Connectivity of the coset poset and the subgroup poset of a group

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Abstract

We study the connectivity of the coset poset and the subgroup poset of a group, focusing in particular on simple connectivity. The coset poset was recently introduced by K. S. Brown in connection with the probabilistic zeta function of a group. We further Brown’s study of the homotopy type of the coset poset, and in particular generalize his results on direct products and classify direct products with simply connected coset posets.

The homotopy type of the subgroup poset \( L(G) \) has been examined previously by Kratzer, Thévenaz, and Shareshian. We generalize some results of Kratzer and Thévenaz, and determine \( \pi_1(L(G)) \) in nearly all cases.

1 Introduction

One may apply topological concepts to any poset (partially ordered set) \( P \) by means of the simplicial complex \( \Delta(P) \) consisting of all finite chains of \( P \). The topology of posets arising from groups has been studied extensively; see for example [1, 2, 3, 4]. The basic topological theory of posets is described in [1] and in [2]. We will often use \( P \) to denote both \( \Delta(P) \) and its geometric realization \( |\Delta(P)| = |P| \).

The coset poset \( C(G) \) of a finite group \( G \) (the poset of all left cosets of all proper subgroups of \( G \), ordered by inclusion) was introduced by Brown [4] in connection with the probabilistic zeta function \( P(G,s) \). (Note that the choice of left cosets over right cosets is irrelevant since each left coset is a right coset \((xH = (xHx^{-1})x) \) and vice versa.) Brown showed that \( P(G,-1) = -\chi(C(G)) \) and used this relationship to prove certain divisibility results about \( P(G,-1) \) (which is always an integer). In fact, Brown proved that for arbitrary integers \( s \), \( P(G,s) \) can be calculated from the coset poset. The connection between \( P(G,-1) \) and \( \chi(C(G)) \) motivates the study of the homotopy type of \( C(G) \). Brown obtained a variety of results along these lines, and in this paper we prove a number of new results about the homotopy-type of \( C(G) \). In particular, we study the connectivity of \( C(G) \) and further Brown’s study of the homotopy type of \( C(G) \) in terms of normal subgroups and quotients. Brown has asked [4, Question 3], “For which finite groups \( G \) is \( C(G) \) simply connected?”
One of the main goals of this paper is to study this question (for both finite and infinite groups). We now gather together our main results on the problem.

1. If $G$ is not a 2-generator group, then $C(G)$ is simply connected (Corollary 2.5).

2. If $G = H \times K$ ($H, K$ non-trivial), then $\pi_1(C(G)) \neq 1$ if and only if both $H$ and $K$ are cyclic of prime-power order (Theorem 3.6).

3. If $G = H \times K$ ($H, K$ non-trivial) where $K$ is not finite cyclic or $H$ is not $K$-cyclic, then $C(G)$ is simply connected (Proposition 3.5).

4. If the subgroup poset $L(G)$ is disconnected, or if $G$ has a maximal subgroup isomorphic to $\mathbb{Z}/p^n$ ($p$ prime), then $C(G)$ is not simply connected (in fact, lower bounds on the rank of $H_1(C(G))$ are obtained in Theorems 4.1 and 4.4).

5. If $G$ is isomorphic to $PSL_2(F_5) \cong A_5$ or $PSL_2(F_7)$, then $C(G)$ is simply connected (Propositions 5.4 and 5.9).

Further results appear in Corollary 2.9 and Propositions 3.5, 3.8, and 3.10.

Brown has given a complete description of the homotopy type of $C(G)$ for any finite solvable group $G$ [4, Proposition 11], and we now take a moment to discuss this result. Recall that a chief series in a group $G$ is a maximal chain in the lattice of normal subgroups of $G$.

**Theorem 1.1 (Brown)** Let $G$ be a finite solvable group, let $\{1\} = N_0 < \cdots < N_k = G$ be a chief series for $G$ and let $c_i$ be the number of complements of $N_i/N_{i-1}$ in $G/N_{i-1}$ ($1 \leq i \leq k$). Then $C(G)$ is homotopy equivalent to a bouquet of spheres, each of dimension $d - 1$ where $d$ is the number of indices $i$ such that $c_i \neq 0$. The number of spheres is given by $\left| \prod_{i=1}^k \left( c_i \frac{|N_i|}{|N_{i-1}|} - 1 \right) \right|$.

The number of spheres may be calculated by induction, using [4, Corollary 3].

We remark that when $G$ is solvable, the number $d$ of complemented factors in a chief series for $G$ is bounded below by $\pi(G)$, the number of distinct primes dividing $a(G)$. This follows from the fact that any solvable group $G$ may be built up by a series of extensions with kernel of prime-power order. In this process, each time we add a new prime the kernel $P$ and quotient $Q$ of the extension are relatively prime and hence $H^2(Q, P) = 0$ [3, Chapter 3, Proposition 10.1] and the extension splits. In light of the above theorem, this shows that if $G$ is a finite solvable group, $C(G)$ is $k$-connected whenever $\pi(G) \geq k + 2$.

The paper is structured as follows. In Section 2, we introduce the notion of an atomized poset. The coset poset of any group and the subgroup poset of any finite group (or any torsion group) are atomized, and the main theorem of this section gives conditions under which an atomized poset is $k$-connected.

The main results on the simple connectivity of $C(G)$ appear in Section 3. We study the coset poset of a semi-direct product $H \rtimes K$ and extend Brown’s analysis of direct products to this setting.
In Section 4, we study the homology of $C(G)$ using Mayer-Vietoris sequences, and obtain conditions under which the coset poset is not simply connected.

The results described so far have little bearing on finite, non-abelian simple groups. In Section 5, we treat the first two non-abelian simple groups and prove that both have simply connected coset posets.

Many of the techniques in this paper apply to the subgroup poset $L(G)$ (the poset of all proper, non-trivial subgroups of $G$, ordered by inclusion) as well as to the coset poset, and in Section 6 we discuss the connectivity of $L(G)$. The subgroup poset of a finite group has been studied previously by Kratzer and Thévenaz [9] and by Thévenaz [19]. Using the homotopy complementation formula of Björner and Walker [2], we generalize some of the results in [9]. Our results on the connectivity of $L(G)$ are based mainly on this complementation formula and on an analysis of the connectedness of $L(G)$. Some of the results could have been proven using methods from earlier sections, but the chosen approach is more elegant and more powerful. After proving general results about the connectivity of $L(G)$, we restrict our attention to finite groups and determine $\pi_1(L(G))$ in nearly all cases.

In the final section of the paper, we discuss a relationship between the homology of the coset poset and that of the subgroup poset of certain groups. Kratzer and Thévenaz [9] have proven an analogue of Theorem 1.1 for the subgroup poset of a finite solvable group (Theorem 6.1). The striking similarity between these two results is part of the motivation behind our discussion, and in particular shows that the relationship holds for finite solvable groups.

Some remarks on notation and conventions: If $\Delta$ is a simplicial complex, we will denote its $k$-skeleton by $\Delta^{\leq k}$. In a poset $P$, a chain $p_1 < p_2 < \cdots < p_n$ is said to have length $n - 1$. A wedge sum of empty topological spaces, or a wedge sum over an empty index set, is defined to be a point. Finally, we consider the empty space to be path-connected but not 0-connected.

The research described in this article began as an undergraduate thesis [14], and certain results are described there in more detail (especially the results in Section 4). I thank Kenneth S. Brown, who served as my thesis advisor, for all his help in connection with this article. In addition to introducing me to the coset poset, Professor Brown suggested many of the techniques in this paper and helped a great deal with the exposition. I also thank Maria Silvia Lucido for providing the answer to a question I asked about the subgroup poset.

2 Connectivity of Atomized Posets

In this section we introduce atomized posets, a class of posets generalizing the coset poset of a group. For finite groups (and in fact torsion groups) the subgroup poset is also atomized. The main result of this section is Theorem 2.2, which gives conditions under which an atomized poset is $k$-connected.

Definition 2.1 Let $P$ be a poset. We call $P$ atomized if every element of $P$
lies above some minimal element and every finite set of minimal elements with
an upper bound has a join.

We call the minimal elements of $P$ atoms, and denote the set of atoms of
$P$ by $\mathcal{A}(P)$. If $S \subset \mathcal{A}(P)$ is finite, we say that $S$ generates its join (or that $S$
generates $P$ if $P \supseteq S$ is empty), and we write $\langle S \rangle$ for the object generated by $S$
(so $\langle S \rangle = P$ is allowed).

The proper part of any finite-length lattice is atomized, but the converse
is not true. (Consider, for example, the five-element poset $\{0 < a, b < c, d\}$.
This poset has a unique minimal element and is thus atomized, but its maximal
elements do not have a meet.) Note that the subgroup poset of $\mathbb{Z}$ has no minimal
elements and is not atomized, and in fact if a group $G$ has an element of infinite
order then $L(G)$ is not atomized.

In the coset poset of any group or the subgroup poset of any torsion group,
the definition of generation coincides with the standard group theoretic defini-
tions, where the coset generated by elements $x_1, x_2, \ldots \in G$ is

$$x_1 \langle x_1^{-1} x_2, x_1^{-1} x_3, \ldots \rangle.$$  

The following lemma shows that, up to homotopy, we can replace any at-
atomized poset $P$ with a smaller simplicial complex, $\mathcal{M}(P)$. This complex has
many fewer vertices, although it has much higher dimension, and will play an
important role in our analysis of the coset poset.

**Lemma 2.2** Let $P$ be an atomized poset and let $\mathcal{M}(P)$ denote the simplicial
complex with vertex set $\mathcal{A}(P)$ and with a simplex for each finite set $S \subset \mathcal{A}(P)$
with $\langle S \rangle \neq P$. Then $\Delta(P) \simeq \mathcal{M}(P)$.

**Proof.** We will apply the Nerve Theorem [1, 10.6]. Consider the collection of
all cones $P \supseteq x$ with $x \in \mathcal{A}(P)$. Since $P$ is atomized, $\bigcup_{x \in \mathcal{A}(P)} \Delta(P \supseteq x) = \Delta(P)$. If $S \subset \mathcal{A}(P)$ is finite and $\langle S \rangle \neq P$, then $\cap_{x \in S} P \supseteq x = P \supseteq \langle S \rangle \simeq \{\ast\}$. So each finite
intersection is either empty or contractible, and the Nerve Theorem tells us that
$\Delta(P)$ is homotopy equivalent to the nerve of this cover, which is exactly
$\mathcal{M}(P)$. $\square$

We call $\mathcal{M}(P)$ the minimal cover of $P$. When $P = C(G)$ for some group $G$,
we denote $\mathcal{M}(C(G))$ by $\mathcal{M}(G)$. This complex has vertex set $G$ and a simplex for each finite subset of $G$ contained in a proper coset.

**Theorem 2.3** Let $P$ be an atomized poset such that no $k$ atoms generate $P$.
Then $P$ is $(k - 2)$-connected. In particular, for any group $G$

1. if $G$ is not $k$-generated, then $C(G)$ is $(k - 1)$-connected;

2. if $G$ is a torsion group in which any $k$ elements of prime order generate a
proper subgroup, then $L(G)$ is $(k - 2)$-connected.
**Proof.** By the lemma, it suffices to check that $\mathcal{M}(P)$ is $(k-2)$-connected. Since no $k$ atoms generate $P$, any $k$ atoms form a simplex in $\mathcal{M}(P)$. So $\mathcal{M}(P)\subseteq (k-1)$ is just the $(k-1)$-skeleton of the full simplex on the vertex set $A(P)$, and thus $\mathcal{M}(P)$ is $(k-2)$-connected. (Note that $A(P)$ may be infinite; in this case the “full simplex” on the set $A(P)$ is the simplicial complex whose simplices are all the finite subsets of $A(P)$.) □

**Remark 2.4** Brown has asked [4, Question 4] whether there exist finite groups $G$ with $\mathcal{C}(G)$ contractible. The above theorem shows that if $G$ is a (necessarily infinite) group which is not finitely generated, then $\mathcal{C}(G)$ is contractible.

Theorem 2.3 specializes to the following result giving conditions under which $\mathcal{C}(G)$ (or any atomized poset) is simply connected.

**Corollary 2.5** If $G$ is not a 2-generator group, then $\mathcal{C}(G)$ is simply connected. More generally, if $P$ is an atomized poset and no three atoms generate $P$, then $|P|$ is simply connected.

This theorem does not characterize finite groups with simply connected coset posets. In fact, the first non-abelian simple group, $A_5$, affords a counterexample (Proposition 2.4).

We now give an alternate proof of Corollary 2.4. This proof will rely solely on the combinatorial structure of $\Delta(P)$ itself (as opposed to that of $\mathcal{M}(P)$). First we need a lemma giving a simple representative for each homotopy class of loops in $|P|$. Recall that an edge cycle $C$ in a graph $\Gamma$ is determined by listing (in order) the vertices through which $C$ passes.

**Lemma 2.6** Let $P$ be an atomized poset in which no two elements generate. Then every loop in $|P|$ is homotopic to an edge cycle in $\Delta(P)^{\leq 1}$ of the form

$$(a_1,(a_1,a_2),a_2,(a_2,a_3),\ldots,a_n,(a_n,a_1),a_1),$$

where the $a_i$ are atoms.

**Proof.** By the Simplicial Approximation Theorem, every loop in $|P|$ is homotopic to an edge cycle. Let $C = (p_1,p_2,\ldots,p_n = p_1)$ be an edge cycle in $\Delta(P)^{\leq 1}$. For each $i$ either $p_i < p_{i+1}$ or $p_i > p_{i+1}$, and it is easy to check that $C$ is homotopic to a cycle $C' = (p'_1,\ldots,p'_n)$ in which these inequalities alternate direction (simply remove the points at which the inequalities do not alternate). We call $p'_i$ lower if $p'_i < p'_{i+1}$ and upper otherwise. The cycle $C'$ is now homotopic to the cycle $C''$ formed by replacing each lower vertex $p'_i$ with an atom $a_i < p'_i$. (The homotopy is obtained by considering the simplices $a_i < p_i < p_{i-1}$.)

Let $C'' = (a_1,q_1,a_2,\ldots,a_n,q_n,a_1)$ (with the $a_i$ atoms). Then $C''$ is homotopic to the edge cycle formed by replacing each upper vertex $q_i$ by $\langle a_i,a_{i+1}\rangle$. (The homotopy is obtained by considering the simplices $a_i < a_{i+1} < q_i$ and $a_{i+1} < a_{i+2} \leq q_i$.) □

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We call edge cycles of the form described in the lemma atomic cycles.

**Alternate Proof of Corollary 2.5.** Let $P$ be an atomized poset in which no three atoms generate. Theorem 2.3 shows that $P$ is connected, so it suffices to show that $\pi_1(P) = 1$. By the lemma any loop $L$ in $|P|$ is homotopic to an atomic cycle $C = (a_1, \langle a_1, a_2 \rangle, a_2, \langle a_2, a_3 \rangle, \ldots, a_n, \langle a_n, a_1 \rangle)$ in which the $a_i$ are atoms. Note that $C$ has even length, and that if the length of $C$ is two or four, $C$ is simply a path followed by its inverse and is thus nullhomotopic.

If the length of $C$ is greater than 4, then $C$ contains the path $\gamma = a_1 < \langle a_1, a_2 \rangle > a_2 < \langle a_2, a_3 \rangle > a_3$, and since $\gamma$ lies in the cone under $\langle a_1, a_2, a_3 \rangle \neq P$ it is homotopic (relative to its endpoints) to any other path between $a_1$ and $a_3$, provided the second path also lies under $\langle a_1, a_2, a_3 \rangle$. In particular, $\gamma$ is homotopic to the path $a_1 \leq \langle a_1, a_3 \rangle \geq a_3$. Thus $C$ is homotopic to a shorter cycle, which is still atomic, and repeating the process will eventually provide a null-homotopy of $C$. \qed

We now present a result guaranteeing simple connectivity of an atomized poset under weaker conditions than those of Corollary 2.5.

The following result is standard; see [17].

**Lemma 2.7** Let $\Delta$ be a simplicial complex, and let $T \subset \Delta^{\leq 1}$ be a maximal tree (i.e. a spanning tree).

Then $\pi_1(\Delta)$ has a presentation with a generator for each (ordered) edge $(u, v)$ with $\{u, v\} \in \Delta^{\leq 1}$, and with the following relations:

1. $(u, v) = 1$ if $\{u, v\} \in T$,
2. $(u, v)(v, u) = 1$ if $\{u, v\} \in \Delta^{\leq 1}$,
3. $(u, v)(v, w)(w, u) = 1$ if $\{u, v, w\} \in \Delta^{\leq 2}$.

When the simplicial complex in question is $M(G)$ for some non-cyclic group $G$, we will always set the tree $T$ to be the collection of all edges $\{1, g\}$ ($g \in G$). (Note that since $G$ is non-cyclic, all such edges exist.) We will refer to the resulting presentation for $\pi_1(M(G)) \cong \pi_1(C(G))$ as the standard presentation.

**Proposition 2.8** Let $P$ be an atomized poset and assume that no two atoms generate $P$. Say there exists an atom $a_0 \in A(P)$ with the following property:

1. For any elements $a_1, a_2 \in A(P)$, there exists an element $a_3 \in A(P)$ such that $\langle a_1, a_2, a_3 \rangle$, $\langle a_0, a_1, a_3 \rangle$, $\langle a_0, a_2, a_3 \rangle \neq P$.

Then $|P|$ is simply connected.

**Proof.** First, note that in the 1-skeleton of $M(P)$ all possible edges between vertices exist (and in particular $M(P)$ is connected). We wish to apply Lemma 2.7 to $M(P)$, so we must choose a maximal tree in $M(P)^{\leq 1}$. Let $T$ be the star at the vertex $a_0$, i.e. $T$ consists of all edges of the form $\{a_0, a\}$ with $a \in A(P)$. The generators given in the lemma consist of ordered edges $(a_1, a_2)$ ($a_1, a_2 \in A(P)$) and we will now show that each generator is trivial.
There are two cases. First, if \( \langle a_0, a_1, a_2 \rangle \neq P \), then these vertices form a simplex in \( M(P) \). So we obtain the relations \( (a_1, a_2) = (a_1, a_0)(a_0, a_2) = 1 \).

Next, say \( \langle a_0, a_1, a_2 \rangle = P \). Let \( a_3 \) be the element guaranteed by the hypothesis. Then, as above, the generators corresponding to the edges \( \{a_1, a_3\} \) and \( \{a_2, a_3\} \) are trivial. Furthermore, since \( \langle a_1, a_2, a_3 \rangle \neq P \) these vertices form a simplex in \( M(P) \), and we have the relations \( (a_1, a_2) = (a_1, a_3)(a_3, a_2) = 1 \).

Thus all generators for \( \pi_1(M(P)) \) are trivial, and hence \( M(P) \) and \( |P| \) are simply connected.

In the case of the coset poset, it is easy to check that Proposition 2.8 specializes to the following result (using the standard presentation, i.e. \( a_0 = 1 \)).

**Corollary 2.9** Let \( G \) be a non-cyclic group in which the following condition holds:

(2) For any \( x, y \in G \) such that \( \langle x, y \rangle = G \), there exists \( z \in G \) such that \( \langle z, x \rangle, \langle z, y \rangle, \langle z^{-1}x, z^{-1}y \rangle \neq G \).

Then \( C(G) \) is simply connected.

We now consider the cases in which these last two results apply. Of course, if \( P \) is not generated by three atoms (respectively \( G \) is not generated by two elements) then (1) (respectively (2)) is satisfied, but Theorem 2.3 also applies. Additionally, if there is an element \( a_0 \in P \) such that \( \langle a_0, x, y \rangle \neq P \) for any \( x, y \in P \), then (1) is satisfied. (In the case of the coset poset, this reduces to saying \( G \) is not a 2-generator group.)

The simple group \( A_5 \) satisfies (2) and hence \( C(A_5) \) is simply connected. The details, which are not difficult, appear in \( 14 \). In Section 3 we will give another proof of simple connectivity, which eliminates some of the computation.

We note one final situation in which (2) is satisfied. Say \( G = H \times K \), where \( K \) and \( H \) are non-cyclic groups. Then if \( x = (h_1, k_1) \) and \( y = (h_2, k_2) \), setting \( z = (h_1, k_2) \) we have \( \langle x, z \rangle, \langle y, z \rangle \) and \( \langle z^{-1}x, z^{-1}y \rangle \neq G \). In Section 3 we will go a bit further and classify direct products with simply connected coset posets (Theorem 3.6).

**3 The Coset Poset of an Extension**

In this section we will consider the coset poset of a (non-simple) group \( G \) in terms of an extension \( N \triangleleft G \rightarrow G/N \).

**Definition 3.1** For any semi-direct product \( G = H \rtimes K \) (\( H \) and \( K \) arbitrary groups), let \( f : G \rightarrow H \) be the function \( f(h, k) = h \), and let \( \pi : G \rightarrow K \) be the quotient map. We call a coset \( gT \in C(G) \) saturating if \( \pi(gT) = K \) and the only \( K \)-invariant subgroup of \( H \) that contains \( f(T) \) is \( H \) itself (a subgroup \( I \triangleleft H \) is called \( K \)-invariant if each element of \( K \) induces automorphisms of \( I \)).
The direct product $P \times Q$ of posets $P$ and $Q$ is the Cartesian product of $P$ and $Q$ together with the ordering $(p, q) \leq (p', q') \iff p \leq p'$ and $q \leq q'$. The join $P*Q$ of $P$ and $Q$ is the disjoint union of $P$ and $Q$ together with the ordering that induces the original orderings on $P$ and $Q$ and satisfies $p < q$ for each $p \in P$, $q \in Q$. There are canonical homeomorphisms $|P \times Q| \cong |P| \times |Q|$ and $|P*Q| \cong |P| \ast |Q|$, so long as one takes the associated compactly generated topology on the right hand side when $P$ and $Q$ are not locally countable (see [20]).

**Lemma 3.2** Let $G = H \rtimes K$, with $H$ and $K$ non-trivial groups. Let $C_0(G)$ be the poset of all non-saturating cosets and let $C_K(H)$ denote the poset of all cosets of proper, $K$-invariant subgroups of $H$. Then $C_0(G)$ is homotopy equivalent to $C_K(H) \ast C(K)$.

**Proof.** Let $C^+(K)$ denote the set of all cosets in $K$ (including $K$ itself) and let $C^+_K(H) = C_K(H) \cup \{H\}$. Then if $C_{00}(H \rtimes K)$ denotes the set of all proper cosets of the form $(g, h)I \rtimes J$ (with $I \leq H$ $K$-invariant and $J \leq K$), one checks that the map

$$(x, y)I \rtimes J \mapsto (xI, yJ)$$

is a well-defined poset isomorphism

$C_{00}(H \rtimes K) \cong C^+_K(H) \times C^+(K) - \{(H, K)\}.$

The latter is homeomorphic to $|C_K(H) \ast C(K)|$ (see [14], Proposition 1.9) or the proof of [8, Proposition 2.5]). Finally, we have an increasing poset map $\Phi$ from $C_0(H \rtimes K)$ onto $C_{00}(H \times K)$ given by

$$\Phi((x, y)T) = (x, y)\hat{f}(T) \rtimes \pi(T)$$

where $\hat{f}(T)$ is the smallest $K$-invariant subgroup containing $f(T)$ (i.e. the intersection of all invariant subgroups containing $f(T)$). This map is a homotopy equivalence by [16, Corollary 10.12].

In the case where the action of $K$ on $H$ is trivial, $f$ becomes the quotient map $H \times K \to H$ and all subgroups of $H$ are $K$-invariant. In this case $C_0(H \times K)$ is the poset of all cosets which do not surject onto both factors, and we obtain the following result which appears as [16, Lemma 5, Proposition 12].

**Lemma 3.3 (Brown)** For any finite groups $H$ and $K$, $C_0(H \times K) \simeq C(H) \ast C(K)$. If $H$ and $K$ are coprime (i.e. have no isomorphic quotients other than the trivial group) then in fact $C(H \times K) \simeq C(H) \ast C(K)$.

Coprimality implies that there are no saturating subgroups (and hence no saturating cosets); see [16, Section 2.4].

We will now use Lemma 3.2 to show that most semi-direct products have simply connected coset posets. First we need the following simple lemma, which appears (for finite groups) as [16, Proposition 14]. The result extends, with the same proof, to infinite groups (the argument for $\mathbb{Z}$ requires a simple modification).

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Lemma 3.4  Let $G$ be a group. Then $C(G)$ is connected unless $G$ is cyclic of prime-power order.

Given a semi-direct product $G = H times K$, we say that $H$ is $K$-cyclic if there is an element $h \in H$ which is not contained in any proper $K$-invariant subgroup of $H$.

Proposition 3.5  Let $G = H \rtimes K$ with $H$ and $K$ non-trivial groups. If $K$ is not finite cyclic, or if $H$ is not $K$-cyclic, then $C(G)$ is simply connected. Furthermore, if $G$ is a torsion group and $K$ is not cyclic of prime-power order, $C(G)$ is simply connected.

Proof. By Lemma 3.2, we have $C_0(G) \simeq C_K(H) \ast C(K)$. We claim that in each of the above cases, $C_0(G)$ is simply connected. The join of a connected space and a non-empty space is always simply connected (see [1]), and both $C_K(H)$ and $C(K)$ are always non-empty ($\{1\} \in C_K(H)$) so it suffices to show that one or the other is connected. If $H$ is not $K$-cyclic, then for every element $h \in H$ there is a $K$-invariant subgroup $I_h < H$ containing $h$, and thus we have a path $hT \geq \{h\} \leq I_h \geq \{1\}$ joining any coset $hT \in C_K(H)$ to the trivial subgroup. In the other cases, Lemma 3.4 shows that $C(K)$ is connected. Thus $\pi_1(C_0(G)) = 1$ in each case.

We now show that every loop in $C(G)$ is nullhomotopic. By Lemma 2.6 it suffices to consider edge cycles of the form

$$C = (\{x_1\}, x_1T_1, \{x_2\}, x_2T_2, \ldots, \{x_n\}, x_nT_n, \{x_1\}),$$

where the $T_i$ are cyclic. If $H$ is not $K$-cyclic, then every cyclic subgroup of $G$ lies in $C_0(G)$ and hence $C$ lies in $|C_0(G)|$ and must be nullhomotopic.

Next, say $K$ is not a finite cyclic group. If none of the vertices of $C$ are saturating cosets, we are done, so assume some coset $x_iT_i$ saturates. Since $T_i$ is cyclic and $K$ is not finite cyclic, we must have $T_i \cong K \cong \mathbb{Z}$. So no subgroup of $T_i$ surjects onto $K$, and hence no cosubset of $x_iT_i$ saturates. Thus adding $x_iT_i$ to $C_0(G)$ cones off a copy of $C(T_i)$, which (by Lemma 3.4) is connected. Since the union of two simply connected spaces with connected intersection is simply connected, we see that $C_0(G) \cup \{x_iT_i\}$ is simply connected ($\Delta(C_0(G)) \cup \{x_iT_i\} = \Delta(C_0(G)) \cup \Delta(C(G)_{x_iT_i})$). Repeating the process eventually shows that $C$ lies in a simply connected poset, and hence is nullhomotopic.

If $K$ is not cyclic of prime-power order and $G$ is a torsion group, each $T_i$ is finite. Thus there are only finitely many cosets in the set $S = \{xT : xT \subset x_iT_i \text{ for some } i\} - C_0(G)$. If $xT$ is a minimum element of $S$, then adding $xT$ to $C_0(G)$ cones of a copy of $C(T)$, which is connected ($T$ surjects onto $K$, which is not cyclic of prime-power order). This process may be repeated until we have added all of $S$, and hence the poset $S \cup C_0(G)$ is simply connected. The cycle $C$ lies in this poset and is thus nullhomotopic.

Theorem 3.6  Let $H$ and $K$ be non-trivial groups. Then $\pi_1(C(H \times K)) \neq 1$ if and only if both $H$ and $K$ are cyclic of prime-power order.
The reader may check that (setting \( m \) of the form · · · 3) \( K \) connecting \( n/2 \) for each \( i, j \) \( S \) subgroups of \( S \) such poset is isomorphic to \( \langle \rangle \). The question of simple connectivity for the coset poset of a finite (non-trivial) semi-direct product is now reduced to the case of products \( H \times \mathbb{Z}/p^n \), where \( p \) is prime and \( H \) is \((\mathbb{Z}/p^n)\)-cyclic. When \( H \) is solvable, Theorem [1.1] applies, so we are most interested in the case where \( H \) is a non-solvable group. The simplest example, then, is \( S_5 \cong A_5 \times \mathbb{Z}/2 \). In this case the coset poset is still simply connected. For the proof we will need the following result of Brown [1, Proposition 10], which we note extends (with the same proof) to infinite groups.

**Lemma 3.7 (Brown)** For any group \( G \) with normal subgroup \( N \), there is a homotopy equivalence \( C(G) \simeq C(G/N) \ast C(G, N) \), where the latter poset is the collection of all cosets \( xH \in C(G) \) which surject onto \( G/N \) under the quotient map.

**Proposition 3.8** For \( n > 3 \), the coset poset of \( S_n \) is simply connected.

**Proof.** We have \( A_n < S_n \) with \( S_n/A_n \cong \mathbb{Z}/2 \), so Lemma [3] gives \( C(S_n) \simeq C(\mathbb{Z}/2) \ast C(S_n, A_n) \cong \text{Susp} (C(S_n, A_n)) \), so it will suffice to show that \( C(S_n, A_n) \) is connected.

Since \( A_n \) has index two in \( S_n \), the elements of \( C(S_n, A_n) \) are exactly the cosets \( xH \) where \( H \) is not contained in \( A_n \). Letting \( S \) denote the set of proper subgroups of \( S_n \) not contained in \( A_n \), we have \( C(S_n, A_n) = \bigcup_{x \in S_n} \{ xH : H \in S \} \).

First we show that for each \( x \) the poset \( \{ xH : H \in S \} \) is connected. Each such poset is isomorphic to \( S \), so it suffices to consider this case. Consider the action of \( S_n \) on the set \( \{1, \ldots, n\} \). For each \( i \), \( \text{Stab}(i) \cong S_{n-1} \), and for any \( i, j \), \( \text{Stab}(i) \cap \text{Stab}(j) \cong S_{n-2} \). All of these groups are in \( S \) (since \( n > 3 \)), and hence for each \( i, j \) there is a path in \( S \) between \( \text{Stab}(i) \) and \( \text{Stab}(j) \).

We now show that for each \( K \in S \) there exists \( i \) such that we have a path (in \( S \)) from \( K \) to \( \text{Stab}(i) \). Since \( K \) is not contained in \( A_n \), there is an element \( k \in K \) such that \( k \notin A_n \). First, say \( k \) is not an \( n \)-cycle. Then if the orbits of \( \langle k \rangle \) have orders \( a_1, \ldots, a_m \), there is a path of the form

\[ K \ni \langle k \rangle \leq S_{a_1} \times \cdots \times S_{a_m} \geq S_{a_1} \leq S_{n-1} \]

connecting \( K \) to the stabilizer of some point. If \( k \) is an \( n \)-cycle, we consider two cases, depending on the parity of \( n/2 \) (note that \( n \) must be even since the \( n \)-cycle \( k \) is not in \( A_n \)). If \( n/2 \) is odd, then \( k^2 \) is a product of two odd-length cycles and hence is not in \( A_n \). Replacing \( k \) by \( k^2 \) reduces to the above case. When \( n/2 \) is even, assume without loss of generality that \( k = (1 \ 2 \cdots n) \). The reader may check that (setting \( m = n/2 \)) the element \( t = (m \ (m+2)(m-1 \ m+3) \cdots (2m) \) lies in the normalizer of \( (1 \ 2 \cdots n) \). This element is a product of \( m-1 \) disjoint transpositions, and hence is not in \( A_n \). Thus we have a path of the form

\[ K \ni \langle k \rangle \leq N(\langle k \rangle) \ni (t) \leq S_2 \times S_2 \times \cdots \times S_2 \geq S_2 \leq S_{n-1}. \]
To complete the proof, we must find a path in $C(S_n, A_n)$ joining $C_x = \{xH : H \in S\}$ to $C_y = \{yH : H \in S\}$ for each $x, y \in S_n$. If $x^{-1}y \notin A_n$ then $x(x^{-1}y)$ lies in $C_x \cap C_y$ and we are done. If $x^{-1}y \in A_n$, then either $x, y \in A_n$ or $x, y \notin A_n$. In the first case, take $z \notin A_n$. Then $x^{-1}z, z^{-1}y \notin A_n$ and hence $C_x \cap C_z \neq \emptyset$ and $C_z \cap C_y \neq \emptyset$. Since $C_z$ is connected, this yields a path joining $C_x$ to $C_y$. If $x, y \notin A_n$, then choosing $z \in A_n$ we may complete the proof in a similar manner.

**Remark 3.9** Analogous arguments show that $C(\mathbb{Z})$ and $C(\mathbb{Z} \times \mathbb{Z}/2)$, where $\mathbb{Z}/2$ acts by inversion, are simply connected (again, the normal subgroup of index two is key).

Turning to the case of a general group extension, we note a simple consequence of Lemma 3.7.

**Proposition 3.10** Let $G$ be a group with quotient $\overline{G}$, and assume that $C(G)$ is $k$-connected. Then $C(\overline{G})$ is $k$-connected as well.

**Proof.** This follows immediately from the lemma, noting that the join of a $k$-connected space with any space is still $k$-connected. $\square$

### 4 Mayer-Vietoris Sequences and $H_*(C(G))$

We will now present some results based on Mayer-Vietoris sequences for the homology of $C(G)$. In particular, we will obtain lower bounds on the rank of $H_1(C(G))$ for certain groups $G$. The results in this section are most interesting for infinite groups, as the classes of finite groups to which the results apply are rather small.

From here on, we let $H_n(X)$ denote the simplicial homology of the space $X$ with coefficients in $\mathbb{Z}$, and we let $\tilde{H}_n(X)$ denote reduced simplicial homology (again with coefficients in $\mathbb{Z}$).

**Theorem 4.1** Let $G$ be a non-cyclic group with a cyclic maximal subgroup $M$ of prime-power order, and let $o(M) = p^n$. Then $H_1(C(G))$ has rank at least $(p - 1)(G : M)$ (or infinite rank if $G$ is infinite), and in particular $C(G)$ is not simply connected.

**Proof.** Let $\{x_i\}_{i \in I}$ be a set of left coset representatives for $M$. Let $X = \Delta(C(G)), \ Y = \Delta(C(G) - \{x_iM\}_{i \in I}), \ Z = \Delta(\cup_{i \in I}C(G) \leq x_iM)$. Then we have $X = Y \cup Z$, yielding a Mayer-Vietoris sequence

$$\cdots \longrightarrow \tilde{H}_1(X) \xrightarrow{\partial} \tilde{H}_0(Y \cap Z) \xrightarrow{h} \tilde{H}_0(Y) \oplus \tilde{H}_0(Z) \longrightarrow \tilde{H}_0(X)$$

on (reduced) homology groups.

By Lemma 3.4, $X$ is connected, so $\tilde{H}_0(X) = 0$ and $h$ is a surjection. Also, $Z$ is a union of $(G : M)$ disjoint cones, so $\tilde{H}_0(Z) \cong \mathbb{Z}^{(G : M) - 1}$ and we claim that $Y$ is connected (so that $\tilde{H}_0(Y) = 0$).
Now,
\[ Y \cap Z = \Delta \left( \bigcup_{i \in I} C(G)_{< M} \right) \cong \coprod_{i \in I} \Delta(C(M)), \]
where \( \coprod \) denotes the disjoint union. First, assume that \( o(G) < \infty \). Then the coset poset of \( M \cong \mathbb{Z}/p^n \) has \( p \) contractible components (the cones under the cosets of the unique maximal subgroup) and thus \( \tilde{H}_0(Y \cap Z) \cong \mathbb{Z}^{p(G:M)-1} \).

Substituting these values into the above sequence yields an exact sequence
\[ \cdots \longrightarrow \tilde{H}_1(X) \xrightarrow{\partial} \mathbb{Z}^{p(G:M)-1} \xrightarrow{h} \mathbb{Z}^{(G:M)-1} \longrightarrow 0. \]

So \( \text{Im} \, \partial = \text{Ker} \, h \cong \mathbb{Z}^{p(G:M)-1-(G:M)-1} \cong \mathbb{Z}^{(p-1)(G:M)} \) and we see that the rank of \( H_1(X) \) is at least \((p-1)(G:M))\), as desired.

If \( G \) is infinite, \( \text{Ker} \, h \) contains all homology classes of the form \([xM' - yM']\) where \( M' < M \) is the unique maximal subgroup of \( M \) and \( x \equiv y \pmod{M} \) (the image of \([xM' - yM']\) in \( \tilde{H}_0(Y) \) is trivial because, as we will show, \( Y \) is connected, and its image in \( Z \) is trivial because \( \partial((yM', xM) + (xM, xM')) = xM' - yM' \), where \( \partial \) denotes the simplicial boundary). Thus \( \text{Im} \, \partial = \text{Ker} \, h \) contains a free abelian group of infinite rank, and hence the rank of \( H_1(X) \) is infinite as well (it is worth noting that maximality of the cyclic group \( M \) implies that \( G \) is a two generator group and hence is countable).

To complete the proof we must show that \( Y \) is connected. Consider a coset \( xH \), where \( H \neq M \). If \( \langle x \rangle \neq M \), then we have a path
\[ xH \supseteq \{x\} \subseteq \langle x \rangle \supseteq \{1\} \]
connecting \( xH \) to the identity. Next, if \( \langle x \rangle = M \) then choose some \( g \in G \), \( g \notin M \). We now have a path
\[ xH \supseteq \{x\} \subseteq x\langle g \rangle \supseteq \{xg\} \subseteq \langle xg \rangle \supseteq \{1\} \]
in \( Y \) connecting \( xH \) to the identity. \( \square \)

The above result does not extend to cyclic groups, since the subcomplex \( Y \) is no longer connected. Nevertheless, Theorem 4.1 can be used to show that the coset poset of a cyclic group \( G \) with a maximal subgroup isomorphic to \( \mathbb{Z}/p^n \) is still not simply connected.

There are many infinite groups to which Theorem 4.1 applies. Such groups are described in [12, Chapter 9]. In particular, for each sufficiently large prime \( p \) there is a continuum of pairwise non-isomorphic groups in which each non-trivial proper subgroup has order \( p \).

As mentioned at the beginning of the section, there are very few finite groups to which the theorem applies. The \( p \)-groups with a cyclic maximal subgroup have been classified (see [3]), and there are only a few types. In fact, any other finite group \( G \) with a cyclic maximal subgroup \( M \) of prime-power order is either a semi-direct product \( A \rtimes \mathbb{Z}/p^n \), where \( A \) is elementary abelian, or of the form \( \mathbb{Z}/p^n \rtimes \mathbb{Z}/q \) where \( q \) is prime. This can be shown as follows. If \( M < G \), then clearly \( G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/q \). Otherwise, Herstein [7] has shown (by very elegant
and elementary methods) that a group with an abelian maximal subgroup is solvable. Now, $M$ is a Sylow $p$-subgroup of $G$ and we have $N_G(M) = M$, so $M$ lies in the center of its normalizer and must have a complement (by Burnside’s Theorem [15, p. 289]). So $G = T \rtimes M$ for some $T < G$, and maximality of $M$ implies that no subgroup of $T$ is invariant under the action of $M$, i.e. $T$ is a minimal normal subgroup. Since $G$ is solvable, this implies that $T$ is in fact elementary abelian. The simplest interesting example of such a group is $A_4 \cong (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/3$, and other examples may be constructed by taking an appropriate generator $x$ of the field extension $F_q \subset F_{q^n}$ and letting $x$ act by multiplication on the additive group of $F_{q^n}$.

The interested reader may now use Theorem 1.1 to compute the exact homotopy type of $\mathcal{C}(G)$ for any finite group $G$ to which the theorem applies (it is worth noting that the bound on the rank of $H_1$ is not very good). The theorem is still useful in this endeavor, though, since it proves that there is exactly one complemented factor in the chief series for $G$ and hence simplifies the computation of the homotopy type of $\mathcal{C}(G)$.

**Problem 4.2** Find a finite non-cyclic group $G$ with $\mathcal{C}(G)$ not simply connected and such that no maximal subgroup of $G$ is cyclic of prime-power order. A non-solvable example would be of particular interest.

The cyclic groups of order $p^nq^m$ ($p, q$ prime) have non-simply connected coset posets (Theorem 3.6) but have no maximal subgroup which is cyclic of prime-power order (unless $n = m = 1$).

The following lemma shows a (somewhat weak) connection between the coset poset and the subgroup poset of a finite group. Further relationships are discussed in the final section.

**Lemma 4.3** Let $G$ be any group, and for $g \in G$, let $\mathcal{C}(G)_g$ denote the poset $\mathcal{C}(G) - \{g\}$.

If there exists $g \in G$ with $\tilde{H}_n(\mathcal{C}(G)_g) = 0$, then there is a surjection

$$\tilde{H}_{n+1}(\mathcal{C}(G)) \twoheadrightarrow \tilde{H}_n(L(G)).$$

Note that for any $g, h \in G$, the posets $\mathcal{C}(G)_g$ and $\mathcal{C}(G)_h$ are isomorphic (consider the map given by left-multiplication by $hg^{-1}$). In particular, $\mathcal{C}(G)_g \cong \mathcal{C}(G)_1$.

**Proof.** Let $X = \Delta(\mathcal{C}(G))$, $Y = \Delta(\mathcal{C}(G)_{\geq (g)}$, and $Z = \Delta(\mathcal{C}(G)_g)$. Note that $Y$ is contractible (so $\tilde{H}_n(Y) = 0$) and that $Y \cap Z \cong L(G)$.

Since $X = Y \cup Z$, we get a Mayer-Vietoris sequence

$$\cdots \rightarrow \tilde{H}_{n+1}(X) \xrightarrow{\partial} \tilde{H}_n(Y \cap Z) \rightarrow \tilde{H}_n(Y) \oplus \tilde{H}_n(Z) \rightarrow \cdots$$

and since $\tilde{H}_n(Y) = \tilde{H}_n(Z) = 0$, $\partial$ is a surjection. \qed
**Theorem 4.4** Let $G$ be a group and let $n$ be the cardinality of the set of path-components of $L(G)$. Then $H_1(C(G))$ has rank at least $n - 1$ (or infinite rank if $n$ is infinite). In particular, if $L(G)$ is disconnected then $C(G)$ is not simply connected.

**Proof.** If $G$ is cyclic then $L(G)$ is connected (unless $G \cong \mathbb{Z}/pq$, $p, q$ prime) and the result is trivial. When $G \cong \mathbb{Z}/pq$, the result follows from Theorem 1.1.

We now assume that $G$ is not cyclic. By the lemma, it will suffice to prove that $C(G)_1$ is connected. Choose some element $x \in G$, $x \neq 1$. Then any vertex $yH$ in $C(G)_1$ can be connected to $\{x\}$ via the path

$$yH \geq \{y\} \leq x(x^{-1}y) \geq \{x\}$$

($yH \neq \{1\}$ so we may assume $y \neq 1$).

For finite groups, it appears that this result is eclipsed by Theorem 4.1. See Lemma 6.8 and discussion following it for details.

There is again an interesting class of infinite groups to which Theorem 4.4 applies (in addition to the groups described above, to which Theorem 4.1 also applies). In [2, Chapter 9], it is shown there that there is a continuum of non-isomorphic infinite groups $G$ all of whose non-trivial proper subgroups are infinite cyclic. Furthermore, in each of these groups, any two maximal subgroups intersect trivially and hence $L(G)$ is disconnected in every case.

## 5 2-transitive Covers and Simple Groups

In this section, we will examine the homotopy type of the coset posets of the finite simple groups $PSL_2(F_5) \cong A_5$ and $PSL_2(F_7)$, utilizing the notion of a 2-transitive cover. Some computational details will be omitted, and a complete presentation of these results (and of all the necessary background information on $PSL_2(F_7)$) may be found in [14].

### 5.1 2-transitive Covers

**Definition 5.1** Let $G$ be a group. We call a collection of subgroups $S \subset L(G)$ a cover of $G$ if every element of $G$ lies in some subgroup $H \in S$. We call $S$ 2-transitive if for each $H \in S$, the action of $G$ on the left cosets of $H$ is 2-transitive.

In addition, we say that a 2-transitive cover $S$ is $n$-regular if for each $H \in S$ there is an element $g \in G$ with $o(g) = n$ whose action on $G/H$ is non-trivial (this is equivalent to requiring that no subgroup $H \in S$ contains all the elements in $G$ of order $n$).

Recall that the standard presentation for $\pi_1(M(G)) \cong \pi_1(C(G))$ is the presentation obtained from Lemma 2.7 using as maximal tree the star at the vertex 1 (assuming $G$ is non-cyclic).
Definition 5.2 Let $G$ be a non-cyclic group, and say $G$ contains elements of order $n$. We say that $M(G)$ is $n$-locally simply connected if each generator $(g, h)$ (in the standard presentation for $\pi_1(M(G))$) with $o(g) = n$ is trivial.

Proposition 5.3 Let $G$ be a non-cyclic group containing elements of order $n$. If $M(G)$ is $n$-locally simply connected and $G$ admits an $n$-regular 2-transitive cover, then $M(G)$ (and hence $C(G)$) is simply connected.

Proof. Let $\{g_1, g_2\}$ be any edge of $M(G)$. We must show that the corresponding generators in the standard presentation for $\pi_1(M(A_5))$ are trivial. It will suffice to show that there exists an element $z$ of order $n$ and a subgroup $K \in L(G)$ such that $g_1 \equiv g_2 \equiv z \pmod{K}$. (Then the set $\{g_1, g_2, z\}$ forms a 2-simplex in $M(G)$ and we have $(g_1, g_2) = (g_1, z)(z, g_2) = 1$ since $M(G)$ is $n$-locally simply connected.)

Let $S$ be an $n$-regular 2-transitive cover of $G$. Then there exists $H \in S$ with $g_1^{-1}g_2 \in H$, and there is an element $z \in G$ with $o(z) = n$ which acts non-trivially on the left cosets of $H$. Let $H, x_1H, \ldots, x_kH$ denote these cosets. Since $z$ acts non-trivially on $G/H$ and $G$ acts 2-transitively, some conjugate of $z$ sends $H$ to $x_iH$ ($1 \leq i \leq k$), i.e. there is an element of order $n$ in every set $\{g \in G : g \cdot H = x_iH\} = x_iH$ ($1 \leq i \leq k$). So we have found an element $z^n \in g_1H = g_2H$, as desired. \hfill\Box

5.2 The Coset Poset of $A_5$

We will establish the following:

Proposition 5.4 The coset poset of $A_5 \cong PSL_2(\mathbb{F}_5)$ has the homotopy type of a bouquet of 1560 two-dimensional spheres.

It can be checked that there are 1018 proper cosets in $A_5$, and hence $C(A_5)$ has 1018 vertices. Evidently, $C(A_5)$ is far too large to admit direct analysis.

Shareshian [unpublished manuscript] has provided a proof of this result using the theory of shellability. The proof given here is somewhat simpler than Shareshian’s argument.

Following Shareshian, we will show that $C = C(A_5)$ has the homotopy type of a two-dimensional complex. For this portion of the proof we will work directly with $C$. We will show that $C$ is simply connected by applying Proposition 5.3.

To show that $C(A_5)$ has the homotopy type of a bouquet of 2-spheres, we appeal to the general fact that a $k$-dimensional complex which is $(k-1)$-connected is homotopy equivalent to a bouquet of $k$-spheres. The number of spheres in the bouquet can be calculated from the Euler characteristic $\chi(C(A_5))$, computed in [4].

As mentioned in Section 2, the reader may also check that $A_5$ satisfies the conditions of Corollary 2.9. This provides a second proof of simple connectivity. The proof given below eliminates some of the computation.
Claim 5.5  Let $C$− denote the poset $C$ with all cosets of all copies of $D_4$ removed. Then $\Delta(C^-)$ is two-dimensional and $C^- \simeq C$.

Proof (Shareshian). Quillen’s Theorem A (see [13, Theorem 1.6] or [1, Theorem 10.5]) shows that the inclusion $C^- \hookrightarrow C$ is a homotopy equivalence (there is a unique subgroup lying above any copy of $D_4$, namely a copy of $A_4$), so it remains to check that $C^-$ is two-dimensional. This follows easily from the fact that each maximal subgroup of $A_5$ is isomorphic $A_4$, $D_{10} \cong \mathbb{Z}/5 \rtimes \mathbb{Z}/2$, or $S_3$. ✷

Claim 5.6 The coset poset of $A_5$ is simply connected.

Proof. We will show that $A_5$ admits a 2-regular 2-transitive cover. The interested reader may check that $\mathcal{M}(A_5)$ is 2-locally simply connected using the method in the proof of Proposition 2.8. Up to automorphism, there are only several cases to check.

We claim that the set $\{H \in L(A_5) : H \cong A_4 \text{ or } H \cong D_{10}\}$ is a 2-regular 2-transitive cover of $A_5$. Each copy of $A_4$ is the stabilizer of a point in $\{1, 2, 3, 4, 5\}$, and the action of $A_5$ on this set is 2-transitive. Each copy of $D_{10}$ is the stabilizer of a point under the action of $\text{PSL}_2(\mathbb{F}_5) \cong A_5$ on $\mathbb{F}_5 \cup \{\infty\}$, and this action is 2-transitive as well (see [3], [4]). In each case, all non-trivial elements act non-trivially, and every element in $A_5$ lies either in a $D_{10}$ or in an $A_4$. ✷

5.3 The Coset Poset of $\text{PSL}_2(\mathbb{F}_7)$

We now consider the finite simple group $G = \text{PSL}_2(\mathbb{F}_7)$, and show that $\mathcal{C}(G)$ is simply connected. Other facts about the homotopy type of $\mathcal{C}(G)$ are discussed at the end of the section.

For basic facts about the groups $\text{PSL}_2(\mathbb{F}_p)$, we refer to [4] (see also [3] and [4]). We write the elements of $\text{PSL}_2(\mathbb{F}_p)$ as “Möbius transformations” $f : \mathbb{F}_p \cup \{\infty\} \to \mathbb{F}_p \cup \{\infty\}, x \mapsto \frac{ax+b}{cx+d}$, where $a, b, c, d \in \mathbb{F}_p$, $\det(f) = ad - bc = 1$, and $\infty$ is dealt with in the usual manner. The action of $\text{PSL}_2(\mathbb{F}_p)$ on $\mathbb{F}_p \cup \{\infty\}$ is 2-transitive. For a proof of the following result, see [3] or [4].

Lemma 5.7 Any maximal subgroup of $G$ is either the stabilizer of a point in $\mathbb{F}_7 \cup \{\infty\}$ (and is isomorphic to $\mathbb{Z}/7 \times \mathbb{Z}/3$) or is isomorphic to $S_4$. The two conjugacy classes of subgroups isomorphic to $S_4$ are interchanged by the transpose-inverse automorphism of $\text{GL}_3(\mathbb{F}_2) \cong G$.

The following lemma will help to minimize the amount of computation in the proof of simple connectivity.

Lemma 5.8 Let $\alpha = \frac{ax+b}{cx+d}$ be any non-trivial element of $\text{PSL}_2(\mathbb{F}_7)$. We define the trace-squared of $\alpha$ to be $tr^2(\alpha) = (a+d)^2$ (note that this is the square of the
trace of either representative of \( \alpha \) in \( SL_2(\mathbb{F}_7) \), and thus is well-defined). The order of \( \alpha \) is then determined as follows:

\[
o(\alpha) = \begin{cases} 2, & tr^2(\alpha) = 0 \\ 3, & tr^2(\alpha) = 1 \\ 4, & tr^2(\alpha) = 2 \\ 7, & tr^2(\alpha) = 4. \end{cases}
\]

**Proof.** If \( \alpha = \frac{ax+b}{cx+d} \in G \), we define \( \text{disc}(\alpha) \) to be the discriminant of the quadratic polynomial \((cx + d)x - (ax + b) = cx^2 + (d - a)x - b \) determined by the equation \( \frac{ax+b}{cx+d} = x \), so that \( \text{disc}(\alpha) = (d - a)^2 - 4(-b)(c) = tr^2(\alpha) - 4 \).

We consider only the elements in \( G \cap \text{Stab}(\infty) \). It is easy to check the result on the remaining elements. The elements of order seven in \( G \) are exactly those which fix one point in \( \mathbb{F}_7 \cup \infty \), and the elements of order three in \( G \) are exactly those which fix two points in \( \mathbb{F}_7 \cup \infty \). Thus the elements of order seven in \( G \cap \text{Stab}(\infty) \) are exactly those with \( \text{disc}(\alpha) = 0 \) and \( tr^2(\alpha) = 4 \) and the elements of order three in \( G \cap \text{Stab}(\infty) \) are those with \( \text{disc}(\alpha) \in (\mathbb{F}_7^*)^2 \). Thus \( o(\alpha) = 3 \iff \text{disc}(\alpha) \in \{1, 2, 4\} \iff tr^2(\alpha) \in \{5, 6, 1\} \iff tr^2(\alpha) = 1 \).

Next, let \( A \in SL_2(\mathbb{F}_7) \) be a matrix representing \( \alpha \in G \), and let \( \lambda_1, \lambda_2 \in \mathbb{F}_7 \) be the eigenvalues of \( A \) (where \( \mathbb{F}_7 \) denotes the algebraic closure). Then \( o(\alpha) = 2 \iff \lambda_1^2 = \lambda_2^2 = \pm 1 \). If \( \lambda_1 = \lambda_2 \) then \( A = \lambda I \) and \( o = 1 \), so \( o(\alpha) = 2 \iff \lambda_1 = -\lambda_2 \iff tr(A) = tr^2(\alpha) = 0 \). A similar but more lengthy calculation shows that \( o(\alpha) = 4 \iff tr^2(\alpha) = 2 \).

**Proposition 5.9** The coset poset of \( PSL_2(\mathbb{F}_7) \) is simply connected.

**Proof.** As in the case of \( A_5 \cong PSL_2(\mathbb{F}_5) \), we will apply Proposition 5.3. First we indicate the proof that \( \mathcal{M}(G) \) is 2-locally simply connected. We must check that each generator \( (g, h) (o(g) = 2) \) in the standard presentation for \( \pi_1(\mathcal{M}(G)) \) is trivial.

Recall that \( (g, h) = 1 \) if \( (g, h) \neq G \). Given a group \( H \), we define an automorphism class of generating pairs to be a set of the form \( \{ (\phi(x), \phi(y)) : \phi \in \text{Aut}(H) \} \) and \( \langle x, y \rangle = H \}. To show that each generator \( (g, h) \) with \( o(g) = 2 \) is trivial, it suffices to check one representative from each automorphism class of generators. Letting \( \Phi_{a, b} \) denote the number of automorphism classes of generators of \( G \) in which all representatives \( (x, y) \) satisfies \( o(x) = a \) and \( o(y) = b \), Möbius inversion allows one to calculate \( \Phi_{a, b} \) using the Möbius function of \( G \) (as calculated in 5). For these computations it is also necessary to know that \( \text{Aut}(G) \cong PGL_2(\mathbb{F}_7) \) has order 336. The method of calculation is described in §1 and §3 of 3.

\( o(h) = 3 \): Möbius inversion shows \( \Phi_{2, 3} = 1 \), and \( g = \frac{x - 2}{x - 1}, h = \frac{4x}{2} \) represents the unique automorphism class of generators because \( o(gh) = 7 \) and no subgroup contains elements of orders two and seven. Letting \( z = \frac{3x - 2}{-2x - 3} \), we have \( o(z) = 2 \) and \( g(-1) = h(-1) = z(-1) = -2 \), so \( (z^{-1}g, z^{-1}h) \leq \text{Stab}(-1) \). Since two elements of order two cannot generate a simple group, \( (g, z) \neq G \), and since \( o(hz) = 3 \), the pair \( (h, z) \) does not fall into the unique automorphism class of
generators with orders two and three. Thus \{z, g, h\} is a two-simplex in \(\mathcal{M}(G)\) and we have the relations \((g, h) = (g, z)(z, h) = 1\).

\(o(h) = 4\): Möbius inversion shows that \(\Phi_{2,4}(G) = 1\). Let \(g = \frac{-1}{x}, h = \frac{4x+1}{2x-3}\) and \(z = \frac{-2}{x}\). The argument is now similar to the previous case (note that \(g(0) = h(0) = z(0) = \infty\)).

\(o(h) = 7\): This time we find that there are three automorphism classes of generating pairs. (Note that any pair of elements with these orders generates \(G\).) Let \(h = x + 1 (o(h) = 7)\), and let \(g = \frac{1}{x}\) \((o(g) = 2)\). We have \(hg = \frac{1}{x} + 1 = \frac{1 + x}{x}\) so \(tr^2(hg) = c^2\), which implies that the pairs \((g_1 = \frac{1}{x}, h), (g_2 = \frac{2}{3x}, h)\), and \((g_3 = \frac{x}{2}, h)\) represent the three generating automorphism classes.

Next, say there exist elements \(z_i \in \text{Stab}(\infty)\) such that \(o(z_i) = 3\) and \(\{h, g_i, z_i\}\) forms a simplex in \(\mathcal{M}(G)\) \((i = 1, 2, 3)\). Then we have \((h, z_i) \leq \text{Stab}(\infty)\) and \((g_i, z_i) = 1\) because \(o(g_i) = 2\) and \(o(z_i) = 3\). So \((g_i, h) = (g_i, z_i)(z_i, h) = 1\).

We will now find such elements \(z_i\). Consider the equations \(h(x) = g_i(x)\), i.e. \(x + 1 = \frac{h}{x}\) \((b = -1, 2, 3)\). Equivalently, (note that \(\infty\) can never be a solution) we want to solve \(x^2 + x + b^2 = 0\), and by examining the discriminant we see that solutions \(x_i \in \mathbb{F}_7\) exist when \(b = -1\) or 3, but not when \(b = 2\). For \(i = 1, 3\) there is an element \(z_i \in \text{Stab}(\infty)\) with \(o(z_i) = 3\) and \(z_i(x_i) = g_i(x_i) = h(x_i)\) \((z_i = \frac{2x + 2(x_i + 1)}{1})\). Thus \((g_1, h) = (g_3, h) = 1\).

Finally, we must show that \((g_2, h) = (\frac{4}{3}, x+1) = 1\). Letting \(z_2 = \frac{2x}{3}\), we have \(o(z_2) = 3\) and \(z_2 \in \text{Stab}(\infty)\), so it suffices to show that these three elements lie in a proper coset, i.e. that \((z_2^{-1}g_2, z_2^{-1}h) \neq G\). We have \(o(z_2^{-1}g_2) = 2, o(z_2^{-1}h) = 3\) and \(o(z_2^{-1}g_2z_2^{-1}h) = 4\), so these elements do not lie in the unique automorphism class found at the start of the proof.

To complete the proof we will check that the set of maximal subgroups of \(G\) is a 2-regular 2-transitive cover. By Lemma 5.3, the maximal subgroups are the stabilizers under the action of \(G\) on \(\mathbb{F}_7 \cup \{\infty\}\), and the copies of \(S_4\). The action of \(G\) on \(\mathbb{F}_7 \cup \{\infty\}\) is 2-transitive and all non-trivial elements act non-trivially. Next, consider the group \(GL_3(\mathbb{F}_2) \cong G\). It is easy to check that the stabilizer of a vector \(v \in \mathbb{F}_2^3\) is isomorphic to \(S_4\), and the action of \(GL_3(\mathbb{F}_2)\) on \(\mathbb{F}_2^3\) is 2-transitive. Additionally, all non-trivial elements act non-trivially on \(\mathbb{F}_2^3\). Since all copies of \(S_4\) are conjugate under \(\text{Aut}(G)\) (Lemma 5.4), this completes the proof.

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There are two main difficulties in extending these results to \(PSL_2(\mathbb{F}_p)\) for primes \(p > 7\). It is not clear how to generalize the ad hoc portion of the proof, in which we showed that \(\mathcal{M}(G)\) is 2-locally simply connected. Also, for large \(p\) the only 2-transitive action of \(PSL_2(\mathbb{F}_p)\) is its standard action on \(\mathbb{F}_p \cup \{\infty\}\). This follows from [10, Exercises 38 and 39, p. 58], the basic problem being that the other subgroups of \(PSL_2(\mathbb{F}_7)\) are too small.

We will now show that \(\mathcal{C}(G)\) has the homotopy type of a three-dimensional complex and that \(H_2(\mathcal{C}(G)) \neq 0\). Any chain of length four (recall that a chain of length four has five vertices) lies under a subgroup isomorphic to \(S_4\), and in
fact must contain a coset $xH$ where $H \cong D_4$ or $\mathbb{Z}/4$. But cosets of copies of $D_4$ and $\mathbb{Z}/4$ may be removed from $C(G)$ without changing the homotopy type (apply Quillen’s Theorem A to the inclusion map). Since the reduced Euler characteristic of $C(G)$ is $17 \cdot 168$ [4, Table I] and the only even dimension in which $C(G)$ has homology is dimension two, it now follows that $H_2(C(G))$ has rank at least $17 \cdot 168$.

The method of removing cosets does not seem to show that $C(G)$ has the homotopy type of a two-dimensional complex. In fact, we expect (see Question 7.1) that since $H_2(L(G))$ is non-zero, $H_3(C(G))$ is non-zero as well. (One shows that $H_2(L(G)) \neq 0$ as follows: We have $\chi(L(G)) = \mu_G(\{1\}) = 0$ [6]. The above argument shows that $L(G)$ has the homotopy type of a two-dimensional complex, and hence rank $H_1(L(G)) = \text{rank } H_2(L(G))$. Shareshian has shown that $H_1(L(G)) \neq 0$ [16, Lemma 3.11], and in fact Shareshian’s argument shows that rank $H_1(L(G)) \geq 21$.)

6 Connectivity of the Subgroup Poset

In this section we will consider the connectivity of the subgroup poset, focusing in particular on $\pi_1(L(G))$ for finite groups $G$. For finite solvable groups, we have the following theorem of Kratzer and Thévenaz [9], which is strikingly similar to Theorem 1.1. As mentioned in [19], this result follows easily by induction from Björner and Walker’s homotopy complementation formula (Lemma 6.4).

**Theorem 6.1** Let $G$ be a finite solvable group, and let

$$1 = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_d = G$$

be a chief series for $G$. Let $c_i$ denote the number of complements of $N_i/N_{i-1}$ in $G/N_{i-1}$. Then $L(G)$ is homotopy equivalent to a bouquet of $c_1 \cdot c_2 \cdots \cdots c_d$ spheres of dimension $d-2$.

The number $d$ in this theorem is at least $\pi(G)$, the number of distinct primes dividing $o(G)$ (this is immediate, since each factor $N_i/N_{i-1}$ is of prime-power order). Note that if $c_i = 0$ for some $i$, then $L(G)$ is in fact contractible. Actually, more is true.

**Lemma 6.2** Let $G$ be a group with a normal subgroup $N$. If $N$ does not have a complement in $G$, then $L(G)$ is contractible.

**Proof.** This is an immediate consequence of [2, Theorem 3.2].

We will say that a group $G$ is complemented if each normal subgroup $N < G$ has a complement. The above result then reduces the study of the homotopy type of the subgroup poset to the case of complemented groups.

**Definition 6.3** For any group $G$, let $d(G)$ denote the length of a chief series for $G$. If $G$ does not possess a chief series, then we write $d(G) = \infty$.
In other words, $d(G)$ is the rank of the lattice of normal subgroups of $G$ (ordered by inclusion).

From Theorem 6.1 we see that if $G$ is a finite, solvable, complemented group, then $L(G)$ is $k$-connected if and only if $d(G) \geq k+3$. Our first goal in this section is to show that the “if” portion of this statement is true for any group (it is clearly true for any non-complemented group).

We now state the homotopy complementation formula of [2], specialized to the case of the proper part of a bounded lattice. Let $\mathcal{L}$ be a bounded lattice, and let $\mathcal{L} = \mathcal{L} - \{0, 1\}$ be its proper part. We say that elements $p, q$ in a poset $\mathcal{P}$ are complements if $p \wedge q = \hat{0}$ and $p \vee q = \hat{1}$, and we denote the set of complements of $p \in \mathcal{P}$ by $p \perp$. We say that a subset $A \subseteq \mathcal{P}$ is an antichain if no two distinct elements of $A$ are comparable.

Lemma 6.4 (Björner-Walker) Let $\mathcal{L}$ be the proper part of a bounded lattice. Say there is an element $x \in \mathcal{L}$ such that $x \perp$ is an antichain. Then $\mathcal{L}$ is homotopy equivalent to $\bigvee_{y \in x^\perp} \text{Susp} \left( \mathcal{L}_{<y} * \mathcal{L}_{>y} \right)$.

In this special case, Björner and Walker’s proof becomes surprisingly simple. For $G$ finite and $N$ abelian, the following result appears as [9, Corollaire 4.8].

Proposition 6.5 Let $G$ be a group with normal subgroup $N \trianglelefteq G$. Then $L(G)$ is homotopy equivalent to $\bigvee_{H \in N^\perp} \text{Susp} \left( L(G/N) * L_H(N) \right)$, where $L_H(N) \subset L(N)$ denotes the poset of $H$-invariant subgroups.

Proof. Note that when $N \triangleleft G$, the lattice-theoretic and group theoretic notions of complement coincide. It is easy to check that the complements of $N$ form an antichain in $L(G)$ (when $G$ is finite, they all have the same order). The result now follows easily from Lemma 6.4, noting that any subgroup $T$ containing a complement $H \in N^\perp$ has the form $T = I \rtimes H$ where $I = T \cap N$ is $H$-invariant, and that $H \cong G/N$ implies $L(G)_{<H} \cong L(G/N)$.

Note that the proposition implies that if $G$ is non-simple, $\pi_1(L(G))$ is free.

Our next result strengthens and generalizes part of [9, Proposition 4.2].

By convention, we will say that a space is $(-1)$-connected if and only if it is non-empty, and that any space is $(-2)$-connected. In addition, we do not consider the empty space to be $0$-connected.

Proposition 6.6 Let $G$ be a group with normal subgroup $N \triangleleft G$. Then if $L(G/N)$ is $k$-connected, $L(G)$ is $(k+1)$-connected. In particular, if $L(G/N)$ is contractible, so is $L(G)$.

Proof. If $L(G/N)$ is $k$-connected, then its join with any space $X$ is $k$-connected and $\text{Susp} \left( L(G/N) * X \right)$ is $(k+1)$-connected [11]. Thus Proposition 6.5 shows that $L(G)$ is $(k+1)$-connected. \[ \square \]
Theorem 6.7  For any group $G$, $L(G)$ in $(d(G) - 3)$-connected. In particular, if $d(G) = \infty$, then $L(G)$ is contractible.

Proof. When $d(G)$ is finite, this follows immediately from Proposition 6.6 by induction. When $d(G) = \infty$, a similar argument shows that $L(G)$ is $k$-connected for any $k$ and hence contractible. \qed

In order to improve on the above result, it is necessary to understand which groups have $L(G)$ path-connected. For non-simple finite groups, we have the following result.

Lemma 6.8  Let $G$ be a non-simple finite group. Then $L(G)$ is disconnected if and only if $G \cong A \rtimes \mathbb{Z}/p$, where $A \neq \{1\}$ is elementary abelian, $p$ is prime, and $L_{\mathbb{Z}/p}(A)$ is empty.

Proof. If $G$ has the above form, then clearly $L(G)$ is disconnected ($\mathbb{Z}/p$ is an isolated vertex).

In the other direction, since $G$ is not simple we may choose a normal subgroup $N \in L(G)$ and Proposition 6.6 gives

$$L(G) \cong \bigvee_{H \in N^\perp} \text{Susp}(L(G/N) \ast L_H(N)).$$

Thus in order for $L(G)$ to be disconnected, $N$ must have a complement $H$ such that $L(G/N) = L(H) = \emptyset$ and $L_H(N) = \emptyset$. Thus $H \cong \mathbb{Z}/p$ for some prime $p$ and $H$ is a maximal subgroup of $G$. If $G$ is a $p$-group, then $o(G) = p^2$ and we must have $G \cong \mathbb{Z}/p \times \mathbb{Z}/p$. If $G$ is not a $p$-group, then the discussion after Theorem 4.1 shows that $N$ is elementary abelian. \qed

Definition 6.9  We will denote the collection of finite groups of the form $A \rtimes \mathbb{Z}/p$ (with $A$ elementary abelian, $p$ prime, and $L_{\mathbb{Z}/p}(A) = \emptyset$) by $F$. The collection of groups in $F$ with $A$ non-trivial will be denoted by $F'$.

Remark 6.10  The above proposition raises the interesting question of when a simple group has a connected subgroup poset. Maria Silvia Lucido [private communication] has shown that $L(G)$ is connected for any finite non-abelian simple group $G$. Her proof relies on the classification of the finite simple groups. Assuming this result, we see that for finite groups $G$, $L(G)$ is disconnected if and only if $G \in F'$.

The following result is immediate from Lemma 6.8 and Proposition 6.6.

Proposition 6.11  If a finite group $G$ has a non-simple quotient $\overline{G}$ which is not in $F$, then $L(G)$ is simply connected.

Of course, one can deduce results about higher connectivity as well, and again (in light of the above remark) the assumption of non-simplicity can presumably be removed.
Our next goal is to study the simple connectivity of $L(G)$ when $G$ can be written as a direct product. Assuming connectedness of $L(G)$ for simple groups, we in fact obtain a characterization of direct products with simply connected subgroup posets. The reader may derive the following analogue of Lemma 6.3 for the subgroup poset of a direct product, using [9, Proposition 2.5] in place of [13, Proposition 1.9]. This result generalizes [1, Proposition 4.4].

**Lemma 6.12** If $H$ and $K$ are coprime groups, then $L(H \times K) \simeq \text{Susp} (L(H) * L(K))$.

**Remark 6.13** The apparent incongruity between Lemma 6.12 and Proposition 6.14 may be explained as follows. The extra terms in the latter decomposition are contractible: these terms correspond to complements $K' \neq K$ of $H$, and if $q_1 : H \times K \to H$ is the projection map, coprimality implies that $I \leq I \cdot q_1(K') \geq q_1(K')$ is a conical contraction of $L_{K'}(H)$.

**Proposition 6.14** Let $G$ be a finite group which admits a non-trivial direct-product decomposition. If $L(G)$ is not simply connected, then either $G \cong H \times \mathbb{Z}/q$ ($q$ prime) with $H \in F'$ or $G \cong S_1 \times S_2$ where $S_1$ and $S_2$ are simple groups. In the former case, $\pi_1(L(G)) \neq 1$.

**Proof.** Let $G = H \times K$. Since $G$ has quotients $H$ and $K$, Proposition 6.11 shows that if $\pi_1(L(G)) \neq 1$, then $H$ and $K$ are either simple or in $F$.

If $H$ and $K$ are both in $F'$, then we may write $H = A_H \times \mathbb{Z}/p$, $K = A_K \times \mathbb{Z}/q$ ($p, q$ prime; $A_H, A_K$ non-trivial elementary abelian groups) and we have a chief series

$$1 \triangleleft A_H \triangleleft H \triangleleft H \times A_K \triangleleft G.$$ 

Theorem 6.4 now shows that $\pi_1(L(G)) = 1$.

Next, say that $H$ is simple and $K \in F'$. If $o(H)$ is not prime, then $H$ and $K$ are coprime and Lemma 6.14 shows that $L(G) \simeq \text{Susp} (L(H) * L(K))$. Since $L(H), L(K) \neq \emptyset$, we have $\pi_1(L(G)) = 1$. If $H \cong \mathbb{Z}/p$ ($p$ prime), then the term $\text{Susp} (L_K(H) * L(K)) = \text{Susp} (L(K))$ appears in the wedge decomposition for $L(G)$, and since $L(K)$ is disconnected $L(G)$ is not simply connected. \hfill $\square$

**Remark 6.15** Assuming Maria Silvia Lucido’s result that $L(G)$ is connected when $G$ is a finite non-abelian simple group, Proposition 6.4 shows that any finite direct product $S \times H$ (S a finite non-abelian simple group) has a simply connected subgroup poset. In addition, $L(\mathbb{Z}/p \times \mathbb{Z}/q)$ is clearly disconnected ($p, q$ prime), and hence we see that if $H$ and $K$ are non-trivial finite groups, $L(H \times K)$ is simply connected unless $H \in F$ and $o(K)$ is prime (or vice versa).

We finish by examining those finite groups $G$ for which we have yet to determine whether or not $L(G)$ is simply connected. From now on we assume that $L(S)$ is connected when $S$ is a finite simple group.

Our results do not apply to non-abelian simple groups $G$, but Shareshian has shown that if $G$ is a minimal simple group (i.e. if $G$ is a finite non-abelian simple
group, all of whose proper subgroups are solvable) then $H_1(L(G)) \neq 0$ [K, Proposition 3.14].

Any non-simple (complemented) finite group for which we have not determined simple connectivity of $L(G)$ may be written as (non-trivial) semi-direct product $H \rtimes K$, where $K \in \mathcal{F}$. We break these groups down into two cases. First we consider groups in which $K \in \mathcal{F}'$, and then we consider groups in which every proper, non-trivial quotient has prime order. The following lemma will be useful, and is a simple consequence of [8, Theorem 7.8].

**Lemma 6.16** If $N \unlhd G$ is a minimal normal subgroup, then $N$ is a direct power of a simple group. In particular, if $G$ is a group in which the only characteristic subgroups are 1 and $G$, then $G$ is a direct power of a simple group ($G$ is a minimal normal subgroup of $G \rtimes \text{Aut}(G)$).

If $S$ is a finite, non-abelian simple group, then in the direct power $S^n$, the $n$ standard copies of $S$ are permuted amongst themselves by $\text{Aut}(S^n)$.

Let $G = H \rtimes K$ with $K = A \rtimes \mathbb{Z}/p \in \mathcal{F}'$. We may assume that $G$ is not also in $\mathcal{F}$. If $H$ has a complement $K'$ such that $L_{K'}(H) = \emptyset$, then the wedge decomposition for $L(G)$ contains the factor $\text{Susp } (L_{K'}(H) \ast L(K')) = \text{Susp } (L(K'))$ and since $K' \cong K$ we see that $L(K')$ is disconnected and $\pi_1(L(G)) \neq 1$ (in fact it is not difficult to check that, if $k$ is the number of complements $K' \in H^\perp$ with $L_{K'}(H) = \emptyset$, then $\pi_1(L(G))$ is a free group on $k(1 + o(A))$ generators). (Note that the groups described in Proposition 6.14 for which $\pi_1(L(G)) \neq 1$ are also of this form.) On the other hand, if $L_{K'}(H) \neq \emptyset$ for each $K' \in H^\perp$, then every term in the wedge decomposition for $L(G)$ is simply connected, so $L(G)$ is simply connected. So in this case, $\pi_1(L(G)) \neq 1$ if and only if some complement of $H$ is a maximal subgroup of $G$.

Finally, we consider finite, non-simple (complemented) groups $G$ in which every proper, non-trivial quotient has prime order. Up to isomorphism, any such group may be written as $G = H \rtimes \mathbb{Z}/p$, and $H$ must be a minimal normal subgroup of $G$. The lemma now shows that $H \cong S^n$ with $S$ simple if $o(S)$ is prime, then $G \in \mathcal{F}'$ so we assume that $S$ is a non-abelian simple group. Now, $\mathbb{Z}/p$ acts on the $n$ standard copies of $S$ in $S^n$ and if this action has more than one orbit then $G$ has a quotient which is not of prime order (and in fact $L(G)$ is simply connected by Proposition 6.11). So we need only consider the cases $G = S \rtimes \mathbb{Z}/p$ and $G = S^p \rtimes \mathbb{Z}/p$, with $\mathbb{Z}/p$ inducing exactly one orbit on the standard copies of $S$. In the latter case, $\pi_1(L(G)) = 1$:

**Proposition 6.17** Let $S$ be a finite, non-abelian simple group and let $p$ be a prime. If $G = S^p \rtimes \mathbb{Z}/p$, with $\mathbb{Z}/p$ inducing exactly one orbit on the standard copies of $S$, then $L(G)$ is simply connected.

**Proof.** Since $L(\mathbb{Z}/p)$ is empty, Proposition 5.5 shows that

$$L(G) \simeq \bigvee_{K \in (S^p)^\perp} \text{Susp } (L_K(S^p)).$$
Hence $\pi_1(L(G)) = 1$ if and only if $L_K(S^p)$ is connected for all $K \leq (S^p)^\perp$. Say $K \leq (S^p)^\perp$. Let $K = \langle k \rangle$ and let $\phi$ be the automorphism of $S^p$ induced by $k$. Also, let $S_1, \ldots, S_p$ denote the standard copies of $S$ in $S^p$ (so that $S^p = S_1 \times \cdots \times S_p$). We may assume without loss of generality that $\phi(S_i) = S_{i+1}$ ($1 \leq i \leq p-1$) and $\phi(S_p) = S_1$.

Let $D_K \in L_K(S^p)$ denote the subgroup consisting of all elements fixed by $\phi$ (and hence by $K$). For each $I \in L_K(S^p)$ we will construct a path (in $L_K(S^p)$) from $I$ to $D_K$. Let $f_i : S^p \to S_i$ denote the $i$th projection map. It is not hard to check that $I \in L_K(S^p)$ implies $I = f_1(I) \times \cdots \times f_p(I) \in L_K(S^p)$ (assuming $I \neq S^p$). Now, since $I$ is non-trivial and $K$-invariant, there is a non-trivial element $s = (s_1, 1, \ldots, 1) \in f_1(I) < I$ and since $I$ is also $K$-invariant, $\phi^i(s) \in I$ for each $i$. Moreover, the product $s' = \prod_{i=0}^{p-1} \phi^i(s) \in I$ is non-trivial and invariant under $\phi$. Thus when $I \neq S^p$ we have a path $I \leq I \geq \langle s' \rangle < D_K$.

If $I = S^p$, then $I$ surjects onto $S_i$ ($i = 1, \ldots, p$) and in particular $I$ is not nilpotent. The automorphism $\phi$ either fixes $I$ pointwise or induces an automorphism of $I$ of prime order. In the latter case, Thompson’s theorem on fixed-point free automorphisms implies that $\phi$ fixes some non-trivial element $i \in I$. Hence we have a path $I \geq \langle i \rangle \leq D_K$ in $L_K(S^p)$, so $L_K(S^p)$ is connected and the proof is complete.

In summary, we have:

**Theorem 6.18** Let $G$ be a finite group which is neither simple nor a semi-direct product $S \rtimes \mathbb{Z}/p$ ($S$ simple and $p$ prime), and assume further that $L(G)$ is $0$-connected (i.e. $G \notin \mathcal{F}$). Then $L(G)$ is simply connected unless $G \cong H \rtimes K$ with $K = A \rtimes \mathbb{Z}/p \in \mathcal{F}$ and $K$ maximal in $G$. In this case $\pi_1(L(G))$ is a free group on $k(1 + o(A))$ generators, where $k$ is the number of complements of $H$ which are maximal in $G$.

## 7 The Homology of $C(G)$ and $L(G)$

We end by discussing a relationship between the homology of the coset poset and the homology of the subgroup poset which exists, at least, for certain groups. This discussion is motivated in part by Lemma 4.3 and Theorem 4.4.

**Question 7.1** If $G$ is a group, then is it true that for any $n > 0$

$$\text{rank } \tilde{H}_n(L(G)) \leq \text{rank } \tilde{H}_{n+1}(C(G))? \quad (1)$$

For $n = 0$, the question is answered affirmatively (for any group) by Theorem 4.4. Additionally, all finite solvable groups satisfy (1). This follows from Theorems 6.1 and 6.2. We leave to the reader the easy task of checking that the number of spheres in the coset poset is greater than the number in the subgroup poset. In light of Lemma 6.2, any non-complemented group satisfies (1) trivially, so (for finite groups at least) we may restrict our attention to non-abelian simple groups and non-trivial semi-direct products $H \rtimes K$ with $K$ simple.

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We will now show that if \( p \equiv \pm 3 \pmod{8} \) and \( p \not\equiv \pm 1 \pmod{5} \) then the simple group \( \text{PSL}_2(\mathbb{F}_p) \) satisfies (1). First, we have the following result due to Shareshian \([16, \text{Lemma 3.8}]\).

**Lemma 7.2 (Shareshian)** Let \( p \) be an odd prime and let \( G \) be a simple group isomorphic to one of the following:

1. \( \text{PSL}_2(\mathbb{F}_p) \) with \( p \equiv \pm 3 \pmod{8} \) and \( p \not\equiv \pm 1 \pmod{5} \),
2. \( \text{PSL}_2(\mathbb{F}_2^p) \),
3. \( \text{PSL}_2(\mathbb{F}_3^p) \),
4. \( \text{Sz}(2^p) \).

Then \( L(G) \) has the homotopy type of a wedge of \( o(G) \) circles.

The proof of the next result is analogous to Shareshian’s proof of Lemma 7.2.

**Lemma 7.3** If \( p \equiv \pm 3 \pmod{8} \) and \( p \not\equiv \pm 1 \pmod{5} \) then \( C(\text{PSL}_2(\mathbb{F}_p)) \) has the homotopy type of a two-dimensional complex.

**Proof (sketch).** Note that for \( p = 5 \), this is just Claim 7.3. Let \( G = \text{PSL}_2(\mathbb{F}_p) \). We begin by removing from \( C(G) \) all cosets \( xH \) which are not intersections of maximal cosets, i.e. we remove all cosets \( xH \) for which \( H \) is not an intersection of maximal subgroups. The resulting poset \( C_0 \) is homotopy equivalent to \( L_0 \) by \([16, \text{Corollary 2.5}]\). Similarly, the poset \( L_0 \) consisting of all subgroups in \( L(G) \) which are intersections of maximal subgroups is homotopy equivalent to \( L(G) \). Now, any chain of length \( k \) in the coset poset corresponds to a chain of length \( k - 1 \) in the subgroup poset (simply take all underlying subgroups, except for identity). By \([16, \text{Lemma 3.4}]\) any chain in \( L_0 \) has length at most two, and hence any chain in \( C_0 \) has length at most three. Shareshian’s argument in the proof of Lemma 7.2 that all two-simplices in \( \Delta(L_0) \) can be removed without changing the homotopy type also shows that all three-simplices may be removed from \( C_0 \) without changing the homotopy type (when removed in the correct order, each corresponds to an “elementary collapse”).

The following computation of the Euler characteristic of \( C(\text{PSL}_2(\mathbb{F}_p)) \) was provided to me by Kenneth S. Brown [private communication].

**Lemma 7.4** If \( p \equiv \pm 3 \pmod{8} \) and \( p \equiv \pm 2 \pmod{5} \), then the Euler characteristic of \( C(\text{PSL}_2(\mathbb{F}_p)) \) is

\[
\chi(C(\text{PSL}_2(\mathbb{F}_p))) = o(\text{PSL}_2(\mathbb{F}_p)) \left( \frac{p}{24}(p-1)(p+1) - p - 4 \right) + 1.
\]

**Proof.** For any finite group \( G \), \( \chi(C(G)) = -P(G,-1) + 1 \), where \( P(G,s) \) is the probabilistic zeta function of \( G \) (see \([3]\)). Möbius inversion allows one to compute \( P(G,-1) \) from the Möbius function of \( G \) \([4, \text{Section 2.1}]\). When \( G = \text{PSL}_2(\mathbb{F}_p) \), the Möbius function has been calculated by Hall \([3]\), and the reader may derive the above result. \(\Box\)
Proposition 7.5 If \( p \equiv \pm 3 \pmod{8} \) and \( p \not\equiv \pm 1 \pmod{5} \), then \( G = PSL_2(\mathbb{F}_p) \) satisfies (1).

Proof. By Lemmas 7.2 and 7.3, it suffices to check that the rank of \( H_2(C(G)) \) is at least \( o(G) \), and for \( p = 5 \) this follows from Proposition 5.4. We now assume \( p > 5 \). Since the Euler characteristic of \( C(G) \) is simply rank \( H_2(C(G)) - rank H_1(C(G)) \), Lemma 7.4 shows that \( H_2(C(G)) \) has rank at least

\[
o(G) \left( \frac{p}{12}(p-1)(p+1) - p - 4 \right) + 1.
\]

The conditions of the proposition force \( p \geq 11 \), so \( \frac{p}{12}(p-1)(p+1) - p - 4 \geq 95 \) (this is an increasing function of \( p \)). Thus rank \( H_2(C(G)) \geq o(G) \), as desired. \( \square \)

At least two other simple groups satisfy (1). These are \( PSL_2(\mathbb{F}_8) \) and the Suzuki group \( Sz(8) \). The proof is analogous to that given above, using [1] Table I for the computation of \( P(G, -1) \) and hence \( \chi(C(G)) \). Presumably, one should be able to answer Question 7.1 for all the groups listed in Lemma 7.2.

We now show that certain direct products satisfy (1). In particular, given any finite collection of non-isomorphic simple groups satisfying (1), their direct product \( \Pi \) also satisfies (1), and if \( G \) is a finite solvable group then \( G \times \Pi \) satisfies (1).

Proposition 7.6 The collection of groups satisfying (1) is closed under coprime direct products.

Proof. Let \( H \) and \( K \) be coprime groups satisfying (1), and let \( G = H \times K \). Recall that Lemmas 6.3 and 6.12 show that \( C(G) \cong C(H) \ast C(K) \) and \( L(G) \cong Susp (L(H) \ast L(K)) \). We need only consider the case in which \( G \) is not solvable, and in light of Theorem 4.3 we need only check condition (1) for \( n \geq 1 \).

If \( L(H) \) and \( L(K) \) are both empty, then \( G \) is solvable and we are done. If \( L(H) \) is empty but \( L(K) \) is not, then we have \( L(G) \cong Susp L(K) \) and hence \( H_i(L(G)) \cong H_{i-1}L(K) \) for \( i \geq 1 \). Letting \( r_i(X) \) denote the rank of the \( i \)th (reduced) homology group of the space \( X \) (\( i \geq 0 \)), we have (for \( n \geq 1 \))

\[
r_n(L(G)) = r_{n-1}(L(K)) \leq r_n(C(K)) \leq \sum_{i+j=n} r_i(C(H)) \cdot r_j(C(K)) = r_{n+1}(C(G)),
\]

the last equality following from [11, Lemma 2.1]. Of course, if \( L(K) \) is empty and \( L(H) \) is not, the situation is symmetric.

Now, assume that \( L(H) \) and \( L(K) \) are each non-empty. Then for any \( n \geq 1 \) we have

\[
r_{n+1}(C(G)) = \sum_{i+j=n} r_i(C(H)) \cdot r_j(C(K))
\]

and

\[
r_n(L(G)) = r_{n-1}(L(H) \ast L(K)) = \sum_{k+l=n-2} r_k(L(H)) \cdot r_l(L(K))
\]

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by [11, Lemma 2.1] (note that $L(G)$ is simply connected so there is no problem when $n = 1$). By assumption, we have $r_{i-1}(L(H)) \leq r_i(C(H))$ and $r_{i-1}(L(K)) \leq r_i(C(K))$, so for each $m$ ($0 \leq m \leq n - 2$) we have

$$r_m(L(H)) \cdot r_{n-2-m}(L(K)) \leq r_{m+1}(C(H)) \cdot r_{n-(m+1)}(C(K))$$

and thus $r_n(L(G)) \leq r_{n+1}(C(G))$ as desired. □

Thévenaz has “found” the spheres in the subgroup poset of a solvable group, i.e. he has provided a proof of Theorem 6.1 by analyzing a certain collection of spherical subposets of $L(G)$ [19]. It would be interesting to explore similar ideas in the coset poset, and in particular such a proof of Theorem 1.1 might allow one to explicitly construct an injection from $H_n(L(G))$ into $H_{n+1}(C(G))$ (for $G$ finite and solvable), and could shed further light on Question 7.1.

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