Superconducting non-Abelian vortices in Weinberg-Salam theory – electroweak thunderbolts

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We present a detailed analysis of classical solutions in the bosonic sector of the electroweak theory which describe vortices carrying a constant electric current $I$. These vortices exist for any value of the Higgs boson mass and for any weak mixing angle, and in the zero current limit they reduce to $Z$ strings. Their current is produced by the condensate of vector $W$ bosons and typically it can attain billions of Amperes. For large $I$ the vortices show a compact condensate core of size $\sim 1/I$, embedded into a region of size $\sim I$ where the electroweak gauge symmetry is completely restored, followed by a transition zone where the Higgs field interpolates between the symmetric and broken phases. Outside this zone the fields are the same as for the ordinary electric wire. An asymptotic approximation of the large $I$ solutions suggests that the current can be arbitrarily large, due to the scale invariance of the vector boson condensate. Finite vortex segments whose length grows with $I$ seem to be perturbatively stable. This suggests that they can transfer electric charge between different regions of space, similarly to thunderbolts. It is also possible that they can form loops stabilized by the centrifugal force – electroweak vortons.

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XI. Special parameter limits
   A. Semilocal limit, $\theta_w = \pi/2$
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Appendix A. Solutions near the symmetry axis

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References

I. INTRODUCTION

More than 20 years ago Witten proposed a field theory model that admits classical solutions describing vortices carrying a constant current – ‘superconducting strings’ [1]. This model has a local U(1)$\times$U(1) invariance and consists of two copies of the Abelian Higgs model for fields $(A_{\mu}^{(1)}, \phi_1)$ and $(A_{\mu}^{(2)}, \phi_2)$. It also includes an interaction between the two complex scalars chosen such that in vacuum one has $\phi_1 \neq 0$ but $\phi_2 = 0$ so that the vector field $A_{\mu}^{(1)}$ is massive while $A_{\mu}^{(2)}$ is massless and can be identified with the electromagnetic field.

The model admits as a solution the Abrikosov-Nielsen-Olesen (ANO) vortex [2] made of the ‘vortex fields’ $(A_{\mu}^{(1)}, \phi_1)$, with vanishing ‘condensate fields’ $(A_{\mu}^{(2)}, \phi_2)$. This embedded vortex is however unstable, but being topological it does not unwind into vacuum and relaxes to a ‘dressed vortex’ which contains a condensate of charged scalar bosons in the core and has $\phi_2 \neq 0$. Giving then a non-trivial phase to the condensate field $\phi_2$ produces a current and promotes the ‘dressed’ vortex to the superconducting string supporting the long-range Biot-Savart field represented by $A_{\mu}^{(2)} \neq 0$. 
The Witten string superconductivity has been much studied [3], [4], mainly in the cosmological context [5], [6], since Witten’s model can be viewed as a sector of some high energy Grand Unification Theory (GUT) [1] that could perhaps be relevant at the early stages of the cosmological evolutions. Using the typical values of the GUT parameters for estimates gives for the string current enormous values of order $10^{20}$ Amperes, which suggests interesting applications [3], [6]. String superconductivity in the GUT-related non-Abelian models has also been studied [7], in which case the string current is produced by a condensate of charged vector bosons.

Although the GUT physics could be important, one may wonder whether a similar string superconductivity could exist also in a less exotic context, at lower energies, as for example in the electroweak sector of Standard Model. In fact, the $U(1) \times U(1)$ symmetry of Witten’s model is contained in the $SU(2) \times U(1)$ electroweak gauge symmetry. In addition, the electroweak theory contains a pair of complex Higgs scalars, one of which could well be responsible for the formation of the vortex while the other one – for the condensate. Since the ANO vortices can be embedded into the electroweak theory in the form of Z strings [8], one might expect that current-carrying generalizations of the latter could exist. These would be electroweak analogues of Witten’s superconducting strings.

However, no attempts to construct such solutions have ever been undertaken. This can probably be explained by the following reasons. The current-carrying Witten strings are usually viewed as excitations over the ‘dressed’ currentless vortex obtained by minimizing the energy of the ‘bare’ embedded ANO vortex. This explains their essential properties, as for example the value of the critical current [5]. Now, the ‘bare’ electroweak Z strings are also unstable [9], and it was conjectured [10, 11] that they could similarly relax to ‘W-dressed Z strings’. However, a systematic search for such solutions gave no result [12]. This can probably be explained by the fact that Z strings are non-topological and can unwind into vacuum [13], so that there is little chance that they could be stabilized by the condensate. But if one does not find currentless ‘dressed’ Z strings, it is natural to think that current-carrying strings in the electroweak theory do not exist either. This presumably explains why there has been almost no activity on electroweak vortices during the last 10-15 years.

At the same time, it is difficult to believe that Z strings are the only possible vortex solutions in the electroweak theory. The theory admits rather non-trivial solutions, such as sphalerons [14], periodic BPS solutions [15], spinning dumbbells [16, 17], oscillons [18],
spinning sphalerons \cite{19}, and others (see \cite{20} for a review). In addition, in the semilocal limit where the SU(2) field decouples the theory admits current-carrying vortices \cite{21}, so that it is plausible that they could exist also for generic values of the weak mixing angle.

We have therefore analysed the problem and found that superconducting vortices indeed exist in the electroweak theory, despite the non-existence of the ‘W-dressed Z strings’. In other words, these two types of solutions are not necessarily related. To construct the solutions, we essentially reverse the standard ‘engineering’ procedure used in Witten’s model. There one starts from the ‘dressed’ currentless vortex and *increases* it ‘winding number density’ that determines the phase of the condensate; in what follows we shall call this parameter ‘twist’. This produces a current that first increases with the twist but then starts to quench and finally vanishes when the solution reduces to the ‘bare’ ANO vortex \cite{5}. The superconducting strings thus comprise a one-parameter family that interpolates between the ‘dressed’ vortex and the ‘bare’ ANO vortex.

We construct this family in the opposite direction, by starting from the ‘bare’ vortex and then *decreasing* its twist. Within Witten’s model this gives of course the same solutions but in the reversed order – their current first increases then starts to quench and vanishes for zero twist when the solution reduces to the ‘dressed’ string. However, the advantage of our method is that it can be applied also within the Weinberg-Salam theory, where there are no ‘dressed’ currentless vortices but only the ‘bare’ ones – Z strings. Their twist is determined by the eigenvalue of the second variation of the energy functional. Decreasing the twist gives us solutions with a non-zero current. Further decreasing it shows that the current always grows and tends to infinity when the twist approaches zero. As a result, contrary to what one would normally expect, and presumably because of the vector character of their current-carriers, the electroweak vortices do not (generically) exhibit the current quenching, so that they do not need to admit the ‘dressed’ currentless limit.

The stability analysis of these vortices shows that their finite segments are perturbatively stable. More precisely, so far this property has been demonstrated only in the semilocal limit \cite{22}, but it is very likely that the result extends for generic values of the weak mixing angle. The length of the stable segments increases with the current and can in principle attain any value, since there is no upper bound for the current.

This may have interesting consequences. First of all, this suggests that loops made of the stable segments and balanced by the centrifugal force could perhaps be stable as well.
Although studying this goes far beyond the scope of the present paper, the possibility to have stable solitonic objects in the Standard Model could be very important, since if such electroweak vortons exist, they could perhaps contribute to the dark matter.

Another, perhaps a more direct consequence, is related to the well-know fact that, since Z strings are non-topological, they can have finite length. This suggests that their current-carrying generalizations could also exist in the form of finite (and stable) segments connecting oppositely charged regions of space. They would then be similar to thunderbolts. For the solutions we could explicitly construct the current can be as large as $10^{10}$ Amperes, which exceeds by several orders of magnitudes the power of the strongest thunderbolts in the Earth atmosphere. This suggests that superconducting vortices could be important for the dynamics of the Standard Model, although analysing this issue in detail would again lead us too far away from our main subject.

The present paper is devoted to a systematic analysis of the string/vortex-type solutions in the electroweak theory (terms string and vortex are assumed to be synonyms). In what follows we shall show that every Z string admits a three-parameter family of non-Abelian, current carrying generalizations. In the gauge where the radial components of the gauge fields vanish, the upper and lower components of the Higgs field doublet behave quite analogously to the vortex field and condensate field in Witten’s model. The new vortices exist for (almost) any value of the weak mixing angle and for any Higgs boson mass. They show a compact, regular core containing the W-condensate that carries a constant electric current, producing the Biot-Savart electromagnetic field outside the vortex. In the comoving reference frame the vortex is characterized by the current $I$ and by the electromagnetic and Z fluxes through its cross section. After performing a Lorentz boost, it develops also a non-zero electric charge density, as well as momentum and angular momentum directed along the vortex.

It seems that the vortex current can be as large as one wants, at least we could not find an upper bound for it. We have also managed to construct a simple approximation in the large current limit which suggests that the current can be arbitrarily large, due to the fact that it is carried by vectors whose condensate exhibits the scale invariance. To the best of our knowledge, a similar effect has never been reported before. For large currents the charged W boson condensate is contained in the vortex core of size $\sim 1/I$, which is surrounded by a large region of size $\sim I$ where the Higgs field is driven to zero by the string magnetic
field so that the electroweak gauge symmetry is completely restored. However, this does not
destroy the vector boson superconductivity, since the scalar Higgs field is not the relevant
order parameter in this case. Outside the symmetric phase region there is a transition zone
where the Higgs field relaxes to the broken phase and the theory reduces to the ordinary
electromagnetism. This reminds somewhat of the electroweak vacuum polarization scenario
discussed by Ambjorn and Olesen \[15\], in which the system stays in the Higgs vacuum if
only the magnetic field is not too strong, while a very strong field restores the full gauge
symmetry.

The rest of the paper is made maximally self-contained. In Sec.II the essential elements of
the Weinberg-Salam theory are introduced. Sec.III presents the reduction to the stationary
sector, the derivation of the conserved quantities, and the discussion of the Bogomol’nyi-
Prasad-Sommerfield (BPS) limit. Sec.IV contains the further reduction to the axially sym-
metric sector, the basic field ansatz (4.4), and the field equations (4.12)–(4.20). In Sec.V the
boundary conditions are discussed, while Sec.VI and Sec.VII contain the derivation of the
charges and a brief description of the known solutions. The new solutions are first considered
in Sec.VIII in the small current limit, when they can be treated as small deformations of Z
strings. In Sec.IX they are presented at the full non perturbative level for generic values of
the current. Sec.X describes the large current limit. Solutions for special parameter values
are considered in Sec.XI, while Sec.XII contains concluding remarks.

There are also three appendices. Appendices A and B contain the derivation of the local
solutions at the symmetry axis and at infinity. Appendix C describes the superconducting
strings in Witten’s model. This Appendix is made self-consistent, but it uses the same
methods and notation as in the main text, so that it illustrates our procedure on a simple
example.

A very brief and preliminary summary of our results has been announced in Ref.\[23\].

Recently there has been an intense activity on the non-Abelian strings with gauge group
SU(N)×U(1) \[24\]. These strings have nothing to do with ours, since they are obtained
within the context of a supersymmetric theory that is different from the electroweak theory
even for N=2. They are currentless and are supposed to be relevant in the QCD context,
as models of gluon tubes. There has also been a report on current-carrying strings \[25\], but
again in a model with a different Higgs sector in which solutions taking values in the Cartan
subalgebra of SU(N)×U(1) are possible.
II. WEINBERG-SALAM THEORY

The bosonic part of the Weinberg-Salam (WS) theory is described by the action

\[ S = \frac{1}{c} \int \left( -\frac{1}{4} W_{\mu\nu}^a W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_{\mu} \Phi)^\dagger D^\mu \Phi - \lambda (\Phi^\dagger \Phi - \Phi_0^2)^2 \right) d^4x. \] (2.1)

Here the complex Higgs field \( \Phi \) is in the fundamental representation of SU(2) and \( W_{\mu\nu}^a = \partial_{\mu} W_{\nu}^a - \partial_{\nu} W_{\mu}^a + g \epsilon_{abc} W_{\mu}^b W_{\nu}^c \), \( B_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} \), \( D_{\mu} \Phi = \left( \partial_{\mu} - \frac{ig'}{2} B_{\mu} - \frac{ig}{2} \tau^a W_{\mu}^a \right) \Phi \),

where \( \tau^a \) are the Pauli matrices and \( c \) in (2.1) is the speed of light. The SU(2) gauge field \( W_{\mu}^a \) and the U(1) field \( B_{\nu} \) are sometimes called isospin and hypercharge fields, respectively.

Here and below we denote dimensionfull quantities by boldfaced symbols. In order to pass to dimensionless variables, let us first introduce the dimensionless coupling constants

\[ g = \frac{g}{g_0} \equiv \cos \theta_W, \quad g' = \frac{g'}{g_0} \equiv \sin \theta_W, \] (2.3)

where the physical value of the Weinberg angle is

\[ g'^2 = \sin^2 \theta_W = 0.23 \ldots \] (2.4)

The parameter

\[ g_0 = \sqrt{g^2 + g'^2} \] (2.5)

is related to the electron charge via \( e = gg'hc g_0 \) so that its dimension is \([g_0] = [1/e] \) and the value

\[ \frac{e^2}{4\pi hc} = \frac{h c}{4\pi (gg'g_0)^2} = \frac{1}{137}. \] (2.6)

The Higgs field vacuum expectation value \( \Phi_0 \) has the dimension \([\Phi_0] = [e/L] \) and the value

\[ \Phi_0 = 54.26 \times 10^9 \text{ Volts}. \] (2.7)

Introducing the length scale \( L = 1/g_0 \Phi_0 = 1.52 \times 10^{-18} \text{ metres} \) (\( \sqrt{2}L \) is the Z boson Compton length) one can define the dimensionless variables

\[ x^\mu = g_0 \Phi_0 x^\mu, \quad B_{\mu} = g' \frac{B_{\mu}}{\Phi_0}, \quad W_{\mu}^a = g \frac{W_{\mu}^a}{\Phi_0}, \quad \Phi = \frac{\Phi}{\Phi_0} \equiv \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right), \quad \beta = \frac{8\lambda}{g_0^2}, \] (2.8)
such that the action (2.1) becomes
\[ S = \frac{1}{c g_0^2} \int \mathcal{L} d^4x. \] (2.9)

Here
\[ \mathcal{L} = -\frac{1}{4g^2} W^a_{\mu
u} W^{a\mu\nu} - \frac{1}{4g^2} B_{\mu\nu} B^{\mu\nu} + (D_\mu \Phi)^\dagger D^\mu \Phi - \frac{\beta}{8} (\Phi^\dagger \Phi - 1)^2, \] (2.10)

and
\[ W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + \epsilon_{abc} W^b_\mu W^c_\nu, \]
\[ B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \]
\[ D_\mu \Phi = \left( \partial_\mu - \frac{i}{2} B_\mu - \frac{i}{2} \tau^a W^a_\mu \right) \Phi. \] (2.11)

The Lagrangian (2.10) is invariant under the SU(2)×U(1) gauge transformations
\[ \Phi \rightarrow U \Phi, \quad W \rightarrow U W U^{-1} + 2i U \partial_\mu U^{-1} dx^\mu, \] (2.12)

with
\[ U = \exp \left( \frac{i}{2} \Theta + \frac{i}{2} \tau^a \theta^a \right) \] (2.13)

where \( \Theta \) and \( \theta^a \) are functions of \( x^\mu \) and
\[ W = (B_\mu + \tau^a W^a_\mu) dx^\mu \] (2.14)

is the SU(2)×U(1) Lie-algebra valued gauge field. Varying the action with respect to the fields gives the field equations,
\[ \partial^\mu B_{\mu\nu} = g^2 \frac{i}{2} ((D_\nu \Phi)^\dagger \Phi - \Phi^\dagger D_\nu \Phi) \equiv g^2 J_\nu^0, \] (2.15)
\[ D^\mu W^a_{\mu\nu} = g^2 \frac{i}{2} ((D_\nu \Phi)^\dagger \tau^a \Phi - \Phi^\dagger \tau^a D_\nu \Phi) \equiv g^2 J_a^\nu, \] (2.16)
\[ D_\mu D^\mu \Phi + \frac{\beta}{4} (\Phi^\dagger \Phi - 1) \Phi = 0, \] (2.17)

with \( D_\mu W^a_{\alpha\beta} = \partial_\mu W^a_{\alpha\beta} + \epsilon_{abc} W^b_\mu W^c_{\alpha\beta} \). Varying the action with respect to the spacetime metric gives the energy momentum tensor,
\[ T^\mu_\nu = 2g^{\alpha\sigma} \frac{\partial \mathcal{L}}{\partial g^{\alpha\nu}} - \delta^\mu_\nu \mathcal{L}, \] (2.18)

which evaluates to
\[ T^\mu_\nu = -\frac{1}{g^2} W^{a\mu\sigma} W^a_{\nu\sigma} - \frac{1}{g^2} B^{\mu\sigma} B_{\nu\sigma} + (D^\mu \Phi)^\dagger D_\nu \Phi + (D_\nu \Phi)^\dagger D^\mu \Phi - \delta^\mu_\nu \mathcal{L}. \] (2.19)
Let us define \textit{vacuum} as
\[ W^a_\mu = B_\mu = 0, \quad \Phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.20} \]
Any field with $T^\mu_\nu = 0$ is a gauge transformation of the vacuum. Let us consider small fluctuations around the vacuum in the unitary gauge, assuming that $W^a_\mu$ and $B_\mu$ are small and that $\Phi = \begin{pmatrix} 1 + \phi \\ 0 \end{pmatrix}$ with $\phi$ real and small. Linearising the field equations (2.15)–(2.17) with respect to the perturbations gives
\[
\begin{align*}
\partial_\mu F^{\mu\nu} &= 0, \\
\partial_\mu Z^{\mu\nu} + m_z^2 Z^\nu &= 0, \\
\partial_\mu W^{\mu\nu}_a + m_w^2 W^\nu_a &= 0 \quad (a = 1, 2), \\
\partial_\mu \partial^\mu \phi + m_\phi^2 \phi &= 0. \tag{2.21}
\end{align*}
\]
Here the electromagnetic and Z fields are defined as
\[ F^{\mu\nu} = \frac{g}{g'} B^{\mu\nu} - \frac{g'}{g} W_3^{\mu\nu}, \quad Z^{\mu\nu} = B^{\mu\nu} + W_3^{\mu\nu}, \tag{2.22} \]
with the potentials
\[ A_\mu = \frac{g}{g'} B_\mu - \frac{g'}{g} W_3^\mu, \quad Z_\mu = B_\mu + W_3^\mu, \tag{2.23} \]
while the field masses
\[ m_z = \frac{1}{\sqrt{2}}, \quad m_w = gm_z, \quad m_\phi = \sqrt{\beta} m_z. \tag{2.24} \]
Multiplying these by $\hbar c g_0 \Phi_0 = e \Phi_0 / (g g')$ gives the dimensionfull masses, for example
\[ m_z c^2 = e \Phi_0 / (\sqrt{2} g g'), \tag{2.25} \]
which evaluates 91.18 GeV [26]. The value of the parameter $\beta$ defining the Higgs field mass is currently unknown, however, there are indications that it belongs to the interval $1.5 \leq \beta \leq 3.5$ [26].

The electromagnetic and Z fields in (2.22) are defined under the assumption that the system is close to the vacuum. In what follows we shall need to define these fields also off the vacuum. This can be done in a gauge invariant way, however there is no \textit{unique} definition
in this case \[27\], although all known definitions reduce to \(2.22\) when the fields approach the vacuum. Unless the otherwise is stated, we shall adopt the definition of Nambu \[16\],

\[
F_{\mu\nu} = \frac{g'}{g} B_{\mu\nu} - \frac{g'}{g} n^a W^a_{\mu\nu}, \quad Z_{\mu\nu} = B_{\mu\nu} + n^a W^a_{\mu\nu},
\]

(2.26)

where

\[
n^a = \xi^+ \tau^a \xi, \quad \xi = \Phi / \sqrt{\Phi^\dagger \Phi}.
\]

(2.27)

It seems that this definition gives the most sensible results as compared to the other definitions. Its only disadvantage is that the 2-forms \(2.26\) are not closed in general, so that there are no field potentials. However, this cannot be considered as a drawback, since there is no reason why the Maxwell equations should hold off the Higgs vacuum. It is worth noting that a similar preference in favour of the Nambu definition was made also in Ref. \[29\].

Another well known definition of the fields is according to ‘t Hooft \[28\],

\[
F^H_{\mu\nu} = \frac{g'}{g} B_{\mu\nu} - \frac{g'}{g} V_{\mu\nu}, \quad Z^H_{\mu\nu} = B_{\mu\nu} + V_{\mu\nu}
\]

(2.28)

where

\[
V_{\mu\nu} = n^a W^a_{\mu\nu} - \epsilon_{abc} n^b D_\mu n^c = \partial_\mu V_\nu - \partial_\nu V_\mu
\]

(2.29)

with

\[
V_\mu = n^a W^a_\mu + 2i \xi^+ \partial_\mu \xi, \quad D_\mu n^a = \partial_\mu n^a + \epsilon_{abc} W^b_\mu n^c.
\]

(2.30)

The field strengths are closed in this case and the potentials are known explicitly. In the unitary gauge where \(n^a = \delta^3_3\) the potentials reduce simply to \(A_\mu = \frac{g'}{g} B_\mu - \frac{g'}{g} W^3_\mu\) and \(Z_\mu = B_\mu + W^3_\mu\) in which form they are most often used (see, e.g., \[15\]). However, as we shall see below, the t’ Hooft definition gives sometimes singular results, while applying instead the Nambu definition makes things perfectly sensible.

### III. SYMMETRY REDUCTION

In what follows we shall be considering stationary and translationally symmetric fields invariant under the action of two spacetime isometries generated by the Killing vectors

\[
K_{(0)} = \frac{\partial}{\partial x^0}, \quad K_{(3)} = \frac{\partial}{\partial x^3}.
\]

(3.1)

Since these vectors commute and all the internal symmetries of the theory are gauged, there exists a gauge where the symmetric fields do not depend on \(x^0 \equiv t\) and \(x^3 \equiv z\) \[30\].
Let $\sigma_\alpha$ be a constant (co)vector in the $(x^0, x^3)$ plane and $\tilde{\sigma}_\alpha = \epsilon_{\alpha\beta}\sigma^\beta$ its dual. These two vectors are orthogonal and have the norm

$$\sigma^2 \equiv -\sigma^\alpha\sigma_\alpha + \tilde{\sigma}^\alpha\tilde{\sigma}_\alpha = -\sigma_0^2 + \sigma_3^2. \quad (3.2)$$

Using these two vectors, the fields respecting the symmetries (3.1) can be written as

$$\mathcal{W} = [u\sigma_\alpha + \tilde{u}\tilde{\sigma}_\alpha]dx^\alpha + B_i dx^i + \tau^a[u_\alpha\sigma_\alpha + \tilde{u}_\alpha\tilde{\sigma}_\alpha]dx^\alpha + \tau^a W^a_i dx^i, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (3.3)$$

where $\alpha = 0, 3$ and $i, k = 1, 2$, while $u, \tilde{u}, u^a, \tilde{u}^a, B_i, W^a_i, \phi_1, \phi_2$ depend on $x^k$, so that there are in total 20 independent real functions. The ansatz (3.3) keeps its form under gauge transformations (2.12) with $\Theta, \theta^a$ depending only on $x^k$.

Inserting this ansatz into (2.10) gives the reduced Lagrangian

$$\mathcal{L} = -\frac{\sigma^2}{2g^2} \left( (\partial_k u)^2 - (\partial_k \tilde{u})^2 \right) - \frac{\sigma^2}{2g^2} \left( (D_k u_a)^2 - (D_k \tilde{u}_a)^2 \right) + \frac{(\sigma^2)^2}{2g^2} \left( (u_a)^2 - (u^a)^2 \right) - \frac{1}{4g^2} (B_{ik})^2 - \frac{1}{4g^2} (W^a_{ik})^2$$

$$- |D_k \Phi|^2 - \frac{\sigma^2}{4} \left( \Phi^\dagger (u + u^a \tau^a)^2 \Phi - \Phi^\dagger (\tilde{u} + \tilde{u}^a \tau^a)^2 \Phi \right) - \frac{\beta}{8} (\Phi^\dagger \Phi - 1)^2 \quad (3.4)$$

where $D_k u_a = \partial_k u_a + \epsilon_{abc} W^b_{k} u^c$ and $D_k \Phi = (\partial_k - \frac{i}{2} B_k - \frac{i}{2} \tau^a W^a_k) \Phi$. Varying this gives the field equations, where, as is evident from the structure of the reduced Lagrangian, one can consistently set

$$\tilde{u} = \tilde{u}^a = 0. \quad (3.5)$$

The remaining non-trivial equations then read

$$\partial_s \partial_s u = \frac{g^2}{2} \phi^\dagger (u + u^a \tau^a) \phi, \quad (3.6a)$$

$$D_s D_s u_a = \frac{g^2}{2} \phi^\dagger (u_a + u^a \tau^a) \phi, \quad (3.6b)$$

$$\partial_s B_{sk} = i \frac{g^2}{2} (\phi^\dagger D_k \Phi - (D_k \Phi)^\dagger \Phi), \quad (3.6c)$$

$$D_s W_{sk}^a - \sigma^2 \epsilon_{abc} u_b D_k u_c = i \frac{g^2}{2} (\phi^\dagger \tau^a D_k \Phi - (D_k \Phi)^\dagger \tau^a \Phi), \quad (3.6d)$$

$$D_s D_s \Phi - \frac{\sigma^2}{4} (u + u^a \tau^a)^2 \Phi = \frac{\beta}{4} (\phi^\dagger \Phi - 1) \phi. \quad (3.6e)$$

Let us discuss the global quantities associated to solutions of these equations – the magnetic fluxes, charge and current, energy and momentum.
A. Fluxes

The electromagnetic and Z fields produce magnetic fluxes through the \( x^1, x^2 \) plane,

\[
\Psi_F = \frac{1}{2} \int \epsilon_{ik} F_{ik} d^2x, \quad \Psi_Z = \frac{1}{2} \int \epsilon_{ik} Z_{ik} d^2x. \tag{3.7}
\]

The dimensionful values are \( \Psi_F = \frac{1}{2} \int \epsilon_{ik} F_{ik} d^2x = \Psi_F / g_0 \) and \( \Psi_Z = \Psi_Z / g_0 \). The fluxes are gauge invariant, but their values depend on the definition of \( F_{\mu\nu} \) and \( Z_{\mu\nu} \) chosen. If the fields are defined according to 't Hooft, then they admit potentials, in which case one can use the Stokes theorem to express the fluxes in terms of integrals over the boundary at infinity. With the Nambu definition one should calculate the surface integrals.

In general there is no reason for the fluxes to be quantized. Indeed, let us briefly recall the standard argument in favour of the vortex flux quantization. Far away from the vortex the condition

\[
(\partial_\mu - \frac{i}{2} B_\mu - \frac{i}{2} \tau^a W^a_\mu) \Phi = 0 \tag{3.8}
\]

should be fulfilled, whose solution is

\[
\Phi(x) = \mathcal{P} \exp \left( \frac{i}{2} \oint_C (B_\mu + \tau^a W^a_\mu) dx^\mu \right) \Phi(x_0). \tag{3.9}
\]

Here \( \mathcal{P} \) stands for the path ordering along a contour \( C \) interpolating between \( x_0 \) and \( x \). Choosing \( C \) to be a large circle around the vortex gives the holonomy condition

\[
\Phi(x_0) = \mathcal{P} \exp \left( \frac{i}{2} \oint_C (B_\mu + \tau^a W^a_\mu) dx^\mu \right) \Phi(x_0). \tag{3.10}
\]

If \( W^a_\mu = 0 \), as for example in the semilocal limit \( g \to 0 \) (see Sec.XIA below), then the condition (3.10) reduces to

\[
\frac{1}{2} \oint_0^{2\pi} B_\varphi d\varphi = 2\pi n \tag{3.11}
\]

and so the hypercharge field flux is quantized, \( \Psi_B = 4\pi n \).

Let us now suppose that \( W^1_\mu = W^2_\mu = 0 \) but \( W^3_\mu \neq 0 \) and let us denote \( \Psi_W = \int_0^{2\pi} W_\varphi d\varphi \). The condition (3.10) then gives

\[
\phi_1(x_0) = e^{\frac{i}{2}(\Psi_B + \Psi_W)} \phi_1(x_0), \quad \phi_2(x_0) = e^{\frac{i}{2}(\Psi_B - \Psi_W)} \phi_2(x_0), \tag{3.12}
\]

and if, for example, \( \phi_2(x_0) = 0 \), then one should have \( \Psi_B + \Psi_W = 4\pi n \) but the difference \( \Psi_B - \Psi_W \) can be arbitrary. The fluxes are therefore not quantized in general.
B. Current

The conserved electromagnetic current is

\[ J_\nu = \partial_\mu F_{\mu\nu}. \quad (3.13) \]

Since nothing depends on \( x^\alpha = (x^0, x^3) \), one has \( J_\alpha = -\partial_k F_{k\alpha} \). This being a total derivative, integrating over the \( x^1, x^2 \) plane gives

\[ I_\alpha = \int J_\alpha d^2x = -\int \partial_k F_{k\alpha} d^2x = -\int \epsilon_{ks} F_{k\alpha} dx^s. \quad (3.14) \]

Here \( I_0 \) is the total electric charge per unit length of the vortex and \( I_3 \) is the total electric current through the \( x^1x^2 \) plane. Since the integration in (3.14) is performed over the boundary at infinity, in the region where all fields approach the vacuum and the electromagnetic field is uniquely defined, the value of \( I_\alpha \) does not depend on the chosen definition of \( F_{\mu\nu} \).

Let us consider the dimensionfull version of (3.13),

\[ \frac{1}{c} J_\nu = \partial_\mu F_{\mu\nu}, \quad (3.15) \]

where \( F_{\mu\nu} = g_0 \Phi_0^2 F_{\mu\nu} \) and the derivative is taken with respect to \( x^\mu = x^\mu/g_0 \Phi_0 \). One has

\[ \int J_\alpha d^2x = -c \int \partial_k F_{k\alpha} d^2x = -c\Phi_0 \int \partial_k F_{k\alpha} d^2x = c\Phi_0 I_\alpha, \quad (3.16) \]

from where it follows that \( I_\alpha \) gives the current in units of

\[ c\Phi_0 = c \times 54.26 \times 10^9 \text{ Volts} = 1.8 \times 10^9 \text{ Amperes}. \quad (3.17) \]

This value is quite large. Below we shall present vortex solutions whose current is typically \( I_3 \sim 1 - 10 \), which looks modest but corresponds in fact to billions of Amperes! Very large currents are typical for superconducting strings. In the GUT-related model of Witten the current can be as large as \( 10^{20} \) Amperes \([1],[3]\) – because the GUT Higgs field vacuum expectation value is much larger than \( \Phi_0 \).

C. Energy and momentum

Contracting the Killing vectors with the energy-momentum tensor \([2.19]\) gives conserved Noether currents,

\[ j^\mu_{(K)} = T^\nu_{\nu} K^\nu \quad \Rightarrow \quad \partial_\mu j^\mu_{(K)} = 0. \quad (3.18) \]
Integrating over the \( x^1, x^2 \) plane gives the conserved charges per unit vortex length – the vortex energy and momentum densities,

\[
E = \int T^\nu_{\nu(0)} d^2 x = \int T^0_0 d^2 x, \quad (3.19)
\]

\[
P = \int T^\nu_{\nu(3)} d^2 x = \int T^3_3 d^2 x. \quad (3.20)
\]

The dimensionfull values are \( E = \Phi^2_0 E \) and \( P = \Phi^2_0 P/c \).

Let us consider the momentum. According to Ref. \[32\], in gauge field theories without global internal symmetries the density \( j_0^\nu(K) \) for a spacelike Killing vectors \( K \) has the total derivative structure. Calculating the energy-momentum tensor (2.19) for the field (3.3) shows that \( T^0_3 \) is indeed a total derivative,

\[
T^0_3 = -\frac{1}{g^2} W_{a0}^\nu W_{3\nu}^a - \frac{1}{g^2} B_{a0}^\nu B_{3\nu}^a + (D^0\Phi)^\dagger D_3 \Phi + (D_3 \Phi)^\dagger D^0 \Phi = \sigma_0 \sigma_3 \partial_k \Xi^a_k \quad (3.21)
\]

with

\[
\partial_k \Xi^a_k = \frac{1}{g^2} (\partial_k u)^2 + \frac{1}{g^2} (D_k u_a)^2 + \frac{1}{2} \Phi^\dagger (u + u_a \tau^a)^2 \Phi
\]

\[
= \partial_k \left( \frac{1}{g^2} u \partial_k u + \frac{1}{g^2} u^a D_k u^a \right), \quad (3.22)
\]

where the field equations (3.6a)–(3.6e) have been used to pass to the second line. For globally regular fields the total momentum is therefore completely determined by the boundary term,

\[
P = \sigma_0 \sigma_3 \oint \epsilon_k \Xi^a_k dx^a. \quad (3.23)
\]

Let us now consider the energy. It is given by a sum of two manifestly positive terms which we shall call, respectively, electric and magnetic energy,

\[
E = \int T^0_0 d^2 x = \int (\mathcal{E}_1 + \mathcal{E}_2) d^2 x = E_1 + E_2. \quad (3.24)
\]

The structure of \( \mathcal{E}_1 \) is very similar to that of \( T^0_3 \) in (3.21),

\[
\mathcal{E}_1 = \frac{\sigma_0^2 + \sigma_3^2}{2} \partial_k \Xi^a_k, \quad (3.25)
\]

so that it is a total derivative. The magnetic energy is

\[
\mathcal{E}_2 = \frac{1}{4g^2} (W^a_{ik})^2 + \frac{1}{4g^2} (B_{ik})^2 + |D_3 \Phi|^2 + \frac{\beta}{8} (|\Phi|^2 - 1)^2. \quad (3.26)
\]
D. Bogomol’nyi bound and the electroweak condensate

It is instructive to mention the existence of the non-trivial lower bound for the magnetic energy, observed by Ambjorn and Olesen [15]. Using the identity

\[ |D_s \Phi|^2 = \frac{1}{2} |D_t \Phi + i\epsilon_{ij}D_j \Phi|^2 + \frac{1}{4} \epsilon_{ik} \Phi^\dagger (B_{ik} + \tau^a W^a_{ik}) \Phi - i\epsilon_{sk} \partial_s (\Phi^\dagger D_k \Phi) \]  

(3.27)

the density \( E_2 \) can be rewritten as

\[
E_2 = \frac{1}{4g^2} \left( W^a_{ik} + \frac{g^2}{2} \epsilon_{ik} \Phi^\dagger \tau^a \Phi \right)^2 + \frac{1}{4g'^2} \left( B_{ik} + \frac{1}{2} \epsilon_{ik} \left( g'^2 \Phi^\dagger \Phi - 1 \right) \right)^2 \\
+ \frac{1}{4g'^2} \epsilon_{ik} B_{ik} - i\epsilon_{sk} \partial_s (\Phi^\dagger D_k \Phi) - \frac{g^2}{8g'^2}.
\]  

(3.28)

From this it follows that for \( \beta \geq 1 \) one has

\[
E_2 \geq \int \left( \frac{1}{4g'^2} \epsilon_{ik} B_{ik} - i\epsilon_{sk} \partial_s (\Phi^\dagger D_k \Phi) - \frac{g^2}{8g'^2} \right) d^2x,
\]

(3.29)

which defines the lower energy bound. This bound is, however, somewhat special. Considering just one vortex and integrating over the whole plane, the first term in the integrand in (3.29) gives the magnetic flux, the second term gives zero, while the third one gives the infinite area of the plane with a negative coefficient. The conclusion is that \( E_2 \) is always larger than minus infinity, which is true but trivial.

A less trivial conclusion can be made if one considers an infinite number of parallel vortices forming a periodic lattice in the \( x^1, x^2 \) plane, since the field \( B_{ik} \) then does not vanish at infinity and its contribution can overcome the effect of the area term. It is then sufficient to integrate just over the elementary periodicity cell with the area \( A \). One uses the relation \( B_{ik} = gg'F_{ik} + g^2Z_{ik} \) where \( Z_{ik} \) does not contribute to the flux since its potentials (defined in [15] according to t’ Hooft) are gauge invariant and therefore periodic if \( Z_{ik} \) is periodic [15]. The bound then reads

\[
E_2 = \int_A E_2 \, d^2x \geq \frac{g}{2g'} \Psi_F - \frac{g^2}{8g'^2} A = \frac{m_W^2}{e}(\Psi_F - \frac{m_W^2}{2e}A)
\]

(3.30)

where \( \Psi_F \) is the flux of \( F_{ik} \) through \( A \). If the system contains only a constant magnetic field \( F_{12} > m_W^2/e \) then this relation gives

\[
E_2 = \frac{1}{2} (F_{12})^2 A \geq \frac{m_W^2}{e}(F_{12} - \frac{m_W^2}{2e})A > 0.
\]

(3.31)
This shows that for large $F_{12}$ the system becomes unstable, since its energy grows as $(F_{12})^2$ while the lower energy bound grows only as $F_{12}$. The bound can be achieved for $\beta = 1$ if the first order Bogomol’nyi equations are fulfilled,

\begin{align}
W_{ik}^a + \frac{g^2}{2} \epsilon_{ik} \Phi^\dagger \tau^a \Phi &= 0, \\
B_{ik} + \frac{1}{2} \epsilon_{ik} (g'^2 \Phi^\dagger \Phi - 1) &= 0, \\
D_i \Phi + i \epsilon_{ij} D_j \Phi &= 0.
\end{align}

(3.32a) (3.32b) (3.32c)

Periodic solutions of these equations describe a non-linear condensate of $W, Z$ and Higgs fields forming an infinite number of non-Abelian vortices that spontaneously appear in the very strong magnetic field \[15\]. It turns out that non-trivial solutions exist if only \[15\]

\[
\frac{m_W^2}{e} < \langle F_{12} \rangle < \frac{m_H^2}{e}
\]

(3.33)

where $\langle F_{12} \rangle$ is the magnetic field averaged over $A$. If the lower inequality here is violated, then the Higgs field falls to the vacuum, while in the opposite limit it is driven to zero \[15\]. This can be interpreted as the electroweak symmetry restoration in the very strong magnetic field.

Since the magnetic fields satisfying the conditions (3.33) are extremely strong, it was suggested to use the Witten superconducting strings of the GUT origin as their source \[15\]. At certain distances from the string core the magnetic field falls into the interval (3.33), so that an electroweak condensate can form within a cylindrical shell around the string. Such systems are sometimes called ‘W-dressed superconducting strings’ \[15\], [31].

Below we shall construct electroweak vortices showing a compact superconducting core embedded into a symmetric phase region that is wrapped in a cylindrical shell where the Higgs field interpolates between the symmetric and broken phases. This reminds very much of the ‘W-dressed superconducting strings’, up to the fact that our strings are of purely electroweak origin and have nothing to do with the GUT physics. They can exist for any value of $\beta$, which is why the Bogomol’nyi equations are of little use for us.

The above arguments go differently in the semilocal limit, where $g = 0$ (see Sec.XI A), since the area term in (3.30) then vanishes and the magnetic flux through the whole plane is no longer constrained to be infinite. The flux is then quantized as $\Psi_B = 4\pi n$ and the energy lower bound is $2\pi n$. Eq.(3.32a) implies then that $W_{ik}^a = 0$ so that the Yang-Mills
field can be gauged away, while the remaining Bogomol’nyi equations (3.32b), (3.32c) admit vortex solutions for any given $n \in \mathbb{Z}$ [33], [36].

IV. AXIAL SYMMETRY

Let us now further restrict our consideration to fields which are invariant with respect to rotations in the $x^1, x^2$ plane. This means that in addition to the two Killing vectors (3.1) there exists a third one. Choosing the polar coordinates in the plane, $x^1 = \rho \cos \varphi$, $x^2 = \rho \sin \varphi$, it is given by

$$K(\varphi) = \frac{\partial}{\partial \varphi}.$$  

(4.1)

Since this vector commutes with the other two, for fields respecting all three symmetries at the same time there exists a gauge where they do not depend neither on $x^\alpha$ nor on $\varphi$. They are then given by Eq. (3.3) with all the functions depending only on $\rho$. The field equations (3.6) become in this case a system of 16 ordinary differential equations for the 16 independent functions left in the ansatz (3.3) after imposing the condition (3.5).

Within the gauge chosen one can still perform gauge transformations (2.12) with $\Theta, \theta^a$ depending only on $\rho$. We use this freedom to set to zero the radial components of the gauge fields,

$$B_\rho = 0, \quad W^a_\rho = 0.$$  

(4.2)

This reduces the number of independent functions to 12. Now, the inspection of the field equations reveals that one can consistently set to zero the imaginary components of the fields. These include the $\tau^2$ projection of the Yang-Mills field and also the imaginary part of the Higgs field. We can therefore set on-shell

$$W^2_\mu = 0, \quad \Im(\Phi) = 0.$$  

(4.3)

As a result, we are left with only 8 independent real functions, in which case we can parametrize the remaining non-trivial components in (3.3) as

$$W = u(\rho) \sigma_\alpha dx^\alpha - v(\rho) d\varphi + \tau^1 [u_1(\rho) \sigma_\alpha dx^\alpha - v_1(\rho) d\varphi]$$

$$+ \tau^3 [u_3(\rho) \sigma_\alpha dx^\alpha - v_3(\rho) d\varphi], \quad \Phi = \begin{pmatrix} f_1(\rho) \\ f_2(\rho) \end{pmatrix},$$  

(4.4)

where $\alpha = 0, 3$ and $f_1, f_2$ are real. This ansatz has the following properties.
a. It is invariant under spacetime symmetries generated by
\[ \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial \varphi}. \] (4.5)

b. It is invariant under complex conjugation,
\[ W = W^*, \quad \Phi = \Phi^*. \] (4.6)

c. It keeps its form under Lorentz rotations in the \( x^0, x^3 \) plane, whose effect on (4.4) is to Lorentz-transform the components of the (co)vector \( \sigma_\alpha \):
\[ \sigma_0 \rightarrow \sigma_0 \cosh \alpha - \sigma_3 \sinh \alpha, \quad \sigma_3 \rightarrow \sigma_3 \cosh \alpha - \sigma_0 \sinh \alpha, \] (4.7)

where \( \alpha \) is the boost parameter. The norm \( \sigma^2 = \sigma_3^2 - \sigma_0^2 \) is Lorentz-invariant, and we shall call the vortex magnetic if \( \sigma^2 > 0 \), electric for \( \sigma^2 < 0 \), and chiral if \( \sigma^2 = 0 \) [4].

d. It keeps its form under gauge transformations (2.12) generated by \( U = \exp \left\{ -\frac{i}{2} \Gamma \tau^2 \right\} \) with constant \( \Gamma \), whose effect is to rotate the field amplitudes,
\[ f_1 \rightarrow f_1 \cos \frac{\Gamma}{2} - f_2 \sin \frac{\Gamma}{2}, \quad f_2 \rightarrow f_2 \cos \frac{\Gamma}{2} + f_1 \sin \frac{\Gamma}{2}, \]
\[ u_1 \rightarrow u_1 \cos \Gamma + u_3 \sin \Gamma, \quad u_3 \rightarrow u_3 \cos \Gamma - u_1 \sin \Gamma, \]
\[ v_1 \rightarrow v_1 \cos \Gamma + v_3 \sin \Gamma, \quad v_3 \rightarrow v_3 \cos \Gamma - v_1 \sin \Gamma, \] (4.8)

whereas \( u, v \) rest invariant. These transformations can also be written in a compact form using the complex variables:
\[ (f_1 + if_2) \rightarrow e^{\frac{i\tau}{2}}(f_1 + if_2), \quad (u_1 + iu_3) \rightarrow e^{-i\tau}(u_1 + iu_3), \quad (v_1 + iv_3) \rightarrow e^{-i\tau}(v_1 + iv_3), \] (4.9)

This symmetry can be fixed by requiring that
\[ v_1(0) = 0. \] (4.10)

e. The ansatz does not change if we multiply the amplitudes \( u, u_1, u_3 \) by a constant dividing at the same time \( \sigma_\alpha \) by the same constant. To fix this symmetry, we impose the condition
\[ u_3(0) = 1, \] (4.11)

which is possible if \( u_3(0) \neq 0 \).

It should be said that the gauge (4.4) is very convenient for calculations, since everything depends only on \( \rho \), but it is not completely satisfactory, because, as we shall see below, the functions \( v, v_1, v_3 \) do not vanish at \( \rho = 0 \) and so the vector fields are not globally defined in this gauge. This problem can be cured by passing to another gauge (Eq.(5.6)).
A. Field equations

With the parametrization (4.4) the U(1) equations (3.6a), (3.6c) reduce to
\[
\frac{1}{\rho} (\rho u')' = \frac{g^2}{2} \left\{ (u + u_3) f_1^2 + 2 u_1 f_1 f_2 + (u - u_3) f_2^2 \right\}, \quad (4.12)
\]
\[
\rho \left( \frac{v'}{\rho} \right)' = \frac{g^2}{2} \left\{ (v + v_3) f_1^2 + 2 v_1 f_1 f_2 + (v - v_3) f_2^2 \right\}, \quad (4.13)
\]
where the prime denotes differentiation with respect to \( \rho \). The Higgs equations (3.6e) become
\[
\frac{1}{\rho} (\rho f_1')' = \left\{ \frac{\sigma^2}{4} [(u + u_3)^2 + u_1^2] + \frac{1}{4\rho^2} [(v + v_3)^2 + v_1^2] + \frac{\beta}{4} (f_1^2 + f_2^2 - 1) \right\} f_1 \\
+ \left( \frac{\sigma^2}{2} u u_1 + \frac{1}{2\rho^2} v v_1 \right) f_2, \quad (4.14)
\]
\[
\frac{1}{\rho} (\rho f_2')' = \left\{ \frac{\sigma^2}{4} [(u - u_3)^2 + u_1^2] + \frac{1}{4\rho^2} [(v - v_3)^2 + v_1^2] + \frac{\beta}{4} (f_1^2 + f_2^2 - 1) \right\} f_2 \\
+ \left( \frac{\sigma^2}{2} u u_1 + \frac{1}{2\rho^2} v v_1 \right) f_1. \quad (4.15)
\]
The Yang-Mills equations (3.6b), (3.6d) reduce to
\[
\frac{1}{\rho} (\rho u_1')' = -\frac{1}{\rho^2} (v_1 u_3 - v_3 u_1) v_3 + \frac{g^2}{2} \left[ u_1 (f_1^2 + f_2^2) + 2 u f_1 f_2 \right], \quad (4.16)
\]
\[
\frac{1}{\rho} (\rho u_3')' = \frac{1}{\rho^2} (v_1 u_3 - v_3 u_1) v_1 + \frac{g^2}{2} \left[ (u_3 + u) f_1^2 + (u_3 - u) f_2^2 \right], \quad (4.17)
\]
\[
\rho \left( \frac{v_1'}{\rho} \right)' = +\sigma^2 (v_1 u_3 - v_3 u_1) u_3 + \frac{g^2}{2} \left[ v_1 (f_1^2 + f_2^2) + 2 v f_1 f_2 \right], \quad (4.18)
\]
\[
\rho \left( \frac{v_3'}{\rho} \right)' = -\sigma^2 (v_1 u_3 - v_3 u_1) u_1 + \frac{g^2}{2} \left[ (v_3 + v) f_1^2 + (v_3 - v) f_2^2 \right]. \quad (4.19)
\]

In addition, a careful inspection reveals that, although we have set to zero the radial and the second isotopic components of the Yang-Mills field, the Yang-Mills equation (3.6d) with \( a = 2 \) and \( k = \rho \) is not satisfied identically but gives the condition
\[
\Lambda = 0, \quad (4.20)
\]
where
\[
\Lambda \equiv \sigma^2 (u_1 u_3' - u_3 u_1') + \frac{1}{\rho^2} (v_1 v_3' - v_3 v_1') - g^2 (f_1 f_2' - f_2 f_1'). \quad (4.21)
\]
Differentiating this quantity and using Eqs. (4.12)–(4.19) we discover the following relation
\[
\Lambda' + \frac{\Lambda}{\rho} = 0, \quad (4.22)
\]
such that
\[ \Lambda = \frac{C}{\rho}, \tag{4.23} \]
where \( C \) is an integration constant. This is a first integral for the above second order equations, so that replacing one of them by Eq. (4.23) would give a completely equivalent system. We thus see that Eq. (4.20) plays a role of constraint restricting the value of \( C \).

The radial energy density for axially symmetric fields is \( \rho T^0_0 = \rho (\mathcal{E}_1 + \mathcal{E}_2) \), where
\[
\mathcal{E}_1 = \frac{\sigma_0^2 + \sigma_3^2}{2} \left\{ \frac{1}{g^0_2} u'^2 + \frac{1}{g^2_2} (u'^2 + u'_3)^2 + \frac{1}{g^2_2 \rho} (v_1 u_3 - v_3 u_1)^2 \right\} + \frac{1}{2} \left[ ((u + u_3) f_1 + u_1 f_2)^2 + ((u - u_3) f_2 + u_1 f_1)^2 \right] \tag{4.24}
\]
and
\[
\mathcal{E}_2 = \frac{1}{2 \rho^2} \left( \frac{1}{g^0_2} v'^2 + \frac{1}{g^2_2} (v'^2 + v'_3)^2 \right) + f_1^2 + f_2^2 \tag{4.25}
\]
and
\[
+ \frac{1}{4 \rho^2} \left[ ((v + v_3) f_1 + v_1 f_2)^2 + ((v - v_3) f_2 + v_1 f_1)^2 \right] + \frac{\beta}{8} (f_1^2 + f_2^2 - 1)^2.
\]
Using the above equations it is not difficult to see that \( \rho \mathcal{E}_1 \) is actually a total derivative,
\[
\rho \mathcal{E}_1 = \frac{\sigma_0^2 + \sigma_3^2}{2} \left( \frac{\rho}{g^0_2} uu' + \frac{\rho}{g^2_2} (u_1 u'_1 + u_3 u'_3) \right), \tag{4.26}
\]
in agreement with Eq. (3.25). Next, one has
\[
\rho \mathcal{E}_2 = \frac{1}{2 \rho g^0_2} (B_1)^2 + \frac{1}{2 \rho g^2_2} [(B_2)^2 + (B_3)^2] + \rho [(B_4)^2 + (B_5)^2] + \rho \frac{\beta - 1}{8} (f_1^2 + f_2^2 - 1)^2
\]
\[
+ \left( \frac{1}{2 g^2_2} v - (v + v_3) \frac{f_1^2}{2} - (v - v_3) \frac{f_2^2}{2} - v_1 f_1 f_2 - \frac{g^2_2}{8 g^0_2} \rho \right)', \tag{4.27}
\]
with
\[
B_1 = v' + \frac{\rho}{2} [g^0_2 (f_1^2 + f_2^2) - 1],
\]
\[
B_2 = v'_1 + g^2_2 \rho f_1 f_2,
\]
\[
B_3 = v'_3 + g^2_2 \rho (f_1 - f_2),
\]
\[
B_4 = f_1' + \frac{1}{2 \rho} [(v + v_3) f_1 + v_1 f_2],
\]
\[
B_5 = f_2' + \frac{1}{2 \rho} [(v - v_3) f_2 + v_1 f_1]. \tag{4.28}
\]
It follows that if \( \beta > 1 \) then \( \int_0^{\infty} \rho \mathcal{E}_2 d\rho \) is bounded from below by the integral of the total derivative in the second line in (4.27). The bound is achieved for \( \beta = 1 \) if \( B_1 = \ldots = \)
Solutions of these Bogomol’nyi equations automatically fulfill the five second order equations (4.13)–(4.15), (4.18), (4.19) with $\sigma^2 = 0$. However, unless for $g = 0$, the Bogomol’nyi configurations are not very interesting, since, as was discussed above, they do not approach the vacuum at infinity and their energy is infinite. In what follows we shall rather focus on solutions of the second order equations (4.12)–(4.20) for generic values of $\beta$ and $g$.

V. BOUNDARY CONDITIONS

In order to construct global solutions of equations (4.12)–(4.20) in the interval $\rho \in [0, \infty)$ we shall need their local solutions in the vicinity of the singular points, $\rho = 0$ and $\rho = \infty$. We shall be considering fields that are regular at $\rho = 0$ and approach the vacuum for $\rho \to \infty$. Let us first consider local solutions at small $\rho$.

A. Boundary conditions at the symmetry axis

Expressions (4.24), (4.25) for the energy density can be used to derive the regularity conditions at $\rho = 0$. For regular fields the energy density at the axis must be bounded, and so the coefficients in front of the negative powers of $\rho$ in (4.24), (4.25) should vanish at $\rho = 0$. This implies that at $\rho = 0$ one should have

$$v' = v'_1 = v'_3 = 0,$$

and also

$$v_1 u_3 - v_3 u_1 = 0,$$

$$(v + v_3) f_1 + v_1 f_2 = 0,$$

$$(v - v_3) f_2 + v_1 f_1 = 0.$$

Using Eq. (4.10) these conditions reduce to

$$v_3 u_1 = 0,$$

$$(v + v_3) f_1 = 0,$$

$$(v - v_3) f_2 = 0.$$
Let us now remember that the azimuthal components of the vector fields should vanish at the symmetry axis for the fields to be defined there. In the gauge (4.4) this would require that \( v(0) = v_3(0) = 0 \), but imposing such a condition would be much too restrictive. A more general possibility is to perform a gauge transformation that gives \( \varphi \)-dependent phases to the scalar fields and shifts the azimuthal components of the vectors, and only after this to impose the regularity conditions for the vectors. Since the two Higgs field components should be single-valued in the new gauge, their phase factors should contain integers. We therefore apply to the ansatz (4.4) the gauge transformation generated by

\[
U = \exp\left\{ \frac{i}{2}(\eta + \psi \tau^3) \right\},
\]

with

\[
\eta = (2n - \nu)\varphi + \sigma_\alpha x^\alpha, \quad \psi = \nu \varphi - \sigma_\alpha x^\alpha,
\]

where \( n, \nu \) are two integer winding numbers. This gives

\[
\mathcal{W} = \left\{ \left. u(\rho) + 1 + \tau^1_P u_1(\rho) + \tau^3[u_3(\rho) - 1] \right\} \sigma_\alpha dx^\alpha
\]

\[
\quad + \{2n - \nu - v(\rho) - \tau^1_P v_1(\rho) + \tau^3[\nu - v_3(\rho)]\} \right\} d\varphi,
\]

\[\Phi = \begin{bmatrix}
e^{in\varphi} f_1(\rho) \\
e^{i(n-\nu)\varphi + i\sigma_\alpha x^\alpha} f_2(\rho)
\end{bmatrix},
\]

with \( \tau^1_P = U\tau^1 U^{-1} = \tau^1 \cos \psi - \tau^2 \sin \psi \). We can assume without loss of generality that \( n \geq 1 \) and we shall see below that \( 1 \leq \nu \leq 2n \). The dependence of the fields on \( x^\alpha \) will be convenient in what follows but not essential for the regularity issue.

The azimuthal components of the vectors in (5.6) will vanish at the axis provided that

\[
v(0) = 2n - \nu, \quad v_3(0) = \nu,
\]

and also if \( v_1(0) = 0 \), as required by Eq.(4.10). This gives a larger set of the allowed boundary values than in gauge (4.4). In addition, the regularity of the scalar fields requires that \( f_1(0) = 0 \) and, unless for \( \nu = n \), that \( f_2(0) = 0 \). The conditions (5.3) now reduce to

\[
\nu u_1(0) = 0, \quad nf_1(0) = 0, \quad (n - \nu) f_2(0) = 0.
\]

Summarizing, the boundary conditions at \( \rho = 0 \) are given by

\[
u_1(0) = 0, \quad u_3(0) = 1, \quad v_1(0) = 0, \quad v_3(0) = \nu, \quad v(0) = 2n - \nu, \quad f_1(0) = 0,
\]

while \( u(0) \) and \( f_2(0) \) can be arbitrary if \( \nu = n \), whereas for \( \nu \neq n \) one should have \( f_2(0) = 0 \).
We can now work out the most general local series solutions of the equations for small \( \rho \). The corresponding analysis is presented in the Appendix A, the result is

\[
\begin{align*}
  u &= a_1 + \ldots, \\
  u_1 &= a_2 \rho^\nu + \ldots, \\
  u_3 &= 1 + \ldots, \\
  v_1 &= O(\rho^{\nu+2}), \\
  v_3 &= \nu + a_3 \rho^2 + \ldots, \\
  v &= 2n - \nu + a_4 \rho^2 + \ldots, \\
  f_1 &= a_5 \rho^n + \ldots, \\
  f_2 &= q\rho^{|n-\nu|} + \ldots,
\end{align*}
\]

(5.10)

where \( a_1, a_2, a_3, a_4, a_5 \) and \( q \) are six integration constants. The dots here stand for the subleading term. As explained in the Appendix, all such terms will be taken into account in our numerical scheme.

**B. Boundary conditions at infinity**

We want the fields for \( \rho \to \infty \) to approach vacuum configurations with zero energy density. The energy density \( \mathcal{E}_1 + \mathcal{E}_2 \) is given by Eqs. (4.24), (4.25) and can be decomposed into the kinetic energy part containing the derivatives of the field amplitudes and the potential energy part that contains no derivatives. The potential energy is a sum of perfect squares that will vanish if only each term in the sum vanishes. For example, the last term in (4.25) will vanish if only \( f_1^2 + f_2^2 = 1 \) and so

\[
  f_1(\infty) = \cos \frac{\gamma}{2}, \quad f_2(\infty) = \sin \frac{\gamma}{2}.
\]

(5.11)

The value of \( \gamma \) is an essential parameter characterising the solutions, we shall call it vacuum angle. It is worth noting that the global symmetry (4.8) acts as \( \gamma \to \gamma + \Gamma \), but this cannot be used to change the value of \( \gamma \), since this symmetry has already been fixed by the condition (4.10) at the origin.

At the same time, just for discussing the local solutions at large \( \rho \), it is convenient to temporarily set \( \gamma = 0 \) to simplify the analysis, since afterwards one can apply the transformation (4.8) to restore the generic value of \( \gamma \). We therefore assume for the time being that
at infinity one has
\[ f_1 = 1, \quad f_2 = 0. \] (5.12)

With this, the potential energy part of Eqs. (4.24), (4.25) will vanish if only
\[ u_1 = v_1 = 0 \] (5.13)

and
\[ u = -u_3, \quad v = -v_3. \] (5.14)

Under these conditions the field equations (4.12)–(4.19) reduce to
\[ (\rho u')' = 0, \quad \left(\frac{v'}{\rho}\right)' = 0, \] (5.15)

from where
\[ u = c_1 + Q \ln(\rho), \quad v = c_2 + A\rho^2, \] (5.16)

where \( c_1, c_2, Q, A \) are integration constants. One should set \( A = 0 \) since otherwise the kinetic part in the energy density diverges at infinity. However, one should keep the logarithmic term in (5.16), since the energy density approaches zero for \( \rho \to \infty \) even if \( Q \neq 0 \). As a result, the conclusion is that for large \( \rho \) the fields approach the following exact solution of the field equations,
\[ \mathcal{W} = (1 - \tau^3) ((c_1 + Q \ln(\rho)) \sigma_\alpha dx^\alpha - c_2 d\phi), \quad \Phi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \] (5.17)

Computing the \( A, Z \) fields defined by Eq. (2.23) gives
\[ A_\mu dx^\mu = \frac{1}{gg'} ((c_1 + Q \ln(\rho)) \sigma_\alpha dx^\alpha - c_2 d\phi), \quad Z_\mu = 0. \] (5.18)

This is the electromagnetic Biot-Savart solution describing fields outside an uniformly charged electric wire. The charge and current are obtained with Eqs. (3.13), (3.14),
\[ I_\alpha = \int \partial^\mu F_{\mu\alpha} d^2 x = -\int \epsilon_{ks} \partial_k A_\alpha dx^s = -2\pi \frac{Q}{gg'} \sigma_\alpha, \] (5.19)

assuming that the charge/current distributions are smooth inside the wire so that the Gauss theorem applies. In addition, there is also a magnetic flux along the wire,
\[ \Psi_F = \oint A_k dx^k = \frac{2\pi}{gg'} c_2. \] (5.20)
In what follows we shall construct smooth vortex solutions in which the wire is represented by a regular distribution of massive non-linear fields, while in the far field zone the massive modes die out and everything reduces to the Biot-Savart configuration (5.17). The energy density at large $\rho$ will then be proportional to $Q^2/\rho^2$ and the total energy per unit vortex length will be logarithmically divergent, just as for the ordinary infinitely long electric wire. The energy of finite vortex pieces, as for example vortex loops, will be finite.

Let us now consider the deviations from the electromagnetic configuration (5.17) within the full system of equations (4.12)–(4.19) in order to determine how the solutions approach their asymptotic form. The corresponding analysis in the linear approximation is carried out in the Appendix B and the result is given by Eq. (B.18). We only need to apply to Eq. (B.18) the phase rotation (4.8) to restore the generic value of $\gamma$ in Eq. (5.11). This gives the large $\rho$ behaviour of the solutions,

\[ u = Q \ln \rho + c_1 + \frac{c_3 g^2}{\sqrt{\rho}} e^{-m_2 \rho} + \ldots \]
\[ v = c_2 + c_4 g^2 \sqrt{\rho} e^{-m_2 \rho} + \ldots \]
\[ u_1 + i u_3 = e^{-i\gamma} \left\{ \frac{c_7}{\sqrt{\rho}} e^{-\int m_\sigma d\rho} + i \left[ -Q \ln \rho - c_1 + \frac{c_3 g^2}{\sqrt{\rho}} e^{-m_2 \rho} \right] \right\} + \ldots \]
\[ v_1 + i v_3 = e^{-i\gamma} \left\{ c_8 \sqrt{\rho} e^{-\int m_\sigma d\rho} + i \left[ -c_2 + c_4 g^2 \sqrt{\rho} e^{-m_2 \rho} \right] \right\} + \ldots \]
\[ f_1 + i f_2 = e^{i2\gamma} \left\{ 1 + \frac{c_5}{\sqrt{\rho}} e^{-m_H \rho} + i \frac{c_6}{\sqrt{\rho}} e^{-\int m_\sigma d\rho} \right\} + \ldots \] (5.21)

These local solutions contain 10 independent parameters $c_1, \ldots, c_8, \gamma, Q$ and they approach (modulo the $\gamma$-phases) the Abelian configuration (5.17) exponentially fast as $\rho \to \infty$, with the rates determined by the masses $m_2, m_H$ and $m_\sigma$. Here

\[ m_\sigma = \sqrt{m_W^2 + \sigma^2 (Q \ln \rho + c_1)^2} \] (5.22)

reduces to the W boson mass $m_W = g/\sqrt{2}$ if $\sigma^2 = 0$. For $\sigma^2 \neq 0$ this could be viewed as the W boson mass ‘screened’ ($\sigma^2 < 0$) or ‘dressed’ ($\sigma^2 > 0$) by the interaction of charged W bosons with the long-ranged Biot-Savart field (5.17). We also note that the solutions (5.21) satisfy the field equations (4.12)–(4.19) but not the constraint (4.20). In fact, the latter has already been imposed on the local solutions (5.10) at small $\rho$, and since it ‘propagates’, it will be automatically enforced by extending the local solution (5.10) to large values of $\rho$. As a result, it would be redundant to impose it again.
We can now outline our strategy for solving the field equations. The local solutions for small and large \( \rho \) are given, respectively, by Eqs.\((5.10)\) and \((5.21)\). Within the numerical multiple shooting method \[33\] we extend these asymptotic solutions to the intermediate values of \( \rho \) and match them. We use the multi-zone version of the method \[34\], with many zones whose number and sizes are adjusted to make sure that for large \( \rho \) the asymptotic solution \((5.21)\) is a good approximation. In order to illustrate the procedure, let us describe the simplest case where there are only two zones and one matching point. To match the solutions at this point we have to fulfill the 16 matching conditions for the 8 field amplitudes and for their first derivatives via adjusting the free parameters in Eqs.\((5.10),(5.21)\). It is therefore essential to have enough of parameters, since otherwise the matching conditions will be incompatible. Counting the parameters we discover that there are 17 of them: 6 integration constants \( a_1, \ldots, a_5 \) and \( q \) in \((5.10)\), then 10 integration constants \( c_1, \ldots, c_8, Q, \gamma \) in \((5.21)\), and finally \( \sigma^2 \). This is enough to fulfill the 16 matching conditions and to have one parameter left free after the matching. This parameter will label the resulting global solutions in the interval \( \rho \in [0, \infty) \). It is convenient to choose it to be \( q \) – the coefficient in front of the amplitude \( f_2 \) in \((5.10)\).

Before integrating the equations, some preliminary considerations are in order.

VI. CONSERVED QUANTITIES

Keeping only the leading terms, the boundary conditions for the field amplitudes for \( 0 \leftarrow \rho \rightarrow \infty \) are

\[
\begin{align*}
\alpha_1 & \leftarrow u \rightarrow c_1 + Q \ln \rho, \\
2n - \nu & \leftarrow v \rightarrow c_2, \\
0 & \leftarrow u_1 \rightarrow -(c_1 + Q \ln \rho) \sin \gamma, \\
1 & \leftarrow u_3 \rightarrow -(c_1 + Q \ln \rho) \cos \gamma, \\
0 & \leftarrow v_1 \rightarrow -c_2 \sin \gamma, \\
\nu & \leftarrow v_3 \rightarrow -c_2 \cos \gamma, \\
a_5 \rho^n & \leftarrow f_1 \rightarrow \cos \frac{\gamma}{2}, \\
q \rho^{2n-\nu} & \leftarrow f_2 \rightarrow \sin \frac{\gamma}{2},
\end{align*}
\]

\((6.1)\)
The knowledge of these boundary conditions is sufficient to calculate most of the global physical quantities associated to the solutions.

A. Fluxes and current

Since the fluxes and currents are gauge invariant, one can compute them in the gauge (4.4), in which case the calculations are simpler. The non-zero components of the field tensors then read

\[
\begin{align*}
B_{\rho\alpha} &= \sigma_\alpha u'_\rho, & B_{\rho\phi} &= -v'_\rho, \\
W^a_{\rho\alpha} &= \sigma_\alpha u''_{\rho a}, & W^a_{\rho\phi} &= -v''_{\rho a}, & a = 1, 3; \\
W^2_{\alpha\phi} &= (u_1 v_3 - u_3 v_1) \sigma_\alpha.
\end{align*}
\]

Introducing the function \(\Omega(\rho)\) defined by

\[
\begin{align*}
\cos \Omega &= \frac{f_1^2 - f_2^2}{f_1^2 + f_2^2}, & \sin \Omega &= \frac{2 f_1 f_2}{f_1^2 + f_2^2},
\end{align*}
\]

such that \(\Omega(\infty) = \gamma\), one can define three orthogonal unit isovectors

\[
\begin{align*}
n^a &= (\sin \Omega, 0, \cos \Omega), \\
k^a &= (\cos \Omega, 0, -\sin \Omega), \\
l^a &= (0, 1, 0).
\end{align*}
\]

Here \(n^a\) corresponds to the definition (2.27). The Nambu electromagnetic and Z fields (2.26) then read

\[
\begin{align*}
F_{\rho\alpha} &= \left( \frac{g'}{g'} u'_\rho - \frac{g'}{g} (u'_1 \sin \Omega + u'_3 \cos \Omega) \right) \sigma_\alpha, \\
F_{\rho\phi} &= -\frac{g}{g'} v'_\rho + \frac{g'}{g} (v'_1 \sin \Omega + v'_3 \cos \Omega), \\
Z_{\rho\alpha} &= (u'_1 + u'_3 \sin \Omega + u'_3 \cos \Omega) \sigma_\alpha, \\
Z_{\rho\phi} &= -v'_\rho - v'_1 \sin \Omega - v'_3 \cos \Omega.
\end{align*}
\]
whereas the 't Hooft (2.28) fields have the total derivative structure,

\[ F^H_{\rho \alpha} = \left( \frac{g}{g'} u - \frac{g'}{g} (u_1 \sin \Omega + u_3 \cos \Omega) \right)' \sigma_\alpha, \]

\[ F^H_{\rho \phi} = \left( -\frac{g}{g'} v + \frac{g'}{g} (v_1 \sin \Omega + v_3 \cos \Omega) \right)' \sigma_\alpha, \]

\[ Z^H_{\rho \alpha} = (u + u_1 \sin \Omega + u_3 \cos \Omega)' \sigma_\alpha, \]

\[ Z^H_{\rho \phi} = -(v + v_1 \sin \Omega + v_3 \cos \Omega)' . \] (6.6)

The electromagnetic current density is

\[ J^\alpha = \partial_\mu F^\mu \alpha = \frac{1}{\rho} (\rho F^\rho \alpha)' \] (6.7)

and integrating over the \( \rho, \varphi \) plane gives the same result for the both definitions of the fields,

\[ I^\alpha = -2\pi \int_0^\infty (\rho F^\rho \alpha)' d\rho = -\frac{2\pi Q}{gg'} \sigma_\alpha . \] (6.8)

For most of the solutions considered below the vector \( \sigma_\alpha = (\sigma_0, \sigma_3) \equiv (\sigma_0, \sigma_z) \) is spacelike, so that there is the rest frame where \( \sigma_\alpha = \sigma_3 \delta_\alpha^3 \). The restframe value of the current is \( I^\alpha = \delta_\alpha^3 I \) with \( I = -2\pi Q\sigma/(gg') \). The restframe components of the electromagnetic field strength and Z field strength are

\[ B_z = \frac{1}{\rho} F_{\rho \phi}, \quad B_\phi = -F_{\rho z}, \quad H_z = \frac{1}{\rho} Z_{\rho \phi}, \quad H_\phi = -Z_{\rho z}, \] (6.9)

such that the magnetic fluxes are

\[ \Psi_F = 2\pi \int_0^\infty \rho B_z d\rho, \quad \Psi_Z = 2\pi \int_0^\infty \rho H_z d\rho . \] (6.10)

In the Nambu case these integrals have to be computed, while in the 't Hooft case they evaluate to

\[ \Psi^H_F = -\frac{2\pi}{gg'} (c_2 + (\nu - 2n)g^2 + \nu g^2 \cos \Omega(0)), \]

\[ \Psi^H_Z = 2\pi (2n + \nu (\cos \Omega(0) - 1)). \] (6.11)

One can also define the W-condensate components by projecting the restframe values of \( W^a_{\rho \nu} \) onto the unit isovectors \( k^a, l^a \) defined in (6.4),

\[ w_z = \frac{1}{\rho} k^a W^a_{\rho \phi}, \quad w_\phi = -k^a W^a_{\rho z}, \quad w_\rho = -\frac{1}{\rho} l^a W^a_{z \varphi} . \] (6.12)
B. Energy, momentum, angular momentum

Using Eqs. (4.25), (4.26) and the boundary conditions (6.1), the total energy evaluates to

$$E = \pi \sigma_0^2 + \frac{\sigma_3^2}{g^2 g'^2} Q u(\infty) + 2\pi \int_0^\infty \rho \mathcal{E} d\rho.$$ (6.13)

Here the first term on the right is the contribution of the total derivative in Eq. (4.26). Comparing Eqs. (3.21), (3.25) and (4.26) one can see that the same total derivative determines the value of the momentum,

$$P = 2\pi \frac{\sigma_0 \sigma_3}{g^2 g'^2} Q u(\infty).$$ (6.14)

We finally notice that, since in the axially symmetric case there is an additional Killing vector $K_\varphi$, associated to this there is the conserved angular momentum

$$M = \int T^0_\varphi K_\varphi d^2 x = \int T^0_\varphi d^2 x.$$ (6.15)

This also has a total derivative structure [32], since one has

$$T^0_\varphi = -\frac{1}{g^2} W_{\mu} W^\mu_\varphi \varphi - \frac{1}{g'^2} B_{\mu} B^\mu_\varphi \varphi + (D^0 \Phi)^\dagger D_\varphi \Phi + (D_\varphi \Phi)^\dagger D^0 \Phi$$

$$= \sigma_0 \left( \frac{1}{g'^2} u' v' + \frac{1}{g^2} u'_a v'_a \right) + 2\sigma_0 \Re \left( \Phi^\dagger (u + u_3 \tau_3) (v + v_3 \tau_3) \Phi \right)$$ (6.16)

with $a = 1, 3$. Using the field equations (4.12), (4.16), (4.17) gives

$$\rho T^0_\varphi = \sigma_0 \left( \frac{\rho}{g^2 g'^2} u' v' + \frac{\rho}{g'^2} (v_1 u_1' + v_3 u_3') \right)'$$ (6.17)

so that

$$M = 2\pi \frac{\sigma_0}{g^2 g'^2} Q u(\infty) = \frac{2\pi Q \sigma_0 c_2}{g^2 g'^2}.$$ (6.18)

Summarizing, the fluxes, current, energy, momentum and angular momentum are given by the above expressions, and in most cases they are totally determined by the boundary conditions (6.1).

VII. KNOWN SOLUTIONS

The only known solutions of Eqs. (4.12)–(4.20) for generic values of $g, g'$ are the embedded ANO vortices. These exist in two different versions, called Z strings [8] and W strings [38], corresponding to two nonequivalent embeddings of U(1) to SU(2) $\times$ U(1).
A. Z strings

These solutions are obtained by setting $u_3 = -u = 1$ and $f_2 = u_1 = v_1 = 0$ after which equations (4.12)–(4.20) reduce to

$$\frac{1}{\rho} (\rho f_1')' = \left( \frac{1}{4 \rho^2} (v + v_3)^2 + \frac{\beta}{4} (f_1^2 - 1) \right) f_1, \quad (7.1a)$$

$$\rho \left( \frac{v'}{\rho} \right)' = \frac{g_2^2}{2} (v + v_3) f_1^2, \quad (7.1b)$$

$$\rho \left( \frac{v_3'}{\rho} \right)' = \frac{g_2^2}{2} (v_3 + v) f_1^2. \quad (7.1c)$$

With

$$A = \frac{1}{2} (g_2 v - g_2^2 v_3), \quad v_{\text{ANO}} = \frac{1}{2} (v + v_3), \quad f_{\text{ANO}} = f_1 \quad (7.2)$$

these equations reduce to the ANO system

$$\frac{1}{\rho} (\rho f_{\text{ANO}}')' = \left( \frac{v_{\text{ANO}}^2}{\rho^2} + \frac{\beta}{4} (f_{\text{ANO}}^2 - 1) \right) f_{\text{ANO}}, \quad (7.3a)$$

$$\rho \left( \frac{v_{\text{ANO}}'}{\rho} \right)' = \frac{1}{2} f_{\text{ANO}}^2 v_{\text{ANO}}, \quad (7.3b)$$

and to

$$\left( \frac{A'}{\rho} \right)' = 0. \quad (7.4)$$

Solutions of the ANO equations (7.3) can be found numerically, they comprise a family labeled by $n = 1, 2, \ldots$ with the boundary conditions

$$0 \leftarrow f_{\text{ANO}} \rightarrow 1, \quad n \leftarrow v_{\text{ANO}} \rightarrow 0 \quad (7.5)$$

for $0 \leftarrow \rho \rightarrow \infty$. The only bounded solution for $A$ is the constant whose value is determined by the boundary conditions (6.1) at the origin, $A = ng^2 - \nu/2$.

Summarizing, Z string profiles are given by

$$u = -1, \quad v = 2g^2(v_{\text{ANO}} - n) + 2n - \nu, \quad u_1 = 0, \quad u_3 = 1, \quad v_1 = 0, \quad v_3 = 2g^2(v_{\text{ANO}} - n) + \nu, \quad f_1 = f_{\text{ANO}}, \quad f_2 = 0, \quad (7.6)$$

from where one can read off the values of the asymptotic parameters (6.1),

$$a_1 = c_1 = -1, \quad Q = 0, \quad c_2 = 2ng^2 - \nu, \quad \Omega = \Omega_0 = \gamma = 0. \quad (7.7)$$
Calculating the fluxes gives the same result for the Nambu and 't Hooft definitions,

\[ \Psi_Z = 4\pi n, \quad \Psi_F = 0. \]  
(7.8)

while the current, momentum and angular momentum vanish, so that the energy is finite. The dependence of the amplitudes in (7.6) on the second winding number \( \nu \) is pure gauge for these solutions, since in the gauge (5.6) it disappears:

\[ W = 2(g^2 + g^2\tau^3)(n - v_{ANO}(\rho)) d\varphi, \quad \Phi = \begin{pmatrix} e^{i\nu\varphi} f_{ANO}(\rho) \\ 0 \end{pmatrix}. \]  
(7.9)

B. W strings

These are obtained by setting in Eqs. (4.12)–(4.20)

\[ u = v = u_1 = u_3 = v_3 = f_2 = 0, \quad v_1 = 2v_{ANO}(\rho), \quad f_1 = f_{ANO}(\rho). \]  
(7.10)

Writing this solution in the gauge (4.4) and applying the gauge transformation generated by \( U = \exp(in\varphi\tau^1) \) gives

\[ W = 2\tau^1(n - v_{ANO}(\rho)) d\varphi, \quad \phi_1 + i\phi_2 = f_{ANO}(\rho)e^{i\nu\varphi} \]  
(7.11)

which is the parametrization used in Ref. [38]. However, throughout this paper we are using different gauge conventions, in particular we impose the condition (4.10). We therefore return to (7.10) and perform the global gauge rotation (4.8) with \( \Gamma = -\pi/2 \), which gives

\[ u = v = u_1 = u_3 = v_1 = 0, \quad v_3 = 2v_{ANO}(\rho), \quad f_1 = -f_2 = \frac{1}{\sqrt{2}} f_{ANO}(\rho). \]  
(7.12)

From here we can read off the values of the asymptotic parameters,

\[ \nu = 2n, \quad Q = c_1 = c_2 = 0, \quad \gamma = \Omega = \Omega_0 = -\frac{\pi}{4}. \]  
(7.13)

Calculating the fluxes (6.10), (6.11) gives, in the both definitions,

\[ \Psi_Z = \frac{4\pi n}{\sqrt{2}}, \quad \Psi_F = -\frac{4\pi n g'}{\sqrt{2} g}. \]  
(7.14)

These values are not the same as in the Z string case, in particular the electromagnetic flux is now different from zero. W strings are therefore physically different from Z strings.
VIII. SMALL CURRENT LIMIT – BOUND STATES AROUND Z STRINGS

Our goal is to construct solutions of equations (4.12)–(4.20) more general than the embedded ANO vortices. Our strategy was briefly summarized above: we numerically extend the asymptotic solutions (5.10) and (5.21) to the intermediate region and impose there the 16 matching conditions, which can be fulfilled by adjusting the 17 free parameters in the local solutions. This leaves one extra parameter, \( q \), which determines the value of the lower component of the Higgs field at the origin.

The matching conditions are resolved iteratively, within the standard method described in [34]. A good choice of the initial configuration is then important, since otherwise the iterations will not converge. The idea is therefore to start the iterations in the vicinity of an already known solution, for which we can only choose the Z string (it is unclear whether W strings can also be used). Let us therefore choose the Z string solution (7.6) as the starting point of our analysis. One has in this case \( f_2(\rho) = 0 \) so that \( q = 0 \).

Suppose now that \( 0 < q \ll 1 \). It is then natural to expect the solution to be a slightly deformed Z string. It can therefore be represented in the form

\[
\begin{align*}
 u &= -1 + \delta u, \quad v = v_z + \delta v, \quad u_1 = \delta u_1, \quad u_3 = 1 + \delta u_3, \\
 v_1 &= \delta v_1, \quad v_3 = v_{z3} + \delta v_3, \quad f_1 = f_z + \delta f_1, \quad f_2 = \delta f_2,
\end{align*}
\]

(8.1)

where \( v_z \equiv 2g^2(v_{ANO} - n) + 2n - \nu \), \( v_{z3} \equiv 2g^2(v_{ANO} - n) + \nu \), \( f_z \equiv f_{ANO} \) and \( \delta u, \ldots, \delta f_2 \) are small deformations. Inserting this to Eqs. (4.12)–(4.19) and linearising with respect to the deformations, the resulting equations split into three independent groups: two coupled equations for \( \delta u, \delta u_3 \) plus three coupled equations for \( \delta f_1, \delta v, \delta v_3 \) plus three coupled equations for \( \delta f_2, \delta u_1, \delta v_1 \). The first two groups do not admit interesting solutions, so that we can set \( \delta u = \delta u_3 = \delta f_1 = \delta v = \delta v_3 = 0 \).

The remaining three equations describe a slightly deformed Z string: \( \mathcal{W} + \delta \mathcal{W}, \Phi + \delta \Phi \), where \( \mathcal{W} \) and \( \Phi \) are given by Eq. (7.9) and

\[
\delta \mathcal{W} = \frac{\tau^+}{2} e^{i\psi} \left[ \delta u_1(\rho) \sigma_\alpha dx^\alpha - \delta v_1(\rho) d\varphi \right] + \text{h.c.}, \quad \delta \Phi = \left( \begin{array}{c} 0 \\ e^{i(n\varphi - \psi)} \delta f_2(\rho) \end{array} \right),
\]

(8.2)

with \( \psi = \nu \varphi - \sigma_\alpha x^\alpha \) and \( \tau^+ = \tau^1 + i\tau^2 \). Here h.c. stands for Hermitian conjugation. The
amplitudes $\delta f_2, \delta v_1, \delta u_1$ fulfill the equations

$$\frac{1}{\rho} (\rho \delta f_2)'' = \left( \frac{(v_z - v_{z3})^2}{4\rho^2} + \frac{\beta}{4} (f_z^2 - 1) + \sigma^2 \right) \delta f_2 + \frac{v_z f_z}{2\rho^2} \delta v_1 - \frac{\sigma^2}{2} f_z \delta u_1, \quad (8.3a)$$

$$\frac{1}{\rho} (\rho \delta u_1)'' = \left( \frac{v_{z3}^2}{\rho^2} + \frac{g^2}{2} f_z^2 \right) \delta u_1 - \frac{v_{z3}}{\rho^2} \delta v_1 - g^2 f_z \delta f_2, \quad (8.3b)$$

$$\rho \left( \frac{\delta v_1'}{\rho} \right)'' = \left( \frac{g^2}{2} f_z^2 + \sigma^2 \right) \delta v_1 - \sigma^2 v_{z3} \delta u_1 + g^2 v_z f_z \delta f_2. \quad (8.3c)$$

This can be viewed as a spectral problem with the eigenvalue $\sigma^2$. In fact, by suitably redefining the variables one can rewrite the equations in the form

$$\Psi'' = (\sigma^2 + V[\beta, \theta_w, n, \nu, \rho])\Psi, \quad (8.4)$$

where $\Psi$ is a 3 component vector and $V$ is a symmetric potential energy matrix depending on the parameters of the background $Z$ string solution, $\beta, \theta_w, n$, and also on $\nu$. Although the dependence on $\nu$ for the $Z$ string background is pure gauge, this is not so for the deformations of this background, since for different values of $\nu$ in (8.3) one obtains different results.

We are interested in localized, bound state solutions of Eqs. (8.3). These equations admit a global symmetry

$$\delta f_2 \rightarrow \delta f_2 + \frac{\Gamma}{2} f_z, \quad \delta u_1 \rightarrow \delta u_1 + \Gamma, \quad \delta v_1 \rightarrow \delta v_1 + \Gamma v_{z3}, \quad (8.5)$$
FIG. 2: The eigenvalue $\sigma^2$ of the spectral problem (8.3) against $\beta$ for $n = \nu = 1$ (left) and for $n = 1, \nu = 2$ (right) for several values of $g^2 = \sin^2 \theta_W$.

which is merely the linearised version of (4.8). Using this symmetry one can impose the gauge condition

$$\delta f_2(\infty) = 0,$$

and then the local behaviour of the solutions at large $\rho$ can be read off from Eq.(5.21),

$$\delta u_1 = \frac{c_7}{\sqrt{\rho}} e^{-m_\sigma \rho} + \ldots, \quad \delta v_1 = c_8 \sqrt{\rho} e^{-m_\sigma \rho} + \ldots, \quad \delta f_2 = c_6 \sqrt{\rho} e^{-m_\sigma \rho} + \ldots.$$  

Here

$$m^2_\sigma = m^2_W + \sigma^2,$$

while the logarithm present in Eq.(5.22) does not appear in the linearized theory. The local behavior at small $\rho$ can be obtained from Eq.(5.10),

$$\delta u_1 = a_4 \rho^\nu + \Gamma + \ldots, \quad \delta v_1 = \Gamma v_3 + \ldots, \quad \delta f_2 = q \rho^{\nu - 1} + \frac{\Gamma}{2} f_2 + \ldots,$$

where we have included the free parameter $\Gamma$, since the symmetry (8.5) is now fixed at large $\rho$ and not at $\rho = 0$. In fact, the gauge condition (8.6) imposed at infinity generically implies that $\delta v_1(0) \neq 0$, which does not agree with the previously adopted in (4.10) condition $v_1(0) = 0$. However, the advantage of this gauge is that $Z$ string deformations are clearly localized around the string core (see Fig.1). One can always apply the symmetry (8.5) to return to the gauge where $\delta v_1(0) = 0$, but then the amplitudes $\delta u_1, \delta v_1, \delta p$ will not vanish at infinity.
The local solutions (8.7), (8.9) contain 6 free parameters: \(c_5, c_6, c_8, a_4, \Gamma, q\). One of them can be absorbed by the overall normalization and so there remain only 5, but together with \(\sigma^2\) their number is again 6, which is just enough to fulfill the 6 matching conditions to construct global solutions of three second order equations (8.3). The solutions obtained are characterized by the following properties.

\[
\sin^2 \theta \quad \sigma^2 (1,1) \quad (1,2) \quad (2,1) \quad (2,2) \quad (2,3) \\
\sin^2 \theta \quad \sigma^2 (4,1) \quad (4,2) \quad (4,3) \quad (4,4) \quad (4,5) \quad (4,6) \quad (4,7) \quad (4,8)
\]

FIG. 3: The eigenvalue \(\sigma^2\) of the spectral problem (8.3) against \(\sin^2 \theta_w\) for \(\beta = 2\). The values of \(n\) and \(\nu\) are shown as \((n, \nu)\). In the \(\sigma^2 < 0\) region the curves terminate when the condition (8.10) is violated. The chiral solutions with \(\sigma^2 = 0\) are possible only for special values of \(\theta_w\).

For given \(\beta\) and for small enough \(\sin \theta_w\) one finds \(2n\) different bound states labeled by \(\nu = 1, 2, \ldots, 2n\) with the eigenvalue \(\sigma^2 = \sigma^2(\beta, \theta_w, n, \nu)\) (see Fig.1). These solutions can be interpreted as small deformations of Z strings by a current \(I_\alpha \sim \sigma_\alpha\), although the current itself appears only in the next order of perturbation theory. If \(\beta > 1\) then \(n\) of these solutions always have \(\sigma^2 > 0\). For \(n - 1\) solutions the eigenvalue \(\sigma^2\) changes sign when \(\theta_w\) increases, and there is one solution with \(\sigma^2 < 0\) for any \(\theta_w > 0\) (see Figs.2,3). If \(\sigma^2\) is negative then it cannot be too large, since solutions are localized if only the mass (8.8) is real, which requires that

\[
\sigma^2 > -m_{\text{w}}^2. \quad (8.10)
\]

Every Z string admits therefore ‘magnetic’ \((\sigma^2 > 0)\) and, unless \(\theta_w\) is close to \(\pi/2\), ‘electric’ \((\sigma^2 < 0)\) linear deformations. The ‘chiral’ \((\sigma^2 = 0)\) deformations are not generic and possible
FIG. 4: The $\sigma^2(\beta, \theta_w, n, \nu) = 0$ curves for several values of $(n, \nu)$. Curves with $\nu \leq n$ are confined in the $\beta \leq 1$ region, while those with $\nu > n$ extend to larger values of $\beta$.

only for special values of $\beta, \theta_w$ such that

$$\sigma^2(\beta, \theta_w, n, \nu) = 0. \quad (8.11)$$

This condition determines a set of curves in the $\beta, \theta_w$ plane (see Fig.4), let us call them chiral curves. They coincide with the curves delimiting the parameter regions of Z string stability [9]. This can be explained by the fact that a simple reinterpretation of the above considerations allows us to reproduce the results of the Z string stability analysis [9]. Let

$$(\delta W, \delta \Phi) \sim \exp\{\pm i(\sigma_0 x^0 + \sigma_3 x^3)\} \quad (8.12)$$

be the Z string perturbation eigenvector (8.2) for an eigenvalue $\sigma^2 = \sigma_3^2 - \sigma_0^2$. One has

$$\sigma_0 = \sqrt{\sigma_3^2 - \sigma^2} \quad (8.13)$$

from where it follows that if $\sigma^2 > 0$ then choosing $\sigma_3 < \sigma$ gives exponentially growing in time solutions, that is unstable modes, since the frequency $\sigma_0$ is then imaginary, while choosing $\sigma_3 > \sigma$ (as was done above) gives stationary Z string deformations, since $\sigma_0$ is then real. If $\sigma^2 < 0$ then $\sigma_0$ is always real. The sign of $\sigma^2$ therefore determines if the negative modes are present or not. As a result, the $\sigma^2(\beta, \theta_w, n, \nu) = 0$ curves separate the parameter regions in the $\beta, \theta_w$ plane in which the $\nu$-th deformation of the $n$-th Z string is of the electric/magnetic type, and at the same time the regions where the $\nu$-th Z string perturbation mode is of stable/unstable type.
IX. GENERIC SUPERCONDUCTING VORTEXES

Having constructed the slightly deformed Z strings in the linear theory, we can promote them to solutions of the full system of equations \((4.12)-(4.19)\), first for small \(q\). In the fully nonlinear theory all 8 field amplitudes deviate from their Z string values, and not only \(u_1, v_1, f_2\). However, if the deviations \(\delta u_1, \delta v_1, \delta f_2\) scale as \(q\) for \(q \ll 1\), those for the remaining 5 amplitudes scale as \(q^2\).

Starting from small values of \(q\), we increase \(q\) iteratively, thereby obtaining fully non-linear superconducting vortices. Equations \((4.12)-(4.19)\) can be viewed in this case as a non-linear boundary value problem with the eigenvalue \(\sigma^2\). It turns out that only the magnetic \((\sigma^2 > 0)\) and chiral \((\sigma^2 = 0)\) solutions are compatible with the boundary conditions at infinity. The main difference with the linearized case is that, once all fields amplitudes are taken into account, the logarithmic term appears in the expression \((5.22)\) for the effective mass,

\[
m^2_\sigma = m^2_w + \sigma^2(Q \ln \rho + c_1)^2.
\]  

(9.1)

This implies that for \(\sigma^2 < 0\) one has the ‘tachyonic mass’ \(m^2_\sigma < 0\) at large \(\rho\), in which case the asymptotic solutions \((5.21)\) oscillate and do not approach the vacuum at infinity (this was not realised in Ref.\[23\]). The energy then diverges faster than \(\ln \rho\) and the interpretation of this is not so clear, since such solutions cannot be considered as field-theoretic models of electric wires or charge distributions. We shall therefore not consider the \(\sigma^2 < 0\) solutions in what follows.

For the magnetic solutions with \(\sigma^2 > 0\) the problem does not arise, since the logarithmic term in the effective mass \((9.1)\) only improves their localization. As a result, the magnetic solutions do generalize within the full, non-linear theory. The described above multiplet structure of the perturbative solutions persists also at the non-linear level, up to the fact that only solutions with \(\sigma^2 \geq 0\) are now allowed. For \(n = 1\) one finds one such solution, with \(\nu = 1\), while for \(n = 2\) there are already three: with \(\nu = 1, 2\) and, provided that \(\theta_w\) is not too large, also with \(\nu = 3\) (see Fig.\[3\]). The general rule seems to be such that for a given \(n\) there are \(2n - 1\) solutions, of which those with \(\nu = 1, 2, \ldots n\) exist for any \(\theta_w\) while those with \(\nu = n + 1, \ldots, 2n - 1\) exist if only \(\theta_w\) is not too close to \(\pi/2\), as shown in Figs.\[2,3\].

The regions of their existence are delimited by the chiral curves in Fig.\[4\]. For each given curve one has \(\sigma^2(\beta, \theta_w, n, \nu) > 0\) in the region below the curve. Let us call it allowed region,
it corresponds to magnetic solutions that generalize within the full non-linear theory. The region above the curve corresponds to the electric solutions with $\sigma^2 < 0$, we therefore call it forbidden region. The chiral curves with $\nu \leq n$ are all contained in the upper left corner of the diagram where $\beta \leq 1$, while those for $\nu > n$ extend to the region where $\beta > 1$. It follows that if $\nu \leq n$ then the allowed region corresponds to any $\theta_w$ if $\beta > 1$, and to $\theta_w$ that is not too close to $\pi/2$ if $\beta < 1$. If $\nu > n$ then the allowed region corresponds to the lower part of the diagram located below the $(n, \nu)$ chiral curve.

The chiral curves in Fig.4 are obtained in the limit of vanishing current, while for finite currents they remain qualitatively similar but shift upward. For large currents they seem to approach the upper boundary of the diagram, such that the allowed regions increase. Therefore, without entering too much into details, one can simply say that the superconducting vortices exist for almost all values of $\beta, \theta_w$.

The typical solution is shown in Fig.5. These solutions describe vortices carrying an electric current – smooth, globally regular field-theoretic analogues of electric wires. At small $\rho$ they exhibit a completely regular core filled with massive fields creating the electric charge and current. At large $\rho$ the massive fields decay and there remains only the Biot-Savart electromagnetic field. Integrating the charge and current densities over the vortex

![Graphs showing profiles of superconducting vortex solutions](image)

**FIG. 5**: Profiles of the superconducting vortex solution for $\beta = 2$, $g'^2 = 0.23$, $n = \nu = 1$ and $\sigma = 0.373$. The ‘electric’ amplitudes (left) show at large $\rho$ the logarithmic growth typical for the Biot-Savart field. The remaining amplitudes (right) are everywhere bounded.
cross section gives the vortex ‘worldsheet current’ \( I_\alpha \sim \sigma_\alpha \) whose components are the vortex electric charge (per unit length), \( I_0 \), and the total electric current through the vortex cross section, \( I_3 \). Since \( \sigma^2 \equiv \sigma_3^2 - \sigma_0^2 > 0 \), the vector \( I_\alpha \) is spacelike, which means that there is a comoving reference frame where \( \sigma_0 = I_0 = 0 \). However, the current \( I_3 \) cannot be boosted away, so that this is an essential parameter, which suggests the term ‘superconductivity’. In what follows we shall denote by \( I \) the value of \( I_3 \) in the comoving frame. We shall also call \( \sigma \) twist, since in the comoving frame one has \( \sigma_3 = \sigma \), which determines the \( z \)-dependent relative phase of the two Higgs field components in the ansatz (5.0).

The solutions exist for any \( \beta > 0 \) and for \( \theta_w \) belonging to the region below the \((n, \nu)\)-chiral curve in Fig.4. Solutions depend on \( \beta, \theta_w \) and also on \( q, n, \nu \) so that \( \sigma^2 \) is in fact a function of five arguments, \( \sigma^2 = \sigma^2(\beta, \theta_w, n, \nu, q) \). In addition, reconstructing the fields (5.6), every solution of the differential equations determines actually a whole family of vortices with fixed \( \sigma^2 = \sigma_3^2 - \sigma_0^2 \) but with different values of \((\sigma_0, \sigma_3)\) related to each other by Lorentz boosts. For given \( \beta, \theta_w \) the superconducting vortices therefore comprise a four parameter family that can be labeled by \( n, \nu, \sigma_0 \) and \( q \). These parameters determine physical quantities associated to the vortex: the electromagnetic and \( Z \) fluxes \( \Psi_F \) and \( \Psi_Z \), the charge \( I_0 \) per unit vortex length and the current \( I_3 \), as well as the momentum along the vortex, \( P \), and the angular momentum \( M \).

![Diagram](image)

**FIG. 6:** The restframe current \( I \) (left) and twist \( \sigma \) (right) as functions of the parameter \( q \) for the solutions with \( \beta = 2, \sin^2 \theta_w = 0.23 \).

The parameter \( q \) measures the deviation from the \( Z \) string limit. Increasing \( q \) starting
from zero increases $I$. It turns out, however, that the dependence $I(q)$ is not one-to-one, since for $q$ not exceeding a certain maximal value $q_*(\beta, \theta_W, n, \nu)$ one finds two different solutions with different currents, as shown in Fig.6 while for $q > q_*$ there are no solutions at all. The solutions thus show two different branches that merge for $q \to q_*$ but have different behaviour for $q \to 0$, when the lower branch reduces to the currentless Z string, while the current of the upper branch solutions grows seemingly without bounds. It is worth noting that this behaviour is drastically different from that found in the model of superconducting strings of Witten (see Fig.24 in the Appendix C), where the maximal value of $q$ corresponds to the ‘dressed’ currentless string.

A similar two-branch structure emerges also for other solution parameters traced against $q$, as for example for $\sigma(q)$ (see Fig.6). This suggests that solutions should be labeled not by $q$ but by a parameter that changes monotonously, as for example the current $I$ or twist $\sigma$ (see Figs.7,8).

The electromagnetic and Z fluxes calculated according to the Nambu definition (6.10) are shown in Fig.9 as functions of the current. We see that in the Z string limit $I \to 0$ the Z flux reduces to $4\pi n$ while the electromagnetic flux vanishes, as they should. As explained above, in the full non-Abelian theory there is no reason for the fluxes to be quantized, and

![Graphs showing the behavior of $\sigma$ and $\gamma$ against $I$](image_url)
FIG. 8: The parameters $c_1$ (left) and $c_2/(2n - \nu)$ (right) against $I$ for the $n = 1, 2$ solutions with $\beta = 2$, $\sin^2 \theta_w = 0.23$. One has $c_2 \to (2\nu - n)g^2$ for $I \to \infty$.

so they vary continuously with $I$.

FIG. 9: The Nambu electromagnetic flux $\Psi_F/\nu$ (left) and $Z$ flux $\Psi_Z/(4\pi n)$ (right) against the current for the vortices with $\beta = 2$, $\sin^2 \theta_w = 0.23$.

It is interesting that the fluxes computed according to the 't Hooft definition (6.11) are completely different and do not reduce to the $Z$ string values in the $Z$ string limit $q \to 0$. This is the consequence of the fact that the function

$$\cos \Omega = \frac{f_1^2 - f_2^2}{f_1^2 + f_2^2}$$

(9.2)
approaches its limiting value for \( q \to 0 \) non-uniformly. Indeed, for \( q = 0 \) one has \( f_2 = 0 \) and \( \cos \Omega = 1 \) everywhere. If \( q \ll 1 \) then \( \cos \Omega \approx 1 \) almost everywhere, apart from the origin where one has \( f_1 \sim \rho^n \) and \( f_2 \sim q \rho^{n-\nu} \) so that \( \cos \Omega = -1 \) at \( \rho = 0 \). Using Eqs.\((6.11)\) then gives the Z flux

\[
\Psi^H_Z = 4\pi n, \quad q = 0,
\]

\[
\Psi^H_Z = 4\pi(n - \nu), \quad q \neq 0, \quad \nu = 1, \ldots, 2n - 1.
\]

The formula \((6.11)\) for the electromagnetic flux contains the parameter \( c_2 \), whose value is known only numerically, but for \( q \to 0 \) it reduces to \( c_2 = 2ng^2 - \nu \), which gives

\[
\Psi^H_F = 0, \quad q = 0,
\]

\[
\Psi^H_F = 4\pi \nu \frac{g'}{g}, \quad q \to 0, \quad \nu = 1, \ldots, 2n - 1.
\]

We see that both electromagnetic and Z fluxes computed according to the 't Hooft definition are discontinuous for \( q \to 0 \) and do not reduce to the Z string values when the solutions approach Z strings in the zero current limit. No such problem arises if one uses the Nambu definition of the fields.

![Graphs showing the 'Coulombian charge' parameter \( Q \) and the magnetic energy against \( I \) for the solutions with \( \beta = 2, \sin^2 \theta_W = 0.23 \).](image)

**FIG. 10:** The 'Coulombian charge' parameter \( Q \) (left) and the magnetic energy (right) against \( I \) for the solutions with \( \beta = 2, \sin^2 \theta_W = 0.23 \).

As the vortices support the long-ranged electromagnetic field, their energy per unit length diverges logarithmically. The total energy consists of the electric and magnetic contributions,
\[ E = E_1 + E_2, \text{ and it is the electric term that diverges,} \]
\[ E_1 = \pi \frac{\sigma_0^2 + \sigma_2^2}{g^2 g'_{\phi}} Q u(\infty), \tag{9.5} \]

since \( u(\rho) \sim Q \ln \rho \) for \( \rho \to \infty \). The ‘Coulombian charge’ \( Q \) is shown against the current in Fig.10 and against \( \theta_w \) in Fig.16. The magnetic energy \( E_2 \) is finite and reaches its minimal value for \( q = q_* \) as shown in Fig.10. This suggests that the value \( q = q_* \) is in some sense ‘energetically preferable’. Since this value determines the ‘bifurcation point’ between the two solution branches, one can expect that the stability of the solutions may change at this point.

\section*{X. LARGE CURRENT LIMIT}

The superconducting strings in the U(1) × U(1) model of Witten exhibit the current quenching – there is an upper bound for their current \( I \). In the Weinberg-Salam theory we \textit{generically} do not observe this phenomenon. Large \( I \) corresponds to small \( \sigma \), and we were able to descend down to \( \sigma \sim 10^{-3} \) (after this the numerics become somewhat involved) always finding larger and larger values of \( I \), without any tendency for this increase to stop.

It seems that the reason why the current can be unbounded in the Weinberg-Salam theory is related to the spin of the current carriers. The existence of the critical current in superconductivity models usually follows from the fact that too large currents should produce strong magnetic fields which will quench the scalar condensate and destroy the superconductivity. This is what happens in Witten’s model where the current is carried by scalars, and so quenching the condensate quenches the current. Specifically, the current defined by Eq.(C.9) of the Appendix C vanishes for \( \phi_2 \to 0 \). In the Weinberg-Salam theory, as we shall see below, the very strong magnetic field also quenches the Higgs field inside the vortex. However, this does not destroy superconductivity, since the current is carried by vectors and so the Higgs field is not the relevant order parameter. In fact, the current defined by Eqs.(2.26), (3.13) does not vanish for \( \Phi \to 0 \). For large currents the vector boson condensate can be described by the pure Yang-Mills field, whose scale invariance implies that the current can be as large as one wants. In the Witten model this does not work since the current carriers are scalars which break the scale invariance.
At the more technical level the current quenching can be related to the ‘dressed’ currentless solutions. If they exist, then changing the twist \( \sigma \) makes the system interpolate between the ‘bare’ and ‘dressed’ strings, and since both of them are currentless, the current passes through a maximum in between. As we shall see in the next section, ‘dressed’ solutions exist in the electroweak theory only for special values of \( \beta, \theta, \) \( \mathcal{W} \), which means that current quenching is non-generic. For generic parameter values there are no ‘dressed’ solutions, and when one starts from the ‘bare’ \( Z \) string and decreases the twist, the current always grows.

For large currents a simple approximation for the solutions can be found that agrees very well with the numerics and suggests that the current can be arbitrarily large.

The large current limit is characterized by the presence of two different length scales. One of them does not depend on the current and is determined by the \( Z \) boson or Higgs boson masses, which are both of the same order of magnitude (for physical values of \( \beta \)), \( R_Z \sim m_Z^{-1} \sim m_H^{-1} \sim 1 \). Another scale is related to the ‘dressed’ \( W \) boson mass \( (5.22) \), which increases with current and becomes large for large currents, \( m_\sigma \sim \sigma Q \sim \mathcal{I} \), so that the corresponding length is small \( R_\sigma \sim m_\sigma^{-1} \sim \mathcal{I}^{-1} \). One can therefore expect the solutions to show a small central region where \( \rho \leq R_\sigma \) that accommodates the very heavy field modes of mass \( m_\sigma \). For \( R_\sigma \leq \rho \leq R_Z \) these ‘supermassive’ modes die out but the other massive modes remain, while the asymptotic region where \( R_Z \leq \rho \) contains only the massless modes.

\[
\begin{align*}
\ln(1+\rho) &
\end{align*}
\]

FIG. 11: Solutions profiles for \( \beta = 2, g', 2 = 0.23, n = \nu = 1 \) and \( \sigma = 0.008 \).

The numerical profiles of the large \( \mathcal{I} \) solutions (see Fig.11) essentially confirm these expectations, but also reveal additional interesting features. For large currents the vacuum
angle $\gamma$ tends to $\pi/2$ (see Fig.7) so that one has $f_1 = f_2 = 1/\sqrt{2}$ for $\rho \to \infty$. However, as is seen in Fig.11, one has $f_1 \approx f_2$ not only for large $\rho$ but everywhere. In particular, there is a region where $f_1 \approx f_2 \approx 0$, which means that the gauge symmetry is completely restored to SU(2)×U(1). In fact, at the origin one still has $f_1 = a_5 \rho$ and $f_2 = q$ (see Eq.(5.10)) but the parameters $q, a_5$ are extremely small. In addition, one has in this region $v_1 \approx \sigma u_3 \approx 0$, $\nu \approx 1$, $\sigma u \approx \text{const.}$, while $v_3$ and $u_1$ change very quickly, and in particular $v_3$ vanishes very rapidly. The latter corresponds to freezing away of the very massive modes.

One can wonder how the existence of the very massive modes can be compatible with the complete symmetry restoration. However, these modes arise not via the Higgs mechanism but rather due to the screening by the current, which is somewhat similar to the colour screening in pure Yang-Mills systems. When $v_3$ vanishes, the Higgs field starts to change and approaches its vacuum value, after which all massive degrees of freedom die out and the solutions reduce to the U(1) electric wire (5.17).

It turns out to be possible to find a relatively simple approximation for the large current solutions that explains all their empirically observed features. Specifically, since the vacuum angle $\gamma$ approaches $\pi/2$ for large currents, let us set $\gamma = \pi/2$. Let us also restrict ourselves to the simplest case where $n = \nu = 1$. The boundary conditions (6.1) then become

\begin{align}
  a_1 &\leftarrow u \rightarrow c_1 + Q \ln \rho, \quad 0 \leftarrow u_1 \rightarrow -(c_1 + Q \ln \rho), \quad 1 \leftarrow u_3 \rightarrow 0, \\
  1 &\leftarrow v \rightarrow c_2, \quad 0 \leftarrow v_1 \rightarrow -c_2, \quad 1 \leftarrow v_3 \rightarrow 0, \\
  a_5 \rho &\leftarrow f_1 \rightarrow 1/\sqrt{2}, \quad q \leftarrow f_2 \rightarrow 1/\sqrt{2}. \quad (10.6)
\end{align}

A. W-condensate region – SU(2) Yang-Mills string

Let us approximate the solution at small $\rho$ by

\begin{align}
  f_1 = f_2 = \sigma u_3 = v_1 = 0, \quad \sigma u = \sigma a_1, \quad v = 1, \quad \sigma u_1(\rho) = \lambda U_1(\lambda \rho), \quad v_3 = V_3(\lambda \rho), \quad (10.7)
\end{align}

where $\lambda$ is a scale parameter. Inserting this to (4.12)–(4.20) the equations reduce to

\begin{align}
  \frac{1}{x}(x U_1')' &= V_3^2 x^2 U_1, \quad (10.8a) \\
  x \left(\frac{V_3'}{x}\right)' &= U_1^2 V_3, \quad (10.8b)
\end{align}
with $x = \lambda \rho$. These equations admit a solution with the following boundary conditions for $0 \leftarrow x \rightarrow \infty$,

$$x + \ldots \leftarrow U_1(x) \rightarrow 0.85 + 0.91 \ln(x) + \ldots$$  \hspace{1cm} (10.9a)

$$1 - 0.45x^2 + \ldots \leftarrow V_3(x) \rightarrow 0.32\sqrt{x} e^{0.06x} x^{-0.91x} + \ldots$$  \hspace{1cm} (10.9b)

for which $V_3$ rapidly approaches zero (see Fig.12). Inserting $U_1(x), V_3(x)$ to (10.7) gives a family of solutions distinguished from each other by values of the scale parameter $\lambda$. This scaling symmetry arises due to the fact that, since the Higgs field is zero, the system is pure Yang-Mills. In fact, Eqs.(10.8) coincide with the Yang-Mills equations for the pure SU(2) system,

$$\mathcal{L}_{SU(2)} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu}$$  \hspace{1cm} (10.10)

provided that the Yang-Mills field is cylindrically-symmetric

$$\tau^a W^a_\mu dx^\mu = \tau^1 \lambda U_1(\lambda \rho) dz + \tau^3 V_3(\lambda \rho) d\varphi.$$

(10.11)

We shall therefore call the solution shown in Fig.12 Yang-Mills string.

The Yang-Mills strings are made of the pure gauge field and describe the charged W-condensate trapped in the region where the amplitude $V_3$ is different from zero. Outside this region there remains the long-range Biot-Savart field generated by the condensate and represented by $U_1$. The right-hand sides of Eqs.(10.8) can be viewed as conserved (although not gauge invariant) current densities. Integrating over the $\rho, \varphi$ plane gives the current along the $z$-direction, $I \sim \lambda$, localized in the region of the size $\sim 1/\lambda$ where $V_3$ is different from zero.

Yang-Mills strings approximate the central current-carrying core of the electroweak vortex. Their scale invariance implies that there is no upper bound for the current $I \sim \lambda$, since the scale parameter $\lambda$ can assume any value.

**B. External U(1)×U(1) region**

Let us now assume that the Yang-Mills string approximation is valid only in the central vortex core, for $x \leq x_0$, with $x_0$ chosen such that $V_3(x_0)$ is sufficiently close to zero, for example $x_0 = 4$ (see Fig.12). This determines the upper value $\rho_0 = x_0/\lambda$. As we shall see, for large $\lambda$ the precise knowledge of $x_0$ is unimportant.
FIG. 12: Left: solution of Eqs. (10.8) with the boundary condition (10.9) and the function $\sin \Omega$ defined by Eqs. (6.3), (10.26). Right: the typical solution of Eqs. (10.17).

For $\rho > \rho_0$ we set

$$v_3 = \sigma u_3 = 0, \quad f_1 = f_2 \equiv f/\sqrt{2},$$

while the remaining field amplitudes can be combined to define the electromagnetic and $Z$ field variables,

$$U = \frac{\sigma}{2}(u + u_1), \quad V = \frac{1}{2}(v + v_1),$$

$$U_A = \sigma(g^2 u - g^2 u_1), \quad V_A = g^2 v - g^2 v_1.$$  (10.13)

More precisely, using formulas of Sec. (VI A) and Eq. (10.12), the electromagnetic and $Z$ fields potentials are

$$A_\mu = \frac{g}{g'} B_\mu - \frac{g'}{g} W^1_\mu, \quad Z_\mu = B_\mu + W^1_\mu, \quad \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - \frac{\beta}{8} (|\phi|^2 - 1)^2$$

with $D_\mu \phi = (\partial_\mu - \frac{1}{2} Z_\mu) \phi$. This contains the free Maxwell theory and the Abelian Higgs model. The fields are $gg' A_\mu dx^\mu = U_A dz + V_A d\varphi$ and $2Z_\mu dx^\mu = Udz + V d\varphi$ while $\phi = f$. Eqs. (4.12)–(4.20) then reduce to the Maxwell equations

$$\left(\rho U^A_\lambda \right)' = 0, \quad \frac{\left(V^A_\lambda \right)'}{\rho} = 0.$$  (10.16)
and to the equations of the Abelian Higgs model

\[ \frac{1}{\rho}(\rho f')' = \left( U^2 + \frac{V^2}{\rho^2} + \frac{\beta}{4}(f^2 - 1) \right) f, \]  
(10.17a)

\[ \rho \left( \frac{V'}{\rho} \right)' = \frac{1}{2} f^2 V, \]  
(10.17b)

\[ \frac{1}{\rho}(\rho U')' = \frac{1}{2} f^2 U. \]  
(10.17c)

The solution of the Maxwell equations (10.16) is

\[ U_A = A + B \ln \rho, \quad V_A = C, \]  
(10.18)

with constant \( A, B, C \). Equations (10.17) admit a one-parameter family of solutions with the boundary conditions

\[ U(\rho_0) \leftarrow U \rightarrow 0, \quad \frac{1}{2} \leftarrow V \rightarrow 0, \quad 0 \leftarrow f \rightarrow 1 \]  
(10.19)

for \( \rho_0 \leftarrow \rho \rightarrow \infty \) (see Fig[12]). The numerics show that if \( U'(\rho_0) \) is large and positive then \( U(\rho_0) \) is large and negative and the derivatives \( f'(\rho_0) \) and \( V'(\rho_0) \) tend to zero such that there is a neighbourhood of \( \rho_0 \) where

\[ V \approx 1/2, \quad f \approx 0, \quad U \approx a + b \ln \rho \]  
(10.20)

with the parameters \( a, b \) related to each other, \( a = a(b) \). As a result,

\[ U(\rho_0) = a(b) + b \ln \rho_0, \quad U'(\rho_0) = b/\rho_0. \]  
(10.21)

C. Matching the solutions

The solutions obtained for \( \rho \leq \rho_0 \) and for \( \rho \geq \rho_0 \) should agree at \( \rho = \rho_0 = x_0/\lambda \) where the functions and their first derivatives should match. Matching \( u, u_1 \) and their derivatives gives four conditions that determine the free parameters \( a_1, b, A, B \) in the above solutions,

\[ B = -0.91g^2\lambda, \]
\[ b = 0.45\lambda, \]
\[ \sigma a_1 = 2a(b) - 0.85\lambda - 2b \ln \lambda, \]
\[ A = B \ln \lambda + g^2a_1 - 0.85g^2\lambda. \]  
(10.22)
It is worth noting that $x_0$ has dropped from these relations. Matching similarly $v, v_1, f$ gives only one condition, $C = g^2$, but the derivatives $v', v'_1, f'$ do not match precisely, since they vanish for $\rho \leq \rho_0$ but not for $\rho \geq \rho_0$. However, this discrepancy tends to zero when $\lambda$ grows, since $U'(\rho_0) \sim \lambda^2$ then increases thus rendering the approximation (10.20) better and better. Similarly, although there is a discrepancy in values of $v_3$, since it vanishes for $\rho \geq \rho_0$ but not for $\rho \leq \rho_0$, increasing $\lambda$ reduces this discrepancy as well.

As a result, by matching the two solutions we can approximate the global solution in the interval $0 \leq \rho < \infty$. The approximate solution depends on $\lambda$ and approaches the true solution when $\lambda$ increases. For $\lambda \to \infty$ the discrepancy of values of $v_3, v', v'_1, f'$ at the matching point vanishes and so the approximate solution becomes exact. There only remains to relate $\lambda$ to the current. Comparing with (10.6) we see that $\sigma Q = B$ (also $\sigma c_1 = A$ and, as noticed in Fig.8 $c_2 = g^2$) and since the current $I = -2\pi\sigma Q/(gg')$ it follows that

$$\lambda = 0.17 \frac{g}{g'} I.$$  

(10.23)

The approximate solution shown in Fig.13 clearly exhibit the same structure as the true one in Fig.11. Although it corresponds to not a very large value of the scale parameter, $\lambda = 7.6$, its current is twice as large as compared to that in Fig.11. The true solution in Fig.11 shows a slowly growing amplitude $u_3$ whereas in Fig.13 one has $\sigma u_3 = 0$. This difference is a subleading effect, since in Fig.11 the vacuum angle is not exactly $\pi/2$ and so there is a growing mode $u_3 \sim \cos \gamma \ln \rho$. For $I \to \infty$ one has $\gamma \to \pi/2$ and this mode disappears,
so that $u_3$ becomes everywhere bounded, and multiplying it by $\sigma \to 0$ will give $\sigma u_3 = 0$, as shown in Fig.13. With some more analysis such subleading effects can be taken into account to determine, for example, $\gamma(I)$, $\sigma(I)$, $q(I)$.

D. **Symmetry restoration inside the vortex**

Comparing Figs.11 and 13 reveals that the zero Higgs field region expands when the current increases. The inner structure of the large current vortex can be schematically represented by Fig.14. The vortex shows a large symmetric phase region of size $\sim I$ where the Higgs field is very close to zero. In the centre of this region there is a compact core of size $\sim I^{-1}$ containing the charged W-condensate and approximated by the Yang-Mills string. Outside the core there live the electromagnetic and Z fields. The symmetric phase is surrounded by a ‘crust’ of thickness $\sim R_z$ where the Higgs field approaches the constant vacuum value while the Z field becomes massive and dies away. The region outside the ‘crust’ is described by the Maxwell electrodynamics and is dominated by the Biot-Savart magnetic field generated by the current in the core.

![FIG. 14: The schematic structure of the large current vortex.](image)

This picture reminds somewhat the ‘W-dressed superconducting string’ scenario of Ambjorn and Olesen [15], briefly described in Sec.III D above. In this scenario Witten’s su-
perconducting string produced at the GUT energies acts as the external source of magnetic field that changes the structure of the electroweak vacuum. There is a cylindrical layer around the string where the field falls within the limits Eq. (3.33) and where the Higgs field is non-trivial. Inside this layer the field is supercritical and the Higgs field is zero, while outside the field is undercritical and the Higgs field is in vacuum. All this is very similar to what is shown in Fig. [15]. This suggests that we have a self-consistent electroweak realization of the Ambjorn-Olesen scenario in which the Witten string is replaced by the Yang-Mills string of completely electroweak origin. The following estimates confirm this.

First of all, the radius of the layer where the Higgs field is non-trivial should scale as $I$, and we shall now see that the crust radius scales precisely in this way. Indeed, inside the crust $f$ is very small and one can neglect the right hand sides in (10.17c) to obtain $U = b \ln(\rho/\rho_*)$, where $\rho_*$ is an integration constant and $b = 0.45 \lambda$. As long as $\rho \ll \rho_*$ the $U^2$ term in (10.17a) is large and makes $f$ stay very close to zero, since $f$ can increase and pass through the inflection point if only $U^2$ is small. The latter follows from the fact that the second derivative of $f$ with respect to $\ln \rho$ vanishes if only the right-hand side of (10.17a) vanishes, which is only possible if $U^2$ is small enough to be canceled by the negative term $\frac{d}{4}(f^2 - 1)$. Therefore, $f$ can vary if only $U \sim \sqrt{\beta}$, but as soon as $f$ deviates from zero, it acts as the mass term for $U$ to drive it to zero. As a result, $f$ interpolates between zero and one close to the place where $U$ vanishes, for $\rho \sim \rho_*$, in an interval of size $\delta \rho \sim R_z$. Multiplying (10.17c) by $\rho$ and integrating from $\rho_0$ to infinity gives $2b = -\int_{\rho_0}^{\infty} \rho f^2 U d\rho$ and since the integrand is almost everywhere zero apart from a small vicinity of $\rho_*$, it follows that $b \sim \sqrt{\beta} \rho_*$. This gives the size of the symmetric phase region (the crust radius)

$$\rho_* = 0.28 \times \frac{g}{g'} \frac{I}{\sqrt{\beta}}, \quad (10.24)$$

where the coefficient is found numerically. As a result, one can approximate the solution of Eqs. (10.17) by

$$f \approx \Theta(\rho - \rho_*), \quad U \approx b (1 - f) \ln(\rho/\rho_*), \quad V \approx \frac{1}{2} (1 - f)(1 - (\rho/\rho_*)^2) \quad (10.25)$$

where $\Theta(\rho - \rho_*)$ is the step-function smoothed over a finite interval $\delta \rho \sim R_z$.

Let us now consider the distribution of the fields inside the vortex. The fields are defined by Eqs. (6.5), (6.12) where there is a hitherto undetermined function $\Omega$. Using Eqs. (6.3), (10.12) one finds that $\Omega = \pi/2$ in the external region, while in the core $\Omega$ is
not yet defined. Let us calculate the first order correction to the core configuration (10.7) assuming that $f_1, f_2$ are small functions of $x = \lambda \rho$ and that $f_1 \sim x$ and $f_2 = O(1)$ for $x \to 0$. Linearising Eqs. (4.14), (4.15) gives for large $\lambda$, with $f_\pm = f_1 \pm f_2$,

$$\frac{1}{x} (x f_\pm')' = \left( \frac{U_1}{2} + p \right)^2 + \frac{1 + V_3^2}{4x^2} f_\pm + \frac{V_3}{2x^2} f_\pm', \quad (10.26)$$

where $p = -\sigma \alpha_1/(2\lambda) \sim \ln \lambda$ and $U_1, V_3$ are given by Eqs. (10.8). Integrating these equations and using Eq. (6.3) determines $\Omega$ inside the core and shows that it approaches $\pi/2$ very rapidly (see Fig. 12).

FIG. 15: The field distributions inside (left) and outside (right) the current-carrying condensate core of the vortex, for the same solution as in Fig. 13.

We have now all the ingredients to compute the fields (6.5), (6.12) and the current density (6.7). The result presented in Fig. 15 shows that the current, W-condensate, and the z-component of the magnetic field are trapped in the core, where the energy density is maximal. Outside the core there extend only the Biot-Savart magnetic field and the Z field, which are both massless as long as the Higgs field stays close to zero. At a distance $\sim I$ away from the core the system enters the ‘crust’ region where the Higgs field approaches the vacuum value. This gives rise to a local maximum in the energy density and drives the Z field very rapidly to zero. Outside the ‘crust’ there remains only the Biot-Savart magnetic field.

The most interesting feature is that the ‘crust’ region where the Higgs field is non-trivial exists only under certain values of the magnetic field. Outside this region, where the field is either too weak or too strong, the Higgs field is either in vacuum or vanishes, in agreement
with the predictions of [15]. At the same time, the magnetic field inside the crust does not quite fall within the limits \((m^2_w/e, m^2_H/e)\) predicted in [15] for the homogeneous magnetic field with \(\beta = 1\). The electroweak condensate in the latter case contains the W, Z and Higgs fields [15]. In our case the crust region does not contain the W field, since it develops the very large effective mass \(m_\sigma\) due to the interaction with the inhomogeneous Biot-Savart field, so that it can be different from zero only in the central core region.

We also notice in Fig.\textsuperscript{15} that, since \(w_\rho, w_\phi\) do not vanish at \(\rho = 0\), the W-condensate field strength is actually singular at the axis. However, this singularity is mild, since the energy density and the field potentials are globally regular. This singularity is absent for solutions with \(\nu > 1\).

\textbf{XI. SPECIAL PARAMETER LIMITS}

The described above properties of the solutions remain qualitatively the same for the generic parameter values. However, for special values of \(\beta, \theta_w\) solutions can exhibit new features. Even though such values do not always belong to the physical region, where \(1.5 \leq \beta \leq 3.5\) and \(\sin^2 \theta_w = 0.23\), it is instructive to consider them, since this helps to understand the structure of the solution space.

For example, the limits \(\theta_w \to 0\) or \(\theta_w \to \pi/2\) are special, because either the hypercharge field \(B_{\mu}\) or isospin field \(W^a_{\mu}\) then become massless and decouple. The long-range mode \(Q \ln \rho\) then disappears since \(Q\) vanishes (see Fig.\textsuperscript{16}) and the total energy of the solutions becomes finite. Solutions for \(\theta_w = \pi/2\) have been studied in the literature [21], but it is worth reviewing them in the context of our general discussion.

Another interesting case corresponds to the chiral solutions, which exist only for special values of \(\theta_w\) given by a certain function \(\theta_w(\beta, n, \nu, q)\). This function can be determined by requiring the solutions to respect the \(\sigma^2 = 0\) condition. Finally, we shall consider infinite Higgs mass limit, \(\beta \to \infty\), in which case the theory reduces to the gauged \(\mathbb{C}P^1\) sigma-model.
mechanism similar to that described above for the W bosons, so that all fields are massive.

but then always bend back to $Q$ = 0 the curves $\sigma(q)$ start at $q = 0$ at their maximum, then go down towards larger values of $q$, but then always bend back to $q = 0$. If $g = 0$ then $\sigma$ decreases monotonously to zero at $q = 1$.  

A. Semilocal limit, $\theta_w = \pi/2$

Since the right hand side of the Yang-Mills equations (2.16) is proportional to $g^2$, one has $W^a_\mu \sim g^2$ for small $g$ so that the field $W^a_\mu$ and its kinetic term in the Lagrangian vanish for $g \to 0$ and the theory (2.10) reduces to

$$
\mathcal{L} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \Phi)\dagger D^\mu \Phi - \frac{\beta}{8} (\Phi\dagger \Phi - 1)^2 ,
$$

(11.27)

with $D_\mu \Phi = (\partial_\mu - \frac{i}{2} B_\mu) \Phi$. This reduced theory is sometimes called semilocal, since it has the local U(1) invariance, while the SU(2) symmetry of the original theory now becomes global [20]. The perturbative mass spectrum of the model (3.4) can be obtained by setting $g = 0$ in Eq.(2.24): it contains a massive vector boson and a Higgs boson, as well as a pair of massless Goldstone scalars. However, the latter acquire a finite mass via the screening mechanism similar to that described above for the W bosons, so that all fields are massive.

The field equations can be obtained by setting in Eqs.(4.12)–(4.20) the SU(2) field amplitudes to be constant and equal to their values at the origin, $u_1 = v_1 = 0$, $u_3 = 1$, $v_3 = \nu$. This solves the Yang-Mills equations (4.16)–(4.20) and implies the following values of the asymptotic parameters in (6.1): $Q = \gamma = 0$, $c_1 = -1$, $c_2 = -\nu$. In the gauge (5.6) the

FIG. 16: Left: the ‘Coulombian charge’ parameter $Q$ against $\theta_w$ for the solutions with $\beta = 2$, $n = \nu = 1$ and $q = 0.2$. Right: the twist $\sigma$ against $q$ for solutions with $n = \nu = 1$, $\beta = 2$. For $g \neq 0$ the curves $\sigma(q)$ start at $q = 0$ at their maximum, then go down towards larger values of $q$, but then always bend back to $q = 0$. If $g = 0$ then $\sigma$ decreases monotonously to zero at $q = 1$.  

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$$
\mathcal{L} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \Phi)\dagger D^\mu \Phi - \frac{\beta}{8} (\Phi\dagger \Phi - 1)^2 ,
$$

(11.27)

with $D_\mu \Phi = (\partial_\mu - \frac{i}{2} B_\mu) \Phi$. This reduced theory is sometimes called semilocal, since it has the local U(1) invariance, while the SU(2) symmetry of the original theory now becomes global [20]. The perturbative mass spectrum of the model (3.4) can be obtained by setting $g = 0$ in Eq.(2.24): it contains a massive vector boson and a Higgs boson, as well as a pair of massless Goldstone scalars. However, the latter acquire a finite mass via the screening mechanism similar to that described above for the W bosons, so that all fields are massive.

The field equations can be obtained by setting in Eqs.(4.12)–(4.20) the SU(2) field amplitudes to be constant and equal to their values at the origin, $u_1 = v_1 = 0$, $u_3 = 1$, $v_3 = \nu$. This solves the Yang-Mills equations (4.16)–(4.20) and implies the following values of the asymptotic parameters in (6.1): $Q = \gamma = 0$, $c_1 = -1$, $c_2 = -\nu$. In the gauge (5.6) the
Yang-Mills field then vanishes and the fields read
\[
\mathcal{W} = (u(\rho) + 1) \sigma_\alpha dx^\alpha + (2n - \nu - v(\rho)) d\varphi, \quad \Phi = \left( \begin{array}{c} e^{in\varphi} f_1(\rho) \\ e^{i(n-\nu)\varphi + i\sigma_\alpha x^\alpha} f_2(\rho) \end{array} \right).
\] (11.28)

Eqs. (4.16)–(4.20) reduce to
\[
\frac{1}{\rho}(\rho u')' = \frac{1}{2} \left( (u + 1)f_1^2 + (u - 1)f_2^2 \right),
\] (11.29)
\[
\frac{1}{\rho}(v')' = \frac{1}{2} \left( (v + \nu)f_1^2 + (v - \nu)f_2^2 \right),
\] (11.30)
\[
\frac{1}{\rho}(f_1')' = \left( \frac{1}{4}(u + 1)^2 + \frac{1}{4\rho^2}(v + \nu)^2 + \frac{\beta}{4}(f_1^2 + f_2^2 - 1) \right) f_1,
\] (11.31)
\[
\frac{1}{\rho}(f_2')' = \left( \frac{1}{4}(u - 1)^2 + \frac{1}{4\rho^2}(v - \nu)^2 + \frac{\beta}{4}(f_1^2 + f_2^2 - 1) \right) f_2.
\] (11.32)

The boundary conditions are obtained from Eqs. (5.10), (5.21),
\[
a_1 + \ldots \leftarrow u \rightarrow -1 + c_3 g^2 \sqrt{\rho} e^{-m_2 \rho} + \ldots,
\]
\[
2n - \nu + a_4 \rho^2 + \ldots \leftarrow v \rightarrow -\nu + c_4 g^2 \sqrt{\rho} e^{-m_2 \rho} + \ldots,
\]
\[
a_3 \rho^n + \ldots \leftarrow f_1 \rightarrow 1 + c_5 \sqrt{\rho} e^{-m_{H} \rho} + \ldots,
\]
\[
q \rho^{n-\nu} + \ldots \leftarrow f_2 \rightarrow c_6 \sqrt{\rho} e^{-m_{\sigma} \rho} + \ldots,
\] (11.33)

where \(m_z, m_H\) are the same as in Eq. (2.24), while Eq. (5.22) now gives \(m_{\sigma} = \sigma\). Since all the fields are massive and approach zero exponentially fast, the vortex energy is finite.

The simplest solution of Eqs. (11.29)–(11.32) is obtained for \(q = 0\), this is the Z string \((7.6)\) restricted to \(\theta_W = \pi/2\), in which case it is called semilocal string \([20],[35]\). Choosing \(q \neq 0\) gives current-carrying vortices called twisted semilocal strings \([21]\), a typical solution being shown in Fig. 17. These solutions exist only for \(\beta > 1\) and for \(\nu = 1, \ldots n\). This is due to the fact that for \(\theta_W = \pi/2\) the allowed region in Fig 4 is \(\beta > 1\) if \(\nu \leq n\) while for \(\nu > n\) the allowed region is empty. For given \(\beta, n, \nu\) the solutions comprise a family that can be labeled by \(q \in [0, 1)\).

The current in this case is not local but global, related to the SU(2) invariance of the model \((3.4)\). It is determined by \(J^a_\mu\) in Eq. (2.16),
\[
I^\mathrm{SU(2)}_a = \int J^a_\alpha d^2x = \pi \sigma_\alpha \int_0^\infty \left\{ (u + 1)f_1^2 + (1 - u)f_2^2 \right\} \rho d\rho,
\] (11.34)
The solution profiles for \( \sigma = 0.1 \) (left) and the total energy \( E/(2\pi n) \) against the restframe current (right).

while the similar integrals of \( J_1^\alpha \) and \( J_2^\alpha \) vanish. The energy \( E = 2\pi \int_0^\infty (\mathcal{E}_1 + \mathcal{E}_2)\rho d\rho \) is finite and can be obtained from Eqs. (4.24), (4.25) by setting the SU(2) amplitudes to the specified above values and omitting terms with their derivatives. For \( \beta > 1 \) and \( g = 0 \) there is a non-trivial lower bound for the energy determined by the total derivative in the second line of Eq. (4.27),

\[
E \geq E_{\text{min}} = 2\pi \int_0^\infty \frac{1}{2} \left\{ v - (v + \nu)f_1^2 - (v - \nu)f_2^2 \right\}' d\rho = 2\pi n. \tag{11.35}
\]

It turns out that the energy of the ‘twisted strings’ monotonously decreases with current (see Fig.17) and approaches the lower bound \( 2\pi n \) when the current tends to infinity \([21]\). The twist \( \sigma \) also decreases with current, as for solutions with \( g \neq 0 \) (see Fig.7). The parameter \( q \) increases monotonously and approaches unit value when the current tends to infinity \([21]\), while for solutions with \( g \neq 0 \) it always passes through a maximum and then tends to zero for large currents, as shown in Fig.6. If one plots \( q \) against \( \sigma \) (or against the current) for different \( g \), then it turns out that the curves approach the limiting curve for \( g = 0 \) pointwise but not uniformly, since the boundary condition at \( \sigma = 0 \) are different, depending on whether \( g = 0 \) or \( g \neq 0 \) (see Fig.16).

The lower energy bound \( E_{\text{min}} \) in Eq. (11.35) is attained for \( \beta = 1 \) if the first order Bogomol’nyi equations \( \mathcal{B}_s = 0 \) with \( \mathcal{B}_s \) defined by Eq. (4.28) are fulfilled. For \( g = 0 \) there are only
three such equations,

\[ v' + \frac{\rho}{2} [(f_1^2 + f_2^2) - 1] = 0, \]
\[ f_1' + \frac{1}{2\rho} (v + \nu)f_1 = 0, \]
\[ f_2' + \frac{1}{2\rho} (v - \nu)f_2 = 0. \]  

(11.36)

Their solutions comprise a family labeled by \( q \in [0, 1) \) \[36\], they also fulfill the three second order equations (11.30)–(11.32) with \( \sigma^2 = 0 \), while the amplitude \( u \) can be obtained from Eq. (11.29). Reconstructing the fields (11.28), solutions with \( \sigma_a = 0 \) are sometimes called ‘skyrmions’, while those with \( \sigma_0 = \pm \sigma_3 \neq 0 \) can be called ‘spinning skyrmions’ \[37\]. Their profiles are qualitatively similar to those for the twisted strings, but since \( m_{\sigma} = \sigma = 0 \) in this case, they show polynomial tails at large \( \rho \). It turns out that the twisted strings with \( \beta > 1 \) approach the ‘spinning skyrmion’ configurations for \( \mathcal{I} \to \infty \) (modulo an infinite rescaling) \[21\].

**B. Isospint limit, \( \theta_W = 0 \)**

If \( g' = \sin \theta_W \) is small then the hypercharge field \( B_\mu \) scales as \( g'^2 \) so that for \( g' \to 0 \) the theory (2.10) reduces to

\[ \mathcal{L} = -\frac{1}{4} W^{a \mu \nu} W_{a \mu \nu} + (D_\mu \Phi)^\dagger D^\mu \Phi - \frac{\beta}{8} (\Phi^\dagger \Phi - 1)^2, \]  

(11.37)

with \( D_\mu \Phi = (\partial_\mu - i \frac{1}{2} \tau^a W^a_\mu) \Phi \). The SU(2) is now local while the U(1) is global. The perturbative mass spectrum contains three massive vector bosons with the mass \( m_Z \) and a Higgs boson with the mass \( m_H \). The field equations in the axially symmetric case are obtained by setting in (4.14)–(4.20) \( u(\rho) = \text{const.}, v(\rho) = 2n - \nu \), which implies that the U(1) part of the field (5.6) vanishes and that one has in Eqs. (5.10), (5.21), (6.1) \( a_1 = c_1 = u, c_2 = 2n - \nu \)
and $Q = c_3 = c_4 = 0$. The boundary conditions for the remaining field amplitudes read

$$
0 \leftarrow u_1 \rightarrow -u \sin \gamma, \\
1 \leftarrow u_3 \rightarrow -u \cos \gamma, \\
0 \leftarrow v_1 \rightarrow (\nu - 2n) \sin \gamma, \\
\nu \leftarrow v_3 \rightarrow (\nu - 2n) \cos \gamma, \\
a_3 \rho^n \leftarrow f_1 \rightarrow \cos \frac{\gamma}{2}, \\
q \rho^{n-\nu} \leftarrow f_2 \rightarrow \sin \frac{\gamma}{2},
$$

(11.38)

The simplest solution is obtained for $q = 0$, this is the Z string restricted to $\theta_W = 0$. For $q \neq 0$ one obtains current-carrying solutions. The typical solution is shown in Fig.18. These solutions relate to the lower boundary of the chiral diagram in Fig.11, which implies that they exist for any $\beta > 0$ and for $\nu = 1, \ldots 2n - 1$. Their current is global, related to the global $U(1)$ invariance of the theory (11.37), it can be expressed in terms of $J_\mu^0$ in Eq.(2.15),

$$I^{(1)}_\alpha = \int J^0_\alpha d^2 x = \pi \sigma_\alpha \int_0^\infty \left\{ (u_3 + u) f_1^2 + 2u_1 f_1 f_2 - (u_3 - u) f_2^2 \right\} \rho d\rho.
$$

(11.39)

As for all solutions described above, there are no apparent restrictions on its values. The energy density is defined by Eqs.(4.24),(4.25) with $u,v$ restricted to the values specified above. The total energy is finite, it always *increases* with current (see Fig.18).

These $\theta_W = 0$ solutions can also be obtained from the generic electroweak vortices in the limit $\theta_W \to 0$. Similarly the $\theta_W = \pi/2$ solutions can be obtained in the limit $\theta_W \to \pi/2$. The generic field configurations then approach the limiting ones pointwise, although non-uniformly at large $\rho$, since they support a long-range field $\sim Q \ln \rho$, while in the limit one has $Q = 0$. Since the electromagnetic current vanishes in the limit, one can wonder how the global currents (11.34) and (11.39) can be obtained via the limiting procedure. The answer is that they can be obtained as limits of the rescaled electromagnetic current $I_\alpha/(gg')$. The following relations take place,

$$
\frac{1}{gg'} J_\alpha = \frac{1}{gg'} \partial^\mu F_{\mu\alpha} = \frac{1}{g^2} \partial^\mu B_{\mu\alpha} - \frac{1}{g^2} \partial^\mu (n^a W^{a\mu}_\alpha) = \\
= J_\alpha^0 - \frac{1}{g^2} \partial^\mu (n^a W^{a\mu}_\alpha) = \\
= \frac{1}{g^2} \partial^\mu B_{\mu\alpha} - n^a J^a_\alpha - \frac{1}{g^2} (D^\mu n^a) W^{a\mu}_\alpha.
$$

(11.40)
FIG. 18: Features of the $n = \nu = 1$, $\beta = 2$ solutions in the isospin limit, $g = 1$. The profiles for $\sigma = 0.5$ (left) and the energy against the restframe current (right). The insertion shows that $E/n$ increases with $\mathcal{I}$ faster for $n = 1$ than for $n = 2$, so that the vortices repel each other for small currents and attract otherwise.

Here we have used Eqs. (2.15), (2.16) and the identity $\partial_\sigma (n^a W^a_{\mu\alpha}) = (D_\sigma n^a) W^a_{\mu\alpha} + n^a D_\sigma W^a_{\mu\alpha}$ where $D_\sigma n^a = \partial_\sigma n^a + \epsilon_{abc} W^b_{\alpha} n^c$. Integrating the first line of these relations over the vortex cross section gives $\frac{1}{gg'} I_\alpha$. Integral of the second line reduces in the limit $g' \to 0$ to the global current (11.39), since the total derivative term in this line gives no contribution, because the field $W^a_{\mu\nu}$ becomes short-ranged in this limit.

Let us now notice that, since the whole expression in (11.40) has the total derivative structure, its integral will not change upon replacing the vector $n^a = (\Phi^\dagger \tau^a \Phi)/(\Phi^\dagger \Phi)$ by any other unit vector with the same value at infinity. Let us therefore replace $n^a$ by $\tilde{n}^a = \delta^a_3$ in the third line in (11.40). One will have $D_\mu \tilde{n}^a = \epsilon_{abc} W^b_\mu$, which scales as $g^2$ for small $g$, so that the last term in the third line vanishes in the limit $g \to 0$. The second term gives upon integration the global current (11.34) (up to the sign), while the first term gives zero, since the field $B_\mu$ becomes massive in the limit. We therefore conclude that

$$I^{(1)}_\alpha \leftarrow \frac{1}{gg'} J_\alpha \to -I^{SU(2)}_\alpha \quad (11.41)$$

for $g' \to 0$ and $g \to 0$, respectively, so that the global currents can indeed be obtained from the local electromagnetic current via the limiting procedure.
C. Special chiral solutions

Superconducting strings in the Witten model are usually constructed starting from the ‘dressed’ \( \sigma^2 = 0 \) chiral/currentless string, viewed as the ‘bare’ ANO vortex stabilized by the condensate in its core. Increasing \( \sigma^2 \) produces the current, while the parameter \( q \) decreases and for \( q \to 0 \) the solutions reduce to the ‘bare’ ANO vortex (see Appendix C).

Superconducting vortices in the Weinberg-Salam theory are obtained in the opposite way, starting from the ‘bare’ ANO vortex (Z string) and then decreasing the twist \( \sigma \). One does not generically find chiral solutions in this case, since the limit \( \sigma^2 \to 0 \) corresponds to infinite and not zero restframe current.

However, chiral solutions can be obtained for special values of the parameters. As was discussed in Sec. VIII for \( q \ll 1 \) they exist if \( \beta, \theta_w, n, \nu \) belong to the chiral curves shown in Fig.4. These curves are obtained by solving the linear eigenvalue problem (8.3). If \( q \) is not small, then one should solve the full system of equations (4.12)-(4.20) under the condition

\[
\sigma^2(\beta, \theta_w, n, \nu, q) = 0. \tag{11.42}
\]

This condition reduces by one the number of free parameters in the local solutions (5.10), (5.21), so that in order to perform the matching one has to consider \( \theta_w \) as one of the shooting parameters. For given \( n, \nu, q \) this determines curves in the \( (\beta, \theta_w) \) plane. For \( q \to 0 \) these curves reduce to those shown in Fig.4, while for \( q \neq 0 \) they look qualitatively similar but shift upwards. As a result, given a point \( \beta, \theta_w \) in an upper vicinity of a curve in Fig.4, one can adjust \( q \) such that the shifted curve will pass through this point. This fine tuning determines the values of \( \beta, \theta_w, q \) for which there is a solution of Eqs. (4.12)-(4.19) with \( \sigma^2 = 0 \). The chiral solutions are therefore not generic but exist only for values of \( \beta, \theta_w \) which are close to the chiral curves in Fig.4.

Reconstructing the fields (5.6), a given solution with \( \sigma^2 = \sigma_3^2 - \sigma_0^2 = 0 \) determines a family of chiral vortices labeled by \( \sigma_0 = \pm \sigma_3 \), different members of this family being related by Lorentz boosts. These vortices carry a chiral current and support a long-range field so that their energy is infinite. However, there is a distinguished solution with \( \sigma_0 = \sigma_3 = 0 \) for which the current is zero and the energy is finite. This special chiral solution is not a Z string or its gauge copy, since it has \( \Psi_A \neq 0 \) (see Fig.19).

Such solutions remind of the ‘W-dressed Z strings’ discussed some time ago [10, 11]. By analogy with the ‘dressed’ vortices in the Witten model, these were supposed to be Z...
FIG. 19: Profiles of the special chiral solution (left) and Z string (right) for $n = 4$, $\nu = 7$, $\beta = 2$, $g'^2 = 0.23$.

strings stabilized by the W-condensate in the core. However, it seems that this stabilization mechanism does not work in the electroweak case because Z strings are non-topological and can unwind into vacuum [13]. The numerical search for the ‘W-dressed Z strings’ gives a negative result [12] and so it seems they do not exist. Fortunately, as we have seen, the ‘W-dressed Z strings’ are not necessary for constructing the superconducting vortices.

The special chiral solutions cannot be considered as the ‘dressed’ Z strings, first because they do not exist for generic values of the parameters, and secondly because they are not always less energetic than the Z string with the same $\beta, \theta_w, n$. Specifically, for $n = 1$ they indeed have lower energy than the corresponding Z strings, but their parameters $\beta, \theta_w$ lie in this case in the non-physical region (see Fig.4), while if they were the true ‘W-dressed Z strings’ they would exist for any parameter values. The special chiral solutions can also be found in the physical region (see Fig.19), but only for higher values of $n, \nu$ starting from $n = 4$, $\nu = 7$ (see Fig.4). In this case they turn out to be more energetic than the corresponding Z strings.

Although the special chiral solutions are not ‘dressed’ Z strings, one can relate them to Z strings very much in the same way as one interpolates between the ‘bare’ and ‘dressed’ vortices in the Witten model. Given a special chiral solution one can iteratively decrease $q$ relaxing at the same time the $\sigma^2 = 0$ condition. This gives solutions with $\sigma^2 > 0$ which in the $q \to 0$ limit reduce to the Z string. The current $I(q)$ then shows (see Figs.20) the
FIG. 20: The current \( I \) and twist \( \sigma \) against \( q \) (left) and the magnetic energy versus current (right) for the family of the \( n = \nu = 1 \) electroweak vortices for \( \beta = 2 \) and \( \sin^2 \theta_w = 0.56 \) that interpolates between the Z string for \( q = 0 \) and the special chiral solution for \( q = q_* = 0.15 \).

quenched behavior very similar for that for the Witten strings (see Fig. 24), but it should be stressed again that this type of behavior is not generic in the electroweak theory, since it requires the fine-tuning of \( \theta_w \). As is seen in Fig. 20, the energy of the special chiral solution can be higher than for the corresponding Z string.

D. Infinite Higgs mass limit, \( \beta \to \infty \)

In this limit the following constraint is enforced,

\[
\Phi^\dagger \Phi = f_1^2 + f_2^2 = 1, \tag{11.43}
\]

so that one can parametrize the Higgs field amplitudes as

\[
f_1 = \cos \Theta(\rho), \quad f_2 = \sin \Theta(\rho). \tag{11.44}
\]

The equations for the gauge field amplitudes \( u, u_1, u_3, v, v_1, v_3 \) are then given by (4.12), (4.13), (4.16)–(4.19) with \( f_1, f_2 \) defined by (11.44), while the equation for \( \Theta \) reads

\[
\frac{1}{\rho} (\rho \Theta')' = \frac{\sigma^2}{2} (u_1 \cos 2\Theta - u_3 \sin 2\Theta) u + \frac{1}{2\rho^2} (v_1 \cos 2\Theta - v_3 \sin 2\Theta) v. \tag{11.45}
\]
The constraint (4.20) becomes

\[ \sigma^2 (u_1 u'_3 - u_3 u'_1) + \frac{1}{\rho^2} (v_1 v'_3 - v_3 v'_1) - g^2 \Theta' = 0. \tag{11.46} \]

FIG. 21: The \( n = \nu = 1, \beta = \infty \) solutions for \( \sigma = 5 \) (left) and for \( \sigma = 0.1 \) (right). The electric amplitudes are not shown.

The boundary conditions for the function \( \Theta \) for \( \nu = n \) read

\[ \frac{\pi}{2} + a_5 \rho^n + \cdots \leftarrow \Theta \rightarrow \frac{\gamma}{2} + \frac{c_5}{\sqrt{\rho}} e^{-\int m_s d\rho} + \cdots, \tag{11.47} \]

which implies that one always has \( q = f_2(0) = 1 \). One cannot therefore use \( q \) to label the solutions, but instead one can use \( \sigma \). The boundary conditions for the gauge field amplitudes are given by Eqs.(5.10),(5.21), provided that \( \nu = n \). For \( \nu \neq n \) the energy density diverges at the origin due to the term \((v - v_3)^2 f_2^2/\rho^2\) in (4.25). Therefore, solutions with \( \nu \neq n \) become singular for \( \beta \to \infty \), which suggests that they can be viewed as excitations over the fundamental regular solution with \( \nu = n \).

The typical solutions are shown in Fig[21]. As usual, the limit where \( \sigma \) is small corresponds to large currents, however, since \( \Phi^\dagger \Phi = 1 \), the solutions do not show in this limit a region of vanishing Higgs field. This can be seen already from Eq.(10.24), since the size of the \( \Phi = 0 \) region scales as \( \rho_* \sim l/\sqrt{\beta} \) and shrinks to zero when \( \beta \to \infty \).

It seems that for \( \beta = \infty \) there is no upper bound for \( \sigma \). Large \( \sigma \)'s correspond to small currents, the solutions then approaching Z strings. The vacuum angle \( \gamma \) then tends to zero,
while the mass term \( \text{(5.22)} \) approaches infinity (since \( \sigma Q \sim I \to 0 \) and \( m_\sigma \approx \sigma \)) in which case \( \Theta \) changes very fast at small \( \rho \) to approach its asymptotic value. For \( \sigma = \infty \) one should set \( \Theta(\rho) = 0 \), and then the solutions reduce to Z strings ‘in the London limit’.

**XII. SUMMARY AND CONCLUDING REMARKS**

In summary, we have presented new solutions in the bosonic sector of the Weinberg-Salam theory which describe straight vortices (strings) carrying a constant electric current. Such superconducting vortices contain a regular central core filled with the condensate of massive W bosons creating the electric current. The current produces the long-range electromagnetic field. The solutions exist for any value of the Higgs boson mass and for (almost) any weak mixing angle, in particular for the physical values \( \beta \in (1.5, 3.5) \) and \( \sin^2 \theta_w = 0.23 \). They comprise a family that can be labeled by the four parameters in the ansatz (5.6): \( \sigma_\alpha, n, \nu \). These parameters determine the vortex electric charge density \( I_0 \sim \sigma_0 \), the electric current \( I_3 \sim \sigma_3 \), the electromagnetic and Z fluxes \( \Psi_F \) and \( \Psi_Z \), as well as the vortex momentum \( P \sim \sigma_0 \sigma_3 \) and angular momentum \( M \sim \sigma_0 \).

The spacetime vector \( I_\alpha = (I_0, I_3) \sim \sigma_\alpha \) is generically spacelike, so that its temporal component can be boosted away to give \( I_\alpha = \delta_3^\alpha I \) and \( \sigma_\alpha = \delta_3^\alpha \sigma \). The restframe current \( I \) can assume any value. In the \( I \to 0 \) limit the solutions reduce to Z strings, while the twist \( \sigma \) reduces to the eigenvalue of the linear fluctuation operator on the Z string background.

For large currents the solutions show a symmetric phase region of size \( \sim I \) where the magnetic field is so strong that it drives the Higgs field to zero. However, this does not destroy the vector boson superconductivity, since the scalar Higgs field is not the relevant order parameter in this case. The current-carrying W-condensate is contained in the very centre of the symmetric phase, in a core of size \( \sim 1/I \), while the rest of this region is dominated by the massless electromagnetic and Z fields. The symmetric phase is surrounded by the ‘crust’ layer where the Higgs field relaxes to the broken phase while the Z field becomes massive and dies away. Outside the crust there remains only the long-range Biot-Savart magnetic field, which produces a mild, logarithmic energy divergence at large distances for their core. However, finite vortex pieces, as for example vortex loops, will have finite energy.

Straight, infinite vortices can have finite energy in special cases: either for \( \theta_w = 0, \pi/2 \) when the massless fields decouple, or for \( \sigma_\alpha = 0 \), when the current vanishes. The latter
case includes Z strings and also the currentless limit of the chiral solutions with $\sigma_3 = \pm \sigma_0$. However, the chiral solutions are not generic and exist only for special values of $\theta_W$.

The W boson condensate producing the vortex current can be visualized as a superposition of two oppositely charged fluids made of $W^+$ and $W^-$, respectively, flowing in the opposite directions. In the vortex restframe the densities of both fluids are the same, which is why the total momentum through the vortex cross section vanishes while the current does not. Passing to a different Lorentz frame the fluid densities will no longer be equal, which will produce a net momentum. The vortex angular momentum can be explained in a similar way if one assumes that the fluids perform the spiral motions in the opposite directions.

Stability of the new solutions is a very important issue. At first one may think that they should be unstable, since in the zero current limit they reduce to unstable Z strings. However, it seems that current can stabilize them. So far this has been checked only in the $\theta_W = \pi/2$ limit, where the complete stability analysis has been carried out [22]. It turns out then that all negative modes of current-carrying vortices can be removed by imposing the periodic boundary conditions, while one cannot do the same in the zero current limit. It seems that the same conclusion applies also for $\theta_W \neq \pi/2$, although the detailed verification of this is still in progress. However, the main argument is simple enough and goes as follows.

As was mentioned above, the Z string perturbation eigenmodes are proportional to $e^{\pm i(\sigma_0 t + \sigma_3 z)}$ (see Eq. (8.2)) with $\sigma_0 = \sqrt{\sigma_3^2 - \sigma^2}$ where $\sigma^2 > 0$ is the eigenvalue of the spectral problem (8.3). It follows that all modes with $\sigma_3 < \sigma$ are unstable, since $\sigma_0$ is imaginary. Since

$$\lambda = \frac{2\pi}{\sigma_3} > \lambda_{\text{min}} = \frac{2\pi}{\sigma}.$$ (12.48)

it follows that all unstable modes are longer than $\lambda_{\text{min}}$. This suggests that one could eliminate them by imposing $z$-periodic boundary conditions with a period $L \leq \lambda_{\text{min}}$, which would leave ‘no room’ for such modes to exist. However, this would not remove one particular homogeneous mode with $\sigma_3 = 0$, since it is independent of $z$ and so can be considered as periodic with any period.

Now, when passing to current-carrying solutions the situation will be essentially the same. It is clear on continuity grounds that, at least for small currents, the solutions will still have negative modes with the wavelength bounded from below by a non-zero value $\lambda_{\text{min}}$. However, the crucial point is that the homogeneous mode will disappear from the spectrum – simply because the background solutions now have a non-trivial $z$-dependence, implying that all
perturbation modes will depend on $z$ as well. This was checked in [22] for $\theta_w = \pi/2$ and it is very plausible that for generic $\theta_w$ the situation will be the same.

Since the homogeneous mode is absent, imposing periodic boundary conditions with the period $L = 2\pi/\sigma$ will remove all negative modes. This is somewhat similar to the hydrodynamical Plateau-Rayleigh instability of a water column [39] or to the gravitational Gregory-Laflamme instability of black strings in the theory of gravity in higher dimensions [40], which manifest themselves only starting from a certain minimal length. Since $\sigma$ decreases with current, $L$ then grows. For large currents one has $\sigma \sim I^{-3}$ (see the caption to Fig.7) so that the length of the stable vortex segment scales as $I^3$ while its thickness grows as $I$ (see Eq.(10.24)).

There could be different ways of imposing periodic boundary conditions on the vortex segment. One possibility is to bend it and identify the extremities to make a loop. If the vortex carries a momentum $P$ then the loop will have an angular momentum $M$ which may balance it against contraction (see Fig.22). One may therefore conjecture that such electroweak analogues of the ‘cosmic vortons’ [41] could exist and could perhaps be stable – if they are made of stable vortex segments. Of course, verification of this conjecture requires serious efforts, since so far vortons have been constructed explicitly only in the global limit of the Witten model [42]. However, even a remote possibility to have stable solitons in the standard model could be very important, since if electroweak vortons exist, they could be a dark matter candidate.

![FIG. 22: Vortex loops balanced by the centrifugal force (left) and vortex segments in a polarized medium (right).](image)

Another possibility to impose periodic boundary conditions is to attach the vortex ends
to something. It is known that Z strings, since they are not topological, do not have to be always infinitely long but can exist in the form of finite segments whose extremities look like a monopole-antimonopole pair \[16,17\]. This suggests that their current-carrying generalizations could perhaps also exist in the form of finite segments joining oppositely polarized regions of space, similar to thunderbolts between clouds. If some processes during the electroweak phase transition polarize the medium, then the ‘vortex thunderbolts’ could provide a very efficient discharge mechanism – in view of the very large currents they can carry. They could be stable in this case and exist as long as the ‘clouds’ they join are not completely discharged.

There is however an important difference with the ordinary atmospheric thunderbolts, which exist only in a polarized medium and emerge when the large electric field between the clouds creates the plasma channel and works against the resistance to drive the charges through. In superconducting vortices on the other hand the current flows without any resistance and no electric field is needed. In fact, inside the (restframe) vortex the electric field is zero. The vortices do not therefore need any polarization of the surrounding medium to exist.

\[
\begin{align*}
\begin{array}{c}
\gamma \\
I \\
W^+ \\
W^- \\
H \\
Z
\end{array}
\end{align*}
\]

FIG. 23: Disintegration of an unstable vortex loop (left) and of an isolated vortex segment (right).

Finite segments of superconducting vortices can probably be created at high temperatures or in high energy particle collisions. Once created, the current will start to leave the segment through its extremities, so that the latter will emit positive and negative charges (see Fig. 23). Since the current is carried by W bosons, it follows that one vortex end will emit the \(W^+\) while the other the \(W^-\), until the segment radiates away all its energy. In addition, if the initial current is large enough, then the vortex will contain a zero Higgs field region, whose
shrinking will be accompanied by a Higgs boson emission. The vortex segment will therefore end up in a blast of radiation, creating two jets of $W^+$ and $W^-$ and a shower of Higgs bosons. One can similarly argue that if the vortex segment is created in the form of an unstable loop (for example without angular momentum) then it will shrink emitting a shower of neutral bosons – Higgs, Z and photons.

The creation of superconducting vortex segments or loops with their subsequent disintegration can presumably be observed in the LHC experiments. It is also possible that processes of this type could be accompanied by a fermion number non-conservation, since already in the zero current limit the vortices (Z strings) admit the sphaleron interpretation \cite{13}. However, a special analysis is needed to work out the details of these processes and their experimental signatures. We leave this and other possible applications of the superconducting vortices for a separate study.

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**Appendix A. Solutions near the symmetry axis**

The boundary conditions at the origin expressed by Eq.\((5.9)\) imply that at small $\rho$ one has

\[
\begin{align*}
    u &= u(0) + \delta u, \\
    u_3 &= 1 + \delta u_3, \\
    u_1 &= \delta u_1, \\
    v &= 2n - \nu + \delta v, \\
    v_3 &= \nu + \delta v_3, \\
    v_1 &= \delta v_1, \\
    f_1 &= \delta f_1, \\
    f_2 &= f_2(0) + \delta f_2, \\
\end{align*}
\]

(A.1)

where the deviations $\delta u, \ldots, \delta f_2$ vanish for $\rho \to 0$ and $u(0)$ is a free parameter, while $f_2(0)$ is also a free parameter if $\nu = n$, whereas $f_2(0) = 0$ if $\nu \neq n$. Inserting this to Eqs.\((4.12)-(4.20)\) and linearising with respect to the deviations the result is as follows.
Let us first consider the case where \( \nu \neq n \) and \( f_2(0) = 0 \). Then the equations read

\[
(\delta u)'' + \frac{1}{\rho} (\delta u)' = 0, \quad (A.2)
\]
\[
(\delta u_3)'' + \frac{1}{\rho} (\delta u_3)' = 0, \quad (A.3)
\]
\[
(\delta v)'' - \frac{1}{\rho} (\delta v)' = 0, \quad (A.4)
\]
\[
(\delta v_3)'' - \frac{1}{\rho} (\delta v_3)' = 0, \quad (A.5)
\]
\[
(\delta f_1)'' + \frac{1}{\rho} (\delta f_1)' - \frac{n^2}{\rho^2} \delta f_1 = \frac{1}{4} (\sigma^2 (u(0) + 1)^2 - \beta) \delta f_1, \quad (A.6)
\]
\[
(\delta f_2)'' + \frac{1}{\rho} (\delta f_2)' - \frac{(n - \nu)^2}{\rho^2} \delta f_2 = \frac{1}{4} (\sigma^2 (u(0) - 1)^2 - \beta) \delta f_2, \quad (A.7)
\]

plus a system of three coupled equations

\[
(\delta u_1)'' + \frac{1}{\rho} (\delta u_1)' = \frac{\nu}{\rho^2} (\nu \delta u_1 - \delta v_1), \quad (A.8)
\]
\[
(\delta v_1)'' - \frac{1}{\rho} (\delta v_1)' = \sigma^2 (\delta v_1 - \nu \delta u_1), \quad (A.9)
\]
\[
\sigma^2 \delta u_1' + \frac{\nu}{\rho^2} \delta v_1' = 0. \quad (A.10)
\]

From Eqs. (A.2)–(A.5) one finds

\[
\delta u = A_1 + A_2 \ln \rho, \quad \delta u_3 = A_3 + A_4 \ln \rho,
\]
\[
\delta v = A_5 + A_6 \rho^2, \quad \delta v_3 = A_6 + A_8 \rho^2, \quad (A.11)
\]

while solution of Eqs. (A.6), (A.7) can be expressed in terms of the Bessel functions, such that for small \( \rho \) one has

\[
\delta f_1 = A_9 \rho^n (1 + \ldots) + A_{10} \rho^{-n} (1 + \ldots),
\]
\[
\delta f_2 = A_{11} \rho^{n-\nu} (1 + \ldots) + A_{12} \rho^{\nu-n} (1 + \ldots). \quad (A.12)
\]

The three coupled equations (A.8), (A.9), (A.10) can be reduced to one third order equation whose solution can be obtained in terms of the hypergeometric functions. One then obtains for small \( \rho \)

\[
\delta u_1 = A_{13} \rho^\nu (1 + \ldots) + A_{14} \rho^{-\nu} (1 + \ldots) + A_{15},
\]
\[
\delta v_1 = - \frac{\sigma^2}{\nu + 2} A_{13} \rho^{\nu+2} (1 + \ldots) + \frac{\sigma^2}{2-\nu} A_{14} \rho^{-\nu} (1 + \ldots) + \nu A_{15}. \quad (A.13)
\]
Eqs. (A.11)–(A.13) give the most general solution of the equations, since it contains the maximal number of the integration constants: $A_1, \ldots, A_{15}$. Now, we have to suppress all the solutions that do not vanish for $\rho \to 0$, so that we set

$$A_1 = A_2 = A_3 = A_4 = A_5 = A_7 = A_{10} = A_{12} = A_{14} = A_{15} = 0,$$  \hspace{1cm} (A.14)

assuming that $n > \nu > 0$, whereas $A_6, A_8, A_9, A_{11}, A_{13}$ remain free. This gives the local solutions in the linear approximation

\begin{align*}
    u &= a_1, \\
    u_1 &= a_2 \rho^n + \ldots, \\
    u_3 &= 1, \\
    v &= 2n - \nu + a_3 \rho^2, \\
    v_1 &= -\frac{\sigma^2 a_2}{\nu + 2} \rho^{n+2} + \ldots, \\
    v_3 &= \nu + a_4 \rho^2, \\
    f_1 &= a_5 \rho^n + \ldots, \\
    f_2 &= q \rho^{n-\nu} + \ldots
\end{align*}

(A.15)

containing 6 free parameters: $a_1 = u(0), a_2 = A_{13}, a_3 = A_6, a_4 = A_8, a_5 = A_9, q = A_{11}$.

If $\nu = n$, $f_2(0) \neq 0$ then the linearized equations for $\delta u, \ldots \delta f_2$ are more complicated. They split into three coupled groups, whose solutions can be constructed in series. This gives essentially the same result as in (A.15),

\begin{align*}
    u &= a_1, \\
    u_1 &= a_2 \rho^n + \ldots, \\
    u_3 &= 1, \\
    v &= n + a_3 \rho^2 + \ldots, \\
    v_1 &= \frac{g^2 q a_5 - \sigma^2 a_2}{n + 2} \rho^{n+2} + \ldots, \\
    v_3 &= n + a_4 \rho^2 + \ldots, \\
    f_1 &= a_5 \rho^n + \ldots, \\
    f_2 &= q, \hspace{1cm} (A.16)
\end{align*}

which also contains 6 free parameters: $a_1, \ldots a_5$ and $q$. Eqs. (A.15) and (A.16) together imply Eq. (5.10) in the main text.
These local solutions are obtained in the linearized approximation. However, they can be used to take into account all the non-linear terms in the equations when starting the numerical integration at $\rho = 0$. This is achieved by rewriting the second order field equations Eqs. (4.12)–(4.19) in the first order form,

$$y'_r(\rho) = F_r(\rho, y_s), \quad r, s = 1, \ldots, 16 \tag{A.17}$$

where

\[
\begin{align*}
y_1 &= u, & y_2 &= \rho u'_1, & y_3 &= u_1/\rho^\nu, & y_4 &= u'_1/\rho^{\nu-1}, \\
y_5 &= u_3, & y_6 &= \rho u'_3, & y_7 &= v, & y_8 &= v'/\rho, \\
y_9 &= v_1, & y_{10} &= v'_1, & y_{11} &= v_3, & y_{12} &= v'_3/\rho, \\
y_{13} &= f_1/\rho^n, & y_{14} &= f'_1/\rho^{n-1}, & y_{15} &= f_2/\rho^{|n-\nu|}, & y_{16} &= f'_2/\rho^{|n-\nu|-1}
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
y_1(0) &= a_1, & y_2(0) &= 0, & y_3(0) &= a_2, & y_4(0) &= \nu a_2, \\
y_5(0) &= 1, & y_6(0) &= 0, & y_7(0) &= 2n - \nu, & y_8(0) &= 2a_3, \\
y_9(0) &= 0, & y_{10}(0) &= 0, & y_{11}(0) &= \nu, & y_{12}(0) &= 2a_4 \\
y_{13}(0) &= a_5, & y_{14}(0) &= na_5, & y_{15}(0) &= q, & y_{16}(0) &= |n - \nu|q.
\end{align*}
\]

The right hand sides $F_r(\rho, y_s)$ in Eqs. (A.17) are defined only for $\rho > 0$. For example, one has $y'_1 = F_1 = y_2/\rho$ which is undefined at $\rho = 0$. However, using the field equations one can check that $y'_r(0) = 0$ for all $r$. One can therefore set by definition $F_r(\rho = 0, y_s) = 0$, and then one can integrate Eqs. (A.17) with the boundary conditions (A.19) starting exactly at $\rho = 0$, thus avoiding any approximations. Such a procedure is sometimes called desingularization of the equations at a singular point.

**Appendix B. Solutions in the asymptotic region**

Solutions of Eqs. (4.12)–(4.19), (4.21) have to approach the Biot-Savart configuration (5.17) far away from the vortex core. Therefore one has

\[
\begin{align*}
u &= u + \delta u, & u_3 &= -u + \delta u_3, & u_1 &= \delta u_1, \\
v &= c_2 + \delta v, & v_3 &= -c_2 + \delta v_3, & v_1 &= \delta v_1, \\
f_1 &= 1 + \delta f_1, & f_2 &= \delta f_2.
\end{align*}
\]
where
\[ u = Q \ln \rho + c_1, \] (B.2)
the deviations \( \delta u, \ldots, \delta f_2 \) tend to zero as \( r \to \infty \) and \( c_1, c_2, Q \) are the same integration constants as in Eq. (5.17). Inserting (B.1) to Eqs. (4.12)–(4.20) and linearising with respect to the deviations, the resulting linear system reduces to five independent equations plus a subsystem of three coupled equations. The first five equations are the following. The linear combinations
\[ \delta u_Z = \delta u + \delta u_3, \quad \delta v_Z = \delta v + \delta v_3 \] (B.3)
fulfill two independent equations
\begin{align*}
(\delta u_Z)'' + \frac{1}{\rho} (\delta u_Z)' &= \frac{1}{2} \delta u_Z, \\
(\delta v_Z)'' - \frac{1}{\rho} (\delta v_Z)' &= \frac{1}{2} \delta v_Z.
\end{align*}
(B.4)
Their solutions vanishing for \( r \to \infty \) are
\[ \delta u_Z = \frac{c_3}{\sqrt{\rho}} e^{-m_2 \rho} + \ldots, \quad \delta v_Z = c_4 \sqrt{\rho} e^{-m_2 \rho} + \ldots, \] (B.5)
where \( c_3, c_4 \) are integration constants and the dots denote subleading terms. This corresponds to the temporal and azimuthal components of the \( Z \) field with the mass \( m_Z = 1/\sqrt{2} \). The linear combinations
\[ \delta u_A = g^2 \delta u - g^2 \delta u_3, \quad \delta v_A = g^2 \delta v - g^2 \delta v_3 \] (B.6)
correspond to the electromagnetic field and satisfy
\begin{align*}
(\delta u_A)'' + \frac{1}{\rho} (\delta u_A)' &= 0, \\
(\delta v_A)'' - \frac{1}{\rho} (\delta v_A)' &= 0.
\end{align*}
(B.7)
The only solution of these equations that vanishes at infinity is the trivial one,
\[ \delta u_A = \delta v_A = 0, \] (B.8)
since the electromagnetic degrees of freedom are already incorporated into the background configuration (5.17). One has also
\[ (\delta f_1)'' + \frac{1}{\rho} (\delta f_1)' = \frac{\beta}{2} \delta f_1, \] (B.9)
whose solution for large $\rho$ is

$$\delta f_1 = \frac{c_6}{\sqrt{\rho}} e^{-m_\mu \rho} + \ldots,$$

(B.10)

which corresponds to the Higgs boson with the mass $m_\mu = \sqrt{\beta/2}$. Let us now consider the equations for $\delta u_1, \delta v_1, \delta f_2$ obtained by linearising Eqs. (4.15), (4.16), (4.18). The analysis turns out to be more involved in this case. It is convenient to replace one of these three equations by the first integral (4.23), which gives a completely equivalent system. Linearising and defining

$$\chi = \frac{g^2}{2} + \sigma^2 u^2 \equiv m^2$$

(B.11)

also

$$\delta f_2 = \sigma^2 u X + \frac{1}{2} Y, \quad \delta u_1 = g^2 X - u Y$$

(B.12)

the resulting equations read

$$X'' + \left(\frac{1}{\rho} + \frac{\chi'}{\chi}\right) X' - \frac{c_2^2}{\rho^2} + \chi X = \frac{Q}{\rho} Y',$$

(B.13a)

$$(\delta v_1)'' - \frac{1}{\rho} (\delta v_1)' - \chi \delta v_1 = c_2 \chi Y,$$

(B.13b)

$$\frac{c_2}{\rho^2} (\delta v_1)' - \chi Y' - \frac{2g^2 \sigma^2 Q}{\rho} X = \frac{C}{\rho},$$

(B.13c)

where $C$ is the same integration constant as in Eq. (4.23). These three equations are equivalent to one fourth order equation

$$Z''' - 2(\mu^2 Z)' + \left(\mu^4 - 3(\mu^2)'' - \frac{6}{\rho} (\mu^2)' + \frac{4(c_2^2 - 1)}{\rho^4}\right) Z = \frac{c_2^2 (5 + 4 \rho^2 \mu^2)}{4 \rho^{5/2}}$$

(B.14)

provided that

$$\delta v_1 = \int \frac{d\rho}{\sqrt{\rho}} Z, \quad \mu^2 = \chi + \frac{4c_2^2 - 5}{4\rho^2}.$$  

(B.15)

Let us first set $C = 0$. Then the asymptotic form of the two independent solutions that vanish at infinity is given by

$$Z = \left(A_1 \mu^{-5/2} (1 + \ldots) + A_2 \rho^{1/2} \mu (1 + \frac{1}{2\rho \mu} + \ldots)\right) \exp(- \int \mu d\rho).$$

(B.16)

The original field amplitudes can be reconstructed with Eqs. (B.13b), (B.13c), (B.15), (B.12),

$$\delta f_2 = \frac{c_6}{\sqrt{\rho}} e^{-f \mu d\rho} + \ldots, \quad \delta u_1 = \frac{c_7}{\sqrt{\rho}} e^{-f \mu d\rho} + \ldots, \quad \delta v_1 = c_8 \sqrt{\rho} e^{-f \mu d\rho} + \ldots,$$

(B.17)
where $c_6, c_7, c_8$ are expressed in terms of the two independent integration constants $A_1, A_2$, for example $c_8 = -A_2/\sqrt{\mu}$.

Let us now consider the remaining four solutions of Eqs. (B.13) – there should be altogether six independent solutions, since the equations contain five derivatives and one integration constant, $C$. Two additional solutions are obtained by choosing the plus sign in the exponent Eq. (B.16), but they are unbounded and should be excluded from the analysis. Yet another solution, always for $C = 0$, is given by $X = 0$, $\delta v_1 = -c_2 Y = \text{const.}$, which does not satisfy the asymptotic conditions and should be excluded as well.

The last solution is obtained by setting $C \neq 0$, which gives rise to a solution that behaves as $Z = C c_2 / (\mu^2 \sqrt{\rho})$ for large $\rho$, so that $\delta v_1 \sim C \ln \ln \rho$. Let us call this solution $C$-mode. One can think at first that it should also be excluded. However, the constraint equation (4.20) requires that $C = 0$, and so, even if the $C$-mode is included, it will be eventually removed by the constraint. The constraint is explicitly imposed on local solutions at small $\rho$, which is sufficient to insure that it holds everywhere, so that it would be redundant to impose it again. One can therefore add the $C$-mode to (B.17), since at the end, when the matching is performed to construct the global solutions, it will be removed anyway.

Since this mode will finally disappear, one may wonder why one should consider it at all. However, including it is essential, since it increases the dimension of space of local solutions. From the technical viewpoint this mode is not very convenient, though, because it is not localized. However, since its main effect is to increase the dimension of function space, any other mode with a non-zero overlap with it can actually be employed. Therefore, we simply use (B.17) with $c_6, c_7, c_8$ considered as three independent parameters, which leads to a good numerical convergence. In fact, these coefficients contain $\mu \sim \ln \rho$, but since this is a slowly varying function, one can treat $c_6, c_7, c_8$ as if they were constants. In addition, without changing the leading asymptotic terms, one can replace $\int \mu d\rho$ in Eq. (B.17) by $\int m_\sigma d\rho$.

Summarizing, the large $\rho$ behaviour of the solutions is given by

$$
\begin{align*}
    u &= Q \ln \rho + c_1 + \frac{c_3 g^2}{\sqrt{\rho}} e^{-m_2 \rho} + \ldots, & u_3 &= -Q \ln \rho - c_1 + \frac{c_3 g^2}{\sqrt{\rho}} e^{-m_2 \rho} + \ldots, \\
    v &= c_2 + \frac{c_4 g^2}{\sqrt{\rho}} e^{-m_2 \rho} + \ldots, & v_3 &= -c_2 + \frac{c_4 g^2}{\sqrt{\rho}} e^{-m_2 \rho} + \ldots, \\
    f_1 &= 1 + \frac{c_5}{\sqrt{\rho}} e^{-m_1 \rho} + \ldots, & f_2 &= \frac{c_6}{\sqrt{\rho}} e^{-\int m_\sigma d\rho} + \ldots, \\
    u_1 &= \frac{c_7}{\sqrt{\rho}} e^{-\int m_\sigma d\rho} + \ldots, & v_1 &= c_8 \sqrt{\rho} e^{-\int m_\sigma d\rho} + \ldots.
\end{align*}
$$

(B.18)
where \( Q, c_1, \ldots, c_8 \) are 9 independent parameters and \( m_\sigma \) is defined in (B.11). This gives rise to Eq. (5.21) in the main text.

### Appendix C. Superconducting strings in the Witten model

Numerical construction of Witten’s superconducting strings was performed by many authors [3] (see [5] for a review). In this Appendix we reproduce these solutions within the same approach and using the same notation as in the main text. We tried to make this Appendix self-consistent in order to illustrate our procedure on a relatively simple example.

The \( U(1) \times U(1) \) Witten model is defined by the Lagrangian [1]

\[
\mathcal{L}_W = -\frac{1}{4} \sum_{a=1,2} F^{(a)}_{\mu\nu} F^{(a)\mu\nu} + \sum_{a=1,2} (D_\mu \phi_a)^* D^\mu \phi_a - U, \tag{C.1}
\]

with \( F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu \) and \( D_\mu \phi_a = (\partial_\mu - ig_a A^{(a)}_\mu) \phi_a \) where \( g_a \) are two gauge coupling constants. The scalar field potential

\[
U = \frac{\lambda_1}{4} (|\phi_1|^2 - \eta_1^2)^2 + \frac{\lambda_2}{4} (|\phi_2|^2 - \eta_2^2)^2 + \gamma |\phi_1|^2 |\phi_2|^2 - \frac{\lambda_2}{4} \eta_2^4 \tag{C.2}
\]

vanishes for \( |\phi_1| = 1 \) and \( \phi_2 = 0 \), and this minimum is global if \( 4\gamma^2 > \lambda_1 \lambda_2 \). The perturbative mass spectrum of the field excitations around this vacuum contains two scalar bosons with masses \( m_1^2 = \lambda_1 \eta_1^2 \) and \( m_2^2 = \gamma \eta_1^2 - \frac{1}{2} \lambda_2 \eta_2^2 \) as well as a vector boson with the mass \( m_v^2 = 2g_1^2 \eta_1^2 \) and a massless vector boson. Pairs \((A^{(1)}_\mu, \phi_1)\) and \((A^{(2)}_\mu, \phi_2)\) are sometimes called vortex fields and condensate fields, respectively.

Passing to cylindrical coordinates, where \( x + iy = \rho e^{i\varphi} \), and making the stationary, axially symmetric ansatz

\[
A^{(1)}_\mu dx^\mu = \frac{1}{g_1} (n - v(\rho)) d\varphi, \quad \phi_1 = f_1(\rho) e^{i n \varphi}, \tag{C.3a}
\]

\[
A^{(2)}_\mu dx^\mu = \frac{1}{g_2} (\sigma_0 dt + \sigma_3 dz) (1 - u(\rho)), \quad \phi_2 = f_2(\rho) e^{i \sigma_0 t + i \sigma_3 z}, \tag{C.3b}
\]

with \( n \in \mathbb{Z} \), the field equations

\[
\partial^\mu F^{(a)}_{\mu\nu} = 2g_a \Re (i \phi^*_a D_\nu \phi_a), \tag{C.4}
\]

\[
D_\mu D^\mu \phi_a = - \frac{\partial U}{\partial \phi^{*}_a}. \tag{C.5}
\]
reduce to
\[ \frac{1}{\rho} (\rho f'_1)' = \left( \frac{v^2}{\rho^2} + \frac{\lambda_1}{2} (f^2_1 - \eta^2_1) + \gamma f^2_1 \right) f_1, \]  
(C.6a)
\[ \frac{1}{\rho} (\rho f'_2)' = \left( \sigma^2 u^2 + \frac{\lambda_2}{2} (f^2_2 - \eta^2_2) + \gamma f^2_1 \right) f_2, \]  
(C.6b)
\[ \rho \left( \frac{v'}{\rho} \right)' = 2g^2_1 f^2_1 v, \]  
(C.6c)
\[ \frac{1}{\rho} (\rho u')' = 2g^2_2 f^2_2 u, \]  
(C.6d)
where \( \sigma^2 = \sigma^2_3 - \sigma^2_0 \). Depending on sign of \( \sigma^2 \), solutions of these equations are called magnetic \((\sigma^2 > 0)\), electric \((\sigma^2 < 0)\), or chiral \((\sigma^2 = 0)\) [4].

Equations (C.6) have singular points for \( \rho = 0, \infty \). Constructing their local series solutions in the vicinity of these points gives the boundary conditions for \( 0 \leftarrow \rho \rightarrow \infty \),

\[ 1 + \cdots \leftarrow u \rightarrow c_1 + Q \ln \rho + \ldots, \]
\[ n + a_2 \rho^2 + \cdots \leftarrow v \rightarrow c_2 \sqrt{\rho} e^{-m_2 \rho} + \ldots, \]
\[ a_1 \rho + \cdots \leftarrow f_1 \rightarrow 1 + \frac{c_3}{\sqrt{\rho}} e^{-m_1 \rho} + \ldots, \]
\[ q + \cdots \leftarrow f_2 \rightarrow \frac{c_4}{\sqrt{\rho}} e^{-\int m_\sigma d\rho} + \ldots, \]  
(C.7)
where \( a_1, a_2, q, Q, c_1, \ldots c_4 \) are integration constants and

\[ m^2_\sigma = m_2^2 + \sigma^2 (c_1 + Q \ln \rho)^2. \]  
(C.8)
These boundary conditions imply that the fields are regular at the symmetry axis, while at infinity the scalars approach vacuum, the massive vector vanishes, whereas the massless vector becomes the Biot-Savart field produced by the string current. The current density is

\[ J_\nu = \partial^\mu F^{(2)}_{\mu\nu} = 2g_2 \Re (i\phi_2^* D_\nu \phi_2), \]  
(C.9)
integrating which over the \( x, y \) plane gives

\[ I_\alpha = \int J_\alpha \, d^2 x = -\frac{2\pi Q \sigma_\alpha}{g_2}, \]  
(C.10)
with \( \alpha = 0, 3 \). Here \( I_0 \) is the charge per unit vortex length while \( I_3 \) is the total current through the vortex cross section. As Eq. (C.8) shows, the long-range Biot-Savart field affects the mass of the second scalar. For \( \sigma^2 \geq 0 \) this only improves the field localization, but
for $\sigma^2 < 0$ the $m_2^2$ term becomes negative at large distances, making the condensate field oscillate at infinity, which renders the energy strongly divergent. Such solutions are more difficult to handle, and so we shall concentrate on the magnetic and chiral cases. Since the structure of the fields (C.3) is invariant with respect to Lorentz boosts along the z-axis, this can be used in the magnetic case to pass to the rest frame where $I_\alpha = \delta_\alpha^3 I$ with $I = -2\pi Q\sigma/g_2$ where the ‘twist’ $\sigma = \sqrt{\sigma^2}$.

To solve equations (C.6) the relaxation method is usually employed [3], [5]. Within this method it is very natural to find first of all the ‘dressed’ currentless solution with $u = \sigma^2 = 0$, $f_2 \neq 0$. After this one can increase $\sigma^2$ to obtain current-carrying strings with $u \neq 0$.

However, we use instead the multiple shooting method in which the local solutions in (C.7) are numerically extended to an intermediate region where the matching conditions are imposed. To fulfil the 8 matching conditions for the 4 functions and their first derivatives we have on our disposal 9 free parameters: 8 integration constants in (C.7) and also $\sigma$. This leaves after the matching one free parameter to label the global solutions in the interval $\rho \in [0, \infty)$. We choose this parameter to be the value of the condensate scalar at the origin, $q = f_2(0)$. Within our approach it is then natural to start not from the ‘dressed’ solution that is apriori unknown, but from the known ANO vortex.

We therefore set $q = 0$ and then one has $f_2 = 0$, $u = 1$ everywhere so that the condensate fields vanish. The remaining equations reduce to the ANO system whose solution is the ANO vortex [2]. For $q \ll 1$ one can expect the solution to be its small deformation that can be considered within the linear perturbation theory. Linearising the equations with respect to the deformations the ANO background, one can see that it is consistent to set the deformations of $u, v, f_1$ to zero, while the deformation of $f_2$ satisfies

$$\frac{1}{\rho} (\rho \delta f_2')' = \left( \sigma^2 - \frac{\lambda_2}{2} \eta_2^2 + \gamma f_1^2 \right) \delta f_2. \quad \text{(C.11)}$$

This can be viewed as the spectral problem with the eigenvalue $\sigma^2$. This problems admits a bound state with $\sigma^2 > 0$, which leads to the dispersion relation

$$\sigma_0 = \sqrt{\sigma_3^2 - \sigma^2}. \quad \text{(C.12)}$$

This interpretation of this is two-fold. If the wave-number $\sigma_3$ is small, $\sigma_3^2 < \sigma^2$, then the frequency $\sigma_0$ is imaginary, implying that the ANO vortex within Witten’s model is unstable [1]. On the other hand, for $\sigma_3^2 \geq \sigma^2$ the frequency is real and the solution of (C.11) describes a small stationary deformation of the ANO vortex by the condensate.
The perturbative bound state can be used as the starting configuration to iteratively construct the solution of the full non-linear system, still for small $q$. After this one can iteratively increase $q$, which gives the generic solutions. The typical solution is shown in Fig.24. One finds then that the restframe current $I$ first increases with $q$ but then starts to quench, and finally both $I$ and $\sigma$ (but not $Q$) vanish for some $q_*$ as shown in Fig.24.

![Diagram](image-url)

**Fig. 24:** Left: profiles of a typical current-carrying solution of Eqs.(C.6) for $\lambda_1 = 0.1$, $\lambda_2 = 10$, $\eta_1 = 1$, $\eta_2 = 0.31$, $g_1 = g_2 = 1$, $\gamma = 0.6$ and $\sigma = 0.12$. Right: the restframe current $I$ and the twist $\sigma$ against the parameter $q$ for the same parameter values as in the left panel.

It is worth reminding that in the Weinberg-Salam theory we had a completely different picture, as shown in Fig.6 – when $q$ approaches its maximal value $q_*$ the current $I(q)$ does not quench but bends back to smaller values of $q$ and continues to grow.

Getting back to Witten’s model, the solution for $q = q_*$ (it looks qualitatively similar to other solutions) has $\sigma^2 = 0$ and so it is chiral, with values of $\sigma_0 = \pm \sigma_3$ related to each other by Lorentz boosts. In this case one has $I = 0$ but the current $I_\alpha$ does not actually vanish and becomes an isotropic vector. The particular member of the chiral family is obtained for $\sigma_0 = \sigma_3 = 0$, in which case one has $I_\alpha = 0$ so that the solution is truly currentless, although it contains a non-trivial condensate. This is the ‘dressed’ string.

Summarizing, when $q$ increases, the Witten strings interpolate between the ‘bare’ ANO vortex for $q = 0$ and the ‘dressed’/chiral vortex for $q = q_*$. Since the restframe current vanishes at the ends of the interval $[0,q_*]$, it passes through a maximum somewhere in
between and so there is an upper bound for it. The magnetic energy, which coincides with the total energy for the currentless strings, decreases along the interpolating sequence, as shown in Fig. 25. The ‘dressed’ vortex is therefore less energetic than the ‘bare’ one.

FIG. 25: Magnetic energy (left) and the restframe current $I$ and $q = f_2(0)$ (right) against the twist $\sigma$ for the same parameter values as in Fig. 24.

Within the standard approach based on the relaxation method the same solutions are obtained in the opposite order. First one finds the ‘dressed’ solution with $q = q_\ast$, $\sigma = 0$, and then one increases the twist $\sigma$. The functions $q(\sigma)$, $I(\sigma)$ then exhibit the characteristic behaviour shown in Fig. 25 (which agrees with Fig. 5.3 in Ref. [5] where $\sigma$ is called ‘winding number density’). The condensate then always decreases, while the current passes through a maximum whose value can be related to the parameters of the ‘dressed’ solution [5].

This scheme does not apply in the Weinberg-Salam theory, where the ‘dressed’ currentless solutions do not generically exist. We are therefore bound to start from the embedded ANO vortex with $q = 0$. Increasing $q$ we find that its maximum value $q_\ast$ exists, although it corresponds not to the ‘dressed’ currentless solution but to a turning point after which $q$ starts decreasing again, while the current continues to grow.

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