by putting $B_2 = \frac{a_{n-1}}{A_1^{-4}}$, we get $a'_{n-1} = 1$. Hence, the algebra $R_2(g^1_{(n,1)}, 1)(1)$ is obtained; 

If $a_{n-1} = 0$, then we obtain $R_2(g^1_{(n,1)}, 1)(4 - n)$.

- Consider the family of algebras $R_3(g^1_{(n,1)}, 1)(\delta_{n-1})$. Then the invariant relation is the following:

$$\delta'_{n-1} = \frac{\delta_{n-1}}{A_1^{-4} B_2}.$$  

If $\delta_{n-1} \neq 0$, then taking $B_2 = \frac{\delta_{n-1}}{A_1^{-4}}$, we can assume $\delta'_{n-1} = 1$ and we obtain $R_3(g^1_{(n,1)}, 1)(1)$; 

If $\delta_{n-1} = 0$, then we derive $R_3(g^1_{(n,1)}, 1)(3 - n)$.

- Let consider the family $R_4(g^1_{(n,1)}, 1)(\delta_n)$. Then

$$\delta'_n = \frac{\delta_n}{A_1^{-4} B_2^2}.$$  

If $\delta_n \neq 0$, then taking $B_2 = \sqrt{\frac{\delta_n}{A_1^{-4}}}$, we can assume $\delta'_n = 1$ and we obtain $R_4(g^1_{(n,1)}, 1)(1)$; 

If $\delta_n = 0$, we derive $R_4(g^1_{(n,1)}, 1)(\frac{3 - n}{2})$.

- For the family of algebras $R_5(g^1_{(n,1)}, 1)(a_n)$ we have the following invariant relation:

$$a'_n = \frac{a_n}{A_1^{-5} B_2^2}.$$  

If $a_n \neq 0$, then by putting $B_2 = \sqrt{\frac{a_n}{A_1^{-5}}}$, we get $a'_n = 1$ and we obtain $R_5(g^1_{(n,1)}, 1)(1)$; 

If $a_n = 0$, then we obtain $R_5(g^1_{(n,1)}, 1)(\frac{3 - n}{2})$.

- Consider the family $R_6(g^1_{(n,1)}, 1)(a_2)$. Then we obtain

$$a'_2 = \frac{a_2 A_1}{B_2}.$$  

If $a_2 \neq 0$, then putting $B_2 = A_1 a_2$ we derive $a'_2 = 1$ and hence, we obtain the algebra $R_6(g^1_{(n,1)}, 1)(1)$; 

If $a_2 = 0$, then we derive $R_6(g^1_{(n,1)}, 1)(1)$.

- Considering the family of algebras $R_7(g^1_{(n,1)}, 1)(b_2)$ we have $b'_2 = b_2$, which show that all algebras of this family are pairwise non-isomorphic.
The next results are established in a similar way for \( R(g_{(n,1)}^2,1) \).

**Theorem 3.4.** Any element of \( R(g_{(n,1)}^2,1) \) admits a basis such that its multiplication table has one of the following types:

\[
R_1(g_{(n,1)}^2,1)(a_1) = [e_i, x] = -[x, e_i] = e_i + \sum_{i=1+2}^{n-1} b_{n-i+2} e_i, \quad 2 \leq i \leq n-1,
\]

\[
R_2(g_{(n,1)}^2,1)(a_2) = \begin{cases} 
\langle e_1, x \rangle = -[x, e_1] = e_1 + a_2 e_2, \\
\langle e_i, x \rangle = -[x, e_i] = (i-1)e_i, \quad 2 \leq i \leq n-1, \\
\langle e_n, x \rangle = -[x, e_n] = -a_2 e_3 + 2e_n,
\end{cases}
\]

\[
R_3(g_{(n,1)}^2,1)(\gamma) = \begin{cases} 
\langle e_1, x \rangle = -[x, e_1] = e_1, \\
\langle e_i, x \rangle = -[x, e_i] = (i+3-n)e_i, \quad 2 \leq i \leq n-1, \\
\langle e_n, x \rangle = -[x, e_n] = \gamma e_{n-1} + 2e_n,
\end{cases}
\]

\[
R_4(g_{(n,1)}^2,1)(\delta) = \begin{cases} 
\langle e_1, x \rangle = -[x, e_1] = e_1, \\
\langle e_i, x \rangle = -[x, e_i] = (i+1-n)e_i, \quad 2 \leq i \leq n-1, \\
\langle e_n, x \rangle = -[x, e_n] = 2e_n, \quad [x, x] = \delta e_{n-1},
\end{cases}
\]

\[
R_5(g_{(n,1)}^2,1)(b_2) = \begin{cases} 
\langle e_1, x \rangle = -[x, e_1] = e_1, \\
\langle e_i, x \rangle = -[x, e_i] = (i-2+b_2)e_i, \quad 2 \leq i \leq n-1, \\
\langle e_n, x \rangle = -[x, e_n] = 2e_n, \quad b_2 \neq 1, 5, n, 3 - n.
\end{cases}
\]

**Theorem 3.5.** Any element of \( R(g_{(n,1)}^2,1) \) is isomorphic to one of the following pairwise non-isomorphic algebras:

\[
R_1(g_{(n,1)}^2,1)(b_1, \ldots, b_{n-1}), R_3(g_{(n,1)}^2,1)(1), R_4(g_{(n,1)}^2,1)(1), R_5(g_{(n,1)}^2,1)(b_2), b_2 \in \mathbb{C}.
\]

Note that the first non-vanishing parameter \( \{b_1, \ldots, b_{n-1}\} \) in the algebra \( R_1(g_{(n,1)}^2,1)(b_1, \ldots, b_{n-1}) \) can be scaled to 1.

Analogously we obtain the descriptions of solvable algebras with nilradicals \( g_{(n,1)}^3, g_{(7,1)}^1, g_{(9,1)}^2, g_{11}^3 \) under the condition that the complemented space is one-dimensional.

**Theorem 3.6.** An arbitrary solvable Leibniz algebra whose nilradical is one of the following: \( g_{(n,1)}^3, g_{(7,1)}^1, g_{(9,1)}^2, g_{11}^3 \) and with one-dimensional complemented space is isomorphic to the following corresponding algebras:

\[
R(g_{(n,1)}^3,1) = \begin{cases} 
\langle e_1, x \rangle = -[x, e_1] = e_1, \\
\langle e_i, x \rangle = -[x, e_i] = (i+1)e_i, \quad 2 \leq i \leq n-1, \\
\langle e_n, x \rangle = -[x, e_n] = 2e_n.
\end{cases}
\]

\[
R(g_{(7,1)}^1,1) = \langle e_i, x \rangle = -[x, e_i] = ie_i, 1 \leq i \leq 7.
\]

\[
R(g_{(9,1)}^2,1) = \langle e_i, x \rangle = -[x, e_i] = ie_i, 1 \leq i \leq 9.
\]

\[
R(g_{(11,1)}^3,1) = \langle e_i, x \rangle = -[x, e_i] = ie_i, 1 \leq i \leq 11.
\]

### 3.3. Descriptions of algebras \( R(g_{(n,1)}^i, 2), i=1, 2 \)

In this subsection, we classify solvable Leibniz algebras whose nilradical is a quasi-filiform Lie algebra of maximal length and the dimension of the complemented space \( Q \) is equal to two.
Due to Proposition 3.1 we conclude that the dimensions of the complementary spaces only for nilradicals $g^i_{(n,1)}$, $i = 1, 2$ can be equal to two.

**Theorem 3.7.** Any element of $R(g^i_{(n,1)}, 2), i = 1, 2$ is isomorphic to one of the following Lie algebras:

\[
R(g^1_{(n,1)}, 2) : \begin{cases}
  [e_1, x] = -[x, e_1] = e_1, \\
  [e_i, x] = -[x, e_i] = (i - 2)e_i, & 2 \leq i \leq n - 1, \\
  [e_n, x] = -[x, e_n] = (n - 4)e_n, \\
  [e_i, y] = -[y, e_i] = e_i, & 2 \leq i \leq n - 1, \\
  [e_n, y] = -[y, e_n] = 2e_n,
\end{cases}
\]

\[
R(g^2_{(n,1)}, 2) : \begin{cases}
  [e_1, x] = -[x, e_1] = e_1, \\
  [e_i, x] = -[x, e_i] = (i - 2)e_i, & 2 \leq i \leq n - 1, \\
  [e_n, x] = -[x, e_n] = 2e_n, \\
  [e_i, y] = -[y, e_i] = e_i, & 2 \leq i \leq n - 1.
\end{cases}
\]

**Proof.** Consider the description of derivations for the algebra $g^i_{(n,1)}$ given in Proposition 3.1. Since two parameters $a_1$, $b_2$ appear on the diagonal, it is clear that we can obtain two nil-independent derivations, such as those corresponding to $(a_1, b_2) = (1, 0)$, and $(0, 1)$, denote them by $\mathcal{R}_x$ and $\mathcal{R}_y$, respectively.

We consider $R(g^i_{(n,1)}, 2) = \langle e_1, ..., e_n, x, y \rangle$, where $\mathcal{R}_x, \mathcal{R}_y$ are the right multiplications on $x$ and $y$, respectively. It is known that the subspace span$\{e_1, ..., e_n, x\}$ forms a subalgebra of the algebra $R(g^i_{(n,1)}, 2)$. Therefore this subalgebra is isomorphic to the algebra $R_7(g^i_{(n,1)}, 1)(0)$ in the list of Theorem 3.2. Then the multiplication table of the algebra $R(g^i_{(n,1)}, 2)$ has the following form:

\[
\begin{cases}
  [e_1, x] = -[x, e_1] = e_1, \\
  [e_i, x] = -[x, e_i] = (i - 2)e_i, & 2 \leq i \leq n - 1, \\
  [e_n, x] = -[x, e_n] = (n - 4)e_n, \\
  [e_1, y] = \sum_{i=2}^{n} a_i e_i, \\
  [e_2, y] = e_2 + \sum_{i=3}^{n} b_i e_i, \\
  [e_i, y] = e_i + \sum_{j=i+1}^{n-1} b_{j-i+2} e_j + (-1)^i a_{n-i+1} e_n, & 3 \leq i \leq n - 1, \\
  [e_n, y] = 2e_n, \\
  [y, e_1] = \sum_{i=1}^{n} \delta^1_{2i} e_i, & 1 \leq i \leq n, \\
  [x, y] = \sum_{k=1}^{n} t^k_{1,2} e_k, \\
  [y, x] = \sum_{k=1}^{n} t^k_{2,1} e_k, \\
  [y, y] = \sum_{k=1}^{n} t^k_{2,2} e_k.
\end{cases}
\]
Taking into account that $e_1, \ldots, e_n \notin \text{Ann}_R(R^{i_1}_{(n, 2)}, 1)$, we derive the products $[y, e_i] = -[e_i, y], [x, y] = -[y, x], [y, y] = 0$ for $1 \leq i \leq n$. Considering the following change of generator basis elements:

$$e'_1 = e_1 + (a_{n-1} - b_n)e_{n-1}, e'_2 = e_2 + (a_{n-1} - b_n)e_n, x' = x - (n-4)(a_{n-1} - b_n)e_{n-2},$$

we have the following:

$$y' = y + b_3e_1 - \sum_{t=3}^{n-2} a_t e_{t-1} - (2a_{n-1} - b_n)e_{n-2},$$

we obtain

$$b_3 = b_n = a_t = 0, \quad 3 \leq t \leq n - 1.$$

Applying the Leibniz identity for triples of elements we obtain

$$\left\{ \begin{array}{ll}
\mathcal{L}(e_2, y, x) = 0 & \Rightarrow t^1_{2,1} = t^n_{2,1} = b_k = 0, 4 \leq k \leq n - 1, \\
\mathcal{L}(e_1, y, x) = 0 & \Rightarrow t^2_{2,1} = a_2 = a_n = 0, 2 \leq k \leq n - 3.
\end{array} \right.$$ 

Putting

$$y' = y - \frac{t^n_{2,1}}{n-3} e_{n-1} - \frac{t^n_{2,1}}{n-4} e_n,$$

we get $t^n_{2,1} = t^2_{2,1} = 0$.

Thus we obtain the following algebra:

$$R^{i_1}_{(n, 1)}(2) : \left\{ \begin{array}{ll}
[e_1, x] = -[x, e_1] = e_1, \\
[e_i, x] = -[x, e_i] = (i-2)e_i, & 2 \leq i \leq n - 1, \\
[e_n, x] = -[x, e_n] = (n-4)e_n, \\
[e_n, y] = -[y, e_n] = e_i, & 2 \leq i \leq n - 1, \\
[e_n, y] = -[y, e_n] = 2e_n.
\end{array} \right.$$ 

The algebra $R^{i_1}_{(n, 1)}(2)$ is obtained by applying similar arguments used for the algebra $R^{i_1}_{(n, 1)}(2)$.

$\square$

### 3.4. Rigidity of algebras $R^{i_1}_{(n, 1)}(2), i=1, 2$

In this part we prove the rigidity of the algebras $R^{i_1}_{(n, 1)}(2)$ and $R^{i_1}_{(n, 1)}(2)$. First we prove the triviality of the second cohomology group with coefficient in itself for these algebras.

**Theorem 3.8.** $H^2(R^{i_1}_{(n, 1)}(2), R^{i_1}_{(n, 1)}(2)) = H^2(R^{i_1}_{(n, 1)}(2), R^{i_1}_{(n, 1)}(2)) = \{0\}$.

**Proof.** Let $R$ be a solvable Lie algebra isomorphic to $R^{i_1}_{(n, 1)}(2), i=1, 2$. Then from the formula (2.2), we have the following:

$$H^2(R, R) \simeq H^2(Q, \mathbb{C}) \otimes H^0(N, R)^Q + H^1(Q, \mathbb{C}) \otimes H^1(N, R)^Q + H^0(Q, \mathbb{C}) \otimes H^2(N, R)^Q.$$ 

From this it is clear that if $H^i(N, R)^Q = \{0\}$ for all $0 \leq i \leq 2$, then we get $H^2(R, R) = \{0\}$. Since the center of $R$ is trivial, then $H^0(N, R)^Q = \{0\}$. Next, we consider the cohomology groups $H^1(N, R)^Q$ and $H^2(NL, R)^Q$. 

Let \( f \in C^1(N, R) \), then \( f \) has the from
\[
f(e_i) = \sum_{i=1}^{n} \alpha_{i,i} e_i + \beta_i x + \gamma_i y, \quad 1 \leq i \leq n, \quad e_i \in N.
\]

Now we rewrite the formula (2.3) for the elements \( x \in \mathbb{Q} \) follows
\[
(d^1f)(x, e_i) = [x, f(e_i)] - f([x, e_i]) = 0.
\]

From the equalities \((d^1f)(x, e_i) = (d^1f)(y, e_i) = 0\) for \( 1 \leq i \leq n \) we deduce
\[
f(e_i) = x_{i,i} e_i, \quad 1 \leq i \leq n, \quad e_i \in N.
\]

this implies \( \dim C^1(N, R) = n \). In order to compute the \( Z^1(N, R)^Q \), we need the general form of a derivation.

From the equalities \((d^1f)(e_i, e_j) = 0\) for \( 1 \leq i, j \leq n \) we obtain
\[
\begin{align*}
  f(e_i) &= x_{1,1} e_1, \\
  f(e_2) &= x_{2,2} e_2, \\
  f(e_i) &= (i - 2)x_{1,1} + x_{2,2} e_i, \quad 3 \leq i \leq n - 1, \\
  f(e_n) &= (\varepsilon x_{1,1} + 2x_{2,2}) e_n,
\end{align*}
\]

where \( \varepsilon = n - 4 \), if \( R = R(g^1_{(n,1)}, 2) \) and \( \varepsilon = 0 \), if \( R = R(g^2_{(n,1)}, 2) \).

Further, from (3.5) we deduce \( \dim Z^1(N, R)^Q = 2 \). It is easily seen that \( R_x \) and \( R_y \) are 1-coboundaries. As a consequence,
\[
\dim H^1(N, R)^Q = 0 \quad \text{and} \quad \dim B^2(N, R)^Q = \dim C^1(N, R) - \dim Z^1(N, R)^Q = n - 2.
\]

We now proceed to compute the 2-cocycles. For an arbitrary fixed cochain \( \phi \in C^2(N, R) \) we set
\[
\phi(e_i, e_j) = \sum_{k=1}^{n} a^k_{i,j} e_k + b_{i,j} x + c_{i,j} y, \quad 1 \leq i, j \leq n.
\]

Using formula (2.3) for the elements \( x \in \mathbb{Q} \) we have
\[
(d^2 \phi)(x, e_i, e_j) = [x, \phi(e_i, e_j)] - \phi([x, e_i], e_j) - \phi(e_i, [x, e_j]) = 0.
\]

in the case when the solvable algebra \( R \) is isomorphic to the algebra \( R(g^1_{(n,1)}, 2) \) by considering the equalities \((d^2 \phi)(x, e_i, e_j) = (d^2 \phi)(y, e_i, e_j) = 0\), we obtain the following
\[
\begin{align*}
  \phi(e_1, e_i) &= -\phi(e_i, e_1) = a_{1,i} e_{i+1}, \quad 2 \leq i \leq n - 2, \\
  \phi(e_i, e_{n-i}) &= -\phi(e_{n-i}, e_i) = a_{i,n-i} e_n, \quad 2 \leq i \leq n - 2.
\end{align*}
\]

Further, the equalities \((d^2 \phi)(e_1, e_i, e_{n-i}) = 0\) imply the following restrictions:
\[
a_{i+1,n-i-1} = (1)^{i+1} (a_{1,n-i-1} - a_{1,i}) - a_{i,n-i}, \quad 2 \leq i \leq \frac{n-1}{2}.
\]

In this case, the free parameters are \( a_{1,2}, a_{1,3}, \ldots, a_{1,n-2}, a_{2,n-2} \).

For the case when the solvable algebra \( R \) is isomorphic to the algebra \( R(g^2_{(n,1)}, 2) \). Then considering the equalities \((d^2 \phi)(x, e_i, e_j) = (d^2 \phi)(y, e_i, e_j) = (d^2 \phi)(e_1, e_i, e_n) = 0\), we obtain the following
\[
\begin{align*}
  \phi(e_1, e_i) &= -\phi(e_i, e_1) = a_{1,i} e_{i+1}, \quad 2 \leq i \leq n - 2, \\
  \phi(e_i, e_n) &= -\phi(e_n, e_i) = a_{i,n} e_{i+2}, \quad 2 \leq i \leq n - 3,
\end{align*}
\]

where
\[ a_{i+1,n} = a_{i,n} - a_{1,i} + a_{1,i+2}, \quad 2 \leq i \leq n - 4. \]

In this case, the free parameters are \( a_{1,2}, a_{1,3}, \ldots, a_{1,n-2}, a_{2,n}. \)

Therefore,

\[ \dim H^2(N, R)^Q = \dim Z^2(N, R)^Q - \dim B^2(N, R)^Q = n - 2 - (n - 2) = 0. \]

\[ \square \]

**Corollary 3.9.** The algebras of Theorem 3.7 are rigid in the variety of Leibniz algebras.

**Proof.** Let \( R \) be a solvable Leibniz algebra of the list of Theorem 3.7. Since Center\( (R) = \{0\} \) and \( H^2(R, R) = \{0\} \). From Theorem 2 in [14] we conclude that \( HL^2(R, R) = \{0\} \). Due to Theorem 2.7 this implies that \( R \) is a rigid algebra.

\[ \square \]

**Theorem 3.10.** Let \( R \) be a solvable Lie algebra isomorphic to one of the algebras from the list of Theorem 3.6. Then \( \dim H^2(R, R) = 1. \)

**Proof.** The following cocycles \( \varphi \) are representatives of \( H^2(R, R) \):

- for the solvable Lie algebra \( R(g^3_{(n, 1)}), 1) : \)

\[
\begin{align*}
\varphi(e_1, e_n) &= e_2, \\
\varphi(e_i, e_n) &= (i - 3)e_{i+2}, & 4 \leq i \leq n - 3, \\
\varphi(e_2, e_i) &= \frac{3}{2}(i - 4)e_{i+3}, & 5 \leq i \leq n - 4, \\
\varphi(e_3, e_i) &= -\frac{3}{2}e_{i+4}, & 4 \leq i \leq n - 5.
\end{align*}
\]

- for the solvable Lie algebra \( R(g^1_7, 1) : \)

\[
\varphi(e_1, e_6) = e_7, \varphi(e_3, e_4) = e_7,
\]

- for the solvable Lie algebra \( R(g^2_9, 1) : \)

\[
\begin{align*}
\varphi(e_1, e_8) &= e_9, \varphi(e_2, e_6) = -24e_8, \varphi(e_3, e_4) = -6e_7, \varphi(e_3, e_5) = -6e_8, \\
\varphi(e_3, e_6) &= 5e_9, \varphi(e_4, e_5) = -7e_9,
\end{align*}
\]

- for the solvable Lie algebra \( R(g^1_{11}, 1) : \)

\[
\begin{align*}
\varphi(e_1, e_{10}) &= e_{11}, \varphi(e_2, e_5) = -\frac{3}{4}e_7, \varphi(e_2, e_6) = -\frac{3}{4}e_7, \varphi(e_2, e_7) = -e_9, \\
\varphi(e_2, e_8) &= \frac{3}{4}e_{10}, \varphi(e_3, e_4) = -\frac{3}{4}e_7, \varphi(e_3, e_5) = -\frac{3}{4}e_8, \varphi(e_3, e_6) = \frac{1}{2}e_9, \\
\varphi(e_3, e_7) &= \frac{7}{4}e_{10}, \varphi(e_4, e_5) = -\frac{5}{4}e_9, \varphi(e_4, e_6) = -\frac{5}{4}e_{10}, \varphi(e_4, e_7) = -e_{11}, \\
\varphi(e_5, e_6) &= 2e_{11}.
\end{align*}
\]

\[ \square \]

Since Center\( (R) = \{0\} \) and \( \dim H^2(R, R) = 1 \) for the algebras in the list of Theorem 3.7, then due to [14, Theorem 2] we conclude \( \dim HL^2(R, R) = 1. \)
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Funding

This work was supported by Agencia Estatal de Investigación (Spain), grant MTM2016-79661-P (European FEDER support included, UE).