Filtered deformations of elliptic algebras

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Abstract

One of the difficulties in doing noncommutative projective geometry via explicitly presented graded algebras is that it is usually quite difficult to show flatness, as the Hilbert series is uncomputable in general. If the algebra has a regular central element, one can reduce to understanding the (hopefully more tractable) quotient. If the quotient is particularly nice, one can proceed in reverse and find all algebras of which it is the quotient by a regular central element (the filtered deformations of the quotient). We consider in detail the case that the quotient is an elliptic algebra (the homogeneous endomorphism ring of a vector bundle on an elliptic curve, possibly twisted by translation). We explicitly compute the family of filtered deformations in many cases and give a (conjecturally exhaustive) construction of such deformations from noncommutative del Pezzo surfaces. In the process, we also give a number of results on the classification of exceptional collections on del Pezzo surfaces, which are new even in the commutative case.

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1 Introduction

The origin of the present work was an attempt to produce elliptic analogues of the algebras of [9], with a particular focus on understanding the associated noncommutative analogues of del Pezzo surfaces. (Note that for our purposes, a “(degenerate) del Pezzo surface” is a smooth projective surface with an almost ample anticanonical bundle; that is, the anticanonical morphism is allowed to contract a finite configuration of −2-curves.) The algebras of [9] were given by very simple explicit presentations (they are generated by three elements that multiply to 1 and have specified minimal polynomials), and are difficult to generalize as such, but they also have (several) filtrations such that the associated graded algebras have nice geometric descriptions. In particular, the associated graded algebras are constructed from the multiplication-by-q map on a nodal curve of genus 1,
and thus their elliptic analogues are relatively easy to describe. One is thus left with the problem of understanding their “filtered deformations”, i.e., for which filtered algebras does one obtain the given associated graded. Although the problem of classifying filtered deformations of a given graded algebra is almost certainly not computable in general, it is still relatively straightforward to do non-rigorous calculations and thus formulate conjectures. In particular, we (P. Etingof and the author) found that the (putative) moduli spaces classifying such deformations were often quite nice: not only were they frequently rational, they were in fact quite frequently weighted projective spaces or products thereof.

Although the resulting conjectures were quite striking, it was not clear how to prove them rigorously; similarly, although the resulting algebras were clearly noncommutative analogues of del Pezzo surfaces, it was unclear how to prove anything much about them. As a result, the project was shelved for quite some time. In the interim, the author developed a completely different approach to noncommutative rational surfaces [20, 19, 21], extending beyond the del Pezzo case and giving a great deal of control over the corresponding representation theory (i.e., the categories and derived categories of sheaves), making much of the original motivation moot. This is not entirely the case, however: the work of [7] in particular suggests that there is independent interest in having representations of such surfaces via graded algebras. The object of the present work is therefore not so much to construct noncommutative del Pezzo surfaces as filtered deformations, but rather to construct filtered deformations via noncommutative del Pezzo surfaces.

As the title indicates, the algebras we are deforming are “elliptic algebras”, which are constructed as follows. The simplest version is simply the graded homogeneous coordinate ring of an elliptic curve: if $E$ is an elliptic curve and $L$ is a line bundle, then there is a corresponding graded algebra

$$B_{E,L} = \bigoplus_{n \in \mathbb{Z}} \Gamma(E; L^n),$$

with multiplication induced by the obvious map $L^m \otimes L^n \cong L^{m+n}$. This has a noncommutative analogue (the twisted homogeneous coordinate rings of [2]), in which we twist by a translation $\tau \in \text{Aut}(E)$:

$$B_{E,L,\tau} = \bigoplus_{n \in \mathbb{Z}} \Gamma(E; L_n),$$

where

$$L_1 = L; \quad L_n = L \otimes (\tau^{-1})^* L_{n-1},$$

with multiplication induced by

$$L_m \otimes (\tau^{-m})^* L_n \cong L_{m+n}.$$  

A somewhat cleaner description is based on the observation that the functor

$$\Psi : M \mapsto L \otimes (\tau^{-1})^* M$$

is actually an autoequivalence of $\text{coh}(E)$. (In fact, any autoequivalence is of this form, except that $\tau$ need not be a translation.) We may thus instead define

$$B_\Psi = \bigoplus_{n \in \mathbb{Z}} \Gamma(E; \Psi^n \mathcal{O}_E),$$

with multiplication given by

$$x_m \cdot x_n = \Psi^m(x_n) \circ x_m$$

where we view $x_n$ as a homomorphism $\mathcal{O}_E \to \Psi^n \mathcal{O}_E$. 

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Although the filtered deformations of these algebras are enough to cover the examples of [7], analogues of the algebras of [9] require a further generalization. For the commutative version, there is a generalization involving a vector bundle $V$, now with

$$B_{E,L,V} = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(V, V \otimes L^n).$$

(1.8)

This also has a twisted version,

$$B_{V,\Psi} := \bigoplus_{n \in \mathbb{Z}} \text{Hom}(V, \Psi^n V).$$

(1.9)

Note that unlike the instance $V = \mathcal{O}_E$, this may fail to be nonnegatively graded, but will always be bounded below.

The basic question we study in the present work is: What are the filtered deformations of $B_{V,\Psi}$? Although a full answer appears out of reach for the moment, we do give some partial results that narrow things down. In one direction, we show that when $\tau = 1$, any filtered deformation arises as the endomorphism ring of a vector bundle $V^+$ on a deformation of the appropriate cone over $E$, with the property that $V^+$ is both rigid ($\text{Ext}^1(V^+, V^+) = 0$) and unobstructed ($\text{Ext}^2(V^+, V^+) = 0$). Thus the problem essentially reduces to understanding such bundles on del Pezzo surfaces. (In principle, one could have a nontrivial filtered deformation of $B_{V,L}$ living on an undeformed cone, but experiment suggests this is not actually possible.) More generally, we classify such bundles on noncommutative del Pezzo surfaces, and recover the phenomenology alluded to earlier by showing that in many cases the resulting family of filtered deformations is a weighted projective space (or a product thereof).

The main gaps in obtaining a full classification are that (a) we only classify rigid, unobstructed bundles on smooth del Pezzo surfaces, while in general for $\tau = 1$ the center is the coordinate ring of a singular del Pezzo surface, (b) when $\dim(V) \neq 1$, the algebra is not a deformation of a del Pezzo surface per se, but rather of an Azumaya algebra on the del Pezzo surface (in particular, there is no a priori reason to expect it to have a well-defined structure sheaf!), and (c) although the results of [20, 19, 21] certainly construct many examples of noncommutative del Pezzo surfaces (obtaining the same family in multiple ways), and some of the results below strongly suggest that these examples are exhaustive, actually proving this is beyond the reach of present techniques.

Note that our focus in the present work has been on the general theory, with little attempt to give explicit presentations for any of the filtered deformations constructed below (especially for $\tau \neq 1$). Those interested in explicit presentations should instead view this work as a guide for future investigations, by showing where one expects the full family to be particularly nice, and letting one predict the degrees of the generators and relations.

The plan of the paper is as follows. In Section 2, we give a general discussion of filtered deformations, in particular not requiring the graded algebra to be nonnegatively graded. The main results for elliptic algebras are (a) the moduli stack of filtered deformations is a closed substack of a weighted projective space, and (b) if $\tau$ is torsion, then the center of any filtered deformation of $B_{V,\Psi}$ is a filtered deformation of the center of $B_{V,\Psi}$. (This section also gives explicit computations of the moduli stack in some cases with $\tau = 1$, and briefly discusses some interesting phenomena arising when viewing $B_{V,\Psi}$ itself as a filtered deformation.)

Any actual algorithm for classifying filtered deformations requires understanding not just the generators and relations but also the syzygies of the relations, so in Section 3, we give an algorithm

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1 We should note here that although the algebras of [9] are analogues of double affine Hecke algebras, the elliptic double affine Hecke algebras of [21] are not in general filtered deformations of elliptic algebras, even in rank 1.
for computing resolutions of the algebras $B_{V,\Psi}$. One particularly nice feature is that qualitative information about the resolution (i.e., the description of each term as a sum of projective modules) can be computed directly from qualitative information about the algebra (the degree of $L$ and the slopes and multiplicities of the constant-slope summands of $V$). In particular, one finds that for nearly every case in which $V$ is semistable, the algebra $B_{V,\Psi}$ is Koszul; in particular, it is a quadratic algebra with syzygies only in degree 3, and thus it is straightforward to write down equations cutting out the family of filtered deformations. This understanding of resolutions also enables us to control the cohomological properties of the deformations, in particular showing that the tautological sheaf associated to any filtered deformation is rigid and unobstructed.

This suggests in general that we should study rigid, unobstructed torsion-free sheaves on non-commutative del Pezzo surfaces, which is done in Section 5 after first reviewing the definition of and main results on such surfaces in Section 4. It is fairly straightforward to show that any such sheaf gives rise to a filtered deformation of an elliptic algebra, and thus in the absence of an understanding of all filtered deformations, we would at least like to understand all the deformations arising from del Pezzo surfaces, or equivalently to understand all such sheaves. In the commutative case, these were studied in [14, 13], and it was in particular shown that any such bundle arises in a natural way from an exceptional collection on the surface. It is straightforward to extend their arguments (especially in the simplified form we give below) to the noncommutative case, and thus the problem of understanding rigid, unobstructed, torsion-free sheaves reduces to that of understanding exceptional collections. In fact, the problem simplifies further: for the cases of interest, any sequence of exceptional bundles with the desired numerical data is automatically an exceptional collection, so one needs only classify exceptional bundles. These are, moreover, uniquely determined by their numerical data, so the only real question is existence. This turns out to be an open and closed condition, so that one can immediately reduce to the commutative case (and further to nondegenerate del Pezzo surfaces of degree 1, as well as to a related question on rational elliptic surfaces). Although this existence question remains open in most cases, we are at least able to settle it for all cases of rank at most 9, which is enough to cover the most interesting examples for filtered deformations.

Finally, in Section 6, we describe how to use the above theory to compute the corresponding moduli spaces as well as their images in the moduli spaces of filtered deformations. Since exceptional collections are unique when they exist, their moduli spaces are finite covers of the moduli spaces of surfaces (the covering arising from the fact that the numerical data breaks symmetries), and the maps from such spaces to moduli spaces of elliptic algebras are straightforward to compute. One finds in general that any family of algebras arising in this way is explicitly parametrized by the quotient of a subgroup of $E^n$ by some Weyl group acting linearly on $E^n$. This partly explains why these spaces are often weighted projective spaces, as results of [15] give a number of cases in which such quotients are weighted projective spaces; the remainder of the explanation involves showing that the natural line bundle on the moduli stack of filtered deformations is the same as the invariant line bundle arising in [16]. Using this, we describe the results of the classification in a large number of cases, as well as explaining some of the consequences arising when the result is a weighted projective space with generators of degree > 1. (For instance, Sklyanin’s original deformation [23] of $\mathbb{P}^3$ arises in our theory from the fact that when $\deg(L) = 4$, the space of deformations of $B_{O_{E,\Psi}}$ has two generators of degree 2, which become the two central degree 2 elements of Sklyanin’s algebra.)

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Poisson structures on centers. The author would also like to thank the organizers of the workshop “Hypergeometry, Integrability and Lie Theory” at the Lorentz Center in Leiden for providing the necessary impetus to finish and write up the present work. In addition, the author would like to thank the referee for extensive comments which led to many improvements in the current version of this paper.

2 Filtered deformations

We first set a convention: for our purposes, a graded algebra $B$ is a $\mathbb{Z}$-graded algebra (over a commutative ring $k$) in the usual sense such that (a) $B_m$ is a finitely generated $k$-module for all $m$, and (b) $B_m = 0$ for $m \ll 0$. (The algebras will typically be algebras over a field $k$ of characteristic 0, but will on occasion need to be algebras over a field of finite characteristic or a discrete valuation ring. We will mainly give arguments in characteristic 0, indicating in remarks where changes need to be made to include finite characteristic.)

Similarly, our convention for filtrations is that they are ascending filtrations, i.e., an assignment of a finitely generated submodule $A_{\leq m} \subset A$ to each integer $m$ such that (a) $A_{\leq m}A_{\leq n} \subset A_{\leq m+n}$, (b) $A_{\leq m} \subset A_{\leq n}$ for $m \leq n$, (c) $A_{\leq m} = 0$ for $m \ll 0$, and (d) $\bigcup_n A_{\leq n} = A$.

Given a filtered algebra, there are two natural ways to produce a graded algebra: the associated graded $\text{gr} A := \bigoplus_i (A_{\leq i}/A_{\leq i-1})$, and the Rees algebra $A^+ := \bigoplus_i A_{\leq i}t^i$, where $t$ is an auxiliary variable. Similarly, a graded algebra may be viewed as a filtered algebra with $B_{\leq i} = \bigoplus_{j \leq i} B_j$; in that case, the associated graded is $B$ and the Rees algebra is $B[t]$.

**Definition 2.1.** Let $B$ be a graded algebra. A filtered deformation of $B$ is a filtered algebra $A$ equipped with an isomorphism $\text{gr} A \cong B$.

**Remark.** Note that the isomorphism is part of the data.

The following fact will later allow us to reduce to the case of nonnegatively graded algebras (for the algebras of interest, at least!).

**Proposition 2.1.** Let $B$ be a graded algebra. For any homogeneous idempotent $e \in B$ and any filtered deformation $A$ of $B$, there is an idempotent $\hat{e} \in A$ with leading term $e$.

**Proof.** An idempotent has degree 0, and thus $\hat{e}$ should be in $A_{\leq 0}$. This is a subalgebra, and our conventions ensure that it is finite-dimensional, and thus Artinian. The ideal $A_{\leq -1}$ is nil, so any idempotent of $A_{\leq 0}/A_{\leq -1} = B_0$ lifts to $A_{\leq 0}$, and thus to $A$. \hfill $\square$

There is a natural action of $\mathbb{G}_m$ on any graded algebra $B$ with $t \in \mathbb{G}_m$ acting by $t^i$ on $B_i$, and this acts on the space of filtered deformations by composing the isomorphism $\text{gr} A \cong B$ with the induced automorphism of $B$.

Any filtered isomorphism $A \cong A'$ of filtered algebras induces an isomorphism $\text{gr} A \cong \text{gr} A'$, and thus if $A$ and $A'$ are both filtered deformations of $B$, the isomorphism induces an automorphism of $B$. The natural notion of equivalence on filtered deformations is an isomorphism such that the induced automorphism is the identity; we also consider the notion of a weak equivalence, in which the induced automorphism comes from the action of $\mathbb{G}_m$.

One example of a filtered deformation is the filtered algebra associated to $B$ itself; in general, we say that a filtered deformation is trivial if it is equivalent to this deformation.

**Definition 2.2.** The moduli stack of filtered deformations of $B$ is the stack classifying nontrivial filtered deformations of $B$ up to weak equivalence.
Remark. There is an even weaker notion of equivalence one may consider, in which one allows all automorphisms of $B$ rather than just $\mathbb{G}_m$. This is not as well-behaved in families, however, and leads to more complicated moduli stacks (esp. in our examples below where the present definition gives a weighted projective space).

**Proposition 2.2.** If $B$ is a finitely presented graded algebra over a field $k$, then the moduli stack of filtered deformations of $B$ is algebraic.

**Proof.** Fix a finite (homogeneous) presentation of $B$, and choose a basis of $B$ consisting of monomials in the generators. Given a filtered deformation $A$ of $B$, since $\text{gr} A \cong B$, each generator $x_i$ of $B$ has a preimage inside $A_{\leq \text{deg}(x_i)}$, which is unique modulo $A_{\leq \text{deg}(x_i) - 1}$. Choose a lift $\hat{x}_i$ of each generator. This induces a basis of $A$ compatible with the filtration by replacing $x_i$ with $\hat{x}_i$ in the chosen monomial basis of $B$.

If we make a similar substitution in a relation $\rho_i$ of $B$, then the result will be an element of $A_{\leq \text{deg}(\rho_i) - 1}$; the degree $\text{deg}(\rho_i)$ term cancels since the relation holds in $\text{gr} A$. This element is then uniquely expressible in terms of our basis of $A$, and thus gives us a unique lift of each relation.

In the other direction, if we start with a collection of deformed relations (i.e., adding lower-order terms from the chosen basis to each relation from the presentation of $B$), the condition that the corresponding finitely presented algebra is a filtered deformation of $B$ is closed. To see this, note that it is equivalent to ask that the chosen monomials in the generators of the filtered algebra be linearly independent. It follows that a given collection of deformed relations gives a filtered deformation iff for any combination of deformed relations in which only the “good” monomials appear, the coefficients all vanish. But this is clearly a collection of polynomial equations, and thus the filtered deformation locus is an intersection of closed subschemes, so is closed.

This shows that the moduli stack of filtered deformations with chosen lifts of the generators is a scheme (an affine scheme, in fact). Two such deformations are equivalent iff one can be obtained from the other by changing the lifts of the generators. Each generator $\hat{x}_i$ can be changed by adding an element of $A_{\leq \text{deg}(\rho_i) - 1}$, which is in turn given by a polynomial in the generators of lower degree.

We thus see that if we order the generators $x_1, \ldots, x_n$ by degree, then after lifting $x_1, \ldots, x_{n-1}$, the set of possible lifts of $x_n$ is a torsor over a unipotent group scheme, so that forgetting the lift takes the quotient by that group, leaving the stack algebraic. We can then continue inductively, and thus find that the moduli stack of filtered deformations up to equivalence is algebraic.

Finally, we can remove the trivial deformations (which correspond to a single closed point of the stack) and quotient out by $\mathbb{G}_m$ to obtain the desired moduli stack, which is therefore still algebraic.

**Remark.** Note that in the above argument, if the degrees of the generators are $\leq$ the degrees of the relations, then the moduli stack is a quotient of a scheme by a group of the form $\mathbb{G}_m \ltimes U$ where $U$ is unipotent. The above induction is needed in general, however, since the coordinatization of the set of lifts of a generator depends on the relations of lower degree, which in turn depend on the lifts of generators of up to that degree.

One caution here is that this argument is highly nonconstructive; indeed, it seems likely that even the question of whether $B$ admits a nontrivial filtered deformation is undecidable in general, even if we are given a finite presentation of $B$. (The situation improves dramatically if we are also given generators of the module of syzygies.)

This moduli stack is nicest under the following condition.

**Proposition 2.3.** Suppose that the finitely presented graded algebra $B$ admits no homogeneous derivations of negative degree. Then the moduli stack of filtered deformations of $B$ is a closed substack in a weighted projective space.
Proof. For each positive integer \(d\), consider the effect of adding terms of degree \(\deg(x_i) - d\) to each lifted generator \(\hat{x}_i\). The resulting effect on the relations is trivial until one reaches degree \(\deg(\rho_i) - d\), where it induces a linear transformation from the space of deformations of generators to the space of deformations of relations. This linear transformation is independent of the choice of filtered deformation of \(B\), and may thus be computed using the trivial filtered deformation. In particular, we find that if we view the linear transformation as its own derivative, then the kernel is precisely the space of degree \(-d\) derivations of \(B\), and thus by hypothesis the linear transformation is injective.

We now choose for each positive integer \(d\) a complement to the image of the linear transformation in the space of degree \(-d\) deformations of relations. We then claim that every filtered deformation is uniquely equivalent to one in which the deformations of every degree lie in the chosen complement. This may be seen by induction on \(d\): if the deformations of relative degree \(> -d\) all lie in their respective subspaces, then (by injectivity and complementarity) there is a unique way to shift the generators by terms of relative degree \(-d\) to make the relations lie in the complementary subspace in degree \(-d\). Since this does not affect the terms of higher degree, the result follows.

It follows that the moduli stack of filtered deformations up to equivalence is a scheme, and is in fact \(\text{Spec}(M)\) for some (commutative) graded algebra \(M\) (with grading coming from the \(\mathbb{G}_m\) action). The desired moduli stack is then obtained by removing the origin (the unique point corresponding to the trivial deformation) and quotienting by \(\mathbb{G}_m\), and is thus closed in the weighted projective space corresponding to the homogeneous generators of \(M\).

Remark. Here we view weighted projective spaces as stacks; if all coordinates of degree prime to \(l\) vanish, then the result will be an orbifold point with stabilizer containing \(\mu_l\). We can recover the usual scheme versions by taking coarse moduli spaces. Note that in either interpretation, weighted projective spaces are proper, and thus so are their closed substacks/subschemes.

An important special case in which it is straightforward to control derivations is when \(B\) is a commutative homogeneous coordinate ring.

Lemma 2.4. Let \(X\) be an irreducible smooth projective scheme of positive dimension with ample line bundle \(\mathcal{L}\), with associated homogeneous coordinate ring \(B := \bigoplus_{i \geq 0} \Gamma(X; \mathcal{L}^i)\). For any integer \(k > 0\), there is a short exact sequence

\[
0 \to \Omega^k_X \to V_k \to \Omega^{k-1}_X \to 0
\]

(2.1)

such that there is an isomorphism

\[
\text{Der}^k(B) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}(V_k, \mathcal{L}^i)
\]

(2.2)

of graded vector spaces.

Proof. If we remove the cone point from \(\text{Spec}(B)\), the resulting scheme is a \(\mathbb{G}_m\)-torsor \(X^+\) over \(X\), so that an element of \(\text{Der}^k(B)\) thus determines an element of \(\text{Der}^k(\mathcal{O}_{X^+})\). This is, in fact, a \(\mathbb{G}_m\)-equivariant isomorphism

\[
\text{Der}^k(B) \cong \text{Der}^k(\mathcal{O}_{X^+}).
\]

(2.3)

Indeed, injectivity follows from the fact that \(X^+\) still contains the generic point of \(\text{Spec}(B)\), while surjectivity follows by observing that any holomorphic section of \(\mathcal{O}_{X^+}\) decomposes via the \(\mathbb{G}_m\) action as a sum of homogeneous elements, each of which induces a section of the corresponding power of \(\mathcal{L}\) and thus gives a homogeneous element of \(B\).
Since \( \pi : X^+ \to X \) is smooth of dimension 1, there is a short exact sequence
\[
0 \to \pi^* \Omega_X \to \Omega_{X^+} \to \Omega_{X^+}/X \to 0,
\]
with \( \Omega_{X^+}/X \) a line bundle. This line bundle is in fact canonically trivial, via the map
\[
\Omega_{X^+}/X \otimes \text{Lie}(G_m) \to \mathcal{O}_{X^+}
\]
coming from the \( G_m \)-torsor structure. We further find that the induced map
\[
\Gamma(X^+; \mathcal{O}_{X^+}) \to H^1(X^+; \pi^* \Omega_X) \cong H^1(X; \pi_* \pi^* \Omega_X) \cong H^1(X; \bigoplus_i \Omega_X \otimes L_i)
\]
is homogeneous of degree 0, and thus corresponds to an element of \( H^1(X; \Omega_X) \). In other words, there is a short exact sequence
\[
0 \to \Omega_X \to V_1 \to \mathcal{O}_X \to 0
\]
such that \( \Omega_{X^+} \cong \pi^* V_1 \). Taking exterior powers gives a similar exact sequence
\[
0 \to \Omega_X^k \to V_k \to \Omega_X^{k-1} \to 0
\]
for \( V_k := \wedge^k V_1 \). We then find
\[
\text{Der}^k(\mathcal{O}_{X^+}) \cong \Gamma(X^+; \mathcal{H}om(\pi^* V_k, \mathcal{O}_{X^+}))
\]
\[
\cong \Gamma(X; \pi_* \mathcal{H}om(\pi^* V_k, \pi^* \mathcal{O}_X))
\]
\[
\cong \Gamma(X; \mathcal{H}om(V_k, \pi_* \pi^* \mathcal{O}_X))
\]
\[
\cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}(V_k, L^i)
\]
as required. \( \square \)

Remark. Note that in this generality, \( B \) can fail to be finitely generated.

Corollary 2.5. Let \( C \) be a smooth projective curve of positive genus, \( \mathcal{L} \) an ample line bundle on \( C \), and \( B \) the corresponding homogeneous coordinate ring. Then any homogeneous element of \( \text{Der}^k(B) \) of negative degree vanishes.

Proof. For \( d > 0 \), the space of homogeneous elements of \( \text{Der}^k(B) \) of degree \(-d\) is
\[
\text{Hom}(V_k, \mathcal{L}^{-d})
\]
where \( V_k \) is the sheaf from the Lemma, so that for \( k = 1 \) there is an exact sequence
\[
0 \to \text{Hom}(\mathcal{O}_C, \mathcal{L}^{-d}) \to \text{Hom}(V_1, \mathcal{L}^{-d}) \to \text{Hom}(\Omega_C, \mathcal{L}^{-d}),
\]
with both spaces on either side of the one of interest vanishing by degree considerations. Vanishing for \( k = 2 \) follows similarly from \( V_2 \cong \Omega_C \), while for \( k > 2 \), \( V_k = 0 \). \( \square \)

Corollary 2.6. With \( B \) as above, the moduli stack of filtered deformations is a closed substack of a weighted projective space, and every filtered deformation of \( B \) is commutative.
Proof. The absence of derivations of negative degree gives the first claim. For the second claim, suppose $A$ were a noncommutative filtered deformation of $B$, and view the Rees algebra of $A$ as a (flat) $k[t]$-module. This gives a 1-parameter family of noncommutative algebras degenerating when $t = 0$ to the commutative algebra $B$, and thus induces a nonzero Poisson bracket on $B$. Since computing the Poisson bracket involves dividing commutators by an appropriate positive power of $t$, this Poisson bracket on $B$ is homogeneous of negative degree. Since a Poisson bracket is in particular a 2-derivation, it must therefore vanish, giving a contradiction. □

This makes it straightforward to compute the moduli space of filtered deformations of commutative elliptic algebras.

Example 2.1. Consider the case of an elliptic curve $E$, with ample line bundle $L = \mathcal{O}_E(\infty)$ of degree 1 coming from the point at infinity. The corresponding commutative graded algebra is generated by elements $z, x, y$ of degree 1, 2, 3 respectively, satisfying a single relation of degree 6:

$$y^2 + a_1xyz + a_3yz^3 = x^3 + a_2x^2z^2 + a_4x^4 + a_6z^6. \quad (2.12)$$

By the corollary, any filtered deformation of this algebra is commutative, and thus simply adds lower degree terms to the single relation; and since there is a single commutative relation, there are no further conditions to give a flat deformation. To get equivalence, we must quotient by changes of variable of the form

$$(y, x, z) \mapsto (y + b_1x + c_1z^2 + b_2z + b_3, x + d_1z + c_2, z + e_1). \quad (2.13)$$

This group acts faithfully, and thus to obtain the structure of the moduli stack, we need simply remove four generators of degree 1, two of degree 2, and one of degree 3. We thus see that in this case, the moduli stack is the weighted projective space $\mathbb{P}[12234456]$ (i.e., with generators of degrees 1, 2, 2, 3, 3, 4, 4, 5, 6). (These are the standard labels of the affine Dynkin diagram of type $E_8$, as explained below).

A similar calculation works if we instead take $L$ to have degree 2, 3, or 4; in the first two cases, the cone is a hypersurface, while for 4, it is a complete intersection, and thus in no case are there any syzygies. The result in each case is thus again a weighted projective space ($\mathbb{P}[11222334]$ ($E_7$); $\mathbb{P}[1111222]$ ($E_6$); and $\mathbb{P}[1111112]$ ($D_5$) respectively).

For higher degrees, there are syzygies, so some actual calculation is required, but one can still easily compute the corresponding moduli space since there is no need to consider noncommutative deformations. One finds that the space is $\mathbb{P}^4$ for degree 5 and $\mathbb{P}^1 \times \mathbb{P}^2$ for degree 6. For degrees 7, 8, 9, the structure is somewhat more complicated (but see below), while for degree $\geq 9$, there are no nontrivial deformations [18, §9]. For degree $\geq 3$, it was shown op. cit. that any nontrivial deformation is rational and anticanonical (albeit possibly singular), i.e., a del Pezzo surface of the given degree. For degrees 3 and 4, this can also be seen by comparing the moduli stack as computed above to the substack coming from del Pezzo surfaces, and the same argument establishes the corresponding claim for degrees 1 and 2. (This can also be seen more directly: that the surface is Fano follows by adjunction from the fact that the curve at infinity has genus 1; that it is rational reduces (via successive blowups of points at infinity) to the degree 1 case, where it follows from the theory of elliptic surfaces.)

Note that the hypotheses above are quite strong, and in particular exclude the cases in which $E$ becomes singular. These are, of course, still quite interesting, but harder to deal with geometrically. (By contrast, it is often quite straightforward to write down explicit presentations in those cases!) As an example of what can go wrong, one has the following family in which the commutative
deformation fails to be a del Pezzo surface in the usual sense (and one obtains deformations for \( \mathcal{L} \) of arbitrarily large degree).

**Example 2.2.** For \( d \geq 3 \), let \( Z \) be the weighted projective space with generators \( x, y, z, w \) of degrees \( d-2, d-2, 1, 1 \) respectively, and let \( X_{ab} \) be the family of hypersurfaces of degree \( d \) with equation

\[
zwx + (az^2 + bw^2)y = z^d + w^d.
\]

(2.14)

The curve \( y = 0 \) in \( X_{ab} \) is isomorphic to the standard nodal cubic, and its image under the embedding corresponding to the invertible sheaf \( O_Z(d-2) \) is a degree \( d \) model in \( \mathbb{P}^{d-1} \); to be precise, it is the image of \( \mathbb{P}^1 \) under the map \((z^d + w^d, zw^{d-1}, \ldots, z^{d-1}w)\) that identifies the points \((0,1)\) and \((1,0)\). In other words, the \((d-2)\)nd Veronese of the homogeneous coordinate ring of \( X_{ab} \) is the Rees algebra of a filtered deformation of that nodal cubic. It is clear that this filtered deformation is nontrivial unless \( a = b = 0 \), and thus gives us a \( \mathbb{P}^1 \) of such deformations for all \( d \). When \( d = 10 \), one can verify by a direct Magma computation that the reduced subscheme of the full deformation scheme is \( \mathbb{P}^1 \), and thus all deformations arise in this way. The presence of such deformations arises from the fact that the cone \( X_{00} \) is not normal, and thus the filtered deformations themselves need not be normal (and, indeed, are not: \( X_{ab} \) is singular along the curve \( z = w = 0 \)). These are examples of the non-normal del Pezzo surfaces studied in \([22]\).

Now, with \( E \) still an elliptic curve and \( \mathcal{L} \) an ample line bundle, we can also obtain noncommutative graded algebras by taking

\[
B = \bigoplus_i \text{Hom}(V, V \otimes \mathcal{L}^i)
\]

(2.15)

for \( V \) a vector bundle on \( E \). For simplicity, we assume that \( B \) is nonnegatively graded; we will see below that this is not truly a constraint. This algebra is finite over its center, the usual homogeneous coordinate ring, and is in fact the ring of global sections of a sheaf of (Azumaya) algebras \( B := \text{End}(\pi^*V) \) on the \( \mathbb{G}_m \)-torsor \( E^+ \) associated to \( \mathcal{L} \).

For any associative algebra \( A \), there is a natural map \( \text{Der}(A) \to \text{Der}(Z(A)) \) coming from the fact that any automorphism (and thus any infinitesimal automorphism) preserves the center.

**Lemma 2.7.** A derivation of \( B \) that is trivial on \( Z(B) \) is inner.

**Proof.** A derivation of \( B \) which is trivial over \( Z(B) \) may be viewed as an \( \mathcal{O}_{\text{Spec}(Z(B))} \)-linear derivation on the corresponding sheaf of algebras, and thus induces an \( \mathcal{O}_{E^+} \)-linear derivation on \( B \) after removing the cone point, or equivalently a holomorphic family of derivations on the fibers of \( B \). Since those fibers are matrix algebras over fields, their derivations are necessarily inner. In particular, we may locally represent the given derivation as the commutator with a traceless matrix, and since this representation is canonical, it extends to a global such representation. As in the proof of Lemma \([2.4]\), a global section of \( B \) is an element of \( B \), and thus the original derivation of \( B \) is inner.

\( \square \)

**Remark.** This argument fails in finite characteristic, since it uses the fact that the map \( \text{Mat}_n \to \text{Inn Mat}_n \) is split by the trace, which fails when \( n \) is a multiple of the characteristic. Of course, any global splitting will do, so it in particular suffices to have some idempotent in \( \text{End}(V) \) with invertible trace, or equivalently for \( V \) to have a summand with invertible rank. (Though even this appears to be stronger than needed for the conclusion to hold.)

**Remark.** An equivalent statement is that the natural map \( \text{HH}^1(B) \to \text{HH}^1(Z(B)) \) is injective.
Corollary 2.8. The algebra $B$ has no homogeneous derivations of negative degree.

Proof. We have already seen that $Z(B)$ has no such derivations, and thus such a derivation of $B$ is trivial on $Z(B)$, so inner, and thus itself vanishes since $B$ is nonnegatively graded. \hfill $\Box$

Remark. Note that $B$ is Noetherian since $B$ is Noetherian, and is in particular finitely presented. We may thus conclude that the moduli stack of filtered deformations of $B$ is a closed substack of a weighted projective space.

The fact that filtered deformations remain commutative when $V = \mathcal{O}_E$ also has an analogue: the center of any filtered deformation of $B$ is a filtered deformation of $Z(B)$. To see this, we use ideas of Hayashi [11] (see also [12, §1] for an explicit discussion), in somewhat more formal terms. We again view the Rees algebra $A^+$ as a flat $k[t]$-algebra. Call an element $a \in A^+$ of this algebra $m$-central if $[a, A^+] \subset t^mA^+$, or equivalently if its image in $A^+/t^mA^+$ is central; in particular, an element is 1-central iff its image in $B$ is central. Now, given an $m$-central element $a$, we may define a derivation $D_{a,m}(x) := t^{-m}[a, x]$ on $A^+$. Since this is $k[t]$-linear, the image of $D_{a,m}(x)$ mod $t$ depends only on the image of $x$ mod $t$, so that $D_{a,m}$ induces a derivation of $B$. We also find
\[ D_{a+t^mb,m}(x) = D_{a,m}(x) + [b, x], \] (2.16)
and thus see that there is an $m+1$-central element of the form $a + t^mb$ iff $D_{a,m}$ is an inner derivation. In other words, the obstruction to lifting an element of $Z(A^+/t^mA^+)$ to an element of $Z(A^+/t^{m+1}A^+)$ is an element of $\text{Der}(B)/\text{Inn}(B) = \text{HH}^1(B)$. Note that since the product of $m$-central elements is $m$-central and $D_{ab,m}(x) = aD_{b,m}(x) + D_{a,m}(x)b$, the obstruction map
\[ Z(A^+/t^mA^+) \to \text{HH}^1(B) \] (2.17)
is itself a derivation.

Now, let $m > 0$ be an integer, and suppose that every central element of $A^+/t^{m-1}A^+$ lifts to an $m$-central element of $A$ (this is vacuously true for $m = 1$). If $a_1$, $a_2$ are $m$-central elements that reduce to the same element of $B$, then
\[ D_{a_1,m}(x) - D_{a_2,m}(x) = t^{-m}[a_1 - a_2, x] = t^{1-m}[t^{-1}(a_1 - a_2), x] = D_{t^{-1}(a_1-a_2),m-1}(x). \] (2.18)
Now, $t^{-1}(a_1-a_2)$ is $(m-1)$-central, and thus agrees modulo $t^{m-1}$ with an $m$-central element, so that $D_{t^{-1}(a_1-a_2),m-1}(x)$ is inner, implying that
\[ D_{a_1,m}(x) - D_{a_2,m}(x) \] (2.19)
is inner. We thus find that the obstruction map factors through a derivation
\[ Z(B) \to \text{HH}^1(B), \] (2.20)
which since it divides by $t^m$ is homogeneous of degree $-m$. (As pointed out in [10] Ex. 2.17(4), this map appears in a spectral sequence for computing the Hochschild cohomology of the filtered deformation.)

For any $a \in Z(B)$, the induced outer derivation of $B$ induces a derivation of $Z(B)$, giving a 2-derivation of $Z(B)$ of negative degree, which vanishes. Since a derivation of $B$ vanishing on $Z(B)$ is inner, we conclude that the obstruction map for degree $m$ vanishes. We thus find the following.

Proposition 2.9. Let $E$ be an elliptic curve, $\mathcal{L}$ an ample line bundle on $E$, and $V$ a vector bundle such that $\text{Hom}(V, V \otimes \mathcal{L}^{-1}) = 0$. The center of any filtered deformation $A$ of $\bigoplus_i \text{Hom}(V, V \otimes \mathcal{L}^i)$ is a filtered deformation of $\bigoplus_i \Gamma(\mathcal{L}^i)$. 

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Proof. Let $\bar{x} \in Z(B)$ be a homogeneous element, and consider the sequence of $m$-central lifts $x_1, x_2, x_3, \ldots, x_m, \ldots$ guaranteed by the above discussion. Then $x_{i+1} - x_i \in t^i A^+$, so that the image of $x_{i+1} - x_i$ in $A$ (i.e., setting $t = 1$) has degree $\leq \deg(\bar{x}) - i$. In particular, we find that any sequence of $m$-central lifts is eventually constant, and thus gives a lift of $\bar{x}$ which is $m$-central for all $m$, thus central. This shows that $\text{gr} Z(A) \supset Z(B)$, while the other inclusion is trivial, so that the claim follows.

As with the commutative case, one can use these results to enable explicit computations of the moduli stacks. One issue is that it is more difficult to write down an explicit presentation of $E$ at once. A useful trick for this purpose involves the observation that $B$ itself has regular central elements of degree 1, so may be viewed as a filtered deformation, and the resulting algebra turns out to be independent of $E$. To be precise, if $w \in Z(B)$ is a degree 1 element corresponding to $d = \deg(L)$ distinct points of $E$, i.e., corresponding to an isomorphism $L \cong \mathcal{L}(D)$ for a reduced divisor $D$, then the quotient $C := B/\langle w \rangle$ has Hilbert series of the form $\dim(B_0) + (n - \dim(B_0))t + nt^2 + nt^3 + \cdots$ where $n = d \text{rank} \mathcal{E}nd(V)$. If $\tilde{C}$ denotes the algebra $\Gamma(D; \mathcal{E}nd(V))$, then every homogeneous component of $C$ may be identified with a subspace of $\tilde{C}$, and thus in particular dimension considerations tell us that $C$ agrees with $\tilde{C}$ in all degrees $> 1$. (Note that the identification with $\tilde{C}$ depends on a choice of trivialization of $\mathcal{L}(D)|_D$, or in other words on a system of uniformizers at the points of $D$. By adjunction, such a system is determined by a choice of holomorphic differential $\omega$ on $E$, so there is an essentially canonical choice of identification.) In degree 0, we of course have $C_0 = B_0 = \text{End}(V) \subset \Gamma(D; \mathcal{E}nd(V))$, so it remains only to determine $C_1$. There is a natural pairing $C_0 \times C_1 \rightarrow \Gamma(\mathcal{L}(D))$ given by $(x, y) \mapsto \text{Tr}(xy)$, which agrees with the $\Gamma(\mathcal{O}_D)$-valued trace pairing on $\tilde{C}$ if we take residues (with the same choice of holomorphic differential). Since the value is a global function, the sum of residues vanishes, and thus any element of $C_1$ must be orthogonal to $C_0$ under the sum-of-traces pairing on $\tilde{C}$. This cuts out a space of the correct dimension, so completely determines $C_1$.

In other words, $C$ is determined by the algebra $\tilde{C}$ and its subalgebra $B_0 \subset \tilde{C}$:

$$C_n = \begin{cases} B_0 & n = 0 \\ \{x : x \in \tilde{C} \mid \text{Tr}(xB_0) = 0\} & n = 1 \\ \tilde{C} & n > 1 \end{cases} \quad (2.21)$$

Here $\tilde{C}$ is (geometrically) just a sum of $d$ copies of $\text{Mat}_{\dim(V)}$, while $B_0 = \text{End}(V)$ depends only on the gross structure of $V$ (the slopes and multiplicities of the indecomposable summands, along with which summands have isomorphic stable constituents). It is then straightforward to determine a presentation of $C$, either by using the results for $B$ below or by using the fact that $C$ has regular central elements (and in particular has a canonical regular central element of degree 2, namely $1 \in \tilde{C} = C_2$), and the quotient by such an element is finite-dimensional.

Example 2.3. It is interesting to consider the commutative case from this perspective. Here the algebra $\tilde{C}$ is just $k^d$, while $B_0$ is the diagonal copy of $k$, so that for $d > 2$, $C$ is the embedded coordinate ring of $d$ points in general position in $\mathbb{P}^{d-2}$. (For $d = 1$, it is $k[y, x]/(y^2 - x^3)$ with $\deg(x) = 2$, $\deg(y) = 3$, and for $d = 2$, it is $k[y, x]/(y^2 - x^2 y)$ with $\deg(x) = 1$, $\deg(y) = 2$. The commutative filtered deformations of such algebras were studied in [15], and in particular shown to always give a (possibly degenerate) genus 1 curve on which the marked degree $d$ divisor is ample. Moreover, the structure of the moduli stack was determined in low degree, giving the weighted projective spaces $\mathbb{P}^{[46]}, \mathbb{P}^{[234]}, \mathbb{P}^{[1223]}, \mathbb{P}^{[11122]}$ in degrees 1, 2, 3, 4, $\mathbb{P}^4$ in degree 5, and the Grassmannian $\text{Gr}(2, 5)$ in degree 6. Not only does this let us write down general versions of $B$ in low degree, but it in fact lets us compute the moduli stack of deformations of $B$: a filtered deformation of $B$ is
associated to B.

d
d
d

of

equipped with an absolute value.

even rank, breaking the proof of Lemma 2.7, and because the exterior algebra over


Example 2.4. Consider the case that V is a stable bundle of rank 2 and degree 1 and the ample bundle is $\mathcal{L}(\infty)$. Then B has Hilbert series $1+4t/(1-t)^2$, while $Z(B)$ has Hilbert series $1+t/(1-t)^2$.

In this case, we get a particularly nice algebra $D$ if we quotient $C$ by the regular central element of degree 2: $D$ is simply the exterior algebra in three generators! The moduli stack of filtered deformations of that exterior algebra is then straightforward to compute, and one finds that the space of filtered deformations is the weighted projective space $\mathbb{P}^{[2222]}$: any filtered deformation of the exterior algebra in three generators is equivalent to a Clifford algebra. (There is a $\mathbb{P}^5$ worth of Clifford algebras, but Clifford algebras have an additional automorphism acting as $-1 \in \mathbb{G}_m$ on the associated graded.) Since a filtered deformation $A$ of $B$ has center a filtered deformation of $Z(B)$, the quotient of the Rees algebra $A^+$ by its two degree 1 elements and one degree 2 element is again the exterior algebra, and thus we may view the filtered deformation of $B$ as a family of filtered deformations of the exterior algebra. We thus see that it is given by a map from the weighted projective space $\mathbb{P}^{[112]}$ to the moduli stack of Clifford algebras, or more precisely by a collection of six homogeneous polynomials of degree 2 in those generators. Of course, the specializations of those polynomials to $t = 0$ are determined by $B$ (that is, together with the specific choices of degree 1 and 2 elements), and thus there are only two free parameters for each of the polynomials, one of degree 1 and one of degree 2. Of those, one each is eliminated by translations of the degree 2 central element of $B$, and one more of degree 1 is eliminated by the translations of the degree 1 central element. We thus conclude that for any elliptic curve and semistable bundle of rank 2, degree 1, the moduli stack of filtered deformations of the corresponding graded algebra is the weighted projective space $\mathbb{P}^{[11112222]}$ ($D_8$). (One caveat: the above calculation assumes 2 is invertible, both because the bundle has even rank, breaking the proof of Lemma 2.7, and because the exterior algebra over $\mathbb{F}_2$ has derivations of negative degree. One can, however, verify by a more complicated calculation that the claims remain true in characteristic 2.)

Remark. For $V$ stable of slope 1/2 and $\mathcal{L}$ of degree $d$, the algebra $C$ is equal to $\text{Mat}_2^d$ in degree $\geq 1$, and the subspace of $d$-tuples of total trace 0 in degree 1, and in particular its center is the algebra considered in Example 2.3. There is an action of $\text{GL}_2^d$ on the corresponding deformation space (of filtered deformations respecting the center), and we find (computationally) the following descriptions after taking that into account:

\begin{align*}
  d &= 1 : S^4(V_1)/((\mathbb{G}_m/\langle \pm 1 \rangle)) \quad (2.22) \\
  d &= 2 : (S^2(V_1) \otimes S^2(V_2))/\mathbb{G}_m \quad (2.23) \\
  d &= 3 : \text{Gr}(2, V_1 \otimes V_2 \otimes V_3), \quad (2.24)
\end{align*}

where $V_1, \ldots$ are the fundamental representations of the respective copies of $\text{GL}_2$. In each case,
In particular, that the universal deformation of this algebra has the form the same way with the idempotents.) An explicit calculation with syzygies of low degree tells us

\[ \text{Remark.} \]

Just as \( A \) may be viewed as a line of filtered deformations of \( C \), we may obtain filtered deformations of \( A^+ \) as either a line of filtered deformations of \( B \) or a plane of filtered deformations of \( C \). (This imposes the additional condition that \( Z(A^+) \) lifts, which is not necessarily automatic.) This at the very least gives rise to a family of (algebras on) Fano 3-folds (embedded in such a way that \(-K_X = O(2)\)), and we could hope to obtain noncommutative Fano 3-folds by replacing \( B \) by a twisted elliptic algebra. (Indeed, the Sklyanin algebra \([23]\) may be obtained by a variation on this idea, see below.) This can, of course, be iterated as long as enough degrees of freedom remain to allow the deformations to be nontrivial. When the moduli stack of deformations of \( B \) is a weighted

\[
\begin{align*}
\text{Example 2.5.} \quad & \text{Similarly, if } V \text{ is a sum of two nonisomorphic degree 0 line bundles, then } C \text{ is generated (over } B_0 \cong k^2 \text{) by elements } F \text{ of degree 1 and } G \text{ of degree 2 interacting with the two idempotents of } B_0 \text{ as } e_1 F = F e_2, \ e_1 G = G e_2, \text{ and satisfying relations } FG = GF, \ G^2 = F^4. \text{ (These both correspond to the matrix } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \tilde{\mathcal{C}}. \text{) Any filtered deformation is equivalent to one in which the relations involving the idempotents remain unchanged, and two such deformations are equivalent iff they are related by a shift } G \mapsto G + b_1 F. \text{ (We can only add lower terms that interact the same way with the idempotents.) An explicit calculation with syzygies of low degree tells us that the universal deformation of this algebra has the form}

\[ FG - GF = a_3(e_1 - e_2), \tag{2.28} \]

\[ G^2 = F^4 + a_2 F^2 + a_4. \tag{2.29} \]

In particular, \( B \) corresponds to a point of the weighted projective space \( \mathbb{P}^{[234]} \), and a filtered deformation of \( B \) extends that point to a map from \( \mathbb{P}^1 \), from which it follows easily that the moduli stack of filtered deformations of \( B \) is a weighted projective space with degrees \( \mathbb{P}^{[1122334]} \) (again \( E_7 \)). (This calculation again assumes 2 is invertible, but can be extended to work over \( \mathbb{Z} \) by waiting until the last step to kill the unipotent symmetry.)

\[ \text{Remark.} \]

Just as \( A \) may be viewed as a line of filtered deformations of \( C \), we may obtain filtered deformations of \( A^+ \) as either a line of filtered deformations of \( B \) or a plane of filtered deformations of \( C \). (This imposes the additional condition that \( Z(A^+) \) lifts, which is not necessarily automatic.) This at the very least gives rise to a family of (algebras on) Fano 3-folds (embedded in such a way that \(-K_X = O(2)\)), and we could hope to obtain noncommutative Fano 3-folds by replacing \( B \) by a twisted elliptic algebra. (Indeed, the Sklyanin algebra \([23]\) may be obtained by a variation on this idea, see below.) This can, of course, be iterated as long as enough degrees of freedom remain to allow the deformations to be nontrivial.
projective space, a particularly interesting variation is the *universal* deformation, in which one adjoins every generator of the weighted projective space to $B$. In the commutative case, the resulting scheme is independent of $E$, and is isomorphic to $[1][122][13][44][5], [1][11][12][22][23], [1][11][11][12][22], [7], \text{Gr}(2, 5)$ for $1 \leq d \leq 5$, so that we obtain noncommutative deformations of those spaces by twisting $B$. (It would be interesting to see if the other cases with nice moduli stacks give rise to recognizable algebras in the untwisted case.)

The claims about the center can be further generalized. Consider an elliptic algebra of the form $B_{V, \Psi} := \bigoplus_i \text{Hom}(V, \Psi^i V)$, where $\Psi$ is an autoequivalence of $\text{coh}(E)$ of the form $M \mapsto \mathcal{L} \otimes (\phi^{-1})^* M$ with $\deg(\mathcal{L}) > 0$. If $\phi$ has finite order, so that $\Psi^d$ is twisting by a line bundle, then $B$ has a large center, namely the homogeneous coordinate ring of the quotient curve $E/\langle \phi \rangle$ relative to the norm of $\mathcal{L}$, with degrees multiplied by the order of $\phi$. When $\phi$ is a translation by a $d$-torsion point $q$, $E' := E/\langle \phi \rangle$ is again an elliptic curve, and thus its homogeneous coordinate ring has no derivations of negative degree. (Note that this could fail when $\phi$ is not a translation, since then $E'$ has genus 0. It would be interesting to understand the filtered deformations in such cases, but they fall outside the scope of the present work.)

Let $E'^+$ be the $\mathbb{G}_m$-torsor associated to the norm line bundle $N_{E/E'}(\mathcal{L})$ on $E'$, modified by pulling back the action of $\mathbb{G}_m$ through the $d$-th power map. (This is to correct for the fact that the degrees in $Z(B)$ are all multiples of $d$, and in particular the degree $dm$ elements are sections of $N_{E/E'}(\mathcal{L})^m$.) Then $B$ is the ring of global sections of a corresponding sheaf of algebras $\mathcal{B}$ on $E'^+$. This sheaf is $\mathbb{G}_m$-equivariant in a somewhat complicated way, but the (geometric) fibers are again just matrix algebras. Indeed, the $\mu_d$-invariant subalgebra of $\mathcal{B}$ is the direct image of the endomorphism ring of a vector bundle on a $\mathbb{G}_m$-torsor over $E$, so is locally a matrix algebra over a cyclic degree $d$ cover of $E'^+$. Relative to this interpretation, the other congruence classes correspond to spaces of linear transformations which are semilinear w.r.t. the appropriate power of $\phi$. We thus find that $\mathcal{B}$ is locally a matrix algebra over a *cyclic* algebra, and is thus an Azumaya algebra.

It is then straightforward to extend the above arguments to show that $B$ has no negative derivations. We can actually extend this to the case that $\tau$ has infinite order, but this will require us to control derivations in finite characteristic, where the argument of Lemma 2.7 can fail. Luckily, if we modify the conclusion somewhat, we obtain a result that holds in general.

**Lemma 2.10.** Let $B$ be an elliptic algebra (possibly in finite characteristic), twisted by a torsion point of exact order $d$. Then any derivation of $B$ of degree not a multiple of $d$ is inner.

**Proof.** Let $D$ be a derivation of $B$ of degree $\delta$, with $\delta$ not a multiple of $d$. Since every element of $Z(B)$ has degree a multiple of $d$, it follows immediately that $D$ is trivial on $Z(B)$. We may then argue as in the proof of Lemma 2.7 that $D$ is locally inner; in other words, there is a collection of representations $D = \text{ad}(z_i^{-1} x_i)$ with $z_i \in Z(B)$, $x_i \in B$, such that the open subsets $U_i := [z_i \neq 0]$ cover $E'$.

Now, since $\text{ad}(z_i^{-1} x_i) = \text{ad}(z_j^{-1} x_j)$, the system of pairs must also satisfy $z_j x_i - z_i x_j \in Z(B)$ for all $i, j$. Since $\deg(z_i^{-1} x_i) = \delta$, we see that

$$\deg(z_j x_i - z_i x_j) \equiv \delta \pmod{d}, \quad (2.30)$$

which (again using the fact that elements $Z(B)$ have degree a multiple of $d$) forces $z_j x_i = z_i x_j$. But this implies that the local representations glue to give a global representation, so that $D$ is inner as required.

**Lemma 2.11.** Let $B$ an elliptic algebra twisted by a translation $\tau$. If $B$ has a nonzero derivation of degree $-d$ with $d > 0$, then $\tau^d = 1$. 

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Proof. Suppose otherwise, so that \( B \) has such a derivation, but \( \tau^d \neq 1 \). If the field of definition \( K \) is infinitely generated, then we can replace it by the field generated by (a) the coefficients of some Weierstrass equation for \( E \), (b) the coordinates of the point by which \( \tau \) is a translation, and (c) the coordinates of the points corresponding to the determinants of stable constituents of \( V \), so that we may assume \( K \) is finitely generated. (See [4] or the next section for a discussion of the structure of the space of derivations of degree \( -d \).)

If \( K \) is finite, then \( \tau \) has finite order, with \( d \) not a multiple of \( \text{ord}(\tau) \), and thus Lemma [2.10] gives a contradiction. Otherwise, let \( \nu \) be a valuation of \( K \), with associated discrete valuation ring \( R \) having residue field \( k \) such that (a) \( E_k \) is smooth, (b) \( \tau^d_k \neq 1 \), and (c) the nonisomorphic stable constituents of \( V \) have nonisomorphic reductions. (Such a place exists because we are imposing finitely many open conditions.) Then \( B_{V_R,\Psi_R} \) is a flat extension of \( B_{V,\Psi} \), so that the dimension of the space of derivations of degree \( -d \) is upper semicontinuous, implying that \( B_k \) itself has a nonzero derivation of degree \( -d \). Since \( \dim(k) = \dim(K) - 1 \), the claim follows by induction on the dimension.

Remark. A similar reduction shows that any elliptic algebra is Noetherian, by a direct generalization of the proof of [1, Thm. 8.3].

This is enough to establish the following for an arbitrary elliptic algebra in characteristic 0.

**Theorem 2.12.** Let \( B \) be an elliptic algebra over a field of characteristic 0, twisted by an arbitrary translation. Then the moduli space of filtered deformations of \( B \) is a closed substack of a weighted projective space.

**Proof.** If \( \tau \) has finite order, then the arguments of Lemma [2.7] and Corollary [2.8] carry over mutatis mutandis to show that \( B \) has no negative derivations. Otherwise, \( \text{ord}(\tau) = \infty \), and we may simply replace Lemma [2.7] by Lemma [2.11].

The argument of Proposition [2.9] also carries over.

**Theorem 2.13.** Let \( B \) be an elliptic algebra of characteristic 0 twisted by translation by a torsion point. The center of any filtered deformation of \( B \) is a filtered deformation of \( Z(B) \).

### 3 Resolutions of elliptic algebras

Let \( E/k \) be an elliptic curve over an algebraically closed field \( k \), let \( V \) be a vector bundle on \( E \), and let \( \Psi : \text{coh}(E) \to \text{coh}(E) \) be an autoequivalence of the form \( M \mapsto \mathcal{L} \otimes (\tau^{-1})^* M \) with \( \tau \in E \subset \text{Aut}(E) \) and \( \mathcal{L} \) an ample line bundle of degree \( d > 0 \). (In fact, most of the arguments below work even if \( \tau \) is not a translation, since they only depend on how \( \Psi \) affects slopes; the only exception is in the explicit calculations of minimal resolutions, where minimality also depends on how it affects determinants.) As mentioned above, these ingredients determine a graded algebra \( B_{V,\Psi} \) such that for \( n \in \mathbb{Z} \),

\[
(B_{V,\Psi})_n = \text{Hom}(V, \Psi^n V),
\]

with the obvious composition. Since \( \mathcal{L} \) is ample, this algebra is bounded below, and we are interested in its filtered deformations.

Any summand of \( V \) determines an idempotent of \( B_{V,\Psi} \), which lifts (nonuniquely) to any filtered deformation by Proposition [2.11]. It follows immediately that applying a power of \( \Psi \) to a summand of \( V \) does not change the space of filtered deformations—it simply shifts the degrees of the corresponding off-diagonal blocks. Since any vector bundle on an elliptic curve is a sum of semistable
bundls (see [3], which also completely determines the structure of indecomposable bundles) and \( \Psi \) adds \( d \) to the slope of a bundle, this freedom allows us to assume that every summand of \( V \) has slope in \([0, d]\) (or, more generally, in any desired half-open interval of length \( d \)). Since a nonzero morphism between semistable bundles forces the slope of the codomain to be at least the slope of the domain, this constraint on \( V \) implies that \((B_{V, \Psi})_n = 0\) for \( n < 0 \).

This, of course, is not quite canonical; if \( V \) has \( c \) different slopes (modulo \( d \)), then there are \( c \) inequivalent choices of nonnegative gradings on \( B_{V, \Psi} \). The most natural way to resolve this involves replacing \( B_{V, \Psi} \) by a category. For each slope \( \mu \in \mathbb{Q} \), let \( V[\mu] \) denote the direct sum of all slope \( \mu \) indecomposable summands of the modules \( \Psi^n V \), so that the construction of the previous paragraph involves replacing \( V \) by \( \bigoplus_{\mu \in [\alpha, \alpha + d]} V[\mu] \). The slopes for which \( V[\mu] \neq 0 \) form a discrete subset of \( \mathbb{Q} \), a union of finitely many cosets of \( d\mathbb{Z} \), and thus we may choose an ordered bijection between \( \mathbb{Z} \) and the set of such slopes. If \( i \mapsto \mu_i \) is that bijection, then we may define \( V_i := V[\mu_i] \). We then let \( B_{V, \Psi} \) be the subcategory of \( \mathcal{C} \) determined by \( V_i \); i.e., it is the category with objects indexed by integers and Hom spaces given by

\[
B_{V, \Psi}(i, j) = \text{Hom}(V_i, V_j).
\]

(This is a nonnegatively graded \( \mathbb{Z} \)-algebra in the sense of [3].) It is then straightforward to see that the category of modules over this category (i.e., contravariant functors to the category of vector spaces) is isomorphic to the category of graded \( B_{V, \Psi} \)-modules. Note that since \( \Psi \) induces an autoequivalence shifting the objects by \( c \), this category induces a graded algebra which in degree \( d \) is

\[
\bigoplus_{1 \leq i, j \leq c} B_{V, \Psi}(i, j + d).
\]

This graded algebra agrees with the naïve grading on \( B_{V, \Psi} \), up to rescaling the grading by \( c \) and shifting the grading on off-diagonal blocks relative to the natural idempotents, and thus gives the same space of filtered deformations. (The only caveat is that the additional terms added to the relations must remain in the same congruence class modulo \( c \), but this is actually forced by consistency with the degree 0 involutions.) With this in mind, where \( B_{V, \Psi} \) appears below, we will always take this canonical grading.

The theory of minimal resolutions, though most familiar in the case of connected graded algebras, applies more generally (following [3]) to nonnegatively graded algebras which are finite-dimensional in every degree. In particular, any \( B_{V, \Psi} \)-module has such a minimal resolution, and thus the same applies to \( B_{V, \Psi} \). The one caveat here is that when \( B_0 \) is not semisimple, the minimal resolution of a finitely graded module need not be convergent; that is, there may be degrees such that the restriction of the complex to those degrees gives an infinite resolution.

**Lemma 3.1.** Let \( B \) be a nonnegatively graded \( k \)-algebra such that \( B_0 \) is finite-dimensional. Then the minimal resolution of \( B_0 \) as a graded \( B \)-module is convergent iff each \( B_n \) has finite homological dimension as a \( B_0 \)-module. Moreover, if this holds, then for any bounded below graded \( B \)-module \( M \), \( M \) has convergent minimal resolution iff each \( M_n \) has finite homological dimension.

**Proof.** First suppose that \( B_n \) and \( M_n \) all have finite homological dimension, and WLOG assume that \( M_n = 0 \) for \( n < 0 \) and \( M_0 \neq 0 \). Then the restriction of the minimal resolution of \( M \) to degree 0 gives a (minimal!) projective resolution of \( M_0 \) as a \( B_0 \)-module, which thus has finite length. In particular, projective modules of degree 0 appear in only finitely many terms, and thus we may consider the partial resolution truncated at the last point where they occur. In each degree, the terms of the resulting complex have finite homological dimension (being homogeneous components of \( B \) or \( M \)) and thus, since the complex is exact except at the very left, the kernel also has finite
homological dimension. This gives us a new graded module $M'$ satisfying the same hypotheses but with a larger lower bound on its degrees, and thus the result follows by induction. This gives the “if” part of the second claim, and thus (taking $M = B_0$) of the first claim as well.

Now, supposing still that the components $B_n$ all have finite homological dimension, let $M$ be a bounded below module with at least one homogeneous component of infinite homological dimension. Then the same argument of the previous paragraph allows us to reduce to the case that $M_n = 0$ for $n < 0$ but $M_0$ has infinite homological dimension. (Here we use the fact that in a partial resolution by modules of finite homological dimension, the kernel has finite dimension iff the cokernel does.) But then the minimal resolution in degree 0 is a projective $B_0$-module resolution of $M_0$, which is infinite by assumption, preventing convergence.

Finally, if $B$ has a homogeneous component of infinite homological dimension, let $n$ be the minimal such degree. Then the degree $n$ part of the minimal resolution of $B_0$ is a resolution of $B_n$ by sums of summands of components $B_i$ for $i < n$, and thus of terms having finite homological dimension. Since $B_n$ has infinite homological dimension, this forces the resolution to have infinite length.

This property holds for the canonical grading on $B_V$, and in fact we have something stronger.

**Lemma 3.2.** Suppose that $V$, $W$ are indecomposable bundles with $\mu(V) < \mu(W)$. Then $\text{Hom}(V, W)$ is free as an $\text{End}(V) \otimes_k \text{End}(W)^\text{op}$-module.

**Proof.** We have $\text{End}(V) \cong k[x]/(x^n)$ and $\text{End}(W) \cong k[y]/(y^n)$ for suitable positive integers $m$ and $n$, and thus we need to show that $\text{Hom}(V, W)$ is a free $k[x, y]/(x^n, y^m)$-module, or equivalently that

$$\dim_k \text{Hom}(V, W) = mn \dim_k (\text{Hom}_{\text{End}(V)}(k, \text{Hom}(V, W) \otimes_{\text{End}(W)} k)), \quad (3.4)$$

or in other words

$$\dim_k \text{Hom}(V, W) = mn \dim_k (\text{Hom}(V \otimes_{\text{End}(V)} k, W \otimes_{\text{End}(W)} k)). \quad (3.5)$$

Since $V \otimes_{\text{End}(V)} k$ is the stable constituent of $V$, and similarly for $W \otimes_{\text{End}(W)} k$, this certainly applies at the level of Euler characteristics, and the difference in slopes forces $\text{Ext}^1$ to vanish (by duality). 

**Lemma 3.3.** Suppose that $V$, $W$ are semistable bundles with $\mu(V) < \mu(W)$. Then $\text{Hom}(V, W)$ is projective as an $\text{End}(V) \otimes_k \text{End}(W)^\text{op}$-module.

**Proof.** Since each of $V$ and $W$ splits naturally as a direct sum over nonisomorphic stable constituents, we may assume that each is built from a single stable bundle. In particular, $V$ has a unique isomorphism class $V_{\text{max}}$ of indecomposable summands of maximal rank, and one has

$$V \cong V_{\text{max}} \otimes_{\text{End}(V_{\text{max}})} M_V \quad (3.6)$$

for some faithful $\text{End}(V_{\text{max}})$-module $M_V$, with a similar description applying to $W$. We then find that

$$\text{Hom}(V, W) \cong \text{Hom}_{\text{End}(V_{\text{max}})}(M_V, \text{Hom}(V_{\text{max}}, W_{\text{max}}) \otimes_{\text{End}(W_{\text{max}})} M_W). \quad (3.7)$$

Since $\text{Hom}(V_{\text{max}}, W_{\text{max}})$ is free, it remains only to show that

$$\text{Hom}_{\text{End}(V_{\text{max}})}(M_V, \text{End}(V_{\text{max}})) \otimes_k M_W \quad (3.8)$$

is projective as an $\text{End}(V) \otimes_k \text{End}(W)^\text{op}$-module.
Here $M_W$ is a faithful module over $\text{End}(W_{\text{max}}) \cong k[x]/(x^n)$, and (since $x$ is central in $\text{End}(W)$) we have $\text{End}(W) \cong \text{End}_{k[x]/(x^n)}(M_W)$. In particular, $M_W \cong \bigoplus_i k[x]/(x^{m_i})$ with $n = m_1 \geq m_2 \geq \cdots$, and thus there is an idempotent cutting out a copy of $k[x]/(x^n)$, so that the corresponding submodule of $\text{End}_{k[x]/(x^n)}(M_W)$ is $M_W$, implying projectivity. The claim for $V$ is analogous (relative now to the left $\text{End}(V)$-module structure on $\text{End}(V))$. \hfill \qed

**Proposition 3.4.** The homogeneous components of $B_{V,\Psi}$ are projective modules over $(B_{V,\Psi})_0$.

**Proof.** This reduces to showing that the Hom spaces $B_{V,\Psi}(i,j)$ are projective over the relevant endomorphism ring. If $j = i$, this is immediate, while if $j > i$, we have in fact shown that it is projective over the tensor product of endomorphism rings. \hfill \qed

We thus see that minimal resolutions of $B_{V,\Psi}$-modules (or, equivalently, $B_{V,\Psi}$-modules) are convergent as long as they are bounded below and have homogeneous components of finite homological dimension. Let $P_i$ denote the projective $B_{V,\Psi}$-module given by the contravariant functor

$$j \mapsto B_{V,\Psi}(j,i),$$

and let $S_i$ be the module agreeing with $P_i$ in degree $i$ but 0 in all other degrees. Since $S_i(i) = \text{End}(V_i)$, $S_i$ has a convergent projective resolution.

An important property of the $\mathbb{Z}$-algebras $B_{V,\Psi}$ is that they are Gorenstein in a suitable sense.

**Proposition 3.5.** The space

$$\text{Ext}^p(S_i, P_j)$$

vanishes unless $i = j$ and $p = 2$, in which case there is a canonical isomorphism

$$\text{Ext}^2(S_i, P_j) \cong \text{Ext}^1(V_i, V_i).$$

**Proof.** The module $S_i$ has a convergent projective resolution

$$\cdots \to Q_{i+2} \to Q_{i+1} \to P_i \to S_i.$$ \hfill (3.12)

For each $j > i$, projectivity of $Q_j$ implies that it is the module of global sections of a vector bundle $W_j$ on $E$ (this is true for any summand of any $P_j$) and thus the complex

$$\cdots \to W_{i+2} \to W_{i+1} \to V_i$$ \hfill (3.13)

of sheaves on $E$ is exact. Now, $R\text{Hom}(S_i, P_j)$ is represented by the complex

$$\text{Hom}(P_i, P_j) \to \text{Hom}(Q_{i+1}, P_j) \to \text{Hom}(Q_{i+2}, P_j) \to \cdots,$$ \hfill (3.14)

or equivalently

$$\text{Hom}(V_i, V_j) \to \text{Hom}(W_{i+1}, V_j) \to \text{Hom}(W_{i+2}, V_j) \to \cdots.$$ \hfill (3.15)

Since the double complex

$$R\text{Hom}(V_i, V_j) \to R\text{Hom}(W_{i+1}, V_j) \to R\text{Hom}(W_{i+2}, V_j) \to \cdots$$ \hfill (3.16)

represents $R\text{Hom}(0, V_j)$, it is acyclic, which, since every column has magnitude contained in $[0, 1]$ (as a $R\text{Hom}$ between sheaves on $E$), implies a quasi-isomorphism between the complex of Hom spaces and a shift of the complex

$$\text{Ext}^1(V_i, V_j) \to \text{Ext}^1(W_{i+1}, V_j) \to \text{Ext}^1(W_{i+2}, V_j) \to \cdots,$$ \hfill (3.17)
or (after fixing an isomorphism $H^1(\mathcal{O}_E) \cong k$)

$$\text{Hom}(V_j, V_i)^* \to \text{Hom}(V_j, W_{i+1})^* \to \text{Hom}(V_j, W_{i+2})^* \to \cdots.$$  \hfill (3.18)

This is dual to the complex

$$\cdots \to \text{Hom}(V_j, W_{i+2}) \to \text{Hom}(V_j, W_{i+1}) \to \text{Hom}(V_j, V_i),$$  \hfill (3.19)

or equivalently

$$\cdots \to \text{Hom}(P_j, Q_{i+2}) \to \text{Hom}(P_j, Q_{i+1}) \to \text{Hom}(P_j, P_i),$$  \hfill (3.20)

which (up to a shift) represents $R\text{Hom}(P_j, S_i) \cong \text{Hom}(P_j, S_i)$. This vanishes unless $i = j$, when it equals $\text{End}(V_i)$.

We thus conclude that $\text{Ext}^p(S_i, P_j) = 0$ unless $i = j$ and (taking into account the shift) $p = 2$, in which case one has a canonical isomorphism

$$\text{Ext}^2(S_i, P_j) \cong \text{Hom}_k(\text{End}(V_i), H^1(\mathcal{O}_E)) \cong \text{Ext}^1(V_i, V_i)$$  \hfill (3.21)

as required.

Now, let $B^+$ be the $\mathbb{Z}$-algebra obtained from $B_{V, \Psi}$ by adjoining a central element $t$ of degree $c$.

In other words,

$$B^+(i, j) = \bigoplus_{l \geq 0} B_{V, \Psi}(i, j - cl),$$  \hfill (3.22)

with $t \in B^+(i, i + c)$ the element corresponding to $1 \in B_{V, \Psi}(i, i)$. Let $P_i^+$ be the natural family of projective $B^+$-modules, with associated modules $S_i^+$, and note that we still have $B^+(i, i) = \text{Hom}(V_i, V_i)$.

**Corollary 3.6.** The space

$$\text{Ext}^p(S_i^+, P_j^+)$$  \hfill (3.23)

vanishes unless $j = i - c$ and $p = 3$, in which case there is a canonical isomorphism

$$\text{Ext}^3(S_i^+, P_j^+) \cong \text{Ext}^1(V_i, V_i).$$  \hfill (3.24)

**Proof.** Since $P_j^+$ restricts to $\bigoplus_{l \geq 0} P_j[-lc]$, we have

$$\text{Ext}^p(S_i[t], P_j^+) \cong \bigoplus_{l \geq 0} \text{Ext}^p(S_i, P_j[-lc]),$$  \hfill (3.25)

where $S_i[t]$ is the induced module. We may thus use the short exact sequence

$$0 \to S_i[t][-c] \to S_i[t] \to S_i^+ \to 0$$  \hfill (3.26)

where $S_i[t]$ is the induced module. We may thus use the short exact sequence

$$0 \to S_i[t][-c] \to S_i[t] \to S_i^+ \to 0$$  \hfill (3.26)

to obtain a long exact sequence

$$\cdots \to \text{Ext}^p(S_i^+, P_j^+) \to \bigoplus_{l \geq 0} \text{Ext}^p(S_i, P_j[-lc]) \to \bigoplus_{l \geq -1} \text{Ext}^p(S_i, P_j[-lc]) \to \cdots$$  \hfill (3.27)

This immediately gives an isomorphism

$$\text{Ext}^p(S_i^+, P_j^+) \cong \text{Ext}^{p-1}(S_i, P_{j+c}),$$  \hfill (3.28)

so the formula reduces to the analogous formula for $B$. \hfill $\square$
Given a graded algebra (or $\mathbb{Z}$-algebra) $A$, there is an associated category $\text{Proj}(A)$, the quotient of the category of all graded modules by the subcategory of “torsion” modules (in which every element generates a finite-dimensional submodule). The Ext groups in that category can then be computed as limits of Ext groups in the category of graded modules, per [3], subject to a certain technical condition (“property $\chi$”) which is satisfied for $B_{V,\Psi}$ by Thm 4.5 op. cit. and thus for the Rees algebra of any filtered deformation by Thm. 8.8 op. cit. Indeed, in our setting, the only nontrivial hypothesis between the two theorems is that $\Psi$ is “ample” relative to $V$: for any coherent $M$, there exists $n$ such that $\text{Ext}^p(\Psi^{-n}V, M) = 0$ for $p > 0$ and $\text{Hom}(\Psi^{-n}V, M) \otimes_k \Psi^{-n}V \to M$ is surjective. Both facts reduce immediately to the case that both $V$ and $M$ are indecomposable, with vanishing of $\text{Ext}^1$ an immediate consequence via duality of the eventual inequality in slopes, and global generation proved in Corollary 3.15 below.

**Theorem 3.7.** Let $\bar{P}_i^+$ denote the object of $\text{Proj}(B^+)$ corresponding to $P_i^+$. Then $\text{Ext}^p(\bar{P}_i^+, \bar{P}_j^+) = 0$ unless $p = 0$ and $j \geq i$ or $p = 2$ and $j \leq i - c$, with

$$\text{Hom}(\bar{P}_i^+, \bar{P}_j^+) \cong \text{Hom}(V_i, V_j)$$

and

$$\text{Ext}^2(\bar{P}_i^+, \bar{P}_j^+) \cong \text{Ext}^1(V_i, V_{j+c}).$$

**Proof.** We have

$$\text{Ext}^p(\bar{P}_i^+, \bar{P}_j^+) \cong \lim_{n \to \infty} \text{Ext}^p((P_i^+)_{\geq n}, P_j^+).$$

Now consider the short exact sequence

$$0 \to (P_i^+)_{\geq n} \to P_i^+ \to P_i^+/(P_i^+)_{\geq n} \to 0.$$  (3.32)

Since $P_i^+$ is projective, $R\text{Hom}(P_i^+, P_j^+) = \text{Hom}(V_i, V_j)$ concentrated in degree 0. Moreover, since the degree $l$ component of $P_i^+$ is a free $\text{End}(V_l)$-module, so $P_i^+/(P_i^+)_{\geq n}$ is a finite extension of such modules, we find that for $n > j - c$,

$$R\text{Hom}(P_i^+/(P_i^+)_{\geq n}, P_j^+) \cong \text{Ext}^1(V_i, V_{j+c})$$  (3.33)

concentrated in degree 3. The desired claim follows immediately.  

For the next claim, we revert to one of the naïve nonnegative gradings of $B_{V,\Psi}$.

**Corollary 3.8.** Let $A^+$ be the Rees algebra of a filtered deformation of $B_{V,\Psi}$, and let $O(l)$ be the objects of $\text{Proj}(A^+)$ corresponding to shifts of $A^+$. Then for $l \leq m$, $\text{Ext}^p(O(l), O(m)) = 0$ for $p \neq 0$ and for $l > m$, $\text{Ext}^p(O(l), O(m)) = 0$ for $p \neq 2$. Moreover, there is a canonical duality

$$\text{Ext}^2(O(l), O(m)) \cong \text{Hom}_k(\text{Hom}(O(m), O(l - 1)), H^1(O_E)).$$

(3.34)

In particular, $O(0)$ has no higher endomorphisms, and if $V$ is stable, then $O(0)$ is exceptional.

**Proof.** The $\mathbb{G}_m$ action on filtered deformations gives rise to a family of graded algebras with generic fiber isomorphic to $A^+$ and special fiber associated to the trivial deformation $B^+ \cong B[t]$. By semicontinuity, it suffices to prove vanishing when $A^+$ is trivial, where it reduces to computing

$$R\text{Hom}(\bigoplus_{0 \leq i < c} P_{i-\ell c}^+, \bigoplus_{0 \leq i < c} P_{i-mc}^+)$$  (3.35)
in \( \text{Proj}(B^+) \). Note that since each case has precisely one nonvanishing cohomology space, the corresponding sheaves on \( \mathbb{A}^1 = \mathbb{G}_m \cup \{0\} \) are flat.

For the duality between nonvanishing Ext spaces, we note that \( \text{coh}(E) \cong \text{Proj}(B_{V, \Psi}) \) embeds in \( \text{Proj}(A^+) \) as the subcategory on which the natural transformation \((-1) \rightarrow \text{id}) associated to the central element of \( A^+ \) vanishes. This embedding factors into a pair \( i_* i^* \) of adjoint functors with \( i^* : \text{Proj}(A^+) \rightarrow \text{coh}(E) \) exact, so that we have distinguished triangles

\[
M(-1) \rightarrow M \rightarrow R i_* i^* M \rightarrow
\]

for any \( M \) and thus distinguished triangles

\[
R \text{Hom}(M,N(-1)) \rightarrow R \text{Hom}(M,N) \rightarrow R \text{Hom}_E(i^* M, i^* N) \rightarrow .
\]

In particular, taking \( M = N = \mathcal{O}(l) \) gives a canonical isomorphism

\[
\text{Ext}^2(\mathcal{O}(l), \mathcal{O}(l-1)) \cong \text{Ext}^1(\Psi^1 V, \Psi^1 V)
\]

and thus a canonical trace

\[
\text{Ext}^2(\mathcal{O}(l), \mathcal{O}(l-1)) \rightarrow H^1(\mathcal{O}_E)(\cong k).
\]

This induces pairings in the usual way, and it remains only to show that the pairings are perfect. But this follows from another semicontinuity argument together with the fact they are perfect in \( \text{Proj}(B^+) \).

\[\square\]

\textbf{Remark.} Note that the trace form produces two pairings:

\[
\text{Hom}(\mathcal{O}(l), \mathcal{O}(m)) \otimes \text{Ext}^2(\mathcal{O}(m), \mathcal{O}(l-1)) \rightarrow \text{Ext}^2(\mathcal{O}(l), \mathcal{O}(l-1)) \rightarrow k
\]

and

\[
\text{Ext}^2(\mathcal{O}(m), \mathcal{O}(l-1)) \otimes \text{Hom}(\mathcal{O}(l), \mathcal{O}(m))
\]

\[
\cong \text{Ext}^2(\mathcal{O}(m), \mathcal{O}(l-1)) \otimes \text{Hom}(\mathcal{O}(l-1), \mathcal{O}(m-1)) \rightarrow \text{Ext}^2(\mathcal{O}(m), \mathcal{O}(m-1)) \rightarrow k.
\]

The corresponding pairings in \( \text{Proj}(B^+) \) agree (since the analogous pairings agree in \( \text{Proj}(B_{V, \Psi}) \cong \text{coh}(E) \)), and thus they differ by a compatible family of automorphisms of \( \text{Hom}(\mathcal{O}(l), \mathcal{O}(m)) \) agreeing with 1 on the associated graded, or in other words by an automorphism of \( A^+ \) acting as 1 on the associated graded. But this implies that the pairings agree: if this automorphism were nontrivial, it would induce a nonzero derivation of \( B_{V, \Psi} \) of negative degree. It follows that (up to a choice of holomorphic differential on \( E \) \(-1)[2] \) is a Serre functor on \( \text{Proj}(A^+) \), since the relevant duality and compatibility relations hold on generators. (In particular, \( S[-2] = -(1) \) is an abelian autoequivalence, so that \( \text{Proj}(A^+) \) is Gorenstein.)

If the components of the slope decomposition of \( V \) have commutative endomorphism rings, we can be more precise about the structure of the minimal resolution. To understand this, we first recall some facts about derived autoequivalences of elliptic curves. In addition to the abelian autoequivalences, which consist of automorphisms of \( E \) along with translations and twists by line bundles, there is an additional autoequivalence \( \Psi_{\mathcal{O}_E} \) which fits into a distinguished triangle

\[
\mathcal{O}_E \otimes_k R \text{Hom}(\mathcal{O}_E, M) \rightarrow M \rightarrow \Psi_{\mathcal{O}_E} M \rightarrow .
\]
We may either use Fourier-Mukai theory or a suitable dg-enhancement to see that this cone can be made functorial. The inverse is also a cone:

$$
\Phi_{OE}^{-1} M \to M \to \text{Hom}_k(\text{Hom}(M, OE), OE) \to,
$$

(3.43)

which can be seen by observing that

$$
R \text{Hom}(OE, \Phi_{OE} E) \cong \text{Ext}^1(OE, OE) \otimes R \text{Hom}(OE, M) \cong R \text{Hom}(M, OE)^* \tag{3.44}
$$

Twisting by the line bundle $OE(z)$ (where $z$ is the identity) has a similar description:

$$
OE \otimes_k R \text{Hom}(OE, M) \to M \to M(z) \to
$$

(3.45)

so that we may call it $\Phi_{OE}$. (These functors generate a central extension of $\text{SL}_2(\mathbb{Z})$.) The following is then standard (essentially Atiyah’s classification of vector bundles [4]):

**Proposition 3.9.** Any semistable sheaf on $E$ is the image of a torsion sheaf under a composition of the functors $\Phi_{OE}$ and $\Phi_z$ and their inverses.

**Remark.** If the semistable sheaf $M$ is not already torsion, then we twist by a suitable power of $OE(z)$ to put its slope in $(0, 1]$ and observe that applying $\Phi_{OE}$ gives a sheaf of smaller rank, so that iterating eventually produces a torsion sheaf.

**Remark.** Since $\Phi_{OE}$ and $\Phi_z$ both preserve the degree 0 divisor class $c_1(M) - \deg(M)[z]$ and a stable sheaf is uniquely determined by its determinant, the torsion sheaf that results is uniquely determined, even though the derived equivalence getting there is not.

**Remark.** Note that since the action of $\Psi_{OE}$ and $\Phi_z$ preserves $\gcd(\text{rank}(M), \deg(M))$, we conclude that $M$ has $\gcd(\text{rank}(M), \deg(M))$ stable constituents (counted with multiplicity), by reduction to the torsion case.

**Lemma 3.10.** Suppose that $M \in \text{coh}(E)$ is such that $\text{End}(M)$ is commutative. Then $M$ is semistable, and each indecomposable summand of $M$ is $S$-equivalent to a power of a different isomorphism class of stable sheaves.

**Proof.** Let $M = \bigoplus M_i$ be the decomposition of $M$ into indecomposable summands. We clearly have $\text{Hom}(M_i, M_j) = 0$ for $i \neq j$, and thus $\mu(M_i) = \mu(M_j)$. Since an indecomposable sheaf is semistable, it follows immediately that $M$ is semistable. Moreover, each indecomposable sheaf is an iterated extension of a unique stable constituent, and the vanishing of $\text{Hom}(M_i, M_j)$ forces the stable constituents to be nonisomorphic.

**Corollary 3.11.** A sheaf $M$ on $E$ has commutative endomorphism ring iff it is the image of the structure sheaf of an effective divisor $D_M$ under an element of the group of derived autoequivalences generated by $\Phi_{OE}$ and $\Phi_z$.

**Proof.** Since the structure sheaf of an effective divisor is commutative, the image of such a sheaf under a derived autoequivalence also has commutative endomorphism ring. Conversely, if $M$ has commutative endomorphism ring, then the torsion sheaf it is derived equivalent to must also have commutative endomorphism ring, so is the structure sheaf of a 0-dimensional subscheme as required.

**Remark.** With this in mind, we call sheaves on $E$ with commutative endomorphism ring “divisorial.”
Now, given a divisorial sheaf $M$, consider the functor $\Phi_M$ on $D^b\text{coh}(E)$ fitting into the distinguished triangle

$$M \otimes_{\text{End}(M)} L \text{Hom}(M, N) \to N \to \Phi_M(N) \to .$$

(3.46)

(Again, we either use a Fourier-Mukai kernel or a dg-enhancement to ensure that this functor is well-defined.)

**Proposition 3.12.** If $M$ is divisorial, then the functor $\Phi_M$ is a derived autoequivalence, and if $N$ is a semistable sheaf then either $\Phi_M(N)$ or $\Phi_M(N)[-1]$ is a semistable sheaf.

**Proof.** The conjugate of the functor $\Phi_M$ under a derived autoequivalence has the same form, and thus to see that it is a derived autoequivalence, it suffices to consider the case that $M$ is the structure sheaf of a divisor. But in that case, $\Phi_{\mathcal{O}_D} N \cong N(D)$, which is clearly an autoequivalence. It also follows from this that the functor $M \otimes_{\text{End}(M)} -$ is exact, and thus we may take the underived tensor product above.

For any sheaf $N$, we then have an exact sequence

$$0 \to h^{-1} \Phi_M(N) \to M \otimes_{\text{End}(M)} \text{Hom}(M, N) \to N \to h^0 \Phi_M(N) \to M \otimes_{\text{End}(M)} \text{Ext}^1(M, N) \to 0$$

(3.47)

with $h^p \Phi_M(N) = 0$ for $p \notin \{-1, 0\}$. If $N$ is stable, then it is simple ($\text{End}(N) \cong k$), and the same applies to $\Phi_M(N)$, forcing $\Phi_M(N)$ to be a shift of a simple (and thus stable!) sheaf. (Since $\text{Ext}^2$ vanishes in $\text{coh}(E)$, any complex in $D^b_{\text{coh}}(E)$ is formal, and thus its endomorphism ring contains at least one idempotent for each nonvanishing cohomology sheaf.) Since there are only two possible shifts, we can distinguish whether it is a sheaf or a shift from its class in the numerical Grothendieck group, so that the claim immediately extends to semistable sheaves.

**Corollary 3.13.** If the divisorial sheaf $M$ has stable constituents $M_1, \ldots, M_m$ (with multiplicity, and in any order), then $\Phi_M \cong \Phi_{M_1} \circ \cdots \circ \Phi_{M_m}$. Moreover, $\Phi_M$ depends only on $\text{rank}(M)$ and $\text{det}(M)$.

**Proof.** When $M$ is torsion, the claims are immediate; here the second claim simply says that the autoequivalence $N \mapsto N(D)$ depends only on the isomorphism class of the line bundle $\mathcal{O}_E(D)$. The claim in general follows by conjugating by the appropriate element of the group generated by $\Phi_{\mathcal{O}_E}$ and $\Phi_{\mathcal{O}_D}$.

**Remark.** Note that the claim as stated descends to smooth genus 1 curves over non-closed fields, where the functor $\Phi_{\mathcal{O}_2}$ is not defined.

The second claim can be restated as saying that $\Phi_M$ only depends on the class of $M$ in $K_0(E)$. If $M$ is stable, then we clearly have

$$[\Phi_M(N)] = [N] - \chi(M, N)[M],$$

(3.48)

where $\chi(M, N) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Ext}^i(M, N)$ is the Mukai pairing on $K_0$. More generally, if $M$ has $m = \gcd(\text{rank}(M), \text{deg}(M))$ stable constituents, then we can use the factorization along with the fact that $\chi(M_i, M_j) = 0$ (since they are stable sheaves of the same slope) to find that

$$[\Phi_M(N)] = [N] - m^{-1}\chi(M, N)[M].$$

(3.49)

Since

$$\chi(M, N) = \text{rank}(M) \text{deg}(N) - \text{deg}(M) \text{rank}(N),$$

(3.50)

this lets us explicitly compute the action of $\Phi_M$ on the Grothendieck group.
Corollary 3.14. Suppose that the divisorial sheaf $M$ has rank $r$, degree $d$, and $m = \gcd(r, d)$ stable constituents. Then the autoequivalence $\Phi_M$ acts on the Grothendieck group of $E$ by

$$\begin{align*}
\text{rank}(\Phi_M(N)) &= \frac{m + dr}{m} \text{rank}(N) - \frac{r^2}{m} \text{deg}(N) \\
\text{deg}(\Phi_M(N)) &= \frac{d^2}{m} \text{rank}(N) + \frac{m - dr}{m} \text{deg}(N)
\end{align*}$$

(3.51)

and

$$\det(\Phi_M(N)) \cong \det(N) \otimes \det(M)^{-\frac{r \text{deg}(N) - d \text{rank}(N)}{m}}. \quad (3.52)$$

Remark. As a function of $m$ with $r/m, d/m$ fixed, this gives a homomorphism from $\mathbb{Z}$ to $\text{SL}_2(\mathbb{Z})$; in particular, the inverse is just the image under negating $m, r, d$.

In the following statement, and below, the convention is that the slope of a (nonzero) torsion sheaf is positive infinity. (This is compatible with the definition of semistability: a semistable sheaf of positive rank has no torsion subsheaf but plenty of torsion quotients.)

Corollary 3.15. Let $V$ be a divisorial vector bundle with rank $V = r$, deg $V = d$. Then for any semistable sheaf $M$, the morphism

$$V \otimes_{\text{End}(V)} \text{Hom}(V, M) \to M$$

(3.53)

is surjective if

$$\mu(M) > \mu(V) + \frac{\gcd(r, d)}{r^2}, \quad (3.54)$$

and is otherwise injective. In either case, the (co)kernel is semistable, of slope $\leq \mu(V) - \frac{\gcd(r, d)}{r^2}$ iff the morphism is surjective.

Proof. The strict lower bound on the slope of $M$ ensures that rank$(\Phi_V(M)) < 0$, and thus $\Phi_V(M)[-1]$ is a semistable sheaf fitting into a short exact sequence

$$0 \to \Phi_V(M)[-1] \to V \otimes_{\text{End}(V)} \text{Hom}(V, M) \to M \to 0. \quad (3.55)$$

If the opposite inequality holds, then rank$(\Phi_V(M)) > 0$, so that $\Phi_V(M)$ is a semistable sheaf and the morphism is injective. Finally, if equality holds, then rank$(M) > 0$ and eliminating deg$(M)$ gives (assuming $V$ has $m$ stable constituents)

$$\deg(\Phi_V(M)) = \frac{m}{r^2} \text{rank}(M), \quad (3.56)$$

which since rank$(\Phi_V(M)) = 0$ again forces $\Phi_V(M)$ to be a semistable sheaf.

If $\mu(M) > \mu(V)$, the map vanishes, so the cokernel is certainly semistable. When $\mu(M) < \mu(V)$, the (co)kernel is a shift of $\Phi_V(M)$, so is semistable by Proposition [3.12] when equality holds (forcing injectivity), this fails, but the morphism is between semistable bundles of the same slope, and thus the cokernel must also have that slope (which also agrees with the slope of $\Phi_V(M)$).

Since $\mu(\Phi_V(M))$ is a linear fractional function of $\mu(M)$, the surjective region in terms of $\mu(\Phi_V(M))$ is the image of the interval $(\mu(V) + \frac{\gcd(r, d)}{r^2}, \infty]$ under that LFT, giving the stated result. \qed
Proposition 3.16. Let $V$ be a divisorial vector bundle on $E$, and suppose $M$ is a coherent sheaf such that $(\Phi_V \Psi)^n M[-n]$ is a sheaf for all $n$. Then $M$ has a natural resolution of the form

$$
\cdots \to \Psi^{-n}V \otimes_{\text{End}(V)} M_n \to \cdots \to \Psi^{-1}V \otimes_{\text{End}(V)} M_1 \to M \to 0
$$

(3.57)

for suitable $\text{End}(V)$-modules $M_i$, and the corresponding complex of $B_V \Psi$-modules is exact in all positive $B$-degrees.

Proof. For the first claim, we first refine to say that there is an exact sequence

$$
0 \to \Psi^{-n}(\Phi_V \Psi)^n M[-n] \to \Psi^{-n}V \otimes_{\text{End}(V)} M_n \to \cdots \to \Psi^{-1}V \otimes_{\text{End}(V)} M_1 \to M \to 0.
$$

(3.58)

for all $n$. This is straightforward: it is trivial for $n = 0$, and we may increase $n$ using the short exact sequence

$$
0 \to \Psi^{-n-1}(\Phi_V \Psi)^{n+1} M[-n-1] \to \Psi^{-n-1}V \otimes_{\text{End}(V)} \text{Hom}(V, \Psi(\Phi_V \Psi)^n M[-n]) \to \Psi^{-n}(\Phi_V \Psi)^n M[-n] \to 0
$$

(3.59)

obtained by applying $\Psi^{-n-1}$ to the short exact sequence associated to $\Phi_V (\Phi_V \Psi)^n M[-n]$.

To see that this also gives a resolution of the (truncated!) $B$-module associated to $M$, observe that every term (except possibly $M$ itself) of the sequence for $n$ is acyclic for $R \text{Hom}(V, \Psi^{n+1}-)$. For the intermediate terms, this is immediate from the fact that $\mu(\Psi^i V) > \mu(V)$, while for the first term it follows from the fact that $(\Phi_V \Psi)^{n+1} M[-n-1]$ is a sheaf. In particular, the spectral sequence computing the (vanishing!) hypercohomology collapses immediately, and thus all homology groups vanish as required.

This is particularly powerful for the following reason: the condition that $(\Phi_V \Psi)^n M[-n]$ should be a sheaf is essentially numerical in nature. Indeed, it follows easily (by considering indecomposable summands of $M$) that this holds iff it holds for every stable constituent of $M$. But for a stable sheaf, we know that $(\Phi_V \Psi)^n M[-n]$ is a sheaf iff $\text{rank}((\Phi_V \Psi)^j M[-j]) > 0$ for $1 \leq j \leq n$, iff $(-1)^j \text{rank}((\Phi_V \Psi)^j M) > 0$ for $1 \leq j \leq n$.

Corollary 3.17. With $V$, $M$ as above, suppose that $M$ is a semistable bundle. Then the above conclusion holds iff the power series

$$
\frac{1 + (r^2 \mu(M) + (m - dr))t/m}{1 - (\deg(L)r^2/m - 2)t + t^2}
$$

(3.60)

has positive coefficients, where $r = \text{rank}(V)$, $d = \deg(V)$, and $m = \gcd(r, d)$ is its number of stable constituents.

Proof. We need $(-1)^j \text{rank}((\Phi_V \Psi)^j M) > 0$ for $j \geq 1$, or equivalently that the power series

$$
\sum_{j \geq 1} (-t)^j \text{rank}((\Phi_V \Psi)^j M)
$$

(3.61)

has positive coefficients. Since $\Phi_V \Psi$ acts linearly on the Grothendieck group, we may rephrase this as saying that all nonconstant coefficients of

$$
\text{rank}(\sum_{j \geq 0} (-t)^j (\Phi_V \Psi)^j M) = \text{rank}((1 + t \Phi_V \Psi)^{-1} M)
$$

(3.62)

are positive. The claim then follows by inverting the relevant $2 \times 2$ matrix and dividing by $\text{rank}(M)$ to rewrite it in terms of the slope. 

\[\square\]
Remark. In the case that $M$ is torsion, we obtain a resolution iff
\[
\frac{1}{1 - (\deg(L)r^2/m - 2)t + t^2} \quad (3.63)
\]
has positive coefficients, or equivalently when
\[
\deg(L)r^2 \geq 4m. \quad (3.64)
\]

Lemma 3.18. If $\tau \geq 2$, then the power series $(1 + \alpha t)/(1 - \tau t + t^2)$ has positive coefficients iff $\alpha \geq -(\tau + \sqrt{\tau^2 - 4})/2$.

Proof. We may write $\tau = \lambda + 1/\lambda$ with $\lambda = (\tau + \sqrt{\tau^2 - 4})/2 \geq 1$. We then find that
\[
\frac{1 - \lambda t}{1 - (\lambda + 1/\lambda)t + t^2} = \frac{1}{1 - t/\lambda} \quad (3.65)
\]
has positive coefficients, as (apart from the constant term) does $1 - (\lambda + 1/\lambda)t + t^2$. Moreover, the coefficients of the latter are unbounded, while the coefficients of the former go to 0, and thus the coefficients of
\[
\frac{1 - (\lambda + \epsilon)t}{1 - (\lambda + 1/\lambda)t + t^2} \quad (3.66)
\]
are either always positive (if $\epsilon \leq 0$) or eventually negative (if $\epsilon > 0$), implying the desired result. \[\square\]

Remark. If $\tau \in \{-1, 0, 1\}$, positivity fails for any $\alpha$.

Theorem 3.19. Let $V$ be a divisorial vector bundle on $E$ with rank $r$, degree $d$ and $m = \gcd(r, d)$ stable constituents. Then $B_{V, \Psi}$ is Koszul unless $m = r$ (so $r|d$) and $\deg(L)r \leq 3$.

Proof. If we apply the corollary to the case $M = V$, we find that the relevant power series is
\[
\frac{1 + t}{1 - (\deg(L)r^2/m - 2)t + t^2}, \quad (3.67)
\]
which has positive coefficients as long as the denominator has positive real roots. This gives the condition
\[
\deg(L)(r/m)^2 m \geq 4 \quad (3.68)
\]
which holds unless $r|d$ and $\deg(L)r \leq 3$.

The resulting complex of $B_{V, \Psi}$-modules is exact in all positive degrees, and thus gives a free resolution of $(B_{V, \Psi})_0$. Since this resolution is manifestly linear, the claim follows. \[\square\]

Something similar applies to minimal resolutions in the category $B_{V, \Psi}$ more generally, again with the assumptions that all objects are divisorial. We may in fact generalize somewhat: we choose for each $i \in \mathbb{Z}$ a divisorial sheaf $M_i$. If $\mu(M_i)$ is an increasing function of $i$, this gives a category as before, but even if not, we can still define a $dg$-category:
\[
B_M(i, j) = \begin{cases} 
0, & i > j \\
\text{End}(M_i), & i = j \\
R\text{Hom}(M_i, M_j) & i < j. 
\end{cases} \quad (3.69)
\]
(Here, of course, $R\text{Hom}(M_i, M_j)$ should be replaced by the appropriate complex.) Let $P_i$ be the associated free modules (which are no longer projective, per se), and $S_i$ be defined analogously to the $B_{V, \Psi}$ case.
Suppose now that \( N \) is a semistable sheaf, and consider the truncated module:

\[
M_{N, \geq 0} : i \mapsto \begin{cases} 0 & i > 0 \\ \text{RHom}(M_i, N) & i \leq 0. \end{cases}
\]  
\[\text{(3.70)}\]

over this dg-category. The computation of the minimal resolution of \( M_{N, \geq 0} \) begins with the morphism

\[
P_0 \otimes_{\text{End}(M_0)} \text{RHom}(M_0, N) \to M_{N, \geq 0},
\]  
\[\text{(3.71)}\]

apart from replacing \( \text{RHom}(M_0, N) \) itself by a minimal resolution (as an \( \text{End}(M_0) \)-module). Let \( M' \) be the corresponding cone, so that we have for each \( d \in \mathbb{Z} \) a distinguished triangle

\[
\text{RHom}(P_{-d}, P_0) \otimes_{\text{End}(M_0)} \text{RHom}(M_0, N) \to \text{RHom}(P_{-d}, M_{N, \geq 0}) \to M'(-d) \to .
\]  
\[\text{(3.72)}\]

For \( d < 0 \), the first two terms vanish, while for \( d = 0 \) the map becomes

\[
\text{End}(M_0) \otimes_{\text{End}(M_0)} \text{RHom}(M_0, N) \cong \text{RHom}(M_0, M_{N, \geq 0}),
\]  
\[\text{(3.73)}\]

so that \( M'(-d) = 0 \) for \( d \leq 0 \), while for \( d > 0 \) we may rewrite the distinguished triangle as

\[
\text{RHom}(M_{-d}, M_0) \otimes_{\text{End}(M_0)} \text{RHom}(M_0, N) \to \text{RHom}(M_{-d}, N) \to M'(-d) \to .
\]  
\[\text{(3.74)}\]

If we pull the tensor product inside the first \( \text{RHom} \), we see that the first morphism is the image under \( \text{RHom}(M_{-d}, -) \) of the natural morphism

\[
M_0 \otimes_{\text{End}(M_0)} \text{RHom}(M_0, N) \to N,
\]  
\[\text{(3.75)}\]

and thus we find that

\[
M'(-d) \cong \text{RHom}(M_{-d}, \Phi_{M_0} N).
\]  
\[\text{(3.76)}\]

In other words, we have a distinguished triangle

\[
P_0 \otimes_{\text{End}(M_0)} \text{RHom}(M_0, N) \to M_{N, \geq 0} \to M_{\Phi_{M_0} N, \geq 1} \to ,
\]  
\[\text{(3.77)}\]

from which it is straightforward to inductively construct the minimal resolution of \( M_{N, \geq 0} \). In particular, we find that the associated graded of the resulting filtered complex is the direct sum

\[
P_0 \otimes_{\text{End}(M_0)} \text{RHom}(M_0, N) \\
\oplus P_1 \otimes_{\text{End}(M_1)} \text{RHom}(M_1, \Phi_{M_0} N) \\
\oplus P_2 \otimes_{\text{End}(M_2)} \text{RHom}(M_2, \Phi_{M_1} \Phi_{M_0} N) \\
\oplus \cdots ,
\]  
\[\text{(3.78)}\]

again with each \( \text{RHom} \) complex replaced by a suitable minimal resolution.

The one caveat above is that it is not a priori obvious that \( \text{RHom}(M_0, N) \) admits a finite projective resolution. If \( \mu(N) > \mu(M_0) \), then this is just \( \text{Hom}(M_0, N) \), which we have shown is already free, and similarly if \( \mu(N) < \mu(M_0) \), we know by duality that \( \text{Ext}^1(M_0, N) \) is free. Thus only the case that equality holds is an issue. We may reduce to the case that \( M_0, N \) are both structure sheaves of divisors, and from that immediately reduce to the case that they are structure sheaves of jets supported at the same point, and may then pass to the complete local ring. In other words, we need to know that

\[
\text{RHom}_{k[[x]]}(k[[x]]/(x^n), k[[x]]/(x^n))
\]  
\[\text{(3.79)}\]

on
is represented by a finite complex of free \(k[[x]]/(x^n)\)-modules. Using the two-term injective resolution
\[
0 \to k[[x]]/(x^m) \to k((x))/x^m k[[x]] \to k((x))/k[[x]] \to 0,
\]
we obtain a two-term free resolution, given by the morphism
\[
x^m : k[[x]]/(x^n) \to k[[x]]/(x^n).
\]

(This morphism vanishes on \(k[[x]]/(x)\), so this is the minimal resolution.) It follows more generally that if \(N\) with \(\mu(N) = \mu(M_0)\) is also divisorial, then \(\text{RHom}(M_0, N)\) is represented by a two-term complex in which both terms are the same direct summand of \(\text{End}(M_0)\). Note that we can also obtain a (non-minimal) free resolution in this case, by taking the direct sum of this complex with a complex having morphism 1 on a complementary direct summand.

We thus obtain the following.

**Proposition 3.20.** The minimal resolution of the module \(S_i\) is given by a filtered complex such that the associated graded is
\[
P_i = P_{i-1} \otimes \text{RHom}(M_{i-1}, M_i)[1] \oplus P_{i-2} \otimes \text{RHom}(M_{i-2}, \Phi_{M_{i-1}}, M_i)[1] \oplus \cdots.
\]

In particular, each \(P_i\) either appears in a single cohomological degree, tensored with a free module, or in two consecutive degrees, tensored with the same direct summand of \(\text{End}(M_i)\).

In particular, we can compute the structure of the minimal resolutions from purely numerical data. Indeed, the object
\[
\Phi_{M_{j+1}} \cdots \Phi_{M_{i-1}} M_i
\]
is always a shift of a sheaf (by induction using Proposition 3.12), with the amount of shift monotone in \(j\) and increasing by at most 1 when we decrease \(j\). Moreover, the jumps occur precisely when the pair \((\text{rank}, \text{deg})\) switches to the other side of \((0, 0)\) in lexicographic ordering, so depend only on the image of \((\text{rank}(M_i), \text{deg}(M_i))\) under the relevant product of elements of \(\text{SL}_2(\mathbb{Z})\). The support of
\[
\text{RHom}(M_j, \Phi_{M_{j+1}} \cdots \Phi_{M_{i-1}} M_i)
\]
is then determined by adding 0, 1, or \([0, 1]\) to that shift, depending on how the two slopes compare.

The above structure of the minimal resolution has the following consequence, using the fact that \(\text{RHom}(P_i, S_j) = 0\) unless \(i = j\), when \(\text{RHom}(P_i, S_j) = \text{End}(M_j)\).

**Proposition 3.21.** One has
\[
\text{RHom}(S_i, S_j) \cong \begin{cases} 0, & i < j \\ \text{End}(M_i), & i = j \\ \text{RHom}_{\text{End}(M_j)}(\text{RHom}(M_j, \Phi_{M_{j+1}} \cdots \Phi_{M_{i-1}} M_i), \text{End}(M_j))[-1], & i > j. \end{cases}
\]

**Remark.** If we define a new sequence of (shifts of) divisorial sheaves by
\[
M'_i = \Phi_{M_0} \cdots \Phi_{M_{i-1}} M_i
\]
for \(i \geq 0\) and
\[
M'_i = \Phi_{M_{i-1}}^{-1} \cdots \Phi_{M_{i}}^{-1} M_i
\]
for \(i \leq 0\), then one has a non-canonical isomorphism
\[
\text{RHom}(M'_i, M'_j) \cong \text{RHom}_{\text{End}(M'_j)}(\text{RHom}(M'_j, M'_i), \text{End}(M'_j))[-1],
\]
which together with the canonical isomorphism

$$R \text{Hom}(M'_j, M'_j) \cong R \text{Hom}(M_j, \Phi_{M_{j+1}} \cdots \Phi_{M_{i-1}} M_i)$$

(3.89)

for $i \geq j$ produces an isomorphism

$$R \text{Hom}(S_i, S_j) \cong \begin{cases} 
0, & i < j \\
\text{End}(M'_i), & i = j \\
R \text{Hom}(M'_i, M'_j), & i > j.
\end{cases}$$

(3.90)

Presumably this quasi-isomorphism of complexes can be made compatible with composition. This is certainly true in any of the cases to which Theorem 3.19 above applies, when the dg-category with Hom complexes $R \text{Hom}(S_i, S_j)$ is essentially just a $\mathbb{Z}$-algebra version of the Koszul dual. The general conjecture is thus that if we relax $B_{M\bar{d}}$ to allow (cohomological) shifts of divisorial sheaves, then the Koszul dual of $B_{M\bar{d}}$ is a dg-category of the same form, and if $B_{M\bar{d}}$ is periodic up to derived autoequivalence (letting us turn the $\mathbb{Z}$-dg-algebra structure into a dg-algebra structure with an additional grading), then so is its Koszul dual.

**Example 3.1.** Consider the case $V = \mathcal{O}_E$, $\text{deg}(\mathcal{L}) = 1$ of our $B_{V, \Psi}$ construction. This is already not generated in degree 1, since $\Phi_{V,-1}(V_0)$ is a sheaf, the structure sheaf of a point. We do, however, get a short exact sequence

$$0 \to \Phi_{V,-2} \Phi_{V,-1}(V_0)[\pm 1] \to V_{-2} \oplus V_{-1} \to V_0 \to 0$$

(3.91)

and find that

$$\Phi_{V,-2} \Phi_{V,-1}(V_0)[\pm 1] \cong V_{-2} \oplus V_{-1}.$$  

(3.92)

In particular, it has slope $-3$, so the standard resolution requires us to add the generic morphism from $V_{-3}$ to $V_0$. The resulting complex

$$V_{-3} \to V_{-3} \oplus V_{-2} \oplus V_{-1} \to V_0$$

(3.93)

represents a shift of $\Phi_{V,-3} \Phi_{V,-2} \Phi_{V,-1} V_0$, and the corresponding complex of $\mathcal{B}$-modules represents the truncation of this module to degree $\geq 4$ (i.e., to the functor taking $V_i$ to 0 if $i > -4$). Apart from a shift by $-3$, this is numerically the same as the case we started with, and we thus obtain a free resolution of $k$ in the following periodic form:

$$\cdots \to [-12, -11, -10, -9] \to [-9, -8, -7, -6] \to [-6, -5, -4, -3] \to [-3, -2, -1] \to [0],$$

(3.94)

where $[i, j, \ldots, k]$ denotes $P_i \oplus P_j \oplus \cdots \oplus P_k$. If $\tau^2 = \text{id}$, then this is minimal, and otherwise the minimal resolution has shape

$$\cdots \to [-14, -13, -12] \to [-12, -11, -10] \to [-8, -7, -6] \to [-6, -5, -4] \to [-2, -1] \to [0].$$

(3.95)

Similarly, for the case $V = \mathcal{O}_E$, $\text{deg}(\mathcal{L}) = 2$, the resolution has shape

$$\cdots \to [-8, -7, -7, -6] \to [-6, -5, -5, -4] \to [-4, -3, -3, -2] \to [-2, -1, -1] \to [0] \to 0,$$

(3.96)

reducing to

$$\cdots \to [-9, -9, -8] \to [-8, -7, -7] \to [-5, -5, -4] \to [-4, -3, -3] \to [-1, -1] \to [0] \to 0$$

(3.97)

unless $\tau^2 = \text{id}$. For $\text{deg}(\mathcal{L}) = 3$, the resolution has shape

$$\cdots \to [-7, -7, -7, -6] \to [-6, -5, -5, -5]$$

$$\to [-4, -4, -4, -3] \to [-3, -2, -2, -2] \to [-1 -1, -1] \to [0] \to 0$$

(3.98)

and is always minimal.
An interesting common property of these three cases is that, in each case, if we omit the highest-degree relation, the resulting algebra still has a nice resolution, and in particular has the same Hilbert series as a free polynomial ring in the generators, with the highest-degree relation corresponding to a central element of this ring. (This will follow once we understand the filtered deformations. In each case, the graded algebra of deformation parameters is a free polynomial ring with one generator of the same degree as the highest-degree relation. So if we set the other deformation parameters to 0, we get a deformation in which the lower-degree relations are undeformed while the top-degree relation has a (nonzero) scalar added. The commutator of the deformed relation with any generator is thus expressible in terms of the undeformed relations, giving the requisite centrality statement, and letting us recover the Hilbert series.)

The other two cases in which the algebra fails to be Koszul have deg(L) = 1 and V a slope 0 divisorial bundle with rank 2 or 3. (This is generically a sum of line bundles of degree 0, except that when the line bundles become isomorphic, they should be replaced by the appropriate self-extension.) Here the free resolution has the same shape as in the previous two cases:

\[ \cdots \rightarrow [-8, -7, -7, -6] \rightarrow [-6, -5, -5, -4] \rightarrow [-4, -3, -3, -2] \rightarrow [-2, -1, -1] \rightarrow [0] \rightarrow 0 \] (3.99)

or

\[ \cdots \rightarrow [-7, -7, -7, -6] \rightarrow [-6, -5, -5, -5] \]
\[ \rightarrow [-4, -4, -4, -3] \rightarrow [-3, -2, -2, -2] \rightarrow [-1, -1, -1] \rightarrow [0] \rightarrow 0; \] (3.100)

the main difference is that minimality can fail in more (and more complicated) ways. Again, in both of these cases, there is a deformation with a single parameter of degree 4 or 3 as appropriate, thus expressing the algebra as a central quotient of another algebra with well-behaved Hilbert series.

One question suggested by the above arguments is: given that we can compute resolutions of \( B_{V, \Psi} \) purely in terms of sheaves on \( E \), can we do something similar for infinitesimal deformations? One could settle a number of cases of Conjecture 1 below (and in many ways the most interesting cases) if one could compute the negative degree part of \( \text{HH}^2( B_{V, \Psi}) \). Certainly, having an explicit resolution of \( B_{V, \Psi} \) makes it feasible to compute this group in special cases, but one would like to do so in general (or at least by hand).

4 Elliptic noncommutative del Pezzo surfaces

We have seen in Corollary 3.8 that any filtered deformation of an elliptic algebra \( B_{V, \Psi} \) is associated to a sheaf on the corresponding compactification (suitably defined) with no higher endomorphisms. This suggests a way to construct such deformations by starting with a suitable sheaf on a (noncommutative) surface. Since (by the same Corollary) the compactification is Gorenstein with dualizing sheaf \( \mathcal{O}(-1) \), this surface should be del Pezzo in a suitable sense. In general, of course, this would be a quite broad sense, since the cone over \( B_{V, \Psi} \) satisfies the same conditions, but this at least suggests that we should obtain a large class of deformations by taking more familiar-looking del Pezzo surfaces. Of course, if the del Pezzo surfaces are too familiar (i.e., commutative), then this will constrain \( \Psi \) to be the tensor product with a line bundle, and thus fail to explain why various properties of the moduli space of filtered deformations extend to more general \( \Psi \).

Luckily, there is a theory of noncommutative surfaces available [19], including a large subclass coming from elliptic curves [20]. This gives two classes of direct constructions, both of which are rather involved. Note that in contrast to the commutative theory, there is no direct geometric interpretation of a noncommutative surface; what one actually studies in general is a category of
(quasi-)coherent sheaves on such a surface (which in our case is a flat deformation of the category of (quasi-)coherent sheaves on a commutative surface). It turns out that the corresponding derived category is quite simple to describe. For simplicity, we will restrict our attention to coherent sheaves (and bounded complexes with coherent cohomology); in particular, here and below, all sheaves on noncommutative surfaces will be assumed coherent. Note that since the noncommutative surfaces we consider are Noetherian \cite{19} Prop. 6.1, coherent sheaves are always Noetherian.

In general, if $X$ is a smooth projective rational surface, then the structure sheaf $\mathcal{O}_X$ is exceptional, and thus determines a semiorthogonal decomposition $(\mathcal{O}_X^\perp, \mathcal{O}_X)$ of $\mathcal{D}^{\text{coh}}_X$. If $X$ has an anticanonical curve $Q$ (e.g., if $X$ is del Pezzo), with corresponding morphism $i : Q \to X$, then one has a functorial distinguished triangle

$$- \otimes \omega_X \to \text{id} \to i_* i^* \to$$

and thus in particular a distinguished triangle

$$R\text{Hom}(M, N \otimes \omega_X) \to R\text{Hom}(M, N) \to R\text{Hom}_Q(M|_Q, N|_Q) \to$$

for any objects $M, N \in D^{\text{coh}}_X$. (One caveat is that the natural transformation $- \otimes \omega_X \to \text{id}$ is only determined up to a scalar by the choice of $Q$, with that scalar corresponding to a choice of isomorphism $H^1(\mathcal{O}_E) \cong H^2(\omega_X) \cong k$.) In particular, if $R\text{Hom}(N, M) = 0$, then Serre duality gives $R\text{Hom}(M, N \otimes \omega_X) = 0$ and thus $R\text{Hom}(M, N) \cong R\text{Hom}_Q(M|_Q, N|_Q)$. As a special case, if $M \in \mathcal{O}_X$, then $R\text{Hom}(M, \mathcal{O}_X) \cong R\text{Hom}_Q(M|_Q, \mathcal{O}_Q)$.

The key observation is that this lets one reconstruct $D^{\text{coh}}_X$ from the (dg-)subcategory $\mathcal{O}_X^\perp$ and the restriction-to-$Q$ functor on that subcategory. Moreover, the dg-functor

$$M \mapsto R\text{Hom}_Q(M|_Q, \mathcal{O}_Q)$$

admits a natural flat deformation: given an invertible sheaf $q$ in the identity component of $\text{Pic}(X)$, we may define $D^{\text{coh}}_X(q)$ to be the dg-category obtained by “gluing” \cite{17} $\mathcal{O}_X^\perp$ to $\mathcal{O}_X$ via the functor

$$M \mapsto R\text{Hom}_Q(M|_Q, q) \cong R\text{Hom}_Q(M|_Q \otimes q^{-1}, \mathcal{O}_Q).$$

(That this is the derived category of a noncommutative surface is shown in \cite{19} Thm. 4.30.) Note that since the forward maps in the semiorthogonal decomposition come from $Q$, the functor $Li^*$ immediately extends to $D^{\text{coh}}_X(q)$, with $Li^* \mathcal{O}_X(q) \cong \mathcal{O}_Q$ and the restriction on $\mathcal{O}_X^\perp$ twisted by $q^{-1}$.

In particular, we see that the Grothendieck group of $D^{\text{coh}}_X(q)$ is independent of $q$ (and largely independent of $X$ itself); this lets one define the rank, first Chern class, and Euler characteristic of a general object (by taking the corresponding homomorphisms on $K_0(X)$). Moreover, since this is a flat deformation, the Mukai pairing $\chi([M], [N]) := \chi R\text{Hom}(M, N)$ is also independent of $q$, so that one immediately has an analogue of Hirzebruch-Riemann-Roch \cite{19} Cor. 7.13 for objects in $D^{\text{coh}}_X(q)$:

$$\chi(M, N) = -\text{rank}(M) \text{rank}(N) + \text{rank}(M) \chi(N) + \text{rank}(N) \chi(M) - c_1(M) \cdot (c_1(N) + \text{rank}(N)Q).$$

We also obtain Serre duality, i.e., the existence of a (canonical) dg-functor $S$ with a (suitably coherent) isomorphism

$$R\text{Hom}(M, N) \cong R\text{Hom}(R\text{Hom}(N, SM), k).$$

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the form action of $\phi$ sheaves to $Q$ where $f$ collection of the form.

Moreover, if we set first embedding $Q$ then the dg-category has the form

$$X \cong X_m \to X_{m-1} \to \cdots \to X_{-1} \cong \mathbb{P}^2$$

where each map $X_i \to X_{i-1}$ blows down a single $-1$-curve. (We take the sequence to end at $-1$ rather than 0 to be compatible with the general case discussed in [19], in which one must typically consider iterated blowups of ruled surfaces.) Then we have a full exceptional collection on $X$ of the form

$$(\mathcal{O}_{e_m}(-1), \mathcal{O}_{e_{m-1}}(-1), \ldots, \mathcal{O}_{e_0}(-1), \mathcal{O}_{p^2}(-2), \mathcal{O}_{p^2}(-1), \mathcal{O}_{p^2}),$$

where we omit the pullback functors from $X_i$ to $X_{i-1}$. (The only case not covered by iterated blowups of $\mathbb{P}^2$ is when $X$ is an even Hirzebruch surface $F_0$ or $F_2$, in which case we have a full exceptional collection of the form

$$(\mathcal{O}_X(-s - f), \mathcal{O}_X(-s), \mathcal{O}_X(-f), \mathcal{O}_X),$$

where $f$ is the divisor class of a fiber of the ruling, while $s$ is the unique divisor class satisfying $s \cdot f = 1, s^2 = 0$. The analogue of the following discussion for that case is straightforward.)

We then see that $D^{b}_{\text{coh}}(X_q)$ may be obtained from the sequence of restrictions of the above sheaves to $Q$: it is the category of perfect modules over a triangular dg-algebra obtained from

$$\text{REnd}(\mathcal{O}_{e_m}(-1)|_Q \oplus \mathcal{O}_{e_{m-1}}(-1)|_Q \oplus \cdots \oplus \mathcal{O}_{e_0}(-1)|_Q \oplus \mathcal{O}_{p^2}(-2)|_Q \otimes q^{-1} \oplus \mathcal{O}_{p^2}(-1)|_Q \otimes q^{-1} \oplus \mathcal{O}_Q)$$

by omitting all “backwards” morphisms and all nontrivial endomorphisms of summands. Since each summand is a simple coherent sheaf on $Q$, it is determined up to isomorphism by its class in $K_0(Q)$, and thus we see that $D^{b}_{\text{coh}}(X_q)$ is determined by the restriction morphism $K_0(X) \to K_0(Q)$. In fact, for any morphism $\phi : K_0(X) \to K_0(Q)$ satisfying

$$\text{rank}(\phi(M)) = \text{rank}(M), \quad \deg(\phi(M)) = c_1(M) \cdot Q, \quad \phi([\mathcal{O}_X]) = [\mathcal{O}_Q],$$

there is a corresponding sequence of simple modules on $Q$ and thus a corresponding dg-category. Moreover, if we set

$$q = \det(\phi([\mathcal{O}_{p^2}(-2)])) \otimes \det(\phi([\mathcal{O}_{p^2}(-1)]))^{-2},$$

then the dg-category has the form $D^{b}_{\text{coh}}(X_q)$. (Indeed, the surface $X$ is obtained from this data by first embedding $Q$ in $\mathbb{P}^2$ using the line bundle $(\det(\phi([\mathcal{O}_{p^2}(-2)])) \otimes q)^{-1} \in \text{Pic}^3(Q)$, then sequentially blowing up the points corresponding to $\det(\phi([\mathcal{O}_{e_i}(-1)])) \in \text{Pic}^1(Q)$.)

If we apply $\theta$ to the sum of exceptional objects before restricting to $Q$, we of course obtain the same endomorphism dg-algebra, and thus (since the summands generate $D^{b}_{\text{coh}}(Q)$) there is necessarily a derived autoequivalence $\xi$ of $Q$ such that $(\theta M)|_{Q} \cong \xi(M)|_{Q}$. Moreover, since the action of $\theta$ on $K_0(X_q)$ follows from the commutative case by flatness, we may deduce the action of
\(\xi\) on \(K_0(Q)\). We thus find that \(\xi\) has the form \(\Psi^{-1}\) where \(\Psi\) is an abelian autoequivalence of the kind considered above.

A choice of blowdown structure induces a basis the Néron-Severi lattice of \(X\) of the form \(h, e_0, \ldots, e_m\), with intersection form given by

\[
h^2 = 1, \quad h \cdot e_i = 0, \quad e_i \cdot e_j = -\delta_{ij},
\]

with \(Q = 3h - e_0 - \cdots - e_m = -K_X\). (In the even Hirzebruch case, this becomes \(s^2 = f^2 = 0\) and \(s \cdot f = 1\) with \(Q = 2s + 2f\).) Changing the blowdown structure by swapping two consecutive blowups acts on the basis as the reflection in \(e_i - e_{i+1}\), while a quadratic transformation in the first three points acts as the reflection in \(h - e_0 - e_1 - e_2\); in either case, the result corresponds to a blowdown structure iff the corresponding \(-2\)-class is ineffective.

One can show that any two blowdown structures are related by a sequence of such operations: see, e.g., [19, Prop. 8.10] for the analogous result for arbitrary blowups of noncommutative ruled surfaces. (Analogously, for even Hirzebruch surfaces, changing the marked ruling acts as the reflection in \(s - f\).)

These reflections are the simple reflections of a natural Coxeter group action. The point is that since \(Q^2 > 0\), the Hodge index theorem tells us that the lattice \(Q^\perp := \{v : v \in \Lambda | v \cdot Q = 0\}\) is negative definite relative to the intersection form, so negating the pairing gives a positive definite integral lattice, which is in particular acted on by reflections in any vectors of square norm 2. For \(Q^2 = 9 - m \leq 6\), the lattice is in fact generated by such vectors, and one recovers familiar root lattices:

\[
E_8, E_7, E_6, D_5, A_4, A_2A_1
\]

(of rank \(m = 9 - Q^2\). (For \(m = 2\), one obtains the unique rank 2 even lattice of determinant 7, while for \(m = 1\), the rank 1 lattice depends on the parity of the Hirzebruch surface \(X\).) In the resulting finite root system, we can take the positive roots to be those which are lexicographically positive, in which case the simple roots are precisely \(h - e_0 - e_1 - e_2, e_0 - e_1, \ldots\). One then finds that a simple root is effective iff its image in \(\text{Pic}^0(Q)\) is trivial.

Given this description of the derived category, we can then recover the actual category of coherent sheaves \(\text{coh}(X_q)\) from a suitable description of the \(t\)-structure. The simplest description involves first defining “line bundles”, inductively defined [19, Defn. 5.1] by starting with \(\mathcal{O}_{X_{-1,q}}(d)\) for \(d \in \{-2, -1, 0\}\) (i.e., the three terms in the semiorthogonal decomposition) and taking the closure under \(\theta^{\pm 1}\) and pullback from \(X_{l-1,q}\) to \(X_{l,q}\). (For the even Hirzebruch surface case, we note that the subcategories generated by the first two and last two exceptional sheaves in the above exceptional collection generate copies of \(D^b_{\text{coh}}(\mathbb{P}^1)\), and we take as our line bundles the images under powers of \(\theta\) of line bundles on \(\mathbb{P}^1\) in those two subcategories.) We caution that the resulting collection of sheaves depends on the choice of blowdown structure (and the choice of ruling when \(X \cong \mathbb{P}^1 \times \mathbb{P}^1\) in general. (The precise dependence is that we may only apply a reflection in a simple root when that simple root is ineffective on the noncommutative surface, which happens unless the corresponding class in \(\text{Pic}^0(Q)\) is a power of \(q\). In particular, each simple root is generically ineffective, but is effective over a dense collection of hypersurfaces in the moduli space of surfaces.)

Then [19, Lem. 5.7], an object \(M \in D^b_{\text{coh}}(X)\) is in \(D^b_{\text{coh}}(X)^{\geq 0}\) iff \(R\text{Hom}(L, M) \in D^b_{\text{coh}}(k)^{\geq 0}\) for every line bundle \(L\); this fully determines the \(t\)-structure, since a \(t\)-structure is determined by its nonnegative part. Moreover, line bundles are themselves in the heart of this \(t\)-structure, and in fact generate the heart. Furthermore, nonzero morphisms between line bundles are injective [19, Prop. 7.21]. The construction of [20] is via a representation of the category of line bundles in which the morphisms are expressed as elliptic difference operators, and such operators cannot
sheaves are necessarily torsion-free.) Furthermore, by the discussion following [19, Defn. 5.1], there is for any $D$ an abelian equivalence of the form $\text{coh}(X_q) \cong \text{coh}(X'_q)$ acting on $K_0$ as $-(D)$ and taking $\mathcal{O}_{X_q}(D')$ to $\mathcal{O}_{X'_q}(D' + D)$ for all $D'$. Note that since twisting by a line bundle changes the surface, there are far fewer graded algebras associated to a noncommutative surface than in the commutative case, and even fewer maps to noncommutative projective spaces.

In particular, we define an elliptic noncommutative (degenerate) del Pezzo surface of degree $d > 0$ to be an abelian category constructed from a map $K_0(X) \to K_0(Q)$ as above, where $Q$ is a smooth genus 1 curve and $K_0(X)$ is the Grothendieck group of a del Pezzo surface. (It will be convenient also to fix the scale of the natural transformation $\alpha : \theta \to \text{id.}$) When $q = \mathcal{O}_Q$, such a surface corresponds to a pair $(X, Q)$, where $X$ is a usual (smooth!) degenerate del Pezzo surface and $Q$ is a smooth anticanonical curve. Note that since these are analogues of smooth surfaces, they are generally not Fano; in the commutative case, the anticanonical embedding contracts $-2$-curves, and the situation is worse in the noncommutative case. (In general, if there is some commutative del Pezzo surface with a $-2$-curve of class $D$, then there is a pure 1-dimensional sheaf of class $[\mathcal{O}_D(l)]$ on $X_q$ iff $\phi[\mathcal{O}_D(l)] = 0$, in which case $Q$ fails to be ample. And we have already seen that this happens for a dense set of $q$!)

We may then use our understanding of the Grothendieck group to extend various concepts from the commutative case: a sheaf is torsion if it has rank 0 (note that every sheaf has nonnegative rank), a divisor class is effective if it is the first Chern class of a torsion sheaf (and [19, Prop. 7.19] if both $D$ and $-D$ are effective, then $D = 0$), a divisor class is nef if it is in the dual cone to the monoid of effective divisors, and ample if it is in the interior of the nef cone, with [19, Thm. 9.10] ample divisors satisfying

$$\text{Ext}^p(\mathcal{O}_X(-nD), M) = 0$$

for $p > 0$ and $\mathcal{O}_X(-nD) \otimes \text{Hom}(\mathcal{O}_X(-nD), M) \to M$ surjective for $n \gg 0$. Note that in our case, $Q$ is always nef: the effective monoid is generated by components of $Q$ along with those $-1$- and $-2$-classes which are effective [19, Thm. 9.2], and thus since $Q$ is irreducible with $Q^2 \geq 0$, it is nef.

Given an ample divisor, there is a corresponding notion of Hilbert polynomial ($p^D(M) := \chi(\mathcal{O}_X(-nD), M)$, which we note is 0 iff $M = 0$ and is otherwise positive for $n \gg 0$), which one may use to define (semi)stable sheaves. One then finds [20, Prop. 11.34] that the corresponding semistable moduli spaces are proper. (One caveat, however: although it is likely that semistable moduli spaces are projective in general, this has only been proved for rank $\leq 1$.)

In general, there are relatively few concepts for commutative surfaces that do not immediately carry over to these noncommutative surfaces. The most basic issue is that the support of a torsion sheaf is not well-defined in general. However, since the Chern class is well-defined, we can often work around this (e.g., replacing the condition that a sheaf is supported on an exceptional curve $e$ by the condition that its Chern class is a multiple of the divisor class $e$). A related issue is that noncommutative surfaces have relatively few points; indeed, on a typical noncommutative surface, the only 0-dimensional sheaves (i.e., with vanishing rank and first Chern class) are in the Serre subcategory generated by 0-dimensional sheaves on $Q$. We have already mentioned the dearth of embeddings in projective space, which breaks arguments involving taking hyperplane sections. In addition, the category of sheaves on a noncommutative surface is no longer a closed monoidal category (i.e., there is no internal Hom or tensor product).

Finally, and most significant for our purposes, although line bundles make sense on a noncommutative surface, there is no fully satisfying definition of vector bundles. For many purposes (and indeed below) it suffices to consider torsion-free sheaves. One could also consider reflexive sheaves, i.e., sheaves $M$ on $X_q$ such that $\text{ad}_q M$ is a sheaf on $X_{q-1}$. (As in the commutative case, such sheaves are necessarily torsion-free.) This has a number of pathologies in general (e.g., there can
be reflexive sheaves of rank 1, trivial Chern class, and arbitrarily small Euler characteristic), but for present purposes those can be ignored, as they do not arise for rigid reflexive sheaves (i.e., with $\text{Ext}^1(M, M) = 0$). In any event, although one can show that the sheaves we construct below are reflexive, we will not need this fact.

Note that when $M$ is torsion-free, $\alpha : \theta M \to M$ is injective, as the kernel is supported on $Q$, so torsion. In particular, we see that $M|_Q$ is a sheaf. (When $M$ is reflexive, we have $(\text{ad } M)|_Q \cong \mathcal{H}om(M|_Q, O_Q)$ and thus $M|_Q$ is a vector bundle.)

### 5 Filtered deformations from del Pezzo surfaces

We return to our considerations of filtered deformations. Let $X$ be a noncommutative elliptic del Pezzo surface, with chosen anticanonical curve $Q$, adjoint functors $i_* : \text{coh}(Q) \to \text{coh}(X)$, $i^* : \text{coh}(X) \to \text{coh}(Q)$, and a choice of natural transformation $\alpha : \theta \to \text{id}$ giving the usual functorial distinguished triangle

$$\theta \to \text{id} \to i_* \text{Li}^* \to .$$ (5.1)

Note that the choice of $\alpha$ can be encapsulated in the observation that there is a canonical natural transformation $\theta \otimes H^1(Q) \to \text{id}$. (In principle, one could develop much of the theory below even when $Q$ is singular, but we assume $Q$ smooth for simplicity. This, of course, is the only case that can produce filtered deformations of elliptic algebras, and is also (see Example [2.2 above] the only case in which we can hope to obtain all filtered deformations using such surfaces. In addition, the argument that the moduli stack is proper breaks down when $Q$ is singular, though the claim appears to still be valid; if so, one can obtain algebras in degenerate cases by taking limits.) Recall that $\theta^{-1}$ induces an autoequivalence of $\text{coh}(Q)$ of the form $\Psi$ considered above, with associated line bundle of degree $Q^2$.

By Corollary [33] we expect there to be a relation between filtered deformations on elliptic algebras and homogeneous endomorphism rings of sheaves on del Pezzo surfaces with no higher endomorphisms, and this is indeed the case.

**Proposition 5.1.** Let $(X, Q)$ be a noncommutative elliptic del Pezzo surface, let $M \in \text{coh}(X)$ be a torsion-free sheaf with $\text{Ext}^1(M, M) = \text{Ext}^2(M, M) = 0$ such that the slopes of $M|_Q$ are constrained to a half-open interval of length $Q^2$, and consider the graded algebra

$$A^+_M := \bigoplus_i \text{Hom}(M, \theta^{-i}M).$$ (5.2)

This algebra is nonnegatively graded, the element $\alpha_{\theta^{-1}M} : M \to \theta^{-1}M$ of degree 1 is regular and central, and the quotient by $\alpha_{\theta^{-1}M}$ is $B_{M|Q, \theta^{-1}}$.

**Proof.** That $\alpha_{\theta^{-1}M}$ is regular and central in $A^+_M$ follows immediately from the facts that it is injective and comes from a natural transformation. For each $i \in \mathbb{Z}$, $\alpha$ induces a distinguished triangle

$$R\text{Hom}(M, \theta^{1-i}M) \to R\text{Hom}(M, \theta^{-i}M) \to R\text{Hom}(M|_Q, \theta^{-i}M|_Q) \to .$$ (5.3)

For $i = 0$, we may use duality to compute $R\text{Hom}(M, \theta M)$, and thus (using the hypotheses on $M$) we obtain isomorphisms

$$\text{End}(M) \cong \text{End}(M|_Q), \quad \text{Ext}^2(M, \theta M) \cong \text{Ext}^1(M|_Q, M|_Q).$$ (5.4)

Moreover, since $\alpha_{\theta^{-1}M}$ is injective for all $i$, the assumption that $\text{Ext}^2(M, M) = 0$ implies that $\text{Ext}^2(M, \theta^i M) = 0$ for all $i \leq 0$, and thus that $\text{Hom}(M, \theta^i M) = 0$ for $i \geq 1$. In addition, the
condition on $M|_Q$ ensures that $\Ext^1(M|_Q, \theta^i M|_Q) = 0$ for $i > 0$. We thus inductively find that $\Ext^1(M, \theta^i M) = 0$ for $i \leq 0$, giving us the desired short exact sequence

$$0 \to \Hom(M, \theta^{i-1} M) \to \Hom(M, \theta^{-1} M) \to \Hom(M|_Q, \theta^{-i} M|_Q) \to 0 \quad (5.5)$$

for all $i$.

\[ \square \]

Remark. Of course, the quotient by $\alpha_{\theta^{-1}M}$ is independent of the choice of scaling on $\alpha_{\theta^{-1}M}$. In addition, although $\theta^{-1}$ appears in the definition of $A^+_M$, we can safely omit it from the notation because $\theta$ is canonical (in both senses of the word).

Note that if we instead quotient by $\alpha_{\theta^{-1}M} - 1$, then the result will be a filtered algebra, and the Proposition tells us that the associated graded algebra is $B_M|_Q, \theta^{-1}$. In other words, the quotient is precisely a filtered deformation of the type we are looking for. The filtered deformation depends only mildly on the choice of scale on $\alpha$: the action of $\G_m$ by rescaling $\alpha$ agrees with the standard action on filtered deformations.

We are thus left with the question of classifying such sheaves. In the commutative case, sheaves with $\Ext^1(M, M) = 0$ on degenerate del Pezzo surfaces were studied in [13]: though the classification was only partial and some of the arguments do not apply in the noncommutative setting, we can use the ideas therein with some supplements to get a fairly complete understanding in general.

A useful simplification comes from the following fact.

**Proposition 5.2.** Suppose $M, N$ are sheaves on a noncommutative del Pezzo surface $(X, Q)$ such that $\Ext^1(N, M) = 0$ and $N$ is torsion-free. Then $\Hom(M, N) \to \Hom(M|_Q, N|_Q)$ is surjective.

**Proof.** We have the exact sequence

$$0 \to \Hom(M, \theta N) \to \Hom(M, N) \to \Hom(M|_Q, N|_Q) \to \Ext^1(M, \theta N), \quad (5.6)$$

but duality gives

$$\Ext^1(M, \theta N) \cong \Ext^1(N, M)^* = 0. \quad (5.7)$$

\[ \square \]

**Corollary 5.3.** If $M$ is a torsion-free sheaf with $\Ext^1(M, M) = 0$, then $\End(M) \to \End(M|_Q)$ is surjective, and any idempotent in $\End(M|_Q)$ lifts to $\End(M)$.

**Proof.** Surjectivity is just the case $N = M$ above, and thus $\End(M|_Q)$ is the quotient of $\End(M)$ by the ideal $\alpha_M \circ \Hom(M, \theta M)$. Since idempotents lift through nil ideals, it remains to show that any element $\beta$ of the ideal is nilpotent.

Since $X$ is Noetherian and $M$ is coherent, the ascending chain $\ker(\beta^i)$ of subsheaves is eventually constant, so that the descending chain $\text{im}(\beta^i) \subseteq M$ is also eventually constant, stabilizing at $N$. Since $\beta|_N$ is an automorphism and $\beta$ factors through $\alpha_M$, we see that $\alpha_N$ must be surjective, and thus $\text{rank}(N) = 0$. But $M$ is torsion-free, and thus $N = 0$, so that $\beta$ is nilpotent as required.

In general, we can write the sheaf $M|_Q$ as a direct sum $M|_Q = \bigoplus_{\mu \in Q \cup \{\infty\}} V_\mu$ where each $V_\mu$ is a semistable sheaf of slope $\mu$. By the Corollary, when $M$ is rigid and torsion-free, this direct sum decomposition lifts (nonuniquely!) to $M$ and thus $M$ itself is a sum of torsion-free sheaves $M_\mu$ such that $(M_\mu)|_Q = V_\mu$. (Note that $\text{rank}(M_\infty) = \text{rank}(V_\infty) = 0$, and thus $M_\infty = 0$.) And, of course, each $M_\mu$ itself splits as a direct sum of sheaves with indecomposable restriction to $Q$.

The key observation of [13] is that such a summand $M_\mu$ is (slope) semistable in a suitable sense.
Proposition 5.4. Let \( M \) be a torsion-free sheaf on \( X \) such that \( M|_Q \) is semistable. Then any nontrivial subsheaf \( N \subset M \) satisfies
\[
\frac{c_1(N) \cdot Q}{\text{rank}(N)} \leq \frac{c_1(M) \cdot Q}{\text{rank}(M)},
\]
(5.8)

If \( M|_Q \) is stable, equality implies \( N = M \).

Proof. We may assume that \( M/N \) is torsion-free, as otherwise replacing \( N \) by the saturated subsheaf can only increase \( c_1(N) \cdot Q \) without increasing \( \text{rank}(N) \). (Here we use the fact that \( Q \) is nef!) But then (since \( (M/N)|_Q \) is a sheaf) we have a short exact sequence
\[
0 \to N|_Q \to M|_Q \to (M/N)|_Q \to 0
\]
(5.9)

and thus by assumption have \( \mu(N|_Q) \leq \mu(M|_Q) \). The first claim then follows by observing that \( \text{rank}(N|_Q) = \text{rank}(N) \) and \( \deg(N|_Q) = c_1(N) \cdot Q \). Moreover, stability of \( M|_Q \) implies that \( \mu(M|_Q) \) has denominator \( \text{rank}(M|_Q) \), and thus \( \mu(N|_Q) = \mu(M|_Q) \) only if \( \text{rank}(N) = \text{rank}(M) \), only if \( N = M \). \( \square \)

Define \( \mu_Q(M) := (c_1(M) \cdot Q)/\text{rank}(M) = \mu(M|_Q) \), and say that the torsion-free sheaf \( M \) is \( \mu_Q \)-semistable if \( \mu_Q(N) \leq \mu_Q(M) \) for nonzero \( N \subset M \) (and similarly for \( \mu_Q \)-stable). In particular, we have just shown that (semi)stability of \( M|_Q \) implies \( \mu_Q \)-(semi)stability of \( M \).

Corollary 5.5. If \( M, N \) are \( \mu_Q \)-semistable torsion-free sheaves, then \( \text{Hom}(M, N) = 0 \) if \( \mu_Q(M) > \mu_Q(N) \) and \( \text{Ext}^2(M, N) = 0 \) if \( \mu_Q(M) < \mu_Q(N) + Q^2 \) (and in particular if \( M = N \)).

Proof. If \( \text{Hom}(M, N) \neq 0 \), then the image \( I \) of such a homomorphism satisfies
\[
\mu_Q(M) \leq \mu_Q(I) \leq \mu_Q(N),
\]
(5.10)

since it is a quotient of \( M \) and a subsheaf of \( N \). By duality, if \( \text{Ext}^2(M, N) \neq 0 \), then \( \text{Hom}(N, \theta M) \neq 0 \), and thus \( \mu_Q(N) \leq \mu_Q(\theta M) = \mu_Q(M) - Q^2 \). \( \square \)

Recall that an exceptional collection in a triangulated category is a sequence \( (E_1, \ldots, E_n) \) of objects such that \( R\text{Hom}(E_i, E_j) \cong k \) and \( R\text{Hom}(E_i, E_j) = 0 \) for \( i > j \). In the commutative case, the following is a special case of [13 Lem. 2.4.4]. (The proof below is somewhat more elementary, as we do not need to use the spectral sequence for \( \text{Ext} \) of filtered sheaves.)

Proposition 5.6. Let \( M \) be a rigid torsion-free sheaf with \( M|_Q \) indecomposable. Then there is an induced filtration \( 0 = F_n \subset F_{n-1} \subset \cdots \subset F_0 = M \) with subquotients \( E_i = F_{i-1}/F_i \) such that \( (E_1, \ldots, E_n) \) is an exceptional collection.

Proof. By assumption, \( \text{Ext}^1(M, M) = 0 \), and slope considerations imply \( \text{Ext}^2(M, M) = 0 \), so that \( \text{End}(M) \cong \text{End}_Q(M|_Q) \cong k[t]/(t^n) \) for some \( n \). We may thus induce a filtration on \( M \) by taking \( F_i = \text{ker}(t^{n-i}) \), and see that the restriction of this filtration to \( Q \) has the same description. In particular, \( \text{rank}(F_i) = (1 - i/n) \text{rank}(M) \), and thus each \( E_i|_Q \) is isomorphic to the unique stable constituent \( V \) of \( M|_Q \). The action of \( t \) certainly respects this filtration, and thus has an induced action on the associated graded. Since \( t^{n-1-i} \) induces an injection \( E_i \to E_n \), we see that \( t : E_i \to E_{i+1} \) is injective for all \( i \), giving a chain \( E_1 \subset \cdots \subset E_n \) of sheaves restricting to the same stable bundle \( V \) on \( Q \).

Slope considerations again tell us that \( \text{Ext}^2(E_i, E_j) = 0 \) for all \( i, j \), and thus \( \text{Hom}(E_i, E_j) \to \text{End}(V) \cong k \) is injective for all \( i, j \). Moreover, the natural map \( \text{Ext}^1(E_i, E_{i+1}) \to \text{Ext}^1(V, V) \cong k \)
is nonzero (since the corresponding subquotient of $M|Q$ is indecomposable), and thus surjective, implying that $\text{Ext}^2(E_i, \theta E_{i+1}) \to \text{Ext}^2(E_i, E_{i+1}) = 0$ is injective. We thus conclude that $\text{Hom}(E_{i+1}, E_i) = 0$ so that $\text{Hom}(E_i, E_j) = 0$ for all $i > j$, and thus (reversing the above argument) $\text{Ext}^1(E_i, E_j) \to \text{Ext}^1(V, V)$ is surjective for $i < j$.

We thus conclude that the Ext spaces satisfy the inequalities

$$\dim \text{Hom}(E_i, E_j) = \delta_{i,j}$$
$$\dim \text{Ext}^1(E_i, E_j) \geq \delta_{i,j}$$
$$\dim \text{Ext}^2(E_i, E_j) = 0$$

and thus

$$\chi(E_i, E_j) \leq \delta_{i,j} - \delta_{i,j} = \delta_{i,j}.$$  

(5.11)

Since

$$\chi(M, M) = \sum_{i,j} \chi(E_i, E_j) \leq n = \chi(M, M),$$

(5.12)

the inequalities must be tight, and thus in particular $(E_1, \ldots, E_n)$ is an exceptional collection as required.

\[ \square \]

**Proposition 5.7.** If $M, N$ are sheaves with $\mu_Q(M) = \mu_Q(N) \in \mathbb{Q} \cup \infty$, then $\chi(M, N) = \chi(N, M)$. In particular, if $E$ and $F$ are exceptional sheaves of the same slope, then $\chi(E, F) = \chi(F, E) = 1 + \rho^2/2 \leq 1$, where $\rho = c_1(E) - c_1(F)$.

**Proof.** From the distinguished triangle

$$R\text{Hom}(M, \theta N) \to R\text{Hom}(M, N) \to R\text{Hom}_Q(M|Q, N|Q) \to,$$

(5.13)

we obtain the identity

$$\chi_Q(M|Q, N|Q) = \chi(M, N) - \chi(M, \theta N).$$

(5.14)

Serre duality gives $\chi(M, \theta N) = \chi(N, M)$, and the fact that $\chi_Q$ is a symplectic pairing on $K_0(Q)$ implies $\chi_Q(M|Q, N|Q) = 0$ and thus $\chi(M, N) = \chi(N, M)$.

In the exceptional case, we have

$$-\rho^2 = \chi([M] - [N], [M] - [N]) = \chi(M, M) + \chi(N, N) - 2\chi(M, N) = 2 - 2\chi(M, N)$$

(5.15)

and solving for $\chi(M, N)$ gives the desired result. The inequality then follows from the fact that $\rho$ is in the negative definite even lattice of divisors orthogonal to $Q$. \[ \square \]

**Lemma 5.8.** Let $M_1 \oplus M_2$ be a rigid torsion-free sheaf such that $M_1|Q$, $M_2|Q$ are indecomposable bundles with isomorphic stable constituents. If $\text{rank}(M_1) \geq \text{rank}(M_2)$, then (up to isomorphism) the exceptional collection associated to $M_1$ contains that associated to $M_2$.

**Proof.** As usual, slope considerations imply that $\text{Ext}^2(M_i, M_j) = 0$. Let the two exceptional collections be $(E_1, \ldots, E_{n_1})$ and $(E'_1, \ldots, E'_{n_2})$, with $n_1 \geq n_2$. Then $\chi(E_i, E'_j) \leq \delta_{E_i \cong E'_j}$, and thus

$$n_2 = \dim \text{Hom}(M_1, M_2) = \chi(M_1, M_2) = \sum_{i,j} \chi(E_i, E'_j) \leq \sum_{i,j} \delta_{E_i \cong E'_j} \leq n_2,$$

(5.16)

since each $E'_j$ is isomorphic to at most one $E_i$. It follows that both inequalities are tight, and in particular that every $E'_j$ is isomorphic to some $E_i$ as required. \[ \square \]
Lemma 5.9. Let $M_1 \oplus M_2$ be a rigid torsion-free sheaf such that $M_1|_Q$ and $M_2|_Q$ are indecomposable bundles with $\mu(M_1|_Q) \leq \mu(M_2|_Q) < \mu(M_1|_Q) + Q^2$. Then the exceptional collections associated to $M_1$ and $M_2$ are contained in a common exceptional collection.

Proof. If $M_1|_Q, M_2|_Q$ have isomorphic stable constituents, then this follows from the previous Lemma. Otherwise, let the two exceptional collections again be $(E_1, \ldots, E_{m_1})$ and $(E_1', \ldots, E_{m_2})$. Then we claim that $(E_1, \ldots, E_n, E_1', \ldots, E_{n_2})$ is an exceptional collection.

The slope constraint implies that $\text{Hom}(M_2, M_1) = \text{Ext}^2(M_2, M_1) = 0$ and (since they have the same slopes) $\text{Hom}(E_i', E_j) = \text{Ext}^2(E_i', E_j) = 0$ for all $i, j$. By hypothesis, $\text{Ext}^1(M_2, M_1) = 0$, so that $\chi(M_2, M_1) = 0$. We then find that

$$0 = \chi(M_2, M_1) = \sum_{i,j} \chi(E_i', E_j) \leq 0$$

(5.17)

and thus $\chi(E_i', E_j) = 0$ for all $i, j$, implying $\text{Ext}^1(E_i', E_j) = 0$ as required. \hfill \square

Corollary 5.10. Let $M$ be a rigid torsion-free sheaf with slopes contained in a half-open interval of length $Q^2$. Then there is an exceptional collection $(E_1, \ldots, E_n)$ and a filtration of $M$ such that the $i$th subquotient has the form $n_iE_i$.

Proof. The two Lemmas tell us that any two indecomposable summands of $M$ have compatible exceptional collections, and thus we immediately conclude that there is a single exceptional collection containing all of them; moreover, that exceptional collection can be split into blocks associated to the different stable constituents of $M|_Q$, and the existence of a filtration as described for each such block is straightforward. \hfill \square

Remark. Since the hypothesis ensures that $\text{Ext}^2(M, M) = 0$, this is (in the commutative case) a special case of [13, Thm. 2.5.1(2)]. The argument there can be extended to the noncommutative case, but we will not need this generalization, so omit the details.

Of course, the above is in a sense the converse of the result we want: we want to know when a given exceptional collection corresponds to a rigid, unobstructed torsion-free sheaf. In the indecomposable case, this is straightforward.

Proposition 5.11. Suppose $(E_1, \ldots, E_n)$ is an exceptional collection on a noncommutative del Pezzo surface, such that each $E_i$ is a torsion-free sheaf and the restrictions to $Q$ are all isomorphic. Then $\dim \text{Hom}(E_i, E_j) = \dim \text{Ext}^1(E_i, E_j) = 1$ for $i < j$, and there is (up to isomorphism) a unique filtered sheaf $0 = F_n \subset \cdots \subset F_0 = M$ with $F_{i-1}/F_i \cong E_i$, $\text{Ext}^1(M, M) = 0$, and $M|_Q$ indecomposable.

Proof. For $i < j$, since $R\text{Hom}(E_j, E_i) = 0$, we have $R\text{Hom}(E_i, E_j) \cong R\text{Hom}(E_i|_Q, E_j|_Q)$, and the sheaves on $Q$ are isomorphic stable sheaves. We then define a sequence $F_i$ by letting $F_{i-1}$ be the generic extension of $E_i$ by $F_i$, giving a sheaf $M$. That this satisfies $\text{Ext}^1(F_i, F_j) = 0$ for all $i$ follows as in [13], giving existence. Moreover, the only way to obtain a nonisomorphic sheaf with the same subquotients is to take a nongeneric extension in some step, but since $\text{Ext}^1(E_i, F_i) \cong \text{Ext}^1(E_i|_Q, F_i|_Q) \cong k$ at the first such step, this would make $F_0|_Q$ decomposable. \hfill \square

Theorem 5.12. Let $(E_1, \ldots, E_n)$ be an exceptional collection of torsion-free sheaves, with $\mu_Q(E_i)$ constant. Then for any sequence $(m_1, \ldots, m_n)$ of nonnegative integers, there is a unique rigid sheaf $M$ having a filtration with subquotients $E_i^{m_i}$.
Proof. For existence, note that for any subset $S \subset \{1, \ldots, n\}$, there is a corresponding sheaf $M_S$ obtained as in Proposition 5.11; at each step, take the generic extension. We then find as above that

$$\chi(M_S, M_T) = |S \cap T|. \quad (5.18)$$

Since $\text{Ext}^2(M_T, M_S) = 0$, we find that $\dim \text{Hom}(M_S, M_T) \leq \dim \text{Hom}(M_S|Q, M_T|Q)$. If $S \subset T$ or $T \subset S$, then $\dim(M_S|Q, M_T|Q) = |S \cap T|$, and

$$|S \cap T| = \chi(M_S, M_T) \leq \dim \text{Hom}(M_S, M_T) \leq |S \cap T|, \quad (5.19)$$

forcing $\dim \text{Ext}^1(M_S, M_T) = 0$ and $\dim \text{Hom}(M_S, M_T) = |S \cap T|$. Existence then follows by taking the sum $\sum_i c_i S_i$ for a suitable chain of subsets and multiplicities $c_i$. (The chain of subsets is determined by a permutation that sorts the sequence $\bar{m}$, and the $c_i$ are then differences of consecutive entries in the sorted sequence. This permutation need not be unique, but the sequence $c_i$ is independent of the choice of permutation, and when $c_i \neq 0$, $S_i$ is also uniquely determined.)

For uniqueness, we may use the fact that $M$ decomposes as a direct sum over different isomorphism classes of stable constituents to reduce to the case that $E_1|Q \cong \cdots \cong E_n|Q$, at which point we conclude from Lemma 5.8 that if both $M_S$ and $M_T$ appear, then $S$ and $T$ are comparable. □

Remark. There is a slight additional technical issue if we want to perform this construction in families, namely the fact that over a field, we are taking generic extensions at various points in the process, and thus run into potential problems in constructing a global extension class that is nonzero whenever possible. Note that since the relevant $\text{Ext}^1$ spaces are preserved by restriction to $Q$, it suffices to show that one can define $M_S|Q$ globally for each $S$. Since this sheaf is semistable, applying a suitable derived autoequivalence of $D^b_{\text{coh}}(E)$ turns it into a torsion sheaf (3.9 above), the fibers of which are isomorphic to structures sheaves of divisors. But such a torsion sheaf can certainly be defined globally: simply view the family of divisors as a divisor on the total space of the family and take its structure sheaf. Inverting the derived autoequivalence gives a global version of $M_S|Q$ as required, and thus the required global extension classes needed to construct $M_S$ itself.

More generally, if we are given an exceptional collection consisting of torsion-free sheaves with nondecreasing $Q$-slopes contained in a half-open interval of length $Q^2$, and for each $\mu$ we construct a rigid sheaf $M_\mu$ from the corresponding subcollection, then $M = \bigoplus \mu M_\mu$ is rigid and unobstructed. Indeed, we have $R\text{Hom}(M_\mu, M_\nu) = 0$ for $\mu < \nu$, and thus $R\text{Hom}(M_\mu, M_\nu) \cong R\text{Hom}(M_\mu|Q, M_\nu|Q) \cong \text{Hom}(M_\mu|Q, M_\nu|Q)$, so that in neither direction are there higher morphisms.

Thus the question of classifying filtered deformations arising from Proposition 5.1 reduces to one of classifying exceptional collections. The basic question here is: given a sequence of classes in $K_0(X)$ satisfying the obvious numerical conditions, is there a corresponding exceptional collection, and is it unique? Unfortunately, the main result of [13] on exceptional collections is not particularly useful in this regard: it says that any exceptional collection can be obtained from a certain standard collection via the braid group action, but it is unclear how to turn that into an existence test, and (worse yet) the arguments depend in a crucial way on the commutativity of the surface. (They involve restriction functors to $D^b_{\text{coh}}(\mathbb{P}^1)$ coming from $-1$-curves.)

Although we do not have a complete answer, we can reduce the problem considerably.

**Proposition 5.13.** Suppose $(E_1, \ldots, E_n)$ is a sequence of exceptional torsion-free sheaves such that $\mu_Q(E_1) \leq \mu_Q(E_2) \leq \cdots \leq \mu_Q(E_n) < \mu_Q(E_1) + Q^2$ and $\chi(E_i, E_i) = 0$ for $i < j$; suppose moreover that if $\mu_Q(E_i) = \mu_Q(E_j)$ for $i < j$, then there is no sheaf of class $[E_i] - [E_j]$. Then $(E_1, \ldots, E_n)$ is an exceptional collection.
Proof. Let $i < j$. Since $\mu_Q(E_j) < \mu_Q(E_i) + Q^2$, we find $\text{Ext}^2(E_j, E_i) = 0$. Since $\chi(E_j, E_i) = 0$, it remains only to show that $\text{Hom}(E_j, E_i) = 0$. If $\mu_Q(E_i) < \mu_Q(E_j)$, this follows from stability, while if the slopes agree, any morphism must be injective, and thus the cokernel is a sheaf of the given class in $K_0(X)$. \qed

Remark. Note that as observed above, when the slopes agree, $c_1(E_i) - c_1(E_j)$ is a root of the lattice $Q^\perp \sim E_{9-Q^2}$. It follows that there is a sheaf of class $[E_i] - [E_j]$ iff $c_1(E_i) - c_1(E_j)$ is a positive root and $[E_i]_Q \cong [E_j]_Q$. In particular, if we arrange for $c_1(E_j) - c_1(E_i)$ to be a positive root for $i < j$, then $[E_i] - [E_j]$ can never be effective.

Uniqueness is even easier.

Proposition 5.14. If $E$, $F$ are exceptional torsion-free sheaves with $[E] = [F] \in K_0(X)$, then $E \cong F$.

Proof. Since $\chi(E, E) = \chi(F, F) = 1$, we have $\chi(E, F) = \chi(F, E) = 1$. Since $E$ is rigid and unobstructed, $k \cong \text{End}(E) \cong \text{End}_Q(E|Q)$, and thus $E|Q$ is stable, so that $E$ is $\mu_Q$-stable. Slope considerations imply that $\text{Ext}^2(E, F) = \text{Ext}^2(F, E) = 0$ and thus there are nonzero morphisms in either direction. Stability forces the morphisms to be injective, and thus so is the composition in either direction, which must therefore be a nonzero multiple of the identity, implying $E \cong F$ as required. \qed

Thus to understand the exceptional collections corresponding to filtered deformations, it suffices to understand (a) which sequences of classes in $K_0(X)$ satisfy the numerical conditions (and positivity of roots, when relevant) and (b) when a given class in $K_0(X)$ is represented by an exceptional torsion-free sheaf.

Lemma 5.15. Let $M$ be a $\mu_Q$-stable torsion-free sheaf, and let $D \in \text{NS}(X)$ be an ample divisor class. Then for all sufficiently large $n$, $M$ is $nQ + D$-stable.

Proof. Suppose otherwise, and for each $n \geq 0$, let $B_n$ be the maximally destabilizing subsheaf of $M$. Since $M$ is $\mu_Q$-stable, we must have $\mu_Q(B_n) < \mu_Q(M)$, and thus there is some $n' > n$ such that

$$n'\mu_Q(B_n) + \mu_D(B_n) < n'\mu_Q(M) + \mu_D(M),$$

(5.20)

and thus in particular

$$n'\mu_Q(B_{n'}) + \mu_D(B_{n'}) > n'\mu_Q(B_n) + \mu_D(B_n)$$

(5.21)

Since $B_n$ is maximally destabilizing for $nQ + D$, we also have

$$n\mu_Q(B_{n'}) + \mu_D(B_{n'}) \leq n\mu_Q(B_n) + \mu_D(B_n),$$

(5.22)

and thus $\mu_Q(B_n) < \mu_Q(B_{n'})$. We thus obtain an infinite sequence of strictly increasing slopes bounded by $\mu_Q(M)$, but since the denominators are bounded by $\text{rank}(M)$, this is impossible. \qed

Remark. If $M$ is only assumed to be $\mu_Q$-semistable, an analogous argument shows that either $M$ is still $nQ + D$-stable for $n \gg 0$ or $B_n$ eventually stabilizes, and thus by induction the corresponding Harder-Narasimhan filtration of $M$ eventually stabilizes.

Proposition 5.16. Let $R$ be a discrete valuation ring with residue field $k$ and field of fractions $K$, and let $X_R$ be a noncommutative del Pezzo surface over $R$. An exceptional torsion-free sheaf on either fiber of $X_R$ extends to a torsion-free sheaf which is exceptional on the other fiber.
Proof. An exceptional sheaf $E_k$ on $X_k$ is simple, so induces a point of the moduli space of simple sheaves. (It is shown in [19, Cor. 11.3] that this moduli problem is represented by an algebraic space, with Prop. 11.1 op. cit. implying that the obstruction theory is as expected.) Since $\text{Ext}^2(E_k, E_k) = 0$, that algebraic space is formally smooth (and thus smooth) at $E_k$, so that the point corresponding to $E_k$ extends to $R$. Since $\gcd(c_1(E) \cdot Q, \text{rank}(E)) = \gcd(\deg(E|_Q), \text{rank}(E|_Q)) = 1$, there is a universal family. (The moduli stack of simple sheaves is a $\mathbb{G}_m$-gerbe over the moduli space of simple sheaves, so that the obstruction to the existence of a universal family is a class in the Brauer group. For any sufficiently ample $D$, this class is represented by the Azumaya algebra $\text{End}(\text{Hom}(O_X(-D), E))$, and the gcd condition ensures that the degrees of those Azumaya algebras generate $\mathbb{Z}$, so the obstruction is trivial.) Thus this point corresponds to a simple sheaf $E_R$, which has exceptional generic fiber by semicontinuity of $\text{Ext}^i$.

For the other direction, fix an ample divisor $D$ on $X_k$, so that an exceptional torsion-free sheaf $E_K$ on $X_K$ is $nQ + rD$-stable for some $n$, with $r = \text{rank}(E_K)$. Since the semistable moduli functor is universally closed, there is thus an extension to a sheaf $E_R$ with $nQ + rD$-semistable special fiber $E_k$. If $\gcd(n, r) = 1$, then the corresponding slope of $E_K$ has denominator $r$, and thus cannot be the slope of a sheaf of smaller rank, so that $E_k$ is actually stable. It follows that $\text{End}(E_k) \cong k$, and thus that $\text{Hom}(E_k, \theta E_k) = 0$, giving $\text{Ext}^2(E_k, E_k) = 0$. Since $\chi(E_k, E_k) = \chi(E_K, E_K) = 1$, it follows that $E_k$ is exceptional as required.

It follows immediately that, given a class $[E] \in K_0(X)$ such that $\text{rank}([E]) > 0$ and $\chi([E], [E]) = 1$, if any noncommutative del Pezzo surface (with the correct value of $Q^2$ and parity if $Q^2 = 8$; i.e., in a given component of the moduli stack of noncommutative surfaces) has an exceptional torsion-free sheaf of class $[E]$, then every noncommutative del Pezzo surface has such a sheaf. (Indeed, every $X$ can be obtained from a finite base change of the generic surface by a descending down a sequence of valuations of the corresponding field, and the existence of a universal family lets us descend through finite field extensions.) We thus in particular see that the existence question reduces to the case of commutative del Pezzo surfaces, which may even be assumed nondegenerate.

In fact, it reduces to the case of (commutative, nondegenerate) del Pezzo surfaces of degree 1. Let $\pi : Y \to X$ be a birational morphism with $Y$ a nondegenerate commutative del Pezzo surface of degree 1. If $[E] \in K_0(X)$ is the class of an exceptional bundle $E$, then $\pi^*[E]$ is the class of the exceptional bundle $\pi^*E$. Conversely, if $\pi^*[E]$ is the class of an exceptional bundle $E'$, then it descends to an exceptional bundle on $X$ by [14, Cor 3.2].

A fairly strong necessary condition for $[E]$ to be representable comes from the following.

Lemma 5.17. If $E$ is an exceptional bundle on $X$, then for any exceptional bundle $E'$ with $\mu(E') - Q^2 < \mu(E) < \mu(E')$, $\chi(E', E) \leq 0$.

Proof. Indeed, the conditions on the slopes ensure that $\text{Hom}(E', E) = \text{Ext}^2(E', E) = 0$ and thus $\chi(E', E) = -\text{dim Ext}^1(E', E) \leq 0$.

Since the twist of an exceptional bundle by a line bundle remains exceptional, any given $E'$ induces a whole family of necessary conditions: $\chi(E'(-D), [E]) \leq 0$ for all $D \in Q^\perp = \{v \in \text{NS}(X)|v \cdot Q = 0\}$. Since (letting $r = \text{rank}([E])$, $r' = \text{rank}(E')$)

$$\chi(E'(-D), [E]) - \frac{rr'}{2}(D - (D_{E'}/r' - D_E/r))^2$$

is independent of $D$, with $D_E := c_1(E) - \frac{\chi(E, Q)}{Q^2}Q$, we see that $\chi(E'(-D), [E])$ is maximized when $D$ is the closest vector (relative to the negative of the intersection form) in $Q^\perp$ to $D_{E'}/r' - D_E/r$; if the maximum is positive, then $[E]$ is not representable. The resulting condition is particularly
This works equally well if $E'$ ranges over line bundles, and is already quite strong in that case.

It turns out it is useful to reformulate the question in terms of one on the elliptic surface $\tilde{X}$ associated to the nondegenerate degree 1 del Pezzo surface $X$. (Note that nondegeneracy of $X$ corresponds to smoothness of the Weierstrass model of $\tilde{X}$.) Here, the representability condition turns out to be essentially trivial.

**Proposition 5.18.** Let $\tilde{X}$ be a smooth rational Weierstrass elliptic surface, and let $[E] \in K_0(\tilde{X})$ be a class with $\operatorname{rank}([E]) > 0$, $\chi([E], [E]) = 1$, and $\gcd(\operatorname{rank}([E]), c_1([E]) \cdot Q) = 1$ (where $Q$ denotes the divisor class of a fiber of $\tilde{X}$). Then $[E]$ is the class of an exceptional vector bundle on $\tilde{X}$, and any two bundles of class $[E]$ are isomorphic.

**Proof.** The derived autoequivalences of elliptic curves extend in a straightforward way to any Weierstrass curve, and thus in particular we can apply them to $\tilde{X}$, viewed as a family of such curves. Those derived autoequivalences act as $\operatorname{SL}_2(\mathbb{Z})$ on $(\operatorname{rank}(M), c_1(M) \cdot Q)$, and thus in particular there is a derived autoequivalence taking $[E]$ to a class with $\operatorname{rank}([E']) = 0$, $c_1([E']) \cdot Q = 1$, $\chi([E'], [E']) = 1$. Any such class is represented by a sheaf $\mathcal{O}_e(\chi([E']) - 1)$ where $e = c_1([E'])$ is a section of the elliptic fibration, and thus we can apply the inverse autoequivalence to obtain an object $E$ representing $[E]$. Since a section $e$ cannot meet the singular point of any fiber, we find that the restriction of $E$ to any fiber is a shift of a vector bundle, and thus $E$ itself is a shift of a vector bundle. This establishes existence, with uniqueness similarly following from the rank 0 case. 

**Proposition 5.19.** Let $\pi : \tilde{X} \rightarrow X$ be the birational morphism contracting the identity section of the smooth rational Weierstrass elliptic surface $\tilde{X}$ with marked smooth fiber $Q$, and let $[E] \in K_0(X)$ be a class with $\operatorname{rank}([E]) > 0$, $\chi([E], [E]) = 1$, and $\gcd(\operatorname{rank}([E]), c_1([E]) \cdot Q) = 1$. Let $E'$ be the exceptional bundle on $\tilde{X}$ representing $\pi^*[E]$. Then $[E]$ is represented by an exceptional bundle iff $R \operatorname{Hom}(E', \mathcal{O}_e(-1)) = 0$, where $e$ is the identity section.

**Proof.** If $[E]$ is representable, then we must have $E' \cong \pi^*E$, and thus $R \operatorname{Hom}(E', \mathcal{O}_e(-1)) = 0$ as required. Conversely, if $R \operatorname{Hom}(E', \mathcal{O}_e(-1)) = 0$, then $E' \cong \pi^*E$ for some sheaf $E$ of class $[E]$, necessarily exceptional.

**Remark.** This reformulation appears particularly amenable to a computational approach: rather than asking about whether an object exists, it asks about the properties of an explicit object $E'$. In addition, the condition $R \operatorname{Hom}(E', \mathcal{O}_e(-1)) = 0$ can be reformulated as $E'|_e \cong \mathcal{O}_e(-1)^r$, so one need not even compute Ext spaces. In addition, it suffices to consider any rational elliptic surface, and the reduction of Proposition 5.16 works equally well when the discrete valuation ring $R$ has mixed characteristic, so that one may feel free to compute in finite characteristic. It is particularly tempting to work with the surface $y^2 = x^3 + z^6 + w^6$ over a suitable finite field. Not only is it easy to find prime fields over which this surface has rational Mordell-Weil group (any prime that splits completely in $\mathbb{Q}(2^{1/3})$ will do), but the Mordell-Weil group is the group of $G_6$-equivariant maps from the Fermat curve $t^6 + u^6 + v^6 = 0$ to the curve $E_0 : y^2 = x^3 - 1$ of $j$-invariant 0, suggesting that it should be possible to reduce to a question about equivariant bundles on the Fermat curve.

More generally, if $E$, $E'$ are two exceptional bundles on $\tilde{X}$ satisfying $R \operatorname{Hom}(E, E') = 0$, then we can apply the standard derived autoequivalences to put $E'$ in the form $\mathcal{O}_e(\chi - 1)$, at which point translating by the corresponding section and twisting by a line bundle makes it $\mathcal{O}_e(-1)$, and thus the image of $E$ under the same operations gives rise to an exceptional bundle on $X$. In fact, this works equally well if $E$, $E'$ are bundles on some blowdown of $\tilde{X}$, since their pullbacks to $\tilde{X}$...
still form an exceptional pair. This gives a powerful technique for constructing exceptional bundles on del Pezzo surfaces.

As an example, consider the case of two line bundles. We may as well take one of the bundles to be $\mathcal{O}_X$, and thus ask when $R\text{Hom}(\mathcal{O}_X, \mathcal{O}_X(-D)) = 0$, with $D \cdot Q > 0$. In particular, we must have $0 = \chi(\mathcal{O}_X(-D)) = (D \cdot (D - Q))/2 + 1$, and since $\chi(\mathcal{O}_X(-D), \mathcal{O}_X) > 0$, we must have $\text{Hom}(\mathcal{O}_X(-D), \mathcal{O}_X) \neq 0$, so that $D$ is effective. In other words, if $(\mathcal{O}_X(-D), \mathcal{O}_X)$ forms an exceptional pair, then $D$ must be the class of a rational curve on $X$. Conversely, if $D$ is the class of a rational curve, then $H^p(\mathcal{O}_D) \cong H^p(\mathcal{O}_X)$, and thus $R\Gamma(\mathcal{O}_X(-D)) = 0$ as required.

There is a contravariant derived autoequivalence $\Phi^*$ of $\tilde{X}$ that swaps $\mathcal{O}_{\tilde{X}}$ and $\mathcal{O}_{\tilde{e}}(-1)$, and thus preserves the triangulated subcategory of objects $M$ on $X$ with $R\Gamma(M) = 0$. This is the composition of the autoequivalence $\Psi$ of [20, Lem. 12.5] with Cohen-Macaulay duality, and thus we may explicitly compute

$$\text{rank}(\Phi^* M) = c_1(M) \cdot Q$$
$$c_1(\Phi^* M) = -c_1(M) + (c_1(M) \cdot Q + \text{rank}(M))(Q + e) + (c_1(M) \cdot e - \chi(M))Q$$
$$\chi(\Phi^* M) = -c_1(M) \cdot e.$$  

(5.24) (5.25) (5.26)

On the subcategory $\mathcal{O}_{\tilde{X}}$ of $D^b_{\text{coh}}(X)$, the action simplifies to

$$\text{rank}(\Phi^* M) = c_1(M) \cdot Q$$
$$c_1(\Phi^* M) = -c_1(M) + (c_1(M) \cdot Q + \text{rank}(M))Q,$$
$$\chi(\Phi^* M) = 0.$$  

(5.27) (5.28) (5.29)

where we note that the $Q$ here is the one on $X$, corresponding to $\tilde{Q} + e$ on $\tilde{X}$.

In particular, since $\mathcal{O}_X(-D)$ is an exceptional object on $X$ with $R\Gamma(\mathcal{O}_X(-D)) = 0$, applying $\Phi^*$ and shifting gives another such object, with invariants

$$\text{rank}(E) = D \cdot Q, \quad c_1(E) = -D + ((D \cdot Q) - 1)Q, \quad \chi(E) = 0.$$  

(5.30)

In this way, we obtain an exceptional bundle of slope $-1/r$ for all $r \geq 2$, and in two different ways for $r = 6$ (corresponding to the two types of rational curve of degree 6: bidegree (1, 2) curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and conics in $\mathbb{P}^2$).

Conversely, suppose we are given an exceptional bundle $E$ of slope $-1/r$, with Chern class of the form $-Q + v$ with $v \in \Lambda_{E_8}$. This has Euler characteristic $(r^2 - r + 2 - v^2)/2r$ (taking the positive inner product on $\Lambda_{E_8}$ rather than the intersection form), which we insist must be negative for any representative of the coset $v + r\Lambda_{E_8}$. We may thus just as well assume that $v/r$ is in the Voronoi cell around 0 (the closure of the set of points closer to 0 than to any other lattice vector), so that $v$ is a minimal coset representative. The maximum of $w^2$ over the Voronoi cell is 1, and thus $v^2 \leq r^2$, forcing $v^2 = r^2 - r + 2$ and $\chi = 0$. We thus obtain an exceptional pair $(E, \mathcal{O}_X)$ on $X$, and thus an exceptional triple $(\mathcal{O}_X(-1), \pi^* E, \mathcal{O}_{\tilde{X}})$ on $\tilde{X}$. Applying $\Phi^*$ turns $\pi^* E$ into (a shift of) a line bundle, and thus $E$ must have come from the above construction.

The above example is somewhat ad hoc, but there is a somewhat more systematic approach available. Consider the case of slope $-2/r$ ($r$ odd). Here $v^2 = r^2 - 2r + 5$, so we still must have $\chi = 0$, but it is difficult to control when $R\text{Hom}(\mathcal{O}_X, V) = 0$ for $V$ an exceptional bundle of rank 2 and very small negative slope. However, since $v^2 > (r - 1)^2$, so that $v/(r - 1)$ must be outside the Voronoi cell, we find that our putative bundle $E$ fits into an exceptional triple on $E$ of the form

$$(E, \mathcal{O}_X(-\alpha), \mathcal{O}_X)$$  

(5.31)
where \(\alpha\) is the highest root. Applying the Cohen-Macaulay dual gives a triple

\[(\omega_X, \omega_X(\alpha), E^D),\] (5.32)

inducing a triple

\[(E^D, \mathcal{O}_X, \mathcal{O}_X(\alpha)),\] (5.33)

in which we can swap \(\mathcal{O}_X\) and \(\mathcal{O}_X(\alpha)\). Lifting to \(\tilde{X}\) and applying \(\Phi^*\) then gives a four-term exceptional collection of the form

\[(\mathcal{O}_e(-1), \mathcal{O}_e'(-1), M, \mathcal{O}_X)\] (5.34)

where \(M\) has rank \(r - 2\) and slope \(-r/(r - 2)\) and (up to the action of \(W(E_8)\)) is the pullback of a sheaf on a degree 2 del Pezzo surface.

We could, of course, do this calculation entirely in the Grothendieck group to get a class \([M]\). Clearly, if \([E]\) is representable, then \([M]\) is representable, but it turns out that the converse applies as well. Indeed, if \([M]\) is representable, then since \(\mu(M) \in (-2, 0)\), we find that \(R\text{Hom}(\mathcal{O}_X, M) = 0\) and thus we indeed obtain a four-term exceptional collection, and can then invert the derived equivalences to deduce that \([E]\) is representable. Moreover, the representability of \([M]\) is equivalent to that of the class \([M](Q_X)\) of slope \(-2/(r - 2)\), and may thus be shown by induction. (To make this explicit, we must reduce \(v/(r - 2)\) to the fundamental alcove, but this requires only a small number of affine reflections for \(r\) large since it is already quite close to the fundamental alcove!) We use the standard coordinatization of the fundamental alcove via inner products with the simple roots. (In terms of the basis of Pic(\(X\)) coming from \(\mathbb{P}^2\), the simple roots are \((h - e_1 - e_2 - e_3, e_1 - e_2, e_2 - e_3, \ldots, e_7 - e_8)\).

**Proposition 5.20.** An exceptional sheaf of slope \(-2/r\) on a degree 1 del Pezzo surface has Chern class \(-2Q + v\) where the image of \(v/r\) in the fundamental alcove is either

\[r^{-1}(0, (r - 5)/2, 0, 0, 1, 0, 0, 0, 0)\]

or one of the following sporadic cases:

\[3^{-1}(1, 0, 0, 0, 0, 0, 0, 0, 0),\]
\[9^{-1}(1, 0, 0, 1, 0, 0, 0, 0, 0),\]
\[11^{-1}(1, 0, 2, 0, 0, 0, 0, 0, 0),\]
\[13^{-1}(3, 2, 0, 0, 0, 0, 0, 0, 0),\]
\[15^{-1}(5, 0, 0, 0, 0, 0, 0, 0, 0)\]

**Proof.** Since \(v^2\) is close to \(r^2\), \(v/r\) must be close to the fundamental weight \((0, 1/2, 0, 0, 0, 0, 0, 0, 0, 0)\), from which it is straightforward to verify that for \(r \gg 0\), the only possibility is as stated. Moreover, one finds that for \(r \geq 5\), the image of \((r/(r - 2))v\) in the fundamental alcove has the same form, while for \(r = 5\), the image is \(3^{-1}(1, 0, 0, 0, 0, 0, 0, 0, 0)\). The only other possibilities for \(r \leq 17\) are as stated, with the sporadic instances for \(r \in \{11, 13, 15\}\) each reducing to the sporadic instance of rank \(r - 2\), and the sporadic instance for \(r = 9\) reducing to the regular instance of rank 7. On the other hand, for the regular instance of rank \(r \geq 9\), there is a unique orbit of \(-1\)-curves on \(X\) (consisting of four orthogonal \(-1\)-curves) such that the slope \(-(r - 2)/r\) bundle has \(\chi(E, \mathcal{O}_e(-1)) = 0\), and thus (by duality) at most one \(W(E_7)\)-orbit of numerical classes of slope \(-(r + 2)/r\) corresponding to exceptional bundles on the degree 2 del Pezzo surface. We thus see that the regular instance can account for at most one type of bundle of rank \(r + 2\), from which the claim follows by induction.
Remark. In both the slope $-1/r$ and $-2/r$ cases, one can verify that there are no other classes with $v^2 = r^2 - r + 2$ or $v^2 = r^2 - 2r + 5$, and thus that the above necessary condition is sufficient for bundles of degree $\pm 1, \pm 2$ modulo $r$.

The interpretation via $\tilde{X}$ also gives rise to one further construction. If $E$ is an exceptional bundle on $X$ of slope $-d/r \in (0, 1)$, then in addition to the pair $(\pi^*E, \mathcal{O}_e(-1))$ on $\tilde{X}$, we also obtain a pair $(\mathcal{O}_e, \pi^*E)$, which itself gives rise to an exceptional bundle $E'$ on $X$. This new bundle has slope $-d'/r \in (0, 1)$ where $d'd = -1(r)$ and satisfies $v' = d'v$. As before, this construction not only gives a new bundle but a new necessary condition. In this way, the slope $-2/r$ construction gives rise to a construction for slope $(r - 1)/2r$ ($r$ odd). Together with Cohen-Macaulay duality, this gives us a complete classification for slopes $-1/r, -2/r, (r - 1)/2r, (r + 1)/2r, (r - 2)/r, (r - 1)/r$. This accounts for every possibility for $r \leq 9$ except $\{-3/8, -5/8\}$. For slope $-3/8$, there are two cases with $\chi \leq 0$, but for one of those cases the putative bundle $E'$ of slope $-d'/r = -5/8$ violates the necessary condition. We thus see that the only possibility for slope $-3/8$ corresponds to $v$ with coordinates

$$8^{-1}(0, 0, 1, 0, 0, 1, 0, 0),$$

and can verify that this reduces to the known bundle of slope $-3/5$.

For rank $\geq 10$, the above techniques no longer suffice to obtain a classification: there is a candidate of slope $-3/10$ with maximal Euler characteristic $-1$. (A direct computation on a suitable Weierstrass surface shows that no such bundle exists.)

6 Classifications

We now return to our original question: given a bundle $V$ and autoequivalence $\Psi$, understand the filtered deformations of $B_{V, \Psi}$. If as usual we constrain the slopes of $V$ to an interval of length $d$, then we can produce such a deformation from the following data: (1) a noncommutative del Pezzo surface $X$ with anticanonical curve $Q$ such that $\theta \otimes H^1(\mathcal{O}_Q) \cong \Psi^{-1}$ on $\mathcal{O}_Q$, so in particular $Q^2 = d$, (2) an abelian equivalence $\psi : \text{coh}(Q) \cong \text{coh}(E)$, and (3) a rigid, unobstructed, torsion-free sheaf $M$ satisfying $\psi M|_Q \cong V$. The discussion above further allows us to replace $M$ by an exceptional collection $(E_1, \ldots, E_n)$ such that the bundles $(\psi E_1|_Q, \ldots, \psi E_n|_Q)$ are the stable constituents of $V$ (with appropriate multiplicities).

Now, a pair $(X, \psi)$ with $X$ a noncommutative del Pezzo surface and $\psi : \text{coh}(Q) \cong \text{coh}(E)$ induces a homomorphism $K_0(X) \to K_0(E)$ by $[M] \mapsto [\psi M|_Q]$, and one can in fact recover the surface from this homomorphism. Indeed, $\psi$ as usual has the form $\psi(M) = f_*(M) \otimes \psi(\mathcal{O}_Q)$ where $f : Q \cong E$ is an automorphism of curves. Since $\psi(\mathcal{O}_Q) \cong \psi(\mathcal{O}_X|_Q)$, we can recover $[\psi(\mathcal{O}_Q)]$ (and thus, up to isomorphism, $\psi(\mathcal{O}_Q)$) from the homomorphism $K_0(X) \to K_0(E)$.

$$[f_*M|_Q] \cong \psi M|_Q] - \text{rank}(\mathcal{M}|_Q)[\mathcal{L}],$$

we can thus compute the homomorphism $[M] \mapsto [f_*M|_Q]$ from the given homomorphism. But this homomorphism is precisely of the kind we considered in Section 4, so lets us recover $X$.

This works for any homomorphism $\phi : K_0(X) \to K_0(E)$, as long as $\text{rank}(\phi([M])) = \text{rank}([M])$ for all $[M] \in K_0(X)$, and $\deg(\phi([M])) = c_1([M]) \cdot Q + \delta \text{rank}([M])$ for some $\delta \in \mathbb{Z}$. (It then follows that $\phi([M]) := \phi([M]) - \text{rank}([M])(\phi(\mathcal{O}_X) - \mathcal{Q})$ gives rise to a noncommutative surface $(X, Q)$ with a marked isomorphism $Q \cong E$, and we recover $\psi$ after twisting by a line bundle of class $\phi(\mathcal{O}_X)$.) We thus find that the moduli stack of pairs $(X, \psi)$ (with blowdown structure) consists of countably many copies of $E^{12-d}$, one for each value of $\deg(\phi(\mathcal{O}_X)) \in \mathbb{Z}$. (More precisely, if one allows $E$ to vary, one obtains the corresponding power of the universal curve over the moduli stack of elliptic curves, since one can act on $\phi$ by $\text{Aut}(E) \ltimes E$.)
Note that the moduli space of possible $\psi$s for a given choice of $(X, Q)$ is 2-dimensional: there is a degree of freedom corresponding to a choice of marked point (to identify $Q$ with an elliptic curve) and a degree of freedom corresponding to the choice of $\psi(\mathcal{O}_Q)$. Thus, for instance, the moduli stack of pairs $(X, Q)$ with $X$ a noncommutative $\mathbb{P}^2$ may be identified with the moduli space of cubic plane curves with a choice of translation, or equivalently by the moduli stack of smooth genus 1 curves with marked line bundles of degrees 3 and 0. Making $Q$ an elliptic curve requires marking a point, at which point we may use $\text{Pic}^2(E) \cong \text{Pic}^0(E) \cong E$ to write the family as $E^2$, where $E$ is the universal elliptic curve. This determines how $\psi$ acts on structure sheaves of points, but it remains to specify its image on $\mathcal{O}_Q$, which is a point of $\text{Pic}(E) \cong \mathbb{Z} \times E$, giving countably many copies of $E^3$.

There is a natural action of $W(E_{9-d})$ on $\text{NS}(X)$, which extends to an action of $W(E_{9-d}) \ltimes \text{NS}(X)$ on $K_0(X)$. This arises from the fact that line bundles span $K_0(X)$ (and, in fact, there is a basis of line bundles), and thus we can define an action by

$$[\mathcal{O}_X(D)] \mapsto [\mathcal{O}_X(wD + D_0)].$$

There is then an induced action of this group on our moduli stack, and we obtain a derived equivalence between the categories of coherent sheaves on any two surfaces in the same orbit. (Indeed, $W(E_{9-d})$ acts on commutative del Pezzo surfaces preserving the structure sheaf, while $\text{NS}(X)$ corresponds to twists by line bundles. One caveat is that in the noncommutative setting, twisting by a line bundle changes the surface!) Similarly, there is an action of $\text{Aut}(E) \ltimes E \times \text{Pic}(E)$ on the other side, and this preserves the surface (since it only changes the $\psi$).

Note that the autoequivalence $\Psi_X$ of $\text{coh}(E)$ corresponding to the action of $\theta^{-1}$ depends only on $\phi([\text{pt}])$, $\phi([\mathcal{O}_Q])$ and the integer $\delta = \deg(\phi([\mathcal{O}_X]))$. Indeed, the automorphism $\tau$ is translation by $q := \det \phi([\text{pt}])$, so that $\Psi_X(\mathcal{O}_x) \cong \mathcal{O}_{x+q}$, while the line bundle is determined by the equation

$$[\mathcal{L}] + \delta q + \phi([\mathcal{O}_X]) = \Psi_X(\phi([\mathcal{O}_X])) = \phi([\mathcal{O}_X(Q)]) = \phi([\mathcal{O}_X]) + \phi([\mathcal{O}_Q]) + dq,$$

(6.3)

giving

$$[\mathcal{L}] = (d-\delta)q + \phi([\mathcal{O}_Q]).$$

(6.4)

(One caveat, which we will discuss further below, is that the isomorphism class of $\mathcal{L}$ is fixed, but $\mathcal{L}$ itself is a more subtle issue.) Thus any given isomorphism class of $\Psi$ cuts out a codimension 2 subscheme of the moduli stack on which $\phi([\text{pt}])$ and $\phi([\mathcal{O}_Q])$ are fixed. For $d < 8$, the two classes $[\text{pt}]$ and $[\mathcal{O}_Q]$ span a saturated sublattice of $K_0(X)$, so that this subscheme is itself a power of $E$, namely $E^{10-d}$; this also holds on the component for $d = 8$ corresponding to odd Hirzebruch surfaces. (For $\mathbb{P}^2$, the subscheme is a coset of $E[3] \times E$, while for $F_0/F_2$ it is a coset of $E[2] \times E^2$.)

It will be useful to give names for the coordinates in $E^{12-d}$. Recall that the expression of $X$ as an iterated blowup of $\mathbb{P}^2$ (i.e., such an expression of the underlying commutative surface) induces a basis of $\text{NS}(X)$ of the form

$$h, e_0, \ldots, e_{8-d},$$

(6.5)

which in turn induces a basis

$$[\mathcal{O}_X], [\mathcal{O}_h(-1)], [\mathcal{O}_{e_0}(-1)], \ldots, [\mathcal{O}_{e_{8-d}}(-1)], [\text{pt}]$$

(6.6)

of $K_0(X)$, so that $(X, \psi)$ is determined by

$$u := \det \phi([\mathcal{O}_X]), \quad h := \det \phi([\mathcal{O}_h(-1)]), \quad q := [\text{pt}]$$

(6.7)

and

$$x_i := \det \phi([\mathcal{O}_{e_i}(-1)].$$

(6.8)
(An expression of $X$ as an iterated blowup of an even Hirzebruch surface gives a similar parametrization, but with $h$ and $e_0$ replaced by $s$ and $f$.)

Now, given a representable numerical exceptional collection $([E_1], \ldots, [E_n])$ in $K_0(X)$ (satisfying the positive root condition where the slopes agree), there is an associated family $(E_1, \ldots, E_n)$ of exceptional collections on the moduli stack, and the bundles $(E_1|_Q, \ldots, E_n|_Q)$ can be read off (up to isomorphism) from the classes $\phi([E_i])$, since any stable sheaf on $E$ is determined by its class in $K_0(E)$. Thus fixing the bundles is again tantamount to fixing the values under the corresponding homomorphisms from $E^{12-d}$ to $E$. An exceptional collection always forms a saturated sublattice (the matrix of Mukai pairings has determinant 1), and thus fixing the bundles gives a family isomorphic to $E^{12-n-d}$.

We thus see in general that the family of surfaces with blowdown structure and choice of $\psi$ such that the given collection gives rise to a given $\mathcal{B}_{V, \psi}$ is the appropriate fiber of a homomorphism $E^{12-d} \to E^{n+2}$. (Of course, the fiber will be empty unless the values are all correct, a constraint on the numerical collection along with the integer $\delta$.) We must then take the union over all numerical collections and quotient by the action of $W(E_{9-d}) \ltimes \text{NS}(X)$ to obtain the “true” moduli space. (Equivalently, we choose one representative from each orbit of valid numerical collections, and for each one, take the quotient of the relevant subscheme of $E^{12-d}$ by the action of the stabilizer, a subgroup of the affine Weyl group $W(E_{9-d}) \ltimes Q^\perp$.)

One caveat here is that reflections do not give abelian autoequivalences; they do not in general preserve the effective cone. In particular, it is not a priori obvious that $W(E_{9-d})$ preserves the graded algebra. Certainly, where none of the roots of $E_{9-d}$ are effective, $W(E_{9-d})$ acts as abelian equivalences, and thus there is no difficulty. In particular, for any coordinate on the appropriate weighted projective space, the two algebras we want to compare induce sections of the same line bundle on the relevant substack of $E^{12-d}$ that agree where no roots are effective (or, more precisely, where the given $w$ does not make any effective root negative). So the only way those coordinates can fail to agree is if the condition on roots always fails. But in that case (which since it holds for all $q$, must come from a case in which the underlying commutative surface has a $-2$-curve in which we wish to reflect), the reflection corresponding to such a root actually acts trivially on the entire component. It follows in either case that the map to the deformation scheme factors through the quotient by $W(E_{9-d}) \ltimes \text{NS}(X)$. (Caveat: the image in the deformation stack is not the quotient stack: weighted projective spaces have cyclic stabilizers, unlike the action of $W(E_{9-d}) \ltimes \text{NS}(X)$ on $E^{12-d}$. This is presumably related to the fact that the quotient by the reflection subgroup of the stabilizer of a point is regular at that point; quotienting by that normal subgroup gives a cyclic group in the cases of interest.)

It remains to understand the morphism from this family to deformation space. The missing piece here is that there is a natural line bundle on deformation space (it is a quotient by $\mathbb{G}_m$, after all), and thus we need to understand the pullback of this line bundle to the relevant family of surfaces.

(The image will then be the corresponding embedding in weighted projective space.) It turns out that this is in fact the restriction of a line bundle on the ambient moduli space of pairs $(X, \psi)$. The main issue is the fact that when defining the moduli space, we only determined the various exceptional sheaves and $\mathcal{L}$ up to isomorphism. For the exceptional sheaves, this is not a problem: rescaling the exceptional sheaves by scalars simply acts on $\mathcal{B}_{V, \psi}$ by an inner automorphism. The line bundle is thus entirely determined by the failure of the induced autoequivalence $\Psi_X$ of $Q$ to be a pullback from the corresponding family over $E^2$.

We thus need to understand this autoequivalence better. For convenience, we consider its inverse instead. A pair $(X, \psi)$ determines an adjoint pair of functors $j_* : \text{coh } E \to \text{coh } X$, $j^* : \text{coh } X \to$
Proposition 6.1. If we fix intersection form, making the analogue for $B$ globally isomorphic to twisting by the given line bundle makes the associated grade $d$ of the universal filtered deformation autoequivalence, and thus we know how $\Psi$ space.

Proof. Twisting by the inverse of this bundle turns $\Psi^{-1} \O_E$ alone. Not only does this imply that $\Psi^{-1}$ is well-defined as a functor on the category of sheaves on the universal curve over the parameter space, but that it is invariant under the action of $\text{NS}(X)$. Similarly, if we compose with an autoequivalence of $\text{coh}(X)$, this simply conjugates $\Psi^{-1}$ by that autoequivalence, and thus we know how $\Psi^{-1} \O_E$ must transform under that action on parameter space.

The action of $E \times \text{Pic}(E)$ suffices to determine the dependence of $\Psi^{-1} \O_E$ on $u$ and $\delta$ (i.e., it tells us that the polarization has the form $uq - \delta(qz + q^2/2)$ plus terms independent of $u$ and $\delta$). The twist-by-$h$ symmetry implies that the polarization is invariant under $(h, u, \delta) \mapsto (h + q, u + 2q + h, \delta + 3)$, which together with the known dependence on $u$ and $\delta$ tells us that the $h$-dependent terms are precisely $-h^2/2 + 3h z$. Similarly, invariance under $(x_i - q, u + x_i, \delta + 1)$ (i.e., twisting by $e_i$) tells us that the $x_i$-dependent terms are precisely $x_i^2/2 - x_i z$. Since the $z$-dependence is determined by the isomorphism classes of the fibers over $E^{12-d}$ (i.e., that they correspond to the point $-3h + x_0 + \cdots + x_{8-d} + \delta q$ of $\text{Pic}^{-d}(E)$), we thus conclude that

$$\Psi^{-1} \O_E \cong \mathcal{L}_{-dz^2/2 + (3h - x_0 - \cdots - x_{8-d} - \delta q) z - h^2/2 + x_0^2/2 + \cdots + x_{8-d}^2/2 + uq - (\delta + a)q^2/2, w},$$

for some unknown $a \in \mathbb{Z}$ and $w \in \mathbb{Z}/12\mathbb{Z}$. (One can also easily verify that this is invariant under $W(E_{0-d})$.

Since changing $a$ and $w$ merely twists by a line bundle pulled back from the curve parametrizing $q$, this is enough to tell us the following. (Note that the dependence on $h, \bar{x}$ is essentially just the intersection form, making the analogue for $F_0/F_2$ straightforward to write down.)

**Proposition 6.1.** If we fix $E, V, \Psi$, then for any family of filtered deformations of $B_{V, \Psi}$ constructed via Proposition 5.1, the pullback of the natural line bundle on deformation space is isomorphic to the restriction of $\mathcal{L}_{-h^2/2 + x_0^2/2 + \cdots + x_{8-d}^2/2 + uq, 0}$.

**Proof.** Twisting by the inverse of this bundle turns $\Psi^{-1} \O_E$ into

$$\mathcal{L}_{-dz^2/2 + (3h - x_0 - \cdots - x_{8-d} - \delta q) z - (\delta + a)q^2/2, w},$$

which is the pullback through the map $(z, 3h - x_0 - \cdots - x_{8-d}, q)$ of a family of line bundles depending only on the isomorphism class of the autoequivalence $\Psi$. In particular, for fixed $\Psi$, twisting by the given line bundle makes the associated graded of the universal filtered deformation globally isomorphic to $B_{V, \Psi}$ as required. 

\[\square\]
Since the polarization is not positive definite, this line bundle is not ample on all of $E^{12-d}$. On the fibers of $(q, 3h - x_0 - \cdots - x_{8-d})$, the polarization becomes positive semidefinite, and becomes positive definite once we fix the image under $\psi$ of some exceptional sheaf of positive rank (as this fixes a linear combination of $u$ and the other parameters). We thus see that we indeed obtain an ample bundle on our family of deformations in this way, and thus a map to a weighted projective space, in such a way that the deformation parameters are homogeneous in the coordinates on weighted projective space.

As an example, let us consider the case that $V$ is semistable of rank $r$ and slope $m \in \mathbb{Z}$ (divisorial, for simplicity) while $\Psi$ increases degrees by $d$. Since changing $\deg \phi([O_X])$ adds the same integer to all slopes in the exceptional collection, the corresponding exceptional collection must consist of line bundles of the same slope. It must therefore have the form

$$(O_X(D - \alpha_1 - \cdots - \alpha_{r-1}), O_X(D - \alpha_1 - \cdots - \alpha_{r-2}), \ldots O_X(D)) \quad (6.12)$$

where $D$ is some divisor class and $\alpha_1, \ldots, \alpha_{r-1}$ are the simple roots of a subsystem of type $A_{r-1}$ inside $E_{9-d}$. The orbits of such collections under $W(E_{9-d}) \ltimes \text{NS}(X)$ are in natural correspondence with the conjugacy classes of $A_{r-1}$ subsystems, and for each orbit, we may take the representative with $D = 0$. (This forces us to take $\delta = m$ in the map $K_0(X) \to K_0(E)$, though of course in this case we can tensor $V$ by a line bundle to obtain an equivalent problem with $m = 0$.)

For $r = 1$, there is (trivially) a unique orbit of $A_0$ subsystems, and we find that, for $d \leq 6$, the base of the corresponding family is the quotient of (a torsor over) $E^{9-d}$ by $W(E_{9-d})$, with line bundle corresponding to the minimal invariant polarization. (Minimality can be read off from the relation of the above polarization to the intersection form, so that any root evaluates to 1.) Per Looijenga [16], such a quotient for an irreducible Weyl group is isomorphic to a weighted projective space, in which the degrees of the generators are the labels on the affine Dynkin diagram. We thus obtain the following descriptions of the quotient subschemes (corresponding to $E_8$, $E_7$, $E_6$, $D_5$, $A_4$, $A_2A_1$):

$$\mathbb{P}^{[12234456]}, \mathbb{P}^{[11222334]}, \mathbb{P}^{[1112223]}, \mathbb{P}^{[111122]}, \mathbb{P}^4, \mathbb{P}^2 \times \mathbb{P}^1. \quad (6.13)$$

For $d = 7$, the Weyl group is too small, so we obtain a quotient of $E^2$ by $\text{Sym}_2$, while for $d = 8$, there are two components (corresponding to the two parities of Hirzebruch surface), one isomorphic to $E$ (the odd case), and one isomorphic to $E[2] \times E/A_1$ (the even case). Finally, for $d = 9$, we obtain $E[3]$.

For $r = 2$, $d \leq 5$, there is again a unique orbit of $A_1$ subsystems, and thus we must replace $E_{9-d}$ by the stabilizer of a root in $E_{9-d}$. This gives $(E_7, D_6, A_5, A_3A_1)$

$$\mathbb{P}^{[11222334]}, \mathbb{P}^{[1111222]}, \mathbb{P}^5, \mathbb{P}^3 \times \mathbb{P}^1 \quad (6.14)$$

for $d \leq 4$. For $d = 5$, the rank is too small, so we have an elliptic factor, and for $d = 6$, there are two orbits of roots, one of which has an elliptic factor. For $d = 7$, we have a unique orbit of roots, with an elliptic factor, while for $d = 8$ only the even case has a root, making the moduli space in that case a torsor over $E[2]$, and for $d = 9$ no configuration exists.

For $r = 3$, we similarly obtain

$$\mathbb{P}^{[1112223]}, \mathbb{P}^5, \mathbb{P}^2 \times \mathbb{P}^2 \quad (6.15)$$

for $d \leq 3$; for $d = 4$, $d = 5$, there are unique orbits but with a rank deficit. For $d = 6$, there is a unique orbit, and the space has the form $E[3] \times \mathbb{P}^1$. For $d > 6$, no configuration exists.

For $r = 4$, we have

$$\mathbb{P}^{[111122]}, \mathbb{P}^3 \times \mathbb{P}^1, \quad (6.16)$$
for \( d = 1, 2 \), a unique orbit with a rank deficit for \( d = 3 \), two orbits for \( d = 4 \) (one with a rank deficit, and one with a torsion factor), and two orbits for \( d = 5 \) (the maximum), again with a rank deficit.

For \( r = 5 \), we have \( \mathbb{P}^4 \) for \( d = 1 \), a unique orbit with a rank deficit for \( d = 2, 3 \), two orbits with a rank deficit for \( d = 4 \), and for \( d = 5 \) obtain \( E[5] \).

For \( r = 6 \), we have \( \mathbb{P}^2 \times \mathbb{P}^1 \) for \( d = 1 \), two orbits (one with a rank deficit, the other with torsion) for \( d = 2 \), two orbits (with torsion) for \( d = 3 \), and nothing for \( d > 3 \). For \( r = 7 \), we have a rank deficit for \( d = 1, 2 \) and nothing for \( d > 2 \), while for \( r = 8 \) we have two components (one with a rank deficit and one with torsion) for \( d = 1 \) and \( E[2] \) for \( d = 2 \), and for \( r = 9 \) we have \( E[3] \) for \( d = 1 \). (In general, if \( r + d > 10 \), then no configuration exists, since the rank of the subsystem is too large.)

The astute reader will have noticed a symmetry above: the description of the given portion of deformation space is invariant under swapping \( r \) and \( d \). Indeed, if we blow up \( d \) further points, then we obtain an exceptional collection on a (deformed!) elliptic surface, and a derived equivalence swaps the roles of the \( r \) line bundles and the \( d \) exceptional curves. Such a symmetry holds in general for the semistable case, with the one caveat that it will in general change the slope from \( a/r \) to \( b/r \) with \( ab = 1(r) \).

In many of the above cases, the image of the family in weighted projective space contains an orbifold point, which leads to additional interesting families. In general, if the space for \( B_{V, \Psi} \) passes through an orbifold point of degree \( g \), then the corresponding deformation extends to one of \( B_{V', \Psi'} \) with

\[
V' = V \oplus \Psi(V) \oplus \cdots \oplus \Psi^{g-1}(V), \quad \Psi' = \Psi^g. \tag{6.17}
\]

We can then further deform \( V', \Psi' \).

For instance, since the family for \( d = r = 1 \) is \( \mathbb{P}^{[122334456]} \), we have orbifold points of order \( 2 \leq g \leq 6 \). The result corresponds to taking \( V \) to be a sum of line bundles of slopes \( 0, 1, \ldots, g-1 \), with \( \Psi \) increasing degrees by \( g \). The differences between the Chern classes of the corresponding line bundles on \( X \) (now a del Pezzo surface of degree \( g \)) must then be rational curves. It is straightforward to classify such configurations, and for \( g = 2, 3, 4 \), we find that the family is the Looijenga-style quotient associated to \( E_6, D_4, A_2 \) respectively. For \( g = 5 \), there is a unique filtered deformation, while for \( g = 6 \), the generic such \( V \) has no filtered deformations, but there is a codimension 2 subfamily of \( V \)'s with unique filtered deformations. Indeed, in this case, we find that the exceptional collection is actually a full exceptional collection, and thus \( \Psi \) is determined by \( V \).

Of course, the cases corresponding to orbifold points are special cases of this scenario, and have an additional symmetry corresponding to shifting the slopes by 1 (and applying \( \Psi \) or \( \Psi^{-1} \) as needed). This relates to the fact that for \( d \leq 6 \), the group \( W(E_{9-d}) \ltimes \text{NS}(X) \) has the affine Weyl group (or product of two affine Weyl groups, for \( d = 6 \)) as a normal subgroup, and thus we have an isomorphism

\[
W(E_{9-d}) \ltimes \text{NS}(X) \cong \mathbb{Z} \ltimes \tilde{W}(E_{9-d}), \tag{6.18}
\]

where the splitting is induced by the requirement that \( \mathbb{Z} \) preserves the set of simple roots. In other words, the group \( W(E_{9-d}) \ltimes \text{NS}(X) \) contains an element \( \pi \) corresponding to a diagram automorphism. Since \( \tilde{W}(E_{9-d}) \) does not change slopes, \( \pi \) must change the slope (increasing it by 1, say). Moreover, \( \pi^d \) commutes with the affine Weyl group, and thus (since it increases slopes by \( d \)) is translation by \( Q \). We can then verify that the divisor classes \( \pi(0), \ldots, \pi^{d-1}(0) \) all correspond to rational curves, and thus induce a configuration as in the previous paragraph. The orbifold case then corresponds to the case that the point in the moduli space is invariant under \( \pi \).

For \( d = 2, r = 1 \), we have decimations for \( g \in \{2, 3, 4\} \). In each case, there is a single configuration (up to the Weyl group action and choice of parity for \( 2g = 8 \)). For \( g = 2, g = 3 \), the result is a Looijenga quotient, with Weyl groups \( D_4 \) and \( A_1 \) respectively, while for \( g = 4 \), the result
is again a full exceptional collection (in an even Hirzebruch surface), and thus there is a unique
decomposition on the appropriate codimension 2 subscheme.

For \( d = 1, r = 2 \), we again have \( D_4 \) and \( A_1 \) for \( g \in \{2, 3\} \) and a unique deformation over a
codimension 2 subscheme of the base for \( g = 4 \), with the only difference from the \((d, r) = (2, 1)\)
case being that the surfaces have degree \( g \) rather than \( 2g \). (In particular, for \( g = 4 \), we obtain a
full exceptional collection of line bundles on a quartic del Pezzo surface.)

For \( d = 3, r = 1 \), the \( g = 2 \) case is the Looijenga quotient of type \( A_2 \), while the \( g = 3 \) case
is a full exceptional collection on \( \mathbb{P}^2 \). The \( g = 3 \) case corresponds to the fact that the standard
noncommutative deformation (the three generator Sklyanin algebra \([\Pi]\) of \( k[x, y, z] \)) is obtained by
removing the degree 3 equation from the \( d = 3, r = 1 \) elliptic algebra.

For \( d = 1, r = 3 \), the \( g = 2 \) case is again the Looijenga quotient of type \( A_2 \), while the \( g = 3 \) case
is a full exceptional collection on a noncommutative cubic surface.

For \( d = 4, r = 1, g = 2 \), the exceptional collection lives on an even Hirzebruch surface, and
\( V \) imposes a single (saturated!) condition on \( \Psi \). So deformations only exist on a codimension
1 subscheme of parameter space, but when they exist are classified by \( \mathbb{P}^1 \cong E/A_1 \). Each such
deformation corresponds to removing a quadratic relation from the algebra with \( d = 4, r = 1 \); since
the scheme is \( \mathbb{P}^1 \) embedded by \( O(1) \), the possibilities span a 2-dimensional subspace of the dual.
Removing both relations gives Sklyanin’s deformation \([23]\) of \( k[x, y, z, w] \) (with the two removed
relations corresponding to the 2-dimensional space of central elements of degree 2.)

For \( d = 1, r = 4, g = 2 \), we again have a \( \mathbb{P}^1 \) of deformations on a codimension 1 subscheme.
Again, we could also remove both relations, which gives something best described as an Azumaya
algebra on a noncommutative Fano 3-fold; it is unclear how to characterize that 3-fold, however.

Finally, for \( d = r = 2, g = 2 \), deformations only exist on a codimension 1 subscheme, while the
general nonempty fiber is the Looijenga quotient of type \( A_3 \), a.k.a. \( \mathbb{P}^3 \).

We now turn to semisimple examples with non-integer slope. Note that in the case of a single
bundle, the process of testing representability of the candidate exceptional classes involves first
putting them in the fundamental alcove. It is of course quite straightforward to find all candidates
in the fundamental alcove, and for each candidate surviving the Euler characteristic test, we can
read off the stabilizer from its standard coordinates. This works particularly well for \( r = d = 1 \); for
larger values of \( r \), we need only classify configurations of \( r - 1 \) roots that have intersection
1 (modulo the denominator) with the fundamental representative and \( -1 \) with each other, while
for larger values of \( d \), we need only find \( d - 1 \) exceptional curves that are orthogonal (modulo the
denominator) and to each other. (And when both are larger, the roots must be orthogonal to the
\( -1 \)-curves.)

For \( \mu = -1/2 \), we obtain the following nice cases (omitting cases where there is either a rank
deficit or torsion) as \( r, d \) vary:

\[
\begin{pmatrix}
D_8 & A_7 & A_5 A_1 \\
A_7 & A_3 A_3 & A_5 A_1 \\
A_5 A_1 & &
\end{pmatrix}
\]

(6.19)

In the case \( r = 9, d = 1 \) (or vice versa), there is a unique combinatorial configuration, with the
corresponding subscheme of parameter space a torsor over \( E[10] \). This case is primarily of note
because there actually are no bundles of slope \(-1/2\) on a noncommutative \( \mathbb{P}^2 \), and thus this case
can only arise by taking \( \delta \neq 0 \).

In the \( r = d = 1 \) case, there are orbifold points, so we can again take a decimation. It turns
out that for bundles of slope \(-3/2, -1/2\) (on a degree 2 del Pezzo surface) to form an exceptional
collection, it is necessary for their Chern classes to differ by \( Q \). This means that a deformation will
only exist on a codimension 1 subscheme of parameter space, and when it does will be classified by the quotient corresponding to $A_7$ (i.e., the same as for $r = 1, d = 2$).

For slope $-1/3$ (or equivalently $-2/3$), we obtain the nice cases

$$
\begin{pmatrix}
A_8 & A_5 A_2 \\
A_5 A_2 & A_2
\end{pmatrix}.
$$

(6.20)

For slope $-1/4$, the $r = d = 1$ case is nice, with corresponding Weyl group $A_7 A_1$. For slope $-1/5$, it appears that no cases are nice, but for $r = d = 1$, slope $-2/5$, the $r = d = 1$ case is nice, with group $A_4 A_4$.

One caveat here is that we have not shown that every invariant section of a power of the ample bundle can be expressed as a polynomial in the coefficients of the associated filtered deformation. For sections of the ample bundle itself, this would reduce to a calculation of the infinitesimal deformation of $\text{Proj}(B[t])$ associated to the $\mathbb{G}_m$ action on the general filtered deformation in the family; this, together with analogous calculations for orbifold points, should be enough to establish the result in general. (Note that this would also establish that the map from the quotient scheme to the deformation scheme is an embedding.)

The fact that precisely these weighted projective spaces arise when computing moduli spaces of filtered deformations of random elliptic algebras suggests the following.

**Conjecture 1.** Any nontrivial filtered deformation of any elliptic algebra $B_{V, \Psi}$ has Rees algebra of the form $A_M^r$ for some rigid, unobstructed, torsion-free sheaf $M$ on a noncommutative del Pezzo surface.

**Remark.** Since the moduli stack is closed in a weighted projective space, the Hilbert series of the moduli stack is upper semicontinuous. As a result, the argument of Lemma 2.11 would let us reduce to the case that $\tau$ has finite order (so the construction involves Azumaya algebras on del Pezzo surfaces), at the cost of having to work in finite characteristic. We also immediately deduce that if the claim holds for a given algebra $B$, then it holds for the generic algebra of the same shape.

In the cases where this would make the deformation space a weighted projective space, there is a possible strategy for proving this: show that for each degree, the space of (nontrivial) infinitesimal deformations agrees with the putative number of generators. This implies that the deformation space is contained in the conjectured weighted projective space, and therefore equal.

Although one could hope to compute the desired Hochschild cohomology spaces by hand, this does at the very least simplify the calculation when done by computer. In particular, for the line bundle cases with $r = 1$ and $d \in \{1, 2\}$, we could carry out the calculation not just for generic parameters, but for general parameters: that is, for any $E$ and any $\Psi$ that increases degrees by 1 or 2, every filtered deformation of the algebra $B_{D, E, \Psi}$ arises from the surface construction. We have also checked the conjecture for random instances (over a moderately large finite field) in every slope 0 case (including those where the family is not nice), as well as for a few of the cases with noninteger slope; as mentioned, this implies the conjecture holds generically in those cases as well.

Although the general twisted case appears out of reach at the moment, there is some hope for the untwisted case, since any deformation has a large center. The first step would be to show that the filtered deformation arises from a sheaf on a singular del Pezzo surface, which would reduce to the following (which would follow from the main conjecture, as would the analogous statement when twisted by a torsion point, since the center of a noncommutative del Pezzo surface with $q$ torsion is a commutative del Pezzo surface).

**Conjecture 2.** Any filtered deformation of an untwisted elliptic algebra $B_{V, \Psi}$ which induces a trivial deformation of the center is trivial.
Remark. This holds for $d = 1$ if $V$ is stable of rank 2 or a sum of two degree 0 line bundles, as in both cases we showed above (Examples 2.4 and 2.5 respectively) that the moduli space is the correct weighted projective space.

The other missing ingredients are showing that the resulting algebra on a singular del Pezzo surface lifts to a flat sheaf of algebras on the minimal desingularization, and showing that the result is still the endomorphism algebra of a rigid, unobstructed object. (In the twisted case, one would also need to show that one obtains the correct class in the Brauer group of the affine del Pezzo surface; this should not be an issue in the untwisted case where the algebra remains Azumaya on the compactification, since the compactification has trivial Brauer group.)

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