Non-splat singularity for the one-phase Muskat problem

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September 10, 2014

Abstract

For the water waves equations, the existence of splat singularities has been shown in [3], i.e., the interface self-intersects along an arc in finite time. The aim of this paper is to show the absence of splat singularities for the incompressible fluid dynamics in porous media.

1 Introduction

The Muskat problem [18] models the evolution of the interface between two fluids of different characteristics in porous media, where the velocity of the fluid is given by Darcy’s law:

$$\frac{\mu}{\kappa} u = -\nabla p - (0, g\rho)$$

where \((x, t) \in \mathbb{R}^2 \times \mathbb{R}^+, u = (u_1(x, t), u_2(x, t))\) is the incompressible velocity (i.e., \(\nabla \cdot u = 0\)), \(p = p(x, t)\) is the pressure, \(\mu = \mu(x, t)\) is the dynamic viscosity, \(\kappa\) is the permeability of the isotropic medium, \(\rho = \rho(x, t)\) is the liquid density and \(g\) is the acceleration due to gravity. The free boundary is caused by the discontinuity between the densities and viscosities of the fluids; the quantities \((\mu, \rho)\) are defined by

$$(\mu, \rho)(x_1, x_2, t) := \begin{cases} (\mu^1, \rho^1) & x \in \Omega^1(t) \\ (\mu^2, \rho^2) & x \in \Omega^2(t) = \mathbb{R}^2 - \Omega^1(t) \end{cases}$$

where \(\mu^1, \rho^1, \mu^2\) and \(\rho^2\) are constants.

We will only study one of the two types of finite time singularities shown for water waves in [3], the splash and splat singularities. The splash-type (Figure 1(a)) corresponds to the case where the fluid interface self-intersects at a single point. This kind of singularity also occurs for the Muskat problem as proved in [2].

In this paper, we will focus on the splat-type (Figure 1(b)). This singularity is a variation of the former in which the fluid interface self-intersects
along an arc. This scenario has been shown to arise for the incompressible Euler equations in the water waves form, see [3], which considers the evolution of the free boundary of a water region in vacuum and irrotational velocity. In [11], these singularities have also been shown to exist for the case with vorticity.

For the Muskat problem, splash singularity cannot be developed in the case in which $\mu_1 = \mu_2$ and $\rho_1 \neq \rho_2$, for more details see [14]. For similar results in the two-fluids interfaces see [13], [10]. However, the splash can be achieved with $\mu_1 = \rho_1 = 0$ where $R_2 - \Omega(t)$ corresponds to the dry region (see [2]).

The aim of this work is to show the absence of splat singularities in the case of an interface between an incompressible irrotational fluid and a dry region in a porous media. Thus, $\mu_1 = \rho_1 = 0$, i.e.,

$$(\mu, \rho)(x_1, x_2, t) := \begin{cases} (\mu^2, \rho^2) & x \in \Omega(t) \\ (0, 0) & x \in \mathbb{R}^2 - \Omega(t). \end{cases}$$

Let the free boundary be parametrized by

$$\partial \Omega = \{ z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)) : \alpha \in \mathbb{R} \}$$

so that the periodic condition holds

$$(z_1(\alpha + 2k\pi, t), z_2(\alpha + 2k\pi, t)) = (z_1(\alpha, t) + 2k\pi, z_2(\alpha, t))$$

with initial data $z(\alpha, 0) = z_0(\alpha)$.

From Darcy’s law, we deduce that the fluid is irrotational, i.e. $\omega = \nabla \times u = 0$, in the interior of the domain $\Omega$. Therefore, the vorticity is concentrated on the free boundary $z(\alpha, t)$ by a Dirac distribution as follows:

$$\omega(x, t) = \nabla^\perp \cdot u(x, t) = \varpi(\alpha, t)\delta(x - z(\alpha, t))$$
where \( \varpi(\alpha, t) \) represents the vorticity strength.

The interface \( z(\alpha, t) \) evolves with an incompressible velocity field satisfying the Biot-Savart law, which can be explicitly computed and is given by the Birkhoff-Rott integral of the amplitude \( \varpi \) along the interface \( z(\alpha, t) \):

\[
BR(z, \varpi)(\alpha, t) = \frac{1}{2\pi} \text{PV} \int \frac{(z(\alpha, t) - z(\beta, t))^\perp}{|z(\alpha, t) - z(\beta, t)|^2} \varpi(\beta, t) d\beta.
\]

(1)

We can subtract any term in the tangential direction to the curve in the velocity field without modifying the geometric evolution of the curve

\[
z_t(\alpha, t) = BR(z, \varpi)(\alpha, t) + c(\alpha, t) \partial_\alpha z(\alpha, t).
\]

(2)

A wise choice of \( c(\alpha, t) \) namely,

\[
c(\alpha, t) = \frac{\alpha + \pi}{2\pi} \int_T \frac{\partial_\beta z(\beta, t)}{||\partial_\beta z(\beta, t)||^2} \cdot \partial_\beta BR(z, \varpi)(\beta, t) d\beta
\]

\[
- \int_{-\pi}^\alpha \frac{\partial_\beta z(\beta, t)}{||\partial_\beta z(\beta, t)||^2} \cdot \partial_\beta BR(z, \varpi)(\beta, t) d\beta
\]

(3)

allows us to accomplish the fact that the length of the tangent vector to \( z(\alpha, t) \) is a function depending only on the variable \( t \) (for more details see [7]):

\[
A(t) = |\partial_\alpha z(\alpha, t)|^2.
\]

We can close the system using Darcy's law and taking the dot product with \( \partial_\alpha z(\alpha, t) \). It is easy to relate \( \varpi \) and the free boundary by (see [7]):

\[
\varpi(\alpha, t) = -2BR(z, \varpi)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) - 2\kappa g \rho^2 \mu^2 \partial_\alpha z_1(\alpha, t).
\]

(4)

For the stability of the problem we consider the Rayleigh-Taylor condition. Rayleigh [19] and Saffman-Taylor [20] gave this condition that must be satisfied for the linearized model in order to have a solution locally in time, namely that the normal component of the pressure gradient jump at the interface has to have a distinguished sign. This condition can be written as

\[
\sigma(\alpha, t) = \frac{\mu^2}{\kappa} BR(z, \varpi)(\alpha, t) \cdot \partial_\alpha^\perp z(\alpha, t) + g\rho^2 \mu^2 \partial_\alpha z_1(\alpha, t) > 0.
\]

Using Hopf’s lemma, the Rayleigh-Taylor condition is satisfied for \( \mu^1 = \rho^1 = 0 \) (see [2]). For the case of equal viscosities (\( \mu^1 = \mu^2 \)), this condition holds when the more dense fluid lies below the interface [4].

This stability has been used to prove local existence in Sobolev spaces, when \( \mu^1 \neq \mu^2 \) and \( \rho^1 \neq \rho^2 \), in [7]. When \( \mu^1 = \mu^2 \) there is local existence and instant analyticity in the stable case, see [4] and [9]. And for small data, the fact that \( \sigma > 0 \) has been used to prove global existence as we can check
in [3, 21, 12] and [16]. Furthermore, there exits initial data with $\sigma > 0$ that in finite time turns to $\sigma < 0$ (see [4] and [15]) and at finite time later the interface breakdown [1].

Finally we introduce the function that measures the arc-chord condition

$$F(z)(\alpha, \beta, t) = \frac{\beta^2}{|z(\alpha) - z(\alpha - \beta)|^2}, \quad \alpha, \beta \in \mathbb{R}$$

with

$$F(z)(\alpha, 0, t) = \frac{1}{|\partial_\alpha z(\alpha, t)|^2}.$$ 

The main theorem of this paper is the following:

**Theorem 1.1.** Let $z_0(\alpha) \in H^k(\mathbb{T})$ for $k \geq 4$ and $F(z_0)(\alpha, \beta) \in L^\infty$. Then the Muskat problem (1-4) will not break down in a splat singularity, i.e., there is no time where there exists disjoint intervals $I_1, I_2 \in \mathbb{R}$ such that $z(I_1, t) = z(I_2, t)$.

In order to prove this theorem we have organized the paper as follows.

In sections 2, 3 and 4 we present several a priori estimates that provide instant analyticity for a single curve that initially satisfies the arc-chord and Rayleigh-Taylor conditions. Section 5 is devoted to prove that the region of analyticity does not collapse to the real axis as long as the curve remains smooth and the arc-chord condition remains bounded.

Instant analyticity and exponential decay of the strip of analyticity is shown in [4] for the case where both fluids have equal viscosities ($\mu_1 = \mu_2$). In such case, the formula for the strength of the vorticity is simpler

$$\varpi(\alpha, t) = -(\rho_2 - \rho_1)\partial_\alpha z_2(\alpha, t).$$

In our scenario, the one-phase Muskat problem, the expression [4] of the strength of the vorticity involves the Birkhoff-Rott integral.

Finally in section 6, we will prove the main theorem using a contradiction argument. The idea of the proof is the following:

Suppose that there exists a splat singularity at time $T$. If the solution $z(\alpha, t)$ is real-analytic at time $T$, the formation of a splat singularity would be impossible. This follows from the fact that we would get a real-analytic curve $z(\alpha, t)$ self-intersecting along an arc, therefore $z(\alpha, t)$ should self-intersect at all points.

Since the curve self-intersects, the arc-chord condition fails in our domain $\Omega$, and thus we have no control on the decay of the strip of analyticity. In order to get around this issue it is necessary to apply a transformation defined by $\tilde{z}(\alpha, t) = P(z(\alpha, t))$ where $P$ is a conformal map (see [3]):

$$P(w) = (\tan\left(\frac{w}{2}\right))^\frac{1}{2}.$$
This conformal map transforms our domain $\Omega$ in $\tilde{\Omega}$ as we can see in Figure 2. The branch of the root will be taken in such a way that it separates the self-intersecting points of the interface.

The new contour evolution equation where we handle the splat singularity is (see [2] for more details):

$$\tilde{z}_t(\alpha, t) = Q^2(\alpha, t)BR(\tilde{z}, \tilde{\omega})(\alpha, t) + \tilde{c}(\alpha)\partial_\alpha \tilde{z}(\alpha, t)$$

where

$$Q^2(\alpha, t) = |\frac{dP}{dw}(z(\alpha, t))|^2 = |\frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha, t)))|^2,$$

$$\tilde{\omega}(\alpha, t) = -2BR(\tilde{z}, \tilde{\omega})(\alpha, t) \cdot \partial_\alpha \tilde{z}(\alpha, t) - \frac{\rho^2}{\mu^2} \partial_\alpha (P_2^{-1}(\tilde{z}(\alpha, t)))$$

and

$$\tilde{c}(\alpha, t) = \frac{\alpha + \pi}{2\pi} \int_{\pi}^{\alpha} \frac{\partial_\beta \tilde{z}(\beta, t)}{|\partial_\beta \tilde{z}(\beta, t)|^2} \cdot \partial_\beta BR(\tilde{z}, \tilde{\omega})(\beta, t) d\beta$$

Finally we find the Rayleigh-Taylor condition in terms of $\tilde{z}$

$$\tilde{\sigma}(\alpha, t) = \frac{\mu^2}{\kappa} BR(\tilde{z}, \tilde{\omega})(\alpha, t) \cdot \partial_\alpha \tilde{z}(\alpha, t) + \rho^2 g \nabla P_2^{-1}(\tilde{z}(\alpha, t)) \cdot \partial_\alpha \tilde{z}(\alpha, t). \quad (5)$$

Our ultimate goal in section 6 is to prove instant analyticity and decay of the strip of analyticity for the Muskat problem in the new domain, which allows us to apply our argument of non-splat, i.e., to prove Theorem 1.1.
2 Estimates on \( z(\alpha, t) \)

Here we show the main estimates that provide instant analyticity into the strip \( S(t) = \{ \alpha + i\zeta : |\zeta| < \lambda t \} \) for each \( t \). To do that we will need the following estimates from [7]:

\[
\| \varpi \|_{H^k} \leq \exp C(\| F(z) \|_{L^\infty}^2 + \| z \|_{H^{k+1}}^2),
\]

for \( k \geq 2 \).

\[
\| BR(z, \varpi) \|_{H^k} \leq \exp C(\| F(z) \|_{L^\infty}^2 + \| z \|_{H^{k+1}}^2),
\]

for \( k \geq 2 \).

These estimates follow also into the complex strip \( S \), since the time derivative plays no role and hence any extra term appears in relation with the terms in [7].

**Remark 2.1.** Inequalities (6) and (7) can also be bounded by a polynomial function, see [8]. In our case, to prove instant analyticity and the decay of the strip, both estimates are valid.

Let \( \lambda \) be given in the definition of \( L^2(S) \) and \( H^k(S) \),

\[
\| z \|_{L^2(S)}^2(t) = \sum_{\pm} \int |z(\alpha \pm i\lambda t, t) - (\alpha \pm i\lambda t, 0)|^2 d\alpha,
\]

\[
\| z \|_{H^k(S)}^2(t) = \| z \|_{L^2(S)}^2(t) + \sum_{\pm} \int |\partial^k_\alpha z(\alpha \pm i\lambda t, t)|^2 d\alpha.
\]

**Remark 2.2.** Above \(|\cdot|\) is the modulus of a vector in \( \mathbb{C}^2 \).

2.1 Estimates for the \( L^2 \) norm of the curve

In order to simplify the exposition we write \( z(\alpha, t) = z(\alpha) \) for a fixed \( t \), and we denote \( \alpha \pm i\lambda t \equiv \gamma \).

Let,

\[
\frac{1}{2} \frac{d}{dt} \int |z(\alpha \pm i\lambda t) - (\alpha \pm i\lambda t, 0)|^2 d\alpha = \frac{1}{2} \frac{d}{dt} \int \overline{z(\gamma)} \cdot (z(\gamma) - (\gamma, 0)) d\alpha
\]

since

\[
\frac{d}{dt} \bar{z} \cdot \bar{\bar{z}} = \frac{dz}{dt} \cdot \bar{z} + z \cdot \frac{d\bar{z}}{dt} = \frac{dz}{dt} \cdot \bar{z} + \bar{z} \cdot \frac{dz}{dt} = 2\Re(\frac{dz}{dt} \cdot \bar{z})
\]

we have,

\[
\frac{1}{2} \frac{d}{dt} \| z \|_{L^2(S)}^2 = \Re \int \overline{z(\gamma) - (\gamma, 0)} \cdot z_t(\gamma) d\alpha \pm \Re \int \overline{z(\gamma) - (\gamma, 0)} \cdot i\lambda \partial_\alpha z(\gamma) d\alpha
\]

\[
\pm \Re \int (z(\gamma) - (\gamma, 0)) \cdot i\lambda (1, 0) d\alpha \equiv I_1 + I_2 + I_3.
\]

\(^1\)At the end of the proof of the theorem [4] we can take any \( \lambda < \frac{\min_{\sigma} |\sigma(\alpha, 0)|}{2} \)
Next we get,

\[ |I_3| \leq \int |z(\gamma) - (\gamma, 0)| d\alpha \leq C \|z\|_{L^2(S)}, \]

\[ |I_2| \leq \int |z(\gamma) - (\gamma, 0)| i\lambda |\partial_\alpha z(\gamma, t)| d\alpha \leq C \|z\|_{L^2(S)} \|z\|_{H^1(S)} \]

and we decompose \( I_1 \) as follow:

\[ |I_1| \leq \int |z(\gamma) - (\gamma, 0)| BR(z, \varpi)(\gamma) |d\alpha + \int |z(\gamma) - (\gamma, 0)| |c(\gamma)| |\partial_\alpha z(\gamma)| d\alpha \]

\[ = J_1 + J_2. \]

Taking \( J_1 \leq \|z\|_{L^2(S)} \|BR(z, \varpi)\|_{L^2(S)} \) and the inequality (7) allows us to estimate \( J_1 \).

On the other hand,

\[ J_2 \leq A^2(t) \|c\|_{L^\infty(S)} \int |z(\gamma) - (\gamma, 0)| d\alpha \]

\[ \leq 2 \int |\partial_\alpha BR(z, \varpi)(\gamma)| d\alpha \int |z(\gamma) - (\gamma, 0)| d\alpha, \]

which yields

\[ |J_2| \leq \exp(C(\|F(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^3(S)})) \]

if we consider the estimate (7). We conclude that

\[ \frac{d}{dt} \|z\|^2_{L^2(S)} \leq \exp(C(\|F(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^3(S)})), \quad \text{for } k \geq 3. \quad (8) \]

### 2.2 Estimates for the \( H^4(S) \) norm

We shall analyze the evolution of \( \|z\|_{H^4(S)}(t) \):

\[ \frac{1}{2} \frac{d}{dt} \int |\partial_\alpha^4 z(\gamma)|^2 d\alpha = \sum_{j=1,2} \frac{1}{2} \frac{d}{dt} \int |\partial_\alpha^4 z_j(\gamma)|^2 d\alpha \]

for \( j = 1, 2 \).

In the same way as before

\[ \frac{1}{2} \frac{d}{dt} \int |\partial_\alpha^4 z_j(\gamma)|^2 d\alpha = \Re \int \overline{\partial_\alpha^4 z_j(\gamma)} (\partial_\alpha^4 z_j(\gamma) \pm i\lambda \partial_\alpha^6 z_j(\gamma)) d\alpha \]

then,

\[ \frac{1}{2} \frac{d}{dt} \int |\partial_\alpha^4 z(\gamma)|^2 d\alpha = \Re \int \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^4 z(\gamma) d\alpha \pm \Re \int \overline{\partial_\alpha^4 z(\gamma)} \cdot i\lambda \partial_\alpha^6 z(\gamma) d\alpha \]

\[ \equiv I_1 + I_2. \]
Let us study $I_2$:

\[ I_2 = \Re \int \partial_{\alpha}^4 z(\gamma) \cdot \bar{\partial}_{\alpha}^4 z(\gamma) i\lambda d\alpha = \int \lambda(-\Re(\partial_{\alpha}^4 z)\Im(\partial_{\alpha}^4 z) + \Im(\partial_{\alpha}^4 z)\Re(\partial_{\alpha}^4 z)) d\alpha \]

\[ = 2\lambda \int \Im(\partial_{\alpha}^4 z)\Re(\partial_{\alpha}^4 z) d\alpha = -2\lambda \int \Im(\partial_{\alpha}^4 z)\Re(\Lambda(H(\partial_{\alpha}^4 z))) d\alpha \]

\[ = -2\lambda \int \Lambda^{\frac{1}{2}}(\partial_{\alpha}^4 z))\Re(\Lambda^{\frac{1}{2}} H(\partial_{\alpha}^4 z)) d\alpha \leq 2\lambda\|\Lambda^{\frac{1}{2}}\Im(\partial_{\alpha}^4 z)\|_{L^2(S)}\|\Lambda^{\frac{1}{2}}\partial_{\alpha}^4 z\|_{L^2(S)} \]

\[ \leq 2\lambda\|\Lambda^{\frac{1}{2}}\partial_{\alpha}^4 z\|^2_{L^2(S)}, \]

where $\Lambda$ is defined by the Fourier transform $\hat{\Lambda}(\xi) = |\xi| \hat{\xi}(\xi)$ and $H$ is the Hilbert transform:

\[ \Lambda(f)(x) = \frac{1}{2\pi} PV \int \frac{f(x) - f(y)}{|x - y|^2} dy, \]

\[ H(f)(x) = \frac{1}{\pi} PV \int \frac{f(y)}{x - y} dy. \]

Therefore,

\[ I_2 \leq 2\lambda\|\Lambda^{\frac{1}{2}}\partial_{\alpha}^4 z\|^2_{L^2(S)}. \]

Since we have $z_1(\gamma) = BR(z, \bar{\varpi})(\gamma) + c(\gamma)\partial_{\alpha} z(\gamma)$, then:

\[ I_1 = \Re \int \partial_{\alpha}^4 z(\gamma) \cdot \bar{\partial}_{\alpha}^4 BR(z, \bar{\varpi})(\gamma) d\alpha + \Re \int \partial_{\alpha}^4 z(\gamma) \cdot \bar{\partial}_{\alpha}^4 (c(\gamma)\partial_{\alpha} z(\gamma)) d\alpha = I_1 + I_2. \]

We will estimate $I_1$ in the subsections $2.2.1$ and $2.2.2$ and $J_2$ in $2.2.3$.

### 2.2.1 Integrable terms in $\partial_{\alpha}^4 BR(z, \bar{\varpi})$

We decompose $J_1 = I_3 + I_4 + I_5 + I_6 + I_7$, where:

\[ I_3 = \frac{1}{2\pi} \Re \int \partial_{\alpha}^4 z(\gamma) \cdot \bar{\partial}_{\alpha}^4 \left( \frac{(z(\gamma) - z(\gamma - \beta))^\perp}{|z(\gamma) - z(\gamma - \beta)|^2} \right) \varpi(\gamma - \beta) d\alpha d\beta, \]

\[ I_4 = \frac{2}{\pi} \Re \int \partial_{\alpha}^4 z(\gamma) \cdot \bar{\partial}_{\alpha}^4 \left( \frac{(z(\gamma) - z(\gamma - \beta))^\perp}{|z(\gamma) - z(\gamma - \beta)|^2} \right) \varpi(\gamma - \beta) d\alpha d\beta, \]

\[ I_5 = \frac{2}{\pi} \Re \int \partial_{\alpha}^4 z(\gamma) \cdot \bar{\partial}_{\alpha}^4 \left( \frac{(z(\gamma) - z(\gamma - \beta))^\perp}{|z(\gamma) - z(\gamma - \beta)|^2} \right) \partial_{\alpha}^2 \varpi(\gamma - \beta) d\alpha d\beta, \]

\[ I_6 = \frac{2}{\pi} \Re \int \partial_{\alpha}^4 z(\gamma) \cdot \bar{\partial}_{\alpha}^4 \left( \frac{(z(\gamma) - z(\gamma - \beta))^\perp}{|z(\gamma) - z(\gamma - \beta)|^2} \right) \partial_{\alpha}^3 \varpi(\gamma - \beta) d\alpha d\beta, \]

\[ I_7 = \frac{1}{2\pi} \Re \int \partial_{\alpha}^4 z(\gamma) \cdot \bar{\partial}_{\alpha}^4 \left( \frac{(z(\gamma) - z(\gamma - \beta))^\perp}{|z(\gamma) - z(\gamma - \beta)|^2} \right) \partial_{\alpha}^4 \varpi(\gamma - \beta) d\alpha d\beta. \]

Below we estimate the highest order term of each $I_j$ for $j = 3, ..., 6$.

The most singular terms for $I_3$ are those in which four derivatives appear. In order to simplify we write $\Delta \partial_{\alpha}^4 z = \partial_{\alpha}^4 z(\gamma) - \partial_{\alpha}^4 z(\gamma - \beta)$. 
One of the two singular terms of $I_3$ is given by

$$K_1 = \frac{1}{2\pi} \Re \iint \frac{\partial^4_\alpha z(\gamma) \cdot (\Delta \partial^4_\alpha z)(\gamma - \beta)}{|\partial^2_\alpha z(\gamma)|^2} \varpi(\gamma - \beta) d\alpha d\beta,$$

which we decompose in $K_1 = L_1 + L_2$, where

$$L_1 = \frac{1}{2\pi} \Re \iint \frac{\partial^4_\alpha z(\gamma) \cdot (\Delta \partial^4_\alpha z)(\gamma - \beta)}{|\partial^2_\alpha z(\gamma)|^2} \varpi(\gamma - \beta) \frac{1}{|z(\gamma) - z(\gamma - \beta)|^2} d\alpha d\beta,$$

$$L_2 = \frac{1}{2\pi} \Re \iint \frac{\partial^4_\alpha z(\gamma) \cdot (\Delta \partial^4_\alpha z)(\gamma - \beta)}{|\partial^2_\alpha z(\gamma)|^2} \varpi(\gamma - \beta) d\alpha d\beta.$$

Let us study $L_1$, if $\psi = \gamma - \beta + s\beta + t\beta - st\beta$, $\phi = \gamma - \beta + s\beta$ and

$$B(\gamma, \beta) \equiv \frac{1}{|z(\gamma) - z(\gamma - \beta)|^2} - \frac{1}{|\partial_\alpha z(\gamma)|^2 \beta^2} = \frac{\beta \int_0^1 \int_0^1 \partial^2_\alpha z(\psi)(1 - s)dtds \cdot \int_0^1 [\partial_\alpha z(\gamma) + \partial_\alpha z(\phi)] ds}{|z(\gamma) - z(\gamma - \beta)|^2 |\partial_\alpha z(\gamma)|^2} - \frac{\beta \int_0^1 \int_0^1 \partial^2_\alpha z(\psi) - \partial^2_\alpha z(\gamma)}{|\psi - \gamma|^2} \beta^2 (1 + s + t - st)^2 (1 - s) dtds \cdot \int_0^1 [\partial_\alpha z(\gamma) + \partial_\alpha z(\phi)] ds}{|z(\gamma) - z(\gamma - \beta)|^2 |\partial_\alpha z(\gamma)|^2} + \frac{\beta \partial^2_\alpha z(\gamma) \cdot \int_0^1 [\partial_\alpha z(\gamma) + \partial_\alpha z(\phi)] ds}{|z(\gamma) - z(\gamma - \beta)|^2 |\partial_\alpha z(\gamma)|^2} \equiv B_1(\gamma, \beta) + B_2(\gamma, \beta)$$

we have,

$$L_1 = \frac{1}{2\pi} \Re \iint \frac{\partial^4_\alpha z(\gamma) \cdot (\Delta \partial^4_\alpha z)(\gamma - \beta)B_1(\gamma, \beta)}{|\partial^2_\alpha z(\gamma)|^2} d\alpha d\beta$$

$$+ \frac{1}{2\pi} \Re \iint \frac{\partial^4_\alpha z(\gamma) \cdot (\Delta \partial^4_\alpha z)(\gamma - \beta)B_2(\gamma, \delta)}{|\partial^2_\alpha z(\gamma)|^2} d\alpha d\beta \equiv M_1 + M_2.$$

It is easy to check,

$$M_1 \leq C \|\mathcal{F}(z)\|_{L^\infty(S)}^2 \|z\|_{L^2(S)} \|\varpi\|_{L^\infty(S)}^2 |\partial^4_\alpha z(\gamma)|^2.$$

Furthermore,

$$B_2(\gamma, \beta) = \frac{\beta^2 \partial^2_\alpha z(\gamma) \int_0^1 \int_0^1 \partial^2_\alpha z(\eta)(s - 1) dtds}{|z(\gamma) - z(\gamma - \beta)|^2 |\partial_\alpha z(\gamma)|^2}$$

$$+ \frac{\beta \partial^2_\alpha z(\gamma) 2\partial_\alpha z(\gamma)}{|z(\gamma) - z(\gamma - \beta)|^2 |\partial_\alpha z(\gamma)|^2} \equiv B_3(\gamma, \beta) + B_4(\gamma, \beta).$$
In the same way, we deal with $M_2$ and we have:

$$
M_2 = \frac{1}{2\pi} \Re \int \int \frac{\partial^4_{\alpha} z(\gamma) \cdot (\Delta \partial^4_{\alpha} z)^{\perp} \varpi(\gamma - \beta) B_4(\gamma, \beta) d\alpha d\beta}{\varpi L} + \frac{1}{2\pi} \Re \int \int \frac{\partial^4_{\alpha} z(\gamma) \cdot (\Delta \partial^4_{\alpha} z)^{\perp} \varpi(\gamma - \beta) B_4(\gamma, \beta) d\alpha d\beta \equiv N_1 + N_2.
$$

It is clear that,

$$
N_1 \leq C ||F(z)||_{L^\infty} \|z\|_{L^2} \|\varpi\|_{L^\infty} \|\partial^4_{\alpha} z\|_{L^2(S)}
$$

and

$$
N_2 = \frac{1}{2\pi} \Re \int \int \frac{\partial^4_{\alpha} z(\gamma) \cdot (\Delta \partial^4_{\alpha} z)^{\perp} \varpi(\gamma - \beta) \frac{\partial^2_{\alpha} z(\gamma) 2 \partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} B(\gamma, \beta) d\alpha d\beta
+ \frac{1}{2\pi} \Re \int \int \frac{\partial^4_{\alpha} z(\gamma) \cdot (\Delta \partial^4_{\alpha} z)^{\perp} \varpi(\gamma - \beta) \frac{\partial^2_{\alpha} z(\gamma) 2 \partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} d\alpha d\beta \equiv N_2^1 + N_2^2.
$$

Directly,

$$
N_2^2 \leq C ||F(z)||_{L^\infty} \|z\|_{L^2} \|\varpi\|_{L^\infty} \|\partial^4_{\alpha} z\|_{L^2(S)}^2
$$

and we decompose,

$$
N_2^2 = \frac{1}{2\pi} \Re \int \int \frac{\partial^4_{\alpha} z(\gamma) \cdot (\partial^4_{\alpha} z(\gamma))^{\perp} \varpi(\gamma - \beta) \frac{\partial^2_{\alpha} z(\gamma) 2 \partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} d\alpha d\beta
- \frac{1}{2\pi} \Re \int \int \frac{\partial^4_{\alpha} z(\gamma) \cdot (\partial^4_{\alpha} z(\gamma - \beta))^{\perp} \varpi(\gamma - \beta) \frac{\partial^2_{\alpha} z(\gamma) 2 \partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} d\alpha d\beta \\
\equiv N_2^{21} + N_2^{22}
$$

where

$$
N_2^{21} = \frac{1}{2} \Re \int \int \frac{\partial^4_{\alpha} z(\gamma) \cdot (\partial^4_{\alpha} z(\gamma))^{\perp} \partial^2_{\alpha} z(\gamma) 2 \partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} H(\varpi) d\alpha
\leq C ||F(z)||_{L^\infty} \frac{1}{2} ||z||_{L^2} ||\varpi||_{L^\infty} \|\partial^4_{\alpha} z\|_{L^2(S)} \|\varpi\|_{C^1}
$$

and

$$\begin{align*}
N_2^{22} &= -\frac{1}{2\pi} \Re \int \int \frac{\partial^4_{\alpha} z(\gamma) \cdot (\partial^4_{\alpha} z(\gamma - \beta))^{\perp} (\varpi(\gamma - \beta) - \varpi(\gamma)) \frac{\partial^2_{\alpha} z(\gamma) 2 \partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} d\alpha d\beta \\
&\leq C ||F(z)||_{L^\infty} \frac{1}{2} ||z||_{L^2} ||\varpi||_{C^1} \|\varpi||_{C^1} \|\partial^4_{\alpha} z\|_{L^2(S)}^2 - \frac{1}{2} \Re \int \int \frac{\partial^4_{\alpha} z(\gamma) \cdot (\varpi(\gamma) \frac{\partial^2_{\alpha} z(\gamma) 2 \partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} H(\varpi) d\alpha
\leq C ||F(z)||_{L^\infty} \frac{1}{2} ||z||_{L^2} \|\varpi||_{C^1} \|\varpi||_{C^1} \|\partial^4_{\alpha} z\|_{L^2(S)}^2
\end{align*}$$
Here we have used
\[ \| H(f) \|_{L^p} \leq C \| f \|_{L^p} \quad \text{for } 1 < p < \infty, \]
\[ \| Hf \|_{L^\infty} \leq \| f \|_{C^\delta} \quad \text{for } f \in C^\delta, \text{ and } 0 < \delta < 1. \]
Hence, using (10)
\[ L_1 \leq \exp C(\| F(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2). \]
For \( L_2 \) we write \( L_2 = M_3 + M_4 \), with
\[
M_3 = \frac{1}{2\pi} \mathfrak{R} \int \int \frac{\partial^4_{\alpha z}(z)}{\partial_{\alpha z}(z)^2} \cdot (\Delta \partial^4_{\alpha} z) \frac{\omega(\gamma - \beta) - \omega(\gamma)}{|\partial_x z(\gamma)|^{2\beta^2}} d\alpha d\beta,
\]
\[
M_4 = \frac{1}{2\pi} \mathfrak{R} \int \int \frac{\partial^4_{\alpha z}(z)}{\partial_{\alpha z}(z)^2} \cdot (\Delta \partial^4_{\alpha} z) \frac{\omega(\gamma)}{|\partial_x z(\gamma)|^{2\beta^2}} d\alpha d\beta.
\]
Next we write
\[
M_3 = \frac{1}{2\pi} \mathfrak{R} \int \int \frac{\partial^4_{\alpha z}(z)}{\partial_{\alpha z}(z)^2} \cdot (\Delta \partial^4_{\alpha} z) \frac{\omega(\gamma - \beta) - \omega(\gamma)}{|\partial_x z(\gamma)|^{2\beta^2}} d\alpha d\beta
\]
\[
- \frac{1}{2\pi} \mathfrak{R} \int \int \frac{\partial^4_{\alpha z}(z)}{\partial_{\alpha z}(z)^2} \cdot (\Delta \partial^4_{\alpha} z) \frac{\omega(\gamma - \beta) - \omega(\gamma)}{|\partial_x z(\gamma)|^{2\beta^2}} d\alpha d\beta \equiv N_3 + N_4
\]
where
\[
N_3 = \mathfrak{R} \int \frac{\partial^4_{\alpha z}(z)}{\partial_{\alpha z}(z)^2} \cdot (\Delta \partial^4_{\alpha} z) \frac{\Lambda(\omega)(\gamma)}{|\partial_x z(\gamma)|^2} d\alpha
\]
\[
\leq C \| F(z) \|_{L^\infty(S)} \| \partial^4_{\alpha z}(z) \|_{L^2(S)} \| \Lambda \omega \|_{L^\infty(S)} \leq \| F(z) \|_{L^\infty(S)} \| \partial^4_{\alpha z}(z) \|_{L^2(S)} \| \omega \|_{C^{1,\delta}(S)}
\]
and
\[
N_4 = -\frac{1}{2\pi} \mathfrak{R} \int \int \frac{\partial^4_{\alpha z}(z)}{\partial_{\alpha z}(z)^2} \cdot (\Delta \partial^4_{\alpha} z) \frac{\omega(\gamma - \beta) - \omega(\gamma)}{|\partial_x z(\gamma)|^2} d\alpha d\beta
\]
\[
- \frac{1}{2\pi} \mathfrak{R} \int \int \frac{\partial^4_{\alpha z}(z)}{\partial_{\alpha z}(z)^2} \cdot (\Delta \partial^4_{\alpha} z) \frac{\partial_x \omega(\gamma) - \partial_x \omega(\gamma)}{|\partial_x z(\gamma)|^2} d\alpha d\beta
\]
\[
\leq C \| F(z) \|_{L^\infty(S)} \| \partial^4_{\alpha z}(z) \|_{L^2(S)} \| \omega \|_{C^2(S)} - \frac{1}{2} \mathfrak{R} \int \frac{\partial^4_{\alpha z}(z)}{\partial_{\alpha z}(z)^2} \cdot (\Delta \partial^4_{\alpha} z) \frac{\partial_x \omega(\gamma)}{|\partial_x z(\gamma)|^2} d\alpha
\]
\[
\leq \exp C(\| F(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2).
\]
For \( M_4 \),
\[
M_4 = \mathfrak{R} \int \frac{\partial^4_{\alpha z}(z)}{\partial_{\alpha z}(z)^2} \cdot \Lambda(\partial^4_{\alpha} z) \frac{\omega(\gamma)}{|\partial_x z(\gamma)|^2} d\alpha
\]
\[
= \int \mathfrak{R} \left( \frac{\omega}{A(t)} \right) (-\mathfrak{R}(\partial^4_{\alpha z}) \cdot \mathfrak{I}(\Lambda(\partial^4_{\alpha} z)) + \mathfrak{R}(\partial^4_{\alpha z}) \cdot \mathfrak{R}(\Lambda(\partial^4_{\alpha} z))) d\alpha
\]
\[
+ \int \mathfrak{R} \left( \frac{\omega}{A(t)} \right) (\mathfrak{R}(\partial^4_{\alpha z}) \cdot \mathfrak{I}(\Lambda(\partial^4_{\alpha} z)) + \mathfrak{I}(\partial^4_{\alpha z}) \cdot \mathfrak{R}(\Lambda(\partial^4_{\alpha} z))) d\alpha
\]
\[
\equiv N_5 + N_6.
\]
Now we take,
\[
N_6 = \int \Re\left( \frac{\omega}{A(t)} \right) (-\Re(\partial_\alpha^4 z_1) \Re(\Lambda(\partial_\alpha^4 z_2)) + \Re(\partial_\alpha^4 z_2) \Re(\Lambda(\partial_\alpha^4 z_1))) d\alpha
\]
\[+ \int \Re\left( \frac{\omega}{A(t)} \right) (-\Im(\partial_\alpha^4 z_1) \Im(\Lambda(\partial_\alpha^4 z_2)) + \Im(\partial_\alpha^4 z_2) \Im(\Lambda(\partial_\alpha^4 z_1))) d\alpha\]
\[= N_6^1 + N_6^2\]
where it is easy to find a commutator formula such that, using (see \[17]\)
\[
\|\Lambda(fg) - g\Lambda(f)\|_{L^2} \leq \|g\|_{C^1} \|f\|_{L^2},
\]
we get
\[
N_6^1 = \int (-\Lambda(\Re\left( \frac{\omega}{A(t)} \right) \Re(\partial_\alpha^4 z_1)) + \Re(\partial_\alpha^4 z_2) \Re(\Lambda(\partial_\alpha^4 z_1))) d\alpha
\]
\[\leq C\|\Re\left( \frac{\omega}{A(t)} \right)\|_{C^1} \|\partial_\alpha^4 z\|_{L^2(S)}^2.
\]
In the same way,
\[
N_6^2 \leq C\|\Re\left( \frac{\omega}{A(t)} \right)\|_{C^1} \|\partial_\alpha^4 z\|_{L^2(S)}^2.
\]
Thus,
\[
N_6 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).
\]
For \(N_5\) we have,
\[
N_5 = \int \Im\left( \frac{\omega}{A(t)} \right) (\Re(\partial_\alpha^4 z \cdot \Re(\Lambda(\partial_\alpha^4 z))) + \Im(\partial_\alpha^4 z) \cdot \Re(\Lambda(\partial_\alpha^4 z))) d\alpha
\]
\[= \int (\Lambda(\Im\left( \frac{\omega}{A(t)} \right) \Re(\partial_\alpha^4 z)) - \Im\left( \frac{\omega}{A(t)} \right) \Re(\Lambda(\partial_\alpha^4 z))) \cdot \Im(\partial_\alpha^4 z) d\alpha
\]
\[+ 2 \int \Im\left( \frac{\omega}{A(t)} \right) \Re(\Lambda(\partial_\alpha^4 z \cdot \cdot)) \cdot \Im(\partial_\alpha^4 z) d\alpha
\]
\[= N_5^1 + N_5^2.
\]
Then,
\[
N_5^1 \leq C\|\Im\left( \frac{\omega}{A(t)} \right)\|_{C^1} \|\partial_\alpha^4 z\|_{L^2(S)}^2 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2)
\]
and
\[
N_5^2 = 2 \int \Lambda \left( \Im\left( \frac{\omega}{A(t)} \right) \Re(\partial_\alpha^4 z) \cdot \Re(\Lambda \left( \partial_\alpha^4 z \cdot \cdot \right)) \right) d\alpha \leq 2\|\Lambda \left( \Im\left( \frac{\omega}{A(t)} \right) \Re(\partial_\alpha^4 z) \cdot \Re(\Lambda \left( \partial_\alpha^4 z \cdot \cdot \right)) \right)\|_{L^2(S)}^2 \Lambda \left( \partial_\alpha^4 z \cdot \cdot \right)\|_{L^2(S)}^2
\]
\[\leq C\|\Im\left( \frac{\omega}{A(t)} \right)\|_{H^2(S)} \|\partial_\alpha^4 z\|_{L^2(S)}^2 + \|\Lambda \left( \partial_\alpha^4 z \cdot \cdot \right)\|_{L^2(S)}^2 \Lambda \left( \partial_\alpha^4 z \cdot \cdot \right)\|_{L^2(S)}^2
\]
\[\leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) + C\|\Im\left( \frac{\omega}{A(t)} \right)\|_{H^2(S)} \|\Lambda \left( \partial_\alpha^4 z \cdot \cdot \right)\|_{L^2(S)}^2.
\]
Concluding,
\[ K_1 \leq \exp\left(C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) + C\|\mathcal{I}\left(\frac{\mathcal{C}}{A(t)}\right)\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)}^2\right). \]

The other singular term with four derivatives inside \( L_3 \) is given by
\[ K_2 = -\frac{1}{\pi} \Re \int \frac{1}{|\Delta z|^4} (\Delta z \cdot \Delta \partial_\alpha^4 z) \varpi (\gamma - \beta) \, d\alpha d\beta. \]

Here we take \( K_2 = L_3 + L_4 + L_5 \) where
\[
L_3 = -\frac{1}{\pi} \Re \int \frac{1}{|\Delta z|^4} (\Delta z \cdot \Delta \partial_\alpha^4 z) \varpi (\gamma - \beta) \, d\alpha d\beta,
\]
\[
L_4 = -\frac{1}{\pi} \Re \int \frac{1}{|\Delta z|^4} (\Delta z \cdot \Delta \partial_\alpha^4 z) \varpi (\gamma - \beta) \, d\alpha d\beta,
\]
\[
L_5 = -\frac{1}{\pi} \Re \int \frac{1}{|\Delta z|^4} (\Delta z \cdot \Delta \partial_\alpha^4 z) \varpi (\gamma - \beta) \, d\alpha d\beta.
\]

We compute
\[
C(\gamma, \beta) = \frac{(\Delta z)^\perp}{|\Delta z|^4} - \frac{\partial_\alpha z(\gamma)^\perp}{|\partial_\alpha z(\gamma)|^4 \beta^3} = \frac{\beta^2 \int_0^1 \int_0^1 \partial_\alpha^2 z(\eta) (s - 1) ds dt}{|\Delta z|^4} + \frac{\beta^2 \partial_\alpha z(\gamma)^\perp \int_0^1 \int_0^1 \partial_\alpha^2 z(\psi) (1 - s) ds dt \cdot \int_0^1 |\partial_\alpha z(\gamma) + \partial_\alpha z(\phi)| ds \int_0^1 |\partial_\alpha z(\gamma)|^2 + |\partial_\alpha z(\phi)|^2| ds}{|\Delta z|^4 |\partial_\alpha z(\gamma)|^4}
\]
\[
\equiv C_1(\gamma, \beta) + C_2(\gamma, \beta)
\]

and
\[
\Delta z - \beta \partial_\alpha z(\gamma) = \beta^2 \int_0^1 \int_0^1 \partial_\alpha^2 z(\eta) (s - 1) ds dt
\]
where \( \eta = \gamma - t\beta + st\beta \), allowing us to obtain the desired estimate for the term \( L_3 \).

Next we split \( L_4 = M_5 + M_6 \) since \( \Delta \partial_\alpha^4 z = \partial_\alpha^4 z(\gamma) - \partial_\alpha^4 z(\gamma - \beta) \):
\[
M_5 = -\frac{1}{\pi} \Re \int \frac{1}{|\Delta z|^4} (\Delta z \cdot C(\gamma, \beta)(\beta \partial_\alpha z(\gamma) \cdot \partial_\alpha^4 z(\gamma)) \varpi (\gamma - \beta) \, d\alpha d\beta,
\]
\[
M_6 = -\frac{1}{\pi} \Re \int \frac{1}{|\Delta z|^4} (\Delta z \cdot C(\gamma, \beta)(\beta \partial_\alpha z(\gamma) \cdot \partial_\alpha^4 z(\gamma - \beta)) \varpi (\gamma - \beta) \, d\alpha d\beta.
\]

By following the same approach for \( L_1 \) we have,
\[ |M_5| \leq C \| \mathcal{F}(z) \|_{L^\infty(S)} \| z \|_{C^2(S)} \| \varpi \|_{L^\infty(S)} \| \partial^4_a z \|_{L^2(S)} \]
\[ + | \Re \int \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} (\partial_a z(\gamma) \cdot \partial_a z(\gamma)) \frac{H(\varpi)(\gamma)}{|\partial_a z(\gamma)|^4} d\alpha | \]
and
\[ |M_6| \leq C \| \mathcal{F}(z) \|_{L^\infty(S)} \| z \|_{C^2(S)} \| \varpi \|_{L^\infty(S)} \| \partial^4_a z \|_{L^2(S)} \]
\[ + | \Re \int \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} (\partial_a z(\gamma) \cdot \partial_a z(\gamma)) \frac{H(\varpi)(\gamma)}{|\partial_a z(\gamma)|^4} d\alpha |. \]

Then, the term \( L_4 \) is controlled.

To conclude the estimates of \( K_2 \), we need to see what happens with the term \( L_5 \)

\[ L_5 = -\frac{1}{\pi} \int \int \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} (\partial_a z(\gamma) \cdot \partial_a z(\gamma)) \frac{\varpi(\gamma - \beta)}{\beta^2} d\alpha d\beta \]
\[ - \frac{1}{\pi} \int \int \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} (\partial_a z(\gamma) \cdot \partial_a z(\gamma)) \frac{\varpi(\gamma - \beta)}{\beta^2} d\alpha d\beta \]
\[ \equiv M_7 + M_8. \]

For \( M_7 \) we proceed in the same way as in \( L_4 \) and we get:

\[ M_7 \leq \exp C(\| \mathcal{F}(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2) \]
\[ - \Re \int \int \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} (\partial_a z(\gamma) \cdot \partial_a z(\gamma)) H(\varpi)(\gamma) d\alpha \]
\[ + \Re \int \int \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} (\partial_a z(\gamma) \cdot \partial_a z(\gamma)) \varpi(\gamma) d\alpha. \]

Then,

\[ M_7 \leq \exp C(\| \mathcal{F}(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2). \]

To control the term \( M_8 \), we decompose it as follows,

\[ M_8 = -\frac{1}{\pi} \int \int \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} (\partial_a z(\gamma) \cdot \partial_a z(\gamma)) \frac{\varpi(\gamma - \beta)}{\beta^2} d\alpha d\beta \]
\[ - \frac{1}{\pi} \int \int \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} (\partial_a z(\gamma) \cdot \partial_a z(\gamma)) \frac{\varpi(\gamma)}{\beta^2} d\alpha d\beta \]
\[ \equiv N_7 + N_8. \]

Since \( \Delta \partial_a^4 z = \partial_a^4 z(\gamma) - \partial_a^4 z(\gamma - \beta) \) we have,

\[ N_7 = -\frac{1}{\pi} \int \int \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} (\partial_a z(\gamma) \cdot \partial_a z(\gamma)) \frac{\varpi(\gamma - \beta)}{\beta^2} d\alpha d\beta \]
\[ + \frac{1}{\pi} \int \int \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{\partial^4_a z(\gamma)}{\partial \alpha z(\gamma)} (\partial_a z(\gamma) \cdot \partial_a z(\gamma - \beta)) \frac{\varpi(\gamma - \beta)}{\beta^2} d\alpha d\beta \]
\[ \equiv O_1 + O_2 \]
where,

\[
O_1 = -2\Re \int \frac{\partial^4 z(\gamma)}{A^2(t)} \cdot \frac{\partial^4 z(\gamma)}{A^2(t)} \, \partial_{\alpha} z(\gamma) \cdot \partial_{\alpha} z(\gamma) \, d\alpha d\beta
\]
\[
\leq C \| F(z) \|_{L^\infty(S)} \| \partial^4 z \|^2_{L^2(S)} \| \varpi \|_{L^\infty(S)} \leq C \| F(z) \|_{L^\infty(S)} \| \partial^4 z \|^2_{L^2(S)} \| \varpi \|_{C^1(S)}
\]
\[
\leq \exp C(\| F(z) \|^2_{L^\infty(S)} + \| z \|^2_{H^4(S)})
\]

and

\[
O_2 \leq \exp C(\| F(z) \|^2_{L^\infty(S)} + \| z \|^2_{H^4(S)})
\]
\[
+ \Re \int \frac{\partial^4 z(\gamma)}{A^2(t)} \cdot \frac{\partial^4 z(\gamma)}{A^2(t)} \, \partial_{\alpha} z(\gamma) \cdot H(\partial^4 z(\gamma)) \partial_{\alpha} \varpi(\gamma) \, d\alpha
\]
\[
\leq \exp C(\| F(z) \|^2_{L^\infty(S)} + \| z \|^2_{H^4(S)}).
\]

Using integration by parts for \( \Lambda \),

\[
N_9 = -2\Re \int \Lambda \frac{\partial^4 z}{A^2(t)} \cdot \frac{\partial^4 z}{A^2(t)} \varpi \partial_{\alpha} z(\gamma) \cdot \partial_{\alpha} z(\gamma) \, d\alpha
\]
\[
= -2\Re \left[ \int \left( \Lambda \frac{\partial^4 z}{A^2(t)} \cdot \frac{\partial^4 z}{A^2(t)} \varpi \partial_{\alpha} z(\gamma) - \partial_{\alpha} z(\gamma) \varpi(\gamma) \frac{\partial^4 z(\gamma)}{A^2(t)} \cdot \Lambda(\frac{\partial^4 z}{A^2(t)})(\gamma) \right) \partial_{\alpha} z(\gamma) \, d\alpha \right]
\]
\[
- 2\Re \int \frac{\partial^4 z(\gamma)}{A^2(t)} \cdot \Lambda(\frac{\partial^4 z}{A^2(t)})(\gamma) \partial_{\alpha} z(\gamma) \cdot \partial_{\alpha} z(\gamma) \varpi(\gamma) \, d\alpha
\]
\[
\equiv O_3 + O_4.
\]

Using the commutator estimate [10],

\[
O_3 \leq \exp C(\| F(z) \|^2_{L^\infty(S)} + \| z \|^2_{H^4(S)}).
\]

Taking three derivatives of \( \Lambda(t) = |\partial_{\alpha} z|^2 \) we take

\[
\partial_{\alpha} z(\alpha) \cdot \partial_{\alpha} z(\alpha) = -3 \partial_{\alpha}^2 z(\alpha) \cdot \partial_{\alpha}^3 z(\alpha).
\]

Together with \( \Lambda = \partial_{\alpha} H \) and integrating by parts

\[
O_4 = -6\Re \int H(\frac{\partial^4 z}{A^2(t)})(\gamma) \cdot \frac{\partial^4 z(\gamma)}{A^2(t)} \cdot \partial_{\alpha} z(\gamma) \cdot \partial_{\alpha} z(\gamma) \varpi(\gamma) \, d\alpha
\]
\[
- 6\Re \int H(\frac{\partial^4 z}{A^2(t)})(\gamma) \cdot \frac{\partial^4 z(\gamma)}{A^2(t)} \partial_{\alpha} z(\gamma) \cdot \partial_{\alpha} z(\gamma) \varpi(\gamma) \, d\alpha
\]
\[
- 6\Re \int H(\frac{\partial^4 z}{A^2(t)})(\gamma) \cdot \frac{\partial^4 z(\gamma)}{A^2(t)} \partial_{\alpha} z(\gamma) \cdot \partial_{\alpha} z(\gamma) \varpi(\gamma) \, d\alpha
\]
\[
- 6\Re \int H(\frac{\partial^4 z}{A^2(t)})(\gamma) \cdot \frac{\partial^4 z(\gamma)}{A^2(t)} \partial_{\alpha} z(\gamma) \cdot \partial_{\alpha} z(\gamma) \partial_{\alpha} z(\gamma) \varpi(\gamma) \, d\alpha
\]
\[
\leq \exp C(\| F(z) \|^2_{L^\infty(S)} + \| z \|^2_{H^4(S)}).
\]
Then,

\[ L_5 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2). \]

All previous discussion shows that \( I_3 \) satisfies,

\[ I_3 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) + C\|\mathcal{F}(\frac{\varpi}{A(t)})\|_{H^2(S)} \|A^{\frac{1}{2}}(\partial_\alpha z)\|_{L^2(S)}^2. \]

For \( I_4 \) the most singular terms are those for which three derivatives are applied to \( z(\gamma) \).

One of those is \( K_3 \)

\[ K_3 = \frac{2}{\pi} \int \int \frac{(\partial_\alpha^3 z(\gamma) - \partial_\alpha^3 z(\gamma - \beta))\partial_\alpha \varpi(\gamma - \beta) d\alpha d\beta}{|z(\gamma) - z(\gamma - \beta)|^2}. \]

We split \( K_3 = L_6 + L_7 + L_8 \)

\[ L_6 = \frac{2}{\pi} \int \int \frac{\partial_\alpha^3 z(\gamma) \partial_\alpha \varpi(\gamma - \beta) - \partial_\alpha \varpi(\gamma)}{|z(\gamma) - z(\gamma - \beta)|^2} d\alpha d\beta, \]

\[ L_7 = \frac{2}{\pi} \int \int \frac{\partial_\alpha^3 z(\gamma) \partial_\alpha \varpi(\gamma)}{|z(\gamma) - z(\gamma - \beta)|^2} \left( 1 - \frac{1}{|\partial_\alpha z(\gamma)|^2\beta^2} \right) d\alpha d\beta, \]

\[ L_8 = \frac{2}{\pi} \int \int \frac{(\partial_\alpha^3 z)^2 \partial_\alpha \varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta^2} d\alpha d\beta. \]

Using that

\[ \Delta \partial_\alpha^3 z = \beta \int_0^1 \partial_\alpha^3 z(\phi) d\phi \quad (12) \]

and \( |\partial_\alpha \varpi(\gamma - \beta) - \partial_\alpha \varpi(\gamma)| \leq \|\varpi\|_{C^2(S)} |\beta|, \)

we have

\[ |L_6| \leq C\|F(z)\|_{L^\infty(S)} \|\varpi\|_{C^2(S)} \|\partial_\alpha^3 z\|_{L^2(S)}^2. \]

Due to (12) and computing \( B(\gamma, \beta) \) as in (9)

\[ |L_7| \leq C\|F(z)\|_{L^\infty(S)} \|\varpi\|_{C^2(S)} \|z\|_{C^1(S)} \|\partial_\alpha^3 z\|_{L^2(S)}^2. \]

And for \( L_8 \)

\[ L_8 = 4\pi \int \partial_\alpha^3 z(\gamma) \left( \frac{\Lambda(\partial_\alpha^3 z)}{|\partial_\alpha z(\gamma)|^2} \partial_\alpha \varpi(\gamma) \right) d\alpha \]

\[ \leq C\|F(z)\|_{L^\infty(S)} \|\varpi\|_{C^2(S)} \|\partial_\alpha^3 z\|_{L^2(S)}^2 \]

Hence,

\[ \frac{1}{\pi} K_3 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2). \]

The other singular term in \( I_4 \) is

\[ K_4 = -\frac{4}{\pi} \int \int \frac{(\partial_\alpha^3 z)^2}{|\Delta z|^2} (\Delta z \cdot \Delta z) \partial_\alpha \varpi(\gamma - \beta) d\alpha d\beta. \]
We split $K_4 = L_9 + L_{10}$

$$L_9 = - \frac{4}{\pi} \Re \int \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \cdot \left( \frac{\Delta \bar{z}(\gamma)}{|\Delta z|^2} - \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \right) \Delta \bar{z}(\gamma) \partial_a \bar{z}(\gamma - \beta) d\alpha d\beta,$$

$$L_{10} = - \frac{4}{\pi} \Re \int \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \cdot \left( \frac{\Delta \bar{z}(\gamma)}{|\Delta z|^2} - \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \right) \Delta \bar{z}(\gamma) \partial_a \bar{z}(\gamma - \beta) d\alpha d\beta.$$

Using the expression $C(\gamma, \beta)$ in [11], [12] and

$$\Delta z = \beta \int_0^1 \partial_\alpha z(\phi) ds$$

we get

$$L_9 \leq \exp C(||F(z)||^2_{L^\infty(S)} + ||z||^2_{H^4(S)}).$$

For $L_{10}$, proceeding as before

$$L_{10} \leq \exp C(||F(z)||^2_{L^\infty(S)} + ||z||^2_{H^4(S)})
- 8 \Re \int \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \cdot \left( \frac{\Delta \bar{z}(\gamma)}{|\Delta z|^2} - \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \right) \Delta \bar{z}(\gamma) \partial_a \bar{z}(\gamma - \beta) d\alpha d\beta.$$

We can conclude that

$$I_4 \leq \exp C(||F(z)||^2_{L^\infty(S)} + ||z||^2_{H^4(S)}).$$

The next term, $I_5$, is decompose in $K_5 + K_6$

$$K_5 = \frac{3}{\pi} \Re \int \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \cdot \left( \frac{\Delta \bar{z}(\gamma)}{|\Delta z|^2} - \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \right) \Delta \bar{z}(\gamma) \partial_a \bar{z}(\gamma - \beta) d\alpha d\beta,$$

$$K_6 = - \frac{6}{\pi} \Re \int \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \cdot \left( \frac{\Delta \bar{z}(\gamma)}{|\Delta z|^2} - \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \right) \Delta \bar{z}(\gamma) \partial_a \bar{z}(\gamma - \beta) d\alpha d\beta.$$

$$K_5 = \frac{3}{\pi} \Re \int \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \cdot \left( \frac{\Delta \bar{z}(\gamma)}{|\Delta z|^2} - \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \right) \Delta \bar{z}(\gamma) \partial_a \bar{z}(\gamma - \beta) d\alpha d\beta
+ \frac{3}{\pi} \Re \int \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \cdot \left( \frac{\Delta \bar{z}(\gamma)}{|\Delta z|^2} - \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \right) \Delta \bar{z}(\gamma) \partial_a \bar{z}(\gamma - \beta) d\alpha d\beta \equiv L_{11} + L_{12},$$

using that

$$D(\gamma, \beta) \equiv \frac{\Delta \bar{z}(\gamma)}{|\Delta z|^2} - \frac{\partial_a \bar{z}(\gamma)}{|\partial_a z(\gamma)|^2} \equiv \frac{\beta^2 \int_0^1 \int_0^1 \partial_a \bar{z}(\gamma)(s-1) t ds}{|\Delta z|^2} + \frac{\beta^2 \partial_a \bar{z}(\gamma) \int_0^1 \int_0^1 \partial_a \bar{z}(\gamma) (1-s) t ds}{|\Delta z|^2 |\partial_a z(\gamma)|^2}.$$
we get

\[ L_{11} = \frac{3}{\pi} 3 \Re \int \frac{\partial^2_\alpha z(\gamma)}{\partial \alpha z(\gamma)} \cdot D(\gamma, \beta) \partial^2_\alpha \varpi (\gamma - \beta) d\alpha d\beta \]

\[ \leq C(\|F(z)\|_{L^\infty(S)} \|z\|_{C^2(S)} \|\partial^4_\alpha z\|_{L^2(S)} + \|F(z)\|_{L^\infty(S)} \|z\|_{C^2(S)} \|\partial^4_\alpha \varpi\|_{L^2(S)} \|\partial^4_\alpha z\|_{L^2(S)}) \]

and

\[ L_{12} = 3 \Re \int \frac{\partial^2_\alpha z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{\partial^2_\alpha z(\gamma)}{[\partial \alpha z(\gamma)]^2} H(\partial^2_\alpha \varpi)(\gamma) d\alpha d\beta \leq C(\|F(z)\|_{L^\infty(S)} \|z\|_{C^2(S)} \|\partial^4_\alpha z\|_{L^2(S)} \|\partial^2_\alpha \varpi\|_{L^2(S)}). \]

Therefore,

\[ K_5 \leq \exp C(\|F(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}). \]

With the same procedure used in \( K_4 \) we have \( K_6 = L_{13} + L_{14} \) where

\[ L_{13} = -\frac{6}{\pi} 3 \Re \int \frac{\partial^2_\alpha z(\gamma)}{\partial \alpha z(\gamma)} \cdot (\Delta z - \frac{\partial_\alpha z(\gamma)}{[\partial \alpha z(\gamma)]^2}) (\Delta z \cdot \Delta \partial^2_\alpha z) \partial^2_\alpha \varpi (\gamma - \beta) d\alpha d\beta, \]

\[ L_{14} = -\frac{6}{\pi} 3 \Re \int \frac{\partial^2_\alpha z(\gamma)}{\partial \alpha z(\gamma)} \cdot (\Delta z \cdot \Delta \partial^2_\alpha z) \partial^2_\alpha \varpi (\gamma - \beta) d\alpha d\beta. \]

Using

\[ \Delta \partial^2_\alpha z = \beta \int_0^1 \partial^3_\alpha z(\phi) d\phi \quad (14) \]

and [13] then

\[ L_{13} \leq C(\|F(z)\|^2_{L^\infty(S)} \|z\|_{C^2(S)} \|z\|_{C^4(S)} \|z\|_{C^2(S)} \|\partial^4_\alpha z\|_{L^2(S)} \|\partial^2_\alpha \varpi\|_{L^2(S)} \]

and

\[ L_{14} \leq \exp C(\|F(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) - 6 \Re \int \frac{\partial^2_\alpha z(\gamma)}{\partial \alpha z(\gamma)} \cdot (\partial_\alpha z(\gamma) \cdot \partial^2_\alpha z(\gamma)) H(\partial^2_\alpha \varpi)(\gamma) d\alpha \]

\[ \leq \exp C(\|F(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + C(\|F(z)\|_{L^\infty(S)} \|z\|_{C^2(S)} \|\partial^4_\alpha z\|_{L^2(S)} \|\partial^2_\alpha \varpi\|_{L^2(S)} \]

Therefore,

\[ I_5 \leq \exp C(\|F(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}). \]

Regarding \( I_6 \), its most singular term is giving by

\[ K_7 = \frac{2}{\pi} 3 \Re \int \frac{\partial^2_\alpha z(\gamma)}{\partial \alpha z(\gamma)} \cdot \frac{(\partial_\alpha z(\gamma) - \partial_\alpha z(\gamma - \beta))}{|z(\gamma) - z(\gamma - \beta)|^2} \partial^3_\alpha \varpi(\gamma - \beta) d\alpha d\beta. \]
We write

\[ K_7 = \frac{2}{\pi} \int \int \frac{\partial_{\alpha} z(\gamma)}{|\Delta z|^2} \cdot \left( \frac{\Delta \partial_{\alpha} z^\perp}{|\Delta z|^2} - \frac{\partial_{\alpha} z(\gamma)^\perp}{|\partial_{\alpha} z|^2} \right) \partial_{\alpha} \varpi(\gamma - \beta) d\alpha d\beta \]

\[ + \frac{2}{\pi} \int \int \frac{\partial_{\alpha} z(\gamma)}{|\Delta z|^2} \cdot \frac{\partial_{\alpha} z(\gamma)^\perp}{|\partial_{\alpha} z|^2} H(\partial_{\alpha} \varpi(\gamma)) d\alpha \]

since

\[ E(\gamma, \beta) \equiv \frac{\Delta \partial_{\alpha} z^\perp}{|\Delta z|^2} - \frac{\partial_{\alpha} z(\gamma)^\perp}{|\partial_{\alpha} z|^2} = \frac{\beta^2 \int_0^1 \int_0^1 \partial_{\alpha} z(\eta)^\perp (s - 1) dt ds}{\Delta z} \]

\[ + \frac{\beta^2 \partial_{\alpha} z(\gamma)^\perp}{\Delta z^2} \int_0^1 \int_0^1 \partial_{\alpha} z(\psi)(1 - s) dt ds \cdot \int_0^1 \partial_{\alpha} z(\gamma) + \partial_{\alpha} z(\phi) d\alpha \]

then

\[ I_6 \leq \exp C(\|\varpi(z)\|_{L^\infty(S)} + \|z\|^2_{H^4(S)}). \]

### 2.2.2 Searching for the Rayleigh-Taylor condition in \( I_7 \)

Let us recall the formula for the Rayleigh-Taylor condition

\[ \sigma(\alpha, t) = \frac{\eta^2}{\kappa} BR(z, \varpi)(\alpha, t) \cdot \partial_\perp z(\alpha, t) + g\rho^2 \partial_\alpha z_1(\alpha, t). \]

We write \( I_7 \) in the form \( I_7 = K_8 + K_9 \) where

\[ K_8 = \frac{1}{2\pi} \int \int \frac{\partial_{\alpha} z(\gamma)}{|\Delta z|^2} \cdot \left( (z(\gamma) - z(\gamma - \beta))^\perp \right) \frac{\partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} \partial_{\alpha} \varpi(\gamma - \beta) d\alpha d\beta, \]

\[ K_9 = \frac{1}{2\pi} \int \int \frac{\partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} \partial_{\alpha} \varpi(\gamma - \beta) d\alpha d\beta. \]

After an integration by parts we obtain:

\[ K_8 = -\frac{1}{2\pi} \int \int \frac{\partial_{\alpha} z(\gamma)}{|\Delta z|^2} \cdot \left( (\Delta z)^\perp \right) \frac{\partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} \partial_{\beta} \partial_{\alpha} \varpi(\gamma - \beta) d\alpha d\beta \]

\[ = \frac{1}{2\pi} \int \int \frac{\partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} \partial_{\beta} \left( (\Delta z)^\perp \right) \frac{\partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} \partial_{\alpha} \varpi(\gamma - \beta) d\alpha d\beta. \]

We decompose

\[ \partial_{\beta} \left( (\Delta z)^\perp \right) \frac{\partial_{\alpha} z(\gamma)}{|\partial_{\alpha} z(\gamma)|^2} \]

\[ = \frac{(\Delta z)^\perp}{|\Delta z|^2} + \frac{1}{|\partial_{\alpha} z(\gamma)|^2} - 2 \frac{(\Delta z)^\perp - \Delta z \cdot \Delta z}{|\Delta z|^4} + \frac{2}{|\Delta z|^4} \]

\[ = \frac{(\Delta z)^\perp}{|\Delta z|^2} + \frac{1}{|\partial_{\alpha} z(\gamma)|^2} - 2 \frac{(\Delta z)^\perp - \Delta z \cdot \Delta z}{|\Delta z|^4} + \frac{2}{|\Delta z|^4} \]

\[ \equiv F_1(\gamma, \beta) + F_2(\gamma, \beta) + F_3(\gamma, \beta) + F_4(\gamma, \beta) + F_5(\gamma, \beta) + F_6(\gamma, \beta) \]  

(15)
Then, we have

\[ K_8 = \frac{1}{2\pi} \Re \int \int \partial^3_\alpha z(\gamma) \cdot F_1(\gamma, \beta) \partial^3_\alpha \varpi(\gamma - \beta) d\alpha d\beta \]
\[ + \frac{1}{2\pi} \Re \int \int \partial^3_\alpha z(\gamma) \cdot F_2(\gamma, \beta) \partial^3_\alpha \varpi(\gamma - \beta) d\alpha d\beta \]
\[ + \frac{1}{2\pi} \Re \int \int \partial^3_\alpha z(\gamma) \cdot F_3(\gamma, \beta) \partial^3_\alpha \varpi(\gamma - \beta) d\alpha d\beta \]
\[ + \frac{1}{2\pi} \Re \int \int \partial^3_\alpha z(\gamma) \cdot F_4(\gamma, \beta) \partial^3_\alpha \varpi(\gamma - \beta) d\alpha d\beta \]
\[ + \frac{1}{2\pi} \Re \int \int \partial^3_\alpha z(\gamma) \cdot F_5(\gamma, \beta) \partial^3_\alpha \varpi(\gamma - \beta) d\alpha d\beta \]
\[ \equiv P_1 + P_2 + P_3 + P_4 + P_5. \]

For \( P_1, P_2, P_3, P_4 \) and \( P_5 \) we can estimates with the same approach as before, and we easily get

\[ P_1 + P_2 + P_3 + P_4 + P_5 \leq \exp C(\| z \|_{L_\infty(S)}^2 + \| z \|_{H^4(S)}^2). \]

Since,

\[ -\frac{1}{2} \int F_6(\gamma, \beta) = \partial^4_\alpha z(\gamma)^{\beta^4 |\partial_\alpha z(\gamma)|^4 - |\Delta z|^4} \]
\[ \frac{\Delta z}{|\partial_\alpha z(\gamma)|^2} \]
\[ = \partial^4_\alpha z(\gamma)^{\beta^4 |\partial_\alpha z(\gamma)|^4 - |\Delta z|^4} \]
\[ \frac{\Delta z}{|\partial_\alpha z(\gamma)|^2} \]
\[ + \partial^4_\alpha z(\gamma)^{\beta^4 |\partial_\alpha z(\gamma)|^4 - |\Delta z|^4} \]
\[ \frac{\Delta z}{|\partial_\alpha z(\gamma)|^2} \]
\[ \equiv U_1(\gamma, \beta) + U_2(\gamma, \beta) + U_3(\gamma, \beta) + U_4(\gamma, \beta) \]

we get,

\[ P_5 = -\frac{1}{\pi} \Re \int \int \partial^3_\alpha z(\gamma) \cdot U_1(\gamma, \beta) \partial^3_\alpha \varpi(\gamma - \beta) d\alpha d\beta \]
\[ -\frac{1}{\pi} \Re \int \int \partial^3_\alpha z(\gamma) \cdot U_2(\gamma, \beta) \partial^3_\alpha \varpi(\gamma - \beta) d\alpha d\beta \]
\[ = Q_1 + Q_2 + Q_3 + Q_4 \]

where

\[ Q_1 \leq C \| \mathcal{F}(z) \|_{L_\infty(S)}^2 \| z \|_{L_\infty(S)} \| z \|_{L^2(S)} \| \partial^3_\alpha z \|_{L^2(S)} \| \partial^3_\alpha \varpi \|_{L^2(S)}, \]
\[ Q_2 \leq C \| \mathcal{F}(z) \|_{L_\infty(S)}^2 \| z \|_{L^2(S)} \| z \|_{L^2(S)} \| \partial^3_\alpha z \|_{L^2(S)} \| \partial^3_\alpha \varpi \|_{L^2(S)}, \]
\[ Q_3 \leq C \| \mathcal{F}(z) \|_{L_\infty(S)}^2 \| z \|_{L^2(S)} \| z \|_{L^2(S)} \| \partial^3_\alpha z \|_{L^2(S)} \| \partial^3_\alpha \varpi \|_{L^2(S)} \]
\[ Q_4 \leq C \]
and if we split,

$$Q_4 = -\frac{1}{\pi} \Re \int \frac{\partial_{a\bar{a}} z(\gamma) \cdot \partial_{a\bar{a}} z(\gamma) \beta^3 \partial_{a\bar{a}} w(\gamma) \partial_{a\bar{a}} \varpi(\gamma - \beta)}{|\Delta z|^4} - \frac{1}{|\partial_{a\bar{a}} z(\gamma)|^4 \beta^4} d\alpha d\beta$$

$$- \frac{1}{\pi} \Re \int \frac{\partial_{a\bar{a}} z(\gamma) \cdot \partial_{a\bar{a}} z(\gamma) \beta^3 \partial_{a\bar{a}} w(\gamma) \partial_{a\bar{a}} \varpi(\gamma - \beta)}{|\Delta z|^4} H(\partial_{a\bar{a}} \varpi(\gamma)) d\alpha.$$  It is clear that

$$Q_4 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^s(S)}^2).$$

Thus,  

$$P_6 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^s(S)}^2).$$

Therefore,  

$$K_9 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^s(S)}^2).$$

We consider now the $K_9$ term which can be written as follows  

$$K_9 = \frac{1}{2} \Re \int \frac{\partial_{a\bar{a}} z(\gamma) \cdot \partial_{a\bar{a}} z(\gamma) \beta^3 \partial_{a\bar{a}} \varpi(\gamma) \partial_{a\bar{a}} \varpi(\gamma)}{|\Delta z|^4} H(\partial_{a\bar{a}} \varpi(\gamma)) d\alpha$$

$$= \frac{1}{2} \Re \int \frac{\Lambda(\partial_{a\bar{a}} z \cdot \partial_{a\bar{a}} z^\perp)(\gamma)}{A(t)} \partial_{a\bar{a}} z(\gamma) \partial_{a\bar{a}} \varpi(\gamma) d\alpha$$

using the formula

$$\varpi(\alpha) = -2B R(z, \varpi)(\alpha, t) \cdot \partial_{a\bar{a}} z(\alpha, t) - 2\kappa g \frac{\rho^2}{\mu^2} \partial_{a\bar{a}} z_2(\alpha, t) = -T(\varpi)(\alpha) - 2\kappa g \frac{\rho^2}{\mu^2} \partial_{a\bar{a}} z_2(\alpha)$$

we separate $K_9$ as a sum of two parts, $P_7$ and $P_8$, where

$$P_7 = -g \kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial_{a\bar{a}} z \cdot \partial_{a\bar{a}} z^\perp)(\gamma)}{A(t)} \partial_{a\bar{a}} z_2(\gamma) d\alpha,$$

$$P_8 = -\frac{1}{2} \Re \int \frac{\Lambda(\partial_{a\bar{a}} z \cdot \partial_{a\bar{a}} z^\perp)(\gamma)}{A(t)} \partial_{a\bar{a}} T(\varpi)(\gamma) d\alpha.$$  

For $P_7$ we decompose further $P_7 = Q_5 + Q_6$ where

$$Q_5 = g \kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial_{a\bar{a}} z \cdot \partial_{a\bar{a}} z^\perp)(\gamma)}{A(t)} \partial_{a\bar{a}} z_2(\gamma) d\alpha,$$

$$Q_6 = -g \kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial_{a\bar{a}} z \cdot \partial_{a\bar{a}} z^\perp)(\gamma)}{A(t)} \partial_{a\bar{a}} z_2(\gamma) d\alpha.$$  

Then $Q_5$ is written as $Q_5 = R_1 + R_2$ with

$$R_1 = g \kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial_{a\bar{a}} z \cdot \partial_{a\bar{a}} z^\perp)(\gamma) \cdot \partial_{a\bar{a}} z_2(\gamma) \Lambda(\partial_{a\bar{a}} z_1)(\gamma)}{A(t)} \partial_{a\bar{a}} z_2(\gamma) d\alpha,$$

$$R_2 = g \kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\partial_{a\bar{a}} z_2(\gamma) \Lambda(\partial_{a\bar{a}} z_1)(\gamma)}{A(t)} \partial_{a\bar{a}} z_2(\gamma) d\alpha.$$
Using the commutator estimate \([10]\), we get

\[
R_1 \leq C \|\mathcal{F}(z)\|_{L^\infty(S)} \|z\|_{C^2(S)} \|\partial_\alpha^4 z\|_{L^2(S)} \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).
\]

The identity

\[
\partial_\alpha z_2(\gamma) \partial_\alpha^4 z_2(\gamma) = \partial_\alpha z_2(\gamma) \cdot \partial_\alpha^4 z_2(\gamma) - \partial_\alpha z_1(\gamma) \partial_\alpha^4 z_1(\gamma)
\]

let us write \(R_2\) as the sum of \(S_1\) and \(S_2\) where

\[
S_1 = g\kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial_\alpha^4 z_1(\gamma))}{A(t)} \partial_\alpha z(\gamma) \cdot \partial_\alpha^4 z(\gamma) d\alpha,
\]

\[
S_2 = -g\kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial_\alpha^4 z_1(\gamma))}{A(t)} \partial_\alpha z_1(\gamma) \partial_\alpha^4 z_1(\gamma) d\alpha.
\]

For \(S_1\) we use an integration by parts and

\[
\partial_\alpha z(\gamma) \cdot \partial_\alpha^4 z(\gamma) = -3 \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^3 z(\gamma)
\]

we get,

\[
S_1 = -3g\kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial_\alpha^4 z_1(\gamma))}{A(t)} \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^3 z(\gamma) d\alpha + 3g\kappa \frac{\rho^2}{\mu^2} \Re \int \frac{H(\partial_\alpha^4 z_1(\gamma))}{A(t)} \partial_\alpha^2 z_1(\gamma) \cdot \partial_\alpha^3 z_1(\gamma) d\alpha \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).
\]

And writing \(Q_6\) in the form

\[
Q_6 = -g\kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial_\alpha^4 z_2 \cdot \partial_\alpha z_1(\gamma)) - \partial_\alpha z_1(\gamma) \Lambda(\partial_\alpha^4 z_2(\gamma))}{A(t)} \partial_\alpha^4 z_2(\gamma) d\alpha
\]

\[
- g\kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\partial_\alpha z_1(\gamma)}{A(t)} \Lambda(\partial_\alpha^4 z_2(\gamma)) \partial_\alpha^4 z_2(\gamma) d\alpha \equiv R_3 + R_4,
\]

by the commutator estimate, we have

\[
R_3 \leq C \|\mathcal{F}(z)\|_{L^\infty(S)} \|z\|_{C^2(S)} \|\partial_\alpha^4 z\|_{L^2(S)} \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).
\]

Since,

\[
S_2 + R_4 = -g\kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\partial_\alpha z_1(\gamma)}{A(t)} \partial_\alpha^4 z(\gamma) \cdot \Lambda(\partial_\alpha^4 z)(\gamma) d\alpha
\]

we obtain finally

\[
P_7 \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) - g\kappa \frac{\rho^2}{\mu^2} \Re \int \frac{\partial_\alpha z_1(\gamma)}{A(t)} \partial_\alpha^4 z(\gamma) \cdot \Lambda(\partial_\alpha^4 z)(\gamma) d\alpha.
\]
In the estimate above we can observe how part of \( \sigma(\gamma) \) appears in the non-integrable terms.

Let us return to \( P_8 = Q_7 + Q_8 + Q_9 + Q_{10} \) where

\[
Q_7 = -9 \Re \int \frac{\Lambda(\partial_a^2 z \cdot \partial_a z^\perp)(\gamma)}{A(t)} \partial_a^3 BR(z, \varpi)(\gamma) \cdot \partial_a z(\gamma) d\alpha,
\]

\[
Q_8 = -3 \Re \int \frac{\Lambda(\partial_a^2 z \cdot \partial_a z^\perp)(\gamma)}{A(t)} \partial_a^2 BR(z, \varpi)(\gamma) \cdot \partial_a^2 z(\gamma) d\alpha,
\]

\[
Q_9 = -3 \Re \int \frac{\Lambda(\partial_a^2 z \cdot \partial_a z^\perp)(\gamma)}{A(t)} \partial_a BR(z, \varpi)(\gamma) \cdot \partial_a^3 z(\gamma) d\alpha,
\]

\[
Q_{10} = -9 \Re \int \frac{\Lambda(\partial_a^2 z \cdot \partial_a z^\perp)(\gamma)}{A(t)} BR(z, \varpi)(\gamma) \cdot \partial_a^4 z(\gamma) d\alpha.
\]

We will control first the terms \( Q_8, Q_7 \) and \( Q_9 \) and then we will show how the rest of \( \sigma(\gamma) \) appears in \( Q_{10} \).

Using \( \Lambda = H \partial_a \) and integrating by parts, we obtain

\[
Q_8 = 3 \Re \int \frac{H(\partial_a^2 z \cdot \partial_a z^\perp)(\gamma)}{A(t)} \partial_a^3 BR(z, \varpi)(\gamma) \cdot \partial_a^2 z(\gamma) d\alpha
\]

\[
+ 3 \Re \int \frac{H(\partial_a^2 z \cdot \partial_a z^\perp)(\gamma)}{A(t)} \partial_a^2 BR(z, \varpi)(\gamma) \cdot \partial_a^3 z(\gamma) d\alpha
\]

\[
\equiv R_5 + R_6.
\]

With (7)

\[
R_5 \leq C \| \mathcal{F}(z) \|_{L^\infty(S)} \| z \|_{C^2(S)} \| \partial_a^3 BR \|_{L^2(S)} \| \partial_a^2 z \cdot \partial_a z \|_{L^2(S)}
\]

\[
\leq \exp C(\| \mathcal{F}(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2)
\]

and

\[
R_6 \leq C \| \mathcal{F}(z) \|_{L^\infty(S)} \| z \|_{C^3(S)} \| \partial_a^2 BR \|_{L^2(S)} \| \partial_a^4 z \cdot \partial_a z \|_{L^2(S)}
\]

\[
\leq \exp C(\| \mathcal{F}(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2).
\]

With \( Q_7 \) we also integrate by parts to obtain \( Q_7 = R_7 + R_8 \) where

\[
R_7 = \Re \int \frac{H(\partial_a^2 z \cdot \partial_a z^\perp)(\gamma)}{A(t)} \partial_a^3 BR(z, \varpi)(\gamma) \cdot \partial_a z(\gamma) d\alpha,
\]

\[
R_8 = \Re \int \frac{H(\partial_a^2 z \cdot \partial_a z^\perp)(\gamma)}{A(t)} \partial_a^2 BR(z, \varpi)(\gamma) \cdot \partial_a^2 z(\gamma) d\alpha.
\]

Easily we have

\[
R_8 \leq C \| \mathcal{F}(z) \|_{L^\infty(S)} \| z \|_{C^2(S)} \| \partial_a^4 z \cdot \partial_a z \|_{L^2(S)} \| \partial_a^3 BR \|_{L^2(S)} \leq \exp C(\| \mathcal{F}(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2).
\]

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In $R_7$ the application of Leibniz’s rule to $\partial_\alpha^3 BR(z, \varpi)$ produces many terms which can be estimated with the same tools used before. For the most singular terms we have the expressions:

$$S_3 = \Re \int \frac{H(\partial_\alpha^3 z \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \partial_\alpha BR(z, \partial_\alpha^3 \varpi)(\gamma) \cdot \partial_\alpha z(\gamma) d\alpha,$$

$$S_4 = \Re \int \frac{H(\partial_\alpha^3 z \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \int \frac{\Delta \partial_\alpha^4 z}{|\Delta z|^2} \cdot \partial_\alpha z(\gamma) \varpi(\gamma - \beta) d\beta d\alpha,$$

$$S_5 = -2\Re \int \frac{H(\partial_\alpha^3 z \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \int \frac{\Delta z^\perp \cdot \partial_\alpha z(\gamma)}{|\Delta z|^4} (\Delta z \cdot \Delta \partial_\alpha^4 z) \varpi(\gamma - \beta) d\alpha d\beta.$$

Let us consider

$$\partial_\alpha BR(z, \varpi)(\gamma) \cdot \partial_\alpha z(\gamma) = \partial_\alpha (BR(z, \varpi)(\gamma) \cdot \partial_\alpha z(\gamma)) - BR(z, \varpi)(\gamma) \cdot \partial_\alpha^2 z(\gamma) = \frac{1}{2} \partial_\alpha T(\varpi)(\gamma) - BR(z, \varpi) \cdot \partial_\alpha^2 z(\gamma)$$

which yields

$$S_3 \leq C \|F(z)\|_{L^\infty(S)} \|\partial_\alpha^3 z\|_{L^2(S)} \|z\|_{C^1(S)} (\|T(\partial_\alpha^2 z)\|_{H^1(S)} + \|BR(z, \partial_\alpha^3 \varpi)\|_{L^2(S)} \|z\|_{C^2(S)})$$

$$\leq \exp(C \|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2)$$

because $\|T\|_{L^2 \to H^1} \leq \|F(z)\|_{L^\infty} \|z\|_{C^2}$ for more details see Lemma 3.1 in [7].

Next we write $S_4 = T_1 + T_2,$

$$T_1 = \Re \int \frac{H(\partial_\alpha^3 z \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \Delta \partial_\alpha^4 z \cdot \partial_\alpha z(\gamma) \varpi(\gamma - \beta) \left(\frac{1}{|\Delta z|^2} - \frac{1}{|\partial_\alpha z(\gamma)|^2 \beta^2}\right) d\beta d\alpha,$$

$$T_2 = \Re \int \frac{H(\partial_\alpha^3 z \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \frac{\Delta \partial_\alpha^4 z \cdot \partial_\alpha z(\gamma) \varpi(\gamma - \beta)}{A(t) \beta^2} d\beta d\alpha.$$

Using $B_2(\gamma, \beta) = B_3(\gamma, \beta) + B_4(\gamma, \beta),$ we split $T_1 = U_1 + U_2 + U_3,$

$$U_1 = \Re \int \frac{H(\partial_\alpha^3 z \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \Delta \partial_\alpha^4 z \cdot \partial_\alpha z(\gamma) \varpi(\gamma - \beta) B_1(\gamma, \beta) d\beta d\alpha,$$

$$U_2 = \Re \int \frac{H(\partial_\alpha^3 z \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \Delta \partial_\alpha^4 z \cdot \partial_\alpha z(\gamma) \varpi(\gamma - \beta) B_3(\gamma, \beta) d\beta d\alpha,$$

$$U_3 = \Re \int \frac{H(\partial_\alpha^3 z \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \Delta \partial_\alpha^4 z \cdot \partial_\alpha z(\gamma) \varpi(\gamma - \beta) B_4(\gamma, \beta) d\beta d\alpha.$$
being
\[
B_1(\gamma, \beta) = \beta \int_0^1 \int_0^1 \frac{\partial_2^2 z(\psi) - \partial_2^2 z(\gamma)}{|\psi - \gamma|^2} \beta^\delta (1 + s + t - st)^3 (1 - s) dt ds \int_0^1 [\partial_\alpha z(\gamma) + \partial_\alpha z(\phi)] ds
\]

\[
B_2(\gamma, \beta) = \beta^2 \partial_\alpha^2 z(\gamma) \int_0^1 \int_0^1 \partial_\alpha^2 z(\eta)(s - 1) dt ds
\]

\[
B_3(\gamma, \beta) = \frac{\beta^2 \partial_\alpha^2 z(\gamma) \int_0^1 \int_0^1 \partial_\alpha^2 z(\eta)(s - 1) dt ds}{|z(\gamma) - z(\gamma - \beta)|^2 |\partial_\alpha z(\gamma)|^2}
\]

therefore
\[
U_1 \leq C \|F(z)\|_{L^\infty(S)} \|z\|_{C^1(S)} \|z\|_{C^2(S)} \|\varpi\|_{L^\infty(S)} \|\partial_\alpha^4 z\|_{L^2(S)},
\]

\[
U_2 \leq C \|F(z)\|_{L^\infty(S)} \|z\|_{C^1(S)} \|z\|_{C^2(S)} \|\varpi\|_{L^\infty(S)} \|\partial_\alpha^4 z\|_{L^2(S)}
\]

and
\[
U_3 = 2\beta \int \int \frac{H(\partial_\alpha^4 z \cdot \partial_\alpha z^\perp(\gamma))}{A^2(t)} \exp\int \int \frac{H(\partial_\alpha^4 z \cdot \partial_\alpha z^\perp(\gamma))}{A^2(t)} \varpi(\gamma - \beta) \beta^2 \partial_\alpha^2 z(\gamma) \partial_\alpha z(\gamma) B(\gamma, \beta) d\beta d\alpha
\]

Recall that
\[
B(\gamma, \beta) \equiv \beta \int_0^1 \int_0^1 \frac{\partial_\alpha^2 z(\psi)(1 - s) dt ds \int_0^1 \partial_\alpha z(\gamma) + \partial_\alpha z(\phi) ds}{|z(\gamma) - z(\gamma - \beta)|^2 |\partial_\alpha z(\gamma)|^2}
\]

then the term $V_1$ is controlled.

We split $V_2 = W_1 + W_2$ where

\[
W_1 = 2\beta \int \int \frac{H(\partial_\alpha^4 z \cdot \partial_\alpha z^\perp(\gamma))}{A^2(t)} \partial_\alpha^4 z(\gamma) \cdot \partial_\alpha z(\gamma) \varpi(\gamma - \beta) \beta^2 \partial_\alpha^2 z(\gamma) \partial_\alpha z(\gamma) d\beta d\alpha
\]

\[
W_2 = -2\beta \int \int \frac{H(\partial_\alpha^4 z \cdot \partial_\alpha z^\perp(\gamma))}{A^2(t)} \partial_\alpha^4 z(\gamma - \beta) \cdot \partial_\alpha z(\gamma) \varpi(\gamma - \beta) \beta^2 \partial_\alpha^2 z(\gamma) \partial_\alpha z(\gamma) d\beta d\alpha.
\]

Easily

\[
W_1 \leq C \|F(z)\|_{L^\infty(S)} \|z\|_{C^1(S)} \|z\|_{C^2(S)} \|H\|_{L^\infty(S)} \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2)
\]

and

\[
W_2 \leq C \|F(z)\|_{L^\infty(S)} \|z\|_{C^1(S)} \|z\|_{C^2(S)} \|\varpi\|_{C^1(S)}
\]

\[
-2\beta \int \int \frac{H(\partial_\alpha^4 z \cdot \partial_\alpha z^\perp(\gamma))}{A^2(t)} H(\partial_\alpha^4 z(\gamma) \cdot \partial_\alpha z(\gamma) \varpi(\gamma - \beta) \beta^2 \partial_\alpha^2 z(\gamma) \partial_\alpha z(\gamma) d\beta d\alpha
\]

\[
\leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2)
\]
Hence,
\[ T_1 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2). \]

We decompose \( T_2 = U_4 + U_5 \),
\[
U_4 = \Re \int \int \frac{H(\partial_{\alpha}^4 z \cdot \partial_{\alpha} z^\perp)(\gamma)}{A^2(t)} \partial_{\alpha}^4 z \cdot \partial_{\alpha} z(\gamma) \frac{\omega(\gamma - \beta) - \omega(\gamma)}{\beta^2} d\beta d\alpha,
\]
\[
U_5 = \Re \int \int \frac{H(\partial_{\alpha}^4 z \cdot \partial_{\alpha} z^\perp)(\gamma)}{A^2(t)} \partial_{\alpha}^4 z \cdot \partial_{\alpha} z(\gamma) \frac{\omega(\gamma)}{\beta^2} d\beta d\alpha.
\]

Then we split
\[
U_4 = \Re \int \int \frac{H(\partial_{\alpha}^4 z \cdot \partial_{\alpha} z^\perp)(\gamma)}{A^2(t)} \partial_{\alpha}^4 z(\gamma) \cdot \partial_{\alpha} z(\gamma) \frac{\omega(\gamma - \beta) - \omega(\gamma)}{\beta^2} d\beta d\alpha
- \Re \int \int \frac{H(\partial_{\alpha}^4 z \cdot \partial_{\alpha} z^\perp)(\gamma)}{A^2(t)} \partial_{\alpha}^4 z(\gamma - \beta) \cdot \partial_{\alpha} z(\gamma) \frac{\omega(\gamma - \beta) - \omega(\gamma)}{\beta^2} d\beta d\alpha
\equiv V_3 + V_4
\]

where
\[ V_3 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) \]
and,
\[ V_4 \leq C(\|F(z)\|_{L^\infty(S)}^2 \|\partial_{\alpha}^4 z\|_{L^2(S)}^2 \|z\|_{C^4(S)} \|\omega\|_{C^2(S)}
- \Re \int \int \frac{H(\partial_{\alpha}^4 z \cdot \partial_{\alpha} z^\perp)(\gamma)}{A^2(t)} H(\partial_{\alpha}^4 z)(\gamma) \cdot \partial_{\alpha} z(\gamma) \partial_{\alpha} \omega(\gamma) d\alpha
\leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).
\]

Then,
\[ U_4 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2). \]

For \( U_5 \) integrating by parts for \( \Lambda \) we have,
\[
U_5 = \Re \int \Lambda \left( \frac{H(\partial_{\alpha}^4 z \cdot \partial_{\alpha} z^\perp)}{A^2(t)} \partial_{\alpha} z \omega \right) \cdot \partial_{\alpha}^4 z(\gamma) d\alpha
= \Re \int \left( \Lambda \left( \frac{H(\partial_{\alpha}^4 z \cdot \partial_{\alpha} z^\perp)}{A^2(t)} \partial_{\alpha} z \omega \right)(\gamma) - \partial_{\alpha} z(\gamma) \Lambda \left( \frac{H(\partial_{\alpha}^4 z \cdot \partial_{\alpha} z^\perp)}{A^2(t)} \right)(\gamma) \right) \cdot \partial_{\alpha}^4 z(\gamma) d\alpha
+ \Re \int \partial_{\alpha} z(\gamma) \Lambda \left( \frac{H(\partial_{\alpha}^4 z \cdot \partial_{\alpha} z^\perp)}{A^2(t)} \right)(\gamma) \cdot \partial_{\alpha}^4 z(\gamma) d\alpha
\leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) - \Re \int \partial_{\alpha} z(\gamma) \frac{\partial_{\alpha} (\partial_{\alpha}^4 z \cdot \partial_{\alpha} z^\perp)(\gamma)}{A^2(t)} \cdot \partial_{\alpha}^4 z(\gamma) d\alpha
\]
now, using \( \partial_\alpha z(\alpha) \cdot \partial_\alpha^4 z(\alpha) = -3\partial_\alpha^2 z(\alpha) \cdot \partial_\alpha^3 z(\alpha) \)
\[
- 2\Re \int \frac{\partial_\alpha (\partial_\alpha^4 z \cdot \partial_\alpha z^2)(\gamma)}{A^2(t)} \partial_\alpha z(\gamma) \cdot \partial_\alpha^4 z(\gamma) d\alpha = 3\Re \int \frac{\partial_\alpha (\partial_\alpha^4 z \cdot \partial_\alpha z^2)(\gamma)}{A^2(t)} \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^3 z(\gamma) d\alpha
\]
\[
= -3\Re \int \frac{\partial_\alpha^2 z \cdot \partial_\alpha z^2(\gamma)}{A^2(t)} \partial_\alpha^3 z(\gamma) \cdot \partial_\alpha^3 z(\gamma) d\alpha - 3\Re \int \frac{\partial_\alpha^4 z \cdot \partial_\alpha z^2(\gamma)}{A^2(t)} \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^3 z(\gamma) d\alpha
\]
\[
\leq \exp(C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|^2_{H^4(S)}).
\]
Therefore,
\[
T_2 \leq \exp(C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|^2_{H^4(S)}).
\]
Thus, \( S_4 \) satisfies identical estimates than \( T_2 \).

To conclude with \( R_7 \), let us estimate \( S_5 \).

We split \( S_5 = T_3 + T_4 \)

\[
T_3 = -2\Re \int \frac{H(\partial_\alpha^4 z \cdot \partial_\alpha z^2)(\gamma)}{A(t)} C(\gamma, \beta) \cdot \partial_\alpha z(\gamma)(\Delta z \cdot \Delta \partial_\alpha^4 z) \varpi(\gamma - \beta) d\alpha d\beta,
\]
\[
T_4 = -2\Re \int \frac{H(\partial_\alpha^4 z \cdot \partial_\alpha z^2)(\gamma)}{A(t)} \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha z(\gamma)(\Delta z \cdot \Delta \partial_\alpha^4 z) \varpi(\gamma - \beta) d\alpha d\beta.
\]

Since \( \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha z(\gamma) = 0 \), for \( (11) \) we have \( C(\gamma, \beta) \cdot \partial_\alpha z(\gamma) = C_1(\gamma, \beta) \cdot \partial_\alpha z(\gamma) \) and \( T_2 = 0 \).

Recall that,
\[
C_1(\gamma, \beta) = \frac{\beta^2 \int_0^1 \int_0^1 \partial_\alpha^2 z^2(\eta)(s - 1) d\eta d\beta}{|\Delta z|^4}
\]

with \( \eta = \gamma - t \beta + st \beta \).

Using \( (13) \) and
\[
C_1(\gamma, \beta) \cdot \partial_\alpha z(\gamma) = \frac{\beta^2 \partial_\alpha^2 z^2(\gamma) \cdot \partial_\alpha z(\gamma)}{|\Delta z|^4} = \frac{\beta^2 \int_0^1 \int_0^1 [\partial_\alpha^2 z^2(\eta) - \partial_\alpha^2 z(\gamma)](s - 1) d\eta d\beta \cdot \partial_\alpha z(\gamma)}{|\Delta z|^4}
\]

we get
\[
S_5 \leq \exp(C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|^2_{H^4(S)})
\]
\[
- 2\Re \int \frac{H(\partial_\alpha^4 z \cdot \partial_\alpha z^2)(\gamma)}{A^3(t)} \partial_\alpha^2 z^2(\gamma) \cdot \partial_\alpha z(\gamma)(\Delta \partial_\alpha^4 z) \varpi(\gamma - \beta) d\alpha d\beta
\]
\[
\leq \exp(C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|^2_{H^4(S)})
\]
\[
- 2\pi \Re \int \frac{H(\partial_\alpha^4 z \cdot \partial_\alpha z^2)(\gamma)}{A^3(t)} \partial_\alpha^2 z^2(\gamma) \cdot \partial_\alpha z(\gamma) \partial_\alpha z(\gamma) \partial_\alpha^4 z(\gamma) H(\varpi)(\gamma) d\alpha
\]
\[
- 4\pi \Re \int \frac{H(\partial_\alpha^4 z \cdot \partial_\alpha z^2)(\gamma)}{A^3(t)} \partial_\alpha^2 z^2(\gamma) \cdot \partial_\alpha z(\gamma) \partial_\alpha z(\gamma) \partial_\alpha^4 z(\gamma) \varpi(\gamma) d\alpha.
\]
Therefore we can control $S_5$.

Let us decompose

$$Q_0 = 3\Re \int \frac{H(\partial^2_{\alpha} z \cdot \partial_{\alpha} z^\perp)(\gamma)}{A(t)} \partial^2_{\alpha} BR(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z(\gamma) d\alpha$$

$$+ 3\Re \int \frac{H(\partial^2_{\alpha} z \cdot \partial_{\alpha} z^\perp)(\gamma)}{A(t)} \partial_{\alpha} BR(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z(\gamma) d\alpha \equiv R_9 + R_{10},$$

using (7)

$$R_9 \leq C \| F(z) \|_{L^\infty(S)} \| z \|_{C^4(S)} \| BR \|_{H^2(S)} \leq \exp C(\| F(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2),$$

$$R_{10} \leq C \| F(z) \|_{L^\infty(S)} \| \partial_{\alpha} BR \|_{L^\infty(S)} \| z \|_{C^4(S)} \| \partial^4_{\alpha} z \|_{H^2(S)}^2 \leq \exp C(\| F(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2).$$

Then $Q_9 \leq \exp C(\| F(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2).$

Finally we have to find the rest of $\sigma(\gamma)$ in $Q_{10}$. To do that let us split

$$Q_{10} = R_{11} + R_{12} + R_{13} + R_{14}$$

where

$$R_{11} = \Re \int \frac{\Lambda(\partial^4_{\alpha} z \partial_{\alpha} z_1)(\gamma)}{A(t)} \Lambda(\partial^4_{\alpha} z_1)(\gamma) \cdot BR_1(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z_1(\gamma) d\alpha,$$

$$R_{12} = \Re \int \frac{\Lambda(\partial^4_{\alpha} z \partial_{\alpha} z_2)(\gamma)}{A(t)} \Lambda(\partial^4_{\alpha} z_2)(\gamma) \cdot BR_2(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z_2(\gamma) d\alpha,$$

$$R_{13} = -\Re \int \frac{\Lambda(\partial^4_{\alpha} z \partial_{\alpha} z_1)(\gamma)}{A(t)} \Lambda(\partial^4_{\alpha} z_1)(\gamma) \cdot BR_1(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z_1(\gamma) d\alpha,$$

$$R_{14} = -\Re \int \frac{\Lambda(\partial^4_{\alpha} z \partial_{\alpha} z_2)(\gamma)}{A(t)} \Lambda(\partial^4_{\alpha} z_2)(\gamma) \cdot BR_2(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z_2(\gamma) d\alpha.$$

Then

$$R_{11} = \Re \int \frac{\Lambda(\partial^4_{\alpha} z \partial_{\alpha} z_2)(\gamma) - \partial_{\alpha} z_2(z, \varpi)(\gamma) \Lambda(\partial^4_{\alpha} z_1)(\gamma) \cdot BR_1(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z_1(\gamma) d\alpha}{A(t)}$$

$$+ \Re \int \frac{\partial_{\alpha} z_2(z, \varpi)(\gamma) \Lambda(\partial^4_{\alpha} z_1)(\gamma)}{A(t)} \cdot BR_1(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z_1(\gamma) d\alpha,$$

and the commutator estimates yields

$$R_{11} \leq \exp C(\| F(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2) + \Re \int \frac{\partial_{\alpha} z_2(z, \varpi)(\gamma) \Lambda(\partial^4_{\alpha} z_1)(\gamma)}{A(t)} \cdot BR_1(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z_1(\gamma) d\alpha.$$

In a similar way we have

$$R_{12} \leq \exp C(\| F(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2) + \Re \int \frac{\partial_{\alpha} z_2(z, \varpi)(\gamma) \Lambda(\partial^4_{\alpha} z_2)(\gamma)}{A(t)} \cdot BR_2(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z_2(\gamma) d\alpha,$$

$$R_{13} \leq \exp C(\| F(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2) - \Re \int \frac{\partial_{\alpha} z_1(z, \varpi)(\gamma) \Lambda(\partial^4_{\alpha} z_1)(\gamma)}{A(t)} \cdot BR_1(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z_1(\gamma) d\alpha,$$

$$R_{14} \leq \exp C(\| F(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2) - \Re \int \frac{\partial_{\alpha} z_1(z, \varpi)(\gamma) \Lambda(\partial^4_{\alpha} z_2)(\gamma)}{A(t)} \cdot BR_2(z, \varpi)(\gamma) \cdot \partial^4_{\alpha} z_2(\gamma) d\alpha.$$
Since,
\[ \partial_\alpha z_2 \partial_\alpha^2 z_2 = \partial_\alpha z \cdot \partial_\alpha^2 z - \partial_\alpha z_1 \partial_\alpha^2 z_1 = -3 \partial_\alpha^2 z \cdot \partial_\alpha^3 z - \partial_\alpha z_1 \partial_\alpha^4 z_1 \]
and \( H \partial_\alpha = \Lambda \), using integration by parts
\[
R_{12} \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|^2_{H^4(S)}) - 3\Re \int \frac{\partial_\alpha (BR_2(z, \bar{\omega}) (\gamma) \partial_\alpha^2 z \cdot \partial_\alpha^3 z \cdot H(\partial_\alpha^{1,1} z_1)(\gamma)}{A(t)} d\alpha \\
- \Re \int \frac{\partial_\alpha z_1(\gamma) \cdot BR_2(z, \bar{\omega})(\gamma) \partial_\alpha^4 z_1(\gamma) \Lambda(\partial_\alpha^{1,1} z_1)(\gamma)}{A(t)} d\alpha \\
\leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|^2_{H^4(S)}) - \Re \int \frac{\partial_\alpha z_1(\gamma) \cdot BR_2(z, \bar{\omega})(\gamma) \partial_\alpha^4 z_1(\gamma) \Lambda(\partial_\alpha^{1,1} z_1)(\gamma)}{A(t)} d\alpha.
\]
And in the same way,
\[
R_{13} \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|^2_{H^4(S)}) + \Re \int \frac{\partial_\alpha z_2(\gamma) \cdot BR_1(z, \bar{\omega})(\gamma) \partial_\alpha^4 z_2(\gamma) \Lambda(\partial_\alpha^{1,1} z_2)(\gamma)}{A(t)} d\alpha.
\]
Therefore,
\[
R_{11} + R_{12} + R_{13} + R_{14} \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|^2_{H^4(S)}) \\
- \Re \int \frac{BR(z, \bar{\omega})(\gamma) \cdot \partial_\alpha^{1,1} z(\gamma) \partial_\alpha^4 z(\gamma) \cdot \Lambda(\partial_\alpha^{1,1} z)(\gamma)}{A(t)} d\alpha.
\]
Then,
\[
P_7 + P_8 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|^2_{H^4(S)}) \\
- \Re \int \frac{BR(z, \bar{\omega})(\gamma) \cdot \partial_\alpha^{1,1} z(\gamma) \partial_\alpha^4 z(\gamma) \cdot \Lambda(\partial_\alpha^{1,1} z)(\gamma) d\alpha}{A(t)} \\
- \Re \int \frac{\partial_\alpha z_1(\gamma) \cdot \partial_\alpha^4 z_1(\gamma) \cdot \Lambda(\partial_\alpha^{1,1} z_1)(\gamma) d\alpha}{A(t)} \\
\]
Let us look at these last two terms,
\[
- \Re \int \frac{\partial_\alpha z_1(\gamma) \cdot \partial_\alpha^4 z_1(\gamma) \cdot \Lambda(\partial_\alpha^{1,1} z_1)(\gamma) d\alpha}{A(t)} \\
= - \Re \int \frac{\sigma(\gamma, t)}{A(t)} \partial_\alpha^4 z(\gamma) \cdot \Lambda(\partial_\alpha^{1,1} z)(\gamma) d\alpha \\
= \int \frac{\sigma(\gamma, t)}{A(t)} \cdot \mathcal{I}(\partial_\alpha^4 z) \cdot \mathcal{I}(\Lambda(\partial_\alpha^{1,1} z)) d\alpha \\
- \int \frac{\sigma(\gamma, t)}{A(t)} \cdot \mathcal{I}(\partial_\alpha^{1,1} z) \cdot \mathcal{I}(\Lambda(\partial_\alpha^4 z)) d\alpha \\
\]
we get
\[
Y_1 = \int \frac{\sigma(\gamma, t)}{A(t)} \cdot \mathcal{I}(\partial_\alpha^4 z) \cdot \mathcal{I}(\Lambda(\partial_\alpha^{1,1} z)) d\alpha \\
\leq C \|\sigma(\gamma, t)\|_{C^1(S)} \|\partial_\alpha^4 z\|^2_{L^2(S)} \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|^2_{H^4(S)})
\]
and
\[ Y_2 = -\int (\mathcal{R}(\frac{\sigma}{A(t)}) - m(t))(\mathcal{R}(\partial_4^4 z) \cdot \mathcal{R}(A(\partial_4^4 z))) + J(\partial_4^4 z) \cdot J(A(\partial_4^4 z))d\alpha \\
- \int m(t)(\mathcal{R}(\partial_4^4 z) \cdot \mathcal{R}(A(\partial_4^4 z))) + J(\partial_4^4 z) \cdot J(A(\partial_4^4 z))d\alpha \\
\equiv Y_3 + Y_4\]

where
\[ m(t) = \min_{\gamma} \sigma(\gamma, t). \]

Since \( \mathcal{R}(\frac{\sigma}{A(t)}) - m(t) > 0 \) using \( 2gA(\gamma) - \Lambda(g^2) \geq 0 \), see [6]

\[ Y_3 \leq \frac{1}{2} \left\| A(\sigma) \right\|_2 \left( A(\partial_4^4 z) \right) \leq C \left\| \frac{\sigma}{A(t)} \right\|_2 \left( A(\partial_4^4 z) \right) \leq \exp C(\Vert \mathcal{F}(z) \Vert_2^2 + \Vert z \Vert^2_{H^4(S)}), \]

\[ Y_4 = -m(t)\left\| \Lambda \partial_4 z \right\|_2^2 \]

Combining all previous estimates
\[ I_7 \leq \exp C(\Vert \mathcal{F}(z) \Vert_2^2 + \Vert z \Vert^2_{H^4(S)}) - m(t)\left\| \Lambda \partial_4^4 z \right\|_2^2 \]

2.2.3 Estimates on \( \partial_4^4(c(\gamma, t) \cdot \partial_4 z(\gamma, t)) \) for \( J_2 \)

In the evolution of the norm of \( \partial_4^4 z(\gamma) \) it remains to control the term
\[ J_2 = \Re \int \partial_4^4 z(\gamma) \cdot \partial_4^4 c(\gamma) \partial_4 z(\gamma) d\alpha + 4\Re \int \partial_4^4 z(\gamma) \cdot \partial_4^3 c(\gamma) \partial_4^2 z(\gamma) d\alpha \\
+ 6\Re \int \partial_4^3 z(\gamma) \cdot \partial_4^2 c(\gamma) \partial_4^3 z(\gamma) d\alpha + 4\Re \int \partial_4^2 z(\gamma) \cdot \partial_4 c(\gamma) \partial_4^4 z(\gamma) d\alpha \\
+ \Re \int \partial_4^1 z(\gamma) \cdot c(\gamma) \partial_4^5 z(\gamma) d\alpha \equiv Q_1 + Q_2 + Q_3 + Q_4 + Q_5. \]

Let us recall the formula
\[ c(\alpha, t) = \frac{\alpha + \pi}{2\pi} \int_T \frac{\partial_\beta z(\beta, t)}{|\partial_\beta z(\beta, t)|^2} \cdot \partial_\beta BR(z, \varpi)(\beta, t) d\beta \]
\[ - \int_{-\pi}^{\alpha} \frac{\partial_\beta z(\beta, t)}{|\partial_\beta z(\beta, t)|^2} \cdot \partial_\beta BR(z, \varpi)(\beta, t) d\beta, \]

then
\[ Q_2 = 4\Re \int \frac{\partial_4^4 z(\gamma)}{A(t)} \cdot \partial_4^3 z(\gamma) \cdot \partial_4^2 z(\gamma) \cdot \partial_4 z(\gamma) d\alpha \]
\[ + 8\Re \int \frac{\partial_4^4 z(\gamma)}{A(t)} \cdot \partial_4^3 z(\gamma) \cdot \partial_4^2 z(\gamma) \cdot \partial_4^2 z(\gamma) d\alpha \]
\[ + 4\Re \int \frac{\partial_4^4 z(\gamma)}{A(t)} \cdot \partial_4^3 z(\gamma) \cdot \partial_4^2 z(\gamma) \cdot \partial_4^3 z(\gamma) d\alpha \equiv N_1 + N_2 + N_3 \]
and

\[
N_1 \leq C \| \mathcal{F}(z) \|_{L^{\infty}}(S) \| z \|_{C^2(S)} \| z \|_{C^3(S)} \| \partial^4_{\alpha} z \|_{L^2(S)},
\]
\[
N_2 \leq C \| \mathcal{F}(z) \|_{L^{\infty}}(S) \| z \|_{C^2(S)} \| \partial^3_{\alpha} B R(z, \varpi) \|_{L^2(S)} \| \partial^3_{\alpha} z \|_{L^2(S)},
\]
\[
N_3 \leq C \| \mathcal{F}(z) \|_{L^{\infty}}(S) \| z \|_{C^2(S)} \| z \|_{C^4(S)} \| \partial^3_{\alpha} B R(z, \varpi) \|_{L^2(S)} \| \partial^4_{\alpha} z \|_{L^2(S)}.
\]

Thus

\[
Q_2 \leq \exp(C(\| \mathcal{F}(z) \|_{L^{\infty}}^2(S) + \| z \|_{H^4(S)}^2)).
\]

In the same way,

\[
Q_3 = -6R \int \partial^4_{\alpha} z(\gamma) \cdot \partial^3_{\alpha} z(\gamma) \cdot \partial^2_{\alpha} z(\gamma) \cdot \partial_{\alpha} B R(z, \varpi)(\gamma) d\alpha
\]
\[
- 6R \int \partial^4_{\alpha} z(\gamma) \cdot \partial^3_{\alpha} z(\gamma) \cdot \partial^2_{\alpha} z(\gamma) \cdot \partial_{\alpha} B R(z, \varpi)(\gamma) d\alpha \equiv N_4 + N_5
\]

where

\[
N_4 \leq C \| \mathcal{F}(z) \|_{L^{\infty}}(S) \| z \|_{C^2(S)} \| z \|_{C^3(S)} \| \partial^4_{\alpha} z \|_{L^2(S)} \| \partial_{\alpha} B R(z, \varpi) \|_{L^2(S)},
\]
\[
N_5 \leq C \| \mathcal{F}(z) \|_{L^{\infty}}(S) \| z \|_{C^2(S)} \| z \|_{C^3(S)} \| \partial^4_{\alpha} z \|_{L^2(S)} \| \partial_{\alpha} B R(z, \varpi) \|_{L^2(S)},
\]

thus

\[
Q_3 \leq \exp(C(\| \mathcal{F}(z) \|_{L^{\infty}}^2(S) + \| z \|_{H^4(S)}^2)).
\]

The term \( Q_4 \) satisfies

\[
Q_4 \leq C \| \partial_{\alpha} c \|_{L^{\infty}}(S) \| \partial^4_{\alpha} z \|_{L^2(S)} \leq C \| \mathcal{F}(z) \|_{L^{\infty}}^2(S) \| \partial_{\alpha} B R \|_{L^{\infty}}(S) \| \partial^4_{\alpha} z \|_{L^2(S)}
\]
\[
\leq \exp(C(\| \mathcal{F}(z) \|_{L^{\infty}}^2(S) + \| z \|_{H^4(S)}^2))
\]

and for \( Q_5 \)

\[
Q_5 = R \int c(\gamma) \partial^4_{\alpha} z(\gamma) \cdot \partial^3_{\alpha} z(\gamma) d\alpha
\]
\[
= \int R(c)(R(\partial^4_{\alpha} z)R(\partial^5_{\alpha} z) + \Im(\partial^4_{\alpha} z)\Im(\partial^5_{\alpha} z)) d\alpha
\]
\[
+ \int R(c)(-R(\partial^4_{\alpha} z)\Im(\partial^5_{\alpha} z) + \Im(\partial^4_{\alpha} z)R(\partial^5_{\alpha} z)) d\alpha
\]
\[
= Q_5^1 + Q_5^2
\]

where,

\[
Q_5^1 = -\frac{1}{2} \int R(\partial_{\alpha} c) \| \partial^4_{\alpha} z \|_{L^2(S)} \leq \| \partial_{\alpha} z \|_{L^{\infty}} \| \partial^4_{\alpha} z \|_{L^2(S)}
\]
\[
\leq C \| \mathcal{F}(z) \|_{L^{\infty}}(S) \| \partial_{\alpha} B R(z, \varpi) \|_{L^{\infty}}(S) \| \partial^4_{\alpha} z \|_{L^2(S)} \leq \exp(C(\| \mathcal{F}(z) \|_{L^{\infty}}^2(S) + \| z \|_{H^4(S)}^2))
\]
and

\[ Q_5^2 = \int \mathcal{H}(\partial_\alpha c)\mathcal{R}(\partial_\alpha^4 z)\mathcal{H}(\partial_\alpha^4 z)\,d\alpha + 2 \int \mathcal{H}(\partial_\alpha c)\mathcal{H}(\partial_\alpha^4 z)\mathcal{R}(\partial_\alpha^5 z)\,d\alpha \]

\[ \leq \|\partial_\alpha c\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2 - 2 \int \mathcal{H}(\partial_\alpha c)\mathcal{H}(\partial_\alpha^4 z)\mathcal{R}(\Lambda(\partial_\alpha^4 z))\,d\alpha \]

\[ \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2) - 2 \int \Lambda^2(\mathcal{H}(\partial_\alpha^4 z))\mathcal{R}(\Lambda^2(\mathcal{H}(\partial_\alpha^4 z)))\,d\alpha \]

\[ \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2) + \|\mathcal{F}(z)\|_{H^1(S)}^2 + K\|\mathcal{H}(c)\|_{H^2(S)}\Lambda^2(\partial_\alpha^4 z)^2\|_{L^2(S)}^2. \]

Finally,

\[ Q_1 = R \int \frac{\partial_\alpha^4 z}{A(t)} \cdot \partial_\alpha z(\gamma) \partial_\alpha^4 z(\gamma) \cdot \partial_\alpha BR(\gamma, \varpi)(\gamma)\,d\alpha \]

\[ - 3R \int \frac{\partial_\alpha^4 z}{A(t)} \cdot \partial_\alpha z(\gamma) \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^2 BR(\gamma, \varpi)(\gamma)\,d\alpha \]

\[ - 3R \int \frac{\partial_\alpha^4 z}{A(t)} \cdot \partial_\alpha z(\gamma) \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^2 BR(\gamma, \varpi)(\gamma)\,d\alpha \]

\[ - R \int \frac{\partial_\alpha^4 z}{A(t)} \cdot \partial_\alpha z(\gamma) \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^2 BR(\gamma, \varpi)(\gamma)\,d\alpha \]

\[ \equiv N_6 + N_7 + N_8 + N_9 \]

where

\[ N_6 \leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\mathcal{H}(c)\|_{L^\infty(S)}\|\partial_\alpha BR\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2, \]

\[ N_7 \leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\mathcal{H}(c)\|_{L^\infty(S)}\|\partial_\alpha^2 BR\|_{L^2(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2, \]

\[ N_8 \leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\mathcal{H}(c)\|_{L^\infty(S)}\|\partial_\alpha^2 BR\|_{L^2(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2. \]

To estimate \( N_9 \), we must proceed in the same way we did with \( J_1 \). We split \( N_9 = I'_3 + I'_4 + I'_5 + I'_6 + I'_7 \)
To study this terms we have to repeat all estimates as in section 2.2.1. We select only the terms with different decompositions and we leave to the reader the remainder easy cases.

If we consider the term corresponding to \( Q_4 \) in section 2.2.1 we have since

\[
\Re(\partial^4 z \cdot \partial_\alpha z) = \Re(\partial^4 z) \cdot \Re(\partial_\alpha z) + \Im(\partial^4 z) \cdot \Im(\partial_\alpha z),
\]

\[
\Im(\partial^4 z \cdot \partial_\alpha z) = -\Re(\partial^4 z) \cdot \Im(\partial_\alpha z) + \Im(\partial^4 z) \cdot \Re(\partial_\alpha z),
\]

\[
\Re(\partial^4 z \cdot \partial_\alpha z) = \Re(\partial^4 z) \cdot \Re(\partial_\alpha z) - \Im(\partial^4 z) \cdot \Im(\partial_\alpha z),
\]

\[
\Im(\partial^4 z \cdot \partial_\alpha z) = \Im(\partial^4 z) \cdot \Re(\partial_\alpha z) + \Re(\partial^4 z) \cdot \Im(\partial_\alpha z),
\]

and

\[
\partial_\alpha z \cdot \partial^4 z = -3\partial^2 z \cdot \partial^3 z.
\]

we can write

\[
\Re(\partial^4 z \cdot \partial_\alpha z) = \Re(-3\partial^2 z \cdot \partial^3 z) + 2\Im(\partial^4 z) \cdot \Im(\partial_\alpha z),
\]

\[
\Im(\partial^4 z \cdot \partial_\alpha z) = \Im(-3\partial^2 z \cdot \partial^3 z) - 2\Re(\partial^4 z) \cdot \Re(\partial_\alpha z).
\]

Thus

\[
Q_4' = -2\pi \Re \int \frac{\partial^4 z(\gamma)}{A^2(t)} \cdot \partial_\alpha z(\gamma) \partial_\alpha z(\gamma) \cdot \Lambda(\partial^4 z^\perp(\gamma)) \bar{\omega}(\gamma) d\alpha.
\]

\[
= -2\pi \int \Re(\partial^4 z \cdot \partial_\alpha z) \Re(\partial_\alpha z \cdot \Lambda(\partial^4 z^\perp) \frac{\bar{\omega}}{A^2(t)}) - \Im(\partial^4 z \cdot \partial_\alpha z) \Im(\partial_\alpha z \cdot \Lambda(\partial^4 z^\perp) \frac{\bar{\omega}}{A^2(t)}) d\alpha.
\]

\[
= -2\pi \int \Re(-3\partial^2 z \cdot \partial^3 z) + 2\Im(\partial^4 z) \cdot \Im(\partial_\alpha z) \Re(\partial_\alpha z \cdot \Lambda(\partial^4 z^\perp) \frac{\bar{\omega}}{A^2(t)}) d\alpha
\]

\[
+ 2\pi \int \Im(-3\partial^2 z \cdot \partial^3 z) - 2\Re(\partial^4 z) \cdot \Re(\partial_\alpha z) \Im(\partial_\alpha z \cdot \Lambda(\partial^4 z^\perp) \frac{\bar{\omega}}{A^2(t)}) d\alpha
\]

\[
\equiv Q_4^1 + Q_4^2.
\]

we have,

\[
Q_4^1 = -2\pi \int \Re(-3\partial^2 z \cdot \partial^3 z) \Re(\partial_\alpha z \cdot \Lambda(\partial^4 z^\perp) \frac{\bar{\omega}}{A^2(t)}) d\alpha
\]

\[
- 4\pi \int \Im(\partial^4 z) \cdot \Im(\partial_\alpha z) \Re(\partial_\alpha z \cdot \Lambda(\partial^4 z^\perp) \frac{\bar{\omega}}{A^2(t)}) d\alpha
\]

\[
\equiv Q_4^{11} + Q_4^{12}.
\]

Clearly,

\[
Q_4^{11} \leq \exp(C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2)).
\]

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Since
\[ R(\partial_\alpha z \cdot \Lambda(\partial_\alpha z) - \Lambda(\partial_\alpha z) \cdot R(\partial_\alpha z)) \]
we take,
\[ Q_4^{12} = -4\pi \int \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\partial_\alpha z) R(\partial_\alpha z \cdot \frac{\partial}{A^2(t)}) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) d\alpha \]
\[ + 4\pi \int \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) d\alpha \]
\[ \leq \exp \left( \| \mathcal{F}(z) \|_L^2(S) + \| z \|_H^2(S) \right) + k \| \mathcal{I}(\partial_\alpha z) R(\partial_\alpha z \cdot \frac{\partial}{A^2(t)}) \| H^2(S) \| \Lambda^\perp \partial_\alpha z \|_L^2(S) \]
\[ + \exp \left( \| \mathcal{F}(z) \|_L^2(S) + \| z \|_H^2(S) \right) + c \| \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z \cdot \frac{\partial}{A^2(t)}) \| H^2(S) \| \Lambda^\perp \partial_\alpha z \|_L^2(S) \].

For \( Q_4^2 \)
\[ Q_4^2 = 2\pi \int \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) d\alpha \]
\[ - 4\pi \int \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) d\alpha \]
\[ \equiv Q_4^{21} + Q_4^{22} \]

Clearly,
\[ Q_4^{21} \leq \exp \left( \| \mathcal{F}(z) \|_L^2(S) + \| z \|_H^2(S) \right). \]

Since,
\[ \mathcal{I}(\partial_\alpha z \cdot \Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) = \mathcal{I}(\partial_\alpha z \cdot \frac{\partial}{A^2(t)}) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) + \mathcal{I}(\partial_\alpha z \cdot \frac{\partial}{A^2(t)}) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) \]
we have,
\[ Q_4^{22} = -4\pi \int \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) d\alpha \]
\[ - 4\pi \int \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) d\alpha \]
\[ \leq \exp \left( \| \mathcal{F}(z) \|_L^2(S) + \| z \|_H^2(S) \right) + k \| \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) \| H^2(S) \| \Lambda^\perp \partial_\alpha z \|_L^2(S) \]
\[ + \exp \left( \| \mathcal{F}(z) \|_L^2(S) + \| z \|_H^2(S) \right) + c \| \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) \| H^2(S) \| \Lambda^\perp \partial_\alpha z \|_L^2(S) \].

Using a similar method for the rest of non-integrable terms we obtain
\[ J_2 \leq \exp \left( \| \mathcal{F}(z) \|_L^2(S) + \| z \|_H^2(S) \right) \]
\[ + C(\| \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) \| H^2(S) + \| \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) \| H^2(S) + \| \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) \| H^2(S) + \| \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\partial_\alpha z) \cdot \mathcal{I}(\Lambda(\partial_\alpha z) \cdot \frac{\partial}{A^2(t)}) \| H^2(S) \].
In conclusion,
\[
I_1 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) + C[\|J(\frac{\varpi}{A(t)})\|_{H^2(S)}\|J(\partial_\alpha z)\|_{H^2(S)}] + \|J(\partial_\alpha z)\|_{H^2(S)} + \|J(c)\|_{H^2(S)} - m(t)]\|A^\frac{1}{2} \partial_\alpha^4 z\|_{L^2(S)}^2
\]
and therefore
\[
\frac{1}{2} \frac{d}{dt} \int \left| \partial_\alpha^4 z(\gamma) \right|^2 d\alpha = I_1 + I_2
\]
\[
\leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) + C[\|J(\frac{\varpi}{A(t)})\|_{H^2(S)}\|J(\partial_\alpha z)\|_{H^2(S)}] + \|J(\partial_\alpha z)\|_{H^2(S)} + \|J(c)\|_{H^2(S)} - m(t) + 2\lambda \|A^\frac{1}{2} \partial_\alpha^4 z\|_{L^2(S)}^2.
\]

3 The evolution of the minimum of $\sigma(\gamma, t)$

Taking the divergence in Darcy’s law we obtain
\[
\Delta p = 0.
\]
Since the pressure is zero on the interface and recalling that the flow is irrotational in the interior of the domain $\Omega$ by the Hopf’s lemma we have
\[
\sigma(\alpha, t) = -\frac{\partial p}{\partial z} > 0.
\]
In spite of this property, we need to get an a priori estimate for the evolution of the minimum of $\sigma$ in the strip $S$ in order to absorb the high order terms in (16).

Recall that
\[
\sigma(\alpha, t) = \frac{H^2}{K} BR(z, \varpi)(\alpha, t) \cdot \partial_\alpha^4 z(\alpha, t) + g\rho^2 \partial_\alpha z_1(\alpha, t).
\]

Lemma 3.1. Let $z(\gamma, t)$ be a solution of the system with $z(\gamma, t) \in C^1([0, T]; H^4)$, and
\[
m(t) = \min_\gamma \sigma(\gamma, t).
\]
Then
\[
m(t) \geq m(0) - \int_0^t \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) ds.
\]
Proof. We may consider $\gamma_t \in \mathbb{C}$ such that
\[
m(t) = \min_\gamma \sigma(\gamma, t) = \sigma(\gamma_t, t).
\]
We may calculate the derivative of \( m(t) \), to obtain
\[
m'(t) = \sigma(\gamma, t).
\]

The identity (17) yields,
\[
s_t(\gamma, t) = \frac{\mu^2}{\kappa} \partial_t BR(z, w)(\gamma, t) \cdot \partial_\alpha z(\gamma, t) + id\frac{\mu^2}{\kappa} \partial_\alpha BR(z, w)(\gamma, t) \cdot \partial_\alpha z(\gamma, t)
\]
\[+ \frac{\mu^2}{\kappa} BR(z, w)(\gamma, t) \cdot \partial_\alpha z(\gamma, t) + \frac{\mu^2}{\kappa} BR(z, w)(\gamma, t) \cdot id\partial_\alpha z(\gamma, t)
\]
\[+ g\rho^2 \partial_\alpha z_1(\gamma, t) + g\rho^2 \partial_\alpha^2 z_1(\gamma, t) \equiv R_1 + R_2 + R_3 + R_4 + R_5 + R_6.
\]

We can easily estimate,
\[
|R_2| \leq d\frac{\mu^2}{\kappa} ||\partial_\alpha BR(z, w)||_{L^\infty(S)} ||\partial_\alpha z||_{L^\infty(S)} \leq \exp C(||\mathcal{F}(z)||_{L^\infty(S)}^2 + ||z||_{H^4(S)}^2),
\]
\[
|R_4| \leq d\frac{\mu^2}{\kappa} ||BR(z, w)||_{L^\infty(S)} ||z||_{C^2(S)} \leq \exp C(||\mathcal{F}(z)||_{L^\infty(S)}^2 + ||z||_{H^4(S)}^2),
\]
\[
|R_6| \leq g\rho^2 ||z||_{C^2(S)} \leq \exp C(||\mathcal{F}(z)||_{L^\infty(S)}^2 + ||z||_{H^4(S)}^2),
\]
and we have,
\[
|R_3| + |R_5| \leq C(||BR(z, w)||_{L^\infty(S)} + 1) ||\partial_\alpha z_t||_{L^\infty(S)}.
\]

Since \( z_t(\gamma) = BR(z, w)(\gamma) + c(\gamma) \partial_\alpha z(\gamma), \)
\[
||\partial_\alpha z_t||_{L^\infty(S)} \leq ||\partial_\alpha BR(z, w)||_{L^\infty(S)} + ||\partial_\alpha c||_{L^\infty(S)} ||\partial_\alpha z||_{L^\infty(S)} + ||c||_{L^\infty(S)} ||\partial_\alpha^2 z||_{L^\infty(S)}
\]
\[
\leq C ||\partial_\alpha BR(z, w)||_{L^\infty(S)} (1 + ||\mathcal{F}(z)||_{L^\infty(S)}^\frac{1}{2}) ||z||_{C^2(S)} \leq \exp C(||\mathcal{F}(z)||_{L^\infty(S)}^2 + ||z||_{H^4(S)}^2).
\]

Then,
\[
|R_3 + R_5| \leq \exp C(||\mathcal{F}(z)||_{L^\infty(S)}^2 + ||z||_{H^4(S)}^2).
\]

Recall that
\[
BR(z, w)(\gamma) = \frac{1}{2\pi} \int \frac{\Delta z^\perp}{|\Delta z|^2} w(\gamma - \beta) d\beta,
\]
then
\[
BR_t(z, w)(\gamma) = \frac{1}{2\pi} \int \frac{\Delta z^\perp}{|\Delta z|^2} w(\gamma - \beta) d\beta - \frac{1}{\pi} \int \frac{\Delta z^\perp (\Delta z \cdot \Delta z_t)}{|\Delta z|^4} w(\gamma - \beta) d\beta
\]
\[+ \frac{1}{2\pi} \int \frac{\Delta z^\perp}{|\Delta z|^2} w_t(\gamma - \beta) d\beta \equiv J_1 + J_2 + J_3.
\]

We get
\[
J_1 = \frac{1}{2\pi} \int \Delta z^\perp w(\gamma - \beta) \left( \frac{1}{|\Delta z|^2} - \frac{1}{A(t) \beta^2} \right) d\beta
\]
\[+ \frac{1}{2\pi} \int \frac{\Delta z^\perp}{A(t) \beta^2} w(\gamma - \beta) d\beta \equiv K_1 + K_2.
\]

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Therefore, and since

\[ K_1 = \frac{1}{2\pi} \int_0^1 \varpi(\gamma - \beta) \beta \int_0^1 \partial_\alpha z(t)(\phi)dsB(\gamma, \beta)d\beta \]

\[ \leq C\|F(z)\|_{L^\infty(S)}^3 \|\varpi\|_{L^\infty(S)} \|\partial_\alpha z\|_{L^\infty(S)} \leq \exp C\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2. \]

Since

\[ \partial_\alpha^2 z = \partial^2 BR(z, \varpi) + \partial_\alpha c\partial_\alpha z + 2\partial_\alpha c\partial_\alpha^2 z + c\partial_\alpha^3 z \]

and

\[ \|\partial_\alpha^2 BR(z, \varpi)\|_{L^\infty(S)} \leq \exp C\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2. \]

\[ \|\partial_\alpha^2 c\partial_\alpha z\|_{L^\infty(S)} = \|\left(\frac{\partial_\alpha^2 z}{\partial_\alpha z}\right)^2 \cdot \partial_\alpha BR(z, \varpi)\partial_\alpha z\|_{L^\infty(S)} + \|\frac{\partial_\alpha z}{\partial_\alpha z}\| \cdot \partial_\alpha^2 BR(z, \varpi)\partial_\alpha z\|_{L^\infty(S)} \]

\[ \leq C\|F(z)\|_{L^\infty(S)}^2 \|\varpi\|_{L^\infty(S)} \|\partial_\alpha BR(z, \varpi)\|_{L^\infty(S)} + \|\partial_\alpha^2 BR(z, \varpi)\|_{L^\infty(S)} \]

\[ \leq \exp C\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2. \]

Thus

\[ 2\|\partial_\alpha c\partial_\alpha^2 z\|_{L^\infty(S)} \leq 4\|F(z)\|_{L^\infty(S)}^2 \|\partial_\alpha BR(z, \varpi)\|_{L^\infty(S)} \|\partial_\alpha^2 z\|_{L^\infty(S)} \]

\[ \leq \exp C\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2. \]

then

\[ \|\partial_\alpha^2 z\|_{L^\infty(S)} \leq \exp C\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2. \]

Thus,

\[ K_2 = \frac{1}{2\pi} \int_0^1 \frac{\partial_\alpha z(t)(\phi)ds}{A(t)\beta} \varpi(\gamma - \beta)d\beta \]

\[ = \frac{1}{2\pi} \int_0^1 \frac{\partial_\alpha^2 z(t)(\psi)ds}{A(t)\beta} \varpi(\gamma - \beta)d\beta + \frac{1}{2\pi} \int_0^1 \frac{\partial_\alpha z(t)(\gamma)}{A(t)\beta} \varpi(\gamma - \beta)d\beta \]

\[ = \frac{1}{2\pi} \int_0^1 \frac{\partial_\alpha^2 z(t)(\psi)ds}{A(t)} \varpi(\gamma - \beta)d\beta + \frac{1}{2\beta} \frac{\partial_\alpha z(t)(\gamma)}{A(t)} H(\varpi(\gamma)) \]

\[ \leq C\|F(z)\|_{L^\infty(S)} \|\partial_\alpha^2 z\|_{L^\infty(S)} \|\varpi\|_{L^\infty(S)} + K\|F(z)\|_{L^\infty(S)} \|\partial_\alpha z\|_{L^\infty(S)} \|\varpi\|_{C^4(S)}. \]

Therefore,

\[ J_1 \leq \exp C\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2. \]

In the same way, it is easy to see that

\[ J_2 \leq \exp C\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2. \]

Finally, since

\[ \frac{\Delta z +}{\Delta z} \frac{\partial_\alpha z}{\partial_\alpha z}(\gamma) = \beta^2 \int_0^1 \frac{\partial_\alpha^2 z(\psi)(s - 1)ds}{A(t)} \frac{\Delta z}{\Delta z} \]

\[ + \beta^2 \int_0^1 \frac{\partial_\alpha^2 z(\psi)(s - 1)ds}{A(t)} \frac{[\partial_\alpha z(\gamma) + \partial_\alpha z(\phi)]ds}{A(t)\Delta z}, \]

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\[ J_3 = \frac{1}{2\pi} \int \left( \frac{\Delta z^+}{|\Delta z|^2} - \frac{\partial_\alpha z^+ (\gamma)}{A(t)} \right) \omega_t (\gamma - \beta) d\beta + \frac{1}{2\pi} \int \frac{\partial_\alpha z^+ (\gamma)}{A(t)} \omega_t (\gamma - \beta) d\beta \]

\[ \equiv K_5 + K_6 \]

where

\[ K_5 \leq C \| F(z) \|_{L^\infty(S)} \| z \|_{C^2(S)} \| \omega_t \|_{L^2(S)}, \]

\[ K_6 = \frac{1}{2} \frac{\partial_\alpha z^+ (\gamma)}{A(t)} H(\omega_t) (\gamma) \leq C \| F(z) \|_{L^\infty(S)} \| \omega_t \|_{C^4(S)}. \]

In order to control \( \| \omega_t \|_{C^4(S)} \) we proceed as in section 9 in [7]. Therefore,

\[ |\sigma_t (\gamma, t)| \leq \exp \left( C (\| F(z) \|_{L^\infty(S)} + \| z \|_{H^4(S)}^2) \right) \]

given us,

\[ m'(t) \geq -\exp \left( C (\| F(z) \|_{L^\infty(S)} + \| z \|_{H^4(S)}^2) \right) \]

for almost every \( t \). And a further integration yields

\[ m(t) \geq m(0) - \int_0^t \exp \left( C (\| F(z) \|_{L^\infty(S)} + \| z \|_{H^4(S)}^2) \right) ds. \]

\[ \square \]

4 Instant Analyticity

**Theorem 4.1.** Let \( z(\alpha, 0) = z_0(\alpha) \in H^4, F(z)(z_0)(\alpha, \beta) \in L^\infty \). Then there exists a solution of the Muskat problem \( z(\alpha, t) \) defined for \( 0 < t \leq T \) that continues analytically into the strip \( S(t) = \{ \alpha \pm i\varsigma : |\varsigma| < \lambda t \} \) for each \( t \). Here, \( d \) and \( T \) are determined by upper bounds of the \( H^4 \) norm and the arc-chord constant of the initial data and a positive lower bound of the \( \sigma(\alpha, 0) \). Moreover, for \( 0 < t \leq T \), the quantity

\[ \sum_{\pm} \int \left( |z(\alpha \pm i\lambda t) - (\alpha \pm i\lambda t)|^2 + |\partial_\alpha^4 z(\alpha \pm i\lambda t)|^2 \right) d\alpha \]

is bounded by a constant determined by upper bounds for the \( H^4 \) norm and the arc-chord constant of the initial data and a positive lower bound of \( \sigma(\alpha, 0) \).

**Proof.** For all estimates in above sections we have finally

\[ \frac{1}{2} \frac{d}{dt} \int |\partial_\alpha^4 z(\alpha \pm i\lambda t)|^2 \leq \exp \left( C (\| F(z) \|_{L^\infty(S)}^2 + \| z \|_{H^4(S)}^2) \right) \]

\[ + (2\lambda + C \| f(\| (t) - m(t) \| \Lambda \partial_\alpha^4 z \|^2_{L^2(S)} (t) \]

\[ \right) \]

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where
\[ \|f\|(t) = \|\mathcal{A}(\omega(t))\|_{H^2(S)} + \|\mathcal{J}(\partial_\alpha z)\mathcal{R}(z\partial_\alpha \omega)\|_{H^2(S)} + \|\mathcal{J}(\partial_\alpha z)\mathcal{J}(\partial_\alpha \omega)\|_{H^2(S)} + \|\mathcal{J}(\omega)\|_{H^2(S)}. \]

Note that \( \|f\|(0) = 0 \). If \( 2\lambda - m(0) < 0 \) we will show that
\[ 2\lambda + K\|f\|(t) - m(t) < 0 \]
for short time. It yields
\[ \frac{1}{2} \frac{d}{dt} \int |\partial_\alpha^4 z(\alpha \pm i\lambda t)|^2 d\alpha \leq \exp C(\|f(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) \]
as long as \( 2\lambda + K\|f\|(t) - m(t) < 0 \). We proceed as in section 8 in [7] to show that
\[ \frac{d}{dt} \|\mathcal{F}(z)\|_{L^\infty(S)} \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2). \]
From the two inequalities above and (8) it is easy to obtain a priori energy estimates that depend upon the negativity of \( 2\lambda + K\|f\|(t) - m(t) \). We denote
\[ \|z\|_{RT}(t) \equiv \|\mathcal{F}(z)\|_{L^\infty(S)}^2(t) + \|z\|_{L^2(S)}^2(t) + \frac{1}{m(t) - 2\lambda - C\|f\|}. \]
At this point it is easy to find
\[ \|f\| \leq \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) \]
and
\[ \frac{d}{dt} \left( \frac{1}{m(t) - 2\lambda - C\|f\|} \right) \leq \frac{\exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2)}{(m(t) - 2\lambda - C\|f\|)^2} \]
then,
\[ \frac{d}{dt} \|z\|_{RT}(t) \leq \exp C(\|z\|_{RT}(t)) \]
and therefore,
\[ \|z\|_{RT} \leq - \log(\exp(-C\|z\|_{RT}(0) - C^2 t)). \]
Now we approximate the problem as follows,
\[ \begin{cases}
  z_\alpha^i(\alpha, t) = BR(z^\epsilon, \omega^\epsilon)(\alpha, t) + c^i(\alpha, t)\partial_\alpha z^\epsilon(\alpha, t) \\
  z^\epsilon(\alpha, 0) = z_0(\alpha)
\end{cases} \]
where
\[ c^i(\alpha, t) = \frac{\alpha + \pi}{2\pi} \int \frac{\partial_\alpha z^\epsilon(\alpha, t)}{|\partial_\alpha z^\epsilon(\alpha, t)|^2} \cdot \partial_\alpha BR(z^\epsilon, \omega^\epsilon)(\alpha, t) d\alpha \]
\[ - \int_{-\pi}^{\pi} \frac{\partial_\alpha z^\epsilon(\beta, t)}{|\partial_\alpha z^\epsilon(\beta, t)|^2} \cdot \partial_\alpha BR(z^\epsilon, \omega^\epsilon)(\beta, t) d\beta, \]

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\( \varpi^k(\alpha, t) = -\phi \ast \phi \ast (2BR(z^\epsilon, \varpi^\epsilon) \cdot \partial_\alpha z^\epsilon)(\alpha) - 2\kappa \frac{\partial^2}{\partial t^2} \phi \ast \phi \ast (\partial_\alpha z^\epsilon)(\alpha) \)

and \( \phi \in C^\infty_c(\mathbb{R}) \), \( \phi(\alpha) \leq 0 \), \( \phi(-\alpha) = \phi(\alpha) \), \( \int \phi(\alpha) \, d\alpha = 1 \), \( \phi(\alpha) = \phi(\frac{\alpha}{\epsilon})/\epsilon \) for \( \epsilon > 0 \).

Picard’s theorem yields the existence of a solution \( z^\epsilon(\alpha, t) \) in \( C([0,\epsilon^2); H^4) \) which is analytic in the whole space for \( z_0 \) satisfying the arc-chord condition and \( \epsilon \) small enough. Using the same techniques we have devoted above we obtain a bound for \( z^\epsilon(\alpha, t) \) in \( H^4 \) in the strip \( S(t) \) for a small enough \( T \) which is independent of \( \epsilon \). We need arc-chord condition, \( z_0 \in H^4 \) and \( 2\lambda - m(0) < 0 \). Then we pass to the limit.

5 Decay estimates on the strip of analyticity

**Theorem 5.1.** Let \( z(\alpha, 0) = z^0(\alpha) \) be an analytic curve in the strip

\[
S = \{ \alpha + i\varsigma \in \mathbb{C} : |\varsigma| < h(0) \},
\]

with \( h(0) > 0 \) and satisfying:

* The arc-chord condition, \( F(z^0)(\alpha + i\varsigma, \beta) \in L^\infty(S \times \mathbb{R}) \)

* The curve \( z^0(\alpha) \) is real for real \( \alpha \)

* The functions \( z^0_1(\alpha) - \alpha \) and \( z^0_2(\alpha) \) are periodic with period \( 2\pi \)

* The functions \( z^0_1(\alpha) - \alpha \) and \( z^0_2(\alpha) \) belong to \( H^4(S) \)

Then there exist a time \( T \) and a solution of the Muskat problem \( z(\alpha, t) \) defined for \( 0 < t \leq T \) that continues analytically into some complex strip for each fixed \( t \in [0, T] \). Here \( T \) is either a small constant depending only on \( \exp C(\|F(z^0)\|_{L^\infty(S)}^2 + \|z^0\|_{L^2(S)}^2) \).

We will use the following:

**Lemma 5.1.** Let \( \psi(\alpha \pm i\xi) = \sum_{k=-N}^{N} A_k(t) e^{ik\alpha} e^{\pm k\xi} \) and \( h(t) > 0 \) be a decreasing function of \( t \). Then

\[
\frac{\partial}{\partial t} \sum_{\pm} \int_T |\psi(\alpha \pm ih(t))|^2 \, d\alpha \leq \frac{h'(t)}{10} \sum_{\pm} \int_T \Lambda \psi(\alpha \pm ih(t)) \overline{\psi(\alpha \pm ih(t))} \, d\alpha \\
- 10h'(t) \int_T \Lambda \psi(\alpha) \overline{\psi(\alpha)} \, d\alpha + 2\Re \sum_{\pm} \int_T \psi(\alpha \pm ih(t)) \overline{\psi(\alpha \pm ih(t))} \, d\alpha.
\]

This lemma is a corollary of the lemma 4.2 in [4] and it allows us to prove the theorem 5.1.
proof of the theorem [5.4] The norms \( \|z\|_{L^2(S)} \) and \( \|z\|_{H^1(S)} \) are defined as before using the new strip \( S(t) \) defined by

\[
S(t) = \{ \alpha + i\xi \in \mathbb{C} : |\xi| < h(t) \}
\]

where \( h(t) \) is a positive decreasing function of \( t \).

We use the Galerkin approximation of equation (2), i.e.

\[
z^N(\gamma, t) = \Pi_N[J[z^N]](\gamma, t)
\]

where \( \gamma \in S(t) \), \( \Pi_N \) will be defined below, and

\[
J[z](\alpha, t) = BR(z, \varpi)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t).
\]

We impose the initial condition

\[
z^N(\alpha, 0) = z^N(\alpha).
\]

Here, for a large enough positive integer \( N \), we define \( z^N(\alpha, 0) \) from \( z^0(\alpha) \) by using the projection

\[
\Pi_N : \sum_{-\infty}^{\infty} A_k e^{ika} \rightarrow \sum_{-N}^{N} A_k e^{ika}.
\]

We defined \( z^N(\alpha) \) by stipulating that

\[
z_1^N(\alpha) - \alpha = \Pi_N[z_1^0(\alpha) - \alpha]
\]

and

\[
z_2^N(\alpha) = \Pi_N[z_2^0(\alpha)].
\]

For \( N \) large enough, the functions \( z^N(\alpha, 0) \) satisfy the arc-chord and Rayleigh-Taylor conditions.

We shall consider the evolution of the most singular quantity

\[
\sum_{\pm} \int |\partial_\alpha^4 z^N(\alpha \pm ih_N(t), t)|^2 d\alpha
\]

where \( h_N(t) \) is a smooth positive decreasing function on \( t \), with \( h_N(0) = h(0) \), which will be given below. Also we denote

\[
S_N(t) = \{ \alpha + i\xi \in \mathbb{C} : |\xi| < h_N(t) \}.
\]

We will drop the dependency on \( N \) from \( z^N \) and \( h_N(t) \) in our notation. Using lemma above,

\[
\frac{d}{dt} \sum_{\pm} \int |\partial_\alpha^4 z_j(\alpha \pm ih(t))|^2 d\alpha \leq \frac{h'(t)}{10} \sum_{\pm} \int \Lambda(\partial_\alpha^4 z_j)(\alpha \pm ih(t))\overline{\partial_\alpha^4 z_j}(\alpha \pm ih(t))
\]

\[
-10h'(t) \int \Lambda(\partial_\alpha^4 z_j)(\alpha)\overline{\partial_\alpha^4 z_j}(\alpha) d\alpha + 2 \sum_{\pm} \Re \int \Pi_N[\partial_\alpha^4 J_j[z]](\alpha \pm ih(t))\overline{\partial_\alpha^4 z_j}(\alpha \pm ih(t)).
\]
Since $\partial_{\alpha}^4 z_j(\alpha \pm ih(t))$ is a trigonometric polynomial in the range of $\Pi_N$

$$2 \sum_{\pm} \Re \int \Pi_N[\partial_{\alpha}^4 J_j(z)](\alpha \pm ih(t)) \partial_{\alpha}^4 z_j(\alpha \pm ih(t))$$

$$= 2 \sum_{\pm} \Re \int \partial_{\alpha}^4 J_j(z)(\alpha \pm ih(t)) \Pi_N[\partial_{\alpha}^4 z_j(\alpha \pm ih(t))]$$

$$= 2 \sum_{\pm} \Re \int \partial_{\alpha}^4 J_j(z)(\alpha \pm ih(t)) \partial_{\alpha}^4 z_j(\alpha \pm ih(t))$$

then,

$$\frac{d}{dt} \sum_{\pm} \int |\partial_{\alpha}^4 z_j(\alpha \pm ih(t))|^2 d\alpha \leq \frac{h'(t)}{10} \sum_{\pm} \int \Lambda(\partial_{\alpha}^4 z_j)(\alpha \pm ih(t)) \partial_{\alpha}^4 z_j(\alpha \pm ih(t))$$

$$- 10 h'(t) \int \Lambda(\partial_{\alpha}^4 z_j)(\alpha) \partial_{\alpha}^4 z_j(\alpha) d\alpha + 2 \sum_{\pm} \Re \int \partial_{\alpha}^4 \Lambda(z, \omega)(\alpha \pm ih(t)) \partial_{\alpha}^4 z_j(\alpha \pm ih(t))$$

$$+ 2 \sum_{\pm} \Re \int \partial_{\alpha}^4(c(\alpha \pm ih(t)) \partial_{\alpha} z_j(\alpha \pm ih(t))) \partial_{\alpha}^4 z_j(\alpha \pm ih(t))$$

$$\equiv M_1 + M_2 + M_3 + M_4.$$ 

To estimate $M_3$ and $M_4$ we have to repeat the arguments in sections 2 with the exception of the term $R_{20} + P_7$.

Following the same way, we will get that

$$M_3 \leq \exp C(\|F(z)\|_{L^\infty(S)} + \|z\|_{H^2(S)}^2) + C\|H(t)\| \Lambda(\partial_{\alpha}^4 z_j) \|_{L^2(S)}^2$$

$$- 2 \Re \int \sigma(\gamma) \partial_{\alpha}^4 z_j(\gamma) \cdot \Lambda(\partial_{\alpha}^4 z_j)(\gamma) d\alpha$$

where $\gamma = \alpha \pm ih(t)$.

In order to avoid problems we write,

$$\sigma(\gamma) = \sigma(\alpha) + h(t) g_{\pm}(\alpha)$$

where $g_{\pm} = \frac{1}{h(t)}(\sigma(\gamma) - \sigma(\alpha))$.

Since

$$\sigma(\alpha) = \frac{\mu^2}{\kappa} BR(z, \omega)(\alpha) \cdot \partial_{\alpha}^4 z(\alpha) + g_{\rho}^2 \partial_{\alpha} z_1(\alpha),$$

we can write,

$$g_{\pm} = \pm \frac{i \mu^2}{\kappa} \int_0^1 \partial_{\alpha}(BR(z, \omega) \cdot \partial_{\alpha}^4 z)(\gamma t + (t - 1) \alpha) dt \pm ig_{\rho}^2 \int_0^1 \partial_{\alpha}^2 z_1(\gamma t + (t - 1) \alpha) dt$$

then

$$\|g_{\pm}\|_{H^2(S)} \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).$$
Thus, we get
\[
- 2\Re \int \frac{\sigma(\gamma)}{A(t)} \overline{\partial_{\alpha}^4 z_j(\gamma)} \Lambda(\overline{\partial_{\alpha}^4 z_j(\gamma)}) d\alpha = -2\Re \int \frac{\sigma(\alpha)}{A(t)} \overline{\partial_{\alpha}^4 z_j(\gamma)} \Lambda(\overline{\partial_{\alpha}^4 z_j(\gamma)}) d\alpha
\]

\[- 2h(t)\Re \int \frac{g(\alpha)}{A(t)} \overline{\partial_{\alpha}^4 z_j(\gamma)} \Lambda(\overline{\partial_{\alpha}^4 z_j(\gamma)}) d\alpha \equiv M_3^1 + M_3^2.
\]

On the one hand, since \(\Re(\frac{\sigma}{A(t)}) > 0\) and \(2\Re(\Lambda(g) - \Lambda(g^2)) \geq 0\)
\[
M_3^1 = - 2 \Re \int \frac{\sigma}{A(t)} \Re(\overline{\partial_{\alpha}^4 z_j(\gamma)}) \Lambda(\overline{\partial_{\alpha}^4 z_j(\gamma)}) d\alpha \\
\leq \|\Lambda(\frac{\sigma}{A(t)})\|_{L^\infty(S)} \|\partial_{\alpha}^4 z\|_{L^2(S)} \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).
\]

On the other hand, like in the term \(N_5\) in section 2.2.1
\[
M_3^2 = - 2h(t)\Re \int \Lambda(\frac{\sigma}{A(t)}) \overline{\partial_{\alpha}^4 z_j(\gamma)} \Lambda(\overline{\partial_{\alpha}^4 z_j(\gamma)}) d\alpha \\
\leq Ch(t) \|\frac{g}{A(t)}\|_{H^2(S)} \|\partial_{\alpha}^4 z\|_{L^2(S)} + \|\Lambda \partial_{\alpha}^4 z\|_{L^2(S)} \|\Lambda \partial_{\alpha}^4 z\|_{L^2(S)} \\
\leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) + Ch(t) \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) \|\Lambda \partial_{\alpha}^4 z\|_{L^2(S)}^2.
\]

For \(M_1\),
\[
M_1 \leq \frac{h'(t)}{10} \|\Lambda \partial_{\alpha}^4 z\|_{L^2(S)}^2
\]

and \(M_4\),
\[
M_4 \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) \\
+ C(\|\partial_{\alpha}^4 z\|_{H^2(S)} + \|\overline{\partial_{\alpha}^4 z}\|_{H^2(S)} + \|\Lambda \partial_{\alpha}^4 z\|_{H^2(S)} + \|\Lambda \partial_{\alpha}^4 z\|_{H^2(S)}^2) \\
\leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) \|\Lambda \partial_{\alpha}^4 z\|_{L^2(S)}^2.
\]

Then,
\[
\frac{d}{dt} \sum_{\pm} \int |\partial_{\alpha}^4 z_j(\alpha \pm ih(t))|^2 d\alpha \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) \\
- 10h'(t) \int \Lambda(\partial_{\alpha}^4 z_j(\alpha \pm ih(t))) d\alpha \\
+ (\exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2) h(t) + \frac{h'(t)}{10} + \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2)) \|\Lambda \partial_{\alpha}^4 z\|_{L^2(S)}^2.
\]

Choosing,
\[
h(t) = \exp(-10 \int_0^t G(r) dr) \left[ \int_0^t -10G(r) \exp(10 \int_0^r G(s) ds) dr + h(0) \right]
\]

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where $G(t) = \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2)(t)$, we eliminate the most dangerous term. The other term in the expression above,

$$\int \Lambda(\partial^4_\alpha z_j)(\alpha)\partial^4_\alpha z_j(\alpha)d\alpha \leq \frac{C}{h(t)} \sum \pm \int |\partial^4_\alpha z_j|^2d\alpha$$

as one sees by examining the Fourier expansion of $\partial^4_\alpha z_j(\alpha, t)$. Thus,

$$|10h'(t)\int \Lambda(\partial^4_\alpha z_j)(\alpha)\partial^4_\alpha z_j(\alpha)d\alpha| \leq C \frac{|h'(t)|}{h(t)}(\|z\|_{H^4(S)}^2 + \|F(z)\|_{L^\infty(S)})$$

$$\leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).$$

And we obtain finally,

$$\frac{d}{dt} \sum \int |\partial^4_\alpha z(\alpha \pm ih(t))|^2d\alpha \leq \exp C(\|F(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2)$$

Recovering the dependency on $N$ in our notation we have that

$$\frac{d}{dt} \sum \int |\partial^4_\alpha z[N](\alpha \pm ih(t))|^2d\alpha \leq \exp C(\|F(z[N])\|_{L^\infty(S_N)}^2 + \|z[N]\|_{L^2(S_N)}^2)$$

(18)

This estimate is true wherever $t \in [0, T_N]$, where $T_N$ is the maximal time of existence of the solution $z[N]$. In addition inequality [18] shows that we can extend these solutions in $H^4(S)$ up to a small enough time $T$ independent of $N$ and dependent on the initial data.

6 Non-splat singularity

As we have said in the introduction, it is necessary to consider a transformed Muskat problem and we need to prove instant analyticity and decay estimates in $\tilde{\Omega}$. We will prove that the energy estimates of the theorems 4.1 and 5.1 holds in $\tilde{\Omega}$ for solutions $\tilde{z}$ of equations:

$$\tilde{z}_t(\alpha, t) = Q^2(\alpha, t)BR(\tilde{z}, \tilde{\omega})(\alpha, t) + \tilde{c}(\alpha)\partial_\alpha \tilde{z}(\alpha, t)$$

(19)

where

$$Q^2(\alpha, t) = \left| \frac{dP}{dw}(\alpha, t) \right|^2 = \left| \frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha, t))) \right|^2,$$

(20)

$$\tilde{\omega}(\alpha, t) = -2BR(\tilde{z}, \tilde{\omega})(\alpha, t) \cdot \partial_\alpha \tilde{z}(\alpha, t) - \frac{P_2^2}{\mu^2} \partial_\alpha (P_2^{-1}(\tilde{z}(\alpha, t)))$$

(21)

and

$$\tilde{c}(\alpha, t) = \frac{\alpha + \pi}{2\pi} \int_{\pi}^{\alpha} \frac{\partial_\beta \tilde{z}(\beta, t)}{|\partial_\beta \tilde{z}(\beta, t)|^2} \cdot \partial_\beta BR(\tilde{z}, \tilde{\omega})(\beta, t)d\beta$$

$$- \int_{-\pi}^{\alpha} \frac{\partial_\beta \tilde{z}(\beta, t)}{|\partial_\beta \tilde{z}(\beta, t)|^2} \cdot \partial_\beta BR(\tilde{z}, \tilde{\omega})(\beta, t)d\beta$$

(22)

with $\tilde{z} \in C([0, T], H^k)$ for $k \geq 4$. 

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6.1 Instant analyticity in $\tilde{\Omega}$ domain

We define

$$q^0 = (0, 0), \quad q^1 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad q^2 = \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right),$$

$$q^3 = \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \quad q^4 = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

which are the singular points of the $P^{-1}$ conformal map. We set $\tilde{z}(\alpha, t)$ to hold $\tilde{z}(\alpha, t) \neq q^l$ for $l = 0, 1, 2, 3, 4$. In order to get this we fix $\Omega(0)$ so that $\frac{dP}{dw}(w) \neq 0$ for any $w \in \Omega(0)$ without loss of generality.

We define the energy

$$\|\tilde{z}\|_{RT} \equiv \|\tilde{z}\|_{H^k(S)}^2 + \left\| \mathcal{F}(\tilde{z}) \right\|_{L^\infty(S)}^2 + \frac{1}{m(Q^2 \tilde{\sigma})(t) - 2\lambda - \|g\|(t)} + \sum_{l=0}^4 \frac{1}{m(q^l)(t)}$$

where

$$\|g\|(t) = C(\|\mathcal{J}(\partial_\alpha \tilde{z})\mathcal{R}(\partial_\alpha \tilde{\nu})Q^2 A^2(t)\|_{H^2(S)} + \|\mathcal{J}(\partial_\alpha \tilde{z})\mathcal{J}(\partial_\alpha \tilde{z})\tilde{\nu} Q^2 A^2(t)\|_{H^2(S)}) + \|\mathcal{J}(\tilde{\nu} Q^2 A^2(t))\|_{H^2(S)} + \|\mathcal{J}(\tilde{\nu})\|_{H^2(S)}$$

and

$$m(Q^2 \tilde{\sigma})(t) = \min_{\alpha} Q^2(\alpha, t)\sigma(\alpha, t), \quad m(q^l)(t) = \min_{\alpha} |\tilde{z}(\alpha, t) - q^l|.$$ 

**Theorem 6.1.** Let $\tilde{z}(\alpha, t)$ be a solution of (19-22). Then, the following estimate holds:

$$\frac{d}{dt} \|\tilde{z}\|_{RT} \leq \exp C(\|\tilde{z}\|_{RT})$$

for $C$ constant.

**Remark 6.1.** We will show the proof for $k = 4$, being the rest of the cases analogous.

**Proof.** We have to estimate

$$\frac{d}{dt} \left\| \partial_\alpha^4 \tilde{z} \right\|_{L^2(S)}^2.$$ 

We quote [3] for dealing with the $Q^2$ term. This factor do not introduce a high order term

$$\|Q^2\|_{H^k(S)} \leq \exp C(\|\tilde{z}\|_{RT}).$$

Then we have to repeat all estimates in section [2] in which $Q^2$ is involved.

We will show below how to deal with them.
We find

\[
\frac{d}{dt} \| \partial_\alpha \tilde{z} \|^2_{L^2(S)} \leq \exp \left( C \left( \| \mathcal{F}(\tilde{z}) \|^2_{L^\infty(S)} + \| \tilde{z} \|^2_{H^4(S)} \right) + 2\lambda \| \Lambda^\frac{1}{2} \partial_\alpha \tilde{z} \|^2_{L^2(S)} \right) + J_1 + J_2
\]

where

\[
J_1 = \Re \int \partial_\alpha^4 \tilde{z}(\gamma) \cdot \partial_\alpha^4 (Q^2(\gamma)BR(\tilde{z}, \tilde{w})(\gamma)) d\alpha,
\]

\[
J_2 = \Re \int \partial_\alpha^4 \tilde{z}(\gamma) \cdot \partial_\alpha^4 (\tilde{c}(\gamma) \partial_\alpha \tilde{z}(\gamma)) d\alpha.
\]

We get

\[
J_1 \leq \exp \left( C \left( \| \mathcal{F}(\tilde{z}) \|^2_{L^\infty(S)} + \| \tilde{z} \|^2_{H^4(S)} \right) \right) + I_7
\]

where

\[
I_7 = \Re \int \partial_\alpha^4 \tilde{z}(\gamma) \cdot Q^2(\gamma) \partial_\alpha \tilde{z} BR(\tilde{z}, \tilde{w})(\gamma) \, d\alpha.
\]

As in [2.2.1] we split \( I_7 = \tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5 + \tilde{I}_6 + \tilde{I}_7 \) in the same way we have

\[
\tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5 + \tilde{I}_6 \leq \exp \left( C \left( \| \mathcal{F}(\tilde{z}) \|^2_{L^\infty(S)} + \| \tilde{z} \|^2_{H^4(S)} \right) \right) + C \| \mathcal{F} \| \exp \left( \| \partial_\alpha \tilde{z} \|^2_{L^2(S)} \right)
\]

and \( \tilde{I}_7 \leq \exp \left( \| \mathcal{F}(\tilde{z}) \|^2_{L^\infty(S)} + \| \tilde{z} \|^2_{H^4(S)} \right) + \tilde{K}_9 \) being

\[
\tilde{K}_9 = \frac{1}{2} \Re \int \partial_\alpha^4 \tilde{z}(\gamma) \cdot \partial_\alpha^4 \tilde{z}(\gamma) \, d\alpha.
\]

Identity \( H(\partial_\alpha) = \Lambda \) allows us to rewrite \( \tilde{K}_9 \) as follows

\[
\tilde{K}_8 = \frac{1}{2} \Re \int \Lambda (\partial_\alpha^4 \tilde{z}) \cdot \partial_\alpha^4 \tilde{z} \, d\alpha.
\]

Using the formula [21], we decompose \( \tilde{K}_9 = \tilde{P}_7 + \tilde{P}_8 \)

\[
\tilde{P}_7 = -\kappa g \rho^2 \Re \int \frac{\Lambda (\partial_\alpha^4 \tilde{z}) \cdot \partial_\alpha^4 (P_2^{-1}(\tilde{z}(\gamma)))}{A(t)} \, d\alpha
\]

\[
\tilde{P}_8 = -\frac{1}{2} \Re \int \frac{\Lambda (\partial_\alpha^4 \tilde{z}) \cdot \partial_\alpha^4 (\tilde{z}(\gamma))}{A(t)} \, d\alpha
\]

where \( \tilde{T}(\tilde{w}) = -2BR(\tilde{z}, \tilde{w}) \cdot \partial_\alpha \tilde{z} \).

The term \( \tilde{P}_8 \) can be estimate as the term \( P_8 \) in subsection [2.2.2.]. An analogous approach provides

\[
\tilde{P}_8 \leq \exp \left( \| \mathcal{F}(\tilde{z}) \|^2_{L^\infty(S)} + \| \tilde{z} \|^2_{H^4(S)} \right)
- \Re \int \frac{Q^2(\gamma)BR(\tilde{z}, \tilde{w})(\gamma) \cdot \partial_\alpha^4 \tilde{z}(\gamma)}{A(t)} \partial_\alpha^4 \tilde{z}(\gamma) \cdot \Lambda^\frac{1}{2} (\partial_\alpha \tilde{z})(\gamma) \, d\alpha.
\]
For $\tilde{P}_t$ we consider the most singular terms: $\tilde{P}_t \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2) + \tilde{P}_t^1$ where

$$\tilde{P}_t^1 = -\kappa g \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial^4_{\alpha} \cdot Q^2 \partial^4_{\alpha} \tilde{z}(\gamma)) \nabla P_{2}^{-1}(\tilde{z}(\gamma)) \cdot \partial^4_{\alpha} \tilde{z}(\gamma) d\alpha.}$$

Then we split $\tilde{P}_t^1 = \tilde{P}_t^{11} + \tilde{P}_t^{12} + \tilde{P}_t^{13} + \tilde{P}_t^{14}$ by writing the component of the curve:

$$\tilde{P}_t^{11} = \kappa g \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial^4_{\alpha} \tilde{z}_1 Q^2 \partial^4_{\alpha} \tilde{z}_2(\gamma)) \partial_{\xi_1} P_{2}^{-1}(\tilde{z}(\gamma)) \partial^4_{\alpha} \tilde{z}_1(\gamma) d\alpha,}$$

$$\tilde{P}_t^{12} = \kappa g \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial^4_{\alpha} \tilde{z}_2 Q^2 \partial^4_{\alpha} \tilde{z}_1(\gamma)) \partial_{\xi_2} P_{2}^{-1}(\tilde{z}(\gamma)) \partial^4_{\alpha} \tilde{z}_2(\gamma) d\alpha,}$$

$$\tilde{P}_t^{13} = -\kappa g \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial^4_{\alpha} \tilde{z}_2 Q^2 \partial^4_{\alpha} \tilde{z}_1(\gamma)) \partial_{\xi_1} P_{2}^{-1}(\tilde{z}(\gamma)) \partial^4_{\alpha} \tilde{z}_1(\gamma) d\alpha,}$$

$$\tilde{P}_t^{14} = -\kappa g \frac{\rho^2}{\mu^2} \Re \int \frac{\Lambda(\partial^4_{\alpha} \tilde{z}_2 Q^2 \partial^4_{\alpha} \tilde{z}_1(\gamma)) \partial_{\xi_2} P_{2}^{-1}(\tilde{z}(\gamma)) \partial^4_{\alpha} \tilde{z}_2(\gamma) d\alpha.}$$

The commutator estimate yields

$$\tilde{P}_t^{11} \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2)$$

$$\tilde{P}_t^{12} \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2)$$

$$\tilde{P}_t^{13} \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2)$$

$$\tilde{P}_t^{14} \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2)$$

Using that

$$\partial_{\alpha} \tilde{z}_2 \partial^4_{\alpha} \tilde{z}_2 = -3 \partial^2_{\alpha} \tilde{z} \cdot \partial^4_{\alpha} \tilde{z} - \partial_{\alpha} \tilde{z}_1 \partial^4_{\alpha} \tilde{z}_1,$$
we get
\[ \dot{P}_1^{12} \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2) \]
\[ - \frac{\kappa}{\mu^2} \text{Re} \int \frac{Q^2(\gamma) \partial_z^2 P_2^{-1}(\tilde{z}(\gamma))}{A(t)} \partial_\alpha \tilde{z}_1(\gamma) \partial_\alpha^4 \tilde{z}_1(\gamma) \Lambda(\partial_\alpha^3 \tilde{z}_1(\gamma)) d\alpha, \quad (26) \]
\[ \dot{P}_1^{13} \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2) \]
\[ + \frac{\kappa}{\mu^2} \text{Re} \int \frac{Q^2(\gamma) \partial_z^2 P_2^{-1}(\tilde{z}(\gamma))}{A(t)} \partial_\alpha \tilde{z}_2(\gamma) \partial_\alpha^4 \tilde{z}_2(\gamma) \Lambda(\partial_\alpha^3 \tilde{z}_2(\gamma)) d\alpha. \quad (27) \]

Adding the inequalities (24), (26), (27), and (25) it is easy to check
\[ \dot{P}_7 \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2) \]
\[ - \frac{\kappa}{\mu^2} \text{Re} \int \frac{Q^2(\gamma) \nabla P_2^{-1}(\tilde{z}(\gamma))}{A(t)} \cdot \partial_\alpha \tilde{z}_1(\gamma) \partial_\alpha^4 \tilde{z}(\gamma) \Lambda(\partial_\alpha^3 \tilde{z}(\gamma)) d\alpha. \]

Above inequality together with (23) let us obtain
\[ \dot{K}_9 \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2) \]
\[ - \text{Re} \int \frac{Q^2(\gamma) \tilde{\sigma}(\gamma)}{A(t)} \partial_\alpha^4 \tilde{z}(\gamma) \Lambda(\partial_\alpha^3 \tilde{z}(\gamma)) d\alpha \]

with \( \tilde{\sigma} \) given in (3).

Considering \( m(Q^2 \tilde{\sigma})(t) \) and the pointwise inequality \( 2f \Lambda(f) \geq \Lambda(f^2) \) we check
\[ \tilde{I}_7 \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2) - m(Q^2 \tilde{\sigma})(t) \|\partial_\alpha^4 \tilde{z}\|_{L^2(S)}^2. \]

For \( J_2 \) it is easy to deal with \( \partial_\alpha^4 \tilde{z} \) in the same way as in section 2.2.3. The analogous approach provides
\[ J_2 \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2) \]
\[ + C(\|J(\partial_\alpha \tilde{z}) \text{Re} (\partial_\alpha^2 \tilde{z}) Q^2 A(t)^{-1}(t))\|H^2(S) + \|J(\partial_\alpha \tilde{z}) \text{Re} (\partial_\alpha^2 \tilde{z}) Q^2 A(t)^{-1}(t))\|H^2(S) \]
\[ + \|J(\tilde{c})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}} \partial_\alpha^4 \tilde{z}\|_{L^2(S)}^2. \]

Finally we obtain,
\[ \frac{d}{dt}\|\partial_\alpha^4 \tilde{z}\|_{L^2(S)}^2 \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2) \]
\[ + (2\lambda + \|g\| - m(Q^2 \tilde{\sigma}))\|\Lambda^{\frac{1}{2}} \partial_\alpha^4 \tilde{z}\|_{L^2(S)}^2. \]

Bearing in mind the singular points of the \( P^{-1} \) together with the estimation for \( m(Q^2 \tilde{\sigma})(t) \), which we can obtain in analogous way as in section 3, we have the desired estimate.
6.2 Decay of the strip of analyticity in the $\tilde{\Omega}$ domain

**Theorem 6.2.** Let $\tilde{z}(\alpha, 0) = \tilde{z}^0(\alpha)$ be an analytic curve in the strip $S = \{\alpha + i\varsigma \in \mathbb{C} : |\varsigma| < h(0)\},$

with $h(0) > 0$ and satisfying:

* The arc-chord condition, $\mathcal{F}(\tilde{z}^0)(\alpha + i\varsigma, \beta) \in L^\infty(S \times \mathbb{R})$

* The curve $\tilde{z}^0(\alpha)$ is real for real $\alpha$

* The functions $\tilde{z}_1^0(\alpha) - \alpha$ and $\tilde{z}_2^0(\alpha)$ are periodic with period $2\pi$

* The functions $\tilde{z}_1^0(\alpha) - \alpha$ and $\tilde{z}_2^0(\alpha)$ belong to $H^4(S)$

Then there exist a time $T$ and a solution of the Muskat problem in $\tilde{\Omega}$, $\tilde{z}(\alpha, t)$ defined for $0 < t \leq T$ that continues analytically into some complex strip for each fixed $t \in [0, T]$. Here $T$ is either a small constant depending only on $\exp C(\|\mathcal{F}(\tilde{z}^0)\|_{L^\infty(S)}^2 + \|\tilde{z}^0\|_{H^4(S)}^2)$.

**Proof.** Here we proceed in the same way that in the proof of the theorem 5.1.

After we use the Galerkin approximation, by Lemma 5.1 we get

$$\frac{d}{dt} \sum_\pm \int |\partial_\alpha^4 \tilde{z}_j(\alpha \pm ih(t))|^2 d\alpha \leq \frac{h'(t)}{10} \sum_\pm \int \Lambda(\partial_\alpha^4 \tilde{z}_j(\alpha \pm ih(t)))\partial_\alpha^4 \tilde{z}_j(\alpha \pm ih(t))$$

$$- 10h'(t) \sum_\pm \int \Lambda(\partial_\alpha^4 \tilde{z}_j(\alpha))\partial_\alpha^4 \tilde{z}_j(\alpha) d\alpha + 2 \sum_\pm \Re \int \partial_\alpha^4 Q^2 BR(\tilde{z}_j(\alpha \pm ih(t)))\partial_\alpha^4 \tilde{z}_j(\alpha \pm ih(t))$$

$$+ 2 \sum_\pm \Re \int \partial_\alpha^4 (\tilde{c}(\alpha \pm ih(t)) \partial_\alpha \tilde{z}_j(\alpha \pm ih(t)))\partial_\alpha^4 \tilde{z}_j(\alpha \pm ih(t)).$$

We write,

$$Q^2(\gamma) \tilde{\sigma}(\gamma) = Q^2(\alpha) \tilde{\sigma}(\alpha) + h(t) g_\pm(\alpha)$$

and we have

$$\frac{d}{dt} \sum_\pm \int |\partial_\alpha^4 \tilde{z}_j(\alpha \pm ih(t))|^2 d\alpha \leq \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2)$$

$$- 10h'(t) \int \Lambda(\partial_\alpha^4 \tilde{z}_j(\alpha))\partial_\alpha^4 \tilde{z}_j(\alpha) d\alpha$$

$$+ (\exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2)h(t) + \frac{h'(t)}{10} + \exp C(\|\mathcal{F}(\tilde{z})\|_{L^\infty(S)}^2 + \|\tilde{z}\|_{H^4(S)}^2))\|\tilde{z}\|_{L^2(S)}^2.$$
6.3 Proof of the main theorem

Suppose that there exists a time $T$ where we have a splat singularity, i.e., the smooth interface collapses along an arc at time $T$.

Initially the curve is real, by theorem 4.1, it instantly becomes analytic. From theorem 5.1, our strip of analyticity is nonzero as long as the regularity of the curve and the arc-chord condition not fail. In $\Omega$ domain, at splat time $T$, the arc-chord condition blow-up so we can not guarantee analyticity at that time.

At this point, we transform the system to the tilde domain $\tilde{\Omega}$.

From theorem 6.1, we can conclude that there exists a solution of the problem $\tilde{z}(\alpha, t)$ defined for $0 < t \leq \tilde{T}$ that continue analytically into the strip $S(t) = \{ \alpha \pm it : |\varsigma| < \lambda t \}$ for each $t$. And, for theorem 6.2, this complex strip decays exponentially to a time that depends on the regularity of the curve and the arc-chord condition.

If we assume that there exists a time $T \in [0, \tilde{T}]$ such that there has a splat singularity in $\tilde{\Omega}$, since in this domain the arc-chord condition and the regularity of the curve are bounded, the strip of analyticity is nonzero and therefore we have analyticity guarantee at the time $T$.

Thus, applying $P^{-1}$, we have the analytic curve self-intersects along an arc, therefore we get a contradiction and the theorem 1.1 is proved.

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