Abstract
Consistent Recalibration (CRC) models have been introduced to capture in necessary generality the dynamic features of term structures of derivatives' prices. Several approaches have been suggested to tackle this problem, but all of them, including CRC models, suffered from numerical intractabilities mainly due to the presence of complicated drift terms or consistency conditions. We overcome this problem by machine learning techniques, which allow to store the crucial drift term’s information in neural network type functions. This yields first time dynamic term structure models which can be efficiently simulated.

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1 Introduction

Term structures of prices exist in many different markets and belong to the most challenging topics for dynamic modelling in mathematical finance or econometrics. Reasons for this are two-fold: first, we have to deal with potentially infinitely many, strongly dependent prices satisfying mutual relations, and, second, the dynamics of those prices is subject to absence of arbitrage conditions. Whence it is a delicate issue to write models which take the full market information as state variables into account, i.e. all available prices, and which evolve at the same time in a way which satisfies all constraints. The problem has been successfully dealt with in the context of bond prices within the Heath-Jarrow-Morton framework [16], but already the term structure of plain vanilla option prices on one underlying \( (S_t) \) has so far been too challenging to come up with fully satisfying solutions despite deep theoretical insights. Of course there are more involved term structures stemming from volatility cubes in treasury or the combined S&P – VIX term structures. We consider our work a first step in the direction of tractable term structure modelling.

Let us be more precise on this point (we assume an environment free of interest rates): the classical approach to modelling term structures consists in choosing a class of stochastic processes \( (S_t) \), which models the price of the underlying with respect to a pricing or physical measure, on a stochastic basis, which encodes the information structure through a filtration. Given market data one model is selected from the given class via calibration, i.e. the procedure guaranteeing that all market prices are reproduced by the model. Then new products are priced and hedged with the calibrated model. In the realm of the term structure of option prices on one underlying, parametric models like SABR, Heston or rough volatility models are used for this purpose; similarly, we can use non-parametric models like local volatility models which are stochastic local volatility models. This works in many respects very well, but suffers from dynamic shortcomings, i.e. newly arriving information leads via recalibration to a new model choice, whence an inconsistency over time in modelling (see, for example, [11], [13], [14], [26]).

Term structure models try to overcome this issue by making market prices state variables of the model: the price to pay for this neo-classical approach is complexity. We shall outline without going into detail some of the suggested
approaches here. Let us denote by \((C_t(T,K))_{0 \leq t \leq T}\) the stochastic process of plain vanilla options on one underlying \((S_t)\). Here \(K\) denotes the option's strike price and \(T\) its maturity; let us ignore interest (and dividend) rates for the moment. The dynamic and static no arbitrage constraints are expressed on the given stochastic basis by the existence of an equivalent measure \(Q\) such that

\[
E_Q[(S_T - K)_+ | \mathcal{F}_t] = C_t(T,K)
\]

and

\[
C_t(T,0) = S_t
\]

for \(0 \leq t \leq T\) hold true. The question has been raised which codebook should be used to facilitate the most in dealing with those constraints. Roughly speaking three suggestions have been made, which we shortly introduce here:

- Given the current market price \(S_t\) there is a unique (implied) volatility \(\sigma_t(T,K)\) such that the Black-Scholes formula BS produces the correct market price

\[
BS(T,K,S_t,\sigma_t(T,K)) = C_t(T,K)
\]

for \(0 \leq t \leq T\). This, however, yields two problematic aspects: first, how to deal with the dynamic absence of arbitrage, and, second, how to express the static absence of arbitrage conditions for implied volatilities. Under some regularity assumptions it turns out that one can write necessary conditions for such a dynamics by imposing that \((\sigma_t(T,K))_{0 \leq t \leq T}\) remains within the set of statically arbitrage free surfaces and that the process

\[
(BS(T,K,S_t,\sigma_t(T,K)))_{0 \leq t \leq T}
\]

is a martingale for all \(T,K\) provided that \(S\) is one. This case was outlined by Schweizer and Wissel in [28] following the work done by Schönbucher in [27].

- Given the current market price \(S_t\) there is a unique local volatility \(\sigma(t,s)\) such that the pricing operator \(P\) for plain vanilla calls of the local volatility equation

\[
dX_r = \sigma(r,X_r) dB_r; \quad X_t = S_t
\]

produces the market prices \(C_t(T,K)\) for \(T \geq t\) and all \(K\). Here situation is simpler, since local variance must simply be non-negative, however, the dynamic absence of arbitrage is more involved: first, we impose that \(S\) has to be a martingale, and, second, we need that

\[
(P(S_t,\sigma_t(\cdot,\cdot),T,K))_{0 \leq t \leq T}
\]

is a martingale, too, for all \(T,K\). This involves a full fledged solution operator of local volatility equations (see [2] for more information).

In a similar context, Wissel builds on top of the cited reference in [30] on a discrete expiry-strike grid an analogous codebook addressing additionally the relevant problem of existence and uniqueness of a solution for the price SDE.
• Given the current market price \( S_t \) there is a unique time-dependent Lévy process \( (b_t, c_t, \nu_t) = \mathcal{L}_t \) characterised by its Lévy triplet such that the pricing operator \( \mathcal{P} \) of the corresponding exponential Lévy martingale produces the market prices \( C_t(T, K) \) for \( T \geq t \) and all \( K \), just as before. Here the situation is slightly more complicated than in the case of local volatility, since Lévy triplets are more complicated objects than non-negative functions of two variables. On the other hand, the pricing operator is considerably simpler due to Fourier pricing. We impose again that \( S \) is a martingale, and that

\[
(\mathcal{P}(S_t, \mathcal{L}_t, T, K))_{0 \leq t \leq T}
\]

is a martingale, too, for all \( T, K \). This approach has been developed at the same time by Carmona and Nadtochiy [2] and by Kallsen and Krühner in [21], still with different choices of codebooks. Therefore, in the following we will refer to this approach as CNKK deploying author’s initials.

It is the goal of this article to make the CNKK approach work in the setting of Consistent Recalibration Models, i.e. where we consider tangent affine models. Basically speaking this amounts to storing the information of a non-linear drift operator in a neural network in an optimal way, in case when the time evolution is locally mimicking a dynamically changing affine model. In our view, this is the simplest way to consistently construct term structure dynamics, which do not come from finite dimensional realisations.

Actually this information, which is stored in the drift, corresponds to solving an inverse problem or a calibration problem, see [7] and the references therein for a general background of this problem: more precisely, it is the inversion of the above mentioned pricing operators given the current market state of the underlyings’ price and the term structure of derivatives’ prices. Let us outline this in case of the Lévy codebook: there the inverse problem corresponds to calculating the time-dependent Lévy triplet \( L_0 \) given the price of the underlying \( S_0 \) and the term structure of derivatives’ prices. Even though the map from model characteristics to prices is usually smooth, it is due to smoothing properties, often hard to invert: existence, uniqueness and stability issues (in the sense of Jacques Hadamard\footnote{The three conditions Hadamard established in the 20\textsuperscript{th} century to define well-posedness of a problem \( X \supseteq U \ni x \mapsto F(x) = y \in Y \) are existence of a solution, i.e. \( \forall x \in U, \exists y \in Y \); uniqueness of the solution; and continuity with respect to the input, that is, if we are in metric spaces with distances \( d_X \) and \( d_Y \), \( \forall \varepsilon > 0, \exists \delta > 0 \) such that \( \forall x_1, x_2 \in U \) and \( y_1, y_2 \in Y \) for which we have \( y_1 = F(x_1) \) and \( y_2 = F(x_2) \) we must have \( d_X(x_1, x_2) < \delta \Rightarrow d_Y(y_1, y_2) < \varepsilon. \) If one (or more) of these conditions is not fulfilled, a problem is said to be ill-posed.}) appear. Machine learning technology provides one way to fix these issues, which otherwise require sophisticated regularization techniques, by implicit regularization, see, e.g. [18]. In other words: learning the map from derivatives prices (given the current price of the underlying) to model characteristics \( L_0 \) and storing the information in a neural network provides an accurate map satisfying Hadamard’s requirements. We shall pursue this approach not in the originally proposed way by solving a supervised learning problem, see,
e.g. [19], but rather by storing first the information of the pricing operator in a neural network and then inverting this network, compare here, e.g. [20].

In the very same years, other applications of ML to the same problem were developed. One of these is [20], where Wiese and co-authors make use of a Generative Adversarial Networks (GANs) to learn to simulate time series of the codebook introduced by Wissel in [21]. Thanks to the particular choice of the codebook, it is relatively easy to learn new plausible prices that satisfy static arbitrage constraints on discrete grids: as we recalled above, non-negativity of the local volatility process is the only requirement in this case. On the other hand, the work we present here does not try to get rid of static arbitrage possibilities alone, but of dynamic arbitrage as well, thus answering in a more complete and satisfactory way to the need of a realistic equity option market simulator, which was the raison d'être of [20].

One could also view our current model as an unusually parameterized neural stochastic differential equation (NSDE) model, see, e.g. [7] for details on this concept. NSDEs, i.e. stochastic differential equations with neural network characteristics, are a wonderful concept to construct non-parametric models, but it is quite delicate to write constraint dynamics with neural networks. Therefore we have chosen Consistent Recalibration Models with tangent affine models, where it is easier to express constraints in terms of neural networks for the drift. A non-parametric approach alluding to NSDEs (and based on random signature methods, see [8]) will be presented in upcoming work.

The remainder of the article is structured as follows. In the second section, we introduce the concept of a Lévy triplet codebook, as shortly alluded to above, and we define consistent recalibration (CRC) models. We also briefly review affine models and embed stochastic volatility affine models in the context of CRC models, outlining some key properties of this codebook. The third section is dedicated to one of the building blocks of the whole theory: generalised Hull-White extensions. We do not simply use the Hull-White extension, as it was exploited when first defined, for the calibration at the initial time of the term structure in the interest rate models, but we think of it as a tool that allows for recalibration of the model parameters. Further, we talk about a generalised extension, since we are replacing the pure drift addition typical of interest rate models with a Lévy process, which naturally encodes a greater calibration power. To make things clearer, an example is laid out in the fourth section, where we analyse how the generalised Hull-White extension is added to the Heston model in order to get a consistent recalibration model, what we call a generalised Bates model. The same example is important because a very similar version of this model has been implemented numerically. The fifth section is devoted to the CNKK equation: how it is derived, defined and how it can be seen as a generalisation of the HJM equation. Section 6 is dedicated to the formal definition of CRC models of stochastic volatility affine models with piecewise constant model parameters for pricing stocks' derivatives. Some numerical considerations are also listed, to show what are the most relevant aspects to deal with in case of a concrete implementation. One of this point is the main subject of Section 7 where we discuss about calibrating the model.
using a neural network. In the end, the conclusion summarises the main results and novelties of the paper and an appendix collects some representative pictures of volatility surfaces obtained with our model.

**Notation.** The set $\mathbb{N}_0$ denotes the set of natural numbers with 0 included; $\mathbb{R}^m_+$ the real vectors in $\mathbb{R}^m$ whose components are greater than 0. Analogously, $\mathbb{R}_{\geq 0}$ denotes the real semi-line starting from 0 (included).

With $L(X)$ we denote the set of $X$-integrable predictable processes for a semimartingale $X$. If we talk about $(X, Y)$ as a $(m+n)$-semimartingale, we mean that $X$ is an $\mathbb{R}^m$-valued semimartingale and $Y$ is an $\mathbb{R}^n$-valued semimartingale.

Whenever we apply complex logarithm of continuous functions $\mathbb{R}^n \ni u \mapsto f(u)$, we use the normalisation $\log f(0) = 0$, so that logarithms are uniquely defined (see Proposition 2.4 in [24]).

For the sake of readability, SDE and SPDE will be used as acronyms for stochastic differential equation and stochastic partial differential equation, respectively.

### 2 Consistent Recalibration Models

We take inspiration from the seminal paper of Kallsen and Krühner [21], but we place ourselves in a more general framework that does not necessarily rely on the infinite divisibility of the processes. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, Q)$ denote a filtered probability space, where $Q$ represents a risk-neutral measure so that discounted asset prices are supposed to be $Q$-local martingales. All expectations, if not differently specified, are also taken with respect to the same probability measure and are denoted by $\mathbb{E}$. We consider an adapted (multivariate) stochastic process $X := (X_t)_{t \geq 0}$ taking values in $\mathbb{R}^n$, which can be considered as logarithms of price processes.

We assume that call options of any strike and maturity are liquidly traded and denote the time $t$ value of a call option with maturity $T$ and strike $K$ by $C_t(T, K)$. Having liquid market prices for all maturities and strikes translates, in mathematical terms, in having the marginal distributions of several (at most $n$) underlying processes (under the pricing measure). Similar to [21], we could at this point require that the given marginal distributions of $X$ are infinitely divisible, that the characteristic functions are absolutely continuous with respect to time and define the *codebook* of our model as a “forward” Lévy exponent and then define dynamics for such (infinite dimensional) codebook. But rather than focusing on the joint behaviour of $X$ and the codebook, whose dynamics are expressed in function of a generic semimartingale $M$, as done in [21], we exploit the intuition behind the choice of such codebook, since it provides easier conditions to avoid dynamic and static arbitrage, but we build around it a new framework.

For this reason, we conveniently define consistent recalibration models as models that keep consistency, which means that future realisations will be in

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2In practice, and for our ensuing numerical algorithm, this is never the case.
a neighbourhood of the current state that can always be reached with positive probability, and that are analytically tractable, thus looking as finite factor models instantaneously. This is reached by introducing stochastic parameters, whose dynamics could be extrapolated by market data, and by means of a Hull-White extension, used to compensate the stochastic updates in the parameters, while leaving the marginal distributions of the state variables unchanged.

We start defining the set of functions that is the base for our theory:

**Definition 2.1** ($\Gamma_n$). The set $\Gamma_n$ denotes the collection of continuous functions $\eta : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{C}$ such that there exists a càdlàg process $Z$ with independent increments and finite exponential moments $\mathbb{E}[\exp((1 + \varepsilon)\|Z_T\|)] < \infty$ for all $T \geq 0$ and for some $\varepsilon > 0$ satisfying

$$\mathbb{E}[\exp(i \langle u, Z_T \rangle)] = \exp \left( i \langle u, Z_0 \rangle + \int_0^T \eta(u, r) dr \right)$$

for $u \in \mathbb{R}^n$.

**Remark 2.2.** Requiring that for some $\varepsilon > 0$, $\mathbb{E}[\exp((1 + \varepsilon)\|Z_T\|)] < \infty$ implies that we can extend the left hand side of (2.1) to the strip $-i[0,1]^n \times \mathbb{R}^n$, thus we could choose, for example, $u = -i$.

**Remark 2.3.** All functions $\eta \in \Gamma_n$ are necessarily of Lévy-Khintchine type at the short end ($r = 0$ in (2.1)). Note that by doing so, we are extending the definitions given in [2] and [21] since we only assume the function $\eta$ to be of Lévy-Khintchine form at the short end.

**Remark 2.4.** Often, elements in $\Gamma_n$ are subject to additional no-arbitrage constraints in order to satisfy, for example, the martingale property for both the price processes $S = \exp(X)$ and call options $C_t(T,K)$ for all $T,K > 0$ (see Theorem 3.7 in [21]). For instance, if we consider $X = (X^i)_{i=1}^d$ as being a log-price process for some $i$, we also assume $\exp(X^i)$ is a martingale, which is equivalent to state that $\eta(e_i, r) = 0$ with $e_i$ being the $i$-th basis vector of $\mathbb{R}^n$, i.e. $\langle e_i, X_T \rangle = X^i_T$. We assume tacitly that such conditions are imposed if necessary. Notice in case of a components of $X$ corresponding to interest rates we do not need to impose such a condition (since we do not need martingality).

We can think of the set $\Gamma_n$ as a chart, in the language of geometry, or codebook, in the language of mathematical finance, for all liquid market prices at one instant of time. If we want to consider their time evolution, we had better define $\Gamma_n$-valued processes:

**Definition 2.5** ($\Gamma_n$-valued semimartingale). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ be a filtered probability space. A stochastic process $\eta$ is called a $\Gamma_n$-valued semimartingale if $(\eta_t(u,T))_{0 \leq t \leq T}$ is a complex-valued semimartingale for $T \geq 0$ and $u \in \mathbb{R}^n$ and if

$$((u, r) \mapsto \eta_t(u, r + t)) \in \Gamma_n.$$

In particular, all trajectories are assumed to be càdlàg.
**Definition 2.6** (Regular decomposition). We say that \( \eta \) allows for a regular decomposition with respect to a \( d \)-dimensional semimartingale \( M \) if there exist predictable processes \((\alpha_t(u,T))_{0 \leq t \leq T}\) taking values in \( \mathbb{C} \) with \( \alpha_t(0,T) = 0 \) for all \( 0 \leq t \leq T \) and \((\beta_i(u,T))_{0 \leq t \leq T}, C^d\)-valued, with \( \beta_i^j(0,T) = 0 \) for all \( i \) and all \( 0 \leq t \leq T \) and for \( T \geq 0 \) and \( u \in \mathbb{R}^n \) such that

\[
\eta_t(u,T) = \eta_0(u,T) + \int_0^t \alpha_s(u,T) \, ds + \sum_{i=1}^d \int_0^t \beta_i^j(u,T) \, dM_s^i
\]  

(2.2)

for \( 0 \leq t \leq T \), and

\[
\left( \sqrt{\int_t^T \| \beta_i(u,r) \|^2 \, dr} \right)_{t \geq 0} \in L(M).
\]

In view of the two new definitions, we can also generalise the condition expressed in (2.1):

**Definition 2.7** (Conditional expectation condition). Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, Q)\) be a filtered probability space, then we say that a tuple \((X, \eta)\) of an \( n \)-dimensional semimartingale \( X \) and of a \( \Gamma_n \)-valued semimartingale \( \eta \) satisfies the conditional expectation condition if

\[
\mathbb{E} \left[ \exp(i \langle u, X_t \rangle) \big| \mathcal{F}_s \right] = \exp \left( i \langle u, X_s \rangle + \int_s^t \eta_s(u,r) \, dr \right)
\]

(2.3)

for \( 0 \leq s \leq t \).

At this point, the link of the whole theory to its discrete-time counterpart exposed in [25] becomes even clearer:

**Definition 2.8** (Forward and process characteristics). Let \( X \) be an adapted semimartingale taking values in \( \mathbb{R}^n \) and \( \eta \) a \( \Gamma_n \)-valued semimartingale with \( \eta_s(0,t) = 0 \) for all \( 0 \leq s \leq t \) and for which the conditional expectation condition [2.3] is satisfied. Then the process \( \eta \) is called forward characteristic process of \( X \). Analogously, the process denoted by \( \kappa_{X_s}(0) = 0 \) for any \( r \geq 0 \), unique-

**Remark 2.9.** Note that both processes are uniquely defined (up to a \( dQ \otimes dt \)-nullset):

1. The normalisation \( \eta_s(0,t) = 0 \) for all \( 0 \leq s \leq t \) ensures that the map \( u \mapsto \eta_s(u,t) \) is continuous and uniquely defined through the use of the complex logarithm.
2. Since the adapted process \( \left( \exp \left( i \langle u, X_s \rangle - \int_0^s \kappa_{X_r}(u) \, dr \right) \right)_{s \geq 0} \) is a local martingale (see Theorem 2.11 below) and \( \kappa_{X_s}(0) = 0 \) for any \( r \geq 0 \), uniqueness follows from Lemma A.5 in [21].

**Definition 2.10** (Term structure for derivatives). We call the tuple \((X, \eta)\) of an \( n \)-dimensional semimartingale \( X \) and of its \( \Gamma_n \)-valued forward characteristic process \( \eta \) a term structure model for derivatives’ prices.
With the following theorems, we are able to characterise which processes $\eta$ can be considered forward processes, given the existence of a regular decomposition. Discrete versions of the same theorems are given in [25], while proofs for the continuous case under examination are in [21].

**Theorem 2.11.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ be a filtered probability space together with a tuple $(X, \eta)$ of an $n$-dimensional semimartingale $X$ and of a $\Gamma_n$-valued semimartingale $\eta$ satisfying the conditional expectation condition, then

- the differentiable, predictable characteristic $\kappa^X$ of the $n$-dimensional semimartingale $X$ exists and is given by $\kappa^X_t(u) = \eta_t(u, t)$ (usually called short end condition) for $t \geq 0$ and $u \in \mathbb{R}^n$, i.e. the process
  \[
  \exp \left( i \langle u, X_t \rangle - \int_0^t \eta_{s-}(u, s) ds \right) \tag{2.4}
  \]
  is a local martingale.

- If $\eta$ allows for a regular decomposition (2.2) with respect to a $d$-dimensional semimartingale $M$, then the drift condition
  \[
  \int_t^T \alpha_t(u, r) dr = \eta_t(u, t) - \kappa^{(X,M)}_t(u, -i \int_t^T \beta_t(u, r) dr) \tag{2.5}
  \]
  holds for $0 \leq t \leq T$ and $u \in -i[0, 1]^n \times \mathbb{R}^n$.

It might be useful for the reader having in mind the following expression for the forward characteristics of the $(n+d)$-semimartingale $(X, M)$: for all $t$ such that $0 \leq t \leq T$, we have

\[
\exp \left( \eta^{(X,M)}_t(u, v; T) \right) = \mathbb{E} \left[ \exp \left( i \langle u, (X_T - X_t) \rangle + i \langle v, (M_T - M_t) \rangle \right) \right]_{\mathcal{F}_t}
\]

from which it is easier to derive the expression for $\kappa^{(X,M)}_t$ for $0 \leq t \leq T$.

**Theorem 2.12.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ be a filtered probability space together with a tuple $(X, \eta)$ of an $n$-dimensional semimartingale $X$ and a $\Gamma_n$-valued semimartingale $\eta$. Furthermore, assume that $\eta$ allows for a regular decomposition (2.2) with respect to a $d$-dimensional semimartingale $M$ such that the predictable characteristics of $X$ satisfy (2.4) and such that the drift condition (2.5) holds, then the conditional expectation condition holds true.

**Corollary 2.13.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ be a filtered probability space together with a tuple $(X, \eta)$ where $X$ is a $n$-dimensional semimartingale and $\eta$, $\Gamma_n$-semimartingale, satisfies the conditional expectation condition. Moreover, assume that $\eta$ allows for a regular decomposition (2.2) with respect to a $d$-dimensional semimartingale $M$ and that the processes $X$ and $M$ are locally independent,

\[
\kappa^{X,M}_t(u_1, u_2) = \kappa^X_t(u_1) + \kappa^M_t(u_2) \tag{2.6}
\]

\footnote{See [21] for a rigorous definition.}
for \( u_1 \in \mathbb{R}^n \) and \( u_2 \in \mathbb{R}^d \). Then
\[
\int_t^T \alpha_t(u,r) \, dr = -\kappa_i^M \left( -i \int_t^T \beta_t(u,r) \, dr \right)
\]
for \( 0 \leq t \leq T \) and \( u \in [0,1] \times \mathbb{R}^n \) and, furthermore, the conditional expectation Condition (2.3) rewrites as
\[
\mathbb{E} \left[ \exp \left( \int_s^t \eta_r(u,r) \, dr \right) \bigg| \mathcal{F}_s \right] = \exp \left( \int_s^t \eta_s(u,r) \, dr \right)
\]
for \( 0 \leq s \leq t \).

Proof. To obtain the new form of the conditional expectation condition is enough to use (2.4).

The two previous theorems basically ratify the equivalence between the conditional expectation condition on one hand, and the short end and drift conditions on the other. In [21], since they assume \( \eta \) being of Lévy-Khintchine type for all times, it is possible to show equivalence with the fact of \( S = \exp(X) \) and \( C_t(T,K) \) being martingales. This is not given for free in our settings, but requires additional assumptions (see Remark 2.4). For example, requiring that \( S = \exp(X) \) is a 1-dimensional martingale is equivalent to the condition \( \eta_s(-i,t) = 0 \) for all \( 0 \leq s \leq t \). In this case, indeed, we can write
\[
\mathbb{E} \left[ e^{iu(X_t-X_s)} \bigg| \mathcal{F}_s \right] = \exp \left( \int_s^t \eta_s(u,r) \, dr \right)
\]
and for \( u = -i \) we have \( \mathbb{E} \left[ e^{X_t-X_s} \bigg| \mathcal{F}_s \right] = 1 \) for all \( 0 \leq s \leq t \).

Remark 2.14. Forward characteristics encode the term structure of distributions of increments of the stochastic process \( X \), i.e. for \( 0 \leq t \leq T \), the distributions of \( X_T - X_t \) conditional on the information \( \mathcal{F}_t \) at time \( t \). Notice that there is redundant information in processes of forward characteristics, which then translates into the drift conditions (2.5).

2.1 Affine processes

In this subsection, we introduce affine processes and give some important results on their forward characteristic processes. Moreover, since we are mainly interested in affine stochastic volatility models, we will state some properties for their particular case.

Let \( D \) be a non empty Borel subset of \( \mathbb{R}^d \) to which we associate the set \( \mathcal{U} := \{ u \in \mathbb{C}^d : \sup_{x \in D} \Re \langle u, x \rangle < \infty \} \).

Definition 2.15 (Affine process). An affine process is a time-homogeneous Markov process \( (X_t, \mathbb{P}^x)_{t \geq 0, x \in D} \) with state space \( D \), whose characteristic function is an exponentially affine function of the state vector. This means that its transition kernel \( p_t \) satisfies the following:
• it is stochastically continuous, i.e. \( \lim_{t \to t} p_s(x, \cdot) = p_t(x, \cdot) \) weakly on \( D \) for every \( t \geq 0 \) and \( x \in D \), and

• its Fourier-Laplace transform has exponential affine dependence on the initial state. This means that there exist functions \( \Phi : U \times \mathbb{R}_{\geq 0} \to \mathbb{C} \) and \( \psi : U \times \mathbb{R}_{\geq 0} \to \mathbb{C}^d \) with

\[
\mathbb{E}_x \left[ e^{\langle u, X_t \rangle} \right] = \Phi(u, t)e^{\langle x, \psi(u, t) \rangle},
\]

for all \( x \in D \), \( u \in U \) and \( t \in \mathbb{R}_{\geq 0} \).

Remark 2.16. The existence of a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}) \) is already included by the notion of Markov process (see [23]).

Remark 2.17. The definition we gave is not the original provided by Duffie et al. in [10] but a slightly more general one: the right hand side of (2.7) is equal to \( e^{\phi(u,t)+(x,\psi(u,t))} \) as long as we know that \( \Phi(u,t) \neq 0 \), but this can be shown (24) and not postulated (as done in [10]). From now on, we assume \( \Phi(u,t) = \exp(\phi(u,t)) \). A-priori we do not even have a unique definition of the functions \( \psi \) and \( \phi \), but we can assume the normalisation \( \phi(u,0) = 0 \) and \( \psi^i(u,0) = u \) for all \( u \in U \) and all \( i = 1, \ldots, d \), which makes the functions unique.

In this subsection we build a generic example for term structure models for derivatives’ prices. Therefore, we define an affine stochastic volatility model:

**Definition 2.18 (Affine stochastic volatility model).** Let us consider a proper convex cone \( C \subset \mathbb{R}^m \) (the stochastic covariance structures). An affine stochastic volatility model is a time-homogenous affine (Markov) process \( (X, Y) \) taking values in \( \mathbb{R}^n \times C \) relative to some filtration \( (\mathcal{F}_t)_{t \geq 0} \) and with state space \( D = \mathbb{R}^n \times C \) such that

• it is stochastically continuous, that is, \( \lim_{s \to t} p_s(x, y, \cdot) = p_t(x, y, \cdot) \) weakly on \( D \) for every \( t \geq 0 \) and \( (x, y) \in D \), and

• its Fourier-Laplace transform has exponential affine dependence on the initial state. This means that there exist (deterministic) functions \( \phi : U \times \mathbb{R}_{\geq 0} \to \mathbb{C} \) and \( \psi_C : U \times \mathbb{R}_{\geq 0} \to \mathbb{C}^m \) with

\[
\mathbb{E} \left[ e^{\langle u, X_t \rangle + (v, Y_s) \mid \mathcal{F}_s} \right] = e^{\phi(u,v,t-s)+\langle (u,X_s) + \psi_C(u,v,t-s),Y_s \rangle},
\]

for all \( (x,y) \in D \), \( 0 \leq s \leq t \) and \( (u,v) \in U \), where

\( U := \{(u,v) \in \mathbb{C}^{n+m} \mid e^{\langle u, \cdot \rangle + \langle v, \cdot \rangle} \in L^\infty(D)\} \),

and the normalisations \( \phi(u,v,0) = 0 \) and \( \psi^i_C(u,v,0) = v \) for all \( (u,v) \in U \) and \( i = 1, \ldots, m \).

Remark 2.19. In line with literature on affine processes there is a \( \mathbb{C}^{n+m} \)-valued function \( \psi \), whose projection onto the \( X \)-directions is \( u \), as exemplified in (2.8). Whence we only need the projection in the \( C \)-directions, which we denote by
ψC. This corresponds to a standard assumption if we consider X as a price process: if we move Xs by a quantity x, then also Xt gets shifted by the same amount.

Functions φ and ψC are important because they allow the introduction of the so-called functional characteristics (because of complete characterisation) of the affine process \((X, Y)\). We define

\[
F(u, v) := \frac{\partial \phi}{\partial t}(u, v, t)|_{t=0}, \quad R_C(u, v) := \frac{\partial \psi_C}{\partial t}(u, v, t)|_{t=0}
\]  

(2.9)

for all \((u, v) \in U\) and continuous in \((0, 0)\) (see [24]). Equations (2.9) are called Riccati equations.

More in general, we can also define the generalised Riccati equations\(^4\) and prove the following theorem (from [22]):

**Theorem 2.20.** Suppose that \(|\phi(u, w, T)| < \infty\) and \(\|\psi_C(u, w, T)\| < \infty\) for some \((u, w, T) \in U \times \mathbb{R}_{\geq 0}\). Then for all \(t \in [0, T]\) and \(v\) with Re\(v\) \leq Re\(w\) the derivatives (2.9) exist. Moreover, for \(t \in [0, T]\), \(\phi\) and \(\psi_C\) satisfy the generalised Riccati equations:

\[
\frac{\partial}{\partial t} \phi(u, v, t) = F(u, \psi_C(u, v, t)), \quad \phi(u, v, 0) = 0 (2.10a)
\]

\[
\frac{\partial}{\partial t} \psi_C(u, v, t) = R_C(u, \psi_C(u, v, t)), \quad \psi_C(u, v, 0) = v. (2.10b)
\]

We can also derive the following proposition:

**Proposition 2.21.** Let \((X, Y)\) be a homogenous affine process taking values in \(D = \mathbb{R}^n \times C\), then for \(t \leq T\) we have that

\[
\phi(u, 0, t) = \int_0^t F(u, \psi_C(u, 0, s)) \, ds
\]

and

\[
\psi_C(u, 0, t) = \int_0^t R_C(u, \psi_C(u, 0, s)) \, ds,
\]

where \((u, v) \mapsto F(u, v)\) and \((u, v) \mapsto (R_C(u, v), y)\) are of Lévy-Khintchine form.

**Proof.** While the first part automatically comes from the definition of generalised Riccati equations, the second can be found in [22]. \(\square\)

**Corollary 2.22.** Let \((X, Y)\) be a homogenous affine process taking values in \(D = \mathbb{R}^n \times C\) and assume that the finite moment condition \(\mathbb{E}[\exp((1 + \varepsilon) \|X_t\|)] < \infty\) holds true for some \(\varepsilon > 0\), then for \(0 \leq t \leq T\)

\[
\eta_t(-iu, T) := F(u, \psi_C(u, 0, T - t)) + \langle R_C(u, \psi_C(u, 0, T - t)), Y_t \rangle
\]

defines a \(\Gamma_n\)-valued semimartingale and the tuple \((X, \eta)\) satisfies the conditional expectation condition.

---

\(^4\)The name comes from the fact that they boil down to the well-known Riccati equations when \((X, Y)\) is a diffusion process.
Proof. The proof follows from the previous proposition and simple algebraic operations. For any \(0 \leq t \leq T\), we have

\[
\mathbb{E}\left[ e^{(u,X_t)} \mid \mathcal{F}_t \right] = e^{\phi(u,0,T-t)+(u,X_t)+(\psi_C(u,0,T-t),Y_t)}
\]

\[
= e^{(u,X_t)+\int_0^{T-t} F(u,\psi_C(u,0,r)) \, dr + \left( \int_0^{T-t} R_C(u,\psi_C(u,0,r)) \, dr, Y_t \right)}
\]

\[
= e^{(u,X_t)+\int_0^{T-t} F(u,\psi_C(u,0,r)) + (R_C(u,\psi_C(u,0,r)),Y_t) \, dr}
\]

\[
= e^{(u,X_t)+\int_0^t F(u,\psi_C(u,0,T-t)) + (R_C(u,\psi_C(u,0,r-t)),Y_t) \, dr}
\]

\[
= e^{(u,X_t)+\int_0^t \eta_t(-iu,r) \, dr}.
\]

\[
\square
\]

In interest rate theory, where affine models proved to be a powerful tool, Hull-White extensions is realised by making the drift term time dependent and plays the fundamental role of allowing the calibration of an initial yield curve to the prescribed model. This will be the topic of the next Section.

We see in the following some applications of such theory.

Example 2.23. **Deterministic term structure of forward characteristics:**

Deterministic forward term structure models correspond to time-dependent Lévy processes. More precisely, let \((X, \eta)\) be a tuple satisfying the conditional expectation condition and assume that \(\eta\) is a deterministic, then \(X\) is an additive process and \(\eta_t(u,T) = \eta_0(u,T)\) is of Lévy-Khintchine form for every \(T \geq 0\) (compare, for example, with Definition 2.6). A particular example would be any time-dependent Lévy model.

Example 2.24. **Interest rate models:**

If the process \(X\) is one-dimensional, pure-drift and absolutely continuous with respect to Lebesgue measure, then we fall in the case treated in Corollary 2.13 and we have

\[
\int_t^T \alpha_t(u,r) \, dr = -\kappa_t^M \left( -i \int_t^T \beta_t(u,r) \, dr \right),
\]

but also

\[
\mathbb{E}\left[ \exp \left( -\int_t^T \eta_s-(u,s) \, ds \right) \bigg| \mathcal{F}_t \right] = \exp \left( -\int_t^T \eta_t(u,r) \, dr \right), \quad (2.11)
\]

from which we obtain

\[
uX_t = uX_0 - \int_0^t \eta_s-(u,s) \, ds.
\]

Equation (2.11) is also well-known in interest rate theory: if we denote with \(P(t,T)\) the price of a risk-less zero coupon bond, with \(f(t,T)\) the forward rate yield prevailing at \(t\) for \(T\) and with \(r(t)\) the short time interest rate at \(t\), then we have

\[
P(t,T) = \mathbb{E}\left[ e^{-\int_t^T r(s) \, ds} \bigg| \mathcal{F}_t \right] = e^{-\int_t^T f(t,S-t) \, dS}
\]
for $0 \leq t \leq T$ and $u \in \mathbb{R}$. Moreover, if we assumed $M$ being a Brownian motion, then we would have $\kappa^M_t(u) = -\frac{u^2}{2}$ and, thus,

$$\int_t^T \alpha_t(u, r) \, dr = - \frac{1}{2} \left( \int_t^T \beta_t(u, r) \, dr \right)^2,$$

from which, differentiating both sides with respect to $T$, we obtain the well-known HJM drift condition

$$\alpha_t(u, T) = -\beta_t(u, T) \int_t^T \beta_t(u, r) \, dr.$$

Notice that $(\eta_s - (u, s))_{s \geq 0}$ is linear in $u$, since $X$ is pure drift.

### 3 Generalised Hull-White extension

Hull-White extension of Vašíček model was performed adding a time-dependent constant drift to the equation for the short term interest rate $r$, in order to have a perfect match with the current ($t = 0$) term structure of forward rates and, thus, to enhance calibration.

In this case, we will take a more general approach and will encode the extension, represented by a Lévy process, in the constant part of the affine process (responsible for the state-independent characteristics thereof). In other words, the function $F$ will become consequently time-inhomogeneous, thus modifying the forward characteristics of the process $X$.

**Corollary 3.1.** Let $(\tilde{X}, Y)$ be a time-inhomogenous, homogeneous càdlàg affine process taking values in $\mathbb{R}^n \times C$ with time-dependent continuous $T \mapsto F_T$, and assume that the finite moment condition $\mathbb{E} \left[ \exp((1 + \varepsilon)\|\tilde{X}_t\|) \right] < \infty$ holds true for some $\varepsilon > 0$, then for $0 \leq t \leq T$

$$\tilde{\eta}_t(-iu, T) := F_T(u, \psi_C(u, 0, T - t)) + \langle R_C(u, \psi_C(u, 0, T - t)), Y_t \rangle$$

defines a $\Gamma_n$-valued semimartingale and the tuple $(\tilde{X}, \eta)$ satisfies the conditional expectation condition.

**Remark 3.2.** Here time-inhomogenous, homogenous affine processes appear as generalisation of the approaches in [2] and [21] (CNKK-approach), since we can calibrate a large variety of (virtually, any) initial term structure into $t \mapsto F_t$.

**Remark 3.3.** Although we are only modifying the forward characteristic process of $X$, the process $Y$, which is Markov in its own filtration, remains the same. This keeps the transformation simple and the processes tractable, since it does not affect the stochastic covariance structure.

The above structure increases the calibration properties of the original model. In addition, since the Lévy process is allowed to change over time, we could calibrate it to match market conditions for other instant of times (apart the initial time).
The main consequence of having such a generalised Hull-White extension is that we could compensate fluctuations (i.e. recalibrations) in the original model’s parameters with a calibration of the Lévy process, so to keep the price/volatility surface unchanged. In other words, we could consider the parameters of the original model as state variables. When this is possible, we will talk about a model that satisfies the consistent recalibration property.

A valid question, at this point, would be to know when this is possible. Are there conditions that we could impose or verify to make sure that such a compensating mechanism can always happen?

Let us denote with \((\nu^L_t)_{t \geq 0}\) the Lévy measure of the time-dependent Lévy process \(L\), with \(p_t\) and \(Z_t\) the set of parameters and state variables belonging to the time-homogeneous model at time \(t\), respectively, and with \(\nu^{p_t, Z_s}\) the Lévy measure that has the same expressive capability as the original model where we made explicit the dependence on the parameters \(p_t\) and state variables \(Z_s\), for \(s \leq t\).

**Proposition 3.4.** Let us assume that the stochastic parameter process \((p_t)_{t \geq 0}\) has trajectories, whose total variation is bounded by a deterministic constant, take values on the compact set \(\Theta\) of admissible parameters. Moreover, assume that \(p \mapsto \nu^{p, \cdot}\) is continuously differentiable and that \(p\) remains constant whenever \(Z\) leaves a prespecified compact set \(K\). Then, if for all \(t \geq 0\) we have the non-negativity condition

\[
\nu^L_t \geq \sum_{0 \leq s \leq t} \nu_t^{p_s, Z_s} - \nu_t^{p_s, Z_s},
\]

(3.1)

the consistent recalibration property holds.

**Proof.** The proof is done by induction on the jumping times of the parameter process \(p\). For more details, see [25].

As already mentioned above, this add-on will transform the functional characteristic \(F\) in a time-dependent function and it might be worth noticing how this happens in practice.

Using the same notation introduced in [25], we can define \(F_T\) as it appears in Corollary 3.1 adding a new time-dependent function \(\mu\):

**Definition 3.5 (Inc\(^D\)).** Let \(Z\) be a generic stochastic process with values in the (state) space \(D\), such that all increments \(\Delta Z_s\) satisfy \(z + \Delta Z_s \in D\) for any \(z \in D\) and any \(s \geq 0\). We denote by \(\text{Inc}^D\) the set which contains all continuous functions \(\mu : \mathcal{U} \times \mathbb{R}_{\geq 0} \to \mathbb{R}\) of the type \(\mu(u, t) := \log \mathbb{E}[\exp(\langle u, \Delta Z_t \rangle)]\) for which \(\mu(0, t) = 0\) for all \(t \geq 0\).

---

5 Since parameters can be considered as state variables, they are allowed to change in time.
6 Here, we mean that the price or volatility surface created by the model and the Lévy measure should be the same.
In other words, we are adding to the “old” $F$ the cumulant generating function of the process $(\Delta Z_s)_{s \geq 0}$, that is $F_s(u, v) := F(u, v) + \mu(u, v, t - s)$ for all $u \in U$ and $t \geq s \geq 0$. This will become even clearer in the following, when we will specify our consistent recalibration model.

Analogously to what already done, we can define $\tilde{\phi}$ and $\tilde{\psi}$ as the time-inhomogeneous versions of $\phi$ and $\psi$ as solutions to the time-inhomogeneous version of the Riccati equations. In particular, for stochastic volatility affine processes, similarly to Theorem 2.20, for $s \leq t$ we can write (compare with [25]):

$$\frac{\partial}{\partial t} \tilde{\phi}(u, v; s, t) = F_t(u, \tilde{\psi}_C(u, v; s, t)), \quad \tilde{\phi}(u, v; 0, 0) = 0$$  \hspace{1cm} (3.2a)

$$\frac{\partial}{\partial t} \tilde{\psi}_C(u, v; s, t) = R_C(u, \tilde{\psi}_C(u, v; s, t)), \quad \tilde{\psi}_C(u, v; 0, 0) = v$$  \hspace{1cm} (3.2b)

with $\tilde{\psi}_C(u, v; s, t) = \psi_C(u, v; t - s)$. At this point, it is also possible to rewrite the expression for the forward characteristics of the time-inhomogeneous process $\tilde{X}$ as

$$\int_t^T \tilde{\eta}(u, r) dr = \tilde{\phi}(iu, 0; t, T) + \left\{ \tilde{\psi}_C(iu, 0; t, T), Y_t \right\}.$$  \hspace{1cm} (3.3)

In particular, for $t = 0$ we recover the characteristic function and we obtain

$$\int_0^T \tilde{\eta}_0(u, r) dr = \tilde{\phi}(iu, 0; 0, T) + \left\{ \tilde{\psi}_C(iu, 0; 0, T), y \right\}$$

and, if we denote with $C(y)$ the set of characteristic functions $\tilde{\eta}_0$, we notice that for any element $\tilde{\eta}_0 \in C(y)$, there exists (at least) one $\mu \in \text{Inc}^D$ that defines $\tilde{\eta}$ itself. It is thus possible to establish a surjective function $g$ between $\text{Inc}^D$ and $C(y)$. The existence of such $g$ is equivalent to nothing but the Condition (3.1) previously stated since we can consider any jump-time as an initial starting point for the process $(X, Y)$ due to the Markovianity of the process.

4 From Heston to Hull-White extended Bates model

Before introducing the consistent recalibration model more mathematically, let us briefly recall the Heston model, which is an affine stochastic volatility model, for $X = \log(S)$ being the log-return of the underlying price

$$dX(t) = \left(r - q - \frac{1}{2} V(t)\right)dt + \sqrt{V(t)}dW_1(t), \quad X(0) = x_0,$$

$$dV(t) = k\left[\theta - V(t)\right]dt + \sigma \sqrt{V(t)}dW_2(t), \quad V(0) = v_0,$$

$$dW_1(t)dW_2(t) = \rho dt, \quad \rho \in [-1, 1],$$  \hspace{1cm} (4.1)

and where $r$ and $q$ represent the instantaneous risk-free and dividend yields respectively and are constant, $\theta > 0$ is the long-term mean of the variance, $k > 0$
is the speed of mean-reversion, $\sigma > 0$ represents the instantaneous volatility of the variance process $V$. In order to ensure positivity of the variance process, we need to satisfy $2k\theta > \sigma^2$ (Feller condition).

For $0 \leq t \leq T$, we have that $\eta$ defines a $\Gamma_1$-semimartingale:

$$\eta_t(u,T) = F(iu, \psi_C(iu,0,T-t)) + R_C(iu, \psi_C(iu,0,T-t)) V_t,$$

where $C$ coincides with $R > 0$ and

$$F(u_1,u_2) = k\theta u_2 + (r-q)u_1,$$

$$R_C(u_1,u_2) = \frac{1}{2} u_1 (u_1 - 1) + \frac{1}{2} \sigma^2 u_2^2 + \sigma \rho u_1 u_2 - ku_2.$$

The Hull-White extension of the Heston model consists in a generalised version of the so-called Bates model in which we add a compensated jump Lévy process $L$ with Lévy measure $\nu(t, dx)$ to the dynamics of the log-return $X$:

$$dX(t) = \left( r - q - \frac{1}{2} V(t) \right) dt + \sqrt{V(t)} dW_1(t) + dL_t.$$

The first consequence that should appear obvious is that we are enriching the space of calibrated volatility surfaces, thanks to the Lévy process, while keeping the same dimensions of the state variables. Accordingly, the functional characteristic $F$ will then change to

$$F_t(u_1,u_2) = k\theta u_2 + (r-q)u_1 + \mu_L(u_1,u_2,t),$$

(4.3)

where $\mu_L$ is the cumulant generating function of $L$. As we will see below, we can establish a bijective relation between $\mu_L$ and the Lévy measure $\nu_L$.

We are talking about a generalised Bates model since the Lévy measure is also allowed to change in time and, as already said, this permits to make also other parameters time dependent.

### 4.1 Consistent recalibration (with words)

Time is mature to explain how the generalised Bates model can be used as a consistent recalibration (CRC) model. Recall that although formally there are only two state variables, now also parameters are free to change in time thanks to the compensation mechanism that the Hull-White extension provides.

Let us start at time $t = t_0$ with a log-price $X_{t_0}$, a set of parameters $p_{t_0} = (r,q,k,\theta_{t_0},\sigma_{t_0},\rho_{t_0})$, an initial variance for the log-price $V_{t_0}$ and the compensated jump Lévy process $L_{t_0}$ which represents the Hull-White extension. Note that some of the parameters are not constant, but change over time and are denoted by the time-index. This particular combination of state variables and parameters fully specifies a particular model $M_1$ among all possible models $M_i$ that can represent an implied volatility surface (IVS) without breaking

\footnote{Compensation is necessary to have a martingale process, as it is often the case in the pricing context.}
any no-arbitrage constraints and should be able to reflect those market conditions that are summarised by the IVS at time \( t_0 \). It is thus natural to write

\[ \text{IVS}_t = \text{IVS}(X_t, V_t; \theta_t, \sigma_t, \rho_t; \{T_i\}, \{K\}) \]

for the volatility surface at time \( t \). The model state variables \( X \) and \( V \) are thus able to evolve in time until \( M_1 \) is able to mirror the market. Eventually, this situation will break at time \( t = t_1 \) and a new calibration will be necessary.

1. Starting from time \( t_0 \), \( X \) and \( V \) can evolve until \( t_1 = t_0 + \Delta t \), where the new volatility surface is given by \( \text{IVS}(X_{t_1}, V_{t_1}; \theta_{t_0}, \sigma_{t_0}, \rho_{t_0}; \{T_i\}, \{K\}) \).

2. Parameters \((\theta_{t_0}, \sigma_{t_0}, \rho_{t_0})\) will move to another configuration \((\theta_{t_1}, \sigma_{t_1}, \rho_{t_1})\), but, to enforce a smooth change between the first configuration and second,

3. Also the Lévy process will be adjusted and will compensate the changes in the parameters \((\theta_t, \sigma_t, \rho_t)\) to reproduce the same IVS.

4. In this way we can represent realistically the behaviour of the market.

The **recalibration** of the Hull-White extension is also preserving the drift-condition of the forward characteristics, thus ensuring that we do not violate no-arbitrage constraints. The new model \( M_2 \) will then specify the evolution in time of the state variables until another recalibration will be needed.

Notice that the evolution of the state variables is determined by the “old” parameters \((\theta_{t_0}, \sigma_{t_0}, \rho_{t_0})\) on the closed interval \([t_i, t_{i+1}]\) and for this reason there is no discontinuity at \( t = t_{i+1} \) in the modelled IVS caused by the parameters movement. Further, since we do not know in advance the exact recalibration times \( t_i \), these are random variables (more in Section 6).
5 CNKK Equation

5.1 Quick heuristics

The mathematical formulation that is needed to describe what we sketched above starts from (2.2). If we rewrite the same equation using Musiela’s parametrisation, defining \( x := T - t \), then the map becomes \((u, t, x) \mapsto \eta_t(u, t + x)\) in the new notation. In addition, let us introduce the strongly continuous semigroup \( \{S(t) | t \geq 0\} \) of right shifts, such that for a proper function \( g \) this is mapped to \( S(t)g(u, \cdot) = g(u, t + \cdot) \). Then we can rewrite Equation (2.2) as

\[
\eta_t(u, t + x) = S(t)\eta_0(u, x) + \int_0^t S(t-s)\alpha_s(u, t+x)ds + \sum_{i=1}^d \int_0^t S(t-s)\beta_i^s(u, t + x)dM^i_s,
\]

in the new notation. In addition, let us introduce the strongly continuous semigroup \( \{S(t) | t \geq 0\} \) of right shifts, such that for a proper function \( g \) this is mapped to \( S(t)g(u, \cdot) = g(u, t + \cdot) \). Then we can rewrite Equation (2.2) as

\[
\eta_t(u, t + x) = S(t)\eta_0(u, x) + \int_0^t S(t-s)\alpha_s(u, t+x)ds + \sum_{i=1}^d \int_0^t S(t-s)\beta_i^s(u, t + x)dM^i_s.
\]

which can be rewritten in terms of \( \theta_t(u, x) := \eta_t(u, t + x) \) and, with abuse of notation, \( \alpha_t(u, x) := \alpha_t(u, t + x) \), \( \beta_t(u, x) := \beta_t(u, t + x) \) as

\[
\theta_t(u, x) = S(t)\theta_0(u, x) + \int_0^t S(t-s)\alpha_s(u, x)ds + \sum_{i=1}^d \int_0^t S(t-s)\beta_i^s(u, x)dM^i_s. \tag{5.2}
\]

Finally, the passage to the limit will justify what written in the next subsection.

5.2 CNKK SPDE

The framework we will develop in the following will allow a thorough analyses of factor models in the CNKK-approach introduced in [2] and [21], yet with crucial differences. For example, as already done by Kallsen and Krühner, we assume that the volatility processes \( \beta^i \) of the forward characteristic \( \eta \) are functions of the present state of \( \eta \) itself, i.e. for all \( i = 1, \ldots, d \)

\[
\beta^i_t(u, T)(\omega) = \sigma(t, \eta_t - (\cdot, \cdot)(\omega))(u, T),
\]

but, since we introduce the right-shift operator, we will obtain an SPDE and not simply an SDE.

**Definition 5.1 (Lévy codebook Hilbert space).** Let \( G \) be a Hilbert space of continuous complex-valued functions defined on the strip \(-i[0, 1]^n \times \mathbb{R}^n\), i.e. \( G \subset C((-i[0, 1]^n) \times \mathbb{R}^n; \mathbb{C})\).

\( H \) is called a Lévy codebook Hilbert space if \( H \) is a Hilbert space of continuous functions \( \eta : \mathbb{R}_{\geq 0} \to G \), i.e. \( H \subset C(\mathbb{R}_{\geq 0}; G) \) such that

- There is a continuous embedding \( H \subset C(\mathbb{R}_{\geq 0} \times (-i[0, 1]^n) \times \mathbb{R}^n; \mathbb{C})\),
- The shift semigroup \( (S_t\eta)(u, x) := \eta(u, t + x) \) acts as strongly continuous semigroup of linear operators on \( H \),
- Continuous functions of finite activity Lévy-Khintchine type

\[
(u, t) \mapsto i \langle a(t), u \rangle - \frac{\langle b(t), u \rangle}{2} + \int_{\mathbb{R}^n} \left( \exp(i \langle \xi, u \rangle) - 1 \right) \nu_t(d\xi)
\]
lie in \( H \), where \( a, b, \nu \) are continuous functions defined on \( \mathbb{R}_{\geq 0} \) taking values in \( \mathbb{R}^n \), the positive-semidefinite matrices on \( \mathbb{R}^n \) and the finite positive measures on \( \mathbb{R}^n \). This is for example the case for processes with independent increments and finite variation.

Remark 5.2. Notice that we do not assume that there are additional stochastic factors outside the considered parametrisation of liquid market prices.

Remark 5.3. Notice that elements of the Hilbert space \( H \) are understood in Musiela parametrisation and therefore denoted by a different letter in the sequel. As already written in Subsection 5.1 we have the relationship \( \eta_t(u, t + x) = \theta_t(u, x) \), with \( x := T - t \). In this sense, we also have the equality \( \theta_t(u, 0) = \kappa^X_t(u) \) for the predictable characteristics of \( X \).

Definition 5.4 (CNKK equation). Let \( H \) be a Lévy codebook Hilbert space. We call the following stochastic partial differential equation

\[
d\theta_t = (A\theta_t + \mu_{\text{CNKK}}(\theta_t)) \, dt + \sum_{i=1}^d \sigma_i(\theta_t) \, dB_i^t
\]  

(5.3)
a CNKK equation \( (\theta_0, \kappa, \sigma) \) with initial term structure \( \theta_0 \) and characteristics \( \kappa \) and \( \sigma \), if

- \( A = \frac{d}{dt} \) is the generator of the shift semigroup on \( H \),

- \( \sigma_i : U \subset H \to H \), \( U \) an open subset of \( H \), are locally Lipschitz vector fields, and

- \( \mu_{\text{CNKK}} : U \to H \) is locally Lipschitz and satisfies that for all \( \eta \in \Gamma_n \) we have

\[
\int_0^{T-t} \mu_{\text{CNKK}}(\theta)(u, r) \, dr = \theta(u, 0) - \kappa_\theta(u, -i \int_0^{T-t} \sigma(\theta)(r, u) \, dr ; 0),
\]  

(5.4)

where \( (\kappa_\theta)_{\theta \in U} \) is \( \Gamma_{n+d} \)-valued for each \( \theta \in \Gamma_n \), such that \( \kappa_\theta(u, 0; 0) = \theta(u, 0) \) and \( \kappa_\theta(0, v; 0) = -\frac{\|v\|^2}{2} \), for \( u \in \mathbb{R}^n \), \( v \in \mathbb{R}^d \).

Remark 5.5. \( \kappa_\theta \) is the forward characteristic process associated to the couple \( (X, B) \), where \( X \) is (still) the log-return price process and \( B \) the driving process of \( \theta \). Moreover, Equation (5.4) can be seen as a drift-condition and it is analogous to Equation (2.5), reformulated under the Musiela's parametrisation.

Remark 5.6. It is evident how Equations (5.3) and (5.2) relate to each other and how the former can be seen as the limit case of the latter.

Remark 5.7. We do not require that all solutions of Equation (5.3) are \( \Gamma_n \)-valued, which would be too strong as a condition and difficult to characterise. In particular, \( \Gamma_n \) is a more general than what we need.
Proposition 5.8. Let \( \theta \) be a \( \Gamma_n \)-valued solution of a CNKK equation and let \( X \) be a semimartingale such that the predictable characteristics satisfy
\[
\kappa_t^{(X,B)}(u,v) = \kappa_0(u,v,t)
\]
for \( u \in \mathbb{R}^n, v \in \mathbb{R}^d \) and \( t \geq 0 \), then the tuple \((X, \theta)\) satisfies the conditional expectation condition.

Proof. The proposition is basically a consequence of Theorem 2.12: the drift condition is satisfied by assumption and
\[
\exp \left( i \langle u, X_t \rangle - \int_0^t \kappa_s^{(X,B)}(u,v;s) \, ds \right)
\]
is a (local) martingale because of the Lévy-Khintchine assumption in Definition 5.1 regarding the functions of the Lévy codebook Hilbert space. \( \square \)

5.3 Generalisation of the HJM equation

Equation (5.3) is very similar to the famous HJM equation\(^8\), but there are relevant differences, for example both the drift and drift condition are different and, what is more, functions have another argument (a strike dimension) that is completely missing in the case of the HJM equation, where only a time-dimension is considered.

We can construct a particular example which corresponds indeed to the HJM equation: let us consider a situation without leverage (where the Brownian motion \( B \) is independent of the return process \( X \)), assuming that \( \kappa_0(u,v;0) = \theta(u,0) - \left\| v \right\|^2 / 2 \), for \( u \in \mathbb{R}^n, v \in \mathbb{R}^d \) and \( t \geq 0 \). Basically, we are looking at functions in a restricted Lévy space, for which the Lévy measure is null. This implies that the CNKK equation is a parameter-dependent HJM equation. In this case, Condition (5.4) can be simplified to
\[
\mu_{CNKK}(\theta)(u,x) = - \sum_{i=1}^d \sigma^i(\theta)(u,x) \int_0^x \sigma^i(\theta)(u,s) \, ds,
\]
for \( x \geq 0 \) and \( u \in \mathbb{R}^n \) (note the analogies with Example 2.24).

Example 5.9. Black-Scholes model: It might be interesting at this point to see a concrete example coming from a simpler model. If we consider asset prices described by a geometric Brownian motion \( dS_t = S_t \sigma dW_t \), where \( \sigma > 0 \) is constant and \( W \) is a standard Brownian motion, then the log-prices \( X \) are given by \( dX_t = d \log S_t = \sigma^2 / 2 \, dt + \sigma dW_t \). We find that \( \eta_t(-iu,r) = 1/2 \sigma^2 u(u - 1) = \)

\(^8\)In the literature, this is also known as HJMM equation, where the last M stands for Musiela.
$F(u)$, in the notation of (2.9). It is easy to see that $\eta$ is pure-drift and that the extended functional characteristic becomes $F_t = \frac{1}{2} \sigma_i^2 u (u - 1) + \mu_L(u, t)$, where $\mu_L$ is the cumulant of the generalised Hull-White extension.

From what has been said so far, it is clear how the CNKK equation is in fact a generalisation of the HJM equation.

Remark 5.10. It is possible to further increase the complexity of the equation, for example considering options on a term structure. In this case, we would need another argument to take into account for both drift and volatility.

Remark 5.11. All these considerations are conceivable only because we are dealing with affine processes. In general, for a return price process $X$, it might not be possible to write the conditional expectation condition and to continue with the following statements.

Remark 5.12. It is possible to generalise what has been said in this section for processes driven by infinite dimensional Brownian motions, e.g. in Equation (5.3) we could replace the sum $\sum_{i=1}^{d}$ with $\sum_{i \in \mathbb{N}}$. The theory has been paved in the book by Da Prato and Zabczyk [9], but also Chapter 2 of [12] by Filipović provides a useful and accessible introduction. We continue considering the finite-dimensional case because this does not really create new hurdles to be solved, in contrast to the main obstacle, the drift $\mu_{\text{CNKK}}$, for which no explicit expression is available.

6 Consistent recalibration (with maths)

We are now ready to face the same considerations we reported above in more rigorous settings. First of all, let us recap the most important equations. For the return process, we have

$$
\begin{align*}
\quad dX(t) & = \delta_t X(t, V_t) \ dt + \gamma_t X(t, V_t) \ dW_1(t) + dL_t, \quad X(0) = x_0, \\
\quad dV(t) & = \delta_t V(t, V_t) \ dt + \gamma_t V(t, V_t) \ dW_2(t), \quad V(0) = v_0,
\end{align*}
$$

with $dW_1(t) dW_2(t) = \rho_t \ dt$, for $\rho_t \in [-1, 1]$. Drifts and volatility coefficients are denoted by $\delta$ and $\gamma$ respectively and can be functions of $X$ and $V$, e.g. $\gamma^X(x, v) = \gamma^V(x, v) = \sqrt{v}$. While for the forward characteristic process, our codebook, we report the CNKK SPDE

$$
\quad d\theta(t) = \left[ A\theta(t) + \mu_{\text{CNKK}}(\theta(t)) \right] dt + \sum_{i=1}^{d} \sigma_i(\theta(t)) dB_i(t), \quad \theta(0) = \theta_0,
$$

where the dependence among the Brownian motions $B^i$ with $i = 1, \ldots, d$ and these with $W_j$ for $j = 1, 2$ is not specified. The key relation that connects the two different systems is given in Corollary 3.1 and is the following (rewritten in the Musiela notation):

$$
\theta_t(-iu, x) = F_{t+x}(u, \psi_C(u, 0; x)) + \langle R_C(u, \psi_C(u, 0; x)), V_t \rangle.
$$
Last but not least, we should also remember that parameters are free to move in time. As such, we consider the process \( p \), whose dynamics are exogenously given, but which are confined inside the space of admissible parameters \( \Theta \subset \mathbb{R}^M \). In the example described in Subsection 4.1 it is defined as

\[
p_t = (\theta_t, \sigma_t, \rho_t), \quad t \geq 0,
\]
given the constraints of positivity and Feller condition for \( \sigma_t \) and \( \theta_t \) and \( \rho_t \in [-1, 1] \). This is the reason why we used the subscript \( t \) in Equations (6.1) above. Let us suppose that at time \( t_0 \) the model can fit well the market surface given by observed call prices (or, equivalently, of implied volatility surface) \( C_{t_0}^{\text{obs}}(T_i, K_j) \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \). In this condition, the process \( (\theta_t, (X, V)) \) are free to evolve in time until the a new calibration is necessary. This is the case when

\[
\Delta C_{t_0} := \sum_{i=1}^{n} \sum_{j=1}^{m} [C_{t_0}^{\text{model}}(T_i, K_j) - C_{t_0}^{\text{obs}}(T_i, K_j)]^2 > \varepsilon, \tag{6.4}
\]

where \( C_{t_0}^{\text{model}} \) is the price of a call option at time \( t \) given by the model and \( \varepsilon \) is a threshold fixed a priori.

Thus, we can define the following hitting times: for \( i \in \mathbb{N} \),

\[
\begin{align*}
\tau_0 &:= \inf \left\{ t > t_0 : \Delta C_t > \varepsilon \right\} \\
\tau_{i+1} &:= \inf \left\{ t > \tau_i : \Delta C_t > \varepsilon \right\}.
\end{align*}
\tag{6.5}
\]

As already underlined in [21], the model price \( C_{t_0}^{\text{model}} \) can be expressed as a measurable function of the codebook \( \theta \) satisfying Equation (6.2). In particular, since it is a progressively measurable process (it is right-continuous on a complete probability space), we can use Début theorem, which tells us that the sequence \( (\tau_i)_{i=\mathbb{N}_0} \) is in fact made by stopping times.

**Remark 6.1.** Since we are in the same framework as [9], and indeed we could generalise all results to infinite dimensional Brownian motion, it is worth mentioning that the solution process \( \theta \) satisfies the strong Markov property.

The strictly increasing sequence \( (\tau_i)_{i=\mathbb{N}_0} \) is important since it is at these random (stopping) times that we have to run a new calibration procedure for the model. Both the “true” state variables \( X \) and \( V \) and the parameter process \( p \) are allowed to change in order to have \( \Delta C_{t_0} \) less than \( \varepsilon \) again. This is just the only first calibration problem we need to solve. Indeed, in order to compensate the changes caused by the new parameters, we have to modify \( F_t \). In particular, we can calibrate the so-called Hull-White extension part, which enters \( F_t \) as the cumulant generating function \( \mu_L \) of the Lévy process \( L \). This re-calibration ensures that we are not breaking the validity of Equation (6.3), while allowing for an “exact” match (in the sense of having \( \Delta C_t < \varepsilon \)) with the observed data. Once \( \mu_L \) is recovered, we are able to write down again the equation for the codebook \( \theta \). We can summarise this last passage more mathematically by introducing an operator \( \mathcal{I} \) such that

\[
\mathcal{I} : \Theta \times \mathbb{R}^{n+m} \to \text{Inc} \mathbb{R}^n \times C, \quad (p, (C^{\text{obs}})) \mapsto \mu_L. \tag{6.6}
\]
Remark 6.2. The cumulant generating function $\mu_L$ identifies uniquely $L$ if and only if the process $L$ has finite moments of order $n$ for all $n \in \mathbb{N}$. If we denote with $\nu_L$ the Lévy measure associated to $L$, then this is true if and only if

$$\forall n \in \mathbb{N}, \int_{\|x\| \geq 1} \|x\|^n \nu_L(t, dx) < \infty.$$ 

Eventually, we are now ready to give the definition of Consistent Recalibration (CRC) model with piecewise constant parameters $p$:

**Definition 6.3** (Consistent Recalibration Model with Piecewise-constant $p$). Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space. The quintuple $(\theta, (X, V), p, L, (\tau_i)_{i \in \mathbb{N}_0})$ is called consistent recalibration model for equity derivative pricing if for the stochastic processes $(\theta, (X, V), p)$ with values in $H \times (\mathbb{R}^n \times C) \times \Theta$ there exists a jump Lévy process $L$ (with finite moments) such that the following conditions are satisfied for all $n \in \mathbb{N}_0$:

(i) The Hull-White extension $L$ on $[\tau_n, \tau_{n+1}]$ is determined by calibration to $\theta(\tau_n)$ through $\mu_L$:

$$\theta(\tau_n)(u, 0) = \kappa \theta(\tau_n)(u, 0; 0),$$

$$\mu_{L(\tau_n)} = I \left( p(\tau_n), (X(\tau_n), V(\tau_n)), C^{obs}_n \right),$$

and for $t \in [\tau_n, \tau_{n+1}]$ we have

$$L(t) = S(t - \tau_n)L(\tau_n).$$

(ii) The evolution of $(X, V)$ on $[\tau_n, \tau_{n+1}]$ corresponds to the Hull-White extended stochastic volatility affine model determined by the parameters $p(\tau_n)$ and by the process $L(\tau_n)$:

$$(X, V)(t) = \left( X^{\tau_n, X(\tau_n), V(\tau_n)} \right)(t) \text{ for } t \in [\tau_n, \tau_{n+1}],$$

where $(X^{s,x}, V^{s,v})$ is the unique solution of the system of SDEs (6.1) on $[s, \infty)$ with initial conditions $X(s) = x$ and $V(s) = v$ and with $L_i$ replaced by $L_{t-s}$. Implicitly, we also assume that all parameters $p$ that enters the model are admissible (with the usual meaning).

(iii) The evolution of $\theta$ on $[\tau_n, \tau_{n+1}]$ is determined by $X$ and $V$ according to the prevailing Hull-White extended stochastic volatility model: for $t \in [\tau_n, \tau_{n+1}]$ and $x \in [0, \tau_{n+1} - \tau_n]$

$$\theta(t)(-iu, x) = F_{\tau_n+x}(u, \psi_C(u, 0; x)) + \langle R_C(u, \psi_C(u, 0; x)), V(t) \rangle.$$ 

For the processes $(X, V)$ and $\theta$, we use the same symbols as in Equations (6.1) and (6.2), with a slight abuse of notation, since these stochastic processes evolve in the intervals $[\tau_n, \tau_{n+1}]$ following the same dynamics, but according to the parameters $p(\tau_n)$ and to the process $L(\tau_n)$. The parameters $p$ remain constant in
each interval of the type \([\tau_n, \tau_{n+1})\) and, by construction, \((\theta, (X, V))\) is continuous on every stopping time \(\tau_n\).

In this sense, any CRC model can be seen as the concatenation of stochastic volatility affine models with static parameters.

We can now prove the following.

**Theorem 6.4.** Let \((\theta, (X, V), p, L, (\tau_i)_{i \in \mathbb{N}_0})\) be a consistent recalibration model as in Definition 6.3 and \(S = \exp(X)\) the discounted price process. Then \(S\) and European call (resp. put) option prices \(C_t(T, K)\) (resp. \(P_t(T, K)\)) are (true) martingales on \(\mathbb{R}_{\geq 0}\). Moreover, if we denote the payoff function of a call option as \(V(y) = (y - K)_+\), then the following pricing formula holds:

\[
C_t(T, K) = e^{rt} \int_{\text{Im} (\tilde{z}) - \infty}^{\text{Im} (\tilde{z}) + \infty} \Psi_T(-z) \hat{V}(z) \, dz, \tag{6.7}
\]

where \(\tau_{n+1} > T \geq \tau_n\) for some \(n \in \mathbb{N}\), \(\tilde{z}\) belongs to the analytic strip for which we have finite exponential moment (cf. Remark 2.2), \(\hat{V}\) denote the Fourier transform of \(V\) and \(\Psi_T|_{\tau_n}\) is the characteristic function of \(X_T\), i.e.

\[
\Psi_T|_{\tau_n}(u) = \mathbb{E} \left[ e^{i(u, X_T)} \mid \mathcal{F}_{\tau_n}\right] = e^{i(u, X_{\tau_n}) + \int_{0}^{T} \theta_{\tau_n}(u, r) \, dr}. \tag{6.8}
\]

**Proof.** From point (ii) of Definition 6.3 we have the couple \((X, V)\) is a stochastic volatility model on each interval of the type \([\tau_n, \tau_{n+1}]\), from which martingality of \(S\) follows by taking the conditional expectation and noting that, by Definition 5.1, we automatically have \(\theta(0, x) = 0\) for any \(x \geq 0\). We can further extend this result on \(\mathbb{R}_{\geq 0}\) because, by our own construction, the concatenation of the entire process is made in a continuous way.

Once it is established that \(S\) is a martingale, the same property follows for European call and put options by use of the tower property of the conditional expectation.

The pricing formula (6.7) comes from Fourier pricing, which was initially introduced by Carr and Madan in [4], while (6.8) comes from Corollary 3.1 where the characteristic process has been expressed in Musiela notation.

**Remark 6.5.** Having shown that the discounted price process \(S\) is a martingale, we automatically rule out arbitrage possibilities for the so-called consistent recalibration models introduced in Definition 6.3.

**Remark 6.6.** Note that the from (6.8) we see that the characteristic process \(\theta\) encodes information on the conditional expectation of \(X\), which is equivalent, as already noted in [25], to the knowledge of the entire derivative-price surface, thanks to Breeden–Litzenberger formulas.

**Remark 6.7.** Equation (6.7) can be generalised to other payoff function and is usually enriched with a dampening factor which is introduced to exploit numerical algorithm ([4]). In our case, we will use Fourier-pricing techniques in order to obtain the dataset for a supervised learning algorithm.
6.1 Numerical considerations

There are practical remarks that we should consider when dealing with CRC models as defined above in a numerical framework.

- **Simulations**: As already mentioned in [15], it is worth noting that simulating \((X, V)\) is much easier than simulating the HJM codebook, that is \(\theta\), in particular when this is infinite dimensional, as it could be the case also here. In fact, with the approach we are outlining, we do not need to simulate anything from any infinite dimensional distribution.

- **Drift term**: Even if we wanted to simulate \(\theta\) solving the SPDE (6.2), we should be able to write down explicitly the drift term \(\mu_{CNKK}(\theta)\), but, apart from some degenerate cases, this is not possible. The only way to overcome this chasm is acknowledging Equation (6.3) as a key relation for the entire construction.

- **Process \(p\)**: If we assume that a piecewise process for the parameters \(p\) is given (or obtained through calibration), then CRC models can be simulated following steps (i) to (iii) in Definition 6.3.

- **Operator \(I\)**: Last but not least, we have not specified precisely how the operator \(I\) is acting. For the moment, we will consider it as an abstract operator. This is anyway of great relevance because once we are able to recover \(L\) (or, alternatively, \(\mu_L\)), we can obtain \(\theta\) through Equation (6.3). Otherwise speaking, we could solve SPDE (6.2).

7 Deep calibration

7.1 An ill-posed inverse problem

If we look more closely to steps (i) – (iii) of Definition 6.3 it is possible to realise that the more complex aspect is given by the application of the operator \(I\). This is basically a calibration conditioned on some (new) parameters and state variables whose complexity depends on the distribution of the Lévy process \(L\). In general, this is not a trivial operation, since it consists in solving an inverse problem that is ill-posed in the sense of Hadamard even for the easiest cases (e.g. Bates model). The inverse problem is ill-posed because of an identifiability issue, which means that the information coming from market data is insufficient to exactly identify the parameters. If we express the quantity \(\Delta C_t\) of Equation (6.4) as a function of the model parameters \(\vartheta\), that is

\[
\Delta C_t(\vartheta) = \sum_{i=1}^{n} \sum_{j=1}^{m} \left| C_t^{\text{mod}}(\vartheta; T_i, K_j) - C_t^{\text{obs}}(T_i, K_j) \right|^2,
\]

we can write the identifiability problem as the fact that the function \(\Delta C_t(\vartheta)\) has many local minima. Furthermore, it is in general unclear whether these minima
can be reached by the adopted algorithm. For example, Cont and Tankov show in [5] that if one had available a set of call options prices (or, equivalently, implied volatilities) for all strikes (in a given time interval!) and a single maturity, then it would be possible to deduce all the parameters of the model and, in particular, the Lévy triplet. But in reality this is never the case, since we only know prices for a finite number of strikes and, in addition, we also have observational errors in the data. As a result, we have a serious identification problem, exemplified by the fact that we can obtain the same prices for (infinitely) many combinations of the parameters. The strategy which they begin to develop in [5] and complete in [6] is the use of the Kullback-Leibler divergence, also called relative entropy, as a regulariser in order to get a well-posed inverse problem.

To overcome this issue, we decided to follow a different strategy, making use of the implicit regularisation present in neural networks.

7.2 Learning the inverse map

Andres Hernandez was among the first who tried neural networks (NN) to address calibration tasks in Finance. In [19], he showed that a feedforward NN can actually approximate the inverse map given by the pricing formula and obtain the two parameters of the Hull-White interest rate model \((a, \sigma)\) as output of a NN. Just as a reminder, the Hull-White model consists of the following SDE

\[
    dr(t) = [\beta(t) - ar(t)] \, dt + \sigma \, dW(t),
\]

where \(a, \sigma > 0\) and \(\beta(t)\) is uniquely determined by the term structure\[7\]. The greatest achievement that he highlighted in the paper is the possibility of replacing the traditional “slow” and cumbersome calibration procedure with a new straightforward deterministic map, which makes calibration itself a very efficient task, since the core of all calculations is offset to the training phase. In fact, once a NN is trained, its application is extremely cheap from a computational point of view, being the most expensive operations simple matrix vector multiplications.

Despite the good results obtained by Hernandez, learning the map from the prices to the parameters can be critical, since this map is not known in explicit form. In principle, we do not even know if the universal approximation theorem could be applied, because the direct map could not be bijective (thus having a discontinuous inverse function). In general, since the inverse map is not known, we lack control on it and it might well be that a NN learns appropriately the map on the given training sample, but is not able to generalise on out-of-sample data. This is actually what happened when we tried to apply this approach to our problem since our situation is considerably more involved than Hernandez’.

For this reason, it is a better idea to learn the direct (or forward) map from the parameters to the prices/implied volatilities. This is done, for example, by

\[\text{This curve is calibrated through the derivative of the instantaneous forward rate } f(t,T) \text{ at time } t = 0, \text{ i.e. } \beta(t) = \frac{\partial f(0,T)}{\partial T} \left( a \right) + af(0,t) + \frac{\sigma^2}{2\pi} \left( 1 - e^{-2\sigma^2 t} \right) \text{ once the other two parameter } a \text{ and } \sigma \text{ have been calibrated.}\]
Horvath and coauthors in [20], where they implemented a feedforward NN to directly get the volatilities from the model parameters. Note that for the training of the networks, the data are artificially generated and the grid of strikes and maturities is fixed at the beginning. Again, the most appealing advantage they see in the application of NN is the possibility of enabling live calibration of derivative instruments, since the application of the NN itself only requires milliseconds avoiding the traditional bottleneck of calibration. This allows making use of new models that were before considered too computationally expansive for use, such as the rough volatility models (e.g. rough Heston or rough Bergomi model, which require Monte Carlo algorithms).

Despite the encouraging results found in [20], there are still some inconveniences using this last approach. If on one hand the advantages on the speed side are evident, on the other there are still downsides that are relevant, but not addressed by this kind of solution. Indeed, problems might come by the second step of this procedure, as denoted in [20], which is the real calibration. Even if we have to face a deterministic optimisation problem, this might not be as easy as it seems, in particular for multidimensional models. In these cases, we might need to use a local optimiser to speed up, which usually requires prior knowledge about the solution, or a global optimiser, which might take longer time.

To overcome these issues, we propose a new method that allows learning the inverse map, as already done by Hernandez, even for ill-posed problems. It is our wish to underline that the same idea could be used to learn the inverse map and solve inverse problems in fields other than mathematical finance. Adopting the same approach as Hernandez did not work out in our case, because of identifiability issues: different combinations of parameters in the stochastic volatility Hull-White extended affine model result in the same volatility surface. Thus, our idea is allowing a neural network to decide autonomously which parameters giving as output knowing that these parameters will then have to give rise to the prescribed volatility surface. In this way, we will also be able to learn the operator $I$ defined in (6.6).

In order to make the system works, we need two neural networks. The first, denoted in the following as $NN_1$, is a map between parameters and volatilities (basically, the usual pricing function, as learnt in [20]), while the second, called $NN_2$, maps volatilities to parameters (but is not trained in the usual way); in our case, to the parameters defining the Lévy process $L$. Then, we compose the two networks, where the first is trained, while the second is not, to obtain a new neural network $NN_3$ which receives in input volatilities and returns the (same) volatilities:

$$NN_3 := NN_2 \circ NN_1.$$  

In other words, $NN_3$ will learn the identity and, during the training phase, $NN_2$ will get trained. In this respect, we can see $NN_2$ as the inverse neural network of $NN_1$. The trick is as simple as that. However, notice that we might not recover exactly the same parameters that gave birth to the original IVS, but an
combination that resulted in the same surface through $NN_1$.

7.3 Numerical implementation

In broad terms, the numerical implementation follows the model outlined in Section 4, where the process $L$ is a compensated compound Poisson process in which the size of jumps is normally distributed. Since the parameters are allowed to change in time, the mean and variance of the Gaussian distribution for the jump-size of $L$ are time-dependent, while the Poisson rate is considered constant in time. The same holds true for the other parameters belonging to the “Heston” part, with the exception of the interest rate $r$, the dividend rate $q$ and $k$, which is the speed of mean reversion for the variance process. We decided to free ourselves from a static framework and to have a variable maturities-strikes grid. More precisely, we used ten different time-to-maturities $\{\tau_i\}_{i=1}^{10}$, with $\tau_1 < \cdots < \tau_{10}$ ranging between 7 and 440 days (extremes included), being more concentrated for short maturities, and thirteen different moneyness $m_1 < \cdots < m_{13}$ ranging between 0.8 and 1.2 (extremes included) in strictly increasing order. One difference with the generalised Bates model described in Section 4 is that we have a maturity-dependent jump distribution: mean and variance depend on the maturity in the sense that they are piecewise constant within two adjacent time-to-maturities, therefore, the process $L$ is here modelled through 11 parameters: the Poisson rate and then 5 tuples mean-variance for the normal distributions.

The first neural network, i.e. $NN_1$, is a 1-cell\textsuperscript{11} residual feedforward NN (see [17] for more information on ResNets) composed by 4 “main” hidden layers with 1024 nodes each. The input layer has dimension 41 and includes $r, q, \{\tau_i\}_{i=1}^{10}, \{m_i\}_{i=1}^{13}, v_0, k, \theta, \sigma, \rho, \lambda, \{\nu_i\}_{i=1}^{5}, \{\delta_i\}_{i=1}^{5}$, where $\nu_i$ and $\delta_i$ are the mean and standard deviations of the normal distributions for the jump size. The output layer has dimension 130 and includes the entire point-valued volatility surface, denoted as $\{IVS_i\}_{i=1}^{130}$.

The activation function used for all layers (apart from the output layer) is ELU. This network is trained first with artificially generated data: all parameters are sampled from uniform distributions whose extremes (parameters) are defined a priori (and are kept fixed throughout the process). Then, QuantLib Python routines (see [1]) are used to obtain in an efficient and fast way all the necessary prices. Implied volatilities are then retrieved through the algorithm outlined by Fabien Le Floc’h in http://chasethedevil.github.

\textsuperscript{10}We could think of the combinations selected by the neural network as the representatives for an equivalence class, where all members originate the same implied volatility surface.

\textsuperscript{11}With this, we mean that between the predefined hidden layers we only find one time the application of the activation function (basically, another hidden layer). The situation might be clearer by looking at Figure 2.
Figure 2: Keras representation of $NN_3$ (with just 2 “main” hidden layers instead of 4 for $NN_2$ to fit the picture in the page). $NN_1$ is summarised as model_1 at the very bottom. Between the 2 wanted hidden layers elu_2 and elu_4, it is possible to find another layer (in this sense the nomenclature 1-cell that we used in the text).
Second, \( NN_2 \) is created, but not (immediately) trained. As already explained, this second neural network will be trained only after being composed with the trained \( NN_1 \), which will be marked as non-trainable in this second phase. This composed NN is called \( NN_3 \). In order to learn the operator \( \mathcal{I} \), \( NN_3 \) will be trained and, as a side result, \( NN_2 \) will be also trained. That is to say that \( NN_3 \) is just used as a mere tool to arrive to get \( NN_2 \) trained as well. Finally, it will be then separated from \( NN_1 \). As already said, the goal of \( NN_3 \) is basically learning the identity function. Thus, the input of \( NN_2 \) is the following:

\[
\theta, \sigma, \rho, \{IVS_i\}_{i=1}^{130},
\]

while the output, since we have to learn the Lévy process \( L \), is

\[
\{\nu_i\}_{i=1}^{5}, \{\delta_i\}_{i=1}^{5}.
\]

From an architectural viewpoint, \( NN_2 \) has 4 “main” hidden layers with 1024 nodes each. The activation function is ELU, as for \( NN_1 \). While training \( NN_3 \) (and, implicitly, \( NN_2 \)), we have to provide as output the entire implied volatility structure \( \{IVS_i\}_{i=1}^{130} \), while as input the concatenation of the complete input of \( NN_2 \), plus the incomplete input of \( NN_1 \), that is everything listed above apart from \( \{\nu_i\}_{i=1}^{5}, \{\delta_i\}_{i=1}^{5} \), which have to be guessed during the training. To obtain a satisfactory training procedure, we tried also different activation functions for the output layer of \( NN_2 \). In the end, the best results were reached using the standard sigmoid function stretched to completely cover the intervals\( ^{12} \) in which \( \nu_i \) and \( \delta_i \) were (randomly) extracted. Without this precaution the training process could not converge to a reliable result. For both training processes, we used the mean squared error on the implied volatilities as loss function, since we were dealing with regression-type tasks (other loss functions were tried, but they gave birth to NNs that were operating more poorly). To obtain better results, it was really helpful also the linear transformation operated on the input and output data: outside of the implied volatilities which were kept unchanged, all other quantities were scaled to reside in the interval \((0, 1)\). Moreover, as it is possible to see from Figure 2, we made use of batch normalisation, while we avoided drop-out. The best batch size for both training processes was 1’000 (out of a database made of around 600’000 elements). All hyperparameters have been selected after tuning the networks, using not only manual adjustments, but also other techniques like randomised search.

The whole neural network architecture was developed using Keras. A schematic representation can be found in Figure 3.

The interested reader who would like to grasp the approximations capabilities of the implemented neural networks is addressed to Appendix A.

\( ^{12} \) Instead of the interval \([0, 1]\) which represents the codomain of the function.
Figure 3: Representation of $N_3$ as a fully-connected feedforward neural network (residual cells are ignored).
7.4 A side result: moving IVS

Having a concrete numerical tool that allows the recalibration of our model online and basically in an instantaneous way has another “cheerful” consequence. Let us imagine, for the moment, that the dynamics of the parameters $p$ are known. If this is the case, then we can model for indefinite time the evolution of an implied volatility surface without breaking any arbitrage constraints, neither static nor dynamic ones. To our knowledge, it is the first time that this achieved in an efficient way, one impressive other implementation has been presented in [3]. The algorithm used to accomplish that is outlined in Algorithm 1.

**Algorithm 1**

1: Pick initial values for the state variables (return of asset $X$ and variance $V$), the Heston parameters ($\theta, \sigma, \rho$ and $k$), the jump-frequency $\sim \text{Pois}(\lambda dt)$ and the jump-size normal distribution $\sim \mathcal{N}(\nu_i, \delta^2_i)$ for $i = 1, \ldots, 5$. Parameters $\lambda$ and $\kappa$ remain fixed throughout the procedure.

2: Compute the implied volatility surface (IVS) given the initial values.

3: **Bates step**: update the two state variables $X$ and $V$ in $X_{\text{new}}$ and $V_{\text{new}}$.

4: Compute the new implied volatility surface IVS$_{\text{new}}$ given $X_{\text{new}}$ and $V_{\text{new}}$.

5: **Heston-parameter step**: update the three parameters $\theta, \sigma, \rho$ according to an exogenously given dynamics and obtain $\theta_{\text{new}}, \sigma_{\text{new}}, \rho_{\text{new}}$.

6: Given IVS$_{\text{new}}$ together with $\theta_{\text{new}}, \sigma_{\text{new}}, \rho_{\text{new}}$, compute the new parameters $(\nu_i^{\text{new}}, \delta_i^{\text{new}})_{i=1}^5$ such that the IVS obtained with $X_{\text{new}}, V_{\text{new}}, \theta_{\text{new}}, \sigma_{\text{new}}, \lambda, k$ and $(\nu_i^{\text{new}}, \delta_i^{\text{new}})_{i=1}^5$ remains constant (equal to IVS$_{\text{new}}$).

7: Overwrite the initial parameters with the new parameters (having the sub/super-script new).

8: Restart from point 3.

As already written in the algorithm and for our purposes, we decided to initially pick randomly the parameters $\theta, \sigma, \rho$, but then letting them evolve according to very simple dynamics, namely adding some noise to the current value to get the new one. The variance of the Gaussian noise has been chosen relatively small and values are scaled if they overcome a certain threshold, so that the relative change (with respect to the initial value) could not exceed 5%. In addition, we made sure that the Feller condition was always satisfied and that the values could not exit the natural domains we assigned them. For example, if the correlation $\rho$ were brought outside of the interval $[-1, 1]$, then we would force it to remain inside by collapsing the value to the closest extreme. For both $\theta$ and $\sigma$ the interval $[0.01, 0.5]$ was chosen.

Notice also that Steps 2, 4 and 6 of Algorithm 1 are made by neural networks, $NN_1$ for 2 and 4, while $NN_2$ for Step 6.

Solving the same problem with the desired precision without neural networks would have required an immense computational power, since the inverse problem.
is notably ill-posed and the regularised inverse problem has to be solved at any point in time along the discretisation grid. This is something possible on a standard laptop only through these techniques. Finally, it is important to underline that we do not break any arbitrage condition because the CNKK drift condition is fully incorporated in the steps of Algorithm 1.

8 Conclusion

In this paper, we tried to set up a new rigorous framework in continuous time for the dynamics of volatility surfaces (or cubes, etc...), so called consistent recalibration models with applications. To do so, we took inspiration from similar work in discrete time by Richter and Teichmann [25] and another paper by Harms et al. [14], which builds the theory in continuous time, but focusing on yield curves modelling. With respect to the latter, in our case we have a more complex setting due to the more complex term structure, which is here enriched with a “strike” dimension. This is reflected in what we called CNKK equation, a generalisation of the more popular HJM equation, but with considerably more involved drift term. It goes without saying that this made the equation intractable from an analytical point of view.

To overcome this issue, we decided to represent the drift term by neural networks. We therefore proposed a new way of solving the (ill-posed) calibration problem, by exploiting the fact that composition of neural networks is still a neural network and by defining, in this sense, a sort of inverse network applying implicit regularisation. The same trick could be used for other applications, also in branches other than mathematical finance, to solve inverse problems. The use of neural networks was crucial to make numerical procedures tractable and to get information on the solution of the CNKK SPDE. In this case, we can say that the neural network helped us solving an equation which we could not even write down (explicitly).

Finally, we could use the same inverse neural network to simulate the evolution in time of an implied volatility surface, in this case generated by a generalised Bates model. To the best of our knowledge, it is the first time this can be achieved for indefinite time without breaking arbitrage constraints. In this way, we are even implicitly building a realistic equity option market simulator capable of avoiding any form of arbitrage.

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A Graphical results

In this appendix, we report some of the pictures produced using the model outlined in Section 6 with Python and the graphical package Matplotlib. In the first case, plotted figures represent a 3D representation of implied volatility surfaces together with a heat-map reporting the (pointwise) differences between the original implied volatility surface (Original IVS) and the one obtained by application of neural network $NN_1$ (New IVS), which takes parameters in input as return the IVS on a grid given by 13 moneyness (or strikes) and 10 maturities. The heat-map was produced using the command `pcolormesh`. All (input) parameters were randomly generated from a uniform distribution, the same used for the generation of training data (but of course not used during the training process). For sake of simplicity, here we calibrated just one couple $(\nu, \delta)$ for each volatility surface.

Figure 4: Parameters: $S_0 = 100$, $r = 0.0205$, $q = 0.03$, $V_0 = 0.0001$, $\kappa = 7.797$, $\theta = 0.247$, $\sigma = 0.280$, $\rho = 0.042$, $\lambda = 0.081$, $\nu = 0.159$, $\delta = 0.205$

\[^{13}\text{J. D. Hunter, "Matplotlib: A 2D Graphics Environment", Computing in Science & Engineering, vol. 9, no. 3, pp. 90-95, 2007}\]
Figure 5: Parameters: $S_0 = 100, r = 0.0068, q = 0.0161, V_0 = 0.0951, \kappa = 5.421, \theta = 0.370, \sigma = 0.224, \rho = 0.242, \lambda = 0.289, \nu = 0.087, \delta = 0.249$

Figure 6: Parameters: $S_0 = 100, r = 0.0111, q = 0.0021, V_0 = 0.0552, \kappa = 8.698, \theta = 0.106, \sigma = 0.391, \rho = -0.12, \lambda = 0.491, \nu = -0.202, \delta = 0.287$
On the other hand, in the next figures we will take into consideration the neural network we called in Section 7, $NN_2$. Figures 7, 8 and 9 were basically obtained after one loop of Algorithm 1, in the sense that we started from an IVS generated by a model in which the price process and the variance process evolved in time (step 3 in the algorithm), we let the 3 parameters $\theta$, $\sigma$ and $\rho$ changing according to exogenous dynamics (essentially, normal noise) and then we exploited $NN_2$ to recover the jump parameters that would give rise to the same IVS (IVS$_{\text{new}}$ in the same algorithm). Note that the ‘Original IVS’ (in the plots) are obtained for Figures 7 and 8 in an analytical way, while in Figure 9 the ‘Original IVS’ is the output of $NN_1$. The fact that the error is zero everywhere, although the starting and derived (by $NN_2$) parameters are not the same, means that the inversion of the neural network is indeed effective.

Figure 7: Parameters: $S_0 = 99.783$, $r = 0.0521$, $q = 0.0082$, $V_0 = 0.0037$, $\kappa = 6.924$, $\theta = 0.146$, $\sigma = 0.328$, $\rho = -0.08$, $\nu = -0.286$, $\delta = 0.211$; after 1 loop and using $NN_2$ $\theta = 0.142$, $\sigma = 0.339$, $\rho = -0.08$, $\nu = -0.238$, $\delta = 0.299$
Figure 8: Parameters: $S_0 = 100.26$, $r = 0.0111$, $q = 0.0021$, $V_0 = 0.0663$, $\kappa = 8.698$, $\theta = 0.106$, $\sigma = 0.391$, $\rho = -0.12$, $\lambda = 0.491$, $\nu = -0.202$, $\delta = 0.287$; after 1 loop and using $NN_2 \theta = 0.102$, $\sigma = 0.408$, $\rho = -0.12$, $\nu = -0.279$, $\delta = 0.300$.

Figure 9: Parameters: $S_0 = 100.83$, $r = 0.0068$, $q = 0.0161$, $V_0 = 0.1046$, $\kappa = 5.421$, $\theta = 0.370$, $\sigma = 0.224$, $\rho = 0.242$, $\lambda = 0.289$, $\nu = 0.087$, $\delta = 0.249$; after 1 loop and using $NN_2 \theta = 0.357$, $\sigma = 0.218$, $\rho = 0.251$, $\nu = 0.289$, $\delta = 0.300$. 