T-duality of Large N QCD.

Z. Guralnik

University of Pennsylvania
Philadelphia PA, 19104
guralnik@ovrut.hep.upenn.edu

We argue that non-supersymmetric large $N$ QCD compactified on $T^2$ exhibits properties characteristic of an $SL(2, Z)$ T-duality. The kahler structure on which this $SL(2, Z)$ acts is given by $\frac{m}{\Lambda^2} + i\Lambda^2 A$, where $A$ is the area of the torus, $m$ is the 't Hooft magnetic flux on the torus, and $\Lambda^2$ is the QCD string tension.
1. Introduction

Following ’t Hooft’s discovery that the large N expansion of QCD is an expansion in the genus of feynman graphs, it has been suspected that large N QCD is a string theory in which the string coupling is given by $\frac{1}{N}$. This correspondence is best understood for pure $QCD_2$. The QCD string in higher dimensions has yet to be constructed, although much progress has been made recently based on the Maldacena conjecture. Assuming the existence of a string description, large N QCD will inherit any self-dualities of this description. In this talk we address the question of whether the QCD string, when compactified on a two torus, has a self T-duality. Such a T-duality would be generated by

$$
\begin{align*}
\tau &\rightarrow \tau + 1 \\
\tau &\rightarrow -\bar{\tau} \\
\tau &\rightarrow -\frac{1}{\tau}
\end{align*}
$$

for $\tau = B + i\Lambda^2 A$, where $A$ is the area of the torus, $B$ is a two form modulus, and $\Lambda^2$ is the string tension. For simplicity we consider only square tori. We shall argue that the two form modulus is $\frac{m}{N}$ where $m$ is the ’t Hooft magnetic flux through the torus. This quantity has the desired properties of periodicity, and continuity in the large N limit. We suspect that pure QCD is not exactly self T-dual, although another theory in the same universality class may be self dual. In two dimensions the partition function of $QCD_2$ is invariant under T-duality after a simple modification. In the case of $QCD_4$ on $T^2 \times R^2$, there are qualitative properties consistent with self T-duality. If $QCD_4$ on $T^2 \times R^2$ is T-dual to another theory in the same universality class, there may be computationally useful consequences. By dualizing pure $QCD_4$ on a very large torus, one would obtain a $QCD_2$-like theory with two adjoint scalars. Such theories have been used as toy models which mimic some of the dynamics of pure $QCD_4$. However these models lack a $U(1) \times U(1)$ symmetry which could generate two extra dimensions. We shall propose a model which does have such a symmetry in the large N limit. This model is obtained by a dimensional reduction of $QCD_4$ preserving the $Z_N \times Z_N$ global symmetry generated by large gauge transformations on the torus.
2. \textit{QCD}$_2$ and T-duality

Consider pure Euclidean \textit{QCD}$_2$ on a two-torus. The partition function for vanishing \textquoteleft t Hooft flux is given by \cite{17,18}

\[ Z = \sum_R e^{g^2 AC_2(R)}, \quad (2.1) \]

where \( C_2(R) \) is the quadratic casimir in the representation \( R \). When the \textquoteleft t Hooft flux \( m \) is non-vanishing, the partition function is given by \cite{19,13}

\[ Z = \sum_R e^{g^2 AC_2(R)} \frac{\text{Tr}_R(D_m)}{d_R}, \quad (2.2) \]

where \( d_R \) is the dimension of the representation. \( \text{Tr}_R(D_m) \) is the trace of the element in the center of \( SU(N) \) corresponding to the \textquoteleft t Hooft flux. For instance in a representation for which the Young Tableaux has \( n_R \) boxes,

\[ D_m = e^{2\pi i \frac{m}{N} n_R}. \quad (2.3) \]

Evaluating the free energy in the planar \( N \to \infty \) limit, as done in \cite{3} for vanishing \( m \), gives

\[ F = \ln \left| \frac{e^{2\pi i \frac{m}{N} \tau}}{\eta(\tau)} \right|^2 \quad (2.4) \]

where \( \eta \) is a Dedekind eta function, and

\[ \tau = \frac{m}{N} - \frac{\lambda A}{2\pi i}, \quad (2.5) \]

with \( g^2 N = \lambda \). This is not quite invariant under \cite{12}. However the modified free energy

\[ \mathcal{F} = F + \frac{1}{24}\lambda A - \frac{1}{2} \ln(\lambda A) \quad (2.6) \]

is modular invariant. The additional term proportional to \( A \) is a local counterterm. However it is not clear what modification of the action could account for the term \( \ln(\lambda A) \). Nonetheless this term is very simple, so it seems that pure \textit{QCD}$_2$ is almost self T-dual.
3. ’t Hooft flux and two form moduli

We have seen that under T-duality, the ’t Hooft flux $m/N$ behaves like a two form modulus of a string description. There are several reasons one could have guessed this correspondance between magnetic flux and the string theory two form. First, $m/N$ is periodic and continuous (for $N \to \infty$). Second, in $QCD_2$ with $\lambda A \to 0$, the partition function is an integral over the moduli space of flat connections. The dimension of this space is invariant under $SL(2, Z)$ transformations acting on the doublet $(m, N)$ [20]. Under these transformations $m/N$ transforms precisely like the Kahler structure $\tau$ of string theory, which for vanishing area is just the two form modulus.

Finally using the correspondance between D-branes and Yang Mills theories one can argue that under the appropriate conditions the magnetic flux is equal to the NS-NS two-form modulus. Consider a system of parallel D-branes stretched between NS5-branes which is described by a theory with a mass gap, such as pure $SU(N) \mathcal{N} = 1$ Yang-Mills [21] [22]. Since parallel D-branes generally give rise to a $U(N)$ bundle, let us construct a $U(N)$ bundle in which the $U(1)$ trace degree of freedom is frozen, so that there are no masseless $U(1)$ degrees of freedom. On a torus the fields are periodic up to $U(N)$ gauge transformations, $U_1$ and $U_2$, which for a $U(N)$ bundle satisfy

$$U_1 U_2 U_1^\dagger U_2^\dagger = I.$$ (3.1)

The $U$’s may be written as products of $U(1)$ and $SU(N)$ pieces, so that the above equation becomes

$$e^{i \int_{T^2} F_{12} e^{-2 \pi i \frac{m}{N}}} = I,$$ (3.2)

where $F_{\mu\nu}$ is the $U(1)$ field strength. If the $U(1)$ degree of freedom is frozen, the locally gauge invariant combination $F_{\mu\nu} - B_{\mu\nu}$ must vanish. Thus one obtains

$$m \frac{1}{N} = \int_{T^2} B.$$ (3.3)

Note this relation was obtained for finite $N$. We shall argue elsewhere [23] that a periodic potential is generated for $B$ when there is a mass gap. In this case one can not obtain a continuous class of theories on non-commutative tori [24] [25] by varying $B$. Having established (3.3), one must still show that $B$ behaves like a two-form modulus of the QCD-string as well as the IIA string. This means that $B$ should be the imaginary part of the action of a QCD string wrapping the torus. This can be argued [23] by lifting the
brane configuration to \(M\) theory, in which case the IIA string and the QCD-string are homotopic [21].

Note that if a QCD T-duality exists, it is not the same T-duality in the IIA theory for a variety of reasons. It can only exist in the \(N \rightarrow \infty\) limit, when \(m_N\) becomes a continuous parameter. Also, there is no D2-brane charge in the brane construction of QCD, so that after a IIA T-duality the D4-brane charge vanishes. Furthermore the Kahler structure of the QCD-string is different from that of the IIA string because the string tensions are different.

4. \(QCD_4\) and T-duality

The four dimensional QCD string is much less understood than the two dimensional QCD string. However there are qualitative properties of large \(N\) \(QCD_4\) on \(T^2 \times R^2\) which are consistent with self T-duality. We take the time direction to lie in \(R^2\). This theory has a \(U(1) \times U(1)\) translation symmetry on the torus. If the theory is self T-dual it must have another \(U(1) \times U(1)\) symmetry corresponding to translations on the dual torus. Large gauge transformations on the torus generate a global \(Z_N \times Z_N\) symmetry, which becomes continuous as \(N \rightarrow \infty\). If this symmetry is a translation symmetry on a dual torus, then eigenstates of this symmetry should have energies proportional to \(1/R_d\), where \(R_d\) is the radius of the small dual torus. This is consistent with electric confinement. The eigenstates of \(Z_N \times Z_N\) transformations carry electric flux [11], which have energy proportional to \(R = 1/(\Lambda^2 R_d)\), where \(R\) is the radius of the original torus. (We are considering the case with vanishing magnetic flux).

If the magnetic flux is non-vanishing, then \(\tau \rightarrow -\frac{1}{\tau}\) does not invert the area of the torus. Thus it would not make sense in this case to exchange the winding number of the QCD string, or the electric flux, with the QCD momentum. In string theory the momentum that gets exchanged with winding number under \(\tau \rightarrow -\frac{1}{\tau}\) is \(P_i = p_i - B_{ij} w^j\). Here \(p_i\) is the velocity of the string, and \(w^j\) is the winding number: \(X^i(t, \sigma) = p^i t + w^i \sigma + \ldots\). Under \(\tau \rightarrow \tau + 1\), \(p_i\) and \(w^j\) are invariant but \(P_i\) is shifted. Therefore the quantity in QCD corresponding to \(P_i\) is \(p_i - \frac{\alpha}{N} \epsilon_{ij} e^j\), where \(p_i\) is the usual momentum and \(e^i\) is the electric flux. The term \(\frac{\alpha}{N} \epsilon_{ij} e^j\) has precisely the form of a cross product of electric and magnetic fluxes and can be thought of as the contribution of 't Hooft fluxes to the momentum.
5. Why there might not be T-duality

If this $Z_N \times Z_N$ symmetry is spontaneously broken for a sufficiently small torus then QCD can not be self T-dual. Shrinking one cycle of the torus while keeping the other fixed would break the $Z_N$ associated with the small torus, just as in a finite temperature deconfinement transition. We do not know what happens if both cycles of the torus are shrunk simultaneously, but we can not rule out the possibility that both $Z_N$’s are spontaneously broken. Even so there may still be some theory in the QCD universality class for which the $Z_N \times Z_N$ symmetry is never spontaneously broken. Recent arguments suggest that large N QCD$_4$ can be described by a critical string theory in a five dimensional background, whose boundary is the QCD world volume$^{[6][7][8][9][10]}$. In this picture, spontaneous breaking of $Z_N \times Z_N$ would require a phase transition to a five dimensional target space geometry in which both cycles of the boundary torus are contractible$^{[10]}$. Perhaps such a transition does not exist.

6. QCD$_4$ from two dimensions

If large N QCD on $T^2 \times R^2$ were self T-dual, pure QCD on $R^4$ would be dual to QCD on $R^2$ with two adjoint scalars. In fact such a model has been used to approximate the dynamics of pure QCD in 4 dimensions $^{[14][15][16]}$. The adjoint scalars in this model play the role of transversely polarized gluons. In $^{[16]}$ the spectrum of this two dimensional model was computed by discrete light cone quantization and compared to the glueball spectrum of pure 4-d QCD computed using Monte-Carlo simulation. The degree of numerical accuracy allows only crude comparison, however the spectra have some qualitatively agreement. Perhaps in the $N \to \infty$ limit the agreement is more than just qualitative. However the usual models with adjoint scalars are incomplete since they lack a $Z_N \times Z_N$ symmetry. A more careful dimensional reduction would give a non-linear sigma model of the form

$$S_{SU(N)} = \frac{N}{\lambda_{2d}} \int d^2 x Tr \left( F_{\mu \nu}^2 + \frac{1}{R_s^2} (h_i D_{\mu} h_i^{\dagger})^2 + \frac{1}{R_s^4} [h_2, h_3][h_2^{\dagger}, h_3^{\dagger}] \right).$$

(6.1)

where $R_s$ is the radius of the cycles of the torus, and $h_i$ an element of $SU(N)$. Writing $h_i = \exp(i R_s X^i)$ and taking $R_s \to 0$ with $X^i$ fixed gives the usual naive reduced action. Note that the naive reduced action requires a mass counterterm for $X^i$. In terms of the $h_i$ fields, such a term would look like $\frac{1}{R_s^2} \sum_i Tr h_i$, which is prohibited by $Z_N \times Z_N$ symmetry. Of course the $Z_N \times Z_N$ symmetry might be spontaneously broken for sufficiently small $R_s$, ...
in which case the large N limit can not generate extra dimensions. At tree level there are flat directions which connect vacua related by the $Z_N$ symmetry. If these flat directions are not lifted quantum mechanically, as in a supersymmetric version of this theory, then long range fluctuations in two dimensions would prevent spontaneous symmetry breaking at finite $N$. Unfortunately this last statement is not always true in the large $N$ limit.

7. Conclusion

We have shown that large $N$ QCD has properties consistent with the existence of self T-duality when compactified on a torus. It may be that QCD is not really self T-dual, but that some QCD-like theory is.

Acknowledgments
I am thankful to Antal Jevicki, Igor Klebanov, Joao Nunez, Burt Ovrut, Sanjaye Ramgoolam, Washington Taylor, and Edward Witten for enlightening conversations.
References

[1] G. 't Hooft, “A planar diagram theory for strong interactions,” Nucl. Phys. B72 (1974) 461.
[2] D. Gross, “Two dimensional QCD as a string theory,” Nucl. Phys. B400 (1993) 161, hep-th/9212149.
[3] D. Gross and W. Taylor, “Two dimensional QCD is a string theory,” Nucl. Phys. B400 (1993) 181, hep-th/9301068.
[4] S. Cordes, G. Moore, S. Ramgoolam, “Large N 2-D Yang-Mills theory and topological string theory,” Commun. Math. Phys.185 (1997) 543-619, hep-th/9402107.
[5] P. Horava, “Topological rigid string theory and two-dimensional QCD,” Nucl. Phys. B463 (1996) 238-286, hep-th/9507060.
[6] A. Polyakov, “String theory and quark confinement,” hep-th/9711002. “Confining strings,” Nucl.Phys.B486 (1997) 23-33, hep-th/9607049.
[7] S. Gubser, I. Klebanov, and A. Polyakov, “Gauge theory correlators from noncritical string theory,” e-Print Archive: hep-th/9802109.
[8] J. Maldacena, “The large N limit of superconformal field theories and supergravity,” e-Print Archive: hep-th/9711200.
[9] E. Witten, “Anti-de Sitter space and holography,” e-Print Archive: hep-th/9802150.
[10] E. Witten, “Anti-de-Sitter space, thermal phase transition, and confinement in gauge theories,” e-Print Archive: hep-th/9803131.
[11] G. 't Hooft, ”A property of electric and magnetic flux in non-abelian gauge theories,” Nucl. Phys. B153 (1979) 141-160.
[12] R. Rudd, ”The string partition function for QCD on the torus,” hep-th/9407176.
[13] Z. Guralnik, “Duality of large N Yang Mills theory on T^2 × R^2,” e-print Archive: hep-th/9804057.
[14] S. Dalley, I. Klebanov, String spectrum of (1 + 1) dimensional large N QCD with adjoint matter,” Phys. Rev. D47 (1993) 2517-2527.
[15] K. Demeterfi, I. Klebanov, Gyan Bhanot, “Glueball spectrum in a (1+1) dimensional model for QCD,” Nucl. Phys.B418 (1994) 15-29, hep-th/9311015.
[16] F. Antonuccio and S. Dalley, “Glueballs from (1+1) dimensional gauge theories with transverse degrees of freedom,” Nucl. Phys. B461 (1996) 275-304, hep-ph/9506456.
[17] B. Rusakov, ”Loop averages and partition functions in U(N) gauge theory on two-dimensional manifolds,” Mod.Phys.Lett.A5 (1990) 693-703.
[18] A. Migdal, “Recursion equations in gauge theories,” Sov.Phys.JETP42 (1975) 413, Zh.Eksp.Teor.Fiz.69 (1975) 810-822.
[19] E. Witten, “Two dimensional gauge theories revisited.” J. Geom. Phys. 9 (1992) 303-368, hep-th/9204083.
[20] Z. Guralnik and S. Ramgoolam, ”From 0-Branes to Torons,” hep-th/9708089.
[21] E. Witten, “Branes and the dynamics of QCD,” Nucl. Phys.B507 (1997) 658-690, e-print Archive: hep-th/9706109.

[22] K. Hori and H. Ooguri, “Strong coupling dynamics of four-dimensional N=1 gauge theories from M theory five-brane.” Adv.Theor.Math.Phys.1 (1998) 1-52, hep-th/9706082.

[23] Z. Guralnik, “Strings and discrete fluxes of QCD,” in preparation.

[24] A. Connes, M. Douglas, A. Schwarz, ”Noncommutative geometry and Matrix theory: compactification on tori,” hep-th/9711162.

[25] M. Douglas and C. Hull, ”D-branes and the noncommutative torus,” hep-th/9711165.