OPTIMAL MEASUREMENTS IN QUANTUM MECHANICS

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Abstract. Four common optimality criteria for measurements are formulated using relations in the set of observables, and their connections are clarified. As case studies, $1-0$ observables, localization observables, and photon counting observables are considered.

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1. Introduction

Any measurement is carried out in order to gain information about an object system. Informationally complete measurements [1] allow a unique determination of the state of the object, and therefore, they are usually regarded as optimal measurements. Informationally complete phase space measurements are well known [2] (also see [3, 4, 5]), and other instances of informationally complete measurements have been found as well; see, for instance, [6, 7]. However, in many practical cases a unique state determination is not attainable. For example, a photodetection or a position measurement does not provide enough information for that purpose. It is still meaningful to seek an optimal measurement in these cases, i.e., a measurement that gives as much information as possible. The optimality of a measurement depends on a specified class of measurements under investigation, and it is therefore a relative property. The specified class of measurements is determined by the requirements and presumptions concerning measurements. Measurements may be, for example, required to be covariant with respect to a relevant symmetry group.

In addition to providing as much information as possible, it would be desirable for a measurement to have as little imprecision as possible. This objective can be thought just as another criterion for an optimal measurement, and it has been investigated in [8, 9, 10].
An imprecise measurement cannot give more information than a more precise counterpart. However, in some cases it may be equally good in state determination or in state distinction. This simple fact is important since imprecision is unavoidable in any real measurement.

In this paper we study measurements only in the aspect of measurement outcome statistics, and therefore, for our purposes a measurement may be described by an observable (normalized positive operator measure). We emphasize that this is only a partial description of a measurement as, for instance, a possible preparative purpose of measurements is ignored. Obviously, consideration of the other aspects of measurements would give different optimality criteria.

The concept of an observable is briefly reviewed in Section 2 where we also recall the description of an observable as an affine mapping from the set of states into the set of probability measures. In Sections 3 and 4 we formulate four common optimality criteria using certain relations on the set of observables. Two of these relations correspond to the state distinction and determination, while the other two are related to the measurement imprecision. These relations are pre-orderings, and thus, they define partial orderings in the respective sets of equivalence classes. The optimality criteria are then defined as maximality requirements for equivalence classes. This approach is suitable also for cases where informationally complete observables do not exist, and connections between different criteria are easily seen. In Section 5 we study the cases of 1−0 observables, photon counting observables, and localization observables.

2. Observables in quantum mechanics

In this section we fix the notation, and for the reader's convenience we briefly recall the basic description of a quantum observable. (For a review see, for example, [3,11,12]).

Let \( \mathcal{H} \) be a complex separable Hilbert space, and denote the set of bounded linear operators on \( \mathcal{H} \) by \( \mathcal{L}(\mathcal{H}) \). Let \( \Omega \) be a set and \( \mathcal{A} \) a \( \sigma \)-algebra on \( \Omega \). The set of probability measures on the measurable space \( (\Omega, \mathcal{A}) \) is denoted by \( M_1^+ (\Omega, \mathcal{A}) \).

Consider a quantum system, described by a Hilbert space \( \mathcal{H} \). States of the system are represented as positive operators of trace one, and observables are represented as normalized positive operator measures. More precisely, an observable with an outcome space \( (\Omega, \mathcal{A}) \) is a mapping \( E : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}) \) such that

(i) \( E(X) \geq O \) for any \( X \in \mathcal{A} \);
(ii) \( E(\Omega) = I \);
(iii) \( E( \cup_i X_i) = \sum_i E(X_i) \) (in the weak sense) for any disjoint sequence \((X_i) \subset A\).

We denote the set of states by \( S(\mathcal{H}) \) and the set of observables with the outcome space \((\Omega, \mathcal{A})\) by \( O(\Omega, \mathcal{A}, \mathcal{H}) \), or just \( O(\Omega, \mathcal{H}) \) when \( \mathcal{A} \) is understood.

Let \( E \in O(\Omega, \mathcal{A}, \mathcal{H}) \) be an observable. For a state \( T \in S(\mathcal{H}) \), we define a probability measure \( p^E_T \) on \((\Omega, \mathcal{A})\) by

\[
p^E_T(X) = \text{tr}[TE(X)], \quad X \in \mathcal{A}.
\]

This is interpreted as the probability distribution of measurement outcomes when the system is in the state \( T \) and the observable \( E \) is measured. The observable \( E \) defines a mapping \( \Phi_E : S(\mathcal{H}) \to M^+_1(\Omega, \mathcal{A}) \) by \( \Phi_E(T) = p^E_T \). The mapping \( \Phi_E \) preserves convex combinations of states: for any \( T_1, T_2 \in S(\mathcal{H}) \) and \( 0 \leq \lambda \leq 1 \), we have

\[
\Phi_E(\lambda T_1 + (1 - \lambda) T_2) = \lambda \Phi_E(T_1) + (1 - \lambda) \Phi_E(T_2).
\]

Conversely, a mapping \( \Phi : S(\mathcal{H}) \to M^+_1(\Omega, \mathcal{A}) \) satisfying (1) defines a unique observable \( E_\Phi \). This correspondence is consistent in the sense that \( E_{\Phi_E} = E \) and \( \Phi_{E_\Phi} = \Phi \). For reviews of the properties of the mapping \( \Phi_E \), we refer to [13, 14, 15].

The representation of an observable via an affine mapping from the set of states \( S(\mathcal{H}) \) into the space of probability measures \( M^+_1(\Omega, \mathcal{A}) \) is physically natural. It captures an intuitive concept of an observable: a specification of the outcome space (possible events in a measurement) and an assignment of a probability distribution to each state of the system. In the following sections we use this representation of observables to make the operational content of the relations and the optimality criteria transparent.

3. Relations on the set \( O(\Omega, \mathcal{A}, \mathcal{H}) \)

3.1. State distinction and state determination. Let us first recall the usual concepts related to the ability of an observable to distinguish and determine states. (For more details, see e.g. [16].)

**Definition 1.** Let \( E \in O(\Omega, \mathcal{A}, \mathcal{H}) \) and \( T_1, T_2 \in S(\mathcal{H}) \).

(i) \( E \) distinguish the states \( T_1 \) and \( T_2 \) if

\[
\Phi_E(T_1) \neq \Phi_E(T_2);
\]

(ii) the state \( T_1 \) is determined by \( E \) if, for all \( T \in S(\mathcal{H}) \),

\[
\Phi_E(T_1) = \Phi_E(T) \Rightarrow T_1 = T.
\]

We denote by \( \mathcal{D}_E \) the set of states determined by \( E \).
The first of these concepts leads to the following relations.

**Definition 2.** Let \( E, F \in \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H}) \). If for all states \( T_1, T_2 \in \mathcal{S}(\mathcal{H}) \),
\[
\Phi_E(T_1) = \Phi_E(T_2) \Rightarrow \Phi_F(T_1) = \Phi_F(T_2),
\]
then we denote \( F \preceq_i E \), and say that the *state distinction power* of \( E \) is greater than or equal to \( F \) (or that \( F \) gives less or the same information than \( E \)). If \( F \preceq_i E \preceq_i F \), we say that \( E \) and \( F \) are *informationally equivalent*, and denote \( E \sim_i F \).

Condition (2) can be written in an equivalent form
\[
\Phi_F(T_1) \neq \Phi_F(T_2) \Rightarrow \Phi_E(T_1) \neq \Phi_E(T_2).
\]
Hence, \( F \preceq_i E \) means that \( E \) distinguish all states that are distinguished by \( F \). It is clear that \( \preceq_i \) is a reflexive and transitive relation, and therefore, \( \sim_i \) is an equivalence relation.

**Definition 3.** Let \( E, F \in \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H}) \). If \( D_F \subseteq D_E \), then we denote \( F \preceq_d E \), and say that the *state determination power* of \( E \) is greater than or equal to \( F \).

It is immediately seen that the relation \( \preceq_d \) is reflexive and transitive, and thus, it defines an equivalence relation \( \sim_d \) in the natural way.

We note that if \( F \preceq_i E \) holds, then \( F \preceq_d E \). Indeed, let \( T_1 \in D_F \), and let \( T \) be a state such that \( \Phi_E(T) = \Phi_E(T_1) \). Relation \( F \preceq_i E \) implies that \( \Phi_F(T) = \Phi_F(T_1) \), and thus, \( T_1 = T \). This means that \( T_1 \in D_E \), and therefore \( D_F \subseteq D_E \).

Examples 1 and 2 show that the converse is, in general, not true: the condition \( F \preceq_d E \) does not imply that \( F \preceq_i E \).

**Example 1.** An observable \( E \in \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H}) \) is *trivial* (or *uninformative*) if it does not distinguish any pair of states, that is,
\[
\Phi_E(T_1) = \Phi_E(T_2) \quad \forall T_1, T_2 \in \mathcal{S}(\mathcal{H}).
\]
Condition (3) is equivalent with the fact that there is a probability measure \( m \in M^+_1(\Omega, \mathcal{A}) \) such that \( E(X) = m(X)I \). If \( E \) is a trivial observable, then obviously \( E \preceq_i F \) for any \( F \in \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H}) \). Moreover, if \( F \preceq_i E \), then also \( F \) is a trivial observable.

**Example 2.** Suppose that \( F \in \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H}) \) is a spectral measure, i.e., \( F(X)^2 = F(X) \) for any \( X \in \mathcal{A} \). It is shown in [16] that \( T \in D_F \) if and only if \( T \) is a one-dimensional spectral projection of \( F \), that is, \( T = F(X) = \langle \psi \rangle \psi \) for some unit vector \( \psi \in \mathcal{H} \). Thus, if \( F \) has no non-degenerate eigenstates, then \( D_F = \emptyset \). For any trivial observable \( E \) we also have \( D_E = \emptyset \), and hence, \( F \sim_d E \).
3.2. **Fuzzy observables and coarse-graining.** Fuzzy sets are used in many different situations to model imprecision and uncertainty, and they are also applicable to describe imprecision in a measurement. We recall that a fuzzy set in $\Omega$ is a function $\tilde{X}$ from $\Omega$ to the interval $[0, 1]$, and the value $\tilde{X}(\omega)$ represents the degree of membership of $\omega$ in $\tilde{X}$. We identify a subset $X$ of $\Omega$ with the characteristic function $\chi_X$, and in this way the subsets of $\Omega$ are (special types of) fuzzy sets.

A fuzzy set is called a fuzzy event if it is measurable, and we denote by $\tilde{A}$ the collection of fuzzy events. If $m \in M^+_1(\Omega, \mathcal{A})$ and $\tilde{X} \in \tilde{A}$, then the probability $m(\tilde{X})$ is defined by the integral

$$m(\tilde{X}) = \int \tilde{X}(\omega) \, dm(\omega).$$

Measurement imprecision may be modelled by a mapping $\Lambda$ from $\mathcal{A}$ to $\tilde{A}$. We require that

(i) $\Lambda(X') = \chi_{\Omega} - \Lambda(X)$;
(ii) $\sum_{i=1}^{\infty} \Lambda(X_i) = \chi_{\Omega}$ if $\bigcup_{i=1}^{\infty} X_i = \Omega$ and $X_i \cap X_j = \emptyset$ for all $i \neq j$.

Condition (i) means that a complement of a set is mapped to a fuzzy complement, while (ii) means that a partition of $\Omega$ is mapped to a fuzzy partition. We call a mapping $\Lambda : \mathcal{A} \to \tilde{A}$ with properties (i) and (ii) a confidence mapping.

Suppose that $\Lambda$ is a confidence mapping and let $m \in M^+_1(\Omega, \mathcal{A})$. In view of (4), the composite mapping $m \circ \Lambda$ makes sense. The properties (i) and (ii) of $\Lambda$ imply that $m \circ \Lambda$ is a probability measure. Our consideration leads to the following definition.

**Definition 4.** Let $E, F \in \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H})$. If there exists a confidence mapping $\Lambda : \mathcal{A} \to \tilde{A}$ such that, for any $T \in \mathcal{S}(\mathcal{H})$,

$$\Phi_F(T) = \Phi_E(T) \circ \Lambda,$$

then we denote $F \preceq_f E$ and say that $F$ is fuzzy version of $E$. If $F \preceq_f E \preceq_f F$, we denote $F \lessdot E$.

There is an equivalent formulation of the relation $\preceq_f$. A mapping $\nu : \Omega \times \mathcal{A} \to [0, 1]$ is a Markov kernel if

(i) for every $\omega \in \Omega$, the mapping $\nu(\omega, \cdot)$ is a probability measure on $(\Omega, \mathcal{A})$;
(ii) for every $X \in \mathcal{A}$, the mapping $\nu(\cdot, X)$ is $\mathcal{A}$-measurable.

It is straightforward to verify that $\nu$ is Markov kernel if and only if the mapping $X \mapsto \nu(\cdot, X)$ is a confidence mapping. Hence, the condition
$F \preceq_f E$ is equivalent to the fact that there exists a Markov kernel $\nu$ such that

$$F(X) = \int \nu(\omega, X) \, dE(\omega), \quad X \in \mathcal{A}. \tag{6}$$

A formulation similar to (6) was introduced in [20, 21, 22], and it has been used, for instance, to investigate joint position-momentum measurements. The relation $\preceq_f$ has been studied in [8] in the case of finite dimensional Hilbert spaces and countable outcome spaces. The general case (with a slightly different relation than ours) has been studied in [9].

The relation $\preceq_f$ is reflexive since the mapping

$$(\omega, X) \mapsto \delta_X(\omega) = \chi_X(\omega)$$

is a Markov kernel and

$$E(X) = \int \chi_X(\omega) \, dE(\omega).$$

**Proposition 1.** The relation $\preceq_f$ is transitive.

*Proof.* Let $F_i \in \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H}), \ i = 1, 2, 3$, and assume that $F_1 \preceq_f F_2$ and $F_2 \preceq_f F_3$, with $\nu_1$ and $\nu_2$ being corresponding Markov kernels, respectively. For any $\omega \in \Omega, X \in \mathcal{A}$, define

$$\nu_3(\omega, X) = \int \nu_1(\omega', X) \nu_2(\omega, d\omega').$$

Let us first note that $\nu_3$ is a Markov kernel. Indeed, for a fixed $X \in \mathcal{A}$, the function $\nu_1(\cdot, X)$ is nonnegative, bounded and measurable. Therefore, there is an increasing sequence $\{h_n\}$ of nonnegative simple functions converging to the function $\nu_1(\cdot, X)$ pointwisely. For each $\omega \in \Omega$, the monotone convergence theorem implies that

$$\int \nu_1(\omega', X) \nu_2(\omega, d\omega') = \lim_{n \to \infty} \int h_n(\omega') \nu_2(\omega, d\omega').$$

For every $n$, the function $\omega \mapsto \int h_n(\omega') \nu_2(\omega, d\omega')$ is measurable and the function $\nu_3(\cdot, X)$ is a pointwise limit of measurable functions. Hence, the function $\nu_3(\cdot, X)$ is measurable. It is easy to see that, for a fixed $\omega \in \Omega$, the mapping $\nu_3(\omega, \cdot)$ is a probability measure. In conclusion, $\nu_3$ is a Markov kernel.
Let $T \in \mathcal{S}(\mathcal{H})$. For any $X \in \mathcal{A}$, we have
\[
\int \nu_3(\omega, X) \, dp^F_T(\omega) = \int \int \nu_1(\omega', X) \, \nu_2(\omega, d\omega') \, dp^F_T(\omega)
= \lim_{n \to \infty} \int \int h_n(\omega') \, \nu_2(\omega, d\omega') \, dp^F_T(\omega)
= \lim_{n \to \infty} \int h_n(\omega') \, dp^F_T(\omega)
= \lim_{n \to \infty} \int \nu_1(\omega', X) \, dp^F_T(\omega')
= \nu_1(\omega', X) \, dp^F_T(\omega).
\]
This shows that $F_1 \approx_f F_3$. \hfill \Box

**Example 3.** Let $E \in \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H})$ be a trivial observable defined by
a probability measure $m \in M^+_1(\Omega, \mathcal{A})$; see Example 1. For any $F \in \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H})$, we then have $E \approx_f F$. Indeed, define
\[
\nu(\omega, X) = m(X), \quad \omega \in \Omega, X \in \mathcal{A}.
\]
Then $\nu$ is a Markov kernel and
\[
\int \nu(\omega, X) \, dF(\omega) = m(X) \int dF(\omega) = m(X) I = E(X).
\]
Moreover, it is easy to see that if $F \approx_f E$, then also $F$ is a trivial observable.

Suppose that $F \approx_f E$, and let $\nu$ be a corresponding Markov kernel such that \[6\] holds. Define a mapping $\Psi_\nu : M^+_1(\Omega, \mathcal{A}) \to M^+_1(\Omega, \mathcal{A})$ by
\[
(7) \quad \Psi_\nu(m)(X) = \int \nu(\omega, X) \, dm(\omega), \quad m \in M^+_1(\Omega, \mathcal{A}), X \in \mathcal{A}.
\]
From equation \[6\] follows that $\Phi_F$ is a composite mapping of $\Phi_E$ and $\Psi_\nu$, that is,
\[
\Phi_F = \Psi_\nu \circ \Phi_E.
\]
Hence, any measurement outcome distribution of the observable $F$ is obtained from the corresponding measurement outcome distribution of $E$ by applying a mapping $\Psi_\nu$, which is independent of a state. This procedure is formulated in the following concept of coarse-graining.

The concept of coarse-graining means, generally speaking, a reduction in the statistical description of a system; see, for instance, \[23\].

**Definition 5.** Let $E, F \in \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H})$. We say that $F$ is a *coarse-graining* of $E$, and denote $F \approx_c E$, if there exists an affine mapping $\Psi : M^+_1(\Omega, \mathcal{A}) \to M^+_1(\Omega, \mathcal{A})$ such that
\[
(8) \quad \Phi_F = \Psi \circ \Phi_E.
\]
The relation \( \preceq_c \) is reflexive as the identity mapping is affine, and the transitivity of \( \preceq_c \) follows from the fact that the composition of affine mappings is affine. The corresponding equivalence relation is denoted by \( \sim_c \).

Our previous discussion shows that if \( F \preceq_f E \), then \( F \preceq_c E \). We note that there are affine mappings on \( M_1^+(\Omega, A) \) which do not have representations via Markov kernels as in (7); see [24]. However, for observables on a finite outcome space the relations \( \preceq_f \) and \( \preceq_c \) are the same, as the following example illustrates.

**Example 4.** Suppose that \( \Omega = \{1, 2, \ldots, n\} \). An observable \( E \in \mathcal{O}(\Omega, \mathcal{H}) \) is determined by the effects \( E_j := E(\{j\}) \), and for each Markov kernel \( \nu \) corresponds a \( n \times n \) row stochastic matrix \( (\nu_{jk}) \), where \( \nu_{jk} = \nu(j, \{k\}) \). Condition (6) can then be written in the form

\[
F_k = \sum_{j=1}^{n} \nu_{jk} E_j, \quad k \in \Omega.
\]

For an affine mapping \( \Psi \) on \( M_1^+(\Omega) \), define \( \nu(j, X) := \Psi(\delta_j)(X) \), where \( \delta_j \) is the point measure concentrated at a point \( j \in \Omega \) and \( X \subseteq \Omega \). Since any probability measure on \( \Omega \) can be written as a convex combination of the point measures, the mapping \( \Psi \) is determined by the Markov kernel \( \nu \). We conclude that \( F \preceq_c E \) if and only if \( F \preceq_f E \), and this is the case exactly when there is a stochastic matrix such that (9) holds.

The condition \( F \preceq_c E \) implies that \( F \preceq_i E \). Indeed, if there is a mapping \( \Psi \) such that (8) holds, then certainly condition (2) is satisfied.

### 4. Optimal Measurements

Let \( \preceq \) be one of the relations \( \preceq_f, \preceq_c, \preceq_i \), or \( \preceq_d \), and let \( \sim \) be the corresponding equivalence relation. Since \( \preceq \) is reflexive and transitive, it defines a partial ordering \( \preceq' \) on the set of equivalence classes \( \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H})/\sim \). Namely, denoting the equivalence class of an observable \( E \) by \([E]\), we define

\[
[E] \preceq' [F] \text{ if and only if } E \preceq F.
\]

Typically, we have some requirements and presumptions for the intended measurements, and therefore, we are interested only on a restricted class \( \mathcal{O} \subseteq \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H}) \) of observables. We are thus led to the following definition.

**Definition 6.** Let \( \mathcal{O} \subseteq \mathcal{O}(\Omega, \mathcal{A}, \mathcal{H}) \). We say that an observable \( E \in \mathcal{O} \) is *optimal* in \( \mathcal{O} \) with respect to preordering \( \preceq \) (or \( \preceq \)-*optimal* in \( \mathcal{O} \)), if
the equivalence class of $E$ is a maximal element of the partially ordered set $\mathcal{O}/\sim$.

In other words, $E$ is $\preceq$-optimal in $\mathcal{O}$ if, for any $F \in \mathcal{O}$, the condition $E \preceq F$ implies that $E \sim F$.

It was shown in the last section that, for observables $E$ and $F$, the following implications hold:

\begin{equation}
F \preceq_f E \Rightarrow F \preceq_c E \Rightarrow F \preceq_i E \Rightarrow F \preceq_d E.
\end{equation}

This means also that the following inclusions hold between the equivalence classes of $E$:

\begin{equation}
[E]_f \subseteq [E]_c \subseteq [E]_i \subseteq [E]_d.
\end{equation}

We emphasize that although the relations have the hierarchy (10), a $\preceq_f$-optimal observable may or may not be optimal with respect to other relations. This is demonstrated in Section 5. However, if an observable $E \in \mathcal{O}$ satisfies a stronger condition that $F \preceq_f E$ for any $F \in \mathcal{O}$ (i.e., the equivalence class $[E]$ is the greatest element), then it follows that $E$ is optimal in $\mathcal{O}$ with respect to all four relations.

We note that the four relations discussed here are not the only interesting relations in the theory of quantum measurements. In the recent paper \cite{10} several other relations were studied, and the notion of a clean measurement was defined similarly to Definition 6. Cleanness property is also a relevant optimality criterion.

5. Examples

5.1. 1-0 observables. The set of effects, denoted by $\mathcal{E}(\mathcal{H})$, is the set of operators $A \in \mathcal{L}(\mathcal{H})$ satisfying $0 \leq A \leq I$. An effect $A$ defines an observable $E^A$ with the outcome space $\Omega = \{0, 1\}$ by

\[ E^A_1 = A, \quad E^A_0 = A' = I - A. \]

These are the most simplest kind of observables, and we call them 1-0 observables.

**Proposition 2.** Let $A, B \in \mathcal{E}(\mathcal{H})$ and let $E^A, E^B$, be the corresponding 1-0 observables. Then $E^A \preceq_f E^B$ if and only if there are numbers $s, t \in [0, 1]$ such that

\begin{equation}
A = tB + sB'.
\end{equation}

**Proof.** As shown in Example 4, the condition $E^A \preceq_f E^B$ means that there is a row stochastic matrix ($\nu_{jk}$) such that

\[ E^A_0 = \nu_{00} E^B_0 + \nu_{10} E^B_1, \]
\[ E^A_1 = \nu_{01} E^B_0 + \nu_{11} E^B_1. \]
Since \( \nu_{11} + \nu_{10} = \nu_{01} + \nu_{00} = 1 \), these equations are equivalent. Therefore, the condition \( E^A \preceq_f E^B \) holds if and only if
\[
A = \nu_{11} B + \nu_{01} B'.
\]
Any \( 2 \times 2 \) row stochastic matrix has the form
\[
\nu_{11} = t, \quad \nu_{10} = 1 - t, \quad \nu_{01} = s, \quad \nu_{00} = 1 - s,
\]
for some numbers \( s, t \in [0, 1] \), and thus, the claim follows. \( \square \)

As a direct consequence of Proposition 2, we note that, for nontrivial observables \( E^A \) and \( E^B \), the equivalence relation \( E^A \sim_f E^B \) holds exactly when \( A = B \) or \( A = B' \).

**Proposition 3.** Let \( A \in \mathcal{E}(\mathcal{H}) \). The observable \( E^A \) is \( \preceq_f \)-optimal in \( \mathcal{O}(\Omega, \mathcal{H}) \) if and only if \( ||A|| = ||A'|| = 1 \).

**Proof.** Let us first assume that \( ||A|| = ||A'|| = 1 \). Suppose that \( B \) is an effect such that \( E^A \preceq_f E^B \). We need to show that \( E^A \not\sim_f E^B \). By Proposition 2 the condition \( E^A \preceq_f E^B \) is equivalent with the fact that there exist numbers \( s, t \in [0, 1] \) such that (12) holds. Since \( ||A'|| = 1 \), for any \( \epsilon > 0 \) there is a unit vector \( \varphi_\epsilon \in \mathcal{H} \) such that
\[
\langle \varphi_\epsilon | (I - A) \varphi_\epsilon \rangle \geq 1 - \epsilon,
\]
and thus,
\[
(\text{13}) \quad \langle \varphi_\epsilon | A \varphi_\epsilon \rangle \leq \epsilon.
\]
From (12) and (13) we get
\[
\epsilon \geq s(1 - \langle \varphi_\epsilon | B \varphi_\epsilon \rangle) + t\langle \varphi_\epsilon | B \varphi_\epsilon \rangle \geq \min(s, t).
\]
Thus, either \( s = 0 \) or \( t = 0 \). If \( s = 0 \), then \( A = tB \). Moreover, as
\[
1 = ||A|| = t||B|| \leq t \leq 1,
\]
we have \( t = 1 \) and \( A = B \). By a similar argument \( t = 0 \) gives \( A = B' \). Thus, \( E^A \not\sim_f E^B \).

Let us then assume that \( ||A|| < 1 \) (the case \( ||A'|| < 1 \) being similar). Denote \( \alpha := ||A|| \) and \( \beta := ||A'|| \). Then
\[
(14) \quad (1 - \beta)I \leq A \leq \alpha I
\]
and
\[
\alpha + \beta = ||A|| + ||A'|| \geq ||A + A'|| = 1.
\]
If \( \alpha + \beta = 1 \), then (14) implies that \( A = \alpha I \). In this case \( E^A \) is a trivial observable, and clearly, not \( \preceq_f \)-optimal. Consider the case \( \alpha + \beta > 1 \).

It follows from (14) that the operator

\[
B := \frac{1}{\alpha + \beta - 1} A + \frac{\beta - 1}{\alpha + \beta - 1} I
\]

is an effect. Moreover,

\[
A = tB + sB',
\]

where \( s = 1 - \beta \) and \( t = \alpha \). Thus, \( E^A \preceq_f E^B \). Since \( 0 < \alpha < 1 \), we have \( B \neq A \neq B' \). This shows that \( E^A \) is not \( \preceq_f \)-optimal. □

The set \( \mathcal{O}(\Omega, \mathcal{H}) \) is convex: if \( E^A, E^B \in \mathcal{O}(\Omega, \mathcal{H}) \) and \( 0 \leq \lambda \leq 1 \), then

\[
\lambda E^A + (1 - \lambda)E^B = E^{\lambda A + (1 - \lambda) B} \in \mathcal{O}(\Omega, \mathcal{H}).
\]

If \( A \neq B \) and \( 0 < \lambda < 1 \), then the convex combination \( E^{\lambda A + (1 - \lambda) B} \) is a randomized observable [11]. An observable is non-randomized if it has no such convex decomposition. The extreme elements of the convex set \( \mathcal{E}(\mathcal{H}) \) are projection operators [23, Lemma 2.3], and hence, an observable \( E^A \) is non-randomized exactly when the respective effect \( A \) is a projection. That kind of observables are \( \preceq_f \)-optimal in \( \mathcal{O}(\Omega, \mathcal{H}) \), but if \( \dim(\mathcal{H}) \geq 3 \), then there are also other \( \preceq_f \)-optimal observables. To give an example, let \( P \) and \( R \) be mutually orthogonal one-dimensional projections. For any \( 0 < t < 1 \), the operator \( A = P + tR \) is an effect but not a projection, and \( ||A|| = ||A'|| = 1 \). The observable \( E^A \) is a convex combination of the non-randomized observables \( E^P \) and \( E^{P+R} \), and all these three observables are \( \preceq_f \)-optimal.

**Remark 1.** The condition \( ||A|| = ||A'|| = 1 \) in Proposition 3 has a physical interpretation. Indeed, if \( P \) is a projection (and not equal to \( O \) or \( I \)), then there exist states \( T_1 \) and \( T_2 \) such that

\[
\text{tr}[T_1 P] = 1, \quad \text{tr}[T_2 P'] = 1.
\]

This means that \( P \) and \( P' \) can be realized in the states \( T_1 \) and \( T_2 \), and thus, they are actualizable properties. On the other hand, the condition \( ||A|| = ||A'|| = 1 \) is equivalent with the fact that for each \( \delta > 0 \) there exist states \( T_1 \) and \( T_2 \) such that

\[
\text{tr}[T_1 A] \geq 1 - \delta, \quad \text{tr}[T_2 A'] \geq 1 - \delta.
\]

This is a relaxation of (15), and we say that the effects \( A \) and \( A' \) are approximately actualizable properties.
5.2. **Photon counting observables.** Let $\mathcal{H}$ be a Hilbert space describing a one-mode of an electromagnetic field. We denote by $\mathbb{N}$ the set of natural numbers (including 0), and $\mathcal{P}(\mathbb{N})$ is the set of all subsets of $\mathbb{N}$. Given an observable $F$ with the outcome space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, we denote $F_n = F(\{n\})$. Also, if $\nu : \mathbb{N} \times \mathcal{P}(\mathbb{N}) \to [0, 1]$ is a Markov kernel, we denote $\nu_{kn} = \nu(k, \{n\})$, $k, n \in \mathbb{N}$.

The number operator $\hat{N} = a^* a$ has a non-degenerate eigenvector $|n\rangle$ for every $n \in \mathbb{N}$. The number observable $E_N$ with the outcome space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ is defined by

$$E_N^n = |n\rangle\langle n|, \ n \in \mathbb{N}.$$ 

A photodetector with efficiency $\epsilon$, $0 \leq \epsilon \leq 1$, may be described by an observable $F$ defined by

$$(17) \quad F_n^\epsilon = \sum_{m=n}^{\infty} \binom{m}{n} \epsilon^n (1 - \epsilon)^{m-n} |m\rangle\langle m|, \ n \in \mathbb{N},$$

see, e.g., [3, Section VII.3]. We denote by $\mathcal{O}_\mathcal{P}$ the set of this kind of observables, and we call them *photon counting observables*. The photon counting observable $F^1$ corresponding to the ideal efficiency $\epsilon = 1$ is the number observable $E_N$, and the observable $F^0$ is the trivial observable given by $F_n^0 = \delta_{0,n} I$.

In the following we investigate the set $\mathcal{O}_\mathcal{P}$ of photon counting observables. Some related results have been discussed in [26, Chapter 7].

**Proposition 4.** Let $F^{\epsilon_1}, F^{\epsilon_2} \in \mathcal{O}_\mathcal{P}$. The condition $F^{\epsilon_1} \preceq F^{\epsilon_2}$ holds if and only if $\epsilon_1 \leq \epsilon_2$.

**Proof.** Let us first assume that $F^{\epsilon_1} \preceq F^{\epsilon_2}$. This means that there exists a Markov kernel $\nu$ such that

$$F_n^{\epsilon_1} = \sum_{k=0}^{\infty} \nu_{kn} F_k^{\epsilon_2}, \ n \in \mathbb{N}.$$ 

For every $m, n \in \mathbb{N}$, we get

$$(18) \quad \langle m | F_n^{\epsilon_1} | m \rangle = \sum_{k=0}^{\infty} \nu_{kn} \langle m | F_k^{\epsilon_2} | m \rangle.$$ 

Substituting (17) into both sides of (18) shows that $\nu_{mm} = \epsilon_1^n \epsilon_2^{m-n}$. Since $\nu$ is a Markov kernel, we have $\nu_{mm} \leq 1$. This can hold only if $\epsilon_1 \leq \epsilon_2$.

Let us then assume that $\epsilon_1 \leq \epsilon_2$. Define

$$\nu_{kn} = \begin{cases} 0 & \text{if } k < n, \\ \binom{k}{n} \epsilon_1^n \epsilon_2^{-k} (\epsilon_2 - \epsilon_1)^{k-n} & \text{if } k \geq n. \end{cases}$$

Then $\nu$ is a Markov kernel, and we have
\[
\sum_{k=0}^{\infty} \nu_{kn} F_k^{\epsilon^2} = \sum_{k=n}^{\infty} \sum_{m=k}^{\infty} \binom{k}{n} \binom{m}{k} \epsilon_1^n \epsilon_2^{-k-n}(1-\epsilon_2)^{m-k} |m\rangle \langle m| \\
= \sum_{m=n}^{\infty} \epsilon_1^n \left( \sum_{k=n}^{m} \binom{k}{n} \binom{m}{k} \epsilon_2^{-k-n}(1-\epsilon_2)^{m-k} \right) |m\rangle \langle m| \\
= \sum_{m=n}^{\infty} \binom{m}{n} \epsilon_1^n (1-\epsilon_1)^{m-n} |m\rangle \langle m| = F_n^{\epsilon_1}.
\]

Thus, $F_n^{\epsilon_1} \preceq_f F_n^{\epsilon^2}$. □

**Corollary 1.** The number observable $E^N$ is an optimal observable in $O_P$ with respect to $\preceq_f, \preceq_c, \preceq_t$ and $\preceq_d$.

Next we show that imprecision in a photon counting measurement does not imply a loss of information.

**Proposition 5.** If $F^\epsilon \in O_P$ and $\epsilon \neq 0$, then $F^\epsilon \simeq E^N$.

**Proof.** As the claim is trivial in the case $\epsilon = 1$, we may assume that $0 < \epsilon < 1$. Moreover, since $F^\epsilon \preceq_f E^N$ by Proposition 4 we have $F^\epsilon \preceq_i E^N$. To prove that $E^N \preceq_i F^\epsilon$, let $T_1, T_2 \in S(H)$ and assume that $\Phi_{F^\epsilon}(T_1) = \Phi_{F^\epsilon}(T_2)$. By (17) this means that, for every $n \in \mathbb{N}$,
\[
\sum_{m=n}^{\infty} \binom{m}{n} (1-\epsilon)^m |m\rangle \langle T_1 - T_2 | m\rangle = 0.
\]

Denote $a_m := (1-\epsilon)^m |T_1 - T_2 | m\rangle$ for every $m \in \mathbb{N}$. Since $|a_m| \leq (1-\epsilon)^m$, the formula
\[
f(z) := \sum_{m=0}^{\infty} a_m z^m
\]
defines a holomorphic function in the region $|z| < \frac{1}{1-\epsilon}$. The $n$th derivative of $f$ is
\[
f^{(n)}(z) = \sum_{m=n}^{\infty} m(m-1) \cdots (m-n+1) a_m z^{m-n},
\]
and hence, (19) implies that $f^{(n)}(1) = 0$ for every $n \in \mathbb{N}$. Thus, $f = 0$, and $a_m = 0$ for every $m \in \mathbb{N}$. We conclude that $\Phi_{E^N}(T_1) = \Phi_{E^N}(T_2)$, and therefore, $E^N \preceq_i F^\epsilon$. □

**Corollary 2.** If $F^\epsilon \in O_P$ and $\epsilon \neq 0$, then $D_{F^\epsilon} = \{ |n\rangle \langle n| \mid n \in \mathbb{N} \}$. 
Proof. For the number observable $E^N$ the claim follows from [16], (see Example 2). Since $F^e \sim E^N$ by Proposition 5 we have $F^e \sim d \sim E^N$, and thus, $D_{F^e} = D_{E^N}$. □

5.3. Localization observables on $\mathbb{R}$. Let us consider a free particle in the real line $\mathbb{R}$. We denote by $U$ and $V$ be the one-parameter unitary representations related to the groups of space translations and velocity boosts, respectively. As shown, for instance, in Chapter III of [14], we may fix $\mathcal{H} = L^2(\mathbb{R})$ and take $U$ and $V$ act on $\varphi \in \mathcal{H}$ as

$$[U(q)\varphi](x) = \varphi(x - q),$$
$$[V(p)\varphi](x) = e^{ipx} \varphi(x).$$

Let $Q$ be the selfadjoint operator such that $V(p) = e^{ipQ}$ for every $p \in \mathbb{R}$. The spectral measure $E^Q$ corresponding to the operator $Q$ is an observable with the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra of $\mathbb{R}$. For any $X \in \mathcal{B}(\mathbb{R})$ and $\varphi \in \mathcal{H}$, we have the usual formula

$$E^Q(X)\varphi = \chi_X \varphi,$$

where $\chi_X$ is the characteristic function of $X$.

The observable $E^Q$ has the property that, for any $q \in \mathbb{R}, X \in \mathcal{B}(\mathbb{R})$,

$$(20) \quad U(q)E^Q(X)U(q)^* = E^Q(X + q).$$

This covariance property justifies to associate the observable $E^Q$ with a localization measurement of the particle. In general, an observable $F$ with the outcome space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a localization observable if it has the covariance property

$$(21) \quad U(q)F(X)U(q)^* = F(X + q), \quad q \in \mathbb{R}, X \in \mathcal{B}(\mathbb{R}).$$

We denote by $\mathcal{O}_L$ the set of localization observables.

Proposition 6. Let $F \in \mathcal{O}_L$. The following conditions are equivalent:

(i) $F \preceq_i E^Q$;
(ii) $F \preceq_{c} E^Q$;
(iii) $F \preceq_{f} E^Q$;
(iv) for every $p \in \mathbb{R}, X \in \mathcal{B}(\mathbb{R})$,

$$(22) \quad V(p)F(X)V(p)^* = F(X);$$

(v) there is a probability measure $\rho \in M_1^+(\mathbb{R})$ such that

$$(23) \quad \Phi_F(T) = \rho \ast \Phi_{E^Q}(T), \quad T \in \mathcal{S}(\mathcal{H}),$$

where $\rho \ast \Phi_{E^Q}(T)$ is the convolution of the measures $\rho$ and $\Phi_{E^Q}(T)$. 
Proof. It is shown in \cite{27} that conditions (ii), (iii), and (v) are equivalent, and (iv) and (v) are equivalent by \cite{28}. Since (ii) \(\Rightarrow\) (i), it is enough to show that (i) \(\Rightarrow\) (iv).

Assume (i). Let \(\psi_1 \in \mathcal{H}\) be a unit vector, \(p \in \mathbb{R}\), and denote \(\psi_2 = V(p)^*\psi_1\). Let \(T_1\) and \(T_2\) be the states corresponding to the vectors \(\psi_1\) and \(\psi_2\), respectively. A short calculation shows that \(\Phi_{EQ}(T_1) = \Phi_{EQ}(T_2)\), and therefore, by the assumption we have \(\Phi_{F}(T_1) = \Phi_{F}(T_2)\). This means that

\begin{equation}
\langle \psi_1 | F(X) \psi_1 \rangle = \langle \psi_1 | V(p)F(X)V(p)^* \psi_1 \rangle
\end{equation}

for all \(X \in \mathcal{B}(\mathbb{R})\). As \(\psi_1\) was an arbitrary unit vector, (iv) follows. \(\square\)

The condition (22) means that the localization observable \(F\) is invariant under velocity boosts. In \(F\) satisfy both (21) and (22), it is called a position observable, \cite{3, 28}. It is clear from Proposition 6 that \(E^Q\) is an optimal position observable. However, not all localization observables are position observables. The localization observables have been characterized in \cite{29, 30}, and it is known that there are localization observables which do not have the invariance property (22). It follows that there are localization observables which do not satisfy the relations (i), (ii) and (iii).

**Proposition 7.** The observable \(E^Q\) is \(\preceq_f\)-optimal in \(\mathcal{O}_L\).

**Proof.** Let \(F \in \mathcal{O}_L\) and assume that \(E^Q \preceq_f F\). By Remark 3 of \cite{9}, we then have \(\text{ran}(E^Q) \subseteq \text{ran}(F)\). Since a projection in the range of \(F\) commutes with the other effects in the range (see e.g. \cite{31}), we get

\[ F(X)E^Q(Y) = E^Q(Y)F(X) \]

for all \(X, Y \in \mathcal{B}(\mathbb{R})\). Thus, by the functional calculus we get

\[ F(X)V(p) = V(p)F(X) \]

for all \(p \in \mathbb{R}, X \in \mathcal{B}(\mathbb{R})\). This and Proposition 6 imply that \(F \preceq_f E^Q\). \(\square\)

To author’s knowledge it is not known whether the observable \(E^Q\) is \(\preceq_c\)-optimal or \(\preceq_i\)-optimal in \(\mathcal{O}_L\). Also, whether the condition \(D_F = \emptyset\) holds for every \(F \in \mathcal{O}_L\) appears to be an open question.

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