Monitoring the edges of a graph using distances
Florent Foucaud, Ralf Klasing, Mirka Miller, Joe Ryan

To cite this version:
Florent Foucaud, Ralf Klasing, Mirka Miller, Joe Ryan. Monitoring the edges of a graph using distances. Proceedings of the 6th International Conference on Algorithms and Discrete Applied Mathematics (CALDAM 2020), Feb 2020, Hyderabad, India. pp.28-40, 10.1007/978-3-030-39219-2_3. hal-02500423

HAL Id: hal-02500423
https://hal.science/hal-02500423
Submitted on 27 Nov 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Monitoring the edges of a graph using distances*,**

Florent Foucaud¹, Ralf Klasing¹, Mirka Miller²,³, and Joe Ryan⁴

¹ Univ. Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR5800, Talence, France. E-mails: florent.foucaud@gmail.com, ralf.klasing@labri.fr
² School of Mathematical and Physical Sciences, The University of Newcastle, Australia
³ Department of Mathematics, University of West Bohemia, Czech Republic
⁴ School of Electrical Engineering and Computing, The University of Newcastle, Australia. E-mail: joe.ryan@newcastle.edu.au

Abstract. We introduce a new graph-theoretic concept in the area of network monitoring. A set \( M \) of vertices of a graph \( G \) is a distance-edge-monitoring set if for every edge \( e \) of \( G \), there is a vertex \( x \) of \( M \) and a vertex \( y \) of \( G \) such that \( e \) belongs to all shortest paths between \( x \) and \( y \). We denote by \( \text{dem}(G) \) the smallest size of such a set in \( G \). The vertices of \( M \) represent distance probes in a network modeled by \( G \); when the edge \( e \) fails, the distance from \( x \) to \( y \) increases, and thus we are able to detect the failure. It turns out that not only we can detect it, but we can even correctly locate the failing edge.

In this paper, we initiate the study of this new concept. We show that for a nontrivial connected graph \( G \) of order \( n \), \( 1 \leq \text{dem}(G) \leq n - 1 \) with \( \text{dem}(G) = 1 \) if and only if \( G \) is a tree, and \( \text{dem}(G) = n - 1 \) if and only if it is a complete graph. We compute the exact value of \( \text{dem} \) for grids, hypercubes, and complete bipartite graphs.

Then, we relate \( \text{dem} \) to other standard graph parameters. We show that \( \text{dem}(G) \) is lower-bounded by the arboricity of the graph, and upper-bounded by its vertex cover number. It is also upper-bounded by five times its feedback edge set number.

Then, we show that determining \( \text{dem}(G) \) for an input graph \( G \) is an NP-complete problem, even for apex graphs. There exists a polynomial-time logarithmic-factor approximation algorithm, however it is NP-hard to compute an asymptotically better approximation, even for bipartite graphs of small diameter and for bipartite subcubic graphs. For such instances, the problem is also unlikely to be fixed parameter tractable when parameterized by the solution size.

* This paper is dedicated to the memory of Mirka Miller, who sadly passed away in January 2016. This research was started when Mirka Miller and Joe Ryan visited Ralf Klasing at the LaBRI, University of Bordeaux, in April 2015.
** The authors acknowledge the financial support from the ANR project HOSI-GRA (ANR-17-CE40-0022), the IFCAM project “Applications of graph homomorphisms” (MA/IFCAM/18/39), and the Programme IdEx Bordeaux – SysNum (ANR-10-IDEX-03-02).
1 Introduction

The aim of this paper is to introduce a new concept of network monitoring using distance probes, called distance-edge-monitoring. Our networks are naturally modeled by finite undirected simple connected graphs, whose vertices represent computers and whose edges represent connections between them. We wish to be able to monitor the network in the sense that when a connection (an edge) fails, we can detect this failure. We will select a (hopefully) small set of vertices of the network, that will be called probes. At any given moment, a probe of the network can measure its graph distance to any other vertex of the network. Our goal is that, whenever some edge of the network fails, one of the measured distances changes, and thus the probes are able to detect the failure of any edge.

Prob es that measure distances in graphs are present in real-life networks, for instance this is useful in the fundamental task of routing [6,11]. They are also frequently used for problems concerning network verification [2,3,4].

We will now proceed with the formal definition of our main concept. In this paper, by graphs we refer to connected simple graphs (without multiple edges and loops). A graph with loops or multiple edges is called a multigraph.

We denote by \( d_G(x, y) \) the distance between two vertices \( x \) and \( y \) in a graph \( G \). When there is no path connecting \( x \) and \( y \) in \( G \), we let \( d_G(x, y) = \infty \). For an edge \( e \) of \( G \), we denote by \( G - e \) the graph obtained by deleting \( e \) from \( G \).

**Definition 1.** For a set \( M \) of vertices and an edge \( e \) of a graph \( G \), let \( P(M, e) \) be the set of pairs \( (x, y) \) with \( x \) a vertex of \( M \) and \( y \) a vertex of \( V(G) \) such that \( d_G(x, y) \neq d_G(x, y) \). In other words, \( e \) belongs to all shortest paths between \( x \) and \( y \) in \( G \).

For a vertex \( x \), let \( M(x) \) be the set of edges \( e \) such that there exists a vertex \( v \) in \( G \) with \( (x, v) \in P(\{x\}, e) \). If \( e \in M(x) \), we say that \( e \) is monitored by \( x \).

A set \( M \) of vertices of a graph \( G \) is distance-edge-monitoring if every edge \( e \) of \( G \) is monitored by some vertex of \( M \), that is, the set \( P(M, e) \) is nonempty. Equivalently, \( \bigcup_{x \in M} M(x) = E(G) \).

We note \( \text{dem}(G) \) the smallest size of a distance-edge-monitoring set of \( G \).

Note that \( V(G) \) is always a distance-edge-monitoring set of \( G \), so \( \text{dem}(G) \) is always well-defined.

Consider a graph \( G \) modeling a network, and a set \( M \) of vertices of \( G \), on which we place probes that are able to measure their distances to all the other vertices. If \( M \) is distance-edge-monitoring, if a failure occurs on any edge of the network (in the sense that the communication between its two endpoints is broken), then this failure is detected by the probes.

In fact, it turns out that not only the probes can detect a failing edge, but they can also precisely locate it (the proof of the following is delayed to Section 2).

**Proposition 2.** Let \( M \) be a distance-edge-monitoring set of a graph \( G \). Then, for any two distinct edges \( e \) and \( e' \) in \( G \), we have \( P(M, e) \neq P(M, e') \).

Thus, assume that we have placed probes on a distance-edge-monitoring set \( M \) of a network \( G \) and initially computed all the sets \( P(M, e) \). In the case a
unique edge of the network has failed, Proposition 2 shows that by measuring the set of pairs \((x, y)\) with \(x \in M\) and \(y \in V(G)\) whose distance has changed, we know exactly which is the edge that has failed.

We define the decision and optimization problem associated to distance-edge-monitoring sets.

\[\text{Distance-Edge-Monitoring Set}\]
Instance: A graph \(G\), an integer \(k\).
Question: Do we have \(\text{dem}(G) \leq k\)?

\[\text{Min Distance-Edge-Monitoring Set}\]
Instance: A graph \(G\).
Task: Build a smallest possible distance-edge-monitoring set of \(G\).

**Related notions.** A weaker model is studied in [2,3] as a network discovery problem, where we seek a set \(S\) of vertices such that for each edge \(e\), there exists a vertex \(x\) of \(S\) and a vertex \(y\) of \(G\) such that \(e\) belongs to some shortest path from \(x\) to \(y\).

Distance-edge-monitoring sets are related to resolving sets and edge-resolving sets, that also model sets of sensors that can measure the distance to all other vertices in a graph. A resolving set is a set \(R\) of vertices such that for any two distinct vertices \(x\) and \(y\) in \(G\), there is a vertex \(r\) in \(R\) such that \(d_G(r,x) \neq d_G(r,y)\). The smallest size of a resolving set in \(G\) is the metric dimension of \(G\) \([12,20]\). If instead, the set \(R\) distinguishes the edges of \(G\), that is, for every pair \(e, e'\) of edges of \(G\), there is a vertex \(r\) of \(R\) with \(d_G(x,e) \neq d_G(x,e')\), where for \(e = uv\), \(d_G(x,e) = \min\{d_G(x,u),d_G(x,v)\}\), we have an edge-resolving set \([15]\).

Another related concept is the one of strong resolving sets \([18,19]\): a set \(R\) of vertices is strongly resolving if for any pair \(x, y\) of vertices, there exists a vertex \(z\) of \(R\) such that \(x\) is on a shortest path from \(z\) to \(y\), or \(y\) is on a shortest path from \(z\) to \(x\). It is related to distance-edge-monitoring sets in the following sense. Given a distance-edge-monitoring set \(M\), for every pair \(x, y\) of adjacent vertices, there is a vertex \(z\) of \(M\) such that either \(x\) is on every shortest path from \(z\) to \(y\), or \(y\) is on every shortest path from \(z\) to \(x\).

Another related concept is the one of (strong) edge-geodetic set. A set \(S\) of vertices is an edge-geodetic set if for every edge \(e\) of \(G\), there are two vertices \(x, y\) of \(S\) such that \(e\) is on some shortest path from \(x\) to \(y\). It is a strong edge-geodetic set if to every pair \(x, y\) of \(S\), we can assign a shortest \(x-y\) path to \(\{x, y\}\) such that every edge of \(G\) belongs to one of these \(\binom{|S|}{2}\) assigned shortest paths \([16]\).

**Our results.** We first derive a number of basic results about distance-edge-monitoring sets in Section 2 (where we also give some useful definitions).

In Section 3 we study \(\text{dem}\) for some basic graph families (like trees and grids) and relate this parameter to other standard graph parameters such as arboricity \(\text{arb}\), vertex cover number \(\text{vc}\) and feedback edge set number \(\text{fes}\). We show that \(\text{dem}(G) = 1\) if and only if \(G\) is a tree. We show that for any graph \(G\) of order \(n\), \(\text{dem}(G) \geq \text{arb}(G)\). Moreover, \(\text{dem}(G) \leq \text{vc}(G) \leq n - 1\) (with equality if and only if \(G\) is complete). We show that for some families of graphs
and hypercubes. Then we show that \( \text{dem}(G) = \text{vc}(G) \), for instance this is the case for complete bipartite graphs and hypercubes. Then we show that \( \text{dem}(G) \leq 5\text{fes}(G) - 5 \) when \( \text{fes}(G) \geq 2 \) (when \( \text{fes}(G) \leq 1, \text{dem}(G) = \text{fes}(G) + 1 \)).

In Section 4, we show that \text{Distance-Edge-Monitoring Set} is \text{NP}-complete, even for apex graphs (graphs obtained from a planar graph by adding an extra vertex). Then, we show that \text{Min Distance-Edge-Monitoring Set} can be approximated in polynomial time within a factor of \( \ln(|E(G)| + 1) \) by a reduction to the set cover problem. Finally, we show that no essentially better ratio can be obtained (unless \( P=\text{NP} \)), even for graphs that are of diameter 4, bipartite and of diameter 5, or bipartite and of maximum degree 3. For the same restrictions, the problem is unlikely to be fixed parameter tractable when parameterized by the solution size. These hardness results are obtained by reductions from the set cover problem.

We conclude our paper in Section 5. Due to the given space constraints, some proofs are not included.

## 2 Preliminaries

We now give some useful lemmas about basic properties of distance-edge-monitoring sets. We start with the proof of Proposition 2.

**Proof (Proof of Proposition 2).** Suppose by contradiction that there are two distinct edges \( e \) and \( e' \) with \( P(M,e) = P(M,e') \). Since by definition, \( P(M,e) \neq \emptyset \), we have \( (x,v) \in P(M,e) \) (and thus \( (x,v) \in P(M,e') \)), where \( x \in M \) and \( v \in V(G) \). Thus, all shortest paths between \( x \) and \( v \) contain both \( e \) and \( e' \). Assume that on these shortest paths, \( e \) is closer to \( x \) than \( e' \), and let \( w \) be the endpoint of \( e \) that is closest to \( v \). Then, we have \( (x,w) \in P(M,e) \) but \( (x,w) \notin P(M,e') \); a contradiction. \( \square \)

An edge \( e \) in a graph \( G \) is a **bridge** if \( G - e \) has more connected components than \( G \). We will now show that bridges are very easy to monitor.

**Lemma 3.** Let \( G \) be a connected graph and let \( e \) be a bridge of \( G \). For any vertex \( x \) of \( G \), we have \( e \in M(x) \).

**Proof.** Assume that \( e = uv \). Since \( e \) is a bridge, we have \( d_G(x,u) \neq d_G(x,v) \). If \( d_G(x,u) < d_G(x,v) \), then \( (x,v) \in P(\{x\}, e) \). Otherwise, we have \( (x,u) \in P(\{x\}, e) \). In both cases, \( e \in M(x) \). \( \square \)

Given a vertex \( x \) of a graph \( G \) and an integer \( i \), we let \( L_i(x) \) denote the set of vertices at distance \( i \) of \( x \) in \( G \).

**Lemma 4.** Let \( x \) be a vertex of a connected graph \( G \). Then, an edge \( uv \) belongs to \( M(x) \) if and only if \( u \in L_i(x) \) and \( v \) is the only neighbour of \( u \) in \( L_{i-1}(x) \).

**Proof.** Let \( uv \in M(x) \). Then, there exists a vertex \( y \) such that all shortest paths from \( y \) to \( x \) go through \( uv \). Thus, one of \( u \) and \( v \) (say, \( u \)) is in a set \( L_i(x) \) and the other (\( v \)) is in a set \( L_{i-1}(x) \), for some positive integer \( i \). Moreover, \( v \) must
be the only neighbour of \( u \) in \( L_{i-1}(x) \), since otherwise there would be a shortest path from \( y \) to \( x \) going through \( u \) but avoiding \( uv \).

Conversely, if \( u \) is a vertex in \( L_{i}(x) \) with a unique neighbour \( v \) in \( L_{i-1}(x) \), then the edge \( uv \) belongs to \( M(x) \) since all shortest paths from \( u \) to \( x \) use it. \( \square \)

We obtain some immediate consequences of Lemma 4.

**Lemma 5.** For a vertex \( x \) of a graph \( G \), the set of edges \( M(x) \) induces a forest.

**Proof.** Let \( G_x \) be the subgraph of \( G \) induced by the edges in \( M(x) \). By Lemma 4, an edge \( e \) belongs to \( M(x) \) if and only if \( e = uv \), \( u \in L_i(x) \) and \( v \) is the only neighbour of \( u \) in \( L_{i-1}(x) \). In this case, we let \( v \) be the parent of \( u \). Each vertex in \( G_x \) has at most one parent in \( G_x \), and each edge of \( G_x \) is an edge between a vertex and its parent. Thus, \( G_x \) is a forest. \( \square \)

**Lemma 6.** Let \( G \) be a graph and \( x \) a vertex of \( G \). Then, for any edge \( e \) incident with \( x \), we have \( e \in M(x) \).

**Proof.** For every vertex \( y \) of \( L_1(x) \) (that is, every neighbour of \( x \)), \( x \) is the unique neighbour of \( y \) in \( L_0(x) = \{ x \} \). Thus, the claim follows from Lemma 4. \( \square \)

### 3 Basic graph families and bounds

In this section, we study \( \text{dem} \) for standard graph classes, and its relation with other standard graph parameters.

**Theorem 7.** Let \( G \) be a connected graph with at least one edge. We have \( \text{dem}(G) = 1 \) if and only if \( G \) is a tree.

**Proof.** In a tree, every edge is a bridge. Thus, by Lemma 4, if \( G \) is a tree, any vertex \( x \) of \( G \) is a distance-edge-monitoring set and we have \( \text{dem}(G) \leq 1 \) (and of course as long as there is an edge in \( G \), \( \text{dem}(G) \geq 1 \)).

For the converse, suppose that \( \text{dem}(G) = 1 \). Then, clearly, \( G \) must have at least one edge. Moreover, since all edges of \( G \) must belong to \( M(x) \), by Lemma 5, \( G \) must be a forest. Since \( G \) is connected, \( G \) is a tree. \( \square \)

Let \( G_{a,b} \) denote the grid of dimension \( a \times b \). By Theorem 7, we have \( \text{dem}(G_{1,a}) = 1 \). We can compute all other values.

**Theorem 8.** For any integers \( a, b \geq 2 \), we have \( \text{dem}(G_{a,b}) = \max\{a, b\} \).

The arboricity \( \text{arb}(G) \) of a graph \( G \) is the smallest number of sets into which \( E(G) \) can be partitioned and such that each set induces a forest. The clique number \( \omega(G) \) of \( G \) is the size of a largest clique in \( G \).

**Theorem 9.** For any graph \( G \) of order \( n \) and size \( m \), we have \( \text{dem}(G) \geq \text{arb}(G) \), and thus \( \text{dem}(G) \geq \frac{m}{n-1} \) and \( \text{dem}(G) \geq \frac{\omega(G)}{2} \).
Theorem 10. In any graph $G$ of order $n$, any vertex cover of $G$ is a distance-edge-monitoring set, and thus $\text{dem}(G) \leq \text{vc}(G) \leq n - 1$. Moreover, we have $\text{dem}(G) = n - 1$ if and only if $G$ is the complete graph of order $n$.

Proof. Let $C$ be a vertex cover of $G$. By Lemma 9, for every edge $e$, there is a vertex in $x$ with $e \in M(x)$, thus $C$ is distance-edge-monitoring.

Moreover, any graph $G$ of order $n$ has a vertex cover of size $n - 1$: for any vertex $x$, the set $V(G) \setminus \{x\}$ is a vertex cover of $G$.

Finally, suppose that $\text{dem}(G) = n - 1$: then also $\text{vc}(G) = n - 1$. If $G$ is not connected, we have $\text{vc}(G) \leq n - 2$ (starting with $V(G)$ and removing any vertex from each connected component of $G$ yields a vertex cover), thus $G$ is connected. Suppose by contradiction that $G$ is not a complete graph. Then, we have three vertices $x, y$ and $z$ in $G$ such that $xy$ and $yz$ are edges of $G$, but $xz$ is not. Then, $V(G) \setminus \{x, z\}$ is a vertex cover of $G$, a contradiction.

This completes the proof. $\square$

We next see that distance-edge-monitoring sets are relaxations of vertex covers. A set $C$ of vertices is a vertex cover of $G$ if every edge of $G$ has one of its endpoints in $C$. The smallest size of a vertex cover of $G$ is denoted by $\text{vc}(G)$.

Observation 11. If, for every vertex $x$ of a graph $G$, $M(x)$ consists exactly of the sets of edges incident with $x$, then a set $M$ is a distance-edge-monitoring set of $G$ if and only if it is a vertex cover of $G$.

Note that Observation 11 does not provide a characterization of graphs with $\text{dem}(G) = \text{vc}(G)$. For example, as seen in Theorem 8 for the grid $G_{a,b}$, we have $\text{dem}(G_{a,b}) = \text{vc}(G_{a,b}) = a$, but for any vertex $x$ of $G_{a,b}$, $M(x)$ consists of the whole row and column of $G_{a,b}$.

Let $K_{a,b}$ be the complete bipartite graph with parts of sizes $a$ and $b$, and let $H_d$ denote the hypercube of dimension $d$. We can deduce the following from Observation 11.

Corollary 12. We have $\text{dem}(K_{a,b}) = \text{vc}(K_{a,b}) = \min\{a, b\}$, and $\text{dem}(H_d) = \text{vc}(H_d) = 2^{d-1}$.

A feedback edge set of a graph $G$ is a set of edges such that removing them from $G$ leaves a forest. The smallest size of a feedback edge set of $G$ is denoted by $\text{fes}(G)$ (it is sometimes called the cyclomatic number of $G$).
Theorem 7 states that for any tree $G$ (that is, a graph with feedback edge set number 0), we have $\text{dem}(G) = \text{fes}(G) + 1$. We now show the same bound for graphs $G$ with $\text{fes}(G) = 1$, that is, unicyclic graphs (graphs with a unique cycle).

**Proposition 13.** If $G$ is a unicyclic graph, then $\text{dem}(G) = 2$.

We will now give a weaker (but similar) bound for any value of $\text{fes}(G)$. But first, we need the following terminology and lemma from [9] about the structure of graphs $G$ with given feedback edge set number.

In a graph, a vertex is a core vertex if it has degree at least 3. The base graph of a graph $G$ is the graph obtained from $G$ by iteratively removing vertices of degree 1 (thus, the base graph of a forest is the empty graph).

The following lemma can be found in Section 5.3.1 of [9].

**Lemma 14 ([9]).** Let $G$ be a graph with $\text{fes}(G) = k \geq 2$. The base graph of $G$ has at most $2k - 2$ core vertices, that are joined by at most $3k - 3$ edge-disjoint paths with internal vertices of degree 2 and whose endpoints are (not necessarily distinct) core vertices.

In other words, Lemma 14 says that the base graph of a graph $G$ with $\text{fes}(G) = k \geq 2$ can be obtained from a multigraph $H$ of order at most $2k - 2$ and size $3k - 3$ by subdividing its edges an arbitrary number of times.

**Theorem 15.** Let $G$ be a graph with $\text{fes}(G) \geq 2$. Then, $\text{dem}(G) \leq 5 \text{fes}(G) - 5$.

**Proof.** Let $\text{fes}(G) = k$ and $G_b$, the base graph of $G$. By Lemma 14, $G_b$ contains at most $2k - 2$ core vertices, joined by at most $3k - 3$ edge-disjoint paths with internal vertices of degree 2 and whose endpoints are core vertices (both endpoints can be the same vertex). Let $\{P_1, \ldots, P_t\}$ be the set of these paths.

By Lemma 8 any non-empty distance-edge-monitoring set of $G_b$ is also one of $G$, since all edges of $G$ not present in $G_b$ are bridges. Thus, it is sufficient to construct a distance-edge-monitoring set $M$ of $G_b$ of size at most $5k - 5$. We build $M$ as follows: first of all, $M$ contains all core vertices of $G_b$. Moreover, for each path $P_i$ of length at least 2 connecting two core vertices $x$ and $y$ of $G_b$, we add to $M$ an internal vertex $z$ of $P_i$. If $x \neq y$ or $P_i$ has length at most 3, we let $z$ be a neighbour of, say, $x$. Otherwise, $x = y$ and $P_i$ has length at least 4; we choose $z$ as a vertex at distance 2 of $x$ on $P_i$. Thus, $|M| \leq 5k - 5$.

Let us prove that $M$ is distance-edge-monitoring in $G_b$. Let $e$ be an edge of $G_b$; necessarily, $e$ belongs to some path $P_i$ connecting two core vertices $x$ and $y$. If $e$ is incident with $x$ or $y$, by Lemma 8 $e$ is monitored by $M$. Thus, suppose this is not the case. So, $P_i$ has length at least 3 and so there is an internal vertex $z$ of $P_i$ in $M$. Again if $e$ is incident with $z$ we are done.

If $x = y$, $P_i$ forms a cycle and for any vertex $v$, $M(v) \cap E(P_i)$ consists of the two edges (if $P_i$ has even length) or the unique edge (if $P_i$ has odd length) that are opposite to $v$ on the cycle. Since $x$ and $z$ are non-adjacent, we have $e \in M(x) \cup M(z)$ and so $z$ is monitored.

If $x \neq y$, assume that $e = uv$ with $u$ the vertex of $e$ closest to $x$. Let $d_x$ be the distance from $x$ to $u$ on $P_i$, and $d_y$ the distance from $y$ to $v$ on $P_i$. If $e \in M(x) \cup M(y)$, we are done. Otherwise, there must exist a path from $x$
to $y$ that is edge-disjoint with $P_i$ (otherwise $e$ would be a bridge and would be monitored by $x$ or $y$). Let $p$ be the length of a shortest such path. Since $e \notin M(x)$, there exists a shortest path from $v$ to $x$ avoiding $e$. This shortest path must go through $y$, and then use a path of length $p$. Thus, we have $d_y + p \leq d_x + 1$. By a similar argument with $u$ and $x$, we have $d_x + p \leq d_y + 1$. We obtain from this that $d_x = d_y$, $|P_i|$ is odd, and $p = 1$. But then, the unique shortest path from $v$ to $e$ goes through $e$, and we have $e \in M(z)$. This completes the proof. 

We do not believe that the bound of Theorem 15 is tight. For the sake of simplicity, we have not tried to optimize the bound of Theorem 15 as we do not believe that this method will provide a tight bound. There are examples of graphs $G$ where $\text{dem}(G) = \text{fes}(G) + 1$, for example this is the case for the grid $G_{a,2}$ by Theorem 5, when $a \geq 2$ we have $\text{dem}(G_{a,2}) = a$, and $\text{fes}(G_{a,2}) = a - 1$.

Note that $G_{a,b}$ for $a, b \geq 2$ provides examples of a family of graphs where for increasing $a, b$ the difference between $\text{fes}(G_{a,b})$ and $\text{dem}(G_{a,b})$ is unbounded by a constant. Indeed by Theorem 5, we have $\text{dem}(G_{a,b}) = \max\{a, b\}$ but $\text{fes}(G_{a,b})$ is linear in the order $ab$.

4 Complexity

A $c$-approximation algorithm for a given optimization problem is an algorithm that returns a solution whose size is always at most $c$ times the optimum. We refer to the books [11][15] for more details. For a decision problem $\Pi$ and for some parameter $p$ of the instance, an algorithm for $\Pi$ is said to be fixed parameter tractable (fpt for short) if it runs in time $f(p)n^c$, where $f$ is a computable function, $n$ is the input size, and $c$ is a constant. In this paper, we will always consider the solution size $k$ as the parameter. The class FPT contains the parameterized decisions problems solvable by an fpt algorithm. The classes W[1] (with $i \geq 1$) denote complexity classes with parameterized decision problems that are believed not to be fpt. We refer to the books [13][17] for more details.

We will now use the connection to vertex covers hinted in Section 3 to derive an NP-hardness result. For two graphs $G$, $H$, $G \Join H$ denotes the graph obtained from disjoint copies of $G$ and $H$ with all possible edges between $V(G)$ and $V(H)$. We denote by $K_1$ the graph on one single vertex.

**Theorem 16.** For any graph $G$, we have $\text{vc}(G) \leq \text{dem}(G \Join K_1) \leq \text{vc}(G) + 1$. Moreover, if $G$ has radius at least 4, then $\text{vc}(G) = \text{dem}(G \Join K_1)$.

We deduce the following from the second part of the statement of Theorem 16 and the NP-hardness of Vertex Cover for planar subcubic graphs with radius at least 4 [10].

**Corollary 17.** Distance-Edge-Monitoring Set is NP-complete, even for graphs obtained from a planar subcubic graph by attaching a universal vertex.

Given a hypergraph $H = (X, S)$ with vertex set $X$ and edge set $S$, a set cover of $H$ is a subset $C \subseteq S$ of edges such that each vertex of $X$ belongs to at least one edge of $C$. We will provide reductions for Min Distance-Edge-Monitoring Set to and from Set Cover and Min Set Cover.
Monitoring the edges of a graph using distances

**Set Cover**
Instance: A hypergraph \( H = (X, S) \), an integer \( k \).
Question: Is there a set cover \( C \subseteq S \) of size at most \( k \)?

**Min Set Cover**
Instance: A hypergraph \( H = (X, S) \).
Task: Build a smallest possible set cover of \( H \).

It is known that **Min Set Cover** is approximable in polynomial time within a factor of \( \ln(|X|+1) \) [14], but, unless \( P=NP \), not within a factor of \( (1-\epsilon) \ln |X| \) (for every positive \( \epsilon \)) [7]. Moreover, **Set Cover** is \( W[2] \)-hard (when parameterized by \( k \)) and not solvable in time \( |X|^{o(k)} |H|^{O(1)} \) unless \( \text{FPT}=\text{W}[1] \) [5].

**Theorem 18.** **Min Distance-Edge-Monitoring Set** is approximable within a factor of \( \ln(|E(G)|+1) \) in polynomial time.

**Proof.** Let \( G \) be a graph and consider the following hypergraph \( H = (X, S) \) with \( X = E(G) \) and such that \( S \) contains, for each vertex \( x \) of \( G \), the set \( S_x = \{ e \in X \mid e \in M(x) \} \). Now, it is not difficult to see that there is a one-to-one correspondence between set covers of \( H \) and distance-edge-monitoring sets of \( G \), where we associate to each vertex \( x \) of a distance-edge-monitoring set of \( G \), the set \( S_x \) in a set cover of \( H \). Thus, the result follows from the \( \ln(|X|+1) \)-approximation algorithm for **Min Set Cover** from [14]. \( \square \)

**Theorem 19.** Even for graphs \( G \) that are (a) of diameter 4, (b) bipartite and of diameter 6, or (c) bipartite and of maximum degree 3, **Min Distance-Edge-Monitoring Set** is not approximable within a factor of \( (1-\epsilon) \ln |E(G)| \) in polynomial time, unless \( P=NP \). Moreover, for such instances, **Distance-Edge-Monitoring Set** cannot be solved in time \( |G|^{o(k)} \), unless \( \text{FPT}=\text{W}[1] \), and it is \( W[2] \)-hard for parameter \( k \).

**Proof.** For an instance \((H, k)\) of **Set Cover**, we will construct in polynomial time instances \((G, k+2)\) or \((G, k+1)\) of **Distance-Edge-Monitoring Set** so that \( H \) has a set cover of size \( k \) if and only if \( G \) has a distance-edge-monitoring set of size at most \( k+2 \) or \( k+1 \).

In our first reduction, the obtained instance has diameter 4, while in our second reduction, the obtained instance is bipartite and has diameter 6, and in our third reduction, the graph \( G \) is bipartite and has maximum degree 3. The three constructions are similar.

The statement will follow from the hardness of approximating **Min Set Cover** proved in [14], the parameterized hardness of **Set Cover** (parameterized by solution size), and the lower bound on its running time [5].

First of all, we point out that we may assume that in an instance \((H = (X, S), k)\) of **Set Cover**, there is no vertex of \( X \) that belongs to a unique set of \( S \). Indeed, otherwise, we are forced to take \( S \) in any set cover of \( H \); thus, by removing \( S \) and all vertices in \( S \), we obtain an equivalent instance \((H', k-1)\). We can iterate until the instance satisfies this property.

We now describe the first reduction, in which the obtained instance has diameter 4. Let \((H, k) = ((X, S), k) \) be an instance of **Set Cover**, where
Let $X = \{x_1, x_2, \ldots, x_{|X|}\}$, $S = \{C_1, C_2, \ldots, C_{|S|}\}$ and $C_i = \{c_{i,j} \mid x_j \in C_i\}$. Construct the following instance $(G, k + 2) = ((V, E), k + 2)$ of **Distance-Edge-Monitoring Set**, where $V = V_1 \cup V_2 \cup \ldots \cup V_5 \cup V'_1 \cup V'_2 \cup V'_3 \cup V'_4$. $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E'_1 \cup E'_2 \cup E'_3 \cup E'_4 \cup E'_5$ and

$V_1 = \{u_1, u_2, u_3\}$, $V_2 = \{v_i \mid 1 \leq i \leq |S|\}$, $V_3 = S$, $V_4 = \{c_{1,j} \mid 1 \leq i \leq |S|, 1 \leq j \leq |X|, x_j \in C_i\}$, $V_5 = X$

$V'_1 = \{u'_1, u'_2, u'_3\}$, $V'_2 = \{v'_i \mid 1 \leq i \leq |S|\}$, $V'_3 = \{c'_{i,j} \mid 1 \leq i \leq |S|, 1 \leq j \leq |X|, x_j \in C_i\}$, $V'_4 = \{w'_i \mid 1 \leq j \leq |X|\}$

$E_1 = \{(u_1, u_2), (u_1, u_3), (u_2, u'_1), (u_3, u'_2)\}$, $E_2 = \{(u_1, v_i), (v_i, C_i) \mid 1 \leq i \leq |S|\}$, $E_3 = \{(C_i, c_{i,j}) \mid 1 \leq i \leq |S|, 1 \leq j \leq |X|, x_j \in C_i\}$, $E_4 = \{(c_{i,j}, x_j) \mid 1 \leq i \leq |S|, 1 \leq j \leq |X|, x_j \in C_i\}$

$E'_1 = \{(u'_1, u'_2), (u'_2, u'_3), (u'_3, u''_1)\}$, $E'_2 = \{(v'_i, v'_j), (v'_j, C'_i) \mid 1 \leq i \leq |S|\}$, $E'_3 = \{(u'_1, c'_{j,i}), (c'_{j,i}, c_{j,i}) \mid 1 \leq i \leq |S|, 1 \leq j \leq |X|, x_j \in C_i\}$, $E'_4 = \{(u'_1, w'_i), (w'_i, x_j) \mid 1 \leq j \leq |X|\}$, $E'_5 = \{(u'_1, v_i) \mid 1 \leq i \leq |S|\}$.

An example is given in Figure 1.

Fig. 1. First reduction from **Set Cover** to **Distance-Edge-Monitoring Set** from the proof of Theorem 19 applied to the hypergraph $\{(x_1, x_2, x_3, x_4), \{C_1 = \{x_1, x_2\}, C_2 = \{x_2, x_3, x_4\}, C_3 = \{x_1, x_3, x_4\}\}\}$. Vertices and edges of $V'_i$ and $E'_i$ for $i = 2, 3, 4$ are only suggested.

To see that $G$ has diameter 4, observe that every vertex of $G$ has a path of length at most 2 to the vertex $v'_i$. Let $C$ be a set cover of $H$ of size $k$. Define $M = C \cup \{u_1, u'_2\}$. Then, by Lemma 2, $u_1$ monitors (in particular) the edges $(u_1, u_2), (u_1, u_3), (u'_1, u'_2)$, and all the edges in $E_2 \cup E_3$. Similarly, $u'_2$ monitors the edges $(u'_1, u'_2), (u'_2, u'_3)$, $(u_3, u'_1), (u_2, u'_1)$ and all the edges in $E'_2 \cup E'_3 \cup E'_4 \cup E'_5$. It thus remains to show that all edges of $E_4$ are monitored. Notice that among those edges, vertex $C_i$ of $S$ monitors exactly all edges $c_{i,j}$ with $x_j \in C_i$. Thus, if $e = (c_{i,j}, x_j)$, there is $i'$ such that $x_j \in C_{i'}$ and $C_{i'} \subset C$, and $e$ is monitored by $C_{i'}$ (either $i = i'$ and...
the only shortest path from \( x_j \) to \( C_i \) contains \( e \), or \( i \neq i' \) and the only shortest path from \( c_{i,j} \) to \( C_{i'} \) contains \( e \). Hence, \( M \) is a distance-edge-monitoring set of \( G \) of size at most \( k + 2 \).

Conversely, let \( M \) be a distance-edge-monitoring set of \( G \) of size at most \( k + 2 \). In order to monitor the edges \((u_2', u_3')\), either \( u_2' \in M \) or \( u_3' \in M \). In order to monitor the edges \((u_1, u_3)\) and \((u_1, u_2)\), there must be a vertex of \( M \) in \( \{u_1, u_2, u_3\} \). We may replace \( M \cap \{u_1, u_2, u_3\} \) by \( u_1 \), and \( M \cap \{u_2', u_3'\} \) by \( u_2' \), as \( u_1 \) monitors the same edges as \( \{u_1, u_2, u_3\} \) (among those not already monitored by \( u_2' \)) and \( u_2' \) monitors the same edges as \( \{u_2', u_3'\} \) (among those not already monitored by \( u_1 \)). As seen in the previous paragraph, all edges of \( E_1, E_1', E_2, E_2', E_3, E_3', E_4 \) and \( E_4' \) are monitored by \( \{u_1, u_2'\} \). However, no edge of \( E_4 \) is monitored by any vertex of \( V_1 \cup V_1' \). Thus, all remaining vertices of \( M \) are needed precisely to monitor the edges of \( E_4 \).

If \( v_i \in M \), let \( M' = M \setminus \{v_i\} \cup \{C_i\} \), and the set of monitored edges does not decrease. If \( \{c_{i,j}\} \in M \), let \( M' = M \setminus \{c_{i,j}\} \cup \{C_i\} \), and the set of monitored edges does not decrease. If \( x_j \in M \), let \( M' = M \setminus \{x_j\} \cup \{C_i\} \), and the set of monitored edges does not decrease. If \( v_i' \in M \), let \( M' = M \setminus \{v_i'\} \cup \{C_i\} \), and the set of monitored edges does not decrease. If \( \{c_{i,j}'\} \in M \), let \( M' = M \setminus \{c_{i,j}'\} \cup \{C_i\} \), and the set of monitored edges does not decrease. If \( w_i'' \in M \), let \( M' = M \setminus \{w_i''\} \cup \{C_i\} \), and the set of monitored edges does not decrease. Iterating this process, we finally obtain a distance-edge-monitoring set \( M' \) of \( G \) with \( |M' \cap V_1| = 1 \), \( |M' \cap V_1'| \geq 1 \), \( |M' \cap V_2| \leq k \), \( M' \cap (V_2 \cup V_4 \cup V_5 \cup V_2' \cup V_2') = \emptyset \). Let \( C = M' \cap V_3 \). If \( C \) is not a set cover of \( H \), then there is \( x_j \in X \) that is not covered by \( C \). Recall that we assumed that each vertex of \( X \) belongs to at least two edges of \( S \). Then, according to Lemma \ref{lem:lower-bound-edges}, any edge \( (c_{i,j}, x_j) \) is not monitored by any of the vertices in \( M' = (M' \cap V_1) \cup (M' \cap V_1') \cup C \), a contradiction. Hence, \( C \) is a set cover of \( H \) of size at most \( k \).

Due to space constraints, we omit the two other reductions. They are very similar to the above one. To obtain a bipartite graph, we remove the odd cycles of the construction at the expense of a higher diameter. To obtain maximum degree 3, we replace the set \( V_2 \) by some suitably defined binary trees. □

5 Conclusion

We have introduced a new graph parameter useful in the area of network monitoring. We have related it to other standard graph parameters by the means of lower and upper bounds. It would be interesting to improve them. In particular, is it true that \( \text{dem}(G) \leq \text{fes}(G) + 1 \)? As we have seen, this bound would be tight.

It would also be interesting to determine graph classes where Distance-Edge-Monitoring Set has a polynomial-time (or parameterized) exact or constant-factor approximation algorithm.
References

1. G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela and M. Protasi. Complexity and Approximation: Combinatorial Optimization Problems and Their Approximability Properties. Springer, 1999.
2. E. Bampas, D. Bilò, G. Drovandi, L. Gualà, R. Klasing and G. Proietti. Network verification via routing table queries. Journal of Computer and System Sciences 81(1):234–248, 2015.
3. Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalák and L. S. Ram. Network discovery and verification. IEEE Journal on Selected Areas in Communications 24(12):2168–2181, 2006.
4. D. Bilò, T. Erlebach, M. Mihalák and P. Widmayer. Discovery of network properties with all-shortest-paths queries. Theoretical Computer Science 411(14–15):1626–1637, 2010.
5. J. Chen, B. Chor, M. Fellows, X. Huang, D. W. Juedes, I. A. Kanj and G. Xia. Tight lower bounds for certain parameterized NP-hard problems. Information and Computation 201(2):216–231, 2005.
6. L. Dall’Asta, J. I. Alvarez-Hamelin, A. Barrat, A. Vázquez and A. Vespignani. Exploring networks with traceroute-like probes: Theory and simulations. Theoretical Computer Science 355(1):6–24, 2006.
7. I. Dinur and D. Steurer. Analytical approach to parallel repetition. Proceedings of the forty-sixth annual ACM symposium on Theory of computing, STOC 2014:624–633, 2014.
8. R. G. Downey and M. R. Fellows. Fundamentals of Parameterized Complexity. Springer, 2013.
9. A. Eiben, A. Kröninger, D. Kröner, R. Mihalík and P. Widmayer. The rectilinear Steiner tree problem is NP-complete. Theoretical Computer Science 377(1):287–296, 2007.
10. R. Govindan and H. Tangmunarunkit. Heuristics for Internet map discovery. Proceedings of the 19th IEEE International Conference on Computer Communications, INFOCOM’00, 2000, pp. 1371–1380.
11. F. Harary and R. A. Melter. On the metric dimension of a graph. Ars Combinatoria 2:191–195, 1976.
12. J. Hromkovič. Algorithmics for Hard Problems - Introduction to Combinatorial Optimization, Randomization, Approximation, and Heuristics, Second Edition. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2004.
13. D. S. Johnson, Approximation algorithms for combinatorial problems. Journal of Computer and System Sciences 9:256–278, 1974.
14. A. Kelenc, N. Tratnik and I. G. Yero. Uniquely identifying the edges of a graph: The edge metric dimension. Discrete Applied Mathematics 251:204–220, 2018.
15. P. Mannel, S. Klavžar, A. Xavier, A. Arkiaraj and E. Thomas. Strong edge geometric problem in networks. Open Mathematics 15:1225–1235, 2017.
16. R. Niedermeier. Invitation to fixed-parameter algorithms. Oxford University Press, 2006.
17. O. R. Oellermann and J. Peters-Fransen. The strong metric dimension of graphs and digraphs. Discrete Applied Mathematics 155:356–364, 2007.
18. A. Sebő and E. Tannier. On metric generators of graphs. Math. Oper. Res. 29(2):383–393, 2004.
19. P. J. Slater. Leaves of trees. Congressus Numerantium 14:549–559, 1975.