On Mean Distance and Girth

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Abstract

We bound the mean distance in a connected graph which is not a tree in function of its order \( n \) and its girth \( g \). On one hand, we show that mean distance is at most \( \frac{n+1}{3} - \frac{g(g^2-4)}{12n(n-1)} \) if \( g \) is even and at most \( \frac{n+1}{3} - \frac{g(g^2-1)}{12n(n-1)} \) if \( g \) is odd. On the other hand, we prove that mean distance is at least \( \frac{ng}{2(n-1)} \) unless \( G \) is an odd cycle.

1 Introduction

In the middle of the 80’s, Fajtlowicz has developed a successful conjecture making program called Graffiti (see [14]). Graffiti is a computer program that checks for relationships among certain graph invariants. It has correctly conjectured a number of new bounds for several graphs but even when conjectures were not true, it has often arisen relations between parameters which were not suspected before. In [8], Chung has showed that the mean distance is at most as large as the independence number, which was a conjecture of Graffiti. In [5], Brewster, Dinneen and Faber have given a list of counterexamples for over forty Graffiti’s conjectures. Quite a few Graffiti’s conjectures remain open and a large number has been refuted.

Later on, Caporossi and Hansen ([7]) have created a system devoted to the same task: auto-generating conjectures, they called it AutoGraphiX (AGX for short).

The first motivation of this work was to solve two AGX’s conjectures dealing with mean distance and proposed in [1], even if at the end these conjectures have not been shown in their original form.

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We use the notations of Berge [3]. In all what follows, we consider simple, finite and undirected graphs. We denote respectively by $V(G)$ and $E(G)$ the set of vertices and the set of edges of $G$. We put $n = |V(G)|$, $m = |E(G)|$ and we denote by $\delta(G)$ the minimum degree of $G$. The distance between two vertices $x$ and $y$ in $V(G)$ is denoted by $d_G(x, y)$ and is defined as the length of a shortest path joining them. The girth of a graph $G$, denoted by $g$, is the length of a shortest cycle in $G$; it is infinite if $G$ is a forest. The neighborhood of a vertex $u \in V(G)$, is denoted by $N_G(u)$ and is defined by $N_G(u) = \{x \in V(G) / (x, u) \in E(G)\}$. We omit the letter $G$ from symbols when only one graph is considered. If $H$ is a subgraph of $G$ then $G - H$ denotes the subgraph of $G$ induced by $V(G) - V(H)$.

The mean distance in a connected graph is denoted by $\bar{d}$ and is given by the average value $\bar{d} = (\sigma(G))/(n(n - 1))$, where $\sigma(G) = \sum_{x,y \in V} d(x, y)$. It is also worth saying that $\sigma(G)$ is called the transmission of $G$ and that mean distance is used (among other) in studying efficiency of networks. Good networks are often characterized by a small mean distance (see for instance [16]). In [12], Doyle and Graver have observed that the least upper bound of the mean distance over all connected graphs on $n$ vertices is $(n + 1)/3$, it is the mean distance of a path on $n$ vertices. In [24], Winkler has showed that for 2-connected graphs the upper bound on mean distance can be sharpened, he has obtained the bound $n^2/(4n - 4)$. In the same paper, he has posed a conjecture dealing with mean distance, called "the four-thirds conjecture" at which Kouider has looked in [19]. She has proved that "any 4-connected graph contains a vertex whose removal increases the mean distance by less than a factor of 4/3". She has moreover showed that the same holds for 2-connected graphs of order at least 150 and minimum degree at least 3. Many authors have used graph parameters to bound mean distance. That was what Kouider and Winkler have done (in [18]) using the minimum degree. They have showed that mean distance is at most $2 + n/\delta + 1$ (which is an asymptotically slightly stronger form of a Graffiti conjecture). Quiet recently, this bound has been improved by Beezer, Riegsecker and Smith in [2]. The three authors have proved that mean distance is at most $(n + 1)/\delta + 1) - \{(2m)/((\delta + 1)(n^2 - n))\}$. In [21] and [17], bounds for mean distance in certain classes of graphs (bipartite, planar, triangle-free, self-complementary) have been given. In [10], Dankelmann and Entringer have studied the problem of finding a spanning tree with small mean distance in a graph. As a consequence they have given bounds on the mean distance in Triangle-free and $C_4$-free graphs. Other approaches using the eigenvalues of either the adjacency or the Laplacian matrix have been used to bound mean distance. Mohar in [20] has bounded above the mean distance in
function of the maximum degree and the second smallest Laplacian eigenvalue of a graph. Later, Rodriguez and Yebra [22] have improved bounds on the mean distance using the eigenvalues of the Laplacian matrix. In [23], results on the existence of a graph with a prescribed mean distance have been presented. Another variant of mean distance, namely weighted mean distance has been investigated by Djelloul and Kouider in [11]. They have obtained among other an upper bound on the mean distance for weighted multigraphs with prescribed edge connectivity. For other results on mean distance in graphs, the reader is invited to consult [4, 6, 9, 13, 21].

In this work, we take interest in 2 conjectures proposed in [1] and that give bounds, in function of the order of \( G \), for the product and the quotient of the mean distance and the girth of \( G \). We show the first conjecture in a stronger form. The second conjecture has already been refuted in case \( \delta \) is equal to 1 ([15]). We improve it when \( \delta \) is at least 2. In all what follows the graphs are supposed to contain at least a cycle.

2 Main Results

We take interest in the following two conjectures ([1]) involving relations between mean distance and girth:

**Conjecture 1.** Let \( G \) be a connected graph of girth \( g \) and mean distance \( l \), then:

\[
7 \cdot g \leq \begin{cases} 
\frac{n^3}{4(n-1)} & \text{if } n \text{ is even} \\
\frac{n^2 + n}{4} & \text{if } n \text{ is odd}
\end{cases}
\]

This bound is reached for cycles.

**Conjecture 2.** Let \( G \) be a connected graph of girth \( g \) and mean distance \( l \), then:

\[
\frac{l}{g} \geq \begin{cases} 
\frac{n}{4(n-1)} & \text{if } n \text{ is even} \\
\frac{n + 1}{4n} & \text{if } n \text{ is odd}
\end{cases}
\]

This bound is reached for cycles.

Notice that the first conjecture holds for 2-connected graphs of order \( n \) at least \( g \). Indeed it is a consequence of Winkler’s result. In [24], Winkler has showed that for every 2-connected graph \( G \), the mean distance \( l \) is at
most $\frac{n^2}{4(n-1)}$.

We prove respectively the following improvements of the aforementioned conjectures:

**Theorem 1.** Let $G$ be a connected graph of girth $g$ and mean distance $\bar{d}$. Then:

\[
\bar{d} \leq \begin{cases} 
\frac{n+1}{3} - \frac{g(g^2-4)}{12n(n-1)} & \text{if } g \text{ is even} \\
\frac{n+1}{3} - \frac{g(g^2-1)}{12n(n-1)} & \text{if } g \text{ is odd}
\end{cases}
\]

The bound is reached for cycles.

**Theorem 2.** Let $G$ be a connected graph of minimum degree at least 2, girth $g$ and mean distance $\bar{d}$. Then $\bar{d} \geq \frac{ng}{4(n-1)}$ unless $G$ is a cycle of odd order in which case $\bar{d} = \frac{g+1}{4}$. The bound $\frac{ng}{4(n-1)}$ is achieved for even cycles.

3 Proofs

3.1 Proof of Theorem 1

To prove Theorem 1, it suffices to show the following proposition:

**Proposition 1.** Let $G$ be a connected graph, of girth $g$, then:

\[
\sigma(G) \leq \begin{cases} 
\frac{n(n^2-1)}{3} - \frac{g(g^2-4)}{12} & \text{if } g \text{ is even} \\
\frac{n(n^2-1)}{3} - \frac{g(g^2-1)}{12} & \text{if } g \text{ is odd}
\end{cases}
\]

**Proof of Proposition 1.**

If $n = g$ then $G$ is a cycle thus $\sigma(C_n) = \frac{g^3}{4}$, if $g$ is even and $\sigma(C_n) = \frac{g(g^2-1)}{4}$, if $g$ is odd. So, in this case, proposition 1 obviously holds.

Now, we suppose that $G$ is of order $n \geq g+1$. We take off edges from $G$ so that we reduce it to a graph $G'$ with exactly one cycle and this cycle is of length $g$. The graph $G'$ is the union of a tree $T$ and an edge $e$ that belongs to the cycle of length $g$. Since $G$ is not itself a cycle then $\delta(G') = 1$, furthermore $\sigma(G) \leq \sigma(G')$. Then it suffices to show proposition 1 for $\sigma(G')$.

Let us denote by $C_g$ the cycle of $G'$ and by $a$ and $b$ the endpoints of the cycle.
edge $e \in C_g$. For $x, y \in V$, put $\delta(x, y) = d_T(x, y) - d_{T \cup e}(x, y)$. Notice that $\delta(x, y) \geq 0$. We have that

$$\sigma(T) - \sigma(T \cup e) = \sum_{x, y \in V} \delta(x, y) \geq \sum_{x, y \in C_g} \delta(x, y) \geq 0. \tag{1}$$

Furthermore (see [12] for instance)

$$\sum_{x, y \in C_g} \delta(x, y) = \sigma(C_g - e) - \sigma(C_g) \tag{2}$$

By (1), (2), (3) and setting $P_g = C_g - e$ we obtain

$$\sigma(G') = \sigma(G) \leq \sigma(P_n) = \frac{n(n^2 - 1)}{3} \tag{3}$$

Theorem 1 is a consequence of proposition 1.

The bounds of Theorem 1 are achieved for cycles. Furthermore, we check that they are better than those of Conjecture 1.

Set $\mu_o = \frac{n + 1}{3} - \frac{g(g^2 - 1)}{2n(n - 1)}$, $\mu_e = \frac{n + 1}{3} - \frac{g(g^2 - 4)}{12n(n - 1)}$ in Theorem 1 and $H_e = \frac{n^3}{4(n - 1)g}$, $H_o = \frac{n^2 + n}{4g}$ in Conjecture 1 (index o for odd and e for even). We have that $\mu_o \leq \mu_e$. If we compare $\mu_e$ with $H_e$, then we have that $\frac{\mu_e}{H_e} \leq \frac{4g}{3n} - \frac{g^4}{3n^4}$. Clearly, $\frac{\mu_e}{H_e} = 1$ if $n = g$, and if $n \neq g$, then we verify that $\frac{\mu_e}{H_e} < 1$, furthermore $\frac{\mu_e}{H_e} \to 0$ if $\frac{n}{g} \to \infty$.

If we compare $\mu_e$ with $H_o$, then if $n \geq \frac{4}{3}g$, we check that $\frac{\mu_e}{H_o} \leq 1$. If $n < \frac{4}{3}g$, then showing that $\frac{\mu_e}{H_o} \leq 1$ amounts to show that the function $f(x) = x(3x - 4)(g^2x^2 - 1) + g^2 - 4$ is positive with $x = \frac{n}{g}$. 


3.2 Proof of Theorem 2

We prove the following proposition which yields Theorem 2:

**Proposition 2.** Let $G$ be a connected graph with girth $g$ and minimum degree $\delta \geq 2$, then for every vertex $x \in V$, we have: $\sigma(x) \geq \frac{g.n}{4}$ unless $G$ is an odd cycle in which case $\sigma(x) = (g^2 - 1)/4$. Furthermore the bound $\frac{g.n}{4}$ is reached for even cycles.

First, we point out some simple properties that will be used to prove proposition 2. Let $x_0$ be a vertex of $G$ and let $T$ be the distances tree of $x_0$. We denote by $N_i$ the set of vertices at distance exactly $i$ from $x_0$. $N_i$ is called the $i^{th}$ level in the distances tree $T$.

**Lemma 1.** Let $G$ be a connected graph of minimum degree $\delta \geq 2$ and of girth $g$. Let $x_0$ be a vertex of $G$ and let $T$ be the distances tree of $x_0$ in $G$. Then every leaf of $T$ is at distance:
- at least $\frac{g-1}{2}$ from $x_0$, if $g$ is odd.
- at least $\frac{g}{2} - 1$ from $x_0$, if $g$ is even, moreover in this case at least one leaf of $T$ is at distance at least $\frac{g}{2}$ from $x_0$.

**Proof of lemma 1.**

We first notice that if $y$ belongs to $N_i$, with $i \geq 1$ then $N_G(y) \subseteq N_{i-1} \cup N_i \cup N_{i+1}$. Indeed, if $z$ be a neighbor of $y$ in $G$ then we have $|d(x_0, z) - d(x_0, y)| \leq 1$.

Now to show lemma 1, we consider two cases according to the parity of $g$.

**Case 1:** $g$ is even.

Suppose that $y$ is a leaf and that $i \leq \frac{g}{2} - 2$. As $d_G(y) \geq \delta \geq 2$ then there exists a neighbor $z$ of $y$ in $G - E(T)$ and necessarily $z \in N_i$ with $t \in \{i - 1, i, i + 1\}$. So $(y, z)$ is contained in a cycle of length at most $g - 2$ and hence we obtain a contradiction. Furthermore if all the leaves of $T$ were in $N_{\frac{g}{2} - 1}$, then their neighbors in $G - E(T)$ would be at distance at most $\frac{g}{2} - 1$ from $x_0$ and hence there would exist a cycle of length at most $g - 1$ which is absurd because $G$ is supposed of girth $g$.

**Case 2:** $g$ is odd.

Suppose that $y$ is a leaf and that $i \leq \frac{g - 1}{2} - 1$. As $\delta \geq 2$, there exists a vertex $z$ neighbor of $y$ in $G - E(T)$. We know that $z \in N_{i-1}$ or $N_i$ or $N_{i+1}$. Then $(y, z)$ is contained in a cycle of length at most $g - 1$ which gives a
contradiction.

Finally, every leaf of $T$ is at distance at least $\frac{g-1}{2}$ from $x_0$ if $g$ is odd, at distance at least $\frac{g}{2} - 1$ from $x_0$ if $g$ is even and in this latter case at least one leaf is at distance at least $\frac{g}{2}$ from $x_0$. This ends the proof of Lemma 1.

We say that a tree $T$ has a minimal configuration if and only if
- When $g$ is odd, all the leaves belong to $N_{\frac{g+1}{2}}$.
- When $g$ is even, all the leaves belong to $\overline{N_{\frac{g}{2}-1}}$ and exactly one leaf belongs to $\overline{N_{\frac{g}{2}}}$.

A path $P = u_1u_2...u_k$ in a tree $T'$ of root $x_0$ is called a 2-path, if and only if $(u_j, u_{j+1}) \in E(T')$ for all $j$, $1 \leq j \leq k-1$, $u_1$ is a leaf and $u_k$ is the farthest vertex from $u_1$ such that $d_{T'}(u_j) = 2$, for all $j$, $2 \leq j \leq k$ and such that $x_0$ is not an internal vertex of $P$. In other words, $P$ crosses vertices of degree 2 in $T'$ and it stops when it meets either $x_0$ or a vertex whose neighbor is of degree $\geq 3$ in $T'$.

Notice that the removal of the vertices of a 2-path does not disconnect the tree $T'$.

In order to prove proposition 2, we show the following statement:

**Lemma 2.** Let $T$ be a tree of order $n \geq g$ and of root $x_0$ with $d_T(x_0) \geq 2$.

1. Suppose that $g$ is even, that the leaves of $T$ are at distance at least $\frac{g+1}{2}$ from $x_0$ and that at least one leaf is at distance at least $\frac{g}{2}$ from $x_0$. Then $\sigma_T(x_0) \geq g.n/4$.

2. Suppose that $g$ is odd, that $|V(T)| \geq g+1$ and that every leaf in $T$ is at distance at least $(g-1)/2$ from $x_0$. Then $\sigma_T(x_0) \geq ng/4$.

**Proof of Lemma 2.** We separate the case $g$ even from the case $g$ odd and we proceed in each case by induction on $n$, the order of the tree $T$.

**Case 1:** $g$ is even.

If $n = g$ then by hypothesis, the tree $T$ has 2 branches and one leaf at distance $\frac{g+1}{2}$ from $x_0$ and the other at distance $\frac{g}{2}$ from $x_0$. In this case $\sigma_T(x_0) = 2 \sum_{i=1}^{g/2-1} i + g/2 = g^2/4$.

Now, suppose that lemma 2 holds for all the trees of order between $g$ and $n-1$ and let us prove that it still holds for a tree of order $n$. We distinguish two cases according to the configuration of $T$.

(a) If $T$ has not a minimal configuration then there exists in $T$ either a leaf $u$ at distance at least $g/2+1$ from $x_0$ or a second leaf $u$ at distance at least
$g/2 \geq 2$ from $x_0$. In both cases, the leaves of the tree $T' = T - \{u\}$, which is of order $n - 1$, are at distance at least $g/2-1$ from $x_0$ and at least one leaf is at distance at least $g/2$. Furthermore, as only $u$ has been removed and as $u$ is not a neighbor of $x_0$ then $d_{T'}(x_0) = d_T(x_0) \geq 2$. By induction hypothesis, we have $\sigma_{T'}(x_0) \geq g(n-1)/4$. Hence $\sigma_T(x_0) \geq \sigma_{T'}(x_0) + d(x_0, u) \geq g(n-1)/4 + g/2$ so $\sigma_T(x_0) \geq \frac{g\cdot n}{4}$.

(b) If $T$ has a minimal configuration, then as $n > g$ the tree $T$ has at least 3 branches. We choose a 2-path $P = u_1u_2...u_k$ of $T$ with $u_1$ in $N_{g-1}$ and such that the length $l(P)$ of $P$ is minimum among all the 2-paths of $T$ with leaf in $N_{g-1}$. Let $T'$ be the tree obtained from $T$ by removing $V(P)$. Put $|V(P)| = k$, then $|V(T')| = n-k \geq n-(\frac{g}{2}-1)$. Notice that as $T$ has at least 3 branches and by the choice of $P$, we have $d_{T'}(x_0) \geq 2$. Indeed, either $d_T(x_0) \geq 3$ and removing $P$ yields $d_{T'}(x_0) \geq 2$; or $d_T(x_0) = 2$ and so $l(P) < g/2 - 2$ and deleting $P$ does not change the degree of $x_0$ so $d_{T'}(x_0) = 2$. Furthermore all the leaves of $T'$ are at distance $g/2-1$ from $x_0$ and one leaf is at distance $g/2$. So by induction hypothesis, we have $\sigma_{T'}(x_0) \geq \frac{g(n-k)}{4}$. On the other hand $\sum_{y \in V(P)} d(x_0, y) = ((g/2-1) - (k-1) + ... + (\frac{g}{2} - 1)) = (k/2)(g-k-1)$.

So $\sigma_T(x_0) \geq \sigma_{T'}(x_0) + \sum_{y \in V(P)} d(x_0, y) \geq \frac{gn}{4} + \frac{k}{2}\frac{g-k-1}{2} \geq \frac{gn}{4}$ because $k \leq g/2-1$.

Case 2: $g$ is odd.

If $n = g+1$ then by hypothesis, the tree $T$ has either 2 branches one with a leaf in $N_{g+1}$ and the other with a leaf in $N_{g+1}$ or $T$ has 3 branches each with a leaf in $N_{g+1}$. In either case, we have $\sigma_T(x_0) \geq 2\sum_{i=1}^{\frac{(g-1)}{2}} i + (g-1)/2 = (g-1)(g+3)/4$ and $(g-1)(g+3)/4 \geq (g+1)g/4$ because $g \geq 3$. So the statement of lemma 2 holds for the smallest value of $n$ if $g$ is odd. Suppose now that it holds for all the trees of order between $g+1$ and $n-1$ and let us prove that it still holds if $T$ is a tree of order $n$. According to the configuration of $T$, we distinguish two cases:

(a) If $T$ has not a minimal configuration, then there exists at least a leaf $u$ at distance at least $(g+1)/2 \geq 2$ from $x_0$. Let $T'$ be the tree obtained from $T$ by removing $u$. Since only $u$ has been removed and because $d_T(x_0, u) \geq 2$ then $d_{T'}(x_0) \geq 2$ in $T'$. On the other hand, all the leaves of $T'$ are at distance at least $(g-1)/2$ from $x_0$. So by induction on $T'$, we have $\sigma_{T'}(x_0) \geq \frac{(n-1)}{4}g$ and it follows that $\sigma_T(x_0) \geq \frac{(n-1)}{4}g + \frac{g+1}{2} \geq \frac{n}{4}g + \frac{g}{4} \geq \frac{n}{4}g$ as desired.
(b) If $T$ has a minimal configuration, then $T$ has at least 3 branches. We choose a 2-path $P = u_1...u_k$ of $T$ such that the length $l(P)$ is minimum among all the 2-paths of $T$. Put $|V(P)| = k$ and $T' = T - P$. Since $T$ has at least 3 branches and by the choice of $P$, either $d_T(x_0) \geq 3$ and so $x_0 \notin V(P)$ or $d_T(x_0) = 2$ and $l(P) < (g-1)/2 - 1$. In either case, removing $P$ gives $d_{T'}(x_0) \geq 2$. The leaves of $T'$ are at distance $(g-1)/2$ from $x_0$. Then by induction hypothesis, $\sigma_{T'}(x_0) \geq \frac(n-k)4g$. Furthermore $\sum_{y \in V(P)} d(x_0,y) = (((g-1)/2 - (k-1)) + ... + (g-1)/2) = (kg/2) - (k^2/2)$. So $\sigma_T(x_0) \geq \frac(n-k)4g + \frac{k^2}{2}g - \frac{k}{2} = \frac{g}{4}n + \frac{k}{4}(g-2k)$. Since $k \leq (g-1)/2$, then the right hand side term in the latter inequality is at least $\frac{n}{4}g$ as desired. 

**Proof of Proposition 2.** First of all notice that if $G$ is an odd cycle then $\sigma(x_0) = \frac{(g^2-1)}4$. In the other cases, proposition 2 derives from lemma 2. Indeed, we can associate to a graph $G$ of order $n$ and girth $g$ a distance tree $T$ of a vertex $x_0$ of $G$. As $\delta \geq 2$, then $d_T(x_0) \geq 2$. By lemma 1, every leaf in $T$ is at distance at least $g-1 \over 2$ from $x_0$ if $g$ is odd. Again by lemma 1, if $g$ is even, all the leaves are at distance at least $g-2 \over 2$ from $x_0$ and at least one leaf is at distance at least $g \over 2$ from $x_0$. The tree $T$ verifies the hypothesis of lemma 2. Notice that as $T$ is a distance tree of root $x_0$ then $\sigma_T(x_0) = \sigma_G(x_0)$. Then proposition 2 follows. 

As a corollary of proposition 2, we obtain Theorem 2.

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