Strong connections between tropical algebraic geometry and the theory of rigid-analytic spaces

1.1. A higher-dimensional theorem of the Newton polygon. Let \( K \) be a field that is complete with respect to a nontrivial non-Archimedean valuation \( \text{val} : K^\times \to \mathbb{R} \) and let \( f_1, \ldots, f_n \) be \( n \) convergent power series (in a sense to be made precise later) in \( n \) variables with coefficients in \( K \). Given \( v \in \mathbb{R}^n \), we will give a formula (11.7) for the number of common zeros (counted with multiplicity) \( \xi = (\xi_1, \ldots, \xi_n) \in (K^\times)^n \) of \( f_1, \ldots, f_n \) such that \( v = \text{trop}(\xi) := (\text{val}(\xi_1), \ldots, \text{val}(\xi_n)) \), in terms of the valuations of the coefficients of the \( f_i \). (The set of all \( v \) such that \( v = \text{trop}(\xi) \) for some common zero \( \xi \in V(f_1) \cap \cdots \cap V(f_n) \) is the tropicalization of \( V(f_1) \cap \cdots \cap V(f_n) \), and can also be effectively calculated.) This theorem generalizes the classical theorem of the Newton polygon, which gives the valuations and multiplicities of the zeros of a convergent power series in one variable; see (11.3).

1.2. Let us discuss (11.1) in more detail. Let \( K \) be as above, and assume for simplicity that \( K = \overline{\mathbb{Q}} \) and that \( \text{val}(\mathbb{K}^\times) = \mathbb{R} \). We (provisionally) define the rigid-analytic unit ball \( B^n \) to be the set of all points \( (\xi_1, \ldots, \xi_n) \in K^n \) such that \( \text{val}(\xi_i) \geq 1 \) for all \( i \). Define a map \( \text{trop} : B^n \to (\mathbb{R}_{\geq 0} \cup \{\infty\})^n \) by \( \text{trop}(\xi_1, \ldots, \xi_n) = (\text{val}(\xi_1), \ldots, \text{val}(\xi_n)) \). Let \( f = \sum a_\nu x^\nu \in K[[x_1, \ldots, x_n]] \) be a function converging on \( B^n \) (i.e. such that \( \text{val}(a_\nu) \to \infty \)) and define \( \text{Trop}(f) \subset (\mathbb{R}_{\geq 0} \cup \{\infty\})^n \) to be the set \( \{\text{trop}(\xi) : f(\xi) = 0\} \). It is a fundamental fact (see [8]) that \( \text{Trop}(f) \cap B^\times_{\geq 1} \) is union of finitely many polyhedra (generically of dimension \( n-1 \)), and that the ultrametric triangle inequality as applied to the equation \( \sum a_\nu \xi^\nu = 0 \)
completely determines the set $\text{Trop}(f)$ (see §7.9). In particular the set $\text{Trop}(f)$ is generally not hard to calculate.

1.3. Now let $f_1, \ldots, f_n \in K[[x_1, \ldots, x_n]]$ be power series converging on $B^n$. If $\xi \in B^n$ is a common zero of $f_1, \ldots, f_n$, then $\text{trop}(\xi)$ is contained in $\bigcap_{i=1}^n \text{Trop}(f_i)$, which is generically a finite set of points. In other words, one gets very strong restrictions on the valuations of the coordinates of the common zeros of $f_1, \ldots, f_n$ via a simple combinatorial calculation, which when $n = 1$ reduces to finding the slopes of the Newton polygon of a power series (see [7.10]).

The tropical hypersurfaces $\text{Trop}(f_i)$ come equipped with multiplicity information (the Newton complex), also determined by the valuations of the coefficients of the $f_i$, which induces a notion of multiplicity on the points of $\bigcap_{i=1}^n \text{Trop}(f_i)$. (When $n = 1$ these multiplicities amount to the horizontal lengths of the line segments in the Newton polygon.) Osserman and Payne [OP10] have proved a very general result relating the multiplicities in the intersection theory of subvarieties of a torus with the multiplicities in the intersection theory of tropical varieties, which when applied to this case gives a formula for the number of common zeros $\xi$ of an $n$-tuple of Laurent polynomials $f_1, \ldots, f_n$ (counted with multiplicity) such that $\text{trop}(\xi)$ is a specified point in $\bigcap_{i=1}^n \text{Trop}(f_i)$. With enough of the framework of tropical analytic geometry in place (see §§6, 7, and 8), a continuity of roots argument (10.2) allows us to formulate and deduce the corresponding result for power series (11.7).

1.4. From the perspective of a tropical geometer, the theory of rigid spaces is useful because the analytic topology on $\mathbb{R}^n$ is much better approximated by the rigid-analytic topology on the torus $G_m^n$. For example, the unit box $[0,1]^n$ is an analytic neighborhood in the Euclidean space $\mathbb{R}^n$, yet $\text{trop}^{-1}([0,1]^n) \subset G_m^n$ is the $n$-fold product of the annulus $\{\xi \in K^\times : \text{val}(\xi) \in (0,1]\}$, which is a very nicely behaved rigid-analytic object (it is a smooth affinoid space), but is not the set of points underlying a subscheme. Similarly, $(\mathbb{R} \cup \{\infty\})^n$ can naturally be regarded as the tropicalization of the affine space $\mathbb{A}^n$ (see §5), under which identification the unit ball $B^n$ is the inverse image under trop of $(\mathbb{R}_{\geq 0} \cup \{\infty\})^n$ (a neighborhood of the point $\infty$).

1.5. The following example is an application of rigid-analytic methods to a tropically-local problem. Let $U_{(1)} = \text{trop}^{-1}(\{1\}) \subset G_m^n$. This is an affinoid space, which implies (see [4]) that $\text{val}(a_n) \to \infty$ as $|v_1| + \cdots + |v_n| \to \infty$. If $a \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is an ideal and $Y = V(a) \subset G_m^n$ is the associated subscheme then $Y \cap U_{(1)}$ is identified with the set of maximal ideals of $K(U_{(1)})$ containing the ideal $aK(U_{(1)})$, so to show that $0 \in \text{Trop}(Y)$ is equivalent to showing that $a$ does not generate the unit ideal in $K(U_{(1)})$. This ends up being equivalent to the well-known criterion that the initial ideal of $a$ at $0$ contain no monomials. The characterization of the tropicalization (or rather the Bieri-Groves set) of a scheme by initial ideals was proved by Einsiedler-Kapranov-Lind [EKL06] using these methods; we give a treatment below (7.9) which also applies to tropicalizations of analytic spaces. (The first complete proof of this theorem was given by Draisma [Dra08, Theorem 4.2] and also uses affinoid algebras, albeit in a different way; see [7.13]).

1.6. A family of tropical varieties parameterized by an interval corresponds to a family of subvarieties of a torus parameterized by a rigid-analytic annulus. We study such families in order to prove the theorem indicated in (1.4 ii); to illustrate the main idea we will sketch a special case. Let $f_1, f_2 \in K[x_1^{\pm 1}, x_2^{\pm 1}]$, and suppose that $\text{Trop}(f_1) \cap \text{Trop}(f_2)$ has a connected component $C$ of positive dimension. In this case there is a notion of the stable intersection multiplicity of $\text{Trop}(f_1)$ and $\text{Trop}(f_2)$ along the component $C$, which is defined by translating $\text{Trop}(f_2)$ in a generic direction by a small amount $\epsilon$ so that $\text{Trop}(f_1) \cap \text{Trop}(f_2)$ is a finite set of points, then taking the limit as $\epsilon$ approaches zero. This corresponds to replacing $f_2(x_1, x_2)$ by $f_2(t^{a_1}x_1, t^{a_2}x_2)$ for generic $a_1, a_2 \in \mathbb{Z}$ and some $t \in K^\times$ and then taking the limit as $t$ approaches 1. The results relating tropical and algebraic intersection multiplicities mentioned above allow us to count the number of common zeros of $f_1(x_1, x_2), f_2(t^{a_1}x_1, t^{a_2}x_2)$ with fixed tropicalization when $\text{val}(t) > 0$.

In order to relate these quantities with the number of common zeros $\xi$ of $f_1, f_2$ such that $\text{trop}(\xi) \in C$ one is led to consider the family of schemes $Y_t = V(f_1(x_1, x_2)) \cap V(f_2(t^{a_1}x_1, t^{a_2}x_2))$ parametrized
by the rigid-analytic annulus \( \{ t \in \mathbb{R}^\times : \text{val}(t) \in (0, \varepsilon] \} \). Under appropriate hypotheses the family \( Y_i \) is automatically finite and flat (at least after passing to an appropriate toric compactification; see \( \S 2.3 \)). In particular, the length of the fiber \( Y_i \) is independent of \( t \), which shows that algebraic intersection multiplicities can be calculated after an analytically small perturbation. We will make this kind of argument precise in \( \S 12 \).

1.7. We now describe in more detail the contents of this paper. As the material in this paper bridges two different fields, we have made an effort to ensure that it be as readable as possible both to tropical geometers (who may not be familiar with affinoid algebras or rigid spaces) as well as to arithmetic geometers (who may not be familiar with convex or tropical geometry). Hence we have included \( \S \S 2-5 \) which are mainly expository, containing many examples and pictures. In \( \S 2 \) we give definitions, basic properties, and pictures of the convex-geometric objects that we will encounter. In \( \S 3 \) we describe the compactification \( N_\mathbb{R}(\Delta) \) of Euclidean space \( N_\mathbb{R} \) associated to a fan \( \Delta \), as introduced by Kajiwara and Payne, which serves as the tropicalization of the toric variety \( X(\Delta) \). We also introduce the notion of a compactified polyhedron inside a space \( N_\mathbb{R}(\Delta) \), which will serve as the tropicalization of a so-called polyhedral affinoid subdomain of \( X(\Delta) \). (The reader who is not familiar with toric varieties will lose little on first reading by assuming throughout that \( X(\Delta) \) is a torus and \( N_\mathbb{R}(\Delta) \) is Euclidean space.) In \( \S 4 \) we define, give examples of, and state the basic properties of affinoid algebras and rigid-analytic spaces. We will emphasize the analogy with the theory of finite-type schemes over a field. In \( \S 5 \) we review Kajiwara and Payne’s notion of extended tropicalizations, in the process defining the analytic spaces. We will emphasize the analogy with the theory of finite-type schemes over a field.

In \( \S 6 \) we introduce the fundamental notion of a polyhedral subdomain of a toric variety. We will show (6.9) that if \( X(\Delta) \) is a toric variety adapted to a polyhedron \( P \subset \mathbb{R}^\Delta \) in an appropriate sense, then the inverse image \( U_P \) of the closure of \( P \) in the compactification \( N_\mathbb{R}(\Delta) \) is the affinoid space associated to an explicitly identified affinoid algebra. This extends the notion of a polytopal subdomain as defined in [EKL06] and [Gub07b] in a nontrivial way. In \( \S 7 \) we define the tropicalization \( \text{Trop}(Y) \) of a closed analytic subspace \( Y \) of \( U_P \) and characterize it in terms of initial ideals (7.9, 7.12). The definition of \( \text{Trop}(Y) \) agrees with Gubler’s notion when \( P \) is a polytope. In \( \S 8 \) we review the canonical polyhedral complex structure on the tropical hypersurface \( \text{Trop}(f) \) associated to a nonzero Laurent polynomial \( f \), as well as introducing the Newton complex \( \text{New}(f) \). We will prove an important finiteness result (8.2) which implies that the tropicalization of a generically non-proper analytic (or algebraic) hypersurface in a polyhedral subdomain \( U_P \) carries a finite polyhedral structure.

In \( \S \S 9-10 \) we prove two “continuity of roots” results which will be useful in \( \S \S 11-12 \). Theorem (9.5) is a tropical criterion for a rigid-analytic family of subvarieties (or analytic subspaces) of a toric variety to be finite and flat. Theorem (10.2) is a local continuity of roots criterion: it says that if \( f_1, t, \ldots, f_n, t \) is a one-parameter family of power series in \( n \) variables such that the specializations \( f_1, 0, \ldots, f_n, 0 \) have finitely many common zeros, then \( f_1, t, \ldots, f_n, t \) has the same number of common zeros when \( |t| \) is small. This result rests on Raynaud’s approach to rigid geometry via formal schemes.

In \( \S 11 \) we prove a rigid-analytic intersection multiplicity formula extending the corresponding result for subschemes of a torus, as described in (1.3). This result is a strict generalization of the theorem of the Newton polygon that applies to convergent power series in several variables. More specifically, \( f_1, \ldots, f_n \) are analytic functions on a polyhedral subdomain \( U_P \) and \( v \in \bigcap_{i=1}^n \text{Trop}(f_i) \) is an isolated point contained in the interior of \( P \) then we will give an explicit formula (11.7) for the number of common zeros \( \xi \) of \( f_1, \ldots, f_n \) such that \( \text{trop}(\xi) = v \).

In \( \S 12 \) we prove a result relating algebraic multiplicities and stable intersection multiplicities along a tropically non-proper complete intersection of hypersurfaces. That is, \( f_1, \ldots, f_n \) are nonzero Laurent polynomials and \( C \subset \bigcap_{i=1}^n \text{Trop}(f_i) \) is a connected component of positive dimension then we will use the intersection multiplicity formula of (11.10) to calculate the number of common zeros of \( f_1, \ldots, f_n \) in an appropriate toric variety \( X(\Delta) \) which lie over the closure of \( C \) in \( N_\mathbb{R}(\Delta) \), in terms of stable tropical intersection multiplicities. The proof will involve families of translations of tropical varieties parametrized by a rigid-analytic base, as indicated above.
1.8. Tropical analytic geometry in the literature. Several papers have already appeared which take advantage of the connections between tropical and rigid-analytic geometry. As mentioned above, Einsiedler, Kapranov, and Lind [EKL06] characterize the Bieri-Groves set of a subvariety of a torus in terms of initial ideals; they also prove its connectedness using rigid-analytic results of Conrad [Con99]. Payne [Pay09a] has proved that the analytic space (in the sense of Berkovich) associated to a subvariety of a toric variety is naturally homeomorphic to the inverse limit of all of its tropicalizations. Gubler [Gub07b] has used the combinatorial structure on the tropicalization of a closed subspace of a polytopal subdomain in order to prove special cases of the Bogomolov conjecture over function fields [Gub07a]. The author has studied the tropicalization of the logarithm of a $p$-divisible formal group in order to show that it has a canonical subgroup if its Hasse invariant is small enough [Rab09].

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1.10. Notation. We will use the following general notation throughout this paper.

- Let $S$ be a set and $T \subseteq S$ a subset, and let $f : S \to \mathbb{R}$ be any function. Define

$$\minset(f, T) := \{t \in T : f(t) = \inf_{t' \in T} f(t')\} \quad \text{and} \quad \maxset(f, T) := \{t \in T : f(t) = \sup_{t' \in T} f(t')\}.$$ 

(These sets may be empty.)

- If $X$ is a scheme (resp. a rigid space; cf. §4) we use $|X|$ to denote the set of closed points of $X$ (resp. the underlying set of $X$). For $\xi \in |X|$ we let $\kappa(\xi) = \mathcal{O}_X,\xi/m_{X,\xi}$ denote the residue field at $\xi$.

- If $Y$ is a topological space and $P \subseteq Y$ is a subspace we let $P^\circ = P \setminus \partial P$ denote the interior of $P$ in $Y$.

- If $\Gamma$ is a subset of $\mathbb{R}$ and $r \in \Gamma$ we set

$$\Gamma_{\geq r} = \{v \in \Gamma : v \geq r\}, \quad \Gamma_{> r} = \{v \in \Gamma : v > r\}, \quad \Gamma_{\leq r} = \{v \in \Gamma : v \leq r\}, \quad \Gamma_{< r} = \{v \in \Gamma : v < r\}.$$
2. Basic notions from convex geometry

2.1. The tropicalization of an algebro-geometric or analytic-geometric object is a convex-geometric object, which is combinatorial in nature. In this section we give definitions of, state some properties of, and draw some pictures of the convex-geometric objects that will appear, for the benefit of the reader who is not familiar with them. Most of this material can be found in [Ful93, §§1.2,1.5] and [Bar02, Chapter VI], although almost all of it is quite easy and instructive to prove on one’s own.

Convex bodies live inside Euclidean space $\mathbb{R}^n$. We prefer not to choose a basis, so we fix the following notation for the rest of this paper:

**Notation 2.2.**

- $N_\mathbb{R} \cong \mathbb{R}^n$ is a real vector space of dimension $n$
- $M_\mathbb{R} = N_\mathbb{R}^*$ is its linear dual
- $\langle \cdot, \cdot \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ is the canonical pairing
- $N \cong \mathbb{Z}^n$ is a full-rank lattice in $N_\mathbb{R}$
- $M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ is the dual lattice in $M_\mathbb{R}$
- $\Gamma \subset \mathbb{R}$ is a nonzero additive subgroup
- $N_\Gamma = N \otimes \mathbb{Z} \Gamma$ is the subgroup of $\Gamma$-rational points of $N_\mathbb{R}$
- $M_\Gamma = M \otimes \mathbb{Z} \Gamma$ likewise for $M_\mathbb{R}$.

The lattice $N \subset N_\mathbb{R}$ is called an integral structure. In the sequel we will take the subgroup $\Gamma$ to be the value group of a field equipped with a nontrivial non-Archimedean valuation.

**Definition 2.3.**

(i) An (affine) half-space in $N_\mathbb{R}$ is a subset of the form

$$H = \{ v \in N_\mathbb{R} : \langle u, v \rangle \leq a \} \quad \text{for some} \quad u \in M_\mathbb{R} \setminus \{0\}, \ a \in \mathbb{R}.$$  

The half-space $H$ is called integral if we can take $u \in M$, and is integral $\Gamma$-affine if we can take $u \in M$ and $a \in \Gamma$. The half-space $H$ is called linear if we can take $a = 0$.

(ii) With $H \subset N_\mathbb{R}$ as above, the complementary half-space of $H$ is

$$H^- = \{ v \in N_\mathbb{R} : \langle u, v \rangle \geq a \}$$

and the boundary of $H$ is its topological boundary

$$\partial H = \{ v \in N_\mathbb{R} : \langle u, v \rangle = a \} = H \cap H^-.$$  

(iii) An affine space in $N_\mathbb{R}$ is a translate of a linear subspace of $N_\mathbb{R}$. Any affine space is of the form $\bigcap_{i=1}^{r} \partial H_i$, where the $H_i$ are half-spaces.

(iv) A polyhedron in $N_\mathbb{R}$ is a nonempty intersection $P = \bigcap_{i=1}^{r} H_i$ of finitely many half-spaces $H_i \subset N_\mathbb{R}$. We say that $P$ is integral (resp. integral $\Gamma$-affine) if we can take the $H_i$ to be integral (resp. integral $\Gamma$-affine). If $P$ is integral $\Gamma$-affine we set $P_\Gamma = P \cap N_\Gamma$.

(v) An integral, resp. integral $\Gamma$-affine, polytope is a bounded integral, resp. bounded integral $\Gamma$-affine, polyhedron.

(vi) Let $P \subset N_\mathbb{R}$ be a polyhedron. The affine span of $P$, denoted $\text{span}(P)$, is the smallest affine subspace of $N_\mathbb{R}$ containing $P$. The dimension $\text{dim}(P)$ of $P$ is the dimension of $\text{span}(P)$. The relative interior of $P$, denoted $\text{relint}(P)$, is the interior of $P$ as a subspace of $\text{span}(P)$.

(vii) Let $S \subset N_\mathbb{R}$ be a subset. The convex hull of $S$ is the intersection $\text{conv}(S)$ of all half-spaces in $N_\mathbb{R}$ containing $S$. It is the smallest convex subset of $N_\mathbb{R}$ containing $S$.

See [Z25] for examples.

**Definition 2.4.** Let $P \subset N_\mathbb{R}$ be a polyhedron. For $u \in M_\mathbb{R}$ we define

$$\text{face}_u(P) = \text{maxset}(u, P) = \{ v \in P : \langle u, v \rangle \geq \langle u, v' \rangle \text{ for all } v' \in P \}.$$
A face of $P$ is a nonempty subset of the form $F = \text{face}_u(P)$ for some $u \in M_{\mathbb{R}}$. We write $F \prec P$ to signify that $F$ is a face of $P$. A vertex of $P$ is a face consisting of a single point; we let $\text{vert}(P)$ denote the set of vertices of $P$.

In other words, a face of $P$ is a subset on which a linear form attains its maximum. Note that using these conventions we have $P \prec P$ but $\emptyset \not\prec P$.

Example 2.5. Let $N = M = \mathbb{Z}^2 \subset \mathbb{R}^2$, and let $(\cdot, \cdot)$ be the dot product. The unit square $S = [0, 1]^2$ is an integral $\mathbb{Z}$-affine polytope in $\mathbb{R}^2$, and the first quadrant $Q = \mathbb{R}^2_{\geq 0}$ is an integral $\mathbb{Z}$-affine polyhedron.

The four edges and four vertices of $S$ are faces; if $u_1 = (-1, 0), u_2 = (-1, -1),$ and $u_3 = (0, -1)$ then the left edge is $\text{face}_{u_1}(S)$, the bottom edge is $\text{face}_{u_2}(S)$, and $\{(0, 0)\} = \text{face}_{u_3}(S)$. The polyhedron $Q$ has four faces: $Q$ itself, two edges $\text{face}_{u_1}(Q)$ and $\text{face}_{u_2}(Q)$, and $\{(0, 0)\} = \text{face}_{u_3}(Q)$. Note that all faces are again integral $\mathbb{Z}$-affine, and that $S$ is the convex hull of its vertices. See Figure 1.

![Figure 1](image_url)

**Figure 1.** The unit square is a polytope and the first quadrant is a polyhedron in $\mathbb{R}^2$.

Many statements about polyhedra can be deduced from the analogous results for cones (2.9) by considering the cone over a polyhedron: see [Ful93, p.24]. Here we collect some of the basic properties of polyhedra:

**Proposition 2.6.** Let $P \subset N_{\mathbb{R}}$ be a polyhedron.

(i) A face $F \prec P$ is a polyhedron in $N_{\mathbb{R}}$. If $F \not\prec P$ then $\dim(F) < \dim(P)$.

(ii) If $F \prec P$ and $P$ is integral (resp. integral $\Gamma$-affine) then $F$ has the same property.

(iii) If $F, F' \prec P$ and $F \subset F'$ then $F \prec F'$. More generally, if $F \cap F' \not= \emptyset$ then $F \cap F'$ is a face of $F$ and of $F'$ (and of $P$).

(iv) $P$ has finitely many faces.

(v) If $P$ is a polytope then $P = \text{conv}(\text{vert}(P))$, and the convex hull of a finite set of points is a polytope [Bar02, Corollary 2.4.3].

Collections of polyhedra will also be of interest:

**Definition 2.7.**

(i) A polyhedral complex is a finite collection II of polyhedra in $N_{\mathbb{R}}$, called the cells or faces of II, satisfying

- (PC1) if $P, P' \in \Pi$ and $P \cap P' \not= \emptyset$ then $P \cap P'$ is a face of $P$ and of $P'$, and
- (PC2) if $P \in \Pi$ and $F \prec P$ then $F \in \Pi$.

The support of II is the set $|\Pi| = \bigcup_{P \in \Pi} P$. The dimension of II is the dimension of its highest-dimensional cell; II has pure dimension $d$ if every maximal cell has dimension $d$. We say that II is integral (resp. integral $\Gamma$-affine) if all of its cells are integral (resp. integral $\Gamma$-affine).

(ii) A polytopal complex is a polyhedral complex whose cells are polytopes.

(iii) A refinement of a polyhedral complex II is a polyhedral complex II’ with the same support, and such that each cell of II is a union of cells of II’.

(iv) Let II, II’ be polyhedral complexes. We define $\Pi \cap \Pi' = \{\text{all faces of } P \cap P' : P \in \Pi, P' \in \Pi', \text{ and } P \cap P' \neq \emptyset\}$. 
It is easy to show that $\Pi \cap \Pi'$ is a polyhedral complex, and that $|\Pi \cap \Pi'| = |\Pi| \cap |\Pi'|$. In particular, if $|\Pi| = |\Pi'|$ then $\Pi \cap \Pi'$ is a common refinement of $\Pi$ and $\Pi'$.

**Example 2.8.** We let $N = M = \mathbb{Z}^2$ as in (2.5). Let

$$
P_1 = \mathbb{R}_{\geq 0}(0,1) \quad P_2 = \mathbb{R}_{\geq 0}(1,0) \quad P_3 = \text{conv}\{(-1, -1), (0, 0)\}
$$

$$
P_4 = (-1, -1) + \mathbb{R}_{\geq 0}(-1, 0) \quad P_5 = (-1, -1) + \mathbb{R}_{\geq 0}(0, -1)
$$

as in Figure 2 and let

$$
\Pi_1 = \{P_1, P_2, P_3, P_4, P_5, \{0, 0\}, \{(-1, -1)\} \}.
$$

Then $\Pi_1$ is an integral $\mathbb{Z}$-affine polyhedral complex of pure dimension 1 in $\mathbb{R}^2$.

Let $Q_1$ denote the triangle $\text{conv}\{(0, 0), (0, 1), (1, 0)\}$ and let $Q_2 = \text{conv}\{(1, 1), (0, 1), (1, 0)\}$, as in Figure 2. These are integral $\mathbb{Z}$-affine polytopes with three vertices and three edges. Let $\Pi_2$ be the collection of all faces of $Q_1$ and $Q_2$. This is an integral $\mathbb{Z}$-affine polytopal complex of pure dimension 2 in $\mathbb{R}^2$. It contains four vertices, five edges, and two faces, and its support $\Pi_2$ is the unit square.

**Figure 2.** An integral $\mathbb{Z}$-affine polyhedral complex $\Pi_1$ of pure dimension 1 and an integral $\mathbb{Z}$-affine polytopal complex $\Pi_2$ of pure dimension 2 in $\mathbb{R}^2$. Only the maximal cells are labeled.

Intersections of linear half-spaces are called cones:

**Definition/Proposition 2.9.**

(i) A (convex polyhedral) cone (resp. integral cone) in $N_R$ is an intersection $\sigma$ of finitely many linear (resp integral linear) half-spaces in $N_R$. Any face of a cone is a cone. We say that $\sigma$ is pointed if 0 is a vertex of $\sigma$, or equivalently if $\sigma$ does not contain any nonzero linear space.

(ii) Let $v_1, \ldots, v_r \in N_R$. The subset $\sigma = \sum_{i=1}^r \mathbb{R}_{\geq 0}v_i$ is a cone in $N_R$, and any cone can be written in this form [Ful93, p.12]. The cone $\sigma$ is integral if $v_i \in N$ for all $i$, and any integral cone can be written $\sigma = \sum_{i=1}^r \mathbb{R}_{\geq 0}v_i$ for $v_1, \ldots, v_r \in N$.

(iii) Let $\sigma = \sum_{i=1}^r \mathbb{R}_{\geq 0}v_i \subset N_R$ be a cone. The (polar) dual cone to $\sigma$ is the cone

$$
\sigma^\vee = \{u \in M_R : \langle u, v \rangle \leq 0 \text{ for all } v \in \sigma\} = \bigcap_{i=1}^r \{u \in M_R : \langle u, v_i \rangle \leq 0\}.
$$

We have $\sigma = \sigma^\vee$ [Ful93, (1.2.1)], and $\sigma$ is integral if and only if $\sigma^\vee$ is integral.

(iv) The annihilator of a cone $\sigma \subset N_R$ is the annihilator of the vector space $\text{span}(\sigma)$:

$$
\sigma^\perp = \{u \in M_R : \langle u, v \rangle = 0 \text{ for all } v \in \sigma\}.
$$

It is a linear space in $M_R$.

(v) Let $\sigma \subset N_R$ be a cone. The map $\tau \mapsto \tau^\perp \cap \sigma^\vee$ is an inclusion-reversing bijection between the faces of $\sigma$ and the faces of $\sigma^\vee$, with inverse $\tau' \mapsto (\tau')^\perp \cap \sigma$ [Ful93, (1.2.10)]. We have $\dim(\tau) + \dim(\tau^\perp \cap \sigma^\vee) = n$. 


3.1. In this section we describe a procedure for constructing a partial compactification \( N_{\mathbb{R}}(\Delta) \) of \( N_{\mathbb{R}}(\Delta) \) associated to a fan \( \Delta \). This procedure is analogous to the construction of the toric variety \( X(\Delta) \) associated to \( \Delta \) (see §5); the space \( N_{\mathbb{R}}(\Delta) \) will serve as the (extended) tropicalization of \( X(\Delta) \). We then describe the closure \( \overline{P} \) of a polyhedron \( P \) in a suitable partial compactification \( N_{\mathbb{R}}(\Delta) \). The

(vi) A fan \( \Delta \) in \( N_{\mathbb{R}} \) is a polyhedral complex whose cells are cones (called the cones of \( \Delta \)). The fan \( \Delta \) is complete if \( \lvert \Delta \rvert = N_{\mathbb{R}} \). The fan \( \Delta \) is pointed if \( \{0\} \in \Delta \), or equivalently if all cells of \( \Delta \) are pointed cones.

(vii) Let \( P \subset N_{\mathbb{R}} \) be a polyhedron. The normal fan to \( P \) is the fan \( \mathcal{N}(P) \) in \( M_{\mathbb{R}} \) whose cells are the cones

\[
\mathcal{N}(P,F) := \{ u \in M : F \subset \text{face}_u(P) \}
\]

where \( F \prec P \).

This fan is integral if \( P \) is integral. See [Ful93, p.26].

(viii) Let \( P \) be a polyhedron. Its normal fan \( \mathcal{N}(P) \) is complete if and only if \( P \) is a polytope, and \( \mathcal{N}(P) \) is pointed if and only if \( \dim(P) = \dim_{\mathbb{R}}(N_{\mathbb{R}}) \).

Example 2.10. Let \( N = M = \mathbb{Z}^2 \) as in (2.5). Let \( \sigma = \mathbb{R}_{\geq 0}(0,1) + \mathbb{R}_{\geq 0}(1,1) \). This is an integral pointed cone in \( \mathbb{R}^2 \). It has four faces: \( \tau \), \( \tau_1 = \mathbb{R}_{\geq 0}(0,1) \), \( \tau_2 = \mathbb{R}_{\geq 0}(1,1) \), and \( \{(0,0)\} \). The dual cone is \( \sigma^\vee = \mathbb{R}_{\geq 0}(-1,0) + \mathbb{R}_{\geq 0}(1,-1) \); its faces are

\[
\begin{align*}
\tau'_1 &= \mathbb{R}_{\geq 0}(-1,0) = \tau_1^\perp \cap \sigma^\vee, \\
\tau'_2 &= \mathbb{R}_{\geq 0}(1,-1) = \tau_2^\perp \cap \sigma^\vee, \\
\sigma^\vee &= \{(0,0)\}^\perp \cap \sigma^\vee = \{0,0\} = \sigma^\perp \cap \sigma^\vee.
\end{align*}
\]

See Figure 3.

\[\begin{array}{c}
\tau_1 \\
\sigma \\
\tau_2
\end{array}\]

\[\begin{array}{c}
\tau'_1 \\
\sigma^\vee \\
\tau'_2
\end{array}\]

Figure 3. An integral pointed cone \( \sigma \) in \( \mathbb{R}^2 \) and its polar dual \( \sigma^\vee \). For \( i = 1, 2 \) we have \( \tau'_i = \tau_i^\perp \cap \sigma^\vee \).

Example 2.11. Let \( N = M = \mathbb{Z}^2 \) and let \( S = [0,1]^2 \), as in (2.5). Label the vertices \( v_1, v_2, v_3, v_4 \) and edges \( F_1, F_2, F_3, F_4 \) of \( S \) as in Figure 4. The normal fan to the polytope \( S \) is drawn in Figure 4. It is a complete integral pointed fan. For \( i = 1, 2, 3, 4 \) the set of \( u \in \mathbb{R}^2 \) such that \( v_i \in \text{face}_u(S) \) is the \( i \)th quadrant \( \sigma_i \). To say that \( F_i \subset \text{face}_u(S) \) is to say that both \( v_i \) and \( v_{i+1} \) are in \( \text{face}_u(S) \), so \( \mathcal{N}(S,F_i) = \sigma_i \cap \sigma_{i+1} = \tau_i \), where the subscripts are taken modulo 4.

\[\begin{array}{c}
v_2 \\
F_1 \\
v_1
\end{array}\]

\[\begin{array}{c}
F_2 \\
S \\
F_3 \quad F_4
\end{array}\]

\[\begin{array}{c}
v_3 \\
v_4
\end{array}\]

Figure 4. The unit square and its normal fan. For \( i = 1, 2, 3, 4 \) we have \( \mathcal{N}(S,v_i) = \sigma_i \) and \( \mathcal{N}(S,F_i) = \tau_i \).

3. Compactification procedures

3.1. In this section we describe a procedure for constructing a partial compactification \( N_{\mathbb{R}}(\Delta) \) of \( N_{\mathbb{R}}(\Delta) \) associated to a fan \( \Delta \). This procedure is analogous to the construction of the toric variety \( X(\Delta) \) associated to \( \Delta \) (see §5); the space \( N_{\mathbb{R}}(\Delta) \) will serve as the (extended) tropicalization of \( X(\Delta) \). We then describe the closure \( \overline{P} \) of a polyhedron \( P \) in a suitable partial compactification \( N_{\mathbb{R}}(\Delta) \). The
compactly. \( \overline{T} \) will correspond to a “polyhedral subdomain” of \( X(\Delta) \); this generalizes [Gub07b] §4 and [EKL08] §3.

The construction of \( N_R(\Delta) \) is originally due to Kajiwara [Kaj08], and was later described by Payne [Pay09a] §3. We follow Payne’s treatment.

**Notation 3.2.** We let \( \overline{R} \) be the additive monoid \( R \cup \{ -\infty \} \), endowed with the topology which restricts to the standard topology on \( R \) and for which the sets of the form \( \{ -\infty, a \} \) for \( a \in R \) constitute a neighborhood basis of \( -\infty \).

**Definition 3.3.** Let \( \sigma \subseteq N_R \) be a cone. The partial compactification of \( N_R \) with respect to \( \sigma \) is the space \( N_R(\sigma) = \text{Hom}_{R_{\geq 0}}(\sigma^\vee, \overline{R}) \) of monoid homomorphisms respecting multiplication by \( R_{\geq 0} \), equipped with the topology of pointwise convergence. We use \( \langle \cdot, \cdot \rangle_\sigma \) to denote the the pairing \( \sigma^\vee \times N_R(\sigma) \rightarrow \overline{R} \).

See (3.7) for an example. Roughly, \( N_R(\sigma) \) is a space that compactifies \( N_R \) in the directions of the faces of \( \sigma \); this statement is made precise in the following Proposition.

**Proposition 3.4.** ([Pay09a] §3) Let \( \sigma \subseteq N_R \) be a cone.

(i) Let \( \tau \prec \sigma \), let \( v \in N_R / \text{span}(\tau) \), and define \( \iota(v) \in N_R(\sigma) \) by

\[
\langle u, \iota(v) \rangle_\sigma = \begin{cases} 
\langle u, v \rangle & \text{if } u \in \tau^\perp \cap \sigma^\vee \\
-\infty & \text{otherwise}
\end{cases}
\]

for \( u \in \sigma^\vee \). Then \( \iota(v) \) is a well-defined element of \( N_R(\sigma) \), and

\[
\iota : \prod_{\tau \prec \sigma} N_R / \text{span}(\tau) \xrightarrow{\sim} N_R(\sigma)
\]

is a bijection. Furthermore, for each \( \tau \prec \sigma \) the restriction \( \iota|_{N_R / \text{span}(\tau)} : N_R(\tau) \rightarrow N_R(\sigma) \) is a topological embedding.

(ii) If \( \sigma^\vee = \sum_{i=1}^n R_{\geq 0} u_i \) then the map

\[
v \mapsto (\langle u_1, v \rangle_\sigma, \ldots, \langle u_r, v \rangle_\sigma) : N_R(\sigma) \rightarrow \overline{\mathbb{R}}^r
\]

is a topological embedding with closed image.

(iii) For \( \tau \prec \sigma \) the inclusion \( \sigma^\vee \subseteq \tau^\vee \) induces a topological embedding \( N_R(\tau) \rightarrow N_R(\sigma) \) with open image.

**Proof.** We will only prove (i). Since \( \tau^\perp \cap \sigma^\vee \) is a face of \( \sigma^\vee \) (2.9), we have \( u_1 + u_2 \in \tau^\perp \cap \sigma^\vee \iff u_1, u_2 \in \tau^\perp \cap \sigma^\vee \), which shows that \( \iota(v) \) is a well-defined element of \( N_R(\sigma) \). We claim that \( \iota \) is injective. For \( v \in N_R / \text{span}(\tau) \) it is clear from the definition that we can recover \( \tau^\perp \cap \sigma^\vee \), and hence that we can recover \( \tau \) from the element \( \iota(v) \), so it suffices to show that \( \iota|_{N_R / \text{span}(\tau)} \) is injective for all \( \tau \prec \sigma \).

This follows from the fact that \( \tau^\perp \cap \sigma^\vee \) spans \( \tau^\perp \) (2.9). As for surjectivity: given any \( v_0 \in N_R(\sigma) \), the set \( \tau^\perp (R) \) is a face of \( \sigma^\vee \), and is hence of the form \( \tau^\perp \cap \sigma^\vee \); by linear algebra, we conclude that \( v_0 = \iota(v) \) for suitable \( v \in N_R / \text{span}(\tau) \).

The topology on \( N_R / \text{span}(\tau) \) coincides with the topology of pointwise convergence, thinking of \( N_R / \text{span}(\tau) \) as the space of linear functions on \( \tau^\perp \). It follows that \( \iota|_{N_R / \text{span}(\tau)} \) is a topological embedding.

From this point on we will identify \( \bigsqcup_{\tau \prec \sigma} N_R / \text{span}(\tau) \) with \( N_R(\sigma) \) without mentioning the map \( \iota \). See (3.7) for an example.

**Remark 3.5.**

(i) Later in this section we will give a “local” description of the topology on \( N_R(\sigma) \); cf. (3.22).

(ii) If \( \sigma \) is a pointed cone then \( N_R = N_R(\{ 0 \} ) \) naturally sits inside of \( N_R(\sigma) \) by (3.4). In this case \( N_R \) is dense in \( N_R(\sigma) \).

(iii) Let \( \sigma \) be a cone and let \( \tau \prec \sigma \). Then for \( v \in N_R(\tau) \) and \( u \in \sigma^\vee \subseteq \tau^\vee \), by definition we have \( \langle u, v \rangle_\sigma = \langle u, v \rangle_\tau \) under the natural inclusion \( N_R(\tau) \rightarrow N_R(\sigma) \).
(iv) We mentioned in the course of the proof of (3.4) that \( v \in N_{\mathbb{R}}/\text{span}(\tau) \) if and only if \( v^{-1}(\mathbb{R}) = \tau^{\perp} \cap \sigma^\vee \).

**Definition 3.6.** Let \( \Delta \) be a pointed fan in \( N_{\mathbb{R}} \). The **partial compactification of \( N_{\mathbb{R}} \) with respect to \( \Delta \)** is the space \( N_{\mathbb{R}}(\Delta) \) obtained by gluing the spaces \( N_{\mathbb{R}}(\sigma) \) for \( \sigma \in \Delta \) using the open immersions \( N_{\mathbb{R}}(\tau) \hookrightarrow N_{\mathbb{R}}(\sigma) \) for \( \tau \prec \sigma \).

It follows from (3.4) that there is a canonical bijection
\[
\prod_{\sigma \in \Delta} N_{\mathbb{R}}/\text{span}(\sigma) \xrightarrow{\sim} N_{\mathbb{R}}(\Delta).
\]

Moreover if \( \Delta \) is the fan whose cones are the faces of a single cone \( \sigma \), then \( N_{\mathbb{R}}(\sigma) \) is canonically identified with \( N_{\mathbb{R}}(\Delta) \). See (3.7).

**Example 3.7.** (The affine and projective planes) Let \( \sigma_1 \) be the first quadrant in \( N_{\mathbb{R}} = \mathbb{R}^2 \) (the toric variety associated to \( \sigma_1 \) is isomorphic to the affine plane). The faces of \( \sigma_1 \) are \( \sigma_1 \) itself, \( \tau_1 = \mathbb{R}_{\geq 0}(1, 0) \), \( \tau_2 = \mathbb{R}_{\geq 0}(0, 1) \), and \( \{0\} \). Therefore
\[
N_{\mathbb{R}}(\sigma_1) = N_{\mathbb{R}}(\mathbb{R}^2/\text{span}(\tau_1)) = (\mathbb{R}^2/\text{span}(\tau_2)) \cup (\mathbb{R}^2/\text{span}(\sigma_1)) = \mathbb{R}^2 \cup \{\infty\} = (\mathbb{R} \cup \{\infty\})^2.
\]

Let \( \sigma_2 = \mathbb{R}_{\geq 0}(1, 0) + \mathbb{R}_{\geq 0}(-1, 1) \) and \( \sigma_3 = \mathbb{R}_{\geq 0}(0, 1) + \mathbb{R}_{\geq 0}(-1, 1) \), let \( \tau_3 = \mathbb{R}_{\geq 0}(1, 0) \), and let \( \Delta = \{\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3, \{0\}\} \). Then \( \Delta \) is a complete integral pointed fan (its associated toric variety is the projective plane; cf. (5.10)). By definition (3.6) we have \( N_{\mathbb{R}}(\Delta) = N_{\mathbb{R}}(\sigma_1) \cup N_{\mathbb{R}}(\sigma_2) \cup N_{\mathbb{R}}(\sigma_3) \). See Figure 5 for a picture.

![Figure 5](image-url)

**Figure 5.** A picture of \( N_{\mathbb{R}}(\Delta) \), where \( \Delta \) is the fan of (3.7).

**3.8.** The construction of \( N_{\mathbb{R}}(\Delta) \) is functorial in \( \Delta \), in the following sense. Let \( N_{\mathbb{R}}, N'_{\mathbb{R}} \) be finite-dimensional real vector spaces and let \( \sigma \) (resp. \( \sigma' \)) be a cone in \( N_{\mathbb{R}} \) (resp. \( N'_{\mathbb{R}} \)). Let \( \varphi : N_{\mathbb{R}} \rightarrow N_{\mathbb{R}} \) be a linear map with \( \varphi(\sigma') \subset \sigma \). The dual map \( \varphi^* : M_{\mathbb{R}} \rightarrow M'_{\mathbb{R}} = (N'_{\mathbb{R}})^* \) induces a monoid homomorphism \( \sigma^\vee \rightarrow (\sigma')^\vee \), and hence a continuous map \( \varphi : N_{\mathbb{R}}(\sigma') \rightarrow N_{\mathbb{R}}(\sigma) \) extending \( \varphi \).
Now let $\Delta$ (resp. $\Delta'$) be a pointed fan in $N_R$ (resp. $N'_R$), and let $\varphi : N'_R \to N_R$ be a linear map respecting the fans $\Delta', \Delta$, i.e., such that for every $\sigma' \in \Delta'$ there exists $\sigma \in \Delta$ such that $\varphi(\sigma') \subset \sigma$. Then we can glue the maps $N'_R(\sigma') \to N_R(\sigma)$ to give a continuous map $\varphi : N_R(\Delta') \to N_R(\Delta)$ extending $\varphi$. 

3.8.1. It is clear from the construction and (3.4)(iii) that if $\Delta'$ is a subfan of $\Delta$ (i.e. if every cone in $\Delta'$ is a cone in $\Delta$) then $N_R(\Delta') \to N_R(\Delta)$ is an open immersion.

Remark 3.9. One can show that $N_R(\Delta)$ is compact if and only if $\Delta$ is a complete fan. More generally, if $\varphi : N' \to N$ is a linear map respecting fans $\Delta'$ and $\Delta$ as above, then the extending map $\varphi : N_R(\Delta') \to N_R(\Delta)$ is proper if and only if $\varphi^{-1}(|\Delta|) = |\Delta'|$. This mirrors the situation for toric varieties. See [Pay09a §3].

Definition 3.10. In the case when $\sigma \subset N$ is an integral pointed cone, we define

$$S_\sigma := \sigma^\vee \cap M$$

and

$$N_\Gamma(\sigma) := \text{Hom}(S_\sigma, \Gamma \cup \{-\infty\}) = \{ v \in N_R(\sigma) : v(S_\sigma) \subset \Gamma \cup \{-\infty\} \subset N_R(\sigma) \}$$

with $\Gamma$ as in (2.2). If $\Delta$ is an integral pointed fan, we define

$$N_\Gamma(\Delta) = \bigcup_{\sigma \in \Delta} N_\Gamma(\sigma) \subset N_R(\Delta).$$

As above we have

$$\prod_{\tau < \sigma} N_\Gamma/(\text{span}(\tau) \cap N_\Gamma) \sim N_\Gamma(\sigma) \quad \text{and} \quad \prod_{\sigma \in \Delta} N_\Gamma/(\text{span}(\sigma) \cap N_\Gamma) \sim N_\Gamma(\Delta).$$

The constructions of $N_\Gamma(\sigma)$ and $N_\Gamma(\Delta)$ are functorial with respect to linear maps $\varphi : N'_R \to N_R$ as in (3.3) such that $\varphi(N') \subset N$.

3.11. We proceed with the compactification of a polyhedron in $N_R$. More specifically, we will take the closure of a polyhedron $P$ inside of a partial compactification $N_R(\sigma)$, but this will only make sense when $\sigma$ partially compactifies $N_R$ in the directions in which $P$ is infinite.

Definition 3.12. Let $P \subset N_R$ be a polyhedron. The cone of unbounded directions or recession cone of $P$ is the cone $U(P)$ which is polar dual to the cone

$$U(P)^\vee := \{ u \in M_R : \text{face}_u(P) \neq \emptyset \} = |\mathcal{N}(P)|.$$ 

We say that $P$ is pointed if $U(P)$ is pointed, or equivalently if $P$ does not contain an affine space.

If $P = \bigcap_{i=1}^r \{ v \in N_R : \langle u_i, v \rangle \leq a_i \}$ then $U(P)^\vee = \sum_{i=1}^r R_{\geq 0} u_i$, and $U(P) = \bigcap_{i=1}^r \{ v \in N_R : \langle u_i, v \rangle \leq 0 \}$. It follows that $U(P)^\vee = M_R$ if and only if $P$ is bounded, and that $U(P)$ is integral when $P$ is integral. See [Bar02, §2.16].

Example 3.13. Let $N_R = R^2$ and let $P \subset R^2$ be the polyhedron

$$P = \{(x, y) \in R^2 : x \geq 1, y \geq 1, x + y \geq 3 \}.$$ 

We have

$$U(P)^\vee = R_{\geq 0}(-1, 0) + R_{\geq 0}(0, -1) + R_{\geq 0}(-1, -1) = R_{\geq 0}(-1, 0) + R_{\geq 0}(0, -1)$$

and therefore its cone of unbounded directions is the first quadrant. See Figure 6. Note that $P = \text{conv}\{(1, 2), (2, 1)\} + U(P)$.

The following lemma is standard:

Lemma 3.14. Let $P \subset N_R$ be a pointed polyhedron and let $\sigma = U(P)$.

(i) $U(P) = \{ v' \in N_R : v + R_{\geq 0}v' \subset P \text{ for all (resp. any) } v \in P \}$.

(ii) If $F_b$ denotes the union of the bounded faces of $P$ then

$$P = F_b + \sigma = \{ u_1 + u_2 : u_1 \in F_b, u_2 \in \sigma \}.$$
Figure 6. A polyhedron $P$, the cone $\mathcal{U}(P)^\vee$, and its cone of unbounded directions $\mathcal{U}(P)$.

**Proof.** We will only prove (ii). Write $P = \bigcap_{i=1}^r \{ v \in \mathbb{R}^d : \langle u_i, v \rangle \leq a_i \}$ for some $u_i \in \sigma^\vee$ and $a_i \in \mathbb{R}$. Let $v_1 \in P$ and $v_2 \in \sigma$. For each $i$ we have

$$\langle u_i, v_1 + v_2 \rangle = \langle u_i, v_1 \rangle + \langle u_i, v_2 \rangle \leq a_i$$

since $\langle u_i, v_2 \rangle \leq 0$ by definition. This shows that $P + \sigma \subset P$, so $F_b + \sigma \subset P$.

For the other inclusion, let $v \in P$ be arbitrary, and let $F$ be the unique face of $P$ such that $v$ is contained in the relative interior of $F$. We will prove by induction on $\dim(F)$ that $v \in F_b + \sigma$. If $\dim(F) = 0$ then we are done because $F$ is bounded, so suppose that $\dim(F) > 0$. Let $u_0 \in \sigma^\vee$ be such that $F = \text{face}_{u_0}(P)$. If $u$ is in the interior of $\sigma^\vee$ then $F$ is bounded, and otherwise there exists nonzero $v_0 \in \sigma$ such that $\langle v_0, u \rangle = 0$. Let $a_0 = \max \{ a \in \mathbb{R} : \langle v - av_0 \in P \} \}$ — this is finite because $\langle u_i, v_0 \rangle \neq 0$ for some $i$ — and let $v_1 = v - a_0v_0$. By construction, $v_1$ is in the boundary of $F$, and hence is contained in the relative interior of a face of strictly smaller dimension. This proves that $P = F_b + \sigma$.

We omit the proofs of the following two lemmas, which follow more or less immediately from (3.14).

**Lemma 3.15.** Let $F$ be a face of a pointed polyhedron $P \subset \mathbb{R}^d$. Then $\mathcal{U}(F) \prec \mathcal{U}(P)$.

**Lemma 3.16.** Let $P, P' \subset \mathbb{R}^d$ be pointed polyhedra such that $P \cap P' \neq \emptyset$. Then $\mathcal{U}(P \cap P') = \mathcal{U}(P) \cap \mathcal{U}(P')$.

3.17. Let $P = \bigcap_{i=1}^r \{ v \in \mathbb{R}^d : \langle u_i, v \rangle \leq a_i \}$ be a pointed polyhedron and let $\sigma = \mathcal{U}(P)$. Then we have

$$(3.17.1) \quad P = \{ v \in \mathbb{R}^d : \langle u, v \rangle \leq \max_{v' \in P} \langle u, v' \rangle \text{ for all } u \in \sigma^\vee \} \quad \text{because } u_1, \ldots, u_r \in \sigma^\vee.$$ 

More generally, let $\tau \prec \sigma$ and let $\pi_{\tau} : \mathbb{R}^d \rightarrow \mathbb{R}^d / \text{span}(\tau)$ denote the projection. Then

$$(3.17.2) \quad \pi_{\tau}(P) = \{ v \in \mathbb{R}^d / \text{span}(\tau) : \langle u, v \rangle_{\sigma} \leq \max_{v' \in P} \langle u, v' \rangle \text{ for all } u \in \sigma^\vee \}.$$ 

This can be seen as follows: one inclusion is clear, so suppose that $v \in \mathbb{R}^d / \text{span}(\tau)$ satisfies $\langle u, v \rangle \leq \max_{v' \in P} \langle u, v' \rangle$ for all $u \in \sigma^\vee \cap \tau^\perp$. If $v_1 \in \mathbb{R}^d$ lifts $v$ and satisfies $\langle u_i, v \rangle \leq a_i$ for all $u_i \notin \tau^\perp$ then $v_1 \in P$ by the above, so $v \in \pi_{\tau}(P)$.

**Definition 3.18.** Let $P \subset \mathbb{R}^d$ be a pointed polyhedron and let $\sigma = \mathcal{U}(P)$. The compactification $\overline{P}$ of $P$ is the closure of $P$ in $N_{\mathbb{R}}(\sigma)$. If $P$ is integral $\Gamma$-affine then we set $\overline{P}_\Gamma = \overline{P} \cap N_\Gamma(\sigma)$.

See (3.20) for an example.

3.18.1. Let $P = \bigcap_{i=1}^r \{ v \in \mathbb{R}^d : \langle u_i, v \rangle \leq a_i \}$ be a pointed polyhedron, and define $f : N_{\mathbb{R}}(\sigma) \rightarrow \mathbb{R}^d / \text{span}(\Gamma^\perp)$ by $f(v) = (\langle u_1, v \rangle_{\sigma}, \ldots, \langle u_r, v \rangle_{\sigma})$ as in (3.4). Then $f(\overline{P})$ is a closed subset of the compact space $\prod_{i=1}^r \mathbb{R}_{\leq a_i}$, so $\overline{P}$ is compact.
Proposition 3.19. Let \( P = \bigcap_{i=1}^{r} \{ v \in N_{R} : \langle u, v \rangle \leq a_{i} \} \) be a pointed polyhedron with cone of unbounded directions \( \sigma \). Then
\[
\overline{P} = \prod_{\tau \prec \sigma} \pi_{\tau}(P) = \prod_{\tau \prec \sigma} \{ v \in N_{R} / \text{span}(\tau) : \langle u, v \rangle_{\sigma} \leq \max_{v' \in P} \langle u, v' \rangle \text{ for all } u \in \sigma^{\vee} \}
\]
\[
= \prod_{\tau \prec \sigma} \{ v \in N_{R} / \text{span}(\tau) : \langle u, v \rangle \leq a_{i} \text{ for all } u \in \tau^{\perp} \}
\]
\[
= \{ u : \sigma^{\vee} \to \overline{R} : \langle u, v \rangle_{\sigma} \leq \max_{v' \in P} \langle u, v' \rangle \text{ for all } u \in \sigma^{\vee} \}.
\]

Proof. The second equality was proved in (3.17.1), the third equality follows from the fact that \( \tau^{\perp} \cap \sigma^{\vee} \) is spanned by the \( u_{i} \) contained in \( \tau^{\perp} \), and the last equality is obvious. The set
\[
\{ u : \sigma^{\vee} \to \overline{R} : \langle u, v \rangle_{\sigma} \leq \max_{v' \in P} \langle u, v' \rangle \text{ for all } u \in \sigma^{\vee} \}
\]
is closed since \( \langle u, v \rangle_{\sigma} \leq \max_{v' \in P} \langle u, v' \rangle \) is a closed condition for fixed \( v \), so \( \overline{P} \) is contained in the right-hand side by (3.17.2). Conversely, let \( \tau \prec \sigma \) be a face of positive dimension, let \( v \in N_{R} / \text{span}(\tau) \) be such that \( \langle u, v \rangle \leq \max_{v' \in P} \langle u, v' \rangle \) for all \( u \in \sigma^{\vee} \cap \tau^{\perp} \), and let \( v_{1} \in P \) be a lift of \( v \). Let \( v_{2} \) be in the relative interior of \( \tau \). Then \( \pi_{\tau}(v_{1} + av_{2}) = v \) for all \( a \in \mathbb{R} \), \( v_{1} + av_{2} \in P \) for all \( a \in \mathbb{R}_{\geq 0} \), and \( v_{1} + av_{2} \to v \) as \( a \to \infty \).

Example 3.20. Let \( P \subset \mathbb{R}^{2} \) be the polyhedron of (3.13) and let \( \sigma = \mathcal{U}(P) \), the first quadrant. Then
\[
N_{R}(\sigma) = \mathbb{R}^{2} \amalg (\{ \infty \} \times \mathbb{R}) \amalg (\mathbb{R} \times \{ \infty \}) \amalg (\{ \infty \} \times \{ \infty \})
\]
as in (3.7.1). According to (3.19), under this identification we have
\[
\overline{P} = P \amalg (\{ \infty \} \times [1, \infty)) \amalg ([1, \infty) \times \{ \infty \}) \amalg (\{ \infty \} \times \{ \infty \}).
\]

See Figure 7.

Remark 3.21. Let \( \Delta \) be a pointed fan in \( N_{R} \). Let \( P \subset N \) be a pointed polyhedron, and suppose that its cone of unbounded directions \( \sigma \) is a cone of \( \Delta \). Then \( N_{R}(\sigma) \subset \Delta \) so \( \overline{P} \) is naturally a subspace of \( N_{R}(\Delta) \); as \( \overline{P} \) is compact, \( \overline{P} \) agrees with the closure of \( P \) in \( N_{R}(\Delta) \). If \( F \prec P \) then it follows from (3.15) that \( \overline{F} \subset \overline{P} \subset N_{R}(\Delta) \).

Remark 3.22. Let \( \Delta \) be a pointed fan in \( N_{R} \). A base for the topology of \( N_{R}(\Delta) \) is given by the interiors of the compactifications of the integral \( \Gamma \)-affine polyhedra whose cone of unbounded directions is a cone of \( \Delta \). See [Pay09a, Remark 3.4].

3.23. This is a convenient place to mention the following construction, which will come up later. Let \( \Delta \) be a pointed fan in \( N_{R} \) and fix \( \sigma \in \Delta \). Let \( N_{R}^{\prime} = N_{R} / \text{span}(\sigma) \) and let \( \Delta_{\sigma} \) be the pointed fan in \( N_{R}^{\prime} \).
whose cones are the images of the cones $\tau \in \Delta$ such that $\sigma \prec \tau$. Then
\[
N'_R(\Delta_\sigma) = \prod_{\sigma \prec \tau} N_R / \text{span}(\tau)
\]
and so we have a natural inclusion $N'_R(\Delta_\sigma) \hookrightarrow N_R(\Delta)$.

Let $P \subset N_R$ be a pointed polyhedron with cone of unbounded directions $\sigma' \in \Delta$, and suppose that $\sigma \prec \sigma'$. Let $P' = \pi_\sigma(P) \subset N'_R$, a polyhedron in $N'_R$. It follows immediately from (3.19) that the compactification $\overline{P'}$ of $P'$ inside of $N'_R(\pi_\sigma(\sigma')) \subset N'_R(\Delta_\sigma)$ is equal to $\overline{P} \cap N'_R(\Delta_\sigma)$.

4. A REVIEW OF AFFINOID ALGEBRAS

4.1. In this section we give a brief introduction of the theory of affinoid algebras for the benefit of the reader who is not familiar with the language of rigid analytic spaces. We will only briefly mention the global theory of rigid analytic spaces as it will not play a major role in the sequel. Our main reference for all thing rigid-analytic is [BGR84], although we refer the reader to [ST08a] for an introduction to the subject in the context of tropical geometry.

We fix the following notation for the rest of this paper:

Notation 4.2.

\begin{align*}
K & \quad \text{is a field that is complete with respect to } | \cdot |, \\
\text{val} : K & \rightarrow \mathbb{R} \cup \{\infty\} \quad \text{a nontrivial non-Archimedean valuation} \\
| \cdot | & = \exp(-\text{val}(\cdot)) \quad \text{the associated absolute value} \\
\mathcal{O}_K & \quad \text{is the valuation ring of } K \\
m_K & \subset \mathcal{O}_K \quad \text{is the maximal ideal} \\
k & = \mathcal{O}_K / m_K \quad \text{is the residue field} \\
\Gamma_K & = \text{val}(K^\times) \quad \text{is the value group of } K \\
\Gamma & = \sqrt{\Gamma_K} = \text{val}(\mathbb{K}^\times) \quad \text{is the saturation of the value group.}
\end{align*}

Note that $\Gamma$ is divisible and hence dense in $\mathbb{R}$. The base of the exponential used in the definition of $| \cdot |$ can be any number greater than 1; we will use the natural exponential for concreteness.

4.3. The theory of rigid analytic spaces was invented by Tate in order to give more structure to his non-Archimedean uniformization of elliptic curves with split multiplicative reduction. It closely parallels the theory of complex analytic spaces, in that it exhibits many of the rigidity characteristics of algebraic geometry while carrying a finer, analytic topology. We will try to emphasize the analogy with the theory of varieties over a field.

4.4. Tate algebras. Rigid spaces are modeled on closed subspaces of the $p$-adic closed unit ball (or polydisc)
\[
B^n_K(\mathbb{K}) = \{ (x_1, \ldots, x_n) \in \mathbb{K}^n : |x_i| \leq 1 \text{ for all } i \},
\]
which plays the same role as affine $n$-space in algebraic geometry. (We use the closed unit ball because the ring of analytic functions on a “compact” space is well-behaved; in any case, $B^n_K(\mathbb{K})$ is still open in the $p$-adic topology.) An infinite sum of elements in a complete non-Archimedean field converges if and only if the absolute values of the summands approaches zero, so one might expect that the holomorphic functions converging on this set would correspond to the formal power series $\sum_{\nu} a_{\nu} x^\nu \in K[x_1, \ldots, x_n]$ such that $|a_\nu| \to 0$ as $|\nu| \to \infty$, where $\nu = (\nu_1, \ldots, \nu_n)$ and $|\nu| = \nu_1 + \cdots + \nu_n$. This leads to the definition of the Tate algebra, which plays the same role as a polynomial ring in algebraic geometry.

Definition 4.5. The Tate algebra in $n$ variables is the $K$-algebra
\[
K\langle x_1, \ldots, x_n \rangle = \left\{ \sum_{\nu} a_{\nu} x^\nu \in K[x_1, \ldots, x_n] : |a_\nu| \to 0 \text{ as } |\nu| \to \infty \right\}.
\]

Theorem 4.6. The Tate algebra $T_n = K(x_1, \ldots, x_n)$ satisfies the following properties:
4.7. The \( K \)-algebra homomorphisms from \( K \langle x_1, \ldots, x_n \rangle \) to \( \overline{K} \) are in bijective correspondence with \( B^n_K(\overline{K}) \) via \( f \mapsto (f(x_1), \ldots, f(x_n)) \). Theorem (4.6(ii)) then allows us to view \( K \langle x_1, \ldots, x_n \rangle \) as a function algebra on \( B^n_K(\overline{K}) \). If we set \( B^n_K = \text{Max}(K(x_1, \ldots, x_n)) \), the maximal spectrum of the Tate algebra, then the map
\[
f \mapsto \ker(f) : B^n_K(\overline{K}) \rightarrow B^n_K
\]
is a surjection whose fibers are the \( \text{Gal}(K^{sep}/K) \)-orbits.

**Definition 4.8.** A \( K \)-affinoid algebra is a \( K \)-algebra that is isomorphic to a quotient of a Tate algebra.

The maximal spectrum of an affinoid algebra is therefore a Zariski-closed subspace of a unit ball \( B^n_K \), defined by some ideal \( \mathfrak{a} \subset K \langle x_1, \ldots, x_n \rangle \). (In general, a closed analytic subspace of a rigid space should be thought of as being Zariski-closed.) By (4.6), an affinoid algebra is a Jacobson ring, and therefore a reduced affinoid algebra \( A \) is a function algebra on the space \( \text{Max}(A)(\overline{K}) := \text{Hom}_K(A, \overline{K}) \).

An affinoid algebra is equipped with a canonical semi-norm\(1\) \( \cdot \) \( | \cdot |_{\text{sup}} \), called the supremum semi-norm, defined by
\[
|f|_{\text{sup}} = \sup_{\xi \in \text{Max}(A)} |f(\xi)| = \sup_{\xi \in \text{Max}(A)(\overline{K})} |f(\xi)|.
\]
(Recall that there is a unique absolute value on any finite extension of \( K \) extending \( | \cdot | \).) If \( A = K \langle x_1, \ldots, x_n \rangle \) and \( f = \sum a_n x^n \) then \( |f|_{\text{sup}} = \max |a_n| \); in this case \( | \cdot |_{\text{sup}} \) is called the Gauss norm.

A form of Gauss’ lemma states that \( | \cdot |_{\text{sup}} \) is multiplicative on \( K \langle x_1, \ldots, x_n \rangle \) (i.e. \( |fg|_{\text{sup}} = |f|_{\text{sup}}|g|_{\text{sup}} \) for all \( f, g \in K \langle x_1, \ldots, x_n \rangle \)). In general the supremum semi-norm may not be multiplicative, but it is always power-multiplicative (i.e. \( |f^m|_{\text{sup}} = |f|_{\text{sup}}^m \) for all \( m \geq 0 \)).

**Theorem 4.9.** (Maximum Modulus Principle) Let \( A \) be a \( K \)-affinoid algebra and let \( f \in A \). Then there exists \( \xi \in \text{Max}(A) \) such that \( |f(\xi)| = |f|_{\text{sup}} \). In particular, \( \xi \mapsto |f(\xi)| \) is bounded and attains a maximum value on \( \text{Max}(A) \).

It is clear that \( |f|_{\text{sup}} = 0 \) if and only if \( f \) is nilpotent. If \( A \) has no nilpotents then \( A \) is complete and separated with respect to \( | \cdot |_{\text{sup}} \); see [BGR84] Theorem 6.2.4/1.

**Remark 4.10.** The Tate algebra satisfies the following universal property (analogous to the universal property satisfied by a polynomial ring): if \( A \) is a \( K \)-affinoid algebra (resp any \( K \)-Banach algebra\(2\)), then to give a \( K \)-algebra homomorphism \( f : K \langle x_1, \ldots, x_n \rangle \rightarrow A \) is equivalent to choosing \( a_1, \ldots, a_n \in A \) with \( |a_i|_{\text{sup}} \leq 1 \) (resp. such that \( \{a_i^m\}_{m \geq 0} \) is bounded). That is, there exists a unique homomorphism \( f \) such \( f(x_i) = a_i \); see [BGR84] Propositions 1.4.3/1 and 6.2.3/2).

**Example 4.11.** (Annuli) Let \( r \in K^\times \) and let \( \rho = |r| \). Suppose that \( \rho \leq 1 \). Consider the affinoid algebra
\[
A = K(x, y)/(xy - r).
\]
Let \( X \subset B^n_K \) be its maximal spectrum and let \( p_1 : X \rightarrow B^n_K \) be the projection onto the first factor. Then \( p_1 \) maps \( X(\overline{K}) \) isomorphically onto \( \{ \xi \in B^n_K(\overline{K}) : |\xi| \geq \rho \} \). We call \( X \) the annulus of inner radius \( \rho \) and outer radius 1. This is an example of a Laurent domain; see [4.1.4].

---

1\ A semi-norm \( | \cdot | \) on a ring \( A \) is a function \( A \rightarrow \mathbb{R}_{\geq 0} \) satisfying the ultrametric triangle inequality and such that \( |1| = 1 \) and \( |fg| \leq |f||g| \) for all \( f, g \in A \). A semi-norm is called a norm if \( |f| = 0 \) if and only if \( f = 0 \).

2\ A \( K \)-algebra that is complete and separated with respect to a norm extending the absolute value on \( K \).
4.12. There is a notion of cofiber (tensor) product in the category of $K$-affinoid algebras. It is constructed as a completion of an ordinary tensor product, but may be described more concretely as follows. If $A = K\langle x_1,\ldots, x_n\rangle/a$ and $B = K\langle y_1,\ldots, y_m\rangle/b$ are affinoid algebras then we set

$$A \hat{\otimes}_K B = K\langle x_1,\ldots, x_n, y_1,\ldots, y_m\rangle/(a+b).$$

This $K$-algebra is visibly affinoid, and satisfies the universal property of the cofiber product in the category of $K$-affinoid algebras.

If $A$ is $K$-affinoid then we set

$$A\langle x_1,\ldots, x_n\rangle = A \hat{\otimes}_K K\langle x_1,\ldots, x_n\rangle.$$

4.13. In order to put a sheaf of rings on the maximal spectrum $\text{Max}(A)$ of an affinoid algebra $A$, one has to understand the analogue of a distinguished affine open subset. As these will be quite a bit more general than the complement of the zero locus of a regular function, it is convenient to define an affinoid open subset by universal property:

**Definition.** Let $A$ be an affinoid algebra and let $U \subset \text{Max}(A)$. If there exists a homomorphism of affinoid algebras $f : A \to B$ such that $f^*$ identifies $\text{Max}(B)$ with $U$, and such that a homomorphism $g : A \to C$ taking Max$(C)$ into $U$ extends uniquely to a homomorphism $B \to C$, then we say that $U = \text{Max}(B)$ is a **affinoid subdomain** of $\text{Max}(A)$.

4.14. If the topology on $\text{Max}(A)$ is going to be “analytic”, one would certainly hope that a subset of the form $\{ \xi : |f(\xi)| \leq 1 \}$ would be an affinoid open for any $f \in A$. In fact we will want to consider the more general kind of analytic subsets:

**Definition.** Let $A$ be an affinoid algebra. A **Laurent domain** is a subset of $\text{Max}(A)$ of the form

$$D(f, g^{-1}) = \{ \xi \in \text{Max}(A) : |f_1(\xi)|, \ldots, |f_n(\xi)| \leq 1, |g_1(\xi)|, \ldots, |g_m(\xi)| \geq 1 \}$$

for some $f = f_1, \ldots, f_n, g = g_1, \ldots, g_m \in A$. If $m = 0$ we call $D(f)$ a **Weierstrass domain**. We set

$$A(f, g^{-1}) = A\langle x_1,\ldots, x_n, y_1,\ldots, y_m\rangle/(x_1 - f_1,\ldots, x_n - f_n, y_1 g_1 - 1,\ldots, y_m g_m - 1).$$

The following proposition follows almost immediately from the universal properties of the Tate algebra and the completed tensor product.

**Proposition 4.15.** The natural map $A \to A(f, g^{-1})$ induces a bijection

$$\text{Max}(A(f, g^{-1})) \sim \to D(f, g^{-1}) \subset \text{Max}(A)$$

that exhibits $D(f, g^{-1})$ as an affinoid subdomain of $\text{Max}(A)$.

**Example 4.16.** The annulus of $\text{Max}(A)$ is by definition the Laurent domain $D((x/r)^{-1})$ inside $B_K^1$. For $m \geq 0$ the Laurent domain $D((x^m/r)^{-1}) \subset B_K^1$ is the set $\{ \xi : |\xi| \leq r^{1/m} \}$. For any $\mu \in \{|K^\times|\}$ with $|\mu| < 1$ we can find $m \geq 1$ such that $\mu^m \in |K^\times|$; thus we can define the annulus of inner radius $\mu$ and outer radius 1 as above. One can identify the coordinate ring of this annulus with the algebra

$$\left\{ \sum_{i \in \mathbb{Z}} a_i x^i \mid a_i \to 0 \text{ as } i \to \infty \text{ and } |a_i| |i| \to 0 \text{ as } i \to -\infty \right\}$$

The supremum norm is $\|\sum a_i x^i\|_{\sup} = \sup\{|a_i|, |a_i| |i| : i \in \mathbb{Z}\}$.

Consider the Weierstrass domain in $B_K^1$

$$B_K^1(\mu) := D(x^m/r) = \{ \xi \in B_K^1 : |\xi| \leq \mu \}.$$ 

This is the ball of radius $\mu$; it is defined for every $\mu \in |K^\times|$ with $\mu \leq 1$. The coordinate ring of $B_K^1(\mu)$ is naturally identified with the modified Tate algebra

$$T_{1,\mu} := \left\{ \sum_{i \geq 0} a_i x^i \mid a_i |i| \to 0 \right\},$$
and the supremum norm is \(| \sum a_i x^i |_{\sup} = \sup |a_i| \mu^i |\). In fact, for arbitrary \( \mu \in \overline{K}^\times \) the algebra \( T_{1,\mu} \) is \( K \)-affinoid with supremum norm \(| \sum a_i x^i |_{\sup} = \sup |a_i| \mu^i |\); we define \( B_K^1(\mu) = \text{Sp}(T_{1,\mu}) \) for any \( \mu \in \overline{K}^\times \).

The constructions above extend in an evident manner to define \( n \)-balls \( \prod_{i=1}^n B_K^1(\mu_i) \) and polyannuli of different radii, and to characterize their affinoid algebras and sup norms; see (4.7) and (4.8). There is a caveat however: if \( \rho \in \mathbb{R}_{>0} \) but \( \rho \not\in \overline{K}^\times \) then \( \{ \xi \in B_K: |\xi| \leq \rho \} \) is not an affinoid subdomain of \( B_K^1 \).

4.17. Here we give a brief sketch of the globalization procedure for rigid spaces. Let \( A \) be an affinoid algebra and let \( X = \text{Max}(A) \). A subset \( U \subset X \) is an admissible open subset if it has a set-theoretic covering \( \{U_i\} \) by affinoid subdomains such that for any map of affinoids \( f : A \to B \) with \( f^*(\text{Max}(B)) \subset U \) the cover \( \{(f^*)^{-1}(U_i)\} \) of \( \text{Max}(B) \) has a finite subcover. A set-theoretic covering \( \{U_i\} \) of an admissible open subset \( U \) is an admissible cover provided that for any map of affinoids \( f : A \to B \) such that \( f^*(\text{Max}(B)) \subset U \) the covering \( \{(f^*)^{-1}(U_i)\} \) of \( \text{Max}(B) \) has a refinement consisting of finitely many affinoid subdomains. In particular, any affinoid subdomain is an admissible open, and any cover by finitely many affinoid subdomains is an admissible cover.

The admissible open subsets of \( X \) form a Grothendieck topology whose covers are the admissible covers. Therefore \( X \) has the structure of a \( G \)-topological space, i.e. a set endowed with a Grothendieck topology on a collection of subsets. (The point-set topology generated by the affinoid opens induces the \( p \)-adic topology on \( X \), which is totally disconnected and therefore too fine — we want \( B_K^1 \) to be connected.) The Tate acyclicity theorem roughly states that there is a sheaf of rings \( \mathcal{O}_X \) on \( X \) such that \( \mathcal{O}_X(\text{Max}(B)) = B \) for every (admissible) affinoid open subset \( \text{Max}(B) \subset X \). The locally ringed \( G \)-topological space \( \text{Max}(A) \) is called an affinoid space and is denoted \( \text{Sp}(A) \). A morphism of affinoid spaces is a morphism as locally ringed \( G \)-topological spaces. Any morphism \( \text{Sp}(B) \to \text{Sp}(A) \) arises from a unique homomorphism \( A \to B \).

A rigid-analytic space is a \( G \)-topological space (satisfying some technical hypotheses) which admits an admissible cover by affinoid spaces, and a morphism of rigid analytic spaces is a morphism in the category of locally ringed \( G \)-topological spaces.

Example 4.18. The rigid-analytic open unit ball is the rigid space \( D_K^1 = \bigcup_{\rho \in \overline{K}^\times, |\rho| < 1} B_K^1(\rho) \), where \( B_K^1(\rho) \) is the ball of radius \( \rho \) defined in (4.16). This cover is admissible by the maximum modulus principle. More generally, we define \( D_K^1(\rho) \) for \( \rho \in \overline{K}^\times \) in an evident manner.

Example 4.19. Rigid-analytic affine \( m \)-space is the rigid space \( A_K^{m, \text{an}} = \bigcup_{\rho \in \overline{K}^\times} B_K^m(\rho) \). Again this cover is admissible by the maximum modulus principle. We can define the rigid-analytic projective space \( P_K^{m, \text{an}} \) by gluing \( m + 1 \) copies of \( A_K^{m+1, \text{an}} \), but in fact \( P_K^{m, \text{an}} \) is covered by the \( m + 1 \) closed unit balls \( B_K^{m+1} \subset A_K^{m+1, \text{an}} \) since we can always normalize \([x_0 : \cdots : x_m]\) so that \( \text{max} |x_i| = 1 \).

4.20. There is an analytification functor \( X \mapsto X^{\text{an}} \) from the category of \( K \)-schemes locally of finite type to the category of rigid analytic spaces. This functor respects most notions common to both categories, such as open and closed immersions, finite, proper (see 9.4), and projective morphisms, fiber products, etc. Furthermore the set underlying \( X^{\text{an}} \) is canonically identified with the set of closed points \(|X| \) of \( X \), and the completed local ring of \( X^{\text{an}} \) at \( \xi \in |X| \) agrees with \( \mathcal{O}_X,\xi \). The analytification can be defined by universal property (as is the case over \( \mathcal{O} \)), but can also be described concretely as follows. The analytification of the algebraic affine space \( A_K^m \) is the analytic affine space \( A_K^{m, \text{an}} \) defined above (4.19). If \( X \subset A_K^m \) is the closed subscheme cut out by a collection of polynomials \( f_1, \ldots, f_m \in K[x_1, \ldots, x_n] \) then \( X^{\text{an}} \subset A_K^{m, \text{an}} \) is the closed subspace defined by the same polynomials \( f_1, \ldots, f_m \in \Gamma(A_K^{m, \text{an}}, \mathcal{O}) \). Finally, if \( X \) is an arbitrary locally-finite-type \( K \)-scheme covered by the affine open subsets \( \{U_i\} \) then \( X^{\text{an}} \) is obtained by pasting the analytifications \( U_i^{\text{an}} \).

The analogues of Serre’s GAGA theorems hold in this context [Kie67]. In particular, any projective rigid space (including any proper curve) has a unique algebraization.
5. Kajiwara-Payne extended tropicalizations

5.1. In this section we set our notation regarding toric varieties and review Kajiwara-Payne’s construction of the tropicalization of a toric variety over a non-Archimedean field \[\text{[Kaj08, Pay09a]}\]. We refer the reader to \[\text{[Ful93]}\] for a general reference for toric varieties.

5.2. **Affine toric varieties are associated to integral pointed cones** \(\sigma\) in \(N_{R}\) as follows. Recall (3.10) that \(S_\sigma = \sigma^\vee \cap M\). This is a finitely-generated monoid by Gordan’s Lemma \[\text{[Ful93 Proposition 1.2.1]}\], and \(S_\sigma\) spans \(\sigma^\vee\).

**Notation.** Let \(\sigma \subset N_{R}\) be an integral pointed cone and let \(K[S_\sigma]\) be the semigroup ring associated to \(S_\sigma\). For \(u \in S_\sigma\) we let \(x_u\) denote the corresponding element of \(K[S_\sigma]\), so

\[
K[S_\sigma] = \left\{ \sum_{u \in S_\sigma} a_u x_u : a_u \in K \text{ and } a_u = 0 \text{ for almost all } u \right\}.
\]

The affine toric variety over \(K\) associated to \(\sigma\) is denoted \(X(\sigma) = \text{Spec}(K[S_\sigma])\).

**Definition 5.3.** Let \(\sigma \subset N_{R}\) be an integral pointed cone. We define the tropicalization map

\[
\text{trop} : |X(\sigma)| \to N_1(\sigma) \quad \text{by} \quad \langle u, \text{trop}(\xi) \rangle_\sigma = -\text{val}(x_u^\sigma(\xi))
\]

with the notation in (3.9) and (3.10). We also define

\[
\text{trop} : X(\sigma)(\overline{K}) \to N_1(\sigma) \quad \text{by} \quad \langle u, \text{trop}(\xi) \rangle_\sigma = -\text{val}(x_u^\sigma(\xi))
\]

and we define

\[
\text{trop} : X(\sigma)_{\text{an}} \to N_1(\sigma)
\]

by identifying the set underlying \(X(\sigma)_{\text{an}}\) with \(|X(\sigma)|\).

The above definition makes sense because the residue field \(K(\xi)\) of a closed point \(\xi \in |X(\sigma)|\) is a finite extension of \(K\), and therefore inherits a unique valuation extending the one on \(K\). Note that \(\text{trop} : X(\sigma)(\overline{K}) \to N_1(\sigma)\) agrees with the composition \(X(\sigma)(\overline{K}) \to |X(\sigma)| \to N_1(\sigma)\). See also \[\text{[Pay09a]}\] §3.

**Remark.** It may seem that we have changed our convention regarding the sign of \(\text{trop}(\xi)\) from the definition given in the introduction. However the latter definition rests on a choice of basis for \(M\); choosing an appropriate basis, we recover the tropicalization map of the introduction. See (5.5) below.

**Remark 5.4.** Let \(\sigma \subset N_{R}\) be a pointed cone and let \(\xi \in |X(\sigma)|\). According to (5.3), we have \((u, \text{trop}(\xi))_\sigma \in R\) if and only if \(x_u^\sigma(\xi) \neq 0\) for \(u \in S_\sigma\). Remark (3.5 iv) then implies that the set \(\{u \in S_\sigma : x_u^\sigma(\xi) \neq 0\}\) is equal to \(S_\sigma \cap \tau^\perp\) where \(\tau < \sigma\) is the face such that \(\text{trop}(\xi) \in N_{R}/\text{span}(\tau) \subset N_{R}(\sigma)\).

**Example 5.5.** Let \(e_1, \ldots, e_n\) be a basis for \(M\) and let \(e_1', \ldots, e_n'\) be the dual basis for \(N\). Let \(\sigma = \{0\}\), so \(\sigma^\vee = \mathbb{Z}^n\) and

\[
T = X(\sigma) = \text{Spec}(K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]) \cong \mathbb{G}_m^n
\]

is a torus, where \(x_i := x_i^{-e_i} \in K[M]\). Let \(\xi = (\xi_1, \ldots, \xi_n) \in T(\overline{K})\) and let \(\text{trop}(\xi) = \sum_{i=1}^{n} v_i e_i' \in \mathbb{R}^n\).

According to (5.3) we have

\[
v_i = (e_i, \text{trop}(\xi)) = \text{val}(x_i(\xi)) = \text{val}(\xi_i).
\]

Hence \(\text{trop} : T(\overline{K}) \to \mathbb{R}^n\) is simply

\[
(\xi_1, \ldots, \xi_n) \mapsto (\text{val}(\xi_1), \ldots, \text{val}(\xi_n)).
\]

For general \(\sigma\), the tropicalization map restricted to the dense torus in \(X(\sigma)\) is of the above form: it is simply the vector of valuations of the coordinates of the point.

**Example 5.6.** Choose bases as in (5.5), and let \(\sigma = \sum_{i=1}^{n} R_{\geq 0} e_i'\). Then \(S_\sigma = \sum_{i=1}^{n} Z_{\leq 0} e_i\), so

\[
X(\sigma) = \text{Spec}(K[x_1, \ldots, x_n]) = \mathbb{A}_K^n.
\]
is affine $n$-space. The partial compactification $\bar{N}_R(\sigma)$ is identified with \((\mathbb{R} \cup \{\infty\})^n\) as in (3.7), and the tropicalization map \(\text{trop} : \mathbb{A}^n_R(\mathbb{K}) = \mathbb{K}^n \to (\mathbb{R} \cup \{\infty\})^n\) is given by (5.5.1) again, where now we allow \(\text{val}(\xi)\) to be \(+\infty\) when \(x_i(\xi) = 0\).

5.7. The definition of the tropicalization map is functorial with respect to equivariant morphisms of affine toric varieties, in the following sense. Let \(\varphi : N'_R \to N_R\) be a homomorphism respecting a choice of integral structures and carrying one integral pointed cone \(\sigma'\) into another integral pointed cone \(\sigma\), as in (3.10). Then \(\varphi^* : M'_R \to M'_R\) maps \(S_\sigma\) into \(S_{\sigma'}\), and therefore induces maps \(K[S_\sigma] \to K[S_{\sigma'}]\) and \(X(\sigma') \to X(\sigma)\) making the following diagram commute:

\[
\begin{array}{ccc}
X(\sigma')(\bar{K}) & \xrightarrow{\text{trop}} & X(\sigma')(\bar{N}) \\
\downarrow & & \downarrow \varphi \\
X(\sigma)(\bar{K}) & \xrightarrow{\text{trop}} & X(\sigma)(\bar{N})
\end{array}
\]

**Notation 5.8.** Let \(\Delta\) be an integral pointed fan in \(N_R\). We let \(X(\Delta)\) denote the toric variety obtained by gluing the affine toric varieties \(X(\sigma)\) along the open immersions \(X(\tau) \hookrightarrow X(\sigma)\) for \(\tau < \sigma\).

**Definition 5.9.** Let \(\Delta\) be an integral pointed fan in \(N_R\). We define the tropicalization map

\[
\text{trop} : |X(\Delta)| \to N_\Gamma(\Delta) \quad \text{or} \quad \text{trop} : X(\Delta)(\bar{K}) \to N_\Gamma(\Delta)
\]

by gluing the maps \(\text{trop} : |X(\sigma)| \to N_\Gamma(\sigma)\) for \(\sigma \in \Delta\) using the diagram (5.7.1). We define

\[
\text{trop} : X(\Delta)^{an} \to N_\Gamma(\Delta)
\]

by identifying the set underlying \(X(\Delta)^{an}\) with \(|X(\Delta)|\).

**Example 5.10.** Let \(\Delta\) be the fan of (3.7). The associated toric variety \(X(\Delta)\) is isomorphic to the projective plane, and we can identify \(N_\Gamma(\Delta)\) with

\[
\left((\mathbb{R} \cup \{\infty\})^3 \setminus \{(\infty, \infty, \infty)\}\right)/\mathbb{R}
\]

where \(\mathbb{R}\) acts by translations. The tropicalization map is then given by

\[
\text{trop} \left[\xi_1 : \xi_2 : \xi_3\right] = \left[\text{val}(\xi_1) : \text{val}(\xi_2) : \text{val}(\xi_3)\right]
\]

with the sign conventions as in (5.5). A similar construction works in higher dimensions.

The above definition of the tropicalization map is functorial with respect to equivariant morphisms of toric varieties; cf. (5.7) and (3.8). See [Pay09a, §3] for more details.

5.11. Let \(\Delta\) be an integral pointed fan in \(N_R\), and let \(T = \text{Spec}(K[M]) \cong G^*_m\) be the dense torus inside \(X(\Delta)\). There is a bijective correspondence [Ful93, §3.1]

\[
\sigma \mapsto T_\sigma : \Delta \to \text{the } T\text{-orbits of } |X(\Delta)|,
\]

defined as follows. For \(\sigma \in \Delta\) we define \(T_\sigma = \text{Spec}(K[\sigma^+ \cap M])\), with the inclusion \(T_\sigma \hookrightarrow X(\sigma) \subset X(\Delta)\) being given by the homomorphism

\[
K[S_\sigma] \to K[\sigma^+ \cap M] \quad \text{defined by} \quad x^u \mapsto \begin{cases} x^u & \text{if } u \in \sigma^+ \cap S_\sigma \\ 0 & \text{otherwise.} \end{cases}
\]

In particular, if \(\xi \in |T_\sigma|\) and \(u \in S_\sigma\) then \(x^u(\xi) \neq 0\) if and only if \(u \in \sigma^+ \cap S_\sigma\), so by (5.4) we have \(\text{trop}(\xi) \in N_R/\text{span}(\sigma)\).

In fact more is true: \(T_\sigma\) is a torus with lattice of characters \(\sigma^+ \cap M\), and the dual of \((\sigma^+ \cap M) \otimes_{\mathbb{Z}} \mathbb{R} = \sigma^+\) is \(N_R/\text{span}(\sigma)\), so (replacing \(M_R\) with \(\sigma^+\) and \(N_R\) with \(N_R/\text{span}(\sigma)\)) we have a tropicalization
map \( \text{trop} : |T_\sigma| \to N_\mathbb{R} / \text{span}(\sigma) \). An elementary compatibility check shows that we have a commuta-
tive square

\[
\begin{array}{ccc}
\prod_{\sigma \in \Delta} |T_\sigma| & \xrightarrow{\text{trop}} & \prod_{\sigma \in \Delta} N_\mathbb{R} / \text{span}(\sigma) \\
\downarrow{\sim} & & \downarrow{\sim} \\
|X(\Delta)| & \xrightarrow{\text{trop}} & N_\mathbb{R}(\Delta).
\end{array}
\]

See (5.11.6) for an example, and see [Pay09a] §3 for more details.

5.11.3. The closure \( T_\sigma \) of \( T_\sigma \) in \( X(\Delta) \) is a \( T \)-equivariant closed subvariety, and the map \( \sigma \mapsto T_\sigma \) is a bijection between the cones of \( \Delta \) and the \( T \)-equivariant closed subvarieties of \( X(\Delta) \). The scheme \( T_\sigma \)
is the toric variety with dense torus \( T_\sigma \) given by the fan \( \Delta_\sigma \) in \( N_\mathbb{R}' = N_\mathbb{R} / \text{span}(\sigma) \) defined in (3.23). If \( \tau \) is a cone of \( \Delta \) such that \( \sigma \prec \tau \) then the inclusion \( T_\sigma \cap X(\tau) \hookrightarrow X(\tau) \) is explicitly given by the map

\[
K[S_\tau] \to K[S_\tau \cap \sigma^\perp] \quad \text{defined by } x^u \mapsto \begin{cases} x^u & \text{if } u \in \sigma^\perp \\ 0 & \text{otherwise}. \end{cases}
\]

We have \( N_\mathbb{R}'(\Delta_\sigma) = \bigsqcup_{\sigma \prec \tau} N_\mathbb{R} / \text{span}(\tau) \) and the following square commutes:

\[
\begin{array}{ccc}
\bigsqcup_{\sigma \prec \tau} \prod_{\sigma \prec \tau} N_\mathbb{R} / \text{span}(\tau) & \xrightarrow{\text{trop}} & \prod_{\tau \in \Delta} N_\mathbb{R} / \text{span}(\tau) \\
\downarrow{\sim} & & \downarrow{\sim} \\
|X(\Delta)| & \xrightarrow{\text{trop}} & N_\mathbb{R}(\Delta).
\end{array}
\]

Example 5.11.6. Let \( \sigma_1 \) be the fan of (3.7), so \( X(\sigma) \cong A^2_K \). The decomposition (3.7.1) corresponds to the decomposition of \( |A^2_K| \) into \( G_m^2 \)-orbits

\[
|A^2_K| = |G_m^2| \bigsqcup |\{0\} \times G_m| \bigsqcup |(G_m \times \{0\})| \bigsqcup \{(0,0)\}
\]

under (5.11.2). The invariant subvariety \( \{0\} \times A^1_K \) corresponds to the cone \( \tau_1 \prec \sigma_1 \), and (5.11.5) expresses the compatibility of the tropicalization \( \text{trop} : \{0\} \times A^1_K \to (\infty) \times (\mathbb{R} \cup \{\infty\}) \) with \( \text{trop} : |A^2_K| \to (\mathbb{R} \cup \{\infty\})^2 \).

6. Polyhedral subdomains of toric varieties

6.1. In this section we introduce a class of admissible affinoid open subdomains of toric varieties
which correspond to polyhedral data inside its tropicalization. These so-called polyhedral subdomains
are generalizations of the polytopal subdomains of affinoid algebras introduced in [EKLO06] §3 and studied in [Gub07b] §4. They enable a local study of the tropicalization of a subvariety of a torus.

Definition 6.2. Let \( P \subset N_\mathbb{R} \) be an integral \( \Gamma \)-affine pointed polyhedron with cone of unbounded directions \( \sigma = u(P) \). The polyhedral subdomain associated to \( P \) is the set \( U_P := \text{trop}^{-1}(P) \subset X(\sigma)^{an} \).

Remark 6.2.1. We will show (6.9) that \( U_P \) is an affinoid open subdomain. However for this to be true it is necessary that \( P \) be integral \( \Gamma \)-affine: see (6.8) and the remark at the end of (4.16).

Remark 6.2.2. If \( P \) is a polytope in \( N_\mathbb{R} \) then \( U_P \) is a polytopal subdomain as defined in [EKLO06] §3 and [Gub07b] Proposition 4.4]. More accurately, Gubler’s polytopal subdomain \( U_P \) is the Berkovich space associated to the affine subdomain \( U_P \) (see (6.2)). We choose to work with classical rigid spaces as opposed to Berkovich spaces simply because rigid spaces are more accessible and they suffice for our purposes.

The subset \( U_P \) of \( X(\sigma)^{an} \) is in fact an affinoid subdomain whose coordinate ring is the following affinoid algebra (see (6.9).
Definition 6.3. Let $P \subset \mathbb{N}_R$ be an integral $\Gamma$-affine pointed polyhedron with cone of unbounded directions $\sigma = \mathcal{U}(P)$. We define

$$K\langle UP \rangle = \left\{ \sum_{u \in S_\sigma} a_u x^u : a_u \in K, \text{val}(a_u) - \langle u, v \rangle \to \infty \text{ for all } v \in P \right\}$$

where the convergence (as always) is taken on the complements of finite subsets of $S_\sigma$. If $A$ is any $K$-affinoid algebra we set $A(UP) = A \otimes_K K\langle UP \rangle$. For $f = \sum a_u x^u \in K\langle UP \rangle$ we define

$$|f|_p = \sup_{u \in S_\sigma, v \in P} |a_u| \exp(\langle u, v \rangle).$$

Remark 6.4. Let $\xi \in |UP|$ and let $v = \text{trop}(\xi) \in \overline{\mathcal{P}}$. Then $\text{val}(x^\nu(\xi))$ is by definition $-\langle u, v \rangle_\sigma$, so if $f = \sum a_u x^u \in K\langle UP \rangle$ then $|a_u x^u(\xi)| \to 0$. In other words, $K\langle UP \rangle$ is precisely the ring of power series that appear to converge on all points of $UP$. This is made precise in (6.9).

Remark 6.5. Let $u \in S_\sigma$ and let face$_u(P) \subset P$ be the associated face. By definition of face$_u(P)$, for any $v \in \text{face}_u(P)$ we have $\langle u, v \rangle = \sup_{v' \in \mathcal{P}} \langle u, v' \rangle$. Since any face contains a vertex, it follows that $f = \sum a_u x^u$ is in $K\langle UP \rangle$ if and only if $\text{val}(a_u) - \langle u, v \rangle \to \infty$ for all $v \in \text{vert}(P)$. Moreover, by (6.19) for any $u \in S_\sigma$ the function $v \mapsto |a_u| \exp(\langle u, v \rangle_\sigma)$ on $\overline{\mathcal{P}}$ takes its maximum on a vertex of $P$. Therefore

(6.5.1) $$|f|_P = \sup_{u \in S_\sigma} |a_u| \exp(\langle u, v \rangle) = \sup_{u \in S_\sigma, v \in \overline{\mathcal{P}}} |a_u| \exp(\langle u, v \rangle_\sigma) < \infty.$$ 

Remark 6.6. The function $| \cdot |_P$ defines a $K$-algebra norm (see footnote 1) on $K[S_\sigma]$ such that $K\langle UP \rangle$ is the completion of $K[S_\sigma]$ with respect to this norm. In other words, $(K\langle UP \rangle, | \cdot |_P)$ is a $K$-Banach algebra.

Example 6.7. Choose bases $e_1, \ldots, e_n$ for $M$ and $e'_1, \ldots, e'_n$ for $N$ and let set $x_i = x^{-e_i} \in K[M]$ as in (5.5). Let $P = [0, \infty)^n \subset \mathbb{N}_R \cong \mathbb{R}^n$. Then $\sigma := \mathcal{U}(P) = P, S_\sigma = \mathbb{Z}_{\geq 0}^n, \overline{\mathcal{P}} = [0, \infty)^n \subset \mathbb{N}_R(\sigma)$, and $X(\sigma) = \mathbb{A}^n_K$ as in (5.6) and (3.20). Hence $U_P = \text{trop}^{-1}(\overline{\mathcal{P}}) = \{(\xi_1, \ldots, \xi_n) \in |K^n| : \text{val}(\xi_i) \geq 0\} = \mathbb{B}^n_K$. This agrees with the fact that

$$K\langle UP \rangle = \left\{ \sum_{v \in \mathbb{Z}_{\geq 0}^n} a_v x^v : |a_v| \to 0 \right\} = K\langle x_1, \ldots, x_n \rangle$$

is a Tate algebra. More generally, if we take $P = \prod_{i=1}^n [r_i, \infty)$ for $r_1, \ldots, r_n \in \Gamma$ then $U_{P'} = \text{trop}^{-1} \left( \prod_{i=1}^n [r_i, \infty) \right) = \prod_{i=1}^n \mathbb{B}_K(\exp(-r_i)) \subset \mathbb{A}_K^n$ and

$$K\langle UP \rangle = \left\{ \sum_{v \in \mathbb{Z}_{\geq 0}^n} a_v x^v : |a_v| \exp(r_1 v_1 + \cdots + r_n v_n) \to 0 \right\}.$$ 

See (4.16).

Example 6.8. With the notation in (6.7), let $r_1, \ldots, r_n, s_1, \ldots, s_n \in \Gamma$ with $r_i \leq s_i$ and let $P = \prod_{i=1}^n [r_i, s_i]$. This $P$ is a polytope, so $\mathcal{U}(P) = \{0\}, S_\sigma = M$, and $\overline{\mathcal{P}} = P$. The polytopal subdomain $U_P = \text{trop}^{-1}(P)$ is the polyannulus \{$(\xi_1, \ldots, \xi_n) \in \mathbb{G}^n_m : r_i \leq \text{val}(\xi_i) \leq s_i$\}. The associated affinoid algebra is

$$K\langle UP \rangle = \left\{ \sum_{v \in \mathbb{Z}_{\geq 0}^n} a_v x^v : |a_v| \mu^v \to 0 \text{ for all } \mu \in \prod_{i=1}^n (\exp(-r_i), \exp(-s_i)) \right\}$$

by (6.5). See (4.16).

The following proposition is due to Einsiedler, Kapranov, and Lind [EKLO6 Proposition 3.1.8] and also to Gubler [Gub07a Proposition 4.1] when $P$ is a polytope. The general case is more difficult since $U_P$ is not a Laurent domain in an easily identifiable affinoid subdomain when $P$ is unbounded.
Proposition 6.9. Let \( P \subseteq \mathbb{N}_\Gamma \) be an integral \( \Gamma \)-affine pointed polyhedron and let \( \sigma \) be its cone of unbounded directions.

(i) The ring \( K\langle U_P \rangle \) is a \( K \)-affine algebra.

(ii) The inclusion \( K[S_\sigma] \hookrightarrow K\langle U_P \rangle \) induces an open immersion \( \text{Sp}(K\langle U_P \rangle) \hookrightarrow X(\sigma)^{an} \), and

(iii) the image of this open immersion is equal to \( U_P \). In particular, \( U_P \) is an admissible affine open subset of \( X(\sigma)^{an} \).

(iv) The supremum norm on \( K\langle U_P \rangle \) agrees with \( | \cdot |_P \), i.e., for \( f \in K\langle U_P \rangle \) we have

\[
|f|_P = \sup_{\xi \in [U_P]} |f(\xi)| = \max_{\xi \in [U_P]} |f(\xi)|.
\]

(v) The ring \( K\langle U_P \rangle \) is a Cohen-Macaulay ring of dimension \( g \).

**Proof.** Write \( P = \bigcap_{i=1}^r \{ v \in \mathbb{N}_\Gamma : \langle u_i, v \rangle \leq b_i \} \) where \( b_i \in \Gamma \) and \( \exp(b_i) = |x^{u_i}|_P \), and assume that \( u_1, \ldots, u_r \) generate \( S_\sigma \). Let \( \varphi : \mathbb{Z}_p^r \to \mathcal{O}_{S_\sigma} \) be the map \( (\nu_1, \ldots, \nu_r) \to \sum_{i=1}^r \nu_i u_i \); this induces a surjective map \( \varphi : [y_1, \ldots, y_r] \to K[S_\sigma] \) given by \( \varphi(y^\nu) = x^\varphi(\nu) \). We identify \( X(\sigma) \) with the image of the associated closed immersion \( X(\sigma) \to \mathcal{A}_K^r \). Letting \( \beta_i = \exp(b_i) \), we have \( U_P = X(\sigma)^{an} \cap \prod_{i=1}^r B_k^1(\beta_i) \) because \( \xi \in [U_P] \) and if only if

\[
\text{val}(y_i(\xi)) = \text{val}(x^{u_i}(\xi)) = -\langle u_i, \text{trop}(\xi) \rangle_{\sigma} \geq -b_i
\]

(see [6.7]). This proves that \( U_P \) is an affine subdomain of \( X(\sigma)^{an} \), and furthermore that \( U_P \) is a closed subspace of \( \prod_{i=1}^r B^1_k(\beta_i) \). Let \( b = (b_1, \ldots, b_r) \) and let

\[
T_{r,b} = \{ \sum_{i=1}^r a_i y^\nu \in K[y_1, \ldots, y_r] : \text{val}(a_i) - \nu \cdot b - \infty \},
\]

so \( T_{r,b} \) is an affine algebra with supremum norm \( |\sum a_i y^\nu|_{sup} = \max |a_i| |\beta_i| \), and \( \text{Sp}(T_{r,b}) = \prod_{i=1}^r B^1_k(\beta_i) \subset \mathcal{A}_K^r \) (see [6.7] and [BGR84 §6.1.5]). The ideal defining \( U_P \subset \prod_{i=1}^r B^1_k(\beta_i) \) is the extension of \( a = \ker(\varphi) \); let \( A = T_{r,b}/aT_{r,b} \), so \( U_P = \text{Sp}(A) \). Since \( |x^{u_i}|_P = \beta_i^\nu \) for all \( m \geq 0 \), the homomorphism \( \varphi \) extends uniquely to a homomorphism \( \varphi : T_{r,b} \to K\langle U_P \rangle \) (see [4.10]). This homomorphism kills \( a \) and therefore descends to \( \overline{\varphi} : A \to K\langle U_P \rangle \). We claim that \( \overline{\varphi} \) is an isomorphism.

First we show that \( \overline{\varphi} \) is injective, i.e. \( \ker(\varphi) \subset aT_{r,b} \). Let \( f = \sum a_i y^\nu \in \ker(\varphi) \); so for \( u \in S_\sigma \), we have \( \sum_{\nu \in \varphi^{-1}(u)} a_i |a_i| = 0 \) (note \( \lim_{\nu \in \varphi^{-1}(u)} \langle \nu, a_i \rangle = 0 \)). Setting \( f^u = \sum_{\nu \in \varphi^{-1}(u)} a_i y^\nu |a_i| \) we have \( f = \sum_{u \in S_\sigma} f^u \varphi(\nu) \) and \( \varphi(\nu) = 0 \); since every ideal in \( T_{r,b} \) is closed [BGR84 Proposition 6.1.1/3] it suffices to show that \( f^u \in aT_{r,b} \) for all \( u \). Thus we may assume that \( f = f^u \) for some \( u \in S_\sigma \). The sum \( f = \sum_{\nu \in \varphi^{-1}(u)} a_{\nu} (y^\nu - y^\nu^0) \) converges for fixed \( \nu_0 \in \varphi^{-1}(u) \), so since \( y^\nu - y^\nu^0 \in a \) for all \( \nu \), we have \( f \in aT_{r,b} \) (again since \( aT_{r,b} \) is closed).

Therefore \( A \subseteq K\langle U_P \rangle \). Next we claim that \( | \cdot |_P \) restricts to the supremum norm \( | \cdot |_{sup} \) on \( A \). For any vertex \( v \) of \( P \) the supremum norm on

\[
K\langle U_P \rangle = \{ \sum_{u \in M} a_u x^u : \text{val}(a_u) - \langle u, v \rangle \to \infty \}
\]

is \( |\sum a_u x^u|_v = \sup_{u \in S_\sigma} |a_u| \exp(\langle u, v \rangle) \) by [Gub07b Proposition 4.1] or using the fact that \( U_P = \text{Sp}(K\langle U_P \rangle) \) is a polyannulus; see [4.16] and [6.8]. Since \( U_P \) is an affinoid subdomain of \( U_P \), for \( f \in K\langle U_P \rangle \) we have

\[
|f|_P = \sup_{\xi \in [U_P]} |f(\xi)| = \max_{\xi \in [U_P]} |f(\xi)| = |f|_P
\]

where the last equality holds by [6.5.1]. To prove the inequality \( |f|_P \leq |f|_P \) we must show that \( |f(\xi)| \leq |f|_P \) for all \( \xi \in [U_P] \). For \( f = \sum_{u \in S_\sigma} a_u x^u \in K\langle U_P \rangle \) we have

\[
|f(\xi)| \leq |\sum_{u \in S_\sigma} a_u x^u(\xi)| = \sup_{u \in S_\sigma} |a_u| \exp(\langle u, \text{trop}(\xi) \rangle_{\sigma}) \leq |f|_P,
\]

where the last inequality comes from [6.5.1].
The reduced affinoid algebra $A$ is complete and separated with respect to $|·|_{\text{sup}} = |·|_p$ by Theorem 6.2.4/1. But $A$ contains $K[S_r]$ which is dense in $K(U_P)$ as noted in (6.6), so $A = K(U_P)$. This proves (i)–(iv). By Hochster’s Theorem [Cox00, Theorem 2.1], $X(\sigma)$ is Cohen-Macaulay of dimension $g$. Assertion (v) follows because the completed local rings of $X(\sigma)$ and $X(\sigma)^{an}$ agree; see [Con99, Appendix A] for details.

Remark 6.10. It follows from the proof of (6.9) that if $u_1, \ldots, u_r$ is a set of generators for $S_r$ such that $P = \bigcap_{i=1}^r \{ v \in N_R : (u_i, v) \leq b_i \}$ and $\beta_i := \exp(b_i) = |x^{u_i}|_p$ then we have a closed immersion

$$U_P \hookrightarrow B_{1_K}(\beta_1) \times \cdots \times B_{1_K}(\beta_r)$$

with the parameter on $B_{1_K}(\beta_i)$ mapping to $x^{u_i} \in K(U_P)$.

6.11. Let $P$ be an integral $\Gamma$-affine pointed polyhedron in $N_R$. The tropicalization map $\text{trop} : X(\sigma)^{an} \to N_I(\sigma)$ restricts to a surjective map $\text{trop} : U_P \to \overline{\Gamma}$. If $\Delta$ is an integral pointed fan containing $\sigma = \mathcal{U}(P)$ then $X(\sigma) \subset X(\Delta)$ and hence we may identify $U_P$ with the admissible affinoid open subset $\text{trop}^{-1}(\overline{\Gamma})$ in $X(\Delta)^{an}$.

7. Tropicalizations of Embedded Subspaces

7.1. In this section we define the tropicalizations of analytic and algebraic subspaces of toric varieties. The definitions are self-contained and illustrated by some examples, but the reader may want to consult [Gat06], for instance, for an introduction to the subject of tropical geometry.

Definition 7.2. Fix an integral $\Gamma$-affine pointed polyhedron $P \subset N_R$ with cone of unbounded directions $\sigma$, and let $Y \subset U_P$ be the closed analytic subspace defined by an ideal $a \subset K(U_P)$. Define

$$\text{Trop}_Y(Y) = \text{trop}(Y) \subset \overline{\Gamma},$$

where $\text{trop} : U_P \to \overline{\Gamma}$ is the tropicalization map (5.3), and let $\text{Trop}(Y) \subset \overline{\Gamma}$ be the closure of $\text{Trop}_Y(Y)$. The set $\text{Trop}(Y)$ is called the tropicalization of $Y$ (as a subspace of $U_P$), and the map $\text{trop} = \text{trop}|_Y : |Y| \to \text{Trop}(Y)$ is again called the tropicalization map. If the ambient space is not clear from context we will write

$$\text{Trop}_Y(Y, \overline{\Gamma}) \quad \text{and} \quad \text{Trop}(Y, \overline{\Gamma}).$$

Remark 7.3. It is more natural to define $\text{Trop}(Y)$ as the image of the Berkovich analytic space $Y^{\text{berk}}$ associated to $Y$ under the natural map $\text{trop} : Y^{\text{berk}} \to N_R$, as in [Gub07b §5] or [Dra08, Definition 4.1]. This approach has several advantages: for instance, there is no need to take closures, the tropicalization inherits topological properties of the Berkovich space (e.g. connectedness), and there is no problem in the case of a trivial valuation. We have chosen the above definition simply in order to avoid discussing Berkovich spaces.

7.3.1. With the above definition it is clear that the tropicalization satisfies the following functoriality property: let $\varphi : N' \to N$ be a homomorphism carrying an integral $\Gamma$-affine pointed polyhedron $P' \subset N'_R$ into another $P \subset N_R$, so $\varphi$ extends to a map $\varphi : N'_R(\mathcal{U}(P')) \to N_R(\mathcal{U}(P))$ taking $\overline{\Gamma}$ into $\overline{\Gamma}$. If $Y' \subset U_{P'}$ and $Y \subset U_P$ are analytic subvarieties such that the induced map $U_{P'} \to U_P$ takes $Y'$ into $Y$, then $Y(\text{Trop}(Y')) \subset \text{Trop}(Y)$. For example, if $N' = N$, $P' \subset P$, and $Y' = Y \cap U_{P'}$, then $\text{Trop}(Y') = \text{Trop}(Y) \cap \overline{\Gamma}$.

It is also clear that the definition of $\text{Trop}(Y)$ is insensitive to finite extension of the base field $K$.

Remark 7.4. The definition of the tropicalization given above agrees with Gubler’s tropicalization [Gub07b]. The point of this section is to show that $\text{Trop}(Y)$ is determined by the valuations of the co-efficients of the power series vanishing on $Y$, thus showing that $\text{Trop}(Y)$ can be effectively calculated and (in certain cases anyway) that it is a well-behaved convex-geometric object. See (7.3).

Example 7.5. In order to illustrate (7.4), we begin with the simplest example. Let $N = M = \mathbb{Z}$ and let $P = [0, \infty)$, so $U_P = B_{1_K}$ as in (6.7). Let $x = x^{(c-1)} \in K[M]$, the character corresponding to $-1 \in M,$
so the coordinate ring of $U_P$ is $K(x)$ with the conventions in (6.7). Let $f = \sum_{u=0}^{\infty} a_u x^u \in K(x)$ be nonzero and let $Y = V(f) \subset B_K$ be the subspace defined by $f$. Then $\text{Trop}(Y) \subset \overline{P} = [0, \infty]$, and $v \in \text{Trop}(Y)$ if and only if there exists $\xi \in \overline{K}$ such that $f(\xi) = 0$ and $\text{val}(\xi) = v$. For a particular choice of $\xi \in \overline{K}$, if there were some $u$ such that $\text{val}(a_u \xi^u) < \text{val}(a_w \xi^w)$ for all $w \neq u$ then by the ultrametric inequality, $\text{val}(f(\xi)) = \text{val}(a_u \xi^u)$, so $f(\xi) \neq 0$. Writing $\text{val}(\xi) = v$ and $\text{val}(\xi') = v'$, this says that if there exists $u \geq 0$ such that $\text{val}(a_u) + uv < \text{val}(a_w) + u'v$ for all $w \neq u$ then $v \notin \text{Trop}(f)$. In other words, a necessary condition for $v \in \text{Trop}(f)$ is that there must exist at least two numbers $u, u'$ such that $\text{val}(a_u) + uv = \text{val}(a_w) + u'v$ for all $u' \geq 0$. (By the theorem of the Newton polygon, or by (7.9) below, this condition is also sufficient. See (7.10).

**7.6.** Let $f = \sum_{u \in S_{\sigma}} a_u x^u \in K(U_P)$ be nonzero and let $\tau \prec \sigma$. Define the *height graph of $f$ with respect to $\tau$* to be

$$H(f, \tau) = \{(u, \text{val}(a_u)) : u \in S_{\sigma} \cap \tau^\perp, a_u \neq 0\} \subset (S_{\sigma} \cap \tau^\perp) \times \mathbb{R}.$$ 

For $v \in N_{\mathbb{R}}/\text{span}(\tau)$ let

$$(7.6.1) \quad \text{vert}_v(f) = \text{minset}((-v, 1), \ H(f, \tau)) \subset H(f, \tau),$$

where we regard $(N_{\mathbb{R}}/\text{span}(\tau)) \times \mathbb{R}$ as a space of linear functionals on $\tau^\perp \times \mathbb{R}$. This is a nonempty finite set by the definition of $K(U_P)$. Define the *initial form of $f$ with respect to $v \in N_{\mathbb{R}}/\text{span}(\tau)$* to be

$$\text{in}_v(f) = \sum_{(u, \text{val}(a_u)) \in \text{vert}_v(f)} a_u x^u.$$ 

In other words, $\text{in}_v(f)$ is the (finite) sum of those monomials $a_u x^u$ such that $u \in S_{\sigma} \cap \tau^\perp$ and

$$(7.6.2) \quad \text{val}(a_u) - \langle u, v \rangle = \min\{\text{val}(a_w) - \langle u', v \rangle : u' \in S_{\sigma} \cap \tau^\perp\}.$$ 

**Example 7.7.** Continuing with (7.5), we have $H(f, \{0\}) = \{(-u, \text{val}(a_u)) : a_u \neq 0\} \subset \mathbb{Z}_{\geq 0} \times \mathbb{R}$, and for $v \in [0, \infty)$ we have

$$\text{vert}_v(f) = \{(-u, \text{val}(a_u)) : \text{val}(a_u) + uv \text{ is minimal among } \{\text{val}(a_w) + u'v : u' \geq 0\}\}.$$ 

Hence by the reasoning in (7.3), if $v \in \text{Trop}(Y)$ then $\# \text{vert}_v(f) \geq 2$, or equivalently $\text{in}_v(f)$ is not a monomial. This is true in general:

**7.8.** Let $\xi \in U_P$, let $v = \text{trop}(\xi)$, and suppose that $v \in N_{\mathbb{R}}/\text{span}(\tau)$, i.e. that $\xi \in |T_{\tau}|$ (5.11). For $u \in S_{\sigma} \cap \tau^\perp$ and $a_u \in K$ we have

$$\text{val}(a_u x^u(\xi)) = \text{val}(a_u) + \text{val}(x^u(\xi)) = \text{val}(a_u) - \langle u, v \rangle,$$

and for $u \in S_{\sigma}$ with $u \notin \tau^\perp$ we have $x^u(\xi) = 0$ by (5.4). Therefore, the initial form $\text{in}_v(f)$ is the sum of those monomials $a_u x^u$ with minimal valuation when evaluated on $\xi$. If $\text{in}_v(f) = a_u x^u$ is a monomial then $\text{val}(f(\xi)) = \text{val}(a_u x^u(\xi)) \neq 0$ by the ultrametric triangle inequality, so $f(\xi) \neq 0$. Therefore, if $f(\xi) = 0$ then $\text{in}_v(f)$ is not a monomial. It is a fundamental fact that in an appropriate sense, the preceding condition is sufficient for there to exist a zero $\xi$ of $f$ with $\text{trop}(\xi) = v$.

**Theorem 7.9.** Let $P \subset N_{\mathbb{R}}$ be an integral $\Gamma$-affine pointed polyhedron with cone of unbounded directions $\sigma$ and let $Y \subset U_P$ be the closed analytic subspace defined by an ideal $a \subset K(U_P)$. Then

(i) $\text{Trop}(Y) = \{v \in \overline{P} : \text{in}_v(f) \text{ is not a monomial for any } f \in a\}.$

(ii) $\text{Trop}_\tau(Y) = \text{Trop}(Y) \cap \overline{T}_{\tau}.$

(iii) For $\tau \prec \sigma$ we have $\text{Trop}(Y) \cap (N_{\mathbb{R}}/\text{span}(\tau)) = \text{Trop}(Y \cap T_{\tau}^{\text{un}})$.

**7.9.1.** Part (iii) requires some explanation. Recall (5.11.3) that $T_{\tau}$ is the affine toric variety defined by the image $\sigma'$ of $\sigma$ in $N_{\mathbb{R}} = N_{\mathbb{R}}/\text{span}(\tau)$, and that $T_{\tau} \cap N_{\mathbb{R}}'(\sigma')$ is the compactification of the integral $\Gamma$-affine pointed polyhedron $P' = P \cap N_{\mathbb{R}}$ by (3.23). Therefore we may consider $Y \cap T_{\tau}^{\text{un}}$ as a closed analytic subspace of $U_{P'} = U_P \cap T_{\tau}^{\text{un}}$, and consider its tropicalization inside $N_{\mathbb{R}}'(\sigma') \subset N_{\mathbb{R}}(\sigma)$. The statement of (iii) is thus an important compatibility that allows us to compute tropicalizations inside toric varieties by reducing to the case of a torus. See [Pay09a, Corollary 3.8].
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Remark 7.9.2. Gubler has pointed out to us that (7.9,i) can be strengthened to show that the local structure of \( \text{Trop}(Y) \) at a point \( v \in P^c \) agrees with the tropicalization of the initial degeneration of \( Y \) at \( v \), as in the algebraic case.

Proof of (7.9). Let \( C = \{ v \in P : \text{in}_v(f) \text{ is not a monomial for any } f \in A \} \). Since the condition for \( \text{in}_v(f) \) not to be monomial is a closed condition for fixed \( f \), and since \( \text{Trop}(Y) \subset C \) by (7.8), we have \( \text{Trop}(Y) \subset C \). Since \( Q \subset \Gamma \) and \( C \) is defined by equations of the form (7.6.2), the closure of \( C \cap \mathcal{T}_f \) is \( C \). Hence it suffices to show that if

\[
(*) \quad v \in \mathcal{T}_f \quad \text{is such that } \text{in}_v(f) \text{ is not a monomial for any } f \in A
\]

then \( v \in \text{Trop}_P(Y) \). Let \( v \) satisfy (*) and suppose that \( v \in P \) (i.e., \( v \in N_\mathbb{R} \)). After possibly making a finite extension of the ground field, we may translate the problem by \(-v\) to assume that \( v = 0 \). Let \( A \) be the ring \( K(U_{(0)}) = \{ \sum_{u \in M} a_u x^u : |a_u| \to 0 \text{ and } | \cdot |_0 \text{ be the supremum norm on } A, \text{ so } |\sum a_u x^u|_0 = \max |a_u| \}. \) To show \( 0 \in \text{Trop}_P(Y) \), we must show that \( U_{(0)} \cap Y \neq \emptyset, \) i.e., that \( a \) does not generate the unit ideal in \( A \). Suppose to the contrary that \( 1 \in aA \). Then there exist \( f_1, \ldots, f_r \in A \) and \( g_1, \ldots, g_r \in A \) such that \( 1 = \sum_{i=1}^r f_i g_i \). Since \( A \) is the completion of \( K[M] \) under \( | \cdot |_0 \), there exist sequences \( \{ f_{i,j} \}_{j \geq 1} \subset K[M] \) such that \( \lim_{j \to \infty} f_{i,j} = f_i \). Since \( K[M] \subset K(U_P) \) we have \( h_j = \sum_{i=1}^r f_{i,j} g_i \in A \) and \( \lim_{j \to \infty} h_j = 1 \), the limit always taken with respect to \( | \cdot |_0 \). Writing \( h_j = \sum_{u \in S} a_{j,u} x^u \) we have

\[
|h_j - 1|_0 = \max\{|a_{j,0} - 1|, |a_{j,u}| : u \in S \setminus \{0\}\}.
\]

Therefore \( \text{val}(a_{j,0}) = 0 \) for \( j \gg 0 \) and \( \text{val}(a_{j,u}) \to \infty \) uniformly for \( u \neq 0, \) so \( \text{in}_v(h_j) = a_{j,0} \) for \( j \gg 0 \). But \( h_j \in A \) and \( \text{in}_0(h_j) \) is a monomial, a contradiction.

Now suppose that \( v \in \mathcal{T}_f \cap (N_\mathbb{R} / \text{span}(\tau)) \) for some \( \tau \prec \sigma \). Let \( N'_\mathbb{R} = N_\mathbb{R} / \text{span}(\tau) \) and \( P' = P \cap N'_\mathbb{R}, \) and let \( \sigma' \) be the image of \( \sigma \) in \( N'_\mathbb{R} \), so \( \mathcal{T}_f = \mathcal{T}_f \cap (N_\mathbb{R} / \text{span}(\sigma')) \) as in (7.9,1). The inclusion \( U_{P'} \to U_P \) corresponds to the surjection \( K(U_P) \to K(U_{P'}) \) defined using the rule (6.11.4). For \( f \in K(U_P) \) let \( f' \) be its image under this map. By construction, we have \( \text{in}_v(f) = \text{in}_v(f') \). Therefore, the above argument (as applied to \( N'_\mathbb{R}, P', \) and \( Y \cap U_{P'} \)) shows that there exists \( \xi \in Y \cap U_{P'} \) such that \( \text{val}(\xi) = v \).

See (7.13) for some remarks on the above proof.

Example 7.10. In this example we explain how (7.9) implies a large part of the theorem of the Newton polygon. Let \( f = \sum_{u=0}^\infty a_u x^u \in K(x) \) as in (7.5), where \( x = x^{(-1)} \) still. By definition the Newton polygon \( \text{NP}(f) \) is the lower convex hull of \( \{(u, \text{val}(a_u)) : a_u \neq 0\} \). In order to maintain our sign conventions we let \( \text{NP}'(f) \) be the lower convex hull of \( H(f, \{0\}) \); this is the Newton polygon of \( f \) flipped over the \( y \)-axis. It is an elementary exercise to show that a line segment \( \text{conv}\{(-u, \text{val}(a_u)), (-u', \text{val}(a_{u'}))\} \) is contained in \( \text{NP}'(f) \) if and only if there exists \( v \geq 0 \) such that

\[
\{(-u, \text{val}(a_u)), (-u', \text{val}(a_{u'}))\} \subset \text{vert}_v(f),
\]

in which case \( v \) is the slope of the line segment. In particular, the line segments in \( \text{NP}'(f) \) are exactly the sets of the form \( \text{conv}(\text{vert}_v(f)) \) for \( v \geq 0 \). See Figure 8. Hence the elementary reasoning of (7.7) translates into the easy direction of the theorem of the Newton polygon: if \( f(\xi) = 0 \) then \( \text{in}_v(f) \) is not a monomial, so \( \#\text{vert}_v(f) \geq 2, \) so \( \text{conv}(\text{vert}_v(f)) \) is a line segment and hence \( \text{val}(\xi) \) is a slope of \( \text{NP}'(f) \).

Theorem (7.9) provides part of the hard direction: if \( \xi \) is a slope of \( \text{NP}'(f) \) then \( \text{conv}(\text{vert}_v(f)) \) is a line segment, so \( \#\text{vert}_v(f) \geq 2, \) so \( \text{in}_v(f) \) is not a monomial and hence there is at least one zero \( \xi \) of \( f \) such that \( \text{val}(\xi) = v \).

The full theorem of the Newton polygon (including information about multiplicities) is the one-dimensional case of the intersection multiplicity formula (11.1); see (11.8). The multiplicity information is encoded in the Newton complex of \( f \) (8.6).

For another example see (8). We now consider tropicalizations of (algebraic) subschemes of toric varieties.
Let \( \Delta \) be an integral pointed fan in \( N_\mathbb{R} \) and let \( Y \subset X(\Delta) \) be a closed subscheme. Define

\[
\text{Trop}_T(Y) = \text{trop}(Y) \subset N_T(\Delta),
\]
and let \( \text{Trop}(Y) \subset N_\mathbb{R}(\Delta) \) be the closure of \( \text{Trop}_T(Y) \). The set \( \text{Trop}(Y) \) is called the tropicalization of \( Y \) (as a subscheme of \( X(\Delta) \)), and the map \( \text{trop} = \text{trop}|_{\{Y\}} : \{Y\} \to \text{Trop}(Y) \) is again called the tropicalization map.

As before if the ambient space is not clear from context we will write

\[
\text{Trop}(Y, N_\mathbb{R}(\Delta)) \quad \text{and} \quad \text{Trop}(Y, N_T(\Delta)).
\]

7.11.1. It follows from the compatibility properties of the tropicalization noted in (7.3.1) that for any integral \( \Gamma \)-affine pointed polyhedron \( P \subset N_\mathbb{R} \) whose cone of unbounded directions \( \sigma \) is contained in \( \Delta \), we have

\[
\text{Trop}(Y^\text{an} \cap U_P) = \text{Trop}(Y) \cap \overline{\sigma} \quad \text{and} \quad \text{Trop}_T(Y^\text{an} \cap U_P) = \text{Trop}_T(Y) \cap \overline{\sigma}_T.
\]

While (7.12) does not follow formally from (7.9), the proof carries over verbatim.

**Theorem 7.12.** Let \( \Delta \) be an integral pointed fan in \( N_\mathbb{R} \) and let \( Y \subset X(\Delta) \) be a closed subscheme. Suppose that for \( \sigma \in \Delta \) the closed subscheme \( Y \cap X(\sigma) \subset X(\sigma) \) is defined by the ideal \( a_\sigma \subset K[S_\sigma] \). Then

(i) \( \text{Trop}(Y) = \bigcup_{\sigma \in \Delta} \{ v \in N_\mathbb{R}(\sigma) : \text{in}_v(f) \text{ is not a monomial for any } f \in a_\sigma \} \).

(ii) \( \text{Trop}_T(Y) = \text{Trop}(Y) \cap \overline{\sigma} \).

(iii) For \( \tau \prec \sigma \) we have \( \text{Trop}(Y) \cap (N_\mathbb{R}/\text{span}(\tau)) = \text{Trop}(Y) \cap \overline{\tau} \).

**Remark 7.13.** Theorem (7.12) is well-known (see [Pay09b, SS04, EKL06] for instance). The characterization of the tropicalization of an analytic subspace of a torus (7.9) has not appeared previously, although it is well-known to the experts. The proof of (7.9) closely resembles the proof of [EKL06 Theorem 2.2.5], which relates the Bieri-Groves set of a subvariety \( Y \) of a torus with its tropicalization \( \text{Trop}(Y) \); in this sense the proof of [EKL06 Theorem 2.2.5] is the “valuation-theoretic” version of the proof of (7.9).

The main piece of machinery that is used in the proof of (7.9) is the interpretation (6.9) of the polyannulus

\[
U_{(0)} = \{ (\xi_1, \ldots, \xi_n) \in |G_m^n| : |\xi_1| = \cdots = |\xi_n| \}
\]
as the set of maximal ideals of the affinoid algebra \( K(U_{(0)}) \); then to show that there is a point \( \xi \) of a subvariety \( Y \) of \( G_m^n \) inside \( U_{(0)} \), i.e. such that \( \text{trop}(\xi) = 0 \), reduces to the algebraic problem of showing that an ideal in \( K(U_{(0)}) \) is not the unit ideal. This approach is quite standard once one is familiar with the theory of affinoid algebras, and is a compelling first application of the theory of rigid spaces to tropical geometry; in fact the author would argue that (7.12) is at heart a theorem in rigid analysis. Another significant advantage of the rigid-analytic approach to this and other tropical...
problems is that the theory of rigid spaces has already been set up to work over fields endowed with a non-discrete valuation, i.e., whose valuation ring is not noetherian.

For a brief history of (7.12) as well as a stronger version, see \cite{Pay09b}. See also \cite{Dra08, Theo} for a (different) proof of (7.12) that uses affinoid algebras.

8. Tropical hypersurfaces and the Newton complex

8.1. When a closed analytic subspace $Y$ of a polyhedral subdomain of a toric variety is defined by a single equation $f$, its tropicalization comes equipped with extra combinatorial structures (as is well-known in the algebraic case): the set $\text{Trop}(Y)$ is the support of a polyhedral complex, which is “dual” to the so-called Newton complex $\text{New}(f)$ also naturally associated to $f$. The Newton complex should be regarded as recording the multiplicity information missing from $\text{Trop}(Y)$. These extra structures render $\text{Trop}(Y)$ easily computable in terms of $f$, and will later be used to compute a local intersection multiplicity formula for rigid-analytic complete intersections \cite{11,17}. The difficulty in setting up the theory is showing that these complexes are in fact finite, so we begin with the key finiteness result.

**Notation.** Let $P \subset \mathbb{N}_R$ be an integral $\Gamma$-affine pointed polyhedron and let $f \in K(U_P)$ be nonzero. For any subset $\Sigma \subseteq P$ we define

$$\text{vert}_\Sigma(f) = \bigcup_{u \in \Sigma} \text{vert}_u(f),$$

where $\text{vert}_u(f)$ is defined in (7.6.1).

**Lemma 8.2.** Let $P \subset \mathbb{N}_R$ be an integral $\Gamma$-affine pointed polyhedron and let $f \in K(U_P)$ be nonzero.

(i) The set $\text{vert}_P(f)$ is finite.

(ii) There exists $\varepsilon > 0$ such that for all $f' \in K(U_P)$ with $|f - f'|_P < \varepsilon$ we have $\text{vert}_P(f) = \text{vert}_P(f')$.

**Proof.** Let $\sigma = \mathcal{U}(P)$ and write $f = \sum_{u \in S_\sigma} a_u u^u$. For fixed $v \in P$ we have $\text{val}(a_u) - \langle u, v \rangle \to \infty$ by definition; let $m(v) = \min_{u \in S_\sigma} \{\text{val}(a_u) - \langle u, v \rangle\}$, so

$$\text{vert}_v(f) = \{(u, \text{val}(a_u)) : \text{val}(a_u) - \langle u, v \rangle = m(v)\}$$

by (7.6.2), and hence

$$\text{vert}_P(f) = \left\{(u, \text{val}(a_u)) : \text{val}(a_u) - \langle u, v \rangle = m(v) \text{ for some } v \in P\right\}.$$

Let $F_b$ be the union of the bounded faces of $P$, so $P = F_b + \sigma$ by (3.14). Let $a \subset K[S_\sigma]$ be the ideal generated by $\{x^u : a_u \neq 0\}$, so since $K[S_\sigma]$ is noetherian, there exist $u_1, \ldots, u_r \in S_\sigma$ such that $a = (x^{u_1}, \ldots, x^{u_r})$. Let

$$\alpha = \max\{\text{val}(a_u) - \langle u_i, v \rangle : i = 1, \ldots, r, v \in F_b\}.$$ 

Let $v \in P$ and write $v = v' + v''$ for $v' \in F_b$ and $v'' \in \sigma$. Note that for any $u_0 \in S_\sigma$ we have

$$m(v) = \min_{u \in S_\sigma} \{\text{val}(a_u) - \langle u, v' \rangle - \langle u, v'' \rangle\}$$

$$\leq \min_{u \in S_\sigma} \{\text{val}(a_u) - \langle u, v' \rangle\} + \min_{u \in S_\sigma} \{-\langle u, v'' \rangle\}$$

$$\leq m(v') - \langle u_0, v'' \rangle \leq \alpha - \langle u_0, v'' \rangle.$$ 

Let $v_1, \ldots, v_s$ be the vertices of $P$, so $F_b \subset \text{conv}\{v_1, \ldots, v_s\}$. Let

$$\Psi = \{u \in S_\sigma : \text{val}(a_u) - \langle u_i, v \rangle \leq \alpha \text{ for some } i = 1, \ldots, s\},$$

so $\Psi$ is a finite set. We will show that $\text{vert}_P(f) \subset \Psi$. Fix $u \in S_\sigma \setminus \Psi$ and assume that $a_u \neq 0$ (since otherwise $(u, \text{val}(a_u)) \notin \text{vert}_P(f)$ by definition), so $x^u \in a$. Fix $i_0 \in \{1, \ldots, s\}$ such that $\langle u, v_{i_0} \rangle = \max_{i=1,\ldots,s}\langle u, v_i \rangle$. Let $v \in F_b$ and write $v = \sum_{i=1}^s t_i v_i$ with $0 \leq t_i \leq 1$ and $\sum_{i=1}^s t_i = 1$. Then we have

$$\text{val}(a_u) - \langle u, v \rangle = \text{val}(a_u) - \sum_{i=1}^s t_i \langle u, v_i \rangle \geq \text{val}(a_u) - \sum_{i=1}^s t_i \langle u, v_{i_0} \rangle = \text{val}(a_u) - \langle u, v_{i_0} \rangle > \alpha$$
where the final inequality holds because \( u \notin \Psi \). Now let \( v \in P \) be arbitrary, and write \( v = v' + v'' \) for \( v' \in F_0 \) and \( v'' \in \sigma \). Since \( x^u \) is contained in the monomial ideal \( a \) we can write \( u = u_{j_0} + u' \) for some \( j_0 = 1, \ldots, r \) and \( u' \in S_\sigma \). We calculate
\[
\text{val}(a_u) - \langle u, v \rangle = (\text{val}(a_u) - \langle u, v' \rangle) - \langle u, v'' \rangle > \alpha - \langle u, v'' \rangle = \alpha - (u_{j_0}, v'') - \langle u', v'' \rangle \geq \alpha - (u_{j_0}, v'')
\]
since \( v' \in F_0 \) and \( \langle u', v'' \rangle \leq 0 \). But \( m(v) \leq \alpha - (u_{j_0}, v'') \) by (8.2.1), so \( \text{val}(a_u) - \langle u, v \rangle > m(v) \) for all \( v \in P \) and hence \( u \notin \text{vert}_P(f) \). This proves (i).

For \( \varepsilon > 0 \) we let \( f' = \sum_{u \in S_\sigma} a'_u x^u \in K(U_P) \) denote a generic power series satisfying \( |f - f'|_p < \varepsilon \); for such an \( f' \) we define \( m'(v) \) as above. Note that \( |f - f'|_p < \varepsilon \) if and only if
\[
\min \{ \text{val}(a_u - a'_u) - \langle u, v \rangle : u \in S_\sigma, v \in P \} > -\log(\varepsilon).
\]

**Step 1.** First we choose \( \varepsilon \) small enough that \( \text{val}(a_u) = \text{val}(a'_u) \) for all \( u \in \Psi \). Since \( \text{vert}_P(f) \subset \Psi \) we have \( m(v) = \min_{u \in \Psi} \{ \text{val}(a_u) - \langle u, v \rangle \} \) for all \( v \in P \), and hence \( m'(v) \leq m(v) \) for all \( v \in P \).

**Step 2.** Decreasing \( \varepsilon \) if necessary we may assume that \( -\log(\varepsilon) > \alpha \). We claim that \( \text{vert}_P(f') \subset \Psi \). Fix \( u \in S_\sigma \setminus \Psi \). For all \( i = 1, \ldots, s \) we have
\[
\text{val}(a_u) - \langle u, v_i \rangle \geq \min \{ \text{val}(a_u) - \langle u, v_i \rangle, \text{val}(a_u - a'_u) - \langle u, v_i \rangle \} > \alpha.
\]
It follows that \( \text{val}(a'_u) - \langle u, v \rangle > \alpha \) for all \( v \in F_0 \) as above. Now let \( v \in P \) be arbitrary, and write \( v = v' + v'' \) for \( v' \in F_0 \) and \( v'' \in \sigma \), so
\[
m'(v) \leq m(v) \leq \alpha - (u, v'') < (\text{val}(a'_u) - \langle u, v' \rangle) - \langle u, v'' \rangle = \text{val}(a'_u) - \langle u, v \rangle,
\]
where we used (8.2.1) for the second inequality. It follows that \( u \notin \text{vert}_P(f') \).

**Step 3.** We claim that we can decrease \( \varepsilon \) further so that \( \text{vert}_P(f) \subset \text{vert}_P(f') \). Choose \( w_1, \ldots, w_t \in P \) such that \( \text{vert}_P(f) = \bigcup_{i=1}^t \text{vert}_{w_i}(f) \), and suppose for the moment that \( m(w_i) = m'(w_i) \) for \( i = 1, \ldots, t \). Let \( u \in \text{vert}_P(f) \), and suppose that \( u, \text{val}(a_u) \in \text{vert}_{w_i}(f), \) so
\[
m'(w_i) = m(w_i) = \text{val}(a_u) - \langle u, w_i \rangle = \text{val}(a'_u) - \langle u, w_i \rangle,
\]
and hence \( u \in \text{vert}_P(f') \). Therefore it suffices to show that \( m(w_i) = m'(w_i) \) for \( i = 1, \ldots, t \) after potentially shrinking \( \varepsilon \) again. In fact, if \( -\log(\varepsilon) > \max \{ \text{m}(w_1), \ldots, m(w_t) \} \) then for \( u \in S_\sigma \) and \( i = 1, \ldots, t \) we have
\[
\text{val}(a'_u) - \langle u, w_i \rangle \geq \min \{ \text{val}(a_u) - \langle u, w_i \rangle, \text{val}(a_u - a'_u) - \langle u, w_i \rangle \} \geq m(w_i),
\]
which shows that \( m'(w_i) \geq m(w_i) \).

**Step 4.** Finally we claim that \( \text{vert}_P(f') \subset \text{vert}_P(f) \) with the above conditions on \( \varepsilon \). We are done if \( \Psi = \text{vert}_P(f) \), so assume that there exists \( u_0 \in \Psi \setminus \text{vert}_P(f) \). Let \( v \in P \), and choose \( u \in \text{vert}_P(f) \) such that \( \text{val}(a_u) - \langle u, v \rangle = m(v) < \text{val}(a_{u_0}) - \langle u_0, v \rangle \). Since \( \text{val}(a_u) = \text{val}(a'_u) \) and \( \text{val}(a_{u_0}) = \text{val}(a'_{u_0}) \) we have \( m(v) \leq \text{val}(a'_u) - \langle u, v \rangle < \text{val}(a'_{u_0}) - \langle u_0, v \rangle \). Since \( v \) was arbitrary, this proves that \( u_0 \notin \text{vert}_P(f) \).

### 8.3. Moving on to tropicalizations of hypersurfaces.

For convenience we use the following piece of Notation. Let \( P \subset N_R \) be an integral \( \Gamma \)-affine pointed polyhedron (resp. let \( \sigma \subset N_R \) be an integral pointed cone) and let \( f \in K(U_P) \) (resp. \( f \in K[S_\sigma] \)). We denote the closed analytic subspace of \( U_P \) (resp. closed subscheme of \( X(\sigma) \)) defined by \( f \) by \( V(f) \), and we set
\[
\text{Trop}_P(f) := \text{Trop}_P(V(f)) \quad \text{and} \quad \text{Trop}(f) := \text{Trop}(V(f)).
\]

As before if the ambient space is not clear from context we write
\[
\text{Trop}_P(f, \mathcal{P}_\Gamma), \quad \text{Trop}(f, \mathcal{P}), \quad \text{Trop}_P(f, N_\Gamma(\sigma)), \quad \text{and} \quad \text{Trop}(f, N_R(\sigma)).
\]

It is clear that if \( \sigma = \mathcal{U}(P) \) and \( f \in K[S_\sigma] \) then \( V(f) \cap U_P = V(f|_{U_P}) \) and \( \text{Trop}(f) \cap \mathcal{P} = \text{Trop}(f|_{U_P}) \), so the ambiguity in the notation should not cause confusion.

We note that \( \text{Trop}(f) \) is determined by \( f \) in the way one might expect:
Lemma 8.4. Let $P \subseteq N^*_R$ be an integral $\Gamma$-affine pointed polyhedron (resp. $\sigma \subseteq N^*_R$ be an integral pointed cone) and let $f \in K(U^0)$ (resp. $f \in K[S^*_\sigma]$) be nonzero. Then
\[ \text{Trop}(f) = \{ v \in \mathcal{P} \mid \text{resp. } v \in N^*_R(\sigma) \} : \text{in}_v(f) \text{ is not a monomial} \} . \]

Proof. The algebraic version follows from the rigid-analytic version, so assume $f \in K(U^0)$. We must show that if $v \in \mathcal{P}$ and $\text{in}_v(f)$ is not a monomial then $v \in \text{trop}(V(f))$. Reducing to the case $v = 0$ as in the proof of (7.9), we would like to show that $f$ is a unit in $A = K(U^0)$ if and only if $\text{in}_0(f)$ is a monomial. Let $A = \{ g \in A : |g_0| \leq 1 \}$ and $\tilde{A} = \{ g \in A : |g_0| < 1 \}$, and let $A = \tilde{A}/\tilde{A} \cong K[M]$. By scaling we may assume $|f_0| = 1$, so its residue $\tilde{f}$ is $A$ is nonzero. If $\tilde{f} \in A^\times$ then there exists $g \in A$ such that $\tilde{f}g = 1$ so $f_0 = 1 - h$ for $h \in \tilde{A}$. Since $|h_0| < 1$ we have $\lim_{m \to \infty} h^m = 0$, so $f_0 \sum_{m=0}^{\infty} h^m = 1$ and hence $f_0 \in \tilde{A}^\times$. But $\text{in}_0(f)$ is a monomial if and only if $\tilde{f}$ is a monomial, in which case $\tilde{f} \in \tilde{A}^\times$, so $f \in A^\times$. \hfill \blacksquare

8.5. The polyhedral complex structure on $\text{Trop}(f)$. Let $\sigma \subseteq N^*_R$ be an integral pointed cone and let $f \in K[S^*_\sigma]$ be nonzero. Let $\tau \prec \sigma$, let $N^*_\tau = N^*_R/\text{span}(\tau)$, and let $H(f, \tau)$ be the height graph of $f$, where we are using the notation of (7.6). Assume that the image of $f$ in $K[S^*_\tau]$ is nonzero. By (5.4) we have $\# \text{vert}_v(f) \geq 2$ (i.e. $\text{in}_v(f)$ is not a monomial) if and only if $v \in \text{Trop}(f)$. For $v \in \text{Trop}(f) \cap N^*_\tau$ define
\[ \gamma_v = \{ v' \in N^*_\tau : \text{vert}_v(f) \supseteq \text{vert}_{v'}(f) \} . \]

It is standard (see for instance [EKL06, Theorem 2.1.11]) that $\{ \gamma_v : v \in \text{Trop}(f) \cap N^*_\tau \}$ is an integral $\Gamma$-affine polyhedral complex in $N^*_R$ of pure codimension 1 (that is, all maximal cells have dimension $\text{dim}_R(N^*_\tau) - 1$), and since $v \in \gamma_v$ the support of this complex is exactly $\text{Trop}(f) \cap N^*_\tau$. We will write $\text{Trop}(f) \cap N^*_\tau$ to denote the polyhedral complex as well as its support. To summarize:

Proposition. Let $\sigma \subseteq N^*_R$ be an integral pointed cone and let $f \in K[S^*_\sigma]$; let $\tau \prec \sigma$ and assume that the image of $f$ in $K[S^*_\tau \cap \tau^\perp]$ is nonzero. Then $\text{Trop}(f) \cap (N^*_R/\text{span}(\tau))$ is the support of a natural polyhedral complex in $N_{\tau^\perp}R^*/\text{span}(\tau)^\perp$ of pure codimension 1.

Example 8.5.1. To illustrate, let $N = M = \mathbb{Z}^2$, let $\sigma = \{ 0 \}$, and let $x = x^{(-1,0)}, y = x^{(0,-1)} \in K[M]$. Let $\lambda \in K$ have valuation 1 and let $f(x,y) = x + y + \lambda \in K[M]$. Then
\[ \gamma_v = \{ v' \in N^*_\tau : \text{vert}_v(f) \supseteq \text{vert}_{v'}(f) \} \]

Each $R_v$ is an open ray in $\mathbb{R}^2$, and $\overline{R}_v = R_v \cup \{(1,1)\} = \gamma_v$ for any $v \in R_v = \text{relint}(\overline{R}_v)$. The vertex $(1,1)$ is equal to $\gamma_{((1,1))}$. Hence $\text{Trop}(f) = \{ \overline{R}_1, \overline{R}_2, \overline{R}_3, \{(1,1)\} \}$ as a polyhedral complex. See Figure 2.

8.5.2. Now let $P \subseteq N^*_R$ be an integral $\Gamma$-affine pointed polyhedron with cone of unbounded directions $\sigma$, let $\tau \prec \sigma$, let $N^*_\tau = N^*_R/\text{span}(\tau)$, and let $P' = \overline{P} \cap N^*_\tau$. Let $f = \sum a_u x^u \in K(U^0)$ have nonzero image in $K(U^P)$. As above we have $\text{vert}_{P'}(f) \subset H(f, \tau)$ and by (5.2) (as applied to the image of $f$ in $K(U^P)$) the set $\text{vert}_{P'}(f)$ is finite. Define
\[ f' = \sum \{ a_u x^u : (u, \text{val}(a_u)) \in \text{vert}_{P'}(f) \} \in K[S^*_\sigma \cap \tau^\perp] , \]

so $f'$ is a Laurent polynomial such that $\text{vert}_{P'}(f') = \text{vert}_{P'}(f)$ for all $v \in P'$. Again since $\text{in}_v(f)$ is a monomial if and only if $\# \text{vert}_v(f) = 1$, by (5.4) we have $\text{Trop}(f) \cap N^*_\tau = \text{Trop}(f') \cap P'$. Therefore,

Proposition. Let $P \subseteq N^*_R$ be an integral $\Gamma$-affine pointed polyhedron with cone of unbounded directions $\sigma$, let $\tau \prec \sigma$, and let $P' = \overline{P} \cap (N^*_R/\text{span}(\tau))$. Let $f = \sum a_u x^u \in K(U^0)$ have nonzero image [BGR84, Proposition 1.2.5/8].
in \(K(U'_P)\). Then \(\text{Trop}(f) \cap (N'_R/\text{span}(\tau))\) is the intersection of the support of a pure-codimension-1 polyhedral complex in \(N'_R/\text{span}(\tau)\) with the polyhedron \(P'\).

**Remark 8.5.3.** In this case \(\text{Trop}(f) \cap N'_R\) is the support of the polyhedral-complex-theoretic intersection (2.7(iv)) of the complex \(\text{Trop}(f') \cap N'_R\) with the complex whose cells are the faces of \(P'\), but this extra structure does not seem very useful.

### 8.6. The Newton complex

We use the notation in (8.5). Let \(\pi : M_R \times R \to M_R\) denote the projection onto the first factor. For \(v \in N'_R\) we define

\[\tilde{\gamma}_v = \pi(\text{conv}(\text{vert}_v(f))).\]

This is an integral \(Z\)-affine polytope in \(\text{span}(\tau) \subset M_R\). Again it is standard [EKL06, Corollary 2.1.2] that \(\text{New}(f, \tau) := \{\tilde{\gamma}_v : v \in N'_R\}\) is an (integral \(Z\)-affine) polytopal complex in \(\text{span}(\tau)\), called the **Newton complex** of \(f\). When \(\tau = \{0\}\) we omit it and simply write \(\text{New}(f)\). It is clear that the support of \(\text{New}(f, \tau)\) is

\[|\text{New}(f, \tau)| = \text{conv}\{u \in S_\tau \cap \tau^\perp : a_u \neq 0\};\]

this is the **Newton polytope** of \(f\).

**Example 8.6.1.** Continuing with (8.5.1), we have

\[\tilde{\gamma}_{(1,1)} = \text{conv}\{(0,0), (-1,0), (0,-1)\}\]

\[v \in R_1 \implies \tilde{\gamma}_v = \text{conv}\{(-1,0), (0,-1)\}\]

\[v \in R_2 \implies \tilde{\gamma}_v = \text{conv}\{(0,0), (-1,0)\}\]

\[v \in R_3 \implies \tilde{\gamma}_v = \text{conv}\{(0,0), (0,-1)\}\].

If \(v\) is in one of the connected components of \(R^2 \setminus \text{Trop}(f)\) then \(\tilde{\gamma}_v\) is one of the vertices \(\{(0,0)\},\{(−1,0)\},\{(0,−1)\}\), so \(\text{in}_v(f)\) is a monomial. See Figure 9.

![Figure 9. The tropicalization and Newton complex of \(f = x + y + \lambda\).](image)

The complexes \(\text{Trop}(f) \cap N'_R\) and \(\text{New}(f, \tau)\) are dual to each other in the following sense:

**Proposition 8.6.2.** We use the notation of (8.6).

1. For \(v, v' \in \text{Trop}(f) \cap N'_R\) we have \(\gamma_v < \gamma_{v'}\) if and only if \(\tilde{\gamma}_v \succ \tilde{\gamma}_{v'}\). In particular, \(\gamma_v = \gamma_{v'} \iff \tilde{\gamma}_v = \tilde{\gamma}_{v'}\).
2. For \(v \in \text{Trop}(f) \cap N'_R\) the cells \(\gamma_v\) and \(\gamma_{v'}\) are orthogonal to each other in the sense that the linear subspace of \(N'_R\) associated to the affine span of \(\gamma_v\) is orthogonal to the linear subspace of \(\text{span}(\tau)\) associated to the affine span of \(\tilde{\gamma}_v\).
3. For \(v \in \text{Trop}(f) \cap N'_R\) we have \(\dim(\gamma_v) + \dim(\tilde{\gamma}_v) = \dim(R(N'_R))\).

For \(v \in \text{Trop}(f) \cap N'_R\) we call \(\tilde{\gamma}_v\) the **dual cell** to \(\gamma_v\). This establishes a bijection between the cells of \(\text{Trop}(f) \cap N'_R\) and the positive-dimensional cells of \(\text{New}(f, \tau)\) (the zero-dimensional cells correspond to the connected components of \(N'_R \setminus \text{Trop}(f)\)). The “duality” between \(\text{Trop}(f)\) and \(\text{New}(f, \tau)\) is not intrinsic (indeed, \(\text{New}(f)\) contains multiplicity information missing from \(\text{Trop}(f)\)); rather, they are related manifestations of the combinatorial structure of the power series \(f\) living in dual vector spaces.
8.6.3. We resume the notation of (8.5.2), so \( f \in K\langle U_P \rangle \) and \( f' \in K[S_\sigma \cap \tau^+] \). For \( v \in P' \) we have vert\(_v\)(\( f \)) = vert\(_v\)(\( f' \)), so
\[
\gamma_v = \pi(\text{conv}(\text{vert}_v(f))) = \pi(\text{conv}(\text{vert}_v(f'))). 
\]
We define
\[
\text{New}(f, \tau) \coloneqq \{ \gamma_v : v \in P' \} \subset \text{New}(f', \tau).
\]
This is not in general a polyhedral complex as it may well happen that there exist \( v \in P' \) and \( v' \in N_{K_\tau}^r \) such that \( \gamma_v \prec \gamma_{v'} \) but \( \gamma_{v'} \) is not a cell of \( \text{New}(f, \tau) \) (i.e. the corresponding cell \( \gamma_{v'} \) is not contained in \( P' \)). We will only use the fact that there is a polytope \( \tilde{\gamma}_v = \pi(\text{conv}(\text{vert}_v(f))) \in \text{New}(f, \tau) \) associated to every \( v \in \text{Trop}(f) \cap N_{K_\tau}^r \).

Remark 8.6.4. If \( |f - g|_P \ll 1 \) then vert\(_v\)(\( f \)) = vert\(_v\)(\( g \)) by (8.2) ii) and therefore \( \text{Trop}(f) = \text{Trop}(g) \) and \( \text{New}(f, \tau) = \text{New}(g, \tau) \).

Remark 8.7. Let \( \sigma \subset N_{K_\tau} \) be a pointed cone and let \( U \subset X(\sigma)^\text{an} \) be an admissible open subset that can be written as a union of polyhedral subdomains \( \{ U_{P_i} \} \) associated to polyhedra \( P_i \) with cone of unbounded directions \( \sigma \). For instance we can take \( U \) to be the rigid-analytic open unit ball \( D_{K_\tau}^+ = \bigcup_{r > 0} \text{trop}^{-1}(r, \infty)^n \) inside of \( A_{K_\tau}^\text{an} \), or we can take \( U \) to be the analytic torus \( T_{K_\tau}^\text{an} = X(\{ 0 \})^\text{an} = \bigcup_{j \geq 0} \text{trop}^{-1}([-j, 0)^n] \). There is an evident tropicalization map \( \text{trop} : [U] \to \bigcup P_i \). Let \( f \) be an analytic function on \( U \) and define \( \text{Trop}(f) \) to be the closure of \( \text{trop}(\{ V(f) \}) \). The finiteness lemma (8.2) implies that \( \text{Trop}(f) \) is a ‘locally finite polyhedral complex’. This complex is not in general finite but is still interesting to study.

Remark 8.8. Let \( P \subset N_{K_\tau} \) be an integral \( \Gamma \)-affine pointed polyhedron and let \( Y \subset U_{P_i} \) be the closed analytic subspace defined by some ideal \( a \subset K\langle U_{P_i} \rangle \). If \( P \) is a polytope then Gubler [Gub07b, Proposition 5.2] has shown using the theory of semistable alterations of rigid spaces that \( \text{Trop}(Y) \) is a finite union of (non-canonical) integral \( \Gamma \)-affine polytopes (among other things), as is the case for subschemes of a torus. Such a result would follow from (8.5.2) for a pointed polyhedron \( P \) if one knew that \( \text{Trop}(Y) = \bigcap_{i=1}^n \text{Trop}(f_i) \) for some finite list of elements \( f_1, \ldots, f_r \in a \). While \( a \) is certainly finitely generated it is not necessarily the case (even for Laurent polynomials) that the intersection of the tropicalizations of a set of generators is equal to \( \text{Trop}(Y) \) (see [12, §1]). What one needs is a theorem that there exists a “universal Gröbner basis” of the ideal \( a \) in \( K\langle U_{P_i} \rangle \); see for instance [SS04, §2]. The author would guess that such a theorem, suitably formulated, would be true. This issue is certainly deserving of further study as such a theorem would form an important part of the foundations of a theory of tropical analytic geometry.

9. Continuity of roots I: the global version

9.1. In this section we give a tropical criterion (7.8) for a family of \( n \)-tuples of power series in \( n \) variables parametrized by a one-dimensional base \( S \) to define a rigid space that is finite and flat over \( S \), so that the number of common zeros of any member of the family is independent of the parameter. This will be a key ingredient in §12. A weaker version of this result has appeared in [Rab09], where it was useful in explicitly counting the number of zeros of a complicated system of power series by deforming the problem to a much simpler one.

The main rigid-analytic ingredient used in this section is the direct image theorem for rigid spaces. The statement is exactly the same as the direct image theorem for algebraic geometry; the subtlety is in the definition of properness for morphisms of rigid spaces, which we review below.

9.2. The intuitive idea behind the continuity of roots theorem is as follows. Suppose for this paragraph that \( K = \overline{K} \) (for simplicity). Let \( f_1, \ldots, f_n \in K\langle x_1, \ldots, x_n, t \rangle \) and let \( Y \subset B_K^r \times B_K^1 \) be the closed analytic subspace defined by the ideal \( (f_1, \ldots, f_n) \). For \( t_0 \in |B_K^1| \) let \( f_{i,t_0} \) be the image of \( f_i \) in \( K\langle x_1, \ldots, x_n \rangle \) and let \( Y_{t_0} \) be the space of common zeros of \( (f_{1,t_0}, \ldots, f_{n,t_0}) \). Let \( \rho \in |K^\times| \) with \( \rho < 1 \) and suppose that \( Y \) is in fact contained in the Weierstrass subdomain \( B_K^r(\rho) \times B_K^1 \) of \( B_K^n \times B_K^1 \) (cf. (4.16)): that is, \( Y \) is a closed subscheme of \( B_K^r(\rho) \times B_K^1 \).
that is simultaneously a closed subscheme of \( B^n_K \times B^1_K \). Tropically, if \( P \) is the polyhedron \( R^{n_0}_\ge \) then our condition is equivalent to \( \text{Trop}(Y_{\mathfrak{b}_0}) \) being contained in the closure of a polyhedron \( R^{n_0}_{\ge r} \) for some \( r > 0 \) and all \( t_0 \).

Roughly, points of \( Y_{\mathfrak{b}_0} \) are “trapped” inside of the smaller ball \( B^n_K(\rho) \) since they cannot escape to the boundary of \( B^n_K \) — that is, no points of \( Y_{\mathfrak{b}_0} \) can enter or leave \( B^n_K(\rho) \) as the parameter \( t_0 \) varies since otherwise we would have points “jumping over” the annulus \( B^n_K \setminus B^n_K(\rho) \). Hence all of the finite rigid spaces (equivalently, finite schemes) \( Y_{\mathfrak{b}_0} \) must have the same length.

9.3. To say that a ball is contained in the “interior” of a larger ball is basically the notion of relative compactness:

**Definition.** \cite{BGR84} §9.6.2) Let \( X = \text{Sp}(A) \) and \( Y = \text{Sp}(B) \) be \( K \)-affinoid spaces, let \( f : X \to Y \) be a morphism, and let \( U \subset X \) be an affinoid subdomain. We say that \( U \) is relatively compact in \( X \) over \( Y \) and we write \( U \Subset_Y X \) provided that we can find a closed immersion \( X \to \mathcal{B}_K \times Y \) over \( Y \) such that \( U \subset B^n_K(\rho) \times Y \) for some \( \rho \in \hat{K}^\times \) with \( \rho < 1 \).

In the above example (9.2), we have \( B^n_K(\rho) \times B^1_K \Subset_B \mathcal{B}_K \times B^1_K \).

9.4. Kiehl’s notion of properness for morphisms of rigid spaces is defined in terms of relative compactness:

**Definition.** \cite{BGR84} Definition 9.6.2(2)] Let \( f : X \to Y \) be a morphism of rigid spaces, and for simplicity assume that \( Y \) is affinoid. We say that \( f \) is proper if it is separated (i.e. the diagonal is closed) and if it satisfies the following condition: there exist two admissible affinoid coverings \( \{U_i\}_{i=1}^n \) and \( \{V_i\}_{i=1}^m \) of \( X \) such that \( U_i \Subset_Y V_i \) for all \( i = 1, \ldots, n \).

Properness over a general base is defined in such a way as to be local on the base.

**Theorem 9.4.1.** \cite{BGR84} Theorem 9.6.3(1)] Let \( f : X \to Y \) be a proper morphism of rigid spaces and let \( \mathcal{F} \) be a coherent sheaf of \( \mathcal{O}_X \)-modules. Then \( f_* \mathcal{F} \) is a coherent sheaf of \( \mathcal{O}_Y \)-modules.

The definition of a coherent sheaf of modules on a rigid space is similar to the analogous definition in algebraic geometry, but the precise definition is not important for our purposes since we will only use the following simple consequence:

**Corollary 9.4.2.** Let \( X = \text{Sp}(A) \) and \( Y = \text{Sp}(B) \) be affinoid spaces and let \( f : X \to Y \) be a proper morphism. Then \( B \) is finite as an \( A \)-module.

**Example 9.4.3.** In the situation of (9.2), the spaces \( Y \) and \( B^1_K \) are affinoid and the map \( Y \to B^1_K \) is proper: in fact \( Y \Subset_B Y \) since

\[
Y \subset B^n_K(\rho) \times B^1_K \Subset_B B^n_K \times B^1_K.
\]

Therefore \( Y \to B^1_K \) is finite; flatness follows from an unmixedness argument as in the proof of (9.3).

The following generalizes the fact (9.2) that \( B^n_K(\rho) \times B^1_K \Subset_B B^n_K \times B^1_K \).

**Lemma 9.5.** Let \( P' \subset P \subset N_R \) be integral \( \Gamma \)-affine pointed polyhedra such that \( \tau = \mathcal{U}(P') \) is a face of \( \sigma = \mathcal{U}(P) \) (so \( \mathcal{P}' \subset \mathcal{P} \subset N_R(\sigma) \)). If \( P' \) is contained in the (topological) interior of \( P \) then \( U_{P'} \Subset_K U_P \).

**Proof.** First note that \( U_{P'} \) is an admissible affinoid open subset of \( X(\tau)^{an} \subset X(\sigma)^{an} \) and is therefore an affinoid subdomain of \( U_P \). \cite{BGR84} Proposition 9.3.1(3). Choose generators \( u_1, \ldots, u_r \) for \( S_{\tau} \) such that we can write \( P = \bigcap_{i=1}^r \{v \in N_R : \langle u_i, v \rangle \le b_i \} \), so \( U_P \) is a closed subspace of \( \prod_{i=1}^r B^n_K(\rho_i) \) as in (6.10) where \( \rho_i = \exp(b_i) \). Since \( P' \) is contained in the interior of \( P \), we can find \( c_i \in \Gamma \) with \( c_i < b_i \) such that \( P' \subset \bigcap_{i=1}^r \{v \in N_R : \langle u_i, v \rangle \le c_i \} \). Setting \( \mu_i = \exp(c_i) \), we have \( U_{P'} \subset \prod_{i=1}^r B^n_K(\mu_i) \Subset_K \prod_{i=1}^r B^n_K(\rho_i) \).

Generalizing (9.4.3) we have the following consequence of (9.5), which is a tropical criterion for an affinoid space to be finite.

**Proposition 9.6.** Let \( P \subset N_R \) be an integral \( \Gamma \)-affine pointed polyhedron and let \( Y \) be a closed analytic subspace of \( U_P \) such that \( \text{Trop}(Y) \) is contained in the interior \( \overline{\text{Trop}}^o \) of \( \overline{P} \). Then \( Y \) is finite.
Suppose that $Y$ the intersection is a completed ray.

Theorem 9.8. (Continuity of roots I) We fix:

(i) $S$ a normal connected rigid space of dimension one.

(ii) $\Delta$ an integral pointed fan in $N_{R}$.

(iii) $P = \{P_1, \ldots, P_r\}$ a collection of integral $\Gamma$-affine polyhedra in $N_{R}$ such that $U(P_i) \in \Delta$ for all $i$.

(iv) $P' = \{P'_1, \ldots, P'_r\}$ a second collection of integral $\Gamma$-affine polyhedra in $N_{R}$ such that $U(P'_i) \cap (U(P_1) \times U(P_2))$ is defined by $n$ equations $f_1, \ldots, f_n 
\in A(U(P_i))$, where $A(U(P)) = A^0 \otimes_K K \langle U(P) \rangle$ as in (6.3).

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(iv) $P' = \{P'_1, \ldots, P'_r\}$ a second collection of integral $\Gamma$-affine polyhedra in $N_{R}$ such that $U(P'_i) \cap (U(P_1) \times U(P_2))$ is defined by $n$ equations $f_1, \ldots, f_n 
\in A(U(P_i))$, where $A(U(P)) = A^0 \otimes_K K \langle U(P) \rangle$ as in (6.3).

**Theorem 9.8.** (Continuity of roots I) We fix:

(i) $S$ a normal connected rigid space of dimension one.

(ii) $\Delta$ an integral pointed fan in $N_{R}$.

(iii) $P = \{P_1, \ldots, P_r\}$ a collection of integral $\Gamma$-affine polyhedra in $N_{R}$ such that $U(P_i) \in \Delta$ for all $i$.

(iv) $P' = \{P'_1, \ldots, P'_r\}$ a second collection of integral $\Gamma$-affine polyhedra in $N_{R}$ such that $U(P'_i) \cap (U(P_1) \times U(P_2))$ is defined by $n$ equations $f_1, \ldots, f_n 
\in A(U(P_i))$, where $A(U(P)) = A^0 \otimes_K K \langle U(P) \rangle$ as in (6.3).
This picture underlies much of §12 where it is essential that we work with families of translations parametrized by rigid-analytic annuli $S_{t}$.

**Figure 10.** Pictures of $\text{Trop}(f_{1,t_{0}})$ and $\text{Trop}(f_{2,t_{0}})$ in $N_{R}(\tau)$ from (9.10) evaluated at a generic value of $t_{0} \in K^{\times}$ with $0 < \text{val}(t_{0}) \ll 1$. The dotted vertical line on the right is $N_{R}/\text{span}(\tau)$ and the solid line segment is $\overline{T}$; the dots in $\overline{T}$ are included in the tropicalizations.

**Proof of (9.8).** By hypothesis $Y \subset U_{P_{i}} \times S$, so since $U_{P_{i}} \times S \subseteq S \times U_{P_{i}} \times S$ by (9.5) we have $Y \cap (U_{P_{i}} \times S) \subseteq S \cap (U_{P_{i}} \times S)$ for all $i$ and hence $\pi : Y \rightarrow S$ is proper. Suppose that $Y_{s}$ is a finite set for all $s \in |S|$. Then $Y$ is finite over $S$ by [BGR84, Corollary 9.6.3/6], so it suffices to prove that $Y$ is $S$-flat. The assertion is local on $Y$ and $S$, so we may assume that $S = \text{Sp}(A)$ is an affinoid algebra that is a Dedekind domain and therefore Cohen-Macaulay. Let $f_{1}, \ldots, f_{n} \in A(U_{P_{i}})$ be a collection of power series defining $Y \cap (U_{P_{i}} \times \text{Sp}(A))$, let $a = (f_{1}, \ldots, f_{n})$, and let $B = A(U_{P_{i}})/a$, so $Y \cap (U_{P_{i}} \times \text{Sp}(A)) = \text{Sp}(B)$. Since $A(U_{P_{i}})$ is a flat $A$-algebra [Con06, Theorem A.1.5] with Cohen-Macaulay fiber rings over maximal ideals (6.9/v), it follows from [Mat89, Theorem 23.9] that $A(U_{P_{i}})$ is itself Cohen-Macaulay. Thus $A(U_{P_{i}})$ is a catenary of dimension $n + 1$, so by Krull's principal ideal theorem, if $p$ is a minimal prime of $B$ then $\dim(B/p) \geq 1$. But the fibers of $\pi$ have dimension zero, so $\dim(B/p) = 1$ and hence by the unmixedness theorem [Mat89, Theorem 17.6], $B$ has no embedded prime ideals. Thus every associated prime of $B$ contracts to the zero ideal of $A$, so since $A$ is a Dedekind domain, $B$ is a flat $A$-module.

In the general case, the theorem on semicontinuity of fiber dimension [Duc07, Theorem 4.9] implies that the set

$$Z = \{ \eta \in |Y| : \dim_{\eta}(Y_{\pi(\eta)}) \geq 1 \}$$

is (Zariski-)closed in $Y$, so the proper mapping theorem [BGR84, Proposition 9.6.3/3] implies that $\pi(Z)$ is a closed subset of $S$, which has dimension zero if $\pi$ has any finite fibers. Deleting $\pi(Z)$ from $S$ does not affect its connectedness, so we are reduced to the case treated above.

**Remark 9.11.** It may be possible to weaken the hypotheses of (9.8) to only require that $\text{Trop}(Y_{s})$ be contained in $\bigcup_{i=1}^{r} \overline{T}_{i}$ for each $s$, or even in the interior of $\bigcup_{i=1}^{r} \overline{T}_{i}$, but it is not immediately obvious how one would do so.

### 10. Continuity of Roots II: the local version

10.1. The purpose of this section is to show that if $f_{1}, \ldots, f_{n}$ is any family of $n$-tuples of power series in $n$ variables parametrized by a one-dimensional rigid space $S$, and if $t \in |S|$ is a point such that the specializations $f_{1,t}, \ldots, f_{n,t}$ at $t$ have finitely many common zeros, then $f_{1}, \ldots, f_{n}$ defines a finite and flat rigid space over a small affinoid neighborhood of $t$ in $S$. This is the rigid-analytic fact that allows us to use a polynomial approximation argument in order to derive the local intersection multiplicity formula for rigid spaces from the analogous theorem for schemes. The proof of (10.2) is more technical than (9.8), and we will assume more familiarity with rigid analytic spaces in it. In particular, we will assume that the reader has some knowledge of Raynaud’s theory of formal models, which we briefly review in (10.4).
We begin with the statement of the theorem we will prove:

**Theorem 10.2.** (Continuity of roots II) Let $A$ be a $K$-affinoid algebra that is a Dedekind domain and let $S = \text{Sp}(A)$. Let $X = \text{Sp}(B)$ be a Cohen-Macaulay affinoid space of dimension $n + 1$, let $f_1, \ldots, f_n \in B$, and let $Y \subset X$ be the subspace defined by the ideal $a = (f_1, \ldots, f_n)$. Suppose that we are given a morphism $\pi : X \to S$ and a point $t \in |S|$ such that the fiber $Y_t = \pi^{-1}(t) \cap Y$ has dimension zero. Then there is an affinoid subdomain $U \subset S$ containing $t$ such that $\pi^{-1}(U) \to U$ is finite and flat.

In particular, the rigid space $Y_s = \pi^{-1}(s) \cap Y$ is finite for all $s \in |U|$ and has the same length as $Y_t$.

**Example 10.3.** The following special case makes (10.2) look very much like a theorem of continuity of roots. Let $X = B^n_K \times B^n_K$ and $S = B^n_K$, with $\pi : X \to S$ the projection onto the second factor. Let $f_1, \ldots, f_n \in K(x_1, \ldots, x_n, t)$. If the specializations $f_{1,0}, \ldots, f_{n,0}$ at 0 have only finitely many zeros in $B^n_K$, then there exists $\varepsilon > 0$ such that $f_{1,s}, \ldots, f_{n,s}$ have the same number of zeros (counted with multiplicity) in $B^n_{K(s)}$ as $f_{1,0}, \ldots, f_{n,0}$ when $|s| < \varepsilon$.

**10.4.** Here we recall some notions used in Raynaud’s theory of formal models. The main reference is Bosch, Lütkebohmert, and Raynaud’s series of papers [BL93a, BL93b, BL95a, BL95b]. The ring of restricted power series in $n$ variables over $\mathcal{O}_K$ is

$$\mathcal{O}_K\langle x_1, \ldots, x_n \rangle = \left\{ \sum_{\nu} a_\nu x^\nu \in \mathcal{O}_K[[x_1, \ldots, x_n]] : |a_\nu| \to 0 \right\} = \{ f \in K\langle x_1, \ldots, x_n \rangle : |f|_{\sup} \leq 1 \}.$$

An $\mathcal{O}_K$-algebra $A$ admitting a surjective homomorphism $\varphi : \mathcal{O}_K\langle x_1, \ldots, x_n \rangle \to A$ for some $n$ is called topologically of finite type or $tf$ type; if we can choose $\varphi$ such that $\ker(\varphi)$ is a finitely generated ideal, we say that $A$ is topologically of finite presentation or $tfp$ presentation. If $A$ is $tf$ type and is $\mathcal{O}_K$-flat we say that $A$ is an admissible $\mathcal{O}_K$-algebra; in this case $A$ is automatically $tf$ presentation and is complete and separated in the $\varpi$-adic topology for any nonzero $\varpi \in m_K$ [BL93a Proposition 1.1]. Note that $A$ is an $\mathcal{O}_K$-flat if and only if it has no $\varpi$-torsion. An admissible formal $\mathcal{O}_K$-scheme is a formal $\text{Sp}(\mathcal{O}_K)$-scheme that is locally isomorphic to the formal spectrum of an admissible $\mathcal{O}_K$-algebra (equipped with the $\varpi$-adic topology).

There is a rigid generic fiber functor $\mathcal{X} \mapsto \mathcal{X}_{\text{rig}}$ from the category of quasi-compact admissible formal $\mathcal{O}_K$-schemes to the category of quasi-compact and quasi-separated rigid spaces over $K$; it becomes an equivalence after inverting so-called admissible formal blow-ups in the source category. If $\mathcal{X} = \text{Sp}(\mathcal{O}_K\langle x_1, \ldots, x_n \rangle/\mathfrak{a})$ is the formal spectrum of an admissible $\mathcal{O}_K$-algebra then $\mathcal{X}_{\text{rig}} = \text{Sp}(K\langle x_1, \ldots, x_n \rangle/K\mathfrak{a})$. The rigid generic fiber functor satisfies many compatibility properties including respecting open immersions and fiber products. If $X$ is a rigid space, an admissible formal scheme $\mathcal{X}$ such that $\mathcal{X}_{\text{rig}} \cong X$ is called a formal model for $X$; such a model always exists when $X$ is quasi-compact and quasi-separated. If $X = \text{Sp}(K\langle x_1, \ldots, x_n \rangle/\mathfrak{a})$ is an affinoid space then $\text{Sp}(\mathcal{O}_K\langle x_1, \ldots, x_n \rangle/\mathfrak{a} \cap \mathcal{O}_K\langle x_1, \ldots, x_n \rangle)$ is a formal model for $X$; however, most formal models for $X$ will not be affine.

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated rigid spaces. The power of Raynaud’s theory lies in the ability to choose formal models $\mathcal{X}$ and $\mathcal{Y}$ for $X$ and $Y$, respectively, along with a morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ such that $\varphi_{\text{rig}} = f$, in such a way that $\varphi$ retains any “nice” properties of $f$ (e.g. flatness). This allows one to use algebraic geometry to prove statements about rigid spaces.

**Notation 10.5.** We fix a nonzero element $\varpi \in m_K$. For $m \geq 0$ we let $\mathcal{O}_{K,m} = \mathcal{O}_K/\varpi^{m+1}\mathcal{O}_K$, and if $\mathcal{X}$ is a formal $\text{Sp}(\mathcal{O}_K)$-scheme we let $\mathcal{X}_m = \mathcal{O}_{K,m} \otimes_{\mathcal{O}_K} \mathcal{X}$.

If $\mathcal{X}$ is an admissible formal $\mathcal{O}_K$-scheme then each $\mathcal{X}_m$ is a flat $\mathcal{O}_{K,m}$-scheme of finite type (having the same underlying topological space as $\mathcal{X}$). The following converse statement is well-known; see [BL93a] §1.

**Lemma 10.6.** Let $\{ A_m \}_{m \geq 0}$ be an inverse system of $\mathcal{O}_K$-algebras such that for all $m \geq 0$ the map $A_{m+1} \to A_m$ identifies $A_m$ with $\mathcal{O}_{K,m} \otimes_{\mathcal{O}_K} A_{m+1}$. If $A_0$ is an $\mathcal{O}_K$-algebra of finite type and $A_m$ is a flat $\mathcal{O}_{K,m}$-algebra for every $m \geq 0$ then $A = \varinjlim A_m$ is an admissible $\mathcal{O}_K$-algebra and the natural maps $\mathcal{O}_{K,m} \otimes_{\mathcal{O}_K} A \to A_m$ are isomorphisms.
10.7. Let $X$ be an admissible formal $\mathcal{O}_K$-scheme. There is a functorial reduction map $\text{red} : |X_{\text{rig}}| \to |X|$, defined as follows. Let $\xi \in |X_{\text{rig}}|$ and let $U = \text{Spf}(A) \subset X$ be a formal affine such that $\xi$ is a point of $U_{\text{rig}} = \text{Sp}(K \otimes_{\mathcal{O}_K} A)$. Then $\xi$ corresponds to a surjective homomorphism $\varphi : K \otimes_{\mathcal{O}_K} A \to \mathcal{K}'$, where $\mathcal{K}' = \kappa(\xi)$ is an extension of $K$. For boundedness reasons we have $\varphi(A) \subset \mathcal{O}_K$; the point $\text{red}(\xi)$ corresponds to the contraction of $\mathcal{K}'$ in $A$.

In the above situation the ring $R = \varphi(A) \subset \mathcal{O}_K$ is a finite admissible local $\mathcal{O}_K$-algebra of dimension one and the closed immersion $\text{Spf}(R) \hookrightarrow X$ is called a rig-point of $X$; see [BL93a, Lemma 3.4]. In this way the rig-points of $X$ correspond naturally to the points of $X_{\text{rig}}$.

**Lemma 10.8.** Let $X$ be a quasi-compact admissible formal $\mathcal{O}_K$-scheme and let $g : \mathcal{Y} \to \mathcal{X}_0$ be an étale morphism of finite-type $\mathcal{O}_{K,0}$-schemes. There is a unique (up to unique isomorphism) admissible formal $\mathcal{O}_{K}$-scheme $\mathcal{Y}$ equipped with a morphism $f : \mathcal{Y} \to X$ such that $\mathcal{Y}_0 \cong \mathcal{Y}$ with $f_0 : \mathcal{Y}_0 \to X$ identified with $g$, and such that $f_m : \mathcal{Y}_m \to \mathcal{X}_m$ is étale for all $m \geq 0$. Moreover, $f$ is flat and $f_{\text{rig}} : \mathcal{Y}_{\text{rig}} \to X_{\text{rig}}$ is an étale morphism of rigid spaces.

**Proof.** The existence and uniqueness of $f : \mathcal{Y} \to X$ is a consequence of the infinitesimal invariance of the étale site [EGAIV, 18.1.2], along with [10.6]. The flatness of $f$ follows from the fibral flatness criterion over general valuation rings [BL93a, Lemma 1.6]. That $f$ is étale is a special case of [BLR95a, Corollary 3.10] — or one can prove it directly by reducing to the case of standard étale morphisms and using the Jacobi criterion.

The following proposition is a translation of the structure theorem for separated finite-type morphisms with a finite fiber [EGAIV, 18.12.3] to rigid spaces, using formal models.

**Proposition 10.9.** Let $f : X \to Y$ be a separated morphism of quasi-compact quasi-separated rigid spaces, and suppose that the fibers of $f$ are finite. For any $\eta \in |Y|$ there is an étale morphism $g : Y' \to Y$ and a point $\eta' \in g^{-1}(\eta)$ such that the product $X' = X \times_\mathcal{X} Y'$ decomposes into a disjoint union of rigid spaces $X' = X_1' \sqcup X_2'$ in such a way that $X_1' \to Y'$ is finite and $X_2' \to Y'$ has empty $\eta'$-fiber.

**Proof.** By [BL93a, Corollary 5.10(b)] there exist formal models $\mathcal{X}$ and $\mathcal{Y}$ for $X$ and $Y$, respectively, along with a morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ with dimension-zero fibers such that $\varphi_{\text{rig}} = f$. The morphism $\varphi_0 : \mathcal{X}_0 \to \mathcal{Y}_0$ is separated by [BL93a, Proposition 4.7], so by [EGAIV, 18.12.3] there is an étale morphism $\psi : \mathcal{Y}' \to \mathcal{Y}_0$ and a point $\eta_0 \in |\mathcal{Y}'|$ over $\eta_0 = \text{red}(\eta)$ such that $\mathcal{X}' = \mathcal{X}_0 \times_{\mathcal{Y}_0} \mathcal{X}_0$ breaks up into a disjoint union $\mathcal{X}'_1 \sqcup \mathcal{X}'_2$ with $\mathcal{X}'_1$ finite over $\mathcal{Y}'$ and $\mathcal{X}'_2 \to \mathcal{Y}'$ having empty $\eta_0'$-fiber. Let $\psi : \mathcal{Y}' \to \mathcal{Y}$ be the unique lift of $\psi_0 : \mathcal{Y}' \to \mathcal{Y}_0$ as in (10.8), and let $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ be the fiber product (in the category of formal $\text{Spf}(\mathcal{O}_K)$-schemes; $\mathcal{X}'$ is then admissible because $\mathcal{Y} \to \mathcal{Y}$ is flat), so $\mathcal{X}_0' = \mathcal{X}'$. Since the topological spaces underlying $\mathcal{X}'$ and $\mathcal{Y}'$ are the same, we have $\mathcal{X}' = \mathcal{X}'_1 \sqcup \mathcal{X}'_2$, where $\mathcal{X}'_i$ is an admissible formal $\mathcal{O}_K$-scheme lifting $\mathcal{X}'_i$ for $i = 1, 2$. It follows from [BL93a, Lemma 1.5] that $\mathcal{X}'_i \to \mathcal{Y}'$ is finite, and certainly $\mathcal{X}'_2 \to \mathcal{Y}'$ has empty $\eta_0'$-fiber. Let $X' = \mathcal{X}'_{\text{rig}}$, $Y' = \mathcal{Y}'_{\text{rig}}$, $g = \psi_{\text{rig}} : Y' \to Y$, and $X'_i = (\mathcal{X}'_i)_{\text{rig}}$ for $i = 1, 2$, so $X' = Y' \times_\mathcal{Y} X = \mathcal{X}'_1 \sqcup \mathcal{X}'_2$. Then $X'_i \to Y'$ is finite, and if $\eta' \in |Y'|$ is any point that reduces to $\eta'_0$ then $X'_2 \to Y'$ has empty $\eta'$-fiber.

It remains to show that there exists $\eta' \in g^{-1}(\eta)$ reducing to $\eta'_0$. Let $\text{Spf}(R) \hookrightarrow \mathcal{Y}$ be the rig-point associated to $\eta$ as in (10.7), so $\text{Spf}(R)_{\text{rig}} = \{\eta\}$. Consider the Cartesian squares

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{g} & \mathcal{Y}' \\
\downarrow & & \downarrow \\
\{\eta\} & \rightarrow & \text{Spf}(R)
\end{array}
\]

where $\mathcal{Z}$ is the fiber product $\mathcal{Y}' \times_{\mathcal{Y}} \text{Spf}(R)$ in the category of formal $\text{Spf}(\mathcal{O}_K)$-schemes; note that $\mathcal{Z}$ is admissible because $\psi$ is flat. Since the rigid generic fiber functor is compatible with fiber products, the left square is canonically identified with the rigid generic fiber of the right square. The result now follows from the surjectivity of the reduction map [BL93a, Proposition 3.5].
Lemma 10.10. Let \( f : A \to B \) be a homomorphism of \( K \)-affinoid algebras and let \( \varphi : \text{Sp}(B) \to \text{Sp}(A) \) be the associated morphism of affinoid spaces. Let \( \xi \in |\text{Sp}(A)| \) be an element not contained in the image of \( \varphi \). Then there is an affinoid subdomain \( U \subset \text{Sp}(A) \) containing \( \xi \) that is disjoint from the image of \( \varphi \).

Proof. Let \( m \subset A \) be the maximal ideal corresponding to \( \xi \), let \( a_1, \ldots, a_r \) generate \( m \), and let \( a_i' = f(a_i) \). Since \( f(m)B = B \), there exist \( b_1, \ldots, b_r \in B \) such that \( \sum_{i=1}^r a_i'b_i = 1 \). Let \( M > \max\{|b_1|_{\text{sup}}, \ldots, |b_r|_{\text{sup}}\} \) with \( M \in \Gamma \). Then for all \( \eta \in |\text{Sp}(B)| \) there is some \( j \) such that \(|a_j(\eta)| > 1/M\), so the Weierstrass subdomain

\[
U = \{ \xi' \in |\text{Sp}(A)| : |a_i(\xi')| \leq 1/M \text{ for all } i = 1, \ldots, r \}
\]
satisfies our requirements.

Proof of (10.2). By the theorem on semicontinuity of fiber dimension for rigid spaces [Duc07, Theorem 4.9], the locus \( Z \) of points \( \eta \in |Y| \) not isolated in its fiber is a Zariski-closed subset of the affinoid space \( Y \), so \( Z \) is the set underlying an affinoid space. By assumption \( Z \cap Y_i = \emptyset \), so by (10.10), after replacing \( S \) with an affinoid subdomain containing \( t \) we may assume that \( Y \to S \) has finite fibers. The flatness of \( Y \to S \) now follows from the unmixedness theorem exactly as in the proof of (9.8).

By (10.2) there is an étale morphism \( g : S' \to S \) and a point \( t' \in |S'| \) in the fiber over \( t \) such that the fiber product \( Y' = Y \times_S S' \) decomposes into a disjoint union \( Y' = Y'_1 \amalg Y'_2 \), where \( Y'_1 \to S' \) is finite and \( Y'_2 \) has empty \( t' \)-fiber. Replacing \( S' \) with an affinoid neighborhood of \( t' \) disjoint from \( g^{-1}(t) \setminus \{ t' \} \) we may assume that \( S', Y', Y'_1, Y'_2 \) are all affinoid and that \( g^{-1}(t) = \{ t' \} \). Then \( Y'_2 \to S \) has empty fiber over \( t \), so again by (10.10), after replacing \( S \) with an affinoid subdomain we may assume that \( Y'_2 = \emptyset \), and therefore that \( Y' \to S' \) is finite. By [BL93b, Corollary 5.11] the image of \( g \) is open, so we again shrink \( S \) to assume that \( g \) is surjective. Then by descent theory for rigid spaces [Con06, Theorems 4.2.7 and 4.2.2], we have that \( Y \to S \) is finite.

11. APPLICATION: A LOCAL INTERSECTION MULTIPLICITY FORMULA FOR RIGID SPACES

11.1. Osserman and Payne [OP10, §5] have proved a general theorem relating the multiplicities of an intersection of subvarieties of a torus with the corresponding multiplicities of the intersection of their tropicalizations. In the case of a dimension-zero complete intersection this theorem becomes a formula for intersection numbers whose history begins with Bernstein [Ber75]; see (11.5.1). We use this multiplicity formula, along with the continuity of roots theorem (10.2) and a polynomial approximation argument, to derive an intersection multiplicity formula (11.7) for rigid spaces in the case of a complete intersection of dimension zero. Theorem (11.7) is a natural generalization of the theorem of the Newton polygon to a higher-dimensional setting; see (11.8).

Tropical intersection multiplicities are calculated in terms of the mixed volume of a collection of polytopes (in the case of a dimension-zero complete intersection):

Definition 11.2. The Minkowski Sum of an \( n \)-tuple of polytopes \( P_1, \ldots, P_n \subset \mathbb{N}_R \) is defined to be

\[
P_1 + \cdots + P_n = \{ v_1 + \cdots + v_n : v_i \in P_i \}.
\]

For \( \lambda \in \mathbb{R}_{\geq 0} \) we let \( \lambda P_1 = \{ \lambda v : v \in P_1 \} \), and we define a function \( V_{P_1, \ldots, P_n} : \mathbb{R}_{\geq 0}^n \to \mathbb{R} \) by

\[
V_{P_1, \ldots, P_n}(\lambda_1, \ldots, \lambda_n) = \text{vol}(\lambda_1 P_1 + \cdots + \lambda_n P_n),
\]

where \( \text{vol} \) is a Euclidean volume form on \( \mathbb{N}_R \cong \mathbb{R}^n \) normalized such that the volume of a fundamental domain for the lattice \( N \) is one. It well-known that \( V_{P_1, \ldots, P_n} \) is a homogeneous polynomial of degree \( n \) in \( \lambda_1, \ldots, \lambda_n \). The mixed volume \( \text{MV}(P_1, \ldots, P_n) \) is defined to be the coefficient of the \( \lambda_1 \cdots \lambda_n \)-term of \( V_{P_1, \ldots, P_n} \).

Example 11.3. Fixing a basis, we identify \( N \) with \( \mathbb{Z}^n \). Suppose that \( P_1 \) is the line segment connecting points \( v_1, v'_1 \in N = \mathbb{Z}^n \), and let \( w_i = v_i - v'_i \). Then

\[
V_{P_1, \ldots, P_n}(\lambda_1, \ldots, \lambda_n) = |\det(\lambda_1 w_1, \ldots, \lambda_n w_n)| = \lambda_1 \cdots \lambda_n |\det(w_1, \ldots, w_n)|,
\]

where \( \det(w_1, \ldots, w_n) \) is the determinant of the matrix whose \( i \)-th column is the column vector \( w_i \in \mathbb{Z}^n \). Therefore \( \text{MV}(P_1, \ldots, P_n) = |\det(w_1, \ldots, w_n)| \) in this case.
Definition 11.4. Let $P \subseteq \mathbb{N}_R$ be an integral $\Gamma$-affine pointed polyhedron, let $f_1, \ldots, f_n \in K \langle U_P \rangle$, let $Y_i = V(f_i)$, and let $Y = \bigcap_{i=1}^n Y_i$. Let $v \in \mathbb{N}_P \cap P$. The intersection multiplicity of $Y_1, \ldots, Y_n$ over $v$, denoted $i(v, Y_1 \cdots Y_n)$, is defined to be the length of the space $Y \cap U_{v}$:

$$i(v, Y_1 \cdots Y_n) := \dim_K \Gamma(Y \cap U_{v}, \partial Y \cap U_{v}).$$

Note that $i(v, Y_1 \cdots Y_n) < \infty$ if and only if $\dim(Y \cap U_{v}) = 0$, in which case

$$i(v, Y_1 \cdots Y_n) = \sum_{\text{trop}(\xi) = v} \dim_K(\partial Y, \xi).$$

Note also that $i(v, Y_1 \cdots Y_n)$ only depends on the images of $f_1, \ldots, f_n$ in $K \langle U_{v} \rangle$. The relation between $i(v, Y_1 \cdots Y_n)$ and the Newton complexes of $f_1, \ldots, f_n$ is as follows:

Theorem 11.5. (Katz; Osserman-Payne) Let $f_1, \ldots, f_n \in K[M]$ and let $v \in \bigcap_{i=1}^n \text{Trop}(f_i)$ be an isolated point. For $i = 1, \ldots, n$ let $Y_i = V(f_i)$ and let $\gamma_i = \pi(\text{conv}(\text{vert}_{v}(f_i))) \in \text{New}(f_i)$ be the polytope corresponding to $v \in \text{Trop}(f_i)$ as in (8.6). Then

$$i(v, Y_1 \cdots Y_n) = \text{MV}(\gamma_1, \ldots, \gamma_n).$$

Note that $v \in \mathbb{N}_P$ since $\{v\}$ is an integral $\Gamma$-affine polytope. The mixed volume $\text{MV}(\gamma_1, \ldots, \gamma_n)$ is the stable tropical multiplicity of the point $v \in \bigcap_{i=1}^n \text{Trop}(f_i)$, as we will discuss in §12.

Remark 11.5.1. Bernstein’s theorem [Ber75] can be seen as the generic coefficient (i.e. trivial valuation) case of (11.5); see also [Stu02, Chapter 3] for a proof in the bivariate case. Theorem (11.5) is due to E. Katz in the case of a nontrivial discrete valuation [Kat09, Theorem 8.8]. Osserman and Payne [OPT0] develop an intersection theory over non-noetherian valuation rings in order to remove the noetherian hypothesis (in addition to proving a very general compatibility theorem).

Example 11.6. The following type of example arises in the analysis of the zeros of the logarithm of a $p$-divisible group over $\mathcal{O}_K$ as in [Rab09]. Let $p$ be a prime number and suppose that $\text{val}(p) = 1$. Let $N = M = \mathbb{Z}^2$ and let $x = x^{(-1,0)}, y = x^{(0,-1)} \in K[M]$ as in (8.5.1). Let

$$f_1 = px + x^p + y^p \quad \text{and} \quad f_2 = y + x^p + y^p \quad \in K[M].$$

The tropicalizations and Newton complexes of $f_1$ and $f_2$ are drawn in Figure 11. They intersect in the two points $v_1 = \left(\frac{-1}{p}, \frac{-1}{p}\right)$ and $v_2 = \left(\frac{-1}{p}, 0\right)$. For $i, j = 1, 2$ let $\gamma_{i,j}$ be the cell in $\text{New}(f_i)$ corresponding to $v_j \in \text{Trop}(f_i)$ as in (8.6). Then

$$\gamma_{1,1} = \text{conv}\{(-1,0), (-p,0), (0,0)\} \quad \gamma_{1,2} = \text{conv}\{(-1,0), (-p,0), (0,-p)\}$$
$$\gamma_{2,1} = \text{conv}\{(0,-1), (-p,0), (0,0)\} \quad \gamma_{2,2} = \text{conv}\{(0,-1), (0,-p)\}$$

as indicated in the figure. Since $\gamma_{1,1}$ and $\gamma_{2,1}$ are line segments, their mixed volume can be calculated as in (11.3):

$$\text{MV}(\gamma_{1,1}, \gamma_{2,1}) = \left|\det\begin{pmatrix} p - 1 & p \\ 0 & -1 \end{pmatrix}\right| = p - 1.$$

By (11.5) there are exactly $p - 1$ common zeros $\xi = (\xi_1, \xi_2)$ of $f_1, f_2$, counted with multiplicity, such that $\text{val}(\xi_1) = \frac{1}{p - 1}$ and $\text{val}(\xi_2) = \frac{p - 1}{p - 1}$. The calculation of $\text{MV}(\gamma_{1,2}, \gamma_{2,2})$ requires some grade-school geometry since $\gamma_{1,2}$ is all of $|\text{New}(f_1)|$: we have

$$\text{vol}(\lambda_1 \gamma_{1,2} + \lambda_2 \gamma_{2,2}) = \lambda_1 \lambda_2 ((p - 1)^2 + p - 1) + \lambda_1^2 \frac{p(p - 1)}{2}$$

for $\lambda_1, \lambda_2 \geq 0$, so $\text{MV}(\gamma_{1,2}, \gamma_{2,2}) = p^2 - p$. Hence there are exactly $p^2 - p$ common zeros $\xi$ of $f_1, f_2$ such that $\text{val}(\xi_1) = \frac{1}{p - 1}$ and $\text{val}(\xi_2) = 0$.

The goal of this section is to derive the following generalization of (11.5):
Figure 11. The tropicalizations and Newton complexes of \( f_1 = px + x^p + y^p \) and \( f_2 = y + x^p + y^p \). See (11.6).

**Theorem 11.7.** Let \( P \subset N_{\mathbb{R}} \) be an integral \( \Gamma \)-affine pointed polyhedron, let \( f_1, \ldots, f_n \in K(U_P) \), and let \( v \in \bigcap_{i=1}^n \text{Trop}(f_i) \) be an isolated point contained in the interior of \( P \). For \( i = 1, \ldots, n \) let \( Y_i = V(f_i) \) and let \( \gamma_i \in \text{New}(f_i) \) be the polytope corresponding to \( v \in \text{Trop}(f_i) \) as in (11.5). Then

\[
i(v, Y_1 \cdots Y_n) = \text{MV}(\gamma_1, \ldots, \gamma_n).
\]

**Example 11.8.** (The theorem of the Newton polygon) Let \( N = M = \mathbb{Z} \) and let \( x = x^{(-1)} \in K[M] \) as in (7.5). Let \( r \in \Gamma \) and \( \rho = \exp(-r) \) and let \( P = [r, \infty) \subset N_{\mathbb{R}} \), so \( U_P = B^1_K(\rho) \) and \( K(U_P) = (\sum_{n \geq 0} a_n x^n : |a_n| \rho^n \to 0) \) as in (6.7). Let \( f \in K(U_P) \) be nonzero, and assume for simplicity that \( f(x) \neq 0 \). As explained in (7.10), a number \( v > r \) is the valuation of a zero of \( f \) if and only if \( \text{conv}(\text{vert}_v(f)) \) is a line segment in the lower convex hull \( \text{NP}^v(f) \) of \( H(f, \{0\}) \), in which case the slope of the segment is \( v \). The polytope \( \gamma = \pi(\text{conv}(\text{vert}_v(f))) \in \text{New}(f) \) is the projection of \( \text{conv}(\text{vert}_v(f)) \) onto the \( x \)-axis; it is a line segment whose length \( L \) is exactly the horizontal length of \( \text{conv}(\text{vert}_v(f)) \). See Figure 8. Therefore (11.7) implies that there are exactly \( L \) zeros \( \xi \) of \( f \), counted with multiplicity, such that \( \text{val}(\xi) = v \).

We will use the following consequence of (10.2):

**Corollary 11.9.** (to 10.2) Let \( P \subset N_{\mathbb{R}} \) be an integral \( \Gamma \)-affine pointed polyhedron and let \( f_1, \ldots, f_n \in K(U_{P,t}) := K(t(t(U_P)) \). Let \( Y_i \subset U_P \times B^1_K \) be the subspace defined by \( f_i \), let \( \pi : Y_i \to B^1_K \) be the projection onto the second factor, and for \( t_0 \in B^1_K \) let \( Y_{i,t_0} = \pi^{-1}(t_0) \subset \kappa(t_0) \otimes_K U_P \). Then for any \( v \in N_{\mathbb{R}} \cap P \) such that \( i(v, Y_{1,t_0} \cdots Y_{n,t_0}) = \infty \) there exists \( \epsilon \in |K^x| \) such that

\[
i(v, Y_{1,t_0} \cdots Y_{n,t_0}) = i(v, Y_{1,0} \cdots Y_{n,0}) \quad \text{whenever} \quad |t_0| \leq \epsilon.
\]

We will also need a device for approximating a power series by a sequence of polynomials fitting into a one-parameter family:

**Lemma 11.10.** Fix a nonzero element \( \varpi \in m_K \). Let \( P \subset N_{\mathbb{R}} \) be an integral \( \Gamma \)-affine pointed polyhedron with cone of unbounded directions \( \sigma \), and let \( f \in K(U_P) \) be nonzero. There is a power series \( g \in K(U_P, t) \) such that \( g_0 = f \) and \( g_m \in K[S_\sigma] \) for all \( m \geq 1 \), where for \( t_0 \in |B^1_K| \) we let \( g_{t_0} \) denote the specialization of \( g \) at \( t = t_0 \). In particular, \( g_{t_0} \to f \) in \( K(U_P) \) as \( m \to \infty \).

**Proof.** For \( m \geq 1 \) define

\[
q_m(t) = (t - \varpi)(t - \varpi^2) \cdots (t - \varpi^m)(t - (-1)^m \varpi^{-m(m+1)/2}) \in K[t],
\]

so \( q_m(\varpi^i) = 0 \) for \( i = 1, \ldots, m \) and \( q_m(0) = 1 \). Write \( f = \sum_{u \in S_\sigma} a_u x^u \). Choose a denumeration \( \delta : S_\sigma \xrightarrow{\sim} \mathbb{Z}_{\geq 0} \), and find a sequence of numbers \( m_N \), tending to \( \infty \) as \( N \to \infty \), such that

\[
|q_{m_N}(u)| \cdot |a_u x^u|_\rho \to 0 \quad \text{as} \quad \delta(u) \to \infty,
\]
where $|q_{m,i(v)}|$ denotes the supremum norm of $q_{m,i(v)}$ in $K(t)$. Set

$$g = \sum_{u \in S_r} q_{m,i(v)}(t) a_u x^n \in K(U_p, t).$$

By construction, $g_{\infty} = g_{\infty} \in K[S_r]$ for all $m \geq 1$ and $g_0 = f$.

**Proof of (11.7).** It follows from (9.6) as applied to a small polytope containing $v$ in its interior that $i(v, Y_1 \cdots Y_n) < \infty$. For $i = 1, \ldots, n$ let $g_i \in K(U_p, t)$ be as in (11.10), so $p_{i,m} := g_i \in K[M]$ for all $m \geq 1$, and $p_{i,m} \to f_i$ as $m \to \infty$. Let $Y_{i,m} = V(p_{i,m})$. By (8.5) we have $\text{vert}_P(f_i) = \text{vert}_P(p_{i,m})$ for $m \gg 0$, so $\text{New}(f_i) = \text{New}(p_{i,m})$ and $\text{Trop}(f_i) \cap P = \text{Trop}(p_{i,m}) \cap P$ for all $i$ and all $m \gg 0$ (see (8.6.4); hence if $\gamma_{i,m} = \pi(\text{conv}(\text{vert}_p(p_{i,m})))$ then $\gamma_{i,m} = \gamma_i$ for $m \gg 0$. By (11.9) we likewise have $i(v, Y_1 \cdots Y_m) = i(v, Y_{1,m} \cdots Y_{n,m})$ for $m \gg 0$. Thus for $m \gg 0$,

$$i(v, Y_1 \cdots Y_m) = i(v, Y_{1,m} \cdots Y_{n,m}) = \text{MV}(\gamma_{1,m}, \ldots, \gamma_{n,m}) = \text{MV}(\gamma_1, \ldots, \gamma_n)$$

by (11.5).

**Remark 11.11.** It would be interesting to investigate a more general relationship between the local intersection theory of tropical varieties with a non-Archimedean toric intersection theory along the lines of Osserman and Payne’s work.

### 12. Application: tropically non-proper complete intersections

**12.1.** Let $f_1, \ldots, f_n \in K[M]$ be nonzero, let $Y = \bigcap_{i=1}^n V(f_i)$, and let $C$ be a connected component of $\bigcap_{i=1}^n \text{Trop}(f_i) \subset N_R$. If $C = \{v\}$ consists of a single point then (11.5) calculates the sum $\sum_{\text{trop}(\xi) = v} \dim_K(\partial Y, \xi)$ in terms of a mixed volume. The main goal of this section is to generalize this result to the case when $C$ is arbitrary. More precisely, after taking the closure $\overline{C}$ of $C$ in an appropriate compactification $N_R(\Delta)$ of $N_R$ and taking the closure $\overline{Y}$ of $Y$ in the corresponding toric variety $X(\Delta)$, the size of the algebraic intersection $\sum_{\text{trop}(\xi) \in \overline{C}} \dim_K(\partial Y, \xi)$ lying above $\overline{C}$ can be calculated in terms of stable tropical intersection multiplicities. See (12.11). The compactification step is necessary: see (12.11). Along the way we will obtain a new proof that the stable tropical intersection multiplicity is well-defined in the case of a dimension-zero torus intersection.

The idea is to translate each $V(f_i)$ by a generic point of the torus in order to reduce our problem to (11.5); the key ingredient is the continuity of roots result (9.3) which allows us to relate the intersection multiplicities before and after the translation. It is important to notice that one is led to work with families of translations parametrized by an affinoid subspace of a torus and not by a scheme; cf. (9.10). This rigid-analytic deformation technique is what makes the algebraic result (12.11) possible.

**Remark 12.1.1.** We have chosen work with Laurent polynomials in this section mainly for simplicity of formulation; most of the ideas also apply to power series.

**12.2. Stable tropical intersection multiplicities.** There is a rich intersection theory of tropical varieties, developed in many papers including [AK10, Kat09a, Mik06, RGST05]. Basic to all of these theories is the notion of the stable tropical intersection, which is entirely combinatorial. As we are restricting ourselves to the case of dimension-zero complete intersections, we will take a pedestrian approach and give a direct definition of the stable tropical intersection multiplicity of $n$ hypersurfaces in an $n$-dimensional torus along a connected component.

**Definition 12.3.** Let $P = \bigcap_{i=1}^r \{v \in N_R : (u_i, v) \leq a_i\}$ be an integral $\Gamma$-affine polyhedron in $N_R$, where $u_i \in M$ and $a_i \in \Gamma$. A **thickening** of $P$ is a polyhedron of the form

$$P' = \bigcap_{i=1}^r \{v \in N_R : (u_i, v) \leq a_i + \varepsilon\}$$
for $\varepsilon > 0$ contained in $\Gamma$. More generally, if $\Pi$ is a polyhedral complex then a thickening $\mathcal{P}$ of $\Pi$ is a set of the form $\mathcal{P} = \{ P' : P \in \Pi \}$, where $P'$ denotes a thickening of $P$. We set

$$|\mathcal{P}| = \bigcup_{P' \in \mathcal{P}} P'$$

and

$$\hat{\mathcal{P}} = \bigcup_{P' \in \mathcal{P}} (P')^o \subset |\mathcal{P}|^o.$$  

If $\mathcal{P}' = \{ P'' : P \in \Pi \}$ is a second thickening of $\Pi$, we say that $\mathcal{P}'$ dominates $\mathcal{P}''$ if $P'' \subset (P')^o$ for all $P \in \Pi$.

**Remark 12.4.**

(i) If $P'$ is a thickening of $P$ then $P$ is contained in the interior $(P')^o$ of $P'$, and hence if $\mathcal{P}$ is a thickening of $\Pi$ then $|\Pi| \subset \bar{\mathcal{P}} \subset |\mathcal{P}|^o$.

(ii) If $P'$ is a thickening of $P$ then $U(P) = U(P')$.

(iii) If $\Pi$ is a polyhedral complex and $C \subset |\Pi|$ is a connected component then $C$ is the support of the subcomplex $\Pi_C$ of $\Pi$ whose cells are contained in $C$. There is a thickening $\mathcal{P}$ of $\Pi_C$ such that $|\mathcal{P}| \cap |\Pi| = C$.

Recall (8.5) that if $f_1, \ldots, f_n \in K[M]$ are nonzero then each $\text{Trop}(f_i)$ is (the support of) a canonical polyhedral complex, and therefore $\bigcap_{i=1}^n \text{Trop}(f_i)$ is also canonically a polyhedral complex. The following lemma is standard, but we include a proof for completeness:

**Lemma 12.5.** (Moving lemma) Let $f_1, \ldots, f_n \in K[M]$ be nonzero, let $C$ be a connected component of $\bigcap_{i=1}^n \text{Trop}(f_i)$, and let $\mathcal{P}$ be a thickening of (the complex underlying) $C$ such that $|\mathcal{P}| \cap \bigcap_{i=1}^n \text{Trop}(f_i) = C$. Then there exist $v_1, \ldots, v_n \in N$ and $\varepsilon \in R_{>0} \cap \Gamma$ such that for all $t \in (0, \varepsilon]$, the intersection

$$|\mathcal{P}| \cap \bigcap_{i=1}^n (\text{Trop}(f_i) + tv_i)$$

is a finite set of points contained in $\hat{\mathcal{P}}$.

**Proof.** Each $\text{Trop}(f_i)$ is a subset of a hyperplane arrangement in $N_R \cong R^n$, so we can find $v_i$ and $\varepsilon$ such that $|\mathcal{P}| \cap \bigcap_{i=1}^n (\text{Trop}(f_i) + tv_i)$ is a finite set of points for $t \leq \varepsilon$, since the intersection of $n$ affine hyperplanes in $R^n$ generically contains zero or one points. Furthermore, the union of the boundaries of the polyhedra in $\mathcal{P}$ is also contained in a hyperplane arrangement, so we can choose $\varepsilon$ such that $|\mathcal{P}| \cap \bigcap_{i=1}^n (\text{Trop}(f_i) + tv_i) \subset \hat{\mathcal{P}}$ for $t \leq \varepsilon$ as well since $n+1$ affine hyperplanes in $R^n$ generically have no points of intersection. \qed

**12.6.** Let $T = \text{Spec}(K[M])$, let $v \in N_T$, and choose $\xi \in T(K')$ with $\text{trop}(\xi) = v$, where $K'$ is a suitable finite extension of $K$. Then $\xi$ induces the translation automorphism $\eta \mapsto \xi \cdot \eta$ of $T_K' = K' \otimes_K T$, which corresponds to the automorphism $x^u \mapsto x^u(\xi)x^u$ of $K'[M]$. We denote the image of $f \in K'[M]$ under this automorphism by $\xi \cdot f$. Since $\text{trop}(\xi \cdot \eta) = \text{trop}(\eta) + v$, we have $\text{Trop}(\xi \cdot f) = \text{Trop}(f) + v$, and since $\text{Trop}(f)$ and $\text{New}(f)$ only depend on the valuations of the coefficients of $f$, the complexes $\text{Trop}(\xi \cdot f)$ and $\text{New}(\xi \cdot f)$ are independent of the choice of $\xi \in \text{trop}^{-1}(v)$.

**Definition 12.7.** Let $f_1, \ldots, f_n \in K[M]$ be nonzero and let $v \in \bigcap_{i=1}^n \text{Trop}(f_i)$ be an isolated point. The stable tropical intersection multiplicity of $\text{Trop}(f_1), \ldots, \text{Trop}(f_n)$ at $v$ is defined to be

$$i(v, \text{Trop}(f_1) \cdots \text{Trop}(f_n)) = \text{MV}(\gamma_1, \ldots, \gamma_n),$$

where $\gamma_i \in \text{New}(f_i)$ is the polytope corresponding to $v \in \text{Trop}(f_i)$ as in (11.5). Now let $C \subset \bigcap_{i=1}^n \text{Trop}(f_i)$ be a connected component and let $\mathcal{P}, v_1, \ldots, v_n \in N$, and $\varepsilon \in R_{>0} \cap \Gamma$ be as in 12.5. The stable tropical intersection multiplicity of $\text{Trop}(f_1), \ldots, \text{Trop}(f_n)$ along $C$ is defined to be

$$i(C, \text{Trop}(f_1) \cdots \text{Trop}(f_n))$$

$$= \sum \left\{ i(v, (\text{Trop}(f_1) + \varepsilon v_1) \cdots (\text{Trop}(f_n) + \varepsilon v_n)) : v \in |\mathcal{P}| \cap \bigcap_{i=1}^n (\text{Trop}(f_i) + \varepsilon v_i) \right\},$$

which makes sense by 12.5 and 12.6.

See 12.13 for an example.
**Remark 12.7.1.** The above definition of $i(C, \text{Trop}(f_1) \cdots \text{Trop}(f_n))$ agrees with the sum of the multiplicities of the points of the stable intersection $\text{Trop}(f_1) \cdots \text{Trop}(f_n)$ contained in $C$; see [ST08] Theorem 4.6. Ordinarily one proves that this number is well-defined using the balancing condition on a tropical variety, but it will also follow from (12.11).

**Notation 12.8.** For a nonzero Laurent polynomial $f = \sum a_u x^u \in K[M]$ we denote the normal fan to $|\text{New}(f)| = \text{conv}\{u : a_u \neq 0\}$ by $\Delta(f)$.

Note that $\Delta(f)$ is a complete fan.

**Example 12.8.1.** Let $M = N = \mathbb{Z}^2$ and let $x = x^{(-1,0)}$, $y = x^{(0,-1)} \in K[M]$ as in (8.5.1). Let $\lambda \in K^\times$ have valuation 2 and let $f = 1 + x + y + \lambda xy \in K[M]$, so $\text{Trop}(f)$ and $\text{New}(f)$ are drawn in Figure 2.

The unbounded cells of $\text{Trop}(f)$ are labeled $P_1$, $P_2$, $P_4$, $P_5$ in the figure; their cones of unbounded directions are the positive and negative coordinate axes. The Newton polytope $|\text{New}(f)|$ is the unit square, so $\Delta(f)$ is the fan of Figure 4. Note that the positive and negative coordinate axes are cones of $\Delta(f)$.

The positive-dimensional cones in $\Delta(f)$ represent the directions in which $\text{Trop}(f)$ is unbounded.

**Lemma 12.9.**

(i) Let $f = \sum a_u x^u \in K[M]$ be a nonzero Laurent polynomial and let $P$ be a cell of $\text{Trop}(f)$. Then $U(P) \in \Delta(f)$.

(ii) Let $f_1, \ldots, f_n \in K[M]$ be nonzero and let $P$ be a cell of $\bigcap_{i=1}^n \text{Trop}(f_i)$. Then $U(P)$ is a cone of $\bigcap_{i=1}^n \Delta(f_i)$.

**Proof.** The second part follows from the first by (3.15) and (3.16), so we proceed with (i). By definition (3.5) there is a point $v \in \text{Trop}(f)$ such that

$$P = \gamma_v = \{v' \in N_R : \text{vert}_v(f) \supset \text{vert}_v(f)\}.$$ 

Let $\tilde{\gamma}_v = \pi(\text{conv}(\text{vert}_v(f))) \in \text{New}(f)$ be the dual cell as in (8.6). We claim that

$$U(P) = \{v \in N_R : \text{face}_v(|\text{New}(f)|) \supset \tilde{\gamma}_v\}.$$ 

First notice that the right side of (12.9.1) is the cone of $\Delta(f)$ corresponding to the minimal face of $|\text{New}(f)|$ containing $\tilde{\gamma}_v$ (so $P$ is unbounded if and only if $\tilde{\gamma}_v$ is contained in the boundary of $|\text{New}(f)|$).

Let $v \in U(P)$ and let $(u, \text{val}(a_u)) \in \text{vert}_v(f)$; we want to show that $\langle u, v \rangle = \max\{\langle u', v \rangle : a_{u'} \neq 0\}$. Fix $v_1 \in P$. For any $\lambda \in R_{\geq 0}$ we have $v_1 + \lambda v \in P$ by (3.14), i.e. $\text{vert}_{v_1 + \lambda v}(f) \supset \text{vert}_v(f)$, so

$$\text{val}(a_u) - \langle u, v \rangle - \lambda(u, v) = \min\{\text{val}(a_{u'}): \langle u', v_1 \rangle - \lambda(u', v) : a_{u'} \neq 0\}.$$ 

If there were some $u'$ with $a_{u'} \neq 0$ and $\langle u', v \rangle > \langle u, v \rangle$ then we could make (12.9.2) false by taking $\lambda \gg 0$. This proves one inclusion of (12.9.1). On the other hand, if $v \in N_R$ satisfies $\text{face}_v(|\text{New}(f)|) \supset \tilde{\gamma}_v$ then a similar argument shows that $v_1 + v \in P$ for any $v_1 \in P$, so the other inclusion also follows from (3.14).

Hence if $f_1, \ldots, f_n \in K[M]$ are nonzero Laurent polynomials then $N_R(\bigcap_{i=1}^n \Delta(f_i))$ is a natural compactification of $N_R$ in which to take the closure of $\bigcap_{i=1}^n \text{Trop}(f_i)$.

**Remark 12.10.** Let $f_1, \ldots, f_n \in K[M]$ be nonzero, and suppose that the fan $\Delta = \bigcap_{i=1}^n \Delta(f_i)$ is not pointed. Then there is a proper subspace $M'_R \subset M_R$ and elements $u_i \in M_R$ such that $\text{New}(f_i) \subset u_i + M'_R$. In this case, a Minkowski sum of cells of the $\text{New}(f_i)$ is also contained in a translate of $M'_R$, so all mixed volumes appearing in the definition of $i(C, \text{Trop}(f_1) \cdots \text{Trop}(f_n))$ are zero for any connected component $C \subset \bigcap_{i=1}^n \text{Trop}(f_i)$. This is the “overdetermined” or “degenerate” case, and for this reason we will generally assume that $\Delta$ is pointed.

The rest of this section is devoted to proving that the purely combinatorially defined quantity $i(C, \text{Trop}(f_1) \cdots \text{Trop}(f_n))$ calculates algebraic intersection multiplicities in the following sense:

**Theorem 12.11.** Let $f_1, \ldots, f_n \in K[M]$ be nonzero Laurent polynomials, and assume that the fan $\Delta = \bigcap_{i=1}^n \Delta(f_i)$ is pointed. Let $C \subset \bigcap_{i=1}^n \text{Trop}(f_i)$ be a connected component and let $C$ be the closure of
Let \( Y_i \) be the closure of \( V(f_i) \) in \( X(\Delta) \) and let \( Y = \bigcap_{i=1}^n Y_i \). Then

\[
i(C, \Trop(f_1) \cdots \Trop(f_n)) = \sum_{\trop(\xi) \in C} \dim_K(\mathcal{O}_{Y, \xi})
\]

if the right side is finite.

**Remark 12.11.2.** If \( C \) is a polyhedron then the right side of (12.11.1) is automatically finite by (9.6) and (12.14) below.

**Corollary 12.12.** Let \( f_1, \ldots, f_n \in K[M] \) be nonzero and let \( C \subset \bigcap_{i=1}^n \Trop(f_i) \) be a connected component. Then \( i(C, \Trop(f_1) \cdots \Trop(f_n)) \) is independent of all choices.

**Remark 12.11.1.** The above corollary is a purely tropical result: it only depends on \( f_1, \ldots, f_n \) through the valuations of their coefficients, and hence can be stated in terms of tropical polynomials. Thus it can be seen as an application of rigid geometry to "pure" tropical geometry.

**Example 12.13.** Let \( M = N = \mathbb{Z}^2 \) and let \( x = x^{(-1,0)}, y = x^{(0,-1)} \in K[M] \) as in (8.5.1). Choose \( \alpha, \beta \in K^\times \) of valuation zero, and let

\[
f_1 = x + y + 1 \quad \text{and} \quad f_2 = \alpha x + \beta y + 1 \quad \in \ K[M].
\]

A picture of \( \Trop(f_1) = \Trop(f_2) \) and \( \New(f_1) = \New(f_2) \) can be found in Figure 2 (with \( \lambda = 1 \)). Hence \( C = \Trop(f_1) \cap \Trop(f_2) \) is a connected component, and one easily calculates that \( i(C, \Trop(f_1) \cdot \Trop(f_2)) = 1 \). The fan \( \Delta = \Delta(f_1) = \Delta(f_2) \) and the completion \( N_R(\Delta) \) are described in (3.7) and drawn in Figure 5 the associated toric variety is \( X(\Delta) = \mathbb{P}_K^2 \). The closure \( \overline{C} \) of \( C = \Trop(f_1) = \Trop(f_2) \) is

\[
\overline{C} = C \amalg \{|0 : 0 : \infty\} \amalg \{|0 : \infty : 0\} \amalg \{|\infty : 0 : 0\}
\]

with the notation in (5.10).

Algebraically, let \( Y_i \) be the closure of \( V(f_i) \) in \( \mathbb{P}_K^2 \times X(\Delta) \) and let \( Y = Y_1 \cap Y_2 \) as in (12.11). Then \( Y \) consists of the single point

\[
(\xi, \eta) = \left( \frac{\beta - 1}{\alpha - \beta}, \frac{\alpha - 1}{\beta - \alpha} \right)
\]

as long as \((\alpha, \beta) \neq (1,1)\). We can choose \( \alpha \) and \( \beta \) so that \( \Trop(Y) \) is located anywhere on \( \overline{C} \):

- If \( \val(\beta - \alpha) \gg 0 \) but \( \val(\beta - 1) = \val(\alpha - 1) = 0 \) then \( \Trop(Y) \) is a point on the ray \( R_1 \) of Figure 9.
- If \( \alpha = \beta \) then \( \Trop(Y) = \{|0 : 0 : \infty\} \).
- If \( \val(\alpha - 1) \gg 0 \) but \( \val(\beta - 1) = 0 \) then \( \Trop(Y) \) is a point on the ray \( R_2 \) of Figure 9.
- If \( \alpha = 1 \) then \( \Trop(Y) = \{|0 : \infty : 0\} \).
- If \( \val(\beta - 1) \gg 0 \) but \( \val(\alpha - 1) = 0 \) then \( \Trop(Y) \) is a point on the ray \( R_3 \) of Figure 9.
- If \( \beta = 1 \) then \( \Trop(Y) = \{|\infty : 0 : 0\} \).

Hence we need to consider all points \( \xi \in Y \) with \( \trop(\xi) \in \overline{C} \) in (12.11.1).

We will prove (12.11) and (12.12) below. First we investigate the relationship between the closure of a subscheme of a torus inside a toric variety and the closure of its tropicalization. For a different treatment see [OP10, §3].

**Proposition 12.14.**

(i) Let \( A \) be an integral domain, let \( f \in A[M] \) be nonzero, let \( \sigma' \in \Delta(f) \), and let \( \sigma \) be an integral pointed cone contained in \( \sigma' \). Then there is a vertex \( u \in M \) of \( |\New(f)| \), depending only on \( |\New(f)| \) and \( \sigma \), such that \( A[S_\sigma]x^{-u}f = (A[M]f) \cap A[S_\sigma] \).

(ii) With the notation in (i), suppose that \( A = K \). Let \( T = \Spec(K[M]) \), let \( Y = V(f) \subset T \), and let \( \overline{Y} \) be the closure of \( Y \) in \( X(\sigma) \). Then \( \overline{Y} \) is cut out by \( x^{-u}f \) and \( \Trop(\overline{Y}, N_R(\sigma)) \) is the closure of \( \Trop(f, N_R) \) in \( N_R(\sigma) \).
(iii) Let $f_1, \ldots, f_n \in K[M]$ be nonzero and suppose that $\Delta = \bigcap_{i=1}^n \Delta(f_i)$ is pointed. Then the closure of $\bigcap_{i=1}^n \Trop(f_i)$ in $N_{\mathbb{R}}(\Delta)$ is equal to $\bigcap_{i=1}^n \overline{Trop}(f_i)$, where $\overline{Trop}(f_i)$ is the closure of $Trop(f_i)$ in $N_{\mathbb{R}}(\Delta)$.

(iv) With the notation in (iii), let $Y_i$ be the closure of $V(f_i)$ in $X(\Delta)$ and let $Y = \bigcap_{i=1}^n Y_i$. Then $\Trop(Y)$ is contained in the closure of $\bigcap_{i=1}^n \Trop(f_i)$ in $N_{\mathbb{R}}(\Delta)$.

**Proof.** We may assume that $\sigma'$ is a maximal cone of $\Delta(f)$, so there is a vertex $u$ of $|\New(f)|$ such that $\sigma' = \{u' \in N_{\mathbb{R}} : u \in \face_{u'}(|\New(f)|)\}$. We claim that this $u$ works. One checks that if $f' = x^{-u}f$ then $f' \in A(S_{\sigma'}) \subset A(S_{\sigma})$. Let $g \in A(M)$, and suppose that $fg \in A(S_{\sigma})$. We have $|\New(f'g)| = |\New(f')| + |\New(g)|$ by Proposition 6.1.2(b), so since $|\New(f'g)| \subset \sigma'$ and $0 \in |\New(f')|$ we have $|\New(g)| \subset \sigma''$, i.e. $g \in A(S_{\sigma})$. This proves (i).

The closure $\overline{\mu}$ of $Y$ is the hypersurface in $X(\sigma)$ cut out by $f'$ by the above. Since $\Trop(f', N_{\mathbb{R}}) = \Trop(f, N_{\mathbb{R}})$ we may replace $f$ by $f'$. Since $\Trop(f, N_{\mathbb{R}}(\sigma))$ contains $\Trop(f, N_{\mathbb{R}})$ it also contains the closure $\Trop(f, N_{\mathbb{R}})$. Let $\tau \prec \sigma$ be nonzero, let $N_{\mathbb{R}}' = N_{\mathbb{R}}/\.span(\tau)$, let $\pi_\tau : N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}'$ be the projection, and let $v_0 \in \Trop(f, N_{\mathbb{R}}(\sigma)) \cap N_{\mathbb{R}}'$. Fix $(u_0, \val(u_0)) \in \vert_{t=0}(f) \subset H(f, \tau)$ in the notation of (7.6), and let $\alpha = \val(u_0) - (u_0, v_0)$. Suppose for the moment that there exists $v_1 \in N_{\mathbb{R}}$ with $\pi_\tau(v_1) = v_0$ such that $\val(u_0) - (u, v_1) > \alpha$ for all $u \in S_{\tau} \setminus \tau^+$ (note $\val(u_0) - (u, v_0) = \val(u_0) - (u, v_0)$ for $u \in \tau^+$). Since $(u, v) \leq 0$ for all $u \in S_{\sigma}$ and $v \in \tau$, we therefore have $\val(u_0) - (u, v_1 + v) > \alpha$ when $u \in S_{\sigma} \setminus \tau^+$, and hence by (7.6.2), $\vert_{t=0}(f) = \vert_{t=0}(f)$ for all $v \in \tau$. But $\vert_{t=0}(f)$ is not a monomial, so $\vert_{t=0}(f)$ is not a monomial, and hence $v_1 + v \in \Trop(f, N_{\mathbb{R}})$ for all $v \in \tau$ by (8.4). But $v_1 + v \rightarrow v_0$ as $v \rightarrow \infty$, so $\Trop(f, N_{\mathbb{R}}) \subset \Trop(f, N_{\mathbb{R}})$.

It remains to prove the existence of such an element $v_1$. Choose any $v_1 \in \pi_{\tau}^{-1}(v_0)$. For $u \in S_{\sigma} \setminus \tau^+$ we can find $v \in \tau$ such that $(u, v) < 0$; replacing $v_1$ with $v_1 + \lambda v$ for $\lambda > 0$ allows us to assume that $\val(u_0) - (u, v_1) > \alpha$. Repeating this procedure for the finitely many $u \in S_{\sigma} \setminus \tau^+$ for which $a_0 \neq 0$ provides the required element $v_1$. This completes the proof of (ii).

Since $N_{\mathbb{R}}(\Delta)$ is covered by the open subspaces $N_{\mathbb{R}}(\sigma)$ for $\sigma \in \Delta$, we will prove (iii) with $N_{\mathbb{R}}(\sigma)$ replacing $N_{\mathbb{R}}(\Delta)$. Let $\tau \prec \sigma$ be nonzero and define $N_{\mathbb{R}}'$ and $\pi_\tau$ as above. The inclusion $\bigcap_{i=1}^n \overline{Trop}(f_i) \subset \bigcap_{i=1}^n \overline{Trop}(f_i)$ is automatic, so let $v_0 \in \bigcap_{i=1}^n \overline{Trop}(f_i) \cap N_{\mathbb{R}}'$. In the proof of (ii) we showed that there exists $v_1 \in \pi_{\tau}^{-1}(v_0)$ such that $v_1 + \tau \subset \Trop(f_i)$ for each $i = 1, \ldots, n$. Since $\tau$ spans $\ker(\pi_{\tau})$, there is some element $v \in \bigcap_{i=1}^n (v_i + \tau)$; then $v + \tau \subset \bigcap_{i=1}^n \Trop(f_i)$, and $v + w \rightarrow v_0$ as $w \rightarrow \infty$, which shows that $v_0 \in \bigcap_{i=1}^n \overline{Trop}(f_i)$. This proves (iii).

The final assertion follows immediately from (ii) and (iii), since $\Trop(Y) \subset \bigcap_{i=1}^n \Trop(Y_i)$.

**Lemma 12.15.** In the situation of (12.5), suppose that $\Delta = \bigcap_{i=1}^n \Delta(f_i)$ is pointed. Then there is a thickening $P'$ of the complex underlying $C$, dominated by $\mathcal{P}$, such that $|P| \cap \bigcap_{i=1}^n (\Trop(f_i) + tv_i) \subset |P'|$ for all $t \in [0, \varepsilon]$.

**Proof.** For $X \subset N_{\mathbb{R}}$ let $\overline{X}$ denote its closure in $N_{\mathbb{R}}(\Delta)$. Let $\Pi$ denote the polyhedral complex underlying $C$. For any $P \in \Pi$ we have $\mathcal{U}(P) \subset \Delta$ by (12.5)ii), and hence $\overline{\mathcal{P}} \subset N_{\mathbb{R}}(\Delta)$. If $P'$ is a thickening of $P$ then $\overline{P'} \subset N_{\mathbb{R}}(\Delta)$ as well, and the interior of $\overline{P'}$ is an increasing union of the closures of smaller thickenings $P''$ of $P$. Hence we can write $\bigcup_{P \in \Pi} (\overline{P'})^\circ$ as an increasing union $\bigcup_{i=1}^\infty \overline{P_i}$, where each $P_i$ is dominated by $\mathcal{P}$. Consider the set

$$D = |P| \cap \bigcup_{t \in [0, \varepsilon]} \bigcap_{i=1}^n (\Trop(f_i) + tv_i).$$

By (12.14)iii) when $t \in (0, \varepsilon]$ we have

$$\bigcap_{j=1}^n (\Trop(f_i) + tv_i) \supset \bigcap_{j=1}^n (\Trop(f_i) + tv_i) \subset \bigcap_{j=1}^n (\Trop(f_i) + tv_i) \subset \bigcap_{j=1}^n (\Trop(f_i) + tv_i) \subset \overline{P_i}.$$
since the right side is a finite set of points contained in \( \mathcal{P} \), and clearly \( \bigcap_{i=1}^{n} \text{Trop}(f_i) \) is contained in \( \bigcup_{\mu \in \mathcal{P}} (\text{Trop}(f_i))^\circ \). Hence \( D \) is covered by \( \bigcup_{i=1}^{\infty} \text{Trop}(f_i) \), so it suffices to show that \( D \) is compact.

For \( i = 1, \ldots, n \) let

\[
D'_i = \bigcup_{t \in [0, \varepsilon]} \{ t \} \times (\text{Trop}(f_i) + tv_i) \subset [0, \varepsilon] \times N_R(\Delta),
\]

so \( D \) is the image of \( ([0, \varepsilon] \times [\mathcal{P}]) \cap \bigcap_{i=1}^{n} D'_i \) under the projection \( [0, \varepsilon] \times N_R(\Delta) \to N_R(\Delta) \). Since \([0, \varepsilon] \times N_R(\Delta)\) is compact, it is enough to show that each \( D'_i \) is closed. But this is clear because

\[
(\text{Trop}(f_i) + tv_i) = \text{Trop}(f_i) + tv_i.
\]

Finally we note that in the case we will be interested in, schematic closure respects fibers:

**Lemma 12.16.** Let \( f = \sum_{u \in M} a_u x^u \in K[M] \) be nonzero and let \( \Delta \) be an integral pointed fan refining \( \Delta(f) \). Let \( v \in N \), and define

\[
g = \sum_{u \in M} a_u x^{u(v)} \in K[M][t^{\pm 1}].
\]

Let \( Y \subset X(\Delta) \times G_m \) be the closure of \( V(g) \), let \( \pi: Y \to G_m \) be projection onto the second factor, and for \( t_0 \in [G_m] \) let \( Y_{t_0} = \pi^{-1}(t_0) \) and let \( g_{t_0} \) be the specialization of \( g \) at \( t_0 \). Then \( Y_{t_0} \) is the closure of \( V(g_{t_0}) \).

**Proof.** Fix \( \sigma \in \Delta \). By (12.14)i there exists \( u_1 \in M \) such that \( Y \cap (X(\sigma) \times G_m) \) is cut out by \( x^{-u_1}g \in K[S_\sigma][t^{\pm 1}] \) and such that the closure of \( V(g_{t_0}) \) is cut out by \( x^{-u_1}g_{t_0} \) (since \( \text{New}(g_{t_0}) = \text{New}(g) \)). But \( Y_{t_0} \cap (X(\sigma) \times G_m) \) is also cut out by \( x^{-u_1}g_{t_0} \).

**Proof of (12.11) and (12.12).** Let \( \mathcal{P} \) be a thickening of \( C \) such that \( |\mathcal{P}| \cap \bigcap_{i=1}^{n} \text{Trop}(f_i) = C \), let \( v_1, \ldots, v_n \in N \) and \( \varepsilon \in R_{>0} \cap \Gamma \) be as in (12.5), and let \( \mathcal{P}' \) be as in (12.15). We may assume without loss of generality that all polyhedra in question are integral \( \Gamma \)-affine (and pointed). Writing \( f_i = \sum_{u \in M} a_{i,u} x^u \), define \( g_i \in K[M][t^{\pm 1}] \). By (12.14), there exist \( u_{i,v} \) such that \( (\text{Trop}(f_i) + tv_i) \subset [0, \varepsilon] \times N_R(\Delta) \). When \( t_0 \) is 1 this implies that \( \text{Trop}(f_i) \subset C \), and \( \text{Trop}(f_i + \delta v_i) \subset [\mathcal{P}] \) when \( \delta \in (0, \varepsilon) \). Therefore the hypotheses of (9.3) are satisfied, so any two finite fibers \( Y_{f,t_0} \) have the same length. By hypothesis \( Y_{f,1} \) is finite, and by (11.7), the length of \( Y_{f,t_0} \) is equal to \( \text{Val}(t_0) \). The corollary is proved as follows. If \( Y_{f,1} \) is finite then we are done, so suppose \( Y_{f,1} \) is not finite. When \( \delta \in (0, \varepsilon) \), the fiber \( Y_{f,t_0} \) is still finite, so by semicontinuity of fiber dimension there exists some \( t_1 \) with \( \text{Val}(t_1) = 0 \) such that \( Y_{f,t_1} \) is finite; we then apply (12.11) to \( g_{t_1} \) to get \( g_{t_1} \) (note that \( \text{Trop}(g_{t_1}) = \text{Trop}(f_i) \) and \( \text{New}(g_{t_1}) = \text{New}(f_i) \)).

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