Remark on the Helmholtz decomposition in domains with noncompact boundary

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Abstract

Let Ω be a domain with noncompact boundary. It is known that the Helmholtz decomposition is not always valid in $L^p(\Omega)$ except for the energy space $L^2(\Omega)$. In this paper we consider a typical unbounded domain whose boundary is given as a Lipschitz graph, and show that the Helmholtz decomposition holds in certain anisotropic spaces which include some infinite energy vector fields.

Keywords: Helmholtz decomposition, weak Neumann problem, noncompact boundary, factorization of divergence form elliptic operators.

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1 Introduction

Let $\Omega \subset \mathbb{R}^{d+1}$ ($d \geq 1$) be a domain with Lipschitz boundary. The Helmholtz decomposition, the decomposition of a given vector field into a solenoidal field and a potential one is the fundamental tool in the mathematical analysis of the incompressible flow. In the energy space $(L^2(\Omega))^{d+1}$ this decomposition is easily derived for any domain $\Omega$ from the standard theory of the Hilbert space. On the other hand, if the space $(L^2(\Omega))^{d+1}$ is replaced by other function spaces such as $(L^q(\Omega))^{d+1}$, then the verification of the Helmholtz decomposition requires detailed analysis in general. In the case when $\Omega$ is a bounded domain or an exterior domain with smooth boundaries, the validity of the decomposition in $(L^q(\Omega))^{d+1}$, $1 < q < \infty$, is shown by [13] and [24] respectively, and then their results are extended to these domains but with $C^1$-boundary by [28]. Moreover, for the bounded Lipschitz domains, the validity is proved around $3/2 < q < 3$ in [8], and for any $1 < q < \infty$ by [14] when the domain is convex. However, even if the boundary is smooth enough, the problem becomes subtle when the boundary is noncompact. Although the decomposition is still valid for $1 < q < \infty$ for some special cases, e.g., aperture domains [9], layers [25], cylinders [29], half spaces and their
small perturbations \cite{28}, it is known that the domain of simple form
\[
\Omega = \left\{ \hat{x} = (x, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \mid x_{d+1} > \eta(x) \right\},
\]  (1.1)
with a given function \( \eta \) does not always admit the Helmholtz decomposition in \((L^q(\Omega))^{d+1}\) if \( q \neq 2 \), even if \( \eta \) is smooth, see \cite{6} and \cite{16}, III.1. Hence it is an important question to ask which function space, other than \((L^2(\Omega))^{d+1}\), admits the Helmholtz decomposition. In \cite{10, 11}, the authors considered \( \tilde{Y}^{q,r}(\Omega) \), with the result in \cite{10}, our function space \( \eta \), with \( q \), \( r \), \( q/r \), \( \eta \), and \( \tilde{Y}^{q,r}(\Omega) \) does not always admit the Helmholtz decomposition in \((L^q(\Omega))^{d+1}\). However, due to the well-known counterexample of the weak Neumann problem in the exterior of the cone-like domain \cite{6}, one cannot expect the validity of the Helmholtz decomposition in the usual \( L^q \) space. On the other hand, compared with the result in \cite{10}, our function space \( Y^{q,2}(\Omega) \) includes a class of functions decaying slowly in the \( x_{d+1} \) direction since Theorem \cite{1.2} allows any \( q \in (2, \infty) \).

**Definition 1.1.** We say that the space \( (X(\Omega))^{d+1} \) admits the Helmholtz decomposition if each \( f \in (X(\Omega))^{d+1} \) has a unique decomposition \( f = u + \nabla p \), \( u \in X_\sigma(\Omega) \), \( \nabla p \in X_G(\Omega) \), satisfying
\[
\|u\|_{X(\Omega)} + \|\nabla p\|_{X(\Omega)} \leq C \|f\|_{X(\Omega)}.
\]  (1.3)
Here \( C \) is a positive constant independent of \( f \).

In order to consider the domain \( \Omega \) of the form \cite{1.1} we define the standard isomorphism \( \Phi : \Omega \ni \hat{x} \mapsto \tilde{y} = \Phi(\hat{x}) \in \mathbb{R}^{d+1} \) by
\[
\Phi_j(\hat{x}) = \left\{ \begin{array}{ll}
x_j & \text{if } 1 \leq j \leq d, \\
x_{d+1} - \eta(x) & \text{if } j = d+1.
\end{array} \right.
\]  (1.4)
Let \( 1 < q, r < \infty \) and let \( Y^{q,r}(\Omega) \) be the Banach space defined by
\[
Y^{q,r}(\Omega) = \left\{ f \in L_{loc}^1(\Omega) \mid \|f\|_{Y^{q,r}(\Omega)} = \|f \circ \Phi^{-1}\|_{L^q_{loc}(\mathbb{R}^d) ; L^r_{loc}(\mathbb{R}^d)} < \infty \right\}
\]  (1.5)
with the norm \( \| \cdot \|_{Y^{q,r}(\Omega)} \). Here we have used the notation \( \tilde{y} = (y, t) \in \mathbb{R}^d \times \mathbb{R}_+ \). Our main result reads as follows:

**Theorem 1.2.** Let \( \Omega \) be a domain of the form \cite{1.1} with uniform Lipschitz boundary. Then the space \((Y^{q,2}(\Omega))^{d+1}\) admits the Helmholtz decomposition for all \( 1 < q < \infty \). Moreover, the constant \( C \) in \cite{1.3} depends only on \( d \), \( q \), and \( \|\nabla \eta\|_{L^\infty(\mathbb{R}^d)} \).

**Remark 1.3.** By the definition it is easy to see that the dual space of \( Y^{q,r}(\Omega) \), denoted by \( Y^{q,r}(\Omega)^* \), is the space \( Y^{q',r'}(\Omega) \) with \( q' = q/(q-1) \) and \( r' = r/(r-1) \). The space \((Y^{q,2}(\Omega))^{d+1}\) coincides with \((L^q(\Omega))^{d+1}\). However, due to the well-known counterexample of the weak Neumann problem in the exterior of the cone-like domain \cite{6}, one cannot expect the validity of the Helmholtz decomposition in the usual \( L^q \) space. On the other hand, compared with the result in \cite{10}, our function space \( Y^{q,2}(\Omega) \) includes a class of functions decaying slowly in the \( x_{d+1} \) direction since Theorem \cite{1.2} allows any \( q \in (2, \infty) \).
As is well-known, the verification of the Helmholtz decomposition is reduced to the unique solvability of the weak Neumann problem in $(Y^{q,2}(\Omega))^{d+1}$:

$$\langle \nabla p, \nabla \varphi \rangle_{L^2(\Omega)} = \langle f, \nabla \varphi \rangle_{L^2(\Omega)} \quad \text{for all } \varphi \in \{ f \in L^1_{loc}(\Omega) \mid \nabla f \in (Y^{q^*,2}(\Omega))^{d+1} \}, \quad (1.6)$$

which is a weak formulation of

$$\Delta p = \nabla \cdot f \quad \text{in } \Omega, \quad n \cdot \nabla p = n \cdot f \quad \text{on } \partial \Omega. \quad (1.7)$$

Here $n$ stands for the exterior unit normal to $\partial \Omega$. Through the isomorphism $\Phi$ defined by (1.4), the problem (1.6) or (1.7) is transformed to the Neumann problem for an elliptic partial differential equation of the divergence form in $\mathbb{R}^{d+1}$. Then our task is to look for the solution of the transformed problem in $L^q(\mathbb{R}_+; L^2(\mathbb{R}^d))$. For the purpose, we will make use of an approach proposed in the companion work [22], where we gave a solution formula for the boundary value problem to divergence form elliptic equations in $\mathbb{R}^{d+1}$ in terms of the Poisson semigroups; see Theorem 2.3 for details. This solution formula combined with the semigroup theory yields a sufficient condition for function spaces to ensure the solvability of the Neumann problem, and it will be verified that $Y^{q,2}$ satisfies this condition.

Before concluding the introduction, we would like to point out the difference of our approach from previous works on the Neumann problem in the domain with regular (e.g. uniformly $C^1$) boundary. In [28, 10, 15], they employed localization procedure for the Neumann problem to reduce the problem to a countable number of the Neumann problems in $\mathbb{R}^{d+1}_+$. Then thanks to the regularity assumption of the boundary, each problem can be dealt with as a small perturbation of the Neumann problem for the Poisson equation. On the other hand, our approach is not relied on this perturbation technique, and therefore one can handle even large perturbation of the Neumann problem for the Poisson equation. On the other hand, our approach is not relied on this perturbation technique, and therefore one can handle even large perturbation of the Neumann problem for the Poisson equation in $\mathbb{R}^{d+1}_+$ with respect to the Lipschitz norm of $\eta$. Instead, we need to use the $L^2$ space in the $x$ direction.

In the next section we will recall the solution formula for the Neumann problem, which is based on the factorization of the elliptic operators in [22]. Then we will prove the main theorem in Section 3.

## 2 Solution formula for the Neumann problem in $\mathbb{R}^{d+1}_+$

Consider the second order elliptic operator of divergence form in $\mathbb{R}^{d+1}_+ = \{ (x,t) \in \mathbb{R}^d \times \mathbb{R}, \}$

$$\mathcal{A} = -\nabla \cdot A \nabla, \quad A = A(x) = (a_{i,j}(x))_{1 \leq i,j \leq d+1}. \quad (2.1)$$

Here $d \in \mathbb{N}$, $\nabla = (\nabla_x, \partial_t)^\top$ with $\nabla_x = (\partial_1, \cdots, \partial_d)^\top$, and each $a_{i,j}$ is always assumed to be $t$-independent. We further assume that $A$ is a real symmetric matrix and each component $a_{i,j}$ is a measurable function satisfying the uniformly elliptic condition

$$\langle A(x)\eta, \eta \rangle \geq \nu_1 |\eta|^2, \quad |\langle A(x)\eta, \zeta \rangle| \leq \nu_2 |\eta||\zeta| \quad (2.2)$$

for all $\eta, \zeta \in \mathbb{R}^{d+1}$ and for some constants $\nu_1, \nu_2$ with $0 < \nu_1 \leq \nu_2 < \infty$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{R}^{d+1}$, i.e., $\langle \eta, \zeta \rangle = \sum_{j=1}^{d+1} \eta_j \zeta_j$ for $\eta, \zeta \in \mathbb{R}^{d+1}$. For later use we set $b = a_{d+1,d+1}$, which satisfies $\nu_1 \leq b \leq \nu_2$ due to (2.2). We also denote by $\mathbf{a}(x)$ the vector $\mathbf{a}(x) = (a_{1,d+1}(x), \cdots, a_{d,d+1}(x))^\top$. 

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We denote by $D_H(T)$ the domain of a linear operator $T$ in a Banach space $H$. Under the condition (2.2) the standard theory of sesquilinear forms gives a realization of $\mathcal{A}$ in $L^2(\mathbb{R}^{d+1})$, denoted again by $\mathcal{A}$, such as

$$D_{L^2}(\mathcal{A}) = \{ w \in H^1(\mathbb{R}^{d+1}) \mid \text{there is } F \in L^2(\mathbb{R}^{d+1}) \text{ such that } \langle A \nabla w, \nabla v \rangle_{L^2(\mathbb{R}^{d+1})} = \langle F, v \rangle_{L^2(\mathbb{R}^{d+1})} \text{ for all } v \in H^1(\mathbb{R}^{d+1}) \},$$

and $Aw = F$ for $w \in D_{L^2}(\mathcal{A})$. Here $H^1(\mathbb{R}^{d+1})$ is the usual Sobolev space and $\langle v, w \rangle_{L^2(\mathbb{R}^{d+1})} = \int_{\mathbb{R}^{d+1}} w(x, t)v(x, t) \, dx \, dt$.

**Definition 2.1.** (i) For a given $h \in S'(\mathbb{R}^d)$ we denote by $M_h : S(\mathbb{R}^d) \rightarrow S'(\mathbb{R}^d)$ the multiplication $M_hu = hu$.

(ii) We denote by $E_\mathcal{A} : H^{1/2}(\mathbb{R}^d) \rightarrow \dot{H}^{1/2}(\mathbb{R}^{d+1})$ the $\mathcal{A}$-extension operator, i.e., $w = E_\mathcal{A}\varphi$ is the solution to the Dirichlet problem

$$\begin{aligned}
Aw &= 0 \quad \text{in } \mathbb{R}^{d+1}, \\
\varphi &= \varphi \quad \text{on } \partial \mathbb{R}_+^d = \mathbb{R}^d.
\end{aligned}$$

The one parameter family of linear operators $\{E_\mathcal{A}(t)\}_{t \geq 0}$, defined by $E_\mathcal{A}(t)\varphi = (E_\mathcal{A}\varphi)(\cdot, t)$ for $\varphi \in H^{1/2}(\mathbb{R}^d)$, is called the Poisson semigroup associated with $\mathcal{A}$.

(iii) We denote by $\Lambda_\mathcal{A} : H^{1/2}(\mathbb{R}^d) \rightarrow \dot{H}^{-1/2}(\mathbb{R}^d) = (\dot{H}^{1/2}(\mathbb{R}^d))^*$ the Dirichlet-Neumann map associated with $\mathcal{A}$, which is defined through the sesquilinear form

$$\langle \Lambda_\mathcal{A}\varphi, g \rangle_{\dot{H}^{-1/2}, \dot{H}^{1/2}} = \langle A\nabla E_\mathcal{A}\varphi, \nabla E_\mathcal{A}g \rangle_{L^2(\mathbb{R}_+^{d+1})}, \quad \varphi, g \in H^{1/2}(\mathbb{R}^d).$$

Here $\langle \cdot, \cdot \rangle_{\dot{H}^{-1/2}, \dot{H}^{1/2}}$ denotes the duality coupling of $\dot{H}^{-1/2}(\mathbb{R}^d)$ and $\dot{H}^{1/2}(\mathbb{R}^d)$.

**Remark 2.2.** (i) As usual, Eq. (2.4) is considered in a weak sense; cf. [22, Section 2.1]. The proof of the existence of the extension operator $E_\mathcal{A}$ is classical, and indeed it is a consequence of the Riesz representation theorem together with the harmonic extension of the function in $H^{1/2}(\mathbb{R}^d)$. As is shown in [22, Proposition 2.4], $\{E_\mathcal{A}(t)\}_{t \geq 0}$ is a strongly continuous and analytic semigroup in $H^{1/2}(\mathbb{R}^d)$. We denote its generator by $-P_\mathcal{A}$, and $P_\mathcal{A}$ is called a *Poisson operator* associated with $\mathcal{A}$. (ii) Since $A$ is Hermite and satisfies the uniformly elliptic condition (2.2), the theory of the sesquilinear forms [20, Chapter VI. §2] shows that $\Lambda_\mathcal{A}$ is extended as a self-adjoint operator in $L^2(\mathbb{R}^d)$.

The following result plays a fundamental role in the derivation of the solution formula for the Neumann problem.

**Theorem 2.3** ([22, Theorem 1.3, Theorem 4.2]). Let $\mathcal{A}$ be the elliptic operator defined in (2.1) with a real symmetric matrix $A$ satisfying (2.2). Then $D_{L^2}(\Lambda_\mathcal{A}) = H^1(\mathbb{R}^d)$ with equivalent norms and the operator $-P_\mathcal{A}$ defined by

$$D_{L^2}(\mathcal{P}_\mathcal{A}) = H^1(\mathbb{R}^d), \quad -P_\mathcal{A}\varphi = -M_{1/b}\Lambda_\mathcal{A}\varphi - M_{a/b} \cdot \nabla_x \varphi,$$

generates a strongly continuous and bounded analytic semigroup in $L^2(\mathbb{R}^d)$. Moreover, the realization $\mathcal{A}'$ in $L^2(\mathbb{R}^d)$ and the realization $\mathcal{A}$ in $L^2(\mathbb{R}^{d+1})$ are respectively factorized as

$$\mathcal{A}' = M_b Q_\mathcal{A} P_\mathcal{A}, \quad Q_\mathcal{A} = M_{1/b}(M_b P_\mathcal{A})^*, \quad \mathcal{A} = -M_b(\partial_t - \mathcal{Q}_\mathcal{A})(\partial_t + \mathcal{P}_\mathcal{A}).$$

Here $(M_b P_\mathcal{A})^*$ is the adjoint of $M_b P_\mathcal{A}$ in $L^2(\mathbb{R}^d)$. 


**Remark 2.4.** The operator $P_A$ is nothing but the Poisson operator $P_A$ associated with $A$. That is, $-P_A \varphi = -P_A \varphi := \lim_{t \downarrow 0} t^{-1} (E_A(t) \varphi - \varphi)$ in $L^2(\mathbb{R}^d)$ for $\varphi \in H^1(\mathbb{R}^d)$.

Now we consider the inhomogeneous Neumann problem

\[
\begin{cases}
Aw = F & \text{in } \mathbb{R}^{d+1}_+,
-(e_{d+1}, A \nabla w) = g & \text{on } \partial \mathbb{R}^{d+1}.
\end{cases}
\] (2.9)

By a direct application of the factorization (2.8), one can easily derive the formal representation of the solution to the Neumann problem (2.9) as follows.

**Theorem 2.5** ([22] Theorem 5.1). Assume that $F, \partial_t F \in \dot{H}^{-1}(\mathbb{R}^{d+1}_+)$, $g \in H^{1/2}(\mathbb{R}^d)$. Assume further that $h = g + M_b \int_0^\infty e^{-s}Q \Lambda M_{1/b}F(s) \, ds$ belongs to the range of $\Lambda_A$ in $L^2(\mathbb{R}^d)$. Let $w \in C([0, \infty); L^2(\mathbb{R}^d)) \cap \dot{H}^1(\mathbb{R}^{d+1}_+)$ be a weak solution to (2.9). Then $w$ is a mild solution, i.e.,

\[
\begin{align*}
&\int_0^t e^{-(t-s)\Lambda_A} M_{1/b} F(s) \, ds + \int_0^t e^{-(t-s)\Lambda_A} M_{1/b} F(\tau) \, d\tau.
\end{align*}
\] (2.10)

We note that $e^{-tQ_A}$ is related with $e^{-tP_A}$ through the formula

\[
e^{-tQ_A} = M_{1/b} (e^{-tP_A})^* M_b.
\] (2.11)

Then the representation (2.10) reduces the inhomogeneous problem (2.9) to the analysis of the semigroup $\{e^{-tP_A}\}_{t \geq 0}$ and the operator $\Lambda_A$.

## 3 Helmholtz decomposition in $(Y^{q,r}(\Omega))^{d+1}$

As stated in the introduction, the Helmholtz decomposition for a given vector field is reduced to the Neumann problem (1.7). Let $\Phi : \Omega \rightarrow \mathbb{R}^{d+1}_+$ be the isomorphism defined by (1.4). By taking the push-forward

\[
w = p \circ \Phi^{-1}, \quad F = (F', F_{d+1}) = f \circ \Phi^{-1},
\] (3.1)

the problem (1.7) is transformed to the Neumann problem in $\mathbb{R}^{d+1}_+$:

\[
\begin{cases}
Aw = -\nabla_x \cdot F' - \partial_t (F_{d+1} + M_\alpha \cdot F') & \text{in } \mathbb{R}^{d+1}_+,
-(e_{d+1}, A \nabla w) = -(F_{d+1} + M_\alpha \cdot F') & \text{on } \partial \mathbb{R}^{d+1}.
\end{cases}
\] (3.2)

Here the matrix $A$ in this case is real symmetric and positive definite with $\alpha = -\nabla_x \eta$, $b = 1 + \|\nabla_x \eta\|^2$, and $A' = (a_{i,j})_{1 \leq i,j \leq d} = I'$ (the identity matrix). Let $1 < q, r < \infty$ and set

\[
Z^{q,r}(\mathbb{R}^{d+1}_+) : = W^{1,q}(\mathbb{R}^+_d; L^r(\mathbb{R}^d)) \cap L^q(\mathbb{R}^+_d; W^{1,r}(\mathbb{R}^d))
\]

\[
= \{ \phi \in L^1_{loc}(\mathbb{R}^{d+1}_+) \mid \partial_t \phi \in L^q(\mathbb{R}^+_d; L^r(\mathbb{R}^d)) \ 1 \leq i \leq d + 1 \}. \quad (3.3)
\]

Let $F \in (L^q(\mathbb{R}^+_d; L^r(\mathbb{R}^d)))^{d+1}$. The weak formulation of (3.2) is then to look for $w \in Z^{q,r}(\mathbb{R}^{d+1}_+)$ such that

\[
\langle A \nabla w, \nabla \phi \rangle_{L^2(\mathbb{R}^{d+1}_+)} = \langle F', \nabla_x \phi + M_\alpha \partial_t \phi \rangle_{L^2(\mathbb{R}^{d+1}_+)} + \langle F_{d+1}, \partial_t \phi \rangle_{L^2(\mathbb{R}^{d+1}_+)} \quad (3.4)
\]
for all $\phi \in Z^{d',x'}(\mathbb{R}^{d+1})$.

In the following paragraphs we abbreviate $\mathcal{P}_A$ ($\mathcal{Q}_A$, $\Lambda_A$) to $\mathcal{P}$ ($\mathcal{Q}$ and $\Lambda$ as well) for simplicity of the notation. The most important step in the analysis of (3.2) is to derive the estimate corresponding with (3.3), which is closely related to the spectral properties of $\mathcal{P}$ and $\Lambda$. To make the essence of our arguments clear, we will give in Section 3.1 natural sufficient conditions for the Helmholtz decomposition to hold in $(Y^{q,\gamma}(\Omega))^{d+1}$ in terms of the properties of $\mathcal{P}$ and $\Lambda$ in $L^r(\mathbb{R}^d)$. Roughly speaking, the following three conditions are required: Let $1 < m < \infty$ and set $m' = m/(m-1)$. (i) boundedness of the semigroups $\{e^{-t\mathcal{P}}\}_{t \geq 0}$, $\{e^{-t\Lambda}\}_{t \geq 0}$ in $L^r(\mathbb{R}^d)$, $r = m, m'$, (ii) coercive estimates for $\mathcal{P}$ and $\Lambda$ in $L^r(\mathbb{R}^d)$, $r = m, m'$, (iii) maximal regularity estimates for $\{e^{-t\mathcal{P}}\}_{t \geq 0}$ in $L^r(\mathbb{R}^d)$, $r = m, m'$. As long as $\nabla_x \eta$ is uniformly bounded, these conditions are shown to hold at least for $m = 2$, which leads to Theorem 1.2 see Section 3.2.

3.1 Sufficient condition for solvability of the Neumann problem

In this section we investigate the relation between the boundedness of the Helmholtz decomposition and the spectral properties of $\mathcal{P}$, $\Lambda$. Let $1 < m < \infty$ and $m' = m/(m-1)$. Let us recall that both $\mathcal{P}$ and $\Lambda$ generate strongly continuous and bounded analytic semigroups in $L^2(\mathbb{R}^d)$. To develop our argument within the general $L^r$ framework we first assume that

(i) The restrictions of $\{e^{-t\mathcal{P}}\}_{t \geq 0}$ and $\{e^{-t\Lambda}\}_{t \geq 0}$ on $L^2(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$ are extended as strongly continuous and bounded semigroups in $L^r(\mathbb{R}^d)$ with $r = m, m'$, i.e.,

$$
\|e^{-t\mathcal{P}}\varphi\|_{L^r(\mathbb{R}^d)} + \|e^{-t\Lambda}\varphi\|_{L^r(\mathbb{R}^d)} \leq C\|\varphi\|_{L^r(\mathbb{R}^d)}, \quad t > 0, \ \varphi \in L^r(\mathbb{R}^d).
$$

(3.5)

In fact, the statement in (i) is always verified at least for $\{e^{-t\Lambda}\}_{t \geq 0}$. While the behavior of the Poisson semigroup in $L^r(\mathbb{R}^d)$ seems to be more difficult to analyze, and the estimate (3.5) for $\{e^{-t\mathcal{P}}\}_{t \geq 0}$ can be obtained at least for the case $r \in [2, \infty)$. We will sketch their proofs in the appendix for reader’s convenience. In addition to (i) we assume in this section the following estimates (ii) - (iii):

(ii) Coercive estimates: Let $r = m, m'$. Then $D_{L^r}(\mathcal{P}) \cup D_{L^r}(\Lambda) \subset W^{1,r}(\mathbb{R}^d)$ and

$$
\|\nabla_x \varphi\|_{L^r(\mathbb{R}^d)} \leq C\|\mathcal{P}\varphi\|_{L^r(\mathbb{R}^d)}, \quad \varphi \in D_{L^r}(\mathcal{P}),
$$

(3.6)

$$
\|\nabla_x \varphi\|_{L^r(\mathbb{R}^d)} \leq C\|\Lambda\varphi\|_{L^r(\mathbb{R}^d)}, \quad \varphi \in D_{L^r}(\Lambda).
$$

(3.7)

(iii) Maximal regularity: Let $r = m, m'$. The function $\Psi_{\mathcal{P}}[\phi](t) = \int_0^t e^{-s\mathcal{P}}\phi(s) \, ds$ satisfies

$$
\|\mathcal{P}\Psi_{\mathcal{P}}[\phi]\|_{L^r(\mathbb{R}^d)} \leq C\|\phi\|_{L^r(\mathbb{R}^d)}, \quad \phi \in L^r(\mathbb{R}^{d+1}).
$$

(3.8)

Remark 3.1. As is well-known, (3.5) and (3.8) imply the analyticity of $\{e^{-t\mathcal{P}}\}_{t \geq 0}$ in $L^r(\mathbb{R}^d)$:

$$
\|t\mathcal{P} e^{-t\mathcal{P}}\varphi\|_{L^r(\mathbb{R}^d)} \leq C\|\varphi\|_{L^r(\mathbb{R}^d)}, \quad t > 0, \ \varphi \in L^r(\mathbb{R}^d).
$$

(3.9)

Remark 3.2. Set $e^{-t\mathcal{P}} = 0$ for $t < 0$ and define the operator $\Psi_{\mathcal{P}}$ by $\Psi_{\mathcal{P}}[\phi](t) = \int_0^t e^{-(t-s)\mathcal{P}}\phi(s) \, ds$. Then (3.8) implies the estimate

$$
\|\mathcal{P}\Psi_{\mathcal{P}}[\phi]\|_{L^r(\mathbb{R}^d)} \leq C\|\phi\|_{L^r(\mathbb{R}^d)}.
$$

(3.10)

From (3.9) and (3.10) the theory of singular integral operators [5] implies that

$$
\|\mathcal{P}\Psi_{\mathcal{P}}[\phi]\|_{L^q(\mathbb{R}^d)} \leq C_q\|\phi\|_{L^q(\mathbb{R}^d)}, \quad 1 < q < \infty.
$$

(3.11)
Remark 3.3. By the assumption (i), $\mathcal{P}$ and $\Lambda$ are sectorial in $L^r(\mathbb{R}^d)$ in the sense of [17, Chapter 2]. Thus the decompositions $L^r(\mathbb{R}^d) = \text{Ker}(\mathcal{P}) \oplus \text{Ran}(\mathcal{P})$ and $L^r(\mathbb{R}^d) = \text{Ker}(\Lambda) \oplus \text{Ran}(\Lambda)$ hold; see [17, Proposition 2.2.1] for the proof. Moreover, since the operators are injective by (i), we see $L^r(\mathbb{R}^d) = \text{Ran}(\mathcal{P}) = \text{Ran}(\Lambda)$ and the inverse operators $\mathcal{P}^{-1}, \Lambda^{-1}$ can be extended to bounded operators from $L^r(\mathbb{R}^d)$ to the homogeneous Sobolev space $\dot{W}^{1,r}(\mathbb{R}^d)$.

Remark 3.4. In Section 3.2 we will see that (ii) and (iii) are always satisfied at least for $r = 2$.

**Proposition 3.5.** Assume that (i) - (iii) hold. Let $F \in (C_0^\infty(\mathbb{R}^{d+1}))^{d+1}$. Then there exists a unique weak solution $w \in \dot{H}^1(\mathbb{R}^{d+1})$ to (3.2) satisfying

$$\|\nabla w\|_{L^q(\mathbb{R}^d; L^r(\mathbb{R}^d))} \leq C \|F\|_{L^q(\mathbb{R}^d; L^r(\mathbb{R}^d))}, \quad 1 < q < \infty, \quad \min\{m, m'\} \leq r \leq \max\{m, m'\},$$

(3.12)

Here $C$ depends only on $m, q, d, \|\nabla x\|_{L^\infty(\mathbb{R}^d)}$, and the constants in the estimates of (i) - (iii).

Note that it suffices to show (3.12) for $r = m, m'$ by the interpolation. We start from the following lemma.

**Lemma 3.6.** Assume that (i) - (iii) hold. Set $e^{-tQ} = 0$ for $t < 0$. Then $\tilde{\Psi}_Q[\phi](t) = \int_\mathbb{R} e^{-(t-s)Q}\phi(s) \, ds$ satisfies

$$\|\mathcal{Q}[\tilde{\Psi}_Q[\phi]]\|_{L^q(\mathbb{R}^d; L^r(\mathbb{R}^d))} \leq C_q \|\phi\|_{L^q(\mathbb{R}^d; L^r(\mathbb{R}^d))}, \quad 1 < q < \infty, \quad r = m, m'.$$

(3.13)

**Proof.** We appeal to the duality argument. Since $e^{-tQ} = M_{1/b}(e^{-t\mathcal{P}})^*M_b$, we have for any $\psi \in C_0^\infty(\mathbb{R}^{d+1})$,

$$\langle \mathcal{Q}[\tilde{\Psi}_Q[\phi]], \psi \rangle_{L^q(\mathbb{R}^d; L^r(\mathbb{R}^d))} = \int_\mathbb{R} \int_\mathbb{R} \langle Qe^{-(t-s)Q}\phi(s), \psi(t) \rangle_{L^2(\mathbb{R}^d)} \, ds \, dt$$

$$= \langle M_b\tilde{\phi}, \mathcal{P}\tilde{\Psi}_Q[M_{1/b}\tilde{\psi}] \rangle_{L^2(\mathbb{R}^d)}.$$

(3.14)

Here $\tilde{\phi}(t) = \phi(-t)$ and $\tilde{\psi}(t) = \psi(-t)$. Then (3.13) follows from (3.11) and the duality. The proof is complete. \qed

**Lemma 3.7.** Assume that (i) - (iii) hold. Then $D_{\mathcal{P}}(\Lambda) \subset D_{\mathcal{P}}(\mathcal{P})$, and $M_{1/b}\nabla x \cdot F'$ and $M_bQe^{-tQ}F_{d+1}$ respectively belong to $\text{Ran}(\mathcal{Q})$ and $\text{Ran}(\Lambda)$ in $L^r(\mathbb{R}^d)$ for any $F = (F', F_{d+1}) \in (C_0^\infty(\mathbb{R}^d))^{d} \times C_0^\infty(\mathbb{R}^d)$ and $t > 0$. Moreover, it follows that

$$\|Q^{-1}M_{1/b}\nabla x \cdot F'\|_{L^r(\mathbb{R}^d)} + \|M_bP\Lambda^{-1}F_{d+1}\|_{L^r(\mathbb{R}^d)}$$

$$+ \|\Lambda^{-1}M_bQe^{-tQ}F_{d+1}\|_{L^r(\mathbb{R}^d)} \leq C \|F\|_{L^r(\mathbb{R}^d)}, \quad r = m, m'.$$

(3.15)

Here

$$M_bP\Lambda^{-1}F_{d+1} := \lim_{\lambda \downarrow 0} M_bP(\Lambda + \lambda)^{-1}F_{d+1} \quad \text{in} \quad L^r(\mathbb{R}^d),$$

$$\Lambda^{-1}M_bQe^{-tQ}F_{d+1} := \lim_{t \to 0} \Lambda^{-1}M_bQe^{-tQ}F_{d+1} \quad \text{in} \quad L^r(\mathbb{R}^d).$$
Proof. We first prove that $M_{1/b} \nabla_x \cdot F'$ belongs to $\text{Ran}(Q)$ in $L^r(\mathbb{R}^d)$ and $Q^{-1}M_{1/b} \nabla_x \cdot F'$ is extended as a bounded operator from $(L^r(\mathbb{R}^d))^d$ to $L^r(\mathbb{R}^d)$. To this end we take any $\lambda > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$ and then use the relation $e^{-tQ} = M_{1/b}(e^{-tP})^*M_b$ to derive

$$\langle (\mathcal{Q} + \lambda)^{-1}M_{1/b} \nabla_x \cdot F', \varphi \rangle_{L^2(\mathbb{R}^d)} = -\langle F', \nabla_x \mathcal{P}(\mathcal{Q} + \lambda)^{-1}M_{1/b} \varphi \rangle_{L^2(\mathbb{R}^d)} = -\langle F', \nabla_x \mathcal{P}^{-1}\mathcal{P}(\mathcal{Q} + \lambda)^{-1}M_{1/b} \varphi \rangle_{L^2(\mathbb{R}^d)}.$$ 

Since $-\nabla_x \mathcal{P}^{-1} : \text{Ran}(\mathcal{P}) \to L^r(\mathbb{R}^d)$ is extended as a bounded operator $K$ from $L^r(\mathbb{R}^d)$ to $(L^r(\mathbb{R}^d))^d$ by the assumptions (cf. Remark 3.3), and since $\mathcal{P}(\mathcal{Q} + \lambda)^{-1}$ is bounded in $L^r(\mathbb{R}^d)$ uniformly in $\lambda > 0$ and $\mathcal{P}(\mathcal{Q} + \lambda)^{-1}h \to h$ as $\lambda \to +0$ in $L^r(\mathbb{R}^d)$ for any $h \in L^r(\mathbb{R}^d)$, we conclude that $g_\lambda = (\mathcal{Q} + \lambda)^{-1}M_{1/b} \nabla_x \cdot F'$ converges to $g = M_{1/b}K^*F'$ as $\lambda \to +0$ weakly in $L^r(\mathbb{R}^d)$. On the other hand, $Qg_\lambda$ converges to $M_{1/b} \nabla_x \cdot F'$ strongly in $L^r(\mathbb{R}^d)$, which implies from the reflexivity of $L^r(\mathbb{R}^d)$ that $g \in D_{L^r}(Q)$ and $Qg = M_{1/b} \nabla_x \cdot F'$. Hence, we have $M_{1/b} \nabla_x \cdot F' \in \text{Ran}(Q)$ and $Q^{-1}M_{1/b} \nabla_x \cdot F' = g = M_{1/b}K^*F'$. This proves the claim. Next we consider $M_b \mathcal{P} \Lambda^{-1}F_{d+1}$. As above, we take $\lambda > 0$ and note that $(\Lambda + \lambda)^{-1}F_{d+1} \in D_{L^r}(\Lambda) \cap D_{L^r}(\Lambda)$ due to the assumption (i). Since $D_{L^2}(\mathcal{P}) = D_{L^2}(\Lambda) = H^1(\mathbb{R}^d)$ and $M_b \mathcal{P} = \Lambda + M_a \nabla_x$ by Theorem 2.3, we have for any $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\langle M_b \mathcal{P}(\Lambda + \lambda)^{-1}F_{d+1}, \varphi \rangle_{L^2(\mathbb{R}^d)} = \langle \Lambda(\Lambda + \lambda)^{-1}F_{d+1}, \varphi \rangle_{L^2(\mathbb{R}^d)} = \langle M_a \cdot \nabla_x \Lambda^{-1}(\Lambda + \lambda)^{-1}F_{d+1}, \varphi \rangle_{L^2(\mathbb{R}^d)}.$$ 

As state in Remark 3.3 the operator $M_a \cdot \nabla_x \Lambda^{-1} : \text{Ran}(\Lambda) \subset L^r(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$ is extended as a bounded operator $L$ in $L^r(\mathbb{R}^d)$. Thus we see

$$\|M_b \mathcal{P}(\Lambda + \lambda)^{-1}F_{d+1}\|_{L^r(\mathbb{R}^d)} \leq C\|\Lambda(\Lambda + \lambda)^{-1}F_{d+1}\|_{L^r(\mathbb{R}^d)} \leq C\|F_{d+1}\|_{L^r(\mathbb{R}^d)}$$

with $C$ independent of $\lambda > 0$. By the assumptions (i) and (iii) this estimate implies $(\Lambda + \lambda)^{-1}F_{d+1} \in D_{L^r}(\mathcal{P})$, and we have $M_b \mathcal{P}(\Lambda + \lambda)^{-1}F_{d+1} = (I + L)\Lambda(\Lambda + \lambda)^{-1}F_{d+1}$. Since $\Lambda(\Lambda + \lambda)^{-1}F_{d+1} \to F_{d+1}$ as $\lambda \to +0$ in $L^r(\mathbb{R}^d)$, we have $M_b \mathcal{P} \Lambda^{-1}F_{d+1} = \lim_{\lambda \to +0} M_b \mathcal{P}(\Lambda + \lambda)^{-1}F_{d+1} = (I + L)F_{d+1}$ in $L^r(\mathbb{R}^d)$. Finally we consider $\Lambda^{-1}M_bQF_{d+1}$. For any $\lambda, t > 0$ we see

$$\langle (\Lambda + \lambda)^{-1}M_bQe^{-tQ}F_{d+1}, \varphi \rangle_{L^2(\mathbb{R}^d)} = \langle F_{d+1}, M_bQe^{-tP}(\Lambda + \lambda)^{-1}\varphi \rangle_{L^2(\mathbb{R}^d)}, \quad \varphi \in C_0^\infty(\mathbb{R}^d).$$

As is proved above, $(\Lambda + \lambda)^{-1}\varphi \in D_{L^r}(\mathcal{P})$ and we have $\|\mathcal{P}(\Lambda + \lambda)^{-1}\varphi\|_{L^r(\mathbb{R}^d)} \leq C\|\varphi\|_{L^r(\mathbb{R}^d)}$ with $C$ independent of $\lambda > 0$ as well as $\mathcal{P}(\Lambda + \lambda)^{-1}\varphi \to M_{1/b}(I + L)\varphi$ (as $\lambda \to +0$) in $L^r(\mathbb{R}^d)$. This implies that for each $t > 0$ the function $(\Lambda + \lambda)^{-1}M_bQe^{-tQ}F_{d+1}$ converges to $(I + L^*)e^{-tQ}F_{d+1}$ as $\lambda \to +0$ weakly in $L^r(\mathbb{R}^d)$, and from the reflexivity of $L^r(\mathbb{R}^d)$ we have $M_bQe^{-tQ}F_{d+1} \in \text{Ran}(\Lambda)$ in $L^r(\mathbb{R}^d)$ and $\Lambda^{-1}M_bQe^{-tQ}F_{d+1} = (I + L^*)e^{-tQ}F_{d+1}$. It is now easy to see the limit $\lim_{t \to 0} \Lambda^{-1}M_bQe^{-tQ}F_{d+1} = (I + L^*)F_{d+1}$ satisfies the desired estimate. The proof is complete. 

\[\square\]

Remark 3.8. From the proof we have $\|\Lambda^{-1}M_bQe^{-tQ}e^{-tQ}f\|_{L^r(\mathbb{R}^d)} \leq C\|e^{-tQ}f\|_{L^r(\mathbb{R}^d)}$ for any $f \in L^r(\mathbb{R}^d)$ by the density argument. In particular, $M_bQe^{-tQ}f \in \text{Ran}(\Lambda)$ in $L^r(\mathbb{R}^d)$ for any $f \in L^r(\mathbb{R}^d)$.

Lemma 3.9. Let $F = (F', F_{d+1}) \in (C_0^\infty(\mathbb{R}_{+}^d))^d \times C_0^\infty(\mathbb{R}_{+}^{d+1})$ and set $G = -(F_{d+1} + M_aF')$. Let $\gamma$ be the trace operator to the boundary $\partial\mathbb{R}_{+}^{d+1}$. Assume that (i) - (iii) hold. Then $\gamma G + \int_0^\infty e^{-sQ}M_{1/b}(-\nabla_x \cdot F + \partial_s G)(s) \, ds$ belongs to $\text{Ran}(\Lambda)$ in $L^r(\mathbb{R}^d)$. 

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Proof. By the integration by parts with respect to the time variable, we see

$$\gamma G + M_b \int_0^\infty e^{-sQ} M_{1/b} (-\nabla_x \cdot F' + \partial_t G) (s) \, ds$$

$$= M_b \int_0^\infty Q e^{-sQ} M_{1/b} G(s) \, ds - M_b \int_0^\infty e^{-sQ} M_{1/b} \nabla_x \cdot F'(s) \, ds$$

$$= M_b Q \int_0^\infty e^{-sQ} \left( M_{1/b} G(s) - Q^{-1} M_{1/b} \nabla_x \cdot F'(s) \right) \, ds.$$  \hspace{1cm} (3.16)

Here we have used Lemma 3.7. Again from Lemma 3.7 and Remark 3.8 the expression (3.16) implies the assertion of Lemma 3.9. The proof is complete.

Corollary 3.10. The conclusion of Lemma 3.9 holds for $r = 2$.

Proof. Proposition 3.14 in the next section and Lemma 3.9 prove the claim. The proof is complete.

Proof of Proposition 3.3. Since $F \in (C_0^\infty (\mathbb{R}^{d+1}_+))^{d+1}$, there exists a unique weak solution $w \in \dot{H}^1(\mathbb{R}^{d+1}_+)$ to (3.2). Then Corollary 3.10 and Theorem 2.5 lead to the following representation:

$$w(t) = e^{-tP} \Lambda^{-1} (\gamma G + M_b \int_0^\infty e^{-sQ} M_{1/b} (-\nabla_x \cdot F' + \partial_t G) (s) \, ds)$$

$$+ \int_0^t e^{-(t-s)P} \int_s^\infty e^{-(\tau-s)Q} M_{1/b} (-\nabla_x \cdot F' + \partial_t G)(\tau) \, d\tau \, ds$$

$$= e^{-tP} \Lambda^{-1} M_b Q \int_0^\infty e^{-sQ} \left( M_{1/b} G - Q^{-1} M_{1/b} \nabla_x \cdot F'(s) \right) \, ds$$

$$+ \int_0^t e^{-(t-s)P} \left( - M_{1/b} G(s) + Q \int_s^\infty e^{-(\tau-s)Q} \left( M_{1/b} G - Q^{-1} M_{1/b} \nabla_x \cdot F'(s) \right) \, d\tau \right) \, ds.$$  \hspace{1cm} (3.17)

Here we have also used (3.16), Lemma 3.7, and the integration by parts. Set

$$v(t) = -M_{1/b} G(t) + Q \int_t^\infty e^{-(s-t)Q} (M_{1/b} G - Q^{-1} M_{1/b} \nabla_x \cdot F')(s) \, ds.$$  

Note that $\gamma G = 0$ for $F \in (C_0^\infty (\mathbb{R}^{d+1}_+))^{d+1}$, and thus, we have $\gamma v = Q \int_0^\infty e^{-sQ} (M_{1/b} G - Q^{-1} M_{1/b} \nabla_x \cdot F')(s) \, ds$. Then the solution of (3.2) is written in the form $w = w_1 + w_2$, where each $w_i$ is given by

$$w_1(t) = \int_0^t e^{-(t-s)P} v(s) \, ds, \quad w_2(t) = e^{-tP} \Lambda^{-1} M_b \gamma v.$$  \hspace{1cm} (3.18)

By the assumption (3.6) it suffices to estimate $\partial_t w_1$ and $Pw_1$.

Step 1: Estimate of $w_1$. From the maximal regularity (3.11) and Lemma 3.6 we have

$$\|Pw_1\|_{L^q(\mathbb{R}_+; L^r(\mathbb{R}^d))} \leq C \|v\|_{L^q(\mathbb{R}_+; L^r(\mathbb{R}^d))}$$

$$\leq C \left( \|G\|_{L^q(\mathbb{R}_+; L^r(\mathbb{R}^d)))} + \|Q^{-1} M_{1/b} \nabla_x \cdot F'\|_{L^q(\mathbb{R}_+; L^r(\mathbb{R}^d))} \right).$$

Thus Lemma 3.7 yields $\|Pw_1\|_{L^q(\mathbb{R}_+; L^r(\mathbb{R}^d))} \leq C_q \|F\|_{L^q(\mathbb{R}_+; L^r(\mathbb{R}^d))}$. The estimate of $\partial_t w_1$ is obtained in the same manner.
Step 2: Estimate of \( w_2 \). We decompose \( \gamma v \) as \( \gamma v = v_1 + v_2 \), where
\[
\begin{align*}
v_1 &= Q \int_0^t e^{-sQ} \left( M_{1/b} G - Q^{-1} M_{1/b} \nabla_x \cdot F' \right)(s) \, ds, \\
v_2 &= Q \int_t^\infty e^{-sQ} \left( M_{1/b} G - Q^{-1} M_{1/b} \nabla_x \cdot F' \right)(s) \, ds.
\end{align*}
\]
Motivated by this decomposition we introduce the linear operators \( T_i, i = 1, 2 \), defined by
\[
\begin{align*}
T_1[\phi](t) &= \mathcal{P} e^{-tP} \Lambda^{-1} M_b Q \int_0^t e^{-sQ} \phi(s) \, ds, \\
T_2[\phi](t) &= e^{-tP} \mathcal{P} \Lambda^{-1} M_b Q \int_t^\infty e^{-sQ} \phi(s) \, ds.
\end{align*}
\]
Each of \( T_i \) makes sense for \( \phi \in L^q(\mathbb{R}^+; L^r(\mathbb{R}^d)) \) due to Lemma 3.7 and the density argument. Clearly we have \( \| P w_2 \|_{L^q(\mathbb{R}^+; L^r(\mathbb{R}^d))} \leq C \| F \|_{L^q(\mathbb{R}^+; L^r(\mathbb{R}^d))} \) since \( \| M_{1/b} G - Q^{-1} M_{1/b} \nabla_x \cdot F' \|_{L^q(\mathbb{R}^+; L^r(\mathbb{R}^d))} \leq C \| F \|_{L^q(\mathbb{R}^+; L^r(\mathbb{R}^d))} \). The estimate of \( \partial_t w_2 \) is the same as \( P w_2 \). The proof of Proposition 3.13 is complete. \( \square \)

Let us recall that the weak formulation of the Neumann problem (1.7) is equivalent with (3.4) through the transformation (3.1). Approximating \( F \in L^q(\mathbb{R}^+; L^r(\mathbb{R}^d))^{d+1} \) by vector fields in \( (C_0^\infty(\mathbb{R}^d))^{d+1} \), we obtain from Proposition 3.3 the following

**Theorem 3.11.** Let \( \eta \) be a globally Lipschitz function and let \( \Omega \) be a domain given in (1.1). Assume that (i) - (iii) hold and let \( 1 < q < \infty \), \( \min\{m,m'\} \leq r \leq \max\{m,m'\} \). Then for any \( f \in (Y_{q,r}(\Omega))^{d+1} \) there exists a unique (up to constant) weak solution \( p \in L^q_{loc}(\Omega) \) to the Neumann problem (1.7) satisfying
\[
\| \nabla p \|_{Y_{q,r}(\Omega)} \leq C \| f \|_{Y_{q,r}(\Omega)}.
\]
Here \( C \) depends only on \( d, q, \| \nabla \eta \|_{L^\infty(\mathbb{R}^d)} \), and the constants of the estimates in (i) - (iii).

**Proof.** The proof of the existence proceeds as described above, and we omit the details. The uniqueness follows from the solvability of (3.4) in the adjoint space \( L^q(\mathbb{R}^+; L^r(\mathbb{R}^d)) \) and the duality. The proof is complete. \( \square \)

By the standard argument as in [16] Lemma III.1.2, we are able to show the validity of the Helmholtz decomposition in \( (Y_{q,r}(\Omega))^{d+1} \) under the assumptions (i) - (iii). Here we give a sketch of the proof for completeness. To this end we first recall a useful lemma in [13] Lemma 7], [16] Lemma III.1.1].
Lemma 3.12. Let \( \Omega \) be a simply connected set in \( \mathbb{R}^{d+1} \). Suppose that \( u \in (L^1_{\text{loc}}(\Omega))^{d+1} \) verifies
\[
\int_{\Omega} u \cdot \varphi = 0 \quad \text{for all} \quad \varphi \in C^\infty_{0,\sigma}(\Omega).
\]
Then there exists a scalar function \( p \in W^{1,1}_{\text{loc}}(\Omega) \) such that \( u = \nabla p \).

Theorem 3.13. Under the assumptions of Theorem 3.11, the space \((Y^{q,r}(\Omega))^{d+1}\) admits the Helmholtz decomposition.

Proof. For \( f \in (Y^{q,r}(\Omega))^{d+1} \) let \( p \in L^1_{\text{loc}}(\Omega) \) be the weak solution to (1.7) given in Theorem 3.11 which satisfies \( \nabla p \in Y^q_G(\Omega) \), and set \( u = f - \nabla p \). Then, we have \( u \in (Y^{q,r}_G(\Omega))^{\perp} \) where \( \mathcal{X}^{\perp} \) denotes the annihilator of the set \( \mathcal{X} \). On the other hand, by Lemma 3.12, we see \((Y^{q,r}_G(\Omega))^{\perp} \subset (Y^{q',r'}_G(\Omega))^{\perp} \), or equivalently, \( Y^{q,r}_G \supset (Y^{q',r'}_G)^{\perp} \).

Thus we have proved that \( u \in Y^{q,r}_G \).

It remains to show that the representation \( f = u + \nabla p \) is unique. This is equivalent to show that the equality
\[
u = \nabla p, \quad u \in Y^{q,r}_G, \quad \nabla p \in Y^{q,r}_G
\]
holds if and only if \( u \equiv \nabla p \equiv 0 \). To this end, we observe that
\[
Y^{q,r}_G \subset (Y^{q',r'}_G)^{\perp},
\]
which implies
\[
\int_{\Omega} \nabla p \cdot \nabla \varphi = 0 \quad \text{for all} \quad \varphi \in \{ f \in L^1_{\text{loc}}(\Omega) \mid \nabla f \in (Y^{q',r'}(\Omega))^{d+1} \}.
\]
By the uniqueness of weak solutions to (1.7), we have \( u \equiv \nabla p \equiv 0 \). The proof is complete. \( \square \)

3.2 Proof of Theorem 1.2

In this section we prove the next proposition, which immediately leads to Theorem 1.2 thanks to Theorem 3.13 in the previous section.

Proposition 3.14. When \( m = m' = r = 2 \), the assumptions (i) - (iii) in Section 3.1 are valid.

Proof. It is well-known that \( \Lambda \) is self-adjoint in \( L^2(\mathbb{R}^d) \), and (2.5) implies the boundedness of \( \{ e^{-t\Lambda} \}_{t \geq 0} \) in \( L^2(\mathbb{R}^d) \). We also observe from Theorem 2.3 that \( D_{L^2}(\mathcal{P}) = D_{L^2}(\Lambda) = H^1(\mathbb{R}^d) \) and \( \mathcal{P} \) generates a strongly continuous and bounded analytic semigroup in \( L^2(\mathbb{R}^d) \). Then it suffices to check the estimates (3.6), (3.7), and (3.8) for \( r = 2 \). In fact, these estimates are already known as a consequence of the Rellich type identity, which is a classical tool in the study of the Dirichlet and Neumann problems for the Laplace equations; see Remark 3.15 below for references. Here we give the detailed proof of them including (3.5) in order to make this paper self-contained as much as possible.

Step 1: Proof of (ii). We will prove
\[
\| M_{\sqrt{\mathcal{P}}} \varphi \|_{L^2(\mathbb{R}^d)} = \| \nabla_x \varphi \|_{L^2(\mathbb{R}^d)} \quad (3.25)
\]
\[
C_1 \| \nabla_x \varphi \|_{L^2(\mathbb{R}^d)} \leq \| \Lambda \varphi \|_{L^2(\mathbb{R}^d)} \leq C_2 \| \nabla_x \varphi \|_{L^2(\mathbb{R}^d)}, \quad (3.26)
\]

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where $C_1$ and $C_2$ are positive constants depending only on $\|\nabla \eta\|_{L^\infty(\mathbb{R}^d)}$. Let us recall that in our case Theorem 2.3 holds with $A' = -\Delta x$. Thus (2.7) yields (3.25). Next by the relation $P = M_{1/b}A - M_{(\nabla x)/b} \cdot \nabla x$ the right-hand side of (3.25) is written as

$$
\|M^{\sqrt{b}} P \varphi\|_{L^2(\mathbb{R}^d)}^2 = \|M^{\sqrt{b}} A \varphi - M_{(\nabla x)/\sqrt{b}} \cdot \nabla x \varphi\|_{L^2(\mathbb{R}^d)}^2
= \|M^{\sqrt{b}} A \varphi\|_{L^2(\mathbb{R}^d)}^2 - 2\langle A \varphi, M_{(\nabla x)/b} \cdot \nabla x \varphi \rangle_{L^2(\mathbb{R}^d)} + \|M_{(\nabla x)/\sqrt{b}} \cdot \nabla x \varphi\|_{L^2(\mathbb{R}^d)}^2.
$$

Thus (3.25) and (3.27) immediately yield

$$
\|M^{\sqrt{b}} A \varphi\|_{L^2(\mathbb{R}^d)}^2 \leq 2\|\nabla x \varphi\|_{L^2(\mathbb{R}^d)}^2.
$$

(3.28)

While, we derive from (3.27) that

$$
\|M^{\sqrt{b}} P \varphi\|_{L^2(\mathbb{R}^d)}^2 \leq (1 + \epsilon^{-1})\|M^{\sqrt{b}} A \varphi\|_{L^2(\mathbb{R}^d)}^2 + (1 + \epsilon) \int_{\mathbb{R}^d} \frac{|\nabla_x \eta|^2}{1 + |\nabla_x \eta|^2} |\nabla x \varphi|^2 \, dx, \quad \epsilon > 0.
$$

Hence, combining this with (3.25) implies

$$
(1 + \epsilon^{-1})\|M^{\sqrt{b}} A \varphi\|_{L^2(\mathbb{R}^d)}^2 \geq \int_{\mathbb{R}^d} \frac{1 - \epsilon |\nabla_x \eta|^2}{1 + |\nabla_x \eta|^2} |\nabla x \varphi|^2 \, dx \geq c\|\nabla x \varphi\|_{L^2(\mathbb{R}^d)}^2,
$$

if $\epsilon > 0$ is small enough. Here $c > 0$ depends only on $\|\nabla_x \eta\|_{L^\infty(\mathbb{R}^d)}$. Thus (ii) is proved.

**Step 2: Proof of (iii).** From Remark 3.2 it suffices to show (3.10) with $r = 2$. Set $w(t) = \int_0^t e^{-(t-s)p} \phi(s) \, ds$. Then $w$ is the solution to $\partial_t w + P w = \phi$ in $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\gamma w = 0$ on $\partial \mathbb{R}^{d+1}_+$. Hence we have from [22, Theorem 1.10, Remark 1.11],

$$
\langle A \nabla w, \nabla w \rangle_{L^2(\mathbb{R}^{d+1}_+)} = \langle \phi, M_{b}(\partial_t + P)w \rangle_{L^2(\mathbb{R}^{d+1}_+)}.
$$

(3.29)

Here we have used the boundary condition $w = 0$ on $t = 0$. By the uniform ellipticity the left-hand side of (3.29) is bounded from below by $c\|\nabla w\|_{L^2(\mathbb{R}^{d+1}_+)}^2$ with some $c > 0$ depending only on $d$ and $\|\nabla_x \eta\|_{L^\infty(\mathbb{R}^d)}$. On the other hand, the right-hand side of (3.29) is calculated as

$$
\text{R.H.S. of (3.29)} = \langle M_{b} \phi, \phi \rangle_{L^2(\mathbb{R}^{d+1}_+)} \leq C\|\phi\|_{L^2(\mathbb{R}^{d+1}_+)}^2.
$$

This complete the proof of (iii).

Finally, we also give the proof of (3.35) for $\{e^{-tP}\}_{t \geq 0}$ for reader’s convenience. It suffices to consider the case $\varphi \in C_0^\infty(\mathbb{R}^d)$. Set $u_0(t) = e^{-tP}\varphi$. Then we see from (3.8) with $r = 2$ (which is proved in Step 2 above) that

$$
\|u_0\|_{L^2(0,T;L^2(\mathbb{R}^d))} \leq C T^{1/2}\|\varphi\|_{L^2(\mathbb{R}^d)}, \quad T > 0,
$$

(3.30)

where $C > 0$ is independent of $T$ and $\varphi$. Set $u_k(t) = (tP)^k u_0(t)$ for $k = 1, 2$. Then $u_k$ satisfies $\partial_t u_k - P u_k = kP u_{k-1}$ with the zero initial data. Hence we have the representation $u_k(t) = \int_0^t e^{(t-s)P} P u_{k-1} \, ds = kP \int_0^t e^{(t-s)P} u_{k-1} \, ds$. Thus (3.8) with $r = 2$ yields

$$
\|u_k\|_{L^2(0,T;L^2(\mathbb{R}^d))} \leq C\|u_{k-1}\|_{L^2(0,T;L^2(\mathbb{R}^d))} \leq C T^{1/2}\|\varphi\|_{L^2(\mathbb{R}^d)}, \quad T > 0.
$$

(3.31)
In particular, (3.30) and (3.31) with $k = 1$ imply that for any $T > 0$ there exists $T_1 \in [T/2, T]$ such that $\left\| u_1(T_1) \right\|_{L^2(\mathbb{R}^d)} \leq C \left\| \varphi \right\|_{L^2(\mathbb{R}^d)}$ with $C$ independent of $T$ and $\varphi$. Next we calculate the evolution of $\left\| M_{\sqrt{T}} u_1(t) \right\|_{L^2(\mathbb{R}^d)}^2$ to get

$$\frac{d}{dt} \left\| M_{\sqrt{T}} u_1(t) \right\|_{L^2(\mathbb{R}^d)}^2 = -2 \langle u_1, A u_1 - M_{\nabla_x \eta} \cdot \nabla_x u_1 \rangle_{L^2(\mathbb{R}^d)} + 2 \langle u_1, M_b \mathcal{P} u_0 \rangle_{L^2(\mathbb{R}^d)}$$

$$\leq 2 \langle u_1, M_{\nabla_x \eta} \cdot \nabla_x u_1 \rangle_{L^2(\mathbb{R}^d)} + 2 t^{-1} \left\| M_{\sqrt{T}} u_1(t) \right\|_{L^2(\mathbb{R}^d)}^2$$

$$\leq C \left\| u_1(t) \right\|_{L^2(\mathbb{R}^d)} \left\| \mathcal{P} u_1(t) \right\|_{L^2(\mathbb{R}^d)} + 2 t^{-1} \left\| M_{\sqrt{T}} u_1(t) \right\|_{L^2(\mathbb{R}^d)}^2$$

$$\leq C t \left\| \mathcal{P} u_1(t) \right\|_{L^2(\mathbb{R}^d)}^2 + C t^{-1} \left\| M_{\sqrt{T}} u_1(t) \right\|_{L^2(\mathbb{R}^d)}^2.$$ 

Here we have used the definition of $u_k$, $\langle u_1, A u_1 \rangle_{L^2(\mathbb{R}^d)} \geq 0$, and (3.6) with $r = 2$. Then by integrating over $[T_1, T]$ and by using the Gronwall inequality, we arrive at

$$\left\| M_{\sqrt{t}} u_1(T) \right\|_{L^2(\mathbb{R}^d)} \leq C \left\| M_{\sqrt{t}} u_1(T_1) \right\|_{L^2(\mathbb{R}^d)} + C \int_{T_1}^T t \left\| \mathcal{P} u_1(t) \right\|_{L^2(\mathbb{R}^d)}^2 \, dt$$

$$\leq C \left\| u_1(T_1) \right\|_{L^2(\mathbb{R}^d)} + C T^{-1} \int_{T_1}^T \left\| u_2 \right\|_{L^2(T_1, T; L^2(\mathbb{R}^d))}^2 \, dt \leq C \left\| \varphi \right\|_{L^2(\mathbb{R}^d)}^2.$$ 

Since $T > 0$ is arbitrary, we have proved

$$\left\| \mathcal{P} e^{-tP} \varphi \right\|_{L^2(\mathbb{R}^d)} \leq C t^{-1} \left\| \varphi \right\|_{L^2(\mathbb{R}^d)}, \quad t > 0, \quad (3.32)$$

Now from (3.30) there is $T_2 \in [T/2, T]$ such that $\left\| u_0(T_2) \right\|_{L^2(\mathbb{R}^d)} \leq C \left\| \varphi \right\|_{L^2(\mathbb{R}^d)}$. Combining this with (3.32) and the equality $u_0(T) = u_0(T_2) - \int_{T_2}^T \mathcal{P} e^{-sP} \varphi \, ds$, we obtain (3.3). The proof of Proposition 3.14 is complete. \hfill $\square$

**Remark 3.15.** As mentioned above, for the case $r = 2$, the coercive estimates for $\mathcal{P}$ and $\Lambda$ as in (3.6) - (3.7) are obtained from a variant of the Rellich identity [27]. These estimates are used to study the behavior of harmonic functions near the boundary [26, 18, 19]. In particular, it works even for nonsmooth domains, and in [18] [19] the Rellich type identity was used in solving the Dirichlet and Neumann problems in bounded Lipschitz domains. We also note that (3.6) and (3.7) are the key to obtain the characterization $D_{L^2}(\mathcal{P}) = H^1(\mathbb{R}^d)$ and $D_{L^2}(\Lambda) = H^1(\mathbb{R}^d)$ for real symmetric but nonsmooth $\Lambda$. For a matrix $\Lambda$ of the form in Section 3, called the Jacobian type, the relation $D_{L^2}(\mathcal{P}) = H^1(\mathbb{R}^d)$ is proved in [7]. The relation $D_{L^2}(\Lambda) = H^1(\mathbb{R}^d)$ is related with the solvability of the Neumann problem for $L^2$ boundary data, and it is solved by [18] in bounded Lipschitz domains. For results in more general class of $\Lambda$ including real symmetric or Hermite ones, see [19] [21] [3] [4] [22] and references therein.

### 3.3 Concluding remark

Recent works [11] [15] revealed that the solvability of the weak Neumann problem in $L^p$ ensures the analyticity of the Stokes semigroup in $L^p$ even if the boundary in noncompact. In view of their results, it is shown that the Stokes semigroup is analytic in the space $Y^{q, 2}$ at least when the boundary is smooth enough. We will address this question in the forthcoming work.

### A Appendix

**A.1 Semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ in $L^r(\mathbb{R}^d)$ for $r \in (1, \infty)$**

**Proposition A.1.** Let $r \in (1, \infty)$. Then the restrictions of $\{e^{-t\Lambda}\}_{t\geq 0}$ on $L^2(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$ is extended as a strongly continuous and bounded semigroup in $L^r(\mathbb{R}^d)$. 

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Proof. Here we give only a sketch of the proof. We first consider the case $r \in [2, \infty)$. Set $u(t) = e^{-tA} f$, $f \in H^1(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$, and set $v(s; t) = e^{-sP} u(t)$, $s \geq 0$. Then we have

$$\frac{d}{dt} \|u(t)\|_{L^r(\mathbb{R}^d)} = -r \langle Au(t), |u(t)|^{r-2}u(t) \rangle_{L^2(\mathbb{R}^d)}$$

$$= -r \int_0^\infty \langle A\nabla v(s; t), \nabla (|v(s; t)|^{r-2}v(s; t)) \rangle_{L^2(\mathbb{R}^d)} \, ds$$

$$\leq -cr \|\nabla (|v(\cdot; t)|^2)\|_{L^2(\mathbb{R}^{d+1})}^2 \leq -cr \|u(t)\|_2^2 \|H^{\frac{r}{2}}(\mathbb{R}^d)\|.$$

In particular, we have $\|e^{-tA} f\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^r(\mathbb{R}^d)}$ when $r \in [2, \infty)$, and thus, when $r \in [2, \infty)$ (by taking the limit $t \to \infty$). By the dual relation $\langle e^{-tA}f, g \rangle_{L^2(\mathbb{R}^d)} = \langle f, e^{-tA}g \rangle_{L^2(\mathbb{R}^d)}$ we have this uniform bound also for $r \in [1, 2)$. Hence, by the density argument $\{e^{-tA}\}_{t \geq 0}$ is extended as a bounded semigroup acting on $L^r(\mathbb{R}^d)$ for all $r \in [1, \infty)$ (note that this uniform bound holds also for $r = 1, \infty$). As for the strong continuity, let $r \in (2, \infty)$, and for any $f \in L^r(\mathbb{R}^d)$ we take $\{f_n\} \subset C_0^\infty(\mathbb{R}^d)$ such that $f_n \to f$ as $n \to \infty$ in $L^r(\mathbb{R}^d)$. Then we see

$$\|e^{-tA} f - f\|_{L^r(\mathbb{R}^d)} \leq \|e^{-tA}(f - f_n)\|_{L^r(\mathbb{R}^d)} + \|f - f_n\|_{L^r(\mathbb{R}^d)} + \|e^{-tA} f_n - f_n\|_{L^r(\mathbb{R}^d)}$$

$$\leq 2\|f - f_n\|_{L^r(\mathbb{R}^d)} + \|e^{-tA} f_n - f_n\|_{L^2(\mathbb{R}^d)}^{\frac{2}{1 - \frac{2}{r}}} + \|e^{-tA} f_n - f_n\|_{L^\infty(\mathbb{R}^d)}^{\frac{2}{1 - \frac{2}{r}}}$$

$$\leq 2\|f - f_n\|_{L^r(\mathbb{R}^d)} + 2\|f_n\|_{L^2(\mathbb{R}^d)}^{\frac{2}{1 - \frac{2}{r}}} + 2\|f_n\|_{L^\infty(\mathbb{R}^d)}^{\frac{2}{1 - \frac{2}{r}}}.$$

Since we have already known that $\{e^{-tA}\}_{t \geq 0}$ is strongly continuous in $L^2(\mathbb{R}^d)$, the last estimate implies the strong continuity in $L^r(\mathbb{R}^d)$ for $r \in (2, \infty)$. The case $r \in (1, 2)$ is proved in the same manner. The proof is complete. \qed

A.2 Semigroup $\{e^{-tP}\}_{t \geq 0}$ in $L^r(\mathbb{R}^d)$ for $r \in [2, \infty)$

**Proposition A.2.** Let $r \in [2, \infty)$. Then the restrictions of $\{e^{-tP}\}_{t \geq 0}$ on $L^2(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$ is extended as a strongly continuous and bounded semigroup in $L^r(\mathbb{R}^d)$.

**Proof.** Again we give only a sketch of the proof. Let $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and set $u(t) = e^{-tP} f$. Then, since $u$ satisfies $Au = 0$ in $\mathbb{R}^{d+1}_+$, the maximum principle implies that

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u\|_{L^\infty(\mathbb{R}^{d+1}_+)} \leq \|f\|_{L^\infty(\mathbb{R}^d)}.$$

This estimate gives the boundedness of $e^{-tP}$ in $L^\infty(\mathbb{R}^d)$. Since $e^{-tP}$ is bounded in $L^2(\mathbb{R}^d)$, the interpolation inequality yields the boundedness of $e^{-tP}$ in $L^r(\mathbb{R}^d)$ for each $r \in (2, \infty)$. The strong continuity in $L^r(\mathbb{R}^d)$ is shown as in the proof of Proposition A.1. The proof is complete. \qed

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