On three-dimensional Type I $\kappa$-solutions to the Ricci flow

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In this short note, we prove that the only simply connected noncompact three-dimensional Type I $\kappa$-solution to the Ricci flow is the shrinking cylinder. This work can be regarded as a generalization of Cao, Chow, and Zhang [2], and a complement of Ding [3] and Ni [10]. Up to this point, three-dimensional $\kappa$-solutions of Type I are completely classified, and it remains interesting to work further towards Perelman’s assertion, that the only remaining possibility of three-dimensional noncompact $\kappa$-solution is the Bryant soliton; see [11]. Brendle [1] is working to that end. The classification of three-dimensional $\kappa$-solution is of importance to the study of four-dimensional Ricci flows, because of a possible dimension-reduction procedure.

We remind the reader of the following definition.

**Definition 1.** An ancient solution to the Ricci flow $(M, g(t))_{t \in (-\infty, 0]}$ is called a $\kappa$-solution if it is $\kappa$-noncollapsed on all scales and has bounded curvature on every time slice. A $\kappa$-solution is called Type I if its Riemann curvature tensor satisfies

$$|Rm|(g(t)) \leq \frac{C}{|t|},$$

for all $t \in (-\infty, 0)$, where $C$ is a constant that does not depend on $t$.

It is well-known that every three-dimensional $\kappa$-solution has uniformly bounded and nonnegative sectional curvature.

Our main theorem is the following.

**Theorem 2.** The only three-dimensional simply connected noncompact Type I $\kappa$-solution is the shrinking cylinder.

It is worth mentioning that Ni [10] has proved that a closed Type I $\kappa$-solution with positive curvature operator of every dimension is a shrinking sphere or one of its quotients. On the other hand, Theorem 2.4 in Ding [3] implies that the only simply
connected noncompact $\kappa$-solution that forms a forward singularity of Type I is the shrinking cylinder, Cao, Chow, and Zhang [2] gave an alternative proof with an additional assumption of backward Type I. Furthermore, the author would like to draw the readers’ attention to Hallgren [4], who also classified three-dimensional Type I $\kappa$-solution to the Ricci flow independently, through a more direct approach.

We recall the notion of an $\varepsilon$-neck.

**Definition 3.** A space-time point $(x_0, t_0)$ in a Ricci flow $(M, g(t))$ is called the center of an $\varepsilon$-neck, where $\varepsilon > 0$, if the Ricci flow $g(t)$ on the space-time neighbourhood $B_g(t_0)(x_0, \varepsilon^{-1}R(x_0, t_0)^{-\frac{1}{2}}) \times [t_0 - R(x_0, t_0)^{-1}, t_0]$ is, after parabolic rescaling by the factor $R(x_0, t_0)$, $\varepsilon$-close in the $C^1$-topology to the corresponding part of a standard shrinking cylinder, or in other words, if there exist diffeomorphisms $\phi_t : S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to B_g(t_0)(x_0, \varepsilon^{-1}R(x_0, t_0)^{-\frac{1}{2}})$, such that

$$\phi_t^{-1}(x_0) \in S^2 \times \{0\},$$

$$\left|R(x_0, t_0)\phi_t^* g(t_0 + tR(x_0, t_0)^{-1}) - g_{cyl}(t)\right|_{C^1(S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}))} < \varepsilon,$$

for any $t \in [-1, 0]$. Here the notation $B_g(t_0)(x_0, r)$ stands for the geodesic ball centered at $x_0$, with radius $r$, and with respect to the metric $g(t_0)$, and $g_{cyl}(t)$ represents the standard shrinking metric on $S^2 \times \mathbb{R}$ with $R(g_{cyl}(0)) \equiv 1$.

We remark here that in the above definition, after parabolic scaling, the space-time neighbourhood $B_g(t_0)(x_0, \varepsilon^{-1}R(x_0, t_0)^{-\frac{1}{2}}) \times [t_0 - R(x_0, t_0)^{-1}, t_0]$ has time expansion 1, and the scalar curvature at $(x_0, t_0)$ is normalized to be 1. This definition is called the strong $\varepsilon$-neck by Perelman [11], whereas we keep consistency with the definition in Kleiner-Lott [7] and call it an $\varepsilon$-neck.

The following neck stability theorem by Kleiner and Lott is of fundamental importance to our proof. Please refer to Theorem 6.1 in [7].

**Theorem 4.** For any $\kappa > 0$, there exists a constant $\delta = \delta(\kappa) > 0$, such that for all $\delta_0$, $\delta_1 \leq \delta$, there is a $T = T(\delta_0, \delta_1, \kappa) \in (-\infty, 0)$, with the following property. Let $(M^3, g(t))_{t \in (-\infty, 0]}$ be a noncompact three-dimensional $\kappa$-solution to the Ricci flow that is not the $\mathbb{Z}_2$-quotient of the shrinking cylinder. Let $(x_0, 0) \in M \times \{0\}$ be such that $R(x_0, 0) = 1$. If $(x_0, 0)$ is the center of an $\delta_0$-neck, then for all $t \leq T$, $(x_0, t)$ is the center of a $\delta_1$-neck.

For the remaining of this paper, we fixed a small positive constant $\varepsilon < \min \left\{\frac{1}{100}, \delta(\kappa), \varepsilon_0(\kappa)\right\}$, where $\delta(\kappa)$ is defined in Theorem [4] and $\varepsilon_0$ is the constant given in Corollary 48.1 of Kleiner and Lott [6]. With such $\varepsilon$ we are guaranteed that the
$\varepsilon$-canonical neighbourhood property holds for all $\kappa$-solutions of dimension three. We will use this $\varepsilon$ as the small positive constant in the definition of the $\varepsilon$-neck.

The following lemma is inspired by Ding [3] and Ni [10].

**Lemma 5.** Let $(M^3, g(t))_{t \in (-\infty, 0]}$ be a three-dimensional noncompact Type I $\kappa$-solution with strictly positive sectional curvature on every time slice. Let $p_0$ be an arbitrary fixed point on $M$. Then for every instance $t \in (-\infty, 0]$, there exists a point $p(t) \in M$ such that $(p(t), t)$ is not the center of a $\varepsilon$-neck. Moreover, $\text{dist}_{g(t)}(p_0, p(t)) \to \infty$ as $t \to -\infty$.

**Proof.** First of all, such $p(t)$ must exist for every $t \in (-\infty, 0]$. We know from the Gromoll-Meyer theorem that $M$ is diffeomorphic to $\mathbb{R}^3$. By Corollary 48.1 in Kleiner and Lott [7], such ancient solution must fall into category $B$, on which there is always a cap (the so-called $M_{\varepsilon}$). In particular, since $M$ is diffeomorphic to $\mathbb{R}^3$, the cap is topologically a disk instead of $\mathbb{R}P^3 \setminus B^3$.

Assume by contradiction that there exists $\{t_i\}_{i=1}^{\infty} \subset (-\infty, 0)$, such that $t_i \searrow -\infty$ but $\text{dist}_{g(t)}(p_0, p(t_i)) \leq C_1$, where $C_1$ is a constant. We prove the following claim.

**Claim.** There exists a constant $C_2 < \infty$, such that
\begin{equation}
\text{dist}_{g(t_i)}(p_0, p(t_i)) \leq C_2 \sqrt{|t_i|} + C_1,
\end{equation}
for every $i$.

**Proof of the Claim.** We recall Perelman’s distance distortion estimate [11]. Suppose on $t_0$-slice of a Ricci flow, around two points $x_0, x_1$ that are not too close to each other, the Ricci curvature tensor is bounded from above, that is, if for some $r > 0$, $\text{dist}_{g(t_0)}(x_0, x_1) \geq 2r$ and $\text{Ric} \leq (n-1)K$ on $B_{g(t_0)}(x_0, r) \cup B_{g(t_0)}(x_1, r)$, then we have
\begin{equation}
\frac{d}{dt} \text{dist}_{g(t)}(x_0, x_1) \geq -2(n-1) \left( \frac{2}{3} Kr + r^{-1} \right)
\end{equation}
at time $t = t_0$. Applying the curvature bound [11] and $r = |t|^{\frac{1}{2}}$ to [3], we have
\begin{equation}
\frac{d}{dt} \text{dist}_{g(t)}(p_0, p(t_i)) \geq -4(C+1)|t|^{-\frac{1}{2}},
\end{equation}
for every $i$, whenever $\text{dist}_{g(t)}(p_0, p(t_i)) > 2|t|^{\frac{1}{2}}$. Integrating [4] from 0 to $t_i \in (-\infty, 0)$ completes the proof of the claim. \qed

Now we recall Perelman’s reduced distance function $l_{(p_0, 0)}(p, t)$ centered at $(p_0, 0)$ and evaluated at $(p, t)$; see [11]. By the estimate of Naber (see Proposition 2.2 in [9]), we have that $l_{(p_0, 0)}(p(t_i), t_i) < C_3$, where $C_3 < \infty$ is a constant. It follows from Perelman [11] that
there exists a subsequence of \( \{(M, |t_i|^{-1}g(|t_i|t, (p(t_i), -1))_{t \in [-2,-1]} \}_{i=1}^{\infty} \) that converges in the pointed smooth Cheeger-Gromov sense to the canonical form of a nonflat shrinking gradient Ricci soliton; see Morgan and Tian \[8\] and Naber \[9\] for details. Notice that the time interval of these scaled flows are taken as \([-1, -\frac{1}{2}]\) in Perelman’s argument, whereas we take the interval to be \([-2, -1]\), so as to keep consistency with the definition of the \(\varepsilon\)-neck. This is valid because \(\sup_{t \in [2t_i, t_i]} l(p_i, 0) (p(t_i), t)\) is bounded uniformly. One may easily verify this bound by using Perelman’s differential inequalities for the reduced distance. The only nonflat three-dimensional shrinking gradient Ricci solitons are the shrinking sphere, the shrinking cylinder, and their quotients; see Perelman \[12\]. The limit shrinking gradient Ricci soliton cannot be flat, since otherwise Perelman’s reduced volume is equal to 1 for all time and the Ricci flow is flat; see \[13\]. The shrinking cylinder is the only one that can arise as the limit of a sequence of Ricci flows that are diffeomorphic to \(\mathbb{R}^3\). However, this yields a contradiction, as we have assumed that \((p(t_i), t_i)\) is not the center of an \(\varepsilon\)-neck.

We are now ready to present the proof of our main theorem.

Proof of Theorem 2. If \(g(t)\) has zero sectional curvature somewhere in space-time, by Hamilton’s strong maximum principle \[5\], \(g(t)\) also has zero sectional curvature everywhere in space at more ancient times, and hence splits locally. Since we assume \(M\) to be simply connected, it must be the shrinking cylinder. Therefore henceforth we assume that \(g(t)\) has strictly positive curvature on every time slice.

We fixed an arbitrary time sequence \(\{t_i\}_{i=1}^{\infty} \subset (-\infty, 0)\) such that \(t_i \searrow -\infty\). For every \(i\), let \(p_i \in M\) be such that \((p_i, t_i)\) is not the center of an \(\varepsilon\)-neck. By Lemma 5 we have that \(\text{dist}_{g(0)}(p_i, p_0) \to \infty\). Since by Perelman \[11\] that every three-dimensional noncompact \(\kappa\)-solution splits as a shrinking cylinder at spacial infinity, we can extract from \(\{(M, R(p_i, 0)g(tR(p_i, 0)^{-1}, (p_i, 0))_{t \in \infty} \}_{i=1}^{\infty}\) a (not relabelled) subsequence that converges in the smooth Cheeger-Gromov sense to the shrinking round cylinder. For the sake of simplicity we denote \(g_i(t) := R(p_i, 0)g(tR(p_i, 0)^{-1})\). It follows that for ever \(i\) large, \((p_i, 0)\) is the center of an \(\varepsilon\)-neck. The following claim is an easy consequence of Theorem 4.

Claim.

\[
\bar{t}_i := t_i R(p_i, 0) \geq T, \tag{5}
\]

for all large \(i\). Where \(T := T(\varepsilon, \varepsilon, \kappa) \in (-\infty, 0)\) is defined in Theorem 4.

Proof of the claim. Suppose the claim is not true, by passing to a subsequence, we can assume \(t_i = t_i R(p_i, 0) < T\) for all \(i\). We consider the scaled Ricci flows \(g_i(t)\), and apply
We continue the proof of the theorem. In the following argument we consider the scaled Ricci flows $g_i(t)$, notice that by our assumption for every $i$ the space-time point $(p_i, \tilde{t}_i)$ is not the center of an $\varepsilon$-neck, where $\tilde{t}_i$ is defined as (5). Since the limit of the sequence $\{(M, g_i(t), (p_i, 0))_{t \in (-\infty, 0]}\}_{i=1}^{\infty}$ is exactly a shrinking round cylinder, we have that for every large $A \in [4|T|, \infty)$, $(B_{g_i(0)}(p_i, A), g_i(t))_{t \in [\tau-\frac{A}{2}, \tau]}$ is as close as we like to the correspondent piece of the shrinking cylinder when $i$ is large enough. In particular, $(p_i, \tilde{t}_i)$ is the center of an $\varepsilon$-neck since $\tilde{t}_i \in [T, 0]$ according to the claim; this is a contradiction. Here we have again taken into account the scaling invariance of the $\varepsilon$-necklike property.

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