Visualizing Topology of Real-Energy Gapless Phase Arising from Exceptional Point

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The discovery of novel topological phase advances our knowledge of nature and stimulates the development of applications. In non-Hermitian topological systems, the topology of band touching exceptional points is very important. Here we propose a real-energy topological gapless phase arising from exceptional points in one dimension, which has identical topological invariants as the topological gapless phase arising from degeneracy points. We develop a graphic approach to characterize the topological phases, where the eigenstates of energy bands are mapped to the graphs on a torus. The topologies of different phases are visualized and distinguishable; and the topological gapless edge state with amplification appropriate for topological lasing exists in the nontrivial phase. These results are elucidated through a non-Hermitian Su-Schrieffer-Heeger ladder. Our findings open new way for identifying topology phase of matter from visualizing the eigenstates.

Introduction.—Topological phase of matter has become a frontier research field due to their novel features for potential applications [1–20]. Majorana [21–24], Dirac [25–30], and Weyl fermions [31–35] predicted in high-energy physics are discovered in this fertile ground. These concepts stimulate an interesting topic of topological gapless phases/semimetals [36–44]. The symmetry protected nodal points in topological gapless phase are topological defects of an auxiliary vector field and are removable until they meets and annihilates in pairs. The topology of gapless bands can be characterized by kink in one-dimension (1D) [45] or vortex of skyrmion in two-dimension (2D) [41–42].

Nowadays, the great efforts have been made to unravel the mystery of non-Hermitian physics [46–61]. Non-Hermitian phase transition occurs at exceptional point (EP) [52–55], which is a unique concept in non-Hermitian systems and has exotic topology related to Riemann surface [62–77]. The investigations of topological physics have been extended to non-Hermitian region [78–107]. The non-Hermitian topological systems under various combined symmetries including the parity-time (PT) symmetry are investigated [108–116]. The PT-symmetric interface states are realized in passive optical systems [120–122]. Furthermore, the edge state lasing is demonstrated in active optical systems [123–128]. The high-order non-Hermitian topological systems [129–133], the breakdown of bulk-boundary correspondence [134–136], and the topological classifications [137–146] are studied.

In non-Hermitian topological systems, the gap closing in energy band is usually associated with EPs instead of degenerate points (DPs) [117–119, 147–148]. The band touching EP pairs split from single DPs are connected by open Fermi arcs [148–151]; alternatively, non-Hermitian semimetals exhibit nodal phases with symmetry protected EPs rings and surfaces [152–158]; the corresponding energy bands in the non-Hermitian gapless phases are all complex. The complex-energy bands associated EPs are systematically studied [146] and they are dramatically different from the gapless phase in a Hermitian system. In contrast to the Hermitian topological gapless phase, a non-Hermitian topological gapless phase is typically characterized by two types of winding numbers; an additional winding number solely for non-Hermitian systems is defined to characterize the topology of Riemann sheet energy bands [118–119, 142]. The real-energy energy band does not exponentially decay or increase as time and has a zero winding in contrast to the complex energy band. Thus, two winding numbers are insufficient to distinguish the gapless phases arising from EPs and DPs.

The aim of this Letter is to propose a real-energy gapless phase associated with band touching EPs. The balanced gain and loss are introduced in chiral symmetric Hermitian topological insulators to create the gapless phase arising from EPs, where robust zero modes with amplification and attenuation are generated for nontrivial topology. We offer a powerful graphic approach to visualize the topological features of different phases. The rich topological phases of either gapped (separable) or gapless, either Hermitian or non-Hermitian band touching, and either trivial or nontrivial phases are all distinguishable from the geometrical topologies of eigenstate graphs. The essence of eigenstate graphs dramatically differs from the knotted or linked nodal lines in semimetals that representing the zero-energy (equal-energy) surface [101, 153–159]. In the graphic approach, the graphic eigenstates of real gapless bands arising from EPs form a network; the topological nature of which is characterized by its nodes, branches, and independent loops. The networks with different links, fixed/movable nodes correspond to different topological phases; other gapless and gapped phases possess distinct geometric graphs.

Gapless phase arising from EPs.—In the theory for topological insulators [1, 2], the energy band is insufficient to determine the full topological character of a phase of matter. The bulk topological features are encoded in the eigenstates. We consider a prototype of non-Hermitian topological system: the 1D two-band model. In the momentum space, the core matrix reads $h_k = B \cdot \sigma$, where \( B \) is a 2x2 matrix and \( \sigma \) is a Pauli matrix. The energy band touching EPs are characterized by two additional independent winding numbers, and the boundary state is a real-energy gapless mode.
where $\mathbf{B} = (B_x, B_y, B_z)$ is an effective magnetic field, and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli matrix; $B_x$ and $B_y$ are real, but $B_z$ is imaginary. The energy bands are tighten under the influence of $B_z$; however, the topological phase transition in $h_k$ is not affected by $B_z$. Thus, the gapless phase is created via increasing $B_z$ in the gapped phase till a non-Hermitian phase transition.

**Graphic eigenstate.**—For $B^2 = B_x^2 + B_y^2 + B_z^2 \geq 0$, the eigenvalues $\varepsilon_k^\pm = \pm B$ are real with corresponding eigenstates

$$|\psi_k^\pm\rangle = \frac{1}{\sqrt{2}}\left(\begin{array}{c} e^{i\varphi_+ (k)} \\ 1 \end{array}\right),$$

where $\varphi_+ (k) = \arctan (-B_y/B_x) + \arctan (-iB_z/B)$ and $\varphi_- (k) = \arctan (-B_y/B_x) \pm \pi - \arctan (-iB_z/B)$ are real functions with a period $2\pi$. The varying directions of $\varphi_\pm (k)$ and $\mathbf{B}$ are the same. Therefore, $\varphi_\pm (k)$ as the kernel of eigenstates $|\psi_k^\pm\rangle$ provides the information of the topological nature of $h_k$. Notably, $|\psi_k^\pm\rangle$ can be represented by a loop on the torus spanned by $\varphi_\pm (k)$ and $k$, referred to as the graphic eigenstates.

In the $k$ space, $h_k$ is $\mathcal{PT}$-symmetric ($\mathcal{PT}^{-1} h_k (\mathcal{PT}) = h_k$) and $\mathcal{CT}$-symmetric ($\mathcal{CT}^{-1} h_k (\mathcal{CT}) = -h_{-k}$), where the operators $\mathcal{PT} = \sigma_x K$, $\mathcal{CT} = \sigma_z K$, and $K$ is the complex conjugation. The $\mathcal{PT}$ symmetry is related to the reality of the spectrum; while the $\mathcal{CT}$ symmetry protects the system topology. For an arbitrary eigenstate $|\psi_k^+\rangle$, the $\mathcal{CT}$ symmetry requires another eigenstate satisfying $|\psi_{-k}^+\rangle = \mathcal{CT}^\dagger |\psi_k^+\rangle$, which leads to

$$\varphi_+ (k) + \varphi_- (-k) = \pm \pi. \quad (2)$$

In particular, when the EP appears at $k_c$, two eigenstates $|\psi_k^\pm\rangle$ and $|\psi_{-k}^\pm\rangle$ coalesce to one denoted as $|\psi_{k_c}^\pm\rangle$. The $\mathcal{CT}$ symmetry ensures $|\psi_{-k_c}^\pm\rangle = \mathcal{CT}^\dagger |\psi_{k_c}^\pm\rangle$, i.e., the existence of a pair of EPs with zero energy for $k_c \neq 0, \pi$; any EP cannot be separately removed, but the EP position changes as the system parameters. When $k_c \neq 0, \pi$, we have one fixed EP with $\varphi_\pm (k_c) = \pi/2$ or $-\pi/2$.

For a Hermitian system with $B_z = 0$, we always have $\varphi_- (k) = \varphi_+ (k) \pm \pi$. In the presence of a DP, two energy bands usually form a single knot [Fig. 1(c)]. In the absence of DPs, $\mathbf{B} \neq 0$, two energy bands form two loops without intersection; this is similar as the gapped non-Hermitian phase shown in Figs. 1(a) and 1(b). For a non-Hermitian system, the effective magnetic field $\mathbf{B}$ is complex. In contrast to two eigenstates of a Hermitian matrix that represented by two opposite points on the torus due to the orthogonality, two eigenstates of the real-energy bands in a non-Hermitian matrix represented by two points can have arbitrary positions on the torus; and they meet when eigenstates coalesce at the EPs, where $\varphi_- (k) = \varphi_+ (k)$. Then, two graphic eigenstates constitute a network [Figs. 1(e)-1(h)]. The network on the $\varphi$-torus provides a complete topological picture for the real-energy gapless phase arising from EPs, the topological features of which are reflected from the network topology. Notably, the graphs represent the eigenstate rather than the zero energy, which essentially differs from the EP links or knotted nodal lines of EPs [138, 139, 152-159].

**Topological characterization.**—In general cases, multiple EPs may appear; accordingly, the graphic eigenstates may form complicate networks. The geometrical topology of any network satisfies a fundamental fact: the numbers of nodes $n$, independent loops $l$, and branches $b$ fulfill

$$n = b - l + 1. \quad (3)$$

An independent loop must contain at least one branch that does not belong to any other loop. For instance, we have $n = 1 (2), b = 2 (4), l = 2 (3)$ for the network shown in Fig. 1(g) [Fig. 1(h)]. This graphic approach is especially helpful in characterizing the real-energy topological gapless phase arising from EPs. Two graphic eigenstate loops are unlinked [Fig. 1(a)] or linked [Fig. 1(b)], which indicate the trivial or nontrivial topology. To verify this assertion, we point out that the varying direction of $\varphi_\pm (k)$ accumulated in a period of $k$ defines a winding number

$$N_\pm = (2\pi)^{-1}\int_0^{2\pi} \nabla_k \varphi_\pm (k) dk. \quad (4)$$

Besides, the average values of Pauli matrices under two eigenstates $|\psi_k^\pm\rangle$ yield $\langle \sigma_x \rangle_\pm = \cos [-\varphi_\pm (k)], \langle \sigma_y \rangle_\pm = \sin [-\varphi_\pm (k)],$ and $\langle \sigma_z \rangle_\pm = 0$ [160], which define a planar vector field that associated with the upper ($|\psi_k^+\rangle$) or lower ($|\psi_k^-\rangle$) band

$$\mathbf{F}_\pm (k) = (\langle \sigma_x \rangle_\pm, \langle \sigma_y \rangle_\pm). \quad (5)$$
The winding number associated with $F_\pm(k)$ is $(2\pi)^{-1}\int_0^{2\pi} \nabla_k \arg F_\pm(k) \, dk$, which equals to the winding number $N_\pm$. The coincidence of varying directions of $\varphi_\pm(k)$ and $F_\pm(k)$ indicates that the topological properties of energy band are reflected from the kernel of eigenstate $\varphi_\pm(k)$. $2\pi N_\pm$ is the nonzero rotation angle of $\varphi_\pm(k)$ accumulated in a period of $k$, indicating the nontrivial topology of the gapped or gapless phase, and the graphic eigenstate loops are linked. Notably, the winding numbers for the two bands are identical $N = N_+ = N_-.$

For the gapless phase with EPs, the topology of EPs arising from entirely real-energy bands are in a striking difference from the EPs in the complex energy bands. The difference is revealed from another winding number $W_{\text{EP}} = (2\pi)^{-1}\int_0^{2\pi} \nabla_k \arg E_\pm(k) \, dk$ that characterizing the Riemann sheets \[118, 119, 122\]. Notably, $W_{\text{EP(DP)}} = 0$ for the EPs (DPs) of real-energy gapless bands in contrast to $W_{\text{EP}} = \pm 1/2$ for the EPs in the complex energy bands \[99, 118, 148\]. However, the topologies of real-energy gapless phases arising from DPs and EPs dramatically differ from each other; the node (network configuration) is absent for the gapless bands with DPs, while the presence of node is a typical feature for the gapless bands with EPs.

**Non-Hermitian SSH ladder.**—As a prototype of one-dimensional non-Hermitian topological system, the $PT$-symmetric non-Hermitian Su-Schrieffer-Heeger (SSH) model is experimentally realized in the coupled waveguides \[82, 121, 122\], coupled resonators \[120\], and polariton micropillars \[123\]. The essential features of the $PT$-symmetric non-Hermitian system are captured even though the system is passive; the active non-Hermitian SSH model is realized with additional pumping \[123-125\]. Candidates for the experimental realization of non-Hermitian topological systems include photonic crystals \[5, 9, 161\], ultracold atomic gasses \[8, 79\], acoustic lattices \[102, 164\], and electric circuits \[130, 165, 167\]. We employ a non-Hermitian SSH ladder [Fig. 2(a)] to elucidate our findings. Introducing additional loss $-2\gamma$ in one sublattice (the pink lattice) generates a passive non-Hermitian SSH ladder \[82\]: an overall decay rate $-i\gamma$ in offset in both sublattices yields a $PT$-symmetric non-Hermitian SSH ladder \[108\]; alternatively, we can consider an active $PT$-symmetric non-Hermitian SSH ladder constituted by two coupled SSH chains incorporated gain and loss \[124\]. The Hamiltonian in the real space reads

$H = \sum_{j=1}^{N} (w a^\dagger_{j+1} b_j + w a^\dagger_{j} b_{j+1} + t a^\dagger_{j} b_j + H.c.) + i\gamma (a^\dagger_{j} a_{j} - b^\dagger_{j} b_j), \tag{6}$

where $a^\dagger_{j}$ ($b^\dagger_{j}$) is the creation operator of the $j$th site in the sublattice $A$ ($B$). $H$ is a two-leg ladder with $2N$ sites, each leg is a $PT$-symmetric SSH chain with staggered real couplings $w$ and $v$ \[120, 125\]; two ladder legs are coupled at the strength $t$ after one leg glided by one site. $H$ is a simple generalization of the $PT$-symmetric non-Hermitian SSH model. Applying the Fourier transformation, the Hamiltonian under periodical boundary condition is rewritten as $H = \sum_{k} h_k = \sum_{k} \mathbf{B} \cdot \sigma$. The core matrix reads

$$h_k = \begin{pmatrix}
we^{-ik} + we^{ik} + t & wi & ve^{-ik} + t \\
wi & ve^{-ik} + t & -i\gamma \\
ve^{ik} + we^{ik} + t & -i\gamma & wi
\end{pmatrix}. \tag{7}$$

The non-Hermitian SSH ladder has $PT$ symmetry and chiral-time ($CT$) symmetry \[81, 108, 110\]. The corresponding energy bands $\varepsilon_k$ for the graphic eigenstates are depicted as a function of $k$ in the lower panel of Fig. 4.

In the gapped phase, the graphic eigenstates are two separated loops on the $\varphi-k$ torus. The two loops are unlinked [Fig. 4(a)] in the topologically trivial phase with $N = 0$; and are linked [Fig. 4(b)] in the topologically nontrivial phase with $N = \pm 1$. The system has the gapless phase arising from DPs in the Hermitian case; at $w + v = \pm t$, we notice one fixed DP [black solid line in Fig. 2(b)], and two movable DPs at $w = v$ [black dashed line in Fig. 2(b)] in region $|w + v| > t$ that characterized by the vector field kinks \[45\]. One fixed DP appears at $k_c = 0$ or $\pi$. The energy bands constitute a single band and the graphic eigenstates form a knot [Fig. 4(c)]. At $w = v$, two components of the magnetic field $\mathbf{B}$ vanish, the band gap closes and two movable DPs appear at $\cos k_c = -t/(2v)$. Two loops on the $\varphi-k$ torus are separated without any intersection [Fig. 4(d)].

As gain and loss increase, the band gap shrinks and the critical non-Hermiticity

$$\gamma_c = |v - w| \sqrt{1 - t^2/(4vw)}, \tag{8}$$

![Figure 2](image_url)

**FIG. 2.** (a) Schematic of the non-Hermitian SSH ladder. The gain (loss) is in sublattice $A$ ($B$) in white (pink). (b) Phase diagram and critical $\gamma_c$ [Eqs. 6 and 7]. (c) $\cos(k_c)$ for the EPs. Red solid curves divide the $v-w$ plane into four regions; the band gap closes at EPs at $k_c = 0$ and $\pi$. Two hyperbola curves are $|w + v| = 4ve/w$. Stars $c$ to $f$ are marked for the cases in Fig. 4.
in the region $|w + v| \leq 4wv/t$, where the EPs are movable in the momentum space and appear at $\cos k_{c} = -t (w + v) / (4wv)$; otherwise, the band gap closes at the critical non-Hermiticity
\[
\gamma_{c} = |w + v \pm t|,
\] (9)
and the EPs are fixed in the momentum space and appear at $k_{c} = 0$ or $\pi$. $\gamma_{c}$ for the band gap closing is depicted in Fig. 2(b); $\cos k_{c}$ for the location of EPs are depicted in Fig. 2(c). In Fig. 2(b), the black dashed line separates two phases with winding numbers $N = 1$ and $N = -1$; while the black solid line separates phases with winding numbers $N = \pm 1$ and $N = 0$.

The hyperbola $|w + v| = 4wv/t$ and line $w + v = 0$ divide the $w$-$v$ plane into four real-energy gapless phases with distinct topological characters [Fig. 2(c)]. The topological feature is clearly revealed by the graphic approach: (i) For $\gamma < \gamma_{c}$, two real-energy bands are separated without any EP; two loops on the $\varphi$-$k$ torus have none intersection [Figs. 3(a) and (b)]. (ii) For $\gamma = \gamma_{c}$ in the region $|w + v| < 4wv/t$ except for $w = v$, two loops on the $\varphi$-$k$ torus have two robust nodes [Fig. 4(h)], which are movable but irremovable as the system parameters $w$ and $v$; two movable nodes merge to a single fixed node at $k_{c} = 0$ or $\pi$ associated with the change of network topology at hyperbola $|w + v| = 4wv/t$ [Fig. 3(g)]. (iii) For $\gamma = \gamma_{c}$ in the region $|w + v| \geq 4wv/t$, two loops have one fixed node except for $w + v = \pm 1$, the location of which does not change as the parameters $w$ and $v$ [Fig. 4(c)]. (iv) For $\gamma = \gamma_{c}$ and $|w + v| = 0$, two loops have two fixed nodes [Fig. 4(f)]. The graphic eigenstates are depicted for $\gamma > \gamma_{c}$ in Supplemental Material [100], where the loops detach the $\varphi$-$k$ torus in the broken $\mathcal{PT}$-symmetric phase.

The two loops are unlinked in Figs. 4(e) and 4(f); while linked in Figs. 4(g) and 4(h) due to the $2\pi$ variation of $\varphi_{\pm}(k)$ for each eigenstate in a $2\pi$ period of $k$. This indicates the topologically nontrivial features and coincides with the winding number $N$ marked in Fig. 2(b). Both two states give identical rotating angle $2\pi$ after $k$ varying a period. Figures 3(a) and 3(b) depict the vector field $\mathbf{F}_{\pm} = (\langle \sigma_{x} \rangle_{\pm}, \langle \sigma_{y} \rangle_{\pm})$. The $2\pi$ varying direction of the vector field yields the nontrivial topology of eigenstates and coincides with the graphic eigenstates shown in Figs. 4(g) and 4(h). The coincidence of the blue and red arrows implies the locations of EPs. The upper panel depicts the situation with single EP located at $k_{c} = \pi$; the lower panel depicts the situation with two EPs located at $\cos k_{c} = -1/4$.

The nontrivial winding of eigenstate predicts the appearance of edge states. In topologically nontrivial gapless phase, the bands touch at zero energy and a pair of zero modes appear in the absence of gain and loss; under the $\mathcal{CT}$ symmetry, the gapless zero modes become imaginary in the presence of gain and loss. As the gain and loss are respectively introduced in two sublattices, thus the left side and right side localized edge states confined in one sublattice experiences either an additional attenuation $-i\gamma$ or an additional amplification $i\gamma$. The probability distribution of edge states is depicted in Fig. 3(c) for the situation in Fig. 3(a). The amplified edge state is appropriate for robust topological lasing.

Conclusion.—Real-energy topological gapless phase arising from EPs in 1D is proposed and investigated through a graphic approach. The graphic eigenstates completely encode the information of system topology and visualize the topologies of different phases in both Hermitian and non-Hermitian systems. The graph of real-energy topological gapless phase arising from EPs forms a network, its geometric topology reflects the topological properties of the ground state phase diagram. The chiral-time symmetry protects the topological gapless phase. A pair of amplification and attenuation topologically zero modes exist in the nontrivial phase. These are elucidated through a non-Hermitian SSH ladder with balanced gain and loss that directly accessible in experiment in many physical systems. Our findings propose a novel non-Hermitian topological gapless phase and provide insights into topological characterization of topological phase of matter.

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SUPPLEMENTAL MATERIAL FOR “VISUALIZING TOPOLOGY OF REAL-ENERGY GAPLESS PHASE ARISING FROM EXCEPTIONAL POINT”

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A: Vector field

The core matrix in the momentum space reads $h_k = B \cdot \sigma = B_x \sigma_x + B_y \sigma_y + B_z \sigma_z$. We consider $B_x$ and $B_y$ are real, but $B_z$ is imaginary, then the eigenstates of $h_k$ within the range of real eigenvalues $\varepsilon_{k}^{\pm}$ can be written in the form of Eq. (1) of the Letter, and the average values of Pauli matrices can be expressed as

$$
\langle \sigma_x \rangle_{\pm} = \langle \psi_{k}^{\pm} | \sigma_x | \psi_{k}^{\pm} \rangle = \cos (-\varphi_{\pm}) \\
\langle \sigma_y \rangle_{\pm} = \langle \psi_{k}^{\pm} | \sigma_y | \psi_{k}^{\pm} \rangle = \sin (-\varphi_{\pm}) \\
\langle \sigma_z \rangle_{\pm} = \langle \psi_{k}^{\pm} | \sigma_z | \psi_{k}^{\pm} \rangle = 0
$$

(10)

The average values of Pauli matrices define a vector field that related to the topological features of the non-Hermitian topological system. The vector field $F_{\pm}(k) = (\langle \sigma_x \rangle_{\pm}, \langle \sigma_y \rangle_{\pm})$ directly relates to the phase factor $\varphi_{\pm}$ of the eigenstates. For the non-Hermitian SSH ladder, the core matrix is in Eq. (6) of the Letter. Within the range of real eigenvalues $\varepsilon_{k}^{\pm}$, the corresponding eigenstates of $h_k$ in Eq. (1) of the Letter have $\varphi_{+} = \theta_1 + \theta_2$ and $\varphi_{-} = \theta_1 - \theta_2 \pm \pi$, and $\tan \theta_1 = [(w-v) \sin k] / [(v+w) \cos k]$, $\tan \theta_2 = \gamma / (|we^{-ik} + ve^{ik} + t|^2 - \gamma^2)^{1/2}$. The complete information of graphic eigenstates are mapped to the phases $\varphi_{\pm}(k)$.

B: Graphic eigenstates in the broken phase

Under the periodical boundary condition of the non-Hermitian SSH ladder, the effective complex magnetic field is $B_x = (v+w) \cos k + t$, $B_y = (v-w) \sin k$, and $B_z = i\gamma$ in the core matrix $h_k$. For large non-Hermiticity $\gamma > \gamma_c$ in the broken parity-time phase, the complex energy levels appear; and the corresponding eigenstates of $h_k$ reduce to the form of

$$
|\psi_{k}^{\pm} \rangle = \left( \begin{array}{c}
sin(\theta_{\pm}/2)e^{i\varphi_{\pm}} \\
\cos(\theta_{\pm}/2)
\end{array} \right)
$$

(11)

![FIG. 4. Graphic eigenstates and energy bands in the broken PT-symmetric phase of the non-Hermitian SSH ladder. (v, w, γ) is (a) (1/4, 1/4, 11/20), (b) (−2, 2, 11/10), (c) (2, 2/7, 7/5), (d) (3, 3/2, 8/5). In all plots, t = 1.](image)
For the complex $\varepsilon_k$, $\varphi_+ = \varphi_- = \arctan (-B_y/B_x)$ and $\theta_\pm = 2\arccos(1/\sqrt{C_\pm})$ where $C_\pm = |B_z \pm B|^2 / (B^2_x + B^2_y) + 1$ is the normalization coefficient. In the $\varphi$-$k$ torus, $R_0$ is the distance from the center of the tube to the center of the torus, and $r_0$ is the radius of the tube. Two loops are plotted on the torus with $R = R_0$ and $r = r_0 + \cos \theta_\pm$. We have $\theta_\pm = \pi/2$ for the real-energy levels, thus the real-energy levels of the two loops are always located on the surface ($r = r_0$) of the $\varphi$-$k$ torus, while the complex energy levels of the two loops are shifted to outside ($r > r_0$) or inside ($r < r_0$) the surface of the $\varphi$-$k$ torus. The graphic eigenstates and energy bands of the non-Hermitian SSH ladder in the broken parity-time symmetric phase are depicted in Supplemental Figures 4(a)-(d) as a comparison with those in the real-energy gapless phase arising from exceptional points shown in Figs. 1(e)-1(h) of the Letter.