SPECTRAL ANALYSIS OF JACOBI OPERATORS AND ASYMPTOTIC BEHAVIOR OF ORTHOGONAL POLYNOMIALS

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Abstract. We find and discuss asymptotic formulas for orthonormal polynomials $P_n(z)$ with recurrence coefficients $a_n, b_n$. Our main goal is to consider the case where off-diagonal elements $a_n \to \infty$ as $n \to \infty$. Formulas obtained are essentially different for relatively small and large diagonal elements $b_n$.

Our analysis is intimately linked with spectral theory of Jacobi operators $J$ with coefficients $a_n, b_n$ and a study of the corresponding second order difference equations. We introduce the Jost solutions $f_n(z), n \geq -1$, of such equations by a condition for $n \to \infty$ and suggest an Ansatz for them playing the role of the semiclassical Liouville-Green Ansatz for solutions of the Schrödinger equation. This allows us to study the spectral structure of Jacobi operators and their eigenfunctions $P_n(z)$ by traditional methods of spectral theory developed for differential equations. In particular, we express all coefficients in asymptotic formulas for $P_n(z)$ as $n \to \infty$ in terms of the Wronskian of the solutions $P_n(z)$ and $f_n(z)$. The formulas obtained for $P_n(z)$ generalize the asymptotic formulas for the classical Hermite polynomials where $a_n = \sqrt{(n+1)/2}$ and $b_n = 0$.

The spectral structure of Jacobi operators $J$ depends crucially on a rate of growth of the off-diagonal elements $a_n$ as $n \to \infty$. If the Carleman condition is satisfied, which, roughly speaking, means that $a_n = O(n)$, and the diagonal elements $b_n$ are small compared to $a_n$, then $J$ has the absolutely continuous spectrum covering the whole real axis. We obtain an expression for the corresponding spectral measure in terms of the boundary values $|f_{-1}(\lambda \pm i0)|$ of the Jost solutions. On the contrary, if the Carleman condition is violated, then the spectrum of $J$ is discrete.

We also review the case of stabilizing recurrence coefficients when $a_n$ tend to a positive constant and $b_n \to 0$ as $n \to \infty$. It turns out that the cases of stabilizing and increasing recurrence coefficients can be treated in an essentially same way.

Contents

1. Overview
  1.1. Two definitions of orthogonal polynomials
  1.2. Jacobi operators

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| Section | Title |
|---------|-------|
| 1.3.   | Asymptotics of orthogonal polynomials. Stabilizing coefficients |
| 1.4.   | Asymptotics of orthogonal polynomials. Increasing coefficients |
| 1.5.   | Difference versus differential operators |
| 1.6.   | Structure of the paper |
| 2.     | Scheme of the approach |
| 2.1.   | Standard relations |
| 2.2.   | Resolvent |
| 2.3.   | Linearly independent solutions of the Jacobi equation |
| 2.4.   | Uniformization |
| 2.5.   | Main steps |
| 2.6.   | An overview of the main results |
| 3.     | Volterra equations |
| 3.1.   | Ansatz |
| 3.2.   | Multiplicative substitution |
| 3.3.   | Volterra integral equation |
| 3.4.   | Solution by iterations |
| 4.     | Jost solutions |
| 4.1.   | An auxiliary difference equation |
| 4.2.   | Main result |
| 4.3.   | Explicit formulas |
| 5.     | Small diagonal elements: orthogonal polynomials on the real axis and the spectral measure |
| 5.1.   | Asymptotics on the continuous spectrum |
| 5.2.   | Resolvent and spectral measure |
| 6.     | Asymptotics in the complex plane |
| 6.1.   | A representation for growing solutions |
| 6.2.   | Large diagonal elements |
| 6.3.   | Small diagonal elements |
| 7.     | Non-Carleman case |
| 7.1.   | Jost solutions and orthogonal polynomials |
| 7.2.   | Essential self-adjointness |
| 7.3.   | Quasiresolvent |
| 7.4.   | Self-adjoint extensions |
| 7.5.   | Resolvent |
| 8.     | Jacobi operators with stabilizing coefficients |
| 8.1.   | Compact perturbations |
| 8.2.   | Jost solutions and orthogonal polynomials |
| 8.3.   | Spectral results |
| 8.4.   | Discussion |
| 8.5.   | Hilbert-Schmidt perturbations |
| 8.6.   | Related research |
| 9.     | Short-range perturbations |
1. Overview

1.1. Two definitions of orthogonal polynomials. There are two a priori different definitions of orthogonal, or more precisely orthonormal, polynomials $P_n(z)$. The first definition proceeds from two sets of coefficients $a_n > 0$, $b_n = \bar{b}_n$ and polynomials $P_n(z)$ are determined by a recurrence relation

$$a_{n-1}P_{n-1}(z) + b_nP_n(z) + a_nP_{n+1}(z) = zP_n(z), \quad n \in \mathbb{Z}_+ = \{0, 1, \ldots\}, \quad z \in \mathbb{C}, \quad (1.1)$$

complemented by boundary conditions

$$P_{-1}(z) = 0, \quad P_0(z) = 1. \quad (1.2)$$

Finding $P_n(z)$ for $n = 1, 2, \ldots$ successively from (1.1), we see that $P_n(z)$ is a polynomial with real coefficients of degree $n$:

$$P_n(z) = k_n(z^n + r_nz^{n-1} \cdots). \quad (1.3)$$

Substituting this expression into (1.1) and comparing the coefficients at $z^{n+1}$, we see that necessarily

$$k_n/k_{n+1} = a_n \quad \text{whence} \quad k_n = (a_0a_1 \cdots a_{n-1})^{-1}. \quad (1.4)$$

Similarly, comparing the coefficients at $z^n$, we see that

$$r_n - r_{n+1} = b_n \quad \text{whence} \quad r_n = -\sum_{m=0}^{n-1} b_m. \quad (1.5)$$

Relations (1.4) and (1.5) allow one to recover the recurrence coefficients $a_n$, $b_n$ from the polynomials $P_n(z)$ satisfying (1.1), (1.2).

Another possibility is to define orthonormal polynomials via a measure $d\rho(\lambda)$ on $\mathbb{R}$. It is supposed that all moments

$$s_n := \int_{-\infty}^{\infty} \lambda^n d\rho(\lambda) < \infty, \quad s_0 = 1, \quad (1.6)$$
and that the support of \(d\rho(\lambda)\) is infinite. One defines polynomials \(P_n(z)\) by the Gram-Schmidt orthonormalization of the monomials \(1, \lambda, \ldots, \lambda^n, \ldots\) in the space \(L^2(\mathbb{R}; d\rho)\) with the scalar product denoted \(\langle \cdot, \cdot \rangle\). Then

\[
\langle P_n, P_m \rangle = \int_{-\infty}^{\infty} P_n(\lambda)P_m(\lambda)d\rho(\lambda) = \delta_{n,m};
\]

as usual, \(\delta_{n,n} = 1\) and \(\delta_{n,m} = 0\) for \(n \neq m\). Such polynomials are defined up to a sign. We accept that the coefficients \(k_n\) in (1.3) are positive. We emphasize that all orthogonal polynomials considered in this paper are normalized. The measure \(d\rho(\lambda)\) is often called the orthogonality measure for the polynomials \(P_0, P_1, \ldots, P_n, \ldots\).

The following statement shows that this definition of orthonormal polynomials implies the first one.

**Proposition 1.1.** The polynomials \(P_n(\lambda)\) defined by equalities (1.7) satisfy recurrence relation (1.1) where

\[
a_n = \int_{-\infty}^{\infty} \lambda P_n(\lambda)P_{n+1}(\lambda)d\rho(\lambda) > 0
\]

and

\[
b_n = \int_{-\infty}^{\infty} \lambda P_n^2(\lambda)d\rho(\lambda).
\]

Proposition 1.1 as well as Propositions 1.2, 1.3, 1.4 stated in the next subsection are checked in Appendix A.

There are three specific classes of polynomials, Jacobi, Hermite and Laguerre, named classical. To a some extent, all modern studies of general orthogonal polynomials can be considered as far reaching generalizations of the results well known for the classical polynomials. Therefore we discuss these important special cases in Appendix B.

### 1.2. Jacobi operators.

Now we proceed from recurrence relations (1.1), (1.2) for the polynomials \(P_0, P_1, \ldots, P_n, \ldots\) and construct their orthogonality measure. To that end, we introduce a Jacobi matrix

\[
\mathcal{J} = \begin{pmatrix}
b_0 & a_0 & 0 & 0 & 0 & \cdots \\
0 & a_1 & b_1 & a_1 & 0 & 0 & \cdots \\
0 & 0 & a_2 & b_2 & a_2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

If \(u = (u_0, u_1, \ldots)^\top =: (u_n)\) is a column, then

\[
(\mathcal{J}u)_0 = b_0u_0 + a_0u_1, \quad (\mathcal{J}u)_n = a_{n-1}u_{n-1} + b_nu_n + a_nu_{n+1} \quad \text{for} \quad n \geq 1,
\]

(1.10)
and equation (1.1) with the boundary condition \( P_{-1}(z) = 0 \) is equivalent to the equation \( \mathcal{J}P(z) = zP(z) \) for the vector \( P(z) = (P_0(z), P_1(z), \ldots) \). Thus, \( P(z) \) is an “eigenvector” of the matrix \( \mathcal{J} \) corresponding to an “eigenvalue” \( z \).

Let us now consider Jacobi operators defined by formula (1.10) in the space \( \ell^2(\mathbb{Z}_+) \). A minimal Jacobi operator \( J_{\text{min}} \) is defined by the equality \( J_{\text{min}}u = \mathcal{J}u \) on a set \( \mathcal{D} \subset \ell^2(\mathbb{Z}_+) \) of vectors \( u = (u_n) \) with only a finite number of non-zero components \( u_n \). Note that \( J_{\text{min}} : \mathcal{D} \to \mathcal{D} \). A maximal operator \( J_{\text{max}} \) is given by the same formula \( J_{\text{max}}u = \mathcal{J}u \) on the set \( \mathcal{D}(J_{\text{max}}) \) of all vectors \( u \in \ell^2(\mathbb{Z}_+) \) such that \( \mathcal{J}u \in \ell^2(\mathbb{Z}_+) \).

Evidently, the operator \( J_{\text{min}} \) is symmetric in the space \( \ell^2(\mathbb{Z}_+) \). We also have the following assertion.

**Proposition 1.2.** The adjoint operator \( J_{\text{min}}^* = J_{\text{max}} \).

It is easy to see that \( J_{\text{min}} \) extends to a bounded operator on \( \ell^2(\mathbb{Z}_+) \) if and only if the sequences \((a_n) \in \ell^\infty(\mathbb{Z}_+)\) and \((b_n) \in \ell^\infty(\mathbb{Z}_+)\). In this paper we are particularly interested in the case \( a_n \to \infty \) when Jacobi operators are unbounded.

Since the operator \( J_{\text{min}} \) commutes with the complex conjugation, its deficiency indices are equal so that \( J_{\text{min}} \) has self-adjoint extensions. Actually, the deficiency indices of \( J_{\text{min}} \) are either \((0,0)\) or \((1,1)\). In the first case the operator \( J_{\text{min}} \) is essentially self-adjoint, that is, its closure \( \text{clos} J_{\text{min}} = J_{\text{max}} \).

A link of orthonormal polynomials with Jacobi operators relies on the following observation.

**Proposition 1.3.** Let \( e_0, e_1, \ldots, e_n, \ldots \) be the canonical basis in the space \( \ell^2(\mathbb{Z}_+) \). Then

\[
e_n = P_n(J_{\text{min}})e_0.
\]

Let \( J \) be an arbitrary self-adjoint extension of the operator \( J_{\text{min}} \), and let \( E_J(\lambda) \) be its spectral family. According to Proposition 1.3 the spectrum of the operator \( J \) is simple with \( e_0 = (1,0,0,\ldots)^\top \) being a generating vector. Therefore it is natural to define the spectral measure of \( J \) by the equality

\[
d\rho_J(\lambda) = d(E_J(\lambda)e_0,e_0).
\]

Let us now state a version of Favard’s theorem.

**Proposition 1.4.** Let polynomials \( P_n(\lambda) \) satisfy recurrence relation (1.1) and boundary conditions (1.2), and let \( J \) be an arbitrary self-adjoint extension of the operator \( J_{\text{min}} \) given by equalities (1.10) on the set \( \mathcal{D} \subset \ell^2(\mathbb{Z}_+) \). Define the measure \( d\rho_J \) by formula (1.12) and set

\[
(\Phi e_n)(\lambda) = P_n(\lambda).
\]

Then the operator

\[
\Phi : \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}; d\rho_J)
\]

is unitary and enjoys the intertwining property

\[
(\Phi Jf)(\lambda) = \lambda(\Phi f)(\lambda), \quad \forall f \in \mathcal{D}(J).
\]
In particular, the spectrum of the operator \( J \) is simple.

**Corollary 1.5.** The following properties are true:

(i) The polynomials \( P_n(\lambda) \) are orthogonal and normalized in the spaces \( L^2(\mathbb{R}; d\rho_J) \), that is, relations (1.7) with \( d\rho(\lambda) = d\rho_J(\lambda) \) are satisfied.

(ii) The set of all polynomials is dense in the spaces \( L^2(\mathbb{R}; d\rho_J) \).

Conversely, suppose that a measure \( d\rho(\lambda) \) satisfying assumptions (1.6) is given. One first constructs orthonormal polynomials \( P_n(\lambda) \) satisfying equalities (1.7) and then defines the recurrence coefficients \( a_n, b_n \) by formulas (1.8) and (1.9). Let \( J_{\text{min}} \) be the minimal Jacobi operator with these coefficients. Taking its arbitrary self-adjoint extension \( J \), one defines its spectral measure \( d\rho_J(\lambda) \) by formula (1.12). It turns out that \( d\rho_J(\lambda) = d\rho(\lambda) \) if the operator \( J_{\text{min}} \) is essentially self-adjoint (then \( J = \text{clos} \ J_{\text{min}} \)). However, this is not true in general; an example of such phenomenon is given in Sect. 1.4 – see the Freud weight (1.21) for \( \beta < 1 \).

This situation can also be described in terms of solutions of the Jacobi equation

\[
a_{n-1} u_{n-1}(z) + b_n u_n(z) + a_n u_{n+1}(z) = zu_n(z), \quad n \geq 1. \tag{1.16}
\]

The Weyl theory developed by him for differential equations can be naturally adapted to difference equations (see, e.g., the book [1] and reference therein). Similarly to differential equations, equation (1.16) for \( \text{Im} z \neq 0 \) always has a non-trivial solution \( f = (f_n) \in \ell^2(\mathbb{Z}_+) \). This solution is either unique (up to a constant factor) or all solutions of equation (1.16) belong to \( \ell^2(\mathbb{Z}_+) \). The first instance is known as the limit point case and the second one – as the limit circle case. It turns out that the operator \( J_{\text{min}} \) is essentially self-adjoint if and only if the limit point case occurs. In the limit circle case, the operator \( J_{\text{min}} \) has deficiency indices \((1,1)\).

Note also that an operator \( J_{\text{min}} \) is essentially self-adjoint if and only if the corresponding moment problem is determinate. This is discussed in Appendix D.

Thus, in the limit point case there is the one-to-one correspondence between Jacobi coefficients \((a_n, b_n)\) and measures \( d\rho(\lambda) \). The reconstruction of the coefficients \((a_n, b_n)\) of a Jacobi operator \( J \) from its spectral measure \( d\rho(\lambda) \) is known as the inverse spectral problem. This procedure (see Proposition 1.1) is stated in terms of the orthonormal polynomials and is quite explicit. This looks quite different from the case of differential operators where a reconstruction of the coefficient \( b(x) \) of a differential operator \( H = D^2 + b(x) \), \( D = -id/dx \), from its spectral measure requires a solution of the integral Gelfand-Levitan equation [24]. Nevertheless, finding one-to-one correspondences between classes of the coefficients \((a_n, b_n)\) and the measures \( d\rho(\lambda) \) (the characterization problem) is as difficult as for differential operators.

Let us, finally, mention an elementary but useful fact.

**Proposition 1.6.** If orthonormal polynomials \( P_n(z) \) are constructed by coefficients \((a_n, b_n)\) and \( \tilde{P}_n(z) \) correspond to the coefficients \((a_n, -b_n)\), then

\[
\tilde{P}_n(z) = (-1)^n P_n(-z).
\]
If $\tilde{J}$ is the Jacobi operator with matrix elements $a_n, -b_n$, then $\tilde{J} = -U^* J U$ where the unitary operator $U$ is defined by $(Uu)_n = (-1)^n u_n$ for $n \in \mathbb{Z}_+$. In particular, if $b_n = 0$ for all $n$, then the operators $J$ and $-J$ are unitarily equivalent. The corresponding spectral measures are linked by the relation $d\tilde{\rho}(\lambda) = d\rho(-\lambda)$.

The comprehensive presentation of the results described shortly here can be found in the books [1, 13, 58, 66] and the surveys [40, 60, 65].

1.3. Asymptotics of orthogonal polynomials. Stabilizing coefficients. We are interested in an asymptotic behavior of the polynomials $P_n(z)$ as $n \to \infty$. Asymptotic properties of $P_n(z)$ can be deduced either from the recurrence coefficients $a_n, b_n$ defining $P_n(z)$ or from the corresponding orthogonality measure $d\rho(\lambda)$:

$$(a_n, b_n) \quad \quad d\rho(\lambda) \quad \quad P_n(z)$$

Asymptotic formulas are very well known for the classical, that is, Jacobi, Hermite and Laguerre, polynomials (see Appendix B), but the first general result is probably due to S. Bernstein (see his pioneering paper [5] or Theorem 12.1.4 in the G. Szegő book [63]). These results were stated in terms of the measure $d\rho(\lambda)$. It was required that $\text{supp} \rho \subset [-1, 1]$, the measure is absolutely continuous, that is

$$d\rho(\lambda) = \tau(\lambda) d\lambda,$$

and the weight $\tau(\lambda)$ satisfies certain regularity conditions, in particular, it does not tend to zero too rapidly as $\lambda \to \pm 1$. Bernstein’s result can be considered as a far reaching generalization of the asymptotic formulas for Jacobi polynomials determined by spectral measure (B.1).

Alternative line of research where asymptotic formulas for the polynomials $P_n(z)$ are deduced from assumptions on the coefficients $a_n, b_n$ was initiated in the book by P. G. Nevai [47] who considered the case

$$\lim_{n \to \infty} (a_n - 1/2) = \lim_{n \to \infty} b_n = 0.$$

Then the “perturbation” $V = J - J_0$ of the “free” Jacobi operator $J_0$ with the elements $a_n = 1/2, b_n = 0$ is compact. Therefore the essential spectrum of the operator $J$ coincides with the interval $[-1, 1]$, and its discrete spectrum consists of simple eigenvalues accumulating, possibly, to the points 1 and $-1$. Under the only assumption (1.18) the spectral structure of the Jacobi operator $J$ can be quite wild.
Therefore Nevai supposed that
\[ \sum_{n=0}^{\infty} (|a_n - 1/2| + |b_n|) < \infty. \] (1.19)

A more general case
\[ \sum_{n=0}^{\infty} (|a_{n+1} - a_n| + |b_{n+1} - b_n|) < \infty \] (1.20)
was studied somewhat later in [44]. Under assumptions (1.18), (1.20) the measure
\( d\rho(\lambda) \) is absolutely continuous on the interval \((-1, 1)\) and the weight \( \tau(\lambda) \) is a continuous and strictly positive function. The corresponding polynomials \( P_n(z) \) satisfy asymptotic relations generalizing formulas (B.4), (B.5) for the Jacobi polynomials.

The paper [44] relies on specific methods of orthogonal polynomials theory. Its initial point is the relation
\[ \lim_{n \to \infty} \frac{P_{n-1}(z)}{P_n(z)} = z - \sqrt{z^2 - 1}, \quad \text{Im } z \neq 0, \quad \sqrt{z^2 - 1} > 0 \quad \text{for } z > 1, \]
established earlier by P. Nevai in the book [47], Theorem 4.1.13, and improving one of Poincaré’s theorems.

Conditions (1.19) and (1.20) are very precise. Indeed, as shown in [46] (see also the preceding paper [54]), there exist coefficients \( b_n \) decaying only slightly worse than \( n^{-1} \) and oscillating as \( n \to \infty \) such that the point spectrum of the corresponding Jacobi operator \( J \) with \( a_n = 1/2 \) is dense in \([-1, 1]\). In this case the limiting absorption principle for the operator \( J \) does not of course hold.

This discussion is continued in Sect. 8.4.

1.4. Asymptotics of orthogonal polynomials. Increasing coefficients. Passing to the case of unbounded Jacobi operators, we first discuss orthogonality measures (1.17) with exponential weights
\[ \tau(\lambda) = k_\beta e^{-|\lambda|^\beta}, \quad \lambda \in \mathbb{R}, \] (1.21)
where \( \beta > 0 \) and \( k_\beta \) are normalization constants. Such weights were introduced in the paper [23] and are known as the Freud weights. Obviously, the value \( \beta = 2 \) yields the Hermite polynomials. Let \( P_n(z) \) be the orthonormal polynomials and let \( a_n, b_n \) be the recurrence coefficients defined by relations (1.8) and (1.9), respectively. It was shown in [41, 42] that the off-diagonal coefficients \( a_n \) have asymptotics
\[ a_n = \alpha n^\ell (1 + o(1)), \quad \ell = 1/\beta, \quad n \to \infty, \] (1.22)
with some explicit constants \( \alpha = \alpha_\ell \). Since \( \tau(-\lambda) = \tau(\lambda) \), the diagonal elements \( b_n = 0 \). The minimal Jacobi operator \( J_{\min} \) with these coefficients is essentially self-adjoint if and only if \( \beta \geq 1 \).

An asymptotics of orthonormal polynomials \( P_n(z) \) defined by measure (1.17), (1.21) is given for all \( z \in \mathbb{C} \) by the Plancherel-Rotach formula generalizing the
corresponding formula for the Hermite polynomials; see [49] for \( \beta = 4 \) and [55, 56] for all \( \beta \geq 1 \). More general results were obtained later with a help of the Riemann-Hilbert problem method combined with the steepest descent method (see the articles [21, 17] and the book [16]). In particular, asymptotic formulas for \( P_n(z) \) were extended in [38] to all \( \beta > 0 \). We emphasize that according to the classical Nevanlinna’s results [51] in the case \( \beta \in (0, 1) \), all self-adjoint extensions \( J \) of the minimal operator \( J_{\text{min}} \) have purely discrete spectra. Note that the Plancherel-Rotach formula yields an asymptotics of \( P_n(z) \) as \( n \to \infty \) for all \( z \in \mathbb{C} \); besides values of \( z \) in this formula are not necessarily fixed.

Probably, the first paper where an asymptotics of \( P_n(z) \) was investigated under conditions on growing recurrence coefficients \( a_n \) (not on the measure \( d\rho(\lambda) \)) is due to Janas and Naboko [33]. It was assumed in this paper that condition (1.22) holds with \( \ell \in (1/2, 1) \), \( b_n = 0 \) and the spectral parameter \( z = \lambda \in \mathbb{R} \). In [33], the authors solve equations (1.1) successively starting from \( n = 0 \). This yields a representation for \( P_n(\lambda) \) in terms of a product of \( n \) two-by-two matrices (the transfer matrices) expressed via \( a_n \) and \( \lambda \). Then one has to study an asymptotics of this product as \( n \to \infty \) which is a non-trivial problem. The proof of the absolute continuity of the spectrum in this approach requires the Gilbert-Pearson subordinacy theory [27] adapted to Jacobi operators in [35]. More general results of this type were obtained in the subsequent paper [2] by Aptekarev and Geronimo where the polynomials \( P_n(z) \) were considered for all \( z \in \mathbb{C} \). The method of [2] relies on a study of auxiliary Jacobi operators \( J^{(N)} \) with the coefficients \( a_n^{(N)} = a_n \), \( b_n^{(N)} = b_n \) for \( n \leq N \) and \( a_N^{(N)} = a_N \), \( b_N^{(N)} = b_N \) for \( n \geq N \). Then one applies to the operators \( J^{(N)} \) the results of [47, 44] and studies the limit \( N \to \infty \). Asymptotic formulas found in papers [33, 2] are consistent with the Plancherel-Rotach formula. Finally, we note a recent paper [62] also devoted to Jacobi operators with increasing coefficients.

1.5. Difference versus differential operators. Our intention is to emphasize and consistently use an analogy between Jacobi operators \( J \) defined by formula (1.10) in the space \( L^2(\mathbb{Z}_+) \) and differential operators

\[
H = Da(x)D + b(x)
\]

(1.23)

(with, for example, the boundary condition \( u(0) = 0 \)) in the space \( L^2(\mathbb{R}_+) \). For Jacobi operators, the parameter \( n \in \mathbb{Z}_+ \) plays the role of the variable \( x \in \mathbb{R}_+ \) and the coefficients \( a_n \), \( b_n \) play the roles of the functions \( a(x) \), \( b(x) \), respectively. In particular, the operator \( H_0 = D^2 \) corresponds to the Jacobi operator \( J_0 \) with the coefficients \( a_n = 1/2 \), \( b_n = 0 \). It is known as the “free” discrete Schrödinger operator. Of course this analogy between difference and differential operators is very well known (see, e.g., [11]), and we are going to use it in a systematic way.

By construction of spectral theory for the operator \( H \), one studies a differential equation

\[
-(a(x)u'(x,z))' + b(x)u(x,z) = zu(x,z), \quad x > 0,
\]

(1.24)
and distinguishes its regular \( \varphi(x, z) \) and Jost \( f(x, z) \) solutions. The regular solution \( \varphi(x, z) \) is fixed by initial conditions \( \varphi(0, z) = 0, \varphi'(0, z) = 1 \) and the Jost solution \( f(x, z) \) is determined by its asymptotics as \( x \to \infty \). In the simplest case when \( a(x) = 1 \) (or another positive constant) and \( b \in L^1(\mathbb{R}_+) \) the asymptotics of the Jost solution is given by a relation \( f(x, z) \sim e^{i\sqrt{zx}} \) (here \( \text{Im} \sqrt{z} \geq 0 \)). For every \( x \geq 0 \), the functions \( f(x, z) \) depend analytically on \( z \in \mathbb{C} \setminus [0, \infty) \) and are continuous up to the cut along \((0, \infty)\). This implies that the integral kernel of the resolvent \( (H - z)^{-1} \) is a continuous function of \( z \) up to the positive half-axis. This fact is known as the limiting absorption principle. It follows that the positive spectrum of the operator \( H \) is absolutely continuous.

If \( \lambda > 0 \), then the solutions \( f(x, \lambda + i0) \) and \( f(x, \lambda - i0) \) are complex conjugate to each other and are linearly independent. If \( z \in \mathbb{C} \setminus [0, \infty) \), then the second solution \( g(x, z) \) of (1.24) can be constructed by an explicit formula stated, for example, in §4.1 of the book \[71\]. It exponentially grows as \( x \to \infty \). Thus for all \( z \in \mathbb{C} \), one has two linearly independent solutions of equation (1.24) with explicit asymptotics as \( x \to \infty \). Since the regular solution \( \varphi(x, z) \) is a linear combination of \( f(x, z) \) and \( g(x, z) \), this gives also an asymptotics of \( \varphi(x, z) \). The scheme described above was realized, for example, in \[73\].

In more general situations, asymptotics as \( x \to \infty \) of the Jost solutions \( f(x, z) \) are given (see, e.g., the book \[53\]) by the Liouville-Green formula

\[
f(x, z) \sim G(x, z)^{-1/2} \exp\left(-\int_{x_0}^{x} G(y, z) dy \right) =: A(x, z). \tag{1.25}\]

Here \( x_0 \) is some fixed number and

\[
G(x, z) = \sqrt{\frac{b(x) - z}{a(x)}}, \quad \text{Re} G(x, z) \geq 0.
\]

Otherwise, under quite general assumptions on the coefficients \( a(x) \) and \( b(x) \) the main steps of the construction of the spectral theory for the Schrödinger operator \( H \) remain basically similar to the particular case \( a(x) = 1, b \in L^1(\mathbb{R}_+) \).

As far as a relation of discrete and continuous problems is concerned, we note also the book \[3\].

1.6. Structure of the paper. Our objective is to develop the same scheme for Jacobi operators \( J \) and the corresponding difference equations (1.16). We concentrate on the case of recurrence coefficients satisfying the assumptions

\[
a_n \to \infty \quad \text{and} \quad -\frac{b_n}{2\sqrt{a_{n-1}a_n}} := \beta_n \to \beta_\infty \quad \text{where} \quad |\beta_\infty| \neq 1 \quad \text{as} \quad n \to \infty. \tag{1.26}\]

Sect. 2. is introductory. Here some general facts about Jacobi operators are discussed.
Sect. 3 and 4 play the central role. In the first of them we introduce and investigate a Volterra equation which is an analytical basis of our method. Then we construct the Jost solutions $f_n(z)$, $n \geq -1$, and study their basic properties. Using these results, we find in Sect. 5 an asymptotic behavior as $n \to \infty$ of the orthonormal polynomials $P_n(z)$ for $z = \lambda \in \mathbb{R}$. At the same time we obtain spectral results for the operators $J$. In the case $|\beta_\infty| < 1$, we check that the spectrum of the operator $J$ coincides with the whole real axis and is absolutely continuous. We also find an expression for the spectral measure of the operator $J$ in terms of boundary values $|f_{-1}(\lambda \pm i0)|$. An asymptotic behavior of the polynomials $P_n(z)$ for $\text{Im} z \neq 0$ is studied in Sect. 6. Our scheme is essentially the same in the cases $|\beta_\infty| < 1$ and $|\beta_\infty| > 1$, but asymptotic formulas and spectral results obtained in these two cases are drastically different.

In Sect. 3-6, we assume that $a_n \to \infty$ but not too rapidly so that the condition (introduced in the book [10] and known as the Carleman condition)

$$\sum_{n=0}^{\infty} a_n^{-1} = \infty$$  \hspace{1cm} (1.27)

is satisfied. The singular case where this condition is violated is studied in Sect. 7. Astonishingly, in this case the asymptotic formulas for the Jost solutions and orthonormal polynomials are particularly simple.

Sect. 8 and 9 are devoted to another limit case of the results of Sect. 3-6 where

$$a_n \to a_\infty > 0 \quad \text{and} \quad b_n \to b_\infty \quad \text{as} \quad n \to \infty.$$  \hspace{1cm} (1.28)

Here our presentation is rather sketchy since this case was already treated by the methods of the present paper in [72, 74]. Note that in terms of the analogy with differential operators conditions (1.19) and (1.20) correspond, respectively, to short-range and long-range perturbations of the operator $D^2$.

We mention that many important topics in the theory of orthogonal polynomials and Jacobi operators such as the critical case $|\beta_\infty| = 1$ and various types of oscillating coefficients are completely out of the scope of this text.

To stress an analogy between differential and difference operators, we often use the “continuous” terminology (Volterra integral equations, integration by parts, etc.) for sequences labelled by the discrete variable $n$. Below $C$, sometimes with indices, and $c$ are different positive constants whose precise values are of no importance; $I$ is the identity operator in various spaces. We usually do not care about precise estimates of various remainders writing $o(1)$.

2. SCHEME OF THE APPROACH

Our plan here is the following. After introducing necessary notations in Sect. 2.1, we construct the resolvents of Jacobi operators in Sect. 2.2 and discuss exponentially growing solutions of the Jacobi equations in Sect. 2.3. In Sect. 2.4 we recall the
uniformization $z \mapsto \zeta$ playing the role of the relation $z \mapsto \zeta = \sqrt{z}$ for differential operators. Crucial steps of our approach are described in Sect. 2.5. Then we briefly state some particular cases of our main results in Sect. 2.6.

2.1. **Standard relations.** Let us consider the difference equation (1.16). Note that the values of $u_{N-1}$ and $u_N$ for some $N \in \mathbb{Z}_+$ determine the whole sequence $u_n$ satisfying (1.16). In particular, this is true if $u_{-1}$ and $u_0$ are given. We often start a construction of solutions $u_n$ of equation (1.16) with large $n$. Then they are extended to all $n \geq -1$ by formula (1.16).

Let $u = (u_n)_{n=-1}^{\infty}$ and $v = (v_n)_{n=-1}^{\infty}$ be two solutions of the Jacobi equation (1.16). A direct calculation shows that their Wronskian

$$\{u, v\} := a_{-1} (u_{n-1}v_n - u_nv_{n-1})$$

(2.1)

does not depend on $n = -1, 0, 1, \ldots$. In particular, for $n = -1$ and $n = 0$, we have

$$\{u, v\} = a_{-1} (u_{-1}v_0 - u_0v_{-1}) \quad \text{and} \quad \{u, v\} = a_0 (u_0v_1 - u_1v_0);$$

here and below $a_{-1}$ is an arbitrary fixed positive number. Clearly, the Wronskian $\{u, v\} = 0$ if and only if the solutions $u$ and $v$ are proportional.

Let $J_{\min}$ be the operator defined in the space $\ell^2(\mathbb{Z}_+)$ by formula (1.10) on the set $\mathcal{D}$ of sequences $u = (u_n)$ such that $u_n = 0$ for sufficiently large $n$. In the limit circle case, all solutions of the Jacobi equation (1.16) are in $\ell^2(\mathbb{Z}_+)$ so that necessarily

$$a_n^{-1} = \{u, v\}^{-1} (u_nv_{n+1} - u_{n+1}v_n) \in \ell^1(\mathbb{Z}_+).$$

It follows that the Carleman condition (1.27) is sufficient for the limit point case and hence for the essential self-adjointness of the operator $J_{\min}$.

It is convenient to introduce a notation

$$u'_n = u_{n+1} - u_n$$

(2.2)

for the “derivative” of a sequence $u_n$. Note a formula

$$(u_nv'_n)' = u'_nv_{n+1} + u_nv'_n.$$  

(2.3)

In particular, this yields the Abel summation formula (“integration-by-parts”):

$$\sum_{n=N}^{M} u_nv'_n = u_Mv_{M+1} - u_{N-1}v_N - \sum_{n=N}^{M} u'_{n-1}v_n;$$

(2.4)

here $M \geq N \geq 0$ are arbitrary, but we have to set $u_{-1} = 0$ so that $u'_{-1} = u_0$.

2.2. **Resolvent.** In this subsection we only suppose that the minimal Jacobi operator $J_{\min}$ is essentially self-adjoint. Then $\text{clo} J_{\min} = J_{\max} =: J$ and equation (1.16) has (see, e.g., §3 of Chapter 1 in the book [1]) a unique, up to a constant factor, non-trivial solution $f_n(z)$ such that

$$f_n(z) \in \ell^2(\mathbb{Z}_+), \quad \text{Im} z \neq 0.$$  

(2.5)
Let us introduce the Wronskian of the solutions $P(z) = (P_n(z))$ and $f(z) = (f_n(z))$:

$$
\Omega(z) := \{P(z), f(z)\} = a_{-1}(P_{-1}(z)f_0(z) - P_0(z)f_{-1}(z)) = -a_{-1}f_{-1}(z).
$$

(2.6)

Observe that $\Omega(z) \neq 0$ if Im $z \neq 0$. Indeed, otherwise $P(z) \in \ell^2(\mathbb{Z}_+)$ so that, in view of equation (1.11), $z$ is an eigenvalue of the operator $J$. This is impossible since $J$ is self-adjoint.

Our goal is to construct the resolvent $R(z) = (J - zI)^{-1}$ of the operator $J$ for Im $z \neq 0$. The following statement is very close to the corresponding result for differential operators.

**Proposition 2.1.** For all $h = (h_n) \in \ell^2(\mathbb{Z}_+)$, we have

$$
(R(z)h)_n = \Omega(z)^{-1}\left(f_n(z) \sum_{m=0}^{n} P_m(z)h_m + P_n(z) \sum_{m=n+1}^{\infty} f_m(z)h_m\right), \quad \text{Im} \ z \neq 0.
$$

(2.7)

**Proof.** Denote the right-hand side of (2.7) by $(R(z)h)_n$. We have to check that

$$
(J - zI)R(z)h = h, \quad \text{Im} \ z \neq 0,
$$

(2.8)
at least for all sequences $h \in D$. Let us set

$$
A_n(z) = \sum_{m=0}^{n} P_m(z)h_m, \quad B_n(z) = \sum_{m=n+1}^{\infty} f_m(z)h_m.
$$

(2.9)

Then

$$
\Omega(z)(R(z)h)_n = f_n(z)A_n(z) + P_n(z)B_n(z).
$$

(2.10)

Note that $A_n(z)$ does not depend on $n$ and $B_n(z) = 0$ for sufficiently large $n$. In view of (2.5), we have $R(z)h \in \ell^2(\mathbb{Z}_+)$. It follows from definition (1.10) of the Jacobi matrix $J$ that

$$
\Omega((J - zI)R(z)h)_n = a_{n-1}(f_{n-1}A_{n-1} + P_{n-1}B_{n-1})
$$

$$
+ (b_n - z)(f_nA_n + P_nB_n) + a_n(f_{n+1}A_{n+1} + P_{n+1}B_{n+1}).
$$

(2.11)

for $n \geq 1$ and

$$
\Omega((J - zI)R(z)h)_0 = (b_0 - z)(f_0A_0 + P_0B_0) + a_0(f_1A_1 + P_1B_1).
$$

(2.12)

According to (2.9) we have

$$
f_{n-1}A_{n-1} + P_{n-1}B_{n-1} = f_{n-1}(A_n - P_nh_n) + P_{n-1}(B_n + f_nh_n), \quad n \geq 1,
$$

and

$$
f_{n+1}A_{n+1} + P_{n+1}B_{n+1} = f_{n+1}A_n + P_{n+1}B_n.
$$

(2.13)

Let us substitute these expressions into the right-hand side of (2.11) and observe that the coefficients at $A_n$ and $B_n$ equal zero by virtue of equation (1.16) for $f_n$ and $P_n$, respectively. It follows that

$$
((J - zI)R(z)h)_n = \Omega^{-1}a_{n-1}(-P_{n-1}f_{n-1} + f_nP_{n-1})h_n = h_n, \quad n \geq 1.
$$

(2.14)
Next, we consider the right-hand side of (2.12). According to equality (2.13) for $n = 0$ it equals
\[(b_0 - z)(f_0 A_0 + P_0 B_0) + a_0 (f_1 A_0 + P_1 B_0) = ((b_0 - z) f_0 + a_0 f_1) A_0 + ((b_0 - z) + a_0 P_1) B_0.\]
The first term on the right is $-a_{-1} f_{-1} A_0 = \Omega h_0$ by equation (1.16) for $f_n$, equality $A_0 = h_0$ and definition (2.6) of the Wronskian $\Omega$. The second term is zero by equation (1.16) for $P_n$. Thus it follows from (2.12) that $((J - z) R(z) h)_0 = h_0$.
Together with (2.14), this implies equality (2.8). In particular, we see that $R(z) h \in \mathcal{D}(J_{\text{max}}) = \mathcal{D}(J)$ so that $(J - z I) R(z) h = h$ whence $R(z) = (J - z I)^{-1}$.

2.3. Linearly independent solutions of the Jacobi equation. Here we suppose that a solution of the Jacobi equation (1.16) is given and describe a general procedure which allows one to construct another solution, linearly independent with the first one. This procedure plays a crucial role in our study of an asymptotic behavior of the orthonormal polynomials $P_n(z)$ as $n \to \infty$ for complex $z$. Recall that the Wronskian of two solutions of equation (1.16) is defined by relation (2.11).

**Theorem 2.2.** Let $f(z) = (f_n(z))$ be an arbitrary solution of the Jacobi equation (1.16) such that $f_n(z) \neq 0$ for sufficiently large $n$, say $n \geq N_0$. Put
\[F_n(z) = \sum_{m=N_0+1}^{n} (a_{m-1} f_{m-1}(z) f_m(z))^{-1}, \quad n \geq N_0 + 1,\]
and
\[g_n(z) = f_n(z) F_n(z), \quad n \geq N_0 + 1.\]
Then the sequence $g(z) = (g_n(z))$ satisfies equation (1.16) and the Wronskian
\[\{f(z), g(z)\} = 1\]
so that the solutions $f(z)$ and $g(z)$ are linearly independent.

**Proof.** First, we check equation (1.16) for $g_n(z)$. According to definition (2.16), we have
\[a_{n-1} g_{n-1} + (b_n - z) g_n + a_n g_{n+1} = a_{n-1} f_{n-1} F_{n-1} + (b_n - z) f_n F_n + a_n f_{n+1} F_{n+1}\]
\[= (a_{n-1} f_{n-1} + (b_n - z) f_n + a_n f_{n+1}) F_n + a_{n-1} f_{n-1} (F_n - F_n) + a_n f_{n+1} (F_{n+1} - F_n).\]
The first term here is zero because equation (1.16) is true for the sequence $f_n$. Since, by (2.15),
\[F_{n+1} = F_n + (a_n f_n f_{n+1})^{-1},\]
the second and third terms equal $-f_n^{-1}$ and $f_n^{-1}$, respectively, which proves equation (1.16) for sequence (2.16).

It also follows from definition (2.16) and relation (2.17) that the Wronskian (2.1) equals
\[\{f(z), g(z)\} = a_n f_n(z) f_{n+1}(z) (F_{n+1}(z) - F_n(z)) = 1,\]
whence the solutions $f(z)$ and $g(z)$ are linearly independent. \qed
As usual, a sequence \( g_n(z) \) constructed for large \( n \) is extended to all \( n \geq -1 \) as a solution of equation (1.16).

2.4. Uniformization. We fix the branch of the analytic function \( \sqrt{z^2 - 1} \) of \( z \in \mathbb{C} \setminus [-1, 1] =: \Pi_0 \) by the condition \( \sqrt{z^2 - 1} > 0 \) for \( z > 1 \). Obviously, this function is continuous up to the cut along \([-1, 1]\), it equals \( \pm i \sqrt{1 - \lambda^2} \) for \( z = \lambda \pm i0 \) where \( \lambda \in (-1, 1) \) and \( \sqrt{z^2 - 1} < 0 \) for \( z < -1 \). Put

\[
\zeta(z) = z - \sqrt{z^2 - 1} = (z + \sqrt{z^2 - 1})^{-1}.
\] (2.18)

The mapping \( \zeta \) of \( \Pi_0 \) onto the unit disc \( D \) is one-to-one and holomorphic. Note also that \( \zeta(-z) = -\zeta(z) \) and \( \zeta(\bar{z}) = \overline{\zeta(z)} \). The function \( \zeta(z) \) maps the half-planes \( \mathbb{C}^{(\pm)} = \{ z \in \mathbb{C} : \pm \text{Im } z > 0 \} \) onto the half-discs \( D^{(\mp)} = D \cap \mathbb{C}^{(\mp)} \); thus, \( \zeta : \mathbb{C}^{(\pm)} \mapsto D^{(\mp)} \). For \( \lambda > 1 \), we have \( \zeta(\lambda) \in (0, 1) \) and \( \zeta(1) = 1, \zeta(\pm \infty) = 0 \). Similarly, \( \zeta(\lambda) \in (-1, 0) \) for \( \lambda < -1 \), \( \zeta(-1) = -1 \) and \( \zeta(-\infty) = 0 \). For \( \lambda \in [-1, 1] \), it is common to set \( \lambda = \cos \theta \) with \( \theta \in [0, \pi] \). Then \( \zeta(\lambda \pm i0) = e^{\mp i\theta} \). Note also that

\[
\zeta(z) + \zeta(z)^{-1} = 2z
\]

so that \( \zeta(z) \) is the inverse Zhukovsky function. Function (2.18) plays the role of the function \( \zeta(z) = \sqrt{z} \) in the theory of the Schrödinger operator. In all estimates below the values of \( |z| \) are bounded. Then the values of function (2.18) are separated from 0.

Let us introduce a notation

\[
\alpha_n = \frac{1}{2\sqrt{a_{n-1}a_n}}, \quad \beta_n = -\frac{b_n}{2\sqrt{a_{n-1}a_n}}
\] (2.19)

and set

\[
z_n = \alpha_n z + \beta_n, \quad \zeta_n = \zeta(z_n).
\] (2.20)

According to assumption (1.26) we have

\[
\lim_{n \to \infty} z_n = \beta_\infty
\] (2.21)

whence

\[
\lim_{n \to \infty} \zeta(z_n) = \zeta(\beta_\infty \pm i0) \quad \text{if } \pm \text{Im } z \geq 0.
\] (2.22)

Note that

\[
\zeta(\beta_\infty + i0) = \zeta(\beta_\infty - i0) = \beta_\infty - \sqrt{\beta_\infty^2 - 1} =: \zeta_\infty \in (-1, 1)
\] (2.23)

if \( |\beta_\infty| > 1 \) and

\[
\zeta(\beta_\infty \pm i0) = \beta_\infty \mp i\sqrt{1 - \beta_\infty^2} =: \zeta_\infty^{(\pm)} \in \mathbb{T} \cap \mathbb{C}^{(\pm)}, \quad \pm \text{Im } z \geq 0,
\] (2.24)

if \( |\beta_\infty| < 1 \). Relations (2.21) and hence (2.23), (2.24) are uniform in \( z \) from compact subsets of \( \mathbb{C} \).
We set \( \Pi = \mathbb{C} \setminus \mathbb{R} \) and denote by \( \text{clos} \Pi \) its closure. Thus, \( \text{clos} \Pi \) is the complex plane with the cut along \( \mathbb{R} \).

2.5. **Main steps.** Our study of an asymptotic behavior of the orthonormal polynomials \( P_n(z) \) as \( n \to \infty \) consists of the following steps. We describe them here for Jacobi operators \( J \) with coefficients satisfying assumptions (1.26) with \( |\beta_\infty| < 1 \) and the Carleman condition (1.27). For Schrödinger operators with short-range coefficients, the scheme used here goes back to the paper [34] by R. Jost.

A. First, we forget about the orthogonal polynomials and distinguish solutions (the Jost solutions) \( f_n(z) \) of the difference equation (1.16) by their asymptotics as \( n \to \infty \). This requires a construction of an Ansatz \( A_n(z) \) for the Jost solutions. Actually, \( A_n(z) \) turns out to be the leading term of \( f_n(z) \) for \( n \to \infty \).

B. We define \( A_n(z) \) by a formula

\[
A_n(z) = a_n^{-1/2} \zeta(z_{N_0}) \zeta(z_{N_0+1}) \cdots \zeta(z_{n-1}), \quad n \geq N_0, \quad z \in \text{clos} \Pi, \quad (2.25)
\]

with \( z_n \) defined by (2.20). Using (2.21) we choose the number \( N_0 = N_0(z) \) in such a way that \( z_n \) for \( n \geq N_0 \) are separated from the points +1 and −1. Clearly, \( N_0 \) can be chosen common for all \( z \) from a given compact subset of \( \text{clos} \Pi \). It turns out that the relative remainder

\[
r_n(z) := (\sqrt{a_{n-1} a_n A_n(z)})^{-1} \left( a_{n-1} A_{n-1}(z) + (b_n - z) A_n(z) + a_n A_{n+1}(z) \right) \quad (2.26)
\]

belongs to \( \ell^1 \) for \( n \to \infty \). Formula (2.25) plays the role of the Liouville-Green Ansatz (1.25) for differential operators. According to (2.22), (2.24) asymptotics of the products

\[
q_n(z) = \zeta_{N_0} \zeta_{N_1} \cdots \zeta_{n-1}, \quad \zeta_n = \zeta(z_n), \quad (2.27)
\]

contain an oscillating factor \( (\zeta(\pm \infty))^n \). Moreover, it can be deduced from the Carleman condition (1.27) that \( |q_n(z)| \to 0 \) as \( n \to \infty \) if \( \text{Im} z \neq 0 \) while \( |q_n(z)| \to c \neq 0 \) if \( \text{Im} z = 0 \).

C. A multiplicative change of variables

\[
f_n(z) = A_n(z) u_n(z) \quad (2.28)
\]

reduces difference equation (1.16) for \( f_n(z) \) to a Volterra “integral” equation for the sequence \( u_n(z) \). This Volterra equation can be standardly solved by iterations which allows us to prove the existence of its solution \( u_n(z) \) such that

\[
\lim_{n \to \infty} u_n(z) = 1. \quad (2.29)
\]

Then the Jost solution \( f_n(z) \) of equation (1.16) is defined by formula (2.28). It follows from (2.29) that

\[
f_n(z) = A_n(z)(1 + o(1)) \quad (2.30)
\]

as \( n \to \infty \). The functions \( u_n(z) \) and therefore \( f_n(z) \) turn out to be analytic in \( z \) in the complex plane with the cut along the real axis and are continuous up to the cut.
We emphasize that the sequences $\mathcal{A}_n(z)$ and hence $f_n(z)$ are defined up to factors depending, possibly, on $z$. These factors can be chosen at our convenience.

D. If $z = \lambda \in \mathbb{R}$, then equation (1.16) has two linearly independent solutions $f_n(\lambda + i0)$ and

$$f_n(\lambda - i0) = \overline{f_n(\lambda + i0)}.$$ 

Their Wronskian equals

$$\{f(\lambda + i0), f(\lambda - i0)\} = 2i\sqrt{1 - \beta^2_\infty} \neq 0.$$ 

Therefore the polynomials $P_n(\lambda)$ are linear combinations of $f(\lambda + i0)$ and $f(\lambda - i0)$:

$$P_n(\lambda) = \frac{\Omega(\lambda - i0)f_n(\lambda + i0) - \Omega(\lambda + i0)f_n(\lambda - i0)}{2i\sqrt{1 - \beta^2_\infty}}, \quad n = 0, 1, 2, \ldots, \quad \lambda \in \mathbb{R},$$ 

where $\Omega(z)$ is Wronskian (2.6). Note that $\Omega(\lambda \pm i0) \neq 0$ for all $\lambda \in \mathbb{R}$ so that formulas (2.30) and (2.31) yield an asymptotics of $P_n(\lambda)$ as $n \to \infty$.

If $\text{Im} z \neq 0$, a solution $g_n(z)$ of equation (1.16) linearly independent with $f_n(z)$ can be constructed by explicit formulas (2.15), (2.16). The sequence $g_n(z)$ rapidly grows as $n \to \infty$ and a limit

$$\lim_{n \to \infty} \left( \mathcal{A}_n(z)g_n(z) \right) = \mp \frac{i}{2\sqrt{1 - \beta^2_\infty}}, \quad \pm \text{Im} z > 0,$$ 

exists. According to Theorem 2.2 we have

$$P_n(z) = \omega(z)f_n(z) - \Omega(z)g_n(z)$$ 

where $\omega(z) = \{P(z), g(z)\}$. Therefore the asymptotics of $P_n(z)$ is given by the relation

$$\lim_{n \to \infty} \left( \mathcal{A}_n(z)P_n(z) \right) = \pm \frac{i}{2\sqrt{1 - \beta^2_\infty}} \frac{\{P(z), f(z)\}}{\sqrt{1 - \beta^2_\infty}}, \quad \pm \text{Im} z > 0.$$ 

E. Our results on the Jost solutions $(f_n(z))$ directly imply that the spectrum of the Jacobi operator $J$ is absolutely continuous and covers the whole axis $\mathbb{R}$. Its spectral measure is given by the formula

$$d\rho(\lambda) = \pi^{-1}\sqrt{1 - \beta^2_\infty} |\Omega(\lambda + i0)|^{-2}d\lambda.$$ 

At the same time, we obtain the limiting absorption principle for the operator $J$ stating that matrix elements $\langle R(z)u, v \rangle$, $\text{Im} z \neq 0$, of its resolvent are continuous functions of $z$ up to the cut along the real axis for all $u, v \in D$.

Let now assumptions (1.26) hold with $|\beta_\infty| > 1$. Then the scheme described above remains the same; in particular, the Ansatz is again defined by equality (2.25). Nevertheless, asymptotic and spectral results are quite different from the case $|\beta_\infty| < 1$. Thus, the functions $\mathcal{A}_n(z)$ and hence $f_n(z)$ exponentially tend to zero as $n \to \infty$ for all $z \in \mathbb{C}$. On the contrary, the solutions $g_n(z)$ and the orthonormal polynomials $P_n(z)$ exponentially tend to infinity as $n \to \infty$. The spectra of the corresponding Jacobi operators $J$ are discrete.
To a large extent, a construction of an Ansatz for difference equations is similar to their formal solutions. Such a procedure was suggested by Birkhoff in [6] and substantially developed in [68]; see the book [19], for a detailed presentation.

We emphasize that the approach of this paper is essentially different from those of [33] and [2] discussed in Sect. 1.4.

2.6. An overview of the main results. Here we state our results under some simplifying assumptions; see Sect. 4.3 for more details. We now suppose that the Carleman condition (1.27) is satisfied but the coefficients \( a_n \) do not tend to infinity too slowly. More precisely, we require that

\[
\sum_{n=0}^{\infty} a_n^{-3} (1 + |b_n|) < \infty. \tag{2.36}
\]

Conditions (1.27) and (2.36) admit a growth of the off-diagonal coefficients \( a_n \) as \( n^p \) for \( p \in (1/2, 1] \) and even for \( p \in (1/3, 1] \) if \( b_n = 0 \).

Recall that the numbers \( \alpha_n, \beta_n \) are defined by equalities (2.19) and \( \beta_{\infty} \) is the limit of \( \beta_n \) as \( n \to \infty \). Suppose first that \( |\beta_{\infty}| < 1 \) and set

\[
\phi_n = \sum_{m=N_0+1}^{n-1} \arccos \beta_m \quad \text{and} \quad \psi_n = \sum_{m=N_0+1}^{n-1} \frac{\alpha_m}{\sqrt{1 - \beta_m^2}}; \tag{2.37}
\]

here \( N_0 \) is so large that \( |\beta_n| < 1 \) for all \( n > N_0 \). Under assumption (2.36) expression (2.25) for the Ansatz \( A_n(z) \) can be simplified which yields asymptotics of the Jost solutions \( f_n(z) \) of the Jacobi equation (1.16) in the following explicit form

\[
f_n(z) = a_n^{-1/2} e^{\mp i \phi_n \pm i \psi_n} (1 + o(1)), \quad \pm \text{Im} \ z \geq 0, \quad n \to \infty. \tag{2.38}
\]

As an example, we note the recurrence coefficients \( a_n = \nu(n+1)^p \) where \( \nu > 0, \ p \in (1/3, 1] \) and \( b_n = 0 \). Then formula (2.38) reads as

\[
f_n(z) = \nu^{-1/2} n^{-p/2} e^{\pm i \gamma n^{1-p}} (1 + o(1)), \quad \gamma = (2(1-p)\nu)^{-1}.
\]

In the case \( p = 1/2 \) this is of course consistent with formulas for the Hermite polynomials when \( a_n = \sqrt{(n+1)/2} \) and \( b_n = 0 \).

In the general case where \( a_n, b_n \) satisfy (1.26) we have

\[
\phi_n = n \arccos \beta_{\infty} + o(n) \quad \text{as} \quad n \to \infty
\]

and \( \psi_n = O(n^{2/3}) \) according to (2.36), whence the phases \( \psi_n \) are negligible compared to \( \phi_n \). However it follows from (2.38) that

\[
|f_n(z)| = a_n^{-1/2} e^{-\text{Im} \ z |\psi_n|} (1 + o(1)), \quad n \to \infty,
\]

and hence the behavior of \( |f_n(z)| \) as \( n \to \infty \) for \( \text{Im} \ z \neq 0 \) is determined by \( \psi_n \). It is easy to show (see Lemma 5.6 below) that \( f_n(z) \in \ell^2(\mathbb{Z}_+) \) for \( \text{Im} \ z \neq 0 \). For each \( n \geq -1 \), the functions \( f_n(z) \) are analytic in the half-planes \( \pm \text{Im} \ z > 0 \) and are continuous up to the real line.
Consider now the orthonormal polynomials \( P_n(z) \). Suppose first that \( z = \lambda \in \mathbb{R} \). It follows from formulas (2.31) and (2.38) that, as \( n \to \infty \), the polynomials \( P_n(\lambda) \) have asymptotics

\[
P_n(\lambda) = -a_n^{-1/2} \left( |\Omega(\lambda+i0)|(1-\beta_\infty^2)^{-1/2} \sin(\phi_n - \lambda \psi_n + \arg \Omega(\lambda+i0)) + o(1) \right)
\]

(2.39)

where \( \Omega(z) \) is Wronskian (2.6). Let us comment on asymptotic coefficients in formula (2.39). For simplicity, we assume that \( a_n = \nu(n+1)^p \) where \( p \in (1/2, 1) \), \( \nu > 0 \), and neglect remainders. It follows from (2.37) that \( \psi_n = \nu n^s(1 + o(1)) \) with \( s = 1 - p \) and \( \nu = (2\nu s \sqrt{1-\beta_\infty^2})^{-1} \). The amplitude in (2.39) equals \( \kappa(\lambda)n^{-r} \) where \( \kappa(\lambda) = \nu^{-1/2}|\Omega(\lambda+i0)|(1-\beta_\infty^2)^{-1/2} \) and \( r = p/2 \). Note that \( 2r + s = 1 \) which is one of the universal relations observed in [76]. Another relation of [76] links the amplitude factor with the spectral weight \( \tau(\lambda) \):

\[
\pi \tau(\lambda)\kappa^2(\lambda) = 2\nu \nu.
\]

(2.40)

Substituting here expressions for \( \nu \) and \( \kappa(\lambda) \), we see that in our case relation (2.40) is equivalent to formula (2.35).

If \( \text{Im} \, z \neq 0 \), then the solution \( g_n(z) \) of the Jacobi equation (1.16) linear independent with \( f_n(z) \) can be constructed by formulas (2.15), (2.16). According to (2.32) it has the asymptotics

\[
g_n(z) = \frac{-i}{2\sqrt{1-\beta_\infty^2}} \frac{1}{\sqrt{a_n}} e^{\pm i\phi_n \frac{1}{2} i z \psi_n(1 + o(1))}, \quad \text{Im} \, z > 0, \quad n \to \infty,
\]

so that \( |g_n(z)| \) grows faster than any power of \( n \) as \( n \to \infty \). The asymptotics of the orthonormal polynomials is given by formula (2.34):

\[
P_n(z) = \frac{-i\Omega(z)}{2\sqrt{1-\beta_\infty^2}} \frac{1}{\sqrt{a_n}} e^{\pm i\phi_n \frac{1}{2} i z \psi_n(1 + o(1))}, \quad \text{Im} \, z > 0, \quad n \to \infty.
\]

(2.41)

Note that formulas (2.39) and (2.41) are consistent with the classical asymptotic relation (1.8) for the Hermite polynomials (see, e.g., Theorems 8.22.6 and 8.22.7 in the G. Szeg"o’s book [63]). Asymptotics (2.39) was obtained earlier in [33], but we are unaware of papers where (2.41) was deduced from assumptions on the Jacobi coefficients (not from properties of the corresponding spectral measure).

Under the Carleman condition (1.27) the Jacobi operator \( J = \text{clos} \, J_{\text{min}} \) is self-adjoint, and its resolvent \( (J - zI)^{-1} \) is given by the general formula (2.7). Since the Jost solutions \( f_n(z) \) depend continuously on \( z \) up to the real axis and \( \Omega(\lambda \pm i0) \neq 0 \) for \( \lambda \in \mathbb{R} \), the spectrum of the Jacobi operator \( J \) is absolutely continuous, covers the whole real line, and its spectral measure can be constructed by relation (2.35).

The main difference between the cases \( |\beta_\infty| > 1 \) and \( |\beta_\infty| < 1 \) is that according to (2.23), (2.24) \( |\zeta(\beta_\infty)| < 1 \) for \( |\beta_\infty| > 1 \) while \( |\zeta(\beta_\infty \pm i0)| = 1 \) for \( |\beta_\infty| < 1 \). Technically these cases are rather similar although many estimates are simpler for \( |\beta_\infty| > 1 \). However the asymptotic behavior of orthonormal polynomials and spectral properties of the Jacobi operators are quite different in these cases. If \( |\beta_\infty| > 1 \),
then for all $z \in \mathbb{C}$ the Jost solution $f_n(z)$ of equation (1.16) is distinguished by the asymptotics
\[ f_n(z) = a_n^{-1/2}(\text{sgn } \beta_\infty)^n e^{-\varphi_n - z\psi_n}(1 + o(1)), \quad n \to \infty, \tag{2.42} \]
where
\[ \varphi_n = \sum_{m=N_0}^{n-1} \arccosh |\beta_m|, \quad n \geq N_0. \tag{2.43} \]
and $\psi_n$ are, as before, defined by formula (2.37). Thus, $f_n(z) \to 0$ exponentially as $n \to \infty$ for all $z \in \mathbb{C}$. Now the functions $f_n(z)$ are analytic in the whole complex plane. As usual, the second solution $g_n(z)$ is constructed by equalities (2.15), (2.16). Its asymptotics as $n \to \infty$ can be deduced from formula (2.42). In view of relation (2.33) this yields an asymptotic formula for the orthonormal polynomials $P_n(z)$:
\[ P_n(z) = -\Omega(z) \frac{(\text{sgn } \beta_\infty)^{n+1} e^{\varphi_n + z\psi_n}}{2\sqrt{\beta_\infty^2 - 1}} \frac{1}{\sqrt{a_n}} (1 + o(1)), \quad n \to \infty, \quad \forall z \in \mathbb{C}. \]
The resolvent of the Jacobi operator $J$ is again determined by formula (2.7), but, in contrast to the case $|\beta_\infty| < 1$, its singularities are due to zeros of the denominator $\Omega(z)$ only. Therefore the spectrum of $J$ is discrete.

### 3. Volterra equations

In this section we begin our study of Jacobi operators with increasing recurrence coefficients in a systematic way. To be precise, we accept assumptions (1.26) and sometimes distinguish the cases $|\beta_\infty| > 1$ and $|\beta_\infty| < 1$. Below $z \in \mathbb{C}$ if $|\beta_\infty| > 1$ and $z \in \text{clos } \Pi$ where $\Pi = \mathbb{C} \setminus \mathbb{R}$ if $|\beta_\infty| < 1$. The Carleman condition (1.27) is not required unless specified explicitly. The function $\zeta(z)$ is defined by formula (2.18).

Here we reduce the Jacobi equation (1.16) for the Jost solutions $f_n(z)$ defined by their asymptotics for $n \to \infty$ to a Volterra equation. Then we solve this equation by iterations. The results of this section give an analytical basis for a construction of the Jost solutions and building of spectral theory of Jacobi operators in subsequent sections.

#### 3.1. Ansatz.

We follow the scheme described in Sect. 2.5. Recall that the numbers $\alpha_n$, $\beta_n$ and $z_n$ are defined by equalities (2.19) and (2.20) and $\zeta_n = \zeta(z_n)$. We also set
\[ \zeta_n = \sqrt{\frac{a_{n+1}}{a_n}}, \quad k_n = \frac{\zeta_{n-1}}{\zeta_n} = \frac{a_n}{\sqrt{a_{n-1}a_{n+1}}}. \tag{3.1} \]

Let us choose $\varepsilon_0 > 0$ so small that $\pm 1 \not\in (\beta_\infty - \varepsilon_0, \beta_\infty + \varepsilon_0)$. We always suppose that the values of the spectral parameter $z$ are bounded, that is, $|z| < \rho_0$ for some $\rho_0 < \infty$ and fix $N_0$ in such a way that $|z_n - \beta_\infty| < \varepsilon_0$ for $n \geq N_0$ and $|z| < \rho_0$. Then $z_n$ are separated from the singular points 1 and $-1$:
\[ |z_n \pm 1| \geq \varepsilon > 0. \tag{3.2} \]
Our first goal is to find an Ansatz $A_n(z)$ for Jost solutions in the case of recurrence coefficients satisfying condition (1.26). Let us seek it in the form

$$A_n(z) = p_n q_n(z)$$

with $q_n(z)$ defined as product (2.27) and a suitable sequence $p_n$. We have to calculate relative remainder (2.26) in equation (1.16). Since

$$A_{n-1} = \frac{p_{n-1} q_{n-1}}{p_n q_n} = \frac{1}{\zeta_n},$$

expression (2.26) equals

$$r_n(z) = s_n^{-1} \zeta_n - 2z_n + k_n s_n \zeta_n \\
= (s_n^{-1} \zeta_n - \zeta_n^{-1}) + (k_n s_n - 1) \zeta_n$$

where

$$s_n = \sqrt{\frac{a_{n+1} p_{n+1}}{a_n p_n}}.$$

Since $\zeta_n^{-1} - \zeta_n^{-1} \to 0$ and $k_n \to 1$ as $n \to \infty$, in order to estimate expression (3.4) we have to set $s_n = 1$. Then $p_n = a_n^{-1/2}$, formula (3.3) coincides with (2.25) and

$$r_n(z) = (\zeta_n^{-1} - \zeta_n^{-1}) + (k_n - 1) \zeta_n, \quad n \geq N_0 + 1.$$

Let us state this auxiliary result.

**Lemma 3.1.** Let the Ansatz $A_n(z)$ be defined by formula (2.25). Then remainder (2.26) admits representation (3.5) where $z_n$ and $k_n$ are given by (2.20) and (3.1), respectively.

### 3.2. Multiplicative substitution.

Let $A_n(z)$ be defined by formula (2.25). We are looking for solutions $f_n(z)$ of the Jacobi equation (1.16) satisfying condition (2.30). We first construct a sequence $f_n(z)$ for large $n$. Then $f_n(z)$ is extended to all $n$ as a solution of difference equation (1.16). By analogy with the continuous case, the sequence $f(z) = (f_n(z))_{n=1}^{\infty}$ will be called the Jost solution of the Jacobi equation (1.16).

The uniqueness of solutions with asymptotics (2.30) is almost obvious.

**Lemma 3.2.** Equation (1.16) may have only one solution $f_n(z)$ satisfying condition (2.30).

**Proof.** Let $\tilde{f}_n(z)$ be another solution of (1.16) satisfying (2.30). Then the Wronskian (2.1) of these solutions equals

$$\{f, \tilde{f}\} = a_n A_n A_{n+1} o(1) = \sqrt{\frac{a_n}{a_{n+1}}} q_n q_{n+1} o(1), \quad n \to \infty,$$

where $q_n = q_n(z)$ are numbers (2.27). Clearly, $|q_n| \leq 1$. Since $a_n \to \infty$, the ratios $a_n/a_{n+1}$ are also bounded, at least for some subsequence of $n \to \infty$. Thus, $\{f, \tilde{f}\} = 0$ whence $\tilde{f} = C f$. Here $C = 1$ by virtue again of condition (2.30). □
Note a symmetry relation
\[ f_n(\bar{z}) = \overline{f_n(z)} \] (3.6)
which is a consequence of the equality \( A_n(\bar{z}) = \overline{A_n(z)} \) and Lemma 3.2.

For a construction of \( f_n(z) \), we make a multiplicative substitution introducing a sequence
\[ u_n(z) = A_n(z)^{-1} f_n(z), \quad n \in \mathbb{Z}_+ . \] (3.7)

Then (2.30) is equivalent to condition (2.29).

We first derive a difference equation for \( u_n(z) \) which will be subsequently reduced to a Volterra “integral” equation.

Lemma 3.3. Let \( A_n(z) \) and \( r_n(z) \) be given by formulas (2.25) and (3.5), respectively. Then equation (1.16) for a sequence \( f_n(z) \) is equivalent to the equation
\[ k_n \zeta_n (u_{n+1}(z) - u_n(z)) - \zeta_{n-1}^{-1}(u_n(z) - u_{n-1}(z)) = -r_n(z)u_n(z) \] (3.8)
for sequence (3.7).

Proof. Substituting expression \( f_n = A_n u_n \) into (1.16) and using definition (2.26) of \( r_n \), we see that
\[
\left( \sqrt{a_{n-1}a_n} A_n \right)^{-1} \left( a_{n-1}f_{n-1} + (b_n - z)f_n + a_n f_{n+1} \right)
\] \[ = r_n u_n + \left( \sqrt{a_{n-1}a_n} A_n \right)^{-1} \left( a_{n-1} A_{n-1}(u_{n-1} - u_n) + a_n A_{n+1}(u_{n+1} - u_n) \right) . \] (3.9)

In view of the equality
\[ \frac{A_{n+1}}{A_n} = \sqrt{\frac{a_n}{a_{n+1}} \zeta_n} , \]
the right-hand side here can be written as
\[ r_n u_n + \zeta_{n-1}^{-1}(u_{n-1} - u_n) + k_n \zeta_n (u_{n+1} - u_n) . \]
Therefore equality (3.9) implies that equations (1.16) and (3.8) are equivalent. Finally, we note that representation (3.5) for the coefficient \( r_n \) follows from Lemma 3.1. □

3.3. Volterra integral equation. Here we reduce difference equation (3.8) to a Volterra equation
\[ u_n(z) = 1 + \sum_{m=n+1}^{\infty} G_{n,m}(z) r_m(z) u_m(z) , \] (3.10)
where the kernels \( G_{n,m}(z) \) are defined as follows. We set
\[ \sigma_n = \zeta_n \zeta_{n-1} , \quad S_n = \sigma_1 \sigma_2 \cdots \sigma_{n-1} , \] (3.11)
and
\[ G_{n,m}(z) = -\zeta_{m-1}^{-1} \zeta_m S_{m+1} \sum_{p=n+1}^{m} \zeta_{p-1}^{-1} S_p^{-1} , \quad m > n , \] (3.12)
where \( \zeta_n = \zeta(z_n) \) and the numbers \( z_n, \kappa_n \) are given by (2.20), (3.1), respectively.

In all estimates below, values of the spectral parameter \( z \) are bounded and \( n \) are sufficiently large. Our estimates of the remainder \( r_n(z) \) are practically the same in the case \(|\beta_\infty| < 1\) and \(|\beta_\infty| > 1\). First we set

\[
\varrho_n = 1 + \frac{z_{n-1} + z_n}{\sqrt{z_{n-1}^2 - 1} + \sqrt{z_n^2 - 1}} \tag{3.13}
\]

and note a representation

\[
\zeta_n^{-1} - \zeta_n^{-1} = (z_{n-1} - z_n) \varrho_n \tag{3.14}
\]

which is a consequence of definitions (2.18) and (2.20). It follows from relation (2.21) where \( \beta_\infty^2 \neq 1 \) that estimate (3.2) is satisfied. Thus (3.13), (3.14) yield an estimate

\[
|\zeta_n^{-1} - \zeta_n^{-1}| \leq C|z_{n-1} - z_n| \leq C_1(|\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n|). \tag{3.15}
\]

In view of representation (3.5), this leads to the following assertion where notation (2.2) is used.

**Lemma 3.4.** Under assumption (1.26) remainder (2.26) satisfies an estimate

\[
|r_n(z)| \leq C(|\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n|) + |k_n - 1|. \tag{3.16}
\]

In particular, if

\[
(\alpha'_n) \in \ell^1(\mathbb{Z}_+), \quad (\beta'_n) \in \ell^1(\mathbb{Z}_+) \tag{3.16}
\]

and

\[
(k_n - 1) \in \ell^1(\mathbb{Z}_+), \tag{3.17}
\]

then

\[
(r_n(z)) \in \ell^1(\mathbb{Z}_+). \tag{3.18}
\]

**Remark 3.5.** Let the numbers \( \kappa_n \) be defined by the first equality (3.1). If inclusion (3.17) is true, then there exists a finite limit

\[
\lim_{n \to \infty} \kappa_n = \kappa_\infty \quad \text{where} \quad \kappa_\infty \geq 1 \tag{3.19}
\]

and

\[
(\kappa'_n) \in \ell^1(\mathbb{Z}_+). \tag{3.20}
\]

Moreover, \( \kappa_\infty = 1 \) if the Carleman condition (1.27) is satisfied.

Indeed, by definition (3.1), we have

\[
\ln \kappa_n - \ln \kappa_0 = - \sum_{m=1}^{n} \ln k_m
\]

where the series on the right converges according to (3.17). It follows that the limit in (3.19) exists. Since

\[
\kappa_n - \kappa_{n-1} = \frac{\kappa_n^2}{\kappa_n + \kappa_{n-1}} (1 + k_n)(1 - k_n),
\]
inclusion (3.20) is a direct consequence of assumption (3.17). If \( \nu_\infty < 1 \), then \( a_n \to 0 \) as \( n \to \infty \) which is excluded by the first assumption (1.26). If \( \nu_\infty > 1 \), then \( a_n \geq c l^n \) for some \( l > 1 \) and \( c > 0 \) so that condition (1.27) is violated.

**Example 3.6.** Condition (3.17) is of course satisfied if \( a_n = \nu (n + 1)^p \) for some \( \nu > 0 \) and \( p > 0 \), but it allows much more rapid growth of \( a_n \) as \( n \to \infty \). Consider, for example, \( a_n = \nu x^n \) where \( x > 1 \). If \( q < 1 \), then

\[
k^2_n = \exp \left( \nu (2n^q - (n + 1)^q - (n - 1)^q) \right) = \exp \left( O(n^{q-2}) \right) = 1 + O(n^{q-2})
\]

so that condition (3.17) holds. In this case \( \nu_\infty = 1 \). If \( q = 1 \), then \( \nu_n = \sqrt{x} \) and \( k_n = 1 \) for all \( n \). On the contrary, condition (3.17) fails and \( \nu_n \to \infty \) as \( n \to \infty \) if \( a_n = \nu x^n \) with \( q > 1 \).

### 3.4. Solution by iterations.

First, we estimate “kernels” (3.12). As before, we use that according to (2.23) or (2.24) the numbers \( \zeta(z_n) \) are separated from the points 1 and \(-1\) for sufficiently large \( n \), say \( n \geq N_0 \). The estimates below are quite straightforward in the case \( |\beta_\infty| > 1 \).

**Lemma 3.7.** Let assumptions (1.26) where \( |\beta_\infty| > 1 \) and

\[
0 < c_1 \leq \nu_n \leq c_2 < \infty
\]

be satisfied. Then kernel (3.12) is bounded uniformly in \( m > n \geq N_0 \):

\[
|G_{n,m}(z)| \leq C < \infty.
\]

**Proof.** Let \( \nu_n = \zeta_n \zeta_{n-1} \). By virtue of (2.23), we have

\[
|\nu_n| \leq \sigma_\infty \quad \text{where} \quad \sigma_\infty < 1 \quad \text{for} \quad n \geq N_0.
\]

Using also (3.21) we see that

\[
|G_{n,m}| \leq C \sum_{p=n+1}^m |\sigma_p \cdots \sigma_m| \leq C_1 \sum_{p=n+1}^m \sigma_{-p+1}^{m-p+1} = C_1 \frac{1 - \sigma_{-m-n}}{\sigma_\infty - 1} \leq C_1 \frac{\sigma_\infty}{1 - \sigma_\infty}
\]

which implies (3.22). \( \square \)

In the case \( |\beta_\infty| < 1 \) we have to “integrate by parts” in (3.12). This requires some additional assumptions.

**Lemma 3.8.** Let assumptions (1.26) where \( |\beta_\infty| < 1 \), (3.10), (3.20) and (3.21) be satisfied. Then kernel (3.12) is bounded uniformly in \( m > n \geq N_0 \), that is, estimate (3.22) holds.

**Proof.** Since \( |\zeta_p| \leq 1 \), it follows from definition (3.11) that

\[
|S_{n+1} S_p^{-1}| = |\sigma_p \cdots \sigma_m| \leq 1, \quad p \leq m.
\]

Since \( \sigma_n \to (\zeta_\infty^{(\pm)})^2 \neq 1 \) as \( n \to \infty \), we see that

\[
|\sigma_n - 1| \geq \varepsilon > 0.
\]
By definitions (2.2) and (3.11) we have
\[(S_p^{-1})' = S_{p+1}^{-1} - S_p^{-1} = (\sigma_p^{-1} - 1)S_p^{-1},\]
and hence integrating by parts (that is, using formula (2.4)), we find that
\[\sum_{p=n+1}^{m} \zeta_{p-1}S_p^{-1} = \sum_{p=n+1}^{m} \zeta_{p-1}(\sigma_p^{-1} - 1)^{-1}(S_p^{-1})' = \zeta_{m-1}(\sigma_m^{-1} - 1)^{-1}S_m^{-1},\]
\[- \zeta_{n-1}(\sigma_n^{-1} - 1)^{-1}S_{n+1}^{-1} = \sum_{p=n+1}^{m} (\zeta_{p-2}(\sigma_{p-2}^{-1} - 1)^{-1})'S_p^{-1}. \quad (3.25)\]

Note that
\[((\sigma_p^{-1} - 1)^{-1})' = (\sigma_p - 1)^{-1}(\sigma_p - 1)^{-1}\sigma_p' \quad (3.26)\]
where \(\sigma_p' \in \ell^1(\mathbb{Z}_+)\) by virtue of (3.15). Using also (3.24), we see that sequence (3.26) belongs to \(\ell^1(\mathbb{Z}_+)\). Finally, taking (3.20) into account we find that
\[(\zeta_{p-1}(\sigma_p^{-1} - 1)^{-1})' \in \ell^1(\mathbb{Z}_+).\]
Let us multiply identity (3.25) by \(S_{m+1}\). According to (3.23) and (3.24) all three terms in the right-hand side of the equality obtained are bounded for \(n \geq N_0\). \(\square\)

Now we are in a position to standardly solve the Volterra equation (3.10) by iterations. First, we estimate these iterations.

**Lemma 3.9.** Set \(u_n^{(0)} = 1\) and
\[u_n^{(k+1)}(z) = \sum_{m=n+1}^{\infty} G_{n,m}(z)r_m(z)u_m^{(k)}(z), \quad k \geq 0, \quad (3.27)\]
for all \(n \in \mathbb{Z}_+\). Then the estimates
\[|u_n^{(k)}(z)| \leq C^k k! \left( \sum_{m=n+1}^{\infty} |r_m(z)| \right)^k, \quad \forall k \in \mathbb{Z}_+. \quad (3.28)\]
are true for all sufficiently large \(n\) with the same constant \(C\) as in (3.22).

**Proof.** Suppose that (3.28) is satisfied for some \(k \in \mathbb{Z}_+\). We have to check the same estimate (with \(k\) replaced by \(k+1\) in the right-hand side) for \(u_n^{(k+1)}\). Set
\[R_m = \sum_{p=m+1}^{\infty} |r_p|, \quad (3.29)\]
According to definition (3.27), it follows from estimates (3.22) and (3.28) that
\[|u_n^{(k+1)}| \leq C \sum_{m=n+1}^{\infty} |r_m||u_m^{(k)}| \leq \frac{C^{k+1}}{k!} \sum_{m=n+1}^{\infty} |r_m|R_m^k. \quad (3.29)\]
Observe that
\[ R_{m+1}^{k+1} + (k+1)|r_m|R_m^k \leq R_{m-1}^{k+1}, \]
and hence, for all \( N \in \mathbb{Z}_+ \),
\[ (k+1) \sum_{m=n+1}^{N} |r_m|R_m^k \leq \sum_{m=n+1}^{N} (R_{m-1}^{k+1} - R_m^{k+1}) = R_n^{k+1} - R_{N+1}^{k+1} \leq R_n^{k+1}. \]
Substituting this bound into (3.29), we obtain estimate (3.28) for \( u_n^{(k+1)} \).

This lemma enables us to construct a solution \( u_n(z) \) of equation (3.10) as a convergent series.

**Theorem 3.10.** Let the assumptions (1.26) as well as (3.16), (3.17) be satisfied. Suppose that \( |z| < \rho_0 \) for some \( \rho_0 < \infty \) and \( n \geq N_0(\rho_0) \) for sufficiently large \( N_0(\rho_0) \). Then equation (3.10) has a unique bounded solution \( u_n(z) \), and condition (2.29) is satisfied. If \( |\beta_\infty| > 1 \), the functions \( u_n(z) \) are analytic in \( z \) for \( |z| < \rho_0 \) and \( n \geq N_0(\rho_0) \). If \( |\beta_\infty| < 1 \), the same is true in the complex plane cut along \( \mathbb{R} \); in this case the functions \( u_n(z) \) are continuous up to the cut.

**Proof.** Set
\[ u_n = \sum_{k=0}^{\infty} u_n^{(k)} \]
where \( u_n^{(k)} = 1 \) and \( u_n^{(k)} \), \( k \geq 1 \), are defined by recurrence relations (3.27). Estimate (3.28) shows that this series is absolutely convergent. Using the Fubini theorem to interchange the order of summations in \( m \) and \( k \), we see that
\[ \sum_{m=n+1}^{\infty} G_{n,m}r_m u_m = \sum_{k=0}^{\infty} \sum_{m=n+1}^{\infty} G_{n,m} r_m u_n^{(k)} = \sum_{k=0}^{\infty} u_{n+1}^{(k)} = -1 + \sum_{k=0}^{\infty} u_n^{(k)} = -1 + u_n. \]
This is equation (3.10) for sequence (3.30). Condition (2.29) for this sequence also follows from (3.28) because \( r_n(z) \) satisfies (3.18).

4. JOST SOLUTIONS

We here use the results of the preceding section to construct the Jost solutions for the Jacobi equation (1.16).

4.1. An auxiliary difference equation. The first step is to transform “integral” equation (3.10) to difference equation (3.8).

**Lemma 4.1.** Let \( r_n(z) \) and \( G_{n,m}(z) \) be given by formulas (3.5) and (3.12), respectively. Then the solution \( u_n(z) \) of integral equation (3.10) satisfies an identity
\[ \kappa_n u_n + \sum_{m=n+1}^{\infty} G_{n,m} S_{m+1}^{-1} \kappa_{m-1} S_{m-1}^{-1} r_m u_m, \]
\[ \kappa_n^{-1}(u_{n+1} - u_n) = \sum_{m=n+1}^{\infty} S_{n+1}^{-1} S_{n}^{-1} \kappa_{m-1} S_{m-1} r_m u_m, \]
and difference equation (3.8).

Proof. It follows from (3.10) that

\[ u_{n+1} - u_n = \sum_{m=n+2}^{\infty} \left( G_{n+1,m} - G_{n,m} \right) r_m u_m - G_{n,n+1} r_{n+1} u_{n+1}. \] (4.2)

Since according to (3.12)

\[ G_{n+1,m} - G_{n,m} = \kappa_n \kappa_{m-1} S_{n+1}^{-1} S_{m+1}^{-1} \]

and difference equation (3.8) and equality (4.2) can be rewritten as (4.1).

Putting together equality (4.2) with the same equality for \( n \) replaced by \( n - 1 \), we see that

\[ \kappa_n^{-1} (u_{n+1} - u_n) - \sigma_n^{-1} \kappa_n^{-1} (u_n - u_{n-1}) \]

\[ = \sum_{m=n+1}^{\infty} S_{n+1}^{-1} S_{m+1} \kappa_m^{-1} r_m u_m - \sigma_n^{-1} \sum_{m=n}^{\infty} S_{n+1}^{-1} S_{m+1} \kappa_m^{-1} r_m u_m. \]

Since \( S_{n+1} = \sigma_n S_n \), the right-hand side here equals \(-\kappa_n^{-1} \kappa_n^{-1} r_n u_n\), and hence the equation obtained coincides with (3.8). \( \square \)

4.2. Main result. Now we are in a position to construct solutions of the Jacobi equation (1.16) with asymptotics (2.30) as \( n \to \infty \). We call them the Jost solutions. Recall that, by Lemma 3.2, equation (1.16) may have only one solution with such asymptotics. We fix some \( \rho_0 > 0 \), suppose that \(|z| < \rho_0\) and choose \( N_0 = N_0(\rho_0)\) such that condition (3.2) is satisfied for \(|z| \leq \rho_0\) and \( n \geq N_0 \). Then using Theorem 3.10 we construct \( u_n(z) \) for \( n \geq N_0 \) as solutions of the integral equation (3.10). According to Lemma 4.1 this sequence satisfies also the difference equation (3.8). Now we define \( f_n(z) \) for \( n \geq N_0 \) by formula (2.28). In view of Lemma 3.3 this sequence satisfies the Jacobi equation (1.16). Then we extend \( f_n(z) \) as a solution of equation (1.16) to all \( n \in \mathbb{R}_+ \). Thus we have the following result.

Theorem 4.2. Suppose that assumptions (1.26) as well as (3.16) and (3.17) are satisfied. Let \( u_n(z) \) be the sequence constructed in Theorem 3.10 and let \( q_n(z) \) be product (2.27). Suppose that \( z \in \mathbb{C} \) if \(|\beta_\infty| > 1\) and \( z \in \text{clos } \Pi \) where \( \Pi = \mathbb{C} \setminus \mathbb{R} \) if \(|\beta_\infty| < 1\). Then the sequence \( f_n(z) \) defined by the equality

\[ f_n(z) = a_n^{-1/2} q_n(z) u_n(z) \]

satisfies equation (1.16), and it has asymptotics

\[ f_n(z) = a_n^{-1/2} q_n(z) (1 + o(1)), \quad n \to \infty. \] (4.3)

If \(|\beta_\infty| > 1\), then all functions \( f_n(z) \) are analytic in \( z \in \mathbb{C} \). If \(|\beta_\infty| < 1\), the same is true for \( z \in \Pi \); in this case the functions \( f_n(z) \) are continuous up to the cut along \( \mathbb{R} \).
We emphasize that the Carleman condition (1.27) is not required in Theorem 4.2. Recall that the sequence \( f_n(z) \) obeys condition (3.6) and, in particular,
\[
f_n(\lambda - i0) = f_n(\lambda + i0), \quad \lambda \in \mathbb{R}.
\] (4.4)

We now define the Jost function \( \Omega(z) \) by formula (2.6) where \( f(z) \) is the Jost solution. The following result is a direct consequence of Theorem 4.2.

**Corollary 4.3.** If \(|\beta_\infty| > 1\), then \( \Omega(z) \) is an analytic function of \( z \in \mathbb{C} \). If \(|\beta_\infty| < 1\), then \( \Omega(z) \) is analytic in \( z \in \Pi \) and is continuous up to the cut along \( \mathbb{R} \). Its limit values on the cut satisfy the identity
\[
\Omega(\lambda - i0) = \overline{\Omega(\lambda + i0)}, \quad \lambda \in \mathbb{R}.
\]

**4.3. Explicit formulas.** We here transform asymptotics (4.3) of the Jost solutions \( f_n(z) \) to a simpler form. This will require an assumption that \( \alpha_n \to \infty \) not too slowly, i.e.
\[
\sum_{n=0}^{\infty} \alpha_n^{K+1} < \infty
\] (4.5)
for some \( K \in \mathbb{Z}_+ \).

We first discuss a behavior of the numbers
\[
\zeta_n = \zeta(z_n) = z_n - \sqrt{z_n^2 - 1}
\] (4.6)
as \( n \to \infty \). Recall that \( z_n \) are defined by equalities (2.19). Differentiating relation (2.18), we see that
\[
-(\ln \zeta(z))' = (z^2 - 1)^{-1/2} =: \vartheta(z).
\] (4.7)
Here the function \( \vartheta(z) \) is defined on \( \mathbb{C} \setminus [-1,1] \) and \( \vartheta(z) > 0 \) for \( z > 1 \). Then \( \vartheta(z) < 0 \) for \( z < -1 \) and \( \vartheta(\lambda \pm i0) = \mp i(1 - \lambda^2)^{-1/2} \) if \( \lambda \in (-1,1) \). We fix the branch of the function \( \ln \zeta(z) \) by the condition \( \ln \zeta(z) < 0 \) for \( z > 1 \). The functions \( \vartheta(z) \) and \( \ln \zeta(z) \) are holomorphic on the set \( \mathbb{C} \setminus [-1,1] \).

We start with an elementary assertion.

**Lemma 4.4.** Let condition (1.26) be satisfied, and let \( n \) be sufficiently large. If \(|\beta_\infty| < 1\) and \( \pm \text{Im} \, z \geq 0 \), then
\[
\zeta(\alpha_n z + \beta_n) = \zeta(\beta_n \pm i0)e^{\psi(\alpha_n z, \beta_n \pm i0)}
\] (4.8)
where, for an arbitrary \( K \geq 1 \),
\[
\psi(\alpha_n z, \beta_n \pm i0) = -\sum_{k=1}^{K} \frac{k!}{(k-1)!} \vartheta^{(k-1)}(\beta_n \pm i0)(\alpha_n z)^k + O(\alpha_n^{K+1})
\] (4.9)
as \( n \to \infty \). If \(|\beta_\infty| > 1\), then relations (4.8) and (4.9) are true for all \( z \in \mathbb{C} \) with \( \beta_n \pm i0 \) replaced by \( \beta_n \).
**Proof.** Let us set
\[
\psi(t, \beta) = \ln \zeta(t + \beta) - \ln \zeta(\beta) \tag{4.10}
\]
where either \(\beta \in (-1, 1)\) and \(\pm \text{Im} t \geq 0\) or \(\beta \in (-\infty, -1) \cup (1, \infty)\) and \(t \in \mathbb{C}\). Then \(\psi(0, \beta) = 0\) and
\[
\zeta(t + \beta) = \zeta(\beta) e^{\psi(t, \beta)}. \tag{4.11}
\]
Differentiating (4.10) in \(t\) and using definition (4.7), we see that
\[
\psi_t(t, \beta) = -\vartheta(t + \beta). \tag{4.12}
\]
Let us take \(K - 1\) terms of the Taylor expansion of the function \(\vartheta(\beta + t)\) at the point \(t = 0\) and then integrate (4.12) over \(t\) using that \(\psi(0, \beta) = 0\). This yields the relation
\[
\psi(t, \beta) = -\sum_{k=1}^{K} \frac{(k!)^{-1}}{(k-1)!} \vartheta^{(k-1)}(\beta \pm i0)t^k + O(|t|^{K+1}) \quad \text{as} \quad t \to 0. \tag{4.13}
\]
Moreover, in the case \(|\beta| > 1\), the numbers \(\beta \pm i0\) can be here replaced by \(\beta\). To obtain relations (4.8) and (4.9), we only have to apply (4.11) and (4.13) to \(t = \alpha_n z\) and \(\beta = \beta_n\).

**Corollary 4.5.** If \(|\beta_n| < 1\), then
\[
|\zeta(\alpha_n z + \beta_n)| = e^{-|\text{Im} z|\alpha_n(1-\beta_n^2)^{-1/2}+O(\alpha_n^2)}. \tag{4.14}
\]

Now we are in a position to find an asymptotics as \(n \to \infty\) of the product \(q_n(z)\) defined by formula (2.27). Let us set
\[
\Theta_n^{(\pm)} = \prod_{m=N_0}^{n-1} \zeta(\beta_m \pm i0), \quad n \geq N_0 + 1, \tag{4.15}
\]
and, for \(K \geq 1\),
\[
L_n^{(\pm)}(z; K) = -\sum_{k=1}^{K} \frac{(k!)^{-1}}{k!} \sum_{m=N_0}^{n-1} \vartheta^{(k-1)}(\beta_m \pm i0)\alpha_m^k \quad \text{for} \quad \pm \text{Im} z \geq 0. \tag{4.16}
\]

If \(|\beta_\infty| > 1\), then all objects labelled by “+” and “−” coincide with each other so that \(\beta_n \pm i0\) can be replaced by \(\beta_n\). In this case (4.15) is true for all \(z \in \mathbb{C}\). We often write \(L_n^{(\pm)}(z)\) instead of \(L_n^{(\pm)}(z; K)\) omitting the dependence on \(K\).

The following statement is a direct consequence of Lemma 4.4.

**Lemma 4.6.** Let assumption (4.5) be satisfied for some \(K \in \mathbb{Z}_+\). Define the sequences \(\Theta_n^{(\pm)}\) and \(L_n^{(\pm)}(z; K)\) by formulas (4.14) and (4.15). Then the limits
\[
\lim_{n \to \infty} q_n(z)/\Theta_n^{(\pm)}, \quad \pm \text{Im} z \geq 0, \quad K = 0, \tag{4.16}
\]
and
\[
\lim_{n \to \infty} \frac{q_n(z)}{\Theta_n^{(\pm)} \exp(L_n^{(\pm)}(z;K))} = \frac{\pm \text{Im} z \geq 0, \ K \geq 1}{}, \tag{4.17}
\]
exist and are not zero. Moreover, if \(|\beta_\infty| > 1\), then (4.16) and (4.17) hold for all \(z \in \mathbb{C}\) and the indices “±” may be omitted.

Using this result we can renormalize definition (2.28) of the Jost solutions \(f_n(z)\) multiplying them by a constant non-zero factor that may depend on \(z\) but not on \(n\). This allows us to state Theorem 4.2 in a more explicit form.

**Theorem 4.7.** In addition to the assumptions of Theorem 4.2 suppose that condition \((4.5)\) is satisfied for some \(K \in \mathbb{Z}_+\). Let the functions \(\Theta_n^{(\pm)}\) be defined by formula \((4.14)\). The functions \(L_n^{(\pm)}(z)\) are defined by formula \((4.15)\) for \(K \geq 1\) and \(L_n^{(\pm)}(z) = 0\) for \(K = 0\). Then for all \(z \in \text{clos } \Pi\), equation \((1.16)\) has a unique solution \(f_n(z)\) with asymptotics
\[
f_n(z) = a_n^{-1/2} \Theta_n^{(\pm)} \exp(L_n^{(\pm)}(z))(1 + o(1)), \quad \pm \text{Im} z \geq 0, \ n \to \infty. \tag{4.18}
\]
Moreover, if \(|\beta_\infty| > 1\), then (4.18) is true for all \(z \in \mathbb{C}\) and the indices “±” may be omitted.

**Remark 4.8.** Under the assumptions of Theorem 4.7 we can simplify definition \((2.25)\) of the Ansatz setting
\[
A_n^{(\pm)}(z) = a_n^{-1/2} \Theta_n^{(\pm)} \exp(L_n^{(\pm)}(z)).
\]
Then expression \((3.5)\) for remainder \((2.26)\) is changed, but otherwise the proof of Theorem 4.7 works with this Ansatz without significant modifications.

**Remark 4.9.** In view of Remark 3.5, under assumption \((3.17)\) condition \((4.5)\) for \(K = 0\) is satisfied if and only if the Carleman condition \((1.27)\) is violated.

According to Theorem 4.7, the asymptotic formulas are simplest in the most singular case when the Carleman condition \((1.27)\) is violated. Indeed, if assumption \((4.5)\) is satisfied for \(K = 0\), then \(L_n^{(\pm)}(z) = 0\) and the asymptotics of \(f_n(z)\) is determined by \(\Theta_n^{(\pm)}\) only. Let us give explicit formulas for this term. In view of definitions \((2.18)\) and \((4.14)\) we have
\[
\Theta_n = (\text{sgn } \beta_\infty)^n \prod_{m=N_0}^{n-1} (|\beta_m| + |\sqrt{\beta_m^2 - 1}|)^{-1}
\]
\[= (\text{sgn } \beta_\infty)^n \exp(-\sum_{m=N_0}^{n-1} \arccosh |\beta_m|), \quad |\beta_\infty| > 1. \tag{4.19}
\]
and
\[ \Theta_n^{(\pm)} = \prod_{m=N_0}^{n-1} (\beta_m \mp i|\sqrt{1-\beta_m^2}|) = \exp(\mp i \sum_{m=N_0}^{n-1} \arccos \beta_m), \quad |\beta_\infty| < 1. \quad (4.20) \]

In particular, for \( b_n = 0 \), (4.20) reduces to \( \Theta_n^{(\pm)} = i^{\pm N_0} e^{\mp n} \). As usual, the constant factor \( i^{\pm N_0} \) can be neglected here.

In the Carleman case formula (4.18) contains the correction \( \exp(L_n^{(\pm)}(z)) \) where \( L_n^{(\pm)}(z) \) is defined by (4.15). This correction gets more complicated as \( K \) increases.

Let us first consider the leading term of \( L_n^{(\pm)}(z) \). This yields a simplified formulation of Theorem 4.7 where an estimate of the remainder is not very precise.

Theorem 4.10. In addition to the assumptions of Theorem 4.2 suppose that the Carleman condition (1.27) is satisfied. Then formula (4.18) holds true with \( \Theta_n^{(\pm)} \) given by (4.19), (4.20) and
\[ L_n(z) = -z \, \text{sgn} \, \beta_\infty \sum_{m=N_0}^{n-1} (\beta_m^2 - 1)^{-1/2} \alpha_m + o(\sum_{m=0}^{n-1} \alpha_m), \quad |\beta_\infty| > 1, \quad (4.21) \]
\[ L_n^{(\pm)}(z) = \pm iz \sum_{m=N_0}^{n-1} (1 - \beta_m^2)^{-1/2} \alpha_m + o(\sum_{m=0}^{n-1} \alpha_m), \quad |\beta_\infty| < 1, \quad \pm \text{Im} \, z \geq 0. \quad (4.22) \]

Note that if \( K = 1 \), then the error terms in (4.21) and (4.22) can be omitted. In this case Theorem 4.10 yields formulas of Sect. 2.6.

Let us write down explicitly two terms of expression (4.15) for \( K = 2 \). If \( |\beta_\infty| > 1 \), then
\[ L_n(z) = -z \, \text{sgn} \, \beta_\infty \sum_{m=N_0}^{n-1} (\beta_m^2 - 1)^{-1/2} \alpha_m + 2^{-1} z^2 \, \text{sgn} \, \beta_\infty \sum_{m=N_0}^{n-1} (\beta_m^2 - 1)^{-3/2} \beta_m \alpha_m^2. \]

If \( |\beta_\infty| < 1 \) and \( \pm \text{Im} \, z \geq 0 \), we have
\[ L_n^{(\pm)}(z) = \pm iz \sum_{m=N_0}^{n-1} (1 - \beta_m^2)^{-1/2} \alpha_m \pm i2^{-1} z^2 \sum_{m=N_0}^{n-1} (1 - \beta_m^2)^{-3/2} \beta_m \alpha_m^2. \]

Moreover, if
\[ \sum_{m=0}^{\infty} |\beta_m| \alpha_m^2 < \infty \]
(in particular, if (4.5) is true with \( K = 1 \)), then the terms with \( z^2 \) in the right-hand sides of (4.23) and (4.24) can be also omitted. Thus we again recover formulas of Sect. 2.6.

In Sect. 5.2, we need the following
Corollary 4.11. If $|\beta_\infty| < 1$, then

$$|f_n(z)|^2 \leq \alpha_n \exp \left( -c \sum_{m=0}^{n-1} \alpha_m \right)$$

(4.25)

for all $z \in \mathbb{C}$ with $\text{Im} \ z \neq 0$ and some $c = c(z) > 0$.

Proof. Observe that $|\Theta_n^{(\pm)}| = 1$ and according to Corollary 4.5

$$\text{Re} L_n^{(\pm)}(z) = -|\text{Im} \ z| \sum_{m=N_0}^{n-1} \frac{\alpha_m}{\sqrt{1 - \beta_m^2}} + o\left(\sum_{m=0}^{n-1} \alpha_m\right).$$

Therefore (4.25) is a direct consequence of (4.18). □

5. Small diagonal elements: orthogonal polynomials on the real axis and the spectral measure

In this section we accept assumption (1.26) where $|\beta_\infty| < 1$ and the Carleman condition (1.27). Our goal is to find an asymptotic behavior of the orthonormal polynomials $P_n(z)$ as $n \to \infty$ for $z = \lambda \in \mathbb{R}$. At the same time we will show that the spectrum of the Jacobi operator $J$ is absolutely continuous, covers the whole real line and obtain an expression for the spectral measure of $J$.

5.1. Asymptotics on the continuous spectrum. Let us proceed from Theorem 4.2 where $z = \lambda \pm i0$ for $\lambda \in \mathbb{R}$. Recall that $q_n(\lambda \pm i0)$ is product (2.27) and

$$\zeta_n(\lambda \pm i0) = \lambda_n \mp i\sqrt{1 - \lambda_n^2} = e^{\mp i \arccos \lambda_n}$$

(5.1)

where

$$\lambda_n = \alpha_n \lambda + \beta_n$$

with the numbers $\alpha_n, \beta_n$ defined by formulas (2.19). According to (2.21) we have

$$\lim_{n \to \infty} \lambda_n = \beta_\infty$$

(5.2)

so that $\lambda_n \in (-1, 1)$ if $n$ is sufficiently large. It follows from (5.1) that

$$q_n(\lambda \pm i0) = e^{\mp i \varphi_n(\lambda)} \text{ where } \varphi_n(\lambda) = \sum_{m=N_0}^{n-1} \arccos \lambda_m.$$  

(5.3)

Let us now state a particular case of Theorem 4.2 for the case $z = \lambda \pm i0$ where $\lambda \in \mathbb{R}$.

Theorem 5.1. Let the assumptions of Theorem 4.2 be satisfied and $|\beta_\infty| < 1$. Then for all $\lambda \in \mathbb{R}$, the equation

$$a_{n-1} f_{n-1} + b_n f_n + a_n f_{n+1} = \lambda f_n$$

...
The following result shows that these solutions are linearly independent. Recall that the Wronskian of \( f(\lambda + i0) \) and \( f(\lambda - i0) \) is defined by the relation

\[
\{ f(\lambda + i0), f(\lambda - i0) \} = a_n (f_n(\lambda + i0) f_{n+1}(\lambda - i0) - f_n(\lambda - i0) f_{n+1}(\lambda + i0)). \tag{5.5}
\]

**Lemma 5.2.** The Wronskian of \( f(\lambda + i0) \) and \( f(\lambda - i0) \) equals

\[
\{ f(\lambda + i0), f(\lambda - i0) \} = 2i \sqrt{1 - \beta_\infty^2} \neq 0, \tag{5.6}
\]

and hence these solutions are linearly independent.

**Proof.** The right-hand side of (5.5) does not depend on \( n \), and so we can calculate this expression for \( n \to \infty \). Therefore using (5.4), we find that

\[
\{ f(\lambda + i0), f(\lambda - i0) \} = \sqrt{\frac{a_n}{a_{n+1}}} (q_n(\lambda + i0) q_{n+1}(\lambda - i0) - q_n(\lambda - i0) q_{n+1}(\lambda + i0) + o(1))
\]

whence

\[
\{ f(\lambda + i0), f(\lambda - i0) \} = \sqrt{\frac{a_n}{a_{n+1}}} (\zeta(\lambda_n - i0) - \zeta(\lambda_n + i0) + o(1))
\]

as \( n \to \infty \). Observe that \( a_n a_{n+1}^{-1} \to 1 \) under the assumptions of Theorem 4.2. Thus, relation (5.6) is a direct consequence of (5.1) and (5.2).

Now we are in a position to find an asymptotic behavior of the polynomials \( P_n(\lambda) \) for \( \lambda \in \mathbb{R} \), that is, on the continuous spectrum of the Jacobi operator \( J \). Since the Jost solutions \( f(\lambda \pm i0) = (f_n(\lambda \pm i0)) \) are linearly independent and \( P_n(\lambda) = P_n(\lambda) \), we see that

\[
P_n(\lambda) = c(\lambda) f_n(\lambda + i0) + c(\lambda) f_n(\lambda - i0)
\]

for some complex constant \( c(\lambda) \). Taking the Wronskian of this equation with \( f(\lambda + i0) \), we can express \( c(\lambda) \) via Wronskian (2.6):

\[
-c(\lambda) \{ f(\lambda + i0), f(\lambda - i0) \} = \{ P(\lambda), f(\lambda + i0) \} = \Omega(\lambda + i0).
\]

In view of Lemma 5.2 this yields the following result.

**Lemma 5.3.** For all \( \lambda \in \mathbb{R} \), we have the representation

\[
P_n(\lambda) = \frac{\Omega(\lambda - i0) f_n(\lambda + i0) - \Omega(\lambda + i0) f_n(\lambda - i0)}{2i \sqrt{1 - \beta_\infty^2}}, \quad n \in \mathbb{Z}_+.
\tag{5.7}
\]

Properties of the Wronskians \( \Omega(\lambda \pm i0) \) are summarized in the following statement.
Theorem 5.4. Let the assumptions of Theorem 4.2 be satisfied and $|\beta_\infty| < 1$. Then the Wronskians $\Omega(\lambda + i0)$ and $\Omega(\lambda - i0) = \Omega(\lambda + i0)$ are continuous functions of $\lambda \in \mathbb{R}$ and

$$\Omega(\lambda \pm i0) \neq 0, \quad \lambda \in \mathbb{R}.$$  \hspace{1cm} (5.8)

Proof. The functions $\Omega(\lambda \pm i0)$ are continuous by Corollary 4.3. If $\Omega(\lambda \pm i0) = 0,$ then according to (5.7) $P_n(\lambda) = 0$ for all $n \in \mathbb{Z}_+$. However, $P_0(\lambda) = 1$ for all $\lambda$. \[ \square \]

Let us set

$$\kappa(\lambda) = |\Omega(\lambda + i0)|, \quad -\Omega(\lambda \pm i0) = \kappa(\lambda)e^{\pm i\eta(\lambda)}.$$  \hspace{1cm} (5.9)

In the theory of short-range perturbations of the Schrödinger operator, the functions $\kappa(\lambda)$ and $\eta(\lambda)$ are known as the limit amplitude and the limit phase, respectively; the function $\eta(\lambda)$ is also called the scattering phase or the phase shift. Definition (5.9) fixes the phase $\eta(\lambda)$ only up to a term $2\pi m$ where $m \in \mathbb{Z}$. We emphasize that the amplitude $\kappa(\lambda)$ and the phase $\eta(\lambda)$ depend on the values of the coefficients $a_n$ and $b_n$ for all $n$, and hence they are not determined by an asymptotic behavior of $a_n$, $b_n$ as $n \to \infty$.

Combined together, relations (5.4) and (5.7) yield asymptotics of the orthonormal polynomials $P_n(\lambda)$.

Theorem 5.5. Let the assumptions of Theorem 4.2 be satisfied and $|\beta_\infty| < 1$. Then, as $n \to \infty$, the polynomials $P_n(\lambda)$ have asymptotics

$$P_n(\lambda) = \kappa(\lambda)(1 - \beta_\infty^2)^{-1/2}a_n^{-1/2}\sin(\varphi_n(\lambda) + \eta(\lambda)) + o(1), \quad \lambda \in \mathbb{R},$$  \hspace{1cm} (5.10)

where the phases $\varphi_n(\lambda)$, $\eta(\lambda)$ and the amplitude $\kappa(\lambda)$ are given by formulas (5.3) and (5.9). Relation (5.10) is uniform in $\lambda$ on compact subintervals of $\mathbb{R}$.

5.2. Resolvent and spectral measure. Under the assumption that the operator $\text{clos} J_{\text{min}} =: J$ is self-adjoint, its resolvent $R(z) = (J - zI)^{-1}$ was constructed in Proposition 2.1. To use formula (2.7), we only need to identify the solution of equation (1.16) satisfying condition (2.5) with the Jost solution.

Lemma 5.6. In addition to the assumptions of Theorem 4.2 suppose that the Carleman condition (1.27) is satisfied and $|\beta_\infty| < 1$. Then, for the Jost solution $f_n(z)$, inclusion (2.5) holds true.

Proof. In view of inequality (4.25), we only have to show that

$$\sum_{n=1}^{\infty} \alpha_n \exp \left( -\sum_{m=0}^{n-1} \alpha_m \right) < \infty.$$  \hspace{1cm} (5.11)
For a proof of the convergence of this series, we can replace here $\alpha_n$ by $1 - e^{-\alpha_n}$ and then observe that
\[
\sum_{n=1}^{N} (1 - e^{-\alpha_n}) \exp\left(-\sum_{m=0}^{n-1} \alpha_m\right)
= \sum_{n=1}^{N} \exp\left(-\sum_{m=0}^{n-1} \alpha_m\right) - \sum_{n=2}^{N+1} \exp\left(-\sum_{m=0}^{n-1} \alpha_m\right) = e^{-\alpha_0} - \exp\left(-\sum_{m=0}^{N} \alpha_m\right) < e^{-\alpha_0}.
\]
This proves (5.11) and hence in view of estimate (4.25), inclusion (2.5).

According to Theorem 4.2, $f_n(z), n = -1, 0, 1, \ldots$, and, in particular, $\Omega(z)$ are analytic functions of $z \in \mathbb{C} \setminus \mathbb{R}$ continuous up to the cut along $\mathbb{R}$. Taking also (5.8) into account, we obtain the following result. Recall that the set $D \subset \ell^2(\mathbb{Z}_+)$ consists of vectors with only a finite number of non-zero elements.

**Theorem 5.7.** In addition to the assumptions of Theorem 4.2 suppose that the Carleman condition (1.27) is satisfied and $|\beta_\infty| < 1$. Then

(i) For $\text{Im} \ z \neq 0$, the resolvent $R(z) = (J - zI)^{-1}$ of the Jacobi operator $J$ acts by formula (2.7).

(ii) For all $u,v \in D$, the functions $\langle R(z)u,v \rangle$ are continuous in $z$ up to the cut along $\mathbb{R}$.

Note that the functions $f_n(z)$ depend on the choice of the parameter $N_0$ in definition (2.27) of $q_n(z)$ but due to the factor $\Omega(z)^{-1}$ expression (2.7) for the resolvent does not depend on it.

Statement (ii) is known as the limiting absorption principle. It implies

**Corollary 5.8.** The spectrum of the operator $J$ is absolutely continuous.

Let us now consider the spectral projector $E(\lambda)$ of the operator $J$. By the Cauchy-Stieltjes-Privalov formula for $u,v \in D$, its matrix elements satisfy the identity
\[
2\pi i \frac{d\langle E(\lambda)u,v \rangle}{d\lambda} = \langle R(\lambda + i0)u,v \rangle - \langle R(\lambda - i0)u,v \rangle. \tag{5.12}
\]

The following assertion is a direct consequence of Theorem 5.7, part (ii).

**Corollary 5.9.** For all $u,v \in D$, the functions $\langle E(\lambda)u,v \rangle$ are continuously differentiable in $\lambda \in \mathbb{R}$.

Now we are in a position to calculate the spectral family $dE(\lambda)$ in terms of the Jost function. Let us proceed from the identity (5.12). Let $e_0, e_1, \ldots, e_n, \ldots$ be the canonical basis in the space $\ell^2(\mathbb{Z}_+)$. It follows from representation (2.7) that
\[
\langle R(z)e_n, e_m \rangle = \Omega(z)^{-1} P_n(z)f_m(z), \quad n \leq m, \tag{5.13}
\]
whence
\[
\langle R(\lambda \pm i0)e_n, e_m \rangle = \Omega(\lambda \pm i0)^{-1} P_n(\lambda)f_m(\lambda \pm i0), \quad n \leq m.
\]
Substituting this expression into (5.12) and using relation (4.4), we find that
\[ 2\pi i \frac{d \langle \mathcal{E}(\lambda)e_n, e_m \rangle}{d\lambda} = P_n(\lambda) \frac{\Omega(\lambda + i0)f_m(\lambda + i0) - \Omega(\lambda + i0)f_m(\lambda - i0)}{\vert \Omega(\lambda + i0) \vert^2} \]
(note that \( \vert \Omega(\lambda + i0) \vert = \vert \Omega(\lambda - i0) \vert \)). Combining this representation with formula (2.31) for \( P_m(\lambda) \), we obtain the following result.

**Theorem 5.10.** Let the assumptions of Theorem 5.7 be satisfied. Then the spectrum of the operator covers the whole line and, for all \( n, m \in \mathbb{Z}_+ \) and all \( \lambda \in \mathbb{R} \), we have the representation
\[ d \langle \mathcal{E}(\lambda)e_n, e_m \rangle = (2\pi)^{-1} \sqrt{1 - \beta_{\infty}^2} |\Omega(\lambda + i0)|^{-2} P_n(\lambda)P_m(\lambda), \quad \lambda \in \mathbb{R}. \]

In particular, the spectral measure of the operator \( J \) equals
\[ d\rho(\lambda) := d \langle \mathcal{E}(\lambda)e_0, e_0 \rangle = \tau(\lambda)d\lambda \]
where the weight \( \tau(\lambda) \) is given by the formula
\[ \tau(\lambda) = (2\pi)^{-1} \sqrt{1 - \beta_{\infty}^2} |\Omega(\lambda + i0)|^{-2}. \]

Putting together Theorem 5.4 and formula (5.16), we obtain

**Theorem 5.11.** Under the assumptions of Theorem 5.7 the weight \( \tau(\lambda) \) is a continuous strictly positive function of \( \lambda \in \mathbb{R} \).

Note that this result was earlier obtained in [2] and [33] by specific methods of the orthogonal polynomials theory.

In view of (5.16) the amplitude \( \kappa(\lambda) \) defined by (5.9) can be expressed via the weight \( \tau(\lambda) \),
\[ \kappa(\lambda) = (2\pi)^{-1/2} (1 - \beta_{\infty}^2)^{1/4} \tau(\lambda)^{-1/2}, \]
and hence asymptotic formula (5.10) can be rewritten as
\[ P_n(\lambda) = (2\pi)^{-1/2} (1 - \beta_{\infty}^2)^{-1/4} \tau(\lambda)^{-1/2} a_n^{-1/2} \sin(\phi_n(\lambda) + \eta(\lambda)) + o(1) \]
as \( n \to \infty \). This form seems to be more common for the orthogonal polynomials literature (cf. Theorem 3 in [2]).

### 6. Asymptotics in the complex plane

In this section, we find (growing) asymptotics of the orthonormal polynomials \( P_n(z) \) as \( n \to \infty \) for \( \text{Im} \, z \neq 0 \). To that end, we first solve the same problem for the solutions \( g_n(z) \) of the Jacobi equation (1.16) defined by formulas (2.15), (2.16). Since \( P_n(z) \) are linear combinations of the solutions \( f_n(z) \) and \( g_n(z) \), this yields asymptotics of \( P_n(z) \). We start in Sect. 6.1 with some general arguments which apply to all Jacobi coefficients satisfying condition (1.16). Then we consider the cases of large (Sect. 6.2) and small (Sect. 6.3) diagonal elements \( b_n \) separately.
6.1. A representation for growing solutions. Recall that the Jost solution $f_n(z)$ of equation (1.16) was defined in Theorem 4.2 by the formula

$$f_n(z) = a_n^{-1/2}q_n(z)u_n(z)$$

(6.1)

where $q_n$ is product (2.27) and $u_n \to 1$ as $n \to \infty$ according to Theorem 3.10.

We define the second solution $g_n(z)$ of (1.16) by equalities (2.15), (2.16) which in view of (6.1) yields a representation

$$g_n(z) = a_n^{-1/2}q_n(z)u_n(z)F_n(z)$$

(6.2)

where

$$F_n(z) = \sum_{m=N_0+1}^{n} \sqrt{\frac{a_m}{a_{m-1}}(q_{m-1}(z)q_m(z))^{-1}(u_{m-1}(z)u_m(z))^{-1}}.$$  

(6.3)

Our goal is to find an asymptotics of this sum. First, integrating by parts we transform representation (6.3) to a more convenient form. Recall that the numbers $\alpha_n, \beta_n$ were defined by equalities (2.19) and the numbers $z_n, \zeta_n$ were defined by equalities (4.6).

Lemma 6.1. Let us set

$$v_n = \zeta_{n-1}\zeta_n(1 - \zeta_{n-1}\zeta_n)^{-1}(u_{n-1}u_n)^{-1}$$

(6.4)

where the numbers $\zeta_n$ are given by formula (3.1) and

$$t_n = (q_{n-1}q_n)^{-1} = \zeta_0^{-2}\zeta_1^{-2} \cdots \zeta_{n-2}\zeta_{n-1}.$$

(6.5)

Then sum (6.3) can be written as

$$F_n = v_n t_{n+1} - v_{N_0} t_{N_0+1} + \tilde{F}_n$$

(6.6)

where

$$\tilde{F}_n = - \sum_{m=N_0+1}^{n} v_m t_m.$$  

(6.7)

Proof. Calculating the derivative of product (6.5), we see that

$$t'_m = (\zeta_{n-1}\zeta_m)^{-1}(1 - \zeta_{n-1}\zeta_m)t_n,$$

whence using notation (6.4), we obtain

$$\sqrt{\frac{a_n}{a_{n-1}}(q_{n-1}q_n)^{-1}(u_{n-1}u_n)^{-1}} = (\zeta_{n-1}\zeta_n)^{-1}(1 - \zeta_{n-1}\zeta_n)v_n t_n = v_n t'_n.$$  

We can now rewrite representation (6.3) as

$$F_n = \sum_{m=N_0+1}^{n} v_m t'_m.$$  

Applying here integration-by-parts formula (2.4), we arrive at relations (6.6), (6.7).
We will see that an asymptotics of the sequence $F_n$ is determined by the first term in the right-hand side of (6.6). Let us calculate it. Recall that $\zeta_\infty$ is given by (2.23) for all $z \in \mathbb{C}$ with $\text{Im } z \neq 0$ if $|\beta_\infty| > 1$. If $|\beta_\infty| < 1$, then $\zeta_\infty = \zeta_\infty^{(\pm)}$ is given by (2.24) for $\pm \text{Im } z > 0$. The number $\kappa_\infty$ is defined as limit (3.19).

**Lemma 6.2.** The asymptotic relation
\[
\lim_{n \to \infty} q_n^2(z)v_n(z)t_{n+1}(z) = \kappa_\infty \zeta_\infty (1 - \zeta_\infty^2)^{-1}
\] (6.8)
holds.

**Proof.** Since $q_{n+1} = \zeta_n q_n$, it follows from equalities (6.4) and (6.5) that
\[
q_n^2 v_n(t_{n+1} = q_n q_{n+1}^{-1} v_n = \kappa_{n-1} (1 - \zeta_{n-1}^{-1})(u_{n-1} - u_n)^{-1}.
\]
This yields (6.8) because $\zeta_n \to \zeta_\infty$, $\kappa_n \to \kappa_\infty$ and $u_n \to 1$ as $n \to \infty$. □

It remains to show that the second and third terms in the right-hand side of (6.6) give no contribution to the asymptotics of $F_n$. This is obvious for $v_{N_0} t_{N_0+1}$ because $q_n(z) \to 0$ as $n \to \infty$.

A relatively difficult part of the proof is to show that the same is true for the sum $\tilde{F}_n$. This requires an inclusion
\[
(u_n') \in \ell^1(\mathbb{Z}_+)
\] (6.9)
for the derivatives $u_n'$ of $u_n$. By its proof, we have to distinguish the cases $|\beta_\infty| > 1$ and $|\beta_\infty| < 1$. An important difference between them is that $|\zeta_\infty| < 1$ for $|\beta_\infty| > 1$ while $|\zeta_\infty^{(\pm)}| = 1$ for $|\beta_\infty| < 1$. This essentially simplifies estimates in the first case.

### 6.2. Large diagonal elements.
Here we suppose that assumption (1.26) is satisfied with $|\beta_\infty| > 1$ and consider arbitrary $z \in \mathbb{C}$.

**Lemma 6.3.** Inclusion (6.9) is true if the assumptions of Theorem 4.2 are satisfied.

**Proof.** It follows from identity (1.1) that
\[
|u_n'| \leq C \sum_{m=n+1}^{\infty} |S_{n+1}^{-1} S_{m+1}| |r_m|.
\] (6.10)
Since
\[
|\zeta_n| \leq \varepsilon < 1
\] (6.11)
for sufficiently large $n$, by definition (3.11), we have
\[
|S_{n+1}^{-1} S_{m+1}| = |\sigma_{n+1} \cdots \sigma_m| \leq C \varepsilon^{2(m-n)}, \quad m > n,
\]
so that according to (6.10)
\[
\sum_{n=0}^{\infty} |u_n'| \leq C \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} \varepsilon^{2(m-n)} |r_m| = C \sum_{m=1}^{\infty} \left( \sum_{n=0}^{m-1} \varepsilon^{2(m-n)} \right) |r_m| \leq C_1 \sum_{m=1}^{\infty} |r_m|.
\]
Since $(r_n) \in \ell^1(\mathbb{Z}_+)$, this implies inclusion (6.9). □
Now we are in a position to estimate $\widetilde{F}_n(z)$.

**Lemma 6.4.** Let $\widetilde{F}_n(z)$ be defined by formula (6.7). Then

$$\lim_{n \to \infty} q_n^2(z)\widetilde{F}_n(z) = 0.$$  (6.12)

**Proof.** By definition (2.27) of $q_n$, we have

$$q_n^2\widetilde{F}_n = -\sum_{m=N_0+1}^n (\zeta_m^2 \cdots \zeta_{n-1}^2)(\zeta_{m-1}v'_m)$$  (6.13)

where the sequence $v_n$ is defined by relation (6.4). Note that $v'_n \in \ell^1$ according to Remark 3.5 and $\zeta'_n \in \ell^1$ according to conditions (3.16). Therefore it follows from Lemma 6.3 that

$$|v'_n| \leq C|u'_n| \text{ whence } v'_n \in \ell^1(\mathbb{Z}_+).$$  (6.14)

Using now (6.11), we find that

$$|q_n^2\widetilde{F}_n| \leq \varepsilon^{n/2} \sum_{m=N_0+1}^{n/2} |v'_{m-1}| + \sum_{m \geq n/2} \varepsilon^{2(n-m)}|v'_{m-1}|$$

$$\leq \varepsilon^n \sum_{m \geq N_0+1} |v'_{m-1}| + \sum_{m \geq n/2} |v'_{m-1}|.$$  

Inclusion (6.14) implies that both terms on the right tend to zero as $n \to \infty$. □

Putting now together equality (6.6) with Lemmas 6.2 and 6.4 and using that

$$\zeta_\infty(1 - \zeta_\infty^2)^{-1} = \frac{1}{2\sqrt{\beta_\infty^2 - 1}},$$

we find an asymptotics of the sequence $g_n(z)$.

**Theorem 6.5.** Let $|\beta_\infty| > 1$, and let $\kappa_\infty$ be defined by (3.19). Under the assumptions of Theorem 4.2 the relation

$$\lim_{n \to \infty} \sqrt{a_n}q_n(z)g_n(z) = \frac{\kappa_\infty}{2\sqrt{\beta_\infty^2 - 1}}$$

is true for all $z \in \mathbb{C}$ with convergence uniform on compact subsets of $\mathbb{C}$.

Let us now use equality (2.33). In view of Theorem 4.2 the term $\omega(z)f_n(z)$ is negligible unless $\Omega(z) = 0$. Therefore Theorem 6.5 yields an asymptotics of the orthonormal polynomials.

**Theorem 6.6.** Let $|\beta_\infty| > 1$. Under the assumptions of Theorem 4.2 the relation

$$\lim_{n \to \infty} \sqrt{a_n}q_n(z)P_n(z) = -\frac{\kappa_\infty\Omega(z)}{2\sqrt{\beta_\infty^2 - 1}}$$

is true for all $z \in \mathbb{C}$ with convergence uniform on compact subsets of $\mathbb{C} \setminus \mathbb{R}$. Moreover, if $\Omega(z) = 0$, then

$$
\lim_{n \to \infty} \sqrt{a_n q_n(z)}^{-1} P_n(z) = \{P(z), g(z)\}
$$

(note that $\{P(z), g(z)\} \neq 0$ if $\Omega(z) = 0$).

6.3. Small diagonal elements. Here we suppose that assumption (1.26) is satisfied with $|\beta_\infty| < 1$ and $z \in \mathbb{C} \setminus \Pi$. In contrast to the previous subsection, the Carleman condition (1.27) is also assumed. We again use identity (4.1) but integrate by parts in its right-hand side which requires additional assumptions. First, we improve estimate (3.18) of the remainder $r_n(z)$. Recall notations (2.19) and (3.1).

**Lemma 6.7.** Suppose that, for some $\delta > 0$,

$$
\alpha' = O(n^{-1-\delta}), \quad \alpha'' = O(n^{-2-\delta}) \quad \beta' = O(n^{-1-\delta}), \quad \beta'' = O(n^{-2-\delta})
$$

and

$$
k_n - 1 = O(n^{-1-\delta}), \quad k' = O(n^{-2-\delta}).
$$

Then remainder (3.5) satisfies estimates

$$
r_n(z) = O(n^{-1-\delta}) \quad \text{and} \quad r'_n(z) = O(n^{-2-\delta}).
$$

**Proof.** Using (3.14), we rewrite relation (3.5) as

$$
r_n = -z' n^{-1} g_n + (k_n - 1) \zeta_n
$$

where $z_n$, $\zeta_n$ and $g_n$ are defined by equalities (2.20) and (3.13). It follows from assumptions (6.15) that

$$
z_n' = O(n^{-1-\delta}), \quad z_n'' = O(n^{-2-\delta}).
$$

In view of (3.2) this also implies that

$$
g_n = O(1) \quad \text{and} \quad g'_n = O(n^{-1-\delta}).
$$

Thus, the first estimate (6.17) is a direct consequence of relation (6.18).

Using (2.3), we differentiate (6.18). Since $|\zeta'_n| \leq C|z'_n|$ according to (3.14), the second bound (6.17) is also a consequence of the estimates obtained. \qed

An asymptotics of $g_n(z)$ is stated in the following assertion.

**Theorem 6.8.** Let $|\beta_\infty| < 1$. Let assumptions (1.26), (1.27) as well as (6.15) and (6.16) be satisfied. Then

$$
\lim_{n \to \infty} \sqrt{a_n q_n(z) g_n(z)} = \pm \frac{1}{2i \sqrt{1 - \beta^2_\infty}}, \quad \pm \text{Im} \, z > 0.
$$

According to representation (6.2) where $u_n(z) \to 1$ as $n \to \infty$ relation (6.19) can be rewritten as

$$
\lim_{n \to \infty} q_n(z)^2 F_n(z) = \pm \frac{1}{2i \sqrt{1 - \beta^2_\infty}}, \quad \pm \text{Im} \, z > 0.
$$
For a proof of (6.20), we again proceed from equality (6.6) and consider the terms on the right separately. According to Lemma 6.2 the asymptotics of
\[ q_n^2 v_n t_{n+1} \]
is given by relation (6.8) where according to (2.24) we now have
\[ \zeta (\pm) (1 - (\zeta (\pm))^2)^{-1} = \pm \frac{1}{2i \sqrt{1 - \beta_\infty^2}}. \] (6.21)
The term \( v_{N_0} t_{N_0+1} \) in (6.6) gives no contribution to the asymptotics of \( F_n \).

It remains to show that the same is true for the sum \( \tilde{F}_n \). We start with estimates of derivatives \( u'_n \). This requires bootstrap arguments.

**Lemma 6.9.** Let the assumptions of Theorem 6.8 be satisfied. Then inclusion (6.9) holds.

**Proof.** Let us proceed from identity (4.1) and integrate by parts in its right-hand side. It follows from definition (3.11) that
\[ S'_n = S_n (\sigma_n - 1). \]
Setting \( \tilde{r}_m = (\sigma_{m+1} - 1)^{-1} \zeta_m r_m \), we see that
\begin{align*}
\chi_n^{-1} u'_n &= - S_{n+1}^{-1} \sum_{m=n+1}^\infty S'_m \tilde{r}_m u_m \\
&= \tilde{r}_{n-1} u_{n-1} - \sum_{m=n+1}^\infty S_{n+1}^{-1} S'_m \tilde{r}_{m-1} u_m - \sum_{m=n+1}^\infty S_{n+1}^{-1} S_m \tilde{r}_{m-1} u'_{m-1}. \tag{6.22}
\end{align*}

Under our assumptions we have estimates (6.17). Since \( u_n \in \ell^\infty (\mathbb{Z}_+) \), the first term in the right-hand side of (6.22) is bounded by \( n^{-1-\delta} \). By definition (3.11), we have \( |S_{n+1}^{-1} S_{m+1}| \leq 1 \) if \( m \geq n + 1 \). Therefore in view of estimate (6.17) on \( \tilde{r}'_n \), the second term in the right-hand side of (6.22) is also bounded by \( n^{-1-\delta} \). Thus identity (6.22) implies that
\[ |u'_n| \leq C n^{-1-\delta} + C \sum_{m=n}^\infty m^{-1-\delta} |u'_m|. \tag{6.23} \]

Suppose now that \( u'_n = O(n^{-\sigma}) \) for some \( \sigma \in [0, 1] \). Then it follows from (6.23) that actually \( u'_n = O(n^{-\sigma-\delta}) \). Since \( u'_n = O(1) \), repeating this argument a sufficient number of times, we find that \( u'_n = O(n^{-1-\delta}) \). This implies (6.9). \( \square \)

Now we are in a position to estimate sum (6.7). As in the case \( |\beta_\infty| > 1 \), we essentially use inclusion (6.9).

**Lemma 6.10.** The function \( \tilde{F}_n (z) \) defined by formula satisfies relation (6.12).

**Proof.** By definition (2.27), we have relation (6.13) where \( v_n \) are given by (6.4). Lemma 6.9 again implies inclusion (6.14). Compared to the case \( |\beta_\infty| > 1 \), the
problem is that now inequality (6.11) is no longer true. Instead, we have a weaker inequality
\[ |ζ_n| ≤ e^{-cn}, \quad c = c(z) > 0, \]
following from Corollary 4.5.

Let us now write (6.13) as
\[ q_n^2 \tilde{F}_n = - \sum_{m=N_0+1}^{∞} X_{m,n}ζ_{m-1}v_m' \]
where
\[ X_{m,n} = ζ_2^m \cdots ζ_{n-1}^2 \quad \text{for} \quad m < n \quad \text{and} \quad X_{m,n} = 0 \quad \text{for} \quad m ≥ n. \]
According to (6.24) we have an estimate
\[ |X_{m,n}| ≤ \exp \left( -c \sum_{k=m}^{n-1} α_k \right). \]
Therefore \( X_{m,n} \to 0 \) as \( n \to ∞ \) for every fixed \( m \) by virtue of the Carleman condition (1.27). Now inclusion (6.14) allows us to use the dominated convergence theorem to show that the limit of sum (6.25) as \( n \to ∞ \) is zero.

Let us come back to the proof of Theorem 6.8. As in Sect. 6.2, we proceed from relation (6.6). Combining Lemma 6.2 and equality (6.21), we find that
\[ \lim_{n \to ∞} q_n^2(z) v_n(z) t_{n+1}(z) = \mp \frac{1}{2i \sqrt{1 - β^2_∞}}. \]
Therefore using Lemma 6.10 we arrive at relation (6.20) or, equivalently, (6.19).

In view of equality (2.33) the following result is a direct corollary of Theorems 4.2 and 6.8. We recall that the Wronskian \( Ω(z) \) is defined by relation (2.6) and \( Ω(z) ≠ 0 \) for \( \text{Im} \ z ≠ 0 \).

**Theorem 6.11.** Under the assumptions of Theorem 6.8 the relation
\[ \lim_{n \to ∞} \sqrt{a_n} q_n(z) P_n(z) = \frac{iΩ(z)}{2 \sqrt{1 - β^2_∞}} \]
is true for all \( z ∈ \mathbb{C} \setminus \mathbb{R} \) with convergence uniform on compact subsets of \( \mathbb{C} \setminus \mathbb{R} \).

7. **Non-Carleman case**

In this section, we specially consider the case when off-diagonal entries \( a_n \) grow so rapidly that
\[ \sum_{n=0}^{∞} a_n^{-1} < ∞; \]
thus the Carleman condition (1.27) fails. Asymptotic properties of orthonormal polynomials are discussed in Sect. 7.1 and spectral properties of the corresponding Jacobi operators - in the following subsections. Some proofs will be omitted since they were already published in the papers [75] and [77].

7.1. Jost solutions and orthogonal polynomials. In the case $|\beta_\infty| > 1$, there is almost no difference between the Carleman and non-Carleman cases. Thus, Theorems 4.2, 6.5 and 6.6 remain true, and their formulations may even be simplified. Indeed, the sequence $q_n(z)$ in all asymptotic formulas may be replaced (cf. Sect. 4.3) by a simpler sequence $\Theta_n$ defined by formula (4.19). We emphasize that $\Theta_n$ do not depend on $z \in \mathbb{C}$. Moreover, in the non-Carleman case, the first condition (3.16) can be omitted. Actually, in view of Remark 3.3 under assumption (3.17) conditions (7.1) and

$$\sum_{n=0}^{\infty} \alpha_n < \infty$$

are equivalent. The last condition implies that the series of $|\alpha_n'|$ converges.

Let us summarize the results of Theorems 4.2, 6.5 and 6.6 for the non-Carleman case. Recall that the sequence $g_n(z)$ is defined by formulas (2.15), (2.16) and $\kappa_\infty – \kappa_\infty$ by formulas (3.1), (3.19).

**Theorem 7.1.** Let condition (7.1) hold true and $|\beta_\infty| > 1$. Suppose that assumptions (1.26), (3.17) and (7.2)

are satisfied. Then for all $z \in \mathbb{C}$, equation (1.16) has a unique solution $f_n(z)$ with asymptotics

$$f_n(z) = a_n^{-1/2} \Theta_n \left(1 + o(1)\right), \ n \to \infty.$$ (7.3)

Besides, the relations

$$\lim_{n \to \infty} \sqrt{a_n} \Theta_n g_n(z) = \frac{\kappa_\infty}{2\sqrt{\beta_\infty^2 - 1}}$$ (7.4)

and

$$\lim_{n \to \infty} \sqrt{a_n} \Theta_n P_n(z) = -\frac{\kappa_\infty \Omega(z)}{2\sqrt{\beta_\infty^2 - 1}}$$

are true for all $z \in \mathbb{C}$ with convergence uniform on compact subsets of $\mathbb{C}$. Moreover, if $\Omega(z) = 0$, then

$$\lim_{n \to \infty} \sqrt{a_n} \Theta_n^{-1} P_n(z) = \{P(z), g(z)\}$$ (note that $\{P(z), g(z)\} \neq 0$ if $\Omega(z) = 0$). All functions $f_n(z)$ are analytic in $z \in \mathbb{C}$.

On the contrary, for $|\beta_\infty| < 1$, the Carleman and non-Carleman cases are significantly different. In particular, for all $z \in \mathbb{C}$, we now have two linearly independent solutions $f_n^{(+)}(z)$ and $f_n^{(-)}(z)$. The leading terms of their asymptotics do not depend on $z$ and are complex conjugate to each other.
Theorem 7.2. Let condition (7.1) hold true and $|\beta_\infty| < 1$. Suppose that assumptions (1.26), (3.17) and (7.2) are satisfied. Define the sequences $\Theta_n^{(\pm)}$ by formula (4.20). Then for each of the signs “$\pm$” and all $z \in \mathbb{C}$, equation (1.16) has a unique solution $f_n^{(\pm)}(z)$ with asymptotics

$$f_n^{(\pm)}(z) = a_n^{-1/2}\Theta_n^{(\pm)}(1 + o(1)), \quad n \to \infty.$$  \hfill (7.5)

These solutions are linked by the relation

$$f(-)(\bar{z}) = \overline{f(+)(z)}$$  \hfill (7.6)

and their Wronskian equals

$$\{f^{(+)}(z), f^{(-)}(z)\} = -2i\xi_\infty^{-1}\sqrt{1 - \beta_\infty^2} \neq 0$$  \hfill (7.7)

so that the solutions $f_n^{(+)}(z)$ and $f_n^{(-)}(z)$ are linearly independent. For all $n \in \mathbb{Z}_+$, the functions $f_n^{(\pm)}(z)$ are analytic in $z \in \mathbb{C}$.

Let us make some comments on the proof. Under assumption (7.1) we can define the Ansatz by one of the formulas $A^{(\pm)} = a_n^{-1/2}\Theta_n^{(\pm)}$. Note that it does not depend on $z$. An easy calculation shows that, for this Ansatz, remainder (2.26) equals

$$r_n^{(\pm)}(z) = \left(\zeta(\beta_{n-1} \pm i0)^{-1} - \zeta(\beta_n \pm i0)^{-1}\right) + (k_n - 1)\zeta(\beta_n \pm i0) - 2z\alpha_n.$$  

Then $(r_n^{(\pm)}(z)) \in \ell^1$, and the whole scheme of the proof of Theorem 1.2 works without any changes. Moreover, all iterations (3.27) are entire functions of $z$ because $r_n^{(\pm)}(z)$ is a linear function of $z$ and the kernels $G_{n,m}$ do not depend on $z$. This ensures that the solutions $u_n^{(\pm)}(z)$ of Volterra equation (3.10) and hence functions (2.28) are analytic in $z \in \mathbb{C}$.

It follows from formulas (1.20) and (7.3) that the Wronskian of the solutions $f^{(+)}(z)$ and $f^{(-)}(z)$ equals

$$\{f^{(+)}(z), f^{(-)}(z)\} = \lim_{n \to \infty} \xi_n^{-1}(\Theta_n^{(+)}\Theta_{n+1}^{(-)} - \Theta_n^{(+)}\Theta_{n+1}^{(-)})$$

$$= \lim_{n \to \infty} \xi_n^{-1}(e^{-i\arccos\beta_n} - e^{i\arccos\beta_n}) = -2i\xi_\infty^{-1}\sin\arccos\beta_\infty$$

which yields relation (7.7). \hfill \square

All solutions of equations (1.16) are linear combinations of the Jost solutions $f^{(+)}(z)$ and $f^{(-)}(z)$, and hence their asymptotics are determined by Theorem 7.2. In particular, this is true for the polynomials $P_n(z)$ defined by conditions (1.2) and the polynomials of the second kind $Q_n(z)$ defined by the boundary conditions

$$Q_0(z) = 0, \quad Q_1(z) = a_0^{-1}.$$  \hfill (7.8)

Set $P(z) = (P_n(z))$, $Q(z) = (Q_n(z))$. Note that

$$\mathcal{J}P(z) = zP(z) \quad \text{and} \quad \mathcal{J}Q(z) = zQ(z) + e_0.$$  \hfill (7.9)
Thus, we have
\[ P_n(z) = \sigma_+(z)f_n^{(+)}(z) + \sigma_-(z)f_n^{(-)}(z) \] (7.10)
and
\[ Q_n(z) = \tau_+(z)f_n^{(+)}(z) + \tau_-(z)f_n^{(-)}(z), \] (7.11)
where the coefficients \( \sigma_\pm(z) \) and \( \tau_\pm(z) \) can be expressed via the Wronskians:
\[ \sigma_+(z) = \frac{i \kappa_\infty}{2 \sqrt{1 - \beta_\infty^2}} \{ P(z), f^{(-)}(z) \}, \quad \sigma_-(z) = -\frac{i \kappa_\infty}{2 \sqrt{1 - \beta_\infty^2}} \{ P(z), f^{(+)}(z) \} \] (7.12)
and
\[ \tau_+(z) = \frac{i \kappa_\infty}{2 \sqrt{1 - \beta_\infty^2}} \{ Q(z), f^{(-)}(z) \}, \quad \tau_-(z) = -\frac{i \kappa_\infty}{2 \sqrt{1 - \beta_\infty^2}} \{ Q(z), f^{(+)}(z) \}. \] (7.13)

Observe that
\[ \sigma_-(\bar{z}) = \sigma_+(z) \quad \text{and} \quad \tau_-(\bar{z}) = \tau_+(z) \]
because \( P_n(\bar{z}) = \overline{P_n(z)} \), \( Q_n(\bar{z}) = \overline{Q_n(z)} \) and \( f_n^{(\pm)}(z) \) satisfy (7.6). Of course, all coefficients \( \sigma_\pm(z) \) and \( \tau_\pm(z) \) are entire functions of \( z \).

According to (7.10) and (7.11) the following result is a direct consequence of Theorem 7.2.

**Theorem 7.3.** Under the assumptions of Theorem 7.2 the orthogonal polynomials \( P_n(z) \) and \( Q_n(z) \) have the asymptotics, as \( n \to \infty \),
\[ P_n(z) = a_n^{-1/2}(\sigma_+(z)\Theta_n^{(+)}) + \sigma_-(z)\Theta_n^{(-)} + o(1)) \] (7.14)
and
\[ Q_n(z) = a_n^{-1/2}(\tau_+(z)\Theta_n^{(+)}) + \tau_-(z)\Theta_n^{(-)} + o(1)). \] (7.15)

In view of conditions (1.2) and (7.8) the Wronskian \( \{ P(z), Q(z) \} = 1 \). On the other hand, we can calculate this Wronskian using relations (7.7) and (7.10), (7.11). This yields an identity
\[ 2i \kappa_\infty^{-1} \sqrt{1 - \beta_\infty^2}(\sigma_-(z)\tau_+(z) - \sigma_+(z)\tau_-(z)) = 1, \quad \forall z \in \mathbb{C}. \] (7.16)

We also note the identity
\[ |\sigma_+(z)|^2 - |\sigma_-(z)|^2 = \text{Im} \ z \kappa_\infty (1 - \beta_\infty^2)^{-1/2} \sum_{n=0}^{\infty} |P_n(z)|^2 \] (7.17)
established in Theorem 4.4 of [75].
7.2. Essential self-adjointness. Here we consider self-adjoint extensions of the minimal Jacobi operator $J_{\text{min}}$ defined in the space $\ell^2(\mathbb{Z}_+)$ by formula (1.10) on the set $\mathcal{D}$ of elements with a finite number of non-zero components.

Recall first of all that the Carleman condition (1.27) is sufficient but not necessary for the essential self-adjointness of the operator $J_{\text{min}}$. Nevertheless it is close to necessary for small diagonal elements $b_n$. Indeed, according to the Berezanskii theorem (see, e.g., page 26 in the book [4]) if $b_n = 0$ (or, more generally, $(b_n) \in \ell^\infty(\mathbb{Z}_+)$), then the Carleman condition is equivalent to the essential self-adjointness of the operator $J_{\text{min}}$ provided $a_n - 1 a_n + 1 \leq a_n^2$. The following example shows that the last condition is very essential.

Example 7.4. Suppose that $b_n = 0$ and that $a_n = n^p(1 + c_1 n^{-1})$ if $n$ is odd and $a_n = n^p(1 + c_2 n^{-1})$ if $n$ is even, at least for sufficiently large $n$. As shown in [37, 32], the corresponding Jacobi operator $J_{\text{min}}$ is essentially self-adjoint if $p > 1$ and $|c_2 - c_1| \geq p - 1$. Since

$$
\frac{a_n}{\sqrt{a_{n-1}a_{n+1}}} = 1 + (-1)^n \frac{c_2 - c_1}{n} + O\left(\frac{1}{n^2}\right), \quad n \to \infty,
$$

(7.18)

the condition $a_{n-1}a_{n+1} \leq a_n^2$ fails in this example.

In addition to the Berezanskii theorem, we have the following result where the case $b_n \to \infty$ is not excluded.

Proposition 7.5. Under the assumptions of Theorem 7.2 the operator $J_{\text{min}}$ has deficiency indices $(1, 1)$.

Proof. Let us consider the solutions $f^{(\pm)}_n(z)$ of the Jacobi equation with asymptotics (7.4). Since $a_n$ satisfy condition (7.1) and $|\Theta_n| = 1$, it follows from (7.5) that $f^{(\pm)}_n(z) \in \ell^2(\mathbb{Z}_+)$. The solutions are linearly independent so that we are in the limit circle case. \hfill \Box

Note that this result does not apply to Example 7.4 because relation (7.18) excludes condition (3.17).

All self-adjoint extensions of the operator $J_{\text{min}}$ will be described in Sect. 7.4.

Next, we consider the case $|\beta_\infty| > 1$.

Proposition 7.6. Under the assumptions of Theorem 7.1 the operator $J_{\text{min}}$ is essentially self-adjoint if and only if

$$
\sum_{n=0}^{\infty} a_n^{-1} \Theta_n^2 = \infty.
$$

(7.19)

Otherwise, $J_{\text{min}}$ has deficiency indices $(1, 1)$.

Proof. According to (7.3) under assumption (7.1) the Jost solution $f_n(z) \in \ell^2(\mathbb{Z}_+)$ for all $z \in \mathbb{C}$. Therefore the operator $J_{\text{min}}$ is essentially self-adjoint if and only if the
solution $g_n(z)$ linearly independent with $f_n(z)$ is not in $\ell^2(\mathbb{Z}_+)$. By virtue of (7.4) this is equivalent to condition (7.19).

Relation (7.19) generalizes the Carleman condition (1.27). Proposition 7.6 shows that in the case $|\beta_\infty| > 1$ the operator $J_{\text{min}}$ is essentially self-adjoint unless the sequence $a_n$ grows very rapidly. Indeed, it follows from definition (2.43) that

$$\varphi_n = n \arccosh |\beta_\infty| + o(n)$$

as $n \to \infty$ whence

$$\exp(n \arccosh |\beta_\infty| - \varepsilon n) \leq \Theta_n \leq \exp(n \arccosh |\beta_\infty| + \varepsilon n)$$

for an arbitrary $\varepsilon > 0$ and sufficiently large $n$. Therefore the operator $J_{\text{min}}$ is essentially self-adjoint if $a_n \leq cn^p$ for some $c > 0$, $p < \infty$ and all $n \geq 1$. On the contrary, $J_{\text{min}}$ has deficiency indices $(1, 1)$ if $a_n \geq cx^{np}$ for some $x > 1$ and $p > 1$.

7.3. Quasiresolvent. In this subsection we do not make any specific assumptions about the coefficients $a_n$ and $b_n$ supposing only that the minimal Jacobi operator $J_{\text{min}}$ is not essentially self-adjoint. Thus, we are in the limit circle case so that

$$\text{clos } J_{\text{min}} \neq J_{\text{max}} = J_{\text{min}}^*$$

and the inclusions

$$P(z) \in \ell^2(\mathbb{Z}_+) \quad \text{and} \quad Q(z) \in \ell^2(\mathbb{Z}_+)$$

are satisfied for all $z \in \mathbb{C}$ (see, for example, Theorem 6.16 in [58]).

In the limit circle case, the operator $J_{\text{max}}$ is of course not symmetric. For all $u,v \in \mathcal{D}(J_{\text{max}})$, we have the identity (the Green formula)

$$\langle Ju, v \rangle - \langle u, Jv \rangle = \lim_{n \to \infty} a_n(u_{n+1} \bar{v}_n - u_n \bar{v}_{n+1})$$

(7.22)

where the limit in the right-hand side exists. Indeed, a direct calculation shows that

$$\sum_{m=0}^{n} (Ju)_m \bar{v}_m - \sum_{m=0}^{n} u_m (J\bar{v})_m = a_n(u_{n+1} \bar{v}_n - u_n \bar{v}_{n+1}).$$

Passing here to the limit $n \to \infty$ and using that $Ju \in \ell^2(\mathbb{Z}_+)$ for $u \in \mathcal{D}(J_{\text{max}})$, we obtain (7.22).

Let us define an operator $R(z)$ playing the role of the resolvent of the operator $J_{\text{max}}$ by an equality

$$(R(z)h)_n = Q_n(z) \sum_{m=0}^{n} P_m(z)h_m + P_n(z) \sum_{m=n+1}^{\infty} Q_m(z)h_m.$$  

(7.23)

In view of inclusions (7.21) the operators $R(z)$ considered in the space $\ell^2(\mathbb{Z}_+)$ belong to the Hilbert-Schmidt class for all $z \in \mathbb{C}$. Note also that $R(z)$ is a holomorphic operator-valued function of $z \in \mathbb{C}$ and $R(z)^* = R(\bar{z})$. It follows from (1.2), (7.8) that

$$(R(z)h)_0 = \langle h, Q(\bar{z}) \rangle.$$  

(7.24)
Our proof of the following statement is close to the construction of the resolvents for self-adjoint Jacobi operators (see Proposition 2.1).

**Theorem 7.7.** Let (7.20) be satisfied. For all $z \in \mathbb{C}$, we have

$$
\mathcal{R}(z) : \ell^2(\mathbb{Z}_+) \to \mathcal{D}(J_{\text{max}})
$$

and

$$(J_{\text{max}} - zI)\mathcal{R}(z) = I. \quad (7.27)$$

**Proof.** Recall that the operator $J$ was defined by equalities (1.10). We will check that

$$((\mathcal{J} - zI)\mathcal{R}(z)h)_n = h_n\quad (7.28)$$

for all $n \in \mathbb{Z}_+$ and $h = (h_n) \in \ell^2(\mathbb{Z}_+)$. For $n = 0$, we have

$$((\mathcal{J} - zI)\mathcal{R}(z)h)_0 = (b_0 - z)(\mathcal{R}(z)h)_0 + a_0(\mathcal{R}(z)h)_1 = h_0$$

according to formulas (7.24) and (7.25). For $n \geq 1$, we rewrite definition (2.10) as

$$(\mathcal{R}(z)h)_n = Q_n(z)x_n(z) + P_n(z)y_n(z)$$

where

$$x_n(z) = \sum_{m=0}^n P_m(z)h_m, \quad y_n(z) = \sum_{m=n+1}^\infty Q_m(z)h_m. \quad (7.29)$$

It now follows from definition (1.10) that

$$((\mathcal{J} - zI)\mathcal{R}(z)h)_n = a_{n-1}(Q_{n-1}x_{n-1} + P_{n-1}y_{n-1})$$

$$+ (b_n - z)(Q_nx_n + P_ny_n) + a_n(Q_{n+1}x_{n+1} + P_{n+1}y_{n+1}), \quad n \geq 1. \quad (7.30)$$

According to (7.29) we have

$$Q_{n-1}x_{n-1} + P_{n-1}y_{n-1} = Q_{n-1}(x_n - P_nh_n) + P_{n-1}(y_n + Q_nh_n)$$

and

$$Q_{n+1}x_{n+1} + P_{n+1}y_{n+1} = Q_{n+1}(x_n + P_{n+1}h_{n+1})$$

$$+ P_{n+1}(y_n - Q_{n+1}h_{n+1}) = Q_{n+1}x_n + P_{n+1}y_n.$$
Let us now collect together all terms containing \( x_n, y_n \) and \( h_n \). Then

\[
((J - zI) \mathcal{R}(z) h)_n = \left( a_{n-1} Q_{n-1} + (b_n - z) Q_n + a_n Q_{n+1} \right) x_n \\
+ \left( a_{n-1} P_{n-1} + (b_n - z) P_n + a_n P_{n+1} \right) y_n + a_{n-1} \left( - P_n Q_{n-1} + P_{n-1} Q_n \right) h_n. \tag{7.31}
\]

The coefficients at \( x_n \) and \( y_n \) equal zero by virtue of the Jacobi equation \( \text{(1.16)} \) for \((Q_n)\) and \((P_n)\), respectively. Since \( \{P,Q\} = 1 \), the right-hand side of \( \text{(7.31)} \) equals \( h_n \). This proves \( \text{(7.28)} \) whence \((J - zI) \mathcal{R}(z) h = h\). In particular, we see that \( \mathcal{R}(z) h \in \mathcal{D}(J_{\max}) \), and hence \( J \) can be replaced here by \( J_{\max} \). This yields both \( \text{(7.26)} \) and \( \text{(7.27)} \).

**Remark 7.8.** In definition \( \text{(7.23)} \), one can replace \( Q_n(z) \) by the polynomials \( \tilde{Q}_n(z) = Q_n(z) + c P_n(z) \) for an arbitrary \( c \in \mathbb{C} \). Then \( \tilde{\mathcal{R}}(z) = \mathcal{R}(z) + c \langle \cdot, P(z) \rangle P(z) \) and formulas \( \text{(7.26)}, \text{(7.27)} \) remain true.

Since \( P(z) \) is a unique (up to a constant factor) solution of the homogeneous equation \((J_{\max} - zI) u = 0\), we can also state

**Corollary 7.9.** Let \( z \in \mathbb{C} \) and \( h \in \ell^2(\mathbb{Z}_+) \). Then all solutions of the equation

\[
(J_{\max} - zI) u = h \quad \text{where}
\]

for \( u \in \mathcal{D}(J_{\max}) \) are given by the formula

\[
u = \Gamma P(z) + \mathcal{R}(z) h \quad \text{for some} \quad \Gamma \in \mathbb{C}. \tag{7.32}
\]

The following asymptotic relation for \( (\mathcal{R}(z) h)_n \) is a direct consequence of definition \( \text{(7.23)} \) and condition \( \text{(7.21)} \):

\[
(\mathcal{R}(z) h)_n = Q_n(z) \langle h, P(z) \rangle + o(|P_n(z)| + |Q_n(z)|) \quad \text{as} \quad n \to \infty \tag{7.33}
\]

for all \( h \in \ell^2(\mathbb{Z}_+) \). This relation can be supplemented by the following result.

**Proposition 7.10.** Set

\[
\mathcal{D} = \text{clos} \, \mathcal{D}(J_{\min}). \tag{7.34}
\]

For all \( z \in \mathbb{C} \) and all \( h \in \ell^2(\mathbb{Z}_+) \), we have

\[
u := \mathcal{R}(z) h - Q(z) \langle h, P(z) \rangle \in \mathcal{D}.
\]

**Proof.** Let first \( h \in \mathcal{D} \). Then \((\mathcal{R}(z) h)_n = Q_n(z) \langle h, P(z) \rangle \) for sufficiently large \( n \), whence \( u \in \mathcal{D} \subset \mathcal{D} \).

Let now \( h \) be an arbitrary vector in \( \ell^2(\mathbb{Z}_+) \). Observe that \( u \in \mathcal{D} \) if and only if there exists a sequence \( u^{(k)} \in \mathcal{D} \) such that

\[
u^{(k)} \to u \quad \text{and} \quad J u^{(k)} \to J u \quad \text{as} \quad k \to \infty. \tag{7.35}
\]

Let us take any sequence \( h^{(k)} \in \mathcal{D} \) such that \( h^{(k)} \to h \) and set

\[
u^{(k)} = \mathcal{R}(z) h^{(k)} - Q(z) \langle h^{(k)}, P(z) \rangle.
\]
Then \( u^{(k)} \in \mathcal{D} \) and \( u^{(k)} \to u \) as \( k \to \infty \) because the operator \( \mathcal{R}(z) \) is bounded. It follows from equalities (7.9) and (7.27) that
\[
(\mathcal{J} - z)u^{(k)} = h^{(k)} - e_0\langle h^{(k)}, P(\bar{z}) \rangle \to h - e_0\langle h, P(\bar{z}) \rangle = (\mathcal{J} - z)u
\]
as \( k \to \infty \). This proves relations (7.35) whence \( u \in \mathcal{D} \). \( \square \)

### 7.4. Self-adjoint extensions

First, we extend the asymptotic formulas of Theorem 7.3 to all vectors \( u \in \mathcal{D}(J_{\max}) \). Using Corollary 7.9 we define the number \( \Gamma(z; u) \) by relation (7.32), that is,
\[
\Gamma(z; u)P(z) = u - \mathcal{R}(z)(J_{\max} - zI)u, \quad z \in \mathbb{C}.
\]

**Theorem 7.11.** Let the assumptions of Theorem 7.2 be satisfied. Then all sequences \((u_n) \in \mathcal{D}(J_{\max})\) have asymptotics
\[
u_n = a_n^{-1/2}(s_+ \Theta_n^{(+)} + s_- \Theta_n^{(-)} + o(1)), \quad n \to \infty,
\]
with some coefficients \( s_\pm = s_\pm(u) \). They can be constructed by the relations
\[
s_+(u) = \Gamma(z; u)\sigma_+(z) + \langle (J_{\max} - zI)u, P(\bar{z}) \rangle \tau_+(z),
\]
\[
s_-(u) = \Gamma(z; u)\sigma_-(z) + \langle (J_{\max} - zI)u, P(\bar{z}) \rangle \tau_-(z),
\]
where the number \( z \in \mathbb{C} \) is arbitrary.

Conversely, for arbitrary \( s_+, s_- \in \mathbb{C} \), there exists a vector \( u \in \mathcal{D}(J_{\max}) \) such that asymptotics (7.36) holds.

**Proof.** According to Corollary 7.9 a vector \( u \in \mathcal{D}(J_{\max}) \) admits representation (7.32) where the operator \( \mathcal{R}(z) \) is defined by equality (2.10). In view of relation (7.33) and asymptotics (7.13) we have
\[
(\mathcal{R}(z)h)_n = a_n^{-1/2}(\tau_+(z)\Theta_n^{(+)} + \tau_-(z)\Theta_n^{(-)})\langle h, P(\bar{z}) \rangle + o(a_n^{-1/2}), \quad n \to \infty,
\]
for all vectors \( h \in \ell^2(\mathbb{Z}_+) \). Therefore it follows from (7.32) and (7.14) that
\[
u_n = a_n^{-1/2}\Gamma(z; (J_{\max} - zI)u)(\sigma_+(z)\Theta_n^{(+)} + \sigma_-(z)\Theta_n^{(-)})
\]
\[]\[+ a_n^{-1/2}(\tau_+(z)\Theta_n^{(+)} + \tau_-(z)\Theta_n^{(-)})\langle (J_{\max} - zI)u, P(\bar{z}) \rangle + o(a_n^{-1/2})
\]
as \( n \to \infty \). This yields relation (7.36) with the coefficients \( s_\pm \) defined by (7.37).

Conversely, given \( s_+ \) and \( s_- \) and fixing some \( z \in \mathbb{C} \), we consider the system of equations
\[
s_+ = \Gamma\sigma_+(z) + \langle h, P(\bar{z}) \rangle \tau_+(z),
\]
\[
s_- = \Gamma\sigma_-(z) + \langle h, P(\bar{z}) \rangle \tau_-(z).
\]
for \( \Gamma \) and \( \langle h, P(\bar{z}) \rangle \). According to (7.14) the determinant of this system is not zero, so that \( \Gamma \) and \( \langle h, P(\bar{z}) \rangle \) are uniquely determined by \( s_+ \) and \( s_- \). Then we take any \( h \) such that its scalar product with \( P(\bar{z}) \) equals the found value of \( \langle h, P(\bar{z}) \rangle \). Finally, we define \( u \) by formula (7.32). The asymptotics as \( n \to \infty \) of \( P_n(z) \) and \( (\mathcal{R}(z)h)_n \).
are given by formulas (7.14) and (7.38), respectively. In view of equations (7.39) this leads to asymptotics (7.36).

Theorem 7.11 yields a mapping $\mathcal{D}(J_{\text{max}}) \to \mathbb{C}^2$ defined by the formula

$$u \mapsto (s_+(u), s_-(u)).$$

(7.40)

The construction of Theorem 7.11 depends on the choice of $z \in \mathbb{C}$, but this mapping is defined intrinsically. In particular, we can set $z = 0$ in all formulas of Theorem 7.11. Note that mapping (7.40) is surjective.

Evidently, (7.40) plays the role of the mapping $f \mapsto (f(0), f'(0))$ for the differential operator $-d^2/dx^2$ in the space $L^2(\mathbb{R}_+)$ and formula (7.22) plays the role of the integration-by-parts formula

$$\int_0^\infty f''(x)g(x)dx - \int_0^\infty f(x)g''(x)dx = f(0)g'(0) - f'(0)g(0).$$

Under the assumptions of Theorem 4.2 the right-hand side of (7.22) can be expressed in terms of the coefficients $s_+$ and $s_-$. Recall that the numbers $\alpha_n, \beta_n$ are defined by equalities (2.19) and $\alpha_\infty, \beta_\infty$ are their limits as $n \to \infty$.

Proposition 7.12. For all $u, v \in \mathcal{D}(J_{\text{max}})$, we have the identity

$$\langle J_{\text{max}}u, v \rangle - \langle u, J_{\text{max}}v \rangle = 2i \alpha^{-1}_\infty \sqrt{1 - \beta^2_\infty} (s_+(u)s_+(v) + s_-(u)s_-(v)).$$

(7.41)

Proof. It follows from formula (7.36) that

$$\sqrt{a_{n_1}a_n(u_{n+1}v_n - u_nv_{n+1})}$$

$$= (s_+(u)\Theta^{(1)}_n + s_-(u)\Theta^{(-1)}_n)(s_+(v)\Theta^{(-1)}_n + s_-(v)\Theta^{(1)}_n)$$

$$- (s_+(u)\Theta^{(1)}_n + s_-(u)\Theta^{(-1)}_n)(s_+(v)\Theta^{(-1)}_n + s_-(v)\Theta^{(1)}_n) + o(1)$$

$$= (s_+(u)s_+(v) - s_-(u)s_-(v))(e^{i \arccos \beta_n} - e^{-i \arccos \beta_n}) + o(1).$$

Passing here to the limit $n \to \infty$ and using equality (7.22), we obtain identity (7.41).

We can now characterize set (7.34).

Proposition 7.13. A vector $v \in \mathcal{D}(J_{\text{max}})$ belongs to $\mathcal{D}$ if and only if $v_n = o(a^{-1/2}_n)$, that is,

$$s_+(v) = s_-(v) = 0.$$  

(7.42)

Proof. A vector $v$ belongs to $\mathcal{D}(J^*_{\text{max}})$ if and only if

$$\langle J_{\text{max}}u, v \rangle = \langle u, J_{\text{max}}v \rangle$$

(7.43)

for all $u \in \mathcal{D}(J_{\text{max}})$. According to Proposition 7.12, equality (7.43) is equivalent to

$$s_+(u)s_+(v) - s_-(u)s_-(v) = 0.$$  

(7.44)
This is of course true if (7.42) is satisfied. Conversely, if (7.44) is satisfied for all \( u \in \mathcal{D}(J_{\max}) \), we use that according to Theorem 7.11 the numbers \( s_+(u) \) and \( s_-(u) \) are arbitrary. This implies (7.42).

This result shows that (7.40) considered as a mapping of the factor space \( \mathcal{D}(J_{\max})/\mathcal{D} \) onto \( \mathbb{C}^2 \) is injective.

All self-adjoint extensions \( J_\omega \) of the operator \( J_{\min} \) are now parametrized by complex numbers \( \omega \in \mathbb{T} \subset \mathbb{C} \). Let the set \( \mathcal{D}(J_\omega) \subset \mathcal{D}(J_{\max}) \) of vectors \( u \) be distinguished by the condition

\[
s_+(u) = \omega s_-(u), \quad |\omega| = 1.
\]

(7.45)

**Theorem 7.14.** Let the assumptions of Theorem 4.2 be satisfied. Then for all \( \omega \in \mathbb{T} \), the operators \( J_\omega \) are self-adjoint. Conversely, every operator \( J \) such that

\[
J_{\min} \subset J = J^* \subset J_{\max}
\]

(7.46)
equals \( J_\omega \) for some \( \omega \in \mathbb{T} \).

**Proof.** We proceed from Proposition 7.12. If \( u, v \in \mathcal{D}(J_\omega) \), it follows from condition (7.45) that \( s_+(u)s_+(v) = s_-(u)s_-(v) \). Therefore according to equality (7.44) \( \langle J_\omega u, v \rangle = \langle u, J_\omega v \rangle \) whence \( J_\omega \subset J_\omega^* \). If \( v \in \mathcal{D}(J_\omega) \), then \( \langle J_\omega u, v \rangle = \langle u, J_\omega^* v \rangle \) for all \( u \in \mathcal{D}(J_\omega) \) so that in view of (7.41) equality (7.44) is satisfied. Therefore \( s_-(u)(\omega s_+(v) - s_-(v)) = 0 \). Since \( s_-(u) \) is arbitrary, we see that \( \omega s_+(v) - s_-(v) = 0 \), and hence \( v \in \mathcal{D}(J_\omega) \).

Suppose that an operator \( J \) satisfies conditions (7.46). Since \( J \) is symmetric, it follows from Proposition 7.12 that equality (7.44) is true for all \( u, v \in \mathcal{D}(J) \). Setting here \( u = v \), we see that \( |s_+(v)| = |s_-(v)| \). There exists a vector \( v_0 \in \mathcal{D}(J) \) such that \( s_-(v_0) \neq 0 \) because \( J \neq \text{clos} J_{\min} \). Let us set \( \omega = s_+(v_0)/s_-(v_0) \). Then \( |\omega| = 1 \) and relation (7.45) is a direct consequence of (7.44).

\[
(7.47)
\]

7.5. **Resolvent.** Now it easy to construct the resolvent of the operator \( J_\omega \) defined in the previous subsection. We previously note that, by definition (7.36),

\[
s_\pm(P(z)) = \sigma_\pm(z) \quad \text{and} \quad s_\pm(Q(z)) = \tau_\pm(z).
\]

(7.48)

**Theorem 7.15.** Let the assumptions of Theorem 4.2 be satisfied. Then for all \( z \in \mathbb{C} \) with \( \text{Im} z \neq 0 \) and all \( h \in \ell^2(\mathbb{Z}_+) \), the resolvent \( R_\omega(z) = (J_\omega - zI)^{-1} \) of the operator \( J_\omega \) is given by the equality

\[
R_\omega(z)h = \gamma_\omega(z)\langle h, P(z) \rangle P(z) + R(z)h
\]

(7.47)

where

\[
\gamma_\omega(z) = \frac{-\tau_+(z) - \omega \tau_-(z)}{\sigma_+(z) - \omega \sigma_-(z)}
\]

(7.48)

and \( \sigma_\pm(z) \), \( \tau_\pm(z) \) are entire functions of \( z \) defined by relations (7.12), (7.13).
Proof. According to Corollary 7.9 the vector \( u = R_\omega(z)h \) is given by formula (7.32) where the coefficient \( \Gamma \) is determined by condition (7.45). It follows from Theorem 7.11 that the components \( u_n \) of \( u \) have asymptotics (7.36) with the coefficients \( s_\pm \) defined by relations (7.39). Thus, \( u \in D(J_\omega) \) if and only if
\[
\Gamma \sigma_+(z) + \tau_+(z) \langle h, P(\bar{z}) \rangle = \omega (\Gamma \sigma_-(z) + \tau_-(z) \langle h, P(\bar{z}) \rangle)
\]
whence
\[
\Gamma = -\frac{\tau_+(z) - \omega \tau_-(z)}{\sigma_+(z) - \omega \sigma_-(z)} \langle h, P(\bar{z}) \rangle.
\]
Substituting this expression into (7.32), we arrive at formulas (7.47), (7.48). □

We emphasize that, for various \( \omega \in \mathbb{T} \), the resolvents \( R_\omega(z) \) of the operators \( J_\omega \) differ from each other only by the coefficient \( \gamma_\omega(z) \) at the rank-one operator \( \langle \cdot, P(\bar{z}) \rangle P(z) \). Observe also that \( \gamma_\omega(z) = \gamma_\omega(\bar{z}) \).

Since \( \langle R(z)e_0, e_0 \rangle = 0 \), we see that \( \langle R_\omega(z)e_0, e_0 \rangle = \gamma_\omega(z) \). Thus, Theorem 7.15 implies the classical Nevanlinna representation obtained in [51] for the Cauchy-Stieltjes transform of the spectral measures \( d\rho_\omega(\lambda) = d(E_\omega(\lambda)e_0, e_0) \) of the operators \( J_\omega \).

**Corollary 7.16.** For all \( z \in \mathbb{C} \) with \( \text{Im} \, z \neq 0 \), we have
\[
\int_{-\infty}^{\infty} (\lambda - z)^{-1} d\rho_\omega(\lambda) = \gamma_\omega(z).
\]

**Corollary 7.17.** If \( z \in \mathbb{C} \) is not an eigenvalue of the operator \( J_\omega \), then its resolvent \( R_\omega(z) \) is in the Hilbert-Schmidt class. Thus, the spectra of the operators \( J_\omega \) are discrete and consist of the points \( z \) where
\[
\sigma_+(z) - \omega \sigma_-(z) = 0.
\]

Note also that according to (7.11) \( \sigma_+(z) \neq \sigma_-(z) \) if \( \text{Im} \, z \neq 0 \), and therefore the zeros \( z \) of equation (7.49) lie on the real axis. This result has of course to be expected since \( z \) are eigenvalues of the self-adjoint operator \( J_\omega \). We finally note that the discreteness of the spectrum of the operators \( J_\omega \) is quite natural because their domains \( D(J_\omega) \) are distinguished by boundary conditions at the point \( n = 0 \) and for \( n \to \infty \). Therefore \( J_\omega \) acquire some features of regular operators.

8. Jacobi operators with stabilizing coefficients

We here study the case of stabilizing recurrence coefficients satisfying condition (1.28). Then the Jacobi operator \( J \) differs by a compact term from the operator with the coefficients \( a_n = a_\infty > 0 \), \( b_n = b_\infty \) for all \( n \in \mathbb{Z}_+ \). Under additional assumptions (1.19) or (1.20) the corresponding orthogonal polynomials are rather close to Jacobi polynomials (see Appendix B). We follow the scheme already presented for the case of increasing coefficients satisfying condition (1.26) where \( |\beta_\infty| < 1 \) and (1.27). Since the case of stabilizing recurrence coefficients was investigated in [74], some technical details will be omitted.
8.1. **Compact perturbations.** As a preliminary remark, we note that, without a loss of generality, we may suppose that $a_\infty = 1/2$ and $b_\infty = 0$ in (1.28), that is, condition (1.18) is satisfied. Indeed, let $\tilde{J}$ be the Jacobi operator with off-diagonal elements $\tilde{a}_n = (2a_\infty)^{-1}a_n$ and diagonal elements $\tilde{b}_n = (2a_\infty)^{-1}(b_n - b_\infty)$. Then

$$J = 2a_\infty \tilde{J} + b_\infty I$$

(8.1)

and $\tilde{a}_n \to 1/2$, $\tilde{b}_n \to 0$ as $n \to \infty$ according to (1.28). In terms of the orthonormal polynomials, relation (8.1) means that

$$\tilde{P}_n(z) = P_n(2a_\infty z + b_\infty).$$

Let $J_0$ be the “free” Jacobi operator. It has the coefficients $a_n = 1/2$, $b_n = 0$ for all $n \in \mathbb{Z}_+$. Under assumption (1.18) the operator $J - J_0$ is compact so that the essential spectrum of $J$ coincides with the interval $[-1, 1]$. Note that (1.18) corresponds to the assumptions $a(x) \to 1/2$, $b(x) \to 0$ as $x \to \infty$ for the differential operator (1.23). We also require conditions (1.19) or (1.20) corresponding, respectively, to “short-range” and “long-range” perturbations of the “free” differential operator $D^2$.

Our notation is very close to that used in the case of increasing coefficients. Recall that $\Pi = \mathbb{C} \setminus \mathbb{R}$. As before, we suppose that the spectral parameter $z$ belongs to a bounded subset of clos $\Pi$, but we now additionally assume that $z \neq \pm 1$. For $z \in \text{clos} \Pi$, we set

$$z_n = \frac{z - b_n}{2a_n}$$

(8.2)

which is now slightly more convenient than (2.20). According to (1.18) we have

$$\lim_{n \to \infty} z_n = z.$$  

(8.3)

It follows that, for sufficiently large $N_0$, the set of points $z_n$ where $n \geq N_0$ is separated from the points 1 and $-1$. Relations (8.3) and (2.21) are quite different but play similar roles in our presentation.

We start with the general case when the long-range condition (1.20) is satisfied. Short-range perturbations will be specially discussed in Sect. 9.

8.2. **Jost solutions and orthogonal polynomials.** Here we suppose that conditions (1.18) and (1.20) are satisfied. The first step is to distinguish the Jost solutions $f_n(z)$ of the Jacobi equation (1.16) by their asymptotic behavior for $n \to \infty$. The corresponding Ansatz $A_n(z)$ can be again defined by formula (2.25), but we will omit the factor $a_n^{-1/2}$ since it tends to a constant. We always suppose that the values of the spectral parameter $z$ are separated from the points $\pm 1$ and fix $N_0$ in such a way that estimate (3.2) holds true for all $n \geq N_0$. Now we set

$$A_n(z) = \zeta_{N_0} \zeta_{N_0+1} \cdots \zeta_{n-1} \text{ where } \zeta_m = \zeta(z_m),$$
so that $A_n(z)$ coincides with the sequence $q_n(z)$ defined by equality (2.27). Then the remainder equals

$$
    r_n(z) := A_n(z)^{-1} \left( a_{n-1} A_{n-1}(z) + (b_n - z) A_n(z) + a_n A_{n+1}(z) \right)
$$

$$
    = a_{n-1} \zeta_{n-1}^{-1} + a_n \zeta_n + b_n - z
$$

$$
    = (a_{n-1} - a_n) \zeta_{n-1}^{-1} + a_n (\zeta_{n-1}^{-1} - \zeta_n^{-1}).
$$

(8.4)

The following statement plays the role of Lemma 3.4.

**Lemma 8.1.** For $z \in \text{clos} \Pi \setminus \{-1, 1\}$, an estimate

$$
    |r_n(z)| \leq C_n |a_n - a_{n-1}| + |b_n - b_{n-1}| \sqrt{z_n^{2} - 1 + \sqrt{z_n^{2} - 1}}
$$

holds. In particular, $(r_n(z)) \in \ell^1(\mathbb{Z}_+)$ if assumptions (1.18) and (1.20) are satisfied.

**Remark 8.2.** The crucial difference between the cases of stabilizing and increasing recurrence coefficients is that $a_n a_{n-1}^{-1} - 1 \in \ell^1(\mathbb{Z}_+)$ in the first case while $a_n a_{n-1}^{-1} = 1 + pn^{-1} + O(n^{-2})$ if, for example, $a_n = (n + 1)^p$ for some $p > 0$. In particular, this is the reason why we had to introduce the factor $a_n^{-1/2}$ in Ansatz (2.25). Indeed, without this factor in the case of increasing coefficients $a_n$, remainder (2.26) differs from (3.5) by a term of order $n^{-1}$ and hence does not belong to $\ell^1(\mathbb{Z}_+)$. Making again the multiplicative change of variables

$$
    f_n(z) = A_n(z) u_n(z) \quad \text{where} \quad A_n(z) = q_n(z),
$$

we see that equation (1.16) for $f_n(z)$ is equivalent to the equation

$$
    a_n \zeta_n (u_{n+1}(z) - u_n(z)) - a_{n-1} \zeta_{n-1}^{-1} (u_n(z) - u_{n-1}(z)) = -r_n(z) u_n(z), \quad n \in \mathbb{Z}_+,
$$

(8.6)

for the sequence $u_n(z)$. This equation is quite similar to (3.8) and plays the same role. The condition

$$
    f_n(z) = q_n(z) (1 + o(1)), \quad n \to \infty,
$$

(8.7)

means of course that $u_n(z) \to 1$ as $n \to \infty$.

Next, we reduce the difference equation (8.6) for $u_n(z)$ with this condition to a “Volterra integral” equation (3.10) with the kernel

$$
    G_{n,m}(z) = q_m(z)^2 \sum_{p=n}^{m-1} (a_p \zeta_p)^{-1} q_p(z)^{-2}, \quad n, m \in \mathbb{Z}_+, \quad m \geq n + 1.
$$

(8.8)

Note that $G_{n,m}(z) = \overline{G_{n,m}(z)}$. The functions $G_{n,m}(z)$ are analytic in $z \in \Pi$ and are continuous up to the real axis.

The following assertion plays the crucial role in our analysis of equation (3.10), in particular, for $z$ lying on the cut along $(-1, 1)$. It shows that kernels (8.8) are bounded uniformly in $n$ and $m$ provided some neighborhoods of the points $\pm 1$ are excluded.
Lemma 8.3. There exist constants $C(z)$ and $N(z)$ such that an estimate
\[ |G_{n,m}(z)| \leq C(z) < \infty, \quad m - 1 \geq n \geq N(z), \]
is true for all $z \in \text{clos} \Pi \setminus \{-1, 1\}$. The constants $C(z)$ and $N(z)$ are common for $z$ in compact subsets of $\text{clos} \Pi \setminus \{-1, 1\}$, that is, for all $z \in \text{clos} \Pi$ such that $|z^2 - 1| \geq \epsilon$ and $|z| \leq \rho$ where $\epsilon > 0$ and $\rho < \infty$ are some fixed numbers.

Lemmas 8.1 and 8.3 allow us to solve the Volterra equation (3.10) with the coefficients (8.4), (8.8) by iterations. Similarly to Sect. 3.4, we set $u^{(0)}_n(z) = 1$ and define $u^{(k)}_n(z)$ recursively by relation (3.27). Then estimates (3.28) hold true and a solution $u_n$ of (3.10) is built as series (3.30). The following result is a direct consequence of Theorem 8.5.

Theorem 8.4. Let assumptions (1.18) and (1.20) be satisfied. For $z \in \text{clos} \Pi \setminus \{-1, 1\}$, equation (3.10) has a (unique) bounded solution $\{u_n(z)\}_{n=0}^{\infty}$. This sequence obeys an estimate
\[ |u_n(z) - 1| \leq C \sum_{m=n}^{\infty} (|a'_m| + |b'_m|) \]
where the constant $C$ is common for $z$ in compact subsets of $\text{clos} \Pi \setminus \{-1, 1\}$. For all $n \in \mathbb{Z}_+$, the functions $u_n(z)$ are analytic in $z \in \Pi$ and are continuous up to the cut along $\mathbb{R}$ with possible exception of the points $z = -1$ and $z = 1$.

Similarly to Lemma 4.1 we check that $u_n(z)$ satisfies also the difference equation (8.6). Then we define $f_n(z)$ by formula (8.5). Since equations (8.6) for $u_n(z)$ and (1.16) for $f_n(z)$ are equivalent, $f_n(z)$ satisfies the Jacobi equation (1.16). Obviously, estimate (8.9) on $u_n(z)$ implies asymptotics (8.7) of $f_n(z)$. Thus we arrive at the following result.

Theorem 8.5. Let assumptions (1.18) and (1.20) be satisfied, let $z \in \text{clos} \Pi \setminus \{-1, 1\}$, and let $q_n(z)$ be defined by equality (2.27). Denote by $u_n(z)$ the sequence constructed in Theorem 8.4. Then the sequence $f_n(z)$ defined by equality (8.5) satisfies equation (1.10), and it has asymptotics (8.7). For all $n \in \mathbb{Z}_+$, the functions $f_n(z)$ are analytic in $z \in \Pi$ and are continuous up to the cut along $\mathbb{R}$ with possible exception of the points $z = -1$ and $z = 1$. Relation (3.6) is satisfied. Asymptotics (8.7) is uniform in $z$ from compact subsets of the set $\text{clos} \Pi \setminus \{-1, 1\}$.

Recall that the polynomials $P_n(z)$ are solutions of equation (1.1) satisfying conditions (1.2). Put $P(z) = (P_n(z))_{n=-1}^{\infty}$, $f(z) = (f_n(z))_{n=-1}^{\infty}$. As before, the sequence $(f_n(z))_{n=-1}^{\infty}$ will be called the Jost solution of equation (1.16), and the Jost function $\Omega(z)$ is defined as Wronskian (2.9). For the operator $J_0$, the Jost solution is $(\zeta(z)^n)_{n=-1}^{\infty}$ and the corresponding Wronskian
\[ \Omega_0(z) = (2\zeta(z))^{-1}. \]

The following result is a direct consequence of Theorem 8.5.
Corollary 8.6. The Wronskian \( \Omega(z) \) depends analytically on \( z \in \Pi \), and it is a continuous function of \( z \) up to the cut along \( \mathbb{R} \) except, possibly, the points \( \pm 1 \).

Remark 8.7. Suppose that \( z \in (-\infty, -1) \cup (1, \infty) \). Then according to (8.3) we also have \( z_n \in (-\infty, -1) \cup (1, \infty) \) if \( n \) is sufficiently large. Since the function \( \zeta(z) \) is real for \( z \in (-\infty, -1) \cup (1, \infty) \), it follows that the numbers \( \zeta(z_n) \), \( q(z_n) \) and hence \( f_n(z) \) are real. Using now relation (3.6), we see that values of \( f_n(z) \) on the upper and lower edges of the cut along \((-\infty, -1) \cup (1, \infty)\) coincide. Therefore the functions \( f_n(z) \) are actually analytic on the whole set \( \Pi_0 = \mathbb{C} \setminus [-1, 1] \).

An asymptotics as \( n \to \infty \) of product (2.27) can be found rather explicitly.

Lemma 8.8. Let assumption (1.18) be satisfied, let \( z \in \text{clos} \Pi \setminus \{-1, 1\} \), and let \( \zeta = \zeta(z) \) be given by equality (2.18). Then
\[
q_n(z) = e^{n(\ln \zeta + o(1))}, \quad n \to \infty.
\] (8.11)

Proof. Let the sequence \( z_n \) be defined by formula (8.2) and \( \zeta_n = \zeta(z_n) \). It follows from (1.18) that \( z_n = z + o(1) \) and hence \( \zeta_n = \zeta(1 + \epsilon_n) \) where \( \epsilon_n \to 0 \) as \( n \to \infty \). For product (2.27), this yields
\[
\ln q_n = n \ln \zeta + \sum_{m=N_0}^{n-1} \ln(1 + \epsilon_m) = n \ln \zeta + o(n)
\]
which is equivalent to (8.11).

Asymptotics in the complex plane. Supposing that \( z \notin \mathbb{C} \setminus [-1, 1] \), we introduce a solution \( g_n(z) \) of equation (1.16) exponentially growing as \( n \to \infty \). A proof of the following result is similar to that of Theorem 6.8. Details can be found in [74].

Theorem 8.9. Let \( z \in \mathbb{C} \setminus [-1, 1] \), and let assumptions (1.18) and (1.20) be satisfied. If \( f_n(z) \) is the Jost solution of equation (1.16), then the solution \( g_n(z) \) of the same equation defined by (2.15), (2.16) satisfies a relation
\[
\lim_{n \to \infty} q_n(z)g_n(z) = \frac{1}{\sqrt{z^2 - 1}}.
\]

Now it is easy to find asymptotics of the orthogonal polynomials \( P_n(z) \) for \( z \in \mathbb{C} \setminus [-1, 1] \). By Theorem 2.2, the Wronskian \( \{f(z), g(z)\} \) of \( f(z) = (f_n(z)) \) and \( g(z) = (g_n(z)) \) equals 1, whence these solutions of equation (1.16) are linearly independent. This yields relation (2.33) where \( \Omega(z) \) is given by (2.6) and \( \omega(z) = \{P(z), g(z)\} \). Obviously, \( \omega(z) \neq 0 \) if \( \Omega(z) = 0 \), that is, \( z \) is an eigenvalue of the operator \( J \). Therefore Theorems 8.5 and 8.9 imply the following result (cf. Theorems 6.6 and 6.11).
Theorem 8.10. Under assumptions \((1.18)\) and \((1.20)\) the relation
\[
\lim_{n \to \infty} q_n(z)P_n(z) = -\left\{ \frac{P(z), f(z)}{\sqrt{z^2 - 1}} \right\}
\]  
(8.12)
is true for all \(z \in \mathbb{C} \setminus [-1, 1]\) with convergence uniform on compact subsets of \(z \in \mathbb{C} \setminus [-1, 1]\). Moreover, if \(\Omega(z) = 0\), then
\[
\lim_{n \to \infty} q_n(z)^{-1}P_n(z) = \{ P(z), g(z) \} \neq 0.
\]  
(8.13)

Asymptotics on the continuous spectrum. Suppose that \(z = \lambda \pm i0\) where \(\lambda \in (-1, 1)\) so that \(\lambda = \cos \theta, \theta \in (0, \pi)\). Now relations \((5.1)\) and \((5.3)\) remain true with the numbers \(\lambda_n\) given by
\[
\lambda_n := \frac{\lambda - b_n}{2a_n}.
\]  
(8.14)

The following result is a direct consequence of Theorem 8.5. It plays the role of Theorem 5.1 stated for the case of increasing coefficients.

Theorem 8.11. Let assumptions \((1.18)\) and \((1.20)\) be satisfied. For \(\lambda \in (-1, 1)\), define the phases \(\varphi_n(\lambda)\) by formula \((5.3)\) where \(\lambda_n\) are numbers \((8.14)\). Then
\[
f_n(\lambda \pm i0) = e^{\mp i\varphi_n(\lambda)}(1 + o(1)), \quad n \to \infty.
\]  
(8.15)

Note that
\[
\varphi_n(\lambda) = n \arccos \lambda + o(n),
\]
but under additional assumptions the error term can be made more explicit. In particular, we see that asymptotics \((8.15)\) of \(f_n(\lambda \pm i0)\) is oscillating as \(n \to \infty\).

To find asymptotic behavior of the polynomials \(P_n(\lambda)\) for \(\lambda \in (-1, 1)\), that is, on the continuous spectrum of the Jacobi operator \(J\), we have to consider two complex conjugate Jost solutions \(f(\lambda \pm i0) = (f_n(\lambda \pm i0))\) for \(\lambda \in (-1, 1)\). Similarly to Lemma 5.2, we have

Lemma 8.12. The Wronskian \((2.1)\) of \(f(\lambda + i0)\) and \(f(\lambda - i0)\) equals
\[
\{ f(\lambda + i0), f(\lambda - i0) \} = i\sqrt{1 - \lambda^2} \neq 0, \quad \lambda \in (-1, 1),
\]  
(8.16)
and hence these solutions are linearly independent.

Note that compared to \((5.6)\), \(\sqrt{1 - \beta_\infty^2}\) is replaced by \(\sqrt{1 - \lambda^2}\) because the role of relation \((2.21)\) is now played by \((8.3)\). Besides, the coefficient 2 has disappeared in \((8.16)\) because \((8.15)\) does not contain an amplitude factor.

Lemma 8.12 implies the following two results (cf. Lemma 5.3 and Theorem 5.4).

Lemma 8.13. For \(\lambda \in (-1, 1)\), the representation
\[
P_n(\lambda) = \frac{\Omega(\lambda - i0)f_n(\lambda + i0) - \Omega(\lambda + i0)f_n(\lambda - i0)}{i\sqrt{1 - \lambda^2}}, \quad n = 0, 1, 2, \ldots
\]  
(8.17)
holds true.
Theorem 8.14. The Wronskians \( \Omega(\lambda + i0) \) and \( \Omega(\lambda - i0) = \Omega(\lambda + i0) \) are continuous functions of \( \lambda \in (-1, 1) \) and
\[
\Omega(\lambda \pm i0) \neq 0, \quad \lambda \in (-1, 1).
\] (8.18)

Let us define the functions \( \kappa(\lambda) \) and \( \eta(\lambda) \) by equalities (5.9). In the theory of short-range perturbations of the Schrödinger operator, these functions are known as the limit amplitude and the limit phase, respectively; the function \( \eta(\lambda) \) is also called the scattering phase or the phase shift.

Combined together, relations (8.15) and (8.17) yield the asymptotics of Bernstein-Szegő type for the polynomials \( P_n(\lambda) \) (cf. Theorem 5.5).

Theorem 8.15. Let assumptions (1.18) and (1.20) be satisfied, let \( \lambda \in (-1, 1) \) and let the phase \( \varphi_n(\lambda) \) be defined by formulas (5.3) and (8.14). Then the polynomials \( P_n(\lambda) \) have asymptotics
\[
P_n(\lambda) = \frac{2\kappa(\lambda)}{\sqrt{1 - \lambda^2}} \sin(\varphi_n(\lambda) + \eta(\lambda)) + o(1) \quad \text{as } n \to \infty.
\] (8.19)
Relation (8.19) is uniform in \( \lambda \) on compact subintervals of \((-1, 1)\).

Note that Theorem 8.15 does not follow from Theorem 8.10 because asymptotics (8.12) is not uniform as \( z \) approaches the cut along \((-1, 1)\).

Asymptotic formulas (8.12) and (8.19) are the classical results of the Bernstein-Szegő theory. They are stated as Theorems 12.1.2 and 12.1.4 in the book [63] where the assumptions are imposed on the spectral measure \( d\rho(\lambda) \); in particular, it is assumed that \( \text{supp} \rho \subset [-1, 1] \). Under assumptions (1.18) and (1.20) on recurrent coefficients \( a_n, b_n \) asymptotic formulas for orthonormal polynomials were established in the paper [44]. We followed here the presentation of [74].
are true with some positive constants that do not depend on \( n, m \) and on \( z \) in compact subsets of the closure of the set \( \mathbb{C} \setminus [-1, 1] \) as long as they are away from the points \( \pm 1 \).

The statement (ii) is known as the limiting absorption principle. It implies that the spectrum of the operator \( J \) on the interval \((-1, 1)\) is absolutely continuous. Using the Cauchy-Stieltjes-Privalov formula (5.12), we also see that matrix elements \( \langle E(\lambda)e_n, e_m \rangle \) of the spectral projector \( E(\lambda) \) of the operator \( J \) are continuously differentiable in \( \lambda \in (-1, 1) \).

Note that the points 1 and \(-1\) may be eigenvalues of \( J \); see Example 4.15 in [72].

A calculation of the spectral family \( dE(\lambda) \) of the operator \( J \) is quite similar to the case of increasing coefficients \( a_n \) so that we again have representation (5.14). Combining this representation with formula (8.17) for \( P_m(\lambda) \), we obtain the following result. We recall that the spectral measure \( d\rho(\lambda) \) of a Jacobi operator \( J \) is defined by relation (1.12); if this measure is absolutely continuous, we define the weight \( \tau(\lambda) \) by equality (1.17).

**Theorem 8.17.** Let assumptions (1.18) and (1.20) hold. Then, for all \( n, m \in \mathbb{Z}_+ \) and \( \lambda \in (-1, 1) \), we have the representation

\[
\frac{\langle E(\lambda)e_n, e_m \rangle}{d\lambda} = (2\pi)^{-1}\sqrt{1-\lambda^2}\Omega(\lambda \pm i0)|^{-2}P_n(\lambda)P_m(\lambda).
\]

(8.20)

In particular, the spectral measure of the operator \( J \) is absolutely continuous on the interval \((-1, 1)\) and the corresponding weight equals

\[
\tau(\lambda) = (2\pi)^{-1}\sqrt{1-\lambda^2}\Omega(\lambda \pm i0)|^{-2}
\]

(8.21)

(the right-hand sides here do not depend on the sign).

Putting together Theorem 8.14 and formula (8.21), we arrive at the next result.

**Theorem 8.18.** Under assumptions (1.18) and (1.20) the weight \( \tau(\lambda) \) is a continuous strictly positive function of \( \lambda \in (-1, 1) \).

Note that this result was earlier obtained in [43] by specific methods of the orthogonal polynomials theory.

Theorems 8.17 and 8.18 are of course quite similar to Theorems 5.10 and 5.11 for the case \( a_n \to \infty \). The difference is that now the factor \( \sqrt{1-\beta^2_{\infty}} \) is replaced by \( \sqrt{1-\lambda^2} \) and we have the restriction \( \lambda \in (-1, 1) \).

According to (8.10) for the operator \( J_0 \), we have

\[
\Omega_0(\lambda \pm i0) = -2^{-1}(\lambda \pm i\sqrt{1-\lambda^2}),
\]

and hence expression (8.21) reduces to (B.3).

In view of (8.21) the amplitude factor in (8.19) equals

\[
\kappa(\lambda) = (2\pi)^{-1/2}(1-\lambda^2)^{1/4}\tau(\lambda)^{-1/2}.
\]

(8.22)
Substituting this expression into (8.19), we can reformulate Theorem 8.15 in a form more common for the orthogonal polynomials literature.

**Theorem 8.19.** Under the assumptions of Theorem 8.15 the polynomials $P_n(\lambda)$ have asymptotics

$$P_n(\lambda) = \left(\frac{2}{\pi}\right)^{1/2}(1 - \lambda^2)^{-1/4}\tau(\lambda)^{-1/2}\sin(\varphi_n(\lambda) + \eta(\lambda)) + o(1)$$

as $n \to \infty$. Relation (8.23) is uniform in $\lambda$ on compact subintervals of $(-1, 1)$.

8.4. **Discussion.** As was already mentioned in Sect. 1.3, under assumptions (1.18), (1.20) asymptotic formulas (8.12) and (8.19) for the orthonormal polynomials were first obtained in paper [44]. However, expressions for the coefficients in the right-hand sides were not, at least in the author’s opinion, very efficient. It was conjectured in [44] that the asymptotic coefficient in (8.19) can be obtained from that in (8.12) as the limit on $(-1, 1)$ from complex values of $z$. This conjecture was later justified in [26]. In our approach this problem does not even arise since both coefficients are expressed in terms of the Wronskian $\{P(z), f(z)\}$ of the polynomial and Jost solutions of the Jacobi equation (1.16).

As far as spectral results are concerned, we note that a large part of Theorem 8.16 can also be obtained by the Mourre method [45]. It was applied to Jacobi operators in [8]; to be precise, the problem in the space $\ell^2(\mathbb{Z})$ was considered in [8], but this is of no importance. However, the Mourre method does not exclude eigenvalues of $J$ embedded in its continuous spectrum. It only shows that these eigenvalues do not have other points of accumulation except 1 and $-1$. The Mourre method applies also to some Jacobi operators with increasing coefficients; see [57]. Note that very general conditions of the absolute continuity of spectrum were obtained in [61] by the subordinacy method of [27].

We also note papers [29, 52] where Bernstein-Szegő results (see Sect. 1.3) were extended to measures with a finite number of point masses away from the interval $[-1, 1]$.

8.5. **Hilbert-Schmidt perturbations.** In addition to (1.20), assume now that the condition

$$\sum_{n=0}^\infty (v_n^2 + b_n^2) < \infty, \quad v_n = a_n - 1/2,$$

is satisfied, that is, $V = J - J_0$ is a Hilbert-Schmidt operator. Then the asymptotic formulas of Theorems 8.10 and 8.15 can be made more explicit. We proceed from the following elementary assertion.

**Lemma 8.20** ([74], Lemma 4.8). Let $z \neq \pm 1$. Under assumption (8.24) there exists a finite limit

$$\lim_{n \to \infty} \left(\zeta(z)^{-n} \exp\left(-\frac{1}{\sqrt{z^2 - 1}} \sum_{m=0}^{n-1} (2zv_m + b_m)\right) q_n(z)\right) \neq 0.$$
Thus, the next statement is a direct consequence of Theorem 8.10.

**Theorem 8.21.** Let assumptions (1.20) and (8.24) be satisfied, and let \( z \in \mathbb{C} \setminus [-1, 1] \). Then there exist finite limits

\[
\lim_{n \to \infty} \left( \zeta(z)^n \exp \left( \frac{1}{\sqrt{z^2 - 1}} \sum_{m=0}^{n-1} (2zv_m + b_m) \right) P_n(z) \right) \neq 0 \quad (8.25)
\]

if \( z \) is not an eigenvalue of the operator \( J \) and

\[
\lim_{n \to \infty} \left( \zeta(z)^{-n} \exp \left( - \frac{1}{\sqrt{z^2 - 1}} \sum_{m=0}^{n-1} (2zv_m + b_m) \right) P_n(z) \right) \neq 0 \quad (8.26)
\]

if \( z \) is an eigenvalue of \( J \).

**Corollary 8.22.** Suppose additionally that the conditions

\[
\sum_{n=0}^{\infty} v_n < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} b_n < \infty \quad (8.27)
\]

(these series should be convergent but perhaps not absolutely) are satisfied. Then the exponential factors in (8.25) and (8.26) may be omitted.

In some cases the exponential factors in (8.25) and (8.26) can be simplified.

**Example 8.23.** Let the conditions

\[
a_n = 1/2 + vn^{-r_1} + \tilde{v}_n, \quad b_n = bn^{-r_2} + \tilde{b}_n \quad (8.28)
\]

be satisfied with some \( r_1, r_2 \in (1/2, 1) \). Then

\[
\sum_{m=0}^{n} (2zv_m + b_m) = 2zv(1 - r_1)^{-1}n^{1-r_1} + b(1 - r_2)^{-1}n^{1-r_2}
\]

\[
+ 2z\gamma_{r_1} + b\gamma_{r_2} + \sum_{m=0}^{\infty} (2z\tilde{v}_m + \tilde{b}_m) + o(1) \quad (8.29)
\]

where \( \gamma_r - (1-r)^{-1} \) is the Euler-Mascheroni constant. With a natural modification, expression (8.29) remains true if \( r_j = 1 \) for one or both \( j \). In this case \((1-r_j)^{-1}n^{1-r_j}\) should be replaced by \( \ln n \) and \( \gamma_1 \) is the Euler-Mascheroni constant.

Asymptotic formula (8.19) can also be simplified. Similarly to Lemma 8.20, we have

**Lemma 8.24** ([74], Lemma 4.12). Under assumption (8.24) there exists a finite limit

\[
\lim_{n \to \infty} \left( \varphi_n(\lambda) - n\theta - (\sin \theta)^{-1} \sum_{m=0}^{n-1} (2 \cos \theta v_m + b_m) \right) =: \gamma(\lambda)
\]

where \( \lambda = \cos \theta \in (-1, 1) \).
Thus, the following statement is a direct consequence of Theorem 8.13.

**Theorem 8.25.** Let assumptions (1.20) and (8.24) be satisfied. Then for $\lambda \in (-1, 1)$, the asymptotic formula

$$P_n(\lambda) = \frac{2\kappa(\lambda)}{\sqrt{1 - \lambda^2}} \sin \left(n\theta + (\sin \theta)^{-1} \sum_{m=0}^{n-1} (2\cos \theta v_m + b_m) + \gamma(\lambda) + \eta(\lambda)\right) + o(1)$$

holds as $n \to \infty$. Relation (8.30) is uniform in $\lambda$ on compact subintervals of $(-1, 1)$.

Under assumption (8.28) the phase in (8.30) can be simplified if one takes relation (8.29) where $z = \cos \theta$ into account.

Of course formulas (8.25) and (8.30) are consistent with asymptotic formulas for Pollaczek polynomials in the Appendix in the book [63].

8.6. **Related research.** Without any additional assumptions, Hilbert-Schmidt perturbations $V$ of the operator $J_0$ were investigated in the deep papers [36] and [15]. In [36], necessary and sufficient conditions in terms of the spectral measure $d\rho(\lambda)$ of the operator $J = J_0 + V$ were found for $V$ to be in the Hilbert-Schmidt class. Asymptotic behavior of the corresponding polynomials $P_n(z)$ was studied in [15]. It was proved in Theorem 5.1 of this paper that the limit of $\zeta(z)^n P_n(z)$ as $n \to \infty$ exists if and only if conditions (8.24) and (8.27) are satisfied. As shown in Theorem 8.1 of [14], assumptions (8.24), (8.27) are sufficient also for the validity of formula (8.30) but only in some averaged sense. Such a regularization seems to be necessary since under these assumptions the structure of the essential spectrum of the operator $J$ can be quite wild.

Condition (1.20) accepted in this paper is different in nature from (8.24), (8.27). On the one hand, it excludes too strong oscillations of the coefficients $v_n$, $b_n$ but, on the other hand, it permits their arbitrary slow decay as $n \to \infty$.

Weights $\tau(\lambda)$ with singularities or zeros inside $(-1, 1)$ were investigated in the papers [47], [67], [22]. In the first of them even weights behaving like $\kappa|\lambda|^\nu$ where $\nu > -1$, $\kappa > 0$ as $\lambda \to 0$ were considered. Such weights are either singular at the point $\lambda = 0$ if $\nu < 0$ or, on the contrary, $\tau(0) = 0$ if $\nu > 0$. It was shown in [47] that the corresponding Jacobi coefficients $a_n$ satisfy the asymptotic relation

$$a_n = 1/2 + (-1)^n \nu/(4n) + o(1/n)$$

(the coefficients $b_n = 0$ if the weight $\tau(\lambda)$ is even). Since $|a_n^\prime| \sim |\nu|/(2n)$, the condition (1.20) is now violated. For such weights, the asymptotic behavior of the polynomials $P_n(\lambda)$ in a neighborhood of the point $\lambda = 0$ differs from (8.19). More general results of this type were obtained in [67] where weights had several exceptional points. The results of [22] for weights with a jump singularity are morally similar.
Thus, condition (1.20) is practically necessary even for our results on the weight \( \tau(\lambda) \). We also note that asymptotics (8.19) obtained under assumption (1.20) is, in some sense, more regular than the asymptotics of \( P_n(\lambda) \) in [47, 67, 22].

9. Short-range perturbations

9.1. Asymptotic formulas. Under the short-range assumption (1.19) the constructions of the previous section can be made more explicit. Instead of (8.7) the Jost solutions are now distinguished by the asymptotics

\[ f_n(z) = \zeta(z)^n(1 + o(1)), \quad z \in \mathbb{C} \setminus [-1, 1], \]

as \( n \to \infty \). Vectors \( f(z) = (f_n(z)) \) are defined as solutions of the discrete Volterra integral equation

\[ f_n(z) = \zeta(z)^n - \frac{1}{\sqrt{z^2 - 1}} \sum_{m=n+1}^{\infty} (\zeta(z)^{n-m} - \zeta(z)^{m-n})(Vf(z))_m. \]

The perturbation \( V = J - J_0 \) acts on vectors \( u = (u_0, u_1, \ldots)^T =: (u_n) \) by the formula

\[ (Vu)_0 = b_0 u_0 + v_0 u_1, \quad (Vu)_n = v_{n-1} u_{n-1} + b_n u_n + v_n u_{n+1} \quad \text{for} \quad n \geq 1, \]

where \( v_n = a_n - 1/2 \). As usual, solutions of equation (9.2) are constructed by iterations. Note that in contrast to the general case there is now the canonical choice of the Ansatz \( A_n(z) = \zeta(z)^n \) which works for all \( z \in \text{clos } \Pi_0 \setminus \{-1, 1\} \) so that there is no need to use the local arguments of Sect. 8.2.

Let us state a relevant particular case of Theorem 8.5.

**Theorem 9.1.** Let assumption (1.19) be satisfied, and let \( z \in \text{clos } \Pi_0, \ z \neq \pm 1. \) Then equation (1.16) has a solution satisfying condition (9.1) as \( n \to \infty \). Every function \( f_n(z) \), \( n = -1, 0, 1, \ldots, \) depends analytically on \( z \in \Pi_0 \), and it is continuous in \( z \) up to the cut along \([-1, 1]\) except, possibly, the points \( \pm 1 \).

**Corollary 9.2.** If \( \lambda \in (-1, 1) \), then, as \( n \to \infty \),

\[ f_n(\lambda \pm i0) = e^{\mp in\theta}(1 + o(1)), \quad \theta = \arccos \lambda \in (0, \pi). \]

With this definition of the Jost solution, the Jost function \( \Omega(z) \) is, as before, defined as the Wronskian \( \Omega(z) = \{P(z), f(z)\} \). We note a representation for \( \Omega(z) \) in terms of the orthonormal polynomials:

\[ \Omega(z) = -\frac{1}{2\zeta(z)} + \sum_{n=0}^{\infty} \zeta(z)^n(VP(z))_n, \quad z \in \text{clos } \Pi, \quad z \neq \pm 1, \]

see Proposition 3.3 in [72] for the proof. A similar representation in the continuous case is quite standard; see, e.g., Lemma 1.12 in Chapter 4 of [71]. The Jost function \( \Omega_0(z) \) for the free operator \( J_0 \) is given by equality (8.10).
Relation (8.17) remains true, and a short-range version of Theorem 8.15 can be stated in the following way.

**Theorem 9.3.** Let the amplitude \( \kappa(\lambda) \) and the phase \( \eta(\lambda) \) be defined by formulas (5.9). Under assumption (1.19) the polynomials \( P_n(\lambda) \) have asymptotics

\[
P_n(\lambda) = \frac{2\kappa(\lambda)}{\sqrt{1 - \lambda^2}} \sin(n \arccos \lambda + \eta(\lambda)) + o(1), \quad \lambda \in (-1, 1),
\]

as \( n \to \infty \). Relation (9.4) is uniform in \( \lambda \) on compact subintervals of \((-1, 1)\).

In the case \( z \in \mathbb{C} \setminus [-1, 1] \) asymptotic formulas (8.12) and (8.13) for \( P_n(z) \) are true with \( q_n(z) \) replaced by \( \zeta(z)^n \).

Representations (5.13) for the resolvent and (8.20) for the spectral measure remain unchanged. We note also an equation for the orthonormal polynomials

\[
P_n(z) = P_n^{(0)}(z) + \frac{1}{\sqrt{z^2 - 1}} \sum_{m=0}^{n-1} (\zeta(z)^{n-m} - \zeta(z)^{m-n})(VP(z))_m, \quad n \geq 1,
\]

where

\[
P_n^{(0)}(z) = \frac{1}{2\sqrt{z^2 - 1}} (\zeta(z)^{-n-1} - \zeta(z)^{n+1})
\]

are the normalized Chebyshev polynomials of the second kind.

**9.2. The perturbation determinant and the spectral shift function.** First we recall abstract definitions of these notions for arbitrary bounded self-adjoint operators \( J_0 \) and \( J \) with a trace class difference \( V = J - J_0 \). We refer to the books [28, 59, 70] for a consistent presentation of this theory. In view of our applications, we suppose that the spectrum of the operator \( J_0 \) coincides with the interval \([-1, 1]\).

If \( V \in \mathcal{S}_1 \) (the trace class), then the perturbation determinant

\[
D(z) := \text{Det} \left( I + VR_0(z) \right)
\]

for the pair \( J_0, J \) is well defined and is an analytic function of \( z \in \mathbb{C} \setminus [-1, 1] \). Obviously, \( D(z) = \overline{D(z)} \) and

\[
D(z) \to 1 \quad \text{as} \quad |z| \to \infty.
\]

Note also the general formula

\[
\text{Tr} \left( R(z) - R_0(z) \right) = -\frac{D'(z)}{D(z)}.
\]

The Kreĭn spectral shift function \( \xi(\lambda) \) is defined in terms of the perturbation determinant (9.6). According to (9.7) we can fix the branch of the function \( \ln D(z) \) for \( \text{Im} \ z \neq 0 \) by the condition

\[
\arg D(z) \to 0 \quad \text{as} \quad |z| \to \infty.
\]
Then
\[ \xi(\lambda) := \pi^{-1} \lim_{\epsilon \to +0} \arg D(\lambda + i\epsilon). \] (9.9)

In the abstract setting, this limit exists for almost every \( \lambda \in \mathbb{R} \),

\[ \int_{-\infty}^{\infty} |\xi(\lambda)| d\lambda \leq \|V\|_1 \] (9.10)

and the representation

\[ \ln D(z) = \int_{-\infty}^{\infty} \xi(\lambda)(\lambda - z)^{-1} d\lambda, \quad \text{Im } z \neq 0, \] (9.11)

holds. The function \( \xi(\lambda) \) is constant on subintervals of \((-\infty, -1)\) and \((1, \infty)\) not containing eigenvalues of \( J \). In particular, \( \xi(\lambda) = 0 \) for \( \lambda \) below the smallest and above the largest eigenvalue of the operator \( J \). If \( \mu \) is an isolated simple eigenvalue of the operator \( J \), then

\[ \xi(\mu + 0) - \xi(\mu - 0) = -1. \] (9.12)

Let us come back to Jacobi operators \( J_0 \) and \( J \).

**Lemma 9.4.** The difference \( V = J - J_0 \) belongs to the trace class \( \mathcal{S}_1 \) if and only if assumption (1.19) is satisfied.

**Proof.** Let \( T \) be the shift in the space \( \ell^2(\mathbb{Z}_+) \) defined by \((Te)_n = e_{n+1}\) and let \( A = \text{diag}\{a_n\}, \ B = \text{diag}\{b_n\} \). Under assumption (1.19) the diagonal operators \( A - I/2 \) and \( B \) are trace class. Therefore the same is true for the operator \( V = A - I/2 + BT + BT^* \). Conversely, if \( V \in \mathcal{S}_1 \), then (see, e.g., Theorem 11.2.3 in the book [7])

\[ \sum_{n=0}^{\infty} |\langle Ve_n, e_n \rangle| + \sum_{n=0}^{\infty} |\langle Ve_n, e_{n+1} \rangle| < \infty. \]

Since \( \langle Ve_n, e_n \rangle = a_n - 1/2 \) and \( \langle Ve_n, e_{n+1} \rangle = b_n \), this proves (1.19). \( \square \)

Let us express the perturbation determinant and the spectral shift function in terms of the Jost function \( \Omega(z) \) and the phase \( \eta(\lambda) \). It is convenient to introduce the corresponding normalized objects by the relations

\[ \Omega(z) := \frac{\Omega(z)}{\Omega_0(z)} = -2\zeta(z)\Omega(z), \quad z \in \Pi_0, \]

and

\[ \eta(\lambda) := \eta(\lambda) - \arccos \lambda, \quad \lambda \in (-1, 1). \]

Using representation (5.13) for the resolvent, one can prove (see Theorem 5.4 in [72]) a relation

\[ \text{Tr} \left( R(z) - R_0(z) \right) = -\frac{\Omega'(z)}{\Omega(z)}. \]
Comparing it with formula (9.8), we see that
\[ D(z) = A \Omega(z), \quad z \in \Pi_0, \] (9.13)
for some constant \( A \in \mathbb{C} \). According to Theorem 5.6 in [72] the normalized Jost function has an asymptotics
\[ \Omega(z) = A^{-1} + O(|z|^{-1}), \quad z \to \infty, \] (9.14)
where
\[ A = \prod_{k=0}^{\infty} (2a_k). \] (9.15)
Note that under assumption (1.19) the infinite product here converges and \( A > 0 \).

Putting together asymptotic relations (9.7) and (9.14), we obtain the next result.

**Theorem 9.5.** Under assumption (1.19), equality (9.13) is true with constant (9.15).

It follows from Theorem 9.1 and formula (9.13) that the perturbation determinant \( D(z) \) is a continuous function of \( z \in \text{clos} \Pi_0 \) except, possibly, the points \( z = \pm 1 \). Moreover, according to (8.18), we have \( D(\lambda \pm i0) \neq 0 \). Therefore the spectral shift function \( \xi(\lambda) \) is a continuous function of \( \lambda \in (-1, 1) \).

Comparing definitions (5.9) and (9.9), we find a link between \( \xi(\lambda) \) and the scattering phase \( \eta(\theta) \).

**Theorem 9.6.** Under assumption (1.19), the relation
\[ \xi(\lambda) = \pi^{-1} \eta(\arccos \lambda) \] (9.16)
holds for all \( \lambda \in (-1, 1) \).

Substituting (9.16) into (2.39) and taking into account relation (8.22), we can reformulate Theorem 8.15 in terms of the weight \( \tau(\lambda) \) and the spectral shift function \( \xi(\lambda) \) (cf. (5.17)). This yields the asymptotic formula as \( n \to \infty \):
\[ P_n(\lambda) = (2/\pi)^{1/2}(1-\lambda^2)^{-1/4}\tau(\lambda)^{-1/2}\sin((n+1) \arccos \lambda + \pi \xi(\lambda)) + o(1), \quad \lambda \in (-1, 1), \]
We emphasize that \( \eta(\theta) \) is a continuous function of \( \theta \in (0, \pi) \), but Theorem 9.1 yields no information about its behavior as \( \theta \to 0 \) and \( \theta \to \pi \). Comparing relations (9.10) and (9.16), we however see that
\[ \int_0^\pi |\eta(\theta)| \sin \theta d\theta \leq \pi \| V \|_1. \]

**9.3. Threshold behavior.** A study of the perturbation determinant in the limits \( z \to 1 \) and \( z \to -1 \) is similar to the same problem for the Schrödinger operators in the space \( L^2(\mathbb{R}_+) \) at zero energy discussed in Sect. 4.3 of [71]. This problem was already considered in Sect. 4 of [72], but here our presentation is closer to [71].

Now an additional assumption on the coefficients \( a_n, b_n \) is required.
Theorem 9.7 ([72], Theorem 4.1). Suppose that
\[ \sum_{n=0}^{\infty} n(|a_n - 1/2| + |b_n|) < \infty. \] (9.17)

Then all functions \( f_n(z) \) are continuous as \( z \to \pm 1 \), the sequence \( f_n(\pm 1) \) satisfies the equation
\[ a_{n-1}f_{n-1}(\pm 1) + b_nf_n(\pm 1) + a_nf_{n+1}(\pm 1) = \pm f_n(\pm 1), \quad n \in \mathbb{Z}_+, \] (9.18)
and \( f_n(\pm 1) = (\pm 1)^n + o(1) \) as \( n \to \infty \).

In particular, the Jost function \( \Omega(z) \) is continuous in \( z \) up to the cut along \([-1, 1]\), including the points \( \pm 1 \). Representation (5.13) for the resolvent \( R(z) \) implies that its matrix elements \( \langle R(z)e_n, e_m \rangle \) are also continuous up to the point \( \pm 1 \) provided \( \Omega(\pm 1) \neq 0 \).

Using Theorem 2.2, we introduce a solution \( g_n(\pm 1) \) of equation (9.18) by the formula
\[ g_n(\pm 1) = f_n(\pm 1) \sum_{m=N_0+1}^{n} (a_{m-1}f_{m-1}(\pm 1)f_m(\pm 1))^{-1}, \quad n \geq N_0 + 1, \]
The following result is a direct consequence of Theorem 9.7.

Lemma 9.8. Let assumption (9.17) hold. Then the sequence \( g_n(\pm 1) \) satisfies equation (9.18),
\[ g_n(\pm 1) = 2(\pm 1)^{n+1}n(1 + o(1)) \]
as \( n \to \infty \) and \( \{f(\pm 1), g(\pm 1)\} = 1 \).

Since neither of linearly independent solutions \( f_n(\pm 1), g_n(\pm 1) \) nor their linear combinations tend to zero as \( n \to \infty \), we obtain

Theorem 9.9. Under assumption (9.17) equations (9.18) do not have solutions tending to zero as \( n \to \infty \). In particular, the operator \( J \) cannot have eigenvalues 1 and \(-1\).

Next, we discuss an asymptotic behavior of the orthonormal polynomials \( P_n(z) \) at the critical points \( z = \pm 1 \). In view of relation (2.33) where \( z = \pm 1 \), the next result is a direct consequence of Theorem 9.7 and Lemma 9.8.

Theorem 9.10. Under assumption (9.17) we have
\[ P_n(\pm 1) = -2\Omega(\pm 1)(\pm 1)^{n+1}n + o(n). \] (9.19)

Passing in (9.5) to the limit \( z \to \pm 1 \) and taking into account that \( P_n(0)(\pm 1) = (n + 1)(\pm 1)^n \), we obtain an equation
\[ (\pm 1)^n P_n(\pm 1) = n + 1 - 2 \sum_{m=0}^{n-1}(\pm 1)^{m+1}(n - m)(VP(\pm 1))_m. \] (9.20)
Similarly, in the limit $z \to \pm 1$ relation (9.3) yields
\[
\Omega(\pm 1) = \mp 2^{-1} + \sum_{n=0}^{\infty} (\pm 1)^n (VP(\pm 1))_n.
\] (9.21)

Our next goal is to consider the exceptional case $\Omega(\pm 1) = 0$. Let us use the same terminology as for the Schrödinger operators.

**Definition 9.11.** Let assumption (9.17) hold. If $\Omega(\pm 1) = 0$, we say that the Jacobi operator $J$ has a threshold resonance at the point $z = \pm 1$.

Clearly, the condition $\Omega(\pm 1) = 0$ is equivalent to the linear dependence of the solutions $P_n(\pm 1)$ and $f_n(\pm 1)$ of equation (9.18) whence
\[
P_n(\pm 1) = f_n(\pm 1)/f_0(\pm 1), \quad n \in \mathbb{Z}_+.
\] (9.22)

Note that $f_0(\pm 1) \neq 0$ if $\Omega(\pm 1) = 0$. In this case Theorem 9.7 allows us to make asymptotic formula (9.19) more precise:
\[
P_n(\pm 1) = (\pm 1)^n/f_0(\pm 1) + o(1), \quad n \to \infty.
\] (9.23)

Conversely, if the sequence $P_n(\pm 1)$ is bounded, then it follows from (9.20) that
\[
(\pm 1)^n P_n(\pm 1) = \left(1 - 2 \sum_{m=0}^{\infty} (\pm 1)^{m+1} (VP(\pm 1))_m \right)n
+ 1 + 2 \sum_{m=0}^{\infty} (\pm 1)^{m+1} m (VP(\pm 1))_m + o(1).
\] (9.24)

The coefficient at $n$ here is necessarily zero so that comparing (9.23) and (9.24), we obtain a relation
\[
f_0(\pm 1)^{-1} = 1 + 2 \sum_{m=0}^{\infty} (\pm 1)^{m+1} m (VP(\pm 1))_m.
\] (9.25)

Thus, the definition of a threshold resonance can be equivalently reformulated in the following way.

**Lemma 9.12.** Let assumption (9.17) hold. Then the Jacobi operator $J$ has a threshold resonance at the point $z = \pm 1$ if and only if the polynomial solution $P_n(\pm 1)$ of equation (9.18) is bounded for $n \to \infty$. In this case asymptotic formula (9.23) holds and $f_0(\pm 1)$ is given by relation (9.25).

Now we find an asymptotic behavior of the Jost function $\Omega(z)$ as $z \to \pm 1$. This requires an estimate on the rate of convergence of $P_n(z)$ to $P_n(\pm 1)$. The following technical assertion is quite similar to Lemma 3.6 in [71] and its proof will be omitted.

**Lemma 9.13.** Let assumption (9.17) hold and $\Omega(\pm 1) = 0$. Then
\[
|P_n(z) - P_n(\pm 1)| \leq Cn \sqrt{|z^2 - 1|} |\zeta(z)|^{-n}.
\]
The next result is a translation to the discrete case of Proposition 3.7 in Chapter 4 of [71] stated there in the continuous framework.

**Theorem 9.14.** Under assumption (9.17) suppose that \( \Omega(\pm 1) = 0 \). Then
\[
\Omega(z) = -2^{-1} f_0(\pm 1)^{-1} \sqrt{z^2 - 1} + o(\sqrt{|z^2 - 1|})
\]  
(9.26)
as \( z \to \pm 1 \).

**Proof.** Let us proceed from representation (9.3) of the Jost function in terms of the orthonormal polynomials. First, we observe that
\[
\sum_{n=0}^{\infty} |\zeta(z)|^n (VP(z)_n - (VP(\pm 1))_n| = o(\sqrt{z^2 - 1})).
\]
Indeed, every term in this sum is \( O(|z^2 - 1|) \), and, by the dominated convergence, the passage to the limit in the sum can be justified by Lemma 9.13. Next, we observe that
\[
\zeta(z)^n = (\pm 1)^n - n(\pm 1)^{n-1} \sqrt{z^2 - 1} + O(z^2 - 1),
\]
so that
\[
\Omega(z) = \mp 2^{-1} - 2^{-1} \sqrt{z^2 - 1} + \sum_{n=0}^{\infty} (\pm 1)^n (VP(\pm 1))_n
\]
\[- \sum_{n=0}^{\infty} n(\pm 1)^{n-1} (VP(\pm 1))_n \sqrt{z^2 - 1} + o(\sqrt{|z^2 - 1|}).
\]  
(9.27)
In view of (9.21) the constant term here is zero because \( \Omega(\pm 1) = 0 \). The coefficient at \( \sqrt{z^2 - 1} \) is \( -2^{-1} f_0(\pm 1)^{-1} \) by virtue of representation (9.25). Therefore (9.27) implies relation (9.26).

Using formula (8.21), we obtain the following consequence for the weight function.

**Corollary 9.15.** Under the assumptions of Theorem 9.14, we have
\[
\tau(\lambda) = 2\pi^{-1} f_0(\pm 1)^2 (1 - \lambda^2)^{-1/2} (1 + o(1)), \quad \lambda \in (-1, 1),
\]
(9.28)
as \( \lambda \to \pm 1 \).

In view of Proposition 2.1 and equality (9.22), Theorem 9.14 implies also the following result.

**Corollary 9.16.** Under the assumptions of Theorem 9.14 for all \( n,m \in \mathbb{Z}_+ \), the representation
\[
\langle R(z)e_n, e_m \rangle = -2 f_n(\pm 1) f_m(\pm 1) + o(1)
\]
\[
\sqrt{z^2 - 1}
\]
as \( z \to \pm 1 \) is satisfied.
Remark 9.17. Under assumption (9.17) Theorems 9.7 and 9.14 ensure that \( \Omega(z) \neq 0 \) for \( z \in (1, 1 + \epsilon) \) and \( z \in (-1 - \epsilon, -1) \) if \( \epsilon > 0 \) is small enough. Therefore the discrete spectrum of the operator \( J \) is finite.

Example 9.18. Let the Jacobi polynomials \( G^{(\alpha, \beta)}(z) \) be defined by spectral measure (B.1). Asymptotics of \( G^{(\alpha, \beta)}(\pm 1) \) are given by formulas (B.6). It follows from relations (B.2) that condition (9.17) is satisfied if and only if \( |\alpha| = |\beta| = 1/2 \). The point \( z = 1 \) (and the point \( z = -1 \)) is regular for the corresponding Jacobi operator \( J^{(\alpha, \beta)}(z) \) if \( \alpha = 1/2 \) (resp., \( \beta = 1/2 \)) and there is a resonance at this point if \( \alpha = -1/2 \) (resp., \( \beta = -1/2 \)). In the regular case formulas (B.6) are consistent with a general relation (9.19), and in the resonant case they are consistent with (9.23).

Formulas (B.6) for the cases \( |\alpha| \neq 1/2 \) or \( |\beta| \neq 1/2 \) show that the asymptotics of the orthonormal polynomials at the edge points is significantly changed if condition (9.17) is even slightly relaxed.

Let us finally consider the case when \( a_n - 1/2 \) and \( b_n \) decay as \( n^{-\varrho} \) with \( \varrho < 2 \). An important example of such coefficients is given by the Pollaczek polynomials when \( \varrho = 1 \); see Appendix C. For the Jacobi polynomials, we have \( \varrho = 2 \). For simplicity, we now assume that \( a_n = 1/2 \) for all \( n \) and, up to sufficiently rapidly decaying terms, \( b_n = \kappa n^{-\varrho} \) where \( \varrho < 2 \) and, for definiteness, \( \kappa > 0 \). In this case the operator \( J \) has infinite discrete spectrum accumulating at the point \( \lambda = 1 \). Relying on an analogy with the continuous case considered in [69] (see also §4.3 in the book [71]), we conjecture that

\[
\ln \tau(\lambda) = -\gamma_{-1}(1 + \lambda)^{-(-2-\varrho)/(2\varrho)}(1 + o(1)) \quad \text{as} \quad \lambda \to -1 + 0
\]

and

\[
\tau(\lambda) = \gamma_1(1 - \lambda)^{-1/4}(1 + o(1)) \quad \text{as} \quad \lambda \to 1 - 0
\]

where \( \gamma_{-1} \) and \( \gamma_1 \) are positive constants depending on \( \varrho \) and \( \kappa \).

According to (9.29) the weight \( \tau(\lambda) \) tends to zero exponentially as \( \lambda \to -1 + 0 \). This result can be interpreted as a virtual shift to the right of the essential spectrum of the operator \( J \) (obviously, it coincides with the interval \([-1 + \kappa, 1 + \kappa] \) if \( \varrho = 0 \)). The spectral point \( \lambda = -1 \) becomes quasiregular in this case. For the Schrödinger operator, this phenomenon is discussed in [69, 71].

It follows from (9.29) that the integral

\[
\int_{-1}^{1} (1 - \lambda^2)^s \ln \tau(\lambda) d\lambda > -\infty
\]

if

\[
s > (2 - 3\varrho)/(2\varrho).
\]

In particular, for all \( \varrho > 1 \), we can take \( s = -1/2 \). This is consistent with the following conjecture of Nevai [50] proved in [36] with earlier partial results obtained in [48] and [25].
Theorem 9.19 ([36], Theorem 2). Under assumption (1.19) condition (9.31) is satisfied for $s = -1/2$.

Condition (9.31) for $s = -1/2$ is known as the Szegő condition. It means that the weight $\tau(\lambda)$ does not tend to zero too rapidly as $\lambda \to \pm 1$.

Under the Hilbert-Schmidt assumption (8.24) a weaker quasi-Szegő condition holds.

Theorem 9.20 ([36], Theorem 1). Under assumption (8.24) condition (9.31) is satisfied for $s = 1/2$.

In the case $\rho > 1/2$ assumption (8.24) is of course satisfied. In the intermediary case $\rho = 1$ formula (9.29) is consistent with relation (C.3) for the Pollaczek polynomials. This example shows that assumption (1.19) in Theorem 9.19 is very precise.

Let us now discuss formula (9.30). Recall that according to (9.31) in the regular case $\Omega(1) \neq 0$, the weight function $\tau(\lambda)$ has a finite positive limit as $\lambda \to 1 - 0$. According to (9.28) it has a singularity $\tau(1 - \lambda)^{-1/2}$, $\tau_1 > 0$, in the resonant case $\Omega(1) = 0$. Formula (9.30) shows that for slowly decaying diagonal elements $b_n \sim \kappa n^{-\theta}$ where $\kappa > 0$ and $\rho < 2$ (the elements $a_n = 1/2$), the behavior of $\tau(\lambda)$ is intermediary between the regular and resonant cases. This can be interpreted as the existence of a weak but stable resonance for slowly decaying recurrence coefficients at the threshold energy $\lambda = 1$.

Of course in the case $\kappa < 0$, the results are exactly the same, but the roles of the thresholds $\lambda = 1$ and $\lambda = -1$ are interchanged.

9.4. Szegő function. We define the Szegő function $S(\zeta)$ by the formula

$$S(\zeta) = \exp \left( \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} \ln \left( \tau(\cos \theta) | \sin \theta | \right) d\theta \right), \quad |\zeta| < 1,$$

(9.32)

where $\tau(\lambda)$ is the weight function (5.15). This is exactly formula (10.2.10) (see also Theorem 12.1.2) in [63], but in contrast to [63] we do not suppose that the Jacobi operator $J$ has no eigenvalues. Of course definition (9.32) requires that the condition

$$\int_{-\pi}^{\pi} | \ln \left( \tau(\cos \theta) | \sin \theta | \right) | d\theta < \infty$$

(9.33)

(or, equivalently, (9.31) for $s = -1/2$) be satisfied.

Recall the standard Jensen-Poisson representation of analytic functions $f(\zeta)$ from the Hardy class $H^1$ on the unit disc $|\zeta| < 1$:

$$f(\zeta) = i \text{Im} f(0) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} \text{Re} f(e^{i\theta}) d\theta.$$
In particular, we have

\[
\ln(1 + \zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} \ln \cos(\theta/2) d\theta,
\]

\[
\ln(1 - \zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} \ln \sin(\theta/2) d\theta.
\]

(9.34)

Here the branches of the functions \(\ln(1 \pm \zeta)\) are fixed by the condition \(\ln 1 = 0\). We also recall that the Nevanlinna class \(N\) of functions \(f(\zeta)\) analytic on the unit disc is distinguished by the condition

\[
\sup_{r<1} \int_{0}^{2\pi} \ln^+ |f(re^{i\theta})| d\theta < \infty
\]

(as usual \(\ln^+ a = \max\{\ln a, 0\}\)).

Let \(D(z)\) be the perturbation determinant (9.9) and \(\Delta(\zeta) = D(z)\) if \(\zeta = \zeta(z)\). It follows from Corollary 8.6 that the function \(\Delta(\zeta)\) is analytic on the unit disc, and it is continuous up to the unit circle with a possible exception of the points \(\pm 1\). Moreover, according to (8.13), \(\Delta(\zeta) \neq 0\) if \(|\zeta| = 1\) but \(\zeta \neq \pm 1\). We also note that \(\Delta(0) = 1\) according to (9.7).

Let \(\lambda_k\) be eigenvalues (lying on \((-\infty, -1) \cup (1, \infty)\)) of the operator \(J\). We suppose that \(|\lambda_1| \geq |\lambda_2| \geq \cdots > 1\) not distinguishing positive and negative eigenvalues in notation. The numbers \(\mu_k := \zeta(\lambda_k) \in (-1, 1)\) are zeros of the function \(\Delta(\zeta)\). It was shown in [30] that under assumption (1.19)

\[
\sum_{k=1}^{\infty} (1 - |\mu_k|) < \infty.
\]

(9.35)

Using this result the inclusion \(\Delta \in N\) was established in [36].

Let us define an outer function

\[
G(\zeta) = \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} \ln|\Delta(e^{i\theta})| d\theta \right)
\]

(9.36)

where the function \(\ln|\Delta(e^{i\theta})|\) belongs to \(L^1(-\pi, \pi)\) because \(\Delta \in N\). This is equivalent to the Szegő condition (9.33) since, by relations (5.16) and (9.13),

\[
|\Delta(e^{i\theta})|^2 = |D(\cos \theta)|^2 = A^2 \frac{2}{\pi} \frac{|\sin \theta|}{\tau(\cos \theta)},
\]

whence

\[
2 \ln |\Delta(e^{i\theta})| = -\ln(\tau(\cos \theta)|\sin \theta|) + \ln(2\pi^{-1}A^2 \sin^2 \theta).
\]

(9.37)

Let us now compare definitions (9.32) and (9.36). Substituting expression (9.37) into (9.32) and taking formulas (9.34) into account, we see that

\[
G(\zeta) = A \frac{1 - \zeta^2}{\sqrt{2\pi} S(\zeta)}
\]

(9.38)
Next, we introduce the Blaschke product
\[ B(\zeta) = \prod_{k=1}^{\infty} \frac{|\mu_k|}{\mu_k} \frac{\mu_k - \zeta}{1 - \mu_k \zeta}, \quad |\zeta| < 1. \] (9.39)
According to condition (9.35) it is well defined. The function \( B(\zeta) \) is continuous on the closed disc \(|\zeta| \leq 1\) except, possibly, the points \( \pm 1 \) and \(|B(\zeta)| = 1\) for \(|\zeta| = 1\).

It is shown in [36] that the function \( \Delta(\zeta) \) does not have a singular inner component. Therefore the classical factorization (see, e.g., Theorem 2.9 in [18]) reads as
\[ \Delta(\zeta) = B(\zeta)G(\zeta). \]
Comparing this relation with (9.38), we arrive at the following result.

**Theorem 9.21.** Let assumption (1.19) be satisfied, and let \(|\zeta| < 1\). Let \( D(z) \) be the perturbation determinant (9.6) and \( \Delta(\zeta) = D(z) \). Define the Blaschke product \( B(\zeta) \) by formula (9.39), the Szegő function \( S(\zeta) \) – by (9.32), and the product \( A \) – by (9.15). Then the factorization
\[ \Delta(\zeta) = AB(\zeta) \frac{1 - \zeta^2}{\sqrt{2\pi}S(\zeta)} \] (9.40)
holds.

We emphasize that Theorem 9.21 is a direct consequence of classical results on the factorization of functions in the Nevanlinna class combined with the analytical results of [36].

According to (9.40) formulas (9.11) and (9.32) provide two different representations of an essentially the same object named the perturbation determinant \( D(z) \) or the Szegő function \( S(\zeta) \). In view of (9.10) the first of them is given in terms of \( \arg D(\lambda + i0) \) while according to (5.16) and (9.13) the second representation is stated in terms of \( \ln |D(\lambda + i0)| \). Obviously, these two functions are harmonic conjugate.

Below it will be convenient to make the change of variables \( \lambda = \cos \theta \) in integral (9.32). Recall that the weight function \( \tau_0(\lambda) \) of the Jacobi operator \( J_0 \) is given by formula (5.6). Taking into account that the function \( w(\cos \theta)|\sin \theta| \) is even and using relations (9.34), we see that
\[ S(\zeta) = \frac{1 - \zeta^2}{\sqrt{2\pi}} \exp \left( \frac{1 - \zeta^2}{2\pi} \int_{-1}^{1} \ln \left( \frac{\tau(\lambda)}{\tau_0(\lambda)} \right) \frac{d\lambda}{1 - 2\zeta \lambda + \zeta^2} \frac{d\lambda}{\sqrt{1 - \lambda^2}} \right). \] (9.41)
In terms of the variable \( z = 2^{-1}(\zeta + \zeta^{-1}) \in \Pi_0 \), the integral in (9.41) becomes the Cauchy integral which yields the representation
\[ S(\zeta(z)) = \zeta(z) \sqrt{\frac{2(z^2 - 1)}{\pi}} \exp \left( - \frac{\sqrt{z^2 - 1}}{2\pi} \int_{-1}^{1} \ln \left( \frac{\tau(\lambda)}{\tau_0(\lambda)} \right) \frac{d\lambda}{\lambda - z} \frac{d\lambda}{\sqrt{1 - \lambda^2}} \right). \]
9.5. **Trace identities.** We discuss two types of trace identities. Both of them are obtained by studying an asymptotic behavior of the perturbation determinant $D(z)$ (more precisely, of $\ln D(z)$) as $z \to \infty$ or, equivalently, $\zeta \to 0$. We will get the first set of identities by considering an asymptotic expansion in powers of $z^{-1}$ while identities of the second type known as the Case sum rules (see [12]) are derived by expanding in powers of $\zeta$. Note that for the Schrödinger operators, trace identities were obtained by Buslaev and Faddeev in [9]; see, e.g., §4.6 in the book [71] for details. In this case the role of the variable (2.18) is played by $\zeta = \sqrt{z}$, and there are also two types of identities: of integer and half-integer orders. The identities of half-integer orders correspond to the Case sum rules.

To find expressions of $\text{Tr}(J^n - J_0^n)$ in terms of the spectral shift function, we only have to compare asymptotic expansions as $|z| \to \infty$ of both sides of representation (9.11). It follows from relation (9.8) that

$$\ln D(z) = -\sum_{n=1}^{\infty} n^{-1} \text{Tr}(J^n - J_0^n) z^{-n}.$$  

Since

$$\int_{-\infty}^{\infty} \xi(\lambda)(\lambda - z)^{-1} d\lambda = -\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \xi(\lambda)\lambda^{-n-1} d\lambda z^{-n},$$

equating the coefficients at $z^{-n}$, we see that

$$\text{Tr}(J^n - J_0^n) = n \int_{-\infty}^{\infty} \xi(\lambda)\lambda^{-n-1} d\lambda.$$  \hspace{1cm} (9.42)

On the discrete spectrum, the spectral shift function can be explicitly calculated. Indeed, let $\lambda_{1}^{(+)} > \lambda_{2}^{(+)} > \cdots > 1$ (and $\lambda_{1}^{(-)} < \lambda_{2}^{(-)} < \cdots < -1$) be eigenvalues of the operator $J$ lying above the point 1 (respectively, below the point $-1$). It follows from formula (9.12) that $\xi(\lambda) = n$ for $\lambda \in (\lambda_{n+1}^{(+)}, \lambda_{n}^{(+)})$ and $\xi(\lambda) = -n$ for $\lambda \in (\lambda_{n}^{(-)}, \lambda_{n+1}^{(-)})$ whence

$$n \int_{1}^{\infty} \xi(\lambda)\lambda^{-n-1} d\lambda = \sum_{k=1}^{\infty} k((\lambda_{k}^{(+)}))^{n} - (\lambda_{k+1}^{(+)}))^{n})$$

and, similarly, for the integral over $(-\infty, -1)$. The series here is convergent by virtue of estimate (9.10). Putting together this relation with (9.42), we obtain the following result.

**Theorem 9.22.** Let assumption (1.19) be satisfied. Then

$$\text{Tr}(J^n - J_0^n) = n \int_{-1}^{1} \xi(\lambda)\lambda^{-n-1} d\lambda + \sum_{k=1}^{\infty} k((\lambda_{k}^{(+)}))^{n} - (\lambda_{k+1}^{(+)}))^{n}) + \sum_{k=1}^{\infty} k((\lambda_{k}^{(-)}))^{n} - (\lambda_{k+1}^{(-)}))^{n}).$$  \hspace{1cm} (9.43)
In view of (9.16), the integral on the right can be expressed in terms of the phase function:
\[
\int_{-1}^{1} \frac{\xi(\lambda)}{\lambda^{n-1}} d\lambda = \frac{1}{\pi} \int_{0}^{\pi} \eta(\theta) \cos^{n-1} \theta \sin \theta d\lambda.
\]

From a somewhat different point of view, formulas of type (9.43) were studied in the book [64], Chapter 6.

The trace formula of zero order (the Levinson theorem) requires a special discussion. Now we assume a stronger condition (9.17) on the coefficients of the operator \(J\). Then according to Theorem 9.7, the corresponding perturbation determinant \(D(z)\) is continuous as \(z \to \pm 1\). One has to distinguish the cases \(D(1) = 0\) and/or \(D(-1) = 0\) when the operator \(J\) has threshold resonances at the points \(\lambda = 1\) and/or \(\lambda = -1\). Note (see Remark 9.17) that under assumption (9.17) the operator \(J\) has only a finite number \(N\) of discrete eigenvalues.

**Theorem 9.23 ([72], Theorem 5.10).** Let assumption (9.17) be satisfied. Then the limits \(\xi(1 - 0)\) and \(\xi(-1 + 0)\) exist and

\[
\xi(1 - 0) - \xi(-1 + 0) = N + p
\]

where \(p = 0\) if \(D(\pm 1) \neq 0\) for both signs, \(p = 1/2\) if \(D(\pm 1) = 0\) for one of the signs and \(p = 1\) if \(D(\pm 1) = 0\) for both signs.

Let us, finally, obtain the Case sum rules for the pair \(J_0, J\). Putting together relations (9.40) and (9.41), we see that

\[
\ln \Delta(\zeta) - \ln B(\zeta) - \ln A = \frac{\zeta^2 - 1}{2\pi} \int_{-1}^{1} \ln \left( \frac{\tau(\lambda)/\tau_0(\lambda)}{1 - 2\zeta \lambda + \zeta^2} \right) \frac{d\lambda}{\sqrt{1 - \lambda^2}}.
\]

(9.45)

First we set here \(\zeta = 0\) and recall that \(\Delta(0) = 1\). Therefore in view of definitions (9.15) and (9.39), relation (9.45) implies the identity

\[
\sum_{k=1}^{\infty} \ln(2\theta_k) + \sum_{k=1}^{\infty} \ln |\mu_k| = \frac{1}{2\pi} \int_{-1}^{1} \ln \left( \frac{\tau(\lambda)/\tau_0(\lambda)}{1 - 2\zeta \lambda + \zeta^2} \right) \frac{d\lambda}{\sqrt{1 - \lambda^2}}.
\]

(9.46)

known as the Case sum rule of zero order. It is of course quite different from the Levinson theorem (9.44).

More generally, we consider the asymptotic expansions of both sides of (9.45) as \(\zeta \to 0\) and compare the coefficients at the same powers of \(\zeta\). According to Theorem 2.13 in [36] we have

\[
\ln \Delta(\zeta) = -2 \sum_{n=1}^{\infty} n \ln \left( T_n(J) - T_n(J_0) \right) \zeta^n
\]
where \( T_n(\lambda) = \cos(n \arccos \lambda) \) are the Chebyshev polynomials of the first kind. It directly follows from definition (9.39) that

\[
\ln B(\zeta) = \sum_{k=1}^{\infty} \ln |\mu_k| + \sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{\infty} (\mu_k^n - \mu_k^{-n}) \zeta^n
\]

where the series over \( k \) are convergent due to the condition (9.35). Finally, we use formula (10.11.29) in [20]:

\[
\frac{1 - \zeta^2}{1 - 2\zeta \lambda + \lambda^2} = 1 + 2 \sum_{n=1}^{\infty} T_n(\lambda) \zeta^n.
\]

Thus the equality of the coefficients at \( \zeta^n \) in the left- and right-hand sides of (9.45) yields the identity

\[
\text{Tr} \left( J_n(H) - J_n(H_0) \right) = -\frac{1}{2} \sum_{k=1}^{\infty} (\mu_k^n - \mu_k^{-n})
\]

\[
+ \frac{n}{2\pi} \int_{-1}^{1} \ln \left( \tau(\lambda)/\tau_0(\lambda) \right) T_n(\lambda) \frac{d\lambda}{\sqrt{1 - \lambda^2}}, \quad n = 1, 2, \ldots \quad (9.47)
\]

The trace identities (9.46) and (9.47) known as the Case sum rules are not new. They were obtained by him in [12] and rigorously proven in [36]. We note that, in the paper [36], the identities (9.46) and (9.47) were first checked for finite rank perturbations \( J - J_0 \) and then (9.46) and (9.47) (for \( n = 1 \)) were used for the proof of the inclusion \( \Delta \in \mathbb{N} \).

**Appendix A. Favard’s theorem**

Here we prove elementary assertions stated in Sect. 1.1 and 1.2.

*Proof of Proposition 1.1* Since \( \lambda P_n(\lambda) \) is a polynomial of degree \( n + 1 \), we have

\[
\lambda P_n(\lambda) = c_{n,n+1} P_{n+1}(\lambda) + c_{n,n} P_n(\lambda) + \cdots + c_{n,1} P_1(\lambda) + c_{n,0} P_0(\lambda).
\]

(A.1)

Taking the scalar product in the space \( L^2(\mathbb{R}; d\rho) \) of this expression with the polynomial \( P_l \), we find that

\[
\int_{-\infty}^{\infty} \lambda P_n(\lambda) P_l(\lambda) d\rho(\lambda) = \sum_{m=0}^{n+1} c_{n,m} \langle P_m, P_l \rangle.
\]

(A.2)

Let \( l = 0, 1, \ldots, n - 2 \). Then the left-hand side here is zero because \( P_n \) is orthogonal in the space \( L^2(\mathbb{R}; d\rho) \) to all polynomials of degree \( \leq n - 1 \). Since the right-hand side of (A.2) equals \( c_{n,l} \), we see that \( c_{n,l} = 0 \) for all \( l = 0, 1, \ldots, n - 2 \). Therefore it follows from (A.1) that

\[
\lambda P_n(\lambda) = c_{n,n+1} P_{n+1}(\lambda) + c_{n,n} P_n(\lambda) + c_{n,n-1} P_{n-1}(\lambda).
\]

(A.3)
Supposing that (1.11) is true for all \( n \) whence the definition (1.1), we have \( g \). If \( P \) taking the scalar product of this expression with \( \ell \). Then we set \( a_n := c_{n,n+1} \) and take the scalar product of (A.3) with \( P_{n+1} \) whence \( a_n \) is given by (1.8). Let us now compare the coefficients at \( \lambda^{n+1} \) in (A.3). According to (1.3) we have \( k_n = a_n k_{n+1} \) whence \( a_n > 0 \) because \( k_n > 0 \) for all \( n \). Finally, taking the scalar product of (A.3) with \( P_{n-1} \), we see that \( c_{n,n+1} = a_{n-1} \). Thus (A.3) yields relation (1.1). □

Proof of Proposition 1.2. If \( f \in \mathcal{D} \) and \( g \in \ell^2(\mathbb{Z}_+) \), then according to (1.10) we have

\[
\langle J_{\min} f, g \rangle = \sum_{n=1}^{\infty} a_{n-1} f_{n-1} g_n + \sum_{n=0}^{\infty} b_n f_n g_n
= \sum_{n=1}^{\infty} a_{n-1} f_{n-1} g_n + \sum_{n=0}^{\infty} b_n f_n g_n + \sum_{n=0}^{\infty} a_n f_{n+1} g_n + \sum_{n=0}^{\infty} a_n f_{n+1} g_n
= \sum_{n=0}^{\infty} f_n (J g)_n.
\]

(A.4)

If \( g \in \mathcal{D}(J_{\max}) \), the right-hand side here equals \( \langle f, J_{\max} g \rangle \) whence \( J_{\max} \subset J_{\min}^* \).

Conversely, suppose that \( g \in \mathcal{D}(J_{\min}) \), that is, \( \langle J_{\min} f, g \rangle = \langle f, g_\ast \rangle \) for some \( g_\ast \in \ell^2(\mathbb{Z}_+) \) and all \( f \in \mathcal{D} \). Comparing this equality with (A.4), we see that \( g_\ast = J g \) whence \( g \in \mathcal{D}(J_{\max}) \) and \( J_{\min}^* \subset J_{\max} \). □

Proof of Proposition 1.3. Relation (1.11) is obvious for \( n = 0 \) when \( P_0(J_{\min}) = I \).

Supposing that (1.11) is true for all \( n \leq N \), we will verify it for \( n = N + 1 \). By definition (1.12), we have

\[
a_N P_{N+1}(J_{\min}) e_0 = (J_{\min} - b_N) P_N(J_{\min}) e_0 - a_{N-1} P_{N-1}(J_{\min}) e_0
= (J_{\min} - b_N) e_N - a_{N-1} e_{N-1}
\]

where we have used (1.11) for \( n = N \) and \( n = N - 1 \). By definition (1.10) of the operator \( J_{\min} \), the right-hand side here equals \( a_N e_{N+1} \). □

Proof of Proposition 1.4. Using (1.11) and (1.12), we see that

\[
d\langle E_J(\lambda) e_n, e_m \rangle = d\langle E_J(\lambda) P_n(J) e_0, P_m(J) e_0 \rangle = P_n(\lambda) P_m(\lambda) d\rho_J(\lambda).
\]

Integrating this equality, we obtain relation (1.7). It now follows from definition (1.13) that

\[
\langle \Phi e_n, \Phi e_m \rangle = \delta_{n,m}
\]

and hence operator (1.14) is isometric.

Next, we check the intertwining property (1.15). It suffices to consider \( f = e_n \).

By definition of \( J \), in this case relation (1.15) means that

\[
(\Phi(a_{n-1} e_{n-1} + b_n e_n + a_n e_{n+1})) (\lambda) = \lambda(\Phi e_n)(\lambda).
\]
In view of definition (1.13) of $\Phi$ this equality is equivalent to relation (1.1) defining $P_n(\lambda)$.

It remains to verify that the operator $\Phi$ is unitary, that is, its range $\text{Ran} \Phi = L^2(\mathbb{R}; d\rho_J)$. Supposing the contrary, we find a vector $g \neq 0$ in $L^2(\mathbb{R}; d\rho_J)$ such that $\langle \Phi f, g \rangle = 0$ for all $f \in L^2(\mathbb{R}+)$. In particular, this is true for all elements $f = E_J(\Lambda)e_0$ where $\Lambda$ is an arbitrary Borelian subset of $\mathbb{R}$. Using the intertwining property (1.15) and equality $\Phi e_0 = 1$, we find that $(\Phi E_J(\Lambda)e_0)(\lambda)$ is the characteristic function of $\Lambda$. Therefore

$$
\langle \Phi E_J(\Lambda)e_0, g \rangle = \int_\Lambda g(\lambda)d\rho_J(\lambda).
$$

Since this integral is zero for all $\Lambda \subset \mathbb{R}$, we see that $g(\lambda) = 0$ on a set of full $d\rho_J$ measure.

It follows from (1.11) and (1.15) that

$$(\Phi e_n)(\lambda) = (\Phi P_n(J)e_0)(\lambda) = P_n(\lambda).$$

Since $e_0, e_1, \ldots$ is the basis in $L^2(\mathbb{Z}+)\), operator (1.14) is unitary, linear combinations of the polynomials $P_n(\lambda)$ are dense in the space $L^2(\mathbb{R}; d\rho_J)$. □

APPENDIX B. CLASSICAL POLYNOMIALS

Here we discuss some basic properties of the Jacobi, Hermite and Laguerre polynomials. Note that for the corresponding recurrence coefficients $a_n$, the Carleman condition (1.27) is satisfied.

It is convenient to define Jacobi polynomials by their spectral measures $d\rho(\lambda) = d\rho^{(\alpha, \beta)}(\lambda)$ where the parameters $\alpha, \beta > -1$. These measures are supported on the interval $[-1, 1]$, $\rho(\{\pm 1\}) = 0$ and

$$
d\rho(\lambda) = k(1-\lambda)^\alpha(1+\lambda)^\beta d\lambda, \quad \lambda \in (-1, 1), \quad \alpha, \beta > -1.
$$

The dependence of various objects on $\alpha$ and $\beta$ is often omitted in notation. The constant $k = k^{(\alpha, \beta)}$ is chosen in such a way that the measure (B.1) is normalized, i.e., $\rho(\mathbb{R}) = \rho((-1, 1)) = 1$. The orthonormal polynomials $G_n(z) = G_n^{(\alpha, \beta)}(z)$ determined by measure (B.1) are known as the Jacobi polynomials. According to Proposition 1.6 we have

$$
G_n^{(\beta, \alpha)}(z) = (-1)^n G_n^{(\alpha, \beta)}(-z).
$$

In some particular but important cases, the polynomials $G_n^{(\alpha, \beta)}(z)$ have special names. They are called Gegenbauer polynomials if $\alpha = \beta$. In particular, $G_n^{(0, 0)}(z)$ are known as the Legendre polynomials; $G_n^{(-1/2, -1/2)}(z)$ and $G_n^{(1/2, 1/2)}(z)$ are the Chebyshev polynomials of the first and second kinds, respectively.

Let $J = J^{(\alpha, \beta)}$ be the Jacobi operator with the spectral measure $d\rho^{(\alpha, \beta)}(\lambda)$. Explicit expressions for its matrix elements $a_n, b_n$ can be found, for example, in the books [20, 63], but we do not need them. We here note only asymptotic formulas

$$
a_n = 1/2 + 2^{-4}(1-2\alpha^2-2\beta^2)n^{-2} + O(n^{-3}), \quad b_n = 2^{-2}(\beta^2-\alpha^2)n^{-2} + O(n^{-3})
$$

(B.2)
for the matrix elements. In the case \( \alpha = \beta = 1/2 \) we have \( a_n = 1/2 \) and \( b_n = 0 \) for all \( n \). The corresponding Jacobi operator denoted \( J^{(1/2, 1/2)} =: J_0 \) is known as the "free" discrete Schrödinger operator. Eigenvectors of \( J_0 \) are normalized Chebyshev polynomials of the second kind, and the corresponding spectral measure \( d\rho_0(\lambda) = d(E_0(\lambda)e_0, e_0) \) is given by the formula

\[
d\rho_0(\lambda) = 2\pi^{-1}\sqrt{1 - \lambda^2} d\lambda, \quad \lambda \in (-1, 1).
\]

(B.3)

In the case \( \alpha = \beta = -1/2 \) we have \( b_n = 0 \) for all \( n \), but \( a_n = 1/2 \) for \( n \geq 1 \) only while \( a_0 = 1/\sqrt{2} \). The corresponding Jacobi operator denoted \( J^{(-1/2, -1/2)} \) is a two-rank perturbation of the operator \( J_0 \). Its eigenvectors are normalized Chebyshev polynomials of the first second kind.

The Jacobi polynomials satisfy asymptotic relations

\[
G_n(\lambda) = 2^{1/2}(\pi k)^{-1/2}(1 - \lambda)^{-(1+2\alpha)/4}(1 + \lambda)^{-(1+2\beta)/4} \times \cos \left( \left( n + \gamma \right) \arcsin \lambda - \pi (2n + \beta - \alpha)/4 \right) + O(n^{-1}), \quad \gamma = (\alpha + \beta + 1)/2,
\]

(see formula (8.21.10) in the book [63]) if \( \lambda \in (-1, 1) \) and

\[
G_n(z) = (2\pi k)^{-1/2}2^{-(\alpha + \beta)/2}(z - 1)^{-(1+2\alpha)/4}(z + 1)^{-(1+2\beta)/4} \times \left( \sqrt{z + 1} + \sqrt{z - 1} \right)^{\alpha + \beta} (z + \sqrt{z^2 - 1})^{n+1/2}(1 + o(1))
\]

(see formula (8.21.9) in [63] if \( z \in \mathbb{C} \setminus [-1, 1] \). Here \( \sqrt{z \pm 1} > 0 \) if \( z \pm 1 > 0 \). Estimates of the remainders in (B.4) and (B.3) are uniform in \( \lambda \) and \( z \) from compact subsets of \((-1, 1)\) and of \( \mathbb{C} \setminus [-1, 1] \), respectively. At the edge points of the spectrum, the asymptotics of the Jacobi polynomials as \( n \to \infty \) are given by the formulas

\[
G_n(1) = k^{-1/2}2^{-\alpha - \beta}\Gamma(\alpha + 1)^{-1}n^{\alpha + 1/2}(1 + O(n^{-1})),
\]

\[
G_n(-1) = (1)^{n}k^{-1/2}2^{-\alpha - \beta}\Gamma(\beta + 1)^{-1}n^{\beta + 1/2}(1 + O(n^{-1})).
\]

(B.6)

The Hermite polynomials \( H_n(z) \) are defined by relations (1.1), (1.2) with the recurrence coefficients

\[
a_n = \sqrt{(n + 1)/2}, \quad b_n = 0.
\]

(B.7)

According to Theorem 8.22.7 in the book [63] asymptotics of \( H_n(z) \) as \( n \to \infty \) is given by the Plancherel-Rotach formula

\[
H_n(z) = 2^{1/2}\pi^{-1/4}e^{-z^2/2}(2n + 1)^{-1/4} \cos \left( \sqrt{2n + 1}z - \pi n/2 \right) + O(e^{\sqrt{2n+1}|\text{Im} z|}n^{-3/4}).
\]

(B.8)

This asymptotics is uniform in \( z \) from compact subsets of \( \mathbb{C} \). Obviously, the right-hand side of (B.8) exponentially grows as \( n \to \infty \) if \( \text{Im} z \neq 0 \), and it is an oscillating function if \( z \in \mathbb{R} \).

Let us consider the Jacobi operator \( J \) with coefficients (B.7). The spectral measure of \( J \) equals

\[
d\rho(\lambda) = \pi^{-1/2}e^{-\lambda^2}d\lambda, \quad \lambda \in \mathbb{R}
\]
(see, e.g., formula (10.13.1) in [20]). Thus, $d\rho(\lambda)$ is absolutely continuous and its support is the whole axis $\mathbb{R}$.

Suppose now that the recurrence coefficients $a_n, b_n$ are given by formulas

$$a_n = a_n^{(p)} = \sqrt{(n + 1)(n + 1 + p)} \quad \text{and} \quad b_n = b_n^{(p)} = 2n + p + 1, \quad p > -1.$$ 

The corresponding Jacobi operator $J = J^{(p)}$ has the absolutely continuous spectrum coinciding with $[0, \infty)$, and the spectral measure is given by the relation (see, e.g., formula (10.12.1) in [20])

$$d\rho^{(p)}(\lambda) = \tau^{(p)}(\lambda)d\lambda \quad \text{where} \quad \tau^{(p)}(\lambda) = \Gamma(p + 1)^{-1}\lambda^p e^{-\lambda}, \quad \lambda \in \mathbb{R}_+.$$ 

The eigenfunctions of $J^{(p)}$ are orthonormal Laguerre polynomials $L_n^{(p)}(z)$ defined by relations (1.1) and (1.2). Note that the normalized polynomials $\tilde{L}_n^{(p)}(z)$ we consider here are related to the Laguerre polynomials $L_n^{(p)}(z)$ defined in §10.12 of the book [20] or in §5.1 of the book [63] by the equality

$$L_n^{(p)}(z) = (-1)^n \sqrt{\frac{\Gamma(1+n)\Gamma(1+p)}{\Gamma(1+n+p)}} \tilde{L}_n^{(p)}(z).$$ 

According to asymptotic formula (10.15.1) in [20] for positive $\lambda$, we have

$$L_n^{(p)}(\lambda) = (-1)^n \sqrt{\frac{\Gamma(1+p)}{\pi}} \lambda^{-p/2-1/4} e^{\lambda^2/4} \cos\left(2\sqrt{n}\lambda - \frac{2p+1}{4}\right) + O(n^{-3/4})$$

as $n \to \infty$. For $z \in \mathbb{C} \setminus [0, \infty)$, one has (see Theorem 8.22.3 in [63])

$$L_n^{(p)}(z) = (-1)^n \sqrt{\frac{\Gamma(1+p)}{\pi}} (-z)^{-p/2-1/4} e^{z^2/4} \cosh^{-1/4} \left(1 + O(n^{-1/2})\right)$$

where $\arg(-z) > 0$ if $z < 0$. Asymptotics (B.9) and (B.10) are uniform in $\lambda$ and $z$ from compact subsets of $\mathbb{R}_+$ and $\mathbb{C} \setminus [0, \infty)$, respectively.

**Appendix C. Pollaczek polynomials**

The normalized Pollaczek polynomials are defined (see, e.g., Appendix in the book [63]) by recurrent relations (1.1), (1.2) with

$$a_n = \frac{n + 1}{\sqrt{(2n + 2\alpha + 1)(2n + 2\alpha + 3)}}, \quad b_n = -\frac{2\beta}{2n + 2\alpha + 1}; \quad (C.1)$$

here the parameters $\alpha, \beta \in \mathbb{R}$ and $\alpha > |\beta|$. It follows that

$$a_n = 2^{-1} - \alpha(2n)^{-1} + O(n^{-2}), \quad b_n = -\beta n^{-1} + O(n^{-2}) \quad \text{as} \quad n \to \infty. \quad (C.2)$$

The spectrum of the corresponding Jacobi operator coincides with the interval $[-1, 1]$, it is absolutely continuous and the weight function is given by the formula

$$\tau(\lambda) = (\alpha + 1/2)e^{(2\theta - \pi)\Xi(\theta)} \left(\cosh(\pi\Xi(\theta))\right)^{-1} \quad \text{where} \quad \Xi(\theta) = (\alpha \cos \theta + \beta)(\sin \theta)^{-1}$$

as $n \to \infty$. For $z \in \mathbb{C} \setminus [0, \infty)$, one has (see Theorem 8.22.3 in [63])

$$L_n^{(p)}(z) = (-1)^n \sqrt{\frac{\Gamma(1+p)}{\pi}} (-z)^{-p/2-1/4} e^{z^2/4} \cosh^{-1/4} \left(1 + O(n^{-1/2})\right)$$

where $\arg(-z) > 0$ if $z < 0$. Asymptotics (B.9) and (B.10) are uniform in $\lambda$ and $z$ from compact subsets of $\mathbb{R}_+$ and $\mathbb{C} \setminus [0, \infty)$, respectively.
and as usual $\lambda = \cos \theta$. It is easy to see that
\[
\ln \tau(\lambda) = -\pi (\alpha + \beta) \theta^{-1} + O(1) \quad (C.3)
\]
as $\lambda \to 1 - 0$ and a similar formula is true as $\lambda \to -1 + 0$.

It follows from (C.2) that $V = J - J_0$ is Hilbert-Schmidt, but the series
\[
\sum_n (a_n - 1/2) \quad \text{and} \quad \sum_n b_n \quad \text{are divergent; in particular, assumption (1.19) is not satisfied.}
\]
According to (C.3) the Szegő condition (9.31) where $s = -1/2$ is violated
for Pollaczek polynomials. This is consistent with the classical theorem
of Szegő, Shohat, Geronimus, Kreĭn and Kolmogorov; see, e.g., Theorem 4 in [36]. On the other hand, relation (C.3) implies condition (9.31) for $s = 1/2$ which is consistent
with Theorem 1 in [36] stated here as Theorem 9.20.

Pursuing an analogy with differential operators, we note that the Jacobi operators with coefficients (C.1) correspond to the Schrödinger operators with Coulomb potentials (see §§36 and 133 in the book [39]). Condition $\alpha > |\beta|$ distinguishes repulsive potentials, and formula (C.3) corresponds to the exponential decay as $\lambda \to 0$ of the corresponding weight function at low energies. If $\alpha < |\beta|$, then the infinite discrete spectrum appears which is also quite similar to the Schrödinger operators with Coulomb attractive potentials; see Sect. 5.4 and 5.5 in the book [31].

Appendix D. Moment problems

For a sequence $s_0, s_1, \ldots, s_n, \ldots$ of positive numbers such that $s_0 = 1$, we set
\[
\mathcal{S}_n = \begin{pmatrix}
  s_0 & s_1 & \ldots & s_n \\
  s_1 & s_2 & \ldots & s_{n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  s_{n-1} & s_n & \ldots & s_{2n-1} \\
  s_n & s_{n+1} & \ldots & s_{2n}
\end{pmatrix} \quad \text{and} \quad \mathcal{P}_n(z) = \begin{pmatrix}
  s_0 & s_1 & \ldots & s_n \\
  s_1 & s_2 & \ldots & s_{n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  s_{n-1} & s_n & \ldots & s_{2n-1} \\
  1 & z & \ldots & z^n
\end{pmatrix} \quad (D.1)
\]

One calls $s_0, s_1, \ldots, s_n, \ldots$ a moment sequence if $\det \mathcal{S}_n > 0$ for all $n \geq 1$. By the Hamburger theorem for an arbitrary moment sequence, there exist measures $d\rho(\lambda)$ with infinite supports satisfying relations (1.6) for all $n \in \mathbb{Z}_+$. Such measures are in general not unique. However the polynomials $P_n(z)$ satisfying conditions (1.7) are the same for all these measures. They can be constructed by a formula
\[
P_n(z) = (d_{n-1}d_n)^{-1/2} \det \mathcal{P}_n(z) \quad (D.2)
\]
where $d_n = \det \mathcal{S}_n$.

According to Proposition 1.1 the polynomials $P_n(z)$ satisfy recurrence relation (1.11) with the coefficients $a_n, b_n$ given by (1.8) and (1.9). Let $J$ be the Jacobi operator (1.10) with these recurrence coefficients. Then the moments $s_n$ can be recovered by the formula
\[
s_n = \langle J^n e_0, e_0 \rangle. \quad (D.3)
\]
Indeed, let $J$ be an arbitrary self-adjoint extension of the operator $J_{\min}$. The right-hand side of (D.3) equals
\[ \langle J^n e_0, e_0 \rangle = \int_{-\infty}^{\infty} \lambda^n d\rho_J(\lambda) = s_n \]
by the spectral theorem and relation (1.6). Conversely, let recurrence coefficients $a_n$, $b_n$ be given. Define the numbers $s_n$ by formula (D.3) and the polynomials $P_n(z)$ by equalities (D.1), (D.2). Then using Proposition 1.1 we can recover coefficients $a_n$, $b_n$. Thus the moments $s_n$ and the recurrence coefficients $a_n$, $b_n$ are in a one-to-one correspondence.

Recall that the moment problem (1.6) is called determinate if the measure satisfying these relations is unique. Otherwise, it is called indeterminate. It is known (see Theorem 2 in [60]) that the determinacy is equivalent to the essential self-adjointness of the operator $J_{\min}$.

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