Two-dimensional Yang-Mills theory: perturbative and instanton contributions, and its relation to QCD in higher dimensions

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Two different scenarios (light-front and equal-time) are possible for Yang-Mills theories in two dimensions. The exact $\bar{q}q$-potential can be derived in perturbation theory starting from the light-front vacuum, but requires essential instanton contributions in the equal-time formulation. In higher dimensions no exact result is available and, paradoxically, only the latter formulation (equal-time) is acceptable, at least in a perturbative context.

1. INTRODUCTION

The non-perturbative structure of non-abelian quantum gauge theories is still a challenging topic in spite of a large amount of efforts in the recent literature. Whereas perturbation theory provides a well-established frame to describe the weak-coupling regime, quantitative predictions for the strong-coupling behaviour are extremely hard to be obtained. Though some non-perturbative features are thought to be transparent, a consistent framework in the continuum is still lacking.

Such problems have often been tackled in the simplified context of two-dimensional gauge theories ($YM_{2}$), taking advantage of the lattice solutions [1]. As far as the continuum is concerned, in two dimensions the theory looks seemingly trivial when quantized in the light-cone gauge (LCG) $A_{-} \equiv A_{0} - A_{1} \sqrt{2} = 0$. As a matter of fact, in the absence of dynamical fermions, it looks indeed free, being described by a Lagrangian quadratic in the fields.

Still topological degrees of freedom occur if the theory is put on a (partially or totally) compact manifold, whereas the simpler behavior on the plane enforced by the LCG condition entails a severe worsening in its infrared structure. These features are related aspects of the same basic issue: even in two dimensions ($D = 2$) the theory contains non-trivial dynamics, as immediately suggested by other gauge choices as well as by perturbative calculations of gauge invariant quantities, typically of Wilson loops[2]. We can say that, in LCG, dynamics gets hidden in the very singular nature of correlators at large distances (IR singularities).

In order to fully appreciate this point and the controversial aspects related to it, it is useful to review briefly the ’t Hooft’s model for $QCD_{2}$ at large $N$, $N$ being the number of colours [3]. In LCG no self-interaction occurs for the gauge fields; in the large-$N$ limit planar diagrams dominate and the $q\bar{q}$ interaction is mediated by the exchange

$$D(x) = -\frac{i}{2} |x^-| \delta(x^+),$$ (1)

which looks instantaneous if $x^+$ is considered a time variable. Eq.(1) is the Fourier transform of the quantity

$$\tilde{D}(k) = \frac{1}{k_-^2},$$ (2)

the singularity at $k_- = 0$ being interpreted as a Cauchy principal value. Such an expression in turn can be derived by quantizing the theory on the light-front (at equal $x^+$), $A_+ \sim$ behaving like a constraint [4].

The full set of planar diagrams can easily be summed, leading to a beautiful pattern of $q\bar{q}$-bound states with squared masses lying on rising Regge trajectories. This was the first evidence,
to our knowledge, of a stringy nature of QCD in its confining regime, reconciling dual models with a partonic field theory.

After this pioneering investigation, many interesting papers followed 't Hooft's approach, blooming into the more recent achievements of QCD2.

Still, if the theory within the same gauge choice is canonically quantized at equal times, a different expression is obtained for the exchange in eq.(1)

\[ D_c(x) = \frac{1}{2\pi} \frac{x^-}{-x^- + i\epsilon x^-}, \]  

and its Fourier transform

\[ \tilde{D}_c(k) = \frac{1}{(k^- + i\epsilon k^-)^2}, \]  

and can now be interpreted as a causal Feynman propagator \[ 2 \].

This expression, first proposed by Wu \[ 3 \], is nothing but the restriction at \( D = 2 \) of the prescription for the LCG vector propagator in four dimensions suggested by Mandelstam \[ 6 \] and Leibbrandt \[ 7 \] (ML), and derived in ref. \[ 8 \] by equal-time canonical quantization of the theory.

In dimensions higher than two, where “physical” degrees of freedom are switched on (transverse “gluons”), this causal prescription is the only acceptable one; indeed causality is mandatory in order to get correct analyticity properties, which in turn are the basis of any consistent renormalization program \[ 9 \]. It has been shown in perturbative calculations \[ 10 \] that agreement with Feynman gauge results can only be obtained if a causal propagator is used in LCG.

This causal behaviour is induced by the propagation of unphysical degrees of freedom (probability ghosts), which can be expunged from the “physical” Hilbert space, but still contribute in intermediate lines as timelike photons do in the QED Gupta-Bleuler quantization. The presence of those ghosts will have far-reaching consequences in our subsequent considerations.

When eq.(4) is used in summing the very same set of planar diagrams considered by 't Hooft, no rising Regge trajectories are found in the spectrum of the \( q\bar{q} \)-system. The bound-state integral equation looks difficult to be solved; early approximate treatments \[ 11,12 \] as well as a more detailed recent study \[ 13 \] indicate the presence of a massless solution, with a fairly obscure interpretation, at least in this context. Confinement seems lost.

Then, how can it be that the causal way to treat the infrared (IR) singularities, which is mandatory in higher dimensions, leads to a disastrous result when adopted at \( D = 2 \)? In order to get an answer we turn to an investigation concerning the \( q\bar{q} \)-potential.

2. THE WILSON LOOP

A very convenient gauge invariant way of looking at the \( q\bar{q} \)-potential is by first considering a rectangular Wilson loop \( \gamma \) with sides parallel to a spatial direction and to the time direction

\[ W_\gamma = \frac{1}{N} \langle 0 | T P \exp \left( i g \int_\gamma dx^u A_\mu(x) \right) | 0 \rangle, \]  

the symbols \( T \) and \( P \) denoting temporal ordering of operators and colour ordering, respectively.

The contour \( \gamma \) can be parametrized as

\[ \gamma_1 : \gamma_1^\mu(s) = (sT, L), \]
\[ \gamma_2 : \gamma_2^\mu(s) = (T, -sL), \]
\[ \gamma_3 : \gamma_3^\mu(s) = (-sT, -L), \]
\[ \gamma_4 : \gamma_4^\mu(s) = (-T, sL), \]

\( -1 \leq s \leq 1, \) describing a (counterclockwise-oriented) rectangle centered at the origin of the plane \((x^1, x^0)\), with sides of length \((2L, 2T)\), respectively.

It is well known that the Wilson loop we have hitherto introduced can be thought to describe the interaction of a couple of static \( q\bar{q} \) at the distance \( 2L \) from each other. If we denote by

\[ M(y, x; \Gamma) = \tilde{q}(y) E[\Gamma] q(x) \]  

the mesonic string operator, \( E[\Gamma] \) representing the gluon phase factor along the path \( \Gamma \) connecting the couple, we can consider the overlap between the states \( \tilde{q}q \) at the times \( t = -T \) and \( t = T \), that is the amplitude

\[ M(L, T) = \langle 0 | M^{(2L, T)} M(2L, -T) | 0 \rangle. \]  

If we insert a complete set of eigenstates \( | \Phi_n \rangle \) which diagonalizes the Hamiltonian of the system
\( \bar{q}q \) at the distance \( 2L \) with eigenvalues \( \mathcal{E}_n(L) \), we easily obtain
\[
M(L,T) = \sum_n |\langle 0 | \mathcal{M}(2L,0)|\Phi_n \rangle|^2 \exp(2i\mathcal{E}_n T). \tag{9}
\]

We can turn to the Euclidean formulation replacing \( T \) with \( iT \). If we denote by \( \mathcal{E}_0(L) \) the ground state energy of the system, we get
\[
M(L,T) = \exp(-2\mathcal{E}_0 T) \int_{\mathcal{E}_0}^{\infty} d\mathcal{E} \rho(\mathcal{L},\mathcal{E}) \exp[-2T(\mathcal{E} - \mathcal{E}_0)]. \tag{10}
\]

Unitarity requires the spectral density \( \rho(\mathcal{L},\mathcal{E}) \) to be a non-negative measure (see eq.\( \text{(9)} \)). Then \( M(L,T) \) is positive and the coefficient of the exponential factor \( \exp(-2\mathcal{E}_0 T) \) is a non-increasing function of \( T \).

In the limit \( T \to \infty \) one can show \([14]\) that the Wilson loop \( \mathcal{W}_\gamma \) is related to \( M(L,T) \) by a threshold factor
\[
\mathcal{W}_\gamma \simeq \exp(4mT)M(L,T), \tag{11}
\]
\( m \) being the static quark mass. In this way we can define the \( \bar{q}q \)-potential
\[
\mathcal{V}(L) = \mathcal{E}_0(L) - 2m.
\]
If the theory confines, \( \mathcal{V}(L) \) is an increasing function of the distance \( L \); if at large distances the increase is linear in \( L \), namely
\[
\mathcal{V}(L) \simeq 2\sigma L,
\]
we obtain an area-law behaviour for the leading exponent, characterized by a string tension \( \sigma \).

For \( D > 2 \) perturbation theory is unreliable in computing the true spectrum of the \( \bar{q}q \)-system. However, when combined with unitarity, it puts an intriguing constraint on the \( \bar{q}q \)-potential. To realize this point, let us consider the formal expansion
\[
\mathcal{V}(L) = g^2 \mathcal{V}_1(L) + g^4 \mathcal{V}_2(L) + \cdots, \tag{12}
\]
g being the QCD coupling constant. When inserted in the expression \( \exp[-2\mathcal{V}(L)T] \), it gives
\[
\exp[-2T\mathcal{V}] = 1 - 2T \left[ g^2 \mathcal{V}_1 + g^4 \mathcal{V}_2 + \cdots \right] + \frac{2T^2}{3} \left[ g^4 \mathcal{V}_1^2 + \mathcal{V}_1 \mathcal{V}_2 + \cdots \right] + \cdots. \tag{13}
\]

At \( \mathcal{O}(g^4) \), the coefficient of the leading term at large \( T \) should be half the square of the term at \( \mathcal{O}(g^2) \). This constraint has often been used as a check of (perturbative) gauge invariance \([4]\).

Therefore, if we denote by \( C_{F(A)} \) the quadratic Casimir expression for the fundamental (adjoint) representation of \( SU(N) \) and remember that \( \mathcal{V}_1 \) is proportional to \( C_F \), at \( \mathcal{O}(g^4) \) the term with the coefficient \( C_FC_A \) should be subleading in the large-\( T \) limit with respect to the Abelian-like term, which is proportional to \( C_F^2 \).

Such a calculation at \( \mathcal{O}(g^4) \) for the loop \( \mathcal{W}_\gamma \) has been performed using Feynman gauge in ref.\([5]\), with the number of space-time dimensions larger than two \( (D > 2) \). The parameter \( D \) acts also as a (gauge invariant) dimensional regulator.

The result depends on the area \( A = LT \) and on the dimensionless ratio \( \beta = \frac{D}{F} \). The \( \mathcal{O}(g^2) \)-term is obviously proportional to \( C_F \); at \( \mathcal{O}(g^4) \) we find that the non-Abelian term is indeed subleading
\[
T^2 \mathcal{V}^{na} \propto C_FC_AA^2 T^{4-2D}, \tag{14}
\]
as expected.

Therefore agreement with exponentiation holds and the validity of previous perturbative tests of gauge invariance in higher dimensions (see ref.\([8]\)) is vindicated. This rather simple way of realizing the exponentiation at \( D > 2 \) might have a deeper justification as well as far-reaching consequences.

The limit of our result when \( D \to 2 \) is finite and depends only on \( A \), as expected on the basis of the invariance of the theory in two dimensions under area-preserving diffeomorphisms. However the non-Abelian term is no longer subleading in the limit \( T \to \infty \), as it is clear from eq.\( \text{(14)} \); we get instead \([7]\)
\[
2T^2 \mathcal{V}^{na} = C_FC_AA^2 \frac{A^2}{16\pi^2} \left( 1 + \frac{\pi^2}{3} \right). \tag{15}
\]

We conclude that the limits \( T \to \infty \) and \( D \to 2 \) do not commute.

This result is confirmed by a calculation of \( \mathcal{W}_\gamma \) performed in LCG with the ML prescription for the vector propagator \([7, 7]\). At odds with Feynman gauge where the vector propagator is not a tempered distribution at \( D = 2 \), in LCG the calculation can also be performed directly in two
space-time dimensions. The result one obtains does not coincide with eq. \((15)\). One gets instead
\[
2T^2 V^{aa} = C_F C_A \frac{A^2}{48}.
\]

The extra term in eq. \((15)\) originates from the self-energy correction to the vector propagator. In spite of the fact that the triple vector vertex vanishes in two dimensions in LCG, the self-energy correction does not. What happens is that the vanishing of triple vertices when \(D \to 2\) is exactly compensated by the loop integration singularity at \(D = 2\) leading, eventually, to a finite result. We would like to stress that this “anomaly-like” contribution is not a pathology of LCG; precisely it is needed to get agreement with the Feynman gauge result.

Perturbation theory is discontinuous at \(D = 2\).

We conclude that the perturbative result, no matter the gauge one adopts, conflicts with unitarity in two dimensions. We recall that any acceptable gauge (including the causal formulation of LCG!) entails the presence of (probability) ghosts at \(D > 2\) (see ref. \([4]\)). At \(D = 2\) in LCG these ghosts are the only surviving degrees of freedom in perturbative YMT \([3]\); therefore a possible violation of unitarity is hardly surprising, although the reason why ghosts are so dangerous precisely in two dimensions is not yet fully clear.

Taking advantage of the invariance under area-preserving diffeomorphisms in dimensions \(D = 2\), Staudacher and Krauth \([18]\) were able to resum the perturbative series at all orders in the coupling constant \(g\) in LCG within the causal formulation, thereby generalizing our \(\mathcal{O}(g^4)\) result (eq. \((15)\)). In the Euclidean formulation, which is possible as the causal propagator can be Wick-rotated, and with a particular choice of the contour (a circumference), they show that the colour factors decouple from geometry and can be summed by the simple matrix integral
\[
\mathcal{W}_\gamma(A) = \frac{1}{Z} \int DF \exp(-\frac{1}{2} Tr F^2)
\]
\[
\frac{1}{N} Tr \exp(i g F \sqrt{\frac{A}{2}}),
\]
where, for \(U(N)\), \(DF\) denotes the flat integration measure on the space of constant Hermitian \(N \times N\) matrices and \(\mathcal{W}_\gamma(0) = 1\). They get
\[
\mathcal{W}_\gamma(A) = \frac{1}{N} \exp\left[-\frac{g^2 A}{4} \right] L_{N-1}^{(1)} \left(\frac{g^2 A}{2}\right),
\]
\((18)\)
the function \(L_{N-1}^{(1)}\) being a Laguerre polynomial.

This result can be easily generalized to a loop winding \(n\)-times around the contour
\[
\mathcal{W}_\gamma = \frac{1}{N} \exp\left[-\frac{g^2 A n^2}{4} \right] L_{N-1}^{(1)} \left(\frac{g^2 A n^2}{2}\right).
\]
\((19)\)
From eq. \((18)\) one immediately realizes that, for even values of \(N\), the result is no longer positive in the large-\(T\) limit. Moreover in the ’t Hooft’s limit \(N \to \infty\) with \(g^2 N = 2\tilde{g}^2\) fixed, the string tension vanishes and eq. \((18)\) becomes
\[
\mathcal{W}_\gamma \to \frac{1}{\sqrt{\tilde{g}^2 A}} J_1(2\sqrt{\tilde{g}^2 A}),
\]
\((20)\)
\(J_1\) being the usual Bessel function.

Confinement is lost.

This explains the failure of the Wu’s approach in getting a bound state spectrum lying on rising Regge trajectories in the large-\(N\) limit. The planar approximation is inadequate when the \(q\bar{q}\)-interaction is described in a causal way.

However in LCG the theory can also be quantized on the light-front (at equal \(x^+\)); with such a choice, in pure YMT and just in two dimensions, no dynamical degrees of freedom occur as the non vanishing component of the vector field does not propagate, but rather gives rise to an instantaneous (in \(x^+\)) Coulomb-like interaction (see eq. \((1)\)). There are no problems with causality and renormalization is no longer a concern.

If the Wilson loop \(\mathcal{W}_\gamma\) is perturbatively computed using expression \((1)\), only planar diagrams contribute for any value of \(N\), thanks to the “instantaneous” nature of such an exchange; the perturbative series can be easily resummed, leading to the result (for imaginary time)
\[
\mathcal{W}_\gamma(A) = \exp\left[-\frac{g^2 N A^4}{4}\right],
\]
\((21)\)
to be compared with eq. \((18)\).

Not only is this result in complete agreement with the exponentiation required by unitarity; it also exhibits, in the ’t Hooft’s limit \(N \to \infty\) with
Wilson loop for a pure "inside" the loop we keep finite in this limit. \( A \) will be sent to \( A \) loop winding around it a number smooth non self-intersecting closed contour and a in computing the spectrum of the \( q \) tension \( \sigma \) is straightforward) \([19]\). On \( S \) choice in higher dimensions (\( D > 2 \)) we are confronted with the following paradox: the absence of ghosts in this formulation; how-

The deep reason of this good behaviour lies in the absence of ghosts in this formulation; however we should stress again that it cannot be derived in a smooth way from any acceptable gauge choice in higher dimensions (\( D > 2 \)). Moreover the confinement exhibited at this stage is, in a sense, trivial, since it occurs also in the Abelian case \( U(1) \), namely QED\(_2\).

We end up with two basically different results for the same model and with the same gauge choice (LCG), according to the different ways in which IR singularities are regularized. Moreover we are confronted with the following paradox: the prescription which is mandatory in dimensions \( D > 2 \) is the one which fails at \( D = 2 \). What is the meaning (if any) of eq.(\ref{eq:23})?

3. THE GEOMETRICAL APPROACH

In order to understand this point, it is worthwhile to study the problem on a compact two-dimensional manifold; possible IR singularities will be automatically regularized in a gauge invariant way. For simplicity, we choose the sphere \( S^2 \). We also consider the slightly simpler case of the group \( U(N) \) (the generalization to \( SU(N) \) is straightforward) \([13]\). On \( S^2 \) we consider a smooth non self-intersecting closed contour and a loop winding around it a number \( n \) of times. We call \( A \) the total area of the sphere, which eventually will be sent to \( \infty \), whereas \( A \) will be the area “inside” the loop we keep finite in this limit.

Our starting point is the well-known heat-kernel expressions \([\text{I}]\) of a non self-intersecting Wilson loop for a pure \( U(N) \) YMT on a sphere \( S^2 \) with area \( A \)

\[
\mathcal{W}_n(A, \mathcal{A}) = \frac{1}{Z(A)} \sum_{r,s} d_{R,s} \exp \left[ -\frac{g^2 A}{4} C_2(R) - \frac{g^2 (A - A)}{4} C_2(S) \right]
\times \int dU \exp \left[ \frac{i}{nN} \chi_n(U) \chi_S^\dagger(U) \right], \quad (22)
\]

\( d_{R,S} \) being the dimension of the irreducible representation \( R(S) \) of \( U(N) \); \( C_2(R) \) (\( C_2(S) \)) is the quadratic Casimir expression, the integral in \((22)\) is over the \( U(N) \) group manifold while \( \chi_{R(S)} \) is the character of the group element \( U \) in the \( R(S) \) representation. In the sequel the partition function \( Z(A) \) will be rescaled by absorbing suitable factors without changing notation. Normalization will always be recovered by setting \( \mathcal{W}_n(A, 0) = 1 \).

We write eq.(\ref{eq:22}) explicitly for \( N > 1 \) and \( n > 0 \) in the form

\[
\mathcal{W}_n(A, \mathcal{A}) = \frac{1}{Z(A)} \sum_{m_i = -\infty}^{\infty} \Delta(m_1, ..., m_N) \times \Delta(m_1 + n, m_2, ..., m_N) \times \exp \left[ -\frac{g^2 A}{4} \sum_{i=1}^{N} (m_i)^2 \right] \times \exp \left[ -\frac{g^2 n}{4} (A - A)(n + 2m_1) \right]. \quad (23)
\]

We have described the generic irreducible representation by means of the set of integers \( m_i = (m_1, ..., m_N) \), related to the Young tableaux, in terms of which we get

\[
C_2(R) = \frac{N}{12} (N^2 - 1) + \sum_{i=1}^{N} (m_i - \frac{N - 1}{2})^2,
\]

\[
d_{R,S} = \Delta(m_1, ..., m_N). \quad (24)
\]

\( \Delta \) is the Vandermonde determinant and the integration in eq.(\ref{eq:22}) has been performed explicitly, using the well-known formula for the characters in terms of the set \( m_i \) and taking symmetry into account.

From eq.(\ref{eq:23}) it is possible to derive, for \( n = 1 \) and in the large-\( A \) decompactification limit, precisely the expression in eq.(\ref{eq:21}) we obtained by re-summing the perturbative series in the ‘t Hooft’s approach \([\text{I}3]\). This is a remarkable result as it has now been derived in a purely geometrical way without even fixing a gauge.

Actually, in the decompactification limit \( A \to \infty \) at fixed \( \mathcal{A} \), from eq.(\ref{eq:23}) one can derive the following expression for any value of \( n \) and \( N \) \([\text{I}3]\)

\[
\mathcal{W}_n(A; N) = \frac{1}{nN} \exp \left[ -\frac{g^2 A}{4} n(N + n - 1) \right]
\]
\[ \sum_{k=0}^n \frac{(-1)^k \Gamma(n + n - k)}{k! \Gamma(n - k) \Gamma(n - k)} \exp\left[\frac{g^2 A}{2} n k\right]. (25) \]

We notice that when \( n > 1 \) the simple abelian-like exponentiation is lost. In other words the theory starts feeling its non-abelian nature as the appearance of different “string tensions” makes clear. The winding number \( n \) probes its colour content. The related light-front vacuum, although simpler than the one in the equal-time quantization, cannot be considered trivial any longer. The above result does not seem related to any simple-minded reduction \( U(N) \sim U(1)^N \), as suggested by the abelianization of the theory in axial gauges.

Eq. (25) exhibits an interesting symmetry between \( N \) and \( n \). More precisely, we have that
\[
\mathcal{W}_n(A; N) = \mathcal{W}_N(\hat{A}; n), \quad \hat{A} = \frac{n}{N} A,
\]
a relation that is far from being trivial, involving an unexpected interplay between the geometrical and the algebraic structure of the theory [24,19].

Looking at eq. (25), the abelian-like exponentiation for \( U(N) \) when \( n = 1 \) appears to be related to the \( U(1) \) loop with \( N \) windings, the “genuine” triviality of Maxwell theory providing the expected behaviour for the string tension. Moreover we notice the intriguing feature that the large-\( N \) limit (with \( n \) fixed) is equivalent to the limit in which an infinite number of windings is considered with vanishing rescaled loop area. Alternatively, this rescaling could be thought to affect the coupling constant \( g^2 \to \frac{\pi}{N} g^2 \).

From eq. (25), in the limit \( N \to \infty \), one can recover the Kazakov-Kostov result [22]
\[
\mathcal{W}_N(A; \infty) = \frac{1}{n} f^{(1)}_{n-1} \left( \frac{g^2 A n}{2} \right) \exp\left[ -\frac{g^2 A n}{4} \right]. (27)
\]

Now, using eq. (24), we are able to perform another limit, namely \( n \to \infty \) with fixed \( n^2 A \)
\[
\lim_{n \to \infty} \mathcal{W}_n(A; N) = \frac{1}{N} \mathcal{L}^{(1)}_{n-1} \left( \frac{g^2 A n^2}{2} \right) \exp\left[ -\frac{g^2 A n^2}{4} \right]. (28)
\]

We remark that this large-\( n \) result reproduces the resummation of the perturbative series (for any \( n \)) (eq. [19]) in the causal formulation of the theory. This is not a coincidence, rather it provides a clue to understand its deep meaning.

We go back to the exact expressions we have found on the sphere for the Wilson loop (eq. 23). As first noted by Witten [23], it is possible to recast \( W_n(A, A) \) (and consequently \( Z(A) \)) as a sum over instable instantons, where each instanton contribution is associated to a finite, but not trivial, perturbative expansion. The easiest way to see it, is to perform a Poisson resummation [24,19]
\[
\sum_{m_i = -\infty}^{+\infty} F(m_1, ..., m_N) = \sum_{f_i = -\infty}^{+\infty} \tilde{F}(f_1, ..., f_N),
\]
\[
\tilde{F}(f_1, ..., f_N) = \int_{-\infty}^{+\infty}dz_1...dz_N F(z_1, ..., z_N) \times \exp\left[ 2\pi i (z_1 f_1 + ... + z_N f_N) \right] (29)
\]
in eq. (24). One gets
\[
\mathcal{W}_n(A, A) = \frac{1}{Z(A)} \exp\left[ \frac{g^2 n^2 (A - 2A)^2}{16 A} \right] \sum_{f_i = -\infty}^{+\infty} \exp\left[ -S_{inst}(f_i) \right] W(f_1, ..., f_N) \times \exp\left[ -2\pi i n f_1 (A - A) \right], (30)
\]
where
\[
S_{inst}(f_i) = \frac{4\pi^2}{g^2 A} \sum_{i=1}^{N} f_i^2, \quad (31)
\]
and
\[
W(f_1, ..., f_N) = \int_{-\infty}^{+\infty} dz_1...dz_N \times \exp\left[ -\frac{1}{g^2 A} \sum_{i=1}^{N} z_i^2 \right] \exp\left[ \frac{inz_1}{2} \right] \times \Delta(z_1 - 2\pi f_1, ..., z_N - 2\pi f_N) \times \Delta(z_1 + 2\pi f_1, ..., z_N + 2\pi f_N), (32)
\]
with
\[
\tilde{f}_i = f_1 + \frac{ig^2 n}{8\pi} (A - 2A).
\]

These formulæ have a nice interpretation in terms of instantons. Indeed, on \( S^2 \), there are non
trivial solutions of the Yang-Mills equation, labelled by the set of integers \( f_i = (f_1, \ldots, f_N) \)
\[
\mathcal{A}_\mu(x) = \begin{pmatrix} f_1A^0_\mu(x) & 0 & \ldots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f_NA^0_\mu(x) \end{pmatrix}
\] (33)
where \( \mathcal{A}^0_\mu(x) = \mathcal{A}^0_\mu(\theta, \phi) \) is the Dirac monopole potential,
\[
\mathcal{A}^0_\mu(\theta, \phi) = 0, \quad \mathcal{A}^0_\phi(\theta, \phi) = \frac{1 - \cos \theta}{2}.
\]
\( \theta \) and \( \phi \) being the polar (spherical) coordinates on \( S^2 \).

The term \( \exp \left[ -2\pi i f_1 \frac{A}{4} \right] \) in eq.(30) corresponds to the classical contribution of such field configurations to the Wilson loop.

From the above representation it is clear why the limit \( A \rightarrow \infty \) should not be performed too early. Indeed \( S_{\text{inst}}(f_i) \), for any finite set of \( f_i \), goes to zero in such a limit and fluctuations around an instanton solution become indistinguishable from fluctuations around the trivial field configuration.

Only the zero instanton contribution should be obtainable in principle by means of a perturbative calculation. Therefore in the following we single out the zero-instanton contribution \( (f_i = 0, \forall q) \) in eqs.(31) to the Wilson loop, obviously normalized to the zero instanton partition function.

The equation, after a suitable rescaling, becomes
\[
\mathcal{W}^0_n(A, A) = \frac{1}{Z^{(0)}(A)} \exp \left[ \frac{g^2n^2(A - 2A)^2}{16A} \right]
\]
\[
\times W_\Delta(0, \ldots, 0)
\] (34)
with
\[
W_\Delta(0, \ldots, 0) = \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \times
\]
\[
\exp \left[ -\frac{1}{2} \sum_{i=1}^{N} z_i^2 \right] \exp \left( \frac{in\sqrt{g^2Az_1}}{2\sqrt{2}} \right) \times
\]
\[
\Delta(z_1 - \frac{in}{4} \sqrt{\frac{2g^2}{A}} (A - 2A), z_2, \ldots, z_N) \times
\]
\[
\Delta(z_1 + \frac{in}{4} \sqrt{\frac{2g^2}{A}} (A - 2A), z_2, \ldots, z_N).
\] (35)

The two Vandermonde determinants can be expressed in terms of Hermite polynomials and then expanded in the usual way. The integrations over \( z_2, \ldots, z_N \) can be performed, taking the orthogonality of the polynomials into account; we get
\[
\mathcal{W}^0_n(A, A) = \exp \left[ \frac{g^2n^2(A - 2A)^2}{16A} \right]
\]
\[
\times \frac{1}{Z^{(0)}(A)} \prod_{n=0}^{N-1} \prod_{k=2}^{N} (jk - 1)! \epsilon_j^{j_1 \ldots j_N} \epsilon_{j_1 \ldots j_N}
\]
\[
\times \int_{-\infty}^{+\infty} dz_1 \exp \left[ -\frac{1}{2} z_1^2 \right] \exp \left( \frac{in\sqrt{g^2Az_1}}{2\sqrt{2}} \right)
\]
\[
H_{j_1-1}(z_1+)H_{j_1-1}(z_1-),
\] (36)
where
\[
z_{1\pm} = z_1 \pm \frac{in}{4} \sqrt{\frac{2g^2}{A}} (A - 2A).
\] (37)

Thanks to the relation
\[
\int_{-\infty}^{+\infty} dz_1 \exp \left[ -\frac{1}{2} z_1^2 \right] \exp \left( \frac{in\sqrt{g^2Az_1}}{2\sqrt{2}} \right)
\]
\[
H_{j_1-1}(z_1+)H_{j_1-1}(z_1-) = \sqrt{2\pi(j_1 - 1)!} \exp \left( -\frac{n^2g^2A}{16} \right) \times
\]
\[
L_{j_1-1} \left( \frac{n^2g^2A(A - 2A)}{2A} \right),
\] (38)
we obtain
\[
\mathcal{W}^0_n(A, A) = \frac{1}{N} L^{(1)}_{N-1} \left( \frac{g^2A(A - 2A)n^2}{2A} \right)
\]
\[
\exp \left[ -\frac{g^2(A - 2A)n^2}{4A} \right].
\] (39)

At this point we remark that, in the decompactification limit \( A \rightarrow \infty, A \) fixed, the quantity in the equation above exactly coincides, for any value of \( N \), with eq.(19), which was derived following completely different considerations. We recall indeed that eq.(19) was obtained by a full resummation of the perturbative expansion of the Wilson loop in terms of causal Yang-Mills propagators in LCG. Its meaning is elucidated by noting that it just represents the zero-instanton contribution to the Wilson loop, a genuinely perturbative quantity.
In turn it also coincides with the expression of the exact result in the large-\( n \) limit, keeping fixed the value of \( n^2A \) (eq.\((28)\)). This feature can be understood if we remember that instantons have a finite size; therefore small loops are essentially blind to them \((21)\).

4. THE GROUP ALGEBRA

Another remarkable result follows using the relation

\[
\int_{-\infty}^{+\infty} dz_1 \exp \left[ -\frac{1}{2} z_1^2 \right] \exp \left( \frac{in\sqrt{g^2A}z_1}{2\sqrt{2}} \right)
\]

\[
He_q(z_1^+)He_r(z_1^-) = \exp \left( -\frac{n^2g^2}{16A} [A - 2A]^2 \right) \times (A - A) \Delta(A - A) \Delta(z_1^+, z_1^-) \times \int_{-\infty}^{+\infty} dz_1 \exp \left( ingz_1 \sqrt{\frac{A(A - A)}{2A}} \right)
\]

\[
\times \exp \left[ -\frac{1}{2} z_1^2 \right] He_q(z_1^+)He_r(z_1^-), \quad n \geq 0. \quad (40)
\]

Thanks to it, for \( q = r = j_1 - 1 \), eqs.\((34,35)\) become

\[
W_n^{(0)}(A, A) = \frac{1}{N Z(0)} \sum_{k=1}^{N} W_k(0, ..., 0) \quad (41)
\]

with

\[
W_k(0, ..., 0) = \int_{-\infty}^{+\infty} dz_1...dz_N \times 
\exp \left[ -\frac{1}{2} \sum_{i=1}^{N} z_i^2 \right] \exp \left( ingz_k \sqrt{\frac{A(A - A)}{2A}} \right)
\]

\[
\left[ \Delta(z_1, z_2, ..., z_N) \right]^2. \quad (42)
\]

This expression is nothing but the matrix integral

\[
W_n^{(0)}(A, A) = \frac{1}{Z(0)} \int \mathcal{D}F \exp \left(-\frac{1}{2} TrF^2 \right)
\]

\[
\frac{1}{N} Tr \left[ \exp(igFE) \right]^n, \quad (43)
\]

where \( E = \sqrt{\frac{A(A - A)}{2A}} \) and \( \mathcal{D}F \) denotes the flat integration measure on the tangent space of constant Hermitian \( N \times N \) matrices.

Eqs.\((34,35)\) are the deep root of eq.\((17)\). They explicitly show that considering the zero instanton sector amounts to integrating over the group algebra. Eq.\((17)\) is then recovered after decompactification \((A \to \infty)\).

The correspondence between zero-instanton sector and group algebra is fairly general. Another example is provided by a Wilson loop calculation for the adjoint representation of \( SU(N) \) (for simplicity we average over different \( \theta \)-sectors)

\[
W_{adj}(A, A) = \frac{1}{Z(A)} \sum_{R,S} d_R d_S \times \quad (44)
\]

\[
\exp \left[ -\frac{g^2 A}{4} C_2(R) - \frac{g^2(A - A)}{4} C_2(S) \right] \times \int \frac{dU}{N^2 - \frac{1}{4} \left[ \text{Tr}U \right]^2} \left( \chi_R(U) \chi_S(U) \right).
\]

\( C_2(R) \) (\( C_2(S) \)) being now the quadratic Casimir expression of the \( SU(N) \) \( R(S) \)-irreducible representation \((26)\).

By repeating the expansion in terms of Young tableaux we get the analog of eq.\((23)\)

\[
W_{adj}(A, A) = \frac{1}{N + 1} \left( 1 + \frac{N}{Z(A)} \int_0^{2\pi} d\alpha \sum_{m_1, ..., m_N = -\infty}^{+\infty} \exp \left[ -\alpha + \frac{2\pi m - 1}{N} \right] \Delta(m_1, ..., m_N) \times 
\Delta(m_1 + 1, m_2 - 1, ..., m_N) \exp \left[ -\frac{g^2 A}{4} C(m_j) \right] \times \exp \left[ -\frac{g^2(A - A)}{2} \left( m_j - m_{j+1} \right) \right] \right), \quad (45)
\]

where \( m = \sum q m_q \) and

\[
C(m_j) = \sum_{j=1}^{N} \left( m_j - \frac{m}{N} \right)^2.
\]

After a Poisson resummation, eq.\((45)\) becomes

\[
W_{adj}(A, A) = \frac{1}{N + 1} + \frac{N \exp \left[ \frac{g^2(A - 2A)^2}{sA} \right]}{Z(A)(N + 1)} \times 
\sum_{f_1, ..., f_{N}} \delta \left( \sum_{j=1}^{N} f_j \right) \exp \left( -\frac{4\pi^2}{g^2A} \sum_{j=1}^{N} f_j^2 \right) \times
\]
exp \left[ 2\pi i(f_2 - f_1) \frac{A - A}{A} \right] \int_{-\infty}^{+\infty} dz_1 \cdots dz_N \\
 \times \exp \left[ -\frac{1}{g^2 A} \sum_{j=1}^{N} z_j^2 \right] \exp \left[ i \frac{1}{2} (z_1 - z_2) \right] (46) \\
 \times \Delta(z_1 - \tilde{f}_1, z_2 - \tilde{f}_2, z_3 - 2\pi f_3, \ldots, z_N - 2\pi f_N) \\
 \times \Delta(z_1 + \tilde{f}_1, z_2 + \tilde{f}_2, z_3 + 2\pi f_3, \ldots, z_N + 2\pi f_N),

\text{where } \tilde{f}_1 = 2\pi f_1 + \frac{ig^2(A-2A)}{4} \text{ and } \tilde{f}_2 = 2\pi f_2 - \frac{ig^2(A-2A)}{4}.

The zero-instanton contribution is again obtained by setting \( f_q = 0, \forall q \). We rescale the variables, express the Vandermonde determinants in terms of Hermite polynomials and then expand them using the completely antisymmetric tensor. We integrate over \( z_3, \ldots, z_N \) taking orthogonality into account. We are left with the expression

\[
\int_{-\infty}^{+\infty} dz_1 \exp \left[ -\frac{1}{2} z_1^2 \right] \exp \left( i \frac{g^2 A z_1}{2\sqrt{2}} \right) \\
He_{q_1} (z_1+) \times \\
H e_{j_1}(z_1-) \times \\
\int_{-\infty}^{+\infty} dz_2 \exp \left[ -\frac{1}{2} z_2^2 \right] \exp \left( i \frac{g^2 A z_2}{2\sqrt{2}} \right)
\]

\[
He_{j_2}(z_2+) \times \]

\[
He_{q_2} (z_2-) \times \]

\[
(47)
\]

Thanks to eq.\((46)\), with \( n = 1 \), the power factors cancel owing to the antisymmetric tensors and eq.\((46)\) for \( f_q = 0 \) becomes

\[
\mathcal{W}_{a0}^{(0)}(A, A) = \frac{1}{N+1} + \frac{N}{(N+1) \mathcal{Z}^{(0)}} \\
\int_{-\infty}^{+\infty} dz_1 \cdots dz_N \exp \left[ -\frac{1}{2} \sum_{j=1}^{N} z_j^2 \right] \times \\
\exp\left[ ig(z_1 - z_2) \sqrt{A(A - A)} \right] \times \\
\left( \Delta(z_1, z_2, z_3, \ldots, z_N) \right)^2 = \\
\frac{1}{2} \frac{1}{\mathcal{Z}^{(0)}} \int DF \exp\left( -\frac{1}{2} \text{Tr} F^2 \right) \times \\
\frac{1}{N^2 - 1} \left( |\text{Tr}\left[ \exp(igFE) \right]|^2 - 1 \right),
\]

(48)
to be compared with eq.\((44)\). The matrix \( F \) is traceless in this case.

5. CONCLUSIONS

We conclude that, in the equal-time formulation at \( D = 2 \), the area-law exponentiation (eq.\((21)\)) follows, after decompactification, only by resumming all instanton sectors, a procedure which changes completely the zero sector behaviour and, in particular, the value of the string tension.

In the equal-time scenario instantons are responsible of the restoration of unitarity, which was threatened by the presence of ghosts.

In the light of these considerations, there is no contradiction between the use of the causal prescription in the light-cone propagator and the exponentiation of eq.\((21)\); this prescription is correct but the ensuing perturbative calculation can only provide us with the expression for \( \mathcal{W}^{(0)} \).

We find quite remarkable that both quantities in eqs.\((18)\) and \((21)\) are (different) analytic functions of \( g^2 \). This is hardly surprising for eq.\((18)\), but not for eq.\((21)\), if it is thought as a sum over instanton contributions. Its analytic behaviour is at the root of the possibility of obtaining it by resumming a perturbative series in the light-front approach.

The result above corroborates a long-standing belief, namely that the light-front vacuum has a much simpler structure than the one in equal-time quantization. In two dimensions such a conjecture receives a precise quantitative support.

On the other hand the full Poisson resummation of non-analytic quantities (the instanton contributions) has to produce the analytic expression one expects for the Wilson loop in the heat kernel representation. The zero-instanton sector instead is analytic on its own and, we have shown, can be obtained by integrating over the group algebra.

We should perhaps conclude with a comment concerning higher dimensional cases, in particular \( D = 4 \).

We believe that most of our results are typical of the two-dimensional case. Perturbation theory at \( D = 2 + \epsilon \) is discontinuous in the limit \( \epsilon \to 0 \); on the other hand the invariance under area-preserving diffeomorphism is lost when \( D > 2 \). In a perturbative picture the presence of massless “transverse” degrees of freedom (the
forces a causal behaviour upon the relevant Green functions, whereas in the soft (IR) limit they get mixed with the vacuum. The light-front vacuum, which also in two dimensions is far from being trivial, in higher dimensions is likely to be simpler only as far as topological degrees of freedom are concerned. Of course there is no reason why it should coincide with the physical vacuum since, after confinement, the spectrum is likely to contain only massive excitations. Moreover, to be realistic, “matter” should be introduced, both in the fundamental and in the adjoint representation. Although many papers have appeared to this regard in the recent literature, we feel that further work is still needed to reach a satisfactory understanding.

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