Point Interaction Controls for the Energy Transfer in 3-D Quantum Systems.

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Abstract

We consider the problem of energy-mass transfer from scattering to bound states for a one body quantum system subject to the action of a time dependent point interaction in 3-D. Under suitable assumptions on the initial state of the particle, we prove a result of local controllability of this process. Our proof exploits the finite time asymptotic analysis of fractional integral equations and the rank theorem for maps defined on Banach spaces.

1 Introduction

A perturbation of the Laplacian supported by a finite set of points \( \{y_i\}_{i=1}^n \) in \( \mathbb{R}^d \) - with \( d \leq 3 \) - defines a special case of singular perturbation referred to as point interaction. At a formal level, the associated Schrödinger operator can be written as:

\[
H = -\Delta + \sum_{i=1}^{n} \alpha_i \delta(x - y_i)
\]  

(1)

These operators appeared first in Theoretical Physics during the 30's. They were introduced in order to realize a model for the interaction of nucleons at low energy [4]. After, they became a natural tool to describe short range forces or "small" obstacles for scattering of waves and particles.

From the point of view of applications, the main reason of interest of this subject rests upon the fact that point interactions often lead to models which are explicitly solvable. It turns out that the spectral characteristics (eigenvalues and eigenfunctions) of operators (1), and then all the physical relevant quantities related to, can be explicitly computed [5]. This circumstance motivates an increasing attention on the application of point interaction models in various sciences, e.g. in physics, chemistry, biology, and in technology.

In this work we build up a point interaction model of a time dependent Schrödinger operator; this interaction will be used as a control for the energy transfer between continuous and discrete spectrum of a one body quantum system. We will investigate the possibility of finding a time dependence profile such that a prescribed part of the energy of a particle, initially placed in a scattering state, moves on a bound state in finite time.

The subject we treat has its natural collocation in the framework of quantum systems control theory. Our analysis may find applications in those contexts where short range quantum potentials can be used as control tools.

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2 Point Interaction Model for Quantum Control Potentials and the Main Result

The rigorous definition of point interactions in Quantum Mechanics - due to F.A. Berezin and L.D. Fadeev [3] - rests upon the theory of selfadjoint extensions of symmetric operators. The Hamiltonians describing point interactions in the origin of $\mathbb{R}^3$ are defined by the selfadjoint extensions of the symmetric operator

$$
\begin{align*}
H &= -\triangle \\
D(H) &= C^\infty(\mathbb{R}^3 \setminus \{0\})
\end{align*}
$$

In [5] it has been shown that these extensions, denoted in the following with $H_\alpha$, are parametrized by a real $\alpha$; for any fixed $\lambda \in \mathbb{R}^+$, the operator $H_\alpha$ can be represented as

$$
\begin{align*}
D(H_\alpha) &= \left\{ \psi \in L^2(\mathbb{R}^3) \mid \psi = \phi^\lambda + qG^\lambda; \phi^\lambda \in H^2(\mathbb{R}^3); \phi^\lambda(0) = q(\alpha + \sqrt{\lambda})^2 \right\} \\
(H_\alpha + \lambda)\psi &= (-\Delta + \lambda)\phi^\lambda 
\end{align*}
$$

where $G^\lambda$ is the Green function of $(-\Delta + \lambda)$

$$
G^\lambda(x) = \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|}
$$

As it follows from [2], every function in $D(H_\alpha)$ is composed of a regular part $\phi^\lambda \in H^2$ plus a singular term $G^\lambda$. It is worthwhile to notice that this representation is not unique, but depends on the choice of $\lambda$. In particular, for $\alpha < 0$, we can fix: $\lambda = (4\pi|\alpha|)^2$, obtaining a null boundary condition: $\phi^\lambda(0) = q(\alpha + \sqrt{\lambda}) = 0$. In this case the operator domain writes as

$$
D(H_\alpha) = \left\{ \psi \in L^2(\mathbb{R}^3) \mid \psi = \phi + \frac{q}{4\pi\sqrt{2|\alpha|}}\psi_\alpha, \phi \in H^2(\mathbb{R}^3), \phi(0) = 0, q \in \mathbb{C} \right\}
$$

where $\psi_\alpha$ is the normalized Green function

$$
\psi_\alpha = \frac{\sqrt{2|\alpha|}e^{-4\pi|\alpha|\frac{|x|}{4\pi}}}{|x|}
$$

This representation, holding for negative values of the parameter $\alpha$, will be extensively used in this work.

We recall the spectral properties of $H_\alpha$: the absolute continuous spectrum coincides, for any $\alpha \in \mathbb{R}$, with the set $(0, +\infty)$; the point spectrum is empty for $\alpha \geq 0$, or it contains a single eigenvalue: $\lambda_\alpha = -16\pi^2\alpha^2$, for $\alpha < 0$. The related eigenstate is $\psi_\alpha$, defined in [5].

A time dependent point interaction Hamiltonian $H_{\alpha(t)}$ is defined by fixing a real valued function, $\alpha(t)$, which characterizes the time dependence profile. The Schrödinger equation

$$
\begin{align*}
i\frac{d}{dt}\psi &= H_{\alpha(t)}\psi \\
\psi(0) &= \psi_0(x) \\
\psi(x, t) &\in D(H_{\alpha(t)})
\end{align*}
$$

1From the physical point of view, the parameter $\alpha$ is linked to the inverse scattering length of $H_\alpha$. [3].
2For the definitions and the properties concerning point interactions operators, we refer to the book [5].
describes the quantum dynamics generated by \( H_{\alpha(t)} \). In [4], D.R. Yafaev has provided an explicit expression for the time propagator associated to \( H_{\alpha(t)} \) (equation (2.6) in the cited paper)

\[
\left\{ \begin{array}{l}
\psi(t, x) = U_t \psi_0(x) + i \int_0^t U(t - s, x)q(s)ds \\
q(t) + 4\sqrt{\pi i} \int_0^t \frac{\alpha(s)q(s)}{\sqrt{t-s}}ds = 4\sqrt{\pi i} \int_0^t U_s \psi_0(0)ds
\end{array} \right. \tag{7}
\]

with

\[
U_t \psi(x) = \int_{R^3} U(t, y - x) \psi(y) dy; \quad U(t, x) = \frac{1}{(4\pi it)^{3/2}} e^{\frac{x^2}{4it}} \tag{8}
\]

Given \( \alpha \in \mathcal{L}^\infty_{loc}(0, \infty) \), the map \( \{7\} \), acting on the initial state \( \psi_0 \), is unitary in \( L^2(R^3) \) and defines, in the weak sense, the solution of (6) at time \( t \) (see [6] and references therein). The auxiliary variable \( q(t) \), solution of the second equation in \( \{7\} \), is usually referred to as the charge associated to the particle. It expresses the coefficient of the singular part of the state \( \psi(x, t) \). We refer to the Appendix A for a study of the charge equation.

Let us fix, from now on, \( \bar{\alpha} < 0 \). We assume, at \( t = 0 \), the particle to be placed in a scattering state of the Hamiltonian \( \bar{H}_\alpha \)

\[
\psi_0 = \phi \in D(H_{\bar{\alpha}}) : (\phi, \psi_{\bar{\alpha}})_{L^2} = 0 \tag{9}
\]

Our purpose is to find a suitable \( \alpha(t) \) in the space of control functions

\[
\alpha \in H^1_0(0, T) \tag{10}
\]

such that the control interaction, \( H_{\alpha(t) + \bar{\alpha}} \), is able to steer the system \( \{7\} \) from \( \psi_0 \) to a state, \( \psi(T) \), whose projection \( (\psi(T), \psi_{\bar{\alpha}}) \) has a previously fixed value in a neighborhood of the origin of \( \mathbb{C} \).

Replacing \( \alpha \) with \( \alpha + \bar{\alpha} \) in \( \{7\} \) and projecting along \( \psi_{\bar{\alpha}} \), we get a complex valued map

\[
\left\{ \begin{array}{l}
F(\alpha) := (U_T \phi + i \int_0^T U(T - s, \cdot)V(\alpha)(s)ds, \psi_{\bar{\alpha}})_{L^2(R^3)} \\
V(\alpha) = q(t) : q(t) + 4\sqrt{\pi i} \int_0^t \frac{\alpha(s)q(s)}{\sqrt{t-s}}ds = 4\sqrt{\pi i} \int_0^t U_s \phi(0)ds
\end{array} \right. \tag{11}
\]

For \( \alpha = 0 \), the system evolves under the action of \( H_{\bar{\alpha}} \); in this case we have

\[
F(0) = (e^{-iT\bar{H}_\alpha} \phi, \psi_{\bar{\alpha}}) = e^{iT\lambda_\alpha} (\phi, \psi_{\bar{\alpha}}) = 0
\]

The condition for the solvability of our problem results to be the local surjectivity of the map

\[
\left\{ \begin{array}{l}
F(\alpha) = z \in \mathbb{C} \\
\alpha \in H^1_0(0, T)
\end{array} \right. \tag{12}
\]

around the point \( \alpha = 0 \); i.e. we aim to prove that there exists a neighborhood of the origin in \( \mathbb{C} \), \( I_0 \), and a \( C^1 \)-class map \( g : I_0 \to H^1_0(0, T) \) such that

\[
F(g(z)) = z \quad \forall z \in I_0 \tag{13}
\]

The main goal of this work is to demonstrate the following result

\[\text{The boundary conditions on } \alpha \text{ being chosen in order to guarantee that at the initial and final times the Hamiltonian is } H_{\bar{\alpha}}.\]
Theorem 1 Let \( \phi \) be a scattering state of the Hamiltonian \( H_{\alpha} \) fulfilling the condition:

\[
\phi \in \{ \gamma \in S(\mathbb{R}^3) | \langle \gamma, \psi_{\alpha} \rangle_{L^2} = 0; \gamma = \gamma(|x|); \gamma(0) = 0 \}
\]

being \( S(\mathbb{R}^3) \) the space of functions of rapid decrease (see e.g. [7]). Then the functional \( F : H^1_0(0,T) \to \mathbb{C} \), defined by (11), (10), is a locally surjective map around the point \( \alpha = 0 \).

Remark 2 The inverse problem related to (11), i.e. find a control \( \alpha \) such that \( F(\alpha) \) has a fixed value in \( \mathbb{C} \), is connected with the investigation of the energy exchanges in non autonomous quantum systems. Although physical intuition suggests that the time dependence of the Hamiltonian can force energy exchanges between the point and the continuous spectrum, the dynamics describing these energy transfers is, in general, rather complex (see, for instance, the study of the ionization problem driven by time periodic delta interactions in [1]-[2]); in particular, we would like to stress that the local controllability of these phenomena is not a simple consequence of the time dependence of the Hamiltonian, but it depends from the specific structure of the interaction considered.

We will adopt here a standard procedure in the analysis of nonlinear systems. First we study the regularity properties of the map \( F : H^1_0(0,T) \to \mathbb{C} \) and prove that \( F \in C^1 \) (Lemma 5). Then we investigate the surjectivity of the linearized map \( d_0F \). To this aim we will study a non controllability condition for the linearized system (Section 4); it will be shown that, under the hypothesis (14) on the initial state, this condition is never satisfied, obtaining, in this way, a controllability result (sections 5-7). Then we conclude using a Rank Theorem for functionals defined on Banach spaces (Section 8).

3 The Nonlinear System

In this section we investigate the differentiability properties of the functional \( F(\alpha) \). We shall denote with \( X \) and \( Y \) two Banach spaces, \( U \) an open subset of \( X \), \( d_\alpha F \) and \( d^G_\alpha F \) respectively the Fréchet and the Gâteaux derivatives of the map \( F : U \to Y \) evaluated in the point \( \alpha \). A differentiable functional \( F : U \to Y \) is said to be of class \( C^1 \) if the map:

\[
F' : U \to L(X,Y), \quad F'(\alpha) = d_\alpha F
\]

is continuous. Next we recall a standard result in the theory of differential calculus in Banach spaces (Theorem 1.9 in [11]):

Theorem 3 Suppose \( F : U \to Y \) is Gâteaux-differentiable in \( U \). If the map:

\[
F'_G : U \to L(X,Y), \quad F'_G(\alpha) = d^G_\alpha F
\]

is continuous at \( \alpha^* \in U \), then \( F \) is Fréchet-differentiable at \( \alpha^* \) and its Fréchet derivative evaluated in \( \alpha^* \) results:

\[
d_{\alpha^*}F = d^G_{\alpha^*}F
\]

Consider the map \( V : H^1_0(0,T) \to C(0,T) \) defined as follows

\[
[V(\alpha)](t) = q(t) : q(t) + 4\sqrt{\pi} \int_0^t \frac{[\alpha(s) + \bar{\alpha}]}{\sqrt{t-s}} q(s) ds = f(t)
\]
with \( f \in C(0, T) \). From the estimate \([62]\), in Lemma \([10]\) we have the following bound for the \( C(0, T) \)-norm of \([V(\alpha)](t)\)

\[
\|V(\alpha)\|_{C(0, T)} \leq \|f\|_{C(0, T)} \Gamma(\alpha, T)
\]

where \( \Gamma(\alpha, T) \) is a positive finite constant depending on \( \alpha \) and \( T \).

**Lemma 4** Let \( f \in C(0, T) \). The map \( H_0^1(0, T) \to C(0, T) \) defined by \((16)\) is of class \( C^1 \).

**Proof.** First we prove that \( V \) is continuous; let \( \alpha, \beta \in H_0^1(0, T) \) and consider the difference \( V(\alpha)(t) - V(\beta)(t) \): it solves the equation

\[
q(t) + 4\sqrt{\pi i} \int_0^t \frac{[\alpha(s) + \bar{\alpha}] q(s)}{\sqrt{t - s}} ds = 4\sqrt{\pi i} \int_0^t \frac{[\beta(s) - \alpha(s)] [V(\beta)](s) ds}{\sqrt{t - s}}
\]

which is of type \((16)\); from the estimate \((17)\) we obtain

\[
\|V(\alpha) - V(\beta)\|_{C(0, T)} \leq \left| 4\sqrt{\pi i} \right| \|V(\beta)\|_{C(0, T)} \|\beta - \alpha\|_{C(0, T)} 2T^{\frac{3}{4}} \Gamma(\alpha, T)
\]

moreover, the Sobolev inequality

\[
\|g\|_{C(0, T)} \leq C\|g\|_{H^1(0, T)}
\]

implies

\[
\|V(\alpha) - V(\beta)\|_{C(0, T)} \leq C\left| 4\sqrt{\pi i} \right| \|V(\beta)\|_{C(0, T)} \|\beta - \alpha\|_{H^1(0, T)} 2T^{\frac{3}{4}} \Gamma(\alpha, T)
\]

From \([62]\), it follows that \( \Gamma(\alpha, T) \) is a continuous function: \( C(0, T) \to \mathbb{R}^+ \). Therefore \( \Gamma(\alpha, T) \) is uniformly bounded for \( \alpha \) close to \( \beta \) in \( H^1 \)-norm. Previous remark and the estimate \([20]\) imply the
continuity of the functional \( V \) from \( H_0^1(0, T) \) to \( C(0, T) \).

Next we introduce the Gâteaux derivative of \( V \) in the point \( \alpha \)

\[
d_\alpha^G V(u) = \lim_{\varepsilon \to 0} \frac{V(\alpha + \varepsilon u) - V(\alpha)}{\varepsilon}
\]

this limit being in the sense of \( C(0, T) \)-topology. The explicit expression of \( d_\alpha^G V(u) \) is\(^4\)

\[
d_\alpha^G V(u) = q : q(t) + 4\sqrt{\pi i} \int_0^t \frac{[\alpha(s) + \bar{\alpha}] q(s)}{\sqrt{t - s}} ds + 4\sqrt{\pi i} \int_0^t \frac{[V(\alpha)](s) u(s) ds}{\sqrt{t - s}} = 0
\]

This is a linear map: \( H_0^1(0, T) \to C(0, T) \). Making use once more of the estimate \((62)\), we have

\[
\|d_\alpha^G V(u)\|_{C(0, T)} \leq \left| 4\sqrt{\pi i} \right| \|V(\alpha)\|_{C(0, T)} \|u\|_{C(0, T)} \Gamma(\alpha, T) 2T^{\frac{3}{4}}
\]

In order to prove the statement of the Lemma we need to show that \( V' : H_0^1(0, T) \to L(H_0^1(0, T), C(0, T)) \), defined in \((15)\), is continuous. From Theorem \([3]\), it is sufficient to prove that the map

\[
V_G' : H_0^1(0, T) \to L(H_0^1(0, T), C(0, T)), \quad V_G'(\alpha) = d_\alpha^G V
\]

\(^4\)This formula can be obtained by exploiting the result of Lemma \([10]\)
is continuous, i.e.: \( \lim_{\alpha \to \beta} \left\| d_\alpha^G V - d_\beta^G V \right\| = 0 \), where \( \| \cdot \| \) is the operator norm in \( L(H^1_0(0,T), C(0,T)) \).

Set \( d_\alpha^G V(u) - d_\beta^G V(u) = q \); from the explicit formula (21), it follows
\[
q(t) + 4\sqrt{\pi} t \int_0^t \frac{[\alpha(s) + \alpha] q(s)}{\sqrt{t - s}} ds = 4\sqrt{\pi} t \int_0^t \left\{ [V(\beta)](s) - [V(\alpha)](s) \right\} u(s) - [\alpha(s) - \beta(s)] [V(\beta)](s) \frac{ds}{\sqrt{t - s}}
\]

This equation is still of the type (16); from the estimates (17) and (19) we get
\[
\|q\|_{C(0,T)} \leq 4\sqrt{\pi} t \int_0^t T^{\frac{1}{2}} C \left( \|u\|_{H^1(0,T)} \|V(\beta) - V(\alpha)\|_{C(0,T)} + \|\alpha - \beta\|_{H^1(0,T)} \|f\|_{C(0,T)} \Gamma(\beta,T) \right) \Gamma(\alpha,T)
\]
then the operator norm \( \left\| d_\alpha^G V - d_\beta^G V \right\| \) can be bounded as follows
\[
\left\| d_\alpha^G V - d_\beta^G V \right\| = \sup_{u \in H^1_0(0,T), \|u\| = 1} \| d_\alpha^G V(u) - d_\beta^G V(u) \|_{C(0,T)} \leq 4\sqrt{\pi} t \int_0^t T^{\frac{1}{2}} C \left( \|V(\beta) - V(\alpha)\|_{C(0,T)} + \|\alpha - \beta\|_{H^1(0,T)} \|f\|_{C(0,T)} \Gamma(\beta,T) \right) \Gamma(\alpha,T)
\]

As already noticed, \( \Gamma(\alpha,T) \) is uniformly bounded when \( \alpha \) is close to \( \beta \) in \( H^1(0,T) \)-norm. Then, the continuity of the map \( V \) allows us to conclude that
\[
\lim_{\|\alpha - \beta\|_{H^1_0(0,T)} \to 0} \left\| d_\alpha^G V - d_\beta^G V \right\| = 0
\]

Our next task is the study of the regularity properties of the map defined in (11) whose explicit expression is recalled here:
\[
\left\{ \begin{array}{l}
F(\alpha) := \left( U_T \phi(z) + i \int_0^T U(T - s, z) V(\alpha)(s) ds, \psi_\alpha(z) \right) \in L^2(\mathbb{R}^3) \\
V(\alpha) = q(t) : q(t) + 4\sqrt{\pi} t \int_0^t \frac{[\alpha(s) + \alpha] q(s)}{\sqrt{t - s}} ds = 4\sqrt{\pi} t \int_0^t \frac{U_T \phi(0)}{\sqrt{t - s}} ds
\end{array} \right. 
\]

Lemma 5 The map \( F: H^1_0(0,T) \to \mathbb{C} \) defined by (23) is of class \( C^1 \).

Proof. First we show that \( F \) is a continuous map. By inverting space and time integrals in (22), the scalar product \( \left( i \int_0^T U(T - s, z) V(\alpha)(s) ds, \psi_\alpha(z) \right) \in L^2(\mathbb{R}^3) \) is given by
\[
\left( i \int_0^T U(T - s, z) V(\alpha)(s) ds, \psi_\alpha(z) \right)_{L^2(\mathbb{R}^3)} = i \int_0^T \int_{\mathbb{R}^3} U(T - s, z) \psi_\alpha(z) d^3z V(\alpha)(s) ds = i \int_0^T \int_{\mathbb{R}^3} U_T \psi_\alpha(\eta) V(\alpha)(s) ds
\]

Let \( \alpha, \beta \in H^1_0(0,T) \); making use of the previous expression, the difference \( F(\alpha) - F(\beta) \) can be written as
\[
F(\alpha) - F(\beta) = i \int_0^T U_T \psi_\alpha(\eta) (V(\alpha)(s) - V(\beta)(s)) ds
\]
Therefore, it satisfies the estimate

\[ |F(\alpha) - F(\beta)| \leq \| V(\alpha)(s) - V(\beta)(s) \|_{C(0,T)} \int_0^T |U_{T-s,\psi}(\Omega)| \, ds \]  

(23)

The continuity of \( F \) then follows directly from Lemma 4. Moreover, from Theorem 3 and definition (15) it follows that \( F \) is of \( C^1 \) class if the map

\[ F_G : H_0^1(0,T) \to L(H_0^1(0,T), \mathbb{C}), \quad F_G(\alpha) = d^G_\alpha F \]

is continuous, i.e.: \( \lim_{\alpha \to \beta} \left\| d^G_\alpha F - d^G_\beta F \right\| = 0 \). The Gâteaux derivative of \( F \) evaluated in the point \( \alpha \) and acting on \( u \) is

\[ d^G_\alpha F(u) = i \left( \int_0^T U(T-s,\bar{x}) d_\alpha V(u)(s) ds, \psi(\bar{x}) \right)_{L^2(\mathbb{R}^2)} \]

(24)

By inverting space and time integrals, we obtain

\[ d^G_\alpha F(u) = i \int_0^T U_{T-s,\psi\bar{\alpha}}(\Omega) d^G_\alpha V(u)(s) ds \]

(25)

Consider the difference \( d^G_\alpha F(u) - d^G_\beta F(u) \); from (25) it results

\[
\left| d^G_\alpha F(u) - d^G_\beta F(u) \right| = \left| \int_0^T U_{T-s,\psi\bar{\alpha}}(\Omega) \left( d^G_\alpha V(u)(s) - d^G_\beta V(u)(s) \right) ds \right| \\
\leq \left\| d^G_\alpha V(u) - d^G_\beta V(u) \right\|_{C(0,T)} \int_0^T \left| U_{T-s,\psi\bar{\alpha}}(\Omega) \right| \, ds \\
\]

and

\[
\left| \left| d^G_\alpha F - d^G_\beta F \right| \right| = \sup_{u \in H_0^1(0,T)} \left| d^G_\alpha F(u) - d^G_\beta F(u) \right| \leq \left| \left| d^G_\alpha V - d^G_\beta V \right| \right| \int_0^T \left| U_{T-s,\psi\bar{\alpha}}(\Omega) \right| \, ds 
\]

The continuity of the map \( F'_G \) easily follows from the continuity of the map \( V'_G \). \( \blacksquare \)

4 The Linearized System

We consider the map:

\[
\left\{ \begin{array}{ll}
d_0 F(u) = z \\
u \in H_0^1(0,T)
\end{array} \right.
\]

(26)

where \( d_0 F \) is the Fréchet derivative of functional \( F \) evaluated in \( \alpha = 0 \). Our aim is to prove the following result:

**Theorem 6** Under the assumptions of Theorem 7 the map defined by (26) is surjective.
The proof of Theorem 6 will be given in Sections 4-7 following an ad absurdum procedure. First we write the functional (26) into an explicit form:

\[
\begin{align*}
d_0 F(u) &= i \left( \int_0^T U(T - s, \cdot) d_0 V(u)(s) ds \right)_{L^2(\mathbb{R}^3)} \\
d_0 V(u) &= q(t) : q(t) + 4\sqrt{\pi i} \bar{\alpha} \int_0^t q(s) \sqrt{t - s} ds = -4\sqrt{\pi i} \int_0^t u(s) V(0)(s) \sqrt{t - s} ds
\end{align*}
\]

Here the dependence of \( d_0 F(u) \) from \( V(0) \) may be emphasized by making use of the relation (77) (in Appendix A), plus the Fubini Theorem (in order to exchange time and space integrations in (27)); proceeding in this way, we get

\[
d_0 F(u) = i \int_0^T U_{T-s} \bar{\psi}_\alpha(0) d_0 V(u)(s) ds
\]

Applying the Dirichlet formula to the double integral we finally obtain

\[
d_0 F(u) = 4\pi i^2 \int_0^T V(0)(s') u(s') \int_s^T U_{T-s} \bar{\psi}_\alpha(0) G(s - s') ds' ds
\]

Let us suppose the map \( d_0 F : H^1_0(0,T) \to \mathbb{C} \) to be non-surjective; then \( d_0 F(u) \) should have a constant direction in the complex plane for any \( u \in H^1_0(0,T) \). We can express this as a non-controllability condition

\[
\exists C \in \mathbb{C} : \left\{ \begin{array}{l}
C \neq 0 \\
C \cdot \int_0^T V(0)(s') u(s') \int_s^T U_{T-s} \bar{\psi}_\alpha(0) G(s - s') ds' ds' = 0 \quad \forall u \in H^1_0(0,T)
\end{array} \right.
\]

where \( \cdot \cdot \cdot \) indicates the scalar product in \( \mathbb{C} \). In particular, being this condition true for any real valued \( u \in C^\infty(0,T) \), the fundamental lemma of calculus of variations implies

\[
\exists C \in \mathbb{C} : \left\{ \begin{array}{l}
C \neq 0 \\
C \cdot V(0)(t) \int_t^T U_{T-s} \bar{\psi}_\alpha(0) G(s - t) ds = 0 \quad \forall t \in [0,T]
\end{array} \right.
\]

Making use of (80) - in Appendix A - we get

\[
\exists C \in \mathbb{C} : \left\{ \begin{array}{l}
C \neq 0 \\
C \cdot V(0)(t) e^{-i(T-t)\lambda} = 0 \quad \forall t \in [0,T]
\end{array} \right.
\]

5 Small Time Asymptotic for the Charge

In this Section we consider the asymptotic behavior of the charge \( V(0) \) for \( t \to 0 \). We shall denote with \( o(t) \) and \( O(t) \) two complex valued functions of the real variable \( t \) satisfying the conditions

\[
\lim_{t \to 0} \frac{o(t)}{t} = 0
\]
\[
\limsup_{t \to 0} \left| \frac{O(t)}{t} \right| < \infty \tag{32}
\]

According to our hypothesis on the initial state, (14), and to definition (11), the function \( V(0) \) is the solution of the equation
\[
g(t) + 4\sqrt{\pi}i\hat{\alpha} \int_0^t \frac{g(s)}{\sqrt{t-s}} ds = 4\sqrt{\pi} \int_0^t \frac{U_s \gamma(\bar{t})}{\sqrt{t-s}} ds
\]
therefore, thanks to (76), it may be represented in the form
\[
V(0)(t) = 4\pi \sqrt{i} \int_0^t G(t-s)U_s \gamma(\bar{t}) ds \tag{33}
\]
Its small time behavior is connected to the limiting behavior of \( G(t) \) and \( U_t \gamma(\bar{0}) \) for \( t \to 0 \). In order to study this problem we need the following Lemmas

**Lemma 7** Let \( \gamma = \gamma(|x|) \) be a radial function belonging to the space of functions of rapid decrease \( S(\mathbb{R}^3) \) (see e.g. [7]). If we assume that
\[
D_0 = \left\{ n \in \mathbb{N} : \left| \frac{d^n}{dt^n} U_t \gamma(\bar{0}) \right|_{t=0} \neq 0 \right\} \tag{34}
\]
is a non empty set, then the function \( U_t \gamma(\bar{0}) \) admits the power expansion:
\[
U_t \gamma(\bar{0}) = a_m t^m + O(t^{m+1}); \quad a_m \neq 0 \tag{35}
\]
\[
a_m = \frac{4\pi}{(2\pi)^{\frac{3}{2}}} (-i)^m m! \int_0^{+\infty} k^{2m+2} \mathcal{F}_\gamma(k) dk \tag{36}
\]
where \( \mathcal{F} \) denotes the Fourier transform, \( m = \min D_0 \) and \( O(t) \in C^\infty[0, +\infty) \).

**Proof.** The Fourier transform operator, \( \mathcal{F} \), is an homeomorphism of the space \( S(\mathbb{R}^3) \) in itself. It acts on \( U_t \gamma(\bar{0}) \) as follows:
\[
\mathcal{F} U_t \gamma(k) = \mathcal{F}_\gamma(k) e^{-ik^2 t}
\]
Then, using \( \mathcal{F}^{-1} \), we can represent \( U_t \gamma(\bar{0}) \) in the form
\[
U_t \gamma(\bar{0}) = \frac{4\pi}{(2\pi)^{\frac{3}{2}}} \int_0^{+\infty} k^2 \mathcal{F}_\gamma(k) e^{-ik^2 t} dk \tag{37}
\]
From the regularity assumptions on \( \gamma \), we have: \( \mathcal{F}_\gamma(k) \in S(\mathbb{R}^3) \) and \( U_t \gamma(\bar{0}) \in C^\infty[0, +\infty) \). Setting \( m = \min D_0 \), the Taylor’s expansion of \( U_t \gamma(\bar{0}) \) up to order \( m \) in a right neighborhood of the origin is
\[
U_t \gamma(\bar{0}) = a_m t^m + O(t^{m+1})
\]
with \( O(t) \in C^\infty[0, +\infty) \). The coefficient \( a_m \) is different from zero - due to the hypothesis (34) - and explicitly given by
\[
\frac{1}{m!} \frac{d^m}{dt^m} U_t \gamma(\bar{0}) \bigg|_{t=0}; \quad \text{form (37)},
\]
this quantity is
\[
a_m = \frac{4\pi}{(2\pi)^{\frac{3}{2}}} \frac{(-i)^m m!}{m!} \int_0^{+\infty} k^{2m+2} \mathcal{F}_\gamma(k) dk.
\]
Lemma 8  Let $G(t)$ be given by (75); then it admits the following representation

$$G(t) = \frac{1}{\sqrt{\pi t}} + b_0 + O(t); \quad b_0 = -4\pi \bar{\alpha} \sqrt{t}$$

(38)

where $O(t) \in C^\infty[0, +\infty)$.

Proof. The proof easily follows from the analytic properties of the $erfc(t)$ function ([8], relation 7.1.5, pag 297)

We will use these results to get an expansion in power of $t^{\frac{1}{2}}$ for the charge (33). If we assume condition (34) to hold, Lemma 7 may be applied to our case, with the only restriction $m \neq 0$ due to the fact that the boundary condition $\gamma(0) = 0$ implies $U_1\gamma(0) = 0$ for $t = 0$. By substitution of (35) and (38) into (33) we obtain

$$V(0)(t) = 4\sqrt{\pi}a_m \int_0^t \frac{s^m}{\sqrt{t-s}} ds + 4\pi \sqrt{i} a_m b_0 \int_0^t s^m ds + 4\pi \sqrt{i} \int_0^t \frac{O(s^{m+1})}{\sqrt{t-s}} ds +$$

$$+ 4\pi \sqrt{i} b_0 \int_0^t O(s^{m+1}) ds + 4\pi \sqrt{i} a_m \int_0^t (t-s) s^m ds + 4\pi \sqrt{i} \int_0^t O(t-s) O(s^{m+1}) ds$$

An explicit calculation of the first terms in this expression leads us to the following expansion

$$V(0)(t) = A_m a_m \sqrt{i} t^{m+\frac{1}{2}} + B_m a_m \sqrt{i} b_0 t^{m+1} + 4\sqrt{\pi} \int_0^t \frac{O(s^{m+1})}{\sqrt{t-s}} ds +$$

$$+ 4\pi \sqrt{i} b_0 \int_0^t O(s^{m+1}) ds + 4\pi \sqrt{i} a_m \int_0^t (t-s) s^m ds + 4\pi \sqrt{i} \int_0^t O(t-s) O(s^{m+1}) ds$$

where $A_m$ and $B_m$ are strictly positive real constants. By definition (32), it exists $C$ such that $|O(s^k)| \leq C |s^k| \leq C |t^k|$ for any $|s| \leq |t|$; using this estimate, the following relations are easily obtained

$$\int_0^t O(s^{m+1}) ds = O(t^{m+2})$$

(39)

$$\int_0^t O(t-s) s^m ds = O(t^{m+2})$$

(40)

$$\int_0^t O(t-s) O(s^{m+1}) ds = o(t^{m+2})$$

(41)

$$\int_0^t \frac{O(s^{m+1})}{\sqrt{t-s}} ds = O(t^{m+\frac{1}{2}})$$

(42)

and the small time asymptotic representation for $V(0)$ can be rewritten as

$$V(0)(t) = A_m a_m \sqrt{i} t^{m+\frac{1}{2}} + B_m a_m \sqrt{i} b_0 t^{m+1} + O(t^{m+\frac{1}{2}})$$

(43)
6 The Non Controllability Condition in the limit $t \to 0$

Here we study condition (30) in a neighborhood $[0, \delta)$ of the origin with $\delta < T$.

Let us first set the condition (30) in the equivalent form

$$\exists K \in [0, 2\pi) : \arg \left( V(0)(t)e^{-i(T-t)\lambda_\alpha} \right) = K \quad \forall t \in [0, T]$$

(44)

where $\arg z$ is defined modulus $2\pi$. Making use of (33), we see that

$$\arg \left( V(0)(t)e^{-i(T-t)\lambda_\alpha} \right) = \arg \left( 4\pi \sqrt{i} \int_0^t G(t-s) U_s \gamma(0) ds e^{-i(T-t)\lambda_\alpha} \right)$$

Then, condition (44) implies

$$\exists K \in [0, 2\pi) : \arg \left( 4\pi \sqrt{i} \int_0^t G(t-s) U_s \gamma(0) ds e^{-i(T-t)\lambda_\alpha} \right) = K \quad \forall t \in [0, T]$$

(45)

In order to analyze (45), we need the following Lemma:

**Lemma 9** Let $a_m$ and $b_0$ be defined by (36) and (38) respectively. Under the assumptions of Lemma 7 and Lemma 8, the function

$$\arg \left( 4\pi \sqrt{i} \int_0^t G(t-s) U_s \gamma(0) ds e^{-i(T-t)\lambda_\alpha} \right)$$

admits the small time expansion

$$\arg \left( 4\pi \sqrt{i} \int_0^t G(t-s) U_s \gamma(0) ds e^{-i(T-t)\lambda_\alpha} \right) = \arg \left( a_m \sqrt{i} + c \sin(\arg b_0) t^{\frac{1}{2}} - (T-t)\lambda_\alpha + o(t^{\frac{1}{2}}) \right)$$

(46)

with $c = \frac{B_m}{A_m} |b_0|$.

**Proof.** Set $z_1 = \rho_1 e^{i\varphi_1}$ and $z_2 = \rho_2 e^{i\varphi_2}$; the first order Taylor expansion of $\arg(z_1 + z_2)$ w.r.t. the ratio $\varepsilon = \frac{z_2}{z_1}$ about the point $\varepsilon = 0$ is

$$\arg(z_1 + z_2) = \varphi_1 + \sin(\varphi_2 - \varphi_1) \varepsilon + o(\varepsilon)$$

(47)

Using (33) and (38) we have

$$\arg \left( 4\pi \sqrt{i} \int_0^t G(t-s) U_s \gamma(0) ds e^{-i(T-t)\lambda_\alpha} \right) = \arg \left( A_m a_m \sqrt{i} t^{m+\frac{1}{2}} + B_m a_m \sqrt{i} b_0 t^{m+1} + O(t^{m+\frac{3}{2}}) \right) - (T-t)\lambda_\alpha$$

(48)
Using (twice) relation (47), the right hand side of (48) can be expanded as

$$\arg \left( A_m a_m \sqrt{t} t^{m+\frac{1}{2}} + B_m a_m \sqrt{t} b_0 t^{m+1} + O(t^{m+2}) \right) =$$

$$= \arg \left( A_m a_m \sqrt{t} t^{m+\frac{1}{2}} + B_m a_m \sqrt{t} b_0 t^{m+1} \right) + O(t) =$$

$$= \arg \left( a_m \sqrt{t} \right) + c \sin(\arg b_0) t^{\frac{1}{2}} + o(t^{\frac{1}{2}})$$

Equation (49) is a straightforward consequence of (48) and (49). 

Lemma 9 leads us to an asymptotic formulation of the non controllability condition for small time. From relations (45) and (46), indeed, we have

$$K = \arg \left( a_m \sqrt{t} \right) + \frac{c}{\sqrt{2}} t^{\frac{1}{2}} - (T - t) \lambda_\bar{\alpha} + o(t^{\frac{1}{2}})$$

where the explicit value \( b_0 = -4\pi \bar{\alpha} \sqrt{t} \) has been taken into account. Recalling that \( c \neq 0 \), relation (50) is an evident contradiction we obtained supposing the system (26) to be non-surjective. This concludes the proof of Theorem 6 for all choices of initial states satisfying condition (34) of Lemma 7.

In the next section we will study an extension of the proof to those cases in which Lemma 7 does not applies.

### 7 Finite Time Asymptotic for the Charge and Proof of Theorem 6

If condition (34) does not hold, we may still recover our results by changing the point in which we perform the expansions of expressions (33) and (44).

To this concern, we consider a radial function \( \gamma = \gamma(|x|) \) in the space \( S(\mathbb{R}^3) \). From (37) in Lemma 7, it follows that, \( U_{t} \gamma(0) \) is a \( C^\infty \)-class function represented by

$$U_{t} \gamma(0) = \frac{4\pi}{(2\pi)^\frac{3}{2}} \int_{0}^{+\infty} k^2 F_{\gamma}(k)e^{-ik^2t} dk$$

where \( F \) denotes the Fourier transform in \( L^2(\mathbb{R}^3) \). Let us define \( f \in L^2(-\infty, +\infty) \)

$$f(y) = \begin{cases} (2\pi)^{-\frac{1}{2}} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

Setting \( k^2 = y \) in the integral (51), we can express \( U_{t} \gamma(0) \) as the Fourier transform of \( f \)

$$U_{t} \gamma(0) = \hat{f}(t) = \int_{0}^{+\infty} f(y)e^{-iyt} dy$$

Making use of relation (53), it is possible to extend \( U_{t} \gamma(0) \) to the complex plane as follows

$$U_{z} \gamma(0) = \int_{0}^{+\infty} f(y)e^{-iyz} dy; \quad z = t + is$$
It is well known that, for any \( f \in L^2(0, +\infty) \), equation (54) defines an holomorphic function in the lower complex half plane \( s < 0 \) ([9], Section 19.1). In order to study the limit of \( U_z \gamma(0) \) as \( z \) approaches the real axis, we notice that this function can be expressed as the Fourier transform of the product of \( f(y) \) times \( e^{\|y\|^s} \in L^2(-\infty, +\infty) \)

\[
U_z \gamma(0) = (f(y) \cdot e^{\|y\|^s})(t)
\]

therefore we have

\[
U_{t+is} \gamma(0) = \hat{f} \ast (e^{\|\cdot\|^s})(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} U_{t-u} \gamma(0) \frac{|s|}{|u'|^2 + s^2} \, du'
\]

(55)

with

\[
\frac{1}{\pi} \frac{|s|}{|u'|^2 + s^2} = (e^{\|\cdot\|^s})(t)
\]

(56)

Taking into account the continuity of \( U_t \gamma(0) \), it can be shown that \( U_z \gamma(0) \) is continuous in the set \( s \leq 0 \).

So far we obtained that \( U_{t+is} \gamma(0) \) is holomorphic in \( \{ t \in \mathbb{R}; \ s < 0 \} \) and continuous in the closure \( \{ t \in \mathbb{R}; \ s \leq 0 \} \). For \( \gamma \neq 0 \), this implies that the zeroes of function \( U_t \gamma(0) \) have to be isolated points. If \( \gamma \) belongs to the domain (14), the origin of the real axis is a zero of \( U_t \gamma(0) \); therefore a time \( t_0 > 0 \), arbitrarily near to the origin, exists such that: \( U_{t_0} \gamma(0) \neq 0 \). Proceeding as in Lemma 7, it is possible to obtain for the function \( U_t \gamma(0) \) the following power expansion around \( t_0 \)

\[
U_t \gamma(0) = a_0 + O((t - t_0)); \ a_0 \neq 0
\]

(57)

with \( a_0 = U_{t_0} \gamma(0) \) and \( O(t) \in C^\infty[0, +\infty) \).

Next, we observe that a simple change of variable

\[
\tau = t - t_0
\]

and the use of (56), provide us with an equation for the charge when the initial time \( t = t_0 \) is assigned

\[
V(0)(\tau) = 4\pi \sqrt{\tau} \int_0^\tau G(\tau - s) U_{t_0 + s} \gamma(0) \, ds
\]

(58)

Using (38) and (57), we get the power expansion of the of the non controllability condition in a right neighbourhood of the point \( t = t_0 \)

\[
K = \arg \left( a_0 \sqrt{\tau} + \frac{c}{\sqrt{2}} (t - t_0) \, \lambda_\tilde{s} + o((t - t_0)^{3/2}) \right)
\]

(59)

with \( c \neq 0 \). As in the previous case, this relation constitutes a contradiction obtained supposing the system (20) to be non-surjective.

This concludes the proof of Theorem 6.

8 Proof of the Main Result and Final Remarks

So far, we succeeded in proving that the functional \( F(\alpha) \), defined by (11)-(14), belongs to \( C^1(H_1^0(0, T), \mathbb{C}) \) and its derivative, evaluated in \( \alpha = 0 \), is surjective. The Rank Theorem (see e.g. in [10], page 336
Theorem 34), then, applied to our case, implies the existence of a neighborhood of the origin in \( C, I_0 \), and a \( C^1 \)-class map \( g : I_0 \to H^1_0(0, T) \) such that:

\[
F(g(z)) = z \quad \forall z \in I_0
\]

(60)

This concludes the proof of Theorem 1.

Our main remark is about the assumptions (14) on the initial state. By considering functions with radial symmetry, we are taking into account only those scattering states which have a null projection along all spherical harmonics excepting the first one. In this choice there is no loss of generality. Indeed, those scattering functions whose expansion in spherical harmonics is:

\[
\phi(r, \theta, \varphi) = \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} f_{lm}(r) Y^m_l(\theta, \varphi)
\]

exhibit the following characterization:

\[
U_t \phi(0) = 0 \quad \forall t
\]

Then it follows from definition (7), and the uniqueness of solution of the charge equation, that a particle, initially placed in such a scattering state, \( \phi \), and subject to the action of any Hamiltonian of type \( H_{\alpha(t)} \), results to have a null charge and to evolve under the action of the free propagator; in other words, starting from this initial condition, the particle doesn’t feel the interaction at all. In this case any transfer of energy is physically impossible. On the other hand our model can be applied to a realistic situation in which an incoming particle - described by a wave function of type: \( \psi = \phi(r)f(\theta, \varphi) \) with \( f(\theta, \varphi) \) null outside a cone - is partially trapped into the attractive potential described by \( H_{\bar{\alpha}} \).

In conclusion, we have proved the local controllability of a process of energy-mass transfer, from scattering to bound states, for a one body quantum system under the action of a time dependent point interaction.

Further development of this studies may concern the global controllability of the same process, as well as the inverse problem of finite time ionization.

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Appendix A: The Integral Equation

We recall some basic properties of the integral equations we deal within this work; some of the relations already used in the previous sections will be here obtained. A detailed analysis of fractional integral equations, which arise in the framework of time dependent point interactions in Quantum Mechanics, is given in [1].

Lemma 10 Let \( \alpha, f \in C(0, T) \); the equation

\[
q(t) + 4\sqrt{\pi i} \int_0^t \frac{[\alpha(s) + \bar{\alpha}] q(s)}{\sqrt{t-s}} ds = f(t); \quad \bar{\alpha} \in \mathbb{R}
\]  

(61)

has an unique continuous solution such that:

\[
\|q\|_{C(0,T)} \leq \|f\|_{C(0,T)} \Gamma(\alpha, T)
\]

(62)

where \( \Gamma(\alpha, T) \) is a finite positive constant depending on \( \alpha \) and \( T \).

Proof. The solution \( q(t) \) may be formally expressed using the Picard series:

\[
\begin{align*}
q(t) & = \sum_{n=0}^{+\infty} q_n(t) \\
q_0(t) & = f(t) \\
q_n(t) & = 4\sqrt{\pi i} \int_0^t \frac{[\alpha(s) + \bar{\alpha}] q_{n-1}(s)}{\sqrt{t-s}} ds
\end{align*}
\]

(63)

which admit the following estimate:

\[
\sum_{n=0}^{+\infty} |q_n(t)| \leq \|f\|_{C(0,T)} \left[ 1 + \sum_{n=1}^{+\infty} |4\sqrt{\pi i}|^n \|\alpha + \bar{\alpha}\|_{C(0,T)}^n A_n \pi^{\frac{n}{2}} t^{\frac{n}{2}} \right]
\]

(64)

with:

\[
A_n = \begin{cases} 
\left(\frac{1}{2}\right)! n! & \text{n even} \\
\left(\frac{n+1}{2}\right)! \frac{1}{n!} & \text{n odd}
\end{cases}
\]

being \( n!! = 1 \cdot 3 \cdot 5 \cdots n \) for \( n \) odd. The strong infinitesimal character of the sequence \( A_n \) allow this sum to converge uniformly for any \( \alpha \in C(0,T) \) and for any finite time interval \([0, T]\). The existence of a unique solution of (61) satisfying the estimate (62) directly follows from (64), with

\[
\Gamma(\alpha, T) = \left[ 1 + \sum_{n=1}^{+\infty} |4\sqrt{\pi i}|^n \|\alpha + \bar{\alpha}\|_{C(0,T)}^n A_n \pi^{\frac{n}{2}} T^{\frac{n}{2}} \right]
\]

(65)

In connection with the charge equation in (7), we claim the following result:

Lemma 11 Let \( \psi_0 \in D(H_\alpha) \) where \( H_\alpha \) is the Hamiltonian associated to a 3-D point interaction placed in the origin and \( \alpha \in \mathbb{R} \); then, the function:

\[
\int_0^t \frac{U_s \psi_0(0)}{\sqrt{t-s}} ds
\]

(66)

is continuous.
Proof. First we consider the case \( \alpha < 0 \).

From definition (4), any function \( \psi_0 \in D(H_\alpha) \) is the sum of a regular part plus a bound state term:

\[
\psi_0 = \varphi + \frac{q}{4\pi \sqrt{2|\alpha|}} \psi_\alpha, \quad \varphi \in H^2(\mathbb{R}^3), \quad \varphi(\mathbf{0}) = 0, \quad q \in \mathbb{C} \quad (67)
\]

Then the function (66) disparts in two contributions:

\[
\int_0^t \frac{U_s \psi_0(0)}{\sqrt{t-s}} ds = \int_0^t \frac{U_s \varphi(0)}{\sqrt{t-s}} ds + \frac{q}{4\pi \sqrt{2|\alpha|}} \int_0^t \frac{U_s \psi_\alpha(0)}{\sqrt{t-s}} ds \quad (68)
\]

We want to prove that (68) defines a continuous function on finite time intervals. To this aim we consider the two contributions of (68) separately.

The first term of the second member is the one half integral of \( U_t \varphi(0) \). Using for the state \( \varphi \) a representation in terms of spherical harmonics, holding for all \( L^2(\mathbb{R}^3) \) functions, we have:

\[
\varphi(r, \theta, \phi) = \chi(r) + \sum_{l \neq 0}^\infty \sum_{m=-l}^l f_{lm}(r) Y_l^m(\theta, \phi) \quad (69)
\]

From the orthogonality relations for \( Y_l^m \), we have

\[
U_t \varphi(0) = U_t \chi(0)
\]

\( \chi \in H^2(\mathbb{R}^3) \) denoting the radial part of \( \varphi \). Thus, the function \( U_t \varphi \) evaluated in \( x = 0 \) may be expressed by the following Fourier integral

\[
U_t \varphi(0) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \mathcal{F}_\chi(k) e^{-i k^2 t} |k|^{-\frac{1}{2}} dk \quad (70)
\]

The Fourier transform of the radial function \( \chi \in H^2(\mathbb{R}^3) \) has the following characterization: \( k^2 \mathcal{F}_\chi(k) \in L^2(\mathbb{R}^3) \). Using this relation and the Schwartz inequality, we get an estimate for the \( L^1 \)-norm of \( \mathcal{F}_\chi \)

\[
\| \mathcal{F}_\chi \|_1 = \int_{\mathbb{R}^3} |\mathcal{F}_\chi(k)| \frac{(1 + k^2)}{1 + k^2} d^3k \leq \| (1 + k^2) \mathcal{F}_\chi(k) \|_2 \left\| \frac{1}{1 + k^2} \right\|_2
\]

from which it follows that \( \mathcal{F}_\chi \in L^1(\mathbb{R}^3) \). This result guarantees that the function (69), as well as the first source term \( \int_0^t \frac{U_s \psi_\alpha(0)}{\sqrt{t-s}} ds \), are continuous.

The second source term of (68) may be evaluated explicitly by using the Laplace transform operator \( \mathcal{L} \); from definition (5), a direct calculation shows that

\[
\mathcal{L} \left[ \int_0^t \frac{U_s \psi_\alpha(0)}{\sqrt{t-s}} ds \right] (p) = \sqrt{\frac{2\pi |\alpha|}{i}} \frac{p^{-\frac{3}{2}}}{p^{\frac{3}{2}} - 4\pi \alpha \sqrt{i}} \quad (70)
\]

Here we recall that

\[
\mathcal{L}^{-1} \left[ \frac{p^{-\frac{3}{2}}}{p^{\frac{3}{2}} - 4\pi \alpha \sqrt{i}} \right] = e^{i 16 \pi^2 \alpha^2 t e r f c(4\pi |\alpha| \sqrt{i t})} \quad (71)
\]

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Then the second contribution of (68) is
\[
\frac{q}{4\sqrt{2}} \int_0^t \frac{U_s \psi_{\alpha}(0)}{\sqrt{t-s}} ds = \frac{q}{4\sqrt{\pi i}} e^{i16\pi^2 \alpha^2 t} \text{erfc}(4\pi |\alpha| \sqrt{it})
\] (72)
which is a continuous function bounded in \( \mathbb{R}^+ \).

The same result holds in the case \( \alpha \geq 0 \), where the only difference consists in the fact that the operator domain does not include any bound state of \( H_\alpha \).

Relation (62) allows us to obtain an estimate for the solution of charge equation in (11)
\[
\|q\|_{C(0,T)} \leq \left\| 4\sqrt{\pi i} \int_0^t \frac{U_s \phi(0)}{\sqrt{t-s}} ds \right\|_{C(0,T)} \Gamma(\alpha,T)
\] (73)
where the boundedness of second member is assured by Lemma 11.

Solving the charge equation

In the particular case \( \alpha = 0 \), the charge \( V(0)(t) \) satisfies a fractional integral equation:
\[
q(t) + 4\sqrt{\pi i} \int_0^t \frac{q(s)}{\sqrt{t-s}} ds = 4\sqrt{\pi i} \int_0^t \frac{U_s \phi(0)}{\sqrt{t-s}} ds
\] (74)
whose solution may be explicitly expressed as a functional of the source term; the Laplace transform of (74), indeed, gives
\[
\tilde{q}(p) \left(1 + \frac{4\pi \bar{\alpha} \sqrt{i}}{\sqrt{p}}\right) = \frac{4\pi \sqrt{i}}{\sqrt{p}} \mathcal{L}(U_t \phi(0))(p) \Rightarrow \tilde{q}(p) = \frac{4\pi \sqrt{i}}{\sqrt{p} + 4\pi \bar{\alpha} \sqrt{i}} \mathcal{L}(U_t \phi(0))(p)
\]
and taking into account the relation:
\[
\mathcal{L}^{-1} \frac{1}{\sqrt{p} + 4\pi \bar{\alpha} \sqrt{i}} = \frac{1}{\sqrt{\pi t}} - (4\pi \bar{\alpha} \sqrt{i}) e^{i(4\pi \bar{\alpha})^2 t} \text{erfc}(4\pi \bar{\alpha} \sqrt{it}) \equiv G(t)
\] (75)
an explicit expression for the charge is obtained:
\[
V(0)(t) = 4\pi \sqrt{i} \int_0^t G(t-s) U_s \phi(0) ds
\] (76)
Due to its similar structure, an analogous expression holds for equation (27):
\[
d_0 V(u)(t) = 4\pi \sqrt{i} \int_0^t G(t-s) u(s) V(0)(s) ds
\] (77)

When \( \phi \) coincide with the bound state \( \psi_{\bar{\alpha}} \) of the point interaction Hamiltonian \( H_{\bar{\alpha}} \), the charge equation (74) is explicitly solvable. Indeed, the time evolution of a quantum particle starting in the state \( \psi_{\bar{\alpha}} \) and moving under the action of \( H_{\bar{\alpha}} \), is
\[
\psi(t) = e^{-itH_{\bar{\alpha}}} \psi_{\bar{\alpha}} = e^{-it\lambda_{\bar{\alpha}}} \psi_{\bar{\alpha}}
\]
Comparing this expression with (41), we get the explicit form of the charge

\[ q(t) = 4\pi \sqrt{2|\alpha|} e^{-it\lambda_\alpha} \]  

(78)

Relations (70) and (78) imply

\[ \sqrt{i} \int_0^t G(t - s) U_s \psi_\alpha(0) \, ds = \sqrt{2|\alpha|} e^{-it\lambda_\alpha} \]  

(79)

Replacing \( t \) with \( T - t \), the previous formula can be written as

\[ \sqrt{i} \int_0^{T-t} G(T - t - s) U_s \psi_\alpha(0) \, ds = \sqrt{2|\alpha|} e^{-i(T-t)\lambda_\alpha} \]

Moreover, setting \( s' = T - s \) in the integral at the l.h.s. and dividing by \( \sqrt{i} \), we get

\[ \int_0^T G(s' - t) U_{T-s'} \psi_\alpha(0) \, ds' = \sqrt{2|\alpha|} e^{-i(T-t)\lambda_\alpha} \]  

(80)

This relation has been used in order to obtain condition (30).

In order to justify the use of Laplace transform in deriving relations (76) and (77), our next task is to prove the following

**Lemma 12** Let \( q(t) \) be the solution of the charge equation:

\[ q(t) + 4\sqrt{\pi}i \int_0^t \frac{\alpha(s) + \bar{\alpha}}{\sqrt{T-s}} q(s) \, ds = 4\sqrt{\pi}i \int_0^t \frac{U_s \psi_0(0)}{\sqrt{T-s}} \, ds \]

with \( \alpha \in L^\infty(\mathbb{R}) \), \( \psi_0 \in \mathcal{D}(H_{\alpha(0)}) \). Then, the Laplace transform \( \mathcal{L}q(p) \) exists and is analytic at least in the open half plane of \( \mathbb{C} \) defined by the condition:

\[ p \in \mathbb{C} : \text{Re} \, p > 16\pi^2 \| \alpha \|^2 \]  

(81)

**Proof.** Consider the function: \( q'(t) = e^{-p\, t} q(t) \) with \( p \in \mathbb{C} \) and \( \text{Re} \, p > 0 \); it satisfies the equation:

\[ q'(t) + 4\sqrt{\pi}i \int_0^t \frac{\alpha(s) e^{-p(t-s)}}{\sqrt{T-s}} q(s) \, ds = 4\sqrt{\pi}i \int_0^t \frac{e^{-p(t-s)}}{\sqrt{T-s}} f(s) \, ds \]  

(82)

where the function \( f \) in the nonhomogeneous term is given by:

\[ f(t) = e^{-p\, t} U_t \psi_0(0) \]  

(83)

Using relations (67) and (69) of the previous proof, we may write (83) in the following form:

\[ f(t) = e^{-p\, t} \left[ \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \mathcal{F}\varphi_r(k) \, e^{-ik^2 \, t} \, dk + \frac{q}{4\pi \sqrt{2|\alpha|}} U_t \psi_\alpha(0) \right] \]  

(84)

with \( \mathcal{F}\varphi_r(k) \in L^1(\mathbb{R}^3) \); again an explicit calculation shows that:

\[ U_t \psi_\alpha(0) = \mathcal{L}^{-1} \left[ \sqrt{\frac{2|\alpha|}{i} \left( \sqrt{p^2 + 4\pi |\alpha|} \right)} \right] = \sqrt{\frac{2|\alpha|}{i}} \left[ \frac{1}{\sqrt{2\pi t}} - 4\pi |\alpha| \sqrt{i} e^{(4\pi |\alpha|)^2 t} \, \text{erfc}(4\pi |\alpha| \sqrt{it}) \right] \]

(84)
from which we deduce that $f(t) \in L^1(0, +\infty)$. 

Next we apply the Young’s inequality:

$$\|f \ast g\|_1 \leq \|f\|_1 \|g\|_1$$

for convolutions of $L^1(0, +\infty)$ functions, to the equation (82), and obtain the following estimate:

$$\|q\|_1 \left(1 - 4\sqrt{\pi} \|\alpha\|_\infty \int_0^{+\infty} \left| \frac{e^{-pt}}{\sqrt{t}} \right| dt \right) \leq 4\sqrt{\pi} \|f\|_1 \int_0^{+\infty} \left| \frac{e^{-pt}}{\sqrt{t}} \right| dt$$

which provides an effective bound for the norm $\|q\|_1$ if the coefficient $\left(1 - 4\sqrt{\pi} \|\alpha\|_\infty \int_0^{+\infty} \left| \frac{e^{-pt}}{\sqrt{t}} \right| dt \right)$ is positive. Recalling that, for $\text{Re} \, p > 0$, holds the equality:

$$\int_0^{+\infty} \left| \frac{e^{-pt}}{\sqrt{t}} \right| dt = \sqrt{\frac{\pi}{\text{Re} \, p}}$$

we get the condition:

$$\left(1 - 4\sqrt{\pi} \|\alpha\|_\infty \int_0^{+\infty} \left| \frac{e^{-pt}}{\sqrt{t}} \right| dt \right) > 0 \Rightarrow 1 > 4\sqrt{\pi} \sqrt{\frac{\pi}{\text{Re} \, p}} \|\alpha\|_\infty \Rightarrow \text{Re} \, p > 16\pi^2 \|\alpha\|^2_\infty$$

(85)

Following the same line, it’s easy to prove that, if the condition (85) holds, the partial derivatives of the function $q'$ w.r.t. the real and the imaginary part of $p$ - both given by:

$$-te^{-pt}q(t)$$

are bounded by integrable functions of $t$:

$$\int_0^{+\infty} \left| te^{-pt}q(t) \right| dt \leq \int_0^{+\infty} \left| te^{-(16\pi^2 \|\alpha\|^2_\infty + \epsilon)}q(t) \right| dt < \infty$$

Then $e^{-pt}q(t)$ is $C^1$ integrable w.r.t. $t \in [0, +\infty)$ for any $p$ in the domain (81); moreover, in the same hypothesis, its partial derivatives w.r.t. $p$ are bounded by measurable functions of $t$. This allows us to conclude that the Laplace integral:

$$\mathcal{L}q(p) = \int_0^{+\infty} q(t)e^{-pt} \, dt$$

defines a $C^1$ class function for $p$ in the domain (81). ■

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