A STATE-SUM FORMULA FOR THE ALEXANDER POLYNOMIAL

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Abstract. We develop a diagrammatic formalism for calculating the Alexander polynomial of the closure of a braid as a state-sum. Our main tools are the Markov trace formulas for the HOMFLY-PT polynomial and Young’s semi-normal representations of the Iwahori-Hecke algebras of type A.

1. Introduction

In [7], Jones gave a construction of the two-variable HOMFLY-PT polynomial invariant using a recursively defined Markov trace on certain representations of the braid group. These representations all factor through the Iwahori-Hecke algebra $\mathcal{H}_n$ (of type A), which enjoys a character theory [4] deforming that of the symmetric group. The Markov trace can be decomposed as a linear combination of irreducible characters $\chi_\lambda$ of the Hecke algebra:

$$\tau = \sum_{\lambda \vdash n} \omega_\lambda(q, z) \chi_\lambda.$$  

The coefficients $\omega_\lambda(q, z)$ in this “Fourier expansion” were calculated by Ocneanu using Schur functions (see, e.g., [5]). By summing only the components corresponding to hook partitions, and specializing $z \to q^{-1}$, the Alexander polynomial $\Delta(L)$ of a link $L$ is recovered. In this paper, we use (1) to develop a diagrammatic formalism for calculating $\Delta(L)$ as a state-sum derived from a braid presentation of the link $L$.

The rest of the paper is organized as follows. In Section 2 we review some of the representation theory of the Hecke algebras of type A, using the Hecke algebra analogue of Young’s semi-normal form from [6]. We introduce some combinatorics specific to tableaux of hook shape, and recast the formulas in this language. In Section 3, we describe the construction of the state-sum and derive the main theorem (Theorem 3.1). An example is calculated explicitly in Section 4.

It is also possible to give a direct proof of our main theorem by verifying that our formulae give a link invariant satisfying the Conway skein relations; see [1]. In subsequent work, I hope to generalize these results to colored braids and the multivariable Alexander polynomial.

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2. Semi-normal representations of Hecke algebras

Much of the following is standard (see e.g. [3] or [3] chapters 4 and 5). We collect some of the definitions here and fix some notation.

2.1. Braid group. Fix \( n \geq 1 \). Let \( \Sigma_n = \{\sigma_1, \ldots, \sigma_{n-1}\} \) be the set of braid generators, and let \( \Sigma_n^\bullet \) denote the set of words in the symbols \( \Sigma_n \sqcup \Sigma_n^{-1} \). Let \( B_n \) be the braid group on \( n \) strands, that is, the quotient of the free group on \( \Sigma_n \) by the relations
\[
\sigma_r \sigma_{r+1} \sigma_r = \sigma_{r+1} \sigma_r \sigma_{r+1} \quad \text{for each } 1 \leq r \leq n-2,
\]
\[
\sigma_r \sigma_s = \sigma_s \sigma_r \text{ when } |r-s| > 1.
\]

Let \( \mathcal{L} \) be the set of isotopy classes of smooth links in \( S^3 \). For each \( n \), there is a canonical map \( \Sigma_n^\bullet \mapsto B_n \). Also there is a map \( B_n \to \mathcal{L} \) defined by closing a braid into a link. The map \( B_n \to \mathbb{Z}, \sigma_r \mapsto 1 \) induces an isomorphism of groups \( B_n / [B_n, B_n] \to \mathbb{Z} \).

The image of \( w \in \Sigma_n^\bullet \) under the composition \( \Sigma_n^\bullet \mapsto B_n \to \mathbb{Z} \) is called the exponent sum of \( w \).

2.2. Iwahori-Hecke algebra. Let \( \mathcal{H}_n \) denote the Iwahori-Hecke algebra associated to \( S_n \). This is the algebra over \( \mathbb{C}(v) \) generated by \( H_1, \ldots, H_{n-1} \), subject to the braid relations
\[
H_r H_{r+1} H_r = H_{r+1} H_r H_{r+1} \quad \text{for each } 1 \leq r \leq n-2,
\]
\[
H_r H_s = H_s H_r \quad \text{when } |r-s| > 1,
\]
and also the quadratic relations
\[
(H_r - v)(H_r + v^{-1}) = 0.
\]

By setting \( q = v^2 \) and \( T_r = vH_r \) for each \( r \), the braid relations look the same in the \( T \) variables (they are homogeneous), and the quadratic relations become
\[
(T_r - q)(T_r + 1) = 0.
\]

Because \( v \) is generic, it is well known that \( \mathcal{H}_n \) is a semisimple algebra and its representation theory is equivalent to that of the symmetric group \( S_n \) over the field \( \mathbb{C}(v) \).

In what follows, \( [r] \in \mathbb{Z}[v, v^{-1}] \) denotes the quantum integer
\[
[r] = \frac{v^r - v^{-r}}{v - v^{-1}}
\]
for any \( r \in \mathbb{Z} \).
2.3. Seminormal representations. This exposition follows \cite{9} section 3, although the results were originally worked out in \cite{6}. For a new point of view and substantial generalization, see \cite{2} section 5.

Let \( \text{Par}(n) \) denote the set of all integer partitions of \( n \). To the partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \), associate its Young diagram, that is, the left-justified diagram with \( \lambda_1 \) boxes on the first row, \( \lambda_2 \) boxes on the second row, etc.

Let \( \text{Tab}(\lambda) \) denote the set of \( \lambda \)-tableaux. These are fillings of the boxes in the Young diagram \( \lambda \) by the numbers \( 1, \ldots, n \). Let \( \text{Std}(\lambda) \) denote the set of standard \( \lambda \)-tableaux, namely those that increase across rows and down columns. The symmetric group \( S_n \) acts on \( \text{Tab}(\lambda) \) via its natural action on the entries, although \( \text{Std}(\lambda) \) is not stable under this action. For a tableau \( T \in \text{Tab}(\lambda) \), its residue sequence \( (i_1, \ldots, i_n) \in \mathbb{Z}^n \) is defined by setting \( i_r = b - a \) where the box labeled \( r \) in \( T \) appears in row \( a \) and column \( b \).

\[
\lambda = \begin{array}{ccc}
1 & 1 \\
2 & 3
\end{array}
\quad T = \begin{array}{ccc}
1 & 2 & 4 \\
3
\end{array}
\quad s_2T = \begin{array}{ccc}
1 & 3 & 4 \\
2
\end{array}
\]

**Figure 1.** The partition \( \lambda = (3,1) \in \text{Par}(4) \) and standard tableaux \( T \) and \( s_2T \). Here, \( T \) has residue sequence \((0,1,-1,2)\).

Fix a partition \( \lambda \in \text{Par}(n) \) and let \( S(\lambda) \) be the \( \mathbb{C}(v) \)-vector space on basis \( \{x_T \mid T \in \text{Std}(\lambda)\} \). Let \( (i_1, \ldots, i_n) \) be the residue sequence of \( T \) and define \( a_r(T), b_r(T) \in \mathbb{C}(v) \) to be

\[
a_r(T) = \frac{v-v^{-1}}{1-v^{2(i_r-i_{r+1})}}, \quad b_r(T) = v^{-1} + a_r(T).
\]

Define actions of the generators \( H_1, \ldots, H_{n-1} \) of \( H_n \) on \( S(\lambda) \) by

\[
H_r x_T = a_r(T)x_T + b_r(T)x_{s_rT},
\]

where we interpret \( x_{s_rT} = 0 \) if \( s_rT \) is not a standard tableau.

**Theorem 2.4** (Seminormal representations). This action extends to make \( S(\lambda) \) into a well-defined \( H_n \)-module. Furthermore, the modules \( \{S(\lambda) \mid \lambda \in \text{Par}(n)\} \) constitute a complete set of pairwise non-isomorphic irreducible modules for \( H_n \).

2.5. Sign sequences and hook partitions. For \( 0 \leq \ell \leq n-1 \), let \( \lambda_\ell \) be the hook partition \((n-\ell,1^\ell)\). We refer to \( \ell \) as leg length.

**Lemma 2.6.** Standard tableaux of shape \( \lambda_\ell \) are in bijection with sign sequences \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm\}^n \) such that \( \varepsilon_1 = + \) and \( \ell \) entries equal \(-\).

**Proof.** Beginning with a standard \( \lambda_\ell \)-tableau, define \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) by

\[
\varepsilon_r = \begin{cases} 
+ & \text{if } r \text{ appears on the first row} \\
- & \text{otherwise}
\end{cases}
\]
Notice that the box labeled 1 has to be in the corner of the hook, so $\varepsilon_1 = +$. Also, $\ell$ numbers are on the leg of the hook, so there are $\ell$ entries equal to $-$. For the inverse, starting with a sign sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_1 = +$ and $\ell$ other entries equal to $-$, construct a standard tableau recursively, as follows. Place 1 in the corner of the diagram. Now, for each $r > 1$, suppose that the numbers $1, \ldots, r - 1$ have been placed. Either add $r$ to the end of the first row or at the bottom of the first column, according to whether $\varepsilon_r$ is $+$ or $-$, respectively.  

Using this bijection, we can adapt the semi-normal representation to the combinatorics of sign sequences.

**Theorem 2.7.** The irreducible module $S(\lambda_\ell)$ has basis $\{x_\varepsilon\}$, where $\varepsilon$ runs over sign sequences having $\varepsilon_1 = +$ and $\ell$ other entries equal to $-$. The generators $H_1, \ldots, H_{n-1}$ of $\mathcal{H}_n$ act by

$$H_r x_\varepsilon = a_r(\varepsilon)x_\varepsilon + b_r(\varepsilon)x_{s_r, \varepsilon}$$

where $s_r \varepsilon$ denotes the sign sequence obtained from $\varepsilon$ by permuting $\varepsilon_r$ and $\varepsilon_{r+1}$, $x_\varepsilon$ is interpreted as zero if $\varepsilon_1 = -$ and

$$a_r(\varepsilon) = \begin{cases} v & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (+, +) \\ -v^{-1} & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (-, -) \\ v^r/[r] & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (-, +) \\ -v^{-r}/[r] & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (+, -) \\ [r+1]/[r] & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (-, +) \\ [r-1]/[r] & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (+, -) \\ 0 & \text{otherwise.} \end{cases}$$

The inverse generators $H_1^{-1}, \ldots, H_{n-1}^{-1}$ act by

$$H_r^{-1} x_\varepsilon = \bar{a}_r(\varepsilon)x_\varepsilon + \bar{b}_r(\varepsilon)x_{s_r, \varepsilon}$$

where $\bar{a}_r(\varepsilon)$ is obtained from $a_r(\varepsilon)$ by replacing $v$ by $v^{-1}$.

**Proof.** This is just a translation of Theorem 2.4 using the bijection from Lemma 2.6. Given a sign sequence $\varepsilon \in \{-, +\}^n$ having $\varepsilon_1 = +$ and $\ell$ other entries equal to $-$, construct the corresponding standard tableau, and let $(i_1, \ldots, i_n)$ be its residue sequence. We have $i_1 = 0$, and for $1 \leq r < n$,

$$i_{r+1} = \begin{cases} i_r + 1 & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (+, +) \\ i_r - 1 & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (-, -) \\ i_r + r & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (-, +) \\ i_r - r & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (+, -). \end{cases}$$

Given this, the formulae (7)–(8) are easily deduced from (3). Finally the formula (9) is easily deduced from (6) since $H_r^{-1} = H_r - (v - v^{-1})$.  \qed
3. Construction

Begin with a word \( w \in \Sigma_n^\ast \) in the braid generators \( \sigma_1, \ldots, \sigma_{n-1} \) and their inverses, which we picture as a diagram drawn up the page as the word is read from right to left.

\[
\sigma_r = \mid \cdots \mid \times_{r+1} \mid \cdots \mid \quad \sigma_r^{-1} = \mid \cdots \mid \times_{r+1} \mid \cdots \mid
\]

Construct permutation diagrams from the braid diagram by replacing each crossing by one of two resolutions:

\[
\begin{array}{c c c c}
\includegraphics{crossing1} & \rightarrow & \includegraphics{crossing2} & \text{or} & \includegraphics{crossing3} \\
\includegraphics{crossing4} & \rightarrow & \includegraphics{crossing5} & \text{or} & \includegraphics{crossing6}
\end{array}
\]

A permutation diagram \( x \) is admissible if

\begin{align*}
\textbf{P1:} & \quad \text{the first (leftmost) strand goes straight through without crossing any other strands, and} \\
\textbf{P2:} & \quad \text{the underlying permutation is the identity.}
\end{align*}

A state is a pair \((x, \epsilon)\), where \( x \) is an admissible permutation diagram and \( \epsilon \) is an assignment of a sign \( \pm \) to each strand such that

\begin{align*}
\textbf{S1:} & \quad \text{the first (leftmost) sign is } +, \text{ and} \\
\textbf{S2:} & \quad \text{no two strands of the same sign cross.}
\end{align*}

Let \( S(w) \) denote the set of states for \( w \). To a state \((x, \epsilon) \in S(w)\), we associate a weight \( M(w, x, \epsilon) \in \mathbb{C}(v) \) defined by multiplying together certain scalars, one for each resolved crossing. The scalar associated to a positive crossing of strands in positions \( r \) and \( r + 1 \) is given in (11). For a negative crossing, replace \( v \) by \( v^{-1} \) in each expression.

\[
\begin{cases}
\begin{array}{c c c c}
\includegraphics{crossing7} & \rightarrow & \includegraphics{crossing8} & \text{or} & \includegraphics{crossing9} \\
\includegraphics{crossing10} & \rightarrow & \includegraphics{crossing11} & \text{or} & \includegraphics{crossing12}
\end{array}
\end{cases}
\]

\[
(11) \quad \begin{cases}
\begin{array}{c c c c}
\includegraphics{crossing13} & \rightarrow & \includegraphics{crossing14} & \text{or} & \includegraphics{crossing15} \\
\includegraphics{crossing16} & \rightarrow & \includegraphics{crossing17} & \text{or} & \includegraphics{crossing18}
\end{array}
\end{cases}
\]

\[
\begin{cases}
\begin{array}{c c c c}
\includegraphics{crossing19} & \rightarrow & \includegraphics{crossing20} & \text{or} & \includegraphics{crossing21} \\
\includegraphics{crossing22} & \rightarrow & \includegraphics{crossing23} & \text{or} & \includegraphics{crossing24}
\end{array}
\end{cases}
\]

\[
\begin{cases}
\begin{array}{c c c c}
\includegraphics{crossing25} & \rightarrow & \includegraphics{crossing26} & \text{or} & \includegraphics{crossing27} \\
\includegraphics{crossing28} & \rightarrow & \includegraphics{crossing29} & \text{or} & \includegraphics{crossing30}
\end{array}
\end{cases}
\]

\[
\begin{cases}
\begin{array}{c c c c}
\includegraphics{crossing31} & \rightarrow & \includegraphics{crossing32} & \text{or} & \includegraphics{crossing33} \\
\includegraphics{crossing34} & \rightarrow & \includegraphics{crossing35} & \text{or} & \includegraphics{crossing36}
\end{array}
\end{cases}
\]

\[
\begin{cases}
\begin{array}{c c c c}
\includegraphics{crossing37} & \rightarrow & \includegraphics{crossing38} & \text{or} & \includegraphics{crossing39} \\
\includegraphics{crossing40} & \rightarrow & \includegraphics{crossing41} & \text{or} & \includegraphics{crossing42}
\end{array}
\end{cases}
\]
Define \( A(w) \in \mathbb{C}(v) \) by
\[
A(w) = \frac{1}{[n]} \sum_{(x,\varepsilon) \in S(w)} \langle \varepsilon \rangle M(w, x, \varepsilon)
\]
where \( \langle \varepsilon \rangle \in \{\pm 1\} \) is the product of the signs \( \varepsilon_1, \ldots, \varepsilon_n \) attached to the strands.

**Theorem 3.1.** Let \( L \) be an oriented link, and let \( w \in \Sigma_n \) represent a braid in \( B_n \) whose closure is \( L \). Then, \( A(w) \) is a polynomial in \( v - v^{-1} \), and
\[
A(w) = \Delta(L),
\]
where \( \Delta(L) \) is the Conway-normalized Alexander polynomial.

**Proof.** To avoid confusion when switching between the generators \( H_r \) and \( T_r = vH_r \) of \( \mathcal{H}_n \), let us write \( \varphi : B_n \to \mathcal{H}_n^\times \) for the group homomorphism given by \( \varphi(\sigma_r) = H_r \) and \( \psi : B_n \to \mathcal{H}_n^\times \) for the one with \( \psi(\sigma_r) = T_r \). Formula (7.2) in [7] gives the Alexander polynomial for a link \( L \) as
\[
\Delta(L) = (-1)^{n-1} \left( \frac{1}{q} \right)^{(e-n+1)/2} \frac{1 - q}{1 - q^n} \sum_{k=0}^{n-1} (-1)^k \chi_{n-1-k}(\psi(w))
\]
where \( \chi_\ell = \text{tr} \circ \rho_\ell \) is the character of \( \mathcal{H}_n \) arising from the irreducible representation \( \rho_\ell : \mathcal{H}_n \to \text{End}(S(\lambda_\ell)) \) indexed by the hook partition \((n-\ell, 1^\ell)\), and \( e \) is the exponent sum of \( w \). Put \( q = v^2 \) and \( T_r = vH_r \) for each \( r \), so that \( \psi(w) = v^e \varphi(w) \). Reindex the sum over \( \ell = n - 1 - k \) to get
\[
\Delta(L) = (-1)^{n-1} \left( \frac{1}{v} \right)^{e-n+1} \frac{1 - v^2}{1 - v^{2n}} \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} \chi_\ell(v^e \varphi(w))
\]

Now we compute \( \chi_\ell \) by using the semi-normal form for \( S(\lambda_\ell) \). The action of the generators \( H_r, H_r^{-1} \) on \( x_\varepsilon \) from Theorem 2.7 are pictured in (15).
Moreover, if \( r = 1 \), the second term on the right hand side should be omitted. Only diagonal entries of the matrix \( \rho_\ell(\varphi(w)) \) contribute to the trace. Hence, for each \( \ell \) we need only consider those permutation diagrams that represent the identity permutation and whose first strand goes straight through. The theorem follows on comparing formulas (7) and (8) with (11). \( \Box \)

4. Example

Let’s use the braid presentation \( w = \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_1 \) for the figure-eight knot, pictured below with its six possible states.

| \( x \) | \( \varepsilon \) | \( M(w, x, \varepsilon) \) |
|---|---|---|
| + + + | \( v^{-1} \cdot v \cdot v^{-1} \cdot v \) | 1 |
| + + - | \( -\frac{v^2}{2} \cdot v \cdot \frac{v^2}{2} \cdot v \) | \( \frac{v^6}{2^2} \) |
| + - + | \( \frac{v^2}{2} \cdot (-v^{-1}) \cdot \frac{v^2}{2} \cdot (-v^{-1}) \) | \( \frac{v^6}{2^2} \) |
| + - - | \( (v) \cdot (-v^{-1}) \cdot (v) \cdot (-v^{-1}) \) | 1 |
| + - - | \( \left[ \frac{3}{2} \right] \cdot (-v^{-1}) \cdot \left[ \frac{1}{2} \right] \cdot v \) | \( -\frac{[3]}{2^2} \) |
| + + - | \( \left[ \frac{1}{2} \right] \cdot v \cdot \left[ \frac{3}{2} \right] \cdot (-v^{-1}) \) | \( -\frac{[3]}{2^2} \) |

Now, we calculate the sum, minding the signs associated to each state and the global rescaling.

\[
A(w) = \frac{1}{[3]} \left( 2 - \frac{v^6 + v^{-6}}{2^2} + \frac{2[3]}{2^2} \right) = -v^2 + 3 - v^{-2} = 1 - (v - v^{-1})^2
\]

References

[1] Samson Black. Representations of Hecke algebras and the Alexander polynomial. PhD in Mathematics, University of Oregon, 2010.
[2] Jonathan Brundan and Alexander Kleshchev. Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras. Invent. Math., 178(3):451–484, 2009.
[3] Charles W. Curtis and Irving Reiner. Methods of representation theory. Vol. I, II. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1987. With applications to finite groups and orders, A Wiley-Interscience Publication.
[4] Meinolf Geck. The character table of the Iwahori-Hecke algebra of the symmetric group: Starkey’s rule. C. R. Acad. Sci. Paris Sér. I Math., 329(5):361–366, 1999.
[5] Meinolf Geck and Nicolas Jacon. Ocneanu’s trace and Starkey’s rule. *J. Knot Theory Ramifications*, 12(7):899–904, 2003.

[6] P. N. Hoefsmit. *Representations of Hecke algebras of finite groups with BN-pairs of classical type*. PhD in Mathematics, University of British Columbia, 1974.

[7] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. of Math. (2)*, 126(2):335–388, 1987.

[8] Christian Kassel and Vladimir Turaev. *Braid groups*, volume 247 of *Graduate Texts in Mathematics*. Springer, New York, 2008. With the graphical assistance of Olivier Dodane.

[9] Arun Ram. Seminormal representations of Weyl groups and Iwahori-Hecke algebras. *Proc. London Math. Soc. (3)*, 75(1):99–133, 1997.

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