Center-vortex dominance after dimensional reduction of $SU(2)$ lattice gauge theory

J. Gattnar, K. Langfeld, A. Schäcke$^a$, H. Reinhardt$^b$

Institut für Theoretische Physik, Universität Tübingen
D–72076 Tübingen, Germany

Abstract

The high-temperature phase of $SU(2)$ Yang-Mills theory is addressed by means of dimensional reduction with a special emphasis on the properties of center vortices. For this purpose, the vortex vacuum which arises from center projection is studied in pure 3-dimensional Yang-Mills theory as well as in the 3-dimensional adjoint Higgs model which describes the high temperature phase of the 4-dimensional $SU(2)$ gauge theory. We find center-dominance within the numerical accuracy of 10%.

---

$^a$ supported in part by Graduiertenkolleg Hadronen und Kerne

$^b$ supported in part by grant DFG Re 856/4-1
1. Introduction:
The idea that vortices are the degrees of freedom which are responsible for confinement dates back to the pioneering work of 't Hooft, Aharonov et al. [1] and Mack et al. [2]. By introducing twisted boundary conditions, 't Hooft introduced topologically stable center vortices winding around the torus (space-time manifold) in order to test the response of the Yang-Mills system to the imprinted magnetic flux. The free energy of such a flux may serve as an order parameter for (de-)confinement and was explicitly calculated in a recent lattice study [3] which is based on the method of reference [4]. Mack and collaborators explicitly extracted center degrees of freedom from the plaquettes and studied the dynamics of the emerging center vortices [2]. In particular, it was observed that randomly distributed center vortices give rise to an area law for the Wilson loop, hence, implying confinement. The same definition of center degrees of freedom was subsequently resumed in [5] and investigated by lattice calculations.

The vortex picture of confinement [1, 2] has recently experienced a revival due to the observation of center dominance of the string tension in the so-called maximal center gauge [6]. After bringing lattice configurations into this gauge, one projects each link to its nearest center element (center projection), thus obtaining an effective $Z_2$ gauge, i.e. the center vortex theory, which accounts for the full string tension [6, 7]. Furthermore, their (area) density as well as their interactions turned out to be meaningful quantities in the continuum limit [7] (see also [8]). In addition, the latter vortex picture also provides an appealing picture of the deconfinement phase transition at finite temperatures: the vortex ensemble undergoes a de-percolation transition from a phase of percolating vortices at low temperatures to a phase of small vortex clusters at high temperatures [8]. In fact, the de-percolation transition is seen in the 3-dimensional hypercubes of the 4-dimensional lattice universe, arising at fixed space slices. In these hypercubes, the vortices partially align parallel to the time axis [8]. On the other hand, the vortices which are detected in the spatial hypercube at a given time are still percolating even at high temperatures.

A detailed understanding of the high temperature phase of Yang-Mills theory is highly desirable for understanding signatures of the quark gluon plasma as it might be produced in near future collider experiments at RHIC and LHC. It was proposed in the early eighties [9] that at asymptotic temperatures $T$, 4-dimensional Yang-Mills theory effectively reduces to the 3-dimensional counterpart coupled to adjoint Higgs matter [10, 11]. The (dimensionful)

\footnote{This vortex type must not be confused with the vortex types discussed in [1] and in [2].}
coupling constant of the latter theory becomes

\[ \frac{1}{g_3^2} = \frac{1}{T g^2(T)}. \]  

(1)

Thereby \( g^2(T) \) denotes the 4-dimensional running coupling constant. Since the dimensional reduction of 4-dimensional Yang-Mills theory yields the 3-dimensional adjoint Higgs Yang-Mills theory in the confining phase [11], one expects that the spatial string tension \( \sigma_s \) (of 4-dimensional) Yang-Mills theory scales with the dimensionful parameter \( g_3^4 = g^4(T) T^2 \). Indeed, a large scale numerical analysis [12] yields

\[ \sigma_s(T) = c g_3^4 = c g^4(T) T^2, \quad c = 0.136 \pm 0.011. \]  

(2)

In this letter, we investigate the high temperature phase of \( SU(2) \) Yang-Mills theory in the center vortex picture defined in [6]. In a previous paper [8], it was shown that the spatial string tension which is calculated after vortex projection increases with increasing temperature according the expectations of dimensional reduction [2]. In order to show that the center vortex scenario is a sensible description of the Yang-Mills vacuum at high temperatures and to reveal the mechanism of dimensional reduction in the vortex picture it is essential to demonstrate that the center vortices of the 3-dimensional Yang-Mills theory coupled with adjoint Higgs matter survive the continuum limit and that a projection of the full 3-dimensional theory onto the center vortex vacuum accounts for the full (spatial) string tension. We will find an agreement of projected and un-projected string tension within the achieved numerical accuracy of 10%.

2. Center projection in the high temperature limit:

For extracting the structure of the two dimensional vortex worldsheets in four space-time dimensions, we adopt the maximal center gauge and subsequently perform center projection [6]. Let \( U_\mu(x), \mu = 0 \ldots 3 \) denote the \( SU(2) \) link variables and \( \Omega(x) \) a gauge transformation. The maximal center gauge condition amounts to maximizing the functional

\[ S_{\text{fix}} = \sum_{\{x\}, \mu} \left[ \text{tr} U_\mu^\Omega(x) \right]^2 \rightarrow \max, \quad U_\mu^\Omega(x) = \Omega(x) U_\mu(x) \Omega^\dagger(x + \mu). \]  

(3)

with respect to \( \Omega(x) \). For calculating the gauge matrices \( \Omega(x) \) for a given link configuration \( U_\mu(x) \) we use an iteration over-relaxation algorithm [6]. Although there is no conceptual Gribov problem with the gauge condition [6], one encounters a practical Gribov problem [13, 14] when looking for the
global maximum of $S_{\text{fix}}$ with numerical techniques. The practical Gribov problem can be alleviated by changing the gauge condition \(3\) to its Laplacian version \([13, 14]\) or to alternative modifications of the maximal center gauge as e.g. proposed in \([17]\). We believe, however, that the naive iteration over-relaxation algorithm is capable to grasp the essential physics of the center vortex vacuum and relegate an investigation of the center vortex properties in Laplacian gauge to future work. Once the gauge condition \(3\) is installed, center-projection of $SU(2) \to Z_2$ is performed by replacing the gauge fixed link variables by their closest center element

$$U^\Omega_\mu(x) \to \text{sign} \{\text{tr} U^\Omega_\mu(x)\} \in \{-1, +1\}.$$  \(4\)

In the following, we will establish a relation between the 4-dimensional vortex world sheets of $SU(2)$ Yang-Mills theory at high temperatures and the vortex world lines of the dimensionally reduced 3-dimensional theory. For this purpose, we first derive the gauge constraint for gauge transformations $\Omega(\vec{x})$ of the spatial hypercube. Decomposing \(3\) as

$$S'_{\text{fix}} = \sum_{\{x\}, k} \left[\text{tr} U^\Omega_k(x)\right]^2 + \sum_{\{x\}} \left[\text{tr} U^\Omega_0(x)\right]^2, \quad k = 1 \ldots 3,$$  \(5\)

and using the fact that $\text{tr} U_0(x)$ is invariant under time independent gauge transformations, the gauge condition \(3\) seamless extends to three dimensions, i.e.

$$S'_{\text{fix}} = \sum_{\{x\}, k} \left[\text{tr} U^\Omega_k(x)\right]^2 \to \max, \quad k = 1 \ldots 3, \quad \Omega = \Omega(\vec{x}).$$  \(6\)

Constructing the dimensionally reduced theory, the integration over the link variables $U_0(x)$ is replaced by an integration over the field $A_0(x)$ which lives in the algebra \([15]\), $A_0 \in \text{su}(2) \cong SO(3) \cong SU(2)/Z_2$, disregarding the center $Z_2$. In the vortex picture, this assumption is consistent with the observation that the vortex world sheets of the high temperature phase are not linked with time-like Wilson loops \([8]\).

The assumption which is inherent in the construction of the dimensionally reduced theory is that the vortex world sheets of high temperature Yang-Mills theory are aligned along the time axis and wrap around the torus in time direction. This assumption is supported by lattice calculations \([8]\). The vortex world lines which are detected in a spatial hypercube at a given time slice can be obtained by the 3-dimensional version of the maximal center
gauge fixing condition (6) and by standard center projection which consider spatial link variables only.

3. Three-dimensional pure SU(2) gauge theory:
Resorting to pure 3-dimensional Yang-Mills theory (i.e. without adjoint Higgs matter), one obtains in the continuum limit [18]

\[ \sigma_s \approx 0.11 g_3^4, \quad \text{(pure 3-D YM-theory)} \]  

This value is remarkably close to the value (2) obtained in the full 4-dimensional theory. This indicates that the static quark correlations within the spatial bulk are dominated by the 3-dimensional Yang-Mills theory. Note, however, that the quark potential of the 4-dimensional Yang-Mills theory at high temperatures is related to correlations of the adjoint Higgs field and sensitively depends on the Higgs gauge field couplings [10].

In a first step, we will investigate 3-dimensional pure SU(2) Yang-Mills theory with Wilson action and we will neglect the coupling to the adjoint Higgs matter. This will be an approximation of the dimensionally reduced 4-dimensional Yang-Mills theory, but will shed light onto the confinement mechanism of 3-dimensional SU(2) gauge theory, which is an interesting issue on its own.

In this section, we will calculate the static quark anti-quark potential in the projected and the un-projected theory as well as the center vortex area density and we will extrapolate the data to continuum limit. The (spatial) string tension \( \sigma_s \) in units of the lattice spacing \( a \) is a function of \( \beta = 4/g_3^2 a \), the only parameter of the theory. High statistics Monte-Carlo simulations [18] show

\[ \sigma_s a^2 = \frac{1.788}{\beta^2} \left( 1 + \frac{1.414}{\beta} + \ldots \right) \quad \text{for} \quad \beta \geq 3. \]  

The continuum limit is approached by taking the limit \( \beta \to \infty \) at a fixed value of the reference scale \( \sigma_s \).

In a first step, we compared the static quark anti-quark potential \( V_{\text{proj}}(r) \) as a function of the distance \( r \) of the quark anti-quark pair with the result which is calculated from the projected link variables (for the derivative of this potential see figure [1]). The calculation were carried out on \( 20^3 \) lattice, and three attempts were made to find the global maximum of (3) with the iteration over-relaxation algorithm. The fit \( V'(r) = \sigma_s + a/r \) to the full result yields \( \sigma_s = 0.11 g_3^4 \) which is in perfect agreement with data presented in [18]. We observe a slight increase of the projected potential with increasing
Figure 1: The derivative of the static quark anti-quark potential of the full 3-dimensional theory and calculated from $Z_2$-projected links.

$r$. The (spatial) string tension obtained from the projected potential is in agreement with the full (spatial) string tension within statistical errors. We recover the same qualitative behavior of the potential as we did in the case of the 4-dimensional theory: the short distance behavior due to gluon radiation is changed by projection while the long range physics of the potential is roughly un-changed. This signals center vortex dominance of the (spatial) string tension in 3-dimensional Yang-Mills theory.

In a second step, we rephrase the $Z_2$ gauge theory which is obtained by projection (4) in terms of vortices. A vortex is said to pierce an elementary plaquette if the plaquette calculated with $Z_2$ links yields $(-1)$. One then shows by virtue of the $Z_2$ Bianchi identity that this vortex material forms closed loops (in three dimension). Guided by the 4-dimensional investigation [7], we investigate whether these vortices extrapolate to the continuum limit $a \to 0$ by calculating the vortex area density $\rho a^2$ for large values of $\beta$. Since it is believed that the vortex area density is the relevant scale for the projected quark anti-quark potential, we calculated the ratio of this density and the derivative of the projected potential, i.e. $V'_{\text{proj}}(r)$, as a function of $r$. This derivative might extrapolate to the value of the full string tension (see figure [4]). The result is shown in figure [2]. The data for this ratio are
Let us compare the asymptotic value of that ratio with the corresponding value calculated in 4-dimensional Yang-Mills theory at $T \approx 2T_c$. Using $\rho_s$ reported in [8] and the spatial string tension provided in [12], one finds

$$\frac{\rho}{\sigma_s} \approx 0.33,$$

(4-D Yang-Mills theory, $T \approx 2T_c$). (9)

There is an agreement of the ratio (9) with the corresponding value estimated from 3-dimensional Yang-Mills theory at the 10% level.

4. Three-dimensional $SU(2)$ adjoint Higgs theory:
It was observed in [10] that static quark anti-quark potential of the high temperature phase of 4-dimensional is well reproduced be an effective 3-dimensional theory described in terms of the action

$$S_{\text{eff}} = S_{YM}(U) + S_{\text{hop}}(U, A_0) + S_{\text{ac}}(A_0),$$

(10)
where $S_{YM}$ is the 3-dimensional Wilson action. The field $A_0$ lives in the space of the $SU(2)$ algebra. Its is a reminder of the zeroth component of the 4-dimensional gauge fields $A_\mu$, $\mu = 0 \ldots 3$. It couples like an adjoint Higgs field in the reduced theory [10], i.e.

$$S_{\text{hop}}(U, A_0) = -\frac{1}{2} \beta \sum_{\{x\}, k} \text{tr} \left[ A_0(x) U_k(x) A_0(x + k) U_k^\dagger(x) \right], \quad (11)$$

$$S_{\text{sc}}(A_0) = \frac{1}{2} \beta \sum_{\{x\}} \left\{ (3 + h/2) \text{tr} A_0^2 + \kappa \left[ \frac{1}{2} \text{tr} A_0^2(x) \right]^2 \right\}. \quad (12)$$

We use hermitian Pauli matrices as generators of the $SU(2)$ algebra. The dimensionally reduced theory is thus described by the partition function

$$Z_{\text{eff}} = \int \mathcal{D}U \mathcal{D}A_0 \exp\left\{ -S_{\text{eff}}(U, A_0) \right\}. \quad (13)$$

The integration over the link variables $U$ (of the 3-dimensional lattice) takes into account the Haar measure, while the integration over the $A_0$ field is carried out with a flat measure. A comparison of the static potential calculated
Table 1: Parameter sets of the effective dimensionally reduced theory for a 24\(^3\) lattice taken from [10].

| set  | \(T/T_c\) | \(\beta_4\) | \(\beta\) | \(h\)  | \(\kappa\) |
|------|----------|----------|--------|------|--------|
| set 1| 2.0      | 2.50     | 12.25  | -.30 | 0.106  |
| set 2| 3.5      | 2.80     | 13.54  | -.26 | 0.094  |
| set 3| 6.0      | 3.00     | 14.48  | -.24 | 0.086  |

with the reduced theory (13) with the same quantity obtained in the high temperature 4-dimensional Yang-Mills theory yields values for the effective coupling constants \(\beta\), \(h\) and \(\kappa\). The values which we will use below were taken from refs. [10, 11] (see table [1]).

Interpreting the effective field theory acting in the spatial hypercube of the full 4-dimensional theory as a field theory in 2 + 1 dimensions, one considers the potential calculated from spatial Wilson loops as the spatial static quark anti-quark potential. The derivative of these potential yields the spatial string tension for asymptotic values of the distance \(r\). Using the effective theory (13) and the parameters of table [1], we calculated the derivative of the spatial static potential. We compare these data with the result for the spatial static potential calculated with center projected configurations (see previous section) in figure 3. An agreement of the spatial string tension calculated from the full and the projected configurations, respectively, as well as a consistency with the asymptotic value (2) of the 4-dimensional theory is observed. Comparing figure 3 with figure 1, we conclude that the adjoint Higgs field yields minor corrections to the spatial static potential.

Finally, we have calculated the ratio of vortex (area) density and (spatial) string tension at temperatures provided by the parameter sets of table [1] using the effective theory (13). The result is shown in figure 4. Comparing the scale of the horizontal axis in figure 2 and figure 4, we conclude that the above ratio has not yet reached its asymptotic value for the parameters given in table [1].

5. Conclusions:

We have studied the spatial string tension in 3-dimensional pure Yang-Mills theory and in the 3-dimensional Yang-Mills theory coupled with adjoint Higgs matter. The latter model applies as the dimensional reduced theory describing the high temperature phase of 4-dimensional \(SU(2)\) Yang-Mills theory [10, 11].
Figure 4: The ratio of the vortex area density $\rho$ and the derivative of the projected quark anti-quark potential at several temperatures (see table I).

In both cases, we compare the (spatial) string tension of the full simulation with the value obtained when the link configurations are reduced to vortex configurations by center projection. We find that the (spatial) string tension is approximated by the vortex configurations within a numerical accuracy of 10%. The numerical error mainly results from the gauge fixing procedure and is largely due to the average over Gribov copies. A modification of the gauge fixing condition [16, 17] is advisable for improving the numerical accuracy.

Furthermore, our results indicate that the vortex area density of 3-dimensional pure $SU(2)$ gauge theory extrapolates to the continuum limit of vanishing lattice spacing. The vortex area density is only slightly changed by the coupling of the 3-dimensional Yang-Mills theory to the adjoint Higgs matter. Using these results, we estimate that $\rho/\sigma_s \approx 0.38 \pm 0.12$ which is consistent with the estimate $0.33 \pm 0.05$ obtained from a calculation using the full 4-dimensional theory.

Our results support the vortex picture of the high temperature phase of 4-dimensional Yang-Mills theory: the center vortices of the latter theory are aligned along the time axis direction by temperature effects [5], while their fingerprint in the spatial hypercube constitutes a percolating vortex cluster.
Hence, we find first evidence that the findings of dimensional reduction \cite{9,10,11,12} extend to the vortex picture.

**Note added:**
During the preparation of this manuscript, ref. \cite{19} appeared. In that work, the ’t Hooft twisted vortices were investigated in 3-dimensional SU(2) Yang-Mills theory. These investigations are complementary to the studies of the present letter, in which the dynamical properties of the center vortices which emerge in center gauge by center projection have been studied.

**References**

[1] G. ’t Hooft, Nucl. Phys. B138 (1978) 1;
   Y. Aharonov, A. Casher and S. Yankielowicz, Nucl. Phys. B146 (1978) 256.

[2] G. Mack and V. B. Petkova, Ann. Phys. (NY) 123 (1979) 442;
   G. Mack, Phys. Rev. Lett. 45 (1980) 1378;
   G. Mack and V. B. Petkova, Ann. Phys. (NY) 125 (1980) 117;
   G. Mack, in: *Recent Developments in Gauge Theories*, eds. G. ’t Hooft et al. (Plenum, New York, 1980);
   G. Mack and E. Pietarinen, Nucl. Phys. B205 [FS5] (1982) 141.

[3] T. G. Kovacs and E. T. Tomboulis, *Computation of the vortex free energy in SU(2) gauge theory*, hep-lat/0002004.

[4] C. Hoelbling, C. Rebbi and V. A. Rubakov, Nucl. Phys. Proc. Suppl. 73 (1999) 527;
   C. Hoelbling, C. Rebbi and V. A. Rubakov, hep-lat/0003010.

[5] T. G. Kovacs and E. T. Tomboulis, Phys. Rev. D57 (1998) 4054.

[6] L. Del Debbio, M. Faber, J. Greensite and S. Olejnik, Phys. Rev. D55 (1997) 2298;
   L. Del Debbio, M. Faber, J. Giedt, J. Greensite and S. Olejnik, Phys. Rev. D58 (1998) 094501.

[7] K. Langfeld, H. Reinhardt and O. Tennert, Phys. Lett. B419 (1998) 317;
   M. Engelhardt, K. Langfeld, H. Reinhardt and O. Tennert, Phys. Lett. B431 (1998) 141.
[8] K. Langfeld, O. Tennert, M. Engelhardt and H. Reinhardt, Phys. Lett. B452 (1999) 301; M. Engelhardt, K. Langfeld, H. Reinhardt and O. Tennert, Phys. Rev. D 61 (2000) 054504.

[9] T. Appelquist and R. D. Pisarski, Phys. Rev. D23 (1981) 2305.

[10] P. LaCock, D. E. Miller and T. Reisz, Nucl. Phys. B369 (1992) 501.

[11] L. Karkkainen, P. Lacock, D. E. Miller, B. Petersson and T. Reisz, Nucl. Phys. B418 (1994) 3.

[12] G. S. Bali, J. Fingberg, U. M. Heller, F. Karsch and K. Schilling, Phys. Rev. Lett. 71 (1993) 3059; G. S. Bali, K. Schilling, J. Fingberg, U. M. Heller and F. Karsch, Int. J. Mod. Phys. C4 (1993) 1179.

[13] T. G. Kovacs and E. T. Tomboulis, Phys. Lett. B463 (1999) 104.

[14] V. G. Bornyakov, D. A. Komarov, M. I. Polikarpov and A. I. Veselov, P-vortices, nexuses and effects of gauge copies, hep-lat/0002017.

[15] J. C. Vink and U. Wiese, Phys. Lett. B289 (1992) 122; A. J. van der Sijs, Prog. Theor. Phys. Suppl. 131 (1998) 149.

[16] C. Alexandrou, M. D’Elia, P. de Forcrand, Jun 1999. 3pp. talk presented at LATTICE 99, Pisa, Italy, Jun 1999; hep-lat/9907028.

[17] K. Langfeld, M. Engelhardt, H. Reinhardt and O. Tennert, talk given by K. Langfeld at LATTICE99, Pisa, Italy, Jun 1999; Nucl. Phys. Proc. Suppl. B83 (2000) 506.

[18] see e.g. M. J. Teper, Phys. Rev. D59 (1999) 014512.

[19] A. Hart, B. Lucini, Z. Schram and M. Teper, Vortices and confinement in hot and cold D=2+1 gauge theories, hep-lat/0005010.