STAR-CUMULANTS OF FREE UNITARY BROWNIAN MOTION

NIZAR DEMNI, MATHIEU GUAY-PAQUET, AND ALEXANDRU NICA

Abstract. We study joint free cumulants of $u_t$ and $u_t^*$, where $u_t$ is a free unitary Brownian motion at time $t$. We determine explicitly some special families of such cumulants. On the other hand, for a general joint cumulant of $u_t$ and $u_t^*$, we “calculate the derivative” for $t \to \infty$, when $u_t$ approaches a Haar unitary. In connection to the latter calculation we put into evidence an “infinitesimal determining sequence” which naturally accompanies any $R$-diagonal element in a tracial $*$-probability space.

1. Introduction

Let $(u_t)_{t \geq 0}$ be a free unitary Brownian motion in the sense of [3], [4] – that is, every $u_t$ is a unitary element in some tracial $*$-probability space $(A_t, \varphi_t)$, with $\varphi_t(u_t) = e^{-t/2}$, and where the rescaled element $v_t := e^{t/2}u_t$ has $S$-transform given by

$$S_{v_t}(z) = e^{tz}, \quad z \in \mathbb{C}. \quad (1.1)$$

Closely related to Equation (1.1), one has a nice formula for the free cumulants of $u_t$, i.e. for the sequence of numbers $(\kappa_n(u_t, \ldots, u_t))_{n=1}^\infty$, where $\kappa_n : A^n_t \to \mathbb{C}$ is the $n$-th free cumulant functional of the space $(A_t, \varphi_t)$. Indeed, these numbers are the coefficients of the $R$-transform $R_{v_t}$. By using the relation between the $S$-transform and the compositional inverse of the $R$-transform (which simply says that $zS(z) = R_{v_t}^{-1}(z)$), one finds that

$$R_{v_t}(z) = \frac{1}{t} W(tz), \quad t > 0, \quad (1.2)$$

where

$$W(y) = y - y^2 + \frac{3}{2} y^3 - \frac{8}{3} y^4 + \cdots + \frac{(-1)^{n-1}}{n!} y^n + \cdots$$

is the Lambert series. Extracting the coefficient of $z^n$ in (1.2) gives the value of $\kappa_n(v_t, \ldots, v_t)$, then rescaling back gives

$$\kappa_n(u_t, \ldots, u_t) = e^{-nt/2} \kappa_n(v_t, \ldots, v_t) = e^{-nt/2} \left( \frac{(-1)^{n-1}}{n!} t^{n-1} \right), \quad n \in \mathbb{N}, \quad t \geq 0. \quad (1.3)$$

In this paper we study joint free cumulants of $u_t$ and $u_t^*$, that is, quantities of the form

$$\kappa_n(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)}), \quad \text{where } n \in \mathbb{N} \text{ and } \omega = (\omega(1), \ldots, \omega(n)) \in \{1, *\}^n.$$

The motivation for paying attention to these joint free cumulants comes from looking at the limit $t \to \infty$, when $u_t$ approaches in distribution a Haar unitary. Recall that a unitary $u$ in a $*$-probability space $(A, \varphi)$ is said to be a Haar unitary when it has the property that $\varphi(u^n) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$. This property trivially implies $\kappa_n(u, \ldots, u) = 0$ for every $n \in \mathbb{N}$, thus the free cumulants of $u$ alone do not look too exciting. However, things become interesting upon considering the larger family of joint free cumulants of $u$ and $u^*$. There we

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get the following non-trivial formula, first found in \[10\]: for \( \omega = (\omega(1), \ldots, \omega(n)) \in \{1, *\}^n \) one has
\[
\kappa_n (u^{\omega(1)}, \ldots, u^{\omega(n)}) = \begin{cases} 
(-1)^{k-1} C_{k-1}, & \text{if } n \text{ is even}, n = 2k, \text{ and } \\
\omega = (1, *, 1, *, \ldots, 1, *) \\
or \omega = (*, 1, *, \ldots, 1, *) \\
otherwise,
\end{cases}
\]
with \( C_{k-1} = (2k-2)!/(k-1)!k! \), the \((k-1)\)-th Catalan number. Formula (1.4) leads to the combinatorial approach to \( R \)-diagonal elements – these are elements in a *-probability space which display, in some sense, free independence in their polar decomposition (\[8\], see also Lecture 15 of \[9\]). For an \( R \)-diagonal element \( a \) in a tracial *-probability space \((\mathcal{A}, \varphi)\), the sequence \((\kappa_{2n}(a, a^*, \ldots, a, a^*))_{n=1}^{\infty}\) is called “determining sequence of \( a^* \)” (and does indeed determine the joint distribution of \( a \) and \( a^* \)); from this point of view, Equation (1.4) says that the determining sequence of a Haar unitary consists of signed Catalan numbers.

Returning to the point of view that, for \( t \to \infty \), the free unitary Brownian motion \( u_t \) is an approximation of \( u \), it then becomes natural to ask what can be said about the joint free cumulants of \( u_t \) and \( u_t^* \). The expressions for these joint cumulants are far more involved than what is on the right-hand side of (1.4), but still seem to have some tractable features. In order to discuss them, it is convenient to start from the fact (easily obtained from the general formula connecting free cumulants to moments in a noncommutative probability space) that for every fixed \( \omega \in \{1, *\}^n \), the cumulant \( \kappa_n (u_1^{\omega(1)}, \ldots, u_t^{\omega(n)}) \) is a quasi-polynomial in \(-t/2\); more precisely, there exists a polynomial \( Z_\omega \in \mathbb{Q}[x, y] \), uniquely determined, such that
\[
\kappa_n (u_1^{\omega(1)}, \ldots, u_t^{\omega(n)}) = Z_\omega(t, e^{-t/2}), \quad \forall t \in [0, \infty).
\]
It is moreover easy to see that the \( y \)-degree of \( Z_\omega(x, y) \) is at most \( n \), and that all the powers of \( y \) that appear in \( Z_\omega \) have exponents of same parity as \( n \). In other words, we can write
\[
Z_\omega(x, y) = Z^{(n)}(x) \cdot y^n + Z^{(n-2)}(x) \cdot y^{n-2} + \cdots,
\]
with \( Z^{(n)}_\omega, Z^{(n-2)}_\omega, \ldots \) in \( \mathbb{Q}[x] \). Note that the formula (1.3) which describes the cumulants of \( u_t \) (without \( u_t^* \)) fits here, and can be read as
\[
Z_{(1, \ldots, 1)}(x, y) = \frac{(-n)^{n-1}}{n!} x^{n-1} y^n, \quad n \in \mathbb{N}.
\]

A less obvious fact about the polynomials \( Z_\omega \) is that the number of relevant terms (counting from the top) in the expansion (1.6) is limited by how many times one switches between the symbols ‘1’ and ‘*’, while going around the string \( \omega \). Thus for \( \omega = (1, \ldots, 1, 1) \) we have \( Z_\omega(x, y) = Z^{(n)}(x) \cdot y^n \) (as just seen above), then for \( \omega \) of the form \((1, \ldots, 1, *, \ldots, *)\) we have \( Z_\omega(x, y) = Z^{(n)}_\omega(x) \cdot y^n + Z^{(n-2)}_\omega(x) \cdot y^{n-2} \), and so on. This fact is stated precisely in Section 3 of the paper, and proved in Theorem 3.8 of that section. It is significant because it gives information on the speed of decay of \( \kappa_n (u_1^{\omega(1)}, \ldots, u_t^{\omega(n)}) \) when \( t \to \infty \), in the case (covering “most” strings \( \omega \in \{1, *\}^n \)) when the right-hand side of Equation (1.4) is equal to 0.

Here are some more details about what we do in this paper, and about how it is organized. Besides the present introduction, we have five sections. After a review of background and notations in Section 2, some general basic properties of the polynomials \( Z_\omega \) are established in Section 3. Then Sections 4 and 5 study two special types of \( \omega \)-s, as follows.

- In Section 4 we look at strings of the form \( \omega = (1, \ldots, 1, *, \ldots, *) \), with \( k \) occurrences of ‘1’ followed by \( \ell \) occurrences of ‘*’. We retrieve by direct calculation the fact mentioned...
above, that the expansion from Equation (1.6) is in this case reduced to its top two terms, and we show moreover how the two polynomials $Z^{(k+\ell)}(x)$ and $Z^{(k+\ell-2)}(x)$ can be written explicitly as Laplace transform integrals.

- In Section 5 we look at the case when $\omega$ is an alternating string of even length; in other words, we pay attention (as suggested by formula (1.4)) to free cumulants $\xi_n(t) := \kappa_n(u_t, u_t^*, \ldots, u_t, u_t^*)$, $n \in \mathbb{N}$, $t \in [0, \infty)$.

The main point of this section is to observe a recursive formula for $\frac{d}{dt}\xi_n(t)$, which amounts to the fact that the generating function $H(t, z) := \frac{1}{2} + \sum_{n=1}^{\infty} \xi_n(t)z^n$ satisfies a quasi-linear partial differential equation of Burgers type,

$$\partial_t H + 2z\partial_z H = z,$$ with initial condition $H(0, z) = 1/2$.

We also show how examining the characteristic curves of the above partial differential equation gives further information on $\xi_n(t)$.

Finally, in Section 6 we look at a general string $\omega$, and we study the behaviour of the corresponding joint cumulant of $u_t$ and $u_t^*$ when $t \to \infty$. We look at the limit

$$\lim_{t \to \infty} \kappa_n(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)}) - \kappa_n(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)}) e^{-t/2},$$

where $u$ is a Haar unitary. This limit turns out to always exist, and to have a very pleasing form, which suggests some kind of “infinitesimal determining sequence” for a Haar unitary. In Section 6 we also outline how the idea of infinitesimal determining sequence can be extended to the framework of a general $R$-diagonal distribution – this is done by considering products $u_t q$ where $q = q^*$ is free from $\{u_t, u_t^*\}$, and then by taking the same kind of “derivative at $t = \infty$” as above.

2. Background and Notation

This section gives a very concise review, intended mostly for setting notations, of free cumulants on a noncommutative probability space. We follow the terminology from [9] (and, for the various definitions and facts stated below, we give specific page references to that monograph).

We start with the structure lying at the basis of the combinatorics of free probability, the lattices $NC(n)$ of non-crossing partitions. We will assume the reader to be familiar with these objects, and we merely list below some basic notations that we will use in connection to them.

**Notation 2.1.** [$NC(n)$-terminology.] Let $n$ be a positive integer, and let us consider the set $NC(n)$ of all non-crossing partitions of $\{1, \ldots, n\}$.

1º Partitions in $NC(n)$ will be denoted by letters like $\pi, \rho, \ldots$. Typical explicit notation for a $\pi \in NC(n)$ is $\pi = \{V_1, \ldots, V_k\}$, where $V_1, \ldots, V_k$ are called the blocks of $\pi$. We sometimes simply write $V \in \pi$ to mean that “$V$ is one of the blocks of $\pi$.”
2° On \( NC(n) \) we consider the partial order given by reverse refinement, where for \( \pi, \rho \in NC(n) \) we have \( \pi \leq \rho \) if and only if every block of \( \rho \) is a union of blocks of \( \pi \). The partially ordered set \((NC(n), \leq)\) turns out to be a lattice – that is, every \( \pi, \rho \in NC(n) \) have a smallest common upper bound \( \pi \lor \rho \) and a greatest common lower bound \( \pi \land \rho \). (See [9], pp. 144-146.)

The minimal and maximal element of \((NC(n), \leq)\) are denoted as 0\(_n\) (the partition of \( \{1, \ldots, n\} \) into \( n \) blocks of 1 element each) and respectively as 1\(_n\) (the partition of \( \{1, \ldots, n\} \) into one block with \( n \) elements).

3° \((NC(n), \leq)\) has a special anti-automorphism called the Kreweras complementation map, which will be denoted as \( K_\pi : NC(n) \to NC(n) \). (See [9], pp. 147-148.)

4° The Möbius function of \( NC(n) \) will be denoted as \( \text{Moeb} \) (or as \( \text{Moeb}_n \), if we need to clarify what \( n \) we are working with). This function is defined on \( \{(\pi, \rho) \mid \pi, \rho \in NC(n), \pi \leq \rho\} \). We will actually use only the special case \( \rho = 1_n \). In that case we have (see [9], pp. 162-164)

\[
\text{Moeb}(\pi, 1_n) = \prod_{W \in K_\pi(\pi)} (-1)^{|W|-1} C_{|W|-1},
\]

where for \( k \in \mathbb{N} \cup \{0\} \) we denote

\[
C_k := \frac{(2k)!}{k!(k + 1)!}
\]

(\( k \)-th Catalan number).

**Notation 2.2. (Restrictions of \( n \)-tuples.)**

In order to write more concisely various formulas that will appear in the paper, it is convenient to use the following natural convention of notation. Let \( \mathcal{X} \) be a non-empty set, let \( n \) be a positive integer, and let \( (x_1, \ldots, x_n) \) be an \( n \)-tuple in \( \mathcal{X}^n \). For a subset \( V = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} \), with \( 1 \leq m \leq n \) and \( 1 \leq i_1 < \cdots < i_m \leq n \), we denote

\[
(x_1, \ldots, x_n) \mid V := (x_{i_1}, \ldots, x_{i_m}) \in \mathcal{X}^m.
\]

We will use this notation in two ways: one of them (already appearing in the next definition) is when \( \mathcal{X} \) is an algebra \( \mathcal{A} \) of noncommutative random variables, and the other is when \( \mathcal{X} = \{1, \ast\} \) and we talk about the restriction \( \omega \mid V \in \{1, \ast\}^m \) of a string \( \omega \in \{1, \ast\}^n \).

**Definition 2.3. (Free cumulant functionals and \( R \)-transforms.)**

Let \( (\mathcal{A}, \varphi) \) be a noncommutative probability space.

1° For every \( n \in \mathbb{N} \), the \( n \)-th moment functional of \((\mathcal{A}, \varphi)\) is the multilinear functional \( \varphi_n : \mathcal{A}^n \to \mathbb{C} \) defined by \( \varphi_n(a_1, \ldots, a_n) := \varphi(a_1 \cdots a_n), \ a_1, \ldots, a_n \in \mathcal{A} \).

2° For every \( n \in \mathbb{N} \), the \( n \)-th free cumulant functional of \((\mathcal{A}, \varphi)\) is the multilinear functional \( \kappa_n : \mathcal{A}^n \to \mathbb{C} \) defined by

\[
\kappa_n(a_1, \ldots, a_n) = \sum_{\pi \in \text{NC}(n)} \left( \text{Moeb}(\pi, 1_n) \cdot \prod_{V \in \pi} \varphi_{|V|}( (a_1, \ldots, a_n) \mid V ) \right).
\]

Equation \((2.1)\) is referred to as the moment–cumulant formula.

3° Let \( a \) be an element of \( \mathcal{A} \). The formal power series

\[
R_a(z) := \sum_{n=1}^{\infty} \kappa_n(a, \ldots, a) z^n
\]
is called the \( R \)-transform of \( a \).

Remark 2.4. Let \((A, \varphi)\) be a noncommutative probability space, and consider its free cumulant functionals \( \kappa_n : A^n \to \mathbb{C} \), as above.

1° Let \( B, C \subseteq A \) be unital subalgebras which are freely independent. The fundamental property of the \( \kappa_n \)'s is that \( \kappa_n (a_1, \ldots, a_n) = 0 \) whenever \( n \geq 2 \) and there are elements from both \( B \) and \( C \) among \( a_1, \ldots, a_n \). Here we record a consequence of this fact – a formula (presented in [9] on pp. 226-227) which expresses an alternating moment \( \varphi(b_1 c_1 \cdots b_n c_n) \) in terms of “free cumulants of the \( b \)'s and moments of the \( c \)'s”:

\[
\varphi(b_1 c_1 \cdots b_n c_n) = \sum_{\pi \in NC(n)} \prod_{V \in \pi} \kappa_{|V|}((b_1, \ldots, b_n) \mid V)) \cdot \prod_{W \in \mathcal{K}_n(\pi)} \varphi_{|W|}((c_1, \ldots, c_n) \mid W). \tag{2.2}
\]

2° We will make essential use of a result of Krawczyk and Speicher [7] (presented in [9] on pp. 178-181), which gives a structured summation formula for “free cumulants with products as entries”, as follows. Let \( \sigma = \{J_1, \ldots, J_k\} \subseteq NC(n) \) be a partition where every block is an interval: \( J_1 = \{1, \ldots, i_1\}, J_2 = \{i_1 + 1, \ldots, i_2\}, \ldots, J_k = \{i_{k-1} + 1, \ldots, i_k\} \) for some \( 1 \leq i_1 < i_2 < \cdots < i_k = n \). Then for every \( a_1, \ldots, a_n \in A \) one has

\[
\kappa_k (a_1 \cdots a_{i_1}, a_{i_1+1} \cdots a_{i_2}, \ldots, a_{i_{k-1}+1} \cdots a_{i_k}) = \sum_{\pi \in NC(n) \text{ such that } \pi \setminus \sigma = 1_n} \prod_{V \in \pi} \kappa_{|V|}((a_1, \ldots, a_n) \mid V). \tag{2.3}
\]

In the special case when \( \sigma = 1_n \), Equation (2.3) becomes a formula expressing the moment \( \varphi(a_1 \cdots a_n) \) in terms of free cumulants; this special case turns out to be equivalent to (2.1), and also goes (same as (2.1)) under the name of “moment–cumulant formula”.

3° We also record two useful properties of free cumulants which follow immediately from their definition, by taking into account some obvious symmetries of the lattices \( NC(n) \).

(a) Invariance under cyclic permutations of entries:

\[
\kappa_n (a_1, \ldots, a_n) = \kappa_n (a_m, \ldots, a_1, a_{m-1}), \quad \forall \ 1 \leq m \leq n \text{ and } a_1, \ldots, a_n \in A.
\]

(b) Left-right symmetry: if \( C \subseteq A \) is a commutative subalgebra, then

\[
\kappa_n (c_1, c_2, \ldots, c_n) = \kappa_n (c_n, \ldots, c_2, c_1) \quad \forall \ n \geq 1 \text{ and } c_1, \ldots, c_n \in C.
\]

We conclude the section with an easy observation concerning the analytic side of free cumulants.

Remark 2.5. Let \((A, \varphi)\) be a \( * \)-probability space. It is immediately seen (by using the Cauchy-Schwarz inequality for the functional \( \varphi \)) that every unitary \( u \in A \) has \(|\varphi(u)| \leq 1\). As a consequence, it follows that the free cumulant functionals of \((A, \varphi)\) satisfy:

\[
\kappa_n (u_1, \ldots, u_n) \leq 16^n, \quad \forall \ n \geq 1 \text{ and } u_1, \ldots, u_n \in A \text{ unitaries.} \tag{2.4}
\]
The constant $16$ in (2.4) appears upon writing cumulants in terms of moments as in Equation (2.1), then by using estimates on the Möbius function – see discussion on p. 219 of [9]. From the bound (2.4) it is clear that for every unitary $u \in A$, the $R$-transform $R_u(z)$ (which was introduced in Definiton 2.3.3 as a formal power series) can also be viewed as an analytic function on the disc $\{z \in \mathbb{C} \mid |z| < 1/16\}.

3. The Polynomials $Z_\omega$

**Proposition and Notation 3.1.** Let $\omega = (\omega(1), \ldots, \omega(n))$ be a string in $\{1, *\}^n$, for some $n \geq 1$. There exists a polynomial $Z_\omega \in \mathbb{Q}[x, y]$, uniquely determined, such that

\[
(3.1) \quad \kappa_n(\omega(1), \ldots, \omega(n)) = Z_\omega(t, e^{-t/2}), \quad \forall t \in [0, \infty).
\]

Moreover, the polynomial $Z_\omega(x, y)$ has the form

\[
(3.2) \quad Z_\omega(x, y) = \sum_{0 \leq j \leq n/2} Z^{(n-2j)}_\omega(x) \cdot y^{n-2j},
\]

where $Z^{(n-2j)}_\omega \in \mathbb{Q}[x]$ for $0 \leq j \leq n/2$.

**Proof.** We will give an explicit formula for the polynomials $Z_\omega$. In order to state it, we introduce some preliminary items of notation. We first recall that Lemma 1 on page 4 of [4] says that the moments of $u_t$ are $\varphi_t(u^n_t) = Q_n(t)e^{-nt/2}$, $n \geq 1$, where

\[
Q_n(t) = \sum_{j=0}^{n-1} \frac{(-n)^{j-1}}{j!} \binom{n}{j+1} t^j.
\]

For a string $\omega$ in $\{1, *\}^n$ which has $k$ occurrences of the symbol “1” and $\ell = n-k$ occurrences of “*”, we then introduce a polynomial $M_\omega \in \mathbb{Q}[x, y]$ defined by

\[
(3.3) \quad M_\omega(x, y) := \begin{cases} Q_{|k-\ell|}(x) y^{|k-\ell|}, & \text{if } k \neq \ell, \\ 1, & \text{if } k = \ell. \end{cases}
\]

[For instance, if $n = 7$ and $\omega = (1, 1, *, 1, 1, *, 1)$ then $M_\omega(x, y) = Q_3(x) y^3 = (1 - 3x + \frac{3}{2}x^2)y^3$.] Based on (3.3), we define the $Z_\omega$'s as follows: for every $\omega \in \{1, *\}^n$ we put

\[
(3.4) \quad Z_\omega := \sum_{\pi \in NC(n)} \text{Moeb}(\pi, 1_n) \cdot \left( \prod_{V \in \pi} M_{\omega[V]} \right),
\]

where the notations related to $NC(n)$ and its Möbius function are as in Section 2.

[A concrete example: if $n = 3$ and $\omega = (1, *, 1)$, then

\[
Z_{(1, *, 1)} := M_{(1, *, 1)} - M_{(1)} M_{(*, 1)} - M_{(1, 1)} M_{(*)} - M_{(1, *)) M_{(1)} + 2M_{(1)} M_{(*)} M_{(1)};
\]

this comes, upon substituting the $M$'s, to $Z_{(1, *, 1)}(x, y) = (x + 1)y^3 - y.$] Fix a $t \in [0, \infty)$, and for every $n \in \mathbb{N}$ let $\varphi_n : \mathbb{A}_t^2 \to \mathbb{C}$ be the $n$-th moment functional of $(\mathcal{A}_t, \varphi_t)$. Then one has

\[
(3.5) \quad \varphi_n(\omega(1), \ldots, \omega(n)) = M_\omega(t, e^{-t/2}), \quad \forall n \in \mathbb{N} \text{ and } \omega \in \{1, *\}^n.
\]
This in turn implies that, for every $n \in \mathbb{N}$ and $\omega \in \{1, *\}^n$:

$$\kappa_n(u_t^{(1)}, \ldots, u_t^{(n)}) = \sum_{\pi \in NC(n)} \operatorname{Mob}(\pi, 1_n) \cdot \left( \prod_{V \in \pi} \varphi_{|V|} \left( (u_t^{(1)}, \ldots, u_t^{(n)}) | V \right) \right)$$

$$= \sum_{\pi \in NC(n)} \operatorname{Mob}(\pi, 1_n) \cdot \left( \prod_{V \in \pi} (M_\omega(t, e^{-t/2}) | V \right) = Z_\omega(t, e^{-t/2})$$

(where we first used the moment–cumulant formula (2.1), then we invoked Equations (3.5) and (3.4)). Thus $Z_\omega$ has the property stated in Equation (3.1).

The uniqueness of $Z_\omega$ with the property stated in Equation (3.1) follows from general considerations (a polynomial of two real variables is determined by its values on pairs $(t, e^{-t/2})$, with $t \in [0, \infty)$).

Finally, let us also verify the specific form of $Z_\omega$ that was indicated in Equation (3.2). It suffices to show that: for every $\pi \in NC(n)$, the term indexed by $\pi$ in the sum on the right-hand side of (3.1) is of the form $y^s \cdot T(x)$, where $s \in \{0, 1, \ldots, n\}$ has same parity as $n$, and where $T \in \mathbb{Q}[x]$. So fix a partition $\pi = \{V_1, \ldots, V_k\} \in NC(n)$, and for every $1 \leq j \leq k$ denote by $p_j$ and by $q_j$ the number of occurrences of “1” and respectively “*” in the restricted word $\omega[V_j]$. The term indexed by $\pi$ in the sum on the right-hand side of (3.1) is $\operatorname{Mob}(\pi, 1_n) \cdot \prod_{j=1}^k Q[p_j - q_j](x)^{p_j} | p_j - q_j|$, where we set $Q_0 := 1$. This is indeed of the form $y^s \cdot T(x)$, with $s := \sum_{j=1}^k |p_j - q_j|$, and we are only left to check that $n - s$ is an even non-negative integer. But the latter fact follows from

$$n - s = \sum_{j=1}^k (p_j + q_j) - \sum_{j=1}^k |p_j - q_j| = \sum_{j=1}^k (p_j + q_j - |p_j - q_j|),$$

where every $p_j + q_j - |p_j - q_j|$ is an even non-negative integer. □

**Example 3.2.** Here are some concrete examples of polynomials $Z_\omega$:

- $Z_{(1, *)}(x, y) = -y^2 + 1$,
- $Z_{(1, 1, *)}(x, y) = (x + 1)y^3 - y$,
- $Z_{(1, 1, 1, *)}(x, y) = -(\frac{3}{2}x^2 + 2x + 1)y^4 + (x + 1)y^2$,
- $Z_{(1, 1, 1, 1, *)}(x, y) = -(x^2 + 2x + 2)y^4 + 2y^2$,  
- $Z_{(1, 1, 1, 1, 1, *)}(x, y) = -(2x + 3)y^4 + 4y^2 - 1$.

If we add to this list the formula for a polynomial $Z_{(1,1,\ldots,1)}$ from Equation (1.7), and if we take into account some obvious invariance properties of the $Z_\omega$’s (as recorded in the next remark), then these examples cover all strings $\omega \in \{1, *\}^n$ for $n \leq 4$.

**Remark 3.3.** The polynomials $Z_\omega$ have some invariance properties which follow directly from their definition.

1° Let $\omega, \omega' \in \{1, *\}^n$ be such that $\omega'$ is obtained from $\omega$ via a cyclic permutation. The invariance of free cumulants under cyclic permutations of entries gives $Z_\omega(t, e^{-t/2}) = Z_{\omega'}(t, e^{-t/2})$, $t \in [0, \infty)$, which implies that the polynomials $Z_\omega$ and $Z_{\omega'}$ coincide.

2° The same conclusion as in 1° holds if we take $\omega'$ to be obtained from $\omega$ by reversing the order of its components, $\omega' = (\omega(n), \ldots, \omega(1))$. (This time we use the invariance property of free cumulants that was reviewed in Remark 2.3(b).)
3. The moments of the free unitary Brownian motion $u_t$ are real numbers (as reviewed at the beginning of the preceding proof). This has the consequence that $u_t^*$ can also serve as free unitary Brownian motion at time $t$, which in turn implies that

$$\kappa_n\left(u_{t_1}^{(1)}, \ldots, u_{t_n}^{(n)}\right) = \kappa_n\left(u_{t_1}^{\prime(1)}, \ldots, u_{t_n}^{\prime(n)}\right), \quad \forall t \in [0, \infty),$$

with $\omega'$ obtained out of $\omega \in \{1, *\}^n$ by swapping the roles of 1 and * (every $\omega(i)$ which is a 1 is replaced by a * , and vice-versa). The uniqueness property of $Z_\omega$ thus implies that $Z_\omega = Z_{\omega'}$ in this situation, too.

We next put into evidence a very useful recursion satisfied by the polynomials $Z_\omega$. This is done in Proposition 3.5. The essence of the argument is a calculation which holds for any unitary in a *-probability space, and is presented in the next lemma.

**Lemma 3.4.** Let $(\mathcal{A}, \varphi)$ be a *-probability space, and let $u \in \mathcal{A}$ be a unitary element. Consider a string $\omega = (\omega(1), \ldots, \omega(n)) \in \{1, *\}^n$ with $n \geq 3$ and where $\omega(1) = 1$, $\omega(n) = *$. Then

$$\kappa_n\left(u_{\omega(1)}, \ldots, u_{\omega(n)}\right) = - \sum_{m=1}^{n-1} \kappa_m\left(u_{\omega(1)}, \ldots, u_{\omega(m)}\right) \cdot \kappa_{n-m}\left(u_{\omega(m+1)}, \ldots, u_{\omega(n)}\right)$$

(where $\kappa_n, \kappa_m, \kappa_{n-m}$ denote free cumulant functionals for $(\mathcal{A}, \varphi)$).

**Proof.** We may assume (by replacing $\mathcal{A}$ with the *-algebra generated by $u$) that $(\mathcal{A}, \varphi)$ is tracial. In particular, we can write

$$\kappa_n\left(u_{\omega(1)}, \ldots, u_{\omega(n)}\right) = \kappa_n\left(u_{\omega(n)}, u_{\omega(1)}, \ldots, u_{\omega(n-1)}\right) = \kappa_n\left(u^*, u, u_{\omega(2)}, \ldots, u_{\omega(n-1)}\right).$$

Now, we know that $\kappa_{n-1}\left(u^* u, u_{\omega(2)}, \ldots, u_{\omega(n-1)}\right) = 0$ (a free cumulant of length $\geq 2$ always vanishes when one of its entries is equal to $1_\mathcal{A}$). On the other hand, the formula (2.5) for cumulants with products as entries gives

$$\kappa_{n-1}\left(u^* u, u_{\omega(2)}, \ldots, u_{\omega(n-1)}\right) = \sum_{\pi \in NC(n) \text{ such that } \pi \cup \sigma = 1_n} \kappa_{\pi}\left(u^* u, u_{\omega(2)}, \ldots, u_{\omega(n-1)}\right),$$

where $\sigma \in NC(n)$ is the partition consisting of the 2-element block $\{1, 2\}$ and of $n-2$ blocks with 1 element.

Let $\pi \in NC(n)$ be such that $\pi \cup \sigma = 1_n$, and let $V'$ and $V''$ be the blocks of $\pi$ which contain 1 and 2, respectively. We observe that $V' \cup V'' = \{1, \ldots, n\}$; indeed, in the opposite case we could consider the partition $\bar{\pi} \in NC(n)$ which is obtained from $\pi$ by joining together the blocks $V$ and $V'$, and this $\bar{\pi}$ would satisfy $\pi, \sigma \leq \bar{\pi} \neq 1_n$, in contradiction with the assumption that $\pi \cup \sigma = 1_n$. If $V' = V''$ then $\pi = 1_n$. If $V' \neq V''$ then either $V' = \{1\}$, $V'' = \{2, 3, \ldots, n\}$, or $V''$ is nested inside $V'$. In the latter case, denoting $|V''| =: m$, we find that $V'' = \{2, \ldots, m+1\}$ and $V' = \{1\} \cup \{m+2, \ldots, n\}$, where $1 \leq m \leq n-2$.

The conclusion of the discussion in the preceding paragraph is that the sum on the right-hand side of (3.7) can be written explicitly as

$$\kappa_n\left(u^*, u, u_{\omega(2)}, \ldots, u_{\omega(n-1)}\right) + \sum_{m=1}^{n} \kappa_{\pi_m}\left(u^*, u, u_{\omega(2)}, \ldots, u_{\omega(n-1)}\right),$$

1 The partition $\bar{\pi}$ thus consists of $V' \cup V''$ and of all the blocks $V \in \pi$ such that $V \neq V', V''$. The detail which prevents $\bar{\pi}$ from having crossings is that $V'$ and $V''$ contain the adjacent points 1 and 2.
with \( \pi_m = \{2, \ldots, m+1\}, \{1, \ldots, n\} \setminus \{2, \ldots, m+1\}\), \(1 \leq m \leq n-1\). It is immediately seen (by doing the suitable cyclic permutation of entries in \(\kappa_{n-m}(u^*, u^{\omega(m+1)}, \ldots, u^{\omega(n-1)})\)) that for every \(1 \leq m \leq n-1\) one has

\[
\kappa_{\pi_m}(u^*, u, u^{\omega(2)}, \ldots, u^{\omega(n)}) = \kappa_m(u^{\omega(1)}, \ldots, u^{\omega(m)}) \cdot \kappa_{n-m}(u^{\omega(m+1)}, \ldots, u^{\omega(n)}).
\]

But the sum in (3.8) is equal to 0 (since it started as an expansion for \(\kappa_{n-1}(u^* u, \ldots) = 0\)), and the statement of the lemma follows.

**Proposition 3.5.** Suppose that \(n \geq 3\) and that \(\omega = (\omega(1), \ldots, \omega(n)) \in \{1,*\}^n\) has \(\omega(1) = 1\) and \(\omega(n) = *\). Then it follows that

\[
Z_\omega = -\sum_{m=1}^{n-1} Z_{(\omega(1), \ldots, \omega(m))} \cdot Z_{(\omega(m+1), \ldots, \omega(n))}
\]

*(equality of polynomials in two variables)*.

**Proof.** Let \(Z \in \mathbb{Q}[x,y]\) be the polynomial which appears on the right-hand side of (3.9). By using Lemma 3.4 one immediately sees that \(Z(t, e^{-t/2}) = \kappa_n(u^{\omega(1)}_t, \ldots, u^{\omega(n)}_t)\), \(t \in [0, \infty)\), and this implies \(Z = Z_\omega\).

As an application of Proposition 3.5 we show the following: the number of relevant terms (counting from the top) in the expansion given for \(Z_\omega\) in Equation (3.12) is limited by how many times we switch between the symbols ‘1’ and ‘∗’, upon going around the string \(\omega\). For instance if \(\omega = (1, 1, \ldots, 1)\) then the expansion (3.12) amounts to just \(Z_\omega(x,y) = Z_\omega^{(n)}(x) \cdot y^n\); if \(\omega = (1, \ldots, 1,*\ldots,*)\) then \(Z_\omega(x,y) = Z_\omega^{(n)}(x) \cdot y^n + Z_\omega^{(n_2)}(x) \cdot y^{n-2}\), and so on. This fact is stated precisely in Theorem 3.8 below. In order to come to it, we first record the (natural) definition for what is the “number of switches between 1 and ∗” in a string \(\omega\).

**Definition and Remark 3.6.** For every \(n \in \mathbb{N}\) and \(\omega \in \{1,*\}^n\) we define the switch-number of \(\omega\) to be

\[
Switch(\omega) := \delta_{\omega(n), \omega(1)} + \delta_{\omega(1), \omega(2)} + \cdots + \delta_{\omega(n-1), \omega(n)},
\]

where the \(\delta\)’s on the right-hand side of the equation are assigned by putting

\[
\delta_{1,*} = \delta_{*,1} = 1 \text{ and } \delta_{1,1} = \delta_{*,*} = 0.
\]

It is easily seen that \(Switch(\omega)\) is an even integer such that \(0 \leq Switch(\omega) \leq n\). Another immediate observation is that \(Switch(\omega) = Switch(\omega')\) whenever \(\omega'\) is obtained out of \(\omega\) via one of the transformations discussed in Remark 3.3.

**Lemma 3.7.** Let \(n \geq 3\) be an integer, and let \(\omega = (\omega(1), \ldots, \omega(n)) \in \{1,*\}^n\) be such that \(\omega(1) = 1\), \(\omega(n) = *\). Let \(m\) be a number in \(\{1, \ldots, n-1\}\), and consider the strings

\[
\omega' := (\omega(1), \ldots, \omega(m)) \in \{1,*\}^m, \quad \omega'' := (\omega(m+1), \ldots, \omega(n)) \in \{1,*\}^{n-m}.
\]

Then we have

\[
Switch(\omega') + Switch(\omega'') \leq Switch(\omega).
\]
Proof. We will prove the required inequality by assuming that $1 < m < n - 1$ (the cases when $m = 1$ or $m = n - 1$ are analogous, and simpler). Look at the difference

$$\text{Switch}(\omega) - \left( \text{Switch}(\omega') + \text{Switch}(\omega'') \right).$$

By cancelling common terms in the expressions which define these three switch-numbers, we find that the above difference is equal to

$$(\delta_{\omega(m),\omega(m+1)} + \delta_{\omega(n),\omega(1)}) - (\delta_{\omega(m),\omega(1)} + \delta_{\omega(n),\omega(m+1)}).$$

Since $\delta_{\omega(n),\omega(1)} = 1$ (while the other $\delta$'s appearing above are 0 or 1), we get that

$$\text{Switch}(\omega) - \left( \text{Switch}(\omega') + \text{Switch}(\omega'') \right) \geq -1.$$

But switch-numbers are always even; so in the latter inequality we are actually forced to have “$\geq 0$”, and (3.11) follows. \hfill \square

**Theorem 3.8.** Let $n$ be a positive integer, and let $\omega$ be a string in $\{1, *\}^n$. Consider the polynomial $Z_\omega(x, y)$ and its expansion as sum of terms $Z_\omega^{(n-2j)}(x) \cdot y^{n-2j}$, with $0 \leq j \leq n/2$, which was obtained in Proposition 3.7. One has

$$Z_\omega^{(n-2j)} = 0 \text{ whenever } 2j > \text{Switch}(\omega).$$

**Proof.** We first observe that the statement of the theorem holds when $\omega$ is of the form $(1, 1, \ldots, 1)$ or $(*, *, \ldots, *)$. In this case we have $\text{Switch}(\omega) = 0$; so Equation (3.12) says that $Z_\omega^{(n-2j)} = 0$ for every $j \neq 0$, i.e. that $Z_\omega(x, y) = Z_\omega^{(n)}(x) \cdot y^n$. This is indeed true, as noticed in Equation (1.7) of the introduction.

We now prove by induction on $n$ that the statement of the theorem holds for general strings $\omega \in \{1, *\}^n$. The case $n = 1$ is included in the preceding paragraph. Let us also verify the case $n = 2$. In this case, the strings $(1, 1)$ and $(*, *)$ are covered by the preceding paragraph, while the strings $(1, *)$ and $(*, 1)$ have switch-number equal to 2 – so for the latter two strings, Equation (3.12) is fulfilled vacuously (there is no $j$ in the range $0 \leq j \leq n/2$ such that $2j > \text{Switch}(\omega)$).

In the remaining part of the proof we do the induction step: we fix an integer $n \geq 3$, we assume that the statement of the theorem holds for strings of length $\leq n - 1$, and we will prove that it also holds for strings of length $n$.

So let us also fix an $\omega = (\omega(1), \ldots, \omega(n))$ in $\{1, *\}^n$, for which we will verify that (3.12) holds. We distinguish three cases.

1. **Case 1.** $\omega = (1, 1, \ldots, 1)$ or $\omega = (*, *, \ldots, *)$.
   
   This case was verified in the first paragraph of the proof.

2. **Case 2.** $\omega$ is such that $\omega(1) = 1$ and $\omega(n) = *$.
   
   Consider a $j \in \mathbb{N}$ such that $0 \leq j \leq n/2$ and such that $2j > \text{Switch}(\omega)$. (We assume that such $j$’s exist, otherwise there is nothing to prove.) We are in a situation where we can invoke Proposition 3.5. By extracting the coefficient of $y^{n-2j}$ on both sides of the recursion provided by that proposition, we find that

$$Z_\omega^{(n-2j)} = - \sum_{m=1}^{n-1} \sum_{0 \leq k \leq m/2, 0 \leq \ell \leq (n-m)/2} \text{such that } k+\ell=j \ Z_{\omega(1),\ldots,\omega(m)}^{(m-2k)} \cdot Z_{\omega(m+1),\ldots,\omega(n)}^{(n-m-2\ell)}.$$
(equality of polynomials in \( \mathbb{Q}[x] \)). We will show that \( Z^{(n-2j)}_\omega = 0 \) by verifying that every term in the double sum on the right-hand side of (3.13) is the zero polynomial. Indeed, let us pick such a term (indexed by an \( m \), and then by a pair \((k, \ell)\)), and let us denote
\[
\omega' := (\omega(1), \ldots, \omega(m)) \in \{1,*\}^m, \quad \omega'' := (\omega(m+1), \ldots, \omega(n)) \in \{1,*\}^{n-m}.
\]
We have
\[
2k + 2\ell = 2j > \text{Switch}(\omega) \geq \text{Switch}(\omega') + \text{Switch}(\omega'')
\]
(where at the last inequality we used Lemma 3.7). So either \( 2k > \text{Switch}(\omega') \) or \( 2\ell > \text{Switch}(\omega'') \), and the induction hypothesis gives us that either \( Z^{(m-2k)}_\omega(\omega(1), \ldots, \omega(m)) = 0 \) or \( Z^{(n+m-2\ell)}_\omega(\omega(m+1), \ldots, \omega(n)) = 0 \). Either way, the product of the latter two polynomials is 0, and this completes the verification of Case 2.

**Case 3.** \( \omega \) does not fall in either Case 1 or Case 2 above.

Since \( \omega \) is not in Case 1, both symbols 1 and * must appear among its components. It is then easy to see that there exists a string \( \omega' \) obtained from \( \omega \) via a cyclic permutation of components, such that \( \omega'(1) = 1 \) and \( \omega'(n) = * \). The string \( \omega' \) has \( Z_{\omega'} = Z_{\omega} \) (Remark 3.31), and has \( \text{Switch}(\omega') = \text{Switch}(\omega) \) (Remark 3.6). For any \( j \) such that \( 2j > \text{Switch}(\omega') = \text{Switch}(\omega') \) we have \( Z^{(n-2j)}_{\omega'} = 0 \), because \( \omega' \) falls in the Case 2 discussed above. It follows that \( Z^{(n-2j)}_\omega = 0 \) as well. This concludes the verification of the induction step, and the proof of the theorem.

\[ \square \]

**4. A special case of \( Z_\omega \)'s**

In the present section we determine what is the polynomial \( Z_\omega \) for a string of the form \( \omega = (1, \ldots, 1, *, \ldots, *) \), having \( k \) occurrences of “1” followed by \( \ell \) occurrences of “*”, for some \( k, \ell \geq 1 \). In this case, Theorem 3.8 says that the expansion from Equation (1.6) is reduced to its top two terms:
\[
Z_\omega(x, y) = Z^{(k+\ell)}_\omega(x) y^{k+\ell} + Z^{(k+\ell-2)}_\omega(x) y^{k+\ell-2}.
\]

We will retrieve this fact, and we will moreover show how the polynomials \( Z^{(k+\ell)}_\omega(x) \), \( Z^{(k+\ell-2)}_\omega(x) \) can be written explicitly as some Laplace transform integrals. We start with a calculation (consequence of the above Lemma 3.4) which holds for any unitary in a *-probability space.

**Lemma 4.1.** Let \( (\mathcal{A}, \varphi) \) be a *-probability space and let \( u \in \mathcal{A} \) be a unitary element. It makes sense to define an analytic function \( F_u : \{(z, w) : \mathbb{C}^2 \mid |z|, |w| < 1/16\} \to \mathbb{C} \) by putting
\[
F_u(z, w) := \sum_{k, \ell=1}^{\infty} \kappa_{k+\ell}(u, \ldots, u, u^*, \ldots, u^*) z^k w^\ell.
\]
Moreover, there exists \( r \in (0, 1/16) \) such that for \( |z|, |w| < r \) one has
\[
F_u(z, w) = \frac{zw - R_u(z)R_{u^*}(w)}{1 + R_u(z) + R_{u^*}(w)},
\]
where \( R_u, R_{u^*} : \{z \in \mathbb{C} \mid |z| < 1/16\} \to \mathbb{C} \) are R-transforms (as discussed in Remark 2.3).
As mentioned in the introduction, the rescaling $v_t = e^{t/2} u_t$ has $R$-transform $R_{v_t}(z) = t^{-1} W(tz)$. But $R_{u_t}(z) = R_{v_t}(e^{-t/2} z)$, so we find the $R$-transform of $u_t$ to be

$$R_{u_t}(z) = \frac{1}{t} W(t^{-t/2} z).$$

The equality (4.6) holds when $|z|$ is small enough so that both sides are defined (one can e.g. use $|z| < 1/16$ on the left-hand side and $|z| < e^{t/2}/(et)$ on the right-hand side). The adjoint
\(u^*_t\) has the same \(R\)-transform as \(u_t\) itself. We replace all this into the result of Lemma 4.1. Upon also requiring the condition that \(|z|, |w|\) are small enough such that

\[
|W(te^{-t/2}z)|, \ |W(te^{-t/2}w)| < t/2
\]

(which ensures that the denominator \(t + W(te^{-t/2}z) + W(te^{-t/2}w)\) does not vanish), we arrive to the formula for \(F_{u_t}\) that was stated in Equation (4.4).

In order to go from (4.4) to (4.5), let us fix \(z, w \in \mathbb{C}\) such that \(|z|, |w|\) satisfy the restrictions mentioned above, and let us denote

\[W(te^{-t/2}z) =: \alpha, W(te^{-t/2}w) =: \beta.\]

From the definition of the Lambert function it follows that \(\alpha e^\alpha = te^{-t/2}z\), \(\beta e^\beta = te^{-t/2}w\), and multiplying together the latter equations gives

\[t^2zw = \alpha \beta e^{t\alpha + \beta}.\]

(4.7)

We write the right-hand side of (4.4) in terms of \(\alpha, \beta\) (where \(t^2zw\) is substituted from Equation (4.7)), and we obtain

\[F_{u_t}(z, w) = \frac{1}{t} \cdot \frac{\alpha \beta e^{t\alpha + \beta} - \alpha \beta}{t + \alpha + \beta} = \frac{\alpha \beta}{t} \int_0^1 e^{x(t + \alpha + \beta)} \, dx.\]

Finally, in the latter integral we make the substitution \(s = 1 - x\), which leads to

\[F_{u_t}(z, w) = \frac{\alpha \beta}{t} \cdot e^{t\alpha + \beta} \cdot \int_0^1 e^{-s(t + \alpha + \beta)} \, ds.\]

The constant \((\alpha \beta e^{t\alpha + \beta})/t\) is (by (4.7)) equal to \(tzw\), hence reverting back from \(\alpha, \beta\) to \(z, w\) takes us precisely to the integral formula stated in Equation (4.5).

\[\square\]

**Proposition 4.3.** Let us fix \(t \in (0, \infty)\) and let \(u_t\) be as above (free unitary Brownian motion at time \(t\)). For every \(k, \ell \in \mathbb{N}\) we have

\[\kappa_{k+\ell}(u_{k1} \ldots u_{k1}, u_{\ell1} \ldots u_{\ell1}) = \frac{(-1)^{k+\ell}}{(k-1)!(\ell-1)!} t^{k+\ell-1} (e^{-t/2})^{k+\ell-2} \cdot I_{k,\ell}(t),\]

(4.8)

where

\[I_{k,\ell}(t) := \int_0^1 e^{-ts} s^{2} (s + k - 1)^{k-2} (s + \ell - 1)^{\ell-2} ds.\]

(4.9)

**Proof.** It is known (see e.g. [5]) that for any \(s \in [0, 1]\) and \(y \in \mathbb{C}\) with \(|y| < 1/e\) one has the series expansion

\[e^{-sW(y)} = 1 - sy + \frac{s(s + 2)}{2!} y^2 - \frac{s(s + 3)}{3!} y^3 + \cdots + (-1)^n \frac{s(s + n)^{n-1}}{n!} y^n + \cdots,\]

which we will find convenient to write concisely as

\[e^{-sW(y)} = \sum_{n=0}^{\infty} \frac{s(s + n)^{n-1}}{n!} (-y)^n.\]

(4.10)

Let us then pick some \(z, w\) with \(|z|, |w|\) small enough (in the sense discussed in Lemma 4.2) and such that moreover \(z, w\) are real negative numbers. By using the expansion (4.10)
in the Equation (4.5) of Lemma 4.2 we infer that

\[ F_{u_t}(z, w) = tzw \int_0^1 e^{-ts} \cdot \sum_{m=0}^{\infty} \frac{s(s+m)^{m-1}}{m!} (-te^{-t/2}z)^m \cdot \sum_{n=0}^{\infty} \frac{s(s+n)^{n-1}}{n!} (-te^{-t/2}w)^n \, ds \]

\[ = \int_0^1 \left( \sum_{m,n=0}^{\infty} tzw \cdot e^{-ts} \cdot \frac{s(s+m)^{m-1}}{m!} (-te^{-t/2}z)^m \cdot \frac{s(s+n)^{n-1}}{n!} (-te^{-t/2}w)^n \right) \, ds. \]

The terms of the infinite double-sum are non-negative, hence the monotone convergence theorem allows us to interchange the double-sum with the integral. When we do that, and we move the powers of \(-z, -w, t, e^{-t/2}\) outside the integral, we come to the fact that (for \(z, w\) picked as above) we have

\[ F_{u_t}(z, w) = \sum_{m,n=0}^{\infty} (-z)^{m+1} (-w)^{n+1} \cdot \frac{t^{m+n+1}(e^{-t/2})^{m+n}}{m! n!} \cdot \left( \int_0^1 s(s+m)^{m-1} s(s+n)^{n-1} \, ds \right). \]

It is convenient to also make here the shift of indices \(m + 1 = k, n + 1 = \ell\), and conclude that

\[ F_{u_t}(z, w) = \sum_{k,\ell=1}^{\infty} (-z)^k (-w)^\ell \cdot \frac{t^{k+\ell-1}(e^{-t/2})^{k+\ell-2}}{(k-1)! (\ell-1)!} \cdot I_{k,\ell}(t), \]

with \(I_{k,\ell}(t)\) defined in (4.9).

Now, it is easy to see that if we put

\[ \lambda_{k,\ell}(t) := \frac{(-1)^{k+\ell}}{(k-1)! (\ell-1)!} t^{k+\ell-1} (e^{-t/2})^{k+\ell-2} \cdot I_{k,\ell}(t), \quad k, \ell \in \mathbb{N}, \]

then the formula

\[ G_t(z, w) := \sum_{k,\ell=1}^{\infty} \lambda_{k,\ell}(t) z^k w^\ell \]

gives an analytic function defined for \(|z|, |w|\) small enough. Indeed, one can simply bound the integrand in \(I_{k,\ell}(t)\) by \(k^{k-2} \ell^{\ell-2}\) to conclude that

\[ 0 \leq I_{k,\ell}(t) \leq k^{k-2} \ell^{\ell-2} \leq \gamma \cdot e^k (k-1)! \cdot e^\ell (\ell-1)!, \]

where \(\gamma > 0\) is an absolute constant (not depending on \(k, \ell\)) – the second inequality displayed above follows from Stirling’s formula. This implies in turn the bound

\[ |\lambda_{k,\ell}(t)| \leq (\gamma e^{t/2}) \cdot (e^{e^{-t/2}})^{k+\ell}, \quad \forall k, \ell \in \mathbb{N}, \]

and gives the claim about the existence of \(G_t(z, w)\).

Finally, Equation (4.11) can be read as saying that \(F_{u_t}(z, w) = G_t(z, w)\) for \(z, w\) real negative numbers of small enough absolute value. This implies that \(F_{u_t}\) and \(G_t\) must have the same series expansion around \((0, 0)\), which is exactly the statement that had to be proved. \(\square\)

The formula for cumulants found in Proposition 4.3 can be re-phrased as a formula for the corresponding polynomials \(Z_\omega\), as follows.
\textbf{Theorem 4.4.} Let $k, \ell$ be positive integers. There exist polynomials $U_{k,\ell}, V_{k,\ell} \in \mathbb{Z}[x]$, uniquely determined, such that
\begin{equation}
\begin{aligned}
U_{k,\ell}(x) &= -x^{k+\ell-1} \int_0^\infty e^{-xs} \left( (s+1)^2(s+k)^{k-2}(s+\ell)^{\ell-2} \right) ds, \\
V_{k,\ell}(x) &= x^{k+\ell-1} \int_0^\infty e^{-xs} \left( s^2(s+k-1)^{k-2}(s+\ell-1)^{\ell-2} \right) ds,
\end{aligned}
\label{eq:4.12}
\end{equation}

One has
\begin{equation}
Z_{(1,\ldots,1 \ast, \ldots, \ast)}(x,y) = \frac{(-1)^{k+\ell}}{(k-1)!(\ell-1)!} \left( U_{k,\ell}(x)y^{k+\ell} + V_{k,\ell}(x)y^{k+\ell-2} \right).
\label{eq:4.13}
\end{equation}

\textbf{Proof.} In order to verify that the function $U_{k,\ell}(x)$ defined by the first integral in (4.12) is indeed a polynomial, we expand the product $(s+1)^2(k+s)^{k-2}(\ell+s)^{\ell-2}$ in powers of $s$, then use the fact that
\[ x^{k+\ell-1} \int_0^\infty e^{-xs} s^m ds = m! x^{(k+\ell)-1-(m+1)}, \quad 0 \leq m \leq k + \ell - 2. \]

A similar calculation shows that $V_{k,\ell}(x)$ is a polynomial as well.

In order to prove that (4.13) holds, it suffices to fix a $t \in [0, \infty)$ and to verify the following fact: when evaluated at $(t, e^{-t/2})$, the polynomial in $(x,y)$ from the right-hand side of (4.13) yields the free cumulant $\kappa_{k+\ell}(u_1, \ldots, u_t, u_1^*, \ldots, u_t^*)$ (with $k$ entries of $u_t$ and $\ell$ entries of $u_t^*$). By comparing this fact against the result of Proposition 4.3 and by doing some obvious simplifications, we see that it is actually sufficient to check that
\[ t^{k+\ell-1} I_{k,\ell}(t) = U_{k,\ell}(t) e^{-t} + V_{k,\ell}(t), \]

where $I_{k,\ell}(t)$ is the integral defined in Equation (4.9). But the latter verification is immediately obtained when one writes the integral “$\int_{1}^{\infty}$” which defines $I_{k,\ell}(t)$ as a difference “$\int_{0}^{\infty} - \int_{1}^{\infty}$” (by using the same integrand). Indeed, the very definition of $V_{k,\ell}$ says that
\[ t^{k+\ell-1} \int_0^\infty e^{-ts} s^2(s+k-1)^{k-2}(s+\ell-1)^{\ell-2} ds = V_{k,\ell}(t), \]
while on the other hand the change of variable $\tilde{s} = s - 1$ gives
\[ t^{k+\ell-1} \int_1^\infty e^{-ts} s^2(s+k-1)^{k-2}(s+\ell-1)^{\ell-2} ds = t^{k+\ell-1} \int_0^\infty e^{-t(\tilde{s}+1)} (\tilde{s}+1)^2(\tilde{s}+k)^{k-2}(\tilde{s}+\ell)^{\ell-2} d\tilde{s}, \]

which is $-e^{-t} U_{k,\ell}(t).$ \hfill \QED

\textbf{Remark 4.5.} Let us illustrate the explicit writing of the polynomials $U_{k,\ell}$ and $V_{k,\ell}$ in the special case $\ell = 1$ (this gives, in some sense, the simplest possible example of free cumulants of $u_t$ and $u_t^*$ that are truly “joint”). The formulas defining $U_{k,\ell}$ and $V_{k,\ell}$ become here
\[ U_{k,1}(x) = -x^k \int_0^\infty e^{-xs}(s+1)(s+k)^{k-2} ds, \quad V_{k,1}(x) = x^k \int_0^\infty e^{-xs}s(s+k-1)^{k-2} ds, \quad k \in \mathbb{N}. \]

We note the special relation
\[ U_{k,1} = -\frac{1}{k} V_{k+1,1}, \quad \forall k \geq 1, \]
which is easily derived by writing $V_{k+1,1}(x) = -x^k \int_0^\infty (e^{-xs})' \cdot s(s + k - 1)^{k-2}ds$, and by doing an integration by parts. We thus only need to write explicitly the $V_{k,1}$’s; this is done in the way shown at the beginning of the preceding proof, which gives $V_{1,1}(x) = V_{2,1}(x) = 1$ and

$$V_{k,1}(x) = \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot (k-1-j)! \cdot (k-1)^j \cdot t^j, \quad k \geq 3.$$ (4.14)

In terms of the actual *-cumulants of $u_t$, the above considerations say that for every $k \in \mathbb{N}$ we have:

$$\kappa_{k+1}(u_t, \ldots, u_t, u_t^*) = Z_{\{1, \ldots, 1,*\}(t,e^{-t/2})}$$

$$\quad = (-1)^{k+1} \frac{(k-1)!}{(k-1)} \left( U_{k,1}(t)(e^{-t/2})^{k+1} + V_{k,1}(t)(e^{-t/2})^{k-1} \right)$$

$$\quad = \frac{(-e^{-t/2})^k}{(k-1)!} \left( -\frac{1}{k} V_{k+1,1}(t)e^{-t} + V_{k,1}(t) \right).$$

Thus, if we consider the sequence of polynomials

$$V_k := V_{k,1}/(k-1)!, \quad k \in \mathbb{N},$$

(with $V_{k,1}$ taken from Equation (4.14)), the conclusion is that for every $k \in \mathbb{N}$ and $t \in [0, \infty)$ we have

$$\kappa_{k+1}(u_t, \ldots, u_t, u_t^*) = (-e^{-t/2})^k V_k(t) - e^{-t} V_{k+1}(t).$$ (4.15)

So for instance for $k \leq 4$ we have

$$\left\{ \begin{array}{ll}
\kappa_2(u_t, u_t^*) & = (-e^{-t/2})^0 (1 - e^{-t}), \\
\kappa_3(u_t, u_t, u_t^*) & = (-e^{-t/2})^1 (1 - e^{-t}(t + 1)), \\
\kappa_4(u_t, u_t, u_t, u_t^*) & = (-e^{-t/2})^2 \left( (t + 1) - e^{-t}(\frac{3}{2} t^2 + 2t + 1) \right), \\
\kappa_5(u_t, u_t, u_t, u_t, u_t^*) & = (-e^{-t/2})^3 \left( (\frac{3}{2} t^2 + 2t + 1) - e^{-t}(\frac{5}{2} t^3 + 4t^2 + 3t + 1) \right).
\end{array} \right.$$

5. Another special case – alternating ω’s

The special form of free joint cumulants for a Haar unitary and its adjoint (reviewed in Equation (1.4) of the introduction) suggests that in our discussion of the polynomials $Z_\omega$ we should consider the case when $\omega$ is an alternating string of even length. The polynomial $Z_\omega$ associated to the alternating string $(1,* \ldots, 1,* \in \{1,*\}^{2k}$ is of the form

$$(-1)^{k-1} \left( C_{k-1} - T_1^{(k)}(x) y^2 + T_2^{(k)}(x) y^4 - \cdots + (-1)^k T_k^{(k)}(x) y^{2k} \right),$$ (5.1)
where $C_{k-1}$ is the $(k - 1)$-th Catalan number, and every $T_j^{(k)}$ (1 ≤ $j$ ≤ $k$) is a polynomial of degree $j - 1$ with strictly positive rational coefficients. Examples for small $k$:

\[
\begin{align*}
Z_{(1,*)}(x, y) &= 1 - y^2, \\
Z_{(1,*,1,*)}(x, y) &= -1 + 4y^2 - (2x + 3)y^4, \\
Z_{(1,*,1,*,1,*)}(x, y) &= 2 - 15y^2 + (12x + 30)y^4 - (6x^2 + 18x + 17)y^6, \\
Z_{(1,*,1,*,1,*,1,*)}(x, y) &= -5 + 56y^2 - 28(2x + 7)y^4 + 8(6x^2 + 26x + 33)y^6 \\
&\quad - (\frac{64}{3}x^3 + 96x^2 + 172x + 119)y^8.
\end{align*}
\]

The inductive verification that the pattern (5.1) holds for general $k$ is not hard (based on the recursion from Proposition 3.5), and is left as exercise to the reader. In this section we do not focus on coefficients of $Z_n$’s, but we find it more interesting to look at the actual cumulants

\[
(5.2) \quad \xi_n(t) := \kappa_{2n}(u_t, u_t^*, \ldots, u_t^*, u_t^*) = Z_{\{(1, \ldots, 1, *)\}}^{2n}(t, e^{-t/2}), \quad n \geq 1,
\]

where $u_t$ is the free unitary Brownian motion at time $t$. The notation introduced in (5.2) emphasizes the dependence on $t$. This is of relevance because the main point of the section is to put into evidence a recursion for the derivative of $\xi_n$ with respect to $t$, as shown next.

**Theorem 5.1.** Let $\xi_n(t)$ be as above. Then for every $n \geq 2$ one has

\[
(5.3) \quad -\frac{1}{n} \frac{d\xi_n}{dt}(t) = \xi_n(t) + \sum_{m=1}^{n-1} \xi_m(t)\xi_{n-m}(t), \quad t \in [0, \infty).
\]

**Proof.** For convenience of notation, throughout this proof we will fix a tracial $*$-probability space $(A, \varphi)$ which is large enough to contain all the unitaries $u_t$ for $t \in [0, \infty)$. By enlarging $(A, \varphi)$ a bit further, we will moreover assume that $A$ contains two families of elements \{\rho_{\theta} | 0 < \theta < 1/2\} and \{\omega_{\theta} | 0 < \theta < 1/2\} such that

(i) $\rho_{\theta}^* = \rho_{\theta}$, $\omega_{\theta}^* = \omega_{\theta}$ and $\rho_{\theta}\omega_{\theta} = \omega_{\theta}\rho_{\theta} = 0$, $\forall \theta \in (0, 1/2)$;

(ii) $\varphi(\rho_{\theta}) = \varphi(\omega_{\theta}) = \theta$, $\forall \theta \in (0, 1/2)$;

(iii) \{\rho_{\theta}, \omega_{\theta}\} is free from \{u_t, u_t^*\}, for all $\theta \in (0, 1/2)$ and $t \in [0, \infty)$.

We consider the rescaled elements $v_t = e^{t/2}u_t$. Following [2], for every $n \in \mathbb{N}$ we define a function $f_{2n} : [0, \infty) \times (0, 1/2) \rightarrow \mathbb{R}$ by

\[
(5.4) \quad f_{2n}(t, \theta) := \varphi(\rho_{\theta} v_t \omega_{\theta} v_t^*)^n, \quad \forall t \geq 0 \text{ and } 0 < \theta < 1/2.
\]

[For instance for $n = 1$ we have $f_2(t, \theta) := \varphi(\rho_{\theta} v_t \omega_{\theta} v_t^*)$, and an immediate application of formula (2.2) for alternating moments yields $f_2(t, \theta) = \theta^2(e^t - 1)$.

**Claim.** For every $n \in \mathbb{N}$, the function $f_{2n}$ is of the form

\[
(5.5) \quad f_{2n}(t, \theta) = \sum_{j=1}^{2n} g_{n,j}(t) \theta^j,
\]

\[\text{For instance we can replace } (A, \varphi) \text{ by the free product } (A, \varphi) \ast (L^\infty[0, 1], dt) \text{, and take } \rho_{\theta}, \omega_{\theta} \in L^\infty[0, 1], dt \text{ to be the indicator functions of the intervals } [0, \theta] \text{ and } [1 - \theta, 1], \text{ respectively.} \]
where the $g_{n,j}$'s are quasi-polynomials, and where (for $j = 2n$) we have

\begin{equation}
(5.6) \quad g_{n,2n}(t) = e^{nt} \xi_n(t).
\end{equation}

**Verification of Claim.** Fix $n \in \mathbb{N}$ for which we will verify that (5.5) and (5.6) hold. We write $f_{2n}(t, \theta)$ as $\varphi(v_t, q_0^t, v_t^* p_\theta^t, \ldots, v_t, q_\theta v_t^*, p_\theta)$, and we expand this alternating moment of order $4n$ in the way indicated in Remark 2.4.1, in terms of moments of $p_\theta, q_\theta$ and of free cumulants of $v_t, v_t^*$. In this way we obtain the formula

\begin{equation}
(5.7) \quad f_{2n}(t, \theta) = \sum_{\sigma \in NC(2n)} g_\sigma(t) \cdot h_\sigma(\theta),
\end{equation}

where for $\sigma \in NC(2n)$ we put

\begin{align*}
  g_\sigma(t) &= \prod_{V \in \sigma} \kappa_{|V|}(v_t, v_t^*), \\
  h_\sigma(\theta) &= \prod_{W \in K_{2n}(\sigma)} \varphi_{|W|}(q_\theta, p_\theta).
\end{align*}

Note that every $g_\sigma$ can be written as

\begin{equation}
(5.8) \quad g_\sigma(t) = e^{nt} \cdot \prod_{V \in \sigma} \kappa_{|V|}(u_t, u_t^*),
\end{equation}

and is thus a quasi-polynomial by Proposition 3.1.

Let us next observe that for every non-empty set $W \subseteq \{1, \ldots, 2n\}$, the moment $\varphi_{|W|}(q_\theta, p_\theta)$ is equal to either 0 or $\theta$. Indeed, if $W$ contains both odd and even numbers, then the moment in discussion vanishes due to the hypothesis that $p_\theta q_\theta = q_\theta p_\theta = 0$. In the opposite case, we are looking either at $\varphi(p_\theta^W)$ or at $\varphi(q_\theta^W)$, and both these moments are equal to $\theta$.

The observation from the preceding paragraph implies that, for every $\sigma \in NC(2n)$, the value of $h_\sigma(\theta)$ is either 0 or $\theta^{j(\sigma)}$, with

\begin{equation*}
j(\sigma) = \left( \begin{array}{c}
\text{# of blocks of } K_{2n}(\sigma) \\
\text{of } \sigma
\end{array} \right) = (2n + 1) - \left( \begin{array}{c}
\text{# of blocks of } \sigma \\
\text{of } \sigma
\end{array} \right).
\end{equation*}

It then becomes clear that we can re-write Equation (5.7) in the form

\begin{equation*}
(5.7) \quad f_{2n}(t, \theta) = \sum_{j=1}^{2n} g_{n,j}(t) \cdot \theta^j,
\end{equation*}

where for $1 \leq j \leq 2n$ we define the quasi-polynomial $g_{n,j}$ to be

\begin{equation}
(5.8) \quad g_{n,j} := \sum_{\sigma \in NC(2n)} g_\sigma(t).
\end{equation}

In the special case $j = 2n$, the only partition involved in the sum from (5.8) is $\sigma = 1_{2n}$, which has

\begin{equation*}
g_{1_{2n}}(t) = \kappa_{2n}(v_t, v_t^*, \ldots, v_t, v_t^*) = e^{nt} \cdot \kappa_{2n}(u_t, u_t^*, \ldots, u_t, u_t^*),
\end{equation*}

and (5.4) also follows. [End of Verif. of Claim]

Besides the $f_2, f_4, \ldots, f_{2n}, \ldots$ introduced in (5.5) we consider, also following [2], the function $f_0 : [0, \infty) \times (0, 1/2) \rightarrow \mathbb{R}$ defined by

\begin{equation*}
f_0(t, \theta) := \theta, \quad \forall t \geq 0 \text{ and } 0 < \theta < 1/2.
\end{equation*}
(Note that the definition of \( f_0 \) is not obtained by extending the range of \( n \) from \( \mathbb{N} \) to \( \mathbb{N} \cup \{0\} \) in Equation (5.4)). Theorem 3.4 in [2] gives us that for every \( n \geq 1 \), the partial derivative \( \partial_t f_{2n} \) satisfies the following recursion:

\[
\partial_t f_{2n}(t, \theta) = - \sum_{1 \leq k < \ell \leq 2n, k=\ell \mod 2} f_{2n-(\ell-k)}(t, \theta) f_{\ell-k}(t, \theta)
\]

\[+ e^t \sum_{1 \leq k < \ell \leq 2n, k=\ell \mod 2} f_{2n-(\ell-k)-1}(t, \theta) f_{\ell-k-1}(t, \theta).\]

(5.9)

For illustration we record that the special cases \( n = 1 \) and \( n = 2 \) of (5.9) come to \( \partial_t f_2 = e^t f_0^2 \), and respectively to \( \partial_t f_4 = -2f_2^2 + e^t \cdot 4f_0 f_2 \).

For a fixed \( t \in [0, \infty) \), both sides of Equation (5.9) are polynomials of degree \( 2n \) in \( \theta \); so it makes sense to extract the coefficient of \( \theta^{2n} \) in this equation. On the left-hand side, the coefficient of \( \theta^{2n} \) is equal to the derivative of \( g_{n,2n}(t) \), thus to

\[
e^{nt} \cdot (n\xi_n(t) + \frac{d\xi_n}{dt}(t)).
\]

(5.10)

On the right-hand side of (5.9) only the terms from the first of the two sums contribute to \( \theta^{2n} \), giving a coefficient equal to

\[- \sum_{1 \leq k < \ell \leq 2n, k=\ell \mod 2} \left(e^{(n-(\ell-k)/2)}\xi_{n-(\ell-k)/2}(t)\right) \cdot \left(e^{(\ell-k)/2}\xi_{(\ell-k)/2}(t)\right) = -e^{nt} \cdot n \sum_{m=1}^{n-1} \xi_m(t)\xi_{n-m}(t).\]

When we equate the latter quantity with the one in (5.10), formula (5.3) follows. \( \square \)

**Corollary 5.2.** Consider the function

\[
H(t, z) := \frac{1}{2} + \sum_{n=1}^{\infty} \xi_n(t)z^n
\]

defined on \( \{(t, z) \mid t \in [0, \infty), z \in \mathbb{C}, |z| < 1/16^2\} \). Then \( H \) satisfies the partial differential equation

\[
\partial_t H + 2z H \partial_z H = z,
\]

(5.12)

with initial condition \( H(z, 0) = 1/2 \).

**Proof.** The domain of \( H \) is considered by taking into account the bounds for \( \xi_n(t) \)'s that follow from Remark 2.5. In order to obtain (5.12), we square both sides of (5.11) and then we take partial derivative \( \partial_z \), to find that

\[
z \cdot \partial_z H^2(t, z) = \xi_1(t)z + 2 \left(\xi_2(t) + \xi_1^2(t)\right) z^2 + \cdots + n \left(\xi_n(t) + \sum_{m=1}^{n-1} \xi_m(t)\xi_{n-m}(t)\right) z^n + \cdots
\]

In view of Theorem 5.1 the latter equation can be written as

\[
z \cdot \partial_z H^2(t, z) = \xi_1(t)z - \xi_2(t) z^2 - \cdots - \xi_n(t) z^n - \cdots,
\]
and (5.12) follows. The initial condition on \( H(0, z) \) is also clear, since we have \( \xi_n(0) = 0 \) for all \( n \in \mathbb{N} \).

**Remark** 5.3. \(^1\) Starting from the fact that \( \xi_1(t) = \kappa_2(u_t, u_t^*) = 1 - e^{-t} \) and from the initial condition \( \xi_n(0) = 0, \forall n \geq 2 \), the recursion found in Theorem 5.1 can be used to calculate all the \( \xi_n \)'s: one has \( \xi_2(t) = -1 + 4e^{-t} - (2t + 3)e^{-2t} \), then \( \xi_3(t) = 2 - 15e^{-t} + 6(2t + 5)e^{-2t} - (6t^2 + 18t + 17)e^{-3t} \), and so on.

\(^2\) It stands to reason that one should also look for a description of the functions \( \xi_n(t) \) that is done by plain algebra (without resorting to the derivative \( \frac{d}{dt} \)), for a given value of \( t \).

That is, we are interested in an algebraic description for the function

\[
H_t : \{ z \in \mathbb{C} \mid |z| < 1/16^2 \} \to \mathbb{C}, \quad H_t(z) := H(t, z) = \frac{1}{2} + \sum_{n=1}^{\infty} \xi_n(t) z^n.
\]  

We will achieve this by examining the characteristic curves of the p.d.e. found in Corollary 5.2.

In order to state precisely what is the algebraic description obtained for \( H_t \), we introduce an auxiliary complex parameter \( c \) and we consider, for every \( t \in [0, \infty) \), the function

\[
\chi_t : \Omega_t \to \mathbb{C}, \quad \chi_t(c) := \frac{c^2(1 - c^2)e^{ct}}{(1 + c - (1 - c)e^{ct})^2},
\]

where \( \Omega_t \) is the open set \( \{ c \in \mathbb{C} \mid 1 + c \neq (1 - c)e^{ct} \} \). One has \( \Omega_t \ni 1 \), with \( \chi_t(1) = 0 \) and \( \chi_t'(1) = e^{-t/2} \neq 0 \). The inverse function theorem thus gives a \( \delta_t > 0 \) such that an analytic inverse for \( \chi_t \) can be defined on \( \{ z \in \mathbb{C} \mid |z| < \delta_t \} \), sending 0 back to 1. We denote this compositional inverse as

\[
\chi_t^{(-1)} : \{ z \in \mathbb{C} \mid |z| < \delta_t \} \to \mathbb{C}.
\]

\( \chi_t^{(-1)} \) is injective and its range-set \( \text{Ran}(\chi_t^{(-1)}) \) is an open subset of \( \Omega_t \).

Without loss of generality, we may assume that in (5.13) we have \( \delta_t < 1/16^2 \), so that \( H_t(z) \) from Equation (5.13) is sure to be defined, too, for \( |z| < \delta_t \).

**Theorem 5.4.** Let \( t \in [0, \infty) \) be fixed, and consider the analytic functions \( H_t \) and \( \chi_t^{(-1)} \) defined in Remark 5.3.2. Then one has

\[
[H_t(z)]^2 = z + \frac{1}{4} [\chi_t^{(-1)}(z)]^2, \quad |z| < \delta_t.
\]

**Proof.** Our strategy will be to prove the following fact.

\[
\begin{align*}
& \text{There exists } \varepsilon_t > 0 \text{ such that for every } c \in (1 - \varepsilon_t, 1 + \varepsilon_t) \subseteq \mathbb{R} \text{ one has:} \\
& \quad \rightarrow c \in \Omega_t \text{ and } |\chi_t(c)| < 1/16^2 \text{ (hence } H_t(\chi_t(c)) \text{ is defined);} \\
& \quad \rightarrow [H_t(\chi_t(c))]^2 = \chi_t(c) + \varepsilon_t^2.
\end{align*}
\]

This fact implies the statement of the theorem. Indeed, let us assume that (5.17) holds. Take a strictly decreasing sequence \( (c_n)_{n=1}^{\infty} \) in \( (1 - \varepsilon_t, 1 + \varepsilon_t) \cap \text{Ran}(\chi_t^{(-1)}) \), with \( \lim_{n \to \infty} c_n = 1 \),
and put \( z_n := \chi_t(c_n) \), \( n \in \mathbb{N} \). Then \( (z_n)_{n=1}^{\infty} \) are distinct points in \( \{ z \in \mathbb{C} \mid |z| < \delta_t \} \), with \( \lim_{n \to \infty} z_n = 0 \), and by applying the last line of (5.17) to the \( c_n \) we get
\[
[H_t(z_n)]^2 = z_n + \frac{1}{4}[\chi_t^{-1}](z_n)]^2, \quad \forall n \in \mathbb{N}.
\]
Hence the analytic functions appearing on the two sides of (5.16) coincide on a subset of \( \{ z \in \mathbb{C} \mid |z| < \delta_t \} \) which has 0 as accumulation point, and (5.16) follows.

We now start towards the proof of the fact stated in (5.17). We consider the rectangular strip
\[
R := [0, \infty) \times (-1/16^2, +1/16^2) \subseteq \mathbb{R}^2,
\]
and we consider the restriction of \( H \) (from its domain stated in Corollary 5.2) to \( R \). This restriction will still be denoted as \( H \), and \( 3 \) we put
\[
\Gamma := \{(s,x,u) \mid (s,x) \in R, u = H(s,x) \in \mathbb{R} \} \quad \text{(graph of restricted \( H \)).}
\]
We also consider the vector field \( V : R \times \mathbb{R} \to \mathbb{R}^3 \) defined by
\[
V(s,x,u) := (1, 2xu, x), \quad \text{for } (s,x) \in R, u \in \mathbb{R}.
\]
The partial differential equation (5.11) says that for every \( (s,x) \in R \), the vector \( V((s,x,H(s,x)) \) is orthogonal to the normal direction \( \left( (\partial_t H)(s,x), (\partial_x H)(s,x), -1 \right) \) to \( \Gamma \) at the point \( (s,x,H(s,x)) \). It follows that \( V((s,x,H(s,x)) \) gives a tangent direction to the graph \( \Gamma \), at the point \( (s,x,H(s,x)) \).

We next pick an \( a \in (-1/16^2, +1/16^2) \) and we consider a path (a.k.a. characteristic curve) \( L_a : [0, \beta(a)) \to \mathbb{R}^3 \) which has
\[
L_a(0) = (0, a, 1/2) \in \Gamma
\]
and follows the vector field \( V \):
\[
L_a'(s) = V(L_a(s)), \quad \forall s \in [0, \beta(a)).
\]
When we write \( L_a \) componentwise,
\[
L_a(s) = (p_a(s), q_a(s), r_a(s)), \quad 0 \leq s < \beta(a),
\]
the Equation (5.20) becomes a system of ordinary differential equations, for which (5.19) gives an initial condition:
\[
\begin{cases}
  p_a'(s) = 1, \quad q_a'(s) = 2q_a(s)r_a(s), \quad r_a'(s) = q_a(s), \\
  \text{with } p_a(0) = 0, q_a(0) = a, r_a(0) = 1/2.
\end{cases}
\]
Luckily, the Cauchy problem from (5.21) can be solved explicitly. More precisely: considering the auxiliary \( 3 \) constants
\[
c = \sqrt{1 - 4a}, \quad \alpha = \frac{1 - c}{1 + c} = \frac{4a}{(1 + c)^2},
\]
we get
\[
\begin{align*}
p_a(s) &= s, \quad q_a(s) = \frac{c^2 \alpha e^{cs}}{(1 - \alpha e^{cs})^2}, \quad r_a(s) = \frac{c}{2} \cdot \frac{1 + \alpha e^{cs}}{1 - \alpha e^{cs}},
\end{align*}
\]
\(3\) Since “\( t \)” is here a specific time that was fixed in the statement of the theorem, we will use the generic letter “\( s \)” for the first component of a point in \( R \).

\(4\) It is useful to keep in mind that \( c \) runs in a neighbourhood of 1 (it satisfies \( \sqrt{66}/8 < c < \sqrt{66}/8 \)), while \( \alpha \) runs in a neighbourhood of 0 (has \( \text{sign}(\alpha) = \text{sign}(a) \) and \( |\alpha| < 4|a| < 1/64 \)).
for $0 \leq s < \beta(a)$. A significant detail which comes up while solving (5.21) (and can, of course, be checked directly on (5.23)) is that one has

$$q_a(s) - (r_a(s))^2 = a - \frac{1}{4}, \quad \forall s \in [0, \beta(a)).$$

(5.24)

It is quite useful if at this point we take a moment to assess what we want to have for “$\beta(a)$” in the discussion from the preceding paragraph. Clearly, $\beta(a)$ must be in any case picked such that

- (i) $1 - ae^{cs} > 0$, $\forall s \in [0, \beta(a))$,
- (ii) $\left| \frac{c^2 ae^{cs}}{(1 - ae^{cs})^2} \right| < \frac{1}{16^2}, \quad \forall s \in [0, \beta(a))$.

The condition (i) ensures that the formulas (5.23) give indeed a well-defined path $L_a(s) = (p_a(s), q_a(s), r_a(s)), \; 0 \leq s < \beta(a)$; then (ii) ensures that $L_a(s) \in R \times \mathbb{R}$ (hence that “$V(L_a(s))$” makes sense) for every $s \in [0, \beta(a))$. We will moreover insist that

- (iii) $\beta(a)$ is a continuous function of $a \in (-1/16^2, +1/16^2)$,
- (iv) $\beta(0) > t$ ( = the time fixed in the statement of the theorem).

We leave it as a routine (though tedious) exercise to the reader to check that all the conditions (i)–(iv) are fulfilled if we go with

$$\beta(a) = \min \{ t + 1, \beta(i)(a), \beta(ii)(a) \} \quad \text{for } |a| < 1/16^2,$$

where $\beta(i), \beta(ii) : (-1/16^2, +1/16^2) \to [0, \infty]$ are continuous functions describing the natural bounds up to which an $s \in [0, \infty)$ fulfills the conditions (i) and (ii), respectively. For instance $\beta(i)(a)$ comes out as

$$\beta(i)(a) = \begin{cases}
\frac{1}{c} \ln \frac{1}{a}, & \text{if } \alpha > 0, \\
\infty, & \text{if } \alpha \leq 0,
\end{cases}$$

with $c = c(a)$ and $\alpha = \alpha(a)$ as defined in Equations (5.22).

We now invoke a basic result from the theory of quasi-linear partial differential equations, which states that: since it starts at a point $L_a(0) \in \Gamma$, the characteristic curve $L_a$ cannot leave the graph $\Gamma$ of $H$. (See e.g. the theorem on page 10 of [6].) In other words, one has

$$H( (p_a(s), q_a(s)) ) = r_a(s), \quad \forall a \in (-1/16^2, +1/16^2) \text{ and } s \in [0, \beta(a)).$$

(5.25)

If we square both sides of (5.25) and take into account the formula (5.24) (also the fact that $p_a(s) = s$), we arrive to

$$\left[ H( (s, q_a(s)) ) \right]^2 = q_a(s) + \left( \frac{1}{4} - a \right), \quad \forall a \in (-1/16^2, +1/16^2) \text{ and } s \in [0, \beta(a)).$$

(5.26)

Finally, let us return to the time $t \in [0, \infty)$ that was fixed in the statement of the theorem. Since $\beta$ is continuous and has $\beta(0) > t$, we can find $0 < \lambda_t < 1/16^2$ such that $\beta(a) > t$ for all $a \in (-\lambda_t, \lambda_t)$. For $|a| < \lambda_t$ we can thus put $s = t$ in (5.26), to obtain that

$$\left[ H( (t, q_a(t)) ) \right]^2 = q_a(t) + \left( \frac{1}{4} - a \right).$$

(5.27)

On the other hand, we make the following claim.

**Claim.** If $|a| < \lambda_t$, then $c := \sqrt{1 - 4a}$ belongs to the domain $\Omega_t$ of the function $\chi_t$, and one has $q_a(t) = \chi_t(c)$.

**Verification of Claim.** We have $t < \beta(a) \leq \beta(i)(a)$, hence (from how $\beta(i)(a)$ is defined) we get $1 - ae^{ct} > 0$, where $\alpha = (1-c)/(1+c)$. This implies $(1+c)-(1-c)e^{ct} > 0$, and it follows
that \( c \in \Omega_t \). The equality \( q_a(t) = \chi_t(c) \) is then immediately obtained by comparing the formulas which describe \( q_a(t) \) and \( \chi_t(c) \) (cf. Equation (5.11) and the case \( s = t \) of (5.28)).

[End of Verif. of Claim]

By using the above claim, we convert Equation (5.27) into

\[
[H_t(\chi_t(c)))]^2 = \chi_t(c) + \frac{c^2}{4}, \quad \text{with } c = \sqrt{1 - 4a},
\]

for every \( a \in (-\lambda_t, \lambda_t) \).

But when \( a \) runs in \((-\lambda_t, \lambda_t)\), the quantity \( c = \sqrt{1 - 4a} \) covers \((\sqrt{1 - 4\lambda_t}, \sqrt{1 + 4\lambda_t})\), which contains an open interval centered at 1. This implies the fact stated in (5.17), and concludes the proof.

**Remark 5.5.** The formula (5.16) from the preceding theorem can be used to calculate the alternating cumulants \( \xi_n(t) \) without doing a derivative \( \frac{d}{dt} \), but rather by starting from the Taylor expansion around 1 of the function \( \chi_t(c) \) defined in Remark 5.3.2:

\[
\chi_t(c) = (c - 1) \chi_t'(1) + \frac{(c - 1)^2}{2} \chi_t''(1) + \ldots
\]

\[
= (-\frac{1}{2} e^t)(c - 1) + \left(\frac{1}{2} e^{2t} - \left(\frac{3}{4} + \frac{t}{2}\right) e^t\right)(c - 1)^2 + \ldots
\]

Indeed, considering the expansion \( \chi_t^{(-1)}(z) = 1 + \lambda_1 z + \lambda_2 z^2 + \cdots \) of \( \chi_t^{(-1)} \) around 0, one can then calculate recursively the \( \lambda_n \) by writing that

\[
(5.28) \quad z = \chi_t(\chi_t^{(-1)}(z)) = (-\frac{1}{2} e^t)(\chi_t^{(-1)}(z) - 1) + \left(\frac{1}{2} e^{2t} - \left(\frac{3}{4} + \frac{t}{2}\right) e^t\right)(\chi_t^{(-1)}(z) - 1)^2 + \ldots
\]

\[
= (-\frac{1}{2} e^t)(\lambda_1 z + \lambda_2 z^2 + \cdots) + \left(\frac{1}{2} e^{2t} - \left(\frac{3}{4} + \frac{t}{2}\right) e^t\right)(\lambda_1 z + \lambda_2 z^2 + \cdots)^2 + \ldots,
\]

and by identifying coefficients. The \( \lambda_n \) come out as quasi-polynomials in \(-t\) (for instance, as immediately seen from the few terms recorded in (5.28), one gets \( \lambda_1 = -2e^{-t} \) and \( \lambda_2 = 4e^{-t} - (4t + 6)e^{-2t} \).

Finally, Equation (5.16) says that

\[
\frac{1}{4} + \xi_1(t) z + (\xi_2(t) + \xi_1^2(t))z^2 + (\xi_3(t) + 2\xi_1(t)\xi_2(t))z^3 + \ldots
\]

\[
= z + \frac{1}{4}\left(1 + 2\lambda_1 z + (2\lambda_2 + \lambda_1^2)z^2 + (2\lambda_3 + 2\lambda_1\lambda_2)z^3 + \cdots\right),
\]

which allows the recursive calculation of the \( \xi_n(t) \) (e.g. \( \xi_1(t) = 1 + \frac{\lambda_1}{2} = 1 - e^{-t} \)), then

\[
\xi_2(t) = \frac{1}{4}(2\lambda_2 + \lambda_1^2) - \xi_1(t)^2
\]

\[
= (2e^{-t} - (2t + 2)e^{-2t}) - (1 - e^{-t})^2 = -1 + 4e^{-t} - (2t + 3)e^{-2t},
\]

which agrees, of course, with the formulas stated in Remark 5.3.1).
6. Behaviour when \( t \to \infty \)

In this section we look at the behaviour of a joint cumulant \( \kappa_n(u_t^{(1)}, \ldots, u_t^{(n)}) \), for general \( \omega \in \{1, *\}^n \), when \( t \to \infty \). Specifically, we discuss how the limit and derivative at \( \infty \) relate to the corresponding polynomial \( Z_\omega \).

When it comes to just taking a plain limit \( t \to \infty \), things are straightforward: we have

\[
\lim_{t \to \infty} \kappa_n(u_t^{(1)}, \ldots, u_t^{(n)}) = \kappa_n(u^{(1)}, \ldots, u^{(n)}),
\]

where \( u \) is a Haar unitary; and the \( * \)-cumulants of a Haar unitary have a very nice form, first found in [10], which puts the spotlight on alternating strings of even length. For later perusal throughout the section, it is convenient to include the latter concept into the following definition.

**Definition 6.1.** 1° Let \( n \) be an even positive integer. A string \( \omega \in \{1, *\}^n \) is said to be **alternating** if it is equal either to \((1, *, 1, *, \ldots, 1, *)\) or to \((*, 1, *, 1, \ldots, *, 1)\).

2° Let \( n \) be an odd positive integer. A string \( \omega \in \{1, *\}^n \) is said to be **alternating** if it is obtained by a cyclic permutation of components from either \((1, *, 1, \ldots, *1)\) or \((*, 1, *, 1, \ldots, *, 1)\).

[So note that we only have 2 alternating strings of length \( n \) when \( n \) is even, but we have \( 2n \) alternating strings of length \( n \) when \( n \) is odd. A concrete example:

\((1, 1, *, 1, *), (*, 1, 1, *, 1), (1, *, 1, 1, *), (*, 1, *, 1, 1), (1, *1, *, 1, *)\)

are 5 of the 10 alternating strings of length 5, and the remaining alternating strings of length 5 are obtained by swapping the roles of ‘1’ and ‘*’ in the above list.]

**Proposition 6.2.** Let \( \omega = (\omega(1), \ldots, \omega(n)) \) be in \( \{1, *\}^n \), for \( n \in \mathbb{N} \).

1° We have

\[
\lim_{t \to \infty} \kappa_n(u_t^{(1)}, \ldots, u_t^{(n)}) = \begin{cases} (-1)^{k-1} C_{k-1}, & \text{if } n \text{ is even, } n = 2k, \text{ and } \\
0, & \text{otherwise,} \end{cases}
\]

with \( C_{k-1} \) the \((k - 1)\)th Catalan number (same as in Equation (1.4) of the introduction).

2° Suppose that \( n \) is even, \( n = 2k \). Consider the polynomial \( Z_\omega \) and its writing (as in Proposition 3.1)

\[
Z_\omega(x, y) = Z_{\omega}^{(2k)}(x) y^{2k} + Z_{\omega}^{(2k-2)}(x) y^{2k-2} + \cdots + Z_{\omega}^{(2)}(x) y^2 + Z_{\omega}^{(0)}(x).
\]

Then \( Z_{\omega}^{(0)}(x) \) is a constant polynomial, where the constant is given by the right-hand side of Equation (6.2).

**Proof.** 1° This is the limit from [6.1], where we also invoke the explicit formula for the \( * \)-cumulants of a Haar unitary that was found in [10]. (See Section 3.4 of [10], or Proposition 15.1 in the monograph [9].)

2° In view of how \( Z_\omega \) is defined, from 1° it follows that \( \lim_{t \to \infty} Z_\omega(t, e^{-t/2}) \) exists. Since \( \lim_{t \to \infty} Z_\omega^{(n-2j)}(t) \cdot (e^{-t/2})^{n-2j} = 0 \) for \( j < n/2 \), we then infer that \( \lim_{t \to \infty} Z_\omega^{(0)}(t) \) exists as.
well. But this can only happen if $Z_\omega^{(0)}$ is a constant (and the constant in question must be the one appearing on the right-hand side of (6.2)). \hfill \Box

The main point of the present section is that when we look at strings of odd length, we get the following analogue of Proposition 6.2.2.

**Theorem 6.3.** Let $\omega = (\omega(1), \ldots, \omega(n))$ be in $\{1, *\}^n$. Suppose that $n$ is odd, $n = 2k - 1$, and consider the polynomial $Z_\omega$ and its writing

$$Z_\omega(x, y) = Z_\omega^{(2k-1)}(x) y^{2k-1} + Z_\omega^{(2k-3)}(x) y^{2k-3} + \cdots + Z_\omega^{(3)}(x) y^3 + Z_\omega^{(1)}(x) y.$$

Then $Z_\omega^{(1)}(x)$ is a constant polynomial, and more precisely:

$$Z_\omega^{(1)}(x) = \begin{cases} (-1)^{k-1}C_{k-1}, & \text{if } \omega \text{ is alternating,} \\ 0, & \text{otherwise.} \end{cases} \quad (6.3)$$

**Proof.** It is easily seen that if $n$ is odd, then every $\omega \in \{1, *\}^n$ has $\text{Switch}(\omega) \leq n - 1$, with equality holding if and only if $\omega$ is alternating. Thus if $n$ is odd and $\omega \in \{1, *\}^n$ is not alternating, then Theorem 3.8 can be applied with $j = (n - 1)/2$, and gives that $Z_\omega^{(1)}(x)$ is constantly equal to 0.

We are left to prove the following:

$$Z_\omega^{(1)}(x) \text{ is constantly equal to } (-1)^{k-1}C_{k-1}. \quad (6.4)$$

We will prove this statement by induction on $k$. The case $k = 1$ is clear, since $Z_\omega^{(1)}(x, y) = Z_\omega^{(*)}(x, y) = y$ (corresponding to the fact that $u_t$ has expectation $e^{-t/2}, \forall t \in [0, \infty)$). The remaining part of the proof is devoted to the induction step: we fix $k \geq 2$, we assume that (6.3) holds for alternating strings of length 1, 3, \ldots, $2k - 3$, and we prove that it also holds for alternating strings of length $2k - 1$.

Since any two alternating strings of length $2k - 1$ can be obtained from each other by operations which do not affect $Z_\omega$’s, it will suffice to verify that, for the $k$ that was fixed, we have

$$Z_\omega^{(1)}(x) = (-1)^{k-1}C_{k-1}. \quad (6.5)$$

In order to verify (6.5), we invoke the recursion from Proposition 3.5 (used for the string $\omega = (1, 1, *, \ldots, 1, *) \in \{1, *\}^{2k-1}$), and we extract the coefficient of $y$ on both sides of that recursion. On the right-hand side of the resulting equation we get a sum which (same as in Equation (3.9) of Proposition 3.5) is indexed by $m$, with $1 \leq m \leq n - 1 = 2k - 2$. By grouping the terms of the sum according to the parity of $m$, we obtain that

$$Z_\omega^{(1)}(1, 1, *, \ldots, 1, *) = -\left( \sum_{\text{odd}} + \sum_{\text{even}} \right), \quad (6.6)$$

where

$$Z_\omega^{(1)}(1, 1, *, \ldots, 1, *) = \sum_{\text{odd}} Z_\omega^{(1)}(1, 1, *, \ldots, 1, *) + \sum_{\text{even}} Z_\omega^{(1)}(1, 1, *, \ldots, 1, *) Z_\omega^{(0)}(1, *) \quad (6.7)$$

In order to simplify the left-hand side of (6.6), we will use the following identity:

$$Z_\omega^{(1)}(1, 1, *, \ldots, 1, *) = \sum_{\text{odd}} Z_\omega^{(1)}(1, 1, *, \ldots, 1, *) + \sum_{\text{even}} Z_\omega^{(1)}(1, 1, *, \ldots, 1, *) Z_\omega^{(0)}(1, *). \quad (6.8)$$
and
\[ \Sigma_{\text{even}} = Z_{(1, 1)}^{(0)} Z_{(*, 1, *, \ldots, 1, *)}^{(1)} Z_{(1, 1, *, \ldots, 1, *)}^{(0)} + Z_{(1, 1, *, \ldots, 1, *)}^{(0)} Z_{(1, 1, *, \ldots, 1, *)}^{(1)} \cdots + Z_{(1, 1, *, \ldots, 1, *)}^{(0)} Z_{(1, 1, *, \ldots, 1, *)}^{(1)} . \]

The sum \( \Sigma_{\text{even}} \) is equal to 0, because of
\[ Z_{(1, 1)}^{(0)} = Z_{(1, 1, *, \ldots, 1, *)}^{(0)} = \cdots = Z_{(1, 1, *, \ldots, 1, *)}^{(0)} = 0 \]
(cf. Proposition 6.2, case of non-alternating strings). On the other hand, the induction hypothesis and the case of alternating strings in Proposition 6.2 give us that
\[ \Sigma_{\text{odd}} = (-1)^0 C_0 \cdot (-1)^{k-2} C_{k-2} + (-1)^1 C_1 \cdot (-1)^{k-3} C_{k-3} + \cdots + (-1)^{k-2} C_{k-2} \cdot (-1)^0 C_0 . \]
Thus Equation (6.6) comes, after all, to
\[ Z_{(1, 1, *, \ldots, 1, *)}^{(1)} (x) = (-1)^{k-1} \sum_{j=0}^{k-2} C_j \cdot C_{k-2-j} . \]

A basic recursion for Catalan numbers says that the sum on the right-hand side of the latter equality is just \( C_{k-1} \), and the required formula (6.5) follows. \( \square \)

We can now follow the same kind of connection as in Proposition 6.2 (but going in reverse order) in order to obtain the “derivative at \( t = \infty \)” for \( * \)-cumulants of the \( u_t \)'s.

**Corollary 6.4.** Let \( n \) be a positive integer and let \( \omega = (\omega(1), \ldots, \omega(n)) \) be a string in \( \{1, *\}^n \). We consider the limit
\[ \lim_{t \to \infty} \frac{\kappa_n (u_t^{\omega(1)}, \ldots, u_t^{\omega(n)}) - \kappa_n (u^{\omega(1)}, \ldots, u^{\omega(n)})}{e^{-t/2}} \]
where \( u_t \) is the free unitary Brownian motion at time \( t \), and \( u \) is a Haar unitary. This limit exists, and is equal to
\[ \begin{cases} 
(-1)^{k-1} C_{k-1}, & \text{if } n \text{ is odd, } n = 2k - 1, \text{ and } \\
0, & \text{otherwise.} 
\end{cases} \]

**Proof.** If \( n \) is even, \( n = 2k \), then the difference on the numerator of the fraction in (6.7) is
\[ Z_{\omega}^{(2k)} (t) \cdot (e^{-t/2})^{2k} + Z_{\omega}^{(2k-2)} (t) \cdot (e^{-t/2})^{2k-2} + \cdots + Z_{\omega}^{(2)} (t) \cdot (e^{-t/2})^{2} , \]
and when divided by \( e^{-t/2} \) this is sure to go to 0 as \( t \to \infty \).

If \( n \) is odd, \( n = 2k - 1 \), then the difference on the numerator of the fraction in (6.7) is
\[ Z_{\omega}^{(2k-1)} (t) \cdot (e^{-t/2})^{2k-1} + Z_{\omega}^{(2k-3)} (t) \cdot (e^{-t/2})^{2k-3} + \cdots + Z_{\omega}^{(3)} (t) \cdot (e^{-t/2})^{3} , \]
and when divided by \( e^{-t/2} \) this converges to the constant \( Z_{\omega}^{(1)} (t) \) described in Theorem 6.3, and the result follows. \( \square \)
Remark 6.5. The limit from Corollary 6.4 points towards an “infinitesimal structure” which accompanies the *-distribution of a Haar unitary, in the sense of the paper of Belinschi and Shlyakhtenko [1]. In order to relate to the framework of [1], one has to do a change a variable: consider the noncommutative probability spaces \((B_s, \psi_s)\), defined for \(s \in [0, 1]\), where for \(s \neq 0\) we put \(B_s = \mathcal{A}_{-2 \log s}, \psi_s = \varphi_{-2 \log s}\), while for \(s = 0\) we take \((B_0, \psi_0)\) to be the space where the Haar unitary lives. With this change of variable, the limit from (6.7) becomes a derivative at 0, as prescribed in [1].

Remark 6.6. The Haar unitary is one of the basic examples which prompted the study of \(R\)-diagonal elements. As reviewed in the introduction, an \(R\)-diagonal element \(a\) in a tracial *-probability space always has a determining sequence, which determines all the joint cumulants of \(a\) and \(a^*\); in the special case of a Haar unitary, the determining sequence consists of signed Catalan numbers. Now, in the case of a Haar unitary \(u\), Corollary 6.4 brings into the picture an additional “infinitesimal determining sequence”, which happens to also consist of signed Catalan numbers, and which determines the derivatives at \(\infty\) of all the joint cumulants of \(u_t\) and \(u_t^*\) (with \(u_t\) seen as an approximation of \(u\)). In the remaining part of the section we will discuss how the concept of infinitesimal determining sequence can be extended to general \(R\)-diagonal elements.

The idea is as follows: any \(R\)-diagonal distribution in a tracial *-probability space can be realized by using an element of the form \(a = u q\), where \(u\) is Haar unitary, \(q = q^*\), and \(q\) is free from \(\{u, u^*\}\). So then elements of the form \(a_t = u_t q\) (where \(q\) is assumed now to also be free from \(\{u_t, u_t^*\}\)) are, as \(t \to \infty\), some kind of “approximate \(R\)-diagonal elements”.

Paralleling Corollary 6.4 we can then study limits

\[
\lambda_\omega := \lim_{t \to \infty} \frac{\kappa_n(a_t^{\omega(1)}, \ldots, a_t^{\omega(n)}) - \kappa_n(a^{\omega(1)}, \ldots, a^{\omega(n)})}{e^{-t/2}}
\]

with \(a = u q\) and \(a_t = u_t q\). The limits \(\lambda_\omega\) turn out to exist for all strings \(\omega\), and the \(\lambda_\omega\)'s which might possibly be non-zero are naturally organized in an “infinitesimal determining sequence” associated to the distribution of \(q\).

In order not to make the discussion excessively long, we will only give an outline, in Remark 6.7, of what is the method for calculating the limits \(\lambda_\omega\) from Equation (6.9). Then we will state the conclusion of the calculation in Proposition 6.8, leaving it to the interested reader to fill in the details of proof for that proposition.

Remark 6.7. 1° For the discussion in this remark, it is convenient that we first introduce some (ad-hoc) terminology which describes the partitions we want to work with.

First of all, for every string \(\omega = (\omega(1), \ldots, \omega(n)) \in \{1,*\}^n\) we denote

\(U_\omega := \{2i - 1 \mid 1 \leq i \leq n, \ \omega(i) = 1\} \cup \{2i \mid 1 \leq i \leq n, \ \omega(i) = *\},\) and

\(Q_\omega := \{1, 2, \ldots, 2n\} \setminus U_\omega = \{2i \mid 1 \leq i \leq n, \ \omega(i) = 1\} \cup \{2i - 1 \mid 1 \leq i \leq n, \ \omega(i) = *\}.
\)

(Thus \(U_\omega, Q_\omega \subseteq \{1, \ldots, 2n\}\), and they have \(n\) elements each.)

Second of all, suppose that \(\omega \in \{1,*\}^n\) where \(n\) is an odd positive integer. Let us agree to say that a partition \(\pi \in NC(n)\) is special-\(\omega\)-alternating when it has the following properties:

\((\text{Sp-}\omega\text{-Alt})\)

\[
\begin{align*}
\text{(i) For every block } & V \in \pi, \text{ the string } \omega \mid V \in \{1,*\}^{\lvert V \rvert} \text{ is alternating; and} \\
\text{(ii) } & \pi \text{ has exactly one block, } V_0, \text{ such that } \lvert V_0 \rvert \text{ is odd.}
\end{align*}
\]
The partitions that we want to look at are in fact in $NC(2n)$. More precisely, for the same $\omega$ as in preceding paragraph, let us say that a partition $\sigma \in NC(2n)$ is $\omega$-good when it has the following properties:

\[
(\omega\text{-Good}) \quad \begin{cases}
(j) & \text{For every } V \in \sigma \text{ we have that either } V \subseteq Q_\omega \text{ or } V \subseteq U_\omega; \\
(jj) & \sigma \cap \{\{1,2\},\{3,4\}, \ldots,\{2n-1,2n\}\} = 1_{2n} \text{ ( = max. of } NC(2n))); \\
(jj) & \text{If } \pi \in NC(n) \text{ is the reindexing of the restriction } \sigma \mid U_\omega \in NC(U_\omega), \text{ then } \pi \text{ is special-$\omega$-alternating.}
\end{cases}
\]

We make the notation

\[
NC_{\omega\text{-good}}(2n) := \{\sigma \in NC(2n) \mid \sigma \text{ is } \omega\text{-good}\}.
\]

For a concrete example illustrating the above terminology, let us assume for a moment that $n = 3$ and $\omega = (1,*1)$. We then have $U_\omega = \{1,4,5\}$ and $Q_\omega = \{2,3,6\}$. The list of partitions in $NC(3)$ which are special-$\omega$-alternating consists of

\[
\pi_1 = 1_3 = \{\{1,2,3\}\}, \quad \pi_2 = \{\{1,2\},\{3\}\}, \text{ and } \pi_3 = \{\{1\},\{2,3\}\}.
\]

In order to find what is $NC_{\omega\text{-good}}(6)$ we take each of $\pi_1, \pi_2, \pi_3$, turn it (in the natural way) into a partition of $U_\omega$, and try to fill that with a partition of $Q_\omega$ such that the $(\omega\text{-Good})$ conditions $(j)$, $(jj)$ and $(jjj)$ are fulfilled. For $\pi_1$ there are two possible ways of doing so, which give the partitions

\[
\sigma_1 = \{\{1,4,5\},\{2,3\},\{6\}\} \text{ and } \sigma'_1 = \{\{1,4,5\},\{2\},\{3\},\{6\}\} \text{ in } NC(6).
\]

For $\pi_2$ it is not possible to find any corresponding partitions in $NC_{\omega\text{-good}}(6)$ – no matter how we try to incorporate $\{1,4\}$ and $\{5\}$ as blocks of a partition in $NC(6)$, the condition $(jj)$ will not be fulfilled. For $\pi_3$ there are three corresponding partitions in $NC_{\omega\text{-good}}(6)$:

\[
\sigma_3 = \{\{1\},\{2,3,6\},\{4,5\}\}, \quad \sigma'_3 = \{\{1\},\{2,3\},\{4,5\},\{6\}\}, \quad \text{and } \sigma''_3 = \{\{1\},\{2,6\},\{3\},\{4,5\}\}.
\]

Hence in this example we find $NC_{\omega\text{-good}}(6) = \{\sigma_1,\sigma'_1,\sigma_3,\sigma'_3,\sigma''_3\}$, with $\sigma_1,\ldots,\sigma''_3$ as listed above.

2° Now let us consider the framework of Remark 6.6 where we have $q = q^*$ such that $q$ is free from $\{u_t,u_t^*\}$, and we examine the joint free cumulants of $a_t$ and $a_t^*$, for $a_t := u_tq$. Let $\omega$ be a string in $\{1,*\}^n$ for some $n \geq 1$ (even or odd), and let us examine the cumulant $\kappa_n(a_t^{\omega(1)},\ldots,a_t^{\omega(n)})$. In this cumulant we replace every $a_t$ as $u_tq$ and every $a_t^*$ as $qu_t^*$, and we use the formula for a cumulant with products of entries. This takes us to a summation over a set of partitions in $NC(2n)$, and a moment’s thought shows us that the set in question is

\[
S_\omega = \left\{ \sigma \in NC(2n) \mid \begin{array}{c}
\text{every } V \in \sigma \text{ is contained in } Q_\omega \text{ or in } U_\omega, \\
\sigma \cap \{\{1,2\},\{3,4\}, \ldots,\{2n-1,2n\}\} = 1_{2n}
\end{array} \right\}.
\]

In the summation indexed by $S_\omega$, every term is a product where some factors are joint cumulants of $u_t$ and $u_t^*$, while some other factors are cumulants of $q$. The dependence on $t$ is coming exclusively from the factors involving $u_t$ and $u_t^*$, which are quasi-polynomials in $t/2$, in the way found earlier in the paper. Another moment’s thought (and an argument analogous to the one used in the proof of Corollary 6.4) show that: if we are interested in the limit $\lambda_\omega$ from Equation (6.9), then the only $\sigma$’s from $S_\omega$ that can contribute to it are those where the restriction $\sigma \mid U_\omega$ is (or rather becomes, after the reordering of $U_\omega$ into $\{1,\ldots,n\}$) a special-$\omega$-alternating partition. This requires in particular that $n$ is odd; and if $n$ is odd, then the various conditions required from a $\sigma$ (in order to contribute to the limit $\lambda_\omega$) amount together to asking that $\sigma \in NC_{\omega\text{-good}}(2n)$. 
In short, what we get is this: if \( n \) is even, then the limit \( \lambda_\omega \) from (6.9) is 0. If \( n \) is odd, then we get a formula
\[
\lambda_\omega = \sum_{\sigma \in NC_{\omega-good}(2n)} \text{term}_\sigma,
\]
with every \( \text{term}_\sigma \) being written as a product in the way described as follows. Let \( V_0 \) be the unique block of \( \sigma \) such that \( V_0 \subseteq U_\omega \) and \( |V_0| \) is odd; let \( V_1, \ldots, V_k \) be the other blocks of \( \sigma \) which are contained in \( U_\omega \); and let \( W_1, \ldots, W_\ell \) be the blocks of \( \sigma \) which are contained in \( Q_\omega \). Then we have
\[
(6.12) \quad \text{term}_\sigma = (-1)^{|V_0|-1}/2 C_{(|V_0|-1)/2} \cdot \prod_{i=1}^{k} (-1)^{|V_i|-2}/2 C_{(|V_0|-2)/2} \cdot \prod_{j=1}^{\ell} \kappa_{|W_j|}(q, \ldots, q).
\]
(The various signed Catalan numbers appearing in Equation (6.12) were chosen according to Proposition 6.2.2 and to Theorem 6.3.)

For instance, in the concrete example \( \omega = (1, *, 1) \) shown at the end of part 1° of this remark, we get
\[
\text{term}_{\sigma_1} = (-1) \cdot \kappa_1(q) \kappa_2(q, q), \quad \text{term}_{\sigma_1'} = (-1) \cdot (\kappa_1(q))^3,
\]
\[
\text{term}_{\sigma_3} = 1 \cdot 1 \cdot \kappa_3(q, q, q), \quad \text{term}_{\sigma_3'} = \text{term}_{\sigma_3''} = 1 \cdot 1 \cdot \kappa_1(q) \kappa_2(q, q),
\]
with the overall effect that
\[
(6.13) \quad \lambda_{(1, *, 1)} = \kappa_3(q, q, q) + \kappa_1(q) \kappa_2(q, q) - (\kappa_1(q))^3.
\]

The next thing to point out is this: the set of partitions \( NC_{\omega-good}(2n) \) was defined for every \( \omega \in \{1, *\}^n \) with \( n \) odd, but actually we have \( NC_{\omega-good}(2n) = \emptyset \) whenever \( \omega \) is not alternating. The proof of this fact is obtained by induction on \( n \) (where we mention that condition (jj) from the \( \omega \)-Good list plays an essential role in the induction step). Since it is immediate that a limit \( \lambda_\omega \) is invariant under cyclic permutations of the components of \( \omega \), we come to the conclusion that the limits \( \lambda_\omega \) (for all \( \omega \)) are completely determined by the sequence \( (\lambda_{\omega_k})_{k=1}^\infty \), where we put
\[
(6.14) \quad \omega_k := (1, *, 1, \ldots, *, 1), \quad k \in \mathbb{N}.
\]
The numbers \( \lambda_{\omega_k} \) are calculated in terms of the distribution of \( q \) in the way described above (for example, (6.13) gives the formula for \( \lambda_{\omega_2} \)). The next proposition records these findings, and also indicates a more economical way of re-writing the formula for \( \lambda_{\omega_k} \), based on joint cumulants of \( q \) and \( q^2 \). The proof of this proposition is left to the reader.

**Proposition 6.8.** Let \( \omega \) be a string in \( \{1, *\}^n \), for some \( n \in \mathbb{N} \), and consider the limit \( \lambda_\omega \), in the framework of Equation (6.4) from Remark 6.7. This limit exists, and is described as follows.

1° If not true that \( n \) is odd and \( \omega \) is alternating, then \( \lambda_\omega = 0 \).

2° If \( n \) is odd and \( \omega \) is alternating, then we have
\[
(6.15) \quad \lambda_\omega = \sum_{\pi \in NC(n)} \text{Moeb}(\pi, 1_n) \prod_{V \in \pi} \kappa_{|V|}((q, q^2, \ldots, q^2) \mid V).
\]
\( \square \)
Recall that the values \( \text{Moeb}(\pi, 1_n) \) of the Möbius function are products of signed Catalan numbers, as reviewed in Section 2. As an illustration of what (6.15) says, for odd \( n \leq 5 \) we have
\[
\lambda_{(1)} = \kappa_1(q), \quad \lambda_{(1,*,1)} = \kappa_2(q,q^2) - \kappa_1(q)\kappa_1(q^2), \\
\lambda_{(1,*,1,*,1)} = \kappa_3(q,q^2,q^2) - 2\kappa_2(q,q^2)\kappa_1(q^2) - \kappa_1(q)\kappa_2(q^2,q^2) + 2\kappa_1(q)\left(\kappa_1(q^2)\right)^2.
\]

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