ON NON-ARCHIMEDEAN RECURRENCE EQUATIONS AND THEIR APPLICATIONS

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Abstract. In the present paper we study stability of recurrence equations (which in particular case contain a dynamics of rational functions) generated by contractive functions defined on an arbitrary non-Archimedean algebra. Moreover, multirecurrence equations are considered. We also investigate reverse recurrence equations which have application in the study of $p$-adic Gibbs measures. Note that our results also provide the existence of unique solutions of nonlinear functional equations as well.

Mathematics Subject Classification: 46S10, 12J12, 39A70, 47H10, 60K35.

Key words: non-Archimedean algebra; recurrence equation; unique solution; tree.

1. INTRODUCTION

In this paper we deal with regulation properties of discrete dynamical systems defined over non-archimedean algebars. Note that the interest in such systems and in the ways in which they can be applied has been rapidly increasing during the last couple of decades (see, e.g., [3, 34]). An example of non-archimedean algebras is a field of $p$-adic numbers (see [8] for more examples). We stress that applications of $p$-adic numbers in $p$-adic mathematical physics [21, 37, 38], quantum mechanics and many others [1, 7, 15, 36] stimulated increasing interest in the study of $p$-adic dynamical systems.

On the other hand, the study of $p$-adic dynamical systems arises in Diophantine geometry in the constructions of canonical heights, used for counting rational points on algebraic vertices over a number field, as in [6]. In [4] dynamical systems (not only monomial) over finite field extensions of the $p$-adic numbers were considered. Other studies of non-Archimedean dynamics in the neighborhood of a periodic and of the counting of periodic points over global fields using local fields appeared in [10, 11, 17, 19, 20, 30]. It is known that the analytic functions play important roles in complex analysis. In the non-Archimedean analysis the rational functions play a role similar to that of analytic functions in complex analysis [8]. Therefore, there naturally arises a question as regards the study the dynamics of these functions in the mentioned setting. In [5, 31] a general theory of $p$-adic rational dynamical systems over complex $p$-adic field $\mathbb{C}_p$ has been developed. Certain rational $p$-adic dynamical systems were investigated in [2, 14, 26, 27], which appear from problems of $p$-adic Gibbs measures [13, 25, 27, 28]. In these investigations it is important to know the regularity or stability of the trajectories of rational dynamical systems.

In the present paper we are going to study stability of recurrence equations (which in particular case contain a dynamics of rational functions) generated by contractive functions defined on an arbitrary non-Archimedean algebra. It is also considered and studied multirecurrence equations. Note that in [35] cetain type of $p$-adic difference equations has been studied. In section 4 we investigate reverse recurrence equations which have application in the study of
$p$-adic Gibbs measures. In the last section 5 we provide applications of the main results. Note that our results also provide the existence of unique solutions of nonlinear functional equations as well.

2. Preliminaries

Let $K$ be a field with a non-Archimedean norm $| \cdot |$, i.e. for all $x, y \in K$ one has

1. $|x| \geq 0$ and $|x| = 0$ implies $x = 0$;
2. $|xy| = |x||y|$;
3. $|x + y| \leq \max\{|x|, |y|\}$.

An example of such kind of field can be considered the $p$-adic field $\mathbb{Q}_p$. Namely, for a fixed prime $p$, the set $\mathbb{Q}_p$ is defined as a completion of the rational numbers $\mathbb{Q}$ with respect to the norm $| \cdot |_p : \mathbb{Q} \to \mathbb{R}$ given by

\begin{equation}
|x|_p = \begin{cases} 
p^{-r}x \neq 0, \\
0, & x = 0,
\end{cases}
\end{equation}

here, $x = p^r \frac{m}{n}$ with $r, m \in \mathbb{Z}$, $n \in \mathbb{N}$, $(m, p) = (n, p) = 1$. A number $r$ is called a $p$-order of $x$ and it is denoted by $ord_p(x) = r$. The absolute value $| \cdot |_p$ is non-Archimedean. There are also many examples of non-Archimedean fields (see for example [18]).

Now let $\mathcal{A}$ be a non-Archimedean Banach algebra over $K$. This means that the norm $\| \cdot \|$ of algebra satisfies the non-Archimedean property, i.e. $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for any $x, y \in \mathcal{A}$. There are many examples of such kind of spaces (see [8, 32]).

Let us consider some basic examples of non-Archimedean Banach algebras.

1. The set

$$K^n = \{x = (x_1, \ldots, x_n) : x_k \in K, k = 1, \ldots, n\}$$

with a norm $\|x\| = \max |x_k|$ and usual pointwise summation and multiplication operations, is a non-Archimedean Banach algebra.

2. Let

$$c_0 = \{x = (x_n) : x_n \in K, x_n \to 0\}.$$

The defined set is endowed with usual pointwise summation and multiplication operations. Put $\|x\| = \max |x_k|$, then $c_0$ is a non-Archimedean Banach algebra.

In what follows, by $\mathcal{A}$ we denote a non-Archimedean Banach algebra.

There is a nice characterization of Cauchy sequence in non-Archimedean spaces.

**Proposition 2.1.** [18] A sequence $\{x_n\}$ in $\mathcal{A}$ is a Cauchy sequence with respect to the norm $\| \cdot \|$ if and only if $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Denote

$$B(a, r) = \{x \in \mathcal{A} : \|x - a\| < r\}, \quad \bar{B}(a, r) = \{x \in \mathcal{A} : \|x - a\| \leq r\},$$

$$S(a, r) = \{x \in \mathcal{A} : \|x - a\| = r\},$$

where $a \in \mathcal{A}$, $r > 0$.

In what follows, we will use the following
Lemma 2.2 ([16]). Let \( \{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset A \) such that \( \|a_i\| \leq 1, \|b_i\| \leq 1, i = 1, \ldots, n, \) then

\[
\left\| \prod_{i=1}^n a_i - \prod_{i=1}^n b_i \right\| \leq \max_{i \leq i \leq n} \{\|a_i - b_i\|\}
\]

Note that the basics of non-Archimedean analysis are explained in [33, ?].

3. A recurrence equations

Let \( A \) be a non-Archimedean Banach algebra and assume that \( C \subset \overline{B}(0, 1) \) be a closed set. A mapping \( f : C^m \rightarrow C \) is called contractive, if there is a constant \( \alpha_f \in [0, 1) \) such that

\[
\|f(x) - f(y)\| \leq \alpha_f \max_{i \leq k \leq m} \|x_k - y_k\| \quad \text{for all } x = (x_i), y = (y_i) \in C^m.
\]

Note that if the function \( f \) does not depend on some variable \( x_k \), then such a variable will be absent in the right hand side of (3.1).

Now assume that we are given several collections \( \{f_i^{(k)}\}_{i=1}^N, k = 1, \ldots, M \) of contractive mappings defined on \( C^m \). Let \( (\ell_1^{(k)}, \ldots, \ell_N^{(k)}) \) such that \( \ell_1^{(k)} = 0, 2 \leq \ell_i^{(k)} - \ell_{i-1}^{(k)} \leq m - 1, i = 2, \ldots, N, k = 1, \ldots, M. \)

Denote \( L := \max\{\ell_m^{(k)} : 1 \leq k \leq M\} + m. \) Take any initial points \( \{x_1, \ldots, x_L\} \subset C \), and consider the following sequence \( \{x_n\} \) defined by the recurrence relations:

\[
x_{n+L} = \sum_{k=1}^M \prod_{i=1}^N f_i^{(k)}(x_{n+\ell_i^{(k)}}, \ldots, x_{n+\ell_i^{(k)}+m-1}), \quad n \in \mathbb{N}.
\]

Lemma 3.1. Let \( \{f_i^{(k)}\}_{i=1}^N, k = 1, \ldots, M \) be collections of contractive mappings defined on \( C^m \) (where \( C \subset \overline{B}(0, 1) \)). Then for any initial points \( \{x_1, \ldots, x_{m+L}\} \subset C \) the sequence \( \{x_n\} \) defined by (3.2) is convergent.

Proof. To prove the lemma it is enough to show that \( \{x_n\} \) is a Cauchy sequence. Due to Proposition 2.1 we need to establish \( \|x_{n+1} - x_n\| \rightarrow 0 \) as \( n \rightarrow \infty \). Let us first denote

\[
\alpha = \max_{k, i} \alpha_{f_i^{(k)}}.
\]

From the contractivity of the functions \( f_i^{(k)} \) we conclude that \( 0 < \alpha < 1 \).
Now from (3.2), $\|f_i^{(k)}(x)\| \leq 1$, $x \in C^m$ and using Lemma 2.2 one finds

$$\|x_{n+L+1} - x_{n+L}\| \leq \max_{1 \leq k \leq M} \left| \prod_{i=1}^{N} f_i^{(k)}(x_{n+1+i^{(k)}}, \ldots, x_{n+1+i^{(k)}+m-1}) - \prod_{i=1}^{N} f_i^{(k)}(x_{n+i^{(k)}}, \ldots, x_{n+i^{(k)}+m-1}) \right|$$

$$\leq \max_{1 \leq k \leq M} \left| f_i^{(k)}(x_{n+1+i^{(k)}}, \ldots, x_{n+1+i^{(k)}+m-1}) - f_i^{(k)}(x_{n+i^{(k)}}, \ldots, x_{n+i^{(k)}+m-1}) \right|$$

$$\leq \alpha \left( \max_{1 \leq i \leq N} \left| x_{n+1+i^{(k)}} - x_{n+i^{(k)}} \right| \right)$$

$$\ldots \ldots \ldots$$

$$\leq \alpha^{n-L}.$$ 

The last inequality yields that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Due to closedness of $C$ we conclude that the limiting element belongs to $C$. This completes the proof. □

**Lemma 3.2.** Let $\{f_i^{(k)}\}_{i=1}^{N}$, $k = 1, \ldots, M$ be collections of contractive functions defined on $C^m$ (where $C \subset \overline{B}(0,1)$). Take any two collections of initial points, i.e. $\{x_1, \ldots, x_{m+L}\} \subset C$ and $\{y_1, \ldots, y_{m+L}\} \subset C$. Then for the corresponding sequences $\{x_n\}$ and $\{y_n\}$, defined by (3.2), one has $\|x_n - y_n\| \to 0$ as $n \to \infty$.

**Proof.** From (3.2), $\|f_i^{(k)}(x)\| \leq 1$, $x \in C^m$ and using Lemma 2.2 one finds

$$\|x_{n+L} - y_{n+L}\| \leq \max_{1 \leq k \leq M} \left| \prod_{i=1}^{N} f_i^{(k)}(x_{n+1+i^{(k)}}, \ldots, x_{n+1+i^{(k)}+m-1}) - \prod_{i=1}^{N} f_i^{(k)}(y_{n+1+i^{(k)}}, \ldots, y_{n+1+i^{(k)}+m-1}) \right|$$

$$\leq \max_{1 \leq k \leq M} \left| f_i^{(k)}(x_{n+1+i^{(k)}}, \ldots, x_{n+1+i^{(k)}+m-1}) - f_i^{(k)}(y_{n+1+i^{(k)}}, \ldots, y_{n+1+i^{(k)}+m-1}) \right|$$

$$\leq \alpha \left( \max_{1 \leq i \leq N} \left| x_{n+1+i^{(k)}} - y_{n+1+i^{(k)}} \right| \right)$$

$$\ldots \ldots \ldots$$

$$\leq \alpha^{n-L}.$$
The last inequality implies that \( \|x_n - y_n\| \to 0 \) as \( n \to \infty \). The proof is complete.  

From these lemmas we infer the following

**Theorem 3.3.** Let \( \{f_i^{(k)}\}_{k=1}^N \), \( k = 1, \ldots, M \) be collections of contractive functions defined on \( C^m \) (where \( C \subset \overline{B}(0, 1) \)). Then there is \( x_\ast \in C \) such that for any initial points \( \{x_1, \ldots, x_L\} \subset C \) the sequence \( \{x_n\} \) defined by (3.2) converges to \( x_\ast \). Moreover, one has
\[
\|x_{n+L} - x_\ast\| \leq \alpha^n \quad \text{for all} \quad n \in \mathbb{N},
\]
where
\[
\alpha = \max_{k,i} \alpha_{f_i^{(k)}}.
\]

**Remark 3.1.** From the last theorem we infer that the sequence (3.2) defines a unique solution (belonging to the set \( C \)) of the equation
\[
x = \sum_{k=1}^M \prod_{i=1}^N f_i^{(k)}(x, \ldots, x).
\]

**Remark 3.2.** If \( f(x) \) a contractive mapping on \( C^m \), then Theorem 3.3 yields that for any \( N > 1 \) the equation
\[
x = (f(x, \ldots, x))^N.
\]
has a unique solution belonging to \( C \). More concrete examples will be given in the final section.

Now let us consider multisequence case.

As before \( \mathcal{A} \) denotes a non-Archimedean Banach algebra and assume that \( C \subset \overline{B}(0, 1) \) be a closed set. Suppose that we are given several collections \( \{F_1^{(k)}, F_2^{(k)}\}_{k=1}^{N_1}, \{G_1^{(k)}, G_2^{(k)}\}_{k=1}^{N_2}, \{H_1^{(k)}, H_2^{(k)}\}_{k=1}^{N_3} \) of contractive mappings defined on \( C^2 \).

Take any initial points \( \{x_1, y_1, z_1\} \subset C \), and consider the following sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) defined by the recurrence relations:
\[
\begin{align*}
x_{n+1} &= \sum_{k=1}^{N_1} F_1^{(k)}(x_n, y_n) F_2^{(k)}(y_n, z_n) \\
y_{n+1} &= \sum_{k=1}^{N_2} G_1^{(k)}(x_{n+1}, y_n) G_2^{(k)}(y_n, z_n) \\
z_{n+1} &= \sum_{k=1}^{N_3} H_1^{(k)}(x_{n+1}, y_{n+1}) H_2^{(k)}(y_{n+1}, z_n)
\end{align*}
\]

**Theorem 3.4.** Let \( \{F_1^{(k)}, F_2^{(k)}\}_{k=1}^{N_1}, \{G_1^{(k)}, G_2^{(k)}\}_{k=1}^{N_2}, \{H_1^{(k)}, H_2^{(k)}\}_{k=1}^{N_3} \) be collections of contractive mappings defined on \( C^2 \) (where \( C \subset \overline{B}(0, 1) \)). Then for any initial points \( \{x_1, y_1, z_1\} \subset C \) the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) defined by (3.4) are convergent. Moreover, the limit does not depend on initial conditions.

**Proof.** First we prove that each sequence is a Cauchy sequence. Let us denote
\[
d_n = \max\{\|x_{n+1} - x_n\|, \|y_{n+1} - y_n\|, \|z_{n+1} - z_n\|\},
\]
\[
\alpha = \max_{k,i} \{\alpha_{F_i^{(k)}}, \alpha_{G_i^{(k)}}, \alpha_{H_i^{(k)}}\}.
\]
Due to condition we have that \( 0 < \alpha < 1 \).
Now from (3.4), \( \| f_i^{(k)}(x) \| \leq 1 \) and using Lemma 2.2 one finds
\[
\| x_{n+1} - x_n \| \leq \max_{1 \leq k \leq N_1} \left\| F_1^{(k)}(x_n, y_n)F_2^{(k)}(y_n, z_n) - F_1^{(k)}(x_{n-1}, y_{n-1})F_2^{(k)}(y_{n-1}, z_{n-1}) \right\|
\leq \max_{1 \leq k \leq N_1} \max\{ \| F_1^{(k)}(x_n, y_n) - F_1^{(k)}(x_{n-1}, y_{n-1}) \|, \| F_2^{(k)}(y_n, z_n) - F_2^{(k)}(y_{n-1}, z_{n-1}) \| \}
\leq \alpha \max\{ \| x_n - x_{n-1} \|, \| y_n - y_{n-1} \|, \| z_n - z_{n-1} \| \}. \tag{3.5}
\]
Using the same argument we obtain
\[
\| y_{n+1} - y_n \| \leq \alpha \max\{ \| x_{n+1} - x_n \|, \| y_n - y_{n-1} \|, \| z_n - z_{n-1} \| \}, \tag{3.6}
\]
\[
\| z_{n+1} - z_n \| \leq \alpha \max\{ \| x_{n+1} - x_n \|, \| y_{n+1} - y_n \|, \| z_n - z_{n-1} \| \}. \tag{3.7}
\]
Hence from (3.5)-(3.7) one finds
\[
d_{n+1} \leq \alpha d_n
\]
for all \( n \in \mathbb{N} \). This means that \( d_n \to 0 \) as \( n \to \infty \). Due to Proposition 2.1 the sequences are Cauchy. The closedness of \( C \) yields that the limiting elements belongs to \( C \), i.e. \( x_n \to x_*, y_n \to y_*, z_n \to z_* \), where \( x_*, y_*, z_* \in C \).

The uniqueness of the limiting elements can by proved by the same argument as the proof of Lemma 3.2. This completes the proof. \( \Box \)

**Remark 3.3.** From Theorem 3.4 we conclude that the sequences (3.4) define a unique solution (belonging to the set \( C \)) of the system of equations
\[
\begin{align*}
x &= \sum_{k=1}^{N_1} f_1^{(k)}(x, y)F_2^{(k)}(y, z) \\
y &= \sum_{k=1}^{N_2} G_1^{(k)}(x, y)G_2^{(k)}(y, z) \\
z &= \sum_{k=1}^{N_3} H_1^{(k)}(x, y)H_2^{(k)}(y, z)
\end{align*}
\tag{3.9}
\]
Note that a priori the existence of the solution of (3.9) is not obvious. Moreover, the proved Theorem 3.4 allows to find solutions of functional equations, when one takes instead of \( \mathcal{A} \) the algebra of analytic functions. In [9] polynomial functional equations have been investigated over \( p \)-adic analytic functions.

**Remark 3.4.** We stress that by modifying (3.4) for arbitrary number of sequences, similar kind of results can be proved by means of the same technique as in the proof of Theorem 3.4.

4. **A reverse recurrence equations**

In this section we consider a reverse recurrence relations to (3.2). To define it, we need some preliminary notions about a \( k \)-ary trees.

Let \((V, L)\) be a graph, here \( V \) is the set of vertices and \( L \) is the set of edges. A pair \( G_k = (V, L) \) is called \( k \)-ary tree if it has a root \( x^0 \) in which each vertex has no more than \( k \) edges. If in a \( k \)-ary tree each vertex has exactly \( k \) edges, then such a tree is called Cayley tree. The vertices \( x \) and \( y \) are called nearest neighbors and they are denoted by \( l = < x, y > \) if there exists an edge connecting them. A collection of the pairs \(< x, x_1 >, \ldots, < x_{d-1}, y > \) is called a path from the point \( x \) to the point \( y \). The distance \( d(x, y) \), \( x, y \in V \), on the tree, is the length of the shortest path from \( x \) to \( y \).
Recall a coordinate structure in $G_k$: every vertex $x$ (except for $x^0$) of $\Gamma_k$ has coordinates $(i_1, \ldots, i_n)$, here $i_m \in \{1, \ldots, k\}, 1 \leq m \leq n$ and for the vertex $x^0$ we put $(0)$. Namely, the symbol $(0)$ constitutes level 0, and the sites $(i_1, \ldots, i_n)$ form level $n$ (i.e. $d(x^0, x) = n$) of the lattice.

For $x \in G_k, x = (i_1, \ldots, i_n)$ denote

\[(4.1)\quad S(x) = \{(x, i) : 1 \leq i \leq k_x\},\]

here $(x, i)$ means that $(i_1, \ldots, i_n, i)$. This set is called a set of direct successors of $x$.

Let $A$ be as usual a non-Archimedean Banach algebra and $C \subset \overline{B}(0, 1)$. Assume that we are given a family $\{f^{(i)}_{x,y}\}_{i=1}^M, \langle x, y \rangle \in L$ of contractive mappinga such that for each $\langle x, y \rangle$ the function $f^{(i)}_{x,y}$ maps $C_{k_x}$ to $C$. Now consider a function $u : V \to C$, i.e. $u = ((u_x)_{x \in G_k}$ such that

\[(4.2)\quad u_x = \sum_{i=1}^M \prod_{y \in S(x)} f^{(i)}_{xy}(u(x,1), \ldots, u(x,k_x)).\]

We are interested how many functions $u$ satisfy the equation (4.2).

Denote

\[\beta = \max_{1 \leq i \leq M} \max_{\langle x, y \rangle \in L} f^{(i)}_{x,y}.\]

**Theorem 4.1.** Let $\{f^{(i)}_{x,y}\}_{i=1}^M, \langle x, y \rangle \in L$ be a family of contractive functions such that $\beta < 1$. Then a solution of the equation (4.2) is not more than one.

**Proof.** If the equation (4.2) has not any solution, then nothing to prove. Therefore, let us assume that the given equation has a solution. To prove Theorem it is enough to show that any two solutions coincide with each other. Namely, if $u = (u_x, x \in V)$ and $v = (v_x, x \in V)$ are solutions of (4.2), then it is sufficient to establish that for any $\varepsilon > 0$ and $x \in V$ the inequality $\|u_x - v_x\| < \varepsilon$ is valid.

Let $x \in V$ be an arbitrary vertex. Then from (4.2), $\|f^{(i)}_{xy}(x)\| \leq 1, x \in C_{k_x}$ and using Lemma 2.2 we obtain

\[
\|u_x - v_x\| \leq \max_{1 \leq i \leq M} \max_{y \in S(x)} \left\|f^{(i)}_{xy}(u(x,1), \ldots, u(x,k_x)) - f^{(i)}_{xy}(v(x,1), \ldots, v(x,k_x))\right\|
\]

\[(4.3)\quad \leq \beta \left(\max_{1 \leq i \leq k_x} \|u(x,i) - v(x,i)\|\right).\]

Let us choose $n_0 \in \mathbb{N}$ such that $\beta^{n_0} < \varepsilon$. Therefore, iterating (5.8) $n_0$-times one gets

\[(4.4)\quad \|u_x - v_x\| \leq \beta^{n_0} < \varepsilon.\]

This completes the proof. □

**Remark 4.1.** We note that particular cases of the present theorem were proved in [13, 26, 27, 28]. The proved theorem generalize and extends all the known results.
5. Application

In the section we consider the $p$-adic field $\mathbb{Q}_p$ ($p \geq 3$). Recall that the $p$-adic logarithm is defined by series
\[
\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n},
\]
which converges for every $x \in B(1, 1)$. And $p$-adic exponential is defined by
\[
\exp_p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!},
\]
which converges for every $x \in B(0, p^{-1/(p-1)})$.

Lemma 5.1. [18] Let $x \in B(0, p^{-1/(p-1)})$ then we have
\[
|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p < 1, \quad |\log_p(1 + x)|_p = |x|_p < p^{-1/(p-1)}
\]
and
\[
\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1 + x)) = 1 + x.
\]

Remark 5.1. Note that, in general, the logarithm and the exponential functions can be defined over the field $K$ with $\text{char}(K) = 0$ (see [33]).

Denote
\[
\mathcal{E}_p = \{x \in \mathbb{Q}_p : |x|_p = 1, \ |x - 1|_p \leq 1/p\}.
\]

Note that from Lemma 5.1 one concludes that if $x \in \mathcal{E}_p$, then there is an element $h \in B(0, p^{-1/(p-1)})$ such that $x = \exp_p(h)$. Therfore, for any $x, y \in \mathcal{E}_p$ one gets $xy \in \mathcal{E}_p$.

1. Assume that $\mathcal{A} = \mathbb{Q}_p$ and $\mathcal{C} = \mathcal{E}_p$. Let us consider a non-linear function:
\[
f(x, y) = \frac{a xy + b(x + y) + c}{a_1 xy + b_1(x + y) + c_1},
\]
where $a, a_1, b, b_1, c, c_1 \in \mathcal{E}_p$.

Proposition 5.2. Let $f$ be given by (5.2). Then one has
(i) $f(x, y) \in \mathcal{E}_p$ for any $x, y \in \mathcal{E}_p$;
(ii) the function $f$ is contractive.

Proof. (i). Take any $x, y \in \mathcal{E}_p$. Then one can see that
\[
|axy + b(x + y) + c|_p = |axy - 1 + b(x - 1 + y - 1) + 2(b - 1) + c - 1 + 4|_p = 1,
\]
since $a, a_1, b, b_1, c, c_1 \in \mathcal{E}_p$. Similarly, we get
\[
|a_1 xy + b_1(x + y) + c_1|_p = 1.
\]
Therefore, $|f(x, y)|_p = 1$. Using the same manner from
\[
|f(x, y) - 1|_p = |(a - a_1)xy + (b - b_1)(x + y) + c - c_1|_p \leq \frac{1}{p}
\]
we find that $f(x, y) \in \mathcal{E}_p$.

(ii) Now take any $(x, y), (x_1, y_1) \in \mathcal{E}_p \times \mathcal{E}_p$. Then using (5.3),(5.4) one finds
\[
|f(x, y) - f(x_1, y_1)|_p = |\Delta_1(x - x_1) + \Delta_2(y - y_1)|_p
\]

\[
\Delta_1 = (a - a_1)\text{ and } \Delta_2 = (b - b_1).
\]
where
\[
\Delta_1 = (ab_1 - a_1b)yy_1 + (ac_1 - a_1c)y_1 + c_1b - cb_1,
\]
\[
\Delta_2 = (ab_1 - a_1b)x_1 + (ac_1 - a_1c)x + c_1b - cb_1.
\]
It is easy to see that \(|\Delta_1|_p \leq 1/p, |\Delta_2|_p \leq 1/p\). Hence, from (5.12) we have
\[
|f(x, y) - f(x_1, y_1)|_p \leq \frac{1}{p} \max\{|x - x_1|_p, |y - y_1|_p\}
\]
which implies the assertion. \(\square\)

Let \(G\) be a Cayley tree of order three, and consider the following functional equation
\[
u_x = f(u(x_1), u(x_2)),
\]
where \((u_x)_{x \in G}\) is unknown function and \(f\) is given by (5.2).

Then due to Theorem 4.1 the equation has a unique solution \(u_x = u_\ast\). Here \(u_\ast\) is a fixed point belonging to \(E_p\) of the function \(f(u, u)\) which exists due to Theorem 3.3. This fact extends the results of the papers [12].

One can consider the following equation
\[
u_x = (f(u(x_1), u(x_2)))^k.
\]
This equation also has a unique solution \(u_x = u_\ast\), where \(u_\ast \in E_p\) is a fixed point of \((f(u, u))^k\) which exists due to Theorem 3.3. This fact implies the main result of the paper [13].

2. Now let us consider another kind of example.
Assume that \(A = \mathbb{Q}_p\) and \(C = S(0, 1)\). Define a non-linear function as follows:
\[
F(x_1, \ldots, x_m) = \frac{P(x_1, \ldots, x_m) + C}{Q(x_1, \ldots, x_m) + C_1}
\]
where
\[
P(x_1, \ldots, x_m) = \sum_{i_1 + \cdots + i_m = 1, \ \ i_k \geq 0, 1 \leq k \leq m}^N A_{i_1, \ldots, i_m} x_1^{i_1} \cdots x_m^{i_m}
\]
\[
Q(x_1, \ldots, x_m) = \sum_{i_1 + \cdots + i_m = 1, \ \ i_k \geq 0, 1 \leq k \leq m}^N B_{i_1, \ldots, i_m} x_1^{i_1} \cdots x_m^{i_m}
\]
and \(A_{i_1, \ldots, i_m}, B_{i_1, \ldots, i_m} \in B(0, 1)\) and \(|C|_p = |C_1|_p = 1\).

**Proposition 5.3.** Let \(F\) be given by (5.9). Then one has
\begin{enumerate}
\item[(i)] \(F(x_1, \ldots, x_m)\) \(\in S(0, 1)\) for any \(x_1, \ldots, x_m \in S(0, 1)\);
\item[(ii)] the function \(F\) is contractive on \(S(0, 1)^m\).
\end{enumerate}

**Proof.** (i). Due to \(A_{i_1, \ldots, i_m}, B_{i_1, \ldots, i_m} \in B(0, 1)\) we immediately find that \(|P(x_1, \ldots, x_m)|_p = |Q(x_1, \ldots, x_m)|_p < 1\) for any \(x_1, \ldots, x_m \in S(0, 1)\), which with \(|C|_p = |C_1|_p = 1\) implies the assertion.
(ii) Now take any \((x_1, \ldots, x_m), (y_1, \ldots, y_m) \in S(0, 1)^m\). Then using (5.10) and Proposition 2.2 one finds

\[
|F(x_1, \ldots, x_m) - F(y_1, \ldots, y_m)|_p = |C_1 P(x_1, \ldots, x_m) + C Q(y_1, \ldots, y_m) + P(x_1, \ldots, x_m)Q(y_1, \ldots, y_m) - C_1 P(y_1, \ldots, y_m) - C Q(x_1, \ldots, x_m) - P(y_1, \ldots, y_m)Q(x_1, \ldots, x_m)|_p
\]

\[
= \left| C_1 \sum_{i_1, \ldots, i_m} A_{i_1, \ldots, i_m} (x_1^{i_1} \cdots x_m^{i_m} - y_1^{i_1} \cdots y_m^{i_m}) \right|
\]

\[
- C \sum_{i_1, \ldots, i_m} B_{i_1, \ldots, i_m} (x_1^{i_1} \cdots x_m^{i_m} - y_1^{i_1} \cdots y_m^{i_m})
\]

\[
+ \sum_{i_1, \ldots, i_m} \sum_{j_1, \ldots, j_m} A_{i_1, \ldots, i_m} B_{j_1, \ldots, j_m} (x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_m^{j_m} - x_1^{j_1} \cdots x_m^{j_m} y_1^{i_1} \cdots y_m^{i_m})
\]

\[
\leq \frac{1}{p} \max\{|x_k - y_k|_p\},
\]

which implies the assertion.

Let us consider the following sequence

\[X_{n+2m} = F(X_n, \ldots, X_{n+m})F(X_{n+1}, \ldots, X_{n+m})F(X_{n+m}, \ldots, X_{n+2m-1}),\]

with initial conditions \(X_1, \ldots, X_{2m} \in S(0, 1)\).

Then due to Theorem 3.3 the sequence \(\{X_n\}\) converges to \(X_* \in S(0, 1)\) which is a solution of the equation

\[X = (F(X, \ldots, X))^3.\]

3. Assume that \(\mathcal{A} = \mathbb{Q}_p^m\) and \(C = \mathcal{E}_p\), here \(m + 1\) is not divisible by \(p\). Define a non-linear mapping \(f : \mathbb{Q}_p^m \to \mathbb{Q}_p^m\) by the following formula:

\[f(x)_k = \frac{\sum_{j=1}^m a_j^{(k)} x_j + a_k}{\sum_{j=1}^m b_j^{(k)} x_j + b_k}, \quad x = (x_1, \ldots, x_m), \quad k = 1, \ldots, m,\]

where \(a_j^{(k)}, b_j^{(k)}, a_k, b_k \in \mathcal{E}_p\).

**Proposition 5.4.** Let \(f\) be given by (5.12). Then one has

(i) \(f(\mathcal{E}_p^m) \subset \mathcal{E}_p^m\);

(ii) the mapping \(f\) is contractive on \(\mathcal{E}_p^m\).

**Proof.** (i) Due to \(a_j^{(k)}, b_j^{(k)}, a_k, b_k \in \mathcal{E}_p\) and \(m + 1 \nmid p\) one finds

\[\left| \sum_{j=1}^m a_j^{(k)} x_j + a_k \right|_p = \left| \sum_{j=1}^m (a_j^{(k)} x_j - 1) + (a_k - 1) + m + 1 \right|_p = 1\]

Similarly, we have

\[\left| \sum_{j=1}^m b_j^{(k)} x_i + b_k \right|_p = 1.\]
This yields that $|f(x)_k|_p = 1$ for all $x \in \mathcal{E}_p^m$, $k \in \{1, \ldots, m\}$.

Using the same argument, one can get $|f(x)_k - 1|_p \leq 1/p$ for all $x \in \mathcal{E}_p^m$. This implies the assertion.

(ii). Now using (5.13),(5.14) we obtain

$$|f(x)_k - f(y)_k|_p = \left| \left( \sum_{j=1}^m a_j^{(k)} x_j + a_k \right) \left( \sum_{j=1}^m b_j^{(k)} y_j + b_k \right) \right|_p$$

$$- \left| \left( \sum_{j=1}^m a_j^{(k)} y_j + a_k \right) \left( \sum_{j=1}^m b_j^{(k)} x_j + b_k \right) \right|_p$$

(5.15)

$$= \left| \sum_{i,j=1}^m a_i^{(k)} b_j^{(k)} (x_i y_j - x_j y_i) \right|_p - \left| \sum_{j=1}^m (a_k b_j^{(k)} - b_k a_j^{(k)}) (x_j - y_j) \right|_p.$$

Now let us rewrite the expression $I$ as follows

$$\sum_{i,j=1}^m a_i^{(k)} b_j^{(k)} (x_i y_j - x_j y_i) = \sum_{i,j=1}^m a_i^{(k)} b_j^{(k)} (x_i (y_j - x_j) + x_j (x_i - y_i))$$

$$= \sum_{i,j=1}^m x_i (b_i^{(k)} a_j^{(k)} - a_i^{(k)} b_j^{(k)}) (x_j - y_j).$$

Therefore, we find that

$$|I|_p \leq \frac{1}{p} \max \{|x_k - y_k|_p\}, \quad |II|_p \leq \frac{1}{p} \max \{|x_k - y_k|_p\},$$

Hence, the last inequallitons with (5.15) implies that

$$\|f(x) - f(y)\| \leq \frac{1}{p} \|x - y\|$$

this proves the proposition. \qed

Let $\Gamma_k$ be a Cayley tree of order $k$ ($k \geq 1$). Let us consider the functional equation

(5.16)

$$u_x = \prod_{y \in S(x)} f(u_y),$$

where $u = (u_x)_{x \in \Gamma_k}$ is unknown function and $f$ is given by (5.12).

It is clear that the equation (5.16) has a solution $u_x = u_*$, where $u_*$ is fixed point of the equation

$$(f(u))^k = u.$$

Note that this solution $u_*$ belongs to $\mathcal{E}_p^m$ which follows from Theorem 3.3.

Now according to Theorem 4.1 we conclude that the equation (5.16) has only one solution which is $u_x = u_*$. This result can be applied to the existence and uniqueness of $p$-adic Gibbs measure associated with $m$-state $p$-adic $\lambda$-model on the Cayley tree of order $k$ (see for the definition of the model [22]).
4. In this example, we assume that $\mathcal{A} = c_0$ and $C = \bar{B}(0,1)$. Define a non-linear mapping $\mathcal{F} : c_0 \to c_0$ as follows:

\[(5.17) \quad (\mathcal{F}(x))_k = \lambda_k F_k(x), \quad x \in c_0,\]

where $\ell = \{\lambda_k\} \in c_0$ with $\|\ell\| \leq 1$, and

\[(5.18) \quad F_k(x) = \frac{ax_k + f_k(x)}{b + f_k(x)}.\]

Here $a, b \in \mathbb{Q}_p$, $\max\{|a|_p, |b|_p\} < 1$ and the functions $\{f_k\}$ such that $|f_k(x)|_p = 1$ for all $x \in \bar{B}(0,1)$, $k \in \mathbb{N}$ and one has

\[(5.19) \quad |f_k(x) - f_k(y)|_p \leq \|x - y\|, \quad \text{for all } x, y \in \bar{B}(0,1).\]

**Proposition 5.5.** Let $\mathcal{F}$ be given by (5.17). Then one has

(i) $\mathcal{F}(\bar{B}(0,1)) \subset \bar{B}(0,1);

(ii) the mapping $\mathcal{F}$ is contractive on $\bar{B}(0,1)$.

**Proof.** (i) From $\max\{|a|_p, |b|_p\} < 1$ and $|f_k(x)|_p = 1$ for all $x \in \bar{B}(0,1)$ we immediately find that $|F_k(x)|_p = 1$ for all $x \in \bar{B}(0,1)$ and $k \in \mathbb{N}$. Therefore, $\|\mathcal{F}(x)\| = \|\ell\| \leq 1$, which is the required assertion.

(ii) Take any $x, y \in \bar{B}(0,1)$. Then we have

\[
|F_k(x) - F_k(y)| = |ab(x_k - y_k) + a(x_k f(y) - y_k f_k(x)) + b(f_k(x) - f_k(y))|_p
\]

\[
= |a(b + f_k(x))(x_k - y_k) + (b - ax_k)(f_k(x) - f_k(y))|_p
\]

\[(5.20) \quad \leq \max\{|a|_p, |b|_p\}\|x - y\|.\]

Hence, from (5.17) and (5.20) one gets

\[
\|\mathcal{F}(x) - \mathcal{F}(y)\| \leq \max\{|a|_p, |b|_p\}\|x - y\|
\]

this completes the proof. \(\square\)

Let $\Gamma_k$ be a Cayley tree of order $k$ ($k \geq 1$). Let us consider the functional equation

\[(5.21) \quad u_{x,i} = \prod_{y \in S(x)} (\mathcal{F}(u_y))_i, \quad \text{for all } i \in \mathbb{N}\]

where $u_x = \{u_{x,k}\} \in C$ for each $x \in \Gamma_k$ is unknown function and $\mathcal{F}$ is given by (5.17). Since $c_0$ is an algebra, then (5.21) can be rewritten as follows

\[(5.22) \quad u_x = \prod_{y \in S(x)} \mathcal{F}(u_y).\]

According to Theorem 4.1 we conclude that the equation (5.22) has only one solution which is $u_x = u_*$. Here $u_*$ is a solution of the equation

\[(\mathcal{F}(u))^k = u.\]

Note that this solution $u_*$ belongs to $C$ which follows from Theorem 3.3.
From this result, as a particular case, we obtain a main result of the paper [24], if one takes
\[ f_k(x) = p \sum_{j=1}^{\infty} x_j + 1 \] for all \( k \in \mathbb{N} \),
and \( a = p(\theta - 1), \ b = \theta - 1 \), where \( \theta \in \mathcal{E}_p \).

Let \( N \geq 2 \) be a fixed natural number. Now consider another kind of the functional equation
\[ u_{x,i} = \prod_{y \in S(x)} (\mathcal{F}(u_y))_{i+j}, \quad \text{for all } i \in \mathbb{N} \]
(5.23)
where as before \( u_x = \{u_{x,k}\} \in C \), for each \( x \in \Gamma_k \), is unknown function and \( \mathcal{F} \) is given by (5.17).

Let us rewrite the last equation in terms of elements of the algebra \( c_0 \). Denote by \( \sigma : c_0 \to c_0 \) the shift operator, i.e.
\[ (\sigma(x))_k = x_{k+1}, \quad k \in \mathbb{N} \]
where \( x = \{x_k\} \in c_0 \). Then (5.23) can be rewritten as follows
\[ u_x = \prod_{y \in S(x)} \prod_{j=1}^{N} \sigma^j(\mathcal{F}(u_y)). \]
(5.24)

Again Theorem 4.1 implies the uniqueness of the solution of (5.22), which is \( u_x = u^* \) for all \( x \in \Gamma_k \). Here \( u^* \) is a solution of the equation
\[ \left( \prod_{j=1}^{N} \sigma^j(\mathcal{F}(u)) \right)^k = u. \]

Note that this solution \( u^* \) belongs to \( C \) which follows from Theorem 3.3.

Acknowledgments

The first named author (F.M.) acknowledges the Scientific and Technological Research Council of Turkey (TUBITAK) for support, and Zirve University (Gaziantep) for kind hospitality. F.M. also thanks the MOHE grant ERGS13-024-0057, the IIUM grant EDW B13-029-0914 and the Junior Associate scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

References

[1] Albeverio S., Khrennikov A. Yu., Shelkovich, V.M. Theory of p-adic Distributions. Linear and Nonlinear Models, Cambridge Univ. Press, Cambridge, 2010.
[2] Albeverio S., Rozikov U., Sattorov I.A. p-adic (2,1)-rational dynamical systems, J. Math. Anal. Appl. 398 (2013) 553–566.
[3] Anashin V., Khrennikov A., Applied Algebraic Dynamics, Walter de Gruyter, Berlin, New York, 2009.
[4] Batra A., Morton P., Algebraic dynamics of polynomial maps on the algebraic closure of a finite field I, II. Rocky Mountain J. of Math., 24(1994), 453-481; 905–932.
[5] Benedetto R., Hyperbolic maps in p-adic dynamics, Ergod. Th. & Dynam. Sys. 21(2001), 1–11.
[6] Call G., Silverman J., Canonical height on varieties with morphisms, Compositio Math. 89(1993), 163–205.
[7] Dragovich B., Khrennikov A. Yu., Kozyrev S.V., Volovich I.V. p-adic mathematical physics, P-Adic Numbers, Ultrametric Anal. Appl. 1 (2009), 1-17.
[8] Escassut A., Ultrametric Banach Algebras, World Scientific, Singapore, 2003.
[9] Escassut A., Ojeda J., Yang C.C. Functional equations in a $p$-adic context, J. Math. Anal. Appl. 351 (2009), 350–359.
[10] Fan A.H., Liao L.M., Wang D., $p$-adic repellers in $\mathbb{Q}_p$, are subsifts of finite type, C.R. Math. Acad. Sci. Paris 344 (2007) 219–224.
[11] Herman M., Yoccoz J.-C., Generalizations of some theorems of small divisors to non-Archimedean fields, In: Geometric Dynamics (Rio de Janeiro, 1981), Lec. Notes in Math. 1007, Springer, Berlin, 1983, pp.408–447.
[12] Khakimov O.N., On $p$-adic Gibbs measures for Ising model with four competing interactions, P-Adic Numbers, Ultram. Anal. Appl. 5 (2013) 194–203.
[13] Khamraev M., Mukhamedov F.M. On $p$-adic $\lambda$-model on the Cayley tree, Jour. Math. Phys. 45 (2004) 4025–4034.
[14] Khamraev M., Mukhamedov F.M., On a class of rational $p$-adic dynamical systems, Jour. Math. Anal. Appl. 315 (2006), 76–89.
[15] Khrennikov A.Yu. Non-Archimedean analysis: quantum paradoxes, dynamical systems and biological models, Kluwer Academic Publisher, Dordrecht, 1997.
[16] Khrennikov A., Mukhamedov F., Mendes J.F.F. On $p$-adic Gibbs measures of countable state Potts model on the Cayley tree, Nonlinearity 20 (2007) 2923-2937.
[17] Khrennikov A.Yu., Nilsson M. $p$-adic deterministic and random dynamical systems, Kluwer, Dordrecht, 2004.
[18] Koblitz N. $p$-adic numbers, $p$-adic analysis and zeta-function, Berlin, Springer, 1977.
[19] Lubin J., Nonarchimedean dynamical systems, Composito Math. 94 (3)(1994), 321–346.
[20] Ledrappier F., Pollicott M., Distribution results for lattices in $\text{SL}(2, \mathbb{Q}_p)$, Bull. Braz. Math. Soc. (N.S.) 36 (2005), 143–176.
[21] Marinary E., Parisi G. On the $p$-adic five point function, Phys. Lett. B 203 (1988) 52–56.
[22] Mukhamedov F., On factor associated with the unordered phase of $\lambda$-model on a Cayley tree. Rep. Math. Phys. 53 (2004), 1–18.
[23] Mukhamedov F.M., On the existence of generalized Gibbs measures for the one-dimensional $p$-adic countable state Potts model, Proc. Steklov Inst. Math. 265 (2009) 165-176.
[24] Khrennikov A.Yu., Mukhamedov F., On uniqueness of Gibbs measure for $p$-adic countable state Potts model on the Cayley tree, Nonlin. Analysis: Theor. Methods Appl. 71 (2009), 5327–5331.
[25] Mukhamedov F., A dynamical system approach to phase transitions $p$-adic Potts model on the Cayley tree of order two, Rep. Math. Phys. 70 (2012), 385–406.
[26] Mukhamedov, F., On a recursive equation over $p$-adic field, Appl. Math. Lett. 20 (2007) 88–92
[27] Mukhamedov F.M., Rozikov U.A. On Gibbs measures of $p$-adic Potts model on the Cayley tree, Indag. Math. N.S. 15 (2004) 85–100.
[28] Mukhamedov F.M., Rozikov U.A. On inhomogeneous $p$-adic Potts model on a Cayley tree, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 8 (2005) 277–290.
[29] Perez-Garcia C., Schikhof W.H., Locally Convex Spaces over Non-Archimedean Valued Fields, Cambridge University Press, 2010.
[30] Qiu W., Wang Y., Yang J., Yin Y., On metric properties of limit sets of contractive analytic non-Archimedean dynamical systems, J. Math. Anal. Appl. (in press)
[31] Rivera-Letelier J., Dynamics of rational functions over local fields, Astérisque 287 (2003), 147–230.
[32] van Rooij A., Non-archimedean functional analysis, Marcel Dekker, New York, 1978.
[33] Schikhof W.H. Ultrametric Calculus, Cambridge University Press, Cambridge, 1984.
[34] Silverman J.H. The arithmetic of dynamical systems, Springer-Verlag, New York, 2007.
[35] van der Put M., Difference equations over $p$-adic fields, Math. Ann. 198 (1972) 189-203.
[36] Vladimirov V.S., Volovich I.V., Zelenov E.I. $p$-adic Analysis and Mathematical Physics, World Scientific, Singapore, 1994.
[37] Volovich I.V. Number theory as the ultimate physical theory, $p$-Adic Numbers, Ultrametric Analysis Appl. 2 (2010), 77-87; // Preprint TH.4781/87, 1987.
[38] Volovich I.V. $p$–adic string, Classical Quantum Gravity 4 (1987) L83-L87.
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