An Interpolation Problem for Functions with Values in a Commutative Ring

Daniel Alpay and Haim Attia

Abstract. It was recently shown that the theory of linear stochastic systems can be viewed as a particular case of the theory of linear systems on a certain commutative ring of power series in a countable number of variables. In the present work we study an interpolation problem in this setting. A key tool is the principle of permanence of algebraic identities.

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1. Introduction

There are numerous connections between classical interpolation problems and optimal control and the theory of linear systems; see for instance [10, 1]. In these settings, the coefficient space is the complex field $\mathbb{C}$, or in the case of real systems, the real numbers $\mathbb{R}$. Furthermore, already from its inception, linear system theory was considered when the coefficient space is a general (commutative) field, or more generally a commutative ring; see [22, 25]. In [8, 6] a new approach to the theory of linear stochastic systems was developed, in which the coefficient space is now a certain commutative ring $\mathfrak{A}$ (see Section 3 below). The results from [22, 25] do not seem to be directly applicable to this theory, and the specific properties of $\mathfrak{A}$ played a key role in the arguments in [8, 6]. We set

$$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}.$$ 

The purpose of this work is to discuss the counterparts of classical interpolation problems in this new setting. To set the problems into perspective, we begin this
introduction with a short discussion of the deterministic case. In the classical theory of linear systems, input-output relations of the form

$$y_n = \sum_{k=0}^{n} h_{n-k} u_k, \quad n = 0, 1, \ldots,$$

(1.1)

where \((u_n)_{n \in \mathbb{N}_0}\) is called the input sequence, \((y_n)_{n \in \mathbb{N}_0}\) is the output sequence, and \((h_n)_{n \in \mathbb{N}_0}\) is the impulse response, play an important role. The sequence \((h_n)_{n \in \mathbb{N}_0}\) may consist of matrices (of common dimensions), and then the input and output sequences consist of vectors of appropriate dimensions. Similarly state space equations

$$x_{n+1} = Ax_n + Bu_n,$$

$$y_n = Cx_n + Du_n, \quad n = 0, 1, \ldots$$

play an important role. Here \(x_n\) denotes the state at time \(n\), and \(A, B, C\) and \(D\) are matrices with complex entries. The transfer function of the system is

$$h(\lambda) = \sum_{n=0}^{\infty} h_n \lambda^n,$$

in the case (1.1), and

$$h(\lambda) = D + \lambda C(I - \lambda A)^{-1}B$$

in the case of state space equations, when assuming the state at \(n = 0\) to be equal to 0. Classical interpolation problems bear various applications to the corresponding linear systems. See for instance [10, Part VI], [21]. To fix ideas, we consider the case of bitangential interpolation problem for matrix-valued functions analytic and contractive in the open unit disk (Schur functions), and will even consider only the Nevanlinna-Pick interpolation problem in the sequel to keep notation simple, but it will be clear that the discussion extends to more general cases. Recall (see [10, §18.5 p. 409]) that the bitangential interpolation problem may be defined in terms of a septuple of matrices \(\omega = (C_+, C_-, A_\pi, A_\zeta, B_+, B_-, \Gamma)\) by the conditions

$$\sum_{\lambda_0 \in \mathbb{D}} \text{Res}_{\lambda = \lambda_0} (\lambda I - A_\zeta)^{-1}B_+ S(\lambda) = -B_-,$$

$$\sum_{\lambda_0 \in \mathbb{D}} \text{Res}_{\lambda = \lambda_0} S(\lambda)C_- (\lambda I - A_\pi)^{-1} = C_+,$$

$$\sum_{\lambda_0 \in \mathbb{D}} \text{Res}_{\lambda = \lambda_0} (\lambda I - A_\zeta)^{-1}B_+ S(\lambda)C_- (\lambda I - A_\pi)^{-1} = \Gamma,$$

where \(A_\zeta\) and \(A_\pi\) have their spectra in the open unit disk, where \((A_\zeta, B_+)\) is a full range pair (that is, controllable) and where \((C_-, A_\pi)\) is a null kernel pair (that is, observable). We send the reader to [10] for the definitions. Moreover, \(\Gamma\) satisfies the compatibility condition

$$\Gamma A_\pi - A_\zeta \Gamma = B_+ C_+ + B_- C_-.$$