Graph topology resulting from addition and deletion of nodes determined by random walk

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Abstract. How the power-law form of the degree distribution of networks can survive in the equilibrium state of number of nodes is an open question, since the deletion of nodes seriously affects the degree distribution. In this paper, we introduce a network evolution rule based on a random walk to describe the replacement of nodes in networks. Numerical results show that model assumptions support the power-law form of degree distribution even when the replacement of vertices achieves an equilibrium state of the number of vertices, and the distribution is described by the distribution of the maximum degree that each node experiences in its life. We showed additionally that the reduction of the addition probability of nodes causes an edge condensation to hub nodes. A novel feature of the model is that the visit frequency and recurrence property of the walker to a vertex determines the degree and lifetime of the vertex in an equilibrium state of the number of vertices.

1. Introduction
Numerous empirical studies of networked systems have revealed structural similarities in different networks [1]. The existence of hub nodes (the scale-free property) is a typical property of such common structures in real networks. The simplest way to reproduce the scale-free property may be the growth property of networks with the preferential attachment rule[2]. However, the evolution of networks in the real world includes the removal of nodes and links, and some networked systems show attenuation rather than growth [3]. Hence, it is worth studying how the power-law can survive in a condition where nodes and links can be deleted.

There are several formulations of model networks which consider replacement of nodes [4, 5, 6]. The simplest model of these model networks is a case of preferential attachment and random deletion of nodes [5]. However, it has been reported that with this case, the scale-free property disappears as the growth rate of the network becomes zero due to the random deletion of nodes. In addition, we showed in a previous work that the scale-free property is sustained by the addition of internal links even if the hub nodes are intentionally deleted [7]. However, this mechanism leads to an unrealistic result in that most nodes tend to become one of the hub nodes just before they are deleted.

In this paper, we propose a model network in which the addition and deletion of nodes are governed by the movement of a random walker in the network. In a preceding work, we proposed a growth model of networks generated by a random walk on the graph, and showed that a fractal and scale-free network can be produced by the model [8]. Newly introduced in this paper is a deletion process of nodes whose lifetime is determined by the latest visit of the walker. The
detailed description of the model including several model parameters will be given in the next section. In this model, the visit frequency and recurrence property of the walker to a node determines the degree and lifetime of the node. We will present in section 3 numerical results for the time dependence of the number of nodes, time dependence of the maximum degree, time dependence of degrees, distribution of degree, and distribution of the lifetime of nodes. As a result, we will show that the power-law form of degree distribution can be sustained by the random movement of a walker even if the number of nodes is in the equilibrium state. As will be shown, the power-law is a result of the distribution of the maximum degree that each node experiences in its life.

2. Model
The examined model is defined as follows (the processes "Initial graph" and "Addition of vertices and edges" are proposed in ref.[8], and process "Deletion of vertices and edges" is newly introduced in this paper.).

Initial graph A complete graph consisting of three vertices is prepared, and a random walker is located at one vertex at initial time \( t = 0 \).

Addition of vertices and edges At each time \( t \), the random walker moves randomly to one of the nearest neighboring vertices (an integer 1 is added to \( t (t = t + 1) \) at next step). For each step of the walker, a vertex and edges are added to the graph by the following rule. A new vertex is created, and the vertex is connected to the two vertices at which the walker stays at times \( t \) and \( t - 1 \) with probability \( p_v \). Additionally, if the vertex where the walker stays at time \( t \) and the vertex where the walker stayed at \( t - 2 \) are not yet joined, a shortcut is created between them with probability \( p_e \). If not, an edge is not added to the graph at this moment. The creation of a loop is also prohibited.

Deletion of vertices and edges A remaining life-span \( L_t = \tau_0 \) is given to the newly added vertex \( i \) at birth time \( t \). The remaining life-span of an existing vertex \( i \) is also updated to the initial value \( L_t = \tau_0 \), if the walker visits vertex \( i \) at time \( t \). At each time step, however, the number 1 is subtracted from the remaining life-span \( L_t \) assigned to each vertex \( i \) (\( L_{t+1} = L_t - 1 \) for all vertices). Vertices whose remaining life-span reaches 0 and edges incident to those vertices are removed.

As the result, each vertex can survive at least time interval \( \tau_0 \) even if the walker does not visit the vertex. In the following sections, we will denote an extended lifetime of a vertex by the visit of the walker as \( \Delta L \). Note that \( \Delta L \) is equal to a time interval between the first visit and last visit of the walker to the vertex.

3. Numerical results
In the numerical calculations, we fixed the value of \( \tau_0 \) as 5,000, and examined the dependence of the results on the values of \( p_v \) and \( p_e \). The values of \( p_v \) and \( p_e \) strongly affect the results, while the value of \( \tau_0 \) mainly determines the size of the resulting graph.

3.1. Stability of power-law form of degree distribution
In graphs in which the replacement of vertices is allowed, a stable degree distribution needs, at least, a balance between the addition and deletion of vertices. Fig. 1 presents the time dependence of the number of vertices and the maximum degree for three cases. When the values of \( p_v \) and \( p_e \) are near to 1 (\( p_v = p_e = 0.8 \)), the number of vertices shows a stable behavior with time. As the value of \( p_v \) becomes small, however, the number of vertices shows a large fluctuation around the mean value (Fig. 1(a)). Further decrease in \( p_v \) causes reduction in the number of vertices from the value initially obtained. The number of vertices decreases until an
equilibrium value is reached. On the contrary, the maximum degree gradually increases with time and approaches the number of vertices in the graph (Fig. 1(b)). This result implies that the hub vertex continues to gain new edges and, in the equilibrium state, links to almost all other vertices.

**Figure 1.** Typical numerical results for time dependence of number of vertices when $\tau_0 = 5,000$ for cases (a) $p_v = p_e = 0.8$ and $p_v = p_e = 0.3$, and (b) $p_v = p_e = 0.1$. The dotted lines indicate the time dependence of the maximum degree for each case.

**Figure 2.** Numerical results for degree $k$ versus number of vertices $V(k)$ with degree $k$ at different times. (a)Log-log plot of $k$ versus $V(k)$ for $p_v = p_e = 0.3$ at $t = 23,320(\bigcirc)$ and $t = 333,320 (\triangle)$. The guide line indicates a power-law form $V(k) \sim k^{-\gamma}$ with $\gamma = 2.1$. The distribution of maximum degree experienced by each vertex, which had been deleted by $t = 333,320$ in their life, is also plotted (\lozenge) with a guide $\gamma = 2.3$. (b)Log-log plot of $k$ versus $V(k)$ for $p_v = p_e = 0.1$ at $t = 25,000(\bigcirc)$, $t = 100,000 (\triangle)$, and $t = 1,000,000 (\lozenge)$. The guide lines (solid lines) indicate power-law exponents with $\gamma = 1.83$, $\gamma = 3.22$ and $\gamma = 5.23$, respectively.

For cases with the same parameter values in Fig. 1(a), the degree distribution maintains a power-law form in the equilibrium state of the number of vertices (Fig. 2(a)). In other cases for which $p_v$ is small enough to cause the continuous increase in the maximum degree as seen in Fig. 1(b), the form of degree distribution changes from a power-law form to a form with a peak. The location of the peak tends to become large with time, while the clear power-law form gradually disappears (Fig. 2(b)). This result implies that edges are condensed into a group of vertices with large degrees which link each other. Note that the edge condensation to hub vertices cannot be realized unless $p_e > 0$, because only the addition process of vertices never link hub vertices with each other.
3.2. Time dependence of degrees

Fig. 3 presents typical results for the time dependence of degrees of each vertex. When the value of \( p_v \) is large enough to support the power-law form of degree distribution, each vertex disappears after the increase and decrease in the degree (Fig. 3(a)). Fig. 3(a) also illustrates that only a few vertices can gain a large degree, while most vertices disappear after gaining only a few edges. We calculated the distribution of the maximum degree that each vertex experiences in their life from the data. The result (Fig. 2(a)) shows that the distribution of the maximum degree that each vertex experiences in its life is similar to the degree distribution obtained at a certain time.

When \( p_e \) is positive and the value of \( p_v \) is too small to sustain the power-law form of degree distribution, a fraction of vertices continues to gain new links without disappearing, while most vertices disappear after gaining only a few links (Fig. 3(b)). It is obvious that the vertices with the monotonic increase in degree correspond to the peak found in the degree distribution in Fig. 2(b).

As expected from Figs. 3(a) and (b), the lifetime of vertices is distributed in a wide range. Numerical results for the distribution of extended lifetime \( \Delta L \) (Fig. 4) indicate that the distribution obeys a power-law \( \sim \Delta L^{-\beta} \) with exponent \( \beta = 1 \) in a certain range of \( \Delta L \). When the values of \( p_v \) and \( p_e \) are near to 1, there is a threshold at which the distribution of \( \Delta L \) becomes destroyed faster than the power-law. As the values of \( p_v \) becomes small, however, some vertices are able to survive longer than expected from the power-law. Further decrease in \( p_v \) leads to further extension of lifetime, corresponding to the edge condensation.

We assumed in the model that the lifetime of vertices would be lengthened by a walker’s visit. According to this assumption, vertices with a larger degree have a longer lifetime than vertices with a smaller degree, because a large degree is the result of frequent visits of the walker that extend the lifetime of vertices. However, this assumption is not always needed to support the power-law form of the degree distribution. As a trial, we examined the degree distribution in the condition where a walker’s visits to vertices do not extend the lifetime of vertices. In this case, all vertices are deleted with age \( \tau_0 \). As indicated in Fig. 5, the same power-law form of degree distribution can be obtained even if all vertices are deleted with age \( \tau_0 \). The effect of the extension of the lifetime of vertices by a walker’s visits is to cause edge condensation to hub vertices, and protection of vertices where the walker stays. (We confirmed numerically that the graph hardly evolves because of the deletion of vertices where the walker stays, if the extension of the lifetime of the vertices by the walker’s visits is not assumed, when \( p_v = p_e = 0.1 \).) As mentioned in the previous subsection, positive \( p_v \) is also a necessary condition for the edge condensation. Fig. 5 also shows that the edge condensation cannot be detected even if \( p_v \) is small (\( p_v = 0.1 \), if \( p_e = 0 \).

![Figure 3. Typical numerical results for time dependence of degree of each vertex for cases, (a) \( p_v = p_e = 0.3 \) and (b) \( p_v = p_e = 0.1 \).](image-url)
4. Summary

We numerically showed that a simple mechanism based on the movement of a random walker sustains the scale-free property of a graph even when the replacement of vertices achieves an equilibrium state of the number of vertices. In the model, for each step of the walker’s movement, a roundabout path via a new vertex and a shortcut are formed with a probability \( p_v \) and \( p_e \), respectively, and the lifetime of each vertex is lengthened by the walker’s visit. These assumptions describe a situation in which the visit frequency and recurrence property of the walker to a vertex determine the degree and lifetime of the vertex. Rank of degrees is naturally generated by this mechanism. As a result, the power-law form of degree distribution is described by the distribution of the maximum degree that each vertex experiences in its life. This result is contrast to a result for a preferential linking rule by which all vertices gain a large degree before they are deleted [7], although the preferential linking rule also can sustain the power-law form of degree distribution in an equilibrium state of number of vertices.

The distribution of extended lifetime \( \Delta L \) is governed by a power-law with exponent 1 in a certain range of \( \Delta L \). \( \Delta L \) is equal to a time interval between the first visit and last visit of the walker to the vertex. As \( p_v \) becomes smaller than a certain value, however, the lifetime of a fraction of vertices is extended anomalously, and the graph shrinks to a nearly complete graph as time passes. Note, however, that extension of lifetime of vertices by the walker’s visit is not a necessary assumption for the maintenance of the power-law form of degree distribution. The main effects of the extension of the lifetime of vertices by the walker’s visit include the protection of vertices where the walker stays and the generation of the edge condensation to hub vertices. The edge condensation to hub vertices requires the extension of the lifetime of vertices by the walker’s visit, a small \( p_v \), and a positive \( p_e \). If \( p_v \) is larger than a certain value, edge condensation does not occur even if \( p_e = 1 \).

[1] Dorogovtsev S N and Mendes J F F 2002 Adv. Phys. 51 1079
[2] Barabási A-L and Albert R 1999 Science 286 509
[3] Barabási A-L 2016 Network Science (Cambridge: Cambridge University Press) chapter 6
[4] Davidsen J, Ebel H, and Bornholdt S 2002 Phys. Rev. Lett. 88 128701
[5] Moore C, Ghoshal G, and Newman M E J 2006 Phys. Rev. E 74 036121
[6] Miura W, Takayasu H, and Takayasu M 2012 Phys. Rev. Lett 108 168701
[7] Ikeda N 2017 Journal of Physics: Conf. Series 936 012039
[8] Ikeda N 2019 Physica A 521 424