I. INTRODUCTION

At low temperatures the physics of crystal surfaces on disordered substrates is dominated by the randomness rather than thermal fluctuations. In 2+1-dimensions this elastic surface is expected to have a roughening phase transition at a critical temperature $T_c$ corresponding to a height-height correlation function $\log(r)$ and $\log^2(r)$ respectively. The log^2(r)-superrough behavior was numerically confirmed at finite temperature via Monte Carlo simulations as well as in the limit of a vanishing temperature via exact ground state calculation using combinatorial optimization methods.

In this paper, we study the stability of the low-temperature (glassy) phase of the solid-on-solid model (SOS) on a disordered substrate with respect to the formation of topological point-like defects. We also consider the density of defects and the screening effect of pre-existing pairs to an introduced extra pair and allow for a vortex-core energy.

The SOS model on a disordered substrate is given by the uniformly distributed substrate height $d_i \in [0, 1]$ and the integer crystal height $n_i$ on a simple cubic $L \times L$ lattice $G$ with periodic b.c. and lattice site $i$ as schematically shown in Fig.1. The $h_i = n_i + d_i$ denotes the total surface height at site $i$ and the SOS model Hamiltonian is defined by

$$H = \sum_{\langle ij \rangle} (h_i - h_j)^2,$$

where the sum runs over all nearest-neighbor pairs $\langle ij \rangle$. To calculate the ground state of the SOS Hamiltonian we introduce the crystal height-differences $n_{kj} = n_k - n_j$ (integer) and substrate height-differences $d_{kj} = d_k - d_j$ ($\in [-1, +1]$) along the links $k = (i, j)$ on the dual lattice.

The minimal (optimal) energy configuration $\{n_i^\ast \}_{\text{min}}$ will just be the closest integer $n_i^\ast$ to $d_k^\ast$ for all links $k = (i, j)$. On the other hand, since the $n_i^\ast$ describe height-differences in the scalar field given by the $n_i$ their sum along any oriented cycle on the surface around site $i$ has to be zero, i.e. the lattice divergence of $n^\ast$ has to vanish for each site $i$:

$$(\nabla \cdot n^\ast)_i = 0.$$  

Note that $n_i$ can be considered to be a potential and $n_i^\ast$ as its force field. Obviously, for a typical disordered substrate the minimal configuration $\{n_i^\ast \}_{\text{min}}$ violates the mass balance constraint. Fig.2 shows an example of a disordered substrate with substrate height $d_i = 0.0$, $0.2$, $0.4$ and $0.6$. Consider the differences $d_k^\ast$: across the dashed line we have $d_k^\ast = 0.6$ and $|d_k^\ast| < 0.5$ elsewhere. Consequently, the absolute minimum-energy configuration without any balance constraint is given by $n_i^\ast = 1$.
II. DEFECT PAIRS IN THE SOS MODEL

The defect pairs in the disordered SOS model are source and sink nodes of strength $+b$ and $-b$, respectively, for the network flow field $n^*_k$, which otherwise fulfills $(\nabla \cdot n^*_k)_i = 0$, i.e. we have to modify the mass balance constraint \( \nabla \cdot n^*_k \) as follows

\[
(\nabla \cdot n^*_k)_i = \begin{cases} 
0 & \text{no dislocation at } i \\
\pm b & \text{dislocation at } i 
\end{cases}
\] (4)

Thus the ground state problem is to minimize the Hamiltonian \( \nabla \cdot n^*_k \) subjected to the mass balance constraint \( \nabla \cdot n^*_k \) which can be solved by the successive-shortest-path algorithm. In the following we concentrate on defect pairs with $b = \pm 1$.

The defect energy $\Delta E$ is the difference of the minimal energy configuration with and without dislocations for each disorder realization, i.e. $\Delta E = E_1 - E_0$. More precisely, for the configuration with $N$ defect pairs of Burgers charge $b = \pm 1$ we introduce two extra nodes $s$ and $t$ with $n_s = +N$ and $n_t = -N$ respectively and connect them via external edges or bonds with particular sites of the lattice depending on the degree of optimization: (a) with two sites separated by $L/2$ (Fig.3(a)), (b) the source node with one site $i$ and the sink node with the sites on a circle of radius $L/2$ around $i$ (Fig.3(b)) and (c) both nodes with the whole lattice. Case (a) corresponds to a fixed defect pair, (b) to a partially optimized pair along a circle, both separated by a distance $L/2$, and (c) to a completely optimized pair with an arbitrary separation. In all cases the energy costs for flow along these external edges are set to a positive value in order to ensure the algorithm to find the optimal defect pair on the chosen sites. These "costs" have no contribution to the ground state energy. In case of multi pairs we always use graph (c). Here, the optimal number $N$ of defects in the system is gradually determined starting with one pair ($N = 1$) with a vortex core energy $2E_c$ and checking whether there is an energy gain or not. If yes, add a further pair (with $2E_c$) and repeat the procedure until there is no energy gain from the difference of the ground state energy between two iterations.

III. SINGLE DEFECT PAIR ($N = 1$)

We study the defect energy $\Delta E$ and its probability distribution $P(\Delta E)$ on a $L \times L$ lattice with $L = 6, 12, 24, 48, 96$ and 192 and $2 \cdot 10^3 - 10^5$ samples for each size and consider the three cases (a)-(c) (see above). With an increasing degree of optimization a negative defect energy $\Delta E$ becomes more probable and its probability distribution $P(\Delta E)$ differs more and more from the Gaussian fit, Fig.4. The resulting disorder averaged defect energy $[\Delta E]_{\text{dis}}$ scales like

\[ d_i = \begin{cases} 
0.6 & \text{if } i = 1 \\
0.4 & \text{if } i = 2 \\
0.2 & \text{if } i = 3 \\
0.0 & \text{otherwise} 
\end{cases}
\]

\[ \text{FIG. 2. Example of a disordered substrate heights } d_i \text{ in a random-surface model with a single dislocation pair connected along a straight line of size } L \text{ (dashed line). The optimal surface without dislocations would be flat, i.e. } n_i = 0 \text{ for all sites } i, \text{ however, allowing dislocations would decrease the ground state energy (see text).} \]
The data collapse for optimally placed dislocations for system size $L = 96$ and $192$. The probability distribution $P(\Delta E)$ of a large-scale topological excitation with a Gaussian fit for different optimizations: (a) for a fixed defect pair, (b) for a partially optimized pair and (c) for a completely optimized pair with different system sizes $L$.

$$[\Delta E]_{\text{dis}} \sim \begin{cases} 
\ln(L), & \text{fixed defect pair} \\
-0.27(7) \cdot \ln^{3/2}(L), & \text{partially optimized} \\
-0.73(8) \cdot \ln^{3/2}(L), & \text{completely optimized}
\end{cases}$$

and its related variance $\sigma$ like

$$\sigma(\Delta E) \sim \begin{cases} 
\ln(L), & \text{fixed defect pair} \\
\ln^{2/3}(L), & \text{partially optimized} \\
\ln^{1/2}(L), & \text{completely optimized}
\end{cases}$$

where the exponents are approximations for the best data collapse. The defect energy indicates that for the optimized cases dislocations can proliferate due to thermal fluctuations and melt the elastic superrough phase. Furthermore, for a growing degree of optimization the scaling amplitude of $[\Delta E]_{\text{dis}}$ increases.

The mean length (mass) $l_{DL}$ of the line connecting the two defects scales with the system size $L$ according to the fractal dimension

$$d_f = 1.28 \pm 0.02$$

for the fixed and partially optimized situation.

For the completely optimized case Fig. 5 shows a probability distribution $P(l_{DL})$, which behaves like

$$P_L(l_{DL}) \sim \frac{1}{L} \cdot p\left(\frac{l_{DL}}{L}\right).$$

**IV. MULTI DEFECT PAIRS ($N > 1$)**

Next, we study the effect of a uniformly given vortex-core energy $E_c$ to the system of multi defect pairs ($N > 1$) as a simplification of the real situation with a distribution of $E_c$. As shown in Fig. 6(a), the density $\rho$ of defects decays exponentially with an increasing $E_c$, i.e.

$$\rho(E_c) \sim e^{(-E_c/E_0)^\alpha}.$$  

For the $E_0$ and $\alpha$ we can distinguish between two intervals of $E_c$ which refers to a stretched and a pure exponential decay, respectively. In detail we have

| $E_c$ | $E_0$ | $\alpha$ |
|-------|-------|---------|
| $[0, \infty]$ | $0.75 \pm 0.15$ | $0.6 \pm 0.2$ |
| $[0, E_c^{\text{max}}(L)]$ | $0.45 \pm 0.03$ | 1 |

The upper limit $E_c^{\text{max}}(L)$ corresponds to the maximal mean length $l_{DL}$ for each system size $L$, c.f. Fig. 6(a) and (b), and scales like

$$E_c^{\text{max}} \approx (\text{const.} + 0.47 \pm 0.02) \cdot \ln(L)^{3/2}.$$  

Moreover, we found the same scaling behavior for the vanishing defect energy, i.e. $[\Delta E]_{\text{dis}} = 0$:

$$E_c^0 \approx (\text{const.} + (0.47 \pm 0.01) \cdot \ln(L))^{3/2}.$$  

From the plot of the maximal mean length $l_{DL}$ (Fig. 6(b)) vs. the system size $L$, i.e. $l_{DL}(E_c^{\text{max}}) \sim L^{d_f}$, the fractal dimension $d_f$ is given by

$$d_f = 1.267 \pm 0.07,$$
FIG. 6. (a) Density $\rho$ of defects with respect to the vortex core energy $E_c$ for different system sizes $L = 6 - 48$ and $10^4$ up to $10^7$ samples. The log-lin plot indicates an exponential decay of $\rho$. Simultaneously, (b) the mean distance $l_{DL}$ of all dislocation pairs vs. the vortex core energy $E_c$. Comparing (a) and (b) one sees that the maximal lengths $l_{DL}$ occurs at the cross-over energy $E_{\text{max}}$ (see text). 

Finally, we focus on the effect of introducing an extra fixed defect pair separated by $L/2$ to an already (completely) optimized configuration with a vortex core energy $E_c$. This extra pair costs

$$\Delta E_{\text{fix}} = E_1' + 2E_c - E_1,$$

where $E_1$ denotes the ground state energy for $N$ (pre-existing optimal) pairs and $E_1'$ the energy for $N + 1$ optimally placed pairs, both for the same disorder configuration $\{d_i\}$. As plotted in Fig. 6, $\Delta E_{\text{fix}}$ is constant in $L$, but linear in $E_c$, i.e.

$$\Delta E_{\text{fix}}(L) = (0.17 \pm 0.02) + (4.35 \pm 0.02) \cdot E_c.$$  

Thus, one obtains a screening effect of the defect-defect interaction due to disorder-induced dislocations. In comparison, Fig. 7 also shows the case for a single pair ($N = 1$) without pre-existing pairs as studied in section III.

V. RELATED MODELS

A similar picture of the effect of dislocations to a randomly pinned elastic media at $T = 0$ were found for other discrete models, the fully-packed loop (FPL) model and the matching model both on a bipartite hexagonal lattices with a linear energetic cost function and periodic b.c.

In the case of a single fixed defect pair we found the same $\ln(L)$-behavior of the defect energy as for the excitation step in Ref. but got a smaller fractal dimension $d_f = 1.28(2)$ rather than $d_f = 1.35(2)$. The disorder-induced dislocations turned out to destroy the quasi-long-range order of the elastic phase due to a negative scaling behavior of defect energy $\Delta E$ with respect to $L$ for optimally placed defects, i.e. $\Delta E \sim -\ln^{3/2}(L)$, in good agreement with the results of the FPL model.

When taking into account screening and an uniform vortex core energy $E_c$ in addition to the energy balance $\Delta E$, one finds that the energetic costs $\Delta E_{\text{fix}}$ of an introduced fixed pair does not depend on the system size $L$, as shown for the FPL and matching model. In addition we found for the disordered SOS model a linear dependence of $\Delta E_{\text{fix}}$ on the vortex core energy $E_c$ (Fig. 7). One concludes that this extra pair is screened by the pre-existing defect pairs. For the exponentially decay of density $\rho$ of the dislocation pairs we distinguished between
(a) the whole range of the vortex core energy \( E_c \) and (b) a range with an upper limit \( E_{\text{max}}^\alpha (L) \), for which (latter case) the mean defect length \( l_{\text{DL}} \) was found to be maximal. The case (a) corresponds to \( \alpha = 1 \) and \( E_0 = 0.45(3) \) and the maximal length \( l_{\text{DL}} \) related to the cross-over energy \( E_{\text{max}}^\alpha (L) \) behaves as \( l_{\text{DL}} \sim L^{d_f} \) with the fractal dimension \( d_f = 1.267(7) \). Both results were also found in Ref.\textsuperscript{[12]}. For the case (b) we get \( \alpha \approx 0.75 \) (close to 2/3) and \( E_0 \approx 0.6 \) in good agreement with Ref.\textsuperscript{[11]}.  

Finally, we relate the SOS model \textsuperscript{[10]} to the continuum description of a randomly pinned elastic medium on large length scales given by the sine-Gordon model Hamiltonian

\[
H = \int d^2 r \left[ \frac{K}{2} (\nabla u(r))^2 - w \cos (2\pi (u(r) - d(r))) \right] 
\]

(15)

where \( K \) is the elastic constant and \( d(r) \) a random field out of [0,1]. The first term represents the elastic energy \( E_{\text{el}} \) and the second one the random pinning energy \( E_{\text{pin}} \). The model is known to describe a weakly disturbed vortex lattice in a thin two-dimensional (2D) superconducting film introduced by a parallel field\textsuperscript{[12]}. Other experimental realizations are charge density wave\textsuperscript{[13]} and Wigner crystallization of electrons\textsuperscript{[14]}.  

The relation to the SOS model is as follows: in the limit of an infinite coupling strength \( w \rightarrow \infty \) and \( T = 0 \) the sine-Gordon model maps onto a lattice SOS model Eq.\textsuperscript{[10]}, as the cosine-term of Eq.\textsuperscript{(15)} forces the displacement field \( u(r) \) to be \( u(r) = d(r) + n(r) \), where \( n(r) \) is an integer. One can identify the \( u(r) \) as the continuous height field \( h(r) \) of the SOS model.

The results of the analytical study of the sine-Gordon model\textsuperscript{[12,15]} are in good agreement with our results, but only refer to the cases of fixed and completely optimized pairs. Furthermore, these studies allow another interpretation of the defect energy \( \Delta E \) and density \( \rho \). From the calculation of the elastic energy \( E_{\text{el}} \) and defect energy \( \Delta E_{\text{el}} \) one gets that for a fixed pair the elastic energy \( E_{\text{el}} \) dominates the pinning energy \( E_{\text{pin}} \), i.e. \( \Delta E \sim E_{\text{el}} \), and for the completely optimized pair the situation is vice versa, i.e. \( \Delta E \sim E_{\text{pin}} \). The resulting scaling behavior is found to be \( \Delta E \sim \ln (L) \) and \( \Delta E \sim -\ln^{3/2} (L) \), respectively.\textsuperscript{[12,16]} The scaling behavior of the fixed dislocation pair in presence of pinning disorder is essentially equivalent to the one of a fixed defect pair at finite temperatures \( T \) without disorder, i.e. \( \Delta E \sim \ln (L) \sim E_{\text{pin}}^\text{pure} (T) \).

The density \( \rho \) of defects can be related to the length scale \( \xi_{\text{DF}} \) beyond which the dislocations become unpaired\textsuperscript{[16]} since for \( \alpha \approx 0.6 \) the density \( \rho \), Eq.\textsuperscript{15}, shows the same scaling behavior as \( \xi_{\text{DF}} \) in the case of low temperatures and large core energy \( E_c \), i.e. \( E_c \gtrsim K \ln (\xi_{\text{DF}}) \). For \( E_c \approx 0 \) we found large densities \( \rho \) and one is probably out of the regime given by \( E_c \gtrsim K \ln (\xi_0) \). This would possibly explain the occurrence of the stretched exponential behavior close to \( E_c \approx 0 \) as seen in Fig.\textsuperscript{1}(a).

To summarize we have studied the effect of dislocation pairs on the ground state properties of the SOS model on a disordered substrate. For a fixed position of the dislocation pair a distance \( L \) apart we found that on average the defect costs an energy proportional to \( \ln (L) \), in agreement with the findings for the energy costs for a step-like excitation step with fixed end points reported earlier\textsuperscript{[14]} and also in agreement with recent results for other two-dimensional lattice models\textsuperscript{[12,15]}. On the other hand if we optimize the position of the dislocation pair we showed that it gains energy, namely an amount proportional to \( \ln (L) \) with an exponent \( \psi \) around 3/2 as predicted by scaling arguments and also observed in the FPL model.\textsuperscript{[12]} When introducing a penalty for the topological defects (i.e. a core energy) we showed that the density of defects vanishes exponentially as a function of this core energy, which is in agreement with the results for the FPL model.\textsuperscript{[12]} Finally we also demonstrated that a dislocation pair is screened by the presence of other dislocations in the system.

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