A note on deformations of moduli spaces of sheaves on K3 surfaces

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Abstract

In this paper we study deformation classes of moduli spaces of sheaves on a projective K3 surface. More precisely, let \((S_1, H_1)\) and \((S_2, H_2)\) be two polarized K3 surfaces, \(m \in \mathbb{N}\), and for \(i = 1, 2\) let \(mv_i\) be a Mukai vector on \(S_i\) such that \(H_i\) is \(mv_i\)-generic. Moreover, suppose that the moduli spaces \(M_{mv_1}(S_1, H_1)\) of \(H_1\)-semistable sheaves on \(S_1\) of Mukai vector \(mv_1\) and \(M_{mv_2}(S_2, H_2)\) of \(H_2\)-semistable sheaves on \(S_2\) with Mukai vector \(mv_2\), have the same dimension. The aim of this paper is to prove that \(M_{mv_1}(S_1, H_1)\) is deformation equivalent to \(M_{mv_2}(S_2, H_2)\), showing a conjecture of Z. Zhang contained in [18].

1 Introduction and notations

Moduli spaces of semistable sheaves on abelian or projective K3 surfaces have been extensively studied since the ’80s, as they are one of the most important tools to produce examples of irreducible symplectic manifolds. In the following, \(S\) will denote a projective K3 surface.

An element \(v \in \tilde{H}(S, \mathbb{Z}) := H^2(S, \mathbb{Z})\) will be written as \(v = (v_0, v_1, v_2)\), where \(v_1 \in H^2(S, \mathbb{Z})\), and \(v_0, v_2 \in \mathbb{Z}\). If \(v_0 \geq 0\) and \(v_1 \in NS(S)\), then \(v\) is called Mukai vector. Recall that \(\tilde{H}(S, \mathbb{Z})\) has a pure weight-two Hodge structure defined as

\[
\tilde{H}^{2,0}(S) := H^{2,0}(S), \quad \tilde{H}^{0,2}(S) := H^{0,2}(S),
\]

\[
\tilde{H}^{1,1}(S) := H^0(S, \mathbb{C}) \oplus H^1(S) \oplus H^4(S, \mathbb{C}),
\]

and a lattice structure with respect to the Mukai pairing \((,\cdot)\). In the following, we let \(v^2 := (v, v)\) for every Mukai vector \(v\).

If \(\mathcal{F}\) is a coherent sheaf on \(S\), we define its Mukai vector to be

\[
v(\mathcal{F}) := ch(\mathcal{F}) \sqrt{td(S)} = (rk(\mathcal{F}), c_1(\mathcal{F}), ch_2(\mathcal{F}) + rk(\mathcal{F})).
\]

Let \(H\) be an ample line bundle on \(S\). For every \(n \in \mathbb{Z}\) and every coherent sheaf \(\mathcal{F}\), let \(\mathcal{F}(nH) := \mathcal{F} \otimes \mathcal{O}_S(nH)\). The Hilbert polynomial of \(\mathcal{F}\) with respect to \(H\) is \(P_H(\mathcal{F})(n) := \chi(\mathcal{F}(nH))\), and the reduced Hilbert polynomial of \(\mathcal{F}\) with respect to \(H\) is

\[
p_H(\mathcal{F}) := \frac{P_H(\mathcal{F})}{\alpha_H(\mathcal{F})},
\]

where \(\alpha_H(\mathcal{F})\) is the coefficient of the term of highest degree in \(P_H(\mathcal{F})\).
Definition 1.1. A coherent sheaf $\mathcal{F}$ is $H$–stable (resp. $H$–semistable) if it is pure and for every proper $\mathcal{E} \subseteq \mathcal{F}$ we have $p_H(\mathcal{E})(n) < p_H(\mathcal{F})(n)$ (resp. $p_H(\mathcal{E})(n) \leq p_H(\mathcal{F})(n)$) for $n \gg 0$.

Let $H$ be a polarization and $v$ a Mukai vector on $S$. We write $M_v(S,H)$ (resp. $M_v^s(S,H)$) for the moduli space of $H$–semistable (resp. $H$–stable) sheaves on $S$ with Mukai vector $v$. If no confusion on $S$ and $H$ is possible, we drop them from the notation.

From now on, we suppose that $H$ is $v$–general (see section 2.1). We write $v = mw$, where $m \in \mathbb{N}$ and $w$ is a primitive Mukai vector on $S$. It is known that if $M^s_v \neq \emptyset$, then $M^s_v$ is a smooth, quasi-projective variety of dimension $v^2 + 2$, which carries a symplectic form (see Mukai [7]). We introduce the following definition:

Definition 1.2. Let $S$ be a projective K3 surface, $v$ a Mukai vector, $H$ an ample line bundle on $S$ and $m, k \in \mathbb{N}$. We say that $(S, v, H)$ is an $(m,k)$–triple if the following conditions are verified:

1. the polarization $H$ is primitive and $v$–generic;
2. we have $v = mw$, where $w$ is a primitive Mukai vector such that $w^2 = 2k$;
3. if $w = (w_0, w_1, w_2)$, we have $w_0 \geq 0$, $w_1 \in NS(S)$, and if $w_0 = 0$ then $w_1$ is the first Chern class of an effective divisor, and $w_2 \neq 0$.

Notice that if $(S, v, H)$ is an $(m,k)$–triple, then the moduli space $M_v(S,H)$ is a normal, irreducible projective variety of dimension $2m^2 k + 2$. In this paper we study the deformation classes of moduli spaces $M_v(S,H)$. Namely, our aim is to show the following:

Theorem 1.3. Let $m, k \in \mathbb{N}$, and let $(S_1, v_1, H_1)$ and $(S_2, v_2, H_2)$ be two $(m,k)$–triples. Then $M_{v_1}(S_1,H_1)$ and $M_{v_2}(S_2,H_2)$ are deformation equivalent.

We add some remarks about this Theorem. If $m = 1$, then the Theorem is true, and the deformation class one obtains is the one of $Hilb^{k+1}(S)$ for some K3 surface $S$: this was shown thanks to the work of several authors (see Mukai [7], Beauville [1], Huybrechts [3], O’Grady [9], Yoshioka [13]).

If $m = 2$ and $k = 1$, then the Theorem is true, as shown in [12] (there, a $(2,1)$–triple is called OLS-triple). Moreover, the deformation class one obtains is the one of the moduli space $M_{10} := M_v(X,H)$, where $X$ is a projective K3 surface such that $Pic(X) = 2 \cdot H$, where $H$ is an ample line bundle such that $H^2 = 2$, and $v = 2(1,0,-1)$; in [10] O’Grady shows that $M_{10}$ admits a symplectic resolution $\tilde{M}_{10}$, which is irreducible symplectic. In [9] it is shown that all the moduli spaces $M_v(S,H)$ admit a symplectic resolutions $\tilde{M}_v(S,H)$; in [12] it is shown that $\tilde{M}_v(S,H)$ is irreducible symplectic and deformation equivalent to $\tilde{M}_{10}$.

In conclusion, we just need to show the Theorem only under the hypothesis that $m = 2$ and $k \geq 2$, or that $m \geq 3$. This was conjectured to be true by Z. Zhang in [18], in which the author proves that the symplectic resolution $\tilde{M}_v(S,H)$ of $M_v(S,H)$ for a $(2,1)$–triple $(S,v,H)$ is an irreducible symplectic manifold which is deformation equivalent to $M_{10}$: the proof of [18] is different from the one of [12] as it uses birational transformations and the results of [4].
Anyway, due to the lack of the results of [4] in the singular case, Zhang could not give a complete proof of the conjecture.

The aim of this paper is to prove Theorem [3, Theorem 1.3] we use the same arguments of [12], namely deformations of the moduli spaces induced by deformations of the underlying surfaces, and isomorphisms between moduli spaces with different Mukai vectors which are induced by Fourier-Mukai transforms (the main ingredient here is given by some results of [15]). As shown in [12], this method allows us to show Theorem [3, Theorem 1.3] holds true even when \( v = (0, v_1, 0) \), and for more general polarizations.

Finally, notice that in [12] we consider only the case \( m = 2 \) and \( k = 1 \): hence the Mukai vector is of the form \( v = 2(r, \xi, a) \), and as \( k = 1 \) it is easy to see that \( r \) and \( \xi \) are prime to each other (see [12]). In the more general situation, anyway, we have that \( k \geq 1 \), so that we no longer have that \( r \) is prime to \( \xi \): this introduces some difficulties that need to be solved, but we show that using some deformation equivalence one can always reduce to the case of \( r \) and \( \xi \) coprime.

## 2 Deformations of moduli spaces

In this section we study how moduli spaces vary under deformation. In section 2.1 we recall the notions of \( v \)-walls and \( v \)-chambers. In section 2.2 we introduce the main deformation we will look at, i.e. the deformation of a moduli space induced by a deformation of an \((m, k)\)-triple along a smooth, connected curve. Finally, in section 2.3 we give explicit deformations of \((m, k)\)-triples whose Mukai vector has positive rank.

### 2.1 Walls and chambers

In this section we recall the notion of walls and chambers associated to a Mukai vector, and the notion of \( v \)-genericity. We need to consider separately the case of positive rank and the case of rank 0. In the following, \( S \) will always denote a projective K3 surface, and \( v = (v_0, v_1, v_2) \) a Mukai vector on \( S \).

#### 2.1.1 Walls and chambers for \( v_0 \geq 2 \)

Suppose \( v_0 \geq 2 \). Let

\[
|v| := \frac{v_0^2}{4}(v, v) + \frac{v_0^4}{2}.
\]

Notice that \(|v|\) depends only on \((v, v)\) and \(v_0\), and as \(v_0 \geq 2\), then \(|v| \geq 1\) (this is true even if \(v_0 = 1\) and \(v^2 \geq 0\)). Hence it makes sense to define

\[
W_v := \{D \in NS(S) \mid -|v| \leq D^2 < 0\}.
\]

**Definition 2.1.** Let \( D \in W_v \). The \( v \)-wall associated to \( D \) is

\[
W^D := \{\alpha \in Amp(S) \mid D \cdot \alpha = 0\}.
\]

Notice that the \( v \)-wall associated to \( D \in W_v \) is a hyperplane in \( Amp(S) \). Moreover, if \( p(S) = 1 \), the generator \( H \) of \( NS(S) \) does not lie on any \( v \)-wall. By Theorem 4.C.2 of [5] the subset \( \bigcup_{D \in W_v} W^D \subset Amp(S) \) is locally finite.
Definition 2.2. Suppose that $\rho(S) \geq 2$. A connected component of the open subset $\text{Amp}(S) \setminus \bigcup_{D \in W_v} W^D$ of $\text{Amp}(S)$ is called $v$–chamber.

2.1.2 Walls and chambers for $v_0 = 0$

First, let $v = (0, v_1, v_2)$ be a Mukai vector on a projective K3 surface $S$. Suppose that $v_2 \neq 0$, and we give the following:

Definition 2.3. Let $E$ be any pure sheaf with Mukai vector $v$, and let $F \subseteq E$ with Mukai vector $u = (0, u_1, u_2)$. The divisor associated to the pair $(E, F)$ is $D := u_2v_1 - v_2u_1$. The set of the non-zero divisors associated to all the possible pairs is denoted $W_v$.

As before, we associate to any element of $W_v$ a hyperplane in the ample cone $\text{con}_S$:

Definition 2.4. Let $D \in W_v$. The $v$–wall associated to $D$ is

$$W^D := \{ \alpha \in \text{Amp}(S) \mid \alpha \cdot D = 0 \}.$$ 

As shown in [14], the set of $v$–walls is finite. Moreover, if $\rho(S) = 1$, the generating divisor $H$ of $\text{NS}(S)$ is not on any $v$–wall.

Definition 2.5. Suppose that $\rho(S) \geq 2$. A connected component of the open subset $\text{Amp}(S) \setminus \bigcup_{D \in W_v} W^D$ of $\text{Amp}(S)$ is called $v$–chamber.

2.1.3 The notion of $v$–generic polarization

We finally give the notion of $v$–generic polarization on a projective K3 surface $S$. Let $v = (v_0, v_1, v_2)$ be a Mukai vector such that wither $v_0 > 0$ or that $v_0 = 0$ and $v_2 \neq 0$.

Definition 2.6. Let $H$ be an ample line bundle on $S$.

1. If $v_0 > 0$, then $H$ is $v$–generic if it does not lie on any $v$–wall.

2. If $v_0 = 0$, then $H$ is $v$–generic if for every $H$–polystable sheaf $E$ of Mukai vector $v$ and for every direct summand $F$ of $E$, we have that $v(F) \in \mathbb{Q} \cdot v$.

First of all, we recall the following (see [12]), allowing us to describe the strictly semistable locus of the moduli space $M_v(S, H)$ when $H$ is $v$–generic:

Lemma 2.7. Let $v = (v_0, v_1, v_2)$ be a Mukai vector on a projective K3 surface $S$ such that either $v_0 \geq 2$ or $v_0 = 0$ and $v_2 \neq 0$. Let $H$ be a polarization $S$ which does not lie on any $v$–wall.

1. If $v_0 \geq 2$, then for every $H$–polystable sheaf $E$ of Mukai vector $v$ and for every direct summand $F$ of $E$, we have that $v(F) \in \mathbb{Q} \cdot v$.

2. If $v_0 = 0$ and $v_2 \neq 0$, then $H$ is $v$–generic.

Another important result is the following, showing that we can change a polarization in a $v$–chamber without changing the moduli space (for a proof, see [12] or [19]):
Proposition 2.8. Suppose that \( \rho(S) \geq 2 \) and that \( v = (v_0, v_1, v_2) \) is such that either \( v_0 \geq 2 \) or \( v_0 = 0 \) and \( v_2 \neq 0 \). Let \( C \) be a \( v \)-chamber.

1. If \( v_0 \geq 2 \) and \( H, H' \in C \), then any sheaf \( \mathcal{E} \) with Mukai vector \( v \) is \( H \)-(semi)stable if and only if it is \( H' \)-(semi)stable, i.e. there is a natural identification between \( M_v(S, H) \) and \( M_v(S, H') \).

2. If \( v_0 = 0 \), \( v_2 \neq 0 \) and \( H, H' \in \overline{C} \) are two \( v \)-generic polarizations, then a sheaf \( \mathcal{E} \) of Mukai vector \( v \) is \( H \)-(semi)stable if and only if it is \( H' \)-(semi)stable, i.e. there is a natural identification between \( M_v(S, H) \) and \( M_v(S, H') \).

We conclude this section with an important property that we will need in the following (see [12] or [18]):

Lemma 2.9. Let \( S \) be a projective K3 surface and \( v = (v_0, v_1, v_2) \) a Mukai vector such that \( v_0 \geq 2 \). The property \( \mathcal{P} := " \text{to be } v \text{-generic} " \) is open.

2.2 Deformations of \((m, k)\)-triples

We introduce the main construction we use in the following. Let \((S, v, H)\) be an \((m, k)\)-triple and \( T \) a smooth, connected curve, and use the following notation: if \( f : Y \to T \) is a morphism and \( \mathcal{L} \in \text{Pic}(Y) \), for every \( t \in T \) we let \( Y_t := f^{-1}(t) \) and \( \mathcal{L}_t := \mathcal{L}|_{Y_t} \).

Definition 2.10. Let \((S, v, H)\) be an \((m, k)\)-triple, where \( v = (r, \xi, a) \) and \( \xi = c_1(L) \). A deformation of \((S, v, H)\) along \( T \) is a triple \((\mathcal{X}, \mathcal{H}, \mathcal{L})\), where:

1. \( \mathcal{X} \) is a projective, smooth deformation of \( S \) along \( T \), i.e. there is a smooth, projective, surjective map \( f : \mathcal{X} \to T \) such that \( \mathcal{X}_t \) is a projective K3 surface for every \( t \in T \), and there is \( 0 \in T \) such that \( \mathcal{X}_0 \simeq S \);

2. \( \mathcal{H} \) is a line bundle on \( \mathcal{X} \) such that \( \mathcal{H}_t \) is ample for every \( t \in T \) and such that \( \mathcal{H}_0 \simeq H \);

3. \( \mathcal{L} \) is a line bundle on \( \mathcal{X} \) such that \( \mathcal{L}_0 \simeq L \);

4. if for every \( t \in T \) we let \( \xi_t := c_1(\mathcal{L}_t) \), \( w_t := (r, \xi_t, a) \) and \( v_t := mw_t \), then we ask that \( \mathcal{H}_t \) is \( v_t \)-generic.

Remark 2.11. Notice that if \((S, v, H)\) is an \((m, k)\)-triple and \((\mathcal{X}, \mathcal{L}, \mathcal{H})\) is a deformation of \((S, v, H)\) along a smooth, connected curve \( T \), then \((\mathcal{X}_t, v_t, \mathcal{H}_t)\) is an \((m, k)\)-triple for every \( t \in T \). Indeed, we have \( v_t = mw_t \), where \( w_t \) is primitive and \( w_t^2 = 2k \). Moreover, if \( r = 0 \), then \( \xi_t \) is effective: we have \( \xi_t^2 = 2k \), hence either \( \xi_t \) or \( -\xi_t \) is effective; as \( \xi \) is effective, then \( -\xi \cdot H < 0 \), so that \( -\xi_t \cdot H_t < 0 \), hence \( \xi_t \) is effective.

Remark 2.12. Consider an \((m, k)\)-triple \((S, v, H)\) where \( v = (r, \xi, a) \) and \( \xi = c_1(L) \). Let \( T \) be a smooth, connected curve. Moreover, consider a smooth, projective deformation \( f : \mathcal{X} \to T \) of \( S \) such that \( \mathcal{X}_0 \simeq S \), and on \( \mathcal{X} \) consider two line bundles \( \mathcal{H} \) and \( \mathcal{L} \) such that \( \mathcal{H}_0 \simeq H \) and \( \mathcal{L}_0 \simeq L \). In general \((\mathcal{X}_t, v_t, \mathcal{H}_t)\) is not an \((m, k)\)-triple: by Remark 2.11 this is the case if and only if \( \mathcal{H}_t \) is ample and \( v_t \)-generic for every \( t \in T \). Up to removing a finite number of points from \( T \), we can always assume that this is the case. Indeed, the set
of \( t \in T \) such that \( \mathcal{H}_t \) is not ample is finite. Moreover, if \( r > 0 \) by Lemma \ref{lem:proposition}, we know that the property of being \( v \)-generic is open, so that there is at most a finite number of \( t \in T \) such that \( \mathcal{H}_t \) is not \( v_t \)-generic, and we are done. If \( r = 0 \), we apply the following Lemma.

**Lemma 2.13.** Let \((S, v, H)\) be an \((m, k)\)-triple where \( v = m(0, \xi, a) \), and let \( \xi = c_1(L) \). Let \( T \) be a smooth, connected curve, \( f: \mathcal{X} \to T \) a smooth, projective deformation of \( S \) such that \( \mathcal{X}_0 \simeq S \), \( \mathcal{H} \in \text{Pic}(\mathcal{X}) \) such that \( \mathcal{H}_t \) is ample for every \( t \in T \) and \( \mathcal{H}_0 \simeq H \), and \( \mathcal{L} \in \text{Pic}(\mathcal{X}) \) such that \( \mathcal{L}_0 \simeq L \). Then the set
\[
T' := \{ t \in T | \mathcal{H}_t \text{ is not } v_t \text{ - generic} \}
\]
is finite.

**Proof.** Consider the relative moduli space of semistable (resp. stable) sheaves
\[
\phi: \mathcal{M} \to T, \quad (\text{resp. } \phi^s: \mathcal{M}^s \to T)
\]
associated to \( f \), where \( \phi \) (resp. \( \phi^s \)) is a projective (resp. quasi-projective) map, such that \( \mathcal{M}_t = M_v(\mathcal{X}_t, \mathcal{H}_t) \) (resp. \( \mathcal{M}^s_t = M^s_v(\mathcal{X}_t, \mathcal{H}_t) \)) for every \( t \in T \). By Theorem 4.3.7 in \[5\], \( \mathcal{M}^s \) is an open subset of \( \mathcal{M} \). Let \( \mathcal{M}^{ss} := \mathcal{M} \setminus \mathcal{M}^s \), which is a closed subset of \( \mathcal{M} \). By definition, \( \mathcal{M}^{ss} \) parameterizes strictly \( \mathcal{H}_t \)-polystable sheaves of Mukai vector \( v_t \) for every \( t \in T \). There is an irreducible component \( \Sigma \) of \( \mathcal{M}^{ss} \) such that \( \Sigma \cap \mathcal{M}_t \) parameterizes strictly \( \mathcal{H}_t \)-polystable sheaves whose direct summands have Mukai vector \( v_t \) for every \( t \in T \). Let \( \Xi \) be the union of all the other irreducible components of \( \mathcal{M}^{ss} \). By definition of \( v \)-genericity we have \( T' = \phi(\Xi) \). As \( \Xi \) is closed in \( \mathcal{M} \) and the morphism \( \phi \) is projective, then \( \phi(\Xi) \) is a closed subset of \( T \). Moreover, notice that as \((\mathcal{X}_0, v_0, \mathcal{H}_0) = (S, v, H) \) is an \((m, k)\)-triple, then \( \mathcal{H}_0 \simeq v_0 \)-generic. Then \( 0 \not\in T' \), and \( T' \) is a proper closed subset of \( T \).

\[\square\]

The reason why we introduce the notion of deformation of an \((m, k)\)-triple, is because it allows us to study how the algebraic structure of the corresponding moduli space varies under variations of the algebraic structure of the base surface. Indeed, we have the following:

**Lemma 2.14.** Let \((S, v, H)\) be an \((m, k)\)-triple, \( T \) a smooth, connected curve, and \((\mathcal{X}, \mathcal{L}, \mathcal{H})\) a deformation of \((S, v, H)\) along \( T \). Then the corresponding relative moduli space \( \phi: \mathcal{M} \to T \) is flat.

**Proof.** Let \( t \in T \), \( T^0 := T \setminus \{t\} \) and \( \mathcal{M}^0 := \phi^{-1}(T^0) \). The morphism \( \phi \) is flat over \( t \) if and only if the fiber \( \mathcal{M}_t \) is the limit of the fibers \( \mathcal{M}_s \) as \( s \to t \), by Lemma II-29 of \[2\]. Now, the limit is the fiber over \( t \) of the closure of the family \( \mathcal{M}^0 \), hence there is an inclusion of the limit in \( \mathcal{M}_t \). Recall that \( \mathcal{M}_t = M_v(\mathcal{X}_t, \mathcal{H}_t) \) is reduced and irreducible, hence it has to coincide with the previous limit.

\[\square\]

In conclusion, if \((S, v, H)\) is an \((m, k)\)-triple, then choosing a non-trivial deformation of it along a smooth, connected curve \( T \) we get a flat, projective deformation \( \phi: \mathcal{M} \to T \) of \( M_v(S, H) \).
2.3 Deformations and Mukai vectors with $v_0 > 0$

In this section we consider $(m, k)$-triples with Mukai vector $v = m(r, \xi, a)$ such that $r > 0$, and we show that the deformation class of $M_v$ depends only on the rank of $r$ and on $g := \gcd(r, \xi)$. To do so, we follow closely the arguments used by O'Grady in [3].

2.3.1 Isomorphism induced by tensoring with a line bundle

As first step, we remark that the tensorization via a line bundle does not change the moduli spaces. Let $S$ be a projective K3 surface.

**Definition 2.15.** Let $v, v' \in \tilde{H}(S, \mathbb{Z})$ be two Mukai vectors, $v = (v_0, v_1, v_2)$, $v' = (v'_0, v'_1, v'_2)$ and $v_0, v'_0 > 0$. We say that $v$ and $v'$ are equivalent if there is a line bundle $L$ on $S$ such that $v' = v \cdot \text{ch}(L)$.

If $(S, v, H)$ and $(S, v', H)$ are two $(m, k)$-triples such that $v' = v \cdot \text{ch}(L)$ for some line bundle $L \in \text{Pic}(S)$, then the tensorization with $L$ defines an isomorphism between $M_v(S, H)$ and $M_v'(S, H)$. This is due to the following, which is Lemma 1.1 of [13]:

**Lemma 2.16.** Let $v$ be a Mukai vector of positive rank on a projective K3 surface $S$, $H$ a $v$-generic polarization and $v' = v \cdot \text{ch}(L)$, where $L \in \text{Pic}(S)$. Then the tensorization with $L$ gives an isomorphism between $M_v(S, H)$ and $M_{v'}(S, H)$.

**Remark 2.17.** This Lemma is originally stated only for stable sheaves, but the argument goes through for semistable sheaves.

In order to give explicit deformations of an $(m, k)$-triple $(S, v, H)$ where $v = m(r, \xi, a)$ is such that $r > 0$, the main idea is to use deformations of the polarized K3 surface $(S, H)$. Hence, it is useful to suppose that $\xi$ is a multiple of the polarization, which is always possible by the following:

**Lemma 2.18.** Let $(S, v, H)$ be an $(m, k)$-triple where $v = m(r, \xi, a)$ is such that $r > 0$, and let $g := \gcd(r, \xi)$. Suppose that $\rho(S) \geq 2$, and let $\mathcal{C}$ be the $v$-chamber such that $H \in \mathcal{C}$. Then there exists a Mukai vector $v' = m(r, \xi', a')$ such that

1. $v'$ is equivalent to $v$;
2. there is a primitive ample line bundle $H' \in \mathcal{C}$ such that $\xi' = gc_1(H')$.

Moreover, we can choose $v'$ so that $(\xi')^2 \gg 0$.

**Proof.** This is a generalization of Lemma II.6 of [3]. Let $\rho := \rho(S)$ be the Picard number of $S$, and let $\{h_1, \ldots, h_\rho\}$ be a basis of $NS(S)$ such that $h_i \in \mathcal{C}$ for $i = 1, \ldots, \rho$ (this is possible as $\mathcal{C}$ is open in $NS(S)$). There is then a line bundle $L' \in \text{Pic}(S)$ such that

$$\xi = \sum_{i=1}^\rho \mu_i h_i + rc_1(L'),$$
where $\mu, a_1, \ldots, a_\rho \in \mathbb{N}$ and $\gcd(a_1, \ldots, a_\rho) = 1$. As $g = \gcd(r, \xi)$, we have $g = \gcd(\mu, r)$. Write $\mu = g\mu'$ and $r = gr'$, where $\mu', r' \in \mathbb{N}$. Finally, define $H' \in \text{Pic}(S)$ to be such that

$$c_1(H') = \sum_{i=1}^{\rho-1} (\mu' a_i + r' n_i) h_i + \mu a_\rho h_\rho,$$

where we have

1. $n_1, \ldots, n_{\rho-1} \in \mathbb{N}$,
2. $\gcd(n_1, \ldots, n_{\rho-1}) = 1$,
3. if $p$ is a prime number dividing $a_\rho$ but not $\mu'$, then $p$ divides every $n_i$.

It is then easy to see that $H'$ is a primitive line bundle, and as every coefficient of $c_1(H')$ is positive, we have that $H' \in \mathcal{C}$. In conclusion, $H'$ is a $v$--generic polarization. Let $L \in \text{Pic}(S)$ be such that

$$c_1(L) = \sum_{i=1}^{\rho} n_i h_i.$$

We then see easily that $\xi + r c_1(L \otimes (L')^{-1}) = gc_1(H')$. Finally, let

$$w' := w \cdot ch(L \otimes (L')^{-1}) = (r, gc_1(H'), a').$$

Then $w'$ is primitive, $v' := mw'$ is equivalent to $v$, and as we can choose $n_1, \ldots, n_\rho \gg 0$, we are done. \hfill \square

### 2.3.2 Deformation to an elliptic K3 surface

An important class of $(m, k)$--triples is given by those on elliptic K3 surfaces, as in this case we have a privileged class of polarizations, called $v$--suitable. We want to prove that if $(S, v, H)$ is an $(m, k)$--triple, where $v = m(r, \xi, a)$ and $r > 0$, then the deformation class of $M_k(S, H)$ depends only on $r$ and on $g := \gcd(r, \xi)$; the strategy will be to deform the $(m, k)$--triple $(S, v, H)$ to an $(m, k)$--triple on an elliptic K3 surface with a $v$--suitable polarization.

Let then $Y$ be an elliptic K3 surface such that $\text{NS}(Y) = \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \sigma$, where $f$ is the class of a fiber and $\sigma$ is the class of a section. Let $v$ be a Mukai vector on $Y$, and recall the following definition (see [9]):

**Definition 2.19.** A polarization $H$ on $Y$ is called $v$--suitable if $H$ is in the unique $v$--chamber whose closure contains $f$.

We have an easy numerical criterion to guarantee that a polarization on $Y$ is $v$--suitable (see Lemma I.0.3 of [9]):

**Lemma 2.20.** Let $Y$ be an elliptic K3 surface such that $\text{NS}(Y) = \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \sigma$, where $\sigma$ is a section and $f$ is a fibre, and let $v = (v_0, v_1, v_2)$ be a Mukai vector on $Y$ such that $v_0 > 0$. Let $H$ be a polarization such that $c_1(H) = \sigma + tf$ for some $t \in \mathbb{Z}$. Then $H$ is $v$--suitable if $t \geq |v| + 1$.

The main result of this section is the following:
**Proposition 2.21.** Let \((S_1, v_1, H_1)\) and \((S_2, v_2, H_2)\) be two \((m, k)\)-triples. Let \(v_i = m(r, \xi_i, a_i)\) for \(i = 1, 2\), and suppose that the following conditions are verified:

1. \(r_1 = r_2 =: r > 0\);
2. \(\gcd(r, \xi_1) = \gcd(r, \xi_2) =: g\);
3. \(a_1 \equiv a_2 \mod g\).

Then \(M_{v_1}(S_1, H_1)\) is deformation equivalent to \(M_{v_2}(S_2, H_2)\).

**Proof.** The argument we present here was first used by O’Grady in [9] and by Yoshioka in [13] for primitive Mukai vectors, and by the author and Rapagnetta in [12] in the case of \(m = 2\) and \(k = 1\). First, we can always assume \(p(S_i) > 1\). Indeed, consider a non-trivial smooth, projective deformation \(\mathcal{X}_i\) of \(S_i\) along an open 1-dimensional disc \(\Delta\), and let \(0 \in \Delta\) be such that \(\mathcal{X}_{i,0} \simeq S_i\). By the Main Theorem of [11], we know that the locus of \(t \in \Delta\) such that \(\rho(\mathcal{X}_{i,t}) > 1\) is dense in the classical topology of \(\Delta\). If \(\mathcal{H}_i \in \text{Pic}(\mathcal{X}_i)\) is a deformation of \(H_i\) and \(\mathcal{L}_i \in \text{Pic}(\mathcal{X}_i)\) is a deformation of the line bundle \(L_i \in \text{Pic}(S_i)\) such that \(c_1(L_i) = \xi_i\), then the triple \((\mathcal{X}_{i,t}, v_{i,t}, \mathcal{H}_{i,t})\) is an \((m, k)\)-triple for all but a finite number of \(t \in \Delta\) (see Remark 2.12): hence there is \(t \in \Delta\) such that \(\rho(\mathcal{X}_{i,t}) > 1\) and \((\mathcal{X}_{i,t}, v_{i,t}, \mathcal{H}_{i,t})\) is an \((m, k)\)-triple.

By Lemma 2.18 and Proposition 2.8 we suppose the triples to be \((S_i, v_i, H_i)\) where \(v_i = m(r, gc_1(H_i), a_i)\), and if \(H_i^2 = 2d_i\), then we suppose \(d_i \gg 0\). Moreover, write \(r = gr'\) for \(r' \in \mathbb{N}\). Now, let \(Y\) be a K3 surface admitting an elliptic fibration and such that

\[
NS(Y) = \mathbb{Z} \cdot \sigma \oplus \mathbb{Z} \cdot f,
\]

where \(f\) is the class of a fiber, and \(\sigma\) is the class of a section. For \(i = 1, 2\), there is a smooth, connected curve \(T_i\) and a deformation \((\mathcal{X}_i, \mathcal{L}_i, \mathcal{H}_i)\) over \(T_i\) of the \((m, k)\)-triple \((S_i, v_i, H_i)\) such that there is \(t \in T_i\) with the property \((\mathcal{X}_{i,t}, v_{i,t}, H_{i,t}) = (Y, v'_i, H'_i)\), where

1. \(c_1(H'_i) = \sigma + p_if\), where \(p_i = d_i + 1 \gg 0\).
2. \(v'_i = m(r, gc_1(H'_i), a_i)\).

Let \(\xi'_i := gc_1(H'_i)\). Notice that \((v'_i)^2 = (v'_i)^2\) and they have the same rank: hence \(|v'_i| = |v'_i|\), so that by Lemma 2.20 a polarization is \(v'_i\)-suitable if and only if it is \(v'_i\)-suitable. Again by Lemma 2.20 we have that \(H'_i\) is \(v'_i\)-suitable for \(i = 1, 2\), as \(p_i \gg 0\). Then \(H'_1\) and \(H'_2\) are in the same chamber \(\mathcal{C}\) (which is clearly a \(v'_i\)-chamber for \(i = 1, 2\)). By Proposition 2.8 we then change to a common generic polarization \(H \in \mathcal{C}\), which is \(v_1\)-generic for \(i = 1, 2\).

As \((v'_i)^2 = (v'_i)^2\), we have \((\xi'_i)^2 - 2ra_1 = (\xi'_i)^2 - 2ra_2\), and as

\[
(\xi'_i)^2 = g^2(\sigma + p_if)^2 = 2g^2(p_i - 1),
\]

we then get the equation

\[
gp_1 = gp_2 + r'(a_1 - a_2).
\]  

(1)

Notice that \(v'_1\) and \(v'_2\) are then equivalent: indeed, by hypothesis we have that \(a_1 - a_2 = lg\) for some \(l \in \mathbb{Z}\), so that

\[
v'_2 \cdot ch(\mathcal{O}_Y(lf)) = m(r, g(\sigma + p_2f), a_2) \cdot (1, lf, 0) =
\]


\[ m(r, g(r + p_1 f), a_1) = v'_1, \]

where the second equality follows from equation (1) and \( r = g r' \). By Lemma 2.16 we are then done.

**Remark 2.22.** We observe that in order to relate \( M_{v_1}(S_1, H_1) \) and \( M_{v_2}(S_2, H_2) \) in the previous proof, we only used deformations of \((m, k)\)-triples along a smooth, connected curve, and isomorphisms between moduli spaces given by tensorization with a line bundle.

### 3 Proof of Theorem 1.3

In this section we finally prove Theorem 1.3: first, we recall two crucial facts, coming from two lemmas due to Yoshioka [15]. Then, we use them to prove Theorem 1.3. In the following, if \( S \) is a projective K3 surface, write \( \Delta \) for the diagonal of \( S \times S \) and \( I_\Delta \) for the ideal sheaf of \( \Delta \).

#### 3.1 Yoshioka’s results

Before proving Theorem 1.3, we need to recall two basic facts originally due to Yoshioka, which will be fundamental in the proof. First, we need the following:

**Remark 3.1.** Let \( (S, v, H) \) be any \((m, k)\)-triple. Hence \( v = mw \), where \( w \) is a primitive Mukai vector such that \( w^2 = 2k \), and we know that \( H \) is \( v \)-generic. Then we know that if \( E \) is any \( H \)-polystable sheaf of Mukai vector \( v \), and \( F \) is a direct summand of \( E \), then \( v(F) \in \mathbb{Q} \cdot v \) (see Lemma 2.7 in the case of positive rank). This property allows us to study the singular locus of \( M_v(S, H) \).

Indeed, if \( m = 1 \), then \( v(F) = v \) (as \( v \) is primitive), so that every \( H \)-polystable sheaf is \( H \)-stable, and \( M_v(S, H) \) is smooth. If \( m > 1 \), then \( v(F) = pw \) for some \( 1 \leq p \leq m \), so that \( F \in M^{sp}_w(S, H) \). Namely, \( M_v(S, H) \) is described in the following manner: let \( \Pi(m) \) be the set of the partitions of \( m \). Let \( \pi := (p_1, \ldots, p_m) \in \Pi(m) \), and define

\[
M_\pi(w) := S^{p_1}(M_w) \times S^{p_2}(M_{2w}^a) \times \cdots \times S^{p_m}(M_{mw}^a).
\]

Hence, we have that

\[
M_v(S, H) = \bigsqcup_{\pi \in \Pi(m)} M_\pi(w).
\]

**Example 3.2.** We give here some examples of descriptions of the singular locus of \( M_v(S, H) \) for \((S, v, H)\) an \((m, k)\)-triple. If \( m = 1 \), we already know that \( M_v = M_v^a \). If \( m = 2 \), then \( \Pi(2) = \{(2, 0), (0, 1)\} \), so that \( M_v = M_v^a \amalg S^2(M_w) \).

If \( m = 3 \), then \( \Pi(3) = \{(3, 0, 0), (1, 1, 0), (0, 0, 1)\} \), so that

\[
M_v(S, H) = M_v^a \amalg S^4(M_w) \amalg (M_w \times M_{2w}^a).
\]

The previous remark will be used to prove the following:
Lemma 3.3. Let \((S, v, H)\) be an \((m, k)\)-triple where \(v = mw, w = (0, \xi, a)\). Let \(\tilde{\omega} := (\alpha, \xi, 0)\) and \(\hat{\omega} := m \tilde{\omega}\), and suppose \(a \gg 0\). Then the Fourier-Mukai transform \(\mathcal{F} : D^b(S) \longrightarrow D^b(S)\) with kernel \(\mathcal{I}_\Delta\) sends any \(H\)-(semi)stable sheaf with Mukai vector \(v\) to an \(H\)-(semi)stable sheaf with Mukai vector \(\hat{\omega}\). In particular, it defines an isomorphism between \(M_v(S, H)\) and \(M_{\hat{\omega}}(S, H)\).

Proof. It is immediate to see that \((S, \hat{\omega}, H)\) is an \((m, k)\)-triple. Now, recall by Remark 3.1 that
\[
M_v = \prod_{\pi \in \Pi(m)} M_\pi(w), \quad M_{\hat{\omega}} = \prod_{\bar{\pi} \in \Pi(m)} M_\bar{\pi}(\hat{\omega})
\]
where if \(\pi = (p_1, ..., p_m)\) is an element of \(\Pi(m)\), we have that
\[
M_\pi(w) = S^{p_1}(M_w) \times S^{p_2}(M_w^2) \times ... \times S^{p_m}(M_w^m),
\]
and similarly for \(\hat{\omega}\). As \(a \gg 0\), by Proposition 3.14 of [15] we have that \(\mathcal{F}\) sends every \(H\)–stable sheaf of Mukai vector \(pw\) to an \(H\)–stable sheaf of Mukai vector \(p\hat{\omega}\) for every \(p \in \mathbb{N}\). Hence \(\mathcal{F}\) defines an isomorphism between \(M_v(w)\) and \(M_{\hat{\omega}}(\bar{\omega})\) for every \(\pi \in \Pi(m)\). In conclusion, the functor \(\mathcal{F}\) defines an isomorphism between \(M_v\) and \(M_{\hat{\omega}}\).

The following lemma is Theorem 3.18 of [15]:

Lemma 3.4. (Yoshioka). Let \(S\) be a projective K3 surface with \(NS(S) = \mathbb{Z} \cdot h\), where \(h = c_1(H)\) is ample and \(h^2 = 2l\). Let \(n, r \in \mathbb{N}\) be such that \(a := (n^2 - k)/r \in \mathbb{Z}\), and suppose that \(n \gg 0\). The Fourier-Mukai transform \(\mathcal{F} : D^b(S) \longrightarrow D^b(S)\) with kernel \(\mathcal{I}_\Delta\) sends \(H\)-(semi)stable sheaves with Mukai vector \(m(r, nh, a)\) to \(H\)-(semi)stable sheaves with Mukai vector \(m(a, nh, r)\). In particular, it defines an isomorphism between \(M_{m(r, nh, a)}(S, H)\) and \(M_{m(a, nh, r)}(S, H)\).

3.2 Conclusion of the proof

We now proceed with the proof of Theorem 1.3. Before, we prove the following:

Lemma 3.5. Let \((S, v, H)\) be an \((m, k)\)-triple, where \(S\) is a projective K3 surface such that \(\text{Pic}(S) = \mathbb{Z} \cdot H\), and \(v = m(r, nh, a)\), where \(h := c_1(H)\). For every \(s \in \mathbb{Z}\), let
\[
v_s := v \cdot ch(G_S(sH)) = m(r, nh, a_s).
\]
Then there is \(s \in \mathbb{Z}\) such that \(n_s \gg 0\) and \(\gcd(n_s, a_s) = 1\).

Proof. Write \(H^2 = 2l\). It is easy to see that we have
\[
n_s = n + rs = n_{s-1} + r
\]
and
\[
a_s = a + 2lns + rls^2 = a_{s-1} + 2lns + 1 + r.
\]
Moreover, recall that the Mukai vector \(w_s := (r, n_s h, a_s)\) verifies the condition \(w_s^2 = 2k\), so that we have
\[
k(n_s^2 - 1) = ra_s - (l - k)n_s^2
\]
In the following, we let \( P(k) \) be the set of the prime numbers dividing \( k \). Moreover, for every \( s \in \mathbb{N} \) we let \( P_s \) be the set of the prime numbers dividing both \( n_s \) and \( a_s \). We show that for every \( s \) there is \( q \geq 0 \) such that \( P_{s+q} = \emptyset \), i.e., such that \( n_{s+q} \) is prime with \( a_{s+q} \), so that we are done.

If \( P_s = \emptyset \), we are done. Otherwise, consider any \( p \in P_s \); then, by equation (4) we see that \( p \) divides \( k(n_s^2 - 1) \). Notice that as \( p \) divides \( n_s \) it surely does not divide \( n_s^2 - 1 \), hence it has to divide \( k \). This means that for every \( s \in \mathbb{Z} \) we have \( P_s \subseteq P(k) \). Clearly, the same argument gives that \( P_{s+q} \subseteq P(k) \) for every \( q \in \mathbb{N} \). Let then

\[
P_s(k) := \bigcup_{q \geq 0} P_{s+q} \subseteq P(k).
\]

Moreover, let \( P_s(k) = \{p_1, ..., p_t\} \).

Now, let \( q \in \mathbb{N} \), and let \( p \in P_s \cap P_{s+q} \). By equation (2) we see that

\[
n_{s+q} = n_s + qr,
\]

hence as \( p \) divides \( n_s \) and \( n_{s+q} \) we have that \( p \) divides \( rq \). If \( p \) divides \( r \), then by equation (2) we see that \( p \) divides \( n_s \), so that by equation (4) it has even to divide \( a \). Then \( p \) has to divide the Mukai vector \( w = (r, nh, a) \), which is not possible as \( w \) is primitive. In conclusion, we see that \( p \) does not divide \( r \).

We then have that if \( p \in P_s \cap P_{s+q} \), then \( p \) divides \( q \). More generally, in the same way one shows that if \( p \in P_{s+q} \cap P_{s+q+q'} \), then \( p \) has to divide \( q' \). As a particular case, we have that \( P_s \cap P_{s+1} = \emptyset \).

We have now several cases.

1. The second case is when \( P_s = P_s(k) \); then \( P_{s+1} = \emptyset \), and we are done.

2. The second case is when \( P_s = \{p_1, ..., p_a\} \) and \( P_{s+1} = \{p_{a+1}, ..., p_t\} \), so that \( P_s(k) = P_s \cup P_{s+1} \). Consider \( q > 0 \), and let \( p \in P_{s+q} \). Then \( p \in P_s(k) \), hence either \( p \in P_s \cap P_{s+q} \) or \( p \in P_{s+1} \cap P_{s+q} \). In any case, we see that either there is \( i \in \{1, ..., a\} \) such that \( p_i \) divides \( q \), or there is \( i \in \{a+1, ..., t\} \) such that \( p_i \) divides \( q - 1 \). Consider now \( q = p_{a+1} \cdot \ldots \cdot p_t \): then for every \( i = 1, ..., a \) we have that \( p_i \) does not divide \( q \), and clearly for every \( i = a+1, ..., t \) we have that \( p_i \) does not divide \( q - 1 \). Then \( P_{s+q} = \emptyset \), and we are done.

3. The more general case is the following: suppose that \( P_s = \{p_1, ..., p_a\} \) and \( P_{s+1} = \{p_{a+1}, ..., p_t\} \), but that \( P_s \cup P_{s+1} \neq P_s(k) \). Let \( q_1 := p_{a+1} \cdot \ldots \cdot p_a \). As seen before, we have that \( P_{s+q_1} \cap (P_s \cup P_{s+1}) = \emptyset \).

If \( P_{s+q_1} = \emptyset \), we are done; hence suppose that \( P_{s+q_1} \neq \emptyset \). Then we have \( P_{s+q_1} = \{p_{a+1}, ..., p_{a_2}\} \), where \( p_{a+1} \in P_s(k) \setminus (P_s \cup P_{s+1}) \) for every \( i = a_1 + 1, ..., a_2 \). Consider now \( q_2 := q_1 p_{a+1} \cdot \ldots \cdot p_{a_2} \). Again, one sees that \( P_{s+q_2} \cap (P_s \cup P_{s+1} \cup P_{s+q_1}) = \emptyset \): indeed, it is clear that if \( p \in P_s \), then \( p \) does not divide \( q_2 \), and that if \( p \in P_{s+1} \) then \( p \) does not divide \( q_2 - 1 \), so that \( P_{s+q_2} \cap (P_s \cup P_{s+1}) = \emptyset \); suppose that \( p \in P_{s+q_1} \cap P_{s+q_2} \); then \( p \) divides \( q_2 - q_1 \). By definition of \( q_2 \) we have that

\[
q_2 - q_1 = q_1 (p_{a+1} \cdot \ldots \cdot p_{a_2} - 1).
\]

As \( p \in P_{s+q_1} \), we see that \( p \) does not divide \( p_{a+1} \cdot \ldots \cdot p_{a_2} - 1 \); as \( p \) divides \( q_2 - q_1 \) we then have that \( p \) divides \( q_1 \). By definition of \( q_1 \), this means
that \( p \in P_{s+1} \): as \( P_{s+1} \cap P_{s+q} = \emptyset \), this is not possible, so that even \( P_{s+q} \cap P_{s+q} = \emptyset \), and we are done.

We continue in this way to the \( i \)-th step: if \( P_{s+q} = \emptyset \) we are done, otherwise we have that \( P_{s+q} = \{p_{a_i+1}, \ldots, p_{a_{i+1}}\} \) is such that

\[
P_{s+q} \cap \left( P_s \cup P_{s+1} \cup \bigcup_{j=1}^{i-1} P_{s+q_j} \right) = \emptyset.
\]

As before, letting \( q_{i+1} := q_p a_{i+1} \cdots p_a_{i+1} \) we have that

\[
P_{s+q_{i+1}} \cap \left( P_s \cup P_{s+1} \cup \bigcup_{j=1}^{i} P_{s+q_j} \right) = \emptyset.
\]

Now, recall that

\[
P_s \cup P_{s+1} \cup \bigcup_{j=1}^{i+1} P_{s+q_j} \subseteq P_s(k).
\]

As they are all disjoint and the set \( P_s(k) \) is finite, there must be \( i \) such that \( P_{s+q} = \emptyset \), and we are done.

In conclusion, we have shown that for every \( \text{Theorem 1.3.} \) Let \( (m, k) \) and isomorphism between moduli spaces.

Proof. Let \( (m, k) \) and isomorphism between moduli spaces. The second step is devoted to show that starting with an \( (m, k) \) triple treated in Step 1. In the third step we show that starting with an \( (m, k) \) triple

\[
\gcd(n_{s+q}, a_{s+q}) = 1.
\]

As this is true for every \( s \), we can then suppose \( s + q \gg 0 \), and we are done.

We can now proceed with the proof of:

**Theorem 1.3.** Let \((S_1, v_1, H_1)\) and \((S_2, v_2, H_2)\) be two \((m, k)\)-triple. Then \(M_{v_1}(S_1, H_1)\) and \(M_{v_2}(S_2, H_2)\) are deformation equivalent.

**Proof.** Let \((S, v, H)\) be an \((m, k)\)-triple, and write \( v = m(r, \xi, a) \). We show that \(M_v(S, H)\) is deformation equivalent to \(M_{m(0, h, 2k)}(X, H)\), where \(X\) is a K3 surface such that \(NS(X) = \mathbb{Z} \cdot h, h = c_1(H)\) is ample and \(H^2 = 2k\). The equivalence is obtained using deformations of the moduli spaces induced by deformations along smooth, connected curves of the corresponding \((m, k)\)-triple, and isomorphism between moduli spaces.

We divide the proof in four major steps: in the first, we show the Theorem only for \((m, k)\)-triples of the form \((X, m(0, h, a), H)\), for every \(a \in \mathbb{Z}\); the second step is devoted to show that starting with an \((m, k)\)-triple of the form \((S, m(r, \xi, a), H)\) with \(r > 0\) and \(gcd(r, \xi) = 1\), then one can reduce to the case treated in Step 1. In the third step we show that starting with an \((m, k)\)-triple of the form \((S, m(r, \xi, a), H)\) with \(r > 0\), we can always reduce to have \(gcd(r, \xi) = 1\), namely to the case treated in Step 2. Finally, the fourth step completes the proof with the remaining cases, namely all the triples \((S, m(0, c_1(H), a), H)\), where \(a \in \mathbb{Z}\) and \(S\) is not necessarily \(X\).

**Step 1:** suppose that \( S = X \), and let \( v = m(0, h, a) \). We show that \(M_v(S, H)\) is deformation equivalent to \(M_{m(0, h, 2k)}(S, H)\). To show this, we first write \(a = p + 2tk\) for some \(0 < p \leq 2k\). Hence \(v = m(0, h, p) \cdot ch(O_S(tH))\), and as tensoring with a multiple of \(H\) does not change \(H-(semi)stability\), we get an isomorphism

\[
M_{m(0, h, p)}(S, H) \rightarrow M_{m(0, h, a)}(S, H), \quad E \mapsto E \otimes O_S(tH).
\]
This means that the isomorphism class of $M_v(S, H)$ depends only on $p$. We now show that $M_{m(0,h,p)}(S, H)$ is deformation equivalent to $M_{m(0,h,2k)}(S, H)$. To do so, let $Z$ be a K3 surface such that $NS(Z) = Z \cdot L \oplus Z \cdot D$, where $L$ is ample and the intersection form is given by $L^2 = 2k$, $D^2 = -2$ and $L \cdot D = 2lk - p$ for some $l \gg 0$. The existence of such a K3 surface is showed in Proposition 2.1.2 of [3], and we have a deformation of $(S, H)$ to $(Z, L)$ along a smooth, connected curve $T$. Such a deformation induces a deformation along $T$ of the $(m, k)$–triple $(S, v, H)$ to $(Z, v', L)$, where $v' := m(0, c_1(L), p)$. We have the following:

**Lemma 3.6.** We have that $(Z, v', L)$ is an $(m, k)$–triple.

**Proof.** We just need to prove that $L$ is $v'$–generic. To do so, let $E$ be an $L$–semistable sheaf of Mukai vector $v'$, and let $F \subseteq E$ be an $L$–destabilizing subsheaf of Mukai vector $(0, C, h)$. As $C$ is an effective curve, we have $C^2 \geq -2$, and as $F$ is a subsheaf of $E$ we have

$$C \cdot L = c_1(F) \cdot L < c_1(E) \cdot L = mL^2 = 2km.$$  

Suppose that $C$ is not a multiple of $L$, and let $c := C^2$, $d := C \cdot L$, $M := Z \cdot L \oplus Z \cdot C$ and $N := NS(Z)$. Then, writing $\Delta(\cdot)$ the discriminant, we have

$$\Delta(M) = 2kc - d^2; \quad \Delta(N) = -4k - (2kl - p)^2.$$  

Now, as $M$ is a sublattice of $N$, there must be an integer $n$ such that $\Delta(M) = n^2 \Delta(N)$. Moreover, as $\Delta(N) < 0$, this means that $\Delta(M) \leq \Delta(N)$. Writing down explicitly this inequality, as we have $d < 2km$ we have the inequality

$$c \leq 2(lp - 1) + 2k(m^2 - l^2) - \frac{p^2}{2k}.$$  

Now, as we choose $l \gg 0$, we then get $c < -2$, which is clearly not possible. In conclusion, we need $C$ to be a multiple of $L$, and this easily implies that $L$ is $v'$–generic. □

Now, let $\mathfrak{c}$ be a $v'$–chamber such that $L \in \mathfrak{c}$, and let $H' := qL + D$ for some $q \gg 0$. Hence we have $H' \in \mathfrak{c}$, and moreover

$$L \cdot H' = L \cdot (qL + D) = 2k(q + l) - p.$$  

By Proposition 2.8 we have that $M_v(Z, L) = M_v(Z, H')$. Now, consider the following

$$v'' := v' \cdot ch(\mathcal{O}_Z(H')) = m(0, c_1(L), p + L \cdot H').$$

Notice that $p + L \cdot H' = 2k(q + l) \equiv 0 \mod 2k$. Moreover, as tensoring with a multiple of the polarization does not change the semistability, we have that $M_{v''}(Z, H') \simeq M_{v''}(Z, H')$. As $L$ and $H'$ are in the same chamber, we then have $M_{v''}(Z, H') = M_v(Z, L)$ by Proposition 2.8. Again, we can deform $(Z, v'', L)$ to $(X, v'''', H)$, where $v'''' = m(0, h, 2k(q + l))$. But we have already seen that $M_{v''''}(X, H) \simeq M_{m(0, h, 2k)}(X, H)$, so that we are done.

**Step 2:** suppose that $(S, v, H)$ is an $(m, k)$–triple such that $r > 0$, and that $gcd(r, \xi) = 1$. By Proposition 2.21 we know that $M_v(S, H)$ is deformation equivalent to $M_{m(0,nh,a)}(X, H)$ for some $n \in \mathbb{Z}$ such that $gcd(r, n) = 1$ (as instance $n \equiv 1 \mod r$). Notice that $a = k(n^2 - 1)/r \in \mathbb{Z}$, hence if $n \gg 0$ we have
a \gg 0$. Tensoring with a sufficiently high multiple of $H$, we get an isomorphism between $M_{m(r, n, a)}(X, H)$ and $M_{m(r, n', a')}((X, H)$, where $n' \gg 0$. Hence $a' \gg 0$, and by Lemma 3.3 we can even suppose that $gcd(n', a') = 1$. As $n' \gg 0$, Lemma 3.1 gives an isomorphism between $M_{m(r, n', a')}(X, H)$ and $M_{m((a', n', h), r)}(X, H)$. As $gcd(a', a'') = 1$, by Proposition 2.21 we have that $M_{m((a', n', h), r)}(X, H)$ is deformation equivalent to $M_{m(a', h, a)}(X, H)$. As $a' \gg 0$, by Lemma 3.4 we have an isomorphism between $M_{m(a', h, a)}(X, H)$ and $M_{m(0, h, a)}(X, H)$. By Step 1, we finally have that $M_{m(0, h, a)}(X, H)$ is deformation equivalent to $M_{m(0, h, d)}(X, H)$, and we are done.

**Step 3**: suppose that $(S, v, H)$ is an $(m, k)$--triple such that $r > 0$, and $y := gcd(r, \xi) > 1$. By Lemma 2.18 and Proposition 2.8 we can suppose that $v = m(r, gc_1(H), a)$. Let $S'$ be a projective K3 surface such that $Pic(S') = \mathbb{Z} H'$, where $H'$ is ample and $(H')^2 = H^2 = 2\ell$: hence $(S, H')$ and $(S', H')$ are two polarized K3 surfaces lying in the same moduli space $\mathcal{M}_2$. By the irreducibility of $\mathcal{M}_2$ there is a deformation of $(S, H)$ to $(S', H')$ along a smooth, connected curve $T$: this defines a deformation of $(S, v, H)$ to $(S', v', H')$ along $T$, where $v' = m(r, gc_1(H'), a)$. As seen in section 2.2, then $M_v(S, H)$ is deformation equivalent to $M_v(S', H')$. Now, tensoring with a sufficiently high multiple of $H'$, we get an isomorphism between $M_{m(r, gc_1(H'), a)}(S', H')$ and $M_{m(r, n, c_1(H'), r)}(S', H')$, where $n' \gg 0$. This clearly implies even that $a' \gg 0$. Moreover, by Lemma 3.3 we can even suppose that $gcd(n', a') = 1$. As $n' \gg 0$, Lemma 3.4 gives an isomorphism between $M_{m(r, n, c_1(H'), a)}(S', H')$ and $M_{m(r, n', c_1(H'), a')}((S', H')$. As $gcd(n', a') = 1$, we are in the situation of Step 2, as that we are done.

**Step 4**: suppose that $(S, v, H)$ is any $(m, k)$--triple such that $r = 0$. Let $d \in \mathbb{N}$ and $v' := v \otimes \mathcal{O}_S(dH)$. As tensoring with a multiple of $H$ does not change $H$--(semi)stability, we have an isomorphism

$$M_v(S, H) \to M_{v'}(S, H), \quad E \mapsto E \otimes \mathcal{O}_S(dH).$$

Notice that

$$v' = m(0, \xi, a) \cdot (1, dH, d^2 H^2/2) = m(0, \xi, a + dH \cdot \xi).$$

As $H$ is ample and $\xi$ is effective, we have $\xi \cdot H > 0$, so that choosing $d \gg 0$ we get $a + d\xi \cdot H \gg 0$. We can then suppose $v = m(0, \xi, a)$ where $a \gg 0$: by point 1 of Lemma 3.3 we have then an isomorphism between $M_v(S, H)$ and $M_v(S, H)$, where $\bar{v} := m(a, \xi, 0)$. We are now in the situation of Step 2, hence we are done.

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