ON THE STATIONARY SOLUTIONS OF DOI-ONSAGER MODEL IN GENERAL DIMENSION

MOHAMMAD NIKSIRAT

Abstract. We give new results of the phase transition of dilute colloidal solutions of rod-like molecules in dimension $D \geq 3$. For the low concentration of particles in a carrier fluid, we prove that the isotropic phase is the unique solution to the Doi-Onsager model with the general potential kernel. In addition, we present the regime of the bifurcation of nematic phases in the class of axially symmetric solutions. Our method is based on a generalization of the classical Leray-Schauder degree we developed for this problem.

1. Introduction

In 1949, L. Onsager [17] proposed a mathematical model for the phase transition of dilute colloidal solutions of rod-like molecules. As the fluid in both phases is homogeneous, Onsager’s theory focuses on a probability density function $f(r)$ over the unit sphere $S^2 \in \mathbb{R}^3$. Let $f(r) : S^2 \to [0, \infty)$ be the probability density function of the directions of the rod-like molecules, that is, for any $A \subset S^2$

$$P(\text{the rod is along } r \in A) = \int_A f(r) d\sigma.$$  

As we are modeling rod-like molecules with no distinction between the two ends, we can further assume $f(r) = f(-r)$. Consequently, the constraints on $f(r)$ are

$$f(r) \geq 0, \quad f(r) = f(-r), \quad \int_{S^2} f(r) d\sigma = 1. \quad (1.2)$$

The mean interaction potential between molecules is expressed by

$$U(f)(r) = \lambda \int_{S^2} K(r, r') f(r') d\sigma(r'), \quad (1.3)$$

where $\lambda$ can be interpreted as either the concentration of the particles in the carrier fluid or equivalently the inverse of the absolute temperature, and the potential kernel $K$ is defined by Onsager as

$$K(r, r') = |r \times r'|. \quad (1.4)$$

With this interaction field, Onsager suggested that the possible phases of a liquid crystal are the critical points of the following energy functional $\mathcal{E}$:

$$\mathcal{E}(f) = \int_{S^2} f(r) \left( \log f(r) + \frac{1}{2} U(f)(r) \right) d\sigma. \quad (1.5)$$

1991 Mathematics Subject Classification. Primary 76A15, Secondary 35Q35.

Key words and phrases. liquid crystal, Onsager model, phase transition, topological methods, degree theory.
By the classical variational method, it is simply seen that the density function $f$ is a minimizer of (1.5) if the functional

$$V(f) := \log f + U(f),$$

is constant. By (1.2), this is in turn equivalent to the equation

$$f(r) = \left( \int_{S^2} e^{-U(f)\,d\sigma} \right)^{-1} e^{-U(f)(r)}.$$

On the other hand, according to the relation

$$\Delta_r f + \text{div}(f \nabla_r U(f)) = \text{div}(f \nabla_r V),$$

it is simply seen that the solutions of (1.7) are the steady state solutions of the Doi equation:

$$\frac{\partial f}{\partial t} = \Delta_r f + \text{div}(f \nabla_r U(f)).$$

Apparently, (1.7) has trivial solutions $\bar{f} = \frac{1}{\text{vol}(S^2)}$ that are correspond to the uniform distribution of molecules without any preferred directional order. This is called an isotropic phase of the fluid. Approximating $K$ by some lower orders terms, Onsager was able to show a transition to a non-uniform state called nematic phases in the case when $\lambda$ passes a critical value.

More quantitative analysis of the system (1.3) - (1.7) with the Onsager kernel turned out to be difficult. On the other hand there are kernels capturing the qualitative behavior of the solution that are more friendly to mathematical analysis. One such kernel, due to Maier and Saupe, reads

$$K(r, r') = |r \cdot r'|^2 - \text{constant},$$

that is usually written as $K(r, r') = \cos^2 \gamma$ if the constant is discarded. The advantage in considering (1.10) instead of the Onsager kernel (1.4) is that the Maier-Saupe kernel is the eigenvector of the Laplace-Beltrami operator on $S^2$ and then lies in a finite dimensional space $\mathbb{R}^3$. This reduces the infinite dimensional problem (1.3) - (1.7) to a finite dimensional nonlinear system of equations. This reduced system, still highly nontrivial, is nevertheless more tractable than the original system. As a consequence, (1.3) - (1.7) with the Maier-Saupe kernel has been well understood through brilliant works of many researchers, see [3, 9, 10, 20, 11, 21] for the model in $\mathbb{R}^3$; also see [5, 9, 1] for the model in $\mathbb{R}^2$, and [19] for the general dimensional case $\mathbb{R}^D$. Inspired by these works, equations (1.3) and (1.7) with other kernels enjoying similar dimension reduction property has also been analyzed, see e.g. [3].

Fatkullin and Slastikov [9, 8] completely classified the solution of the Onsager equation with the Maier-Saupe kernel and for the anti-symmetric kernel

$$K(r, r') = -r \cdot r'$$

on $S^1$ and $S^2$. Instead of $\lambda$, they presented their results in terms of the temperature $\tau$, however since $\lambda$ and $\tau$ are inversely proportional their results hold for the original case. In particular they obtained the exact nematic solutions in $S^2$ for the problem with the Maier-Saupe kernel as:

$$f(\varphi, \theta) = \beta^{-1} e^{-r \cdot \varphi(\tau)(3 \cos^2 \theta - 1)},$$

and for the anti-symmetric kernel they obtained

$$f(\varphi, \theta) = \beta^{-1} e^{-r(\tau) \cos \theta}.$$
In addition they presented some results of the stability of the above solutions. Luo et al [14] considered the Maier-Saupe interaction kernel on \( S^1 \) and proved that for the potential strength \( \lambda \leq 4 \), the isotropic solution \( \bar{f} = \frac{1}{2\pi} \) is the unique solution of the equation. The nematic solution will bifurcates when the liquid crystal cool down or equivalently the potential strength increase to \( \lambda > 4 \). They also proved that all nematic solutions are obtained by an arbitrary rotation from a \( \pi \) periodic nematic solution. At the same time, Liu et al [12] obtained an explicit solution to (1.3) and (1.7) on \( S^2 \) with the Maier-Saupe kernel and determined the bifurcation regime of the solution. The solution is of the following form for a director \( y \) and constant \( k \) and

(1.13) \[
 f(x) = ke^{-\eta(x,y)^2}.
\]

With the Maier-Saupe model understood, interest in the original Onsager model was resurrected. Much progress has been made in the past few years in the case \( D = 2 \). In [3], the axisymmetry of all possible solutions is proved in the sense that for any solution \( f(\theta) \), there is \( \theta_0 \) such that \( f(\theta_0) = f(\theta_0 + \pi) \). It is also proved in [3] that for appropriate \( \lambda \), there are solutions of arbitrary periodicity. In [19] the authors rewrite (1.3) and (1.7) into an infinite system of nonlinear equations for the Fourier coefficients of \( f(\theta) \) and calculated numerically the first few bifurcations. Chen et al [3] observed that for even integers \( l = 2n \), the the interaction potential

(1.14) \[
 U(f)(\theta) = \int_{S^1} \sin^l(\theta - \theta') f(\theta')d\theta',
\]

behave completely similar to the Maier-Saupe original potential and can be reduced to a model in finite dimensional space, while for odd \( l = 2n - 1 \) the obtained equation will be a nonlinear partial differential equation. By reducing the Onsager equation to a system of ordinary differential equations, they could prove the existence of auxiliary symmetric nematic solution for the Onsager equation on \( S^1 \) and for all odd power potential kernel. More recently, in [13] the authors studied the case \( D = 2 \) through cutting-off the Onsager kernel and reducing (1.3) and (1.7) to a finite dimensional system of nonlinear equations, and obtain local bifurcation structure for this finite dimensional approximation. In particular, they used a result of bifurcation by Crandall and Rabinowitz [6] for the general truncated trigonometric kernel

(1.15) \[
 K(\theta, \theta') = -\sum_{n=0}^{N} k_n \cos 2n(\theta - \theta').
\]

The original Onsager kernel \( |\sin(\theta - \theta')| \) on \( S^1 \) is approximated by the above kernel for special

(1.16) \[
 k_n = \frac{1}{\pi} \left( n^2 - \frac{1}{4} \right)^{-1}.
\]

In this case the problem is reduced to finding the zeros of a finite dimensional nonlinear problem.

In a new study of this problem in \( D = 2 \), X. Yu and the author [15] obtained the following results for a general potential kernel (still covering the original Onsager kernel):
The problem has a unique solution, which must be the constant solution, when \(0 < \lambda < \lambda_0 := \frac{1}{|K_{\infty}| - k_0}\) where \(k_0\) is the first mode of the Fourier expansion

\[
K(\theta) = \sum_{k=0}^{\infty} k_m \cos(2m\theta).
\]

Two solutions bifurcate from the trivial solution at every \(\lambda_m = -\frac{2}{k_m}\). The bifurcation is supercritical if \(\frac{2k_m^2}{k_m^2} < 1\) and subcritical if \(\frac{2k_m^2}{k_m^2} > 1\). Furthermore, in the former case, the first pair of bifurcated solutions are stable and other bifurcated solutions are unstable, while in the latter case all bifurcated solutions are unstable.

In particular, for the Onsager’s model all bifurcations are supercritical. The first pair of bifurcated solutions are stable and other bifurcated solutions are unstable.

2. Reformulation of the problem

Let us formulate the problem in general dimension \(D\) in terms of a nonlinear map in a suitable space. The general Onsager model in \(\mathbb{R}^D\) reads

\[
U(f)(r) = \lambda \int_{S^{D-1}} K(r, r') f(r') d\sigma(r'),
\]

\[
f(r) = \left( \int_{S^{D-1}} e^{-U(f)} d\sigma \right)^{-1} e^{-U(f)(r)},
\]

where the potential kernel \(K\) in (2.1) is assumed to satisfy the following properties:

\[
K(r, r') = K(-r, r') = K(r', r) = K(O(r), O(r'))
\]

for any rotation matrix \(O(\mathbb{R}^D)\). It is simply verified that this conditions are satisfied by the original Onsager kernel (1.4).

We reformulate the system (2.1) and (2.2) into an abstract equation involving a bounded, \((S)_+\) mapping, and then we generalize the classical Leray-Schauder degree to prove the existence and multiplicity of the solution. Substitution (2.2) in (2.1) and canceling \(f\) gives an equation for the potential \(U(r)\):

\[
U(r) = \lambda \left( \int_{S^{D-1}} e^{-U} d\sigma \right)^{-1} \int_{S^{D-1}} K(r, r') e^{-U(r')} d\sigma(r'),
\]

where \(U\) enjoys the symmetric property \(U(r) = U(-r)\). Note that once (2.4) is solved, \(f(r)\) can be recovered from

\[
f(r) = \frac{e^{-U(r)}}{\int_{S^{D-1}} e^{-U(r)} d\sigma}.
\]

Thus (2.4) is equivalent to the original problem (2.1) - (2.2). Further reduction of the problem needs the following lemma.

**Lemma 2.1.** Under the symmetry assumptions (2.3) on \(K\), we have

\[
K(r, r') = F(|r - r'|),
\]
for some function $F$. In particular, this gives
\begin{equation}
\bar{K} = \frac{1}{|S^{D-1}|} \int_{S^{D-1}} K(r, r') d\sigma(r') = \frac{1}{|S^{D-1}|} \int_{S^{D-1}} K(r, r') d\sigma(r)
\end{equation}
is a constant.

For a proof, we refer to [15]. Now we define
\begin{equation}
\bar{K}(r, r') = K(r, r') - \bar{K}
\end{equation}
with $\bar{K}$ is defined in (2.13). Now, for
\begin{equation}
\hat{U}(r) = U(r) - \lambda \bar{K}.
\end{equation}
it is simply verified that (2.4) is equivalent to the following system:
\begin{equation}
V(r) = \frac{\lambda \int_{S^{D-1}} \bar{K}(r, r') e^{-V(r')} d\sigma(r')}{\int_{S^{D-1}} e^{-V(r')} d\sigma}, \quad V(-r) = V(r), \quad \int_{S^{D-1}} V(r) d\sigma = 0.
\end{equation}

Summarizing the above we reach:

**Lemma 2.2.** The original problem (2.1) is equivalent to the following problem.
\begin{equation}
V(r) = \frac{\lambda \int_{S^{D-1}} \bar{K}(r, r') e^{-V(r')} d\sigma(r')}{\int_{S^{D-1}} e^{-V(r')} d\sigma}, \quad V(-r) = V(r), \quad \int_{S^{D-1}} V(r) d\sigma = 0.
\end{equation}

From now on, we will work with (2.10) which is naturally a fixed point problem. Let $G$ be the operator
\begin{equation}
G(V)(r) = \beta(V)^{-1} \int_{S^{D-1}} \bar{K}(r, r') e^{-V(r')} d\sigma(r'),
\end{equation}
where $\beta$ is
\begin{equation}
\beta(V) = \int_{S^{D-1}} e^{-V(r')} d\sigma(r),
\end{equation}
and $V$ belongs to $H_0(S^{D-1})$ where
\begin{equation}
H_0(S^{D-1}) = \{ u \in L^2(S^{D-1}), u(-r) = u(r), \int_{S^{D-1}} u(r) d\sigma = 0 \}\.
\end{equation}

We employ the topological degree argument to study the structure of the solutions of the equation
\begin{equation}
A(V)(r) := V(r) - \lambda G(V)(r),
\end{equation}
in terms of the parameter $\lambda$. We carry out the calculations in terms of spherical harmonics on $S^{D-1}$, which are the eigenfunctions of the Laplace-Beltrami operator $-\Delta$ on $S^{D-1}$. For $D = 2$, these functions are just the usual trigonometric functions. Alternatively, for $r \in S^{D-1}$, the spherical harmonics $S_{nj}(D, r)$ can be defined by the restriction of harmonic polynomials to the unit sphere $S^{D-1}$. As it is simply verified, see e.g. [2], for $D \geq 3$ and given $n$, there are exactly
\begin{equation}
N(D, n) = \frac{(2n + D - 2)(n + D - 3)!}{(D - 2)! n!}
\end{equation}
spherical harmonics $\{S_{nj}(D, r)\}_{j=1}^{N(D, n)}$. An important class of spherical harmonics on $S^{D-1}$ consists of ones that are invariant under the rotation of $S^{D-2}$. These
are usual Legendre polynomial $P_n(D,t)$ for $t \in [-1,1]$. Alternatively, the Legendre polynomial $P_n$ can be defined through the expansion of the potential $V(\rho) = (1 + \rho^2 - 2\rho \cos \gamma) \cdot \frac{d}{d\rho}$ that is generated by a unit mass located at distance $r = \frac{1}{\rho} > 1$ in terms of $\rho$ as

$$(2.16) \quad V = \sum_{n=0}^{\infty} P_n(D, \cos \gamma) \rho^n.$$  

**Proposition 2.3.** The Onsager kernel in $\mathbb{R}^D$ defined by

$$(2.17) \quad K(r, r') = |\sin \gamma|,$$

where $\gamma$ is the angle between $r, r'$ has the expansion

$$(2.18) \quad K(\gamma) = -\sum_{n=1}^{\infty} k_n P_{2n}(D, \cos \gamma) + k_0,$$

where $k_0 > 0$ and for $n \geq 1$, $k_n$ are positive and make a decreasing sequence, i.e.

$$(2.19) \quad k_n > 0, \quad k_n > k_{n+1}.$$  

For the proof we need the following lemma.

**Lemma 2.4.** Let $C_n^{(\alpha)}(t)$ denote the Gegenbauer polynomials of order $n$ and $\alpha = \frac{D-2}{2}$. We have

$$(2.20) \quad \int_{-1}^{1} (1-t^2)^\alpha C_{n+2}^{(\alpha)}(t) = \frac{(n-1)(n+2\alpha)}{(n+2)(n+2\alpha+3)} \int_{-1}^{1} (1-t^2)^\alpha C_n^{(\alpha)}(t).$$  

**Proof.** Recall that $C_n^{(\alpha)}$ satisfies the Gegenbauer differential equation

$$(2.21) \quad \frac{d}{dt} \left[(1-t^2)^{\alpha+\frac{1}{2}} \frac{d}{dt} C_n^{(\alpha)}\right] + n(n+2\alpha)(1-x^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)} = 0.$$  

Multiply both sides of the Gegenbauer equation by $(1-t^2)^{1/2}$ and integrate in $(-1,1)$ to obtain

$$(2.22) \quad \int_{-1}^{1} (1-t^2)^\alpha t \frac{d}{dt} C_n^{(\alpha)} = -n(n+2\alpha) \int_{-1}^{1} (1-t^2)^\alpha C_n^{(\alpha)}.$$  

The same result holds for $n+2$, that is,

$$(2.23) \quad \int_{-1}^{1} (1-t^2)^\alpha t \frac{d}{dt} C_{n+2}^{(\alpha)} = -(n+2)(n+2\alpha+2) \int_{-1}^{1} (1-t^2)^\alpha C_{n+2}^{(\alpha)}.$$  

Subtract formula (2.22) from (2.23) and use the identity

$$(2.24) \quad \frac{d}{dt}(C_{n+2}^{(\alpha)} - C_n^{(\alpha)}) = 2(n+\alpha+1)C_{n+1}^{(\alpha)}.$$  

to reach

$$2(n+\alpha+1) \int_{-1}^{1} (1-t^2)^\alpha t C_{n+1}^{(\alpha)} = n(n+2\alpha) \int_{-1}^{1} (1-t^2)^\alpha C_n^{(\alpha)} - (n+2)(n+2\alpha+2) \int_{-1}^{1} (1-t^2)^\alpha C_{n+2}^{(\alpha)}.$$  

Use the identity

$$(2.25) \quad 2(n+\alpha+1)t C_{n+1}^{(\alpha)} = (n+2)C_{n+2}^{(\alpha)} + (n+2\alpha)C_n^{(\alpha)},$$
to conclude
\[
\int_{-1}^{1} (1 - t^2)^\alpha [(n + 2)C_{n+2}^{(\alpha)} + (n + 2\alpha)C^{(\alpha)}] = n(n + 2\alpha) \int_{-1}^{1} (1 - t^2)^\alpha C_{n}^{(\alpha)} - (n + 2)(n + 2\alpha + 2) \int_{-1}^{1} (1 - t^2)^\alpha C_{n+2}^{(\alpha)}.
\]

Now, a simple algebraic calculation gives (2.20). □

Now let us return and prove the proposition (2.3).

**Proof. (of proposition (2.3))** The set \( P_n(D, \cos \gamma) \), \( n \geq 0 \) forms a complete system for the functions of \( \gamma \) on \( S^{D-1} \) and thus the Onsager kernel has an expansion in terms of the functions in this set. On the other hand, since (2.17) is even, the coefficients of the odd terms in the expansion are zero. Thus, the Onsager kernel has the expansion of the form (2.18). Since \( K \) has a positive average, \( k_0 > 0 \). In order to show that \( k_n \) enjoy (2.19), we use the explicit formula for \( k_n \)

\[
k_n = -\frac{\sigma_{D-1}N(D, 2n)}{\sigma_D} \int_{-1}^{1} (1 - t^2)^\frac{D-2}{2} P_{2n}(D, t) dt.
\]

We obtain

\[
k_{n+1} = -\frac{\sigma_{D-1}N(D, 2n + 2)}{\sigma_D} \int_{-1}^{1} (1 - t^2)^\frac{D-2}{2} P_{2n+2}(D, t) dt.
\]

Use the formula (2.20) to obtain

\[
k_{n+1} = -\frac{\sigma_{D-1}N(D, 2n + 2)}{\sigma_D C_{2n+2}^{(\alpha)}(1)} \int_{-1}^{1} (1 - t^2)^\alpha C_{2n+2}^{(\alpha)}(t) =
-\frac{\sigma_{D-1}N(D, 2n + 2)}{\sigma_D C_{2n+2}^{(\alpha)}(1)} \frac{(2n - 1)(n + \alpha)}{(n + 1)(2n + 2\alpha + 3)} \int_{-1}^{1} (1 - t^2)^\alpha C_{2n}^{(\alpha)}(t) =
-\frac{\sigma_{D-1}N(D, 2n + 2)}{\sigma_D C_{2n+2}^{(\alpha)}(1)} \frac{(2n - 1)(n + \alpha)C_{2n}^{(\alpha)}(1)}{(n + 1)(2n + 2\alpha + 3)} \int_{-1}^{1} (1 - t^2)^\frac{D-2}{2} P_{2n}(D, t).
\]

Now use (2.20) for the last integral above to write

\[
k_{n+1} = \frac{C_{2n}^{(\alpha)}(1)N(D, 2n + 2)}{C_{2n+2}^{(\alpha)}(1)N(D, 2n)} \frac{(2n - 1)(n + \alpha)}{(n + 1)(2n + 2\alpha + 3)} k_n.
\]

Direct computation shows that \( k_1 > 0 \) and then all \( k_n, n \geq 1 \) are positive. Substitution \( N(D, n) \) from (2.19) and the following formula

\[
\frac{C_{2n}^{(\alpha)}(1)}{C_{2n+2}^{(\alpha)}(1)} = \frac{(2n + 1) \cdots (2n + D - 3)}{(2n + 3) \cdots (2n + D - 1)},
\]

into (2.20) we obtain

\[
k_{n+1} = \frac{(2n - 1)(4n + D + 2)(2n + D - 2)}{(2n + 1)(4n + D - 1)(2n + D + 1)} k_n.
\]

The coefficient of \( k_n \) in (2.29) is simply verified to be less than 1, and therefore \( (k_n), n \geq 1 \) forms a positive decreasing sequence. Direct computation shows that \( k_0 > 0 \) and therefore \( k_n, n \geq 1 \) satisfy conditions (2.19). □
3. Main Results

In sequel, we assume $D \geq 3$, and that the potential kernel $K$ has the expansion \[ (2.15) \]. As we observed above, this assumption covers the original Onsager kernel. We systematically use the topological degree argument and its bifurcation consequences for the equation

\[ A(u) := u - g(u) = 0, \]

where $g = \lambda G$ and $G$ is defined in \((2.11)\). Notice that the classical Leray-Schauder degree fails to apply here because the map $A : H_0(S^{D-1}) \to H_0(S^{D-1})$ is not continuous in any neighborhood of $0 \in H_0(S^{D-1})$. Let us show this by a simple example in $D = 3$. For fixed $\bar{r} \in S^2$, let $u_n$ be the sequence

\[ u_n(r) = \begin{cases} \log(2\pi(1 - \cos(1/n))) & \cos^{-1}(r, \bar{r}) \in \{0, \frac{1}{n}\} \\ 0 & \text{otherwise} \end{cases}. \]

Obviously, $u_n \to 0$, while

\[ \lim_n G(u_n) = \lim_n \frac{1}{2\pi(1 - \cos(1/n))} \int_0^{1/n} \int_0^{2\pi} K(\gamma)d\sigma \neq G(0) = 0. \]

**Theorem 3.1.** Assume that $K(\gamma)$ belongs to the class of Holder continuous maps, then the map $G : \Omega(\lambda) \subset H_0 \to H_0$ is continuous and compact where

\[ \Omega(\lambda) = \{ u \in H_0(S^{D-1}), |u(r)| \leq \lambda \|K\|_{\infty} \}. \]

**Proof.** Notice that the fixed point set of $g$ has the a priori bound

\[ |u(r)| \leq \lambda \|K\|_{\infty} \beta(u)^{-1} \int_{S^{D-1}} e^{-u(r')}d\sigma = \lambda \|\hat{K}\|_{\infty}. \]

The continuity of $g : \Omega \to H_0(S^{D-1})$ simply follows from the dominant convergence theorem. To prove that $g : \Omega(\lambda) \to H_0(S^{D-1})$ is compact, we show that it is the limit of a sequence of compact maps in $\Omega$. Let $\hat{K}_N$ be the truncated kernel

\[ \hat{K}_N(\gamma) = -\sum_{n=1}^{N} k_n P_{2n}(D, \cos \gamma). \]

and also let $g_N : \Omega(\lambda) \to H_0(S^{D-1})$ be the finite range operators as

\[ g_N(u)(r) = \lambda \beta(u)^{-1} \int_{S^{D-1}} \hat{K}_N(\gamma)e^{-u(r')}d\sigma(r'). \]

It is seen that for $u \in \Omega$ we have

\[ \|g(u) - g_N(u)\|^2 = \lambda^2 \beta(u)^{-2} \int_{S^{D-1}} \left( \int_{S^{D-1}} (\hat{K}(\gamma) - \hat{K}_N(\gamma))e^{-u(r')}d\sigma(r') \right)^2 d\sigma(r) \leq \max_{\gamma} |\hat{K}(\gamma) - \hat{K}_N(\gamma)|2\pi \lambda^2 \overset{N \to \infty}{\to} 0. \]

Thus, $g$ is the uniform limit of a sequence of finite range operators $g_N$ on the bounded set $\Omega \subset H_0$ and therefore a compact map on $\Omega$. \(\square\)
3.1. Degree Argument. We give a generalization of the classical Leray-Schauder degree for the map \( A : \Omega \to H_0(S^{D-1}) \) where \( A \) is the map \((3.1)\) and \( \Omega \) is given in \((3.3)\). A generalization of the Browder degree for \((S)_+\) mappings will appear soon \([16]\).

Let \( H \) be a separable Hilbert space with a fixed orthonormal basis \( \mathcal{H} = \{ u^1, u^2, \cdots \} \) and the inner product \((.,.)\). The finite dimensional subspaces \( H_n \subset H \) for \( n \geq 1 \) are naturally defined by \( H_n := \text{span}\{ u^1, \ldots, u^n \} \). For a given map \( f : H \to H \), the finite rank approximation \( f_n : H \to H_n \) is defined by the projection of \( f(u) \) into \( H_n \), that is,

\[
(f_n(u), u^k) = \sum_{k=1}^{n} (f(u), u^k) u^k.
\]

(3.7)

It is simply verified that \((f(u), v) = (f_n(u), v)\) for arbitrary \( v \in H_n \). For our case, \( H \) is the space \( H_0(S^{D-1}) \) defined in \((2.13)\) and we chose \( \mathcal{H} \) the set of spherical harmonics \( \{ S_{2n} j(D, r) \} \). Notice that the following conditions are satisfied

- For any \( n \geq 1 \), the set \( \Omega_n := \Omega \cap H_n \) has a non-empty interior in \( H_n \),
- The map \( g : \Omega \to H \) is continuous and relatively compact,
- The solution set of the equation \( A(u) = 0 \) lies in \( \Omega \) and furthermore \( 0 \not\in A(\partial \Omega) \) where

\[
\partial \Omega := \{ u \in \Omega; \| u \|_\infty = \lambda \| \hat{K} \|_\infty \}.
\]

Theorem 3.2. There exists \( N_0 > 0 \) such that

\[
\deg(A_n, \Omega_n, 0) = \deg(A_{n+1}, \Omega_{n+1}, 0), \quad \forall n \geq N_0.
\]

(3.9)

Proof. Notice first that \( \Omega_n \) has an open interior in \( H_n \) for all \( n \) and \( A_n : \Omega_n \to H_n \) is continuous. Next we show that for sufficiently large \( n \), there is no solution of the equation \( A_n(u) = 0 \) for \( u \in \partial \Omega_n \). Assuming contrary, there is a sequence \( (z_n) \), \( z_n \in \partial \Omega \) such that \( z_n = g_n(z_n) \). Since \( g \) is completely continuous on \( \Omega \), the sequence \( g(z_n) \) converges (in a subsequence) to some \( \zeta \in H \). Since \( g_n(z_n) = \text{Pr}_{H_n} g(z_n) \), it implies that \( g_n(z_n) \overset{H}{\to} \zeta \) (in a subsequence) and then \( z_n \overset{H}{\to} \zeta \). It is verified by the embedding of \( L^\infty \) into \( L^2 (S^{D-1}) \) that \( \zeta \in \Omega_n \). Since \( g \) is continuous in \( \Omega_n \), we conclude \( g(z_n) \overset{H}{\to} g(\zeta) \) and then \( \zeta = g(\zeta) \). By a fact from the measure theory, we conclude \( z_n \overset{\text{pointwise}}{\to} \zeta \) almost everywhere. For arbitrary \( \varepsilon > 0 \), choose \( n, r \) such that \( |z_n(r) - \zeta(r)| < \frac{\varepsilon}{2} \) and \( |z_n(r)| > \lambda \| \hat{K} \|_\infty - \frac{\varepsilon}{2} \). This implies that \( |\zeta(r)| > \lambda \| \hat{K} \|_\infty - \varepsilon \) and then \( \zeta \in \partial \Omega_n \), a contradiction. This establishes that for sufficiently large \( n \), the solution of \( A_n = 0 \) does not occur on \( \partial \Omega_n \). This allows us to define the classical Brouwer degree of the map \( A_n \) restricted to \( \Omega_n \cap H \) for sufficiently large \( n \). Now define the map \( B_{n+1} \) as follows:

\[
B_{n+1}(u) = (A_n(u), (u, u^{n+1}) u^{n+1}).
\]

(3.10)

Clearly by the classical properties of the Brouwer degree we have

\[
\deg(B_{n+1}, \Omega_{n+1}, 0) = \deg(A_n, \Omega_{n+1}, 0).
\]

(3.11)

Now, consider the convex homotopy \( h : [0,1] \times \Omega_{n+1} \to H_{n+1} \)

\[
h(t) = (1 - t)A_{n+1} + tB_{n+1}.
\]

(3.12)

Note first that \( 0 \not\in h(t)(z) \) for \( z \in \partial \Omega_{n+1} \) and \( t = 0,1 \). If \((3.8)\) does not hold then there exists a sequence \( (t_n) \) for \( t_n \in (0,1) \) and \( z_n \in \partial \Omega_n \) such that \( h(t_n)(z_n) = 0 \).
This implies that
\[(3.13)\quad A_n(z_n) + t_n(g(z_n), u^n)u^n = 0.\]
Since \(\{z_n\} \subset \Omega\) is bounded and \(g\) is compact on \(\Omega\) then \((g(z_n), u^n) \to 0\) and then \(A_n(z_n) \to 0\) which implies in turn \(z_n \to \zeta \in \partial \Omega\) and \(A(z) = 0\), a contradiction. This completes the proof. \(\square\)

By the aid of the theorem \(3.2\), we define the degree of \(A\) on \(\Omega\) at 0 by the limit in the following definition.

**Definition 3.3.** Under the above settings, the degree of \(A\) in \(\Omega\) at 0 is defined as
\[(3.14)\quad \deg(A, \Omega, 0) = \lim_{n \to \infty} \deg(A_n, \Omega_n, 0).\]

In addition, we need to define the class of admissible homotopy for the generalized equation. In particular, we are interested in the homotopy invariance and the solvability properties.

**Definition 3.4.** The map \(h : [0, 1] \times \Omega \to H\) defined by the relation \(h(t)(u) = u - G(t)(u)\) is called an admissible homotopy if \(h\) satisfies the following conditions
- \(h\) is continuous in \([0, 1] \times \Omega\),
- The solution set of the equation \(h(t)(u) = 0\) lies in \(\Omega\) and furthermore \(0 \notin h(t)(\partial \Omega)\) for all \(t \in [0, 1]\),
- The map \(g\) is compact, i.e., \(g([0, 1]) : \Omega \to H\) is compact, and thus for every \(\Omega' \subset \Omega\) and \(\epsilon > 0\) there exist \(\delta = \delta(\epsilon, \Omega')\) such that
\[(3.15)\quad |t - s| < \delta \Rightarrow \|g(t)(x) - g(s)(x)\| < \epsilon, \quad x \in \Omega'.\]

The above definition of admissible homotopy is very similar to one defined for \((S)_+\) mappings by F. Browder [22]. We verify that the definition \(3.3\) satisfies the classical properties of a topological degree. In particular, we are interested in the homotopy invariance and the solvability properties.

**Theorem 3.5.** Under the above setting, the degree \(h_t : \Omega \to H\) is constant with respect to \(t \in [0, 1]\). Furthermore, if \(\deg(h_t, \Omega, 0) \neq 0\) for some \(t \in [0, 1]\) then there exist \(u = u(t) \in \Omega\) such that \(h(t)(u(t)) = 0\).

**Proof.** We show first that for some \(N_0 > 0\) and for \(n \geq N_0\), the finite rank approximation \(h_n\) of the homotopy \(h\) satisfies the condition \(0 \notin h_n([0, 1])(\partial \Omega_n)\). Assuming contrary, there exists a sequence \(t_n\) and \(z_n \in \partial \Omega_n\) such that \(h_n(t_n)(z_n) = 0\). Since \(t_n\) converges (in a subsequence) to some \(t \in [0, 1]\) and since \(g(t) : \Omega \to H\) is compact, we conclude \(g(t)(z_n)\) converges (in a subsequence) to some \(\zeta \in H\). Hence, we can write
\[\|z_n - \zeta\| = \|g_n(t_n)(z_n) - g(t)(z_n) + g(t)(z_n) - \zeta\| \leq \|g_n(t_n)(z_n) - g(t)(z_n)\| + \|g(t)(z_n) - \zeta\| = \|g_n(t_n)(z_n) - g(t)(z_n)\| + O(1).\]

On the other hand, since \(g\) is a compact transformation, we have
\[\|g_n(t_n)(z_n) - g(t)(z_n)\| = \|g_n(t_n)(z_n) - g(t_n)(z_n) + g(t_n)(z_n) - g(t)(z_n)\| \leq \|g_n(t_n)(z_n) - g(t_n)(z_n)\| + \|g(t_n)(z_n) - g(t)(z_n)\| \leq O(1).\]

and thus \(\|g(t_n)(z_n) - g(t)(z_n)\| \to 0\). Recall that \(g_n = \text{Pr}_{H_n} g\), and then by the relation \(\|g_n(t_n)(z_n) - g(t_n)(z_n)\| \to 0\) we conclude \(z_n H\to \zeta\) (in a subsequence). A
measure theoric argument that we have used in the proof of the Theorem (3.2) implies $\zeta \in \partial \Omega$. We conclude finally $0 = h(\bar{t})(\zeta)$ for $\zeta \in \partial \Omega$ which contradicts the second assumption on $h$. Furthermore, one can employ an argument similar to one presented in the proof of the Theorem (3.2) to shows that the degree $\deg(h(t), \Omega, 0)$ is stable with respect to $n$ for any $t \in [0, 1]$. This allows us to write

$$
(3.16) \quad \deg(h(t), \Omega, 0) = \deg(h_n(t), \Omega_n, 0),
$$

for sufficiently large $n$. Now assume that for $t_1, t_2 \in [0, 1]$, we have

$$
(3.17) \quad \deg(h(t_1), \Omega, 0) \neq \deg(h(t_2), \Omega, 0).
$$

Thus we can choose $n$ so large that

$$
(3.18) \quad \deg(h_n(t_1), \Omega_n, 0) \neq \deg(h_n(t_2), \Omega_n, 0).
$$

On the other hand, since $0 \not\in h_n([0,1])(\partial \Omega)$, the homotopy invariance property of the Brouwer degree implies

$$
(3.19) \quad \deg(h_n(t_1), \Omega_n, 0) = \deg(h_n(t_2), \Omega_n, 0).
$$

which is a contradiction.

Now we prove the second part of the theorem. Assume $\deg(h(t), \Omega, 0) = 0$ for a fixed $t \in [0, 1]$. According to the definition (3.14) and the solvability property of the Brouwer degree, there exists a sequence $(u_n(t)) \subset \Omega_n$ such that $h_n(t)(u_n(t)) = 0$. Since $g$ is compact, the sequence $(g(t, u_n(t)))$ converges (in a subsequence) to some $u(t) \in H$. This implies that $g_n(t, u_n(t))$ converges to $u(t)$ and therefore $u_n(t) \xrightarrow{H} u(t) \in \Omega$. Since $h$ is continuous, we have $h(t)(u(t)) = 0$ and this completes the proof. \qed

### 3.2. Phase Transition

We prove that thres is $\lambda_0 > 0$ such that the isotropic phase is the unique solution to the system (2.1) and (2.2) for $\lambda < \lambda_0$. In addition we derive a sequence of critical values $\lambda_n$ for which the axisymmetric nematic phases will bifurcate from the trivial solution. First the following lemma.

**Lemma 3.6.** Let $L : H_0(S^{D-1}) \to H_0(S^{D-1})$ be the map

$$
(3.20) \quad L(u)(r) = -\frac{1}{\sigma_D} \int_{S^{D-1}} K(\gamma)u(r')d\sigma(r').
$$

If $\lambda$ is not a characteristic value of $L$ then $\bar{u} = 0$ is an isolated solution of $A(u) = 0$.

**Proof.** It is simply verified by the dominant convergence theorem that if $u \in 2\Omega(\lambda)$ then

$$
(3.21) \quad \|G(u) - L(u)\|_{L^2(S^{D-1})} = o(\|u\|_{L^2(S^{D-1})}).
$$

Fix $\lambda$ a non-characteristic value of $L$. If $\bar{u} = 0$ is not isolated then choose a sequence $(\lambda_n, u_n)$ such that $\lambda_n \to \lambda$, $u_n \to 0$ and $u_n = g(u_n)$. On the other hand, we have

$$
(3.22) \quad 0 = \|u_n - g(u_n)\| \geq \|u_n - \lambda L(u_n)\| - |\lambda - \lambda_n|\|L\|\|u\| - \|g(u_n) - \lambda_n L(u_n)\|.
$$

Since $\lambda$ is not a characteristic value of $L$, there exist $k > 0$ such that

$$
\|u_n - \lambda L(u_n)\| > k\|u_n\|.
$$

Take $|\lambda_n - \lambda|$ very small and then

$$
(3.23) \quad k\|u_n\| + o(\|u_n\|) < 0,
$$

that is a contradiction. \qed
Theorem 3.7. Under the above setting, there exist \( \lambda_0 > 0 \), such that the equation 
\[ A(u) = 0 \]
has the unique solution \( \bar{u} = 0 \) in the class of axially symmetric solutions for \( 0 < \lambda < \lambda_0 \).

Proof. Let \( \sigma_D \) stands for the surface of the unit sphere in \( \mathbb{R}^D \). For \( R = \lambda \sqrt{\sigma_D} \| \hat{K} \|_\infty \) the equation
\[ u - \tan(u) = 0, \]
has no solution on the sphere \( S_R \) for \( t \in [0, 1] \). In fact we have \( ||u||_\infty \leq R/\sqrt{\sigma_D} \) and \( ||u||_{L^2} \leq R \). By the homotopy invariance property of degree we conclude:
\[ \text{deg}(\text{Id} - g, B_R, 0) = \text{deg}(\text{Id} - \tan, B_R, 0) = \text{deg}(\text{Id}, B_R, 0) = +1 \]
We show that the index of the trivial solution \( \bar{u} \) is +1. We use the expansion of functions in \( H^0(S^D-1) \) in terms of the orthonormal spherical harmonics \( \{ S_{2n_j}(D, r) \} \) for \( j = 1, \ldots, N(D, n) \). Let us write for \( u \in H^0(S^D-1) \) the expansion
\[ u(r) = \sum_{n=1}^{\infty} \sum_{j=1}^{N(D, 2n)} u_{nj} S_{2n_j}(D, r), \]
for some coefficients \( u_{nj} \). With regards to (3.26), it is more convenient in our calculations to write \( u = u(u_{nj}) \). Calculation of the entries of the Jacobian matrix of \( g \) at \( u = \bar{u} \) gives
\[ \frac{\partial}{\partial u_{nj}} g(\bar{u}) = -\frac{\lambda}{\sigma_D} \int_{S^{D-1}} \hat{K}(\gamma) S_{2n_j}(D, r')d\sigma(r') = \frac{\lambda k_n}{\sigma_D} \int_{S^{D-1}} P_{2n}(D, \cos\gamma) S_{2n_j}(D, r')d\sigma(r') = \frac{\lambda k_n}{N(D, 2n)} S_{2n_j}(D, r). \]
This implies that the infinite dimensional Jacobian matrix of \( g(\bar{u}) \) in the bases of \( \{ S_{nj} \} \) with \( n \) an even number has the form
\[ J_G = \text{diag} \left( \frac{\lambda k_n}{N(D, 2n)} \right). \]
Therefore, if \( \tilde{\lambda}_0 \leq Dk^{-1}_n \), then \( \text{ind}(\bar{u}, \lambda) = 1 \) if \( 0 < \lambda < \tilde{\lambda}_0 \). The axially symmetric solutions are functions which are symmetric with respect to the rotation of \( S^{D-2} \) around any point of \( S^{D-1} \). For the fixed \( r \in S^{D-1} \), if \( \theta \) denotes the angel between arbitrary point \( r' \in S^{D-1} \) and \( r \), then \( u \) can be expanded in terms of Legendre polynomials \( P_{2n} \) as
\[ u(\theta) = \sum_{n=1}^{\infty} u_n P_{2n}(D, \cos\theta). \]
In this case \( g \) has the simpler form:
\[ g(u)(\theta) = \lambda \int_0^\theta \hat{K}(\gamma) \tilde{g}(\theta')d\theta', \]
where \( \tilde{g} \) is defined as
\[ \tilde{g}(\theta) = \frac{e^{-u(\theta)} \sin^{D-2}(\theta)}{\int_0^\theta e^{-u(\theta)} \sin^{D-2}(\theta)d\theta}. \]
The calculations reduces to what we have carried out for the Onsager problem in case $D = 2$, see [15]. The Jacobian matrix entries for the trivial solution is obtained as:

$$
\frac{\partial}{\partial u_n} g(\bar{u}) = -\lambda \sigma D - 1 \frac{\sigma D}{\sigma D} \int_0^{\pi} \hat{K}(\gamma) P_{2n}(D, \cos \theta') \sin D - 2(\theta') d\theta' = \frac{\lambda k_n}{N(D, 2n)} P_{2n}(D, \cos \theta).
$$

The calculation for a non-trivial solutions also is carried out as

$$
\left\langle \frac{\partial}{\partial u_n} g(u), P_{2m}(D, \cos \theta) \right\rangle = \lambda k_m \left\{ \int_0^{\pi} \tilde{g}(\theta) P_{2n}(D, \cos \theta) P_{2m}(D, \cos \theta) d\theta - \int_0^{\pi} \tilde{g}(\theta) P_{2n}(D, \cos \theta) d\theta \int_0^{\pi} \tilde{g}(\theta) P_{2m}(D, \cos \theta) d\theta \right\}.
$$

The bracket in the right hand side of the above identity can be calculated using a Gruss type inequality [7] which states that for functions $a, b \in L^\infty(D)$ and the probability measure $\mu$, we have the inequality

$$
\left| \int_D a(x) b(x) d\mu - \left( \int_D a(x) d\mu \right) \left( \int_D b(x) d\mu \right) \right| \leq \|a\|_L \|b\|_L.
$$

According to the above inequality we obtain

$$
\left| \int_0^{\pi} \tilde{g}(\theta) P_{2n}(D, \cos \theta) P_{2m}(D, \cos \theta) d\theta - \int_0^{\pi} \tilde{g}(\theta) P_{2n}(D, \cos \theta) d\theta \int_0^{\pi} \tilde{g}(\theta) P_{2m}(D, \cos \theta) d\theta \right| \leq 1,
$$

and then finally we reach

$$
\left| \left\langle \frac{\partial}{\partial u_n} g(u), P_{2m}(D, \cos \theta) \right\rangle \right| \leq \lambda k_m.
$$

The above calculation establishes the fact $\text{ind}(u, \lambda) = +1$ for any axially symmetric solution $u$ of $A(u) = 0$ and for $0 < \lambda < \lambda_0$ where

$$
\lambda_0 = \left( \sum_{m=1}^{\infty} k_m \right)^{-1}.
$$

The bound (3.34) is obtained based on the assumption $k_n > 0$ in (2.19). Since the degree of $A$ is $+1$ according to (3.29), we conclude that the trivial solution $\bar{u}$ is the unique solution in the class of axially symmetric solutions for $0 < \lambda < \lambda_0$. \[\square\]

It can be shown that the above theorem holds in general case (not necessarily for axially symmetric solutions). The calculation in this case gives a bound $\lambda_0 < \lambda_0$. We have the following theorem.

**Theorem 3.8.** The equation $A(u) = 0$ has no non-trivial solution for $0 < \lambda < \hat{\lambda}_0$ where $\hat{\lambda}_0 = \frac{1}{\|K\|_{\infty}^{-1}}$.

The proof is based on the fact that the index of every solution of $(A(u) = 0$ is $+1$ for $0 < \lambda < \hat{\lambda}_0$. The calculation is given in the appendix.

To prove the existence of nematic phases for the Onsager model (2.1) and (2.2), we use the following lemma and the degree argument we established in the previous section.

**Lemma 3.9.** Let $\sigma(L)$ be the spectrum of the map $L$ defined in (3.20) and let $\hat{\lambda} \in \sigma(L)$. If for the pair $(\lambda, \mu)$ where $0 < \lambda < \hat{\lambda} < \mu$ we have

$$
\text{ind}(\text{Id} - \lambda L, 0) \text{ind}(\text{Id} - \mu L, 0) < 0,
$$

then...
then $\bar{\lambda}$ is a bifurcation point for (3.1).

Proof. Assuming contrary, the value $(\bar{\lambda}, 0)$ is isolated due to the lemma (3.6) and then there exists $\varepsilon > 0$ such that for $\lambda \in (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon)$, the trivial solution $\bar{u}$ is the unique solution of $A(u) = 0$ for $u \in B_{\varepsilon}(\bar{u})$. This implies that there is no solution lying on $\partial B_{\varepsilon/2}$ for $\lambda \in (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon)$ and then by the homotopy invariance property of the degree (3.14), the index $\text{ind}(\bar{u})$ is constant that is a contradiction. $\square$

By the Lemma (3.9) and calculations (3.31) and (3.33), we are able to prove the existence of infinitely many axially symmetric nematic phases of the Onsager model. All nematic phases are bifurcation solutions of (3.1) from the trivial solution $\bar{u} = 0$. The argument is completely similar to one we employed for the model in $D = 2$, see [15]. In particular we have the following theorem.

Theorem 3.10. There exists a sequence of axially symmetric solution of (3.1) in $H_0(S^{D-1})$ bifurcating from the trivial solution $\bar{u}$ at the critical values

$$\lambda_n = N(D, 2n)k_n^{-1}.$$

The multiplicity of the bifurcating solutions at each bifurcating point $\lambda_n$ is exactly equal 2 and the first bifurcation solution is stable.

Proof. Notice that $\lambda_n$ are the eigenvalues of the operator $L$ defined in (3.20). According to the calculation (3.31), the $\text{ind}(\bar{u}, \lambda)$ changes the sign when $\lambda$ passes through $\lambda_n$, i.e., for sufficiently small $\varepsilon > 0$ and $\lambda \in (\lambda_n - \varepsilon, \lambda_n)$ and $\mu \in (\lambda_n, \lambda_n + \varepsilon)$ we have

$$\text{ind}(\bar{u}, \lambda)\text{ind}(\bar{u}, \mu) = -1.\quad (3.36)$$

Therefore, due to the lemma (3.9), the values $\lambda_n$ are bifurcation points. Since $L$ is a self-adjoint operator, the algebraic and geometric multiplicity of eigenvalues of $L$ coincide. It is simply seen that the unique eigenfunction (up to the normalization) of $L$ at $\lambda_n$ is $P_{2n}(D, \cos \theta)$ and then $\lambda_n$ is a simple eigenvalue for $L$. Due to the Theorem 4.2 in [15], we conclude that there exist exactly two solutions bifurcating from the trivial solution $\bar{u}$ at all critical values $\lambda_n$ and furthermore the first bifurcation solution at $\lambda_1$ is stable; see also [15]. $\square$

Appendix A. Proof of the theorem 3.8

Let $u$ be an arbitrary solution to $A(u) = 0$, we have

$$\left< \frac{\partial}{\partial u_{nj}} g(u), S_{2ml}(D, r) \right> = -\lambda \int_{S^{D-1}} \int_{S^{D-1}} \hat{K}(\gamma) f(r') S_{2nj}(D, r') S_{2ml}(D, r) d\sigma(r') +$$

$$+ \lambda \int_{S^{D-1}} f(r) S_{2nj}(D, r) \int_{S^{D-1}} \hat{K}(\gamma) f(r') S_{2ml}(D, r) d\sigma(r') =$$

$$= \frac{\sigma_D \lambda k_m}{N(D, 2m)} \int_{S^{D-1}} f(r) S_{2nj}(D, r) S_{2ml}(D, r) -$$

$$- \frac{\sigma_D \lambda k_m}{N(D, 2m)} \int_{S^{D-1}} f(r) S_{2nj}(D, r) \int_{S^{D-1}} f(r) S_{2ml}(D, r).$$

Let $b_{nj}^{ml}$ denote the following expression:

$$b_{nj}^{ml} = \int_{S^{D-1}} f(r) S_{2nj}(D, r) S_{2ml}(D, r) - \int_{S^{D-1}} f(r) S_{2nj}(D, r) \int_{S^{D-1}} f(r) S_{2ml}(D, r).$$
ON THE STATIONARY SOLUTIONS OF DOI-ONSAGER MODEL IN GENERAL DIMENSION

An estimate for $b_{nj}^m$ (not necessarily optimal) using the a priori estimate (3.4) is as follows:

(A.1) \[ |b_{nj}^m| \leq \frac{2e^{4\lambda||K||_\infty}}{\sigma_D}. \]

This implies in turn

\[ \frac{\partial}{\partial u_{nj}} g(u)(r) = \sum_{m=1}^\infty N(D, 2m) \sum_{l=1} a_{ml} S_{2ml}(D, r), \]

where $a_{ml}$ has the following bound:

(A.2) \[ |a_{ml}| \leq \frac{2\lambda k_m e^{4\lambda||K||_\infty}}{N(D, 2m)}. \]

The following estimate gives a bound for which $\text{ind}(u, \lambda) = +1$ for any possible solution of (3.1):

(A.3) \[ \lambda e^{4\lambda||K||_\infty} \sum_{m=0}^{\infty} N(D, 2m) \sum_{l=1} k_m \frac{k_m}{N(D, 2m)} = \lambda e^{4\lambda||K||_\infty} \sum_{m=1}^{\infty} k_m < \frac{1}{2}. \]

Using the estimate (A.3) we conclude that the index of every possible solution of (3.1) is $+1$ for $0 < \lambda < \lambda_0$ where

(A.4) \[ \lambda_0 = \frac{1}{5} ||\hat{K}||^{-1}_\infty. \]

REFERENCES

[1] H. Zhang C. Luo and P. Zhang. The structure of equilibrium solutions of the one-dimensional doi equation, Nonlinearity, 18:397–389, 2005.
[2] H. Calf. On the expansion of a function in terms of spherical harmonics in arbitrary dimensions, Bull. Belg. Math. Soc., 2:361–380, 1995.
[3] W. Chen, C. Li and G. Wang. On the stationary solutions of the 2d Doi-Onsager model, Nonlinear analysis, Theory, Methods & Applications, 73:2410–2425, 2010.
[4] P. Constantin, I. G. Kevrekidis and E. S. Titi. Asymptotic states of a smoluchowski equation, Arch. Rational Math. Anal., 174:365–384, 2004.
[5] P. Constantin and J. Vukadinovic. Note on the number of steady states for a two-dimensional smoluchowski equation, Nonlinearity, 18:441–443, 2005.
[6] M Crandall and P. Rabinoowitz. Bifurcation from simple eigenvalues, Journal of Functional Analysis, 8(2):321–340, 1971.
[7] S. S. Dragomir. Some integral inequalities of Grüss type, Indian J. Pure Appl. Math, 31(4):397–415, 2000.
[8] I Fatkullin and V Salistikov. Critical points of the onsager functional on a sphere, Nonlinearity, 18:2565–2580, 2005.
[9] I. Fatkullin and V. Salistikov. A note on the onsager model of nematic phase transitions, COMM. MATH. SCI., 3:21–26, 2005.
[10] H. Zhang H. Liu and P. Zhang. Axial symmetry and classification of stationary solutions of doi-onsager equation on the sphere with maier-saupe potential, Comm. Math. Sci., 3:201–218, 2005.
[11] H. Liu. Global orientation dynamics for liquid crystalline polymers, Physica D., 228:122–129, 2007.
[12] H. Liu, H. Zhang and P. Zhang. Axial symmetry and classification of stationary solutions of doi-onsager equation on the sphere with maier-saupe potential, COMM. MATH. SCI., 3:201–218, 2005.
[13] M. Lucia and J. Vukadinovic. Exact multiplicity of nematic states for an onsager model, Nonlinearity, 23:3157–3185, 2010.
[14] C. Luo, H. Zhang and P. Zhang. The structure of equilibrium solutions of the one-dimensional doi equation, Nonlinearity, 18:379–389, 2005.

[15] M. Niksirat and X. Yu. Note on stationary solutions of the 2D Doi-Onsager model, J. Math Anal. Appl., 430:152–165, 2015.

[16] M. Niksirat. A topological invariant for (s)+ mappings, Appea soon.

[17] L. Onsager. The effect of shape on the interaction of colloidal particles, Ann. New York Acad. Sci., 51:627–59, 1949.

[18] D. Sattinger. Topics in stability and bifurcation theory, Lecture notes in mathematics, Springer-Verlag, 1973.

[19] H. Wang and H. Zhou. Multiple branches of ordered states of polymer ensembles with the Onsager excluded volume potential, Physics Letters A, 372:3423–3428, 2008.

[20] H. Zhou, H. Wang, Q. Wang and M. G. Forest. A new proof on axisymmetric equilibria of a three-dimensional smoluchowski equation, Nonlinearity, 18:2815–2825, 2005.

[21] H. Zhou, H. Wang, Q. Wang and M. G. Forest. Characterization of stable kinetic equilibria of rigid, dipolar rod ensembles for coupled dipole-dipole and Maier-Saupe potentials, Nonlinearity, 20:277–297, 2007.

[22] F. Browder. The theory of degree of mapping for nonlinear mappings of monotone type, Nonlinear partial differential equations and their applications, 6:165-177,1982