A comparison of solutions of two convolution-type unidirectional wave equations

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\section*{ABSTRACT}
In this work, we prove a comparison result for a general class of nonlinear dispersive unidirectional wave equations. The dispersive nature of one-dimensional waves occurs because of a convolution integral in space. For two specific choices of the kernel function, the Benjamin–Bona–Mahony equation and the Rosenau equation that are particularly suitable to model water waves and elastic waves, respectively, are two members of the class. We first prove an energy estimate for the Cauchy problem of the non-local unidirectional wave equation. Then, for the same initial data, we consider two distinct solutions corresponding to two different kernel functions. Our main result is that the difference between the solutions remains small in a suitable Sobolev norm if the two kernel functions have similar dispersive characteristics in the long-wave limit. As a sample case of this comparison result, we provide the approximations of the hyperbolic conservation law.

\section*{1. Introduction}
In this paper, we establish a comparison result for solutions to the Cauchy problem associated to the one-dimensional non-local nonlinear wave equation

\begin{equation}
    u_t + \alpha \ast (u + u^{p+1})_x = 0,
\end{equation}

under the assumption that kernel functions have similar dispersive characteristics in the long-wave limit. Here, $u = u(x, t)$ is a real-valued function, $p$ is a positive integer, $\alpha(x)$ is a general kernel function and the symbol $\ast$ denotes convolution in the $x$-variable. The linear dispersion relation $\xi \mapsto \omega(\xi) = \hat{\xi} \hat{\alpha}(\xi)$, where $\hat{\alpha}$ represents the Fourier transform of $\alpha$, shows that the dispersive nature of waves is directly related to the kernel function. Since we intend to confine our interest to long-wave solutions, we rewrite (1) in the form

\begin{equation}
    u_t + \alpha_{\delta} \ast (u + u^{p+1})_x = 0
\end{equation}

by utilizing the transformation $(x, t) \rightarrow (x/\delta, t/\delta)$ with small parameter $\delta > 0$ and by introducing the family of kernels as $\alpha_{\delta}(x) = \frac{1}{\delta} \alpha(\frac{x}{\delta})$. It is worth noting that, as $\delta \to 0$, the kernels $\alpha_{\delta}$ will converge to the Dirac measure in the distribution sense and (2) will formally approach the hyperbolic conservation law $u_t + (u + u^{p+1})_x = 0$. Indeed, (2) is a dispersive regularization of the hyperbolic conservation law which was widely studied.
The non-local equation (2) with a general kernel function was given in [1] to provide a numerical treatment of nonlinear unidirectional waves with a non-local dispersion relation. With particular kernel functions, (2) has been proposed as a model in a wide range of physical contexts. The Benjamin–Bona–Mahony (BBM) equation [2]

\[ u_t + u_x - \delta^2 u_{xxx} + (u^{p+1})_x = 0 \]  

is the most well-known member of the one parameter family (2). It corresponds to the exponential kernel

\[ \alpha_\delta(x) = \frac{1}{2\delta} e^{-|x|/\delta} \]

and models unidirectional propagation of small amplitude long waves in shallow water. Another well-known member of (2) is the Rosenau equation [3]

\[ u_t + u_x + \delta^4 u_{xxxx} + (u^{p+1})_x = 0, \]

which corresponds to the kernel function

\[ \alpha_\delta(x) = \frac{1}{2\sqrt{2\delta}} e^{-|x|/\sqrt{2\delta}} \left( \cos \left( \frac{|x|}{\sqrt{2\delta}} \right) + \sin \left( \frac{|x|}{\sqrt{2\delta}} \right) \right) \]

and models propagation of longitudinal waves on a one-dimensional dense chain of particles. It is worth to mention here that the BBM and Rosenau equations might be viewed as degenerate cases of the family (2), because in both cases \( \alpha_\delta \) is Green’s function of a differential operator. However, in a ‘genuinely non-local’ case, this is not the case and (2) cannot be transformed into a partial differential equation. We underline that we will establish our comparison result for (2) in which the kernel functions may or may not be the Green’s function of a differential operator and we pose minimal restrictions on the kernel.

The primary purpose of this work is to prove a comparison result of solutions to (2) in the weak dispersive regime and also to show that the behavior of solutions is determined by the dispersive character of the kernel in the long-wave limit rather than the shape of the kernel function. In a recent work [4], a similar comparison result was given for the non-local bidirectional wave equations. Based mainly on an energy estimate with no loss of derivative, our present comparison result extends basically the notion of ‘kernel-based comparison’ introduced in [4] to the non-local unidirectional wave equation (2).

We first start by considering two different kernel functions with the same dispersive nature in the long-wave limit. Then, for these two kernels, we consider the corresponding solutions to the Cauchy problem with the same initial value. We basically prove that the difference between the two solutions remains small in a suitable norm. We refer the reader to [5–9] and the references therein for a detailed discussion of similar comparison results of many different physical models.

The structure of the paper is as follows. Section 2 is devoted to the proof of an energy estimate. In Section 3, we start with the moment conditions to be satisfied by the two different kernels and prove the main result that establishes an estimate on the difference between the corresponding solutions. In Section 4, we illustrate our comparison result through a particular case in which one of the kernels is the Dirac measure.

Throughout the paper, we will use the standard notation for Lebesgue and Sobolev spaces. The \( L^p \) \((1 \leq p < \infty)\) norm of \( u \) on \( \mathbb{R} \) is represented by \( \| u \|_{L^p} \) and the notation \( L^p = L^p(\mathbb{R}) \) is used. To denote the inner product of \( u \) and \( v \) in \( L^2 \), the symbol \( \langle u, v \rangle \) is used. The Fourier transform of \( u \) is defined as \( \hat{u}(\xi) = \int_{\mathbb{R}} u(x) e^{-i\xi x} \, dx \). The notation \( H^s \) \(= H^s(\mathbb{R}) \) is used to denote the \( L^2 \)-based Sobolev space of order \( s \) on \( \mathbb{R} \), with the norm \( \| u \|_{H^s} = \left( \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2} \). All the integrals in the paper will be over \( \mathbb{R} \), so we will omit the limits of integration. \( C \) is a generic positive constant. We denote partial differentiations by \( D_x \) etc.

2. An energy estimate

The current section is devoted to the derivation of an energy estimate. Throughout this work we will assume that the kernel \( \alpha \) is an even function in \( L^1(\mathbb{R}) \) with \( \int \alpha(x) \, dx = 1 \), or more generally a finite
Borel measure on \( \mathbb{R} \) with \( \int dx = 1 \). For convenience, we will use the notation \( Ku = \alpha * u \). We note that the assumptions on \( \alpha \) being an even \( L^1(\mathbb{R}) \) function, or more generally a finite Borel measure, imply that \( K \) is bounded and self-adjoint operator on \( H^s \) for any \( s \).

As \( K \) also commutes with the derivative operator, for all \( f \in H^s, s \geq 1 \) we have

\[
\int (Kf)x dx = \int (Kf_x)f dx = 0.
\]

This identity follows from the self-adjointness of \( K \) on \( L^2(\mathbb{R}) \) and integration by parts:

\[
\int (Kf)x dx = \int f(Kf_x) dx = \int f(Kf_x) dx = -\int f_x(Kf) dx.
\]

In the rest of the work, we will use the notations \( \Lambda^s = (1 - D^2)^{s/2} \) and \( [\Lambda^s,f]g = \Lambda^s(fg) - f \Lambda^s g \). Furthermore, to prove our estimates below and in the next section, we will need the following commutator estimates [10]:

**Lemma 2.1:** Let \( s > 0 \). Then for all \( f, g \) satisfying \( f \in H^s, D_x f \in L^\infty, g \in H^{s-1} \cap L^\infty \),

\[
\|[\Lambda^s,f]g\|_{L^2} \leq C(\|D_x f\|_{L^\infty} \|g\|_{H^{s-1}} + \|f\|_{H^s} \|g\|_{L^\infty}).
\]

In particular, when \( s > 3/2 \), due to the Sobolev embeddings \( H^{s-1} \subset L^\infty \), for all \( f, g \in H^s \)

\[
\|[\Lambda^s,f]D_x g\|_{L^2} \leq C\|f\|_{H^s} \|g\|_{H^s}.
\]

We will consider the related linear equation

\[
u_t + K((1 + w)u_x) = F + G(u), \quad x \in \mathbb{R}, \quad t > 0,
\]

with the given functions \( w(x,t), F(x,t) \) and the linear map \( G(u) \). Clearly, the original Equation (1) is to be obtained from (7) by setting \( w = (p + 1)u^p \) and \( F = G = 0 \); hence (7) is a linearization of (1).

We define the \( H^s \)-energy functional by

\[
\mathcal{E}^2_s(t) = \frac{1}{2} \int (1 + w(x,t))(\Lambda^s u(x,t))^2 dx.
\]

Note that when \( w \) is assumed to satisfy

\[
0 < c_1 \leq 1 + w(x,t) \leq c_2,
\]

where \( c_1 \) and \( c_2 \) are constants, \( \mathcal{E}^2_s(t) \) is equivalent to the norm \( \|u(t)\|_{H^s}^2 \).

**Lemma 2.2:** Let \( s > 3/2, T > 0 \) and \( F \in C([0,T],H^s), w \in C([0,T],H^s) \cap C^1((0,T],H^{s-1}) \) with \( 0 < c_1 \leq 1 + w(x,t) \leq c_2 \) for all \( (x,t) \in \mathbb{R} \times [0,T] \). Let \( G \) be a linear map on \( C([0,T],H^s) \), satisfying \( \|G(u(t))\|_{H^s} \leq C_G \|u(t)\|_{H^s} \) for all \( t \in [0,T] \). Suppose \( u \in C([0,T],H^s) \) satisfies (7) on \( \mathbb{R} \times [0,T] \) with \( u(x,0) = u_0(x) \). Then we have the estimate

\[
\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} e^{At} + \frac{B}{A}(e^{At} - 1)
\]

for \( 0 \leq t \leq T \), where

\[
A = \sup_{0 \leq t \leq T} C(\|w(t)\|_{L^\infty} + (1 + \|w(t)\|_{L^\infty})(C_G + \|\alpha\|_{L^1} \|w(t)\|_{H^s})),
\]

\[
B = \sup_{0 \leq t \leq T} C(1 + \|w(t)\|_{L^\infty})\|F(t)\|_{H^s}.
\]
Proof: Differentiating (8) and using (7), we get

\[
\frac{d}{dt} \mathcal{E}_t^2(t) = \int \left( \frac{1}{2} w_t (\Lambda^s u)^2 + (1 + w)(\Lambda^s u_t)(\Lambda^s u) \right) dx
\]

\[
= \int \left( \frac{1}{2} w_t (\Lambda^s u)^2 + (1 + w)(\Lambda^s F)(\Lambda^s u) + (1 + w)(\Lambda^s G(u))(\Lambda^s u) \right.
\]

\[
- (1 + w)(\Lambda^s K((1 + w) u_x))(\Lambda^s u) \big) dx.
\]

(13)

We handle the last term on the right-hand side separately as follows:

\[
I(t) = \langle (1 + w)\Lambda^s K((1 + w) u_x), \Lambda^s u \rangle
\]

\[
= \langle K\Lambda^s ((1 + w) u_x), (1 + w)\Lambda^s u \rangle
\]

\[
= \langle \Lambda^s((1 + w) u_x), K((1 + w)\Lambda^s u) \rangle
\]

\[
= \langle (1 + w)\Lambda^s u_x, K((1 + w)\Lambda^s u) \rangle + \langle [\Lambda^s, 1 + w] u_x, K((1 + w)\Lambda^s u) \rangle
\]

\[
= \langle (1 + w)\Lambda^s u_x, K((1 + w)\Lambda^s u) \rangle - \langle w_x\Lambda^s u, K((1 + w)\Lambda^s u) \rangle
\]

\[
+ \langle [\Lambda^s, 1 + w] u_x, K((1 + w)\Lambda^s u) \rangle
\]

\[
= -\langle w_x\Lambda^s u, K((1 + w)\Lambda^s u) \rangle + \langle [\Lambda^s, w] u_x, K((1 + w)\Lambda^s u) \rangle,
\]

where we have used \([\Lambda^s, 1 + w] u_x = [\Lambda^s, w] u_x + [\Lambda^s, 1] u_x = [\Lambda^s, w] u_x \) and (6) with \( f = (1 + w)\Lambda^s u \). As \( ||K\varphi||_2 \leq ||\varphi||_{L^1} ||v||_{L^2} \) and \( \|\Lambda^s u\|_{L^2} = \|u\|_{H^s} \),

\[
||I(t)|| \leq ||w_x\Lambda^s u\|_{L^2} ||K((1 + w)\Lambda^s u)||_{L^2} + ||[\Lambda^s, w] u_x||_{L^2} ||K((1 + w)\Lambda^s u)||_{L^2}
\]

\[
\leq ||\alpha||_{L^1} (||w_x\Lambda^s u||_{L^2} ||(1 + w)\Lambda^s u||_{L^2} + ||[\Lambda^s, w] u_x||_{L^2} ||(1 + w)\Lambda^s u||_{L^2})
\]

\[
\leq ||\alpha||_{L^1} (1 + ||w||_{L^\infty})(||w_x||_{L^\infty} ||\Lambda^s u||_{L^2}^2 + ||[\Lambda^s, w] u_x||_{L^2} ||\Lambda^s u||_{L^2})
\]

\[
\leq ||\alpha||_{L^1} (1 + ||w||_{L^\infty})(||w_x||_{L^\infty} ||u||_{H^s}^2 + ||[\Lambda^s, w] u_x||_{L^2} ||u||_{H^s}).
\]

Finally, since \( s > 3/2 \), by the commutator estimate in Lemma 2.1 we have

\[
||[\Lambda^s, w] u_x||_{L^2} \leq C||w||_{H^s} ||u||_{H^s}.
\]

So we get

\[
|I(t)| \leq ||\alpha||_{L^1} (1 + ||w||_{L^\infty})(||w_x||_{L^\infty} + ||w||_{H^s}) ||u||_{H^s}^2.
\]

Then, from (13), we have

\[
\frac{d}{dt} \mathcal{E}_t^2(t) = \int \left( \frac{1}{2} w_t (\Lambda^s u)^2 + (1 + w)(\Lambda^s F)(\Lambda^s u) + (1 + w)(\Lambda^s G(u))(\Lambda^s u) \right) dx - I(t)
\]

\[
\leq \frac{1}{2} ||w_t||_{L^\infty} ||u||_{H^s}^2 + (1 + ||w||_{L^\infty})(||F||_{H^s} ||u||_{H^s} + ||G(u)||_{H^s} ||u||_{H^s})
\]

\[
+ ||\alpha||_{L^1} (1 + ||w||_{L^\infty})(||w_x||_{L^\infty} + ||w||_{H^s}) ||u||_{H^s}^2.
\]

Since \( ||G(u(t))||_{H^s} \leq C_G ||u(t)||_{H^s} \), we have

\[
\frac{d}{dt} \mathcal{E}_t^2(t) \leq \left( \frac{1}{2} ||w_t||_{L^\infty} + (1 + ||w||_{L^\infty})(C_G + ||\alpha||_{L^1} (||w_x||_{L^\infty} + ||w||_{H^s})) \right) ||u||_{H^s}^2
\]

\[
+ (1 + ||w||_{L^\infty}) ||F||_{H^s} ||u||_{H^s}.
\]

(14)
But \( \|u(t)\|_{H^s} \leq \frac{2}{c_1} e^{c_2} (t) \) due to (8)–(9), so

\[
\frac{d}{dt} E_s(t) \leq C(\|w_t\|_{L^\infty} + (1 + \|w\|_{L^\infty})(C_G + \|\alpha\|_{L^1}(\|w_x\|_{L^\infty} + \|w\|_{H^r}))) E_s(t)
\]

\[+ C(1 + \|w\|_{L^\infty})\|F\|_{H^r},
\]

\[
\frac{d}{dt} E_s(t) \leq C(\|w_t\|_{L^\infty} + (1 + \|w\|_{L^\infty})(C_G + \|\alpha\|_{L^1}(\|w_x\|_{L^\infty} + \|w\|_{H^r}))) E_s(t)
\]

\[+ C(1 + \|w\|_{L^\infty})\|F\|_{H^r},
\]

\[
\leq A E_s(t) + B,
\]

where

\[
A = \sup_{0 \leq t \leq T} C(\|w(t)\|_{L^\infty} + (1 + \|w(t)\|_{L^\infty})(C_G + \|\alpha\|_{L^1}(\|w(t)\|_{H^r}))),
\]

\[
B = \sup_{0 \leq t \leq T} C(1 + \|w(t)\|_{L^\infty})\|F(t)\|_{H^r}.
\]

We note that when \( \alpha \) is a measure, the quantity \( \|\alpha\|_{L^1} \) should be replaced by \( |\alpha|([0,\infty)) \). An application of Gronwall’s inequality to (15) then yields

\[
E_s(t) \leq E_s(0) e^{At} + \frac{B}{A} (e^{At} - 1).
\]

Since \( E_s(t) \approx \|u(t)\|_{H^r} \), this completes the proof of (10).

The local well-posedness of the Cauchy problem for (1) follows from the standard hyperbolic approach [11], where the main tool is the energy estimate of Lemma 2.2. To be precise, we have the following theorem.

**Theorem 2.3:** Suppose \( s > 3/2, \ u_0 \in H^s \) with sufficiently small \( \|u_0\|_{H^r} \). Then there exists \( T > 0 \); so that the Cauchy problem for (1) with initial data \( u_0 \) has a unique solution \( u \in C([0, T], H^r) \cap C^1([0, T], H^{r-1}) \). Moreover, the existence time \( T \) is of the form \( T = T(\|u_0\|_{H^r}, \|\alpha\|_{L^1}) \); in other words it depends only on \( \|u_0\|_{H^r} \) and \( \|\alpha\|_{L^1} \).

The idea of the proof is as follows. Roughly speaking, the nonlinear problem (1) can be considered as the linearized problem (7) with \( w = (p + 1)u^p \) and \( F = G = 0 \). The condition \( 0 < c_1 \leq 1 + (p + 1)u^p \leq c_2 \) is then achieved at \( t = 0 \) by the smallness assumption of \( \|u_0\|_{H^r} \) and carried on by continuity for short times. From Lemma 2.2, we see that this can be achieved for small \( At \); in other words, the existence time should satisfy \( T = O(A^{-1}) \). Again when \( t = 0, A \) depends on \( \|u_0\|_{H^r} \) and \( \|\alpha\|_{L^1} \); by continuity, this can be carried over short times using the energy estimate, yielding \( T = T(\|u_0\|_{H^r}, \|\alpha\|_{L^1}) \).

**Remark 2.1:** Consider the parameter-dependent family of (2). Since \( \|\alpha_\delta\|_{L^1} = \|\alpha\|_{L^1}, \) the solutions \( u_\delta \) with the same initial data \( u_0 \) will have a uniform existence time \( T \) independent of \( \delta \). Moreover, again by Lemma 2.2, \( \|u_\delta(t)\|_{H^r} \) will be uniformly bounded in \( \delta \) and \( t \in [0, T] \).

**Remark 2.2:** If the above theorem is applied to the equation \( u_t + \alpha \ast (u + \epsilon^p u^{p+1})_x = 0 \) with small nonlinearity \( \epsilon^p u^{p+1} \), then Lemma 2.2 with \( F = G = 0 \) gives \( A = O(\epsilon^p) \). In other words, the existence time turns out to be \( T^\epsilon = O(\frac{1}{\epsilon^p}) \).
3. Comparison of solutions

In this section, we restrict ourselves to the kernel functions that have the same dispersive nature in the long-wave limit and we prove our main theorem, which states that the corresponding solutions with the same initial data are ‘close’ in a sense which will be made precise below.

Suppose that \( \alpha^{(1)} \) and \( \alpha^{(2)} \) are two different kernel functions satisfying the following conditions for some \( k \geq 1 \):

\[
\begin{align*}
\text{(C1)} \quad &\text{\( \alpha^{(1)} \) and \( \alpha^{(2)} \) have the same first (2k − 1)-order moments, namely} \\
&\int x^j \alpha^{(1)}(x) \, dx = \int x^j \alpha^{(2)}(x) \, dx \quad \text{for } 0 \leq j < 2k - 1, \\
\text{(C2)} \quad &\text{\( x^{2k} \alpha^{(i)}(x) \in L^1(\mathbb{R}) \) for } i = 1, 2.
\end{align*}
\]

Notice that if \( \alpha = \mu \) is a finite measure, we should replace the moment integral by \( \int x^j \, d\mu \). We now consider the Cauchy problem for (2) with initial data \( u_0 \). Let \( u_1^\delta \) and \( u_2^\delta \) be solutions of the Cauchy problem, corresponding to the kernels \( \alpha_1^\delta \) and \( \alpha_2^\delta \), respectively. In the following main result of this paper, we prove that the solutions \( u_1^\delta \) and \( u_2^\delta \) are ‘close’ to each other. The idea of the proof is similar to that already sketched in Theorem 3.2 of [4].

**Theorem 3.1:** Let \( \alpha^{(1)} \) and \( \alpha^{(2)} \) be two kernels satisfying the conditions (C1) and (C2) for some \( k \geq 1 \). Let \( s > 3/2 \) and \( u_0 \in H^{s+2k+1} \) with sufficiently small \( \|u_0\|_{H^{s+2k+1}} \). Then there are some constants \( C \) and \( T > 0 \) independent of \( \delta \) so that the solutions \( u_1^\delta \) of the Cauchy problems

\[
\begin{align*}
(u_i)_t + \alpha_1^{(i)}(u_i + u_i^{p+1})_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \\
u_i(x, 0) = u_0(x), \quad x \in \mathbb{R},
\end{align*}
\]

for \( i = 1, 2 \) are defined for all \( t \in [0, T] \) and satisfy

\[
\|u_1^\delta(t) - u_2^\delta(t)\|_{H^s} \leq C\delta^{2k} t \quad \text{for all } t \leq T.
\]

**Proof:** The proof will be split into several steps.

**Step 1.** Let \( u_0 \in H^{s+2k+1} \). By Theorem 2.3 and Remark 2.1 applied with \( s \) replaced by \( s + 2k + 1 \), solutions \( u_i^\delta \) exist in \( C([0, T], H^{s+2k+1}) \cap C([0, T], H^{s+k}) \), and we have the uniform existence time and the uniform bound

\[
\|u_i^\delta(t)\|_{H^{s+2k+1}} \leq C \quad \text{for all } \delta \quad \text{and} \quad t \leq T,
\]

for both families of solutions.

**Step 2.** From now on, we will drop the superscript \( \delta \) to simplify the presentation. We will use the following form of (17)–(18)

\[
(u_i)_t + K_1^{(i)}(u_i + u_i^{p+1})_x = 0, \quad u_i(x, 0) = u_0(x)
\]

for \( i = 1, 2 \). Let \( r \) denote the difference between the solutions \( u_1 \) and \( u_2 \), i.e. \( r = u_1 - u_2 \). Then \( r \) satisfies

\[
r_t + K_1^{(1)}((1 + \nu)r_x) = F + G(r), \quad r(x, 0) = 0,
\]

where

\[
F = (K_1^{(2)} - K_1^{(1)})(u_2)_x + (p + 1)u_1^p(u_2)_x,
\]

\[
G(r) = \int K_1^{(2)}(u_1^p - u_1) \, d\mu.
\]
We have
\[ G(r) = G(u_1 - u_2) = (p + 1)K^{(2)}_\delta ((u_2^p - u_1^p)(u_2)_x), \] (23)
\[ w = (p + 1)u_1^p. \] (24)

**Step 3.** Under the conditions (C1) and (C2) described above, the Fourier transforms of the kernels satisfy $\alpha^{(i)} \in C^{2k}$ for $i = 1, 2$ and
\[ \frac{d^j}{d\xi^j} (\hat{\alpha}^{(2)}(\xi) - \hat{\alpha}^{(1)}(\xi)) \bigg|_{\xi = 0} = 0 \quad \text{for } 0 \leq j < 2k - 1. \] (25)

Then $\hat{\alpha}^{(2)}(\xi) - \hat{\alpha}^{(1)}(\xi) = \mathcal{O}(\xi^{2k})$ near the origin and we have
\[ \hat{\alpha}^{(2)}(\xi) - \hat{\alpha}^{(1)}(\xi) = \xi^{2k} m(\xi) \]
for some continuous function $m$. Since both $\hat{\alpha}^{(1)}(\xi)$ and $\hat{\alpha}^{(2)}(\xi)$ are bounded, $m(\xi)$ is also bounded. With $\alpha^{(i)}_\delta(\xi) = \hat{\alpha}^{(i)}(\delta \xi)$ for $i = 1, 2$, the associated operators $K^{(1)}_\delta$ and $K^{(2)}_\delta$ will satisfy
\[ K^{(2)}_\delta = K^{(1)}_\delta + (-1)^k \delta^{2k} M_\delta, \]
where $M_\delta$ is the operator with symbol $m(\delta \xi)$. By the boundedness of $m$, it follows that
\[ \| (K^{(2)}_\delta - K^{(1)}_\delta) u \|_{H^p} = \delta^{2k} \| D_x^{2k} M_\delta u \|_{H^p} \leq C \delta^{2k} \| u \|_{H^{p+2k}}, \] (26)
with the constant $C$ independent of $\delta$.

**Step 4.** We now estimate the terms $w, F$ and $G$ in (21). From the definition $w = (p + 1)u_1^p$, we get
\[ \| w(t) \|_{H^p} \leq C \| u_1(t) \|_{H^p}^p \leq C, \] (27)
\[ \| w(t) \|_{H^{p+1}} = \| p(p + 1)u_1^{p-1}(t)(u_1)_x(t) \|_{H^{p+1}} \]
\[ \leq C \| u_1(t) \|_{H^{p+1}}^p \| (u_1)_x(t) \|_{H^{p+1}} \leq C. \] (28)

We have
\[ \| F(t) \|_{H^p} = \| (K^{(2)}_\delta - K^{(1)}_\delta)((u_2)_x + (p + 1)u_1^p(u_2)_x) \|_{H^p} \]
\[ \leq C \delta^{2k} \| (u_2)_x \|_{H^{p+2k}} + (p + 1) \| u_1^p(u_2)_x \|_{H^{p+2k}} \]
\[ \leq C \delta^{2k} \| u_2(t) \|_{H^{p+2k+1}} + \| u_1(t) \|_{H^{p+2k}}^p \| u_2(t) \|_{H^{p+2k+1}} \]
\[ \leq C \delta^{2k}, \] (29)

where $\| u_i(t) \|_{H^{p+2k+1}}$ for $i = 1, 2$ are bounded. Similarly,
\[ \| G(r)(t) \|_{H^p} = \| (p + 1)K^{(2)}_\delta ((u_2^p - u_1^p)(u_2)_x)(t) \|_{H^p} \]
\[ \leq C \| \alpha^{(2)}_\delta \|_{L_1} \| r(t) \|_{H^p} \| u_2(t) \|_{H^{p+1}} \leq C G \| r(t) \|_{H^p}, \] (30)

where we have used the facts that $\| u_i(t) \|_{H^p} \leq C$ and
\[ u_2^p - u_1^p = (u_2 - u_1)(u_2^{p-1} + u_2^{p-2}u_1 + \cdots + u_2u_1^{p-2} + u_1^{p-1}). \]

Step 5 We now apply Lemma 2.2 to the solution $r$ of the initial-value problem (21). Since $\| r(0) \|_{H^p} = 0$, by (10) we have the estimate
\[ \| r(t) \|_{H^p} \leq \frac{B}{A} (e^{At} - 1) \quad \text{for all } t \leq T, \] (31)
where \( A \) and \( B \) are given by (11) (with \( \alpha \) replaced by \( \alpha^{(1)} \)) and (12), respectively. Using the estimates on \( \|w\|_{H^s}, \|w_t\|_{H^{s-1}}, \|F\|_{H^s} \) and \( \|G\|_{H^s} \) given by (27), (28), (29) and (30), respectively, in (11) and (12), we get from (31) that
\[
\|r(t)\|_{H^s} = \|u_1(t) - u_2(t)\|_{H^s} \leq C\delta^{2k}t \quad \text{for all } t \leq T.
\]
This completes the proof.

4. Convergence to the hyperbolic conservation law

In this section, we provide a justification of the convergence of the non-local (dispersive) models to the hyperbolic conservation law in the limit of vanishing non-locality, that is, in the zero-dispersion limit.

When the kernel \( \alpha \) is taken as the Dirac measure, (2) becomes the hyperbolic conservation law
\[
\frac{du}{dt} + u_x + (u^{p+1})_x = 0.
\]
This equation is also called the inviscid Burgers equation or the Hopf equation and arises in various fields such as fluid dynamics, traffic flow, acoustics. The non-local equation (2) with a general kernel function may be considered as a dispersive regularization of (32). From now on, we take one of the two kernel functions considered in the previous section as the Dirac measure. One question for (32) is whether solutions of the non-local wave equation (2) converge to the solution of the non-dispersive equation (32) when the dispersion parameter \( \delta \) tends to zero. Under rather general assumptions this question has been answered in the following comparison result which is an immediate consequence of Theorem 3.1. So, for sufficiently smooth initial conditions, the solutions of (2) and (32) approximate each other with an approximation error of order \( \delta^{2k} \) as the kernel \( \alpha \) approaches the Dirac distribution.

**Theorem 4.1:** Suppose that the zeroth-order moment of \( \alpha \) is 1 and that the first \((2k - 1)\)-order moments of \( \alpha \) for some \( k \geq 1 \) are 0. Let \( s > 3/2 \) and \( u_0 \in H^{s+2k+1} \) with sufficiently small \( \|u_0\|_{H^{s+2k+1}} \). Suppose also that \( u^\delta \) and \( u \) satisfy (2) and (32), respectively, with the initial data \( u^\delta(x,0) = u(x,0) = u_0(x) \). Then there are some constants \( C \) and \( T > 0 \) independent of \( \delta \) so that
\[
\|u^\delta(t) - u(t)\|_{H^s} \leq C\delta^{2k}t \quad \text{for all } t \leq T.
\]

We will consider some examples of the kernel functions for which one would expect convergence of the non-local (dispersive) solution of (2) to the solution of (32). As is expected, the smaller the dispersion parameter, the closer the non-local (dispersive) solution of (2) is to the solution of (32). We start with considering the kernel function \( \alpha_\delta(x) \) whose Fourier transform is
\[
\hat{\alpha}_\delta(\xi) = (1 + \delta^{2k}\xi^{2k})^{-1}
\]
with \( k \geq 1 \). The dispersive wave equation corresponding to \( \alpha_\delta(x) \) is
\[
Lu_t + u_x + (u^{p+1})_x = 0, \quad L = 1 - (-1)^{k-1}\delta^{2k}D_x^{2k}.
\]
As both of the Fourier transforms of \( \alpha_\delta(x) \) and the Dirac measure have exactly the same Taylor expansion (around the origin) to order \( 2k-1 \), the solutions of (34) and (32) approximate each other with an error of order \( \delta^{2k} \) by Theorem 4.1. We recall that if \( (\hat{\alpha}_\delta(\xi))^{-1} \) is a polynomial in \( \xi \), then (2) takes the form of a differential equation rather than an integro-differential equation. When \( k = 1 \), \( \alpha_\delta(x) \) is the exponential kernel \( \alpha_\delta(x) = \frac{1}{2\pi} e^{-|x|/\delta} \) and (34) reduces to the BBM equation (3). Similarly, when \( k = 2 \), \( \alpha_\delta(x) \) is given by (5) and (34) becomes the Rosenau equation (4). So the approximation error in Theorem 4.1 will be of the order of \( \delta^2 \) and \( \delta^4 \) for the BBM and Rosenau approximations of (32), respectively.
More generally, we can consider the fractional BBM-type equation
\[ u_t + u_x + \delta^{2\gamma}(-D_x^2)^\gamma u_t + (u^{p+1})_x = 0, \]  
(35)
where \( \gamma > 0 \) is not necessarily an integer. This equation is of the form (2) with the kernel defined by \( \widehat{\alpha}_\delta(\xi) = (1 + \delta^{2\gamma}|\xi|^{2\gamma})^{-1} \).

Since \( \widehat{\alpha}_\delta(\xi) = 1 - \delta^{2\gamma}|\xi|^{2\gamma} + O(|\xi|^{2\gamma+1}) \) near the origin, the approximation error between solutions of the fractional equation (35) and the hyperbolic conservation law (32) is order \( O(\delta^{2\gamma} t) \) by Theorem 4.1.

We now consider the rectangular kernel defined by
\[ \alpha_\delta(x) = \frac{1}{\delta} \begin{cases} 1 & \text{for } |x| \leq \frac{\delta}{2}, \\ 0 & \text{for } |x| > \frac{\delta}{2}, \end{cases} \]
for which \( \widehat{\alpha}_\delta(\xi) = \frac{2}{\delta \xi} \sin(\frac{\delta \xi}{2}) \). Then (2) becomes the differential-difference equation
\[ u_t + \nabla^d_\delta (u + u^{p+1}) = 0 \]
(36)
if we use the difference operator
\[ \nabla^d_\delta v(x, t) = \frac{1}{\delta} \left( v \left( x + \frac{\delta}{2}, t \right) - v \left( x - \frac{\delta}{2}, t \right) \right). \]  
(37)
As a member of the class (2), the differential-difference equation (36) is the standard second-order (a three-point stencil) central finite-difference scheme in space for the hyperbolic conservation law (32). Since \( \widehat{\alpha}_\delta(\xi) = 1 - \delta^2 \xi^2/24 + \cdots \) about the origin, Theorem 4.1 tells us that the difference of the solutions of (36) and (32) with the same initial data tends to zero as \( \delta \to 0 \) with a rate of \( \delta^2 \).

As another example, we now consider the kernel defined by
\[ \alpha_\delta(x) = \frac{1}{6\delta} \begin{cases} -1 & \text{for } -\delta \leq x < -\frac{\delta}{2}, \\ 7 & \text{for } -\frac{\delta}{2} \leq x < \frac{\delta}{2}, \\ -1 & \text{for } \frac{\delta}{2} \leq x < \delta, \\ 0 & \text{otherwise} \end{cases} \]
for which \( \widehat{\alpha}_\delta(\xi) = \frac{8}{3\delta \xi} \sin(\frac{\delta \xi}{2}) - \frac{1}{3\delta \xi} \sin(\delta \xi) \). Then (2) becomes the differential-difference equation
\[ u_t + \tilde{\nabla}^d_\delta (u + u^{p+1}) = 0 \]
(38)
if we use the difference operator
\[ \tilde{\nabla}^d_\delta v(x, t) = \frac{1}{6\delta} \left( -v(x+\delta, t) + 8v \left( x + \frac{\delta}{2}, t \right) - 8v \left( x - \frac{\delta}{2}, t \right) + v(x-\delta, t) \right). \]
(39)
As a member of the class (2), the differential-difference equation (38) is the standard fourth-order (a five-point stencil) central finite-difference scheme in space for the hyperbolic conservation law (32). Since \( \widehat{\alpha}_\delta(\xi) = 1 - \delta^4 \xi^4/480 + \cdots \) about the origin, by Theorem 4.1 we know that the difference between the solutions of (38) and (32) with the same initial data tends to zero at a quartic rate as \( \delta \to 0 \).

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References

[1] Erbay HA, Erbay S, Erkip A. A semi-discrete numerical method for convolution type unidirectional wave equations. J Comput Appl Math. 2021;387:112496.

[2] Benjamin TB, Bona JL, Mahony JJ. Model equations for long waves in nonlinear dispersive systems. Philos Trans R Soc Lond A: Math Phys Sci. 1972;272:47–78.

[3] Rosenau P. Dynamics of dense discrete systems: high order effects. Prog Theor Phys. 1988;79:1028–1042.

[4] Erbay HA, Erbay S, Erkip A. Comparison of nonlocal nonlinear wave equations in the long-wave limit. Appl Anal. 2020;99:2670–2679.

[5] Bona JL, Colin T, Lannes D. Long wave approximations for water waves. Arch Ration Mech Anal. 2005;178:373–410.

[6] Constantin A, Lannes D. The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations. Arch Ration Mech Anal. 2009;192:165–186.

[7] Duchene V, Israwi S, Talhouk R. A new fully justified asymptotic model for the propagation of internal waves in the Camassa-Holm regime. SIAM J Math Anal. 2015;47:240–290.

[8] Lannes D. The water waves problem: mathematical analysis and asymptotics. Providence (RI): American Mathematical Society; 2013. (AMS mathematical surveys and monographs; vol. 188).

[9] Lannes D, Linares F, Saut JC. The Cauchy problem for the Euler–Poisson system and derivation of the Zakharov–Kuznetsov equation. Prog Nonlinear Differ Equ Appl. 2013;84:181–213.

[10] Kato T, Ponce G. Commutator estimates and the Euler and Navier–Stokes equations. Commun Pure Appl Math. 1988;41:891–907.

[11] Taylor ME. Partial differential equations II: qualitative studies of linear equations. 2nd ed. New York (NY): Springer; 2011.