Measuring model risk in financial risk management and pricing

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Abstract

Risk measurement and pricing of financial positions are based on modeling assumptions, which are common assumptions on the probability distribution of the position’s outcomes. We associate a model with a probability measure and investigate model risk by considering a model space. First, we incorporate model risk into market risk measures by introducing model weighted and superposed market risk measures. Second, we quantify model risk itself and propose axioms for model risk measures. We introduce superposed model risk measures that quantify model risk relative to a reference model, which is the financial institution’s model of choice. Several risk measures that we propose require a probability distribution on the model space, which can be obtained from data by applying Bayesian analysis. Examples and a case study illustrate our approaches.

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Contents

1 Introduction 2
2 Risk Measures 3
  2.1 Monetary risk measures 3
3 Measures of market risk that account for model risk 4
  3.1 Model weighted market risk measures 5
  3.2 Superposed market risk measures 6
  3.3 Robust representation of superposed risk measures 7
4 Measuring model risk 9
  4.1 Requirements on measures of model risk 9
  4.2 Measures of model risk for market risk measures 10
  4.3 Model risk measures for pricing contingent claims 11
  4.4 Example: Model risk of prices for barrier options 14
5 Case study: Model risk of market risk 16
  5.1 Bayesian methods for estimating model distributions 16
  5.2 Data and model set 17
  5.3 Reversible Jump MCMC specification 18
  5.4 Results 19

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1 Introduction

Mathematical methods for financial risk measurement and pricing are based on assumptions that form models. Models capture only some particular characteristics of real-world phenomena. Therefore, one should not expect that they always give us a decent answer to the problem at hand. Improper modeling assumptions can lead to wrong business decisions and significant financial losses. For example, China Aviation Oil (Singapore) Ltd faced losses of USD 550 million in November 2004 from using inadequate modeling approaches for derivatives. Their valuation models only considered the intrinsic value of derivatives and ignored the time value, see e.g., Matulich and Currie [2008], Chapter 14, De Jonghe and Schrans [2006]. J.P. Morgan lost over USD 6 billion in 2012 after the European sovereign debt crisis due to a wrong value at risk measurement methodology, see e.g., Aggarwal et al. [2016].

While the roots of credit and market risk management can be found three decades ago, scholars and practitioners start paying special attention to model risk only after the recent financial crisis, see e.g., Alexander [2009], Chapter 6. Model risk is also a current issue from a regulatory perspective. The Basel Committee on Banking Supervision and the Federal Reserve explicitly require active management of model risk, see Federal Reserve [2011]; BIS [2010,0,0,0].

The aim of this paper is to analyze and quantify the risk arising from misspecification of models in the context of financial market risk management and contingent claim pricing.

Standard market risk measures, such as value at risk or expected shortfall, depend on modeling assumptions, which are in this case the probability distributions of financial positions. We depart from the standard setting by considering a space of models.

First, we propose monetary market risk measures that incorporate model risk. These are, for example, model weighted market risk measures or superposed risk measures, which are risk measures on the space of financial positions combined (superposed) with risk measures on the model space. For robust representations of superposed risk measures, we derive their penalty function from the penalty functions of their building blocks.

Second, as opposed to measures of market risk that account for model risk, we measure model risk on its own. We quantify model risk in relation to a reference model that is the usual model chosen by the financial institution. We start by introducing basic axioms for measures of model risk, where we have to distinguish between model risk of market risk and model risk of contingent claim pricing. We then propose superposed model risk measures and give representative examples. For certain superposed risk measures, a distribution on the model space is required, which calls for Bayesian methods.

A case study for a portfolio consisting of major stock indices illustrates our proposed risk measures and shows that model risk has a significant impact on the financial institution’s estimates of market risk. The results of the case study show that market risk measures that heavily depend on the tail distribution of a financial position are more exposed to model risk, than risk measures less exposed to tail risk.

In the literature, there are various approaches to model risk. Derman [1996] studies the assumptions of financial models and underlines the importance of model risk management. Based on a simulation study, Green and Figlewski [1999] show that for an options trading desk model risk is significant. Hull and Suo [2002]; An et al. [2007] analyze the impact of stochastic and local volatility models on the pricing of exotic options. They find that the model risk is particularly high for path dependent contracts. Schoutens et al. [2004] calibrate stochastic volatility and jump models to market prices of vanilla options and shows that, even with almost perfect calibration, the prices of exotic options strongly depend on the model choice. Alexander and Sheedy [2008] show that constant parameter risk models fail to forecast short-term risk and propose an efficient stress testing methodology to resolve this shortcoming of risk management. The last credit crisis shows that for credit portfolios model risk plays a crucial role, see e.g., Jorion [2009]; Heitfield [2009]; Tarashev [2010]; Morini [2011]; Hellmich et al. [2013]. Branger et al. [2012] demonstrate that model misspecification can induce substantial hedging errors. Further, when the true model includes stochastic volatility and a finite number of jumps, losses arising from underestimating stochastic volatility are larger than losses arising from underestimating jump risk. Bernard and Tang [2016]; Bayraktar and Zhou [2017] highlight the model risk in the hedging context of path-dependent and American derivatives.

Cont [2006] develops a framework for measuring the model risk of contingent claim pricing based on coherent and convex risk measures. Assuming that the model set includes econometric methods, Kerkhof et al. [2010] investigate model risk in the context of market risk management and quantifies it by the worst case value at risk.
across a set of models. Alexander and Sarabia [2012] study model risk of the value at risk concept and propose a regulatory capital correction based on maximum relative entropy. Breuer and Csiszár [2013]; Glasserman and Xu [2014]; Breuer and Csiszár [2016] suggest a methodology for measuring model risk based on relative entropy. Provided a distribution on the parameter space is available, Bannör and Scherer [2013] introduce the notion of risk-capturing functionalities and apply convex risk measures to incorporate parameter risk into risk-neutral pricing. Boucet al. [2014] propose to apply backtesting the performance of models to get a risk adjustment for model risk. Detering and Packham [2015,0] consider model risk in the context of contingent claim pricing. Even though prices across models are the same, there could be still substantial model risk, as hedging strategies might be different. The authors propose measures of model risk based on value at risk and expected shortfall of the hedging error. Barrieu and Scandolo [2015] propose absolute, relative and local measures of model risk, which are useful to rank models but could not be applied to risk-covering capital calculations. Bernard and Vanduffel [2015] study model risk in the high dimensional portfolio context. They show that uncertainty in the portfolio’s dependence is a source of huge model risk. Coqueret and Tavin [2016] investigate model risk for variance swaps and forward start options in a market with jumps and stochastic volatility and measured model risk based on the worst case risk measure. 

Market practitioners, however, still seem to stick mainly to scenario analysis and stress testing for assessing model risk. As pointed out by Alexander and Sheedy [2008], stress testing is only a complementary tool of market risk measurement. 

Our work is related to the papers by Cont [2006]; Kerkhof et al. [2010]; Boucet al. [2014]; Detering and Packham [2015,0]. We aim at developing a general concept of consistent market and model risk measurement. Our methodology is also compatible with recent regulatory requirements BIS [2012,0].

The paper is organized as follows. In Section 2 we review the common framework for market risk measurement. Section 3 introduces a theoretical concept for measuring market risk that incorporates model risk. In Section 4 we introduce measures for quantifying model risk on its own. In particular, we distinguish between model risk of market risk and model risk of pricing contingent claims. Section 5 illustrates our approach in a case study on major stock indices. We apply Bayesian methods for estimating distributions on the model space and calculate various model risk measures and market risk measures that incorporate model risk. Section 6 concludes.

## 2 Risk Measures

In this section, we review market risk measures. Let \((\Omega, \mathcal{F})\) be a measurable space. Let \(W\) be a random variable, \(W : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B})\), where \(\mathcal{B}\) denotes the Borel \(\sigma\)-algebra in \(\mathbb{R}\). Economically, \(W(\omega)\) describes the realized payout of the financial position \(W\) if the scenario \(\omega \in \Omega\) has occurred. We denote by \(L^0(\Omega, \mathcal{F})\) the linear space of random variables and by \(L^\infty(\Omega, \mathcal{F})\) the space of bounded random variables. In the sequel, \(X \subseteq L^0(\Omega, \mathcal{F})\) is some linear space of random variables containing the constants. The space \(X\) will be the domain of definition for our risk measures.

### 2.1 Monetary risk measures

In risk management practice one important goal is to calculate adequate capital to act as a buffer against the risk of a financial position. Traditional risk measures, such as variance or lower partial moments, are not the best choices to calculate capital for a risky position as these measures do not directly give us the value of potential losses. Risk measures that are appropriate for calculating capital adequacy are so-called monetary risk measures, see e.g., Föllmer and Schied [2016], Chapter 4.

**Definition 2.1.** A monetary risk measure \(\rho\) is a mapping \(\rho : X \rightarrow \mathbb{R}\) that satisfies the following properties,

(i) Monotonicity: If \(W_1 \leq W_2\), then \(\rho(W_1) \geq \rho(W_2)\).

(ii) Translation invariance: For \(\alpha \in \mathbb{R}\), \(\rho(W + \alpha) = \rho(W) - \alpha\).

The economic interpretation of these properties is intuitive. Monotonicity ensures that financial positions that promise larger payouts in all scenarios have lower risk. The translation invariance property means that adding a risk free (cash) position to a risky position reduces the risk by the amount of this risk free position. Monetary risk measures are extensively used by financial institutions and regulators for defining capital requirements. The most widespread monetary risk measures are value at risk and expected shortfall, see e.g., BIS [2012,0,0].
Definition 2.2. Value at risk (VaR) of a financial position \( W \) at confidence level \( \alpha \in (0, 1) \) under the probability distribution \( P \) on \( (\Omega, \mathcal{F}) \) is defined as

\[
\text{VaR}_{\alpha,P}(W) = \inf \{ x \in \mathbb{R} : P(W < -x) \leq 1 - \alpha \}. \tag{1}
\]

Definition 2.3. Assume that \( X \subseteq L^\infty(\Omega, \mathcal{F}) \). Expected shortfall (ES) of a financial position \( W \in X \) at the confidence level \( \alpha \) under the probability distribution \( P \) on \( (\Omega, \mathcal{F}) \) is defined as

\[
\text{ES}_{\alpha,P}(W) = -\frac{1}{1 - \alpha} \mathbb{E}(W \mathbb{I}(W \leq -\text{VaR}_{\alpha,P}(W))) = \frac{1}{1 - \alpha} \int_{\alpha}^{1}\text{VaR}_{\gamma,P}(W) \, d\gamma. \tag{2}
\]

In general monetary risk measures do not necessarily account for diversification benefits. An important family of monetary risk measures that account for diversification benefits are coherent risk measures, introduced by Artzner et al. [1999].

Definition 2.4. A monetary risk measure \( \rho : X \to \mathbb{R} \) is called coherent if it satisfies the following conditions,

(i) Subadditivity: \( \rho(W_1 + W_2) \leq \rho(W_1) + \rho(W_2) \),

(ii) Positive homogeneity: \( \rho(\alpha W) = \alpha \rho(W), \alpha \geq 0 \).

Subadditivity states that diversification is beneficial. In general, VaR for non-elliptical distributions is not subadditive and therefore not coherent. On the other hand, expected shortfall is a coherent risk measure Acerbi and Tasche [2002]. Positive homogeneity states a linear scaling of risk. In practice, because of liquidity issues, scaling of financial positions increases the associated risk in a nonlinear way. Therefore, positive homogeneity could be an unrealistic property in practice. Convex risk measures are more flexible to capture the liquidity issue and simultaneously account for diversification benefits.

Definition 2.5. A monetary risk measure \( \rho : X \to \mathbb{R} \) is called convex if

\[
\rho(\alpha W_1 + (1 - \alpha) W_2) \leq \alpha \rho(W_1) + (1 - \alpha) \rho(W_2), \text{ for every } \alpha \in [0, 1].
\]

A coherent risk measure is obviously convex. A convex risk measure is coherent if we additionally require the positive homogeneity property. Closely related to coherent risk measures are spectral risk measures, introduced by Acerbi [2002].

Definition 2.6. The spectral risk measure \( \text{SPR}_{\phi,P}(W) \) of a financial position \( W \) with weight function \( \phi \) and the probability distribution \( P \) is defined as

\[
\text{SPR}_{\phi,P}(W) = -\int_0^1 F_{W,P}^{-1}(\gamma) \phi(\gamma) \, d\gamma, \tag{3}
\]

where \( F_{W,P}^{-1} \) denotes the quantile function of \( W \) under the probability measure \( P \) and \( \phi \) is a non-increasing, non-negative, right-continuous, integrable weight function such that \( \int_0^1 \phi(\gamma) \, d\gamma = 1 \).

Expected shortfall and value at risk are special cases of the spectral risk measures. The spectral risk measure with weight function,

\[
\phi(\gamma) = \frac{1}{1 - \alpha} \mathbb{I}_{\gamma \geq \alpha}, \tag{4}
\]

is the expected shortfall with confidence level \( \alpha \). In the degenerate case when the weight function takes the form of Dirac function \( \phi(p) = \delta_p \), the corresponding spectral risk measure is value at risk.

A risk measure \( \rho \) is called normalized, if \( \rho(0) = 0 \). Throughout this paper, we assume that all risk measures under consideration are normalized.

3 Measures of market risk that account for model risk

The examples of monetary risk measures discussed in the previous section are based on a given probability distribution \( P \) on \( (\Omega, \mathcal{F}) \). From now on, we identify a model with a probability measure \( P \) on \( (\Omega, \mathcal{F}) \). To account for model risk we consider a set \( M \) of probability measures on the measurable space \( (\Omega, \mathcal{F}) \).
For example suppose that the model set \( M \) is a set of parameterized probability distributions of \((\Omega, \mathcal{F})\),
\[
M = \{ P_{\theta} : \theta \in \Theta \},
\]
where \( \Theta \subset \mathbb{R}^n \) denotes the parameter space. We define a \( \sigma \)-algebra \( \mathcal{M} \) on \( M \) by
\[
\mathcal{M} = \{ \{ P_{\theta} : \theta \in D \}, D \in \mathcal{B}(\Theta) \},
\]
where \( \mathcal{B}(\Theta) \) denotes the Borel \( \sigma \)-algebra on \( \Theta \). In this case, we even have a measurable model space \((M, \mathcal{M})\).

In the following sections, we take a set \( M \) of models under consideration as given. For each model \( m \in M \) let \( \rho_m : X \to \mathbb{R} \) be a monetary risk measure for financial positions \( W \in X \). Denote by \( \rho = (\rho_m)_{m \in M} \) the family of these risk measures. We call \( \rho \) coherent (convex) if \( \rho_m \) is coherent (convex) for every \( m \in M \). Further, \( \rho \) is called \( \mathcal{M} \)-measurable if \( \rho_m(W) : M \to \mathbb{R} \) is \( \mathcal{M} \)-measurable for every \( W \). We call \( \rho \) bounded if \( \rho_m(W) \) is bounded on \( M \), for every \( W \in X \). In case of \( X \subseteq L^\infty(\Omega, \mathcal{F}) \), by monotonicity, translation invariance and normalization, \( \rho \) is bounded for every \( W \in X \).

The simplest market risk measure that accounts for model risk is the worst case market risk measure.

**Definition 3.1.** Let \( \rho \) be bounded. The worst case market risk measure \( \rho_{WC} \) is defined as
\[
\rho_{WC}(W) = \sup_{m \in M} (\rho_m(W)), \ W \in X. \tag{7}
\]

Clearly, \( \rho_{WC} \) is a monetary risk measure. Moreover, \( \rho_{WC} \) is coherent (convex) if \( \rho \) is coherent (convex). The worst case market risk measure tends to overestimate the actual risk and is not appropriate for use in practice.

### 3.1 Model weighted market risk measures

Let \((M, \mathcal{M})\) be a measurable model space and \( \mu \) a probability measure on \((M, \mathcal{M})\). Assume that \( \rho \) is \( \mathcal{M} \)-measurable and bounded.

**Definition 3.2.** Define the model weighted market risk measure \( \rho \ast \mu \) as
\[
\rho \ast \mu(W) = \int_M \rho_m(W) d\mu(m). \tag{8}
\]

Definition 3.2 can be viewed as a weighted market risk over the model space \((M, \mathcal{M})\). For example, if \( \rho \) is the family of market value at risk under the different models, then the risk measure \( \rho \ast \mu \) is a model weighted market value at risk.

**Remark 1.** The boundedness of \( \rho \) could be released as long as the integral on the right-hand side of (8) is well-defined.

The proposition below states that the model weighted market risk measure \( \rho \ast \mu \) preserves the properties of the family of risk measures \( \rho \). The proof of the proposition is straightforward.

**Proposition 3.1.** The model weighted market risk measure \( \rho \ast \mu \) is a monetary risk measure. If \( \rho \) is coherent (convex), then \( \rho \ast \mu \) is coherent (convex) as well.

**Remark 2.** Let \((M, \mathcal{M})\) be a measurable model space and \( \mu \) be a measure on \((M, \mathcal{M})\). An alternative approach might be to use the model weighted probability measure \( \hat{P} \) on \((\Omega, \mathcal{F})\), defined by
\[
\hat{P}(A) = \int_M m(A) d\mu(m), A \in \mathcal{F}, \tag{9}
\]
to construct market risk measures with respect to the distribution \( \hat{P} \), such as, for example, \( VaR_{\alpha, \hat{P}}(W) \). Clearly
\[
VaR_{\alpha, \hat{P}}(W) \neq \int_M VaR_{\alpha, m}(W) d\mu(m) = (VaR_{\alpha, \ast \mu})(W).
\]

\( VaR_{\alpha, \hat{P}}(W) \) is not a reasonable risk measure accounting for model risk since it is not based on the variation of the model-based risk measures \( VaR_{\alpha, m}(W) \) as the models \( m \) vary in the model set \( M \).
Remark 3. Our model weighted risk measures can be slightly generalized by employing a capacity $\mu$ on $(M, M)$ in place of a measure, see e.g., Sriboonchita et al. [2009] Chapter 5. Then the model weighted market risk measure is defined as the Choquet integral,

$$\rho \ast \mu(W) = \int_M \rho_m(W) \, d\mu(m)$$

$$= \int_{-\infty}^0 [\mu(\{m: \rho_m(W) > x\}) - 1] \, dx + \int_0^{\infty} \mu(\{m: \rho_m(W) > x\}) \, dx.$$

Proposition 3.1 still holds if the capacity $\mu$ is submodular. For more details on capacities and Choquet integrals we refer to Denneberg [1994]; Sriboonchita et al. [2009], Chapter 5.

3.2 Superposed market risk measures

In this section, we propose a general class of market risk measures that account for model risk. They are superpositions of risk measures on $X \subseteq L^0(\Omega, \mathcal{F})$ and on $L^\infty(M, M)$. One example will be quantile-based market risk measures, like value at risk or expected shortfall that account for model risk.

Let $\zeta : L^\infty(M, M) \to \mathbb{R}$ be a monetary risk measure on $L^\infty(M, M)$. Throughout this section, we assume that $\rho$ is $M$-measurable and bounded.

Definition 3.3. We define the superposed risk measure $\zeta \circ \rho$ by,

$$\zeta \circ \rho(W) = \zeta(-\rho(W)), \ W \in X.$$  \hfill (10)

Proposition 3.2. The risk measure $\zeta \circ \rho$ is a monetary risk measure on $X$, which is coherent (convex) if $\zeta$ and $\rho$ are both coherent (convex).

Proof. Let $\alpha \in \mathbb{R}$, then

$$\zeta \circ \rho(W + \alpha) = \zeta(-\rho(W) + \alpha) = \zeta \circ \rho(W) - \alpha.$$

Further let, $W_1$ and $W_2$ be financial positions such that $W_1 \leq W_2$. Then $-\rho_m(W_1) \leq -\rho_m(W_2), m \in M$, and, as $\zeta$ is monotone,

$$\zeta \circ \rho(W_1) \geq \zeta \circ \rho(W_2).$$

Thus, $\zeta \circ \rho$ is a monetary risk measure.

Now let $\zeta$ and $\rho$ be both coherent. The positive homogeneity of $\zeta \circ \rho$ is obvious. To verify subadditivity, by monotonicity and subadditivity of $\zeta$ and $\rho$,

$$\zeta \circ \rho(W_1 + W_2) = \zeta(-\rho(W_1 + W_2))$$

$$\leq \zeta(-\rho(W_1) - \rho(W_2))$$

$$\leq \zeta(-\rho(W_1)) + \zeta(-\rho(W_2))$$

$$= \zeta \circ \rho(W_1) + \zeta \circ \rho(W_2).$$

Analogously, $\zeta \circ \rho$ is convex if both, $\zeta$ and $\rho$ are convex. \hfill \square

Superposed market risk measures provide a quite general approach to define market risk measures that account for model risk.

In particular, the worst case market risk measure $\rho_{WC}$ from Definition 3.1 and the model weighted market risk measure $\rho \ast \mu$ from Definition 3.2 are special cases for $\zeta(.) = \sup_{m \in M}(-.)$ and $\zeta(.) = -\mathbb{E}_\mu(.)$, respectively.

Other examples, introduced below, are value at risk, expected shortfall and spectral risk measures that account for model risk.

Definition 3.4. Let $\mu$ be a probability measure on the measurable model space $(M, M)$.

(i) Value at risk of $\rho(W)$ of the financial position $W$, with confidence level $\alpha \in (0, 1)$ is defined as

$$\text{VaR}_{\alpha, \rho, \mu} = \text{VaR}_{\alpha, \mu}(-\rho(W)) = \inf \{x \in \mathbb{R} : \mu(\{m : \rho_m(W) > x\}) \leq 1 - \alpha\}. \hfill (11)$$

(ii) Expected shortfall of $\rho(W)$ of the financial position $W$, with confidence level $\alpha \in (0, 1)$ is defined as

$$\text{ES}_{\alpha, \rho, \mu} = \text{ES}_{\alpha, \mu}(-\rho(W)) = \frac{1}{1 - \alpha} \int_0^1 \text{VaR}_{\gamma, \rho, \mu}(W) \, d\gamma \hfill (12)$$
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(iii) Let, φ be a non-increasing, non-negative, right-continuous, integrable function such that \( \int_0^1 \phi(\gamma) \, d\gamma = 1 \). The spectral risk measure of \( \rho(W) \) of the financial position \( W \) with the weight function \( \phi \) is given by,

\[
\text{SPR}_{\phi, \rho, \mu}(W) = \text{SPR}_{\phi, -\rho}(W) = \int_0^1 \text{VaR}_{\gamma, \rho, \mu}(W) \phi(\gamma) \, d\gamma.
\] (13)

According to Proposition 3.2, \( \text{ES}_{\alpha, \rho, \mu} \) and \( \text{SPR}_{\phi, \rho, \mu} \) are coherent risk measures if \( \rho \) is coherent.

The family \( \rho \) and \( \zeta \) can be quite general. For example, \( \rho = (\rho_m)_{m \in M} \) could be a family of value at risks, expected shortfalls or spectral risk measures under different models \( m \in M \), each of which can then be superposed with a value at risk or expected shortfall or spectral risk measure on \( L^\infty(M, \mathcal{M}) \). In the special case when \( \rho = (\text{VaR}_{\alpha, m})_{m \in M} \) is itself a family of value at risk measures, \( \text{VaR}_{\alpha, \rho, \mu}(W) \) becomes value at risk under the model distribution \( \mu \) of market value at risk of the financial position \( W \); it is a value at risk measure that accounts for model risk, which is particularly interesting when it comes to the calculation of model robust economic or regulatory capital requirements.

### 3.3 Robust representation of superposed risk measures

Robust representations of risk measures are related to model risk. In these representations the risk of a financial position \( X \), such as expected shortfalls or spectral risk measures under different models \( \mu \), can be quite general. For example, \( \rho = (\rho_m)_{m \in M} \) is a family of value at risk measures, \( \text{VaR}_{\alpha, m} \), and the risk measures from \( \rho \) are law invariant in the sense, \( \rho_m(W_1) = \rho_m(W_2) \), if \( W_1 = W_2 \) \( \mathcal{P} \)-a.s., \( W_1, W_2 \in \mathcal{X} \).

We say that \( \rho \) is (pointwise) continuous, if for a sequence \( W_n \to W \), \( \mathcal{P} \)-a.s., \( W_n, W \in \mathcal{X} \), we have \( \lim_{n \to \infty} \rho_m(W_n) = \rho_m(W) \), for all \( m \in M \).

A risk measure \( \rho \) on \( \mathcal{X} \) is said to satisfy the Fatou property, if for a bounded sequence \( W_n \to W \), \( \mathcal{P} \)-a.s., \( W_n, W \in \mathcal{X} \), we have \( \rho(W) \leq \liminf_{n \to \infty} \rho(W_n) \).

Denote by \( M_1(\mathcal{P}) \) the set of all probability measures on \((\Omega, \mathcal{F})\) that are absolutely continuous w.r.t. \( \mathcal{P} \), so \( M \subset M_1(\mathcal{P}) \). Proposition below provides a version of the well-known robust representation theorem for convex risk measures.

**Proposition 3.3.** (see, e.g., Theorem 4.33 Föllmer and Schied [2016]) Let \( \rho : \mathcal{X} \to \mathbb{R} \) be a convex risk measure that satisfies the Fatou property. Then \( \rho \) can be represented as

\[
\rho(W) = \sup_{Q \in M_1(\mathcal{P})} \left( \mathbb{E}_Q(-W) - \alpha_{\text{min}}(Q) \right), \quad W \in \mathcal{X},
\] (15)

with the (minimal) penalty function \( \alpha_{\text{min}} \),

\[
\alpha_{\text{min}}(Q) = \sup_{W \in \mathcal{X}} \{ \mathbb{E}_Q(-W) - \rho(W) \}.
\] (16)

To study robust representations for a superposed risk measure \( \zeta \circ \rho \), we assume a reference measure \( \mu \) on the model space \((M, \mathcal{M})\) and denote by \( N_1(\mu) \) the set of all probability measures on \((M, \mathcal{M})\) that are absolutely continuous w.r.t. \( \mu \). Also, we assume that the risk measure \( \zeta \) on \( L^\infty(M, \mathcal{M}) \) is law invariant, that is, \( \zeta(X_1) = \zeta(X_2) \) if \( X_1 = X_2 \) \( \mu \)-a.s.

If \( \zeta \) is convex and satisfies the Fatou property, then according to Proposition 3.3, for \( X \in L^\infty(M, \mathcal{M}) \) we have the representation

\[
\zeta(X) = \sup_{\nu \in N_1(\mu)} \left( \mathbb{E}_\nu(-X) - \beta_{\text{min}}(\nu) \right)
\] (17)

\[
\beta_{\text{min}}(\nu) = \sup_{X \in L^\infty(M, \mathcal{M})} \{ \mathbb{E}_\nu(-X) - \zeta(X) \}.
\] (18)

In the same way, if \( \rho = (\rho_m)_{m \in M} \) is convex and (pointwise) continuous,

\[
\rho_m(W) = \sup_{Q \in M_1(\mathcal{P})} \left( \mathbb{E}_Q(-W) - \alpha_{\text{min}}^m(Q) \right)
\] (19)

\[
\alpha_{\text{min}}^m(Q) = \sup_{W \in \mathcal{X}} \{ \mathbb{E}_Q(-W) - \rho_m(W) \}.
\] (20)
The following proposition states a robust representation of a superposed risk measure $\zeta \circ \rho$ and provides the link between its penalty function and the penalty function $\beta^\text{min}$ of $\zeta$ and the penalty functions $\alpha^\text{min}_m$ of $\rho_m, m \in M$.

**Proposition 3.4.** Let $\zeta$ be a convex risk measure on $L^\infty(M, M)$ that satisfies the Fatou property. Further, let $\rho$ be a family of convex risk measures that is (pointwise) continuous. Then,

$$
\zeta \circ \rho(W) = \sup_{Q \in M_1(P)} \{E_Q(-W) - \gamma(Q)\}, W \in \mathcal{X}
$$

where,

$$
\gamma(Q) = \inf_{\nu \in N_1(\mu)} (\alpha^\nu(Q) - \beta^\text{min}(\nu)),
$$

$$
\alpha^\nu(Q) = \inf_{Q: M \rightarrow M_1(P)} \int_M \alpha^\text{min}_m(Q_m)\nu(dm)
$$

and $\alpha^\text{min}_m$ and $\beta^\text{min}$ are given by Equations (20) and (18), respectively.

**Proof.** It is straightforward to see that $\zeta \circ \rho$ is convex and satisfies the Fatou property, so it admits a robust representation.

Combining (17), (19),

$$
\zeta \circ \rho(W) = \sup_{\nu \in N_1(\mu)} \left( \int_M \rho_m(W)\nu(dm) - \beta^\text{min}(\nu) \right)
$$

$$
= \sup_{\nu \in N_1(\mu)} \left( \int_M \sup_{Q \in M_1(P)} \{E_Q(-W) - \alpha^\text{min}_m(Q)\} \nu(dm) - \beta^\text{min}(\nu) \right).
$$

To interchange the order of the supremum and the integral we need an auxiliary result.

**Lemma 3.5.** Let $f(m, Q), m \in M, Q \in M_1(P)$ be a bounded measurable function. Then

$$
\sup_{Q: M \rightarrow M_1(P)} \int_M f(m, Q_m)\nu(dm) = \int_M \sup_{Q \in M_1(P)} f(m, Q)\nu(dm),
$$

where the supremum on the left hand side is taken over all measurable functions $Q = (Q_m, m \in M)$ from $M$ to $M_1(P)$.

**Proof.** For all $m \in M, Q_m \in M_1(P)$,

$$
f(m, Q_m) \leq \sup_{Q \in M_1(P)} f(m, Q)
$$

and thus

$$
\sup_{Q: M \rightarrow M_1(P)} \int_M f(m, Q_m)\nu(dm) \leq \int_M \sup_{Q \in M_1(P)} f(m, Q)\nu(dm).
$$

To show the opposite inequality, for $m \in M, \epsilon > 0$ there exists some $\tilde{Q}_m \in M_1(P)$ such that

$$
\sup_{Q \in M_1(P)} f(m, Q) - \epsilon \leq f(m, \tilde{Q}_m).
$$

By integrating over the model set $M$ we get

$$
\int_M \sup_{Q \in M_1(P)} f(m, Q)\nu(dm) - \epsilon \leq \int_M f(m, \tilde{Q}_m)\nu(dm)
$$

$$
\leq \sup_{Q: M \rightarrow M_1(P)} \int_M f(m, Q_m)\nu(dm).
$$
To proceed with the proof of Proposition 3.4, we apply Lemma 3.5 with \( f(m, Q) = \mathbb{E}_Q(-W) - \epsilon_m^{\min}(Q) \) to get
\[
\zeta \circ \rho(W) = \sup_{\nu \in N_1(\mu)} \left( \sup_{Q \mid M \rightarrow M_1(P)} \int_M \left( \mathbb{E}_Q(-W) - \epsilon_m^{\min}(Q_m) \right) \nu(dm) - \beta^{\min}(\nu) \right).
\]

For every \( Q \mid M \rightarrow M_1(P) \) we have \( \int_M Q \nu(dm) \in M_1(P) \) and, vice versa, since \( Q = \int_M Q \nu(dm) \), every \( Q \in M_1(P) \) is of this form. For \( Q \in M_1(P) \) define
\[
\alpha'(Q) = \inf_{Q \mid M \rightarrow M_1(P)} \int_M \epsilon_m^{\min}(Q_m) \nu(dm).
\]

Then
\[
\zeta \circ \rho(W) = \sup_{\nu \in N_1(\mu)} \left( \sup_{Q \in M_1(P)} \left( \mathbb{E}_Q(-W) - \alpha'(Q) \right) - \beta^{\min}(\nu) \right)
= \sup_{Q \in M_1(P)} \left( \mathbb{E}_Q(-W) - \inf_{\nu \in N_1(\mu)} (\alpha'(Q) - \beta^{\min}(\nu)) \right),
\]
which concludes the proof. \( \square \)

Righi [2018] considers combinations of risk measures, which can also be interpreted as superposed market risk measures. However, the robust representation result of combinations of risk measures in Righi [2018] differs from our Proposition 3.4.

4 Measuring model risk

4.1 Requirements on measures of model risk

So far we investigated market risk measures on \( \mathcal{X} \subseteq L^0(\Omega, \mathcal{F}) \) that account for market risk. However, we have not yet addressed the problem of measuring model risk itself.

We are given a measurable model space \((M, \mathcal{M})\) and a family \( \rho = (\rho_m)_{m \in M} \) of mappings \( \rho_m : \mathcal{X} \rightarrow \mathbb{R} \). We suppose that \( \rho \) is \( \mathcal{M} \)-measurable.

We take a slightly generalized view on model risk and do not necessarily assume that \( \rho_m \) is a monetary risk measure under the model \( m \). Instead, \( \rho_m(W) \) could be the price of the financial position \( W \) under the pricing model \( m \), which allows us to incorporate model risk for pricing contingent claims as, for example, in Cont [2006].

We are looking for a measure of model risk, that is a mapping \( P_\rho : \mathcal{X} \rightarrow \mathbb{R} \), which, for every \( W \in \mathcal{X} \), quantifies the uncertainty of \( \rho_m(W) \) associated with the fact that \( m \) varies in the model set \( M \). Further, the model risk \( P_\rho(W) \) should be expressed in monetary units and normalized to make it comparable to the market risk resp. price of the position \( W \).

In the following, we denote by \( H \) the set of model invariant financial positions,
\[
H = \{ W \in L^0(\Omega, \mathcal{F}) : \rho_{m_1}(W) = \rho_{m_2}(W), \forall m_1, m_2 \in M \}.
\]

**Definition 4.1.** We call \( P_\rho : \mathcal{X} \rightarrow \mathbb{R} \) a model risk measure if it satisfies the following two conditions:

(i) The model risk of a bounded family \( \rho \) is bounded, more precisely, if \( \rho_m(W) \in [a, b], m \in M \) then \( P_\rho(W) \leq b - a \).

(ii) The model risk of a model invariant financial position vanishes, more precisely, \( P_\rho(W) = 0 \) for \( W \in H \).

Further, we call \( P_\rho \) positively homogeneous if \( P_\rho(\alpha W) = \alpha P_\rho(W) \), for every \( \alpha \geq 0 \).

The economic interpretation of these properties is straightforward. If \( \rho \) gets values from a bounded interval, then the associated model risk should be less or equal to the length of this interval. The model risk of prices axiom in Cont [2006], Equation 4.4. is a special case of condition (i): if the financial position is a liquidly traded benchmark instrument, then the model risk of the price should be less than the bid-ask spread.

If the financial position is model invariant, then there is no model risk. The model risk of prices axiom in Cont [2006], Equation 4.6, is a special case of condition (ii): if a contingent claim is replicable in a model free way, then there is no model risk.

Positive homogeneity together with (i) ensures that model risk is measured in the same monetary units as \( \rho \).
Remark 4. Since \( P_m(W) \) is supposed to measure the variability of \( \rho_m(W) \), \( m \in M \), requiring monotonicity or subadditivity in \( W \) does not make sense for model risk measures. Also, reasonable model risk measures do not satisfy the cash invariance property.

The simplest model risk measure is obviously the worst case risk measure:

**Definition 4.2.** The worst case risk measure \( P_{\rho,WC} \) for the model risk of the financial position \( W \) is defined as

\[
P_{\rho,WC}(W) = \sup_{m \in M} \rho_m(W) - \inf_{m \in M} \rho_m(W)
\]  

\((24)\)

**Proposition 4.1.** The worst case risk measure \( P_{\rho,WC} \) is a model risk measure. If \( \rho \) is positively homogeneous, then \( P_{\rho,WC} \) is positively homogeneous too.

Proof. If \( \rho_m(W), m \in M \) is bounded taking values in the interval \([a,b]\), then \( \sup_{m \in M} \rho_m(W) \leq b \) and \( \inf_{m \in M} \rho_m(W) \geq a \). Therefore, \( P_{\rho,WC}(W) \leq b - a \). Further, if \( W \in H \), i.e. \( \rho_m(W) = \rho_m(W) \) for all \( m_1, m_2 \in M \), then obviously \( \sup_{m \in M} \rho_m(W) = \inf_{m \in M} \rho_m(W) \) and \( P_{\rho,WC}(W) = 0 \). Further, the worst case risk measure clearly satisfies the positive homogeneity property if \( \rho \) is positively homogeneous.

The worst case model risk measure tends to overestimate the true model risk, as it measures the maximum distance of the market risk measures resp. prices.

In the following, we introduce more reasonable model risk measures. We investigate and distinguish model risk of market risk and model risk of prices in separate subsections, where we add further requirements on model risk measures if appropriate.

### 4.2 Measures of model risk for market risk measures

For prudent risk management, it is desirable to decompose the total financial risk into market and model risk components. This decomposition helps us to understand what the main sources of risk are and how capital should be allocated to each risk type. In this subsection, we specifically study model risk measures for market risk measures.

Again, we are given a measurable model space \((M, M)\) and a family \( \rho = (\rho_m)_{m \in M} \) of monetary risk measures on \( X \subseteq L^0(\Omega, \mathcal{F}) \). We suppose that \( \rho \) is \( M \)-measurable and bounded.

In order to get more reasonable model risk measures we quantify the variability of \( \rho_m(W), m \in M \) relative to some reference market risk measure \( \rho^0 \), for example, \( \rho^0 = \rho_k \) for some \( k \in M \). In practice, \( \rho^0 \) would be the financial institution’s model of choice.

For the reference market risk measure \( \rho^0 \) to be aligned with the family \( \rho \) we assume that

\[
\inf_{m \in M} \{ \rho_m(W) \} \leq \rho^0(W) \leq \sup_{m \in M} \{ \rho_m(W) \}, W \in X.
\]  

\((25)\)

Let \( \zeta : L^\infty(M,M) \rightarrow \mathbb{R} \) be a monetary risk measure on \( L^\infty(M,M) \).

**Definition 4.3.** We define the superposed model risk measure with respect to the reference market risk measure \( \rho^0 \) as,

\[
P_{\rho^0,\zeta}(W) = \zeta(\rho^0(W) - \rho(W)).
\]  

\((26)\)

An important special case is to choose \( \rho^0 = \zeta \circ \rho \). In this case, the reference market risk measure corresponds to the superposed market risk measure from Definition 3.3. The overall risk of a financial position \( W \) can now be consistently decomposed into a pure market risk component \( \zeta \circ \rho(W) \) and a pure model risk component \( P_{\rho,\rho^0,\zeta}(W) \). This decomposition is an important part of the bottom-up analysis and optimal capital planning.

Observe that \( \zeta \circ \rho \) satisfies condition \((25)\) if \( \zeta \) is normalized, i.e., \( \zeta(0) = 0 \).

**Proposition 4.2.** Suppose \( \zeta \) is normalized, i.e., \( \zeta(0) = 0 \). Then the superposed model risk \( P_{\rho,\rho^0,\zeta} \) is a model risk measure. If \( \rho, \rho^0 \) and \( \zeta \) are positively homogeneous then \( P_{\rho,\rho^0,\zeta} \) is positively homogeneous too.

Proof. Let \( \rho_m(W), m \in M \) take values in the interval \([a,b]\). By assumption \((25)\), \( \rho^0(W) \in [a,b] \). Then, \( \rho^0(W) - \rho_m(W) \geq a - b \) and by monotonicity of \( \zeta \),

\[
P_{\rho,\rho^0,\zeta}(W) = \zeta(\rho^0(W) - \rho(W)) \leq \zeta(a - b) \leq b - a,
\]

where the last equality is due to the normalization and translation invariance of \( \zeta \). If \( W \in H \), then \( \inf_{m \in M} \{ \rho_m(W) \} = \sup_{m \in M} \{ \rho_m(W) \} \) and according to \((25)\), \( \rho^0(W) = \rho(W) \). Therefore

\[
P_{\rho,\rho^0,\zeta}(W) = \zeta(\rho^0(W) - \rho(W)) = \zeta(0) = 0.
\]

The positive homogeneity assertion is straightforward.

\(\square\)
We list some special cases of Definition 4.3.

**Definition 4.4.** Let $\mu$ be a probability measure on the measurable model space $(M, M)$ and assume that $\rho$ is bounded. We define the expected model risk $\text{EMR}_{\rho, \rho^0, \mu}$ relative to the reference market risk measure $\rho^0$ and with respect to the weighting $\mu$ as,

$$\text{EMR}_{\rho, \rho^0, \mu}(W) = \int_M \left[ \rho_m(W) - \rho^0(W) \right] d\mu(m), \ W \in L^0(\Omega, \mathcal{F}). \quad (27)$$

**Remark 5.** Observe that $\text{EMR}_{\rho, \rho^0, \mu}(W) = \rho \ast \mu(W) - \rho^0(W)$ with $\rho \ast \mu$ from Definition 3.2.

**Definition 4.5.** Let $\mu$ be a probability measure on the model space $(M, M)$.

(i) The model value at risk (MVaR) of the family $\rho$ with confidence level $\alpha \in (0, 1)$ and reference market risk measure $\rho^0$ is defined as

$$\text{MVaR}_{\alpha, \rho, \rho^0, \mu}(W) = \inf \{ x \in \mathbb{R} : \mu(\rho^0(W) - \rho(W) < -x) \leq 1 - \alpha \}. \quad (28)$$

(ii) The model expected shortfall (MES) of the family $\rho$ with confidence level $\alpha \in (0, 1)$ and reference market risk measure $\rho^0$ is defined as

$$\text{MES}_{\alpha, \rho, \rho^0, \mu}(W) = \frac{1}{1 - \alpha} \int_0^1 \text{MVaR}_{\alpha, \rho, \rho^0, \mu}(W) d\gamma. \quad (29)$$

(iii) The model spectral risk measure (MSPR) of the family $\rho$ with the non-increasing, non-negative, right-continuous, integrable weight function $\phi$, such that $\int_0^1 \phi(\gamma) d\gamma = 1$, is given by

$$\text{MSPR}_{\phi, \rho, \rho^0, \mu}(W) = \int_0^1 \text{MVaR}_{\alpha, \rho, \rho^0, \mu}(W) \phi(\gamma) d\gamma. \quad (30)$$

In Definitions 4.4 and 4.5 a reasonable choice for the reference market risk measure $\rho^0$ would be the model weighted market risk measure from Definition 3.2, $\rho^0 = \rho \ast \mu$.

Quantile-based market risk measures are widely used for determining economic and regulatory capital for market risk. Now, quantile-based measures of model risk help financial institutions to build a consistent market and model risk measurement methodology. Of course, the financial institution might also choose different types of quantile-based risk measures for market and model risk, for example, the VaR-approach for market risk and the MES-approach to measure model risk.

The model risk measures introduced in this subsection will be illustrated in a case study in Section 5.

**Remark 6.** The total risk of holding a financial position $W$ is the sum of its market risk and model risk,

$$\rho^0(W) + \rho_{\rho, \rho^0, \mu}(W).$$

Minimizing the overall risk, the financial institution might consider the following optimization problem regarding the choice of the reference risk measure $\rho^0$,

$$\inf_{\rho^0} \left\{ \rho^0(W) + \rho_{\rho, \rho^0, \mu}(W) \right\}.$$ 

### 4.3 Model risk measures for pricing contingent claims

In this section, we focus on the model risk associated with pricing contingent claims based on different models. Throughout this section $\rho = \{ \rho_m \}_{m \in M}$, $\rho_m : \mathcal{X} \rightarrow \mathbb{R}$, is a family of pricing functions for different models $m \in M$. For example, the price $\rho_m(W)$ of the contingent claim $W$ under the model $m$ could be the discounted expectation of $W$ under a model dependent risk neutral distribution.

It is natural to suppose that the pricing function $\rho_m$ is linear for each model $m$:

$$\rho_m(\alpha U + \beta V) = \alpha \rho_m(U) + \beta \rho_m(V). \quad (31)$$

**Definition 4.6.** Let $\rho_{\text{price}}^\rho : \mathcal{X} \rightarrow \mathbb{R}$ be a model risk measure. We call $\rho_{\text{price}}^\rho$ model risk measure of prices if it satisfies the following additional properties:
(i) The model risk of prices is not affected by adding a model invariant position: More precisely, \( P_{\rho}^{\text{price}}(W + V) = P_{\rho}^{\text{price}}(W) \) for \( V \in H \).

(ii) The model risk of prices satisfies the subadditivity property, \( P_{\rho}^{\text{price}}(W_1 + W_2) \leq P_{\rho}^{\text{price}}(W_1) + P_{\rho}^{\text{price}}(W_2) \).

The first property states that a model risk measure of price should not change if we add a contingent claim whose price does not depend on the chosen model. Cont [2006] claims that the model risk of derivative prices should remain the same if we add a certain hedging portfolio in the underlying securities. This statement is close to our property (i) required for a model risk measure of prices, as the prices of hedging portfolios are model invariant. The second property emphasizes that model risk can be reduced due to diversification.

Remark 7. In practice, vega hedging of a derivative instrument \( W \) can be seen as an application of subadditivity. Assume that the trader believes that, at option maturity, the underlying of the financial position has a lognormal distribution \( \mathcal{P}_\sigma \) with some volatility parameter \( \sigma \). The model set for pricing is then,

\[
M = \{ \mathcal{P}_\sigma : \sigma \in [\sigma_{\min}, \sigma_{\max}] \},
\]

where \( \sigma_{\min}, \sigma_{\max} \) are taken from a calibration or the beliefs of the trader. In order to hedge model risk, practitioners calculate vega, which is the sensitivity of the price of \( W \) with respect to the parameter \( \sigma \). Model risk is then reduced (“hedged”) by entering into a liquidity traded option with opposite vega.

Definition 4.7. We call \( P_{\rho}^{\text{price}} : \mathcal{X} \rightarrow \mathbb{R} \) coherent model risk measure of prices if \( P_{\rho}^{\text{price}} \) is a model risk measure of prices and satisfies the positive homogeneity property, \( P_{\rho}^{\text{price}}(\alpha W) = \alpha P_{\rho}^{\text{price}}(W), \alpha \geq 0 \).

Proposition 4.3. The worst case risk measure,

\[
P_{\rho, WC}^{\text{price}}(W) = \sup_{m \in M} \rho_m(W) - \inf_{m \in M} \rho_m(W),
\]

is a model risk measure of prices. Further, if \( \rho \) is positive homogenous, then \( P_{\rho, WC}^{\text{price}} \) is a coherent model risk measure of prices.

Proof. Due to Proposition 4.1 the worst case risk measure is a model risk measure. Let \( V \in H \), then \( \rho_m(V) = c \) for all \( m \in M \) with some constant \( c \in \mathbb{R} \) and

\[
P_{\rho, WC}^{\text{price}}(W + V) = \sup_{m \in M} \rho_m(W + V) - \inf_{m \in M} \rho_m(W + V)
\]

\[
= \sup_{m \in M} (\rho_m(W) + \rho_m(V)) - \inf_{m \in M} (\rho_m(W) + \rho_m(V))
\]

\[
= \sup_{m \in M} \rho_m(W) - \inf_{m \in M} \rho_m(W)
\]

\[
= P_{\rho, WC}^{\text{price}}(W).
\]

Now let \( W_1, W_2 \in \mathcal{X} \), then

\[
P_{\rho, WC}^{\text{price}}(W_1 + W_2) = \sup_{m \in M} \rho_m(W_1 + W_2) - \inf_{m \in M} \rho_m(W_1 + W_2)
\]

\[
= \sup_{m \in M} \left( \rho_m(W_1) + \rho_m(W_2) \right) - \inf_{m \in M} \left( \rho_m(W_1) + \rho_m(W_2) \right)
\]

\[
\leq \sup_{m \in M} \rho_m(W_1) + \sup_{m \in M} \rho_m(W_2) - \left( \inf_{m \in M} \rho_m(W_1) + \inf_{m \in M} \rho_m(W_2) \right)
\]

\[
= P_{\rho, WC}^{\text{price}}(W_1) + P_{\rho, WC}^{\text{price}}(W_2).
\]

The positive homogeneity statement is already part of Proposition 4.1. \( \square \)

Cont [2006], Section 4.2, introduced a measure of model uncertainty that is a special case of the worst case risk measure \( P_{\rho, WC}^{\text{price}} \).

Again, the worst case measure of model risk usually overestimates the true risk. In order to come up with more reasonable model risk measures, we again chose a reference pricing model \( \rho^0 \), which is compatible with the family \( \rho \) in the sense that,

\[
\inf_{m \in M} \{ \rho_m(W) \} \leq \rho^0(W) \leq \sup_{m \in M} \{ \rho_m(W) \}, W \in \mathcal{X}.
\]

(32)

Of course, \( \rho^0 \) is also assumed to be a linear price function (see (31)).

Let \( \zeta : L^\infty(M, M) \rightarrow \mathbb{R} \) be a monetary risk measure on \( L^\infty(M, M) \).
**Definition 4.8.** Define the superposed model risk measure of prices, $P_{\rho, \rho^0, \zeta}^\text{price}$, as

$$P_{\rho, \rho^0, \zeta}^\text{price} (W) = \zeta \left( - |\rho(W) - \rho^0(W)| \right).$$

**Proposition 4.4.** If $\zeta$ is subadditive and normalized then $P_{\rho, \rho^0, \zeta}^\text{price}$ is a model risk measure of prices. Further, if in addition $\rho, \rho^0$ and $\zeta$ are positive homogenous, then $P_{\rho, \rho^0, \zeta}^\text{price}$ is a coherent model risk measure of prices.

**Proof.** Let $\rho_m(W), m \in M$, take values in the interval $[a, b]$. By assumption (32), $\rho^0 \in [a, b]$. Then, by monotonicity, normalization and translation invariance of $\zeta$,

$$P_{\rho, \rho^0, \zeta}^\text{price} (W) = \zeta \left( - |\rho(W) - \rho^0(W)| \right) \leq \zeta (-|b - a|) \leq b - a.$$ 

If $V \in \mathcal{H}$, then according to assumption (32), $\rho^0(V) = \rho(V)$. Then,

$$P_{\rho, \rho^0, \zeta}^\text{price} (V) = \zeta \left( - |\rho(V) - \rho^0(V)| \right) = \zeta (0) = 0.$$ 

Further,

$$P_{\rho, \rho^0, \zeta}^\text{price} (W + V) = \zeta \left( - |\rho(W + V) - \rho^0(W + V)| \right) = \zeta \left( - |\rho(W) + \rho(V) - \rho^0(W) - \rho^0(V)| \right) = P_{\rho, \rho^0, \zeta}^\text{price} (W).$$

Now let $W_1, W_2 \in \mathcal{X}$ and let $\zeta$ be subadditive, then by monotonicity

$$P_{\rho, \rho^0, \zeta}^\text{price} (W_1 + W_2) = \zeta \left( - |\rho(W_1 + W_2) - \rho^0(W_1 + W_2)| \right) = \zeta \left( - |\rho(W_1) + \rho(W_2) - \rho^0(W_1) - \rho^0(W_2)| \right) \leq \zeta \left( - |\rho(W_1) - \rho^0(W_1)| - |\rho(W_2) - \rho^0(W_2)| \right) \leq \zeta \left( - |\rho(W_1) - \rho^0(W_1)| \right) + \zeta \left( - |\rho(W_2) - \rho^0(W_2)| \right).$$

The assertion of positive homogeneity is straightforward. $\Box$

**Remark 8.** In Definition 4.8 the deviation from the reference price $\rho^0$ is measured by the absolute value. Let $g: \mathbb{R} \to \mathbb{R}_+$ be a function, such that $g(x) \leq x, g(-x) \leq x, x \in \mathbb{R}_+$ and $g(x+y) \leq g(x) + g(y), x, y \in \mathbb{R}$. Now, Definition 4.8 can be generalized by,

$$P_{\rho, \rho^0, \zeta}^\text{price} (W) = \zeta \left( - g(\rho(W) - \rho^0(W)) \right), W \in \mathcal{X}.$$ (33)

It is easy to check that $P_{\rho, \rho^0, \zeta}^\text{price}$ is model risk measure of price. However, in the remainder of the paper we stick to the superposed model risk measure of prices as introduced in Definition 4.8.

We present some interesting special cases of $P_{\rho, \rho^0, \zeta}^\text{price}$.

**Definition 4.9.** Let $\mu$ be a probability measure on the measurable model space $(\mathcal{M}, \mathcal{M})$. We define the following model risk measures of prices:

(i) Expected model risk of price is defined as

$$\text{EMR}_{\rho, \rho^0, \mu}^\text{price}(W) = \int_{\mathcal{M}} |\rho(W) - \rho^0(W)| \ d\mu(m).$$ (34)

(ii) The model value at model risk of price with the confidence level $\alpha \in [0, 1]$ is defined as

$$\text{MVaR}_{\alpha, \rho, \rho^0, \mu}^\text{price}(W) = \inf \{x : \mu\left(|\rho_m(W) - \rho^0(W)| > x\right) \leq 1 - \alpha\}.$$ (35)

(iii) The model expected shortfall of price with a confidence level $\alpha \in [0, 1]$ is defined as

$$\text{MES}_{\alpha, \rho, \rho^0, \mu}^\text{price}(W) = \frac{1}{1 - \alpha} \int_{\alpha}^1 \text{MVaR}_{\gamma, \rho, \rho^0, \mu}^\text{price}(W) \ d\gamma.$$ (36)
(iv) The model spectral risk measure of price with weight function $\phi$ is given by

$$
\text{MSPR}_{\phi, \rho, \mu}^{\text{price}} (W) = \int_0^1 \text{MVaR}_{\gamma, \rho, \mu}^{\text{price}} (W) \phi (\gamma) \, d\gamma.
$$

(37)

A natural choice for the reference pricing model $\rho^0$ in the definition above would be the model weighted price, $\rho^0(W) = \int_{\mathcal{M}} \rho_m(W) \, d\mu(m)$.

**Remark 9.** In the context of contingent claim pricing via risk-neutral expectations under complete models, Detering and Packham [2016] suggest an interesting alternative approach for quantifying model risk. They investigate model risk with respect to a set of (real world) measures $\mathbb{P}_a, a \in \Theta$, which plays the role of the model space$^5$. Detering and Packham [2016] propose model risk measures that are based on the hedging error if hedging is done with a strategy originating from some preferred reference model. Introducing a distribution $\mu$ on the model space, they define various model risk measures, such as value at risk and expected shortfall of the hedging error with respect to the distribution $\int \mathbb{P}_a(.) \mu(da)$. These risk measures satisfy requirements of model risk measures as introduced by Cont [2006] and from our Definition 4.1 and the first property from Definition 4.6.

### 4.4 Example: Model risk of prices for barrier options

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q})$ be a filtered probability space. Consider a non-dividend paying stock $S$ with the following price process,

$$
dS_t = S_t \left[ r \, dt + \sigma(t) \, dW_t \right],
$$

(38)

where $r$ denotes the constant risk free interest rate, $\sigma$ is a time varying volatility function and $W$ denotes a standard Brownian motion under $\mathbb{Q}$.

Let us assume that there is a liquidly traded call option on the underlying $S$ with maturity $T$ traded at the Black-Scholes implied volatility $\Sigma$. The model (38) is calibrated to the market price of the call option by requiring the condition

$$
\frac{1}{T} \int_0^T \sigma(t)^2 \, dt = \Sigma^2.
$$

(39)

We assume a piecewise constant volatility function where $\sigma(t)$ is constant on the interval $[0, T_1]$ and on the interval $[T_1, \infty)$, where $T_1 \in (0, T)$. The solutions of the calibration condition (39) are

$$
\sigma_a(t) = a 1_{[0, T_1]}(t) + \sqrt{\frac{T \Sigma^2 - T_1 a^2}{T - T_1}} 1_{(T_1, \infty)}(t),
$$

(40)

with

$$
a \in \Theta = \left( 0, \sqrt{\frac{T}{T_1}} \Sigma \right). 
$$

(41)

Let $\mathbb{Q}_a$ be the probability measure such that (38) holds with volatility function $\sigma = \sigma_a$. We define the measurable model space $(\mathcal{M}, \mathcal{M})$ by

$$
\mathcal{M} = \{ \mathbb{Q}_a : a \in \Theta \},
$$

(42)

$$
\mathcal{M} = \{ \{ \mathbb{Q}_a : a \in D \}, D \in \mathcal{B}(\Theta) \}.
$$

(43)

The volatility setting in this example is a simplified version of Example 4.1. in Cont [2006].

We investigate the model risk of prices of barrier options with maturity less than $T$. Consider a concrete example where $S_0 = 100, T = 1, T_1 = 0.5, \Sigma = 0.3$ and $r = 0.01$. Consider a European up-and-out call (UOC) on the underlying $S$ with maturity 0.7, strike 110 and barrier 140 and a European down-and-out put (DOP) with maturity 0.7, strike 90 and barrier 60. For pricing European barrier options under a time dependent volatility function, we utilize a technique proposed by Lo et al. [2003].

The left and middle graph in Figure 1 show the dependence of the corresponding option prices on the model parameter $a$. The right graph refers to the strangle position UOC + DOP.

---

$^5$Due to completeness, the set of real world measures maps to a corresponding set of risk-neutral pricing measures, $\mathbb{Q}_a, a \in \Theta$. 

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Table 1 summarizes the range of prices and risk parameters (Greeks) for the model set $M$. The results show that the prices, Greeks and thus also standard hedging strategies of barrier options are quite sensitive to the chosen model.

|           | UOC                    | DOP                     |
|-----------|------------------------|-------------------------|
| Price     | [1.2324, 2.0577]       | [2.8122, 3.5893]        |
| Delta     | [0.0119, 0.1106]       | [-0.2078, -0.0843]      |
| Gamma     | [-0.0043, -0.0011]     | [-0.0950, 0.1128]       |
| Vega$^2$  | [0.0592, 3.1901]       | [-3.5709, 2.1853]       |
| Theta     | [-0.0228, 2.3756]      | [-0.3191, 9.6019]       |
| Rho       | [1.5951, 7.6760]       | [-16.7259, -9.8545]     |

Table 1: Ranges for prices and risk parameters for European up-and-out call (UOC) and European down-and-out put (DOP).

Given liquidly traded call and put options, barrier options can be effectively hedged in a model free way through a static hedging strategy in calls and puts with different strikes, see e.g. Carr et al. [1998]. In the setting of our example, for all call options with maturity of one year, the option implied volatility is $\Sigma$. Therefore there is no model risk for barrier options with maturity equal to one year and that is why we have chosen a maturity of 0.7 years to illustrate model risk.

We illustrate superposed model risk measures of prices by calculating the model risk measures from Definition 4.9 for our example. To this end we need a distribution $\mu$ on the model space $(M, M)$. For simplicity, we assume that $\mu$ is a uniform distribution on $(0, \sqrt{\frac{T}{\pi} \Sigma})$. For the reference pricing model $\rho^0$ we employ the model weighted price. Table 2 summarizes the results for our example. As expected, the worst case model risk highly overestimates model risk. Further, the results for the strangle illustrate the subadditivity property of model risk of prices, see Definition 4.6.

|           | UOC | DOP | Strangle |
|-----------|-----|-----|----------|
| $P^{\text{price}}_{\rho^0, WC}$ | 0.8252 | 0.7771 | 0.4902 |
| EMR$^p_{\rho^0, \mu}$ | 0.2249 | 0.2527 | 0.1173 |
| MVar$_{\mu}$ | 0.4754 | 0.4545 | 0.2161 |
| MES$_{\mu}$ | 0.5206 | 0.4675 | 0.2883 |

Table 2: Superposed model risk measures of prices for European up-and-out call (UOC) and European down-and-out put (DOP) and strangle positions.

$^2$Vega is defined here as $\frac{\partial \text{Price}}{\partial a}$.  

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5 Case study: Model risk of market risk

In Sections 3 and 4 we introduce market risk measures that incorporate model risk and model risk measures of market risk. Several risk measures there rely on a distribution $\mu$ on the model space $(M,M)$. Bayesian analysis is a standard technique that attempts to estimate such distributions based on observations. In the Bayesian framework, the starting point is a prior distribution on the model space (e.g., a subjective believe) that is then transformed into a posterior distribution using a density, which depends on the observations.

In this case study, we apply Bayesian methods to estimate the distribution $\mu$ on the model space for a position consisting of five major stock indices. We then illustrate various measures of model risk for market risk of this position.

5.1 Bayesian methods for estimating model distributions

Let $(M,M)$ be a measurable model space. Recall that a model $m \in M$ is a probability measure on $(\Omega,\mathcal{F})$. In general, $M$ could contain models from different distribution classes. Assume that the model set $M$ consists of $N$ distribution classes $\Lambda^1, \ldots, \Lambda^N$. Every model $m \in M$ corresponds to a pair $(\Lambda^j, \theta^j)$, where $\theta^j \in \Theta^j \subseteq \mathbb{R}^{n_j}$ stands for the parameter vector within the distribution class $\Lambda^j$; in this case, we write $m_{(\Lambda^j, \theta^j)}$.

Let $\Pi(\Lambda^j)$ denote the prior probability of distribution class $\Lambda^j$ and let $\pi_j$ be the prior density function of the parameter $\theta^j \in \Theta^j$. Observation $y$ of a random variable $Y \in \mathcal{X}$, $\mathcal{X} \subseteq L^0(\Omega,\mathcal{F})$, is then,

$$P(Y = y) = \sum_j \int_{\Theta^j} m_{(\Lambda^j, \theta^j)}(Y = y)\pi_j(\theta^j) d\theta^j \Pi(\Lambda^j), \quad (44)$$

where, $m_{(\Lambda^j, \theta^j)}(Y = y)$ denotes likelihood under the model $m_{(\Lambda^j, \theta^j)}$. According to Bayes theorem, the posterior distribution $\mu$ on the model space is given by,

$$\mu((\Lambda^j', \theta^j')|Y = y) = \frac{m_{(\Lambda^j, \theta^j)}(Y = y)\pi_j(\theta^j)\Pi(\Lambda^j)}{\sum_i \int_{\Theta^i} m_{(\Lambda^i, \theta^i)}(Y = y)\pi_i(\theta^i) d\theta^i \Pi(\Lambda^i)}. \quad (45)$$

The posterior probability of model class $\Lambda^j$ is then,

$$\mu(\Lambda^j|Y = y) = \frac{\int_{\Theta^j} m_{(\Lambda^j, \theta^j)}(Y = y)\pi_j(\theta^j) d\theta^j \Pi(\Lambda^j)}{\sum_i \int_{\Theta^i} m_{(\Lambda^i, \theta^i)}(Y = y)\pi_i(\theta^i) d\theta^i \Pi(\Lambda^i)}, \quad (46)$$

and the posterior density of the parameter vector $\theta^j$ under the model class $\Lambda^j$ is given by,

$$\mu(\theta^j|Y = y) = \frac{m_{(\Lambda^j, \theta^j)}(Y = y)\pi_j(\theta^j)}{\int_{\Theta^j} m_{(\Lambda^j, \theta^j)}(Y = y)\pi_j(\theta^j) d\theta^j}. \quad (47)$$

In general, calculating the posterior distribution $(45)$ is numerically challenging because of the integral appearing in the denominator. We employ Markov chain Monte Carlo (MCMC) simulation methods to estimate the posterior distribution.

MCMC is a very powerful method for sampling from a high-dimensional probability distribution and for calculating the expected value, in the case when, direct simulations or analytical methods are not available, see e.g., Geweke [1999]. MCMC generates a simulated random sequence such that each state of the sequence depends only on the previous one, i.e., the sequence is a Markov chain. The main idea consists in constructing a Markov chain whose stationary distribution is some given distribution, in our case, the posterior distribution. The actual simulations are generated by a two step procedure: given the actual state of the Markov chain, in a first step a proposal for the next state is randomly sampled, where the proposal distribution is to a large extent arbitrary. In a second step, the proposed state is accepted with some probability that involves the ratio of the posterior probabilities $(45)$ for the current and the proposed state; otherwise, the chain stays in its current state. Observe that by taking ratios of $(45)$, we get rid of the integral in the denominator. The acceptance probability is defined such that the so-called detailed balance condition is satisfied, which implies that the posterior distribution is the stationary distribution of the chain. A popular specification of the MCMC simulation is the so-called Metropolis-Hastings algorithm, Metropolis et al. [1953]; Hastings [1970]. For more information see, e.g., Geweke [1999]; Brooks et al. [2011].

For the case of different distribution classes, Green [1995] proposed a reversible jump MCMC algorithm that enables us to construct a Markov chain with stationary distribution equal to the joint posterior distribution $(45)$ of the distribution class and the corresponding parameter vector. The reversible jump MCMC simulation
is an extension of the Metropolis-Hastings algorithm that involves the following two steps. In the first step, the parameter vector given the current fixed distribution class is updated. This update can be done based on a Metropolis-Hastings algorithm over the corresponding parameter vector space. The second step consists of a simultaneous update of the distribution class and the associated parameter vector. For the details of implementation, we refer to Brooks et al. [2011], Section 3.2 or Green [1995]; Green and Hastie [2009]; Green and Worden [2015]. For the comprehensive review of MCMC theory we refer to Roberts et al. [2004]; Robert [2007] Chapter 6; Gelman et al. [2014] Part 3.

5.2 Data and model set

We consider the following four major stock indexes: DAX 30, S&P 500, FTSE 100, Nikkei 225 and the commodity index S&P GSCI. The data consists of daily closing prices between Jan. 04, 2010 and Apr. 30, 2018 which have been transformed into Euros. A shows some descriptive statistics of the data, which call for models that allow fat tails and skewness.

We consider multivariate GARCH models for the index returns. GARCH models capture many stylized facts of asset returns and for this reason, they are very popular in risk management, see e.g., Cont [2001]; Sewell [2011]. GARCH models for the volatility of asset returns capture volatility clustering phenomena and can generate heavy tails.

Consider a $n$-dimensional discrete time stochastic process $\{y_t\}_{t=0,1,...}$. We assume that $E(y_t) = 0$ and

$$y_t = H_t^{1/2} \epsilon_t,$$

where $H_t^{1/2}$ is the Cholesky decomposition of the conditional covariance matrix $H_t$ of $y_t$ given $y_0, \ldots, y_{t-1}$ and $\epsilon_t$ is the vector of innovations, with $E(\epsilon_t) = 0$ and $\text{var}(\epsilon_t) = I_n$, where $I_n$ denotes the identity matrix of order $n$.

The conditional covariance matrix $H_t$ of $y_t$ can be decomposed into

$$H_t = D_t R_t D_t,$$

where $R_t$ denotes the conditional correlation matrix and $D_t$ is a diagonal matrix with the conditional variances $\sqrt{h_{ii,t}}$ on the diagonal.

Each conditional variance follows a univariate GARCH(p,q) process,

$$h_{ii,t} = \omega_i + \sum_{k=1}^{q} \alpha_{i,k} y_{i,t-k}^2 + \sum_{j=1}^{p} \beta_{i,j} h_{ii,t-j},$$

where $\omega \in \mathbb{R}^n_+$, $\alpha \in (0,1)^{n \times q}$, $\beta \in (0,1)^{n \times p}$, $\sum_{k=1}^{q} \alpha_{i,k} + \sum_{j=1}^{p} \alpha_{i,j} < 1$, for every $i = 1, \ldots, n$.

Engle and Sheppard [2001] propose a dynamic conditional correlation GARCH model (DCC-GARCH). The DCC-GARCH(1,1) model has the following representation for the correlation matrix,

$$R_t = \text{diag}(Q_t^{1/2}) Q_t \text{diag}(Q_t^{-1/2}),$$

with symmetric positive definite matrix $Q_t$ defined by

$$Q_t = (1 - a - b)V + a D_{t-1}^{-1} y_{t-1} y_{t-1}^T D_{t-1}^{-1} + b Q_{t-1},$$

where $V$ stands for unconditional covariance matrix of $D_{t-1}^{-1} y_t$ and $a, b > 0, a + b < 1$. For more details about the multivariate GARCH models see Bauwens et al. [2006].

We consider DCC-GARCH(1,1) models with standard multivariate skew normal and standard multivariate skew Student’s $t$ distribution for the innovations $\epsilon_t$. The probability density function of a standard $n$-dimensional skew distribution as proposed by Bauwens and Laurent [2005] is given by,

$$f(y | \gamma) := 2^n \left( \prod_{i=1}^{n} \frac{\gamma_i}{1 + \gamma_i^2} \right) g(\kappa),$$

where, $g$ is the density function of a symmetric distribution. The vector $\kappa$ is defined by $\kappa_i = y_i / \gamma_i$ if $y_i > 0$ and $y_i \gamma_i$ otherwise. The parameter $\gamma_i$ controls the skewness for margin $i$: $\gamma_i > 1$ corresponds to positive skewness.

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and $\gamma_i < 1$ implies negative skewness. If $\gamma$ is a unit vector we get a symmetric multivariate distribution. We employ for $g$ a standard $n$-variate normal or Student’s $t$ distribution with $\nu$ degrees of freedom.

From the model setup (50), (52), (53), our model set $M$ is finally,

\[
M = M_1 \cup M_2,
\]

where,

\[
M_1 = \{ \text{N-DCC-GARCH}(1,1) | \omega \in \mathbb{R}_+^5, \alpha \in (0,1)^5, \beta \in (0,1)^5, \gamma \in \mathbb{R}_+^5, a \in (0,1), b \in (0,1) \},
\]

\[
M_2 = \{ \text{t-DCC-GARCH}(1,1) | \omega \in \mathbb{R}_+^5, \alpha \in (0,1)^5, \beta \in (0,1)^5, \gamma \in \mathbb{R}_+^5, a \in (0,1), b \in (0,1), \nu \in (2, \infty) \}.
\]

### 5.3 Reversible Jump MCMC specification

Let $\Lambda^N$ and $\Lambda^t$ denote the DCC-GARCH(1,1) models with the multivariate skew normal distribution and multivariate skew Student’s $t$-distribution classes, respectively. For the prior model class distribution we assume that both model classes are equally likely, $\Pi(\Lambda^N) = \Pi(\Lambda^t) = \frac{1}{2}$.

The reversible jump MCMC algorithm is constructed as follows.

At simulation step $t$, the Markov chain is in state $(\Lambda_i, \theta_i)$ which is some $(\Lambda^N, \theta_i^N)$ or $(\Lambda^t, \theta_i^t)$. We use the following prior distributions.

\[
\pi(\omega_i^N|\Lambda^N) = N(0,0.5)\mathbb{1}_{\omega_i^N > 0}, \pi(\omega_i^t|\Lambda^t) = N(0,0.5)\mathbb{1}_{\omega_i^t > 0}, \quad l = 1, \ldots, 5,
\]

\[
\pi(\alpha_i^N|\Lambda^N) = N(0,0.5)\mathbb{1}_{0 < \alpha_i^N < 1}, \pi(\alpha_i^t|\Lambda^t) = N(0,0.5)\mathbb{1}_{0 < \alpha_i^t < 1}, \quad l = 1, \ldots, 5,
\]

\[
\pi(\beta_i^N|\Lambda^N) = N(0,0.5)\mathbb{1}_{0 < \beta_i^N < 1}, \pi(\beta_i^t|\Lambda^t) = N(0,0.5)\mathbb{1}_{0 < \beta_i^t < 1}, \quad l = 1, \ldots, 5,
\]

\[
\pi(\gamma_i^N|\Lambda^N) = N(0,0.5)\mathbb{1}_{\gamma_i^N > 0}, \pi(\gamma_i^t|\Lambda^t) = N(0,0.5)\mathbb{1}_{\gamma_i^t > 0}, \quad l = 1, \ldots, 5,
\]

\[
\pi(\alpha_i^N|\Lambda^N) = N(0,0.5)\mathbb{1}_{0 < \alpha_i^N < 1}, \pi(\alpha_i^t|\Lambda^t) = N(0,0.5)\mathbb{1}_{0 < \alpha_i^t < 1}, \quad l = 1, \ldots, 5,
\]

\[
\pi(\beta_i^N|\Lambda^N) = N(0,0.5)\mathbb{1}_{0 < \beta_i^N < 1}, \pi(\beta_i^t|\Lambda^t) = N(0,0.5)\mathbb{1}_{0 < \beta_i^t < 1}, \quad l = 1, \ldots, 5,
\]

\[
\pi(\nu_i|\Lambda^t) = N(10,10)\mathbb{1}_{\nu_i > 2},
\]

where $N(\mu, \sigma)$ denotes normal distribution with mean $\mu$ and standard deviation $\sigma$. In a first step we sample a proposal for the parameter $\theta_{i+1}$ within the actual model class using Metropolis Hastings algorithm described by Fiorucci et al. [2014] and implemented in the R package ‘BayesDecGarch’. Denote by $\theta_{i+1}$ the result of the acceptance/rejection decision.

In the next step, the distribution class is updated. If there is a mismatch in the parameter dimensions, a simultaneous update of parameters might be necessary. For a move from class $\Lambda_i = \Lambda^N$ to class $\Lambda_{i+1} = \Lambda^t$, the proposal parameters are $\theta_i^N = h(\theta_i^N)$, where $u$ has truncated normal distribution, $u \sim N(10,10)\mathbb{1}_{\nu > 2}$ and $\theta_i^N = (\omega_i^N, \alpha_i^N, \beta_i^N, \gamma_i^N, a_{i+1}, b_{i+1})$ comes from the first step. The Jacobian of the transformation $h$ is 1. The Metropolis-Hastings-Green ratio becomes

\[
A_{N,t} = \frac{m(N,\theta_{i+1}^N)(Y = y)}{m(N,\theta_i^N)(Y = y)}. \tag{61}
\]

We define the acceptance probability for the move from $(\Lambda^N, \theta_{i+1}^N)$ to $(\Lambda^t, \theta_{i+1}^t)$ as $\min \{ 1, A_{N,t} \}$. For the reverse move, from class $\Lambda_i = \Lambda^t$ to class $\Lambda_{i+1} = \Lambda^N$, the associated parameter vector is specified as $\theta_i^N = h^{-1}(\theta_i^t) = (\omega_i^t, \alpha_i^t, \beta_i^t, \gamma_i^t, a_{i+1}, b_{i+1})$ with $\theta_{i+1}$ from the first step, and the acceptance probability for the move is $\min \{ 1, A_{t,N} \}$, with

\[
A_{t,N} = \frac{m(N,\theta_{i+1}^N)(Y = y)}{m(N,\theta_i^N)(Y = y)}. \tag{62}
\]

We run reversible jump MCMC simulations with 250,000,000 iterations and burn-in the first 225,000,000 iterations. This is a very conservative approach. We observe that the chains are good mixed and the autocorrelation of chains are relatively small. For a review of convergence and diagnostic methods, we refer to Cowles and Carlin [1996]; Castellino and Zimmerman [2002].

**Remark 10.** In the case study, we apply MCMC simultaneously for two model classes with up to 23 parameters. For the case, of only one high dimensional model class one can significantly improve the sampling performance by applying more efficient adaptive schemes discussed by Haario et al. [2006] or by applying differential evolution algorithms, see, e.g., Vrugt et al. [2009]; Vrugt [2016]. These algorithms can also be easily adapted to our reversible jump MCMC sampling.

18
5.4 Results

The posterior probabilities for the DCC-GARCH(1,1) with multivariate skew normal and multivariate skew Student’s $t$-distribution classes given the observed return data $Y = y$ are $\mu(\Lambda^n|Y = y) = 0.1384$, $\mu(\Lambda^t|Y = y) = 0.8616$. The Bayes factor is $\frac{\mu(\Lambda^n|Y = y)\Pi(\Lambda^n)}{\mu(\Lambda^t|Y = y)\Pi(\Lambda^t)} = 6.2254$. So, given the observed data there is strong evidence for the DCC-GARCH(1,1) with multivariate skew Student’s $t$-distribution class, see e.g., Wasserman [2000].

In the following example, we illustrate market and model risk measures that make use of the obtained posterior distribution on the model space. The underlying financial position $W$ is a EUR 1000 investment in the equally weighted portfolio of the indices described above.

Market risk that accounts for model risk

We compute the 1-day VaR and ES at confidence levels of 95% and 99% for the MCMC simulated model outcomes $(\Lambda_i, \theta_i)$. The risk measures are calculated via Monte Carlo simulation with 100,000 iterations. Figure 2 shows the posterior distribution and Table 3 provides the posterior descriptive statistics for the outcomes of these market risk measures. All obtained distributions have positive skewness and heavy tails. ES has a much larger standard deviation, skewness, and kurtosis than VaR. Further, risk measures with 99% confidence level have higher standard deviations, skewness and kurtosis then corresponding market risk measures with 95% confidence level. These results show that market risk measures that account for the tail of the distribution are more model sensitive.

| Risk Measure | VaR$_{0.95}$ | VaR$_{0.99}$ | ES$_{0.95}$ | ES$_{0.99}$ |
|--------------|---------------|---------------|-------------|-------------|
| Mean         | 24.0131       | 39.7324       | 33.2941     | 49.6187     |
| Median       | 22.3131       | 38.1723       | 32.0351     | 49.0533     |
| Standard deviation | 6.5245       | 8.0822       | 7.0304     | 8.8499     |
| Skewness     | 1.4972        | 1.2887        | 1.3178      | 1.0661      |
| Kurtosis     | 5.7435        | 5.7708        | 6.1068      | 7.5276      |
| Max          | 57.0771       | 91.0999       | 82.4595     | 130.4805    |
| Min          | 13.6715       | 18.2893       | 17.4313     | 22.4281     |

Table 3: Posterior descriptive statistics for VaR$_{0.95}$, VaR$_{0.99}$, ES$_{0.95}$ and ES$_{0.99}$.

Figure 2: Histograms for VaR and ES with different confidence level.

Table 4 shows the worst case market risk measures $\rho_{WC}$, see (7),$^4$ and the model weighted market risk

$^4$The supremum over the model space in (7) is calculated as the supremum over the MCMC simulations, so it is approximately the essential supremum with respect to the posterior model distribution $\mu$. 

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measures $\rho \ast \mu$, see (8), that account for model risk. Not surprisingly, the model weighted market risk measures are significantly smaller than the respective worst case market risk measures.

|             | $\text{VaR}_{0.95}$ | $\text{VaR}_{0.99}$ | $\text{ES}_{0.95}$ | $\text{ES}_{0.99}$ |
|-------------|----------------------|----------------------|---------------------|---------------------|
| $\rho_{\text{WC}}$ | 57.0771              | 91.0999              | 82.4505             | 130.4805            |
| $\rho \ast \mu$    | 24.0131              | 39.7324              | 33.2941             | 49.6187             |

Table 4: The worst case market risk measures and model weighted market risk measures.

Next, consider superposed market risk measures that account for model risk. Since the posterior distributions of $\text{VaR}_{0.99}, \text{VaR}_{0.95}$ and $\text{ES}_{0.99}, \text{ES}_{0.95}$ are right skewed with small heavy tails it is appropriate to look into value at risk (11) and expected shortfall (12) of these market risk measures with respect to the posterior distribution $\mu$ on the model space. Figure 3 shows the superposed market risk measures $\text{VaR}_\alpha, \text{VaR}_{0.95, \mu}, \text{VaR}_\alpha, \text{ES}_{0.95, \mu}$, $\text{VaR}_\alpha, \text{ES}_{0.99, \mu}$ as well as $\text{ES}_\alpha, \text{VaR}_{0.95, \mu}, \text{ES}_\alpha, \text{VaR}_{0.99, \mu}, \text{ES}_\alpha, \text{ES}_{0.95, \mu}, \text{ES}_\alpha, \text{ES}_{0.99, \mu}$ for different confidence levels $\alpha$. These risk measures examine the tail of the market risk measures and they are of course far smaller than the respective worst case market risk measures.

![Figure 3: Superposed market risk measures with different significance level $\alpha$. Left up: $\text{VaR}_\alpha, \text{VaR}_{0.95, \mu}$ and $\text{VaR}_\alpha, \text{ES}_{0.95, \mu}$. Right up: $\text{ES}_\alpha, \text{VaR}_{0.95, \mu}$ and $\text{ES}_\alpha, \text{ES}_{0.95, \mu}$. Left down: $\text{VaR}_\alpha, \text{VaR}_{0.99, \mu}$ and $\text{VaR}_\alpha, \text{ES}_{0.99, \mu}$. Right down: $\text{ES}_\alpha, \text{VaR}_{0.99, \mu}$ and $\text{ES}_\alpha, \text{ES}_{0.99, \mu}$.](image)

**Model risk of market risk**

Next we illustrate measures of model risk for market risk. We start with the worst case model risk measure (24); the results are summarized in Table 5.

| $P_{\text{VaR}_{0.95, \text{WC}}}$ | $P_{\text{VaR}_{0.99, \text{WC}}}$ | $P_{\text{ES}_{0.95, \text{WC}}}$ | $P_{\text{ES}_{0.99, \text{WC}}}$ |
|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 43.4055                       | 72.8106                       | 65.0282                       | 108.0525                      |

Table 5: Worst case model risk measures

These risk measures clearly overestimate the true model risk. As an alternative consider superposed model risk measure with respect to some reference market risk measure $\rho^0$ as introduced in Definition 4.3. As reference market risk measure we chose the model weighted market risk measures $\rho_{\text{WC}}^0 = \text{VaR} \ast \mu$ and $\rho_{\text{ES}}^0 = \text{ES} \ast \mu$ with respective confidence levels.

Figure 4 shows the superposed model risk measures $\text{MVaR}$ from (28) and $\text{MES}$ from (29) with different significance levels $\alpha$. We observe that model risk is significantly higher for the expected shortfall family compared

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5The supremum resp. infimum over the model space in (24) is calculated as the supremum resp. infimum over the MCMC simulations, so it is approximately the essential supremum resp. infimum with respect to the posterior model distribution $\mu$. 

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to the value at risk family. Further, MES is significantly larger than MVaR for all four risk measures. The results show that model risk is an important issue particularly for market risk measures that account for the tail risk of a financial position. Bernard and Vanduffel [2015] get similar results and show that VaR at very high confidence levels is prone to huge model risk.

Figure 4: Model risk measures with different significance level $\alpha$. left up: $\text{MVar}_\alpha, \text{VaR}_{0.95}, \rho\text{VaR};^\text{MD}$ and $\text{MVar}_\alpha, \text{ES}_{0.95}, \rho\text{ES};^\text{MD}$, right up: $\text{MES}_\alpha, \text{VaR}_{0.95}, \rho\text{VaR};^\text{MD}$ and $\text{MES}_\alpha, \text{ES}_{0.95}, \rho\text{ES};^\text{MD}$, left down: $\text{MVar}_\alpha, \text{VaR}_{0.95}, \rho\text{VaR};^\text{MD}$ and $\text{MVar}_\alpha, \text{ES}_{0.95}, \rho\text{ES};^\text{MD}$, right down: $\text{MES}_\alpha, \text{VaR}_{0.95}, \rho\text{VaR};^\text{MD}$ and $\text{MES}_\alpha, \text{ES}_{0.95}, \rho\text{ES};^\text{MD}$.

6 Conclusion

Model risk is of paramount importance for financial institutions and actual regulatory requirements. We investigate model risk in the context of financial risk measurement and contingent claim pricing. Risk measures and pricing models are based on assumptions about the random evolution, in particular, the probability distributions, of the underlying financial positions. Clearly, those assumptions are subject to misspecification, that is, model risk. To investigate model risk we consider a space $M$ of models, which consists of probability measures $m \in M$ on the state space $(\Omega, \mathcal{F})$. Given a family of market risk measures $(\rho_m)_{m \in M}$ depending on the model $m \in M$, we first introduce market risk measures $\rho$ that account for model risk. In particular we investigate the worst case risk measure, model weighted risk measures and superposed market risk measures. Superposed risk measures are an appropriate superposition of the family of risk measures $(\rho_m)_{m \in M}$ with some risk measure on the model set $M$. In the spirit of Föllmer and Schied [2016] we provide a robust representation for superposed market risk measures and derive their penalty function from the penalty functions of their building blocks.

Even though the introduced risk measures incorporate model risk into market risk measurement they do provide a measure for model risk on its own. To this end, we investigate model risk measures. A model risk measure should quantify the variation of $\rho_m$ as $m$ varies in the model set. We set up axioms for model risk measures, where we distinguish between model risk for market risk measurement and for contingent claim pricing. Examples of model risk measures are the worst case model risk measure and superposed model risk measures, which quantify model risk relative to a reference market risk measure resp. reference contingent claim pricing function.

Model risk for contingent claim pricing is illustrated by an example analyzing barrier options in a set of models that have been calibrated to a benchmark instrument. The results show that model risk is substantial as prices, risk parameters (Greeks) and hedging strategies vary largely in the model set.

Some important examples of superposed risk measures require a probability distribution on the model space. Bayesian analysis is the natural choice to quantify such distributions from observed data. In a case study, using data from the five major stock and commodity indices, we illustrate market and model risk measures. The model set we consider consists of dynamic conditional correlation DCC-GARCH(1,1) models with multivariate skew normal and multivariate skew Student’s $t$-distributions for the innovations of the portfolio daily returns. We apply reversible jump Markov chain Monte Carlo simulation methods to determine a distribution on the
model space from the data. For this portfolio, we calculate market risk measures that account for model risk as well as measure of pure model risk. It turns out that risk measures exposed to the tail distribution of financial positions are particularly sensitive with respect to model risk.

A Descriptive statistics for index data

Figure 5 presents log-return evolutions and the histograms for the index data in Section 5.2. We see that the return volatilities are not time homogenous. The patterns in Figure 5 indicate volatility clustering.

![Figure 5: Euro-denominated log-returns and histograms of the indices.](image)

Table 6 summarizes the descriptive statistic of daily log-returns. The returns have small negative skewness and heavy tails. Table 7 represents the historical correlation matrix for daily log-returns. There are significant positive correlations between the indices.

|                | DAX 30 | S&P 500 | FTSE 100 | Nikkei 225 | S&P GSCI |
|----------------|--------|---------|----------|------------|----------|
| Mean           | 0.0003 | 0.0005  | 0.0002   | 0.0004     | 0.0002   |
| Median         | 0.0008 | 0.0007  | 0.0006   | 0.0005     | 0.0001   |
| Standard deviation | 0.0123 | 0.0099  | 0.0107   | 0.0133     | 0.0118   |
| Skewness       | -0.2770| -0.4159 | -0.4789  | -0.3206    | -0.0531  |
| Kurtosis       | 5.5822 | 6.4005  | 8.0048   | 7.2886     | 4.9750   |
| Max            | 0.0521 | 0.0457  | 0.0508   | 0.0775     | 0.0658   |
| Min            | -0.0707| -0.0670 | -0.0890  | -0.1014    | -0.0602  |

Table 6: Descriptive statistics for Euro-denominated daily log-returns.
Table 7: Historical correlation matrix for Euro-denominated daily log-returns.

|          | DAX 30 | S&P 500 | FTSE 100 | Nikkei 225 | S&P GSCI |
|----------|--------|---------|----------|------------|----------|
| DAX 30   | 1.0000 | 0.5389  | 0.8115   | 0.1278     | 0.3058   |
| S&P 500  | 0.5389 | 1.0000  | 0.6080   | 0.1668     | 0.4181   |
| FTSE 100 | 0.8115 | 0.6080  | 1.0000   | 0.2275     | 0.4443   |
| Nikkei 225| 0.1278 | 0.1668  | 0.2275   | 1.0000     | 0.1207   |
| S&P GSCI | 0.3058 | 0.4181  | 0.4443   | 0.1207     | 1.0000   |

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