Constructing expander graphs by 2-lifts and discrepancy vs. spectral gap

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Abstract

We present a new explicit construction for expander graphs with nearly optimal spectral gap. The construction is based on a series of 2-lift operations. Let $G$ be a graph on $n$ vertices. A 2-lift of $G$ is a graph $H$ on $2n$ vertices, with a covering map $\pi : H \to G$. It is not hard to see that all eigenvalues of $G$ are also eigenvalues of $H$. In addition, $H$ has $n$ “new” eigenvalues. We conjecture that every $d$-regular graph has a 2-lift such that all new eigenvalues are in the range $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ (If true, this is tight, e.g. by the Alon-Boppana bound). Here we show that every graph of maximal degree $d$ has a 2-lift such that all “new” eigenvalues are in the range $[-c\sqrt{d\log^3 d}, c\sqrt{d\log^3 d}]$ for some constant $c$. This leads to a polynomial time algorithm for constructing arbitrarily large $d$-regular graphs, with second eigenvalue $O(\sqrt{d\log^3 d})$.

The proof uses the following lemma (Lemma 3.3): Let $A$ be a real symmetric matrix with zeros on the diagonal. Let $d$ be such that the $l_1$ norm of each row in $A$ is at most $d$. Suppose that $\frac{|x^TAy|}{\|x\|\|y\|} \leq \alpha$ for every $x, y \in \{0,1\}^n$ with $\langle x, y \rangle = 0$. Then the spectral radius of $A$ is $O(\alpha(\log(d/\alpha) + 1))$. An interesting consequence of this lemma is a converse to the Expander Mixing Lemma.

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1 Introduction

An $d$-regular graph is called a $\lambda$-expander, if all its eigenvalues but the first are in $[-\lambda, \lambda]$. Such graphs are interesting when $d$ is fixed, $\lambda < d$, and the number of vertices in the graph tends to infinity. Applications of such graphs in computer science and discrete mathematics are many, see for example [22] for a survey.

It is known that random $d$-regular graphs are good expanders ([12], [17], [14]), yet many applications require an explicit construction. Some known construction appear in [24], [19], [7], [23], [5], [25], [1] and [28]). The Alon-Boppana bound says that $\lambda \geq 2\sqrt{d-1} - o(1)$ (cf. [27]). The graphs of [23] and [25] satisfy $\lambda \leq 2\sqrt{d-1}$, for infinitely many values of $d$, and are constructed very efficiently. However, the analysis of the eigenvalues in these construction relies on deep mathematical results. Thus, it is interesting to look for construction whose analysis is elementary.

The first major step in this direction is a construction based on iterative use of the zig-zag product [28]. This construction is simple to analyze, and is very explicit, yet the eigenvalue bound falls somewhat short of what might be hoped for. The graphs constructed with the zig-zag product have second eigenvalue $O(d^{3/4})$, which can be improved, with some additional effort to $O(d^{2/3})$. Here we introduce an iterative construction based on 2-lifts of graphs, which is close to being optimal and gives $\lambda = O(\sqrt{d \log^3 d})$.

A graph $\hat{G}$ is called a $k$-lift of a “base graph” $G$ if there is a $k : 1$ covering map $\pi : V(\hat{G}) \to V(G)$. Namely, if $y_1, \ldots, y_d \in G$ are the neighbors of $x \in G$, then every $x' \in \pi^{-1}(x)$ has exactly one vertex in each of the subsets $\pi^{-1}(y_i)$. See [9] for a general introduction to graph lifts.

The study of lifts of graphs has focused so far mainly on random lifts [9, 10, 11, 24, 16]. In particular, Amit and Linial show in [10] that w.h.p. a random $k$-lift has a strictly positive edge expansion. It is not hard to see that the eigenvalues of the base graph are also eigenvalues of the lifted graph. These are called by Joel Friedman the “old” eigenvalues of the lifted graph. In [16] he shows that w.h.p. a random $k$-lift of a $d$-regular graph on $n$ vertices is “weakly Ramanujan”. Namely, that all eigenvalues but, perhaps, those of the base graph, are, in absolute value, $O(d^{3/4})$. In both cases the probability tends to 1 as $k$ tends to infinity.

Here we study 2-lifts of graphs. We conjecture that every $d$ regular graph has a 2-lift with all new eigenvalues at most $2\sqrt{d-1}$ in absolute value. It
is not hard to show (e.g., using the Alon-Boppana bound [27]) that if this conjecture is true, it is tight. We prove (in Theorem 3.1) a slightly weaker result: every graph of maximal degree \(d\) has a 2-lift with all new eigenvalues \(O(\sqrt{d \log^3 d})\) in absolute value. Under some natural assumptions on the base graph, such a 2-lift can be found efficiently. This leads to a polynomial time algorithm for constructing families of \(d\)-regular expander graphs, with second eigenvalue \(O(\sqrt{d \log^3 d})\).

A useful property of expander graphs is the so-called Expander Mixing Lemma. Roughly, this lemma states that the number of edges between two subsets of vertices in an expander graph is what is expected in a random graph, up to an additive error that depends on the second eigenvalue.

A key lemma in this paper (Lemma 3.3) shows a close connection between the combinatorial discrepancy in a symmetric matrix, and its spectral radius. This key lemma implies the following converse to the Expander Mixing Lemma: Let \(G\) be a \(d\)-regular graph on \(n\) vertices, such that for every two subsets of vertices, \(A\) and \(B\), \(|e(A, B) - d|A||B|/n| \leq \alpha \sqrt{|A||B|}\) for some \(\alpha < d\). Then all eigenvalues of \(G\) but the first are, in absolute value, \(O(\alpha \log(d/\alpha))\).

The fact that the bound is tight up to a logarithmic factor is surprising. It is known that expansion implies a spectral gap (cf. [3]), but the actual bounds are weak, and indeed expansion alone does not imply strong bounds on the spectral gap [2].

The paper is organized as follows. After defining the basic objects - expander graphs, signed graphs and 2-lifts - in section 2 we present the main results in section 3. In sub-section 3.1 we observe that the spectrum of 2-lifts has a simple characterization, which suggests an iterative construction of expander graphs. It reduces the problem of constructing expander graphs to finding a signing of the edges with a small spectral radius. In sub-section 3.2 we show that such a signing always exists. In sub-section 3.3 we show how to find such a signing efficiently. An alternative method is given in section 4, which leads to a somewhat stronger notion of explicitness. Finally, in section 5 we prove the converse to the Expander Mixing Lemma mentioned above.

## 2 Definitions

Let \(G = (V, E)\) be a graph on \(n\) vertices, and let \(A\) be its adjacency matrix. Let \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\) be the eigenvalues of \(A\). We denote by \(\lambda(G) = \)
max_i=2,...,n |\lambda_i|. We say that G is an \((n, d, \mu) - expander\) if G is d-regular, and \(\lambda(G) \leq \mu\). If \(\lambda(G) \leq 2\sqrt{d-1}\) we say that G is Ramanujan.

A signing of the edges of G is a function \(s : E(G) \to \{-1, 1\}\). The signed adjacency matrix of a graph G with a signing s has rows and columns indexed by the vertices of G. The \((x, y)\) entry is \(s(x, y)\) if \((x, y) \in E\) and 0 otherwise.

A 2-lift of G, associated with a signing s, is a graph \(\hat{G}\) defined as follows. Associated with every vertex \(x \in V\) are two vertices, \(x_0\) and \(x_1\), called the fiber of \(x\). If \((x, y) \in E\), and \(s(x, y) = 1\) then the corresponding edges in \(\hat{G}\) are \((x_0, y_0)\) and \((x_1, y_1)\). If \(s(x, y) = -1\), then the corresponding edges in \(\hat{G}\) are \((x_0, y_1)\) and \((x_1, y_0)\). The graph G is called the base graph, and \(\hat{G}\) a 2-lift of G. By the spectral radius of a signing we refer to the spectral radius of the corresponding signed adjacency matrix. When the spectral radius of a signing of a d-regular graph is \(O(\sqrt{d})\) we say that the signing (or the lift) is Quasi-Ramanujan.

For \(v, u \in \{-1, 0, 1\}^n\), denote \(S(u) = \text{supp}(u)\), and \(S(u, v) = \text{supp}(u) \cup \text{supp}(v)\).

It will be convenient to assume throughout that \(V(G) = \{1, \ldots, n\}\).

### 3 Quasi-Ramanujan 2-Lifts and Quasi-Ramanujan Graphs

#### 3.1 Preliminaries

The eigenvalues of a 2-lift of G can be easily characterized in terms of the adjacency matrix and the signed adjacency matrix:

**Lemma 3.1** Let \(A\) be the adjacency matrix of a graph G, and \(A_s\) the signed adjacency matrix associated with a 2-lift \(\hat{G}\). Then every eigenvalue of \(A\) and every eigenvalue of \(A_s\) are eigenvalues of \(\hat{G}\). Furthermore, the multiplicity of each eigenvalue of \(\hat{G}\) is the sum of its multiplicities in \(A\) and \(A_s\).

**Proof:** It is not hard to see that the adjacency matrix of \(\hat{G}\) is:

\[
\hat{A} = \begin{pmatrix}
A_1 & A_2 \\
A_2 & A_1
\end{pmatrix}
\]
Where $A_1$ is the adjacency matrix of $(V, s^{-1}(1))$ and $A_2$ the adjacency matrix of $(V, s^{-1}(-1))$. (So $A = A_1 + A_2$, $A_s = A_1 - A_2$). Let $v$ be an eigenvector of $A$ with eigenvalue $\mu$. It is easy to check that $\hat{v} = (v, v)$ is an eigenvector of $\hat{A}$ with eigenvalue $\mu$.

Similarly, if $u$ is an eigenvector of $A_s$ with eigenvalue $\lambda$, then $\hat{u} = (u - u)$ is an eigenvector of $\hat{A}$ with eigenvalue $\lambda$.

As the $\hat{v}$'s and $\hat{u}$'s are perpendicular and $2n$ in number, they are all the eigenvectors of $\hat{A}$. 

We follow Friedman’s ([16]) nomenclature, and call the eigenvalues of $A$ the old eigenvalues of $\hat{G}$, and those of $A_s$ the new ones.

Consider the following scheme for constructing $(n, d, \lambda)$-expanders. Start with $G_0 = K_{d+1}$, the complete graph on $d + 1$ vertices \textsuperscript{1}. Its eigenvalues are $d$, with multiplicity 1, and $-1$, with multiplicity $d$. We want to define $G_i$ as a 2-lift of $G_{i-1}$, such that all new eigenvalues are in the range $[-\lambda, \lambda]$. Assuming such a 2-lifts always exist, the $G_i$ constitute an infinite family of $(n, d, \lambda)$-expanders.

It is therefore natural to look for the smallest $\lambda = \lambda(d)$ such that every graph of degree at most $d$ has a 2-lift, with new eigenvalues in the range $[-\lambda, \lambda]$. In other words, a signing with spectral radius $\leq \lambda$.

We note that $\lambda(d) \geq 2\sqrt{d - 1}$ follows from the Alon-Boppana bound. We next observe:

**Proposition 3.1** Let $G$ be a $d$-regular graph which contains a vertex that does not belong to any cycle of bounded length, then no signing of $G$ has spectral radius below $2\sqrt{d - 1} - o(1)$.

To see this, note first that all signing of a tree have the same spectral radius. This follows e.g., from the easy fact that any 2-lift of a tree is a union of two disjoint trees, isomorphic to the base graph. The assumption implies that $G$ contains an induced subgraph that is a full $d$-ary tree $T$ of unbounded radius. The spectral radius of $T$ is $2\sqrt{d - 1} - o(1)$. The conclusion follows now from the interlacing principle of eigenvalues.

There are several interesting examples of arbitrarily large $d$-regular graphs for which there is a signing with spectral radius bounded away from $2\sqrt{d - 1}$. One such example is the 3-regular graph $R$ defined as follows. $V(R) =$

\textsuperscript{1}We could start with any small $d$-regular graph with a large spectral gap. Such graphs are easy to find.
{0, \ldots, 2k - 1} \times \{0, 1\}. For i \in [2k], j \in \{0, 1\}, the neighbors of (i, j) \in R are ((i - 1) \mod 2k, j), ((i + 1) \mod 2k, j) and (i, 1 - j). Define s, a signing of R, to be \(-1\) on the edges ((2i, 0), (2i, 1)), for i \in \{0, \ldots, k - 1\}, and 1 elsewhere (see Figure 1). Let A_s be the signed adjacency matrix. It is easy to see that A_s^2 is a matrix with 3 on the diagonal, and two 1’s in each row and column. Thus, its spectral radius is 5, and that of A_s is \sqrt{5} < 2\sqrt{2}.

### 3.2 Quasi-ramanujan 2-lifts for every graph

We conjecture that every graph has a signing with small spectral radius:

**Conjecture 3.1** Every d-regular graph has a signing with spectral radius at most 2\sqrt{d - 1}.

We have numerically tested this conjecture quite extensively. In this subsection we show a close upper bound:

**Theorem 3.1** Every graph of maximal degree d has a signing with spectral radius \(O(\sqrt{d \cdot \log^3 d})\).

The theorem is an easy consequence of the following two lemmata (along with Lemma 3.1). The first shows that with positive probability the Rayleigh quotient is small for vectors in \(v, u \in \{-1, 0, 1\}^n\). The second shows how to conclude from this that all eigenvalues are small.

**Lemma 3.2** For every graph of maximal degree d, there exists a signing s such that for all \(v, u \in \{-1, 0, 1\}^n\) the following holds:

\[
\frac{|v^t A_s u|}{||v|| ||u||} \leq 10\sqrt{d \log d},
\]

(1)
where $A_s$ is the signed adjacency matrix.

**Lemma 3.3** Let $A$ be an $n \times n$ real symmetric matrix such that the $l_1$ norm of each row in $A$ is at most $d$. Assume that for any two vectors, $u, v \in \{0, 1\}^n$, with $\text{supp}(u) \cap \text{supp}(v) = \emptyset$:

$$\frac{|u^t A v|}{||u|| ||v||} \leq \alpha,$$

and that all diagonal entries of $A$ are, in absolute value, $O(\alpha (\log(d/\alpha) + 1))$. Then the spectral radius of $A$ is $O(\alpha (\log(d/\alpha) + 1))$.

**Proof:** (Lemma 3.2) First note that it’s enough to prove this for $u$’s and $v$’s such that the set $S(u, v)$ spans a connected subgraph. Indeed, assume that the claim holds for all connected subgraphs and suppose that $S(u, v)$ is not connected. Split $u$ and $v$ according to the connected components of $S(u, v)$, and apply the claim to each component separately. Summing these up and using the Cauchy-Schwartz inequality, we conclude that the claim for $u$ and $v$ as well. So henceforth we assume that $S(u, v)$ is a connected.

Consider some $u, v \in \{-1, 0, 1\}^n$. Suppose we choose the sign of each edge uniformly at random. Denote the resulting signed adjacency matrix by $A_s$, and by $E_{u,v}$ the “bad” event that $\frac{|v^t A_s u|}{||v|| ||u||} > 10 \sqrt{d \log d}$. Assume w.l.o.g. that $|S(u)| \geq \frac{1}{2} |S(u, v)|$. By the Chernoff inequality ($v^t A_s u$ is the sum of independent variables, attaining values of either $\pm 1$ or $\pm 2$):

$$\Pr[E_{u,v}] \leq 2 \exp\left(-\frac{100d \log d |S(u)||S(v)|}{8 |e(S(u), S(v))|}\right)$$

$$\leq 2 \exp\left(-\frac{100d \log d |S(u)||S(v)|}{8d |S(v)|}\right)$$

$$< d^{-10|S(u,v)|}$$

We want to use the Lovász Local Lemma [13], with the following dependency graph on the $E_{u,v}$: There is an edge between $E_{u,v}$ and $E_{u',v'}$ if $S(u, v) \cap S(u', v') \neq \emptyset$. Denote $k = |S(u, v)|$. How many neighbors, $E_{u',v'}$, does $E_{u,v}$ have, with $|S(u', v')| = l$?

Since we are interested only in connected subsets, this is clearly bounded by the number of rooted directed subtrees on $l$ vertices, with a root in $S(u, v)$. 

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It is known (cf. [20]) that there are at most \( k^{d(l-1)} \approx kd^{l-1} \) such trees (a similar argument appears in [18]. The bound on the number of trees is essentially tight by [4]).

In order to apply the Local Lemma, we need to define for such \( u \) and \( v \) a numbers \( 0 \leq X_{u,v} < 1 \). It is required that:

\[
X_{u,v} \prod_{(u',v') : E_u,v \sim E_{u',v'}} (1 - X_{u',v'}) \geq d^{-10|S(u,v)|},
\]

(2)

Observe that for \( S \subseteq [n] \) there are at most \( 2^{|S|} \) distinct pairs \( v, u \in \{-1, 0, 1\}^n \) with such that \( S(u, v) = S \).

For all \( u, v \) set \( X_{u,v} = d^{-3k}, \) where \( k = |S(u, v)|. \) Then in (2) we get:

\[
X_{u,v} \cdot \prod_{(u',v') : E_u,v \sim E_{u',v'}} (1 - X_{u',v'}) = d^{-3k} \prod_{l=1}^{n} (1 - d^{-3l}) kd^{2l} \approx
d^{-3k} \exp(-k \sum_{l=1}^{n} d^{-3l} d^{2l}) \geq d^{-3k} e^{-2k} > d^{-10k}
\]
as required.

**Proof:** (Lemma 3.3) For simplicity, assume that all diagonal entries of \( A \) are zeros. We explain at the end of the proof how to deal with a general matrix (but in fact it does not matter for the purpose of this paper).

First note that our assumptions imply that for any \( u \in \{0, 1\}^n \),

\[
\frac{|uAu|}{||u||^2} \leq 2\alpha:
\]

For any \( u_1, u_2 \in \{0, 1\}^n \) such \( S(u_1) \cap S(u_2) = \emptyset \) we have that

\[
|u_1Au_2| \leq \alpha ||u_1|| ||u_2||.
\]

(3)

Let \( u \in \{0, 1\}^n \), and denote \( k = |S(u)|. \) Set \( K = \binom{k}{k/2} \). Summing up inequality (3) over all subsets of \( S(u) \) of size \( k/2 \), we have that:

\[
\sum_{u_1 : S(u_1) \subseteq S(u), |S(u_1)| = k/2} u_1A(u - u_1) \leq K\alpha k/2.
\]

For each \( i \neq j \in S(u) \), \( a_{i,j} \) is added up \( \binom{k-2}{k/2-1} \) times in the sum on the LHS, hence (since diagonal entries are zero):

\[
\left( \binom{k-2}{k/2-1} \right) uAu \leq K\alpha k/2,
\]

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or:

\[ uAu \leq \alpha 2k. \]

Next, it follows that for any \( u, v \in \{-1, 0, 1\}^n \), such that \( S(u) = S(v) \), or \( S(u) \cap S(v) = \emptyset \):

\[ \frac{|uAv|}{||u|| ||v||} \leq 4\alpha. \]

Fix \( u, v \in \{-1, 0, 1\}^n \). Denote \( u = u^+ - u^- \) and \( v = v^+ - v^- \), where \( u^+, u^-, v^+, v^- \in \{0, 1\}^n \), and \( S(u^+) \cap S(u^-) = S(v^+) \cap (v^-) = \emptyset \).

\[
|uAv| = |(u^+ - u^-)A(v^+ - v^-)| \\
\leq 2\alpha(||u^+||||v^+|| + ||u^-||||v^-|| + ||u^+||||v^-|| + ||u^-||||v^+||) \\
\leq 4\alpha \sqrt{||u^+||^2||v^+||^2 + ||u^-||^2||v^-||^2 + ||u^+||^2||v^-||^2 + ||u^-||^2||v^+||^2} \\
= 4\alpha \sqrt{(||u^+||^2 + ||u^-||^2)(||v^+||^2 + ||v^-||^2)} = 4\alpha ||u|| ||v||
\]

The first inequality follows from our assumption on vectors in \( \{0, 1\}^n \), and the second from the \( l_2 \) to \( l_1 \) norm ratio.

Fix \( x \in \mathbb{R}^n \). We need to show that \( \frac{|x^T Ax|}{||x||^2} = O(\alpha \log(d/\alpha)) \). By losing only a multiplicative factor of 2, we may assume that the absolute value of every non-zero entry in \( x \) is a negative powers of 2: Clearly we may assume that \( ||x||_\infty < \frac{1}{2} \). To bound the effect of rounding the coordinates, denote \( x_i = \pm (1 + \delta_i)2^{t_i} \), with \( 0 \leq \delta_i \leq 1 \) and \( t_i < -1 \), an integer. Now round \( x \) to a vector \( x' \) by choosing the value of \( x'_i \) to be \( \text{sign}(x_i) \cdot 2^{t_i+1} \) with probability \( \delta_i \) and \( \text{sign}(x_i) \cdot 2^{t_i} \) with probability \( 1 - \delta_i \). The expectation of \( x'_i \) is \( x_i \). As the coordinates of \( x' \) are chosen independently, and the diagonal entries of \( A \) are 0’s, the expectation of \( x'Ax' \) is \( xAx \). Thus, there’s a rounding, \( x' \), of \( x \), such that \( |xAx| \leq |x'Ax'| \). Clearly \( ||x'||^2 \leq 2||x||^2 \), so \( \frac{|x^T Ax|}{||x||^2} \leq \frac{|x'Ax'|}{||x'||^2} \).

Denote \( S_i = \{ j : x_j = \pm 2^{-i} \} \), \( s_i = |S_i| \). Denote by \( k \) the maximal index \( i \) such that \( s_i > 0 \). Denote by \( x^i \) the sign vector of \( x \) restricted to \( S_i \), that is, the vector whose \( j \)’th coordinate is the sign of \( x_j \) if \( j \in S_i \), and zero otherwise. By our assumptions, for all \( 1 \leq i \leq j \leq k \):

\[ |x^iAx^j| \leq \alpha \sqrt{s_i s_j}. \]  \hspace{1cm} (4)

Also, since the \( l_1 \) norm of each row is at most \( d \), for all \( 1 \leq i \leq k \):

\[ \sum_j |x^iAx^j| \leq ds_i. \]  \hspace{1cm} (5)
We wish to bound:

\[
\frac{|xAx|}{||x||^2} \leq \frac{\sum_{i,j=1}^{k} |x_i A x_j| 2^{-(i+j)}}{\sum_i 2^{-2i} s_i},
\]

Denote \(\gamma = \log(d/\alpha)\), \(q_i = s_i 2^{-2i}\) and \(Q = \sum_i q_i\). Add up inequalities (4) and (5) as follows. For \(i = j\) multiply inequality (4) by \(2^{-2i}\). When \(i < j \leq i + \gamma\) multiply it by \(2^{-(i+j)+1}\). Multiply inequality (5) by \(2^{-(2i+\gamma)}\). (We ignore inequalities (4) when \(j > i + \gamma\).)

We get that:

\[
\sum_i 2^{-2i} |x_i A x_i| + \sum_i \sum_{i < j \leq i + \gamma} 2^{-(i+j)+1} |x_i A x_j| + \sum_i 2^{-(2i+\gamma)} \sum_j |x_i A x_j| \\
\leq \sum_i \alpha q_i + \sum_i \sum_{i < j < i + \gamma} 2\sqrt{q_i q_j} + \sum_i 2^{-\gamma} d \cdot q_i \\
\leq \alpha \sum_i q_i + \alpha \sum_i \sum_{i < j < i + \gamma} (q_i + q_j) + 2^{-\gamma} d \sum_i q_i \\
< (2^{-\gamma} d + 2\gamma \alpha) \sum_i q_i = (\alpha + \alpha \log(d/\alpha)) Q.
\]

Note that the denominator in (6) is \(Q\), so to prove the lemma it’s enough to show that the numerator,

\[
\sum_{i < j} 2^{-(i+j+1)} |x_i A x_j| + \sum_i 2^{-2i} |x_i A x_i|,
\]

is bounded by

\[
\sum_i 2^{-2i} |x_i A x_i| + \sum_i \sum_{i \leq j \leq i + \gamma} 2^{-(i+j)+1} |x_i A x_j| + \sum_i 2^{-(2i+\gamma)} \sum_j |x_i A x_j|.
\]

Indeed, let us compare the coefficients of the terms \( |x_i A x_j|\) in both expressions (Since \( |x_i A x_j| = |x_j A x_i|\), it’s enough to consider \(i \leq j\)). For \(i = j\) this coefficient is \(2^{-2i}\) in (7), and \(2^{-2i} + 2^{-(2i+\gamma)}\) in (8). For \(i < j \leq i + \gamma\), it is \(2^{-(i+j)+1}\) in (7), and in (8) it is \(2^{-(i+j)+1} + 2^{-(2i+\gamma)} + 2^{-(2j+\gamma)}\). For \(j > i + \gamma\), in (7) the coefficient is again \(2^{-(i+j)+1}\). In (8) it is:

\[
2^{-(2i+\gamma)} + 2^{-(2j+\gamma)} > 2^{-(2i+\gamma)} \geq 2^{-(i+j)+1}.
\]

It remains to show that the lemma holds when the diagonal entries of \(A\) are not zeros, but \(O(\alpha(\log(d/\alpha) + 1))\) in absolute value. Denote \(B = A - D\), with
$D$ the matrix having the entries of $A$ on the diagonal, and zero elsewhere.
We have that for any two vectors, $u, v \in \{0, 1\}^n$, with $\text{supp}(u) \cap \text{supp}(v) = \emptyset$:
\[
\frac{|uAv|}{||u|| ||v||} \leq \alpha.
\]
For such vectors, $uBv = uAv$, so by applying the lemma to $B$, we get that its spectral radius is $O(\alpha(\log(d/\alpha) + 1))$. By the assumption on the diagonal entries of $A$, the spectral radius of $D$ is $O(\alpha(\log(d/\alpha) + 1))$ as well. The spectral radius of $B$ is at most the sum of these bounds - also $O(\alpha(\log(d/\alpha) + 1))$.

Note that Lemma 3.3 is tight up to constant factors. To see this, consider the $n$-dimensional vector $x$ whose $i$'th entry is $1/\sqrt{i}$. Let $A$ be the outer product of $x$ with itself, that is, the matrix whose $(i,j)$'th entry is $1/\sqrt{i \cdot j}$. Clearly $x$ is an eigenvector of $A$ corresponding to the eigenvalue $||x||^2 = \Theta(\log(n))$. Also, the sum of each row in $A$ is $O(\sqrt{n})$. To prove that the lemma is essentially tight, we need to show that $\max_{u,v \in \{0,1\}^n} \frac{uAv}{||u|| ||v||}$ is constant. Indeed, fix $k, l \in [n]$. Let $u, v \in \{0, 1\}^n$ be such that $||u|| = k$ and $||v|| = l$. As the entries of $A$ are decreasing along the rows and the columns, $uAv$ is maximized for such vectors when their support is the first $k$ and $l$ coordinates. For these optimal vectors, $uAv = \Theta(\sqrt{k \cdot l})$. Thus,
\[
\max_{u,v \in \{0,1\}^n} \frac{uAv}{||u|| ||v||} = \Theta(1).
\]
A more elaborate example appears in section 5.

### 3.3 An explicit construction of quasi-ramanujan graphs

For the purpose of constructing expanders, it is enough to prove a weaker version of Theorem 3.1. Roughly, that every expander graph has a 2-lift with small spectral radius. In this sub-section we show that when the base graph is a good expander (in the sense of the definition below), then w.h.p. a random 2-lift has a small spectral radius. We then derandomize the construction to get a deterministic polynomial time algorithm for constructing arbitrarily large expander graphs.

**Definition 3.1** We say that a graph $G$ on $n$ vertices is $(\beta, t)$-sparse if for every $u, v \in \{0, 1\}^n$, with $|S(u,v)| \leq t$,
\[
uAv \leq \beta ||u|| ||v||.
\]
Lemma 3.4 Let $A$ be the adjacency matrix of a $d$-regular $(\gamma(d), \log n)$-sparse graph on $n$ vertices, where $\gamma(d) = 10\sqrt{d \log d}$. Then for a random signing of $G$ (where the sign of each edge is chosen uniformly at random) the following hold w.h.p.:

1. $\forall u,v \in \{-1,0,1\}^n : |uA_s v| \leq \gamma(d) ||u|| ||v||$. 
2. $\hat{G}$ is $(\gamma(d), 1 + \log n)$-sparse

where $A_s$ is the random signed adjacency matrix, and $\hat{G}$ is the corresponding 2-lift.

Proof: Following the same arguments and notations as in the proof of Lemma 3.2, we have that there are at most $n \cdot d^k$ connected subsets of size $k$. With probability at most $d^{-10k}$ requirement (1) is violated for a given pair $u,v$ such that $|S(u,v)| = k$. Since for each $S$ there are at most $2^{4|S|}$ pairs $u,v$ such that $S(u,v) = S$, by the union bound, w.h.p. no pair $u,v$ such that $|S(u,v)| > \log n$ violates (1). If $|S(u,v)| \leq \log n$ then by (2) there are simply not enough edges between $S(u)$ and $S(v)$ for (1) to be violated.

Next, we show that w.h.p. (2) holds as well. Let $s$ be a signing, and define $A_1$, $A_2$ and $\hat{A}$ as in Lemma 3.1. Given $u = (u_1 u_2)$, $v = (v_1 v_2) \in \{0,1\}^n \times \{0,1\}^n$, we wish to prove that $uA_s v \leq \gamma(d) ||u|| ||v||$. As in the proof of Lemma 3.2 we may assume that $S(u,v)$ is connected - in fact, that it is connected via the edges between $S(u)$ and $S(v)$. Hence, we may assume that the ratio of the sizes of these subsets is at most $d$. Define $x = u_1 \lor u_2$, $y = v_1 \lor v_2$, $x' = u_1 \land u_2$, and $y' = v_1 \land v_2$ (the characteristic vectors of $S(u_1, u_2)$, $S(v_1, v_2)$, $S(u_1) \cap S(u_2)$ and $S(v_1) \cap S(v_2)$). It is not hard to verify that:

$$u\hat{A}v = u_1 A_1 v_1 + u_1 A_2 v_2 + u_2 A_2 v_1 + u_2 A_1 v_2 \leq xAy + x'Ay'. \quad (9)$$

If $|S(x,y)| \leq \log n$, then clearly $|S(x',y')| \leq \log n$ and from the assumption that $G$ is $(\gamma(d), \log n)$-sparse

$$xAy + x'Ay' \leq \gamma(d)(\sqrt{|S(x)||S(y)|} + \sqrt{|S(x')||S(y')|})$$

Observe that $|S(u)| = |S(x)| + |S(x')|$ and $|S(v)| = |S(y)| + |S(y')|$, so in particular $u\hat{A}v \leq \gamma(d)\sqrt{|S(u)||S(v)|}$, and requirement (2) holds.

So assume $|S(x,y)| = |S(u,v)| = \log n + 1$. It is not hard to see that this entails $S(u_1, v_1) \cap S(u_2, v_2) = \emptyset$. In other words, $S(u,v)$ contains at
most one vertex from each fiber. Hence, \( x' = y' = 0 \) and \(|S(u)| = |S(x)|, |S(v)| = |S(y)|\).
Denote \( S = S(x, y) \), and assume w.l.o.g. that \(|S(y)| > \frac{1}{2} \log n\). From (9) \( u\hat{A}v \leq \gamma(d)\sqrt{|S(x)||S(y)|} \). If this is not the case, we can bound the ratio between \(|S(x)|\) and \(|S(y)|\): Since the graph is of maximal degree \( d \) we have \( u Ay \leq d|S(X)| \). Hence, \( \frac{|S(x)|}{|S(y)|} > \frac{\gamma(d)^2}{d^2} = \frac{100 \log d}{d} \).
Observe that the edges between \( S(u) \) and \( S(v) \) in \( \hat{G} \) originate from edges between \( S(x) \) and \( S(y) \) in \( G \) in the following way - for each edge between \( S(x) \) and \( S(y) \) in \( G \) there is, with probability \( \frac{1}{2} \), an edge between \( S(u) \) and \( S(v) \) in \( \hat{G} \).
Next we bound \( u Ay \). Averaging over all \( S \setminus \{i\} \), for \( i \in S(y) \) we have that:
\[
(|S(y)| - 2)u Ay \leq |S(y)|\gamma(d)\sqrt{(|S(y)| - 1)|S(x)|}.
\]
Hence the expectation of \( u \hat{A}v \) is at most \( \frac{1}{2} c \gamma(d)||u||||v|| \), where
\[
c = \frac{\sqrt{|S(y)|(|S(y)| - 1)}}{|S(y)| - 2} \leq 1.1
\]
(assuming \( n \) is not very small). By the Chernoff bound, the probability that
\[
u \hat{A}v > \gamma(d)||u||||v||\]
is at most:
\[
2 \exp\left(-\frac{0.9}{2.2} \gamma(d)||u||||v||\right) \leq 
\exp\left(-0.2 \gamma(d)(\log n + 1)(10\sqrt{\log d})\right) = 
\exp\left(-20 \log d(\log n + 1)\right),
\]
Since \( \frac{|S(x)|}{|S(y)|} > \frac{100 \log d}{d} \), and \(|S(v)| > \frac{1}{2} \log n\).
There are at most \( d^{\log n + 1}4^{\log n + 1} \) pairs \( u, v \) with \( S(u, v) \) connected and of size \( \log n + 1 \), so by the union bound, w.h.p., requirement (2) holds.

**Corollary 3.1** Let \( A \) be the adjacency matrix of a \( d \)-regular \((\gamma(d), \log n)\)-sparse \( G \) graph on \( n \) vertices, where \( \gamma(d) = 10\sqrt{d \log d} \). Then there is a deterministic polynomial time algorithm for finding a signifi cing \( s \) of \( G \) such that the following hold:
1. The spectral radius of $A_s$ is $O(\sqrt{d \log^3 d})$.

2. $\hat{G}$ is $(\gamma(d), 1 + \log n)$-sparse,

where $A_s$ is the signed adjacency matrix, and $\hat{G}$ is the corresponding 2-lift.

**Proof:** Consider a random signing $s$. For each closed path $p$ in $G$ of length $l = 2\lceil \log n \rceil$ define a random variable $Y_p$ equal to the product of the signs of its edges. From lemmas 3.3 and 3.4 the expected value of the trace of $A_s^l$, which is the expected value of the sum of these variables, is $O(\sqrt{d \log^3 d})^l$ (since $l$ is even the sum is always positive). For each $u, v \in \{0, 1\}^n$, with $|S(u, v)| = \log n + 1$, and $S(u, v)$ connected, define $Z_{u,v}$ to be $d^l$ if $uAv \leq \gamma(d)||u||||v||$, and 0 otherwise. In the proof of Lemma 3.4 we've seen that the probability that $Z_{u,v}$ is not 0 is at most $d^{-10 \log n}$, thus the expected value of $Z_{u,v}$ is at most $d^{-8 \log n}$. Let $Z$ be the sum of the $Z_{u,v}$'s. Recall that there are at most $n(4d)^{\log n + 1}$ pairs $(u, v)$ such that $|S(u, v)| = \log n + 1$, and $S(u, v)$ connected. Hence, the expected value of $Z$ is less than $d^{-6 \log n}$.

Let $X = Y + Z$. Note that the expected value of $X$ is approximately that of $Y$, $O(\sqrt{d \log^3 d})^l$. The expectation of $Y_p$ and $Z_{u,v}$ can be easily computed even when the sign of some of the edges is fixed, and that of the other is chosen at random. As there is only a polynomial number of variables, using the method of conditional probabilities (cf. [8]) one can find a signing $s$ such that the value of $X$ is at most its expectation. For this value of $X$, $tr(A_s^l) = Y = O(\sqrt{d \log^3 d})^l$, and $Z = 0$ since if $Z \neq 0$ then $Z \geq d^l$. Clearly, the spectral radius of $A_s$ is $O(\sqrt{d \log^3 d})$.

In the proof of Lemma 3.4 we've seen that if $G$ is $(\gamma(d), \log n)$-sparse then so is $\hat{G}$, for any signing of $G$. For our choice of $s$ all $Z_{u,v} = 0$, hence $\hat{G}$ is actually $(\gamma(d), \log n + 1)$-sparse.

An alternative method for derandomization, using an almost $k$-wise independent sample space, is given in the appendix.

Recall the construction from the beginning of this section. Start with a $d$-regular graph $G_0$ which is an $(n_0, d, \mu)$-expander expander, for $\mu = 10\sqrt{d \log d}$ and $n_0 > d \log^2 n_0$. From the Expander Mixing Lemma (cf. [8]), $G_0$ is $(\mu, \log n_0)$-sparse. Iteratively chose $G_{i+1}$ to be a 2-lift of $G_i$ according to Corollary 3.1 for $i = 1, \ldots, \log(n/n_0)$. Clearly this is a polynomial time algorithm that yields a $(n, d, O(\sqrt{d \log^3 d}))$-expander graph.
3.4 Random 2-lifts

Theorem 3.1 states that for every graph there exists a signing such that the spectral radius of the signed matrix is small. The proof shows that for a random signing, this happens with positive, yet exponentially small, probability. The following example shows the limitations of this argument and in particular, that there exist graphs for which a random signing almost surely fails to give a small spectral radius.

Consider a graph composed of \( n/(d+1) \) disjoint copies of \( K_{d+1} \) (the complete graph on \( d+1 \) vertices). If all edges in one of the components are equally signed, then \( A_s \) has spectral radius \( d \). For \( d \) fixed and \( n \) large, this event will occur with high probability. Note that connectivity is not the issue here - it is easy to modify this example and get a connected graph for which, w.h.p., the spectral radius of \( A_s \) is \( \Omega(d) \).

However, for a random \( d \)-regular graph, it is true that a random 2-lift will, w.h.p., yield a signed matrix with small spectral radius. This follows from the fact that, w.h.p., a random \( d \)-regular graph is an \((n, d, O(\sqrt{d}))\)-expander ([17]). In particular, by the Expander Mixing Lemma, it is \((O(\sqrt{d}), \log n)\)-sparse. By Lemma 3.4, w.h.p., a random 2-lift yields a signed matrix with small spectral radius.

4 A stronger notion of explicitness

In this section we suggest an alternative derandomization scheme of section 3.3. We use the construction of Naor and Naor [26] of a small, almost \( k \)-wise independent sample space. This derandomization scheme leads to a construction which, in a sense, is more explicit than that in section 3.3.

4.1 Derandomization using an almost \( k \)-wise independent sample space

Recall that in the proof of Corollary 3.1 we defined two types of random variables: Let \( l = 2[\log n] \). For each closed path \( p \) of length \( l \), \( Y_p \) is the product of the signs of the edges of \( p \). For each \( u, v \in \{0, 1\}^n \), with \( |S(u, v)| = \log n + 1 \), and \( S(u, v) \) connected, let \( Z_{u,v} \) be \( d_l \) if \( uAv \leq \gamma(d)||u||||v|| \), and 0 otherwise. Define \( X \) to be the sum of all these random variables.
For brevity it will be convenient to make the following ad-hoc definitions:

**Definition 4.1** A signing $s$ of a $d$-regular graph $G$ is $(n, d)$-good, if the spectral radius of $A_s$ is $O(\sqrt{d \log 3} d)$ and $G$ is $(\gamma(d), 1 + \log n)$-sparse.

A $d$-regular graph $G$ is a $(n, d)$-good expander, if it’s an $(n, d, O(\sqrt{d \log 3} d))$-expander, and is $(\gamma(d), 1 + \log n)$-sparse.

The proof showed that finding a good signing is equivalent to finding a signing such that the value of $X$ is at most its expected value. We now show that this conclusion is also true when rather than choosing the sign of each edge uniformly and independently, we choose the signing from an $(\epsilon, k)$-wise independent sample space, with $k = d \log n$ and $\epsilon = d^{-2d \log n}$.

**Definition 4.2** ([26]) Let $\Omega_m$ be a sample space of $m$-bit strings, and let $S = s_1 \ldots s_m$ be chosen uniformly at random from $\Omega_m$. We say that $\Omega_m$ is an $(\epsilon, k)$-wise independent sample space if for any $k' \leq k$ positions $i_1 < i_2 \ldots < i_{k'}$,

$$\sum_{\alpha \in \{-1, 1\}^{k'}} | \Pr[s_{i_1} \ldots s_{i_{k'}} = \alpha] - 2^{-k'}| < \epsilon.$$ 

Naor and Naor [26] suggest an explicit construction of such sample spaces. When $k = O(\log m)$ and $1/\epsilon = poly(m)$, the size of the sample space is polynomial in $m$ (simpler constructions are also given in [6]).

We shall immediately see that the expected value of $X$ does not change significantly when the signing is chosen from such a sample space. Hence, an alternative way of efficiently finding a good signing is to go over the entire sample space. For at least one point in it, the value of $X$ is at most its expected value, and thus the signing is good.

**Lemma 4.1** Let $m = dn/2$, $k = d \log n$ and $\epsilon = d^{-2d \log n}$. Let $\Omega_m$ be an $(\epsilon, k)$-wise independent sample space. Let $X$ be as in the proof of Corollary 3.7 Let $U_m$ be the uniform distribution on $m$ bits. Then

$$|\mathbb{E}_{\Omega_m}[X] - \mathbb{E}_{U_m}[X]| = o(1).$$
Proof: Recall that $X = \sum_p Y_p + \sum_{u,v} Z_{u,v}$, where $Y_p$ and $Z_{u,v}$ are as above, the first sum is over all closed paths of length $l = 2\lceil \log n \rceil$, and the second sum is over all $u,v \in \{0,1\}^n$, with $|S(u,v)| = \log n + 1$ and $S(u,v)$ connected. Hence

$$|E_{\Omega_m}[X] - E_{U_m}[X]| \leq \sum_p |E_{\Omega_m}[Y_p] - E_{U_m}[Y_p]| + \sum_{u,v} |E_{\Omega_m}[Z_{u,v}] - E_{U_m}[Z_{u,v}]|.$$

Let $p$ be a path of length $l$, and denote the edges that appear in it an odd number of times by $i_1, \ldots, i_{l'}$, for some $l' < l$. Let $s_{i_1}, \ldots, s_{i_{l'}}$ be the signs of these edges. Then the value of $Y_p$ is $\prod_{j=1}^{l'} s_{i_j}$, and (for every distribution)

$$E[Y_p] = \sum_{\alpha \in \{-1,1\}^{l'}} \Pr[s_{i_1}, \ldots, s_{i_{l'}} = \alpha] \cdot \prod_{j=1}^{l'} \alpha_j.$$

Thus,

$$|E_{\Omega_m}[Y_p] - E_{U_m}[Y_p]| =$$

$$|\sum_{\alpha \in \{-1,1\}^{l'}} (\prod_{j=1}^{l'} \alpha_j)(\Pr_{\Omega_m}[s_{i_1}, \ldots, s_{i_{l'}} = \alpha] - \Pr_{U_m}[s_{i_1}, \ldots, s_{i_{l'}} = \alpha])| \leq$$

$$|\sum_{\alpha \in \{-1,1\}^{l'}} (\Pr_{\Omega_m}[s_{i_1}, \ldots, s_{i_{l'}} = \alpha] - 2^{-l'})| < \epsilon.$$

As there are less than $d^l$ closed paths $p$ of length $l$, $\sum_p |E_{\Omega_m}[Y_p] - E_{U_m}[Y_p]| < \epsilon d^l = o(1)$. A similar argument shows that $|\sum_{u,v} |E_{\Omega_m}[Z_{u,v}] - E_{U_m}[Z_{u,v}]| = o(1)$ as well.

In fact, it follows that w.h.p. (say, $1 - \frac{1}{n^c}$ for an appropriate choice of $\epsilon$), choosing an element uniformly at random from $\Omega_m$ leads to a good signing.

4.2 A probabilistic strongly explicit construction

The constructions of section 4.1 and of the previous subsection are explicit in the sense that given $n$ and $d$ they suggest a polynomial (in $n$) time algorithm for constructing an $(n, d)$-good expander. However, in some applications a stronger notion of explicitness is required. Namely, an algorithm that given $n, d$ and $i,j \in [n]$, decides in time polylog($n$) whether $i$ and $j$ are adjacent. We do not know how to achieve such explicitness using the 2-lifts schema, but we can do so probabilistically. Consider the following:
**Definition 4.3** Let $f_n : \{0, 1\}^t \times [n] \times [n] \to \{0, 1\}$, with $t = O(\log n)$. Given $r \in \{0, 1\}^t$, $f_n$ defines a graph $G_{f_n}(r)$, on $n$ vertices, where $i$ and $j$ are adjacent iff $f_n(r, i, j) = 1$. We say that $f_n$ is a $\delta$-probabilistic strongly explicit description of an $(n, d)$-good expander graph, if given $n$, $f_n$ can be computed in time $\text{polylog}(n)$, and, with probability at least $1 - \delta$ (over a uniform choice of $r$), $G_{f_n}(r)$ is an $(n, d)$-good expander graph.

We show that this notion of explicitness can be achieved by our construction. It will be convenient to give a similar definition for a signing of a graph, and for a composition of such functions:

**Definition 4.4** Let $h_n : \{0, 1\}^t \times [n] \times [n] \to \{-1, 1\}$, with $t = O(\log n)$. Given $r \in \{0, 1\}^t$, and a graph $G$ on $n$ vertices, $h_n$ defines a signing $s_{h_n}$ of $G$ by $s_{h_n}(r)(i, j) = h_n(r, i, j)$. We say that $h_n$ is a $\delta$-probabilistic strongly explicit description of an $(n, d)$-good signing, if given $n$, $h_n$ can be computed in time $\text{polylog}(n)$, and, for any $(\log n, \gamma(d))$-sparse $d$-regular graph $G$ on $n$ vertices, with probability at least $1 - \delta$ (over a uniform choice of $r$), $h_n$ defines an $(n, d)$-good signing.

**Definition 4.5** Let $f_n : \{0, 1\}^{t_1} \times [n] \times [n] \to \{0, 1\}$, and $h_n : \{0, 1\}^{t_2} \times [n] \times [n] \to \{-1, 1\}$ be as above. Their composition, $f_{2n} : \{0, 1\}^t \times [2n] \times [2n] \to \{0, 1\}$, with $t = \max\{t_1, t_2\}$ is as follows. For $r \in \{0, 1\}^t$, let $r_1$ be the first $t_1$ bits of $r$, and $r_2$ the first $t_2$ bits in $r$. $f_{2n}$ is such that the graph $G_{f_{2n}}(r)$ is the 2-lift of $G_{f_n}(r_1)$ described by the signing $s_{h_n}(r_2)$.

The following lemma is easy, and we omit the proof:

**Lemma 4.2** Let $f_n$ be a $\delta_1$-probabilistic strongly explicit description of an $(n, d)$-good expander, and $h_n$ a $\delta_2$-probabilistic strongly explicit description of an $(n, d)$-good signing. Then their composition is a $(\delta_1 + \delta_2)$-probabilistic strongly explicit description of an $(2n, d)$-good expander.

Think of an $(\epsilon, k)$-wise independent space $\Omega_m$ as a function $\omega : \{0, 1\}^t \to \{-1, 1\}^m$, where $|\Omega_m| = 2^t$. It follows from the work of Naor and Naor (26), that not only can $\omega$ be computed efficiently, but that given $r \in \{0, 1\}^t$, $p \in [m]$ $\omega(r)_p$ (the $p$’th coordinate of $\omega(r)$) can be computed efficiently (i.e. in time $\text{polylog}(m)$). Take $m = \binom{n}{2}$, and think of the elements of $\{-1, 1\}^m$ as being indexed by unordered pairs $(i, j) \in \binom{[n]}{2}$. Define $h_n(r, i, j) = \omega(r)_{i,j}$. It follows from the above discussion that $h_n$ is a $\frac{1}{\pi^2}$-probabilistic strongly explicit
We now describe how to construct a $\delta$-probabilistic strongly explicit description of an $(N, d)$-good expander graph. Let $G$ be an $(n, d)$-good expander, with $n \geq \frac{1}{\delta}$. For $i = 0, \ldots, l = \log(N/n)$, define $n_i = n \cdot 2^i$ and $m_i = \binom{n_i}{2}$. Define $k_i = d \log n_i$. Let $\omega_i : \{0, 1\}^{t_i} \rightarrow \{-1, 1\}^{m_i}$ be a description of an $(\epsilon_i, k_i)$-wise independent space of bit strings of length $m_i$, where $\epsilon_i$ is such that an element chosen uniformly at random from this space yields an $(n_i, d)$-good signing with probability at least $1 - \frac{1}{n_i^2}$.

The functions $h_{n_i}(r, p, q) = \omega_i(r)_{p,q}$ are $\frac{1}{n_i^2}$-probabilistic strongly explicit descriptions of $(n_i, d)$-good signings. Let $f_n$ be a description of $G$. For simplicity, assume that adjacency in $G$ can be decided in time polylog($n$). Thus, $f_n$ is, trivially, a 0-probabilistic strongly explicit description of an $(n, d)$-good expander. Define $f_{n_i}$ as the composition of $f_{n_{i-1}}$ and $h_{n_{i-1}}$. It follows from this construction and Lemma 4.2 that:

**Lemma 4.3** $f_{n_l}$ is an $\frac{1}{n}$-probabilistic strongly explicit description of an $(N, d)$-good expander graph.

### 5 A converse to the Expander Mixing Lemma

So far, we have concentrated on the algebraic approach to expansion in graphs. Namely, that a graph is an $(n, d, \lambda)$-expander if all eigenvalues but the largest are, in absolute value, at most $\lambda$. A combinatorial definition says that a $d$-regular graph on $n$ vertices is an $(n, d, c)$-edge expander if every set of vertices, $W$, of size at most $n/2$, has at least $c|W|$ neighbors emanating from it.

The two notions are closely related. Thus (cf. [5]), an $(n, d, \lambda)$-expander is also an $(n, d, \frac{d-\lambda}{2})$-edge expander. Conversely, an $(n, d, c)$-edge expander is also an $(n, d, d - \frac{c^2}{2d})$-expander. Thus, though the two notions of expansion are qualitatively equivalent, they are far from being quantitatively the same. While algebraic expansion yields good bounds on edge expansion, the reverse implications are very weak. It is also known that this is not just a failure of the proofs and indeed this estimate is nearly tight [2]. Is there, we ask,

\[\text{A related result, showing that vertex expansion implies spectral gap appears in [6]. The implication from edge expansion is easier, and the proof we are aware of is also due to Noga Alon.}\]
another combinatorial property that is equivalent to spectral gaps? We next answer this question.

For two subsets of vertices, $S$ and $T$, let $e(S, T)$ denote the number of edges between them. We follow the terminology of [29]:

**Definition 5.1** A $d$-regular graph $G$ on $n$ vertices is $(d, \alpha)$-jumbled, if for every two subsets of vertices, $A$ and $B$,

\[
|e(A, B) - d|A||B||n| \leq \alpha \sqrt{|A||B|}.
\]

A very useful property of $(n, d, \lambda)$-expanders, known as the Expander Mixing Lemma (cf. [8]), is that an $(n, d, \lambda)$-expander is $(d, \lambda)$-jumbled. Lemma 3.3 implies the promised converse to this well known fact:

**Corollary 5.1** Let $G$ be a $d$-regular graph on $n$ vertices. Suppose that for any $S, T \subset V(G)$, with $S \cap T = \emptyset$

\[
|e(S, T) - \frac{|S||T|d}{n}| \leq \alpha \sqrt{|S||T|}
\]

Then all but the largest eigenvalue of $G$ are bounded, in absolute value, by $O(\alpha(1 + \log(d/\alpha)))$.

**Note 5.1** In particular, this means that for a $d$-regular graph $G$, $\lambda(G)$ is a $\log d$ approximation of the “jumblesness” parameter of the graph.

**Proof:** (Corollary 5.1) Let $A$ be the adjacency matrix of $G$. Denote $B = A - \frac{d}{n}J$, where $J$ is the all ones $n \times n$ matrix. Clearly $B$ is symmetric, and the sum of the absolute value of the entries in each row is at most $2d$. The first eigenvalue of $A$ is $d$. The other eigenvalues of $A$ are also eigenvalues of $B$. Thus, for the corollary to follow from Lemma 3.3 it suffices to show that for any two vectors, $u, v \in \{0, 1\}^n$:

\[
|uBv| = |u - \frac{d}{n}Jv - uAv| \leq \alpha ||u||||v||.
\]

This is exactly the hypothesis for the sets $S(u)$ and $S(v)$.

The corollary is actually tight, up to a constant multiplicative factor, as we now show:
Theorem 5.1 For any large enough \( d \), and \( 7\sqrt{d} < \alpha < d \), there exist infinitely many \((d, \alpha)\)-jumbled graphs with second eigenvalue \( \Omega(\alpha(\log(d/\alpha) + 1)) \).

It will be useful to extend Definition 5.1 to unbalanced bipartite graphs:

Definition 5.2 A bipartite graph \( G = (U, V, E) \) is \((c, d, \alpha)\)-jumbled, if the vertices in \( U \) have degree \( c \), those in \( V \) have degree \( d \), and for every two subsets of vertices, \( A \subset U \) and \( B \subset V \),

\[
|e(A, B) - d|A||B|/|U|| \leq \alpha \sqrt{|A||B|}.
\]

We note that such bipartite graphs exist:

Lemma 5.1 For \( c|d \) and \( \alpha = 2\sqrt{d} \), there exist \((c, d, \alpha)\)-jumbled graphs.

Proof: Let \( G' = (U', V', E') \) be a \( c\)-regular Ramanujan bipartite graph, such that \(|U'| = |V'| = n\). Let \( G = (U, V, E) \) be a bipartite graph obtained from \( G' \) by partitioning \( V' \) into subsets of size \( d/c \), and merging each subset into a vertex, keeping all edges (so this is a multi-graph).

Let \( A \subset U \) and \( B \subset V \). Let \( A' = A \), and \( B' \) the set of whose merger gives \( B \). Clearly \( e(A, B) = e(A', B') \), \( |B'| = d/c|B| \). As \( G' \) is Ramanujan, by the expander mixing lemma

\[
|e(A', B') - c|A'||B'|/n| \leq 2\sqrt{c|A'||B'|},
\]

or:

\[
|e(A, B) - d|A||B|/|U|| \leq 2\sqrt{d|A||B|}.
\]

We shall need the following inequality, that can be easily proven by induction:

Lemma 5.2 For \( i = 0, \ldots, t \) let \( a_i \) be numbers in \([0, 2^{2i} \cdot N]\), for some \( N > 0 \). Then

\[
(\sum a_i2^{-i})^2 \leq 3N \sum a_i.
\]
Proof: (Theorem 5.1) Fix $d$, and $\sqrt{d} < \Delta < d$. Set $t = \frac{1}{2}\log(\frac{n}{\Delta})$, $\tau = \sum_{i=0}^{t} 4^i = \frac{d}{\Delta} < \Delta$. Let $N$ be some large number and $n = \tau N$ (this will be the number of vertices). For $i, j = 0, \ldots, t$ set $d_{i,j} = \frac{4}{\tau} 2^j + \Delta 2^{j-i}$ when $i, j < t$ or $i = j = t$, and $d_{i,j} = \frac{4}{\tau} 2^j - \Delta 2^{j-i}$ otherwise. Note that in this case $\frac{4}{\tau} 2^j - \Delta 2^{j-i} \geq 2^i(\Delta 2^j - \Delta 2^{-i}) > 0$. Set $\alpha_{i,j} = 2\sqrt{\min\{d_{i,j}, d_{j,i}\}}$.

Let $|V_i|$ be subsets of size $4^iN$, and $G$ a graph on vertices $V = \bigcup_{i=0}^{t} V_i$ (hence, $|V| = n$). For $0 < i, j \leq t$ construct a $(d_{i,j}, d_{i,j}, \alpha_{i,j})$-jumbled graph between $V_i$ and $V_j$ (or $(d_{i,j}, \alpha_{i,j})$-jumbled if $i = j$).

The theorem follows from the following two lemmata.

**Lemma 5.3** $G$ is $(d, 7\Delta)$-jumbled.

Proof: It is not hard to verify that $G$ is indeed $d$ regular. Take $A, B \subset V$, and denote their size by $a$ and $b$. Denote $A_i = A \cap V_i$, $B_i = B \cap V_i$, and their size by $a_i$ and $b_i$. We want to show that:

$$|e(A, B) - dab/n| \leq 3\Delta\sqrt{ab}.$$  

For simplicity we show that $e(A, B) \leq dab/n + 7\Delta\sqrt{ab}$. A similar argument bounds the number of edges from below. From the construction, $|e(A_i, B_j)| - d_{i,j}|A_i||B_j|/|V_j| \leq \alpha_{i,j}\sqrt{|A_i||B_j|}$, or:

$$e(A_i, B_j) \leq d_{i,j}a_ib_j/(4^iN) + \alpha_{i,j}\sqrt{a_ib_j}.$$  

Summing up over $i, j = 0, \ldots, t$ we get:

$$e(A, B) \leq \sum d_{i,j}a_ib_j/(4^iN) + \alpha_{i,j}\sum \alpha_{i,j}\sqrt{a_ib_j} \leq \frac{d}{n}\sum a_ib_j + \frac{\Delta}{N}\sum a_ib_j2^{-(i+j)} + \sum \alpha_{i,j}\sqrt{a_ib_j}.$$  

$\sum a_ib_j = ab$, so it remains to bound the error term. As $a_i, b_i \in [0, N \cdot 2^{2i}]$, by Lemma 5.2

$$\frac{\Delta}{N}\sum a_ib_j2^{-(i+j)} = \frac{\Delta}{N}(\sum a_i2^{-i})(\sum b_i2^{-i}) \leq \frac{\Delta}{N}(\sqrt{3N}\sum a_i)(\sqrt{3N}\sum b_i) = 3\Delta\sqrt{ab}$$

so it remains to show that $\sum \alpha_{i,j}\sqrt{a_ib_j} \leq 4\Delta\sqrt{ab}$, and hence it’s enough to show that:

$$\sum 2\sqrt{\frac{d}{\tau}a_ib_j2^{\max\{i,j\}}} + \sum 2\sqrt{\Delta 2^{j-i}a_ib_j} \leq 4\Delta\sqrt{ab}.$$
Indeed, as \( i, j \leq t \), and \( 2^{2t} < \tau \),
\[
\sum \sqrt{\frac{d}{\tau} a_i b_j 2^{2 \max\{i, j\}}} < \sqrt{d} \sum a_i b_j < \Delta \sqrt{ab}.
\]
Similarly, \( 2^{j-i} \leq 2^t < \sqrt{\tau} < \sqrt{\Delta} \), and so:
\[
\sum \Delta 2^{j-i} a_i b_j < \Delta \sum a_i b_j = \Delta \sqrt{ab}.
\]

\textbf{Lemma 5.4} \( \lambda(G) \geq \Delta(t+1) \)

\textbf{Proof:} Take \( x \in \mathbb{R}^n \) to be \( -2^{-t} \) on vertices in \( V_t \), and \( 2^{-i} \) on vertices in \( V_i \), for \( i < t \). It is easy to verify that \( x \perp \vec{1} \), and that \( ||x||^2 = N \cdot (t+1) \). Let \( M \) be the adjacency matrix if \( G \). Since \( \vec{1} \) is an eigenvector of \( M \) corresponding to the largest eigenvalue, by the variational characterization of eigenvalues, \( \lambda(G) \geq \frac{x M x}{||x||^2} \). Hence, to prove the lemma it suffices to show that \( x M x \geq \Delta N(t+1)^2 \). Indeed:
\[
\begin{align*}
\sum d i,j A^i A^j & = \sum_{i,j=0}^t d_{i,j} 2^{-i-j} + \Delta N \sum_{i,j=0}^t 2^{i+j} + 4 \Delta N \sum_{i=0}^{t-1} 2^{i+t} + 4 \Delta N \sum_{i=0}^{t-1} 1 \\
& = \frac{d}{\tau} N (2^{2(t+1)} - 4 \cdot 2^t) + \Delta N ((t+1)^2 + 4t) > \Delta N (t+1)^2.
\end{align*}
\]

\textbf{6 Reflections on Lemma 3.3}

\textbf{6.1 Finding the proof: LP-duality}

As the reader might have guessed, the proof for Lemma 3.3 was discovered by formulating the problem as a linear program. Define \( \Delta_{i,j} = |x^i A x^j| \). Our assumptions translate to:
\[
\forall 1 \leq i \leq j \leq k : |\Delta_{i,j}| \leq \alpha \sqrt{s_i s_j},
\]
∀1 ≤ i ≤ k : \[ \sum_j |\Delta_{i,j}| \leq d_s_i. \]

We want to deduce an upper bound on \( |xAx| \). In other words, we are asking, under these constraints, how big

\[
\frac{|xAx|}{||x||^2} \leq \frac{\sum_{i,j=1}^k \Delta_{i,j} 2^{-(i+j)}}{\sum_i 2^{-2i}s_i}
\]

can be.

The dual program is to minimize:

\[
\frac{\alpha \sum_{i \leq j} b_{i,j} \sqrt{s_is_j} + d \sum_i c_is_i}{\sum_i 2^{-2i}s_i}
\]

under the constraints:

\[
\begin{align*}
&\forall 1 \leq i < j \leq k, \quad b_{i,j} + c_i + c_j \geq 2^{-(i+j)+1} \\
&\forall 1 \leq i \leq k, \quad b_{i,i} + c_i \geq 2^{-2i} \\
&\forall 1 \leq i \leq j \leq k, \quad b_{i,j} \geq 0 \\
&\forall 1 \leq i \leq k, \quad c_i \geq 0
\end{align*}
\]

The following choice of \( b \)'s and \( c \)'s satisfies the constraints, and gives the desired bound. These indeed appear in the proof above:

\[
\begin{align*}
&\forall 1 \leq i < j \leq k, \quad j < i + \gamma, \quad b_{i,j} = 2^{-(i+j)+1} \\
&\forall 1 \leq i < j \leq k, \quad j \geq i + \gamma, \quad b_{i,j} = 0 \\
&\forall 1 \leq i \leq k, \quad b_i = 2^{-2i} \\
&\forall 1 \leq i \leq k, \quad c_i = 2^{-2i-\gamma+1}
\end{align*}
\]

Note that we can further constrain the \( \delta \)'s, and perhaps get better bounds this way.

### 6.2 Algorithmic aspect

Lemma 3.3 is algorithmic, in the sense that given a matrix with a large eigenvalue, we can efficiently construct, from its eigenvector, a pair \( u, v \in \{0, 1\}^n \)
such that $S(u) \cap S(v) = \emptyset$, and $|uAv| \geq \alpha||u||||v||$ (There is a small caveat - in the proof we used a probabilistic argument for rounding the coordinates. This can be easily derandomized using the conditional probabilities method). Taking into consideration Note 5.1 given a $d$-regular graph $G$ where $\lambda(G)$ is large, one can efficiently find disjoint subsets $S$ and $T$, such that $e(S,T) - \frac{d}{n}|S||T| \geq c \cdot \lambda(G)/\log d\sqrt{|S||T|}$ (for some constant $c$). We speculate that this might be useful in designing graph partitioning algorithms.

In Lemma 3.2 we showed the with positive probability a random signing has a small spectral radius. It is interesting to find a sample space where this happens with high probability. We conjecture that the following algorithm works: Choose a signing at random. If some eigenvalue of the signed matrix is big, use Lemma 3.3 to find a pair $u, v \in \{0,1\}^n$ such that $|uAv| \geq \alpha||u||||v||$. Choose a signing at random for the edges between $S(u)$ and $S(v)$. Repeat until all eigenvalues are small.

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