\(L^p(\mathbb{R}^2)\)-boundedness of Hilbert Transforms and Maximal Functions along Plane Curves with Two-variable Coefficients

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Abstract

In this paper, for general plane curves \(\gamma\) satisfying some suitable smoothness and curvature conditions, we obtain the single annulus \(L^p(\mathbb{R}^2)\)-boundedness of the Hilbert transforms \(H_{U,\gamma}^0\) along the variable plane curves \((t, U(x_1, x_2)\gamma(t))\) and the \(L^p(\mathbb{R}^2)\)-boundedness of the corresponding maximal functions \(M_{U,\gamma}^0\), where \(p > 2\) and \(U\) is a measurable function. The range on \(p\) is sharp. Furthermore, for \(1 < p \leq 2\), under the additional conditions that \(U\) is Lipschitz and making a \(\epsilon_0\)-truncation with \(\gamma(2\epsilon_0) \leq 1/4\|U\|_{\text{Lip}}\), we also obtain similar boundedness for these two operators \(H_{U,\gamma}^{\epsilon_0}\) and \(M_{U,\gamma}^{\epsilon_0}\).

1 Introduction

The main purpose of this article is to study the \(L^p(\mathbb{R}^2)\)-boundedness of the Hilbert transform and corresponding maximal function along variable plane curve. Let us first recall some backgrounds of this topic. The so-called Zygmund conjecture is a long-standing open problem, which can be stated as follows. Denote

\[M_{U}^{\epsilon_0}f(x_1, x_2) := \sup_{0<\epsilon<\epsilon_0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x_1 - t, x_2 - U(x_1, x_2)t)| \, dt.\]

Zygmund conjecture: Let \(U : \mathbb{R}^2 \to \mathbb{R}\) be a Lipschitz function and \(\epsilon_0 > 0\) small enough depending on \(\|U\|_{\text{Lip}}\). Is the operator \(M_{U}^{\epsilon_0}\) bounded on \(L^p(\mathbb{R}^2)\) for some \(p \in (1, \infty)\)?

Much later Stein [35] raised the singular integral variant of Zygmund conjecture. Denote

\[H_{U}^{\epsilon_0}f(x_1, x_2) := \text{p.v.} \int_{-\epsilon_0}^{\epsilon_0} f(x_1 - t, x_2 - U(x_1, x_2)t) \, \frac{dt}{t}.\]

Stein conjecture: Let \(U : \mathbb{R}^2 \to \mathbb{R}\) be a Lipschitz function and \(\epsilon_0 > 0\) small enough depending on \(\|U\|_{\text{Lip}}\). Is the operator \(H_{U}^{\epsilon_0}\) bounded on \(L^p(\mathbb{R}^2)\) for some \(p \in (1, \infty)\)?

A trivial fact is that \(M_{U}^{\epsilon_0}\) is bounded on \(L^\infty(\mathbb{R}^2)\) and \(H_{U}^{\epsilon_0}\) is unbounded on \(L^\infty(\mathbb{R}^2)\) if \(U\) is measurable. Furthermore, a counterexample based on a construction of the Besicovitch-Kakeya set shows that we can not expect any \(L^p(\mathbb{R}^2)\)-boundedness of \(M_{U}^{\epsilon_0}\) and \(H_{U}^{\epsilon_0}\) for \(1 < p < \infty\) if \(U\)

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is only assumed to be Hölder continuous \( C^\alpha \) with \( \alpha < 1 \). Both Zygmund conjecture and Stein conjecture are very difficult conjectures. Indeed, it is known that if the Stein Conjecture holds for \( C^2 \) vector fields, then Carleson’s Theorem on the pointwise convergence of Fourier series \([11]\) would follow.

Let us state some partial progresses toward understanding the above two open problems. For any real analytic function \( U \), Bourgain \([6]\) first obtained the \( L^2(\mathbb{R}^2) \)-boundedness of \( M_U^{0} \), and the \( L^p(\mathbb{R}^2) \)-boundedness can also been obtained with some standard modifications. The corresponding result for \( H_U^{\varepsilon} \) can be found in \([37]\). For \( U \in C^\infty \) with some additional curvature conditions, Christ, Nagel, Stein and Wainger \([14]\) proved the \( L^p(\mathbb{R}^2) \)-boundedness of \( H_U^{\varepsilon} \) and \( M_U^{\varepsilon} \) for \( p > 1 \). Furthermore, Lacey and Li \([27]\) established the \( L^2(\mathbb{R}^2) \)-boundedness of \( H_U^{\varepsilon} \) if \( U \in C^\alpha \) with \( \alpha > 1 \) and a suitable Kakeya maximal operator is bounded on \( L^2(\mathbb{R}^2) \). For further progress towards these two conjectures, we refer to \([1, 6, 18, 19, 20]\) and the references therein. Moreover, Lacey and Li \([27]\) brought tools from time-frequency analysis into the problem of Hilbert transforms along vector fields, which made the major breakthrough in terms of the regularity of \( U \). To state their results, we first introduce some definitions.

Let \( \psi : \mathbb{R} \to \mathbb{R} \) be a smooth function supported on \( \{t \in \mathbb{R} : 1/2 \leq |t| \leq 2\} \) with the property that \( 0 \leq \psi(t) \leq 1 \) and \( \sum_{k \in \mathbb{Z}} \psi_k(t) = 1 \) for any \( t \neq 0 \), where \( \psi_k(t) := \psi(2^{-k}t) \). Here and hereafter, for any \( k \in \mathbb{Z} \) and \( j = 1, 2 \), we denote \( P_k^{(j)} \) the Littlewood-Paley projection in the \( j \)-th variable corresponding to \( \psi_k \)

\[
(1.1) \quad P_k^{(1)} f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1 - z, x_2) \psi_k(z) \, dz, \quad P_k^{(2)} f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1, x_2 - z) \psi_k(z) \, dz.
\]

Lacey and Li obtained the following results.

**Theorem 1.1.** \((27)\) Let \( U : \mathbb{R}^2 \to \mathbb{R} \) be a measurable function. For any \( p \geq 2 \), there exists a positive constant \( \tilde{C}_p \), independent of \( k \), such that

\[
\left\| H_U^{\infty} P_k^{(2)} f \right\|_{L^2(\mathbb{R}^2)} \leq \tilde{C}_p \left\| P_k^{(2)} f \right\|_{L^2(\mathbb{R}^2)} \quad \text{for any } k \in \mathbb{Z},
\]

and for any \( p > 2 \),

\[
\left\| H_U^{\infty} P_k^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \leq \tilde{C}_p \left\| P_k^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \quad \text{for any } k \in \mathbb{Z}.
\]

It should be pointed out that the weak \( L^2(\mathbb{R}^2) \) estimates are sharp for measurable vector fields \( U \).

Let \( \gamma : \mathbb{R} \to \mathbb{R} \) be a continuous curve with \( \gamma(0) = 0 \). We next will focus on the study of the Hilbert transform and corresponding maximal function along the variable plane curve \((t, U(x_1, x_2)\gamma(t))\). Similarly, we define:

\[
H_{U, \gamma}^{\varepsilon} f(x_1, x_2) := \text{p. v.} \int_{-\infty}^{\varepsilon} f(x_1 - t, x_2 - U(x_1, x_2)\gamma(t)) \, \frac{dt}{t};
\]

\[
M_{U, \gamma}^{\varepsilon} f(x_1, x_2) := \sup_{0 < \varepsilon < \varepsilon_0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x_1 - t, x_2 - U(x_1, x_2)\gamma(t))| \, dt.
\]

Denote \( |t|^\beta := |t|^\alpha \) or \( \text{sgn}(t)|t|^\alpha \). The following results were obtained in \([31, 21]\).
Theorem 1.2. Let $\alpha > 0$ and $\alpha \neq 1$. Suppose that $U : \mathbb{R}^2 \to \mathbb{R}$ is measurable, then for any $2 < p < \infty$,

\[(1.2) \quad \left\| M_{U,[T]}^\infty f \right\|_{L^p(\mathbb{R}^2)} \leq C_{p,\alpha} \left\| f \right\|_{L^p(\mathbb{R}^2)} \]

and

\[(1.3) \quad \left\| H_{U,[T]}^\infty P_k^2 f \right\|_{L^p(\mathbb{R}^2)} \leq \tilde{C}_{p,\alpha} \left\| P_k^2 f \right\|_{L^p(\mathbb{R}^2)}, \]

where $C_{p,\alpha}, \tilde{C}_{p,\alpha}$ are positive constants that depend only on $p$ and $\alpha$.

We note that (1.2) was first proved by Marletta and Ricci [31], in which the authors used Bourgain’s result on the circular maximal operator [5] as a black box. Later, Guo, Hickman, Lie and Roos [21] adopted another approach that is more self-contained to reprove (1.2). They also proved (1.3). Moreover, under the condition that $U$ is Lipschitz, Guo et al. [21] obtained the following result for $1 < p \leq 2$.

Theorem 1.3. Let $\alpha > 0$ and $\alpha \neq 1$. Suppose that $U : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz, then there exists $\varepsilon_0 > 0$ depending only on $\left\| U \right\|_{Lip}$ and $\alpha$ such that

\[(1.3) \quad \left\| M_{U,[T]}^\infty f \right\|_{L^p(\mathbb{R}^2)} \leq C_{p,\alpha} \left\| f \right\|_{L^p(\mathbb{R}^2)} \]

for any $1 < p \leq 2$, where $C_{p,\alpha}$ is a positive constant that depends only on $p$ and $\alpha$.

As the development of the Hilbert transforms along curves $(t, \gamma(t))$, an interesting question is whether these results of Theorems 1.2 and 1.3 can be extended to more general curves $\gamma$ obeying some suitable smoothness and curvature conditions.

In the present paper, we have obtained some results for this question. Firstly, we state the needed conditions on the curves.

Hypothesis on curves (H.). Assume $\gamma \in C(\mathbb{R}) \cap C^N(\mathbb{R}^+)$ with $N \in \mathbb{N}$ large enough and $\gamma(0) = 0$. And $\gamma$ is either odd or even, and increasing on $\mathbb{R}^+$. Moreover, $\gamma$ satisfies the following three conditions:

(i) there exists a positive constant $C_1$ such that $|\left(\frac{\gamma'}{\gamma}\right)'(t)| \geq C_1$ for any $t \in \mathbb{R}^+$;

(ii) there exist positive constants $\left\{C_{2j}^i\right\}_{j=1}^N$ such that $\left|\left(\frac{\gamma^{(j)}(t)}{\gamma^{(j)}}\right)'\right| \geq C_{2j}^i$ for any $t \in \mathbb{R}^+$;

(iii) there exist positive constants $\left\{C_{3j}^i\right\}_{j=1}^N$ such that $\left|\frac{\gamma^{(j)}(t)}{\gamma^{(j)}}\right| \leq C_{3j}^i$ for any $t \in \mathbb{R}^+$.

Remark 1.4. We give some remarks for the hypothesis above. (i) of (H.), including some other similar forms, has been applied in the study of the Hilbert transforms along the variable curves $(t, u(x_1) \gamma(t))$ with one-variable coefficients (e.g., [30, 13]), and the bilinear Hilbert transform along the curves $(t, \gamma(t))$ ([29]). (ii) and (iii) of (H.) imply the following “doubling property” of the curves. For $t > 0$, there holds

\[(1.4) \quad e^{C_{2t}^{(i)}} \leq \frac{\gamma(2t)}{\gamma(t)} \leq e^{C_{3t}^{(i)}} \]
and

\[(1.5) \quad \text{either } e^{C_2^{(2)}/2C_2^{(1)}} \leq \frac{\gamma'(2t)}{\gamma(t)} \leq e^{C_3^{(2)}/C_2^{(1)}} \quad \text{or} \quad e^{-C_2^{(2)}/C_2^{(1)}} \leq \frac{\gamma'(2t)}{\gamma(t)} \leq e^{-C_2^{(2)}/2C_2^{(1)}}.\]

In fact, it follows from (ii) and (iii) of (H.) that

\[C_2^{(1)} \leq \frac{\gamma'(t)}{\gamma(t)} \leq C_3^{(1)} \quad \text{and} \quad \frac{C_2^{(2)}/C_3^{(1)}}{\gamma'(t)} \leq \gamma''(t) \leq \frac{C_2^{(2)}/C_3^{(1)}}{\gamma(t)} \quad \text{for any } t \in \mathbb{R}^+.\]

Let \(F(t) := \ln \gamma(t)\) for any \(t \in \mathbb{R}^+\), then we have \(C_2^{(1)}/t \leq F'(t) \leq C_3^{(1)}/t\) for any \(t \in \mathbb{R}^+\). On the other hand, by the Lagrange mean value theorem, there exists \(\theta \in [1, 2]\) such that \(F(2t) - F(t) = F'(\theta)t\). Hence, we have \(C_2^{(1)}/2 \leq F(2t) - F(t) \leq C_3^{(1)}\), which further implies (1.4). Similarly, if consider \(G(t) := \ln \gamma'(t)\) for any \(t \in \mathbb{R}^+\), we can get (1.5). We note that some similar “doubling properties” of curves have occurred in obtaining the boundedness of the Hilbert transforms along curves \((t, \gamma(t))\) (cf. [12]).

**Example 1.5.** Let us list some examples of curves satisfying (H.). Since \(\gamma(t)\) is odd or even and \(\gamma(0) = 0\), we write only the part for \(t > 0\).

1. for any \(t > 0\), \(\gamma(t) := t^\alpha\), where \(\alpha \in (0, \infty)\) and \(\alpha \neq 1\);
2. for any \(k \in \mathbb{N}\) and \(t > 0\), \(\gamma(t) := \sum_{i=1}^{k} t^\alpha_i\), where either \(\alpha_i \in (0, 1)\) for all \(i = 1, 2, \cdots, k\), or \(\alpha_i > 1\) for all \(i = 1, 2, \cdots, k\).
3. for any \(t > 0\), \(\gamma(t) := t^\alpha \log(1 + t)\), where \(\alpha > 1\);
4. for any \(t > 0\), \(\gamma(t) := (t \sin t)1_{[0 < t < \epsilon_0]}(t)\), or \((t - \sin t)1_{[0 < t < \epsilon_0]}(t)\), or \((1 - \cos t)1_{[0 < t < \epsilon_0]}(t)\), where \(\epsilon_0\) is small enough.

Next, we will state two main results of this paper. The first one is as follows.

**Theorem A.** Let \(U : \mathbb{R}^2 \to \mathbb{R}\) be a measurable function, and the curve \(\gamma\) satisfies (H.). Then, for any \(p > 2\), there exists a positive constant \(C\), independent of \(U\), such that the following estimates hold for all \(f \in L^p(\mathbb{R}^2)\),

\[(i) \quad \left\| H_{U, \gamma}^{\mathcal{P}^{(2)}} k \frac{f}{\gamma(t)} \right\|_{L^p(\mathbb{R}^2)} \leq C \left\| P^{(2)} f \right\|_{L^p(\mathbb{R}^2)}, \quad \text{where } C \text{ does not depend on } k \in \mathbb{Z};
(ii) \quad \left\| M_{U, \gamma}^\mathcal{P} f \right\|_{L^p(\mathbb{R}^2)} \leq C \left\| f \right\|_{L^p(\mathbb{R}^2)}.\]

Observe that Theorem A is the generalization of Theorem [22] from the special curve \([t]^\alpha\) to more general curves \(\gamma(t)\). As for (i) of Theorem A, the reason that we consider single annulus \(L^p(\mathbb{R}^2)\)-boundedness, in place of \(L^p(\mathbb{R}^2)\)-boundedness, is that the latter fails for every \(p \in (1, \infty)\) even in the case \(\gamma(t) = t^2\). It will follow from a straightforward modification of Karagulyan’s counter-example in the case \(\gamma(t) = t^2\) ([26]). The range on \(p\) in Theorem A is sharp. It can be seen that Theorem A fails for \(p \leq 2\), if we assume \(f\) to be the characteristic function of the unit ball.

As a direct corollary of Theorem A and linearization, we have the results below about the directional Hilbert transforms along curves \(\gamma\).
Corollary 1.6. If $\gamma$ satisfies (H). Then, for any $p > 2$, there exists a positive constant $C$, such that the following estimates hold for all $f \in L^p(\mathbb{R}^2)$,

\begin{align*}
(i) & \quad \left\| \sup_{t \in \mathbb{R}} |H_{\lambda, \gamma}^n f| \right\|_{L^p(\mathbb{R}^2)} \leq C \left\| P_k f \right\|_{L^p(\mathbb{R}^2)} \quad \text{uniformly in } k \in \mathbb{Z}; \\
(ii) & \quad \left\| \sup_{t \in \mathbb{R}} |M_{\lambda, \gamma}^n f| \right\|_{L^p(\mathbb{R}^2)} \leq C \| f \|_{L^p(\mathbb{R}^2)}.
\end{align*}

The second main result of this paper, Theorem B, will study the analogue of Theorem A for the case $1 < p \leq 2$ under the additional conditions that $U$ is Lipschitz and making a $\varepsilon_0$-truncation. (ii) of Theorem B is also an extension of Theorem 1.3 to more general curves $\gamma$.

Theorem B. Let $U : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Lipschitz function and the curve $\gamma$ satisfies (H). Then, for any $1 < p \leq 2$, there exist constants $C > 0$ and $\varepsilon_0 > 0$ with $\gamma(2\varepsilon_0) \leq 1/4\|U\|_{\text{Lip}}$, such that the following estimates hold for all $f \in L^p(\mathbb{R}^2)$,

\begin{align*}
(i) & \quad \left\| H_{U, \gamma}^n f \right\|_{L^p(\mathbb{R}^2)} \leq C \left\| P_k f \right\|_{L^p(\mathbb{R}^2)}; \\
(ii) & \quad \left\| M_{U, \gamma}^n f \right\|_{L^p(\mathbb{R}^2)} \leq C \| f \|_{L^p(\mathbb{R}^2)},
\end{align*}

where $C$ is independent of $U$ and $k \in \mathbb{Z}$.

Remark 1.7. We now point out the differences of proofs between our theorems and Theorems 1.2 and 1.3. In the homogeneous curve case $\gamma(t) = |t|^\alpha$, the following special property

\begin{equation}
\gamma(ab) = \gamma(a)\gamma(b), \quad \text{for any } a > 0, b > 0,
\end{equation}

plays a very important role in the proofs of Theorems 1.2 and 1.3. More precisely, it is convenient to use

\begin{equation}
1 = \sum_{k \in \mathbb{Z}} \psi(2^{-l}(u_z^{(0)})^{\beta} t), \quad \text{see } [21] (4.9), \quad \text{and} \quad 1 = \sum_{k \in \mathbb{Z}} \psi(2^{-l/2}\widetilde{u}_z^{(0)} t), \quad \text{see } [21] (5.2)
\end{equation}

to split these operators considered. Here the purpose of adding $2^{-l/2}$ in [21] (5.2) is to make the terms of $u_z$ and $\gamma(2^{-l}t)$ cancel out, after applying this special property (1.7). However, in the general curve case, we can’t continue to use these partition of unity (1.7) to split our operators, since $\gamma(2^l(u_z^{(0)})^{\beta} t) \neq \gamma(2^l)(u_z^{(0)})^{\beta} \gamma(t)$ and $\gamma(2^l 2^{-l/2}(u_z^{(0)})^{\beta} t) \neq \gamma(2^l)\gamma(2^{-l/2})(u_z^{(0)})^{\beta} \gamma(t)$ in general. As a result, we have to use the classical partition of unity, i.e.,

\begin{equation}
1 = \sum_{k \in \mathbb{Z}} \psi(2^{-l} t),
\end{equation}

to split our operators, though it will greatly increase the difficulty of the proof. Indeed, even though we have used the classical partition of unity (1.8), we still will encounter the difficulty of $\gamma(2^l t) \neq \gamma(2^l)\gamma(t)$. In order to overcome this difficulty and separate $2^l$ from $\gamma(2^l t)$, we replace $\gamma(2^l t)$ by $\Gamma(t) := \gamma(2^l t)/\gamma(2^l)$ in our proof, which is based on the observation that the main properties of $\gamma$ are very similar to that of $\gamma$.

On the other hand, based on this special property (1.6) of $\gamma(t) = |t|^\alpha$, Guo et al. [21] can reduce the proof of (1.3) to obtaining a local smoothing estimate to

\begin{equation}
A_{u, \alpha} f(x, y) := \int_{-\infty}^{\infty} f(x - ut, y - ut^\alpha) \psi_0(t) \, dt.
\end{equation}
In general curves case considered, we may not reduce our theorems to some local smoothing estimate to $A_{u,y}(t)$ since the lack of this special property (1.6). Furthermore, the critical point of the phase function in $A_{u,v}$ is independent of $u$, but the critical point in general curve case will depend on $u$, which leads to essential difficulties. In this paper, we combine the theory of oscillatory integrals and interpolation with the result of Beltran, Hickman and Sogge [3] Proposition 3.2 to show a kind of variable coefficient local smoothing estimate. Our proofs of Theorems A and B rely on this variable coefficient local smoothing estimate, the Littlewood-Paley theory and a bootstrapping argument similar that of Nagel, Stein and Wainger [32].

We should point out that the study of the boundedness properties of the Hilbert transforms along curves when $U$ is a constant, first appeared in the work of Jones [24] and Fabes and Rivièreme [17] for studying the behavior of the constant coefficient parabolic differential operators. Later, the study has been extended to more general classes of curves; see, for example, [38, 33, 15, 7, 9].

In the case of $U(x_1, x_2) = u(x_1)$, many important results have been obtained. For example, when $U(x_1, x_2) = x_1$, Carbery, Wainger and Wright [10] obtained the $L^p(\mathbb{R}^2)$-boundedness of $H^\infty_{U,y}$ and $M^\infty_{U,y}$, for $p \in (1, \infty)$, where $y \in C^2(\mathbb{R})$ is either an odd or even, convex on $\mathbb{R}^+$ satisfying $y(0) = y'(0) = 0$ and $\frac{y''(x)}{y'(x)}$ is decreasing and bounded below on $\mathbb{R}^+$. It is worth noting that these conditions allow the curve to be flat at the origin (e.g. $\gamma(t) = e^{-1/t^2}$). Then, Bennett [4] extended the $L^2(\mathbb{R}^2)$ results of [10] to the case $U(x_1, x_2) = P(x_1)$, where $P(x_1)$ is a polynomial. Some other related results about the one-variable coefficient case, we refer to [8, 30, 13, 39].

For the general two-variable case, we would like to mention more useful results besides the relevant results introduced at the beginning of this paper. Seeger and Wainger [34] obtained the $L^p(\mathbb{R}^2)$-boundedness of $H^\infty_{U,y}$ and $M^\infty_{U,y}$, for $p \in (1, \infty)$, where $U(x_1, x_2)\gamma(t)$ was written as $\Gamma(x_1, x_2, t)$, under some convexity and doubling hypothesis about $\Gamma$. Recently, for $\gamma(t) := |t|^\alpha$ (where $0 < \alpha < 1$ or $\alpha > 1$), Di Plinio, Guo, Thiele and Zorin-Kranich [16] obtained the $L^p(\mathbb{R}^2)$-boundedness, $p \in (1, \infty)$, of $H^\infty_{U,y}$ from some positive constant $\epsilon_0$ and a Lipschitz function $U : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\|U\|_{\text{Lip}} \leq 1$, where Jones’s beta numbers from [25] play an important role in their proof.

The layout of the article is as follows. In Section 2, we get a kind of variable coefficient local smoothing estimate based on Beltran, Hickman and Sogge [3] Proposition 3.2. In Subsection 3.1, we show (i) of Theorem A, whose proof relies heavily on the variable coefficient local smoothing estimate in Section 2. In Subsection 3.2, we prove (ii) of Theorem A. Its proof is less difficult than (i) of Theorem A, since the maximal function in the proof don’t need a summation process relative to Hilbert transform. In Section 4, we first obtain the $L^p(\mathbb{R}^2)$-boundedness, $p \in (1, \infty)$, of the maximal function associated with plane curve $(t, 2^j \gamma(t))$ in lacunary coefficient, which will play a key role in our proof. Then we prove (i) of Theorem B in Subsection 4.1 by bootstrapping an iterated interpolation argument in the spirit of Nagel, Stein and Wainger [32]. Finally, in Subsection 4.2 we prove (ii) of Theorem B by following the approach of (ii) of Theorem A and (i) of Theorem B.

Throughout this paper, the letter “$C$” will denote (possibly different) constants that are independent of the essential variables. $a \leq b$ (or $a \geq b$) means that there exist a positive constant $C$ such that $a \leq Cb$ (or $a \geq Cb$). $a \approx b$ means $a \leq b$ and $b \leq a$. $\hat{f}$ and $\check{f}$ shall denote the Fourier transform and the inverse Fourier transform of $f$, respectively. For $1 < q \leq \infty$, we will denote $q'$ the adjoint number of $q$, i.e. $1/q + 1/q' = 1$. For any set $E$, we use $1_E$ to denote the characteristic
function of \( E \). For any \( a < b \), we will denote \( \int_{a}^{b} f(t) \, dt := \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \).

## 2 A key local smoothing estimate

In this section, the aim is to prove a kind of variable coefficient local smoothing estimate, which will be used to prove Theorem A in Section 3. In the proof of this kind of variable coefficient local smoothing estimate, \([2] \text{ Proposition 3.2}\) is a powerful tool.

Let \( \Gamma_r(t) := \gamma(2^r t) / \gamma(2^r) \) with \( r \in \mathbb{R} \). Assume that \( \phi, \psi \) are smooth functions supported on \( \{ t \in \mathbb{R} : 1/2 \leq |t| \leq 2 \} \). For any \( u \in [1, 2) \), denote

\[
T_u f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u\Gamma_r(t)) \phi(t) \, dt
\]

and

\[
P_u f(x_1, x_2) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x_1 \xi + x_2 \eta)} \hat{f}(\xi, \eta) \psi (2^{-\kappa}(\xi^2 + \eta^2)\frac{1}{2}) \, d\xi \, d\eta.
\]

**Proposition 2.1.** For any \( p > 2 \), there exist positive constants \( \delta \) and \( C \), independent of \( r \) and \( \kappa \), such that

\[
\left\| \sup_{u \in [1, 2)} |T_u P_\kappa f| \right\|_{L^p(\mathbb{R}^2)} \leq C 2^{-\delta\kappa} \| f \|_{L^p(\mathbb{R}^2)} \quad \text{for all } \kappa \in \mathbb{N}.
\]

Firstly, by using the fundamental theorem of calculus to \( |T_u|^p \) and Hölder’s inequality, we have

\[
\sup_{u \in [1, 2)} |T_u g|^p \leq |T_1 g|^p + \left( \int_1^2 |T_u g|^p \, du \right)^{\frac{1}{p}} \left( \int_1^2 |\partial_u T_u g|^p \, du \right)^{\frac{1}{p}},
\]

which enables us to reduce the proof of (2.3) to prove the following (2.4)-(2.6):

\[
\|T_1 P_\kappa f\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta\kappa} \| f \|_{L^p(\mathbb{R}^2)} \quad \text{for all } \kappa \in \mathbb{N} \text{ and } p \in (2, \infty);
\]

\[
\left( \int_1^2 \|T_u P_\kappa f\|_{L^p(\mathbb{R}^2)}^p \, du \right)^{\frac{1}{p}} \leq 2^{-\delta \kappa \kappa} \| f \|_{L^p(\mathbb{R}^2)} \quad \text{for all } \kappa \in \mathbb{N} \text{ and } p \in (2, \infty);
\]

and

\[
\left( \int_1^2 \|\partial_u T_u P_\kappa f\|_{L^p(\mathbb{R}^2)}^p \, du \right)^{\frac{1}{p}} \leq 2^{-\delta \kappa \kappa} 2^\kappa \| f \|_{L^p(\mathbb{R}^2)} \quad \text{for all } \kappa \in \mathbb{N} \text{ and } p \in (2, \infty).
\]

Next, we provide the proofs of (2.4)-(2.6) in turn.

**Proof of (2.4):** It is clear that

\[
\|T_1 P_\kappa f\|_{L^\infty(\mathbb{R}^2)} \leq \| f \|_{L^\infty(\mathbb{R}^2)}.
\]
By interpolation, it only needs to prove (2.4) for $p = 2$.

Consider the case of $p = 2$. Let the multiplier of $T_1 P_s f$ be

$$m_1(\xi, \eta) := \psi\left(\frac{2^{-\delta}(\xi^2 + \eta^2)^{1/2}}{\int_{-\infty}^{\infty} e^{-it\xi - \Gamma_r(t)\eta}} \phi(t) dt\right),$$

and the corresponding phase function be $\Phi_1(t) := -t\xi - \Gamma_r(t)\eta$, we then have $\Phi_1'(t) = -\xi - \Gamma_r'(t)\eta$ and $\Phi_1''(t) = -\Gamma_r''(t)\eta$. Assume $|\xi| \approx 2^M$ and $|\eta| \approx 2^N$, then we have $\max\{M, N\} \approx \kappa$. From (ii) of (H.) and (1.4), we have $|\Gamma_r'(t)| \geq 1$ with a bound independent of $r$, which further yields $|\Phi_1'(t)| \geq 2^N$. We apply van der Corput’s lemma to obtain

$$|m_1(\xi, \eta)| \leq 2^{-\frac{1}{2}N}. \quad (2.7)$$

If $\max\{M, N\} = N$, we use Plancherel’s theorem to obtain (2.4) with $\delta = 1/2$ and $p = 2$. Consider the case $\max\{M, N\} = M$. If $M \leq 32C_3^{(1)} e^{c_3^{(1)}} N$, from (2.7), we also obtain (2.4) with $p = 2$ and some $\delta > 0$; If $M \geq 32C_3^{(1)} e^{c_3^{(1)}} N$, by (iii) of (H.) and (1.4), we have $|\Gamma_r'(t)| \leq 2C_3^{(1)} e^{c_3^{(1)}}$. Therefore, $|\Phi_1'(t)| \geq |\xi|/2 - |\Gamma_r'(t)\eta| \geq 2^M \approx 2^k$. This, combined with van der Corput’s lemma and the fact that $\Phi_1'(t)$ is monotonic, implies $|m_1(\xi, \eta)| \leq 2^{-\delta}$. Applying Plancherel’s theorem again, we obtain (2.4) with $\delta = 1$ and $p = 2$.

We complete the proof of (2.4).

Before giving the proof of (2.5), we state and show some necessary lemmas.

**Lemma 2.2.** Let $t \in [1/2, 2]$. We have the following inequalities hold uniformly in $r$,

(i) $e^{-c_3^{(1)}} \leq \Gamma_r(t) \leq e^{c_3^{(1)}}$;

(ii) $C_2^{(1)} / 2e^{c_3^{(1)}} \leq |\Gamma_r'(t)| \leq 2e^{c_3^{(1)}} C_3^{(1)}$;

(iii) $C_2^{(2)} / 4e^{c_3^{(1)}} \leq |\Gamma_r''(t)| \leq 4e^{c_3^{(1)}} C_3^{(2)}$;

(iv) $|\Gamma_r^{(j)}(t)| \leq 2^{j}e^{C_3^{(j)}} C_3^{(j)}$ for all $2 \leq j \leq N$;

(v) $|((\Gamma_r')^{-1})^{(k)}(t)| \leq 1$ for all $0 \leq k < N$, where $(\Gamma_r')^{-1}$ is the inverse function of $\Gamma_r$.

**Proof of Lemma 2.2.** (i), (ii), (iii) and (iv) are straightforward to verify by (1.2), (ii) and (iii) of (H.). As for (v), noting that $\gamma'$ is strictly monotonic on $\mathbb{R}^+$ with $\gamma'(\mathbb{R}^+) = \mathbb{R}^+$ from (ii) of (H.) and (1.5), then there exists the inverse function $(\Gamma_r')^{-1}$ of $\Gamma_r'$. It is easy to see that (1.5) and Lemma 2.2(ii) imply $|((\Gamma_r')^{-1})^{(k)}(t)| \approx 1$. For $k = 1$, as Lemma 2.2(iii), we have

$$|\Gamma_r'((\Gamma_r')^{-1}(t))| \approx 1, \quad (2.8)$$

which further leads to $((\Gamma_r')^{-1})'(t) = 1/\Gamma_r''((\Gamma_r')^{-1}(t)) \leq 1$. On the other hand, as Lemma 2.2(iv), we also have

$$|\Gamma_r^{(k)}((\Gamma_r')^{-1}(t))| \leq 1 \quad \text{for all } 2 \leq k \leq N. \quad (2.9)$$

By simple calculation, it is easy to see that (2.8) and (2.9) imply Lemma 2.2(v). \qed
Lemma 2.3. Let \( c \in C^M(\mathbb{R}) \) supported on \( \{ t \in \mathbb{R} : |t| \leq 1 \} \), \( M \in \mathbb{N} \). Assume that \( |c^{(k)}(t)| \leq |t|^{2m-k} \) for all \( k \leq M \), where \( m, k \in \mathbb{N} \) and \( m + 1 \leq M \). Then, for all \( \lambda > 0 \), we have

\[
\left| \int_{-\infty}^{\infty} e^{-i\lambda t^2} c(t) \, dt \right| \leq \lambda^{-\frac{1}{2} - m}.
\]

Proof of Lemma 2.3. We will adopt a similar argument to that in [36, page 335, Proposition 3]. Let \( \sigma : \mathbb{R} \to \mathbb{R} \) be a smooth function supported on \( \{ t \in \mathbb{R} : |t| \leq 1 \} \) such that \( \sigma(t) = 1 \) on \( \{ t \in \mathbb{R} : |t| \leq 1 \} \), then write

\[
\int_{-\infty}^{\infty} e^{-i\lambda t^2} c(t) \, dt = \int_{-\infty}^{\infty} e^{-i\lambda t^2} c(t) \sigma(t/\varepsilon) \, dt + \int_{-\infty}^{\infty} e^{-i\lambda t^2} c(t)(1 - \sigma(t/\varepsilon)) \, dt,
\]

where \( \varepsilon > 0 \) will be set later. The first integral is bounded by

\[
\int_{|t| \leq 2\varepsilon} |t|^{2m} \, dt \leq e^{2m+1}.
\]

The second integral can be written as

\[
\int_{-\infty}^{\infty} e^{-i\lambda t^2} D^{(k)} (c(\cdot)(1 - \sigma(\cdot/\varepsilon))) (t) \, dt,
\]

where \( D^{(k)}(f)(t) := \frac{1}{i2\pi} \frac{d^k}{dr^k} \left( \frac{f(t)}{r} \right) \). By calculation, the second integral can be further bounded by

\[
\lambda^{-k} \int_{|t| \geq \varepsilon} |t|^{2m-2k} \, dt = \lambda^{-k} e^{2m-2k+1}, \quad \text{if } 2m - 2k + 1 < 0.
\]

Letting \( \varepsilon := \lambda^{-1/2} \) with \( k = m + 1 \) yields the desired estimate. \( \square \)

Lemma 2.4. Let

\[
a(\lambda, t_0) := \int_{-\infty}^{\infty} e^{-i\lambda x^2 T^{(n)}(x) + i\lambda \frac{1}{2} \int_{0}^{1} (1-\theta)^2 T^{(n)}(x) \, d\theta} \tilde{\phi}(t + t_0) \, dt,
\]

where \( \tilde{\phi}(t) := \phi(t) \chi_{(-\infty,0)}(t) \). Then, for |\lambda| \geq 1 and \( t_0 < 0 \) with |t_0| \approx 1, we have

\[
\left| \partial_{\lambda}^\alpha \partial_{\lambda_0}^\beta a(\lambda, t_0) \right| \leq |\lambda|^{-\frac{1}{2} - |\alpha|} \quad \text{for all } \alpha, \beta \in \mathbb{N} \text{ and } \beta < N - 2,
\]

with a bound independent of \( r \).

Proof of Lemma 2.4. We first verify |\partial_{\lambda} \partial_{\lambda_0} a(\lambda, t_0)| \leq |\lambda|^{-3/2}. A computation gives

\[
\partial_{\lambda} \partial_{t_0} a(\lambda, t_0) = \int_{-\infty}^{\infty} e^{-i\lambda x^2 T^{(n)}(x) + i\lambda \frac{1}{2} \int_{0}^{1} (1-\theta)^2 T^{(n)}(x) \, d\theta} c_{1,1}(\lambda, t, t_0) \, dt,
\]
where $c_{1,1}(\lambda, t, t_0)$ can be expressed as the sum of the following four terms:

\[
\begin{align*}
&-\left(\frac{\lambda^2}{2} \Gamma''(t_0)\right) \left(\frac{t^2}{2} \Gamma''(t_0) + \frac{t^3}{2} \int_0^1 (1-\theta)^2 \Gamma''(\theta t + t_0) d\theta\right) \phi(t + t_0); \\
&-\left(\frac{\lambda^3}{2} \int_0^1 (1-\theta)^2 \Gamma^{(4)}(\theta t + t_0) d\theta\right) \left(\frac{t^2}{2} \Gamma''(t_0) + \frac{t^3}{2} \int_0^1 (1-\theta)^2 \Gamma''(\theta t + t_0) d\theta\right) \phi(t + t_0); \\
&-i \left(\frac{t^2}{2} \Gamma'(t_0) + \frac{t^3}{2} \int_0^1 (1-\theta)^2 \Gamma'(\theta t + t_0) d\theta\right) \phi(t + t_0); \\
&-i \left(\frac{t^2}{2} \Gamma''(t_0) + \frac{t^3}{2} \int_0^1 (1-\theta)^2 \Gamma''(\theta t + t_0) d\theta\right) \phi'(t + t_0).
\end{align*}
\]

The fact that $t_0 < 0$ with $|t_0| \approx 1$ and $t + t_0 < 0$ with $|t + t_0| \approx 1$ imply that $|\lambda| \leq 1$, $\theta(t + t_0) = \theta(t_0 + t) + (1-\theta) t_0 < 0$ and $|\theta t_0| \approx 1$. By Lemma 2.2 we have $|\Gamma^{(n)}(\theta t + t_0)| \leq 1$ for all $n \leq N$ and $|\Gamma'(\theta t + t_0)| \approx 1$. By simple calculation, one has

\[
\left|\partial_t^k e^{-it\frac{1}{2} \int_0^1 (1-\theta)^2 \Gamma''(\theta t + t_0) d\theta} c_{1,1}(\lambda, t, t_0)\right| \leq |\lambda| |\alpha|^{\frac{k}{2}} + |t|^{\frac{k}{2}}
\]

for all $k \in \mathbb{N}$ with $k < N - 3$.

Then, by Lemma 2.3 we get

\[
|\partial_t^\alpha \partial_0^\beta \alpha(\lambda, t_0)| \leq |\lambda| |\alpha|^{\frac{1}{2}} + |\lambda| |\alpha|^{\frac{1}{2}} - 1 \leq |\lambda|^{\frac{1}{2}} - 1.
\]

Similarly, one can write

\[
\partial_t^\alpha \partial_0^\beta \alpha(\lambda, t_0) = \int_{-\infty}^{\infty} e^{-i(\lambda^{\frac{1}{2}} \Gamma''(t_0) + \frac{1}{2} \int_0^1 (1-\theta)^2 \Gamma''(\theta t + t_0) d\theta)} c_{\alpha, \beta}(\lambda, t, t_0) dt,
\]

where

\[
|\partial_t^k e^{-it\frac{1}{2} \int_0^1 (1-\theta)^2 \Gamma''(\theta t + t_0) d\theta} c_{\alpha, \beta}(\lambda, t, t_0)| \leq |\lambda|^{\frac{k}{2}} |\alpha|^{2\theta - k} + |t|^{2\theta - k}
\]

for all $k \in \mathbb{N}$ with $k < N - 2 - \beta$.

By Lemma 2.3 for $N$ large enough, we have

\[
|\partial_t^\alpha \partial_0^\beta \alpha(\lambda, t_0)| \leq |\lambda|^{-\frac{1}{2} + \beta} \quad \text{for all } \alpha, \beta \in \mathbb{N} \text{ with } \beta < N - 2.
\]

This finishes the proof of Lemma 2.4

\[
\square
\]

In the following, we will prove (2.5).

**Proof of (2.5).** Let $\Omega(u) \in C_c^{\infty}(1/2, 5/2)$ and $\Omega(u) = 1$ if $u \in [1, 2]$, and define

\[
\tilde{T}_u f(x_1, x_2) := \Omega(u) \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u\Gamma(t)) \phi(t) dt.
\]

It suffices to prove

\[
\left( \int_{\mathbb{R}^2} \|\tilde{T}_u P_s f\|_{L_p(\mathbb{R}^2)}^p \, dt \right)^{\frac{1}{p}} \leq 2^{-\left(\delta + \frac{1}{2}\right) \kappa} \|f\|_{L_p(\mathbb{R}^2)}.
\]
By taking Fourier transform, we have
\[
\hat{T}_u P_x f(x_1, x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i x_1 \xi + i x_2 \eta} m(u, \xi, \eta) \hat{f}(\xi, \eta) \, d\xi \, d\eta,
\]
where the multiplier of \( \hat{T}_u P_x f \) is defined as
\[
m(u, \xi, \eta) := \Omega(u) \psi \left( 2^{-\kappa} \left( \xi^2 + \eta^2 \right)^{\frac{1}{2}} \right) \int_{-\infty}^{\infty} e^{-i (\xi + u \Gamma_r(t)) \eta} \phi(t) \, dt.
\]
Denote the corresponding phase function by
\[
\Phi_u(t) := t \xi + u \Gamma_r(t) \eta,
\]
and we have
\[
\Phi_u'(t) = \xi + u \Gamma_r(t) \eta \quad \text{and} \quad \Phi_u''(t) = u \Gamma_r''(t) \eta.
\]
By (ii) of Lemma 2.2, we get \( C_2^{(1)} / 2 \varepsilon^{(1)} \leq \Gamma_r'(t) \leq 2 \varepsilon^{(1)} C_3^{(1)} \). Then, if \( |\xi| \geq 6 \varepsilon^{(1)} C_3^{(1)} |\eta| \), it follows that \( |\Phi_u'(t)| \geq |\xi| - |u \Gamma_r'(t) \eta| \geq |\xi| + |\eta| \). Similarly, if \( |\eta| \geq (6 \varepsilon^{(1)} / C_2^{(1)}) |\xi| \), we obtain \( |\Phi_u'(t)| \geq |\xi| + |\eta| \). Integration by parts yields
\[
\left| \int_{-\infty}^{\infty} e^{-i (\xi + u \Gamma_r(t) \eta)} \phi(t) \, dt \right| \leq (|\xi| + |\eta|)^{-n} \quad \text{for } n \leq 4.
\]
Furthermore, noting that \( 2^\kappa = \sqrt{\xi^2 + \eta^2} \leq \max\{|\xi|, |\eta|\} \) and \( \kappa \in \mathbb{N} \), one can obtain
\[
(2.10) \quad \left| \partial_\xi^\alpha \partial_\eta^\beta \left( 1 - \chi(|\xi|/|\eta|) \right) m(u, \xi, \eta) \right| \leq 2^{-\kappa}(1 + |\xi| + |\eta|)^{-3} \quad \text{for all } \alpha, \beta \in \mathbb{N} \text{ and } \alpha + \beta \leq 3,
\]
where \( \chi \in C_c^\infty(\mathbb{R}^+) \) such that \( \chi = 1 \) on \( [C_2^{(1)} / 6 \varepsilon^{(1)}, 6 \varepsilon^{(1)} C_3^{(1)}] \).

Denote
\[
\hat{T}_u^1 P_x f(x_1, x_2) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i x_1 \xi + i x_2 \eta} \chi(|\xi|/|\eta|) m(u, \xi, \eta) \hat{f}(\xi, \eta) \, d\xi \, d\eta
\]
and
\[
\hat{T}_u^2 P_x f(x_1, x_2) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i x_1 \xi + i x_2 \eta} (1 - \chi(|\xi|/|\eta|)) m(u, \xi, \eta) \hat{f}(\xi, \eta) \, d\xi \, d\eta.
\]
Since \( \mathcal{F}_{\xi, \eta}(2^\kappa (1 - \chi(|\xi|/|\eta|)) m(u, \xi, \eta)) \in L^1(\mathbb{R}^2) \) by (2.10), then
\[
||\hat{T}_u^2 P_x f||_{L^p(\mathbb{R}^2)} \leq 2^{-\kappa} ||f||_{L^p(\mathbb{R}^2)},
\]
which yields
\[
\left( \int_2^\infty \left( \int_{ \frac{2^\kappa}{4} } ||\hat{T}_u^2 P_x f||_{L^p(\mathbb{R}^2)}^p \, dt \right)^{\frac{1}{p}} \right)^{\frac{1}{2}} \leq 2^{-\kappa \left( 1 - \frac{1}{p} \right)} ||f||_{L^p(\mathbb{R}^2)} \quad \text{for all } \kappa \in \mathbb{N} \text{ and } p \in (2, \infty).
\]

\( ^1 \)We denote \( \mathcal{F}_{\xi, \eta} \) means the Fourier transform about \( (\xi, \eta) \in \mathbb{R}^2 \).
Thus, it remains to prove

\[
\left( \int_{\frac{1}{2}}^{\frac{3}{2}} \left\| \hat{T}_{u}^{1} P_{f} \right\|_{L^p(\mathbb{R}^{2})}^p \, du \right)^{\frac{1}{p}} \leq 2^{-\left(\frac{\gamma + \frac{1}{p}}{p}\right)\kappa} \|f\|_{L^p(\mathbb{R}^{2})} \quad \text{for all } \kappa \in \mathbb{N} \text{ and } p \in (2, \infty).
\]

Without loss of generality, we may assume that $\xi, \eta > 0$ in (2.11), and the other cases can be treated similarly. If $\gamma$ is odd, then $\Gamma_r$ is also odd. Furthermore, we have $\Phi'_\gamma(t) \sim \xi + \eta$. Then a similar argument to that of $\hat{T}_{u}^{2} P_{f}$ gives (2.13) in this case. If $\gamma$ is an even function, it implies that $\Gamma_r$ is also an even function. We will consider the following two cases. Define

\[
\begin{align*}
\bar{m}_1(u, \xi, \eta) &:= \chi(|\xi|/|\eta|)\Omega(u)\psi \left(2^{-\kappa}(\xi^2 + \eta^2)^\frac{1}{2}\right) \left(\int_{-\infty}^{0} e^{-i(\xi \xi + \eta \eta)\theta} \phi(t) \, dt\right) 1_{\mathbb{R}^{+}}(\xi) 1_{\mathbb{R}^{+}}(\eta), \\
\bar{m}_1(u, \xi, \eta) &:= \chi(|\xi|/|\eta|)\Omega(u)\psi \left(2^{-\kappa}(\xi^2 + \eta^2)^\frac{1}{2}\right) \left(\int_{0}^{\infty} e^{-i(\xi \xi + \eta \eta)\theta} \phi(t) \, dt\right) 1_{\mathbb{R}^{+}}(\xi) 1_{\mathbb{R}^{+}}(\eta),
\end{align*}
\]

and the associated operators are $\hat{T}_{u}^{1} - P_{f}$ and $\hat{T}_{u}^{1} + P_{f}$, respectively.

For $\hat{T}_{u}^{1} - P_{f}$, it is easy to see that $\Phi'_\gamma(t) \geq \xi + \eta$ in $\bar{m}_1$. Therefore, as $\hat{T}_{u}^{2} P_{f}$, we may also obtain

\[
\left( \int_{\frac{1}{2}}^{\frac{3}{2}} \left\| \hat{T}_{u}^{1} P_{f} \right\|_{L^p(\mathbb{R}^{2})}^p \, du \right)^{\frac{1}{p}} \leq 2^{-\left(\frac{\gamma + \frac{1}{p}}{p}\right)\kappa} \|f\|_{L^p(\mathbb{R}^{2})} \quad \text{for all } \kappa \in \mathbb{N} \text{ and } p \in (2, \infty).
\]

For $\hat{T}_{u}^{1} + P_{f}$, let $\Phi'_\gamma(t_0) = \xi + u \Gamma_r(t_0) \eta = 0$, then $t_0 < 0$ is the critical point. Noting that $\gamma'$ is strictly monotonic on $\mathbb{R}^+$ with $\gamma'(\mathbb{R}^+) = \mathbb{R}^+$, we may write $t_0 := (\Gamma_r')^{-1}(-\xi/\eta)$. By Taylor’s formula, we have

\[
\begin{align*}
\Phi(u(t + t_0) &= t_0 \xi + u \Gamma_r(t_0) \eta + \frac{u \eta}{2} \Gamma''_r(t_0) + \frac{u \eta}{2} t_0 \int_{0}^{1} (1 - \theta)^2 \Gamma'''_r(\theta t + t_0) \, d\theta. \\
\end{align*}
\]

Denote $\tilde{\phi}(t) := \phi(t) 1_{(-\infty, 0)}(t)$. One can write

\[
\bar{m}_1(u, \xi, \eta) = e^{i\Phi(u, \xi, \eta)} c(u, \xi, \eta),
\]

where

\[
\Phi(u, \xi, \eta) := -\Phi_u(t_0) = -t_0 \xi - u \Gamma_r(t_0) \eta;
\]

\[
c(u, \xi, \eta) := \chi(|\xi|/|\eta|)\Omega(u)\psi \left(2^{-\kappa}(\xi^2 + \eta^2)^\frac{1}{2}\right) a(u \eta, t_0) 1_{\mathbb{R}^{+}}(\xi) 1_{\mathbb{R}^{+}}(\eta)
\]

and

\[
a(\lambda, t_0) := \int_{-\infty}^{\infty} e^{-\frac{i}{2}(\lambda^2 \Gamma''_r(t_0) + \int_{0}^{1} (1 - \theta)^2 \Gamma'''_r(\theta t + t_0) \, d\theta)\theta} \tilde{\phi}(t + t_0) \, dt.
\]

As a result,

\[
\hat{T}_{u}^{1} P_{f}(x_1, x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ix_1 \xi + ix_2 \eta} e^{i\Phi(u, \xi, \eta)} c(u, \xi, \eta) \tilde{f}(\xi, \eta) \, d\xi \, d\eta,
\]
and it suffices to prove that there exists a positive constant $\delta$, independent of $r$, such that
\begin{equation}
\left( \int_{\mathbb{R}^2} \left\| \tilde{T}_{u_r}^1 P_k f \right\|_{L^p(\mathbb{R}^2)}^p \, du \right)^{\frac{1}{p}} \lesssim 2^{-\left(\delta + \frac{1}{p}\right)k} \left\| \mathcal{F} f \right\|_{L^p(\mathbb{R}^2)} \quad \text{for all } k \in \mathbb{N} \text{ and } p \in (2, \infty).
\end{equation}

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function supported on $\{(x_1, x_2) \in \mathbb{R}^2 : |(x_1, x_2)| \leq 2\}$ such that $\varphi(x_1, x_2) = 1$ on $\{(x_1, x_2) \in \mathbb{R}^2 : |(x_1, x_2)| \leq 1\}$. We write
\begin{equation}
b(x_1, x_2, u, \xi, \eta) := \varphi(x_1, x_2)c(u, \xi, \eta) \quad \text{and} \quad \Psi(x_1, x_2, u, \xi, \eta) := x_1 \xi + x_2 \eta + \Phi(u, \xi, \eta).
\end{equation}

Define
\begin{equation}
A f(x_1, x_2, u) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i \Psi(x_1, x_2, u, \xi, \eta)} b(x_1, x_2, u, \xi, \eta) \hat{f}(\xi, \eta) \, d\xi \, d\eta.
\end{equation}

We first prove that there exists a positive constant $\delta$ independent of $r$ such that
\begin{equation}
\| A f \|_{L^p(\mathbb{R}^2)} \lesssim 2^{-\left(\delta + \frac{1}{p}\right)k} \| \mathcal{F} f \|_{L^p(\mathbb{R}^2)} \quad \text{for all } k \in \mathbb{N} \text{ and } p \in (2, \infty).
\end{equation}

Recall that $t_0(u, \xi, \eta) = \left( \Gamma_{r}^* \right)^{-1} \left( -\xi/u \eta \right)$. By (v) of Lemma 2.2, we have
\begin{equation}
\left| \partial^\alpha_a \partial^\beta_\xi \partial^\gamma_\eta t_0(\xi, \eta) \right| \lesssim (1 + |\xi| + |\eta|)^{-\beta - \gamma} \quad \text{for all } \alpha, \beta, \gamma \in \mathbb{N} \text{ with } \alpha + \beta + \gamma < N.
\end{equation}

By Lemma 2.2, for $|\lambda| \geq 1$ and $t_0 < 0$ with $|t_0| \approx 1$, we obtain
\begin{equation}
\left| \partial^\alpha_a \partial^\beta_\xi a(\lambda, t_0) \right| \lesssim |\lambda|^{-\frac{1}{2} - \alpha} \quad \text{for all } \alpha, \beta \in \mathbb{N} \text{ with } \beta < N - 2.
\end{equation}

This, combined with (2.16), yields
\begin{equation}
\left| \partial^\alpha_a \partial^\beta_\xi a(\lambda, \xi, \eta) \right| \lesssim (1 + |\xi| + |\eta|)^{-\frac{1}{2} - \beta - \gamma} \quad \text{for all } \alpha, \beta, \gamma \in \mathbb{N} \text{ with } \alpha + \beta + \gamma < N,
\end{equation}

which implies that
\begin{equation}
\left| \partial^\alpha_a \partial^\beta_\xi \partial^\gamma_\eta b(x_1, x_2, u, \xi, \eta) \right| \lesssim (1 + |\xi| + |\eta|)^{-\alpha_1 - \alpha_4} \quad \text{for all } \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathbb{N} \text{ with } \alpha_2 + \alpha_4 + \alpha_5 < N.
\end{equation}

It is easy to see that supp $b \subset \{(x_1, x_2) \in \mathbb{R}^2 : |(x_1, x_2)| \leq 2\} \times (1/2, 5/2) \times \{(\xi, \eta) \in \mathbb{R}^2 : \xi \approx \eta \approx \xi, \eta > 0, \eta > 0\}$.

(2.17) implies the multiplier of $\tilde{T}_{u_r}^1 P_k$ can be bounded from above by $2^{-s/2}$. We apply Plancherel’s theorem to obtain
\begin{equation}
\left( \int_{\mathbb{R}^2} \left\| \tilde{T}_{u_r}^1 P_k f \right\|_{L^2(\mathbb{R}^2)}^2 \, du \right)^{\frac{1}{2}} \lesssim 2^{-\frac{s}{2}} \| \mathcal{F} f \|_{L^2(\mathbb{R}^2)}.
\end{equation}

By interpolation, it suffices to prove (2.14) for $p \in [6, \infty)$. We will first prove (2.15) for $p \in [6, \infty)$ by using local smoothing estimates provided in [3], then prove (2.14) by using (2.15) in the case $p \in [6, \infty)$.
For the convenience of writing, we denote $e_1 := (1, 0)$, $B(0, 2) := \{(x_1, x_2) \in \mathbb{R}^2 : |(x_1, x_2)| \leq 2\}$, and $Z := B(0, 2) \times (1/2, 5/2)$ and

$$
\Gamma := \left\{ Q^{-1}(\xi, \eta) \in \mathbb{R}^2 : 1/4 \leq \xi^2 + \eta^2 \leq 4, C_2^{(1)}/6e^{c_3} \leq |\xi|/|\eta| \leq 6e^{c_3} C_3^{(1)}, \xi > 0, \eta > 0 \right\},
$$

where $Q$ is the orthogonal matrix so that $Qe_1 = (\sqrt{2}/2, \sqrt{2}/2)$. We write $\Phi_Q(u, \xi, \eta) := \Phi(u, Q(\xi, \eta))$, $b_Q(x_1, x_2, u, \xi, \eta) := b(x_1, x_2, u, Q(\xi, \eta))$ and $\Psi_Q(x_1, x_2, u, \xi, \eta) := \Psi(x_1, x_2, u, Q(\xi, \eta))$. For $\alpha, \beta, \gamma \in \mathbb{N}$ with $\alpha + \beta + \gamma < N$ and $a_1, \ldots, a_5 \in \mathbb{N}$ with $a_3 + a_4 + a_5 < N$, we now claim the following four terms hold:

(2.19) \begin{equation}
|\partial_{\xi}^2 \partial_{\eta}^2 \partial_{\xi} \Phi_Q(u, \xi, \eta)| \leq 1 \quad \text{on } Z \times \Gamma;
\end{equation}

(2.20) \begin{equation}
|\partial_{\xi}^3 \partial_{\eta} \partial_{\xi}^2 \partial_{\xi} \partial_{\eta} b_Q(x_1, x_2, u, \xi, \eta)| \leq (1 + |\xi| + |\eta|)^{-\frac{1}{2} - \alpha_4 - \alpha_5} \quad \text{on } Z \times \Gamma;
\end{equation}

(2.21) \begin{equation}
\begin{pmatrix}
\partial_{\xi}[\Psi_Q(x_1, x_2, u, \xi, \eta)] \\
\partial_{\eta}[\Psi_Q(x_1, x_2, u, \xi, \eta)]
\end{pmatrix} = Q^T \quad \text{on } Z \times \Gamma;
\end{equation}

and

(2.22) \begin{equation}
|\partial_{\xi}^2 \partial_{\eta} \Psi_Q(x_1, x_2, u, \xi, \eta)| \geq 1 \quad \text{on } Z \times \Gamma,
\end{equation}

where the bounds are independent of $r$.

Recall that $\Psi(x_1, x_2, u, \xi, \eta) = x_1 \xi + x_2 \eta + \Phi(u, \xi, \eta)$, $\Phi(u, \xi, \eta) = -t_0 \xi - u \Gamma_r(t_0) \eta$ and $t_0 = (\Gamma_r')^{-1}(\xi/\eta)$. Then, (2.19) follows from (2.16) and Lemma 2.2, (2.20) follows from (2.18) and Lemma 2.2, (2.21) is easy to check. We now turn to verify (2.22). It suffices to verify

$$
|\partial_{\xi}^2 \partial_{\eta} \Phi(u, \xi, \eta) - 2\partial_{\xi} \partial_{\eta} \partial_{\xi} \Phi(u, \xi, \eta) + \partial_{\eta}^2 \partial_{\xi} \Phi(u, \xi, \eta)| \geq 1 \quad \text{on } Z \times Q \Gamma.
$$

Indeed, by noting that $\xi + u \Gamma_r'(t_0) \eta = 0$, we have the following results:

$$
\partial_{\eta} t_0 = \frac{\xi}{u^2 \eta \Gamma_r''(t_0)}, \quad \partial_{\xi} t_0 = -\frac{1}{u \eta \Gamma_r''(t_0)}, \quad \partial_{\eta} \Phi(u, \xi, \eta) = -\Gamma_r(t_0) \eta;
$$

$$
\partial_{\eta} \partial_{\eta} \Phi(u, \xi, \eta) = \frac{\Gamma_r'(t_0)}{u \eta \Gamma_r''(t_0)} - \Gamma_r(t_0);
$$

$$
\partial_{\xi}^2 \partial_{\eta} \Phi(u, \xi, \eta) = \frac{\Gamma_r''(t_0) \Gamma_r''''(t_0) - \Gamma_r''(t_0)^2}{u^2 \eta \Gamma_r'''(t_0)^3};
$$

$$
\partial_{\xi} \partial_{\eta} \partial_{\xi} \Phi(u, \xi, \eta) = -\frac{\xi \Gamma_r'(t_0)}{u \eta \Gamma_r''(t_0)} - \Gamma_r(t_0);
$$

$$
\partial_{\eta}^2 \partial_{\xi} \Phi(u, \xi, \eta) = \frac{\Gamma_r''(t_0) \Gamma_r''''(t_0) - \Gamma_r''(t_0)^2}{u^2 \eta \Gamma_r'''(t_0)^3};
$$

Note that $\xi \approx \eta \approx 1$ and $\xi > 0$ and $\eta > 0$ if $(\xi, \eta) \in Q \Gamma$. By Lemma 2.2 and (i) of (Hl), we then have

$$
|\partial_{\xi}^2 \partial_{\eta} \Phi(u, \xi, \eta) - 2\partial_{\xi} \partial_{\eta} \partial_{\xi} \Phi(u, \xi, \eta) + \partial_{\eta}^2 \partial_{\xi} \Phi(u, \xi, \eta)|
$$
\[
\frac{\Gamma'(t_0) \Gamma'''(t_0) - \Gamma''(t_0)^2}{u^2 \eta \Gamma''(t_0)^3} |\xi + \eta|^2 \geq 1 \quad \text{on } Z \times Q \Gamma.
\]

Combining (2.19), (2.22), we can apply [3] Proposition 3.2 to obtain, for any \( p \in [6, \infty) \) and \( \kappa \in \mathbb{N} \), there exists \( \delta \in (0, 1/p) \) independent of \( r \) such that

\[
\|A f\|_{L^p(\mathbb{R}^3)} = 2^{-(\delta + \frac{1}{p})} \left\| 2^{(\delta + \frac{1}{p})} A f \right\|_{L^p(\mathbb{R}^3)} \leq 2^{-(\delta + \frac{1}{p})} \|f\|_{L^p(\mathbb{R}^2)}
\]

with a bound independent of \( r \). This completes the proof of (2.15) for \( p \in [6, \infty) \).

We now will use (2.15) to prove (2.14) for any \( p \in [6, \infty) \). Note that

\[
\varphi(x_1, x_2) \tilde{T}_{u,-} P_k f(x_1, x_2) = \int_{\mathbb{R}^4} e^{-ix\xi} e^{-in\eta} e^{i\xi x_1} e^{i\eta x_2} \varphi(x_1, x_2) e^{\Phi(u, \xi, \eta)} f(x, y) \, dx \, dy \, d\xi \, d\eta.
\]

Denote \( L_{\xi} f := (1 - \Delta_{\xi})(1 + |x|^2)^{-1} \) and \( L_{\eta} f := (1 - \Delta_{\eta})(1 + |y|^2)^{-1} \). One can write

\[
L_{\xi} L_{\eta} \left( e^{i\xi x_1} e^{i\eta x_2} e^{\Phi(u, \xi, \eta)} \varphi(x_1, x_2) c(u, \xi, \eta) f(x, y) \right) = e^{i\xi x_1} e^{i\eta x_2} e^{\Phi(u, \xi, \eta)} \tilde{c}(x_1, x_2, u, \xi, \eta) f(x, y) (1 + |x|^2)^{-1} (1 + |y|^2)^{-1}
\]

for some \( \tilde{c} \in S^{-1/2} \) satisfying

\[
\left| \partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta \partial_{\eta_1}^\gamma \partial_{\eta_2}^\delta \tilde{c}(x_1, x_2, u, \xi, \eta) \right| \leq (1 + |\xi| + |\eta|)^{-\frac{1}{2} - \alpha_1 - \alpha_2}
\]

for all \( \alpha, \beta, \gamma, \delta \in \mathbb{N} \) with \( \alpha_1 + \alpha_2 < \alpha \), where the constant is independent of \( r \), and \( \text{supp} \tilde{c} \subset B(0, 2) \times (1/2, 5/2) \times \{(\xi, \eta) \in \mathbb{R}^2 : \xi \approx \eta \approx 2^k, \xi > 0, \eta > 0 \} \).

Integration by parts shows that \( \varphi(x_1, x_2) \tilde{T}_{u,-} P_k f(x_1, x_2) \) can be written as

\[
\int_{\mathbb{R}^4} e^{-ix\xi} e^{-in\eta} e^{i\xi x_1} e^{i\eta x_2} e^{\Phi(u, \xi, \eta)} \tilde{c}(x_1, x_2, u, \xi, \eta) f(x, y) \, dy \, d\xi \, d\eta
\]

\[
= \int_{\mathbb{R}^2} e^{i\xi x_1} e^{i\eta x_2} e^{\Phi(u, \xi, \eta)} \tilde{c}(x_1, x_2, u, \xi, \eta) g(\xi, \eta) \, d\xi \, d\eta,
\]

where \( g(x, y) := f(x, y)(1 + |x|^2)^{-1} (1 + |y|^2)^{-1} \). As in (2.23), we get

\[
\left\| \varphi \tilde{T}^1_{u,-} P_k f \right\|_{L^p(\mathbb{R}^3)} = 2^{-(\delta + \frac{1}{p})} \|g\|_{L^p(\mathbb{R}^2)} \leq 2^{-(\delta + \frac{1}{p})} \int_{\mathbb{R}^2} \frac{|f(x, y)|^p}{(1 + |x|^2 + |y|^2)^p} \, dx \, dy.
\]

Then, because \( \tilde{T}^1_{u,-} P_k f(x_1 + x_0, x_2 + y_0) = \tilde{T}^1_{u,-} P_k (f(\cdot + x_0, \cdot + y_0))(x_1, x_2) \), we obtain

\[
\int_{\mathbb{R}^3} |\varphi(x_1 + x_0, x_2 + y_0) \tilde{T}^1_{u,-} P_k f(x_1, x_2, u)|^p \, dx_1 \, dx_2 \, du
\]

\[
\leq 2^{-(\delta + \frac{1}{p})} \int_{\mathbb{R}^2} \frac{|f(x, y)|^p}{(1 + |x - x_0|^2 + |y - y_0|^2)^p} \, dx \, dy.
\]

By integration in \( x_0, y_0 \), we have

\[
\int_{\mathbb{R}^2} |\tilde{T}^1_{u,-} P_k f(x_1, x_2, u)|^p \, dx_1 \, dx_2 \, du \leq 2^{-(\delta + \frac{1}{p})} \int_{\mathbb{R}^2} |f(x, y)|^p \, dx \, dy.
\]
We have proved (2.14) for any $p \in [6, \infty)$ and thus finish the proof of (2.5).

**Proof of (2.6):** The proof of (2.6) is very similar as that of (2.5) with two slight modifications. Firstly, we will replace $\phi(t)$ in (2.1) by $-i\ell_{\gamma}(t)\phi(t)$. Note that the latter maybe belongs to $C^N$ on $\{t \in \mathbb{R} : 1/2 \leq |t| \leq 2\}$ with $N \in \mathbb{N}$ large enough, but it doesn’t affect our proof. Secondly, we write $\eta\psi(2^{-x}(\xi^2 + \eta^2)^{1/2})$ in (2.2) as

$$2^x \times \left(2^{-x}(\xi^2 + \eta^2)^{1/2} \psi(2^{-x}(\xi^2 + \eta^2)^{1/2})\right) \times \left(\frac{\eta}{(\xi^2 + \eta^2)^{1/2}}\right).$$

Note that $2^{-x}(\xi^2 + \eta^2)^{1/2} \psi(2^{-x}(\xi^2 + \eta^2)^{1/2})$ just like $\psi(2^{-x}(\xi^2 + \eta^2)^{1/2})$, and $\eta/(\xi^2 + \eta^2)^{1/2}$ is a harmless quantity. By repeating the process of proving (2.5), we can control the LHS of (2.6) by $2^{-(\delta + \frac{1}{p})}\|f\|_{L^p(\mathbb{R}^2)}$ times a extra factor $2^x$, which is just the RHS of (2.6).

### 3 Proof of Theorem A

We will give the proof of Theorem A in the following two subsections.

#### 3.1 Proof of (i) of Theorem A

The main strategy of our proof is to decompose the operator into a sum of the low frequency part and high frequency part by defining a measurable function $l_\gamma : \mathbb{R}^2 \to \mathbb{R}$ in (3.3). Here, replacing $U_\gamma$ by $2\Gamma_l$ is a useful observation. For the low frequency part, we bound it by the sum of the Hardy-Littlewood maximal operator and maximal truncated Hilbert transform. For the high frequency part, we further split it into two parts by the Littlewood-Paley projection in the first variable. The first part can also be controlled by the Hardy-Littlewood maximal operator. For the second part, we prove it by a local smoothing estimate which has been obtained in Section 2.

We first assume that $U(x_1, x_2) > 0$ for almost every $(x_1, x_2) \in \mathbb{R}^2$ and the other case $U(x_1, x_2) < 0$ can be handled similarly. Let $V : \mathbb{R}^2 \to \mathbb{Z}$ be a measurable function satisfying

$$2^{V(x_1, x_2)} \leq U(x_1, x_2) < 2^{V(x_1, x_2) + 1}. \quad (3.1)$$

For any given $k_0 \in \mathbb{Z}$, we define

$$U^{(k_0)}(x_1, x_2) := 2^{k_0} \frac{U(x_1, x_2)}{2^{V(x_1, x_2)}}. \quad (3.2)$$

It is easy to see that $U^{(V(x_1, x_2))}(x_1, x_2) = U(x_1, x_2)$. Recall that $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth function supported on $\{t \in \mathbb{R} : 1/2 \leq |t| \leq 2\}$ with the property that $0 \leq \psi(t) \leq 1$ and $\sum_{k \in \mathbb{Z}} \psi_k(t) = 1$ for any $t \neq 0$, where $\psi_k(t) = \psi(2^{-k}t)$. For any given $l \in \mathbb{Z}$, set

$$H^l_{U^{(\gamma)}} f(x_1, x_2) := \text{p. v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - U(x_1, x_2)\gamma(t))\psi_l(t) \frac{dt}{t}. \quad (2)$$
Then, we can write
\[ H_{U_\gamma} P_k^{(2)} f = \sum_{l \in \mathbb{Z}} H_{U_\gamma}^l P_k^{(2)} f. \]

Furthermore, for simplicity, let \( z := (x_1, x_2), U_\gamma := U(x_1, x_2) \) and \( V_\gamma := V(x_1, x_2). \) Note that \( \gamma \) is strictly increasing on \( \mathbb{R}^+ \) with \( \gamma(\mathbb{R}^+) = \mathbb{R}^+ \) from (ii) of (H) and (1.4). For these \( k \in \mathbb{Z} \) and \( V_\gamma \in \mathbb{Z}, \) we can denote \( l_z : \mathbb{R}^2 \to \mathbb{R} \) be a measurable function satisfying
\[ (3.3) \]
\[ 2^k 2^{V_\gamma} (2^l) = 1. \]

We then further split \( H_{U_\gamma} P_k^{(2)} f \) into the following low frequency part \( H_{U_\gamma}^l P_k^{(2)} f \) and the high frequency part \( H_{U_\gamma}^H P_k^{(2)} f, \) where
\[ H_{U_\gamma}^l P_k^{(2)} f := \sum_{l \leq l_z} H_{U_\gamma}^l P_k^{(2)} f \quad \text{and} \quad H_{U_\gamma}^H P_k^{(2)} f := \sum_{l > l_z} H_{U_\gamma}^l P_k^{(2)} f. \]

Consider \( H_{U_\gamma}^l P_k^{(2)} f. \) We compare it with the following operator \( \mathbb{H}_{U_\gamma}^l P_k^{(2)} f, \)
\[ \mathbb{H}_{U_\gamma}^l P_k^{(2)} f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} P_k^{(2)} f(x_1 - t, x_2) \phi_l(t) \frac{dt}{t}, \]
where \( \phi_l(t) := \sum_{l \leq l_z} \psi_l(t). \) We can bound \( |\mathbb{H}_{U_\gamma}^l P_k^{(2)} f| \) by
\[ \left| \text{p.v.} \int_{-\infty}^{\infty} P_k^{(2)} f(x_1 - t, x_2) (\phi_l(t) - 1) \frac{dt}{t} \right| + \left| \text{p.v.} \int_{-\infty}^{\infty} P_k^{(2)} f(x_1 - t, x_2) \frac{dt}{t} \right| \leq M^{(1)} P_k^{(2)} f(x_1, x_2) + H^{(1)} P_k^{(2)} f(x_1, x_2). \]

Here and hereafter, \( M^{(j)}, j = 1, 2, \) denotes the Hardy-Littlewood maximal operator applied in the \( j \)-th variable. \( H^{(1)} \) denotes the maximal truncated Hilbert transform applied in the first variable. Since both \( M^{(1)} \) and \( H^{(1)} \) are known to be bounded on \( L^p(\mathbb{R}^2) \), we may conclude that
\[ (3.4) \]
\[ \left\| \mathbb{H}_{U_\gamma}^l P_k^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \leq \left\| P_k^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \quad \text{uniformly in } k \in \mathbb{Z} \text{ for all } p \in (1, \infty). \]

The difference between \( H_{U_\gamma}^l P_k^{(2)} f(x_1, x_2) \) and \( \mathbb{H}_{U_\gamma}^l P_k^{(2)} f(x_1, x_2) \) can be written as
\[ (3.5) \]
\[ \int_{-\infty}^{\infty} \left[ \int_0^{U_\gamma(t)} \tilde{\partial}_s \left( P_k^{(2)} f(x_1 - t, x_2 - s) \right) dt \right] \phi_l(t) \frac{dt}{t} \]
\[ = - \int_{-\infty}^{\infty} \left[ \int_0^{U_\gamma(t)} \tilde{P}_k^{(2)} f(x_1 - t, x_2 - s) dt \right] 2^k U_\gamma(t) \phi_l(t) \frac{dt}{t}, \]
where \( \tilde{P}_k^{(2)} f \) denotes the Littlewood-Paley projection in the second variable corresponding to \( (\cdot) \psi(\cdot)_k. \) From (3.1) and (3.3), we have \( 2^k U_\gamma \leq 1 / \gamma(2^l). \) Noticing (1.4) implies \( |\gamma(2^{l-\gamma}) / \gamma(2^l)| \leq e^{-C_l^{(1)} / 2}, \) we can bound the last expression in (3.5) by
\[ \sum_{j \in \mathbb{N}} \int_{l_z \geq 2^l} \int_0^{U_\gamma(t)} \tilde{P}_k^{(2)} f(x_1 - t, x_2 - s) \left| \frac{\gamma(2^{l-\gamma})}{\gamma(2^l)} \right| dt \leq M^{(1)} M^{(2)} \tilde{P}_k^{(2)} f(x_1, x_2). \]
From the $L^p(\mathbb{R})$-boundedness of $M^{(1)}$ and $M^{(2)}$, we have

\begin{equation}
\left\|H_{U,L}^{l}f^{(2)}_k - \mathbb{E}H_{U,L}^{l}f^{(2)}_k\right\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)}\quad \text{uniformly in } k \in \mathbb{Z} \text{ for all } p \in (1, \infty),
\end{equation}

By replacing $f$ in (3.6) with $P^{(2)}_k f$ and the fact that $P^{(2)}_k f = P^{(2)}_k P^{(2)}_k f$ essentially, we obtain the single annulus $L^p(\mathbb{R}^2)$-boundedness of $H_{U,L}^{l} - \mathbb{E}H_{U,L}^{l}$. This, combined with (3.4), leads to

\begin{equation}
\left\|H_{U,L}^{l}f^{(2)}_k\right\|_{L^p(\mathbb{R}^2)} \leq \|P^{(2)}_k f\|_{L^p(\mathbb{R}^2)}\quad \text{uniformly in } k \in \mathbb{Z} \text{ for all } p \in (1, \infty).
\end{equation}

Consider $H_{U,L}^{l}f^{(2)}_k$. We rewrite it as $\sum_{l=0}^{\infty} H_{U,L}^{l}f^{(2)}_k$. Hence, for $l \in \mathbb{N}$, it is enough to show that there exists a positive constant $\delta$ such that

\begin{equation}
\left\|H_{U,L}^{l}f^{(2)}_k\right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta l}\|f\|_{L^p(\mathbb{R}^2)}\quad \text{for all } l \in \mathbb{N} \text{ and } p \in (2, \infty).
\end{equation}

We split

\begin{equation}
H_{U,L}^{l}f^{(2)}_k = \sum_{j \leq 0} H_{U,L}^{l}f^{(1)}_{j-l-1}P^{(2)}_k f + \sum_{j \geq 1} H_{U,L}^{l}f^{(1)}_{j-l-1}P^{(2)}_k f =: H_1P^{(2)}_k f + H_2P^{(2)}_k f.
\end{equation}

Consider $H_1P^{(2)}_k f$. Let $Q^{(1)}_{-l-1} := \sum_{j \leq 0} P^{(1)}_{j-l-1}$ then we bound

\begin{equation}
|H_1P^{(2)}_k f| \leq \sup_{k_0 \in \mathbb{Z}} \sup_{u \in (1,2)} \left|H_{2k_0u,Y}^{l+\lambda}Q^{(1)}_{-l-1}P^{(2)}_k f\right| \quad \text{with } 2^{-\lambda} = 1.
\end{equation}

The multiplier of $H_{2k_0u,Y}^{l+\lambda}Q^{(1)}_{-l-1}P^{(2)}_k f$ is equal to

\[
\int_{-\infty}^{\infty} e^{-i\xi(t)}\gamma(2^l\xi)\psi(t) \, dt \cdot \frac{d}{d^l} \psi(\xi) = \frac{\psi(\xi)}{2^k \xi}.
\]

where $\phi_{-l}(\cdot) := \sum_{j \leq 0} \psi_j \phi_{j-l}(\cdot)$. Integration by parts about $e^{-i\xi(2^l\gamma(t))}$ shows that we can further write it as the sum of the following two parts:

\[
\begin{align*}
& \left\{ \frac{1}{2^k2^l\gamma(2^l+1)} \int_{-\infty}^{\infty} e^{-i\xi(t)}\gamma(2^l+1)\psi(t) \, dt \cdot \frac{d}{d^l} \psi(\xi) \right. \\
& \left. \frac{i}{2^k2^l\gamma(2^l+1)} \int_{-\infty}^{\infty} e^{-i\xi(t)}\gamma(2^l+1) \psi(t) \, dt \cdot \frac{d}{d^l} \psi(\xi) \right\}
\end{align*}
\]

We will only consider the first term above, since the second term above can be handled similarly by noticing that $|t^2\gamma(t)|/\gamma(2^l+1) \leq C_3$ for any $t > 0$. With abusing notions, we write $H_{2k_0u,Y}^{l+\lambda}Q^{(1)}_{-l-1}P^{(2)}_k f(x_1, x_2)$ as

\[
\frac{1}{2^k2^l\gamma(2^l+1)} \int_{-\infty}^{\infty} \tilde{Q}^{(1)}_{-l-1}P^{(2)}_k f(x_1-t, x_2-2^l\gamma(t)) \gamma(2^l+1)\psi_{j-l}(t) \, dt.
\]
where \( F(\tilde{Q}^{(1)}_{-\nu_{-l}} f)(\xi,\eta) := \sum_{j \geq 0} 2^j \tilde{\psi}(2^{-j+\nu_{-l}+1} \xi) \hat{f}(\xi,\eta) \) with \( \tilde{\psi}(\cdot) := \cdot \psi(\cdot) \), and \( F(P_k^{(2)} f)(\xi,\eta) := \frac{\psi_{\nu_{-l}}}{2^{\nu_{-l}}} f(\xi,\eta). \) Let \( H^{l+0}_{2^0 u,\gamma_{-l}} \tilde{Q}^{(1)}_{-\nu_{-l}} P_k^{(2)} f(1) \) be

\[
(3.11) \quad \frac{1}{2^k 2^{\nu_{-l}} u \gamma(2^{\nu_{-l}})} \text{ p. v. } \int_{-\infty}^{\infty} \tilde{Q}^{(1)}_{-\nu_{-l}} \tilde{F}_k^{(2)} f(x_1, x_2 - 2^{\nu_{-l}} u \gamma(t)) \frac{\gamma(2^{\nu_{-l}}) \psi_{\nu_{-l}}(t)}{2^{2^{\nu_{-l}} \gamma(t)}} \, dt.
\]

After changing of variable \( 2^{\nu_{-l}} u \gamma(t) =: w \), we apply (1.4), (ii) of (II) and the fact \( (\gamma^{-1})'(t) \gamma'(\gamma^{-1}(t)) = 1 \) to obtain

\[
\frac{1}{\gamma'(\gamma^{-1}(w/2^{\nu_{-l}} u))} \left( \frac{\gamma^{-1}(w/2^{\nu_{-l}} u)}{\gamma^{-1}(w/2^{\nu_{-l}} u)} \right) \leq \frac{2^{\nu_{-l} + 1}}{2^{2^{\nu_{-l}}}}.
\]

On the other hand, note that \( 2^k 2^{\nu_{-l}} u \gamma(2^{\nu_{-l}}) = 1 \) and (1.4) imply \( 1/2^k 2^{\nu_{-l}} u \gamma(2^{\nu_{-l} + 1}) \leq e^{-C^{(1)}_{l+1/2}} \). Then we have

\[
\left\| \sum_{j \geq 0} 2^j 2^{\nu_{-l}} u \gamma(2^{\nu_{-l}}) \tilde{Q}^{(1)}_{-\nu_{-l}} P_k^{(2)} f(x_1, x_2) \right\|_{L^p(\mathbb{R}^2)} \leq e^{-C^{(1)}_{l+1/2}} M(2) \gamma(2^{\nu_{-l}}) \psi_{\nu_{-l}}(t) \leq e^{-C^{(1)}_{l+1/2}} M(2) \gamma(2^{\nu_{-l}}) \psi_{\nu_{-l}}(t).
\]

Therefore, the same boundedness also holds for \( \sup_{P_{k_0} \in \mathbb{Z}} \sup_{w \in [1,2]} \left\| \sum_{j \geq 0} 2^j 2^{\nu_{-l}} u \gamma(2^{\nu_{-l}}) \tilde{Q}^{(1)}_{-\nu_{-l}} P_k^{(2)} f(x_1, x_2) \right\|_{L^p(\mathbb{R}^2)} \) for all \( p \in (1, \infty) \).

The difference between \( H^{l+0}_{2^0 u,\gamma_{-l}} \tilde{Q}^{(1)}_{-\nu_{-l}} P_k^{(2)} f(x_1, x_2) \) and \( H^{l+0}_{2^0 u,\gamma_{-l}} \tilde{Q}^{(1)}_{-\nu_{-l}} P_k^{(2)} f(x_1, x_2) \) can be written as

\[
(3.13) \quad \sum_{j \geq 0} \frac{2^j}{2^k 2^{\nu_{-l}} u \gamma(2^{\nu_{-l}})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{F}_k^{(2)} f(x_1 - w, x_2 - 2^{\nu_{-l}} u \gamma(t))
\times \left( \tilde{\psi}_{j-l_{-l} - l(w-t)} - \tilde{\psi}_{j-l_{-l} - l(w)} \right) \, dw \frac{\gamma(2^{\nu_{-l}}) \psi_{\nu_{-l}}(t)}{2^{2^{\nu_{-l}} \gamma(t)}} \, dt.
\]

The mean value theorem gives

\[
(3.14) \quad \left| \tilde{\psi}_{j-l_{-l} - l(w-t)} - \tilde{\psi}_{j-l_{-l} - l(w)} \right| \leq |t| 2^{2(j-l_{-l} - l)} 2^{-2m}
\]

if \( |t| \leq 2^{-j+l_{-l} + 1} \) and \( 2^{-j+l_{-l} + 1} \leq |w| \leq 2^{-j+l_{-l} + 1} \) for \( m \in \mathbb{N} \). For \( m = 0 \) the above estimate holds for all \( |w| \leq 2^{-j+l_{-l} + 1} \). Since \( t \in \text{supp} \psi_{l_{-l} + l} \) and \( j \leq 0 \) imply \( |t| \leq 2^{-j+l_{-l} + 1} \), thus (3.14) holds.

Therefore the absolute value of (3.13) can be controlled by

\[
\sum_{m \in \mathbb{N}} \sum_{j \geq 0} \frac{2^j}{2^k 2^{\nu_{-l}} u \gamma(2^{\nu_{-l}})} \int_{-\infty}^{\infty} \int_{|w| \leq 2^{-j+l_{-l} + 1} + m} \left| \tilde{F}_k^{(2)} f(x_1 - w, x_2 - 2^{\nu_{-l}} u \gamma(t)) \right| \, dt.
\]

\[\text{We denote } F(f) \text{ means the Fourier transform of } f.\]

\[\text{We denote } \tilde{F}(f) \text{ means the inverse Fourier transform of } \tilde{\psi}_{j-l_{-l} - l}(\cdot).\]
By changing of variable and noticing that $2^k 2^{b_0} \gamma(2^k) = 1$, we apply (1.4) and (ii) of (H.) to bound the absolute value of (3.13) by
\[
\sum_{m \in \mathbb{N}} \sum_{j \geq 0} \frac{2^{2j-m}}{2^k 2^{b_0} u \gamma(2^k)} M^{(1)} M^{(2)} \tilde{P}_k^2 f(x_1, x_2) \leq e^{-C_1^2 l/2} M^{(1)} M^{(2)} \tilde{P}_k^2 f(x_1, x_2).
\]
Thus, by the $L^p(\mathbb{R})$-boundedness of $M^{(1)}$, $M^{(2)}$ and $\tilde{P}_k^2$, we further obtain
\[
\left\| \sup_{k_0 \in \mathbb{Z}} \sup_{u \in [1, 2]} \left| H^{l+b}_0 \, \mathcal{Q}_{-l+b}^{(1)} P_k^2 \right| \right\|_{L^p(\mathbb{R}^2)} \leq e^{-C_1^2 l/2} \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]
This, combined with (3.12), gives
\[
\left\| \sup_{k_0 \in \mathbb{Z}} \sup_{u \in [1, 2]} \left| H^{l+b}_0 \, \mathcal{Q}_{-l+b}^{(1)} P_k^2 \right| \right\|_{L^p(\mathbb{R}^2)} \leq e^{-C_1^2 l/2} \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]
By (3.10), we conclude that
\[
(3.15) \quad \left\| H_l P_k^2 f \right\|_{L^p(\mathbb{R}^2)} \leq e^{-C_1^2 l/2} \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } l \in \mathbb{N} \text{ and } p \in (1, \infty).
\]

Consider $H_2 P_k^2 f$. It suffices to prove that there exists $\delta > 0$ such that
\[
(3.16) \quad \left\| H^{l+b}_{l+b} P_k^2 \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta(\epsilon+1)} \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (2, \infty).
\]
The expression inside the $L^p(\mathbb{R}^2)$ norm on the LHS of (3.16) can be majorized by
\[
\sup_{k_0 \in \mathbb{Z}} \left| H^{l+b}_{k_0} \, \mathcal{Q}_{-l+b}^{(1)} P_k^2 \right| \leq \left( \sum_{k_0 \in \mathbb{Z}} \left| H^{l+b}_{k_0} \, \mathcal{Q}_{-l+b}^{(1)} P_k^2 \right|^p \right)^{\frac{1}{p}} \quad \text{with } 2^k 2^{b_0} \gamma(2^k) = 1.
\]
Noting that the commutation relation of $l^p$ and $L^p$ norms, and the $l^p$ norm can be controlled by the $l^2$ norm for any $p \in (2, \infty)$, we will only prove
\[
(3.17) \quad \left\| H^{l+b}_{k_0} \, \mathcal{Q}_{-l+b}^{(1)} P_k^2 \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta(\epsilon+1)} \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (2, \infty).
\]
To get (3.17), we will use the local smoothing estimate (Proposition 2.1). Indeed, by denoting $\Gamma_{l+b}(t) \equiv \gamma(2^l t) / \gamma(2^b t)$ and $\phi(t) \equiv \psi(t) / t$, we can rewrite $H^{l+b}_{k_0} \mathcal{Q}_{-l+b} P_k^2 f(x_1, x_2)$ as
\[
\int_{-\infty}^{\infty} P_k^2 \, f \left( x_1 - 2^l t, x_2 - 2^b t \right) \phi(t) \, dt.
\]
The reason of replacing \( \gamma(2^{l_0}t) \) by \( \Gamma_{l+l_0}(t) \) is that the main properties of \( \Gamma_{l+l_0}(t) \) is independent of \( l \) and \( l_0 \) from Lemma 2.2. Let

\[
T^{l+l_0} f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1 - t, x_2 - U^{(0)}(x_1, x_2) \Gamma_{l+l_0}(t)) \phi(t) dt,
\]

and for any given real numbers \( a, b \), define

\[
\Omega_{a,b} f(x_1, x_2) := f(2^a x_1, 2^b x_2).
\]

Then

\[
H^{l+l_0}_{2 \Omega U, \gamma} P^{(1)}_{j-l-l_0} P^{(2)}_k f = \Omega_{-l-l_0,-k_0 - \log_2 \gamma(2^{l_0})} (T^{l+l_0} f) \Omega_{l+l_0,k_0 + \log_2 \gamma(2^{l_0})} P^{(1)}_{j-l} P^{(2)}_k f.
\]

Note that

\[
\Omega_{l+l_0,k_0 + \log_2 \gamma(2^{l_0})} P^{(1)}_{j-l} P^{(2)}_k f = P^{(1)}_j P^{(2)}_k \Omega_{l+l_0,k_0 + \log_2 \gamma(2^{l_0})} f,
\]

so we can write

\[
H^{l+l_0}_{2 \Omega U, \gamma} P^{(1)}_{j-l-l_0} P^{(2)}_k f = \Omega_{-l-l_0,-k_0 - \log_2 \gamma(2^{l_0})} (T^{l+l_0} f) \Omega_{l+l_0,k_0 + \log_2 \gamma(2^{l_0})} f,
\]

which further gives

\[
\left\| H^{l+l_0}_{2 \Omega U, \gamma} P^{(1)}_{j-l-l_0} P^{(2)}_k f \right\|_{L^p(\mathbb{R}^2)} = \left\| T^{l+l_0} f \right\|_{L^p(\mathbb{R}^2)}.
\]

Note that the frequency support of \( P^{(1)}_j P^{(2)}_k f \) is contained in the annulus \( \{ (\xi, \eta) \in \mathbb{R}^2 : \sqrt{\xi^2 + \eta^2} \approx 2^{\max(j,k_0 + \log_2 \gamma(2^{l_0}))} \} \). We use \( 2^{2k_0} \gamma(2^{k_0}) = 1 \), (i) of (H.) and Proposition 2.1 to obtain that there exists \( \delta > 0 \) such that

\[
\left\| T^{l+l_0} f \right\|_{L^p(\mathbb{R}^2)} \leq 2\delta \max\{j,k_0 + \log_2 \gamma(2^{l_0})\} \left\| f \right\|_{L^p(\mathbb{R}^2)} \leq 2\delta \left\| f \right\|_{L^p(\mathbb{R}^2)}
\]

for all \( p \in (2, \infty) \). Therefore,

\[
\left\| H^l_{2 \Omega U, \gamma} f \right\|_{L^p(\mathbb{R}^2)} \leq e^{-C_l \delta/4} \left\| f \right\|_{L^p(\mathbb{R}^2)} \quad \text{for all } l \in \mathbb{N} \text{ and } p \in (2, \infty).
\]

This, combined with (3.19), yields (3.8). Therefore, we finish the proof of (i) of Theorem A.

3.2 Proof of (ii) of Theorem A

Since the operator \( M_{U,\gamma} \) that we are dealing is positive, we may assume that \( f \) is non-negative. Furthermore, we may assume that \( U(x_1, x_2) > 0 \) for all \( (x_1, x_2) \in \mathbb{R}^2 \) and also adopt the notation

\[
U^{(k_0)}(x_1, x_2) = 2^{k_0} \frac{U(x_1, x_2)}{2^{V(x_1, x_2)}} \quad \text{for } k_0 \in \mathbb{Z},
\]
where \( V : \mathbb{R}^2 \to \mathbb{Z} \) is a measurable function satisfying \( 2^{V(x_1,x_2)} \leq U(x_1,x_2) < 2^{V(x_1,x_2)+1} \). Recall that \( \psi : \mathbb{R} \to \mathbb{R} \) is a smooth function supported on \( \{ t \in \mathbb{R} : 1/2 \leq |t| \leq 2 \} \) with the property that \( 0 \leq \psi(t) \leq 1 \) and \( \sum_{k \in \mathbb{Z}} \psi_k(t) = 1 \) for any \( t \neq 0 \), where \( \psi_k(t) = \psi(2^{-k} t) \). Hence, \( M_{U,Y} \) can be bounded by

\[
\sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \sum_{|l| \leq \varepsilon} 2^{|l|} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - U(x_1, x_2) \gamma(t)) \psi_l(t) \frac{dt}{|t|} \leq \sup_{l \in \mathbb{Z}} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - U(x_1, x_2) \gamma(t)) \psi_l(t) \frac{dt}{|t|}.
\]

For any \( u > 0 \) and \( l \in \mathbb{Z} \), let

\[
S_u f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u \gamma(t)) \psi_l(t) \frac{dt}{|t|}.
\]

As before, for simplify, we denote \( z := (x_1, x_2) \), \( U_z := U(x_1, x_2) \) and \( V_z := V(x_1, x_2) \). Then, by linearization, the \( L^p(\mathbb{R}^2) \)-boundedness of \( M_{U,Y} \) can be reduced to that of \( S_{U_z} \), where \( l_z := l(x_1, x_2) \) and \( l_z : \mathbb{R}^2 \to \mathbb{Z} \) is a measurable function.

For these \( V_z, l_z \in \mathbb{Z} \), we denote \( k_z : \mathbb{R}^2 \to \mathbb{R} \) be a measurable function satisfying

\[
2^{k_z} 2^{V(x_1,x_2)} \gamma(2^{l_z}) = 1,
\]

then split \( S_{U_z} f(z) \) as

\[
S_{U_z} f(z) = \sum_{k \leq k_z} S_{U_z} P^{(2)}_k f(z) = \sum_{k \leq k_z} S_{U_z} P^{(2)}_k f(z) + \sum_{k > k_z} S_{U_z} P^{(2)}_k f(z)
\]

\[
=: S^{a}_{U_z} f(z) + S^{b}_{U_z} f(z).
\]

Consider \( S^{a}_{U_z} f(z) \). Let us set

\[
S^{a}_{U_z} f(z) := \sum_{k \leq k_z} \int_{-\infty}^{\infty} P^{(2)}_k f(x_1 - t, x_2) \psi_{l_z}(t) \frac{dt}{|t|},
\]

which can be majorized by \( M_{L}(\sum_{k \leq k_z} P^{(2)}_k f)(z) \). Note that the operator \( \sum_{k \leq k_z} P^{(2)}_k f \) is bounded on \( L^p(\mathbb{R}^2) \) for any given \( p \in (1, \infty) \) by multiplier theory, then we can use the \( L^p \) boundedness of \( M_{L} \) to get

\[
\| S^{a}_{U_z} f \|_{L^p(\mathbb{R}^2)} \leq \| f \|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]

As in \( (3.6) \), we obtain the \( L^p(\mathbb{R}^2) \)-boundedness of \( |S^{a}_{U_z} f - S^{a}_{U_z} f| \). This, combined with \( (3.21) \) shows that

\[
\| S^{a}_{U_z} f \|_{L^p(\mathbb{R}^2)} \leq \| f \|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]
Consider $S_{U_l,z}^b f(z)$. We rewrite it as $\sum_{\ell>0} S_{U_l,z} P_{\ell+1}^{1} P_{\ell+1}^{2} f(z)$. Furthermore, we split $S_{U_l,z}^b f(z)$ as the sum of $S_{U_l,z}^{1b} f(z)$ and $S_{U_l,z}^{2b} f(z)$, where

$$S_{U_l,z}^{1b} f(z) := \sum_{m \leq -l} \sum_{k > 0} S_{U_l,z} P_{m}^{1} P_{k+l}^{2} f(z) \quad \text{and} \quad S_{U_l,z}^{2b} f(z) := \sum_{m > -l} \sum_{k > 0} S_{U_l,z} P_{m}^{1} P_{k+l}^{2} f(z).$$

We first estimate $S_{U_l,z}^{1b} f(z)$. Let

$$S_{U_l,z}^{1b} f(z) := \sum_{m \leq -l} \sum_{k > 0} \int_{-\infty}^{\infty} P_{m}^{1} P_{k+l}^{2} f(x_1, x_2 - U_z \gamma(t)) \psi_m(w - t) \psi_l(t) \frac{dt}{|t|}. $$

After changing of variable as in (3.11), we have

$$\left\| S_{U_l,z}^{1b} f(z) \right\|_{L^p(\mathbb{R}^2)} \leq M^2 \left( \sum_{m \leq -l} \sum_{k > 0} P_{m}^{1} P_{k+l}^{2} \right) f(z),$$

which gives

$$\left(3.23\right) \quad \left\| S_{U_l,z}^{1b} f(z) \right\|_{L^p(\mathbb{R}^2)} \leq \| f \|_{L^p(\mathbb{R}^2)} \quad \text{for all} \quad p \in (1, \infty).$$

As in (3.13), the difference between $S_{U_l,z}^{1b} f(z)$ and $S_{U_l,z}^{1b} f(z)$ can be written as

$$\sum_{m \leq -l} \sum_{k > 0} \int_{-\infty}^{\infty} P_{m}^{1} P_{k+l}^{2} f(x_1 - w, x_2 - U_z \gamma(t)) \psi_m(w - t) \psi_l(t) \frac{dt}{|t|}$$

which can be controlled by $M^1 M^2 (\sum_{m \leq -l} \sum_{k > 0} P_{m}^{1} P_{k+l}^{2}) f(z)$. Therefore, we obtain the $L^p(\mathbb{R}^2)$-boundedness of $S_{U_l,z}^{1b} f - S_{U_l,z}^{1b} f$ for $p > 1$. This, combined with (3.23), yields

$$\left(3.24\right) \quad \left\| S_{U_l,z}^{1b} f \right\|_{L^p(\mathbb{R}^2)} \leq \| f \|_{L^p(\mathbb{R}^2)} \quad \text{for all} \quad p \in (1, \infty).$$

Next we turn to $S_{U_l,z}^{2b} f(z)$. We rewrite it as $\sum_{m>0} \sum_{k > 0} S_{U_l,z} P_{m}^{1} P_{k+l}^{2} f(z)$. Then, it suffices to prove that there exists a positive constant $\delta$ such that

$$\left(3.25\right) \quad \left\| S_{U_l,z} P_{m}^{1} P_{k+l}^{2} f \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta (k+m)} \| f \|_{L^p(\mathbb{R}^2)} \quad \text{for all} \quad p \in (2, \infty).$$

We further bound the LHS of (3.25) by

$$\left\| \left( \sum_{b \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left| S_{U_l,z} P_{m}^{1} P_{k+l}^{2} f \right|^p \right)^{\frac{1}{p}} \right\|_{L^p(\mathbb{R}^2)},$$

where $2^{k +\gamma(b)}(2^l) = 1$.

The commutation relation of $L^p$ and $L^p$ norms, together with the fact that the $L^p$ norm can be controlled by the $L^2$ norm for all $p \in (2, \infty)$, implies that (3.25) follows from

$$\left(3.26\right) \quad \left\| S_{U_l,z} P_{m}^{1} P_{k+l}^{2} f \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta (k+m)} \| f \|_{L^p(\mathbb{R}^2)} \quad \text{uniformly in} \quad l \in \mathbb{Z} \quad \text{and for all} \quad p \in (2, \infty).$$
As in (3.19), the LHS of (3.26) is equal to
\[ \left\| T^l p_m^{(1)} p_m^{(2)} \right\|_{L^p(\mathbb{R}^2)}, \]
where the definition of \( T^l \) can be found in (3.18). Noting that \( 2^K 2^{k_0} \gamma(2^l) = 1 \) and the frequency support of \( p_m^{(1)} p_m^{(2)} \) is contained in the annulus \( \{(\xi, \eta) \in \mathbb{R}^2 : \sqrt{\xi^2 + \eta^2} \approx \max\{2^m, 2^{k_0 + k_0 + \log_2 \gamma(2^l)} \} \approx 2^{\max(m, k)} \} \), we can apply Proposition 2.1 to get that there exists some \( \delta_0 > 0 \) such that
\[ \left\| T^l p_m^{(1)} p_m^{(2)} \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta_0 \max(m, k)} \| f \|_{L^p(\mathbb{R}^2)}, \]
which implies (3.26). Therefore,
\[ (3.27) \quad \left\| S_{U_0, \mathcal{E}}^{a,b} f \right\|_{L^p(\mathbb{R}^2)} \leq \| f \|_{L^p(\mathbb{R}^2)} \quad \text{for all} \ p \in (2, \infty). \]

Combining (3.22), (3.24) and (3.27), we obtain
\[ \left\| M_{U_0, \mathcal{E}} f \right\|_{L^p(\mathbb{R}^2)} \leq \| f \|_{L^p(\mathbb{R}^2)} \quad \text{for all} \ p \in (2, \infty), \]
which finishes the proof of (ii) of Theorem A.

4 Proof of Theorem B

Before starting the proof of Theorem B, we first study the \( L^p(\mathbb{R}^2) \)-boundedness, \( 1 < p < \infty, \) of the following maximal function associated with plane curve \( (t, 2^j \gamma(t)) \) in lacunary coefficient,
\[ (4.1) \quad M_{L, \gamma} f(x_1, x_2) := \sup_{j \in \mathbb{Z}} \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x_1 - t, x_2 - 2^j \gamma(t))| dt \]
under some weaker assumptions on \( \gamma \) than \( (H.) \), which will play an important role in obtaining Theorem B. For the corresponding results about the operator \( M_{L, \gamma} \), we refer to [22, 23] for the case \( \gamma \) is a polynomial, and [21] Lemma 5.1 for the case \( \gamma(t) = |t|^\alpha \) with \( \alpha > 0 \) and \( \alpha \neq 1 \). Here we will provide a different and slightly simpler proof.

**Proposition 4.1.** Let \( \gamma \in C(\mathbb{R}) \cap C^2(\mathbb{R}^+) \) be either odd or even, \( \gamma(0) = 0 \), and increasing on \( \mathbb{R}^+ \), and satisfying

(i) there exist \( C_2^{(1)} > 0, C_3^{(1)} > 0 \) such that \( C_2^{(1)} \leq \left| \frac{\gamma'(t)}{\gamma(t)} \right| \leq C_3^{(1)} \) for any \( t \in \mathbb{R}^+ \);

(ii) there exists \( C_2^{(2)} > 0 \) such that \( \frac{\gamma''(t)}{\gamma(t)} \geq C_2^{(2)} \) for any \( t \in \mathbb{R}^+ \).

Then, for any \( p > 1 \), there exists \( C > 0 \) such that
\[ \| M_{L, \gamma} f \|_{L^p(\mathbb{R}^2)} \leq C \| f \|_{L^p(\mathbb{R}^2)} \]
for any \( f \in L^p(\mathbb{R}^2) \).
Proof of Proposition 4.1. By linearization, it can be seen that $M_{L^r}$ is the special situation of $M_{U^r}$ with $U(x_1, x_2) := 2^j(x_1, x_2)$, where $j(x_1, x_2) : \mathbb{R}^2 \to \mathbb{Z}$ is a measurable function. Therefore, we may use some ideas of the proof of (ii) of Theorem A. Recall that

$$S_{u, f}f(x_1, x_2) = \int_{-\infty}^{\infty} f(x_1 - t, x_2 - uy(t))\psi(t) \frac{dt}{|t|}$$

in (3.20). Repeating the proof of (ii) of Theorem A, from (3.22) and (3.24), as in (3.25), it suffices to prove that there exists a positive constant $\delta$ such that

$$\left\|S_{2^k, f} F_{\rho(1)} P_{m-l, k+k_f} f \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta(k+m)} \left\|f\right\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, \infty)$, where $k, m \in \mathbb{N}$ and $2^k 2^j \gamma(2^j) = 1$. By interpolation, it is enough to prove

$$\left\|S_{2^k, f} F_{\rho(1)} P_{m-l, k+k_f} f \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta(k+m)} \left\|f\right\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (2, \infty)$,

and

$$\left\|S_{2^k, f} F_{\rho(1)} P_{m-l, k+k_f} f \right\|_{L^p(\mathbb{R}^2)} \leq \left\|f\right\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, 2]$.

Proof of (4.2). Indeed, we have obtained (4.2) in (3.25) by a local smoothing estimate, where $\gamma$ satisfies (H). For $M_{L^r}$, we want to explore a simpler method to get (4.2). We first bound the LHS of (4.2) by

$$\left\|\left(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left\|S_{2^j, f} P_{m-l, k+k_f} f \right\|^{p(1)}_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{p}} \right\|_{L^p(\mathbb{R}^2)}$$

where $2^k 2^j \gamma(2^j) = 1$.

As before, it remains to prove

$$\left\|S_{2^j, f} P_{m-l, k+k_f} f \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta(k+m)} \left\|f\right\|_{L^p(\mathbb{R}^2)}$$

uniformly in $l \in \mathbb{Z}$ for all $p \in (2, \infty)$.

As in (3.19), $2^k 2^j \gamma(2^j) = 1$ implies that the LHS of (4.4) equals $\|\mathcal{D} P_{m} f \|_{L^p(\mathbb{R}^2)}$, where

$$\mathcal{D} f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \Gamma(t))\psi(t) \frac{dt}{|t|} \quad \text{and} \quad \Gamma(t) := \frac{\gamma(2^j t)}{\gamma(2^j)}.$$

Therefore, it suffices to show

$$\left\|\mathcal{D} P_{m} f \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta(k+m)} \left\|f\right\|_{L^p(\mathbb{R}^2)}$$

uniformly in $l \in \mathbb{Z}$ for all $p \in (2, \infty)$.

Repeating this process of proving (2.4), we obtain (4.5).

Proof of (4.3). By linearization, we bound the expression inside the $L^p(\mathbb{R}^2)$ norm on the LHS of (4.3) by $\sup_{l \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} \left\|S_{2^j, f} P_{m-l, k+k_f} f \right\|$ with $2^k 2^j \gamma(2^j) = 1$. Therefore, it suffices to show

$$\left\|\left(\sum_{k \in \mathbb{Z}} \left\|S_{2^j, f} P_{m-l, k+k_f} f \right\|^2_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \leq \left\|\left(\sum_{k \in \mathbb{Z}} \left\|P_{m-l, k+k_f} f \right\|^2_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)}.$$
We will follow the method of bootstrapping an iterated interpolation argument in the spirit of Nagel, Stein and Wainger \[32\]. Now consider the more general estimate

\[
(4.7) \quad \left\| \left( \sum_{l, j_0 \in \mathbb{Z}} \left| S_{2j_0, l} P_{m-l, f}^{1(2)} \right|^{q_1} \right)^{\frac{1}{q_1}} \right\|_{L^2(\mathbb{R}^2)} \leq \left\| \left( \sum_{l \in \mathbb{Z}} \left| P_{m-l, f}^{(1)} \right|^{q_1} \right)^{\frac{1}{q_1}} \right\|_{L^2(\mathbb{R}^2)}
\]

for some \( q_1 \in (1, \infty) \) and \( q_2 \in (1, \infty) \).

If \( q_1 = \infty \) and \( q_2 > 2 \), by linearization, we have that the expression inside the \( L^2(\mathbb{R}^2) \) norm on the LHS of \( (4.7) \) can be bounded by \( M L_\gamma M_k(\sup_{l \in \mathbb{Z}} |P_{m-l, f}^{(1)}|) \). It is clear that \( (4.2) \) implies the \( L^{q_2}(\mathbb{R}^2) \)-boundedness of \( M_\gamma \) for all \( q_2 > 2 \). This along with the \( L^{q_2}(\mathbb{R}^2) \)-boundedness of \( M_k^{(2)} \) yields

\[
(4.8) \quad (4.7) \text{ holds for } q_1 = \infty \text{ and } q_2 > 2.
\]

We claim

\[
(4.9) \quad \left\| \sup_{j_0 \in \mathbb{Z}} \left| S_{2j_0, l} P_{m-l, f}^{1(2)} \right| \right\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]

If the claim above has been proved, then we can obtain \( (4.7) \) for the case \( q_1 = q_2 \in (1, \infty) \) by replacing \( f \) as \( P_{m-l, f}^{(1)} \), and using the commutation relation of \( L^p \) and \( L^\infty \) norms. Interpolation this with \( (4.8) \) implies that \( (4.6) \) holds for all \( p \in (4/3, 2] \). Repeating the interpolation argument and using \( q_1 = \infty \) and \( q_2 \in (4/3, 2] \), we can prove \( (4.6) \) holds for all \( p \in (8/7, 2] \). Reiterating this process sufficiently many times, we thus show \( (4.6) \) holds for all \( p \in (1, 2] \).

Therefore, it suffices to show \( (4.9) \). Furthermore, note that \( \sup_{j_0 \in \mathbb{Z}} \left| S_{2j_0, l} P_{m-l, f}^{1(2)} \right| \) can be bounded by \( M_\gamma M_k^{(1)} M_k^{(2)} f \), and the \( L^p(\mathbb{R}^2) \)-boundedness of \( M_\gamma, M_k^{(1)} \) and \( M_k^{(2)} \) for all \( p > 2 \), thus we may restrict \( p \in (1, 2] \) in the proof of \( (4.9) \). This further reduces to showing

\[
(4.10) \quad \left\| \left( \sum_{j_0 \in \mathbb{Z}} \left| S_{2j_0, l} P_{m-l, f}^{1(2)} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \leq \left\| \left( \sum_{j_0 \in \mathbb{Z}} \left| P_{k, f}^{(2)} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, 2].
\]

Repeating the method of bootstrapping an iterated interpolation argument above, it suffices to prove

\[
(4.11) \quad \left\| S_{2j_0, l} P_{m-l, f}^{1(2)} \right\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]

By Minkowski’s inequality, it is easy to see that the LHS of \( (4.11) \) can be bounded by

\[
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left| P_{m-l, f}^{1(2)} x_1 - t, x_2 - 2j_0 \gamma(t) \right|^p \, dx_1 \, dx_2 \right)^{\frac{1}{p}} \psi(t) \, dt \leq \|f\|_{L^p(\mathbb{R}^2)}.
\]

We then obtain \( (4.11) \). This finishes the proof of Proposition 4.1. \( \square \)
4.1 Proof of (i) of Theorem B

Recall the definition of $H_{U,y}^l$ in (3.2) and denote $\Delta := \{l \in \mathbb{Z} : 2^l \leq \varepsilon_0\}$, we then write

$$H_{U,y}^l P_k^{(2)} f = \sum_{l \in \Delta} H_{U,y}^l P_k^{(2)} f.$$ 

We further split $H_{U,y}^l P_k^{(2)} f$ as the sum of the following low frequency part $I_{U,y}^l P_k^{(2)} f$ and the high frequency part $I_{U,y}^{l+1} P_k^{(2)} f$, 

$$I_{U,y}^{l+1} P_k^{(2)} f := \sum_{l \in \Delta, l \leq l_z} H_{U,y}^l P_k^{(2)} f \quad \text{and} \quad I_{U,y}^l P_k^{(2)} f := \sum_{l \in \Delta, l > l_z} H_{U,y}^l P_k^{(2)} f.$$ 

We only consider the case $l_z < \log_2 \varepsilon_0$, since the other case $l_z \geq \log_2 \varepsilon_0$ can be handled similarly. The proof of $I_{U,y}^l P_k^{(2)} f$ can be found in (3.7). As for $I_{U,y}^{l+1} P_k^{(2)} f$, we may rewrite it as

$$\sum_{0 < l < \log_2 \varepsilon_0 - l_z} H_{U,y}^{l+1} P_k^{(2)} f. \quad \text{Therefore, for } l \in \mathbb{N} \text{ with } 0 < l < \log_2 \varepsilon_0 - l_z, \text{ it suffices to show that there exists a positive constant } \delta \text{ such that}$$

$$\left\| H_{U,y}^{l+1} P_k^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta l} \left\| f \right\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, 2].$$

By interpolation, it is enough to prove that there exists a positive constant $\delta$ such that

$$\left\| H_{U,y}^{l+1} P_k^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta l} \left\| f \right\|_{L^p(\mathbb{R}^2)} \quad \text{for all } l \in \mathbb{N} \text{ and } p \in (2, \infty),$$

and

$$\left\| H_{U,y}^{l+1} P_k^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \leq \left\| f \right\|_{L^p(\mathbb{R}^2)} \quad \text{for all } l \in \mathbb{N} \text{ with } 0 < l < \log_2 \varepsilon_0 - l_z \text{ and } p \in (1, 2].$$

The proof of (4.12) has been obtained in (3.8). So it remains to show (4.13). From (3.9) and (3.15), it suffices to prove that there exists a positive constant $\delta$ such that

$$\left\| H_{U,y}^{l+1} P_k^{(1)} j^j l^j P_k^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \leq 2^{-\delta j} \left\| f \right\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, 2],$$

where $j, l \in \mathbb{N}$ with $0 < l < \log_2 \varepsilon_0 - l_z$ and $j \geq 1$, and $2^j 2^{V(2^j)} = 1$.

If we can prove

$$\left\| H_{U,y}^{l+1} P_k^{(1)} j^j l^j P_k^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \leq \left\| f \right\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, 2],$$

then (4.14) will follow by interpolation between (4.16) and (4.15).

Next, we will prove (4.14). Define a new measurable function $U^{(k_0)}_\varepsilon : \mathbb{R}^2 \to [2^k, 2^{k+1}]$ as

$$U^{(k_0)}_\varepsilon := U_\varepsilon \text{ if } V_\varepsilon = k_0, \quad \text{and} \quad U^{(k_0)}_\varepsilon := 2^k \text{ if } V_\varepsilon \neq k_0.$$ 

We note that the property that both $U^{(k_0)}_\varepsilon$ and $U^{(k_0)}_\varepsilon$ are in $[2^k, 2^{k+1})$ is very important to us. Then the expression inside the $L^p(\mathbb{R}^2)$ norm on the LHS of (4.14) can be bounded by

$$\sup_{k_0 \in \mathbb{Z}} \left\| H_{U^{(k_0)}_\varepsilon}^{l+1} P_k^{(1)} j^j l^j P_k^{(2)} f \right\|_{L^p(\mathbb{R}^2)}, \quad \text{where } 2^k 2^{V(2^k)} = 1.$$
Furthermore, let \( \Lambda := \{ l \in \mathbb{Z} : 0 < l < \log_2 \varepsilon_0 - l_0 \} \), we then bound the LHS of (4.15) by
\[
\left\| \left( \sum_{k_0 \in \mathbb{Z}} \left| 1_{\Lambda}(l) \mathcal{H}^{l_0}_{l_0} \mathcal{P}^{(1)}_{j_0-1} P^{(2)}_k f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} , \quad \text{where } 2^l 2^{l_0} \gamma(2^{l_0}) = 1.
\]

Therefore, for \( p \in (1, 2] \), it is enough to prove that
\[
(4.17) \quad \left\| \left( \sum_{k_0 \in \mathbb{Z}} \left| 1_{\Lambda}(l) \mathcal{H}^{l_0}_{l_0} \mathcal{P}^{(1)}_{j_0-1} P^{(2)}_k f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \leq \left\| \sum_{k_0 \in \mathbb{Z}} \left| P^{(1)}_{j_0-1} f \right|^2 \right\|_{L^p(\mathbb{R}^2)} .
\]

Note that \( |1_{\Lambda}(l) \mathcal{H}^{l_0}_{l_0} \mathcal{P}^{(1)}_{j_0-1} P^{(2)}_k f| \) can be bounded by \( M_{U \gamma} M_{(2)} (\sup_{k_0 \in \mathbb{Z}} |P^{(1)}_{j_0-1} f|) \) and apply (ii) of Theorem A, then we can repeat this process from (4.6) to (4.9). Thus estimate (4.17) reduces to proving
\[
(4.18) \quad \left\| \left( \sum_{k_0 \in \mathbb{Z}} \left| 1_{\Lambda}(l) \mathcal{H}^{l_0}_{l_0} \mathcal{P}^{(1)}_{j_0-1} P^{(2)}_k f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]

In fact, we will prove the following stronger version,
\[
\left\| 1_{\Lambda}(l) \mathcal{H}^{l_0}_{l_0} \mathcal{P}^{(1)}_{j_0-1} f \right\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]

We now define another new measurable function \( \tilde{U}^{(k_0)}_{\varepsilon} : \mathbb{R}^2 \to [2^{k_0}, 2^{k_0+1}] \) as follows:
\[
(4.19) \quad \tilde{U}^{(k_0)}_{\varepsilon} := U_{\varepsilon} \text{ if } V \varepsilon = k_0, \quad \text{and extend } \tilde{U}^{(k_0)}_{\varepsilon} \text{ to the whole } \mathbb{R}^2 \text{ space with } \|\tilde{U}^{(k_0)}_{\varepsilon}\|_{\text{Lip}} \leq 2 \|U\|_{\text{Lip}}.
\]

We remark that both \( \tilde{U}^{(k_0)}_{\varepsilon} \) and \( U^{(k_0)}_{\varepsilon} \) and are in \( [2^{k_0}, 2^{k_0+1}] \), but the former is Lipschitz and the latter is not. By the definitions of \( M_{U \gamma}, \tilde{U}^{(k_0)}_{\varepsilon} \) and \( U^{(k_0)}_{\varepsilon} \), one has the following pointwise estimate
\[
(4.20) \quad 1_{\Lambda}(l) \mathcal{H}^{l_0}_{l_0} \mathcal{P}^{(1)}_{j_0-1} f(\varepsilon) \leq M_{U \gamma} f(\varepsilon) + 1_{\Lambda}(l) \mathcal{H}^{l_0}_{l_0} \mathcal{P}^{(1)}_{j_0-1} \tilde{U}^{(k_0)}_{\varepsilon} f(\varepsilon).
\]

By Proposition 4.1, it suffices to prove
\[
(4.21) \quad \left\| 1_{\Lambda}(l) \mathcal{H}^{l_0}_{l_0} \mathcal{P}^{(1)}_{j_0-1} \tilde{U}^{(k_0)}_{\varepsilon} f \right\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]

Indeed, by Minkowski’s inequality, the LHS of (4.21) can be bounded by
\[
(4.22) \quad 1_{\Lambda}(l) \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| f(x_1 - t, x_2 - \tilde{U}^{(k_0)}_{\varepsilon} \gamma(t)) \right|^p \, dx_1 \, dx_2 \right)^{\frac{1}{p}} \psi_{l_0}^p(t) \frac{dt}{|t|}.
\]

Let
\[
(4.23) \quad X_1 := x_1 - t \quad \text{and} \quad X_2 := x_2 - \tilde{U}^{(k_0)}_{\varepsilon} \gamma(t),
\]
we then write the corresponding Jacobian determinant as
\[
\frac{\partial (X_1, X_2)}{\partial (x_1, x_2)} = \begin{vmatrix} 1 - \frac{\partial}{\partial x_1} \tilde{S}_z^{(k_0)} \gamma(t) & 1 - \frac{\partial}{\partial x_2} \tilde{S}_z^{(k_0)} \gamma(t) \\ \frac{\partial}{\partial x_1} \tilde{S}_z^{(k_0)} \gamma(t) & \frac{\partial}{\partial x_2} \tilde{S}_z^{(k_0)} \gamma(t) \end{vmatrix} = 1 - \frac{\partial}{\partial x_2} \tilde{S}_z^{(k_0)} \gamma(t). 
\]

Noting that \( t \in \text{supp} \psi_{1, l_0} \), \( l \in \Delta \) and \( |\frac{\partial}{\partial x_2} \tilde{S}_z^{(k_0)}| \leq \| \tilde{S}_z^{(k_0)} \|_{\text{Lip}} \), we can assert that
\[
\left| \frac{\partial}{\partial x_2} \tilde{S}_z^{(k_0)} \gamma(t) \right| \leq 2 \| \tilde{S}_z^{(k_0)} \gamma(2\varepsilon_0) \|.
\]

Furthermore, let the positive constant \( \varepsilon_0 \) satisfying \( \gamma(2\varepsilon_0) \leq 1/4 \| U \|_{\text{Lip}} \), then we conclude that
\[
\left| \frac{\partial (X_1, X_2)}{\partial (x_1, x_2)} \right| \geq \frac{1}{2}.
\]

This implies that the change of variables in (4.25) is valid and thus (4.22) can be bounded by \( \| f \|_{L^p(\mathbb{R}^2)} \) for all \( p \in (1, \infty) \). Therefore, we obtain (4.21), which completes the proof of (i) of Theorem B.

### 4.2 Proof of (ii) of Theorem B

From (3.22) and (3.24), we have obtained the \( L^p(\mathbb{R}^2) \)-boundedness of \( S_{U, z}^{p} f(z) \) and \( S_{U, z}^{1,b} f(z) \) for all \( p \in (1, \infty) \). But it is possible that the operator \( S_{U, z}^{2,b} f(z) \) is unbounded on \( L^p(\mathbb{R}^2) \) for any \( p \in (1, 2] \), if we only assume that \( U \) is a measurable function. Therefore, for the case \( p \in (1, 2] \), instead of \( M_{U, \gamma} \), we consider \( M_{U, \gamma}^{\varepsilon_0} \) with \( U \) is a Lipschitz function. The truncation \( \varepsilon_0 \) depending only on \( \| U \|_{\text{Lip}} \) plays a crucial role in the proof of the \( L^p(\mathbb{R}^2) \)-boundedness of \( S_{U, z}^{2,b} f(z) \) for all \( p \in (1, 2] \), which implies that the measurable function \( l_z : \mathbb{R}^2 \to \mathbb{Z} \) satisfies \( 2l_z \leq \varepsilon_0 \).

We now turn to \( S_{U, z}^{2,b} \), i.e.,
\[
S_{U, z}^{2,b} f(z) = \sum_{m>0} \sum_{k>0} S_{U, z}^{p_{m-l_z} k_k} f(z).
\]

Indeed, by interpolation with (3.25), it suffices to show
\[
\left\| S_{U, z}^{p_{m-l_z} k_k} f \right\|_{L^p(\mathbb{R}^2)} \leq \| f \|_{L^p(\mathbb{R}^2)} \text{ for all } p \in (1, 2].
\]

Recall the measurable function \( \tilde{U}^{(k_0)}_z : \mathbb{R}^2 \to [2^{k_0}, 2^{k_0+1}) \) defined in (4.16). Then the expression inside the \( L^p(\mathbb{R}^2) \) norm on the LHS of (4.24) can be bounded by
\[
\sup_{k_0 \in \mathbb{Z}} \left| S_{U, z}^{p_{m-l_z} k_k} f \right|, \quad \text{where } 2^{k_0} 2^{k_0} \gamma(2l_z) = 1.
\]

Noting \( 2l_z \leq \varepsilon_0 \) and letting \( \Delta := \{ l \in \mathbb{Z} : 2l \leq \varepsilon_0 \} \), we bound the LHS of (4.24) by
\[
\left\| \left( \sum_{l \in \Delta} \left| l \right| \sup_{k_0 \in \mathbb{Z}} \left| S_{U, z}^{p_{m-l_z} k_k} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)}, \quad \text{where } 2^{k_0} 2^{k_0} \gamma(2l_z) = 1.
\]
Therefore, for \( p \in (1, 2] \), it is enough to prove that

\[
\left\| \left( \sum_{k_0 \in \mathbb{Z}} |\Delta(l) \sup_{k_0 \in \mathbb{Z}} |S_{U^{(k_0)}_j} P^{(1)}_{m-l} P^{(2)}_{k+K} f |^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \leq \left\| \left( \sum_{k_0 \in \mathbb{Z}} |P^{(1)}_{m-l} f |^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)}.
\]

Notice that \( \sup_{k_0 \in \mathbb{Z}} |S_{U^{(k_0)}_j} P^{(1)}_{m-l} P^{(2)}_{k+K} f | \) can be bounded by \( M_{U, \gamma} M^{(2)}(\mathbb{R}^2) \), and the \( L^p(\mathbb{R}^2) \)-boundedness, \( p \in (2, \infty) \), of \( M_{U, \gamma} \) has been obtained in (ii) of Theorem A. Repeating this process from (4.6) to (4.9), it is enough to show

\[
\left\| \left( \sum_{k_0 \in \mathbb{Z}} |\Delta(l) S_{U^{(k_0)}_j} P^{(1)}_{m-l} P^{(2)}_{k+K} f |^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]

Furthermore, it suffices to prove (4.26) for \( p \in (1, 2] \). This is because \( \sup_{k_0 \in \mathbb{Z}} S_{U^{(k_0)}_j} \) can be bounded by \( M_{U, \gamma} \) and the result in (ii) of Theorem A. As in (4.25), we only need to show that

\[
\left\| \left( \sum_{k_0 \in \mathbb{Z}} |\Delta(l) S_{U^{(k_0)}_j} P^{(1)}_{m-l} P^{(2)}_{k+K} f |^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \leq \left\| \left( \sum_{k_0 \in \mathbb{Z}} |P^{(2)}_{k+K} f |^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \quad \text{where } 2K^2 \gamma(2') = 1.
\]

By the same argument as in (4.25)-(4.26), it remains to show that

\[
\left\| \Delta(l) S_{U^{(k_0)}_j} P^{(1)}_{m-l} P^{(2)}_{k+K} f \right\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]

In fact we will prove the following stronger version,

\[
\left\| \Delta(l) S_{U^{(k_0)}_j} f \right\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]

As in (4.20), we have the following pointwise estimate

\[
\Delta(l) S_{U^{(k_0)}_j} f(z) \leq M_{L, \gamma} f(z) + \Delta(l) S_{U^{(k_0)}_j} f(z).
\]

From Proposition 4.1, we only have to show that

\[
\left\| \Delta(l) S_{U^{(k_0)}_j} f \right\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for all } p \in (1, \infty).
\]

As in (4.21), noting that \( t \in \text{supp } \psi \), \( l \in \Delta \) and \( |\partial_{\xi_2} \tilde{U}^{(k_0)}_j| \leq ||\tilde{U}^{(k_0)}||_{L^{\infty}} \leq 2||U||_{L^{\infty}} \) imply \( |\partial_{\xi_2} \tilde{U}^{(k_0)}_j \gamma| \leq 2||U||_{L^{\infty}} \gamma(2\epsilon_0) \), and the fact that \( \epsilon_0 \) satisfies \( \gamma(2\epsilon_0) \leq 1/4||U||_{L^{\infty}} \), we then obtain (4.27). This completes the proof of (ii) of Theorem B.

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