NONASYMPTOTIC AND ASYMPTOTIC LINEAR CONVERGENCE OF AN ALMOST CYCLIC SHQP DYKSTRA’S ALGORITHM FOR POLYHEDRAL PROBLEMS

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Abstract. We show that an almost cyclic (or generalized Gauss-Seidel) Dykstra’s algorithm which incorporates the SHQP (supporting halfspace-quadratic programming) strategy can achieve nonasymptotic and asymptotic linear convergence for polyhedral problems.

Contents

1. Introduction 1
2. On the least squares lasso 4
3. Preliminaries 5
4. Algorithm statement 6
5. Asymptotic linear convergence 1: Adapting [LP90] 9
6. Asymptotic linear convergence 2: Adapting [DH94] 13
7. Nonasymptotic convergence properties in polyhedral problems 20
8. References 24

1. INTRODUCTION

We consider the following problem, known as the best approximation problem (BAP).

\[
(BAP) \quad \min \ f(x) := \frac{1}{2} \|x - d\|^2 \\
\text{s.t.} \quad x \in C := C_1 \cap \cdots \cap C_m,
\]

where \(d\) is a given point and \(C_i, i = 1, \ldots, m\), are closed convex sets in a Hilbert space \(X\). The BAP is equivalent to projecting \(d\) onto \(C\). We shall assume throughout that \(C \neq \emptyset\). We now give an introduction of the background and techniques of this paper.
1.1. Alternating projections and the dual of the BAP. The BAP is often associated with the set intersection problem (SIP)

\[(SIP) \quad \text{Find} \quad x \in C := C_1 \cap \cdots \cap C_m. \quad (1.2)\]

A well studied method for the SIP is the method of alternating projections (MAP). We recall material from [BC11, Deu01a, Deu01b, ER11] on material on the MAP. As its name suggests, the MAP projects the iterates in a cyclic or non-cyclic manner so that the iterates converge to a point in the intersection of these sets.

Remark 1.1. (MAP on linear subspaces) For future discussions, we recall that rate of convergence of the MAP when all the \(C_i\)s are linear subspaces is studied in [DH97], which builds on the work of [SSW77, KW88]. See Theorem 6.9 for a corollary of [DH97, Theorem 2.7].

As remarked by several authors, the MAP does not converge to the solution of the BAP in the general case. Dykstra’s algorithm [Dyk83] solves the best approximation problem through a sequence of projections onto each of the sets in a manner similar to the MAP, but correction vectors are added before every projection. The proof of convergence to \(P_{C}(d)\) was established in [BD85] and sometimes referred to as the Boyle-Dykstra theorem. For a closed convex set \(D \subset X\), recall that \(\delta^*(\cdot, D) : X \to \mathbb{R}\) is the support function defined by

\[\delta^*(z, D) = \sup_{x \in D} \langle z, x \rangle. \quad (1.3)\]

As pointed out in [Han88] and [GM89], the dual problem of the BAP is defined as follows.

Definition 1.2. (Dual problem of the BAP) Let \(X\) be a Hilbert space, \(d \in X\), and \(C_i \subset X\) be closed convex sets such that \(C := \cap_{i=1}^{m} C_i \neq \emptyset\). Following [Han88], we recall the (Fenchel) dual of the BAP \((1.1)\):

\[(D') \quad \inf_{y_1, \ldots, y_m} v(y_1, \ldots, y_m), \quad (1.4)\]

where \(v : X^m \to \mathbb{R}\) is defined by

\[v(y) = v(y_1, \ldots, y_m) = \frac{1}{2} \left\| d - P_C(d) - \sum_{i=1}^{m} y_i \right\|^2 + \sum_{i=1}^{m} \delta^*(y_i, C_i - P_C(d)), \quad (1.5)\]

where \(P_C(d)\) denotes the projection of \(d\) onto \(C\), and \(y \in X^m\) is the dual variable.

1.2. Alternating minimization and variants. Note that in \((1.5)\), the underbraced term \((A)\) is smooth, while \((B)\) is a nonsmooth term that is block separable. The method of alternating minimization (AM) applied to minimizing \((1.5)\) is to minimize the coordinates \(y_i, i \in \{1, \ldots, m\}\) one at a time in a cyclic manner while holding all other block coordinates fixed. The papers [Han88] and [GM89] also pointed out that Dykstra’s algorithm is AM on \((1.5)\). AM is also referred to as the block-nonlinear Gauss Seidel method or block coordinate descent method.

Since the Hessian of the smooth portion of the subproblem of solving for one block \(y_i\) while keeping all other blocks fixed is a multiple of the identity matrix, the block coordinate (proximal) gradient descent algorithm (BCGD) in [TY09b, TY09a] is identical to AM.
For a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$ where $m << n$, the least squares lasso problem is

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \| Ax - b \|^2 + \lambda \| x \|_1. \quad (1.6)$$

The least squares lasso problem is an example of a problem where AM is a competitive method. A notable but dated paper on applying AM for this problem is ST13.

1.3. Asymptotic linear convergence of Dykstra’s algorithm and Alternating Minimization. We first recall results on the asymptotic linear convergence of Dykstra’s algorithm when the sets $C_i$ are all polyhedral.

The first proof of asymptotic linear convergence of a variant of Dykstra’s algorithm was presented in [IP90] for the case when $C_i$ are halfspaces (Dykstra’s algorithm coincides with Hildreth’s algorithm for this case). Deutsch and Hundal [DH94] refined the linear convergence rate in [IP90] (also for the case when $C_i$ are halfspaces) by applying results mentioned in Remark 1.1.

Luo and Tseng [LT93] used a more general framework to give a different proof of the asymptotic linear convergence of Dykstra’s algorithm when $C_i$ are polyhedral. They showed that if $g : \mathbb{R}^m \to \mathbb{R}$ is strongly convex, $E \in \mathbb{R}^{m \times n}$ is a matrix with no zero column, $q \in \mathbb{R}^n$ and $X$ is a polyhedral set, then first order methods (which also includes AM) applied to

$$\min_{x \in X} g(Ex) + \langle q, x \rangle \quad (1.7)$$

has asymptotic linear convergence. (They mentioned that (1.5) can be transformed into the form (1.7). This transformation is explicitly stated in [Yun14].) See also [TY09a]. The proofs in [LT93, TY09b] are vastly different from that of [IP90, DH94].

The method in [LT93] is superior in some ways compared to the approach of [IP90, DH94]. First, [LT93] allows for multiple coordinates $y_i$ in (1.5) to be minimized at a time instead of just one coordinate at a time. Secondly, their approach allows for $C_i$ to be polyhedra rather than halfspaces. But the original approach in [IP90] allows for an almost cyclic sampling: More precisely, the approach of [LT93] requires each coordinate to be minimized exactly once in each cycle, but the approach of [IP90] allows for each coordinate to be minimized at least once in each cycle instead.

1.4. Nonasymptotic convergence rates. Rather than the asymptotic convergence rates, a measure of the effectiveness of alternating minimization is the nonasymptotic convergence rates (or absolute rate of convergence). Nonasymptotic rates hold from the very first iteration, and are more useful than the asymptotic rates for large scale problems, which can take many iterations to achieve the asymptotic convergence rates. These rates are typically sublinear, like $O(1/k)$ for example. A modern elementary reference on the nonasymptotic convergence of first order methods is Nes83.

The papers [BT13, Bec15] gave a summary of the history behind AM and showed that AM has an $O(1/k)$ nonasymptotic rate of convergence for the cases when there are multiple blocks but no proximal terms (i.e., the term corresponding to $(B)$ in (1.5) is zero), and when there are proximal terms but only two blocks. See also [HWRL17]. For the dual problem corresponding to Dykstra’s algorithm, [CPI15] showed that the techniques in [BT13, Bec15] give a $O(1/k)$ convergence rate. More
can be said for BCGD in general. For example, [Yun14] showed that BCGD has an $O(1/k)$ nonasymptotic rate of convergence.

As explained in [Nes83], a typical condition needed for the nonasymptotic linear convergence of first order methods is the strong convexity of the objective function. Wang and Lin [WL14] showed that first order methods for problems of the form (1.7) achieve nonasymptotic linear convergence, and [Yun14] showed that AM for (1.5) achieves nonasymptotic linear convergence.

1.5. Other notable results on Dykstra’s algorithm. Another aspect of Dykstra’s algorithm useful for future discussions is that Hundal and Deutsch [HD97] showed that Dykstra’s algorithm converges when the sets in Dykstra’s algorithm are sampled in a random order provided that each set is projected onto infinitely often. (The same paper also showed that Dykstra’s algorithm converges for the case of infinitely many sets, but we will not make use of this property in this paper.)

A method studied in [Pan15] and [Pan16] to improve convergence of the MAP and Dykstra’s algorithm respectively is to notice that each projection onto a set $C_i$ generates a supporting halfspace of $C_i$, which in turn contains $C$, and that the projection onto the intersection of these halfspaces is relatively easy using quadratic programming. We call this the SHQP strategy. The SHQP strategy can be seen as a greedy step, as explained in Remark 4.6. For the case when $m$ is small and the $C_i$s are halfspaces in the BAP (1.1), one could apply the SHQP strategy and solve the BAP in one step.

1.6. Contributions of this paper. We provide more context behind our contribution. On the one hand, the approach in [IP90, DH94] gives asymptotic linear convergence for almost cyclic sampling, but only for $C_i$ being halfspaces. On the other hand, the approach in [LT93, TY09b] give asymptotic linear convergence for polyhedral problems (i.e., $C_i$ can be any polyhedra), but requires a restricted Gauss-Seidel sampling and not almost cyclic sampling. It doesn’t seem easy to improve the general strategy in [LT93, TY09b] mentioned in Subsection 1.3 to get asymptotic linear convergence. (In fact, [TY09b] proved that BCGD with almost cyclic sampling, which they called unrestricted Gauss-Seidel, has global convergence, but they did not address asymptotic linear convergence.)

Our approach is to build on the techniques of [IP90, DH94] together with results in various directions in [HD97, DH97] to obtain asymptotic linear convergence for almost cyclic sampling for the case when the sets $C_i$ are polyhedral and not just halfspaces. We also show that we can incorporate the SHQP step and still have both asymptotic and nonasymptotic linear convergence.

1.7. Notation. For integers $l_1$ and $l_2$ such that $l_1 \leq l_2$, we write $\{l_1, l_1+1, \ldots, l_2-1, l_2\}$ as $[l_1, l_2]$ in order to simplify notation.

2. On the least squares lasso

To further motivate this paper, we first point out a rather elementary fact that the least squares lasso problem (1.6) is a special case of (1.5), the dual of the BAP, before the recalling preliminaries for the rest of the paper.

Recall the lasso problem (1.3). Denote the $i$th column of $A$ to be $A_i$. We can assume that none of the $A_i$s are zero since if $A_i$ is zero, the $i$th component of any
optimal vector \( x \) has to be zero. Consider the following problems

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \frac{1}{2} \left\| b - \sum_{i=1}^{n} A_i x_i \right\|_2^2 + \lambda \left\| x_i \right\|_1 \\
\min_{x \in \mathbb{R}^n} & \frac{1}{2} \left\| b - \sum_{i=1}^{n} \frac{A_i}{\|A_i\|} x_i \right\|_2^2 + \sum_{i=1}^{n} \frac{\lambda}{\|A_i\|} |x_i| \\
\min_{(y_1, \ldots, y_n) \in (\mathbb{R}^m)^n} & \frac{1}{2} \left\| b - \sum_{i=1}^{n} y_i \right\|_2^2 + \sum_{i=1}^{n} \delta^*(y_i, S_i),
\end{align*}
\]

where the slab \( S_i \subset \mathbb{R}^m \) in (2.1c) is defined by

\[
S_i = \left\{ z : \left( \frac{A_i}{\|A_i\|} \right)^T z \in \left[ -\frac{\lambda}{\|A_i\|}, \frac{\lambda}{\|A_i\|} \right] \right\}.
\]

The problem (2.1a) is equivalent to the least squares lasso problem in (1.6). The problems (2.1a) and (2.1b) are equivalent up to a scaling of the coordinates of \( x \). Lastly, we show the equivalence of the problems (2.1b) and (2.1c). If \( y_i \) is a multiple of \( \frac{A_i}{\|A_i\|} \), say \( y_i = \frac{A_i}{\|A_i\|} x_i \), then

\[
\delta^*(y_i, S_i) = \delta^* \left( \frac{A_i}{\|A_i\|} x_i, S_i \right) = \max \left\{ \left( \frac{A_i}{\|A_i\|} x_i \right)^T z : z \in S_i \right\} = |x_i| / \|A_i\|.
\]

Next, \( y_i \) not being a multiple of \( \frac{A_i}{\|A_i\|} \) would mean that \( \delta^*(y_i, S_i) = \infty \). So throughout an algorithm where the objective value in (2.1c) is finite, the \( y_i \)'s are multiples of \( \frac{A_i}{\|A_i\|} \) and identical to (2.1a). Since (2.1c) is of the form (1.5), we are done.

3. Preliminaries

For the BAP (1.1), we point out a few known facts on the dual function \( v(\cdot) \) defined in (1.5).

**Theorem 3.1.** (Known results on dual functions) It is known that (1.5) is the dual function of the BAP. (See for example [Han88, GM89].) Suppose \( P_C(d) = 0 \). Then

1. For \( y \in X^m \), let \( x := d - \sum_{i=1}^{m} y_i \). Then \( \frac{1}{2} \| x - P_C(d) \|_2^2 \leq v(y) \).
2. \( \inf_y v(y) = 0 \).
3. If \( y \) is a minimizer of \( v(\cdot) \), then \( d - \sum_{i=1}^{m} y_i \) is the primal minimizer of the BAP (1.1).

**Proof.** Since \( 0 \in C_i - P_C(d) \), it follows that \( \delta^*(y_i, C_i - P_C(d)) \geq 0 \) for all \( i \), and statement (1) follows. Statement (2) can be obtained from [GM89] pages 32–33. (For more details on the elementary steps needed to convert the material in [GM89] pages 32–33 to statement (2), see [Pan16].) Note that if \( y \) is a minimizer of \( v(\cdot) \), then (1) and (2) imply that \( \frac{1}{2} \| d - \sum_{i=1}^{m} y_i - P_C(d) \|_2^2 = 0 \), from which we get statement (3). \( \square \)

As pointed out in [Han88, GM89], Dykstra’s algorithm corresponds to alternating minimization on the dual problem \( (D') \) in (1.3). This detail will be elaborated in (4.8). After we introduce our extended Dykstra’s algorithm.

We make our assumptions of the polyhedral structure of \( C_i \) in (1.1).

**Assumption 3.2.** (Polyhedral setting) Let \( X \) be a Hilbert space and let \( C_1, \ldots, C_m \) be \( m \) polyhedra in \( X \) with nonempty intersection \( C = \bigcap_{i=1}^{m} C_i \). Let \( d \in X \).
Suppose that \( x_\infty := P_C(d) \), and assume without loss of generality that \( x_\infty = 0 \). Suppose each polyhedron \( C_i \) is defined by
\[
C_i = \bigcap_{r=1}^{K_i} H_{i,r},
\]
where \( H_{i,r} \) are the halfspaces
\[
H_{i,r} = \{ x \in X : \langle x, f_{i,r} \rangle \leq c_{i,r} \},
\]
where \( f_{i,r} \in X \setminus \{0\} \) and \( c_{i,r} \in \mathbb{R} \cup \{\infty\} \). By scaling, we may assume that \( \|f_{i,r}\| = 1 \).

Define the affine space \( H_{i,r} \) to be the boundary of \( H_{i,r} \), i.e.,
\[
H_{i,r} = \{ x \in X : \langle x, f_{i,r} \rangle = c_{i,r} \}.
\]

For each \( i \in [1, m] \), consider the polyhedron \( C'_i \) to be the set defined similarly to \( C_i \) such that the halfspaces that are not tight at \( x_\infty \) are removed, i.e.,
\[
C'_i = \bigcap_{r \in \{1, \ldots, K_i\}} H_{i,r},
\]
where the halfspaces \( H_{i,r} \) for \( r \in \{1, \ldots, K_i\} \) are all active at \( x_\infty \), and the halfspaces \( H_{i,r} \) for \( r \in \{K_i + 1, \ldots, K\} \) are all not active at \( x_\infty \).

When \( c_{i,r} = \infty \), then \( H_{i,r} = X \) and \( H_{i,r} = \emptyset \). It is clear to see that Assumption 3.2 do not lose any generality.

4. Algorithm statement

We state our extended Dykstra’s algorithm in Algorithm 4.1

**Algorithm 4.1.** (Main algorithm) Suppose \( P_C(d) = 0 \). Let \( y^0 \in X^m \) be such that \( y^0 \in X \) are the starting dual variables to \( C_i \) for \( i \in \{1, \ldots, m\} \) (for the dual function \( (1.5) \)). The iterates \( y^k \in X^m \) of the algorithm are such that \( d - \sum_{i=1}^{m} y^k_i \) tries to approximate \( P_C(d) \).

01 For \( k = 1, 2, \ldots \)
02 Run Algorithm 4.2 with input \( y^{k-1} \) to get \( y^k \).
03 End for

We now describe the subroutine in Algorithm 4.2 using the ideas in [HD97]. See the remarks following the algorithm for more insight.

The SHQP step can be omitted in first reading in order to understand Algorithm 4.2. (That would correspond to the case when \( Q_j = \emptyset \) for all \( j \).) We now comment on Algorithm 4.2.
Algorithm 4.2. (One cycle in almost cyclic Dykstra’s algorithm with SHQP) Recall the assumptions stated in Algorithm 4.1.

**Input:** $y^0 \in X^m$

**Output:** $y^+ \in X^m$

01 Choose $w'$ such that $m \leq w'$.

02 Define $s : [1 - m, w'] \to [1, m]$ so that $\cup_{j=1}^{w'} \{ s(j) \} = [1, m]$ and

$$s(i - m) = i \text{ for all } i \in [1, m].$$

(4.1)

03 Define $\pi : [1, w'] \times [1, m] \to [1 - m, w']$ to be

$$\pi (j, i) = \max \{ j' : s(j') = i, j' \leq j \}.$$ (4.2)

04 Define $p : [1, w'] \to [1 - m, w']$ to be

$$p(j) = \pi(j - 1, s(j)).$$ (4.3)

05 Define $\varepsilon_i^{-m,1} := y_i^0$ for all $i \in [1, m]$.

06 Let $x_0^+ \leftarrow d - y_0^0 - \cdots - y_m^0$.

07 For $j = 1, 2, \ldots, w'$

08 $z \leftarrow x_j^+ + e_{p(j),j}$

09 $x_j^+ \leftarrow P_{\mathcal{C}_j}(z)$

10 $e_{j,j} \leftarrow z - x_j^+$

11 SHQP greedy step:

12 Choose a subset $Q_j$ of $\{1, \ldots, m\}$.

13 For all $i \in Q_j$, let $P_{i,j} \supset \mathcal{C}_i$ be polyhedra such that

$$\delta^*(e_{p(j),i,j}, P_{i,j}) = \delta^*(e_{p(j),i,j}, \mathcal{C}_i).$$ (4.4)

14 Let $\{e_{p(j),i,j+1} \}_{i=1}^m$ be defined by

$$(e_{p(j),1,j+1}, \ldots, e_{p(j),m,j+1}) = \arg \min_{(\tilde{y}_1, \ldots, \tilde{y}_m)} \left\{ \frac{1}{2} \left\| d - \sum_{i=1}^m \tilde{y}_i \right\|^2 + \sum_{i=1}^m \delta^*(\tilde{y}_i, P_{i,j}) \right\}
\text{s.t. } \tilde{y}_i = e_{p(j),i,j} \text{ if } i \notin Q_j.$$

(In other words, only the components in $Q_j$ are changed from before.)

15 $x_j^+ = d - \sum_{i=1}^m e_{p(j),i,j+1}$.

16 End for

17 Let the vector $y^+ \in X^m$ be defined by $y_i^+ = e_{p(w',i),w'+1}$ for all $i \in [1, m]$.

**Remark 4.3.** (On $s(\cdot)$, $\pi(\cdot, \cdot)$ and $p(\cdot)$) The definitions of $s(\cdot)$, $\pi(\cdot, \cdot)$ and $p(\cdot)$ come from [HD97]. For $j \in [1, w']$, the index $s(j) \in [1, m]$ gives the index of the set being projected onto at the $j$th iteration. Once we substitute the definition of $\pi(\cdot, \cdot)$ in (4.2) onto the definition of the variable $p(j)$ in (4.3), we see that $p(j)$ is the most recent past index $j'$ for which $s(j') = s(j)$. To model the original Dykstra’s algorithm where the variables are sampled in a cyclic order, we can set $w' = m$ and $s(i) = i$ for all $i \in [1, m]$.

**Remark 4.4.** (Warmstart solutions) As studied in [Pan16], the definition of $\{e_{i^{-m,1}} \}_{i=1}^m$ that will allow for a warmstart iterate $y^0 \in X^m$. The case $y^0 = 0$ reduces to the original Dykstra’s algorithm with random order as explained in [HD97].

**Remark 4.5.** (Known properties of Dykstra’s algorithm) We could have written Algorithm 4.2 in terms of the vector $y \in X^m$, with this vector $y$ produced at the
jth iteration (before the SHQP step) being

\[(e_{\pi(j,1),j}, e_{\pi(j,2),j}, \ldots, e_{\pi(j,m),j}).\]

But the notations \(s(\cdot), \pi(\cdot, \cdot)\) and \(p(\cdot)\) used in [HD97] and adopted here allow us to reference intermediate calculations easily. From Algorithm 4.2 we have

\[x_j^+ \text{ Lines 6, 15, Alg. 4.2 } d - m \sum_{i=1}^m e_{\pi(j,i),j+1} \text{ for all } j \in [0, w'] \] (4.5)

and \(x_j^o \text{ Line 10, Alg. 4.2 } d - m \sum_{i=1}^m e_{\pi(j,i),j} \text{ for all } j \in [1, w']. \) (4.6)

Furthermore

\[x_0^+ \text{ Line 6, Alg. 4.2 } d - m \sum_{i=1}^m y_i^o, \text{ and } x_{w'}^+ \text{ Line 17, Alg. 4.2 } d - m \sum_{i=1}^m y_i^+ . \] (4.7)

The variable \(e_{j,j}\) can be written as

\[e_{j,j} \text{ Line 10, Alg. 4.2 } x_j^{+1} + e_{p(j),j} - PC_{s(j)}(x_{j-1}^+ + e_{p(j),j}) \]

\[\text{arg min}_e \frac{1}{2} \| x_j^{+1} + e_{p(j),j} - e \|^2 + \delta^*(e, C_{s(j)}) \]

\[\geq \text{arg min}_e \| d - \sum_{1 \leq i \leq m \atop i \neq s(j)} e_{\pi(i,j),j} - e \|^2 + \delta^*(e, C_{s(j)}). \]

(As is known [Han88, GM89], the second equation of (4.8) comes from the fact that the optimization problem in the second statement is the dual of

\[\min x \frac{1}{2} \| x_{j-1}^+ + e_{p(j),j} - x \|^2 + \delta(x, C_{s(j)}), \]

which has primal solution \(x = PC_{s(j)}(x_{j-1}^+ + e_{p(j),j})\) and dual solution \(e_{j,j}.\) So recalling the definition of \(v(\cdot)\) (see (1.3)) and matching the last formula in (4.8), we get the known result that evaluating \(e_{j,j}\) corresponds to minimizing the \(s(j)\)th coordinate while holding all other coordinates fixed. (The formula (4.8) also coincides with the BCGD algorithm mentioned in Subsection 1.2.)

If \(w' = m\) and \(s(i) = i\) for all \(i \in [1, m]\), then such a strategy corresponds to alternating minimization as discussed in Subsection 1.2. Hence if \(j_2 = j_1 + 1,\) then

\[v(e_{\pi(j_2,1),j_1}, e_{\pi(j_2,2),j_1}, \ldots, e_{\pi(j_2,m),j_1}) \leq v(e_{\pi(j_1,1),j_1}, e_{\pi(j_1,2),j_1}, \ldots, e_{\pi(j_1,m),j_1}). \] (4.9)

Thus \(v(\cdot)\) is nonincreasing as Algorithm 4.2 progresses.

**Remark 4.6.** (SHQP step) The supporting halfspace quadratic programming (SHQP) step in lines 12 to 15 of Algorithm 4.2 comes from the observation that the projection onto each set \(C_i\) performed in line 9 generates a supporting halfspace of the set \(C_i,\) and that the projection of a point onto the intersection of halfspaces is a relatively easy problem. See [Pan16] for more details.

We give two examples motivating the design of Algorithm 4.2.

**Example 4.7.** (Many sets of orthogonal constraints) Consider the problem

\[\min_x \frac{1}{2} \| x - d \|^2 \]

s.t. \(l_i \leq A_i x \leq u_i \text{ for } i \in \{1, \ldots, m\}.\)

Let the \(A_i \in \mathbb{R}^{m_i \times n}\) be such that the rows of \(A_i\) are orthonormal. This is the setting of the Algebraic Reconstruction Technique (ART). (See for example [CCC+12]...
Lemma 5.1. (Behavior when \( v(\cdot) \) sufficiently small) Suppose Assumption \( \ref{Assumption:Approximation} \) holds. Consider Algorithm \( \ref{Algorithm:ExtendedDykstra} \) with \( y^0 \in X^m \) as input and \( y^+ \in X^m \) as output. We have the following:

(A) For all \( i \in [1,m] \) and \( v \in X \), \( \delta^*(v,C_i) = 0 \) if and only if \( v \) lies in the normal cone of \( C_i \) at 0. For \( i \in I \) (see \( \ref{Assumption:BoundaryAssumption} \)), this means that \( v = 0 \), and for \( i \notin I \), it means that \( v \) lies in the positive hull of \( \{ f_{i,r} : r \in [1,K'] \} \).

Moreover, there is an \( \bar{\epsilon} > 0 \) such that if \( v(y^0) \leq \bar{\epsilon} \), then

1. For all \( j \in \{1,\ldots,w^0\} \), \( P_{C_{s(j)}}(x^e_{j-1} + e_{p(j),j}) = P_{C_{s(j)}}(x^e_{j-1} + e_{p(j),j}) \).
2. For all \( j \in \{1,\ldots,w^0\} \), \( \delta^*(e_{j,j},C_{s(j)} - P_C(d)) = 0 \).

Proof. We first prove the first statement in (A). If \( v \) lies in the normal cone of \( C_i \) at 0, then when you recall the definition of the support function \( \delta^*(\cdot,\cdot) \) in \( \ref{Definition:SupportFunction} \), we see that \( P_C(d) = 0 \) is a maximizer. Thus \( \delta^*(v,C_i) = \langle v,0 \rangle = 0 \). For the converse, suppose \( \delta^*(v,C_i) = 0 \). The definition of the support function tells us that the halfspace \( \{ x : \langle v,x \rangle \leq 0 \} \) contains \( C_i \). Moreover, 0 \( \in C_i \) lies on the boundary of this halfspace. It follows that \( v \) lies in the normal cone of \( C_i \) at 0. The second statement in (A) is elementary.

Next, we prove property (1). For each \( i \in [1,m] \), we write \( C_i \) as \( C_{i'} \cap \bar{C}_i \), where \( \bar{C}_i \) is the intersection of the halfspaces defining \( C_i \) that contain \( x_\infty \) in their interior. There is a \( \gamma > 0 \) such that \( B(x_\infty,\gamma) \), the ball with center \( x_\infty \) and radius \( \gamma \), is contained in \( \bar{C}_i \) for all \( i \in [1,m] \). Recall that the iterates in Algorithm \( \ref{Algorithm:ExtendedDykstra} \) give nonincreasing dual objective values. (See \( \ref{Lemma:NonincreasingDualObjective} \)). Hence if \( v(y^0) \leq \bar{\epsilon} \), then

\[
\frac{1}{2} \left\| d - \sum_{i=1}^{m} e_{\pi(j,i),j} - P_C(d) \right\|^2 \quad \leq \quad v(e_{\pi(j,1),j},e_{\pi(j,2),j},\ldots,e_{\pi(j,m),j}) \quad \leq \quad v(y^0) \quad \leq \quad \bar{\epsilon},
\]

which gives \( \| d - \sum_{i=1}^{m} e_{\pi(j,i),j} \|^2 \leq 2\bar{\epsilon} \), or in other words \( \| x^0_\infty - x_\infty \| \leq \sqrt{2\bar{\epsilon}} \) through \( \ref{Thm:PropertiesOfExtendedDykstra} \). Recall that \( x_j = P_{C_{s(j)}}(x_{j-1}^e + e_{p(j),j}) \). If \( \bar{\epsilon} \) is chosen to be such that \( \sqrt{2\bar{\epsilon}} \leq \gamma \), then \( \| x_j^0 - x_\infty \| \leq \sqrt{2\bar{\epsilon}} \) implies that \( x_j^0 \) cannot be on the boundaries of the halfspaces defining \( C_{s(j)} \) which contain \( x_\infty \) in their interior. Thus property (1) holds.
To get property (2), first observe that

\[ e_{j,j} \stackrel{\text{Line 10, Alg 4.2}}{=} \text{Alg} \quad (x_{j-1}^+ + e_{j}) - P_{C_{i}(j)}(x_{j-1}^+ + e_{j}) \quad (5.1) \]

\[ d \quad \text{Property} \quad \Rightarrow (1) \quad (x_{j-1}^+ + e_{j}) - P_{C_{i}(j)}^L(x_{j-1}^+ + e_{j}), \quad (5.2) \]

Hence \( e_{j,j} \) lies in the normal cone of \( C_{i}(j) \) at 0. We then apply (A). \( \square \)

The following is adapted from [DH94, Lemma 3.4]. (This is similar to [IP90, Lemma 2].)

**Lemma 5.2.** (Adaptation of [DH94] Lemma 3.4) Recall Assumption 5.2. Suppose

\[ x = d - \sum_{i \in I} \sum_{r=1}^{K'} \tilde{e}_{i,r}, \]

where \( \tilde{e}_{i,r} = \lambda_{i,r} f_{i,r} \) and \( \lambda_{i,r} \geq 0 \). There is a \( \hat{\epsilon} > 0 \) such that if \( \|x\| \leq \hat{\epsilon} \), then \( x \in L \) and \( d \in L \), where

\[ L = \text{span}\{f_{i,r} : (i, r) \in T\}, \]

and \( T = \{(i, r) : i \in [1, m], r \in [1, K'], \lambda_{i,r} > 0\} \).

Furthermore, if \( d \neq x_{\infty} \), then \( T \neq \emptyset \).

**Proof.** We now prove the first part. We have, by the definition of \( L \),

\[ x = d - \sum_{i \in I} \sum_{r=1}^{K'} \tilde{e}_{i,r} \in d + L. \]

Define

\[ \hat{\epsilon} := \min_{F \in \mathcal{F}} d(0, F) = \min_{F \in \mathcal{F}} d(x_{\infty}, F), \]

where \( d(p, D) \) is the distance of \( p \) to the set \( D \) and

\[ \mathcal{F} = \{ F = \text{span}\{f_{i,j} : (i, j) \in S\} + d : \]

\[ S \subset \{1, \ldots, m\} \times \{1, \ldots, K'\} \text{ and } d(0, F) > 0\}. \]

Note that \( \hat{\epsilon} \) exists and \( \hat{\epsilon} > 0 \) because there are only a finite number of subsets of \( \{1, \ldots, m\} \times \{1, \ldots, K'\} \) and \( \emptyset \subset \{1, \ldots, m\} \times \{1, \ldots, K'\} \). By the definition of \( \hat{\epsilon} \), either \( d(0, d + L) \geq \hat{\epsilon} \) or \( d(0, d + L) = 0 \).

From \( x \in d + L \) and \( \|x\| < \hat{\epsilon} \), we must have \( d(0, d + L) = 0 \). That is, \( d \in L \). It follows that \( x \in L \). This proves the first statement.

We now prove the last statement. If \( T \) were empty, then \( \tilde{e}_{i,r} = 0 \) for all \( i \in \{1, \ldots, m\} \) and \( r \in \{1, \ldots, K'\} \). This would imply \( d = x_{\infty} \), a contradiction. \( \square \)

We prove a proposition about the SHQP step.

**Proposition 5.3.** (Decrease in dual function) We have

\[ v(\epsilon_{\pi_1(j),j}, \ldots, \epsilon_{\pi_1(m),j}) \leq v(\epsilon_{\pi_1(j-1),j}, \ldots, \epsilon_{\pi_1(j-1,m),j}) - \frac{1}{2} \|x_j^0 - x_{j-1}^+\|^2 \text{ for all } j \in [1, w'], \]

and

\[ v(\epsilon_{\pi_1(j),j+1}, \ldots, \epsilon_{\pi_1(m),j+1}) \leq v(\epsilon_{\pi_1(j),j}, \ldots, \epsilon_{\pi_1(m),j}) - \frac{1}{2} \|x_j^+ - x_{j}^0\|^2 \text{ for all } j \in [1, w']. \]
Suppose further that \( \delta^*(e_{\pi(j),j}, C_i) = \delta^*(e_{\pi(j),j+1}, C_i) = 0 \) for all \( i \in [1, m] \). Then

\[
v(e_{\pi(1),1}, \ldots, e_{\pi(m),m}) = \frac{1}{2} \| x_j^+ \|^2 \text{ for all } j \in [1, w'], \tag{5.4a}
\]

and

\[
v(e_{\pi(1),1+1}, \ldots, e_{\pi(m),m+1}) = \frac{1}{2} \| x_j^+ \|^2 \text{ for all } j \in [0, w']. \tag{5.4b}
\]

**Proof.** The formulas in (5.4) are straightforward from (1.3) and the assumptions. In view of (1.3), (5.3a) is equivalent to

\[
v(e_{\pi(1),1}, \ldots, e_{\pi(m),m}) \leq v(e_{\pi(j-1),1}, \ldots, e_{\pi(j-1,m),m}) - \frac{1}{2} \| e_{p(j),j} - e_{j,j} \|^2. \tag{5.5}
\]

We now show that inequality (5.5) holds. Let \( i^* = s(j) \), which implies \( \pi(i, i^*) = j \). We also note that \( e_{\pi(i),j} = e_{\pi(j-1),j} \) if \( i \neq i^* \). Then (5.5) can be written as

\[
\frac{1}{2} \| d - e_{j,j} - \sum_{1 \leq i \leq m, i \neq i^*} e_{\pi(i,i),i} \|^2 + \delta^*(e_{j,j}, C_i^*) \leq \frac{1}{2} \| d - e_{p(j),j} - \sum_{1 \leq i \leq m, i \neq i^*} e_{\pi(i,i),i} \|^2 + \delta^*(e_{p(j),j}, C_i^*) - \frac{1}{2} \| e_{p(j),j} - e_{j,j} \|^2.
\]

Since \( e_{j,j} \) is the minimizer to the function

\[
e \mapsto \frac{1}{2} \| d - e - \sum_{1 \leq i \leq m, i \neq i^*} e_{\pi(i,i),i} \|^2 + \delta^*(e, C_i^*),
\]

(see remark (4.5), which is strongly convex with modulus 1, we see that (5.5) holds. To prove that (5.3b) holds, we look at the following chain of inequalities:

\[
\frac{1}{2} \| d - \sum_{1 \leq i \leq m, i \in Q_j} e_{\pi(i,i),i} \sum_{1 \leq i \leq m, i \in Q_j} e_{\pi(i,i),i+1} \|^2 + \sum_{1 \leq i \leq m, i \in Q_j} \delta^*(e_{\pi(i,i),i+1}, C_i) \leq \frac{1}{2} \| d - \sum_{1 \leq i \leq m, i \in Q_j} e_{\pi(i,i),i+1} \|^2 + \sum_{1 \leq i \leq m, i \in Q_j} \delta^*(e_{\pi(i,i),i+1}, P_{i,j}) \leq \frac{1}{2} \| d - \sum_{1 \leq i \leq m, i \in Q_j} e_{\pi(i,i),i} \|^2 + \sum_{1 \leq i \leq m, i \in Q_j} \delta^*(e_{\pi(i,i),i}, P_{i,j}) - \frac{1}{2} \| \sum_{1 \leq i \leq m, i \in Q_j} (e_{\pi(i,i),i} - e_{\pi(i,i),i+1}) \|^2.
\]

(5.3)

The first inequality holds because \( P_{i,j} \supset C_i \) implies that \( \delta^*(., P_{i,j}) \geq \delta^*(., C_i) \). The second inequality holds because the variables \( e_{\pi(j,i),j+1} \) are the minimizers of a block coordinate minimization problem whose smooth function is quadratic. Notice that by the definition of \( x_j^+ \) and \( x_j^0 \) from (4.5) and (4.6) that

\[
\sum_{1 \leq i \leq m, i \in Q_j} (e_{\pi(i,i),i+1} - e_{\pi(i,i),i}) = x_j^+ - x_j^0.
\]

(5.7)

Combining (5.6) and (5.7) gives (5.3b). \( \square \)

The following result is immediate from Proposition 5.3.
Corollary 5.4. Recall the iterates of Algorithm 4.2. If \( \delta^*(e_{(j,i),j}, C_i) = \delta^*(e_{(j,i),j+1}, C_i) = 0 \) for all \( i \in [1,m] \) and \( j \in [1,w'] \), then
\[
\|x^+_w - 0\|^2 \leq \|x^+_0 - 0\|^2 - \sum_{j=1}^{w'} \|x^+_j - x^+_0\|^2 + \|x^+_0 - x^+_{j-1}\|^2.
\]

Proof. Sum up the terms in (5.3). Recalling the definition of \( v(\cdot) \) in (1.5), substitute in (5.4) and multiply by 2. \( \square \)

Recall the definition \( I \) in (3.3). We make the following definitions
\[
\bar{A} := \{(i,r) \in [1,m] \times [1,K'] : (f_{i,r},0) = c_{i,r}\} = ([1,m] \setminus I) \times [1,K'],
\]
(5.8)
\[
L_{\bar{A}} := \text{span}\{f_{i,r} : (i,r) \in \bar{A}\}.
\]
We thus have \( L_{\bar{A}} = \{x : \langle f_{i,r}, x \rangle = 0 \text{ for all } (i,r) \in \bar{A}\} = \cap_{(i,r) \in \bar{A}} H_{i,r}. \) The following result is well known. (For example, [IP90] cited [Go80].)

Lemma 5.5. (Regularity in system of equations) There is a constant \( \bar{\mu} > 0 \) such that for all \( x \), \( \max_{(i,r) \in \bar{A}} d(x, H_{i,r}) > \bar{\mu}d(x, L_{\bar{A}}^+) \).

We continue with the proof of the asymptotic linear convergence.

Theorem 5.6. (Asymptotic linear convergence) Recall the conditions in Lemmas 5.1 and 5.2 and the constant \( \mu \) from Lemma 5.4. Suppose in Algorithm 4.2, we have \( \|x_0\| \leq \min(\bar{\epsilon}, \sqrt{2\bar{\epsilon}}) \) and \( \delta^*(y^k_i, C) = 0 \) for all \( i \). Suppose further that
\( \text{(1)} \) For all \( i \in [1,m], j \geq \pi(w',i) \) implies that the polyhedron \( P_{i,j} \) is the half-space \( P_{i,j} = \{x : \langle e_{\pi(w',i),\pi(w',i),j}, x \rangle \leq 0\} \).
\( \text{(2)} \) For all \( i \in [1,m], j < \pi(w',i) \) implies that the polyhedron \( P_{i,j} \) is the intersection of halfspaces with 0 on their boundaries.

Then we have
\[
\sqrt{1 + \frac{\bar{\epsilon}}{\mu^2}} \|x^+_w - 0\| \leq \|x^+_0 - 0\|.
\]
(5.9)

Proof. Our assumptions ensure that we can apply the conclusions of those lemmas. Consider the point \( x^+_w \), which can be written as \( x^+_w = d - \sum_{i \in I} e_{\pi(w',i), w'+1} \) by making use of (5.5) and Lemma 5.1.

From Lemma 5.4, the condition that \( \delta^*(y^k_i, C) = 0 \) for all \( i \), and conditions (1) and (2), we see that \( \delta^*(e_{j,k}, C_{(j)}') = 0 \) for all \( k \in [1, w'+1] \) and \( j \leq k \). Moreover, for \( j \in [1, w'] \) and \( k \in [j, w'+1] \), we can write
\[
e_{j,k} = \sum_{r=1}^{K'} \bar{e}_{j,r,k}, \quad (5.10)
\]
where \( \bar{e}_{j,r,k} = f_{s_{(j)}, r, \lambda_j, r, k} \) for some \( \lambda_j, r, k \geq 0 \). We can assume that \( \bar{e}_{j,r,j} \) are chosen so that they are the multipliers to the projection step \( x^+_j = P_{C_{(j)}'}(\epsilon) \) in line 9 of Algorithm 4.2 which would give
\[
\bar{e}_{j,r,j} \neq 0 \text{ implies } x^+_j \in H_{s_{(j)}, r}. \quad (5.11)
\]

So
\[
x^+_w \overset{\text{def.}}{=} d - \sum_{i \in I} f_{i,r} \epsilon_{\pi(w',i), r, w'+1}. \quad (5.12)
\]

Let \( A = \{(i,r) : \bar{e}_{\pi(w',i), r, w'+1} \neq 0\} \), and let \( L_A = \text{span}\{f_{i,r} : (i,r) \in A\} \). It is clear that \( x^+_w \in d + L_A \). By Lemma 5.1 we have \( A \subset \bar{A} \). By Lemma 5.2 we have
lies in the hyperplane $H$. The Cauchy Schwarz inequality gives

$$\left\| x_{w',i}^+ - 0 \right\| = d(x_{w',i}^+, L_A^+) \quad (5.13)$$

By Lemma 5.5, there is a $\mu > 0$ such that $d(x, L_A^+) \leq \frac{1}{\mu} \max_{(i,r) \in A} d(x, H_{i,r})$ for all $x$. (The $\mu > 0$ can be chosen to be independent of $A$ by taking the infimum over all $A \subset A$.) Thus

$$d(x_{w',i}^+, L_A^+) \leq \frac{1}{\mu} \max_{(i,r) \in A} d(x_{w',i}^+, H_{i,r}). \quad (5.14)$$

Let $H_{i,r}$ be a hyperplane such that the maximum in (5.14) is attained. By the condition (1) in the theorem statement, the vectors in $H$ are all multiples of $e_{\pi(w',\tilde{r}),\pi(w',\tilde{r})}$. Since $(\tilde{i}, \tilde{r}) \in A$, we have (from the definition of $A$) that $e_{\pi(w',\tilde{r}),\pi(w',\tilde{r})} \neq 0$, which implies $e_{\pi(w',\tilde{r}),\pi(w',\tilde{r})} \neq 0$. Therefore, the point $x_{\pi(w',\tilde{r})}^+$ lies in the hyperplane $H_{i,r}$ by Lemma 6.1. The usual triangular inequality implies that

$$d(x_{w',i}^+, H_{i,r}) \leq \left\| x_{\pi(w',\tilde{r})}^+ - x_{\pi(w',\tilde{r})}^0 \right\|$$

$$\leq \sum_{j=\pi(w',\tilde{r})+1}^{w'} \left[ \left\| x_j^+ - x_j^0 \right\| + \left\| x_j^0 - x_{j-1}^+ \right\| \right]$$

$$\leq \sum_{j=1}^{w'} \left[ \left\| x_j^+ - x_j^0 \right\| + \left\| x_j^0 - x_{j-1}^+ \right\| \right].$$

The Cauchy Schwarz inequality gives

$$\left\| x_{w',i}^+ - 0 \right\|^2 \leq \frac{1}{\mu^2} \left( \sum_{j=1}^{w'} \left[ \left\| x_j^+ - x_j^0 \right\| + \left\| x_j^0 - x_{j-1}^+ \right\| \right] \right)^2 \quad (5.16)$$

$$\leq \frac{2w'\mu}{\mu^2} \left( \sum_{j=1}^{w'} \left[ \left\| x_j^+ - x_j^0 \right\|^2 + \left\| x_j^0 - x_{j-1}^+ \right\|^2 \right] \right).$$

Therefore,

$$\left\| x_{w',i}^+ - 0 \right\|^2 \leq \left\| x_0^+ - 0 \right\|^2 - \sum_{j=1}^{w'} \left[ \left\| x_j^+ - x_j^0 \right\|^2 + \left\| x_j^0 - x_{j-1}^+ \right\|^2 \right]$$

$$\leq \left\| x_0^+ - 0 \right\|^2 - \frac{\mu^2}{2w'} \left\| x_{w',i}^+ - 0 \right\|^2.$$

Rearranging the above gives us (5.19) as needed.

6. Asymptotic Linear Convergence 2: Adapting [DH94]

In this section, we show that the iterations in Algorithm 4.2 result in an asymptotic linear convergence of the primal objective value when the sets $C_i$ are polyhedral.

6.1 Preliminaries and results from [DH94]. In this subsection, we list the assumptions we make, and also recall some results from [DH94] useful for the proof of asymptotic linear convergence.

Lemma 6.1. (See [DH94] Lemma 3.2) If $H$ is a closed linear variety in a Hilbert space $X$, then $P_H(\cdot)$ is "affine", that is,

$$P_H \left( \sum_{i=1}^{n} \alpha_i x_i \right) = \sum_{i=1}^{n} \alpha_i P_H(x_i)$$
for all \( x_i \in X \) and any \( \alpha_i \in \mathbb{R} \) which satisfy \( \sum_{i=1}^{n} \alpha_i = 1 \).

In particular, if \( A \) is any nonempty subset of \( X \), then
\[
P_H(\text{co}(A)) = \text{co}(P_H(A)).
\]

We have the following result that was proved within bigger results in [DH94].

**Lemma 6.2.** (Local behavior of Dykstra-like iterations) Let \( X \) be a Hilbert space. Let \( H \subseteq X \) be the halfspace \( \{ x : \langle x, f \rangle \leq 0 \} \), where \( \|f\| = 1 \), and let \( H \) be the hyperplane \( \{ x : \langle x, f \rangle = 0 \} \).

1. For any \( x \in X \) and \( \lambda \geq 0 \), we have \( P_H(x + \lambda f) \in \text{co}\{x, P_H(x)\} \).
2. If \( x + \lambda f - P_H(x + \lambda f) \neq 0 \), then \( P_H(x + \lambda f) = P_H(x + \lambda f) \).

**Proof.** We can easily check that \( P_H(x + \lambda f) = P_H(x) \). If \( P_H(x + \lambda f) = P_H(x) \), then (1) holds. If \( P_H(x + \lambda f) \neq P_H(x + \lambda f) = P_H(x) \), then the only possibility is that \( x + \lambda f \in \text{int}H \), so that \( P_H(x + \lambda f) = x + \lambda f \). The conclusion (1) can also be easily checked. Conclusion (2) is also easy to check. \( \square \)

### 6.2. Proof of result

In this subsection, we present the asymptotic linear convergence result and its proof.

We have the following generalization of [DH94] Theorem 3.3.

**Theorem 6.3.** (An estimate of iterates in Dykstra’s algorithm) Let \( m_1 \) and \( m_2 \) be two integers. For \( i = 1, 2 \), define \( H_j^{(i)} \), where \( j \in \{1, \ldots, m_i\} \), are subspaces of a Hilbert space \( X \). Define sets \( E^{(i)} = \{0, 1\}^{m_i} \) and \( D^{(i)} = \{0, 1\}^{m_i} \) such that \( E^{(i)} \supseteq D^{(i)} \). Let \( x_1, x_2 \) and \( x_3 \) be points in \( X \). Define \( K(E^{(i)}, D^{(i)}, x_i) \) by
\[
K(E^{(i)}, D^{(i)}, x_i) := \text{co}\{\bar{P}_{S^{(i)}}(x_i) : S^{(i)} \in \{0, 1\}^{m_i} \text{ and } E^{(i)} \supseteq S^{(i)} \supseteq D^{(i)} \},
\]
where
\[
\bar{P}_{S^{(i)}}(x_i) = Q_{S^{(i)}, m_i}Q_{S^{(i)}, m_i - 1} \cdots Q_{S^{(i)}, 1}(x_i),
\]
and
\[
Q_{S^{(i)}, j} = \begin{cases} P_{H_j^{(i)}} & \text{if } S_j^{(i)} = 1 \\ I & \text{if } S_j^{(i)} = 0. \end{cases}
\]

If \( x_{i+1} \in K(E^{(i)}, D^{(i)}, x_i) \) for \( i \in \{1, 2\} \), then \( x_3 \in K([E^{(1)}, E^{(2)}], [D^{(1)}, D^{(2)}], x_1) \), where \([E^{(1)}, E^{(2)}]\) and \([D^{(1)}, D^{(2)}]\) are the concatenation of the respective vectors, and the hyperplanes are relabeled accordingly.

**Proof.** Note from the definition of \( K(\cdot, \cdot, \cdot) \) that our result would hold if we can prove that
\[
\text{co}\{\bar{P}_{S^{(2)}}(\text{co}\{\bar{P}_{S^{(1)}}(x_1) : E^{(1)} \supseteq S^{(1)} \supseteq D^{(1)} \}) : E^{(2)} \supseteq S^{(2)} \supseteq D^{(2)} \} \subseteq \text{co}\{\bar{P}_{S^{(2)}}\bar{P}_{S^{(1)}}(x_1) : E^{(i)} \supseteq S^{(i)} \supseteq D^{(i)} \text{ for } i = 1, 2\}.
\]

We write down a claim whose proof is embedded within its statement.

**Claim:** Suppose \( P_1 \) and \( P_2 \) are affine operators in the sense of Lemma 6.1. Let \( A \) be any set in \( X \). From the fact that \( P_2(P_1(x)) = (P_1P_2)(x) \), we have
\[
(P_2P_1)(\text{co}(A)) = P_2(\text{co}(P_1(A)))
\]
\[
\text{Lem. 6.1} \quad P_2(\text{co}(P_1(A))) \subseteq P_2(\text{co}(P_1(A)))
\]
\[
\text{Lem. 6.1} \quad \text{co}(P_2(P_1(A))) = \text{co}(P_2(P_1(A))).
\]
By making use of the principle in (6.4), the term in (6.3a) can be seen to be

\[ \text{co}\left\{ \text{co}\{P_{S^{(2)}}(x_1)\} : E^{(1)} \geq S^{(1)} \geq D^{(1)} \} : E^{(2)} \geq S^{(2)} \geq D^{(2)} \} \text{.} \quad (6.5) \]

To prove (6.5) ⊂ (6.3b), note that this inclusion can be phrased as

\[ \text{co}\left\{ \text{co}\{p_{i,j} : i \in I^*\} : j \in J^* \} \subset \text{co}\{p_{i,j} : i \in I^*, j \in J^* \}, \]

where \( I^* \) and \( J^* \) are two index sets and \( p_{i,j} \in X \) corresponds to \( P_{S^{(2)}}(x_1) \). It is clear every element on the left hand side of (6.6) can be written as a convex combination of the \( p_{i,j} \), so (6.6) holds. Thus we are done.

**Proposition 6.4.** (Dual problem in breaking up \( C_i \)) Recall Assumption 3.2 and Algorithm 4.2. Suppose that \( s(j) = i \). Recall from Remark 4.7 that

\[ e_{j,i} = \arg\min_{\bar{e}} \left[ \frac{1}{2} \left\| d - \sum_{1 \leq i' \leq m, i' \neq i} e_{\pi(i,i'), j} - e_{\bar{e}} \right\|^2 + \delta^*(e, C_i) \right] \text{.} \quad (6.7) \]

Recall also that \( C_i = \bigcap_{r=1}^{K} H_{i,r} \). Define the function \( \nu' : X^K \to \mathbb{R} \cup \{\infty\} \) by

\[ \nu'(y''_1, \ldots, y''_K) := \frac{1}{2} \left\| x_{j-1}^+ + e_{\pi(j,i'), j} \right\|^2 + \sum_{r=1}^{K} \delta^*(y'', H_{i,r}) \text{.} \quad (6.8) \]

If \( y''_r \) are chosen such that

\[ \sum_{r=1}^{K} y''_r = e_{\pi(j,i)}, \text{ and } \sum_{r=1}^{K} \delta^*(y'', H_{i,r}) = \delta^*(e_{\pi(j,i)}, C_i) \text{,} \quad (6.9) \]

then

\[ \nu'(y''_1, \ldots, y''_K) + \sum_{1 \leq i' \leq m, i' \neq i} \delta^*(e_{\pi(i,i'), j}, C_\nu) = \nu(e_{\pi(j,i)}, \ldots, e_{\pi(j,m), j}) \text{.} \quad (6.10) \]

Moreover, let \( (y'_1, \ldots, y'_K) \) be such that

\[ (y'_1, \ldots, y'_K) = \arg\min_{(y''_1, \ldots, y''_K)} \nu'(y''_1, \ldots, y''_K) \text{.} \quad (6.11) \]

Then the term \( e_{j,i} \) is equal to \( e_{j,i} = \sum_{r=1}^{K} y'_r \).

**Proof.** The equality (6.10) follows directly from (6.9) and (6.7), and how \( \nu(\cdot) \) and \( \nu'(\cdot) \) are defined in (1.5) and (6.8). The formula for \( e_{j,i} \) follows from the background theory of Dykstra’s algorithm.

We recall a warmstart Dykstra’s algorithm for finding \( e_{j,i} \) from \( e_{\pi(j,i)} \).

**Algorithm 6.5.** (Warmstart Dykstra’s algorithm) Consider the problem of finding \( e_{j,i} \) from \( e_{\pi(j,i)} \) in lines 8 to 10 of Algorithm 4.2, and suppose \( s(j) = i \). Let \( e_{\pi(j,i)} = \sum_{r=1}^{K} \lambda_r \cdot \bar{f}_{i,r} \), where \( \lambda_r \geq 0 \) for all \( r \in [1 - K', 0] \). The sequence \( \lambda_r \) is defined as follows.

\begin{align*}
01 & \quad x_0 = x_{j-1}^+ \\
02 & \quad \text{for } t = 1, 2, \ldots \\
03 & \quad x_t = P_{H_{i,[t]}}(x_{t-1}^+ + \lambda_t \cdot \bar{f}_{i,[t]}) \\
04 & \quad \text{let } \lambda_t \text{ be such that } x_t^+ + \lambda_t \cdot \bar{f}_{i,[t]} = x_{t-1}^+ + \lambda_t \cdot \bar{f}_{i,[t]} \\
05 & \quad \text{end for}
\end{align*}
If the $\lambda_r$ were chosen so that $\lambda_r = 0$ for all $r \in [1 - K', 0]$, then Algorithm 6.3 reduces to Dykstra’s algorithm. The $\{\lambda_r\}_{r=1-K'}$ are warmstarts to Dykstra’s algorithm, so we refer to Algorithm 6.3 as a warmstart Dykstra’s algorithm. The fact that $\lim_{t \to \infty} x_t^r = P_{C_r}(x_0^r + e_p(j))$ (even with the nonzero warmstart values $\lambda_r$ for $r \in [1 - K, 0]$) follows from some simple changes to the Boyle-Dykstra theorem (see [Pan16].)

The next result is a relationship between $x_j^0$ and $x_{j-1}^+$.

**Proposition 6.6.** (Dykstra steps leads to projection form) Suppose that in Algorithm 6.3, we have $\|x_0^r\| \leq \min(\hat{t}, \sqrt{2t})$ and that $\delta^*(y_i^r, C_i) = 0$ for all $i \in [1, m]$. Recall the definition of $K(\cdot, \cdot, \cdot)$ in Theorem 7.5. We can find $E, D \in \{0, 1\}^2$ such that

$$x_j^0 \in K(E, D, x_{j-1}^+),$$

$$E_t = 1 \text{ for all } t \in [1, \hat{t}] \text{ and } D_t = \begin{cases} 1 & \text{if } \lambda_t > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

1. For $t \in \{1, \ldots, \hat{t} - 1\}$, the $t$th hyperplane is $H_{i,[t]}$ (see 3.3), where $[t]$ is the integer in $\{1, \ldots, K'\}$ such that $K'$ divides $t - [t]$.
2. The $t$th hyperplane (i.e., the last hyperplane), which we call $H_j$, is the intersection of some of the hyperplanes $H_{i,[t]}$ such that $E_t = D_t = 1$.

Furthermore, the dual vector $e_{j,j}$ is in the conical hull of $\{f_{i,[t]} : E_t = D_t = 1\}$.

**Proof.** We make use of Proposition 6.4 and solve (6.11) using Algorithm 6.5. By Lemma 5.1(2), $y_{i,r}^r = 0$ for all $r \in [K' + 1, K]$, so we can ignore the indices $K' + 1$ to $K$. For example, the summands “$\sum_{r=1}^{K'}$” in (6.8) can be replaced by “$\sum_{r=1}^{K'}$” instead. The starting variables $(y_i^1, \ldots, y_i^{K'})$ are chosen so that $y_i^r = \lambda_r - K'f_{i,[r]}$ for $r \in [1, K']$, $\lambda_r = 0$, $\lambda_r = 1 - K'$, are nonnegative numbers like in Algorithm 6.3 and

$$\sum_{r=1}^{K'} y_i^r = e_{p(j),j},$$

which would satisfy (6.9). Let $\{x_t^r\}_{t=0}^\infty$ be the iterates generated by Algorithm 6.3. The choice of $\{y_i^r\}_{r=1}^{K'}$ gives $x_0 = x_{\hat{t}-1}^+$. By the convergence properties of a warmstart Dykstra’s algorithm, we have $\lim_{t \to \infty} x_t^r = x_0^r$.

Through Lemma 6.2 line 3 of Algorithm 6.5 implies that $x_t^r \in K(\{1\}, \{0\}, x_{t-1}^r)$, where the hyperplane involved is $H_{i,[t]}$. Also, there are indices $t$ for which $\lambda_t > 0$, which implies that $x_t^r = P_{H_{i,[t]}}(x_{t-1}^r + \lambda_t - K'f_{i,[t]})$, or $x_t^r \in K(\{1\}, \{1\}, x_{t-1}^r)$. Define $T_t^r$ and $T_t^r$ by

$$T_t^r := \{t' : t' \leq t, \lambda_{t'} > 0\},$$

$$T_t^r := \{t' : t' \leq t, \lambda_{t'} > 0\}.$$

Since $T_t^r \subset \{1, \ldots, K'\}$ and is monotonically increasing, there is some $\hat{t}$ such that $T_t^r = T_{\hat{t}-1}$ for all $t \geq \hat{t} - 1$. Let $\hat{T}^r$ be $T_{\hat{t}-1}$. Using Theorem 6.3 we obtain

$$x_{\hat{t}-1}^r \in K(E, D, x_{\hat{t}}^r),$$

where $E, D \in \{0, 1\}^{\hat{t}-1}$ are as defined in (6.12). (We still have to resolve $E_t$ and $D_t$.) By Lemma 5.2 we can increase $\hat{t}$ if necessary so that if $t \geq \hat{t} - 1$, then

$$x_j^0 - x_t^r \in \text{span}\{f_{i,[t]} : t' \in [t - K' + 1, t], \lambda_{t'} > 0\}.$$
Moreover, \( \bar{t} \) can be large enough so that if \( t \geq \bar{t} - 1 \), then \( \lambda_{t'} > 0 \) and \( t' \in [t-K'+1,t] \)
implies \( x^+_j \in H_{i,[t']} \). This implies that the projection of \( x^+_{t-1} \) onto \( H_j \), where \( H_j \) is defined by
\[
H_j = \cap \{ H_{i,[t]} : t \in [\bar{t} - K', \bar{t} - 1], \lambda_t > 0 \}.
\]
equals \( x^+_j \). In other words \( x^+_j = P_{H_j}(x^+_{t-1}) \). Since
\[
\{ [t] : t \in [\bar{t} - K', \bar{t} - 1], \lambda_t > 0 \} \subset T^*,
\]
(2) holds, and the previous discussions show that (1) holds.

Lastly, we show that \( e_{j,j} = x^+_{j-1} + e_{p(j),j} - x^+_j \) is in cone \( \{ f_i,[t'] : t' \in T^* \} \). Note that
\[
x^+_{j-1} + e_{p(j),j} - x^+_j \quad \text{equals} \quad \sum_{r=K+1}^{\infty} \lambda_r f_i,[t]
\]
which lies in cone \( \{ f_i,[t'] : D_{t'} = 1 \} \) if \( t \) is large enough. Since cone \( \{ f_i,[t'] : t' \in T^* \} \) is a closed convex cone and
\( \lim_{t \to \infty} x^+_t = x^+_j \) by Dykstra’s algorithm, we have that \( e_{j,j} = x^+_{j-1} + e_{p(j),j} - x^+_j \) is
in cone \( \{ f_i,[t] : [t] \in T^* \} \) as needed. \( \square \)

Next, we show that the similar thing happens to SHQP steps.

**Proposition 6.7.** (SHQP steps leads to projection form) Consider the assumptions in Proposition 6.6. Suppose \( \|x^+_0\| \leq \min(\bar{t}, \sqrt{2}) \) and that \( \delta^*(\tilde{y}_i, C_i) = 0 \) for all \( i \in [1,m] \). Suppose that in the SHQP step of Algorithm 4.2 whenever \( i \in Q_j \), the polyhedron \( P_{t,j} \) is the intersection of halfspaces of the form

(H1) \( H_{i,r} \), where \( r \in [1,K'] \) or

(H2) \( \{ x : \langle e_{j,j'}, x \rangle \geq 0 \} \) where \( j' \leq j \) and \( s(j') = i \).

Then we have \( x^+_j \in K(E,D,x^+_0) \) for some \( E,D \in \{0,1\}^i \), where \( E_t = 1 \) for all \( t \in [1,i] \), and

1. For \( t \in \{1,\ldots,i-1\} \), the \( t \)th hyperplane, which we call \( \tilde{H}_t \), is the boundary of a halfspace of either the kind (H1) or (H2) above.

2. The \( t \)th hyperplane (i.e., last hyperplane) is the intersection of some of the hyperplanes mentioned in the previous point (1) for which \( D_t = 1 \).

**Proof.** The proof is almost exactly the same as in Proposition 6.6. We show how to find the dual variables \( e_{\pi(i,1),j+1}, \ldots, e_{\pi(i,m),j+1} \) from \( e_{\pi(i,1),j}, \ldots, e_{\pi(i,m),j} \). Recall the definition of \( Q_j \). The SHQP step can be phrased as the problem of finding \( e_{\pi(i,j),j+1} \) in \( Q_j \) from \( e_{\pi(i,j),j} \).

By repeating halfspaces defining each \( P_{t,j} \), where \( i \in Q_j \), if necessary, we can assume that each \( P_{t,j} \) is the intersection of \( K' \) halfspaces. We label the halfspaces used to form each \( P_{t,j} \) by \( H_{i,r} \), where \( r \in [1,K'] \). (Note that these halfspaces can be of the type (H2), and hence the tilde.) Consider the optimization problem

\[
\min_{\{ \hat{y}_{i,r} \}_{i \in Q_j, r \in [1,K]}} \frac{1}{2}\|d - \sum_{i \in Q_j} e_{\pi(i,j),i} - \sum_{i \in Q_j} \hat{y}_{i,r}\|^2 + \sum_{i,r=1}^{K} \delta^*(\hat{y}_{i,r}, \tilde{H}_{i,r}) \quad (6.14)
\]

Let the starting \( \{ \hat{y}_{i,r} \}_{i \in Q_j, r \in [1,K]} \) be such that
\[
\sum_{r=1}^{K} \hat{y}_{i,r} = e_{\pi(i,j),j} \quad \text{and} \quad \sum_{r=1}^{K} \delta^*(\hat{y}_{i,r}, \tilde{H}_{i,r}) = \delta^*(e_{\pi(i,j),j}, C_i).
\]

An optimal solution of (6.14), say \( \{ \hat{y}^+_i \}_{i \in Q_j, r \in [1,K]} \), would allow us to reconstruct \( e_{\pi(i,j),j+1} \) by
\[
e_{\pi(i,j),j+1} = \sum_{r=1}^{K} \hat{y}^+_i.
The primal iterates can be estimated using a warmstart Dykstra’s algorithm similar to that in the proof of Proposition 6.6.

Let $M$ and $N$ be closed subspaces in the Hilbert space $X$. The angle between $M$ and $N$ is the angle between 0 and $\pi/2$ whose cosine is given by

$$c(M, N) := \sup\{ |\langle x, y \rangle| : x \in M \cap [M \cap N]^{\perp}, \|x\| \leq 1, \quad y \in N \cap [M \cap N]^{\perp}, \|y\| \leq 1\}. $$

This definition is due to Friedrichs [Fri37].

We take the following two results concerning angles.

**Theorem 6.8. (Properties of $c(M, N)$)** Let $M$ and $N$ be closed subspaces of a Hilbert space $X$. We have the following results regarding $c(M, N)$.

1. [Deu01b, Corollary 9.37] If $M$ and $N$ are closed subspaces, one of which has finite codimension, in the Hilbert space $X$, then $c(M, N) < 1$.
2. [Deu01b, Lemma 9.5(8)] $c(M, N) = 0$ if $M \subset N$ or $N \subset M$.

We have the following result on the convergence rate of the method of alternating projections when the sets involved are linear subspaces.

**Theorem 6.9. (Consequence of [DH97, Theorem 2.7])** Let $M_i$ be linear subspaces for $i \in [1, k]$, $M = \bigcap_{i=1}^{k} M_i$, and

$$\alpha := \left[ 1 - \prod_{i=1}^{k-1} \left( 1 - c^2(M_i, \bigcap_{i=1}^{i-1} M_i) \right) \right]^{1/2}.$$  

Then $\left\| P_{M_k}P_{M_{k-1}} \cdots P_{M_1} - P_M \right\| \leq \alpha$.

**Proof.** This is easily seen to be a particular case of [DH97, Theorem 2.7]. We refer to their result for the most general version. □

We have the following theorem, adapting the proof of [DH94] Lemma 3.7.

**Theorem 6.10. (Asymptotic linear convergence)** Recall the conditions in Lemmas 5.1 and 5.2. Suppose in Algorithm 4.2, we have $\|x_0^+\| \leq \min(\hat{\epsilon}, \sqrt{2\bar{\epsilon}})$ and $\delta^*(y_i^+, C_i) = 0$ for all $i$. Suppose further that

1. For all $i \in [1, m]$, the polyhedron $P_{i,j}$ is the intersection of halfspaces of the form $H_{i,r}$ in (3.2) and halfspaces of the form $\{ x : \langle e_{j'}, j, x \rangle \leq 0 \}$, where $j' < j$ is such that $s(j') = i$.

Then there is a constant $\rho \in [0, 1)$ such that $\|x_{w^+}\| \leq \rho \|x_0^+\|$. 

**Proof.** Each $y_i^+$ can be written in terms of $\{y_{i,r}^+\}_{r=1}^{K_i}$ so that

$$y_i^+ = \sum_{r=1}^{K_i} y_{i,r}^+ \quad \text{and} \quad \delta^*(y_i^+, C_i) = \sum_{r=1}^{K_i} \delta^*(y_{i,r}^+, H_{i,r}).$$

Let

$$T = \{(i, r) : y_{i,r}^+ \neq 0 \}.$$ 

From Lemma 5.2, we have $x_{w^+}^+ \in \text{span}\{ f_{i,r} : (i, r) \in T \}$.

Next, we make use of Theorem 6.3 and Propositions 6.6 and 6.8 to see that

$$x_{w^+}^+ \in K(E, D, x_0^+)$$
for some \( E, D \in \{0, 1\}^l \). Let the hyperplanes defined in Propositions 6.6 and 6.7 for each \( t \in \{1, \ldots, l\} \) be \( \bar{H}_t \). Moreover, if \( \bar{H}_t \) equals \( H_{t_i} \) for some \( i \in [1, m] \) and \( r \in [1, K'] \), we refer to the normal vector \( f_{i,r} \) as \( \bar{f}_t \). Since \( x_{w^t}^+ \in \text{span}\{f_{i,r} : (i, r) \in T\} \), we recall the definition of \( K(\cdot, \cdot, \cdot) \) to get

\[
x_{w^t}^+ \in \text{span}\{f_{i,r} : (i, r) \in T\}K(E, D, x_{0^t}^+)
\]

Therefore, if \( \bar{H}_t \) is a hyperplane of the type in Proposition 6.6(2), or of the type in Proposition 6.7(1) as well, this time making use of the fact that \( \bar{f}_t \) gives \( t_i, r \in [1, m] \), we recall the definition of \( \text{span}\{f_{i,r} : (i, r) \in T\} \) to get

\[
x_{w^t}^+ \in \text{span}\{f_{i,r} : (i, r) \in T\}K(E, D, x_{0^t}^+)
\]

Reordering the sequence \( \{ t \in \{1, \ldots, l\} : S_t = 1 \} \) as \( \{ t_1, t_2, \ldots, t_{\bar{H}_t(S)} \} \) gives

\[
\|P_{\text{span}\{H_{t_i} : S_{t_i} = 1\}}\bar{P}_S\| \leq \|P_{\text{span}\{H_{t_i} : S_{t_i} = 1\}}\bar{P}_S - P_{\bar{H}_t(S)}\| \leq \|P_{\text{span}\{H_{t_i} : S_{t_i} = 1\}}\bar{P}_S - P_{\bar{H}_t(S)}\|.
\]

One can retrace from conditions (1) and (2) of both Propositions 6.6 and 6.7 that span\{f_{i,r} : (i, r) \in T\} \subset \text{span}\{H_{t_i} : D_t = 1\}. Also, if \( S \in \{0, 1\}^l \) is such that \( S \geq D \), then span\{H_{t_i} : D_t = 1\} \subset \text{span}\{H_{t_i} : S_t = 1\}. Combining with (6.16) gives

\[
\|x_{w^t}^+\| \leq \|x_{0^t}^+\| \max\{\|\text{span\{H_{t_i} : S_{t_i} = 1\}}\bar{P}_S\| : E \geq S \geq D\}. \tag{6.16}
\]

Now, since span\{H_{t_i} : S_t = 1\} = [\cap\{H_{t_i} : S_t = 1\}]^\perp, we have

\[
P_{\text{span\{H_{t_i} : S_{t_i} = 1\}}} = I - P_{\cap\{H_{t_i} : S_{t_i} = 1\}}. \tag{6.17}
\]

Therefore,

\[
P_{\text{span\{H_{t_i} : S_{t_i} = 1\}}}\bar{P}_S = I - P_{\cap\{H_{t_i} : S_{t_i} = 1\}}\bar{P}_S = \bar{P}_S - P_{\cap\{H_{t_i} : S_{t_i} = 1\}}. \tag{6.18}
\]

Reordering the sequence \( \{ t \in \{1, \ldots, l\} : S_t = 1 \} \) as \( \{ t_1, t_2, \ldots, t_{\bar{H}_t(S)} \} \) gives

\[
\|P_{\text{span\{H_{t_i} : S_{t_i} = 1\}}\bar{P}_S}\| = \|P_{\bar{H}_t(S)}P_{H_{t_{\bar{H}_t(S)-1}}} \cdots P_{H_{t_{2}}} P_{H_{t_{1}}} - P_{\cap\{H_{t_i} : S_{t_i} = 1\}}\| \leq \|P_{\bar{H}_t(S)}P_{H_{t_{\bar{H}_t(S)-1}}} \cdots P_{H_{t_{2}}} P_{H_{t_{1}}} - P_{\cap\{H_{t_i} : S_{t_i} = 1\}}\|. \tag{6.19}
\]

We now apply Theorem 6.9 to estimate the last term in (6.19). We look at what the \( \alpha \) in (6.15) would be for (6.19). For this sequence \( \{ t_1, \ldots, t_{\bar{H}_t(S)} \} \), let \( Q \) be the subset defined by

\[
Q := \{ l : \bar{H}_t = H_{t_i} \text{ for some } i \in [1, m] \setminus I \text{ and } r \in [1, K'] \}.
\]

**Claim:** If \( l \notin Q \), then

\[
\bar{H}_t \supset \cap_{i=1}^{l-1}\bar{H}_{t_i}. \tag{6.20}
\]

If \( l \notin Q \), then \( \bar{H}_t \) is a hyperplane of the type in Proposition 6.6(2), or of the type in Proposition 6.7(1)/(2). If \( \bar{H}_t \) is a hyperplane of the type in Proposition 6.6(2), then (6.20) holds since the last hyperplane is the intersection of hyperplanes for which \( D_t = 1 \). Condition (6.20) holds for the hyperplanes of the type in Proposition 6.7(1) as well, this time making use of the fact that \( e_{j,j} \) is in the conical hull of \( \{ f_{i,t} : E_t = D_t = 1 \} \) at the end of Proposition 6.6. Lastly, Condition (6.20) holds for the hyperplanes of the type in Proposition 6.7(2) for the same reason that it holds for the hyperplane of the type in Proposition 6.6(2). The proof of the claim is complete.

If \( l \notin Q \), then

\[
s_t^2 = 1 - \|H_{t_i, \cap_{i=1}^{l-1}H_{t_i}}\|^2. \tag{6.20}, \text{ Thm 6.3(2)} = 1. \tag{6.21}
\]
If \( l \in Q \), then
\[
\text{Estimate of decrease in dual objective function} \quad (6.19)
\]
\[
s_l^2 \leq 1 - c^2(H_{l_1}, \cap_{i=1}^{l-1} H_{l_i}) \leq 1 - c^2(H_{l_1}, \cap_{i \in Q} H_{l_i}). \quad (6.22)
\]
The formulas and (6.21) and (6.22) for when \( l \notin Q \) and \( l \in Q \), together with Theorem (6.8)(2), implies that only finitely many of the \( s_l^2 \) are less than 1, and that each \( s_l^2 \) takes only finitely many possibilities. Thus the term \( c \) in (6.19) takes on only finitely many possibilities in \([0, 1]\), so there is a constant \( \rho \in [0, 1] \) such that the last term in (6.19) lies in \([0, \rho]\). This ends our proof. \( \square \)

7. Nonasymptotic convergence properties in polyhedral problems

In this section, we recall Assumption [7.2] and look at the nonasymptotic convergence properties in polyhedral problems.

We prove a lower bound on the decrease in the dual objective function in one cycle.

**Proposition 7.1. (Estimate of decrease in dual objective function) Recall Assumption [7.2] and Algorithm [4.2].** Recall also that \( x_0^+ = d - \sum_{i=1}^{m} y_i^c \). Suppose \( y_i^c \) can be written in terms of \( \tilde{y}_{i,r} \) (where \( i \in [1, m], r \in [1, K]\)) so that
\[
y_i^c = \sum_{r=1}^{K} \tilde{y}_{i,r}, \quad (7.1a)
\]
and \( \delta^*(y_i^c, C_i) = \sum_{r=1}^{K} \delta^*(\tilde{y}_{i,r}, H_{i,r}) \). \( (7.1b) \)

Then
\[
v(y^o) - v(y^+) \geq \frac{1}{2w-1} \max_{r \in [1,\ldots,K]} \left\{ d(x_0^+, \mathcal{H}_{i,r}), \min_{i,r} \{ d(x_0^+, H_{i,r}), \|\tilde{y}_{i,r}\| \} \right\}^2. \quad (7.2)
\]

**Proof.** For each \( i \in \{1, \ldots, m\} \) and \( r \in \{1, \ldots, L\} \), we seek to show that \( v(y^o) - v(y^+) \geq \frac{1}{2w-1} d(x_0^+, \mathcal{H}_{i,r})^2 \). Define \( n(i) \) to be \( n(i) = \min \{ j' \geq i : s(j') = i \} \). (I.e., \( n(i) \) is the first positive index \( j' \) such that \( s(j') = i \).) Note that \( n(i) \leq w' \). We thus have
\[
v(y^+) - v(y^o) \leq \sum_{j=1}^{w'} \left[ \|x_{0,j}^+ - x_{j}^-\|^2 + \|x_{j-1}^+ - x_{j}^-\|^2 \right]
\]
\[
\leq -\|x_{0,j}^+ - x_{j}^-\|^2 - \sum_{j=1}^{n(i)-1} \left[ \|x_{0,j}^+ - x_{j}^-\|^2 + \|x_{j}^- - x_{j+1}^-\|^2 \right]
\]
\[
\leq -\frac{1}{2n(i)-1} \left[ \|x_{0,j}^+ - x_{j}^-\|^2 + \sum_{j=1}^{n(i)-1} [\|x_{0,j}^+ - x_{j}^-\| + \|x_{j}^- - x_{j+1}^-\|] \right]^2
\]
\[
\leq -\frac{1}{2n(i)-1} \|x_{0,j}^+ - x_{n(i)}^-\|^2. \quad (7.3)
\]
For any \( r \in [1, K'] \), note that the primal iterate \( x_{n(i)}^o \) lies in \( C_i \), and hence \( H_{i,r} \), so \( \|x_{0,j}^+ - x_{n(i)}^o\| \geq d(x_0^+, H_{i,r}) \). Together with (7.2), we have \( v(y^o) - v(y^+) \geq \frac{1}{2w-1} d(x_0^+, \mathcal{H}_{i,r})^2 \) for all \( i \), which addresses the first term in the maximum in (7.2).

Next, we show that for each \( i \in \{1, \ldots, m\} \) and \( r \in \{1, \ldots, L\} \),
\[
v(y^o) - v(y^+) \geq \frac{1}{2w-1} \min_{i,r} \{ d(x_0^+, H_{i,r}), \|\tilde{y}_{i,r}\| \}^2, \quad (7.4)
\]
which would complete the proof of this result. Fix some \((i, r)\) such that \(\|\bar{y}_{i,r}\| > 0\). When \(x^+_0 \notin \mathcal{H}_{i,r}\), we recall that \(x^0_{n(i)} \in C_i \subset \mathcal{H}_{i,r}\). This would imply that \(\|x^+_0 - x^0_{n(i)}\| \geq d(x^+_0, \mathcal{H}_{i,r})\), which gives
\[
\sum_{j=1}^{w'} [\|x^+_0 - x^+_j\|^2 + \|x^+_j - x_j\|] \geq d(x^+_0, \mathcal{H}_{i,r}). \tag{7.5}
\]
Another case when \((7.5)\) holds is when \(x^+_0 \in \mathcal{H}_{i,r}\) and there is some \(j^* \in \{1, \ldots, w'\}\) such that \(x^+_j \notin \mathcal{H}_{i,r}\) or \(x^+_j \notin \mathcal{H}_{i,r}\). If \((7.5)\) holds, an argument similar to \((7.3)\) gives
\[
v(y^o) - v(y^+) \geq \frac{1}{2w'} d(x^+_0, \mathcal{H}_{i,r})^2 \text{ for all } i, \tag{7.6}
\]
which implies \((7.4)\).

It remains to prove \((7.4)\) for the case when both \(x^+_0\) and \(x^+_j\) lie in \(\mathcal{H}_{i,r}\) for all \(j \in \{1, \ldots, w'\}\). Recall that the term \(x^0_{n(i)}\) is found from \(e_{n(i),n(i)}\), where \(e_{n(i),n(i)}\) is obtained by the method in Proposition 6.4.

The problem \((6.8)\) can be solved by a warmstart Dykstra’s algorithm (i.e., a block coordinate minimization of the coordinates) where one warms starts with \((y^0_1, \ldots, y^0_K) = (\bar{y}_{i,1}, \ldots, \bar{y}_{i,K})\). Suppose we now minimize the \(r\)th coordinate to get
\[
y^+_r := \arg \min_y v'(y^0_1, \ldots, y^0_{r-1}, y_r, y^0_{r+1}, \ldots, y^0_K). \tag{7.7}
\]
For convenience, let \(j = n(i)\). We label the resulting primal variable as \(x^r_{n(i)}\), which can be written in two ways
\[
x^r_{n(i)} = d - \sum_{1 \leq i' \leq K \atop i' \neq r} \bar{y}_{i',r} - y^r_{i,r} - \sum_{1 \leq i' \leq m \atop i' \neq i} e_{(j,i'\prime),j} = P_{\mathcal{H}_{i,r}} (d - \sum_{1 \leq i' \leq K \atop i' \neq r} \bar{y}_{i',r} - \sum_{1 \leq i' \leq m \atop i' \neq i} e_{(j,i'\prime),j}). \tag{7.8}
\]
Define \((y^0_1, \ldots, y^0_K)\) to be a minimizer of \(v'(\cdot)\) like in \((6.11)\). We have
\[
v'(y^0_1, \ldots, y^0_K) \leq v'(y^0_1, \ldots, y^0_{r-1}, y_r, y^0_{r+1}, \ldots, y^0_K). \tag{7.9}
\]
From the definitions of \(v(\cdot)\) and \(v'(\cdot)\) in \((1.5)\) and \((6.8)\), we have
\[
v'(y^0_1, \ldots, y^0_K) + \sum_{1 \leq i' \leq m \atop i' \neq i} \delta^*(e_{(j-1,i'\prime),j}, C_{i'}) = v(e_{(j-1,1),j}, \ldots, e_{(j-1,m),j}) \tag{7.9a}
\]
\[
v'(y^0_1, \ldots, y^0_K) + \sum_{1 \leq i' \leq m \atop i' \neq i} \delta^*(e_{(j,i'\prime),j}, C_{i'}) = v(e_{(j,1),j}, \ldots, e_{(j,m),j}). \tag{7.9b}
\]
Note that since \(i' \neq i = s(j)\), we have \(e_{(j,i'\prime),j} = e_{(j-1,i'\prime),j}\), which implies
\[
\sum_{1 \leq i' \leq m \atop i' \neq i} \delta^*(e_{(j-1,i'\prime),j}, C_{i'}) = \sum_{1 \leq i' \leq m \atop i' \neq i} \delta^*(e_{(j,i'\prime),j}, C_{i'}). \tag{7.10}
\]
By using the methods in Proposition \((6.8)\) we have
\[
v'(y^0_1, \ldots, y^0_{r-1}, y_r^+, y^0_{r+1}, \ldots, y^0_K) - v'(y^0_1, \ldots, y^0_K) \leq -\|x^r_{n(i)} - x^r_{n(i)-1}\|^2. \tag{7.11}
\]
These give the following chain of inequalities

\[ v(y^+) - v(y^0) \leq v(e_{(j,1)} + \ldots + e_{(j,m)_j}) - v(y^0) \]

(7.9a)

\[ v'(y_1, \ldots, y_K) + \sum_{i', j' \leq m, i' \neq j'} \delta^* (\pi_{(j,i')_j}, C_{i'}) - v(y^0) \]

(7.9b)

\[ \leq v'(y_1', \ldots, y_{r-1}') + \sum_{i', j' \leq m, i' \neq j'} \delta^* (\pi_{(j,i')_j}, C_{i'}) - v(y^0) \]

(7.9c)

\[ \leq v'(y_1'', \ldots, y_r'') - v'(y_1', \ldots, y_r') + v(e_{(j',1,1)} + \ldots + e_{(j',m,1)}) - v(y^0) \]

(7.9d)

To simplify discussions, let \( d' \) be the point marked in (7.7). The point \( d' \) also equals \( x_{n(i)-1}' + \tilde{y}_{i,r} \). If \( d' \in H_{i,r} \), then \( x_{n(i)}' = d' \) and \( x_{n(i)-1}' - x_{n(i)}' = r \), in which case

\[ v(y^+-y^0) \leq -\|\tilde{y}_{i,r}\|^2 \leq -\frac{1}{2w-1} \|\tilde{y}_{i,r}\|^2. \]

But if \( d' \notin H_{i,r} \), then \( x_{n(i)}' \in H_{i,r} \), in which case the term \( \gamma \) marked above satisfies \( \gamma \geq d(x_0^+, H_{i,r}) \), and the argument in (7.9) can be repeated to prove \( v(y^+)-v(y^0) \leq -\frac{1}{2w-1} d(x_0^+, H_{i,r})^2 \). This ends our proof.

Next, we prove the following.

**Proposition 7.2.** (Decrease in dual function when sufficiently far from 0) Recall \( x_0^+ = d - \sum_{i=1}^m y_i^+ \), \( \tilde{y}_{i,r} \) are defined as in (7.11), and \( P_C(d) = 0 \). For any \( \delta_0 > 0 \), we can find \( \delta_2 > 0 \) such that if \( \|x_0^+\| > \delta_1 \), then the formula (7.12) is bounded from below by a constant \( \delta_2 > 0 \).

**Proof.** Recall

\[ H_{i,r} := \{ \bar{x} : (f_{i,r}, \bar{x}) \leq c_{i,r} \}. \]

Let \( \delta_1 > 0 \) and suppose \( x_0^+ \) is such that \( \|x_0^+\| > \delta_1 \). We recall the following fact that can be inferred from Lemma 7.3:

(1) For all \( I' \subset \{1, \ldots, m\} \times \{1, \ldots, K\} \), let \( \bar{x}_{I'} \) be defined by \( P_{\gamma (i',\inj H_{i,r})} (d) \).

For any choice of \( I' \subset \{1, \ldots, m\} \times \{1, \ldots, K\} \) such that \( \{f_{i,r} : (i, r) \in I'\} \) is linearly independent, \( e \in \mathbb{R}^n \) and \( \hat{c}_{i,r} \in \mathbb{R} \) for all \( (i, r) \in I' \), let \( x \) be defined to be the projection of \( d - e \) onto

\[ \cap_{(i, r) \in I'} \{ \bar{x} : (f_{i,r}, \bar{x}) \leq \hat{c}_{i,r} \}. \]

(7.12)

Then for any \( \delta_1 > 0 \), there exists \( \delta_2 > 0 \) such that

\[ \|e\| \leq mK \delta_2 \text{ and } |\hat{c}_{i,r} - c_{i,r}| \leq \delta_2 \text{ for all } (i, r) \in I' \text{ implies } \|x - \bar{x}_{I'}\| \leq \delta_3. \]

(7.13)

Let

\[ \bar{x}_{I'} = P_{\gamma (i',\inj H_{i,r})} (d). \]
If $\bar{x}_{i′} \neq 0$, then $\bar{x}_{i′} \notin C$. Then for these $I′$, there are halfspaces $H_{i,r}$, where $i \in \{1, \ldots, m\}$ and $r \in \{1, \ldots, K\}$, for which $d(\bar{x}_{i′}, H_{i,r}) > 0$. Let

$\delta_4 := \min \{d(\bar{x}_{i′}, H_{i,r}) : d(\bar{x}_{i′}, H_{i,r}) > 0, i \in \{1, \ldots, m\}, r \in \{1, \ldots, K\}\}.$

Let $\delta_3 = \min(\delta_4/2, \delta_1)$, and let $\delta_2 > 0$ be chosen such that (7.3) holds.

Let the formula in the right hand side of (7.2) be $F$. We now prove that if $\|x_0^+\| > \delta_1$, then $F > \delta_2$. Seeking a contradiction, suppose $F \leq \delta_2$. For the decomposition of $y_i^0$ satisfying (7.1), let $I^o := \{(i, r) : \|\tilde{y}_{i,r}\| > \delta_2\}$. Since (7.2) is satisfied, we must have

$$d(x_0^+, H_{i,r}) \leq \delta_2 \text{ for all } (i, r) \in I^o.$$ 

Recall $x_0^+ = d - \sum_{i=1}^{m} y_i^0$, and that the $y_i^0$ can be decomposed as $y_i^0 = \sum_{r=1}^{K} \tilde{y}_{i,r}$ satisfying (7.1). We then write

$$x_0^+ = d - \sum_{i=1}^{m} \sum_{r=1}^{K} \tilde{y}_{i,r} = d - \sum_{(i,r) \in I^o} \tilde{y}_{i,r} - \sum_{(i,r) \notin I^o} \tilde{y}_{i,r}. \quad (7.14)$$

By Caratheodory’s theorem, we can find a subset $I′ \subset I^o$ and $\alpha_{i′} \geq 0$ for all $i \in I′$ such that $y_{i,r}^0 = \alpha_{i,r} f_{i,r}$ for all $(i, r) \in I′$, $\{f_{i,r} : (i, r) \in I′\}$ is linearly independent, and

$$\sum_{(i,r) \in I^o} \tilde{y}_{i,r} = \sum_{(i,r) \in I^o} y_{i,r}^0. \quad (7.15)$$

Hence

$$x_0^+ = d - \sum_{(i,r) \in I^o} \tilde{y}_{i,r} - \sum_{(i,r) \notin I^o} \tilde{y}_{i,r} = d - \sum_{(i,r) \in I^o} y_{i,r}^0 - \sum_{(i,r) \notin I^o} \tilde{y}_{i,r}. \quad (7.16)$$

Then $x_0^+$ is the projection of $d - \sum_{(i,r) \notin I^o} \tilde{y}_{i,r}$ onto halfspaces $\cap_{(i,r) \in I^o} \{\tilde{x} : \langle f_{i,r}, \tilde{x} \rangle \leq \langle f_{i,r}, x_0^+ \rangle\}$. (To see this, note that the nonzero multipliers $y_{i,r}^0$ correspond to halfspaces tight at $x_0^+$ and that (7.15) is satisfied.) Moreover, recall that the halfspaces $H_{i,r}$ are of the form

$$H_{i,r} := \{\tilde{x} : \langle f_{i,r}, \tilde{x} \rangle \leq c_{i,r}\}.$$

Since $d(x_0^+, H_{i,r}) \leq \delta_2$ for all $(i, r) \in I^o$, we have $\|\langle f_{i,r}, x_0^+ \rangle - c_{i,r} \| \leq \delta_2$ for all $(i, r) \in I^o$. $(\langle f_{i,r}, x_0^+ \rangle)$ plays the role of $\tilde{c}_{i,r}$. Note that

$$\left\|d - \left(\sum_{(i,r) \notin I^o} \tilde{y}_{i,r}\right)\right\| = \left\|\sum_{(i,r) \notin I^o} \tilde{y}_{i,r}\right\| \leq mK \delta_2.$$ 

By the choice of $\delta_2 > 0$ that satisfies property (1) above and the definition of $\bar{x}_{i′}$, we have $\|x_0^+ - \bar{x}_{i′}\| \leq \min(\delta_4/2, \delta_1)$. If $\bar{x}_{i′} \neq 0$, then there is a halfspace $H_{i,r}$ such that $d(\bar{x}_{i′}, H_{i,r}) \geq \delta_4$, in which case

$$d(x_0^+, H_{i,r}) \geq d(\bar{x}_{i′}, H_{i,r}) - \|x_0^+ - \bar{x}_{i′}\| \geq \delta_4/2.$$ 

In the case where $\bar{x}_{i′} = 0$, then $\|x_0^+\| \leq \delta_1$, which contradicts the choice of $\|x_0^+\| > \delta_1$. Thus we are done.

We now prove an elementary lemma involving projections onto polyhedra.

**Lemma 7.3.** (Sensitivity analysis of projections onto polyhedra) Let $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^m$, and assume that $A$ has linearly independent rows. Define the set $S$ by $\{x : Ax \leq b\}$. For $\tilde{b} \in \mathbb{R}^m$, define $\tilde{S}$ by $\{x : Ax \leq \tilde{b}\}$. Let $d$ and $\tilde{d}$ be in $\mathbb{R}^m$. For any $\delta_3 > 0$, there exists $\delta_2 > 0$ such that if $\|b - \tilde{b}\|_{\infty} \leq \delta_2$ and $\|d - \tilde{d}\|_2 \leq \delta_2$, then $\|P_S(d) - P_{\tilde{S}}(\tilde{d})\| \leq \delta_3$. \qed
Proof. The well known result on the nonexpansiveness of the projections gives us $\|P_S(d) - P_S(\tilde{d})\| \leq \|d - \tilde{d}\| \leq \delta_2$. Suppose $\delta_2 \leq \delta_3/2$.

Next, we prove $\|P_S(d) - P_S(\tilde{d})\| \leq \delta_3/2$. Let $e \in \mathbb{R}^m$ be the vector of all ones. The smallest feasible region under the condition $\|\tilde{b} - b\|_\infty \leq \delta_2$ is attained when $\tilde{b} = b - \delta_2 e$. This gives an upper bound of $\|d - P_S(\tilde{d})\|$, which we call $U$. Similarly, when $\tilde{b} = b + \delta_2 e$, then we get the lower bound of $\|d - P_S(d)\|$, which we call $L$. Let the $P_S(d)$ obtained in this case be $p^*$. For all other possible cases, $\|d - P_S(d)\| \in [L, U]$. Therefore both $P_S(d)$ and $P_S(\tilde{d})$ lie in the sphere with center $d$ and radius $U$, and in the halfspace $\{x : \langle d - p^*, x - p^* \rangle \leq 0\}$. One can use trigonometry to calculate that this region has diameter $2\sqrt{L^2 - U^2}$. This quantity goes to 0 as $\delta_2 \searrow 0$, so we can make $\delta_2$ small enough so that $\|P_S(d) - P_S(\tilde{d})\| \leq \delta_3/2$. Combining the previous paragraph completes the proof of this result. \hfill \Box

The next result shows the nonasymptotic convergence rate of the main algorithm, Algorithm 4.1.

Proposition 7.4. (Transitioning to asymptotic linear convergence) Consider Algorithm 4.2 being run on an instance of (1.1). Suppose Algorithm 4.2 is run so that

1. For all $i \in [1, m]$, if $i \in Q_1$, then the polyhedra $P_{i,j}$ are chosen to be the intersection of halfspaces that were produced by the projection process so far.
2. The $w'$ over all calls in Algorithm 4.2 are uniformly bounded by some $w$.

Then for an instance of the BAP (1.1) and a starting $y^0$ in Algorithm 4.2, there are $\delta_2 > 0$, $\bar{k} > 0$ and $\rho \in [0, 1)$ such that

(A) If $k < \bar{k}$, then $v(y^k) < v(y^{k-1}) - \delta_2$, and
(B) If $k > \bar{k}$, then $v(y^k) \leq \rho v(y^{k-1})$.

Proof. Let $\bar{\epsilon}$ and $\bar{\delta}$ be as defined in Lemmas 5.1 and 5.2 respectively. Suppose $\delta_1 > 0$ and $\delta_2 > 0$ are chosen to be small enough so that they satisfy Proposition 7.2 and $\frac{1}{2} \delta_1^2 + \bar{D} \delta_2 < \min(\bar{\epsilon}, \frac{1}{2} \bar{\delta}^2)$, where

$$D = \sum_{(i,r) \in G} d(0, H_{i,r}), \text{ and } G = \{(i,r) : c_{i,r} < \infty\}.$$ 

Denote the right hand side of (7.2) by $F(y^0)$. (Note the $\bar{y}_{i,r}$ and $x_0^+$ are derived from $y^0$.) We simplify $F(y^k)$ to be $F_k$. As long as $F_{k-1} > \delta_2$, we have $v(y^k) \leq v(y^{k-1}) - F_{k-1} < v(y^{k-1}) - \delta_2$.

Let $\bar{k}$ be the first $k$ such that $F_{k-1} \leq \delta_2$. By the condition $\frac{1}{2} \delta_1^2 + \bar{D} \delta_2 < \bar{\epsilon}$ and how $D$ is defined, we have that $v(y^{k-1}) < \bar{\epsilon}$. Lemma 6.1 implies that $\delta^*(e_j, C_{e(j)}) = 0$ for all $j > 0$, and property (1) implies that $\delta^*(e_{j,j'}, C_{e(j)}) = 0$ for all $j$ and $j'$ such that $j \leq j' \leq w'$. This in turn implies that $\delta^*(y^+, C_{e}) = 0$ for all $i \in [1, m]$.

The local convergence results (Theorems 5.6 and 6.10) would ensure local linear convergence. \hfill \Box

References

[BC11] H.H. Bauschke and P.L. Combettes, Convex analysis and monotone operator theory in Hilbert spaces, Springer, 2011.
[BD85] J.P. Boyle and R.L. Dykstra, *A method for finding projections onto the intersection of convex sets in Hilbert spaces*, Advances in Order Restricted Statistical Inference, Lecture notes in Statistics, Springer, New York, 1985, pp. 28–47.

[Bec15] A. Beck, *On the convergence of alternating minimization for convex programming with applications to iteratively reweighted least squares and decomposition schemes*, SIAM J. Optim. 25 (2015), no. 1, 185–209.

[BT13] A. Beck and L. Tetruashvili, *On the convergence of block coordinate descent type methods*, SIAM J. Optim. 23 (2013), no. 4, 2037–2060.

[CCC+12] Y. Censor, W. Chen, P. L. Combettes, R. Davidi, and G.T. Herman, *On the effectiveness of projection methods for convex feasibility problems with linear inequality constraints*, Comput. Optim. Appl. 51 (2012), 1065–1088.

[CP15] A. Chambolle and T. Pock, *A remark on accelerated block coordinate descent for computing the proximity operators of a sum of convex functions*, manuscript.

[Deu01a] F. Deutsch, *Accelerating the convergence of the method of alternating projections via a line search: A brief survey*, Inherently Parallel Algorithms in Feasibility and Optimization and their Applications (D. Butnariu, Y. Censor, and S. Reich, eds.), Elsevier, 2001, pp. 203–217.

[Deu01b], *Best approximation in inner product spaces*, Springer, 2001, CMS Books in Mathematics.

[DH94] F. Deutsch and H. Hundal, *The rate of convergence of Dykstra’s cyclic projections algorithm: the polyhedral case*, Numer. Funct. Optimiz. 15 (1994), no. 5-6, 536–565.

[DH97] F. Deutsch and H. Hundal, *The rate of convergence of the method of alternating projections II*, J. Math. Anal. Appl. 205 (1997), 381–405.

[Dyk83] R.L. Dykstra, *An algorithm for restricted least-squares regression*, J. Amer. Statist. Assoc. 78 (1983), 837–842.

[ER11] R. Escalante and M. Raydan, *Alternating projection methods*, SIAM, 2011.

[Fri37] K. Friedrichs, *On certain inequalities and characteristic value problems for analytic functions and for functions of two variables*, Trans. Amer. Math. Soc. 41 (1937), 321–364.

[GM89] N. Gaifke and R. Mathar, *A cyclic projection algorithm via duality*, Metrika 36 (1989), 29–54.

[Gof80] J.L. Goffin, *The relaxation method for solving systems of linear inequalities*, Mathematics of Operations Research 5 (1980), 388–414.

[Han88] S.P. Han, *A successive projection method*, Math. Programming 40 (1988), 1–14.

[HC08] G.T. Herman and W. Chen, *A fast algorithm for solving a linear feasibility problem with application to intensity-modulated radiation therapy*, Linear Algebra Appl. 428 (2008), 1207–1217.

[HD97] H.S. Hundal and F. Deutsch, *Two generalizations of Dykstra’s cyclic projections algorithm*, Math. Programming 77 (1997), 335–355.

[HWRL17] M. Hong, X. Wang, M. Razaviyayn, and Z. Luo, *Iteration complexity analysis of block coordinate descent methods*, Math. Program. 163 (2017), 85–114.

[KW88] S. Kayalar and H. Weinert, *Error bounds for the method of alternating projections*, Math. Control Signal Systems 1 (1988), 43–59.

[LP90] A.N. lusem and A.R. De Pierro, *On the convergence rate of Hildreth’s quadratic programming algorithm*, Mathematical Programming 47 (1990), 37–51.

[LT93] Z.-Q. Luo and P. Tseng, *Error bounds and convergence analysis of feasible descent methods: A general approach*, Ann. Oper. Res. 46 (1993), 157–178.

[Nes83] Y. Nesterov, *A method for solving a convex programming problem with rate of convergence O(1/k3/2)*, Soviet Math. Doklady 269 (1983), no. 3, 543–547, (in Russian).

[Pan15] C.H.J. Pang, *Set intersection problems: Supporting hyperplanes and quadratic programming*, Math. Program. Ser. A 149 (2015), 329–359.

[Pan16] C.H.J. Pang, *The supporting halfspace - quadratic programming strategy for the dual of the best approximation problem*, SIAM J. Optim. 26 (2016), no. 4, 2591–2619.

[SSW77] K.T. Smith, D.C. Solmon, and S.L. Wagner, *Practical and mathematical aspects of the problem of reconstructing objects from radiographs*, Bull. Amer. Math. Soc. 83 (1977), 1227–1270.
[ST13] A. Saha and A. Tewari, *On the nonasymptotic convergence of cyclic coordinate descent methods*, SIAM J. Optim. **23** (2013), no. 1, 576–601.

[TY09a] P. Tseng and S. Yun, *Block-coordinate gradient descent method for linearly constrained nonsmooth separable optimization*, J. Optim. Theory Appl. **140** (2009), 513–535.

[TY09b] ———, *A coordinate gradient descent method for nonsmooth separable minimization*, Math. Program. Ser. B **117** (2009), no. 117, 387–423.

[WL14] P.W. Wang and C.J. Lin, *Iteration complexity of feasible direction methods for convex optimization*, Journal of Machine Learning Research **15** (2014), 1523–1548.

[Yun14] S. Yun, *On the iteration complexity of cyclic coordinate gradient descent methods*, SIAM J. Optim. **24** (2014), no. 3, 1567–1580.

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