TEMPERED CURRENTS AND THE COHOMOLOGY OF THE REMOTE FIBER OF A REAL POLYNOMIAL MAP

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Abstract. Let $p : \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial map. Consider the complex $\mathcal{S}'(\mathbb{R}^n)$ of tempered currents on $\mathbb{R}^n$ with the twisted differential $d_p = d - dp$ where $d$ is the usual exterior differential and $dp$ stands for the exterior multiplication by $dp$. Let $t \in \mathbb{R}$ and let $F_t = p^{-1}(t)$. In this paper we prove that the reduced cohomology $\tilde{H}^{k+1}(\mathcal{S}'(\mathbb{R}^n), d_p)$ is isomorphic to $H^k(F_t; \mathbb{C})$ in the case when $p$ is homogeneous and $t$ is any positive real number. We conjecture that this isomorphism holds for any polynomial $p$, for $t$ large enough (we call the $F_t$ for $t \gg 0$ the remote fiber of $p$) and we prove this conjecture for polynomials that satisfy certain technical condition (cf. Theorem 1.9). The result is analogous to that of A. Dimca and M. Saito ([2]), who give a similar (algebraic) way to compute the reduced cohomology of the generic fiber of a complex polynomial.

1. Introduction

1.1. The Dimca-Saito theorem. Let $p : \mathbb{C}^n \rightarrow \mathbb{C}$ be a complex polynomial. Let $F$ denote the generic fiber of $p$ (it is well-defined as a topological space). In [2], A. Dimca and M. Saito have given the following algebraic way to compute the cohomology of $F$. Let $\Omega^\bullet$ denote the De Rham algebra of polynomial differential forms on $\mathbb{C}^n$. Define a differential $d_p$ on $\Omega^\bullet$ by

$$d_p(\omega) = d\omega - dp \wedge \omega$$

Theorem 1.2 (Dimca-Saito). There exists an isomorphism

$$H^{k+1}(\Omega^\bullet, d_p) \simeq \tilde{H}^{k}(F; \mathbb{C}) \quad \text{for } k = 0, 1, \ldots , n - 1$$

where $\tilde{H}^\bullet(F; \mathbb{C})$ denotes the reduced cohomology of $F$ with coefficients in $\mathbb{C}$.

1.3. The main result. The main purpose of this paper is to describe certain real analogue of Theorem 1.2. Namely, let now $p : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real polynomial. Then (cf. [4]) the topological type of the fiber $F_t = p^{-1}(t)$ does not depend upon $t$ provided $t$ is large enough. We shall refer to $F_t$ as to remote fiber of $p$ and we shall be interested in the cohomology of $F_t$.

Let $\mathcal{O} := \mathbb{C} [x_1, \ldots , x_n]$ denote the ring of complex polynomials on $\mathbb{R}^n$ and let $\mathcal{S}'(\mathbb{R}^n)$ denote the $\mathcal{O}$ module of tempered complex valued distributions on $\mathbb{R}^n$. Let $\Omega^\bullet$ be the complex of global algebraic differential forms on $\mathbb{R}^n$. Consider the space

$$\mathcal{S}'(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n) \otimes \mathcal{O}^\bullet$$
of tempered currents and define a differential \( d_p : S'\Omega^\bullet(\mathbb{R}^n) \to S'\Omega^{\bullet+1}(\mathbb{R}^n) \) on \( S'\Omega^\bullet(\mathbb{R}^n) \) by
\[
d_p(\omega) = d\omega - dp \wedge \omega \quad \text{for} \quad \omega \in S'\Omega^\bullet(\mathbb{R}^n).
\]

In this paper we discuss the following

**Conjecture 1.4.** For any real polynomial \( p : \mathbb{R}^n \to \mathbb{R} \) the following isomorphism holds:
\[
H^{k+1}(S'\Omega^\bullet(\mathbb{R}^n), d_p) = \tilde{H}^k(F_t; \mathbb{C}), \quad k = 0, 1, \ldots, n - 1
\]  
(1.1)
where \( \tilde{H} \) denotes reduced cohomology.

In particular, we prove the following

**Theorem 1.5.** Assume that \( p : \mathbb{R}^n \to \mathbb{R} \) is a homogeneous polynomial map of degree \( m \), i.e. \( p(sx) = s^mp(x) \) for any \( x \in \mathbb{R}^n \), \( s \in \mathbb{R} \). For any \( t > 0 \), the isomorphism (1.1) holds.

**Remark 1.6.** a) By definition, the reduced cohomology of any topological space \( X \) is the cohomology of the complex
\[
0 \to \mathbb{C} \xrightarrow{\varepsilon} H^0(X; \mathbb{C}) \xrightarrow{0} \cdots \xrightarrow{0} H^k(X; \mathbb{C}) \to \cdots
\]
where \( \varepsilon = 0 \) if \( X \) is empty and \( \varepsilon \) is the natural map \( \mathbb{C} = H^0(pt; \mathbb{C}) \to H^0(X; \mathbb{C}) \) coming from the projection \( X \to pt \) otherwise. Therefore in the case when \( X \) is empty one should have \( \tilde{H}^{-1}(X; \mathbb{C}) = \mathbb{C} \) (but \( \tilde{H}^{-1}(X; \mathbb{C}) = 0 \) if \( X \) contains at least one point). With this convention, Theorem 1.4 remains true also for \( k = -1 \).

b) Note that if instead of the complex \( S'\Omega^\bullet(\mathbb{R}^n) \) we considered the complex of all currents with the same differential \( d_p \), then we would get a complex quasi-isomorphic to the usual complex of currents on \( \mathbb{R}^n \) with the ordinary exterior differential \( d \) (since \( d \) and \( d_p \) are conjugate to one another by means of the function \( e^p \)). Therefore if we do not impose any growth conditions on our currents we will not get any interesting cohomology.

**1.7. Sketch of the proof.** Step 1. Let \( U_t = \{ x \in \mathbb{R}^n : \ p(x) > t \} \) (note that \( U_t \) might be empty). Then, \( U_t \) is diffeomorphic to the product \( F_t \times (0, \infty) \), for any \( t > 0 \). Using the long cohomological sequence of the pair \( (\mathbb{R}^n, U_t) \) one can easily see that \( \tilde{H}^{k-1}(F_t, \mathbb{C}) = H^k(\mathbb{R}^n, U_t; \mathbb{C}) \).

Step 2. Let \( D'\Omega^\bullet(U_t) \) denote the complex of all currents on \( U_t \). In Section 2.2 we define certain subcomplex \( S'\Omega^\bullet(U_t) \) of \( D'\Omega^\bullet(U_t) \) (the complex of tempered currents on \( U_t \)) and prove that its natural inclusion into \( D'\Omega^\bullet(U_t) \) is a quasi-isomorphism.

Step 3. Let \( D'\Omega^\bullet(\mathbb{R}^n) \) denote the complex of all currents on \( \mathbb{R}^n \). Let \( \theta(s) \) be a smooth function on \( \mathbb{R} \), such that \( \theta(s) = s \) for \( s < 1 \) and \( \theta(s) = 0 \) for \( s > 2 \). Define \( \tilde{\phi} : \mathbb{R}^n \to \mathbb{R} \) by \( \tilde{\phi}(x) = \theta(p(x)) \). Let \( S'_p\Omega^\bullet(\mathbb{R}^n) \) denote the space of all currents \( \omega \) on \( \mathbb{R}^n \), such that \( e^{\tilde{\phi}}\omega \in S'\Omega^\bullet(\mathbb{R}^n) \). Then we show in Lemma 3.3 that \( S'_p\Omega^\bullet(\mathbb{R}^n) \) is a subcomplex of \( D'\Omega^\bullet(\mathbb{R}^n) \) and the natural embedding \( S'_p\Omega^\bullet(\mathbb{R}^n) \hookrightarrow D'\Omega^\bullet(\mathbb{R}^n) \) is a quasi-isomorphism.

Let now \( \rho \) denote the natural map from \( S'_p\Omega^\bullet(\mathbb{R}^n) \) to \( S'\Omega^\bullet(U_t) \) (restriction to \( U_t \)). It follows from step 2 and from the above statement that the complex
\[
\text{Cone}^\bullet(\rho) = S'_p\Omega^\bullet(\mathbb{R}^n) \oplus S'\Omega^{\bullet-1}(U_t)
\]
computes the relative cohomology $H^\bullet(\mathbb{R}^n, U_t; \mathbb{C})$.

Step 4. The map $\Phi_1 : \omega \rightarrow e^{-p}\omega$ defines a morphism of complexes $S'\Omega^\bullet(\mathbb{R}^n) \rightarrow S'_p\Omega^\bullet(\mathbb{R}^n)$. Moreover, every element in the image of $\Phi_1$ is rapidly decreasing along the rays $R_x = \{sx : s > 0\}$, for any $x \in F_t$. This enables us to extend $\Phi_1$ to an explicit map $\Phi : S'\Omega^\bullet(\mathbb{R}^n) \rightarrow \text{Cone}^\bullet(\rho)$. In order to do that we need the following notations.

Let $\mu_s : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the multiplication by $s$ and let $\mu_s^* : \mathcal{D}'\Omega^\bullet(\mathbb{R}^n) \rightarrow \mathcal{D}'\Omega^\bullet(\mathbb{R}^n)$ be the corresponding pull-back map.

Consider the Euler vector field $\mathcal{R} = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$ and let $\iota_\mathcal{R}$, $\mathcal{L}_\mathcal{R}$ denote the interior multiplication by $\mathcal{R}$ and the Lie derivative along $\mathcal{R}$. Then

$$\frac{d}{ds}\mu_s^*(\omega) = \mu_s^*(\mathcal{L}_\mathcal{R}\omega)s^{-1} \quad \text{for any} \quad \omega \in \mathcal{D}'\Omega^\bullet(\mathbb{R}^n). \quad (1.2)$$

We define the map $\Phi : S'\Omega^\bullet(\mathbb{R}^n) \rightarrow \text{Cone}^\bullet(\rho) = S'_p\Omega^\bullet(\mathbb{R}^n) \oplus S'\Omega^{*-1}(U_t)$ by the formula

$$\Phi : \omega \mapsto (\Phi_1\omega, \Phi_2\omega) = \left(e^{-p}\omega, -\int_1^\infty \mu_s^*(e^{-p}\iota_\mathcal{R}\omega)\, ds\right). \quad (1.3)$$

The integral in $(1.3)$ converges since $e^{-p(sz)}$ decreases exponentially in $s$ as $s$ tends to infinity. One uses $(1.3)$ to show that $\Phi$ commutes with differentials.

Finally we prove by an explicit calculation that $\Phi$ is a quasi-isomorphism. Therefore $S'\Omega^\bullet(\mathbb{R}^n)$ computes $H^\bullet(\mathbb{R}^n, U_t; \mathbb{C})$, which is isomorphic to $\tilde{H}^{*-1}(F_t, \mathbb{C})$ by step 1.

1.8. The general case. Let now $p : \mathbb{R}^n \rightarrow \mathbb{R}$ be an arbitrary polynomial. Set $v = \frac{\nabla p}{|\nabla p|^2}$. Then the Lie derivative of $p$ along $v$ is equal to 1. In the Appendix we show that the flow of $v$ is globally defined on $U_t$ if $t$ is large enough. We denote this flow by $g^t_s : U_t \rightarrow U_t$ and let $g^t_s : \mathcal{D}'\Omega^\bullet(U_t) \rightarrow \mathcal{D}'\Omega^\bullet(U_t)$ be the corresponding pull-back of currents. Then $g^t_s(p) = p + s$. In particular, we obtain a new proof of topological equivalence of the fibers $\tilde{F}_t$ with $t \gg 0$.

Denote

$$\tilde{v} = pv$$

and let $\iota_{\tilde{v}}$, $\mathcal{L}_{\tilde{v}}$ denote the interior multiplication by $\tilde{v}$ and the Lie derivative along $\tilde{v}$. The flow $\tilde{g}_s$ of $\tilde{v}$ is defined on $U_t$, $t \gg 0$. The flow $\tilde{g}_s$ and the vector field $\tilde{v}$ are connected by the formula

$$\frac{d}{ds}\tilde{g}_s^*(\omega) = \tilde{g}_s^*(\mathcal{L}_{\tilde{v}}\omega) \quad \text{for any} \quad \omega \in \mathcal{D}'\Omega^\bullet(\mathbb{R}^n),$$

which is similar to $(1.2)$ (if $p$ is a homogeneous polynomial of degree $m$ then $\mu_s = g^t_{m\text{ln}s}$). One can easily check that $\tilde{g}_s^*(p) = e^s p$.

One can try to define a map $\Phi : S'\Omega^\bullet(\mathbb{R}^n) \rightarrow \text{Cone}^\bullet(\rho)$ by formula

$$\Phi : \omega \mapsto (\Phi_1\omega, \Phi_2\omega) = \left(e^{-p}\omega, -\int_1^\infty \tilde{g}_s^*(e^{-p}\iota_{\tilde{v}}\omega)\, ds\right),$$

similar to $(1.3)$. The only problem here is that we were not able to prove that the integral in the definition of $\Phi_2$ converges to a tempered current. However, if the map $\Phi_2 :$
$S'\Omega^\bullet (\mathbb{R}^n) \to S'\Omega^{\bullet -1}(\mathbb{R}^n)$ is well defined a verbatim repetition of the proof of Theorem 1.5 gives the following

**Theorem 1.9.** Suppose that $p : \mathbb{R}^n \to \mathbb{R}$ is a real polynomial and $\tilde{g}_s, s > 0$ is a one-parameter semigroup of diffeomorphisms $U_t \to U_t$ such that $\tilde{g}_s^*(p) = e^{ms}p$. Let $\bar{v} = \frac{d}{ds}|_{s=0} g_s$. If for any tempered current $\omega$ the integral

$$\int_1^\infty \tilde{g}_s^*(e^{-p\bar{v}}) \omega \, ds$$

converges to a tempered current, then the isomorphism (1.1) holds.

1.10. **Example.** Consider the polynomial of two variables $p(x, y) = x^2 - x - y$. Set $U_0 = \{(x, y) \in \mathbb{R}^2 : p(x, y) > 0\}$ and define a one parameter semigroup $g_s$ of diffeomorphisms of $U_0$ by the formula

$$g_s(x, y) = \left(e^{s/2}x, e^s x - e^{s/2}x + e^s y\right).$$

Then $g_s^* p = e^s p$. Clearly, all other conditions of Theorem 1.9 are satisfied. Hence, the isomorphism (1.1) holds for $p(x, y)$.

**Acknowledgments.** It is a great pleasure for us to express our gratitude to J. Bernstein and M. Farber; the paper was considerably influenced by communications with them. It was M. Farber who suggested to use the map (1.3) for the study of $H^\bullet(\Omega^\bullet, d_p)$.

We are also thankful to N. Zobin, M. Zaidenberg and S. Kaliman.

The first author would like to thank Institute for Advance Study for hospitality.

## 2. Complexes of Currents

In this section we review some facts about complexes of currents which will be used in the proof of Theorem 1.5.

Let $p : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial map of degree $m$, i.e. $p(sx) = s^m p(x)$. Let $U_t = \{x \in \mathbb{R}^n : p(x) > t\}$, where $t \in \mathbb{R}$.

2.1. **The complex of currents.** By $\Omega^\bullet_c(U_t)$ we denote the De Rham complex of compactly supported complex valued $C^\infty$-forms on $U_t$. The cohomology of $\Omega^\bullet_c(U_t)$ is called the **compactly supported cohomology** of $U_t$.

Recall that if $0 \to C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots \xrightarrow{d} C^n \to 0$ is a complex of topological vector spaces then the dual complex to $(C^\bullet, d)$ is, by definition, the complex

$$0 \to (C^0)^* \xrightarrow{d^*} (C^1)^* \xrightarrow{d^*} \cdots \xrightarrow{d^*} (C^n)^* \to 0,$$

where $(C^i)^*$ denotes the topological dual of the space $C^i$ and $d^*$ denotes the adjoint operator of $d$.

The complex of currents $\mathcal{D}'\Omega^\bullet(U_t)$ on $U_t$ is the complex dual to $\Omega^\bullet_c(U_t)$. By the Poincaré duality for non-compact manifolds (cf. [I]), the cohomology of $\mathcal{D}'\Omega^\bullet(U_t)$ is equal to the cohomology of $U_t$.

Analogously, one defines the complex $\mathcal{D}'\Omega^\bullet(\mathbb{R}^n)$ of currents on $\mathbb{R}^n$. 
Let \( r : \mathcal{D}' \Omega^* (\mathbb{R}^n) \to \mathcal{D}' \Omega^* (U_t) \) be the restriction. Recall that the cone \( \text{Cone}^* (r) \) of \( r \) is the complex
\[
\text{Cone}^* (r) = \mathcal{D}' \Omega^* (\mathbb{R}^n) \oplus \mathcal{D}' \Omega^{-1} (U_t), \quad d : (\omega, \alpha) \mapsto (d\omega, \omega - d\alpha).
\]
The cohomology of \( \text{Cone}^* (r) \) is equal to the relative cohomology \( H^* (\mathbb{R}^n, V; \mathbb{C}) \) of the pair \((\mathbb{R}^n, V)\).

2.2. The complex of tempered currents. The space \( \mathcal{S}(\mathbb{R}^n) \) of Schwartz (rapidly decreasing) functions on \( \mathbb{R}^n \) is the set of all \( \phi \in C^\infty (\mathbb{R}^n) \) such that for any linear differential operator \( L : C^\infty (\mathbb{R}^n) \to C^\infty (\mathbb{R}^n) \) with polynomial coefficients
\[
\sup_{x \in \mathbb{R}^n} |L \phi (x)| < \infty.
\] (2.1)
The topology in \( \mathcal{S}(\mathbb{R}^n) \) defined by the semi-norms in the left-hand side of (2.1) makes \( \mathcal{S}(\mathbb{R}^n) \) a Fréchet space.

Recall that by \( \Omega^* \) we denote the De Rham complex of global algebraic differential forms on \( \mathbb{R}^n \). The complex of Schwartz forms on \( \mathbb{R}^n \) is the complex
\[
\mathcal{S} \Omega^* (\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n) \otimes \Omega^*
\]
with natural differential. By the complex of Schwartz forms on \( U_t \) we will understand the subcomplex of \( \mathcal{S} \Omega^* (\mathbb{R}^n) \) consisting of the forms \( \omega \) such that there exists a real number \( \varepsilon = \varepsilon (\omega) \) such that the support of \( \omega \) lies in \( U_t + \varepsilon \).

The complex \( \mathcal{S}' \Omega^* (\mathbb{R}^n) \) of tempered currents on \( \mathbb{R}^n \) is, by definition, the dual complex to \( \mathcal{S} \Omega^* (\mathbb{R}^n) \). Similarly, the complex \( \mathcal{S}' \Omega^* (U_t) \) of tempered currents on \( U_t \) is, the dual complex to \( \mathcal{S} \Omega^* (U_t) \).

Lemma 2.3. For any \( t_1 > t_2 > 0 \) the natural map \( i : \mathcal{S}' \Omega^* (U_{t_2}) \to \mathcal{S}' \Omega^* (U_{t_1}) \) is a homotopy equivalence of complexes.

Proof. Let \( \mu_s : \mathbb{R}^n \to \mathbb{R}^n \) denote the multiplication by \( s \) and let \( \mu_s^* : \mathcal{D}' \Omega^* (\mathbb{R}^n) \to \mathcal{D}' \Omega^* (\mathbb{R}^n) \) be the corresponding pull-back map. Clearly, \( \mu_s^* \) preserves the space of tempered currents.

Set \( \tau = (t_1/t_2)^{1/m} \). Then \( \mu_{\tau} (U_{t_2}) = U_{t_1} \). In particular, we can consider \( \mu_{\tau}^* \) as a map from \( \mathcal{S} \Omega^* (U_{t_1}) \) to \( \mathcal{S} \Omega^* (U_{t_2}) \). To prove the lemma we will show that \( \mu_{\tau}^* \) is a homotopy inverse of \( i \).

Consider the Euler vector field
\[
\mathcal{R} = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}
\]
on \( \mathbb{R}^n \) and let \( \iota_\mathcal{R}, \mathcal{L}_\mathcal{R} \) denote the interior multiplication by \( \mathcal{R} \) and the Lie derivative along \( \mathcal{R} \). Then
\[
\frac{d}{ds} \mu_s^* (\omega) = \mu_s^* (\mathcal{L}_\mathcal{R} \omega) s^{-1} \quad \text{for any} \quad \omega \in \mathcal{D}' \Omega^* (\mathbb{R}^n).
\] (2.2)
Note that if \( \omega \) is a tempered current so are \( \iota_\mathcal{R} \omega \) and \( \mathcal{L}_\mathcal{R} \omega \).
For any current $\omega$, set
\[ H_\omega = \int_1^\tau \mu_s^*(\iota_R \omega) \frac{ds}{s}. \]
The operators $\mu_s^*$ and $\iota_R$ preserve the space of tempered currents. Hence, so does $H$. Using (2.2) and the Cartan homotopy formula
\[ \mathcal{L}_R = d\iota_R + \iota_R d \quad (2.3) \]
we obtain
\[ (dH + Hd)\omega = \mu_s^*\omega - \omega, \]
for any current $\omega$. The lemma is proven. \(\square\)

**Lemma 2.4.** For any $t > 0$ the embedding $S'\Omega^\bullet(U_1) \hookrightarrow \mathcal{D}'\Omega^\bullet(U_1)$ is a homotopy equivalence of complexes. In particular, the cohomology of the complex $S'\Omega^\bullet(U_1)$ is equal to the cohomology $H^\bullet(U_t; \mathbb{C})$ of $U_t$.

**Proof.** By Lemma 2.3 it is enough to show that the embedding $i : S'\Omega^\bullet(U_1) \hookrightarrow \mathcal{D}'\Omega^\bullet(U_1)$ is a quasi-isomorphism.

Let $h_s : U_1 \to U_1$, $s > 0$ denote the map defined by the formula
\[ h_s : x \mapsto \frac{1 + s}{1 + s|x|} \cdot x. \]
Here $|x|$ denotes the norm of the vector $x \in \mathbb{R}^n$. Let $h_s^* : \mathcal{D}'\Omega^\bullet(U_1) \to \mathcal{D}'\Omega^\bullet(U_1)$ denote the corresponding pull-back. Then (cf. (2.2))
\[ \frac{d}{ds} h_s^*(\omega) = \frac{1 - |x|}{(1 + s)(1 + s|x|)} h_s^*(\mathcal{L}_R \omega), \quad (2.4) \]
for any current $\omega$. Note also that $h_s^*$ preserves the space of tempered currents.

Clearly, $h_0$ is the identity map. The image of $h_1 : U_1 \to U_1$ lies in the compact set
\[ \{ x \in \mathbb{R}^n : |x| \leq 2 \}. \]
Hence, $h_s^*\omega$ is a tempered current for any $\omega \in \mathcal{D}'\Omega^\bullet(\mathbb{R}^n)$. Note also that $h_s^*$ preserves the space of tempered currents for any $s > 0$. We will prove that the map $h_1^* : \mathcal{D}'\Omega^\bullet(U_1) \to S'\Omega^\bullet(U_1)$ is a homotopy inverse of the embedding $i : S'\Omega^\bullet(U_1) \hookrightarrow \mathcal{D}'\Omega^\bullet(U_1)$.

For any $\omega \in \mathcal{D}'\Omega^\bullet(U_1)$, set
\[ H_\omega = \int_0^1 \frac{1 - |x|}{(1 + s)(1 + s|x|)} h_s^*(\iota_R \omega) \, ds. \]
Here, $\iota_R$ denote the operator of interior multiplication by $R$.

Using (2.4) and (2.3) we obtain
\[ (dH + Hd)\omega = h_1^*\omega - \omega, \quad \omega \in \mathcal{D}'\Omega^\bullet(U_1). \quad (2.5) \]
Thus the map $i \circ h_1^* : \mathcal{D}'\Omega^\bullet(U_1) \to \mathcal{D}'\Omega^\bullet(U_1)$ homotopic to the identity map.
Since the operators $h_s^*$ and $v_R$ preserve the space of tempered currents, so does $H$. Hence, (2.3) implies that the map $h_s^* \circ i : S'_p \Omega^*(U_1) \to S'_p \Omega^*(U_1)$ is also homotopic to the identity map.

2.5. The complex $S'_p \Omega^*(\mathbb{R}^n)$. We will need the following twisted version of the complex of tempered currents on $\mathbb{R}^n$.

Fix a smooth function $\theta : \mathbb{R} \to \mathbb{R}$ such that
\[
\theta(s) = \begin{cases} 
  s & \text{if } s < 1, \\
  0 & \text{if } s > 2.
\end{cases}
\]
and define $\tilde{\theta}(x) = \theta(p(x)), x \in \mathbb{R}^n$. Note that the current $d\tilde{\theta} \wedge \omega$ is tempered for any tempered current $\omega$.

**Lemma 2.6.** The space $S'_p \Omega^*(\mathbb{R}^n) = \{ \omega \in \mathcal{D}' \Omega^*(\mathbb{R}^n) : e^\tilde{\theta} \omega \in S'_p \Omega^*(\mathbb{R}^n) \}$ is a subcomplex of $\mathcal{D}' \Omega^*(\mathbb{R}^n)$ and the embedding $S'_p \Omega^*(\mathbb{R}^n) \hookrightarrow \mathcal{D}' \Omega^*(\mathbb{R}^n)$ is a quasi-isomorphism.

**Proof.** Suppose $\omega \in S'_p \Omega^*(\mathbb{R}^n)$, i.e. $e^\tilde{\theta} \omega \in S'_p \Omega^*(\mathbb{R}^n)$. Then
\[
e^\tilde{\theta} d\omega = d(e^\tilde{\theta} \omega) - d\tilde{\theta} \wedge e^{-\tilde{\theta}} \omega \in S'_p \Omega^*(\mathbb{R}^n),
\]
i.e. $d\omega \in S'_p \Omega^*(\mathbb{R}^n)$. Hence $S'_p \Omega^*(\mathbb{R}^n)$ is a subcomplex of $\mathcal{D}' \Omega^*(\mathbb{R}^n)$.

Clearly, the embedding $S'_p \Omega^*(\mathbb{R}^n) \hookrightarrow \mathcal{D}' \Omega^*(\mathbb{R}^n)$ induces an isomorphism of 0-cohomology. To prove Lemma 2.6 it remains to show that the $k$-th cohomology $H^k(S'_p \Omega^*(\mathbb{R}^n)), k > 0$ of $S'_p \Omega^*(\mathbb{R}^n)$ vanishes.

We will use the notation introduced in the proof of Lemma 2.3. In particular, $\mu_s : \mathbb{R}^n \to \mathbb{R}^n$ is the multiplication by $s$ and $R$ is the Euler vector field on $\mathbb{R}^n$.

Let $\omega$ be a closed current, $d\omega = 0$. Using (2.3) and the Cartan homotopy formula $L_R = dv_R + v_R d$, we obtain
\[
\omega - \mu_0^*(\omega) = d \int_0^1 \mu_0^*(v_R \omega) \frac{ds}{s},
\]
(Note that the integral in the left hand side converges, because $R$ vanishes at 0). If $\omega$ is a $k$-current, $k > 0$, then $\mu_0^*(\omega) = 0$. Hence, to finish the proof we need only to show that
\[
\int_0^1 \mu_s^*(v_R \omega) \frac{ds}{s} \in S'_p \Omega^*(\mathbb{R}^n) \quad (2.6)
\]
for any $\omega \in S'_p \Omega^*(\mathbb{R}^n)$.

Set $\beta = e^\tilde{\theta} \omega \in S'_p \Omega^*(\mathbb{R}^n)$. Then
\[
e^\tilde{\theta} \int_0^1 \mu_s^*(v_R \omega) \frac{ds}{s} = \int_0^1 e^{\tilde{\theta}(x) - \tilde{\theta}(sx)} \mu_s^*(v_R \beta) \frac{ds}{s}. \quad (2.7)
\]
Since, for any $s \in [0, 1]$, the function $\tilde{\theta}(x) - \tilde{\theta}(sx)$ is bounded from above, all the derivatives of the function $e^{\tilde{\theta}(x) - \tilde{\theta}(sx)}$ are bounded by polynomials. It follows that $s \mapsto e^{\tilde{\theta}(x) - \tilde{\theta}(sx)} \mu_s^*(v_R \beta)s^{-1}$ defines a continuous map $[0, 1] \to S'_p \Omega^*(\mathbb{R}^n)$. Hence the current (2.7) is tempered and (2.6) holds. \qed
2.7. For any \( \omega \in S'_p\Omega^*(\mathbb{R}^n) \), \( t > 0 \) the restriction of \( \omega \) on \( U_t \) is a tempered current on \( U_t \). Hence, the restriction map \( \rho : S'_p\Omega^*(\mathbb{R}^n) \rightarrow S'\Omega^*(U_t) \) is defined.

**Lemma 2.8.** The complexes \( \text{Cone}^\bullet(\rho) \) and \( \text{Cone}^\bullet(r) \) (cf. Section 2.4) are quasi-isomorphic. In particular, the cohomology of \( \text{Cone}^\bullet(\rho) \) equals the relative cohomology of the pair \( (\mathbb{R}^n, U_t) \).

**Proof.** Let \( i : S'_p\Omega^*(\mathbb{R}^n) \rightarrow \mathcal{D}'\Omega^*(\mathbb{R}^n) \), \( j : S'\Omega^*(U_t) \rightarrow \mathcal{D}'\Omega^*(U_t) \) denote the natural inclusions. Consider the commutative diagram

\[
\begin{array}{ccc}
S'_p\Omega^*(\mathbb{R}^n) & \xrightarrow{\rho} & S'\Omega^*(U_t) \\
\downarrow{i} & & \downarrow{j} \\
\mathcal{D}'\Omega^*(\mathbb{R}^n) & \xrightarrow{r} & \mathcal{D}'\Omega^*(U_t)
\end{array}
\] (2.8)

According to Lemmas 2.4 and 2.6 the vertical arrows of this diagram are quasi-isomorphisms. Hence, (2.8) induces a quasi-isomorphism between \( \text{Cone}^\bullet(\rho) \) and \( \text{Cone}^\bullet(r) \).

In the proof of Lemma 3.4 we will also need the following

**Lemma 2.9.** Suppose \( t_1 > t_2 > 0 \) and let

\[
\rho_1 : S'_p\Omega^*(\mathbb{R}^n) \rightarrow S'\Omega^*(U_{t_1}), \quad \rho_2 : S'_p\Omega^*(\mathbb{R}^n) \rightarrow S'\Omega^*(U_{t_2})
\]

denote the corresponding restrictions. The natural map \( \text{Cone}^\bullet(\rho_2) \rightarrow \text{Cone}^\bullet(\rho_1) \) is a quasi-isomorphism.

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
S'_p\Omega^*(\mathbb{R}^n) & \xrightarrow{\rho_2} & S'\Omega^*(U_{t_2}) \\
\| & & \downarrow \\
S'_p\Omega^*(\mathbb{R}^n) & \xrightarrow{\rho_1} & S'\Omega^*(U_{t_1})
\end{array}
\] (2.9)

By Lemma 2.3, the right vertical arrow of this diagram is a quasi-isomorphism. Hence, (2.9) induces a quasi-isomorphism between \( \text{Cone}^\bullet(\rho_2) \) and \( \text{Cone}^\bullet(\rho_1) \).

3. **Proof of Theorem 0.2**

3.1. **Cohomology of \( F_t \) as relative cohomology.** Fix \( t > 0 \) and set

\[ F_t = p^{-1}(t); \quad U_t = \{ x \in \mathbb{R}^n : p(x) > t \}. \]

Then \( U_t \) is diffeomorphic to the the product \( F_t \times (0, +\infty) \). In particular, \( U_t \) has the same cohomology as \( F_t \). Using the long exact sequence of the pair \( (\mathbb{R}^n, U_t) \), we obtain

\[ \widetilde{H}^k(F_t; \mathbb{C}) = H^{k+1}(\mathbb{R}^n, U_t; \mathbb{C}), \quad k = 0, 1, \ldots, n - 1, \] (3.1)

where \( H^\bullet(\mathbb{R}^n, U_t; \mathbb{C}) \) denotes the relative cohomology of the pair \( (\mathbb{R}^n, U_t) \) and \( \widetilde{H}^\bullet(F_t; \mathbb{C}) \) denotes the reduced cohomology of \( F_t \).
3.2. A map from \( S'\Omega^\bullet(\mathbb{R}^n) \) to \( \text{Cone}^\bullet (\rho) \). Recall that by \( \mu_s : \mathbb{R}^n \to \mathbb{R}^n \) we denote the multiplication by \( s \in \mathbb{R} \). Then \( \mu_s(U_t) = U_{s\cdot t} \). In particular, if \( s \geq 1 \), then \( \mu_s \) may be considered as a map from \( U_t \) to itself. Let \( \mu^*_s : \mathcal{D}'\Omega^\bullet(U_t) \to \mathcal{D}'\Omega^\bullet(U_t) \) denote the corresponding pull-back map. Then \( \mu^*_s p(x) = p(\mu_s x) = s^n p(x) \).

Recall also that \( t_R \) denote the interior multiplication by the Euler vector field \( \mathcal{R} = \sum x_i \frac{\partial}{\partial x_i} \). Note that if \( \omega \) is a tempered current on \( U_t \) then so are \( t_R \omega \) and \( \mu^*_s \omega \).

Recall from Section 2.7 that \( \rho : S'_p \Omega^\bullet(\mathbb{R}^n) \to S'\Omega^\bullet(U_t) \) denotes the restriction. We define the map

\[
\Phi : S'\Omega^\bullet(\mathbb{R}^n) \to \text{Cone}^\bullet (\rho) = S'_p \Omega^\bullet(\mathbb{R}^n) \oplus S'\Omega^{\bullet-1}(U_t)
\]

by the formula

\[
\Phi : \omega \mapsto (\Phi_1 \omega, \Phi_2 \omega) = \left( e^{-p} \omega, - \int_1^\infty \mu^*_s (e^{-p} t_R \omega) \frac{ds}{s} \right).
\]

The integral in (3.2) converges since \( e^{-p(sx)} \) decreases exponentially in \( s \) as \( s \) tends to infinity. It follows from (2.2), (2.3) that the map \( \Phi : S'\Omega^\bullet(\mathbb{R}^n) \to \text{Cone}^\bullet (\rho) \) commutes with differentials, i.e.

\[
\Phi_1 d_p \omega = d\Phi_1 \omega; \quad \Phi_2 d_p \omega = \Phi_1 \omega|_U - d\Phi_2 \omega.
\]

**Lemma 3.3.** The map \( H^\bullet(S'\Omega^\bullet(\mathbb{R}^n), d_p) \to H^\bullet(\text{Cone}^\bullet (\rho)) \) induced by \( \Phi \) is injective.

**Proof.** Suppose that \( \omega \) is a tempered current on \( \mathbb{R}^n \) and that \( \Phi \omega \) is a coboundary in \( \text{Cone}^\bullet (\rho) \). Then there exists \( \alpha \in S'_p \Omega^\bullet(\mathbb{R}^n) \) and \( \beta \in S'\Omega^{\bullet-1}(U_t) \) such that

\[
e^{-p} \omega = d\alpha; \quad - \int_1^\infty \mu^*_s (e^{-p} t_R \omega) \frac{ds}{s} = \alpha|_U - d\beta.
\]

Choose \( j \in C^\infty(\mathbb{R}) \) such that \( j(s) = 0 \) if \( s \leq t + 1 \) and \( j(s) = 1 \) if \( s \geq t + 2 \) and set \( \phi(x) = j(p(x)), \ (x \in \mathbb{R}^n) \). Then the support of \( \phi \) is contained in \( U_t \). Hence \( \phi \beta \) may be considered as a current on \( \mathbb{R}^n \). Since all the derivatives of \( \phi \) are bounded by polynomials, \( \phi \beta \in S'_p \Omega^\bullet(\mathbb{R}^n) \).

Define \( \alpha = \alpha - d(\phi \beta) \). Then \( \omega = d_p (e^p \alpha) \). So to prove the lemma we only need to show that \( e^p \alpha \) is a tempered current.

Let \( \psi(x) = j(p(x) - 1) \). It is enough to prove that \( e^p (1 - \psi) \alpha \) and \( e^p \psi \alpha \) are tempered currents.

Since \((1 - \psi) \alpha \in S'_p \Omega^\bullet(\mathbb{R}^n)\) vanishes when \( p(x) > t + 2 \), we see from the definition of \( S'_p \Omega^\bullet(\mathbb{R}^n) \) that \( e^p (1 - \psi) \alpha \) is a tempered current.
On the support of $\psi$ the function $\phi$ is identically equal to 1. Hence $\psi\bar{\alpha} = \psi(\alpha - d\beta)$ and
\[
e^{p}\psi\bar{\alpha} = e^{p}\psi(\alpha - d\beta) = -\psi \int_{1}^{\infty} e^{p} \mu_{s}^{*}(e^{-p\nu_{\mathcal{R}}}\omega) \frac{ds}{s} =
-\psi \int_{1}^{\infty} e^{p(x)-p(\mu_{x})} \mu_{s}^{*}(\nu_{\mathcal{R}}\omega) \frac{ds}{s} \in S'\Omega^{*}(\mathbb{R}^{n}).
\]

Lemma 3.4. The map $H^{*}(S'\Omega^{*}(\mathbb{R}^{n}), d_{p}) \rightarrow H^{*}(\text{Cone}^{*}(\rho))$ induced by $\Phi$ is surjective.

Proof. Choose $\varepsilon > 0$ such that $t - \varepsilon > 0$ and set $U_{t-\varepsilon} = \{x \in \mathbb{R}^{n} : p(x) > t - \varepsilon\}$. Let $\rho_{\varepsilon} : S'\Omega^{*}(\mathbb{R}^{n}) \rightarrow S'\Omega^{*}(U_{t-\varepsilon})$ be the restriction. By Lemma 2.9, any cohomology class $\xi$ of the complex $\text{Cone}^{*}(\rho)$ may be represented by a pair $(\alpha, \beta)$, where $\alpha \in S'\Omega^{*}(\mathbb{R}^{n})$ and $\beta \in S'\Omega^{*}(U_{t-\varepsilon})$.

Fix $j \in C^{\infty}(\mathbb{R})$ such that $j(s) = 0$ if $s \leq t - \varepsilon/2$ and $j(s) = 1$ if $s \geq t$ and set $\phi(x) = j(p(x))$, $(x \in \mathbb{R}^{n})$. Then all the derivatives of $\phi$ are bounded by polynomials and the support of $\phi$ is contained in $U_{t-\varepsilon}$. Hence, $\phi\beta$, considered as a current on $\mathbb{R}^{n}$, belongs to the space $S'\Omega^{*}(\mathbb{R}^{n})$.

The cohomology class of the pair $(\alpha - d(\phi\beta), 0) \in \text{Cone}^{*}(\rho)$ equals $\xi$. Set $\omega = e^{p}(\alpha - d(\phi\beta))$. Since the current $\alpha - d(\phi\beta)$ vanishes on $U_{t}$, we see from the definition of the space $S'\Omega^{*}(\mathbb{R}^{n})$ that $\omega$ is a tempered current. Clearly, $\Phi\omega = (\alpha - d(\phi\beta), 0)$. Hence, $\xi$ belongs to the image of the map $H^{*}(S'\Omega^{*}(\mathbb{R}^{n}), d_{p}) \rightarrow H^{*}(\text{Cone}^{*}(\rho))$.

From Lemmas 3.3 and 3.4, we see that the complexes $(S'\Omega^{*}(\mathbb{R}^{n}), d_{p})$ and $\text{Cone}^{*}(\rho)$ are quasi-isomorphic. Theorem 1.5 follows now from Lemma 2.8 and (3.1).

APPENDIX A. GRADIENT VECTOR FIELD NEAR INFINITY

In this appendix we show that on the set $U_{T}, T \gg 0$ the gradient vector field $\nabla p$ of $p$ is bounded from below by $c/|x|$. That means that the vector field $v = \frac{\nabla p}{|\nabla p|}$ defined in Section 1.8 grows at most linearly. Hence, it generates a globally defined one parameter semigroup of diffeomorphisms $g_{s} : U_{T} \rightarrow U_{T}$. In particular, since $g_{s}^{*}(p) = p + s$, it proves that, for all $t_{1}, t_{2} > T$ the fibers $F_{t_{1}}$ and $F_{t_{2}}$ are diffeomorphic.

A.1. Let $p : \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a polynomial map. Set $U_{t} = \{x \in \mathbb{R}^{n} : p(x) > t\}$.

Theorem A.2. There exist $T, c > 0$ such that
\[
|\nabla p(x)| > \frac{c}{|x|} \quad \text{for any} \quad x \in U_{T}.
\]

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Theorem A.2. There exist $T, c > 0$ such that
\[
|\nabla p(x)| > \frac{c}{|x|} \quad \text{for any} \quad x \in U_{T}.
\]

The rest of this appendix is devoted to the proof of Theorem A.2.
A.3. **Semialgebraic sets.** We will use the following results about semialgebraic sets, cf. [3, Appendix A].

Recall that a subset of \( \mathbb{R}^n \) is called *semialgebraic* if it is a finite union of finite intersection of sets defined by polynomial equation or inequality.

Recall also that a *Puiseux series* in a neighborhood of infinity is a series of the form

\[
t(r) = \sum_{k=N}^{\infty} c_k r^{-\frac{k}{p}}
\]

where \( p \) is an integer and \( r \) is a positive variable. Here \( N \) may be a positive or negative integer or 0. An important consequence of the expansion (A.2) is that if we choose \( N \) so that \( c_N \neq 0 \) (which is possible unless \( t(r) \equiv 0 \)) then

\[
t(r) = c_N r^{-N} (1 + o(1)), \quad r \to \infty.
\]

The proof of Theorem A.2 is based on the following theorem, cf. [3, Th. A.2.6]

**Theorem A.4.** Suppose \( E \) is a semialgebraic set in \( \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \) such that the image of the projection

\[
E \ni (r, t, x) \mapsto r \in \mathbb{R}
\]

contains all large positive \( r \). Then one can find Puiseux series

\[
t(r), \ x(r) = (x_1(r), \ldots, x_n(r))
\]

converging for large positive \( r \) such that \( (r, t(r), x(r)) \in E \).

If

\[
f(r) = \sup \{ t : \text{there exists } x \in \mathbb{R}^n \text{ such that } (r, t, x) \in E \}
\]

is finite and the supremum is attained for large positive \( r \), one can take \( t(r) = f(r) \).

A.5. **Proof of Theorem A.2.** Let \( E \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \) denote the set of solutions of the following system of algebraic equations and inequalities:

\[
\begin{cases}
|\nabla p|^2 \leq \frac{1}{|x|^2}, \\
p(x) \geq t, \\
|x|^2 = r^2.
\end{cases}
\]  

(A.3)

Set

\[
t(r) = \sup \{ t : \text{there exists } x \in \mathbb{R}^n \text{ such that } (r, t, x) \in E \}
\]

Since the supremum is essentially taken over the compact set \( \{x \in \mathbb{R}^n : |x| = r, x \in E \} \), it is clear that \( t(r) \) is finite.

By Theorem A.4, there exist a rational number \( \alpha \) and a constant \( c \) such that

\[
t(r) = cr^\alpha (1 + o(1)) \quad \text{as} \quad t \to \infty.
\]  

(A.4)

It also follows from Theorem A.4, that there exist a function \( x(r) = (x_1(r), \ldots, x_n(r)) \) defined for large \( t > 0 \), rational numbers \( \beta_1, \ldots, \beta_n \) and constants \( c_1, \ldots, c_n \) such that

\[
(r, t(r), x(r)) \in E, \quad \text{and} \quad x_i(r) = c_i r^{\beta_i} (1 + o(1)), \quad i = 1, \ldots, n.
\]  

(A.5)
From (A.3), we know that $|x(r)|^2 = r^2$. Hence, it follows from (A.5) that $\beta_i \leq 1$ for any $i = 1, \ldots, n$. It follows that there exists a constant $A > 0$ such that

$$\frac{dx(r)}{dr} < A.$$  \hfill (A.6)

Suppose now that the statement of Theorem (A.2) is wrong. Then $t(r)$ tends to infinity as $r \to \infty$. Hence, it follows from (A.4), that $\alpha > 0$. Consider the function $f(r) = p(x(r))$. Then, using the second inequality in (A.3), we obtain

$$\lim_{r \to \infty} f(r) = \infty.$$  \hfill (A.7)

Also, from (A.6), we obtain

$$\left| \frac{df}{dr} \right| = \left| \langle \nabla p, \frac{dx}{dr} \rangle \right| \leq |\nabla p| \cdot \left| \frac{dx}{dr} \right| < A |\nabla p|.$$  \hfill (A.8)

Using (A.8), (A.4) and the first inequality in (A.3), we get

$$\left| \frac{df(r)}{dr} \right| < \frac{A}{r^{1+\alpha}}.$$  \hfill (A.9)

Since, $\alpha > 0$, this inequality contradicts (A.7). \hfill \Box

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