Negaton and Positon solutions of the soliton equation with self-consistent sources

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Abstract

The KdV equation with self-consistent sources (KdVES) is used as a model to illustrate the method. A generalized binary Darboux transformation (GBDT) with an arbitrary time-dependent function for the KdVES as well as the formula for \( N \)-times repeated GBDT are presented. This GBDT provides non-auto-Bäcklund transformation between two KdV equations with different degrees of sources and enable us to construct more general solutions with \( N \) arbitrary \( t \)-dependent functions. By taking the special \( t \)-function, we obtain multisoliton, multipositon, multinegaton, multisoliton-POSITON, multinegaton-POSITON and multisoliton-negaton solutions of KdVES. Some properties of these solutions are discussed.

1 Introduction

The soliton equations with self-consistent sources (SESCS) have attracted some attention (see, for example, [1]-[14]). The SESCS can be solved by the inverse scattering method and \( N \)-soliton solutions of some SESCSs were obtained [1]-[15]. However, since the explicit time-part of the Lax representation for SESCS was not found, the determination of the evolution for scattering date was quite complicated in [1]-[14]. In recent years, we presented the time-part of the Lax representation for SESCS by means of the adjoint representation of soliton equation [16, 17]. This enable us to determine the evolution of scattering date in a simple and natural way [15] and to construct the Darboux transformation for SESCS [18, 19]. It was pointed out in [18, 19] that the normal Darboux transformation for SESCS which provides
auto-Bäcklund transformation can not be used to construct solution of SESCS from the trivial solution. In [18, 19] we presented special kind of Binary Darboux transformation for some SESCSs which offers non-auto-Bäcklund transformation between soliton equations with different degrees of sources and can be used to obtained the \( N \)-soliton solutions. To our knowledge, no other kinds of solution, except soliton solution, for SESCS are investigated.

In recent years positon and negaton solution of soliton equations have been wide studied (see [20] and references therein). The positon solutions of soliton equation are long-range analogues of solitons and slowly decreasing, oscillating solutions, and possesses so-called supertransparent property: the corresponding reflection coefficient is zero and the transmission coefficient is unity [20]. The negaton solution of KdV equation was studied in [21].

In this letter, we use the KdV equation with self-consistent sources (KdVES) as model to illustrate the idea. We present generalized binary Darboux transformation (GBDT) with arbitrary \( t \)-dependent functions for KdVES and the formula for \( N \)-times repeated GBDT which contains \( N \) arbitrary \( t \)-dependent functions. This GBDT offers a non-auto-Bäcklund transformation between KdV equations with different degrees of sources and enables us to find the more general solution with arbitrary \( t \)-functions for KdVES. By taking the special \( t \)-function, we obtain multisoliton, multipositon, multinegaton, multisoliton-positon, multisoliton-negaton and multipositon-negaton solutions of KdVES.

This paper is organized as follows. In section 2, we derive the GBDT with an arbitrary \( t \)-dependent function for the KdVES and the formula for \( N \)-times repeated GBDT with \( N \) arbitrary \( t \)-dependent function. Using this GBDT gives rise to some kind of general solutions of KdVES including multi-soliton solution as a special case. In section 3 and 4, multi-positon and multi-negaton solutions of KdVES are obtained, respectively. Finally, in section 5, multisoliton-positon, multisoliton-negaton and multipositon-negaton solutions of KdVES are presented.

2 The generalized binary Darboux transformation

The KdV equation with sources of degree \( n \) (KdVES) is defined by [3, 5, 14, 15, 18]

\[ u_t + 6uu_x + u_{xxx} + 4 \sum_{j=1}^{n} \varphi_j \varphi_{j,x} = 0, \]  

\( (2.1a) \)

\[ \varphi_{j,xx} + (\lambda_j + u) \varphi_j = 0, \quad j = 1, \ldots, n, \]  

\( (2.1b) \)
where \(\lambda_j\) are distinct real constants. Let \(\Phi_n = (\phi_1, \ldots, \phi_n)\). The Lax representation for (2.1) can be found from the adjoint representation for KdV equation [16, 17, 18]

\[
\begin{align*}
\phi_{xx} + (\lambda + u)\phi &= 0, \\
\phi_t &= A_n(\lambda, u, \Phi_n)\phi,
\end{align*}
\]

(2.2a)

(2.2b)

where

\[
A_n(\lambda, u, \Phi_n)\phi = u_x\phi + (4\lambda - 2u)\phi_x + \sum_{j=1}^{n} \frac{\phi_j}{\lambda_j - \lambda} W(\phi_j, \phi),
\]

and \(W(\varphi_j, \phi) \equiv \varphi_j \phi_x - \varphi_j x \phi\) is the usual Wronskian determinant. It is shown that the well-known Darboux transformation (DT) for the KdV equation can be applied to the KdVES [18]. Let \(f\) be a solution of (2.2) with \(\lambda = \xi\), then (2.2) is covariant under the DT defined as [18]

\[
\begin{align*}
\tilde{\phi} &= \frac{W(f, \phi)}{f}, \\
\tilde{u} &= u + 2\partial_x^2 \ln f, \\
\tilde{\phi}_j &= \frac{1}{\sqrt{\lambda_j - \xi}} \frac{W(f, \phi_j)}{f}, \quad j = 1, \ldots, n,
\end{align*}
\]

(2.3a)

(2.3b)

(2.3c)

i.e., \(\tilde{\phi}, \tilde{u}\) and \(\tilde{\Phi}_n = (\tilde{\phi}_1, \ldots, \tilde{\phi}_n)\) satisfy

\[
\begin{align*}
\tilde{\phi}_{xx} + (\lambda + \tilde{u})\tilde{\phi} &= 0, \\
\tilde{\phi}_t &= A_n(\lambda, \tilde{u}, \tilde{\Phi}_n)\tilde{\phi},
\end{align*}
\]

(2.4a)

(2.4b)

and \(\tilde{u}, \tilde{\Phi}_n\) is a new solution of (2.1). Through this DT, we can find two linearly independent solutions of (2.4) with \(\lambda = \xi\). First, (2.3a) gives a solution of (2.4) with \(\lambda = \xi\):

\[
\tilde{f}_1 = \frac{C}{f},
\]

(2.5)

where \(C\) is some constant. Second, let \(g\) be a solution of (2.2) with \(\lambda = \eta \neq \xi\), we define

\[
\omega(f, g) = \frac{W(f, g)}{\xi - \eta}.
\]

According to (2.3a),

\[
\tilde{g} = \frac{1}{\xi - \eta} \frac{W(f, g)}{f} = \frac{1}{f} \omega(f, g)
\]
is a solution of (2.4) with \( \lambda = \eta \). For analytic \( f = f(\xi) \) and let \( g = f(\eta) \), we have

\[
\omega(f, f) \equiv \lim_{\eta \to \xi} \frac{W(f(\xi), f(\eta))}{\xi - \eta} = -W(f, \partial_\xi f),
\]

and

\[
\tilde{f} = \frac{1}{f} \omega(f, f)
\]

is another solution of (2.4) with \( \lambda = \xi \). Therefore

\[
\tilde{h} \equiv \tilde{f} + \tilde{f}_t = \frac{1}{f} \left[ C + \omega(f, f) \right]
\]

is also a solution of (2.4) with \( \lambda = \xi \). Using \( f \) and \( \tilde{h} \) consecutively, the two-times action of DT (2.3) yields the following binary DT

\[
\bar{\phi} = \frac{1}{\lambda - \xi} \frac{W(\tilde{h}, \bar{\phi})}{\tilde{h}} = \phi - \frac{f}{C + \omega(f, f)} \omega(f, \phi),
\]

(2.7a)

\[
\bar{u} = \bar{u} + 2 \partial_\xi^2 \ln \bar{h} = u + 2 \partial_\xi^2 \ln [C + \omega(f, f)],
\]

(2.7b)

\[
\bar{\varphi}_j = \frac{1}{\sqrt{\lambda_j - \xi}} \frac{W(\tilde{h}, \bar{\varphi}_j)}{\tilde{h}} = \varphi_j - \frac{f}{C + \omega(f, f)} \omega(f, \varphi_j), \quad j = 1, \ldots, n,
\]

(2.7c)

then the system (2.2) is covariant under the binary DT (2.7) and \( \bar{u}, \bar{\Phi}_n \equiv (\bar{\varphi}_1, \ldots, \bar{\varphi}_n) \) satisfies the KdVES (2.1).

Note that \( \partial_\xi \omega(f, f) = f^2 \), \( \partial_\xi \omega(f, \phi) = f \phi \) and \( \partial_\xi \omega(f, \varphi_j) = f \varphi_j \), the binary DT (2.7) can be transformed into the original binary DT given in [18].

Substitution of (2.7) into (2.2b) gives

\[
\bar{\phi}_t = \left[ \phi - \frac{f}{C + \omega(f, f)} \omega(f, \phi) \right]_t
\]

\[
= \phi_t - \frac{f \omega(f, \phi)}{C + \omega(f, f)} + \frac{f \partial_\xi \omega(f, f)}{[C + \omega(f, f)]^2} \omega(f, \phi) - \frac{f \partial_\xi \omega(f, \phi)}{C + \omega(f, f)} = A_n(\lambda, \bar{u}, \bar{\Phi}_n) \bar{\phi}.
\]

(2.8)

When substituting (2.7) into \( A_n(\lambda, \bar{u}, \bar{\Phi}_n) \bar{\phi} \), the last equality holds for any constant \( C \). In the expression of \( A_n(\lambda, \bar{u}, \bar{\Phi}_n) \phi \), there is no derivatives with respect to \( t \). So the last equality holds when \( C \) is replaced by \( e(t) \), an arbitrary \( t \)-function. We have the following lemma.

**Lemma 2.1** Given \( u, \Phi_n \) a solution of (2.1), if \( f \) is a solution of (2.2) with \( \lambda = \xi \), then the last equality of (2.8) holds for \( C = e(t) \)

Obviously, under DT defined by (2.7) with \( C \) replaced by \( e(t) \), (2.2a) is still covariant, however, (2.2b) is no longer covariant. In fact, we have
Theorem 2.1  Given $u$, $\Phi_n$ as a solution of (2.1), let $f$ be a solution of the system (2.2) with $\lambda = \lambda_{n+1}$. Then, the generalized binary DT with an arbitrary $t$-function defined by

$$\tilde{\phi} = \phi - \frac{f}{e(t) + \omega(f, f)} \omega(f, \phi),$$  \hspace{1cm} (2.9a)$$

$$\tilde{u} = u + 2\partial_t^2 \ln[e(t) + \omega(f, f)],$$  \hspace{1cm} (2.9b)$$

$$\tilde{\varphi}_j = \varphi_j - \frac{f}{e(t) + \omega(f, f)} \omega(f, \varphi_j), \quad j = 1, \ldots, n,$$  \hspace{1cm} (2.9c)$$

and

$$\tilde{\varphi}_{n+1} = \frac{\sqrt{e'(t)} f}{e(t) + \omega(f, f)},$$  \hspace{1cm} (2.9d)$$

transforms (2.2) into

$$\tilde{\phi}_{xx} + (\lambda + \tilde{u}) \tilde{\phi} = 0,$$  \hspace{1cm} (2.10a)$$

$$\tilde{\phi}_t = A_{n+1}(\lambda, \bar{u}, \bar{\Phi}_{n+1}) \bar{\phi},$$  \hspace{1cm} (2.10b)$$

and $\bar{u}, \bar{\Phi}_{n+1} \equiv (\bar{\varphi}_1, \ldots, \bar{\varphi}_{n+1})$, satisfy the KdV equation with sources of degree $n + 1$

$$\bar{u}_t + 6\bar{u}\bar{u}_x + \bar{u}_{xxx} + 4 \sum_{j=1}^{n+1} \bar{\varphi}_j \bar{\varphi}_{j,x} = 0,$$  \hspace{1cm} (2.11a)$$

$$\bar{\varphi}_{j,xx} + (\lambda_j + \bar{u}) \bar{\varphi}_j = 0, \quad j = 1, \ldots, n + 1,$$  \hspace{1cm} (2.11b)$$

Proof. $\tilde{h}$ defined by (2.6) with $C$ replaced by $e(t)$ still satisfies (2.4a). This implies that (2.10a) and (2.11b) hold. Substituting (2.9a) into the left-hand side of (2.10b) and using the Lemma 2.1 gives rise to

$$\tilde{\phi}_t = \left[ \phi - \frac{f}{e(t) + \omega(f, f)} \omega(f, \phi) \right]_t = \phi_t - \frac{f \omega(f, \phi)}{e(t) + \omega(f, f)} + \frac{f [e'(t) + \partial_t \omega(f, f)]}{[e(t) + \omega(f, f)]^2} \omega(f, \phi)$$

$$- \frac{f \partial_t \omega(f, \phi)}{e(t) + \omega(f, f)} = A_n(\lambda, \bar{u}, \bar{\Phi}_n) \bar{\phi} + \frac{e'(t) \omega(f, \phi) f}{[e(t) + \omega(f, f)]^2} = A_{n+1}(\lambda, \bar{u}, \bar{\Phi}_{n+1}) \bar{\phi},$$

where we have use the formula

$$W(\bar{\varphi}_{n+1}, \bar{\phi}) = \frac{\sqrt{e'(t)} \omega(f, \phi)}{e(t) + \omega(f, f)}.$$

Then the compatibility condition of (2.10) leads to (2.11a). This completes the proof.

The generalized binary DT (GBDT) defined by (2.9) contains an arbitrary $t$-function. The flexibility of the choices of $e(t)$ and $f$ enables us to construct some kind of general
solutions with arbitrary $t$-functions of the KdVES some of that can not be constructed through the original binary DT.

For $m$ solutions of (2.2), $g_1, \ldots, g_m$ and $m$ arbitrary $t$-functions $e_1(t), \ldots, e_m(t)$, we define two types of Wronskian determinant

$$W_1(g_1, \ldots, g_m; e_1, \ldots, e_m) = \det F, \quad W_2(g_1, \ldots, g_m; e_1, \ldots, e_{m-1}) = \det G$$

where

$$F_{ij} = \delta_{ij} e_i(t) + \omega(g_i, g_j), \quad i, j = 1, \ldots, m,$$

$$G_{ij} = \delta_{ij} e_i(t) + \omega(g_i, g_j), \quad i = 1, \ldots, m-1, \quad j = 1, \ldots, m,$$

$$G_{mj} = g_j, \quad j = 1, \ldots, m.$$

We have the following formula of $N$-times repeated GBDT.

**Theorem 2.2** Given $u$, $\Phi_n$ as a solution of (2.1), let $f_1, \ldots, f_N$ be solutions of (2.2) with $\lambda = \lambda_{n+1}, \ldots, \lambda_{n+N}$, respectively. Then the $N$-times repeated GBDT with $N$-arbitrary $t$-functions $e_1(t), \ldots, e_N(t)$ defined by

$$\bar{\phi} = \frac{W_2(f_1, \ldots, f_N; \phi; e_1, \ldots, e_N)}{W_1(f_1, \ldots, f_N; e_1, \ldots, e_N)}, \quad (2.12a)$$

$$\bar{u} = u + 2\partial_x^2 \ln W_1(f_1, \ldots, f_N; e_1, \ldots, e_N), \quad (2.12b)$$

$$\bar{\varphi}_j = \frac{W_2(f_1, \ldots, f_N; \varphi_j; e_1, \ldots, e_N)}{W_1(f_1, \ldots, f_N; e_1, \ldots, e_N)}, \quad j = 1, \ldots, n, \quad (2.12c)$$

and

$$\bar{\varphi}_{n+j} = \sqrt{e_j'(t)} W_2(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_N; f_j; e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_N) W_1(f_1, \ldots, f_N; e_1, \ldots, e_N), \quad j = 1, \ldots, N \quad (2.12d)$$

transforms (2.2) into (2.2) with $n$ replaced by $n + N$ and $\bar{u}$, $\bar{\Phi}_{n+N}$ satisfy the KdVES of degree $n + N$, i.e. (2.1) with $n$ replaced by $n + N$.

The proof of this theorem is completely similar to that given in [18] and we omit it.

**Example:** $N$-soliton solution.

We take $u = 0$ as the initial solution of (2.1) with $n = 0$ and let $\lambda_j = -\kappa_j^2 < 0, \kappa_j > 0$, $j = 1, \ldots, N$,

$$f_j = e^{\kappa_j x - 4\kappa_j^3 t}, \quad e_j(t) = e^{2\alpha_j t}, \quad j = 1, \ldots, N,$$
then
\[ \omega(f_i, f_j) = \frac{1}{\kappa_i + \kappa_j} e^{(\kappa_i + \kappa_j)x - 4(\kappa_i^3 + \kappa_j^3)t}, \quad i, j = 1, \ldots, N, \]
the \( N \)-soliton solutions of (2.1) with \( n = N \) and \( \lambda_j = -\kappa_j^2 < 0, \ j = 1, \ldots, N, \) is given by
\[ u = 2\partial_x^2 \ln W_1(f_1, \ldots, f_N; e_1, \ldots, e_N), \]
\[ \varphi_j = \sqrt{a_j} \frac{W_2(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_N, f_j; e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_N)}{W_1(f_1, \ldots, f_N; e_1, \ldots, e_N)}, \quad j = 1, \ldots, N, \]
which was obtained in [14, 15, 18].

### 3 Positon solutions

Hereafter we always take simple and special choice of \( e(t) \) as
\[ e_j(t) = a_j t + b_j, \tag{3.1} \]
where \( a_j \neq 0 \) and \( b_j \) are real constants.

#### 3.1 One-positon solution and the supertransparency.

We take \( u = 0 \) as the initial solution of (2.1) with \( n = 0 \). Let \( f \) be an oscillating solution of (2.2) with \( u = 0, n = 0 \) and \( \lambda = \lambda_1 = \kappa^2 > 0, \kappa > 0, \)
\[ f = \sin \Theta, \quad \Theta = \kappa(x + x_1 + 4\kappa^2t), \tag{3.2} \]
where \( x_1 = x_1(\kappa) \) is an real differential function of \( \kappa \). Then the GBDT (2.9) gives
\[ u = 2\partial_x^2 \ln(2\kappa\gamma - \sin 2\Theta) = \frac{32\kappa^2 \sin \Theta(\kappa\gamma \cos \Theta - \sin \Theta)}{(2\kappa\gamma - \sin 2\Theta)^2}, \tag{3.3a} \]
\[ \varphi_1 = \frac{4\kappa\sqrt{a} \sin \Theta}{2\kappa\gamma - \sin 2\Theta}, \tag{3.3b} \]
with
\[ \gamma = \frac{\partial_x \Theta + 2e(t)}{2} = x + \bar{x}_1 + (12\kappa^2 + 2a)t + 2b, \quad \bar{x}_1 = x_1 + \kappa \partial_x x_1(\kappa), \]
which gives the one-positon solution of KdVES (2.1) with \( n = 1, \lambda_1 = \kappa^2 \) corresponding to the one-positon solution for the KdV equation in [20, 21].

The solution of linear system (2.2) with \( n = 1, \lambda = k^2, \lambda_1 = \kappa^2 \) \( u \) and \( \varphi_1 \) given by (3.3) is
\[ \phi(x, k) = \left( -k^2 + \frac{4ik\kappa \sin^2 \Theta}{\sin 2\Theta - 2\kappa \gamma} - \frac{2\kappa \sin 2\Theta + 2\kappa \gamma}{\sin 2\Theta - 2\kappa \gamma} \right) e^{ikx + 4ik^3t}. \tag{3.4} \]
Based on formulas (3.3) and (3.4), we can analyze the basic features of the one-positon solution of (2.1) in the same way as in [20]. We can conclude that the one-positon solution of (2.1) with \( n = 1 \) has the same shape, the same asymptotic behavior when \( x \to \pm \infty \) and the same scattering data as the one positon solution of the KdV equation: it is long-range analogues of solitons of the KdVES and is slowly decreasing, oscillating solutions. Similarly under a proper choice of the scattering data, the corresponding reflection coefficient is zero and the transmission coefficient is unity.

### 3.2 Two-positon solution and multi-positon solutions.

The two-positon solution of (2.1) with \( n = 2, \lambda_j = \kappa_j^2 > 0, \kappa_j > 0, j = 1, 2 \), is given by (2.13) with \( N = 2, e_j = a_j t + b_j \)

\[
f_j = \sin \theta_j, \quad \theta_j = \kappa_j(x + x_j + 4\kappa_j^2t), \quad j = 1, 2,
\]

\[
W_1(f_1, f_2; e_1, e_2) = (16\kappa_1\kappa_2)^{-1}(2\kappa_1\gamma_1 - \sin 2\Theta_1)(2\kappa_2\gamma_2 - \sin 2\Theta_2)
\]

\[
-(\kappa_1^2 - \kappa_2^2)^{-1}(\kappa_2 \sin \Theta_1 \cos \Theta_2 - \kappa_1 \sin \Theta_2 \cos \Theta_1)^2,
\]

\[
W_2(f_2, f_1; e_2) = (4\kappa_2)^{-1}\sin \Theta_1(2\kappa_2\gamma_2 - \sin 2\Theta_2)
\]

\[
-(\kappa_1^2 - \kappa_2^2)^{-1}\sin \Theta_2(\kappa_2 \sin \Theta_1 \cos \Theta_2 - \kappa_1 \sin \Theta_2 \cos \Theta_1),
\]

\[
W_2(f_1, f_2; e_1) = (4\kappa_1)^{-1}\sin \Theta_2(2\kappa_1\gamma_1 - \sin 2\Theta_1)
\]

\[
-(\kappa_1^2 - \kappa_2^2)^{-1}\sin \Theta_1(\kappa_2 \sin \Theta_1 \cos \Theta_2 - \kappa_1 \sin \Theta_2 \cos \Theta_1),
\]

\[
\gamma_j = x + \bar{x}_j + (12\kappa_j^2 + 2a_j t + 2b_j, \quad \bar{x}_j = x_j + \kappa_j \partial_{\kappa_j} x_j(\kappa_j), \quad j = 1, 2.
\]

Using (3.5), we obtain the asymptotic behavior of the solution for fixed \( \gamma_1 \) as \( t \to \pm \infty \) (which implies \( \gamma_2 \to \infty \))

\[
u = 2\partial_x^2 \ln(2\kappa_1\gamma_1 - \sin 2\Theta_1)[1 + O(\gamma_2^{-1})],
\]

\[
\varphi_1 = \frac{4\kappa_1\sqrt{a_1} \sin \Theta_1}{2\kappa_1\gamma_1 - \sin 2\Theta_1}[1 + O(\gamma_2^{-1})], \quad \varphi_2 = O(\gamma_2^{-1}).
\]

In the asymptotic domain where \( \gamma_2 \) is fixed and \( t \to \pm \infty(\gamma_1 \to \infty) \), we have

\[
u = 2\partial_x^2 \ln(2\kappa_2\gamma_2 - \sin 2\Theta_2)[1 + O(\gamma_1^{-1})],
\]

\[
\varphi_1 = O(\gamma_1^{-1}), \quad \varphi_2 = \frac{4\kappa_2\sqrt{a_2} \sin \Theta_2}{2\kappa_2\gamma_2 - \sin 2\Theta_2}[1 + O(\gamma_1^{-1})].
\]

Thus we have proved that the two positons are totally insensitive to the mutual collision, even without additional phase shifts which is intrinsic for the collision of two solitons. Calculating the corresponding solution of system (2.2), we can prove that potential is also supertransparent.
The $N$-positon solution of (2.1) with $n = N, \lambda_j = \kappa_j^2 > 0, \kappa_j > 0, j = 1, \ldots, N,$ is given by (2.13) with $e_j = a_j t + b_j$,

$$f_j = \sin \Theta_j, \quad \Theta_j = \kappa_j(x + x_j + 4\kappa_j^2 t), \quad j = 1, \ldots, N,$$

$$\omega(f_i, f_j) = (\kappa_i^2 - \kappa_j^2)^{-2}(\kappa_j \sin \Theta_i \cos \Theta_j - \kappa_i \sin \Theta_j \cos \Theta_i)^2.$$

Analogously, we will see that the $N$-positon solution at large time decays into the sum of $N$ free positons and it is also supertransparent.

4 Negaton solutions

4.1 One-negaton solution. Let $\lambda_1 = -\kappa^2 < 0, \kappa > 0$ and $f$ be a solution of (2.2) with $u = 0, n = 0$ and $\lambda = \lambda_1$,

$$f = \sinh \Theta, \quad \Theta = \kappa(x + x_1 - 4\kappa^2 t). \quad (4.1)$$

Then the GBDT (2.9b) and (2.9d) with $e(t) = at + b$ gives

$$u = 2\partial_x^2 \ln(\kappa\gamma - \sinh \Theta \cosh \Theta) = \frac{8\kappa^2 \sinh \Theta(\sinh \Theta - \kappa\gamma \cosh \Theta)}{(\kappa\gamma - \sinh \Theta \cosh \Theta)^2}, \quad (4.2a)$$

$$\varphi_1 = \frac{2\kappa\sqrt{a} \sinh \Theta}{\kappa\gamma - \sinh \Theta \cosh \Theta}, \quad (4.2b)$$

where

$$\gamma = x + x_1 + \kappa \partial_x x_1 - (12\kappa^2 - 2a)t + 2b. \quad (4.2c)$$

(4.2) gives the [S] one-negaton solution of (2.1) with $n = 1$ and $\lambda_1 = -\kappa^2 < 0$ which corresponds to the [S] one-negaton solution for the KdV equation in [21].

When $t$ is fixed, then we have

$$u \sim 8\kappa^2\left(\frac{1}{\cosh^2 \Theta} - \frac{\kappa\gamma}{\sinh \Theta \cosh \Theta}\right) \to 0, \quad \varphi_1 \sim -\frac{\kappa\sqrt{a}}{\cosh \Theta} \to 0, \quad x \to \pm \infty.$$ 

For fixed $x$, we have the same formula when $t \to \pm \infty$.

As a function of $x$, $u$ has a second-order pole and $\varphi_1$ has a first-order pole which locate at the same point $x = x_p(t)$ determined by the equation $\sinh \Theta \cosh \Theta - \kappa \gamma = 0$. Also it is easy to that $u(x, t)$ has two zeros and $\varphi_1(x, t)$ has one zero. The shape and the motion of $u(x, t)$ is the same as that described in [21].

Similarly, if we take $f = \cosh \Theta$, we can obtain the [C] one-negaton.
4.2 Two-negaton solution and multi-negaton solutions.

The [S] two-negaton solution of (2.1) with \( n = 2 \) and \( \lambda_j = -\kappa_j^2 < 0, \kappa_j > 0, j = 1, 2 \) is given by (2.13) with \( N = 2 \)

\[ f_j = \sinh \Theta_j, \quad \Theta_j = \kappa_j(x + x_j - 4\kappa_j^2t), \quad e_j(t) = a_j t + b_j, \quad j = 1, 2, \]

\[ W_1(f_1, f_2; e_1, e_2) = (4\kappa_1\kappa_2)^{-1}(\kappa_1\gamma_1 - \sinh \Theta_1 \cosh \Theta_1)(\kappa_2\gamma_2 - \sinh \Theta_2 \cosh \Theta_2), \]

\[ -(\kappa_1^2 - \kappa_2^2)^{-2}(\kappa_2 \sinh \Theta_1 \cosh \Theta_2 - \kappa_1 \sinh \Theta_2 \cosh \Theta_1)^2, \quad (4.3) \]

\[ W_2(f_2, f_1) = (2\kappa_2)^{-1} \sinh \Theta_1 (\kappa_2 \gamma_2 - \sinh \Theta_2 \cosh \Theta_2) \]

\[ + (\kappa_1^2 - \kappa_2^2)^{-1} \sinh \Theta_2 (\kappa_1 \sinh \Theta_1 \cosh \Theta_2 - \kappa_1 \sinh \Theta_2 \cosh \Theta_1), \quad (4.4) \]

\[ W_2(f_1, f_2) = (2\kappa_1)^{-1} \sinh \Theta_2 (\kappa_1 \gamma_1 - \sinh \Theta_1 \cosh \Theta_1) \]

\[ + (\kappa_1^2 - \kappa_2^2)^{-1} \sinh \Theta_1 (\kappa_2 \sinh \Theta_1 \cosh \Theta_2 - \kappa_1 \sinh \Theta_2 \cosh \Theta_1), \quad (4.5) \]

\[ \gamma_j = x + \tilde{x}_j - 12\kappa_j^2 t + 2a_j t + b_j, \quad \tilde{x}_j = x_j + \kappa_j \partial_{\kappa_j} x_j(\kappa_j), \quad \text{Im} x_j = 0, \quad j = 1, 2. \]

In the domain where \( x + x_1 - 4\kappa_1^2 t \) is fixed and \( t \to \pm \infty \), the asymptotic solution is

\[ u = 2 \frac{\partial^2}{\partial x^2} \ln(\kappa_1 \gamma_1 - \sinh \Theta_1 \cosh \Theta_1)[1 + O(t^{-1})], \]

\[ \varphi_1 = \frac{2\kappa_1 \sqrt{\alpha_1} \sinh \Theta_1}{\kappa_1 \gamma_1 - \sinh \Theta_1 \cosh \Theta_1}[1 + O(t^{-1})], \quad \varphi_2 = O(t^{-1}). \]

In the asymptotic domain where \( x + x_2 - 4\kappa_2^2 t \) is fixed and \( t \to \pm \infty \), we have

\[ u = 2 \frac{\partial^2}{\partial x^2} \ln(\kappa_2 \gamma_2 - \sinh \Theta_2 \cosh \Theta_2)[1 + O(t^{-1})], \]

\[ \varphi_1 = O(t^{-1}), \quad \varphi_2 = \frac{2\kappa_2 \sqrt{\alpha_2} \sinh \Theta_2}{\kappa_2 \gamma_2 - \sinh \Theta_2 \cosh \Theta_2}[1 + O(t^{-1})]. \]

This estimates show that in the indicated domain, the leading terms of the asymptotic [S] two-negaton solution is a standard [S] one-negaton solution. In other words, negatons are totally insensitive to the mutual collision, even without additional phase shifts in contrast to the solitons collision case.

Similarly we can construct the [C] two-negaton and [SC] two-negaton and find the same property.

The [S] \( N \)-negaton solution of (2.1) with \( n = N \) and \( \lambda_j = -\kappa_j^2 < 0, \kappa_j > 0, j = 1, \ldots, N \) is given by (2.13) with \( e_j = a_j t + b_j \)

\[ f_j = \sinh \Theta_j, \quad \Theta_j = \kappa_j(x + x_j - 4\kappa_j^2 t), \quad \text{Im} x_j = 0, \quad j = 1, \ldots, N. \]

Analogously, we will see that the [S] \( N \)-negaton solution at large time decays into the sum of \( N \) [S] free negatons.
5 Multisoliton-positon, Multisoliton-negaton and Multipositon-negaton solutions

Like the KdV equation, the KdVES also has Multisoliton-positon, Multisoliton-negaton and Multipositon-negaton solutions. The $N$-positon $M$-soliton solutions of (2.1) with $n = N + M$ and $\lambda_j = \kappa_j^2 > 0$, $j = 1, \ldots, N$, $\lambda_{N+j} = -\kappa_{N+j}^2 < 0$, $j = 1, \ldots, M$ are given by (2.13) with $N$ replaced by $N + M$ and

$$f_j = \sin \Theta_j, \quad \Theta_j = \kappa_j (x + x_j + 4\kappa_j^2 t), \quad \kappa_j > 0, \quad j = 1, \ldots, N,$$

$$f_{N+j} = e^{\kappa_{N+j} (x - 4\kappa_{N+j}^2 t)}, \quad \kappa_{N+j} > 0, \quad j = 1, \ldots, M.$$

The $N$-negaton $M$-soliton solution of (2.1) with $n = N + M$ and $\lambda_j = -\kappa_j^2 > 0$, $j = 1, \ldots, N + M$, is given by (2.13) with $N$ replaced by $N + M$ and

$$f_j = \sinh(\cosh) \Theta_j, \quad \Theta_j = \kappa_j (x + x_j - 4\kappa_j^2 t), \quad \kappa_j > 0, \quad j = 1, \ldots, N,$$

$$f_{N+j} = e^{\kappa_{N+j} (x - 4\kappa_{N+j}^2 t)}, \quad \kappa_{N+j} > 0, \quad j = 1, \ldots, M.$$

The $N$-positon $M$-negaton solution of (2.1) with $n = N + M$ and $\lambda_j = \kappa_j^2 > 0$, $j = 1, \ldots, N$, $\lambda_{N+j} = -\kappa_{N+j}^2 < 0$, $j = 1, \ldots, M$ is given by (2.13) with $N$ replaced by $N + M$ and

$$f_j = \sin \Theta_j, \quad \Theta_j = \kappa_j (x + x_j + 4\kappa_j^2 t), \quad \kappa_j > 0, \quad j = 1, \ldots, N,$$

$$f_{N+j} = \sinh(\cosh) \Theta_{N+j}, \quad \Theta_{N+j} = \kappa_{N+j} (x + x_{N+j} - 4\kappa_{N+j}^2 t), \quad \kappa_{N+j} > 0, \quad j = 1, \ldots, M.$$

We can analyze the interaction of the soliton and the positon, the soliton and the negaton, the positon and the negaton in a similar way as in [20]. We would like to point out that the results of the analysis will be almost the same as in [20] and we omit it.

6 Conclusions

We present $N$-times repeated GBDT with $N$ arbitrary $t$-functions which provides no-auto-Bäcklund transformation between two KdV equations with $n$-degrees of sources and $n + N$-degrees of sources. This $N$-times repeated GBDT enables us to construct some kind of general solutions with $N$ arbitrary $t$-functions for KdVES. By taking special choice of the $t$-functions we obtain the Multi-soliton, Multi-negaton Multi-positon, Multisoliton-positon, Multisoliton-negaton and Multipositon-negaton solutions for KdVES. The method can be applied to other SESCSs.
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