Linearization of Time-Varying Nonlinear Systems Using A Modified Linear Iterative Method

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Abstract—The linearization of nonlinear systems is an important digital enhancement technique. In this paper, a real-time capable post- and pre-linearization method for the widely applicable time-varying discrete-time Volterra series is presented. To this end, an alternative view on the Volterra series is established, which enables the utilization of certain modified linear iterative methods for linearization. For one particular linear iterative method, the Richardson iteration, the corresponding post- and pre-linearizers are discussed in detail. It is motivated that the resulting algorithm can be regarded as a generalization of some existing methods. Furthermore, a simply verifiable condition for convergence is presented, which allows the straightforward evaluation of applicability. The proposed method is demonstrated by means of the linearization of a time-varying nonlinear amplifier, which highlights its capability of linearizing significantly distorted signals, illustrates the advantageous convergence behavior, and depicts its robustness against modeling errors.

Index Terms—Linearization, equalization, digital predistortion, nonlinear systems, time-varying systems, Volterra series, iterative methods, Richardson iteration, \textit{P}-order inverse.

I. INTRODUCTION

D IGITAL enhancement techniques became an effective approach to improve the performance of analog systems due to rapid advances in semiconductor technology [1]. Linearization of nonlinear systems via real-time capable methods, as investigated in this paper, is a particular digital enhancement technique. It is applied, e.g., to sensor linearization [2], amplifier predistortion [3], channel equalization [4], and loudspeaker linearization [5]. The purpose of a linearizer is to compensate for the nonlinear behavior of a nonlinear system, i.e., the nonlinear system in cascade with a corresponding linearizer results in a defined linear behavior. A special case thereof is equalization, where the targeted linear behavior is the identity function.

Due to the lack of a unifying model for nonlinear systems, linearizers are generally limited to a particular class of systems and, of course, linearization is only possible if the nonlinear system preserves all information (cf., e.g., [6]). In this paper, nonlinearities described by a \textit{time-varying discrete-time Volterra series} are considered. The Volterra series is a widely used approximator for weakly nonlinear systems [7]–[9]. The objective of this paper is the construction of a linearizer for a \textit{known} time-varying discrete-time Volterra series. For the time-invariant Volterra series, the special case of equalization has already been considered by Schetzen [10] via the concept of a \textit{P}-th-order inverse. A \textit{P}-th-order inverse is constructed by constraining the Volterra operators of the overall system, i.e., the cascade of the nonlinear system and the \textit{P}-th-order inverse. The constraint applies to the Volterra operators up to order \textit{P}, whereas the operators of higher order are arbitrary.

Sarti and Pupolin [11] utilized the fact that orders greater than \textit{P} of the overall system are not constrained to derive a recursive synthesis scheme for a \textit{P}-th-order inverse that is less complex compared to the \textit{P}-th-order inverse in [10]. However, the analysis of the existence and convergence of a \textit{P}-th-order inverse is nontrivial and addressed, e.g., in [12] for signals with finite energy. An approach to linearization of the time-invariant Volterra series is discussed by Nowak and Van Veen in [13], where the linearization problem is reformulated as a nonlinear fixed-point equation, which is solved via successive approximation. They provide an analysis of convergence with respect to a “windowed $l^2$ norm,” which, again, turns out as a nontrivial task. Aschbacher et al. [14] (cf. [15] as well) reduce the linearization problem for a time-invariant Volterra series to the root-finding problem, which is solved using the Newton method. However, for this iterative algorithm the crucial analysis of convergence is even more involved and the corresponding conditions have not been reported yet. As all these methods are limited to time-invariant nonlinear systems, it is worthwhile to mention that for linear time-varying (LTV) systems equalization techniques have already been introduced. They may be divided roughly into two classes [16], explicitly designed correction filters, where the equalization problem is posed as a filter design problem [17], [18], and iterative correction filters, where the desired equalization result is approximated iteratively [19]–[21].

A. Contributions and Outline

Real-world nonlinear systems often vary with time, e.g., due to temperature variations or other environmental changes. However, all existing methods for nonlinear systems reviewed above consider only a \textit{time-invariant} Volterra series. Although it is possible to extend the methods in [10], [11] and [13] to the time-varying Volterra series [22], they potentially become prohibitively complex in computational terms. This stems from the fact that the \textit{P}-th-order inverse in [10] and [11] as well as the method in [13] require a (stable) inverse filter for the first-order Volterra operator. This time-varying
inverse filter is usually not known and, in practice, has to be approximated using filter design techniques as discussed, e.g., in [16]. Therefore, a change of the first-order Volterra operator implies the need for a computationally costly filter design of its inverse. Furthermore, the condition for convergence of all aforementioned methods is either missing, very restrictive on the input signal, or rather complicate to evaluate. Finally, it must be pointed out that in [22] a post-equalizer for a time-varying Volterra series based on a nonlinear fixed point iteration is discussed briefly, however, it completely lacks the critical analysis of convergence. In this paper, the issues above are addressed via the following contributions:

a) Alternative view on the Volterra series: In Section II, an alternative description of the Volterra series is established, which provides a framework for the derivation of linearization methods based on certain modified linear iterative methods.

b) Post- and pre-linearization: Using this system model, a modification of the Richardson iteration is proposed in Section III, which permits its application for post- and pre-linearization of a time-varying discrete-time Volterra series as discussed in Sections IV and V. The presented method is independent of the inverse of the first-order Volterra operator and, therefore, offers a computational advantage compared to the methods in [10], [11], and [13] since the repeated and computationally costly inverse filter design is not necessary.

c) Condition for convergence: In Section VI, a sufficient condition for convergence is presented, which is particularly simple to evaluate and only requires a bounded input signal. Therewith, the applicability of the introduced linearization method is easily verified.

d) Generalization: In Section VII, it is shown that the proposed method is a generalization of the equalization method for LTV systems in [21]. Furthermore, it is motivated that the presented approach can be regarded as a generalization of the post-linearization method in [13] as well as the $P$th-order inverse.

Section VIII presents simulation results, which demonstrate the proposed method by means of the linearization of a time-varying nonlinear amplifier and highlight its properties. Finally, Section IX concludes the paper. The theory presented in this paper requires two results for the time-varying discrete-time Volterra series which have not been established yet and, therefore, are contributed via the appendix of this paper:

e) Properties of a time-varying discrete-time Volterra series:

In Appendix A, the conditions for the convergence of a time-varying discrete-time Volterra series are presented. Furthermore, in Appendix B, it is proven that a convergent time-varying discrete-time Volterra series is Lipschitz continuous.

B. Relation to Adaptive Nonlinear Equalization

In practical application scenarios, the nonlinear system is usually not known and, as a consequence, two approaches towards linearization emerge, i.e., (a) the direct identification of the linearizer, and (b) the identification of the nonlinear system and construction of the linearizer. For (a), adaptive nonlinear filters may be utilized, e.g., [23]. However, the identification is complicated by the model selection as the structure of the linearizing system is usually not known. In contrast, for (b) the nonlinear system is identified, whose structure is often known, e.g., in terms of its circuit schematics, topology, or physical properties. This simplifies the model selection and, consequently, the identification, which motivates the utilization of the approach in (b) that may use the construction of the linearizer discussed in this paper. The identification of a Volterra series is discussed, e.g., in [9], [24]–[27] and is not addressed in this paper.

II. SYSTEM MODEL

The nonlinear system is modeled by a time-varying discrete-time Volterra series, i.e., its complex-valued output sample $y[n]$ at time instant $n \in \mathbb{Z}$ is given by [7]–[9]

$$y[n] = \sum_{p=1}^{\infty} \sum_{k_1, \ldots, k_p \in \mathbb{Z}} h_{p,n}[k_1, \ldots, k_p] \prod_{i=1}^{p} x[n - k_i]$$

(1)

where $x[n]$ is the complex-valued input signal and $h_{p,n}$ are the complex-valued time-varying Volterra kernels.1

Throughout this paper, it is assumed that the Volterra series converges for the given input signal, cf. Appendix A for a discussion of convergence. As a simplified representation, the time-varying Volterra series operator $H_n$ is defined to describe the relation in (1), i.e.,

$$y[n] = H_{n}[x[n]]$$

(2)

and the nonlinear system is referred to as the Volterra system $H_n$ in the remainder of the text, cf. Fig. 1. For the derivation of the linearization algorithm proposed in this paper, a new view on the Volterra system is established. To this end, the sum over $k_1$ in (1) is evaluated as the outermost sum and $x[n - k_1]$ is factored out, which permits the reformulation

$$y[n] = \sum_{k_1 \in \mathbb{Z}} g_{x,n}[k_1] x[n - k_1]$$

(3)

where $g_{x,n}[k_1] = h_{1,n}[k_1] + \sum_{p=2}^{\infty} \sum_{k_2, \ldots, k_p \in \mathbb{Z}} h_{p,n}[k_1, \ldots, k_p] \prod_{i=2}^{p} x[n - k_i]$. (4)

The equivalent description in (3) of the Volterra system in (1) resembles an LTV system with the time-varying impulse response in (4). However, the pretended impulse response $g_{x,n}[k_1]$ is not only time-varying by means of a dependence

1In its most general form, the Volterra series includes a term of order 0, i.e., a time-varying offset $h_{0,n}$. However, to simplify the discussion it is common to require that the offset is compensated separately [23] and, therefore, it is assumed that $h_{0,n} = 0$. 

Fig. 1. Volterra system $H_n$ with input signal $x[n]$ and output signal $y[n]$. 


of the coefficients on the time index $n$, which is denoted by the subscript $n$, but depends on the input signal $x[n]$ as well, which is indicated by the subscript $x$ and symbolizes its nonlinear nature. The system description in (3) can also be cast in a matrix equation. Let $\mathbb{C}^Z$ denote the space of bi-infinite complex-valued sequences. The input vector $x \in \mathbb{C}^Z$ is defined as

$$x = (\ldots, x[n+1], x[n], x[n-1], \ldots)^T$$

(5)

and comprises the samples of the input signal $x[n]$. Analogously, the output vector $y \in \mathbb{C}^Z$ is defined as

$$y = (\ldots, y[n+1], y[n], y[n-1], \ldots)^T$$

(6)

and comprises the samples of the output signal $y[n]$. Furthermore, an infinite coefficient matrix $A_x$ is defined, whose elements $(A_x)_{ij}$ are given by

$$(A_x)_{ij} = g_{x[i]}[i-j]$$

(7)

in which $i, j \in \mathbb{Z}$ denote the row and column index, respectively, and where the subscript $x$ denotes the dependence on the input vector $x$. Therewith, the matrix equation

$$y = A_x x$$

(8)

constitutes an equivalent description of (3) and, consequently, of the Volterra system $H_n$ in (1).

A. Problem Statement

Consider the case of an equalizer that is connected to the output of the Volterra system $H_n$ in Fig. 1. Then, equalization is the task of finding the input vector $x$ given the output vector $y$ and the Volterra system $H_n$. With the previously introduced system model, the unknown input vector $x$ can be found by solving the “system of linear equations” in (8). Indeed, post- and pre-linearization can be recast as a problem with such a structure, which is discussed later on. However, the coefficient matrix $A_x$ depends on the solution $x$ and, therefore, is unknown. Furthermore, if the resulting algorithm should be real-time capable and, thus, reconstruct the signal sample by sample, it has to operate row by row with respect to the matrix equation. Consequently, in order to solve the linearization problem at hand, an algorithm to solve the “system of linear equations” in (8) is required, which (a) operates row by row and (b) determines the coefficient matrix $A_x$ alongside the solution $x$ by exploiting the structural knowledge.

III. MODIFIED LINEAR ITERATIVE METHOD

There exist certain linear iterative methods [28]–[30] for solving systems of linear equations, whose iteration steps operate row by row and, therefore, address problem (a) in Section II-A. These methods reformulate the problem of solving a system of linear equations as a linear fixed-point problem, which is solved using successive approximation [28]. A fixed-point equation comprises a function $T$, where the image of the solution $x$ of the system of linear equations under $T$ is $x$ [28], i.e., $x = T(x)$. There exist various approaches to rewrite a matrix equation of the form in (8) as a fixed-point equation, which eventually leads to different linear iterative methods [28]–[30]. This paper focuses on the Richardson iteration, but other linear iterative methods, e.g., the Jacobi and Gauss-Seidel iteration, are applicable as well. Let $I$ denote the identity matrix, then adding $(I - A_x)x$ to the left and right hand side of (8) results in the fixed-point equation

$$x = (I - A_x)x + y$$

(9)

If the fixed-point $x$ is determined using successive approximation, the Richardson iteration is obtained [28], i.e.,

$$x^{(r+1)} = (I - A_x)x^{(r)} + y$$

(10)

in which $r$ is the iteration index. Therefore, given an initial approximation $x^{(0)}$ of the solution $x$, this iteration provides a sequence of approximations, which, under certain conditions, converges to the fixed-point $x$, i.e., $\lim_{r \to \infty} x^{(r)} = x$.

A. Modified Richardson Iteration

The Richardson iteration in (10) requires the knowledge of the coefficient matrix $A_x$ and, consequently, cannot overcome problem (b) in Section II-A. It reconstructs the unknown input vector $x$ by iteratively improving an initial approximation $x^{(0)}$ using the output vector $y$ and the unknown coefficient matrix $A_x$. However, in iteration $r + 1$ the approximation $x^{(r)}$ is already available and may be used to approximate the coefficient matrix. To this end, in analogy to (7) the approximation $A_{x^{(r)}}$ of $A_x$ based on $x^{(r)}$ is defined in terms of its elements $(A_{x^{(r)}})_{ij}$ in row $i$ and in column $j$ as

$$(A_{x^{(r)}})_{ij} = g_{x^{(r)}[i]}[i-j]$$

(11)

where $i, j \in \mathbb{Z}$ and

$$g_{x^{(r)}[i]}[k_i] = h_{1,n}[k_i]$$

$$+ \sum_{p=2}^{\infty} \sum_{k_2, \ldots, k_p \in \mathbb{Z}} h_{p,n}[k_1, \ldots, k_p] \prod_{i=2}^{p} x[n - k_i]^{(r)}.$$  

(12)
Substitution of the coefficient matrix $A_x$ in the Richardson iteration in (10) with the approximation $A_{x(r)}$ yields

$$x^{(r+1)} = (I - A_{x(r)})x^{(r)} + y.$$ \hspace{1cm} (13)

This modified Richardson iteration does not only generate approximations of the input vector $x$, but also of the coefficient matrix $A_x$ and, consequently, overcomes problem (b) in Section II-A as well. Indeed, it provides a solution to a significant class of systems as shown by the condition for convergence discussed in Section VI later on.

B. Richardson Equalizer

The modified Richardson iteration in (13) is a matrix equation, but for a real-time capable algorithm, the iteration needs to be sample-based. Therefore, consider the evaluation of (13) row by row. Using (5), (6), and (11), this results in

$$x^{(r+1)}[n] = x^{(r)}[n] + y[n] - \sum_{k_1 \in \mathbb{Z}} g_x[n,k_1]x[n-k_1]^{(r)}.$$ \hspace{1cm} (14)

A comparison of the convolution of $g_x[n,k]$ with $x[n]^{(r)}$ to (3) reveals that it equals the response of the Volterra system $H_n$ to the input signal $x[n]^{(r)}$. Hence, a real-time capable algorithm based on the modified Richardson iteration, for convenience called Richardson equalizer in the remainder, is described by the iteration

$$x^{(r+1)}[n] = x^{(r)}[n] + y[n] - H_n\{x^{(r)}[n]\}.$$ \hspace{1cm} (15)

If $H_n$ is causal, (14) is indeed realizable as a sample-based iteration, i.e., the reconstruction $x^{(r+1)}[n]$ of the sample $x[n]$ depends only on previous reconstructions $x[k]^{(r)}$, where $k \leq n$. While the approximation $x[n]^{(r)}$ will equal the desired input sample $x[n]$ if $r \to \infty$ and if the iteration converges, a practical system can, of course, only implement a finite number of iterations. This is not a limitation per se as the number of iterations can be chosen so that the accuracy of the approximation suffices for the specific application. However, this statement assumes that the approximation improves in every iteration or, in other words, the error with respect to the solution decreases in every iteration. This iterative reduction of the error is indeed ensured by the conditions for convergence discussed in Section VI. Still, the finite number of iterations introduces another issue, i.e., the initialization $x^{(0)}[n]$ influences the approximation accuracy. While the initialization may be chosen arbitrarily, e.g., $x^{(0)}[n] = 0$, it should be as close to the solution $x[n]$ as possible to improve the approximation result. In Section VI, it is shown that the iteration in (14) converges for moderately nonlinear systems. Under these circumstances, the output is a rough approximation of the input and it turns out to be advantageous to use the initialization

$$x^{(0)}[n] = y[n].$$ \hspace{1cm} (16)

Concluding, a Richardson equalizer based on three iterations of (14) with the initialization in (15) is depicted in Fig. 2.

IV. POST-LINEARIZATION

Linearization is the problem of correcting the nonlinear behavior of a given system to a defined linear behavior by cascading it with another system, where the latter system is called linearizer. As the cascade of two nonlinear systems, in general, exhibits a different behavior depending on the ordering of the systems, two configurations arise, i.e., post- and pre-linearization. In the case of post-linearization of a Volterra system $H_n$, the linearizer is connected to the output of $H_n$ as depicted in Fig. 3 and termed post-linearizer to distinguish it from the pre-linearizer discussed in Section V. The desired behavior of the cascade is described by the linear time-invariant (LTI) filter $L$, i.e., the output signal $x[n]$ of the cascade is given by

$$x[n] = L\{u[n]\}$$ \hspace{1cm} (17)

in which $u[n]$ is the input signal of the cascade. Consequently, the task of the post-linearizer is to reconstruct the signal $x[n]$ by observation of the signal $y[n]$ and knowledge of the Volterra system $H_n$. Given that the LTI filter $L$ is minimum-phase and, thus, possesses a stable inverse filter $L^{-1}$, i.e.,

$$u[n] = L^{-1}\{x[n]\}$$ \hspace{1cm} (18)

the signal $y[n]$ in Fig. 3 can be regarded as the response of an augmented Volterra system $H_n$ to the input signal $x[n]$. To this end, consider that the Volterra system $H_0$ is the cascade of the inverse filter $L^{-1}$ and the Volterra system $H_n$, i.e.,

$$H_n = L^{-1} \circ H_0$$ \hspace{1cm} (19)
Richardson equalizer provides the means to synthesize the output, i.e., any decrease in the approximation error is directly observable. Consequently, the Richardson equalizer described by the iteration (14) based on the Volterra system \( \hat{H}_n \) in (18) with the initialization in (15) constitutes a post-linearizer for the Volterra system \( \hat{H}_n \).

**V. Pre-Linearization**

In the case of pre-linearization of a Volterra system \( \hat{H}_n \), often called digital predistortion as well, the linearizer is connected to the input of \( H_n \) as depicted in Fig. 5 and termed pre-linearizer. The desired behavior of the cascade is described by the LTI filter \( L \), i.e., the output signal \( v[n] \) of the cascade is given by

\[
v[n] = L\{y[n]\} \tag{19}
\]

in which \( y[n] \) is the input signal of the cascade. Consequently, the task of the pre-linearizer is to reconstruct the signal \( x[n] \) by observation of the signal \( y[n] \) and knowledge of the Volterra system \( \hat{H}_n \) so that \( x[n] \) filtered by \( H_n \) results in the desired output signal \( v[n] \) in (19). Analogous to Section IV, the signal \( y[n] \) in Fig. 5 can be regarded as the response of an augmented Volterra system \( H_n \) to the input signal \( x[n] \) if the LTI filter \( L \) possesses a stable inverse filter \( L^{-1} \). To this end, consider that the Volterra system \( H_n \) is the cascade of the Volterra system \( \hat{H}_n \) and the inverse filter \( L^{-1} \), i.e.,

\[
H_n = \hat{H}_n \circ L^{-1} \tag{20}
\]

Then, the signal \( y[n] \) can be described as

\[
y[n] = L^{-1}\{v[n]\} = L^{-1}\{\hat{H}_n\{x[n]\}\} = H_n\{x[n]\}
\]

which is illustrated in Fig. 6. Again, it can be observed that this corresponds to the relation in (2) and, thus, the Richardson equalizer provides the means to synthesize the signal \( x[n] \). Consequently, the Richardson equalizer described by the iteration (14) based on the Volterra system \( H_n \) in (20) with the initialization in (15) constitutes a pre-linearizer for the Volterra system \( H_n \).

It is important to recognize that there is a difference in the approximation mechanism between post- and pre-linearization if the Richardson equalizer is utilized with a finite number of iterations. With post-linearization, the approximation generated by the Richardson iteration appears directly at the output, i.e., any decrease in the approximation error is directly visible. In the case of pre-linearization, the approximation generated by the Richardson iteration traverses the Volterra system \( \hat{H}_n \) before appearing at the output of the cascade, i.e., the approximation is subject to a nonlinear filtering operation. However, as shown in Appendix B, a time-varying discrete-time Volterra series is Lipschitz continuous if it converges. This implies that if the approximation error at the input decreases, the upper bound on the approximation error at the output decreases as well. In other words, an improvement in approximation accuracy at the input results in an improvement of the worst-case approximation accuracy at the output and, consequently, the application of the pre-linearizer is indeed appropriate.

**VI. Conditions for Convergence**

The application of the Richardson equalizer or any other iterative method is only reasonable if the iteration converges to the solution. In order to study the conditions under which convergence of the Richardson equalizer can be guaranteed, the error \( e^{(r)} \) in iteration \( r \) is defined as

\[
e^{(r)} = x - x^{(r)} \tag{21}
\]

in which \( x \) is the solution that satisfies the fixed-point equation in (9) and \( x^{(r)} \) is the approximation in iteration \( r \) of the Richardson equalizer (13) in matrix notation. The subtraction of (13) from (9) and utilization of (21) results in

\[
e^{(r+1)} = (I - A_x)x - (I - A_{x,(r)})x^{(r)} = (I - A_x)(x^{(r)} + e^{(r)}) - (I - A_{x,(r)})x
\]

\[
= (I - A_x)e^{(r)} + (A_{x,(r)} - A_x)x^{(r)} \tag{22}
\]

which depicts the influence of the error \( e^{(r)} \) in the previous iteration and the approximation error \( A_{x,(r)} - A_x \) of the coefficient matrix. The Richardson equalizer converges to \( x \) if the error decays to zero, i.e., \( \lim_{r \to \infty} e^{(r)} = 0 \). An even more restrictive requirement is

\[
\|e^{(r+1)}\|_\infty < \|e^{(r)}\|_\infty \tag{23}
\]

which has to hold for all iterations \( r \geq 0 \). In (23), \( \|\cdot\|_\infty \) denotes the supremum norm [31], i.e., it requires the supremum of the error signal to be strictly monotonically decreasing with respect to the iteration index \( r \). In this case, the approximation error needs to decrease in every iteration, which corresponds to the requirement on the Richardson equalizer for a finite number of iterations identified in Section III-B. Using (22), it is shown in Appendix C that if the function

\[
\psi_{x,n} = \sum_{k_1 \in Z} |\delta[k_1] - h_{1,n}[k_1]| + \sum_{p=2}^\infty \|h_{p,n}\|_1 \cdot \omega(p) \tag{24}
\]

of a Volterra system \( H_n \) satisfies the condition for convergence

\[
\sup_{n \in Z} \psi_{x,n} < 1 \tag{25}
\]

then (23) holds for the Richardson equalizer in (14) with the initialization in (15). In (24), \( \delta[k_1] \) denotes the unit impulse sequence, i.e.,

\[
\delta[k_1] = \begin{cases} 1, & \text{if } k_1 = 0 \\ 0, & \text{if } k_1 \neq 0 \end{cases}
\]

\( \|h_{p,n}\|_1 \) is defined as the sum of the absolute coefficients of the \( p \)-th order Volterra kernel at time instant \( n \), i.e.,

\[
\|h_{p,n}\|_1 = \sum_{k_1, \ldots, k_p \in Z} |h_{p,n}[k_1, \ldots, k_p]| \tag{27}
\]

Note that any valid norm may be used in (23) and that the choice has an impact on the derivation and the resulting condition for convergence. Due to its beneficial structure, the supremum norm is employed.
and the weighting factor \( w_x(p) \) is given by
\[
  w_x(p) = (2^p - 1)\|x\|_{\infty}^{p-1}.
\]
(28)

For practical systems, which operate only for a finite time, the condition in (25) is particularly simple to verify as it suffices to ensure that \( \psi_{x,n} < 1 \) holds at every time instant \( n \). This is simply a threshold on a weighted sum of the absolute kernel coefficients, where the weights depend on the input amplitude range which is usually known.

It follows from the condition in (25) that the rate of time-variance of the Volterra system has no impact on whether the Richardson equalizer converges as long as \( \psi_{x,n} \) consistently remains below the threshold. Furthermore, it is worthwhile to mention that the coefficient \( h_{1,n}[0] \), i.e., the coefficient of the first-order Volterra kernel at time lag zero, is of particular significance since only its difference from one contributes to the sum in (24). Considering that the threshold on \( \psi_{x,n} \) in (25) is one, this implies that the coefficient \( h_{1,n}[0] \) is restricted to the open interval \((0,1)\) and, in general, it must be dominant, i.e., all other coefficients must be small compared to \( h_{1,n}[0] \). However, by appropriately delaying signals and matching time indices, this restriction may be loosened to some arbitrary coefficient of the first-order Volterra kernel, instead of being mandatory for the coefficient at time lag 0. As the corresponding structural modifications equal those for the method in [21], a detailed discussion thereof is omitted here (see also Section VII-A).

A. Remarks to the Condition for Linearization

If the Richardson equalizer is utilized for post- and pre-linearization, the Volterra system \( H_n \) is the cascade of an LTI filter and a Volterra system as given by (18) and (20). In order to discuss the implications thereof on the condition for convergence, let the inverse filter \( L^{-1} \) of the minimum-phase LTI filter \( L \) be characterized by the impulse response \( q[n] \), i.e.,
\[
  L^{-1}\{x[n]\} = \sum_{l \in \mathbb{Z}} q[l]x[n-l].
\]
(29)

As shown in Appendix G, the kernels of the Volterra system \( H_n \) in (18) for post-linearization are given by
\[
  h_{p,n}[k_1, \ldots, k_p] = \sum_{l_1, \ldots, l_p \in \mathbb{Z}} h_{p,n}[l_1, \ldots, k_p - l_p] \prod_{j=1}^{p} q[l_j]
\]
and the kernels of the Volterra system \( H_n \) in (20) for pre-linearization are given by
\[
  h_{p,n}[k_1, \ldots, k_p] = \sum_{l \in \mathbb{Z}} h_{p,n-1}[k_1 - l, \ldots, k_p - l]q[l].
\]
(31)
Those results show that the rather restrictive condition for convergence in (25) is mitigated as it applies to the Volterra systems \( H_n \) and \( \hat{H}_n \) only relative to the LTI filter \( L \). To exemplify this, consider \( H_n \) and \( \hat{H}_n \) to model a nonlinear amplifier of gain \( K > 0 \), where the desired behavior \( L \) is an ideal amplifier with gain \( K \). Thus, \( L^{-1} \) is characterized by
\[
  q[n] = \begin{cases} 
  1/K, & \text{if } n = 0 \\
  0, & \text{if } n \neq 0.
  \end{cases}
\]
(32)
In case of pre-linearization, it follows from (31) that the kernels \( h_{p,n} \) equal the kernels \( \hat{h}_{p,n} \), weighted by \( 1/K \). Therefore, the coefficients are weighted so that only the nonlinearity relative to the linear gain has impact on the condition for convergence. In case of post-linearization, it follows from (30) that the kernels \( h_{p,n} \) equal the kernels \( \hat{h}_{p,n} \), weighted by \( 1/K^p \). However, in this setting the linearizer operates on the amplified signal, cf. Fig. 3. To investigate the implications in terms of the unamplified signal \( u[n] \), it is recognized from (16) that \( \|x\|_{\infty} = K\|u\|_{\infty} \). It can be seen from the weighting factor in (28) that this amplification results in an additional factor \( K^{p-1} \) for \( p \geq 2 \). Therefore, the weighting of the kernels \( h_{p,n} \) does not only relate them to \( L \) by weighting with \( 1/K \), but also accounts for the change in signal amplitude by including the factor \( 1/K^{p-1} \).

VII. RELATION TO OTHER METHODS

A. Equalization of Linear Weakly Time-Varying Systems

Soudan and Vogel [21] proposed an equalizer for linear weakly time-varying systems which is based on the Richardson iteration. The method proposed in this paper can be regarded as the generalization of the method in [21] from linear to nonlinear systems and from equalization to linearization. In particular, if the Volterra system \( H_n \) comprises only a linear (first-order) kernel, the Richardson equalizer equals the iteration in [21]. Furthermore, for a linear system the condition for convergence in (25) reduces to the criterion provided in [21].

B. Nonlinear Iterative Methods

Instead of applying modified linear iterative methods to (8), it is possible to directly formulate a nonlinear fixed-point equation based on (1) and solve it via successive approximation as presented by Nowak and Van Veen [13]. However, let the first-order Volterra operator \( \hat{H}_{1,n} \) of the Volterra system \( \hat{H}_n \) be defined as
\[
  \hat{H}_{1,n}\{x[n]\} = \sum_{k_1 \in \mathbb{Z}} \hat{h}_{1,n}[k_1]x[n-k_1]
\]
and possess an inverse \( \hat{H}_{1,n}^{-1} \), which is a fundamental assumption in [13]. Then, the post-linearizer in [13] for \( \hat{H}_n \), which is realized as a post-equalizer followed by an LTI filter, equals
the post-linearizer in Section IV followed by the same LTI filter, if the latter linearizer is based on the Volterra system

$$H_n = H_n \circ H_{1,n}^{-1} \quad (33)$$

and \( L \) is set to the identity function, see Fig. 7. In fact, it equals the extension of the post-linearizer in [13] to time-varying systems and also illustrates the dependence on the inverse \( H_{1,n}^{-1} \) considered in Section I-A. Consequently, the presented method may be regarded as a generalization of the post-linearizer in [13], as the latter amounts to the application of the proposed post-linearizer to the augmented Volterra system \( H_n \) in (33).

C. \( P \)th-Order Inverse

Due to the fact that the definition of a \( P \)th-order inverse does not constrain the Volterra kernels of order greater than \( P \) of the overall system, different realizations exist [10], [11], [32]. If the post-linearizer in Section IV, with \( L \) set to the identity function, is applied to \( H_n \) in (33) using the initialization

$$x[n]^{(0)} = H_{1,n}^{-1}\{y[n]\} \quad (34)$$

the resulting iteration equals the extension of the recursive synthesis technique for a \( P \)th-order inverse in [32, ch. 5.2.3] to time-varying systems, cf. Fig 7.\(^3\) That is, the reconstruction after \( r \) iterations corresponds to the reconstruction of the \((r + 1)\)th-order inverse. Consequently, the presented method may be regarded as a generalization of the \( P \)th-order inverse as well, as the latter constitutes a particular application of the presented post-linearizer.

VIII. Simulation Results

In the following, the post- and pre-linearization methods introduced in this paper are demonstrated by means of the linearization of a nonlinear amplifier with time-varying gain and dynamic saturation. This amplifier is modeled by the Volterra system \( H_n \) comprising the kernels

$$\begin{align*}
\hat{h}_{1,n}(k_1) &= \kappa_n c_1(k_1) \\
\hat{h}_{3,n}(k_1, k_2, k_3) &= \kappa_n c_3(k_1) c_2(k_2) c_3(k_3) \\
\hat{h}_{5,n}(k_1, k_2, k_3, k_4, k_5) &= \kappa_n c_5(k_1) c_5(k_2) c_5(k_3) c_5(k_4) c_5(k_5)
\end{align*}$$

in which the coefficient vectors \( c_1, c_3, \) and \( c_5 \) with zero-based element indexing are given by

$$\begin{align*}
c_1 &= (1.00, 0.03, 0.015) \\
c_3 &= (-0.38, -0.07, -0.03) \\
c_5 &= (-0.27, -0.06)
\end{align*} \quad (35)$$

and the time-varying gain \( \kappa_n \) is defined as

$$\kappa_n = K \cdot [1 + 0.03 \cos(4\pi n/N)] .$$

In the latter, \( K = 50 \) is the fundamental gain of the amplifier and \( N = 500 \) denotes the number of samples used for the

\(^3\)The \( P \)th-order inverse in [32] is specified by the recursive scheme (5.24) therein. Adding \( u_{p-1}[n] = H_{1,n}^{-1}\{H_1[u_{p-1}[n]]\} = 0 \) to this equation, utilizing the linearity of \( H_{1,n}^{-1} \), and recognizing that \( u_1[n] = H_{1,n}^{-1}\{y[n]\} \) leads to \( u_p[n] = u_{p-1}[n] + H_{1,n}^{-1}\{y[n]\} - H_{1,n}^{-1}\{H_1[u_{p-1}[n]]\} \), in which \( H_1[u_{p-1}[n]] = H_1[u_{p-1}[n]] + H_{NL}[u_{p-1}[n]] \). This recursive scheme corresponds to the iteration implemented by the post-linearizer in Fig. 7.

simulation. For the sake of consistent notation, an equivalent Volterra system \( \hat{H}_n = H_n \) is defined for pre-linearization. The desired behavior of the amplifier is an ideal gain of factor \( K \), i.e., the LTI filter \( L \) implements \( L\{x[n]\} = K x[n] \). Consequently, its inverse \( L^{-1} \) is characterized by the impulse response \( \tilde{q}[n] \) in (32). The input to the amplifier shall be bounded by \( B \) and, therefore, it follows from Fig. 3 and (16) that for post-linearization

$$\|x\|_\infty = K \|u\|_\infty = KB \quad (36)$$

and from Fig. 5 that for pre-linearization

$$\|x\|_\infty = B . \quad (37)$$

The input signal to the nonlinear amplifier is the modulated sine wave

$$s[n] = B \sin(2\pi n/N) \sin(3\pi n/N) . \quad (38)$$

Consequently, the desired output signal is \( K s[n] \). To achieve this output, the input is set to \( u[n] = s[n] \) for post-linearization in Fig. 3 and to \( y[n] = s[n] \) for pre-linearization in Fig. 5. Depending on the bound \( B \) on the input of the nonlinear amplifier, the distortion of the output signal varies and, in the following, the linearization is studied for mildly, moderately, and severely distorted output signals. Subsequently, the section concludes with an investigation of the influence of modeling errors.

A. Mild Distortion

For \( B = 0.75 \) the output signal of the nonlinear amplifier is only mildly distorted as shown in Fig. 8. The applicability of
the post- and pre-linearization methods is verified using (36) and (37) and the definition of \( H_n \) in (30) and (31) in (24), respectively, to determine \( \psi_{x,n} \). The maxima of \( \psi_{x,n} \) are at 0.5644, which is significantly less than one, and thus the condition in (25) guarantees convergence. The linearization performance is measured with the signal-to-noise ratio (SNR)

\[
\text{SNR}\{x[n]\} = 10 \cdot \log \left( \frac{\sum_{n=0}^{N-1} |Ks[n]|^2}{\sum_{n=0}^{N-1} |Ks[n] - x[n]|^2} \right)
\]

which is a logarithmic measure for the deviation from the desired output signal \( Ks[n] \). Post- and pre-linearization is performed with the Richardson equalizer in (14) using \( H_n \) in (18) and (20), respectively, and the initialization in (15). In Fig. 9, the linearization performance is depicted in terms of SNR with respect to the number of iterations employed in the Richardson equalizer. It can be observed that both linearizers converge very fast. The improvement in SNR per iteration is significant and it increases approximately linear with the number of iterations. The performance for pre-linearization is somewhat inferior to that of post-linearization, which is primarily a consequence of the nonlinear filtering of the approximation as discussed in Section V.

B. Moderate Distortion

For \( B = 1 \) the output signal of the nonlinear amplifier is moderately distorted as shown in Fig. 10. In this case, the maxima of \( \psi_{x,n} \) are at 0.9987, which is just below one, and thus the condition in (25) still guarantees convergence. The linearization performance is depicted in Fig. 11. It can be observed that the improvement in SNR per iteration is still significant, but less compared to the performance for the mildly distorted signal in Fig. 9. This behavior is a consistent property of the Richardson equalizer, i.e., the closer the bound imposed by the condition in (25) is attained, the slower is the convergence. Another characteristic observable in Fig. 11 is the deterioration in performance of the pre-linearizer compared to the post-linearizer. Although this is, in part, explained by the argument provided in the previous section, another issue becomes evident here. In particular, the pre-linearizer operates on the signal \( y[n] = s[n] \) and convergence is ensured for \( x[n] \) bounded by (37). Thus, it is implicitly assumed that the maximum gain of the pre-linearizer is one. For mild distortions this is approximately true, but for moderate and severe distortions the pre-linearizer needs to compensate the saturation effect by amplification of the input signal. Consequently, some samples are outside the bound of guaranteed convergence and deteriorate the performance, a case which is investigated in more detail in the next section.

C. Severe Distortion

For the moderate distortion discussed in the previous section, the bound imposed by the condition in (25) is nearly attained. Therefore, it represents the amount of distortion for which convergence is guaranteed by this condition. However, (25) is derived by the repeated application of the triangle inequality, utilization of the supremum norm as an upper bound on individual samples, and the upper bound in (52), cf. Appendix C. As the latter bound is not exact and the worst case in terms of the other bounds appears to be quite improbable, it is reasonable to try to linearize more severely distorted signals. To this end, consider the input of the nonlinear amplifier to be bounded by \( B = 1.3 \). The corresponding output signal is depicted in Fig. 12. In this case, \( \psi_{x,n} \) is between 1.6788
and 1.7808, which is significantly above one, and thus the condition in (25) cannot guarantee convergence. However, the linearization performance in Fig. 13 illustrates that the post-linearizer still converges. In case of the pre-linearizer, the issue identified in the previous section becomes more severe. Due to the strong saturation, the pre-linearizer needs to substantially amplify the signal peaks. The SNR initially improves because the iteration converges for the majority of samples, but finally it starts to deteriorate because of the divergence at the signal peaks as illustrated in Fig. 14. In this context, it is important to recognize that due to the structure of the Richardson equalizer in (14) and the memory in \( H_n \), the divergence can propagate to neighboring samples with repeated iterations. Concluding, the condition for convergence is rather conservative and the linearization method presented in this paper may be utilized in cases of more severe distortion. However, it should be kept in mind that the rate of convergence decreases and that it may involve the risk of divergence induced by signal peaks.

D. Modeling Errors

The previous examples assumed that the nonlinear amplifier is perfectly known. However, in practice the model is usually only an approximation of the actual nonlinear system and, therefore, the impact of modeling errors on the linearization performance is of interest. In the following, this is investigated by employing the erroneous coefficient vectors

\[
\begin{align*}
c_1 &= (0.99, 0.025, 0.03) \\
c_3 &= (-0.37, -0.1, -0.01) \\
c_5 &= (-0.29, -0.03)
\end{align*}
\]

in the Volterra system used for linearization, which represents a significant modeling error with respect to the nonlinear amplifier based on the coefficient vectors in (35). The corresponding linearization performance is depicted in Fig. 15 for \( B = 1 \). For the augmented Volterra system with modeling errors, the maxima of \( \psi_{x,n} \) are at 0.9809 and, therefore, convergence is guaranteed. Indeed, the SNR increases in the first two iterations but, subsequently, the convergence stalls. This stems from the fact that the linearizers effectively linearize a different Volterra system, i.e., they converge to a different solution and the linearization performance is limited by this deviation. Consequently, the proposed linearization method is robust against modeling errors if the condition for convergence is satisfied and the limitation in linearization performance is determined by the severity of the modeling errors.

IX. Conclusion

In this paper, a novel real-time capable method for the linearization of nonlinear systems modeled by a time-varying discrete-time Volterra series was presented. To this end, an alternative view on the Volterra series was established, which resembles the description of an LTV system. Based on this system model, a systematic approach to the modification of certain linear iterative methods was proposed that permits their use for linearization. The modification was presented for the Richardson iteration and its utilization for post- and pre-linearization was discussed in detail. It was shown that the resulting method is a generalization of the equalizer for linear weakly time-varying systems in [21] and that it may be regarded as a generalization of the post-linearizer in [13] and the \( P \)-th-order inverse. Due to the iterative structure of the proposed linearizers, their computational cost scales with the required accuracy via the employed number of iterations. With the presentation of a simply verifiable condition for convergence, a practical tool to determine the applicability of the method was established. By means of the linearization of a time-varying nonlinear amplifier, the application of the proposed method was exemplified and properties thereof were discussed. It was shown that the condition for convergence can guarantee the applicability for mildly to moderately distorted signals. In this case, the linearizers perform very well and the iteration converges fast. Consequently, one or two iterations of the underlying fixed-point iteration may already suffice to achieve a practically relevant accuracy. It was demonstrated that the method is also applicable to severely distorted signals, however, by trading slower convergence and the risk of divergence. Finally, it was shown that the method is robust against modeling errors and that the performance penalty is determined by the severity of the modeling errors.

The proposed method offers considerable potential for future research. Specifically, the method was presented on the
basis of the Richardson iteration, but it is not limited to this particular linear iterative method. Therefore, other linear iterative methods like the Jacobi or Gauß-Seidel iteration may be explored as well, which includes the derivation of the corresponding modified iteration and condition for convergence. Additionally, a preconditioner in terms of a relaxation parameter might be incorporated to improve the convergence behavior. In specific scenarios, where further information about the input signal is available, a more elaborate performance analysis might be performed by means of the derivation of a worst-case and average rate of convergence. These results also aid the design of practical systems as they support the choice of the employed number of iterations.

**APPENDIX A**

**CONVERGENCE OF A TIME-VARYING DISCRETE-TIME VOLterra SERIES**

A time-varying discrete-time Volterra series \( H_n \) is convergent if the output of the system is finite for a given input signal \( x[n] \). For a further analysis of convergence, let the supremum norm \( \|x\|_\infty \) be the bound on the input signal \( x[n] \). Using the triangle inequality, the bound \( \|x\|_\infty \) on the input, and \( \|h_{p,n}\|_1 \) in (27), it follows from \( y[n] \) in (1) that

\[
|y[n]| = \left| \sum_{p=1}^{\infty} \sum_{k_1, \ldots, k_p \in \mathbb{Z}} h_{p,n}[k_1, \ldots, k_p] \prod_{i=1}^{p} x[n-k_i] \right| \\
\leq \sum_{p=1}^{\infty} \sum_{k_1, \ldots, k_p \in \mathbb{Z}} |h_{p,n}[k_1, \ldots, k_p]| \prod_{i=1}^{p} |x[n-k_i]| \\
\leq \sum_{p=1}^{\infty} \|h_{p,n}\|_1 \cdot \|x\|_\infty^p.
\]

Let the bound function \( f_n(\|x\|_\infty) \) at time instant \( n \) be defined as (cf. [33])

\[
f_n(\|x\|_\infty) = \sum_{p=1}^{\infty} \|h_{p,n}\|_1 \cdot \|x\|_\infty^p.
\]

Then it follows from (39) that

\[
\|y\|_\infty = \sup_{n \in \mathbb{Z}} |y[n]| \leq \sup_{n \in \mathbb{Z}} f_n(\|x\|_\infty).
\]

Consequently, if a time-varying discrete-time Volterra series \( H_n \) satisfies the condition

\[
\sup_{n \in \mathbb{Z}} f_n(\|x\|_\infty) < \infty
\]

it converges for all input signals bounded by \( \|x\|_\infty \). The bound function \( f_n(\|x\|_\infty) \) in (40) is a power series with non-negative coefficients and, therefore, is finite for \( \|x\|_\infty < R_n \), where the radius of convergence \( R_n \) is given by [31], [33]

\[
R_n = \left[ \limsup_{p \to \infty} \|h_{p,n}\|_1^{1/p} \right]^{-1}.
\]

This implies that a time-varying discrete-time Volterra series \( H_n \) satisfies (41) and, thus, converges if

\[
\|x\|_\infty < R = \inf_{n \in \mathbb{Z}} R_n
\]

in which \( R \) is the radius of convergence.

**APPENDIX B**

**LIPSCHITZ CONTINUITY OF A TIME-VARYING DISCRETE-TIME VOLterra SERIES**

A time-varying discrete-time Volterra series \( H_n \) is Lipschitz continuous if

\[
\|A_{x(r)} \cdot x(r) - A_x \cdot x\|_\infty \leq \kappa \cdot \|x(r) - x\|_\infty
\]

holds, where the system model in Section II is used, \( \kappa \) is non-negative and finite, and \( x \) and \( x(r) \) are two input signal vectors with the corresponding coefficient matrices \( A_x \) and \( A_{x(r)} \) in (7) and (11), respectively. In the following, it is shown that a convergent time-varying discrete-time Volterra series \( H_n \) is Lipschitz continuous.\(^4\) The approach below is an adaptation of the corresponding proof for the time-invariant continuous-time Volterra series in [33]. Let \( x \) and \( x(r) \) be two input vectors, where the difference is given by \( e(r) \) in (21). Furthermore, let

\[
\|x\|_\infty + \|e(r)\|_\infty < R
\]

in which \( R \) is the radius of convergence of \( H_n \). Thus, \( H_n \) is convergent for \( x \) and \( x(r) \) because \( \|e(r)\|_\infty \geq 0 \) and

\[
\|x(r)\|_\infty \leq \|x\|_\infty + \|e(r)\|_\infty
\]

respectively, where the latter is obtained from the definition of \( e(r) \) in (21) by taking the supremum norm and applying the triangle inequality. Using (7), (11), \( \gamma_{x,n}^{(1)}[p, k_1, \ldots, k_p] \) defined in (62) in Appendix F, and the triangle inequality, the upper bound

\[
\|A_{x(r)} \cdot x(r) - A_x \cdot x\|_\infty
\]

\[
= \sup_{n \in \mathbb{Z}} \sum_{p=1}^{\infty} \sum_{k_1, \ldots, k_p \in \mathbb{Z}} h_{p,n}[k_1, \ldots, k_p] \gamma_{x,n}^{(1)}[p, k_1, \ldots, k_p]
\]

\[
\leq \sup_{n \in \mathbb{Z}} \sum_{p=1}^{\infty} \sum_{k_1, \ldots, k_p \in \mathbb{Z}} |h_{p,n}[k_1, k_2, \ldots, k_p] \cdot \gamma_{x,n}^{(1)}[p, k_1, \ldots, k_p]|
\]

is obtained. Using the upper bound (65) in Appendix F on \( \gamma_{x,n}^{(1)}[p, k_1, \ldots, k_p] \) as well as \( \|h_{p,n}\|_1 \) in (27) and the bound function \( f_n(\|x\|_\infty) \) in (40) enables

\[
\|A_{x(r)} \cdot x(r) - A_x \cdot x\|_\infty
\]

\[
\leq \sup_{n \in \mathbb{Z}} \sum_{p=1}^{\infty} \|h_{p,n}\|_1 \cdot \left( (\|x\|_\infty + \|e(r)\|_\infty)^p - \|x\|_\infty^p \right)
\]

\[
= \sup_{n \in \mathbb{Z}} f_n(\|x\|_\infty + \|e(r)\|_\infty) - f_n(\|x\|_\infty).
\]

From the mean value theorem it follows that [31], [33]

\[
f_n(\|x\|_\infty + \|e(r)\|_\infty) - f_n(\|x\|_\infty) = f_n'(\zeta) \cdot \|e(r)\|_\infty
\]

where \( f_n' \) is the derivative of \( f_n \) and

\[
\|x\|_\infty \leq \zeta \leq \|x\|_\infty + \|e(r)\|_\infty.
\]

\(^4\) In fact, for a time-varying discrete-time Volterra series \( H_n \), with a radius of convergence \( R > 0 \), \( H_n \) stimulated by the input vector \( x \) is continuous if \( \|x\|_\infty < R \) and Lipschitz continuous if \( \|x\|_\infty < R' < R \), cf. the proof for the continuous-time Volterra series in [33].

Using this relation in (45) yields
\[ \|A_{x(r)}x^{(r)} - A_{x}x\|_{\infty} \leq \|e^{(r)}\|_{\infty} \sup_{n \in Z} f'_{n}(\zeta) \]
which corresponds to (42) where
\[ \kappa = \sup_{n \in Z} f'_{n}(\zeta). \quad (47) \]
Due to (41), (43), and (46), \( \kappa \) in (47) is indeed non-negative and finite, which completes the proof.

**APPENDIX C
CONDITION FOR CONVERGENCE FOR THE RICHARDSON EQUALIZER**

In this appendix, it is proven that (25) guarantees convergence of the Richardson equalizer in (14) with the initialization in (15) by showing that it is a sufficient condition for (23) to hold.

**A. Problem Statement**

Using \( e^{(r+1)} \) in (22) and the definition of \( A_{x} \) and \( A_{x}(r) \) in (7) and (11), respectively, the error \( e[n](r+1) \) in iteration \( r + 1 \) at instant time \( n \) can be expressed as
\[
e[n](r+1) = \sum_{k_{1} \in Z} (\delta[k_{1}] - g_{x,n}[k_{1}])e[n - k_{1}]^{(r)} + \sum_{k_{1} \in Z} (g_{x,n}(r)[k_{1}] - g_{x,n}[k_{1}])x[n - k_{1}]^{(r)}.
\]

Therewith, the supremum norm of \( e^{(r+1)} \) is upper bounded using the triangle inequality as
\[
\|e^{(r+1)}\|_{\infty} = \sup_{n \in Z} |e[n](r+1)| = \sup_{n \in Z} \left[ \sum_{k_{1} \in Z} (\delta[k_{1}] - g_{x,n}[k_{1}])e[n - k_{1}]^{(r)} + \sum_{k_{1} \in Z} (g_{x,n}(r)[k_{1}] - g_{x,n}[k_{1}])x[n - k_{1}]^{(r)} \right] \leq \sup_{n \in Z} \left[ \alpha_{x,n}(e^{(r)}) + \beta_{x,n}(e^{(r)}) \right]
\]

where the first and second sum is given by \( \alpha_{x,n}(e^{(r)}) \) and \( \beta_{x,n}(e^{(r)}) \) \( n \in Z \) and \( \beta_{x,n}(e^{(r)}) \) \( n \in Z \) and Appendix D and E, respectively. Using the upper bounds (57) and (61) for \( \|e^{(r)}\|_{\infty} \) and \( \|\beta_{x,n}(e^{(r)})\| \) derived in Appendix D and E, respectively, it follows that
\[
\|e^{(r+1)}\|_{\infty} \leq \|e^{(r)}\|_{\infty} \sup_{n \in Z} \eta_{x,n}(\|e^{(r)}\|_{\infty}) \quad (48)
\]
where
\[
\eta_{x,n}(\|e^{(r)}\|_{\infty}) = \sum_{k_{1} \in Z} |\delta[k_{1}] - h_{n,[k_{1}]}| + \sum_{p=2}^{\infty} \|h_{p,n}\|_{1} \cdot \|x\|_{p-1} \quad (49)
\]
and
\[
\tilde{w}_{x}(p, \|e^{(r)}\|_{\infty}) = \|x\|_{p-1}^{-1} + \|x\|_{\infty}^{p-1} \sum_{l=1}^{p-1} \left( \frac{p-1}{l} \right) \|x\|_{p-1-l}^{-1}\|e^{(r)}\|_{\infty}^{-1}. \quad (50)
\]

It can be observed that \( \eta_{x,n}(\|e^{(r)}\|_{\infty}) \) is monotonically increasing with respect to the non-negative argument \( \|e^{(r)}\|_{\infty} \), which is relevant later on. Indeed, only \( \tilde{w}_{x}(p, \|e^{(r)}\|_{\infty}) \) depends on \( \|e^{(r)}\|_{\infty} \) and, as can be seen in (50), it is a polynomial of degree \( p-2 \) with non-negative coefficients and, therefore, monotonically increasing. According to (48), if
\[
\sup_{n \in Z} \eta_{x,n}(\|e^{(r)}\|_{\infty}) < 1 \quad (51)
\]
holds for all iterations \( r \geq 0 \), the condition for convergence in (23) holds as well. In the following, sufficient conditions for (51) to hold in the first iteration are derived. Subsequently, this result is used for an inductive proof of convergence under the same conditions.

**B. Error Reduction in First Iteration**

Due to the initialization in (15) and the system model in (8), the initial error \( e^{(0)} \) is given by
\[
e^{(0)} = x - x^{(0)} = x - y = (I - A_{x})x.
\]

With the definition of \( A_{x} \) in (7), the supremum norm of the initial error can be identified as
\[
\|e^{(0)}\|_{\infty} = \sup_{n \in Z} |\alpha_{x,n}(x)|.
\]
Due to the upper bound on \( |\alpha_{x,n}(x)| \) in (57) in Appendix D, this norm is upper bounded by
\[
\|e^{(0)}\|_{\infty} \leq \|x\|_{\infty} \sup_{n \in Z} \tilde{\eta}_{x,n}
\]
where
\[
\tilde{\eta}_{x,n} = \sum_{k_{1} \in Z} |\delta[k_{1}] - h_{1,n}[k_{1}]| + \sum_{p=2}^{\infty} \|h_{p,n}\|_{1} \cdot \|x\|_{p-1} \quad (52)
\]
A comparison of \( \tilde{\eta}_{x,n} \) to \( \eta_{x,n}(\|e^{(r)}\|_{\infty}) \) in (49) reveals that \( \tilde{\eta}_{x,n} \leq \eta_{x,n}(\|e^{(r)}\|_{\infty}) \) for all \( \|e^{(r)}\|_{\infty} \geq 0 \). Consequently, any condition that ensures (51) enforces
\[
\sup_{n \in Z} \tilde{\eta}_{x,n} < 1 \quad (53)
\]
as well. Therefore, it can be assumed that the initial error is bounded by
\[
\|e^{(0)}\|_{\infty} < \|x\|_{\infty}
\]
because a contradiction in this inequality would also invalidate (51). Due to the fact that \( \eta_{x,n}(\|e^{(r)}\|_{\infty}) \) in (49) is a monotonically increasing function for non-negative arguments, it follows that
\[
\eta_{x,n}(\|e^{(0)}\|_{\infty}) \leq \eta_{x,n}(\|x\|_{\infty}) \quad (54)
\]
Consequently, requiring
\[
\sup_{n \in Z} \eta_{x,n}(\|x\|_{\infty}) < 1
\]
ensures that (51) holds for the first iteration and, therefore, \( \|e^{(1)}\| < \|e^{(0)}\| \). Rewriting (21), taking the supremum norm, and applying the triangle inequality leads to (44) and permits the bound
\[
\|x^{(0)}\|_{\infty} \leq \|x\|_{\infty} + \|e^{(0)}\|_{\infty} < 2\|x\|_{\infty}
\]
Using this upper bound in (50) for the argument \( \|x\|_\infty \) gives\(^5\)

\[
\hat{w}_x(p, \|x\|_\infty) < \|x\|_\infty^{-p} \left[ 1 + 2 \sum_{i=1}^{p-1} \binom{p-1}{i} \right]
\]

= \|x\|_\infty^{-p} \left( 2^p - 1 \right).

Finally, utilizing this upper bound on \( \hat{w}_x(p, \|x\|_\infty) \) in (49) to obtain an upper bound on \( \eta_{x,n}(\|x\|_\infty) \) and, subsequently, using the result in (54) leads to the condition for convergence in (25).

### C. Inductive Proof of Convergence

Convergence of the Richardson equalizer can be ensured by induction if

\[
\sup_{n \in \mathbb{Z}} \eta_{x,n}(\|e^{(r+1)}\|_\infty) \leq \sup_{n \in \mathbb{Z}} \eta_{x,n}(\|e^{(r)}\|_\infty)
\]

holds, as, due to (48), this implies that (23) holds. The condition for convergence in (25) establishes the basis

\[
\sup_{n \in \mathbb{Z}} \eta_{x,n}(\|e^{(0)}\|_\infty) < 1
\]

which follows from (53) and (54). As a consequence of (48), this basis implies \( \|e^{(r)}\|_\infty < \|e^{(0)}\|_\infty \). As \( \eta_{x,n}(\|e^{(r)}\|_\infty) \) is a monotonically increasing function for non-negative arguments, it follows that (55) holds for \( r = 0 \) and, due to (48), (23) holds for \( r = 1 \). This induction step can be repeated ad infinitum and, therefore, completes the proof.

### APPENDIX D

**Upper Bound for \( |\alpha_{x,n}(e^{(r)})| \)**

In this appendix, an upper bound for the absolute value of \( \alpha_{x,n}(e^{(r)}) = \sum_{k_1 \in \mathbb{Z}} (\delta[k_1] - g_{x,n}[k_1]) e[n - k_1] e^{(r)} \) is derived. Using the triangle inequality and the supremum norm \( \|e^{(r)}\|_\infty \) as an upper bound on \( e[n - k_1] e^{(r)} \) yields

\[
|\alpha_{x,n}(e^{(r)})| \leq \sum_{k_1 \in \mathbb{Z}} |\delta[k_1] - g_{x,n}[k_1]| \cdot |e[n - k_1] e^{(r)}|
\]

\[
\leq \|e^{(r)}\|_\infty \sum_{k_1 \in \mathbb{Z}} |\delta[k_1] - g_{x,n}[k_1]|.
\]

Substitution of \( g_{x,n}[k_1] \) with (4) and application of the triangle inequality permits the upper bound

\[
|\alpha_{x,n}(e^{(r)})| \leq \|e^{(r)}\|_\infty \sum_{k_1 \in \mathbb{Z}} |\delta[k_1] - h_{1,n}[k_1]|
\]

\[
+ \sum_{p=2}^{\infty} \sum_{k_1 \in \mathbb{Z}} |h_{p,n}[k_1, \ldots, k_p]| \prod_{i=2}^{p} |x[n - k_i] |.
\]

With the supremum norm \( \|x\|_\infty \) as an upper bound on \( x[n-k_1] \) and \( \|h_{p,n}\|_1 \) in (27), \( |\alpha_{x,n}(e^{(r)})| \) is upper bounded by

\[
|\alpha_{x,n}(e^{(r)})| \leq \|e^{(r)}\|_\infty \left[ \sum_{k_1 \in \mathbb{Z}} |\delta[k_1] - h_{1,n}[k_1]| + \sum_{p=2}^{\infty} \|h_{p,n}\|_1 \cdot \|x\|_\infty^{-p} \right].
\]

\[
(57)
\]

\(^5\) A comparison of \( \sum_{p=0}^{p-1} \binom{p-1}{i} \) to the binomial theorem shows that it corresponds to \( 2^{p-1} - 1 \).

### APPENDIX E

**Upper Bound for \( |\beta_{x,n}(e^{(r)})| \)**

In this appendix, an upper bound for the absolute value of \( \beta_{x,n}(e^{(r)}) = \sum_{k_1 \in \mathbb{Z}} (g_{x,n}[k_1] - g_{x,n}[k_1]) x[n - k_1] e^{(r)} \) is derived. Using the triangle inequality and the supremum norm \( \|e^{(r)}\|_\infty \) as an upper bound on \( x[n - k_1] e^{(r)} \) yields

\[
|\beta_{x,n}(e^{(r)})| \leq \sum_{k_1 \in \mathbb{Z}} |g_{x,n}[k_1] - g_{x,n}[k_1]| \cdot |x[n - k_1] e^{(r)}|.
\]

\[
\leq \|x\|_\infty \sum_{k_1 \in \mathbb{Z}} |g_{x,n}[k_1] - g_{x,n}[k_1]|.
\]

(59)

Substituting the impulse responses with (4) and (12), respectively, applying the triangle inequality, and utilizing \( \gamma_{x,n}(p, k_1, \ldots, k_p) \) defined in (62) in Appendix F results in the upper bound

\[
\sum_{k_1 \in \mathbb{Z}} |g_{x,n}[k_1] - g_{x,n}[k_1]| = \sum_{k_1 \in \mathbb{Z}} \left( \sum_{p=2}^{\infty} \sum_{k_2, \ldots, k_p \in \mathbb{Z}} h_{p,n}[k_2, \ldots, k_p] \gamma_{x,n}(p, k_2, \ldots, k_p) \right)
\]

\[
\leq \sum_{p=2}^{\infty} \sum_{k_1, \ldots, k_p \in \mathbb{Z}} |h_{p,n}[k_1, \ldots, k_p]| \cdot |\gamma_{x,n}(p, k_2, \ldots, k_p)|.
\]

Using the upper bound (64) on \( |\gamma_{x,n}(p, k_2, \ldots, k_p)| \) in Appendix F and \( \|h_{p,n}\|_1 \) in (27) yields

\[
\sum_{k_1 \in \mathbb{Z}} |g_{x,n}[k_1] - g_{x,n}[k_1]| \leq \|e^{(r)}\|_\infty \sum_{p=2}^{\infty} \|h_{p,n}\|_1 \prod_{l=1}^{p-1} \left( \frac{p-1}{l} \right) \|x\|_\infty^{p-1-l} \|e^{(r)}\|^{l-1}.
\]

(60)

Finally, using (60) in (59) permits the upper bound

\[
|\beta_{x,n}(e^{(r)})| \leq \|e^{(r)}\|_\infty \sum_{p=2}^{\infty} \|h_{p,n}\|_1 \cdot \|x\|_\infty \prod_{l=1}^{p-1} \left( \frac{p-1}{l} \right) \|x\|_\infty^{p-1-l} \|e^{(r)}\|^{l-1}.
\]

(61)

### APPENDIX F

**Upper Bound for \( |\gamma_{x,n}(p, q, \ldots, k_p)| \)**

In this appendix, an upper bound for the absolute value of \( \gamma_{x,n}(p, q, \ldots, k_p) = \prod_{i=q}^{p} x[n - k_i] e^{(r)} - \prod_{i=q}^{p} x[n - k_i] \) is derived, where \( 1 \leq q \leq p \). Using the definition of \( e^{(r)} \) in (21), the absolute value of \( \gamma_{x,n}(p, q, \ldots, k_p) \) can be expressed as

\[
|\gamma_{x,n}(p, q, \ldots, k_p)| = \left| \prod_{i=q}^{p} (x[n - k_i] - e[n - k_i] e^{(r)}) - \prod_{i=q}^{p} x[n - k_i] \right|.
\]

(63)
If the first product therein is expanded, it contains a summand that cancels with the second product. In order to find an upper bound on the remaining terms, the first product is analyzed. Using the triangle inequality and $\|x\|_\infty$ and $\|e^{(r)}\|_\infty$ as upper bounds on $x[n-k_i]$ and $e[n-k_i]^{(r)}$, respectively, enables
\[
\left| \prod_{i=q}^p (x[n-k_i] - e[n-k_i]^{(r)}) \right| 
\leq \prod_{i=q}^p \left( |x[n-k_i]| + |e[n-k_i]^{(r)}| \right) 
\leq (\|x\|_\infty + \|e^{(r)}\|_\infty)^{p-q+1}.
\]
For this bound, the binomial theorem gives
\[
(\|x\|_\infty + \|e^{(r)}\|_\infty)^N = \|x\|_\infty^N + \sum_{i=1}^N \binom{N}{i} \|x\|_\infty^{N-i} \|e^{(r)}\|_\infty^i
\]
in which $N = p - q + 1$. It can be recognized that $\|x\|_\infty^N$ corresponds to the upper bound of the term that cancels with the second product in (63) and, therefore,
\[
|\gamma(q)_{x,n}[p, k_q, \ldots, k_p]| 
\leq \sum_{l=1}^{p-q+1} \left( \frac{p-q+1}{l} \right) \|x\|_\infty^{p-q+1-l} \|e^{(r)}\|_\infty^l.
\]
Equivalently, this bound can be stated as
\[
|\gamma(q)_{x,n}[p, k_q, \ldots, k_p]| 
\leq (\|x\|_\infty + \|e^{(r)}\|_\infty)^{p-q+1} - \|x\|_\infty^{p-q+1}.
\]

**APPENDIX G**

**KERNELS OF THE VOLterra SYSTEM $H_n$ FOR POST- AND PRE-LINEARIZATION**

1) **Post-Linearization**: For post-linearization, the Volterra system $H_n$ is given by (18). From Fig. 4 and the definition of the Volterra system in (1) it follows that
\[
y[n] = \sum_{p=1}^\infty \sum_{\nu_1, \ldots, \nu_p \in Z} \hat{h}_{p,n}[\nu_1, \ldots, \nu_p] \prod_{i=1}^p u[n - \nu_i].
\]
Using the definition of $u[n]$ in (17) and $L^{-1}$ in (29) results in
\[
y[n] = \sum_{p=1}^\infty \sum_{\nu_1, \ldots, \nu_p \in Z} \hat{h}_{p,n}[\nu_1, \ldots, \nu_p] 
\times \prod_{i=1}^p \sum_{l \in Z} q[l] x[n - \nu_i - l]
\]
\[
= \sum_{p=1}^\infty \sum_{\nu_1, \ldots, \nu_p \in Z} \sum_{l_1, \ldots, l_p \in Z} \hat{h}_{p,n}[\nu_1, \ldots, \nu_p]
\times \prod_{i=1}^p q[l_i] x[n - \nu_i - l_i].
\]

The substitution $k_i = \nu_i + l_i$, for $i = 1, \ldots, p$, and partition of the product yields
\[
y[n] = \sum_{p=1}^\infty \sum_{k_1, \ldots, k_p \in Z} \left[ \sum_{l_1, \ldots, l_p \in Z} \hat{h}_{p,n}[k_1 - l_1, \ldots, k_p - l_p]
\times \prod_{j=1}^p q[l_j] \right] \prod_{i=1}^p x[n - k_i].
\]
A comparison to (1) shows that $y[n]$ is given by (1) with the Volterra kernels in (30).

2) **Pre-Linearization**: For pre-linearization, the Volterra system $H_n$ is given by (20). From Fig. 6 and the definition of $L^{-1}$ in (29) it follows that
\[
y[n] = \sum_{l \in Z} q[l] v[n - l].
\]
Furthermore, from Fig. 6 and the definition of the Volterra system in (1) it follows that $v[n]$ is given by
\[
v[n] = \sum_{p=1}^\infty \sum_{\nu_1, \ldots, \nu_p \in Z} \hat{h}_{p,n}[\nu_1, \ldots, \nu_p] \prod_{i=1}^p x[n - \nu_i].
\]
Using (67) in (66) yields
\[
y[n] = \sum_{p=1}^\infty \sum_{\nu_1, \ldots, \nu_p \in Z} \sum_{l \in Z} \hat{h}_{p,n-l}[\nu_1, \ldots, \nu_p] q[l]
\times \prod_{i=1}^p x[n - l - \nu_i].
\]
The substitution $k_i = l + \nu_i$, for $i = 1, \ldots, p$, results in
\[
y[n] = \sum_{p=1}^\infty \sum_{k_1, \ldots, k_p \in Z} \left[ \sum_{l \in Z} \hat{h}_{p,n-l}[k_1 - l, \ldots, k_p - l] q[l] \right]
\times \prod_{i=1}^p x[n - k_i].
\]
A comparison to (1) shows that $y[n]$ is given by (1) with the Volterra kernels in (31).

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