A POSSIBLE MECHANISM OF GRAVITY

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Abstract:

We consider systems of two free particles in de Sitter invariant quantum theory and calculate the mean value of the mass operator for such systems. It is shown that, in addition to the well known relativistic contribution (and de Sitter antigravity which is small when the de Sitter radius is large), there also exists a contribution caused by the fact that certain decomposition coefficients have different phases. Such a contribution is negative and proportional to the particle masses in the nonrelativistic approximation. In particular, for a class of two-body wave functions the mean value is described by standard Newtonian gravity and post Newtonian corrections in General Relativity. This poses the problem whether gravity can be explained without using the notion of interaction at all. We discuss a hypothesis that gravity is a manifestation of Galois fields in quantum physics.

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# Contents

1 Introduction ........................................... 4  
1.1 Motivation ........................................... 5  
1.2 de Sitter invariance .................................... 8  

2 Basic properties of de Sitter invariant quantum theories 12  
2.1 UIRs of the so(1,4) algebra .............................. 12  
2.2 Poincare limit .......................................... 17  
2.3 de Sitter antigravity ..................................... 19  
2.4 de Sitter antigravity in quasiclassical approximation .... 21  
2.5 Free and interacting systems ............................ 23  

3 de Sitter invariant quantum theory in su(2)×su(2) basis 29  
3.1 UIRs in the su(2)×su(2) basis ........................... 29  
3.2 Implementation of UIRs in the space of functions on the SU(2) group 34  
3.3 Matrix elements in quasiclassical approximation ........... 38  
3.4 Free two-body mass operator ............................ 40  
3.5 Internal Hilbert space for the two-body system .......... 44  

4 Mean value of the two-body mass operator 49  
4.1 Preliminary discussion ................................... 49  
4.2 Standard relativistic mass operator ........................ 51  
4.3 Effect of phase difference ................................ 59  

5 Quantum theory over a Galois field 63  
5.1 What mathematics is most suitable for quantum physics? 63  
5.2 Modular representations of Lie algebras .................. 70
Chapter 1
Introduction

In standard local quantum field theories (LQFT) the Lagrangian of the system under consideration is usually written as $L = L_m + L_g + L_{int}$ where $L_m$ is the Lagrangian of 'matter', $L_g$ is the Lagrangian of gauge fields and $L_{int}$ is the interaction Lagrangian. The symmetry conditions do not define $L_{int}$ uniquely since at least the interaction constant is arbitrary. Nevertheless, such an approach has turned out to be highly successful in QED, electroweak theories and QCD. By analogy with those theories it is usually believed that the gravitational interaction is a consequence of the graviton exchange.

A strong indirect indication on the existence of gravitons has been obtained in measurements of binary pulsars (see e.g. Ref. [1] and the discussion in Refs. [2, 3, 4]), although some physicists express reservations about such a conclusion (see e.g. Ref. [5]). It is also worth noting that the analysis performed in the above references is based only on classical General Relativity (GR) while difficulties in constructing quantum gravity have not been overcome yet. One cannot exclude a possibility that, since gravity is the only known universal interaction, its mechanism is not similar to that for other interactions. On the other hand, the history of physics knows many examples when approaches, which seemed considerably different at first glance, actually did not contradict each other.

In the present paper we investigate whether standard gravitational effects can be obtained without introducing any interaction. Since there exist different approaches to quantum theory in general
and de Sitter invariant quantum theory in particular, we explain our approach in Sects. 1.1 and 1.2. The basic facts of de Sitter invariant quantum theories are discussed in Chapter 2. Chapter 3 is preparatory for calculating the mean value of the mass operator in Chapter 4. Finally, in Chapter 5 we discuss a hypothesis that gravity is a manifestation of Galois fields in quantum physics.

1.1 Motivation

The physical meaning of spacetime is one of the main problems in modern physics. In the standard approach to elementary particle theory it is assumed from the beginning that there exists a background spacetime (e.g. Minkowski or de Sitter spacetime), and the system under consideration is described by local quantum fields defined on that spacetime. Then by using Lagrangian formalism and Noether theorem, one can (at least in principle) construct global quantized operators (e.g. the four-momentum operator) for the system as a whole. It is interesting to note that after this stage has been implemented, one can safely forget about spacetime and concentrate his or her efforts on calculating S-matrix and other physical observables.

There exist two essentially different approaches to quantum theory — the standard operator approach and the path integral approach. We accept the operator approach. In this case, to be consistent, we should assume that any physical quantity is described by a selfadjoint operator in the Hilbert space of states for our system (we will not discuss the difference between selfadjoint and Hermitian operators). Then the first question which immediately arises is that, even in nonrelativistic quantum mechanics, there is no operator corresponding to time [6]. It is also well known that, when quantum mechanics is combined with relativity, there is no operator satisfying all the properties of the spatial position operator (see e.g. Ref. [7]). For these reasons the quantity $x$ in the Lagrangian density $L(x)$ is only a parameter which becomes the coordinate in the classical limit.

These facts were well known already in 30th of the last cen-
tury and became very popular in 60th (recall the famous Heisenberg S-matrix program). In the first section of the well-known textbook [8] it is claimed that spacetime and local quantum fields are rudimentary notions which will disappear in the ultimate quantum theory. Since that time, no arguments questioning those ideas have been given, but in view of the great success of gauge theories in 70th and 80th, such ideas became almost forgotten.

The problem of whether the empty classical spacetime has a physical meaning, has been discussed for a long time. In particular, according to the famous Mach’s principle, the properties of space at a given point depend on the distribution of masses in the whole Universe. As described in a wide literature (see e.g. Refs. [9, 10] and references therein), Mach’s principle was a guiding one for Einstein in developing GR, but when it has been constructed, it has been realized that it does not contain Mach’s principle at all! As noted in Ref. [9, 10], this problem is not closed.

At present, the predictions of the standard model are in agreement with experiment with an unprecedented accuracy. At the same time, the well-known difficulties of the LQFT have not been overcome. For this reason there exist several approaches (string theory, noncommutative quantum theory etc.) in which the product of local interacting quantum fields at the same spacetime point is somehow avoided.

Consider now the problem of how one should define the notion of elementary particles.

Although particles are observable and fields are not, in the spirit of the LQFT, fields are more fundamental than particles, and a possible definition is as follows [11]: 'It is simply a particle whose field appears in the Lagrangian. It does not matter if it’s stable, unstable, heavy, light — if its field appears in the Lagrangian then it’s elementary, otherwise it’s composite’.

Another approach has been developed by Wigner in his investigations of unitary irreducible representations (UIRs) of the Poincare group [12]. In view of this approach, one might postulate that a particle is elementary if the set of its wave functions is the space of a UIR
of the symmetry group in the given theory (see also Ref. [13]).

Although in standard well-known theories (QED, electroweak theory and QCD) the above approaches are equivalent, the following problem arises. The symmetry group is usually chosen as a group of motions of some classical manifold. How does this agree with the above discussion that quantum theory in the operator formulation should not contain spacetime? A possible answer is as follows. One can notice that for calculating observables (e.g. the spectrum of the Hamiltonian) we need in fact not a representation of the group but a representation of its Lie algebra by Hermitian operators. After such a representation has been constructed, we have only operators acting in the Hilbert space and this is all we need in the operator approach. The representation operators of the group are needed only if it is necessary to calculate some macroscopic transformation, e.g. time evolution. In the approximation when classical time is a good approximate parameter, one can calculate evolution, but nothing guarantees that this is always the case (e.g. at the very early stage of the Universe). An interesting discussion of this problem can be found in Ref. [14]. Let us also note that in the stationary formulation of scattering theory, the S-matrix can be defined without any mentioning of time (see e.g. Ref. [15]). For these reasons we can assume that on quantum level the symmetry algebra is more fundamental than the symmetry group.

In other words, instead of saying that some operators satisfy commutation relations of a Lie algebra $A$ because spacetime $X$ has a group of motions $G$ such that $A$ is the Lie algebra of $G$, we say that there exist operators satisfying commutation relations of the Lie algebra $A$ such that: for some operator functions $\{O\}$ of them, the classical limit is a good approximation, a set $X$ of the eigenvalues of the operators $\{O\}$ represents a classical manifold with the group of motions $G$ and its Lie algebra is $A$. This is not of course in the spirit of famous Klein’s Erlangen program or LQFT.

Consider for illustration the well-known example of nonrelativistic quantum mechanics. Usually the existence of the Galilei spacetime is assumed from the beginning. Let $(r, t)$ be the space-time coordi-
coordinates of a particle in that spacetime. Then the particle momentum operator is $-i\partial/\partial r$ and the Hamiltonian describes evolution by the Schrödinger equation. In our approach one starts from an irreducible representation of the Galilei algebra. The momentum operator and the Hamiltonian represent four of ten generators of such a representation. If it is implemented in a space of functions $\psi(p)$ then the momentum operator is simply the operator of multiplication by $p$. Then the position operator can be defined as $i\partial/\partial p$ and time can be defined as an evolution parameter such that evolution is described by the Schrödinger equation with the given Hamiltonian. Mathematically the both approaches are equivalent since they are related to each other by the Fourier transform. However, the philosophies behind them are essentially different. In the second approach there is no empty spacetime (in the spirit of Mach’s principle) and the spacetime coordinates have a physical meaning only if there are particles for which the coordinates can be measured.

Summarizing our discussion, we assume that, by definition, on quantum level a Lie algebra is the symmetry algebra if there exist physical observables such that their operators satisfy the commutation relations characterizing the algebra. Then, a particle is called elementary if the set of its wave functions is a space of irreducible representation of this algebra by Hermitian operators. Such an approach is in the spirit of that considered by Dirac in Ref. [16].

In the literature, irreducible representations of Lie algebras by Hermitian operators are often called UIRs meaning that the representation of the Lie algebra can be extended to an UIR of the corresponding Lie group. Although we are interested in representations of Lie algebras, it is sometimes convenient to investigate their properties by using well known facts about UIRs of the corresponding Lie groups.

### 1.2 de Sitter invariance

As already mentioned, our goal is to investigate whether the standard gravitational effects can be obtained in the framework of a free the-
ory. In the standard nonrelativistic approximation gravity can be described by adding the term $-Gm_1m_2/r$ to the nonrelativistic Hamiltonian, where $G$ is the gravitational constant, $m_1$ and $m_2$ are the particle masses and $r$ is the distance between the particles. Since the kinetic energy is always positive, the free nonrelativistic Hamiltonian is positive definite and therefore there is no way to obtain gravity in the framework of the free theory. Analogously, in Poincare invariant theory the spectrum of the free two-body mass operator belongs to the interval $[m_1 + m_2, \infty)$ while the existence of gravity necessarily requires that the spectrum should contain values less than $m_1 + m_2$.

In theories where the invariance group is the anti de Sitter (AdS) group $SO(2,3)$, the structure of UIRs of the $so(2,3)$ algebra is well known (see e.g. Ref. [17]). In particular, for positive energy UIRs the AdS Hamiltonian has the spectrum in the interval $[m, \infty)$ and $m$ has the meaning of the mass. Therefore the situation is pretty much analogous to that in Poincare invariant theories. In particular, the free two-body mass operator again has the spectrum in the interval $[m_1 + m_2, \infty)$ and therefore there is no way to reproduce gravitational effects in the free AdS invariant theory.

Consider now the case when the de Sitter (dS) group $SO(1,4)$ is chosen as the symmetry group or the dS algebra $so(1,4)$ is chosen as the symmetry algebra. It is well known that in UIRs of the dS algebra, the dS Hamiltonian is not positive definite and has the spectrum in the interval $(-\infty, +\infty)$ see e.g. Refs. [18, 19, 20, 21, 22, 23]). Note also that in contrast to the AdS algebra $so(2,3)$, the dS one does not have a supersymmetric generalization. For this and other reasons it was believed that the dS group or algebra were not suitable for constructing elementary particle theory.

In the framework of LQFT in curved spacetime (see e.g. Ref. [24] and references therein) the choice of $SO(1,4)$ as the symmetry group encounters serious difficulties which are intensively discussed in the literature. Our approach considerably differs from that in Refs. [24]. In particular, we do not require the existence of empty spacetime (see the discussion in Sect. [24]). However, in Ref. [25] we come to the same
conclusion that in the standard approach the dS group cannot be a symmetry group. Nevertheless, the standard approach can be modified as follows. Instead of requiring that UIR should describe an elementary particle, we now require that one UIR should describe a particle and its antiparticle simultaneously. Then the theory with the dS symmetry become consistent (see Ref. [25] for details).

It is well known that the group SO(1,4) is the symmetry group of the four-dimensional manifold in the five-dimensional space, defined by the equation

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -R^2$$  \hspace{1cm} (1.1)

where a constant $R$ has the dimension of length. The quantity $R^2$ is often written as $R^2 = 3/\Lambda$ where $\Lambda$ is the cosmological constant. The nomenclature is such that $\Lambda < 0$ for the AdS symmetry while $\Lambda > 0$ - for the dS one. Elements of a map of the point $(0, 0, 0, 0, R)$ (or $(0, 0, 0, 0, -R)$) can be parametrized by coordinates $(x_0, x_1, x_2, x_3)$. If $R$ is very large then such a map proceeds to Minkowski space and the action of the dS group on this map — to the action of the Poincare group. The recent astronomical data show that, although $\Lambda$ is very small, it is probably positive (see e.g. Ref. [26]). For this reason the interest to dS theories has increased. Nevertheless, as noted above, the existing difficulties have not been overcome.

In the present paper it will be convenient for us to work with the units $\hbar/2 = c = 1$. Then the spin of any particle is always an integer. For the normal relation between spin and statistics, the spin of fermions is odd and the spin of bosons is even. In this system of units the representation generators of the SO(1,4) group $M^{ab}$ ($a, b = 0, 1, 2, 3, 4, M^{ab} = -M^{ba}$) should satisfy the commutation relations

$$[M^{ab}, M^{cd}] = -2i(\eta^{ac}M^{bd} + \eta^{bd}M^{ac} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad})$$  \hspace{1cm} (1.2)

where $\eta^{ab}$ is the diagonal metric tensor such that $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = -\eta^{44} = 1$.

An important observation is as follows. If we accept that the symmetry on quantum level means that proper commutation relations
are satisfied (see Sect. 1.1) then Eq. (1.2) can be treated as the definition of the dS symmetry on that level. In our system of units all the operators $M^{ab}$ are dimensionless, in contrast with the situation in Poincare invariant theories, where the representation generators of the Lorentz group are dimensionless while the momentum operators have the dimension $(\text{length})^{-1}$. For this reason it is natural to believe that the dS or AdS symmetries are more fundamental than the Poincare symmetry.

Note that our definition of the dS symmetry on quantum level does not involve the cosmological constant at all. It appears only if one is interested in interpreting results in terms of the dS spacetime or in the Poincare limit. Since all the operators $M^{ab}$ are dimensionless in units $\hbar/2 = c = 1$, dS (or AdS) invariant quantum theories can be formulated only in terms of dimensionless variables. In particular one might expect that the gravitational and cosmological constants are not fundamental in the framework of such theories. Mirmovich has proposed a hypothesis [27] that only quantities having the dimension of the angular momentum can be fundamental.

If one assumes that spacetime is fundamental then in the spirit of GR it is natural to think that the empty space is flat, i.e. that the cosmological constant is equal to zero. This was the subject of the well-known dispute between Einstein and de Sitter described in a wide literature (see e.g. Refs. [10, 28] and references therein). In the modern approach to the LQFT, the cosmological constant is given by a contribution of vacuum diagrams, and the problem is to explain why it is so small. On the other hand, if we assume that symmetry on quantum level in our formulation is more fundamental, then the problem of the cosmological constant does not exist at all. Instead we have a problem of why nowadays Poincare symmetry is so good approximate symmetry. It seems natural to involve the anthropic principle for the explanation of this phenomenon (see e.g. Ref. [29] and references therein).
Chapter 2

Basic properties of de Sitter invariant quantum theories

2.1 UIRs of the so(1,4) algebra

There exists a wide literature devoted to UIRs of the dS group and algebra (see e.g. Refs. [30, 31, 32, 33, 34, 19, 20, 18, 35, 36, 21, 22]). In particular the first complete mathematical classification of the UIRs has been given in Ref. [30], three well known realizations of the UIRs have been first considered in Ref. [31] and their physical context has been first discussed in Ref. [32].

It is well known that for classification of UIRs of the dS group, one should, strictly speaking, consider not the group SO(1,4) itself but its universal covering group. The investigation carried out in Refs. [30, 31, 32, 33, 20] has shown that this involves only replacement of the SO(3) group by its universal covering group SU(2). Since this procedure is well known then for illustrations we will work with the SO(1,4) group itself and follow a very elegant presentation for physicists in terms of induced representations, given in the book [19] (see also Refs. [18, 38, 33]). The elements of the SO(1,4) group can be described in the block form

\[ g = \begin{pmatrix} g_0^0 & a^T & g_4^0 \\ b & r & c \\ g_0^4 & d^T & g_4^4 \end{pmatrix} \]  

\[ (2.1) \]
where

\[ a = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} \quad b^T = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} \quad r \in SO(3) \quad (2.2) \]

(the subscript $^T$ means a transposed vector).

UIRs of the SO(1,4) group are induced from UIRs of the subgroup $H$ defined as follows \[33, 19, 18\]. Each element of $H$ can be uniquely represented as a product of elements of the subgroups SO(3), $A$ and $T$: $h = r\tau_A a_T$ where

\[
\tau_A = \begin{pmatrix} \cosh(\tau) & 0 & \sinh(\tau) \\ 0 & 1 & 0 \\ \sinh(\tau) & 0 & \cosh(\tau) \end{pmatrix}
\]

\[
a_T = \begin{pmatrix} 1 + a^2/2 & -a^T & a^2/2 \\ -a & 1 & -a \\ -a^2/2 & a^T & 1 - a^2/2 \end{pmatrix}
\quad (2.3)
\]

The subgroup $A$ is one-dimensional and the three-dimensional group $T$ is the dS analog of the conventional translation group (see e.g. Ref. [19]). We hope it should not cause misunderstandings when 1 is used in its usual meaning and when to denote the unit element of the SO(3) group. It should also be clear when $r$ is a true element of the SO(3) group or belongs to the SO(3) subgroup of the SO(1,4) group.

Let $r \rightarrow \Delta(r; s)$ be a UIR of the group SO(3) with the spin $s$ and $\tau_A \rightarrow \exp(i\mu \tau)$ be a one-dimensional UIR of the group $A$, where $\mu$ is a real parameter. Then UIRs of the group $H$ used for inducing to the SO(1,4) group, have the form

\[ \Delta(r\tau_A a_T; \mu, s) = \exp(i\mu \tau) \Delta(r; s) \quad (2.4) \]

We will see below that $\mu$ has the meaning of the dS mass and therefore UIRs of the SO(1,4) group are defined by the mass and spin, by analogy with UIRs in Poincare invariant theory.

Let $G=SO(1,4)$ and $X = G/H$ be a factor space (or coset space) of $G$ over $H$. The notion of the factor space is well known (see e.g.Refs. [37, 18, 33, 19, 38]). Each element $x \in X$ is a class containing the elements $x_G h$ where $h \in H$, and $x_G \in G$ is a representative of the class $x$. The choice of representatives is not unique since if $x_G$ is a representative of the class $x \in G/H$ then $x_G h_0$, where $h_0$ is an arbitrary
element from $H$, also is a representative of the same class. It is well known that $X$ can be treated as a left $G$ space. This means that if $x \in X$ then the action of the group $G$ on $X$ can be defined as follows: if $g \in G$ then $gx$ is a class containing $gx_G$ (it is easy to verify that such an action is correctly defined).

As noted above, although we can use well known facts about group representations, our final goal is the construction of the generators. The explicit form of the generators $M^{ab}$ depends on the choice of representatives in the space $G/H$. As explained in several papers devoted to UIRs of the SO(1,4) group (see e.g. Ref. [19]), to obtain the possible closest analogy between UIRs of the SO(1,4) and Poincare groups, one should proceed as follows. Let $v_L$ be a representative of the Lorentz group in the factor space SO(1,3)/SO(3) (strictly speaking, we should consider $SL(2,c)/SU(2)$). This space can be represented as the well known velocity hyperboloid with the Lorentz invariant measure

$$d\rho(v) = d^3v/v_0$$

(2.5)

where $v_0 = (1 + v^2)^{1/2}$. Let $I \in SO(1,4)$ be a matrix which formally has the same form as the metric tensor $\eta$. One can show (see e.g. Ref. [19] for details) that $X = G/H$ can be represented as a union of three spaces, $X_+$, $X_-$ and $X_0$ such that $X_+$ contains classes $v_Lh$, $X_-$ contains classes $v_LIh$ and $X_0$ is of no interest for UIRs describing elementary particles since it has measure zero relative to the spaces $X_+$ and $X_-$. 

As a consequence of these results, the space of UIR of the so(1,4) algebra can be implemented as follows. If $s$ is the spin of the particle under consideration, then we use $||...||$ to denote the norm in the space of UIR of the su(2) algebra with the spin $s$. Then the space of UIR in question is the space of functions $\{f_1(v), f_2(v)\}$ on two Lorentz hyperboloids with the range in the space of UIR of the su(2) algebra with the spin $s$ and such that

$$\int [||f_1(v)||^2 + ||f_2(v)||^2]d\rho(v) < \infty$$

(2.6)

We see that, in contrast with UIRs of the Poincare algebra (and AdS one), where UIRs are implemented on one Lorentz hyper-
boloid, UIRs of the dS algebra can be implemented only on two Lorentz hyperboloids, $X_+$ and $X_-$. As shown in Ref. [25], this fact (which is well known) has a natural explanation if it is required that one UIR should describe a particle and its antiparticle simultaneously.

In the case of the Poincare and AdS algebras, the positive energy UIRs are implemented on an analog of $X_+$ and negative energy UIRs - on an analog of $X_-$. Since the Poincare and AdS groups do not contain elements transforming these spaces to one another, the positive and negative energy UIRs are fully independent. At the same time, the dS group contains such elements (e.g. $I$ [19, 18, 35]) and for this reason its UIRs cannot be implemented only on one hyperboloid.

In Ref. [25] we have described all the technical details needed for computing the explicit form of the generators $M^{ab}$. In our system of units the results are as follows. The action of the generators on functions with the supporter in $X_+$ has the form

$$M^{(+)} = 2l(v) + s, \quad N^{(+)} = -2i\nu_0 \frac{\partial}{\partial v} + \frac{s \times v}{v_0 + 1},$$

$$B^{(+)} = \mu v + 2i[\frac{\partial}{\partial v} + v(\frac{\partial}{\partial v}) + \frac{3}{2}v] + \frac{s \times v}{v_0 + 1},$$

$$M^{(+)}_{04} = \mu v_0 + 2i\nu_0(\frac{v}{v_0}v + \frac{3}{2}) \quad (2.7)$$

where $M = \{M^{23}, M^{31}, M^{12}\}$, $N = \{M^{01}, M^{02}, M^{03}\}$, $B = -\{M^{14}, M^{24}, M^{34}\}$, $s$ is the spin operator, and $l(v) = -i v \times \partial/\partial v$.

At the same time, the action of the generators on functions with the supporter in $X_-$ is given by

$$M^{(-)} = 2l(v) + s, \quad N^{(-)} = -2i\nu_0 \frac{\partial}{\partial v} + \frac{s \times v}{v_0 + 1},$$

$$B^{(-)} = -\mu v - 2i[\frac{\partial}{\partial v} + v(\frac{\partial}{\partial v}) + \frac{3}{2}v] - \frac{s \times v}{v_0 + 1},$$

$$M^{(-)}_{04} = -\mu v_0 - 2i\nu_0(\frac{v}{v_0}v + \frac{3}{2}) \quad (2.8)$$

In view of the fact that SO(1,4)=SO(4)AT and $H=SO(3)AT$, there also exists a choice of representatives which is probably even more
natural than that described above [19] [18] [20]. Namely, we can choose as representatives the elements from the coset space SO(4)/SO(3). Since the universal covering group for SO(4) is SU(2)×SU(2) and for SO(3) — SU(2), we can choose as representatives the elements of the first multiplier in the product SU(2)×SU(2). Elements of SU(2) can be represented by the points $u = (u, u_4)$ of the three-dimensional sphere $S^3$ in the four-dimensional space as $u_4 + i\sigma u$ where $\sigma$ are the Pauli matrices and $u_4 = \pm (1 - u^2)^{1/2}$ for the upper and lower hemispheres, respectively. Then the calculation of the generators is similar to that described above and the results are as follows.

The Hilbert space is now the space of functions $\varphi(u)$ on $S^3$ with the range in the space of the UIR of the su(2) algebra with the spin $s$ and such that

$$\int ||\varphi(u)||^2 du < \infty$$

(2.9)

where $du$ is the SO(4) invariant volume element on $S^3$. The explicit calculation shows that the generators for this realization have the form

$$M = 2l(u) + s, \quad B = 2nu_4 \frac{\partial}{\partial u} - s,$$

$$N = -2i[\frac{\partial}{\partial u} - u(\frac{\partial}{\partial u})] + (\mu + 3i)u - u \times s + u_4 s,$$

$$M_{04} = (\mu + 3i)u_4 + 2nu_4u \frac{\partial}{\partial u}$$

(2.10)

Since Eqs. (2.6-2.8) on the one hand and Eqs. (2.9) and (2.10) on the other are the different realization of one and the same representation, there exists a unitary operator transforming functions $f(v)$ into $\varphi(u)$ and operators (2.7,2.8) into operators (2.10). For example in the spinless case the operators (2.7) and (2.10) are related to each other by a unitary transformation

$$\varphi(u) = exp(-\frac{i}{2} \mu ln v_0)v_0^{3/2} f(v)$$

(2.11)

where $u = v/v_0$. 

16
2.2 Poincare limit

A general notion of contraction has been developed in Ref. [39]. In our case it can be performed as follows. Let us assume that $\mu > 0$ and denote $m = \mu / 2R$, $P = B / 2R$ and $E = M_0 / 2R$. Then, as follows from Eq. (2.7), in the limit when $R \to \infty$, $\mu \to \infty$ but $\mu / R$ is finite, one obtains a standard representation of the Poincare algebra for a particle with the mass $m$ such that $P = mv$ is the particle momentum and $E = mv_0$ is the particle energy. In that case the generators of the Lorentz algebra have the same form for the Poincare and dS algebras. Analogously the operators given by Eq. (2.8) are contracted to ones describing negative energy UIRs of the Poincare algebra.

In the standard interpretation of UIRs it is assumed that each element of the full representation space represents a possible physical state for the elementary particle in question. It is also well known (see e.g. Ref. [18, 19, 20, 35]) that the dS group contains elements (e.g. $I$) such that the corresponding representation operator transforms positive energy states to negative energy ones and vice versa. Are these properties compatible with the fact that in the Poincare limit there exist states with negative energies?

One might say that the choice of the energy sign is only a matter of convention. For example, in the standard theory we can define the energy not as $(m^2 + p^2)^{1/2}$ but as $-(m^2 + p^2)^{1/2}$. However, let us consider, for example, a system of two free noninteracting particles. The fact that they do not interact means mathematically that the representation describing the system is the tensor product of single-particle UIRs. The generators of the tensor product are equal to sums of the corresponding single-particle generators. In the Poincare limit the energy and momentum can be chosen diagonal. If we assume that both positive and negative energies are possible then a system of two free particles with the equal masses can have the same quantum numbers as the vacuum (for example, if the first particle has the energy $E$ and momentum $p$ while the second one has the energy $-E$ and the momentum $-p$) what obviously contradicts experiment. For this and
other reasons it is well known that in the Poincare invariant theory all the particles in question should have the same energy sign.

We conclude that UIRs of the dS algebra cannot be interpreted in the standard way since such an interpretation is physically meaningless even in the Poincare limit. Although our approach considerably differs from that in LQFT in curved spacetime, this conclusion is in agreement with that in Refs. [24] and references therein (see Sect. 1.1).

In the framework of our assumption that one UIR should describe a particle and its antiparticle simultaneously, one could try to interpret the operators (2.8) as those describing a particle while the operators (2.9) as those describing the corresponding antiparticle. Such a program has been implemented in Ref. [25]. If one requires that the dS Hamiltonian should be positive definite in the Poincare limit, then, as shown in Ref. [25], the annihilation and creation operators for the particle and antiparticle in question can satisfy only anticommutation relations, i.e. the particle and antiparticle can be only fermions. If the normal spin-statistics connection takes place then it follows that elementary particles can have only a half-integer spin (in conventional units).

At present the phenomenon of gravity has been observed only on macroscopic level, i.e. for particles which cannot be treated as elementary. Then the question arises whether they can be described by using the results on UIRs. The usual assumption is as follows. In the approximation when it is possible to neglect the internal structure of the particles (e.g. when the distance between them is much greater than their sizes), the structure of the internal wave function is not important and one can consider only the part of the wave function describing the motion of the particle as the whole. This part is described by the same parameters as the wave function of the elementary particle. Even if only fermions can be elementary, this does not mean of course that the external wave function of the composite system can be used only if the total spin is half-integer (in conventional units). Moreover, the rotation of a macroscopic system as a whole is usually the effect which does not
play an important role in the gravitational interaction. For this reason it is usually sufficient to describe the motion of the macroscopic system as a whole by using wave functions of UIRs with zero spin.

2.3 de Sitter antigravity

Consider now the Poincare limit in the approximation when $R$ is large but the first order corrections in $1/R$ to the conventional energy and momentum are taken into account. By using the definitions of the Poincare mass, energy and momentum from the preceding section and taking the dS generators in the form (2.17), one can obtain the expressions for the conventional energy and momentum in first order in $1/R$. For simplicity we assume that the particles is spinless and nonrelativistic. Consider a system of two free particles with the masses $m_1$ and $m_2$. Then the momentum and energy operators for each particle are given by

$$P_j = p_j + \frac{im_j}{R} \frac{\partial}{\partial p_j}$$
$$E_j = m_j + \frac{p_j^2}{2m_j} + i\frac{1}{R}(p_j \frac{\partial}{\partial p_j} + \frac{3}{2})$$

(2.12)

where $p_j = mv_j$ and $j = 1, 2$.

As noted in the preceding section, the fact that the particle do not interact with each other implies that the generators for the two-body system are equal to sums of the corresponding single-particle generators. Adding the corresponding operators and introducing the standard total and relative momenta

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \quad \mathbf{q} = (m_2\mathbf{p}_1 - m_1\mathbf{p}_2)/(m_1 + m_2)$$

(2.13)

one can obtain the expressions for the momentum $\mathbf{P}$ and energy $E$ of the two-body system as a whole. Let $M$ be the mass operator of the two-body system defined as $M^2 = E^2 - \mathbf{P}^2$. Then a simple calculation shows that in our approximation

$$M = m_1 + m_2 + \frac{q^2}{2m_{12}} + i\frac{1}{R}(q \frac{\partial}{\partial q} + \frac{3}{2})$$

(2.14)
where \( m_{12} = \frac{m_1 m_2}{m_1 + m_2} \) is the reduced mass.

In spherical coordinates the nonrelativistic mass operator can be written as

\[
M_{nr} = \frac{q^2}{2m_{12}} + V, \quad V = \frac{i}{R} \left( q \frac{\partial}{\partial q} + \frac{3}{2} \right)
\]  

where \( q = |q| \). Although this expression has been obtained in first order in \( 1/R \), let us consider for illustrative purposes the spectrum of this operator. It acts in the space of functions \( \psi(q) \) such that

\[
\int_0^\infty |\psi(q)|^2 q^2 dq < \infty
\]  

(2.16)

and the eigenfunction \( \psi_K \) of \( M_{nr} \) with the eigenvalue \( K \) satisfies the equation

\[
q \frac{d\psi_K}{dq} = \frac{iRq^2}{2m_{12}} \psi_K - \left( \frac{3}{2} + iRK \right) \psi_K
\]  

(2.17)

The solution of this equation is

\[
\psi_K = \sqrt{\frac{R}{2\pi}} q^{-3/2} e^{\frac{iRq^2}{4m_{12}} - iRKlnq}
\]  

(2.18)

and the normalization condition is

\[
(\psi_K, \psi_{K'}) = \delta(K - K')
\]  

(2.19)

The spectrum of the operator \( M_{nr} \) obviously belongs to the interval \((-\infty, \infty)\) and one might think that this is unacceptable. Suppose however that \( f(q) \) is a wave function of some state. As follows from Eq. (2.18), the probability to have the value of the kinetic energy \( K \) in this state is given by

\[
c_K = \sqrt{\frac{R}{2\pi}} \int_0^\infty \exp\left( -\frac{iRq^2}{4m_{12}} + iRKlnq \right) f(q) \sqrt{qdq}
\]  

(2.20)

If \( f(q) \) does not contain a rapidly oscillating exponent depending on \( R \) (in particular when it does not depend on \( R \)) and \( R \) is very large then \( c_K \) will practically be different from zero only if the integrand in Eq.
has a stationary point \( q_0 \). It is obvious that the stationary point is defined by the condition
\[
K = \frac{q_0^2}{2m_1}.
\]
Therefore, for negative \( K \), when the stationary point is absent, the value of \( c_K \) will be very small.

We see that if one works only with a subset of wave functions not depending on \( R \) (which is typically the case), then the existence of the points of the spectrum of the two-body mass operator with the values less than \( m_1 + m_2 \) does not play an important role.

### 2.4 de Sitter antigravity in quasiclassical approximation

In conventional quantum mechanics the motion of a particle is quasiclassical if at each moment of time \( t = t_0 \) the particle wave function satisfies the following conditions (see e.g. Ref. [40]). In the coordinate representation the function has a sharp maximum at some \( r = r_0 \), and the uncertainty of the position \( \Delta r \) is much less than \( r_0 \). At the same time, in the velocity representation it should have a sharp maximum at some \( v = v_0 \), and the uncertainty of the velocity \( \Delta v \) should be much less than \( v_0 \). In particular, the particle cannot be quasiclassical if it is at rest, i.e. \( v_0 = 0 \).

As follows from this definition, the notion of quasiclassical approximation necessarily implies that the position and velocity operators are well-defined and have a clear physical meaning. This is indeed the case in conventional nonrelativistic quantum mechanics. It is well known (see also Sect. 1.1) that in relativistic quantum theory there is no operator satisfying all the requirements for the position operator [7]. In dS invariant theories there exists an analogous problem. In particular, as seen from Eq. (2.7), the operator \( v \) by itself does not define the dS momentum (which is a physical operator) uniquely, and the operator \( i\partial/(m\partial v) \), which in nonrelativistic quantum mechanics is the position operator in velocity representation, does not define the physical Lorentz boost operators uniquely. However, as noted in Sect. 2.2 when \( R \) is very large, the generators (2.7) can be contracted to standard generators of the UIR of the Poincare group. In this case the
momentum $P$ is exactly proportional to $v$ and the proportionality coefficient is the mass. Moreover, when the particle is nonrelativistic, then, as follows from Eq. (2.7), the Lorentz boost operators are proportional to the corresponding coordinate operators in velocity representation.

We conclude that, at least when $R$ is large and $|v| \ll 1$, there exists a well-defined quasiclassical approximation in the representation when the generators are given by Eq. (2.7). For example, the wave function can be chosen in the form

$$f(v) = a(v) e^{i m v r_0} \quad (2.21)$$

where $a(v)$ has a sharp maximum at $v = v_0$ with a width $|\Delta v| \ll |v_0|$ and such that $|\partial a(v)/\partial v| \ll m |r_0|$. A possible choice of $a(v)$ is

$$a(v) = cv_0^{1/2} e^{i b^2 (v - v_0)^2} \quad (2.22)$$

where the function is normalized to one (see Eq. (2.6)) if

$$c = \sqrt{2b^{3/2}/\pi^{3/4}}.$$ 

Then the condition $|\Delta v| \ll |v_0|$ is satisfied if $b |v_0| \gg 1$ and the condition $|\Delta r| \ll |r_0|$ is satisfied if $b \ll m |r_0|$. For macroscopic particles there exists a wide range of values $b$ such that these conditions can be satisfied.

On classical level the effect of the additional term in Eq. (2.14) in comparison with the standard free nonrelativistic expression can be investigated as follows. We define the position operator $r$ as $r = i(\partial/\partial q)$. Then the classical Hamiltonian of the internal motion corresponding to the operator (2.14) is

$$H(r, q) = \frac{q^2}{2m_{12}} + \frac{r q}{R} \quad (2.23)$$

From classical equations of motion it follows that $d^2r/dt^2 = r/R^2$. It is well known that in classical dS space there exists a universal repulsion (antigravity) the force is which is proportional to the distance between particles. Therefore the operator $V$ indeed corresponds to the dS antigravity.
Although the example of the dS antigravity considered in the preceding and this sections is extremely simple, we can draw the following three very important conclusions.

The first conclusion is that the standard classical dS antigravity has been obtained from a quantum operator without introducing any classical background. When the position operator is defined as \( r = i(\partial/\partial q) \) and time is defined by the condition that the Hamiltonian is the evolution operator then we recover the classical result obtained by considering a motion of particles in the classical dS spacetime. This is an illustration of the discussion in Sect. 1.1 about the difference between the standard approach, where the classical dS spacetime is introduced from the beginning, and our one.

The second conclusion is as follows. We have considered the particles as free, i.e. no interaction into the two-body system has been introduced. However, we have realized that when the two-body system in the dS invariant theory is considered from the point of view of the Galilei invariant theory, the particles interact with each other. Although the reason of the effective interaction in our example is obvious, the existence of the dS antigravity poses the problem whether other interactions, e.g. gravity, can be treated as a result of transition from a higher symmetry to Poincare or Galilei one.

Finally, the third conclusion is that if in a free theory the spectrum of the mass operator has values less than \( m_1 + m_2 \), this does not necessarily mean that the theory is unphysical.

### 2.5 Free and interacting systems

The above discussion poses the following problem. We treated a system of two particles as free if the two-body generators are sums of the corresponding single-particle generators, and in the preceding sections we assumed that the single-particle generators are given by Eq. (2.7). In Sect. 2.1 we have shown that there also exists a realization of the single-particle generators in the form of Eq. (2.10) and in general there exist infinitely many unitarily equivalent realizations of single-particle
generators. We have noted that in the spinless case the realizations (2.7) and (2.10) are related to each other by a unitary transformation (2.11). Suppose now that we treat a two-body system as free if the two-body generators are sums of the single-particle generators in the form (2.10), not (2.7). Then it is easy to see that in terms of the generators (2.7) the system is not free but the sums of the two-body generators (2.7) are unitarily equivalent to the corresponding sums of the two-body generators (2.10). Therefore the problem arises whether one should treat a system as free or interacting when the generators are unitarily equivalent to free ones.

In Poincare invariant theory a system is treated as free or interacting depending on whether the S matrix is the identity operator or not. For a pair of Hamiltonians \((H, H_0)\) the S matrix is defined as follows. First one defines the Moeller wave operators \(W_\pm\) as

\[
W_\pm = s - \lim_{t \to \pm \infty} \exp(iHt)\exp(-iH_0t)P_0
\]

where \(t \to \pm \infty\), \(s - \lim\) means the strong limit and \(P_0\) is the projector onto the subspace corresponding to the (absolutely) continuous spectrum of the operator \(H_0\). (see e.g. Ref. [15]). Then the S matrix is defined as \(S = W_+^*W_-\). The Moeller operators are not necessarily unitary but the S matrix is.

The usual assumption is that the operator \(H_0\), which is treated as free, has only the continuous spectrum and therefore \(P_0 = 1\). The interacting Hamiltonian \(H\) in general can have both, the continuous and discrete spectrum. It is well known that two selfadjoint operators can be unitarily equivalent if and only if they have the same spectrum (see e.g. Ref. [15]). Therefore, if \(H\) has the discrete spectrum, the operators \(H\) and \(H_0\) cannot be unitarily equivalent. In that case the Moeller operators are not unitary. On the contrary, if \(H\) has only continuous spectrum then a typical situation is that the spectra of \(H\) and \(H_0\) are the same. In that case the Moeller operators are unitary and

\[
H = W_\pm H_0 W_\pm^{-1}
\]

but nevertheless in the general case the S matrix is not the identity op-
erator. Therefore in general, the case when the generators are unitarily equivalent to the free ones should be treated as interacting.

The philosophy of quantum mechanics is such that if one has a set of states and operators acting on these states then only probabilities of different experiment outcomes are important. If one applies a unitary transformation to the given set of states and transforms the operators accordingly then the probabilities remain the same. From this point of view it is not quite clear why in the case when $H$ and $H_0$ are unitarily equivalent they nevertheless are not on equal footing (one of them is treated as free and the other as interacting). In the full Hilbert space we can always find states on which $H_0$ acts as $H$ and *vice versa*. It is often assumed that the particles are free not only if the two-body generators are the sums of the corresponding single-particle generators but also when the wave function of the system as a whole is a product of single-particle wave functions. However, the Hilbert space for the two-body system is the tensor product of the single-particle Hilbert spaces, and therefore any linear combination of the products of the single-particle wave functions is an allowable wave function. The example considered at the end of this section will shed some light on this problem. In any case, if $H$ and $H_0$ are unitarily equivalent then, the interaction can be introduced not only as $H = H_0 + V$ but also by using the unitary operator realizing the equivalence. For example, one can use the Moeller operators which relate the free and interacting Hamiltonians according to Eq. (2.25). Such an approach has been proposed by Sokolov and Shatny in Ref. 41, 42.

Let us also note that in Poincare invariant theory there is no unique way of introducing interaction into the $N$-body system. Dirac has proposed a notion of forms of relativistic dynamics and singled out three forms: instant, front and point ones 16. As shown by Sokolov and Shatny 42, all the three forms have unitarily equivalent S matrix and therefore they are physically equivalent.

While in Poincare invariant theory the free operators are defined in fact uniquely (up to unitary transformations of particle spin variables), in dS invariant theory it is not clear what choice of opera-
tors is most justified and there is no S matrix. As already noted, the choice (2.7) corresponds to our intuition in Poincare invariant theory but this does not mean that such a choice is more physical than say the choice (2.10). We will argue below that the choice (2.10) is in fact more physical.

In any case, dS invariant theories have a feature which is specific only for such theories, and the reason is as follows (see Ref. [22] for details). Although the dS antigravity is typically small when $R$ is large, as follows from Eq. (2.15), the universal repulsion at asymptotically large distances is dominant in comparison with any reasonable interaction. Therefore there is no bound states in the dS invariant theory: any state which in Poincare invariant theory is bound becomes quasibound with a large lifetime. As a result, the free and interacting operators have the same spectrum and therefore they are unitarily equivalent. This again poses the problem about the meaning of interactions in dS invariant theories.

For illustration, consider a radial motion of two quasiclassical particles with the relative momentum $q_0$ and such that the distance between them is $r_0$. This implies that the wave function in $q$ representation has a sharp maximum at $q = q_0$, and the wave function in $r$ representation has a sharp maximum at $r = r_0$. The sharpness of the maximum in $q$ representation is defined by the quantity $(\Delta q)/q \ll 1$ where $\Delta q$ is the width of the maximum. Analogously the sharpness of the maximum in $r$ representation is defined by the quantity $(\Delta r)/r \ll 1$. Let us take the radial wave function for such a system and multiply it by the factor

$$\eta(q, r_0) = \exp(iRGm_1m_2lnq/r_0)$$  \hspace{1cm} (2.26)

Then, instead of Eq. (2.20), the mass distribution is defined by

$$c_K = \sqrt{\frac{R}{2\pi}} \int_0^\infty \exp\left(\frac{i}{r_0}RGm_1m_2\ln q - \frac{iRq^2}{4m_{12}} + iRKlnq\right)f(q)\sqrt{q}dq$$  \hspace{1cm} (2.27)

The exponent index has the stationary point at $q$ satisfying the condition

$$K = K(q) = \frac{q^2}{2m_{12}} - \frac{Gm_1m_2}{r_0}$$  \hspace{1cm} (2.28)
and hence the mass distribution has a sharp maximum about $K = K(q_0)$.

Therefore, although no operator corresponding to gravitational interaction has been introduced, for some class of wave functions the mean value of the mass operator is compatible with the standard classical gravity. However, the problem arises what is the physical meaning of wave functions containing the factor \((2.26)\). Indeed, if the position operator is defined as above then the distance between the particles will be not $r_0$ but an anomalously large value. One can try to modify the position operator, but the usual approach is such that the position operators are always the same regardless of whether an interaction is present or not.

For this reason we reformulate the problem as follows. We will not assume anymore that the two-particle wave function contains the factor \((2.26)\) but instead introduce an interaction into the two-body mass operator as follows. Define the selfadjoint operator

$$A = \frac{1}{2} G m_1 m_2 \{ lnq, \frac{1}{r} \}$$

(2.29)

where \{...\} is used to denote the anticommutator, and define the nonrelativistic mass operator as

$$\hat{M} = exp(-iRA) [\frac{q^2}{2m_{12}} + \frac{i}{R} (q \frac{\partial}{\partial q} + \frac{3}{2})] exp(iRA)$$

(2.30)

With such a mass operator, its eigen functions are given by (compare with Eq. \((2.18)\))

$$\hat{\psi}_K = exp(-iRA) \sqrt{\frac{R}{2\pi}} q^{-3/2} exp(\frac{iRq^2}{4m_{12}} - iRKlnq)$$

(2.31)

Let us now consider wave functions for which the $r$-distribution is much sharper than the $q$-distribution, i.e. $(\Delta r/r) \ll (\Delta q/q)$ (recall that $\Delta r$ is a characteristic of the wave function describing the motion as a whole; for the classical particle it has nothing to do with the particle dimension). Since the uncertainty of the quantity $r^{-1}$ is $\Delta r^{-1} = (\Delta r)/r^2$, we have that $(\Delta r^{-1})/r^{-1} = (\Delta r)/r$. Therefore
the distribution in $r^{-1}$ is much sharper than the distribution in $q$. As a result, with a good accuracy the action of $A$ on such wave functions $\psi$ can be written as $A\psi = (Gm_1m_2\ln q/r_0)\psi$ and the mass distribution is again given by Eq. (2.27).

Although the above example is mainly illustrative, it gives grounds to draw the following conclusions. Even if no interaction into the two-body mass operator is introduced, for some class of wave functions the mean value of the mass operator is compatible with the standard classical gravity. This result can be interpreted in a standard way by introducing an interaction by means of a unitary operator as in Eq. (2.30). On classical level the operator (2.30) results in the same observable consequences as the operator

$$M_{\text{standard}} = \frac{q^2}{2m_{12}} + \frac{i}{R}(q\frac{\partial}{\partial q} + \frac{3}{2}) - \frac{Gm_1m_2}{r}$$ \hspace{1cm} (2.32)

which is usually expected to be a quantum generalization of the conventional nonrelativistic gravity. Since at present gravity is known only on classical level, one cannot conclude that the operator (2.30) is inferior with respect to the operator (2.32). These operators are essentially different only on quantum level.

Below we consider in detail the case when the generators are defined by Eq. (2.10) and the two-body mass operator is not decomposed in powers of $1/R$. 

28
Chapter 3

de Sitter invariant quantum theory in su(2) × su(2) basis

3.1 UIRs in the su(2) × su(2) basis

Proceeding from the method of su(2) × su(2) shift operators, developed by Hughes [43] for constructing UIRs of the group SO(5), and following Ref. [23], we now give a pure algebraic description of UIRs of the so(1,4) algebra. It will be convenient for us to deal with the set of operators \((J', J'', R_{ij})\) \((i,j = 1,2)\) instead of \(M^{ab}\). Here \(J'\) and \(J''\) are two independent su(2) algebras (i.e. \([J', J''] = 0\)). In each of them one chooses as the basis the operators \((J_+ + J_-, J_3)\) such that in our system of units \(J_1 = J_+ + J_-\), \(J_2 = -i(J_+ - J_-)\) and the commutation relations have the form

\[
[J_3, J_+] = 2J_+, \quad [J_3, J_-] = -2J_-, \quad [J_+, J_-] = J_3 \quad (3.1)
\]

The commutation relations of the operators \(J'\) and \(J''\) with \(R_{ij}\) have the form

\[
[J'_3, R_{1j}] = R_{1j}, \quad [J'_3, R_{2j}] = -R_{2j}, \quad [J''_3, R_{i1}] = R_{i1}, \\
[J''_3, R_{i2}] = -R_{i2}, \quad [J'_+, R_{2j}] = R_{1j}, \quad [J''_+, R_{i2}] = R_{i1}, \\
[J'_-, R_{1j}] = R_{2j}, \quad [J''_-, R_{i1}] = R_{i2}, \quad [J'_+, R_{1j}] = \\
[J''_+, R_{i1}] = [J'_-, R_{2j}] = [J''_-, R_{i2}] = 0, \quad (3.2)
\]
and the commutation relations of the operators $R_{ij}$ with each other have the form

\[
\begin{align*}
[R_{11}, R_{12}] &= 2J'_+, \quad [R_{11}, R_{21}] = 2J''_+, \\
[R_{11}, R_{22}] &= -(J'_3 + J''_3), \quad [R_{12}, R_{21}] = J'_3 - J''_3 \\
[R_{11}, R_{22}] &= -2J'_, \quad [R_{21}, R_{22}] = -2J'_-
\end{align*}
\] (3.3)

The relation between the sets $(J', J'', R_{ij})$ and $M^{ab}$ is given by

\[
\begin{align*}
M &= J' + J'', \quad B = J' - J'', \quad M_{01} = i(R_{11} - R_{22}), \\
M_{02} &= R_{11} + R_{22}, \quad M_{03} = -i(R_{12} + R_{21}), \\
M_{04} &= R_{12} - R_{21}
\end{align*}
\]

(3.4)

Then it is easy to see that Eq. (1.2) follows from Eqs. (3.2-3.4) and vice versa.

Consider the space of maximal $su(2) \times su(2)$ vectors, i.e. such vectors $x$ that $J'_+ x = J''_- x = 0$. Then from Eqs. (3.2) and (3.3) it follows that the operators

\[
\begin{align*}
A^{++} &= R_{11}, \quad A^{+-} = R_{12}(J''_3 + 1) - J'_- R_{11}, \\
A^{-+} &= R_{21}(J'_3 + 1) - J'_- R_{11}, \\
A^{--} &= -R_{22}(J'_3 + 1)(J''_3 + 1) + J''_- R_{21}(J'_3 + 1) + \\
&\quad J'_- R_{12}(J''_3 + 1) - J''_- R_{11}
\end{align*}
\]

(3.5)

act invariantly on this space. The notations are related to the property that if $x^{kl}$ $(k, l > 0)$ is the maximal $su(2) \times su(2)$ vector and simultaneously the eigenvector of operators $J'_3$ and $J''_3$ with the eigenvalues $k$ and $l$, respectively, then $A^{++} x^{kl}$ is the eigenvector of the same operators with the values $k + 1$ and $l + 1$, $A^{+-} x^{kl}$ - the eigenvector with the values $k + 1$ and $l - 1$, $A^{-+} x^{kl}$ - the eigenvector with the values $k - 1$ and $l + 1$ and $A^{--} x^{kl}$ - the eigenvector with the values $k - 1$ and $l - 1$.

As follows from Eq. (3.1), the vector $x_{ij}^{kl} = (J'_i)^i(J''_j)^j x^{kl}$ is the eigenvector of the operators $J'_3$ and $J''_3$ with the eigenvalues $k - 2i$ and $l - 2j$, respectively. Since

\[
J^2 = J''_3 - 2J_3 + 4J'_+ J_- = J''_3 + 2J_3 + 4J_- J_+
\]

30
is the Casimir operator for the $J$ algebra, and the Hermiticity condition can be written as $J^*_- = J_+$, it follows in addition that

$$J'^2x^{kl}_{ij} = k(k + 2)x^{kl}_{ij}, \quad J''^2x^{kl}_{ij} = l(l + 2)x^{kl}_{ij} \quad (3.6)$$

$$J'^+x^{kl}_{ij} = i(k + 1 - i)x^{kl}_{i-1,j}, \quad J''^+x^{kl}_{ij} = j(l + 1 - j)x^{kl}_{i,j-1} \quad (3.7)$$

$$(x^{kl}_{ij}, x^{kl}_{ij}) = \frac{i!j!k!l!}{(k - i)!(l - j)!}(x^{kl}, x^{kl}) \quad (3.8)$$

where $(...,...)$ is the scalar product in the representation space. From these formulas it follows that the action of the operators $J'$ and $J''$ on $x^{kl}_{ij}$ generates a space with the dimension $(k + 1)(l + 1)$ and the basis $x^{kl}_{ij}$ ($i = 0, 1, ...k$, $j = 0, 1, ...l$). Note that the vectors $x^{kl}_{ij}$ are orthogonal but in this section we do not normalize them to one (the reason will be clear below).

The Casimir operator of the second order for the algebra (1.2) can be written as

$$I_2 = -\frac{1}{2} \sum_{ab} M_{ab}M^{ab} =$$

$$4(R_{22}R_{11} - R_{21}R_{12} - J'^3_3) - 2(J'^2 + J''^2) \quad (3.9)$$

A direct calculation shows that for the generators given by Eqs. (2.7), (2.8) and (2.10), $I_2$ has the numerical value

$$I_2 = w - s(s + 2) + 9 \quad (3.10)$$

where $w = \mu^2$. As noted in Sect. 2.2, $\mu = 2mR$ where $m$ is the conventional mass. If $m \neq 0$ then $\mu$ is very large since $R$ is very large. We conclude that for massive UIRs the quantity $I_2$ is a large positive number.

The basis in the representation space can be explicitly constructed assuming that there exists a vector $e^0$ which is the maximal $su(2) \times su(2)$ vector such that

$$J'^3e^0 = n_1e^0 \quad J''^3e^0 = n_2e^0 \quad (3.11)$$
and $n_1$ is the minimum possible eigenvalue of $J_3'$ in the space of the maximal vectors. Then $e_0$ should also satisfy the conditions

$$A^{-}e_0 = A^{+}e_0 = 0$$

(3.12)

We use $\tilde{I}$ to denote the operator $R_{22}R_{11} - R_{21}R_{12}$. Then as follows from Eqs. (3.2), (3.3), (3.5), (3.9), (3.11) and (3.12),

$$\tilde{I}n_1e_0 = 2n_1(n_1 + 1)e_0.$$ 

Therefore, if $n_1 \neq 0$ the vector $e_0$ is the eigenvector of the operator $\tilde{I}$ with the eigenvalue $2(n_1 + 1)$ and the eigenvector of the operator $I_2$ with the eigenvalue

$$-2[(n_1 + 2)(n_2 - 2) + n_2(n_2 + 2)].$$

The latter is obviously incompatible with Eq. (3.10) for massive UIRs. Therefore the compatibility can be achieved only if $n_1 = 0$. In that case we use $s$ to denote $n_2$ since it will be clear soon that the value of $n_2$ indeed has the meaning of spin. Then, as follows from Eqs. (3.10) and (3.11), the vector $e_0$ should satisfy the conditions

$$J' e^0 = J_+'' e^0 = 0, \quad J_3'' e^0 = se^0,$$

$$I_2 e^0 = [w - s(s + 2) + 9]e^0$$

(3.13)

where $w, s > 0$ and $s$ is an integer.

Define the vectors

$$e^{nr} = (A^{+})^n(A^{-})^r e^0$$

(3.14)

Then a direct calculation taking into account Eqs. (3.1)-(3.3), (3.5), (3.6), (3.9), (3.12) and (3.13) gives

$$A^{-}A^{+}e^{nr} = -\frac{1}{4}(n+1)(n+s+2)[w + (2n + s + 3)^2]e^{nr}$$

(3.15)

$$A^{+}A^{-}e^{nr} = -\frac{1}{4}(r+1)(s-r)[w + 1 + (2r - s)(2r + 2 - s)]e^{nr}$$

(3.16)
\[(e^{n+1,r}, e^{n+1,r}) = \frac{(n + 1)(n + s + 2)[w + (2n + s + 3)^2]}{4(n + r + 2)(s - r + n + 2)}(e^{nr}, e^{nr}) \] (3.17)

\[(e^{n,r+1}, e^{n,r+1}) = \frac{1}{4}(r + 1)(s - r)[w + 1 + (2r - s)(2r + 2 - s)]\]
\[
\frac{s - r + n + 1}{s - r + n + 2}(e^{nr}, e^{nr}) \] (3.18)

As follows from Eqs. (3.15) and (3.17), the possible values of \(n\) are \(n = 0, 1, 2, \ldots\) while, as follows from Eqs. (3.16) and (3.18), \(r\) can take only the values of 0, 1, ..., \(s\) (and therefore \(s\) indeed has the meaning of the particle spin). Since \(e^{nr}\) is the maximal \(su(2) \times su(2)\) vector with the eigenvalues of the operators \(J'\) and \(J''\) equal to \(n + r\) and \(n + s - r\), respectively, then as a basis of the representation space one can take the vectors \(e_{ij}^{nr} = (J'_+)^i(J''_-)^j e^{nr}\) where, for the given \(n\) and \(s\), the quantity \(i\) can take the values of 0, 1, ..., \(n + r\) and \(j\) - the values of 0, 1, ...\(n + s - r\).

In the subsequent section we consider in detail the spinless case and show that the basis discussed in this section is an implementation of the generators (2.10) but not (2.7) and (2.8). If \(s = 0\) then there exist only the maximal \(su(2) \times su(2)\) vectors \(x^{kl}\) with \(k = l\) and therefore the basis of the representation space is formed by the vectors \(e_{\alpha\beta}^n \equiv e_{\alpha\beta}^{n0}\) where \(n = 0, 1, 2, \ldots; \alpha, \beta = 0, 1, \ldots n\). The explicit expressions for the action of operators \(R_{ij}\) in this basis can be calculated by using
Eq. (3.2), and the result is

\[ R_{11}e_{\alpha\beta}^n = \frac{(n + 1 - \alpha)(n + 1 - \beta)}{(n + 1)^2} e_{\alpha\beta}^{n+1} + \]

\[ \frac{\alpha\beta n}{4(n+1)} [w + (2n + 1)^2] e_{\alpha-1,\beta-1}^{n-1}, \]

\[ R_{12}e_{\alpha\beta}^n = \frac{n + 1 - \alpha}{(n + 1)^2} e_{\alpha,\beta+1}^{n+1} - \frac{\alpha n}{4(n+1)} [w + (2n + 1)^2] e_{\alpha-1,\beta}^{n-1}, \]

\[ R_{21}e_{\alpha\beta}^n = \frac{n + 1 - \beta}{(n + 1)^2} e_{\alpha+1,\beta}^{n+1} - \frac{\beta n}{4(n+1)} [w + (2n + 1)^2] e_{\alpha,\beta-1}^{n-1}, \]

\[ R_{22}e_{\alpha\beta}^n = \frac{1}{(n + 1)^2} e_{\alpha+1,\beta+1}^{n+1} + \]

\[ \frac{n}{4(n+1)} [w + (2n + 1)^2] e_{\alpha\beta}^{n-1} \]  (3.19)

As follows from Eqs. (3.8) and (3.17)

\[ (e_{\alpha\beta}^n, e_{\alpha\beta}^n) = \frac{(n!)^2\alpha!\beta!}{4^n(n+1)(n-\alpha)!(n-\beta)!} \prod_{j=1}^{n} [w + (2j + 1)^2] \]  \quad (3.20)

### 3.2 Implementation of UIRs in the space of functions on the SU(2) group

As already noted, the three dimensional unit sphere \( S^3 \) represents the group space of the SU(2) group. The elements \( u \in SU(2) \) are often parametrized by the Euler angles as follows (see e.g. Ref. [44])

\[ u_{11} = \cos(\theta/2)\exp(i(\chi + \psi)/2) \quad u_{12} = \sin(\theta/2)\exp(i(\chi - \psi)/2) \]

\[ u_{21} = u_{12}^* \quad u_{22} = u_{11}^* \]  (3.21)

where * is used to denote the complex conjugation. On the other hand, since the matrix \( u \) can also be represented as \( u = u_4 + i\sigma u \), where \( (u, u_4) \) are the components of the four-dimensional vector belonging to \( S^3 \), one can express these components in terms of the Euler angles. Note that in mathematical literature usually the notation \( \varphi \) instead of \( \chi \) is used, but we reserve \( \varphi \) to denote the conventional polar angle, which, as follows
from Eq. (3.21), is equal to \( (\psi - \chi) / 2 \). Also in physical literature the notation \( \theta \) has another meaning than in Eq. (3.21) since the ”true” azimuthal angle \( \theta \) is defined in such a way that \( \cos \theta = u_z / |u| \). For simplicity we do not rename the \( \theta \) in Eq. (3.21).

As a result of direct calculations, the form of representation generators (2.10) for the \( SU(2) \times SU(2) \) subgroup of the dS group are as follows

\[
J'_+ = i \exp(-i\chi) \left( \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \psi} - i \tan \theta \frac{\partial}{\partial \chi} \right)
\]

\[
J'_- = -i \exp(i\chi) \left( -\frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \psi} - i \tan \theta \frac{\partial}{\partial \chi} \right)
\]

\[
J''_+ = -i \exp(i\psi) \left( \frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \chi} + i \tan \theta \frac{\partial}{\partial \psi} \right)
\]

\[
J''_- = i \exp(-i\psi) \left( -\frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \chi} + i \tan \theta \frac{\partial}{\partial \psi} \right)
\]

\[
J'_z = 2i \frac{\partial}{\partial \chi} \quad J''_z = -2i \frac{\partial}{\partial \psi} \quad (3.22)
\]

Let \( e_{\alpha \beta}^n(\chi, \theta, \psi) \) be a function corresponding to the basis element \( e_{\alpha \beta}^n \) in the preceding section for a spinless UIR. Since \( e_{\alpha \beta}^n \) is the eigenvector of \( J'_3 \) with the eigenvalue \( n - 2\alpha \) and the eigenvector \( J'_3 \) with the eigenvalue \( n - 2\beta \), then, as follows from Eq. (3.22), the dependence of \( e_{\alpha \beta}^n(\chi, \theta, \psi) \) on \( \chi \) and \( \psi \) is in the form

\[
e_{\alpha \beta}^n(\chi, \theta, \psi) = \exp[-i(\rho \chi + \nu \psi)] e_{\alpha \beta}^n(\theta) \quad (3.23)
\]

where \( \rho = (n - 2\alpha) / 2, \nu = -(n - 2\beta) / 2 \).

As noted in the preceding section, \( e_{\alpha \beta}^n \) is also the eigenvector of the operators \( J'^2 \) and \( J''^2 \) with the eigenvalue \( n(n + 2) \). We denote \( l = n / 2 \) and \( z = \cos \theta \). Then, as follows from Eqs. (3.22) and (3.23)

\[
(1 - z^2) \frac{d^2 e_{\alpha \beta}^n(z)}{dz^2} - 2z \frac{de_{\alpha \beta}^n(z)}{dz} - \frac{\rho^2 + \nu^2 - 2\rho \nu z}{1 - z^2} e_{\alpha \beta}^n(z) = -l(l + 1) e_{\alpha \beta}^n(z) \quad (3.24)
\]

This is the equation for functions \( P_{\rho \nu}^l(z) \) describing matrix elements of the regular representation of the SU(2) group (see e.g. Ref. [44]). One
of the convenient forms of $P_{\rho\nu}^{l}(z)$ is as follows \[44\]

$$P_{\rho\nu}^{l}(\theta) = i^{-\rho-\nu}(l-\rho)!/(l+\rho)!(l+\nu)! \left(\frac{\sin\gamma}{\sin\theta/2}\right)_{\rho+\nu}^{1/2}$$

$$\sum_{j=\max(\rho,\nu)}^{l} \frac{(l+j)!l^{2j}}{(l-j)!(j-\rho)!(j-\nu)!} \left(\sin\gamma\right)^{2j}$$

(3.25)

where $\gamma = \theta/2$. In some cases it is convenient to use the relation between the functions $P_{\rho\nu}^{l}(z)$ and the Jacobi polynomials \[44\]

$$P_{k}^{(i,j)} = 2^\rho l^{\nu-\rho}[\left(\frac{l-\nu}{l+\rho}\right)!/(l-\rho)!/(l+\rho)!]^{1/2}(1-z)^{(\nu-\rho)/2}(1+z)^{-(\nu+\rho)/2}P_{\rho\nu}^{l}(z)$$

(3.26)

where

$$l = k + \frac{i+j}{2} \quad \rho = \frac{i+j}{2} \quad \nu = \frac{j-i}{2}$$

(3.27)

We conclude that

$$e_{\alpha\beta}^{n}(\chi, \theta, \psi) = c_{\alpha\beta}^{n} \exp[-i(\rho\chi + \nu\psi)] P_{\rho\nu}^{l}(z)$$

(3.28)

and our next task is to find the coefficients $c_{\alpha\beta}^{n}$. As follows from Eq. (3.22),

$$J_{\nu} = -ie^{i\chi} [(1-z^2)^{1/2} \frac{\partial}{\partial z} + \frac{i}{(1-z^2)^{1/2}} \frac{\partial}{\partial \psi} - \frac{iz}{(1-z^2)^{1/2}} \frac{\partial}{\partial \chi}]$$

$$J_{-\nu} = ie^{-i\psi} [(1-z^2)^{1/2} \frac{\partial}{\partial z} - \frac{i}{(1-z^2)^{1/2}} \frac{\partial}{\partial \psi} + \frac{iz}{(1-z^2)^{1/2}} \frac{\partial}{\partial \chi}]$$

(3.29)

By using the relations \[44\]

$$(1-z^2)^{1/2} \frac{\partial P_{\rho\nu}^{l}(z)}{\partial z} - \frac{\rho z - \nu}{(1-z^2)^{1/2}} P_{\rho\nu}^{l}(z) = -i[(l+\rho)(l-\rho+1)]^{1/2} P_{\rho-1,\nu}^{l}(z)$$

$$+ i[(l+\nu)(l-\nu+1)]^{1/2} P_{\rho,\nu-1}^{l}(z)$$

(3.30)
and the properties $J_{-}^{\prime}e_{\alpha\beta}^{n} = e_{\alpha+1,\beta}^{n}$ and $J_{-}^{"}e_{\alpha\beta}^{n} = e_{\alpha,\beta+1}^{n}$, we now conclude that

$$e_{\alpha\beta}(\chi, \theta, \psi) = c_{n}(-1)^{\alpha}[\frac{\alpha!\beta!}{(n-\alpha)!(n-\beta)!}]^{1/2}exp[-i(\rho\chi + \nu\psi)]P_{l}^{\rho\nu}(z)$$

(3.31)

Our final goal in this section is to find the coefficients $c_{n}$ from the fact that the maximal $su(2) \times su(2)$ vectors $e^{n}$ are defined in such a way that $R_{11}e^{n} = e^{n+1}$. Since $e^{n} = e^{n}_{00}$ then it follows from Eq. (3.31) that $e^{n}$ depends on $\chi$ and $\psi$ only via $exp(in\varphi)$ and on $z$ only via $P_{l}^{l,-l}(z)$. We use the fact that

$$P_{l}^{l,-l}(z) = i^{2l}(\frac{1-z}{2})^{l}$$

(3.32)

Then, as follows from Eqs. (2.10), (3.4) and (3.21), the action of $R_{11}$ on functions depending on $\chi$ and $\psi$ only via $\varphi$ is given by

$$R_{11} = e^{i\varphi}[cos\gamma \frac{\partial}{\partial \gamma} + \frac{i}{sin\gamma} \frac{\partial}{\partial \varphi} + \frac{i}{2}(-\mu + 3i)sin\gamma]$$

(3.33)

Our conclusion is that if $e^{0}(\chi, \theta, \psi) = 1$ then the final expression for $e_{\alpha\beta}^{n}(\chi, \theta, \psi)$ can be written in the form

$$e_{\alpha\beta}^{n}(\chi, \theta, \psi) = \frac{n!}{2^{n}}(-1)^{\alpha}[\frac{\alpha!\beta!}{(n-\alpha)!(n-\beta)!}]^{1/2}\{\prod_{j=1}^{n}[\mu + i(2j + 1)]\}$$

$$exp[-i(\rho\chi + \nu\psi)]P_{l}^{\rho\nu}(cos\theta)$$

(3.34)

where $\rho = (n-2\alpha)/2$, $\nu = -(n-2\beta)/2$, $l = n/2$. The norm of $e_{\alpha\beta}^{n}$ defined in such a way is compatible with Eq. (3.20) since the volume element on the SU(2) group in terms of the Euler angles is

$$dV = \frac{1}{16\pi^{2}}sin\theta d\chi d\theta d\psi$$

(3.35)

$$0 \leq \chi < 2\pi, 0 < \theta < \pi, -2\pi \leq \psi < 2\pi$$ and

$$\int |P_{l}^{l}(cos\theta)|^{2}sin\theta d\theta = \frac{2}{2l + 1}$$

(3.36)
3.3 Matrix elements in quasiclassical approximation

As follows from Eqs. (2.11) and (2.21), in the representation when the generators are given by Eq. (2.10) the quasiclassical wave function can be written as

\[ f(u) = a(u/u_4)u_4^{-3/2} \exp(imRnu_4 - imu_0/u_4) \]  

(3.37)
since \( \mu = 2mR \) (see Sect. 2.2). Note that if \( R \) is very large then the exponent in this expression is a rapidly oscillating function of \( u_4 \). Therefore, in that case the dependence on \( u_4 \) is always quasiclassical even if the dependence on the other variables is not.

Instead of the Euler angles, one can also use the following parametrization of the points on \( S^3 \):

\[ u_1 = \sin\gamma\cos\varphi \quad u_2 = \sin\gamma\sin\varphi \quad u_3 = \cos\gamma\cos\delta \quad u_4 = \cos\gamma\sin\delta \]  

(3.38)

where \( \varphi, \delta \in [0, 2\pi] \) and \( \gamma \in [0, \pi/2] \). Then, as follows from Eq. (3.34), the coefficient \( c^{n\alpha\beta} \) defining the probability of the state (3.37) to have the quantum numbers \((n\alpha\beta)\) is given by

\[ c^{n\alpha\beta} = \int \exp[imR ln(cos\gamma sin\delta) - imv_0(x_0 \sin\gamma \cos\varphi + y_0 \sin\gamma \sin\varphi + \quad z_0 \cos\gamma \cos\delta) - i(n - \alpha - \beta)\varphi - i(\beta - \alpha)\delta)] \quad P_{\nu\nu}(\gamma)f_1(\varphi, \gamma, \delta)d\varphi d\delta d\gamma \]  

(3.39)

where \( f_1 \) is not a rapidly oscillating function of its variables. Here \((x_0, y_0, z_0)\) are the components of \( r_0 \).

As follows from Eq. (3.4), the \( z \) component of the angular momentum operator is given by \( J_z = J'_z + J_3'' \) and the operator \( M_4^3 \), which is equal to \( 2RP_z \) in the Poincare limit, is given by \( M_3^3 = J'_z - J''_z \). Therefore the state \( e^{n\alpha\beta} \) is the eigenvector of the operator \( J_z \) with the eigenvalue \( 2n - 2\alpha - 2\beta \) and the eigenvector of the operator \( M_4^3 \) with the eigenvalue \( 2\beta - 2\alpha \). We will now show that these results are compatible.
with the quasiclassical approximation. Indeed, the integral over $\varphi$ in Eq. (3.39) has a stationary point if
\[ m(x_0v_y - y_0v_x) = n - \alpha - \beta \] (3.40)
and the integral over $\delta$ in Eq. (3.39) has a stationary point if
\[ 2mRv_z = 2(\beta - \alpha) \] (3.41)
The result (3.40) is compatible with the quasiclassical approximation since in our units $\hbar/2 = 1$.

As follows from Eq. (3.41), the maximum value of $v_z$ is equal to $n/mR$. For this reason one might think that the meaning of the quantum number $n$ when $R$ is large is such that $|v| = n/(mR)$. To prove that this is the case one has to use asymptotic expressions for $P_{l\rho\nu}^l$ when the numbers $(l\rho\nu)$ are large. As seen from Eq. (3.25), in the general case the functions $P_{l\rho\nu}^l$ contain many oscillating terms and therefore the problem of finding their asymptotic expressions is not easy. As follows from Eqs. (3.26) and (3.27), one can rewrite Eq. (3.39) in the form
\[ c_{\alpha\beta}^n = \int \exp \left[ imR \ln(\cos\gamma\sin\delta) - imv_0(x_0\sin\gamma\cos\varphi + y_0\sin\gamma\sin\varphi + z_0\cos\gamma\cos\delta) - i(n - \alpha - \beta)\varphi - i(\beta - \alpha)\delta \right] P_{N}^{(k_1,k_2)}(\gamma) \sin^{k_1}(\gamma)\cos^{k_2}(\gamma)f_1(\varphi, \gamma, \delta)d\varphi d\delta d\gamma \] (3.42)
where
\[ k_1 = |n - \alpha - \beta| \quad k_2 = |\beta - \alpha| \quad N = (n - k_1 - k_2)/2 \] (3.43)
One of the cases when it is possible to obtain the asymptotic expression is when $N$ is large while $k_1$ and $k_2$ are fixed (in particular this means that $|v_z| << |v|$). In that case the result is (see e.g. Ref. [46])
\[ P_{N}^{(k_1,k_2)}(\gamma) = \cos[(n + 1)\gamma - (2k_1 + 1)\pi/4] \left[ (\pi N)^{1/2} (\sin\gamma)^{k_1+1/2} (\cos\gamma)^{k_2+1/2} \right]^{-1} \] (3.44)
As follows from this expression, the integral over $\gamma$ in Eq. (3.42) has a stationary point if $n \approx mR|v|$. We will see in subsequent sections that this result is valid in the general case, i.e. when $|v_z|$ is not necessarily small.
3.4 Free two-body mass operator

In contrast with Sect. 2.3, we now assume that the two-body system is free if its generators are sums of the single-particle generators in the form (2.10) (see the discussion in Sect. 2.3). This implies that \( M_{ab} = M_{ab}^{(1)} + M_{ab}^{(2)} \) where \( M_{ab}^{(1)} \) are the generators for the first particle and \( M_{ab}^{(2)} \) - for the second one. Each generator acts over the variables of its ”own” particle, as described in Sect. 3.1, and over the variables of another particle it acts as the identity operator. In other words, the representation describing the two-body system is the tensor product of single-particle UIRs.

Denote by \( \mu_1 \) and \( \mu_2 \) (\( \mu_1, \mu_2 > 0 \)) the dS masses of the corresponding particles and assume that they are spinless. Then, as follows from Eq. (3.10), \( I_2^{(1)} = 2(w_1 + 9) \), \( I_2^{(2)} = 2(w_2 + 9) \), where \( w_1 = \mu_1^2 \), \( w_2 = \mu_2^2 \). The tensor product of UIRs can be decomposed into the direct integral of UIRs and there exists a well elaborated general theory [37]. In terminology of the theory of induced UIRs, UIRs discussed in Sect. 2.1 belong to the principal series of UIRs. In general, the decomposition of the tensor product of UIRs belonging to the principal series may contain not only UIRs of the principal series (i.e. it may also contain UIRs not having ”rest states” defined by Eq. (3.13)). We will consider only a part of the tensor product containing the ”rest states” and show that even for this part the spectrum of the mass operator is not bounded below by the value of \((\mu_1 + \mu_2)^2\).

It is clear that only UIRs with \( s = 0, 2, 4... \) (in our system of units) can enter the tensor product of two spinless representations. Therefore in order to find which values of \( w \) are possible for the given \( s \) one can act as follows. Construct \( H_s \) — a space of elements \( x \), satisfying the condition (compare with Eq. (3.13))

\[
J^\prime x = J_+^\prime x = 0, \quad J^{\prime 2}x = s(s + 2)x
\]  

Since \( I_2 \) commutes with all the representation operators then \( H_s \) is invariant under the action of \( I_2 \). Let \( I_2^s \) be the reduction of \( I_2 \) onto \( H_s \). Define the operator \( W \) such that its reduction onto \( H_s \) — \( W^s \) is defined
from the relation $I_z^2 = 2[W^s - s(s + 2) + 9]$. Then the spectrum of the operator $W^s$ defines the possible values of $w$ for the given $s$.

Note that although $W$ is the dS analog of the mass operator squared in Poincare invariant theory, it is not a square of any operator. Therefore one cannot exclude a possibility that $W$ has even a negative part of the spectrum. However, the part of $W$ corresponding to principles series UIRs has only the positive spectrum.

To construct a basis in the space $H_s$, we have first to ascertain which linear combinations of the elements $e^{(1)\alpha_1\beta_1}_{\alpha_2\beta_2}$ belong to $H_s$. Since $e^{(1)\alpha_1\beta_1}_{\alpha_2\beta_2}$ is the spinor with the spin $n_1$ with respect to the algebra $J^{(1)}$, as well as to $J^{(1)}$, and analogously for $e^{(2)\alpha_2\beta_2}_{\alpha_2\beta_2}$, then zero eigenvalues of the operator $J'$ can be obtained only if $n_1 = n_2$, and the value $s$ for the spin relative to the $J''$ algebra can be obtained only if $n_1,n_2 \geq s/2$. The explicit finding of the required linear combinations can be performed by using Clebsch-Gordan coefficients for the $su(2) \times su(2)$ algebra but it is also possible to verify directly that the vectors

$$
\Phi_n = \frac{4^{n+j}(1+2j)!}{[(n+j)!]^2 j!} \sum_{\alpha=0}^{n+j} \sum_{\beta=0}^{n} (-1)^{\alpha+\beta} \times 
\frac{(j+\beta)!(n+j-\beta)!}{\beta!(n-\beta)!} e^{(1)\alpha\beta}_{\alpha_1\beta_1} e^{(2)\alpha_2\beta_2}_{\alpha_2\beta_2} 
$$

where $j = s/2$ and $n = 0, 1, 2, ...$, belong to $H_s$. The value of $j$ is obviously equal to the spin of the two-body system in conventional units. The fact that the vectors $e^{(1)}$ and $e^{(2)}$ enter Eq. (3.46) with the same value of the quantum number $n$ supports the interpretation of $n/R$ as the magnitude of the momentum (see the preceding section) since, by analogy with Poincare invariant theory, one would expect that the magnitudes of particle momenta in their common c.m. frame are equal to each other.

It is obvious that the vectors $\Phi_n$ with different $n$'s are orthogonal to each other. The result of the calculation of the norm of the
vector $\Phi_n$ (see Ref. [23] for details) is

$$(\Phi_n, \Phi_n) = \left\{ \prod_{l=1}^{n+j} \left[ w_1 + (2l + 1)^2 \right] \left[ w_2 + (2l + 1)^2 \right] \right\} \times \frac{(n + 2j + 1)!(1 + 2j)!}{(n + j + 1)n!}$$

(3.47)

(we have fixed a misprint in Eq. (32) of Ref. [23]).

Our next goal is to find how the operator $W^s$ acts in the basis \{ $\Phi_n$ \}. As follows from Eq. (3.9),

$$I_2^s \Phi_n = 8\left( (R_{22}^{(1)} + R_{22}^{(2)}) (R_{11}^{(1)} + R_{11}^{(2)}) - (R_{21}^{(1)} + R_{21}^{(2)}) \times (R_{12}^{(1)} + R_{12}^{(2)}) \right) \Phi^n - 4s(s + 2)\Phi_n,$$  

(3.48)

and an analogous formula takes place for $I_2^e e_{a\beta}^{(1)n}$.

Taking into account Eqs. (3.19), (3.46), (3.48), (3.49) and the definition of the operator $W^s$, a direct calculation shows that

$$W^s \Phi_n = \sum_{l=0}^{\infty} \Phi_l W^s_{ln}$$

(3.50)

where the matrix $\|W^s_{ln}\|$ has only the following components different from zero:

$$W^s_{n+1,n} = \frac{n + 1}{n + 1 + j}, \quad W^s_{nn} = w_1 + w_2 + 8(n + 1)^2 + 2s(4n + 3) + s^2 + 1, \quad W^s_{n,n+1} = \frac{n + 2j + 2}{n + j + 2} \times$$

$$[w_1 + (2n + 2j + 3)^2][w_2 + (2n + 2j + 3)^2]$$

(3.51)

Such a matrix is called three-diagonal and in fact, only the terms with $l = n - 1, n, n + 1$ contribute to the sum (3.50). Note that the operator $W^s$ is certainly Hermitian, but since the basis elements are not
normalized to one, the Hermiticity condition has not the usual form $W_{nl} = W_{ln}^{\ast s}$, but $||\Phi_n||^2 W_{nl} = ||\Phi_l||^2 W_{ln}^{\ast s}$.

The matrix of the operator $W^s - \lambda$ has the matrix elements $||W_{nl} - \lambda \delta_{nl}||$. We use $\Delta^n(\lambda)$ to denote the determinant of the matrix obtained from this one by taking into account only the rows and columns with the numbers $0, 1, \ldots, n$. It is well known (and can be verified directly) that for the three-diagonal matrix the following relation is valid:

$$\Delta^{n+1}(\lambda) = (W^s_{n+1,n+1} - \lambda) \Delta^n(\lambda) - W^s_{n+1,n} W^s_{n,n+1} \Delta^{n-1}(\lambda)$$

(3.52)

where it is formally assumed that $\Delta^{-1}(\lambda) = 1$.

Since we consider only representations of the principle series, we are interesting only in the region of positive $\lambda$'s. Therefore we can represent $\lambda$ as $\lambda = (\mu_1 + \mu_2 + \sigma)^2$ where $\sigma$ has the meaning of the dS kinetic energy. However as we will see below, not only $\sigma \geq 0$, but also $\sigma < 0$ is possible.

Consider the vector

$$\chi^s = \sum_{n=0}^{\infty} (-1)^n \Delta^n(\lambda) \prod_{l=1}^{n} W^s_{l,l+1}^{-1}\Phi_n$$

(3.53)

As follows from Eqs. (3.51) and (3.52), it is formally an eigenvector of the operator $W^s$ with the eigenvalue $\lambda$. If the sum in Eq. (3.53) converges, i.e. $(\chi^s, \chi^s)$ is finite, then $\lambda$ is the true eigenvalue i.e. it belongs to the discrete spectrum of the operator $W^s$. It is also possible that $\chi^s$ is the generalized eigen vector, i.e. $(\chi^s, \chi^s)$ is proportional to $\delta(\lambda - \lambda')$. Then $\lambda$ belongs to the continuous spectrum of the operator $W^s$. A detailed calculation has been carried out in Ref. [23]
and the result is

\[
(\chi_\lambda, \chi_{\lambda'}) = \frac{\pi^{1/2} \Gamma((3 + s)/2)}{2^s (1 + s/2)(1 + s/2)!} \prod_{j=1}^{s/2} \left[ w_1 + (2j + 1)^2 \right] \left[ w_2 + (2j + 1)^2 \right] \delta(\lambda - \lambda') \Gamma((3 + s - i\mu_1)/2) \Gamma((3 + s - i\mu_2)/2) \Gamma((3 + s - i\mu_1 + \mu_2 + \sigma)/2) \Gamma((3 + s - i(2\mu_1 + 2\mu_2 + \sigma))/4) \Gamma((3 + s - i(2\mu_1 + \sigma))/4) | \Gamma((3 + s - i(2\mu_2 + \sigma))/4) |^{-2}
\]

where \( \Gamma \) is the gamma function. Therefore the discrete spectrum is absent and continuous one fills all the interval \( \lambda \in (0, \infty) \).

In Ref. [21], where the explicit expression for the two-body mass operator in the form of a differential operator in some space of functions has been found, a similar result has been obtained. The result (3.54), however (obtained in Ref. [23]) is in fact algebraic since the operator \( W^s \) has been considered in the form of an infinite matrix. In Sect. 2.3 we have shown that the dS mass operator has the infinite spectrum in the range \( (-\infty, \infty) \). However, this result has been obtained in the nonrelativistic approximation and in first order in \( 1/R \). On the contrary, no approximation has been assumed in deriving Eq. (3.54).

### 3.5 Internal Hilbert space for the two-body system

When a two-body system is considered in Galilei and Poincare invariant theories, it is often convenient to describe its wave function not only in terms of individual particle variables but also in terms of external and internal variables. External variables describe the motion of the system as a whole while internal variables describe the relative motion. In Galilei invariant theories the two-body mass operator acts only in the Hilbert space of wave functions describing the relative motion. In Poincare invariant theories the two-body mass operator is unitarily equivalent to one acting only in the internal Hilbert space (see e.g. Ref.
The problem arises what is the internal Hilbert space in dS invariant theories.

Let us consider the vector $\Phi_n$ (see Eq. (3.46)) in the case when the vectors $e_{\alpha\beta}^n$ are treated as functions on the SU(2) group space (see Eq. (3.34)). We denote

$$T_{\rho \nu}(u) = \exp[-i(\rho \chi + \nu \psi)]P_{\rho \nu}(\theta)$$  \hspace{1cm} (3.55)

where $u \in SU(2)$ is defined by the Euler angles $(\chi, \theta, \psi)$. Eq. (3.55) defines the matrix of the representation operator $T^l$ in the case when the regular representation of the SU(2) group is reduced onto the subspace of functions which are the eigenfunctions of the operators $J^2$ and $J^2$ with the eigenvalue $n(n + 2)$, $n = 2l$ (see e.g. Ref. [44]).

The functions $e^{(1)}$ and $e^{(2)}$ depend on the corresponding single-particles variables, which we denote as $u_1$ and $u_2$, respectively. Then, as follows from Eqs. (3.34), (3.46) and (3.55)

$$\Phi_n(u_1, u_2) = (-1)^{n+j} \left\{ \prod_{k=1}^{n+j} [\mu_1 + i(2k + 1)][\mu_2 + i(2k + 1)] \right\}$$

$$\sum_{\alpha=0}^{n+j} \sum_{\beta=0}^{n} (-1)^{\alpha+\beta} \left[ (j + \beta)!(n + j - \beta)! \right]^{1/2}$$

$$T_{\rho_1 \nu_1}^{l+j/2}(u_1) T_{\rho_2 \nu_2}^{l+j/2}(u_2)$$ \hspace{1cm} (3.56)

where

$$\rho_1 = \frac{1}{2}(n + j - 2\alpha) \hspace{0.5cm} \nu_1 = -\frac{1}{2}(n + j - 2\beta) \hspace{0.5cm} \rho_2 = \frac{1}{2}(2\alpha - n - j)$$

$$\nu_2 = -\frac{1}{2}(2\beta + j - n)$$  \hspace{1cm} (3.57)

Consider now the following question. If the $T_{\rho \nu}(u)$ are the matrix elements of the representation operator $T^l(u)$ then what are the matrix elements of the operator $T^l(u^{-1})$? We use $g(\varphi)$ and $h(\theta)$ to denote the following SU(2) elements

$$g(\chi) = \begin{pmatrix} \exp(i\chi/2) & 0 \\ 0 & \exp(-i\chi/2) \end{pmatrix} \hspace{1cm} h(\theta) = \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$  \hspace{1cm} (3.58)
If \( u \in SU(2) \) is characterized by the Euler angles \((\chi \theta \psi)\) then \( u = g(\chi)g(\theta)g(\psi) \) and therefore \( u^{-1} = g(-\psi)g(-\theta)g(-\chi) \). Then, as follows from Eq. (3.55)

\[
T^l_{\rho \nu}(u^{-1}) = \exp[i(\rho \psi + \nu \chi)]P^l_{\rho \nu}(-\theta)
\]

(3.59)

Note that \( P^l_{\rho \nu}(-\theta) \) is not defined yet since the argument should be in the range \((0, \pi)\). We can use the fact that \( h(\theta) = g(\pi)h(\theta)g(-\pi) \). Then, as follows from Eqs. (3.55) and (3.59)

\[
T^l_{\rho \nu}(u^{-1}) = (-1)^{\nu - \rho}T^l_{-\nu, -\rho}(u)
\]

(3.60)

As follows from Eq. (3.60) and the property of representation operators \( T^l(u_1u_2) = T^l(u_1)T^l(u_2) \), Eq. (3.56) can now be written in the form

\[
\Phi_n(u_1, u_2) = \frac{(-1)^j}{j!}\left\{ \prod_{k=1}^{n+j} \left[ \mu_1 + i(2k + 1) \right] \left[ \mu_2 + i(2k + 1) \right] \right\}
\]

\[
\sum_{\nu=-l}^{l} \frac{(l + j + \nu)!(l + j - \nu)!}{(l + \nu)!(l - \nu)!} \frac{1}{l^{l+j/2}} T^{l+j/2}_{\nu+j/2, \nu-j/2}(u^{-1}_2u_1)
\]

(3.61)

We conclude that the function \( \Phi_n \) describing an internal state, actually depends not on single-particle variables \( u_1 \) and \( u_2 \) independently but only on their combination \( u_2^{-1}u_1 \). Therefore the latter can be treated as a variable describing the internal motion of the two-body system. This is a generalization of the well known fact that in nonrelativistic quantum mechanics the internal momentum \( q \) is defined as in Eq. (2.13). Indeed, when the velocities are small, \( u_1 \) is defined by the three-dimensional vector \( u_1 \approx \mathbf{v}_1 \) such that \( |u_1| \ll 1 \) and \( u_1 \approx 1 + i\sigma u_1 \). Analogously \( u_2 \approx 1 + i\sigma u_2 \). Therefore \( u_2^{-1}u_1 \) in this approximation is defined by \( u = u_1 - u_2 \). On the other hand, \( p_k = m_k u_k \), \( (k = 1, 2) \) and therefore the relative momentum \( q \) in the nonrelativistic approximation is equal to \( m_{12}u \) where \( m_{12} \) is the reduced mass.

Let us stress that our conclusion that \( u_2^{-1}u_1 \) is the internal variable in dS invariant theories is obtained without involving any approximation. For this reason it is possible to derive the relation between
\( u \) and \( q \) in the general case. Recall that \( u_j = v_j/(v_0)_j \) \((j = 1, 2)\) (see Sect. 2.1). Therefore, in the c.m. frame, where \( p_1 = q \) and \( p_2 = -q \), the elements of the SU(2) group corresponding to \( u_1 \) and \( u_2 \) are given by

\[
\begin{align*}
    u_1 &= \frac{m_1 + i\sigma q}{(m_1^2 + q^2)^{1/2}} \\
    u_2 &= \frac{m_2 - i\sigma q}{(m_2^2 + q^2)^{1/2}}
\end{align*}
\] (3.62)

where \( q = |q| \). We now use \( u \) to denote the internal variable \( u_2^{-1}u_1 \).

Since the element of the SU(2) group corresponding to \( u \) is \( u_{\alpha} + i\sigma u \), it follows from Eq. (3.62) that

\[
\begin{align*}
    u &= \frac{(m_1 + m_2)q}{[(m_1^2 + q^2)(m_2^2 + q^2)]^{1/2}} \\
    u_4 &= \frac{m_1m_2 - q^2}{[(m_1^2 + q^2)(m_2^2 + q^2)]^{1/2}}
\end{align*}
\] (3.63)

The vector \( \Phi_n \) in Eq. (3.61) depends only on \( u \in SU(2) \), and we can parametrize \( u \) by the Euler angles, for which we will again use the notations \((\chi \theta \psi)\). Then the internal polar angle is \( \varphi = (\psi - \chi)/2 \) (see Sect. 3.2) and, as follows from Eq. (3.55), the dependence of \( \Phi_n \) on \( \varphi \) is in the form \( \exp(ij\varphi) \). This is in agreement with the fact that \( \Phi_n \) is the eigenvector of the \( z \) component of the two-body spin operator \( S \) with the eigenvalue \( j \) (in conventional units).

Since \( \Phi_n \) is also the eigenvector of \( S^2 \) with the eigenvalue \( j(j + 1) \), one might expect that \( \Phi_n \) depends on \( \varphi \) and the "true" azimuthal angle \( \tilde{\theta} \) (see Sect. 3.2) in the form \( Y_{jj}(\tilde{\theta}, \varphi) = \exp(ij\varphi)P^j_j(\tilde{\theta}) \), where \( Y_{jj} \) is the spherical function and \( P^j_j \) is the associated Legendre function. However, since the main goal of the present paper is to demonstrate the idea that gravitational effects can be obtained in the free theory, we will concentrate our attention on the simplest case when the internal two-body spin is equal to zero, i.e. \( j = 0 \). Then Eq. (3.61) has a much simpler form:

\[
\Phi_n(u) = \left\{ \prod_{k=1}^n [\mu_1 + i(2k + 1)] [\mu_2 + i(2k + 1)] \right\} \sum_{\nu=-l}^l T^l_{\nu\nu}(u) \quad (3.64)
\]

The sum in Eq. (3.64) is the trace of the representation operator \( T^l(u) \), and in the literature it is called the character of the representation \( u \to T^l(u) \). The character can be calculated in a much simpler
way than the general result when \( j \neq 0 \) [44]. Define a parameter \( t \) such that
\[
\cos t = \cos \frac{\theta}{2} \cos \frac{\chi + \psi}{2}
\] (3.65)

Then
\[
\sum_{\nu=-l}^{l} T_{\nu
u}^l (u) = \frac{\sin(n + 1)t}{\sin t}
\] (3.66)

As follows from Eq. (3.21), \( \cos t = u_4 \) and therefore \( t \) has a clear meaning: since \( u_1^2 + u_2^2 = 1 \), \( t \) is the angle between the four-dimensional unit vector \( u \) and the 4 axis. In particular, the result (3.66) shows that the character depends only on \( |u| \) and does not depend on \( \varphi \) and \( \tilde{\theta} \), as expected.

We conclude that in the case when the internal angular momentum of the two-body system is zero, the functions \( \Phi_n \) have the following simple form:
\[
\Phi_n(u) = \left\{ \prod_{k=1}^{n} [\mu_1 + i(2k + 1)][\mu_2 + i(2k + 1)] \right\} \frac{\sin(n + 1)t}{\sin t}
\] (3.67)

It is easy to show that the characters are normalized to one [44] and therefore the normalization of states is compatible with Eq. (3.47).
Chapter 4

Mean value of the two-body mass operator

4.1 Preliminary discussion

Let $\Psi(u_1, u_2)$ be a wave function of the two-body system and one wants to calculate the distribution of possible values of the mass in the state $\Psi$. In Poincare invariant theory an analogous problem has been investigated in different approaches [41, 45] and the results are similar. Namely, instead of individual particle variables $(p_1, p_2)$ one can introduce the total momentum $P$, the relative momentum $q$ and decompose the two-body Hilbert space into a direct integral of Hilbert spaces $H(P)$ corresponding to a given value of $P$. The mass operator can be decomposed into a direct integral of operators $M(P)$ acting in corresponding spaces $H(P)$. The spectra of all the operators $M(P)$ are the same, which is the manifestation of the fact that $P$ is only a kinematical variable while dynamics is fully defined by the mass operator. Therefore the dynamics is fully defined by the operator $M(0)$ in $H(0)$. In turn, since $M(0)$ commutes with the two-body spin operator $S$, its spectrum can be investigated by decomposing $H(0)$ into subspaces $H_{jm}(0)$ corresponding to the eigenvalue of $S^2$ equal to $j(j + 1)$ and the eigenvalue of $S_z$ equal to $m$.

In the preceding chapter we have investigated the structure of the Hilbert space $H_s$, which is the analog of $H_{jj}(0)$ in Poincare invariant theory. By analogy with the construction of UIRs in Sect.
one can construct the full two-body Hilbert space as a direct sum of spaces identical to $H_s$. Since the mass operator commutes with all the representation operators, it also commutes with all the projectors onto such spaces and its spectra in each space are the same. Therefore to investigate the spectrum of the two-body mass operator it is sufficient to consider its spectrum in $H_s$. As shown in Sect. 3.4, the the operator $W$, which is the analog of the mass operator squared in Poincare invariant theory, has the spectrum containing the interval $(0, \infty)$ (and possibly other points) i.e. the spectrum is not bounded below by $(\mu_1 + \mu_2)^2$ as one would expect. Therefore one has to investigate whether the states having the values of the mass less than $\mu_1 + \mu_2$ are physical or not. For this purpose it is sufficient to investigate not all possible wave functions $\Psi(u_1,u_2)$ but only their projections onto $H_s$.

Instead of the states $\Phi_n$ in $H_s$, we now introduce the states

$$\Psi_n = \Phi_n(\Phi_n, \Phi_n)^{-1/2} \quad (4.1)$$

Then the elements $\{\Psi_n\}$ form the orthonormal basis in $H_s$. As follows from Eqs. (3.47) and (3.51), the nonzero matrix elements of the operator $W^*$ in the basis $\{\Psi_n\}$ are given by

$$W_{n+1,n} = W_{n,n+1} = \left[ w_1 + (2n + 2j + 3)^2 \right] \left[ w_2 + (2n + 2j + 3)^2 \right] \frac{(n + 2j + 2)(n + 1)}{(n + 1 + j)(n + 2 + j)}^{1/2}$$

$$W_{nn} = w_1 + w_2 + 8(n + 1)^2 + 4j(4n + 3) + 4j^2 + 1 \quad (4.2)$$

where $j = s/2$.

Let

$$\Psi = \sum_{n=0}^{\infty} c_n \Psi_n \quad (\sum_{n=0}^{\infty} |c_n|^2 = 1) \quad (4.3)$$

be a state in $H_s$. Then the mean value of the operator $W$ in the state $\Psi$ is given by

$$(\Psi, W \Psi) = \sum_{n=0}^{\infty} [W_{nn}|c_n|^2 + 2W_{n,n+1} Re(c_{n+1}c_n^*)] \quad (4.4)$$
where $W_{nn}$ and $W_{n,n+1}$ are given by Eq. (4.3). Since we are interested in comparing the results with Poincare invariant theory, the mass operator squared in Poincare terms should be defined as $M^2 = W/4R^2$ (see Sects. 2.2 and 3.4). We also take into account that typical values of $n$ are very large since, as noted in the preceding chapter, $n$ is of order $qR$ where $q$ is the relative momentum and $q = |\mathbf{q}|$. In particular, $n \gg j$ since for quasiclассical particles $j$ is of order $qr$ where $r$ is the distance between the particles. Then taking into account the relation between the dS and Poincare masses (see Sect. 2.2), we get from Eq. (4.4)

\[
(\Psi, M^2 \Psi) = \sum_{n=0}^{\infty} \left\{ \left[ m_1^2 + m_2^2 + 2(n/R)^2 \right] |c_n|^2 + 2\left[ m_1^2 + (n/R)^2 \right]^{1/2} \left[ m_2^2 + (n/R)^2 \right]^{1/2} Re(c_{n+1}c_n^*) \right\}
\]

(4.5)

i.e. the dependence on $s$ formally dissapears.

4.2 Standard relativistic mass operator

Before proceeding to calculations, let us look at this expression more carefully. If the particles are quasiclassical then one would expect that the dependence on $n$ of the coefficients $c_n$ has a maximum around $qR$ and the width of the maximum is much less than $qR$. Suppose also that the difference between the consecutive coefficients is small, i.e. $c_n \approx c_{n+1}$. Then it follows from Eq. (4.5) that the mean value of the operator $M^2$ is given by the standard relativistic expression for the free mass operator:

\[
\bar{M}^2 = M_0^2 = \left[ (m_1^2 + q^2)^{1/2} + (m_2^2 + q^2)^{1/2} \right]^2
\]

(4.6)

Suppose now that $c_n$ and $c_{n+1}$ have approximately the same magnitudes but the difference between their phases - $\delta$ - cannot be neglected. If $\delta$ does not change significantly when $n$ is inside the width of the maximum then instead of Eq. (4.6) we have

\[
\bar{M}^2 = M_0^2 - 4\left( \sin \frac{\delta}{2} \right)^2 [m_1^2 + q^2]^{1/2} [m_2^2 + q^2]^{1/2}
\]

(4.7)
The contribution of the last term in this expression is always negative. In the nonrelativistic approximation it is proportional to the particle masses. In particular, if \( \delta \ll 1 \) then in the nonrelativistic approximation

\[
\bar{M}^2 = M_0^2 - m_1 m_2 \delta^2
\] (4.8)

where \( M_0 \) should also be taken in the nonrelativistic approximation: \( M_0 = m_1 + m_2 + q^2/(2m_1) \).

If Newtonian gravity is taken into account then the nonrelativistic mass of the classical two-body system is

\[
M = M_0 - \frac{G m_1 m_2}{r_0}
\] (4.9)

where \( r_0 \) is the distance between the particles. As a result, in the nonrelativistic approximation

\[
M^2 = M_0^2 - \frac{2G(m_1 + m_2)}{r_0} m_1 m_2
\] (4.10)

This expression is compatible with Eq. (4.8) if

\[
\delta = \delta_0(r_0) = \left[ \frac{2G(m_1 + m_2)}{r_0} \right]^{1/2} = \left( \frac{r_{1G} + r_{2G}}{r_0} \right)^{1/2}
\] (4.11)

where \( r_{1G} = 2Gm_1 \) and \( r_{2G} = 2Gr_2 \). In the literature the quantity \( r_G = 2Gm \) is called the gravitational radius of the particle with the mass \( m \). Therefore the condition \( \delta \ll 1 \) has a clear meaning: the sum of the gravitational radii of the particles in the two-body system should be much less than the distance between them. This is precisely the condition when Newtonian gravity is a good first approximation to General Relativity. Another important observation is as follows. If gravity is a consequence of Eqs. (4.6) and (4.7) then the gravitational potential is always nonsingular since at small values of \( r \) the condition \( \delta \ll 1 \) is not satisfied.

As noted in Sect. 1.2, any dS invariant quantum theory can be constructed only in terms of dimensionless quantities. In particular, in
terms of dS masses $\delta$ can be written as $\tilde{G}(\mu_1 + \mu_2)/\varphi$ where the angular variable $\varphi$ can be treated as $r/(2R)$ and $\tilde{G} = G/(4R^2)$. However, since we wish to compare our results with the standard ones, in this chapter we will use the standard gravitational constant and masses.

We now proceed to actual calculations. Consider a system of two spinless quasiclassical particles and assume that their wave functions can be taken as in Eqs. (2.21) and (2.22). We also assume (see the discussion in Sect. 2.3) that if the particles do not interact with each other then the two-body wave function is the product of the single-particle wave functions. As noted in the preceding chapter, all the information about the mass operator can be obtained by considering its action only in the spaces $H_s$. For this reason we consider only the case when the total momentum of the two-body system is equal to zero.

We first consider the nonrelativistic approximation. As noted above, in that case $u \approx v$, $|u| \ll 1$ and hence $u_4 \approx 1 - u^2/2$. Therefore, taking into account Eqs. (2.21) and (2.22), one can write the two-body wave function as

$$
\Psi(u_1, u_2) = \text{const} \exp\{-\frac{iR}{2}(m_1u_1^2 + m_2u_2^2) - i(m_1u_1r_1 + m_2u_2r_2) - \frac{1}{2}[b_1^2(u_1 - g_1)^2 + b_2^2(u_2 - g_2)^2]\} \tag{4.12}
$$

where we use $g_1$ and $g_2$ to denote the quantity $v_0$ for particles 1 and 2, respectively.

As noted in the preceding chapter, the quantity $u = u_1 - u_2$ has the meaning of the relative velocity in the nonrelativistic approximation. The question arises what is the correct choice of the external two-body variables. Recall that for calculating the mass distribution for the two-body system with zero total momentum, one has to calculate the coefficients $c_n = (\Psi_n, \Psi)$. Since $\Psi_n$ depends only on internal variables, the dependence on external variables in the integral for $c_n$ will be integrated out. Therefore, any variable $U$, such that there exists a one-to-one relation between the sets $(Uu)$ and $(u_1u_2)$, can be taken as the external variable. By analogy with the nonrelativistic theory we choose $U = (m_1u_1 + m_2u_2)/(m_1 + m_2)$ as the external velocity variable.
Then
\[ u_1 = U + \frac{m_2}{m_1 + m_2} \quad u_2 = U - \frac{m_1}{m_1 + m_2} \] (4.13)
and Eq. (4.12) can be rewritten as
\[
\Psi(U, u) = \text{const} \exp\left\{-\frac{iR}{2}((m_1 + m_2)U^2 + m_{12}u^2) - \right.
\]
\[ i((m_1 + m_2)UR + m_{12}ur) - \frac{1}{2}\left[b_1^2(U + \frac{m_2}{m_1 + m_2} - 
\]
\[ g_1)^2 + b_2^2(U - \frac{m_1}{m_1 + m_2} - g_2)^2]\right\} \] (4.14)
where
\[ R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \quad r = r_1 - r_2 \] (4.15)
Therefore Eq. (4.14) can be represented as
\[
\Psi(U, u) = \text{const} \exp\left\{-\frac{U^2}{2}\right\}[b_1^2 + b_2^2 + i(m_1 + m_2)R] + 
\]
\[ U[b_1^2(g_1 - \frac{m_2 u}{m_1 + m_2}) + b_2^2(g_2 + \frac{m_1 u}{m_1 + m_2}) - 
\]
\[ i(m_1 + m_2)R]\} \Psi(u) \] (4.16)
where
\[ \Psi(u) = \exp\left\{-\frac{i}{2}Rmu^2 - \frac{b_2^2}{2}(\frac{mu}{m_1} - g_1)^2 - 
\]
\[ \frac{b_2^2}{2}(\frac{mu}{m_2} + g_2)^2 - imurm\} \] (4.17)
and we write \( m = m_{12} \) for simplicity.
Since our goal is to calculate \( c_n \), and the vectors \( \Psi_n \) do not depend on the external variables, we can integrate \( \Psi(U, u) \) over \( U \).
Since we assume that \( R \) is very large, it is clear from Eq. (4.16) that
\[
\int \Psi(U, u)d^3U = \text{const} \quad \Psi(u) \] (4.18)
where const does not depend on the parameters characterizing the quasiclassical wave functions, i.e. does not depend on whether the functions
\( a(\mathbf{v}) \) in Eq. (2.21) are taken as in Eq. (2.22) or in other forms. This result shows that \( \Psi(\mathbf{u}) \) is proportional to \( \Psi(0, \mathbf{u}) \). As a consequence,

\[
c_n = \text{const} \int \Psi(\mathbf{u})\Psi_n(\mathbf{u})^* d^3\mathbf{u}
\]

(4.19)

where \( \Psi_n \) belongs to a certain \( H_s \).

We will describe all the necessary steps in the case when the internal two-particle spin is equal to zero. One might think that this special case is rather representative since, at least in the nonrelativistic approximation, the standard gravitational interaction operator \(-Gm_1m_2/r\) does not act over the internal angular variables of the two-body system. From the technical point of view the case \( s = 0 \) is the simplest since, as noted in the preceding chapter, the expressions for the functions \( \Phi_n \) are defined by the characters of the UIR of the SU(2) group.

For quasiclassical particles the internal angular momentum is equal to \( \mathbf{r} \times \mathbf{q} \). Since we consider the case when the total momentum of the two-body system and the internal angular momentum are equal to zero, we can assume that \( g_1 = (q/m_1)e_z \), \( g_2 = -(q/m_2)e_z \) and \( \mathbf{r} = r\mathbf{e}_z \) where \( \mathbf{e}_z \) is the unit vector along the \( z \) axis. Then the function \( \Psi(\mathbf{u}) \) in Eq. (4.17) can be written as

\[
\Psi(\mathbf{u}) = \exp[-i\frac{\mathbf{R}\mathbf{u}}{2} - \frac{m^2}{2}(\frac{b_1^2}{m_1^2} + \frac{b_2^2}{m_2^2})(u^2 + \frac{q^2}{m^2}) + \eta \mathbf{h}(\mathbf{q}, \mathbf{r})]
\]

(4.20)

where \( u = |\mathbf{u}| \),

\[
\mathbf{h}(\mathbf{q}, \mathbf{r}) = mq(\frac{b_1^2}{m_1^2} + \frac{b_2^2}{m_2^2}) - imr
\]

(4.21)

and \( \eta \) is the cosine of the angle between \( \mathbf{u} \) and the \( z \) axis.

As noted in the preceding chapter, the functions \( \Phi_n \) in \( H_0 \) depend only on \( u \) (see Eq. (3.67)). Therefore to calculate \( c_n \) in Eq. (4.19) one has first to calculate

\[
\int_{-1}^{1} \exp[\eta \mathbf{h}(\mathbf{q}, \mathbf{r})] d\eta
\]
As noted in Sect. 3.3, the meaning of the quantity \( b \) in Eq. (2.22) is that \( 1/b \) is the uncertainty of the particle velocity. Therefore \( (b_1/m_1)^2 \) can be written as \( 1/(\Delta p_1)^2 \) where \( \Delta p_1 \) is the uncertainty of the momentum for particle 1. Since particle 1 is quasiclassical, our choice of the momentum variables implies that \( muq \approx |p_1|^2 \). Therefore \( mqu(b_1/m_1)^2 \gg 1 \) and analogously \( mqu(b_2/m_2)^2 \gg 1 \). We conclude that \( \text{Re}(uh(r)) \gg 1 \).

Then, as follows from Eqs. (3.67), (4.1), (4.19) and (4.20)

\[
c_n = \frac{\text{const}}{h(q,r)} \left\{ \prod_{k=1}^{n} \left[ \mu_1 - i(2k+1) \right] \left[ \mu_2 - i(2k+1) \right] \right\} \\
\times \left\{ \prod_{k=1}^{n} \left[ w_1 + (2k+1)^2 \right] \left[ w_2 + (2k+1)^2 \right] \right\}^{-1/2} \\
\int_0^\infty \exp\left( -\frac{i}{2} Rmu^2 - imur \right) A(u) u \frac{\sin(n+1)t}{\sin t} du \quad (4.22)
\]

where

\[
A(u) = \exp\left[ -\frac{m^2}{2} \left( \frac{b_1^2}{m_1^2} + \frac{b_2^2}{m_2^2} \right) (u - \frac{q}{m})^2 \right] \quad (4.23)
\]

Since we assume that the particles are nonrelativistic, it follows from Eq. (4.23) that the main contribution to the integral in Eq. (4.22) is given by the region where \( u \approx (q/m) \ll 1 \) and therefore \( t \approx u \). One might expect that Eq. (4.22) is rather general in the sense that for any reasonable choice of the quasiclassical wave functions the expression for \( A(u) \) is not necessarily given by Eq. (4.23) but in any case \( A(u) \) has a sharp maximum around \( u \approx (q/m) \).

We now write

\[
\sin[(n+1)u] = \frac{1}{2i} \left( \exp[i(n+1)u] - \exp[-i(n+1)u] \right) \quad (4.24)
\]

and notice that the contribution of the second term to the integral in Eq. (4.22) is small since the rapidly oscillating exponent in this expression does not have a stationary point. At the same time the expression corresponding to the first contribution has the stationary point at

\[
u_n = \frac{n+1}{Rm} - \frac{r}{R} \quad (4.25)
\]
Therefore, calculating the integral in Eq. (4.22) by the stationary phase method we obtain

\[ c_n = \frac{\text{const}}{h(r)} \left\{ \prod_{k=1}^{n} [\mu_1 - i(2k + 1)][\mu_2 - i(2k + 1)] \right\} \]

\[ \{ \prod_{k=1}^{n} [w_1 + (2k + 1)^2][w_2 + (2k + 1)^2] \}^{-1/2} \]

\[ A(u_n) \exp \left( \frac{i}{2} mR u_n^2 \right) \]

As follows from this expression, \( |c_n| \approx |c_{n+1}| \). It is also easy to see that the phases of \( c_n \) and \( c_{n+1} \) also are approximately the same. Indeed, when \( |u_n| \ll 1 \) implies that \( n \ll \mu_1 \) and \( n \ll \mu_2 \). Therefore the phase difference is approximately equal to

\[ \frac{mR}{2} (u_{n+1}^2 - u_n^2) - 2n\left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \]

As follows from Eq. (4.25), this quantity is indeed negligible.

We conclude that \( c_n \approx c_{n+1} \) and, as follows from Eq. (4.25), the dependence of the coefficients \( c_n \) on \( n \) has a sharp maximum near \( (n/R) = q + (mr/R) \). The second term on the r.h.s. of this equality obviously represents the correction caused by the dS antigravity (see Sect. 2.3). When \( R \) is large, this correction is small and, as a result, in the nonrelativistic approximation we have the situation discussed after Eq. (4.5). Therefore the mean value of the mass operator squared in the nonrelativistic approximation is indeed given by the standard formula (4.6) assuming that \( q \ll m_1 \) and \( q \ll m_2 \).

Let us now show that in fact the situation discussed after Eq. (4.5) takes place in the general case, and therefore the mean value of the mass operator squared is precisely given by Eq. (4.6). Note first that the above derivation in the nonrelativistic case was not quite accurate for the following reason. We expanded the exponent index in powers of \( u \). However, since the index contains a large parameter \( R \), the terms which have been dropped are by no means small. Nevertheless, such a
procedure is legitimate if the manipulations with the exponent index are used for calculations in the stationary phase method.

As argued above, the result that $\Psi(u)$ is proportional to $\Psi(0,u)$ is quite general and therefore Eq. (4.19) is correct in the general case. It has been also shown that for the considered class of wave functions the contribution of the dS antigravity to the mean value of the mass operator is small and for this reason we neglect this contribution.

In Sect. 3.4 it has been shown that there exists a one-to-one correspondence between the quantities $u$ and $q$ (see Eq. (3.63)). Therefore, as follows from Eq. (2.11), a natural generalization of Eq. (4.22) is

$$c_n = \text{const} \left\{ \prod_{k=1}^n \left[ \mu_1 - i(2k + 1) \right] \left[ \mu_2 - i(2k + 1) \right] \right\}$$

$$\left\{ \prod_{k=1}^n [w_1 + (2k + 1)^2][w_2 + (2k + 1)^2] \right\}^{-1/2}$$

$$\int_0^\infty \exp\left[ -\frac{i}{4} \mu_1 \ln(m_1^2 + q^2) - \frac{i}{4} \mu_2 \ln(m_2^2 + q^2) \right]$$

$$\sin(n + 1)t A(q, r)qdq$$

(4.27)

where now $q$ is the integration variable while the average magnitude of the relative momentum will be denoted as $q_0$. Then one might expect that, as a function of $q$, $A(q, r)$ has a sharp maximum in the region where $q$ is close to $q_0$.

We will again calculate the integral in the stationary phase method. Therefore, taking into account the definition of $t$, we have to consider the rapidly oscillating exponent $\exp[i f(q, n)]$ where

$$f(q, n) = -\frac{R}{2} m_1 \ln(m_1^2 + q^2) - \frac{R}{2} m_2 \ln(m_2^2 + q^2) + (n + 1) \cos u_4$$

(4.28)

A direct calculation using Eq. (3.63) gives

$$\frac{\partial f(q, n)}{\partial q} = (n + 1 - Rq) \frac{(m_1 + m_2)(m_1 m_2 + q^2)}{(m_1^2 + q^2)(m_2^2 + q^2)}$$

(4.29)
Therefore the stationary point is defined by

\[ q_n = \frac{n + 1}{R} \]  

(4.30)

This result is compatible with Eq. (4.25) (if the contribution of the dS antigravity is neglected) and shows that the quantum number \( n \) has the meaning of the magnitude of the relative momentum times \( R \) not only in the nonrelativistic approximation but exactly.

The next step is to show that \( c_n \) and \( c_{n+1} \) have the same phases if \( n \) and \( R \) are large. As follows from Eqs. (4.29) and (4.30), \( f(q_{n+1}, n) - f(q_n, n) \) is of order \( 1/R \) and therefore, as follows from Eq. (4.27), \( c_n \) and \( c_{n+1} \) will have approximately the same phases if

\[ acosu_4 = asin \left( \frac{q}{(m_1^2 + q^2)^{1/2}} \right) + asin \left( \frac{q}{(m_2^2 + q^2)^{1/2}} \right) \]  

(4.31)

As follows from Eq. (3.63), this expression is indeed valid.

### 4.3 Effect of phase difference

Before discussing the problem of interactions in dS invariant theories, we recall well-known facts about such a problem in Poincare invariant theory. It is much more complicated than in nonrelativistic quantum mechanics since if the Hamiltonian is interaction dependent then some other generators are necessarily interaction dependent too. For this reason, even for a two-body system, the problem arises how one could introduce interaction into the generators without breaking commutation relations of the Poincare group Lie algebra. For the first time this problem has been solved by Bakamdjian and Thomas [47]. They showed that if \( V \) is added not to the Hamiltonian but to the mass operator (or mass operator squared), and \( V \) commutes with the internal two-body angular momentum operator then the required commutation relations are preserved. As shown by several authors (see e.g. Refs. [48, 41, 45]), the Bakamdjian-Thomas procedure can be extended to the case when the number of particles is arbitrary (but fixed).
By analogy with the Bakamdjian-Thomas procedure, it is clear from the construction of Sects. 3.4 and 3.5 that the mass operator in dS invariant theory is fully defined by its action in \( H_s \). Therefore one can modify the free dS mass operator by introducing an interaction operator in \( H_s \) commuting with the reduction of the two-body spin operator onto \( H_s \). In that case the commutation relations (1.2) will be preserved. However, the important difference between Poincare and dS invariant theories is as follows. While in the former the free and interacting mass operators can have different spectra (the free mass operator has only the continuous spectrum while the interacting mass operator can also have the discrete spectrum), in the latter they have the same spectra and the free mass operator is not bounded below by \( m_1 + m_2 \). These facts have been discussed in Chap. 2 and it has been noted that such features pose the problem whether the very notion of interaction is needed at all.

As it has been shown in the preceding section, if the two-body wave function is the product of the single-particle wave functions having the form (2.21), then the mean value of the mass operator is given by Eq. (4.6) when \( R \) is large. Although this fact could be expected, the above derivation shows that if the Poincare invariant theory is treated as a special case of the dS invariant one in the limit when \( R \) is large, then rather delicate cancellations are required to ensure the condition \( c_n \approx c_{n+1} \).

We again consider a quasiclassical two-body system such that the internal wave function in momentum space has a sharp maximum at \( q = q_0 \) and the internal wave function in coordinate space has a sharp maximum at \( r = r_0 \). Since each space \( H_s \) has the basis characterized by the functions \( \Psi_n \), the projection of each wave function \( \Psi \) onto \( H_s \) is fully characterized by the coefficients \( c_n = (\Psi_n, \Psi) \). Let \( c_n^{(0)} \) be the coefficients calculated in the preceding section.

Consider a wave function defined as follows. Its projection onto \( H_s \) is characterized by the coefficients \( c_n = c_n^{(0)} \exp[i \eta_n(r_0)] \) where \( \eta_n(r_0) \) are real functions such that \( \eta_{n+1}(r_0) - \eta_n(r_0) = \pm \delta(r_0) \) and \( \delta(r_0) \) is given by Eq. (4.11). Since \( c_{n+1}^{(0)} \approx c_n^{(0)} \), we have precisely the situation
discussed in Sect. 4.2 when the mean value of the mass operator squared is given by Eq. (4.10).

The class of wave functions for which the above property is satisfied is rather wide. For example, a possible choice of \( \eta_n(r_0) \) is \( \eta_n(r_0) = \pm n\delta_0(r_0) \). Since the coefficients \( c_n \) are defined up to an arbitrary overall phase factor, an interesting scenario is such that \( \eta_n(r_0) = 0 \) if \( n \) is even and \( \eta_{n}(r_0) = \delta_0(r_0) \) if \( n \) is odd. In that case the phase difference between \( c_n \) and \( c_n^{(0)} \) does not exceed \( \delta_0(r_0) \) for all values of \( n \).

The problem arises whether the wave functions with such coefficients \( c_n \) are compatible with the fact that the two-body system is quasiclassical. Since there exist many scenarios when the result (4.10) takes place, this problem seems to be rather complicated. In the subsequent section we consider a hypothesis that a natural explanation of the phase difference between \( c_n \) and \( c_n^{(0)} \) can be obtained in quantum theory over a Galois field.

In the framework of General Relativity, Newtonian gravity is the first approximation valid when the particles are nonrelativistic and \( \delta_0(r_0) \ll 1 \). The internal classical two-body Hamiltonian taking into account the corrections of order \( (q/m)^2 \) and \( \delta_0(r)^2 \) is given by (see e.g. problem 3 in section 106 of Ref. [49])

\[
M = m_1 + m_2 + \frac{q^2}{2m} - \frac{Gm_1m_2}{r} - \frac{q^4}{8}(\frac{1}{m_1^3} + \frac{1}{m_2^3}) - \frac{G}{2r}[3q^2(m_1 + m_2)^2 + \frac{m_2}{m_1}] + 7q^2 + \frac{(qr)^2}{r^2} + \frac{G^2}{2r^2}m_1m_2(m_1 + m_2) \tag{4.32}
\]

In quasiclassical approximation the angular momentum is equal to \( \mathbf{r} \times \mathbf{q} \) and therefore \( q^2r^2 - (qr)^2 \approx j^2 \) where \( j \) is the quasiclassical internal angular momentum. Therefore with the same accuracy

\[
M^2 = M_0^2 - \frac{2Gm_1m_2(m_1 + m_2)}{r} + \frac{G^2}{r^2}m_1m_2[(m_1 + m_2)^2 + m_1m_2] - \frac{3G(m_1 + m_2)}{m_1m_2r}q^2[(m_1 + m_2)^2 + m_1m_2] + \frac{G}{r^3}j^2(m_1 + m_2) \tag{4.33}
\]
where \( M_0^2 \) is given by Eq. (4.6).

In quantum theory Eq. (4.33) should be treated as the mean value of the mass operator squared, and \( q \) and \( r \) should be treated as the mean values of the corresponding quantities, i.e. \( q \) should be replaced by \( q_0 \) and \( r \) by \( r_0 \). Since we consider the mass distribution in each space \( H_s \) separately, \( j^2 \) should be replaced by the eigenvalue of the internal angular momentum squared in \( H_s \), i.e. \( j(j + 1) \).

By analogy with the nonrelativistic case, one can show that the result compatible with Eq. (4.33) can be obtained for wave functions characterized by the coefficients \( c_n = c_n^{(0)} \exp[i\eta_n(r_0, q_0)] \) such that \( \eta_{n+1}(r_0, q_0) - \eta_n(r_0, q_0) = \pm \delta(r_0, q_0) \),

\[
\delta(r_0, q_0) = \delta_0(r_0)[1 - a_1 \delta_0(r_0)^2 - \frac{j(j + 1)}{8m_1m_2r_0^2} + a_2q_0^2] \tag{4.34}
\]

and the coefficients \( a_1 \) and \( a_2 \) are given by

\[
a_1 = \frac{1}{12} + \frac{m_1m_2}{8(m_1 + m_2)^2} \quad a_2 = \frac{1}{2m_1^2} + \frac{1}{2m_2^2} + \frac{9}{4m_1m_2} \tag{4.35}
\]

(note that in the nonrelativistic approximation \( \delta(r_0, q_0) = \delta_0(r_0) \)).
Chapter 5

Quantum theory over a Galois field

5.1 What mathematics is most suitable for quantum physics?

In the preceding chapters it has been shown that the dS invariant quantum theory has unusual features which have no analogs in Poincare and AdS invariant theories. As a consequence of the fact that the spectrum of the dS mass operator is not bounded below by the value of $m_1 + m_2$, for a class of two-body wave functions, the mean value of the mass operator is compatible with Newtonian gravity and post-Newtonian corrections. This poses a very interesting problem, whether gravity can be explained without using the notion of interaction at all. However, such wave functions contain additional phase factors and it is not clear whether they are compatible with our understanding of the position operator in quantum theory. Another problem in the standard quantum theory is that the notion of interaction depends on the choice of the form of the single-particle generators.

In the present chapter we argue that quantum theory over a Galois field is more natural than the standard quantum theory based on complex numbers. We believe that some properties of the GFQT give indications that gravity can be explained in the framework of this theory. Since the absolute majority of physicists are not familiar with Galois fields, our first goal in this chapter is to convince the reader that the notion of Galois fields is not only very simple and elegant,
but also is a natural basis for quantum physics. We will follow mainly our arguments given in Ref. [50]. If a reader wishes to learn Galois fields on a more fundamental level, he or she might start with standard textbooks (see e.g. Ref. [51]).

The existing quantum theory is based on standard mathematics containing the notions of infinitely small and infinitely large.

The notion of infinitely small is based on our everyday experience that any macroscopic object can be divided by two, three and even a million parts. But is it possible to divide by two or three the electron or neutrino? It seems obvious that the very existence of elementary particles indicates that the standard division has only a limited sense. Indeed, let us consider, for example, the gram-molecule of water having the mass 18 grams. It contains the Avogadro number of molecules $6 \cdot 10^{23}$. We can divide this gram-molecule by ten, million, billion, but when we begin to divide by numbers greater than the Avogadro one, the division operation loses its sense.

The notion of infinitely large is based on our belief that in principle we can operate with any large numbers. Suppose we wish to verify experimentally whether addition is commutative, i.e. whether $a + b = b + a$ is always satisfied. If our Universe is finite and contains not more than $N$ elementary particles then we shall not be able to do this if $a + b > N$. In particular, if the Universe is finite then it is impossible in principle to build a computer operating with any large number of bits.

It is interesting to note that the number of elementary particles in the Universe might be not so immense as one could think. Indeed, if the radius of the Universe is 20 billion light years, and the average density of matter is $10^{-29} g/cm^3$ then the mass of the Universe is of order $10^{57} g$. If we assume that the average energy of elementary particles is $1eV$ then the number of elementary particles in the Universe is ”only” of order $10^{90}$. Note, for example, that the largest known prime number has 4033946 digits [52].

As noted in Sect. 2.2, the dS and Poincare masses are related to each other as $\mu = 2Rm$. In our units $m = 2/l_C$ where $l_C$ is the
particle Compton wave length. Therefore $\mu$ is roughly the ratio of the dS radius to the Compton wave length. Hence even the dS masses of elementary particles are very large. For example, if $R$ is, say, 20 billion light years then the dS mass of the electron is of order $10^{39}$. My observation is that physicists are usually surprised by this fact and their first reaction is that the dS mass is something unrealistic. However, the dS mass is dimensionless and therefore more fundamental than the Poincare mass which depends on the system of units. The electron dS mass might be an indication that the electron is not a truly elementary particle, but at present such an assumption is highly speculative.

As shown in Sect. 2.1 the UIRs defined by Eqs. (2.7) and (2.10) are related to each other by the unitary transformation (2.11). Note that the exponent index in Eq. (2.11) is very large. For example, if $R$ is again taken to be 20 billion light years then the dS mass of the Earth is of order $10^{94}$. If the exponent is defined in a standard way by the Taylor series then no existing computer can compute the exponent with such a large index. One can notice that in fact the index in Eq. (2.11) should be taken modulo $2\pi$, but in that case the result will be sensitive to the accuracy $\pi$ is known with. Suppose, however, that this problem is solved. Then we are facing another serious difficulty. The quasiclassical approximation implies that all the integrals in question can be calculated by the stationary phase method. It is well known that the idea of the method is to perturb the contour of integration in such a way that in some vicinity of the stationary point $x_0$ the integral can be approximated by

$$\int_{-\infty}^{\infty} e^{(x-x_0)^2} \mathrm{d}x \quad (a > 0).$$

In the case of Eq. (4.22) the role of $x$ is played by the nonrelativistic velocity and $b$ is the dS mass. Therefore the integral in Eq. (4.22) is mainly defined by a very narrow region of velocities having the width of order $10^{-47}c$. It is difficult to imagine that such an accuracy is meaningful.

Many physicists believe that (in contrast with the Dirac hypothesis [53]) the gravitational constant $G$ is fundamental and therefore
the Planck mass is fundamental too. The electron Planck mass is very small (of order $10^{-23}$) in agreement with the assumption that gravity is unified with the other interactions at the Planck scale $10^{-5}g$. On the other hand, as noted in Sect. 1.2, in quantum dS (and AdS) invariant theories there is no place for quantities having the dimension $(length)^2$ in units $\hbar/2 = c = 1$.

Let us look at mathematics from the point of view of the famous Kronecker expression: ‘God made the natural numbers, all else is the work of man’. Indeed, the natural numbers 0, 1, 2... (we treat zero as a natural number) have a clear physical meaning. However only two operations are always possible in the set of natural numbers: addition and multiplication.

In order to make addition reversible, we introduce negative integers -1, -2 etc. Then, instead of the set of natural numbers we can work with the ring of integers where three operations are always possible: addition, subtraction and multiplication. However, the negative numbers do not have a direct physical meaning (we cannot say, for example, ‘I have minus two apples’). Their only role is to make addition reversible.

The next step is the transition to the field of rational numbers in which all four operations excepting division by zero are possible. However, as noted above, division has only a limited sense.

In mathematics the notion of linear space is widely used, and such important notions as the basis and dimension are meaningful only if the space is considered over a field or body. Therefore if we start from natural numbers and wish to have a field, then we have to introduce negative and rational numbers. However, if, instead of all natural numbers, we consider only $p$ numbers 0, 1, 2, ... $p-1$ where $p$ is prime, then we can easily construct a field without adding any new elements. This construction, called Galois field, contains nothing that could prevent its understanding even by pupils of elementary schools.

Let us denote the set of numbers 0, 1, 2, ... $p-1$ as $F_p$. Define addition and multiplication as usual but take the final result modulo $p$. For simplicity, let us consider the case $p = 5$. Then $F_5$ is the set
0, 1, 2, 3, 4. Then $1 + 2 = 3$ and $1 + 3 = 4$ as usual, but $2 + 3 = 0$, $3 + 4 = 2$ etc. Analogously, $1 \cdot 2 = 2$, $2 \cdot 2 = 4$, but $2 \cdot 3 = 1$, $3 \cdot 4 = 2$ etc. By definition, the element $y \in F_p$ is called opposite to $x \in F_p$ and is denoted as $-x$ if $x + y = 0$ in $F_p$. For example, in $F_5$ we have $-2=3$, $-4=1$ etc. Analogously $y \in F_p$ is called inverse to $x \in F_p$ and is denoted as $1/x$ if $xy = 1$ in $F_p$. For example, in $F_5$ we have $1/2=3$, $1/4=4$ etc.

It is easy to see that addition is reversible for any natural $p > 0$ but for making multiplication reversible we should choose $p$ to be a prime. Otherwise the product of two nonzero elements may be zero modulo $p$. If $p$ is chosen to be a prime then indeed $F_p$ becomes a field without introducing any new objects (like negative numbers as fractions). For example, in this field each element can obviously be treated as positive and negative simultaneously!

One might say: well, this is beautiful but impractical since in physics and everyday life $2+3$ is always 5 but not 0. Let us suppose, however that fundamental physics is described not by ‘usual mathematics’ but by ‘mathematics modulo $p$’ where $p$ is a very large number. Then, operating with numbers much smaller than $p$ we shall not notice this $p$, at least if we only add and multiply. We will feel a difference between ‘usual mathematics’ and ‘mathematics modulo $p$’ only while operating with numbers comparable with $p$.

We can easily extend the correspondence between $F_p$ and the ring of integers $Z$ in such a way that subtraction will also be included. To make it clearer we note the following. Since the field $F_p$ is cyclic (adding 1 successively, we will obtain 0 eventually), it is convenient to visually depict its elements by the points of a circle of the radius $p/2\pi$ on the plane $(x, y)$. In Fig. 1.1 only a part of the circle near the origin is depicted. Then the distance between neighboring elements of the field is equal to unity, and the elements 0, 1, 2,... are situated on the circle counterclockwise. At the same time we depict the elements of $Z$ as usual such that each element $z \in Z$ is depicted by a point with the coordinates $(z, 0)$. We can denote the elements of $F_p$ not only as 0, 1,... $p - 1$ but also as 0, ±1, ±2,...±$(p - 1)/2$, and such a set is called the set of minimal residues. Let $f$ be a map from $F_p$ to $Z$, such
that the element $f(a) \in Z$ corresponding to the minimal residue $a$ has the same notation as $a$ but is considered as the element of $Z$. Denote $C(p) = p^{1/(\ln p)^{1/2}}$ and let $U_0$ be the set of elements $a \in F_p$ such that $|f(a)| < C(p)$. Then if $a_1, a_2, \ldots, a_n \in U_0$ and $n_1, n_2$ are such natural numbers that

$$n_1 < (p - 1)/2C(p), \ n_2 < ln((p - 1)/2)/(\ln p)^{1/2}$$

then

$$f(a_1 \pm a_2 \pm \ldots a_n) = f(a_1) \pm f(a_2) \pm \ldots f(a_n)$$

if $n \leq n_1$ and

$$f(a_1 a_2 \ldots a_n) = f(a_1)f(a_2)\ldots f(a_n)$$

if $n \leq n_2$. Thus though $f$ is not a homomorphism of rings $F_p$ and $Z$, but if $p$ is sufficiently large, then for a sufficiently large number of elements of $U_0$ the addition, subtraction and multiplication are performed according to the same rules as for elements $z \in Z$ such that $|z| < C(p)$. Therefore $f$ can be regarded as a local isomorphism of rings $F_p$ and $Z$.

The above discussion has a well known historical analogy. For many years people believed that our Earth was flat and infinite, and only after a long period of time they realized that it was finite and had a curvature. It is difficult to notice the curvature when we deal only with distances much less than the radius of the curvature $R$. Analogously one might think that the set of numbers describing physics has a curvature defined by a very large number $p$ but we do not notice it when we deal only with numbers much less than $p$.

Our arguments imply that the standard field of real numbers will not be fundamental in future quantum physics (although there is
no doubt that it is relevant for macroscopic physics). Let us discuss this question in a greater detail (see also Ref. [54]). The notion of real numbers can be fundamental only if the following property is valid: for any $\epsilon > 0$, it is possible (at least in principle) to build a computer operating with a number of bits $N(\epsilon)$ such that computations can be performed with the accuracy better than $\epsilon$. It is clear that this is not the case if, for example, the Universe is finite.

Let us note that even for elements from $U_0$ the result of division in the field $F_p$ differs generally speaking, from the corresponding result in the field of rational number $Q$. For example the element $1/2$ in $F_p$ is a very large number $(p+1)/2$. For this reason one might think that physics based on Galois fields has nothing to with the reality. We will see in the subsequent section that this is not so since the spaces describing quantum systems are projective.

By analogy with the field of complex numbers, we can consider a set $F_{p^2}$ of $p^2$ elements $a + bi$ where $a, b \in F_p$ and $i$ is a formal element such that $i^2 = 1$. The question arises whether $F_{p^2}$ is a field, i.e. we can define all the four operations excepting division by zero. The definition of addition, subtraction and multiplication in $F_{p^2}$ is obvious and, by analogy with the field of complex numbers, we can try to define division as $1/(a + bi) = a/(a^2 + b^2) - ib/(a^2 + b^2)$. This definition can be meaningful only if $a^2 + b^2 \neq 0$ in $F_p$ for any $a, b \in F_p$ i.e. $a^2 + b^2$ is not divisible by $p$. Therefore the definition is meaningful only if $p$ cannot be represented as a sum of two squares and is meaningless otherwise. We will not consider the case $p = 2$ and therefore $p$ is necessarily odd. Then we have two possibilities: the value of $p \pmod{4}$ is either 1 or 3. The well known result of the number theory (see e.g. the textbooks [51]) is that a prime number $p$ can be represented as a sum of two squares only in the former case and cannot in the latter one. Therefore the above construction of the field $F_{p^2}$ is correct only if $p \pmod{4} = 3$. The first impression is that if Galois fields can somehow replace the conventional field of complex numbers then this can be done only for $p$ satisfying $p \pmod{4} = 3$ and therefore the case $p \pmod{4} = 1$ is of no interest for this purpose. We will see in the subsequent section that
correspondence between complex numbers and Galois fields containing $p^2$ elements can also be established if $p \pmod{4} = 1$. In general, it is well known (see e.g. Ref. [51]) that any Galois field consists of $p^n$ elements where $p$ is prime and $n > 0$ is natural. The numbers $p$ and $n$ define the field $F_{p^n}$ uniquely up to isomorphism and $p$ is called the characteristic of the Galois field.

### 5.2 Modular representations of Lie algebras

A well-known historical fact is that quantum mechanics has been originally proposed by Heisenberg and Schroedinger in two forms which seemed fully incompatible with each other. While in the Heisenberg matrix formulation quantum states are described by infinite columns and operators — by infinite matrices, in the Schroedinger wave formulations the states are described by functions and operators — by differential operators. It has been shown later by von Neumann and others that the both formulations are mathematically equivalent.

Quantum theory over a Galois field (GFQT) can be treated as a version of the matrix formulation when complex numbers are replaced by elements of a Galois field. We will see below that in that case the columns and matrices are automatically truncated in a certain way, and therefore the theory becomes finite-dimensional (and even finite since any Galois field is finite).

In conventional quantum theory the state of a system is described by a vector $\tilde{x}$ from a separable Hilbert space $H$. We will use a "tilde" to denote elements of Hilbert spaces and complex numbers while elements of linear spaces over Galois fields and elements of the fields will be denoted without a "tilde".

Let $(\tilde{e}_1, \tilde{e}_2, ...)$ be a basis in $H$. This means that $\tilde{x}$ can be represented as

$$\tilde{x} = \tilde{c}_1 \tilde{e}_1 + \tilde{c}_2 \tilde{e}_2 + ... \quad (5.2)$$

where $(\tilde{c}_1, \tilde{c}_2, ...)$ are complex numbers. It is assumed that there exists a complete set of commuting selfadjoint operators $(\tilde{A}_1, \tilde{A}_2, ...)$ in $H$ such that each $\tilde{e}_i$ is the eigenvector of all these operators: $\tilde{A}_j \tilde{e}_i = \lambda_{ji} \tilde{e}_i$. Then
the elements \((\tilde{e}_1, \tilde{e}_2, \ldots)\) are mutually orthogonal: \((\tilde{e}_i, \tilde{e}_j) = 0\) if \(i \neq j\) where \((\ldots, \ldots)\) is the scalar product in \(H\). In that case the coefficients can be calculated as
\[
\tilde{c}_i = \frac{(\tilde{e}_i, \tilde{x})}{(\tilde{e}_i, \tilde{e}_i)} \quad (5.3)
\]
Their meaning is that \(|\tilde{c}_i|^2(\tilde{e}_i, \tilde{e}_i)/(\tilde{x}, \tilde{x})\) represents the probability to find \(\tilde{x}\) in the state \(\tilde{e}_i\). In particular, when \(\tilde{x}\) and the basis elements are normalized to one, the probability is exactly equal to \(|\tilde{c}_i|^2\).

Let us note first that the Hilbert space contains a big redundancy of elements, and we do not need to know all of them. Indeed, with any desired accuracy we can approximate each \(\tilde{x} \in H\) by a finite linear combination
\[
\tilde{x} = \tilde{c}_1 \tilde{e}_1 + \tilde{c}_2 \tilde{e}_2 + \ldots \tilde{c}_n \tilde{e}_n \quad (5.4)
\]
where \((\tilde{c}_1, \tilde{c}_2, \ldots \tilde{c}_n)\) are rational complex numbers. In turn, the set Eq. (5.4) is redundant too. Indeed, we can use the fact that Hilbert spaces in quantum theory are projective: \(\psi\) and \(c\psi\) represent the same physical state. Then we can multiply both parts of Eq. (5.4) by a common denominator of the numbers \((\tilde{c}_1, \tilde{c}_2, \ldots \tilde{c}_n)\). As a result, we can always assume that in Eq. (5.4) \(\tilde{c}_j = \tilde{a}_j + i\tilde{b}_j\) where \(\tilde{a}_j\) and \(\tilde{b}_j\) are integers. If it is convenient, we can approximate all physical states by another sets of elements. For example, we can use Eq. (5.4), where \(\tilde{c}_j = \tilde{a}_j + \sqrt{2}i\tilde{b}_j\) and \(\tilde{a}_j\) and \(\tilde{b}_j\) are integers.

The meaning of the fact that Hilbert spaces in quantum theory are projective is very clear. The matter is that not the probability itself but the relative probabilities of different measurement outcomes have a physical meaning. We believe, the notion of probability is a good illustration of the Kronecker expression about natural numbers (see Sect. 5.1). Indeed, this notion arises as follows. Suppose that conducting experiment \(n\) times we have seen the first event \(n_1\) times, the second event \(n_2\) times etc. such that \(n_1 + n_2 + \ldots = n\). We introduce the quantities \(w_i(n) = n_i/n\) (these quantities depend on \(n\)) and \(w_i = \lim w_i(n)\) when \(n \to \infty\). Then \(w_i\) is called the probability of the \(ith\) event. We see that all the information about the experiment is given
by a set of natural numbers. However, in order to define probabilities, people introduce additionally the notion of rational numbers and the notion of limit. Of course, the standard notion of probability can be used even if quantum theory is based entirely on natural numbers, but one should realize that this is only a convention on how to describe the measurement outcomes.

The Hilbert space is an example of a linear space over the field of complex numbers. Roughly speaking this means that one can multiply the elements of the space by the elements of the field and use the properties $\tilde{a}(\tilde{b}\tilde{x}) = (\tilde{a}\tilde{b})\tilde{x}$ and $\tilde{a}(\tilde{b}\tilde{x} + \tilde{c}\tilde{y}) = \tilde{a}\tilde{b}\tilde{x} + \tilde{a}\tilde{c}\tilde{y}$ where $\tilde{a}, \tilde{b}, \tilde{c}$ are complex numbers and $\tilde{x}, \tilde{y}$ are elements of the space. The fact that complex numbers form a field is important for such notions as linear dependence and the dimension of spaces over complex numbers.

In general, it is possible to consider linear spaces over any fields (see any textbook on modern algebra, e.g. [51]). It is natural to assume that in the GFQT the states of physical systems should be described by elements of linear spaces over a Galois field. Since we wish to have a correspondence with the conventional theory, we assume that the Galois field in question contains $p^2$ elements where $p$ is the characteristic of the field. In Sect. 5.1 we discussed the correspondence between the ring of integers and the field $F_p$. It has been also noted that if $p = 3 (\text{mod} 4)$ then the elements of $F_{p^2}$ can be written as $a + bi$ where $a, b \in F_p$. We will now discuss the construction of the field $F_{p^2}$ in more general cases.

The field $F_{p^2}$ can be constructed by means of the standard extension of the field $F_p$. Let the equation $\kappa^2 = -a_0$ ($a_0 \in F_p$) have no solutions in $F_p$. Then $F_{p^2}$ can be formally described as a set of elements $a + b\kappa$, where $a, b \in F_p$ and $\kappa$ satisfies the condition $\kappa^2 = -a_0$. The actions in $F_{p^2}$ are defined in the natural way. The condition that the equation $\kappa^2 = -a_0$ has no solutions in $F_p$ is important in order to ensure that any nonzero element from $F_{p^2}$ has an inverse. Indeed, the definition $(a + b\kappa)^{-1} = (a - b\kappa)/(a^2 + b^2a_0)$ is correct since the denominator can be equal to zero only if both, $a$ and $b$ are equal to zero.
Consider three cases.

(1) $a_0 = 1$. In that case it is natural to write $i$ instead of $\kappa$. We must be sure that the element -1 in $F_p$ cannot be represented as a square of another element from $F_p$. In the number theory this is expressed by saying that -1 is not a quadratic residue in $F_p$. It is well known (see e.g. Ref. [51]) that this is the case when $p = 3 \pmod{4}$ and is not the case when $p = 1 \pmod{4}$. This fact is closely related to one mentioned in Sect. 5.1 that a prime $p$ can be represented as a sum of two squares only if $p = 1 \pmod{4}$.

(2) $a_0 = 2$. Assume that $p = 1 \pmod{4}$. Since -1 is a quadratic residue in $F_p$ in this case, then the problem is that 2 should not be a quadratic residue in this field. This is the case (see e.g. Ref. [51]) if $p = 5 \pmod{8}$.

(3) $a_0 = 3$. Again, assume that $p = 1 \pmod{4}$. Then 3 should not be a quadratic residue in $F_p$. This is, for example, the case (see e.g. Ref. [51]), when $p$ is the prime of Fermat’s type, i.e. $p = 2^n + 1$.

In the general case, the field $F_p$ can be extended to a field containing $p^2$ elements as follows. First we note a well known fact [51] that $F_p$ is a cyclic group with respect to multiplication. There exists at least one element $r \in F_p$ such that $(r, r^2, \ldots r^{p-1} = 1)$ represent the set $(1, 2, \ldots p - 1)$ in some order. The element $r$ is called the primitive root of the field $F_p$. It is clear from this observation that $F_p$ contains the equal number $(p - 1)/2$ of quadratic residues and nonresidues: the elements represented as odd powers of $r$ are nonresidues while those represented by even powers of $r$ — residues. In particular, $r$ is not a quadratic residue. Now we can formally introduce $\kappa$ as an element satisfying the condition $\kappa^2 = r$, and $F_{p^2}$ as a set of elements $a + b\kappa$ ($a, b \in F_p$). In this case the definition $(a + b\kappa)^{-1} = (a - b\kappa)/(a^2 - b^2r)$ is correct since $a^2 - b^2r \neq 0$ if $a \neq 0$ or $b \neq 0$.

The above observation shows that if $F_p$ is extended to $F_{p^2}$ then any element of $F_p$ has a square root belonging to $F_{p^2}$.

It is well known (see e.g. Ref. [51]) that the field of $p^2$ elements has only one nontrivial automorphism. In the above cases it can be defined as $a + b\kappa \rightarrow a - b\kappa$ or as $a + b\kappa \rightarrow (a + b\kappa)^p$. We will use a bar
to denote this automorphism. This means that if \( c = a + b\kappa \) \((a, b \in F_p)\) then \( \bar{c} = a - b\kappa \).

By analogy with conventional quantum theory, we require that linear spaces \( V \) over \( F_{p^2} \), used for describing physical states, are supplied by a scalar product \((.,.)\) such that for any \( x, y \in V \) and \( a \in F_{p^2}, (x, y) \) is an element of \( F_{p^2} \) and the following properties are satisfied:

\[
(x, y) = (y, x), \quad (ax, y) = \bar{a}(x, y), \quad (x, ay) = a(x, y)
\]  

(5.5)

In Sect. 5.1 we discussed a map \( f \) from \( F_p \) to the ring of integers \( Z \). We can extend the map in such a way that it maps \( F_{p^2} \) to a subset of complex numbers. For example, if \( p = 3 \pmod{4}, c \in F_{p^2}, c = a + bi \) \((a, b \in F_p)\) then we can define \( f(c) \) as \( f(a) + f(b)i \). If \( a_0 = -2 \) and \( c = a + b\kappa \), we define \( f(c) = f(a) + \sqrt{2}f(b)i \). Analogously, if \( a_0 = -3 \) and \( c = a + b\kappa \), we define \( f(c) = f(a) + \sqrt{3}f(b)i \). Let \( U \) be a subset of elements \( c = a + b\kappa \) where \( a, b \in U_0 \) (see Sect. 5.1). Then, for the elements from \( U \), addition, subtraction and multiplication look the same as for the elements \( f(c) \) of the corresponding subset in the field of complex numbers. This subset is of the form \( a + bi \) in the first case, \( a + \sqrt{2}bi \) in the second case and \( a + \sqrt{3}bi \) in the third one. Here \( a, b \in Z \).

We will always consider only finite dimensional spaces \( V \) over \( F_{p^2} \). Let \((e_1, e_2,...e_N)\) be a basis in such a space. Consider subsets in \( V \) of the form \( x = c_1e_1 + c_2e_2 + ...c_ne_n \) where for any \( i, j \)

\[
c_i \in U, \quad (e_i, e_j) \in U
\]  

(5.6)

On the other hand, as noted above, in conventional quantum theory we can describe quantum states by subsets of the form Eq. (5.4). If \( n \) is much less than \( p \),

\[
f(c_i) = \tilde{c}_i, \quad f((e_i, e_j)) = (\tilde{e}_i, \tilde{e}_j)
\]  

(5.7)

then we have the correspondence between the description of physical states in projective spaces over \( F_{p^2} \) on the one hand and projective Hilbert spaces on the other. This means that if \( p \) is very large, then for a large number of elements from \( V \), linear combinations with the
coefficients belonging to $U$ and scalar products look in the same way as for the elements from a corresponding subset in the Hilbert space.

In the general case a scalar product in $V$ does not define any positive definite metric, and thus there is no probabilistic interpretation for all the elements from $V$. In particular, $(e, e) = 0$ does not necessarily imply that $e = 0$. However, the probabilistic interpretation exists for such a subset in $V$ that the conditions (5.7) are satisfied. Roughly speaking this means that for elements $c_1 e_1 + ... c_n e_n$ such that $(e_i, e_i), c_i c_i \ll p, f((e_i, e_i)) > 0$ and $c_i c_i > 0$ for all $i = 1, ... n$, the probabilistic interpretation is valid. It is also possible to explicitly construct a basis $(e_1, ... e_N)$ such that $(e_j, e_k) = 0$ for $j \neq k$ and $(e_j, e_j) \neq 0$ for all $j$ (see the subsequent section). Then $x = c_1 e_1 + ... c_N e_N$ ($c_j \in \mathbb{F}_p^2$) and the coefficients are uniquely defined by $c_j = (e_j, x)/(e_j, e_j)$.

As usual, if $A_1$ and $A_2$ are linear operators in $V$ such that

$$(A_1 x, y) = (x, A_2 y) \quad \forall x, y \in V$$

(5.8)

they are said to be conjugated: $A_2 = A_1^*$. It is easy to see that $A_1^{**} = A_1$ and thus $A_2^* = A_1$. If $A = A^*$ then the operator $A$ is said to be Hermitian.

If $(e, e) \neq 0$, $A e = a e$, $a \in \mathbb{F}_p^2$, and $A^* = A$, then it is obvious that $a \in \mathbb{F}_p$. At the same time, there exist possibilities (see e.g. Ref. [55]) when $e \neq 0$, $(e, e) = 0$ and the element $a$ is imaginary, i.e. $a = b \kappa$, $b \in \mathbb{F}_p$. Further, if

$$A e_1 = a_1 e_1, \quad A e_2 = a_2 e_2, \quad (e_1, e_1) \neq 0,$$

$$(e_2, e_2) \neq 0, \quad a_1 \neq a_2$$

(5.9)

then as in the usual case, one has $(e_1, e_2) = 0$. At the same time, the situation when

$$(e_1, e_1) = (e_2, e_2) = 0, \quad (e_1, e_2) \neq 0,$$

$$a_1 = b_1 \kappa, \quad a_2 = b_2 \kappa \quad (b_1, b_2 \in \mathbb{F}_p)$$

(5.10)

is also possible [55].

Let now $(A_1, ... A_k)$ be a set of Hermitian commuting operators in $V$, and $(e_1, ... e_N)$ be a basis in $V$ with the properties described above,
such that $A_j e_i = \lambda_{ji} e_i$. Further, let $(\tilde{A}_1, \ldots, \tilde{A}_k)$ be a set of Hermitian commuting operators in some Hilbert space $H$, and $(\tilde{e}_1, \tilde{e}_2, \ldots)$ be some basis in $H$ such that $\tilde{A}_j e_i = \tilde{\lambda}_{ji} \tilde{e}_i$. Consider a subset $c_1 e_1 + c_2 e_2 + \ldots + c_n e_n$ in $V$ such that, in addition to the conditions (5.7), the elements $e_i$ are the eigenvectors of the operators $A_j$ with $\lambda_{ji}$ belonging to $U$ and such that $f(\lambda_{ji}) = \tilde{\lambda}_{ji}$. Then the action of the operators on such elements have the same form as the action of corresponding operators on the subsets of elements in Hilbert spaces discussed above.

Summarizing this discussion, we conclude that if $p$ is large then there exists a correspondence between the description of physical states on the language of Hilbert spaces and selfadjoint operators in them on the one hand, and on the language of linear spaces over $F_{p^2}$ and Hermitian operators in them on the other.

The field of complex numbers is algebraically closed (see standard textbooks on modern algebra, e.g. Ref. [51]). This implies that any equation of the $n$th order in this field always has $n$ solutions. This is not, generally speaking, the case for the field $F_{p^2}$. As a consequence, not every linear operator in the finite-dimensional space over $F_{p^2}$ has an eigenvector (because the characteristic equation may have no solution in this field). One can define a field of characteristic $p$ which is algebraically closed and contains $F_{p^2}$. However such a field will necessarily be infinite and we will not use it. For the same reason we will not discuss theories based on $p$-adic or adelic fields (see e.g. Ref. [56]) although they have several interesting features (for example, in the adelic case there is no distinguished value of $p$). We will see in the subsequent section that uncloseness of the field $F_{p^2}$ does not prevent one from constructing physically meaningful representations describing elementary particles in the GFQT.

In physics one usually considers Lie algebras over $R$ and their representations by Hermitian operators in Hilbert spaces. It is clear that the analogs of such representations in our case are the representations of Lie algebras over $F_p$ by Hermitian operators in spaces over $F_{p^2}$. Representations in spaces over a field of nonzero characteristics are called modular representations. There exists a wide literature devoted
to such representations; detailed references can be found for example in Ref. [57] (see also Ref. [55]). In particular, it has been shown by Zassenhaus [58] that all modular irreducible representations (IRs) are finite-dimensional and many papers have dealt with the maximum dimension of such representations. For classical Lie algebras this has been done in Refs. [59, 60, 61, 62]. The complete classification of IRs has been obtained only for \( A_1 \) algebras [60]; important results for \( A_2 \) and \( B_2 \) algebras are obtained in Refs. [63, 64]; results for \( A_n \) algebras can be found in Refs. [64, 65, 66], and results obtained for some other algebras can be found in Ref. [64]. At the same time, it is worth noting that usually mathematicians consider only representations over an algebraically closed field.

From the previous, it is natural to expect that the correspondence between ordinary and modular representations of two Lie algebras over \( \mathbb{R} \) and \( \mathbb{F}_p \), respectively, can be obtained if the structure constants of the Lie algebra over \( \mathbb{F}_p - c_{ij}^{kl} \), and the structure constants of the Lie algebra over \( \mathbb{R} - \tilde{c}_{ij}^{kl} \), are connected by the relations \( f(c_{ij}^{kl}) = \tilde{c}_{ij}^{kl} \) (the Chevalley basis [67]), and all the \( c_{ij}^{kl} \) belong to \( U_0 \). In Refs. [55, 23] there have been considered modular analogs of IRs of \( \text{su}(2), \text{sp}(2), \text{so}(2,3), \text{so}(1,4) \) algebras and the \( \text{osp}(1,4) \) superalgebra. Also modular representations describing strings have been briefly mentioned. In all these cases the quantities \( \tilde{c}_{ij}^{kl} \) take only the values 0, \( \pm 1, \pm 2 \) and there have been shown that the correspondence under consideration does have a place.

One might wonder how continuous transformations (e.g. time evolution or rotations) can be described in the framework of the GFQT. A general remark is that if theory \( \mathcal{B} \) is a generalization of theory \( \mathcal{A} \) then the relation between the theories is not always straightforward. For example, quantum mechanics is a generalization of classical mechanics, but in quantum mechanics the experiment outcome cannot be predicted unambiguously etc. As noted in Sect. 1.1, even in the framework of standard quantum theory, the time evolution is well-defined only if \( t \) is a good approximate parameter. Suppose that this is the case, and the Hamiltonian \( H_1 \) in standard theory is a good approximation for the
Hamiltonian $H$ in the GFQT. Then one might think that $\exp(-iH_1t)$ is a good approximation for $\exp(-iHt)$. However, such a straightforward conclusion is not correct for the following reasons. First, there can be no continuous parameters in the GFQT. Second, even if $t$ is somehow discretized, the notion of $\exp$ in the GFQT is not clear since in standard mathematics $\exp$ is defined by an infinite Taylor series containing infinitely small quantities. Therefore a direct correspondence between the standard quantum theory and the GFQT in the sense described in this section exists only between Lie algebras but not Lie groups. However, at some conditions (e.g. when $t$ is discretized and only a finite number of the Taylor series for $\exp$ is important) one can define $\exp(-iHt)$ in the GFQT. Analogously one can define rotations, Lorentz boosts etc.

5.3 Modular analogs of single-particle and two-particle representations of the so(1,4) algebra

As noted in Sect. 1.1, in the framework of our approach the dS invariance on quantum level implies by definition that the operators describing the system satisfy the commutation relations (1.2). Therefore in the GFQT one could postulate that the same relations should be valid if the operators act in a space over $F_{p^2}$. However, in that case the following question arises. The relations in Eq. (1.2) contain $i$ explicitly while, as noted in Sect. 5.1, the field $F_{p^2}$ contains the element which can be identified with $i$ only if $p = 3 \mod 4$. A possible solution is as follows. As noted in Sect. 3.1 in terms of the operators $(J'J''R_{ij})$ the commutation relation (1.2) can be written in the form of Eqs. (3.1-3.3) which do not contain $i$ explicitly. One might think that Eqs. (3.1-3.3) are rather chaotic, but in fact they are very natural in the Weyl basis of the so(1,4) algebra. We therefore can postulate that the dS invariance in the GFQT implies by definition that Eqs. (3.1-3.3) are satisfied.

The next step is the construction of modular analogs of UIRs of the so(1,4) algebra. One can define the operators $A^{++}$, $A^{+-}$, $A^{-+}$ and $A^{-}$ by Eq. (3.5) which now should treated in spaces over $F_{p^2}$. By analogy with the construction in Sect. 3.1 one can require that in a
space over $F_p^2$ there exists a vector $e_0$ satisfying Eq. (3.13) but $w$ and $s$
should now be treated as elements of $F_p$. Then, proceeding as in Sect. 3.1, one can derive Eq. (3.15).

As noted in Sect. 3.1, in the standard theory Eq. (3.15) would imply that the quantum number $n$ can take the values $n = 0, 1, \ldots$ and therefore the representation is infinite dimensional. However, in the modular case $n$ can take only the values $0, 1, \ldots n_{\text{max}}$ where the maximum value of $n$, $n_{\text{max}}$ can be found as follows. By definition of the operator $A^{++}, A^{++}e^{nr} = 0$ if $n = n_{\text{max}}$. This relation should not contradict Eq. (3.15) if $n = n_{\text{max}}$. Therefore $n_{\text{max}}$ should satisfy the condition

$$(n_{\text{max}} + 1)(n_{\text{max}} + s + 2)[w + (2n_{\text{max}} + s + 3)^2] = 0 \quad (5.11)$$

As follows from Eq. (5.11), $n_{\text{max}}$ cannot be greater than $p - s - 2$ and therefore the irreducible representation is necessarily finite dimensional in agreement with the Zassenhaus theorem [58]. In view of the correspondence principle discussed in the preceding section, it is natural to require that $s \ll p$ and then $n_{\text{max}}$ is of order $p$. However, there also exists a possibility that $n_{\text{max}}$ is much less than $p$ if $w + (2n_{\text{max}} + s + 3)^2 = 0$ in $F_p$. In the standard theory $w = \mu^2$ where $\mu$ is the particle dS mass. In the GFQT the element $w \in F_p$ is not necessarily a square in $F_p$ but suppose that this is the case. As noted in Sect. 5.1, the prime number $p \neq 2$ can be represented as a sum of two squares if and only if $p = 1 \pmod{4}$. Therefore, if $w$ is a square in $F_p$ and $p = 3 \pmod{4}$ then the equality $w + (2n_{\text{max}} + s + 3)^2 = 0$ in $F_p$ is impossible and $n_{\text{max}} = p - 2 - s$. However, if $w$ is a square in $F_p$ and $p = 1 \pmod{4}$ then the equality $w + (2n_{\text{max}} + s + 3)^2 = 0$ in $F_p$ is possible and typically $n_{\text{max}}$ is less than $p - 2 - s$. There exist scenarios (see Ref. [23]) when $n_{\text{max}}$ is of order $p^{1/2}$, but in any case, if $p$ is large, the dimension of the irreducible representation is quite sufficient to ensure the correspondence principle.

The reader can notice that if the dS invariance is postulated in the form of Eqs. (3.1-3.3) then the consideration in Sect. 3.1 does not involve $i$ at all: we only assume that, according to the principles of quantum theory, the operators should act in complex Hilbert spaces. To
ensure the correspondence with standard quantum theory, we assume that in GFQT the operators act in spaces over $F_{p^2}$. However, if GFQT is treated as a generalization of standard quantum theory then there should be deeper reasons for that. If we were able to find reasons why the operators in GFQT should act in spaces $F_{p^2}$ this could explain why standard quantum theory is based on the field of complex numbers. This problem will be considered elsewhere.

By analogy with the consideration in Sect. 3.4, one can consider a system of two spinless particles in the GFQT. In particular, the element $\Phi_n$ can be again defined by Eq. (3.46) assuming that the operators act in the tensor product of two modular IRs. Then the $(\Phi_n, \Phi_n)$ is formally given by Eq. (3.47) where the r.h.s. should be understood as an element of $F_p$. Analogously one can define the operator $W$ such that its action on $\Phi_n$ is given by Eq. (3.50) and its matrix elements are given by Eq. (3.51).

In contrast with the situation in the standard theory, where the space $H_s$ is infinite dimensional, in the GFQT it is always finite dimensional. This is already clear from the fact (see Eq. (3.46)) that $\Phi_n$ is a linear combination of the products of $e^{(1)n+j}_{\alpha\beta}$ and $e^{(2)n+j}_{\alpha\beta}$. Therefore if $N_{1max}$ is the maximum value of $n$ in $e^{(1)n}_{\alpha\beta}$ and $N_{2max}$ is the maximum value of $n$ in $e^{(2)n}_{\alpha\beta}$ then the maximum value of $n$ for $\Phi_n$ is

$$N_{max} = \min(N_{1max} - \frac{s}{2}, N_{2max} - \frac{s}{2})$$  \hspace{1cm} (5.12)

A detailed discussion of the spectrum of $W$ in the modular case is given in Ref. [23].

As noted in Sect. 5.1, division in $F_p$ in general considerably differs from that in the field of rational numbers. Therefore the important problem arises whether the presence of division in Eq. (3.51) is compatible with the correspondence between the GFQT and the standard theory for the mean value of the operator $W$.

Let us consider the notion of the mean value in the GFQT. In the standard theory the wave function $\Psi$ of any state can be always normalized to one, and in that case the mean value of the operator $A$
in the state $\Psi$ is defined as $\bar{A} = (\Psi, A\Psi)$. As noted in the preceding section, normalization to one is assumed only for convenience and does not have any fundamental meaning. If the state $\Psi$ is not normalized to one then the mean value of the operator $A$ in the standard theory is defined as

$$\bar{A} = \frac{(\Psi, A\Psi)}{(\Psi, \Psi)} \quad (5.13)$$

In the GFQT it is also possible to normalize states to one, but, since $1/n$ is typically a big number in $F_p$, such a normalization is meaningless. However, as discussed in the preceding section, if $|f(\Psi, A\Psi)| \ll p$, $|f(\Psi, \Psi)| \ll p$ and $f(\Psi, \Psi) > 0$ then one can define probabilities as in the standard theory. In that case the numerator and denominator in Eq. (5.13) can be treated as usual integers and their ratio should be calculated in a standard way (i.e. as in the field of rational numbers).

Suppose that $\Psi$ is a state in the modular analog of $H_s$ and

$$\Psi = \sum_{n=0}^{N_{max}} c_n \Phi_n \quad (5.14)$$

where the $c_n$ are elements of $F_p^2$. Suppose also that the quantities $c_n$ are different from zero only for $n \ll p$ and for such values of $n$ the conditions $|f(c_n\bar{c}_n)| \ll p$ and $f(c_n\bar{c}_n) > 0$ are satisfied. Then there exists the correspondence between the quantities $c_n$ in the standard theory and GFQT. If $|f(w_j)| \ll p$, $f(w_j) > 0$ ($j = 1, 2$) and $n \ll p$ then the correspondence between the quantities $(\Phi_n, \Phi_n)$ exists as a consequence of Eq. (3.47) and the fact that

$$\frac{(n+s+1)!}{(n+1+s/2)n!} = (n+1) \cdots (n+s/2)(n+s/2+2) \cdots (n+s+1) \quad (5.15)$$

i.e. the denominator in Eq. (3.47) is cancelled out by the numerator. Since

$$\langle \Psi, \Psi \rangle = \sum_{n=0}^{N_{max}} c_n\bar{c}_n(\Phi_n, \Phi_n) \quad (5.16)$$

the correspondence for $(\Psi, \Psi)$ in the standard theory and GFQT exists too. Note that if the coefficients $c_n$ are such as discussed above then actually the upper limit in the sum is much less than $N_{max}$. 

81
Consider now the correspondence between the quantities \((\Psi, W\Psi)\) in the standard theory and GFQT. As follows from Eqs. (3.47), (3.50), (3.51) and (5.14)

\[
(\Psi, W\Psi) = \sum_{n=0}^{N_{\text{max}}} W_{nn}(\Phi_n, \Phi_n)c_n\bar{c}_n + \\
\sum_{n=0}^{N_{\text{max}}} \frac{n + 2 + s}{n + 2 + s/2}(\Phi_n, \Phi_n)[w_1 + (2n + s + 3)^2] \\
[w_1 + (2n + s + 3)^2](c_{n+1}\bar{c}_n + c_n\bar{c}_{n+1})
\]

One can notice again that actually the upper limit is much less than \(N_{\text{max}}\). Since there is no division in \(W_{nn}\) (see Eq. (3.51)), no problem arises with the correspondence for the first sum in the r.h.s. of this expression. Since \((\Phi_n, \Phi_n)\) contains \((n + 1 + s)!/[(n + 1 + s/2)n!]\), the second sum in Eq. (5.17) contains

\[
\frac{(n + 2 + s)!}{(n + 1 + s/2)(n + 2 + s/2)n!} = \frac{(n + 1) \cdots (n + 2 + s)}{(n + 1 + s/2)(n + 2 + s/2)}
\]

It is obvious that the denominator is again cancelled out by the numerator and therefore there exists the correspondence for the quantity \((\Psi, W\Psi)\).

Our final conclusion in this section is that the presence of division in nondiagonal matrix elements of \(W\) is not an obstacle for the correspondence of the mean values of this operator in the standard theory and GFQT.

### 5.4 Discussion

In the present paper we have investigated several aspects of the property that, in contrast with the Poincare and AdS invariant theories, the mass operator in the dS invariant theory is not bounded below by the value of \(m_1 + m_2\).

We treat the dS invariance on quantum level as the requirement that the operators describing the system should satisfy the commutation relations (1.2). In contrast with the approach based on gauge
theories in curved spacetime, we do not require the existence of any classical background (i.e. the dS space).

In Chap. 2 we have considered the results which can be obtained with rather simple calculations. They are based on the form of UIRs of the dS algebra resembling the main features of UIRs of the Poincare algebra. It has been shown that the unusual features of the dS invariant theory are manifested already in first order in $1/R$, where $R$ is the dS radius. Although the phenomenon of the dS antigravity is well known, we pay attention to the fact that the interaction caused by the dS antigravity has not been introduced as an interaction operator. On the level of dS invariance the particles are treated as free since they are described by the tensor product of UIRs; the interaction arises only when one treats the particles in Poincare (or Galilei) invariant terms. This observation poses the problem whether all interactions in nature can be obtained as a result of transition from a higher symmetry to a lower one. Analogous ideas have been proposed in Kaluza-Klein theories, superstring theories etc. In our approach no extra dimensions or new objects are introduced.

As discussed in Sects. 2.3 and 2.4, the fact that the free two-particle mass operator in the dS invariant theory is not bounded below by the value of $m_1 + m_2$ does not mean that the theory is unphysical. Moreover, this fact opens new possibilities for understanding the physical meaning of interactions. As argued in Sect. 2.5, one might expect that for any reasonable interaction, the free and interacting mass operators are unitarily equivalent.

In Chap. 3 we develop an approach based on the $su(2) \times su(2)$ basis, and the results are used in Chap. 4. In particular, we derive an important Eq. (4.5) for the mean value of the two-body mass operator. Then we show that the standard relativistic result (4.6) is a consequence of Eq. (4.5) in a special case when the decomposition coefficients $c_n$ and $c_{n+1}$ are approximately the same.

If the phases of $c_n$ and $c_{n+1}$ are different, then, as follows from Eq. (4.5), there exists a correction to the standard relativistic result, which is always negative and proportional to the particle masses in
the nonrelativistic approximation. In particular, the correction corresponding to the standard Newtonian potential $-Gm_1 m_2 / r$ is obtained in the case where the phase difference between $c_n$ and $c_{n+1}$ is equal to 
\[
\delta = [2G(m_1 + m_2)/r]^{1/2}.
\]
Since the quantity $2Gm$ is the gravitational radius for the particle with the mass $m$, the condition $\delta \ll 1$ is exactly the well-known requirement that Newtonian gravity is the good first approximation to General Relativity if the particles are nonrelativistic and their gravitational radii are much less than the distance between them. As shown in Sect. 4.3, analogous results take place not only in the nonrelativistic but also in post-Newtonian approximation.

As we have already mentioned, a very interesting possibility is that gravity can be explained without introducing any interaction at all. Our hypothesis is that this can be implemented in quantum theory over a Galois field (GFQT). One of the reasons is as follows. Since the spectrum of the two-body mass operator in dS invariant theories is not bounded below by $m_1 + m_2$, the nonrelativistic mass operator is not positive definite. Consider a quasiclassical two-body system with the relative momentum $q$ and relative distance $r$. If the rules of calculating probabilities are not exactly the same as in the standard theory then the most probable value of the two-body mass might be $q^2/2m_{12} - Gm_1 m_2 / r$ rather than $q^2/2m_{12}$.

In Chap. 5 the basics of the GFQT is described. We argue that it is not only simple and elegant but also a more natural quantum theory than the standard one. As noted in Sect. 5.1, the GFQT can be treated as a version of the Heisenberg matrix formulation of quantum theory when the field of complex numbers is replaced by a Galois field. In that case the theory does not contain actual infinity and therefore the problem of divergencies does not exist in principle.

A well known point of view on the hierarchy of physical theories is that (see e.g. Ref. 68 and references therein) any quantum theory with a lower symmetry can be obtained from some theory with a higher symmetry by using the contraction procedure. For example, the nonrelativistic quantum mechanics can be obtained from relativistic one by means of contraction $c \to \infty$. In turn, the latter can be
obtained from dS or AdS invariant theories by means of contraction $R \to \infty$. Since the dS and AdS groups are semisimple, the dS and AdS invariant theories cannot be obtained from another theory by means of any contraction. However, this statement is based on the assumption that the number field used in all the theories is the same. As noted in Chap. 5 the transition from a theory based on $F_p$ to one based on the ring of integers can be treated as a contraction $p \to \infty$.

One might wonder whether the GFQT could be treated as a version of quantum theory with a fundamental length. For the first time the notion of fundamental length has been probably discussed by Snyder [69]. He has considered a possibility that different coordinate operators do not commute with each other and their commutator is proportional to a quantity with the dimension (length). At present this idea has become popular in view of quantum theories on noncommutative spaces (see e.g. Ref. [70] and references therein). In the GFQT all physical quantities are dimensionless and take values in a Galois field. Nevertheless, on a qualitative level the GFQT can be thought to be a theory with the fundamental length in the following sense. The maximum value of the angular momentum in the GFQT cannot exceed the characteristic of the Galois field $p$. Therefore the Poincare momentum cannot exceed $p/R$. This can be qualitatively interpreted in such a way that the fundamental length in the GFQT is of order $R/p$.

As noted in Chap. 2, in the standard approach there exist infinitely many forms of UIRs and they are related to each by unitary transformations (for example, the forms (2.7) and (2.10) are related to each other by the unitary transformation (2.11) containing a large exponent index). Since the interaction depends on the form, the question arises, what form is physically preferable. In the framework of the GFQT the answer is clear: the form (2.10) is preferable since it can be reformulated in terms of modular representations.

As shown in Sect. 5.3 starting from modular irreducible representations one can extend the results of the standard theory and calculate the mean value of the two-body mass operator in the GFQT. We will now discuss our hypothesis that in such a way it will be possible
to find a natural explanation of gravity.

The problem of cosmological constant is one of the most outstanding in modern quantum theory and there exists a wide literature on this problem (see e.g. Ref. [71, 28] and references therein). The explanation of the problem "for pedestrians" is rather simple. Naive calculations of the cosmological constant $\Lambda$ in quantum field theory produce anomalously large values and this could be expected from the beginning. Indeed, the Planck length is of order $l_P = 10^{-33} \text{cm}$ and therefore if $\Lambda$ is of order unity in the Planck units then it is of order $10^{66}/\text{cm}^2$ in the units $\hbar/2 = c = 1$. Meanwhile, since $\Lambda = 3/R^2$, for realistic values of $R$ it is much smaller (for example, if $R$ is 20 billion light years (in qualitative agreement with the data of Ref. [26]) then $\Lambda$ is of order $10^{-56}/\text{cm}^2$ and the discrepancy is 123 orders of magnitude).

As noted in Sect. 1.1, in our approach we postulate that on quantum level the dS invariance implies that the expressions in Eq. (1.2) are valid. They do not contain the cosmological constant at all and therefore in such a formulation the problem of cosmological constant does not exist. This is clear from the fact that all the operators describing the dS algebra are angular momenta and therefore in units $\hbar/2 = c = 1$ there are no quantities with the dimensions $(\text{length})^n$ ($n = \pm 1, \pm 2, ...$). Such quantities arise only if one wishes to reformulate the results in terms of Poincare invariant theories or classical dS space. In other words, on quantum level any dS invariant theory is described only in terms of angular momentum and/or angular variables (a discussion on whether angular or radial variables are more fundamental can be found, for example, in Ref. [72]).

In units $\hbar/2 = c = 1$ the gravitational constant $G$ is of order $10^{-66} \text{cm}^2$ since $G = l_P^2$. Let us rewrite the Newtonian potential in dS terms. Since the dS energy is $M_{04} = 2RE$ and the dS masses are related to Poincare ones as $\mu = 2Rm$ (see Sect. 2.2) then the Newtonian potential $-Gm_1m_2/r$ in dS terms is $-\tilde{G}\mu_1\mu_2/\varphi$ where the angular variable $\varphi$ has the meaning of $r/(2R)$ and $\tilde{G} = G/(4R^2)$.

If one accepts that dS invariance is more fundamental than Poincare one then it is $\tilde{G}$ which should be treated as a "true" gravit-
tional constant. Note that $\tilde{G}$ is proportional to $GA$. This quantity is dimensionless and extremely small (for example, if $R$ is 20 billion light years then $\tilde{G}$ is of order $10^{-123}$). Therefore the fact that $G$ cannot be neglected in our world can be explained as a consequence of the circumstance that $G = 4R^2\tilde{G}$, i.e. a very small quantity $\tilde{G}$ is multiplied by a very large one $4R^2$.

If one accepts that $G$ and $\Lambda$ are no longer fundamental then there is no reason to believe that they are really constants. Since $G$ is proportional to $R^2$ and $\Lambda$ is proportional to $1/R^2$, it seems reasonable to suggest that at present $G$ is increasing and $\Lambda$ is decreasing (in contrast with Dirac’s hypothesis [53]).

Our hypothesis is that $\tilde{G}$ is different from zero as a consequence of the fact that quantum theory is based on a Galois field rather than the field of complex numbers. In other words, $\tilde{G} = 1/p_0$ where the large number $p_0$ is somehow related to the characteristic of the Galois field $p$. In that case if formally $p \to \infty$ then $\tilde{G} \to 0$. Therefore, the transition $\tilde{G} \to 0$ in our hypothesis has the meaning of contraction from the GFQT to the standard quantum theory.

To verify this hypothesis, one has to calculate the mean value of the two-body mass operator in the framework of the GFQT. As shown in Sect. 5.3 such a calculation is in principle possible since the correspondence between the standard theory and the GFQT is not broken as a consequence of the fact that nondiagonal matrix elements of the $W$ operator contain division in $F_p$. The main technical problems are that quasiclassical states should be constructed in the framework of the GFQT, and all the calculations should be performed exactly (since there is no GFQT analog of the stationary phase method).

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