Entanglement and quantum discord in coherent Ising machines

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We present analytical and numerical simulation results for squeezing, entanglement, and quantum discord in a dissipatively coupled coherent Ising machine (CIM). Both the analytical solutions and numerical simulation results, obtained with positive-$P$, truncated-Wigner and truncated-Husimi representations for the density operator, predict the presence of entanglement and quantum discord below and above the threshold of dissipatively coupled CIM. The entanglement criteria and the degree of quantum discord are evaluated as a function of the dissipative coupling strength relative to the background loss. The degrees of both the entanglement and quantum discord are maximum at the threshold. While the entanglement disappears as the linear loss exceeds the Ising coupling strength, the quantum discord remains even in a large linear loss case.

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I. INTRODUCTION

A coherent Ising machine (CIM) with dissipatively coupled degenerate optical parametric oscillators (DOPOs) has been theoretically studied$^{[1]}$ and experimentally demonstrated$^{[5]}$. In contrast to a CIM with measurement-feedback-coupled DOPOs$^{[6,12]}$, a CIM with dissipatively coupled DOPOs has been shown numerically to satisfy entanglement criterion$^{[2,3]}$. It employs linear Liouvillian coupling of squeezed states which resembles the entanglement generation experiment using linear Hamiltonian coupling of squeezed states$^{[13,14]}$. The previous numerical results showed that a CIM satisfies Duan’s entanglement criterion$^{[15]}$, not only below the threshold but even above it$^{[2,3]}$. However, the degree of entanglement above the threshold depends on the system parameters: for some cases the entanglement criterion is violated before the threshold is reached$^{[2]}$. The quantum discord$^{[17,18]}$ was also calculated$^{[2]}$, but its detailed behavior have not been clarified. To obtain the condition for entanglement and to understand behavior of the quantum discord in a CIM for a wide range of system parameters, a comprehensive theoretical description which passes the numerical test is required.

The previous theoretical investigation of CIM highly depends on the numerical approach, particularly in the phase space. A rigorous numerical simulation of cavity quantum electrodynamics (C-QED) system can be performed when the field density operator can be expressed using Fock states with discrete spectra$^{[19,20]}$. The Fock state approach is actually effective for expanded by using Fock states with discrete spectra$^{[19,20]}$. The Fock state approach is actually effective for the field density operator. For lasers with a vastly increasing number of photons, the field density operator can be expanded using diagonal coherent states$^{[24]–27}$, and the Fokker-Planck equation for the Glauber’s $P$ function is derived. By subsequently using the Itô rule, the $c$-number stochastic differential equation (SDE) can be derived. For a DOPO, however, the Fokker-Planck equation of the $P$ function has negative diffusion, and this approach also failed.

A CIM with dissipatively coupled DOPOs was first studied by using $c$-number Langevin equations$^{[1]}$, which constitute a $c$-number counterpart to the $q$-number Heisenberg-Langevin equations$^{[23]}$. Later, an equivalence between the $c$-number Heisenberg-Langevin equations and SDEs in Wigner representation was established with the truncation of Fokker-Planck equation in the Wigner representation$^{[3]}$. The phase space treatments of DOPOs$^{[27,33]}$ were, on the other hand, performed with positive-$P$ SDEs$^{[34,35]}$ or complex-$P$ SDEs$^{[35]}$. These phase-space methods overcome the difficulties in Glauber’s $P$ representation and simultaneously deal with a varying photon number over many orders of magnitude. Various quantum features can be computed from such SDEs if a sufficient number of trajectories are ensemble averaged. Positive-$P$ and truncated Wigner approaches produce similar values for entanglement criterion if DOPOs are operating in a so-called weak noise limit$^{[31]}$.

On the analytical side, quantum statistical properties of a single DOPO at a steady state have been rigorously obtained by the integration of the Fokker-Planck equation in the complex-$P$ representation$^{[23,30]}$. In coupled quantum nonlinear optical systems, some recent theoretical studies have used the mean field approxi-

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ation for the coupling part between nonlinear optical subsystems[26][38]. Such treatment motivated by condensed matter theory[36], is useful in searching for a macroscopic or global order, but neglects quantum correlation, including entanglement, between subsystems induced by the noise terms. To elucidate the quantum correlation in a CIM, we consider a weak noise limit[31] of a DOPO, instead of the general solution of a DOPO[29][30]. In such a weak noise case, we can analytically calculate entanglement and quantum discord. To check the analytical results, through numerical simulation, we compute the squeezing, entanglement and quantum discord. To check the analytical results, through numerical simulation, we present the analytical and numerical results on the degrees of squeezing/anti-squeezing, entanglement, and quantum discord. Analytical results are verified by numerical results. Section IV examines a one-dimensional (1D) lattice of DOPOs with only nearest-neighbor coupling. Section V summarizes the paper. Appendix A summarizes the numerical simulation methods based on the positive-\(P\), truncated Wigner and truncated Husimi SDEs for a single DOPO. Appendix B summarizes the analytical method for a single DOPO. Appendix C shows the traveling wave model for 1D nearest-neighbor-coupled DOPOs.

II. MASTER EQUATION OF A DOPO

In this section, we present the theoretical model for a single DOPO. The system of a single DOPO consists of the pump-mode operator \(\hat{a}_p\) and the signal-mode operator \(\hat{a}_s\). The quantum master equation of the system is represented as follows:

\[
\frac{\partial \hat{\rho}}{\partial t} = \mathcal{L}\hat{\rho} = \varepsilon [\hat{a}_p^{\dagger} \hat{a}_p - \hat{a}_p \hat{a}_p^{\dagger}, \hat{\rho}] + \frac{\kappa}{2} [\hat{a}_s^{\dagger} \hat{a}_p - \hat{a}_p^{\dagger} \hat{a}_s, \hat{\rho}] + (\gamma_p [\hat{a}_p, \hat{\rho}] + \gamma_s [\hat{a}_s, \hat{\rho}], \text{h.c.}) \tag{1}
\]

This equation has Hamiltonian and dissipation (Liouvillian) parts. The Hamiltonian part has two terms \(\hat{H}_1 + \hat{H}_2\), where \(\hat{H}_1 = i\hbar \varepsilon (\hat{a}_p^{\dagger} - \hat{a}_p)\) is the coherent excitation of the pump mode by an external injection field \(\varepsilon\), and \(\hat{H}_2 = i\hbar \frac{\kappa}{2} (\hat{a}_s^{\dagger} \hat{a}_p - \hat{a}_p^{\dagger} \hat{a}_s)\) is the nonlinear parametric coupling between the signal and pump modes via the coupling Liouvillian which represents the degenerate two-photon absorption that describes the saturation of the parametric gain. This master equation is, in the Heisenberg picture, equivalent to the following Heisenberg-Langevin equation[1]:

\[
\frac{d\hat{a}}{dt} = -\gamma_s \hat{a} + S\hat{a}^\dagger - B\hat{a}^\dagger \hat{a}^2 + \sqrt{2B}\hat{a}^\dagger \xi_1 + \sqrt{\gamma_s} \xi_2. \tag{6}
\]

The \(e\)-number counterpart of this equation is equivalent to the SDE in the truncated Wigner representation[3].

III. TWO COUPLED DOPOS

A. Model

We now consider a CIM consisting of two DOPOs with ferromagnetic dissipative coupling, as shown in Fig[1]. For the two signal modes represented by \(\hat{a}_1\) and \(\hat{a}_2\), the Liouvillian of a whole system is represented as the summation of a single DOPO part and coupling part:

\[
\frac{d\hat{\rho}}{dt} = \sum_r \mathcal{L}_r^{(DOPO)}\hat{\rho} + \mathcal{L}_C\hat{\rho}. \tag{7}
\]

Here, \(\mathcal{L}_r^{(DOPO)}\) operates on only \(r\)-th DOPO and \(\mathcal{L}_C\hat{\rho}\) is the coupling Liouvillian. We consider the following Liouvillian which represents the dissipative ferromagnetic coupling[2]:

\[
\mathcal{L}_C\hat{\rho} = J[\hat{a}_1 - \hat{a}_2, \hat{\rho} (\hat{a}_1^\dagger - \hat{a}_2^\dagger)] + \text{h.c.} \tag{7}
\]

This Liouvillian is derived when the two signal modes \(\hat{a}_1\) and \(\hat{a}_2\) with eigenfunctions \(E_{s1}\) and \(E_{s2}\), respectively,
FIG. 1. Standing-wave model for dissipative ferromagnetically coupled DOPOs. The two cavities have pump and signal modes, and the two signal modes $\hat{a}_r (r = 1, 2)$ are coupled via the dissipative mode $\hat{a}_C$. The eigenfunction of $\hat{a}_C$ has a node at the center of the cavity. When the two pump modes are excited with the same phase, $\hat{a}_1$ and $\hat{a}_2$ can have the same or opposite phases. When $\hat{a}_2$ has the same phase as $\hat{a}_1$ has, the dissipative mode $\hat{a}_C$ is not excited, so that the dissipation induced by $\gamma_C$ disappears.

are coupled via the standing-wave cavity mode $\hat{a}_C$ with eigenfunction $E_C$, which has a node at the center of the cavity and has the opposite sign at the interface of $\hat{a}_1$ and $\hat{a}_2$:

$$\hat{H}_C = \hbar \chi (\hat{a}_1^\dagger \hat{a}_C + \hat{a}_C^\dagger \hat{a}_1) - \hbar \chi (\hat{a}_2^\dagger \hat{a}_C + \hat{a}_C^\dagger \hat{a}_2).$$

Here, the coupling mode $\hat{a}_C$ has a dissipation rate (HWWM) $\gamma_C$. When the two signal modes $\hat{a}_1$ and $\hat{a}_2$ are excited with the same phase (ferromagnetic order), the dissipation induced by $\gamma_C$ does not occur, at least at the mean amplitude level. When $\gamma_C$ is large, mode $\hat{a}_C$ can be adiabatically eliminated via the Lindblad procedure, and then the equation (8) is obtained with $J \sim \chi^2/\gamma_C$.

The positive-$P$ SDE for two coupled DOPOs is obtained, as the extension of that for a single DOPO, as shown in Appendix A, and is represented as

$$\frac{d}{dt} \begin{bmatrix} \alpha_1 \\ \alpha_1^* \end{bmatrix} = - \begin{bmatrix} \gamma_s + J + B \alpha_1^\dagger \alpha_1 - S \\ -S & \gamma_s + J + B \alpha_1^\dagger \alpha_1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_1^* \end{bmatrix} + J \begin{bmatrix} \sqrt{S - B \alpha_1^\dagger \xi_{R1}} \\ \sqrt{S - B \alpha_1^\dagger \xi_{R1}} \end{bmatrix},$$

$$\frac{d}{dt} \begin{bmatrix} \alpha_2 \\ \alpha_2^* \end{bmatrix} = - \begin{bmatrix} \gamma_s + J + B \alpha_2^\dagger \alpha_2 - S \\ -S & \gamma_s + J + B \alpha_2^\dagger \alpha_2 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_2^* \end{bmatrix} + J \begin{bmatrix} \sqrt{S - B \alpha_2^\dagger \xi_{R2}} \\ \sqrt{S - B \alpha_2^\dagger \xi_{R2}} \end{bmatrix}. $$

Here $\alpha_r (r = 1, 2)$ and $\alpha_1^r (r = 1, 2)$ are independent $\alpha$-numbers satisfying $\langle \alpha_r \rangle = \langle \alpha_1^r \rangle^*$ and $\xi_{Rr}$ and $\xi_{Rr}^*$ are independent real number random variables satisfying $\langle \xi_{Rr}(t) \xi_{Rr}(t') \rangle = 6 \delta_{rr} \delta(t - t')$. When the system has the ferromagnetic order $\alpha_1 \sim \alpha_2$, each cavity has no additional loss due to coupling. Therefore, the lasing occurs with the same excitation as for a single DOPO i.e., at $p = S/\gamma_s = 1$.

The dissipative coupling Liouvilinan does not introduce new noise terms into the positive-$P$ SDE. In the Wigner and Husimi SDEs, however, the dissipative coupling introduces new noise terms. When the coupling parts of the SDE are written as $\frac{d\alpha_1}{dt} |C$ and $\frac{d\alpha_2}{dt} |C$, they are represented as:

$$\frac{d\alpha_1}{dt} |C = -J \alpha_1 + J \alpha_2 + \sqrt{A} J \xi_C,$$

$$\frac{d\alpha_2}{dt} |C = -J \alpha_2 + J \alpha_1 - \sqrt{A} J \xi_C.$$  

Here, $A = 1$ for the Husimi representation and $A = \frac{1}{2}$ for the Wigner representation. The complex noise source $\xi_C$ is common for $\alpha_1$ and $\alpha_2$, since it comes from the vacuum noise of the single dissipative mode $\hat{a}_C$, and satisfies the correlation function: $\langle \xi_C^*(t) \xi_C(t') \rangle = 2 \delta(t - t')$.

B. Duan’s entanglement criterion

We can consider the entanglement criterion for two coupled DOPOs by applying the fluctuation analysis in the positive-$P$ representation, which, for a single DOPO, is represented in Appendix B. If two modes have the same excitations, then they have symmetry under the exchange of modes. We assume that the fluctuation products of the positive-$P$ amplitudes satisfy $\langle \Delta \alpha_1 \rangle \langle \Delta \alpha_2 \rangle \geq \langle \Delta \alpha_1^* \rangle \langle \Delta \alpha_2^* \rangle$ and $\langle \Delta \alpha_1^* \rangle \langle \Delta \alpha_1 \rangle \geq \langle \Delta \alpha_2^* \rangle \langle \Delta \alpha_2 \rangle$. Below the lasing threshold $(p < 1)$, where $\langle \alpha_1 \rangle = \langle \alpha_2 \rangle = 0$, the steady-state squared amplitudes satisfy the following:

$$\begin{bmatrix} 1 + j & -p & -j & 0 \\ -p & 1 + j & 0 & -j \\ -j & 0 & 1 + j & -p \\ 0 & -j & -p & 1 + j \end{bmatrix} \begin{bmatrix} \langle \Delta \alpha_1^2 \rangle \\ \langle \Delta \alpha_1^* \Delta \alpha_1 \rangle \\ \langle \Delta \alpha_1^* \Delta \alpha_2 \rangle \\ \langle \Delta \alpha_2^* \Delta \alpha_2 \rangle \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Here, $p = S/\gamma_s$ and $j = J/\gamma_s$ represent normalized excitation and normalized coupling coefficient respectively. $\langle \Delta \alpha_1^2 \rangle$ and $\langle \Delta \alpha_1^* \Delta \alpha_1 \rangle$ represent the averaged fluctuation products for the same DOPO, while $\langle \Delta \alpha_1 \Delta \alpha_2 \rangle$ and $\langle \Delta \alpha_1^* \Delta \alpha_2 \rangle$ represent the averaged fluctuation products for different DOPOs. Next, Duan’s entanglement criterion is represented as

$$\mathcal{D} = \langle \Delta \hat{u}^2 \rangle + \langle \Delta \hat{v}^2 \rangle,$$

where $\hat{u} = \hat{X}_1 - \hat{X}_2$, $\hat{v} = \hat{P}_1 + \hat{P}_2$, $\hat{X}_r = \hat{a}_r + \hat{a}_r^\dagger \rangle (r = 1, 2)$ and $\hat{P}_r = \hat{a}_r - \hat{a}_r^\dagger \rangle (r = 1, 2)$. If $\mathcal{D}/2 < 1$, then the system’s density matrix is not separated into the product states of the individual DOPOs: $\rho \neq \rho_0 \otimes \rho_2$. Duan’s criterion can be calculated using the fluctuation products as

$$\frac{\mathcal{D}}{2} = 1 + 2(\Delta \alpha_1^* \Delta \alpha_1) - 2(\Delta \alpha_1 \Delta \alpha_2).$$
Below the threshold, Duan’s criterion is represented as follows:

\[
\frac{\mathcal{D}}{2} = 1 - \frac{p(j - p)}{(1 + p)(1 + 2j - p)}.
\]

We can thus see that Duan’s criterion is satisfied if \( j > p \) is satisfied. This means that when \( j < 1 \), Duan’s entanglement criterion is violated before the threshold is reached as seen in [2].

Above the threshold, the mean-squared fluctuations are given by \( \langle \alpha_1 \rangle = \langle \alpha_2 \rangle = \sqrt{2p/(p - 1)} \). We consider the fluctuation around this value using the linearization technique [30]. This procedure in a single DOPO is shown in Appendix B. Under a steady-state condition, the fluctuation amplitude products are represented as follows:

\[
\sigma = \Lambda \sigma^T, \quad \text{where} \quad \Lambda = \text{diag}(1,1,1,-1).
\]

The symplectic eigenvalues of \( \sigma \) are eigenvalues of \( \sigma \Omega \sigma^T \) and satisfy \( S'\sigma S'^T = \text{diag}(\nu_+,\nu_+,\nu_-,\nu_-) \) \((\nu_+ > \nu_-)\), for a symplectic transformation \( S' \). The separable state must satisfy \( \nu_- \geq 1 \) [41].

For the system of two DOPOs below the threshold, all the nonzero components of the covariance matrix are described as follows.

\[
\mathcal{D} = \text{diag} \left[ 1 + \frac{p(1 + j - p)}{(1 - p)(1 - p + 2j)}, 1 - \frac{p(p + 1)}{(1 + p)(1 + p + 2j)} \right].
\]

These produce the symplectic eigenvalues of \( \sigma \). The two independent symplectic eigenvalues of \( \sigma \) are calculated as eigenvalues \( \lambda \) of \( \Omega \sigma \), and are reduced to \( \lambda^2 = (a_1 + c_1)(a_2 + c_2), (a_1 - c_1)(a_2 - c_2) \). Substituting Eqs. (21) (22) into \( \lambda^2 \), we can obtain \( \nu_+^2 = 1 + \frac{p^2}{p - p^2} \) and \( \nu_-^2 = 1 + \frac{p^2}{(1 + 2j + p^2)(p + 2j - p)} \). To obtain Simon’s entanglement criterion, two independent symplectic eigenvalues of \( \sigma \) are calculated as eigenvalues \( \lambda' \) of \( \Omega \sigma \), and are reduced to \( \lambda'^2 = (a_1 + c_1)(a_2 + c_2), (a_1 - c_1)(a_2 + c_2) \). The smallest symplectic eigenvalue of \( \sigma \) is

\[
\tilde{\nu}^2 = 1 - \frac{p(2j - p)}{(1 + p)(1 + 2j - p)}.
\]

When \( \tilde{\nu}_- < 1 \), the two coupled DOPOs are inseparable. This entanglement condition for below the threshold is satisfied when \( j > p/2 \).

We then consider the above-threshold characteristics described by the covariance matrix \( \sigma_{ij} \) with the following nonzero components:

\[
\mathcal{D} = \text{diag} \left[ 1 + \frac{2p - 2 + j}{4(p - 1)(p + j - 1)}, 1 - \frac{2p + j}{4p(p + j)} \right].
\]

C. Simon’s entanglement criterion

We now consider Simon’s entanglement criterion [40]. Simon’s theory is described by the covariance matrix defined for the vectors \( \vec{R} = \sqrt{2}[\hat{X}_1, \hat{P}_1, \hat{X}_2, \hat{P}_2] \),

\[
\sigma_{ij} = \frac{1}{2} \langle [\hat{R}_i, \hat{R}_j]_+ \rangle - \langle \hat{R}_i \hat{R}_j \rangle.
\]

Here, \( [\hat{A}, \hat{B}]_+ = \hat{A}\hat{B} + \hat{B}\hat{A} \). For two DOPOs with dissipative coupling, this matrix can be written as follows:

\[
\sigma = \begin{pmatrix}
\alpha & \gamma \\
\gamma & \alpha
\end{pmatrix}.
\]

Here, \( \alpha = \text{diag}(a_1, a_2) \) and \( \gamma = \text{diag}(c_1, c_2) \). \( a_1 = 2\langle \Delta \hat{X}_1^2 \rangle = 2\langle \Delta \hat{P}_1^2 \rangle \), and \( a_2 = 2\langle \Delta \hat{X}_2^2 \rangle = 2\langle \Delta \hat{P}_2^2 \rangle \) represent the single-mode characteristics of \( \hat{X} \) and \( \hat{P} \), respectively. On the other hand, \( c_1 = 2\langle \Delta \hat{X}_1 \Delta \hat{X}_2 \rangle \), and \( c_2 = 2\langle \Delta \hat{P}_1 \Delta \hat{P}_2 \rangle \) represent the intermode correlation of \( \hat{X} \) and \( \hat{P} \) respectively. The covariance matrix can be diagonalized with a symplectic transformation \( S \) as \( S\sigma S^T = \text{diag}(\nu_+,\nu_+,\nu_-,\nu_-) \), where \( \nu_+ > \nu_- \). The symplectic eigenvalues are related to the uncertainty relation, and a physical state must satisfy \( \nu_- \geq 1 \) [41]. These are obtained by the eigenvalues of \( \Omega \sigma \) where \( \Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). Simon’s entanglement criterion is defined on the partially transposed covariance matrix
The symplectic eigenvalues of $\sigma$ are $\nu_+^2 = 1 + \frac{1}{4(p+1)}$ and $\nu_-^2 = 1 + \frac{1}{4(p+j-1)}$. The smallest symplectic eigenvalue of $\sigma$ is represented as

$$\hat{\nu}^2 = 1 - \frac{2j - 1}{4p + j - 1}. \quad (26)$$

Therefore, Simon’s entanglement criterion for above the threshold is satisfied for $j > 1/2$. The required $j$ to satisfy the entanglement criterion is thus smaller than that for Duan’s criterion above the threshold ($j > 1$).

**D. Quantum discord**

Next, we consider the quantum discord of two dissipatively coupled DOPOs. When the total system density operator with $\hat{\rho}$ is composed of the first DOPO with $\hat{\rho}_1$ and the second DOPO with $\hat{\rho}_2$, the von Neumann entropy of the total system is defined by $S(\hat{\rho}) = -\text{Tr} \hat{\rho} \log \hat{\rho}$. The total entropy also seems to be written as the summation $S(\hat{\rho}_1) + S(\hat{\rho}_2)$, where $S(\hat{\rho}_1)$ is the reduced entropy of first DOPO, and $S(\hat{\rho}_2)$ is the conditional (or residual) entropy of the second DOPO with the subtraction of mutual information. In classical information theory, these two formula for the total entropy are identical, but when the systems have quantumness they generally differ\[17, 18\]. The quantum discord is defined as the difference between these two expressions for entropies. Nonzero quantum discord is identified as a figure of merit for quantifying the quantum resource in a specific quantum computational model (deterministic quantum computation with one pure qubit DQC1\[32, 43\]).

For a system of two DOPOs, using the theory of two-mode Gaussian states [44, 45], quantum discord is written as follows:

$$D(\sigma) = f(\sqrt{a_1a_2}) + \inf_{\sigma} f(\sqrt{\text{det} \sigma}) - f(\nu_+) - f(\nu_-). \quad (27)$$

Here $f(x) = \frac{1}{2} \log \frac{x + 1}{x - 1} - \frac{1}{2} \log \frac{x^2 - 1}{x} + \log x$, and $\sigma = \rho - 2(\rho_0 + \rho_{\text{f}})^{-1}$. The first two terms on the right hand side of Eq. (27) represent the entropies $S(\hat{\rho}_1)$ and $S(\hat{\rho}_2)$. The last two terms represent the entropy $S(\hat{\rho})$ consisting of two DOPOs\[43\]. In the calculation of $S(\hat{\rho}_2)$, we consider only an optical homodyne detector which measures $X$ and discards all information about $\hat{X}$

In our two-DOPO system, it seems to be reasonable that the residual entropy is minimized for $\lambda \to 0$. Actually, we have not obtained the minimum value of $\text{det} \sigma$ at $\lambda \neq 0$. All results of quantum discord below were obtained with Eq. (29). The quantum discord is always nonzero in an optical delay-line coupled CIM, no matter how large the dissipation is, because of the nonzero $c_1$ and $c_2$ values.

For comparison, we can present the theory for mean-field coupled CIMs. The mean-field coupling theory has been developed in the context of the Bose-Hubbard model \[39\]. In such approaches, the Hamiltonian coupling between different sites, $\hat{H}_C = \hbar t \sum_{i,j}(\hat{a}_i^\dagger \hat{a}_j + \hat{a}_j^\dagger \hat{a}_i)$, is approximated by the mean-field coupling via $\hat{a}_i^\dagger \hat{a}_j \to \hat{a}_i^\dagger \langle \alpha_j \rangle + \langle \alpha_i \rangle \hat{a}_j$. Similarly, we can consider two dissipatively coupled DOPOs, where the Liouvillian coupling [Eq. (7)] is replaced by $[\hat{a}_1 - a_2^* \hat{a}_2, \rho(a_1^\dagger - a_1^* \rangle) \to [\hat{a}_1 - a_2^*, \hat{a}_2^\dagger - (\alpha_2^* \rangle)] + [\hat{a}_2 - \langle \alpha_1 \rangle, \rho(a_2^\dagger - (\alpha_1^* \rangle)]$. Assuming that $\langle \alpha_r \rangle (r = 1, 2)$ are real, $L_C \hat{\rho}$ in Eq. (7) is replaced as

$$L_C \hat{\rho} = J \sum_{r=1,2}\langle [\alpha_r, \rho \hat{a}_r^\dagger] + \text{h.c.} \rangle + J(\alpha_2^* \langle \hat{a}_1^\dagger - \hat{a}_1, \hat{\rho} \rangle + J(\alpha_1^* \langle \hat{a}_2^\dagger - \hat{a}_2, \hat{\rho} \rangle. \quad (30)$$

Analytically, such a model has a covariance matrix $\text{Eq. (29)}$ with $a_1 = 1 + \frac{p}{1 + p + j}$, $a_2 = 1 - \frac{1}{1 + p + j}$ below the threshold and that with $a_1 = 1 + \frac{1}{2p(1 + p + j)}$, $a_2 = 1 - \frac{1}{2p + j}$ above the threshold. Since $c_1 = 0$ both below and above the threshold, the mean field coupled model has zero quantum discord.

**E. Numerical simulation**

To test the theoretical results for a ferromagnetically coupled two-DOPO system, we performed a numerical simulation based on the positive-P, truncated Wigner, and truncated Husimi SDEs. In the simulation, we assumed that the system has a small two-photon absorption loss $B/\gamma_s = 2 \times 10^{-3}$. The mean fluctuation products were calculated with both the trajectory average and the time average, assuming ergodicity around the mean amplitude. We generated $9 \times 10^4$ trajectories. For each trajectory, the time development was computed for a period of $2 \times 10^4/\gamma_s$. The time development started from $\alpha_1 = \alpha_2 = \alpha_1^* = \alpha_2^* = 0$ in the positive-P calculation. In the truncated Wigner SDE calculation, it started from vacuum noise with $|\langle \alpha_1^2 \rangle | = |\langle \alpha_2^2 \rangle | = \frac{1}{2}$. In the truncated Husimi SDE calculation, the time development started from vacuum noise with $|\langle \alpha_1^2 \rangle | = |\langle \alpha_2^2 \rangle | = \frac{1}{2}$. The time step was $\Delta t = 10^{-3}/\gamma_s$, $\Delta t = 10^{-4}/\gamma_s$, and $\Delta t = 5 \times 10^{-5}/\gamma_s$, respectively for positive-P, truncated Wigner and truncated Husimi SDEs. In the first time period of $t_f = 10^3/\gamma_s$, the time average was not taken, and the pump excitation depended on time via $p(t) = p \sqrt{t/t_f}$. After the first period of $t_f$, the pump
excitation was held constant at \( p \) and the time average was taken.

We calculated the degrees of squeezing/anti-squeezing, Duan’s entanglement criterion, Simon’s entanglement criterion, and quantum discord for varying pump excitations \( p \) with a constant dissipative coupling \( j = J/\gamma_s = 7/3 \). When \( p > 1 \), all trajectories had the expected ferromagnetic order after the initial time development over \( t_f \). There are two degenerate ground state configurations, however, with two opposite signs of the amplitude \( \langle \alpha_1 \rangle \). Duan’s criterion can be calculated as the direct trajectory average. It does not depend on the sign of \( \langle \alpha_1 \rangle \), because it cannot remain in \( \hat{u}_1 = X_1 - X_2 \) when \( \langle \alpha_1 \rangle \sim \langle \alpha_2 \rangle \). The components of the covariance matrix \( \sigma_{ij} \) in Eq. (19) were calculated as averages around the mean amplitude \( \langle \alpha \rangle > 0 \) when \( p > 1 \). We first took the time average (represented as \( \langle \cdots \rangle \)). When its sign \( \langle \alpha_j \rangle_t \) was negative, a sign factor \(-1\) was added before taking the trajectory average.

We first present the results for \( \langle \Delta X^2 \rangle = a_1/2 \) and \( \langle \Delta P^2 \rangle = a_2/2 \) for the two-DOPO system in Fig. 2(a). The filled circles represent \( \langle \Delta X^2 \rangle \), while the open circles represent \( \langle \Delta P^2 \rangle \). The theoretical values are shown by dashed lines. Far below the threshold (\( p \ll 1 \)) and far above the threshold (\( p \gg 1 \)), the DOPO field should be in a vacuum state and a coherent state, respectively, so that the variance \( \langle \Delta X^2 \rangle \) and \( \langle \Delta P^2 \rangle \) are expected to approach an asymptotic value \( \langle \Delta X^2 \rangle = \langle \Delta P^2 \rangle = \frac{1}{2} \). On the other hand, the degree of squeezing and anti-squeezing should be maximum at the threshold (\( p = 1 \)). The numerical simulations based on the positive-\( P \), truncated Wigner and truncated Husimi SDEs reproduced these theoretical predictions well and thus verified the analytical results presented in the previous sections.

Duan’s entanglement criterion \( (2/2) \) is shown in Fig. 2(b). When this value is smaller than one, the system is inseparable and entangled. We can see that this value was smallest at the threshold and smaller than one even above the threshold, because the entanglement condition above the threshold, \( j > 1 \), is satisfied. The theoretical values shown by dashed line were calculated from Eqs. (16) and (17), and these match the numerical ones. Next, Fig. 2(c) shows the quantum discord as a function of \( p \), calculated from the covariance matrix \( \sigma_{ij} \). The theoretical values were obtained from Eqs. (27) and (29) and are shown by a dashed line. As shown by filled circles, the quantum discord takes a maximum value at the threshold and remains finite even at the pump rate far below and far above it. If the mutual coupling term is replaced by the mean-field approximation model in Eq. (30), however, the quantum discord disappears in all the pump rates as shown by open circles. The mean fields \( \langle \alpha_1 \rangle \) and \( \langle \alpha_2 \rangle \) in Eq. (31) were calculated at each time step as the ensemble average of \( 9 \times 10^4 \) trajectories. Figure 2(d) shows the analytical quantum discord of dissipatively coupled DOPOs at three different pump rates as a function of normalized dissipative coupling \( j = J/\gamma_s \). A dashed vertical line represents the Simon’s entanglement criterion for above threshold \( j = 1/2 \). For a large loss limit \( j \ll 1 \), the entanglement disappears, but the quantum discord survives particularly around \( p \sim 1 \).

IV. 1D LATTICE OF \( N \) DOPOS

A. Models

We consider a CIM consisting of \( N \)-DOPOs \( (\hat{a}_{1,r}(r = 0, \cdots , N - 1)) \) which represents the ferromagnetic 1D lattice with nearest neighbor coupling. The dissipative coupling Liouvillian is obtained from the extension of Eq. (7) as follows:

\[
\mathcal{L}_C \hat{\rho} = \sum_{r=1}^{N} \frac{J}{2} [\hat{a}_r - \hat{a}_{r+1}, \hat{\rho}(\hat{a}_r^\dagger - \hat{a}_{r+1}^\dagger)] + h.c.. \tag{31}
\]

Here, we consider a periodic system such that \( \hat{a}_{N+1} \) is identical to \( \hat{a}_1 \). Such a 1D lattice can be constructed with \( N \) dissipative standing-wave modes with half width \( \gamma_C \):

\[
\hat{H}_C = \hbar \chi \sum_{r=1}^{N} (\hat{a}_r^\dagger - \hat{a}_{r+1}^\dagger) \hat{a}_{C,r} + h.c.. \tag{32}
\]

Here, the \( r \)-th DOPO is coupled to the \( (r - 1) \)-th DOPO via the dissipative mode \( \hat{a}_{C,r-1} \), and is also coupled to the \( (r + 1) \)-th DOPO via the dissipative mode \( \hat{a}_{C,r} \). When all dissipative modes \( \hat{a}_{C,r} \) have the same loss \( \gamma_C \) with \( \gamma_C \gg \gamma_s \), the coefficient of the Liouvillian follows \( J/2 \sim \chi^2/\gamma_C \). In the Wigner and Husimi SDE, from Eq. (32), the coupling part in the SDE for the \( r \)-th DOPO is represented as follows:

\[
\frac{d\alpha_r}{dt} = -J \alpha_r + \frac{J}{2} (\alpha_{r-1} + \alpha_{r+1}) + \sqrt{\frac{A J}{2}} \xi_{C,r} - \sqrt{\frac{A J}{2}} \xi_{C,r+1}. \tag{33}
\]

Here, \( A = 1 \) for the Husimi SDE and \( A = \chi/\gamma_C \) for the Wigner SDE. The noise sources satisfy \( \langle \xi_{C,r}(t)\xi_{C,r'}(t') \rangle = 2\delta_{r,r'}\delta(t-t') \). This model was obtained as the extension of the standing wave model in Fig. 1. The theory of the traveling wave model of a CIM \( \hat{C} \) implemented in the Wigner representation is shown in Appendix C.

B. Extended Duan criterion

First, we take the analytical approach using the positive-\( P \) SDE. For the \( r \)-th signal mode \( \hat{a}_r(r = 1, \cdots , N) \), the positive-\( P \) SDE is written as:

\[
\frac{d\alpha_r}{dt} = -(\gamma_s + J)\alpha_r + \frac{J}{2} (\alpha_{r-1} + \alpha_{r+1}) + S \alpha_r^\dagger - B \alpha_r^\dagger \alpha_r^2 + \sqrt{S - B \alpha_r^2} \xi_{R,r}. \tag{34}
\]
FIG. 2. Entanglement and quantum discord of two dissipatively coupled DOPOs. (a) Variances in canonical components $\langle \Delta \hat{X}^2 \rangle$ (filled circles) and $\langle \Delta \hat{P}^2 \rangle$ (open circles) as a function of the normalized excitation $p$. (b) Duan’s inseparability criterion as a function of $p$. (c) Quantum discord in dissipatively coupled (filled circles) and mean-field coupled (open circles) DOPOs as a function of $p$. (d) Quantum discord as a function of normalized coupling coefficient $j = J/\gamma_s$.

We can introduce the Fourier-transformed fluctuation amplitude $\Delta \hat{\alpha}_k$:

$$
\Delta \alpha_{1+r} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \Delta \hat{\alpha}_k e^{i \theta_k r}.
$$

(35)

Here, $\theta_k = \frac{2\pi}{N} k$. Below the threshold, we can obtain the following SDE for the Fourier components:

$$
d\Delta \hat{\alpha}_k/dt = - (\gamma_s + J(1 - \cos \theta_k)) \Delta \hat{\alpha}_k + S \Delta \hat{\alpha}_k^\dagger + \sqrt{S} \xi_{R,k}.
$$

(36)

The Fourier-transformed random noise sources are defined as

$$
\xi_{R,k} = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \xi_{R,r} e^{-i \theta_k r},
$$

(37)

which satisfy $\langle \xi_{R,k}(t) \xi_{R,-k}(t') \rangle = \delta_{k,k'} \delta(t-t')$. We obtain the steady-state fluctuation of Fourier components as follows.

$$
\langle \Delta \hat{\alpha}_k \Delta \hat{\alpha}_{-k} \rangle = \frac{p}{2} \frac{1 + j(1 - \cos \theta_k)}{(1 + j(1 - \cos \theta_k))^2 - p^2}.
$$

(38)

We now calculate the extended Duan criterion $[3]$, denoted as $\mathcal{D}' = \langle \Delta \hat{u}^2 \rangle + \langle \Delta \hat{v}^2 \rangle$, where $\hat{u} = \sum_{r=1}^{N} (-1)^{r+1} \hat{X}_r$ and $\hat{v} = \sum_{r=1}^{N} \hat{P}_r$. When $\mathcal{D}'/N < 1$, the entire system is not separated into the product state: $\hat{\rho} \neq \otimes_{r=1}^{N} \hat{\rho}_r$. The definition of $\mathcal{D}'$ implicitly assumes an even number $N$ of DOPOs in the system. The extended Duan’s criterion can be written with Fourier components as

$$
\frac{\mathcal{D}'}{N} = 1 + \langle \Delta \hat{\alpha}_k^\dagger \Delta \hat{\alpha}_{k=0} \rangle - \langle \Delta \hat{\alpha}_k^2 \rangle + \langle \Delta \hat{\alpha}_k^\dagger \Delta \hat{\alpha}_{k=0} \rangle + \langle \Delta \hat{\alpha}_k \Delta \hat{\alpha}_{k=0}^\dagger \rangle.
$$

(40)

We can see that this expression is reduced to $\mathcal{D}/2$ for two DOPOs given in Eq.(16): $\langle \Delta \hat{u}^2 \rangle = \langle \Delta \hat{u}^2 \rangle + \langle \Delta \hat{v}^2 \rangle$. Note that this $\mathcal{D}'$ contains the case of $N = 2$ as a special case.

Above the threshold, we can obtain the following SDE for the Fourier components:

$$
d\Delta \hat{\alpha}_k/dt = - (\gamma_s + J(1 - \cos \theta_k)) \Delta \hat{\alpha}_k + S \Delta \hat{\alpha}_k^\dagger - 2B\langle \alpha \rangle^2 \Delta \hat{\alpha}_k - B\langle \alpha \rangle^2 \Delta \hat{\alpha}_k^\dagger + \sqrt{S - B\langle \alpha \rangle^2} \xi_{R,k}.
$$

(41)
Here, \( S - B(\alpha)^2 \sim \gamma_s \) above the threshold. A similar equation seems to be obtained in the dissipative Bose Hubbard model with the nonlinear Kerr effect \(^{18}\). From Eq. (41), we can obtain the steady-state fluctuation correlations as follows.

\[
\langle \Delta \hat{\alpha}_k \Delta \hat{\alpha}_{-k} \rangle = \frac{1}{2} \left( \frac{2p - 1 + j(1 - \cos \theta_k)}{(2p - 1 + j(1 - \cos \theta_k))^2 - 1} \right), \tag{42}
\]

\[
\langle \Delta \hat{\alpha}^\dagger_k \Delta \hat{\alpha}_k \rangle = \frac{1}{2} \left( \frac{1}{(2p - 1 + j(1 - \cos \theta_k))^2 - 1} \right). \tag{43}
\]

The extended Duan criterion \( \mathcal{D}'/N \) is given by Eq. (40), and the result is reduced to \( \mathcal{D}'/2 \) [Eq. (18)] for two DOPOs: \( \mathcal{D}' = 1 - \frac{j^2}{4(p + j - 1)} \). Therefore, \( j = 1 \) is required to satisfy the extended Duan criterion above the threshold.

To understand these results, we introduce an intuitive picture based on the loss spectrum. The loss matrix of fluctuation in a 1D DOPO system with nearest-neighbor coupling is described by

\[
\begin{bmatrix}
\Delta \hat{\alpha}_k^\dagger & \Delta \hat{\alpha}_k \\
\Delta \hat{\alpha}^\dagger_{-k} & \Delta \hat{\alpha}_{-k}
\end{bmatrix} =
\begin{bmatrix}
1 + j(1 - \cos \theta_k) & -p \\
-p & 1 + j(1 - \cos \theta_k)
\end{bmatrix}
\]

below the threshold, and by

\[
\begin{bmatrix}
\Delta \hat{\alpha}_k^\dagger & \Delta \hat{\alpha}_k \\
\Delta \hat{\alpha}^\dagger_{-k} & \Delta \hat{\alpha}_{-k}
\end{bmatrix} =
\begin{bmatrix}
2p - 1 + j(1 - \cos \theta_k) & -1 \\
-1 & 2p - 1 + j(1 - \cos \theta_k)
\end{bmatrix}
\]

above the threshold. When the loss matrix is diagonalized, the \( \hat{X} \)-like (\( \Delta \hat{\alpha}_k + \Delta \hat{\alpha}^\dagger_{-k} \)) mode has a normalized loss spectrum represented by \( \Gamma_X(k)/\gamma_s = 1 - p + j(1 - \cos \theta_k) \) (below threshold) and \( \Gamma_X(k)/\gamma_s = 2(p - 1) + j(1 - \cos \theta_k) \) (above threshold). Likewise, the \( \hat{P} \)-like (\( \Delta \hat{\alpha}_k - \Delta \hat{\alpha}^\dagger_{-k} \)) mode has a normalized loss spectrum of \( \Gamma_P(k)/\gamma_s = 1 + p + j(1 - \cos \theta_k) \) (below threshold) and \( \Gamma_P(k)/\gamma_s = 2(p + j) + j(1 - \cos \theta_k) \) (above threshold). Figure 3(a) shows these spectra with \( j = 7/3 \) for three excitation strengths \( p = 0.3, p = 1 \), and \( p = 3 \). Here, the filled circles represent \( \hat{X} \)-like modes, while the open circles represent \( \hat{P} \)-like modes. As the fluctuation term does not depend on \( \theta_k \) and the mode type (i.e., \( \hat{X} \)-like or \( \hat{P} \)-like), the fluctuation magnitude is represented by the loss function. When the loss is smaller, the fluctuation is larger in proportion to \( \Gamma(k) \). We can see that when \( p = 1 \), the \( \hat{X} \)-like mode becomes lossless at \( k = 0 \) and lasing occurs at \( k = 0 \). The extended Duan criterion for the entire lattice is represented by fluctuations at two symmetric points in wavenumber space [Eq. (40)]. The extended Duan criterion \( \mathcal{D}'/N \) is satisfied when the loss of the \( \hat{X} \)-like band at \( k = \pm \frac{\pi}{4} \) is larger than that of the \( \hat{P} \)-like band at \( k = 0 \). This is always the case far below the threshold. Above the threshold, however, the criterion is satisfied only when \( j > 1 \). In Fig. 3(b), we present the wavenumber dependent loss for \( j = 2/3 \). In this case, the extended Duan criterion is satisfied for \( p = 0.3 \), but not satisfied for \( p = 1 \) and \( p = 3 \).

We performed a numerical simulation to confirm the analytical results for the 1D DOPO ring with nearest-neighbor coupling. The simulation was performed with a small two-photon absorption loss \( B/\gamma_s = 10^{-4} \). The fluctuations were calculated with the time average for a single trajectory. The total time development was a period of \( 10^3/\gamma_s \). In the first time period of \( t_f = 10^3/\gamma_s \), the time average was not taken, and the pump excitation depended on time via \( p(t) = p(t)/t \). After the first period of \( t_f \), the excitation was held constant to \( p \), and the time average was taken. We considered \( N = 32 \) DOPOs with \( j = 7/3 \) or \( j = 2/3 \). To distinguish the ground state from the first excited states in Fig 3(a) \( t_f \) should be larger than the inverse of \( \Gamma = J(\Delta t^2/2) \) by \( N^2 \). The adopted \( t_f \) satisfies this condition. Fig 3(a) shows the extended Duan entanglement criterion for an entire lattice. As seen in Fig 3(b) by the loss spectrum, extended Duan entanglement criterion is always satisfied for \( j = 7/3 \). However, for \( j = 2/3 \) it is satisfied only for \( p < 2/3 \).

C. Duan’s criterion between two DOPOs

Instead of the entire lattice, we can consider the entanglement for two DOPOs, after tracing out the remaining \((N - 2)\) DOPOs. Duan’s entanglement criterion for a pair with distance \( r \) is calculated as follows:

\[
\mathcal{D}(r) = 1 + \frac{2}{N} \sum_k (\langle \Delta \hat{\alpha}^\dagger_k \Delta \hat{\alpha}_k \rangle - \langle \Delta \hat{\alpha}_k \Delta \hat{\alpha}^\dagger_k \rangle \cos(r \theta_k)).
\]

We first consider the \( r = 1 \) case (nearest neighbor pair). In the limit \( N \to \infty \), we can compute the above summation by replacing it with the integration \( \frac{1}{N} \sum_{k=0}^{N-1} f(\theta_k) \to \int_{-\pi}^{\pi} f(\theta) \frac{d\theta}{2\pi} \). After the integration, we obtain

\[
\mathcal{D}(1) = 1 - p \left[ \left[ 1 + 2 \sqrt{\frac{1 - p}{1 - p + 2j}} + \frac{1}{2} \sqrt{\frac{1 + p + 2j}{1 + p}} \right] \right] \tag{44}
\]

below the threshold, and

\[
\mathcal{D}(1) = 1 - \frac{1}{2} \left[ \left[ \frac{p - 1}{p + j - p} + \frac{1}{2} \sqrt{\frac{j + p}{p}} \right] \right] \tag{45}
\]
FIG. 4. Entanglement and quantum discord characteristics of N-DOPOs with 1D nearest-neighbor ferromagnetic coupling. (a) Numerical result of extended Duan criterion for $N = 32$ lattice as a function of $p$ with $j = 7/3$ (filled circles) or $j = 2/3$ (open circles). (b) Numerical result of Duan’s inseparability criterion for a nearest-neighbor ($r = 1$) pair as a function of $p$ ($N = 32$, $j = 7/3$). Black dashed lines represent the entanglement criterion, below which the inseparability is shown. In (a) and (b), gray dashed lines represent the theoretical result. (c) Theoretical Duan inseparability criterion as a function of $r$ ($j = 9$). (d) Theoretical quantum discord as a function of $r$ ($j = 9$). In (c) and (d), filled circles represent results from Fourier components ($N = 32$) and dotted lines represent results from parabolic approximation with $N \to \infty$.

above it. We can see that $\cD(1)/2 < 1$ is, far below the threshold ($p \ll 1$) always satisfied, at the threshold ($p \to 1$) satisfied when $j > 3$, and far above the threshold ($p \gg 1$) satisfied when $j > 2$. When $2 < j < 3$, Duan’s entanglement criterion is not satisfied at $p \sim 1$, even though it is satisfied well below and well above the threshold. We performed the numerical simulation as done in Fig. 4(a). Figure 4(b) shows the Duan’s entanglement criterion for a nearest-neighbor pair in $N = 32$ DOPOs with $j = 7/3$. The theoretical values [Eqs. (44) and (45)] were obtained by assuming $N \to \infty$. We can see that the entanglement criterion ($\cD^2(r) < 1$) is not satisfied at the threshold but is satisfied below and above it.

In the large $N$ limit, Duan’s entanglement criterion for a pair with distance $r$ ($\cD(r)$) can be calculated using the parabolic approximation $\cos \theta \sim 1 - \frac{\theta^2}{2}$ and the integration $\int \frac{\cos \theta}{m^2 + \theta^2}$. Duan’s entanglement criterion for a pair is calculated as

$$\frac{\cD(r)}{2} = 1 + \frac{\psi(p)}{2j} \left[ \frac{1 - e^{-m_-r}}{m_-} - \frac{1 + e^{-m_+r}}{m_+} \right].$$  (46)

Here, $\psi(p) = pH(1 - p) + H(p - 1)$, $H(p)$ is the Heaviside’s step function, $m_- = \frac{2(1 - p)}{j}$ below the threshold and $m_- = \frac{2(2p - 1 + 1)}{j}$ above the threshold. The theoretical Duan’s criterion between DOPOs with distance $r$ is shown in Fig.4(c), where $j = 9$. Dotted lines represent the values with parabolic approximation [Eq.(46)], and closed circles are calculated from Fourier components [Eqs. (38) (39) (42) and (43)] with $N = 32$. Duan’s entanglement criterion is satisfied only for small $r$. Particularly, at the threshold ($p = 1$), $\cD(r)$ is proportional to $r$ due to small $m_-$. 

D. Quantum discord

We next consider the quantum discord between two DOPOs with distance $r$. The small-$k$ part of the loss spectrum is also important in calculating the spatial correlation function of the in-phase components in covariance matrix [Eq.(20)]

$$c_1 = 2\langle \Delta \hat{X}_1 \Delta \hat{X}_{1+r} \rangle = \delta_{r,0} + \frac{2}{N} \sum_k (\langle \Delta \hat{a}_k \Delta \hat{a}_{-k} \rangle + \langle \Delta \hat{a}_k^\dagger \Delta \hat{a}_{-k} \rangle) \cos(\theta_k r).$$

In the limit
We have presented analytical results on the degrees of squeezing/anti-squeezing, entanglement and quantum discord for a network of DOPOs both below and above threshold. We confirmed the validity of the analytical results through numerical simulations based on the positive-$P$, truncated Wigner, and truncated Husimi SDEs. The comparison well establishes the validity of the phase-space methods based on these SDEs and the validity of the analytical result. The $c$-number SDEs are essentially identical to the truncated Wigner SDE and valid for DOPOs below and above threshold. The analytical methods and insights obtained here for a 1D lattice of DOPOs with nearest-neighbor coupling are applicable to other cases, with measurement feedback\cite{[1]}, and with more complicated DOPO lattices with frustration\cite{[4]} or topological nontriviality of the loss spectrum. Similar approach is also applicable in searching for entanglement and quantum discord in optical networks with other nonlinear components in the weak noise limit. For example, in the coherent XY machine\cite{[51]} with dissipative coupling, the entanglement has not been considered. Some lasers, for example Raman lasers, are known to have photon number squeezed state (sub-Poissonian state) only far above the threshold\cite{[52],[53]}. The coherent XY machines would also satisfy entanglement criterion above the threshold, due to coupling of the photon number squeezed states. In a high-loss CIM based on optical delay-line coupling, entanglement among DOPO pulses disappear before reaching the threshold\cite{[2]}. It does, however, have quantum discord, indicating the existence of correlations among the quantum fluctuations of DOPO pulses below and above threshold. Such correlations disappear if we use a mean-field coupling\cite{[36]} that does not account for the fluctuations in the coupling fields. The methods in this paper are effective for weak noise limit, and will not be applied directly for large quantum noise case with $B \gtrsim \gamma_s$. When investigating global order or the impact of quantum tunneling\cite{[32]} in the large quantum noise case, pursuing novel numerical and analytical methods would be still important.

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Appendix A: Phase-space method

We can expand the density operator in the quantum master equation by using the positive-$P$, Wigner, and Husimi quasi-distribution functions. First, we discuss Glauber’s $P$ distribution function, which is represented as $\hat{\rho} = \int P(\alpha|\alpha)\langle\alpha|d^2\alpha$. For a DOPO, the Fokker-Planck equation of the $P$ function is derived as:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \alpha}[\gamma_\alpha \alpha - S\alpha^* + B|\alpha|^2 P] + \frac{1}{2} \frac{\partial^2}{\partial \alpha^2}[(S-B\alpha^2)P]+c.c..$$

(A1)

In Glauber’s $P$ distribution, the Fokker-Planck equation exists for a DOPO. The $P$ function is not positive, however, and it does not have a corresponding $\epsilon$-number SDE because of the difficulty in adding noise term to $d(\alpha^2)/dt$ without adding it to $d(|\alpha|^2)/dt$. The positive-$P$ representation is a modified Glauber’s $P$ number representation and is known to have a similar Fokker-Planck equation. We can write the distribution function in positive-$P$ representation as

$$\hat{\rho} = \int P(\alpha, \alpha^\dagger)|\alpha\rangle\langle\alpha|d^2\alpha d^2\alpha^\dagger.$$

(A2)
Here, \( \alpha \) and \( \alpha^\dagger \) are two independent complex numbers satisfying \( \langle \alpha^* \rangle = \langle \alpha \rangle \). Using Eq. (A2) for Eq. (5), we obtain the quantum-mechanical Fokker-Planck equation for \( P(\alpha, \alpha^\dagger) \). This Fokker-Planck equation for the positive-\( P \) representation is the same as that for Glauber’s \( P \) distribution, except for the replacement \( \alpha^* \rightarrow \alpha^\dagger \). Applying the Ito rule to the resulting Fokker-Planck equation, we can obtain the following \( c \)-number SDE for the signal mode [11]:

\[
\frac{d\alpha}{dt} = -\gamma_s\alpha + S\alpha^\dagger - B\alpha^\dagger \alpha^2 + \sqrt{S - B^2\xi_R}, \quad (A3)
\]

\[
\frac{d\alpha^\dagger}{dt} = -\gamma_s\alpha^\dagger + S\alpha - B\alpha^2\alpha^* + \sqrt{S - B^2\xi_R^\dagger}. \quad (A4)
\]

Here, \( \xi_R \) and \( \xi_R^\dagger \) are independent real-number random noise sources satisfying \( \langle \xi_R(t)\xi_R(t') \rangle = \delta(t - t') \) and \( \langle \xi_R^\dagger(t)\xi_R^\dagger(t') \rangle = \delta(t - t') \). This SDE under positive-\( P \) representation is defined with no truncation and is equivalent to the quantum master equation (5). In numerical simulation, however, these equations have some difficulty for large \( B/\gamma_s \). It is known that when \( B \) is large, positive-\( P \) calculation becomes unstable [54, 55]. This is because a negative photon number is allowed in positive-infinity because of two-photon absorption. In this paper, we thus consider the case of small \( B \) and a sufficiently slow adiabatic excitation schedule. In our simulations, the divergence problem was not observed.

As noted above, the SDEs with positive-\( P \) representation are equivalent to the Liouville equation [5]. On the other hand, for Eq. (5) the Wigner and the Husimi functions have more than third order derivatives in their Fokker-Planck equations. As these higher order derivatives are proportional to \( B \), we can neglect higher-order derivatives when \( B \) is small, and we can obtain truncated SDEs. In return for the possible incorrectness due to truncation, the truncated Wigner and Husimi SDEs represent the signal boson operator \( \hat{a} \) by only one \( c \)-number amplitude. The Wigner distribution function is written with a density matrix as the following [56]:

\[
W(\alpha) = \frac{1}{\pi^2} \int \text{tr} \rho e^{(\hat{a}^\dagger - \alpha^*) - (\hat{a} - \alpha)} d^2 \eta. \quad (A5)
\]

The Fokker-Planck equation can be obtained from the time derivative of this equation. The truncated Wigner SDE is then obtained from the Fokker-Planck equation after neglecting third-order derivatives and terms resulting from the Weyl ordering in the drift and diffusion terms with the assumption of \( B \ll \gamma_s \):

\[
\frac{d\alpha}{dt} = -\gamma_s\alpha + S\alpha^\dagger - B|\alpha|^2\alpha + \sqrt{\frac{\gamma_s}{2}} + B|\alpha|^2\xi_C. \quad (A6)
\]

We can see that this Wigner SDE resembles the \( c \)-number counterpart of the Heisenberg-Langevin equation after eliminating the pump mode [1], but Eq. (A6) is the \( c \)-number SDE in the Schrödinger picture rather than the \( q \)-number equation in the Heisenberg picture.

The Husimi distribution function is then written as \( Q(\alpha) = \frac{1}{\gamma} \langle \alpha | \hat{\rho} | \alpha \rangle \). The Fokker-Planck equation of the Husimi distribution function for degenerate two-photon absorption is presented in [35], and it has third- and fourth-order derivatives. Here, we consider the truncated Husimi Fokker-Planck equation of Eq. (5) only up to second-order derivatives. As this approximation assumes small \( B \), we also neglect small \( B \) dependent terms resulting from the anti-normal ordering of drift and diffusion terms. The truncated Husimi SDE is obtained through the Fokker-Planck equation as the following:

\[
\frac{d\alpha}{dt} = -\gamma_s\alpha + S\alpha^* - B|\alpha|^2\alpha + \sqrt{\gamma_s - \frac{3}{2}} + 2B|\alpha|^2\xi_C
\]

\[
+ i\sqrt{\xi_R} + \sqrt{B}\xi_{R^2}. \quad (A7)
\]

Here, \( \xi_C \) is a complex-number noise source with \( \langle \xi_C(t)\xi_C(t') \rangle = 2\delta(t - t') \), and \( \xi_R \) is a real-number noise source with \( \langle \xi_R(t)\xi_R(t') \rangle = \delta(t - t') \). We can use Eqs. (A3) and (A4) for the first principle numerical simulation of a CIM, and Eqs. (A6) and (A7) for the approximated simulation of a CIM with \( B \ll \gamma_s \).

Appendix B: Analytical method for single DOPO

Here, we derive the analytical solutions for the quantum noise of a CIM. We use the positive-\( P \) representation, while noting that the truncated Wigner and truncated Husimi representations can produce identical results in the weak noise limit. Below the threshold, we neglect the two-photon absorption terms with \( B \) in Eqs. (A3) and (A4). We can then obtain the following positive-\( P \) SDEs:

\[
\frac{d\alpha}{dt} = -\gamma_s\alpha + S\alpha^\dagger + \sqrt{S}\xi_R \quad (B1)
\]

\[
\frac{d\alpha^\dagger}{dt} = -\gamma_s\alpha^\dagger + S\alpha + \sqrt{S}\xi_R^\dagger \quad (B2)
\]

The term with \( B \) in the square root of Eqs. (A3) and (A4) can contribute even below the threshold when \( B \sim \gamma_s \), e.g., as \( d\langle \alpha^2 \rangle/dt = -(2\gamma_s + B)\langle \alpha^2 \rangle + \cdots \). When \( B \) is large, the saturation term \(-B\alpha^2\alpha^* \) in Eq. (A3) is also not negligible even below the threshold, because the rate of two-photon absorption is enhanced by the intensity correlation function \( g^{(2)}(0) = \langle \hat{\alpha}^2 \hat{\alpha}^2 \rangle \gamma_s \), and a squeezed vacuum state in a DOPO can have large \( g^{(2)}(0) \) proportional to \( 2|\alpha|/\gamma_s \). We can neglect the saturation term, however, by assuming that \( B \ll \gamma_s \). Below the threshold, the mean amplitude of the system is \( \langle \alpha \rangle = 0 \). Using the normalized excitation \( p = S/\gamma_s \), which is equal to \( \varepsilon/\varepsilon_{\text{thr}} \), we can obtain the following steady-state equation:

\[
\begin{pmatrix}
1 & -p & 0 \\
-p & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
|\Delta \alpha^2 \rangle \\
|\Delta \alpha \Delta \alpha^* \rangle \\
1
\end{pmatrix} = \frac{1}{p} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (B3)
\]
We assume that $\langle \Delta \alpha^2 \rangle$ is real, because from Eqs. (111) and (112), $\alpha$ and $\alpha^\dagger$ are real: the time development starts from the vacuum state $\alpha = \alpha^\dagger = 0$, and both the coefficients and random noises are real. We define the canonical coordinate and momentum as $\hat{X} = \frac{\hat{X} + \hat{P}}{\sqrt{2}}$ and $\hat{P} = \frac{\hat{X} - \hat{P}}{\sqrt{2}}$, respectively. In the positive-$P$ representation, the fluctuations are calculated as $\langle \Delta \hat{X}^2 \rangle = \langle \Delta \alpha^2 \rangle^2 + \langle \Delta \alpha^2 \rangle^2 + \frac{1}{2}$ or $\langle \Delta \hat{P}^2 \rangle = \langle \Delta \alpha^2 \rangle - \langle \Delta \alpha^2 \rangle + \frac{1}{2}$. In a DOPO below the threshold under the steady-state condition, these canonical components have the following fluctuations: $\langle \Delta \hat{X}^2 \rangle = \frac{1}{2} + \frac{p}{2(1+p)}$ and $\langle \Delta \hat{P}^2 \rangle = \frac{1}{2} - \frac{p}{2(1+p)}$. We can see that $\langle \Delta \hat{X}^2 \rangle$ has a singularity at the threshold. On the other hand, $\langle \Delta \hat{P}^2 \rangle = \frac{1}{4}$, which is half the vacuum fluctuation. Although squeezing exists above the threshold, the squeezing becomes weaker for large $p$ [30]. We can see that Eqs. (113) and (114) are reduced to an identical equation at the threshold (the limit $p \to 1$). For the canonical momentum $\hat{P}$, the mean-squared fluctuations are the same at $p = 1$ below and above the threshold. For the canonical coordinate $\hat{X}$, the mean-squared fluctuations at $p \to 1^-$ and $p \to 1^+$, correspond to the replacement $1 - p \to 2(p - 1)$.

Next, we consider the fluctuations of a DOPO above the threshold. We separate the signal amplitudes into the mean amplitudes $\langle \alpha \rangle = \langle \alpha^\dagger \rangle = \sqrt{\frac{1}{2}(p - 1)}$ and small fluctuations ( $\Delta \alpha = \alpha - \langle \alpha \rangle$ and $\Delta \alpha^\dagger = \alpha^\dagger - \langle \alpha^\dagger \rangle$ ). Such linearization treatment was previously applied in quantum nonlinear optics for degenerate two-photon absorption [30] and an above-threshold DOPO [27, 28]. We obtain the positive-$P$ SDE for the fluctuation part as follows.

$$\frac{d\Delta \alpha}{dt} = -\gamma_c \Delta \alpha + S \Delta \alpha^\dagger - 2B\langle \alpha \rangle^2 \Delta \alpha$$

$$-B\langle \alpha \rangle^2 \Delta \alpha^\dagger + \sqrt{S - B\langle \alpha \rangle^2}\xi_R.$$  

(B4)

$$\frac{d\Delta \alpha^\dagger}{dt} = -\gamma_c \Delta \alpha^\dagger + S \Delta \alpha - 2B\langle \alpha^\dagger \rangle^2 \Delta \alpha^\dagger$$

$$-B\langle \alpha^\dagger \rangle^2 \Delta \alpha + \sqrt{S - B\langle \alpha^\dagger \rangle^2}\xi_R.$$  

(B5)

Using the normalized excitation $p = \varepsilon/\varepsilon_{thr}$, we can then obtain the following equation under the steady-state condition:

$$\begin{bmatrix} 2p - 1 & -p \\ -1 & 2p - 1 \end{bmatrix} \begin{bmatrix} \langle \Delta \alpha^2 \rangle \\ \langle \Delta \alpha \rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}. $$  

(B6)

The squared fluctuations of the canonical coordinate and momentum are represented as $\langle \Delta \hat{X}^2 \rangle = 1 + \frac{1}{4(p - 1)}$ and $\langle \Delta \hat{P}^2 \rangle = \frac{1}{2} - \frac{p}{4(p - 1)}$, respectively. Above the threshold, the canonical coordinate and momentum monotonically reach the vacuum fluctuation. Although squeezing exists above the threshold, the squeezing becomes weaker for large $p$. We can see that Eqs. (113) and (114) are reduced to an identical equation at the threshold (the limit $p \to 1$). For the canonical momentum $\hat{P}$, the mean-squared fluctuations are the same at $p = 1$ below and above the threshold. For the canonical coordinate $\hat{X}$, the mean-squared fluctuations at $p \to 1^-$ and $p \to 1^+$, correspond to the replacement $1 - p \to 2(p - 1)$.

**Appendix C: Traveling-wave model of 1D lattice**

We also consider the traveling-wave model [5] of the dissipatively nearest-neighbor-coupled 1D lattice of DOPOs, shown in Fig. 5. This model uses traveling optical pulses in a single ring cavity with a nonlinear crystal, in which the pulses are coupled with an optical delay-line. We assume that the two beam splitters (BSs) have reflectance $R_B$, the two half-beam splitters (HBSs) have 50% reflectance, and the mirrors (Ms) have 100% reflectance. The setting shown in Fig. 5 has HBSs in the delay-line and is, as we show below, equivalent to the Liouvillian coupling of Eq. (31). We note that some models [4, 5] discussed in earlier papers are not equivalent to Eq. (31) because they have only one delay component. This traveling-wave model is easily implemented with the Heisenberg Langevin model or SDE for the Wigner representation.

In the Wigner representation, the $r$th signal pulse $\alpha_r$ is mixed with the vacuum noise $f_r$, at a beam splitter and split into the transmitted $\alpha_{T,r}$ and reflected $\alpha_{R,r}$ modes: $\alpha_{T,r} = \sqrt{1 - R_B}\alpha_r + \sqrt{R_B}f_r$ and $\alpha_{R,r} = \sqrt{R_B}\alpha_r - \sqrt{1 - R_B}f_r$. The reflected light is then separated by an HBS into $\alpha_{D,r} = \sqrt{1/2}f_{2r} + \sqrt{1/2}\alpha_{R,r}$ and $\alpha_{A,r} = \frac{1}{\sqrt{2}}f_{2r} - \frac{1}{\sqrt{2}}\alpha_{R,r}$, where $f_{2r}$ is the independent vacuum noise. Here, $\alpha_{A,r}$ has a 1-bit delay and $\alpha_{A,r}$ has a -1-bit delay. The optical delayline injects $\alpha_{A,r+1}$ and $\alpha_{D,r+1}$ into the $r$th pulse $\alpha_{T,r}$. $\alpha_{F,r} = \frac{1}{\sqrt{2}}\alpha_{D,r+1} - \frac{1}{\sqrt{2}}\alpha_{A,r+1}$ is the mixed injection field from the delayline. The combined field at the injection beam splitter is represented by $\alpha'_{s} = \sqrt{1-R_B}\alpha_{T,r} + \sqrt{R_B}\alpha_{F,r}$. Here, $f_{1r}$ and $f_{2r}$ are vacuum fields in the Wigner representation, satisfying $\langle f_{sr}'f_{br} \rangle = \frac{1}{2}\delta_{ab}\delta_{r,r'}$. Next, we assume that the DOPO cavity round-trip time is $\Delta t$, and that the delay-line’s reflectance is effectively simulated by a distributed coupling constant $J$, via $R_B = J\Delta t$. The optical delay-line coupling model of Eq. (31) is then represented as

$$\frac{d\alpha_r}{dt} = -J\alpha_r + \frac{1}{2}(\alpha_{r-1} + \alpha_{r+1}) + \sqrt{\frac{J}{2}}\xi_{C1,r} - \sqrt{\frac{J}{4}}\xi_{C1,r-1}$$

$$- \sqrt{\frac{J}{4}}\xi_{C1,r+1} - \sqrt{\frac{J}{4}}\xi_{C2,r+1} + \sqrt{\frac{J}{4}}\xi_{C2,r-1}.$$  

(C1)

Here, $\xi_{C1,r}$ and $\xi_{C2,r}$ come from the input noise of the BS and HBS respectively, and satisfy $\langle \xi_{C1,r}(t)\xi_{C2,r}(t') \rangle = 2\delta_{ab}\delta_{r,r'}(t - t')$. We can see that this traveling-wave
model produces the same diffusion terms in the Fokker-Planck equation, as Eq. (33) does. We assume here that $R_B = J \Delta t \ll 1$. To satisfy Duan’s criterion at the threshold, with $J/\gamma_s > 1$, we need $\gamma_s \Delta t < J \Delta t \ll 1$. Therefore, the cavity lifetime ($\gamma_s^{-1}$) must be much longer than the round-trip time $\Delta t$. 
