1. Introduction.

A billiard is a dynamical system describing the motion of a point particle in a connected, compact domain $Q \subset \mathbb{T}^d$, $d \geq 2$. In general, the boundary of the domain $Q$ is assumed to be piecewise $C^3$-smooth, however later on we impose further restrictions on the boundary (cf. Section 2). Smooth components of the boundary $\partial Q$ are called scatterers. If the smooth components of the boundary are strictly convex, the billiard is called dispersing or a Sinai billiard. Inside $Q$ the particle moves with constant velocity with elastic reflections at the boundary. The kinetic energy of a point is a first integral of the motion. Therefore the phase space of the billiard is $M = Q \times S^{d-1} = \{(q, v) | q \in Q, ||v|| = 1\}$. The Liouville probability measure $\mu$ on $M$ is a product of the Lebesgue measures on $Q$ and $S^{d-1}$, i.e. $d\mu = \text{const} \cdot dq dv$. The resulting dynamical system $(M, \{S^t, t \in \mathbb{R}\}, \mu)$ is the billiard flow.

The boundary $\partial Q$ defines a natural cross-section for the billiard flow. Let $n(q)$ be the unit normal vector of a smooth component of the boundary $\partial Q$ at the point $q \in \partial Q$, directed inwards. The billiard map $T$ is defined as the first return map on $\partial M$, where

$$\partial M = \{(q, v) | q \in \partial Q, (v, n(q)) \geq 0\}.$$ 

The invariant measure for the map $T$ is denoted by $\mu_1$, and we have

$$d\mu_1 = \text{const} \cdot |(v, n(q))|dq dv.$$ 

Contrary to the case of smooth dynamical systems, the billiard map $T$ is discontinuous, which makes the application of classical methods of analysis of stochastic properties of a system significantly more difficult. Nevertheless, in 1970 Sinai demonstrated [S1] that at least in the case $d = 2$ dispersing billiards are hyperbolic, ergodic and even $K$-mixing. The main tool in the proof of ergodicity of the billiard system was the so-called fundamental theorem for the dispersing billiards. This theorem ensures an abundance of geometrically nicely situated and sufficiently large stable and unstable manifolds. For the case $d > 2$ the fundamental theorem was first stated and proved in [SCh] and generalized in [KSSz].

In the proof of the fundamental theorem for dispersing billiards one makes some assumptions concerning the structure of the set of points, where the billiard map and its iterations are discontinuous. These sets are called singularity manifolds. They were assumed to be smooth manifolds with boundary until recently, when it was found in [BCST1] that in a typical situation there are special points where these manifolds may have singularities in the sense of the singularity theory. At these points the singularity manifolds don’t have a tangent plane. Therefore one has to analyze the singularities to show that the arguments in the proof of the
fundamental theorem remain valid. It was shown in [BCST2] that in the case of algebraic semi-dispersing billiards the proof of the fundamental theorem remains valid.

In this paper we show that in the case of strictly dispersing scatterers in $Q \subset \mathbb{T}^d$, $d \geq 3$ the assumption about algebraicity is not needed and the fundamental theorem remains valid if the boundaries of the scatterers are $C^{8d-7}$ smooth.

The paper consists of two parts. First, we analyze the singularity manifolds of the billiard map. These manifolds can be obtained as the images of some smooth manifold $M$ under several iterations of the billiard map. This map is irregular at the the points of tangency, so to understand the structure of the singularity one needs to study the billiard map near the points of tangency. We compute partial derivatives of the map and show (Lemma 4.3), that after a singular, but simple change of variables the billiard map becomes a local diffeomorphism. In Lemma 5.1 we show that the singularity manifold is a level set of a smooth function with a non-vanishing germ. The Theorem 7.1 implies that the singularity manifold has a certain regularity property required in the proof of the fundamental theorem.

In the second part we show that the proof of [SCh] of the fundamental theorem remains valid for typical configurations of strictly dispersing scatterers. The important observation here is Lemma 6.1 which states that for a typical configuration strictly dispersing scatterers the trajectories can have at most $2d - 2$ tangencies.

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2. Preliminaries

2.1. Assumptions about the scatterers. We make the following assumptions about the the scatterers:

1. There are finitely many scatterers $R_1, \ldots, R_K$.
2. Each scatterer is a strictly convex $C^{8d-7}$ hypersurface,
3. Finite horizon: the length of the trajectory between two successive reflections is uniformly bounded,
4. Time between two successive reflections is bounded from below by some positive constant $\tau_0 > 0$.

Strictly convex here means that the second fundamental form of each scatterer is positively defined at every point.

Throughout the paper we consider the configuration space $Q \subset \mathbb{T}^d$ as a subset of the $\mathbb{R}^d$, the universal cover of $\mathbb{T}^d$. The configuration of scatterers in $\mathbb{T}^d$ lifts to a periodic configuration of scatterers in $\mathbb{R}^d$. We denote the phase space of the corresponding billiard system in $\mathbb{R}^d$ by $\partial M$ and use the same notation for the billiard map in $\mathbb{R}^d : T : \partial M \to \partial M$.

To fix the notations for the scatterers we choose $K$ scatterers $\hat{R}_1, \ldots, \hat{R}_K$ in $\mathbb{R}^d$, so that these scatterers satisfy the following two properties:

- for every scatterer $R_m$ on $\mathbb{T}^d$ there exists a unique scatterer $\hat{R}_m$ in $\mathbb{R}^d$, which projects to $R_m$
- $\hat{R}_m$ is the closest to the origin among the preimages of $R_m$

Fix coordinate system in $\mathbb{R}^d$ and assume that $R_m$ is given as zero level set of a function $R_m \in C^{8d-7}(\mathbb{R}^d)$. The rest of scatterers are then given as zero level sets...
of the shifts of functions \( R_k \) by elements of the lattice which generates the torus. The assumption \( \Delta \) on the scatterers implies that for each \( k = 1, \ldots K \) the hessian of \( R_k \) is positive definite at each \( z \in \mathbb{R}^d \), such that \( R_k(z) = 0 \).

2.2. **Singularities.** Consider the set of tangential reflections\
\[ S := \{(q, v) \in \partial M | (v, n(q)) = 0 \}. \]

It is easy to see, that the map \( T \) is not continuous at the set \( T^{-1}S \). As a result, the singularity set for any iterate \( T^n \) is\
\[ S^{(n)} = \bigcup_{i=-1}^{n} S^{-i}, \]
where in general \( S^k = T^kS \). The set of regular trajectories, i.e. the trajectories, for which \( T^i x \notin S \), \( -\infty < i < \infty \), will be denoted by\
\[ \partial M^R := \partial M \setminus \bigcup_{n \in \mathbb{Z}} S^n. \]

In the case when the boundary \( \partial Q \) is only piecewise smooth, additional singularities - multiple collisions - arise. They correspond to points \( q \in \partial Q \), which belong to several smooth components of the boundary. At these points the dynamics is not well-defined. The assumption \( \Delta \) implies that we consider only tangential singularities. It is enough to consider this situation, since the blow-up of the derivative of the map \( T \) - the effect found in [BCST1] doesn’t occur for the singularities corresponding to multiple collisions.

3. **Space of lines**

The dynamics in a billiard system consists of two parts: the motion along a straight line and a reflection. It is the reflection part that causes discontinuities and singularities, so it is natural to isolate it. One way to do so is to look at the dynamics on the space of oriented lines.

In this and next section we assume, that the dynamics takes place in \( \mathbb{R}^d \). Consider a point \( x = (q, v) \in \partial M \) in the phase space of a billiard. The pair \( (q, v) \) uniquely determines an oriented line \( l \) in \( \mathbb{R}^d \). The image \( x' = (q', v') = T(x) \) determines a new line \( l' \), which is by definition the image of \( l \). This map on the space of oriented lines will be called the billiard map on the space of oriented lines. Since the same oriented line can intersect several scatterers, the dynamics is globally not well-defined. However, if we are interested in what happens near a fixed trajectory in a finite number of iterations of the billiard map, then we know from which scatterers the lines are reflected and so the dynamics is well-defined.

3.1. **Coordinates on the space of oriented lines.** A line \( l \) in \( \mathbb{R}^d \) can be characterized by its closest point to the origin \( p \), and by its direction \( v \). Vector \( v \) can be normalized to have length one, since multiplying \( v \) by any positive constant still gives the same direction of the line:

\[
||v|| = 1
\]

Clearly,

\[
(p, v) = 0
\]

These coordinates give the identification of he space of oriented lines in \( \mathbb{R}^d \) with the (co)tangent bundle of unit sphere \( S^d \). We denote this space by \( \Omega \). The symplectic structure given by this identification differs from the standard symplectic structure on \( T^*S^d \), by sign. For a fixed \( v \) we shall denote the plane \( \Omega_v \) in \((\mathbb{R}^d, p)\) by \( \Pi_v \).
Lemma 3.1. Suppose \( x_0 = (q_0, v_0) \in \partial M \) and \( U(x_0) \) is a neighborhood of \( x_0 \) in \( \partial M \), such that for every \( x = (q, v) \in U(x_0) \) we have

\[
(v, n(q)) \geq c > 0,
\]

where \( c \) is some positive constant. Then there exists a diffeomorphism \( \Phi \in C^{s^d-7}(U(x_0)) \) from the neighborhood \( U(x_0) \) to the neighborhood of the line \( t_0 \) determined by \( x_0 \), such that \( \Phi(x) = t_0 \) and \( |\det D\Phi| \geq c \).

Proof. The diffeomorphism \( \Phi \) is given by the formula

\[
\Phi(q, v) = (\pi_\omega(q), v),
\]

where \( \pi_\omega : \mathbb{R}^d \to \mathbb{R}^d \) is a projection to the plane \( \Pi_\omega \) in the direction of vector \( v \). It follows from (3.3), that \( |\det D\Phi| \geq c \). \( \square \)

4. Local structure of a reflection near tangency

In this section we consider the reflection of trajectories from a fixed scatterer given by the equation \( F(x_1, \ldots, x_d) = 0 \). We start by showing that the set of lines tangent to this scatterer is a smooth hypersurface in the space of oriented lines.

Consider a unit tangent bundle \( T_1 \mathbb{R}^d \). It is obviously a bundle over \( \mathbb{R}^d \) with fibers isomorphic to \( S^d \). Any function \( F(x_1, \ldots, x_d) \) lifts to a function \( \tilde{F} \) on \( T_1\mathbb{R}^d \), which constant along the fibers. The space \( T_1 \mathbb{R}^3 \) is also a bundled over \( \mathbb{R}^d \), the space of lines in \( \mathbb{R}^d \). Denote by \( \pi \) the projection corresponding to this bundle. The fibers \( \pi^{-1}(\omega) \), \( \omega \in \Omega \) are the points in \( \mathbb{R}^d \) which lie on a given line with the direction determined by \( \omega \).

Denote the coordinate in the fiber by \( t \). If \( x = (p, v) \) are coordinates in \( \Omega \), then \( (x, t) \) are the coordinates in \( T_1 \mathbb{R}^d \). The set of lines tangent to the scatterer is the projection of the discriminant set \( \Sigma \) given by

\[
\tilde{F}(x, t) = 0, \quad \frac{\partial \tilde{F}}{\partial t}(x, t) = 0
\]
on \( \Omega \).

Lemma 4.1. Let \( (x_0, t_0) \in \Sigma \). There exists a neighborhood \( U \) of \( (t_0, x_0) \) in \( \mathbb{R} \times \Omega \), such that \( \pi(\Sigma \cap U) \) is a parameterized \( 2d - 3 \) dimensional \( C^{s^d-9} \) manifold in \( \Omega \).

Proof. Consider a mapping \( G : \mathbb{R}^{2d-2} \times \mathbb{R} \to \mathbb{R}^2 \), given by \( G(x, t) = (\tilde{F}, \frac{\partial \tilde{F}}{\partial t})(x, t) \). Then \( \Sigma = G^{-1}(0) \). Jacobi matrix for \( G \) is equal to

\[
\begin{pmatrix}
\frac{\partial \tilde{F}}{\partial x_1} & \cdots & \frac{\partial \tilde{F}}{\partial x_{2d-2}} & \frac{\partial \tilde{F}}{\partial t} \\
\frac{\partial \tilde{F}}{\partial x_1 t} & \cdots & \frac{\partial \tilde{F}}{\partial x_{2d-2} t} & \frac{\partial^2 \tilde{F}}{\partial x^2 t}
\end{pmatrix}
\]

Notice that from the assumption \( \Box \) on the scatterers, it follows that

\[
\frac{\partial^2 \tilde{F}}{\partial t^2}(x_0, t_0) \neq 0
\]

Also, there exists an index \( i, 1 \leq i \leq d - 1 \), such that \( \frac{\partial \tilde{F}}{\partial x_i}(x_0, t_0) \neq 0 \). Assume that \( i = d - 1 \). Then by the Implicit Function Theorem applied to \( G \) we can express \( t \) and \( x_{d-1} \) as the functions of \( x_1, \ldots, x_{d-1}, x_{2d-3} \) on \( \Sigma \cap U \) for a neighborhood \( U \) of \( (t_0, x_0) \). Hence, on \( \pi(\Sigma \cap U) \), \( x_{d-1} \) is a smooth function of \( x_1, \ldots, x_{d-1}, x_{2d-2} \). \( \square \)
Function $\tilde{F}: \Omega \times \mathbb{R} \to \mathbb{R}$ induces a function $F: \Omega \to \mathbb{R}$. For a given $\omega \in \Omega$

$$F(\omega) = \inf_{(x,t) \in \pi^{-1}(\omega)} \tilde{F}(x,t).$$

It follows from the assumption 2 on the scatterers that for $\omega$ sufficiently close to $\partial\Omega^+$ the infimum is attained at a unique point $(\omega,t(\omega))$. The zero level set of $F$ coincides with the set of lines tangent to the scatterer. Let $\Omega^+ = \{\omega \in \Omega | F(\omega) \leq 0\}$ be the set of lines that experience a reflection from the scatterer. Its boundary $\partial\Omega^+$ consists of lines tangent to the scatterer. On $\Omega^+$ the billiard map $T$ is defined. It sends the lines before the reflection from the scatterer to the reflected lines.

Let $x_0 \in \partial\Omega^+$, by the previous Lemma there exists a neighborhood $U(x_0) \subset \Omega$, such that the $\partial\Omega^+ \cap U(x_0)$ is a hypersurface parameterized by $(x_1, \ldots, x_{d-1}, x_{2d-2})$. In particular, this implies that for any fixed $v_1$ sufficiently close to $v_0$ the set of lines $\{(p,v) \in U(x_0) \cap \partial\Omega^+ \mid v = v_1\}$ is a parameterized $d-2$ dimensional manifold in $\Pi_{v_1}$. We can assume that it is parameterized by $(p_1, \ldots, p_{d-2})$. For every point on this manifold we consider a line passing through this point in the direction perpendicular to $\partial\Omega^+ \cap U(x_0)$. Let’s choose a parameter $\tau$ on these lines so that $\tau = 0$ corresponds to a point on $\partial\Omega^+$ and so that $\tau > 0$ corresponds to reflection. For any $x \in U(x_0)$ there exists a point $x_1 \in \partial\Omega^+ \cap U(x_0)$ and $\tau$, such that $x$ lies on the line passing through $x_1$ perpendicular to $\partial\Omega^+$ and $\text{distance}(x_1, \partial\Omega^+ \cap U(x_0)) = \tau$. Let

$$\lambda_0 = \min_{x \in \mathbb{R}^d: F(x) = 0} \min_{w \in \mathbb{R}^d: ||w|| = 1} (\text{Hess}(F)(x)w, w),$$

where $\text{Hess}(F)(x)$ is the Hessian of $F$ at $x$. Pick a neighborhood $U_1(x_0) \subset U_0(x_0)$ such that for every $x \in U_1(x_0)$ we have $\text{distance}(x, \partial\Omega^+) < \lambda_0/2$. Then the neighborhood $U_1(x_0)$, is parameterized by $(p_1, \ldots, p_{d-2}, \tau, v)$.

**Definition 4.2.** Suppose $H^+ = \{(x_0, \ldots, x_n) | x_n > 0\}$. The mapping

$$\begin{cases}
  y_0 = x_0, \\
  \ldots \\
  y_{n-1} = x_{n-1} \\
  y_n = \sqrt{x_n}
\end{cases}$$

of $H^+$ into itself will be called a quasi-regular change of variables.

**Lemma 4.3.** After a quasi-regular change of variables $v = \sqrt{\tau}$ in a semi-neighborhood $U_1(x_0) \cap \Omega^+$ parameterized as above the billiard map becomes a $C^{4d-4}$ diffeomorphism.

**Proof.** The lemma follows from the following decomposition of the billiard map.

Consider again the unit tangent bundle $T_1\mathbb{R}^3$ as a bundle over the space of lines $\Omega$. The scatterer $F = 0$ defines a section of this bundle over the points corresponding to lines that experience a reflection: to each such lines one associates the point on the scatterer where the line hits the scatterer and the direction of this line. The points of this section are the pairs $\{(p,v) | p \in \mathbb{R}^d, R(p) = 0, ||v|| = 1\}$ form a surface $C$ topologically equivalent to $S^3 \times S^d$. On this surface the following smooth map is defined. To each $(p,v) \in C$ one associates $(p', v')$, where $p = p'$ and $v'$ is a reflection of $v$ with respect to $\nabla R(p)$. This map is smooth on the surface $C$, but the projection from $\Omega$ to $C$ is not smooth at the points corresponding to the lines tangent to $R = 0$. The only thing one needs to show is that this projection is smooth after a quasi-regular change of variable.
Fix any direction \( v \in S^d \) close to \( v_0 \). The projection from \( \Omega \) to \( C \) for a fixed \( v \) is the projection from the hyperplane \( \Pi_v \) to the scatterer in \( \mathbb{R}^d \). Let \( t \) be the coordinate in \( \mathbb{R}^d \) in the direction perpendicular to \( \Pi_v \). For fixed coordinates \( p_1, \ldots, p_{d-2} \) the projection is a function \( t = f(\tau) \). Assume that \( f(0) = 0 \). Since the scatterer is strictly convex at each point, we get that \( \tau = t^2 g(t) \), where \( g(0) \neq 0 \) and so \( v = \pm t \sqrt{g(t)} \). We choose the sign so that it corresponds to the trajectories coming to the scatterer. Then by the Implicit Function Theorem \( t \) is a smooth function of \( v \) and so after a quasi-regular change of coordinated the billiard map becomes smooth.

5. Resolution of the singularity

Using the description of the billiard map given by Lemma 4.3, we show that the preimage of a level set of some \( F \in C^k(\Omega) \) function under the billiard map can be represented as a level set of a function of the same class of smoothness.

The billiard map (and its inverse) is smooth away from tangencies, so in a neighborhood of any point corresponding to a non-tangential reflection the pullback of \( F \) is smooth. We show that in a (semi)neighborhood of a tangency the preimage of a level set of a smooth function can still be given as a level set of a smooth function. Consider the coordinates from the Lemma 4.3 and suppose that the set of lines (the example in mind is the set of lines tangent to some other scatterer or its preimage) is given by the equation

\[
F(x_1, \ldots, x_n) = 0
\]

In the the decomposition given by the lemma, only the substitution \( \tau \to \sqrt{\tau} \) is singular, so it is enough to look at this part separately.

**Lemma 5.1.** Let \( U \) be a neighborhood of zero in \( \mathbb{R}^n \), \( F \in C^k(U \cap \{(x_1, \ldots, x_n)|x_n \geq 0\}) \), then there exists a neighborhood \( U_1 \subset U \) of zero and functions \( G_+, G_- \in C^k(U_1) \), such that

\[
F(x_1, \ldots, x_n) = G_+(x_1, \ldots, x_n^2) + x_n G_-(x_1, \ldots, x_n^2)
\]

**Proof.** Extend \( F \) (for example, by Whitney extension theorem) to small full neighborhood of zero and denote the continuation by \( \hat{F} \). Let

\[
F_+(x) = \frac{\hat{F}(x) + \hat{F}(-x)}{2}, \quad F_- = \frac{\hat{F}(x) - \hat{F}(-x)}{2}
\]

be the decomposition of \( \hat{F} \) in even and odd parts. By a theorem of Glaeser [BL], a function, which is even in some variable is a function of the square of this variable, so there exists a neighborhood \( U_1 \), and functions \( G_+, G_- \) as in the statement of the lemma, such that \( F_+(x_1, \ldots, x_n) = G_+(x_1, \ldots, x_n^2) \) and \( F_-(x_1, \ldots, x_n) = x_n G_-(x_1, \ldots, x_n^2) \). This implies the existance of the decomposition (5.5).

**Corollary 5.2.** The level set \( \{F(x_1, \ldots, \sqrt{x_n}) = 0\} \cap U_1 \) is given by the equation \( G_+^2 = x_n G_-^2 \).

It follows from (5.5) that if \( F \) has a non-zero \( m \)-jet, \( 0 < m < k/2 \) then \( G_+^2 - x_n G_-^2 \) has non-zero \( 2m \)-jet.
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6. No more than $2d - 2$ tangencies for generic scatterers

Since the dimension of the phase space is equal to $2d - 2$, it is natural to assume that, in a certain sense, no degenerations of codimension greater than $2d - 2$ can appear in a typical billiard system. Here we formulate and prove one such statement that will be used in the next section.

Recall that the billiard system in $\mathbb{R}^d$ consists of $K$ scatterers given by the equation $R_i = 0$, $i = 1, \ldots, K$ and their shifts by the elements of the lattice which generates the torus $\mathbb{T}^d$. The collection of functions $\{R_1, \ldots, R_K\}$ is a point in $[C^{4d-4}(\mathbb{R}^d)]^K$.

Proposition 6.1. For any $0 \leq m \leq 8d - 7$ and $\varepsilon > 0$ there exists a collection of scatterers $\{R_1(\varepsilon), \ldots, R_K(\varepsilon)\} \in [C^{8d-7}(\mathbb{R}^d)]^K$, such that

- for the billiard system defined by $\{R_1(\varepsilon), \ldots, R_K(\varepsilon)\}$ there are no trajectories with more than $2d - 2$ (not necessarily successive) tangencies
- $\|R_1(\varepsilon) - R_1\|_{C^m(\mathbb{R}^d)} + \ldots + \|R_K(\varepsilon) - R_K\|_{C^m(\mathbb{R}^d)} < \varepsilon$

Proof. We prove the proposition for the case $d = 3$ and some fixed $m \geq 2$. The proof proceeds by induction on the number of reflections that can occur between tangencies. At the step $k$ we make perturbations of the scatterer, so that

- The set of trajectories with three tangencies and at most $k$ proper reflections in between form a finite union of ruled surfaces in the configuration space.
- Each ruled surface intersects every scatterer transversally.
- There are no 5-tangencies with at most $k$ reflections in between.

Let’s first describe two types of perturbations used in the proof.

Type 1 Suppose that a ruled surface in $\mathbb{R}^3$ is tangent to a smooth strictly convex surface given by $F(x, y, z) = 0$ along a curve $\gamma$. Then for every $\varepsilon_1, \varepsilon_2 > 0$ there exists a smooth strictly convex function $\tilde{F}$, such that $\|F - \tilde{F}\|_{C^m(\mathbb{R}^3)} < \varepsilon_1$, $F(x, y, z) = \tilde{F}$, if the distance from $(x, y, z)$ to $\gamma$ in $\mathbb{R}^3$ is greater or equal to $\varepsilon_2$ and such that the ruled surface intersects the surface $\tilde{F} = 0$ transversally at each point. In a degenerate situation, when $\gamma$ is a point, this type includes the case when a ruled surface has a non-transversal intersection with the convex surface at one point.

Type 2 Suppose that a line $l$ is tangent to a smooth strictly convex surface given by $F(x, y, z) = 0$ at a point $p$. Then for every $\varepsilon_1, \varepsilon_2 > 0$ there exists a smooth strictly convex function $\tilde{F}$, such that $\|F - \tilde{F}\|_{C^m(\mathbb{R}^3)} < \varepsilon_1$, $F(x, y, z) = \tilde{F}$, if the distance from $(x, y, z)$ to $p$ in $\mathbb{R}^3$ is greater or equal to $\varepsilon_2$ and such that the line $l$ intersects the surface $\tilde{F} = 0$ transversally.

Let us call by a combinatorial type of a trajectory of length $L$ a sequence of $L$ scatterers in $\mathbb{R}^n$ labelled by "what happens" at this scatterer (a reflection or a tangency).

We know from Lemma [11], that the set of trajectories tangent to a given scatterer form a smooth hypersurface. In the same way, the set of trajectories which have two successive tangencies to given two scatterers is a smooth 2-dimensional manifold $T$ with a boundary. If we allow additional reflections between the two tangencies, then additional components appear. Due to the effect described in [11], the manifold may behave irregular at certain points of new components of the boundary. However, the set of these points $S$ is a finite union of smooth
one-dimensional manifolds. Hence by small perturbation of type 1 of the scatterers we can make the ruled surfaces which correspond to these manifolds transversal to the scatterers and so that no points of $S$ belong to more than one component of the boundary.

This implies, that if we consider the set of trajectories tangent to the previous two scatterers (with a number of reflections in between) and a new scatterer, then (possible after an additional perturbation of type 1 of the new scatterer) the set of tangents to the new scatterer intersect $T$ along a curve which has no points at which $T$ is irregular.

This shows that the set of trajectories with three tangencies of any combinatorial type of fixed length is a one-parameter family of trajectories which form a ruled surface in the configuration space. The set of trajectories with 4 tangencies of a fixed combinatorial type maybe non-discrete if and only if this ruled surface (or it’s image under a billiard map) is tangent to some scatterer along some curve on the scatterer. We make the perturbations of Type 2 of the scatterers so that all such ruled surfaces are transversal to all scatterers using perturbations of type 1.

Thus for every fixed length of combinatorial type we get finitely many trajectories with 4 tangencies. Suppose there exists a 5-tangency among them. Consider a line which corresponds to the trajectory “right before” the fifth tangency. By choosing a sufficiently small perturbation of the scatterer at the point of the fifth tangency we can make the line transversal to the scatterer and not create any new 5-tangencies. Thus we can eliminate 5-tangencies for a fixed length of the combinatorial class one by one.

To make this work we use the induction on the number of reflections between the first and the last tangency.

Let $k$ denote the length of the considered trajectories. We start with $k = 3$. Fix some $\varepsilon_0 > 0$ First, we find some ruled surface of order 0, which is tangent to some scatterer along a curve. There is a finite number of ruled surfaces of order 0 associated with this scatterer. Among them some of the ruled surfaces are transversal to each scatterer at each point of intersection. We find a perturbation which destroys this non-discrete tangency and keeps all ruled surface of order 0, which are associated to this scatterer transversal to other scatterers. Now the number of ruled surfaces of order 0 which are tangent to some scatterer along a curve decreased by one. In the same way we eliminate all non-transversal intersections of ruled surfaces with the scatterers at points. This way we get that the set of trajectories with four successive tangencies is finite.

Notice that there exists a positive number $\alpha$, such that any perturbation of size less that $\alpha$, all ruled surfaces for the perturbed system are transversal to every scatterer. Here $\alpha$ depends on the angles between ruled surfaces of order zero and the scatterers.

We now make additional perturbations to eliminate all five successive tangencies. Fix one successive 5-tangency. Since the set of successive 4-tangencies is finite, there exists a neighborhood $U$ of the point of tangency on the last scatterer, in which the minimal angle between the directions of incoming 4-tangencies and the trajectories tangent to the scatterer at the points in $U$ is greater or equal to $c_1$. Then by perturbing (Type 2) the scatterer in $U$ so that the directions of the lines tangent to the scatterer at the points of $U$ change by at most $c_1/2$ we make the scatterer
transversal (locally) to the trajectory and we don’t create additional successive 5-
tangencies are created. By making a finite number of such perturbations we can
eliminate all successive 5-tangencies.

The same argument applies to the induction step.

Notice, that at each step a finite number of perturbations of three types is
made: the ones used to get that 3-tangencies are smooth 1-parameter families of
trajectories, the ones which eliminate non-transversal intersections of ruled surfaces
with the scatterers and the ones which eliminate 5-tangencies. One can choose
the size of the perturbations at each step in such a way that the perturbations
made after step \(N\) keep the transversal manifolds considered at the previous steps
transversal.

Indeed, let \(\alpha_n\) be such that any perturbation of size less than \(\alpha_n\) keeps 3-
tangencies being smooth 1-parameter families of trajectories, \(\beta_n < \alpha_n/2K^{n+5}\) be
such that any perturbation of size less than \(\beta_n\) keeps ruled surfaces of order \(\leq n\)
transversal and let \(\gamma_n < \beta_n/2K^{n+3}\) be such that any perturbation of size less
than \(\gamma_k\) doesn’t introduce 5-tangencies of order \(\leq k\). One can choose
\(\alpha_j, \beta_j, \gamma_j\), \(j = 1, 2, \ldots\) in such a way that for every \(N > 0\):

\[
\sum_{n \geq N} K^{n+4} \alpha_n + K^{n+5} \beta_n + K^{n+3} \gamma_n \leq \gamma_{N-1}/2
\]

and

\[
\sum_{n \geq 0} K^{n+4} \alpha_n + K^{n+5} \beta_n + K^{n+3} \gamma_n < \varepsilon_0.
\]

Then the series of perturbations converges, the total perturbations is less than \(\varepsilon_0\)
and there are no trajectories with more that 4 tangencies for the obtained collection
of scatterers.

From now on assume that for the configuration of the scatterers we consider
there are no trajectories with more that 2d – 2 tangencies. \(\square\)

7. Estimates on the measure

Let \(B(x, r)\) denote the open ball of radius \(r\) about \(x\) in \(\mathbb{R}^n\). The following theorem
is proved in [BF].

**Theorem 7.1.** Let \(F\) be a real-valued \(C^m\) function on \(B(0, 1)\), with

(1) \(c_0 < \max_{|\alpha|=m-1} |\partial^\alpha F(0)| < C_0\), and with

(2) \(|\partial^\alpha F| \leq C_1\) on \(B(0, 1)\) for \(|\alpha| = m\).

Let

(3) \(V(F) = \{x \in B(0, 1) : F(x) = 0\}\), and let

(4) \(V(F, \delta) = \{x \in B(0, c_1) : \text{distance}(x, V(F)) < \delta\}\),

where \(c_1\) is a small enough constant determined by \(c_0, C_0, C_1, m, n\).

Then

\[\text{Vol}\{V(F, \delta)\} \leq C_2 \delta\text{ for } 0 < \delta < c_1,\]

where \(C_2\) is a large constant determined by \(c_0, C_0, C_1, m, n\).

Thus, if \(F\) is a smooth function that never vanishes to infinite order, then the
set of points within the distance \(\delta\) from the zeroes of \(F\) has volume \(O(\delta)\).
Corollary 7.2. Let \( x_0 \in \partial M \setminus S \). There exists a neighborhood \( U(x_0) \subset \partial M \setminus S \), such that the measure of the \( \delta \)-neighborhood of any component of \( T^{-k}S \cap U \) is of order \( O(\delta) \).

Proof. By Lemma 8.1 there is a \( C^{8d-7} \) diffeomorphism of a neighborhood of \( x_0 \) to the neighborhood of the corresponding line in the space of lines in \( \mathbb{R}^d \) with the jacobian bounded away from zero.

From Lemma 8.3 Lemma 8.4 and Proposition 8.1 it follows that in the neighborhood of any of its points in the space of lines the singularity manifold \( T^{-k}S \) is given by a \( C^{4d-3} \) function with a non-zero \( 4d - 4 \)-jet. The theorem 8.1 gives the estimate in a neighborhood of any point of \( T^{-k}S \cap U \). \( \square \)

8. The fundamental theorem

In this section we formulate a version of the fundamental theorem for multidimensional dispersing billiards. This theorem was first formulated and proved in [S1] for plane billiards. It allows to apply the modified Hopf argument to prove local ergodicity of the system. The derivation of the local ergodicity from the fundamental theorem is done in details in [KSS2], [LW] and is omitted here.

In the second part of this section we give the definitions required in the proof of the fundamental theorem and show that under our main assumption in the case of strictly dispersing scatterers one needs only finitely smooth scatterers. The fundamental theorem for multidimensional dispersing billiards was first formulated and proved by Chernov and Sinai [SCh] and generalized in [KSS2]. We follow here the exposition in [BCST2].

8.1. The statement of the Fundamental Theorem. Let \( x \in \partial M \) be such a point, that the trajectory of \( x \) has at most one tangency. Consider a small neighborhood \( U(x) \). A parameterized family of finite coverings

\[
G^s_i = \{ G^s_i | i = 1, \ldots, I(\delta) \} \quad 0 < \delta < \delta_0
\]

is a family of regular coverings iff:

1. each \( G^s_i \) is a "parallelepped", that is the image of a 2\(d - 2\) dimensional cube under the linear map \( \mathbb{R}^{2d-2} \rightarrow \partial M \),
2. the faces of each \( G^s_i \) are parallel to tangent planes of stable and unstable manifolds \( W^s(w(G^s_i)) \) and \( W^u(w(G^s_i)) \) passing through the center of the parallelepiped \( (w(G^s_i)) \),
3. for any point, the number of parallelepipeds covering it is at most \( 2^{2d-2} \),
4. if \( G^s_i \cap G^s_j \neq \emptyset \), then

\[
\mu(G^s_i \cap G^s_j) \geq c_1 \delta^{2d-2}
\]

where \( c_1 \) is independent of \( \delta \).

For a given \( G^s_i \) s-faces are those parallel to \( W^s(w(G^s_i)) \) and u-faces are those parallel to \( W^u(w(G^s_i)) \). For \( y \in G^s_i \) we say that a stable manifold \( W^s(y) \) intersects \( G^s_i \) correctly, if \( \partial(G^s_i \cap W^s(y)) \) belongs to the union of u-faces and that an unstable manifold \( W^u(y) \) intersects \( G^s_i \) correctly, if \( \partial(G^s_i \cap W^u(y)) \) belongs to the union of u-faces.

Theorem 8.1. Let \( x \in M \) be such a point, that the trajectory of \( x \) has at most one tangency and \( 0 < \varepsilon_1 < 1 \) be a fixed constant. Then there exists a sufficiently small neighborhood \( U_{\varepsilon_1} \), such that for any family of regular coverings \( G^s \) of \( U_{\varepsilon_1} \), the
covering $G^\delta$ can be divided into two disjoint subsets, $G_g^\delta$ and $G_b^\delta$ (called 'good' and 'bad'), in such a way that:

- For any $G^\delta_i \in G_b^\delta$ and any $s$-face $E^s$ of it, the set

  \[ \{ y \in G_i^\delta | \rho(y, E^s) < \varepsilon_1 \delta \text{ and } \gamma_s(y) \text{ intersects correctly} \} \]

  has positive relative $\mu_1$ measure in $G_i^\delta$.

- $\mu_1(\bigcup G_i^\delta \in G_b^\delta) = o(\delta)$

8.2. The proof of the Fundamental theorem. Let us remind the main ideas of the proof of the theorem. Informally, the parallelepiped is good if sufficiently many (in the sense of the relative measure in the parallelepiped) stable manifolds are not cut inside this parallelepiped. A given parallelepiped can be bad either because it intersects the preimages of the singularity manifolds $T^{-1}S, T^{-2}S, \ldots T^{-F(\delta)}S$ or because it intersects $T^{-F(\delta)-1}, \ldots$. The set of bad parallelepipeds of the second type is small because of hyperbolicity of the system. This is a so-called tail estimate, which becomes rather simple in the setting of strictly dispersing scatterers (see \[SCh\] for the proof). As it was noticed in \[BCST2\] this part doesn’t require information about the structure of the preimages of the singularity manifolds. To estimate the measure of the remaining part of bad parallelepipeds we notice that

1. for a sufficiently small $\delta$ there are no parallelepipeds which intersect more than four singularity manifold from $T^{-1}S, T^{-2}S, \ldots T^{-F(\delta)}S$.
2. a parallelepiped intersected by at most four singularity manifolds cannot be bad because by choosing a sufficiently small neighborhood of $x$ we can assume, that the angle between stable manifolds and singularity manifolds is arbitrary small.

As it is explained in \[BCST2\] it is here that the additional information about singularity manifolds is needed.

We formulate the lemma about parallelization (see \[KSSz\], Lemma 4.9).

**Lemma 8.2.** Given any $x \in \partial M^0$ and any $\varepsilon > 0$ there is a neighborhood $U(x) \subset \partial M$ such that for every $\gamma_1^v, \gamma_2^v$ and any 3-dimensional manifold $T^{-n}S$ ($n > 0$) intersecting $U(x)$ with points $y_1, y_2$ and $\hat{y}$, lying on three manifolds, respectively, so that $T_yT^{-n}S$ exists:

\[
\angle(T_{y_1}\gamma_1^v, T_{y_2}\gamma_2^v) < \varepsilon
\]

\[
\angle(T_{y_1}\gamma_1^v, T_\hat{y}T^{-n}S) < \varepsilon
\]

We introduce some more notations. The following two quantities measure the hyperbolicity near the point $y \in \partial M^0$. Let

\[
\kappa_{n,0}(y) = \inf_{\Sigma} \| (D^n_{-T^n y}T^n y, \Sigma) \|^{-1}_p,
\]

where the infimum is taken over all convex local orthogonal manifolds passing through $-T^n y$. Furthermore denote

\[
\kappa_{n,\delta}(y) = \inf_{\Sigma} \inf_{w \in \Sigma} \| (D^n w, \Sigma) \|^{-1}_p
\]

Here the infimum is taken for the set of convex fronts $\Sigma$ passing through $-T^n y$ such that $T^n$ is continuous on $\Sigma$ and $T^n \Sigma \subset B_\delta(-y)$. 


The following subsets of the neighborhood \( U(x) \) depend on the constant \( \delta \):

\[
U^g = \left\{ y \in U | \forall n \in \mathbb{Z}_+, z(T^n y) \geq (\kappa_{n,c_3 \delta}(y))^{-1} c_3 \delta \right\},
\]

\[
U^b = U \setminus U^g
\]

\[
U^b_n = \left\{ y \in U | z_{n \text{lab}}(T^n y) < (\kappa_{n,c_3 \delta}(y))^{-1} c_3 \delta \right\}
\]

**Definition 8.3.** A function \( F : \mathbb{R}_+ \to \mathbb{Z}_+ \) defined in a neighborhood of the origin is called permitted if \( F(\delta) \uparrow \infty \) as \( \delta \downarrow 0 \). For a fixed permitted function \( F(\delta) \) we define \( U^b_n = \bigcup_{n > \delta} U^b_n \).

**Lemma 8.4 (Tail bound).** For any permitted function \( F(\delta) \):

\[
\mu_1(U^b_\omega) = \tilde{o}(\delta)
\]

The proof of the fundamental theorem follows from the following estimates:

- The estimate from below on the measure of \( G^s_\delta \cap (E^s)^{[\varepsilon_4 \delta]} \), where \( E^s \) is an \( s \)-face for the bad parallelepiped \( G^s_\delta \):

\[
\mu_1(G^s_\delta \cap (E^s)^{[\varepsilon_4 \delta]}) \geq c \varepsilon_4^{d-1} \mu_1(G^s_\delta) \geq \varepsilon_3 \mu_1(G^s_\delta)
\]

in case \( \varepsilon_3 = \varepsilon_3(\varepsilon_1) \) is chosen sufficiently small.

- Similarly,

\[
\mu_1(G^s_\delta \cap (\partial G^s_\delta)^{[\varepsilon_4 \delta]}) \leq \frac{\varepsilon_3}{4} \mu_1(G^s_\delta)
\]

for \( \varepsilon_4 = \varepsilon_4(\varepsilon_3) \) chosen sufficiently small.

- The estimate on the measure of the \( \varepsilon_4 \delta \)-neighborhood of the singularity manifold inside the parallelepiped. The Corollary \( \square \) implies that

\[
\mu_1(G^s_\delta \cap (S^n)^{[\varepsilon_4 \delta]}) \leq \varepsilon_3/16 \mu_1(G^s_\delta)
\]

for \( \varepsilon_4 = \varepsilon_4(\varepsilon_3) \) chosen sufficiently small.

- We now choose \( \varepsilon_2(\varepsilon_4) \) small enough, so that by Lemma \( \square \) the stable manifolds and singularity components are 'almost parallel'. Namely, the smallness of \( \varepsilon_2 \) should guarantee that for any \( y \in G^s_\delta \) for which \( \gamma^s(y) \) doesn’t intersect correctly we have:

\[
y \in (G^s_\delta \cap (S^n)^{[\varepsilon_4 \delta]}) \cup (G^s_\delta \cap (\partial G^s_\delta)^{[\varepsilon_4 \delta]}) \cup U^b
\]

Let us now consider a bad parallelepiped \( G^s_\delta \) with an \( s \)-face \( E^s \) for which

\[
\mu_1(G^s_\delta \cap (E^s)^{[\varepsilon_4 \delta]} \cap U_{ic}) \leq \frac{\varepsilon_3}{4} \mu_1(G^s_\delta)
\]

Here \( U_{ic} \) is the set of points in \( G^s_\delta \) with correctly intersecting local stable manifolds.

By the estimates above

\[
\mu_1(G^s_\delta \cap U^b_n) \geq \mu_1(G^s_\delta \cap (E^s)^{[\varepsilon_4 \delta]} \cap U^b_n) \geq \frac{\varepsilon_3}{4} \mu_1(G^s_\delta).
\]

Now recall that in a regular covering there are at most \( 2^{d-2} \) parallelepipeds with a non-empty common intersection. Thus:
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\[ 2^{d-2} \mu_1(U^b_w) \geq \sum' \mu_1(G^b_i \cap U^b_w) \geq \frac{\varepsilon_3}{4} \sum' \mu_1(G^b_i), \]

where \( \sum' \) denotes the sum over bad parallelepipeds for which (8.10) holds for some s-face. By the Tail Bound (Lemma 8.4) we have \( \sum' \mu_1(G^b_i) = \bar{o}(\delta) \) thus the proof of the theorem 8.1 is complete.

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