A constructive proof presenting languages in $\Sigma^P_2$ that cannot be decided by circuit families of size $n^k$.

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1 The long and complicated history of this result

I proved this result in 1998 when I was a doctoral student at Rutgers University in New Jersey; the lecturer in charge of the Complexity Theory course, Professor Eric Allender, presented a non-constructive proof of this result, but said that, at the time, he did not know of any constructive proof of the result. He promised an “instant A” in the course (198:538 Complexity Theory) to any student who could constructively prove this result.

I devised this constructive proof of the result at the time. As far as I know, I was the only student in the class to succeed in constructively proving this result. I wrote up a draft of the result, nearly in form suitable for publication, printed it out and presented it to Allender: It was eight pages long, including the cover sheet. There were a few minor mistakes, which I corrected by hand on the printed-out result, and an incomplete reference to some chapters by Allender, Louis and Regan in the CRC Handbook of Algorithms and Theory of Computation. However, because of various personal circumstances at the time, I did not have the opportunity to attempt to publish it.

I left the USA and moved to the UK in 1999, and, again, because of personal circumstances (I went through a period of financial hardship), did not have the opportunity to attempt to publish it. In 2001 I was fully occupied with an IT job in the UK unrelated to Complexity Theory, and hence did not have the time to attempt to publish the result. At that time, I incorrectly assumed that I did not have a copy of the result with me; I had left behind several suitcases of personal possessions with friends in New Jersey, and I wrongly assumed that I had left all copies of the result in New Jersey.

However, I managed to obtain a Council flat in the UK in November 2002 (I was again unable to find a job, and wanted the financial security of living in social housing), and so had more financial stability, but, by this time, I had almost completely forgotten about the result that I proved in 1998 in New Jersey: I was a bit occupied with looking for work, and carrying out some unpaid research in areas unrelated to Complexity Theory.
Then, in January 2004, I became curious about the fate of this result; I e-mailed Allender, and was told that the same result had been proved by Cai and Watanabe and presented at COCOON 2003\cite{CW03}; apparently a copy of my result had been faxed to the COCOON 2003 conference and shown to Cai and Watanabe, who agreed to acknowledge it in a subsequent publication. It appears that Cai and Watanabe’s COCOON 2003 paper was also published as technical report C-161 at Tokyo Institute of Technology\cite{CW02}. I do not remember getting any response from Cai and Watanabe for several years after this; in 2008, however, in response to a bit of pressure to rewrite my CV, I e-mailed Allender about this again. It appears that Cai and Watanabe responded by withdrawing technical report C-161 and replacing it with report C-256\cite{CW08}. Technical report C-256 acknowledges my result.

However, Allender claimed to have only ever had a hard-copy of the result (he e-mailed me a copy of the faxed image in 2004, if I remember rightly), and so the only way that I could have published it properly at the time would have been to re-typeset the whole result, which seemed to be a bit tedious; the mediocre resolution of the faxed image only compounded the problem.

I ultimately moved back to New Zealand (I originally graduated from Massey University in 1993 and then the University of Auckland in 1998) in November 2008, but was still unaware of the fact that I had the \LaTeX{} source of the result until a few weeks ago (in August 2014).

I have five Iomega Zip cartridges of data from my time as a student at Rutgers, but, since I ceased to have easy access to an Iomega Zip drive when I left Rutgers in 1999, and I had never owned a Zip drive until a few weeks ago, I was unaware of the fact that one of the Zip cartridges contained the \LaTeX{} source for my 1998 result. The fate of the \LaTeX{} source of my 1998 result was still unclear to me until a few weeks ago.

I received an Iomega Zip drive (for free) from a programmer, Ed Church, here in Auckland, a few weeks ago. I installed it in one of the old Linux computers that I use, and looked at the contents of the five cartridges, expecting to find some interesting old files, but certainly not expecting to find my 1998 result! I was amazed to find the complete \LaTeX{} source of the old 1998 result on one of the Zip cartridges. (Note added later: Since finding the original 1998 result on a Zip cartridge, I found a mostly-complete attempt to re-typeset the result, dated 21 July 2005, in a collection of files copied from the hard drive of the computer that I had in the UK at the time. I now vaguely recall corresponding with Allender about attempting to publish the result at the time, but I don’t clearly remember why I did not complete this attempt to re-typeset the result at the time; I remember getting involved in an unpaid IT project for a local business at about that time).

My result from 1998 (the version from the original \LaTeX{} source) is presented here, with the minor mistakes corrected, and the references fixed.

2 Acknowledgements

Thanks to Professor Eric Allender for originally making me aware of this, at the time, unsolved problem back in 1998. Thanks for his assistance with my apparent previous attempt to publish it in 2005.

Thanks to Professors Jin-Yi Cai, at the University of Wisconsin at Madison,
and Osamu Watnabe at Tokyo Institute of Technology for acknowledging this result in 2008. Thanks also to Cai and Watanabe for giving this result a name (in the bibliography of [CW08]), which I have used as inspiration for the title of this article: my original unpublished manuscript in 1998 had no title. (I feel that my title more accurately reflects the exact form in which I state my result in section 1). And thanks finally to Ed Church, a programmer here in Auckland, for giving me the Iomega Zip drive which enabled me to re-discover and retrieve the original \LaTeX source of the result.

3 The Result to be Proved

Theorem 1. Take any $k \in \mathbb{N}$. Then there is a language $L_k \in \Sigma^P_2$ (with $L_k \in \{0,1\}^*$) that is not decidable by any circuit family of size $n^k$.

Proof: We will define $L_k$ in terms of two other languages $\Gamma$ and $\Lambda_k$. We will define the language $\Gamma$ in section 5. We will then define the language $\Lambda_k$ in section 10, after we have made some necessary preliminary definitions.

4 The first $q$ strings of length $n$:

For any $n \in \mathbb{N}$, it is obviously possible to list all of the strings over $\{0,1\}$ in increasing lexicographic order. For example, if $n = 3$, the following is a list of all of the strings over $\{0,1\}$ of length $n$ in increasing lexicographic order:

\[
000, 001, 010, 011, 100, 101, 110, 111
\]

Now, for any $q \in \{0,1,\ldots,2^n\}$, we can form the set of the first $q$ strings from this list of all of the strings in $\{0,1\}^n$. Let $\diamondsuit^q_n$ denote the set formed in this way. For example:

\[
\diamondsuit^3_3 = \{000, 001, 010, 011, 100\}
\]

5 The language $\Gamma$:

Let $\Gamma$ be the language defined by:

\[
\Gamma = \left\{ 0x \mid x \in \{0,1\}^* \text{ \ is a satisfiable Boolean expression} \right\}
\]

Here, we assume the existence of some intuitively-simple method of representing Boolean expressions by bit strings. In section 8, we will give a particular example of such a mapping. We will then assume that the mapping used in the definition of $\Gamma$ is the mapping given in section 8. Now, it is a well-known fact that the satisfiability problem is in NP, so there is a non-deterministic polynomial-time Turing Machine $M_\alpha$ to solve it. It is obviously possible to modify the machine $M_\alpha$ to discard the initial zero before starting the task of solving the satisfiability problem. Hence, there is a non-deterministic polynomial-time (and hence certainly $\Sigma^P_2$) Turing machine $M_\beta$ that accepts $\Gamma$.  

3
6 A problem that can be solved with a non-deterministic Turing Machine and a satisfiability circuit

Let $B'$ be any language over the alphabet $\{0,1\}$ in which all of the strings are of the same length $n \in \mathbb{N}$. Then define the language $K$ as follows:

$$K = \left\{ (1^n, 1^m, B') \mid n, m \in \mathbb{N}, n \leq m, B' \subseteq \hat{\hat{\bigwedge}}_m^n, \exists \text{n-input circuit } C' \text{ of size } m, L(C') \cap \hat{\hat{\bigwedge}}_m^n = B' \right\}$$

Here, $B'$ is represented in the input string as a comma-delimited list of all of the strings in $B'$. We claim that:

Claim 1 The language $K$, as defined above, is Karp-reducible to the satisfiability problem. (A definition of Karp-reducibility is given by, for example, Allender et al[ALR98]).

Now, it should be fairly obvious that the following algorithm is in NP and accepts $K$, and hence $K \in \text{NP}$:

1. Check that $n \leq m$. If we find that $n \not\leq m$, we reject the input.
2. Run through all of the strings in $B'$, checking that each one is in $\hat{\hat{\bigwedge}}_m^n$ (i.e. checking that each one is of length $n$ and lexicographically precedes the binary representation of $m$). If we find some string $B'$ to not be in $\hat{\hat{\bigwedge}}_m^n$, we reject the input.
3. Existentially guess a circuit $C'$ of size $m$.
4. Run through all of the strings $x \in \hat{\hat{\bigwedge}}_m^n$, checking that, for each such string $x$, $x \in B'$ if and only if $C'$ accepts $x$. If we find some string $x \in \hat{\hat{\bigwedge}}_m^n$ that does not satisfy this condition, then we reject the input. Otherwise, we accept the input.

Now, there is a well-known result called Cook’s Theorem that states that every language in NP is Karp-reducible to the satisfiability problem. For a proof of Cook’s Theorem, see, for example, Allender et al[ALR98]. The proof of Cook’s Theorem given by Allender, et. al[ALR98] is constructive; it explicitly gives a procedure for constructing the “transformation function” $\tau$ referred to in the definition of Karp-reducibility. Part of the definition of Karp-reducibility implies that some polynomial function $p(|x|)$ is an upper bound on the length of $\tau(x)$, for any string $x \in \{0,1,\ldots\}^*$ of length $n$. But, since we assume that $n \leq m$, the lengths of the strings $1^n, 1^m$ and $B'$ are all polynomial in $m$. Hence, the length of $x$ is also polynomial in $m$. Since the composition of two polynomial functions is also a polynomial function, this implies that there is a polynomial

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1 I called the transformation function $\tau$ in my original 1998 version of this result: I presume that this was the name for the transformation function used in lectures. However, it appears that Allender et al’s article[ALR98] uses the name $f$ for the transformation function. I prefer to stick with my original name, $\tau$, for the transformation function.

2 i.e. the comma-delimited list of all elements in $B'$
function $q$ for which $q(m)$ is an upper bound on the length of $\tau((1^n,1^m, B'))$, for any $(1^n,1^m, B') \in K$. Since $q$ is a polynomial function, there is some $\delta \in \mathbb{N}$ for which $q(m) \leq m^\delta$ for all $m \geq 2$.

7 Recursive Definition of the Satisfiability Problem

In this section, we will discuss a property of the satisfiability problem that will later be important to our proof. In order to do this, we will start by giving a formal definition of the satisfiability language SAT. This language is defined to be the language over $\Sigma_B = \{0, 1, (,), \lor, \land, v\}$ for which, for any $x \in \Sigma_B^*$:

1. Boolean expressions are constructed from the symbols in $\Sigma_B$. $\lor$, $\land$, ( and ) denote disjunction, conjunction and precedence in the usual way. Variables are denoted by strings of 0s and 1s following the symbol $v$ (e.g. “$v101$”, “$v11001$”, “$v0011$”). Literals are denoted by 1 and 0, for “true” and “false”, respectively. For any $x \in \Sigma_B^*$ that is not a well-formed Boolean expression, $x \not\in$ SAT.

2. For any $x \in \Sigma_B^*$ that represents a well-formed Boolean expression, $x$ is defined to be in SAT if and only if this Boolean expression is satisfiable.

Now, given this definition, and the intuitively-obvious properties of the satisfiability problem, it should be clear that SAT is the unique language $Q$ over $\Sigma_B$ that satisfies the following three properties:

1. For any $x \in \Sigma_B^*$ that does not represent a well-formed Boolean expression, $x \not\in Q$.

2. For any $x \in \Sigma_B^*$ that represents a well-formed Boolean expression with no variables, $x \in Q$ if and only if $x$ evaluates to true.

3. For any $x \in \Sigma_B^*$ that represents a well-formed Boolean expression with variables, let $v_1$ be the variable in $x$ that has the lexicographically-first name. Let $x_0$ be the expression resulting from substituting the literal “0” for every occurrence of $v_1$ in $x$. Similarly, let $x_1$ be the expression resulting from substituting the literal “1” for every occurrence of $v_1$ in $x$. Then $x \in Q$ if and only if either $x_0 \in Q$ or $x_1 \in Q$.

8 Binary representation of Boolean expressions:

In section 4, we represented Boolean expressions as strings over the alphabet:

$$\Sigma_B = \{0, 1, (,), \lor, \land, v\}$$

This is actually a special case of a property of languages known as self-reducibility. For a definition of self-reducibility, see, for example, Ko[1983] (Note added in 2014: I don’t clearly remember exactly which definition of self-reducibility I was using in 1998: it appears that I incorrectly assumed that it was defined in Allender et al[1998] at the time. Presumably I was using a definition given in lectures. However, Ko[1983] gives several different precise definitions of self-reducibility, including $d$-self-reducibility, which he claims SAT to satisfy. It appears that $d$ in this context stands for “disjunctive”).
Clearly, then, Boolean expressions can be represented as bit strings just by replacing each of the symbols in $\Sigma_B$ by a different three-bit string. For example, we could use the following mapping:

| Symbol in $\Sigma_B$ | Replacement Bit String |
|----------------------|------------------------|
| 0                    | 000                    |
| 1                    | 001                    |
| (                    | 010                    |
| )                    | 011                    |
| $\lor$               | 100                    |
| $\land$              | 101                    |
| $\forall$            | 110                    |

Note that the three-bit sequence 111 does not represent any symbol of $\Sigma_B$. This allows us to use a string of 1s to pad out the binary representation of any given string over $\Sigma_B$ to any desired length (whether or not the desired length is a multiple of three).

We will assume, from here on, that this method of representing Boolean expressions is used in the definition of $\Gamma$ in section 5.

9 Upper bound on number of circuits of given polynomial size

We will refer to the number of gates in a given circuit as the *size* of that circuit. Now, fix any natural number $\eta$. Now, take any natural number $n$, and consider the task of constructing an arbitrary $n$-input circuit of size at most $n^\eta$. There are clearly at most $(n^\eta)^2 = n^{2\eta}$ different (source,destination) pairs for wires in the circuit. Hence, a circuit can be described by:

1. Choosing one of the $2^{n^{2\eta}}$ different subsets of the (source,destination) pairs for the wires in the circuit.

2. Choosing, for each of the $\leq n^\eta$ non-input gates, one of a finite number $c$ of types (and, or, not, etc) for the gate. Clearly there are at most $c^{n^\eta}$ different ways in which this can be done.

Therefore, the total number of ways in which the circuit can be constructed is at most:

$$c^{n^\eta}2^{n^{2\eta}} \leq c^{n^\eta}2^{n^{2\eta}} = (2c)^n2^{n^\eta}$$

Hence, there is clearly some natural number $\mu$ for which, for all $n \geq 2$, there strictly fewer than $2^n\mu$ ways of constructing a circuit of size $n^\eta$.

10 The language $\Lambda_k$:

Let $k$ be any given natural number. Clearly there is some natural number $\eta$ for which $n^\eta \geq (n+1)^k$ for every natural number $n \geq 2$. Let $\mu$ be the number $\mu$ derived from $\eta$ in the way described in section 5. Now, let $\Lambda_k$ be the language accepted by a $\Sigma_2$ Turing machine $M_{\Lambda_k}$. The machine $M_{\Lambda_k}$ is defined to operate as follows:
1. Check that the input string is of the form $1x$, where $x$ is a string of binary digits. If the input string is not of this form, then halt and reject. Clearly this step can be performed in time linear (and hence polynomial) in the input length. Let $l$ denote the length $|x|$ of $x$.

2. Existentially guess a circuit $G$ of size $(l^\delta + 1)^k$ with $l^\delta$ inputs. Here, $\delta$ is as defined in section 6, under the assumption that $m$ is always equal to $l^\mu$. Since $\delta$, $\mu$ and $k$ are all independent of $l$, and the input length to $M_\Lambda$ is $l + 1$, then the amount of time taken to existentially guess $G$ is polynomial in the length of the input to $M_\Lambda$.

3. Existentially guess a subset $B_l$ of $\mathcal{O}_\mu$. Since $\mathcal{O}_\mu$ contains $l^\mu$ strings, each of length $l$, the amount of time required for this existential guessing step is proportional to $l^\mu$, which is polynomial in $l$.

4. Perform a four-outcome universal choice operation. Depending upon the outcome of the universal choice, branch ahead to either step 5, step 7, step 9 or step 10.

5. Universally guess a circuit $C$ of size $(l + 1)^k$. This universal guessing step will clearly require time at most proportional to $(l + 1)^k$ to guess the types (and, or, not, etc) of the gates, plus an additional time at most proportional to $(l + 1)^{2k}$ to determine the pairs $A, B_l$ of gates for which to run a wire from $A$ to $B_l$. Hence, the time required for this universal guessing step is at most polynomial in the size of the input to $M_\Lambda$.

6. Check that, for the circuit $C$, $L(C) \cap \mathcal{O}_\mu$ is not equal to $B_l$. To do this, all we have to do is run through all of the strings in $\mathcal{O}_\mu$, and check that one of them is either in $B_l$ but not in $L(C)$, or in $L(C)$ but not in $B_l$. Now, $\mathcal{O}_\mu$ contains $l^\mu$ strings, each of which is of length $l$. So computing all of the strings in $\mathcal{O}_\mu$ requires time at most proportional to $l^\mu$. But, for each string $y \in \mathcal{O}_\mu$, we perform two checks:

   (a) Check whether $y \in B_l$. If we represent $B_l$ as the list of all of its elements, then we can use sequential search to check whether or not $y \in B_l$. Since $B_l$ is a subset of $\mathcal{O}_\mu$, then the list representation of $B_l$ will be of length at most $l^\mu$. Hence, the time required for the search will be at most polynomial in $l$.

   (b) Check whether $y \in L(C)$. This just requires the circuit $C$ to be simulated for the input $y$. Clearly this can be done in time at most proportional to the number of wires in $C$, i.e. at most proportional to the square of the size of $C$. Since the size of $C$ is at most proportional to $(l + 1)^k$, the square of the size of $C$ is at most proportional to $(l + 1)^{2k}$, which is polynomial in $l$.

So, this step involves a polynomial number of checks, each of which takes polynomial time. So the total time required for this step is polynomial in the size of the input to $M_\Lambda$. If it turns out that $L(C) \cap \mathcal{O}_\mu$ is not equal to $B_l$, then we halt and accept. If, on the other hand, $L(C) \cap \mathcal{O}_\mu = B_l$, then we halt and reject.

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4We regard the size of a circuit as the number of gates that it contains.
7. Universally guess a subset $B'$ of $\mathcal{B}_{lu}$ that lexicographically precedes $B_l$. Since $\mathcal{B}_{lu}$ contains $ll^u$ strings, each of which is of length $l$, the amount of time required to perform this universal guessing operation is proportional to $ll^u$ (clearly one pass through $B_l$ and $B'$ is sufficient to determine that $B'$ lexicographically precedes $B_l$).

8. Take the deterministic polynomial-time Turing Machine for computing the function $\tau$ mentioned in section 6. Simulate the execution of this Turing Machine on the input $(1^l, 1^l, B')$. (We established in section 6 that the function $\tau$ can be computed by a deterministic Turing machine in time polynomial in its input length. Since $B' \subseteq 1^l$, it is clearly also true that the size of the argument to $\tau$ is polynomial in the length of the input to $M_{\Lambda_k}$. Hence, this simulation can be performed in time polynomial in $M_{\Lambda_k}$’s input length. Since $l \leq l^u$ and $B' \subseteq 1^l$, $\tau((1^l, 1^l, B'))$ will be satisfiable if and only if there is some $n$-input circuit $C'$ of size $m$ for which $L(C') \cap 1^l = B'$. Now, since the second argument to $\tau$ is $1^l$, the properties of $\tau$ shown in section 6 show that $l^u$ is an upper bound on the length of $\tau((1^l, 1^l, B'))$. Hence, it is possible to evaluate the circuit $G$ on the input $\tau((1^l, 1^l, B'))$ (if $\tau((1^l, 1^l, B')$ is a string of less than $l^u$ bits, then we pad it out to $l^u$ bits by adding 1s to the end of it.) If $G$ accepts this input, then we halt and accept. If $G$ rejects this input, then we halt and accept. (Clearly we can evaluate the circuit $G$ on this input in polynomial time).

9. Check that $G$ computes the language SAT for its given input length. By the result in section 6, all we need to do in order to do this is to:

(a) Universally guess a bit string $y$ whose size equals the number of inputs of $G$.

(b) If $y$ does not represent a well-formed Boolean expression, see whether $G$ rejects $y$. (If $G$ rejects $y$, halt and accept. Otherwise halt and reject.)

(c) If $y$ represents a well-formed Boolean expression with no variables, then halt and accept if $y$ evaluates to true, halt and reject if $y$ evaluates to false.

(d) If $y$ represents a well-formed Boolean expression with variables, then try substituting 0 for the lexicographically-first variable in $y$ and evaluating $G$ on the resulting expression. Also try substituting 1 for the lexicographically-first variable in $y$ and evaluating $G$ on the resulting expression. Check that $G$ accepts $y$ if and only if $G$ accepts at least one of the expressions resulting from these substitutions. (If so, halt and accept. If not, halt and reject).

Clearly, the amount of time required to perform this check is at most polynomial in $G$’s input length, which is at most polynomial in $M_{\Lambda_k}$’s input length.

10. Check whether $x \in B_l$. If so, halt and accept. Otherwise halt and reject. (Obviously, from the construction of $B_l$, this check can be performed in time polynomial in $M_{\Lambda_k}$’s input length.)
11 Definition of $L_k$ itself

Now, we define (for each $k \in \mathbb{N}$) $L_k$ to be the union of $\Gamma$ and $\Lambda_k$. It is now trivial to construct a $\Sigma_2^p$ machine to accept $L_k$: this machine just looks at the first bit of the input string, and runs the machine for $\Gamma$ or $\Lambda_k$ accordingly.

12 Conclusion of Proof

Now, assume for the sake of a contradiction that Theorem 1 is false. Hence, for some $k \in \mathbb{N}$, $L_k$ is decidable by a circuit family of size $n^k$.

12.1 Deriving a Satisfiability Circuit

Then, we can obviously hardwire the first input of each circuit in the family to 0, and thereby obtain a circuit family of size $(n+1)^k$ that decides whether a given input bit string represents a satisfiable Boolean expression.

Now, as in section 10, let $\eta$ be a natural number for which $n^\eta \geq (n+1)^k$ for every natural number $n \geq 2$. Let $\mu$ be the number $\mu$ derived from $\eta$ in the way described in section 9. Then, for every natural number $l \geq 2$, we can set $n = l^\delta \mu$ and thereby conclude that there is a circuit of size $(l^\delta + 1)^k$ with $l^\delta$ inputs that decides if the input bit string represents a satisfiable Boolean expression.

12.2 Deriving the $\Omega_k$ circuit

Also, by taking each circuit in the circuit family for $L_k$, and hardwiring its first input to 1, we can obtain a circuit family of size $(n+1)^k$ that decides the language $\Omega_k$ defined by:

$$\Omega_k = \{ x \in \{0, 1\}^* : 1x \in \Lambda_k \}$$

Now, consider the result of running the machine $M_{\Lambda_k}$ on the input $1x$, for an arbitrary bit string $x \in \{0, 1\}^*$. Obviously step 1 will not reject the input string $1x$, so execution will pass through to step 2.

Now, look at steps 2, 3 and 4. From section 12.1 above, we know that there is a circuit of size $(l^\delta + 1)^k$ that solves the satisfiability problem for inputs of size $l^\delta$. Furthermore, one of the places that step 4 universally branches ahead to is step 9. But step 9 halts and rejects unless the circuit $G$ solves the satisfiability problem. Hence, we can assume that the circuit $G$ existentially guessed in step 2 solves the satisfiability problem for all inputs of size $l^\delta$.

Now, consider the subset $B_l$ of $\Omega_k$ existentially guessed in step 3. We have already established that $l^n > (l+1)^k$ for every $l \geq 2$. Then, by the result in section 9 there are strictly fewer than $2^{2^n}$ ways of constructing a $l$-input circuit of size $l^n$, and hence certainly fewer than $2^{2^n}$ ways of constructing an $l$-input circuit $E$ of size $(l+1)^k$. Hence, there are strictly fewer than $2^{2^n}$ different sets $L(E) \cap \Omega_k$ that can be constructed from such $l$-input circuits $E$. Hence (since $2^{2^n} = 2^{l^n}$), there must be at least one subset of $\Omega_k$ that does not equal $L(E) \cap \Omega_k$ for any $l$-input circuit $E$ of size $(l+1)^k$, let’s call every such subset of $\Omega_k$ a non circuit-constructible subset. Now, clearly the check in steps 5 and 6 ensures that the machine $M_{\Lambda_k}$ will halt and reject unless the set $B_l$ guessed in step 3 is non circuit-constructible. Now, our assumption that the circuit
\(G\) solves the satisfiability problem clearly implies that steps 7 and 8 have the effect of checking that \(B_l\) is the lexicographically-first non-circuit-constructible subset of \(\mathcal{O}_{l,\mu}\); step 8 halts and rejects unless \(B_l\) is the lexicographically-first such set. Hence, we can assume that the set \(B_l\) existentially guessed in step 3 is the lexicographically-first non circuit-constructible subset of \(\mathcal{O}_{l,\mu}\). But then clearly step 10 has the effect of making \(M_{\Lambda_k}\) halt and reject if and only if \(x\) is in the lexicographically-first non-circuit-constructible subset of \(\mathcal{O}_{l,\mu}\). Hence, the language \(\mathcal{U}_k\) defined at the beginning of this section is the set of all strings \(x \in \{0, 1\}^*\) for which \(x\) is in the lexicographically-first non-circuit-constructible subset of \(\mathcal{O}_{l,\mu}\) (where \(l = |x|\), but \(\mu\) is the same for all \(x\)). This, together with our earlier definition of non-circuit-constructibility, implies that \(\mathcal{U}_k\) is not decided by any circuit family of size \((n + 1)^k\). This is a contradiction, since we established earlier in this section that \(\mathcal{U}_k\) does have a circuit family of size \((n + 1)^k\).

Therefore, we conclude that theorem 1 is true.

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