SHARP COMPLEX CONVEXITY ESTIMATES

ALEXANDER LINDENBERGER, PAUL F. X. MÜLLER, AND MICHAEL SCHMUCKENSCHLÄGER

Abstract. In this paper we determine the value of the best constants in the 2-uniform
PL-convexity estimates of $\mathbb{C}$. This solves a problem posed by W. J. Davis, D. J. H.
Garling and N. Tomczak-Jaegermann [2].

1. Introduction

Quoting from the book of Pisier [7, section 11.9], “Haagerup was the first who
noticed the importance of the (sharp) inequality

$$
\left( |x|^2 + \frac{1}{2} |y|^2 \right)^{\frac{1}{2}} \leq \int_{T} |x + \zeta y| \, dm(\zeta) \quad \text{for } x, y \in \mathbb{C},
$$

where $T$ denotes the complex unit circle and $m$ the usual Haar measure on $T$ with
$m(T) = 1$. W. J. Davis, D. J. H. Garling and N. Tomczak-Jaegermann [2] Proposition
3.1] present a proof of (1) based on the power series representation of elliptic in-
tegrals. U. Haagerup and G. Pisier obtained an extension of (1) to non-commutative
$H^1$-spaces in [3].

W. J. Davis, D. J. H. Garling and N. Tomczak-Jaegermann [2, Problem 4] conjec-
ture, that, for $\alpha \in (0, 2]$,

$$
\left( |x|^2 + \alpha |y|^2 \right)^{\frac{1}{2}} \leq \left( \int_{T} |x + \zeta y|^\alpha \, dm(\zeta) \right)^{\frac{1}{\alpha}} \quad \text{for } x, y \in \mathbb{C},
$$

holds true and that the constant $\frac{1}{2}$ is sharp. In this paper we give a proof of this
conjecture.

In [2] the notion of uniform PL-convexity is introduced for continuously quasi-
normed vector spaces over the complex numbers. For $r \in [2, \infty)$, such a space
$(E, \| \cdot \|)$ is called $r$-uniformly PL-convex, if and only if for any $\alpha \in (0, \infty)$ there exists
a $\lambda > 0$, such that

$$
\left( \|x\|^r + \lambda \|y\|^r \right)^{\frac{1}{r}} \leq \left( \int_{T} \|x + \zeta y\|^\alpha \, dm(\zeta) \right)^{\frac{1}{\alpha}} \quad \text{for } x, y \in E.
$$

The largest possible value of $\lambda$ satisfying (3) is then denoted by $I_{r, \alpha}(E)$. Using this
notion, $\mathbb{C}$ itself is 2-uniformly PL-convex and it is easily seen, that $I_{2, \alpha}(\mathbb{C}) = 1$ for
$\alpha \geq 2$. We thus identify the value of $I_{2, \alpha}(\mathbb{C})$ for each $\alpha \in (0, \infty)$ by proving the
following theorem:

Theorem 1.1. Let $\alpha \in (0, 2]$ and $\lambda \leq \frac{1}{2}$. We have

$$
\left( |x| + \lambda |y| \right)^{\frac{1}{2}} \leq \left( \int_{T} |x + \zeta y|^\alpha \, dm(\zeta) \right)^{\frac{1}{\alpha}} \quad \text{for } x, y \in \mathbb{C},
$$

2010 Mathematics Subject Classification. Primary 60G42; Secondary 32U05.
Key words and phrases. martingale inequalities, complex convexity, best constants.

The research for this paper was conducted while the first named author was preparing his Master’s
thesis at the Department of Analysis, Johannes Kepler University Linz. The first and the second named
authors were supported by the Austrian Science Foundation (FWF) Pr.Nr P28352-N32.
and \( \frac{\alpha}{2} \) is the best (i.e. largest) real constant satisfying \( \frac{4}{x} \).

2. Preliminaries

In this section we collect three main ingredients employed in the proof of Theorem 1.1.

The proof of \( I_{2,1} = \frac{1}{2} \) given in [2] relies on the power series representation of elliptic integrals and seems difficult to generalize. In 1994 the third named author gave an alternative proof, using techniques of stochastic analysis. The main idea was rewriting the line integral on the right-hand side of (1) as an area integral. For general \( \alpha \) this reads as follows.

Lemma 2.1. For \( \alpha \in (0, \infty) \) and \( y > 0 \), we have

\[
\int_T |1 + y \zeta|^\alpha \, d\mu(\zeta) = 1 + \alpha \frac{2 \gamma^2}{2} \int_D |1 + y \zeta|^{\alpha - 2} G(z) \, dz,
\]

where \( D \) denotes the complex unit disk and \( G \) Green’s function of this domain and pole \( 0 \), so \( \frac{1}{\pi} \ln \frac{1}{|z|} \) for each \( z \in D \).

Next we recall the inequality of arithmetic and geometric means in the form used below. For details we refer to [4, section 6.7].

Theorem 2.1 (AM-GM inequality). Let \( f : \mathbb{T} \to \mathbb{R} \) be non-negative almost everywhere on \( \mathbb{T} \) and measurable. Let \( r \) be a positive real number. Then we have

\[
\left( \int_{\mathbb{T}} f^{-r} \, dm \right)^{-\frac{1}{r}} \leq \exp \int_{\mathbb{T}} \ln f \, dm \leq \left( \int_{\mathbb{T}} f^r \, dm \right)^{\frac{1}{r}},
\]

if all these integrals exist.

The usefulness of the AM-GM inequality is due to the fact, that for our specific functions, the geometric mean is easier to calculate than the arithmetic means. Indeed we can evaluate the geometric mean using the familiar Jensen formula (see [1, p. 207-208]). For integrals over the complex unit circle, it has the following form.

Theorem 2.2 (Jensen’s formula). Let \( f : U \to \mathbb{C} \) be analytic on a domain \( U \subseteq \mathbb{C} \), where \( U \) contains the closed unit disk \( \overline{D} \). Let \( a_1, a_2, \ldots, a_n \) be the zeros of \( f \) in the interior of \( \overline{D} \) (repeated according to multiplicity). Assuming \( f(0) \neq 0 \), we have

\[
\int_{\mathbb{T}} \ln |f(\zeta)| \, d\mu(\zeta) = \ln |f(0)| + \sum_{i=1}^{n} \ln \frac{1}{|a_i|}.
\]

3. Proof of Theorem 1.1

Using the tools collected in section 2, we can now give a proof of Theorem 1.1.

Note that for \( x = 0 \) or \( y = 0 \), the inequality \( \frac{4}{x} \) is trivial. Hence, by rescaling and rotation, it suffices to show, that

\[
\left( \int_{\mathbb{T}} |1 + y \zeta|^\alpha \, d\mu(\zeta) \right)^{\frac{1}{\alpha}} \geq \left( 1 + \lambda y^2 \right)^{\frac{1}{2}} \quad \text{for each } y > 0
\]

holds for \( \lambda = \frac{\alpha}{2} \) and does not hold for \( \lambda > \frac{\alpha}{2} \).
We first show, that \( \lambda = \alpha \) satisfies (6). To this end, we define functions \( a, h, g : [0, \infty) \to \mathbb{R} \) by

\[
a(y) := \int_T |1 + y\zeta|^\alpha \, dm(\zeta), \\
h(y) := \begin{cases} 
1 + \frac{\alpha^2}{4} y^2, & y^2 \in [0, 1] \\
\frac{\alpha^2}{4} + \left( y^2 \right)^{\frac{\alpha}{2}} - \frac{\alpha}{2} \left( 1 - \frac{\alpha}{2} \right) \ln y^2, & y^2 \in \left( 1, \left( 1 - \frac{\alpha}{2} \right)^{-1} \right), \\
y^\alpha, & y^2 \geq \left( 1 - \frac{\alpha}{2} \right)^{-1}
\end{cases}
\]

\[
g(y) := \left( 1 + \frac{\alpha}{2} y^2 \right)^{\frac{\alpha}{2}}
\]

and show

(7) \( a(y) \geq h(y) \geq g(y) \) for \( y > 0 \).

Therefore, we fix \( y > 0 \) and distinguish three cases, according to the definition of \( h \).

**Case** \( y^2 \geq \left( 1 - \frac{\alpha}{2} \right)^{-1} \): Using the AM-GM inequality and applying Jensen's formula to the analytic function \( z \mapsto 1 + yz \) yield

\[
\left( \int_T |1 + y\zeta|^\alpha \, dm(\zeta) \right)^{\frac{1}{\alpha}} \geq \exp \int_T \ln |1 + y\zeta| \, dm(\zeta) = \exp(\max(0, \ln y)) = \max(1, y).
\]

In this case, we particularly have \( y \geq 1 \), so we conclude

\[
a(y) \geq y^\alpha = h(y).
\]

On the other hand, \( y^2 \geq \left( 1 - \frac{\alpha}{2} \right)^{-1} \) implies

\[
h(y) = y^\alpha \geq \left( 1 + \frac{\alpha}{2} y^2 \right)^{\frac{\alpha}{2}} = g(y).
\]

For the other two cases, we use the alternative representation of \( a \) obtained by Lemma 2.1, i.e.

\[
a(y) = 1 + \frac{\alpha^2 y^2}{2} \int_D |1 + yz|^\alpha G(z) \, dz
\]

\[
= 1 + \alpha^2 y^2 \int_0^1 \int_T |1 + yr\zeta|^\alpha \, dm(\zeta) r \ln \frac{1}{r} \, dr.
\]

We note \( \alpha - 2 \leq 0 \), so the AM-GM inequality for negative exponents and Jensen’s formula, applied to the analytic function \( z \mapsto 1 + yrz \), yield

\[
\left( \int_T |1 + yr\zeta|^{\alpha-2} \, dm(\zeta) \right)^{\frac{1}{\alpha-2}} \leq \exp \int \ln |1 + yr\zeta| \, dm(\zeta) = \max(1, yr).
\]
Hence, we arrive at
\[ a(y) \geq 1 + a^2 y^2 \int_0^1 \min\{1, (yr)^{\alpha-2}\} r \ln \frac{1}{r} dr. \]

**Case \( y^2 \in \left(1, \left(1 - \frac{\alpha}{2}\right)^{-1}\right) \):** By calculation, we obtain
\[
\int_0^1 \min\{1, (yr)^{\alpha-2}\} r \ln \frac{1}{r} dr = \int_0^{\frac{1}{2}} r \ln \frac{1}{r} dr + y^{\alpha-2} \int_0^1 r^{\alpha-1} \ln \frac{1}{r} dr
\]
\[= \frac{1 + 2 \ln y}{4y^2} + y^{\alpha-2} \left(\frac{1}{\alpha^2} - \frac{1 + \alpha \ln y}{\alpha^2 y^\alpha}\right).\]

Therefore, we have
\[ a(y) \geq 1 + \frac{a^2}{4} y^2 = h(y). \]

The inequality \( h(y) \geq g(y) \) is an exercise in elementary calculus, carried through in the appendix (Lemma 4.2 with \( \beta := \frac{\alpha}{2} \) and \( t := y^2 \)).

**Case \( y \leq 1 \):** Here, the integral simplifies to
\[
\int_0^1 \min\{1, (yr)^{\alpha-2}\} r \ln \frac{1}{r} dr = \int_0^1 r \ln \frac{1}{r} dr = 1 - \frac{1}{4}.
\]

Consequently, we have
\[ a(y) \geq 1 + \frac{a^2}{4} y^2 = h(y). \]

The inequality \( h(y) \geq g(y) \) is again verified by elementary calculus (Lemma 4.1 with \( \beta := \frac{\alpha}{2} \) and \( t := y^2 \)).

It remains to show, that \( \frac{\alpha}{4} \) is the largest constant satisfying \( \text{[6]} \). To this end, we fix \( \lambda \in \mathbb{R} \), define \( b(y) := (1 + \lambda y^2)^2 \) for each \( y \geq 0 \), and assume
\[ a(y) \geq b(y) \quad \text{for } y > 0. \]

Obviously, we have \( a(0) = b(0) = 1 \) and a short calculation shows \( a'(0) = b'(0) = 0 \). From the assumption, we now conclude \( a''(0) \geq b''(0) \). Using \( a''(0) = \frac{a^2}{2} \) and \( b''(0) = \alpha \lambda \), we obtain \( \lambda \leq \frac{\alpha}{2} \). \( \square \)

### 4. Appendix

#### 4.1. Proof of Lemma 2.1

We give a proof of the identity obtained and exploited by the third named author, i. e.

\[ \int_T |1 + y\zeta|^\alpha \, dm(\zeta) = 1 + \frac{a^2 y^2}{2} \int_D |1 + yz|^{\alpha-2} G(z) \, dz \quad \text{for } y > 0, \alpha \in (0, \infty). \]

Here we use techniques of stochastic analysis. Alternatively, one could obtain \( \text{[8]} \) using Gauss’s divergence theorem. See for instance \( \text{[6]} \) p. 355-356.

Let \( (B_t)_{t \geq 0} = (B_{t}^{(1)} + iB_{t}^{(2)})_{t \geq 0} \) be a complex Brownian motion started at the origin and \( \tau := \inf \{t > 0 : |B_t| > 1\} \). We then have
\[
\int_T |1 + y\zeta|^\alpha \, dm(\zeta) = \mathbb{E}[|1 + yB_\tau|^\alpha].
\]
Applying Itô’s formula to the continuous martingale \((1 + yB_t)_{t \geq 0}\) gives
\[
|1 + yB_t|^\alpha = 1 + \alpha y \int_0^t \left( |1 + yB_s^{(1)}|^\alpha - |1 + yB_s^{(2)}|^\alpha \right) dB_s^{(2)} + \frac{\alpha^2}{2} y^2 \int_0^t |1 + yB_s|^\alpha ds,
\]
whenever \(t < \tau\), and hence
\[
\mathbb{E}[|1 + yB_t|^\alpha] = 1 + \frac{\alpha^2}{2} y^2 \mathbb{E} \left[ \int_0^\tau |1 + yB_s|^\alpha ds \right].
\]

Let \(G\) denote Green’s function of the unit disk with pole 0. Using the occupation time formula for the Brownian motion \([5, \text{Theorem 3.32, section 3.3, p. 80}]\), we obtain
\[
1 + \frac{\alpha^2}{2} y^2 \mathbb{E} \left[ \int_0^\tau |1 + yB_s|^\alpha ds \right] = 1 + \frac{\alpha^2}{2} y^2 \int_{\mathbb{D}} |1 + yz|^\alpha G(z) dz.
\]

\[\square\]

4.2. Elementary estimates. Here we collect and prove the elementary estimates used in section 3. We begin with a special case of the classical Bernoulli inequality for real exponents.

**Lemma 4.1** (Bernoulli’s inequality). Let \(\beta \in [0,1]\) and \(t > 0\). We have
\[1 + \beta^2 t \geq (1 + \beta t)^\beta.\]

**Proof.** Taylor’s theorem, applied to \(\phi(t) := (1 + \beta t)^\beta\), guarantees the existence of a \(\xi \in [0,t]\), such that
\[
\phi(t) = \phi(0) + \phi'(0)t + \phi''(\xi)\frac{t^2}{2} = 1 + \beta^2 t + \beta^3 (\beta - 1)(1 + \beta \xi)^{\beta - 2} \frac{t^2}{2}.
\]
Using this \(\xi\), we obtain
\[1 + \beta^2 t - (1 + \beta t)^\beta = \beta^3 (1 - \beta)(1 + \beta \xi)^{\beta - 2} \frac{t^2}{2} \geq 0.
\]

\[\square\]

**Lemma 4.2.** Let \(\beta \in [0,1]\) and \(t \in \left[1, \frac{1}{1-\beta}\right]\). We have
\[
\beta^2 + t^\beta \geq 1 + \beta t)^\beta.
\]

**Proof.** Taylor’s theorem, applied to \(\phi(t) := \beta^2 + t^\beta - (1 + \beta t)^\beta\), guarantees the existence of a \(\xi \in [1,t]\), such that
\[
\phi(t) = \phi(1) + (t-1)\phi'(\xi) = \beta^2 + 1 - (1 + \beta)^\beta + (t-1)^\beta \left( \xi^{\beta - 1} - (1 - \beta)\xi^{-1} - \beta(1 + \beta \xi)^{\beta - 1} \right).
\]
From Lemma 4.1 we know
\[\beta^2 + 1 - (1 + \beta)^\beta \geq 0.
\]
\(\xi \in \left[1, \frac{1}{1-\beta}\right]\) implies \(\xi \leq 1 + \beta \xi\) and thus
\[
\xi^{\beta - 1} - (1 - \beta)\xi^{-1} - \beta(1 + \beta \xi)^{\beta - 1} \geq \xi^{\beta - 1} - (1 - \beta)\xi^{-1} - \beta \xi^{\beta - 1} = (1 - \beta) \left( \xi^{\beta - 1} - \xi^{-1} \right) \geq 0.
\]

\[\square\]
References

[1] L. V. Ahlfors. Complex analysis. McGraw-Hill Book Co., New York, third edition, 1978. An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[2] W. J. Davis, D. J. H. Garling, and N. Tomczak-Jaegermann. The complex convexity of quasinormed linear spaces. J. Funct. Anal., 55(1):110–150, 1984.

[3] U. Haagerup and G. Pisier. Factorization of analytic functions with values in noncommutative \( L_1 \)-spaces and applications. Canad. J. Math., 41(5):882–906, 1989.

[4] G. H. Hardy, J. E. Littlewood, and G. Pólya. Inequalities. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. Reprint of the 1952 edition.

[5] P. Mörters and Y. Peres. Brownian motion, volume 30 of Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.

[6] P. F. X. Müller. Isomorphisms between \( H^1 \) spaces, volume 66 of Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)]. Birkhäuser Verlag, Basel, 2005.

[7] G. Pisier. Martingales in Banach spaces, volume 155 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016.

A. Lindenberger, Institute of Analysis, Johannes Kepler University Linz, Altenberger Strasse 69, 4040 Linz, Austria
E-mail address: alexander.lindenberger@jku.at

P. F. X. Müller, Institute of Analysis, Johannes Kepler University Linz, Altenberger Strasse 69, 4040 Linz, Austria
E-mail address: paul.mueller@jku.at

M. Schmuckenschläger, Institute of Analysis, Johannes Kepler University Linz, Altenberger Strasse 69, 4040 Linz, Austria
E-mail address: michael.schmuckenschlaeger@jku.at