Graviton multi-point amplitudes for higher-derivative gravity in anti-de Sitter space

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Abstract

We calculate graviton multi-point amplitudes in an anti-de Sitter black brane background for higher-derivative gravity of arbitrary order in numbers of derivatives. The calculations are performed using tensor graviton modes in a particular regime of comparatively high energies and large scattering angles. The regime simplifies the calculations but, at the same time, is well suited for translating these results into the language of the dually related gauge theory. After considering theories of up to eight derivatives, we generalize to even higher-derivative theories by constructing a “basis” for the relevant scattering amplitudes. This construction enables one to find the basic form of the \(n\)-point amplitude for arbitrary \(n\) and any number of derivatives. Additionally, using the four-point amplitudes for six and eight-derivative gravity, we re-express the scattering properties in terms of the Mandelstam variables.

1 Introduction

The gauge–gravity duality enables one to describe a \(d\)-dimensional gauge theory in terms of a \((d+1)\)-dimensional gravitational theory \[1\]. Importantly, the duality relates a strongly coupled field theory to a
weakly coupled theory of gravity. Since strongly coupled gauge theories are not very well understood, the duality provides a means for making analytical statements about them. One application of this framework [2] is the correspondence between stress-energy tensor correlation functions in the relevant gauge theory and graviton scattering amplitudes in its gravitational dual [3].

In the earliest investigations into the duality — which mostly focused on 5-dimensional anti-de Sitter (AdS) space and 4-dimensional super Yang–Mills theory — the rank of the gauge theory $N$ is taken to infinity, which then corresponds to Einstein’s theory of gravity [4]. (On the other hand, the ’t Hooft coupling $\lambda = g_s^2 N$ is regarded as large but finite in the standard limit.) With deviations to large but finite values of $N$, the gravitational dual can be expected to include higher-derivative corrections in addition to Einstein’s (two-derivative) Lagrangian [5]. If the interest is only in gauge-invariant quantities (such as scattering amplitudes), then one can limit considerations to the multi-derivative terms in the Lagrangian which are strictly composed of contractions between “proper” four-index Riemann tensors (i.e., two-index tensors and scalars are excluded). This simplification was recently discussed in [6] and can be shown through a gauge transformation of the graviton that involves the Ricci scalar and tensor [7, 8].

In this paper, we calculate graviton scattering amplitudes for higher-derivative theories of gravity in an AdS black brane background. Our approach is similar to that of [9], where the focus is on Einstein and four-derivative gravity. However, those scattering amplitudes could be substantially simplified by enforcing on-shell conditions. In the
case of Einstein’s gravity, the equations of motion can be used to eliminate amplitudes with two derivatives acting on single graviton. Meanwhile, four-derivative gravity can always be gauge transformed to a Gauss–Bonnet theory by using the “inverse” of the aforementioned transformation. As a Lovelock extension of Einstein’s theory, Gauss–Bonnet gravity has equations of motion that have at most two derivatives. Thus, the very same logic and simplification applies in this case as well. (See the Appendix in [9] for further details.) In the current work, we do not have the luxury of restricting to Lovelock theories or theories that are related to Lovelock via a gauge transformation; meaning that the previous simplification can no longer be relied upon. It follows that terms with two derivatives acting on a graviton must now be included in the calculation of amplitudes.

Starting with six and eight-derivative theories in Section 2, we utilize what could be called “basis amplitudes” to construct the \( n \)-point amplitudes for arbitrarily large \( n \). The generalization of these basis amplitudes mitigates the task of finding multi-point amplitudes for theories with an arbitrary number of derivatives (see Section 3). Finally, using 4-point amplitudes in Section 4, we re-express the scattering properties of the six and eight-derivative theories in terms of the Mandelstam variables. (Section 5 contains a brief conclusion.)

Our intention is, in a later paper, to map the current results to stress–tensor correlation functions in the dual gauge theory. The motivation for such a treatment is to learn about the gravitational dual to the gauge theory describing the quark–gluon plasma and other

1This follows from any \( 2k \)-derivative Lovelock extension vanishing identically in \( 2k − 1 \) or less dimensions.
strongly coupled gauge theories [10]. These findings could conceivably be tested thanks to the high-energy hadron-collider laboratories at Brookhaven and CERN.

Because our long-term goal is to apply the gauge–gravity duality for the purpose of making experimentally viable predictions, we are mainly interested in the boundary limit of the amplitudes and, moreover, only those contributions that would survive holographic renormalization [11] and could be discernible in gauge-theory correlation functions. With this in mind, our calculations are limited to a special kinematic region which was referred to as the “high-momentum regime” in [9]. As discussed later, this regime is particularly well suited for discriminating the higher-derivative contributions in the graviton scattering amplitudes and then, ultimately, in the dual correlation functions.

1.1 Summary of formalism

Much of groundwork for the following analysis has already been laid out in [9, 12]. Here, we will summarize some of the key elements that are necessary for the task at hand.

The AdS black brane background in a five-dimensional spacetime has a metric of the form

\[ ds^2 = -f(r) dt^2 + \frac{1}{g(r)} dr^2 + \frac{r^2}{L^2} (dx^2 + dy^2 + dz^2) , \tag{1} \]

where the radial coordinate ranges from the black brane horizon at \( r = r_h \) to the AdS boundary limit of \( r \to \infty \), the parameter \( L \) is

\(^2\)Generalizations to other dimensionalities are straightforward.
the AdS radius of curvature, and the functions \( f(r), g(r) \) must satisfy \( \lim_{r \to r_h} f(r) = g(r) = 0 \) and \( \lim_{r \to \infty} f(r) = g(r) = \frac{r^2}{L^2} \).

Metric fluctuations are represented by \( \bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \), where \( g_{\mu\nu} \) is the AdS background metric of Eq. (1) and \( h_{\mu\nu} \) is a small perturbation of the background metric (i.e., a graviton). In this analysis, the gravitons will propagate along the \( z \)-coordinate and so

\[
h_{ab} = \phi(r)e^{i(\omega t - kz)},
\]

where \( \phi(r) \) is some well-behaved function.

We assume weak coupling and consider a perturbative expansion of the Lagrangian in terms of \( \epsilon = \frac{\ell_s^2}{L^2} \ll 1 \) (\( \ell_s \) is the string length) as appropriate for the case of large \( N \) and large but relatively smaller \( \lambda \). The expansion can be expressed schematically as

\[
\mathcal{L} = \sqrt{-g} \frac{R}{16\pi G_5} \left( 1 + \epsilon L^2 R + \epsilon^2 L^4 R^2 + \cdots + \epsilon^k L^{2k} R^k + \cdots \right),
\]

where we have suppressed indices (all terms are contractions of 4-index Riemann tensors as previously discussed) and \( G_5 \) is Newton’s constant in five dimensions. For the sake of brevity, the Lagrangian can also be written as

\[
\mathcal{L} = \sum_{k=1} L_k,
\]

where \( L_k \sim R^k \) is meant to describe a contraction of \( k \) Riemann tensors. So that, when referring to (e.g) six-derivative gravity, we really mean the Lagrangian \( L_3 = \frac{\sqrt{-g} R}{16\pi G_5} [\epsilon^2 L^4 R^2] \).

Gravitational scattering amplitudes must be holographically renormalized if we are to make a connection with the gauge–gravity duality. As shown in [9], this process eliminates the need to consider the
radial derivatives of gravitons as these would lead to divergences in
the amplitude even after renormalization. Such terms would then be
discarded in the gauge theory after the standard techniques of holo-
graphic renormalization have been applied \[11\]. Hence, for the current
work, only the $t$ and $z$ derivatives of gravitons are considered.

Let us now clarify what is meant by the “high-momentum regime”.
Our motivation is the idealized case of a kinematic region that is po-
tentially accessible via experiment \[3\] and that allows one to distinguish
between the contributions of the different $L_k$’s in Eq. (4). The basic
idea is that larger values of momenta can compensate for larger pow-
ers of $\epsilon$ as one probes theories which are higher order in numbers of
derivatives. This is accomplished by insisting that a contribution to
amplitudes of order $\epsilon^q$ always include $2q+2$ derivatives acting on gravi-
tons. Other contributions are to be discarded. So that, if $L^2\epsilon \omega^2$ is not
too small a number, the surviving contributions should be prominent
in the scattering profiles (and ultimately in the gauge-theory correla-
tors).

Let us explain further what classifies a value of momentum as
“high” in this context. Firstly, background derivatives go as $\nabla_r \nabla_r \sim \frac{1}{L^2}$, while graviton momenta go as $\nabla_t \nabla_t \sim \omega^2$ and $\nabla_z \nabla_z \sim k^2$; therefore, one requirement is that $\omega^2, k^2 \gg \frac{1}{L^2}$. Furthermore, to apply
the tools of the gauge–gravity duality, the hydrodynamic regime must
necessarily be in effect. This means that $\omega \ll T$ and, consequently,

$$1 \ll L \omega \ll TL.$$  \[
(5)
\]

\[3\]The viability of experimental accessibility hinges on having $\epsilon$ not too small. See 13
for further details.
This range is indeed viable because, according to the duality, \( TL \sim \frac{\rho}{\rho} \gg 1 \) \cite{3}. One then needs only to hope that \( \epsilon(\omega L)^2 \) is sufficiently large.

We also work in the radial gauge, which means that \( h_{ra} = 0 \) for any \( a \). This gauge divides the gravitonal perturbations into three distinct sectors: scalar, vector and tensor \cite{14}. However, scalar modes do not contribute to scattering amplitudes in the high-momentum regime. This is because scalars need to be sourced and sources can be expected, on general grounds, to introduce an additional factor of \( \epsilon \). Meanwhile, vector modes are analogous to the electromagnetic potential and, as such, can either be gauged away or require a source when appearing in gauge-invariant combinations. Hence, these modes can also be discounted. Other fields can only couple to the gravitons through a stress-tensor, which will invoke additional powers of \( \epsilon \). This leaves only the tensor modes \( h_{xy} \) as relevant in the high-momentum regime.

Given that the number of derivatives acting on gravitons has been maximized, the only other type of tensor-mode amplitude requiring suppression is one that includes \( \Box h_{xy} = 0 + \mathcal{O}(\epsilon) \), since this adds an extra factor of \( \epsilon \). Note though that an expression like \( g^{ab} g^{cd} \nabla_a \nabla_c h_{xy} \) would survive (with the free indices suitably contracted). We should also add that, to maintain general covariance, tensor modes must come in pairs (see below). And so, with the restriction to tensor modes, any non-trivial scattering amplitude will necessarily be even.

Finally, we will perturbatively expand the metric determinant and contravariant metrics by adopting the conventions of \cite{15}. To briefly
review, a perturbed metric such as

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$  \(6\)

would have a contravariant counterpart of the form

$$\bar{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + h^{\rho}_{\mu} h^{\nu}_{\rho} + \mathcal{O}(h^3)$$  \(7\)

and the metric determinant is then

$$\sqrt{-\bar{g}} = \sqrt{-g} \left[ 1 + \frac{1}{2} h^{\mu}_{\mu} - \frac{1}{4} h^{\rho}_{\mu} h^{\rho}_{\mu} + \frac{1}{8} (h^{\mu}_{\mu})^2 + \mathcal{O}(h^3) \right].$$  \(8\)

Since our interest is in the tensor modes $h_{xy}$, only expressions with an even number of gravitons will survive in Eqs. \(7\) and \(8\), and undifferentiated gravitons will have to come from either

$$\bar{g}^{xx} = g^{xx} + h^{xy}_{y} h^{yx}_{x} + (h^{xy}_{y} h^{yx}_{x})^2 + \cdots + (h^{xy}_{y} h^{yx}_{x})^p$$  \(9\)

or

$$\sqrt{-g} = \sqrt{-g} \left[ 1 - \frac{1}{2} h^{xy}_{y} h^{yx}_{x} - \frac{1}{22!} (h^{xy}_{y} h^{yx}_{x})^2 + \cdots + \Theta(p)(h^{xy}_{y} h^{yx}_{x})^p \right],$$  \(10\)

where

$$\Theta(p) = -\frac{\Gamma[p - \frac{1}{2}]}{2\sqrt{\pi}p!}, \quad \text{for } p \in \mathbb{Z}.$$  \(11\)

As for the differentiated gravitons, these will of course come from expansions of the Riemann tensors (see below).
2 Multi-point amplitudes

2.1 Basis multi-point amplitudes

We start here by introducing some shorthand notation for perturbations of the Riemann tensor,

\[ \delta^{(1)} R_{abcd} = \nabla_b \Gamma_{dac}(h) - \nabla_c \Gamma_{dab}(h), \quad (12) \]

\[ \delta^{(2)} R_{abcd} = g^{ef} [\Gamma_{ecb}(h)\Gamma_{fda}(h) - \Gamma_{ead}(h)\Gamma_{fbc}(h)], \quad (13) \]

where

\[ \Gamma_{abc}(h) = \frac{1}{2} (\nabla_b h_{ac} + \nabla_c h_{ab} - \nabla_a h_{bc}). \quad (14) \]

Given that the high-momentum regime is in effect (so that the background contributions from any Riemann tensor can be ignored), then each Riemann tensor effectively contributes

\[ R_{abcd} \rightarrow R_{abcd} \equiv \delta^{(1)} R_{abcd} + \delta^{(2)} R_{abcd}. \quad (15) \]

Recalling that gravitons can only be differentiated with respect to \( t \) and \( z \), one can observe that any perturbed Riemann tensor must have a specific arrangement of indices. Up to symmetries, these would be \( \delta^{(1)} R_{axby} \) for \( a, b = \{t, z\} \) and \( \delta^{(2)} R_{cxdx} \) for \( c, d = \{t, z, y\} \), where \( x \) and \( y \) are interchangeable.

With the high-momentum regime in mind, we will define basis amplitudes as multi-point amplitudes for which all of the included gravitons are differentiated. Each such basis amplitude then represents a different combination of \( \delta^{(1)} R \)'s and \( \delta^{(2)} R \)'s. Furthermore, in constructing a general \( 2n \)-point amplitude, we will use the notation

\[ R_{abcd} = R_{cdab} \quad \text{and} \quad R_{abcd} = -R_{abcd}. \]

\[ ^4 \text{In particular, } R_{abcd} = R_{cdab} \quad \text{and} \quad R_{abcd} = -R_{abcd}. \]
\[ \langle h^{2n}\rangle_{2p} \] to indicate a 2n-point amplitude constructed from a 2p-point basis amplitude \((n \geq p)\). For instance, the basis amplitudes themselves are denoted by \(\langle h^{2p}\rangle_{2p}\).

### 2.2 Multi-point amplitudes from Riem\(^3\) (\(\epsilon^2\)-order) gravity

Six-derivative or Riem\(^3\) gravity can be defined, up to gauge transformations, as

\[
L_3 = \sqrt{-g} \left[ \alpha R_{abcd} R^{ab}_{\phantom{ab}mn} R^{mncd} + \beta R_{abcd} R^{ad}_{\phantom{ad}mn} R^{mncb} \right],
\]

where \(\frac{\epsilon^2 L^4}{16\pi G}\) has now been absorbed into the model-dependent constants \(\alpha\) and \(\beta\).

In this theory, there are two types of basis amplitudes; \(^5\) the 4 and 6-point amplitudes. The basis 4-point amplitude has 6 derivatives and 4 gravitons; schematically it can be written in terms of the previously introduced shorthand (cf. Eqs. (12) and (13)),

\[
\langle h^4\rangle_4 \sim 3(\alpha + \beta) R \delta^{(1)} R \delta^{(1)} R \delta^{(2)} R ,
\]

where the tensor indices have been suppressed and the 3 counts the number of ways of choosing one of the tensors to carry two gravitons.

Similarly, the 6-point amplitude with 6 derivatives and 6 gravitons takes the schematic form

\[
\langle h^6\rangle_6 \sim (\alpha + \beta) \delta^{(2)} R \delta^{(2)} R \delta^{(2)} R .
\]

\(^5\)Here, we will be disregarding 2-point amplitudes because they are not really of interest from the viewpoint of someone discriminating between different theories.
Now more explicitly, the 4-point amplitude of Eq. (17) can be expressed as

\[
\langle h_4 \rangle_4 = 3(\alpha + \beta)^2 \left[ R_{txty} R_{txty} R_{tx} + R_{txzy} R_{txzy} R_{tx} + R_{txzy} R_{txzy} R_{tx} + \{ t \leftrightarrow z \} \right] \\
+ \{ x \leftrightarrow y \},
\]

(19)

where \( \{ a \leftrightarrow b \} \) is shorthand for the interchange of \( a \) and \( b \) in the preceding expression and the \( 2^3 \) accounts for the symmetries of the Riemann tensors. Even more explicitly, in terms of gravitons (and with the help of the equations in Subsection 2.1),

\[
\langle h_4 \rangle_4 = -6(\alpha + \beta) \sqrt{-g}(g^{xx} g^{yy})^2 \left[ (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2)(\omega_1 g^{tt} \omega_3 + k_1 g^{zz} k_3)(\omega_2 g^{tt} \omega_4 + k_2 g^{zz} k_4) \right] \\
\times h^{(1)}_{xy} h^{(2)}_{xy} h^{(3)}_{xy} h^{(4)}_{xy},
\]

(20)

where each factor of \( (\omega_i g^{tt} \omega_j + k_i g^{zz} k_j) \) is a product of the momenta for gravitons \( h^{(i)}_{xy} \) and \( h^{(j)}_{xy} \), and the symmetrization of the gravitons is always implied.

Using similar reasoning, one finds that the basis 6-point amplitude translates into

\[
\langle h_6 \rangle_6 = -\frac{3}{2}(\alpha + \beta) \sqrt{-g}(g^{xx} g^{yy})^3 \left[ (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2)(\omega_3 g^{tt} \omega_4 + k_3 g^{zz} k_4)(\omega_5 g^{tt} \omega_6 + k_5 g^{zz} k_6) \right] \\
\times h^{(1)}_{xy} h^{(2)}_{xy} h^{(3)}_{xy} h^{(4)}_{xy} h^{(5)}_{xy} h^{(6)}_{xy}.
\]

(21)

The utility of the basis amplitudes in Eqs. (20) and (21) is that one can use these to construct \( 2n \)-point amplitudes by drawing out additional pairs of gravitons from the metric determinant and the contravariant metrics \( g^{xx} \), \( g^{yy} \). Let us begin with Eq. (20) and suppose that \( p \) pairs are drawn from the metric determinant and \( n - 2 - p \)
pairs from the four contravariant metrics. Recalling that the number of ways of drawing $q$ identical objects from $m$ distinct boxes is
\[ \binom{q + m - 1}{m - 1} , \]
we then have that, for any $n \geq 2$,
\[
\langle h_{2n} \rangle_4 = -6(\alpha + \beta) \binom{2n}{4} \sum_{p=0}^{n-2} \binom{n + 1 - p}{3} \Theta(p) \sqrt{-g(g_{xx} g_{yy})²}
\times \left[ (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2)(\omega_1 g^{tt} \omega_3 + k_1 g^{zz} k_3)(\omega_2 g^{tt} \omega_4 + k_2 g^{zz} k_4) \right]
\times h_{xy}^{(1)} h_{xy}^{(2)} h_{xy}^{(3)} h_{xy}^{(4)} \prod_{j=3}^{n} \left[ (h_{xy}^{(2j-1)}(h_{xy}^{(2j)}) \right] , (22)
\]
where the combinatorial factor before the summation represents the number of ways of choosing the four differentiated gravitons from the total of $2n$ and the summation itself accounts for all possible ways of drawing gravitons from the contravariant metrics and metric determinant.

In similar fashion, $2n$-point amplitudes can be constructed from the basis 6-point amplitude of Eq. (21) for any $n \geq 3$. The result of this is
\[
\langle h_{2n} \rangle_6 = -\frac{3}{2}(\alpha + \beta) \binom{2n}{6} \sum_{q=0}^{n-3} \binom{n - q + 2}{5} \Theta(q) \sqrt{-g(g_{xx} g_{yy})³}
\times \left[ (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2)(\omega_3 g^{tt} \omega_4 + k_3 g^{zz} k_4)(\omega_5 g^{tt} \omega_6 + k_5 g^{zz} k_6) \right]
\times h_{xy}^{(1)} h_{xy}^{(2)} h_{xy}^{(3)} h_{xy}^{(4)} h_{xy}^{(5)} h_{xy}^{(6)} \prod_{k=4}^{n} \left[ (h_{xy}^{(2k-1)}(h_{xy}^{(2k)}) \right] . (23)
\]
As Eqs. (22) and (23) now make clear, the 4 and 6-point amplitudes of Eqs. (20) and (21) form the basis for the $2n$-point amplitudes of Riem$^3$ gravity in the high-momentum regime. The complete $2n$-point
amplitude for Riem\(^3\) gravity is then given by the linear combination
\[
\langle h^{2n} \rangle_{\text{Riem}^3} = \langle h^{2n} \rangle_4 + \langle h^{2n} \rangle_6 .
\] (24)

2.3 Multi-point amplitudes from Riem\(^4\) (\(\epsilon^3\)-order) gravity

Eight-derivative or Riem\(^4\) gravity is expressible, up to gauge transformations, as
\[
L_4 = \left[ \alpha R_{abcd} R^{abcd} R_{mn} R_{pq} R_{mp} R_{pq} + \beta R_{abcd} R_{mn} R_{pq} R_{mp} R_{pq} + \gamma R_{abcd} R_{pq} R_{mp} R_{pq} + \mu R_{abcd} R_{mpq} R_{mpq} R_{npq} \right].
\] (25)

Here, like before, \(\frac{3f^6}{16\pi G_5}\) has been absorbed into the model-dependent constants.

With the very same reasoning as in the previous section, we can call on Eq. (25) to construct three types of basis amplitudes; the 4, 6 and 8-point amplitudes. In terms of the Riemann tensor in Eq. (15), these can be schematically written as
\[
\langle h^4 \rangle_4 \sim (\alpha + \beta + \gamma + \mu + \nu + \rho) \delta^{(1)} R \delta^{(1)} R \delta^{(1)} R \delta^{(1)} R ,
\] (26)
\[
\langle h^6 \rangle_6 \sim 6(\alpha + \beta + \gamma + \mu + \nu + \frac{1}{3} \rho) \delta^{(1)} R \delta^{(1)} R \delta^{(2)} R \delta^{(2)} R ,
\] (27)
\[
\langle h^8 \rangle_8 \sim (\alpha + \beta + \gamma + \mu + \nu + \rho) \delta^{(2)} R \delta^{(2)} R \delta^{(2)} R \delta^{(2)} R .
\] (28)

Expanding Eq. (26), one finds that
\[
\langle h^4 \rangle_4 = 4(\alpha + \beta + \gamma + \mu + \nu + \rho) \sqrt{-g} (g^{tx} g^{xy})^2 \left[ (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2) (\omega_2 g^{tt} \omega_3 + k_2 g^{zz} k_3) \right. \\
\times \left. (\omega_3 g^{tt} \omega_4 + k_3 g^{zz} k_4) (\omega_1 g^{tt} \omega_4 + k_1 g^{zz} k_4) \right] h_{x}^{(1)} h_{x}^{(2)} h_{x}^{(3)} h_{x}^{(4)} ,
\] (29)
which can then be used to construct a 2n-point amplitude for any
\( n \geq 2 \),
\[
\langle h^{2n} \rangle_4 = (\alpha + \beta + \gamma + \mu + \nu + \rho) \binom{2n}{4} \sum_{p=0}^{n-2} \binom{n + 1 - p}{3} \Theta(p) \sqrt{-g} (g^{xx} g^{yy})^2 \\
\times \left[ (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2)(\omega_2 g^{tt} \omega_3 + k_2 g^{zz} k_3)(\omega_3 g^{tt} \omega_4 + k_3 g^{zz} k_4)(\omega_4 g^{tt} \omega + k_1 g^{zz} k_4) \right] \\
\times h_{xy}^{(1)} h_{xy}^{(2)} h_{xy}^{(3)} h_{xy}^{(4)} h_{xy}^{(5)} h_{xy}^{(6)} \prod_{j=3}^{n} \left[ (h_y^{(2j-1)})(h_y^{(2j)}) \right], \quad (30)
\]
where the combinatoric factors (here and below) are handled similarly
to those in Eq. (22).

As for the basis 6-point amplitude, this goes as
\[
\langle h^6 \rangle_6 = (4\rho + 6(\alpha + \beta + \gamma + \mu + \nu)) \sqrt{-g} (g^{xx} g^{yy})^3 \\
\times \left[ (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2) \omega_3 g^{tt} \omega_4 + k_3 g^{zz} k_4) (\omega_5 g^{tt} \omega_6 + k_5 g^{zz} k_6) \right] \\
\times h_{xy}^{(1)} h_{xy}^{(2)} h_{xy}^{(3)} h_{xy}^{(4)} h_{xy}^{(5)} h_{xy}^{(6)} \prod_{k=3}^{n} \left[ (h_y^{(2k-1)})(h_y^{(2k)}) \right], \quad (31)
\]
from which one can construct a 2n-point amplitude for any \( n \geq 3 \),
\[
\langle h^{2n} \rangle_6 = (4\rho + 6(\alpha + \beta + \gamma + \mu + \nu)) \binom{2n}{6} \sum_{q=0}^{n-3} \binom{n - q + 2}{5} \Theta(q) \sqrt{-g} (g^{xx} g^{yy})^3 \\
\times \left[ (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2) \omega_3 g^{tt} \omega_4 + k_3 g^{zz} k_4) (\omega_5 g^{tt} \omega_6 + k_5 g^{zz} k_6) \right] \\
\times h_{xy}^{(1)} h_{xy}^{(2)} h_{xy}^{(3)} h_{xy}^{(4)} h_{xy}^{(5)} h_{xy}^{(6)} \prod_{k=4}^{n} \left[ (h_y^{(2k-1)})(h_y^{(2k)}) \right]. \quad (32)
\]
Finally, the basis 8-point amplitude is of the form
\[
\langle h^8 \rangle_8 = (\rho + \frac{1}{2}(\alpha + \beta + \gamma + \mu + \nu))(g^{xx} g^{yy})^4 (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2)(\omega_3 g^{tt} \omega_4 + k_3 g^{zz} k_4) \\
\times (\omega_5 g^{tt} \omega_6 + k_5 g^{zz} k_6)(\omega_7 g^{tt} \omega_8 + k_7 g^{zz} k_8) h_{xy}^{(1)} h_{xy}^{(2)} h_{xy}^{(3)} h_{xy}^{(4)} h_{xy}^{(5)} h_{xy}^{(6)} h_{xy}^{(7)} h_{xy}^{(8)}, \quad (33)
\]
and the corresponding 2\(n\)-point amplitude for any \(n \geq 4\) is then
\[
\langle h_{2n} \rangle_8 = \left( p + \frac{1}{2}(\alpha + \beta + \gamma + \mu + \nu) \right) \left( \frac{2n}{8} \right) \sum_{r=0}^{n-4} \left( \frac{n-r+3}{7} \right) \Theta(r) \sqrt{-g(g_{xx}g_{yy})^4} \\
\times (\omega_1 g_{11}^{tt}\omega_2 + k_1 g_{zz}k_2)(\omega_3 g_{11}^{tt}\omega_4 + k_3 g_{zz}k_4)(\omega_5 g_{11}^{tt}\omega_6 + k_5 g_{zz}k_6)(\omega_7 g_{11}^{tt}\omega_8 + k_7 g_{zz}k_8) \\
\times h_{x_1 y_1}^{(1)} h_{x_2 y_2}^{(2)} h_{x_3 y_3}^{(3)} h_{x_4 y_4}^{(4)} h_{x_5 y_5}^{(5)} h_{x_6 y_6}^{(6)} h_{x_7 y_7}^{(7)} h_{x_8 y_8}^{(8)} \prod_{l=5}^{n} \left( h_{x_l y_l}^{(2l-1)} h_{x_l y_l}^{(2l)} \right).
\]

(34)

Like before, the complete 2\(n\)-point amplitude for the Riem\(4\) theory in the high-momentum regime is a linear combination of the basis amplitudes in Eqs. (30), (32) and (34). That is,
\[
\langle h_{2n} \rangle_{\text{Riem}^4} = \langle h_{2n} \rangle_4 + \langle h_{2n} \rangle_6 + \langle h_{2n} \rangle_8.
\]

(35)

### 3 Multi-Point amplitudes from Riem\(q\)

(\(\epsilon^{q-1}\)-order) gravity

In this section, we will find the basic form of the 2\(n\)-point amplitude in the high-momentum regime when the most general type of gravitational theory is considered. Namely, one whose Lagrangian is composed of \(q\) contracted Riemann tensors for arbitrary \(q\).

This task appears to be quite arduous, as one would expect that the number of gauge-invariant terms in the Lagrangian grows exponentially with \(q\). On the other hand, as shown in Subsection 2.3, each of the six invariants makes essentially the same contribution to

\[\text{It is amusing to note that the number of gauge-invariant terms grows exactly as } (q-1)! \text{ for } q \leq 4 \text{, which then grows roughly as } e^q \text{ for large } q.\]

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any one of the three basis amplitudes (cf, Eqs. (29), (31) and (33)) — although the different basis amplitudes will indeed have different forms. It is not difficult to convince oneself that the relative simplicity of the high-momentum regime is enough to ensure that these trends will persist to higher orders in $q$.

Since any single graviton can carry zero, one or two derivatives, there are many ways to obtain a $2n$-point amplitude from a Riemann theory depending on the size and parity of $q$. Let us suppose that this is a $q$-odd theory; then the set of basis amplitudes (i.e., those with only differentiated gravitons) is the set $P_q = \{ \langle h^{q+1} \rangle_{q+1}, \langle h^{q+3} \rangle_{q+3}, \ldots, \langle h^{2q} \rangle_{2q} \}$ with cardinality $\frac{q+1}{2}$. The reasoning here is that, for an odd value of $q$, $\langle h^{q+1} \rangle_{q+1}$ has the maximum possible number of gravitons carrying two derivatives with the remainder carrying one, whereas $\langle h^{2q} \rangle_{2q}$ has all gravitons carrying a single derivative.

To elaborate further, let us consider the following arrangements of $q$ contracted Riemann tensors (while keeping in mind Eqs. (12)
and (13) and that the gravitons come in pairs):

\begin{align}
\langle h^{q+1} \rangle_{q+1} &\sim \frac{\delta^{(1)} \mathcal{R} \delta^{(1)} \mathcal{R} \cdots \delta^{(1)} \mathcal{R} \delta^{(2)} \mathcal{R}}{q \text{ products of Riemann tensors}} \tag{36} \\
\langle h^{q+3} \rangle_{q+3} &\sim \frac{\delta^{(1)} \mathcal{R} \delta^{(1)} \mathcal{R} \cdots \delta^{(2)} \mathcal{R} \delta^{(2)} \mathcal{R}}{q \text{ products of Riemann tensors}} \tag{37} \\
\langle h^{2q} \rangle_{2q} &\sim \frac{\delta^{(2)} \mathcal{R} \delta^{(2)} \mathcal{R} \cdots \delta^{(2)} \mathcal{R} \delta^{(2)} \mathcal{R}}{q \text{ products of Riemann tensors}} \tag{38}
\end{align}

A single \( \delta^{(2)} \mathcal{R} \) term and an even number of \( \delta^{(1)} \mathcal{R} \) terms,

three \( \delta^{(2)} \mathcal{R} \) terms and an even number of \( \delta^{(1)} \mathcal{R} \) terms,

all \( \delta^{(2)} \mathcal{R} \) terms.

A similar argument can be used for \( q \)-even theories; in which case, the set of basis amplitudes is \( Q_q = \{ \langle h^q \rangle_q, \langle h^{q+2} \rangle_{q+2}, \ldots, \langle h^{2q} \rangle_{2q} \} \) with cardinality \( \frac{4 q^2}{2} \). In both cases, the key point is that there should either be zero or an even number of \( \delta^{(1)} \mathcal{R} \)’s in any basis amplitude.

The next step involves reformulating the different basis amplitudes in terms of gravitons rather than Riemann tensors. It is clear that each such basis amplitude will be a polynomial in \( \omega \)'s and \( k \)'s of degree \( 2q \).

Since either one or two derivatives can act on a graviton, each term in the polynomial must contain one of \( \omega_i \), \( k_i \), \( \omega_i k_i \), \( \omega^2_i \), \( k^2_i \) for each graviton \( h^{(i)}_{xy} \). Then as long as \( q \leq 2p < 2q \) for some \( p \in \mathbb{N}^+ \), a basis amplitude for Riem\( ^q \) can be expressed somewhat schematically as

\begin{align}
\langle h^{2p} \rangle_{2p} &= A_q \sqrt{-g} (g^{xx} g^{yy})^p C_{(1,2)} C_{(3,4)} \cdots C_{(2p-1,2p)} C_{(1,2p)} \cdots \\
&\times h^{(1)}_{xy} h^{(2)}_{xy} h^{(3)}_{xy} \cdots h^{(2p-1)}_{xy} h^{(2p)}_{xy} . \tag{39}
\end{align}

This form assumes that the gravitons can be freely labeled.
where \( C(i,j) \equiv K_i \cdot K_j = (\omega_i g^{tt} \omega_j + k_i g^{zz} k_j) \) with \( K_i = (\omega_i, 0, 0, k_i) \) and \( A_q \sim O(e^q) \) is a numerical coefficient. For \( 2p = 2q \), the basis amplitude should rather be written as

\[
\langle h^{2q} \rangle_{2p} = A_q 2^{2-q} \sqrt{-g(g^{xx} g^{yy})^q} \left( C_{(1,2)} C_{(3,4)} \cdots C_{(2q-1,2q)} \right) \right.
\end{array} \right.

\times h_x h^{(1)}_{xy} h^{(2)}_{xy} h^{(3)}_{xy} \cdots h^{(2q-1)}_{xy} h^{(2q)}_{xy}.
\]

Now each basis amplitude \( \langle h^{2p} \rangle_{2p} \) — for which \( q \leq 2p \leq 2q \) if \( q \) is even or \( q + 1 \leq 2p \leq 2q \) if \( q \) is odd — will contribute to the \( 2n \)-point amplitude in accordance with

\[
\langle h^{2n} \rangle_{2p} = A_q 2^{q+2-2p} \left( \begin{array}{c} 2n \\ 2p \end{array} \right) \sum_{r=0}^{n-p} \left( \begin{array}{c} n + p - r + 1 \\ 2p - 1 \end{array} \right) \Theta(r) \sqrt{-g(g^{xx} g^{yy})^p}
\end{array} \right.

\times \left( C_{(1,2)} C_{(2q-1,2q)} \cdots \right) \left( \begin{array}{c} n \\ m+1 \end{array} \right) \left( h^{(2m-1)}_{y}(h^{(2m)}_{x}) \right),
\]

for all \( n \geq p \), and the combinatoric factors follow the same logic as in the analysis from Section 2.

Finally, we can use Eq. (42) to express the complete \( 2n \)-point for any Riem\( ^q \) theory gravity as the sum of contributions from the various basis amplitudes,

\[
\langle h^{2n} \rangle_{\text{Riem}^q} = \begin{cases} 
\sum_{m=0}^{q} \langle h^{2n} \rangle_{2m+q} & \text{with } q \text{ even and } 2n \geq 2m + q \text{ for every } m, \\
\sum_{m=0}^{q-1} \langle h^{2n} \rangle_{2m+q+1} & \text{with } q \text{ odd and } 2n \geq 2m + q + 1 \text{ for every } m.
\end{cases}
\]

(42)
It should again be emphasized that the validity of these results depends upon the restriction to the high-momentum regime.

4 Scattering properties of $2n$-point amplitudes

Our next order of business is to look at the scattering properties of some of these amplitudes. We will restrict to the cases with four gravitons (all of which are differentiated) as then the results can be expressed directly in terms of the familiar Mandelstam variables. However, it can be expected that the same basic theme — each theory carrying its own characteristic signature for scattering experiments in the high-momentum regime — will persist to more complicated scenarios.

Let us begin with the 4-point amplitude of Riem$^3$ gravity as depicted in Subsection 2.2. It is convenient to express the products of momenta in terms of the condensed notation $K_i \cdot K_j = (\omega_i g^{tt} \omega_j + k_i g^{zz} k_j)$; in which case,

$$\langle h^4 \rangle = -6(\alpha + \beta) \sqrt{-g}(g^{xx} g^{yy})^2 K_1 \cdot K_2 K_3 \cdot K_4 K_1 \cdot K_4 h^{(1)}_{xy} h^{(2)}_{xy} h^{(3)}_{xy} h^{(4)}_{xy}.$$  \hspace{1cm} (43)

It should be noted that the symmetrized product of momenta in Eq. (43) is really a shorthand for the symmetrization of all possible products in a particular way,

$$K_1 \cdot K_2 K_3 \cdot K_4 K_1 \cdot K_4 \rightarrow K_1 \cdot K_2 K_3 \cdot K_4 K_1 \cdot K_4 + K_1 \cdot K_2 K_3 \cdot K_4 K_1 \cdot K_2 + K_1 \cdot K_2 K_3 \cdot K_4 K_1 \cdot K_3 + K_1 \cdot K_2 K_3 \cdot K_4 K_2 \cdot K_3 + K_1 \cdot K_2 K_3 \cdot K_4 K_2 \cdot K_4 + K_1 \cdot K_2 K_3 \cdot K_4 K_3 \cdot K_4.$$  \hspace{1cm} (44)
To better appreciate Eq. (44), one can take note that each of the momenta has an equal opportunity of appearing either once or twice in any given permutation of the four gravitons.

Let us now recall the Mandelstam variables,

\[ s = 2K_1 \cdot K_2 = 2K_3 \cdot K_4, \]  
\[ t = -2K_1 \cdot K_3 = -2K_2 \cdot K_4, \]  
\[ u = -2K_1 \cdot K_4 = -2K_2 \cdot K_3, \]  

with \( s + t + u = 0 \) for this case of massless particles. In terms of its dependency on the Mandelstam variables, Eq. (43) can be expressed as

\[ \langle h^4 \rangle_{\text{Riem}^4} \propto (s - t - u)(s^2 + t^2 + u^2)h^{(1)}_{xy}h^{(2)}_{xy}h^{(3)}_{xy}h^{(4)}_{xy}. \]  

(45)

(46)

(47)

In the interest of making a connection with the gauge theory, it should be emphasized that Eq. (48) is only valid at the AdS boundary where \( g_{zz} = |g_{tt}| \) holds true. But, since the idea is to translate these expressions into statements in the dual gauge theory (as in [12]), the boundary limit is sufficient. Indeed, we expect to observe a related signature for the stress–energy correlators in the gauge theory. This is because of the observation in [9] that the amplitudes which survive in the high-momentum regime are mostly unaffected by the process of holographic renormalization.

We can similarly apply this process to the 4-point amplitude from Riem\(^4\) gravity, which leads to

\[ \langle h^4 \rangle_{\text{Riem}^4} \propto (s^4 + t^4 + u^4 + s^2t^2 + s^2u^2 + u^2t^2)h^{(1)}_{xy}h^{(2)}_{xy}h^{(3)}_{xy}h^{(4)}_{xy}. \]  

(48)

(49)
In view of Eqs. (48) and (49), two conclusions immediately follow: The first is that the two theories have very distinct scattering signatures and the second is that $s$, $t$ and $u$ appear democratically in both cases. The latter is a consequence of the high-momentum regime favoring no particular scattering channel. Meaning that, in general, this is also a regime of large-angle scattering.

5 Conclusion

In this paper, we have computed graviton multi-point scattering amplitudes for higher-derivative theories in an AdS black brane background. All computations were carried out in the so-called high-momentum regime as was first introduced in [9]. This regime allows for higher-curvature corrections to contribute significantly to higher-point amplitudes provided that $s \epsilon \lesssim 1$, where $\epsilon$ is the perturbative ($\alpha'$ or Regge slope) expansion parameter. Along with explicit calculations for six and eight-derivative theories, we were able to generalize the formalism to higher-derivative gravity of arbitrary order. A critical element of this generalization was the construction of a certain class of basis amplitudes.

We proceeded to demonstrate the scattering properties of Riem$^3$ and Riem$^4$ gravity in terms of the Mandelstam variables $s$, $t$ and $u$ by using their respective 4-point amplitudes. Our expectation is that this procedure can be generalized to higher-point scattering amplitudes and higher-derivative theories with some amount of work.

The graviton multi-point amplitudes in this paper should correspond to stress–tensor correlators in the gauge theory. This means,
for instance, that the stress–energy tensor 4-point correlators should include, in addition to the distinct signatures of Einstein gravity and four-derivative gravity \[9, 12\], those of six and eight-derivative gravity as depicted in Eqs. (48) and (49). Note, however, that, to make contact with actual experiments, it is the connected functions in the gauge theory that are required, whereas the amplitudes in this paper would correspond to 1PI functions. This point will be addressed at a later time \[16\].

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References

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Int. J. Theor. Phys. 38, 1113 (1999) [Adv. Theor. Math. Phys. 2, 231 (1998)] doi:10.1023/A:1026654312961 [hep-th/9711200].

[2] D. M. Hofman and J. Maldacena, “Conformal collider physics: Energy and charge correlations,” JHEP 0805, 012 (2008) doi:10.1088/1126-6708/2008/05/012 [arXiv:0803.1467 [hep-th]].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [hep-th/9802150].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323, 183 (2000) doi:10.1016/S0370-1573(99)00083-6 [hep-th/9905111].

[5] C. Cheung and G. N. Remmen, “Positivity of Curvature-Squared Corrections in Gravity,” Phys. Rev. Lett. 118, no. 5, 051601 (2017) doi:10.1103/PhysRevLett.118.051601 [arXiv:1608.02942 [hep-th]].

[6] S. Deser, “One-loop gravity divergences in $D > 4$ cannot all be removed,” Gen. Rel. Grav. 48, no. 12, 157 (2016) doi:10.1007/s10714-016-2151-1 [arXiv:1609.04432 [gr-qc]].

[7] G. ’t Hooft, “An algorithm for the poles at dimension four in the dimensional regularization procedure,” Nucl. Phys. B 62, 444 (1973) doi:10.1016/0550-3213(73)90263-0.
[8] M. D. Pollock, “On the application of the field-redefinition theorem to the heterotic superstring theory,” Eur. Phys. J. Plus 130, no. 5, 87 (2015) doi:10.1140/epjp/i2015-15087-3.

[9] R. Brustein and A. J. M. Medved, “Graviton n-point functions for UV-complete theories in Anti-de Sitter space,” Phys. Rev. D 85, 084028 (2012) doi:10.1103/PhysRevD.85.084028 [arXiv:1202.2221 [hep-th]].

[10] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal and U. A. Wiedemann, “Gauge/String Duality, Hot QCD and Heavy Ion Collisions,” book:Gauge/String Duality, Hot QCD and Heavy Ion Collisions. Cambridge, UK: Cambridge University Press, 2014 doi:10.1017/CBO9781139136747 [arXiv:1101.0618 [hep-th]].

[11] K. Skenderis, “Lecture notes on holographic renormalization,” Class. Quant. Grav. 19, 5849 (2002) doi:10.1088/0264-9381/19/22/306 [hep-th/0209067].

[12] R. Brustein and A. J. M. Medved, “Graviton multipoint functions at the AdS boundary,” Phys. Rev. D 87, no. 2, 024005 (2013) doi:10.1103/PhysRevD.87.024005 [arXiv:1211.0109 [hep-th]].

[13] R. Brustein and A. J. M. Medved, “Universal stress-tensor correlation functions of strongly coupled conformal fluids,” Phys. Lett. B 724, 144 (2013) doi:10.1016/j.physletb.2013.06.002 [arXiv:1207.5388 [hep-th]].

[14] G. Policastro, D. T. Son and A. O. Starinets, “From AdS /
CFT correspondence to hydrodynamics,” JHEP **0209**, 043 (2002) doi:10.1088/1126-6708/2002/09/043 [hep-th/0205052].

[15] G. ’t Hooft and M. J. G. Veltman, “One loop divergencies in the theory of gravitation,” Ann. Inst. H. Poincare Phys. Theor. **A20**, 69-94 (1974).

[16] M. M. W. Shawa and A. J. M. Medved, *work in progress.*