3-SASAKIAN MANIFOLDS IN DIMENSION SEVEN, THEIR SPINORS AND G_2-STRUCTURES

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Abstract. It is well-known that 7-dimensional 3-Sasakian manifolds carry a one-parametric family of compatible G_2-structures and that they do not admit a characteristic connection. In this note, we show that there is nevertheless a distinguished cocalibrated G_2-structure in this family whose characteristic connection ∇^c along with its parallel spinor field Ψ_0 can be used for a thorough investigation of the geometric properties of 7-dimensional 3-Sasakian manifolds. Many known and some new properties can be easily derived from the properties of ∇^c and of Ψ_0, yielding thus an appropriate substitute for the missing characteristic connection.

1. Introduction

3-Sasakian manifolds have been studied by the Japanese school in Differential Geometry decades ago [13]. They are Einstein spaces of positive scalar curvature carrying three compatible orthogonal Sasakian structures. In the middle of the 80-ties, a relation between 3-Sasakian manifolds and the spectrum of the Dirac operator was discovered [10], [11]. Indeed, they admit three Riemannian Killing spinors, which realize the lower bound for the eigenvalues of the Dirac operator [6]. Seven-dimensional, regular 3-Sasakian manifolds are classified in [10]. In the 90-ties, many new families of non-regular 3-Sasakian manifolds have been constructed specially in dimension seven [4]. This dimension is important because the exceptional Lie group G_2 admits a 7-dimensional representation and any 3-Sasakian-structure on a Riemannian manifold induces a family of adapted, non-integrable G_2-structures. A deformation of one of these G_2-structures—we call it the canonical G_2-structure—yields examples of 7-dimensional Riemannian manifolds with precisely one Killing spinor [12]. The whole family of underlying G_2-structures has been investigated from the viewpoint of spin geometry in [2], section 8. In particular, they are solutions of type II string theory with 4-fluxes (see [1] for more background and motivation).

We will show that the canonical G_2-structure of a 3-Sasakian manifold is cocalibrated. Consequently, there exists a unique connection with totally skew-symmetric torsion preserving it, see [5], [6]. The aim of this note is to study this characteristic connection ∇^c as well as the corresponding ∇^c-parallel spinor field Ψ_0. This point of view allows us to prove many properties of 3-Sasakian manifolds in a unified way. For example, the Riemannian Killing spinors are the Clifford products of the canonical spinor Ψ_0 by the three unit vectors defining the 3-Sasakian structure: in this sense, the ∇^c-parallel spinor field Ψ_0 is more fundamental than the Killing spinors. Finally we study the

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spinorial field equations and the deformations of the canonical $G_2$-structure in more detail.

2. 3-Sasakian manifolds in dimension seven

A 7-dimensional Sasakian manifold is a Riemannian manifold $(M^7, g)$ equipped with a contact form $\eta$, its dual vector field $\xi$ as well as with an endomorphism $\varphi : TM^7 \to TM^7$ such that the following conditions are satisfied:

$$\eta \wedge (d\eta)^3 \neq 0, \quad \eta(\xi) = 1, \quad g(\xi, \xi) = 1,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X) \cdot \eta(Y), \quad \varphi^2 = \text{Id} + \eta \otimes \xi,$$

$$\nabla^g_X \xi = -\varphi X, \quad (\nabla^g_X \varphi)(Y) = g(X, Y) \cdot \xi - \eta(Y) \cdot X.$$

These conditions imply several further relations, for example

$$\varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad d\eta(X, Y) = 2 \cdot g(X, \varphi Y).$$

A 7-dimensional 3-Sasakian manifold is a Riemannian manifold $(M^7, g)$ equipped with three Sasakian structures $(\xi_\alpha, \eta_\alpha, \varphi_\alpha), \alpha = 1, 2, 3$, such that

$$[\xi_1, \xi_2] = 2 \xi_3, \quad [\xi_2, \xi_3] = 2 \xi_1, \quad [\xi_3, \xi_1] = 2 \xi_2$$

and

$$\varphi_1 \circ \varphi_2 = -\varphi_1 + \eta_2 \otimes \xi_3, \quad \varphi_2 \circ \varphi_3 = \varphi_1 + \eta_3 \otimes \xi_2,$$

$$\varphi_1 \circ \varphi_3 = -\varphi_2 + \eta_3 \otimes \xi_1, \quad \varphi_3 \circ \varphi_1 = \varphi_2 + \eta_1 \otimes \xi_3,$$

$$\varphi_2 \circ \varphi_1 = -\varphi_3 + \eta_1 \otimes \xi_2, \quad \varphi_1 \circ \varphi_2 = \varphi_3 + \eta_2 \otimes \xi_1.$$

The vertical subbundle $T^v \subset TM^7$ is spanned by $\xi_1, \xi_2, \xi_3$, its orthogonal complement is the horizontal subbundle $T^h$. Both subbundles are invariant under $\varphi_1, \varphi_2, \varphi_3$.

The properties as well as examples of Sasakian and 3-Sasakian manifolds are the topic of the book [3]. 3-Sasakian manifolds are always Einstein with scalar curvature $R = 42$. If they are complete, they are compact with finite fundamental group. Therefore we shall always assume that $M^7$ is compact and simply-connected. The frame bundle has a topological reduction to the subgroup $SU(2) \subset SO(7)$. In particular, $M^7$ is a spin manifold. Moreover, there exists locally an orthonormal frame $e_1, \ldots, e_7$ such that $e_1 = \xi_1, e_2 = \xi_2, e_3 = \xi_3$ and the endomorphisms $\varphi_\alpha$ acting on the horizontal part $T^h := \text{Lin}(e_4, e_5, e_6, e_7)$ of the tangent bundle are given by the following matrices

$$\varphi_1 := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \varphi_2 := \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \varphi_3 := \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$  

We will identify vector fields with 1-forms via the Riemannian metric, thus obtaining a coframe $\eta_1, \eta_2, \ldots, \eta_7$, and shall use throughout the abbreviation $\eta_{ij\ldots} := \eta_i \wedge \eta_j \wedge \ldots$. In this frame, we compute the differentials $d\eta_\alpha$,

$$d\eta_1 = -2 (\eta_{23} + \eta_{45} + \eta_{67}),$$

$$d\eta_2 = 2 (\eta_{13} - \eta_{46} + \eta_{57}),$$

$$d\eta_3 = -2 (\eta_{12} + \eta_{47} + \eta_{56}).$$

Each of the three Sasaki structures on $M^7$ admits a characteristic connection, i.e. a metric connection with antisymmetric torsion; however, this torsion is well-known to
be η_i ∧ dη_i \[8, Thm 8.2\], and these do not coincide for i = 1, 2, 3. Thus, a 3-Sasakian manifold has no characteristic connection \[1, §2.6\].

3. The canonical \(G_2\)-structure of a 3-Sasakian manifold

Consider the following 3-forms,

\[ F_1 := η_1 ∧ η_2 ∧ η_3, \quad F_2 := \frac{1}{2}(η_1 ∧ dη_1 + η_2 ∧ dη_2 + η_3 ∧ dη_3) + 3η_1 ∧ η_2 ∧ η_3. \]

Then

\[ ω := F_1 + F_2 = η_{123} − η_{145} − η_{167} − η_{246} + η_{257} − η_{347} − η_{356} \]

is a generic 3-form defined globally on \(M^7\). It induces a \(G_2\)-structure on \(M^7\).

**Definition 3.1.** The 3-form \(ω = F_1 + F_2\) is called the **canonical **\(G_2\)-structure of the 7-dimensional 3-Sasakian manifold.

We investigate now the type of this canonical \(G_2\)-structure from the point of view of \(G_2\)-geometry \[5\], \[8\]. It is basically described by the differential of the \(G_2\)-structure \(ω\). We compute directly \[12\]

\[ dF_1 = 2 \cdot (∗F_2), \quad dF_2 = 12 \cdot (∗F_1) + 2 \cdot (∗F_2), \quad d∗F_1 = d∗F_2 = 0. \]

In particular, the canonical \(G_2\)-structure is cocalibrated. Equivalently, it is of type \(W_1 ⊕ W_3 = Λ^3_1 ⊕ Λ^3_{27}\) in the Fernandez/Gray notation, see \[5\], \[8\], \[9\],

\[ d∗ω = 0, \quad ∗dω = 4(3F_1 + F_2). \]

There exists a unique connection \(∇^c\) preserving the \(G_2\)-structure with totally skew-symmetric torsion \(T^c\) \[8\], \[9\]. For a cocalibrated \(G_2\)-structure \(ω\) this characteristic torsion form \(T^c\) is given by the formula

\[ T^c = −dω + \frac{1}{6}(dω, ∗ω) · ω. \]

We express the characteristic torsion by the data of the 3-Sasakian structure,

\[ T^c = −6F_1 + 2F_2 = η_1 ∧ dη_1 + η_2 ∧ dη_2 + η_3 ∧ dη_3 = 2ω − 8F_1. \]

Thus, we see that \(T^c\) is the sum of the three characteristic torsion forms of the Sasakian structures \(η_i\).

Let us decompose the characteristic torsion \(T^c = T^c_1 + T^c_{27}\) into the \(W_1 = Λ^3_1\) and the \(W_3 = Λ^3_{27}\) -part, respectively. Then we obtain

\[ T^c_1 = \frac{6}{7}(F_1 + F_2) = \frac{6}{7}ω, \quad T^c_{27} = \frac{8}{7}(F_2 − 6F_1). \]

In particular, the canonical \(G_2\)-structure of a 3-Sasakian manifold is never of pure type \(W_1\) or \(W_3\).

We will now prove that the canonical \(G_2\)-structure has parallel characteristic torsion, \(∇^cT^c = 0\), and realizes one type of cocalibrated \(G_2\)-structures with characteristic holonomy contained in the maximal, six-dimensional subalgebra \(su(2) ⊕ su_c(2)\) of \(g_2\) \[7\]. Later, we shall see that its holonomy algebra coincides with \(su(2) ⊕ su_c(2)\).
Theorem 3.1. The canonical $G_2$-structure $\omega$ of a 7-dimensional 3-Sasakian manifold is cocalibrated, $d \ast \omega = 0$. Its characteristic torsion is given by the formula

$$T^c = - d \omega + 6 \omega.$$ 

Moreover, we have $(d \omega, \ast \omega) = 36$, $|T^c|^2 = 60$ and

$$d \ast T^c = 0, \quad dT^c = - 4 \ast T^c, \quad d \omega = \frac{1}{2} d \ast d \omega - 12 \ast \omega.$$ 

The characteristic connection preserves the splitting $TM^7 = T^v \oplus T^h$ and the characteristic torsion is $\nabla^c$-parallel, $\nabla^c T^c = 0$.

Proof. Since $\xi_1$ is a Killing vector field, we have

$$\nabla^g_X \eta_1 = \frac{1}{2} X \ast d \eta_1.$$

Then we obtain

$$\nabla^c_X \eta_1 = \nabla^g_X \eta_1 + \frac{1}{2} T^c(X, \eta_1, -) = \frac{1}{2} X \ast d \eta_1 - \frac{1}{2} X \ast (\eta_1 \ast T^c).$$

The formula $T^c = \eta_1 \wedge d \eta_1 + \eta_2 \wedge d \eta_2 + \eta_3 \wedge d \eta_3$ yields directly

$$\eta_1 \ast T^c = d \eta_1 + (\eta_1 \ast d \eta_2) \wedge \eta_2 + (\eta_1 \ast d \eta_3) \wedge \eta_3.$$

Moreover, the formulas for the differential $d \eta_\alpha$ imply that

$$\eta_1 \ast d \eta_2 = 2 \eta_3, \quad \eta_1 \ast d \eta_3 = - 2 \eta_2$$

holds. Thus we obtain

$$\nabla^c_X \eta_1 = 2 X \ast (\eta_2 \wedge \eta_3),$$

i.e. $\nabla^c$ preserves the subbundle $T^v$. Finally we have

$$(\nabla^c_X \eta_1) \wedge \eta_2 \wedge \eta_3 = 0$$

and then $\nabla^c(\eta_1 \wedge \eta_2 \wedge \eta_3) = 0$. Since $T^c = 2 \omega - 8 \eta_1 \wedge \eta_2 \wedge \eta_3$ and $\nabla^c \omega = 0$ we conclude that $\nabla^c T^c = 0$ holds, too. \qed

4. The canonical spinor of a 3-Sasakian manifold

Since the spin representation of Spin(7) is real, let us consider the real spinor bundle $\Sigma$. Any $G_2$-structure $\omega$ acts via the Clifford multiplication on $\Sigma$ as a symmetric endomorphism with eigenvalue $(-7)$ of multiplicity one and eigenvalue 1 of multiplicity seven. Consequently, any $G_2$-structure on a simply-connected manifold $M^7$ defines a canonical spinor field $\Psi_0$ such that (see [12], [8])

$$\omega \cdot \Psi_0 = - 7 \Psi_0, \quad |\Psi_0| = 1.$$ 

If $(M^7, \omega)$ is cocalibrated and $\nabla^c$ is its characteristic connection, we obtain [8], [3]

$$\nabla^c \Psi_0 = 0, \quad T^c \cdot \Psi_0 = - \frac{1}{6} (d \omega, \ast \omega) \cdot \Psi_0, \quad \text{Scal}^g = \frac{1}{18} (d \omega, \ast \omega)^2 - \frac{1}{2} |T^c|^2,$$

We apply the general formulas to the canonical spinor of a 3-Sasakian manifold $M^7$. Then we obtain a spinor field such that

$$\omega \cdot \Psi_0 = - 7 \Psi_0, \quad T^c \cdot \Psi_0 = - 6 \Psi_0, \quad \nabla^g_X \Psi_0 + \frac{1}{4} (X \ast T^c) \cdot \Psi_0 = 0.$$ 

Using the explicit formulas for $\omega$ and $T^c$, a direct algebraic computation in the real spin representation yields the following
Lemma 4.1.

\[ T^c \cdot X \cdot \Psi_0 = -\frac{5}{3} X \cdot T^c \cdot \Psi_0 = 10 X \cdot \Psi_0 \quad \text{if} \quad X \in T^v, \]
\[ T^c \cdot X \cdot \Psi_0 = X \cdot T^c \cdot \Psi_0 = -6 X \cdot \Psi_0 \quad \text{if} \quad X \in T^h, \]

The equation \( \nabla^g \Psi_0 = 0 \) can be written as
\[ \nabla^g_X \Psi_0 - \frac{1}{8} (X \cdot T^c + T^c \cdot X) \cdot \Psi_0 = 0. \]

We apply now the algebraic Lemma and obtain a differential equation involving the canonical spinor of a 3-Sasakian manifold.

**Theorem 4.1.** The canonical spinor field \( \Psi_0 \) of a 7-dimensional 3-Sasakian manifold satisfies the following differential equation:
\[ \nabla^g_X \Psi_0 = \frac{1}{2} X \cdot \Psi_0 \quad \text{if} \quad X \in T^v, \]
\[ \nabla^g_X \Psi_0 = -\frac{3}{2} X \cdot \Psi_0 \quad \text{if} \quad X \in T^h. \]

In particular, \( \Psi_0 \) is an eigenspinor for the Riemannian Dirac operator, \( D^g \Psi_0 = \frac{9}{2} \Psi_0. \)

**Remark 4.1.** This equation has already been discussed in [7], section 10. It follows essentially from the formula \( T^c = 2 \omega - 8 F_1. \)

5. \( \nabla^c \)-parallel vectors and spinors of the canonical \( G_2 \)-structure

The spinor bundle splits into three subbundles, \( \Sigma = \Sigma_1 \oplus \Sigma_3 \oplus \Sigma_4, \) where
\[ \Sigma_1 := \mathbb{R} \cdot \Psi_0, \quad \Sigma_3 := \{ X \cdot \Psi_0 : X \in T^v \}, \quad \Sigma_4 := \{ X \cdot \Psi_0 : X \in T^h \}. \]

The characteristic connection preserves this splitting. Obviously, the 3-form \( \omega \) acts as the identity on \( \Sigma_3 \oplus \Sigma_4, \) while the torsion form satisfies

**Lemma 5.1.** The torsion form \( T^c \) acts on \( \Sigma_3 \) as a multiplication by 10 and it acts on \( \Sigma_1 \oplus \Sigma_4 \) as a multiplication by \(-6).\)

Given the definition of \( \Sigma_4, \) it is now a crucial observation that \( \nabla^c \)-parallel vector fields cannot be horizontal:

**Proposition 5.1.** Horizontal, \( \nabla^c \)-parallel vector fields
\[ \nabla^c X = 0, \quad 0 \neq X \in \Gamma(T^c) \]
do not exist.

**Proof.** Let \( 0 \neq X \) be the vector field. Then \( \Psi := X \cdot \Psi_0 \) is a \( \nabla^c \)-parallel spinor, too. Moreover, the torsion form acts on \( \Psi_0 \) and on \( \Psi \) by the same eigenvalue,
\[ T^c \cdot \Psi_0 = -6 \Psi_0, \quad T^c \cdot \Psi = -6 \Psi. \]

The holonomy algebra \( \mathfrak{so}(\nabla^c) \) is contained in \( \mathfrak{su}(2) \oplus \mathfrak{su}_c(2) \subset \mathfrak{g}_2 \subset \mathfrak{so}(7) \) and the linear holonomy representation splits into \( \mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^3. \) The vector field \( X \) is an element of \( \mathbb{R}^4 \) such that \( \mathfrak{so}(\nabla^c) \cdot X = 0. \) In [7] we explicitly realized the Lie algebra \( \mathfrak{su}(2) \oplus \mathfrak{su}_c(2) \) inside \( \mathfrak{so}(7). \) Using these formulas, an easy computation yields that the holonomy algebra is contained in \( \mathfrak{so}(3) \subset \mathfrak{su}(3) \subset \mathfrak{g}_2 \) and the linear holonomy representation splits into \( \mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^1. \) Consequently, the \( G_2 \)-manifold \( (M^7, \omega) \) is cocalibrated, its characteristic holonomy is contained in \( \mathfrak{so}(3) \) and the characteristic torsion \( T^c \) acts on both \( \nabla^c \)-parallel spinors with the same eigenvalue. It turns out that \( M^7 \) cannot be an Einstein manifold with positive scalar curvature by [7] Thm 7.1, a contradiction. \( \square \)
In general, the Casimir operator of a metric connection with parallel characteristic
torsion is given by the following formulas [3]
\[ \Omega = (D^{1/3})^2 - \frac{1}{16}(2 \text{Scal}^g + |T^c|^2) = \Delta_{T^c} + \frac{1}{16}(2 \text{Scal}^g + |T^c|^2) - \frac{1}{4}(T^c)^2. \]
Its kernel contains the space of all \( \nabla^c \)-parallel spinor fields. In particular, any \( \nabla^c \)-parallel spinor field \( \Psi \) satisfies the algebraic condition [5, 3]
\[ 4(T^c)^2 \cdot \Psi = (2 \text{Scal}^g + |T^c|^2) \cdot \Psi. \]
For the canonical \( G_2 \)-structure of a 3-Sasakian manifold we have \( 2 \text{Scal}^g + |T^c|^2 = 144 \). Consequently, any \( \nabla^c \)-parallel spinor field is a section in the subbundle \( \Sigma_1 \oplus \Sigma_4 \), i.e. of the form \( \Psi = a \cdot \Psi_0 + X \cdot \Psi_0 \), where \( a \) is constant and \( X \in \Gamma(T^h) \) is a horizontal, parallel vector field. But horizontal, \( \nabla^c \)-parallel vector fields do not exist. This argument proves:

**Theorem 5.1.** Any \( \nabla^c \)-parallel spinor field is proportional to \( \Psi_0 \). Moreover, the holonomy algebra is the six-dimensional maximal subalgebra \( \mathfrak{hol}(\nabla^c) = \mathfrak{su}(2) \oplus \mathfrak{su}_e(2) \) of \( \mathfrak{g}_2 \).

The latter argument proves that vertical, \( \nabla^c \)-parallel vector fields do not exist. Indeed, if \( \nabla^c X = 0 \), then \( X \cdot \Psi_0 \in \Gamma(\Sigma_3) \) is a parallel spinor in \( \Sigma_3 \). We conclude that \( X \cdot \Psi_0 = 0 \) and \( X = 0 \). Together with Proposition [5, 1] and the splitting of the tangent bundle, one concludes:

**Theorem 5.2.** There are no non-trivial \( \nabla^c \)-parallel vector fields.

### 6. Riemannian Killing spinors on 3-Sasakian manifolds

Consider the spinor fields \( \Psi_1 := \xi_1 \cdot \Psi_0, \Psi_2 := \xi_2 \cdot \Psi_0, \Psi_3 := \xi_3 \cdot \Psi_0 \). These spinors are sections in the bundle \( \Sigma_3 \).

**Theorem 6.1.** The spinor fields \( \Psi_\alpha \) are Riemannian Killing spinors, i.e.
\[ \nabla^g_X \Psi_\alpha = \frac{1}{2} X \cdot \Psi_\alpha, \quad \alpha = 1, 2, 3. \]

**Corollary 6.1 (10, 11).** Any simply-connected 3-Sasakian manifold admits at least three Riemannian Killing spinors.

**Proof.** We use the differential equation
\[ \nabla^g_X \Psi_0 = \frac{1}{8} (X \cdot T^c + T^c \cdot X) \cdot \Psi_0 \]
as well as the properties of Sasakian structures. Then we obtain
\[ \nabla^g_X (\xi_1 \cdot \Psi_0) = (\nabla^g_X \xi_1) \cdot \Psi_0 + \xi_1 \cdot \nabla^g_X \Psi_0 \]
\[ = -\varphi_1(X) \cdot \Psi_0 + \frac{1}{8} \xi_1 \cdot (X \cdot T^c + T^c \cdot X) \cdot \Psi_0 \]
\[ = \frac{1}{2} (X \cdot d\eta_1) \cdot \Psi_0 + \frac{1}{8} \xi_1 \cdot (X \cdot T^c + T^c \cdot X) \cdot \Psi_0 \]
\[ = -\frac{1}{4} (X \cdot d\eta_1 - d\eta_1 \cdot X) \cdot \Psi_0 + \frac{1}{8} \xi_1 \cdot (X \cdot T^c + T^c \cdot X) \cdot \Psi_0. \]

A direct algebraic computation yields now that
\[ -\frac{1}{4} (X \cdot d\eta_1 - d\eta_1 \cdot X) \cdot \Psi_0 + \frac{1}{8} \xi_1 \cdot (X \cdot T^c + T^c \cdot X) \cdot \Psi_0 = \frac{1}{2} X \cdot \xi_1 \cdot \Psi_0 \]
holds specially for the spinor $\Psi_0$. This proves the statement of the Theorem. \hfill \Box

In general, any real spinor field $\Phi$ of length one defined on a 7-dimensional Riemannian manifold induces a $G_2$-structure $\omega_\Phi$ (see [12]). Moreover, if two spinor fields $\Phi_2 = \xi \cdot \Phi_1$ are related via Clifford multiplication by some vector field $\xi$, then

$$\omega_{\Phi_2} = -\omega_{\Phi_1} + 2(\xi \cdot \omega_{\Phi_1}) \wedge \xi$$

holds [12, Remark 2.3]. Denote by $\omega_\alpha$ the nearly parallel $G_2$-structure induced by the Riemannian Killing spinor $\Psi_\alpha = \xi_\alpha \cdot \Psi_0$ ($\alpha = 1, 2, 3$). Then we obtain

$$\omega_\alpha = -\frac{1}{2} (\eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3) - 2 \eta_1 \wedge \eta_2 \wedge \eta_3 + 2(\xi_\alpha \cdot \omega) \wedge \eta_\alpha.$$ 

Consider, for example, the case $\alpha = 1$. Then

$$\xi_1 \cdot \omega = \frac{1}{2} d\eta_1 + \frac{1}{2} (\xi_1 \cdot d\eta_2) \wedge \eta_2 + \frac{1}{2} (\xi_1 \cdot d\eta_3) \wedge \eta_3 + 4 \eta_{23} = \frac{1}{2} d\eta_1 + 2 \eta_{23}.$$ 

Inserting the latter formula, we obtain

$$\omega_1 = \frac{1}{2} \eta_1 \wedge d\eta_1 - \frac{1}{2} \eta_2 \wedge d\eta_2 - \frac{1}{2} \eta_3 \wedge d\eta_3 = \eta_{23} - \eta_{145} - \eta_{167} + \eta_{246} - \eta_{257} + \eta_{347} + \eta_{356}.$$ 

**Theorem 6.2.** The nearly parallel $G_2$-structures $\omega_1, \omega_2, \omega_3$ induced by the Killing spinors of a 3-Sasakian manifold are given by the formulas

$$\omega_1 = \frac{1}{2} \eta_1 \wedge d\eta_1 - \frac{1}{2} \eta_2 \wedge d\eta_2 - \frac{1}{2} \eta_3 \wedge d\eta_3,$$

$$\omega_2 = -\frac{1}{2} \eta_1 \wedge d\eta_1 + \frac{1}{2} \eta_2 \wedge d\eta_2 - \frac{1}{2} \eta_3 \wedge d\eta_3,$$

$$\omega_3 = -\frac{1}{2} \eta_1 \wedge d\eta_1 - \frac{1}{2} \eta_2 \wedge d\eta_2 + \frac{1}{2} \eta_3 \wedge d\eta_3.$$ 

All three nearly parallel $G_2$-structures satisfy the equation $d\omega_\alpha = -4 (\ast \omega_\alpha)$.

**Remark 6.1.** The nearly parallel structures $\omega_\alpha$ admit characteristic connections, too. Their characteristic torsions $T^s_\alpha$ are proportional to $\omega_\alpha$ [3]. Moreover, the existence of a nearly parallel $G_2$-structure or—equivalently—of a Riemannian Killing spinor implies that $M^7$ is Einstein [6]. Consequently, our construction explains why 3-Sasakian manifolds are Einstein manifolds.

7. **Deformations of the canonical $G_2$-structure**

Deformations of 3-Sasakian metrics from the viewpoint of $G_2$-geometry have been studied in [12] and [7]. We once again describe the construction of these particular $G_2$-structures and their properties in a unified way, and add some more. Fix a positive parameter $s > 0$ and consider a new Riemannian metric $g^s$ defined by

$$g^s(X, Y) := g(X, Y) \text{ if } X, Y \in T^h, \quad g^s(X, Y) := s^2 \cdot g(X, Y) \text{ if } X, Y \in T^v.$$ 

Then $s\eta_1, s\eta_2, s\eta_3, \eta_4, \ldots, \eta_7$ is an orthonormal coframe and we replace the 3-forms

$$F_1 = \eta_1 \wedge \eta_2 \wedge \eta_3,$$

$$F_2 = \frac{1}{2} (\eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3) + 3 \eta_1 \wedge \eta_2 \wedge \eta_3$$

$$= -\eta_{145} - \eta_{167} - \eta_{246} + \eta_{257} - \eta_{347} - \eta_{356}.$$
by the new forms
\[ F_1^s := s^3 F_1, \quad F_2^s := s F_2, \quad \omega^s := F_1^s + F_2^s. \]

\((M^7, g^s, \omega^s)\) is a Riemannian 7-manifold equipped with a \(G_2\)-structure \(\omega^s\). Denote by \(*_s\) the corresponding Hodge operator acting on forms. We summarize some well-known properties of these \(G_2\)-structures that follow from a straightforward computation.

**Theorem 7.1** ([12], Theorem 5.4 and [7], §10).

1. The \(G_2\)-manifold \((M^7, g^s, \omega^s)\) is cocalibrated, \(d*_s \omega^s = 0\).
2. The differential of the \(G_2\)-structure is given by the formula
   \[ d\omega^s = 12s (*_s F_1^s) + (2s + 2s)(*_s F_2^s). \]
3. The characteristic torsion \(T^c_s\) is given by the formula
   \[ T^c_s = \left( \frac{2}{s} - 10s \right) (s\eta_1) \land (s\eta_2) \land (s\eta_3) + 2s \omega^s. \]
4. The Riemannian Ricci tensor is given by the formula
   \[ \text{Ric}^g_s = 6 \left( 2 - s^2 \right) \text{Id}_{T^h} \oplus \frac{2 + 4s^4}{s^2} \text{Id}_{T^v}. \]
   In particular, the scalar curvature equals
   \[ \text{Scal}^g_s = 6 \left( 8 + \frac{1}{s^2} - 2s^2 \right). \]
5. The canonical spinor field \(\Psi_0\) satisfies the differential equation
   \[ \nabla^g_s \Psi_0 = -3 \frac{s}{2} X \cdot \Psi_0 \quad \text{if} \quad X \in T^h, \]
   \[ \nabla^g_s \Psi_0 = \left( -\frac{1}{2s} + s \right) X \cdot \Psi_0 \quad \text{if} \quad X \in T^v. \]

**Corollary 7.1** ([12], Theorem 5.4). For \(s = 1/\sqrt{5}\) the \(G_2\)-structure is nearly parallel and \(\Psi_0\) is a Riemannian Killing spinor,
\[ d\omega^s = \frac{12}{\sqrt{5}} (*_s \omega^s), \quad \text{Ric}^g_s = \frac{54}{5} \text{Id}, \quad \nabla^g_s \Psi_0 = -3 \frac{2}{\sqrt{5}} X \cdot \Psi_0. \]

\(\Psi_0\) is the unique Riemannian Killing spinor of the metric.

**Remark 7.1.** The Ricci tensor of the characteristic connection of \((M^7, g^s, \omega^s)\) is given by the formula [7]
\[ \text{Ric}^{\nabla^c} = 12 \left( 1 - s^2 \right) \text{Id}_{T^h} \oplus 16 \left( 1 - 2s^2 \right) \text{Id}_{T^v}. \]
If \(s = 1\) (the 3-Sasakian case), then \(\text{Ric}^{\nabla^c}\) vanishes on the subbundle \(T^h\). For \(s = 1/\sqrt{5}\), the Ricci tensor is proportional to the metric, \(\text{Ric}^{\nabla^c, 1/\sqrt{5}} = (48/5) \text{Id}_{TM^7}\). From this point of view there is a third interesting parameter, namely \(s^2 = 1/2\). Then the \(\nabla^c\)-Ricci tensor vanishes on the subbundle \(T^v\) and the canonical spinor field \(\Psi_0\) is parallel in vertical directions. It is a transversal Killing spinor with respect to the three-dimensional foliation and
\[ (Dg^s)^2 \Psi_0 = 18 \Psi_0 = \frac{1}{44 - 1} \text{Scal}^g s \Psi_0. \]
In particular, \(\Psi_0\) is the first known example to realize the lower bound for the basic Dirac operator of the foliation, see the recent work by Habib and Richardson [13].
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