Homogenization of linear parabolic equations with a certain resonant matching between rapid spatial and temporal oscillations in periodically perforated domains

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Abstract

In this article, we study homogenization of a parabolic linear problem governed by a coefficient matrix with rapid spatial and temporal oscillations in periodically perforated domains with homogeneous Neumann data on the boundary of the holes. We prove results adapted to the problem for a characterization of multiscale limits for gradients and very weak multiscale convergence.

Keywords: Homogenization, two-scale convergence, multiscale convergence, periodically perforated domains

MSC 2000: 35B27; 35K10

1 Introduction

Homogenization theory deals with the question of finding effective properties and microvariations in heterogeneous materials. However, it is difficult to handle the rapid periodic oscillations of coefficients that govern partial differential equations describing processes in such materials. In this paper, we study a parabolic problem with a certain resonant matching between rapid oscillations in space and time in periodically perforated domains with homogeneous Neumann data on the boundary of the holes. Homogenization for linear parabolic problems with rapid oscillations of similar kind as in this paper was achieved already by Bensoussan, Lions and Papanicolaou in [7] using asymptotic expansions for domains without perforations. See also the pioneering work [11] from 1977 by Colombini and Spagnolo, where homogenization of linear parabolic equations with rapid spatial oscillations is performed. A further development of parabolic homogenization problems applying techniques of two-scale convergence type was presented by Holmbom in 1997, see [24], where the first compactness result of
very weak multiscale convergence type was shown for one rapid scale in space and time each. A similar result with the setting of $\Sigma$-convergence was obtained in 2007 by Nguetseng and Woukeng in [31]. Later in [20] from 2010 was proven a compactness result for the case with $n$ well-separated spatial scales by Flodén et. al. Multiscale convergence techniques for linear parabolic problems for two rapid time scales with one of them identical to the single rapid spatial scale were achieved by Flodén and Olsson in 2007, see [22]. In 2009, these results were extended by Woukeng, who studied non-linear parabolic problems with the same choice of scales in [41]. Also [37], by Persson, deals with monotone parabolic problems, but with an arbitrary number of temporal microscales, where none of them has to be identical with the rapid spatial scale or even has to be a power of $\varepsilon$. In [21] we return to the case of linear parabolic homogenization for arbitrary numbers of spatial and temporal scales benefitting from the concept of jointly separated scales introduced in [35].

Perforated domain means facing a further difficulty of a different kind than for oscillating coefficients. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ and $\Omega_{\varepsilon}$ a corresponding perforated domain with small holes which occur with a period $\varepsilon$. Advanced extension techniques to maintain a priori estimates in perforated domains for a linear elliptic problem with Neumann data on the boundary of the perforations is used by Cioranescu and Saint Jean Paulin in [9]. Examples of further developed extension techniques for a larger class of perforated domains is presented by Acerbi et. al. in [1]. In [5] and [3] Allaire developed methods which are independent of advanced extension techniques. [3] introduces methods of two-scale convergence type and these methods we adapt to the time-dependent problem in the present paper. With inhomogeneous Neumann data on the boundary of the holes the problem becomes more complicated as the assumptions must be adapted to the fact that the area of interface between holes and domain increases when $\varepsilon$ decreases, see [10], [13].

An early study of evolution problems in perforated domains is found in [14], where a parabolic problem with fast oscillations in one spatial scale is studied using advanced extension techniques and in a slightly more general setting in [15]. In 2016 Donato and Yang [18] performed a generalization of [14] by using the time-dependent unfolding method adapted to perforated domains. [14], [15] and [18] deal with homogenous Neumann data on the boundary of the holes. See also [2]. In e.g. [38] the case of non-homogenous Neumann data is studied for a nonlinear parabolic problem with oscillations in one rapid spatial scale.

In this paper we study homogenization of the parabolic linear problem with spatial and temporal oscillations

$$\partial_t u_\varepsilon(x,t) - \nabla \cdot \left( A \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u_\varepsilon(x,t) \right) = f_\varepsilon(x,t) \text{ in } \Omega_\varepsilon \times (0,T),$$

$$u_\varepsilon(x,t) = 0 \text{ on } \partial \Omega \times (0,T),$$

(1)

$$A \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u_\varepsilon(x,t) \cdot n = 0 \text{ on } (\partial \Omega_\varepsilon - \partial \Omega) \times (0,T),$$

$$u_\varepsilon(x,0) = u_0(x) \text{ in } \Omega_\varepsilon,$$
where $f_{\varepsilon} \in L^2(\Omega_{\varepsilon} \times (0, T))$ and $u_{\varepsilon}^0 \in L^2(\Omega_{\varepsilon})$. We develop a method without nontrivial extensions that generalize the approach in [3] and bring forth a gradient characterization adapted to the problem. In particular, we show a result of very weak multiscale convergence type and perform the homogenization procedure for the problem. [19] is sharing similarities with our problem but includes methods that are based on nontrivial extension techniques. See also [17].

The paper is organized as follows. In Section 2 we give a brief description of two-scale convergence and its generalization to cases with larger numbers of scales, see Definition 2 and Definition 12. We take a look at the recently developed idea of very weak multiscale convergence, see Definition 15. Section 3 is dedicated to the multiscale convergence for sequences of time-dependent functions in perforated domains and the answering concept for very weak multiscale convergence with one rapid scale in space and time each. We define perforated domains $\Omega_{\varepsilon}$. Then we prove in Proposition 18 an essential result for the regularity of the $(2,2)$-scale limit for bounded sequences in $L^2(0, T; H^1(\Omega_{\varepsilon}))$. In Theorem 23 we find a characterization of the $(2,2)$-scale limit for $\{\nabla u_{\varepsilon}\}$ under certain assumptions and in Corollary 24 we consider a version of very-weak multiscale convergence for the same choice of scales. Finally, in Section 4 we state a homogenization result which is proven by applying the results from Section 3.

**Notation 1** Below we introduce a list of sets and function spaces.

- $\Omega$: Open bounded subset of $\mathbb{R}^N$ with a smooth boundary.
- $\Omega_{\varepsilon}$: A domain with small holes situated periodically in $\Omega$. See more about notation for perforated domains in Section 3.
- $Y$: The unit cube $(0, 1)^N$.
- $Y^*$: An open subset of $Y$.
- $E_{Y^*}^g(Y^*)$: The $Y^*$ periodic extension of $Y^*$ infinitely along all principal directions of $\mathbb{R}^N$.
- $S$: The intervall $(0, 1)$.
- $Y_{n,m}$: The set $Y^n \times S^m$.
- $A, B$: Subsets of $\mathbb{R}^N$.
- $G(A)$: A space of real valued functions defined on $A$.
- $G(A)/\mathbb{R}$: The space of functions $\{u \in G(A) \mid \int_A u(y) dy = 0\}$.
- $D(\Omega)$: The space of $C^\infty(\Omega)$ - functions with compact support in $\Omega$.
- $C^\infty_0(Y)$: The space of $Y$-periodic functions in $C^\infty(\mathbb{R}^N)$.
- $H^1_2(Y)$: The closure of $C^\infty_0(Y)$ with respect to the $H^1(Y)$ - norm.
- $C^\infty_0(Y^*)$: $C^\infty(E_{Y^*}^g(Y^*))$ - functions that are periodic with respect to $Y^*$.
- $D_2(Y^*)$: The functions in $C^\infty_0(Y^*)$ with support contained in $E_{Y^*}^g(Y^*)$.
- $H^1_2(Y^*)$: $H^1_{loc}(E_{Y^*}^g(Y^*))$ - functions that are periodic with respect to $Y^*$.
- $L^2(a, b; G(A))$: The set of functions $\{u: (a, b) \rightarrow G(A) \mid \int_a^b \|u\|_{G(A)}^2 dt < \infty\}$.
- $D(B; G(A))$: The space of infinitely differentiable functions $\{u \mid u : B \rightarrow G(A)\}$ with compact support in $B$.  

3
2 The two-scale convergence

The concept was originally introduced by Nguetseng [30] and later in 90’s further developed by Allaire [3]. A review of classical two-scale convergence from 2002 can be found in [26]. Yet it works for more than two scales, see [4]. A quite attractive generalization of two-scale convergence, scale convergence, is introduced by Mascarenhas and Toader in [27]. Moreover, [39] adapted ideas of scale-convergence from [27] and from a different kind of generalization of two-scale convergence in [25] to develop the concept of $\lambda$-scale convergence. See also [36]. Moreover, Nguetseng also introduced a quite sophisticated concept, $\Sigma$-convergence, which goes beyond the periodic setting, see e.g. [32]. Another important improvement in the two-scale convergence theory was made by Pak [33] in 2005. He adapted it to differential forms and manifolds. We would also like to mention [8].

Let us begin with the classical definition by Nguetseng and Allaire which was shown for the case of bounded sequences in $L^2$.

**Definition 2** We say that a bounded sequence of functions $\{u_\varepsilon\}$ in $L^2(\Omega)$ two-scale converges to a limit $u_0 \in L^2(\Omega \times Y)$, if

$$\lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon(x)v(x, \frac{x}{\varepsilon}) \, dx = \int_{\Omega} \int_{Y} u_0(x,y)v(x,y) \, dy \, dx$$

holds for all $v \in D(\Omega; C_0^\infty(Y))$. We write $u_\varepsilon \rightharpoonup u_0$. If, in addition,

$$\lim_{\varepsilon \to 0} \|u_\varepsilon(x) - u_0(x, \frac{x}{\varepsilon})\|_{L^2(\Omega)} = 0,$$

we say that $\{u_\varepsilon\}$ two-scale converges strongly to $u_0$.

**Remark 3** The strong two-scale convergence is also called a corrector type result, according to the vocabulary of homogenization.

**Theorem 4** Two-scale limits are unique.

**Proof.** See reasoning after Definition 1 in [26].

**Theorem 5** Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then $\{u_\varepsilon\}$ is compact with respect to the two-scale convergence, i.e. there exist a subsequence $\{u_{\varepsilon'}\}$ two-scale converging to a function $u_0 \in L^2(\Omega \times Y)$.

**Proof.** See proof of Theorem 4.1 in [29].

The next theorem shows relations between norms for weak $L^2(\Omega)$-limits and two-scale limits.

**Theorem 6** Let $\{u_\varepsilon\}$ be a sequence in $L^2(\Omega)$ that two-scale converges to $u_0 \in L^2(\Omega \times Y)$. Then

$$\liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^2(\Omega)} \geq \|u_0\|_{L^2(\Omega \times Y)} \geq \|u\|_{L^2(\Omega)},$$
where
\[ u_\varepsilon(x) \to u(x) = \int_Y u_0(x, y) dy \text{ in } L^2(\Omega). \]

**Proof.** See Theorem 10 in [26]. ■

Phenomenons of two-scale convergence type may appear under certain conditions also when neither of the involved sequences originates from an admissible test function.

**Theorem 7** Let \( \{ u_\varepsilon \} \) be a bounded sequence in \( L^2(\Omega) \) which two-scale converges to \( u_0 \in L^2(\Omega \times Y) \) and assume that
\[ \lim_{\varepsilon \to 0} \| u_\varepsilon(x) \|_{L^2(\Omega)} = \| u_0(x, y) \|_{L^2(\Omega \times Y)}. \]

Then
\[ \lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)v(x)\tau(x)dx = \int_\Omega \int_Y u_0(x, y)v(x, y)\tau(x)dydx \]
for any \( \tau \in D(\Omega) \).
Moreover, if the \( Y \)-periodic extension of \( u_0 \) belongs to \( L^2(\Omega; C_1(Y)) \), then
\[ \lim_{\varepsilon \to 0} \left\| u_\varepsilon(x) - u_0(x, x/\varepsilon) \right\|_{L^2(\Omega)} = 0. \]

**Proof.** See Theorem 11 in [26]. ■

The two-scale convergence can be used in many applications due to the compactness property.

**Remark 8** In [3] Allaire demonstrates proof of Theorem 5 for the test functions from \( L^2(\Omega; C_1(Y)) \).

We also define the concept of very weak two-scale convergence using a smaller class of test functions than usual two-scale convergence.

**Definition 9** Let \( \{ \varphi_\varepsilon \} \) be a sequence of functions in \( L^1(\Omega) \). We say that \( \{ \varphi_\varepsilon \} \) two-scale converges very weakly to \( \varphi_1 \in L^1(\Omega \times Y) \) if
\[ \lim_{\varepsilon \to 0} \int_\Omega \varphi_\varepsilon(x)v_1(x)v_2 \left( \frac{x}{\varepsilon} \right) dx = \int_\Omega \int_Y \varphi_1(x, y)v_1(x)v_2(y) dydx \]
for all \( v_1 \in D(\Omega) \) and all \( v_2 \in C^\infty_2(Y)/\mathbb{R} \) and
\[ \int_Y \varphi_1(x, y)dy = 0. \]
We write
For \( \{u_\varepsilon\} \) bounded in \( H^1(\Omega) \) there is a characterization of the two-scale limit for \( \{\nabla u_\varepsilon\} \) which means that, up to a subsequence,

\[
\nabla u_\varepsilon(x) \rightharpoonup \nabla u(x) + \nabla_y u_1(x, y),
\]

where \( u \) is the weak \( H^1(\Omega) \)-limit for \( \{u_\varepsilon\} \) and \( u_1 \in L^2(\Omega, H^1_0(Y)/\mathbb{R}) \).

In fact, there is a connection between the very weak two-scale limit for \( \{\varepsilon^{-1} u_\varepsilon\} \) and the two-scale limit for \( \{\nabla u_\varepsilon\} \). It is possible to establish a compactness result for a sequence \( \{\varepsilon^{-1} u_\varepsilon\} \), though it is not bounded in any Lebesgue space. We have that

\[
\varepsilon^{-1} u_\varepsilon \rightharpoonup \nabla u_1,
\]

up to a subsequence, if \( \{u_\varepsilon\} \) is bounded in \( H^1_0(\Omega) \).

Just as with regular two-scale convergence we can generalize this result to be valid for several scales and to the evolution setting. See e.g. [21].

For regular multiscale convergence, we have certain assumptions about how the scales relate to each other. Assuming that the scales in the lists \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) and \( \{\varepsilon'_1, \ldots, \varepsilon'_m\} \) are microscopic, i.e. \( \varepsilon_k, \varepsilon'_k \) goes to zero when \( \varepsilon \) does. We say, according to the definition in [4], that the scales in one list are separated if

\[
\lim_{\varepsilon \to 0} \varepsilon_k + 1 = 0
\]

and well-separated if there exists a positive integer \( \ell \) such that

\[
\lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k}^\ell = 0
\]

where \( k = 1, \ldots, n - 1 \). For the evolution setting we need the equivalent for multiscale convergence with time-dependent effect. Following [37] we provide the concept in the next definition.

**Definition 10** Let \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) and \( \{\varepsilon'_1, \ldots, \varepsilon'_m\} \) be lists of well-separated scales (see [4]). Collect all elements from both lists in one common list. If from possible duplicates, where by duplicates we mean scales which tend to zero equally fast, one member of each such pair is removed and the list in order of magnitude of all the remaining elements is well-separated, the lists \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) and \( \{\varepsilon'_1, \ldots, \varepsilon'_m\} \) are said to be jointly well-separated. Moreover, \( y^n = (y_1, y_2, \ldots, y_n) \) and \( s^m = (s_1, s_2, \ldots, s_m) \).

**Remark 11** For some more examples and a technically strict definition, see Section 2.4 in [37].

Below we provide a characterization of multiscale limits for gradients.
**Definition 12** A sequence \( \{u_\varepsilon\} \) in \( L^2(\Omega \times (0, T)) \) is said to \((n + 1, m + 1)\)-scale converge to \( u_0 \in L^2(\Omega \times (0, T) \times Y_{n,m}) \) if
\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega u_\varepsilon(x,t)v \left( x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}, \frac{t}{\varepsilon_1}, \ldots, \frac{t}{\varepsilon_m} \right) \, dx \, dt = \int_0^T \int_\Omega \int_{Y_{n,m}} u_0(x,t,y^n,s^m)v(x,t,y^n,s^m) \, dy^n \, ds^m \, dx \, dt
\]
for all \( v \in L^2(\Omega \times (0, T); C_2(Y_{n,m})) \). This is denoted by
\[
u_\varepsilon(x,t) \Rightarrow_{n+1,m+1} u_0(x,t,y^n,s^m).
\]

We give the compactness result for \((n + 1, m + 1)\)-scale convergence.

**Theorem 13** Let \( \{v_\varepsilon\} \) be a bounded sequence in \( L^2(\Omega_T) \) and assume that the lists \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) and \( \{\varepsilon_1', \ldots, \varepsilon_m'\} \) are jointly separated. Then there exists a function \( u_0 \in L^2(\Omega_T \times Y_{n,m}) \) such that
\[
u_\varepsilon(x,t) \Rightarrow_{n+1,m+1} u_0(x,t,y^n,s^m),
\]
up to a subsequence.

**Proof.** See Theorem 17 in [21] and also Theorem 2.66 in [37].

**Theorem 14** Let \( \{u_\varepsilon\} \) be a bounded sequence in \( W^1_2(0,T; H^1_2(\Omega), L^2(\Omega)) \) and suppose that the lists \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) and \( \{\varepsilon_1', \ldots, \varepsilon_m'\} \) are jointly well-separated. Then there exists a subsequence such that
\[
\begin{align*}
\nu_\varepsilon(x,t) &\to u(x,t) \text{ in } L^2(\Omega \times (0, T)), \\
u_\varepsilon(x,t) &\to u(x,t) \text{ in } L^2(0,T; H^1_0(\Omega))
\end{align*}
\]
and
\[
\nabla \nu_\varepsilon(x,t) \Rightarrow_{n+1,m+1} \nabla u(x,t) + \sum_{k=1}^n \nabla y_k u_k(x,t,y^k,s^m),
\]
where \( u \in W^1_2(0,T; H^1_0(\Omega), L^2(\Omega)) \), \( u_1 \in L^2(\Omega \times (0, T) \times S^m; H^1_0(Y_1)/\mathbb{R}) \) and \( u_k \in L^2(\Omega \times (0, T) \times Y_{k-1,m}; H^1_0(Y_k)/\mathbb{R}) \) for \( k = 2, \ldots, n \).

**Proof.** See Theorem 2.74 in [37] or the Appendix of [21].

We define very weak evolution multiscale convergence.

**Definition 15** A sequence \( \{\varphi_\varepsilon\} \) in \( L^1(\Omega \times (0, T)) \) is said to \((n + 1, m + 1)\)-scale converge very weakly to \( \varphi_0 \in L^1(\Omega \times (0, T) \times Y_{n,m}) \) if
\[
\int_0^T \int_\Omega \int \varphi_\varepsilon(x,t) v_1 \left( x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n-1} \right) v_2 \left( \frac{x}{\varepsilon_n} \right) c \left( t, \frac{t}{\varepsilon_1}, \ldots, \frac{t}{\varepsilon_m} \right) \, dx \, dt \\
\to \int_0^T \int_\Omega \int_{Y_{n,m}} \varphi_0(x,t,y^n,s^m) v_1(x,y^n) v_2(y^n) c(t,s^m) \, dy^n \, ds^m \, dx \, dt
\]
for all \( v_1 \in D(\Omega; C^\infty_4(Y^{n-1})) \), \( v_2 \in C^\infty_4(Y_n)/\mathbb{R} \) and \( c \in D(0, T; C^\infty_2(S^m)) \).

A unique limit is provided by requiring that
\[
\int_{Y_n} \varphi_n(x, t, y^n, s^m) dy_n = 0.
\]

We write
\[
\varphi_n(x, t) \xrightarrow{wu} \varphi_n(x, t, y^n, s^m).
\]

Remark 16 Note that the decoupling of the function \( v_2 \) governed by fastest spatial variable from \( v_1 \) depending on the remaining local spatial variables and the global variable \( x \) is important when proving the compactness result in Theorem 17 below because \( v_2 \) has to be found by means of a certain kind of Poisson equation.

We are now ready to state compactness result for very weak \((n+1, m+1)\)-scale convergence.

**Theorem 17** Let \( \{u_\varepsilon\} \) be a bounded sequence in \( W^{1,2}(0, T; H^1_0(\Omega); L^2(\Omega)) \) and assume that the lists \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) and \( \{\varepsilon'_1, \ldots, \varepsilon'_m\} \) are jointly well-separated. Then there exists a subsequence such that
\[
\frac{u_\varepsilon(x, t)}{\varepsilon_n} \xrightarrow{wu} u_n(x, t, y^n, s^m),
\]
where, for \( n = 1 \), \( u_1 \in L^2(\Omega \times (0, T) \times S^m; H^1_2(Y_1)/\mathbb{R}) \) and, for \( n = 2, 3, \ldots \), \( u_n \in L^2(\Omega \times (0, T) \times Y_{n-1,m}; H^1_2(Y_n)/\mathbb{R}) \).

**Proof.** See Theorem 8 in [21].

In Sections 3-4 we consider a special case of very weak multiscale convergence, where the fast spatial scale is \( \varepsilon_1 = \varepsilon \) and the rapid temporal scale is chosen as \( \varepsilon'_1 = \varepsilon^r \), \( r > 0 \), \( n = m = 1 \) and make the necessary modifications to suit homogenization in perforated domains.

### 3 An adaptation to perforated domains

By knowing that two-scale convergence can handle homogenization problems in perforated domains, let us define periodically perforated domains \( \Omega_\varepsilon \) in a setting suitable for our problem.

We define \( \Omega \) as an open bounded domain in \( \mathbb{R}^N \), \( N \geq 2 \), with smooth boundary \( \partial \Omega \). We choose \( Y_1^\varepsilon \), to be disjoint open cubes with side-length \( \varepsilon \) such that
\[
\Omega \subset \bigcup_{i=1}^{N(\varepsilon)} Y_i^\varepsilon.
\]

We need to define
\[
\mathcal{A} = \{ i \in \mathbb{N} | Y_i^\varepsilon \cap \partial \Omega \neq \emptyset \}.
\]
We let $Y^H \subset \subset Y$, where $Y^H$ has smooth boundary; $Y^* = Y - Y^H$ and $Y^*_i$ are miniatures with side-length $\varepsilon$ of $Y^*$ such that $Y^*_i \subset Y^*_i$. We define

$$\hat{\Omega}_\varepsilon = \left( \bigcup_{i=1}^{N(\varepsilon)} Y^*_i \right) \cap \Omega.$$

Furthermore, $S_\varepsilon$ is defined as

$$S_\varepsilon = \left( \bigcup_{i \in \mathcal{A}} Y^*_i \right) \cap \Omega.$$

Analogously, we define

$$R_\varepsilon = \left( \bigcup_{i \in \mathcal{A}} Y^*_i \right) \cap \Omega.$$

We let

$$\Omega_\varepsilon = \left( \hat{\Omega}_\varepsilon - S_\varepsilon \right) \cup R_\varepsilon.$$

This means that we can define $\Omega_\varepsilon$ as

$$\Omega_\varepsilon = \left\{ \left\{ x \in \Omega \, | \, \chi_{Y^*}(\frac{x}{\varepsilon}) = 1 \right\} - S_\varepsilon \right\} \cup R_\varepsilon,$$

where $\chi_{Y^*}$ is the $Y$-periodic repetition of a function defined on $Y$ that is equal to one on $Y^*$ and zero elsewhere. Hence, the perforations do not cut $\partial \Omega$.

**Proposition 18** Let $\{ u_\varepsilon \}$ be bounded in $L^2(0, T; H^1(\Omega_\varepsilon))$ and assume that

$$\tilde{u}_\varepsilon(x, t) \overset{2,2}{\rightharpoonup} u(x, t) \chi_{Y^*}(y),$$

where $u \in L^2(\Omega \times (0, T))$ and $\tilde{u}_\varepsilon$ is an extension by zero of $u_\varepsilon$ from $\Omega_\varepsilon$ to $\Omega$. Then $u \in L^2(0, T; H^1(\Omega))$.

**Definition 19** Let $(D^*)^N = D[\Omega; D\sharp(Y^*_1 \times \ldots \times Y^*_n)]^N$ be the set of smooth functions $v : \Omega \times Y^*_1 \times \ldots \times Y^*_n \to \mathbb{R}^N$ periodic in $(y_1, \ldots, y_n)$ with compact support in $\Omega$ for $x$ and with their support contained in each of $E_\sharp(Y^*_k)$ for the respective variable $y_k$, $k = 1, \ldots, n$.

Next Lemma is cited from [4].

**Lemma 20** For any $k \in \{1, \ldots, n\}$, let $D^*_k$ be the subset of $(D^*)^N$ composed of functions satisfying a “generalised” divergence-free condition, i.e.

$$D^*_n = \left\{ v \in (D^*)^N; \, \text{div}_{y_n} v = 0 \right\}, \quad D^*_k = \left\{ v \in (D^*)^N; \, \text{div}_{y_n} v = 0 \text{ and } \int_{Y^*_1} \ldots \int_{Y^*_k} \text{div}_{y_j} v = 0 \, \forall k \leq j \leq n - 1 \right\}.$$

These spaces have the following property:
(i) Any function \( \theta \in D_1(\Omega; D_k(Y_1 \times \ldots \times Y_k))^N \) can be expressed as the average of a function in \( D_k^* \), i.e. there exists \( v(x, y_1, \ldots, y_n) \in D^*_k \) such that
\[
\theta = \int_{Y_{k+1}} \ldots \int_{Y_n} v \, dy_{k+1} \ldots dy_n.
\]
(ii) Any function \( \theta \in D(\Omega)^N \) can also be expressed as the average of a function \( v \) in \( D^*_n \), such that
\[
\theta = \int_{Y_1} \ldots \int_{Y_n} v \, dy_1 \ldots dy_n \quad \text{and} \quad \| v \|_{L^2(\Omega \times Y_1 \times \ldots \times Y_n)}^N \leq C \| \theta \|_{L^2(\Omega)}^N,
\]
where the constant \( C \) is independent of \( v \) and \( \theta \).

**Proof.** See proof of Lemma 4.13 (ii) in [4].

Further we need to state Corollary 21 in order to make a proof of the Proposition 18. Hence, the next Corollary, which means the special case of Lemma 20 (ii) for \( n = 1 \). See also Lemma 2.10 in [3].

**Corollary 21** For \( n = 1 \) the space \( D^*_1 \) in Lemma 20 has the following property: any function \( \theta \in D(\Omega)^N \) can also be expressed as the average of a function \( v \) in \( D^*_1 \), such that
\[
\theta = \int_{Y_1} v(x, y) dy \quad \text{and} \quad \| v \|_{L^2(\Omega \times Y)}^N \leq C \| \theta \|_{L^2(\Omega)}^N.
\]

**Proof.** (Proof of Proposition 18) We’ll show that the limit \( u \in L^2(0, T; H^1(\Omega)) \). We choose \( v \in D(\Omega \times (0, T); C^\infty(\mathbb{R}^d))^N \), such that \( v \in D^*_1 \) for any \( t \in (0, T) \), in
\[
\int_0^T \int_{\Omega} \nabla u_e(x, t) \cdot v(x, t, x \cdot \frac{1}{\varepsilon}) dx dt.
\]
We assume that \( \{u_e\} \) is bounded in \( L^2(0, T; H^1(\varepsilon)) \). Then \( \{\nabla u_e\} \) is bounded in \( L^2(\Omega \varepsilon \times (0, T))^N \) and can be extended with zero to \( \{\nabla u_e\} \) that is bounded in \( L^2(\Omega \times (0, T))^N \). We can now write (3) as
\[
\int_0^T \int_{\Omega} \nabla u_e(x, t) \cdot v(x, t, x \cdot \frac{1}{\varepsilon}) dx dt
\]
and obtain by Theorem 13, up to a subsequence,
\[
\int_0^T \int_{\Omega} \nabla u_e(x, t) \cdot v(x, t, x \cdot \frac{1}{\varepsilon}) dx dt
\rightarrow \int_0^T \int_{\Omega} \int_{Y^*} w_0(x, t, y) v(x, t, y) dy dy dx dt
\rightarrow \int_0^T \int_{\Omega} \int_{Y^*} \left( \int_0^1 w_0(x, t, y, s) ds \right) v(x, t, y) dy dx dt
= \int_0^T \int_{\Omega} \eta_0(x, t, y) v(x, t, y) dy dx dt
\]
for some \( w_0 \in L^2(\Omega \times (0, T) \times Y \times S)^N \) and \( \eta_0 = \int_0^1 w_0 ds \in L^2(\Omega \times (0, T) \times Y^* \times S)^N \), when \( \varepsilon \) goes to zero.

We now integrate by parts in (3) and let \( \varepsilon \) go to 0. Further, we let \( \tilde{u}_\varepsilon \) be an extension of \( u_\varepsilon \) from \( \Omega_\varepsilon \times (0, T) \) to \( \Omega \times (0, T) \). Clearly, \( \{ \tilde{u}_\varepsilon \} \) is bounded in \( L^2(\Omega \times (0, T)) \) if \( \{ u_\varepsilon \} \) is bounded in \( L^2(\Omega_\varepsilon \times (0, T)) \). We get

\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} \nabla u_\varepsilon(x, t) \cdot v(x, t, \frac{x}{\varepsilon}) dxdt = \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon(x, t) \nabla \cdot v(x, t, \frac{x}{\varepsilon}) dxdt
\]

\[
= \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega} -\tilde{u}_\varepsilon(x, t) \left( \nabla_x \cdot v(x, t, \frac{x}{\varepsilon}) + \varepsilon^{-1} \nabla_y \cdot v(x, t, \frac{x}{\varepsilon}) \right) dxdt
\]

\[
= \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega} -\tilde{u}_\varepsilon(x, t) \nabla_x \cdot v(x, t, \frac{x}{\varepsilon}) dxdt
\]

since \( \nabla_y \cdot v = 0 \). When \( \varepsilon \) tends to zero we get from assumption (2)

\[
- \int_0^T \int_{\Omega} \int_{Y^*} u(x, t) \chi_{Y^*}(y) \nabla_x \cdot v(x, t, y) dy dxdt
\]

\[
= - \int_0^T \int_{\Omega} \int_{Y^*} u(x, t) \nabla_x \cdot v(x, t, y) dy dxdt.
\]

This means that

\[
\int_0^T \int_{\Omega} \int_{Y^*} \eta_0(x, t, y) \cdot v(x, t, y) dy dxdt
\]

\[
= - \int_0^T \int_{\Omega} \int_{Y^*} u(x, t) \nabla_x \cdot v(x, t, y) dy dxdt
\]

\[
= \int_0^T \int_{\Omega} u(x, t) \left( \nabla_x \cdot \int_{Y^*} v(x, t, y) dy \right) dxdt.
\]

A simple modification of the proof of Lemma 4.13 (ii) in [4] (see our Lemma 20 and Corollary 21) means that any \( \theta \in D(\Omega \times (0, T))^N \) can be expressed as

\[
\theta(x, t) = \int_{Y^*} v(x, t, y) dy
\]

for some testfunction \( v \) of the type we use. Moreover,

\[
||v||_{L^2(\Omega \times (0, T) \times Y^*)^N} \leq C ||\theta||_{L^2(\Omega \times (0, T))^N}.
\]

Hence,

\[
\left| - \int_0^T \int_{\Omega} u(x, t) \left( \nabla_x \cdot \theta(x, t) \right) dxdt \right| = \left| \int_0^T \int_{\Omega} \int_{Y^*} \eta_0(x, t, y) \cdot v(x, t, y) dy dxdt \right|
\]

\[
\leq C ||\eta_0||_{L^2(\Omega \times T \times Y^*)} ||\theta||_{L^2(\Omega \times (0, T))^N}.
\]
This means that

\[ F(\theta) = - \int_0^T \int_\Omega u(x,t) \nabla_x \cdot \theta(x,t) dxdt \]

is a bounded linear functional on \( L^2(\Omega \times (0,T))^N \) for arbitrary \( \theta \in D(\Omega \times (0,T))^N \). By continuous extension, (e.g. Theorem 6.14 in [23]) this also applies to all \( \theta \in L^2(\Omega \times 0,T)^N \). This means that, for some \( r_0 \in L^2(\Omega \times (0,T))^N \),

\[ F(\theta) = - \int_0^T \int_\Omega u(x,t) \nabla_x \cdot \theta(x,t) dxdt = \int_0^T \int_\Omega r_0(x,t) \cdot \theta(x,t) dxdt \]

and hence the distributional gradient \( \nabla u \) of \( u \) belongs to \( L^2(\Omega \times (0,T))^N \). We already knew that \( u \in L^2(\Omega \times (0,T)^N) \) and hence \( u \in L^2(0,T;H^1(\Omega)) \).

We also cite the Lemma 4.14 of Allaire and Briane in [4], because we will use it in the following theorem.

**Lemma 22** Let \( H^* \) be the space of 'generalised' divergence-free functions in \( L^2[\Omega;L^2(Y^*_1 \times \cdots \times Y^*_n)^n] \) defined by

\[ v \in H^* \iff \begin{cases} \text{div}_{y_n} v = 0 \text{ in } Y^*_n \\ v \cdot n = 0 \text{ on } \partial Y^*_n - \partial Y_n \end{cases} \]

and

\[ \begin{cases} \int_{Y_{k+1}} \cdots \int_{Y_n} \text{div}_{y_k} v = 0 \text{ in } Y^*_k \\ \int_{Y_{k+1}} \cdots \int_{Y_n} v \cdot n = 0 \text{ on } \partial Y^*_k - \partial Y_k \end{cases} \]

for all \( 1 \leq k \leq n - 1 \).

The subspace \( H^* \) has the following properties:

(i) \( (D^*)_N \cap H^* \) is dense into \( H^* \).

(ii) The orthogonal of \( H^* \) is

\[ (H^*)^\perp = \left\{ \sum_{k=1}^n \nabla_{y_k} q_k(x, y_1, \ldots, y_k) \text{ with } q_k \in L^2[\Omega \times Y^*_1 \times \cdots \times Y^*_k; H_0^1(Y^*_k)/\mathbb{R}] \right\} . \]

We first find a characterization of the \((2,2)\)-scale limit for \( \{\nabla u_\varepsilon\} \) under certain assumptions.

**Theorem 23** Assume that \( \{u_\varepsilon\} \) is bounded in \( L^2(0,T;H^1(\Omega_\varepsilon)) \) and for any \( v_1 \in D(\Omega) \), \( c_1 \in D(0,T) \), \( c_2 \in C_\infty^\infty(0,1) \)

\[ \varepsilon^r \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon(x,t)v_1(x) \partial_t \left( c_1(t)c_2 \left( \frac{t}{\varepsilon^r} \right) \right) dxdt \to 0 \quad (7) \]

for some \( r > 0 \).
Indeed, integration by parts gives us

\[ \int_0^T \int_{\Omega^N} \nabla u_\varepsilon(x,t) \cdot v(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}) \, dx \, dt \]

Then, up to a subsequence,

\[ \int_0^T \int_{\Omega^N} \nabla u_\varepsilon(x,t) \cdot v(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}) \, dx \, dt \to \int_0^T \int_{\Omega^N} (\nabla u(x,t) + \nabla_y u_1(x,t,y,s)) \cdot v(x,t,y,s) \, dyds \, dx \, dt, \]

where \( u \in L^2(0,T; H^1(\Omega)) \), \( u_1 \in L^2(\Omega \times (0,T) \times (0,1); H^1_2(Y^*)/\mathbb{R}) \), for any \( v \in L^2(\Omega \times (0,T); C_2(Y_{1,1}))^N \).

**Proof.** We first show that the \((2,2)\)-scale limit \( u_0 \) for \( \tilde{u}_\varepsilon \) does not depend on \( y \). Indeed, integration by parts gives us

\[
\int_0^T \int_{\Omega^N} \nabla u_\varepsilon(x,t) \cdot v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) c_1(t)c_2 \left( \frac{t}{\varepsilon^r} \right) \, dx \, dt
\]

\[
= - \int_0^T \int_{\Omega^N} \tilde{u}_\varepsilon(x,t) \left[ v_1(x) \nabla_y \cdot v_2 \left( \frac{x}{\varepsilon} \right) + \varepsilon \nabla_x v_1(x) \cdot v_2 \left( \frac{x}{\varepsilon} \right) \right]
\]

\[
\times \left( c_1(t)c_2 \left( \frac{t}{\varepsilon^r} \right) \right) \, dx \, dt,
\]

for any \( v_1 \in D(\Omega) \) and \( v_2 \in C^\infty_2(Y^N) \) with \( v_2(y) = 0 \) for \( y \in Y - Y^* \) and \( c_1 \in D(0,T) \), \( c_2 \in C^\infty_2(0,1) \). Passing to the limit on both sides leads to

\[
- \int_0^T \int_{\Omega} \int_{Y^N} u_0(x,t,y,s) v_1(x) \nabla_y \cdot v_2(y) c_1(t)c_2(s) \, dyds \, dx \, dt = 0.
\]

This implies that \( u_0 \) does not depend on \( y \) in \( Y^* \), i.e. there exists \( u \in L^2(\Omega \times (0,T) \times (0,1)) \) such that

\[
u_0(x,t,y,s) = u(x,t,s) \chi_{Y^*}(y).
\]

Let us then show that, by assumption

\[
\varepsilon^r \int_0^T \int_{\Omega^N} u_\varepsilon(x,t) v_1(x) \partial_t \left( c_1(t)c_2 \left( \frac{t}{\varepsilon^r} \right) \right) \, dx \, dt \to 0,
\]

\( u_0 \) does not depend of \( s \). We rewrite the form (7) as

\[
\int_0^T \int_{\Omega} \varepsilon^r \tilde{u}_\varepsilon(x,t) v_1(x) ( \partial_t c_1(t) ) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega} \tilde{u}_\varepsilon(x,t) v_1(x) c_1(t) \left( \partial_s c_2 \left( \frac{t}{\varepsilon^r} \right) \right) \, dx \, dt,
\]

where \( v_1 \in D(\Omega) \), \( c_1 \in D(0,T) \) and \( c_2 \in C^\infty_2(0,1) \).

As \( \varepsilon \) tends to zero, we obtain that

\[
\int_0^T \int_{\Omega} \int_{Y^N} u(x,t,y,s) v_1(x) c_1(t) \partial_s c_2(s) \, dyds \, dx \, dt = 0.
\]
Finally, applying the variational lemma, we get

\[ \int_0^1 u(x, t, s) \partial_s c_2(s) ds = 0, \]

for a.e. \((x, t) \in \Omega \times (0, T)\). We deduce now that \(u_0\) does not depend on the local time variable \(s\). Hence,

\[ \tilde{u}_\epsilon(x, t) \overset{2\to 2}{\rightharpoonup} u(x, t) \chi_{Y^*}(y) \]

and Proposition 18 yields that \(u \in L^2(0, T; H^1(\Omega))\). We now let \(v \in (D^*)^N \cap H^*\) for \(n = 1\) (see Lemma 22) and again \(c_1 \in D(0, T), c_2 \in C^\infty_c(0, 1)\). Integrating by parts in \(\Omega_\epsilon\) gives

\[ \int_0^T \int_{\Omega_\epsilon} \nabla u_\epsilon(x, t) \cdot v \left( x, \frac{x}{\epsilon} \right) c_1(t) c_2 \left( \frac{t}{\epsilon^2} \right) dx dt = - \int_0^T \int_{\Omega_\epsilon} u_\epsilon(x, t) \nabla_x \cdot v \left( x, \frac{x}{\epsilon} \right) c_1(t) c_2 \left( \frac{t}{\epsilon^2} \right) dx dt. \]

Hence, passing to the two-scale limit yields that, for some \(w_0 \in L^2(\Omega \times (0, T) \times Y^* \times (0, 1))^N\),

\[ \int_0^T \int_{\Omega} \int_0^1 \int_{Y^*} w_0(x, t, y, s) \cdot v(x, y)c_1(t)c_2(s)dydxdsdt = - \int_0^T \int_{\Omega} \int_0^1 \int_{Y^*} u(x, t) \nabla_x \cdot v(x, y)c_1(t)c_2(s)dydxdsdt = \int_0^T \int_{\Omega} \int_0^1 \int_{Y^*} \nabla u(x, t) \cdot v(x, y)c_1(t)c_2(s)dydxdsdt. \]

We have

\[ \int_0^T \int_{\Omega} \int_0^1 (w_0(x, t, y, s) - \nabla u(x, t)) \cdot v(x, y)c_1(t)c_2(s)dydsdxdt = 0, \]

which means that a.e. on \((0, T) \times (0, 1)\)

\[ \int_{\Omega} \int_{Y^*} (w_0(x, t, y, s) - \nabla u(x, t)) \cdot v(x, y)dydx = 0. \]

Moreover, the orthogonal of \(H^*\) are gradients. See Lemma 22 for \(n = 1\). See also the proof of Theorem 2.9 in [3]. This implies that there exists a unique function \(u_1\) in \(L^2 \left[ \Omega \times (0, T) \times (0, 1); H^1_2(Y^*)/\mathbb{R} \right]\) such that

\[ \nabla \tilde{u}_\epsilon(x, t) \overset{2\to 2}{\rightharpoonup} (\nabla u(x, t) + \nabla_y u_1(x, t, y, s)) \chi_{Y^*}(y). \]

See also [24]. ■

From the Theorem 23 above, as a consequence, we have the following corollary.
Corollary 24 Assume that \( \{u_\varepsilon\} \) is bounded in \( L^2(0, T; H^1(\Omega_\varepsilon)) \) and that (7) holds in Theorem 23. Then

\[
\int_0^T \int_{\Omega_\varepsilon} \varepsilon^{-1}u_\varepsilon(x, t)v_1(x)c_2 \left( \frac{t}{\varepsilon} \right) c_1(t) \frac{1}{\varepsilon} \ dx \ dt \tag{12}
\]

\[
- \int_0^T \int_{\Omega_\varepsilon} \nabla u_\varepsilon(x, t)v_1(x)c_1(t)c_2 \left( \frac{t}{\varepsilon} \right) \cdot \nabla \frac{1}{\varepsilon} \ dx \ dt
\]

for \( v_1 \in D(\Omega) \), \( v_2 \in C^\infty_t(Y^*)/\mathbb{R} \), \( c_1 \in D(0, T) \) and \( c_2 \in C^\infty_t(0, 1) \).

**Proof.** First we note that any \( v_2 \in C^\infty_t(Y^*)/\mathbb{R} \) can be expressed as

\[
v_2(y) = \Delta_y \rho(y) = \nabla_y \cdot \left( \nabla_y \rho(y) \right) \tag{13}
\]

for some \( \rho \in C^\infty_t(Y^*)/\mathbb{R} \). Furthermore, we note that we can find \( \rho \in C^\infty_t(Y^*)/\mathbb{R} \) through

\[
\Delta \rho(y) = v_2(y), \quad y \in Y^*,
\]

\[
\nabla_y \rho(y) \cdot n = 0, \quad y \in \partial Y^* - \partial Y.
\]

By (13) the left-hand side of (12) can be expressed as

\[
\int_0^T \int_{\Omega_\varepsilon} \varepsilon^{-1}u_\varepsilon(x, t)v_1(x)c_2 \left( \frac{t}{\varepsilon} \right) \left( \nabla_y \cdot \nabla \frac{1}{\varepsilon} \right) \ dx \ dt
\]

\[
\int_0^T \int_{\Omega_\varepsilon} u_\varepsilon(x, t)v_1(x)c_2 \left( \frac{t}{\varepsilon} \right) \nabla \cdot \left( \nabla_y \rho \left( \frac{x}{\varepsilon} \right) \right) \ dx \ dt.
\]

Integrating by parts with respect to \( x \) we obtain

\[
- \int_0^T \int_{\Omega_\varepsilon} \nabla u_\varepsilon(x, t)v_1(x)c_1(t)c_2 \left( \frac{t}{\varepsilon} \right) \cdot \nabla_y \rho \left( \frac{x}{\varepsilon} \right) \ dx \ dt
\]

\[
+ u_\varepsilon(x, t)\nabla v_1(x)c_1(t)c_2 \left( \frac{t}{\varepsilon} \right) \cdot \nabla_y \rho \left( \frac{x}{\varepsilon} \right) \ dx \ dt.
\]

Passing to the limit in the first term we get, up to a subsequence, that

\[
- \int_0^T \int_{\Omega_\varepsilon} \nabla u(x, t) + \nabla_y u_1(x, t, y, s)v_1(x)c_1(t)c_2(s) \cdot \nabla_y \rho(y)
\]

\[
+ u(x, t)\nabla v_1(x)c_1(t)c_2(s) \cdot \nabla_y \rho(y) \ dy ds \ dx dt.
\]

Integration by parts in the last term with respect to \( x \), gives

\[
- \int_0^T \int_{\Omega_\varepsilon} \int_0^1 \int_{Y^*} \nabla u_1(x, t, y, s)v_1(x)c_1(t)c_2(s) \cdot \nabla_y \rho(y) \ dy ds \ dx dt.
\]

Finally, integrating by parts with respect to \( y \) we obtain

\[
\int_0^T \int_{\Omega_\varepsilon} \int_0^1 \int_{Y^*} u_1(x, t, y, s)v_1(x)c_1(t)c_2(s) \nabla_y \cdot \left( \nabla_y \rho(y) \right) dy ds \ dx dt
\]

\[
= \int_0^T \int_{\Omega_\varepsilon} \int_0^1 \int_{Y^*} u_1(x, t, y, s)v_1(x)c_1(t)c_2(s)v_2(y) dy ds \ dx dt,
\]

which is the right-hand side of (12). □
4 Homogenization result

We will investigate the parabolic problem with spatial and temporal oscillations

$$\frac{\partial}{\partial t} u(\xi, x, t) - \nabla \cdot \left( A \left( \frac{x}{\xi}, \frac{t}{\xi^2} \right) \nabla u(\xi, x, t) \right) = f(\xi, x, t) \text{ in } \Omega_x \times (0, T),$$

$$u(\xi, x, t) = 0 \text{ on } \partial \Omega \times (0, T),$$

$$A \left( \frac{x}{\xi}, \frac{t}{\xi^2} \right) \nabla u(\xi, x, t) \cdot n = 0 \text{ on } (\partial \Omega - \partial \Omega_x) \times (0, T),$$

$$u(\xi, x, 0) = u^0(\xi, x) \text{ in } \Omega_x,$$

where

$$\tilde{u}^0_\xi \rightharpoonup u^0 \text{ in } L^2(\Omega)$$

and

$$\tilde{f}_\xi \rightharpoonup f \text{ in } L^2(\Omega \times (0, T)).$$

Moreover, we assume that

(H1) $A \in C^\#(Y_1, 1)^N \times N$

(H2) $A(y, s)\xi \cdot \xi \geq \alpha |\xi|^2$

for all $(y, s) \in \mathbb{R}^N \times \mathbb{R}$, all $\xi \in \mathbb{R}^N$ and some $\alpha > 0$.

We introduce the space

$$V_\xi = \{ v \in H^1(\Omega_x) \mid v(x) = 0 \text{ on } \partial \Omega \}$$

with the $H^1(\Omega_x)$-norm.

Under these conditions the problem (15) allows a unique solution $\{u_\xi\}$ bounded in $L^\infty(0, T; L^2(\Omega_x))$ and $L^2(0, T; V_\xi)$, see Section 3 in [14].

We are now prepared to prove the following theorem.

**Theorem 25** Let $\{u_\xi\}$ be a sequence of solutions in $L^2(0, T; V_\xi)$ to (15). Then it holds that

$$\tilde{u}_\xi(x, t) \rightharpoonup u^0 \text{ in } L^2(\Omega)$$

and

$$\tilde{f}_\xi \rightharpoonup f \text{ in } L^2(\Omega \times (0, T)).$$

is the unique solution of the following two-scale homogenized system:

$$\mu(Y^*) \partial_t u(x, t) - \nabla \cdot (b \nabla u(x, t)) = f(x, t) \text{ in } \Omega \times (0, T),$$

$$u(x, t) = 0 \text{ on } \partial \Omega \times (0, T),$$

$$u(x, 0) = (\mu(Y^*))^{-1} u^0(x) \text{ in } \Omega,$$

where

$$b \nabla u = \int_0^1 \int_{Y^*} A(y, s)(\nabla u(x, t) + \nabla_y u_1(x, t, y, s))dyds.$$
and \( u_1 \) solves our local problem

\[
\partial_s u_1(x,t,y,s) - \nabla_y \cdot (A(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s))) = 0 \text{ in } Y^* \times (0,1) \quad (19)
\]

\[
(A(y,s) [\nabla u(x,t) + \nabla_y u_1(x,t,y,s)]) \cdot n = 0 \text{ on } (\partial Y^* - \partial Y) \times (0,1) \quad (20)
\]

for a.e. \((x,t) \in \Omega \times (0,T)\).

**Remark 26** Let us point out, that by the uniqueness of solutions to the above system, the entire sequences \(\{u_\varepsilon\}, \{\nabla u_\varepsilon\} \) two-scale converge (see [28]). Two-scale homogenized system (18) can be decoupled to a familiar type of homogenized system. Using the ansatz

\[
u_1(x,t,y,s) = \nabla u(x,t) \cdot z(y,s),
\]

the variable separated version of the local problem becomes

\[
\partial_s z_j(y,s) - \nabla_y \cdot (A(y,s)(e_j + \nabla_y z_j(y,s))) = 0 \text{ in } Y^* \times (0,1),
\]

where \( j = 1, \ldots, N \). Analogously for (20), we obtain

\[
A(y,s)(e_j + \nabla_y z_j(y,s)) \cdot n = 0 \text{ on } (\partial Y^* - \partial Y) \times (0,1).
\]

We get the expression for the homogenized coefficients

\[
b_{ij} = \int_0^1 \int_{Y^*} A_{ij}(y,s) + \sum_{k=1}^N A_{ik}(y,s) \partial_y z_j(y,s) dy ds.
\]

**Proof.** We carry out a homogenization procedure for (15). The corresponding weak form states that we are searching for a unique \( u_\varepsilon \) in \( L^2(\Omega \times [0,T]; \mathcal{V}_\varepsilon) \) such that

\[
\int_0^T \int_{\Omega_\varepsilon} -u_\varepsilon(x,t) v(x) \partial_t c(t) + A \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u_\varepsilon(x,t) \cdot \nabla v(x) c(t) dx dt = \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon(x,t) v(x) c(t) dx dt, \quad v \in \mathcal{V}_\varepsilon, \quad c \in D(0,T). \quad (21)
\]

We want to prove the weak form of the homogenized problem (18). To see that (7) is satisfied for \( r > 0 \) and hence for \( r = 2 \), we conclude that for the choice of test functions in (7):

\[
-\varepsilon^r \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon(x,t) v(x) \partial_t \left( c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) dx dt = -\varepsilon^r \int_0^T \int_{\Omega_\varepsilon} A \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u_\varepsilon(x,t) \cdot \nabla v(x) c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) dx dt \rightarrow 0.
\]
We choose \( v(x, t) = v_1(x)c_1(t), v_1 \in D(\Omega), c_1 \in D(0, T) \) in (21). When passing to the limit and letting \( \varepsilon \to 0 \), we obtain through Theorem 23 and (17)

\[
\int_0^T \int_{\Omega} \int_0^1 \int_{Y^*} -u(x, t)v_1(x)\partial_t c_1(t) \\
+ A(y, s)(\nabla u(x, t) + \nabla_y u_1(x, t, y, s)) \cdot \nabla v_1(x)c_1(t)dydsdxdt
\]

\( \Rightarrow \int_0^T \int_{\Omega} f(x, t)v_1(x)c_1(t)dxdt. \) (22)

To find the local problem we choose the test functions in (21) as

\[
v(x) = \varepsilon v_1(x)v_2 \left( \frac{t}{\varepsilon^2} \right) \\
c(t) = c_1(t)c_2 \left( \frac{t}{\varepsilon^2} \right)
\]

where \( v_1 \in D(\Omega), v_2 \in C^\infty_0(\mathcal{Y}^*)/\mathbb{R}, c_1 \in D(0, T) \) and \( c_2 \in C^\infty_0(0, 1) \). This gives us

\[
\int_0^T \int_{\Omega_x} -u(x, t)\varepsilon v_1(x)v_2 \left( \frac{x}{\varepsilon} \right) \partial_t \left( c_1(t)c_2 \left( \frac{t}{\varepsilon^2} \right) \right) \\
+ A \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u_\varepsilon(x, t) \cdot \nabla \left( \varepsilon v_1(x)v_2 \left( \frac{x}{\varepsilon} \right) \right) c_1(t)c_2 \left( \frac{t}{\varepsilon^2} \right) dxdt \]

\( \Rightarrow \int_0^T \int_{\Omega_x} f_\varepsilon(x, t)\varepsilon v_1(x)v_2 \left( \frac{x}{\varepsilon} \right) c_1(t)c_2 \left( \frac{t}{\varepsilon^2} \right) dxdt. \) (23)

Carring out the differentiations yields

\[
\int_0^T \int_{\Omega_x} -u_\varepsilon(x, t)v_1(x)v_2 \left( \frac{x}{\varepsilon} \right) \partial_x c_1(t)c_2 \left( \frac{t}{\varepsilon^2} \right) + \varepsilon^{-1}c_1(t)\partial_x c_2 \left( \frac{t}{\varepsilon^2} \right) + A \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u_\varepsilon(x, t) \cdot \left( \varepsilon \nabla v_1(x)v_2 \left( \frac{x}{\varepsilon} \right) + v_1(x)\nabla_y v_2 \left( \frac{x}{\varepsilon} \right) \right) c_1(t)c_2 \left( \frac{t}{\varepsilon^2} \right) dxdt
\]

\( \Rightarrow \int_0^T \int_{\Omega_x} f_\varepsilon(x, t)\varepsilon v_1(x)v_2 \left( \frac{x}{\varepsilon} \right) c_1(t)c_2 \left( \frac{t}{\varepsilon^2} \right) dxdt. \)

If we keep only terms that do not tend to zero there remains

\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_x} -\varepsilon^{-1}u_\varepsilon(x, t)v_1(x)v_2 \left( \frac{x}{\varepsilon} \right) c_1(t)\partial_x c_2 \left( \frac{t}{\varepsilon^2} \right) \\
+ A \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u_\varepsilon(x, t) \cdot v_1(x)\nabla_y v_2 \left( \frac{x}{\varepsilon} \right) c_1(t)c_2 \left( \frac{t}{\varepsilon^2} \right) dxdt = 0. \) (24)

So far, we know by the Corollary 24 that we can get

\[
\int_0^T \int_{\Omega_x} -\varepsilon^{-1}u_\varepsilon(x, t)v_1(x)v_2 \left( \frac{x}{\varepsilon} \right) c_1(t)\partial_x c_2 \left( \frac{t}{\varepsilon^2} \right) dxdt \\
\Rightarrow \int_0^T \int_{\Omega_x} \int_0^1 \int_{Y^*} -u_1(x, t, y, s)v_1(x)v_2(y)c_1(t)\partial_s c_2(s)dydsdxdt \) (25)
and by Theorem 23
\begin{align*}
\int_0^T \int_{\Omega} A\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \nabla u_\varepsilon(x,t) \cdot v_1(x) \nabla_y v_2 \left(\frac{x}{\varepsilon}\right) c_1(t) c_2 \left(\frac{t}{\varepsilon^2}\right) \, dxdt \\
\quad \rightarrow \int_0^T \int_{\Omega} \int_0^1 \int_{Y^*} A(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s)) \\
\quad \cdot v_1(x) \nabla_y v_2(y) c_1(t) c_2(s) \, dydsdxdt.
\end{align*}
(26)

Hence, putting (24), (25) and (26) together we arrive at the local problem
\begin{align*}
\int_0^T \int_{\Omega} \int_0^1 \int_{Y^*} -u_1(x,t,y,s)v_1(x)v_2(y)c_1(t)\partial_s c_2(s) \\
\quad + A(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s)) \\
\quad \cdot v_1(x) \nabla_y v_2(y) c_1(t) c_2(s) dydsdxdt = 0,
\end{align*}
which is the weak form of (19). The proof is complete. ■

**Remark 27** The corresponding limit for the initial condition (16) is found in the same way as in the proof of Theorem 4.1 in [14].

**Acknowledgement 28** The author thanks Prof. Anders Holmbom, Mid Sweden University, Östersund, for the useful comments, remarks and also for assistance with the proof of Proposition 18. She is also immensely grateful to Dr. Lotta Flodén and Dr. Marianne Olsson Lindberg, Mid Sweden University, Östersund, for theirs insightful comments on an earlier version of the paper.

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