On Generalized $m$-th Root Finsler Metrics

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Abstract

In this paper, we characterize locally dually flat generalized $m$-th root Finsler metrics. Then we find a condition under which a generalized $m$-th root metric is projectively related to a $m$-th root metric. Finally, we prove that if a generalized $m$-th root metric is conformal to a $m$-th root metric, then both of them reduce to Riemannian metrics.

Keywords: Generalized $m$-th root metric, locally dually flat metric, projectively related metrics, conformal change.

1 Introduction

An $m$-th root metric $F = \sqrt[m]{A}$, where $A := a_{i_1...i_m}(x)y^{i_1}...y^{i_m}$, is regarded as a direct generalization of Riemannian metric in a sense, i.e., the second root metric is a Riemannian metric. The theory of $m$-th root metrics has been developed by Matsumoto-Shimada [15][19], and applied by Antonelli to Biology as an ecological metric [3]. The third and fourth root metrics are called the cubic metric and quartic metric, respectively.

For quartic metrics, a study of the geodesics and of the related geometrical objects is made by Balan, Brinzei and Lebedev [6][9][13]. Also, Einstein equations for some relativistic models relying on such metrics are studied by Balan-Brinzei in two papers [10][11]. In four-dimension, the special quartic metric in the form $F = \sqrt[4]{y^1y^2y^3y^4}$ is called the Berwald-Moór metric [7][8]. In the last two decades, physical studies due to Asanov, Pavlov and their co-workers emphasize the important role played by the Berwald-Moór metric in the theory of space-time structure and gravitation as well as in unified gauge field theories [4][16][17]. In [7], Balan prove that the Berwald-Moór structures are pseudo-Finsler of Lorentz type and for co-isotropic submanifolds of Berwald-Moór spaces present the Gauss-Weingarten, Gauss-Codazzi, Peterson-Mainardi and Ricci-Kühne equations.

In [20], tensorial connections for $m$-th root Finsler metrics have been studied by Tamassy. Li-Shen study locally projectively flat fourth root metrics under irreducibility condition [14]. Yu-You show that an $m$-th root Einstein Finsler

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metrics are Ricci-flat [23]. In [21], Tayebi-Najafi characterize locally dually flat and Antonelli $m$-th root metrics. They prove that every $m$-th root metric of isotropic mean Berwald curvature (resp, isotropic Landsberg curvature) reduces to a weakly Berwald metric (resp, Landsberg metric). They show that $m$-th root metric with almost vanishing $H$-curvature has vanishing $H$-curvature [22].

Let $(M, F)$ be a Finsler manifold of dimension $n$, $TM$ its tangent bundle and $(x^i, y^j)$ the coordinates in a local chart on $TM$. Let $F$ be a scalar function on $TM$ defined by $F = \sqrt{A^{2/m} + B}$, where $A$ and $B$ are given by

$$A := a_{i_1 \ldots i_m}(x)y^{i_1} \ldots y^{i_m}, \quad B := b_{ij}(x)y^iy^j.$$  

(1)

Then $F$ is called generalized $m$-th root Finsler metric. Put

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad B_i = \frac{\partial B}{\partial y^i}, \quad B_{ij} = \frac{\partial^2 B}{\partial y^i \partial y^j},$$

$$A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i}y^i, \quad B_{x^i} = \frac{\partial B}{\partial x^i}, \quad B_0 = B_{x^i}y^i.$$ 

Suppose that the matrix $(A_{ij})$ defines a positive definite tensor and $(A^{ij})$ denotes its inverse. Then the following hold

$$g_{ij} = \frac{A^{ij}}{m^2} - \frac{2}{m} [mA_{ij} + (2 - m)A_iA_j] + b_{ij},$$  

(2)

$$y^jA_i = mA, \quad y^iA_{ij} = (m - 1)A_j, \quad y_i = \frac{1}{m}A^{j}_{jk}A_{ji},$$  

(3)

$$A^{ij}A_{jk} = \delta^i_k, \quad A^{ij}A_i = \frac{1}{m - 1}y^j, \quad A_iA_jA^{ij} = \frac{m}{m - 1}A.$$  

(4)

Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory [1][2]. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which plays a very important role in studying flat Finsler information structure [15]. A Finsler metric $F$ on a manifold $M$ is said to be locally dually flat, if at any point there is a standard coordinate system $(x^i, y^j)$ in $TM$ such that $(F^2)_{x^i}y^j y^k = 2(F^2)_{x^i}$. In this case, the coordinate $(x^i)$ is called an adapted local coordinate system. In this paper, we characterize locally dually flat generalized $m$-th root Finsler metrics. More precisely, we prove the following.

**Theorem 1.** Let $F = \sqrt{A^{2/m} + B}$ be a generalized $m$-th root metric on an open subset $U \subset \mathbb{R}^n$. Suppose that $A$ is irreducible. Then $F$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_i(x)y^i$ on $U$ such that the following holds

$$B_{0i} = 2B_{x^i},$$

(5)

$$A_{x^i} = \frac{1}{3m} [mA\theta_i + 2\theta A_i],$$

(6)

where $B_{0i} = B_{x^k}y^ky^i$. 
In local coordinates \((x^i, y^i)\), the vector field \(G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}\) is a global vector field on \(TM_0\), where \(G^i = G^i(x, y)\) are local functions on \(TM_0\) given by following
\[
G^i := \frac{1}{4} g^{ij} \left\{ \frac{\partial^2 F^2}{\partial x^i \partial y^j} - \frac{\partial F^2}{\partial x^j} \right\}, \quad y \in T_x M.
\]
The vector field \(G\) is called the associated spray to \((M, F)\). Two Finsler metrics \(F\) and \(\bar{F}\) on a manifold \(M\) are called projectively related if there is a scalar function \(P(x, y)\) defined on \(TM_0\) such that
\[
\bar{G}^i = G^i + Py^i,
\]
where \(\bar{G}^i\) and \(G^i\) are the geodesic spray coefficients of \(\bar{F}\) and \(F\), respectively.

**Theorem 2.** Let \(\bar{F} = \sqrt{A^{2m}} + B\) and \(F = A^{1/m}\) are generalized \(m\)-th root and \(m\)-th root Finsler metrics on an open subset \(U \subset \mathbb{R}^n\), respectively, where \(A := a_{i_1 \ldots i_m}(x)y^{i_1} \cdots y^{i_m}\) and \(B := c_{i_1 \ldots i_m}(x)y^{i_1} \cdots y^{i_m}\) with \(c_{i_1 \ldots i_m} = c_{i_1} \cdots c_{i_m}\). Suppose that the following holds
\[
(1 + c_k d^k) A^{ij}(B_{0j} - B_{xj}) - d^k [2\Delta_k + (B_{0k} - B_{xk})] A^{ij} c_j = 0, \quad (7)
\]
where
\[
\Delta_k = \frac{A^{2 - 2}}{m} \left[ \frac{2}{m} - 1 \right] A_k A_0 + AA_{0k} - AA_{xk},
\]
d\(^k = g^{ik} d_i\) and \(g^{ik} = \left[ \frac{1}{2} (F^2) y^i y^k \right]^{-1}\). Then \(\bar{F}\) is projectively related to \(F\). Moreover, suppose that the following holds
\[
2A^{\frac{2}{m} - 2} d^k c_i d^l [\frac{2}{m} - 1] A_j A_0 + AA_{0j} - AA_{xj} - md^k [B_{0i} - B_{xk}] \neq 0. \quad (8)
\]
Then \(B = 0\). In this case, \(\bar{F} = F\).

The first to treat the conformal theory of Finsler metrics generally was Knebelman. He defined two metric functions \(F\) and \(\bar{F}\) as conformal if the length of an arbitrary vector in the one is proportional to the length in the other, that is if \(\bar{g}_{ij} = \varphi g_{ij}\). The length of vector \(\varepsilon\) means here the fact that \(\varphi g_{ij}\), as well as \(g_{ij}\), must be Finsler metric tensor, he showed that \(\varphi\) falls into a point function. In this paper, we show that if a generalized \(m\)-th root is conformal to a \(m\)-th root Finsler metric, then both of them reduce to Riemannian metrics. More precisely, we prove the following.

**Theorem 3.** Let \(\bar{F} = \sqrt{A^{2/m}} + B\) and \(F = A^{1/m}\) are generalized \(m\)-th root and \(m\)-th root Finsler metric on an open subset \(U \subset \mathbb{R}^n\), respectively, where \(A := a_{i_1 \ldots i_m}(x)y^{i_1} \cdots y^{i_m}\) and \(B := b_{i_1 \ldots i_m}(x)y^{i_1} \cdots y^{i_m}\). Suppose that \(\bar{F}\) is conformal to \(F\). Then \(\bar{F}\) and \(F\) reduce to Riemannian metrics.
2 Preliminaries

Let $M$ be a n-dimensional $C^\infty$ manifold. Denote by $T_x M$ the tangent space at $x \in M$, by $TM = \cup_{x \in M} T_x M$ the tangent bundle of $M$ and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle. A Finsler metric on $M$ is a function $F : TM \to [0, \infty)$ which has the following properties: (i) $F$ is $C^\infty$ on $TM_0$, (ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$, (iii) for each $y \in T_x M$, the following quadratic form $g_y$ on $T_x M$ is positive definite,

$$g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]_{s, t = 0}, \quad u, v \in T_x M.$$  

Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ by

$$C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} [g_y + tw(u, v)]_{t = 0}, \quad u, v, w \in T_x M.$$  

The family $C := \{C_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if $F$ is Riemannian.

Given a Finsler manifold $(M, F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^j)$ for $TM_0$ is given by $G^i = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where $G^i(y)$ are local functions on $TM$ given by

$$G^i := \frac{1}{4} y^j \left\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l} y^k \right\}_{y^l}^{x^l}, \quad y \in T_x M. \quad (9)$$

$G$ is called the associated spray to $(M, F)$. The projection of an integral curve of $G$ is called a geodesic in $M$. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates ($c^i(t)$) satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_x M_0$, define $B_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$ and $E_y : T_x M \otimes T_x M \to \mathbb{R}$ by $B_y(u, v, w) := B^i_{jkl}(y) u^j v^k w^l |_{x}$ and $E_y(u, v) := E_{jk}(y) u^j v^k$, where

$$B^i_{jkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y), \quad E_{jk}(y) := \frac{1}{2} B^m_{jkm}(y),$$

$$u = u^i \frac{\partial}{\partial x^i}|_x, \quad v = v^i \frac{\partial}{\partial x^i}|_x \quad \text{and} \quad w = w^i \frac{\partial}{\partial x^i}|_x.$$  

$B$ and $E$ are called the Berwald curvature and mean Berwald curvature, respectively. A Finsler metric is called a Berwald metric if $B = 0$ or $E = 0$, respectively.

Define $D_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$ by $D_y(u, v, w) := D^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ where

$$D^i_{jkl} := B^i_{jkl} - \frac{2}{n + 1} \{ E_{jk} \delta^i_l + E_{jl} \delta^i_k + E_{kl} \delta^i_j + E_{jkli} y^i \}. $$

We call $D := \{D_y\}_{y \in TM_0}$ the Douglas curvature. A Finsler metric with $D = 0$ is called a Douglas metric. It is remarkable that, the notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics \[5\].
3 Proof of the Theorem

To prove Theorem 1, we need the following.

Lemma 1. Let $F = \sqrt{A^{2/m} + B}$ be a generalized $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Then, $F$ is a locally dually flat metric if and only if the following holds

$$A_x^l = \frac{1}{2A} \left( \frac{2}{m} - 1 \right) A_0 A_l + A A_{0l} + \frac{m}{2} A^{2\frac{m-2}{m}} \left( B_{0l} - 2B_x^l \right).$$  \hspace{1cm} (10)

Proof. Let $F$ be a locally dually flat metric

$$[F^2]_{x^l y^k} = 2[F^2]_x^l.$$ \hspace{1cm} (11)

We have

$$(A^\frac{2}{m} + B)_x^l = \frac{2}{m} A^{\frac{2-m}{m}} \left[ A_x^l + \frac{m}{2} A^{\frac{m-2}{m}} B_x^l \right],$$ \hspace{1cm} (12)

$$(A^\frac{2}{m} + B)_x^l y^k = \frac{2}{m} A^{\frac{2-m}{m}} \left[ \frac{2-m}{m} A_0 A_l A^{-1} + A_{0l} + \frac{m}{2} A^{\frac{m-2}{m}} B_{0l} \right].$$ \hspace{1cm} (13)

By (11)-(13), we have (10). The converse is trivial. \hfill $\square$

Proof of Theorem 1. Now, suppose that $A$ is irreducible. One can rewrite (10) as follows

$$(1 - \frac{2}{m}) A_0 A_l - A [A_{0l} - 2A_x^l] = \frac{m}{2} A^{2-\frac{2}{m}} [B_{0l} - 2B_x^l].$$ \hspace{1cm} (14)

The left hand side of (14) is a rational function in $y$, while its right hand side is an irrational function in $y$. Thus, (14) reduces to following

$$(2 - m) A_0 A_l = m A [2A_x^l - A_{0l}],$$ \hspace{1cm} (15)

$$B_{0l} - 2B_x^l = 0.$$ \hspace{1cm} (16)

By (15), the irreducibility of $A$ and $deg(A_l) = m - 1$, it follows that there exists a 1-form $\theta = \theta_l y^l$ on $U$ such that

$$A_0 = \theta A.$$ \hspace{1cm} (17)

This implies that

$$A_{0l} = A \theta_l + \theta A_l - A_x^l.$$ \hspace{1cm} (18)

By plugging (17) and (18) in (15), we get (10). The converse is a direct computation. This completes the proof. \hfill $\square$

By Lemma 1 and Theorem 1, we get the following.

Corollary 1. Let $F = A^{1/m}$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Then $F$ is a locally dually flat metric if and only if the following holds

$$A_x^l = \frac{1}{2A} \left( \frac{2}{m} - 1 \right) A_l A_0 + A A_{0l}.$$ \hspace{1cm} (19)

Moreover, suppose that $A$ is irreducible. Then $F$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_l(x) y^l$ on $U$ such that (17) holds.
4 Proof of the Theorem

Two Finsler metrics $F$ and $\bar{F}$ on a manifold $M$ are called projectively related if any geodesic of the first is also geodesic for the second and vice versa. Thus, there is a scalar function $P(x, y)$ defined on $TM_0$ such that $G^i = G^i + Py^i$, where $G^i$ and $\bar{G}^i$ are the geodesic spray coefficients of $F$ and $\bar{F}$, respectively.

**Lemma 2.** Let $A = [A_{ij}]$ be an $n \times n$ invertible and symmetric matrix, $C = [C_i]$ and $D = [D_j]$ are two non-zero $n \times 1$ and $1 \times n$ vector, such that $C_iD_j = C_jD_i$. Suppose that $1 + A^{pq}C_pD_q \neq 0$. Then the matrix $B = [B_{ij}]$ defined by $B_{ij} := A_{ij} + C_iD_j$ is invertible and

$$B^{ij} := (B_{ij})^{-1} = A^{ij} - \frac{1}{1 + A^{pq}C_pD_q} A^{ki}A^{lj}C_kD_l,$$  \hspace{1cm} (20)

where $A^{ij} := (A_{ij})^{-1}$.

**Lemma 3.** Let $\bar{F} = \sqrt{A^{2/m} + B}$ and $F = A^{1/m}$ are generalized $m$-th root and $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, respectively, where $A := a_{i_1...i_m}(x)y^{i_1}...y^{i_m}$ and $B := c_i d_j y^i y^j$ with $c_i d_j = c_j d_i$ and $c_i d_i \neq -1$. Suppose that the following holds

$$mA^{m-2} A^{ij} \mathbb{G}_i - [4\Upsilon + kd^k] A^i = 0,$$  \hspace{1cm} (21)

where

$$\mathbb{G}_i = B_{0i} - B_{xi},$$

$$\Upsilon = \frac{kd^k}{4} \left\{ [F^2]_{x+y}^k y^k - [F^2]_{xy} \right\},$$

$$k = \frac{1}{1 + c_i d_i},$$

$$A^i = mA^{m-2} A^{ij}c_j,$$

d^k = g^{ik} d_k$ and $g^{ik} = \left[ \frac{1}{2} (F^2)_{y^i y^k} \right]^{-1}$. Then $\bar{F}$ is projectively related to $F$.

**Proof.** By assumption, we have

$$\bar{F}^2 = F^2 + B,$$  \hspace{1cm} (22)

where $F = A^{1/m}$ be an $m$-th root Finsler metric, $A := a_{i_1...i_m}(x)y^{i_1} y^{i_2} \ldots y^{i_m}$ is symmetric in all its indices and $B = c_i d_j y^i y^j$. Then we have

$$\bar{g}_{ij} = g_{ij} + c_i d_j,$$  \hspace{1cm} (23)

where

$$g_{ij} = \frac{A^{\pm 2}}{m^2} [mA_{ij} + (2 - m)A_i A_j].$$  \hspace{1cm} (24)
Then by Lemma 2 we get
\[ \bar{g}^{ij} = g^{ij} - \frac{1}{1 + c_m d^m} c^i d^j, \quad (25) \]
where \( d^m = g^{mi} d_i, \quad c^m = g^{mi} c_i \) and
\[ g^{ij} = A^{-\frac{2}{m}} \left[ m A A^{ij} + \frac{m - 2}{m - 1} y^i y^j \right]. \quad (26) \]

Then by (9), (22) and (25), we have
\[ \bar{G}^i = \frac{1}{4} g^{il} \left[ \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right] = \frac{1}{4} \left[ g^{il} - kc^i d^l \right] \left[ \frac{\partial^2 (F^2 + B)}{\partial x^k \partial y^l} y^k - \frac{\partial (F^2 + B)}{\partial x^l} \right], \]
where \( k = \frac{1}{1 + c_m d^m} \). Then
\[ \bar{G}^i = \frac{1}{4} \left[ g^{il} - kc^i d^l \right] \left[ \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right] + \frac{1}{4} \left[ g^{il} - kc^i d^l \right] \left[ \frac{\partial^2 B}{\partial x^k \partial y^l} y^k - \frac{\partial B}{\partial x^l} \right] = G^i - [\Upsilon + 1 \frac{kd^l \mathfrak{B}_l}{4}] c^i + \frac{1}{4} g^{il} \mathfrak{B}_l, \quad (27) \]
where
\[ \Upsilon = \frac{kd^l}{4} \left\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \right\}, \quad \mathfrak{B}_l = B_{0l} - B_{xl}. \quad (28) \]

Put
\[ \Phi := \frac{m - 2}{m - 1} A^{-\frac{2}{m}} y^p c_p, \quad \mathcal{A}^i := mA^i A^{-\frac{m - 2}{m}} y^p c_p. \quad (29) \]

Then we have
\[ c^i = g^{ip} c_p = A^{-\frac{2}{m}} \left[ m A A^{ip} + \frac{m - 2}{m - 1} y^i y^p \right] c_p = \mathcal{A}^i + \Phi y^i, \quad (30) \]

By (26), (27) and (30), we get
\[ \bar{G}^i = G^i + \left[ A^{-\frac{2}{m}} \frac{m - 2}{4(m - 1)} y^l \mathfrak{B}_l - (\Upsilon + 1 \frac{kd^l \mathfrak{B}_l}{4}) c_p \right] y^i \]
\[ - \left[ \Upsilon + 1 \frac{kd^l \mathfrak{B}_l}{4} \right] \mathcal{A}^i + \frac{m}{4} A^{-\frac{m - 2}{m}} A^{il} \mathfrak{B}_l. \quad (31) \]

If the relation (21) holds, then by (31) the Finsler metric \( \bar{F} \) is projectively related to \( F \).
Lemma 4. Let \( \bar{F} = \sqrt{A^{2/m} + B} \) and \( F = A^{1/m} \) are generalized \( m \)-th root and \( m \)-th root Finsler metric on an open subset \( U \subset \mathbb{R}^n \), respectively, where \( A := a_{i_1...i_m}(x)y^{i_1}...y^{i_m} \) are symmetric in all its indices and \( B := c_{i}d_{j}y^{i}y^{j} \) with \( c_{i}d_{j} = c_{j}d_{i} \) and \( c_{i}d_{i} \neq -1 \) is a 2-form on \( M \). Suppose that (21) and (8) hold. Then \( B = 0 \).

Proof. Let (21) holds

\[
[4g + k\partial^\theta \mathfrak{B}_l] A^i = mA^{1-2}A^l \mathfrak{B}_l. \tag{32}
\]

Then by (28) and (29) we have

\[
g^{ij} d_j \left[ (F^2)_{x+y^k} - (F^2)_{x} + \mathfrak{B}_l \right] A^l p c_p = A^l \mathfrak{B}_l g^{qr} c_q d_q + A^l \mathfrak{B}_l, \tag{33}
\]

or equivalently

\[
g^{ij} d_j \left[ (F^2)_{x+y^k} - (F^2)_{x} \right] A^l p c_p - A^l \mathfrak{B}_l = [A^l d^j - A^l d^j] c_j \mathfrak{B}_l. \tag{34}
\]

The following holds

\[
(F^2)_{x+y^k} - (F^2)_{x} = \frac{2}{m} A^{1-2} \left[ \left( \frac{2}{m} - 1 \right) A l A_0 + A A_0 l - A A_{x l} \right]. \tag{35}
\]

Contracting (35) with \( g^{ij} \) yields

\[
g^{ij} [(F^2)_{x+y^k} - (F^2)_{x}] = \frac{2}{m} A^{1-2} \mathcal{H}^{ij} \left[ \left( \frac{2}{m} - 1 \right) A l A_0 + A A_0 l - A A_{x l} \right], \tag{36}
\]

where \( \mathcal{H}^{ij} := [m A A^{ij} + \frac{m-2}{m-1} y^{i}y^{j}] \). By considering (36), the left hand side of (41) is a rational function in \( y \), while its right hand side is a irrational function in \( y \). Then (34) reduces to following

\[
g^{ij} d_j A^l p c_p \mathfrak{F}_l = A^l \mathfrak{B}_l, \tag{37}
\]

\[
A^l d^j c_j \mathfrak{B}_l = A^l d^j d^j c_j \mathfrak{B}_l, \tag{38}
\]

where \( \mathfrak{F}_l := [(F^2)_{x+y^k} - (F^2)_{x}] \). Contracting (37) with \( A_{si} \) implies that

\[
\mathfrak{B}_s = (d^l \mathfrak{F}_l)c_s. \tag{39}
\]

By multiplying (38) with \( A_{is} \), we have

\[
d^j c_j \mathfrak{B}_s = d^l \mathfrak{B}_l c_s. \tag{40}
\]

\[
d^j c_j \times (39)- (40) \text{ yields } ((d^j c_j)d^j \mathfrak{F}_l - d^j \mathfrak{B}_l)c_s = 0. \tag{41}
\]

By assumption, (3) holds and then \( (d^j c_j)d^j \mathfrak{F}_l - d^j \mathfrak{B}_l \neq 0 \). Thus \( c_s = 0 \) and \( B = 0 \) which implies that \( \bar{F} = F \). This completes the proof. \( \square \)
Proof of Theorem 2. By Lemmas 3 and 4, we get the proof.

Recently, Zu-Zhang-Li proved that every Douglas $m$-th root Finsler metric $F = A^{1/m}$ ($m > 4$) with irreducibility of $A$, is a Berwald metric [24]. Then by Theorem 2, we have the following.

Corollary 2. Let $\bar{F} = \sqrt{A^{2/m} + B}$ and $F = A^{1/m}$ are generalized $m$-th root and $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, respectively, where $m > 4$, $A := a_{i_1...i_m}(x)y^{i_1}...y^{i_m}$ is irreducible and $B := c_i(x)d_j(x)y^iy^j$ with $c_id_j = c_jd_i$ and $c_id^j \neq -1$. Suppose that (7) holds and $\bar{F}$ is a Douglas metric. Then $F$ reduces to a Berwald metric.

5 Proof of the Theorem 3

Let $(M, F)$ and $(\tilde{M}, \tilde{F})$ be two Finsler spaces on same underlying $n$-dimensional manifold $M$. A Finsler space $(M, F)$ is conformal to a Finsler space $(\tilde{M}, \tilde{F})$, if and only if there exists a scalar field $\alpha(x)$ satisfying $\bar{F} = e^{\alpha}F$ (see [12]). The conformal change $\alpha(x)$ is called homothetic and isometry if $\alpha_i = \frac{\partial \alpha}{\partial x^i} = 0$ and $\alpha = 0$, respectively. In these section, we will prove a generalized version of Theorem 3. Indeed, we are going to consider two generalized $m$-th root metrics $\bar{F} = \sqrt{A^{2/m} + B}$ and $\tilde{F} = \sqrt{A^{2/m} + \tilde{B}}$ which are conformal and prove the following.

Theorem 4. Let $\bar{F} = \sqrt{A^{2/m} + B}$ and $\tilde{F} = \sqrt{A^{2/m} + \tilde{B}}$ are two generalized $m$-th root metrics on an open subset $U \subset \mathbb{R}^n$, where $B := b_{ij}(x)y^iy^j$ and $\tilde{B} := \tilde{b}_{ij}(x)y^iy^j$. Suppose that $\bar{F}$ is non-isometry conformal to $\tilde{F}$. Then $F = A^{1/m}$ is a Riemannian metric.

Proof. Let

$$F = e^{\alpha}\bar{F},$$

where $\bar{F} = \sqrt{A^{2/m} + B}$ and $\tilde{F} = \sqrt{A^{2/m} + \tilde{B}}$ are generalized $m$-th root Finsler metrics on an open subset $U \subset \mathbb{R}^n$, where $B := b_{ij}(x)y^iy^j$ and $\tilde{B} := \tilde{b}_{ij}(x)y^iy^j$. By assumption $\bar{F}$ is conformal to $\tilde{F}$. Then, we have

$$\bar{g}_{ij} = e^{2\alpha}\tilde{g}_{ij}.\quad (43)$$

Then we have

$$g_{ij} + \tilde{b}_{ij} = e^{2\alpha}(g_{ij} + \tilde{b}_{ij}),\quad (44)$$

where $g_{ij} = \frac{1}{2}(A^{1/m})_{y^iy^j}$ is the fundamental tensor of $F := A^{1/m}$. Since $\alpha$ is not isometry, i.e., $\alpha \neq 0$, then by (43) and (44), we get

$$g_{ij} = \frac{1}{1 - e^{2\alpha}}(e^{2\alpha}\tilde{g}_{ij} - \bar{b}_{ij}).\quad (45)$$

This implies that $C_{ijk} = 0$ and then $F$ is Riemannian.
By [44], we get the following.

**Corollary 3.** Let \( \bar{F} = \sqrt{A^{2/m} + B} \) and \( \tilde{F} = \sqrt{A^{2/m} + \tilde{B}} \) are two generalized \( m \)-th root metrics on an open subset \( U \subset \mathbb{R}^n \), where \( F := A^{1/m} \) is not Riemannian, \( \bar{B} := \bar{b}_{ij}(x)y^iy^j \) and \( \tilde{B} := \tilde{b}_{ij}(x)y^iy^j \). Suppose that \( \bar{F} \) is conformal to \( \tilde{F} \). Then \( \bar{F} = \tilde{F} \) or equivalently \( \bar{B} = \tilde{B} \).

**Proof of Theorem 3** In Theorem [4] put \( \tilde{B} = 0 \) and \( \tilde{F} := F \). Suppose that the generalized \( m \)-th root metric \( \bar{F} = \sqrt{A^{2/m} + B} \) is conformal to the \( m \)-th root Finsler metric \( F = A^{1/m} \). By Theorem [4] \( F \) is Riemannian and then \( C_{ijk} = 0 \). Since \( \bar{g}_{ij} = e^{2\alpha}g_{ij} \) then \( \bar{g}_{ij} = g_{ij} + b_{ij} \), which yields \( \bar{C}_{ijk} = C_{ijk} \).

Thus \( \bar{C}_{ijk} = 0 \), which implies that \( \bar{F} \) reduces to a Riemannian metric. This completes the proof. \( \square \)

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