ON INVARIANTS AND ENDOMORPHISM RINGS OF CERTAIN LOCAL COHOMOLOGY MODULES

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Abstract. Let \((R, \mathfrak{m})\) denote an \(n\)-dimensional Gorenstein ring. For an ideal \(I \subset R\) with grade \(I = c\) we define new numerical invariants \(\tau_{i,j}(I)\) as the socle dimensions of \(H^i_m(H^{n-j}_m(R))\). In case of a regular local ring containing a field these numbers coincide with the Lyubeznik numbers \(\lambda_{i,j}(R/I)\). We use \(\tau_{d,d}(I), d = \dim R/I,\) to characterize the surjectivity of the natural homomorphism \(f : \hat{R} \to \text{Hom}_{\hat{R}}(H^c_{I\hat{R}}(\hat{R}), H^c_{I\hat{R}}(\hat{R}))\). As a technical tool we study several natural homomorphisms. Moreover we prove a few results on \(\tau_{i,j}(I)\).

1. Introduction

Let \((R, \mathfrak{m})\) denote a local ring. For an ideal \(I \subset R\) let \(H^i_I(R)\), \(i \in \mathbb{Z}\), denote the local cohomology modules of \(R\) with respect to \(I\) (see [4] and [2] for the definitions). In recent research there is an interest in the study of the endomorphism rings \(\text{Hom}_R(H^i_I(R), H^i_I(R))\) for certain \(i\) and \(I\). This was started by Hochster and Huneke (see [8]) for \(i = \dim R\) and \(I = \mathfrak{m}\). See also [3] for a generalization to an arbitrary ideal \(I\) and \(i = \dim R\).

The case of a cohomologically complete intersection with \(i = \text{grade } I\) was investigated at first by Hellus and Stückrad (see [7]). They have shown that the natural map \(f : \hat{R} \to \text{Hom}_{\hat{R}}(H^c_{I\hat{R}}(\hat{R}), H^c_{I\hat{R}}(\hat{R}))\) is an isomorphism. In the case of a Gorenstein ring \((R, \mathfrak{m})\) it follows (see [17]) that \(\text{Hom}_{\hat{R}}(H^c_{I\hat{R}}(\hat{R}), H^c_{I\hat{R}}(\hat{R}))\) is a commutative ring. Here we continue with the investigations of \(f\).

Theorem 1.1. Let \((R, \mathfrak{m})\) denote an \(n\)-dimensional Gorenstein ring. For an ideal \(I \subset R\) with grade \(I = c\) and \(d = \dim R/I\) the following conditions are equivalent:

(i) \(\dim_k \text{Hom}_R(k, H^d_m(H^c_I(R))) = 1\).

(ii) The natural map \(f : \hat{R} \to \text{Hom}_{\hat{R}}(H^c_{I\hat{R}}(\hat{R}), H^c_{I\hat{R}}(\hat{R}))\) is surjective.

Let \(S = \cap (R \setminus \mathfrak{p})\) where the intersection is taken over all associated prime ideal \(\mathfrak{p} \in \text{Ass}_R R/I\) with \(\dim R/\mathfrak{p} = \dim R/I\). Then \(\ker f = 0R_S \cap R\). That is, if \(\hat{R}\) is a domain then \(\text{Ann} H^c_{I\hat{R}}(\hat{R}) = 0\).

As a consequence of Theorem 1.1 there is a numerical condition for \(f\) being an isomorphism. In general the socle dimension of \(H^d_m(H^c_I(R))\) is not finite. In case of \((R, \mathfrak{m})\) a regular local ring containing a field it is known that \(\dim_k \text{Hom}_R(k, H^d_m(H^c_I(R)))\) is equal
to the Lyubeznik number \( \dim_k \mathrm{Ext}_R^d(k, H^i_f(R)) \). Another part of these investigations is the relation of the Bass-Lyubeznik number to the socle dimension of \( H^d_m(H^i_f(R)) \). To this end we consider several re-interpretations of the natural map \( f \) by Local and Matlis Duality resp. This is related to the problem whether the natural homomorphism \( \mathrm{Ext}_R^d(k, H^i_f(R)) \to k \) is non-zero, a question originally posed by the second author and Hellus (see [6]). Under the additional assumption of \( \text{inj dim}_R H^i_f(R) \leq d \) we are able to prove that if \( \mathrm{Ext}_R^d(k, H^i_f(R)) \to k \) is non-zero (resp. isomorphic), then \( f \) is injective (resp. isomorphic). Note that if \((R, \mathfrak{m})\) is a regular local ring containing a field \( \text{inj dim}_R H^i_f(R) \leq d \) holds by the results of Huneke and Sharp resp. Lyubeznik (see [10] resp. [13]). It is unknown to us whether this is true for any regular local ring. In the final section we discuss the \( \tau \)-numbers in more detail.

2. Preliminaries

Let \((R, \mathfrak{m})\) denote a (local) Gorenstein ring of dimension \( n \). Let \( I \subset R \) be an ideal of \( \text{grade}(I) = c \). Then we have that \( \text{height}(I) = c \) and \( d := \dim_R (R/I) = n - c \). For the definition and basic results on local cohomology theory we refer to [4] and the textbook [2]. In the following we need a slight extension of the Local Duality Theorem as its was proved at first by Grothendieck (see [4]).

For an arbitrary local ring \((R, \mathfrak{m})\) let \( \hat{R} \) denote the \( \mathfrak{m} \)-adic completion. Moreover \( E = E_R(k), k = R/\mathfrak{m}, \) denotes the injective hull of the residue field \( k \).

Let \( \underline{x} = x_1, \ldots, x_r \in I \) denote a system of elements of \( R \) such that \( \text{Rad} I = \text{Rad}(\underline{x})R \).

We consider the \( \check{\text{Cech}} \) complex \( \check{C} \) with respect to \( \underline{x} = x_1, \ldots, x_r \) (see [2] and also [13] for its definition). Then

\[
H^i(\check{C}_\underline{x} \otimes_R M) \simeq H^i_I(M)
\]

for \( i \in \mathbb{Z} \) and an \( R \)-module \( M \), where \( H^i_I(M), i \in \mathbb{Z}, \) denote the local cohomology of \( M \) with support in \( I \).

**Lemma 2.1.** Let \((R, \mathfrak{m})\) denote a Gorenstein ring. Let \( M \) denote an arbitrary \( R \)-module. Then there are the following natural isomorphisms

(a) \( \text{Tor}_n^R(M, E) \simeq H^0_n(M) \) and

(b) \( \text{Hom}_R(H^i_m(M), E) \simeq \text{Ext}^{n-i}_R(M, \hat{R}) \)

for all \( i \in \mathbb{Z} \).

**Proof.** Let \( \underline{x} = x_1, \ldots, x_n \) denote a system of parameters of \( R \). Then \( H^i_n(R) \simeq H^i(\check{C}_{\underline{x}}) \) and therefore

\[
H^i_m(R) \simeq H^i(\check{C}_{\underline{x}}) = 0
\]

for all \( i \neq n \) and \( H^0_n(R) \simeq E \). Note that \( R \) is a Gorenstein ring. That is, \( \check{C}_{\underline{x}}[-n] \) provides a flat resolution of \( E \). Whence by the definitions

\[
\text{Tor}_n^R(M, E) \simeq H^i(\check{C}_{\underline{x}} \otimes_R M) \simeq H^i_n(M)
\]

for all \( i \in \mathbb{Z} \) and any \( R \)-module \( M \). This proves the first statement. By applying the Matlis duality functor it follows that

\[
\text{Hom}_R(\text{Tor}_n^R(M, E), E) \simeq \text{Hom}_R(H^i_n(M), E)
\]
Therefore the duality between ”Ext” and ”Tor” yields the isomorphism
\[ \text{Hom}_R(\text{Tor}_{i-\cdot}^R(M, E), E) \simeq \text{Ext}_{i-\cdot}^R(\text{Hom}_R(E, E)). \]
By Matlis duality we get \( \text{Hom}_R(E, E) \simeq \mathcal{R} \), which completes the proof. \( \square \)

Note that the proof of Lemma 2.1 is well-known (see for example [9]). The new aspect - we need in the following - is its validity for an arbitrary \( R \)-module \( M \).

Now let us recall the definition of the truncation complex as it was introduced in [15]. Assume that \((R, m)\) is an \( n\)-dimensional Gorenstein ring. Let \( E_R(R) \) denote the minimal injective resolution of \( R \). Let \( \Gamma_I(\cdot) \) denote the section functor with support in \( I \). Then \( \Gamma_I(E_R(R))^i = 0 \) for all \( i < c \) (see [15] for more details).

**Definition 2.2.** Let \( C_R(I) \) be the cokernel of the embedding \( H^c_I(R)[-c] \to \Gamma_I(E_R(R)) \) considered as a morphism of complexes. It is called the truncation complex of \( R \) with respect to \( I \). Then there ia a short exact sequence of complexes
\[ 0 \to H^c_I(R)[-c] \to \Gamma_I(E_R(R)) \to C_R(I) \to 0. \]
In particular \( H^i(C_R(I)) = 0 \) for all \( i \leq c \) or \( i > n \) and \( H^i(C_R(I)) \simeq H^i_I(R) \) for all \( c < i \leq n \).

The advantage of the truncation complex is the separation of the properties of the cohomology module \( H^c_I(R) \) of those of the other cohomology modules \( H^i_I(R) \) for \( i \neq c \). The truncation complex is helpful in order to construct a few natural homomorphisms. But as a first application of the truncation complex we have the following result.

**Lemma 2.3.** Let \((R, m)\) be a local ring. Let \( X \) denote an \( R \)-module such that \( \text{Supp}_R(X) \subseteq V(I) \) for an ideal \( I \) of grade \( I = c \). Then there is a natural isomorphism
\[ \text{Hom}_R(X, H^c_I(R)) \simeq \text{Ext}_R^c(X, R) \]
and \( \text{Ext}_R^i(X, R) = 0 \) for all \( i < c \).

**Proof.** For the proof see [16, Theorem 2.3]. \( \square \)

The truncation complex provides a few natural maps that arise also as edge homomorphisms of certain spectral sequences.

**Lemma 2.4.** With the previous notation and Definition 2.2 there are the following natural homomorphisms
\[ \begin{align*}
(\text{a}) & \ H^d_m(H^c_I(R)) \to E, \\
(\text{b}) & \ \hat{R} \to \text{Ext}_R^c(H^c_I(R), \hat{R}), \\
(\text{c}) & \ \text{Tor}_c(R, H^c_I(R)) \to E, \\
(\text{d}) & \ \text{Ext}_R^d(k, H^c_I(R)) \to k, \text{ and} \\
(\text{e}) & \ \text{Tor}_c(R, k, H^c_I(R)) \to k.
\end{align*} \]

**Proof.** Let \( x_1, \ldots, x_n \in m \) denote a system of parameters of \( R \). Let \( \check{C}_x \) denote the \( \check{C}ech \) complex with respect to \( x \). It induces a homomorphism of complexes of \( R \)-modules
\[ \check{C}_x \otimes_R H^c_I(R)[-c] \to \check{C}_x \otimes_R \Gamma_I(E_R(R)) \]
By view of [14, Theorem 3.2] we have a morphism of complexes
\[ \Gamma_m(\Gamma_I(E_R(R))) \to \hat{C}_\omega \otimes_R \Gamma_I(E_R(R)) \]
that induces an isomorphism in cohomology. Moreover there is an identity of functors \( \Gamma_m(\Gamma_I(-)) = \Gamma_m(-) \). Because \( R \) is a Gorenstein ring it follows that \( \Gamma_m(E_R(R)) \cong E[-n] \). Therefore the above homomorphism considered in degree \( n \) induces the homomorphism in \( (a) \).

By view of the Local Duality (see [2.1]) we have the isomorphism
\[ H^d_m(H^c_I(R)) \cong \text{Tor}^R_c(E, H^c_I(R)). \]
Therefore the homomorphism in \( (a) \) implies the homomorphism in \( (c) \).

Again by Local Duality (see [2.1]) there is the isomorphism
\[ \text{Ext}^c_R(H^c_I(R), \hat{R}) \cong \text{Hom}_R(H^d_m(H^c_I(R)), E) \]
By the Matlis Duality it follows that the homomorphism in \( (a) \) provides the homomorphism in \( (b) \).

Now let \( F \) denote a free resolution of the residue field \( k \). Then the truncation complex induces – by tensoring with \( F \) – the following morphism of complexes of \( R \)-modules
\[ (F \otimes_R H^c_I(R))[-c] \to F \otimes_R \Gamma_I(E_R(R)). \]
Now let \( y = y_1, \ldots, y_r \) a generating set for the ideal \( I \). Let \( \hat{C}_\omega \) denote the Čech complex with respect to \( \omega \). Because of the natural morphisms \( \Gamma_I(E_R(R)) \to \hat{C}_\omega \otimes_R E_R(R) \) (see [14]) induces an isomorphism in cohomology. We get the quasi-isomorphism of complexes
\[ F \otimes_R \Gamma_I(E_R(R)) \sim \to F \otimes_R \hat{C}_\omega \otimes_R E_R(R) \]
But now we have a quasi-isomorphism \( F \otimes_R \hat{C}_\omega \sim \to F \). Then the homology in degree 0 induces the homomorphism in \( (e) \). A similar argument with \( \text{Hom}_R(F, -) \) implies the homomorphism in \( (d) \). This completes the proof. \( \Box \)

3. NATURAL HOMOMORPHISMS

Let \( M \) denote an \( R \)-module. Then there is the natural homomorphism
\[ \Phi : R \to \text{Hom}_R(M, M), r \to f_r \]
where \( f_r : M \to M, m \to rm, \) for all \( m \in M \) and \( r \in R \). This homomorphism of \( R \) into the endomorphism ring is in general neither injective nor surjective. Of course the endomorphism ring is in general not a commutative ring.

Remark 3.1. (A) For a local Gorenstein ring \( (R, \mathfrak{m}) \) we use the above notations and conventions. By view of Lemma [2.4] there are the following natural homomorphisms
\[ \varphi_1 : H^d_m(H^c_I(R)) \to E, \]
\[ \varphi_2 : \text{Tor}^R_c(E, H^c_I(R)) \to E, \]
\[ \varphi_3 : \hat{R} \to \text{Ext}^c_R(H^c_I(R), \hat{R}), \]
\[ \varphi_4 : \hat{R} \to \text{Hom}_R(H^c_I(R), H^c_I(R)). \]

(B) It is known (see [17, Theorem 3.2 (a)]) that \( S = \text{Hom}_R(H^c_I(R), H^c_I(R)) \) is a commutative ring for an ideal \( I \) of a Gorenstein ring \( (R, \mathfrak{m}) \) and \( c = \text{grade} I \).
In the following it will be our intention to investigate the homomorphisms of Remark 3.1 in more detail. In particular we have the following result:

**Theorem 3.2.** Let $I \subset R$ denote an ideal of height $c$ of a Gorenstein ring $(R, \mathfrak{m})$. With the previous notation we have:

(a) $\varphi_i$, $i = 1, \ldots, 4$, are all non-zero homomorphisms.

(b) The following conditions are equivalent:

(i) $\varphi_1$ is injective (resp. surjective).

(ii) $\varphi_2$ is injective (resp. surjective).

(iii) $\varphi_3$ is surjective (resp. injective).

(iv) $\varphi_4$ is surjective (resp. injective).

(c) The natural map $\psi : \text{Hom}_R(k, H^d_m(H^c_I(R))) \to k$ is non-zero and therefore onto.

**Proof.** First of all we prove that there is a natural isomorphism

$$\text{Ext}^c_R(H^c_I(R), \hat{R}) \simeq \text{Hom}_R(H^c_{\hat{I}}(\hat{R}), H^c_{\hat{I}}(\hat{R}))$$

This follows from Lemma 2.3 since $\text{Ext}^c_R(H^c_I(R), \hat{R}) \simeq \text{Ext}^c_{\hat{R}}(H^c_{\hat{I}}(\hat{R}), \hat{R})$ and $\hat{R}$ is a flat $R$-module. Moreover we have $c = \text{grade } I \hat{R}$.

Because $\varphi_4$ is non-zero and the previous isomorphism is natural we get that $\varphi_3$ is non-zero too. By Lemma 2.1 the module $\text{Ext}^c_R(H^c_I(R), \hat{R})$ is the Matlis dual of $H^d_m(H^c_I(R))$. So $\varphi_1$ is also a non-zero homomorphism. By Local Duality (see Lemma 2.1) the last module is isomorphic to $\text{Tor}^c_{\hat{R}}(H^c_I(R), E)$. This finally proves also that $\varphi_2 \neq 0$. Therefore the statement in (a) is shown to be true. With the same arguments the equivalence of the conditions in the statement (b) follows by Matlis Duality.

For the proof of (c) first note that $\varphi_4$ is non-zero and $1 \mapsto \text{id}$. So the reduction modulo $\mathfrak{m}$ induces a non-zero homomorphism $k \to k \otimes \text{Hom}_R(H^c_{\hat{I}}(\hat{R}), H^c_{\hat{I}}(\hat{R}))$. But this is the Matlis dual of $\psi$ as it follows by Local Duality. □

In order to investigate the natural homomorphisms (d), (e) of Lemma 2.4 we need a few more preliminaries. First note that by Lemma 2.4 (see also [17, Section 6]) there is a commutative diagram

$$
\begin{array}{ccc}
\text{Ext}^d_R(k, H^c_I(R)) & \to & k \\
\downarrow & & \downarrow \\
H^d_m(H^c_I(R)) & \to & E
\end{array}
$$

here the right vertical morphism is - by construction - the natural inclusion and the left vertical morphism is the direct limit of the natural homomorphisms

$$\lambda_\alpha : \text{Ext}^d_R(k, K(R/I^\alpha)) \to H^d_m(K(R/I^\alpha))$$

for all $\alpha \in \mathbb{N}$, where $K(M)$ denote the canonical module of $M$. Then it induces the following commutative diagram

$$
\begin{array}{ccc}
\text{Ext}^d_R(k, H^c_I(R)) & \xrightarrow{\varphi} & k \\
\lambda \downarrow & & \| \\
\text{Hom}_R(k, H^d_m(H^c_I(R))) & \xrightarrow{\psi} & k
\end{array}
$$
It is of some interest to investigate the homomorphism $\varphi$. In fact, it was conjectured that $\varphi$ is in general non-zero. This was shown to be true for $R$ a regular local ring containing a field (see [17, Theorem 1.3]) while $\psi$ is non-zero in a Gorenstein ring as shown above (see Theorem 3.2 (c)).

For our purposes here let us summarize the main results of Huneke and Sharp resp. Lyubeznik (see [10] and [11]).

**Theorem 3.3.** Let $(R, m)$ be a regular local ring containing a field. Let $I \subset R$ be an ideal of grade $c$. Then for all $i, j \in \mathbb{Z}$ the following results are true:

(a) $H^d_m(H^1_I(R))$ is an injective $R$-module.

(b) $\text{inj dim}_R(H^1_I(R)) \leq \dim_R(H^1_I(R)) \leq \dim R - i$.

(c) $\text{Ext}^i_R(k, H^1_I(R)) \cong \text{Hom}_R(k, H^1_m(H^{n-j}_I(R)))$ and its dimension is finite.

**Proof.** For the proof we refer to [10] and [11]. \hfill \square

In particular the result of Theorem 3.3 (c) is a motivation for the following Definition.

**Definition 3.4.** If $I \subset R$ is an ideal of grade $c$ in $(R, m)$ a Gorenstein ring. Then

$$\tau_{i,j}(I) := \dim_k \text{Hom}_R(k, H^i_m(H^{n-j}_I(R)))$$

is called $\tau$-number of type $(i, j)$ of $I$.

**Remark 3.5.** Note that $\tau_{i,j}(I)$ is the socle dimension of the local cohomology module $H^i_m(H^{n-j}_I(R))$. In the case of a regular local ring $(R, m)$ containing a field it was shown by Huneke and Sharp resp. Lyubeznik (see Theorem 3.3) that

$$\text{Ext}^i_R(k, H^{n-j}_I(R)) \cong \text{Hom}_R(k, H^i_m(H^{n-j}_I(R)))$$

for all $i, j \in \mathbb{Z}$. These Bass numbers are also called Lyubeznik numbers. In fact, Lyubeznik showed that they are invariants of $R/I$ and finite.

By the example of Hartshorne (see Example 5.3) it turns out that for a Gorenstein $(R, m)$ the integers $\tau_{i,j}(I)$ need not to be finite.

As a first result on the integers $\tau_{i,j}(I)$ let us consider its vanishing resp. non-vanishing.

**Theorem 3.6.** Let $I \subset R$ denote an ideal of the local Gorenstein ring $R$ and $c = \text{grade } I$.

(a) $\tau_{d,d}(I) \neq 0$, where $d = \dim R/I$.

(b) $\tau_{i,j}(I) = 0$ if and only if $H^1_m(H^{n-j}_I(R)) = 0$, where $n = \dim R$.

(c) $\tau_{i,j}(I) < \infty$ if and only if $H^i_m(H^{n-j}_I(R))$ is an Artinian $R$-module.

**Proof.** The statement in (a) is a consequence of Theorem 3.2 (c). For the proof of (b) first note that any element of $H^1_m(H^{n-j}_I(R))$ is annihilated by a power of $m$. Then the statement is true by the well-known fact that the module is zero if and only if its socle vanishes.

In order to prove (c) recall again that $\text{Supp} H^1_m(H^{n-j}_I(R)) \subseteq V(m)$. Then the finiteness of the socle dimension is equivalent to the Artinianness of a module. \hfill \square
4. The endomorphism ring

As above let \( I \subset R \) denote an ideal of grade \( c \) in the Gorenstein ring \((R, \mathfrak{m})\). In this section we shall investigate the natural homomorphism

\[
\varphi_4 : \hat{R} \to \text{Hom}_R(H^\ell_{I_R} \hat{(R)}, H^\ell_{I_R}(\hat{R})).
\]

As a first step we want to characterize when it will be surjective. To this end we need a few preparations. For an ideal \( I \) let

\[\text{Assh } R/I = \{ p \in \text{Ass } R/I | \dim R/p = \dim R/I \}.\]

Then we define the multiplicatively closed set

\[S = \bigcap (R \setminus p),\]

where the intersection is taken over all \( p \in \text{Assh } R/I \). For \( n \in \mathbb{N} \) the \( n \)-th symbolic power \( I^{(n)} \) of \( I \) is defined as

\[I^{(n)} = I^n R_S \cap R.\]

Definition 4.1. With the previous notation put

\[u(I) = \bigcap_{n \geq 1} I^{(n)} = 0R_S \cap R.\]

Note that \( u(I) \) is equal to the intersection of all \( p \)-primary components \( q \) of a minimal primary decomposition of the zero ideal 0 in \( R \) satisfying \( S \cap p = \emptyset \).

Lemma 4.2. Let \( I \subset R \) be an ideal of grade \( c \) in \((R, \mathfrak{m})\) a Gorenstein ring. Then \( \ker \varphi_4 = u(I \hat{R}) \) and \( \text{Ann}_R H^c_{I \hat{R}}(\hat{R}) = u(I \hat{R}) \).

Proof. It is known (see [17, Theorem 3.2]) that \( \ker \varphi_4 = u(I \hat{R}) \). Then clearly \( u(I \hat{R}) = \text{Ann}_R H^c_{I \hat{R}}(\hat{R}) \).

\( \square \)

It follows that \( H^c_{I \hat{R}}(\hat{R}) \) is a torsion-free \( \hat{R} \)-module if \( \hat{R} \) is a domain. In the next we investigate when \( \varphi_4 \) is surjective.

Theorem 4.3. Let \((R, \mathfrak{m})\) denote an \( n \)-dimensional Gorenstein ring. Let \( I \subset R \) be an ideal and \( \text{grade}(I) = c \). Set \( d := n - c \) then the following conditions are equivalent:

(i) \( \tau_{d,d}(I) = 1 \).

(ii) The natural homomorphism \( \text{Hom}_R(k, H^d_m(H^\ell_I(R))) \to k \) is an isomorphism.

(iii) The natural homomorphism \( \hat{R} \to \text{Hom}_R(H^c_{I \hat{R}}(\hat{R}), H^c_{I \hat{R}}(\hat{R})) \) is surjective.

Proof. First of all note that we may assume that \( R \) is complete without loss of generality.

(iii)\( \Rightarrow \) (ii): Fix the above notation. If \( \varphi_4 \) is surjective it follows (see Theorem 3.2) that

\[\varphi_1 : H^d_m(H^\ell_I(R)) \to E\]

is injective. Since \( \psi \) is non-zero it provides (by applying \( \text{Hom}_R(k, -) \)) that

\[\psi : \text{Hom}_R(k, H^d_m(H^\ell_I(R))) \to k\]

is an isomorphism.

(ii)\( \Rightarrow \) (i): This is obviously true by the definition of \( \tau_{d,d} \).

(i)\( \Rightarrow \) (iii): In order to simplify notation we put \( S = \text{Hom}_R(H^\ell_I(R), H^\ell_I(R)) \). Then we have to show that the natural homomorphism \( R \to S \) is surjective. Note that \( \varphi_4 \) is non-zero since \( 1 \to \text{id} \) and \( H^\ell_I(R) \neq 0 \). Moreover

\[S \simeq \text{Hom}_R(H^d_m(H^\ell_I(R)), E)\]
as follows because of $S \cong \text{Ext}^d_R(k, H^d_I(R))$ (see Lemma 2.3) and by the Local Duality Theorem. Therefore we have isomorphisms

$$S/\mathfrak{m}S \cong k \otimes_R \text{Hom}_R(H^d_m(H^d_I(R)), E) \cong \text{Hom}_R(\text{Hom}_R(k, H^d_m(H^d_I(R))), E)$$

By the assumption and Matlis Duality this implies that $\dim_k S/\mathfrak{m}S = 1$.

Now we claim that $S$ is $\mathfrak{m}$-adically complete. With the same duality argument as above there are isomorphisms

$$S/\mathfrak{m}^\alpha S \cong \text{Hom}_R(\text{Hom}_R(R/\mathfrak{m}^\alpha R, H^d_m(H^d_I(R))), E).$$

for all $\alpha \in \mathbb{N}$. Now both sides form an inverse system of modules. Since inverse limits commute with direct limits into $\text{Hom}_R(\_,-)$ at the first place. It provides an isomorphism

$$\varprojlim S/\mathfrak{m}^\alpha S \cong \text{Hom}_R(H^0_m(H^d_m(H^d_I(R))), E)$$

as follows by passing to the inverse limit. But $H^0_m(H^d_m(H^d_I(R))) \cong H^d_m(H^d_I(R))$ since $H^d_m(H^d_I(R))$ is an $R$-module whose support is contained in $V(\mathfrak{m})$. Finally this provides an isomorphism

$$\varprojlim S/\mathfrak{m}^\alpha S \cong S$$

and hence $S$ is $\mathfrak{m}$-adically complete.

By virtue of [12, Theorem 8.4] $S$ is a finitely generated $R$-module, in fact a cyclic module $S \cong R/J$. This proves the statement in (iii).

As shown above there is the following commutative diagram

$$\begin{array}{ccc}
\text{Ext}^d_R(k, H^d_I(R)) & \xrightarrow{\varphi} & k \\
\downarrow \lambda & & \parallel \\
\text{Hom}_R(k, H^d_m(H^d_I(R))) & \xrightarrow{\psi} & k.
\end{array}$$

Whence there is a relation between the Bass number $\mu_d(\mathfrak{m}, H^d_I(R))$ and $\tau_{d,d}(I)$. In fact, if $(R, \mathfrak{m})$ is regular local ring containing a field, then $\lambda$ is an isomorphism (see [17]). It is conjectured ([6]) that $\varphi$ is non-zero. Note that if $\varphi$ is non-zero then $\psi$ is non-zero too and both maps are surjective. In the next we want to continue with the investigations of $\varphi$.

**Theorem 4.4.** Let $(R, \mathfrak{m})$ be Gorenstein ring. Suppose that $I \subset R$ is an ideal of grade $c$ such that $\text{inj dim}_R(H^1_I(R)) \leq d = n - c$. Suppose that the natural homomorphism

$$\varphi : \text{Ext}^d_R(k, H^d_I(R)) \to k$$

is non-zero (resp. isomorphic). Then

$$\varphi_a : \hat{R} \to \text{Hom}_{\hat{R}}(H^e_{I\hat{R}}(\hat{R}), H^e_{I\hat{R}}(\hat{R}))$$

is injective (resp. isomorphic).

**Proof.** Since $\hat{R}$ is a flat $R$-module, the injective dimension of $H^d_I(R)$ is finite if and only if the injective dimension of $H^e_{I\hat{R}}(\hat{R})$ is finite. Moreover $\varphi$ is non-zero (resp. isomorphic) if and only if $\text{Ext}^d_R(k, H^e_{I\hat{R}}(\hat{R})) \to k$ is non-zero (resp. isomorphic). So without loss of generality we may assume that $R$ is complete.

It is a consequence of the truncation complex that there are natural homomorphisms

$$\varphi_a : \text{Ext}^d_R(R/\mathfrak{m}^\alpha, H^d_I(R)) \to \text{Hom}_R(R/\mathfrak{m}^\alpha, E)$$
Then there is a short exact sequence
$$0 \rightarrow \frac{m^\alpha}{m^{\alpha+1}} \rightarrow R/m^{\alpha+1} \rightarrow R/m^\alpha \rightarrow 0$$
induces a commutative diagram with exact rows
$$\begin{array}{cccc}
\text{Ext}^d_R(R/m^\alpha, H) & \rightarrow & \text{Ext}^d_R(R/m^{\alpha+1}, H) & \rightarrow & \text{Ext}^d_R(m^\alpha/m^{\alpha+1}, H) & \rightarrow & 0 \\
\downarrow \varphi_\alpha & & \downarrow \varphi_{\alpha+1} & & \downarrow f & & \\
0 & \rightarrow & \text{Hom}_R(R/m^\alpha, E) & \rightarrow & \text{Hom}_R(R/m^{\alpha+1}, E) & \rightarrow & \text{Hom}_R(m^\alpha/m^{\alpha+1}, E) & \rightarrow & 0
\end{array}$$
Recall that the homomorphism $\text{Ext}_R^d(R/m^{\alpha+1}, H) \rightarrow \text{Ext}_R^d(m^\alpha/m^{\alpha+1}, H)$ is onto since $\text{inj dim } H \leq d$.

Now suppose that $\varphi$ is non-zero, that is, it is surjective. So assume that $\varphi$ is surjective (resp. isomorphic). Then the natural homomorphism $f$ is surjective (resp. isomorphic). By passing to the direct limit of the direct systems it induces the homomorphism
$$\varphi_1 : H^d_m(H^c_I(R)) \rightarrow E$$
is surjective (resp. isomorphic). Note that the direct limit is exact on direct systems. So the statement follows by virtue of Theorem 3.2.

**Remark 4.5.** (A) The assumption $\text{inj dim } H^c_I(R) \leq d$ implies that $\text{inj dim } H^c_I(R) = d$. This follows since $d = \dim H^c_I(R) \leq \text{inj dim } H^c_I(R)$ as it is easily seen.

(B) In the case of a regular local ring $(R, m)$ containing a field it is shown (see Theorem 3.3) that $\text{inj dim } H^c_I(R) \leq d$. We do not know whether this holds in any regular local ring.

5. THE $\tau$-NUMBERS

In the following we shall give two applications concerning the $\tau$-numbers of the ideal $I \subset R$ of a Gorenstein ring $(R, m)$. To this end let us recall the following result:

**Lemma 5.1.** Let $(R, m)$ denote an $n$-dimensional Gorenstein ring. Let $I \subset R$ denote an ideal with $c = \text{grade } I$. Let $C_R(I)$ denote the truncation complex of $I$ (see Definition 2.2). Then there is a short exact sequence
$$0 \rightarrow H^{n-1}_m(C_R(I)) \rightarrow H^d_n(H^c_I(R)) \rightarrow E \rightarrow H^d_m(C_R(I)) \rightarrow 0,$$
isomorphisms $H^i_m(C_R(I)) \simeq H^i_m(C_R(I))$ for $i < n$ and the vanishing $H^i_m(C_R(I)) = 0$ for $i > n$.

**Proof.** For the proof we refer to [6, Lemma 2.2]. It is easily seen a consequence of the truncation complex.

**Theorem 5.2.** Let $I \subset R$ denote a 1-dimensional ideal in the Gorenstein ring $(R, m)$.

(a) $\tau_{i,j}(I) = 0$ for all $(i, j) \notin \{(0, 0), (1, 1)\}$.

(b) $\tau_{0,0}(I) = \dim_k \text{Ext}_R^1(k, H^1_m(H^{n-1}_m(R))) < \infty$ and $\tau_{1,1}(I) = 1$.

(c) $\text{Hom}_R(H^c_{IR}(\bar{R}), H^c_{IR}(\bar{R})) \simeq \bar{R}/u(IR)$. 
Proof. Since $H^i(C_R(I)) \simeq H^i_I(R)$ for all $c < i \leq n$ and zero otherwise there is a naturally defined morphism $C_R(I) \to H^i_I(R)[-n]$ that induces an isomorphism in cohomology. Therefore $H^i_m(C_R(I)) \simeq H^i_m(H^i_I(R))$ for all $i \in \mathbb{Z}$. Whence there are an exact sequence

$$0 \to H^i_m(H^i_I(R)) \to H^i_A(H^i_I(R)^{-1}(R)) \to E \to H^0_m(H^i_I(R)) \to 0$$

and isomorphisms $H^i_m(H^i_I(R)) \simeq H^{i+2-n}_m(H^{i-1}_I(R))$ for all $i \neq n, n-1$ as it is a consequence of Lemma 5.1. This provides all the vanishing of the $\tau$-numbers as claimed in the statement. To this end recall that $\dim H^2_I(R) \leq n-i$ for $i = n-1, n$. Moreover it implies a short exact sequence

$$0 \to H^1_m(H^{n-1}_I(R)) \to E \to H^0_m(H^n_I(R)) \to 0.$$ 

By applying $\text{Ext}_R(k, -)$ it induces isomorphisms

$$\text{Hom}_R(k, H^m_m(H^i_I(R))) \simeq k \text{ and } \text{Hom}_R(k, H^0_m(H^i_I(R))) \simeq \text{Ext}_R(k, H^0_m(H^i_I(R))).$$

To this end recall the the natural homomorphism $\text{Hom}_R(k, H^m_m(H^i_I(R))) \to k$ is not zero (see Theorem 3.3 (c)). Because $H^i_I(R)$ is an Artinian $R$-module (see e.g. [3]) we have $H^0_m(H^i_I(R)) = H^i_I(R)$ and its socle dimension is finite. This proves the statements in (a) and (b). The claim in (c) is a particular case of Theorem 4.3. □

In his paper (see [1]) Blickle has shown a certain duality statement for Lyubeznik numbers in the case of cohomologically isolated singularities in a regular local ring containing a field (see also [17, Corollary 5.4] for a slight extension). Here we prove a corresponding result for the $\tau$-numbers for an ideal in a Gorenstein ring.

**Theorem 5.3.** Let $I \subset R$ denote an ideal of grade $c$ in the Gorenstein ring $(R, \mathfrak{m})$. Suppose that $c < n-1$ and $\text{Supp} H^i_I(R) \subseteq V(\mathfrak{m})$ for all $i \neq c$. Then

$$\tau_{0,j}(I) = \tau_{d-j+1,d}(I) - \delta_{d-j+1,d},$$

where $d = \dim R/I$.

**Proof.** Under the assumption of $\text{Supp} H^i_I(R) \subseteq V(\mathfrak{m})$ for all $i \neq c$ it follows that there are isomorphisms $H^i_m(C_R(I)) \simeq H^i(C_R(I))$ for all $i \in \mathbb{Z}$. The exact sequence and the isomorphisms shown in Lemma 5.1 provide the statement of the claim. Note that we have an exact sequence

$$0 \to H^{n-1}_I(R) \to H^d_m(H^i_I(R)) \to E.$$ 

This in particular provides (as above) that $\tau_{d,d}(I) = \tau_{0,1}(I) + 1$. □

Even under the assumptions of Theorem 5.3 it may happen that $\tau_{d,d}$ is infinite. To this end consider the following example of Hartshorne (see [5]).

**Example 5.4.** (cf. [5] Section 2 and [17] Example 3.5) Let $k$ be a field and let $A = k[[u, v, x, y]]$ be the formal power series ring in four variables. Consider the complete Gorenstein ring $R = A/fA$, where $f = xu - yu$. Let $I = (x, y)R$. Then $n = 3, d = 2, c = 1$ and $\text{Supp} H^2_I(R) = V(\mathfrak{m}), H^3_I(R) = 0$. It was shown by Hartshorne (see [5] Section 2) that

$$\dim_k \text{Hom}_R(k, H^2_I(R)) = \tau_{0,1}(I) = \tau_{2,2}(I) - 1$$

is not finite.
Problem 5.5. It would be of some interest to get a criterion for $\tau_{d,d}(I)$ to be finite for an ideal $I \subset R$ with $d = \dim R/I$. By Theorem 3.6 this is equivalent to the Artinianess of $H^d_{m}(H^f_{I}(R))$.

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