Soliton solutions of a generalized Gross-Pitaevskii equation via Darboux transform

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Abstract

The nonlinear Schrödinger equation (NLS) is widely used in different branches of physics. The GP equation is a kind of NLS equation with potential function term. In this paper, the soliton solutions of a generalized GP equation are explored. Firstly, we construct the Lax pair and Darboux transform (DT) of the equation. Then, we solve the single and double soliton solutions of the equation. Lastly, we draw the images of the single and double soliton solutions, and investigate the properties of the solitons. It is found that the amplitude of a single solitary wave does not change, and the shape and amplitude of the double solitary wave remain unchanged after the collision.

1. Introduction

The nonlinear Schrödinger (NLS) equation

\[ i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2 |u|^2 u = 0, \]  

is one of the most important soliton equations, which is widely used in different branches of physics, such as plasma physics, nonlinear fiber optics and Bose-Einstein Condensates (BECs). Soliton equations preserve some balance between the dispersion and the nonlinear effect, and contain good symmetries. Solitons, which are the exact solutions of soliton equations, have wave particle duality. It means that solitons can propagate stably over long distances and retain their shapes after interactions[1].

There are a vast variety of the powerful methods to find the soliton solutions of soliton equations. For example, inverse scattering method, Hirota bilinear method[2], Riemann-Hilbert formulation[3], Darboux transformation[4-8], and so on[9-10].

Darboux transformation is an effective tool for finding another explicit solutions of partial differential equation from a seed solution. It was discovered by French mathematician Darboux in 1882 when he studied the eigenvalue problem of one-dimensional Schrödinger equation. The basic idea of DT is to construct the exact solution of the integrable equation by the solution of the linear partial differential equations associated with the Lax pair of the integrable equation.

The GP equation is a kind of NLS equation with potential function term, which is often used to simulate the evolution of BEC wave function in the low temperature regime. Therefore, solving the soliton solution of GP equation is very important to the study of BECs properties and other physical problems.
In this paper, based on DT, we consider the following GP equation

\[ i\frac{\partial u}{\partial t} + \beta \frac{\partial^2 u}{\partial x^2} + 2\alpha\beta |u|^2 u + 2v(t)u = 0, \tag{2} \]

where \( u(x,t) \) is the complex envelope wave function with respect to the space variable \( x \) and time variable \( t \), and \( |u|^2 \) denotes the power of atoms in the BECs. The parameters \( \alpha \neq 0, \beta \) are arbitrary real constants, and \( v(t) \) is a function of \( t \). The second and third terms stand for the dispersion and nonlinear effects, respectively. The fourth term describes external harmonic trap potential in the BECs.

The paper is arranged as follows. In section 2, we derive the Lax pair and DT of the equation. In section 3, we explore the properties of soliton solutions.

2. Darboux transform
We can easily derive the Lax pair of equation (2), which is the linear spectral problem,

\[ \Phi_x = U\Phi, \Phi_t = V\Phi, \tag{3} \]

where

\[ U = -i\lambda J + P, \quad V = -2i\beta\lambda J + 2\beta\lambda P + iQ, \]

\[ J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & \sqrt{\alpha u} \\ -\sqrt{\alpha u} & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \alpha\beta|u|^2 + v(t) & \sqrt{\alpha\beta u} \\ \sqrt{\alpha\beta u} & -\alpha\beta|u|^2 - v(t) \end{pmatrix}, \]

and \( \bar{u} \) stands for the complex conjugate of \( u(x,t) \), \( \lambda \) is the spectral parameter. The vector \( \Phi = (f,g)^T \) is an eigenfunction associated with the parameter \( \lambda \) and potential function \( u(x,t) \), which consists of two complex functions \( f = f(x,t) \) and \( g = g(x,t) \).

It can be verified that the compatibility condition \( U_t - V_x + [U,V] = 0 \), which is called zero curvature equation, exactly gives equation (2). The square brackets denote the usual matrix commutator, namely, \([U,V] = UV - VU\).

Let us introduce the one-fold DT \( \Phi[1] \) of equation (3), which is a special gauge transformation,

\[ \Phi[1] = D[1]\Phi, \tag{4} \]

with Darboux matrix \( D[1] = \lambda E - S[0] \), where \( E \) denotes the \( 2 \times 2 \) identity matrix, and \( S[0] = S[0](x,t) \) is a \( 2 \times 2 \) matrix with respect to \( x \) and \( t \). The DT (4) transforms the Lax pair (3) into a new Lax pair

\[ \Phi[1]_x = U[1]\Phi[1], \quad \Phi[1]_t = V[1]\Phi[1], \tag{5} \]

where

\[ U[1] = -i\lambda J + P[1], \quad V[1] = -2i\beta\lambda J + 2\beta\lambda P[1] + iQ[1], \]

\[ J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P[1] = \begin{pmatrix} 0 & \sqrt{\alpha u[1]} \\ -\sqrt{\alpha u[1]} & 0 \end{pmatrix}, \quad Q[1] = \begin{pmatrix} \alpha\beta|u[1]|^2 + v(t) & \sqrt{\alpha\beta u[1]} \\ \sqrt{\alpha\beta u[1]} & -\alpha\beta|u[1]|^2 - v(t) \end{pmatrix}, \]

and \( U[1],V[1] \) have the same forms as those of \( U,V \) in equation (3) except for replacing \( u \) with a new potential function \( u[1] \).

Substituting \( U[1] = -i\lambda J + P[1], \quad D[1] = \lambda E - S[0] \) into the relation (5), we can get
\[ P[1] = P - i[J, S[0]]. \]  

Let
\[ S[0] = H \Lambda H^{-1}, \]
where \( \Lambda \) is the diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \lambda_2) \), and \( H = H(x,t) \) is a nonsingular matrix.

If the eigenfunctions \( (f_i, g_i)^T \) of the Lax pair (3) at \( \lambda = \lambda_i \) are given, one can easily prove that \( (-g_1, f_1)^T \) is also an eigenfunction of the Lax pair (3) at \( \lambda_1 \). We can choose the matrix function
\[ H = \begin{pmatrix} f_1 & -g_1 \\ g_1 & f_1 \end{pmatrix}, \]
and
\[ S[0] = \begin{pmatrix} s_{11}[0] & s_{12}[0] \\ s_{21}[0] & s_{22}[0] \end{pmatrix} = \begin{pmatrix} 1 & (\lambda_1 - \lambda_1)g_1^{\frac{1}{2}}f_1^{\frac{1}{2}} \\ 0 & \lambda_1^{\frac{1}{2}}g_1^{\frac{1}{2}}f_1^{\frac{1}{2}} \end{pmatrix}, \]

Thus we can derive that the one-fold DT and the expression of the single soliton solution of equation (2)
\[ \Phi[1] = D[1]\Phi = (\lambda E - S[0])\Phi, \]
\[ u[1] = u - \frac{2i\alpha_{12}[0]}{\sqrt{\lambda}} = u - \frac{2i(\lambda_1 - \lambda_1)f_1^{\frac{1}{2}}g_1^{\frac{1}{2}}}{\sqrt{\lambda}(|f_1|^2 + |g_1|^2)}. \]

By the similar process as the one-fold DT, we let
\[ \Phi[2] = D[2]\Phi[1] = (\lambda E - S[1])\Phi[1] = \begin{pmatrix} \lambda - s_{11}[1] & -s_{12}[1] \\ -s_{21}[1] & \lambda - s_{22}[1] \end{pmatrix}\Phi[1] = D[2]D[1]\Phi, \]
where
\[ S[1] = \begin{pmatrix} s_{11}[1] & s_{12}[1] \\ s_{21}[1] & s_{22}[1] \end{pmatrix} = \begin{pmatrix} f_2[1] & -g_2[1] \\ g_2[1] & \bar{f}_2[1] \end{pmatrix}^{-1}, \]
\[ f_2[1] = (\lambda_2 - s_{11}[0])f_2 - s_{12}[0]g_2, \]
\[ g_2[1] = (\lambda_2 - s_{22}[0])g_2 - s_{21}[0]f_2, \]
and \( (f_2, g_2)^T \) is the eigenfunctions of the Lax pair (3) at \( \lambda = \lambda_2 \), \( (-g_2, f_2)^T \) is an eigenfunctions of the Lax pair (3) at \( \lambda_2 \).

Thus we can derive that the expression of the double soliton solutions of equation (2)
\[ u[2] = u[1] - \frac{2i\alpha_{12}[1]}{\sqrt{\lambda}} = u[1] - \frac{2i(\lambda_2 - \lambda_2)g_2^{\frac{1}{2}}f_2^{\frac{1}{2}}}{\sqrt{\lambda}(|f_2|^2 + |g_2|^2)}. \]

After continuing the above iteration process \( n \) times, the N-fold DT can be obtained
\[ \Phi[N] = D[N]\Phi[N-1] = (\lambda E - S[N-1])\Phi[N-1] \]
\[ = \begin{pmatrix} \lambda - s_{11}[N-1] & -s_{12}[N-1] \\ -s_{21}[N-1] & \lambda - s_{22}[N-1] \end{pmatrix}\Phi[N-1] \]
\[ = D[N] \cdots D[2]D[1]\Phi, \]
where
\[ S[N-1] = \begin{pmatrix} s_1[N-1] & s_2[N-1] \\ s_{21}[N-1] & s_{22}[N-1] \end{pmatrix} = \begin{pmatrix} f_N[N-1] & -g_N[N-1] \\ g_N[N-1] & f_N[N-1] \end{pmatrix} \begin{pmatrix} \lambda_N & 0 \\ 0 & \lambda_N \end{pmatrix} \begin{pmatrix} f_N[N-1] & -g_N[N-1] \\ g_N[N-1] & f_N[N-1] \end{pmatrix}^{-1}, \]
\[ f_N[N-1] = \prod_{j=0}^{N-2} \left( (\lambda_N - s_1[j])f_N - s_2[j]g_N \right), \]
\[ g_N[N-1] = \prod_{j=0}^{N-2} \left( (\lambda_N - s_2[j])g_N - s_1[j]f_N \right), \]
and \((f_N, g_N)^T\) is the eigenfunctions of the Lax pair (3) at \(\lambda = \lambda_N\), \((-g_N, f_N)^T\) is an eigenfunctions of the Lax pair (3) at \(\lambda = \lambda_N\).

And we can derive that the expression of the N-soliton solutions of equation (2)
\[ u[N] = u[N-1] - \frac{2i\kappa_{12}[N-1]}{\sqrt{\alpha}} = u - \frac{2i}{\sqrt{\alpha}} \sum_{j=1}^{N} (\lambda_j - \bar{\lambda}_j) f_j[j-1] g_j[j-1] (|f_j[j-1]|^2 + |g_j[j-1]|^2). \]  

3. **Soliton solution**

In this section, we construct the explicit multisoliton solutions of equation (2) by DT. Taking a zero seed solution \(u = 0\), the corresponding Lax pair equation (3) is
\[ \Phi_j = \begin{pmatrix} -i\lambda_j & 0 \\ 0 & i\lambda_j \end{pmatrix}, \Phi_j = \begin{pmatrix} -2i\beta^2 + iv(t) & 0 \\ 0 & 2i\beta^2 - iv(t) \end{pmatrix}, \]  

Then we can easily solve the linear spectral problem equation (16) and obtain the eigenfunctions with regard to the spectral parameters \(\lambda_j = \alpha_j + ib_j\),
\[ \Phi_j = (f_j, g_j)^T, f_j = e^{-ip_j}, g_j = e^{ip_j}, \rho_j = \lambda_j x + 2\beta \lambda_j t - \int_0^t v(t)dt, (j = 1, 2, \cdots, N). \]  

3.1. **Single soliton solution**

Taking \(j = 1\), and substituting (17) into one-fold DT(11), we can get the single soliton solution of equation (2),
\[ u[1] = \frac{2h \sec h(2h x + 8\beta a_h t) e^{j\gamma_1}}{\sqrt{\alpha}}, \]  

with
\[ \gamma_1 = -2\alpha x - 4\beta (a_h^2 - h^2) + 2\int_0^t v(t)dt. \]

From equation (18), we can see that the power of atoms is
\[ |u|^2 = \frac{4}{\alpha} b_h^2 \sec h^2(2h x + 8\beta a_h t). \]  

From Figure 1, we can see that a bright single solitary wave is simulated, and the amplitude of the solitary wave is unchanged. According to Figure 1 and Figure 2, it can be concluded that the value of \(\alpha\) affects the wave height, while the value of \(\beta\) affects the propagation direction and wave width.
3.2 Double soliton solution

Taking \( j = 2 \), and substituting (17) into two-fold DT(13), we can get the double soliton solution of equation (2),

\[ u[2] = \frac{2}{\sqrt{\alpha}} [h \sec h(2h_1 x + 8\beta h_1 t) e^{i\phi_1} + \frac{2h_2 f_2[1]g_2[1]}{|f_2[1]|^2 + |g_2[1]|^2}], \quad (20) \]

with

\[ f_2[1] = [a_2 - a_1 + ib_2 - ib_1 \tanh(2h_1 x + 8\beta h_1 t)] e^{-i\phi_2} - ib_1 \sec h(2h_1 x + 8\beta a h_1 t) e^{i\phi_2 + i\phi_1}, \]
\[ f_2[1] = [a_2 - a_1 - ib_2 + ib_1 \tanh(2h_1 x + 8\beta a h_1 t)] e^{i\phi_2} + ib_1 \sec h(2h_1 x + 8\beta a h_1 t) e^{-i\phi_2 + i\phi_1}, \]
\[ g_2[1] = [a_2 - a_1 + ib_2 - ib_1 \tanh(-2h_1 x - 8\beta a h_1 t)] e^{i\phi_2} - ib_1 \sec h(2h_1 x + 8\beta a h_1 t) e^{-i\phi_2}, \]
\[ g_2[1] = [a_2 - a_1 - ib_2 + ib_1 \tanh(-2h_1 x - 8\beta a h_1 t)] e^{-i\phi_2} + ib_1 \sec h(2h_1 x + 8\beta a h_1 t) e^{i\phi_2 + i\phi_1}. \]

From equation (20), we can see that the power of atoms is

Figure 1. The dynamic evolution of the single soliton solution \( |u|^2 \) via equation (2) with the parameters \( a_i = b_i = 1, \beta = 1 \).

Figure 2. Contour diagram of single soliton solution \( |u|^2 \) via equation (2) with the parameters \( a_i = b_i = \alpha = 1 \).
\[ |u| = \frac{4}{\alpha} \left| b \right|^2 \sec^2 b \left( 2h x + 8 \beta a_i b_i \phi t \right) + \frac{4b^2 \left| f_2 (1) \right|^2 \cdot \left| g_2 (1) \right|^2 \left( |f_2 (1)|^2 + |g_2 (1)|^2 \right)^{3/2}}{\left( |f_2 (1)|^2 + |g_2 (1)|^2 \right)^{3/2}} \]

\[ + \frac{2h \cdot b \sec b \left( 2h x + 8 \beta a_i b_i \phi t \right)}{|f_2 (1)|^2 + |g_2 (1)|^2} \left( f_2 (1) g_2 (1) e^{-e t} + g_2 (1) f_2 (1) e^{e t} \right). \]

From Figure 3 and Figure 4, we can see the shape and amplitudes of two solitons have remained unchanged after collision. We can also find that the angle of two solitons are related to the parameters \( a_i, a_2 \), and the wave height is related to the values of \( b_i, b_2 \).

Figure 3. The dynamic evolution of the double soliton solution \( |u| \) via equation (2) with the parameters \( b_1 = b_2 = 1, \alpha = \beta = 1, v(t) = e^t \).

Figure 4. The dynamic evolution of the double soliton solution \( |u| \) via equation (2) with the parameters \( a_1 = 1, a_2 = -1, \alpha = \beta = 1, v(t) = e^t \).

4. Conclusions
DT is an effective method to find the soliton solutions. In terms of DT, we explored a generalized GP equation, and we derived the single soliton solution, double soliton solution. The propagation characteristics of solitary waves were also investigated. This research will help us better understand the behavior of nonlinear waves in systems. We hope that this research can be further applied to BEC and other physical topics.
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