On some Limit Theorem for Markov Chain

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Abstract

The goal of this paper is to describe conditions which guarantee a central limit theorem for random variables, which distributions are controlled by hidden Markov chains. We proved that when a Markov chain is ergodic and random variables fulfill Lindeberg’s condition then the Central Limit Theorem is true.

Keywords: Regime switching models, Central Limit Theorem

1. Introduction

Regime switching models have been used extensively in econometric time series analysis. In most of these models, two regimes are introduced with a state process determining one of the regimes to take place in each period. The bivalued state process is typically modeled as a Markov chain. The autoregressive model with this type of Markov switching was first considered by Hamilton (1989), and later analyzed by Kim (1994). Markov-switching models with endogenous explanatory variables have been considered by Kim (2004, 2009). The most authors assume that the Markov chain, which determine regimes, is completely independent from all other parts of the model. Diebold et al. (1994) and Kim (2009) considers a Markov-switching driven by a set of observed variables. Chang et al. (2016) introduces a new approach to model regime switching using an autoregressive latent factor, which determines
regimes depending upon whether it takes a value above or below some threshold level.

Despite numerous generalizations of this type of models, there is still little known about their theoretical properties. For example, one of the problems is the likelihood ratio test and other tests for comparing two regime switching models, the second is the indication of regularity conditions for the efficiency of maximum likelihood estimator of unknown model parameters. Various statistical properties of the model have been studied by Hansen (1992), Hamilton (1996), Garcia (1998), Timmermann (2000), and Cho and White (2007), among others. The overview of the literature is in monograph by Kim and Nelson (1999). In order to solve many problems related to testing hypothesis or some estimator efficiency, it is enough to prove Central Limit Theorem. In regime switching models the random variables, which distributions are controled by hidden Markov chains we have dependent variables.

The central limit theorem has been extended to the case of dependent random variables by several authors. The conditions under which these theorems are stated either are very restrictive or involve conditional distributions, which makes them difficult to apply. Hoeffding and Robbins (1994) prove central limit theorems for sequences of dependent random variables of a certain special type which occurs frequently in mathematical statistics.

In this paper we prove Central Limit Theorem for random variables, which distributions are controled by hidden Markov chains. We prove that when a Markov chain is ergodic and random variables fullfiled Lindeberg’s condition then the Central Limit Theorem is true.

2. Asymptotic independence

Let consider a process:

$$(S_t, X_t)_{t=0}^\infty$$

where

- $S_t$ is an unobservable hidden Markow chain with $N$ states;
• realizations of process \(X_t\) are observed;

• the distribution of \(X_t\) conditional on history \(\mathcal{R}_{t-1} = (x_0, \ldots, x_{t-1})\) has a form:

\[
f(x_t \mid \mathcal{R}_{t-1}; \theta) = \sum_{j=1}^{l} f(x_t \mid S_t = j, \mathcal{R}_{t-1}; \theta) P(S_t = j \mid \mathcal{R}_{t-1}; \theta).
\]

We prove the following lemma:

**Lemma 1.** Let consider a process \((S_t, X_t)\) where the conditional distribution of \(X_t \mid \mathcal{R}_{t-1}\) is defined as (2). Let assume that \(S_t\) is an ergodic process. Then the random variables \(\{X_t\}\) are asymptotically independent, i.e.

\[
\lim_{s_1, \ldots, s_k \to \infty} \left| P\left( \bigcap_{\nu=1}^{k} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) - \prod_{\nu=1}^{k} P(X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}) \right| = 0
\]

**Proof:** At first we estimate the difference

\[
\left| P\left( \bigcap_{\nu=1}^{k} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) - \prod_{\nu=1}^{k} P(X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}) \right| \leq P\left( \bigcap_{\nu=1}^{k} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) - P\left( \bigcap_{\nu=1}^{k-1} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) P(X_{t+\sum_{\rho=0}^{k} s_{\rho}} \in A_{\rho}) + \left| \prod_{\nu=1}^{k} P(X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}) \right| - \left| \prod_{\nu=1}^{k-1} P(X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}) \right| = I_1 + I_2
\]

\[
I_1 = \left| P\left( \bigcap_{\nu=1}^{k} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) - P\left( \bigcap_{\nu=1}^{k-1} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) P(X_{t+\sum_{\rho=0}^{k} s_{\rho}} \in A_{\rho}) \right|
\]

\[
= \left| P\left( \bigcap_{\nu=1}^{k} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) - P\left( \bigcap_{\nu=1}^{k-1} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) P(X_{t+\sum_{\rho=0}^{k} s_{\rho}} \in A_{\rho}) \right| - P\left( \bigcap_{\nu=1}^{k} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) P\left( \bigcap_{\nu=1}^{k-1} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right)
\]

\[
\leq \left| P\left( \bigcap_{\nu=1}^{k} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) - P\left( \bigcap_{\nu=1}^{k-1} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) \right| - P\left( \bigcap_{\nu=1}^{k} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right) P\left( \bigcap_{\nu=1}^{k-1} \{X_{t+\sum_{\rho=0}^{\nu} s_{\rho}} \in A_{\rho}\} \right)
\]

Denote

\[
T = t + \sum_{\rho=0}^{k-1} \tau_{\rho}
\]
• \( \tau = \tau_k \)
• \( A = A_k \)
• \( B = A_{k-1} \)

We notice, that

\[
P(X_{T+\tau} \in A, X_T \in B) = P(X_{T+\tau} \in A|X_T \in B)P(X_T \in B)
\]

Moreover for \( A = \{s_1, \ldots, s_p\} \times \mathfrak{A}, B = \{s_0\} \times \mathfrak{B} \)

\[
P(X_{T+\tau} \in A|X_T \in B) = \sum_{i=1}^p p_{s_0, s_i}(\tau) \int_{\mathfrak{A}} f_i(x)dx \quad (4)
\]

and

\[
P(X_{T+\tau} \in A) = \sum_{\sigma \in \mathcal{S}} P(X_{T+\tau} \in A|X_T \in \{\sigma\} \times \mathbb{R}^N)P(X_T \in \{\sigma\} \times \mathbb{R}^N)
\]

\[
= \sum_{\sigma \in \mathcal{S}} \sum_{i=1}^p p_{\sigma, s_i}(\tau) \int_{\mathfrak{A}} f_i(x)dxP(X_T \in \{\sigma\} \times \mathbb{R}^N) \quad (5)
\]

From the ergodic theorem follows, that

\[
\lim_{s \to \infty} p_{ij}(s) = p^*_j(s)
\]

and

\[
|p_{ij}(s) - p^*_j(s)| \leq \alpha^s
\]

for some \( \alpha < 1 \).

From (5), (4) and the obvious equality

\[
\sum_{\sigma \in \mathcal{S}} P(X_T \in \{\sigma\} \times \mathbb{R}^N) = 1
\]

follows, that

\[
P(X_{T+\tau} \in A|X_T \in B) - P(X_{T+\tau} \in A)
\]

\[
= \sum_{i=1}^p p_{s_0, s_i}(\tau) \int_{\mathfrak{A}} f_i(x)dx - \sum_{\sigma \in \mathcal{S}} \sum_{i=1}^p p_{\sigma, s_i}(\tau) \int_{\mathfrak{A}} f_i(x)dxP(X_T \in \{\sigma\} \times \mathbb{R}^N)
\]

\[
= \sum_{i=1}^p [p_{s_0, s_i}(\tau) - p^*_i(\tau)] \int_{\mathfrak{A}} f_i(x)dx + \sum_{\sigma \in \mathcal{S}} \sum_{i=1}^p [p^*_i(\tau) - p_{\sigma, s_i}(\tau)] \int_{\mathfrak{A}} f_i(x)dxP(X_T \in \{\sigma\} \times \mathbb{R}^N)
\]
and in consequence
\[ |P(X_{T+\tau} \in A|X_T \in B) - P(X_{T+\tau} \in A)| \leq 2\alpha^r \]

Hence \( I_1 \leq 2\alpha^r \). Now, notice, that

\[ I_2 = \left| P\left( \left\{ X_{t+k}+\sum_{\rho=0}^{k} \tau_{\rho} \in A_{\rho} \right\} \right) \right| P\left( \bigcap_{\mu=1}^{k-1} \left\{ X_{t+\sum_{\nu=0}^{\nu} \tau_{\nu} \in A_{\nu} \right\} \right) - \prod_{\mu=1}^{k-1} P\left( X_{t+\sum_{\nu=0}^{\nu} \tau_{\nu} \in A_{\nu} \right) \right| \leq \left| P\left( \bigcap_{\nu=1}^{k-1} \left\{ X_{t+\sum_{\rho=0}^{\rho} \tau_{\rho} \in A_{\rho} \right\} \right) - \prod_{\nu=1}^{k-1} P\left( X_{t+\sum_{\rho=0}^{\rho} \tau_{\rho} \in A_{\rho} \right) \right| \]

By simple induction we can conclude, that

\[ \left| P\left( \bigcap_{\nu=1}^{k} \left\{ X_{t+\sum_{\rho=0}^{\rho} \tau_{\rho} \in A_{\rho} \right\} \right) - \prod_{\nu=1}^{k} P\left( X_{t+\sum_{\rho=0}^{\rho} \tau_{\rho} \in A_{\rho} \right) \right| \leq k\alpha \sum_{\rho=0}^{k} \tau_{\rho} \]

which completes the proof.

3. Property of \( \varepsilon \)-independence

Next, let define the notion of \( \varepsilon \)-independence. This concept will be useful to prove central limit theorem.

**Definition 1.** The random variables sequence \( \{X_k\}_{k \in \mathbb{N}} \) are \( \varepsilon \)-independent, when for any \( n \in \mathbb{N} \) and any sets \( A_1, \ldots, A_n \) we have the following inequality:

\[ |P(X_1 \in A_1, \ldots, X_n \in A_n) - P(x_1 \in A_1) \ldots P(x_n \in A_n)| \leq \varepsilon. \quad (6) \]

We prove, that for \( \varepsilon \)-independent variables the following lemma is true.

**Lemma 2.** When random variables \( X_1, \ldots, X_n \) are \( \varepsilon \)-independent, then

\[ |\varphi_{X_1+\ldots+X_n}(t) - \varphi_{X_1}(t) \ldots \varphi_{X_n}(t)| \leq 2\varepsilon. \]

**Proof:** From the formula (6) follows, that

\[ |E(\mathbb{1}_{A_1}(X_1) \ldots \mathbb{1}_{A_n}(x_n)) - E\mathbb{1}_{A_1}(X_1) \ldots E\mathbb{1}_{A_n}(x_n)| \leq \varepsilon. \]
Since every continuous function can be approximated by simple functions consider at first the real function

\[ f = \sum_{j=1}^{m} c_j \mathbb{1}_{A_j}, \quad (7) \]

where the sets \( A_j \) are pairwise disjoint and \( |c_j| \leq 1 \). In the first step, we estimate the real part of \((Ef(X_1) \cdots f(X_n) - Ef(X_1) \cdots Ef(X_n))\). Since \( |c_j| \leq 1 \) we get that:

\[
\Re(Ef(X_1) \cdots f(X_n) - Ef(X_1) \cdots Ef(X_n)) \\
= \sum_{j_1, j_2, \ldots, j_n=1}^{n} \Re(c_{j_1} c_{j_2} \cdots c_{j_n}) [P(X_1 \in A_{j_1}, \ldots, X_n \in A_{j_n}) - P(X_1 \in A_{j_1}) \cdots P(X_n \in A_{j_n})] \\
\leq \sum_{j_1, j_2, \ldots, j_n=1}^{n} [P(X_1 \in A_{j_1}, \ldots, X_n \in A_{j_n}) - P(X_1 \in A_{j_1}) \cdots P(X_n \in A_{j_n})].
\]

Since the sets \( A_j \) are pairwise disjoint then:

\[
\sum_{j_1, j_2, \ldots, j_n=1}^{n} P(X_1 \in A_{j_1}, \ldots, X_n \in A_{j_n}) = P(X_1 \in \mathbb{A}, \ldots, x_n \in \mathbb{A}), \\
\sum_{j=1}^{m} P(X_k \in A_j) = P(X_k \in A),
\]

where

\[
A = \bigcup_{j=1}^{m} A_j.
\]

From it follows that

\[
\Re(Ef(X_1) \cdots f(X_n) - Ef(X_1) \cdots Ef(X_n)) \leq P(X_1 \in \mathbb{A}, \ldots, x_n \in \mathbb{A}) - P(X_1 \in \mathbb{A}) \cdots P(X_n \in \mathbb{A}) \leq \varepsilon.
\]

Analogously

\[
\Re(Ef(X_1) \cdots f(X_n) - Ef(X_1) \cdots Ef(X_n)) \geq -P(X_1 \in \mathbb{A}, \ldots, x_n \in \mathbb{A}) + P(X_1 \in \mathbb{A}) \cdots P(X_n \in \mathbb{A}) \geq -\varepsilon
\]

In the second step, we estimate the imaginary part of \((Ef(X_1) \cdots f(X_n) - Ef(X_1) \cdots Ef(X_n))\).

So, we have:

\[
\Im(Ef(X_1) \cdots f(X_n) - Ef(X_1) \cdots Ef(X_n)) \leq P(X_1 \in \mathbb{A}, \ldots, x_n \in \mathbb{A}) - P(X_1 \in \mathbb{A}) \cdots P(X_n \in \mathbb{A}) \leq \varepsilon
\]

\[
\Im(Ef(X_1) \cdots f(X_n) - Ef(X_1) \cdots Ef(X_n)) \geq -P(X_1 \in \mathbb{A}, \ldots, x_n \in \mathbb{A}) + P(X_1 \in \mathbb{A}) \cdots P(X_n \in \mathbb{A}) \geq -\varepsilon.
\]
In consequence we obtain:

\[ |Ef(X_1) \cdots f(X_n) - Ef(X_1) \cdots Ef(X_n)| \leq \sqrt{2}\varepsilon. \]

The last inequality is true for any \( X_1, \ldots, X_n \) for any \( n \) and for any \( f \) Nour, let fix \( t \in \mathbb{R} \), \( n \) and \( \eta > 0 \). Let define the function \( f_{\eta} \) which can be presented as \( \mathbb{C} \) and satisfying the inequality

\[ |f_{\eta}(x) - e^{itx}| \leq \eta \]

for every \( x \in \mathbb{R} \). Let estimate the difference

\[ \varphi_{X_1 + \ldots + X_n}(t) - Ef_{\eta}(X_1) \cdots f_{\eta}(X_n) \]

This difference is equal to

\[ Ee^{itX_1} \cdots e^{itX_n} - Ef_{\eta}(X_1) \cdots f_{\eta}(X_n) = E(e^{itX_1} - f_{\eta}(X_1))e^{itX_2} \cdots e^{itX_n} + Ef_{\eta}(X_1)(e^{itX_2} - f_{\eta}(X_2))e^{itX_3} \cdots e^{itX_n} + \ldots \]

Each component of the sum \( \mathbb{C} \) has the form \( EZ(e^{itX_k} - f_{\eta}(X_k)) \) where \( |Z| \leq 1 \).

From this fact implies that

\[ |Ee^{itX_1} \cdots e^{itX_n} - Ef_{\eta}(X_1) \cdots f_{\eta}(X_n)| \leq n\eta. \]

By conducting a similar reasoning, we obtain that:

\[ |Ee^{itX_1} \cdots e^{itX_n} - E(f_{\eta}(X_1) \cdots f_{\eta}(X_n))| \leq n\eta. \]

From it follows, that

\[ |Ee^{itX_1} \cdots e^{itX_n} - Ee^{itX_1} \cdots e^{itX_n}| \]

\[ \leq |Ee^{itX_1} \cdots e^{itX_n} - E(f_{\eta}(X_1) \cdots f_{\eta}(X_n))| + Ef(X_1) \cdots f(X_n) - Ef(X_1) \cdots Ef(X_n) + Ee^{itX_1} \cdots e^{itX_n} - Ef_{\eta}(X_1) \cdots f_{\eta}(X_n)| \]

\[ \leq \sqrt{2}\varepsilon + 2n\eta \]

Choosing such \( \eta \) that \( 2n\eta \leq 2\varepsilon - \sqrt{2}\varepsilon \) we complete the proof.
4. Central Limit Theorem

Using the thesis of the above lemma, one can prove a central limit theorem for \( \varepsilon \)-independent variables.

**Theorem 1.** Let assume that for each \( n \), \( X_{1n}, X_{2n}, \ldots, X_{rn} \) are \( \varepsilon \)-independent random variables with expected value equal 0 and:

\[
\sum_{k=1}^{rn} \mathbb{E}X_{kn}^2 \xrightarrow{n \to \infty} 1.
\]

(10)

Additionally, let us assume that Lindeberg’s condition is fulfilled:

\[
\sum_{k=1}^{rn} \mathbb{E}X_{kn}^2 \mathbb{1}_{\{|X_{kn}| > \eta\}} \xrightarrow{n \to \infty} 0 \quad \text{for each } \eta > 0.
\]

Let \( Y_n \) be a sum:

\[
= X_{1n} + X_{2n} + \ldots + X_{rn}.
\]

Then

\[
\limsup |\varphi_{Y_n}(t) - e^{-\frac{1}{2}t^2}| < \varepsilon.
\]

**Proof:** For Lindeberg Theorem (Loeve, 1977) implies that:

\[
\lim_{n \to \infty} \varphi_{X_{1n}}(t) \ldots \varphi_{X_{rn}}(t) = e^{-\frac{1}{2}t^2}
\]

From last Lemma implies that:

\[
|\varphi_{Y_n}(t) - \varphi_{X_{1n}}(t) \ldots \varphi_{X_{rn}}(t)| < \varepsilon.
\]

Now, our goal is to prove the fact that asymptotically independent variables also satisfy the Central Limit Theorem.

**Theorem 2.** Assume, that

1. Random Variables \( X_1, \ldots, X_n \) are asymptotically independent i.e., they satisfy [3].
2. \( EX_i = 0 \) for \( i = 1, \ldots, n. \)
3. \[
E \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{i+k} \right)^2 \to 1 \text{ uniformly in } i.
\]

(11)
Then, the sequence of the distributions of the random variables
\[ \frac{1}{\sqrt{n}} (X_1 + \cdots + X_n) \]
tend to the standard normal distribution.

**Proof:** At first we fix some arbitrary \( \varepsilon > 0 \). From (3) follows, that there exists such \( m \), that
\[ |P(X_{k_1} \in A_1, \ldots, X_{k_r} \in A_r) - P(X_{k_1} \in A_1) \cdots P(x_{k_r} \in A_r)| \leq \varepsilon, \]
where \( k_{j+1} > k_j + m \).

Fix \( n \) and let \( 0 < \alpha < \frac{1}{4} \). Let define \( k = \lfloor n^\alpha \rfloor \) and \( \nu = \lceil \frac{k}{n} \rceil \), so clearly \( k \leq n^\alpha \) and \( n = k\nu + r \).

Let define:
\[ U_i = X_{ik-k+1} + \cdots + X_{ik-m}. \]
Because \( (i+1)k-k+1 - (ik-m) = m+1 \) so \( U_i \) are \( \varepsilon \)-independent. Let us consider a sum:
\[ X_1 + \cdots + X_n. \]
This sum we can separate into two parts: a sum of \( U_1 + \ldots, U_\nu \) and the rest. We can notice that each component \( U_i \) includes \( k+1-m \) elements, so the sum \( U_1 + \ldots + U_\nu \) consists of \( (k+1-m)\nu = n - r - (m-1)\nu \) elements. So the rest includes \( r + (m-1)\nu \) ingredients, which we denote as \( Z_1^{(n)}, \ldots, Z_p^{(n)} \), for fixed \( n \). So:
\[ X_1 + \cdots + X_n = U_1 + \ldots + U_\nu + Z_1^{(n)} + \ldots Z_p^{(n)} = \sqrt{n}U_\nu + \sqrt{n}3_n. \]

From Schwarz and Hölder inequality (Vuong, 1989) implies that:
\[ E(Z_1^{(n)} + \ldots + Z_p^{(n)})^2 \leq \rho^2 R^2, \]
where \( R = E|X_i|^3 \), so:
\[ E\left( \frac{1}{\sqrt{n}}(Z_1^{(n)} + \ldots + Z_p^{(n)}) \right)^2 \leq \frac{\rho^2}{n} R^2. \]
From a fact that \( p = r + (m-1)\nu \leq m\nu \leq m \frac{k}{n} \leq m \frac{n^\alpha}{n} \) implies that \( \frac{\rho^2}{n} \leq m^2 \frac{n^{3\alpha}}{n^\gamma} \).
So, the sum
\[ \sum_{n=1}^{\infty} \frac{\rho^2}{n} R^2 \]
is consistent, so with probability 1, we have:

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} (Z^{(n)}_1 + \ldots + Z^{(n)}_p) = 0.$$  \hspace{1cm} (12)

Now, we need estimate the

$$\frac{1}{\sqrt{\nu}} (U^{(n)}_1 + \ldots + U^{(n)}_\nu)$$

$$\frac{1}{\sqrt{n}} (U^{(n)}_1 + \ldots + U^{(n)}_\nu) = \frac{\sqrt{k\nu}}{\sqrt{n}} \frac{1}{\sqrt{\nu}} (U^{(n)}_1 + \ldots + U^{(n)}_\nu) = \frac{\sqrt{k\nu}}{\sqrt{n}} \left( \frac{1}{\sqrt{k}} U^{(n)}_1 + \ldots + \frac{1}{\sqrt{k}} U^{(n)}_\nu \right).$$

Moreover

$$U^{(n)}_i = X_{ik-k+1} + \ldots + X_{ik-m},$$

from which follows, that

$$E \left( \frac{1}{\sqrt{k}} U^{(n)}_i \right)^2 = \frac{k-m}{k} E \left( \frac{1}{\sqrt{k-m}} U^{(n)}_i \right)^2.$$

From the assumption (11) follows, that

$$\lim_{k \to \infty} E \left( \frac{1}{\sqrt{k}} U^{(n)}_i \right)^2 = 1.$$

In consequence we obtain that:

$$\lim_{n \to \infty} \frac{1}{\nu} \left( \sum_{i=1}^{\nu} E \left( \frac{1}{\sqrt{k}} U^{(n)}_i \right)^2 \right) = 1.$$

Having above estimates we go to the next step of the proof. We have to estimate the difference:

$$\left| \varphi_{\mathcal{H}}(X_1 + \ldots + X_n)(t) - e^{-\frac{t^2}{2}} \right|. \hspace{1cm} (13)$$

So,

$$\left| \varphi_{\mathcal{H}}(X_1 + \ldots + X_n)(t) - e^{-\frac{t^2}{2}} \right| = \left| \varphi_{\mathcal{H}}(X_1 + 3_n)(t) - e^{-\frac{t^2}{2}} \right| =$$

$$= \left| \varphi_{\mathcal{H}}(X_1 + 3_n)(t) - \varphi_{\mathcal{H}}(t) + \varphi_{\mathcal{H}}(t) - e^{-\frac{t^2}{2}} \right| \leq \left| \varphi_{\mathcal{H}}(X_1 + 3_n)(t) - \varphi_{\mathcal{H}}(t) \right| + \left| \varphi_{\mathcal{H}}(t) - e^{-\frac{t^2}{2}} \right|$$

$$\leq E \left| e^{it(3_n)} - e^{it} \right| + \left| \varphi_{\mathcal{H}}(t) - e^{-\frac{t^2}{2}} \right| = E \left| e^{it(3_n)} (e^{it} - 1) \right| + \left| \varphi_{\mathcal{H}}(t) - e^{-\frac{t^2}{2}} \right|$$

$$\leq E \left| e^{it3_n} - 1 \right| + \left| \varphi_{\mathcal{H}}(t) - e^{-\frac{t^2}{2}} \right|$$
From (12) follows, that
\[
\lim_{n \to \infty} E \left| e^{\mu n} - 1 \right| = 0
\]
and from lemma follows, that
\[
\limsup_{n \to \infty} \left| \phi_U n(t) - e^{-\frac{t^2}{2}} \right| < 2\varepsilon
\]
Then
\[
\limsup_{n \to \infty} \left| \phi \frac{1}{\sqrt{n}}(X_1 + \ldots + X_n)(t) - e^{-\frac{t^2}{2}} \right| \leq 2\varepsilon
\]
Since the last estimation can be proved for any \( \varepsilon \)
\[
\lim_{n \to \infty} \left| \phi \frac{1}{\sqrt{n}}(X_1 + \ldots + X_n)(t) - e^{-\frac{t^2}{2}} \right| = 0
\]
which completes the proof.

Conclusion

In this paper we prove a Central Limit Theorem. The assumptions to this theorem is not restrictive and they are not difficult to apply in the practical. We proved that when a Markov chain is ergodic and random variables fulfilled Lindeberg’s condition then the Central Limit Theorem is true.

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