A variational approach to the regularity of minimal surfaces of
annulus type in Riemannian manifolds

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Abstract

Given two Jordan curves in a Riemannian manifold, a minimal surface of annulus type
bounded by these curves is described as the harmonic extension of a critical point of some
functional (the Dirichlet integral) in a certain space of boundary parametrizations. The $H^{2,2}$-
regularity of the minimal surface of annulus type will be proved by applying the critical points
theory and Morrey’s growth condition.

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Morrey’s growth condition

1 Introduction

Extending the Ljusternik-Schnirelman Theory on convex sets in Banach spaces, a general
theory of critical points was developed in 1983 ([St1], see also [St2] [St3]), and an approach
to unstable solutions and Morse theory for Plateau’s problem of disc or annulus type in $\mathbb{R}^n$
was given. Here a minimal surface is described as the harmonic extension of a critical point
of the following functional, defined on a set of boundary parametrizations:

$$\mathcal{E}(x) := \frac{1}{2} \int |\mathcal{H}(x)|^2 d\omega,$$

where $\mathcal{H}$ denotes the harmonic extension in $\mathbb{R}^n$. $H^{2,2}$-regularity of the above minimal surface
was proved in the setting normalized by the integral condition (see [St1]). In [IS], further
details were given and similar results were obtained for the setting normalized by the three-
points condition.
Recently, in [Ho], the existence of unstable minimal surfaces of higher topological structure with one boundary in a nonpositively curved Riemannian manifold was studied by applying the method introduced in [St2], and the regularity of minimal surfaces was discussed.

In this paper, we want to give a similar regularity result for a minimal surface of annulus type in manifolds satisfying some appropriate conditions, namely, we will consider two boundary curves $\Gamma_1, \Gamma_2$ in a Riemannian manifold $(N, h)$ such that one of the following conditions holds.

(C1) There exists a point $p \in N$ with $\Gamma_1, \Gamma_2 \subset B(p, r)$, where $B(p, r)$ lies within the normal range of all of its points. Here we assume $r < \pi/(2\sqrt{\kappa})$, where $\kappa$ is an upper bound of the sectional curvature of $(N, h)$.

(C2) $N$ is compact with nonpositive sectional curvature.

These conditions are related to the existence and the uniqueness of the harmonic extension for a given boundary parametrization.

We first construct suitable spaces of functions, the sets of boundary parametrizations, where we have to distinguish the cases of (C1) and (C2). Then, following some idea of Struwe, we introduce a convex set which, in fact, serves as a tangent space for the given boundary parametrization. Moreover, we consider the following functional:

$$E(x) := \frac{1}{2} \int |d\mathcal{F}(x)|_h^2 d\omega,$$

where $\mathcal{F}(x)$ denotes the harmonic extension of annulus type in a manifold $N$ with metric $h$.

We may then describe a minimal surface as the harmonic extension of a critical point of $E$.

We will always use the fact that $N$ can be properly embedded into some $\mathbb{R}^k$ as a closed submanifold (see [Gr]).

Then we compute the $H^{2,2}$-regularity of our surfaces using the Morrey growth condition, see Section 3.2. We generalize the idea in [St1] to a minimal surface of annulus type in Riemannian manifolds of the above property.

## 2 Preliminaries

### 2.1 Some definitions

Let $(M, g)$ be a manifold of dimension 2 with boundary $\partial M$, metric $(g_{ij})$, and $(N, h)$ a connected, oriented, complete Riemannian manifold with metric $(h_{\alpha\beta})$ of dimension $n \geq 2$, embedded isometrically and properly into some $\mathbb{R}^k$ as a closed submanifold by $\eta$ (see [Gr]). Moreover, $\nabla$ and $\tilde{\nabla}$ denote the covariant derivative in $(N, h)$ and $\mathbb{R}^k$, respectively.

We use the summation convention for indices and a colon denotes the ordinary derivative with $i = 1, 2$, $\alpha = 1, \cdots, n$. Moreover, $d\omega$ and $d_0$ denote the area element in $\Omega \subset \mathbb{R}^2$ in $\partial\Omega$, respectively.
• The energy of \( f \in C^2((M, g), (N, h)) \) is defined by

\[
E(f) := \frac{1}{2} \int_M |df|^2 dM_g = \frac{1}{2} \int_M g^{ij} h_{\alpha\beta} \circ f f_\alpha^i f_\beta^j dM_g.
\]

The Euler-Lagrange equation of \( E \) for \( f \in C^2((M, g), (N, h)) \), called the tension field along \( f \), is as follows:

\[
\tau_h(f) := \langle \nabla \partial_z \partial \circ f, dz \rangle = g^{ij} (\nabla df)_{ij} = g^{ij} (f_\alpha^i f_\beta^j - f_\alpha^k \Gamma_{ij}^k + f_\alpha^i f_\beta^j \Gamma_{ij}^\gamma \circ f) \frac{\partial}{\partial y^\gamma} \circ f.
\]

Further, \( f \in C^2((M, g), (N, h)) \) is called harmonic if \( \tau_h(f) = 0 \).

For \( f = (f^a)_{a=1,\ldots,k} \), the second fundamental form of \( \eta \) is:

\[
II \circ f(df, df) := \langle \tilde{\nabla} \partial \partial_z df - \nabla \partial \partial_z df, dz \rangle \in T_{f(z)}^\perp \eta(N).
\]

• A weak Jacobi field \( J \) with boundary \( \xi \) along a harmonic function \( f \) is a vector field along \( f \) as a weak solution of

\[
\int_M \langle \nabla J, \nabla X \rangle + \langle \text{tr} R(J, df) df, X \rangle d\omega = 0
\]

for all \( X \in H^{1,2} \cap L^\infty(M, f^*TN) \) with \( X|_\partial M = \xi \).

• For \( B := \{ w \in \mathbb{R}^2 | ||w|| < 1 \} \),

\[
H^{1,2} \cap C^0(B, N) := \{ f \in H^{1,2} \cap C^0(B, \mathbb{R}^k) | f(B) \subset N \},
\]

with the norm, \( ||f||_{1,2,0} := ||\nabla f||_{L^2} + ||f||_{C^0} \).

Let \( \Gamma \) be a Jordan curve in \( N \) that is diffeomorphic to \( S^1 := \partial B \), and observe that \( N \) can be equipped with another metric \( \tilde{h} \) such that \( \Gamma \) is a geodesic in \((N, \tilde{h})\). Note that \( H^{1,2} \cap C^0((B, \partial B), (N, \Gamma)_{\tilde{h}}) \) and \( H^{1,2} \cap C^0((B, \partial B), (N, \Gamma)_h) \) coincide as sets. Using the exponential map in \((N, \tilde{h})\), we define the following spaces.

\[
H^{1,2} \cap C^0(\partial B; \Gamma) := \{ u \in H^{1,2} \cap C^0(\partial B, \mathbb{R}^k) | u(\partial B) = \Gamma \}
\]

with the norm \( ||u||_{1,2,0} := ||\nabla \mathcal{H}(u)||_{L^2} + ||u||_{C^0} \), here \( \mathcal{H}(u) \) is the harmonic extension in \( \mathbb{R}^k \), and

\[
T_u H^{1,2} \cap C^0(\partial B; \Gamma) := \{ \xi \in H^{1,2} \cap C^0(\partial B, u^*TN) | \xi(z) \in T_u(z) \Gamma, \text{ for all } z \in \partial B \} = H^{1,2} \cap C^0(\partial B, u^*T\Gamma).
\]
2.2 The setting

Let \( \Gamma_1, \Gamma_2 \) be two Jordan curves of class \( C^3 \) in \( N \) with diffeomorphisms \( \gamma^i : \partial B \rightarrow \Gamma_i, i = 1, 2 \), and \( \text{dist}(\Gamma_1, \Gamma_2) > 0 \). Moreover, for \( \rho \in (0, 1) \),

\[
A_\rho = \{ w \in B \mid \rho < |w| < 1 \}, \ C_1 = \{ w \mid |w| = 1 \}, \ C_2 = \{ w \mid |w| = \rho \}.
\]

Let further

\[
X^i_{\text{mon}} := \{ x^i \in H^{\frac{1}{2}, 2} \cap C^0(\partial B; \Gamma_i) \mid x^i \text{ is weakly monotone onto } \Gamma_i \}.
\]

I) We first consider the following condition for \((N, h) \cap \Gamma_1, \Gamma_2)\):

(C1) There exists a point \( p \in N \) with \( \Gamma_1, \Gamma_2 \subset B(p, r) \), where \( B(p, r) \) lies within the normal range of all of its points. Here we assume \( r < \pi/(2\sqrt{\kappa}) \), where \( \kappa \) is an upper bound of the sectional curvature of \((N, h)\).

In this paper, \( B(p, r) \) denotes a geodesic ball of \( p \in N \) with the properties in the condition (C1).

Remark 2.1. If \( \Gamma_1, \Gamma_2 \subset N \) satisfy (C1), for each \( x^i \in H^{\frac{1}{2}, 2} \cap C^0(\partial B; \Gamma_i) \) and \( \rho \in (0, 1) \) there exist \( g_\rho \in H^{1, 2} \cap C^0(A_\rho, B(p, r)) \) and \( g^i \in H^{1, 2} \cap C^0(B, B(p, r)) \) with \( g_\rho|_{C_1} = x^1 \), \( g_\rho|_{C_2}(\cdot) = x^2(\cdot) \) and \( g^i|_{\partial B} = x^i, i = 1, 2 \).

Proof. Let \( \Omega := \exp^{-1}(B(p, r)) \subset B(0, \tilde{r}) \subset \mathbb{R}^n \) for some \( \tilde{r} > 0 \). For \( \tilde{x}^i := \exp^{-1}(x^i) \), we have an Euclidean harmonic extension \( h_\rho(\tilde{x}^1, \tilde{x}^2) \) of finite energy, whose image is in \( B(0, \tilde{r}) \). The map \( \exp \) is a diffeomorphism and \( \Omega \) is star shaped, so there exists a retraction \( \delta : B(0, \tilde{r}) \rightarrow \Omega \) with \( \delta|_{\Omega} = Id \) in the class of \( H^{1, 2} \). Then the map \( g_\rho := \exp(\delta(h_\rho(\tilde{x}^1, \tilde{x}^2))) : A_\rho \rightarrow \Omega \) is an \( H^{1, 2} \cap C^0(\overline{A_\rho}, B(p, r)) \)-extension with boundary \( x^1 \) and \( x^2(\cdot) \). We may also find an \( H^{1, 2} \cap C^0(B, B(p, r)) \)-extension. \( \square \)

From the results in [HKW], [JK] and the above remark, we obtain a unique harmonic map of annulus and of disc type in \( B(p, r) \subset N \) for a given boundary mapping in the class of \( H^{\frac{1}{2}, 2} \cap C^0 \). Now we define,

\[
M^i := \{ x^i \in H^{\frac{1}{2}, 2} \cap C^0(\partial B; \Gamma_i) \mid x^i \text{ is weakly monotone, orientation preserving} \}.
\]

Then \( M^i \) is complete, since the \( C^0 \)-norm preserves the monotonicity.

We now investigate another alternative condition for \((N, h)\).

(C2) \( N \) is compact with nonpositive sectional curvature.

A compact Riemannian manifold is homogeneously regular and the condition of nonpositive sectional curvature for \( N \) implies \( \pi_2(N) = 0 \).

In order to define \( M^i \), we need some preparation. First, we consider for \( \rho \in (0, 1) \),

\[
G_\rho := \{ f \in H^{1, 2} \cap C^0(\overline{A_\rho}, N) \mid f|_{C_1} \text{ is continuous and weakly monotone onto } \Gamma_i \}.
\]
We may take a continuous homotopy class, denoted by \( F_\rho \subset G_\rho \), so that every two elements \( f, g \) in \( F_\rho \) are continuous homotopic (not necessarily relative), denoted by \( f \sim g \), more exactly:

\[
f \sim g \Leftrightarrow \text{there exists a continuous mapping } H : [0,1] \times \overline{\mathcal{A}_\rho} \to N \text{ with } H(0,\cdot) = f(\cdot), H(1,\cdot) = g(\cdot).
\]

Now define

\[
M^1 := \{ f \mid f \in C_{\pi}(\cdot) \in H^{1/2}_c \cap C^0(\partial B; \Gamma_1) \text{ orientation preserving, } f \in \mathcal{F}_\rho \},
\]

\[
M^2 := \{ f \mid f \in C_{\pi}(\cdot) \in H^{1/2}_c \cap C^0(\partial B; \Gamma_2) \text{ orientation preserving, } f \in \mathcal{F}_\rho \}.
\]

Then, for \( x^i \in M^i \), there exists a unique harmonic extension to \( A_\rho \) with \( x^1(\cdot) \) on \( C_1 \) and \( x^2(\overline{\rho}) \) on \( C_2 \) by [Le], [ES], [Hm].

**Definition** For \( x^i \in M^i, i = 1,2 \), let \( \mathcal{F}_\rho(x^1,x^2) \) be the unique solution of the following Dirichlet problem:

\[
\begin{align*}
\tau_h(\mathcal{F}_\rho(x^1,x^2)) &= 0 \text{ in } A_\rho \\
\mathcal{F}_\rho(x^1,x^2)(e^{i\theta}) &= x^1(e^{i\theta}) \text{ on } C_1 \\
\mathcal{F}_\rho(x^1,x^2)(\rho e^{i\theta}) &= x^2(e^{i\theta}) \text{ on } C_2(= \partial B_\rho),
\end{align*}
\]

and define \( \mathcal{E} : \mathcal{M} \longrightarrow \mathbb{R} \) with

\[
x \mapsto E(\mathcal{F}(x)) := \frac{1}{2} \int_{A_\rho} |d\mathcal{F}_\rho(x^1,x^2)|_h^2 d\omega.
\]

**II** Now let \( (N,h) \) and \( \Gamma_i, i = 1,2 \), satisfy (C1) or (C2).

We will introduce a kind of tangent space of \( x^i \in M^i \).

For a given oriented \( y^i \in X^i \text{mon} \), there exists a weakly monotone map \( w^i \in C^0(\mathbb{R}, \mathbb{R}) \) with \( w^i(\theta + 2\pi) = w^i(\theta) + 2\pi \) such that \( y^i(\theta) = \gamma^i(\cos(w^i(\theta)), \sin(w^i(\theta))) =: \gamma^i \circ w^i(\theta) \).

We note that \( w^i = \tilde{w}^i + Id \) for some \( \tilde{w}^i \in C^0(\partial B, \mathbb{R}) \). Roughly speaking, \( w^i \) can be viewed as a map in \( C^0(\partial B, \partial B) \) and then \( w^i \) is unique for given \( y^i \), whereas \( w^i \in C^0(\mathbb{R}, \mathbb{R}) \) is unique up to \( 2\pi l, l \in \mathbb{Z} \). Whether \( w^i \) is in \( C^0(\partial B, \partial B) \) or \( C^0(\mathbb{R}, \mathbb{R}) \) will be determined according to a given situation, simply denoted by \( y^i = \gamma^i \circ w^i \).

Denoting the Dirichlet integral by \( D \) and the \( \mathbb{R}^k \)-harmonic extension by \( \mathcal{H} \), let

\[
W^i_{\mathbb{R}^k} := \{ w^i \in C^0(\mathbb{R}, \mathbb{R}) \mid \text{weakly monotone, } w^i(\theta + 2\pi) = w^i(\theta) + 2\pi; D(\mathcal{H}(\gamma^i \circ w^i)) < \infty \}.
\]

Clearly, \( W^i_{\mathbb{R}^k} \) is convex. For further details, we refer to [SU].

**Definition** For \( x^i \in M^i \), considering \( w - w^i \) as a tangent vector along \( \tilde{w}^i \), let

\[
T_{x^i} = \{ d\gamma^i((w - w^i) \frac{d}{d\theta} \circ \tilde{w}^i) \mid w \in W^i_{\mathbb{R}^k} \text{ and } \gamma^i \circ w^i = x^i \}.
\]

\( T_{x^i} \) is convex in \( T_{x^i} H^{1/2}_c \cap C^0(\partial B; \Gamma_i) \), since \( W^i_{\mathbb{R}^k} \) is convex.

Let \( \tilde{\exp} \) denote the exponential map with respect to the metric \( \tilde{h} \). Then we note the following.
Remark 2.2. In case of (C1), \( \tilde{\exp}_x \xi \in M^i \) for \( \xi \in T_{x^i} \), \( i = 1, 2 \).
For the case (C2), there exist \( l_i > 0 \), depending on \( \gamma^i \) such that for any \( x^i \in M^i \), \( \tilde{\exp}_x \xi \in M^i \), if \( \| \xi \|_{T_{x^i}} < l_i \), \( i = 1, 2 \).

Proof For (C1) it is clear. In the case of (C2), for some small \( \delta > 0 \), there exists a retraction \( r \) from the \( \delta \)-neighborhood of \( N \) in \( \mathbb{R}^k \) onto \( N \), since \( N \) is compact. Then, letting \( \| x^i - x^i_0 \|_{1,2,0} < \delta \),

\[
\int_{A^i_\rho} |d(r(f_\rho + J_\rho(x^1 - x^1_0)))|^2 d\omega \leq C(\| f_\rho \|_{C^0}, \varepsilon, N) \left( \int_{A_\rho} |df_\rho|^2 d\omega + \int_B |dJ_\rho(x^1 - x^1_0)|^2 d\omega \right) \leq C(\| f_\rho \|_{1,2,0}, \delta, N).
\]

Then we have some \( l_i > 0 \) with the desired property, since \( \tilde{\exp}_x \xi = \gamma^i(w) \) for \( \xi = d\gamma^i((w - w^i) \frac{d}{d\theta} \circ \tilde{w}^i) \in T_{x^i} \).

Lemma 2.1. \( \mathcal{E} \) is continuously partially differentiable in \( x^1 \) and \( x^2 \) with respect to variations \( \xi^1 \in T_{x^1} \) and \( \xi^2 \in T_{x^2} \) respectively with

\[
\langle \delta_x \mathcal{E}, \xi^1 \rangle = \int_{A_\rho} \langle df_\rho(x^1, x^2), \nabla J_\rho(\xi^1, 0) \rangle_h d\omega.
\]

A similar result is obtained for the second variation.
Moreover, the derivatives are continuous in \( M^1 \times M^2 \).

Proof See [Ki].

3 \( H^{2,2} \)- Regularity of minimal surfaces

3.1 A result

Now we define for \( x = (x^1, x^2, \rho) \in M^1 \times M^2 \times (0,1), \)

\[
g_i(x) := \sup_{\xi^i \in T_{x^i}} (-\langle \delta_x \mathcal{E}, \xi^i \rangle), \quad i = 1, 2.
\]

Then we have the following result.

Theorem 3.1. Let \( x = (x^1, x^2, \rho) \in M^1 \times M^2 \times (0,1) \) with \( g_i(x) = 0, i = 1, 2 \). Then \( \mathcal{F}_\rho(x^1, x^2) \) is in the class of \( H^{2,2}(A_\rho, N) \).

Remark 3.1. In addition to the above conditions in Theorem 3.1 let us require that \( g_3(x) := \rho \cdot \partial_\rho \mathcal{E} = 0 \). Then, \( x = (x^1, x^2, \rho) \) is defined as a critical point of \( \mathcal{E} \) such that \( \mathcal{F}_\rho(x^1, x^2) \) is a minimal surface of annulus type in \( N \). For details we refer to [Ki].
Lemma 3.1. Let \( \mathcal{F}_\rho := \mathcal{F}_\rho(x^1, x^2) : A_\rho \to N \overset{\gamma}{\to} \mathbb{R}^k \) and \( \mathcal{F}_\rho \in H^{1,2}(A_\rho, \mathbb{R}^k) \). If \( \int_{A_\rho} |\partial_\theta d\mathcal{F}_\rho|^2 d\omega \leq C < \infty \), then \( \mathcal{F}_\rho(x^1, x^2) \in H^{2,2}(A_\rho, \mathbb{N}) \).

**Proof** By Young’s inequality it holds in polar coordinates with \( \Delta \mathcal{F}_\rho := \Delta_{\mathbb{E}^k} \mathcal{F}_\rho \) that
\[
|\nabla^2 \mathcal{F}_\rho|^2 = |\partial_r d\mathcal{F}_\rho|^2 + \frac{1}{r^2} |\partial_\theta d\mathcal{F}_\rho|^2
\]
\[
= |\Delta \mathcal{F}_\rho - \frac{1}{r^2} \partial_\theta\partial_r \mathcal{F}_\rho - \frac{1}{r} \partial_r \mathcal{F}_\rho|^2 + \frac{1}{r^2} |\partial_\theta\partial_r \mathcal{F}_\rho|^2 + \frac{1}{r^4} |\partial_r \mathcal{F}_\rho|^2 - 2 \frac{1}{r^3} \partial_\theta \mathcal{F}_\rho \partial_r \mathcal{F}_\rho + \frac{1}{r^2} |\partial_\theta d\mathcal{F}_\rho|^2
\]
\[
\leq C(\varepsilon) |\Delta \mathcal{F}_\rho|^2 + (1 + \varepsilon) \frac{1}{r^2} |\partial_\theta\partial_r \mathcal{F}_\rho|^2 + \frac{1}{r^4} |\partial_r \mathcal{F}_\rho|^2 - 2 \frac{1}{r^3} \partial_\theta \mathcal{F}_\rho \partial_r \mathcal{F}_\rho + C(\varepsilon) \frac{1}{r^4} |\partial_\theta \mathcal{F}_\rho|^2 + \frac{1}{r^2} |\partial_\theta d\mathcal{F}_\rho|^2
\]
\[
\leq C(\varepsilon, \rho, A_\rho) |d\mathcal{F}_\rho|^2 + C(\varepsilon, \rho) |\partial_\theta d\mathcal{F}_\rho|^2,
\]
since \( \mathcal{F}_\rho \) is harmonic in \( N \overset{\gamma}{\to} \mathbb{R}^k \), i.e., \( \tau_h(f) = 0 \). \( \square \)

### 3.2 The Morrey growth condition

We introduce a lemma from [Mo].

**Lemma 3.2.** Let \( G \) be a bounded domain in \( \mathbb{R}^2 \). Suppose \( \varphi \in H_0^{1,2}(G) \), and \( \psi \in L^1(G) \) satisfies the Morrey growth condition
\[
\int_{B_r(z_0)} |\psi| d\omega \leq C_0 r^\mu, \text{ for all } B_r(z_0).
\]

Then \( \psi \varphi^2 \in L^1(G) \) and for all \( B_r(z_0) \) it holds:
\[
\int_{B_r(z_0) \cap G} |\psi \varphi^2| d\omega \leq C_1 C_0 r^{\mu/2} \int_G |d\varphi|^2 d\omega
\]
for some uniform constant \( C_1 \).

Let \( x^i \) be as in Theorem 3.1 with \( x^i = \gamma^i \circ w^i \), and \( w^i = \tilde{w}^i + 1d \), \( \tilde{w}^i \in H^{1,2}_0(\partial B, \mathbb{R}) \), \( i = 1, 2 \) (recall the construction in Section 2.2). Moreover, for a given function \( f \) on \( \mathbb{R} \), \( f_+(-) \) and \( f_-(\cdot) \) denote the function \( f_+(- + h) \) and \( f_-(\cdot - h) \), for \( h \in \mathbb{R} \) respectively.

For \( x^i \in M^i \) let \( \mathcal{H}_\rho(x^1, x^2) \) denote the unique \( \mathbb{R}^k \)-harmonic extension with boundary \( x^i \) on \( C_i \), \( i = 1, 2 \), and \( \mathcal{H}(\cdot) \) the \( \mathbb{R}^k \)-harmonic extension of disc type.

Then we have the following growth condition.

**Lemma 3.3.** For each \( P_0 \in \partial A_\rho \) there exist \( C_0, \mu, r_0 > 0 \) such that, for all \( r \in [0, r_0] \), it holds that
\[
\int_{A_\rho \cap B_r(P_0)} (|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(w^1, 0)|^2) d\omega \leq C_0 r^\mu \int_{A_\rho} (|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(w^1, 0)|^2) d\omega.
\]
Remark 3.2. We also obtain the same result as in Lemma 3.3 for |d\mathcal{F}_\rho| \ (\text{resp. } |d\mathcal{F}_{\rho-}|) and |d\mathcal{H}_\rho(w^1_+, w^2_+)| \ (\text{resp. } |d\mathcal{H}_\rho(w^1_-, w^2_-)|).

As in [Ho], we observe the following.

Remark 3.3. (i) Let \mathcal{F}_\rho : \mathbb{A}_\rho \to \mathbb{N} be harmonic, we have then for \(X \in H^{1,2}_0(\mathbb{A}_\rho, \mathbb{R}^k)\),

\[- \int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), X \rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_\rho, dX \rangle d\omega = 0.\]

(ii) This means, for \(X \in H^{1,2}(\mathbb{A}_\rho, \mathbb{R}^k)\) the above expression only depends on the boundary of \(X\). Thus, for \(\phi = (\phi^1, \phi^2) \in H^{1,2}_\mathbb{F} \times H^{1,2}_\mathbb{F}(\tilde{\rho})\) we define

\[(4) \quad \mathbf{A}(\mathcal{F}_\rho)(\phi) := - \int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), X \rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_\rho, dX \rangle d\omega,\]

where \(X\) is any mapping in \(H^{1,2}(\mathbb{A}_\rho, \mathbb{R}^k)\) with \(X|_{\partial A_\rho} = \phi\).

Specially for \(\phi^i \in H^{1,2}_\mathbb{F} \cap C^0(\partial B, \mathbb{R}^i \times T\Omega), i = 1, 2,\) we take \(X := \mathbf{J}_{\mathcal{F}_\rho}(\phi^1, \phi^2),\) which is tangent to \(\mathbb{N}\) along \(\mathcal{F}_\rho\), then \(\langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), \mathbf{J}_{\mathcal{F}_\rho}(\phi^1, \phi^2) \rangle \equiv 0\) from the definition of the second fundamental form, so

\[(5) \quad \mathbf{A}(\mathcal{F}_\rho)(\phi) = \int_{A_\rho} \langle d\mathcal{F}_\rho, d\mathbf{J}_{\mathcal{F}_\rho}(\phi^1, \phi^2) \rangle d\omega \]

\[= \int_{A_\rho} \langle d\mathcal{F}_\rho, d\mathbf{J}_{\mathcal{F}_\rho}(\phi^1, 0) \rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_\rho, d\mathbf{J}_{\mathcal{F}_\rho}(0, \phi^2) \rangle d\omega \]

\[= \langle \partial_x \mathcal{E}, \phi^1 \rangle + \langle \partial_x \mathcal{E}, \phi^2 \rangle.\]

Hence, for a critical point \(x = (x^1, x^2, \rho)\) of \(\mathcal{E}\), we obtain that \(\mathbf{A}(\mathcal{F}_\rho)(\xi) \geq 0\), for all \(\xi = (\xi^1, \xi^2) \in \mathcal{F}_{x^1} \times \mathcal{F}_{x^2} \).

Proof of Lemma 3.3 We will show (3) in several steps.

1) Let \(P_0 \in C_1\) fixed, \(B_r := B_r(P_0)\), and

\[(6) \quad \tilde{w}^1 := \mathcal{Q}^{-1}\int_{(B_2r \setminus B_r) \cap \partial B} \tilde{w}^1 d\omega, \quad w^1 := w^1_0 + I d : \mathbb{R} \to \mathbb{R},\]

where \(\int_{\partial B \cap (B_2r \setminus B_r)} d\omega := \mathcal{Q},\)

\[\hat{\xi}_\phi := -[\phi(|e^{i\theta} - P_0|)]^2 (w^1 - w^1_0) \frac{\partial}{\partial \theta} \circ w^1 \in H^{1,2} \cap C^0(\partial B, \tilde{w}^1 * T(\partial B)),\]

where \(\tilde{w}^1\) means the map from \(\partial B\) into itself, and \(\phi \in C^\infty\) is a non-increasing function of \(|z|\) satisfying the conditions \(0 \leq \phi(z) \leq 1, \phi \equiv 1\) if \(|z| \leq 2r, \phi \equiv 0\) if \(|z| \geq 3r, |d\phi| \leq \frac{C}{r}, |d^2 \phi| \leq \frac{C}{r^2}\) for some \(C\), fixed \(r\).
Since \((1 - \phi^2)w^1 + \phi^2w_0^1 \in W^1_{2k}\), \(d\gamma^1(\xi_\phi) \in \mathcal{F}_{x^1}\), hence
\(\tag{7} \mathbf{A}(\mathcal{F}_\rho)(-d\gamma^1(\xi_\phi), 0) \geq 0.\)

Let \(x_0^1 := \gamma^1(w_0^1)\), then
\[
x^1 - x_0^1 = d\gamma^1(w^1 - w_0^1) - \int_{w_0^1}^{w^1} d^2\gamma^1(s') ds'' ds' = d\gamma^1(w^1 - w_0^1) - \alpha(w^1),
\]
and for small \(r > 0,\)
\[
\mathbf{A}(\mathcal{F}_\rho)(\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)|_{c_1}, 0) = \mathbf{A}(\mathcal{F}_\rho)(\phi^2 d\gamma^1(w^1 - w_0^1), 0) - \mathbf{A}(\mathcal{F}_\rho)(\phi^2 \alpha(w^1), 0)
\leq -\mathbf{A}(\mathcal{F}_\rho)(\phi^2 \alpha(w^1), 0),
\]
where \(\mathcal{F}_\rho^0(A_\rho) \equiv x_0^1 \in \Gamma_1.\)

On the other hand, for small \(r > 0, \phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)|_{c_2} \equiv 0,\) we can take \(\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)|\) in the definition of \(\mathbf{A}(\mathcal{F}_\rho)\). Hence,
\[
\mathbf{A}(\mathcal{F}_\rho)(\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)|_{c_1}, 0)
= \int_{A_\rho} (\phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho) d\omega + \int_{A_\rho} (2\phi d\phi(\mathcal{F}_\rho - \mathcal{F}_\rho^0), d\mathcal{F}_\rho) d\omega - \int_{A_\rho} (\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0), II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho)) d\omega
\leq -\mathbf{A}(\mathcal{F}_\rho)(\phi^2 \alpha(w^1), 0),
\]
and
\[
\int_{A_\rho} (\phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho) d\omega \leq \int_{A_\rho} (\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0), II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho)) d\omega
\]
\(\tag{8} - \int_{A_\rho} (2\phi d\phi(\mathcal{F}_\rho - \mathcal{F}_\rho^0), d\mathcal{F}_\rho) d\omega - \mathbf{A}(\mathcal{F}_\rho)(\phi^2 \alpha(w^1), 0).\)

For the estimate of \(-\mathbf{A}(\mathcal{F}_\rho)(\phi^2 \alpha(w^1), 0),\) consider
\[
\mathcal{H} := \phi^2 \int_{w_0^1}^{T^1(w^1)} \int_{s'}^{T^1(w^1)} d^2\gamma^1(s'') ds'' ds' \in H^{1.2}(A_\rho, \mathbb{R}^k)
\]
with \(\mathcal{H}|_{c_1} = \phi^2 \alpha(w^1), \mathcal{H}|_{c_2} \equiv 0,\) where \(w_0^1(r, \theta) = w_0^1 + Id(r, \theta) = w_0^1 + \theta, (r, \theta) \in [\rho, 1) \times \mathbb{R}.\)

By simple computation we obtain
\[
|\mathcal{H}| \leq C(\gamma^1, x^1)\phi^2|H_\rho(w^1, 0) - \tilde{w}_0^1|^2,
|\mathcal{D}\mathcal{H}| \leq C(\gamma^1, x^1)|H_\rho(w^1, 0) - \tilde{w}_0^1|^2\phi|d\phi| + C(\gamma^1, x^1)|dH_\rho(w^1, 0)||H_\rho(w^1, 0) - \tilde{w}_0^1|^2\phi^2,
\]
\[ \text{9} \]
and from \( \mathbf{S} \) by Young’s inequality,

\[
\int_{A_{\rho}} \langle \phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho \rangle d\omega \leq \int_{A_{\rho}} |d\mathcal{F}_\rho|^2 |\mathcal{F}_\rho - \mathcal{F}_\rho^0| \phi^2 d\omega
\]

\[
+ \varepsilon \int_{A_{\rho}} |d\mathcal{F}_\rho|^2 \phi^2 d\omega + C(\varepsilon) \int_{A_{\rho}} |\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 |d\phi|^2 d\omega
\]

\[
+ C\|\mathcal{H}_\rho(w^1, 0) - w_0^1\|_{L^\infty(B_{r^0})} \int_{A_{\rho}} (|d\mathcal{F}_\rho|^2 \phi^2 + |\mathcal{H}_\rho(w^1, 0) - w_0^1|^2 |d\phi|^2) d\omega
\]

Thus, for \( r \in (0, r_0) \), sufficiently small, dependent on \( \varepsilon, C \), modulus of continuity of \( \mathcal{F}_\rho - \mathcal{F}_\rho^0 \) and \( \mathcal{H}_\rho(w^1, 0) - w_0^1 \) we have the following estimate:

\[
\int_{A_{\rho}} \langle \phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho \rangle d\omega \leq \varepsilon \int_{A_{\rho}} (|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(w^1, 0)|^2) \phi^2 d\omega
\]

\[
+ C(\varepsilon) \int_{A_{\rho}} (|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |\mathcal{H}_\rho(w^1, 0) - w_0^1|^2) |d\phi|^2 d\omega.
\]

II) We will estimate \( \int_{A_{\rho}} |d\mathcal{H}_\rho(w^1, 0)|^2 \phi^2 d\omega. \)

- First, we obtain

\[
D[(\mathcal{H}_\rho(w^1, 0) - w_0^1)\phi] = \int_{A_{\rho}} [ |d\mathcal{H}_\rho(w^1, 0)|^2 \phi^2 + |(\mathcal{H}_\rho(w^1, 0) - w_0^1)|^2 |d\phi|^2
\]

\[
+ 2d\mathcal{H}_\rho(w^1, 0)(\mathcal{H}_\rho(w^1, 0) - w_0^1)\phi d\phi \] d\omega,
\]

and by Young’s inequality

\[
\int_{A_{\rho}} |d\mathcal{H}_\rho(w^1, 0)|^2 \phi^2 d\omega \leq D[(\mathcal{H}_\rho(w^1, 0) - w_0^1)\phi]
\]

\[
+ \varepsilon \int_{A_{\rho}} |d\mathcal{H}_\rho(w^1, 0)|^2 \phi^2 d\omega + C(\varepsilon) \int_{A_{\rho}} (|\mathcal{H}_\rho(w^1, 0)|^2 + |w_0^1|^2) |d\phi|^2 d\omega.
\]

- The estimate of \( D[(\mathcal{H}_\rho(w^1, 0) - w_0^1)\phi] \):

On \( C^1 \), we have \( \mathcal{F}_\rho - \mathcal{F}_\rho^0 = d\gamma_1(w^1 - w_0^1) - \int_{w_0^1}^{w_1} \int_{s'} d^2\gamma_1(s')ds' ds \), and \( \phi|_{\partial B_{3r}(P_0)} \equiv 0 \). Hence, on \( \partial(A_{\rho} \cap B_{3r}(P_0)) \),

\[
(\mathcal{H}_\rho(w^1, 0) - w_0^1)\phi = |d\gamma_1(T^1(w^1))|^{-2} [d\gamma_1(T^1(w^1)) \cdot (\mathcal{F}_\rho - \mathcal{F}_\rho^0)
\]

\[
+ d\gamma_1(T^1(w^1)) \cdot \int_{w_0^1}^{w_1} \int_{s'} d^2\gamma_1(s')ds' \phi.
\]

10
We denote the latter map on $A_\rho$ by $\Psi$.

Moreover, it holds that

$$\Delta [(\mathcal{H}_\rho (w^1, 0) - \tilde{w}_0^1)\phi] = 2d\mathcal{H}_\rho (w^1, 0) \cdot d\phi + (\mathcal{H}_\rho (\tilde{w}_0^1) - \tilde{w}_0^1)\Delta \phi =: f. \quad (11)$$

Note that for a solution $\varphi \in C^2(\Omega, \mathbb{R})$ of $\Delta \varphi = f$ it holds, with a boundary data $\varphi_0$, that

$$D\varphi \leq D\psi - \int f(\varphi - \psi), \text{ for all } \psi \in \varphi_0 + H^1_{0,2}(\Omega).$$

Hence, by the variation characterization of equation (11), we obtain

$$D\left[(\mathcal{H}_\rho (w^1, 0) - \tilde{w}_0^1)\phi\right] \leq D(\Psi) - \int_{A_\rho \cap B_{3\rho}} f\left[(\mathcal{H}_\rho (\tilde{w}_0^1) - \tilde{w}_0^1)\phi - \Psi\right]d\omega. \quad (12)$$

Let

$$\Psi := \frac{d\gamma (T^1(w^1)) \cdot (\mathcal{F}_\rho - \mathcal{F}_0^\rho) + d\gamma (T^1(w^1)) \cdot \int_{w_0^1}^{T^1(w^1)} \int_{s'}^{T^1(w^1)} d^2\gamma (s''d's')}{|d\gamma (T^1(w^1))|^2} \phi$$

$$= \frac{\Theta}{|d\gamma (T^1(w^1))|^2} \phi,
$$

d$[d\gamma (T^1(w^1)) \cdot (\mathcal{F}_\rho - \mathcal{F}_0^\rho)] = d^2\gamma (T^1(w^1))d(T^1(w^1))(\mathcal{F}_\rho - \mathcal{F}_0^\rho) + d\gamma (T^1(w^1))d\mathcal{F}_\rho =: a,$$

$$d \left( \int_{w_0^1}^{T^1(w^1)} \int_{s'}^{T^1(w^1)} d^2\gamma (s''d's')d's' \right) = d^2\gamma (T^1(w^1))d\mathcal{H}_\rho (\tilde{w}_0^1, 0)(\mathcal{H}_\rho (\tilde{w}_0^1, 0) - \tilde{w}_0^1) =: b,$$

$$d|d\gamma (T^1(w^1))|^2 = -2|d\gamma (T^1(w^1))|^{-4}(d^2\gamma (T^1(w^1)), d^1\gamma (T^1(w^1)))d\mathcal{H}_\rho (\tilde{w}_0^1, 0) =: c,$$

that we have

$$|d\Psi|^2 = \frac{|a + b|^2\phi^2 + \Theta^2\phi^2c^2 + \Theta^2|d\phi|^2 + (a + b)c\phi\Theta + (a + b)\phi\Theta d\phi + \Theta^2\phi c d\phi}{|d\gamma (T^1(w^1))|^2},$$

and we compute further, from the property of $\phi$, that

$$\int_{A_\rho} |d\Psi|^2d\omega \leq C \int_{A_\rho} |d\mathcal{F}_\rho|^2\phi^2d\omega + C \int_{A_\rho} \left[|\mathcal{H}_\rho - \mathcal{H}_0|^2 + |\mathcal{H}_\rho (\tilde{w}_0^1, 0) - \tilde{w}_0^1|^2\right]|d\phi|^2d\omega$$

$$+ C\delta \int_{A_\rho} \left[|\mathcal{H}_\rho (\tilde{w}_0^1, 0) - \tilde{w}_0^1|^2|d\phi|^2 + |d\mathcal{H}_\rho (\tilde{w}_0^1, 0)|^2\phi^2\right]d\omega,$$

where $\delta = \left\|\mathcal{F}_\rho - \mathcal{F}_0^\rho + |\mathcal{H}_\rho (\tilde{w}_0^1, 0) - \tilde{w}_0^1|\right\|_{L^\infty(A_\rho \cap B_{3\rho})}.$
We can also compute that

\[ - \int_{A_{\rho} \cap B_{3r}} f \left[ (\mathcal{H}_\rho(w^1, 0) - w_0^1) \phi - \Psi \right] d\omega \]

\[ \leq \int_{A_{\rho} \cap B_{3r}} \left[ 2d\mathcal{H}_\rho(w^1, 0)d\phi|\mathcal{H}_\rho(w^1, 0) - w_0^1| + |\mathcal{H}_\rho(w^1, 0) - w_0^1|^2|\Delta \phi| \right] d\omega \]

\[ + C|d\mathcal{H}_\rho(w^1, 0)|d\phi|\mathcal{F}_\rho - \mathcal{F}_\rho^0||d\phi| + C|\mathcal{H}_\rho(w^1, 0) - w_0^1||\mathcal{F}_\rho - \mathcal{F}_\rho^0||\Delta \phi|d\omega \]

\[ + C|\mathcal{H}_\rho(w^1, 0) - w_0^1||(d\mathcal{H}_\rho(w^1, 0))d\phi|\mathcal{H}_\rho(w^1, 0) - w_0^1||d\phi| + |\mathcal{H}_\rho(w^1, 0) - w_0^1|^2|\Delta \phi| \]d\omega

\[ \leq \int_{A_{\rho} \cap B_{3r}} \left[ C(|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |\mathcal{H}_\rho(w^1, 0) - w_0^1|^2)(|d\phi|^2 + |\Delta \phi| \right)

\[ + \left( \frac{\varepsilon}{2} + C\|\partial B \cap (A_{\rho} \cap B_{3r})\|L^\infty(\partial B \cap b_{\partial B_{2r}}) \right) \int_{A_{\rho}} |d\mathcal{H}_\rho(w^1, 0)|^2 d\omega. \]

Now the estimate of \( D[(\mathcal{H}_\rho(w^1, 0) - w_0^1)\phi] \) follows from (12).

- From (10) and the above estimates, we derive

\[ \int_{A_{\rho}} |d\mathcal{H}_\rho(w^1, 0)|^2 d\omega \leq C \int_{A_{\rho}} |\mathcal{F}_\rho|^2 d\omega \]

\[ + C(\varepsilon) \int_{A_{\rho}} (|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |\mathcal{H}_\rho(w^1, 0) - w_0^1|^2)(|d\phi|^2 + |\Delta \phi|) d\omega \]

\[ + \left( \frac{3\varepsilon}{4} + C\|\partial B \cap (A_{\rho} \cap B_{3r})\|L^\infty(\partial B \cap b_{\partial B_{2r}}) \right) \int_{A_{\rho}} |d\mathcal{H}_\rho(w^1, 0)|^2 d\omega. \]

\[ (13) \]

III) From (9), (13), for \( r \leq r_0 \), where \( r_0 \) is dependent on \( \varepsilon, C(x^1, \rho) \) and the modulus of continuity of \( \mathcal{F}_\rho - \mathcal{F}_\rho^0 \) and \( \mathcal{H}_\rho(w^1, 0) - w_0^1 \), the definition of \( \phi \) yields

\[ \int_{A_{\rho} \cap B_{3r}} (|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(w^1, 0)|^2) d\omega \leq C r^{-2} \int_{A_{\rho} \cap B_{3r} \setminus B_{2r}} (|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |\mathcal{H}_\rho(w^1, 0) - w_0^1|^2) d\omega \]

\[ \leq C r^{-2} \int_{A_{\rho} \cap B_{3r} \setminus B_{2r}} (|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |\mathcal{H}_\rho(w^1, 0) - w_0^1|^2) d\omega \]

(Poincaré inequality)

\[ \leq C \int_{A_{\rho} \cap B_{3r} \setminus B_r} (|d\mathcal{F}_\rho|^2 + |d\mathcal{H}_\rho(w^1, 0)|^2) d\omega \]

\[ + C r^{-2} \left( \int_{\partial B \cap B_{2r} \setminus B_r} (\mathcal{F}_\rho - \mathcal{F}_\rho^0) d\sigma \right)^2 + C r^{-2} \left( \int_{\partial B \cap B_{2r} \setminus B_r} (\mathcal{H}_\rho(w^1, 0) - w_0^1) d\sigma \right)^2, \]

where the last term is 0 from the definition of \( w_0^1 \).

On \( \partial B \), we have

\[ \mathcal{F}_\rho - \mathcal{F}_\rho^0 = d\gamma^1(w^1_0)(w^1 - w_0^1) + \int_{\partial B \cap B_{2r} \setminus B_r} \int_{w_0^1}^{w^1} d^2\gamma^1(s')(ds'') ds', \]

12
so, from the estimate in the integration and by the second inequality in Lemma 3.4,

\[
\int_{\partial B \cap B_r \setminus B_r} (\mathcal{F}_\rho - \mathcal{F}_\rho^0) \, d_o
\]

\[
= \int_{\partial B \cap B_r \setminus B_r} d \gamma^1(\tilde{w}_1^1)(\tilde{w}_1^1 - \tilde{w}_0^1) \, d_o + \int_{\partial B \cap B_r \setminus B_r} \int_{w_1^1}^{w_1^0} \int_{w_1}^{w_1^0} d^2 \gamma^1(\tilde{s}) \, d s' \, d s''
\]

\[
\leq C \int_{\partial B \cap (B_r \setminus B_r)} |\tilde{w}_1^1 - \tilde{w}_0^1|^2 \, d_o
\]

\[
\leq C r \int_{B \cap (B_r \setminus B_r)} |d \mathcal{H}_\rho(\tilde{w}_1^1, 0)|^2 \, d \omega + \frac{C}{r} \left( \int_{\partial B \cap (B_r \setminus B_r)} (\tilde{w}_1^1 - \tilde{w}_0^1) \, d_o \right)^2.
\]

Here, the last term is again zero by the definition of \( \tilde{w}_0^1 \).
Thus,

\[
C r^{-2} \left( \int_{\partial B \cap (B_r \setminus B_r)} (\mathcal{F}_\rho - \mathcal{F}_\rho^0) \, d_o \right)^2
\]

\[
\leq C \left( \int_{B \cap (B_r \setminus B_r)} |d \mathcal{H}_\rho(\tilde{w}_1^1, 0)|^2 \, d \omega \right)^2 \leq C(x^1, \rho) \int_{B \cap (B_r \setminus B_r)} |d \mathcal{H}_\rho(\tilde{w}_1^1, 0)|^2 \, d \omega,
\]

hence

\[
\int_{A_{\rho} \cap B_r} (|d \mathcal{F}_\rho|^2 + |d \mathcal{H}_\rho(\tilde{w}_1^1, 0)|^2) \, d \omega \leq C \int_{A_{\rho} \cap B_{3r} \setminus B_r} (|d \mathcal{F}_\rho|^2 + |d \mathcal{H}_\rho(\tilde{w}_1^1, 0)|^2) \, d \omega.
\]

Let \( \Upsilon(r) := \int_{A_{\rho} \cap B_r} (|d \mathcal{F}_\rho|^2 + |d \mathcal{H}_\rho(\tilde{w}_1^1, 0)|^2) \, d \omega \), then the above inequality means that

\[
\Upsilon(r) \leq C(\Upsilon(3r) - \Upsilon(r)),
\]

where \( C \) is independent of \( r \leq r_0 \), for some small \( r_0 \).
Then the inequality \( 3.4 \) follows from the Iteration-lemma.

\[\square\]

3.3 The proof of the main theorem

We will give here the proof of Theorem 3.1. We begin with Poincaré inequality as follows (see [St1] Lemma 5.5):

Lemma 3.4. Let \( z_0 \in \partial A_{\rho} \), \( B_r := B_r(z_0) \), \( G_r := A_{\rho} \cap (B_{3r} \setminus B_r) \), \( K_r := A_{\rho} \cap (B_{2r} \setminus B_r) \) and \( S_r := \partial A_{\rho} \cap (B_{2r} \setminus B_r) \). Then, for some small \( r_0 > 0 \), there exists a uniform constant \( C \) independent of \( z_0 \) such that for all \( r \leq r_0 \) and for each \( \varphi \in H^{1,2}(G_r) \):

\[
\int_{G_r} |\varphi|^2 \, d \omega \leq C r^2 \int_{G_r} |d \varphi|^2 \, d \omega + C \left( \int_{S_r} \varphi \, d_o \right)^2, \text{ and}
\]

\[
\int_{S_r} |\varphi|^2 \, d_o \leq C r \int_{K_r} |d \varphi|^2 \, d \omega + \frac{C}{r} \left( \int_{S_r} \varphi \, d_o \right)^2,
\]

where \( d_o \) is the one-dimensional area element.
Proof. Let \( z_0, r \) be fixed. Suppose by contradiction that for a sequence \( \varphi_m \in H^{1,2}(G_r) \)
\[
1 \equiv \int_{G_r} |\varphi_m|^2 \, d\omega \geq m r^2 \int_{G_r} |d\varphi_m|^2 \, d\omega + m \left( \int_{S_r} \varphi_m \, d\sigma \right)^2.
\]

Then \( \{\varphi_m\} \) is bounded in \( H^{1,2}(G_r) \) and some subsequence, denoted again by \( \{\varphi_m\} \), converges weakly to some \( \varphi \in H^{1,2}(G_r) \) but strongly in \( L^2(G_r) \) by Rellich-Kondrakov. From the above assumption, \( d\varphi_m \to 0 \) strongly.

Thus, \( \{\varphi_m\} \) converges strongly to some constant \( C \) in \( H^{1,2}(G_r) \) and \( \varphi_m \to C \) in \( L^2(S_r) \).

On the other hand, \( \int_{S_r} \varphi_m \, d\sigma \to 0 \), so \( \varphi \equiv 0 \) in \( G_r \), contradicting the assumption, since \( \varphi_m \to \varphi \) in \( L^2 \).

The second inequality can be proved similarly, supposing by contradiction that

\[
1 \equiv \int_{S_r} |\varphi_m|^2 \, d\sigma \geq m r \int_{K_r} |d\varphi_m|^2 \, d\omega + \frac{m}{r} \left( \int_{S_r} \varphi_m \, d\sigma \right)^2
\]

and applying the above result for \( \int_{K_r} |\varphi_m|^2 \, d\omega \).

By scaling, one can see that \( C \) is independent of \( z_0, r \). \( \square \)

**Proof of Theorem 3.1**

From Lemma 3.1 and by a well known result in [GT], it suffices to show that

\[
(14) \quad \int_{A_p} |\Delta_h d\mathcal{F}_p|^2 \, d\omega \leq C < \infty,
\]

where \( \Delta_h d\mathcal{F}_p : = \frac{d\mathcal{F}_p(x,\varrho+h)-d\mathcal{F}_p(x,\varrho)}{h}, h \neq 0 \), and \( C \) is independent of \( h \).

We show (14) in several steps. The same notations as in the preceding sections will be used.

(I) With \( \Delta_{-h} \Delta_h \mathcal{F}_p|_{\partial B} = \Delta_{-h} \Delta_h \gamma^1 \circ e^{iw^1} \) and \( \Delta_{-h} \Delta_h \mathcal{F}_p|_{\partial B_p}(\cdot, \rho) = \Delta_{-h} \Delta_h \gamma^2 \circ e^{iw^2}(\cdot) \),

\[
\int_{A_p} |\Delta_h d\mathcal{F}_p|^2 \, d\omega = -\int_{A_p} \langle d\mathcal{F}_p, d(-\Delta_h \Delta_h \mathcal{F}_p) \rangle \, d\omega
\]

\[
= -\int_{A_p} \langle II \circ \mathcal{F}_p(d\mathcal{F}_p, d\mathcal{F}_p), \Delta_{-h} \Delta_h \mathcal{F}_p \rangle \, d\omega - A(\mathcal{F}_p)(\Delta_{-h} \Delta_h \mathcal{F}_p|_{\partial A_p}).
\]

Denoting \( \gamma^1 \circ e^{iw^1} \) and \( \gamma^2 \circ e^{iw^2} \) by \( \gamma^1(w^i(\theta)) \), further \( w^i(\cdot + h) \) and \( w^i(\cdot - h) \) by \( w^i_+ \) and \( w^i_- \) respectively, we have:

\[
\Delta_{-h} \Delta_h \gamma^i(w^i) = \Delta_{-h} \left[ \frac{\gamma^i(w^i_+) - \gamma^i(w^i_-)}{h} \right]
\]

\[
= \Delta_{-h} \left[ d\gamma^i(w^i) \left( w^i_+ - w^i_- \right) + \frac{1}{h} \int_{w^i_-}^{w^i_+} \int_{w^i_-}^{s'} d^2 \gamma^i(s'')ds'' ds' \right]
\]

\[
= d\gamma^i(w^i)(\Delta_{-h} \Delta_h w^i) - \frac{1}{h} \int_{w^i_-}^{w^i_+} d^2 \gamma^i(s') ds' \cdot \Delta_h w^i_- + \Delta_{-h} \left( \frac{1}{h} \int_{w^i_-}^{w^i_+} \int_{w^i_-}^{s'} d^2 \gamma^i(s'')ds'' ds' \right)
\]

\[
= d\gamma^i(w^i)(\Delta_{-h} \Delta_h w^i) + P^i.
\]
Since $\gamma^i$ is smooth, clearly $d\gamma^i(w^i)(\Delta_{-h}\Delta_hw^i) \in H^{1,2}_\mathbb{R} \cap C^0(\partial B, (x^i)^*TT_i)$.

We write $w^i = \tilde{w}^i + Id$ for some $\tilde{w}^i \in H^{1,2}_\mathbb{R} \cap C^0(\partial B, \mathbb{R})$ and define a real valued map of $(r, \theta) \in [\rho, 1] \times \mathbb{R}$ as follows: for $i = 1$,

$$T^1(w^1)(r, \theta) := H_\rho(\tilde{w}, 0)(r, \theta) + Id(r, \theta) \quad \text{with} \quad Id(r, \theta) = \theta,$$

where $H_\rho(\tilde{w}, 0)$ is the harmonic extension to $A_\rho \approx [\rho, 1] \times \mathbb{R}/2\pi$ with $\tilde{w}$ on $\partial B$ and 0 on $\partial B_\rho$.

Then it holds that

$$T^1(w^1)(r, \theta + 2\pi) = T^1(w^1)(r, \theta) + 2\pi, \quad \text{for} \quad (r, \theta) \in [\rho, 1] \times \mathbb{R},$$

and $e^{iT^1(w^1)}$ can be considered as a map from $\partial B$ into itself.

Now define a map $S(P^1, 0)(\cdot) : A_\rho \to \mathbb{R}^k$ with the boundary $P^1$ (resp. 0) on $C_1$ (resp. $C_2$) as follows:

$$S(P^1, 0)(\cdot) := -\frac{1}{h} \int_{T^1(w^1)} d^2\gamma^1(s')ds' \cdot H_\rho(\Delta_{-h}w^1, 0)(\cdot) + \Delta_{-h} \left( \frac{1}{h} \int_{T^1(w^1)} \int s' d^2\gamma^1(s'')ds''ds' \right).$$

Similarly, a map $S(0, P^2)(\cdot) : A_\rho \to \mathbb{R}^k$ with the boundary 0 (resp. $P^2$) on $C_1$ (resp. $C_2$):

$$S(0, P^2)(\cdot) := -\frac{1}{h} \int_{T^2(w^2)} d^2\gamma^2(s')ds' \cdot H_\rho(0, \Delta_{-h}w^2_2)(\cdot) + \Delta_{-h} \left( \frac{1}{h} \int_{T^2(w^2)} \int s' d^2\gamma^2(s'')ds''ds' \right),$$

where $T^2(w^2)(\cdot) = H_\rho(0, \tilde{w})(\cdot) + Id(\cdot)$, and $S(0, P^2)|_{C_1} = 0, S(0, P^2)|_{C_2}(\rho) = P^2(\cdot)$.

Clearly $S(P^1, 0), S(0, P^2) \in H^{1,2}(A_\rho, \mathbb{R}^k)$, so letting $S(P^1, P^2) := S(P^1, 0) + S(0, P^2)$, we have a map in $H^{1,2}(A_\rho, \mathbb{R}^k)$ with boundary $(P^1, P^2)$.

By computation, $\frac{h^2}{2} \Delta_{-h}\Delta_hw^i = \frac{1}{2}(w^i_+ + w^i_-) - w^i$. And $\frac{1}{2}(w^i_+ + w^i_-) \in W^i_{\mathbb{R}^k}$ which is convex. Thus, by the definition of $\mathcal{F}^i$,

$$\frac{h^2}{2} d\gamma^i(w)(\Delta_{-h}\Delta_hw^i) \in \mathcal{F}^i,$$

and $\gamma^i(w^i)(\Delta_{-h}\Delta_hw^i)$ is in $H^{1,2}_\mathbb{R}$, for which $A(\mathcal{F}_\rho)$ is well defined, recall Remark 3.3.

From (1) and Remark 3.3 since $g^1(x) = g^2(x) = 0$,

$$\frac{h^2}{2} A(\mathcal{F}_\rho) (d\gamma^i(w^i)(\Delta_{-h}\Delta_hw^i), 0) = A(\mathcal{F}_\rho) \left( \frac{h^2}{2} d\gamma^1(w^1)(\Delta_{-h}\Delta_hw^1), 0 \right) \geq 0,$$

so $A(\mathcal{F}_\rho) (d\gamma^i(w^i)(\Delta_{-h}\Delta_hw^i), 0) \geq 0.$
Similarly, for the second variation, \( A(\mathcal{F}_\rho) \left( 0, d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2)(\varphi) \right) \geq 0. \)

From now on we will omit the scaling term \( (\varphi) \) for the second variation.

Moreover, from the definition of \( A(\mathcal{F}_\rho) \), clearly it follows that
\[
A(\mathcal{F}_\rho)(\phi^1 + \xi^1, \phi^2 + \xi^2) = A(\mathcal{F}_\rho)(\phi^1, \phi^2) + A(\mathcal{F}_\rho)(\xi^1, \xi^2),
\]
if there exist \( H^{1,2} \) extensions of \( (\phi^1, \phi^2) \) and \( (\xi^1, \xi^2) \).

Hence, we have that
\[
A(\mathcal{F}_\rho) \left( d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1), d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2) \right)
= A(\mathcal{F}_\rho) \left( d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1), 0 \right) + A(\mathcal{F}_\rho) \left( 0, d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2) \right) \geq 0.
\]

Now we can compute:
\[
\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega = - \int_{A_\rho} \left\langle II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h \mathcal{F}_\rho \right\rangle d\omega - A(\mathcal{F}_\rho)(\Delta_{-h}\Delta_h \mathcal{F}_\rho |_{\partial A_\rho})
= - \int_{A_\rho} \left\langle II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h \mathcal{F}_\rho \right\rangle d\omega
- A(\mathcal{F}_\rho)(P^1, P^2) - A(\mathcal{F}_\rho) \left( d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1), d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2) \right)
\leq - \int_{A_\rho} \left\langle II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h \mathcal{F}_\rho \right\rangle d\omega
- A(\mathcal{F}_\rho)(P^1, P^2)
\]
\[
\int_{A_\rho} \left\langle II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h \mathcal{F}_\rho \right\rangle d\omega
= - \int_{A_\rho} \left\langle II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h \mathcal{F}_\rho \right\rangle d\omega
+ \int_{A_\rho} \left\langle II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho), S(0, P^2) \right\rangle d\omega
\]
\[
\int_{A_\rho} \left\langle d\mathcal{F}_\rho, dS(P^1, 0) \right\rangle d\omega - \int_{A_\rho} \left\langle d\mathcal{F}_\rho, dS(0, P^2) \right\rangle d\omega.
\]

(II) For the estimates of the above terms we need some preparation.

First, let \( s(\tau) := \tau \mathcal{F}_{\rho^+} + (1 - \tau) \mathcal{F}_\rho, 0 \leq \tau \leq 1, \) then
\[
|\Delta_h II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho)| = \left| \frac{1}{h} (II \circ \mathcal{F}_{\rho^+} (\mathcal{F}_{\rho^+}, \mathcal{F}_{\rho^+}) - II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho)) \right|
= \left| \frac{1}{h} \left\{ II \circ \mathcal{F}_{\rho^+} (d\mathcal{F}_{\rho^+}, d\mathcal{F}_{\rho^+}) - II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho) \right\} \right|
+ II \circ \mathcal{F}_\rho (d\mathcal{F}_{\rho^+}, d\mathcal{F}_{\rho^+}) - II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho)
\]
\[
= \left| \frac{1}{h} \left\{ (dII(\mathcal{F}_\rho) \cdot (\mathcal{F}_{\rho^+} - \mathcal{F}_\rho)) + \int_0^1 \int_0^t d^2 II(s(\tau))(\mathcal{F}_{\rho^+} - \mathcal{F}_\rho)^2 d\tau dt \right\} \right|
+ II \circ \mathcal{F}_\rho (d\mathcal{F}_{\rho^+}, d\mathcal{F}_{\rho^+}) - II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho)
\]
\[
= \left| dII(\mathcal{F}_\rho) \cdot \Delta_h \mathcal{F}_\rho (d\mathcal{F}_{\rho^+}, d\mathcal{F}_{\rho^+}) + \frac{1}{h} \left\{ \int_0^1 \int_0^t d^2 II(s(\tau))(\mathcal{F}_{\rho^+} - \mathcal{F}_\rho)^2 d\tau dt \right\} \right|
+ II \circ \mathcal{F}_\rho (d\mathcal{F}_{\rho^+}, d\mathcal{F}_{\rho^+}) - II \circ \mathcal{F}_\rho (d\mathcal{F}_\rho, d\mathcal{F}_\rho)
\]
\[
\leq C(|\mathcal{F}_\rho|_{C^1(A_\rho)})(|\Delta_h \mathcal{F}_\rho| d\mathcal{F}_{\rho^+} + |\Delta_h \mathcal{F}_\rho| d\mathcal{F}_\rho + |\Delta_h \mathcal{F}_\rho| (d\mathcal{F}_{\rho^+} + d\mathcal{F}_\rho)).
\]
Now let
\[-\frac{1}{h} \int_{T^1(w_1)}^{T^1(w_1^{\perp})} d^2 \gamma_1(s') ds' := \star \quad \text{and} \quad \frac{1}{h} \int_{T^1(w_1)}^{T^1(w_1^{\perp})} \int_{T^1(w_1)}^{s'} d^2 \gamma_1(s'') ds'' ds' := \star^*,\]
then we have
\[|\star| \leq C(\gamma_1)|H_{\rho}(\Delta_{-h} w_1, 0)|, \quad |\star^*| \leq C(\gamma_1)|H_{\rho}(\Delta_{-h} w_1, 0)|,
\]
and
\[
|d \star| = \left| -\frac{1}{h} \left[ d^2 \gamma_1(T^1(w_1^{\perp}))dT^1(w_1^{\perp}) - d^2 \gamma_1(T^1(w_1))dT^1(w_1) \right] \right| \\
= \left| -\frac{1}{h} \left[ \frac{d^2 \gamma_1(T^1(w_1^{\perp})) - d^2 \gamma_1(T^1(w_1))}{T^1(w_1^{\perp}) - T^1(w_1)} \right] (T^1(w_1^{\perp}) - T^1(w_1))dT^1(w_1) \\
\quad +d^2 \gamma_1(T^1(w_1))dT^1(w_1^{\perp}) - dT^1(w_1) \right| \\
\leq C(||\gamma_1||_{C^3})(|H_{\rho}(\Delta_{-h} w_1, 0)||dH_{\rho}(w_1^{-}, 0)| + |dH_{\rho}(\Delta_{-h} w_1, 0)|),
\]
\[
|d \star^*| = \left| d \left[ \frac{1}{h} \left( \int_{T^1(w_1^{\perp})}^{T^1(w_1^{\perp})} d^2 \gamma_1(s') ds' - \int_{T^1(w_1^{\perp})}^{T^1(w_1)} d^2 \gamma_1(T^1(w_1^{\perp})) ds' \right) \right] \right| \\
= \frac{1}{h} \left[ \frac{d^2 \gamma_1(T^1(w_1^{\perp})) - d^2 \gamma_1(T^1(w_1))}{T^1(w_1^{\perp}) - T^1(w_1)} \right] (T^1(w_1^{\perp}) - T^1(w_1))dT^1(w_1) \\
\quad -d^2 \gamma_1(T^1(w_1))dT^1(w_1^{\perp}) - dT^1(w_1) \right| \\
\leq C(||\gamma_1||_{C^2})(|H_{\rho}(\Delta_{-h} w_1, 0)||dH_{\rho}(w_1^{-}, 0)| + |dH_{\rho}(w_1^{-}, 0)|).
\]
Using the above results, we estimate (16), (17), (18) for some $C \in \mathbb{R}$, independent of $h$.

First,
\[
\text{(16)} \quad \leq \int_{A_{\rho}} |\langle \Delta_{h} II \circ F_\rho(dF_{\rho}', dF_{\rho}), \Delta_{h} F_{\rho} \rangle| d\omega \\
\leq C \int_{A_{\rho}} (|\Delta_{h} F_{\rho}|^2 |dF_{\rho, +}|^2 + |\Delta_{h} dF_{\rho}|(|dF_{\rho, +}| + |dF_{\rho}|) |\Delta_{h} F_{\rho}|) d\omega \\
\leq C \int_{A_{\rho}} |dF_{\rho, +}|^2 |\Delta_{h} F_{\rho}|^2 d\omega + \varepsilon \int_{A_{\rho}} |\Delta_{h} dF_{\rho}|^2 d\omega + C(\varepsilon) \int_{A_{\rho}} (|dF_{\rho, +}|^2 + |dF_{\rho}|^2) |\Delta_{h} F_{\rho}|^2 d\omega.
\]
For the estimate of (17),

\[
\int_{A_0} \langle II \circ F_\rho (dF_\rho, dF_\rho), S(P^1, 0) \rangle d\omega 
\]

\[
\leq \int_{A_0} \left\{ |\langle II \circ F_\rho (dF_\rho, dF_\rho), (\star) H_\rho (\Delta h w^1, 0) \rangle| + |\langle \Delta h II \circ F_\rho (dF_\rho, dF_\rho), (\star\star) \rangle \rangle \right\} d\omega 
\]

\[
\leq C \int_{A_0} |dF_\rho|^2 |H_\rho (\Delta h w^1, 0)|^2 d\omega 
+ C \int_{A_0} \{ |\Delta h F_\rho|^2 |H_\rho (\Delta h w^1, 0)| + |\Delta h dF_\rho|(|dF_\rho| + |dF_\rho||H_\rho (\Delta h w^1, 0)|) \} d\omega 
\]

\[
\leq C \int_{A_0} |dF_\rho|^2 |H_\rho (\Delta h w^1, 0)|^2 d\omega + C \int_{A_0} |dF_\rho|^2 (|\Delta h F_\rho|^2 + |H_\rho (\Delta h w^1, 0)|^2) d\omega 
+ \varepsilon \int_{A_0} |\Delta h dF_\rho|^2 d\omega + C(\varepsilon) \int_{A_0} (|dF_\rho|^2 + |H_\rho (\Delta h w^1, 0)|^2 + |H_\rho (\Delta h w^1, 0)|^2 + |H_\rho (\Delta h w^1, 0)|^2) d\omega.
\]

note that \( \Delta h w^1 = \Delta h w^1 \), and we obtain a similar estimate for the second term of (17).

Thus, we have that

(17) \[
\leq \varepsilon C \int_{A_0} |\Delta h dF_\rho|^2 d\omega + C(\varepsilon) \int_{A_0} (|dF_\rho|^2 + |H_\rho (\Delta h w^1, 0)|^2 + |H_\rho (\Delta h w^1, 0)|^2 + |H_\rho (\Delta h w^1, 0)|^2 + |H_\rho (\Delta h w^1, 0)|^2) d\omega.
\]

For the estimate of (18),

\[
- \int_{A_0} \langle dF_\rho, dP^1, 0 \rangle d\omega \leq \int_{A_0} |\langle dF_\rho, d(\star) H_\rho (\Delta h w^1, 0) \rangle| d\omega 
+ \int_{A_0} |\langle dF_\rho, (\star) dH_\rho (\Delta h w^1, 0) \rangle| d\omega + \int_{A_0} |\langle \Delta h dF_\rho, d(\star\star) \rangle| d\omega 
\]

\[
\leq \varepsilon C \int_{A_0} |\Delta h dF_\rho|^2 d\omega + \varepsilon C \int_{A_0} |dH_\rho (\Delta h w^1, 0)|^2 d\omega 
+ C(\varepsilon) \int_{A_0} (|dF_\rho|^2 + |dH_\rho (\Delta h w^1, 0)|^2 + |dH_\rho (\Delta h w^1, 0)|^2) \cdot
\]

\[
(|H_\rho (\Delta h w^1, 0)|^2 + |H_\rho (\Delta h w^1, 0)|^2 + |H_\rho (\Delta h w^1, 0)|^2) d\omega.
\]

We obtain a similar estimate for the second term of (18):

(18) \[
\leq \varepsilon C \int_{A_0} |\Delta h dF_\rho|^2 d\omega + \varepsilon C \int_{A_0} |dH_\rho (\Delta h w^1, 0)|^2 d\omega 
+ C(\varepsilon) \int_{A_0} (|dF_\rho|^2 + |dH_\rho (\Delta h w^1, 0)|^2 + |dH_\rho (\Delta h w^1, 0)|^2 + |dH_\rho (\Delta h w^1, 0)|^2 + |dH_\rho (\Delta h w^1, 0)|^2) \cdot
\]

\[
(|H_\rho (\Delta h w^1, 0)|^2 + |H_\rho (\Delta h w^1, 0)|^2 + |H_\rho (\Delta h w^1, 0)|^2 + |H_\rho (\Delta h w^1, 0)|^2) d\omega.
\]
Similarly, we obtain an estimate

\begin{equation}
\left(19\right) \int_{A_{\rho}} |\Delta_{h} d\mathcal{F}_{\rho}|^2 \, d\omega = \varepsilon C \int_{A_{\rho}} |\Delta_{h} d\mathcal{F}_{\rho}|^2 \, d\omega + \varepsilon C \int_{A_{\rho}} |d\mathcal{H}_{\rho}(\Delta_{h} w^{1}, 0)|^2 \, d\omega + C(\varepsilon) \Xi,
\end{equation}

where

\[ \Xi := \int_{A_{\rho}} \left( |d\mathcal{F}_{\rho}|^2 + |d\mathcal{F}_{\rho}|^2 + |d\mathcal{H}_{\rho}(\tilde{w}^{1}, 0)|^2 + |d\mathcal{H}_{\rho}(\tilde{w}^{1}, 0)|^2 + |d\mathcal{H}_{\rho}(\tilde{w}^{1}, 0)|^2 \right. \]
\[ + |d\mathcal{H}_{\rho}(0, \tilde{w}^{2}_{+})|^2 + |d\mathcal{H}_{\rho}(0, \tilde{w}^{2}_{+})|^2 + |d\mathcal{H}_{\rho}(0, \tilde{w}^{2})|^2 \cdot \]

\[ \left. \left( |\Delta_{h} \mathcal{F}_{\rho}|^2 + |\mathcal{H}_{\rho}(\Delta_{h} w^{1}, 0)|^2 + |\mathcal{H}_{\rho}(\Delta_{h} w^{1}, 0)|^2 + |\mathcal{H}_{\rho}(0, \Delta_{h} w^{2} - 2) + |\mathcal{H}_{\rho}(0, \Delta_{h} w^{2})|^2 \right) \right) \, d\omega \]

\[ \text{(III) On } \partial B, \text{ it holds that } \Delta_{h}(\gamma^{i} \circ w^{i}) = d\gamma^{i}(w^{i}) \Delta_{h} w^{i} + \frac{1}{h} \int_{w^{i}}^{w^{i}} \frac{d^{2} \gamma^{i}(s'' d')}{ds'' d'}. \]

Using \( T^{i}(w^{i}) \) at the right hand side of \( \text{(20)} \), we obtain an \( H^{1,2}(A_{\rho}, \mathbb{R}^{k}) \)- extension with boundary \( \Delta_{h} w^{i} \) on \( C^{1} \) and 0 on \( C_{2} \), and by the D-minimality of the harmonic extension between the maps with the same boundary, we have

\begin{equation}
\int_{A_{\rho}} |d\mathcal{H}_{\rho}(\Delta_{h} w^{1}, 0)|^2 \, d\omega
\end{equation}

\[ \leq C \int_{A_{\rho}} \left[ |d\mathcal{H}_{\rho}(w^{1}, 0)| \left( |\Delta_{h} \mathcal{F}_{\rho}| + |\star| \right) + |d\Delta_{h} \mathcal{F}_{\rho}| + |d\star| \right]^2 \, d\omega
\end{equation}

\[ \leq C \int_{A_{\rho}} \left\{ |d\mathcal{H}_{\rho}(w^{1}, 0)|^2 |\Delta_{h} \mathcal{F}_{\rho}|^2 + |d\mathcal{H}_{\rho}(\Delta_{h} w^{1}, 0)|^2 |\mathcal{H}_{\rho}(\Delta_{h} w^{1}, 0)|^2 + |d\Delta_{h} \mathcal{F}_{\rho}|^2 \right. \]
\[ + |\mathcal{H}_{\rho}(\Delta_{h} w^{1}, 0)|^2 \left( |d\mathcal{H}_{\rho}(\tilde{w}^{1}, 0)| + |d\mathcal{H}_{\rho}(\tilde{w}^{1}, 0)|^2 \right) \]
\[ + |d\mathcal{H}_{\rho}(w^{1}, 0)|^2 |\Delta_{h} w^{1}, 0| + |d\mathcal{H}_{\rho}(w^{1}, 0)| |\Delta_{h} \mathcal{F}_{\rho}| |d\Delta_{h} \mathcal{F}_{\rho}| \]
\[ + |d\mathcal{H}_{\rho}(w^{1}, 0)| |\mathcal{H}_{\rho}(\Delta_{h} w^{1}, 0)| \left( |d\mathcal{H}_{\rho}(\tilde{w}^{1}, 0)| + |d\mathcal{H}_{\rho}(\tilde{w}^{1}, 0)|^2 \right) |\Delta_{h} \mathcal{F}_{\rho}| \]
\[ + |d\mathcal{H}_{\rho}(w^{1}, 0)| |\mathcal{H}_{\rho}(\Delta_{h} w^{1}, 0)| |d\Delta_{h} \mathcal{F}_{\rho}| \]
\[ + |d\mathcal{H}_{\rho}(w^{1}, 0)| |\mathcal{H}_{\rho}(\Delta_{h} w^{1}, 0)| |d\mathcal{H}_{\rho}(\tilde{w}^{1}, 0)| + |d\mathcal{H}_{\rho}(w^{1}, 0)| \right\} \, d\omega
\end{equation}

\[ \leq C \int_{A_{\rho}} |d\Delta_{h} \mathcal{F}_{\rho}|^2 \, d\omega + C \Xi. \]
Using the estimate \(^{(19)}\) for \(\int_{A_{\rho}}|d\Delta_{\rho}\mathcal{F}|^2\,d\omega\) and from \(^{(21)}, (22)\)

\[
\int_{A_{\rho}}|d\Delta_{\rho}\mathcal{F}|^2\,d\omega + \int_{A_{\rho}}|d\mathcal{H}_{\rho}(\Delta_{\rho}w^1, 0)|^2\,d\omega + \int_{A_{\rho}}|d\mathcal{H}_{\rho}(0, \Delta_{\rho}w^2)|^2\,d\omega \\
\leq \varepsilon C \int_{A_{\rho}}|d\Delta_{\rho}\mathcal{F}|^2\,d\omega + \varepsilon C \int_{A_{\rho}}|d\mathcal{H}_{\rho}(\Delta_{\rho}w^1, 0)|^2\,d\omega + \varepsilon C \int_{A_{\rho}}|d\mathcal{H}_{\rho}(0, \Delta_{\rho}w^2)|^2\,d\omega + C(\varepsilon)\Xi.
\]

Since \(\frac{1}{2}(a^2 + b^2) \leq (a + b)^2 \leq \frac{3}{2}(a^2 + b^2)\), \(a, b \in \mathbb{R}\) and \(H_{\rho}(f, g) = H_{\rho}(f, 0) + H_{\rho}(0, g)\), for some small \(\varepsilon > 0\) in the above estimate we finally obtain the following inequality:

\[
\int_{A_{\rho}}|\Delta_{\rho}d\mathcal{F}|^2\,d\omega + \int_{A_{\rho}}|d\mathcal{H}_{\rho}(\Delta_{\rho}w^1, \Delta_{\rho}w^2)|^2\,d\omega \\
\leq C(\varepsilon) \int_{A_{\rho}}(|d\mathcal{F}|^2 + |d\mathcal{F}_+|^2 + |d\mathcal{F}_-|^2 \\
+ |d\mathcal{H}_{\rho}(w^1, w^2)|^2 + |d\mathcal{H}_{\rho}(w^1_+, w^2_+)|^2 + |d\mathcal{H}_{\rho}(w^1_-, w^2_-)|^2) \cdot (|\Delta_{\rho}\mathcal{F}|^2 + |H(\Delta_{-\rho}w^1, \Delta_{-\rho}w^2)|^2 + |H(\Delta_{\rho}w^1, \Delta_{\rho}w^2)|^2)\,d\omega.
\]

\(^{(23)}\)

**IV** Now extend \(\mathcal{F}\) to \(\mathbb{R}^2 \setminus B_{\rho}^2\) by conformal reflection as follows

\[
\mathcal{F}_\rho(z) = \mathcal{F}_\rho\left(\frac{z}{|z|^2}\right), \text{ if } 1 \leq |z|
\]

\[
\mathcal{F}_\rho(z) = \mathcal{F}_\rho\left(\frac{2}{|z|^2}\right), \text{ if } \rho^2 \leq |z| \leq \rho.
\]

Choose \(r \in (0, \min\{\frac{\rho^2}{2}, r_0\})\), and \(\varphi \in C_0^\infty(B_{2r}(0))\) with \(\varphi \equiv 1\) on \(B_r(0)\).

We may cover \(A_{\rho}\) with balls of radius \(r\) in such a way that at most \(k\) balls of the covering intersect at any point \(p \in A_{\rho}\), for any \(r\) as above (\(\mathbb{R}^2\) is metrizable). Let \(B^1\) denote the balls of the covering with centers \(p_i\) and \(\varphi_i(p) := \varphi(p - p_i)\).

Then, from \(^{(23)}\)

\[
\int_{A_{\rho}}|\Delta_{\rho}d\mathcal{F}|^2\,d\omega + \int_{A_{\rho}}|d\mathcal{H}_{\rho}(\Delta_{\rho}w^1, \Delta_{\rho}w^2)|^2\,d\omega \\
\leq C \Sigma_{\rho} \int_{\mathbb{R}^2 \setminus A_{\rho}} \left(|\Delta_{\rho}\mathcal{F}_\rho|^2 + |H(\Delta_{\rho}w^1, \Delta_{\rho}w^2)|^2 + |H(\Delta_{\rho}w^1, \Delta_{\rho}w^2)|^2\right)\varphi_i^2 \cdot \\
\frac{(|d\mathcal{F}_\rho|^2 + |d\mathcal{F}_\rho|^2 + |d\mathcal{H}_{\rho}(\tilde{w}^1, \tilde{w}^2)|^2 + |d\mathcal{H}_{\rho}(\tilde{w}^1_+, \tilde{w}^2_+)|^2 + |d\mathcal{H}_{\rho}(\tilde{w}^1_-, \tilde{w}^2_-)|^2)}{\varphi_i^2} \,d\omega.
\]

According to Lemma \(^{(33)}\) and Remark \(^{(3.2)}\) \(\chi\) satisfies the Morrey growth condition, so apply the Morrey Lemma with \(\chi\) and \((\Delta_{\rho}\mathcal{F}_\rho)\varphi_i\) resp.\(H(\Delta_{\rho}w^1, \Delta_{\rho}w^2)\varphi_i\) resp. \(H(\Delta_{\rho}w^1, \Delta_{\rho}w^2)\varphi_i\).
Then we obtain
\[
\int_{B_{2r}(p_i)} \chi(|\Delta_h \mathcal{F}_\rho|^2 + |H(\Delta_{-h} \omega^1, \Delta_{-h} \omega^2)|^2 + |H(\Delta_h \omega^1, \Delta_h \omega^2)|^2) \varphi_i^2 d\omega \\
\leq C r^2 \int_{B_{2r} \setminus B_{r2}} \chi d\omega \int_{B_{2r}(p_i)} \left( |d \Delta_h \mathcal{F}_\rho|^2 + |d H(\Delta_{-h} \omega^1, \Delta_{-h} \omega^2)|^2 + |d H(\Delta_h \omega^1, \Delta_h \omega^2)|^2 \right) d\omega \\
+ C r^2 \int_{B_{2r} \setminus B_{r2}} \chi d\omega \int_{B_{2r}(p_i)} \left( |\Delta_h \mathcal{F}_\rho|^2 + |H(\Delta_{-h} \omega^1, \Delta_{-h} \omega^2)|^2 + |H(\Delta_h \omega^1, \Delta_h \omega^2)|^2 \right) d\omega.
\]

Summing over \(i\) yields a constant \(C\), independent of \(r\), such that
\[
\int_{A_\rho} |\Delta_h \mathcal{F}_\rho|^2 d\omega + \int_{A_\rho} |d \mathcal{H}_\rho(\Delta_h \omega^1, \Delta_h \omega^2)|^2 d\omega \\
\leq C r^2 \int_{B_{2r} \setminus B_{r2}} \left( |d \Delta_h \mathcal{F}_\rho|^2 + |d H(\Delta_{-h} \omega^1, \Delta_{-h} \omega^2)|^2 + |d H(\Delta_h \omega^1, \Delta_h \omega^2)|^2 \right) d\omega \\
+ C r^2 \int_{B_{2r} \setminus B_{r2}} \left( |\Delta_h \mathcal{F}_\rho|^2 + |H(\Delta_{-h} \omega^1, \Delta_{-h} \omega^2)|^2 + |H(\Delta_h \omega^1, \Delta_h \omega^2)|^2 \right) d\omega.
\]

Since \(d \mathcal{F}_\rho, d H(\omega^1, \omega^2) \in L^2\), choosing small \(r > 0\), we obtain \(C \in \mathbb{R}\), independent of \(|h| \leq h_0\) with
\[
\int_{A_\rho} |\Delta_h \mathcal{F}_\rho|^2 d\omega \leq C.
\]

\[\square\]

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