Some remarks about normal rings

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Abstract

We give a constructive proof that \( R[X] \) is normal when \( R \) is normal. We apply this result to an operation needed for studying the henselization of a local ring. Our proof is based on the case where \( R \) is without zero divisors, which is more involved than the case where \( R \) is an integral domain. We have to use a constructive deciphering technique that replaces the use of minimal primes (in classical mathematics) by suitable explicit localizations in a suitable tree.

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An integrally closed domain \( R \) is an integral domain whose integral closure in its field of fraction is \( R \) itself. An element \( b \) is integral over an ideal \( I \) iff \( b \) satisfies an integral relation

\[ b^n + u_1 b^{n-1} + \cdots + u_n = 0 \]

with \( u_i \) in \( I \). We can reformulate the definition of being integrally closed by stating that whenever \( b \) is integral over \( \langle a \rangle \) then \( b \) belongs to \( \langle a \rangle \). In this form, this definition makes sense even if \( R \) is an arbitrary ring (not necessarily a domain) and this characterizes the notion of normal ring. It can be checked that this is equivalent to the following: any localization of \( R \) at a prime ideal is an integrally closed integral domain (Ducos et al., 2004, Proposition 6.4).

This paper is mainly concerned with the analysis of the following classical result: if \( R \) is an integrally closed domain then so is \( R[X] \). We first recall a proof which reduces this result to Kronecker’s Theorem (Lombardi and Quitté, 2015, Theorem 3.3). Interestingly, the argument depends in a crucial how we interpret constructively the notion of “integral domain”. Logically, to be an integral domain can be stated as

\[ \forall x. \forall y. xy = 0 \rightarrow [x = 0 \lor y = 0] \]

which is classically, but not constructively, equivalent to

\[ \forall x. x = 0 \lor [\forall y. xy = 0 \rightarrow y = 0] \]

On this form, this means that any element is 0 or is regular. This Definition (2) is actually the usual definition of integral domain in constructive mathematics Lombardi and Quitté (2015);

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Mines et al. (1988). With this definition the argument using Kronecker’s Theorem makes sense constructively.

The definition (1) also has been considered in constructive algebra: a ring satisfying this condition is called a ring without zero divisors Lombardi and Quitté (2015). The main part of this paper presents a proof that if \( R \) is a normal ring without zero divisors than so is \( R[X] \). What is surprising is that this proof seems to require a technique which is used for analyzing argument involving minimal prime ideal (Lombardi and Quitté, 2015, Section XV-7). Furthermore, the proof involves the introduction of the notion of gcd tree of two polynomials, which is important in other context Alonso et al. (2014). Going from Definition (2) to Definition (1), classically equivalent, requires a much more complex argument.

The advantage of Definition (2) is that it is now relatively easy to conclude from this that, more generally, if \( R \) is normal (without any integrality condition) then so is \( R[X] \). The last section analyzes a connected operation useful for studying the henselization of a local ring.

1 Constructible and Gcd trees

Given a reduced ring \( R \) we define the notion of constructible tree for \( R \). This is a binary tree. To each node of this tree is associated a reduced ring, and \( R \) is associated to the root of the tree. Such a tree can only grow in the following way: we choose a leaf, and an element \( a \) of the ring \( S \) associated to this leaf. We add then two sons to this node: to the left branch we associate the ring \( S[1/a] \) and to the right branch the ring \( S/\sqrt{\langle a \rangle} \). Any such tree defines a partition of the constructible spectrum of \( R \) Johnstone (1986).

If we look at the leftmost branch of this tree, the leaf is of the form \( R[1/(a_1 \cdots a_n)] \) and so is a localization of the ring \( R \).

The main proofs in this note will be by induction on the size of a given constructible tree.

If we have two polynomials \( P \) and \( Q \) in \( R[X] \) we can associate a constructible tree which corresponds to the formal computation of the \( \text{gcd} \) of \( P \) and \( Q \). To each leaf \( S \) are also associated polynomials \( A, B, G, P_1, Q_1 \) in \( S[X] \), with \( G \) monic, which witness the computation of the \( \text{gcd} \) of \( P \) and \( Q \)

\[
AP_1 + BQ_1 = 1 \quad P = GP_1 \quad Q = GQ_1.
\]

Notice that for building this tree, \( R \) does not need to be discrete (i.e. to have a decidable equality). Here is a simple example: \( P = X^2 \) and \( Q = aX + b \). We start by the two branches \( S_0 = R[1/a] \) and \( S_1 = R/\sqrt{\langle a \rangle} \). Over \( S_0 \) we have the two branches \( S_{00} = S_0[1/b] \) and \( S_{01} = S_0/\sqrt{\langle b \rangle} \). Over \( S_1 \) we have the two branches \( S_{10} = S_1[1/b] \) and \( S_{11} = S_1/\sqrt{\langle b \rangle} \). The gcd is 1 over \( S_{00} \) and \( S_{10} \), and is \( X \) over \( S_{01} \) and is \( X^2 \) over \( S_{11} \).

This tree is called the \( \text{gcd tree} \) of \( P \) and \( Q \).

If one of the polynomial is monic, one can reduce the size of this tree by using subresultants Apéry and Jouanolou (2006).

2 Kronecker’s Theorem

We shall only need a simple case of Kronecker’s Theorem (Lombardi and Quitté, 2015, Theorem 3.3).

**Theorem 2.1** Let \( R \) be a ring, if \( X^n + a_1X^{n-1} + \cdots + a_m \) divides a polynomial of the form \( X^n + b_1X^{n-1} + \cdots + b_n \) in \( R[X] \) then \( a_1, \ldots, a_m \) are integral over the subring of \( R \) generated by \( b_1, \ldots, b_n \).
Proof. We introduce the splitting algebra\(^1\) \(T = X^n + b_1 X^{n-1} + \cdots + b_n\) Lombardi and Quitté (2015) so that \(X^n + b_1 X^{n-1} + \cdots + b_n = (X - t_1) \cdots (X - t_n)\) in \(T[X]\). The ring \(R\) embeds in \(T\) and \(a_i\) is a polynomial in \(t_1, \ldots, t_n\).

**Corollary 2.2** Let \(R\) be a ring, if \(Y + a_0 X^m + \cdots + a_m\) divides a polynomial of the form \(Y^n + b_1 Y^{n-1} + \cdots + b_n\) in \(R[X, Y]\) then all coefficients \(a_0, \ldots, a_m\) are roots of polynomials of the form \(Z^i + p_1 Z^{i-1} + \cdots + p_i\) where \(p_i\) is a homogeneous polynomial of degree \(i\) in \(b_1, \ldots, b_n\) where \(b_j\) has weight \(j\).

**Proof.** It is enough to look at the case where \(a_0, \ldots, a_m\) are indeterminates, and \(R\) is a polynomial ring on \(a_0, \ldots, a_m\) and some other indeterminates. By replacing \(Y\) by \(X^n\) for \(N\) big enough, we get that each \(a_0, \ldots, a_m\) is integral over \(\mathbb{Z}[b_1, \ldots, b_n]\) and hence each \(a_k\) is root of a polynomial of the form \(Z^i + p_1 Z^{i-1} + \cdots + p_i\) where \(p_i\) is a polynomial in \(b_1, \ldots, b_n\). By replacing \(Y\) by \(Y/c\) where \(c\) is another indeterminate, we get that \(p_i\) is homogeneous of degree \(i\) in \(b_1, \ldots, b_n\) where \(b_j\) has weight \(j\).

**Corollary 2.3** Let \(R\) be a normal integral domain, then \(R[X]\) is a normal integral domain.

**Proof.** We assume given \(P\) and \(Q\) in \(R[X]\) such that \(P\) is integral over \(\langle Q \rangle\) and we want to show that \(P\) is in \(\langle Q \rangle\) in \(R[X]\). Let \(K\) be the total fraction field of \(R\). Since \(R\) is an integral domain, we can consider \(R\) to be a subring of \(K\). Since \(K[X]\) is euclidean, we know that \(P\) is in \(\langle Q \rangle\) in \(K[X]\) and we have \(cP = HQ\) for some regular element \(c\). Since \(P\) is integral over \(\langle Q \rangle\) we have a relation

\[
P^n + A_1 Q P^{n-1} + \cdots + A_n Q^n = 0
\]

with \(A_1, \ldots, A_n\) in \(R[X]\) and so we can write

\[
Y^n + A_1 c Y^{n-1} + \cdots + A_n c^n = (Y - H) S(X, Y)
\]

where \(S(X, Y)\) is a monic polynomial in \(R[X][Y]\). Using Corollary 2.2, it follows that all coefficients of \(H\) are integral over \(\langle c \rangle\) and hence are in \(\langle c \rangle\) since \(R\) is normal. We can then write \(H = cH_1\) and so \(c(P - QH_1) = 0\) in \(R[X]\). It follows that we have \(P = QH_1\) and hence \(P\) is in \(\langle Q \rangle\) in \(R[X]\).\(\square\)

This is the argument we are going to adapt in the case where \(R\) is normal and without zero divisors.

### 3 Polynomial ring

We assume that \(R\) is normal without zero divisors and we show that \(R[X]\) is normal. We assume given \(P\) and \(Q\) in \(R[X]\) such that \(P\) is integral over \(\langle Q \rangle\) and we want to show that \(P\) is in \(\langle Q \rangle\) in \(R[X]\).

**Lemma 3.1** If \(R\) is normal then \(R\) is reduced.

**Proof.** If \(b^2 = 0\) then \(b\) is integral over \(\langle 0 \rangle\) and so is in \(\langle 0 \rangle\).\(\square\)

**Lemma 3.2** If \(R\) is normal then so is \(R[1/a]\).

\(^1T = R[x_1, \ldots, x_n]/\langle f \rangle = R[x_1, \ldots, x_n]\) where \(J(f)\) is the ideal of symmetric relators necessary to identify \(\prod_{i=1}^{n} (X - x_i)\) with \(f(X)\) in \(T[X]\). We let \(x = x_1\) and the quotient ring \(R[x] = R[X]/\langle f \rangle\) is identified with a subring of \(T\). If \(g(X, Y) = \frac{f(X) - f(Y)}{X - Y}\) then \(g(x_1, X) = \prod_{i=2}^{n} (X - x_i)\) in \(T[X]\) and \(g(x_1, x_j) = 0\) for \(j \geq 2\).
Proof. For $c$ and $b$ in $R$, if $c$ is integral over $\langle b \rangle$ in $R[1/a]$ we have a relation $(a^Nc)^n + u_1b(a^Nc)^{n-1} + \cdots + u_nb^n = 0$ with $u_1, \ldots, u_n$ in $R$. Since $R$ is normal we have $a^Nc$ in $\langle b \rangle$. \hfill \Box

Lemma 3.3 If $R$ is without zero divisors then so is $R[1/a]$.  

Proof. We take two elements $v = b/a^n$ and $w = c/a^m$ of $R[1/a]$ with $b$ and $c$ in $R$. If we have $vw = 0$ in $R[1/a]$ we have $a^mbc = 0$ in $R$ for some $p \geq 0$. We have then $a^mb = 0$ in $R$ or $a^pc = 0$ in $R$, which implies that $v = 0$ or $w = 0$ in $R[1/a]$. \hfill \Box

From now on in this section, we assume $R$ to be a normal ring without zero divisors.

Lemma 3.4 If $P$ is integral over $\langle Q \rangle$ and is in $\langle Q \rangle$ in $R[1/a][X]$ then $a = 0$ or $P$ is in $\langle Q \rangle$ in $R[X]$.  

Proof. We have $H$ in $R[X]$ such that $a^NP = QH$ for some $N$. We write $c = a^N$. Since $P$ is integral over $\langle Q \rangle$ we have a relation

$$P^n + A_1QP^{n-1} + \cdots + A_nQ^n = 0$$

with $A_1, \ldots, A_n$ in $R[X]$ and so

$$Q^n(H^n + A_1cH^{n-1} + \cdots + A_nc^n) = 0$$

in $R[X]$. Hence either $Q = 0$ in which case $P = 0$ is in $\langle Q \rangle$ or we can write

$$Y^n + A_1cY^{n-1} + \cdots + A_nc^n = (Y - H)S(X,Y)$$

where $S(X,Y)$ is a monic polynomial in $R[X][Y]$. Using the corollary of Kronecker’s Theorem 2.2, it follows that all coefficients of $H$ are integral over $\langle c \rangle$ and hence are in $\langle c \rangle$ since $R$ is normal. We can then write $H = cH_1$ and so $c(P - QH_1) = 0$ in $R[X]$. It follows that we have $c = 0$, that is equivalent to $a = 0$, or $P = QH_1$ and hence $P$ is in $\langle Q \rangle$ in $R[X]$. \hfill \Box

Lemma 3.5 If we have $P$ and $Q$ in $R[X]$ and a constructible tree for $R$ such as, at all leaves $S$ of this tree, we have $P$ in $\langle Q \rangle$ in $S[X]$. Then $P$ is in $\langle Q \rangle$ in $R[X]$.

Proof. We look at the leftmost branch of this tree, indexed by elements $a_1, \ldots, a_l$, so that $S = S'[1/a_l]$ where $S' = R[1/(a_1 \cdots a_{l-1})]$ is without zero divisors by Lemma 3.3 and is normal by Lemma 3.2. Using Lemma 3.4 we get that $a_l = 0$ in $S'$ or $P$ is in $\langle Q \rangle$ in $S'[X]$. In the second case, we can shorten the leftmost branch to $a_1, \ldots, a_l$ and get a smaller tree. In the first case where $a_l = 0$ in $S'$, this means that the right son $S'/\langle a \rangle$ of $S'$ is equal to $S'$ and we also can shorten the tree. We conclude by tree induction. \hfill \Box

Theorem 3.6 If $R$ is normal and without zero divisors then so is $R[X]$.  

Proof. We take $P$ and $Q$ in $R[X]$ and we assume that we have a relation

$$P^n + A_1QP^{n-1} + \cdots + A_nQ^n = 0$$

with $n \geq 1$ and $A_1, \cdots, A_n$ in $R[X]$. We have to show that $P$ is in $\langle Q \rangle$ in $R[X]$.

We look now at the gcd tree of $P$ and $Q$ as defined in the first section. At all leaves $S$ of this tree, we have $P_1, Q_1, G, A, B$ in $S[X]$ satisfying

$$P = GP_1, \quad Q = GQ_1, \quad AP_1 + BQ_1 = 1$$
in $S[X]$ and $G$ is monic. Since $G$ is monic and

$$P^n + A_1QP^{n-1} + \cdots + A_nQ^n = G^n(P_1^n + A_1Q_1P_1^{n-1} + \cdots + A_nQ_1^n) = 0$$

we have

$$P_1^n + A_1Q_1P_1^{n-1} + \cdots + A_nQ_1^n = 0$$

and $Q_1$ divides $P_1^n$. With $AP_1 + BQ_1 = 1$ this implies that $Q_1$ is a unit and so $P$ is in $\langle Q \rangle$ in $S[X]$. We can now apply Lemma 3.5. \qed

4 Normal ring

We say that the ring is \textit{locally} without zero divisors (Lombardi and Quitté, 2015, Lemma VIII-3.2) if, and only if, whenever $ab = 0$ then there exists $u$ such that $ua = 0$ and $(1 - u)b = 0$. These rings are often called \textit{pf}-rings. In this note, only the notion of rings without zero divisors and locally without zero divisors will play a role.

**Lemma 4.1** If $R$ is normal then $R$ is locally without zero divisors.

**Proof.** If $ab = 0$ then $b^2 - (a + b)b = 0$ so $b$ is integral over $\langle a + b \rangle$ and so is in $\langle a + b \rangle$. We can write $b = (a + b)u$ and so $ua = (1 - u)b$. This implies $ua^2 = (1 - u)ba = 0$ and so $ua = (1 - u)b = 0$ since $R$ is reduced. \qed

**Theorem 4.2** If $R$ is normal then so is $R[X]$.

**Proof.** By Lemma 4.1, $R$ is locally without zero divisors. Assume then that a polynomial $P \in R[X]$ is integral over $\langle Q \rangle$ in $R[X]$. Following the proof of Theorem 3.6, each time we use $ab = 0 \rightarrow a = 0$ or $b = 0$, we split the “current ring $R[1/v]$” in two rings $R[1/vu]$ and $R[1/v(1 - u)]$ by using an $u$ such that $ua = 0$ and $(1 - u)b = 0$. We find finally $u_1, \ldots, u_m$ in $R$ such that $\langle u_1, \ldots, u_m \rangle = 1$ and $P$ belongs to $\langle Q \rangle$ in each $R[1/u_j][X]$. It follows that $P$ is in $\langle Q \rangle$ as required. \qed

5 The ring $R_{\{f\}}$

Let $f$ be a monic polynomial in $R[X]$. We can consider the extension $S = R[X]/\langle f \rangle$ where $f$ has a root $x$. We let $f_X$ be the formal derivative of $f$ w.r.t. $X$, and we define $R_{\{f\}}$ to be the localization $S[1/f_X(x)]$. This construction is important to study the properties of henselization of a local ring Raynaud (1970).

The goal of this section is to show that $R_{\{f\}}$ is normal whenever $R$ is normal. As in the previous section, we can first assume that $R$ is without zero divisors, and then use that a normal ring is locally without zero divisors to conclude. So in the rest of the section, we assume that $R$ is a normal ring without zero divisors.

If $f = gh$ is the product of two monic polynomials $g$ and $h$ we have $R_{\{f\}}$ isomorphic to $R_{\{g\}}[1/h(x)] \times R_{\{h\}}[1/g(x)]$. This remark is important since by using Lemma 3.2 we can reason by induction on the degree of $f$ to show that $R_{\{f\}}$ is normal if $R$ is normal.

**Lemma 5.1** If $R$ is normal without zero divisors, and $a$ in $R$ and $T = R[1/a]$ and $f = gf_1$ with $g$ and $f_1$ monic in $T[X]$ then we have $g$ and $f_1$ in $R[X]$ or $a = 0$.

**Proof.** Using Kronecker’s Theorem 2.1, each coefficient of $g$ and $f_1$ is integral over $R$. Since $R$ is normal and without zero divisors, this implies that $a = 0$ or $g$ and $f_1$ are in $R[X]$. \qed
We have the trace map $\text{tr} : S \to R$. If we introduce the splitting algebra (Lombardi and Quitté, 2015, Definition III-4.1) of $f$ and write $f = (X - x_1) \cdots (X - x_n)$ with $x = x_1$ then the trace of $h(x) \in S$ is $h(x_1) + \cdots + h(x_n)$. If $v = h(x)$ in $S$ is integral over $\langle a \rangle_S$ with $a \in R$ then all elements $h(x_1), \ldots, h(x_n)$ are integral over $\langle a \rangle_R$ and so $\text{tr}(v)$ is also integral over $\langle a \rangle_X$ and so is in $\langle a \rangle_X$ since $R$ is normal. Also if we write $f(x) - f(Y) = (X - Y)g(X,Y) = \sum_i g_i(Y)X^i$, we have $f_X(T) = g(T,T)$ and $g(x_1,x_j) = 0$ for $j \neq 1$. So we get for all $v = h(x) = h(x_1)$ in $S$

$$f_X(x)h(x) = g(x_1,x_1)h(x_1) = \sum_i g_i(x_1)h(x_1) \, x_1^i$$

and for $j \neq 1$

$$0 = g(x_1,x_j)h(x_j) = \sum_i g_i(x_j)h(x_j) \, x_1^i$$

so that, by summation, we get Tate’s formula (Raynaud, 1970, Chapter VII, 1)

$$f_X(x) = \sum_i \text{tr}(g_i(x)) \, x^i.$$ 

Since each $g_i(x)$ is integral over $R$, we can state the following lemma.

**Lemma 5.2** If $R$ is normal, if $a$ in $R$ and if $v$ in $S$ is integral over $\langle a \rangle_S$ then $f_X(x)v$ is in $\langle a \rangle_S$.

**Theorem 5.3** If $R$ is normal then $R_{(f)}$ is normal.

*Proof.* We assume given $p,q$ in $R[X]$ such that $p(x)$ is integral over $\langle q(x) \rangle$ in $S$ so that we have a relation $p(x)^n + u_1(x)q(x)p(x)^{n-1} + \cdots + u_n(x)q(x)^n = 0$. The goal is to show that $p(x)$ is in $q(x)S[f_X(x)^{-1}]$.

We look at the gcd tree of $q$ and $f$, and the leftmost branch of this tree. At the leaf of this branch we have a list of elements that we force to be invertible $a_1, \ldots, a_n$ and $g_1, f_1, g, A, B$ in $R[a^{-1}]$ with $a = a_1 \ldots a_n$ such that

$$f = gf_1 \quad q = gq_1 \quad 1 = Af_1 + Bq_1.$$ 

Furthermore $g$ and hence $f_1$ are monic since $f$ is monic.

If $f = f_1$ we have $g = 1$ and $q = q_1$. In this case we have $c = Af + Bq$ where $c = (a_1 \ldots a_n)^m$ for some $m$ and so we have $c = B(x)q(x)$ in $S = R[X]/(f)$. We then have a relation

$$(p(x)B(x))^n + u_1(x)c(p(x)B(x))^{n-1} + \cdots + u_n(x)c^n = 0$$

and hence, by Lemma 5.2, we get that $p(x)B(x)$ is in $\langle c \rangle$ in $S[f_X(x)^{-1}]$. Hence we have $l(x)$ in $S$ and $N$ such that

$$f_X(x)^N p(x)B(x) = cl(x) = q(x)B(x)l(x)$$

and so

$$c(p(x)f_X(x)^N - l(x)q(x)) = 0.$$ 

We have then $c = 0$ or $p(x)$ is in $\langle q(x) \rangle$ in $R[f_X(x)^{-1}]$. So either we have the desired conclusion that $p(x)$ is in $\langle q(x) \rangle$ or we have $a_n = 0$ in $R[1/(a_1 \cdots a_{n-1})]$ and we can shorten the computation tree of the gcd of $f$ and $q$.

If $f$ and $f_1$ have not the same degree, we have found a proper decomposition $f = gf_1$ of $f$ in $R[1/a]$ with $a = a_1 \cdots a_n$. In this case, since $R$ is normal, by Lemma 5.1, we have two subcases

- either $g$ and $f_1$ are in $R[X]$ and we can conclude by induction on the degree of $f$, using that $R_{(f)}$ isomorphic to $R_{(g)}[1/f] \times R_{(f_1)}[1/g]$,
or $a = 0$ and as in the previous case, we can shorten the computation tree of the gcd of $f$ and $q$.

As in (Lombardi and Quitté, 2015, VIII-4.4), we say that a ring is a Prüfer ring if it is arithmetic and reduced. A coherent Prüfer ring is an arithmetic $pp$-ring.

**Corollary 5.4** If $R$ is a Prüfer ring of Krull dimension $\leq 1$ then so is $R_{(f)}$.

*Proof.* We use the fact that a ring is a normal coherent ring of Krull dimension $\leq 1$ if, and only if, it is Prüfer and of Krull dimension $\leq 1$ Ducos et al. (2004). We have shown that $R_{(f)}$ is normal. Since $S = R[X]/\langle f \rangle$ is an integral extension of $R$ it is also of Krull dimension $\leq 1$ Coquand et al. (2009) and so is its localization $R_{(f)}$. Finally, $S$ is a finite free $R$-module, and so it is coherent if $R$ is coherent and so is its localization $R_{(f)}$.

This gives an alternative proof to the main result of Coquand et al. (2010), that $R_{(f)}$ is Prüfer when $R = k[X]$, in the case where $f$ is monic in $Y$. It is possible however to reduce the general case to this case, by a change of variables.

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