Representation Theory of Analytic Holonomy

\textit{C*} Algebras

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Abstract

Integral calculus on the space $\mathcal{A}/\mathcal{G}$ of gauge equivalent connections is developed. Loops, knots, links and graphs feature prominently in this description. The framework is well-suited for quantization of diffeomorphism invariant theories of connections.

The general setting is provided by the abelian \textit{C*}–algebra of functions on $\mathcal{A}/\mathcal{G}$ generated by Wilson loops (i.e., by the traces of holonomies of connections around closed loops). The representation theory of this algebra leads to an interesting and powerful “duality” between gauge–equivalence classes of connections and certain equivalence classes of closed loops. In particular, regular measures on (a suitable completion of) $\mathcal{A}/\mathcal{G}$ are in 1–1 correspondence with certain functions of loops and diffeomorphism invariant measures correspond to (generalized) knot and link invariants. By carrying out a non–linear extension of the theory of cylindrical measures on topological vector spaces, a faithful, diffeomorphism invariant measure is introduced. This measure can be used to define the Hilbert space of quantum states in theories of connections. The Wilson–loop functionals then serve as the configuration operators in the quantum theory.

1 Introduction

The space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations plays an important role in gauge theories, including certain topological theories and
In a typical setup, $\mathcal{A}/\mathcal{G}$ is the classical configuration space. In the quantum theory, then, the Hilbert space of states would consist of all square-integrable functions on $\mathcal{A}/\mathcal{G}$ with respect to some measure. Thus, the theory of integration over $\mathcal{A}/\mathcal{G}$ would lie at the heart of the quantization procedure. Unfortunately, since $\mathcal{A}/\mathcal{G}$ is an infinite dimensional non-linear space, the well-known techniques to define integrals do not go through and, at the outset, the problem appears to be rather difficult. Even if this problem could be overcome, one would still face the issue of constructing self-adjoint operators corresponding to physically interesting observables. In a Hamiltonian approach, the Wilson loop functions provide a natural set of (manifestly gauge invariant) configuration observables. The problem of constructing the analogous, manifestly gauge invariant “momentum observables” is difficult already in the classical theory: these observables would correspond to vector fields on $\mathcal{A}/\mathcal{G}$ and differential calculus on this space is not well-developed.

Recently, Ashtekar and Isham [2] (hereafter referred to as A–I) developed an algebraic approach to tackle these problems. The A–I approach is in the setting of canonical quantization and is based on the ideas introduced by Gambini and Trias in the context of Yang-Mills theories [3] and by Rovelli and Smolin in the context of quantum general relativity [4]. Fix a $n$–manifold $\Sigma$ on which the connections are to be defined and a compact Lie group $G$ which will be the gauge group of the theory under consideration. The first step is the construction of a $C^*$–algebra of configuration observables—a sufficiently large set of gauge–invariant functions of connections on $\Sigma$. A natural strategy is to use the Wilson–loop functions—the traces of holonomies of connections around closed loops on $\Sigma$—to generate this $C^*$–algebra. Since these are configuration observables, they commute even in the quantum theory. The $C^*$–algebra is therefore Abelian. The next step is to construct representations of this algebra. For this, the Gel’fand spectral theory provides a natural setting since any given Abelian $C^*$–algebra with identity is naturally isomorphic with the $C^*$–algebra of continuous functions on a compact, Hausdorff space, the Gel’fand spectrum $sp(C^*)$ of that algebra. (As the notation suggests, $sp(C^*)$ can be constructed directly from the given $C^*$–algebra: its elements are homomorphism from the given $C^*$–algebra to the $*$–algebra of complex numbers.) Consequently, every (continuous) cyclic representation of the A–I $C^*$–algebra is of the following type: The carrier Hilbert space is $L^2(sp(C^*), d\mu)$ for some regular measure $d\mu$ and the Wilson–loop operators act (on square–integrable functions on $sp(C^*)$) simply by multi-

\footnote{In typical applications, $\Sigma$ will be a Cauchy surface in an $n+1$ dimensional space–time in the Lorentzian regime or an $n$–dimensional space–time in the Euclidean regime. In the main body of this paper, $n$ will be taken to be 3 and $G$ will be taken to be $SU(2)$ both for concreteness and simplicity. These choices correspond to the simplest, non–trivial applications of the framework to physics.}
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A–I pointed out that, since the elements of the $C^*$–algebra are labelled by loops, there is a 1–1 correspondence between regular measures on $sp(C^*)$ and certain functions on the space of loops. Diffeomorphism invariant measures correspond to knot and link invariants. Thus, there is a natural interplay between connections and loops, a point which will play an important role in the present paper.

Note that, while the classical configuration space is $A/\mathcal{G}$, the domain space of quantum states is $sp(C^*)$. A–I showed that $A/\mathcal{G}$ is naturally embedded in $sp(C^*)$ and Rendall [5] showed that the embedding is in fact dense. Elements of $sp(C^*)$ can be therefore thought of as generalized gauge equivalent connections. To emphasize this point and to simplify the notation, let us denote the spectrum of the A–I $C^*$–algebra by $\mathcal{A}/\mathcal{G}$. The fact that the domain space of quantum states is a completion of—and hence larger than—the classical configuration space may seem surprising at first since in ordinary (i.e. non–relativistic) quantum mechanics both spaces are the same. The enlargement is, however, characteristic of quantum field theory, i.e. of systems with an infinite number of degrees of freedom. A–I further explored the structure of $\mathcal{A}/\mathcal{G}$ but did not arrive at a complete characterization of its elements. They also indicated how one could make certain heuristic considerations of Rovelli and Smolin precise and associate “momentum operators” with (closed) strips (i.e. $n–1$ dimensional ribbons) on $\Sigma$. However, without further information on the measure $d\mu$ on $\mathcal{A}/\mathcal{G}$, one can not decide if these operators can be made self–adjoint, or indeed if they are even densely defined. A–I did introduce certain measures with which the strip operators “interact properly.” However, as they pointed out, these measures have support only on finite dimensional subspaces of $\mathcal{A}/\mathcal{G}$ and therefore “too simple” to be interesting in cases where the theory has an infinite number of degrees of freedom.

In broad terms, the purpose of this paper is to complete the A–I program in several directions. More specifically, we will present the following results:

1. We will obtain a complete characterization of elements of the Gel’fand spectrum. This will provide a degree of control on the domain space of quantum states which is highly desirable: Since $A/\mathcal{G}$ is a non–trivial, infinite dimensional space which is not even locally compact, while $\mathcal{A}/\mathcal{G}$ is a compact Hausdorff space, the completion procedure is quite non–trivial. Indeed, it is the lack of control on the precise content of the Gel’fand spectrum of physically interesting $C^*$ algebras that has prevented the Gel’fand theory from playing a prominent role in quantum field theory so far.

2. We will present a faithful, diffeomorphism invariant measure on $\mathcal{A}/\mathcal{G}$. The theory underlying this construction suggests how one might introduce other diffeomorphism invariant measures. Recently, Baez [6] has exploited this general strategy to introduce new measures. These
developments are interesting from a strictly mathematical standpoint because the issue of existence of such measures was a subject of controversy until recently. They are also of interest from a physical viewpoint since one can, e.g., use these measures to regulate physically interesting operators (such as the constraints of general relativity) and investigate their properties in a systematic fashion.

3. Together, these results enable us to define a Rovelli–Smolin “loop–transform” rigorously. This is a map from the Hilbert space $L^2(\mathcal{A}/\mathcal{G}, d\mu)$ to the space of suitably regular functions of loops on $\Sigma$. Thus, quantum states can be represented by suitable functions of loops. This “loop representation” is particularly well–suited to diffeomorphism invariant theories of connections, including general relativity.

We have also developed differential calculus on $\mathcal{A}/\mathcal{G}$ and shown that the strip (i.e. momentum) operators are in fact densely–defined, symmetric operators on $L^2(\mathcal{A}/\mathcal{G}, d\mu)$; i.e. that they interact with our measure correctly. However, since the treatment of the strip operators requires a number of new ideas from physics as well as certain techniques from graph theory, these results will be presented in a separate work.

The methods we use can be summarized as follows.

First, we make the (non–linear) duality between connections and loops explicit. On the connection side, the appropriate space to consider is the Gel’fand spectrum $\mathcal{A}/\mathcal{G}$ of the $A$–$I$ $C^*$–algebra. On the loop side, the appropriate object is the space of hoops—i.e., holonomically equivalent loops. More precisely, let us consider based, piecewise analytic loops and regard two as being equivalent if they give rise to the same holonomy (evaluated at the base point) for any $(G)$–connection. Call each “holonomic equivalence class” a $(G)$–hoop. It is straightforward to verify that the set of hoops has the structure of a group. (It also carries a natural topology [7], which, however, will not play a role in this paper.) We will denote it by $\mathcal{H}G$ and call it the hoop group. It turns out that $\mathcal{H}G$ and the spectrum $\mathcal{A}/\mathcal{G}$ can be regarded as being “dual” to each other in the following sense:

1. Every element of $\mathcal{H}G$ defines a continuous, complex–valued function on $\mathcal{A}/\mathcal{G}$ and these functions generate the algebra of all continuous functions on $\mathcal{A}/\mathcal{G}$;

2. Every element of $\mathcal{A}/\mathcal{G}$ defines a homomorphism from $\mathcal{H}G$ to the gauge group $G$ (which is unique modulo an automorphism of $G$) and every homomorphism defines an element of $\mathcal{A}/\mathcal{G}$.

In the case of topological vector spaces, the duality mapping respects the topology and the linear structure. In the present case, however, $\mathcal{H}G$
has the structure only of a group and \( \mathcal{A}/\mathcal{G} \) of a topological space. The "duality" mappings can therefore respect only these structures. Property 2 above provides us with a complete (algebraic) characterization of the elements of Gel'fand spectrum \( \mathcal{A}/\mathcal{G} \) while property 1 specifies the topology of the spectrum.

The second set of techniques involves a family of projections from the topological space \( \mathcal{A}/\mathcal{G} \) to certain finite dimensional manifolds. For each sub-group of the hoop group \( \mathcal{H}\mathcal{G} \), generated by \( n \) independent hoops, we define a projection from \( \mathcal{A}/\mathcal{G} \) to the quotient, \( G^n/\text{Ad} \), of the \( n \)-th power of the gauge group by the adjoint action. (An element of \( G^n/\text{Ad} \) is thus an equivalence class of \( n \)-tuples, \( (g_1, \ldots, g_n) \), with \( g_i \in G \), where, \( (g_1, \ldots, g_n) \sim (g_o^{-1}g_1g_o, \ldots, g_o^{-1}g_ng_o) \) for any \( g_o \in G \).) The lifts of functions on \( G^n/\text{Ad} \) are then the non-linear analogs of the cylindrical functions on topological vector spaces. Finally, since each \( G^n/\text{Ad} \) is a compact topological space, we can equip it with measures to integrate functions in the standard fashion. This in turn enables us to define cylinder measures on \( \mathcal{A}/\mathcal{G} \) and integrate cylindrical functions thereon. Using the Haar measure on \( G \), we then construct a natural, diffeomorphism invariant, faithful measure on \( \mathcal{A}/\mathcal{G} \).

Finally, for completeness, we will summarize the strategy we have adopted to introduce and analyse the "momentum operators," although, as mentioned above, these results will be presented elsewhere. The first observation is that we can define "vector fields" on \( \mathcal{A}/\mathcal{G} \) as derivations on the ring of cylinder functions. Then, to define the specific vector fields corresponding to the momentum (or, strip) operators we introduce certain notions from graph theory. These specific vector fields are "divergence-free" with respect to our cylindrical measure on \( \mathcal{A}/\mathcal{G} \), i.e., they leave the cylindrical measure invariant. Consequently, the momentum operators are densely defined and symmetric on the Hilbert space of square-integrable functions on \( \mathcal{A}/\mathcal{G} \).

The paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3, we obtain the characterization of elements of \( \mathcal{A}/\mathcal{G} \) and also present some examples of those elements which do not belong to \( \mathcal{A}/\mathcal{G} \), i.e. which represent genuine, generalized (gauge equivalence classes of) connections. Using the machinery introduced in this characterization, in Section 4 we introduce the notion of cylindrical measures on \( \mathcal{A}/\mathcal{G} \) and then define a natural, faithful, diffeomorphism invariant, cylindrical measure. Section 5 contains concluding remarks.

In all these considerations, the piecewise analyticity of loops on \( \Sigma \) plays an important role. That is, the specific constructions and proofs presented here would not go through if the loops were allowed to be just piecewise smooth. However, it is far from being obvious that the final results would not go through in the smooth category. To illustrate this point, in Appendix A we restrict ourselves to \( U(1) \) connections and, by exploiting the Abelian
character of this group, show how one can obtain the main results of this paper using piecewise $C^1$ loops. Whether similar constructions are possible in the non–Abelian case is, however, an open question. Finally, in Appendix B we consider another extension. In the main body of the paper, $\Sigma$ is a 3–manifold and the gauge group $G$ is taken to be $SU(2)$. In Appendix B, we indicate the modifications required to incorporate $SU(N)$ and $U(N)$ connections on an n–manifold (keeping however, piecewise analytic loops).

2 Preliminaries

In this section, we will introduce the basic notions, fix the notation and recall those results from the A–I framework which we will need in the subsequent discussion.

Fix a 3–dimensional, real, analytic, connected, orientable manifold $\Sigma$. Denote by $P(\Sigma, SU(2))$ a principal fibre bundle $P$ over the base space $\Sigma$ and the structure group $SU(2)$. Since any $SU(2)$ bundle over a 3–manifold is trivial we take $P$ to be the product bundle. Therefore, $SU(2)$ connections on $P$ can be identified with $su(2)$–valued 1–form fields on $\Sigma$. Denote by $A$ the space of smooth (say $C^1$) $SU(2)$ connections, equipped with one of the standard (Sobolev) topologies (see, e.g., [8]). Every $SU(2)$–valued function $g$ on $\Sigma$ defines a gauge transformation in $A$,

$$A \cdot g := g^{-1}Ag + g^{-1}dg. \tag{2.1}$$

Let us denote the quotient of $A$ under the action of this group $G$ (of local gauge transformations) by $A/G$. Note that the affine space structure of $A$ is lost in this projection; $A/G$ is a genuinely non–linear space with a rather complicated topological structure.

The next notion we need is that of closed loops in $\Sigma$. Let us begin by considering continuous, piecewise analytic ($C^\omega$) parametrized paths, i.e., maps

$$p : [0, s_1] \cup \ldots \cup [s_{n-1}, 1] \to \Sigma \tag{2.2}$$

which are continuous on the whole domain and $C^\omega$ on the closed intervals $[s_k, s_{k+1}]$. Given two paths $p_1 : [0, 1] \to \Sigma$ and $p_2 : [0, 1] \to \Sigma$ such that $p_1(1) = p_2(0)$, we denote by $p_2 \circ p_1$ the natural composition:

$$p_2 \circ p_1(s) = \begin{cases} p_1(2s), & \text{for } s \in [0, \frac{1}{2}] \\ p_2(2s - 1), & \text{for } s \in [\frac{1}{2}, 1]. \end{cases} \tag{2.3}$$

The “inverse” of a path $p : [0, 1] \to \Sigma$ is a path

$$p^{-1}(s) := p(1 - s). \tag{2.4}$$

A path which starts and ends at the same point is called a loop. We will be interested in based loops. Let us therefore fix, once and for all, a point $x^o \in \Sigma$. Denote by $L_{x^o}$ the set of piecewise $C^\omega$ loops which are based at
Given a connection $A$ on $P(\Sigma, SU(2))$, a loop $\alpha \in \mathcal{L}_{x^0}$, and a fixed point $\hat{x}^0 \in \mathcal{L}_{x^0}$, we denote the corresponding element of the holonomy group by $H(\alpha, A)$. (Since $P$ is a product bundle, we will make the obvious choice $\hat{x}^0 = (x^0, e)$, where $e$ is the identity in $SU(2)$.) Using the space of connections $\mathcal{A}$, we now introduce a key equivalence relation on $\mathcal{L}_{x^0}$:

Two loops $\alpha, \beta \in \mathcal{L}_{x^0}$ will be said to be holonomically equivalent, $\alpha \sim \beta$, iff $H(\alpha, A) = H(\beta, A)$, for every $SU(2)$–connection $A$ in $\mathcal{A}$.

Each holonomic equivalence class will be called a hoop. Thus, for example, two loops (which have the same orientation and) which differ only in their parametrization define the same hoop. Similarly, if two loops differ just by a retraced segment, they belong to the same hoop. (More precisely, if paths $p_1$ and $p_2$ are such that $p_1(1) = p_2(0)$ and $p_1(0) = p_2(1) = x^0$, so that $p_2 \circ p_1$ is a loop in $\mathcal{L}_{x^0}$, and if $\rho$ is a path starting at the point $p_1(1) = p_2(0)$, then $p_2 \circ p_1$ and $p_2 \circ \rho \circ \rho^{-1} \circ p_1$ define the same hoop.) Note that in general, the equivalence relation depends on the choice of the gauge group, whence the hoop should in fact be called a $SU(2)$–hoop. However, since the group is fixed to be $SU(2)$ in the main body of this paper, we drop this suffix. The hoop to which a loop $\alpha$ belongs will be denoted by $\tilde{\alpha}$.

The collection of all hoops will be denoted by $\mathcal{H}G$; thus $\mathcal{H}G = \mathcal{L}_{x^0}/\sim$.

With this machinery at hand, we can now introduce the $A$–I $C^*$–algebra. We begin by assigning to every $\tilde{\alpha} \in \mathcal{H}G$ a real–valued function $T_{\tilde{\alpha}}$ on $\mathcal{A}/G$, bounded between $-1$ and $1$:

$$T_{\tilde{\alpha}}(\tilde{A}) := \frac{1}{2} \text{Tr} H(\alpha, A),$$

(2.5)

where $\tilde{A} \in \mathcal{A}/G$; $A$ is any connection in the gauge equivalence class $\tilde{A}$; $\alpha$ any loop in the hoop $\tilde{\alpha}$; and where the trace is taken in the fundamental representation of $SU(2)$. ($T_{\tilde{\alpha}}$ is called the Wilson loop function in the physics literature.) Due to $SU(2)$ trace identities, product of these functions can be expressed as sums:

$$T_{\tilde{\alpha}}T_{\tilde{\beta}} = \frac{1}{2}(T_{\tilde{\alpha} \tilde{\beta}} + T_{\tilde{\alpha} \tilde{\beta}^{-1}}),$$

(2.6)

The hoop equivalence relation seems to have been introduced independently by a number of authors in different contexts (e.g. by Gambini and collaborators in their investigation of Yang–Mills theory and by Ashtekar in the context of 2+1 gravity). Recently, the group structure of $\mathcal{H}G$ has been systematically exploited and extended by Di Bartolo, Gambini and Griego [9] to develop a new and potentially powerful framework for gauge theories.
where, on the right side, we have used the composition of hoops in $\mathcal{H}G$. Therefore, the complex vector space spanned by the $T_{\tilde{\alpha}}$ functions is closed under the product; it has the structure of a $\star$-algebra. Denote it by $\mathcal{H}A$ and call it the holonomy algebra. Since each $T_{\tilde{\alpha}}$ is a bounded continuous function on $A/G$, $\mathcal{H}A$ is a subalgebra of the $C^*$-algebra of all complex-valued, bounded, continuous functions thereon. The completion $\overline{\mathcal{H}A}$ of $\mathcal{H}A$ (under the sup norm) has therefore the structure of a $C^*$-algebra. This is the $A$–I $C^*$-algebra.

As in [2], one can make the structure of $\mathcal{H}A$ explicit by constructing it, step by step, from the space of loops $L_{x_0}$. Denote by $FL_{x_0}$ the free vector space over complexes generated by $L_{x_0}$. Let $K$ be the subspace of $FL_{x_0}$ defined as follows:

$$\sum_i a_i [\alpha_i] K = 0, \quad \forall \tilde{A} \in A/G,$$

(2.7)

It is then easy to verify that $\mathcal{H}A = FL_{x_0}/K$. (Note that the $K$–equivalence relation subsumes the hoop equivalence.) Thus, elements of $\mathcal{H}A$ can be represented either as $\sum_i a_i T_{\tilde{\alpha}_i}$ or as $\sum_i [a_i] K$. The $\star$–operation, the product and the norm on $\mathcal{H}A$ can be specified directly on $FL_{x_0}/K$ as follows:

$$\sum_i (a_i [\alpha_i])^* = \sum_i \bar{a}_i [\alpha_i],$$

$$\left(\sum_i a_i [\alpha_i] K\right) \cdot \left(\sum_j b_j [\beta_j] K\right) = \sum_{ij} a_i b_j ([\alpha_i \circ \beta_j] K + [\alpha_i \circ \beta_j^{-1}] K),$$

$$\| \sum_i a_i [\alpha_i] K \| = \sup_{\tilde{A} \in A/G} |\sum_i a_i T_{\tilde{\alpha}_i}(\tilde{A})|,$$

(2.8)

where $\bar{a}_i$ is the complex conjugate of $a_i$. The $C^*$–algebra $\overline{\mathcal{H}A}$ is obtained by completing $\mathcal{H}A$ in the norm–topology.

By construction, $\overline{\mathcal{H}A}$ is an Abelian $C^*$–algebra. Furthermore, it is equipped with the identity element: $T_{\tilde{\alpha}} \equiv [\alpha] K$, where $\alpha$ is the trivial (i.e. zero) loop. Therefore, we can directly apply the Gel’fand representation theory. Let us summarize the main results that follow [2]. First, we know that $\overline{\mathcal{H}A}$ is naturally isomorphic to the $C^*$ algebra of complex–valued, continuous functions on a compact, Hausdorff space, the spectrum $sp(\overline{\mathcal{H}A})$. Furthermore, one can show [5] that the space $A/G$ of connections modulo gauge transformations is densely embedded in $sp(\overline{\mathcal{H}A})$. Therefore, we can denote the spectrum as $A/G$. Second, every continuous, cyclic representation of $\overline{\mathcal{H}A}$ by bounded operators on a Hilbert space has the following form: The representation space is the Hilbert space $L^2(A/G, d\mu)$ for some regular measure $d\mu$ on $A/G$ and elements of $\overline{\mathcal{H}A}$, regarded as functions on $A/G$, act simply by (pointwise) multiplication. Finally, each of these representations is uniquely determined (via the Gel’fand–Naimark–Segal construction) by a continuous, positive linear functional $<,>$ on the $C^*$–
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The functionals $\langle . \rangle$ naturally define functions $\Gamma(\alpha)$ on the space $L^\infty$ of loops— which in turn serve as the “generating functionals” for the representation— via: $\Gamma(\alpha) \equiv \langle T_\alpha \rangle$. Thus, there is a canonical correspondence between regular measures on $A/G$ and generating functionals $\Gamma(\alpha)$.

The question naturally arises: can we write down necessary and sufficient conditions for a function on the loop space $L^\infty$ to qualify as a generating functional? Using the structure of the algebra $\mathcal{HA}$ outlined above, one can answer this question affirmatively:

A function $\Gamma(\alpha)$ on $L^\infty$ serves as a generating functional for the GNS construction if and only if it satisfies the following two conditions:

$$\sum_i a_i T_{\bar{\alpha}_i} = 0 \Rightarrow \sum_i a_i \Gamma(\alpha_i) = 0; \text{ and}$$

$$\sum_{i,j} \bar{a}_i a_j (\Gamma(\alpha_i \circ \alpha_j) + \Gamma(\alpha_i \circ \alpha_j^{-1}) \geq 0. \quad (2.9)$$

The first condition ensures that the functional $\langle . \rangle$ on $\mathcal{HA}$, obtained by extending $\Gamma(\alpha)$ by linearity, is well-defined while the second condition (by (2.6)) ensures that it satisfies the “positivity” property, $\langle B^* B \rangle \geq 0, \forall B \in \mathcal{HA}$. Thus, together, they provide a positive linear functional on the $*$-algebra $\mathcal{HA}$. Finally, since $\mathcal{HA}$ is dense in $\mathcal{HA}$ and contains the identity, it follows that the positive linear functional $\langle . \rangle$ extends to the full $C^*$-algebra $\mathcal{HA}$.

Thus, a loop function $\Gamma(\alpha)$ satisfying (2.9) determines a cyclic representation of $\mathcal{HA}$ and therefore a regular measure on $A/G$. Conversely, every regular measure on $A/G$ provides (through vacuum expectation values) a loop functional $\Gamma(\alpha) \equiv \langle T_\alpha \rangle$ satisfying (2.9). Note, finally, that the first equation in (2.9) ensures that the generating function factors through the hoop equivalence relation: $\Gamma(\alpha) \equiv \Gamma(\bar{\alpha})$.

We thus have an interesting interplay between connections and loops, or, more precisely, generalized gauge equivalent connections (elements of $A/G$) and hoops (elements of $H(G)$). The generators $T_\alpha$ of the holonomy algebra (i.e., configuration operators) are labelled by elements $\bar{\alpha}$ of $H(G)$. The elements of the representation space (i.e. quantum states) are $L^2$ functions on $A/G$. Regular measures $d\mu$ on $A/G$ are, in turn, determined by functions on $H(G)$ (satisfying (2.9)). The group of diffeomorphisms on $\Sigma$ has a natural action on the algebra $\mathcal{HA}$, its spectrum $A/G$, and the space of hoops, $H(G)$. The measure $d\mu$ is invariant under this induced action on $A/G$ iff $\Gamma(\bar{\alpha})$ is invariant under the induced action on $H(G)$. Now, a diffeomorphism invariant function of hoops is a function of generalized knots— generalized because our loops are allowed to have kinks, intersections
and segments that may be traced more than once. Hence, there is a 1–1 correspondence between generalized knot invariants (which in addition satisfy (2.9)) and regular diffeomorphism invariant measures on generalized gauge equivalent connections.

3 The spectrum

The constructions discussed in Section 2 have several appealing features. In particular, they open up an algebraic approach to the integration theory on the space of gauge equivalent connections and bring to forefront the duality between loops and connections. However, in practice, a good control on the precise content and structure of the Gel’fand spectrum $\mathcal{A}/\mathcal{G}$ of $\mathcal{H}\mathcal{A}$ is needed to make further progress. Even for simple algebras which arise in non–relativistic quantum mechanics—such as the algebra of almost periodic functions on $R^n$—the spectrum is rather complicated; while $R^n$ is densely embedded in the spectrum, one does not have as much control as one would like on the points that lie in its closure (see, e.g., [5]). In the case of a $C^*$–algebra of all continuous, bounded functions on a completely regular topological space, the spectrum is the Stone–Cech compactification of that space, whence the situation is again rather unruly. In the case of the $\Lambda$–I algebra $\mathcal{H}\mathcal{A}$, the situation seems to be even more complicated: Since the algebra $\mathcal{H}\mathcal{A}$ is generated only by certain functions on $\mathcal{A}/\mathcal{G}$, at least at first sight, $\mathcal{H}\mathcal{A}$ appears to be a proper sub–algebra of the $C^*$–algebra of all continuous functions on $\mathcal{A}/\mathcal{G}$, and therefore outside the range of applicability of standard theorems.

However, the holonomy $C^*$–algebra is also rather special: it is constructed from natural geometrical objects—connections and loops—on the 3–manifold $\Sigma$. Therefore, one might hope that its spectrum can be characterized through appropriate geometric constructions. We shall see in this section that this is indeed the case. The specific techniques we use rest heavily on the fact that the loops involved are all piecewise analytic.

This section is divided into three parts. In the first, we introduce the key tools that are needed (also in subsequent sections), in the second, we present the main result and in the third we give examples of “generalized gauge equivalent connections”, i.e., of elements of $\mathcal{A}/\mathcal{G} - \mathcal{A}/\mathcal{G}$. To keep the discussion simple, generally we will not explicitly distinguish between paths and loops and their images in $\Sigma$.

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6 If the gauge group were more general, we could not have expressed products of the generators $T_\alpha$ as sums (Eq. (2.6)). For $SU(3)$, for example, one can not get rid of double products; triple (and higher) products can be however reduced to linear combinations of double products of $T_\alpha$ and the $T_\alpha$ themselves. In this case, the generating functional is defined on single and double loops, whence, in addition to knot invariants, link invariants can also feature in the definition of the generating function.
3.1 Loop decomposition

A key technique that we will use in various constructions is the decomposition of a finite number of hoops, \( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k \), into a finite number of independent hoops, \( \tilde{\beta}_1, \ldots, \tilde{\beta}_n \). The main point here is that a given set of loops need not be “holonomically independent”: every open segment in a given loop may be shared by other loops in the given set, whence, for any connection \( A \) in \( \mathcal{A} \), the holonomies around these loops could be inter–related. However, using the fact that the loops are piecewise analytic, we will be able to show that any finite set of loops can be decomposed into a finite number of independent segments and this in turn leads us to the desired independent hoops. The availability of this decomposition will be used in the next sub–section to obtain a characterization of elements of \( \mathcal{A}/\mathcal{G} \) and in Section 4 to define “cylindrical” functions on \( \mathcal{A}/\mathcal{G} \).

Our aim then is to show that, given a finite number of hoops, \( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k \), (with \( \tilde{\alpha}_i \neq \text{identity} \) in \( \mathcal{HG} \) for any \( i \)), there exists a finite set of loops, \( \{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\} \subset \mathcal{L}_{x^0} \), which satisfies the following properties:

1. If we denote by \( \mathcal{HG}(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m) \) the subgroup of the hoop group \( \mathcal{HG} \) generated by the hoops \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_m \), then, we have
   \[ \mathcal{HG}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) \subset \mathcal{HG}(\tilde{\beta}_1, \ldots, \tilde{\beta}_n), \]
   where, as before, \( \tilde{\beta}_i \) denotes the hoop to which \( \beta_i \) belongs.

2. Each of the new loops \( \beta_i \) contains an open segment (i.e., an embedding of an interval) which is traced exactly once and which has at most a finite number of intersections
   intersects any of the remaining loops \( \beta_j \) (with \( i \neq j \)) at most at a finite number of points.

The first condition makes the sense in which each hoop \( \tilde{\alpha}_i \) can be decomposed in terms of \( \tilde{\beta}_j \) precise while the second condition specifies the sense in which the new hoops \( \tilde{\beta}_1, \ldots, \tilde{\beta}_n \) are independent.

Let us begin by choosing a loop \( \alpha_i \in \mathcal{L}_{x^0} \) in each hoop \( \tilde{\alpha}_i \), for all \( i = 1, \ldots, k \) such that none of the \( \tilde{\alpha}_i \) has a piece which is immediately retraced (i.e., none of the loops contain a path of the type \( \rho \cdot \rho^{-1} \)). Now, since the loops are piecewise analytic, any two which overlap do so either on a finite number of finite intervals and/or intersect in a finite number of points. Ignore the isolated intersection points and mark the end points of all the overlapping intervals (including the end points of self overlaps of a loop). Let us call these marked points vertices. Next, divide each loop \( \alpha_i \) into paths which join consecutive end points. Thus, each of these paths is piecewise–analytic, is part of at least one of the loops, \( \alpha_1, \ldots, \alpha_k \), and, has a non–zero (parameter) measure along any loop to which it belongs. Two distinct paths intersect at a finite set of points. Let us call these (oriented) paths edges. Denote by \( n \) the number of edges that result. The edges and
vertices form a (piecewise analytically embedded) graph. By construction, each loop in the list is contained in the graph and, ignoring parametrization, can be expressed as a composition of a finite number of its \( n \) edges. Finally, this decomposition is “minimal” in the sense that, if the initial set \( \alpha_1, ..., \alpha_k \) of loops is extended to include \( p \) additional loops, \( \alpha_{k+1}, ..., \alpha_{k+p} \), each edge in the original decomposition of \( k \) loops is expressible as a product of the edges which feature in the decomposition of the \( k + p \) loops.

The next step is to convert this edge–decomposition of loops into a decomposition in terms of elementary loops. This is easy to achieve. Connect each vertex \( v \) to the base point \( x_0 \) by oriented, piecewise analytic curve \( q(v) \) such that these curves overlap with any edge \( e_i \) at most at a finite number of isolated points. Consider the closed curves \( \beta_i \), starting and ending at the base point \( x_0 \),

\[
\beta_i := q(v_i^+ \circ e_i \circ (q(v_i^-)))^{-1},
\]

where \( v_i^\pm \) are the end points of the edge \( e_i \), and denote by \( S \) the set of these \( n \) loops. Loops \( \beta_i \) are not unique because of the freedom in choosing the curves \( q(v) \). However, we will show that they provide us with the required set of independent loops associated with the given set \( \{ \alpha_1, ..., \alpha_k \} \).

Let us first note that the decomposition satisfies certain conditions which follow immediately from the properties of the segments noted above:

1. \( S \) is a finite set and every \( \beta_i \in S \) is a non–trivial loop in the sense that the hoop \( \tilde{\beta}_i \) it defines is not the identity element of \( HG \);
2. every loop \( \beta_i \) contains an open interval which is traversed exactly once and no finite segment of which is shared by any other loop in \( S \);
3. every hoop \( \tilde{\alpha}_1, ..., \tilde{\alpha}_k \) in our initial set can be expressed as a finite composition of hoops defined by elements of \( S \) and their inverses (where a loop \( \beta_i \) and its inverse \( \beta_i^{-1} \) may occur more than once);

and,

4. if the initial set \( \alpha_1, ..., \alpha_k \) of loops is extended to include \( p \) additional loops, \( \alpha_{k+1}, ..., \alpha_{k+p} \), and if \( S' \) is the set of \( n' \) loops \( \beta'_1, ..., \beta'_{n'} \) corresponding to \( \alpha_1, ..., \alpha_{k+p} \), such that the paths \( q'(v') \) agree with the paths \( q(v) \) whenever \( v = v' \), then each hoop \( \tilde{\beta}_i \) is a finite product of the hoops \( \tilde{\beta}'_j \).

These properties will play an important role throughout the paper. For the moment we only note that we have achieved our goal: the property 2 above ensures that the set \( \{ \beta_1, ..., \beta_n \} \) of loops is independent in the sense we specified in the point 2 (just below Eq. (3.1)) and that 3 above implies that the hoop group generated by \( \alpha_1, ..., \alpha_k \) is contained in the hoop group generated by \( \beta_1, ..., \beta_n \), i.e. that our decomposition satisfies (3.1).

We will conclude this sub–section by showing that the independence of loops \( \{ \tilde{\beta}_1, ..., \tilde{\beta}_n \} \) has an interesting consequence which will be used repeatedly:
**Lemma 3.1** For every \((g_1, ..., g_n) \in [SU(2)]^n\), there exists a connection \(A_o \in A\) such that:

\[ H(\beta_i, A_o) = g_i \forall i = 1, ..., n \quad (3.3) \]

**Proof** Fix a sequence of elements \((g_1, ..., g_n)\) of \(SU(2)\), i.e., a point in \([SU(2)]^n\). Next, for each loop \(\beta_i\), pick a connection \(A'_{(i)}\) such that its support intersects the \(i\)-th segment \(s_i\) in a finite interval, and, if \(j \neq i\), intersects the \(j\)-th segment \(s_j\) only on a set of measure zero (as measured along any loop \(\alpha_m\) containing that segment). Consider the holonomy of \(A'_{(i)}\) around \(\beta_i\). Let us suppose that the connection is such that the holonomy \(H(\beta_i, A'_{(i)})\) is not the identity element of \(SU(2)\). Then, it is of the form:

\[ H(\beta_i, A'_{(i)}) = \exp w, \quad w \in su(2), \quad w \neq 0 \]

Moreover, let the connection be abelian; i.e. may be written as \(A'_{(i)} = wa'_{(i)}, a'_{(i)}\) being a real valued 1-form on \(\Sigma\). Now, consider a connection

\[ A_{(i)} = t g^{-1} \cdot A'_{(i)} \cdot g, \quad g \in SU(2), \quad t \in R \]

Then, we have:

\[ H(\beta_i, A_{(i)}) = \exp t(g^{-1}wg). \]

Hence, by choosing \(g\) and \(t\) appropriately, we can make \(H(\beta_i, A_{(i)})\) coincide with any element of \(SU(2)\), in particular \(g_i\). Because of the independence of the loops \(\beta_i\) noted above, we can choose connections \(A_{(i)}\) independently for every \(\beta_i\). Then, the connection

\[ A_o := A_{(1)} + ... + A_{(n)} \tag{3.4} \]

satisfies (3.3).

\[ \square \]

1. The result (3.3) we just proved can be taken as a statement of the independence of the \((SU(2))-\)hoops \(\tilde{\beta}_1, ..., \tilde{\beta}_n\); it captures the idea that the loops \(\tilde{\beta}_1, ..., \tilde{\beta}_n\) we constructed above are holonomically independent as far as \(SU(2)\) connections are concerned. This notion of independence is, however, weaker than the one contained in the statement 2 above. Indeed, that notion does not refer to connections at all and is therefore not tied to any specific gauge group. More precisely, we have the following. We just showed that if loops are independent in the sense of 2, they are necessarily independent in the sense of (3.3). The converse is not true. Let \(\alpha\) and \(\gamma\) be two loops in \(\mathcal{L}_x\) which do not share any point other than \(x^0\) and which have no self-intersections or self-overlaps. Set \(\beta_1 = \alpha\) and \(\beta_2 = \alpha \cdot \gamma\). Then, it is easy to check that \(\tilde{\beta}_i, \ i = 1, 2\) are independent in the sense of (3.3) although they are obviously not independent in the sense of 2. Similarly, given \(\alpha\), we can set \(\beta = \alpha^2\). Then, \(\{\beta\}\) is independent in the sense of (3.3) (since square-root is well-defined in \(SU(2)\)) but not
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in the sense of 2. From now on, we will say that loops satisfying 2 are strongly independent and those satisfying (3.3) are weakly independent. This definition extends naturally to hoops. Although we will not need them explicitly, it is instructive to spell out some algebraic consequences of these two notions. Algebraically, weak independence of hoops $\hat{\beta}_i$ ensures that $\hat{\beta}_i$ freely generate a sub-group of $H\mathcal{G}$. On the other hand, if $\hat{\beta}_i$ are strongly independent, then every homomorphism from $H\mathcal{G}(\hat{\beta}_1, ..., \hat{\beta}_n)$ to any group $G$ can be extended to every finitely generated subgroup of $H\mathcal{G}$ which contains $H\mathcal{G}(\hat{\beta}_1, ..., \hat{\beta}_n)$. Weak independence will suffice for all considerations in this section. However, to ensure that the measure defined in Section 4 is well-defined, we will need hoops which are strongly independent.

2. The availability of the loop decomposition sheds considerable light on the hoop equivalence: One can show that two loops in $L_{x_0}$ define the same ($SU(2)$)-hoop if and only if they differ by a combination of reparametrization and (immediate) retracings. This second characterization is useful because it involves only the loops; no mention is made of connections and holonomies. This characterization continues to hold if the gauge group is replaced by $SU(N)$ or indeed any group which has the property that, for every non-negative integer $n$, the group contains a subgroup which is freely generated by $n$ elements.

3.2 Characterization of $A/G$

We will now use the availability of this hoop decomposition to obtain a complete characterization of all elements of the spectrum $A/G$.

The characterization is based on results due to Giles [10]. Recall first that elements of the Gel’fand spectrum $A/G$ are in $1$–$1$ correspondence with (multiplicative, $*$–preserving) homomorphisms from $H\mathcal{A}$ to the $*$–algebra of complex numbers. Thus, in particular, every $\bar{A} \in A/G$ acts on $T_{\tilde{\alpha}} \in H\mathcal{A}$ and hence on any hoop $\tilde{\alpha}$ to produce a complex number $\bar{A}(\tilde{\alpha})$. Now, using certain constructions due to Giles, A–I [2] were able to show that:

Lemma 3.2 Every element $\bar{A}$ of $A/G$ defines a homomorphism $\hat{H}_A$ from the hoop group $H\mathcal{G}$ to $SU(2)$ such that, $\forall \tilde{\alpha} \in H\mathcal{G}$,

$$\bar{A}(\tilde{\alpha}) = \frac{1}{2} \text{Tr} \hat{H}_A(\tilde{\alpha}). \quad (3.5a)$$

They also exhibited the homomorphism explicitly. (Here, we have paraphrased the A–I result somewhat because they did not use the notion of hoops. However, the step involved is completely straightforward.)

Our aim now is to establish the converse of this result. Let us make a small digression to illustrate why the converse can not be entirely straightforward. Given a homomorphism $\hat{H}$ from $H\mathcal{G}$ to $SU(2)$ we can simply define $\bar{A}$ via: $\bar{A}(\tilde{\alpha}) = \frac{1}{2} \text{Tr} H(\tilde{\alpha})$. The question is whether this $\bar{A}$ can
qualify as an element of the spectrum. From the definition, it is clear that:
\[ |\bar{A}(\tilde{\alpha})| \leq \|T_{\tilde{\alpha}}\| \equiv \sup_{\tilde{A} \in \mathcal{A}/\mathcal{G}} |T_{\tilde{\alpha}}(\tilde{A})| \]
for all hoops \( \tilde{\alpha} \). On the other hand, to qualify as an element of the spectrum \( \mathcal{A}/\mathcal{G} \), the homomorphism \( \bar{A} \) must be continuous, i.e., should satisfy:
\[ |\bar{A}(f)| \leq \|f\| \]
for every \( f(\equiv \sum a_i[\tilde{\alpha}_i]_K) \in \mathcal{H}\mathcal{A} \). Since, a priori, (3.6b) appears to be stronger requirement than (3.6a), one would expect that there are more homomorphisms \( \bar{H} \) from the hoop group \( \mathcal{H}\mathcal{G} \) to \( SU(2) \) than the \( \bar{H}_A \), which arise from elements of \( \mathcal{A}/\mathcal{G} \). That is, one might expect that the converse of the A–I result would not be true. It turns out, however, that if the holonomy algebra is constructed from piecewise analytic loops, the apparently weaker condition (3.6a) is in fact equivalent to (3.6b), and the converse does hold. Whether it would continue to hold if one used piecewise smooth loops—as was the case in the A–I analysis—is not clear (see Appendix A).

We begin our detailed analysis with an immediate application of Lemma 3.1:

**Lemma 3.3** For every homomorphism \( \bar{H} \) from \( \mathcal{H}\mathcal{G} \) to \( SU(2) \), and every finite set of hoops \( \{\tilde{\alpha}_1, ..., \tilde{\alpha}_k\} \) there exists an \( SU(2) \) connection \( A_o \) such that for every \( \tilde{\alpha}_i \) in the set,
\[ \bar{H}(\tilde{\alpha}_i) = H(\alpha_i, A_o), \]
where, as before, \( H(\alpha_i, A_o) \) is the holonomy of the connection \( A_o \) around any loop \( \alpha \) in the hoop class \( \tilde{\alpha} \).

**Proof** Let \( (\beta_1, ..., \beta_n) \) be, as in Section 3.1, a set of independent loops corresponding to the given set of hoops. Denote by \( (g_1, ..., g_n) \) the image of \( (\beta_1, ..., \beta_n) \) under \( H \). Use the construction of Section 3.1 to obtain a connection \( A_o \) such that \( g_i = H(\beta_i, A_o) \) for all \( i \). It then follows (from the definition of the hoop group) that for any hoop \( \tilde{\gamma} \) in the sub–group \( \mathcal{H}\mathcal{G}(\tilde{\beta}_1, ..., \tilde{\beta}_n) \) generated by \( \tilde{\beta}_1, ..., \tilde{\beta}_n \), we have \( \bar{H}(\tilde{\gamma}) = H(\tilde{\gamma}, A_o) \). Since each of the given \( \tilde{\alpha}_i \) belongs to \( \mathcal{H}\mathcal{G}(\tilde{\beta}_1, ..., \tilde{\beta}_n) \), we have, in particular, (3.7).

We now use this lemma to prove the main result. Recall that each \( \bar{A} \in \mathcal{A}/\mathcal{G} \) is a homomorphism from \( \mathcal{H}\mathcal{A} \) to complex numbers and as before denote by \( \bar{A}(\tilde{\alpha}) \) the number assigned to \( T_{\tilde{\alpha}} \in \mathcal{H}\mathcal{A} \) by \( \bar{A} \). Then, we have:

**Lemma 3.4** Given a homomorphism \( \bar{H} \) from the hoop group \( \mathcal{H}\mathcal{G} \) to \( SU(2) \) there exists an element \( \bar{A}_{\bar{H}} \) of the spectrum \( \mathcal{A}/\mathcal{G} \) such that
\[ \bar{A}_{\bar{H}}(\tilde{\alpha}) = \frac{1}{2} \text{Tr} \bar{H}(\tilde{\alpha}). \]
Two homomorphisms \( \bar{H} \) and \( \bar{H}' \) define the same element of the spectrum
if and only if \( \hat{H}'(\hat{\alpha}) = g^{-1} \cdot \hat{H}(\hat{\alpha}) \cdot g \), \( \forall \hat{\alpha} \in \mathcal{H}_G \) for some (\( \hat{\alpha} \)-independent) \( g \) in \( SU(2) \).

**Proof** The idea is to define the required \( \hat{A}_H \) using (3.5b), i.e. to show that the right side of (3.5b) provides a homomorphism from \( \mathcal{H}\mathcal{A} \) to the \( * \)-algebra of complex numbers. Let us begin with the free complex vector space \( \mathcal{F}\mathcal{H}_G \) over the hoop group \( \mathcal{H}_G \) and define on it a complex–valued function \( h \) as follows: \( h(\sum a_i \hat{\alpha}_i) := \frac{1}{\varepsilon} \sum a_i \text{Tr} \hat{H}(\hat{\alpha}_i) \). We will first show that \( h \) passes through the \( K \)-equivalence relation of \( \mathcal{H} \)-I (noted in Section 2) and is therefore a well–defined complex–valued function on the holonomy \( * \)-algebra \( \mathcal{H}\mathcal{A} = \mathcal{F}\mathcal{H}_G / K \). Note first that, Lemma 3.3 immediately implies that, given a finite set of hoops, \( \{\hat{\alpha}_1, ..., \hat{\alpha}_k\} \), there exists a connection \( A_o \) such that,

\[
|h(\sum_{i=1}^k a_i \hat{\alpha}_i)| = |\sum_{i=1}^k a_i T_{\hat{\alpha}_i}(\hat{A}_o)| \leq \sup_{\hat{\alpha} \in \mathcal{A}/\mathcal{G}} |\sum_{i=1}^k a_i T_{\hat{\alpha}_i}(\hat{A})|.
\]

Therefore, we have:

\[
\sum_{i=1}^k a_i T_{\hat{\alpha}_i}(\hat{A}) = 0 \forall \hat{A} \in \mathcal{A}/\mathcal{G} \quad \Rightarrow \quad h(\sum_{i=1}^k a_i \hat{\alpha}_i) = 0.
\]

Since the left side of this implication defines the \( K \)-equivalence, it follows that the function \( h \) on \( \mathcal{F}\mathcal{H}_G \) has a well–defined projection to the holonomy \( * \)-algebra \( \mathcal{H}\mathcal{A} \). That \( h \) is linear is obvious from its definition. That it respects the \( * \)-operation and is a multiplicative homomorphism follows from the definitions (2.8) of these operations on \( \mathcal{H}\mathcal{A} \) (and the definition of the hoop group \( \mathcal{H}_G \)). Finally, the continuity of this function on \( \mathcal{H}\mathcal{A} \) is obvious from (3.8). Hence \( h \) extends uniquely to a homomorphism from the \( C^* \)-algebra \( \mathcal{H}\mathcal{A} \) to the \( * \)-algebra of complex numbers, i.e. defines a unique element \( \hat{A}_H \) via \( \hat{A}_H(B) = h(B) \), \( \forall B \in \mathcal{H}\mathcal{A} \).

Next, suppose \( \hat{H}_1 \) and \( \hat{H}_2 \) give rise to the same element of the spectrum. In particular, then, \( \text{Tr} \hat{H}_1(\hat{\alpha}) = \text{Tr} \hat{H}_2(\hat{\alpha}) \) for all hoops. Now, there is a general result: Given two homomorphisms \( \hat{H}_1 \) and \( \hat{H}_2 \) from a group \( G \) to \( SU(2) \) such that \( \text{Tr} \hat{H}_1(g) = \text{Tr} \hat{H}_2(g) \), \( \forall g \in G \), there exists \( g_o \in SU(2) \) such that \( \hat{H}_2(g) = g_o^{-1} \cdot \hat{H}_1(g) \cdot g_o \), where \( g_o \) is independent of \( g \). Using the hoop group \( \mathcal{H}_G \) for \( G \), we obtain the desired uniqueness result. \( \square \)

To summarize, by combining the results of the three lemmas, we obtain the following characterization of the points of the Gel’fand spectrum of the holonomy \( C^* \)-algebra:

**Theorem 3.5** Every point \( \hat{A}/\mathcal{G} \) of \( \mathcal{A}/\mathcal{G} \) gives rise to a homomorphism \( \hat{H} \) from the hoop group \( \mathcal{H}_G \) to \( SU(2) \) and every homomorphism \( \hat{H} \) defines a point of \( \mathcal{A}/\mathcal{G} \), such that \( \hat{A}(\hat{\alpha}) = \frac{1}{\varepsilon} \text{Tr} \hat{H}(\hat{\alpha}) \). This is a 1–1 correspondence modulo the trivial ambiguity that homomorphisms \( H \) and \( g^{-1} \cdot H \cdot g \) define
We conclude this sub-section with some remarks:

1. It is striking that the Theorem 3.5 does not require the homomorphism \( H \) to be continuous; indeed, no reference is made to the topology of the hoop group anywhere. This purely algebraic characterization of the elements of \( \mathcal{A}/\mathcal{G} \) makes it convenient to use it in practice.

2. Note that the homomorphism \( \hat{H} \) determines an element of \( \mathcal{A}/\mathcal{G} \) and not of \( \mathcal{A}/\mathcal{G} \); \( \hat{A}_H \) is not, in general, a smooth (gauge equivalent) connection. Nonetheless, as Lemma 3.3 tells us, it can be approximated by smooth connections in the sense that, given any finite number of hoops, one can construct a smooth connection which is indistinguishable from \( \hat{A}_H \) as far as these hoops are concerned. (This is stronger than the statement that \( \mathcal{A}/\mathcal{G} \) is dense in \( \overline{\mathcal{A}/\mathcal{G}} \).) Necessary and sufficient conditions for \( \hat{H} \) to arise from a smooth connection were given by Barrett [11] (see also [7]).

3. There are several folk theorems in the literature to the effect that given a function on the loop space \( \mathcal{L} \) satisfying certain conditions, one can reconstruct a connection (modulo gauge) such that the given function is the trace of the holonomy of that connection. (For a summary and references, see, e.g., [4].) Results obtained in this sub-section have a similar flavor. However, there is a key difference: Our main result shows the existence of a generalized connection (modulo gauge), i.e., an element of \( \overline{\mathcal{A}/\mathcal{G}} \) rather than a regular connection in \( \mathcal{A}/\mathcal{G} \). A generalized connection can also be given a geometrical meaning in terms of parallel transport, but in a generalized bundle [7].

### 3.3 Examples of \( \hat{A} \)

Fix a connection \( A \in \mathcal{A} \). Then, the holonomy \( H(\alpha, A) \) defines a homomorphism \( \hat{H}_A : \mathcal{H}\mathcal{G} \to SU(2) \). A gauge equivalent connection, \( A' = g^{-1}Ag + g^{-1}dg \), gives rise to the homomorphism \( \hat{H}_{A'} = g^{-1}(x^o)\hat{H}_Ag(x^o) \). Therefore, by Theorem 3.5, \( A \) and \( A' \) define the same element of the Gel'fand spectrum; \( \mathcal{A}/\mathcal{G} \) is naturally embedded in \( \overline{\mathcal{A}/\mathcal{G}} \). Furthermore, Lemma 3.3 now implies that the embedding is in fact dense. This provides an alternate proof of the A–I and Rendall[7] results quoted in Section 1. Had the gauge group been different and/or had \( \Sigma \) a dimension greater than 3, there would exist non–trivial \( G \)–principal bundles over \( \Sigma \). In this case, even if we begin with a specific bundle in our construction of the holonomy \( C^* \)–algebra, connections on all possible bundles belong to the Gel’fand spectrum (see Appendix B). This is an one illustration of the

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7 Note, however, that while this proof is tailored to the holonomy \( C^* \)–algebra \( \mathcal{H}\mathcal{A} \), Rendall’s [5] proof is applicable in more general contexts.
non-triviality of the completion procedure leading from $\mathcal{A}/\mathcal{G}$ to $\overline{\mathcal{A}/\mathcal{G}}$.

In this subsection, we will illustrate another facet of this non-triviality: We will give examples of elements of $\overline{\mathcal{A}/\mathcal{G}}$ which do not arise from $\mathcal{A}/\mathcal{G}$. These are the \textit{generalized} gauge equivalent connections $\bar{A}$ which have a “distributional character” in the sense that their support is restricted to sets of measure zero in $\Sigma$.

Recall first that the holonomy of the zero connection around any hoop is identity. Hence, the homomorphism $\hat{H}_\alpha$ from $\mathcal{H}\mathcal{G}$ to $SU(2)$ it defines sends every hoop $\tilde{\alpha}$ to the identity element $e$ of $SU(2)$ and the multiplicative homomorphism it defines from the $C^*$-algebra $\mathcal{H}\mathcal{A}$ to complex numbers sends each $T_{\tilde{\alpha}}$ to $1 = (1/2)Tr(e)$. We now use this information to introduce the notion of the support of a generalized connection. We will say that $\bar{A} \in \overline{\mathcal{A}/\mathcal{G}}$ has support on a subset $S$ of $\Sigma$ if for every hoop $\tilde{\alpha}$ which has at least one representative loop $\alpha$ which fails to pass through $S$, we have: i) $\hat{H}_\alpha(\tilde{\alpha}) = e$, the identity element of $SU(2)$; or, equivalently, ii) $\bar{A}(\tilde{\alpha}|_K) = 1$. (In ii), $\bar{A}$ is regarded as a homomorphism of $\ast$-algebras. While $\hat{H}_\alpha$ has an ambiguity, noted in Theorem 3.5, conditions i) and ii) are insensitive to it.)

We are now ready to give the simplest example of a generalized connection $\bar{A}$ which has support at a single point $x \in \Sigma$. Note first that, due to piecewise analyticity, if $\alpha \in \mathcal{L}_x$ passes through $x$, it does so only a finite number of times, say $N$. Let us denote the incoming and outgoing tangent directions to the curve at its $j$th passage through $x$ by $(v_j^-, v_j^+)$. The generalized connections we want to now define will depend only on the set of tangent directions at $x$. Let $\phi$ be a (not necessarily continuous) mapping from the 2-sphere $S^2$ of directions in the tangent space at $x$ to $SU(2)$. Set:

$$\Phi_j = \phi(-v_j^-)^{-1} \cdot \phi(v_j^+)$$

so that, if at the $j$-th passage, $\alpha$ arrives at $x$ and then simply returns by retracing its path in a neighborhood of $x$, we have $\Phi_j = e$, the identity element of $SU(2)$. Now, define $\hat{H}_\phi : \mathcal{H}\mathcal{G} \rightarrow SU(2)$ via:

$$\hat{H}_\phi(\tilde{\alpha}) := \begin{cases} e, & \text{if } \tilde{\alpha} \notin \mathcal{H}\mathcal{G}_x, \\ \Phi_N \ldots \Phi_1, & \text{otherwise} \end{cases} \quad (3.9)$$

where $\mathcal{H}\mathcal{G}_x$ is the space of hoops every loop in which passes through $x$. For each choice of $\phi$, the mapping $\hat{H}_\phi$ is well-defined, i.e., is independent of the choice of the loop $\alpha$ in the hoop class $\tilde{\alpha}$ used in its construction. (In particular, we used tangent directions rather than vectors to ensure this.) It defines a homomorphism from $\mathcal{H}\mathcal{G}$ to $SU(2)$ which is non-trivial only on $\mathcal{H}\mathcal{G}_x$, and hence a generalized connection with support at $x$. Furthermore, one can verify that (3.9) is the most general element of $\overline{\mathcal{A}/\mathcal{G}}$ which has support at $x$ and depends only on the tangent vectors at that point. Using analyticity of the loops, we can similarly construct elements which depend on the higher order behavior of loops at $x$. (There is no generalized connec-
tion with support at \( x \) which depends only on the zeroth order behavior of the curve, i.e., only on whether the curve passes through \( x \) or not.)

One can proceed in an analogous manner to produce generalized connections which have support on 1 or 2–dimensional sub–manifolds of \( \Sigma \). We conclude with an example. Fix a 2–dimensional, oriented, analytic sub–manifold \( M \) of \( \Sigma \) and an element \( g \) of \( SU(2) \), \(( g \neq e )\). Denote by \( \mathcal{H}G_M \) the subset of \( \mathcal{H}G \) consisting of hoops, each loop of which intersects \( M \) (non–tangentially, i.e., such that least one of the incoming or the outgoing tangent direction to the curve is transverse to \( M \).) As before, due to analyticity, any \( \alpha \in \mathcal{L}_x^\alpha \) can intersect \( M \) only a finite number of times. Denote as before the incoming and the outgoing tangent directions at the \( j \)–th intersection by \( v_j^– \) and \( v_j^+ \) respectively. Let \( \epsilon(\alpha_j^\pm) \) equal 1 if the orientation of \(( v_j^\pm, M )\) coincides with the orientation of \( \Sigma \); –1 if the two orientations are opposite; and 0 if \( v_j^\pm \) is tangential to \( M \). Define \( \hat{H}_{(g, M)} : \mathcal{H}G \rightarrow SU(2) \) via:

\[
\hat{H}_{(g, M)}(\tilde{\alpha}) := \begin{cases} 
   e, & \text{if } \tilde{\alpha} \notin \mathcal{H}G_M, \\
   g^{\epsilon(\alpha_j^–)} \cdot g^{\epsilon(\alpha_j^+)} \cdot g^{\epsilon(\alpha_1^–)} \cdot g^{\epsilon(\alpha_1^+)} \cdots \cdot g^{\epsilon(\alpha_{N}^–)} \cdot g^{\epsilon(\alpha_{N}^+)} , & \text{otherwise,}
\end{cases} \tag{3.10}
\]

where \( g^0 \equiv e \), the identity element of \( SU(2) \). Again, one can check that (3.10) is a well–defined homomorphism from \( \mathcal{H}G \) to \( SU(2) \), which is non–trivial only on \( \mathcal{H}G_M \) and therefore defines a generalized connection with support on \( M \). One can construct more sophisticated examples in which the homomorphism is sensitive to the higher order behavior of the loop at the points of intersection and/or the fixed \( SU(2) \) element \( g \) is replaced by \( g(x) : M \rightarrow SU(2) \).

4 Integration on \( \mathcal{A}/\mathcal{G} \)

In this section we shall develop a general strategy to perform integrals on \( \mathcal{A}/\mathcal{G} \) and introduce a natural, faithful, diffeomorphism invariant measure on this space. The existence of this measure provides considerable confidence in the the ideas involving the loop–transform and the loop representation introduced in [3,4].

The basic strategy is to carry out an appropriate generalization of the theory of cylindrical measures [12,13] on topological vector spaces [1]. The key idea in our approach is to replace the standard linear duality on vector spaces by the duality between connections and loops. In the first part of this section, we will define cylindrical functions and in the second part, we will present a natural cylindrical measure.

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\( ^*\)This idea was developed also by Baez [6] using, however, an approach which is somewhat different from the one presented here. Chronologically, the authors of this paper first found the faithful measure introduced in this section and reported this result in the first draft of this paper. Subsequently, Baez and the authors independently realized that the theory of cylindrical measures provides the appropriate conceptual setting for this discussion.
4.1 Cylindrical functions

Let us begin by recalling the situation in the linear case. Let $V$ denote a topological vector space and $V^*$ its dual. Given a finite dimensional subspace $S^*$ of $V^*$, we can define an equivalence relation on $V$: $v_1 \sim v_2$ if and only if $< v_1, s^* > = < v_2, s^* >$ for all $v_1, v_2 \in V$ and $s^* \in S^*$, where $< v, s^* >$ is the action of $s^* \in S^*$ on $v \in V$. Denote the quotient $V/\sim$ by $S$. Clearly, $S$ is a finite-dimensional vector space and we have a natural projection map $\pi(S^*): V \mapsto S$. A function $f$ on $V$ is said to be cylindrical if it is a pull-back via $\pi(S^*)$ to $V$ of a function $\tilde{f}$ on $S$, for some choice of $S^*$. These are the functions we will be able to integrate. Equip each finite-dimensional vector space $S$ with a measure $d\tilde{\mu}$ and set

$$\int_V d\mu f := \int_S d\tilde{\mu} \tilde{f}. \quad (4.1)$$

For this definition to be meaningful, however, the set of measures $d\tilde{\mu}$ that we chose on vector spaces $S$ must satisfy certain compatibility conditions. Thus, if a function $f$ on $V$ is expressible as a pull-back via $\pi(S^*)$ of functions $\tilde{f}_j$ on $S_j$, for $j = 1, 2$, say, we must have

$$\int_{S_1} d\tilde{\mu}_1 \tilde{f}_1 = \int_{S_2} d\tilde{\mu}_2 \tilde{f}_2. \quad (4.2)$$

Such compatible measures do exist. Perhaps the most familiar example is the (normalized) Gaussian measure.

We want to extend these ideas, using $\mathcal{A}/G$ in place of $V$. An immediate problem is that $\mathcal{A}/G$ is a genuinely non-linear space whence there is no obvious analog of $V^*$ and $S^*$ which are central to the above construction of cylindrical measures. The idea is to use the results of Section 3 to find suitable substitutes for these spaces. The appropriate choices turn out to be the following: we will let the hoop group $H \mathcal{G}$ to play the role of $V^*$ and its subgroups, generated by a finite number of independent hoops, the role of $S^*$. (Recall that, $S^*$ is generated by a finite number of linearly independent basis vectors.)

More precisely, we proceed as follows. From Theorem 3.5, we know that each $\bar{A}$ in $\mathcal{A}/G$ is completely characterized by the homomorphism $\bar{H}_A$ from the hoop group $H \mathcal{G}$ to $SU(2)$, which is unique up to the freedom $H_A \mapsto g_o^{-1} H_A g_o$ for some $g_o \in SU(2)$. Now suppose we are given $n$ loops, $\beta_1, \ldots, \beta_n$, which are (strongly) independent in the sense of Section 3. Denote by $S^*$ the sub-group of the hoop group $H \mathcal{G}$ they generate. Using this $S^*$, let us introduce the following equivalence relation on $\mathcal{A}/G$: $\bar{A}_1 \sim \bar{A}_2$ if $\bar{H}_{\bar{A}_1}(\bar{\gamma}) = g_o^{-1} \bar{H}_{\bar{A}_2}(\bar{\gamma}) g_o$ for all $\bar{\gamma} \in S^*$ and some (hoop independent) $g_o \in Su(2)$. Denote by $\pi(S^*)$ the projection from $\mathcal{A}/G$ onto the quotient space $[\mathcal{A}/G]/\sim$. The idea is to consider functions $f$ on $\mathcal{A}/G$ which are pull-backs under $\pi(S^*)$ of functions $\tilde{f}$ on $[\mathcal{A}/G]/\sim$ and define the
integral of $f$ on $\mathcal{A}/G$ to be equal to the integral of $\tilde{f}$ on $[\mathcal{A}/G]/\sim$. However, for this strategy to work, $[\mathcal{A}/G]/\sim$ should have a simple structure that one can control. Fortunately, this is the case:

**Lemma 4.1** $[\mathcal{A}/G]/\sim$ is isomorphic with $[SU(2)]^n/Ad$.

**Proof** By definition of the equivalence relation $\sim$, every $\{A\}$ in $[\mathcal{A}/G]/\sim$ is in 1–1 correspondence with the restriction to $S^\star$ of an equivalence class $\{H_A\}$ of homomorphisms from the full hoop group $\mathcal{H}G$ to $SU(2)$. Now, it follows from Lemma 3.3 that every homomorphism from $S^\star$ to $SU(2)$ extends to a homomorphism from the full hoop group to $SU(2)$. Therefore, there is a 1–1 correspondence between equivalence classes $\{A\}$ and equivalence class of homomorphisms from $S^\star$ to $SU(2)$ (where, as usual, two are equivalent if they differ by the adjoint action of $SU(2)$). Next, since $S^\star$ is generated by the $n$ hoops $\tilde{\beta}_1,\ldots,\tilde{\beta}_n$, every homomorphism from $S^\star$ to $SU(2)$ is completely determined by its action on the $n$ generators. Hence, $\{A\}$ is in 1–1 correspondence with the equivalence class of $n$–tuples, $\{\hat{H}_A(\tilde{\beta}_1),\ldots,\hat{H}_A(\tilde{\beta}_n)\} = \{g_\circ^{-1}\hat{H}_A(\tilde{\beta}_1)\ g_\circ,\ldots,g_\circ^{-1}\hat{H}_A(\tilde{\beta}_n)\ g_\circ\}$, which in turn defines a point of $[SU(2)]^n/Ad$. Thus, we have an injective map from $[\mathcal{A}/G]/\sim$ to $[SU(2)]^n/Ad$. Finally, it follows from Lemma 3.3 that the map is also surjective. Thus $[\mathcal{A}/G]/\sim$ is isomorphic with $[SU(2)]^n/Ad$. □

Some remarks about this result are in order:

1. Although we began with (strongly) independent loops $\{\beta_1,\ldots,\beta_n\}$, the equivalence relation we introduced makes direct reference only to the subgroup $S^\star$ of the hoop group they generate. This is similar to the situation in the linear case where one may begin with a set of linearly independent elements $s^\star_1,\ldots,s^\star_n$, let them generate a subspace $S^\star$ and then introduce the equivalence relation on $S$ using $S^\star$ directly. There is, however, a difference. In the linear case, $S^\star$ admits other bases and we should make sure that our subsequent constructions refer only to $S^\star$ and not to any specific basis therein. In the case under consideration, the situation is simpler: it follows from the notion of (strong) independence of loops that the only set of independent hoops which generates (precisely) $S^\star$ is the set $\{\tilde{\beta}_1,\ldots,\tilde{\beta}_n\}$ we began with.

2. An important consequence of this remark is that, given the sub–group $S^\star$ of the hoop group $\mathcal{H}G$, the isomorphism between $[\mathcal{A}/G]/\sim$ and $[SU(2)]^n/Ad$ is natural. That is, although we have used the independent generators $\tilde{\beta}_1,\ldots,\tilde{\beta}_n$ in Lemma 4.1, since these generators are canonical, we do not have to worry about how the identification between points of $[\mathcal{A}/G]/\sim$ and $[SU(2)]^n/Ad$ would change if we change the generators. Consequently, the projection map $\pi(S^\star)$ can now be taken to map $\mathcal{A}/G$ to $[SU(2)]^n/Ad$. In the case of topological vector spaces, by contrast, one does have to worry about this issue.
In that case, it is only when a basis is specified in $S^*$ that we can identify points in $V/\sim$ with vectors in $\mathbb{R}^n$; if the basis is rotated, the identification changes accordingly. Consequently, the projection map from $V$ to $n$–dimensional vector spaces also depends on the choice of basis. Although the space of cylindrical functions is unaffected by this ambiguity, care is needed in, e.g., the specification of the measure. Specifically, one must make sure that the values of integrals of cylindrical functions are basis–independent.

3. The equivalence relation $\sim$ of Lemma 4.1 says that two generalized connections are equivalent if their actions on the finitely generated subgroup $S^*$ of the hoop group are indistinguishable. It follows from Lemma 3.3 that each equivalence class $\{\bar{A}\}$ contains a regular connection $A$. Finally, if $A$ and $A'$ are two regular connections whose pull–backs to all $\beta_i$ are the same for $i = 1,\ldots,n$, then they are equivalent (where $\beta_i$ is any loop in the hoop class $\tilde{\beta}_i$). Thus, in effect, the projection $\pi(S^*)$ maps the domain space $A/G$ of the continuum quantum theory to the domain space of a lattice gauge theory where the lattice is formed by the $n$ loops $\beta_1,\ldots,\beta_n$. The initial choice of the independent hoops is, however, arbitrary and it is this arbitrariness that is responsible for the richness of the continuum theory.

We can now define cylindrical functions on $A/G$. As remarked already, one can regard $\pi(S^*)$ as a projection map from $\overline{A/G}$ onto $[SU(2)]^n/Ad$. By a cylindrical function $f$ on $\overline{A/G}$, we shall mean the pull–back to $\overline{A/G}$ of a continuous function $\tilde{f}$ on $[SU(2)]^n/Ad$ under $\pi(S^*)$ for some finitely generated subgroup $S^*$ of the hoop group. These are the functions we want to integrate. Let us note an elementary property which will be used repeatedly. Let $S^*$ be generated by $n$ strongly independent hoops $\tilde{\beta}_1,\ldots,\tilde{\beta}_n$ and $(S^*)'$ by $n'$ strongly independent hoops $\tilde{\beta}_1',\ldots,\tilde{\beta}_n'$ such that $(S^*)' \subseteq (S^*)$. (Thus, $n' \leq n$.) Then, it is easy to check that if a function $f$ on $\overline{A/G}$ is cylindrical with respect to $(S^*)'$, it is also cylindrical with respect to $S^*$. Furthermore, there is now a natural projection from $[SU(2)]^n$ on to $[SU(2)]^{n'}$ and $f$ is the pull–back of $(f)'$ via this projection.

In the remainder of this sub–section, we will analyse the structure of the space $C$ of cylindrical functions. First we have:

**Lemma 4.2** $C$ has the structure of a $\star$–algebra.

**Proof** It is clear by definition of $C$ that if $f \in C$, then any complex multiple $\lambda f$ as well as the complex–conjugate $\bar{f}$ also belongs to $C$. Next, given two elements, $f_i$ with $i = 1, 2$, of $C$, we will show that their sum and their product also belong to $C$. Let $S_i^*$ be the finitely generated subgroups of the hoop group $HG$ such that $f_i$ are the pull–backs to $\overline{A/G}$ of continuous functions $\tilde{f_i}$ on $[SU(2)]^n/Ad$. Consider the set of $n_1 + n_2$ independent hoops generating $S_1^*$ and $S_2^*$. While the first $n_1$ and the last $n_2$ hoops in
this set are independent, there may be relations between hoops belonging to the first set and those belonging to the second. Using the technique of Section 3.1, find the set of independent hoops, say, \(\beta_1, \ldots, \beta_n\) which generate the given \(n_1 + n_2\) hoops and denote by \(S^*\) the subgroup of the hoop group generated by \(\hat{\beta}_1, \ldots, \hat{\beta}_n\). Then, since \(S^*_1 \subseteq S^*\), it follows that \(f_i\) are cylindrical also with respect to \(S^*_1\). Therefore, \(f_1 + f_2\) and \(f_1 \cdot f_2\) are also cylindrical with respect to \(S^*\). Thus \(C^*\) has the structure of a \(\star\)-algebra.

Now, since \([SU(2)]^n/\text{Ad}\) is compact for any \(n\), and since elements of \(C^*\) are pull–backs to \(\mathcal{A}/G\) of all continuous functions on \([SU(2)]^n/\text{Ad}\), it follows that one can use the \(\sup\)–norm to endow \(C^*\) the structure of a \(C^*\)-algebra.

A natural class of functions one would like to integrate on \(\mathcal{A}/G\) is given by the (Gel'fand transforms of the) traces of holonomies. The question then is if these functions are contained in \(C^*\). Not only is the answer affirmative, but in fact the traces of holonomies generate all of \(C^*\). More precisely, we have:

**Theorem 4.3** The completion \(\overline{C^*}\) of \(C^*\) is isomorphic to the \(C^*\)-algebra \(\overline{\mathcal{A}}\).

**Proof** Let us begin by showing that \(\overline{\mathcal{A}}\) is embedded in \(\overline{C^*}\). Given an element \(F\) with \(F(A) = \sum a_i T_{\alpha_i}(A)\) of \(\overline{\mathcal{A}}\), its Gel'fand transform is a function \(f\) on the spectrum \(\mathcal{A}/G\), given by \(f(\hat{A}) = \sum a_i \hat{A}(\hat{\alpha}_i)\). Consider the set of hoops \(\hat{\alpha}_1, \ldots, \hat{\alpha}_k\) that feature in the sum, decompose them into independent hoops \(\hat{\beta}_1, \ldots, \hat{\beta}_n\) as in Section 3.1 and denote by \(S^*\) the subgroup of the hoop group they generate. Then, it is clear that the function \(f\) is the pull–back via \(\pi(S^*)\) of a continuous function \(\tilde{f}\) on \([SU(2)]^n/\text{Ad}\). Hence we have \(f \in C^*\). Finally, since the norms of the functions \(f\) and \(\tilde{f}\) are given by

\[
||f|| = \sup_{A \in \mathcal{A}/G} |f(\hat{A})| \quad \text{and} \quad ||\tilde{f}|| = \sup_{\tilde{a} \in [SU(2)]^n/\text{Ad}} |\tilde{f}(\tilde{a})|,
\]

it is clear that the map from \(f\) to \(\tilde{f}\) is isometric. The first of these norms features in the definition of the (Gel'fand transform of the) \(C^*\)-algebra \(\overline{\mathcal{A}}\) while the second features in the \(C^*\)-algebra \(\overline{C^*}\). Finally, since \(\overline{\mathcal{A}}\) is obtained by a \(C^*\)-completion of the space of functions of the form \(F\), we have the result that \(\overline{\mathcal{A}}\) is embedded in the \(C^*\)-algebra \(\overline{C^*}\).

Next, we will show that there is also an inclusion in the opposite direction. We first note a fact about \(SU(2)\). Fix \(n\)-elements \((g_1, \ldots, g_n)\) of \(SU(2)\). Then, from the knowledge of traces (in the fundamental representation) of all elements which belong to the group generated by these \(g_i, i = 1, \ldots, n\), we can reconstruct \((g_1, \ldots, g_n)\) modulo an adjoint map. It follows from this fact that the space of functions \(\tilde{f}\) on \([SU(2)]^n/\text{Ad}\) which is the image under the projection \(\pi(S^*)\) of \(\overline{\mathcal{A}}\) suffices to separate points of
\[ [SU(2)]^n/Ad. \] Since this space is compact, it follows (from the Weirstrass theorem) that the \( C^* \)-algebra generated by these projected functions is the entire \( C^* \)-algebra of continuous functions on \([SU(2)]^n/Ad\). Now, suppose we are given a cylindrical function \( f' \) on \( A/G \). Its projection under the appropriate \( \pi(S^*) \) is, by the preceding result, contained in the projection of some element of \( \mathcal{HA} \). Thus, \( C \) is contained in \( \mathcal{HA} \).

Combining the two results, we have the desired result: the \( C^* \) algebras \( \mathcal{HA} \) and \( \mathcal{C} \) are isomorphic. \( \square \)

**Remark** Theorem 4.3 suggests that from the very beginning we could have introduced the \( C^* \)-algebra \( \mathcal{C} \) in place of \( \mathcal{HA} \). This is indeed the case. More precisely, we could have proceeded as follows. Begin with the space \( A/G \), introduce the notion of hoops, and hoop decomposition in terms of independent hoops. Then given a subgroup \( S^* \) generated by \( n \) independent hoops, we could have introduced an equivalence relation on \( A/G \) (not \( A/G \) which we do not yet have!) as follows: \( A_1 \sim A_2 \) iff \( H(\tilde{\gamma}, A_1) = g^{-1}H(\tilde{\gamma}, A_2)g \) for all \( \tilde{\gamma} \in S^* \) and some (hoop independent) \( g \in SU(2) \). It is then again true (due to Lemma 3.3) that the quotient \( [A/G]/\sim \) is isomorphic to \( [SU(2)]^n/Ad. \) (This is true inspite of the fact that we are using \( A/G \) rather than \( A/G \) because, as remarked immediately after Lemma 4.1, each \( \{A\} \in [A/G]/\sim \) contains a regular connection.) We can therefore define cylindrical functions, but now on \( A/G \) as the pull-backs of continuous functions on \([SU(2)]^n/Ad\). These functions have a natural \( C^* \)-algebra structure. We can use it as the starting point in place of the A–I holonomy algebra. While this strategy seems natural at first from an analysis viewpoint, it has two drawbacks, both stemming from the fact that the Wilson loops are now assigned a secondary role. For physical applications, this is unsatisfactory since Wilson loops are the basic observables of gauge theories. From a mathematical viewpoint, the relation between knot/link invariants and measures on the spectrum of the algebra would now be obscure. Nonetheless, it is good to keep in mind that this alternate strategy is available as it may make other issues more transparent.

### 4.2 A natural measure

In this sub-section, we will discuss the issue of integration of cylindrical functions on \( A/G \). Our main objective is to introduce a natural, faithful, diffeomorphism invariant, cylindrical measure on \( A/G \).

The idea is similar to the one used in the case of topological vector spaces discussed in the beginning of the previous sub-section. Thus, for each \( n \), we wish to equip the spaces \([SU(2)]^n/Ad\) with a measure \( d\tilde{\mu}(n) \) and, as in (4.1), to set

\[
\int_{A/G} d\mu f := \int_{[SU(2)]^n/Ad} d\tilde{\mu}(n) \tilde{f}.
\] (4.3)
It is clear that for the integral to be well-defined, the set of measures we choose on \([SU(2)]^n/Ad\) should be compatible so that the analog of (4.2) holds. We will now exhibit a natural choice for which the compatibility is automatically satisfied. Denote by \(d\mu\) the normalized Haar measure on \(SU(2)\). It naturally induces a measure on \([SU(2)]^n/Ad\) which is invariant under the adjoint action of \(SU(2)\) and therefore projects down unambiguously to \([SU(2)]^n/Ad\). This is our choice of \(d\tilde{\mu}(n)\). This is a natural choice since \(SU(2)\) comes equipped with the Haar measure. In particular, we have not introduced any additional structure on the underlying 3–manifold \(\Sigma\) (on which the initial connections \(A\) are defined); hence the resulting cylindrical measure will be automatically invariant under the action of \(Diff(\Sigma)\).

We will first show that the measures \(d\tilde{\mu}(n)\) satisfy the appropriate compatibility conditions and then explore the properties of the resulting cylindrical measure on \(\mathcal{A}/\mathcal{G}\). For simplicity of presentation, in what follows we shall not draw a distinction between functions on \([SU(2)]^n/Ad\) and those on \([SU(2)]^n\) which are invariant under the adjoint \(SU(2)\)–action. Thus, instead of carrying our integrals over \([SU(2)]^n/Ad\) of functions thereon, we will often integrate their lifts to \([SU(2)]^n\).

**Theorem 4.4**

a) If \(f\) is a cylindrical function on \(\mathcal{A}/\mathcal{G}\) with respect to two different finitely generated subgroups \(S^*_i\), \(i = 1, 2\) of the hoop group, and \(\tilde{f}_i\), its projections to \([SU(2)]^n/Ad\), then we have:

\[
\int_{[SU(2)]^n_1/Ad} d\tilde{\mu}(n_1) \tilde{f}_1 = \int_{[SU(2)]^n_2/Ad} d\tilde{\mu}(n_2) \tilde{f}_2; \quad (4.4)
\]

b) The functional \(v: \mathcal{H}\mathcal{A} \rightarrow C\) on the \(A\)–\(C^*\)-algebra \(\mathcal{H}\mathcal{A}\) defined below is linear, strictly positive and \(Diff(\Sigma)\)-invariant:

\[
v(F) = \int_{\mathcal{A}/\mathcal{G}} d\mu f , \quad (4.5)
\]

where the cylindrical function \(f\) on \(\mathcal{A}/\mathcal{G}\) is the Gel'fand transform the element \(F\) of \(\mathcal{H}\mathcal{A}\); and
c) the cylindrical ‘measure’ \(d\mu\) defined by (4.3) is a genuine, regular and strictly positive measure on \(\mathcal{A}/\mathcal{G}\); \(d\mu\) is \(Diff(\Sigma)\) invariant.

**Proof**: As in the proof of Theorem 4.3, let us first consider the \(n_1\) generators of \(S^*_1\) and \(n_2\) generators of \(S^*_2\) and use the construction of Section 3.1 to carry out the decomposition of these \(n_1 + n_2\) hoops in terms of independent hoops, say \(\tilde{\beta}_1,...,\tilde{\beta}_n\). Thus, the original \(n_1 + n_2\) hoops are contained in the group \(S^*\) generated by the \(\tilde{\beta}_i\). (Note that \(n_i < n\) in the non–trivial case when the original subgroups \(S^*_1\) and \(S^*_2\) are distinct.) Clearly, \(S^*_i \subset S^*\). Hence, the given \(f\) is cylindrical also with respect to \(S^*\), i.e., is the pull–back of a function \(\tilde{f}\) on \([SU(2)]^n/Ad\). We will now establish (4.4) by showing that each of the two integrals in that equation equals the
analogous integral of $\tilde{f}$ over $[SU(2)]^n/\text{Ad}$.

To deal with quantities with suffixes 1 and 2 simultaneously, for notational simplicity we will let a prime stand for either the suffix 1 or 2. Then, in particular, we have finitely generated subgroups $S^*$ and $(S^*)'$ of the hoop group, with $(S^*)' \subset S^*$, and a function $f$ which is cylindrical with respect to both of them. The generators of $S^*$ are $\beta_1, ..., \beta_n$ while those of $(S^*)'$ are $\beta'_1, ..., \beta'_{n'}$ (with $n' < n$). From the construction of $S^*$ (Section 3.1) it follows that every hoop in the second set can be expressed as a composition of the hoops in the first set and their inverses. Furthermore, since the hoops in each set are strongly independent, the construction implies that for each $i' \in \{1, ..., n'\}$, there exists $K(i') \in \{1, ..., n\}$ with the following properties: i) if $i' \neq j'$, $K(i') \neq K(j')$; and, ii) in the decomposition of $\beta'_{i'}$ in terms of unprimed hoops, the hoop $\tilde{\beta}_{K(i')}$ (or its inverse) appears exactly once and if $i' \neq j'$, the hoop $\tilde{\beta}_{K(j')}$ does not appear at all. For simplicity, let us assume that the orientation of the primed hoops is such that it is $\beta_{K(i')}$ rather than its inverse that features in the decomposition of $\beta'_{i'}$. Next, let us denote by $(g_1, ..., g_n)$ a point of the quotient $[SU(2)]^n$ which is projected to $(g'_1, ..., g'_{n'})$ in $[SU(2)]^{n'}$. Then $g'_{i'}$ is expressed in terms of $g_i$ as $g'_{i'} = ..g_{K(i')}..$, where .. denotes a product of elements of the type $g_m$ where $m \neq K(j)$ for any $j$. Since the given function $f$ on $A/G$ is a pull–back of a function $\tilde{f}$ on $[SU(2)]^n$ as well as of a function $\tilde{f}'$ on $[SU(2)]^{n'}$, it follows that:

$$\int f(g_1, ..., g_n) = \int \tilde{f}(g'_1, ..., g'_{n'}) = \int \tilde{f}'(..g_{K(1)}.., ..., ..g_{K(n')}..).$$

Hence,

$$\int d\tilde{\mu}_1 \cdots d\tilde{\mu}_n \tilde{f}(g_1, ..., g_n) = \int \prod_{m \neq K(1) \cdots K(n')} d\tilde{\mu}_m \left( \int d\tilde{\mu}_{K(1)} \cdots \int d\tilde{\mu}_{K(n')} \tilde{f}'(..g_{K(1)}.., ..., ..g_{K(n')}..) \right)$$

$$= \int \prod_{m = n' + 1}^n d\tilde{\mu}_m \left( \int d\tilde{\mu}_1 \cdots \int d\tilde{\mu}_{n'} \tilde{f}'(g_1, ..., g_{n'}) \right)$$

$$= \int d\tilde{\mu}_1 \cdots d\tilde{\mu}_{n'} \tilde{f}'(g_1, ..., g_{n'}),$$

where, in the first step, we simply rearranged the order of integration, in the second step we used the invariance property of the Haar measure and simplified notation for dummy variables being integrated, and, in the third step, we have used the fact that, since the measure is normalized, the integral of a constant produces just that constant\footnote{In the simplest case, with $n = 2$ and $n' = 1$, we would for example have $f(g_1, g_2) =$}. This establishes the
required result (4.4).

Next, consider the “vacuum expectation value functional” $v$ on $\mathcal{H}A$ defined by (4.5). The continuity and linearity of $v$ follow directly from its definition. That it is positive, i.e., satisfies $v(F^* F) \geq 0$ is equally obvious. In fact, a stronger result holds: if $F \neq 0$, then $v(F^* F) > 0$ since the value $v(F^* F)$ of $v$ on $F^* F$ is obtained by integrating a non-negative function $(\tilde{f})^* \tilde{f}$ on $[SU(2)]^n/\text{Ad}$ with respect to a regular measure. Thus, $v$ is strictly positive.

To see that we have a regular measure on $A/G$, recall first that the $C^*$-algebra of cylindrical functions is naturally isomorphic with the Gel’fand transform of the $C^*$-algebra $\mathcal{H}A$, which, in turn is the $C^*$-algebra of all continuous functions on the compact, Hausdorff space $A/G$. The “vacuum expectation value function” $v$ is therefore a continuous linear functional on the algebra of all continuous functions on $A/G$ equipped with the $L_\infty$ (i.e., sup-)norm. Hence, by the Reisz–Markov theorem, it follows that $v(f) \equiv \int d\mu f$ is in fact the integral of the continuous function $f$ on $A/G$ with respect to a regular measure.

Finally, since our construction did not require the introduction of any extra structure on $\Sigma$ (such as a metric or a volume element), it follows that the functional $v$ and the measure $d\mu$ are $\text{Diff}(\Sigma)$ invariant.

We will refer to $d\mu$ as the “induced Haar measure” on $A/G$. We now explore some properties of this measure and of the resulting representation of the holonomy $C^*$-algebra $\mathcal{H}A$.

First, the representation of $\mathcal{H}A$ can be constructed as follows: The Hilbert space is $L^2(A/G, d\mu)$ and the elements $F$ of $\mathcal{H}A$ act by multiplication so that we have $F \Psi = f \cdot \Psi$ for all $\Psi \in L^2(A/G, d\mu)$, where $f \in \mathcal{C}$ is the Gel’fand transform of $F$. This representation is cyclic and the function $\Psi_0$ defined by $\Psi_0(A) = 1$ on $A/G$ is the “vacuum” state. Every generator $T_\alpha$ of the holonomy $C^*$-algebra is represented by a bounded, self-adjoint operator on $L^2(A/G, d\mu)$.

Second, it follows from the strictness of the positivity of the measure that the resulting representation of the $C^*$-algebra $\mathcal{H}A$ is faithful. To our knowledge, this is the first faithful representation of the holonomy $C^*$-algebra that has been constructed explicitly. Finally, the diffeomorphism group $\text{Diff}(\Sigma)$ of $\Sigma$ has an induced action on the $C^*$-algebra $\mathcal{H}A$ and hence also on its spectrum $A/G$. Since the measure $d\mu$ on $A/G$ is invariant under this action, the Hilbert space $L^2(A/G, d\mu)$ carries a unitary representation of $\text{Diff}(\Sigma)$. Under this action, the “vacuum state” $\Psi_0$ is left invariant.

Third, restricting the values of $v$ to the generators $T_\alpha \equiv [\alpha]_K$ of $\mathcal{H}A$, we obtain the generating functional $\Gamma(\alpha)$ of this representation: $\Gamma(\alpha) := f'(g_1 \cdot g_2) \equiv f'(g')$. Then $\int d\mu(g_1) \int d\mu(g_2) f(g_1, g_2) = \int d\mu(g_1) \int d\mu(g_2) \tilde{f}(g_1 \cdot g_2) = \int d\mu(g') \tilde{f}(g')$, where in the second step we have used the invariance and normalization properties of the Haar measure.
\[ \int d\mu \bar{A}(\tilde{a}) \]. This functional on the loop space \( \mathcal{L}_{x^o} \) is diffeomorphism invariant since the measure \( d\mu \) is. It thus defined a generalized knot invariant. We will see below that, unlike the standard invariants, \( \Gamma(\alpha) \) vanishes on all smoothly embedded loops. It’s action is non–trivial only when the loop is re–traced, i.e. only on generalized loops.

Let us conclude this section by indicating how one computes the integral in practice. Fix a loop \( \alpha \in \mathcal{L}_{x^o} \) and let

\( \alpha = [\beta_1]^{\epsilon_1} \cdots [\beta_n]^{\epsilon_n} \quad (4.6) \)

be the minimal decomposition of \( \alpha \) into (strongly) independent loops \( \beta_1, \ldots, \beta_n \) as in Section 3.1, where \( \epsilon_k \in \{-1,1\} \) keeps track of the orientation and \( i_k \in \{1, \ldots, n\} \). (We have used the double suffix \( i_k \) because any one independent loop, \( \beta_1, \ldots, \beta_n \) can arise more than once in the decomposition. Thus, \( N \geq n \).) The loop \( \alpha \) defines an element \( T_{\tilde{a}} \) of the holonomy \( C^\ast \)-algebra \( \overline{\mathcal{H}A} \) whose Gel’fand transform \( \tilde{t}_{\tilde{a}} \) is a function on \( \mathcal{A}/G \). This is the cylindrical function we wish to integrate. Now, the projection map of Lemma 4.1 assigns to \( \tilde{t}_{\tilde{a}} \) the function \( \tilde{t}_{\tilde{a}}(g_1, \ldots, g_n) := \text{Tr} (g_1^{\epsilon_1} \cdots g_n^{\epsilon_N}) \).

Thus, to integrate \( \tilde{t}_{\tilde{a}} \) on \( \mathcal{A}/G \), we need to integrate this \( \tilde{t}_{\tilde{a}} \) on \( [SU(2)]^n/Ad \). Now, there exist in the literature useful formulas for integrating polynomials of the “matrix element functions” over \( SU(2) \) (see, e.g., [14] ). These can be now used to evaluate the required integrals on \( [SU(2)]^n/Ad \). In particular, one can readily show the following result:

\[ \int_{\mathcal{A}/G} d\mu \ t_{\tilde{a}} = 0, \]

unless every loop \( \beta_i \) is repeated an even number of times in the decomposition (4.6) of \( \alpha \).

5 Discussion

In this paper, we completed the A–I program in several directions. First, by exploiting the fact that the loops are piecewise analytic, we were able to obtain a complete characterization of the Gel’fand spectrum \( \mathcal{A}/G \) of the holonomy \( C^\ast \)-algebra \( \overline{\mathcal{H}A} \). A–I had shown that every element of \( \mathcal{A}/G \) defines a homomorphism from the hoop group \( \mathcal{H}G \) to \( SU(2) \). We found that every homomorphism from \( \mathcal{H}G \) to \( SU(2) \) defines an element of the spectrum and two homomorphisms \( \hat{H}_1 \) and \( \hat{H}_2 \) define the same element of \( \mathcal{A}/G \) only if they differ by an \( SU(2) \) automorphism, i.e., if and only if \( \hat{H}_2 = g^{-1} \cdot \hat{H}_1 \cdot g \) for some \( g \in SU(2) \). Since this characterization is rather simple and purely algebraic, it is useful in practice. The second main result also intertwines the structure of the hoop group with that of the Gel’fand spectrum. Given a subgroup \( S^\ast \) of \( \mathcal{H}G \) generated by a finite number, say
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For \( n \), of (strongly) independent hoops, we were able to define a canonical projection \( \pi(S^*) \) from \( \mathcal{A}/G \) to the compact space \( [SU(2)]^n/Ad \). This family of projections enables us to define cylindrical functions on \( \mathcal{A}/G \): these are the pull-backs to \( \mathcal{A}/G \) of the continuous functions on \( [SU(2)]^n/Ad \). We analysed the space \( C \) of these functions and found that it has the structure of a \( C^* \)-algebra which, moreover, is naturally isomorphic with the \( C^* \)-algebra of all continuous functions on the compact Hausdorff space \( \mathcal{A}/G \) with which we began. We then carried out a non-linear extension of the theory of cylindrical measures to integrate these cylindrical functions on \( \mathcal{A}/G \). Since \( C \) is the \( C^* \)-algebra of all continuous functions on \( \mathcal{A}/G \), the cylindrical measures are in fact regular measures on \( \mathcal{A}/G \). Finally, we were able to introduce explicitly such a measure on \( \mathcal{A}/G \) which is natural in the sense that it arises from the Haar measure on \( SU(2) \). The resulting representation of the holonomy \( C^* \)-algebra has several interesting properties. In particular, the representation is faithful and diffeomorphism invariant. Hence, its generating function \( \Gamma \)—which sends loops \( \alpha \) (based at \( x^o \)) to real numbers \( \Gamma(\alpha) \)—defines a generalized knot invariant (generalized because we have allowed loops to have kinks, self-intersections and self-overlaps). These constructions show that the A–I program can be realized in detail.

Having the induced Haar measure \( d\mu \) on \( \mathcal{A}/G \) at one’s disposal, we can now make the ideas on loop transform \( T \) of [3,4] rigorous. The transform \( T \) sends states in the connection representation to states in the so-called loop representation. In terms of machinery introduced in this paper, a state \( \Psi \) in the connection representation is a square-integrable function on \( \mathcal{A}/G \); thus \( \Psi \in L^2(\mathcal{A}/G, d\mu) \). The transform \( T \) sends it to a function \( \psi \) on the space \( \mathcal{H}_G \) of hoops. (Sometimes, it is convenient to lift \( \psi \) canonically to the space \( L_{x^o} \) and regard it as a function on the loop space. This is the origin of the term “loop representation.”) We have: \( T \circ \Psi = \psi \) with

\[
\psi(\tilde{\alpha}) = \int_{\mathcal{A}/G} d\mu \, \overline{t_{\tilde{\alpha}}(A)} \Psi(A) = \langle t_{\tilde{\alpha}}, \Psi \rangle,
\]

(5.1)

where, \( t_{\tilde{\alpha}} \) (with \( t_{\tilde{\alpha}}(A) = \overline{A(\tilde{\alpha})} \)) is the Gel’fand transform of the trace of the holonomy function, \( T_{\tilde{\alpha}} \), on \( \mathcal{A}/G \), associated with the hoop \( \tilde{\alpha} \), and \( \langle \ldots \rangle \) denotes the inner product on \( L^2(\mathcal{A}/G, d\mu) \). Thus, in the transform the traces of holonomies play the role of the “integral kernel.” Elements of the holonomy algebra have a natural action on the connection states \( \Psi \). Using \( T \), one can transform this action to the hoop states \( \psi \). It turns out that the action is surprisingly simple and can be represented directly in terms of elementary operations on the hoop space. Consider a generator \( T_{\tilde{\gamma}} \) of \( \mathcal{H}_A \).

In the connection representation, we have: \( (T_{\tilde{\gamma}} \circ \Psi)(A) = T_{\tilde{\gamma}}(A) \cdot \psi(A) \). On
the hoop states, this action translates to:

\[(T_\gamma \circ \psi)(\tilde{\alpha}) = \frac{1}{2}(\psi(\tilde{\alpha} \cdot \tilde{\gamma}) + \psi(\tilde{\alpha} \cdot \tilde{\gamma}^{-1})).\] (5.2)

where \(\tilde{\alpha} \cdot \tilde{\gamma}\) is the composition of hoops \(\tilde{\alpha}\) and \(\tilde{\gamma}\) in the hoop group. Thus, one can forgo the connection representation and work directly in terms of hoop states. We will show elsewhere that the transform also interacts well with the “momentum operators” which are associated with closed strips, i.e. that these operators also have simple action on the hoop states.

The hoop states are especially well-suited to deal with diffeomorphism (i.e. Diff(\(\Sigma\)) invariant theories. In such theories, physical states are required to be invariant under Diff(\(\Sigma\)). This condition is awkward to impose in the connection representation and, it is difficult to control the structure of the space of resulting states. In the loop representation, by contrast, the task is rather simple: Physical states depend only on generalized knot and link classes of loops. Because of this simplification and because the action of the basic operators can be represented by simple operations on hoops, the loop representation has proved to be a powerful tool in quantum general relativity in 3 and 4 dimensions.

The use of this representation, however, raises several issues which are still open. Perhaps the most basic of these is that, without referring back to the connection representation, we do not yet have a useful characterization of the hoop states which are images of connection states, i.e. of elements of \(L^2(\overline{A}/G, d\mu)\). Neither do we have an expression of the inner product between hoop states. It would be extremely interesting to develop integration theory also over the hoop group \(HG\) and express the inner product between hoop states directly in terms of such integrals, without any reference to the connection representation. This may indeed be possible using again the idea of cylindrical functions, but now on \(HG\), rather than on \(\overline{A}/G\) and exploiting, as before, the duality between these spaces. If this is achieved and if the integrals over \(\overline{A}/G\) and \(HG\) can be related, we would have a non-linear generalization of the Plancherel theorem which establishes that the Fourier transform is an isomorphism between two spaces of \(L^2\)-functions. The loop transform would then become an extremely powerful tool.

**Appendix A:** \(C^1\) loops and \(U(1)\)–connections

In the main body of the paper, we restricted ourselves to piecewise analytic loops. This restriction was essential in Section 3.1 for our decomposition of a given set of finite number of loops into independent loops which in turn is used in every major result contained in this paper. The restriction is not as severe as it might first seem since every smooth 3–manifold admits an unique analytic structure. Nonetheless, it is important to find out if our arguments can be replaced by more sophisticated ones so that the main results would continue to hold even if the holonomy \(C^*\)–algebra were
constructed from piecewise smooth loops. In this appendix, we consider $U(1)$ connections (rather than $SU(2)$) and show that, in this theory, our main results do continue to hold for piecewise $C^1$ loops. However, the new arguments make a crucial use of the Abelian character of $U(1)$ and do not by themselves go over to non–Abelian theories. Nonetheless, this Abelian example is an indication that analyticity may not be indispensible even in the non–Abelian case.

The appendix is divided into three parts. The first is devoted to certain topological considerations which arise because $3$–manifolds admit non–trivial $U(1)$–bundles. The second proves the analog of the spectrum theorem of Section 3. The third introduces a diffeomorphism invariant measure on the spectrum and discusses some of its properties. Throughout this appendix, by loops we shall mean continuous, piecewise $C^1$ loops. We will use the same notation as in the main text but now those symbols will refer to the $U(1)$ theory. Thus, for example, $\mathcal{H}_G$ will denote the $U(1)$–hoop group and $\mathcal{H}_A$, the $U(1)$ holonomy $\star$–algebra.

A.1 Topological considerations

Fix a smooth $3$–manifold $\Sigma$. Denote by $\mathcal{A}$ the space of all smooth $U(1)$ connections which belong to an appropriate Sobolev space. Now, unlike $SU(2)$ bundles, $U(1)$ bundles over $3$–manifolds need not be trivial. The question therefore arises as to whether we should allow arbitrary $U(1)$ connections or restrict ourselves only to the trivial bundle in the construction of the holonomy $C^*$–algebra. Fortunately, it turns out that both choices lead to the same $C^*$–algebra. This is the main result of this sub–section.

Denote by $\mathcal{A}^0$, the sub–space of $\mathcal{A}$ consisting of connections on the trivial $U(1)$ bundle over $\Sigma$. As in common in the physics literature, we will identify elements $A^0$ of $\mathcal{A}^0$ with real–valued $1$–forms. Thus, the holonomies defined by any $A^0 \in \mathcal{A}^0$ will be denoted by: $H(\alpha, A^0) = \exp(i \oint_\alpha A^0) \equiv \exp(i\theta)$, where $\theta$ takes values in $(0, 2\pi)$ and depends on both the connection $A^0$ and the loop $\alpha$. We begin with the following result:

**Lemma A.1** Given a finite number of loops, $\alpha_1, ..., \alpha_n$, for every $A \in \mathcal{A}$, there exists $A^0 \in \mathcal{A}^0$ such that:

$$H(\alpha_i, A) = H(\alpha_i, A^0), \quad \forall i \in \{1, ..., n\}.$$

**Proof** Suppose the connection $A$ is defined on the bundle $P$. We will show first that there exists a local section $s$ of $P$ which contains the given loops $\alpha_i$. To construct the section we use a map $\Phi : \Sigma \rightarrow S_2$ which generates the bundle $P(\Sigma, U(1))$ (see Trautman [15]). The map $\Phi$ carries each $\alpha_i$ into a loop $\hat{\alpha}_i$ in $S_2$. Since the loops $\hat{\alpha}_i$ cannot form a dense set in $S_2$, we may remove from the 2–sphere an open ball $B$ which does not intersect any of $\hat{\alpha}_i$. Now, since $S_2 - B$ is contractible, there exists a smooth section $\sigma$ of
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the Hopf bundle $U(1) \to SU(2) \to S_2$, 

$$\sigma : (S_2 - B) \to SU(2).$$

Hence, there exists a section $s$ of $P(\Sigma, U(1))$ defined on a subset $V = \Phi^{-1}(S_2 - B) \subset \Sigma$ which contains all the loops $\alpha_i$. Having obtained this section $s$, we can now look for the connection $A^0$. Let $\omega$ be a globally defined, real $1$–form on $\Sigma$ such that $\omega|V = s^*A$. Hence, the connection $A^0 := \omega$ defines on the given loops $\alpha_i$ the same holonomy elements as $A$; i.e.

$$H(\alpha_i, A^0) = H(\alpha_i, A) \quad (A.1)$$

for all $i$. 

This lemma has several useful implications. We mention two that will be used immediately in what follows.

1. Definition of hoops: A priori, there are two equivalence relations on the space $L_x$ of based loops that one can introduce. One may say that two loops are equivalent if the holonomies of all connections in $A$ around them are the same, or, one could ask that the holonomies be the same only for connections in $A^0$. Lemma A.1 implies that the two notions are in fact equivalent; there is only one hoop group $HG$. As in the main text, we will denote by $\tilde{\alpha}$ the hoop to which a loop $\alpha \in L_x$ belongs.

2. Sup norm: Consider functions $f$ on $A$ defined by finite linear combinations of holonomies on $A$ around hoops in $HG$. (Since the holonomies themselves are complex numbers, the trace operation is now redundant.) We can restrict these functions to $A^0$. Lemma A.1 implies that:

$$\sup_{A \in A} |f(A)| = \sup_{A^0 \in A^0} |f(A^0)| \quad (A.2)$$

As a consequence of these implications, we can use either $A$ or $A^0$ to construct the holonomy $C^*$–algebra; inspite of topological non–trivialities, there is only one $\mathcal{HA}$. To construct this algebra we proceed as follows. Let $\mathcal{HA}$ denote the complex vector space of functions $f$ on $A^0$ of the form

$$f(A^0) = \sum_{j=1}^{n} a_j H(\alpha, A^0) \equiv \sum_{j=1}^{n} a_j \exp(i \oint_{\alpha} A^0) \quad (A.3)$$

where $a_j$ are complex numbers. Clearly, $\mathcal{HA}$ has the structure of a $\star$–algebra with the product law $H(\alpha, A^0) \circ H(\beta, A^0) = H(\alpha \cdot \beta, A^0)$ and the $\star$–relation given by $(H(\alpha, A^0))^\star = H(\alpha^{-1}, A^0)$. Equip it with the sup–norm and take completion. The result is the required $C^*$–algebra $\overline{\mathcal{HA}}$.

We conclude by noting another consequence of Lemma A.1: the (Abelian) hoop group $HG$ has no torsion. More precisely, we have the following result:
Lemma A.2 Let $\tilde{\alpha} \in \mathcal{H}G$. Then if $(\tilde{\alpha})^n = e$, the identity in $\mathcal{H}G$, for $n \in \mathbb{Z}$, then $\tilde{\alpha} = e$.

Proof Since $(\tilde{\alpha})^n = e$, we have, for every $A^o \in A^o$,
$$H(\alpha, A^o) \equiv H(\alpha^n, \frac{1}{n} A^o) = 1,$$
where $\alpha$ is any loop in the hoop $\tilde{\alpha}$. Hence, by Lemma A.1, it follows that $\tilde{\alpha} = e$. $\Box$

Although the proof is more complicated, the analogous result holds also in the $SU(2)$-case treated in the main text. However, in that case, the hoop group is non-Abelian. As we will see below, it is the Abelian character of the $U(1)$-hoop group that makes the result of this lemma useful.

A.2 The Gel’fand spectrum of $\mathcal{H}A$

We now want to obtain a complete characterization of the spectrum of the $U(1)$ holonomy $C^*$-algebra $\mathcal{H}A$ along the lines of Section 3. The analog of Lemma 3.2 is easy to establish: Every element $\tilde{A}$ of the spectrum defines a homomorphism $\tilde{H}_\tilde{A}$ from the hoop group $\mathcal{H}G$ to $U(1)$ (now given simply by $\tilde{H}_\tilde{A}(\tilde{\alpha}) = \tilde{A}(\tilde{\alpha})$). This is not surprising. Indeed, it is clear from the discussion in Section 3 that this lemma continues to hold for piecewise smooth loops even in the full $SU(2)$ theory. However, the situation is quite different for Lemma 3.3 because there we made a crucial use of the fact that the loops were (continuous and) piecewise analytic. Therefore, we must now modify that argument suitably. An appropriate replacement is contained in the following lemma.

Lemma A.3 For every homomorphism $\tilde{H}$ from the hoop group $\mathcal{H}G$ to $U(1)$, every finite set of hoops $\{\tilde{\alpha}_1, ..., \tilde{\alpha}_k\}$ and every $\epsilon > 0$, there exists a connection $A^o \in A^o$ such that:
$$|\tilde{H}(\tilde{\alpha}_i) - H(\alpha_i, A^o)| < \epsilon, \ \forall i \in \{1, ..., k\} \quad (A.4)$$

Proof Consider the subgroup $\mathcal{H}G(\tilde{\alpha}_1, ..., \tilde{\alpha}_n) \subset \mathcal{H}G$ generated by $\tilde{\alpha}_1, ..., \tilde{\alpha}_k$. Since $\mathcal{H}G$ is Abelian and since it has no torsion, $\mathcal{H}G(\tilde{\alpha}_1, ..., \tilde{\alpha}_n)$ is finitely and freely generated by some elements, say $\tilde{\beta}_1, ..., \tilde{\beta}_n$. Hence, if (A.4) is satisfied by $\tilde{\beta}_i$ for a sufficiently small $\epsilon'$ then it will also be satisfied by the given $\tilde{\alpha}_j$ for the given $\epsilon$. Consequently, without loss of generality, we can assume that the hoops $\alpha_i$ are (weakly) $U(1)$-independent, i.e., that they satisfy the following condition:
$$\text{if } (\tilde{\alpha}_1)^{k_1}...(\tilde{\alpha}_n)^{k_n} = e, \text{ then, } k_i = 0 \ \forall i.$$  

Now, given such hoops $\tilde{\alpha}_i$, the homomorphism $\tilde{H} : \mathcal{H}G \to U(1)$ defines a point $(\tilde{H}(\tilde{\alpha}_1), ..., \tilde{H}(\tilde{\alpha}_k)) \in [U(1)]^k$. On the other hand, there also exists
a map from $A^o$ to $[U(1)]^k$, which can be expressed as a composition:

$$A^o \ni A^o \mapsto \left( \oint A^o, \ldots, \oint A^o \right) \mapsto \left( e^{i \oint A^o}, \ldots, e^{i \oint A^o} \right) \in [U(1)]^k.$$  
(A.5)

The first map in (A.5) is linear and its image, $V \subset \mathbb{R}^k$, is a vector space. Let $0 \neq m = (m_1, \ldots, m_k) \in \mathbb{Z}^k$. Denote by $V_m$ the subspace of $V$ orthogonal to $m$. Now if $V_m$ were to equal $V$ then we would have

$$(\tilde{\alpha}_1)^{m_1} \ldots (\tilde{\alpha}_k)^{m_k} = 1,$$

which would contradict the assumption that the given set of hoops is independent. Hence we necessarily have: $V_m < V$. Furthermore, since a countable union of subsets of measure zero has measure zero, it follows that

$$\bigcup_{m \in \mathbb{Z}^k} V_m < V.$$

This strict inequality implies that there exists a connection $B^o \in A^o$ such that

$$\mathbf{v} := \left( \oint B^o, \ldots, \oint B^o \right) \in (V - \bigcup_{m \in \mathbb{Z}^k} V_m).$$

In other words, $B^o$ is such that every ratio $v_i/v_j$ of two different components of $\mathbf{v}$ is irrational. Thus, the line in $A^o$ defined by $B^o$ via $A^o(t) = tB^o$ is carried by the map (A.5) into a line which is dense in $[U(1)]^k$. This ensures that there exists an $A^o$ on this line which has the required property (A.4).

Armed with this substitute of Lemma 3.3, it is straightforward to show the analog of Lemma 3.4. Combining these results, we have the $U(1)$ spectrum theorem:

**Theorem A.4** Every $\tilde{A}$ in the spectrum of $\mathcal{H}\mathcal{A}$ defines a homomorphism $\tilde{H}$ from the hoop group $\mathcal{H}\mathcal{G}$ to $U(1)$, and, conversely, every homomorphism $\tilde{H}$ defines an element $\tilde{A}$ of the spectrum such that $\tilde{H}(\tilde{\alpha}) = \tilde{A}(\tilde{\alpha})$. This is a 1–1 correspondence.

### A.3 A natural measure

Results presented in the previous sub–section imply that one can again introduce the notion of cylindrical functions and measures. In this subsection, we will exhibit a natural measure. Rather than going through the same constructions as in the main text, for the sake of diversity, we will adopt here a complementary approach: We will present a (strictly–) positive linear functional on the holonomy $C^*$–algebra $\mathcal{H}\mathcal{A}$ whose properties suffice to ensure the existence of a diffeomorphism invariant, faithful, regular measure on the spectrum of $\mathcal{H}\mathcal{A}$.
We know from the general theory outlined in Section 2 that, to specify a positive linear functional on $\mathcal{HA}$, it suffices to provide an appropriate generating functional $\Gamma[\alpha]$ on $L_x^\ast$. Let us simply set

$$\Gamma(\alpha) = \begin{cases} 1, & \text{if } \hat{\alpha} = \hat{o} \\ 0, & \text{otherwise} \end{cases} \quad (A.6)$$

where $\hat{o}$ is the identity hoop in $\mathcal{HG}$. It is clear that this functional on $L_x^\ast$ is diffeomorphism invariant. Indeed, it is the simplest of such functionals on $L_x^\ast$. We will now show that it does have all the properties to be a generating function.

**Theorem A.5** $\Gamma(\alpha)$ extends to a continuous positive linear function on the holonomy $C^\ast$–algebra $\mathcal{HA}$. Furthermore, this function is strictly positive.

**Proof** By its definition, the loop functional $\Gamma(\alpha)$ admits a well–defined projection on the hoop space $\mathcal{HG}$ which we will denote again by $\Gamma$. Let $\mathcal{FHG}$ be the free vector space generated by the hoop group. Extend $\Gamma$ to $\mathcal{FHG}$ by linearity. We will first establish that this extension satisfies the following property: Given a finite set of hoops $\hat{\alpha}_j, j = 1, ..., n$ such that, if $\hat{\alpha}_i \neq \hat{\alpha}_j$ if $i \neq j$, we have

$$\Gamma((\sum_{j=1}^n a_j \hat{\alpha}_j)(\sum_{j=1}^n a_j \hat{\alpha}_j)) < \sup_{A^o \in A^o} (\sum_{j=1}^n a_j H(\hat{\alpha}_j, A^o))^\ast (\sum_{j=1}^n a_j H(\hat{\alpha}_j, A^o)). \quad (A.7)$$

To see this, note first that, by definition of $\Gamma$, the left side equals $\sum |a_j|^2$. The right side equals $(\sum a_j \exp i\theta_j)^\ast (\sum a_j \exp i\theta_j)$ where $\theta_j$ depend on $A^o$ and the hoop $\hat{\alpha}_j$. Now, by Lemma A.3, given the $n$ hoops $\hat{\alpha}_j$, one can find a $A^o$ such that the angles $\theta_j$ can be made as close as one wishes to a pre–specified $n$–tuple $\theta^o_j$. Finally, given the $n$ complex numbers $a_i$, one can always find $\theta^o_j$ such that $\sum |a_j|^2 < |(\sum a_j \exp i\theta^o_j)^\ast (\sum a_j \exp i\theta^o_j)|$. Hence we have $\Gamma(A.7)$.

The inequality $\Gamma(A.7)$ implies that the functional $\Gamma$ has a well-defined projection on the holonomy $\ast$–algebra $\mathcal{HA}$ (which is obtained by taking

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\[\text{Since this functional is so simple and natural, one might imagine using it to define a positive linear functional also for the SU}(2) \text{ holonomy algebra of the main text. However, this strategy fails because of the SU}(2) \text{ identities. For example, in the SU}(2) \text{ holonomy algebra we have } T_{\alpha, \beta} \cdot a_{\alpha} + T_{\alpha, \beta} \cdot a_{-\beta}^{-1} - T_{\alpha, \beta} \cdot a_{\beta} = 1. \text{ Evaluation of the generating functional (A.6) on this identity would lead to the contradiction } 0 = 1. \text{ We can define a positive linear function on the SU}(2) \text{ holonomy } C^\ast–\text{algebra via:} \]

$$\Gamma'(\alpha) = \begin{cases} 1, & \text{if } T_{\alpha}(A) = 1, \forall A \in A^o \\ 0, & \text{otherwise} \end{cases}$$

where, we have regarded $A^o$ as a subspace of the space of SU$(2)$ connections. However, on the SU$(2)$ holonomy algebra, this positive linear functional is not strictly positive: the measure on $\mathcal{AG}$ it defines is concentrated just on $A^o$. Consequently, the resulting representation of the SU$(2)$ holonomy $C^\ast–\text{algebra fails to be faithful.}
the quotient of $\mathcal{F}H\mathcal{G}$ by the subspace $K$ consisting of $\sum b_i\tilde{\alpha}_i$ such that $\sum b_i H(\tilde{\alpha}_i, A^o) = 0$ for all $A^o \in \mathcal{A}^o$.) Furthermore, the projection is positive definite; $\Gamma(f^* f) \geq 0$ for all $f \in \mathcal{H}\mathcal{A}$, equality holding only if $f = 0$. Next, since the norm on this $*$-algebra is the sup–norm on $\mathcal{A}^o$, it follows from (A.7) that the functional $\Gamma$ is continuous on $\mathcal{H}\mathcal{A}$. Hence it admits a unique continuous extension to the $C^*$–algebra $\overline{\mathcal{H}\mathcal{A}}$. Finally, (A.7) implies that the functional continues to be strictly positive on $\overline{\mathcal{H}\mathcal{A}}$. □

Theorem A.5 implies that the generating functional $\Gamma$ provides a continuous, faithful representation of $\mathcal{H}\mathcal{A}$ and hence a regular, strictly positive measure on the Gel’fand spectrum of this algebra. It is not difficult to verify by direct calculations that this is precisely the $U(1)$ analog of the induced Haar measure discussed in Section 4.

Appendix B: $C^\omega$ loops, $U(N)$ and $SU(N)$–connections

In this appendix, we will consider another extension of the results presented in the main paper. We will now work with analytic manifolds and piecewise analytic loops as in the main text. However, we will let the manifold have any dimension and let the gauge group $G$ be either $U(N)$ or $SU(N)$. Consequently, there are two types of complications: algebraic and topological. The first arise because, e.g., the products of traces of holonomies can no longer be expressed as sums while the second arise because connections in question may be defined on non–trivial principal bundles. Nonetheless, we will see that the main results of the paper continue to hold in these cases as well. Several of the results also hold when the gauge group is allowed to be any compact Lie group. However, for brevity of presentation, we will refrain from making digressions to the general case. Also, since most of the arguments are rather similar to those given in the main text, details will be omitted.

Let us begin by fixing a principal fibre bundle $P(\Sigma, G)$, obtain the main results and then show, at the end, that they are independent of the initial choice of $P(\Sigma, G)$. Definitions of the space $L^c_{x^o}$ of based loops, and holonomy maps $H(\alpha, A)$, are the same as in Section 2. To keep the notation simple, we will continue to use the same symbols as in the main text to denote various spaces which, however, now refer to connections on $P(\Sigma, G)$. Thus, $\mathcal{A}$ will denote the space of (suitably regular) connections on $P(\Sigma, G)$, $\mathcal{H}\mathcal{G}$ will denote the $P(\Sigma, G)$–loop group, and $\overline{\mathcal{H}\mathcal{A}}$ will be the holonomy $C^*$–algebra generated by (finite sums of finite products of) traces of holonomies of connections in $\mathcal{A}$ around loops in $\mathcal{H}\mathcal{G}$.

B.1 Holonomy $C^*$–algebra

We will set $T_\alpha(A) = \frac{1}{N} \text{Tr} H(\alpha, A)$, where the trace is taken in the fundamental representation of $U(N)$ or $SU(N)$, and regard $T_\alpha$ as a function on $\mathcal{A}/\mathcal{G}$, the space of gauge equivalent connections. The holonomy algebra
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$\mathcal{H}A$ is, by definition, generated by all finite linear combinations (with complex coefficients) of finite products of the $T_{\tilde{a}}$, with the $\star$-relation given by $f^* = \bar{f}$, where the “bar” stands for complex-conjugation. Being traces of $U(N)$ and $SU(N)$ matrices, $T_{\tilde{a}}$ are bounded functions on $\mathcal{A}/\mathcal{G}$. We can therefore introduce on $\mathcal{H}A$ the sup–norm

$$||f|| = \sup_{A \in \mathcal{A}} |f(A)|,$$

and take the completion of $\mathcal{H}A$ to obtain a $C^*$–algebra. We will denote it by $\overline{\mathcal{H}A}$. This is the holonomy $C^*$–algebra.

The key difference between the structure of this $\mathcal{H}A$ and the one we constructed in Section 2 is the following. In Section 2, the $\star$–algebra $\mathcal{H}A$ was obtained simply by imposing an equivalence relation ($K$, see Eq. (2.7)) on the free vector space generated by the loop space $L_{x^*}$. This was possible because, due to $SU(2)$ Mandelstam identities, products of any two $T_{\tilde{a}}$ could be expressed as a linear combination of $T_{\tilde{a}}$ (Eq. (2.6)). For groups under consideration, the situation is more involved. Let us summarize the situation in the slightly more general context of the group $GL(N)$. In this case, the Mandelstam identities follow from the contraction of $N + 1$ matrices $-H(\alpha_1, A),...,H(\alpha_{N+1}, A)$ in our case— with the identity

$$\delta_{[i_1, \ldots, i_{N+1}]}^{j_1, \ldots, j_N+1} = 0$$

where $\delta_i^j$ is the Kronecker delta and the bracket stands for the antisymmetrization. They enable one to express products of $N + 1$ $T_{\tilde{a}}$–functions as a linear combination of traces of products of $1, 2, \ldots, N$ $T_{\tilde{a}}$–functions. Hence, we have to begin by considering the free algebra generated by the hoop group and then impose on it all the suitable identities by extending the definition of the kernel (2.7). Consequently, if the gauge group is $GL(N)$, an element of the holonomy algebra $\mathcal{H}A$ is expressible as an equivalence class $[a\alpha, b\alpha_1, \alpha_2, \ldots, c\gamma_1, \ldots, \gamma_N]$, where $a, b, c$ are complex numbers; products involving $N + 1$ and higher loops are redundant. (For sub-groups of $GL(N)$—such as $SU(N)$— further reductions may be possible.)

Finally, let us note identities that will be useful in what follows. These result from the fact that the determinant of $N \times N$ matrix $M$, can be expressed in terms of traces of powers of that matrix, namely

$$\text{det } M = F(\text{Tr}M, \text{Tr}M^2, \ldots, \text{Tr}M^N)$$

for a certain polynomial $F$. Hence, if the gauge–group $G$ is $U(N)$, we have, for any hoop $\tilde{a}$,

$$F(T_{\tilde{a}}, \ldots, T_{\tilde{a}N})F(T_{\tilde{a}-1}, \ldots, T_{\tilde{a}-N}) = 1. \quad (B.1a)$$
In the $SU(N)$ case a stronger identity holds:

$$F(T_{\tilde{\alpha}},...,T_{\tilde{\alpha}^N}) = 1. \quad (B.1b)$$

**B.2 Loop decomposition and the spectrum theorem**

The construction of Section 3.1 for decomposition of a finite number of loops into (strongly) independent loops makes no direct reference to the gauge group and therefore carries over as is. Also, it is still true that the map

$$\mathcal{A} \ni A \mapsto (H(\beta_1, A),...,H(\beta_n, A)) \in G^n$$

is onto if the loops $\beta_i$ are independent. (The proof requires some minor modifications which consist of using local sections and a sufficiently large number of generators of the gauge group). A direct consequence is that the analog of Lemma 3.3 continues to hold.

As for the Gel'fand spectrum, the A–I result that there is a natural embedding of $\mathcal{A}/G$ into the spectrum $\overline{\mathcal{A}/G}$ goes through as before and so does the general argument due to Rendall [5] that the image of $\mathcal{A}/G$ is dense in $\overline{\mathcal{A}/G}$. (This is true for any group and representation for which the traces of holonomies separate the points of $\mathcal{A}/G$.) Finally, using the analog of Lemma 3.3, it is straightforward to establish the analog of the main conclusion of Lemma 3.4.

The converse —the analog of the Lemma 3.2— on the other hand requires further analysis based on Giles’ results along the lines used by A–I in the $SU(2)$ case. For the gauge group under consideration, given an element of the spectrum $\overline{\mathcal{A}}$—i.e., a continuous homomorphism from $\overline{H\mathcal{A}}$ to the $*$ algebra of complex numbers—Giles’ theorem [10] provides us with a homomorphism $\hat{H}_{\overline{\mathcal{A}}}$ from the hoop group $H\mathcal{G}$ to $GL(N)$.

What we need to show is that $\hat{H}_{\overline{\mathcal{A}}}$ can be so chosen that it takes values in the given gauge group $G$. We can establish this by considering the eigenvalues $\lambda_1, ..., \lambda_N$ of $H_{\overline{\mathcal{A}}} (\tilde{\alpha})$. (After all, the characteristic polynomial of the matrix $H_{\overline{\mathcal{A}}} (\tilde{\alpha})$ is expressible directly by the values of $\tilde{\alpha}$ taken on the hoops $\tilde{\alpha}, \tilde{\alpha}^2, ..., \tilde{\alpha}^n$.) First, we note from Eq. $(B.1a)$ that in particular $\lambda_i \neq 0$, for every $i$. Thus far, we have used only the algebraic properties of $\tilde{\alpha}$. From continuity it follows that for every hoop $\tilde{\alpha}$, $\text{Tr}H_{\overline{\mathcal{A}}} (\tilde{\alpha}) \leq N$. Substituting for $\tilde{\alpha}$ its powers we conclude that

$$\|\lambda_1^k + ... \lambda_N^k\| \leq N$$

for every integer $k$. This suffices to conclude that

$$\|\lambda_i\| = 1.$$
by the identity \((B.1b)\), we have

\[
\det H_{\hat{A}} = 1,
\]

so that the analog of Lemma 3.2 holds.

Combining these results, we have the spectrum theorem: Every element \(\hat{A}\) of the spectrum \(\overline{A/G}\) defines a homomorphism \(\hat{H}\) from the hoop group \(\mathcal{H} G\) to the gauge group \(G\), and, conversely, every homomorphism \(\hat{H}\) from the hoop group \(\mathcal{H} G\) to \(G\) defines an element \(\hat{A}\) of the spectrum \(\overline{A/G}\) such that \(A(\tilde{\alpha}) = \frac{1}{N} \text{Tr} \hat{H}(\tilde{\alpha})\) for all \(\tilde{\alpha} \in \mathcal{H} G\). Two homomorphisms \(\hat{H}_1\) and \(\hat{H}_2\) define the same element of the spectrum if and only if \(\text{Tr} \hat{H}_2 = \text{Tr} \hat{H}_1\).

### B.3 Cylindrical functions and the induced Haar measure

Using the spectrum theorem, we can again associate with any subgroup \(S^* \equiv \mathcal{H} G(\tilde{\beta}_1, \ldots, \tilde{\beta}_n)\) generated by \(n\) independent hoops \(\tilde{\beta}_1, \ldots, \tilde{\beta}_n\), an equivalence relation \(\sim\) on \(\overline{A/G}\): \(A_1 \sim A_2\) iff \(A_1(\tilde{\gamma}) = A_2(\tilde{\gamma})\) for all \(\tilde{\gamma} \in S^*\). This relation provides us with a family of projections from \(\overline{A/G}\) onto the compact manifolds \(G^n/\text{Ad}\) (since, for groups under consideration, the traces of all elements of a finitely generated sub-group in the fundamental representation suffice to characterize the sub-group modulo an overall adjoint map.) Therefore, as before, we can define cylindrical functions on \(\overline{A/G}\) as the pull–backs to \(\overline{A/G}\) of continuous functions on \(G^n/\text{Ad}\). They again form a \(C^*\)-algebra. Using the fact that traces of products of group elements suffice to separate points of \(G^n/\text{Ad}\) when \(G = U(N)\) or \(G = SU(N)\) [16], it again follows that the \(C^*\)-algebra of cylindrical functions is isomorphic with \(\mathcal{H} A\). Finally, the construction of Section 4.2 which led us to the definition of the induced Haar measure goes through step by step. The resulting representation of the \(C^*\)-algebra \(\mathcal{H} A\) is again faithful and diffeomorphism invariant.

### B.4 Bundle dependence

In all the constructions above, we fixed a principal bundle \(P(\Sigma, G)\). To conclude, we will show that the \(C^*\)-algebra \(\mathcal{H} A\) and hence its spectrum are independent of this choice. First, as noted at the end of Section 3.1, using the hoop decomposition one can show that two loops in \(L_{x^*}\) define the same hoop if and only if they differ by a combination of reparametrization and (an immediate) retracing of segments for gauge groups under consideration. Therefore, the hoop groups obtained from any two bundles are the same.

Fix a gauge group \(G\) from \(\{U(N), SU(N)\}\) and let \(P_1(\Sigma, G)\) and \(P_2(\Sigma, G)\) be two principal bundles. Let the corresponding holonomy \(C^*\)-algebras be \(\mathcal{H} A^{(1)}\) and \(\mathcal{H} A^{(2)}\). Using the spectrum theorem, we know that their spectra are naturally isomorphic: \(\overline{A/G}^{(1)} = \text{Hom}(\mathcal{H} G, G) = \overline{A/G}^{(2)}\). We wish to show that the algebras are themselves naturally isomorphic, i.e. that the map \(\mathcal{I}\) defined by
\[ \mathcal{I} \circ \left( \sum_{i=1}^{n} a_i \hat{T}_{\hat{\alpha}_i}^{(1)} \right) = \sum_{i=1}^{n} a_i \hat{T}_{\hat{\alpha}_i}^{(2)} \quad (B.2) \]

is an isomorphism from \( \overline{\mathcal{H}A}^{(1)} \) to \( \overline{\mathcal{H}A}^{(2)} \). Let us decompose the given hoops, \( \hat{\alpha}_1, \ldots, \hat{\alpha}_k \), into (strongly) independent hoops \( \hat{\beta}_1, \ldots, \hat{\beta}_n \). Let \( S^* \) be the subgroup of the hoop group they generate and, as in Section 4.1, let \( \pi^{(i)}(S^*) \), \( i = 1, 2 \), be the corresponding projection maps from \( \overline{A/G} \) to \( G^n/\text{Ad} \). From the definition of the maps it follows that the projections of the two functions in \( (B.2) \) to \( G^N/\text{Ad} \) are in fact equal. Hence, we have:

\[ || \sum_{i=1}^{n} a_i \hat{T}_{\hat{\alpha}_i}^{(1)} ||_1 = || \sum_{i=1}^{n} a_i \hat{T}_{\hat{\alpha}_i}^{(2)} ||_2, \]

whence it follows that \( \mathcal{I} \) is an isomorphism of \( C^* \)-algebras. Thus, it does not matter which bundle we begin with; we obtain the same holonomy \( C^*-\text{algebra } \overline{H\mathcal{A}} \). Now, fix a bundle, \( P_1 \) say, and consider a connection \( A_2 \) defined on \( P_2 \). Note, that the holonomy map of \( A_2 \) defines a homomorphism from the (bundle–independent) hoop group to \( G \). Hence, \( A_2 \) defines also a point in the spectrum of \( \overline{H\mathcal{A}}^{(1)} \). Therefore, given a manifold \( M \) and a gauge–group \( G \) the spectrum \( \overline{A/G} \) automatically contains all the connections on all the principal \( G \)-bundles over \( M \).

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