W¹,p REGULARITY OF SOLUTIONS TO KOLMOGOROV EQUATION WITH GILBARG-SERRIN MATRIX

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Abstract. In \( \mathbb{R}^d, d \geq 3 \), consider the divergence and the non-divergence form operators
\[
-\Delta - \nabla \cdot (a - I) \cdot \nabla + b \cdot \nabla, \tag{i}
-\Delta - (a - I) \cdot \nabla^2 + b \cdot \nabla, \tag{ii}
\]
where the second order perturbations are given by the matrix
\[
a - I = c|x|^{-2}x \otimes x, \quad c > -1.
\]
The vector field \( b : \mathbb{R}^d \to \mathbb{R}^d \) is form-bounded with the form-bound \( \delta > 0 \) (this includes a sub-critical class \([L^d + L^\infty]^d\), as well as vector fields having critical-order singularities). We characterize quantitative dependence on \( c \) and \( \delta \) of the \( L^q \to W^{1,qd/(d-2)} \) regularity of the resolvents of the operator realizations of (i), (ii) in \( L^q, q \geq 2 \lor (d-2) \) as (minus) generators of positivity preserving \( L^\infty \) contraction \( C_0 \) semigroups.

Consider in \( \mathbb{R}^d, d \geq 3 \), the operator
\[
-\Delta + b \cdot \nabla \equiv -\sum_{i=1}^d \partial^2_{x_i} + \sum_{k=1}^d b_k(x) \partial_{x_k}, \quad b \in F_{\delta},
\]
where \( F_{\delta} \equiv F_{\delta}(-\Delta), \delta > 0, \) is the class of form-bounded vector fields \( \mathbb{R}^d \to \mathbb{R}^d \), that is, \( |b| \in L^2_{\text{loc}} \equiv L^2_{\text{loc}}(\mathbb{R}^d) \) and there exist a constant \( \lambda = \lambda_{\delta} > 0 \) such that
\[
\|b(\lambda - \Delta)^{-\frac{1}{2}}\|_{2\to 2} \leq \sqrt{\delta}
\]
(see examples below). By [KS, Lemma 5] (see also [KiS, Theorems 3.7-3.10]), if \( \delta < 1 \lor (\frac{2}{d-2})^2 \), then for
\[
u = (\mu + \Lambda_q(b))^{-1} f, \quad f \in L^q, \quad q \in [2 \lor (d-2), \frac{2}{\sqrt{\delta}}],
\]
one has
\[
\|\nabla u\|_{\frac{qd}{d-2}} \leq K \|f\|_q, \tag{*}
\]
where \( \mu > \frac{\lambda^d}{2(q-1)} \), \( K = K(\mu, \delta, q) \). Here \( \Lambda_q(b) \) is an operator realization of \( -\Delta + b \cdot \nabla \) as the (minus) generator of a positivity preserving \( L^\infty \) contraction \( C_0 \) semigroup in \( L^q \). The estimate (⋆) and the iteration method in [KS] (see also [KiS, sect. 3.6]) allow to construct a Feller process associated with \( -\Delta + b \cdot \nabla \).

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In this paper we are concerned with a second order perturbation of $-\Delta$,

$$-\Delta - \nabla \cdot (a - I) \cdot \nabla, \quad a_{ij}(x) = \delta_{ij} + c|x|^2 x_i x_j, \quad c > -1, c \neq 0.$$  \hspace{1cm} (1)

This is a model example of a divergence form operator that is not accessible by classical means such as the parametrix [F], [LSU, Ch. IV]. Although the matrix $a$ is discontinuous at $x = 0$, it is uniformly elliptic, so by the De Giorgi-Nash theory, the solution $u \in W^{1,2}(\mathbb{R}^d)$ to the corresponding elliptic equation $(\mu - \nabla \cdot a \cdot \nabla)u = f$, $\mu > 0$, $f \in L^p \cap L^2$, $p \in ]\frac{d}{2}, \infty[$, is in $C^{0,\gamma}$, where the Hölder continuity exponent $\gamma \in ]0,1[$ depends only on $d$ and $c$. The operator (1) and its modifications both of divergence and non-divergence type have been investigated by many authors in order to precise the connection between the regularity properties of the solution and the continuity properties of the matrix, see [GS], [M], [LU, Ch. 1.2], [ABT], [OG], [A] and references therein.

In Theorem 1 below (the principal result) we show that the perturbation $-\nabla \cdot (a - I) \cdot \nabla$ of $-\Delta$ preserves, under the appropriate assumptions on $c$, the essential properties of $-\Delta$ that allow to establish the estimate $(\ast)$ for $u = (\mu + \Lambda_q(a,b))^{-1}f$, where $\Lambda_q(a,b)$ is an operator realization of

$$-\Delta - \nabla \cdot (a - I) \cdot \nabla + b \cdot \nabla, \quad b \in F_\delta$$

as the (minus) generator of a positivity preserving $L^\infty$ contraction $C_0$ semigroup in $L^q$ (Theorem 2).

The class $F_\delta$ contains a sub-critical class $|b| \in L^d + L^\infty$, with $\delta > 0$ arbitrarily small, as well as vector fields having critical-order singularities, e.g. $b(x) = \frac{d-2}{d}\sqrt{d}|x|^{-2}x$. More generally, if $|b|$ is in $L^{d,\infty}$, the Campanato-Morrey class or the Chang-Wilson-Wolff class, then $b \in F_\delta$ with $\delta$ depending on the norm of $|b|$ in these classes, see e.g. [KiS] for details. We also note that, for the class of uniformly elliptic matrices as a whole, the class $F_\delta$ of first order perturbations $b \cdot \nabla$ of $-\nabla \cdot a \cdot \nabla$ destroys the $C^{0,\gamma}$ regularity of bounded solutions.

Set $(\nabla a)_k := \sum_{i=1}^d(\partial_{x_i} a_{ik})$, $1 \leq k \leq d$. Then $\nabla a = c(d-1)|x|^{-2}x$, and so $\nabla a \in F_\delta$, $\delta = \frac{4c^2(d-1)^2}{(d-2)^2}$. The latter allows us to construct an operator realization of the non-divergence form operator

$$-a \cdot \nabla^2 + b \cdot \nabla \equiv -\sum_{i,j=1}^d a_{ij}(x)\partial_{x_i} \partial_{x_j} + \sum_{k=1}^d b_k(x)\partial_{x_k}, \quad b \in F_{\delta_1}$$

in $L^q$ as $\Lambda_q(a,\nabla a + b)$ (formally, $-a \cdot \nabla^2 + b \cdot \nabla \equiv -\nabla \cdot a \cdot \nabla + (\nabla a) \cdot \nabla + b \cdot \nabla$) and then characterize quantitative dependence of the $W^{1,p}$ regularity of $u \equiv (\mu + \Lambda_q(a,\nabla a + b))^{-1}f$, $f \in L^q$, on $c, d, q, \mu$ and $\delta_1$ (Theorems 3 and 4). This result (including the class of first order perturbations $b \cdot \nabla$, $b \in F_{\delta_1}$ of $-a \cdot \nabla^2$) cannot be achieved on the basis of the Krylov-Safonov a priori estimates [Kr, Ch. 4.2]. (We note that the operator $-a \cdot \nabla^2$ with $\partial_{x_k} a_{ij} \in L^{d,\infty}$ has been studied earlier in [AT]; see also [ABT].)

The method of proof of the results of this paper admits immediate extension to

$$a_{ij}(x) = \delta_{ij} + \sum_l c_l \kappa_{ij}(x - x^l), \quad \kappa_{ij}(x) = |x|^2 x_i x_j,$$

\hspace{1cm} (2)

$$c_+ := \sum_{c_l > 0} c_l < \infty, \quad c_- := \sum_{c_l < 0} c_l > -1,$$

where $\{x^l\}$ is an arbitrary countable subset of $\mathbb{R}^d$.

The arguments in this paper can be transferred without significant changes from $\mathbb{R}^d$ to the ball $B(0,1)$.
Theorem 2 and the iteration method in [KS] allow to construct a Feller process associated with 
\(-\nabla \cdot a \cdot \nabla + b \cdot \nabla, b \in \mathbf{F}_b\). The method of this paper seems to be suited to treat classes of second-order perturbations 
\(-\nabla \cdot (a - I) \cdot \nabla, -(a - I) \cdot \nabla^2\) of \(-\Delta\) more general than (2), for example, given by 
a - I = vv, where (bounded) \(v : \mathbb{R}^d \to \mathbb{R}^d, v \in W^{1,2}_\text{loc}(\mathbb{R}^d, \mathbb{R}^d)\) satisfies \((\sum_{k}(\nabla v_k)^2)^{\frac{1}{2}} \in F_\delta\). We plan to address this matter in another paper.

1. We now state our results in full. First, consider the divergence form operator.

**Theorem 1** \((-\nabla \cdot a \cdot \nabla\). Let \(d \geq 3\), \(a(x) = I + c|x|^{-2}x \otimes x\), \(c > -1\).

(i) The formal differential operator \(-\nabla \cdot a \cdot \nabla\) has an operator realization \(A_q\) on \(L^q\), \(q \in [1, \infty]\), as 
the (minus) generator of a positivity preserving \(L^\infty\) contraction \(C_0\) semigroup.

Set \(u := (\mu + A_q)^{-1}f\), \(\mu > 0\), \(f \in L^q\).

(ii) Let \(d \geq 4\). Assume that \(q \geq d - 2\) and 
\[-\frac{(q-1)(d-2)^2}{q^2\ell_2} < c < \frac{(q-1)(d-2)^2}{q^2\ell_1},\]

where 
\[\ell_1 \equiv \ell_1(q, d) := d - 1 - d\frac{d - 2}{q} + (1 + \theta)\frac{(d - 2)^2}{q^2}, \quad \ell_2 \equiv \ell_2(q, d) := d - 1 + (q - 1)\frac{(d - 2)^2}{q^2}.\]

Then \(u \in \bigcap_{q \leq p \leq \frac{qd}{d-2}} W^{1,p}\), and there exist constants \(K_l = K_l(d, q, c), l = 1, 2\), such that 
\[
\|\nabla u\|_q \leq K_1\mu^{-\frac{1}{q}}\|f\|_q, \\
\|\nabla u\|_{\frac{qd}{d-2}} \leq K_2\mu^{\frac{1}{q} - \frac{1}{2}}\|f\|_q. 
\]

The dependence on \(q\) and \(\mu\) in (*) is the best possible.

(iii) Let \(d \geq 3\). Assume that 
\[-\left(1 + \frac{4(d-1)}{(d-2)^2}\right)^{-1} < c < \frac{(d-2)^2}{4}.\]

Then \(u \in W^{2,2}\) and (*) holds with \(q = 2\).

Of special interest are the minimal assumptions on \(c\) such that the second estimate in (*) holds with 
some \(q > d - 2\).

**Corollary 1.** For \(d = 3\) and \(-\frac{1}{9} < c < \frac{1}{2}\), 
\[(\mu + A_2)^{-1}L^2 \subset C^{0,\gamma}, \quad \gamma = \frac{1}{2}.
\]

For all \(d \geq 4\), \(-\frac{1}{2 + \frac{2}{d-3}} < c < 2(d-1)(d-3)\) and \(q > d - 2\) sufficiently close to \(d - 2\), 
\[(\mu + A_q)^{-1}L^q \subset C^{0,\gamma}, \quad \gamma = 1 - \frac{d-2}{q}.
\]
Theorem 2 \((-\nabla \cdot a \cdot \nabla + b \cdot \nabla)\). Let \(d \geq 3\), \(a(x) = I + c|x|^{-2}x \otimes x\), \(c > -1\), \(b \in \mathbf{F}_\delta\).

(i) If \(\delta_1 := [1 \vee (1 + c)^{-2}] \delta < 4\), then \(-\nabla \cdot a \cdot \nabla + b \cdot \nabla\) has an operator realization \(\Lambda_q(a, b)\) in \(L^q\), \(q \in \left[\frac{2}{1 - \sqrt[4]{3}}, \infty\right]\), as the (minus) generator of a positivity preserving \(L^\infty\) contraction \(C_0\) semigroup.

(ii) Let \(d \geq 4\). Assume that \(q \geq d - 2\), \(\delta < 1 \land \frac{4}{(d-2)^2}\) and \(c\) satisfy one of the following two conditions:

1) \(c > 0\) and \(1 + c \left(1 - \frac{1}{2(d-1)} - \frac{2\sqrt{\delta}}{4}\right) \geq 0\), and

\[
(q - 1) \frac{(d-2)^2}{q^2} - L_1(c, \delta) > 0,
\]

2) \(-1 < c < 0\) and \(1 + c \left(1 + \frac{2\sqrt{\delta}}{4}\right) \geq 0\), and

\[
(q - 1) \frac{(d-2)^2}{q^2} - L_2(-c, \delta) > 0,
\]

where

\[
L_1(c, \delta) = L_1(c, \delta, q, d) := c \left[1 + \frac{q-2}{q} (d-2) - \left(q - 1 - \frac{1}{2(d-1)}\right) \frac{(d-2)^2}{q^2}\right]
\]

\[
+ \frac{c\sqrt{\delta}}{2} \left[\frac{(d-2)^2}{q} + (d+3)(d-2)\right] + \left[q\delta^2 + (q-2)q\sqrt{3}\right] \frac{(d-2)^2}{q^2},
\]

\[
L_2(c, \delta) = L_2(c, \delta, q, d) := c \left[-d + 1 + \frac{q-2}{q} (d-2) + (q - 1) \frac{(d-2)^2}{q^2}\right]
\]

\[
+ \frac{c\sqrt{\delta}}{2} \left[\frac{(d-2)^2}{q} + (d+3)(d-2)\right] + \left[q\delta^2 + (q-2)q\sqrt{3}\right] \frac{(d-2)^2}{q^2}.
\]

Then there exist constants \(\mu_0 = \mu_0(d, q, c, \delta) > 0\) and \(K_l = K_l(d, q, c, \delta)\), \(l = 1, 2\), such that for all \(\mu > \mu_0\), \(u := (\mu + \Lambda_q(a, b))^{-1} f\), \(f \in L^q\), is in \(W^{1,q} \cap W^{1,\frac{4d}{d-2}}\), and

\[
\|\nabla u\|_q \leq K_1(\mu - \mu_0)^{-\frac{1}{2}} \|f\|_q,
\]

\[
\|\nabla u\|_{\frac{4d}{d-2}} \leq K_2(\mu - \mu_0)^{\frac{1}{2} - \frac{1}{2}} \|f\|_q.
\]

(**) \(\|\nabla u\|_{\frac{4d}{d-2}} \leq K_2(\mu - \mu_0)^{\frac{1}{2} - \frac{1}{2}} \|f\|_q\).

(iii) Let \(d \geq 3\). Assume that \(\delta < 1 \vee (1 + c)^{-2}\) and

\[
c > 0, \quad 1 - \frac{4c}{(d-2)^2} - c\sqrt{\delta} \left(\frac{2d+3}{d-2} + 1\right) - \delta > 0,
\]

or

\[
-1 < c < 0, \quad 1 - |c| - |c| \frac{4(d-1)}{(d-2)^2} - |c|\sqrt{\delta} \left(\frac{2d+3}{d-2} + 1\right) - \delta > 0.
\]

Then \(u \in W^{2,2}\) and (**) holds with \(q = 2\).

Corollary 2. Let \(d \geq 3\), \(a(x) = I + c|x|^{-2}x \otimes x\), \(c > -1\), \(b \in \mathbf{F}_\delta\). If

\[
\left\{ \begin{array}{ll}
-\frac{1}{2+\sqrt[4]{3}} < c < 2(d-1)(d-3), & d \geq 4, \\
-\frac{1}{3} < c < \frac{1}{4}, & d = 3,
\end{array} \right. \quad \text{and \(\delta > 0\) is sufficiently small}
\]

or

\[
|c| \text{ is sufficiently small and } \delta < 1 \lor \frac{4}{(d-2)^2},
\]

\[
\right.
\]
then, for all \(d \geq 4\) and \(q > d - 2\) sufficiently close to \(d - 2\),
\[
(\mu + \Lambda_q(a, b))^{-1}L^q \subset C^{0, \gamma}, \quad \gamma = 1 - \frac{d - 2}{q};
\]
and, for \(d = 3\),
\[
(\mu + \Lambda_2(a, b))^{-1}L^2 \subset C^{0, \gamma}, \quad \gamma = \frac{1}{2}.
\]

**Remark.** In Theorem 2, if \(\delta = 0\), then the assumptions on \(q\) and \(c\) coincide with the ones in Theorem 1. On the other hand, if \(c = 0\), then the assumptions on \(\delta\) are reduced to \(\delta < 1 \wedge \frac{4}{(d-2)^2}\), so we recover the result in [KS, Lemma 5], [KiS, Theorem 3.7].

2. Next, we consider the non-divergence form operator.

**Theorem 3** \((-a \cdot \nabla^2)\). Let \(d \geq 3\), \(a(x) = I + c|x|^{-2}x \otimes x\), \(c > -1\).

(i) \(-a \cdot \nabla^2\) has an operator realization \(\Lambda_q(a, \nabla a)\) in \(L^q\), \(q \in [(1 - \frac{d-1}{d-2} \frac{c}{1+c})^{-1}, \infty]\) if \(0 < c < d - 2\), and \(q \in [1, \infty]\) if \(-1 < c < 0\), as the (minus) generator of a positivity preserving \(L^\infty\) contraction \(C_0\) semigroup.

Set \(u := (\mu + \Lambda_q(a, \nabla a))^{-1}f\), \(\mu > 0\), \(f \in L^q\).

(ii) Let \(d \geq 4\). Assume that \(q > d - 2\) and
\[
-\left(1 + \frac{1}{q} \frac{(q-2)^2}{4(d-2)(q-1)(q+d-3)}\right)^{-1} < c < \frac{d-3}{2} \wedge \frac{d-2}{q-d+2}.
\]

Then \(u \in W^{1,q} \cap W^{1,\frac{q}{d-2}}\), and there exist constants \(K_l = K_l(d, q, c), l = 1, 2\), such that \((*)\) holds (for \(u = (\mu + \Lambda_q(a, \nabla a))^{-1}f\)).

(iii) Let \(d \geq 3\) and \(q = 2\). Assume that \(-1 < c < \frac{(d-2)^2}{2(2+(d-2)(d-3))}\). Then \(u \in W^{2,2}\).

**Corollary 3.** (i) Let \(d \geq 4\). For all \(c \in [0, \frac{d-2}{2}]\) and \(q \in [d-2, d+\frac{2}{d-2}]\), or for all \(c \in -\frac{1}{1+\frac{1}{2(d-2)(d-3)}}[0, \frac{d-2}{2}]\) and \(q > d - 2\) sufficiently close to \(d - 2\),
\[
(\mu + \Lambda_q(a, \nabla a))^{-1}L^q \subset C^{0, \gamma}, \quad \gamma = 1 - \frac{d - 2}{q}.
\]

(ii) For \(d = 3\) and all \(c \in [-1, \frac{1}{3}]\),
\[
(\mu + \Lambda_2(a, \nabla a))^{-1}L^2 \subset C^{0, \gamma}, \quad \gamma = \frac{1}{2}.
\]

**Remark.** Set \(a^\varepsilon := I + c|x|^{-2}x \otimes x\), \(\|x\| := \sqrt{|x|^2 + \varepsilon}\), \(\varepsilon > 0\). Let \(d \geq 4\). Then, in the assumptions of Theorem 3, we have
\[
(\mu + \Lambda_q(a, \nabla a))^{-1} = s-L^q \lim_{\varepsilon \downarrow 0}(\mu + \Lambda_q(a^\varepsilon, \nabla a^\varepsilon))^{-1}.
\]
(3)
See Theorem A.2 for details. In particular, \(u := (\mu + \Lambda_q(a, \nabla a))^{-1}f\), \(f \in L^q\), is a good solution of \((\mu - a \cdot \nabla^2)u = f\) in the sense of [CEF].

**Theorem 4** \((-a \cdot \nabla^2 + b \cdot \nabla)\). Let \(d \geq 3\), \(a(x) = I + c|x|^{-2}x \otimes x\), \(-1 < c < d - 2\), \(b \in F_\delta\).

Set \(\delta_1 := [1 \vee (1 + c)^{-2}]\), and
\[
\sqrt{\delta_2} := \left\{\begin{array}{ll}
\sqrt{\delta_1} + \frac{d-1}{d-2} \frac{c}{1+c}, & 0 < c < d - 2, \\
\sqrt{\delta_1}, & -1 < c < 0.
\end{array}\right.
\]
(i) Assume that $\delta_2 < 4$. Then $-a \cdot \nabla^2 + b \cdot \nabla$ has an operator realization $\Lambda_q(a, \nabla a + b)$ in $L^q$, $q \in \left[\frac{2}{2-\sqrt{\delta_2}}, \infty \right]$, as the (minus) generator of a positivity preserving $L^\infty$ contraction $C_0$ semigroup.

(ii) Let $d \geq 4$. Assume that $q > d - 2$, $\delta < 1 + \frac{4}{(d-2)^2}$ and $c$ satisfy one of the following two conditions:

1) $c > 0$ and $1 + c(1 - \frac{q}{d-2} - \frac{2\sqrt{q}}{q}) \geq 0$, and

$$
(q - 1) \frac{(d - 2)^2}{q^2} - L_1^{nd}(c, \delta) > 0,
$$

$$
L_1^{nd}(c, \delta) \equiv L_1^{nd}(c, \delta, q, d) := c(1 + \theta) \frac{(d - 2)^2}{q^2} + \frac{c\sqrt{\delta}}{2} \left[ \frac{(d-2)^2}{q} + (d+3)(d-2) \right] + \left[ q^2 \delta \frac{4}{4} + (q-2) q \sqrt{\delta} \right] \frac{(d-2)^2}{q^2}, \quad \theta = \frac{q}{d-2}.
$$

2) $-1 < c < 0$ and $1 + c(1 + \frac{2\sqrt{q}}{q}) \geq 0$, and

$$
(q - 1) \frac{(d - 2)^2}{q^2} - L_2^{nd}(-c, \delta) > 0,
$$

$$
L_2^{nd}(c, \delta) \equiv L_2^{nd}(c, \delta, q, d) := c \left[ 1 + (q - 2)(1 + \theta) \right] \frac{(d - 2)^2}{q^2} + \frac{c\sqrt{\delta}}{2} \left[ \frac{(d-2)^2}{q} + (d+3)(d-2) \right] + \left[ q^2 \delta \frac{4}{4} + (q-2) q \sqrt{\delta} \right] \frac{(d-2)^2}{q^2}, \quad \theta := \frac{1 - q}{4d - 2q + d - 3}.
$$

Then there exist constants $\mu_0 = \mu_0(d, q, c, \delta) > 0$ and $K_l = K_l(d, q, c, \delta)$, $l = 1, 2$, such that $u := (\mu + \Lambda_q(a, \nabla a + b))^{-1}f$, $f \in L^q$, is in $W^{1,q} \cap W^{1,\frac{d}{d-2}}$ for all $\mu > \mu_0$, and $(\star \star)$ hold.

(iii) Let $d \geq 3$ and $q = 2$. Assume that

$$
c > 0, \quad \sqrt{\delta} + 2 \frac{d-1}{d-2} \frac{c}{1 + c} < 1, \quad 1 - \frac{4c}{(d-2)^2} \left( 1 + \frac{(d-2)(d-3)}{2} \right) - c\sqrt{\delta} \left( \frac{2d+3}{d-2} + 1 \right) - \delta > 0
$$
or

$$
-1 < c < 0, \quad \delta < (1 + |c|)^{-2}, \quad 1 - |c| - |c| \sqrt{\delta} \left( \frac{2d+3}{d-2} + 1 \right) - \delta > 0.
$$

Then $u \in W^{2,2}$.

**Corollary 4.** Let $d \geq 3$, $a(x) = I + c|x|^{-2}x \otimes x$, $b \in F_\delta$. Assume that

$$
\begin{cases}
-\frac{1}{1 + \frac{1}{d-2}} < c < \frac{d-3}{2}, & d \geq 4, \\
-1 < c < \frac{1}{3}, & d = 3,
\end{cases}
$$

and $\delta > 0$ is sufficiently small,

or

$$
|c| \text{ is sufficiently small and } \delta < 1 + \frac{4}{(d-2)^2}.
$$

Let $d \geq 4$. Then, for all $q \in [d-2, d + \frac{2}{d-3}]$ in case of positive $c$, and for a $q > d - 2$ sufficiently close to $d - 2$ in case of negative $c$, we have

$$(\mu + \Lambda_q(a, \nabla a + b))^{-1}L^q \subset C^{0,\gamma}$$. \quad \gamma = 1 - \frac{d-2}{q}$$
Let \( d = 3 \). Then
\[
(\mu + \Lambda_2(a, \nabla a + b))^{-1}L^2 \subset C^{0,\gamma}, \quad \gamma = \frac{1}{2}.
\]

In conclusion, we mention that in Theorems 2-4 we tried to find optimal constraints on \( c \) and \( \delta \) such that \((*)\), \((**)\) hold. The weaker result that there exist sufficiently small \( c \) and \( \delta \) such that \((*)\), \((**)\) are valid (still not accessible by the existing results prior to our work) can be obtained with considerably less effort.

We have included Appendix A to make the paper self-contained.

1. Proof of Theorem 1

In what follows, we use notation
\[
\langle h \rangle := \int_{\mathbb{R}^d} h(x)dx, \quad \langle h, g \rangle := \langle h \tilde{g} \rangle.
\]

Define \( t[u,v] := \langle \nabla u \cdot a \cdot \nabla v \rangle, D(t) = W^{1,2} \). There is a unique self-adjoint operator \( A \equiv A_2 \geq 0 \) on \( L^2 \) associated with the form \( t: D(A) \subset D(t), \langle Au, v \rangle = \langle t[u,v], u \rangle, u \in D(A), v \in D(t) \). \( -A \) is the generator of a positivity preserving \( L^\infty \) contraction \( C_0 \) semigroup \( T_t^2 \equiv e^{-tA}, t \geq 0, \) on \( L^2 \).

By interpolation, \( T^t_r := \left[ T^t_2 \mid L^r \cap L^2 \right]^{\text{clos}}_{L^1 \rightarrow L^r} \) determines a \( C_0 \) semigroup on \( L^r \) for all \( r \in [2, \infty] \) and hence, by self-adjointness, for all \( r \in [1, \infty] \). The (minus) generator \( A_r \) of \( T^t_r (\equiv e^{-tA_r}) \) is the desired operator realization of \( \nabla \cdot a \cdot \nabla \) on \( L^r, r \in [1, \infty] \). One can furthermore show that \( T^t_1 := \left[ T^t_2 \mid L^1 \cap L^2 \right]^{\text{clos}}_{L^1 \rightarrow L^1} \) is a \( C_0 \) semigroup. This completes the proof of the assertion \((i)\) of the theorem.

To prove \((ii)\), we will need the following notation and auxiliary results. Define the smoothed out matrices \( a^\varepsilon = (a^\varepsilon_{ij}) \), \( 1 \leq i, j \leq d, \varepsilon > 0 \) by
\[
a^\varepsilon_{ij} := \delta_{ij} + c|x|_\varepsilon^{-2}x_i x_j, \quad |x|_\varepsilon := \sqrt{|x|^2 + \varepsilon}.
\]

Set \( u \equiv u^\varepsilon = (\mu + A^\varepsilon_q)^{-1}f, \ A^\varepsilon_q := A_q(a^\varepsilon), 0 \leq f \in C^1_c \). Clearly \( a^\varepsilon \in C^\infty \) and \( 0 \leq u^\varepsilon \in W^{3,q} \). Denote \( w \equiv w^\varepsilon := \nabla u^\varepsilon \),
\[
I_q := \sum_{r=1}^d \langle (\nabla_r w)^2 |w|^{q-2} \rangle, \quad J_q := \langle (|\nabla w|^2 |w|^{q-2}) \rangle,
\]
\[
\bar{I}_{q,\chi} := \langle (x \cdot \nabla) w \rangle \chi |x|_\varepsilon^{-2} |w|^{q-2}, \quad \bar{J}_{q,\chi} := \langle (x \cdot \nabla |w|^2 \chi)|x|_\varepsilon^{-2} |w|^{q-2}, \quad \chi := |x|^2 |x|_\varepsilon^{-2},
\]
\[
H_{q,\chi} := \langle \chi |x|_\varepsilon^{-2} |w|^{q} \rangle, \quad H_{q,\chi^2} := \langle \chi^2 |x|_\varepsilon^{-2} |w|^{q} \rangle, \quad G_{q,\chi^2} := \langle \chi^2 |x|_\varepsilon^{-2} |w|^{q} \rangle.
\]

1. The following inequality plays an important role in the proof of Theorem 1.

**Lemma 1** (Hardy-type inequality).
\[
\frac{d^2}{4} H_{q,\chi} - (d + 2)H_{q,\chi^2} + 3H_{q,\chi^3} \leq \frac{q^2}{4} \bar{J}_{q,\chi} \quad \text{ (HI)}
\]

**Proof.** Set \( F := |x|_\varepsilon^{-1} |w|^{\frac{q}{2}} \). Then
\[
\frac{q^2}{4} \bar{J}_{q,\chi} = \langle (|x|_\varepsilon^{-1} x \cdot \nabla |w|^\frac{q}{2})^2 \rangle = \langle (x \cdot \nabla F + \chi F)^2 \rangle = \langle (x \cdot \nabla F)^2 + \chi^2 F^2 \rangle + 2 \langle x \cdot \nabla F, \chi F \rangle.
\]

(HI) follows from the inequality \( \langle (x \cdot \nabla F)^2 \rangle \equiv \| x \cdot \nabla F \|_2^2 \geq \frac{d^2}{4} \| F \|_2^2 \equiv \frac{d^2}{4} H_{q,\chi} \) and the equalities
\[
2 \langle x \cdot \nabla F, \chi F \rangle = -d \langle \chi F^2 \rangle - \langle F^2, x \cdot \nabla \chi \rangle, \quad x \cdot \nabla \chi = 2 (|x|^2 |x|_\varepsilon^{-2} - |x|_\varepsilon^4) = 2 \chi (1 - \chi).
\]

The following equalities are crucial steps in the proof of Theorem 1.
Lemma 2 (The basic equalities).
\[
\mu(|w|^q) + I_q + c\bar{J}_{q,\chi} + (q - 2)(J_q + c\bar{J}_{q,\chi}) - c\left(1 + (q - 2)\frac{d}{q}\right)H_{q,\chi} + 2c(d - 1)G_{q,\chi^2} + 2c\frac{g - 2}{q}H_{q,\chi^2} + 8c\varepsilon(|x|^{-6}(x \cdot w)^2|w|^{-q^2}) = \beta_1 + \langle f, \phi \rangle, \quad (\text{BE}_+)\]
\[
\mu(|w|^q) + I_q + c\bar{J}_{q,\chi} + (q - 2)(J_q + c\bar{J}_{q,\chi}) - c\left(1 + (q - 2)\frac{d}{q}\right)H_{q,\chi} + cdG_{q,\chi^2} + 2c\frac{g - 2}{q}H_{q,\chi^2} + 4c\varepsilon(|x|^{-6}(x \cdot w)^2|w|^{-q^2}) = -\frac{1}{2}\beta_2 + \langle f, \phi \rangle, \quad (\text{BE}_-)
\]
where \(\phi = -\nabla \cdot (w|w|^{-q^2})\),
\[
\beta_1 := -2c\langle |x|^{-4}x \cdot w, x \cdot (x \cdot \nabla |w|)|w|^{-q^2} \rangle, \quad \beta_2 := -2c(q - 2)\langle |x|^{-4}(x \cdot w)^2x \cdot \nabla |w|, |w|^{-q^2} \rangle.
\]

Remark. Below we use the representation (\text{BE}_+) in case \(c > 0\), and the representation (\text{BE}_-) in case \(c < 0\). (One could still use (\text{BE}_+) for \(c < 0\) or (\text{BE}_-) for \(c > 0\), but this would lead to more restrictive constraints on \(c\).

Proof of Lemma 2. Set \([F, G]_- := FG - GF\). We multiply \(\mu u + A^\varepsilon_q u = f\) by \(\phi\) and integrate:
\[
\mu(|w|^q) + \langle A^\varepsilon_q w, w|w|^{-q^2} \rangle + \langle [\nabla, A^\varepsilon_q]_- u, w|w|^{-q^2} \rangle = \langle f, \phi \rangle,
\]
\[
\mu(|w|^q) + I_q + c\bar{J}_{q,\chi} + (q - 2)(J_q + c\bar{J}_{q,\chi}) + \langle [\nabla, A^\varepsilon_q]_- u, w|w|^{-q^2} \rangle = \langle f, \phi \rangle. \quad (4)
\]
The term to evaluate: \(\langle [\nabla, A^\varepsilon_q]_- u, w|w|^{-q^2} \rangle \equiv \langle [\nabla_r, A^\varepsilon_q]_- u, w_r|w|^{-q^2} \rangle := \sum_{r=1}^d \langle [\nabla_r, A^\varepsilon_q]_- u, w_r|w|^{-q^2} \rangle\).

Note that
\[
[\nabla_r, A^\varepsilon_q]_- = -\nabla \cdot (\nabla_r \varepsilon^r) \cdot \nabla, \quad (\nabla_r \varepsilon^r)_i = c|x|^{-2}\delta_{ri}x_k + c(|x|^{-2}\delta_{rk}x_i - 2|x|^{-4}x_i x_k x_r),
\]
\[
\langle [\nabla_r, A^\varepsilon_q]_- u, w_r|w|^{-q^2} \rangle = -c\langle w_k\nabla_i(|x|^{-2}\delta_{ri}x_k) + |x|^{-2}\delta_{ri}x_k \nabla_i w_k, w_r|w|^{-q^2} \rangle + c\langle (|x|^{-2}\delta_{rk}x_i - 2|x|^{-4}x_i x_k x_r)w_k, \nabla_i(w_r|w|^{-q^2}) \rangle
\]
\[
= \alpha_1 + \alpha_2,
\]
\[
\alpha_1 = -c\langle (|x|^{-2}\delta_{rk} - 2|x|^{-4}\delta_{ri}x_k x_r)w_k + |x|^{-2}x \cdot \nabla w_r, w_r|w|^{-q^2} \rangle
\]
\[
= -c\langle |x|^{-2}|w|^{q^2} \rangle + 2c\langle |x|^{-4}(x \cdot w)^2|w|^{-q^2} \rangle - c\langle |x|^{-2}x \cdot \nabla |w|, |w|^{-q^1} \rangle.
\]

Then
\[
\alpha_1 = -c\left(1 - \frac{d - 2}{q}\right)H_{q,\chi} + 2cG_{q,\chi^2} + 2c\varepsilon(|x|^{-4}|w|^{q})
\]
due to \(\langle |x|^{-2}x \cdot \nabla |w|, |w|^{-q^1} \rangle = \frac{1}{q}\langle |x|^{-2}x \cdot \nabla |w|^{q^1} \rangle = -\frac{1}{q}\langle |w|^q \nabla \cdot (x |x|^{-2}) \rangle = -\frac{d}{q}H_{q,\chi} + \frac{2}{q}\langle |x|^2|x|^{-4}|w|^q \rangle = -\frac{d - 2}{q}H_{q,\chi} - \frac{2}{q}\varepsilon(|x|^{-4}|w|^{q})\), and
\[
\alpha_2 = c\langle |x|^{-2}w, x \cdot \nabla (w|w|^{-q^2}) \rangle - 2c\langle |x|^{-4}x \cdot w, x \cdot (x \cdot \nabla (w|w|^{-q^2})) \rangle
\]

Then
\[
\alpha_2 = \beta_1 + \beta_2 + c\langle |x|^{-2}x \cdot \nabla |w|, |w|^{-q^1} \rangle + c(q - 2)\langle |x|^{-2}x \cdot \nabla |w|, |w|^{-q^1} \rangle
\]
\[
= \beta_1 + \beta_2 + c(q - 1)\left(\frac{d - 2}{q}H_{q,\chi} + \frac{2}{q}\varepsilon(|x|^{-4}, |w|^{q})\right).
\]
In view of 
\[
\beta_1 = -\frac{1}{2}\beta_2 + c(d-2)G_{q,\chi^2} + 4ce \langle |x|^6(x \cdot w)^2|w|^{q-2}\rangle,
\]
we rewrite \(\alpha_1 + \alpha_2 = \langle [\nabla, A^\varepsilon]_u, w|w|^{q-2}\rangle\) in two ways:
\[
\langle [\nabla, A^\varepsilon]_u, w|w|^{q-2}\rangle = -\beta_1 - c\left(1 + \left(q-2\right)\frac{d-2}{q}\right)H_{q,\chi} + 2c(d-1)G_{q,\chi^2}
- 2c\frac{q-2}{q}\varepsilon\langle |x|^{-4}|w|^q\rangle + 8ce\langle |x|^6(x \cdot w)^2|w|^{q-2}\rangle
\]
and
\[
\langle [\nabla, A^\varepsilon]_u, w|w|^{q-2}\rangle = \frac{1}{2}\beta_2 - c\left(1 + \left(q-2\right)\frac{d-2}{q}\right)H_{q,\chi} + c\varepsilon G_{q,\chi^2}
- 2c\frac{q-2}{q}\varepsilon\langle |x|^{-4}|w|^q\rangle + 4ce\langle |x|^6(x \cdot w)^2|w|^{q-2}\rangle.
\]
The last two identities applied in (4) yield (BE\(_+\)), (BE\(_-\)).  

2. Next, we estimate from above the term \(\langle f, \phi \rangle\) in the right-hand side of (BE\(_+\)), (BE\(_-\)).

**Lemma 3.** For each \(\varepsilon_0 > 0\) there exists a constant \(C(\varepsilon_0) < \infty\) such that
\[
\langle f, \phi \rangle \leq \varepsilon_0(I_q + J_q + H_q) + C(\varepsilon_0)\|w\|_{q-2}^2\|f\|_q^2,
\]
where \(H_q := \langle |x|^{-2}|w|^q\rangle\).

**Proof of Lemma 3.** Clearly,
\[
\langle f, \phi \rangle = \langle f, (-\Delta u)|w|^{q-2} \rangle - \left(q-2\right)\langle f, |w|^{q-3}w \cdot \nabla |w| \rangle =: F_1 + F_2.
\]
Since \(-\Delta u = \nabla \cdot (a^\varepsilon - 1) \cdot w - \mu u + f\) and
\[
F_1 = \langle \nabla \cdot (a^\varepsilon - 1) \cdot w, |w|^{q-2} f \rangle + \langle (-\mu u + f), |w|^{q-2} f \rangle
\]
(we expand the first term using \(\nabla a^\varepsilon = c(d+1)x|x|^{-2} - 2c|x|^2|x|^{-4}x\))
\[
= c(d+1)\langle |x|^{-2}x \cdot w, |w|^{q-2} f \rangle - 2c\langle \chi \langle |x|^{-2}x \cdot w, |w|^{q-2}(-b \cdot w + f) \rangle + c\langle |x|^{-2}x \cdot (x \cdot \nabla w), |w|^{q-2} f \rangle + \langle (-\mu u + f), |w|^{q-2} f \rangle.
\]
We bound from above \(F_1\) and \(F_2\) by applying consecutively the following estimates:

1) \(\langle |x|^{-2}x \cdot w, |w|^{q-2} f \rangle \leq \frac{H_{q,\chi}^2}{q^2} \|w\|_{q-2}^2\|f\|_q^2\).

2) \(\langle \chi |x|^{-2}x \cdot w, |w|^{q-2} f \rangle \leq \frac{H_{q,\chi}^2}{q^2} \|w\|_{q-2}^2\|f\|_q^2\).

3) \(\langle |x|^{-2}x \cdot (x \cdot \nabla w), |w|^{q-2} f \rangle \leq \langle \bar{I}_{q,\chi} \rangle^2 \frac{1}{q^2} \|w\|_{q-2}^q\|f\|_q^2\).

4) \(\langle -f, |w|^{q-2} \mu u \rangle \leq 0\).

5) \(\langle f, |w|^{q-2} f \rangle \leq \|w\|_{q-2}^2\|f\|_q^2\).

6) \(\langle q-2 \rangle \langle -f, |w|^{q-3}w \cdot \nabla |w| \rangle \leq \langle q-2 \rangle J_{q,\chi}^2 \|w\|_{q-2}^2\|f\|_q^2\).

1)-6) and the standard quadratic estimates now yield the lemma.  

\(\square\)
We choose \( \varepsilon_0 > 0 \) in Lemma 3 so small that in the estimates below we can ignore \( \varepsilon_0(I_q + J_q + H_q) \).

3. We will use (\BE_+), (\BE_-) and Lemma 3 to prove the following inequality

\[
\mu(\|w\|^q) + \eta J_q \leq C\|w\|_q^{q-2}\|f\|_q^2, \quad C = C(\varepsilon_0)
\]  

(5)

for some \( \eta = \eta(q, d, \varepsilon_0) > 0 \).

**Case** \( c > 0 \). In (\BE_+) we omit the term \( 8c \varepsilon \langle |x|^{-6} (x \cdot w)^2|w|^{q-2} \rangle \), obtaining

\[
\mu(\|w\|^q) + I_q + c I_{q, \chi} + (q - 2)(J_q + c I_{q, \chi}) - c \left( 1 + \frac{q - 2}{q} d \right) H_{q, \chi} + 2c(d - 1) G_{q, \chi^2}
\]

\[
+ 2c\frac{q - 2}{q} H_{q, \chi^2} \leq \beta_1 + \langle f, \phi \rangle.
\]

Estimating \( \beta_1 \) from above using the standard quadratic estimates,

\[
\beta_1 \leq 2c \langle |x|^{-4} (x \cdot w)^2|w|^{q-2} \rangle \frac{1}{2} \langle |x|^{-4} |x|^2 (x \cdot \nabla w)^2|w|^{q-2} \rangle \frac{1}{2} \leq 2c(G_{q, \chi^2} I_{q, \chi^2})^\frac{1}{2} \leq c \theta I_{q, \chi} + c \theta^{-1} G_{q, \chi^2}
\]

(\( \theta > 0 \)), and then applying Lemma 3, we have

\[
\mu(\|w\|^q) + I_q + c(1 - \theta) I_{q, \chi} + (q - 2)(J_q + c I_{q, \chi}) - c \left( 1 + \frac{q - 2}{q} d \right) H_{q, \chi} + c(2(d - 1) - \theta^{-1}) G_{q, \chi^2}
\]

\[
+ 2c\frac{q - 2}{q} H_{q, \chi^2} \leq C\|w\|_q^{q-2}\|f\|_q^2.
\]

Let \( 0 < \theta < 1 \). Using the inequalities \( J_q \leq I_q, \bar{I}_{q, \chi} \leq I_{q, \chi} \) and \( \frac{1}{q^2} \left( \frac{d^2}{4} H_{q, \chi} - (d + 2) H_{q, \chi^2} + 3 H_{q, \chi^3} \right) \leq \bar{I}_{q, \chi}, \) see (\HI), we have

\[
\mu(\|w\|^q) + \eta J_q + (-\eta + q - 1) J_q + [2c(d - 1) - c \theta^{-1}] G_{q, \chi^2} + c \langle M(\chi)|x|^{-2}|w|^q \rangle \leq C\|w\|_q^{q-2}\|f\|_q^2,
\]

where

\[
M(\chi) := \left[ (q - 1 - \theta) \frac{4}{q^2} \left( \frac{d^2}{4} - (d + 2) \chi + 3 \chi^2 \right) - \left( 1 + \frac{q - 2}{q} d \right) + 2q - 2 \frac{q - 2}{q} \chi \right] \chi,
\]

i.e.

\[
M(\chi) := [a \chi^2 + b \chi + c_0] \chi,
\]

where

\[
a = \frac{12}{q^2}(q - 1 - \theta), \quad b = -4(q - 1 - \theta)^2 + 2q - 2 \frac{q - 2}{q}, \quad c_0 = \frac{d^2}{q^2}(q - 1 - \theta) + 2d - 1 - d.
\]

Elementary arguments show that the choice \( \theta := \frac{1}{2(d - 1)} \) is the best possible. In particular,

\[
\min_{0 \leq t \leq 1} M(t) = M(1) < 0.
\]

Since \( -\eta + q - 1 > 0 \) for all \( q > 0 \) sufficiently small, we can use \( J_q \geq \frac{(d - 2)^2}{q^2} H_q \), obtaining

\[
\mu(\|w\|^q) + \eta J_q + \left( (-\eta + q - 1) \frac{(d - 2)^2}{q^2} + c M(1) \right) H_q \leq C\|w\|_q^{q-2}\|f\|_q^2,
\]

Recalling the assumption \( (q - 1) \frac{(d - 2)^2}{q^2} - c \ell_1 > 0, \ell_1 = -M(1), \) it is seen that there exists \( \eta > 0 \) such that \( (-\eta + q - 1) \frac{(d - 2)^2}{q^2} \geq c \ell_1. \) (5) is proved for \( 0 < c < \frac{(q - 1)(d - 2)^2}{q^2 \ell_1} \).

The choice of \( \theta \in [1, 1 + c^{-1}] \) leads to sub-optimal constraints on \( c \) and \( q \).
Case $-1 < c < 0$. Set $s := |c|$. In (BE$_-$), we estimate ($\theta > 0$)

$$|\beta_2| \leq 2s(q-2)(\theta J_{q,\chi} + 4^{-1}\theta^{-1}G_{q,\chi^2}).$$

By (BE$_-$) and Lemma 3,

$$\mu(|w|^q) + I_q - s\tilde{I}_{q,\chi} + (q-2)(J_q - s(1+\theta)J_{q,\chi}) + s\left(1 + (q-2)\frac{d}{q}\right)H_{q,\chi}$$

$$- 2s\frac{q-2}{q}H_{q,\chi^2} - sdG_{q,\chi^2} - 4sG_{q,\chi^2} + 4sG_{q,\chi^2} - s(q-2)\frac{1}{4\theta}G_{q,\chi^2} \leq C\|w\|_{q-2}^2\|f\|_{q}^2.$$  

Clearly, $I_q - s\tilde{I}_{q,\chi} + (q-2)(J_q - s(1+\theta)J_{q,\chi}) \geq (q-1 - s(q-2)(1+\theta))J_q$. Therefore

$$\mu(|w|^q) + (q-1 - s(q-2)(1+\theta))J_q + s\left(1 + (q-2)\frac{d}{q}\right)H_{q,\chi}$$

$$- 2s\frac{q-2}{q}H_{q,\chi^2} - sdG_{q,\chi^2} - 4sG_{q,\chi^2} + 4sG_{q,\chi^2} - s(q-2)\frac{1}{4\theta}G_{q,\chi^2} \leq C\|w\|_{q-2}^2\|f\|_{q}^2,$$

i.e.

$$\mu(|w|^q) + \eta J_q + (-\eta + q - 1 - s(q-2)(1+\theta))J_q + s(M(\chi)|x|^{-4}(x \cdot w)^2|w|^{q-2}) \leq C\|w\|_{q-2}^2\|f\|_{q}^2,$$

where

$$M(\chi) := \left[1 + (q-2)\frac{d-2}{q} + \left(-d - 4 + 4\chi - (q-2)\frac{1}{4\theta}\right)\chi\right]$$

i.e.

$$M(\chi) = [a\chi^2 + b\chi + c_0] \chi,$$

where

$$a = 4, \quad b = -d - 4 - \frac{1}{2}(q-2)\frac{d-2}{q}, \quad c_0 = 1 + (q-2)\frac{d-2}{q}.$$  

Select $\theta := \frac{\frac{1}{2}\|f\|_{q}^2}{\frac{q}{q-2}}$. (Motivation: Below we estimate $I_q - s\tilde{I}_{q,\chi} + (q-2)(J_q - s(1+\theta)J_{q,\chi}) \geq (q-1 - s(q-2)(1+\theta))J_q \geq (q-1 - s(q-2)(1+\theta))\frac{(d-2)^2}{q^2}G_q$, so estimating the terms involving $\theta$ in the resulting inequality as $-s(q-2)\theta\frac{(d-2)^2}{q^2}G_q - (q-2)\frac{1}{4\theta}G_{q,\chi^2} \geq -(s(q-2)\theta\frac{(d-2)^2}{q^2} - (q-2)\frac{1}{4\theta})G_q$, we arrive clearly at $\theta = \frac{\frac{1}{2}\|f\|_{q}^2}{\frac{q}{q-2}}$.)

Elementary arguments show that $\min_{0 \leq t \leq 1} M(t) = M(1) < 0$. By the assumptions of the theorem,

$$(-\eta + q - 1 - s(q-2)(1+\theta))\frac{(d-2)^2}{q^2} + sM(1) \geq 0.$$  

Thus, by $J_q \geq \frac{(d-2)^2}{q^2}H_q$ and $H_q \geq G_q$,

$$\mu(|w|^q) + \eta J_q + \left[(-\eta + q - 1 - s(q-2)(1+\theta))\frac{(d-2)^2}{q^2} + sM(1)\right]G_q \leq C\|w\|_{q-2}^2\|f\|_{q}^2,$$
or, setting $\ell_2 := [1 + (q - 2)(1 + \theta)]^{(d-2)^2 q}/q^2 - M(1)$,
\[
\mu \langle |w|^q \rangle + \eta J_q + \left[ (\eta + q - 1) \frac{(d-2)^2}{q^2} + c \ell_2 \right] G_q \leq C \|w\|_{q}^{q-2} \|f\|_{q}^2.
\]
(5) is proved.

4. By (5), $\mu \|w\|_q^q \leq C \|w\|_{q}^{q-2} \|f\|_q^2$, $w = \nabla u^\varepsilon$, $\varepsilon > 0$, and so
\[
\|\nabla u^\varepsilon\|_q \leq K_1 \mu^{-\frac{1}{2}} \|f\|_q, \quad K_1 := C^\varepsilon.
\]
Again by (5), $\eta J_q \leq C \|w\|_{q}^{q-2} \|f\|_q^2$, $J_q = \frac{4}{q^2} \|\nabla |w|\|_2^2$, so in view of the previous inequality $\eta \|\nabla \nabla u^\varepsilon\|_2^2 \leq \frac{q^2}{4} CK_1^{q-2} \mu^{1-\frac{1}{2}} \|f\|_q^2$. The Sobolev Embedding Theorem now yields
\[
\|\nabla u^\varepsilon\|_{qj} \leq K_2 \mu^{-\frac{1}{2} - \frac{1}{2}}, \quad K_2 := C_3 \eta^{-\frac{1}{2}} (q^2/4) \frac{1}{2} C^\frac{1}{1} K_1^{\frac{2-2}{q}}.
\]
Since the weak gradient in $L^q$ is closed, Theorem A.2(i) (with $b = 0$) yields $\|\nabla u\|_q \leq K_1 \mu^{-\frac{1}{2}} \|f\|_q$, $\|\nabla u\|_{qj} \leq K_2 \mu^{-\frac{1}{2} - \frac{1}{2}} \|f\|_q$ for $u = (\mu + A)^{-1} f$, $0 \leq f \in C^\infty$, and thus for all $f \in L^q$.

We have proved (ii).

**Proof of (iii).** Let $q = 2, d \geq 3$. The arguments above yield $(I_2 \equiv I_2(u^\varepsilon))$

(a) For $c > 0$,
\[
\langle \nabla, A^\varepsilon_2 \rangle u, w \rangle = -\beta_1 - c H_{2,\varepsilon} + 2c(d-1) G_{2,\varepsilon},
\mu \|w\|_2^2 + I_2 + c i I_2 - \beta_1 - c H_{2,\varepsilon} + 2c(d-1) G_{2,\varepsilon} = \langle f, -\nabla \cdot w \rangle,
\beta_1 = -2c(|x|^{-4} x \cdot w, x \cdot (x \cdot \nabla w)).
\]

By $\beta_1 \leq 2c(\sqrt{G_{2,\varepsilon}} H_{2,\varepsilon} I_{2,\varepsilon} \leq c i I_2 + c G_{2,\varepsilon}^2, G_{2,\varepsilon} \leq H_{2,\varepsilon}$ and $I_2 \geq (d-2)^2 H_{2,\varepsilon}$, it follows that $1 - c^2(4\sqrt{d-2}) > 0 \Rightarrow I_2 \leq K \|f\|_2^2$;

(b) For $c < 0$,
\[
\langle \nabla, A^\varepsilon_2 \rangle u, w \rangle = \frac{1}{2} \beta_2 - c H_{2,\varepsilon} + c d G_{2,\varepsilon}^2, \quad \beta_2 = 0,
\mu \|w\|_2^2 + I_2 + c i I_2 - c H_{2,\varepsilon} + c d G_{2,\varepsilon}^2 = \langle f, -\nabla \cdot w \rangle, \quad I_2 \geq I_2,
\mu \|w\|_2^2 + (1 - |c|) I_2 + |c| H_{2,\varepsilon} - |c| d G_{2,\varepsilon}^2 \leq \langle f, -\nabla \cdot w \rangle.
\]

Thus, by $G_{2,\varepsilon} \leq H_{2,\varepsilon} \leq (d-2)^2 I_2$, we conclude that $1 - |c| + |c|(1 - d)(d-2)^2 > 0 \Rightarrow I_2(u^\varepsilon) \leq K \|f\|_2^2$.

By passing to the limit $\varepsilon \downarrow 0$, using Theorem A.2, we obtain $I_2(u) \leq K \|f\|_2$. Therefore, $u \in W^{2,2}$.

The proof of Theorem 1 is completed.

2. **Proof of Theorem 2**

**Proof of (i).** A vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $\mathbf{F}_{\delta_1}(A)$, $\delta_1 > 0$, the class of form-bounded vector fields (with respect to $A \equiv A_2$), if $b_a^2 := b \cdot a^{-1} \cdot b \in L^1_{\text{loc}}$, and there exists a constant $\lambda = \lambda_{\delta_1} > 0$ such that
\[
\|b_a(e + A)^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta_1}.
\]
It is easily seen that if $b \in \mathbf{F}_{\delta}$, then $b \in \mathbf{F}_{\delta_1}(A)$, where $\delta_1 := \delta$ if $c > 0$, and $\delta_1 := (1 + c)^{-2}$ if $-1 < c < 0$. By our assumption, $\delta_1 < 4$. Therefore, by [KGS, Theorem 3.2], $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ has an operator realization $\Lambda_q(a, b)$ in $L^q$, $q \in \left[ \frac{2}{d} - \frac{1}{2}, \infty \right]$, as the (minus) generator of a positivity preserving
contraction quasi contraction $C_0$ semigroup. Moreover, $(\mu + \Lambda_q(a,b))^{-1}$ is well defined on $L^q$ for all $\mu > \frac{M}{2(q-1)}$. This completes the proof of (i).

Proof of (ii). Set $a^\varepsilon := I + c|x|^{-2}\otimes x$, $|x| := |x|^2 + \varepsilon$, $\varepsilon > 0$. Put $A^\varepsilon = A(a^\varepsilon)$. It is clear that $b \in F_\delta(A^\varepsilon)$ for all $\varepsilon > 0$.

Let $1_n$ denote the indicator of $\{ x \in \mathbb{R}^d \mid |x| \leq n, |b(x)| \leq n \}$, and set $b_n := \gamma_\varepsilon * 1_n b \in C^{\infty}$, where $\gamma_\varepsilon$ is the K.Friedrichs mollifier, $\varepsilon n \downarrow 0$. Since our assumptions on $\delta$ and thus $\delta_1$ involve strict inequalities only, we can select $\varepsilon_n \downarrow 0$ so that $b_n \in F_\delta(A^\varepsilon)$, $\varepsilon > 0$, $n \geq 1$. Therefore, in view of the previous discussion, $(\mu + \Lambda_q(a^\varepsilon, b_n))^{-1}$ is well defined on $L^q$, $\mu > \frac{M}{2(q-1)}$, $\varepsilon > 0$, $n \geq 1$. Here $\Lambda_q(a^\varepsilon, b_n) = -\nabla \cdot a^\varepsilon \cdot \nabla + b_n \cdot \nabla$, $D(\Lambda_q(a^\varepsilon, b_n)) = W^{2,q}$.

Define $0 \leq u \equiv u^\varepsilon, n := (\mu + \Lambda_q(a^\varepsilon, b_n))^{-1} f$, $0 \leq f \in C^1$. Then $u \in W^{3,q}$. Set $w \equiv w^\varepsilon, n := \nabla u^\varepsilon, n$ and

$$I_q := (\nabla w)^2 |w|^{q-2}, \quad J_q := (|\nabla w|^2 |w|^{q-2})^2,$$

$$\bar{I}_{q, \chi} := (\chi |x|^{-2} |w|^{q-2}), \quad \bar{J}_{q, \chi} := (\chi |x|^{-2} |w|^{q-2}), \quad \chi = |x|^2 |x|^{-2},$$

$$H_{q, \chi} := (\chi |x|^{-2} |w|^q), \quad H_{q, \chi}^2 := (\chi |x|^{-2} |w|^q), \quad G_{q, \chi}^2 := (\chi |x|^{-2} |w|^q).$$

Below we follow closely the proof of Theorem 1.

1. We repeat the proof of Lemma 2, where in the right-hand side of (BE$_+$), (BE$_-$) we now get an extra term $\langle -b_n \cdot w, \phi \rangle$:

$$\mu(|w|^q) + I_q + c\bar{I}_{q, \chi} + (q - 2)(J_q + c\bar{J}_{q, \chi}) - c \left( 1 + (q - 2)\frac{d}{q} \right) H_{q, \chi} + 2c(d - 1)G_{q, \chi}^2 + 2c\frac{q - 2}{q} H_{q, \chi}^2 + 4c\varepsilon (|x|^{-6} (x \cdot w)^2 |w|^{q-2}) = \beta_1 (\langle -b_n \cdot w, \phi \rangle + \langle f, \phi \rangle), \quad (\text{BE}_+, \beta)$$

$$\mu(|w|^q) + I_q + c\bar{I}_{q, \chi} + (q - 2)(J_q + c\bar{J}_{q, \chi}) - c \left( 1 + (q - 2)\frac{d}{q} \right) H_{q, \chi} + cdG_{q, \chi}^2 + 2c\frac{q - 2}{q} H_{q, \chi}^2 + 4c\varepsilon (|x|^{-6} (x \cdot w)^2 |w|^{q-2}) = \frac{1}{2} \beta_2 (\langle -b_n \cdot w, \phi \rangle + \langle f, \phi \rangle), \quad (\text{BE}_-, \beta)$$

where, recall,

$$\beta_1 := -2c(|x|^{-4} x \cdot w, x \cdot \nabla w |w|^{q-2}), \quad \beta_2 := -2c(q - 2)(|x|^{-6} (x \cdot w)^2 x \cdot \nabla |w|, |w|^{q-3}).$$

We estimate $\langle -b_n \cdot w, \phi \rangle$ as follows.

**Lemma 4.** There exist constants $C_i$ $(i = 1, 2)$ such that

$$\langle -b_n \cdot w, \phi \rangle \leq |c|(d + 3)\frac{q\sqrt{\delta}}{2} G_{q, \chi}^2 J_q^2 + |c| \frac{q\sqrt{\delta}}{2} L_{q, \chi}^2 J_q^2 + \left( \frac{q\delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2} \right) J_q + C_1 ||w||_q^2 + C_2 ||w||_{q-2}^2 ||f||_q^2.$$

**Proof of Lemma 4.** For brevity, below $b \equiv b_n$. We have:

$$\langle -b \cdot w, \phi \rangle = \langle -\Delta u, |w|^{q-2} (-b \cdot w) \rangle + (q - 2)\langle |w|^{q-3} w \cdot \nabla |w|, -b \cdot w \rangle$$

$$=: F_1 + F_2.$$

Set $B_q := (|b \cdot w|^2 |w|^{q-2})$. We have

$$F_2 \leq (q - 2)B_q^\frac{3}{2} J_q^\frac{1}{2}.$$
Next, we bound $F_1$. We represent $-\Delta u = \nabla \cdot (a^2 - 1) \cdot w - \lambda u - b \cdot w + f$, and evaluate

$$ F_1 = \langle \nabla \cdot (a^2 - 1) \cdot w, |w|^{q-2}(b \cdot w) \rangle + \langle (-\lambda u - b \cdot w + f), |w|^{q-2}(b \cdot w) \rangle $$

(by expanding the first term using $\nabla a^e = c(d+1)|x|^{-2}x - 2c|x|^2|x|^{-4}x$)

$$ = c(d+1)\langle |x|^{-2}x \cdot w, |w|^{q-2}(b \cdot w) \rangle $$

$$ - 2c\langle \chi |x|^{-2}x \cdot w, |w|^{q-2}(b \cdot w) \rangle $$

$$ + c\langle |x|^{-2}x \cdot (x \cdot \nabla w), |w|^{q-2}(b \cdot w) \rangle $$

$$ + \langle (-\lambda u - b \cdot w + f), |w|^{q-2}(b \cdot w) \rangle. $$

We bound $F_1$ from above by applying consecutively the following estimates:

1) $\langle |x|^{-2}x \cdot w, |w|^{q-2}(b \cdot w) \rangle \leq G^\frac{1}{2}_{q,\chi} B^\frac{1}{q}_{q}.$

2) $\langle \chi |x|^{-2}x \cdot w, |w|^{q-2}(b \cdot w) \rangle \leq G^\frac{1}{2}_{q,\chi} B^\frac{1}{q}_{q} \leq G^\frac{1}{2}_{q,\chi} B^\frac{1}{q}_{q}.$

3) $\langle |x|^{-2}x \cdot (x \cdot \nabla w), |w|^{q-2}(b \cdot w) \rangle \leq \frac{1}{4} B^\frac{1}{q}_{q}.$

4) $\langle (-\lambda u), |w|^{q-2}(b \cdot w) \rangle \leq \frac{\lambda}{\delta} B^\frac{1}{q}_{q} \|w\|^{\frac{q-2}{2}} \|f\|_q$ (here $\frac{2}{2-\delta} < q \Rightarrow \|u_n\|_q \leq (\lambda - \omega q)^{-1}\|f\|_q$).

5) $\langle b \cdot w, |w|^{q-2}(b \cdot w) \rangle = B_q.$

6) $\langle f, |w|^{q-2}(b \cdot w) \rangle \leq B^\frac{1}{q}_{q} \|w\|^{\frac{q-2}{2}} \|f\|_q.$

In 4°) and 6° we estimate $B^\frac{1}{q}_{q} \|w\|^{\frac{q-2}{2}} \|f\|_q \leq \varepsilon_0 B_q + \frac{1}{4\varepsilon_0} \|w\|^{\frac{q-2}{2}} \|f\|_q^2 (\varepsilon_0 > 0).$

Therefore,

$$ \langle -b \cdot w, \phi \rangle $$

$$ \leq |c|(d + 3)G^\frac{1}{2}_{q,\chi} B^\frac{1}{q}_{q} + |c|I^\frac{1}{q}_{q} B^\frac{1}{q}_{q} + B_q + (g - 2)B^\frac{1}{q}_{q} \|J^\frac{1}{q}_{q} + \varepsilon_0 B_q + C_2(\varepsilon_0) \|w\|^{\frac{q-2}{2}} \|f\|_q^2.$$

It is easily seen that $b \in F_\delta$ is equivalent to the inequality

$$ \langle b^2 |\varphi|^2 \rangle \leq \delta \langle |\nabla \varphi|^2 \rangle + \lambda \delta \langle |\varphi|^2 \rangle, \quad \varphi \in W^{1,2}. $$

Thus,

$$ B_q \leq \|b\|_2^2 \|w\|_2 \leq \delta \langle |\nabla \varphi|^2 \rangle + \lambda \delta \|w\|_q^2 = \frac{q^2\delta}{4} J_q + \lambda \delta \|w\|_q^2, $$

and then selecting $\varepsilon_0 > 0$ sufficiently small, and noticing that the assumption on $\delta$ in the theorem is a strict inequality, we can and will ignore below the terms multiplied by $\varepsilon_0$. The proof of Lemma 4 is completed.

2. We estimate $\langle f, \phi \rangle$ in (BE+), (BE-), by an evident analogue of Lemma 3:

$$ \langle f, \phi \rangle \leq \varepsilon_0 (I_q + J_q + H_q + \|w\|_q^2) + C(\varepsilon_0) \|w\|^{q-2}_q \|f\|_q^2 $$

(selecting $\varepsilon_0 > 0$ sufficiently small so that we will ignore below the terms multiplied by $\varepsilon_0$).

Applying Lemma 4 and the last inequality in (BE+), (BE-), and using $\beta_1 \leq c\theta I_{q,\chi} + c\theta^{-1} G_{q,\chi^2}$,

$$ |\beta_2| \leq 2|c|(q - 2)(\theta J_{q,\chi} + 4^{-1}\theta^{-1} G_{q,\chi^2}), \quad \theta > 0, $$

we obtain:
If $c > 0$, then
\[
\mu(\|w\|^q) + I_q + c(1 - \theta)\overline{I}_{q,\chi} + (q - 2)(J_q + c\overline{I}_{q,\chi}) - c\left(1 + \frac{q - 2}{q}d\right)H_{q,\chi} + c(2(d - 1) - \theta^{-1})G_{q,\chi^2} + 2c\frac{q - 2}{q}H_{q,\chi^2}
\]
\[
\leq c(d + 3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}^2J_q^2 + c\frac{q\sqrt{\delta}}{2}I_{q,\chi}^\frac{1}{2}J_q^\frac{1}{2} + \left(\frac{q^2\delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|^q_q + C_2\|w\|^{q - 2}_q\|f\|^2_q.
\]
\[
\text{(6)}
\]
If $-1 < c < 0$, then (set $s := |c|$)
\[
\mu(\|w\|^q) + I_q - s\overline{I}_{q,\chi} + (q - 2)(J_q - s(1 + \theta)\overline{J}_{q,\chi}) + s\left(1 + (q - 2)\frac{d}{q}\right)H_{q,\chi}
\]
\[
- 2s\frac{q - 2}{q}H_{q,\chi^2} - sdG_{q,\chi^2} - 4sG_{q,\chi^2} + 4sG_{q,\chi^2} - s(q - 2)\frac{1}{4}\theta G_{q,\chi^2}^2
\]
\[
\leq s(d + 3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}^2J_q^2 + s\frac{q\sqrt{\delta}}{2}I_{q,\chi}^\frac{1}{2}J_q^\frac{1}{2} + \left(\frac{q^2\delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|^q_q + C_2\|w\|^{q - 2}_q\|f\|^2_q.
\]
\[
\text{(7)}
\]
3. Employing (6), (7) we will prove the following inequality
\[
\mu(\|w\|^q) + \eta J_q \leq C_1\|w\|^q_q + C_2\|w\|^{q - 2}_q\|f\|^2_q, \quad C_i = C_i(\varepsilon_0), \quad i = 1, 2,
\]
\[
\text{(8)}
\]
for some $\eta = \eta(q, d, \varepsilon_0) > 0$.

**Case** $c > 0$. In the LHS of (6) we select $\theta := \frac{q - 2}{2(d - 1)} < 1$. Consider two subcases:

C1) $1 - \frac{cq\sqrt{\delta}}{4} \geq 0$. Arguing as in the proof of Theorem 1, using $H_q \geq G_{q,\chi^2}$, we obtain from (6):
\[
\mu(\|w\|^q) + I_q + (q - 2)J_q + cM(1)H_q
\]
\[
\leq c(d + 3)\frac{q\sqrt{\delta}}{2}H_{q,\chi^2}J_q^\frac{1}{2} + c\frac{q\sqrt{\delta}}{2}I_{q,\chi}^\frac{1}{2}J_q^\frac{1}{2} + \left(\frac{q^2\delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|^q_q + C_2\|w\|^{q - 2}_q\|f\|^2_q,
\]
where $M(1) := \left(q - 1 - \frac{1}{2(d - 1)}\left(\frac{d - 2)^2}{q^2} - 1 + \frac{q - 2}{q}(d - 2)\right)\right) < 0$.

Using the quadratic estimates, we obtain ($\theta_2, \theta_3 > 0$)
\[
\mu(\|w\|^q) + I_q + (q - 2)J_q + cM(1)H_q
\]
\[
\leq c(d + 3)\frac{q\sqrt{\delta}}{4}(\theta_2 J_q + \theta_2^{-1}H_q) + c\frac{q\sqrt{\delta}}{4}(\theta_3 I_{q,\chi} + \theta_3^{-1}J_q) + \left(\frac{q^2\delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q
\]
\[
+ C_1\|w\|^q_q + C_2\|w\|^{q - 2}_q\|f\|^2_q.
\]

We select $\theta_2 = \frac{d - 2}{q}, \theta_3 = 1$, so
\[
\mu(\|w\|^q) + I_q + (q - 2)J_q + cM(1)H_q
\]
\[
\leq c(d + 3)\frac{q\sqrt{\delta}}{4}\left(\frac{q - 2}{d - 2}J_q + \frac{d - 2}{q}H_q\right) + c\frac{q\sqrt{\delta}}{4}(\overline{I}_{q,\chi} + J_q) + \left(\frac{q^2\delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q
\]
\[
+ C_1\|w\|^q_q + C_2\|w\|^{q - 2}_q\|f\|^2_q.
\]

Since $1 - \frac{cq\sqrt{\delta}}{4} \geq 0$, we have $I_q - \frac{cq\sqrt{\delta}}{4}\overline{I}_{q,\chi} \geq 1 - \frac{cq\sqrt{\delta}}{4}J_q$, so using $J_q \geq \frac{(d - 2)^2}{q^2}H_q$ we obtain
\[
\mu(\|w\|^q) + \eta J_q + \left[\left(-\eta + q - 1\right)\frac{(d - 2)^2}{q^2} - L_1(c, \delta)\right]H_q \leq C_1\|w\|^q_q + C_2\|w\|^{q - 2}_q\|f\|^2_q.
\]
where

\[ L_1(c, \delta) = \left[ c \frac{q \sqrt{\delta}}{4} + c(d+3) \frac{q \sqrt{\delta}}{4} \frac{q}{d-2} + c \frac{q \sqrt{\delta}}{4} + \frac{q^2 \delta}{4} + (q-2) \frac{q \sqrt{\delta}}{2} \right] \frac{(d-2)^2}{q^2} \]

\[ + c \left[ -M(1) + (d+3) \frac{q \sqrt{\delta} d - 2}{q} \right]. \]

By the assumptions of the theorem, \((- \eta + q - 1) \frac{(d-2)^2}{q^2} - L_1(c, \delta) \geq 0\) for all \(\eta > 0\) sufficiently small. (8) is proved.

C2) \(1 - \frac{q \sqrt{\delta}}{4} < 0\). Arguing as above, we obtain

\[
\mu(|w|^q) + I_q + (q - 2)I_q + c(1 - \theta) \bar{I}_{q,\chi} + cM(1)H_q
\leq c(d+3) \frac{q \sqrt{\delta}}{4} \left( \frac{q}{d-2}I_q + \frac{d-2}{q}H_q \right) + c \frac{q \sqrt{\delta}}{4}(\bar{I}_{q,\chi} + J_q) + \left( \frac{q^2 \delta}{4} + (q-2) \frac{q \sqrt{\delta}}{2} \right) J_q
\]

\[ + C_1||w||^2_q + C_2||w||^q_{q-2}||f||^2_q, \]

where \(M(1) := (q - 2) \frac{(d-2)^2}{q^2} - (1 + \frac{q^2}{q} - 2) < 0\).

If \(1 - \frac{1}{2(d-1)} - \frac{q \sqrt{\delta}}{4} < 0\), then clearly \(c(1 - \frac{1}{2(d-1)}) \bar{I}_{q,\chi} - c \frac{q \sqrt{\delta}}{4} \bar{I}_{q,\chi} \geq c(1 - \frac{1}{2(d-1)} - \frac{q \sqrt{\delta}}{4}) I_q \). By the assumption of the theorem, \(1 + c(1 - \frac{1}{2(d-1)} - \frac{q^2 \delta}{4}) \geq 0\), so in the previous estimate \([1 + c(1 - \frac{1}{2(d-1)} - \frac{q \sqrt{\delta}}{4})] I_q \geq [1 + c(1 - \frac{1}{2(d-1)} - \frac{q \sqrt{\delta}}{4})] J_q\), and thus

\[
\mu(|w|^q) + \eta J_q + \left[ (- \eta + q - 1) \frac{(d-2)^2}{q^2} - L_1(c, \delta) \right] H_q \leq C_1||w||^q_q + C_2||w||^q_{q-2}||f||^2_q. \tag{9}
\]

where

\[
L_1(c, \delta) = \left[ -c \left( 1 - \frac{1}{2(d-1)} - \frac{q \sqrt{\delta}}{4} \right) + c \frac{q \sqrt{\delta}}{4} + c(d+3) \frac{q \sqrt{\delta}}{4} \frac{q}{d-2} + \frac{q^2 \delta}{4} + (q-2) \frac{q \sqrt{\delta}}{2} \right] \frac{(d-2)^2}{q^2} \]

\[ + c \left[ -M(1) + (d+3) \frac{q \sqrt{\delta} d - 2}{q} \right]. \]

(\text{another representation for } L_1(c, \delta)). By the assumptions of the theorem, \((- \eta + q - 1) \frac{(d-2)^2}{q^2} - L_1(c, \delta) \geq 0\) for all \(\eta > 0\) sufficiently small. (8) is proved.

If \(1 - \frac{1}{2(d-1)} - \frac{q \sqrt{\delta}}{4} \geq 0\), then clearly \(I_q + c(1 - \frac{1}{2(d-1)} - \frac{q \sqrt{\delta}}{4}) \bar{I}_{q,\chi} \geq J_q + c(1 - \frac{1}{2(d-1)} - \frac{q \sqrt{\delta}}{4}) \bar{I}_{q,\chi}\). Arguing as above, we obtain (9) and therefore (8).

\textbf{Case} \(-1 < c < 0\). In the LHS of (7) we select \(\theta = \frac{1}{2(\alpha - 2)}\). Arguing as in the proof of Theorem 1, we obtain from (7):

\[
\mu(|w|^q) + I_q - s \bar{I}_{q,\chi} + (q - 2)(J_q - s(1 + \theta) \bar{J}_{q,\chi}) + sM(1)G_q \leq s(d+3) \frac{q \sqrt{\delta}}{2} G_q J_q^\frac{1}{q} + s \frac{q \sqrt{\delta}}{2} \bar{I}_{q,\chi} J_q^\frac{1}{q} + \left( \frac{q^2 \delta}{4} + (q-2) \frac{q \sqrt{\delta}}{2} \right) J_q + C||w||^q_q + C||w||^q_{q-2}||f||^2_q, \tag{10}
\]

where \(M(1) := -d + 1 + \frac{1}{2} \frac{(q-2)(d-2)}{q} < 0\). In the RHS of (7) we have used \(G_q \geq G_{q,\chi^2}\).
Further \((\theta_2, \theta_3 > 0)\),

\[
\mu(\|w\|^q) + I_q - s\bar{I}_{q,\chi} + (q - 2)(J_q - s(1 + \theta)\bar{J}_{q,\chi}) + sM(1)G_q
\leq s(d + 3)\frac{q\sqrt{\delta}}{4}(\theta_2J_q + \theta_2^{-1}G_q) + s\frac{q\sqrt{\delta}}{4}(\theta_3\bar{I}_{q,\chi} + \theta_3^{-1}J_q) + \left(\frac{q^2\delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q
+ C_1\|w\|_q^q + C_2\|\cdot\|^2\|f\|_q^2.
\]

Selecting \(\theta_2 = \frac{q}{d-2}\), \(\theta_3 = 1\) and using the inequalities \(I_q \geq \bar{I}_{q,\chi}, J_q \geq \bar{J}_{q,\chi}\), we obtain

\[
\mu(\|w\|^q) + \left[1 - s\left(1 + \frac{q\sqrt{\delta}}{4}\right)\right]I_q
+ \left[(q - 2)\left(1 - s\left(1 + \frac{1}{2}\frac{q}{d - 2}\right)\right) - s\frac{q\sqrt{\delta}}{4} - s(d + 3)\frac{q\sqrt{\delta}}{4} - \frac{q^2\delta}{4} - (q - 2)\frac{q\sqrt{\delta}}{2}\right]J_q
+ s\left[M(1) - (d + 3)\frac{q\sqrt{\delta}}{4} - \frac{2}{4}d - 2\right]G_q
\leq C_1\|w\|_q^q + C_2\|\cdot\|^2\|f\|_q^2.
\]

By the assumptions of the theorem, \(1 - s\left(1 + \frac{2\sqrt{\delta}}{4}\right) \geq 0\). So, using the inequalities \(J_q \leq I_q, J_q \geq \frac{(d - 2)^2}{q^2}G_q\), we arrive at

\[
\mu(\|w\|^q) + \eta J_q + \left[(-\eta + q - 1)\frac{(d - 2)^2}{q^2} - L_2(s, \delta)\right]G_q \leq C_1\|w\|_q^q + C_2\|\cdot\|^2\|f\|_q^2,
\]

where

\[
L_2(s, \delta)
= \left[s\left(1 + \frac{q\sqrt{\delta}}{4}\right) + (q - 2)s\left(1 + \frac{1}{2}\frac{q}{d - 2}\right) + s\frac{q\sqrt{\delta}}{4} + s(d + 3)\frac{q\sqrt{\delta}}{4} - \frac{q^2\delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right]\frac{(d - 2)^2}{q^2}
+ s\left[-M(1) + (d + 3)\frac{q\sqrt{\delta}}{4} - \frac{2}{4}d - 2\right].
\]

By the assumptions of the theorem, \((-\eta + q - 1)\frac{(d - 2)^2}{q^2} - L_2(s, \delta) \geq 0\) for all \(\eta > 0\) sufficiently small. (8) is proved.

4. The Sobolev Embedding Theorem and Theorem A.2(i) now yield \((**\)) (cf. the proof of Theorem 1, step 4).

Proof of (iii). Let \(q = 2, d \geq 3\). Recall that since \(b \in \mathbf{F}_\delta\), then \(b \in \mathbf{F}_{\delta_1}(A)\), where \(\delta_1 := \delta\) if \(c > 0\), and \(\delta_1 := \delta(1 + c)^{-2}\) if \(-1 < c < 0\) By our assumptions, \(\delta_1 < 1\), so \(\Lambda_2(a^\delta, b_n) = A_2^\delta + b_n \cdot \nabla\) are well defined on \(L^2\). Following the proof of Theorem 1(iii), we obtain: for \(c > 0\)

\[
\langle [\nabla, A_2^\delta]_\mu u, w \rangle = -\beta_1 - cH_{2,\chi} + 2c(d - 1)G_{2,\chi^2},
\]

\[
\|w\|_2^2 + I_2 + cI_{2,\chi} - \beta_1 - cH_{2,\chi} + 2c(d - 1)G_{2,\chi^2} = \langle -b_n \cdot w, -\nabla \cdot w \rangle + \langle f, -\nabla \cdot w \rangle
\]

\[
\beta_1 = -2c\langle |x|^{-4} w \cdot x, x \cdot (x \cdot \nabla w) \rangle;
\]
for $c < 0$

\[
\langle [\nabla, A^5]_{-} u, w \rangle = \frac{1}{2} \beta_2 - cH_{2, x} + cdG_{2, x^2}, \quad \beta_2 = 0,
\]

\[
\mu \|w\|^2_2 + I_2 + cf_2 - cH_{2, x} + cdG_{2, x^2} = \langle -b_n \cdot w, -\nabla \cdot w \rangle + \langle f, -\nabla \cdot w \rangle, \quad I_2 \geq I_2,
\]

\[
\mu \|w\|^2_2 + (1 - |c|) I_2 + |c| H_{2, x} - |c| dG_{2, x^2} \leq \langle -b_n \cdot w, -\nabla \cdot w \rangle + \langle f, -\nabla \cdot w \rangle.
\]

Now, applying Lemma 4 and arguing as in step 3 above, we arrive at $\sup_{\varepsilon > 0, n \geq 1} I_2(u^{\varepsilon, n}) \leq K \|f\|_2$, and so (by passing to the limit $\varepsilon \downarrow 0$, using Theorem A.2, we arrive at $I_2(u) \leq K \|f\|_2 \Rightarrow u \in W^{2, 2}$.

The proof of Theorem 2 is completed. \hfill \square

3. PROOF OF THEOREM 3

Recall that a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $F_{d_1}(A)$, $d_1 > 0$, the class of form-bounded vector fields (with respect to $A \equiv A_2$), if $b_a^2 := b \cdot a^{-1} \cdot b \in L^1_{\text{loc}}$ and there exists a constant $\lambda = \lambda_{d_1} > 0$ such that $\|b_a(\bar{\lambda} + A)^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta_1}$.

We will need the following auxiliary results. Recall: \( \langle \nabla a \rangle_k = \sum_{i=1}^d (\partial_x a_{ik}), 1 \leq k \leq d \).

**Lemma 5.** $\nabla a = c(d - 1)|x|^{-2}x \in F_{d}(A)$, where $\delta := 4 \left( \frac{d - 1}{d - 2} \right)^2$.

**Proof.** It is easy to see that $b := \nabla a = c(d - 1)|x|^{-2}x$, $a^{-1} = I - \frac{c}{c+1}|x|^{-2}x \otimes x$ and $b_a^2 := b \cdot a^{-1} \cdot b = \frac{(d-1)c^2}{c+1} |x|^{-2}$. Now, that $b \in F_{d}(A)$ is immediate from the following Hardy-type inequality:

\[
(c + 1) \left( \frac{d - 2}{4} \right)^2 \| |x|^{-1} h \|^2_2 \leq \langle \nabla h \cdot a \cdot \nabla \bar{h} \rangle, \quad h \in W^{1, 2}(\mathbb{R}^d).
\]

(*)

It remains to prove (*). Since $\langle \phi, x \cdot \nabla \phi \rangle = -\frac{d}{2} \langle \phi, \phi \rangle$, $\phi \in C^\infty$, we have

\[
\langle \phi, -\nabla \cdot (a - 1) \cdot \nabla \phi \rangle = c \langle |x| \cdot \nabla (|x|^{-1} \phi) \|_2^2 - (d - 1) \| |x|^{-1} \phi \|_2^2 \rangle
\]

(11)

Next, the following inequality (with the sharp constant) is valid:

\[
\| x \cdot \nabla f \|_2 \geq \frac{d}{2} \| f \|_2, \quad (f \in D(D)),
\]

(12)

where the operator $D = (D \upharpoonright C^\infty_c)_{L^2 \to L^2}, D \upharpoonright C^\infty_c = \frac{\sqrt{d}}{2} (x \cdot \nabla + \nabla \cdot x)$, is selfadjoint.

Indeed, by the Spectral Theorem, $\| (D - \zeta)^{-1} \|_{2 \to 2} = \frac{1}{|\text{Im} \zeta|}$ for $\text{Re} \zeta = 0$, and hence

\[
\| x \cdot \nabla f \|_2 = \| \frac{d}{2} (x \cdot \nabla + \nabla \cdot x - d) f \|_2 = \| (D - \sqrt{-1} d \frac{d}{2}) f \|_2 \geq \frac{d}{2} \| f \|_2, \quad (f \in C_c^\infty).
\]

(12) is proved.

Let $c > 0$. By (11) and (12),

\[
\langle \phi, -\nabla \cdot (a - 1) \cdot \nabla \phi \rangle \geq c \left( \frac{d - 2}{4} \right)^2 \| |x|^{-1} \phi \|^2_2 \quad (\phi \in C^\infty_c).
\]

(*) follows now from the equality $\langle \phi, -\nabla \cdot a \cdot \nabla \phi \rangle = \langle \phi, -\nabla \cdot (a - 1) \cdot \nabla \phi \rangle + \langle \phi, -\Delta \phi \rangle$ and Hardy’s inequality $\langle \phi, -\Delta \phi \rangle \geq \frac{(d-2)^2}{4} \| |x|^{-1} \phi \|^2_2$. Finally, the obvious inequality $(1 + c) \langle \phi, -\Delta \phi \rangle \geq \langle \phi, -\nabla \cdot a \cdot \nabla \phi \rangle$ shows that the constant in (*) is sharp.

If $-1 < c < 0$, (*) is a trivial consequence of Hardy’s inequality. \hfill \square
If $0 < c < d - 2$, then $\nabla a \in F_\delta(A)$ with $\delta < 4$ by Lemma 5, so $\Lambda_\delta(a, \nabla a)$ is well defined for all $q \in (1 - \frac{d-1}{4} \frac{c}{d-2}, 1 + c)^{-1}, \infty$, [see [KiS, Theorem 3.2].

If $-1 < c < 0$, then $\Lambda_\delta(a, \nabla a)$ is well defined for all $q \in \mathbb{R}$ by Theorem A.1 (there take $b = 0$). We have proved assertion (i) of the theorem.

In order to prove assertion (ii), we will need the following result. Set

$$|x|_\epsilon := \sqrt{|x|^2 + \epsilon}, \epsilon > 0, \quad \chi := |x|^2|x|_{\epsilon}^{-2}.$$  

**Lemma 6.** Set $a^\epsilon(x) := I + c|x|^{-2} x \cdot x$, $A^\epsilon \equiv [-\nabla \cdot a^\epsilon \cdot \nabla \mid C_c^\infty \mid_{2 \rightarrow 2}$. If $d \geq 4$, $-1 < c \leq \frac{d-3}{2}$, or if $d = 3, -1 < c < 0$, then

$$\nabla a^\epsilon \in F_\delta(A^\epsilon) \quad \text{with} \quad \delta := 4 \left(\frac{d-1}{d-2} \frac{c}{1+c}\right)^2.$$  

**Proof.** 1. First, let $c > 0$. Note that $(a^\epsilon)^{-1}(x) = I - \frac{\epsilon}{1+\epsilon} x \otimes x$, $(\nabla a^\epsilon) = c\chi (d+1-2\chi)|x|^{-2} x$ and

(a) $(\nabla a^\epsilon) \cdot (a^\epsilon)^{-1} \cdot (\nabla a^\epsilon) = \left(\frac{c^2}{1+\epsilon}\right)^2 |x|^{-2}$.

(b) $\langle -\nabla \cdot a^\epsilon \cdot \nabla h, h \rangle = \langle (\nabla h)^2 \rangle + c\chi |x|^{-2} \langle (\nabla h)^2 \rangle \geq \frac{(d-2)^2}{4} |x|^{-2} \langle (\nabla h)^2 \rangle + c\left(\frac{d^2}{4} - (d+2)\chi + 3\chi^2\right) |x|^{-2} \langle (\nabla h)^2 \rangle$

$h \in C_c^\infty$, see (HI).

Combining (a) and (b), we obtain that $\nabla a^\epsilon \in F_\delta(A^\epsilon)$ for any $\delta^\epsilon$ such that

$$\delta^\epsilon \left[ \left(\frac{d-2}{4}\right)^2 + c \left(\frac{d^2}{4} - (d+2)\chi + 3\chi^2\right) \chi \right] \langle |x|^{-2} \rangle \geq \left(\frac{c\chi^2}{1+\epsilon}\right)^2 \langle (d+1-2\chi)^2 \rangle |x|^{-2} \langle h^2 \rangle, \quad h \in C_c^\infty; $$

we can take

$$\delta^\epsilon := \sup_{0 \leq t \leq 1} \left(\frac{ct^2}{1+ct}\right)^2 \left[\left(\frac{d-2}{4}\right)^2 + ct \left(\frac{d^2}{4} - (d+2)\chi + 3\chi^2\right) \chi \right].$$

Let us show that

$$\delta^\epsilon = 4 \left(\frac{d-1}{d-2} \frac{c}{1+c}\right)^2 \delta,$$

which would imply that $\nabla a^\epsilon \in F_\delta(A^\epsilon)$, as claimed.

Note that $\frac{d^2}{4} - (d+2)\chi + 3\chi^2 \geq \frac{(d-2)^2}{4}$ for all $0 \leq t \leq 1$ and $d \geq 4$. Thus

$$\delta \leq \delta^\epsilon \leq 4 \sup_{0 \leq t \leq 1} \frac{c^2 \chi^2}{(d-2)^2(1+ct)} = 4 \left[\frac{c}{d-2} \sup_{0 \leq t \leq 1} \frac{(d+1-2t)t}{1+ct}\right]^2 \delta.$$  

2. Let $-1 < c < 0$. Then $\langle -\nabla \cdot a^\epsilon \cdot \nabla h, h \rangle \geq \frac{(d-2)^2}{4} (1+c) \langle |x|^{-2} h^2 \rangle$, so by (a) above $\nabla a^\epsilon \in F_\delta(A^\epsilon)$ for any $\delta^\epsilon$ such that

$$\delta^\epsilon \left(\frac{d-2}{4}\right)^2 (1+c) \langle |x|^{-2} h^2 \rangle \geq \left(\frac{c^2 |(d+1-2\chi)^2 |}{1+c\chi} \chi^2 |x|^{-2} \langle h^2 \rangle \right), \quad h \in C_c^\infty; $$

we can take

$$\delta^\epsilon := \sup_{0 \leq t \leq 1} \frac{c^2 (d+1-2t)^2 t^2}{(d-2)^2(1+c)(1+ct)}.$$
Finally, since $1 + c \leq 1 + ct$,

$$\delta \leq \delta_{\varepsilon} \leq 4\left[\frac{c}{(1 + c)(d - 2)} \sup_{0 < t < 1} (d + 1 - 2t)^2\right] = \delta.$$ \hfill \Box

1. We start the proof of assertion (ii) of the theorem. Let $d \geq 4$. We follow closely the proof of Theorem 1. Set $u_{\varepsilon} = (\mu + \Lambda_{\varepsilon}(a_{\varepsilon}, \nabla a_{\varepsilon}))^{-1} f$, $0 \leq f \in C^{1}_{c}$. Since $a_{\varepsilon} \in C^{\infty}$, we have $0 \leq u_{\varepsilon} \in W^{3,q}$. Below

$$w \equiv w_{\varepsilon} := \nabla u_{\varepsilon}, \quad \phi := -\nabla \cdot (w|w|^{q-2}),$$

$$I_{q} := \langle (\nabla, w)^{2}|w|^{q-2} \rangle, \quad J_{q} := \langle (\nabla|w|)^{2}|w|^{q-2} \rangle,$$

$$\tilde{I}_{q,\chi} := \langle (x \cdot \nabla w)^{2}\chi|x|^{-2}|w|^{q-2} \rangle, \quad \tilde{J}_{q,\chi} := \langle (x \cdot \nabla|w|)^{2}\chi|x|^{-2}|w|^{q-2} \rangle,$$

$$H_{q,\chi} := \langle \chi|x|^{-2}|w|^{q} \rangle, \quad G_{q,\chi} := \langle \chi^{2}|x|^{-4}(x \cdot w)^{2}|w|^{q-2} \rangle,$$

where $\chi = |x|^{2}|x|^{-2}$. We will need

**Lemma 7** (The basic equalities, non-divergence form).

\[
\begin{align*}
\mu\langle |w|^{q}\rangle &+ I_{q} + c\tilde{I}_{q,\chi} + (q - 2)(J_{q} + c\tilde{J}_{q,\chi}) - c\frac{d(d - 1)}{q}H_{q,\chi} \\
&+ 2c\frac{2d + 1}{q}H_{q,\chi} - 8c\frac{q}{q}H_{q,\chi} = \beta_1 + \langle f, \phi \rangle, & (\text{BE}_{+}^{\text{nd}}) \\
\mu\langle |w|^{q}\rangle &+ I_{q} + c\tilde{I}_{q,\chi} + (q - 2)(J_{q} + c\tilde{J}_{q,\chi}) - c\frac{d(d - 1)}{q}H_{q,\chi} - c(d - 2)G_{q,\chi} \\
&+ 2c\frac{2d + 1}{q}H_{q,\chi} - 8c\frac{q}{q}H_{q,\chi} - 4\varepsilon\langle |x|^{-6}(x \cdot w)^{2}|w|^{q-2} \rangle = -\frac{1}{2}\beta_2 + \langle f, \phi \rangle, & (\text{BE}_{-}^{\text{nd}})
\end{align*}
\]

where

$$\beta_1 := -2c\langle |x|^{-4}x \cdot w, x \cdot (x \cdot \nabla w)|w|^{q-2} \rangle, \quad \beta_2 := -2c(q - 2)\langle |x|^{-4}(x \cdot w)^{2}x \cdot \nabla |w|, |w|^{q-3} \rangle.$$

**Proof of Lemma 7.** We modify the proof of Lemma 2. In the left-hand side of (BE$_{+}$, BE$_{-}$) we have the extra term $\langle (\nabla a_{\varepsilon}) \cdot w, -\nabla \cdot (w|w|^{q-2}) \rangle$, which we evaluate as follows:

$$\langle (\nabla a_{\varepsilon}) \cdot w, -\nabla \cdot (w|w|^{q-2}) \rangle$$

we integrate by parts

$$= c(d - 1)(H_{q,\chi} + \langle |x|^{-2}x \cdot \nabla |w|, |w|^{q-1} \rangle - 2G_{q,\chi}^{2})$$

$$+ 2\varepsilon\langle \langle |x|^{-4}|w|^{q} \rangle + \langle |x|^{-4}x \cdot \nabla |w|, |w|^{q-1} \rangle - 4\langle |x|^{-6}(x \cdot w)^{2}|w|^{q-2} \rangle \rangle.$$

Note that

$$\langle |x|^{-2}x \cdot \nabla |w|, |w|^{q-1} \rangle = \frac{1}{q}\langle |x|^{-2}x \cdot \nabla |w|^{q} \rangle = -\frac{d - 2}{q}H_{q,\chi} - \frac{2}{q}\varepsilon\langle |x|^{-4}|w|^{q} \rangle,$$

$$\langle |x|^{-4}x \cdot \nabla |w|, |w|^{q-1} \rangle = \frac{1}{q}\langle |x|^{-4}x \cdot \nabla |w|^{q} \rangle = -\frac{1}{q}\langle |w|^{q}\nabla \cdot (x|x|^{-4}) \rangle = -\frac{d - 4}{q}\langle |x|^{-4}|w|^{q} \rangle - \frac{4}{q}\varepsilon\langle |x|^{-6}|w|^{q} \rangle.$$
Thus,
\[
\langle (\nabla \alpha^e) \cdot w, -\nabla \cdot (|w|^q|w|^{-2}) \rangle \nabla \alpha^e \cdot \nabla w + q(1 - \theta)\langle \theta H \rangle q \leq C(w)\| w \|_q^q \| f \|_q^q \| w \|_q^q. \]

The latter, added to the left-hand side of (BE\_+), (BE\_−) yields (BE\_+), (BE\_−).

\[\square\]

2. We estimate from above the term \( \langle f, \phi \rangle \) in the right-hand side of (BE\_+), (BE\_−) employing an evident analogue of Lemma 3:

\[
\langle f, \phi \rangle \leq \varepsilon_0 (I_q + J_q + H_q) + C(\varepsilon_0)\| w \|_q^q \| f \|_q^q. \tag{13}
\]

Again we choose \( \varepsilon_0 > 0 \) so small that in the estimates below we can ignore the terms multiplied by \( \varepsilon_0 \).

3. We will use (BE\_+), (BE\_−) and (13) to establish the inequality

\[
\mu \langle |w|^q \rangle + \eta J_q \leq C\| w \|_q^q \| f \|_q^q, \quad C = C(\varepsilon_0), \quad \eta = \eta(q, d, \varepsilon_0) > 0. \tag{14}
\]

**Case** \( c > 0 \). By the assumptions of the theorem, \( c < \frac{d - 2}{2} \land \frac{d - 2}{q - d + 2} \). In (BE\_+), we estimate

\[
\beta_1 \leq c\phi \bar{I}_{q,\chi} + c\theta^{-1}G_{q,\chi^2}, \quad \theta > 0,
\]

and then apply (13) to obtain

\[
\mu \langle |w|^q \rangle + I_q + q(1 - \theta)\bar{I}_{q,\chi} + (q - 2)J_q + c(q - 2)\bar{J}_{q,\chi} - c\frac{d(d - 1)}{q}H_{q,\chi}
\]

\[
+ 2c\frac{2d + 1}{q}H_{q,\chi^2} - \frac{8c}{q}H_{q,\chi^3} - \frac{c}{\theta}G_{q,\chi^2} \leq C\| w \|^q \| f \|^2_\| q \|^2.
\]

We exclude the case \( 0 < \theta \leq 1 \) by noting that \( \bar{I}_{q,\chi} \geq \frac{(d - 2)^2}{q}G_{q,\chi^2} \) and \( f(\theta) = (1 - \theta)^\frac{(d - 2)^2}{q^2} - 1 \) achieves its maximum at \( \theta = \frac{q}{d - 2} > 1 \).

Let \( \theta > 1 \). Clearly we have to assume now that \( 1 + c(1 - \theta) > 0 \). Since \( I_q + c(1 - \theta)\bar{I}_{q,\chi} \geq (1 + c(1 - \theta))I_q \geq (1 + c(1 - \theta))J_q \) and \( H_{q,\chi^2} \geq G_{q,\chi^2} \) we have

\[
\mu \langle |w|^q \rangle + (q - 1 + c(1 - \theta))J_q + c(q - 2)\bar{J}_{q,\chi} - c\frac{d(d - 1)}{q}H_{q,\chi}
\]

\[
+ 2c\frac{2d + 1}{q}H_{q,\chi^2} - \frac{8c}{q}H_{q,\chi^3} - \frac{c}{\theta}H_{q,\chi^2} \leq C\| w \|^q \| f \|^2_\| q \|^2.
\]

Using \( \bar{J}_{q,\chi} \geq \frac{4}{q^2} \left( \frac{d^2}{4}H_{q,\chi} - (d + 2)H_{q,\chi^2} + 3H_{q,\chi^3} \right) \), see (HI), we obtain

\[
\mu \langle |w|^q \rangle + (q - 1 + c(1 - \theta))J_q + c(q - 2)\frac{4}{q^2} \left( \frac{d^2}{4}H_{q,\chi} - (d + 2)H_{q,\chi^2} + 3H_{q,\chi^3} \right) - c\frac{d(d - 1)}{q}H_{q,\chi}
\]

\[
+ 2c\frac{2d + 1}{q}H_{q,\chi^2} - \frac{8c}{q}H_{q,\chi^3} - \frac{c}{\theta}H_{q,\chi^2} \leq C\| w \|^q \| f \|^2_\| q \|^2.
\]

Thus, by \( J_q \geq \frac{(d - 2)^2}{q^2} \langle |x|^{-2} |w|^q \rangle \), for all \( \eta > 0 \) sufficiently small,

\[
\mu \langle |w|^q \rangle + \eta J_q + \left( (-\eta + q - 1 + c(1 - \theta))\frac{(d - 2)^2}{q^2} + cM(\chi) \right) |x|^{-2} |w|^q \leq C\| w \|^q \| f \|^2_\| q \|^2,
\]
where
\[ M(\chi) := \left[ (q - 2) \frac{4}{q^2} \left( \frac{d^2}{4} - (d + 2)\chi + 3\chi^2 \right) - \frac{d(d - 1)}{q} + \frac{2d + 1}{q} \chi - \frac{8}{q} \chi^2 - \frac{1}{\theta} \chi \right] \chi. \]

Select \( \theta := \frac{d}{d - 2}. \) (Motivation: estimating the terms involving \( \theta \) from below by \( -c\theta \frac{(d - 2)^2}{q^2} - \frac{c}{q} \)) \( H_q \) and maximizing the latter in \( \theta \), we arrive at \( \theta = \frac{d}{d - 2}. \) Then, since \( c < \frac{d - 2}{q - d + 2} \), we have \( 1 + c(1 - \theta) > 0 \). Elementary arguments show that
\[
\min_{0 \leq t \leq 1} M(t) = M(1) < 0,
\]
and so
\[
\mu(|w|^q) + \eta J_q + \left[ (-\eta + q - 1) \frac{(d - 2)^2}{q^2} - c\ell_1 \right] H_q \leq C||w||_q^{q-2}||f||_q^2,
\]
where \( \ell_1 := \frac{d - 2}{q} (d - 2) \). By the assumption \( c < \frac{d - 2}{q^2} \) of the theorem, there exists \( \eta > 0 \) such that
\[
(-\eta + q - 1) \frac{(d - 2)^2}{q^2} - c\ell_1 \geq 0.
\]
Thus (14) is proved.

**Case** \(-1 < c < 0. \) By the assumptions of the theorem, \(-(1 + \frac{4}{d - 2}\frac{q - 2}{q - 2}\frac{q - 2}{q - 2})^{-1} < c < 0. \) Set \( s := |c| \). In (BE\(_{\text{end}}\)) we estimate \( (\theta > 0) \)
\[
|\beta_2| \leq 2s(q - 2)(\theta J_q + 4^{-1}\theta^{-1} G_{q,\chi^2}),
\]
obtaining
\[
\mu(|w|^q) + I_q - sI_{q,\chi} + (q - 2)(J_q - s(1 + \theta)J_{q,\chi}) + \frac{s}{q} H_{q,\chi} \frac{d(d - 1)}{q} \]
\[
- 2s \frac{2d + 1}{q} H_{q,\chi^2} + \frac{8s}{q} H_{q,\chi^3} + s \left( d + 2 - (q - 2) \frac{1}{4\theta} \right) G_{q,\chi^2} - 4sG_{q,\chi^3} \leq C||w||_q^{q-2}||f||_q^2.
\]
Then by the obvious inequalities \( I_q - sI_{q,\chi} \geq (1 - s)J_q \) and \( J_q - s(1 + \theta)J_{q,\chi} \geq (1 - s(1 + \theta))J_q, \)
\[
\mu(|w|^q) + (q - 1 - s - s(q - 2)(1 + \theta))J_q + s \frac{d(d - 1)}{q} H_{q,\chi} \]
\[
- 2s \frac{2d + 1}{q} H_{q,\chi^2} + \frac{8s}{q} H_{q,\chi^3} + s \left( d + 2 - (q - 2) \frac{1}{4\theta} \right) G_{q,\chi^2} - 4sG_{q,\chi^3} \leq C||w||_q^{q-2}||f||_q^2.
\]
Note that \( d(d - 1) - 2(2d + 1)t + 8t^2 \geq 0, \) \( (d \geq 3, \ 0 \leq t \leq 1), \) and so
\[
\frac{d(d - 1)}{q} H_{q,\chi} - 2\frac{2d + 1}{q} H_{q,\chi^2} + \frac{8s}{q} H_{q,\chi^3} \geq \frac{d(d - 1)}{q} G_{q,\chi} - 2\frac{2d + 1}{q} G_{q,\chi^2} + \frac{8s}{q} G_{q,\chi^3}.
\]
Therefore, we obtain from \( (*) \)
\[
\mu(|w|^q) + (q - 1 - s - s(q - 2)(1 + \theta))J_q + s \frac{d(d - 1)}{q} G_{q,\chi} \]
\[
- 2s \frac{2d + 1}{q} G_{q,\chi^2} + \frac{8s}{q} G_{q,\chi^3} + s \left( d + 2 - (q - 2) \frac{1}{4\theta} \right) G_{q,\chi^2} - 4sG_{q,\chi^3} \leq C||w||_q^{q-2}||f||_q^2,
\]
i.e.
\[
\mu(|w|^q) + (q - 1 - s - s(q - 2)(1 + \theta))J_q - s \langle M(\chi)|x|^{-4}(x \cdot w)^2|w|^{q-2} \rangle \leq C||w||_q^{q-2}||f||_q^2,
\]
where
\[
M(\chi) := q^{-1}[a\chi^2 + b\chi + c_0] \chi.
\]
\[ a := 4(q - 2), \quad b := 2(2d + 1) - q \left( d + 2 - (q - 2) \frac{1}{4d} \right), \quad c_0 := -d(d - 1). \]

Select \( \theta := \frac{1}{4} \frac{d - 2}{q - d - 2} \) if \( q > 2 \). Then \( M(0) = M(1) = \max_{0 \leq t \leq 1} M(t) = 0 \). This is the best possible choice of \( \theta \). (Selecting a larger \( \theta \), so that \( \max_{0 \leq t \leq 1} M(t) < 0 \), decreases the term \([ \ldots ]_q \). On the other hand, selecting a smaller \( \theta \), so that \( \max_{0 \leq t \leq 1} M(t) > 0 \), leads to constraints on \( c \) which are sub-optimal, i.e. which can be improved by selecting a larger \( \theta \).)

Note that \( q - 1 - s - s(q - 2)(1 + \theta) > 0 \) by the assumptions of the theorem. Thus,

\[
\mu(\|w\|^q) + (q - 1 - s - s(q - 2)(1 + \theta)) J_q \leq C\|w\|^{q-2}\|f\|_q^2,
\]

and hence (14) is proved for \( q > 2 \).

We are left to treat the case \( d = 4 \) and \( q = 2 \). Note that the proof above still works. See also a proof of (iii) below.

4. For \( d \geq 4 \), the Sobolev Embedding Theorem and Theorem A.2(ii) (with \( \delta = 0 \)) now yield estimates (\( \ast \)) and convergence (3). The proof of (ii) is completed.

**Proof of (iii).** Let \( q = 2, \ d \geq 3 \). If \( c < 0 \), then we can argue as in steps 1-3 obtaining

\[
\sup_{\varepsilon > 0} I_2(u^\varepsilon) \leq K \|f\|_2, \quad \text{and so} \quad I_2(u) \leq K \|f\|_2 \quad \Rightarrow u \in W^{2,2}.
\]

Now, let \( c > 0 \). By Lemma 5, \( \nabla a \in \mathbf{F}_\delta(A), \ \delta = 4\left(\frac{d+1}{d-2} - \frac{1}{d-2}c\right)^2 \). Since \( c < \frac{d+2}{d} \), we have \( \delta < 1 \), and so \( \Lambda_2(a, \nabla a) \) is well defined. By the Miyadera Perturbation Theorem and Theorem 1, \( D(\Lambda_2(a, \nabla a)) = D(A_2) \subset W^{2,2} \), and \( u := (\mu + \Lambda_2(a, \nabla a))^{-1} f, \ \mu > 0, \ f \in L^2 \), belongs to \( W^{2,2} \). Multiplying \( (\mu + A_2 + \nabla a \cdot \nabla) u = f \) by \( \phi_m := -E_m \nabla \cdot w \), where \( w := \nabla u, \ E_m = (1 - m^{-1} \Delta)^{-1}, \ m \geq 1, \) and integrating by parts we have (omitting the summation sign in the repeated indices):

\[
\mu(\|w\|^2) + \langle a \cdot \nabla w, E_m \nabla w \rangle + \langle -\nabla_r a \cdot w, E_m \nabla w \rangle + \langle \nabla a \cdot w, \phi_m \rangle = \langle f, \phi_m \rangle. \tag{\ast}
\]

Now we pass in (\( \ast \)) to the limit \( m \to \infty \).

Then following closely the proof of (BE\(_{\ast}^d\)) for \( q = 2 \) we obtain:

\[
\mu(\|w\|^2) + I_2 + cI_2 - \frac{c}{2}(d - 2)(d - 3)H_2 = \beta + \langle f, \phi \rangle,
\]

where \( I_2 := \langle \nabla w, \nabla w \rangle, \ I_2 := \langle (x \cdot \nabla w)^2 |x|^{-2}, \ H_2 := \langle |x|^{-2} |w|^2 \rangle, \ \beta := -2c\langle |x|^{-4} x \cdot w, x \cdot (x \cdot \nabla w) \rangle \).

Using the inequalities \( \beta \leq cI_2 + cH_2, \ \frac{(d-2)^2}{4} H_2 \leq I_2 \), we have

\[
\mu(\|w\|^2) + \left[ 1 - \frac{4}{(d - 2)^2} \left( 1 + \frac{(d - 2)(d - 3)}{2} c \right) \right] I_2 \leq \langle f, \phi \rangle.
\]

The proof of (iii) follows.

The proof of Theorem 3 is completed. \( \square \)

4. **Proof of Theorem 4**

We follow closely the proofs of Theorems 2, 3.

**Proof of (i).** It is easily seen that if \( b \in \mathbf{F}_\delta \), then \( b \in \mathbf{F}_{\delta_1}(A) \), where \( \delta_1 := \delta \) if \( c > 0 \), and \( \delta_1 := (1+c)^{-2} \) if \( -1 < c < 0 \). Further, by Lemma 5, \( \nabla a = c(d - 1)|x|^{-2} x \in \mathbf{F}_{\delta_0}(A) \), where \( \delta_0 := 4\left(\frac{d-1}{d-2} - \frac{1}{d-2}c\right)^2 \). Now, set
as in the formulation of assertion (i),

$$\sqrt{\delta_2} := \begin{cases} \sqrt{\delta_1} + \sqrt{\delta_0}, & 0 < c < d - 2, \\
\sqrt{\delta_1}, & -1 < c < 0. \end{cases}$$

For $c > 0$, we have by our assumption $\delta_2 < 4$ if $c > 0$, so by [KiS, Theorem 3.2] the formal differential expression $-a \cdot \nabla^2 + b \cdot \nabla \left( \equiv -\nabla \cdot a \cdot \nabla + (\nabla a) \cdot \nabla + b \cdot \nabla \right)$ has an operator realization $\Lambda_q(a, \nabla a + b)$ in $L^q$, $q \in \left[ \frac{d-2}{\sqrt{\lambda_0}}, \infty \right]$, as the (minus) generator of a positivity preserving $L^\infty$ contraction quasi-contraction $C_0$ semigroup; moreover, $(\mu + \Lambda_q(a, b))^{-1}$ is well defined on $L^q$ for all $\mu > \frac{\lambda_2}{2(q-1)}$. In case $c < 0$, we apply Theorem A.1. This completes the proof of (i).

**Proof of (ii).** Let $d \geq 4$. Set $a^\varepsilon := I + c|x|^{-2}x \otimes x$, $|x| := \sqrt{|x|^2 + \varepsilon}$, $\varepsilon > 0$. Put $A^\varepsilon = A(a^\varepsilon)$. It is clear that $b \in F_{\delta_1}(A^\varepsilon)$ for all $\varepsilon > 0$.

Let $1_n$ denote the indicator of $\{x \in \mathbb{R}^d \mid |x| \leq n, |b(x)| \leq n\}$, and set $b_n := \gamma_n b \in C^\infty$, where $\gamma_n$ is the K. Friedrichs mollifier, $\gamma_n \downarrow 0$. Since our assumptions on $\delta$ and thus $\delta_1$ involve strict inequalities only, we can select $\gamma_n \downarrow 0$ so that $b_n \in F_{\delta_1}(A^\varepsilon)$, $\varepsilon > 0$, $n \geq 1$. Next, note that by the assumptions of the theorem $-1 < c < \frac{d-3}{2}$, and hence by Lemma 6, $\nabla a^\varepsilon \in F_{\delta_1}(A^\varepsilon)$, $\varepsilon > 0$. Thus, in view of the discussion above, $(\mu + \Lambda_q(a^\varepsilon, \nabla a^\varepsilon + b_n))^{-1}$ is well defined on $L^q$, $\mu > \frac{\lambda_2}{2(q-1)}$, $\varepsilon > 0$, $n \geq 1$. Here

$$\Lambda_q(a^\varepsilon, \nabla a^\varepsilon + b_n) = -\nabla \cdot a^\varepsilon \cdot \nabla + (\nabla a^\varepsilon) \cdot \nabla + b_n \cdot \nabla, \ D(\Lambda_q(a^\varepsilon, \nabla a^\varepsilon + b_n)) = W^{2,q}.$$

Set $u \equiv u^{\varepsilon,n} = (\mu + \Lambda_q(a^\varepsilon, \nabla a^\varepsilon + b_n))^{-1}f$, $0 \leq f \in C^{1}_c$. Then $u \in W^{3,q}$. Below

$$w \equiv w^{\varepsilon,n} := \nabla u^{\varepsilon,n}, \quad \phi := -\nabla \cdot (w|w|^{-2}),$$

$$I_q := \langle (\nabla w)^2 |w|^{q-2} \rangle, \quad J_q := \langle (\nabla |w|)^2 |w|^{q-2} \rangle,$$

$$I_{q,\chi} := \langle (x \cdot \nabla w)^2 \chi |x|^{-2} |w|^{q-2} \rangle, \quad J_{q,\chi} := \langle (x \cdot \nabla |w|)^2 \chi |x|^{-2} |w|^{q-2} \rangle,$$

$$H_{q,\chi} := \langle \chi |x|^{-2} |w|^q \rangle, \quad G_{q,\chi} := \langle \chi^2 |x|^{-4} (x \cdot w)^2 |w|^{q-2} \rangle,$$

where $\chi = |x|^2 |x|^{-2}$.

**1.** We repeat the proof of Lemma 7, where in the right-hand side of $(\text{BE}^{nd}_{+})$, $(\text{BE}^{nd}_{-})$ we now get an extra term $\langle -b_n \cdot w, \phi \rangle$:

$$\mu\langle |w|^q \rangle + I_q + cI_{q,\chi} + (q - 2)(J_q + cJ_{q,\chi}) - c\frac{d(d - 1)}{q}H_{q,\chi}$$

$$+ 2c \frac{2d + 1}{q}H_{q,\chi^2} - \frac{8c}{q}H_{q,\chi^3} = \beta_1 + \langle -b_n \cdot w, \phi \rangle + \langle f, \phi \rangle,$$

$$(\text{BE}^{nd}_{+})$$

$$\mu\langle |w|^q \rangle + I_q + cI_{q,\chi} + (q - 2)(J_q + cJ_{q,\chi}) - c\frac{d(d - 1)}{q}H_{q,\chi} - c(d - 2)G_{q,\chi^2}$$

$$+ 2c \frac{2d + 1}{q}H_{q,\chi^2} - \frac{8c}{q}H_{q,\chi^3} - 4c\varepsilon\langle |x|^{-6} (x \cdot w)^2 |w|^{q-2} \rangle = -\frac{1}{2}\beta_2 + \langle -b_n \cdot w, \phi \rangle + \langle f, \phi \rangle,$$

$$(\text{BE}^{nd}_{-})$$

where

$$\beta_1 := -2c\langle |x|^{-4} x \cdot w, x \cdot (x \cdot \nabla w) |w|^{q-2} \rangle, \quad \beta_2 := -2c(q - 2)\langle |x|^{-4} (x \cdot w)^2 x \cdot \nabla |w|, |w|^{q-3} \rangle.$$
2. By Lemma 4,
\[
\langle -b_n \cdot w, \phi \rangle 
\leq |c|(d + 3)\frac{q\sqrt{\delta}}{2} G_{q,\lambda x}^\frac{1}{2} J_q \|w\|^\frac{2}{q} + |c|\frac{q\sqrt{\delta}}{2} I_{q,\lambda x}^\frac{1}{2} J_q + \left(\frac{q^2 \delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|^\frac{2}{q} + C_2\|w\|_{q^{-2}}\|f\|^2_q.
\]
Next, by an evident analogue of Lemma 3,
\[
\langle f, \phi \rangle \leq \varepsilon_0(I_q + J_q + H_q + \|w\|^\frac{2}{q}) + C(\varepsilon_0)\|w\|_{q^{-2}}\|f\|^2_q.
\]
Again we choose \(\varepsilon_0\) so small that in the estimates below we can ignore the terms multiplied by \(\varepsilon_0\).
Applying the last two inequalities in (BE\(_\text{nd}\)) + (BE\(_\text{sc}\)), and using \(|\beta_1| \leq c\theta I_q + c\theta^{-1}G_{q,\lambda x}^2\), \(|\beta_2| \leq 2|c|(q - 2)(\theta J_q + 4^{-1}\theta^{-1}G_{q,\lambda x}^2)\), we obtain:
If \(c > 0\), then:
\[
\mu(|w|^q) + I_q + c(1 - \theta)I_q + (q - 2)(J_q + c\tilde{I}_{q,\lambda x}) - \frac{c}{q} \frac{d(d - 1)}{q} H_{q,\lambda x}
\]
\[
+ 2c \frac{2d + 1}{q} H_{q,\lambda x}^2 - \frac{8c}{q} H_{q,\lambda x}^3 + \frac{c}{q} G_{q,\lambda x}^2
\]
\[
\leq c(d + 3)\frac{q\sqrt{\delta}}{2} G_{q,\lambda x}^\frac{1}{2} J_q \|w\|^\frac{2}{q} + \frac{q\sqrt{\delta}}{2} I_{q,\lambda x}^\frac{1}{2} J_q + \left(\frac{q^2 \delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|^\frac{2}{q} + C_2\|w\|_{q^{-2}}\|f\|^2_q.
\]
If \(-1 < c < 0\), then (set \(s := |c|\)):
\[
\mu(|w|^q) + I_q - sI_{q,\lambda x} + (q - 2)(J_q - s(1 + \theta)\tilde{I}_{q,\lambda x}) + s \frac{d(d - 1)}{q} H_{q,\lambda x}
\]
\[
- 2s \frac{2d + 1}{q} H_{q,\lambda x}^2 + \frac{8s}{q} H_{q,\lambda x}^3 + s \left(d + 2 - (q - 2)\frac{1}{4\theta}\right) G_{q,\lambda x}^2 - 4sG_{q,\lambda x}^3
\]
\[
\leq s(d + 3)\frac{q\sqrt{\delta}}{2} G_{q,\lambda x}^\frac{1}{2} J_q \|w\|^\frac{2}{q} + \frac{q\sqrt{\delta}}{2} I_{q,\lambda x}^\frac{1}{2} J_q + \left(\frac{q^2 \delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|^\frac{2}{q} + C_2\|w\|_{q^{-2}}\|f\|^2_q.
\]
3. We will use (15), (16) to prove the following inequality
\[
\mu(|w|^q) + \eta J_q \leq C_1\|w\|_{q^{-2}}^2 + C_2\|w\|_{q^{-2}}\|f\|^2_q,
\]
\[
C_i = C_i(\varepsilon_0), \quad i = 1, 2,
\]
for some \(\eta = \eta(q, d, \varepsilon_0) > 0\).

**Case** \(c > 0\). In (15), select \(\theta = \frac{q}{2} > 1\). By the assumptions of the theorem, \(1 + c(1 - \theta) > 0\). Since \(I_q + c(1 - \theta)\tilde{I}_{q,\lambda x} \geq (1 + c(1 - \theta))I_q\) and \(H_{q,\lambda x} \geq G_{q,\lambda x}^2\), we have
\[
\mu(|w|^q) + (1 + c(1 - \theta))I_q + (q - 2)(J_q + c\tilde{I}_{q,\lambda x}) - \frac{c}{q} \frac{d(d - 1)}{q} H_{q,\lambda x}
\]
\[
+ 2c \frac{2d + 1}{q} H_{q,\lambda x}^2 - \frac{8c}{q} H_{q,\lambda x}^3 - \frac{c}{q} H_{q,\lambda x}^2
\]
\[
\leq c(d + 3)\frac{q\sqrt{\delta}}{2} G_{q,\lambda x}^\frac{1}{2} J_q \|w\|^\frac{2}{q} + \frac{q\sqrt{\delta}}{2} I_{q,\lambda x}^\frac{1}{2} J_q + \left(\frac{q^2 \delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|^\frac{2}{q} + C_2\|w\|_{q^{-2}}\|f\|^2_q.
\]
Arguing as in the proof of Theorem 3, we arrive at
\[
\mu(|w|^q) + (1 + c(1 - \theta))I_q + (q - 2)J_q + cM(1)H_q
\]
\[
\leq c(d + 3)\frac{q\sqrt{\delta}}{2} G_{q,\lambda x}^\frac{1}{2} J_q \|w\|^\frac{2}{q} + \frac{q\sqrt{\delta}}{2} I_{q,\lambda x}^\frac{1}{2} J_q + \left(\frac{q^2 \delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|^\frac{2}{q} + C_2\|w\|_{q^{-2}}\|f\|^2_q,
where \(M(1) = -2\frac{(d-2)^2}{q^2}\). Using \(I_q \geq I_{q,\chi}, H_q \geq G_{q,\chi^2}\) in the RHS, and applying the standard quadratic estimates, we obtain \((\theta_2, \theta_3 > 0)\),

\[
\mu(|w|^q) + (1 + c(1 - \theta))I_q + (q - 2)J_q + cM(1)H_q
\]

\[
\leq c(d + 3)\frac{q\sqrt{\delta}}{4}(\theta_2J_q + \theta_2^{-1}H_q) + c\frac{q\sqrt{\delta}}{4}(\theta_3I_q + \theta_3^{-1}J_q) + \left(\frac{q^2\delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q
\]

\[
+ C_1\|w\|_{q}^q + C_2\|w\|_{q}^{q-2}\|f\|_{q}^2.
\]

We select \(\theta_2 = \frac{2}{d-2}, \theta_3 = 1\). By the assumptions of the theorem, \(1 + c(1 - \theta - \frac{q\sqrt{\delta}}{4}) \geq 0\), so that \((1 + c(1 - \theta - \frac{q\sqrt{\delta}}{4}))I_q \geq (1 + c(1 - \theta - \frac{q\sqrt{\delta}}{4}))J_q\). Thus, we arrive at

\[
\mu(|w|^q) + \left[\left(1 + c(1 - \theta \right) - \frac{q\sqrt{\delta}}{2}\right) - c(d + 3)\frac{q\sqrt{\delta}}{4}\frac{q}{d-2} - \frac{q^2\delta}{4} \right]J_q
\]

\[
+ \left[cM(1) - c(d + 3)\frac{q\sqrt{\delta}}{4}\frac{d - 2}{q}\right]H_q \leq C_1\|w\|_{q}^q + C_2\|w\|_{q}^{q-2}\|f\|_{q}^2.
\]

So, by \(J_q \geq \frac{(d-2)^2}{q^2}H_q\),

\[
\mu(|w|^q) + \eta J_q + \left[\left(-\eta + q - 1\right)\frac{(d-2)^2}{q^2} - L_1^m(c, \delta)\right]H_q \leq C_1\|w\|_{q}^q + C_2\|w\|_{q}^{q-2}\|f\|_{q}^2,
\]

where

\[
L_1^m(c, \delta) = \left[-c\left(1 - \theta - \frac{q\sqrt{\delta}}{2}\right) + c(d + 3)\frac{q\sqrt{\delta}}{4}\frac{q}{d-2} + \frac{q^2\delta}{4} \right] \frac{(d-2)^2}{q^2}
\]

\[
+ c\left[-M(1) + (d + 3)\frac{q\sqrt{\delta}}{4}\frac{d - 2}{q}\right].
\]

By the assumptions of the theorem, \((-\eta + q - 1)\frac{(d-2)^2}{q^2} - L_1(c, \delta) \geq 0\) for all \(\eta > 0\) sufficiently small. (17) is proved.

**Case** \(-1 < c < 0\). Following the proof of Theorem 3, we obtain from (16)

\[
\mu(|w|^q) + (1 - s)I_q + (q - 2)(1 - s(1 + \theta))J_q + s\frac{d(d - 1)}{q}G_{q,\chi}
\]

\[
- 2s\frac{2d + 1}{q}G_{q,\chi^2} + \frac{8s}{q}G_{q,\chi^3} + s\left(d + 2 - (q - 2)\frac{1}{4\theta}\right)G_{q,\chi^2} - 4sG_{q,\chi^3}
\]

\[
\leq s(d + 3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}J_q + \frac{s\sqrt{\delta}}{2}G_{q,\chi^3} + \left(\frac{q^2\delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|_{q}^q + C_2\|w\|_{q}^{q-2}\|f\|_{q}^2.
\]

In the RHS, we use \(\frac{q^2}{(d-2)^2}J_q \geq G_{q,\chi^2}, I_q \geq I_{q,\chi^1}, \frac{1}{2}(I_q + J_q) \geq I_q^\frac{1}{q}, J_q^\frac{1}{q}\)

\[
\mu(|w|^q) + (1 - s)I_q + (q - 2)(1 - s(1 + \theta))J_q + s\frac{d(d - 1)}{q}G_{q,\chi}
\]

\[
- 2s\frac{2d + 1}{q}G_{q,\chi^2} + \frac{8s}{q}G_{q,\chi^3} + s\left(d + 2 - (q - 2)\frac{1}{4\theta}\right)G_{q,\chi^2} - 4sG_{q,\chi^3}
\]

\[
\leq s(d + 3)\frac{q\sqrt{\delta}}{2}\frac{d}{d-2}J_q + \frac{s\sqrt{\delta}}{4}(I_q + J_q) + \left(\frac{q^2\delta}{4} + (q - 2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|_{q}^q + C_2\|w\|_{q}^{q-2}\|f\|_{q}^2.
\]
Arguing as in the proof of Theorem 3, and selecting \( \theta := \frac{1}{4} \frac{q}{d-2} \frac{q-2}{q-d} \), we arrive at

\[
\mu \langle \|w\|^q \rangle + (1-s)I_q + (q-2)(1-s(1+\theta))J_q
\leq s(d+3) \frac{q\sqrt{\delta}}{2} \frac{q}{d-2} J_q + s \frac{q\sqrt{\delta}}{4} (I_q + J_q) + \left( \frac{q^2\delta}{4} + (q-2) \frac{q\sqrt{\delta}}{2} \right) J_q + C_1 \|w\|_{q_1}^g + C_2 \|w\|_{q-2}^g \|f\|_q^2.
\]

By the assumptions of the theorem, \( 1-s(1+\frac{q\sqrt{\delta}}{4}) \geq 0 \). Therefore, since \( J_q \leq I_q \),

\[
\mu \langle \|w\|^q \rangle + \left[ q - 1 - s(q-2)(1+\theta)-(s(d+3)\frac{q\sqrt{\delta}}{2} \frac{q}{d-2} - s \frac{q\sqrt{\delta}}{4} - (q-2) \frac{q\sqrt{\delta}}{2} \right] J_q
\leq C_1 \|w\|_{q}^g + C_2 \|w\|_{q-2}^g \|f\|_q^2.
\]

By the assumptions of the theorem, \( q-1-s(q-2)(1+\theta)-(s(d+3)\frac{q\sqrt{\delta}}{2} \frac{q}{d-2} - s \frac{q\sqrt{\delta}}{4} - (q-2) \frac{q\sqrt{\delta}}{2} \geq 0 \). Hence (17) is proved.

4. For \( d \geq 4 \), the Sobolev Embedding Theorem and Theorem A.2(ii) now yield (**). We have proved (ii).

Proof of (iii). Let \( q = 2, d \geq 3 \). If \( c < 0 \), then we can argue as in steps 1-3 obtaining

\[
\sup_{\varepsilon > 0, n} I_2(\varepsilon, n) \leq K \|f\|_2, \quad \text{and so} \quad I_2(u) \leq K \|f\|_2 \quad \Rightarrow u \in W^{2,2}.
\]

Now, let \( c > 0 \). We have \( b + \nabla a \in F_{\delta_2}(A), \sqrt{\delta_2} := \sqrt{\delta} + 2 \frac{d-1}{d-2} \frac{c}{1+c} \) (cf. beginning of the proof). By the assumptions of the theorem, \( \delta_2 < 1 \), and so \( \Lambda_2(a, \nabla a) \) is well defined. By the Miyadera Perturbation Theorem and Theorem 1, \( D(\Lambda_2(a, \nabla a + b)) = D(A_2) \subset W^{2,2}, \) and \( u := (\mu + \Lambda_2(a, \nabla a + b))^{-1} f, \mu > 0, f \in L^2, \) belongs to \( W^{2,2}. \) Multiplying \( (\mu + A_2 + (\nabla a + b) \cdot \nabla) u = f \) by \( \phi_m := -E_m \nabla \cdot w, \) \( E_m = (1-m^{-1} \Delta)^{-1}, m \geq 1 \) and integrating by parts we have (omitting the summation sign in the repeated indices):

\[
\mu \langle \|w\|^2 \rangle + \langle a \cdot \nabla w_r, E_m \nabla w_r \rangle + \langle -(\nabla_r a) \cdot w, E_m \nabla w_r \rangle + \langle \nabla a \cdot w, \phi_m \rangle = \langle -b \cdot w, \phi_m \rangle + \langle f, \phi_m \rangle, \quad (*)
\]

Now we pass in (*) to the limit \( m \to \infty \). We obtain an analogue of \( (BE^\text{nd})_+ \) for \( q = 2 \):

\[
\mu \langle \|w\|^2 \rangle + I_2 + c I_2 - \frac{c}{2} (d-2)(d-3)H_2 = \beta + \langle -b \cdot w, \phi \rangle + \langle f, -\nabla \cdot w \rangle
\]

where \( I_2 := \langle \nabla w_r, \nabla w_r \rangle, \quad I_2 := \langle (|x| \nabla |x|)^2 |x|^{-2} \rangle, \quad H_2 := \langle (|x|^{-2} |w|^2) \rangle, \quad \beta := -2c \langle |x|^{-4} x \cdot w, x \cdot (x \cdot \nabla w) \rangle. \)

Using the inequalities \( \beta \leq c I_2 + c H_2, \frac{d-2)^2-2}{4} H_2 \leq I_2, \) we have

\[
\mu \langle \|w\|^2 \rangle + \left[ 1 - \frac{4}{(d-2)^2} \left( 1 + \frac{(d-2)(d-3)}{2} \right) \right] I_2 \leq \langle -b \cdot w, \phi \rangle + \langle f, \phi \rangle,
\]

so by

\[
\langle -b \cdot w, \phi \rangle \leq c(d+3) \frac{\sqrt{\delta}}{2} \left( \frac{2}{d-2} J_2 + \frac{d-2}{2} H_2 \right) + c \frac{\sqrt{\delta}}{2} (I_2 + J_2 + \delta J_2 \quad (J_2 \leq I_2)
\]

\[
\leq \left( 2c \sqrt{\delta} \frac{d+3}{d-2} + c \sqrt{\delta} \right) I_2.
\]

Therefore,

\[
\mu \langle \|w\|^2 \rangle + \left[ 1 - \frac{4}{(d-2)^2} \left( 1 + \frac{(d-2)(d-3)}{2} \right) c - c \sqrt{\delta} \left( \frac{2}{d-2} + 1 \right) \right] I_2 \leq \langle f, \phi \rangle,
\]
where the coefficient of $I_2$ is positive by the assumptions of the theorem. The proof of (iii) follows.

The proof of Theorem 4 is completed. □

**Appendix A.**

**Theorem A.1.** Let $d \geq 3$. Let $a = I + c|x|^{-2}x \otimes x$, $-1 < c < 0$, $a^\varepsilon := I + c|x|^{-2}x \otimes x$, $|x|_2^2 := |x|^2 + \varepsilon$, $\varepsilon > 0$. Set $a_n := a^\varepsilon_n$, $\varepsilon_n \downarrow 0$. Let $b \in F_0(A)$, $0 < \delta < 4$. Let $1_n$ denote the indicator of $\{x \in \mathbb{R}^d \mid |x| \leq n, |b(x)| \leq n\}$, and set $b_n := \gamma_{\varepsilon_n} \ast 1_n b$, where $\gamma_{\varepsilon_n}$ is the K. Friedrichs mollifier, $\varepsilon_n \downarrow 0$.

Then $-\nabla \cdot a \cdot \nabla + (\nabla a + b) \cdot \nabla$ has an operator realization $\Lambda_n(a, b)$ in $L^r$, $r \geq \frac{2}{2 - \sqrt{\delta}}$, as the (minus) generator of a positivity preserving, $L^\infty$ contraction, quasi contraction $C_0$ semigroup on $L^r$, and, for each $r > \frac{2}{2 - \sqrt{\delta}}$,

$$e^{-t \Lambda_n(a, \nabla a + b)} = s-L^r \lim_{n \to \infty} e^{-t \Lambda_n(a, \nabla a_n + b_n)}.$$  

Proof. We modify the proof of [KiS, Theorem 3.2] replacing $b_n$ there with $\nabla a_n + b_n$.

In comparison with [KiS, Theorem 3.2], essentially we have an extra term $\langle \nabla a_n \cdot \nabla v, v^{r-1} \rangle$ to deal with. Since $\nabla a_n = c(d - 1)|x|^{-2}x + 2\varepsilon_n|x|^{-4}x$, we have (write $\varepsilon = \varepsilon_n$)

$$\langle \nabla a_n \cdot \nabla v, v^{r-1} \rangle = c(d - 1)\langle |x|^{-2}x \cdot \nabla v, v^{r-1} \rangle + 2\varepsilon \langle |x|^{-4}x \cdot \nabla v, v^{r-1} \rangle,$$

and

$$\frac{r}{2} \langle |x|^{-2}x \cdot \nabla v, v^{r-1} \rangle = \langle |x|^{-2}x \cdot \nabla v^\sharp, v^\sharp \rangle = -\frac{d - 2}{2} \langle |x|^{-2}v^r \rangle - \varepsilon \langle |x|^{-4}v^r \rangle,$$

$$\frac{r}{2} \varepsilon \langle |x|^{-4}x \cdot \nabla v, v^{r-1} \rangle = \varepsilon \langle |x|^{-4}x \cdot \nabla v^\sharp, v^\sharp \rangle = -\frac{d - 4}{2} \varepsilon \langle |x|^{-4}v^r \rangle - 2\varepsilon^2 \langle |x|^{-6}v^r \rangle.$$

Thus, since $c < 0$ and $v > 0$, we have $\langle \nabla a_n \cdot \nabla v, v^{r-1} \rangle \geq 0$ for all $d \geq 3$. This inequality allows us to discard the extra term.

The rest of the proof repeats [KiS, Theorem 3.2]. □

**Theorem A.2.** Let $d \geq 3$. Let $a = I + c|x|^{-2}x \otimes x$, $a^\varepsilon := I + c|x|^{-2}x \otimes x$, $|x|_2^2 := |x|^2 + \varepsilon$, $\varepsilon > 0$. Set $a_n := a^\varepsilon_n$, $\varepsilon_n \downarrow 0$. Let $b \in F_0$, let $b_n$’s be as in Theorem A.1.

(i) Assume that $q > d - 2$, $d \geq 4$, $c, \delta$ satisfy the assumptions of Theorem 2(ii), or $q = 2$, $d \geq 3$, $c, \delta$ satisfy the assumptions of Theorem 2(iii). Then $(\mu + \Lambda_q(a, b))^{-1}$, $(\mu + \Lambda_q(a_n, b_n))^{-1}$, $\mu > \omega_q$, are well defined, and

$$(\mu + \Lambda_q(a, b))^{-1} = s-L^q \lim_n (\mu + \Lambda_q(a_n, b_n))^{-1}.$$  

Here $\Lambda_q(a_n, b_n) = -\nabla \cdot a_n \cdot \nabla + b_n \cdot \nabla$, $D(\Lambda_q(a_n, b_n)) = W^{2,q}$.

(ii) Assume that $q > d - 2$, $d \geq 4$, $c, \delta$ satisfy the assumptions of Theorem 4(ii), or $q = 2$, $d \geq 3$, $c < 0$, $\delta$ satisfy the assumptions of Theorem 4(iii). Then $(\mu + \Lambda_q(a, \nabla a + b))^{-1}$, $(\mu + \Lambda_q(a_n, \nabla a_n + b_n))^{-1}$, $\mu > \omega_q$, are well defined, and

$$(\mu + \Lambda_q(a, \nabla a + b))^{-1} = s-L^q \lim_n (\mu + \Lambda_q(a_n, \nabla a_n + b_n))^{-1}.$$  

Proof. We modify the proof of [KiS, Theorem 3.2] but will work with resolvents instead of semigroups.

To prove (i), set $\delta := [1 \vee (1 + c)^{-2}] \delta$, $b := b$, $b_n := b_n$.

To prove (ii), set

$$\sqrt{\delta} := \begin{cases} \sqrt{\delta} + \frac{2d - 1}{d - 2} \frac{c}{|x|_\varepsilon^2}, & 0 < c < d - 2, \\ (1 + c)^{-1} \sqrt{\delta}, & -1 < c < 0. \end{cases}$$
To show that \( Z \triangleq \arg \text{arguing as in [KiS], we arrive at the inequality}

\[
\left\| (\mu + \Lambda_\theta(a, \tilde{b}_n))^{-1} f - (\mu + \Lambda_\theta(a, \tilde{b}_n))^{-1} f \right\|_q = 0, \quad f \in L^q.
\]

Thus, it suffices to show that

\[
\lim_n \left\| (\mu + \Lambda_\theta(a, \tilde{b}_n))^{-1} f - (\mu + \Lambda_\theta(a, \tilde{b}_n))^{-1} f \right\|_q = 0.
\]

1. First, prove (i) and (ii) for \( c > 0 \). Set \( A^n \equiv [-\nabla \cdot a_n \cdot \nabla \mid C^\infty_{c(2)}] \). Then \( \tilde{b} \in F_{\delta_n}(A), \tilde{b}_n \in F_{\delta_n}(A^n) \) (for details see the proofs of Theorems 2 and 4, respectively), where, by our assumptions, \( \delta_n < 4 \). Therefore, by [KiS, Theorem 3.2] \((\mu + \Lambda_\theta(a, \tilde{b}))^{-1} f \geq 0, \tilde{b}_n \geq 0 \). Let \( v := \zeta u_n \geq 0, \) where \( \zeta = \zeta(R), 0 \leq \zeta \leq 1, \zeta \equiv 0 \) on the complement of an open ball of radius \( \simeq R \), \( \zeta \equiv 1 \) on the complement of an open ball of radius \( \simeq 2R, \) is defined in [KiS, proof of Theorem 3.2, step 1]. Note that

\[
\langle \zeta(\mu + A + \tilde{b}_n \cdot \nabla)u_n, v^{q-1} \rangle = \langle \zeta f, v^{q-1} \rangle.
\]

Now, proceeding as in [KiS, proof of Theorem 3.2, step 1], we arrive at the following. For every \( \varepsilon > 0 \) there exists \( R > 0 \) such that

\[
\| \zeta u_n \|_q \leq \varepsilon, \quad n \geq 1, \quad \mu > \omega_q.
\]

Similarly,

\[
\| \zeta \tilde{u}_n \|_q \leq \varepsilon, \quad n \geq 1, \quad \mu > \omega_q.
\]

(b) Set \( g_n := u_n - \tilde{u}_n \). For the \( R \) determined above, set \( v := \zeta g_n, \) where \( 0 \leq \zeta \leq 1, \zeta \equiv 1 \) on an open ball of radius \( \simeq R, \zeta \equiv 0 \) on the complement of an open ball of radius \( \simeq 2R, \) is defined in [KiS, proof of Theorem 3.2, step 2]. Subtracting the equations for \( u_n \) and \( \tilde{u}_n \), we have

\[
\langle \zeta (\mu + A + \tilde{b}_n \cdot \nabla)g - \zeta \nabla \cdot (a - a_n) \cdot \nabla \tilde{u}_n, v|v|^{q-2} \rangle = 0.
\]

Arguing as in [KiS], we arrive at the inequality

\[
\| \mu |v|_q^{q-1} \|_q \leq M(q, d, \delta_n) R^{-q} \| f \|_q + |\langle \nabla \cdot (a - a_n) \cdot \nabla \tilde{u}_n, v|v|^{q-2} \rangle|.
\]

To show that \( \zeta (u_n - \tilde{u}_n) \rightarrow 0 \) strongly in \( L^q \) as \( n \rightarrow 0 \), it remains to prove that \( \lim n |Z| = 0 \), where

\[
Z := \langle (a - a_n) \cdot \nabla \tilde{u}_n, \nabla (\zeta v|v|^{q-2}) \rangle.
\]

The latter is possible due to the bounds \( \| \nabla u_n \|_q \leq K \| f \|_q, \| \nabla \tilde{u}_n \|_q \leq K \| f \|_q \) (cf. the proof of Theorem 3 (steps 1-3) for (i), the proof of Theorem 4 (steps 1-3) for (ii)). Indeed,

\[
Z = q \langle \zeta^{q-1} \nabla \cdot (a - a_n) \cdot \nabla \tilde{u}_n, g_n \rangle + \langle \nabla g_n \cdot (a - a_n) \cdot \nabla \tilde{u}_n, \zeta |g_n| |g_n|^{q-2} \rangle
\]

\[
+ (q - 3) \langle \nabla |g_n| \cdot (a - a_n) \cdot \nabla \tilde{u}_n, \zeta^2 |g_n|^2 |g_n|^{q-3} \rangle
\]

\[
\equiv q Z_1 + Z_2 + (q - 3) Z_3,
\]

and \( q := \frac{qd}{d-2} > 2 \)

\[
\| Z_1 \|_{q} \leq \| \nabla \cdot (a - a_n) \|_{q} \| \nabla \tilde{u}_n \|_{q} \| g_n \|_{q}^{q-1},
\]

\[
\| Z_2 \| \leq \| \nabla g_n \cdot (a - a_n) \| \| \nabla \tilde{u}_n \|_{q} \| g_n \|_{q}^{q-2},
\]

\[
\| Z_3 \| \leq \| \nabla g_n \cdot (a - a_n) \| \| \nabla \tilde{u}_n \|_{q} \| g_n \|_{q}^{q-2},
\]

\[
\| \nabla |g_n| \|_{q} \leq 2K \| f \|_{q}, \quad \sup_{i,j} \| \zeta (a_{ij} - a_{ij}^\varepsilon) \|_{q} \rightarrow 0 \quad \text{as} \quad n \rightarrow 0.
\]
Thus, by Hölder’s inequality and $|a_{ij} - \tilde{a}_{ij}| \leq c|x|^2 \downarrow 0$ a.e. as $n \to \infty$,
\[
\lim_{n} |Z| = 0.
\]
It follows that $\zeta(u_n - \tilde{u}_n) \to 0$ in $L^q$.

Combining the results of (a) and (b), we obtain the required.

2. To prove (ii) with $c < 0$, we repeat the proof above but taking into account the proof of Theorem A.1. □

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