STRATEGY-PROOF AGGREGATION RULES IN MEDIAN SEMILATTICES WITH APPLICATIONS TO PREFERENCE AGGREGATION

ERNESTO SAVAGLIO AND STEFANO VANNucci

Abstract. Two characterizations of the whole class of strategy-proof aggregation rules on rich domains of locally unimodal preorders in finite median join-semilattices are provided. In particular, it is shown that such a class consists precisely of generalized weak sponsorship rules induced by certain families of order filters of the coalition poset. It follows that the co-majority rule and many other inclusive aggregation rules belong to that class. The co-majority rule for an odd number of agents is characterized and shown to be equivalent to a Condorcet-Kemeny median rule. Applications to preference aggregation rules including Arrowian social welfare functions are also considered. The existence of strategy-proof anonymous, weakly neutral and unanimity-respecting social welfare functions which are defined on arbitrary profiles of total preorders and satisfy a suitably relaxed independence condition is shown to follow from our characterizations.

JEL Classification: D71

1. Introduction

Aggregation rules are procedures to choose outcomes by taking outcome-profiles as inputs. They are often encountered in the collective decision literature under the labels of ‘voting schemes’, ‘voting rules’, ‘social choice rules’, ‘aggregators’, ‘consensus functions’, etc. The outcomes to be aggregated usually are alternative items in some relevant outcome space (e.g. scores, grades, signals, preferences, judgements, nodes of an abstract network, etc.) which are submitted by the agents. Since agents are stakeholders endowed with nonverifiable ‘preferential attitudes’ on the different items of the outcome space, then a reliable and effective decision protocol should be strategy-proof, i.e. immune to advantageous individual manipulations through submission of false information. The celebrated Gibbard-Satterthwaite theorem implies that if the domain of admissible individual preferences includes every possible linear order on the outcome space and the range of a strategy-proof aggregation rule consists of at least three alternatives, then that aggregation rule must be a dictatorial rule, certainly a scarcely appealing protocol. Nevertheless, there exists a much wider and interesting class of strategy-proof aggregation rules whenever rich, i.e. suitably large, domains of certain single peaked preference preorders are considered. Single peaked preferences are preorders with a unique maximum (the top element) which naturally arise if each agent’s representation of the outcome space is endowed with a plausible notion of compromises (betweenness relation), meaning that the outcome lying between the top and another outcome cannot be worse than the latter. The current literature has identified the outcome spaces (domains), on which single peaked total preorders are defined, that allow strategy-proof rules.

Date: August 17, 2022.

2000 Mathematics Subject Classification. Primary 05C05; Secondary 52A21, 52A37.

Key words and phrases. Strategy-proofness, single peakedness, median join-semilattice, social welfare function.
We study here the case in which single peaked preorders are defined on outcome spaces endowed with a (join-)semilatticial structure. Thus, the present work is devoted to characterizing those aggregation rules in finite median join-semilattices which are strategy-proof on some rich domains of locally unimodal (or single peaked) total preorders on their outcomes.

We also characterize the co-majority (or median) rule as the only one, within the class of such strategy-proof rules, that is anonymous and bi-idempotent when the number of agents is odd. Applications of our characterization results to preference aggregation rules are also provided, focusing on the classic case of Arrowian social welfare functions, namely aggregation rules taking arbitrary profiles of total preference preorders as inputs and returning total preference preorders as outputs. Thus, the existence of anonymous, unanimity-respecting and strategy-proof social welfare functions on the full domain of total preorders is established. In particular, a new characterization of generalized Condorcet-Kemeny rules for an odd number of agents is proved. Similar applications to preference aggregation rules arising from other, less regular preference structures, including generalized tournaments, reflexive relations, and irreflexive relations are also briefly presented and discussed.

Now, addressing strategy-proofness issues for aggregation rules of any sort requires a suitable specification of the agents’ preferences on outcomes, namely their preferences on preferences. It is well-known, in view of the Gibbard-Satterthwaite ‘impossibility theorem’, that (i) some domain-restriction on the foregoing ‘preferences on preferences’ is required in order to open up the possibility to design interesting and non-dictatorial strategy-proof preference-aggregation rules, and (ii) some form of single-peakedness is a most natural and plausible domain-restriction to that effect. But single-peakedness notions typically rely in turn on an underlying ternary betweenness relation defined on the preference space which is supposedly shared by the relevant agents and thus should presumably be ‘naturally’ embedded in that space. Therefore, when it comes to the application of our basic model to preference aggregation we are immediately confronted with a list of key issues to address, namely:

- What sort of preference relations are to be aggregated? Arbitrary total preorders, linear orders or even wider domains containing them? (Of course, the answer has a significant impact on the general structure of the outcome space to focus on).
- Are the preference profiles to be aggregated of an arbitrary but fixed finite size (fixed population approach), or of every possible finite size (variable population approach)?

---

1We recall that a partial order over a set is a reflexive, antisymmetric and transitive binary relation and that a partial order is a (join-)semilattice if every pair of elements in the set has a least upper bound, named the join or supremum of the pair, with respect to the underlying order relation.

2A join-semilattice is median if the ternary partial operation μ such that μ(x, y, z) = (x ∨ y) ∧ (y ∨ z) ∧ (x ∨ z) is a well-defined operation (see Section 2 for more details).

3Namely, a rule selecting an outcome between the two proposals advanced by a perfectly polarized population.

4It is worth emphasizing here that our usage of the term ‘Arrowian social welfare function’, while arguably sound and well-grounded, is by no means widely established. Sometimes that term is also used to denote aggregation rules for profiles of linear orders, possibly with the additional conditions of Idempotence and the Arrowian ‘Independence of Irrelevant Alternatives’ requirement (see e.g. Sethuraman, Teo, Vohra (2003)).
• What type of aggregation protocol are we to consider? Namely, given a profile of preference relations of the prescribed type as input, what kind of object is the output required to be? A single preference relation of the prescribed type (aggregation with no qualifier, namely exact or pure aggregation), one or more preference relations of the prescribed type (multi-aggregation), a single preference relation belonging to a class which includes but does not reduce to the prescribed type for the input ((domain) restricted aggregation), or a single preference relation of the same type as that prescribed for inputs but enjoying some additional requirements ((codomain) constrained aggregation)?

• What sort of single-peakedness property for the relevant ‘preferences on preferences’ are we to focus on, or equivalently, what is the most natural/plausible notion of betweenness on the basic ‘preference space’ to refer to?

This paper relies on a definite choice of focus for each one of the foregoing issues, namely:

(a) the basic preference domain should include all the total preorders;
(b) the preference profiles to be aggregated are of some fixed size;
(c) the type of aggregation protocol to focus on is (pure) aggregation (and possibly constrained aggregation);
(d) the betweenness relation to be used in order to define single-peaked ‘preferences on preferences’ should be the one ‘naturally’ dictated by the underlying basic preference domain of the aggregation rule under consideration.

Within such a framework, the main results established in the present work on strategy-proof aggregation rules in median semilattices as applied to preference aggregation consists in:

(i): (see Corollary 1 and Proposition 1) proving the existence of a large class of (‘full-domain!’) social welfare functions that are strategy-proof on the domain of all single-peaked ‘preferences on preferences’, and characterizing them as those aggregation rules $f$ for total preorders whose behaviour is dictated by a family $F_m$ of superset-closed collections of agent-coalitions, one for each bipartite total preorder $m$ of $A$ (namely, a preorder having just two indifference classes of good -respectively, bad- alternatives), in the following manner. At every preference profile of total preorders $R_N$, $f(R_N)$ is just the intersection of all the bipartite total preorders $m$ of $A$ that -at preference profile $R_N$- are consistent with the preferences of all the agents of at least one coalition in $F_m$. Such a large class of strategy-proof social welfare functions includes quorum systems, fixed-majority collegial rules, weakly neutral rules, quota rules and (within the latest subclass) the co-majority (median) rule (all of them can also be described as weak sponsorship rules);

(ii): (see Proposition 2) proving that the co-majority rule $\hat{f}_{\partial maj}$ is the only social welfare function that is strategy-proof on the domain of preferences on the set of all total preorders of a finite set of alternative social states that are single-peaked, anonymous (namely, invariant with respect to agent-relabeling) and bi-idempotent (namely, capable to choose one of the pair of proposed preorders at any perfectly bipolarized profile). Furthermore, under the present hypotheses the co-majority rule turns out to be identical to the generalized Condorcet-Kemeny aggregation rule, namely a rule that, at each profile of total preorders, selects one of the total preorders
of the finite set of alternative social states having a minimum sum of Kemeny distances\footnote{Recall that the Kemeny distance between two binary relations is the size of the symmetric difference between them (see section 4 and Appendix C for the relevant definitions and more details).} from the preorders of the given profile.

Thus, the foregoing results \textit{jointly} address several related issues: the characterization of the \textit{entire class} of strategy-proof aggregation rules in median finite semilattices, a specific characterization of ‘median’ rules amongst them, and -as a by-product- both a general characterization of strategy-proof preference aggregation rules including social welfare functions, and a specific characterization of the Condorcet-Kemeny aggregation rule in a fixed population setting.

As it turns out, such issues have been previously considered in the literature, but most typically from mutually ‘disconnected’ perspectives. For instance, aggregation rules in semilattices and lattices have been studied in depth in seminal contributions mostly due to Monjardet and his co-workers (see e.g. Monjardet (1990)), but with no reference to strategy-proofness issues. Condorcet-Kemeny aggregation rules have been characterized at least for the case of linear orders and in a variable population setting (see e.g. Young, Levenglick (1978)), but again with no reference to strategy-proofness properties. By contrast, Bossert, Sprumont (2014) \textit{does} consider strategy-proofness issues, and also provides characterizations of some strategy-proof preference aggregation rules in a variable population setting, but focusses in fact on \textit{restricted aggregation rules} which admit arbitrary total preorders as possible outputs while being \textit{only} defined on profiles of linear orders. Most recently, Bonifacio, Massó (2020) essentially characterizes the \textit{sub-class of anonymous and unanimity-respecting aggregation rules} in arbitrary join-semilattices which are strategy-proof on ‘single-peaked’ domains of total preorders (according to a notion of ‘single-peakedness’ that reflects the structure of the underlying join-semilattice). Since the relevant join-semilattice may not be median, however, ‘median’ rules such as Condorcet-Kemeny rules are in general not available, and a \textit{specific} application to the case of total preorders and consequently to classical Arrowian social welfare functions is in fact out of reach in that general framework \textit{unless} some further structure is adjoined.

On the contrary, focussing on the case of median join-semilattices makes it possible and natural to address \textit{jointly} all of the previous issues: that is precisely what is done in the present work. The application of its main results to two prominent examples of median join-semilattices (namely, the median semilattice of total preorders) and the distributive lattice of reflexive binary relations on a finite set) provides a simple way out of Arrow’s ‘impossibility theorem’ that also ensures strategy-proofness for a quite large class of anonymous and idempotent social welfare functions. It consists in combining full retention of transitivity and a \textit{basic} version of Pareto optimality for social preferences with a considerable \textit{relaxation of Arrow’s Independence of Irrelevant Alternatives}.

The rest of the paper is organized as follows. Section 2 is devoted to the basic definitions and preliminaries of our model. Section 3 includes the main results. Section 4 presents the application of the foregoing results to preference aggregation rules. Section 5 collects some concluding remarks. Finally, Appendix A includes the proof of the main result of the present work, Appendix B contains some useful auxiliary notions concerning median join-semilattices, and Appendix C provides a detailed discussion of related literature.
2. Notation, definitions and preliminaries

Let \( N = \{1, \ldots, n\} \) denote the finite population of voters, with \( n \geq 3 \) in order to avoid tedious qualifications. The subsets of \( N \) are also referred to as coalitions, and \( (\mathcal{P}(N), \subseteq) \) denotes the partially ordered set (poset) of coalitions induced by set-inclusion preorder on the set \( \mathcal{P}(N) \) of all possible subsets of \( N \). A set \( F \subseteq \mathcal{P}(N) \) of coalitions such that, for any \( S \in F \) and any \( T \subseteq N \), if \( S \subseteq T \) then \( T \in F \) is called order filter. The set of inclusion-minimal elements/coalitions of \( F \) are the basis of the order filter \( F \) and it is denoted as \( F^{\text{min}} \).

Let \( X \) be an arbitrary nonempty finite set of alternatives and \( \leq \) a partial order, i.e. a reflexive, transitive and antisymmetric binary relation on \( X \). Let \( \mathcal{X} = (X, \leq) \) be the corresponding partially ordered set on \( X \). We denote by \( \vee \) and \( \wedge \) the least-upper-bound (or join) and greatest-lower-bound (or meet) binary partial operations on \( X \) as induced by \( \leq \), respectively and by \( \forall Y \) and \( \forall Y \) the least-upper-bound and greatest-lower-bound of \( Y \) (whenever they exist), for any \( Y \subseteq X \). The order filter of a partially ordered set \( (X, \leq) \) is a set \( Y \subseteq X \) such that, for any \( y, z \in X \), if \( y \leq z \) and \( y \in Y \), then \( z \in Y \). For any \( x \in X \), we then denote with \( \uparrow x = \{ y \in X : x \leq y \} \) the principal order filter generated by \( x \). An element \( x \in X \) is meet-irreducible (join-irreducible) if for any \( y, z \in X \), \( x \wedge y \) entails \( x \in Y \) (\( x = \forall Y \) entails \( x \in Y \)). Moreover, for any \( Y \subseteq X \), \( \forall Y \) (\( \forall Y \), respectively) is well-defined if and only if there exists \( z \in X \) such that \( z \leq y \) (\( y \leq z \), respectively) for all \( y \in Y \), namely the elements of \( Y \) have a common lower (upper) bound. The set of all meet-irreducible elements (join-irreducible elements) of \( \mathcal{X} = (X, \leq) \) will be denoted by \( M_X \) (\( J_X \), respectively). Notice that, by construction, for every \( x \in X \), \( x = \wedge M(x) \) where \( M(x) := \{ m \in M_X : x \leq m \} \) and, dually \( x = \vee J(x) \) where \( J(x) := \{ j \in J_X : j \leq x \} \).

The partially ordered set \( \mathcal{X} = (X, \leq) \) is said to be a (finite) join-semilattice (meet-semilattice, respectively) if and only if the least upper bound or joint \( x \vee y \) (the greatest lower bound or meet \( x \wedge y \)) is well-defined in \( X \) for all \( x, y \in X \), so that \( \forall : X \times X \to X \) (\( \forall : X \times X \to X \)) is a function. \( \mathcal{X} = (X, \leq) \) is a lattice if it is both a join-semilattice and a meet-semilattice. Notice that a finite join-semilattice \( \mathcal{X} = (X, \leq) \) has a (unique) universal upper bound or top element \( 1 = \forall X \), and its co-atoms are those elements \( x \in X \) such that \( x \ll 1 \), with \( \ll \) denoting the so-called cover relation, meaning that there is no \( y \neq x \) such that \( x \leq y \leq 1 \) (see Appendix C). The set of co-atoms of \( \mathcal{X} = (X, \leq) \) is denoted by \( C_X \). Dually, a finite meet-semilattice \( \mathcal{X} = (X, \leq) \) has a (unique) universal lower bound or bottom element \( 0 = \wedge X \), and its atoms are those elements \( x \in X \) such that \( 0 \ll x \). The set of atoms of \( \mathcal{X} \) is denoted by \( A_X \). Notice that a co-atom (atom, respectively) is also a meet-irreducible (join-irreducible, respectively) element. When co-atoms and meet-irreducibles (atoms and join-irreducibles) do in fact coincide the join-semilattice (meet-semilattice) is said to be coatomistic (atomistic, respectively).

Let us now introduce the class of finite join-semilattices which is the focus of the present paper.

**Definition 1.** A (finite) join-semilattice \( \mathcal{X} = (X, \leq) \) is median if it satisfies the following pair of conditions:

\(^6\)From now on, in order to avoid tedious repetitions and if any ambiguity is excluded, we will basically only define one of the two main notions of (joint or meet) semilattice used in the paper, assuming that the other can simply be obtained by duality.
(i): **upper distributivity:** for all \( u \in X \), and for all \( x, y, z \in X \) such that \( u \) is a lower bound of \( \{x, y, z\} \), \( x \lor (y \land z) = (x \lor y) \land (x \lor z) \) (or, equivalently, \( x \land (y \lor z) = (x \land y) \lor (x \land z) \)) holds. Namely, \((\uparrow u, \leq_{\uparrow u})\), where \( \leq_{\uparrow u} \) denotes the restriction of \( \leq \) to \( \uparrow u \), is a **distributive lattice**.

(ii): **co-coronation** (or meet-Helly): for all \( x, y, z \in X \) if \( x \land y \), \( y \land z \), and \( x \land z \) exist, then \((x \land y \land z)\) also exists.

A well-known property of (finite) upper distributive join-semilattices that will be used below is in the following:

**Claim 1.** Let \( m \in M_X \) be a meet-irreducible element of an upper distributive finite join-semilattice \( \mathcal{X} = (X, \leq) \) and \( Y \subseteq X \) such that \( \land Y \) exists. If \( \land Y < m \) then there also exists some \( y \in Y \) such that \( y \leq m \) (see e.g. Monjardet (1990)).

It is easily checked that if \( \mathcal{X} = (X, \leq) \) is a median join-semilattice, then the partial function \( \mu : X^3 \to X \) defined as follows: for all \( x, y, z \in X \),

\[
\mu(x, y, z) = (x \lor y) \land (y \lor z) \land (x \lor z)
\]

is indeed a **well-defined ternary operation** on \( X \), the **median** of \( \mathcal{X} \) which satisfies the following two characteristic properties (see Sholander (1952, 1954):

\[
\begin{align*}
(\mu_1) \mu(x, x, y) &= x \quad \text{for all } x, y \in X \\
(\mu_2) \mu(\mu(x, y, v), \mu(x, w, z), x, y) &= \mu(\mu(v, w, z), x, y) \quad \text{for all } x, y, v, w, z \in X.
\end{align*}
\]

Relying on \( \mu \), we define a **ternary (median-induced) betweenness** relation:

\[
B_\mu = \{(x, z, y) \in X^3 : z = \mu(x, y, z)\}
\]

on \( \mathcal{X} \), and, for any \( x, y \in X \), the **interval** induced by \( x \) and \( y \), namely:

\[
I_\mu(x, y) := B_\mu(x, .., y) = \{z \in X : z = \mu(x, y, z)\}
\]

Therefore, for any \( x, y, z \in X \), \((x, z, y) \in B_\mu\) (also written \( B_\mu(x, z, y) \)) if and only if \( z \in I_\mu(x, y) \). We recall here that the most appropriate interpretation of the betweenness relation consists of considering an 'outcome \( z \)' that lies between outcomes \( x \) and \( y \)' as a 'natural compromise' between \( x \) and \( y \), namely, the betweenness relation is meant to represent a shared structure of compromises between outcomes.

Furthermore, a (finite) median join-semilattice \( \mathcal{X} = (X, \leq) \) admits a **rank-based metric** \( d_r : X \times X \to \mathbb{Z}_+ \) (with \( r : X \to \mathbb{Z}_+ \) denoting a **rank function** (see Appendix C for the definition)), defined, for any \( x, y \in X \), as \( d_r(x, y) = 2r(x \lor y) - r(x) - r(y) \).

---

7We recall that a poset \((Y, \leq)\) is a **distributive lattice** if and only if, for any \( x, y, z \in X \), \( x \land y \) and \( x \lor y \) exist, and \( x \land (y \lor z) = (x \land y) \lor (x \land z) \) (or, equivalently, \( x \lor (y \land z) = (x \lor y) \land (x \lor z) \)). Moreover, a (distributive) lattice \( \mathcal{X} \) is said to be **lower (upper) bounded** if there exists \( \bot \in \mathcal{X} (\top \in \mathcal{X}) \) such that \( \bot \leq x (\top \leq x) \) for all \( x \in \mathcal{X} \), and **bounded** if it is both lower bounded and upper bounded.

8We recall that a **metric** on \( X \) is a real-valued function \( \delta : X \times X \to \mathbb{R}_+ \) such that for any \( x, y, z \in X \): (i) \( \delta(x, y) = 0 \) iff \( x = y \); (ii) \( \delta(x, y) = \delta(y, x) \); (iii) \( \delta(x, z) \leq \delta(x, y) + \delta(y, z) \).

9The metric median induced by this metric is strictly related to some aggregation rules to be discussed in the present work. More details on this topic are provided in Appendix C.
Let \( \succ \) be a preorder, namely a reflexive and transitive binary relation, on \( X \) \(^{10}\) then we denote with \( \text{Top}(\succ) \) the possibly empty set of its maxima, and with \( || \) the set of its incomparable ordered pairs, i.e. \( x \parallel y \) if and only if neither \( x \succ y \) nor \( y \succ x \) hold. Then, we say that:

**Definition 2.** \( \succ \) is **locally unimodal** with respect to the betweenness relation \( B_\mu \), or \( B_\mu - \text{lu} \), if and only if:

(i): there exists a unique maximum of \( \succ \) in \( X \), its top outcome, denoted \( \text{Top}(\succ) \), and

(ii): for all \( x, y, z \in X \), if \( z \in \text{I}^\mu(\text{Top}(\succ), y) \setminus \{ \text{Top}(\succ) \} \) then (not \( y \succ z \)).

The local unimodality of \( \succ \) amounts to the requirement that individual preference relations have a unique maximum or top outcome and be such that an outcome located between the maximum and another distinct outcome is invariably regarded as not worse than the latter. Thus, local unimodality is in fact nothing else than the specific notion of **single-peakedness** we are going to use in the present work.

We denote by \( U_{B_\mu} \) the set of all \( B_\mu - \text{lu} \) preorders on \( X \) and by \( U_{B_\mu}^N \) the set of all \( N \)-profiles of \( B_\mu - \text{lu} \) preorders, where an \( N \)-profile of \( B_\mu - \text{lu} \) preorders is a mapping from \( N \) into \( U_{B_\mu} \). Moreover, we call a set \( D_X \subseteq U_{B_\mu}^N \) of locally unimodal preorders with respect to \( B_\mu \) **rich** if for all \( x, y \in X \) there exists \( \succ \in D_X \) such that \( \text{Top}(\succ) = x \) and \( \text{UC}(\succ, y) = \text{I}^\mu(x, y) \) (where \( \text{UC}(\succ, y) := \{ y \in X : x \succ y \} \) is the upper contour of \( \succ \) at \( y \)).

An aggregation rule for \((N, X)\) is a function \( f: X^N \rightarrow X \). We (occasionally) consider both weaker and stricter versions of aggregation rules, namely:

(i) a **restricted aggregation rule** for \((N, X)\) is a function \( f: D \rightarrow X \) for some \( D \subseteq X^N \);

(ii) a **multi-aggregation rule** for \((N, X)\) is a function \( f: X^N \rightarrow \mathcal{P}(X) \setminus \{\emptyset\} \) \(^{11}\)

(iii) (by contrast), a **constrained aggregation rule** for \((N, X)\) is a function \( f: X^N \rightarrow C \) for some \( C \subseteq X \).

All of the aforementioned notions of aggregation rule have been considered in the relevant literature. In the present paper we shall focus on ‘pure’ aggregation rules (and occasionally on constrained ones).

To proceed we define two compelling conditions on aggregation rules for median joint-semi lattice which will play a crucial role in our main characterization result.

**Definition 3.** An aggregation rule \( f: X^N \rightarrow X \) is said to be:

(i) **strategy-proof** on \( U_{B_\mu}^N \) if and only if, for all \( B_\mu \)-unimodal \( N \)-profiles \((\succ_i)_{i \in N} \in U_{B_\mu}^N \), and for all \( i \in N \), \( y_i \in X \), and \((x_j)_{j \in N} \in X^N \) such that \( x_j = \text{Top}(\succ_i) \) for each \( j \in N \), not \( f((y_i, (x_j)_{j \in N \setminus \{i\}})) \succ_i f((x_j)_{j \in N}) \);

(ii) **\( B_\mu \)-monotonic** if and only if, for all \( i \in N \), \( y_i \in X \), and \((x_j)_{j \in N} \in X^N \),

\[
\text{f}((x_j)_{j \in N}) \in \text{I}^\mu(x_i, f(y_i, (x_j)_{j \in N \setminus \{i\}})) \] \(^{12}\)

\(^{10}\)We denote with \( \succ \) and \( \sim \) the asymmetric and symmetric components of \( \succ \), respectively.

\(^{11}\)Notice that a multi-aggregation aggregation rule for \((N, X)\) can also be regarded as an instance of a restricted aggregation rule for \((N, \mathcal{P}(X))\).
Indeed, a reliable and effective decision protocol should be reputedly strategy-proof, i.e. immune to advantageous individual manipulations through submission of false information, and $B_{\mu}$-monotonic, i.e. the outcome that an agent obtains by submitting a certain outcome $x$ lies between $x$ itself and the outcome that the agent would obtain by submitting another outcome (for any fixed profile of proposals/submissions on the part of the other agents). Thus, both strategy-proofness on single-peaked domains and $B_{\mu}$-monotonicity of an aggregation rule defined on a median semilattice are properties that are by construction strictly related to the median-induced betweenness of the semilattice (it will be shown below that they are in fact equivalent).

We further observe that non-trivial strategy-proof aggregation rules should be, at least to some extent, input-responsive and both input-unbiased and output-unbiased. A few requirements can be deployed to present several versions, degrees and combinations of input-responsiveness, input-unbiasedness and output-unbiasedness of aggregation rules, namely:

**Definition 4.** An aggregation rule $f$ for $(N, X)$ is:

- **inclusive** if and only if, for each voter $i \in N$, there exist $x^N \in X^N$ and $y_i \in X$ such that $f(x^N \backslash \{i\}, y_i) \neq f(x^N)$;
- **anonymous** if, for each $x^N \in X^N$ and each permutation $\sigma$ of $N$, $f(x^N) = f(x^{\sigma(N)})$ (where $x^{\sigma(N)} = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$);
- **idempotent** (or unanimity-respecting) if $f(x, \ldots, x) = x$ for each $x \in X$;
- **sovereign** if, for each $y \in X$, there exists $x^N \in X^N$ such that $f(x^N) = y$ i.e. $f$ is an onto function;
- **neutral** if, for each $x^N \in X^N$ and each permutation $\pi$ of $X$, $f(\pi(x^N)) = \pi(f(x^N))$ (where $\pi(x^N) = (\pi(x_1), \ldots, \pi(x_k))$).

Anonymity, Idempotence and Neutrality have a straightforward standard meaning. Inclusiveness means that for each agent there exists at least one profile of outcomes at which her own outcome turns out to be pivotal. Sovereignty entails that every element of the aggregation rule’s codomain, i.e. each possible outcome, is the image of at least one element of its domain, i.e. of each possible outcome profile.

In particular, let $X = (X, \leq)$ be a finite join-semilattice and $M_X$ the set of its meet-irreducible elements, and for any $x^N \in X^N$, and any $m \in M_X$, posit $N_m(x^N) := \{i \in N : x_i \leq m\}$. Then, the following properties of an aggregation rule can also be introduced:

**Definition 5.** An aggregation rule $f : X^N \rightarrow X$ is:

(i) $M_X$-independent if and only if, for all $x_N, y_N \in X^N$ and all $m \in M_X$, if $N_m(x_N) = N_m(y_N)$ then $f(x_N) \leq m$ if and only if $f(y_N) \leq m$;

(ii) Isotonic if $f(x_N) \leq f(x'_N)$ for all $x_N, x'_N \in X^N$ such that $x_N \leq x'_N$ (i.e. $x_i \leq x'_i$ for each $i \in N$).

---

13 Notice that both Idempotence and Neutrality imply Sovereignty (but not conversely), while Anonymity and Sovereignty jointly imply Inclusiveness (but not conversely). However, it is easily checked that if Strategy-proofness holds, Sovereignty and Idempotence are in fact equivalent.
Thus, an $M_X$-independent rule ensures that at any pair of profiles having the same set of agents proposing an outcome consistent (respectively, inconsistent) with a certain join-irreducible element, the social outcome will also be either consistent or inconsistent with the latter in both cases. An aggregation rule $f$ is isotonic if it is an order-preserving function. It can be easily shown (see Monjardet (1990)) that the conjunction of $M_X$-Independence and Isotony is equivalent to the following condition:

**Definition 6.** An aggregation rule $f : X^N \to X$ is **monotonically $M_X$-independent** if and only if, for all $x_N, y_N \in X^N$ and all $m \in M_X$, if $N_m(x_N) \subseteq N_m(y_N)$ then $f(x_N) \leq m$ implies $f(y_N) \leq m$.

It should be noticed that $M_X$-Independence amounts to a weakening of the Arrovian Independence of Irrelevant Alternatives (more on this in Example 3 of Section 4).

### 3. Main results

We are now ready to state the main result of this paper concerning strategy-proofness of aggregation rules on rich domains of locally unimodal profiles in median join-semilattices.

**Theorem 1.** Let $X = (X, \leq)$ be a finite median join-semilattice, $B_\mu$ its median-induced betweenness, and $f : X^N \to X$ an aggregation rule for $(N, X)$. Then, the following statements are equivalent:

- $(i)$ $f$ is strategy-proof on $D^N$ for any rich domain $D \subseteq U_{B_\mu}$ of locally unimodal preorders w.r.t. $B_\mu$ on $X$;
- $(ii)$ $f$ is $B_\mu$-monotonic;
- $(iii)$ $f$ is monotonically $M_X$-independent.

A similar argument is used for the case of total preorders on (not necessarily finite) bounded distributive lattices in Savaglio and Vannucci (2019), and in Vannucci (2019). It should also be emphasized here that, obviously, (finite) distributive lattices are a prominent special subclass of (finite) median join-semilattices.

As a consequence of Theorem 1, we obtain the following:

**Corollary 1.** Let $X = (X, \leq)$ be a finite median join-semilattice, $B_\mu$ its median-induced betweenness, and $f : X^N \to X$ an aggregation rule. Then, the following statements are equivalent:

- $(i)$ $f$ is strategy-proof on $D^N$ for every rich domain $D \subseteq U_{B_\mu}$ of locally unimodal preorders with respect to. $B_\mu$ on $X$;
- $(ii)$ for each $m \in M_X$ there exists an order filter $F_m$ of $(\mathcal{P}(N), \subseteq)$ such that:

$$f(x_N) = f_{\{F_m : m \in M_X\}}(x_N) := \bigwedge \{ m \in M_X : N_m(x_N) \in F_m \}$$

for all $x_N \in X^N$.

14 The notions of $J_X$-Independence and Monotonic $J_X$-Independence are defined similarly by dualization for a finite median inf-semilattice $X = (X, \leq)$ as follows: for all $x_N, y_N \in X^N$ and all $j \in J_X$, if $N_j(x_N) := \{ i \in N : j \leq x_i \} \subseteq N_j(y_N) := \{ i \in N : j \leq y_i \}$, then $j \leq f(x_N)$ implies $j \leq f(y_N)$.

15 A similar result holds for finite median meet-semilattices, and can be easily established by dualization of the relevant arguments.
Proof. Immediate from Theorem 1 and duality of Proposition 1.4 of Monjardet (1990). In particular, each order filter $F_m$ consists of the **locally m-winning coalitions for** $f$, namely for every $m \in M_X$, 

$$F_m := \left\{ T \subseteq N : \text{there exists } x_N \in X^N \text{ such that} \right.$$

\[ \{ i \in N : x_i \leq m \} = T \text{ and } f(x_N) \leq m \left. \right\}.$$

It should be emphasized that the class of aggregation rules $f_{\{F_m : m \in M_X\}}$ identified by Corollary 1 is in principle very comprehensive indeed. More specifically, Corollary 1.(ii) allows a broad description of such rules as those returning the strictest consensus among the admissible alternatives actually sponsored by the agents of the relevant coalitions (as specified by the order filters $F_m$). In particular, the class of aggregation rules thus characterized encompasses a lot of suitably ‘inclusive’ and/or ‘unbiased’ rules, including the following:

- **Quorum system aggregation rules**, namely functions $f_{\{F_m : m \in M_X\}}$ such that every order filter $F_m$ is transversal, i.e. $S \cap T \neq \emptyset$ for all $S, T \in F_m$.
- **Inclusive aggregation rules**, namely functions $f_{\{F_m : m \in M_X\}}$ such that $\bigcup_{m \in M_X} F_m^\min = N$.
- **Collegial aggregation rules**, namely functions $f_{\{F_m : m \in M_X\}}$ such that for some $m \in M_X$, there exists a non-empty $S_m \subseteq N$ with $F_m \subseteq \{ T \subseteq N : S_m \subseteq T \}$.
- **Outcome-biased aggregation rules**, namely functions $f_{\{F_m : m \in M_X\}}$ where $F_m = \emptyset$ for some $m \in M_X$.
- **Weakly neutral (or $M_X$-neutral) aggregation rules**, namely functions $f_{\{F_m : m \in M_X\}}$ where $F_m = F_m'$ whenever $m \land m'$ exists.
- **Quota aggregation rules**, namely anonymous aggregation rules i.e. functions $f_{\{F_m : m \in M_X\}}$ such that for each $m \in M_X$ there exists an integer $q_{[m]} \leq |N|$ with $F_m = \{ T \subseteq N : q_{[m]} \leq |T| \}$. In particular, a quota aggregation rule is also $M_X$-neutral (or weakly neutral) if and only if $q_{[m]} = q_{[m']}$ whenever $m \land m'$ exists.

A prominent instance of a weakly neutral quota aggregation rule is co-majority as defined below.

**Definition 7. (Co-majority rule)** Let $\mathcal{X} = (X, \leq)$ be a finite median join-semilattice, and $N$ a finite set. Then, the **co-majority rule** $f^{\partial maj}$ for $(N, X)$ is defined as follows: for all $x_N \in X^N$,

$$f^{\partial maj}(x_N) := \bigwedge_{S \in W^{maj}} \left( \bigvee_{i \in S} x_i \right)$$

where $W^{maj} = \left\{ S \subseteq N : |S| \geq \left\lfloor \frac{|N|+2}{2} \right\rfloor \right\}$.

It is easily seen, and left to the reader to check, that the co-majority rule is in particular a positive instance of an idempotent, inclusive and transversal aggregation rule.

As a further corollary of Theorem 1 and Corollary 1 we obtain a new characterization of the co-majority rule via strategy-proofness, anonymity as defined above and the following well-known general property for aggregation rules, namely:

**Definition 8.** An aggregation rule $f : X^N \to X$ is **Bi-Idempotent** whenever, for any $x_N \in X^N$ and $y, z \in X$, if $x_i \in \{ y, z \}$ for all $i \in N$, then $f(x_N) \in \{ y, z \}$. 
Clearly, Bi-Idempotence amounts to a local requirement combining ‘decisiveness’ (the ability to select a single outcome) and ‘faithfulness’ (the ability to select the outcome among the proposals actually advanced) both under perfect binary polarization and under perfect agreement.

Thus, we have the following characterization result of the co-majority rule.

**Proposition 1.** Let \( X = (X, \leq) \) be a finite median join-semilattice, \( B_\mu \) its median betweenness relation, \( D \subseteq U_{B_\mu} \) a rich domain of locally unimodal preorders with respect to \( B_\mu \). Then, an aggregation rule \( f : X^N \to X \) satisfies Anonymity, Bi-Idempotence and is Strategy-proof on \( D^N \) with \( |N| \) odd if and only if \( f \) is the co-majority rule \( f^{\text{omaj}} \).

**Proof.** Immediate from Theorem 1 above and a straightforward dualization of Corollary 7.4 of Monjardet (1990).

The co-majority rule can also be seen as a way to compute a certain metric-median of the outcome profiles to be aggregated. Of course, this also applies to the case of total preorders to be discussed in the next section (see Appendix C for more details on this important topic).

We are now ready to consider a most significant application of the previous results that involves *strategy-proof aggregation of preferences* including (Arrowian) *social welfare functions* and their strategy-proofness properties: the next section is entirely devoted to that topic.

## 4. Applications to strategy-proof preference aggregation

The major examples of finite median join-semilattices we are going to analyze involve the set of all total preorders on a finite set and consider the corresponding (pure) aggregation rules, that are of course *social welfare functions*. There are several but subtly distinct ways of relating the set of total preorders to a median join-semilattice, regard such collection as a subset of a larger collection of admissible preference relations (e.g. reflexive and connected, or even just reflexive binary relations). We focus on the first and more straightforward example, while discussing only briefly the similar results that obtain from the application of Theorem 1 and its Corollaries to the latter cases.

**Total preorders and social welfare functions.**

**Example 1.** The join-semilattice of total preorders on a finite set.

Let \( A \) be a nonempty finite set of alternative social states, \( \mathcal{R}^T_A \) the set of all total preorders (i.e. reflexive, transitive and connected binary relations) on \( A \). Let us define the join of two total preorders on \( A \) as the *transitive closure* \( \overline{\cup} \) of their set-theoretic union. Then, by construction, \( \mathcal{X}' := (\mathcal{R}^T_A, \overline{\cup}) \) is a join-semilattice, and satisfies both *upper distributivity* (by Claim (P.1) of Janowitz (1984)), and *co-coronation* (by Claims (P.3) and (P.5) of Janowitz (1984)). It follows that \( (\mathcal{R}^T_A, \overline{\cup}) \) thus defined is indeed a median join-semilattice whose median ternary operation is denoted here \( \mu \), and its meet-irreducibles are the total preorders \( R_{A_1,A_2} \in \mathcal{R}^T_A \) having just two (non-empty) indifference classes \( A_1, A_2 \) such that (i) \( (A_1, A_2) \) is a two-block ordered partition of \( A \), written \( (A_1, A_2) \in \Pi^{(2)}_A \), namely \( A_1 \cup A_2 = A \), \( A_1 \cap A_2 = \emptyset \) and (ii) \( [xR_{A_1,A_2}y \text{ and not } yR_{A_1,A_2}x] \) if and only if \( x \in A_1 \) and \( y \in A_2 \). Such total preorders \( R_{A_1,A_2} \) with \( (A_1, A_2) \in \Pi^{(2)}_A \) are also the co-atoms of \( (\mathcal{R}^T_A, \overline{\cup}) \), hence, the join-semilattice of total preorders is in particular *co-atomistic*. 

Thus, a most interesting application of our main result involves aggregation rules for preference profiles of total preorders namely social welfare functions \( f : \mathcal{R}_A^N \to \mathcal{R}_A \) in the classic Arrowian sense. Such an application is made precise by the following proposition.

**Proposition 2.** Let \( A \) be a nonempty finite set of alternative social states, \( \mathcal{R}_A \) the set of all total preorders of \( A \), \( \mathcal{X} := (\mathcal{R}_A, \sqcup) \) the join-semilattice of total preorders of \( A \), \( \mu \) the median ternary operation of \( \mathcal{X} \), \( B_\mu \) the corresponding betweenness relation as previously defined, \( M_\mathcal{X} \) the set of all meet-irreducible elements of \( \mathcal{X} \), and \( f : \mathcal{R}_A^N \to \mathcal{R}_A \) an aggregation rule for \((N, \mathcal{R}_A)\). Then, the following statements are equivalent:

(i) \( f \) is strategy-proof on \( D^N \) for every rich domain \( D \subseteq \cup B_\mu \) of locally unimodal preorders w.r.t. \( B_\mu \) on \( \mathcal{R}_A \);

(ii) for each \( m \in M_\mathcal{X} \) there exists an order filter \( F_m \) of \((\mathcal{P}(N), \subseteq)\) such that, for all \( R_N \in \mathcal{R}_A^N \),
\[
  f(R_N) = f_{\{F_m : m \in M_\mathcal{X}\}}(R_N) := \bigcap \{m \in M_\mathcal{X} : \{i \in N : R_i \subseteq m\} \in F_m\}.
\]

Proof. Immediate, from Theorem 1 and Example 1. \(\Box\)

Notice that, as a consequence of the previous characterization result, there exist a large class of ‘classical’ Arrowian social welfare functions on \((N, A)\) which are inclusive and idempotent (or unanimity-respecting) as well as strategy-proof on an arbitrary rich domain of locally unimodal preorders with respect to the betweenness relation \( B_\mu \) of \((\mathcal{R}_A^N, \sqcup)\) or \((T_A, \sqcup)\)). Such a large class includes aggregation rules which are respectively neither anonymous nor neutral, just anonymous, just neutral, or both anonymous and neutral. To see this, consider the following list of examples:

- **Inclusive quorum systems**, namely functions \( f_{\{F_m : m \in M_\mathcal{X}\}} \) such that every order filter \( F_m \) is transversal i.e. \( S \cap T \neq \emptyset \) for all \( S, T \in F_m \) and \( \bigcup_{m \in M_\mathcal{X}} F_m = N \). Observe that such a class includes any rule such that for every \( R_m \in M_\mathcal{X} \), \( F_m \) is simple-majority collegial i.e. there exists a minimal simple majority coalition \( S_m \subseteq N \), \(|S_m| = \left\lfloor \frac{N+2}{2} \right\rfloor \) with \( F_m = \{T \subseteq N : S_m \subseteq T\} \). Generally speaking, inclusive quorum systems need not be anonymous or neutral.

- **Outcome-biased aggregation rules**, namely functions \( f_{\{F_m : m \in M_\mathcal{X}\}} \) where \( F_m = \emptyset \) for some \( m \in M_\mathcal{X} \). Observe that they include the subclass of those aggregation rules such that for some total preorder \( T \in \mathcal{R}_A^N \), including possibly a linear order, \( F_m = \emptyset \) for every \( m \in M_\mathcal{X} \) such that \( T \subseteq m \).

- **(Weakly) Neutral aggregation rules**, namely functions \( f_{\{F_m : m \in M_\mathcal{X}\}} \) where \( F_m = F_m^\prime \) whenever \( R_m \land R_m^\prime \) exists.
• Quota aggregation rules, namely functions \( f_{\{F_m: m \in M_X\}} \) such that for each \( m \in M_X \) there exists an integer \( q[m] \leq |N| \) with \( F_m = \{ T \subseteq N : q[m] \leq |T| \} \). Such rules are clearly anonymous, but not necessarily neutral: they are of course neutral as well if, furthermore, \( F_m = F_m' \) whenever \( m \cap m' \) exists.

It is worth noticing that a large subclass of such aggregation rules \( f_{\mathcal{M}_X} \) (including positive quota aggregation rules and inclusive quorum systems) satisfy the Basic Pareto Principle (BP), as made precise by the following definition and claim.

**Definition 9.** (Basic Pareto Principle (BP)) An aggregation rule \( f : \mathcal{R}^N \to \mathcal{R} \) with \( \mathcal{R} \in \{ \mathcal{R}_A^T, \mathcal{T}_A \} \) satisfies BP if for every \( x, y \in A \) and \( R_N \in \mathcal{R}_A^N \), if \( xR_Ny \) for every \( i \in N \) then \( xf(R_N)y \).

**Claim 2.** Let \( \mathcal{R} \in \mathcal{R}_A^T \) and \( f_{\mathcal{M}_X} : \mathcal{R}^N \to \mathcal{R} \) be an aggregation rule as defined above such that \( F_m \) is a nontrivial proper order filter (i.e. \( \varnothing \not\in F_m \neq \varnothing \)) for every \( m \in M_X \). Then \( f_{\mathcal{M}_X} \) satisfies BP.

**Proof.** Suppose that \( x, y \in A \) and \( R_N \in \mathcal{R}_A^N \) are such that \( xR_Ny \) for every \( i \in N \), yet not \( xf_{\mathcal{M}_X}y \). Namely, by construction,

\[
(x, y) \notin \bigcap \{ m \in M_X : \{ i \in N : R_i \subseteq m \} \in F_m \}.
\]

Hence, there exists \( m \in M_X \) such that \( \{ i \in N : R_i \subseteq m \} \in F_m \) and \( (x, y) \notin m \). However, by assumption, \( F_m \) is nonempty and every \( T \subseteq F_m \) is itself nonempty: thus, \( N \in F_m \). But then \( (x, y) \in R_i \subseteq m \) for any \( i \in T \), a contradiction. \( \square \)

A remarkable family of anonymous but typically not neutral aggregation rules for \( (N, \mathcal{R}) \) (with \( \mathcal{R} \in \{ \mathcal{R}_A^T, \mathcal{T}_A \} \) ) is that of Condorcet-Kemeny rules, as defined below (see also Young, Levenglick (1978), Young (1995)).

**Definition 10.** (Generalized Condorcet-Kemeny aggregation rules) Let \( \mathcal{X} := (\mathcal{R}, \sqcup) \in (\mathcal{R}_A^T, \sqcup) \) be the join-semilattice of total preorders on finite set \( A \), \( C(\mathcal{X}) \) its covering graph, \( \delta_{C(\mathcal{X})} \) the shortest-path metric on \( C(\mathcal{X}) \), \( \mathcal{L}_A \subseteq \mathcal{R} \) the set of linear orders on \( A \), \( N \) a finite set, and \( \leq \) a linear order on \( \mathcal{R} \).

The generalized Condorcet-Kemeny aggregation rule for \( (N, \mathcal{R}) \) induced by \( \leq \) is the function \( f_{\leq}^{\mathcal{C}K} : \mathcal{R}^N \to \mathcal{R} \) defined as follows: for all \( R_N \in \mathcal{R}^N \),

\[
f_{\leq}^{\mathcal{C}K}(R_N) := \min_{\leq} \left\{ R \in \mathcal{R} : \sum_{i \in N} \delta_{C(\mathcal{X})}(R, R_i) \leq \sum_{i \in N} \delta_{C(\mathcal{X})}(R', R_i) \right\}.
\]

In particular, the (strict) Condorcet-Kemeny aggregation rule for \( (N, \mathcal{R}) \) induced by \( \leq \) is the function \( f_{\leq}^{\mathcal{C}K} : \mathcal{R}^N \to \mathcal{L}_A \) defined as follows: for all \( R_N \in \mathcal{R}^N \),

\[
f_{\leq}^{\mathcal{C}K}(R_N) := \min_{\leq} \left\{ R \in \mathcal{L}_A : \sum_{i \in N} \delta_{C(\mathcal{X})}(R, R_i) \leq \sum_{i \in N} \delta_{C(\mathcal{X})}(R', R_i) \right\}.
\]

17See Appendix B for a precise definition of the covering graph and the shortest-path metric of \( \mathcal{X} \).
Notice that a (strict) Condorcet-Kemeny rule amounts to a constrained generalized Condorcet-Kemeny rule. It should also be emphasized that generalized Condorcet-Kemeny aggregation rules require a prefixed linear order \( \preceq \) as a tie-breaker device whenever the remoteness function \( \sum_{i \in N} \delta_{C(\mathcal{X})}(\cdot, R_i) \) of a profile \( R_N \) admits several distinct minima: that is the only role of \( \preceq \) in \( f^{CK}_N \) and \( f^{CK}_N^* \), and the source of the typical failure of Condorcet-Kemeny rules to satisfy Neutrality. It follows that, to the extent that uniqueness of minima of the remoteness function is warranted, the outcome of Condorcet-Kemeny rules is unaffected by the choice of \( \preceq \) and Neutrality is restored. That is precisely the case when the size \( n \) of the set of agents \( N \) is odd, as implied by the following characterization result:

**Proposition 3.** Let \( \mathcal{X} := (\mathcal{R}_A, \cup) \) be the join-semilattice of total preorders on finite set \( A \) as defined above, \( \mu \) its median ternary operation and \( B_\mu \) the corresponding betweenness relation as previously defined, \( N \) a finite set such that \( |N| \) is an odd number, and \( f : \mathcal{R}_A^N \to \mathcal{R}_A \) an aggregation rule for \((N, \mathcal{R}_A)\). Then, the following statements are equivalent:

(i) \( f \) satisfies Anonymity and Bi-Idempotence, and is strategy-proof on \( D^N \) for every rich domain \( D \subseteq U_{B_\mu} \) of locally unimodal preorders w.r.t. \( B_\mu \) on \( \mathcal{R}_A \);

(ii) \( f \) is defined, \( \mu \) its median ternary operation and \( B_\mu \) the corresponding betweenness relation as previously defined, \( N \) a finite set such that \( |N| \) is an odd number, and \( f : \mathcal{R}_A^N \to \mathcal{R}_A \) an aggregation rule for \((N, \mathcal{R}_A)\);

(iii) \( f = f^{\partial \text{maj}} = f^{CK}_N \) for \((N, \mathcal{R}_A)\) for any pair of linear orders \( \preceq, \preceq' \) on \( \mathcal{R} \).

**Proof.** Immediate from Theorem 1, Proposition 1, Claim 1, and Example 1 above.

Thus, in particular, when the size of \( N \) is odd the generalized Condorcet-Kemeny rule for \((N, \mathcal{R}_A)\) is precisely the same as the co-majority rule, and can be characterized as the unique aggregation rule for \((N, \mathcal{R}_A)\) (or, in other terms, the unique Arrowian social welfare function) which is Anonymous, Bi-Idempotent and strategy-proof on \( U_{B_\mu} \) (and any of its rich subdomains).

Moreover, notice that (for an odd \( n \)) \( f^{\partial \text{maj}} = f^{CK}_N \) satisfies a weak version of the so-called Condorcet principle, namely for every \((R_i)_{i \in N} \in \mathcal{R}_A \) and \( x \in A \), if \( x \) is a Condorcet winner, that is

\[ \sum_{i \in N} \delta_{C(\mathcal{X})}(\cdot, R_i) \]

The computational complexity issues raised by computation of the Condorcet-Kemeny aggregation rule will not be addressed in the present work. However, it is worth mentioning here that the computation of median total preorders for arbitrary profiles of total preorders is a NP-complete problem (i.e. it belongs to the class of the hardest problems whose solutions are polynomial-time verifiable or ‘easy’ to verify, but apparently worst-case ‘hard’ to compute). Specifically, if the size of \( N \) is suitably larger than the size of \( A \), computing a median total preorder is NP-complete for arbitrary profiles of total preorders or linear orders (Hudry (2012)) and NP-hard (i.e. ‘easy’ to reduce to a NP-complete problem) for arbitrary profiles of binary relations (Wakabayashi (1998)).

\[ \sum_{i \in N} \delta_{C(\mathcal{X})}(\cdot, R_i) \]

It should be noticed that the requirement that \( n = |N| \) be odd is not at all as restrictive as it might seem at first sight. In fact, for \( n \) even our aggregation rule \( f \) for \((N, \mathcal{R}_A^N)\) might be embedded in a natural way into a more comprehensive aggregation rule \( f \) for \((N, \mathcal{R}_A^N \times \mathbb{Z})\) (where \( \mathbb{Z} \) denotes the set of integer numbers) as supplemented with the natural projection from \( \mathbb{Z} \) to the finite additive group \( \mathbb{Z}_n \) of integers modulo \( n \). Such an aggregation rule implements a pseudo-random ‘anonymous’ selection of a ‘president’ in \( N \) to the effect of producing an artificially but fairly augmented ‘electorate’ of odd size. Furthermore, a similar construct obtained by replacing \( \mathbb{Z}_n \) with \( \mathbb{Z}_k \) (where \( k := |A| \)) results in a further aggregation rule \( \tilde{f} \) for \((N, \mathcal{R}_A^N \times \mathbb{Z})\) which implements a pseudorandom ‘neutral’ choice of one linear order among those consistent with the total preorder selected by \( f \) at any profile. Such an aggregation rule \( \tilde{f} \) is in fact constrained (actually an \( L_A^T \)-constrained one), since its values are constrained to lie in \( L_A^T \subseteq \mathcal{R}_A^T \).
\( \{ i \in N : xR_iy \text{ and not } yR_ix \in W^{maj} \text{ for every } y \in A \setminus \{x\}, \text{ then } x \in Top(\hat{f}^{maj}(\{R_i\}_{i \in N})) \text{ (where for any } R \in R_A, \text{ Top}(R) := \{x \in A : xR_y \text{ for all } y \in A\}) \).

To check this, suppose \( x \) is indeed a Condorcet winner, yet \( x \notin \text{ Top}(\hat{f}^{maj}(\{R_i\}_{i \in N})) \). Thus, there exist \( y \in X \setminus \{x\} \) and a meet-irreducible \( R_{[y][x]} \) of the join-semilattice \( (R_A, \sqcup) \), (i.e. a two-indifference-class total preorder having \( y \) among its maxima and \( x \) among its minima), such that \( \text{ Top}(\hat{f}^{maj}(\{R_i\}_{i \in N})) \subseteq R_{[y][x]} \). But then, upper distributivity of \( (R_A, \sqcup) \) entails that \( \bigcup_{i \in T} R_i \subseteq R_{[y][x]} \) for some \( T \in W^{maj} \) whence \( R_i \subseteq R_{[y][x]} \) for each \( i \in T \in W^{maj} \), a contradiction.

It is easily checked that \( \hat{f}^{maj} \) is also \( M_X \)-Neutral if \( n := |N| \) is odd. It follows that for any odd \( n \) there exists an Arrowian social welfare function on the full domain of total preorders on an arbitrary finite set which is anonymous, neutral, idempotent (because Bi-Idempotence clearly implies Idempotence), satisfies a monotonic independence property w.r.t. the meet-irreducible total preorders (which are the co-atoms of the join-semilattice \( (R_A, \sqsubseteq) \), i.e. the total preorders having just two indifference classes) and is strategy-proof on any rich locally unimodal preference domain on \( R_A \). Therefore, \( \hat{f}^{maj} \) is in particular a social welfare function that satisfies all the properties required by Arrow's (Im)Possibility Theorem except for the Independence of Irrelevant Alternatives (IIA) condition. \(^{20}\)

What is then the relationship between \( M_X \)-Independence (\( M_X \)-I) and IIA? It is quite clear that under Idempotence \( M_X \)-I is definitely weaker than IIA because, as a consequence of Proposition 1, the former is consistent with Anonymity and Neutrality of an (Arrowian) unanimity-respecting social welfare function while the latter is not. Indeed, as established by Hansson (1969), IIA in combination with Anonymity and Neutrality provides a characterization of the constant social welfare function having the universal indifference relation \( A \times A \) as its unique value (hence in particular the former combination of properties is inconsistent with Idempotence). In other terms, strengthening \( M_X \)-Independence to IIA is just impossible for unanimity-respecting, anonymous and neutral Arrowian social welfare functions.

**Remark 1. (The case of weak orders)** A weak order on \( A \) is a binary relation \( W \) that satisfy asymmetry (\( xWy \) entails not \( yWx \) for every \( x, y \in A \)) and negative transitivity (not \( xWy \) and not \( yWz \) entail not \( xWz \) for all \( x, y, z \in A \)). It is easily checked that the partially ordered set \( (W_A, \leq^\partial) \) of weak orders on \( A \) (where \( W \subseteq^\partial W' \) if and only if \( W' \subseteq W \)) is isomorphic to \( (R_A, \sqsubseteq) \) hence a median join-semilattice. It follows that versions of Corollary 1 and Proposition 1 also hold true for weak orders on a finite set.

As it turns out, reconciling unanimity-respecting and strategy-proof preference aggregation to IIA is however possible by moving away from the domain of total preorders, towards some more comprehensive preference domains. This observation brings us to a few other examples, to which we now turn.\(^{21}\)

\(^{20}\)Recall that Arrow’s IIA (in binary form) is a condition on social welfare functions \( f : (R_A^T)^N \to R_A^T \) defined as follows: for every \( x, y \in A \) and any \( R_N, R'_N \in (R_A^T)^N \) such that \( xR_iy \) if and only if \( xR'_iy \) for each \( i \in N \), \( xf(R_N)y \) entails \( xf(R'_N)y \).

\(^{21}\)A further relevant example is the median join-semilattice \( R_A^T \times P(A) \), which is particularly convenient when it comes to addressing squarely agenda-manipulation issues. Such semilattice will be discussed in some detail elsewhere.
Other types of preference relations and preference aggregation rules.

To begin with, let us consider the collections of *generalized weak tournaments* and of *generalized strict tournaments*, namely the set of all reflexive (respectively, irreflexive) connected relations on a finite set, no matter if transitive or not.

**Example 2. Two isomorphic join-semilattices: the join-semilattices of generalized weak tournaments and of generalized strict tournaments on a finite set.**

Let $A$ be a nonempty finite set of alternative social states with $|A| = m$, $T_A$ the set of all generalized weak tournaments namely reflexive and connected binary relations on $A$. Let us define the join of two total relations on $A$ as their set-theoretic union $\cup$. Then, by construction, $X' := (T_A, \cup)$ is a join-semilattice, and its meet-irreducibles are the $m \cdot (m - 1)$ total relations $T_{x,y} := \{(a,b) \in A^2 : (a,b) \neq (x,y)\}$ with $x, y \in A$, $x \neq y$: namely, total relations whose asymmetric components consist of some single ordered pair $(x,y)$. It can also immediately checked that such $T_{x,y}$ total relations are indeed the co-atoms of $(T_A, \cup)$. It turns out that validity of the following claim can be easily established. The join-semilattice $X'' := (T_A^\circ, \cup)$ is defined similarly on the set $T_A^\circ$ of all generalized strict tournaments, namely irreflexive and connected binary relations on $A$. It is easily checked that $X'$ and $X''$ are isomorphic join-lattices, with an isomorphism $\psi : T_A \to T_A^\circ$ between them being defined by the rule $\psi(T) := T \setminus \Delta_A$ for any $T \in T_A$, with $\Delta_A := \{(x,x) : x \in A\}$.

**Claim 3.** For any nonempty finite set $A$, the join-semilattices $(T_A, \cup)$ and $(T_A^\circ, \cup)$ are median.\(^{22}\)

Clearly, versions of Propositions 2 and 3 (and of Claim 2) also hold true for both generalized weak tournaments and generalized strict tournaments.

Next, we proceed to remove the connectedness requirement as well.

**Example 3. Two isomorphic lattices: the lattices of reflexive binary relations and of irreflexive binary relations on a finite set.**

Clearly enough, any distributive lattice $(X, \lor, \land)$ also provides an example of a median join-semilattice.

In particular, let $A$ be a nonempty finite set of alternative social states, $B_A^r$ the set of all reflexive binary relations on $A$, $(B_A^r, \subseteq)$ the set-inclusion poset on $B_A^r$. Let us then define the join $\lor$ and meet $\land$ of two reflexive binary relations on $A$ as their set-theoretic union $\cup$ and intersection $\cap$, respectively. Hence, $X' := (B_A^r, \cup, \cap)$ is indeed, by construction, a (bounded) *distributive lattice*. It follows that $\cup$-closedness of $B_A^r$ and both upper-distributivity and co-coronation trivially hold in $X'$, i.e. $(B_A^r, \cup)$ is in particular a *median join-semilattice* whose median $\mu'$ is precisely the median of the distributive lattice $(B_A^r, \cup, \cap)$. Namely, for any $R_1, R_2, R_3 \in B_A^r$,

$$\mu''(R_1, R_2, R_3) = (R_1 \cup R_2) \cap (R_2 \cup R_3) \cap (R_3 \cup R_1) = (R_1 \cap R_2) \cup (R_2 \cap R_3) \cup (R_3 \cap R_1).$$

Moreover, it can be easily shown (and left to the reader to check) that

$$M_{X'} = C_{X'} = \left\{\{(a,b)\} : a, b \in A, a \neq b\right\},$$

\(^{22}\)The proof is available from the authors upon request.
and

\[
J_{\mathcal{X}'} = A_{\mathcal{X}'} = \{\Delta_A \cup \{(a,b) : a, b \in A, a \neq b\},
\]

where \(\Delta_A := \{(a,a) : a \in A\}\).

Hence, \(\mathcal{X}'\) is in particular a \textit{co-atomistic} and \textit{atomistic} lattice.

It should be emphasized that the set of all total preorders on \(A\) is clearly a \textit{subset}, but \textit{not} a sub-join semilattice of the join-semilattice reduct \((B_A^r, \cup)\) of the lattice \((B_A^r, \cup, \cap)\), since the union of two total preorders may \textit{not} be transitive.\(^{23}\)

Let us turn now to the set \(B_A^r\) the set of all \textit{irreflexive} binary relations on \(A\) (namely, the binary relations \(R \subseteq A^2\) such that \((x,x) \notin R\) for every \(x \in A\)) and to \((B_A^r, \subseteq)\), namely the set-inclusion poset on \(B_A^r\). Since both the intersection and the union of two irreflexive relations are also irreflexive, \(\mathcal{X}'' := (B_A^r, \cup, \cap)\) is indeed, by construction, a (bounded) \textit{distributive lattice}. Moreover, \((B_A^r, \cup, \cap)\) and \((B_A^r, \cup, \cap)\) are isomorphic lattices, an obvious isomorphism \(\varphi : B_A^r \to B_A^r\) between them being defined by the rule \(\varphi(R) := R \setminus \Delta_A\).

Let us now introduce the strenghtening of \textit{Monotonic M\textsubscript{X}-Independence} which results from substituting IIA for \(M\textsubscript{X}\)-Independence.

\begin{definition}
(Monotonic IIA): Let \(A\) be a nonempty finite set of alternative social states, \(B_A \in \{B_A^r, B_A^r\}\), \(\mathcal{X}' := (B_A, \cup, \cap)\) the (bounded) distributive lattice induced on \(B_A\) by \(\cup\) and \(\cap\), and \(f : (B_A)^N \to B_A\) an aggregation rule for \((N, B_A)\). Then, \(f\) is \textit{monotonically IIA} if, for all \(R_N, R'_N \in (B_A)^N\) and all \((u, v) \in A^2\): if \(\{i \in N : uR_iv\} \subseteq \{i \in N : uR_iv\}\) then \(u f(x_N)v\) implies \(u f(y_N)v\).
\end{definition}

We are now ready to show that when the relevant join-semilattice is (the join-reduct of) \(\mathcal{X}' \in \{(B_A^r, \cup, \cap), (B_A^r, \cup, \cap)\}\) we can rely on the full force of a counterpart of Theorem 1 for bounded distributive lattices (see e.g. Savaglio, Vannucci (2019)) to obtain the following result.

\begin{proposition}
Let \(A\) be a nonempty finite set of alternative social states, \(B_A \in \{B_A^r, B_A^r\}\), \(\mathcal{X}' := (B_A, \cup, \cap)\) the (bounded) distributive lattice induced on \(B_A\) by \(\cup\) and \(\cap\), \(\mu^r\) its median ternary operation and \(B_{\mu^r}\) the corresponding betweenness as previously defined, and \(f : (B_A)^N \to B_A\) an aggregation rule for \((N, B_A)\). Then, the following statements are equivalent:

(i) \(f\) is strategy-proof on \(D^N\) for every rich domain \(D \subseteq U_{B_{\mu^r}}\) of locally unimodal preorders w.r.t. \(B_{\mu^r}\) on \(B_A\);

(ii) \(f\) is \(B_{\mu^r}\)-monotonic;

(iii) \(f\) is monotonically \(M\textsubscript{X}'\)-independent;

(iv) \(f\) is monotonically \(J\textsubscript{X}'\)-independent;

(v) \(f\) is monotonically IIA;

(vi) there exists an order filter \(\mathcal{F}\) of \((\mathcal{P}(N), \subseteq)\) and a family \(\{R_S \in B_A : S \in \mathcal{F}\}\) of relations in \(B_A\) such that \(f(R_N) = \bigcap_{S \in \mathcal{F}} ((\cup_{i \in S} R_i) \cup R_S)\) for all \(R_N \in (B_A^r)^N\).
\end{proposition}

\(^{23}\)To see this, consider e.g. \(A = \{a, b, c, d\}\), and the linear orders, \(R_1 := abcd, R_2 := dcba\) (written according to the usual ‘decreasing’ notation). Now, \(R_1 \cup R_2 = \{(x,x) : x \in A\} \cup \{(a, b), (b, a), (a, c), (c, a), (a, d), (d, a), (b, c), (b, d), (d, b), (c, d), (d, c)\}\) which is not transitive since \(\{(c, a), (a, b)\} \subseteq R_1 \cup R_2\) but \((c, b) \notin R_1 \cup R_2\).
(vii) there exists an order filter \( \mathcal{F} \) of \( (\mathcal{P}(N), \subseteq) \) and a family \( \{R_S \in \mathcal{B}_A : S \in \mathcal{F}\} \) of relations in \( \mathcal{B}_A \) such that \( f(R_N) = \bigcup_{S \in \mathcal{F}} ((\cap_{i \in S} R_i) \cap R_S) \) for all \( R_N \in (\mathcal{B}_A)^N \).

It goes without saying that the strategy-proof aggregation rules for \((N, \mathcal{B}_A)\) characterized above (with \( \mathcal{B}_A \in \{\mathcal{B}_A', \mathcal{B}_A^\prime\} \)) comprise counterparts to inclusive quorum systems, quota rules and all the other aggregation rules for total preorders mentioned above. The co-majority rule \( f^\partial_{\text{maj}} : (\mathcal{B}_A)^N \rightarrow \mathcal{B}_A \) is defined by the identity \( f^\partial_{\text{maj}}(R_N) = \bigcap_{S \in \mathcal{W}^\text{maj}} (\cup_{i \in S} R_i) \) for each \( R_N \in (\mathcal{B}_A)^N \), which is obtained from the general formula under statement (vi) of the previous proposition by setting \( \mathcal{F} = \mathcal{W}^\text{maj} := \{S \subseteq N : |S| \geq \lfloor |N|/2 \rfloor \} \) and \( R_S = \Delta_A \) for each \( S \in \mathcal{W}^\text{maj} \).

But new possibilities arise here. To begin with, a version of the majority rule \( f^\text{maj} : (\mathcal{B}_A)^N \rightarrow \mathcal{B}_A \) is now well-defined by the identity \( f^\text{maj}(R_N) = \bigcup_{S \in \mathcal{W}^\text{maj}} (\cap_{i \in S} R_i) \) for each \( R_N \in (\mathcal{B}_A)^N \), which is obtained from the general formula under statement (vii) of the previous proposition by setting \( \mathcal{F} = \mathcal{W}^\text{maj} \) and \( R_S = A \times A \) for each \( S \in \mathcal{W}^\text{maj} \).

Of course, the outputs of \( f^\text{maj} \) and \( f^\partial_{\text{maj}} \), or for that matter of any idempotent aggregation rule for \((N, \mathcal{B}_A)\) may well be nontransitive or even intransitive (i.e. include cycles with asymmetric components). To see this, just consider a profile consisting of identical elements. It then follows that the strategy-proof aggregation rules for \((N, \mathcal{B}_A)\) resulting from idempotent ones by just removing cycles from their outputs through a minimal number of pair-deletions are also \( B_{\mu^\prime}-\text{monotonic} \) (though, of course, not idempotent but rather weakly idempotent in the following sense: for any profile \( R_N \in (\mathcal{B}_A)^N \) such that \( R_i = R_j = R \) for all \( i, j \in N \), \( f(R_N) \subseteq R \). Thus, here is a new (sub)class of interesting strategy-proof aggregation rules for \((N, \mathcal{B}_A)\) whose output for any profile of total preorders is indeed a total preorder (let us call them minimal monotonic retracts just for ease of reference).

Furthermore, for an odd-sized \( N \) the majority rule for \((N, \mathcal{B}_A)\) turns out to coincide with the co-majority rule. This is made precise by the following:

**Proposition 5.** Let \( A \) be a nonempty finite set of alternative social states, \( \mathcal{B}_A \in \{\mathcal{B}_A', \mathcal{B}_A^\prime\} \), \( \mathcal{X} := (\mathcal{B}_A', \cup, \cap) \) the (bounded) distributive lattice induced on \( \mathcal{B}_A \) by \( \cup \) and \( \cap \), \( \mu' \) its median ternary operation and \( B_{\mu^\prime} \) the corresponding betweenness relation, \( N \) a finite set such that \( |N| \) is an odd number, and \( f : (\mathcal{B}_A)^N \rightarrow \mathcal{B}_A \) an aggregation rule for \((N, \mathcal{B}_A)\). Then, the following statements are equivalent:

(i) \( f \) satisfies Anonymity and Bi-Idempotence, and is strategy-proof on \( D^N \) for every rich domain \( D \subseteq U_{B_{\mu^\prime}} \) of locally unimodal preorders w.r.t. \( B_{\mu^\prime} \) on \( \mathcal{B}_A \);

(ii) \( f = f^\text{maj} = f^\partial_{\text{maj}} \);

(iii) \( f = f^{\text{CK}'} \) i.e. the generalized Condorcet-Kemeny aggregation rule for \((N, \mathcal{B}_A)\) for any pair of linear orders \( \preceq, \preceq' \) on \( \mathcal{B}_A' \).

**Proof.** Immediate from Corollary 1 and Example 3. \( \square \)

---

\(^{24}\)We omit the proof, which relies on a result mentioned in Davey, Priestley (1990), p. 178, and is available from the authors upon request.
Thus, when $N$ has an odd size, generalized Condorcet-Kemeny aggregation rules for $(N, \mathcal{R}_T)$, $(N, T_A)$, $(N, B_R)$ and $(N, B_{IR})$ are amenable to the same sort of simple characterization via Anonymity, Bi-Idempotence and Strategy-Proofness on certain rich single-peaked domains. Moreover, in both cases strict Condorcet-Kemeny rules are also available.

5. Concluding remarks

The results of the present work imply that, at least for an odd-sized population of agents, even anonymous and weakly neutral social welfare functions on the full domain of total preference preorders on a finite set do exist, and are indeed strategy-proof on suitably defined single-peaked domains of ‘preferences on preferences’ (i.e. arbitrary rich locally unimodal domains).

Arguably, such social welfare functions may also be regarded as a positive solution to a suitably reformulated version of the classic Arrowian preference aggregation problem. Namely, the focus is restricted to strategic as opposed to structural manipulation, and the Arrowian Independence condition IIA is accordingly replaced with a most ‘natural’ and milder independence requirement tightly related to the intrinsic order-theoretic structure of $\mathcal{R}_T$. In other words, we have here a first explicit escape route from Arrow’s ‘impossibility’ theorem on preference aggregation, which relies on retention of the ‘transitivity plus totality’ format requirement for preference relations as combined with a considerable weakening of IIA that relies on the (semi)latticial structure of the set of total preorders. Such a weakening is totally unrelated to other sorts of weakenings of IIA previously proposed in the literature including several versions of Positionalist Independence, as introduced and discussed by Hansson (1973) with no reference whatsoever to nonmanipulability issues. One of the strongest of them, labelled as Strong Positionalist Independence (SPI) by Hansson himself, requires invariance of aggregate preference between any two alternatives $x, y$ for any pair of preference profiles whose restrictions to $\{x, y\}$ are identical whenever for every agent/voter the supports of the respective closed preference intervals having $x$ and $y$ as their extrema are also identical. SPI has been recently rediscovered -and relabeled as Modified IIA- by Maskin (2020). Maskin motivates it in terms of resistance to certain sorts of ‘vote splitting’ effects, hence broadly speaking with reference to manipulation issues, including strategic manipulation. Notice, however, that what is at stake in that proposal is strategy-proofness of the ‘maximizing’ social choice function induced by a certain social welfare function (as opposed to strategy-proofness of the social welfare function itself).

\textsuperscript{25}Indeed, consider $\hat{f}^{maj}$ for $(N, A)$ with $|N| = |A| = 3$, and profiles $R_N, R_N'$ of linear orders with (under the usual permutation-based notation for linear orders, and square-bracket notation to denote indifference):

$R_1 = R_1' = xyz; \quad R_2 = R_2' = yzx; \quad R_3 = xzx, R_3' = xzy.$

Note that $R_3$ and $R_3'$ are adjacent. Nevertheless, as it is easily checked, $\hat{f}^{maj}(R_N) = [xyz]$, while $\hat{f}^{maj}(R_N') = x[zy]$. It is then immediately seen that the co-majority rule (which clearly satisfies $M_X$-Independence with respect to $X = (\mathcal{R}_T, \mathcal{T_0})$) does not satisfy IIA with respect to $X$. In fact, $R_i| \{x, y\} = R_i'| \{x, y\}$ for every $i \in N$. Yet, $yf(R_N)x$ while not $yf(R_N')x$.

\textsuperscript{26}Notice that the version of Proposition 3 that applies to generalized tournaments and previously mentioned in this work also implies that dropping transitivity and retaining just totality for both individual and social preferences is still another way out of Arrowian impossibility results. It goes without saying that such an escape route would leave ample scope for manipulation activities through agenda-structure control.
In a similar vein, another weakening of IIA that is even stronger than SPI has been proposed by Saari under the label ‘Intensity form of IIA’ (IIIA). IIIA requires invariance of aggregate preference between any two alternatives \(x, y\) for any pair of preference profiles such that \(\text{for every agent/voter the rank (or score) difference between } x \text{ and } y \text{ is left unchanged from one profile to the other}\) (see Saari (1995)). Arguably, Saari’s IIIA can also be regarded as a formalization of the criticism of IIA originally advanced by Dahl (1956) with his advocacy of aggregation rules based on intensity of individual preferences. Notice that IIIA is indeed satisfied by some positional aggregation rules such as the Borda Count. Moreover, both IIIA and \(M_X\)-Independence are also satisfied by majority judgment as discussed in Vannucci (2019), which provides an alternative approach to include intensity of preferences while preserving strategy-proofness. A further weakening of IIA in a quite different vein is due to Huang (2014), under the label \(\text{Weak Arrow’s Independence (WIIA)}\). In plain words, a social welfare function \(f\) satisfies WIIA if, for any pair \(R_N, R_N'\) of profiles of total preorders and any pair \(x, y\) of alternatives such that the preferences between \(x\) and \(y\) of every agent \(i\) in \(N\) are the same in \(R_N\) and \(R_N'\), the following condition holds: if \(x\) is strictly preferred to \(y\) according to social preference \(f(R_N)\) then \(x\) is preferred (i.e. either strictly preferred or indifferent) to \(y\) according to social preference \(f(R_N')\). Notice the main difference between WIIA and virtually all of the other weakenings of IIA considered in the present work: while the other weakenings strengthen the hypothetical clause of IIA and leave its consequent unaltered, WIIA keeps the hypothetical clause of IIA unaltered and weakens its consequent.

As mentioned above, even at a first glance one conspicuous difference between \(M_X\)-Independence and SPI (or IIIA and WIIA) stands out immediately: the former relies heavily on the structure of the outcome set, while SPI, IIIA and WIIA only impinge upon the relevant preference profiles, completely disregarding any specific feature/structure of the relevant outcome set (namely the set of all total preorders of the set \(A\) of basic alternatives). The \(M_X\)-Independence condition makes most sense if (to the contrary of IIA) it is divorced from agenda manipulation issues, and relies in fact on a fixed agenda setting. From a mechanism-design perspective, it amounts to a sort of divide-and-conquer approach to collective choice problems: the agenda-formation process and the underlying protocols (if any) are to be taken for granted, and thereby ignored.  

6. Appendix A

**Theorem 1.** Let \(\mathcal{X} = (X, \leq)\) be a finite median join-semilattice, \(B_\mu\) its median-induced betweenness, and \(f : X^N \to X\) an aggregation rule for \((N, X)\). Then, the following statements are equivalent:

(i) \(f\) is strategy-proof on \(D^N\) for any rich domain \(D \subseteq U_{B_\mu}\) of locally unimodal preorders w.r.t. \(B_\mu\) on \(X\);
(ii) $f$ is $B_{\mu}$-monotonic;
(iii) $f$ is monotonically $M_X$-independent.

Proof. (i) $\implies$ (ii) By contraposition. Let us assume that $f : X^N \to X$ is not $B_{\mu}$-monotonic. Thus, there exist $i \in N$, $x'_i \in X$ and $x_N = (x_i)_{i \in N} \in X^N$ such that $f(x_N) \not\in \{x_i, f(x'_i, x_{N \setminus \{i\}})\}$. Then, consider a preorder $\succ^*$ on $X$ defined as follows: $x_i = \text{top}(\succ^*)$ and for all $y, z \in X \setminus \{x_i\}$, $y \succ^* z$ if and only if (a) $\{y, z\} \subseteq [x_i, f(x'_i, x_{N \setminus \{i\}})] \setminus \{x_i\}$ or (b) $y \in [x_i, f(x'_i, x_{N \setminus \{i\}})] \setminus \{x_i\}$ and $z \not\in [x_i, f(x'_i, x_{N \setminus \{i\}})]$ or (c) $y \not\in [x_i, f(x'_i, x_{N \setminus \{i\}})]$ and $z \not\in [x_i, f(x'_i, x_{N \setminus \{i\}})]$. Clearly, by construction, $\succ^*$ consists of three indiffERENCE classes with $\{x_i\}$, $[x_i, f(x'_i, x_{N \setminus \{i\}})] \setminus \{x_i\}$ and $X \setminus [x_i, f(x'_i, x_{N \setminus \{i\}})]$ as top, medium and bottom indiffERENCE classes, respectively. Now, observe that $\succ^* \in U_{B_{\mu}}$ (which is by definition a rich locally unimodal domain w.r.t. $B_{\mu}$). To check that such a statement holds true, take any $y, z, v \in X$ such that $y \not\succ z$ and $v \in [y, z]$, i.e. $\mu(y, v, z) = v$ (if $y = z$, then $v = y = z$ and there is in fact nothing to prove). If $\{y, z\} \subseteq [x_i, f(x'_i, x_{N \setminus \{i\}})]$, then by definition $\mu(x_i, f(x'_i, x_{N \setminus \{i\}}), y) = y$ and $\mu(x_i, f(x'_i, x_{N \setminus \{i\}}), z) = z$. Thus, by property (\mu_2) of $\mu$,

$$
\mu(\mu(x_i, f(x'_i, x_{N \setminus \{i\}}), y), \mu(x_i, f(x'_i, x_{N \setminus \{i\}}), z), v) =
$$

$$
= \mu(\mu(y, z, v), x_i, f(x'_i, x_{N \setminus \{i\}})),
$$

whence

$$
\mu(\mu(x_i, f(x'_i, x_{N \setminus \{i\}}), y), \mu(x_i, f(x'_i, x_{N \setminus \{i\}}), z), v) = \mu(y, z, v) = v
$$

implies

$$
\mu(\mu(y, z, v), x_i, f(x'_i, x_{N \setminus \{i\}})) = \mu(y, z, v) = v,
$$

i.e. $v \in [x_i, f(x'_i, x_{N \setminus \{i\}})]$.

Clearly, $\{y, z\} \neq \{x_i\}$ since $y \not\succ z$. Now, assume without loss of generality that $y \neq x_i$ : thus $v \succ^* y$ by definition of $\succ^*$. If on the contrary $\{y, z\} \cap (X \setminus [x_i, f(x'_i, x_{N \setminus \{i\}})]) \neq \emptyset$, then clearly by definition of $\succ^*$ there exists $w \in \{y, z\}$ such that $v \succ^* w$. Thus, $\succ^* \in U_{B_{\mu}}$ as claimed. Also, by assumption $f(x_N) \in X \setminus [x_i, f(x'_i, x_{N \setminus \{i\}})]$ whence by construction $f(x'_i, x_{N \setminus \{i\}}) \succ^* f(x_N)$. But then, $f$ is not strategy-proof on $U_{B_{\mu}}^N$.

(ii) $\implies$ (i) Conversely, let $f$ be $B_{\mu}$-monotonic. Now, consider any $\succ = (\succ_j)_{j \in N} \in U_{B_{\mu}}^N$ and any $i \in N$. By definition of $B_{\mu}$-monotonicity, $f(\text{top}(\succ), x_{N \setminus \{i\}}) \in [\text{top}(\succ), f(x_i, x_{N \setminus \{i\}})]$ for all $x_{N \setminus \{i\}} \in X^{N \setminus \{i\}}$ and $x_i \in X$. But then, since clearly $\text{top}(\succ) \succ_i f(\text{top}(\succ), x_{N \setminus \{i\}})$, either $f(\text{top}(\succ), x_{N \setminus \{i\}}) \succ_i f(\text{top}(\succ), x_{N \setminus \{i\}})$ by local unimodality of $\succ_{i\text{w.r.t.} B_{\mu}}$. Hence, not $f(x_i, x_{N \setminus \{i\}}) \succ_i f(\text{top}(\succ), x_{N \setminus \{i\}})$ in any case. It follows that $f$ is indeed strategy-proof on $U_{B_{\mu}}^N$.

(iii) $\implies$ (ii) Hence, that $f$ is $B_{\mu}$-monotonic. Hence, for all $i \in N$, $y_i \in X$, and $(x_j)_{j \in N} \in X^N$, $f((x_j)_{j \in N}) \in I^p(x_i, f(y_i, (x_j)_{j \in N \setminus \{i\}}))$, i.e. $f((x_j)_{j \in N}) = \mu(x_i, f((x_j)_{j \in N}), f(y_i, (x_j)_{j \in N \setminus \{i\}}))$. Therefore, for any meet-irreducible element $m \in M_X$, $f((x_j)_{j \in N}) \leq m$ if and only if

$$
\mu(x_i, f((x_j)_{j \in N}), f(y_i, (x_j)_{j \in N \setminus \{i\}})) =
$$

$$
= (x_i \lor f((x_j)_{j \in N})) \land (f((x_j)_{j \in N}) \lor f(y_i, (x_j)_{j \in N \setminus \{i\}})) \land (x_i \lor f(y_i, (x_j)_{j \in N \setminus \{i\}})) \leq m.
$$
It follows that if $f((x_j)_{j \in N}) \leq m$ then $[x_i \leq m$ or $f(y_i, (x_j)_{j \in N \setminus \{i\}}) \leq m]$. Indeed, suppose that $f((x_j)_{j \in N}) \leq m$, yet $[x_i \not\leq m$ and $f(y_i, (x_j)_{j \in N \setminus \{i\}}) \not\leq m]$. Then, $(x_i \lor f((x_j)_{j \in N})) \not\leq m$, $f((x_j)_{j \in N}) \lor f(y_i, (x_j)_{j \in N \setminus \{i\}}) \not\leq m$, and $x_i \lor f(y_i, (x_j)_{j \in N \setminus \{i\}}) \not\leq m$. Therefore, since $\mathcal{X}$ is upper distributive, $(x_i \lor f((x_j)_{j \in N})) \land (f((x_j)_{j \in N}) \lor f(y_i, (x_j)_{j \in N \setminus \{i\}})) \not\leq m$ whence, by upper distributivity again, $(x_i \lor f((x_j)_{j \in N})) \land (f((x_j)_{j \in N}) \lor f(y_i, (x_j)_{j \in N \setminus \{i\}})) \land (x_i \lor f(y_i, (x_j)_{j \in N \setminus \{i\}})) \not\leq m$, i.e. $\mu(x_i, f((x_j)_{j \in N}), f(y_i, (x_j)_{j \in N \setminus \{i\}})) = f((x_j)_{j \in N}) \not\leq m$, a contradiction.

Now, suppose that $m \in M_X$, $f(x_N) \leq m$ and $N_m(x_N) \subseteq N_m(y_N)$ for some $x_N := (x_j)_{j \in N}$, $y_N := (y_j)_{j \in N} \in X^N$: we need to establish the claim that $f(y_N) \leq m$ as well.

By $B_\mu$-monotonicity of $f$, $x_i \leq m$ or $f(y_i, (x_j)_{j \in N \setminus \{i\}}) \leq m$ for any $i \in N$. Thus, if $x_i \leq m$, then also $y_i \leq m$, by assumption. Hence, $f((x_j)_{j \in N}) \leq m$ and $B_\mu$-monotonicity of $f$ entail $f(y_i, (x_j)_{j \in N \setminus \{i\}}) \leq m$: indeed, by $B_\mu$-monotonicity $f(y_i, (x_j)_{j \in N \setminus \{i\}}) \in I^\mu(y_i, f((x_j)_{j \in N}))$, i.e.

$$f(y_i, (x_j)_{j \in N \setminus \{i\}}) = (y_i \lor f(y_i, (x_j)_{j \in N \setminus \{i\}})) \land (f((y_i)_{j \in N \setminus \{i\}}) \lor f((x_j)_{j \in N})) \land (y_i \lor f((x_j)_{j \in N})) \leq (y_i \lor f((x_j)_{j \in N})) \leq m.$$  

It follows that $f(y_i, (x_j)_{j \in N \setminus \{i\}}) \leq m$ in any case.

But then, from $B_\mu$-monotonicity of $f$ and $f(y_i, (x_j)_{j \in N \setminus \{i\}}) \leq m$, it similarly follows that $x_{i+1} \leq m$ or $f((y_i, y_{i+1}, (x_j)_{j \in N \setminus \{i, i+1\}}) \leq m$. Now, $x_{i+1} \leq m$ entails $y_{i+1} \leq m$ as well, hence $f(y_i, (x_h)_{h \in N \setminus \{i\}}) \leq m$ and $B_\mu$-monotonicity jointly imply $f((y_i, y_{i+1}, (x_j)_{j \in N \setminus \{i, i+1\}}) \leq m$, by the same argument previously employed. Repeating the argument, we eventually obtain $f((y_i)_{i \in N}) \leq m$, which implies that $f$ is indeed monotonically $M_X$-independent as required.

(iii) $\implies$ (ii) Suppose that $f$ is monotonically $M_X$-independent but not $B_\mu$-monotonic. Thus, there exist $i \in N$, $(x_j)_{j \in N} \in X^N$, $y_i \in X$ such that $f((x_j)_{j \in N}) \not\in \mu(x_i, f((x_j)_{j \in N}), f(y_i, (x_j)_{j \in N \setminus \{i\}}))$, i.e. there must exist $m \in M_X$ such that $f((x_j)_{j \in N}) \not\leq m$ but $(x_i \lor f((x_j)_{j \in N})) \land (f((x_j)_{j \in N}) \lor f(y_i, (x_j)_{j \in N \setminus \{i\}})) \not\leq m$ or $(x_i \lor f(x_j)_{j \in N \setminus \{i\}}) \lor f(y_i, (x_j)_{j \in N \setminus \{i\}}) \not\leq m$ and $(x_i \lor f(y_i, (x_j)_{j \in N \setminus \{i\}}) \not\leq m$. Thus, suppose that $f((x_j)_{j \in N}) \leq m$ and $(x_i \lor f((x_j)_{j \in N})) \land (f((x_j)_{j \in N}) \lor f(y_i, (x_j)_{j \in N \setminus \{i\}}) \land (x_i \lor f(y_i, (x_j)_{j \in N \setminus \{i\}}) \not\leq m$. Then, it must be the case that $x_i \not\leq m$ and $f(y_i, (x_j)_{j \in N \setminus \{i\}}) \not\leq m$ whence by construction $N_m((x_j)_{j \in N}) \subseteq N_m((y_i, (x_j)_{j \in N \setminus \{i\}}))$ and therefore $f(y_i, (x_j)_{j \in N \setminus \{i\}}) \leq m$ by monotonic $M_X$-independence, a contradiction. Next, suppose that $(x_i \lor f((x_j)_{j \in N})) \land (f((x_j)_{j \in N}) \lor f(y_i, (x_j)_{j \in N \setminus \{i\}} \lor f(y_i, (x_j)_{j \in N \setminus \{i\}}) \not\leq m$ and $f((x_j)_{j \in N}) \not\leq m$.

Since, by upper distributivity of $\mathcal{X}$, it must be the case that either $(x_i \lor f((x_j)_{j \in N}) \leq m$ or $(f((x_j)_{j \in N}) \lor f(y_i, (x_j)_{j \in N \setminus \{i\}}) \leq m$ or else $(x_i \lor f(y_i, (x_j)_{j \in N \setminus \{i\}}) \leq m$, it follows that $(x_i \lor f(y_i, (x_j)_{j \in N \setminus \{i\}}) \leq m$ hence in particular both $x_i \leq m$ and $f(y_i, (x_j)_{j \in N \setminus \{i\}}) \leq m$. Thus,

$$N_j((y_i, (x_j)_{j \in N \setminus \{i\}}) \subseteq N_j((x_j)_{j \in N}) \land f(y_i, (x_j)_{j \in N \setminus \{i\}}) \leq m$$

whence, by monotonic $M_X$-independence, $f((x_j)_{j \in N}) \leq m$, a contradiction again, and the thesis is established. □
7. Appendix B

We present here further properties of a median semi-lattice.

A chain of poset \( X = (X, \leq) \) is a set \( Y \subseteq X \) such that for any distinct \( u, v \in Y \) either \( u \leq v \) or \( v \leq u \) holds, and its length \( l(Y) \) is \( |Y| - 1 \) (where \( |Y| \) denote its size). A chain \( Y \) of \( (X, \leq) \) having \( x \) as its \( \leq \)-minimum and \( y \) as its \( \leq \)-maximum is maximal if there is no \( z \in X \setminus Y \) such that \( x \leq z \leq y \). For any \( x, y \in X \) such that \( x < y \) (i.e. \( x \leq y \) and not \( y \leq x \)) the length of the order-interval \( [x, y] := \{ z \in X : x \leq z \leq y \} \), written \( l([x, y]) \), is the length of a (maximal) chain of maximum length having \( x \) as its \( \leq \)-minimum and \( y \) as its \( \leq \)-maximum. In particular, \( x \in X \) is said to be covered by \( y \in X \), written \( x \ll y \), iff \( x < y \) and \( [x, y] = \{ x, y \} \), namely \( l(\{ x, y \}) = 1 \). The covering graph \( C(X) = (X, E_{\leq}) \) of \( X \) is the undirected graph having \( X \) as vertex-set and \( E_{\leq} := \{ \{ x, y \} \subseteq X : x \ll y \text{ or } y \ll x \} \) as edge-set. A path \( \pi_{xy} \) of \( C(X) \) connecting two vertices \( x \) and \( y \) is a maximal chain \( \{ z_0, ..., z_k \} \) of \( X \) such that \( \{ z_0, z_k \} = \{ x, y \} \) and \( z_i \ll z_{i+1} \) for any \( i = 1, ..., k - 1 \), and is of length \( l(\pi_{xy}) = k \). The set of all paths of \( C(X) \) connecting \( x \) and \( y \) is denoted by \( \Pi_{xy} \). A geodesic from \( x \) to \( y \) on \( C(X) \) is a path of minimum length (or equivalently a shortest path) connecting \( x \) and \( y \). It can be easily proved (and left to the reader to check) that the shortest length function \( \delta_{C(X)} : X \times X \to \mathbb{Z}_+ \) such that, for any \( x, y \in X \), \( \delta_{C(X)}(x, y) := l(\pi_{xy}) \) (where \( \pi_{xy} \) is a path of minimum length in \( \Pi_{xy} \)) is indeed a metric namely for any \( x, y, z \in X \): (i) \( \delta_{C(X)}(x, y) = 0 \) iff \( x = y \), (ii) \( \delta_{C(X)}(x, y) = \delta_{C(X)}(y, x) \), (iii) \( \delta_{C(X)}(x, z) \leq \delta_{C(X)}(x, y) + \delta_{C(X)}(z, y) \).

The following properties of finite upper-distributive semi-lattices are also to be recalled:

**Claim 4.** (i) a finite upper distributive join-semilattice \( X = (X, \leq) \) is graded i.e. it admits a rank function namely a function \( r : X \to \mathbb{Z}_+ \) such that for any \( x, y \in X \) if \( x \ll y \) then \( r(y) = r(x) + 1 \) (see Barbut, Monjardet (1970), Leclerc (1994));

(ii) the rank function of a finite upper distributive join-semilattice is a valuation, namely for any \( x, y \in X \) the following condition holds: if the meet \( x \land y \) exists then \( r(x) + r(y) = r(x \lor y) + r(x \land y) \) (Leclerc (1994)).

In order to appreciate the actual content of the co-majority rule and its specialization to the case of total preorders, some further properties of median join-semilattices are to be introduced and discussed here.

**Claim 5.** (Barthélemy (1978), Monjardet (1981), Leclerc (1994)). Let \( X = (X, \leq) \) be a finite upper distributive join-semilattice, 1 its top element, \( C(X) \) its covering graph, and \( r \) its normalized rank function defined as follows: for any \( x \in X \), \( r(x) := r(1) - l([x, 1]) \). Then,

(i) the function \( d_r : X \times X \to \mathbb{Z}_+ \) such that for any \( x, y \in X \) \( d_r(x, y) := 2r(x \lor y) - r(x) - r(y) \) is a metric on \( X \);

(ii) \( d_r = \delta_{C(X)} \);

(iii) \( \delta_{C(X)}(x, y) = \delta_{C(X)}(x, z) + \delta_{C(X)}(z, y) \) for any \( x, y, z \in X \) such that \( z \in \pi_{xy} \) for some geodesics \( \pi_{xy} \) from \( x \) to \( y \) on \( C(X) \).

**Proof.** See Barthélemy (1978) (Proposition 1), Monjardet (1981) (Theorem 8), and Leclerc (1994) (Theorem 3.1).

\( \square \)
As a consequence, for any nonnegative integer \( n \in \mathbb{N} \) a nonempty metric median set \( \mathbf{m}(x_1, \ldots, x_n) \) can be defined on any finite family -or profile- of \( n \) elements \( x_i \), \( i = 1, \ldots, n \) of a finite median join-semilattice \( \mathcal{X} = (X, \preceq) \) with rank function \( r \) and covering graph \( C(\mathcal{X}) \) as follows:

\[
\mathbf{m}(x_1, \ldots, x_n) := \left\{ z \in X : z \in \arg\min_{x \in X} \sum_{i=1}^{n} d(x, x_i) \right\},
\]

where \( d = d_r = \delta_{C(\mathcal{X})} \) as defined above.

Thus, a metric median function \( \mathbf{m} : \bigcup_{n \in \mathbb{N}} X^n \to \mathcal{P}(X) \) (where \( \mathcal{P}(X) \) denotes the power set of \( X \), and all of its restrictions \( \mathbf{m}_{(n)} : X^n \to \mathcal{P}(X) \) to a fixed \( n \in \mathbb{N} \), are well-defined and nonempty-valued. In particular, if \( n \) is odd then it is well-known and easily proved that \( \mathbf{m}_{(n)} \) is single-valued, hence it can also be regarded as an \( n \)-ary algebraic operation on \( X \), written \( \hat{\mathbf{m}}_{(n)} : X^n \to X \) (see e.g. Bandelt, Barthélémy (1984), Monjardet, Raderanirina (2004), Hudry, Leclerc, Monjardet, Barthélémy (2009)).

The following well-known key result clarifies the tight connection (indeed, the equivalence) between the co-majority aggregation rule and the foregoing restrictions of the metric median function on a finite median join-semilattice \( \mathcal{X} = (X, \preceq) \).

**Claim 6.** \((\text{Bandelt, Barthélémy (1984)})\) Let \( \mathcal{X} = (X, \preceq) \) be a finite median join-semilattice and \( \mathbf{m} \) its metric median function. Then, for any \( n \in \mathbb{N} \), and any \( (x_1, \ldots, x_n) \in X^n \),

\[
f_{\partial \text{maj}}(x_N) := \bigwedge_{S \in \mathcal{W}_{\text{maj}}} \left( \bigvee_{i \in S} x_i \right) \subseteq \mathbf{m}_{(n)}(x_1, \ldots, x_n).
\]

Moreover, if \( n \) is odd then

\[
f_{\partial \text{maj}}(x_N) := \bigwedge_{S \in \mathcal{W}_{\text{maj}}} \left( \bigvee_{i \in S} x_i \right) = \hat{\mathbf{m}}_{(n)}(x_1, \ldots, x_n).
\]

**Proof.** Immediate, by Proposition 5 and dualization of Corollaries 1 and 2 of Bandelt, Barthélémy (1984).

Thus, the co-majority rule is essentially the same as a metric median rule, namely a rule that selects a metric median.

Finally, some further facts concerning betweenness relations in finite median join-semilattices are worth mentioning here in order to appreciate the naturalness and robustness of the betweenness relation involved in our main characterization Theorem.

Generally speaking, at least three distinct betweenness relations can be defined in a natural way on any finite graded join-semilattice (see e.g. Sholander (1952, 1954), Avann (1961), Van de Vel (1993)), namely:

(i) **median betweenness** \( B_\mu \) (for all \( x, y, z \in X \), \( B_\mu(x, y, z) = y \) where is the possibly partial median operation as defined above);

(ii) **interval betweenness** \( B_I \) (for all \( x, y, z \in X \), \( B_I(x, y, z) = y \preceq x \lor z \) and \( x \preceq y \) or \( z \preceq y \)),

(iii) **metric betweenness** \( B_d \) (for all \( x, y, z \in X \), \( B_d(x, y, z) \) iff \( d(x, y) + d(y, z) = d(x, z) \), with \( d = d_r = \delta_{C(\mathcal{X})} \) the interval-length-based metric as defined above).

Now, it turns out that if a finite join-semilattice is median then the relationships among \( B_\mu, B_I \) and \( B_d \) is very tight, as made precise by the following claim.
Claim 7. (Sholander (1952, 1954), Avann (1961), Barbut, Monjardet (1970), Leclerc (1994)). Let $\mathcal{X} = (X, \leq)$ be a finite median join-semilattice. Then, $B_I \subseteq B_\mu = B_d$. Moreover, if $\mathcal{X} = (X, \leq)$ is in particular a distributive lattice, then

(i) $B_I(x, y, z)$ holds if and only if $x \land z \leq y \leq x \lor z$;
(ii) $B_I = B_\mu = B_d$;
(iii) $d_r(x, y) = r(x \lor y) - r(x \land y) = |(M(x) \setminus M(y)) \cup (M(y) \setminus M(x))|$ (where $M(z) := \{ m \in M_X : z \leq m \}$).

It is thus confirmed, in particular, that local unimodality (the notion of single-peakedness introduced above and used in Theorem 1) rests on a very natural and robust notion of betweenness on the underlying poset $\mathcal{X} = (X, \leq)$, which is in turn tightly anchored to an ‘intrinsic’ metric of $\mathcal{X}$ itself. It should also be emphasized here that if $\mathcal{X}$ is in particular a median join-semilattice of binary relations on a finite set (as discussed in the next section), then point (iii) of the previous Claim establishes that $d_r$ is precisely the so-called Kemeny distance for binary relations as defined below (see Kemeny (1959)).

Kemeny distance on binary relations. Let $A$ be a finite set and $(B_A, \subseteq)$ the poset of all binary relations on $A$. Then the Kemeny distance on $(B_A, \subseteq)$ is the function $d_K : B_A \to \mathbb{Z}_+$ defined as follows: for any $R, R' \in B_A$,

$$d_K(R, R') := |\{(x, y) \in A \times A : xRy \text{ and not } xR'y\} \cup \{(x, y) \in A \times A : xR'y \text{ and not } xRy\}|$$

8. Appendix C

Related literature

The study of aggregation rules for ordered sets, semilattices, and lattices was pioneered by Monjardet and his co-workers, whose contributions provide characterizations of several classes of such rules mostly within a fixed population setting but also, occasionally, within a variable population framework (see e.g. Barthélemy, Monjardet (1981), Bandelt, Barthélemy (1984), Monjardet (1990), Barthélemy, Janowitz (1991), Leclerc (1994), Monjardet, Raderanirina (2004), Hudry et al. (2009)). In particular, characterizations of the simple majority and co-majority rules (sometimes also denoted as ‘median’ rules) are established in several latticial and semilatticial settings both as aggregation rules within a fixed population framework (see e.g. Monjardet (1990)) and as multi-aggregation rules within a variable population framework (see e.g. Barthélemy, Janowitz (1991), Monjardet, Raderanirina (2004)).

Concerning the special case of preference aggregation, an early characterization of (a version of) the Condorcet-Kemeny rule, regarded as a multi-aggregation rule for linear orders in a variable population setting is due to Young, Levenglick (1978). Indeed, Young and Levenglick prove that the Condorcet-Kemeny multi-aggregation rule is in fact the unique function $f : \bigcup_{n \in \mathbb{N}} (\mathcal{L}_A)^n \to \mathcal{P}(\mathcal{L}_A) \setminus \{\emptyset\}$ that satisfies the following three properties: neutrality, a version of the Condorcet principle, and ‘consistency’ across the

\footnote{Observe that $(B_A^d, \subseteq)$ is indeed a distributive lattice since it is obviously closed with respect to both intersection $\cap$ and union $\cup$.}
committees/electorates (i.e. for any pair of profiles $R_N, R_M$ such that $N \cap M = \emptyset$, if $f(R_N) \cap f(R_M) \neq \emptyset$ then $f((R_N, R_M)) = f(R_N) \cap f(R_M)$).  

In a similar vein, but in a much more general setting and building partly on Barthélémy, Janowitz (1991), McMorris, Mulder, Powers (2000) establishes a further elegant characterization of the median function as a multi-aggregation rule $f : \bigcup_{n \in \mathbb{N}} X^n \rightarrow (\mathcal{P}(X) \setminus \{\emptyset\})$ for a median meet-semilattice $(X, \preceq)$ in a variable population framework, using suitably generalized counterparts of a weaker version of Condorcet principle (labelled as $\frac{1}{2}$-Condorcet property) and ‘consistency’ across populations/electorates as presented above, and a very mild ‘faithfulness’ condition simply requiring $f((x)) = \{x\}$ for each $x$ in $X$.

The present paper obviously owes much to that most remarkable body of literature. Notice, however, that the contributions mentioned above do not consider at all strategy-proofness properties of aggregation rules (or, for that matter, nonmanipulability properties of any sort).

Some previous joint works of Nehring and Puppe (see in particular Nehring, Puppe (2007),(2010)) have also several significant connections to the present contribution. To be sure, Nehring, Puppe (2007) is mainly concerned with strategy-proof social choice functions as defined on profiles of total preorders on finite sets. Conversely, Nehring, Puppe (2010) is focussed on an ‘abstract’ class of Arrowian aggregation problems including preference aggregation and, more specifically, social welfare functions, but it does not address issues concerning their strategy-proofness properties. However, social choice functions with the tops-only property may be regarded as aggregation rules endowed with a specific domain of total preorders, and the class of Arrowian aggregation rules considered in Nehring, Puppe (2010) does include the case of preference aggregation rules in finite median semilattices. Specifically, Nehring and Puppe attach to any finite outcome space a certain finite hypergraph $\mathbb{H} = (X, \mathcal{H})$ denoted as property space, where the set $\mathcal{H} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ of (nonempty) hyperedges or properties of outcomes/states in $X$ is complementation-closed and separating (namely $X \setminus h \in \mathcal{H}$ whenever $h \in \mathcal{H}$, and for every two distinct $x, y \in X$ there exists $H_{x\leftrightarrow y} \in \mathcal{H}$ such that $x \in H_{x\leftrightarrow y}$ and $y \notin H_{x\leftrightarrow y}$). Such a property space $\mathbb{H}$ models the set of all binary properties of outcomes that are regarded as relevant for the decision problem at hand. Then, a betweenness relation $B_\mathbb{H} \subseteq X^3$ is introduced by stipulating that $B_\mathbb{H}(x, y, z)$ holds precisely when $y$ satisfies all the properties shared by $x$ and $z$.

Moreover, single-peaked preference domains on $X$ can be defined relying on $B_\mathbb{H}$. In particular, $B_\mathbb{H}$ is said to be median if for every $x, y, z \in X$ there exists a unique $m_{xyz} \in X$ such that $B_\mathbb{H}(x, m_{xyz}, y)$.

\footnote{In subsequent work (see e.g. Young (1995)), it is emphasized that at any profile $R_N$ of linear orders on a finite $A$ the linear orders selected by the Condorcet-Kemeny rules can also be regarded as the maximum likelihood rankings according to the evidence provided by $R_N$. It should be noted that Young’s argument is quite general and also applies to wider classes of preference relations on $A$ including the set of all total preorders $\mathcal{R}_A^{\uparrow}$ and the set of all reflexive relations $\mathcal{R}_A^\uparrow$.}

\footnote{A social choice function for $(N, A)$ is a function $f : \mathcal{D}^N \rightarrow A$ where $\mathcal{D} \subseteq \mathcal{R}_A^\uparrow$: it satisfies the tops-only property if $f(R_N) = f(R'_N)$ whenever $t(R_i) = t(R'_i)$ for each $i \in N$, and $|t(R_i)| = |t(R'_i)| = 1$ for all $i \in N$ (with $t(R_i) := \{x \in A : xR_iy$ for all $y \in A\}$).}

\footnote{In particular, a nonempty subset $Y \subseteq X$ is said to be convex for $\mathbb{H} = (X, \mathcal{H})$ if for every $x, y \in Y$ and $z \in X$, if $B_\mathbb{H}(x, z, y)$ then $z \in Y$, and prime (or a halfspace) for $\mathbb{H}$ if both $Y$ and $X \setminus Y$ are convex for $\mathbb{H}$ and $\{Y, X \setminus Y\} \subseteq \mathcal{H}$.}
The following key results are obtained by Nehring and Puppe: (i) the class of all idempotent social choice functions which are strategy-proof on the domain of single-peaked preferences thus defined are characterized in terms of voting by binary issues through a certain combinatorial property of the families of winning coalitions for the relevant issues and (ii) if the property space is median then such combinatorial property is definitely met, and consequently non-dictatorial neutral and/or anonymous strategy-proofs aggregation rules including the majority voting rule are available (Nehring, Puppe (2007), Theorems 3 and 4). Furthermore, in Nehring, Puppe (2010) the very same theoretical framework is deployed to analyze preference aggregation and social welfare functions. In particular, several ‘classical’ properties for social welfare conditions including the Arrowian Independence of Irrelevant Alternatives (IIA) property can be reformulated in more general terms which depend on the specification of the relevant property space, it follows that several versions of IIA can be considered. But then, as it turns out, (iii) the versions of IIA attached to median property spaces are consistent with anonymous and neutral social welfare functions including those induced by majority-based aggregation rules (Nehring, Puppe (2010), Theorem 4). Interestingly, a specific example of a median property space for the set of all total preorders is also provided, namely the one whose issues consist in asking for each non-empty \( Y \subseteq X \) and any total preorder \( R \) whether or not \( Y \) is a lower contour of \( R \) with respect some outcome \( x \in X \).

The overlappings between such results and those presented here are remarkable, along with some sharp differences which make them mutually independent. Since finite median semilattices are indeed an example of a finite median algebra, and are consequently representable as median property spaces, all of the Nehring and Puppe’s results mentioned above do apply to finite median semilattices as a special case. Notice however that our results provide a characterization of strategy-proof aggregation rules for finite median join-semilattices which is both more explicit (it includes a polynomial

\[ B_0(x, m_{xyz}, z) \text{ and } B_0(y, m_{xyz}, z) \text{ hold} \]

in that case, \( \mathbb{H} \) is said to be a median property space, \( (X, m^\mathbb{H}) \) (where \( m^\mathbb{H} : X^3 \to X \) is defined by the rule \( m^\mathbb{H}(x, y, z) = m_{x,y,z} \) for every \( x, y, z \in X \)) is a median algebra, and for each \( u \in X \) the pair \( (X, \lor_u) \) (where \( x \lor_u y = y \) iff \( m^H(x, y, u) = y \) for some \( u \in X \)) is a median join-semilattice having \( u \) as its maximum.

The combinatorial property mentioned in the text is the so-called ‘Intersection Property’ which requires that for every minimally inconsistent set of properties, it must be the case that any selection of winning coalitions for the corresponding binary issues has a non-empty intersection.

Specifically, given a property space \( \mathbb{H} = (\mathcal{R}_A, H) \), such a generalized IIA for a social welfare function \( f : (\mathcal{R}_A)^N \to \mathcal{R}_A \) can be defined as follows: for every \( H \in \mathbb{H} \) and \( R_N, R'_N \in (\mathcal{R}_A)^N \) such that \( \{ i \in N : R_i \in H \} = \{ i \in N : R'_i \in H \} \), if \( f(R_N) \in H \) then \( f(R'_N) \in H \) as well. Of course the original Arrowian version of such a generalized IIA is obtained by taking \( H := \{ H(x, y) : x, y \in A \} \) with \( H(x, y) := \{ R \in \mathcal{R}_A : xRy \} \).

Thus, the property space suggested here is \( \mathbb{H}^\mathcal{O} := \{ \mathcal{R}_A, H^\mathcal{O} \} \), where \( H^\mathcal{O} := \{ H_L : \emptyset \neq L \subseteq A \} \) and \( H_L := \{ R \in \mathcal{R}_A : \text{for some } x \in A, L = \{ y \in A : xRy \} \} \).

Specifically, a finite median join-semilattice can be regarded as a generic instance of a finite median algebra with one of its elements singled out (that point corresponds to the top element of the semilattice).

For instance, it is always possible represent a (finite) median algebra as a (finite) property space by taking as properties its prime sets as defined through its median betweenness (see e.g. Bandelt, Hedlichková (1983), Theorem 1.5, and note 19 above for a definition of prime sets). It is important to observe that in general a finite median algebra or ternary space admits of several representations by distinct median property spaces (and other non-median as well). By contrast, a ternary (finite) algebra or space which is not median can only be represented by (finite) property spaces which are not median.
description of some such rules) and more comprehensive (it is a complete characterization in that it is not limited to sovereign and idempotent rules). Concerning alternative representations of the semilattice of total preorders on a finite set, our treatment can also be translated in terms of a median property space, though a different one from that considered by Nehring and Puppe. In fact, in our case the set of relevant properties corresponds to the meet-irreducibles of that semilattice, namely the total preorders having just two indifference classes, or equivalently a binary ordered classification of outcomes as good or bad, respectively. Accordingly, the collection of relevant issues consist in asking, for each binary good/bad classification of outcomes and any total preorder \( R \), whether the latter is consistent with the given binary classification. Summing up, while Nehring and Puppe’s contributions do not address explicitly strategy-proofness issues for social welfare functions, their approach via property spaces provides an additional and helpful perspective to appreciate the content and significance of the results of the present work.

The issue of strategy-proofness for preference aggregation rules has been indeed explicitly addressed in the previous literature, but never -to the best of the authors’ knowledge- with respect to the ‘full’ domain of all total preorders on a set. Under the heading ‘social welfare functions’, Bossert and Storcken (1992) study in fact aggregation rules for linear orders on a finite set (hence what we refer to as strict social welfare functions) and their coalitional strategy-proofness properties with respect to topped metric total preference preorders (on the set of linear orders) as induced by a suitably ‘renormalized’ version of the Kemeny distance to be further discussed below. They prove an impossibility theorem for those coalitionally strategy-proof and sovereign strict social welfare functions that also satisfy a certain condition of independence from extrema.

Working within a variable population framework, Bossert and Sprumont (2014) offer several possibility results concerning restricted strategy-proof aggregation rules (mapping profiles of linear orders on a finite set \( A \) into total preorders on \( A \) ) which are strategy-proof on the domain of topped preferences (on the set of total preorders) that are single-peaked with respect to the median betweenness of the distributive lattice of reflexive binary relations on \( A \) (which amounts to an outcome space \( B^A \) which is far more comprehensive than the ‘small’ domain-base \( L_A \) or even the larger codomain \( \mathcal{R}^T_A \) of the aggregation rule). That paper identifies some (variable-population) strategy-proof restricted aggregation rules on \( \mathcal{R}^T_A \) including (strict) Condorcet-Kemeny rules, a class of variable-population counterparts of our monotonic retracts of the majority relation as introduced above, and a family of rules denoted as status-quo rules that are related to the class of outcome-biased rules mentioned above as one family of examples covered by Proposition 2. An (implicit) characterization of such

---

39 Thus, the appropriate version of generalized IIA in our own model is \( H^* := (\mathcal{R}^T_A, H^*) \) with \( H^* := \left\{ H_{A_1, A_2} : A_1 \neq \emptyset \neq A_2 \right\} \)

\[
H_{A_1, A_2} := \left\{ R \in \mathcal{R}^T_A : R \subseteq R_{A_1, A_2} \right\}
\]

and \( R_{A_1, A_2} \) is of course the two-indifference-class total preorders having \( A_1 \) and \( A_2 \) as top and bottom indifference classes, respectively. Notice that both \( \Xi^* \) and Nehring-Puppe’s \( \Xi^* \) as previously defined (see footnote 22 above) are median property spaces, while the original Arrowian \( \Xi \) is not.

40 Thus, in a sense, the median-induced betweenness relation under consideration (and the resulting single-peakedness property) is not the one ‘naturally’ dictated by the codomain \( \mathcal{R}^T_A \) (let alone the strictly smaller domain-base \( L_A \)) of the aggregation rule.
monotonic majority-retracts is also provided, and the family of status-quo rules is explicitly characterized (but the strict Condorcet-Kemeny rules are not). Thus, the present paper provides extensions of the fixed-population counterparts of such strategy-proof restricted aggregation rules to strategy-proof exact aggregation rules for total preorders, and a unified joint characterization of all of them (see in particular Corollary 1, Propositions 2 and Proposition 3 above), as well as a specific characterization of generalized Condorcet-Kemeny rules for the case of odd-dimensional domains. Notice, however, that the notion of betweenness underlying the relevant notion of single-peakedness for ‘preferences on preferences’ that guarantees the strategy-proofness of such exact rules in the present paper is in fact a most ‘natural’ one, namely the median betweenness which is characteristic of their domain-base $R^T_A$ (but is not well-defined on its subdomain $L_A$).

The issue of strategy-proof aggregation in arbitrary (possibly infinite) join-semilattices is addressed in Bonifacio, Massó (2020) within a fixed population framework. To be sure, that work focuses in fact on so-called ‘simple rules’, namely anonymous and unanimity-respecting social choice functions with the ‘tops-only’-property. But then, such ‘simple rules’ are essentially equivalent to anonymous and idempotent aggregation rules which are endowed with an explicitly pre-defined domain of preference profiles of total preorders. In particular, the Authors consider a restriction on total topped preference preorders they denote (join-)semilattice-single-peakedness which results in a maximal domain that is consistent with the existence of strategy-proof ‘simple rules’. Then, they proceed to characterize the subclass of anonymous and idempotent strategy-proof aggregation rules, establishing that they are precisely the ‘supremum’ rule $f^\vee$ and a family of ‘generalized quota-supremum’ rules. It should also be noticed that such a comparatively weak notion of semilattice-single-peakedness is admittedly consistent with the notion of single-peakedness induced by metric-betweenness according to the shortest-path-metric on the covering graph of the semilattice.

However, semilattice-single-peakedness is clearly bound to relinquish any connection not only to a median-induced betweenness if the relevant semilattice is not median, but also to the most natural rank-based metric betweenness if the semilattice also happens to be not even graded. Therefore, in the latter case there is no natural metric to ground the claim that a certain type of single-peakedness describes a sort of ‘preferences on preferences’ that are induced in a ‘natural’ -hence plausibly shared-way by the actual basic preferences of agents.

It is also worth mentioning here that, in any case, strategy-proofness only concerns strategic manipulation of a preference-aggregation process, namely manipulation of the outcome of a certain game.

---

41See footnote 31 above.
42The notion of semilattice-single-peakedness (SSP) for total preorders on a join-semilattice $(X, \leq)$ was first introduced in Chatterji, Massó (2018). A total preorder $R$ on $X$ is SSP in $(X, \leq)$ if and only if: (i) $R$ has a unique maximum element $x^*$ in $X$; (ii) $yRx$ for each $y, z \in X$ such that $x^* \leq y \leq z$; (iii) $(x^* \vee u)Ru$ for each $u \in X$ such that $x \not\in u$.
43The ‘supremum’ (or join n-projection) rule $f^\vee$ for $(N, X)$ is defined as follows: $f^\vee(x_N) := \vee_{i \in X} x_i$. A generalized quota-supremum rule returns a certain prefixed alternative $x^*$ if $x^*$ reaches a prespecified quota, and $\vee_{i \in X} x_i$ otherwise.
44That is so because (if the join-semilattice $(X, \leq)$ is discrete i.e. it has no bounded infinite chain) for any pair of elements $x, y \in X$ which are not $\leq$-comparable the join $x \vee y$ must lie on a shortest path from $x$ to $y$ of the covering graph of the semilattice.
45Lattices (hence, of course, semilattices) which are not graded are quite common: in the present context, the lattice of partial preorders is perhaps the most obvious example (see e.g. Barbut, Monjardet (1970)).
by means of an appropriate choice of strategy in the available strategy-set(s). In other terms, a given game is implicitly being taken for granted, including of course the population of its players and the set of its possible alternative outcomes, or its agenda. But then, manipulation of the agenda (or, for that matter, of the relevant population of players itself) can also be considered: notice, however, that from a game-theoretic perspective, that is a kind of structural (as opposed to strategic) manipulation since it amounts to a change of the game itself.

Such a broader perspective on manipulation issues in preference-aggregation is apparent (if mostly implicit) in Sato (2015). Indeed, Sato’s contribution relies on a fixed population framework and is mainly focussed on strict social welfare functions as defined on some connected domain of linear orders over a finite set \( A \). However, it also considers the family of social choice functions which are induced by any such strict social welfare function on the subsets of \( A \) through maximization -at each preference profile- of the ‘social’ linear order selected at that profile (as restricted to the relevant subset of \( A \)). In that connection, four notions of nonmanipulability for strict social welfare functions are considered, with the primary aim to address issues of strategic manipulation. Then, relying on a renormalized and ‘contracted’ version \( \hat{d}_K \) of the Kemeny distance as defined previously, Sato introduces a ‘continuity-type’ condition for strict social welfare functions called Bounded Response. A strict social welfare function \( f \) satisfies Bounded Response if \( \hat{d}_K(f(R_N), f(R'_N)) \leq 1 \) whenever two preference profiles \( R_N, R'_N \) are the same except for the preference of a single agent \( i \), and \( R_i \) and \( R'_i \) are adjacent (i.e. \( R'_i \) is obtained from \( R_i \) by permuting the \( R_i \)-ranks of a single pair of alternatives with consecutive \( R_i \)-ranks). In a similar vein, a very mild Adjacency-Restricted Monotonicity condition for strict social welfare functions is considered. The main result established by Sato (2015) is the equivalence of the following statements for a strict social welfare function \( f \) on a connected domain of linear orders on \( A \): (1) \( f \) satisfies Bounded Response and at least one of the four nonmanipulability

---

46 This is also, arguably, Arrow’s own perspective on manipulation issues (see Arrow (1963)), except that he overtly renounces to address strategic manipulation issues, while acknowledging their substantial import (see e.g. Arrow (1963), chpt. 1). By contrast, agenda-manipulation issues play a key role in the arguments offered by Arrow to support his own proposal of the Independence of Irrelevant Alternatives (IIA) condition for social welfare functions (a more detailed discussion of the relationship of IIA to agenda manipulation will be provided elsewhere).

47 A connected domain of linear orders over \( A \) is a set \( \mathcal{D} \subseteq \mathcal{L}_A \) such that for any \( R, R' \in \mathcal{D} \) there exists a finite family \( \{R_1, ..., R_k\} \subseteq \mathcal{D} \) such that (i) \( R_1 = R \); (ii) \( R_k = R' \); (iii) for every \( i = 1, ..., k-1 \), \( R_i \) and \( R_{i+1} \) can be mutually obtained by reversing the respective ranks of two adjacent (or consecutive) elements of \( A \) that are ‘adjacent’ (i.e. consecutive) according to the other. Thus, a (strict) social welfare function on a connected domain is a function \( f : \mathcal{D}^N \rightarrow \mathcal{L}_A \) (clearly, it is also restricted for \( (N, \mathcal{L}_A) \) if \( \mathcal{D} \neq \mathcal{L}_A \)). Observe that \( \mathcal{L}_A \) itself is of course a connected domain.

48 One of them is akin to the notion of strategy-proofness for aggregation rules proposed by Bossert, Sprumont (2014) as discussed above, and another one relies on the ‘renormalized’ Kemeny distance for linear orders. By contrast, the last two nonmanipulability notions invoke the induced maximizing choices on \( A \), and on its subsets, respectively (and are also most suitable to address certain agenda-manipulation issues).

49 Namely, the ‘halved’ Kemeny distance for linear orders. That is essentially the distance between rankings due to Kendall, given by the minimal number of transpositions of adjacent elements that is necessary to obtain one linear order starting from another one (see e.g. Kendall (1955)).

50 It is worth recalling here that 1 is the minimum positive value of both \( d_K \) and \( \hat{d}_K \).
conditions mentioned above; (2) \( f \) satisfies Bounded Response and each one of the four nonmanipulability conditions mentioned above; (3) \( f \) satisfies Adjacency-Restricted Monotonicity and the Arrowian Independence of Irrelevant Alternatives (IIA) condition.

As a corollary of that result (and of arguments from standard proofs of the Arrowian ‘impossibility’ theorem for strict social welfare functions) a new characterization of dictatorial strict social welfare functions in terms of Bounded Response, one of the four equivalent nonmanipulability conditions mentioned above, and Sovereignty (or Ontoness) is established. Furthermore, the set \( D^{sp}(Q) \) of linear orders on \( A \) which are single-peaked with respect to some fixed linear order \( Q \) on \( A \) can also be shown to be a connected domain, and the strict social welfare function \( f^{wmaj} \) induced by the method of ‘(weak) majority decision’ clearly satisfies both Adjacency-Restricted Monotonicity and IIA. Hence, it immediately follows that \( \tilde{f}^{wmaj} : D^{sp}(Q) \to \mathcal{L}_A \) is a (restricted) strict social welfare function which satisfies both Bounded Response and all of the four nonmanipulability conditions mentioned above (hence, in particular, the strategy-proofness properties implied by the first two conditions from that list).

Thus, at least when applied to strict social welfare functions, the combination of Bounded Response and standard nonmanipulability conditions (including, more specifically, strategy-proofness requirements) tends apparently to reproduce a well-known pattern. Namely, ‘impossibility’ theorems on the full domain of linear orders, and some ‘possibility’ results on suitably restricted domains of linear orders (to the effect that e.g. several versions of the simple majority rule provide well-defined and strategy-proof restricted strict social welfare functions on certain single-peaked domains of linear orders).

By contrast, the existence issue for strategy-proof social welfare functions as aggregation rules on the full domain of total preorders or even larger sets of reflexive and possibly nontransitive binary relations on a finite set has never been addressed explicitly in previously published work, as mentioned above.

References

[1] Arrow K.J. (1963): Social Choice and Individual Values (2nd ed.). Yale University Press, New Haven.
[2] Avann S.P. (1961): Metric ternary distributive semilattices, Proceedings of the American Mathematical Society 12, 407-414.
[3] Bandelt H.J., J.-P. Barthélémy (1984): Medians in median graphs, Discrete Applied Mathematics 8, 131-142.
[4] Bandelt H.J., J. Hedlǐková (1983): Median algebras, Discrete Mathematics 45, 1-30.
[5] Barbut M., B. Monjardet (1970): Ordre et Classification. Algèbre et Combinatoire, Vol. 1,2. Hachette, Paris.
[6] Barthélémy J.-P. (1978): Remarques sur les propriétés métriques des ensembles ordonnés, Mathématiques et Sciences Humaines 61, 39-60.
[7] Barthélémy J.-P., M.F. Janowitz (1991): A formal theory of consensus, SIAM Journal on Discrete Mathematics 4, 305-322.

---

51 A strict social welfare function \( f \) is sovereign if for any \( L \in \mathcal{L}_A \) there exists \( R_N \in \mathcal{L}_A^N \) such that \( f(R_N) = L \).
52 Namely, for any \( R_N \in \mathcal{L}_A^N \), and \( x, y \in A \), \( xf^{wmaj}(R_N)y \) if and only if \( |N_x(R_N)| \geq |N_y(R_N)| \).
53 The significant body of literature devoted to the elaboration of such two related themes is extensively reviewed in the fourth chapter of Gaertner (2001).
[8] Barthélémy J.-P., B. Monjardet (1981): The median procedure in cluster analysis and social choice theory, *Mathematical Social Sciences* 1, 235-268.

[9] Bonifacio A.G., J. Massó (2020): On strategy-proofness and semilattice single-peakedness, *Games and Economic Behavior* 124, 219-238.

[10] Bossert W., Y. Sprumont (2014): Strategy-proof preference aggregation: possibilities and characterizations, *Games and Economic Behavior* 85, 109-126.

[11] Bossert W., T. Storcken (1992): Strategy-proofness of social welfare functions: the use of the Kemeny distance between preference orderings, *Social Choice and Welfare* 9, 345-360.

[12] Chatterji S., J. Massó (2018): On strategy-proofness and the salience of single-peakedness, *International Economic Review* 59, 163-189.

[13] Dahl R.A. (1956): *A Preface to Democratic Theory*. University of Chicago Press, Chicago.

[14] Danilov V.I. (1994): The structure of non-manipulable social choice rules on a tree, *Mathematical Social Sciences* 27, 123-131.

[15] Davey B.A., H.A. Priestley (1990): *Introduction to Lattices and Order*. Cambridge University Press, Cambridge UK.

[16] Dutta B., M.O. Jackson, M. Le Breton (2001): Strategic candidacy and voting procedures, *Econometrica* 69, 1013-1037.

[17] Gaertner W. (2001): *Domain Conditions in Social Choice Theory*. Cambridge University Press, Cambridge UK.

[18] Hansson B. (1973): The independence condition in the theory of social choice, *Theory and Decision* 4, 25-49.

[19] Huang W.-S. U. (2014): Singularity and Arrow’s paradox, *Social Choice and Welfare* 42, 671-706.

[20] Hudry O. (2012): On the computation of median linear orders, of median complete preorder and of median weak orders, *Mathematical Social Sciences* 64, 2-10.

[21] Hudry O., B. Leclerc, B. Monjardet, J.-P. Barthélémy (2009): Metric and latticial medians, in D. Bouyssou, D. Dubois, M. Pirlot, H. Prade (eds.): *Decision-making Process: Concepts and Methods*. Wiley, New York.

[22] Janowitz M.F. (1984): On the semilattice of weak orders of a set, *Mathematical Social Sciences* 8, 229-239.

[23] Kemeny J.G. (1959): Mathematics without numbers, *Daedalus (Quantity and Quality* 88)), 577-591.

[24] Kendall M.G. (1955): *Rank Correlation Methods* (2nd ed.). Griffin, London.

[25] Leclerc B. (1994): Medians for weight metrics in the covering graphs of semilattices, *Discrete Applied Mathematics* 49, 281-297.

[26] Maskin E. (2020): A modified version of Arrow’s IIA condition, *Social Choice and Welfare* 54, 203-209.

[27] McMorris F.R., H.M. Mulder, R.C. Powers (2000): The median function on median graphs and semilattices, *Discrete Applied Mathematics* 101, 221-230.

[28] Monjardet B. (1981): Metrics on partially ordered sets- A Survey, *Discrete Mathematics* 35, 173-184.

[29] Monjardet B. (1990): Arrowian characterizations of latticial federation consensus functions, *Mathematical Social Sciences* 20, 51-71.

[30] Monjardet B., V. Raderanirina (2004): Lattices of choice functions and consensus problems, *Social Choice and Welfare* 23, 349-382.

[31] Nehring K., C. Puppe (2007): The structure of strategy-proof social choice-part I: general characterization and possibility results on median spaces, *Journal of Economic Theory* 135, 269-305.

[32] Nehring K., C. Puppe (2010): Abstract Arrowian aggregation, *Journal of Economic Theory* 145, 467-494.

[33] Saari D.G. (1995): *Basic Geometry of Voting*. Springer, New York.

[34] Sato S. (2015): Bounded response and the equivalence of nonmanipulability and independence of irrelevant alternatives, *Social Choice and Welfare* 44, 133-149.

[35] Savaglio E., S. Vannucci (2019): Strategy-proof aggregation rules and single peakedness in bounded distributive lattices, *Social Choice and Welfare* 52: 295-327.

[36] Sethuraman J., C.-P. Teo, R.V. Vohra (2003): Integer programming and Arrovian social welfare functions, *Mathematics of Operations Research* 28, 309-326.
STRATEGY-PROOF AGGREGATION IN MEDIAN SEMILATTICES

[37] Sholander M. (1952): Trees, lattices, order, and betweenness, *Proceedings of the American Mathematical Society* 3, 369-381.

[38] Sholander M. (1954): Medians and betweenness, *Proceedings of the American Mathematical Society* 5, 801-807.

[39] Van de Vel M.L.J. (1993): *Theory of Convex Structures*. North Holland, Amsterdam.

[40] Vannucci S. (2019): Majority judgment and strategy-proofness: a characterization, *International Journal of Game Theory* 48, 863-886.

[41] Wakabayashi Y. (1998): The complexity of computing medians of relations, *Resenhas IME-USP* 3 (n.3), 323-349.

[42] Young H.P. (1995): Optimal voting rules, *Journal of Economic Perspectives* 9, 51-64.

[43] Young H.P., A. Levenglick (1978): A consistent extension of Condorcet’s election principle, *SIAM Journal on Applied Mathematics* 35, 285-300.

DEA, UNIVERSITY OF CHIETI-PESCARA, ITALY

DEPS, UNIVERSITY OF SIENA, ITALY