Nucleon matrix element of Weinberg’s CP-odd gluon operator from the instanton vacuum

C. Weiss

Theory Center, Jefferson Lab, Newport News, VA 23606, USA

Abstract

We calculate the nucleon matrix element of Weinberg’s dimension-6 CP-odd gluon operator $f^{abc}(\tilde{F}_{\mu\nu})^a(G_{\rho\pi})^b(G_{\sigma}^\rho)^c$ in the instanton vacuum. In leading order of the instanton packing fraction, the dimension-6 operator is effectively proportional to the topological charge density $(\tilde{F}_{\mu\nu})^a(G_{\rho\pi})^b(G_{\sigma}^\rho)^c$, whose nucleon matrix element is given by the flavor-singlet axial charge and constrained by the $U(1)_A$ anomaly. The nucleon matrix element of the dimension-6 operator is obtained substantially larger than in other estimates, because of the strong localization of the nonperturbative gluon fields in the instanton vacuum. We argue that the neutron electric dipole moment induced by the dimension-6 operator is nevertheless of conventional size.

Keywords: QCD composite operators, instanton vacuum, topological charge, CP-violation, neutron electric dipole moment

1. Introduction

The gluonic structure of the nucleon has become a topic of great interest in nuclear physics. It is expressed in matrix elements of gauge-invariant composite QCD operators between nucleon states and provides insight into the emergence of hadrons from color fields. Gluonic operators arise in the factorization of deep-inelastic scattering (DIS) processes and in the interaction of heavy quark systems with nucleons. Gluonic operators also appear as the result of processes at the electroweak scale and beyond, when the short-distance degrees of freedom are integrated out and represented by QCD operators, which are then transported to the hadronic scale by the renormalization group equation. In the context of CP-violation, one gluonic operator of particular interest is the dimension-6 spin-0 CP-odd Weinberg operator $O_{P6}$

$$O_{P6} = (g^3/16\pi^2) f^{abc}(G_{\mu\nu})^a(G_{\rho\pi})^b(G_{\sigma}^\rho)^c,$$

where $g$ is the QCD coupling, $f^{abc}$ are the $SU(3)$ structure constants, $(G_{\mu\nu})^a$ is the QCD field strength tensor and $(G_{\rho\pi})^b \equiv (1/2)\epsilon_{\rho\sigma\pi\nu}(G^{\sigma\nu})^b$ its dual (our conventions are given below; our definition of the operator follows Ref. [2]). The operator Eq. (1) appears in a scenario of CP-violation that avoids the strong CP problem [3] and can be connected to a nonzero electric dipole moment (EDM) of the nucleon [4], whose experimental study is the object of extensive efforts [4]. Because the operator cannot be measured directly in other processes, the program relies on theoretical estimates of the hadronic matrix elements. The operator Eq. (1) also appears in the $1/m_Q$ expansion of heavy quark contributions to hadron structure \[3,4,5,6\].

Estimating hadronic matrix elements of gluonic operators such as Eq. (1) is a challenging problem. Lattice QCD calculations are limited by the fact that higher-dimensional operators mix with lower-dimensional ones of the same quantum numbers with power-divergent coefficients, requiring accurate nonperturbative treatment of the operator mixing [7]. Estimates using effective models of hadron structure such as the quark model [10] are uncertain because these models usually do not specify how the effective degrees of freedom match with the non-perturbative gluon fields of QCD. Methods based on QCD vacuum structure, such as the estimate of Ref. [2] or the QCD sum rule calculations of Refs. [11,12], can trace the connection between QCD fields and hadronic structure and are appropriate to the problem, but face difficulties in implementing the strong correlations affecting matrix elements of higher-dimensional operators. The approach of Refs. [13,14] connects Eq. (1) with operators appearing in higher-twist corrections to the nucleon spin structure functions, but relies on additional assumptions for estimating the matrix element (see below).

The instanton-based description of the QCD vacuum is a framework that can provide realistic estimates of hadronic matrix elements of gluon operators such as Eq. (1). Its basic assumption is that the non-perturbative gluon fields relevant for hadron structure are localized fluctuations carrying topological charge, instantons and antiinstantons, with an average size $\bar{\rho} \sim 0.3$ fm much smaller than their average distance $\bar{R} \sim 1$ fm in 4-dimensional Euclidean spacetime \[15,16,17\]. This is supported by a large body of empirical evidence from Euclidean correlation functions and hadron structure, and by lattice QCD studies of field configurations using cooling and other techniques; see Refs. [18,19] for reviews. In particular, instantons cause the spontaneous breaking of chiral symmetry, through the fermionic zero modes associated with the topological charge, which explains their unique role in hadron structure [20]. Correlation functions and hadronic matrix elements can be evaluated in an expansion of the instanton packing fraction in the vacuum, $\bar{\rho}/\bar{R}$ (diluteness parameter), which provides both a formal calculational scheme and an intuitive physical picture.

A method for evaluating gluonic operators in the instanton vacuum was formulated in Ref. [21] and used to calculate nucleon matrix elements of operators such as the topo-
logical charge [21], higher-twist operators appearing in DIS [22][23][24], or the QCD energy-momentum tensor [25], as well as higher-dimensional vacuum condensates [26][27]. The normalization scale of the operators in these calculations is set by the average instanton size, \( \mu \sim \bar{p}^{-1} \approx 0.6 \text{ GeV} \), which defines the boundary between perturbative and nonperturbative modes in the instanton vacuum. The expansion in the packing fraction ensures that the method preserves general operator relations of QCD such as the low-energy theorems for the action density \( (G_{\mu\nu}^a)^a(G^{\mu\nu})^a \) from the scale anomaly [21] (see Ref. [28] for a recent review), the relation between the topological charge density \( g^2(G_{\mu\nu})^a(G^{\mu\nu})^a \) and the flavor-singlet axial current from the \( U(1)_A \) anomaly [21], and relations between higher-dimensional operators resulting from the QCD equations of motion [22]. These properties are critically important in estimating matrix elements of higher-dimensional gluonic operators such as Eq. (1).

In this note we evaluate the nucleon matrix element of the gluonic operator Eq. (1) in the instanton vacuum. In leading order of the packing fraction (in the field of a single instanton), the operator Eq. (1) is effectively proportional to the topological charge density, whose matrix element is given by the nucleon flavor-singlet axial charge and constrained by the \( U(1)_A \) anomaly of QCD. This circumstance allows us to estimate the nucleon matrix element of Eq. (1) with minimal model dependence. Our result is substantially larger than the estimate of Ref. [2], because of the strong localization of the nonperturbative gluon fields in the instanton vacuum. We comment on the results of other approaches in light of our findings.

The object of the present study is the nucleon matrix element of the operator Eq. (1), not its role in CP-violation or the neutron EDM. Even so, our framework allows us to comment on the neutron EDM induced by Eq. (1). We argue that, in spite of the large value of the nucleon matrix element of Eq. (1), the instanton vacuum suggests that the neutron EDM induced by the operator is suppressed and of similar size as obtained in earlier estimates [2]. These comments should be regarded as speculative and inviting of further study.

2. Calculation

We consider the dimension-6 spin-0 CP-odd gluon operator as defined by Eq. (1). The field strength tensor is \( (G_{\mu\nu})^a \equiv \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + ig f^{abc} B_\mu^b B_\nu^c \), with the gauge potential \( B_\mu^a \) defined such that the covariant derivative for fermions is \( \nabla_\mu = \partial_\mu - ig B_\mu^a (\sigma^a/2) \). The dual field strength tensor is \( (\tilde{G}_{\mu\nu})^a \equiv (1/2)\epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma})^a \) with \( \epsilon^{0123} = 1 \) and Minkowski metric (++--). As is standard in nonperturbative calculations, we use an alternative definition of the field strength and potential, in which the coupling constant is included in the fields, \( F_{\mu\nu} \equiv g G_{\mu\nu}, A_\mu \equiv g B_\mu, \) and \( F^\mu_\nu ≡ g \tilde{G}^\mu_\nu \); these fields correspond to those in Ref. [21] and the instanton literature. In terms of these fields the operator Eq. (1) is identically expressed as

\[
O_{P6} = (1/16\pi^2) f^{abc}(F_{\mu\nu}^a)^i(F^{\rho\sigma}^b)(F^{\rho\sigma}^c)^j.
\]  

(2)

The matrix element of the operator (at the space-time point \( x = 0 \)) between nucleon states with momenta \( p \) and \( p' \) is parametrized as

\[
\langle N(p')|O_{P6}|N(p)\rangle = A_{P6}(q^2) m_N U' i\gamma_5 U,
\]  

(3)

where \( q \equiv p' - p \) is the 4-momentum transfer, \( m_N \) is the nucleon mass, \( U' \) and \( U \) are the nucleon 4-spinors, and \( \gamma_5 \equiv -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \) (sign opposite to the Bjorken-Drell convention). The bilinear form vanishes at \( q = 0 \) and can be represented as

\[
m_N U' i\gamma_5 U = (i/2) S^\mu \tilde{q}_\mu, \quad S^\mu \equiv U' \gamma^\mu \gamma_5 U,
\]  

(4)

where \( S \) is the spin 4-vector of the \( N \to N' \) transition. The form factor \( A_{P6} \) is a function of the invariant \( q^2 \) and has dimension \((\text{mass})^3\). We prefer to define it as a dimensionful quantity and not absorb the dimension by powers of \( m_N \), as \( m_N \) is not the natural dynamical scale for this matrix element (see below).

Parallel to the dimension-6 operator, Eqs. (1) and (2), we consider the well-known dimension-4 operator

\[
O_{P4} \equiv (g^2/16\pi^2) (G_{\mu\nu}^a)^i(G^{\mu\nu})^a = (1/16\pi^2)(\tilde{F}_{\mu\nu}^a)(F^{\mu\nu}),
\]  

(5)

whose matrix element is parametrized as

\[
\langle N(p')|O_{P4}|N(p)\rangle = A_{P4}(q^2) m_N U' i\gamma_5 U,
\]  

(6)

where the form factor \( A_{P4} \) has dimension \((\text{mass})^0\). The matrix element can be inferred from the \( U(1)_A \) anomaly in QCD. This operator relation states that the divergence of the flavor-singlet axial current is given by an anomalous term proportional to the gluonic operator Eq. (5), and a regular term proportional to the quark mass and pseudoscalar density,

\[
\sum_f \partial_\mu (\bar{\psi}_f \gamma^\mu \gamma_5 \psi_f) = N_f O_{P4} + 2 \sum_f m_f \bar{\psi}_f i\gamma_5 \psi_f,
\]  

(7)

where \( \psi_f \) is the quark field, and the sum runs over the \( N_f \) light quark flavors. The nucleon matrix element of the flavor-singlet axial current is parameterized as

\[
\langle N(p')|\sum_f \bar{\psi}_f \gamma^\mu \gamma_5 \psi_f |N(p)\rangle = \mathcal{O}_f^0 \left( \gamma^\mu \gamma_5 G^0_A(q^2) - \frac{q^\mu q^\nu}{2m_N} G^0_P(q^2) \right) U,
\]  

(8)

where \( G^0_A \) and \( G^0_P \) are the axial and pseudoscalar form factors. Combining Eq. (7) and Eq. (8), neglecting the quark mass dependent term, one obtains

\[
A_{P4}(q^2) = \frac{2}{N_f} \left( \frac{G^0_A(q^2) - \frac{q^2}{4m_N^2} G^0_P(q^2)}{G^0_A(q^2)} \right).
\]  

(9)

The flavor-singlet pseudoscalar form factor \( G^0_P \) does not have a Goldstone boson pole, so that the second term is suppressed at small \( q^2 \) even in the limit of zero light quark masses. The form factor at \( q^2 = 0 \) is therefore given by (see e.g. Refs. [2][21])

\[
A_{P4}(0) = 2 g_A^0 \sqrt{N_f},
\]  

(10)

where \( g_A^0 \equiv G^0_A(0) \) is the nucleon flavor-singlet axial coupling. For evaluation in the instanton vacuum one expresses the operators Eqs. (2) and (5) in terms of the fields of the Euclidean
theory, using the standard conventions \((x^0, x^i) \rightarrow (\bar{x}_i, \bar{x}_4)\) with \(x^0 = -i\bar{x}_4, x^i = \bar{x}_i, \bar{x}_\mu = \bar{x}_4^\mu + \bar{x}_\mu^4\). The Minkowskian fields are related to the Euclidean ones (denoted by a bar) as

\[
F_{\mu \nu} = i\bar{F}_{\mu \nu}, \quad F_{ij} = \bar{F}_{ij}, \quad \bar{F}_{\mu \nu} = \bar{F}_{\mu \nu}, \quad \bar{F}_{ij} = -i\bar{F}_{ij},
\]

where \(\bar{F}_{\mu \nu}\) is related to \(F_{\mu \nu}\) with \(\epsilon_{1234} = 1\). We obtain

\[
\begin{align*}
(-i)\bar{O}_{\Phi_6} &= (1/16\pi^2) f^{abc} F^{\alpha c}_{\mu \nu} \bar{F}^{\alpha \mu}_{\rho \sigma} \bar{F}^{\rho \nu}_{\sigma \tau}, \\
(-i)\bar{O}_{\Phi_4} &= (1/16\pi^2) \bar{F}^{\alpha \mu}_{\nu \rho} F^{\alpha \mu}_{\nu \rho}.
\end{align*}
\]

In the following all coordinates and fields are understood as Euclidean, and the bar is dropped.

We now calculate the nucleon matrix elements of the operators in the instanton vacuum following the approach of Ref. \[21\]. The framework is the variational approximation to the interacting instanton ensemble \[17\], where all dynamical scales emerge from the scale inherent in the QCD coupling, consistent with the renormalization properties of QCD. The fermionic zero modes of the instantons cause the spontaneous breaking of chiral symmetry and give rise to an effective theory of massive quarks with multifermionic interactions \[20\]. Hadronic correlation functions are computed using the \(1/N_c\) expansion (saddle point approximation to the functional integral), and the nucleon emerges as a saddle point solution of massless quarks bound by a classical Goldstone boson field (chiral soliton) \[29\]. Gluon operators in the original instanton ensemble are systematically converted to "effective operators" in the effective theory of massive quarks, where they can be inserted in hadronic correlation functions for calculation of the matrix elements. The method relies on the smallness of the instanton packing fraction \(\bar{\rho}/\bar{R}\) and preserves the basic properties of the QCD operators at the level of the effective operators. We outline the steps of the calculation and refer to Ref. \[21\] for a detailed description of the method.

In the first step, in leading order of \(\bar{\rho}/\bar{R}\), the gluonic operators Eqs. \((12)\) and \((13)\) are evaluated in the field of a single instanton or antiinstanton (I or \(\bar{I}\)). The \(I(\bar{I})\) fields in singular gauge are (with size \(\rho\), standard color orientation, and center at \(x = 0\))

\[
F^{\alpha \mu}_{\nu \rho}(x)_{I(\bar{I})} = (\eta^\alpha)^{\rho \sigma} \left( \frac{x_{\mu} x_{\rho}}{\lambda} \delta_{\nu \sigma} + \frac{x_{\nu} x_{\rho}}{\lambda^2} \delta_{\mu \sigma} - \frac{1}{2} \delta_{\mu \nu} \delta_{\rho \sigma} \right) \times \frac{g_{\rho \sigma}}{(\lambda^2 + \rho^2)^2},
\]

\[
\bar{F}^{\alpha \mu}_{\nu \rho}(x)_{I(\bar{I})} = \mp F^{\alpha \mu}_{\nu \rho}(x)_{I(\bar{I})},
\]

where \((\eta^\alpha)^{\rho \sigma} = \eta_D^{\rho \sigma}\), \(\eta^\alpha_{\nu \rho}\) are the 't Hooft symbols. The \(I(\bar{I})\) fields take values in an \(SU(2)\) subalgebra of the \(SU(N_c)\) color algebra, in which the structure constants are \(f^{abc} = e^{abc}\) (with \(a, b, c\) in \(\{1, 2, 3\}\)). Computing the color sum in the dimension-6 operator Eq. \((12)\), we obtain

\[
f^{abc} \bar{F}^{\alpha \mu}_{\nu \rho} F^{\beta \rho}_{\kappa \lambda} \bar{F}^{\kappa \lambda}_{\mu \nu}(x)_{I(\bar{I})} = \mp \frac{3 \cdot 512}{\lambda^8} \rho^6 \approx \pm f_6(x | \rho).
\]

This should be compared with the well-known result for the dimension-4 operator Eq. \((13)\) in the \(I(\bar{I})\) field

\[
\bar{F}^{\mu \nu}_{\rho \sigma} F^{\rho \sigma}_{\mu \nu}(x)_{I(\bar{I})} = \mp \left( \frac{3 \cdot 64}{\lambda^8} \rho^4 \right) \approx \pm f_4(x | \rho),
\]

which, up to a factor, is the \(I(\bar{I})\)’s topological charge density.

In the second step, the effective fermion operators are constructed, by multiplying the gluonic operators in the \(I(\bar{I})\) field with the projectors on the fermionic zero modes, and averaging over the collective coordinates of the \(I(\bar{I})\) (size, color orientation, center coordinate) \[21\]. The average over the \(I(\bar{I})\) size is computed with the variational size distribution of Refs. \[17, 21\], which describes the suppression of large \(\rho\) by the instanton interactions in the ensemble; small \(\rho\) are suppressed by the vanishing free instanton weight. The distribution is concentrated around \(\rho \approx \bar{\rho}\), and its width is parametrically small, so that the average is computed simply by replacing the actual size \(\rho\) by the average size \(\bar{\rho}\) in the operators. Performing the average over the color orientation and the center coordinate \(z\), we obtain the effective operators corresponding to \(\bar{O}_{\Phi_6}\) and \(\bar{O}_{\Phi_4}\) (denoted by quotation marks) as

\[
\begin{align*}
\bar{O}_{\Phi_6(x)} &= \frac{\lambda}{16\pi^2} \int d^4z \ f_6(x - z | \bar{\rho}) \times [\det I_{J_1}(z) + \det I_{J_2}(z)], \\
\bar{O}_{\Phi_4(x)} &= \frac{\lambda}{16\pi^2} \int d^4z \ f_4(x - z | \bar{\rho}) \times [\det I_{J_1}(z) - \det I_{J_2}(z)],
\end{align*}
\]

\(\lambda \equiv (2V/N_c)^{5/4} M_{\pi}^N\).

Here \(\lambda\) is a dimensionful factor composed from the average instanton density \(N(2V)\) and the dynamical quark mass generated by chiral symmetry breaking, \(M\); the factor represents the dynamical scales emerging from the instanton vacuum and provides the normalization of the effective operator \(\bar{O}_{\Phi_4}\). (The present calculation is performed with an equal number of \(I\) and \(\bar{I}\) in the ensemble, \(N_c = N_c = N/2\), which is sufficient for connected correlation functions of the operators at nonzero momentum transfer; the role of topological fluctuations \(N_c \neq N_c\) is discussed below.) The functions

\[
J_{\lambda}(z)f_6 = \psi_7(z) F(\partial \psi(z) \psi_6(z)
\]

are the left- and right-handed chiral quark densities, where \(\psi_7 \equiv \bar{\psi} \psi\) are the Euclidean quark fields, and \(F(\psi)\) denote the

1. Because the gluon operators in Eqs. \((16)\) and \((17)\) are gauge-invariant and do not explicitly involve fermions, they can as well be evaluated using the instanton field in regular gauge, which has a simpler form than Eq. \((13)\).

2. The dimension-6 operator in the instanton field, Eq. \((16)\), scales as \(f_6(x | \rho) \sim \rho^{-6}\) when evaluated at \(x = \rho\), inside the instanton. The instanton size distribution at small \(\rho\) behaves as \(f_6(\rho) \sim \rho^{-5} \log(\Lambda_p)/\Lambda_p\), where \(b = (11/3) N_c\) is the coefficient of the \(\beta\) function in gluodynamics. Because \(b \sim N_c\), is parametrically large in the approach of Refs. \[13, 21\], the integral over \(\rho\) converges at small \(\rho\), and the contributions from \(\rho \ll \bar{\rho}\) should be suppressed if the QCD operator is regarded as normalized at the scale \(\mu = \bar{\rho}^{-1}\).
form factors resulting from the $I(\bar{I})$ fermionic zero modes of the $I(\bar{I})$ [21]. The $L_\epsilon(z)$ are $N_f \times N_f$ matrices in flavor, and the fermionic vertices appearing in the effective operators Eqs. (18) and (19) have the same flavor determinant form as the 't Hooft vertices appearing in the effective fermion action derived from instantons [21]. The differences between the left- and right-handed vertices appear because the gluonic operators are CP-odd and take opposite values in the $I$ and $\bar{I}$.

In the final step, the effective operators are inserted in hadronic correlation functions evaluated in the effective theory of massive quarks. If the momentum transfer to the gluonic operator is of the order $q \sim B^{-1} \ll \rho^{-1}$, the non-locality of the instanton field in Eqs. (18) and (19) can be neglected, and we can approximate the profile functions $f_6$ and $f_4$ by 4-dimensional $\delta$ functions, with coefficients fixed by the integral over $x$,

$$f_{6,4}(x-z) \rightarrow C_{6,4} \delta^4(x-z), \quad C_{6,4} \equiv \int d^4x \, f_{6,4}(x); \quad (22)$$

the difference between the infinite integral and a finite integral over a domain $x \sim B$ is a correction in $\rho/B$ because of the localization of the instanton field. Integrating the profile functions Eqs. (18) and (17) over $x$, we obtain

$$C_6 = 3 \cdot 128 \rho^2 / (5 \rho^2), \quad C_4 = 32 \pi^2; \quad (23)$$

the value of $C_4$ is the well-known instanton action. In this approximation the effective operators become

$$O_{P6}(x') = \left\{ -C_6 \right\} \times \frac{\lambda}{16\pi^2} \left[ \det I_\epsilon(x) - \det I_\epsilon(x^-) \right], \quad (24)$$

and one sees that the effective operators for $O_{P6}$ and $O_{P4}$ are the same up to the coefficients. Without calculating the nucleon correlation function of the operators, we therefore immediately conclude that the nucleon matrix elements are related by

$$\frac{A_{P6}(q^2)}{A_{P4}(q^2)} = -\frac{C_6}{C_4} = -\frac{12}{5\rho^2} \quad (q \sim B^{-1}). \quad (25)$$

This conclusion relies only on the fact that the operators enter in the correlation functions by coupling to a single $I(\bar{I})$, and on the specific value of the operators in the $I(\bar{I})$ fields. Now combining Eq. (25) with the value of $A_{P4}(0)$ obtained from the $U(1)_A$ anomaly, Eq. (2), we obtain

$$A_{P6}(0) = -\frac{12}{5\rho^2} \times \frac{2g_A^{(0)}}{N_f}. \quad (26)$$

This estimate uses the instanton vacuum only to relate the dimension-6 and 4 operators at the instanton scale but does not require the calculation of $g_A^{(0)}$ in the instanton vacuum.

Alternatively, we could estimate $A_{P6}(0)$ by performing the actual calculation of the nucleon matrix element of the effective operators $O_{P6}$ in the instanton vacuum. The result would be identical to Eq. (26), only with $g_A^{(0)}$ replaced by the value obtained from the instanton vacuum. This happens because the nucleon matrix element of the gluonic operator $O_{P4}$ calculated in the instanton vacuum is proportional to that of the divergence of the flavor-singlet axial current in the instanton vacuum, leading to the same relation as obtained from the $U(1)_A$ anomaly in QCD [21]. This remarkable result comes about as follows. The effective action of massive quarks emerging from chiral symmetry breaking in the instanton vacuum has the form

$$-S_{\text{eff}} = \int d^4x \left\{ \psi \partial^\mu \psi \psi + \lambda \left[ \det I_\epsilon(x) + \det I_\epsilon(x^-) \right] \right\}, \quad (27)$$

where $\lambda$ is the dynamical scale Eq. (20). The flavor-singlet axial current in the effective theory and its divergence can be derived from the response to a space-time dependent $U(1)_A$ chiral transformation of the quark fields,

$$\psi, \psi^\dagger \rightarrow \exp(i\gamma_5 x) \psi, \psi^\dagger \exp(i\gamma_5 x), \quad \epsilon \equiv e(x), \quad (28)$$

and one obtains

$$J_{\mu \epsilon}(x)_{\text{eff}} = \sum_f \psi^\dagger \gamma_5 \gamma_\mu \psi \frac{\lambda}{16\pi^2} \left[ \det I_\epsilon(x) + \det I_\epsilon(x^-) \right]. \quad (29)$$

This divergence has the same form as the effective operator for $O_{P4}$ in the effective theory, Eq. (24), with $C_4$ given by Eq. (23).

The instanton vacuum thus provides the same relation between the flavor-singlet axial current and the gluonic operator $O_{P4}$ as the $U(1)_A$ anomaly in QCD. Eq. (27). Note, however, that the nature of $U(1)_A$ symmetry breaking is very different in the two cases. In QCD the $U(1)_A$ symmetry is broken anomalously, by the integration over high-momentum modes of the fermion fields. The effective theory derived from instantons contains only low-momentum modes (momenta $k \lesssim \rho^{-1}$) and has no anomaly, but the $U(1)_A$ symmetry is broken explicitly by the dynamical scales embedded in the theory.

We can use Eq. (26) to estimate the numerical value of $A_{P6}(0)$. The ratio between the dimension-6 and 4 matrix elements is (using $\rho^{-1} = 0.60\text{ GeV}$)

$$12/(5\rho^2) = 0.86 \text{ GeV}^2 = (0.22 \text{ fm})^{-2}. \quad (31)$$

This represents a large value on the hadronic scale and attests to the strength of the localized non-perturbative gluon fields in the instanton vacuum. The flavor-singlet axial coupling $g_A^{(0)}$ can be determined in polarized DIS experiments (where it appears as the total light quark contribution to the nucleon spin, $\Delta\Sigma$) or in theoretical calculations. The polarized DIS analysis of Ref. [20] with three light flavors ($N_f = 3$) obtains $g_A^{(0)} = 0.36$ at the scale $\mu = 1\text{ GeV}$; other analyses report values in the range 0.20–0.25 with uncertainties up to $\sim 100\%$; see Ref. [31] for a compilation. Recent Lattice QCD calculations obtain values of $g_A^{(0)}$ in the range $-0.3$–$0.45$ [32, 33, 34]. A calculation in the chiral soliton model of the nucleon, based on the effective action Eq. (27), gives $g_A^{(0)} = 0.36$ at the scale $\mu \sim \rho^{-1}$. For a conservative estimate covering these values we assume

$$g_A^{(0)} = 0.36^{+0.09}_{-0.36} \quad (N_f = 3). \quad (32)$$

\^It is worth emphasizing that the $U(1)_A$ anomaly in QCD involves only the dimension-4 gluon operator $O_{P4}$, not any higher-dimensional operators. The relation between $O_{P6}$ and $O_{P4}$ observed in the instanton vacuum, Eq. (26), is specific to the effective theory with cutoff $\rho^{-1}$ and does not imply that $O_{P6}$ would somehow participate in the QCD anomaly along with $O_{P4}$. 

4
and neglect the scale dependence compared to the uncertainties. With these parameters we obtain

$$A_{P6}(0) = -(0.21^{+0.05}_{-0.21}) \text{GeV}^2. \quad (33)$$

This estimate refers to the instanton vacuum scale $\mu \sim \tilde{\rho}^{-1}$.

To justify our instanton vacuum calculation of the $O_{P6}$ and $O_{P4}$ matrix elements, we need to comment on the role of topological charge fluctuations. In Ref. [21] the instanton vacuum was constructed as a grand canonical ensemble with fluctuating numbers of $I$'s and $\bar{I}$'s, including topological charge fluctuations with $\Delta \equiv N_+ - N_- \neq 0$. It was shown that the topological susceptibility $(\Delta^2)$ is qualitatively affected by the fermion determinant and vanishes if at least one quark flavor becomes massless, in accordance with general expectations. The effective operator formalism was developed including $A$ fluctuations, which are essential for correlation functions of topological operators at zero momentum transfer (integrated over the system volume $V$), such as the topological charge $\int_V d^4x \tilde{F}\tilde{F}(x)$. In the present study we consider the connected correlation functions of the local operator $\tilde{F}\tilde{F}(x)$ at non-zero momentum transfer, which are not affected by $\Delta$ fluctuations and can be computed in the effective theory at $\Delta = 0$; the fluctuations experienced by the operator in a finite subvolume do not depend on the boundary conditions of the system [21]. In this sense the present calculation is consistent with the general treatment of Ref. [21]. Note, however, that when one considers correlation functions of $O_{P6}$ and $O_{P4}$ integrated over the system volume, one has to revert back to the ensemble with $A$ fluctuations [34].

3. Discussion

It is interesting to compare our estimate with those of other approaches. In Ref. [2] $A_{P4}(0)$ was estimated by assuming that the ratio of the nucleon matrix elements of the dimension-6 and dimension-4 pseudoscalar gluon operators is the same as the ratio of the vacuum matrix elements of the corresponding scalar gluon operators; in short-hand notation:

$$\frac{A_{P6}(0)}{A_{P4}(0)} = \frac{\langle N|\tilde{F}\tilde{F}F|N\rangle}{\langle N|\tilde{F}\tilde{F}|N\rangle} \approx \frac{\langle 0|\tilde{F}\tilde{F}F|0\rangle}{\langle 0|\tilde{F}\tilde{F}|0\rangle}. \quad (34)$$

The vacuum ratio was then evaluated using the empirical values of the gluon condensates determined in QCD sum rule calculations, resulting in $A_{P6}(0)/A_{P4}(0) \approx 0.13 \text{GeV}^2$. In our instanton-based approach Eq. (34) is indeed realized in leading order of the packing fraction, if the vacuum condensates are saturated by the instanton fields; this comes about because the $I(\bar{I})$ fields are self-dual (anti-self-dual), cf. Eq. (14). However, the numerical value of the ratio is obtained as in Eqs. (25) and (31). ~7 times larger than the estimate of Ref. [2]. The enhancement is due to the strong localization of the non-perturbative gluon fields in the instanton vacuum, which increases the value of higher-dimensional condensates relative to expectations based on a uniform distribution.

In Refs. [13, 14] the gluon operator $O_{P6}$ was related to other dimension-6 operators using the operator identity [expressed here in our conventions and for Minkowskian fields, cf. Eq. (2)]

$$\left(1/16\pi^2\right)\sum_f \delta^a_b \left[ \bar{\psi}_f(\tilde{F}\mu\nu)(\gamma^\mu/2) \gamma^\nu \psi_f \right] = \left(1/16\pi^2\right) f^{abc} F^{\mu\nu\rho\sigma} F_{\mu\nu}^{\rho\sigma}$$

$$- (1/32\pi^2) (\tilde{F}_{\mu\nu})^{(ab)}(D_{\rho})^{(cd)}(D_{\sigma})^{(ef)} F^{\mu\nu\rho\sigma}.$$  

$$\equiv O_{P6} + O_{P4}, \quad (35)$$

where $(D_{\rho})^{(ab)} \equiv \delta^{ab} \partial_\rho + f^{abc} A^c_\rho$ is the covariant derivative in the adjoint representation. Equation (35) is obtained from the QCD equations of motion for the fields and the algebraic identities for covariant derivatives. The dimension-5 spin-1 quark-gluon operator (the operator under the total derivative) appears in the twist-4 corrections to the isoscalar nucleon spin structure functions [37], and its matrix element can be inferred from DIS data and theoretical calculations. The matrix element of $O_{P6}$ was then estimated in Ref. [14] assuming that the contributions of all three operators are of the same magnitude at the hadronic scale. The instanton vacuum suggests that this assumption is not justified, for the following reasons: (i) The matrix element of $O_{P6}$ estimated in Ref. [14] is substantially smaller than our instanton vacuum result. Expressed in our convention, the estimate of Ref. [14] is

$$A_{P6}(0) = -(g^2/16\pi^2) m_Q^2 E(0) \approx -(0.18 \text{GeV}^2) \times E(0), \quad (36)$$

where $E$ is the form factor introduced in Ref. [14]; in the last equation we have evaluated the QCD coupling $g^2$ using the well-known LO expression $(N_f = 3, \Lambda = 0.2 \text{GeV}, \mu = \tilde{\rho}^{-1} = 0.6 \text{GeV})$. With $E(0) \approx 0.01$ as used in Ref. [14], $A_{P6}(0)$ would be two orders of magnitude smaller than our result Eq. (35). (ii) The instanton vacuum predicts that the nucleon matrix elements of $O_{P6}$ and $O_{P4}$ are individually large but cancel each other in Eq. (35). Extending our calculation of the matrix element from $O_{P6}$ to $O_{P4}$, we obtain

$$A_{P4}(0) = -A_{P6}(0) \quad (\text{in leading order of } \tilde{\rho}/R). \quad (37)$$

In fact, the cancellation of $O_{P6}$ and $O_{P4}$ in one-instanton approximation is mathematically necessary, because the instanton is a (complex) solution to the free Yang-Mills equations without quark sources, and using the free equations in deriving the operator identity one would obtain zero instead of the quark-gluon operator in Eq. (35). We note that the cancellation of $O_{P6}$ and $O_{P4}$ is consistent with the earlier instanton vacuum result for the nucleon matrix element of the flavor-singlet twist-4 quark-gluon operator, which is of the order $M_{Q}^2 \sim \tilde{\rho}^2/R^4$ rather than $\tilde{\rho}^{-2}$ and thus parametrically suppressed [24]. (In contrast, the flavor-non-singlet nucleon matrix element is of order $\tilde{\rho}^{-2}$ and parametrically large [22].) In summary, the instanton vacuum shows that the matrix elements of the operators in Eq. (35) have very unequal magnitude, and that one cannot infer the “large” gluonic ones from the “small” quark-gluon one.

We also want to comment on the implications of our results for CP-violation and the neutron EDM. The scenario of Ref. [1] considers a CP-violating Lagrangian of the form

$$\delta L_{CP} = a_0 O_{P6}, \quad (38)$$
where the coefficient $a_0$ results from the CP-violating short-distance processes and includes the effects of renormalization group evolution to the hadronic scale. The nucleon EDM appears through the correlation function
\[ i \int d^4x \langle N| \mathcal{T} O_{P6}(x) J^\mu_{\text{em}}(0) |N \rangle, \tag{39} \]
which describes the electromagnetic vertex of the nucleon under the influence of the CP-odd Lagrangian. The correlation function can be computed by inserting a complete set of hadronic intermediate states, which include the nucleon state and “other” states (whose precise composition is not relevant),
\[ \sum_X |X \rangle\langle X| = |N \rangle\langle N| + \sum_{X \neq N} |X' \rangle\langle X'|. \tag{40} \]
In Ref. [2] it was assumed that Eq. (39) can be approximated by the contribution of the nucleon intermediate state, i.e., that there are no strong cancellations between the nucleon and the other intermediate states. This made it possible to compute the correlation function in terms of the nucleon matrix elements $\langle N'| O_{P6} |N \rangle$ and $\langle N'| J^\mu_{\text{em}} |N \rangle$; see Ref. [14] for details. The instanton vacuum suggests that this assumption is likely not valid. When Eq. (39) is computed in the instanton vacuum, we can suppose that the effective operator representing $O_{P6}$ is proportional to that of $O_{P4}$, see Eq. (42). The EDM extracted from the correlation function is thus proportional to that induced by $O_{P4}$, the traditional $\theta$ angle term. The latter is known to be proportional to the quark mass and suppressed in the chiral limit [5]. This chiral suppression requires the exact cancellation of the contributions of the nucleon and other intermediate states in Eq. (39) and is incompatible with the intermediate-nucleon approximation, as was already pointed out in Ref. [2]. One therefore should not use our result for the nucleon matrix element of $O_{P6}$, Eq. (39), to estimate the EDM in the approach of Refs. [2,14]; doing so would grossly overestimate it.

Instead, our findings suggest a different way to estimate the neutron EDM induced by $O_{P6}$. Assuming that the proportionality between the effective operators for $O_{P6}$ and $O_{P4}$ holds when computing the full correlation function Eq. (39) in the instanton vacuum, we can estimate the EDM induced by $O_{P6}$ by “converting” previous results for the EDM induced by $O_{P4}$. The CP-violating Lagrangian associated with $O_{P4}$ is
\[ \delta \mathcal{L}_{\text{CP}} = \langle \theta/2 \rangle O_{P4}, \tag{41} \]
where $\theta$ is the vacuum angle. Choosing $\theta/2 = (-12/5\pi^2) a_0$, we effectively match the strength of Eq. (41) to Eq. (38) in the instanton vacuum correlation functions; see Eqs. (23) and (25). For a numerical estimate of the neutron EDM induced by $O_{P6}$ we use the well-known result of Ref. [5] based on the non-analytic chiral term, $|a_0| = 3.6 \times 10^{-3} [\theta/\text{fm}]$, which takes into account the chiral suppression of the EDM. In this way we estimate the neutron EDM induced by $O_{P6}$ in the instanton vacuum as
\[ |a_0| \sim 6 \times 10^{-3} [\theta/\text{fm}] \tag{42} \]
This estimate is 4 times smaller than that of Ref. [2] obtained with the intermediate-nucleon approximation, even though our nucleon matrix element of $O_{P6}$ is larger by almost an order of magnitude (see above). While we do not propose Eq. (42) as a serious estimate of the neutron EDM, it illustrates the limits of the intermediate-nucleon approximation and demonstrates that the large nucleon matrix element of $O_{P6}$ obtained from instantons is compatible with a normal size of the neutron EDM.

A quantitative estimate of the EDM induced by $O_{P6}$ should be performed by calculating the full correlation function Eq. (39) in the instanton vacuum and studying its quark mass dependence. Such a calculation should confirm the chiral suppression of the $O_{P6}$ EDM conjectured here and determine both analytic and non-analytic terms in the quark mass dependence.

In summary, the instanton vacuum suggests an interesting picture of hadronic CP-violation induced by the dimension-6 operator $O_{P6}$. The nonperturbative dynamics effectively connects the operator with the topological charge density $O_{P4}$. The nucleon matrix element of $O_{P6}$ is large because of the strong localization of the instanton field. However, the neutron EDM induced by $O_{P6}$ is of conventional size, because it is subject to the same chiral suppression as that induced by $O_{P4}$. The picture should be explored in further studies.

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