REACTION-DIFFUSION EQUATIONS WITH FRACTIONAL DIFFUSION ON NON-SMOOTH DOMAINS WITH VARIOUS BOUNDARY CONDITIONS

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Abstract. We investigate the long term behavior in terms of finite dimensional global attractors and (global) asymptotic stabilization to steady states, as time goes to infinity, of solutions to a non-local semilinear reaction-diffusion equation associated with the fractional Laplace operator on non-smooth domains subject to Dirichlet, fractional Neumann and Robin boundary conditions.

1. Introduction. The main concerns in the present paper are to investigate the existence, the regularity and the long-time behavior of solutions to some non-local reaction-diffusion equations associated with the fractional Laplace operator with Dirichlet, fractional Neumann and fractional Robin type boundary conditions on non-smooth subsets of \( \mathbb{R}^N \). In order to introduce the fractional Laplacian, let \( 0 < s < 1 \), \( \Omega \subset \mathbb{R}^N \) an arbitrary open set and set

\[
\mathcal{L}^1(\Omega) := \{ u : \Omega \to \mathbb{R} \text{ measurable}, \quad \int_{\Omega} \frac{|u(x)|}{(1 + |x|)^{N+2s}} \, dx < \infty \}. 
\]

For \( u \in \mathcal{L}^1(\mathbb{R}^N) \), \( x \in \mathbb{R}^N \) and \( \varepsilon > 0 \), we write

\[
(-\Delta)^s_x u(x) = C_{N,s} \int_{\{y \in \mathbb{R}^N, |y-x| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy
\]

with the normalized constant \( C_{N,s} \) given by

\[
C_{N,s} = s^{2s} \Gamma \left( \frac{N+2s}{2} \right) \frac{2}{\pi s^2 \Gamma(1-s)}.
\]
where $\Gamma$ denotes the usual Gamma function. The fractional Laplacian $(-\Delta)^s u$ of the function $u$ is defined by the formula

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy = \lim_{\varepsilon \downarrow 0} (-\Delta)^s_{\varepsilon} u(x), \quad x \in \mathbb{R}^N,$$

provided that the limit exists. We notice that if $0 < s < 1/2$ and $u$ is smooth (for example, Lipschitz continuous), then the integral in (1.1) is in fact not singular near $x$. We also recall that in the whole space $\mathbb{R}^N$, using the Fourier transform, $(-\Delta)^s$ can be also defined as a pseudo-differential operator with symbol $|\xi|^{2s}$. If one wishes to consider the fractional Laplace operator $(-\Delta)^s$ on open subsets $\Omega$ of $\mathbb{R}^N$ it cannot be used on $\Omega$ automatically due to its nonlocal character. In order to give a proper definition, we follow [28, 29, 30, 48] in the following fashion. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. We restrict the integral kernel of the fractional Laplacian to the open set $\Omega$. For $u \in \mathcal{L}^1(\Omega)$, $x \in \Omega$ and $\varepsilon > 0$, we let

$$A_{\Omega,\varepsilon}^s u(x) = C_{N,s} \int_{\{y \in \Omega : |y - x| > \varepsilon\}} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy,$$

and we define the operator

$$A_{\Omega}^s u(x) = C_{N,s} \text{P.V.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy = \lim_{\varepsilon \downarrow 0} A_{\Omega,\varepsilon}^s u(x), \quad x \in \Omega,$$

provided that the limit exists. As in the case $\mathbb{R}^N$, if $s \in (0,1/2)$ and $u$ is smooth then the integral in (1.2) is not singular near $x$. We call the operator $A_{\Omega}^s$ the regional fractional Laplacian (cf. [28, 29, 30]). The regional fractional $p$-Laplace operator with $p \in (1,\infty)$ has been also introduced in [49]. Let now $u \in \mathcal{D}(\Omega)$, the space of infinitely continuously differentiable functions with compact support in $\Omega$. Since $u = 0$ on $\mathbb{R}^N \setminus \Omega$, a simple calculation gives

$$A_{\Omega}^s u(x) := C_{N,s} \text{P.V.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy$$

$$= C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy - C_{N,s} \int_{\mathbb{R}^N \setminus \Omega} \frac{u(x)}{|x - y|^{N+2s}} \, dy$$

$$= (-\Delta)^s u(x) - V_{\Omega}(x) u(x),$$

where the potential $V_{\Omega}$ is given by

$$V_{\Omega}(x) := C_{N,s} \int_{\mathbb{R}^N \setminus \Omega} |x - y|^{-N-2s} \, dy, \quad x \in \Omega.$$

More precisely, we have

$$(-\Delta)^s u(x) = A_{\Omega}^s u(x) + V_{\Omega}(x) u(x), \quad \text{for all } u \in \mathcal{D}(\Omega).$$

The operator $A_{\Omega}^s$ describes a particle jumping from one point $x \in \Omega$ to another $y \in \Omega$ with intensity proportional to $|x - y|^{-N-2s}$. Parabolic problems associated with the operator $A_{\Omega}^s$ have been intensively studied in [6, 11, 29, 30] and the references therein, employing probabilistic approaches, and in [48] by using the method of Dirichlet forms on non-smooth domains. Based on (1.4), we then view the fractional Laplacian $(-\Delta)^s$ with domain $\mathcal{D}(\Omega)$ as a perturbation of the regional fractional operator $A_{\Omega}^s$ with the non-negative potential $V_{\Omega}$. Recently, various elliptic and parabolic equations associated with the fractional Laplace operator with Dirichlet boundary conditions were also investigated by Caffarelli et al. [8, 9, 10].
In this paper, we shall be concerned with non-local diffusion processes associated with the fractional Laplace operator with various boundary conditions. To be more precise, we consider diffusion processes described by the following systems

$$
\begin{align*}
\partial_t u + d(-\Delta)^s u + f(u) &= 0 \quad \text{in } \Omega \times (0, \infty), \\
u &= 0 \quad \text{on } (\mathbb{R}^N \setminus \Omega) \times (0, \infty), \\
u(0) &= u_0 \quad \text{in } \Omega,
\end{align*}
$$

(1.5)

and

$$
\begin{align*}
\partial_t u + dA^s_\Omega u + f(u) &= 0 \quad \text{in } \Omega \times (0, \infty), \\
u &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
u(0) &= u_0 \quad \text{in } \Omega,
\end{align*}
$$

(1.6)

and

$$
\begin{align*}
\partial_t u + dA^s_\Omega u + f(u) &= 0 \quad \text{in } \Omega \times (0, \infty), \\
dB_{N,s} N^{2-2s} u + \gamma u &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
u(0) &= u_0 \quad \text{in } \Omega.
\end{align*}
$$

(1.7)

In (1.5), (1.6) and (1.7), $f = f(u)$ plays the role of nonlinear source, not necessarily monotone and $d > 0$ is a diffusion coefficient. In (1.5), the operator $(-\Delta)^s$ denotes the fractional Laplace operator defined in (1.1) and in (1.6) and (1.7), $A^s_\Omega$ is the regional fractional Laplacian given by (1.2). Finally in (1.7), $N^{2-2s} u$ denotes the fractional normal derivative of the function $u$ in direction of the outer normal vector (see Section 2 below), $B_{N,s}$ is a normalized constant (see (2.17) below) and $\gamma \in L^\infty(\partial \Omega)$ is a non-negative function.

Our motivation for considering such problems is two-fold. First, the systems (1.5)-(1.7) and their stationary versions have been used recently to describe the motion of nonlinear defects in crystalline materials in the field of dislocation dynamics (see, e.g., [26, 36]). In the theory of phase-field and interfacial dynamics, these equations are usually referred as the fractional Allen-Cahn equation (see, e.g., [31, 41]). Moreover, nonlocal reaction-diffusion equations have been also considered in the monograph [4] but the integral operators there are generally smooth or only mildly singular (i.e., the kernel is at least integrable over $\mathbb{R}^N$). On the other hand, the linear parabolic equation $\partial_t u + (-\Delta)^s u = 0$, $s \in (0, 1)$, instead of the usual parabolic equation $\partial_t u - \Delta u = 0$, is a much studied topic of anomalous diffusion in physics, probability and finance (see, e.g., [1, 33, 40, 42]). We also refer the reader to an interesting tutorial in [47] which introduces the main concepts behind normal and anomalous diffusion. Second, our work is further motivated by the need to develop a complete dynamical theory for these problems where not much seems to be known about basic issues, such as global existence and regularity, uniqueness, blow-up phenomena and longtime behavior of solutions, as time goes to infinity. This seems to be due to the fact that the parabolic structure of (1.5)-(1.7) has not been exploited before, an issue which is intimately connected with an $L^2-L^\infty$ smoothing result in $(0, \infty) \times \Omega$ of solutions. This is essential to the study of the asymptotic behavior of these systems, in terms of global attractors and $\omega$-limit sets. We also emphasize the generality of our results by assuming only minimal conditions on the regularity of $\Omega$: we shall assume $\Omega$ to be simply an open subset of $\mathbb{R}^N$ in the case of problems (1.5)-(1.6), while in the third case (1.7) it suffices to assume that $\partial \Omega$ is Lipschitz continuous. Our current contribution is also motivated by our recent work on parabolic equations with classical diffusion on rough domains and nonlocal boundary conditions (see [24]).
The main novelties of the present paper are the following:

(I) We show the existence of global and unique strong solutions (which are Lipschitz continuous in time for $t \in (0, \infty)$) for any of the preceding problems provided that the nonlinearity $f \in C^1(\mathbb{R})$ obeys

$$\liminf_{|\tau| \to +\infty} \frac{f(\tau)}{\tau} > -\lambda_*,$$

for some constant $\lambda_* \in [0, C_*]$, where roughly speaking $C_* = C(\Omega, N, s) > 0$ is the best Sobolev/Poincaré constant in the embedding $W^{s,2}(\Omega) \subset L^2(\Omega)$ (see Section 3). This condition turns out to be optimal in the sense that if it is violated by some function $f$ then blow-up of some strong solutions occurs in finite time (see Section 6). For the latter issue, our approach is based on the concavity and eigenvalue methods inspired by [37] and [35], respectively. Global existence of strong solutions of (1.5)-(1.7) under assumption (1.8) is deduced by performing a Moser-type iteration procedure as in, e.g., [24] for the case of the Laplace operator. Here a new inequality of Poincaré type and a comparison lemma for various energy forms allows to control the iteration at every step (see Section 3.1). These facts together with some classical arguments allow us also to develop a complete weak solution theory, say when $f$ is a polynomial of arbitrary degree satisfying the additional condition $f'(\tau) \geq -c_f$, for some $c_f > 0$, for all $\tau \in \mathbb{R}$.

(II) We prove that every weak solution of the problems (1.5)-(1.7) "instantaneously" regularizes to a strong solution in both space and time. Then, taking advantage of this smoothness, we can show that our problems have a gradient structure, and as a result establish the existence of a finite-dimensional global attractor in the phase space $L^2(\Omega)$. In some cases, owing to the $L^2-L^\infty$ smoothing property of these equations we can also establish explicit both sided estimates for the fractal dimension of the global attractor. In particular, for problem (1.5) we establish the (sharp) two-sided estimate for the fractal dimension of the global attractor $A_s$:

$$c_1 \left( -\frac{f'(0)}{d} \right) \frac{N}{|\Omega|} \leq \text{dim}_F(A_s, L^2(\Omega)) \leq c_2 \left( \frac{c_f}{d} \right) \frac{N}{|\Omega|},$$

provided that $f$ is a polynomial density function of arbitrary growth such that $f'(\tau) \geq -c_f$, for any $\tau \in \mathbb{R}$, assuming $f'(0) < 0$ as well. Here, the constants $c_1, c_2 > 0$ depend on the shape of $\Omega$ and $N$ only, but are independent of $s$. We recall that the dimension of the global attractor can be used in practice to indicate the number of degrees of freedom needed to simulate the given dynamical system since this dimension is usually associated with the temporal and spatial complexity of the long-time dynamics. Thus, the bound (1.9) indicates that the “permanent regime” for problem (1.5) is indeed more structurally complex when compared to that of the classical reaction-diffusion equation for the Laplace operator $\Delta$. It is also worth noting that the foregoing bounds also stabilize as $s \to 1^-$ to the (classical) dimension bounds for the well-established reaction-diffusion equation associated with the operator $\Delta$.

(III) The $\omega$-limit sets of the problems (1.5)-(1.7) can exhibit a complicated structure if the function $f$ is non-monotone. This can happen if the stationary problems associated with (1.5)-(1.7) possess a continuum of nonconstant solutions. We show the validity of the so-called Łojasiewicz-Simon inequality for our problems under the assumption that a certain elliptic boundary value problem has Hölder continuous (up to the boundary $\partial \Omega$) solutions. To be more precise, we need only require
that the bounded solution of

$$A_K w = h \in L^\infty(\Omega),$$

has the property: $w \in C^{0,\nu}(\Omega)$, for some $\nu \in (0, 1)$. We also give an example when such a condition is met. Then, for any of these problems we prove the convergence of a given trajectory $u = u(t; u_0)$, $u_0 \in L^2(\Omega)$, as time goes to infinity, to a single equilibrium which solves the corresponding stationary version associated with (1.5)-(1.7). More precisely, assuming also that $f$ is a real analytic function over $\mathbb{R}$, any weak solution $u$ to problems (1.5)-(1.7) satisfies

$$\|u(t) - u_*\|_{L^\infty(\Omega)} \sim (1 + t)^{-\frac{1}{\zeta}}$$

as $t \to \infty$, where $\zeta \in (0, 1)$ depends on $u_* \in L^\infty(\Omega) \cap W^{s,2}(\Omega)$, such that $u_*$ solves the corresponding stationary problems.

Finally, we can also mention that the present analysis can be exploited to extend and establish existence and existence of finite dimensional attractor results for systems of reaction-diffusion equations for a vector valued function $\vec{u} = (u_1, \ldots, u_k)$ ($k \geq 2$). For instance, our framework requires only minor modifications: the function spaces become product spaces, and the principal dissipation operators become block operators on these product spaces, typically with block diagonal form. The nonlinearities in these models can be treated in a similar way as in Section 3. We also remark that one can also allow for time-dependent external forces $h(t)$, $h \in C_b(\mathbb{R}; L^2(\Omega))$, acting on the right-hand side of these systems. In this case, one can generalize the notion of global attractor and replace it by the notion of pullback attractor, for example. One can still study the set of all complete bounded trajectories, that is, trajectories which are bounded for all $t \in \mathbb{R}_+$. We leave the details to the interested reader.

The plan of the paper goes as follows. In Section 2, we introduce the functional analytic framework associated with (1.5)-(1.7). Then, in Section 3 (and corresponding subsections) we prove well-posedness and regularity results for any of the problems (1.5)-(1.7). In Section 4 we establish the existence of a compact semiflow in $L^2(\Omega)$ and derive optimal estimates on the global attractor, while in the remaining Sections 5, 6 we deal with convergence of solutions and with blow-up phenomena, respectively.

2. Preliminaries. In this section we introduce the function spaces needed to investigate our problems and we show some intermediate results. In particular we introduce the fractional normal derivative of a function $u$ mentioned in the introduction. We also give the integration by parts formula for the regional fractional Laplacian and we introduce the fractional Neumann and Robin type boundary conditions associated with the operator $A^\alpha_{\Omega}$. Some generation of semigroup results, and the regularity of weak solutions of elliptic equations associated with these operators are also given.

2.1. The functional setup. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. For $s \in (0, 1)$, we denote by

$$W^{s,2}(\Omega) := \{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy < \infty \}$$
the fractional order Sobolev space endowed with the norm
\[ \|u\|_{W^{s,2}(\Omega)} := \left( \int_\Omega |u|^2 \, dx + \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}. \]

If \( \Omega \) is a bounded open set with a Lipschitz continuous boundary, then by [16, Theorem 6.7], there exists a constant \( C > 0 \) such that for every \( u \in W^{s,2}(\Omega) \),
\[ \|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{s,2}(\Omega)}, \quad (2.1) \]
for all \( q \) satisfying
\[ q \in [1, 2^*] \text{ with } 2^* := \frac{2N}{N - 2s} \text{ if } N > 2s \text{ and } q \in [1, \infty) \text{ if } N = 2s. \]

By [16, Section 7], (2.1) implies that for every \( q \in [1, 2^*] \), the embedding \( W^{s,2}(\Omega) \hookrightarrow L^q(\Omega) \) is compact. Moreover, by [14, Theorem 11.1], there exists a constant \( C > 0 \) such that for every \( u \in W^{s,2}(\Omega) \),
\[ \|u\|_{L^r(\partial \Omega)} \leq C\|u\|_{W^{s,2}(\Omega)}, \quad (2.2) \]
for all \( r \) satisfying
\[ r \in [1, 2] \text{ with } 2^* := \frac{2(N - 1)}{N - 2s} \text{ if } N > 2s \text{ and } r \in [1, \infty) \text{ if } N = 2s. \]

If \( N < 2s \), then
\[ W^{s,2}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}) \text{ with } \alpha := s - \frac{N}{2}. \]

For an arbitrary open set \( \Omega \subset \mathbb{R}^N \), we let
\[ W^{s,2}_0(\Omega) = \overline{D(\Omega)}^{W^{s,2}(\Omega)}. \]
By definition, \( W^{s,2}_0(\Omega) \) is the smaller closed subspace of \( W^{s,2}(\Omega) \) containing \( D(\Omega) \).

By [16, Remark 6.6], there exists a constant \( C > 0 \) such that for every \( u \in W^{s,2}_0(\Omega) \),
\[ \|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{s,2}(\Omega)}, \quad \forall q \in [2, 2^*]. \quad (2.3) \]

In particular, if \( \Omega \) is bounded, we have that (2.3) holds for every \( q \in [1, 2^*] \). In that case, by [16, Section 7] again, (2.3) also implies that for every \( q \in [1, 2^*] \), the embedding \( W^{s,2}_0(\Omega) \hookrightarrow L^q(\Omega) \) is compact. Let \( \tilde{W}^{s,2}_0(\Omega) = \overline{D(\Omega)}^{W^{s,2}(\mathbb{R}^N)} \). By [2, Theorem 10.1.1], it can be characterized as follows:
\[ \tilde{W}^{s,2}_0(\Omega) = \{ u \in W^{s,2}(\mathbb{R}^N) : \tilde{u} = 0 \text{ on } \mathbb{R}^N \setminus \Omega \}, \quad (2.4) \]
where \( \tilde{u} \) is the quasi-continuous version (with respect to the capacity defined with the space \( W^{s,2}(\mathbb{R}^N) \)) of \( u \). Then \( \tilde{W}^{s,2}_0(\Omega) \subset W^{s,2}_0(\Omega) \) and it is well known that they coincide if \( s \neq 1/2 \) and they may be different if \( s = 1/2 \). Finally, we mention that in the case where \( \tilde{W}^{s,2}_0(\Omega) = W^{s,2}_0(\Omega) \), we have that for \( u \in \tilde{W}^{s,2}_0(\Omega) \),
\[ \|u\|^2 = \frac{C_{N,2}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_\Omega V_0|u|^2 \, dx \quad (2.5) \]
\[ = \frac{C_{N,2}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \]
defines an equivalent norm on the space $W^{s,2}_0(\Omega)$. In fact, by [16, Lemma 6.1], there exists a constant $C = C(\Omega, N, s) > 0$ such that $V_\Omega(x) \geq C$ for every $x \in \Omega$. This implies that for every $u \in W^{s,2}_0(\Omega)$,

$$
\|u\|^2_{W^{s,2}_0(\Omega)} = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\Omega} |u|^2 \, dx
$$

$$
\leq C \left( \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\Omega} V_\Omega |u|^2 \, dx \right)
$$

$$
= C\|u\|^2.
$$

Let $u \in W^{s,2}_0(\Omega)$ and let $\tilde{u}$ be the extension of $u$ by 0 on $\mathbb{R}^N \setminus \Omega$. Then $\tilde{u} \in W^{s,2}(\mathbb{R}^N)$ and since the extension operator is continuous from $W^{s,2}_0(\Omega)$ into $W^{s,2}(\mathbb{R}^N)$ (see e.g. [16, Section 6]) we have that there exists a constant $C > 0$ such that $\|\tilde{u}\|_{W^{s,2}(\mathbb{R}^N)} = |||u||| \leq C\|u\|_{W^{s,2}_0(\Omega)}$ for every $u \in W^{s,2}_0(\Omega)$. This completes the proof of (2.5).

Remark 1. We notice that, if $\Omega$ has a Lipschitz continuous boundary, then by [6] (see also [48, Theorem 4.8] for a more general version), $W^{s,2}_0(\Omega) = W^{s,2}(\Omega)$ for every $0 < s \leq \frac{1}{2}$.

For more information on the fractional order Sobolev spaces we refer to [2, 14, 16, 27, 34, 38, 48] and their references.

2.2. The fractional Laplacian with Dirichlet boundary conditions. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded domain. Let $0 < s < 1$ and let $V_\Omega$ be the potential given in (1.3). Throughout the remainder of the article, if we write $u \in \tilde{W}^{s,2}_0(\Omega)$, we mean that $u \in W^{s,2}(\mathbb{R}^N)$ and $u = 0$ on $\mathbb{R}^N \setminus \Omega$.

Let $\mathcal{E}_E$ be the bilinear symmetric closed form with domain $D(\mathcal{E}_E) = \tilde{W}^{s,2}_0(\Omega)$ and defined for $u, v \in \tilde{W}^{s,2}_0(\Omega)$ by

$$
\mathcal{E}_E(u, v) = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy
$$

$$
+ \int_{\Omega} V_\Omega(x)u(x)v(x) \, dx
$$

$$
= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy.
$$

Let $A_E$ be the closed linear selfadjoint operator on $L^2(\Omega)$ associated with $\mathcal{E}_E$ in the sense that

$$
\begin{cases}
D(A_E) := \{ u \in \tilde{W}^{s,2}_0(\Omega), \exists v \in L^2(\Omega), \mathcal{E}_E(u, \varphi) = (v, \varphi)_{L^2(\Omega)} \forall \varphi \in \tilde{W}^{s,2}_0(\Omega) \}
\end{cases}
A_E u = v.
$$

(2.6)

We call $A_E$ a realization of the fractional Laplace operator $(-\Delta)^s$ on $L^2(\Omega)$ with the Dirichlet boundary condition. We have the following more explicit description of the operator $A_E$.

**Proposition 2.1.** Let $A_E$ be the operator defined in (2.6). Then

$$
D(A_E) = \{ u \in \tilde{W}^{s,2}_0(\Omega), (\Delta)^s u \in L^2(\Omega) \}, \quad A_E u = (\Delta)^s u.
$$

(2.7)
Proof. Set \( \tilde{D} := \{ u \in \tilde{W}^{s,2}_{0}(\Omega), \ (\Delta)^s u \in L^2(\Omega) \} \). Let \( u \in D(A_E) \). Then by definition, there exists a function \( v \in L^2(\Omega) \), such that for every \( \varphi \in \tilde{W}^{s,2}_{0}(\Omega) \),

\[
\int_{\Omega} v \varphi dx = \int_{\mathbb{R}^N} v \varphi dx = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))((\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} dxdy
= \int_{\mathbb{R}^N} (\Delta)^{s/2} u (\Delta)^{s/2} \varphi dx = \int_{\mathbb{R}^N} \varphi (\Delta)^s u dx.
\]

We have shown that \( u \in \tilde{D} \) and \( A_E u := v = (\Delta)^s u \). Now, let \( u \in \tilde{D} \) and set \( v := (\Delta)^s u \in L^2(\Omega) \). Let \( \varphi \in \tilde{W}^{s,2}_{0}(\Omega) \). Then

\[
\int_{\Omega} v \varphi dx = \int_{\Omega} \varphi (\Delta)^s u dx = \int_{\mathbb{R}^N} \varphi (\Delta)^s u dx = \int_{\mathbb{R}^N} (\Delta)^{s/2} u (\Delta)^{s/2} \varphi dx
= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} dxdy = \mathcal{E}_E(u, \varphi).
\]

We have shown that \( \tilde{D} \subseteq D(A_E) \) and the proof of (2.7) is complete. \( \square \)

Next, let \( \mathcal{E}_D \) be the bilinear symmetric closed form with domain \( D(\mathcal{E}_D) = W^{s,2}_{0}(\Omega) \) and defined for \( u, v \in W^{s,2}_{0}(\Omega) \) by

\[
\mathcal{E}_D(u, v) = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dxdy.
\]

Let \( A_D \) be the closed linear selfadjoint operator on \( L^2(\Omega) \) associated with \( \mathcal{E}_D \) in the sense that

\[
\begin{align*}
D(A_D) & := \{ u \in W^{s,2}_{0}(\Omega), \exists v \in L^2(\Omega), \ \mathcal{E}_D(u, v) = (v, \varphi)_{L^2(\Omega)} \forall \varphi \in W^{s,2}_{0}(\Omega) \} \\
A_D u & = v.
\end{align*}
\]

We call \( A_D \) a realization of the regional fractional Laplace operator \( A^s_{\Omega} \) on \( L^2(\Omega) \) with the Dirichlet boundary condition. We have the following more explicit description of the operator \( A_D \).

**Proposition 2.2.** Let \( A_D \) be the operator defined in (2.8). Then

\[
D(A_D) = \{ u \in W^{s,2}_{0}(\Omega), \ A^s_{\Omega} u \in L^2(\Omega) \} \quad \text{and} \quad A_D u = A^s_{\Omega} u. \tag{2.9}
\]

Proof. The proof follows as the proof of Proposition 2.1 by using also the integration by part formula \( \int_{\Omega} v A^s_{\Omega} u dx = \mathcal{E}_D(u, v) \) for every \( u, \varphi \in W^{s,2}_{0}(\Omega) \) with \( A^s_{\Omega} u \in L^2(\Omega) \). \( \square \)

We need not make any confusion between the operator \( A_E \) and \( A_D \). They are different and coincide only if \( \Omega \) has capacity zero with respect to the capacity defined with the space \( W^{s,2}(\mathbb{R}^N) \) (of course, this cannot be the case since \( \Omega \) is bounded). On one hand, we have shown that \( A_E u = A_D u + V_\Omega u \), for every \( u \in D(\Omega) \). On the other hand, the potential \( V_\Omega \) is in general very difficult to describe. For example, if \( \Omega \) has a Lipschitz continuous boundary then it has been shown in [27, Formula (1.3.2.12), p. 19] that there exist some constants \( 0 < C_1 \leq C_2 \) such that for every \( x \in \Omega \),

\[ C_1 \rho^{-2s}(x) \leq V_\Omega(x) \leq C_2 \rho^{-2s}(x), \]

where \( \rho(x) := \text{dist}(x, \partial \Omega) \), \( x \in \Omega \). Instead of the fractional Laplace operator \( (\Delta)^s \), whose definition is independent of the open set \( \Omega \), the regional fractional Laplace operator \( A^s_{\Omega} \) depends on \( \Omega \) and hence on the potential \( V_\Omega \). But we have the following convergence result.
Proposition 2.3. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set. Then for every $u \in \mathcal{D}(\Omega)$ and $v \in W^{1,2}_0(\Omega)$, we have that

$$\lim_{s \uparrow 1} \int_{\Omega} v A^s_\Omega u dx = \lim_{s \uparrow 1} \int_{\Omega} v(-\Delta)^s u dx = -\int_{\Omega} v \Delta u dx. \quad (2.10)$$

Proof. First, let $u \in \mathcal{D}(\Omega)$. Then using [7], the fact that $\lim_{s \uparrow 1} (1-s)\Gamma(1-s) = 1$ and the classical integration by part formula for the Laplace operator, we get that

$$\lim_{s \uparrow 1} \int_{\Omega} u A^s_\Omega u dx = \lim_{s \uparrow 1} \frac{s^{2s-1}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}} (1-s) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy \quad (2.11)$$

$$= \int_{\Omega} |\nabla u|^2 dx = -\int_{\Omega} u \Delta u dx.$$

Proceeding as in (2.11), we also have that

$$\lim_{s \uparrow 1} \int_{\Omega} u(-\Delta)^s u dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx = -\int_{\Omega} u \Delta u dx = -\int_{\Omega} u \Delta u dx. \quad (2.12)$$

We have show (2.10) for $u = v \in \mathcal{D}(\Omega)$. Replacing $u$ by $u + v$ in (2.11) and (2.12) for $u, v \in \mathcal{D}(\Omega)$, we get (2.10) for every $u, v \in \mathcal{D}(\Omega)$. Finally we get (2.10) for every $u \in \mathcal{D}(\Omega)$ and $v \in W^{1,2}_0(\Omega)$ by density and using that $W^{1,2}_0(\Omega)$ is continuously embedded into $W^{1,2}_0(\Omega)$. The proof is finished. □

2.3. The fractional normal derivative. In this subsection, we introduce the fractional normal derivative mentioned in the introduction. This will be used to define the fractional Neumann and Robin boundary conditions for the operator $A^s_\Omega$.

Throughout the remainder of this subsection, $\Omega \subset \mathbb{R}^N$ denotes a bounded open set of class $C^{1,1}$ and we will also use the following notations:

- $\rho(x) = \text{dist}(x, \partial \Omega) = \inf\{|y-x| : y \in \partial \Omega\}, \forall x \in \Omega,$
- $\Omega_\delta = \{x \in \Omega : 0 < \rho(x) < \delta\}, \delta > 0$ is a real number,
- $\vec{n}(z)$ the inner normal vector of $\partial \Omega$ at the point $z \in \partial \Omega$,
- $\nu(z) = -\vec{n}(z)$ the outer normal vector of $\partial \Omega$ at the point $z \in \partial \Omega$.

The following definition is taken from [28, Definition 2.1] (see also [29, Definition 7.1] for the one-dimensional case).

Definition 2.1. For $u \in C^1(\Omega)$, $z \in \partial \Omega$ and $0 \leq \alpha < 2$, we define the boundary operator $\mathcal{N}^\alpha$ by

$$\mathcal{N}^\alpha u(z) = -\lim_{t \downarrow 0} \frac{du(z + \vec{n}(z)t)}{dt} t^\alpha, \quad (2.13)$$

whenever the limit exists.

Remark 2. Let $0 \leq \alpha < 2$ and let $\mathcal{N}^\alpha$ be the boundary operator defined in (2.13).

(a) If $\alpha = 0$, then $\mathcal{N}^0 u(z) = -\nabla u \cdot \vec{n}(z) = \frac{\partial u(z)}{\partial \nu}$ for every $u \in C^1(\Omega)$ and $z \in \partial \Omega$.

(b) If $0 < \alpha < 2$, then $\mathcal{N}^\alpha u(z) = 0$ for every $u \in C^1(\Omega)$ and $z \in \partial \Omega$.

Next, let $\beta > 0$. By [28, p.294], there exist a real number $\delta > 0$ (depending only on $\Omega$) and a function $h_\beta \in C^2(\Omega)$ (depending on $\Omega$ and $\beta$) such that

$$h_\beta(x) = \begin{cases} \rho(x)^{\beta-1}, & \forall x \in \Omega_\delta, \text{ when } \beta \in (0,1) \cup (1,\infty); \\ \ln(\rho(x)), & \forall x \in \Omega_\delta, \text{ when } \beta = 1. \end{cases} \quad (2.14)$$
For $\beta > 0$ we define the space
\[
C^2_{\beta}(\Omega) = \{ u : u(x) = f(x)h_\beta(x) + g(x), \ \forall \ x \in \Omega, \ \text{for some } f, g \in C^2(\Omega) \}.
\]
When $\beta > 1$, we always assume that $u \in C^2_{\beta}(\Omega)$ is defined on $\overline{\Omega}$ by continuous extension. The following explicit representation of the operator $\mathcal{N}^\alpha$ is taken from [48, Lemma 6.3].

**Lemma 2.2.** Let $1 < \beta \leq 2$, $u := fh_\beta + g \in C^2_{\beta}(\overline{\Omega})$ be the representation of $u$. Let $u_0 := fh_\beta$ so that $u = u_0 + g$. Then the following assertions hold.

(a) If $\beta \in (1, 2)$, then for $z \in \partial \Omega$,
\[
\mathcal{N}^{2-\beta}u(z) = (1 - \beta) \lim_{\Omega \ni x \to z} \frac{u(x) - u(z)}{\rho(x)^{\beta-1}} = (1 - \beta) \lim_{\Omega \ni x \to z} \frac{u_0(x)}{\rho(x)^{\beta-1}}.
\]

(b) If $\beta = 2$, then for $z \in \partial \Omega$,
\[
\mathcal{N}^0u(z) = -\lim_{\Omega \ni x \to z} \frac{u(x) - u(z)}{\rho(z)}.
\]

Next, let
\[
C_s := \frac{C_{1,s}}{2s(2s-1)} \int_0^\infty |\tau - 1|^{1-2s} - (\tau \lor 1)^{1-2s} \frac{d\tau}{\tau^{2s-2}}, \quad \frac{1}{2} < s < 1,
\]
and let the constant $B_{N,s}$ be such that
\[
\frac{C_{1,s}}{C_{N,s}} B_{N,s} := \begin{cases} 
C_s & \text{if } N = 1 \\
\frac{2\pi N-1}{\Gamma(N-1)} \int_0^{\pi/2} \cos^{2s}(\theta) \sin^{N-2}(\theta) \ d\theta, & \text{if } N \geq 2.
\end{cases}
\]

We have the following fractional Green type formula for the regional fractional Laplace operator.

**Theorem 2.3.** Let $\frac{1}{2} < s < 1$ and let $A^s_\Omega$ be the nonlocal operator defined in (1.2). Then, for every $u := fh_{2s} + g = u_0 + g \in C^2_{2s}(\overline{\Omega})$ and $v \in W^{s,2}(\Omega)$,
\[
\int_\Omega v(x) A^s_\Omega u(x) \, dx = \frac{1}{2} C_{N,s} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy \\
- B_{N,s} \int_{\partial \Omega} v \mathcal{N}^{2-2s} u \, d\sigma \\
= \frac{1}{2} C_{N,s} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy \\
+ B_{N,s}(2s-1) \int_{\partial \Omega} v(z) \lim_{z \to x} \frac{u(x) - u(z)}{\rho(x)^{2s-1}} \, d\sigma_z \\
= \frac{1}{2} C_{N,s} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy \\
+ B_{N,s}(2s-1) \int_{\partial \Omega} v(z) \left( \frac{u_0}{\rho^{2s-1}} \right)(z) \, d\sigma_z,
\]
where by $\left( \frac{u_0}{\rho^{2s-1}} \right)(z)$ at the point $z \in \partial \Omega$, we mean
\[
\left( \frac{u_0}{\rho^{2s-1}} \right)(z) = \lim_{\Omega \ni x \to z} \frac{u_0(x)}{\rho(x)^{2s-1}}.
\]
We mention that the first identity in Theorem 2.3 has been obtained in [28, Theorem 3.3] under the assumption that \( v \) also belongs to \( C_{2s}^{2}(\Omega) \). Its validity for every \( v \in W^{s,2}(\Omega) \) and the second and third identities have been proved in [48, Theorem 5.7].

**Definition 2.4.** For \( \frac{1}{2} < s \leq 1 \) and \( u \in C_{2s}^{2}(\Omega) \), we call the function \( B_{N,s}^{2-2s}u \) the fractional normal derivative of the function \( u \) in direction of the outer normal vector.

We make some comments about the fractional normal derivative introduced above.

**Remark 3.** We mention that another definition of fractional normal derivative, called non-local normal derivative, has been introduced in [18, 32] (see also [17]) for functions \( u \) defined on \( \mathbb{R}^{N} \). More precisely, for \( 0 < \alpha < 1 \) and \( u \in L^{1}(\mathbb{R}^{N}) \), the non-local normal derivative is defined by

\[
N_{\alpha}u(x) = C_{N,s} \int_{\mathbb{R}^{N} \setminus \Omega} \frac{u(x) - u(y)}{|x - y|^{N + 2\alpha}} dy, \quad x \in \mathbb{R}^{N} \setminus \Omega.
\]  

(2.18)

The definition of \( N_{\alpha}u \) in (2.18) requires that the function is defined on all \( \mathbb{R}^{N} \). This is different from the fractional Normal derivative \( N^{\alpha}u \) given in (2.13) where the function \( u \) is defined only on \( \Omega \). Starting with a function defined only on \( \Omega \), it seems impossible to deal with \( N_{\alpha}u \). For example if \( u \in W^{s,2}(\Omega) \) and letting \( \tilde{u} \in W^{s,2}(\mathbb{R}^{N}) \) be an extension to all \( \mathbb{R}^{N} \), then the relation (2.18) can make sense but the definition cannot be independent of the extension, except in the case where there is only one such possible extension. This shows that the expression \( N_{\alpha}u \) cannot be used in our context since we consider functions defined a priori only on \( \Omega \). We recall that it has been shown in [17, Proposition 5.1] (see also [18, 32]) that if \( \Omega \subset \mathbb{R}^{N} \) is a bounded domain with Lipschitz continuous boundary \( \partial \Omega \), then for every \( u, v \in C_{0}^{2}(\mathbb{R}^{N}) \),

\[
\lim_{\alpha \uparrow 1} \int_{\mathbb{R}^{N} \setminus \Omega} v N_{\alpha}u \, dx = \int_{\partial \Omega} \frac{\partial u}{\partial \nu} v \, d\sigma.
\]

As we have seen in Remark 2, the fractional normal derivative \( N^{\alpha}u \) is continuous with respect to \( \alpha \), so that for every \( u \in C_{2}^{2}(\Omega) = C^{1}(\Omega) \) we have that \( N^{1}u = \frac{\partial u}{\partial \nu} \), i.e., the classical normal derivative of the function \( u \) in direction of the outer normal vector \( \nu \). Next, let \( B_{N,s} \) be the constant given in (2.17). First, we notice that using a change of variable, we get that

\[
\int_{0}^{\pi/2} \cos^{2} s(\theta) \sin^{N-2} s(\theta) d\theta = \frac{1}{2} \int_{0}^{1} t^{s+\frac{1}{2}-1} (1 - t)^{\frac{N-1}{2}} \, dt \quad (2.19)
\]

\[
= \frac{1}{2} B \left( \frac{2s + 1}{2}, \frac{N - 1}{2} \right)
\]

\[
= \frac{1}{2} \frac{\Gamma \left( \frac{2s + 1}{2} \right) \Gamma \left( \frac{N - 1}{2} \right)}{\Gamma \left( \frac{N + 2s}{2} \right)},
\]

where \( B \) denotes the usual Beta function. Replacing this expression (2.19) in (2.17), we get that in fact \( B_{N,s} = C_{s} \) and hence, it is independent of \( N \). Moreover, we have that \( \lim_{s \uparrow 1} C_{s} = 1 \). This shows that the integration by parts formula given in Theorem 2.3 is consistent with the well-known integration by part formula for the Laplace operator where there is no constant depending on the dimension in the boundary integral.
2.4. The fractional Neumann boundary conditions. Throughout this section, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz continuous boundary $\partial \Omega$. We consider the bilinear symmetric closed form $\mathcal{E}_N$ with domain $D(\mathcal{E}_N) = W^{s,2}(\Omega)$ and given for $u, v \in W^{s,2}(\Omega)$ by
\[
\mathcal{E}_N(u, v) = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy.
\]
Since $W^{s,2}(\Omega) = W_0^{s,2}(\Omega)$ for all $0 < s \leq 1/2$ (by Remark 1), we have that $\mathcal{E}_N = \mathcal{E}_D$ if $0 < s \leq 1/2$. Therefore, we assume that $1/2 < s < 1$.

Let $A_N$ be the closed linear selfadjoint operator associated with $\mathcal{E}_N$ in the sense that
\[
\begin{aligned}
D(A_N) &:= \{ u \in W^{s,2}(\Omega), \exists v \in L^2(\Omega), \mathcal{E}_N(u, \varphi) = (v, \varphi)_{L^2(\Omega)}, \forall \varphi \in W^{s,2}(\Omega) \} \\
A_Nu &= v.
\end{aligned}
\]

We call $A_N$ a realization of the regional fractional Laplace operator $A^s_\Omega$ on $L^2(\Omega)$ with the fractional Neumann type boundary conditions. In fact, we have the following more explicit description of the operator $A_N$ which has been proved in [48, Proposition 6.1] exploiting Theorem 2.3.

**Proposition 2.4.** Let $A_N$ be the operator defined in (2.20). Assume also that $\Omega$ is a bounded open set of class $C^{1,1}$. Then
\[
D(A_N) \cap C^2_{2s}(\overline{\Omega}) = \{ u \in C^2_{2s}(\overline{\Omega}), \mathcal{N}^{2-2s}u = 0 \text{ on } \partial \Omega \}, \quad A_Nu = A^s_\Omega u.
\]

The following result shows in particular, that as $s \uparrow 1$, the operator $A_N$ converges (in some sense) to the realization $\Delta_N$ in $L^2(\Omega)$ of the Laplace operator with the classical Neumann boundary conditions.

**Proposition 2.5.** For every $u \in C^2(\overline{\Omega})$ and $v \in W^{1,2}(\Omega)$ we have that
\[
\lim_{s \uparrow 1} \int_{\Omega} v A^s_\Omega u \, dx = \int_{\Omega} \nabla u \nabla v \, dx - \int_{\Omega} v \Delta u \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial \nu} v \, d\sigma.
\]

**Proof.** Since $\Omega$ has a Lipschitz continuous boundary, we have that $W^{1,2}(\Omega) \hookrightarrow W^{s,2}(\Omega)$ (see e.g. [16, Proposition 2.2]). Let $u \in C^2(\overline{\Omega})$. Since $u \in C^2(\overline{\Omega})$, then by Remark 2, we have that $\mathcal{N}^{2-2s}u(z) = 0$ for every $z \in \partial \Omega$. Then, using the definition, the integration by part formula for the operator $A^s_\Omega$ given in Theorem 2.3, the convergence result of fractional order Sobolev spaces contained in [7], the fact that $\lim_{s \uparrow 1} (1-s)\Gamma(1-s) = \Gamma(1) = 1$ and the integration by part formula for the Laplace operator, we have that
\[
\lim_{s \uparrow 1} \int_{\Omega} v A^s_\Omega u \, dx = \frac{1}{2} \lim_{s \uparrow 1} C_{N,s} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[
\quad = \frac{1}{2} \lim_{s \uparrow 1} s^{2s} \Gamma\left(\frac{N+2s}{2}\right) (1-s) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[
\quad = \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} u \Delta u \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial \nu} v \, d\sigma.
\]
It follows from (2.22) that for every $u, v \in C^2(\overline{\Omega})$, we have that
\[
\lim_{s \uparrow 1} \int_{\Omega} v A^s_\Omega u \, dx = - \int_{\Omega} v \Delta u \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial \nu} v \, d\sigma.
\]
Now we obtain (2.21) for every $u \in C^2(\overline{\Omega})$ and $v \in W^{1,2}(\Omega)$ by density. \qed
As we have mentioned in Remark 3, since our functions are a priori defined only on \( \Omega \), we have that \( \mathcal{N}^{2-2s}u = 0 \) is the right fractional homogeneous Neumann type boundary conditions for the regional fractional Laplace operator.

2.5. **The fractional Robin boundary conditions.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with Lipschitz continuous boundary \( \partial \Omega \) and \( s \in (0, 1) \). Let \( \gamma \in L^\infty(\partial \Omega) \) satisfy

\[
\gamma(x) \geq \gamma_0 \quad \text{for} \quad \sigma - \text{a.e.} \quad x \in \partial \Omega,
\]

for some constant \( \gamma_0 > 0 \). We consider the bilinear symmetric form \( \mathcal{E}_R \) with domain \( D(\mathcal{E}_R) = W^{s,2}(\Omega) \) and defined for \( u, v \in D(\mathcal{E}_R) \) by

\[
\mathcal{E}_R(u, v) = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy + \int_{\partial \Omega} \gamma uv \, d\sigma.
\]

We assume that \( s \in (0, 1) \) is such that \( W_0^{s,2}(\Omega) \neq W^{s,2}(\Omega) \), that is, \( 1/2 < s < 1 \), otherwise we are in the situation of the Dirichlet boundary condition. It follows from [48, Theorem 6.4] that the form \( \mathcal{E}_R \) is closed on \( L^2(\Omega) \).

Let \( A_R \) be the closed linear selfadjoint operator associated with the form \( \mathcal{E}_R \) in the sense that

\[
\begin{cases}
D(A_R) := \{ u \in W^{s,2}(\Omega), \exists v \in L^2(\Omega), \mathcal{E}_R(u, \varphi) = (v, \varphi)_{L^2(\Omega)}, \forall \varphi \in W^{s,2}(\Omega) \} \\
A_Ru = v.
\end{cases}
\]

We call \( A_R \) a realization of the regional fractional Laplace operator \( A_0^s \) on \( L^2(\Omega) \) with the fractional Robin type boundary conditions. The following result has been proved in [48, Proposition 6.5] by using the integration by parts formula given in Theorem 2.3.

**Proposition 2.6.** Let \( A_R \) be the operator defined in (2.24). Assume also that \( \Omega \) is a bounded open set of class \( C^{1,1} \). Then

\[
D(A_R) \cap C^2_{\text{loc}}(\Omega) = \{ u \in C^2_{\text{loc}}(\Omega), B_{N,s} \mathcal{N}^{2-2s}u + \gamma u = 0 \text{ on } \partial \Omega \}, \quad A_Ru = A_0^s u,
\]

where \( B_{N,s} \) is the constant given in (2.17).

We refer to [28, 29, 48] for more details. We notice that it also follows from Proposition 2.5 that, as \( s \uparrow 1 \), the operator \( A_R \) converges (in some sense) to the realization \( \Delta_R \) in \( L^2(\Omega) \) of the Laplace operator with the classical Robin boundary conditions.

2.6. **Generation of semigroup.** Let \( 0 < s < 1 \) and let \( A_K, K \in \{ E, D, \mathcal{N}, R \} \) be the operators introduced above. We also let \( W_0^{s,2}(\Omega) = W^{s,2}_{E}(\Omega) = W^{s,2}_D(\Omega) = W_0^{s,2}(\Omega) \) and \( W^{s,2}_K(\Omega) = W^{s,2}(\Omega) \) if \( K \in \{ \mathcal{N}, R \} \). We introduce the following assumption.

\( (H) \): If \( K = \mathcal{N} \) or \( K = R \), we assume that \( \frac{1}{2} < s < 1 \) and \( \Omega \subset \mathbb{R}^N \) is a bounded domain with Lipschitz continuous boundary. If \( K = E \) or \( K = D \), then \( \Omega \subset \mathbb{R}^N \) is an arbitrary bounded open set.

Indeed recall that by the foregoing considerations, \( W_0^{s,2}(\Omega) = W^{s,2}(\Omega) \) provided that \( 0 < s \leq 1/2 \). We have the following result.

**Theorem 2.5.** Let \( 0 < s < 1 \) and let assumption \( (H) \) be satisfied. Then the following assertions hold.
(a) The operator $-A_K$ generates a submarkovian semigroup $(e^{-tA_K})_{t \geq 0}$ on $L^2(\Omega)$ and hence, can be extended to contraction strongly continuous semigroups on $L^p(\Omega)$ for every $p \in [1, \infty)$, and to a contraction semigroup on $L^\infty(\Omega)$.

(b) The operator $A_K$ has a compact resolvent, and hence has a discrete spectrum. The spectrum of $A_K$ is an increasing sequence of real numbers $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ that converges to $+\infty$. Moreover, $0$ is an eigenvalue of $A_N$ and is not an eigenvalue of $A_K$ for $K \in \{E, D, R\}$, and if $u_n$ is an eigenfunction associated with $\lambda_n$, then $u_n \in D(A_K) \cap L^\infty(\Omega)$.

(c) Denoting the generator of the semigroup on $L^p(\Omega)$ by $A_{p,K}$, so that $A_K = A_{2,K}$, then the spectrum of $A_{p,K}$ is independent of $p$ for every $p \in [1, \infty]$. 

(d) Let $\theta \in (0, 1)$. Then $D(A_{p,K})$ embeds continuously into $L^\infty(\Omega)$ provided that $\theta > \frac{N}{4s}$. Let $p \in (2, \infty)$ and assume that $\theta > \frac{N}{4s} \left(1 - \frac{2}{p}\right)$. Then also $D(A_{p,K}) \subset L^p(\Omega)$ continuously.

Proof. Let $0 < s < 1$ and let $A_K, K \in \{E, D, N, R\}$ be the operators introduced above. Assume the assumption $(H)$.

(a) The proof of this part is contained in [48, Theorems 6.2 and 6.6]. We notice that in [48] the operator $A_K$ for $K \in \{N, R\}$ has been considered. The proof of the corresponding result for $A_K, K \in \{E, D\}$ follows similarly.

(b) By [48, Theorems 6.2 and 6.6], the operator $A_K$ for $K \in \{N, R\}$ has a compact resolvent. We have shown above that the embedding $W_0^{1,2}(\Omega) \to L^2(\Omega)$ is compact. Hence, the operator $A_K$ for $K \in \{E, D\}$ also has a compact resolvent. Since $A_K$ is a nonnegative self-adjoint operator and has a compact resolvent, then it has a discrete spectrum which is an increasing sequence of real numbers $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, that converges to $+\infty$. It is easy to see that $0$ is an eigenvalue of $A_N$ and is not an eigenvalue of $A_K$ for $K \in \{E, D, R\}$. Next, let $u_n \in W_0^{1,2}(\Omega)$ be an eigenfunction associated with $\lambda_n$. Then, $A_Ku_n = \lambda_n u_n$. Let $\alpha > 0$ be a real number. Since $\alpha \in \rho(-A_K)$, we have that $\alpha I + A_K$ is invertible. From $A_Ku_n = \lambda_n u_n$ we have that

$$u_n = (\alpha I + A_K)^{-1}(\lambda_n + \alpha)u_n = (\lambda_n + \alpha)(\alpha I + A_K)^{-1}(u_n).$$

By [48, Theorems 6.2 and 6.6], the semigroup $(e^{-tA_K})_{t \geq 0}$, for $K \in \{N, R\}$, is ultracontractive in the sense that it maps $L^2(\Omega)$ into $L^\infty(\Omega)$. It also follows from (2.3) that the semigroup $(e^{-tA_K})_{t \geq 0}$, for $K \in \{E, D\}$, is ultracontractive. More precisely, there is a constant $C > 0$ such that for every $f \in L^p(\Omega)$ and $t > 0$,

$$\|e^{-tA_K}f\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N}{2s}}\|f\|_{L^p(\Omega)}, \quad K \in \{E, D\},$$

and

$$\|e^{-tA_K}f\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N}{2s}}e^t\|f\|_{L^p(\Omega)}.$$  \hfill (2.25)

Since for every $f \in L^2(\Omega)$ and $\alpha > 0$,

$$(\alpha I + A_K)^{-1}f = \int_0^\infty e^{-\alpha t}e^{-tA_K}f dt,$$

it follows from (2.25) and (2.26) and the fact that $u_n \in L^p(\Omega)$ for some $p > \frac{N}{2s}$ that there exists a constant $M > 0$ such that

$$\|u_n\|_{L^\infty(\Omega)} \leq M(\lambda_n + \alpha)\|u_n\|_{L^p(\Omega)}.$$

This completes the proof of part (b).
(c) Let $p \in [1, \infty]$ and let $A_{p,K}$ be the generator of the semigroup on $L^p(\Omega)$. Since $A_K = A_{2,K}$ has a compact resolvent and $\Omega$ is bounded, it follows from the ultracontractivity that each semigroup has a compact resolvent on $L^p(\Omega)$ for $p \in [1, \infty]$. Now it follows from [15, Corollary 1.6.2] that the spectrum of $A_{p,K}$ is independent of $p$.

(d) Since $I + A_K$ is invertible we have that the $L^2$-norm of $(I + A_K)^\theta$ defines an equivalent norm on $D\left(A_K^{\theta}\right)$. Besides, for every $f \in L^2(\Omega)$,

\[ (\alpha I + A_K)^{-\theta} f = \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta - 1} e^{-\alpha t} e^{-tA_K} f dt, \quad \alpha > 0. \]

We shall prove the first claim in the case $K \in \{\mathcal{N}\}$ (the argument in the cases $K \in \{E,D,R\}$ is similar). Using (2.26) for $t \in (0,1)$ and the contractivity of $e^{-tA_N}$ for $t > 1$, for $u \in D\left(A_K^{\theta}\right)$, we deduce

\[ \|u\|_{L^\infty(\Omega)} \leq C \|u\|_{D\left(A_K^{\theta}\right)} \int_0^1 t^{-\frac{\theta}{2} + \theta - 1} dt + C \|u\|_{D\left(A_K^{\theta}\right)} \int_1^\infty e^{-t} dt. \]

The first integral is finite if and only if $\theta > N/4s$. For the second claim, we begin by interpolating the inequality (2.26) with the $L^2(\Omega)$-contractivity of $e^{-tA_N}$ to obtain that

\[ \|e^{-tA_N} f\|_{L^p(\Omega)} \leq C t^{-\frac{\theta}{2}} \left(1 - \frac{\theta}{2}\right) e^{\frac{t}{2}} \|f\|_{L^2(\Omega)} \] (2.27)

for every $p \in (2, \infty)$. As above with $\alpha > 1 - 2/p \in (0,1)$, the $L^2$-norm of $(\alpha I + A_K)^\theta$ defines an equivalent norm on $D\left(A_K^{\theta}\right)$ so that (2.27) for $t \in (0,1)$ and the contractivity of $e^{-tA_N}$ for $t > 1$, for $u \in D\left(A_K^{\theta}\right)$, allow us to deduce once again that

\[ \|u\|_{L^p(\Omega)} \leq C \|u\|_{D\left(A_K^{\theta}\right)} \int_0^1 t^{-\frac{\theta}{2}} \left(1 - \frac{\theta}{2}\right) e^{\frac{t}{2}} \|f\|_{L^2(\Omega)} \int_1^\infty e^{-\alpha t} dt < \infty \]

provided that the first integral is finite, i.e., $\theta > N (1 - 2p^{-1}) / (4s)$. The proof in the remaining cases $K \in \{E,D,R\}$ is analogous and thus omitted.

We notice that the assumption $\frac{1}{2} < s < 1$ if $K = \mathcal{N}$ or $K = R$ in (H) is not a restriction, since, otherwise Dirichlet, fractional Neumann and Robin boundary conditions coincide, that is, $A_N \equiv A_R \equiv A_D$ if $0 < s \leq \frac{1}{2}$.

We conclude the section by the following regularity result taken from [44, Proposition 1.1].

**Proposition 2.7.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary, $f \in L^\infty(\Omega)$ and let $u \in W_0^{s,2}(\Omega)$ be a weak solution of the elliptic problem

\[ (-\Delta)^s u = f \quad \text{in} \quad \Omega. \] (2.28)

Then, $u \in C^{0,s}(\mathbb{R}^N)$.

**Remark 4.** We notice that even if $(-\Delta)^s$ and $A_{\Omega}^s$ are related by the relation (1.4), due to the effect of the potential $V_\Omega$ one cannot immediately deduce from Proposition 2.7 a similar result for the elliptic problem associated with $A_{\Omega}^s$. To have such a result, one needs to give a complete proof.
3. Well-posedness and regularity. Without loss of generality, we take \( d = 1 \) in this section. We consider all problems (1.5)-(1.7) in unified form

\[
\begin{align*}
\begin{cases}
\partial_t u + A_K u + f(u) &= 0, & \text{in } \Omega \times (0, \infty), \\
u(0) &= u_0, & \text{in } \Omega,
\end{cases}
\end{align*}
\]

where \( A_K, K \in \{D, N, R, E\} \), is the self-adjoint operator associated with the regional fractional Laplacian, subject to Dirichlet, Neumann, Robin and the fractional Laplace operator associated with Dirichlet (i.e., \( u = 0 \) on \( \mathbb{R}^N \setminus \Omega \)) boundary conditions, respectively, as introduced in the previous section.

We recall that we have set \( W^{s,2}_E(\Omega) := W^{s,2}_0(\Omega), W^{s,2}_D(\Omega) := W^{s,2}_0(\Omega) \) and \( W^{s,2}_K(\Omega) := W^{s,2}(\Omega) \) when \( K \in \{N, R\} \). We also use the notation \( W^{-s,2}(\Omega) := (W^{s,2}(\Omega))^* \), as the topological dual of \( W^{s,2}(\Omega) \), \( 0 < s < 1 \). Furthermore, we endow the domain \( D(A_K) \) of the operator \( A_K \) with the graph norm \( \|A_K u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \). Finally, the symbol \( \langle \cdot, \cdot \rangle \) stands for the duality pairing between any Banach space \( X \) and its dual \( X^* \).

Our assumptions on the nonlinearity \( f \) that we will need in this section are as follows.

**(H1):** When \( K \in \{D, R, E\} \) the function \( f \in C^1_{\text{loc}}(\mathbb{R}) \) satisfies

\[
\liminf_{|\tau| \to +\infty} \frac{f(\tau)}{\tau} > -\lambda_*,
\]

for some constant \( \lambda_* \in [0, C_{K,s}] \), where \( C_{K,s} = C(K, \Omega, s) > 0 \) is the best Sobolev/Poincaré constant in

\[
\|\varphi\|_{L^2(\Omega)}^2 \leq C_{K,s} \mathcal{E}_K(\varphi, \varphi), \quad \text{for all } \varphi \in W^{s,2}_K(\Omega).
\]

When \( K = N \), we assume the function \( f_N(\tau) := f(\tau) - \chi \tau \), for some \( \chi > 0 \), also satisfies (3.2) with \( \lambda_* \in [0, C_{N,s}] \), where \( C_{N,s} = C(N, s, \Omega) > 0 \) is such that

\[
\|\varphi\|_{L^2(\Omega)}^2 \leq C_{N,s} \left( \mathcal{E}_N(\varphi, \varphi) + \|\varphi\|_{L^1}^2 \right), \quad \text{for all } \varphi \in W^{s,2}(\Omega).
\]

**(H2):** \( f \in C^1_{\text{loc}}(\mathbb{R}) \) satisfies

\[
\tilde{C}_f |\tau|^p - c_f \leq f(\tau) |\tau| \leq \tilde{C}_f |\tau|^p + \tilde{c}_f, \quad \text{for all } \tau \in \mathbb{R},
\]

for some appropriate positive constants \( \tilde{C}_f, c_f, \tilde{c}_f \) and some \( p > 1 \).

**(H3):** \( f \in C^1(\mathbb{R}) \) satisfies

\[
f'(\tau) \geq -C_f, \quad \text{for all } \tau \in \mathbb{R},
\]

for some positive constant \( C_f \).

Concerning regularity conditions for the domain \( \Omega \) we assume the following.

**(H4):** \( \Omega \) is an arbitrary bounded open set if \( K \in \{D, E\} \) and \( \Omega \) is a bounded domain with Lipschitz continuous boundary if \( K \in \{N, R\} \).

In what follows we shall use classical (linear/nonlinear semigroup) definitions of strong solutions to the unified problem (3.1). “Strong” solutions are defined via nonlinear semigroup theory for bounded initial data and satisfy the differential equations almost everywhere in \( t > 0 \). We first introduce the rigorous notion of (global) weak solutions to the problem (3.1) as in the classical case for the semilinear parabolic equation with ”Laplacian” diffusion. Throughout the remainder of this article the solution of our system is a function that depends on both time and spatial variables but in our proofs we sometime omit the dependence in \( x \).
Definition 3.1. Let \( u_0 \in L^2(\Omega) \) be given and assume (H2) holds for some \( p > 1 \). The function \( u \) is said to be a weak solution of (3.1) if, for a.e. \( t \in (0, T) \), for any \( T > 0 \), the following properties are valid:

- Regularity:

\[
\begin{cases}
  u \in L^\infty \left((0, T); L^2(\Omega)\right) \cap L^p \left((0, T) \times \Omega\right) \cap L^2((0, T); W^{s,2}_K(\Omega)),
  \\
  \partial_t u \in L^2((0, T); W^{-s,2}_K(\Omega)) + L^p \left((0, T) \times \Omega\right),
\end{cases}
\]  

(3.4)

where \( p' = p/(p - 1) \).

- Variational identity: for the weak solutions the following equality

\[
\langle \partial_t u(t), \xi \rangle + \mathcal{E}_K(u(t), \xi) + \langle f(u(t)), \xi \rangle = 0
\]

holds for all \( \xi \in W^{s,2}_K(\Omega) \cap L^p(\Omega) \), a.e. \( t \in (0, T) \). Finally, we have, in the space \( L^2(\Omega) \), \( u(0) = u_0 \) almost everywhere.

- Energy identity: weak solutions satisfy the following identity

\[
\frac{1}{2} \|u(t)\|_{L^2(\Omega)}^2 + \int_0^t \mathcal{E}_K(u(\tau), u(\tau)) + \langle f(u(\tau)), u(\tau) \rangle \, d\tau = \frac{1}{2} \|u(0)\|_{L^2(\Omega)}^2.
\]

(3.6)

Remark 5. Note that by (3.4), \( u \in C_w\left([0, T]; L^2(\Omega)\right) \), that is the space of all \( L^2(\Omega) \)-valued weakly continuous functions on the interval \([0, T] \). Therefore the initial value \( u(0) = u_0 \) is meaningful when \( u_0 \in L^2(\Omega) \).

Finally, our notion of (global) strong solution is as follows.

Definition 3.2. Let \( u_0 \in L^\infty(\Omega) \) be given. A weak solution \( u \) is "strong" if, in addition, it fulfills the regularity properties:

\[
u \in W^{1,\infty}_{\text{loc}} ((0, T]; L^2(\Omega)) \cap C\left([0, T]; L^\infty(\Omega)\right),
\]

(3.7)

such that \( u(t) \in D(A_K) \), a.e. \( t \in (0, T) \), for any \( T > 0 \).

This section consists of two main parts. At first we will establish the existence and uniqueness of a (local) strong solution on a finite time interval using the theory of monotone operators exploited and developed in [24]. Then exploiting a modified Moser iteration argument we show that the strong solution is actually a global solution. In the second part, we will show the existence of (globally-defined) weak solutions which satisfy the energy identity (3.6) and the variational form (3.5). Then combining the energy method with another refined iteration scheme we also show that any weak solution with initial data in \( L^2(\Omega) \) acquires additional smoothness in an infinitesimal time.

3.1. Weak and strong solutions. We first prove a Poincaré-type inequality in the space \( W^{s,2}(\Omega) \), \( 0 < s < 1 \). Its application is crucial in the proof of strong solutions to semilinear parabolic equations with fractional diffusion.

Lemma 3.3. Let \( 0 < s < 1 \) and let \( \Omega \subset \mathbb{R}^N \) be as in (H4) with \( K \in \{D, N, R, E\} \). Then for all \( \epsilon \in (0, 1) \) there is \( \zeta > 0 \) such that

\[
\|u\|_{L^2(\Omega)}^2 \leq \epsilon \mathcal{E}_K(u, u) + \epsilon^{-\zeta} \|u\|_{L^1(\Omega)}^2,
\]

for all \( u \in W^{s,2}_K(\Omega) \).
The foregoing inequality implies that the resulting sequence \((u_k)\) converges strongly in \(L^2(\Omega)\). By a scaling argument it suffices to prove the inequality for \(u \in W_K^{s, 2} (\Omega)\). Suppose that there is no \(\zeta > 0\) such that the inequality holds for a given \(\epsilon \in (0, 1)\). Then for any \(k \in \mathbb{N}\) there is \(u_k \in W_K^{s, 2} (\Omega)\) such that
\[
\|u_k\|_{L^2(\Omega)}^2 = 1 > \epsilon \mathcal{E}_K (u_k, u_k) + \epsilon^{-k} \|u_k\|_{L^1(\Omega)}^2.
\]
The foregoing inequality implies that the resulting sequence \((u_k)\) is bounded in \(W_K^{s, 2} (\Omega)\). Since the identity operator is a compact map from \(W_K^{s, 2} (\Omega)\) into \(L^2(\Omega)\) and into \(L^1(\Omega)\), respectively, we find a subsequence, again denoted by \((u_k)\), that converges strongly in \(L^2(\Omega)\) and in \(L^1(\Omega)\) to some limit function \(u \in W_K^{s, 2} (\Omega)\). By assumption we have \(\|u\|_{L^2(\Omega)} = 1\). On the other hand, the inequality shows that \(\|u_k\|_{L^1(\Omega)}^2 \leq \epsilon^k\) for all \(k\), such that \(\|u\|_{L^1(\Omega)} = 0\) and thus \(u = 0\) a.e. in \(\Omega\). This is a contradiction which altogether completes the proof of the lemma.

We notice that it follows from Lemma 3.3 that for \(u \in W_K^{s, 2} (\Omega)\),
\[
(\mathcal{E}_K (u, u))^{1/2} + \|u\|_{L^1(\Omega)}
\]
defines an equivalent norm on \(W_K^{s, 2} (\Omega)\). In fact, it is clear that there exists a constant \(C > 0\) such that \((\mathcal{E}_K (u, u))^{1/2} + \|u\|_{L^1(\Omega)} \leq \|u\|_{W_K^{s, 2} (\Omega)}\). Using Lemma 3.3 we get the converse inequality.

The next inequality is essential in comparing various energy forms.

**Lemma 3.4.** Let \(\mathcal{E}\) be the energy given by
\[
\mathcal{E} (u, v) = \int_\Omega \int_\Omega (u(x) - u(y))(v(x) - v(y))K(x, y) \, dx \, dy,
\]
for some positive kernel \(K : \Omega \times \Omega \to \mathbb{R}_+\). Then
\[
\frac{4p}{(p + 1)^2} \mathcal{E} (|u|^{\frac{p+1}{p}}, |u|^{\frac{p+1}{p}}) \leq \mathcal{E} (u, |u|^{p-1} u),
\]
for all functions \(u\) for which the terms in (3.8) make sense and all \(p > 1\).

**Proof.** We prove the inequality by elementary analysis. Define the function \(g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) by
\[
g(z, t) = (z - t) \left(|z|^{p-1} z - |t|^{p-1} t\right) - \frac{4p}{(p + 1)^2} \left(|z|^{\frac{p-1}{p}} z - |t|^{\frac{p-1}{p}} t\right)^2.
\]
Using the definition of \(\mathcal{E}\), we first notice that (3.8) is equivalent to showing that
\[
g(z, t) \geq 0 \quad \text{for all} \quad (z, t) \in \mathbb{R}^2.
\]
Indeed, assume this were true so that for \(u : \Omega \to \mathbb{R}\) there holds
\[
\frac{1}{p} (u(x) - u(y)) \left(|u(x)|^{p-1} u(x) - |u(y)|^{p-1} u(y)\right) \geq \frac{4}{(p + 1)^2} \left(|u(x)|^{\frac{p-1}{p}} u(x) - |u(y)|^{\frac{p-1}{p}} u(y)\right)^2.
\]
Then (3.8) is an immediate consequence of (3.10). We now prove our claim. First, we observe that
\[
g(z, t) = g(t, z), \quad g(z, 0) \geq 0, \quad g(0, t) \geq 0 \quad \text{and} \quad g(z, t) = g(-z, -t).
\]
Therefore, without any restriction, we may assume that \( z \geq t \). A simple calculation shows that
\[
\frac{2}{p+1} \left[ |z|^{\frac{p}{p-1}} z - |t|^{\frac{p}{p-1}} t \right] = \int_t^z |\tau|^{\frac{p}{p-1}} \, d\tau.
\]
Since the function \( \varphi : \mathbb{R} \to \mathbb{R} \) given by \( \varphi(\tau) = |\tau|^p \) \((p \geq 2)\) is convex, then using the well-known Jensen inequality, it follows that
\[
\frac{4p}{(p+1)^2} |z|^{\frac{p-1}{p}} z - |t|^{\frac{p-1}{p}} t \mid^2 = p \left[ \frac{2}{p+1} \left[ |z|^{\frac{p}{p-1}} z - |t|^{\frac{p}{p-1}} t \right] \right]^2
\]
\[
= p \left[ \int_t^z |\tau|^{\frac{p}{p-1}} \, d\tau \right]^2
\]
\[
= p(z - t)^2 \left[ \int_t^z |\tau|^{\frac{p}{p-1}} \frac{d\tau}{z - t} \right]^2
\]
\[
\leq p(z - t)^2 \int_t^z |\tau|^{p-1} \frac{d\tau}{z - t}
\]
\[
= p(z - t) \left( |z|^{p-1} z - |t|^{p-1} t \right).
\]
We have shown the claim (3.9) and this completes the proof of lemma.

We will now state a well-known result for the non-homogeneous Cauchy problem
\[
\begin{cases}
  u'(t) + A(u) \ni g(t), & t \in [0, T], \\
  u_{t=0} = u_0.
\end{cases}
\]  
(3.11)

**Theorem 3.5.** ([43, Chapter IV, Theorem 4.3]) Let \( \varphi : H \to (-\infty, +\infty) \) be a proper, convex, and lower-semicontinuous functional on the Hilbert space \( H \) and set \( A = \partial \varphi \), where \( \partial \varphi \) denotes the subdifferential of the functional \( \varphi \). Let \( u \) be the generalized solution of (3.11) with \( g \in L^2(0, T; H) \) and \( u_0 \in \overline{D(A)} \). Then \( \varphi(u) \in L^1(0, T), \sqrt{t}u'(t) \in L^2((0, T); H) \) and \( u(t) \in D(A) \) for a.e. \( t \in [0, T] \).

The second one is a more general version of [43, Chapter IV, Proposition 3.2] and was proved in [24, Theorem 6.3 and Corollary 6.4].

**Theorem 3.6.** Let the assumptions of Theorem 3.5 be satisfied. Assume that \( A = \partial \varphi \) is strongly accretive in \( H \), that is, \( A - \omega I \) is accretive for some \( \omega > 0 \) and, in addition,
\[
g \in L^\infty([\delta, \infty); H) \cap W^{1,2}([\delta, \infty); H),
\]
for every \( \delta > 0 \). Let \( u \) be the unique generalized solution of (3.11) for \( u_0 \in \overline{D(A)} \). It follows that
\[
u \in L^\infty([\delta, \infty); D(A)) \cap W^{1,\infty}([\delta, \infty); H).
\]

Now we state the first main theorem of this section.

**Theorem 3.7.** Assume that the nonlinearity \( f \) obeys (H1) and assume \( \Omega \) satisfies (H4). For every \( u_0 \in L^\infty(\Omega) \), there exists a unique strong solution of (3.1) in the sense of Definition 3.2.
Proof. Step 1. (Local existence). Fix $K \in \{D, \mathcal{N}, R, E\}$. Let $u_0 \in L^\infty(\Omega) \subset L^2(\Omega) = D(A_K)^{L^2(\Omega)}$. Let $\varphi_K$ be the functional on $L^2(\Omega)$ with domain $D(\varphi_K) = W^{1,2}_0(\Omega)$ and defined by $\varphi_K(u) = \frac{1}{2} \mathcal{E}_K(u, u)$ for $u \in W^{1,2}_0(\Omega)$. From Theorem 2.5 we know that $-A_K = -\partial \varphi_K$ generates a strongly continuous (linear) semigroup $(e^{-tA_K})_{t \geq 0}$ of contraction operators on $L^2(\Omega)$. Finally, $e^{-tA_K}$ is non-expansive on $L^\infty(\Omega)$, that is,

$$
\|e^{-tA_K}u_0\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}, \quad t \geq 0 \text{ and } u_0 \in L^\infty(\Omega),
$$

and $A_K$ is strongly accretive only in the case $K \in \{D, R, E\}$ whereas $A_{\mathcal{N}}$ is only accretive on $L^2(\Omega)$ (this follows from the definition of $A_{\mathcal{N}}$ and from Theorem 2.5). Thus, the operator version of the original problem (3.1) reads

$$
\partial_t u = -A_{\mathcal{N}}u - f_K(u), \quad K \in \{D, E, R\}
$$

and

$$
\partial_t u = -A_{\mathcal{N}, K}u - f_{\mathcal{N}}(u) + \chi I, \quad K \in \{D, E, R\}
$$

where we have set $f_K(\tau) = f(\tau)$, $K \in \{D, E, R\}$ and $f_{\mathcal{N}}(\tau) = f(\tau) - \chi \tau$. Clearly, $A_{\mathcal{N}, K}$ is also strongly accretive on $L^2(\Omega)$ by construction and satisfies (3.12).

We adapt an argument we have developed in [24, Theorem 3.4]. We prove the existence of a (locally-defined) strong solution to (3.13), (3.14) by a fixed point argument. We shall focus on the case $K \in \{D, R, E\}$ as the case $K = \mathcal{N}$ is similar. To this end, fix $0 < T^* \leq T$, consider the space

$$
\mathcal{X}_{T^*, R^*} \equiv \left\{ u \in C([0, T^*]; L^\infty(\Omega)) : \|u(t)\|_{L^\infty(\Omega)} \leq R^* \right\}
$$

and define the following mapping

$$
\Sigma(u)(t) = e^{-tA_K}u_0 - \int_0^t e^{-(t-\tau)A_K}f(u(\tau))d\tau, \quad t \in [0, T^*].
$$

Note that $\mathcal{X}_{T^*, R^*}$, when endowed with the norm of $C([0, T^*]; L^\infty(\Omega))$, is a closed subset of $C([0, T^*]; L^\infty(\Omega))$, and since $f$ is continuously differentiable, $\Sigma(u)(t)$ is continuous on $[0, T^*]$. We will show that, by properly choosing $T^*, R^* > 0$, $\Sigma : \mathcal{X}_{T^*, R^*} \to \mathcal{X}_{T^*, R^*}$ is a contraction mapping with respect to the metric induced by the norm of $C([0, T^*]; L^\infty(\Omega))$. The appropriate choices for $T^*, R^* > 0$ will be specified below. First, we show that $u \in \mathcal{X}_{T^*, R^*}$ implies that $\Sigma(u) \in \mathcal{X}_{T^*, R^*}$, that is, $\Sigma$ maps $\mathcal{X}_{T^*, R^*}$ to itself. From (3.12) and the fact that $f \in C^1_{loc}(\mathbb{R})$, we observe that the mapping $\Sigma$ satisfies the following estimate

$$
\|\Sigma(u)(t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + \int_0^t \left\|e^{-(t-\tau)A_K}(f(0) + (f(u(\tau)) - f(0)))\right\|_{L^\infty(\Omega)}d\tau
$$

$$
\leq \|u_0\|_{L^\infty(\Omega)} + \varepsilon \left(\|f(0)\| + Q_{f'}(R^*) R^*\right),
$$

for some positive continuous function $Q_{f'}$ which depends only on the size of the nonlinearity $f'$. Thus, provided that we set $R^* = 2 \|u_0\|_{L^\infty(\Omega)}$, we can find a sufficiently small time $T^* > 0$ such that

$$
2T^* \left(\|f(0)\| + Q_{f'}(R^*) R^*\right) \leq R^*,
$$

(3.16)
in which case \( \Sigma (u(t)) \in X_{T^*, R^*} \), for any \( u(t) \in X_{T^*, R^*} \). Next, we show that by possibly choosing \( T^* > 0 \) smaller, \( \Sigma : X_{T^*, R^*} \to X_{T^*, R^*} \) is also a contraction. Indeed, for any \( u_1, u_2 \in X_{T^*, R^*} \), exploiting again (3.12), we estimate
\[
\| \Sigma (u_1(t)) - \Sigma (u_2(t)) \|_{\infty, \Omega} \\
\leq Q_{f'} (R^*) \int_0^T \left\| e^{-(t-\tau)A_K} (u_1(\tau) - u_2(\tau)) \right\|_{\infty, \Omega} d\tau \\
\leq t Q_{f'} (R^*) \| u_1 - u_2 \|_{C([0, T^*]; L^\infty(\Omega))}.
\] (3.17)

This shows that \( \Sigma \) is a contraction on \( X_{T^*, R^*} \) provided that \( T^* < 1 \) is much smaller than the one determined by (3.16) and \( T^* Q_{f'} (R^*) < 1 \). Therefore, owing to the contraction mapping principle, we conclude that problem (3.13) has a unique local solution \( u \in X_{T^*, R^*} \). This solution can certainly be (uniquely) extended on a right maximal time interval \( [0, T_{\max}) \), such that \( \Sigma (u(t)) \) is also a contraction on \( X_{T^*, R^*} \).

Indeed, if \( T_{\max} < \infty \) and the latter condition does not hold, we can find a sequence \( t_n \nearrow T_{\max} \) such that \( \| u(t_n) \|_{L^\infty(\Omega)} \leq C \). This would allow us to extend \( u \) as a solution to Equation (3.13) to an interval \( [0, t_n + \delta) \), for some \( \delta > 0 \) independent of \( n \). Hence \( u \) can be extended beyond \( T_{\max} \) which contradicts the construction of \( T_{\max} > 0 \). To conclude that the solution \( u \) belongs to the class in Definition 3.2, let us further set \( G(t) := -f(u(t)) \), for \( u \in C([0, T_{\max}); L^\infty(\Omega)) \) and notice that \( u \) is the “generalized” solution of
\[
\partial_t u + A_K u = G(t), \ t \in [0, T_{\max}),
\] (3.18)
such that \( u(0) = u_0 \in L^\infty(\Omega) \subset L^2(\Omega) = D(A_K) \). By Theorem 3.5, the “generalized” solution \( u \) has the additional regularity \( \partial_t u \in L^2 ([\delta, T_{\max}); L^2(\Omega)) \), which together with the facts that \( u \) is continuous on \( [0, T_{\max}) \) and \( f \in C^1_{loc}(\mathbb{R}) \), yield

\[
G \in W^{1,2} ([\delta, T_{\max}); L^2(\Omega)) \cap L^\infty ([\delta, T_{\max}); L^2(\Omega)).
\] (3.19)

Thus, we can apply Theorem 3.6 to deduce that
\[
u \in L^\infty ([\delta, T_{\max}); D(A_K)) \cap W^{1,\infty} ([\delta, T_{\max}); L^2(\Omega)),
\] (3.20)
such that the solution \( u \) is Lipschitz continuous on \( [\delta, T_{\max}) \), for every \( \delta > 0 \). Thus, we have obtained a locally-defined strong solution in the sense of Definition 3.2. As to the variational equality in Definition 3.1, we note that this equality is satisfied even pointwise (in time \( t \in (0, T_{\max}) \)) by the strong solutions. Our final point is to show that \( T_{\max} = \infty \), because of condition (H1). This ensures that the strong solution constructed above is also global.

Step 2. (Energy estimate) Let \( m \geq 1 \) and consider the function \( E_m : (0, \infty) \to [0, \infty) \) defined by \( E_m(t) := \| u(t) \|_{L^{m+1}(\Omega)}^{m+1} \). First, notice that \( E_m \) is well-defined on \( (0, T_{\max}) \) because \( u \) is bounded in \( \Omega \times (0, T_{\max}) \) and because \( \Omega \) has finite measure. Since \( u \) is a strong solution on \((0, T_{\max}) \), see Definition 3.2 (or (3.20)), recall that \( u \) is continuous from \((0, T_{\max}) \to L^\infty(\Omega) \) and Lipschitz continuous on \([\delta, T_{\max}) \) for every \( \delta > 0 \). Thus, \( u \) (as function of \( t \)) is differentiable a.e., whence, the function \( E_m(t) \) is also differentiable for a.e. \( t \in (0, T_{\max}) \).

For strong solutions and \( t \in (0, T_{\max}) \), integration by parts procedure yields the following standard energy identity:

\[
\frac{1}{2} \frac{d}{dt} E_1(t) + E_K(u(t), u(t)) + \int_{\Omega} f(u(t)) u(t) \, dx = 0.
\] (3.21)
Gronwall’s inequality, (3.23) gives the following estimate for \( t \) where
\[
K \in all cases
\]
Assumption (H1) in the case \( K \in \{ D, R, E \} \) implies that
\[
f(\tau) \tau \geq -\lambda_s \tau^2 - C_f, \tag{3.22}
\]
for some \( C_f > 0 \) and for all \( \tau \in \mathbb{R} \). This inequality allows us to estimate the nonlinear term in (3.21). We have (by using an equivalent norm in \( W_{K,2}^{s,2}(\Omega) \)) that
\[
\frac{1}{2} \frac{d}{dt} E_1(t) + C_{K,s} \| u(t) \|_{W_{K,2}^{s,2}(\Omega)}^2 \leq C_f |\Omega| + \lambda_s E_1(t), \tag{3.23}
\]
where \(|\Omega|\) denotes the \( N \)-dimensional Lebesgue measure of \( \Omega \). In view of (3.3) and Gronwall’s inequality, (3.23) gives the following estimate for \( t \in (0, T_{\max}) \),
\[
\| u(t) \|_{L^2(\Omega)}^2 + 2 (C_{K,s} - \lambda_s) \int_0^t \| u(\tau) \|_{W_{K,2}^{s,2}(\Omega)}^2 d\tau \\
\leq \| u_0 \|_{L^2(\Omega)}^2 e^{-\rho t} + C (f, |\Omega|),
\]
for some constants \( \rho = \rho(N, \Omega) > 0 \), \( C(f, |\Omega|) > 0 \). The proof of the energy inequality in the case \( K = \mathcal{N} \) is analogous (in this case, \( f_k \) obeys (3.22)).

**Step 3. (The iteration argument).** In this step, \( c > 0 \) will denote a constant that is independent of \( t, T_{\max}, m, k \) and initial data, which only depends on the other structural parameters of the problem. Such a constant may vary even from line to line. Moreover, we shall denote by \( Q_\tau (m) \) a monotone nondecreasing function in \( m \) of order \( \tau \), for some nonnegative constant \( \tau \), independent of \( m \). More precisely, \( Q_\tau (m) \sim cm^\tau \) as \( m \to +\infty \). We begin by showing that \( E_m(t) \) satisfies a local recursive relation which can be used to perform an iterative argument. Testing the variational equation (3.5) for the strong solution with \( \|u\|_{m-1}^m u, m \geq 1 \), gives on account of (3.22) and Lemma 3.4 the following inequality:
\[
\frac{d}{dt} E_m(t) + \frac{4m}{m+1} E_K(\|u(t)\|_{m+1}^m u, \|u(t)\|_{m+1}^m u) \tag{3.25}
\]
\[
\leq Q_1 (m + 1) \left( E_m(t) + (E_m(t))^{\frac{m}{m+1}} \right),
\]
in all cases \( K \in \{ D, \mathcal{N}, R, E \} \). Next, set \( m_k + 1 = 2^k, k \in \mathbb{N} \), and define
\[
M_k := \sup_{t \in (0, T_{\max})} \int_\Omega |u(t, x)|^{2^k} dx = \sup_{t \in (0, T_{\max})} E_{m_k}(t). \tag{3.26}
\]
Our goal is to derive a recursive inequality for \( M_k \) using (3.25). In order to do so, for \( q > 1 \) fixed that we will choose below, we define
\[
\bar{\eta}_k := \frac{m_k - m_{k-1}}{q(1 + m_k) - (1 + m_{k-1})} = \frac{1}{2q - 1} < 1, \bar{\eta}_k := 1 - \bar{\eta}_k = \frac{2q - 1}{2q - 1}. 
\]
We aim to estimate the terms on the right-hand side of (3.25) in terms of the \( L^{1+m_k-1}(\Omega) \)-norm of \( u \). First, the Hölder inequality and the Sobolev inequality (i.e., \( W_{K,2}^{s,2}(\Omega) \subset L^{q} (\Omega) \), with \( q = q(N, s) \in (1, N/(N-2s)] \), if \( N > 2s \) and \( q \in (1, \infty) \) if \( N = 2s \), see (2.3) and (2.1)) yield
\[
\int_\Omega |u|^{1+m_k} dx \leq \left( \int_\Omega |u|^{(1+m_k)q} dx \right)^{\bar{\eta}_k} \left( \int_\Omega |u|^{-m_k-1} dx \right)^{\bar{\eta}_k} \tag{3.27}
\]
\[
\leq c \left[ E_K(\|u\|_{m_k-1}^2 u, \|u\|_{m_k-1}^2 u) \right]^{\bar{\eta}_k} \left( \int_\Omega |u|^{-m_k-1} dx \right)^{\bar{\eta}_k},
\]
with \( \overline{s}_k = \overline{q}_k q \equiv q/(2q - 1) \in (0, 1) \). Applying Young’s inequality on the right-hand side of (3.27), we get for every \( \epsilon > 0 \),

\[
Q_1 (m_k + 1) \int_{\Omega} |u|^{1+m_k} \, dx \leq \epsilon \mathcal{E}_K(|u|^{\frac{m_k-1}{2}} u, |u|^{\frac{m_k-1}{2}} u) \\
+ Q_\alpha (m_k + 1) \left( \int_{\Omega} |u|^{1+m_k-1} \, dx \right)^2,
\]

(3.28)

for some \( \alpha > 0 \) independent of \( k \), since \( z_k := \overline{q}_k/(1-\overline{s}_k) \equiv 2 \). In order to estimate the last term on the right-hand side of (3.25), we define two decreasing and increasing sequences \( (l_k)_{k \in \mathbb{N}} \) and \( (w_k)_{k \in \mathbb{N}} \), respectively, such that

\[
l_k := \frac{m_k + 1}{\overline{s}_k m_k} \quad \text{and} \quad w_k := \frac{\overline{q}_k m_k}{m_k (1-\overline{s}_k) + 1},
\]

and observe that they satisfy

\[ 1 < l_k \leq 2 \left( 2 - \frac{1}{q} \right), \quad \frac{2(q-1)}{3q-2} \leq w_k \leq 2 \]

for all \( k \in \mathbb{N} \) (in particular, \( w_k \to 2 \) as \( k \to \infty \)). The application of Young’s inequality in (3.28) yields again

\[
Q_1 (m_k + 1) \left( \int_{\Omega} |u|^{1+m_k} \, dx \right)^{\frac{m_k}{m_k+1}} \leq \epsilon \mathcal{E}_K(|u|^{\frac{m_k-1}{2}} u, |u|^{\frac{m_k-1}{2}} u) \\
+ Q_{\beta_k} (m_k + 1) \left( \int_{\Omega} |u|^{1+m_k-1} \, dx \right)^{w_k},
\]

(3.29)

for every \( \epsilon > 0 \), where now

\[
Q_{\beta_k} (m_k + 1) \sim \frac{c}{\epsilon^{1/(l_k-1)}} (1+m_k)^{\beta_k}
\]

with

\[
\beta_k := \frac{m_k + 1}{m_k (1-\overline{s}_k) + 1} \rightarrow \frac{2q-1}{q-1}, \quad \text{as} \quad k \to \infty.
\]

Hence, inserting (3.28), (3.29) into inequality (3.25), choosing a sufficiently small \( 0 < \epsilon \leq \epsilon_0 := \frac{1}{2} \), and simplifying, we obtain for \( t \in (0, T_{\text{max}}) \),

\[
\frac{d}{dt} \int_{\Omega} |u(t,x)|^{1+m_k} \, dx + \epsilon_0 \frac{\mathcal{E}_K(|u(t)|^{\frac{m_k-1}{2}} u, |u(t)|^{\frac{m_k-1}{2}} u)}{2} \\
\leq Q_\delta (m_k + 1) \left( \int_{\Omega} |u|^{1+m_k-1} \, dx \right)^2,
\]

(3.30)

for some positive constant \( \delta > 0 \) independent of \( k \).

Next, since \( u(t) \in W^s_{K} (\Omega) \cap L^\infty (\Omega) \), for a.e. \( t \in (0, T_{\text{max}}) \), it follows from [48, Remark 2.5] that \( |u(t)|^{1+m_k} \in W^{s}_{K}(\Omega) \). Thus, we can apply Lemma 3.3 to infer that

\[
\epsilon_0 \mathcal{E}_K(|u|^{\frac{m_k-1}{2}} u, |u|^{\frac{m_k-1}{2}} u) \geq \int_{\Omega} |u|^{m_k+1} \, dx - \epsilon_0^\zeta \left( \int_{\Omega} |u|^{1+m_k-1} \, dx \right)^2,
\]

(3.31)

for some \( \zeta > 0 \) independent of \( u, k \). We can now combine (3.31) with (3.30) to deduce

\[
\frac{d}{dt} \int_{\Omega} |u(t,x)|^{2^s} \, dx + \frac{1}{2} \int_{\Omega} |u(t,x)|^{2^s} \, dx \leq Q_\delta (2^k) M_{k-1}^2,
\]

(3.32)
for $t \in (0, T_{\text{max}})$. Integrating (3.32) over $(0, t)$, we infer from Gronwall-Bernoulli’s inequality [13, Lemma 1.2.4] that there exists yet another constant $c > 0$, independent of $k$, such that

$$M_k \leq \max \left\{ \int_\Omega |u_0|^{2^k} \, dx, c2^{k\delta} M_{k-1}^2 \right\}, \text{ for all } k \geq 2. \quad (3.33)$$

On the other hand, let us observe that there exists a positive constant $C_\infty = C_\infty (\|u_0\|_{L^\infty(\Omega)}) \geq 1$, independent of $k$, such that

$$\|u_0\|_{L^2(\Omega)} \leq C_\infty.$$ 

Taking the $2^k$-th root on both sides of (3.33), and defining $X_k := \sup_{t \in (0, T_{\text{max}})} \|u(t)\|_{L^2(\Omega)}$, we easily arrive at

$$X_k \leq \max \left\{ C_\infty, (c2^{k\delta})^{\frac{1}{2^k}} X_{k-1} \right\}, \text{ for all } k \geq 2. \quad (3.34)$$

By straightforward induction in (3.34) (see [3, Lemma 3.2]; cf. also [13, Lemma 9.3.1]), we finally obtain the estimate

$$\sup_{t \in (0, T_{\text{max}})} \|u(t)\|_{L^\infty(\Omega)} \leq \lim_{k \to +\infty} X_k \leq c \max \left\{ C_\infty, \sup_{t \in (0, T_{\text{max}})} \|u(t)\|_{L^2(\Omega)} \right\}. \quad (3.35)$$

It remains to notice that (3.35) together with the bound (3.24) shows that the norm $\|u(t)\|_{L^\infty(\Omega)}$ is uniformly bounded for all times $t > 0$ with a bound, independent of $T_{\text{max}}$, depending only on $\|u_0\|_{L^\infty(\Omega)}$, the “size” of the domain and the non-linear function $f$. This gives $T_{\text{max}} = +\infty$ so that strong solutions are in fact global. This completes the proof of the theorem.

**Remark 6.** Strong solutions to the system (3.1) exhibit an improved regularity in time, we have

$$u \in C ([0, T]; L^\infty (\Omega)) \cap C((\delta, T]; W^{s,2}_K (\Omega)), \quad (3.36)$$

for any $T > \delta > 0$. This follows from the fact that the non-linear function $f$ is continuously differentiable. Note that the second regularity in (3.36) is a consequence of the first one, the time regularity in (3.7) (see Definition 3.2) and the variational identity (3.5).

The following result is immediate.

**Corollary 3.1.** Suppose that the assumptions of Theorem 3.7 are satisfied and let $K \in \{D, N, R, E\}$. The reaction-diffusion system (3.1) defines a (nonlinear) continuous semigroup

$$T_K (t) : L^\infty (\Omega) \to L^\infty (\Omega),$$

given by

$$T_K (t) u_0 = u(t), \quad (3.37)$$

where $u$ is the (unique) strong solution in the sense of Definition 3.2.

In the final part of this section, we aim to prove the existence of weak solutions in the sense of Definition 3.1.

**Theorem 3.8.** Assume that the non-linearity $f$ obeys (H2), (H3) and $\Omega$ satisfies (H4). Then, for any initial data $u_0 \in L^2 (\Omega)$, there exists a unique (globally-defined) weak solution

$$u \in C ([0, T]; L^2 (\Omega))$$

in the sense of Definition 3.1.
Proof. We divide the proof into three steps. For practical purposes, $C$ will denote a positive constant that is independent of time, $T$, $\epsilon > 0$ and initial data, but which only depends on the other structural parameters. Such a constant may vary even from line to line.

Step 1. (Approximation scheme). First, we consider a sequence $u_{0\epsilon} \in L^\infty(\Omega) \cap W^{s,2}_K(\Omega)$ such that $u_{0\epsilon} \to u_0 = u(0)$ in $L^2(\Omega)$. Next, for each $K \in \{D, N, R, E\}$ let $u_\epsilon(t)$ be a strong solution, in the sense of Definition 3.2, of the system

$$\begin{cases}
\partial_t u_\epsilon + A_K u_\epsilon + f(u_\epsilon) = 0, & \text{in } \Omega \times (0, \infty), \\
u_\epsilon(0) = u_{0\epsilon}, & \text{in } \Omega.
\end{cases}$$

(3.38)

Note that such a smooth solution exists since every function that satisfies (H2) also obeys (3.22). Testing the weak formulation associated with problem (3.38), cf. (3.5), with $\xi = u_\epsilon(t)$ we find

$$\frac{d}{dt} \int_\Omega u_\epsilon(t) dx = 0,$$

for all $t \in (0, T)$. Invoking assumption (H2), we infer

$$\frac{d}{dt} \int_\Omega u_\epsilon(t) dx + 2\mathcal{E}_K(u_\epsilon(t), u_\epsilon(t)) + 2\tilde{C}_f \|u_\epsilon(t)\|_{L^p(\Omega)}^p \leq C|\Omega|. \quad (3.39)$$

Integrate the foregoing inequality over $(0, T)$ we deduce

$$\|u_\epsilon(t)\|_{L^2(\Omega)}^2 + \int_0^t \left(2\mathcal{E}_K(u_\epsilon(\tau), u_\epsilon(\tau)) + 2\tilde{C}_f \|u_\epsilon(\tau)\|_{L^p(\Omega)}^p \right) d\tau \leq \|u_\epsilon(0)\|_{L^2(\Omega)}^2 e^{-\rho t} + C \quad (3.40)$$

for all $t \in (0, T)$, for some $\rho > 0$ independent of $\epsilon > 0$. As usual, on account of (3.40), we deduce the following uniform (in $\epsilon > 0$) bounds

$$u_\epsilon \in L^\infty((0, T); L^2(\Omega)) \cap L^p((0, T); L^p(\Omega)),$$

(3.41)

$$u_\epsilon \in L^2((0, T); W^{s,2}_K(\Omega)),$$

for any $T > 0$. Hence, by (3.41) we also get

$$A_K u_\epsilon \in L^2((0, T); W^{s,2}_K(\Omega)), \quad f(u_\epsilon) \in L^p((0, T); L^{p'}(\Omega)),$$

(3.42)

uniformly in $\epsilon > 0$. Here, recall that $A_K$ is the nonnegative self-adjoint operator associated with the bilinear form $\mathcal{E}_K$. Comparison in (3.5) then gives

$$\partial_t u_\epsilon \in L^2((0, T); W^{s,2}_K(\Omega)) + L^p((0, T); L^{p'}(\Omega)) \quad (3.43)$$

uniformly in $\epsilon > 0$.

Step 2. (Passage to limit). From the above properties (3.41)-(3.43), we see that as $\epsilon \to 0^+$,

$$u_\epsilon \to u \text{ weakly star in } L^\infty((0, T); L^2(\Omega)),$$

$$u_\epsilon \to u \text{ weakly in } L^p((0, T); L^p(\Omega)),$$

(3.44)

$$\partial_t u_\epsilon \to \partial_t u \text{ weakly in } L^2((0, T); W^{s,2}_K(\Omega)) + L^p((0, T); L^{p'}(\Omega)),$$

along a subsequence. Since the continuous embedding $W^{s,2}_K(\Omega) \subset L^2(\Omega)$ is compact, then we can exploit standard embedding results for vector valued functions (see, e.g., [12]), to deduce

$$u_\epsilon \to u \text{ strongly in } L^2((0, T); L^2(\Omega)). \quad (3.45)$$
By refining in (3.45), $u_\epsilon$ converges to $u$ a.e. in $\Omega \times (0,T)$. Then, by means of known results in measure theory [12], the continuity of $f$ and the convergence of (3.45) imply that $f(u_\epsilon)$ converges weakly to $f(u)$ in $L^p((0,T) \times \Omega)$, while from (3.41)-(3.42) and the linearity of $A_K$, we further see that

$$A_K u_\epsilon \rightarrow A_K u \text{ weakly in } L^2((0,T); W^{-s,2}_K(\Omega)). \quad (3.46)$$

We can now pass to the limit as $\epsilon \rightarrow 0$ in the weak form (3.5) for $u_\epsilon$ to deduce the desired weak solution $u$, satisfying the variational identity (3.5) and the regularity properties (3.4). The energy identity (3.6) is an immediate consequence of [12, Theorem II.1.8] and the simple observation by which the distributional derivative $\partial_t u(t)$ from $D'([0,T]; W^{-s,2}_K(\Omega) + L^p(\Omega))$ can be represented as $\partial_t u(t) = Z_1(t) + Z_2(t)$, with

$$Z_1(t) := -A_K u(t) \in L^2((0,T); W^{-s,2}_K(\Omega)),$$

$$Z_2(t) := -f(u(t)) \in L^p((0,T); L^p(\Omega)).$$

These spaces are precisely the dual of the space $L^2((0,T); W^{s,2}_K(\Omega))$, and the space $L^p((0,T); L^p(\Omega))$, respectively. In particular, we obtain that every weak solution $u \in C([0,T]; L^2(\Omega))$, and that the map $t \mapsto \|u(t)\|_{L^2(\Omega)}$ is absolutely continuous on $[0,T]$, such that $u$ satisfies the energy identity (3.6).

**Step 2. (Uniqueness and continuous dependence).** As usual, consider any two weak solutions $u_1, u_2$, and set $u(t) = u_1(t) - u_2(t)$. According to the energy identity (3.6) and assumption (H3), we obtain

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + 2\mathcal{E}_K(u(t), u(t)) \leq 2C_f \|u(t)\|^2_{L^2(\Omega)}. \quad (3.47)$$

Upon integration over $(0,t)$, we infer

$$\|u_1(t) - u_2(t)\|^2_{L^2(\Omega)} \leq C e^{Ct} \|u_1(0) - u_2(0)\|^2_{L^2(\Omega)}. \quad (3.48)$$

This yields the desired continuous dependence result with respect to the initial data. The proof of the theorem is finished.

Consequently, problem (3.1) defines a dynamical system in the classical sense.

**Corollary 3.2.** Let the assumptions of Theorem 3.8 be satisfied. The semilinear parabolic equation (3.1) defines a (nonlinear) continuous semigroup $\mathcal{S}_K(t) : L^2(\Omega) \rightarrow L^2(\Omega)$, $K \in \{D,N,R,E\}$, given by

$$\mathcal{S}_K(t) u_0 = u(t), \quad (3.49)$$

where $u$ is the (unique) weak solution in the sense of Definition 3.1.

### 3.2. Regularity of weak solutions.

The main result of this section is concerned with proving that any weak solution with initial condition in $L^2(\Omega)$ acquires additional smoothness in an infinitesimal time; more precisely, it becomes a strong solution in the sense of Definition 3.2. Moreover, the same result also establishes the existence of an absorbing ball for the semigroup $\mathcal{S}_K$ in the space $W^{s,2}_K(\Omega) \cap L^\infty(\Omega)$.

The latter is an essential property in the theory of attractors (see the next section). Fix now $K \in \{D,N,R,E\}$.

**Theorem 3.9.** Let the assumptions of Theorem 3.8 be satisfied. Then, for $u_0 \in L^2(\Omega)$ any orbit $u(t) = \mathcal{S}_K(t) u_0$ of (3.1) satisfies

$$u \in L^\infty((\rho, \infty); W^{s,2}_K(\Omega) \cap L^\infty(\Omega)) \cap W^{1,2}((0, \infty); L^2(\Omega)), \quad (3.50)$$
for every $\rho > 0$, and the following estimate holds:

$$
\sup_{t \geq \rho} \left( \|u(t)\|^2_{W^{s,2}_K(\Omega) + L^\infty(\Omega)} + \int_0^t \|\partial_t u(\tau)\|^2_{L^2(\Omega)} d\tau \right) \leq C_\delta,
$$

(3.50)

for some constant $C_\rho = C(\rho) > 0$, independent of $t$ and initial data. Moreover, $u(t) \in D(A_K)$ for a.e. $t > \rho$. The constant $C_\rho$ in (3.50) is uniformly bounded in $\rho$ if $\rho \geq 1$.

**Proof.** Step 1. (The bound in $L^\infty(\Omega)$). In this case, as in the proof of Theorem 3.8, we can use strong solutions in order to provide sufficient regularity to justify all the calculations performed in the proof below. At the very end one can pass to the limit and obtain the estimate even for the weak solutions.

Let now $\tau' > \tau > 0$ and fix $\mu := \tau' - \tau$. We claim that there exists a positive constant $C = C(\mu) \sim \mu^{-\eta}$ (for some $\eta > 0$), independent of $t$ and the initial data, such that

$$
\sup_{t \geq \tau'} \|u(t)\|_{L^\infty(\Omega)} \leq C \sup_{\sigma \geq \tau} \|u(\sigma)\|_{L^2(\Omega)},
$$

(3.51)

The argument leading to (3.51) follows exactly as in [21, Theorem 2.3] (cf. also [22, 24]). It is based on the following recursive inequality for $E_{m_k}(t)$, which is a consequence of (3.32) and (3.27)-(3.30):

$$
\sup_{t \geq t_{k-1}} E_{m_k}(t) \leq C \left(2^k\right)^t \left(\sup_{\sigma \geq t_k} E_{m_{k-1}}(\sigma)\right)^2,
$$

(3.52)

where the sequence $\{t_k\}_{k \in \mathbb{N}}$ is defined recursively $t_k = t_{k-1} - \mu/2^k$, $k \geq 1$, $t_0 = \tau'$.

Here we recall that $C = C(\mu) > 0$, $l > 0$ are independent of $k$ and that $C(\mu)$ is uniformly bounded in $\mu$ if $\mu \geq 1$ (see [21, Theorem 2.3]). We can iterate in (3.52) with respect to $k \geq 1$ and obtain that

$$
\sup_{t \geq t_{k-1}} E_{m_k}(t) \leq \left(C \left(2^k\right)^\frac{1}{l}\right) \left(C \left(2^{k-1}\right)^\frac{1}{l}\right)^2 \cdots \left(C \left(2\right)^\frac{1}{l}\right)^{2^k} \left(\sup_{\sigma \geq \tau} \|u(\sigma)\|_{L^2(\Omega)}\right)^{2^k}
\leq C \left(2^k \sum_{i=1}^{\infty} \frac{1}{2^i}\right) \left(2 \sum_{i=1}^{\infty} \frac{1}{2^i}\right) \left(\sup_{\sigma \geq \tau} \|u(\sigma)\|_{L^2(\Omega)}\right)^{2^k}.
$$

(3.53)

Therefore, we can take the $2^k$-th root on both sides of (3.53) and let $k \to +\infty$. Using the facts that $\zeta := \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty$, $\zeta := \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty$, we easily deduce (3.51). From the inequality (3.51) together with the energy estimate (3.40), which is also satisfied by the weak solution $u(t)$ with initial datum $u_0 \in L^2(\Omega)$, we deduce

$$
\sup_{t \geq \rho} \|u(t)\|_{L^\infty(\Omega)} \leq C_\delta,
$$

(3.54)

with $C_\rho \sim \rho^{-\eta}$, for some $\eta > 0$, for each $\rho > 0$. This yields the first part in (3.50).

**Step 2.** (The bound in $W^{s,2}_K(\Omega)$). The argument relies on using the test function $\xi = \partial_t u(t)$ into the variational equation (3.5). However, in order to further justify this choice in (3.5) we actually need to require more regularity of the strong solution, in particular we need to have

$$
u \in W^{1,q}_{loc}((0,\infty);W^{s,2}_K(\Omega)), \text{ for some } q > 1.
$$

(3.55)

Due to the non-smooth nature of the domain $\Omega$ and its boundary $\partial \Omega$, one generally lacks any further information on both weak and strong solutions than the one provided by Definitions 3.1 and 3.2. In order to overcome this difficulty, we need to further truncate the strong solutions resulting in approximate solutions which
Thus, the key choice of the test function $u$ (3.5) is now allowed for these truncated solutions bounded attractor in $D$ since the latter problem is uniquely solvable by the Cauchy-Lipschitz theorem since $\dim t$ \text{for all} $d \in D$. We recall that the latter problem is uniquely solvable by the Cauchy-Lipschitz theorem since $f \in C^1$, and that the solution $u_{t,n}$ has the desired regularity. Thus, they key choice of the test function $\xi = \partial_t u_{t,n,e}$ into the variational formulation (3.5) is now allowed for these truncated solutions $u_{t,n}$. We infer

\begin{equation}
\frac{d}{dt} \left( \frac{1}{2} \mathcal{E}_K (u_{t,n} (t) , u_{t,n} (t)) + (F (u_{t,n} (t)) , 1)_{L^2(\Omega)} \right) + \| \partial_t u_{t,n} (t) \|_{L^2(\Omega)}^2 = 0, \tag{3.57}
\end{equation}

for all $t \geq 0$. As usual, $F$ denotes the primitive of $f$, i.e., $F (\sigma) = \int_0^\sigma f (y) dy$. Multiply the foregoing equation by $t \geq \rho > 0$ and integrate over $(0, t)$ to get

\begin{align*}
t \left( \frac{1}{2} \mathcal{E}_K (u_{t,n} (t) , u_{t,n} (t)) + (F (u_{t,n} (t)) , 1)_{L^2(\Omega)} \right) + \int_0^t \tau \| \partial_t u_{t,n} (\tau) \|_{L^2(\Omega)}^2 d\tau, \\
= \int_0^t \left( \frac{1}{2} \mathcal{E}_K (u_{t,n} (t) , u_{t,n} (t)) + (F (u_{t,n} (\tau)) , 1)_{L^2(\Omega)} \right) d\tau,
\end{align*}

for all $t \geq \rho$. Recalling that, due to (H2)-(H3), $F$ is bounded from below, independently of $u$, and $|F (\sigma)| \leq C (1 + |\sigma|^p)$, we infer from (3.40) (which is also satisfied by $u_{t,n}$) and (3.54),

\begin{equation}
\mathcal{E}_K (u_{t,n} (t) , u_{t,n} (t)) + \int_0^t \| \partial_t u_{t,n} (\tau) \|_{L^2(\Omega)}^2 d\tau \leq c \left( 1 + \frac{1}{t} \right), \tag{3.58}
\end{equation}

for some constant $c > 0$ independent of $t, n, \epsilon$. On the basis of standard lower-semicontinuity arguments, we can now pass to the limit, first with respect to $n \to \infty$ and then as $\epsilon \to 0^+$, to obtain the desired inequality (3.50), owing once more to estimates (3.40)-(3.54) and uniqueness (cf. Theorem 3.8). The proof is finished.

4. Finite dimensional attractors. The first main result of this section is the following. As before, we fix $K \in \{D, N, R, E\}$.

**Theorem 4.1.** Let the assumptions of Theorems 3.8 be satisfied for some $f \in C^2(\mathbb{R})$. There exists a compact attractor $A_K \subseteq L^2(\Omega)$ for the parabolic system (3.1) which attracts the bounded sets of $L^2(\Omega)$. Moreover, $A_K$ is the maximal bounded attractor in $D (A_K) \cap L^\infty (\Omega)$ and has finite fractal dimension, that is, $\dim_F (A_K, L^2 (\Omega)) < \infty$. 

Proof. Step 1. (Global attractor). By the proof of Theorem 3.8, (3.40), there is a ball $B_K$ in $L^2(\Omega)$ which is absorbing in $L^2(\Omega)$, meaning that for any bounded set $U \subset L^2(\Omega)$ there exists $t_0 = t_0(\|U\|_{L^2(\Omega)}) > 0$ such that $S_K(t)U \subset B_K$ for all $t \geq t_0$. Moreover, by Theorem 3.9, (3.50) and (3.54), we infer the existence of a new time $t_1 \geq 1$ such that
\[
\sup_{t \geq t_1} \left( \|u(t)\|_{W_K^{1,2}(\Omega)} + \|u(t)\|_{L^\infty(\Omega)} \right) \leq C, \tag{4.1}
\]
for some positive constant $C$ independent of time and the initial data. We also observe that the Galerkin truncated solutions $u_{\epsilon,n}$ satisfy (in the weak sense of Definition 3.1) the following "time-differentiated" version of the original problem
\[
\partial_t \pi_{\epsilon,n} + A_K \pi_{\epsilon,n} + f'(u_{\epsilon,n}) \pi_{\epsilon,n} = 0 \quad \text{in } \Omega \times (0, \infty),
\]
where we have set $\pi_{\epsilon,n} = \partial_t u_{\epsilon,n}$. In particular, testing the aforementioned equation with $2t \pi_{\epsilon,n}$ we deduce upon integrating over $(0, t)$ that
\[
t \|\pi_{\epsilon,n}(t)\|^2_{L^2(\Omega)} + 2 \int_0^t \tau \|\pi_{\epsilon,n}(\tau)\|^2_{W_K^{1,2}(\Omega)} \, d\tau = \int_0^t \left(\int_0^\infty \left( f'(u_{\epsilon,n}(\tau)) \pi_{\epsilon,n}(\tau) , \pi_{\epsilon,n}(\tau) \right) \, d\tau \right) \, d\tau + \|\pi_{\epsilon,n}(\tau)\|^2_{L^2(\Omega)} \, d\tau \
\leq (C_f + 1) \int_0^t \|\pi_{\epsilon,n}(\tau)\|^2_{L^2(\Omega)} \, d\tau,
\]
where in the last line we have used assumption (H3). Exploiting (3.58) we obtain in the limit as $(\epsilon, n) \to (0, \infty)$ that
\[
\sup_{t \geq t_1} \|\partial_t u(t)\|^2_{L^2(\Omega)} \leq C.
\]
The usual comparison argument in equation (3.38) together with the uniform bound (4.1) yields $u \in L^\infty((t_1, \infty); D(A_K))$ uniformly with respect to time. Thus, for any bounded set $U \subset L^2(\Omega)$, we have that $\cup_{t \geq t_1} S_K(t)U$ is relatively compact in $L^2(\Omega)$, when endowed with the metric topology of $L^2(\Omega)$. Finally, applying [46, Theorem I.1.1] we have that the set
\[
A_K = \cap_{\tau \geq 0} \cup_{t \geq \tau} S_K(t)B_K
\]
is a compact attractor for $S_K$, and the rest of the result is immediate.

Step 2. (Uniform differentiability on $A_K$). We show that the bound obtained in (4.1) is sufficient to show the uniform differentiability of $S_K$ on the attractor $A_K$ with $T(t; u_0) \xi := S_K'(t) \xi$ as a solution of
\[
\partial_t U + A_K U + f'(u(t)) U = 0, \quad U(0) = \xi. \tag{4.2}
\]
To this end, consider two solutions $u_1, u_2$ of problem (3.1) with initial conditions $u_i(0) \in A_K$, $i = 1, 2$ and let $U$ be the solution of (4.2) with $\xi = u_1(0) - u_2(0)$. Then the function $\omega(t) = u_1(t) - u_2(t) - U(t)$ satisfies the equation
\[
\partial_t \omega + A_K \omega + f'(u) \omega + g = 0, \quad \omega(0) = 0, \tag{4.3}
\]
with $g := f(u_1) - f(u_2) - f'(u)(u_1 - u_2)$. Next, by Taylor’s theorem and the fact that $u_i \in L^\infty(\Omega)$ (as both $u_1, u_2$ lie on the attractor $A_K \subset L^\infty(\Omega)$), we infer that $|g(x)| \leq C|u_1(x) - u_2(x)|^r$, for some $C > 0$. Let $r$ be the conjugate exponent
to $2^*$ from the Sobolev embedding inequality (2.1) such that $W^{s,2}_{2^*} (\Omega) \subset L^{2^*} (\Omega)$. Therefore, if we write $g(t) = g(u(x, t))$ it follows that

$$
\|g(t)\|_{L^r(\Omega)} \leq C \int_\Omega |u_1(t) - u_2(t)|^{2r-2+\delta} |u_1(t) - u_2(t)|^{2-\delta} \, dx
$$

\[\leq C \|u_1(t) - u_2(t)\|^{2-\delta}_{L^2(\Omega)},\]

owing to Hölder’s inequality and the $L^\infty (\Omega)$ bound on $u_1, u_2$. Choosing now $\delta = 2 - r (1 + \varepsilon)$, for some $\varepsilon \in (0, (2 - r)/r)$, we easily deduce from (3.48) that

$$
\|g(t)\|_{L^r(\Omega)} \leq C e^{C(1+\varepsilon)t} \|u_1(0) - u_2(0)\|^{1+\varepsilon}_{L^2(\Omega)}.
$$

Testing now equation (4.3) by $\omega(t)$ in $L^2 (\Omega)$ yields

$$
\frac{d}{dt} \|\omega\|_{L^2(\Omega)}^2 + 2\mathcal{L}_K (\omega, \omega)
\leq 2C f_t \|\omega\|_{L^2}^2 + C e^{C(1+\varepsilon)t} \|u_1(0) - u_2(0)\|^{2(1+\varepsilon)}_{L^2(\Omega)}
$$

which upon integrating over $(0, t)$ gives

$$
\|\omega(t)\|_{L^2(\Omega)}^2 \leq C (t) \|u_1(0) - u_2(0)\|^{2(1+\varepsilon)}_{L^2(\Omega)},
$$

for some function $C (t) > 0$. Thus, the flow $\mathcal{S}_K(t)$ is indeed differentiable on $A_K$ and the derivative $\mathcal{S}_K'(t)$ is given by the solution of the linearized equation (4.2).

Finally, we also observe that for $\xi \in L^2 (\Omega)$ the set $\Upsilon (t; u_0)$ $\xi$ is relatively bounded in $W^{s,2}_{2^*} (\Omega) \cap L^\infty (\Omega)$, whence the mapping $\Upsilon (t; u_0)$ is also compact in $L^2 (\Omega)$ for each $t > 0$. The desired finite dimensionality of the global attractor $A_K$ follows from standard results in the theory of infinite dimensional dynamical systems (see, e.g., [12, 46]). The proof of the theorem is now complete. \qed

The following lemma states other basic properties of the dynamical system associated with problem (3.1). In particular, it shows that $(\mathcal{S}_K(t), L^2 (\Omega))$ is a “gradient” system, namely, we have the following.

**Lemma 4.2.** Let the assumptions of Theorem 3.9 be satisfied. Then the functional $\mathcal{L}_K : W^{s,2}_{2^*} (\Omega) \cap L^\infty (\Omega) \to \mathbb{R}$, given by

$$
\mathcal{L}_K (u(t)) := \frac{1}{2} \mathcal{E}_K (u(t), u(t)) + (F(u(t)), 1)_{L^2(\Omega)}.
$$

has along the strong solutions of (3.1), the derivative

$$
\frac{d}{dt} \mathcal{L}_K (u(t)) = - \partial_u u(t, \omega) \|u(t)\|^2_{L^2(\Omega)}, \text{ a.e. } t > 0.
$$

In other words, the functional $\mathcal{L}_K$ is decreasing, and becomes stationary exactly on equilibria $u_*$, which are solutions of the system:

$$
A_K u + f(u) = 0 \text{ in } \Omega. \quad (4.4)
$$

**Proof.** The proof is a consequence of the calculation (3.57) and the fact that strong solutions are smooth enough, see Definition 3.2 and Remark 6. \qed

The foregoing Lemma 4.2 can now be used to study the asymptotic behavior of the solutions of (3.1) by means of the LaSalle’s invariance principle. To this end, to any (weak) trajectory of (3.1) we associate the respective $\omega$-limit set $\omega_{L^2}$:

$$
\omega_{L^2} := \{ y \in L^2 (\Omega) : \exists n \to \infty, \ z_n \in L^2 (\Omega) \text{ such that } \mathcal{S}_K(t_n) z_n \to y \text{ in } L^2\text{-topology} \}.
$$
The following lemma states some basic properties of the $\omega$-limit sets associated with the dynamical system $(\mathcal{S}_K(t), L^2(\Omega))$.

**Lemma 4.3.** (i) Any $\omega$-limit set $\omega_{L^2}$ is nonempty, compact and connected. 
(ii) The trajectory approaches its own limit set in the norm of $L^2(\Omega)$, i.e.,
$$\lim_{t \to \infty} \text{dist}_{L^2(\Omega)}(\mathcal{S}_K(t) u_0, \omega_{L^2}) = 0.$$ 
(iii) Any $\omega$-limit set is invariant: new trajectories which start at some point in $\omega_{L^2}$ remain in $\omega_{L^2}$ for all times $t > 0$.

**Proof.** The proof is immediate owing to the continuity properties of the strong solution and the compactness of the embedding $W^{s,2}_K(\Omega) \subset L^2(\Omega)$.

The second main result is concerned with the parabolic problem

$$\begin{cases}
\partial_t u + dA_E u + f(u) = 0, & \text{in } \Omega \times (0, \infty), \\
u(0) = u_0, & \text{in } \Omega,
\end{cases}$$

(4.5)

in the case $K = E$ (recall that $u = 0$ in $\mathbb{R}^N \setminus \Omega$, $N \geq 1$), with $d > 0$ playing the role of a diffusion coefficient. Recall that $0 < s < 1$.

**Theorem 4.4.** Let the assumptions of Theorem 3.8 be satisfied. The fractal dimension of $A_E$ admits the estimate

$$\dim_F(A_E, L^2(\Omega)) \leq \frac{c_s}{C_E^{N/2s}} \left( \frac{C_f}{d} \right)^{N/2s} |\Omega|,$$

(4.6)

as either $C_f \to \infty$ or $d \to 0^+$, where $c_s$ depends on the shape of $\Omega$ and $N$ only. Here we have set $C_E = (4\pi)^s \Gamma(1 + N/2)^{2s/N}$ and $C_f > 0$ is such that $(H3)$ is satisfied.

**Proof.** In order to deduce (4.6), it is sufficient (see, e.g., [12, Chapter III, Definition 4.1]) to estimate the $j$-trace of the operator $L(t, U(t)) := -A_E u - f'(u(t)) U$, for $u \in \mathcal{A}_E$. We have

$$\text{Trace}(L(t, U(t)) Q_m) = \sum_{j=1}^m \langle L(t, U(t)) \varphi_j, \varphi_j \rangle_{L^2(\Omega)}$$

$$= -\sum_{j=1}^m \left( f'(u(t)) \varphi_j, \varphi_j \right)_2 - d \sum_{j=1}^m \lambda_j \langle \varphi_j, \varphi_j \rangle_{L^2(\Omega)},$$

where the set of real-valued functions $\varphi_j \in L^2(\Omega) \cap W^{s,2}_E(\Omega)$ is an orthonormal basis in $Q_m(L^2(\Omega))$. Since the family $\varphi_j$ is orthonormal in $Q_m(L^2(\Omega))$, using assumption (H3) on $f$ (i.e., $f'(\sigma) \geq -C_f$, for all $\sigma \in \mathbb{R}$), we find

$$\text{Trace}(L(t, U) Q_m) \leq -d \sum_{j=1}^m \lambda_j + C_f m.$$

We now consider the eigenvalue problem $(-\Delta)^s \varphi = \lambda \varphi$ in $\Omega$ and $\varphi = 0$ in $\mathbb{R}^N \setminus \Omega$, which is equivalent to the eigenvalue problem $A_E \varphi = \lambda \varphi$, $\varphi \in D(A_E)$. By [5], the eigenvalues $\lambda_j$ obey the following Weyl asymptotic formula:

$$\lambda_j = (4\pi)^s \left( \frac{j \Gamma(1 + N/2)}{|\Omega|} \right)^{\frac{2s}{N}} + o \left( j^{2s/N} \right) \text{ as } j \to \infty.$$
From (4.7), since \( \lambda_j \geq C_E |\Omega|^{-\frac{N}{2s}} j^{\frac{N}{s}} \) with \( C_E = (4\pi)^s \Gamma (1 + N/2)^{2s/N} \) we obtain

\[
\text{Trace} (L(t, U) Q_m) \leq -dC_E |\Omega|^{-\frac{N}{2s}} \sum_{j=1}^{m} j^{\frac{N}{s}} + C_f m
\]

\[
\leq -dC_0 C_E |\Omega|^{-\frac{N}{2s}} m^{\frac{N}{s} + 1} + C_f m,
\]

for some \( c_0 > 0 \) which only depends on the shape of \( \Omega \) and \( N \). Let us define the function on the right-hand side as \( \rho(m) \). The function \( \rho \) is concave and the non-zero root of the equation \( \rho(m) = 0 \) is

\[
m^* = \left( \frac{C_f}{dC_0 C_E} \right)^{N/2s} |\Omega|.
\]

Thus, we can apply [12, Corollary 4.2 and Remark 4.1] to deduce that

\[
\dim_F (A_E, L^2(\Omega)) \leq \max \{ m^*, 1 \},
\]

from which (4.6) follows.

**Remark 7.** It is worth emphasizing that when \( s = 1 \), in (4.7) we obtain the classical Weyl’s formula for the Dirichlet Laplacian eigenvalue problem \( -\Delta \varphi = \lambda \varphi \) in \( \Omega \), \( \varphi = 0 \) on \( \partial \Omega \). Moreover, the upper bound in (4.6) also stabilizes as \( s \to 1 \) to the corresponding upper bound for the fractal dimension of the parabolic equation \( \partial_t u - \Delta u + f(u) = 0 \) in \( \Omega \times (0, \infty) \), and \( u = 0 \) on \( \partial \Omega \times (0, \infty) \), see, for instance, [46, Chapter VI], [12]. We conjecture that a similar bound on the dimension of \( A_K \) also holds in the remaining cases \( K \in \{ D, N, R \} \) where at the moment Weyl asymptotic formulas are not yet available.

**Remark 8.** One can also provide a lower bound on the dimension of \( A_E \):

\[
\dim_F (A_E, L^2(\Omega)) \geq \tau \left( \frac{-f'(0)}{dC_E} \right)^{N/2s} |\Omega|,
\]

for some \( \tau > 0 \) which depends only on the shape of \( \Omega \) and \( N \), if \( f'(0) < 0 \) is sufficiently large or \( d > 0 \) is sufficiently small. This estimate is obtained in the same spirit of [46, Chapter VII] (see also [22]) and relies on the fact that, owing to the boundedness of \( u \in L^\infty(\Omega) \), the semigroup \( S_E(t) \) is uniformly differentiable with derivative of Hölder class \( C^\alpha, \alpha \in (0, 1) \) (in fact, in our case \( \alpha = 1 \)). More precisely, there exists a smooth manifold \( W^{1,\text{loc}}(u*) \) (of class \( C^{1,\alpha} \)) localized in an open neighborhood of the hyperbolic equilibrium \( u_* = 0 \), with finite instability dimension \( \dim X_*(u_*) < \infty \). Here, \( X_*(u_*) \) is the unstable subspace of \( -A_K - f'(u_*) \) which is tangent to \( W^{1,\text{loc}}(u_* \) at the point \( u_* \) and we recall that the global attractor always contains localized unstable manifolds.

5. **Asymptotic stabilization to single equilibria.** Let \( u \) be a weak solution of (3.1) according to Theorem 3.8. We show that any such weak solution converges (in a certain sense) to a single steady state as time tends to infinity. Recall that any weak solution of (3.1) regularizes in finite time to a strong solution by Theorem 3.9. Moreover, observe that, by virtue of (3.50) and the proof of Theorem 4.1, all stationary solutions \( u_* \in \omega_{L^2}(u) \) of problem \( A_K u_* + f(u_*) = 0 \) in \( \Omega \), are bounded in \( L^\infty(\Omega) \cap D(A_K) \). Setting now, for each \( K \in \{ D, E, N, R \} \),

\[
L_K(u) := \frac{1}{2} \mathcal{E}_K(u, u) + \int_0^t F(u) \, dx, \quad F(\sigma) := \int_0^\sigma f(t) \, dt,
\]
it is easy to see that $L_K \in C^1 (L^2 (\Omega), \mathbb{R})$ but $L_K \notin C^2 (L^2 (\Omega), \mathbb{R})$ no matter how smooth $F$ is due to the nature of the nonlocal term $E_K$ (cf. also [19, 23, 39], where the same issue occurs for other nonlocal problems).

Consequently, we shall employ a generalized version of the Lojasiewicz-Simon theorem which is well-suited for our nonlocal problem (3.1). As usual, we fix $K \in \{D, E, N, R\}$ and set $\mathcal{F} (r) := |r|^{-N-2s}$. Recall that the Lojasiewicz-Simon result applies in principle to functionals which can be written as a maximal monotone operator plus a linear compact perturbation. The version that applies to our cases $K \in \{D, E, N, R\}$ is formulated in the subsequent lemma requiring the following condition:

\textbf{(H-er):} Let $w$ be a bounded solution of the elliptic boundary value problem $A_K w = h$ in $\Omega$, for some $h \in L^\infty (\Omega)$. Then, $w \in C^{0,\nu} (\overline{\Omega})$ for some $\nu \in (0, 1)$.

\textbf{Lemma 5.1.} Let $F \in C^2$ be a real analytic function satisfying (H1), (H3) and let $\Omega$ obey condition (H4). Assume condition (H-er). Then, there exist constants $\theta \in (0, \frac{1}{2}]$, $C > 0$, $\varepsilon > 0$ such that the following inequality holds:

$$|L_K (u) - L_K (u_*)|^{1-\theta} \leq C \|A_K u + f (u)\|_{L^2(\Omega)}$$

(5.1)

for all $u \in W^{s,2}_k (\Omega) \cap L^\infty (\Omega)$ provided that $\|u - u_*\|_{L^2(\Omega)} \leq \varepsilon$.

We employ an abstract version of the Lojasiewicz-Simon inequality provided by [20, Theorem 6] and which has been specifically tailored to our needs. We need the following assumptions:

\textbf{A1:} Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be Banach spaces densely and continuously embedded into the Hilbert space $(H, \|\cdot\|_H)$ and its dual $(H^*, \|\cdot\|_{H^*})$, respectively. We assume that the restriction $j_V$ of the duality map $j \in \mathcal{L} (H, H^*)$ to $V$ is an isomorphism from $V$ onto $W = j(V)$.

\textbf{A2:} Let $\mathcal{T} \in \mathcal{L} (H, H^*)$ be a self-adjoint and completely continuous operator such that its restriction $\mathcal{T}_V$ to $V$ is a completely continuous operator in $\mathcal{L} (V, W)$. For fixed $\pi \in W$ and $d \in \mathbb{R}$ consider the quadratic functional $\Psi : H \to \mathbb{R}$ given by

$$\Psi (u) = \langle T u, u \rangle + \langle \pi, u \rangle + d, \ u \in H.$$  

\textbf{A3:} Let $\mathcal{U}$ be an open subset of $V$ and $\Phi : \mathcal{U} \to \mathbb{R}$ be a Fréchet differentiable function. Additionally, assume that the Fréchet derivative $D\Phi : \mathcal{U} \to \mathbb{R}$ is a real analytic operator which satisfies

$$\langle D\Phi (u) - D\Phi (v), u - v \rangle \geq c_1 \|u - v\|^2_H,$$

$$\|D\Phi (u) - D\Phi (v)\|_{H^*} \leq c_2 \|u - v\|_H$$

for all $u, v \in \mathcal{U}$, for some constants $c_1, c_2 > 0$. Moreover, assume that the second Fréchet derivative $D^2\Phi (u) \in \mathcal{L} (V, W)$ is an isomorphism for all $u \in \mathcal{U}$.

\textbf{Theorem 5.2.} ([20, Theorem 6]) Let the assumptions (A1)-(A3) hold for the energy $\mathcal{F}$ defined by $\mathcal{F} = \Phi + \Psi$. Let $u_* \in \mathcal{U}$ be a critical point of $\mathcal{F}$, i.e., $u_*$ is a solution of the Euler-Lagrange equation $D\mathcal{F} (u_*) = 0$. Then there exist constants $\theta \in (0, \frac{1}{2}]$, $C > 0$, $\varepsilon > 0$ such that for each $u \in \mathcal{U}$ which satisfies $\|u - u_*\|_H \leq \varepsilon$ we have the following inequality:

$$|\mathcal{F} (u) - \mathcal{F} (u_*)|^{1-\theta} \leq C \inf \{\|D\mathcal{F} (z)\|_H : z \in H\}.$$
Proof of Lemma 5.1. We will apply the abstract result of Theorem 5.2 to the energy functional $\mathcal{L}_K$ with

$$V = W = L^\infty(\Omega), \ H = L^2(\Omega).$$

According to its definition we can split $\mathcal{L}_K$ into the sum of a convex (entropy) functional $\Phi : L^2(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$, with a suitable effective domain, and a non-local interaction functional $\Psi : \text{dom}(\Psi) \rightarrow \mathbb{R}$. To this end, we define the lower-semicontinuous and strongly convex functional

$$\Phi(u) := \begin{cases} \int_{\Omega} \left( F(u) + \frac{\mu}{2} u^2 \right) dx, & \text{if } u \in L^\infty(\Omega) \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\mu > C_f$ (with $C_f > 0$ as in assumption (H3)), with closed effective domain $\text{dom}(\Phi) = L^\infty(\Omega)$, and the quadratic functional $\Psi : L^2(\Omega) \rightarrow \mathbb{R}$, given by

$$\Psi(u) := ((A_K - \mu/2) u, u)_{L^2(\Omega)},$$

for all $u \in W^{3,2}_K(\Omega) = \text{dom}(\Psi)$.

We have that $\Phi$ is Fréchet differentiable on any open subset $\mathcal{U}$ of $L^\infty(\Omega)$ with Fréchet derivative $D\Phi : \mathcal{U} \rightarrow L^\infty(\Omega)$ having the form

$$\langle D\Phi(u), \xi \rangle = \int_{\Omega} (F'(u) + \mu u) \cdot \xi dx,$$

for all $u \in \mathcal{U}$ and $\xi \in L^\infty(\Omega)$. The analyticity of $D\Phi$ as a mapping on $L^\infty(\Omega)$ is standard and can be proved exactly as in, e.g., [20, Remark 3]. Moreover, due to assumption (H3), there holds

$$\langle D\Phi(u_1) - D\Phi(u_2), u_1 - u_2 \rangle_{L^2(\Omega)} \geq \varsigma \|u_1 - u_2\|_{L^2(\Omega)}^2,$$

for some $\varsigma \in (0, \mu - C_f)$ (which exists since $\mu > C_f$), for all $u_1, u_2 \in \mathcal{U}$, and

$$\|D\Phi(u_1) - D\Phi(u_2)\|_{L^2(\Omega)} \leq C \|u_1 - u_2\|_{L^2(\Omega)},$$

for some positive constant C. Moreover, computing the second Fréchet derivative $D^2\Phi$ of $\Phi$,

$$\langle D^2\Phi(u), \xi_1, \xi_2 \rangle = \int_{\Omega} (F''(u) + \mu) \xi_1 \cdot \xi_2 dx$$

yields that $D^2\Phi \in \mathcal{L}(L^\infty(\Omega), L^\infty(\Omega))$ is an isomorphism for every $\varphi \in \mathcal{U}$. Next, defining the linear operator $T_{\mathcal{J}} : L^2(\Omega) \rightarrow L^2(\Omega)$, we have

$$\Psi(u) = \langle T_{\mathcal{J}} u, u \rangle_{L^2(\Omega)} = (A_K u, u)_{L^2(\Omega)} + (\pi, u)_{L^2(\Omega)},$$

with $\pi := -\mu/2 \in L^\infty(\Omega)$ (indeed, every weak solution of the nonlinear elliptic problem $A_K u + f(u) = 0$, belongs to $L^\infty(\Omega) \cap D(A_K)$, owing to assumption (H1)). By the regularity results provided in the previous section the operator $T_{\mathcal{J}} \in \mathcal{L}(L^2(\Omega), W^{3,2}_K(\Omega))$ and hence it is compact from $L^2(\Omega)$ to itself. Moreover, by assumption (H-er), we have that $T_{\mathcal{J}} \in \mathcal{L}(L^\infty(\Omega), C^{0,\nu}(\Omega))$ is also compact from $L^\infty(\Omega)$ to $C(\overline{\Omega})$. Hence, the hypotheses of Theorem 5.2 are satisfied and the sum

$$\mathcal{L}_K = \Phi + \Psi : L^2(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$$

is a well defined, bounded from below functional with nonempty, closed, and convex effective domain $\text{dom}(\mathcal{L}_K)$. Noting that the Fréchet derivative $D\mathcal{L}_K(u) = A_K u + F'(u)$ (here $D\mathcal{L}_K : L^2(\Omega) \rightarrow (L^2(\Omega))^*$, with $L^2(\Omega)$ being identified with its dual by means of the Riesz isometry), inequality (5.1) follows from Theorem 5.2. The proof is finished. \qed
We make some comments on the assumption \((H-er)\).

**Remark 9.** We notice that by Proposition 2.7, if \(\Omega\) is a bounded domain with Lipschitz continuous boundary, then the assumption \((H-er)\) is satisfied for the operator \(A_E\). It is also satisfied by the operator \(A_K, K \in \{D, N, R\}\) for bounded domains in dimension \(N = 1\) and if \(s > 1/2\) (cf. Section 2). We think that assumption \((H-er)\) is also satisfied for \(A_K, K \in \{D, N, R\}\) for bounded domains with Lipschitz boundary, in any space dimension and for any \(s \in (0, 1)\). Such a result is not yet available in the literature, and is interesting on its own independently of the application given in the present paper, and also necessitates a careful study. Since this is not the main objective of the paper, we will not go into details.

We can now prove the following convergence result by the application of Lemma 5.1.

**Theorem 5.3.** Let the assumptions of Theorem 3.8 hold and let \(F\) be real analytic. Then, for any initial datum \(u_0 \in L^2(\Omega)\) the corresponding weak solution \(u\) to problem (3.1) satisfies
\[
\lim_{t \to \infty} \|u(t) - u_*\|_{L^2(\Omega)} = 0,
\]
where \(u_* \in D(A_K) \cap L^\infty(\Omega)\) is a solution to (4.4), i.e., \(A_K u_* + f(u_*) = 0\) in \(\Omega\). Moreover, the following convergence rate holds:
\[
\|u(t) - u_*\|_{L^2(\Omega)} \sim (1 + t)^{-\frac{\theta}{1-2\theta}}\text{ as } t \to \infty,
\]
where \(\theta \in (0, \frac{1}{2}]\) is the constant given in (5.1).

**Proof.** By the energy identity in Lemma 4.2, we have
\[
L_K(u(t)) \to L_* = L_K(u_*), \text{ as } t \to \infty,
\]
and the limit energy \(L_*\) is the same for every steady-state solution \(u_* \in \omega_{L^2}(u)\). Moreover, we can integrate the energy equality in Lemma 4.2,
\[
\frac{d}{dt} L_K(u(t)) = -\|\partial_t u(t)\|_{L^2(\Omega)}^2
\]
over \((t, \infty)\) to get
\[
\int_t^\infty \|\partial_t u(\tau)\|_{L^2(\Omega)}^2 d\tau = L_K(u(t)) - L_* = L_K(u(t)) - L_K(u_*).
\]
By virtue of Lemma 5.1 and equation (3.1), we have
\[
|L_K(u(t)) - L_K(u_*)|^{1-\theta} \leq C \|A_K u(t) + f(u(t))\|_{L^2(\Omega)}
\]
\[
\leq C \|\partial_t u(t)\|_{L^2(\Omega)}
\]
provided that
\[
\|u - u_*\|_{L^2(\Omega)} \leq \varepsilon.
\]
This, combined with the previous identity, yields
\[
\int_t^\infty \|\partial_t u(\tau)\|_{L^2(\Omega)}^2 d\tau \leq C \|\partial_t u(t)\|_{L^2(\Omega)}^{1-\theta},
\]
for all \(t > 0\), as long as (5.5) holds. Note that, in general, the quantities \(\theta, C\) and \(\varepsilon\) above may depend on \(u_*\). Finally, let us set
\[
\mathcal{M} = \cup \{\mathcal{I} : \mathcal{I} \text{ is an open interval on which (5.5) holds}\}.
\]
Clearly, $\mathcal{M}$ is nonempty since $u_* \in \omega_{L^2}(u)$. We can now use (5.6), the fact that $\partial_t u(t) \in L^2([t_*; t])$ (cf. Lemma 4.2), and exploit [19, Lemma 6.1] with $\alpha = 2(1 - \theta)$ to deduce that $\|\partial_t u(\cdot)\|_{L^2(\Omega)} \in L^1(\mathcal{M})$ and

$$\int_\mathcal{M} \|\partial_t u(\tau)\|_{L^2(\Omega)} d\tau \leq C(u_*) < \infty.$$  

(5.7)

We now claim that we can find a sufficiently large time $\tau > 0$ such that $(\tau, \infty) \subset \mathcal{M}$. To this end, recalling (5.4) and the above bounds, we also have that $\partial_t u \in L^2((0, \infty); L^2(\Omega))$ and, furthermore, for any $\delta > 0$ there exists a time $t_* = t_*(\delta) > 0$ such that

$$\|\partial_t u\|_{L^1((\tau, \infty); L^2(\Omega))} \leq \delta, \quad \|\partial_t u\|_{L^2((\tau, \infty); L^2(\Omega))} \leq \delta.$$  

(5.8)

Next, observe that by Theorem 3.9 (see also (4.1)), there is a time $t_1 > 0$ such that

$$\sup_{t \geq t_1} \|u(t)\|_{W^{s,2}(\Omega) + L^\infty(\Omega)} \leq C.$$  

(5.9)

Now, let $(t_0, t_2) \subset \mathcal{M}$ for some $t_2 > t_0 \geq t_*(\delta)$, $|t_0 - t_2| \geq 1$ such that (5.9) holds (w.l.o.g., we shall assume that $t_* \geq t_1$). Using (5.8) and (5.9), we obtain

$$\|u(t_0) - u(t_2)\|_{L^2(\Omega)}^2 \leq 2 \int_{t_0}^{t_2} \|\partial_t u(\tau)\|_{L^2(\Omega)}^2 d\tau \leq C \|\partial_t u\|_{L^1((t_0, t_2); L^2(\Omega))} \|u\|_{L^\infty((\tau, \infty); L^\infty(\Omega))} \leq C\delta.$$  

(5.10)

Therefore we can choose a time $t_*(\delta) = \tau < t_0 < t_2$, such that

$$\|u(t_0) - u(t_2)\|_{L^2(\Omega)} < \frac{\varepsilon}{3}$$  

(5.11)

provided that (5.5) holds for all $t \in (t_0, t_2)$. Since $u_* \in \omega_{L^2}(u)$, a large (refined) $\tau$ can be chosen such that

$$\|u(\tau) - u_*\|_{L^2(\Omega)} < \varepsilon/3;$$  

(5.12)

hence, (5.11) yields $(\tau, \infty) \subset \mathcal{M}$. Indeed, taking

$$\tilde{\tau} = \inf \left\{ t > \tau : \|u(t) - u_*\|_{L^2(\Omega)} \geq \varepsilon \right\},$$

we have $\tilde{\tau} > \tau$ and $\|u(\tilde{\tau}) - u_*\|_{L^2(\Omega)} \geq \varepsilon$ if $\tilde{\tau}$ is finite. On the other hand, in view of (5.11) and (5.12), we have

$$\|u(t) - u_*\|_{L^2(\Omega)} \leq \|u(t) - u(\tau)\|_{L^2(\Omega)} + \|u(\tau) - u_*\|_{L^2(\Omega)} < \frac{2}{3} \varepsilon,$$

for all $\tilde{\tau} > t \geq \tau$, and this leads to a contradiction. Therefore, $\tilde{\tau} = \infty$ and by (5.8) the integrability of $\partial_t \varphi$ in $L^1((\tau, \infty); L^2(\Omega))$ follows. Hence, $\omega_{L^2}(u) = \{u_*\}$ and (5.2) holds. The convergence rate (5.3) is an immediate consequence of the definition of $L^\infty_K$ and (5.7). We leave the details to the interested reader. The proof is finished. 

Remark 10. Exploiting the $L^2(\Omega) \to (L^\infty(\Omega) \cap D(A_K))$ smoothing property of the weak and stationary solutions together with a similar argument from Theorem 3.9, and the convergence rate (5.3) it is possible to show the convergence rate:

$$\|u(t) - u_*\|_{L^\infty(\Omega)} \sim (1 + t)^{-\frac{\theta}{2}}, \quad \text{as } t \to \infty,$$
for some positive constant \( \kappa = \kappa (\theta, u_*) \in (0, 1) \). Indeed, we can also prove (3.51) for the difference \( u - u_* \) owing to the boundedness of \( u, u_* \in L^\infty (\Omega) \).

6. Some blow-up results. Our goal in this section is to show that assumption (H1) is in fact quite optimal for global well-posedness of strong solutions of the problem (3.1). We recall that this condition implies in particular that if \( f \) is a source with a bad sign at infinity then it can only be of at most linear growth at infinity. Indeed, we will show below by the concavity method of Levine–Payne [37] that as soon as \( f \) has superlinear growth and a bad sign at infinity as \(|\sigma| \to \infty\), as provided by the example \( f (\sigma) = -|\sigma|^{p-1} \sigma, \sigma \in \mathbb{R}, \) with \( p > 1 \), then blowup in finite time of some strong solutions occurs.

**Theorem 6.1.** Let \( \Omega \) satisfy condition (H4) and suppose that \( f \in C^1_{loc} (\mathbb{R}) \) obeys

\[
F (\sigma) - \frac{1}{2} f (\sigma) \sigma \geq 0, \text{ for all } \sigma \in \mathbb{R} \tag{6.1}
\]

and \( u_0 \in L^\infty (\Omega) \cap W^{s,2}_K (\Omega) \) is an initial datum such that

\[
\frac{1}{2} E_K (u_0, u_0) + \int_\Omega F (u_0) \, dx < 0, \quad K \in \{ D, N, R, E \}, \tag{6.2}
\]

then the strong solution of (3.1) must blow up in finite time.

**Proof.** For any strong solution of problem (3.1), which exists locally on some interval \( t \in (0, T_{\text{max}}) \) by the proof of Theorem 3.7 (Step1), we have

\[
\frac{1}{2} \frac{d}{dt} \| u (t) \|^2_{L^2 (\Omega)} + E_K (u(t), u(t)) = - \int_\Omega f (u(t)) u(t) \, dx \tag{6.3}
\]

and

\[
Q'_{K} (t) = \int_\Omega (\partial_t u (t))^2 \, dx,
\]

where we have set

\[
Q_{K} (t) := -\frac{1}{2} E_K (u(t), u(t)) - \int_\Omega F (u(t)) \, dx.
\]

In particular, one has

\[
Q_{K} (t) = Q_K (0) + \int_0^t \| \partial_t u(\tau) \|^2_{L^2 (\Omega)} \, d\tau. \tag{6.4}
\]

Defining as usual the function

\[
\mathcal{V}_{K} (t) := \int_0^t \int_\Omega u^2 (\tau) \, dxd\tau + A,
\]

for some constant \( A > 0 \) to be determined later on, we see that

\[
\mathcal{V}''_{K} (t) = \| u (t) \|^2_{L^2 (\Omega)} , \quad \mathcal{V}'_{K} (t) = -2 \left( E_K (u(t), u(t)) + \int_\Omega f (u(t)) u(t) \, dx \right),
\]

owing to (6.3). Furthermore, by assumption (6.1) and (6.4), it holds

\[
\mathcal{V}'_{K} (t) \geq 4 Q_{K} (t) = 4 \left( Q_K (0) + \int_0^t \| \partial_t u(\tau) \|^2_{L^2 (\Omega)} \, d\tau \right). \tag{6.5}
\]

Clearly, since

\[
\mathcal{V}'_{K} (t) = 2 \int_0^t \int_\Omega u(\tau) \partial_t u(\tau) \, dxd\tau + \int_\Omega u_0^2 \, dx
\]
it follows for any $\varepsilon > 0$, that
\[
V_K'(t)^2 \leq 4(1 + \varepsilon) \left( \int_0^t \int_{\Omega} u^2(\tau)d\tau \right) \left( \int_0^t \int_{\Omega} (\partial_\tau u)^2(\tau)d\tau \right) + \left( \frac{1}{\varepsilon} + 1 \right) \left( \int_{\Omega} u_0^2dx \right)^2.
\]
For $\alpha > 0$, combining these estimates together yields
\[
V''_K(t)V_K(t) - (1 + \alpha)V'_K(t)^2 > 0
\]
provided that $\varepsilon$ and $\alpha$ are small enough such that $(1 + \alpha)(1 + \varepsilon) \leq 1$ and $A > 0$ is large enough (since $Q_K(0) > 0$, by assumption). The foregoing inequality implies as usual that
\[
\frac{V'_K(t)}{V_{K+\alpha}^2(t)} > \frac{V'_K(0)}{V_{K+\alpha}^2(0)} \text{ for } t > 0,
\]
which yields that the quantity $V_K(t)$ cannot remain finite for all time $t > 0$. The proof is finished.

**Remark 11.** Let $f(\sigma) = -|\sigma|^{p-1}\sigma$ with $p > 1$ and note that it satisfies (6.1) but it fails to verify condition (H1). Therefore, for any initial condition $u_0$ which satisfies (6.2), blow-up must occur.

**Remark 12.** The same blow-up result also holds for some initial datum for which $E_K(u_0, u_0) > 0$, see [25].

We conclude this section with another blow-up result by exploiting the well-known eigenvalue method of Kaplan [35]. Let $K \in \{E, D\}$.

**Theorem 6.2.** Assume that $h \in C^1(\mathbb{R})$ is a concave decreasing function and
\[
\limsup_{\sigma \to \infty} \left( h'(\sigma) \right) < -\lambda_1,
\]
where $\lambda_1 > 0$ is the first eigenvalue of the self-adjoint operator $A_E$ and such that
\[
\frac{1}{h(\sigma)}d\sigma < \infty \text{ and } f(\sigma) - \kappa V_{\Omega}(x) \sigma \leq h(\sigma) < 0 \text{ for all } \sigma > \sigma_0, \text{ a.e. in } \Omega,
\]
for some $\sigma_0 > 0$. Here $\kappa = 0$ if $K = E$ and $\kappa = 1$ if $K = D$. There exist strong solutions of problem (3.1) that blow up in finite time.

**Proof.** Let $\phi$ be the positive solution (cf. [45, Proposition 9]) of the eigenvalue problem
\[
A_E \phi = \lambda_1 \phi \text{ in } \Omega, \quad \phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
\]
with $\int_{\Omega} \phi(x)dx = 1$, and where $\lambda_1 > 0$ is the first eigenvalue of $A_E$. Next recall that the potential $V_{\Omega}(x) \geq C |\Omega|^{-2s/N}$ a.e. in $\Omega$, see [48, Lemma 5.10]. Let $y_K(t) := \int_{\Omega} u(x, t) \phi(x)dx$. Then we get, using (3.1) and Jensen’s inequality for the function
−h that
\[ y_K'(t) = \int_\Omega \partial_t u \phi dx = \int_\Omega (-A_K u - f(u)) \phi dx \]
\[ = - \int_\Omega u (A_K \phi + \kappa V_\Omega \phi) dx - \int_\Omega (f(u) - \kappa V_\Omega u) \phi dx \]
\[ \geq -\lambda_1 \int_\Omega u (x, t) \phi(x) dx - h \left( \int_\Omega u (x, t) \phi(x) dx \right) \]
\[ = -\lambda_1 y_K(t) - h(y_K(t)), \]
for as long as it exists, owing to the fact that
\[ (A_K u + \kappa V_\Omega u, \phi)_{L^2} = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y)) (\phi(x) - \phi(y))}{|x - y|^{N+2s}} \, dx \, dy \]
\[ = (u, A_E \phi)_{L^2}. \]
If \( u(t) \) remains finite for all time then \( y_K(t) \) is well-defined for all time \( t > 0 \). However, from the ODE theory of equation \( y_K' = -h(y_K) \), \( y_K \) will blow up in finite time provided that \( y_K(0) \) is sufficiently large. Therefore, the solution of (3.1) must blow up in finite time under the given assumptions.

**Remark 13.** In particular, any function \( f \) obeying
\[ \limsup_{\sigma \to \infty} \frac{f(\sigma)}{\sigma (\ln(\sigma))^p} < 0, \]
for some \( p > 1 \), satisfies the above conditions.

**Remark 14.** By symmetry we can get the analogue of Theorem 6.2 in the case when solutions of (3.1) blow up in finite time toward \(-\infty\) provided that \( h \) is a decreasing, convex function such that
\[ \limsup_{\sigma \to -\infty} \left( h'(\sigma) \right) < -\lambda_1 \]
and
\[ \int_{-\infty}^{\sigma_0} \frac{1}{|h(\sigma)|} \, d\sigma < \infty \text{ and } f(\sigma) - \kappa V_\Omega (x) \sigma \geq h(\sigma) > 0 \text{ for all } \sigma < \sigma_0, \text{ a.e. in } \Omega, \]
for some \( \sigma_0 \leq 0 \). We only need to apply the foregoing theorem on the equation satisfied by \(-u\).

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