Research Article

On the Construction of the Reflexive Vertex $k$-Labeling of Any Graph with Pendant Vertex

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1. Introduction

We consider a simple and finite graph $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. We motivate the readers to refer Chartrand et al. [1], for detailed definition of the graph. A topic in graph theory which has grown fast is the labeling of graphs. The concept of graph labeling, firstly, was introduced by Wallis in [2]. He defined a labeling of graphs. The concept of graph labeling, firstly, was introduced by Chartrand et al. [1], for detailed definition of the graph. A total labeling, is determined by summing the incident edge labels and the label of the vertex itself. By this definition, we can have a vertex label, edge label, or both of them. Baca et al. [3] introduced the total labeling, and they defined the vertex weight as the sum of all incident edge labels along with the label of the vertices. Many types of labeling have been studied by researchers, namely, graceful labeling, magic labeling, antimagic labeling, irregular labeling, and irregular reflexive labeling.

Furthermore, labeling known as a vertex irregular total $k$-labeling and total vertex irregularity strength of graph is the minimum $k$ for which the graph has a vertex irregular total $k$-labeling. The bounds for the total vertex irregularity strength are given in [3]. In [4], Tanna et al. identified the concept of vertex irregular reflexive labeling of graphs. In this paper, we continue to study a vertex irregular reflexive labeling as there are still many open problems. By irregular reflexive labeling, we mean a labeling of graph which the vertex labels are assigned by even numbers from 0, 2, . . . , $2k$ and the edge labels are assigned by 1, 2, 3, . . . , $k$, where $k$ is positive integer. The weight of each vertex, under a total labeling, is determined by summing the incident edge labels and the label of the vertex itself.

A $k$-labeling assigns numbers $\{1, 2, \ldots, k\}$ to the elements of graph. Let $k$ be a natural number, a function $f: V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, k\}$ is called total $k$-irregular labeling. Hinding et al. [5] defined that a total labeling $\phi: V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, k\}$ is called vertex irregular total $k$-labeling of graph $G$ if the vertex weight $\omega_\phi(x) = \phi(x) + \sum_{x,y \in E(G)} \phi(xy)$ is distinct for every two different vertices, $\omega_\phi(x) \neq \omega_\phi(y)$ for $x, y \in V(G), x \neq y$. The minimum $k$ for which graph $G$ has a vertex irregular total $k$-labeling is called total vertex irregularity strength, denoted by tvs($G$).

The concept of vertex irregular total $k$-labeling extends to a vertex irregular reflexive total $k$-labeling. The definition of
total $k$-labeling is a function $f_e$ from the edge set to the first natural number $k_e$, and a function $f_v$ from the vertex set to the nonnegative even number up to $2k_e$, where $k = \max\{k_e, 2k_e\}$. A vertex irregular reflexive $k$-labeling of the graph $G$ is the total $k$-labeling, for every two different vertices $x$ and $x'$ of $G$, $w_v(x) \neq w_v(x')$, where $w_v(x) = f_v(x) + \sum_{y \in E(G)} f_e(xy)$. The minimum $k$ for graph $G$ which has a vertex irregular reflexive $k$-labeling is called the reflexive vertex strength of the graph $G$, denoted by $r(vs(G))$.

Some results related to vertex irregular reflexive labeling have been studied by several researchers. Tanna et al. [4] have studied the vertex irregular reflexive of prism and wheel graphs. Ahmad and Bacá [6] have studied the total vertex irregularity strength for two families of graphs, namely, Jahangir graphs and circulant graphs, and Agustin et al. [7] also study the concept of vertex irregular reflexive labeling of cycle graph and generalized friendship. Another results of irregular labeling can be seen on [8-15]. In this paper, we have found the lower bound of vertex irregular reflexive strength of any graph $G$ and determined the vertex irregular reflexive strength of graphs with pendant vertex. Our results are started by showing one lemma and theorem which describe a general construction of the existence of vertex irregular reflexive $k$-labeling of graph with pendant vertex.

2. Result and Discussion

The following lemma and theorem will be used as a base construction of analysing the reflexive vertex strength of any graph with pendant vertex, namely, sunlet graph, helm graph, subdivided star graph, and broom graph.

**Lemma 1.** For any graph $G$ of order $p$, the minimum degree $\delta$, and the maximum degree $\Delta$,

$$r(vs(G)) \geq \left\lceil \frac{p + \delta - 1}{\Delta + 1} \right\rceil.$$  \hspace{1cm} (1)

**Proof.** Let $G$ be a graph of order $p$, the minimum degree $\delta$, and the maximum degree $\Delta$. The total $k$-labeling which labeling $f$ defined $f_e: E(G) \rightarrow \{1, 2, \ldots, k_e\}$ and $f_v: V(G) \rightarrow \{0, 2, \ldots, 2k_e\}$ such that $f(v) = f_v(x)$ if $x \in V(G)$ and $f(v) = f_v(x)$ if $x \in E(G)$, where $k = \max\{k_e, 2k_e\}$. The total $k$-labeling $f$ is called a vertex irregular reflexive $k$-labeling of the graph $G$ if every two different vertices $x$ and $x'$, and it holds $w_t(x) \neq w_t(x')$, where $w_t(x) = f_v(x) + \sum_{y \in E(G)} f_e(xy)$. Furthermore, since we require $k$-minimum for the graph $G$ which is a vertex irregular reflexive labeling, the set of a vertex weight should be consecutive, otherwise it will not give a minimum $r(vs)$. Thus, the set of a vertex weight is $W_t(x) = \{0, \delta + 1, \delta + 2, \ldots, 1(2k_e) + \Delta k_e\}$. Since the minimum $k = \max\{k_e, 2k_e\}$ is the reflexive vertex strength, the maximum possible vertex weight of graph $G$ is at most $k(1 + \Delta)$. It implies $2k_e + \Delta k_e \geq \delta + (p - 1) \implies k + \Delta k_e \geq \delta + (p - 1) \implies k \geq \delta + p - 1/\Delta + 1$. Since $r(vs(G))$ should be integer and we need a sharpest lower bound, it implies

$$r(vs(G)) \geq \left\lceil \frac{\delta + p - 1}{\Delta + 1} \right\rceil.$$  \hspace{1cm} (2)

It completes the proof.

**Theorem 1.** Let $G$ be a graph of order $n$ and contains $l$ pendant vertex. If $l \geq n - l$, then

$$r(vs(G)) = \begin{cases} \left\lfloor \frac{l}{2} \right\rfloor + 1, & \text{for } l \text{ even and } \frac{l}{2} \text{ odd}, \\ \left\lfloor \frac{l}{2} \right\rfloor, & \text{otherwise}. \end{cases} \hspace{1cm} (3)$$

**Proof.** Given that a graph $G$ of order $n$ is with $l$ pendant vertices. The labeling of graph $G$ is with respect to two components, namely, the pendant vertices and otherwise vertices. Thus, we will split our proof into two cases.

**Case 1.** Let $V_l$ be a set of pendant vertices and the number of $V_l$ is $l$. A pendant vertex consists of two elements, i.e., a vertex and an edge. The vertex weight on each pendant vertex must be different. Suppose we choose those vertex weights are $1, 2, \ldots, l$. Those vertex weights are obtained by summing the vertex and edge labels. To prove the above $r(vs(G))$, let us suppose the maximum vertex weight of $l$. Let

$$l \left\lfloor \frac{l}{2} \right\rfloor = \begin{cases} \frac{l + 1}{2}, & \text{for } l \text{ odd}, \\ \frac{l}{2}, & \text{for } l \text{ even}. \end{cases} \hspace{1cm} (4)$$

Define an injection $f$ as the labels. Since the weight $l$ is contributed by one vertex and one edge labels, it will give four possibilities.

1. If $l + 1/2$ is odd number, then $f(v) = (l + 1/2 - 1)$ and $f(e) = l + 1/2$, such that the vertex weight is $(l + 1/2 - 1) + (l + 1/2) = l + 1 = l - 1 = l + 1/2$.
2. If $l + 1/2$ is even number, then $f(v) = (l + 1/2)$ and $f(e) = l + 1/2$, such that the vertex weight is $(l + 1/2) + (l + 1/2) = l + 1$.
3. If $l/2$ is odd number, then $f(v) = (l/2 - 1)$ and $f(e) = l/2$, such that the vertex weight is $(l/2 - 1) + (l/2) = l - 1$.
4. If $l/2$ is even number, then $f(v) = (l/2)$ and $f(e) = l/2$, such that the vertex weight is $(l/2) + (l/2) = l$.

The vertex weight of point (1), (2), and (4) are, respectively, $l$, and $l + 1$. It will give all weights are different, whereas, point (3) has a vertex weight of $l - 1$. Since the number of pendants is $l$, we will have at least two vertices which have the same weight. Therefore, for $l$ is even and $l/2$ is odd, we need to add 1 for the largest vertex or edge labels. Thus, we will have a different weight for every pendant. Thus, the labels of vertex and edge of the pendant are the following.
From Table 1, it is easy to see that all vertex weights are different.

\[
    k = \begin{cases} 
        \left\lceil \frac{l}{2} \right\rceil + 1, & \text{for } l \text{ even and } \frac{l}{2} \text{ odd,} \\
        \left\lfloor \frac{l}{2} \right\rfloor, & \text{otherwise.} 
    \end{cases} \tag{5}
\]

Case 2. The vertex set apart from pendant vertices \( V(G) - V_l \) must have a degree of at least two. The cardinality of \( V(G) - V_l \) is less than or equal to the cardinality of \( V_l \). It implies that the vertex or edge labels of pendant vertices can be re-used on labels of \( V(G) - V_l \). Thus, the vertex weight of \( V(G) - V_l \) will be different with the vertices of \( V_l \) since it has more combination, namely, \( 2k + 1, 2k + 2, \ldots, n \).

Based on Case 1 and Case 2, the reflexive vertex strength of graph \( G \) is

\[
    \text{rvs}(G) = \begin{cases} 
        \left\lceil \frac{l}{2} \right\rceil + 1, & \text{for } l \text{ even and } \frac{l}{2} \text{ odd,} \\
        \left\lfloor \frac{l}{2} \right\rfloor, & \text{otherwise.} 
    \end{cases} \tag{6}
\]

It concludes the proof.

**Corollary 1.** Let \( S_n \) be a sunlet graph, and for every \( n \geq 3 \),

\[
    \text{rvs}(S_n) = \begin{cases} 
        \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } n \text{ even and } \frac{n}{2} \text{ odd,} \\
        \left\lfloor \frac{n}{2} \right\rfloor, & \text{otherwise.} 
    \end{cases} \tag{7}
\]

**Proof.** Moreover, to determine the label of vertices \( V(S_n) = \{u_i, v_i; 1 \leq i \leq n\} \) and edge set \( E(S_n) = \{u_i v_i, 1 \leq i \leq n\} \cup \{u_{i+1}, 1 \leq i \leq n-1\} \cup \{u_1 u_n\} \), we will use (Algorithm 1)

\[
    k = \begin{cases} 
        \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } n \text{ even and } \frac{n}{2} \text{ odd,} \\
        \left\lfloor \frac{n}{2} \right\rfloor, & \text{otherwise.} 
    \end{cases} \tag{8}
\]

It concludes the proof.

**Theorem 2.** Let \( H_n \) be a helm graph, and for every \( n \geq 3 \),

\[
    \text{rvs}(H_n) = \begin{cases} 
        \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } n \text{ even and } \frac{n}{2} \text{ odd,} \\
        \left\lfloor \frac{n}{2} \right\rfloor, & \text{otherwise.} 
    \end{cases} \tag{9}
\]

**Proof.** Let \( H_n \) be a helm graph with vertex set \( V(H_n) = \{A, u_i, v_i; 1 \leq i \leq n\}, |V(H_n)| = 2n + 1 \) and edge set \( E(H_n) = \{u_i v_i, 1 \leq i \leq n\} \cup \{u_{i+1}, 1 \leq i \leq n-1\} \cup \{u_1 u_n\}, |E(H_n)| = 3n \). Helm graph has \( n \) pendant vertices and one central vertex of degree \( n \). Since the central vertex has degree of much greater than the other vertices, it must have a different vertex weight than the others. Based on Theorem 1, we have the following lower bound:

\[
    \text{rvs}(H_n) \geq \begin{cases} 
        \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } n \text{ even and } \frac{n}{2} \text{ odd,} \\
        \left\lfloor \frac{n}{2} \right\rfloor, & \text{otherwise.} 
    \end{cases} \tag{10}
\]

Furthermore, we will show the upper bound of vertex irregular reflexive \( k \)-labeling by defining the injection \( f \) and \( g \) in the following:

\[
    f(A) = 0, \\
    f(u_i) = f(v_i), \\
    f(v_i) = \begin{cases} 
        0, & \text{if } 1 \leq i \leq k, \\
        \left\lceil \frac{i - k}{2} \right\rceil, & \text{if } k + 1 \leq i \leq n, 
    \end{cases} \tag{11}
\]

\[
    g(Au_i) = g(u_{i+1}) = g(u_1 u_n) = k, \\
    g(u_i) = \begin{cases} 
        i, & \text{if } 1 \leq i \leq k, \\
        k - 1, & \text{if } k + 1 \leq i \leq n, \text{ (i odd for } k \text{ even) and (i even for } k \text{ odd),} \\
        k, & \text{if } k + 1 \leq i \leq n, \text{ (i even for } k \text{ even) and (i odd for } k \text{ odd),} 
    \end{cases}
\]
According to Theorem 1, let \( S_n \) be a subdivided star graph, and for every vertex \( v \) there be a subdivided star graph with vertex set \( V(\kappa) \) and edge set \( E(\kappa) \). The vertices' and edges' labels:

- For labeling of the graph elements, assign the labels of vertices and edges of pendants \( v_i \), otherwise show a different vertex weight. Do the following.
- Apart from pendants, assign the label of vertices \( v_i \) and \( v_j \), but the labels of edges \( u_iu_j \) with \( k \) and \( k+1 \) or \( k+2 \) to each vertex or edge by 1.
- When \( l \) is even and \( l/2 \) is odd, add the label of each vertex or edge by 1.
- Otherwise, it will exist two types of vertices which have the same weights. Do the following.
- Based on Lemma 1, we have the following lower bound:

\[
\text{rws}(SS_n) = \left\lfloor \frac{2n}{3} \right\rfloor. \tag{14}
\]

**Theorem 3.** \( SS_n \) be a subdivided star graph, and for every \( n \geq 3 \),

\[
\text{rws}(SS_n) = \left\lfloor \frac{2n}{3} \right\rfloor. \tag{14}
\]

It is easy to see that the above elements of set are all different. It concludes the proof.

**Algorithm 1:** The vertices' and edges' labels.

\[
k = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor + 1, & \text{if } n \text{ even and } \frac{n}{2} \text{ odd,} \\
\left\lfloor \frac{n}{2} \right\rfloor, & \text{otherwise.} 
\end{cases} \tag{12}
\]

Based on the above injection, the overall vertex weight sets are

\[
w(v_i) = i, \\
w(v_j) = i + 3k, \\
w(A) = nk.
\tag{13}
\]

**Figure 1:** The illustration of labeling on \( S_7 \) and \( S_{12} \).

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
v_i & v_1 & v_2 & v_3 & \cdots & v_k & v_{k+1} & v_{k+2} & v_{k+3} & \cdots & v_{l-1} & v_l \\
\hline
f(v) & 0 & 0 & 0 & \cdots & 0 & 2 & 2 & 4 & \cdots & k & k \\
f(e) & 1 & 2 & 3 & \cdots & k & k+1 & k & k & \cdots & k+1 & k \\
w(v_i) & 1 & 2 & 3 & \cdots & k & k+1 & k+2 & k+3 & \cdots & 2k-1 & 2k \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
v_i & v_1 & v_2 & v_3 & \cdots & v_k & v_{k+1} & v_{k+2} & v_{k+3} & \cdots & v_{l-1} & v_l \\
\hline
f(v) & 0 & 0 & 0 & \cdots & 0 & 2 & 2 & 4 & \cdots & k & k \\
f(e) & 1 & 2 & 3 & \cdots & k & k+1 & k & k & \cdots & k+1 & k \\
w(v_i) & 1 & 2 & 3 & \cdots & k & k+1 & k+2 & k+3 & \cdots & 2k-1 & 2k \\
\hline
\end{array}
\]

Proof. Let \( SS_n \) be a subdivided star graph with vertex set \( V(SS_n) = \{ A, x_i, y_j; 1 \leq i \leq n \} \) and edge set \( E(SS_n) = \{ Ay_j, x_jy_i; 1 \leq i \leq n \} \). The maximum degree of \( SS_n \) is \( n \). The graph \( SS_n \) has one central vertex of degree \( n \). Since the central vertex has degree of much greater than the other vertices, it must have a different vertex weight than the others. Based on Lemma 1, we have the following lower bound:

\[
\text{rws}(G) \geq \left\lfloor \frac{p + \delta - 1}{\Delta + 1} \right\rfloor = \left\lfloor \frac{2n + 1 - 1}{2 + 1} \right\rfloor = \left\lfloor \frac{2n}{3} \right\rfloor. \tag{15}
\]
For the illustration of the vertex irregular reflexive, k-labeling of SS3 and SS4 can be depicted in Figure 2.
Furthermore, we will show the upper bound of vertex irregular reflexive k-labeling by defining the injection f and g. For n ≥ 5, we have the following:

\[
f(A) = 0,
\]

\[
f(x_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq k, \\ 2 \left\lfloor \frac{i - k}{2} \right\rfloor, & \text{if } k + 1 \leq i \leq n. \end{cases}
\] (16)

For otherwise n, we have

\[
f(y_i) = \begin{cases} 2 \left\lfloor \frac{n - k}{2} \right\rfloor, & \text{if } 1 \leq i \leq k, \\ 2 \left\lfloor \frac{n - k}{2} \right\rfloor + 2 \left\lfloor \frac{i - k}{2} \right\rfloor, & \text{if } k + 1 \leq i \leq n - 1, \\ 2 \left\lfloor \frac{n - k}{2} \right\rfloor + 2 \left\lfloor \frac{n - 1 - k}{2} \right\rfloor, & \text{if } i = n, \end{cases}
\] (17)

\[
g(Ay_i) = \begin{cases} k - 1, & \text{if } 1 \leq i \leq n - 2, \\ k, & \text{if } n - 1 \leq i \leq n. \end{cases}
\]
Theorem 4. Let $B_{r,n}$ be a broom graph, and for every $n \geq 2$, $n+1 \leq m$,

\[
\text{rvs}(B_{r,n}) = \begin{cases} 
\left\lceil \frac{n+m}{3} \right\rceil + 1, & \text{if } n+m \equiv 3 \pmod{6}, \\
\left\lfloor \frac{n+m}{3} \right\rfloor, & \text{otherwise.}
\end{cases}
\]

Proof: Let $B_{r,n}$ be a broom graph with vertex set $V(B_{r,n}) = \{A, v_i, u_j; 1 \leq i \leq n, 1 \leq j \leq m\}$, $|V(B_{r,n})| = n + m + 1$, and edge set $E(B_{r,n}) = \{A v_i, u_j, v_i u_j; 1 \leq i \leq n, 1 \leq j \leq m\}, |E(H_n)| = n + m$.

The Broom graph $B_{r,n}$ has $n$ pendant vertices and one central vertex of degree $n$. Since the central vertex has a degree much greater than the other vertices, it must have a different vertex weight than the others. Based on Lemma 1, we have the following lower bound:

\[
\text{rvs}(B_{r,n}) \geq \left\lceil \frac{\rho + \delta - 1}{\Delta + 1} \right\rceil = \left\lceil \frac{n+m}{3} \right\rceil.
\]

for $n + m \equiv 3 \pmod{6}$, $n + m = 3k$ and $k$ is odd. Since the vertices $u_j$ apart from vertex $A$ have degree of at most $2$, the labels of $u_j$ are $(n + m/3 - 1)$, and the label of edges, which are incident to $u_j$, are $(n + m/3)$. Thus, the vertex weight $u_i$ is $3(n + m/3 - 1) = n + m - 1$. Furthermore, since the number of vertices of Broom graph is $n + m$, there must be at least two vertices with the same vertex weight. Thus, we need to add $1$ on the sharpest lower bound:

\[
\text{rvs}(B_{r,n}) \geq \left\lceil \frac{n+m}{3} \right\rceil + 1, \quad \text{if } (n + m) \equiv 3 \pmod{6}.
\]

Furthermore, we will show that $k$ is an upper bound of the reflexive vertex strength of Broom graph $B_{r,n}$. Let

\[
k = \begin{cases} 
\left\lceil \frac{n+m}{3} \right\rceil + 1, & \text{if } n + m \equiv 3 \pmod{6}, \\
\left\lfloor \frac{n+m}{3} \right\rfloor, & \text{otherwise.}
\end{cases}
\]

Define an injection $f$ and $g$ of the vertex irregular reflexive $k$-labeling of Broom graph $\text{rvs}(B_{r,n})$ as follows:
(1) Given that the vertex weight \( V(G) = \{u_j, 2 \leq j \leq m\} \) by \( w(u_j) = n + j, 2 \leq j \leq m \).

(2) Assign the labels of vertices \( u_i \) by \( f(u_i) = f(U - j - 1) \) and assign the labels of edges which are incident to \( u_i \) by 1, 2, 3, \ldots, \( k \) such that it meets with given vertex weight.

(3) When on point (ii), the label of edges is more than \( k \), relabel the vertices with \( f(u_i) = f(u_{i-1}) + 2 \) as well as relabel the edges which are incident to \( u_i \) by 1, 2, 3, \ldots, \( k \) such that it meets with given vertex weight \( w(u_i) = n + j, 2 \leq j \leq m \).

(4) STOP.

**Algorithm 2: The vertices' and edges' labels.**

\[
\begin{align*}
f(A) &= \begin{cases} 
  k - 1, & \text{if } k \text{ odd,} \\
  k, & \text{if } k \text{ even,}
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
f(v_i) &= \begin{cases} 
  0, & \text{if } 1 \leq i \leq k, \\
  2 \left[ \frac{i - k}{2} \right], & \text{if } k + 1 \leq i \leq n + 1,
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
f(u_i) &= f(v_{n+1}). \\
\end{align*}
\]

\[
\begin{align*}
g(A v_i) &= \begin{cases} 
  i, & \text{if } 1 \leq i \leq k, \\
  k - 1, & \text{if } k + 1 \leq i \leq n + 1, (i \text{ odd for } k \text{ even}) \text{ and } (i \text{ even for } k \text{ odd}), \\
  k, & \text{if } k + 1 \leq i \leq n + 1, (i \text{ even for } k \text{ even}) \text{ and } (i \text{ odd for } k \text{ odd}),
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
g(u_i u_j) &= g(A v_{n+1}).
\end{align*}
\]

Based on the above injection, the overall vertex weight sets of \( Br_{n,m} \) for \( v_i; 1 \leq i \leq n + 1 \) is

\[
w(v_i) = i, \quad 1 \leq i \leq n + 1. \tag{26}
\]

Moreover, to determine label of vertices \( V(G) = \{u_j, 2 \leq j \leq m\} \) and \( E(G) = \{Au_m\} \cup \{u_ju_{j+1}, 1 \leq j \leq m - 1\} \), we will use Algorithm 2.

It is easy to see that the above elements of set \( w(v_i) \) and \( w(u_j) \) are all different. It concludes the proof.

### 3. Concluding Remark

In this paper, we have studied the construction of the reflexive vertex \( k \)-labeling of any graph with pendant vertex. We have determined a sharp lower bound of the reflexive vertex strength of any graph \( G \) in Lemma 1, as well as obtained the exact value the reflexive vertex strength of any graph \( G \) in Theorem 1. By this lemma and theorem, we finally determined the reflexive vertex \( k \)-labeling of some families of graph with a pendant vertex. However, we need to find an upper bound of the reflexive vertex strength of any graph and study the reflexive vertex \( k \)-labeling of other families of graph or some graph operations. Therefore, we propose the following open problems:

1. Determine an upper bound of reflexive vertex strength of any graph to find the gap between lower bound and upper bound, and continue to determine

   the exact values for reflexive vertex strength of any other special graphs

2. Determine the construction of the reflexive vertex \( k \)-labeling of any regular graph, planar graph, or some graph operations

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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