Stability of a Dynamically Collapsing Gas Sphere

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(Received 1999 May 28; accepted 1999 November 17)

Abstract

We discuss the stability of dynamically collapsing gas spheres. We use a similarity solution for a dynamically collapsing sphere as the unperturbed state. In the similarity solution the gas pressure is approximated by a polytrope of $P = K \rho^{5/3}$. We examined three types of perturbations: bar ($\ell = 2$) mode, spin-up mode, and Ori–Piran mode. When $\gamma < 1.097$, it is unstable against the bar mode. It is unstable against the spin-up mode for any $\gamma$. When $\gamma < 0.961$, the similarity solution is unstable against the Ori–Piran mode. The unstable mode grows in proportion to $|t - t_0|^{1/5}$, while the central density increases in proportion to $\rho_c \propto (t - t_0)^{-2}$ in the similarity solution. The growth rate, $\sigma$, is obtained numerically as a function of $\gamma$ for the bar mode and the Ori–Piran mode. The growth rate of the bar mode is larger for a smaller $\gamma$. The spin-up mode has a growth rate of $\sigma = 1/3$ for any $\gamma$.

Key words: gravitation — hydrodynamics — ISM: clouds — stars: formation

1. Introduction

As well as the majority of nearby pre-main-sequence stars, most young stars have companions (see, e.g., Mathieu 1994). Since even young protostars have companions, fragmentation is more plausible for the formation of multiple star systems than capture and other processes. Fragmentation during the collapse of molecular cloud cores has been extensively studied, and a number of clarifying numerical simulations have been performed. Nonetheless, the results are not conclusive. Some early simulations have only limited spatial resolution and their results concerning fragmentation are unreliable (Truelove et al. 1997). As Truelove et al. (1997) have demonstrated, there is a tendency for artificial fragmentation to occur in numerical simulations which do not adequately resolve the Jeans length. If we exclude artificial fragmentation, fragmentation proceeds, but only slowly in numerical simulations. Fragmentation competes with the total collapse of a cloud. A cloud collapses and the density increases before it fragments. Only when an initial model was either very much elongated or very cold, did the simulation show fragmentation of a cloud at a relatively low density [see, e.g., Sigalotti and Klapp (1997) for fragmentation criteria of initially elongated gas clouds and Tsuribe and Inutsuka (1999) for those of initially cold spherical gas clouds]. These initial models are, however, not in force balance with the assumed purely hydrodynamical physics in the absence of a magnetic field.
In this paper we extend the linear-stability analyses of Hanawa and Matsumoto (1999) for collapsing nonisothermal spheres. For simplicity we employ the polytropic relation, \( P = K \rho^n \), for the model cloud. The polytrope model can describe the temperature change during collapse in the most simple form. The polytrope model of \( \gamma \approx 4/3 \) can also be applied to a collapsing iron core, resulting in a Type II-supernova. In section 2 we present a similarity solution for a gravitationally collapsing sphere of a polytropic gas cloud. This section is essentially a review of Yahil (1983) and Suto and Silk (1988). We summarize their results in order to investigate the stability. In section 3 we examine the stability of the similarity solution against the bar modes. The growth rate of the bar mode depends on \( m \) and the sign of the perturbation when \( \gamma < 0.961 \). The analysis given in section 5 is based on Ori and Piran (1988). A short summary is given in section 6.

2. Similarity Solution

For simplicity we consider gas of which the equation of state is expressed by the polytrope relation,

\[
P = K \rho^n,
\]

where \( P \) and \( \rho \) denote the pressure and density, respectively. The hydrodynamical equations are then expressed as

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2}
\]

and

\[
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla P + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \rho \nabla \Phi = 0, \tag{3}
\]

where \( \mathbf{v} \) and \( \Phi \) denote the velocity and gravitational potential, respectively. The gravitational potential is related to the density distribution by the Poisson equation,

\[
\Delta \Phi = 4\pi G \rho, \tag{4}
\]

where \( G \) denotes the gravitational constant.

For later convenience, we introduce the zooming coordinates of Bouquet et al. (1985) to solve equations (1) through (4). The zooming coordinates, \((\xi, \tau)\), are related to the ordinary coordinates, \((r, t)\), by

\[
\left( \begin{array}{c} \xi \\ \tau \end{array} \right) = \left( \begin{array}{c} r \\ \frac{r}{c_0 |t - t_0|} \end{array} \right), \tag{5}
\]

where \( c_0 \) denotes a standard sound speed and is a function of time, \( t \). The symbol \( t_0 \) denotes an epoch at the instant of protostar formation. The density in the zooming coordinates, \( \rho \), is related to that in the ordinary coordinates, \( \rho \), by

\[
\rho(\xi, \tau) = 4\pi G \rho (t - t_0)^2. \tag{6}
\]

We define the standard sound speed, \( c_0 \), so that it denotes the sound speed at a given \( t \) when \( \rho = 1 \). Thus, it is expressed as

\[
c_0 = \sqrt{\gamma K (4\pi G)^{(1-\gamma)/2} |t - t_0|^{1-\gamma}}. \tag{7}
\]

The pressure in the zooming coordinates, \( p \), is related to that in the ordinary coordinates, \( P \), by

\[
p = \frac{\rho^\gamma}{\gamma}. \tag{8}
\]

Substituting equations (6) and (8) into equation (1), we obtain the polytrope relation in the zooming coordinates,

\[
p = \frac{\rho^\gamma}{\gamma}. \tag{9}
\]

The velocity in the zooming coordinates, \( \mathbf{u} \), is defined as

\[
\mathbf{u} = \frac{\mathbf{v}}{c_0} + (2 - \gamma) \frac{r}{c_0 |t - t_0|}. \tag{10}
\]

This velocity denotes that with respect to the zooming coordinates, and includes the apparent motion, the last term in equation (10). The gravitational potential in the zooming coordinates, \( \phi \), is related to that in the ordinary coordinates, \( \Phi \), by

\[
\phi = \frac{\Phi}{c_0^2}. \tag{11}
\]

In the zooming coordinates, the hydrodynamical equations are expressed as

\[
\frac{\partial \rho}{\partial \tau} + \nabla' \cdot (\rho \mathbf{u}) = (4 - 3\gamma) \rho, \tag{12}
\]

\[
\frac{\partial (\rho \mathbf{u})}{\partial \tau} + \nabla' \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla' p + \rho \nabla' \phi
\]

\[
= (2 - \gamma) (\gamma - 1) \rho \xi + (7 - 5\gamma) \rho \mathbf{u}, \tag{13}
\]

and

\[
\Delta' \phi = \rho \tag{14}
\]

for \( t < t_0 \). The symbols \( \nabla' \) and \( \Delta' \) denote the gradient and Laplacian in \( \xi \)-space, respectively.

Assuming stationarity in the zooming coordinates \( \partial / \partial \tau = 0 \) and spherical symmetry \( \partial / \partial \Phi = \partial / \partial \phi = 0 \), we seek a similarity solution. Under these assumptions equations (12), (13), and (14) reduce to

\[
\frac{\partial \mathbf{u}}{\partial \xi} + \frac{u_r}{\rho} \frac{\partial \rho}{\partial \xi} = (4 - 3\gamma) - \frac{2u_r}{\xi} \tag{15}
\]
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\[ u_r \frac{\partial u_r}{\partial \xi} + \frac{1}{\rho} \left( \frac{dp}{d\rho} \right) \frac{\partial \rho}{\partial \xi} + \frac{\partial \phi}{\partial \xi} = (2 - \gamma)(\gamma - 1) \xi + (3 - 2\gamma) u_r, \tag{16} \]

and

\[ \frac{\partial \phi}{\partial \xi} = \frac{1}{\xi^2} \int_0^\xi \rho(\zeta) \zeta^2 d\zeta = \frac{\rho u_r}{4 - 3\gamma}, \tag{17} \]

where \( \xi = |\xi| \). After some algebra we can rewrite equations (15) and (16) as

\[ (\gamma^{-1} - u_r^2) \left( \frac{d\rho}{d\xi} \right) = \rho \left[ -\frac{\rho u_r}{4 - 3\gamma} \right. \]

\[ + (2 - \gamma)(\gamma - 1) \xi u_r - (3 - 2\gamma) u_r^2 \]

\[ \left. + (4 - 3\gamma) \rho^{-1} - \frac{2u_r}{\xi} \right], \tag{18} \]

and

\[ (\gamma^{-1} - u_r^2) \left( \frac{du_r}{d\xi} \right) = \frac{\rho u_r^2}{4 - 3\gamma} \]

\[ - (2 - \gamma)(\gamma - 1) \xi u_r - (3 - 2\gamma) u_r^2 \]

\[ + (4 - 3\gamma) \rho^{-1} - \frac{2u_r}{\xi} \rho^{-1} \]. \tag{19} \]

Equations (18) and (19) are singular at the sonic point, \( u_r^2 = \gamma^{-1} \). We obtain the similarity solution by integrating equations (18) and (19) by the Runge-Kutta method. In the numerical integration we used the auxiliary variable of Whitworth and Summers (1985), \( s \), defined by

\[ \frac{d\xi}{ds} = \gamma^{-1} - u_r^2. \tag{20} \]

Using equation (20), we rewrite equations (18) and (19) as

\[ \frac{d\rho}{ds} = \rho \left[ -\frac{\rho u_r}{4 - 3\gamma} \right. \]

\[ + (2 - \gamma)(\gamma - 1) \xi u_r + \left. \frac{2u_r}{\xi} \right], \tag{21} \]

and

\[ \frac{du_r}{ds} = \frac{\rho u_r^2}{4 - 3\gamma} - (2 - \gamma)(\gamma - 1) \xi u_r \]

\[ - (3 - 2\gamma) u_r^2 + (4 - 3\gamma) \rho^{-1} - \frac{2u_r}{\xi} \rho^{-1}, \tag{22} \]

respectively.

Similarity solutions exist for \( \gamma < 4/3 \). Figure 1 shows the similarity solutions for \( \gamma = 0.9, 1.0, \) and 1.1. The solid curves denote \( \rho \) while the dashed curves denote the infall velocity, \( -u + (2 - \gamma) \xi \). These solutions are the same as those obtained by Yahil (1983) and Suto and Silk (1988).

![Fig. 1. Similarity solutions for \( \gamma = 0.9, 1.0, \) and 1.1.](image)

The solid curves denote the density, \( \rho \). The dashed curves denote the infall velocity, \( -u + (2 - \gamma) \xi \). The infall velocity does not include the apparent motion in the zooming coordinates and is positive for inward flow.

3. Bar Mode

In this section we consider a non-spherical perturbation around the similarity solution. The density perturbation is assumed to be proportional to the spherical harmonics, \( Y^m_l(\theta, \varphi) \). Then, the density and velocity are expressed as

\[ \rho = \rho_0 + \delta \rho(\xi) e^{\sigma \tau} Y^m_l(\theta, \varphi), \tag{23} \]

\[ u_r = u_{r0} + \delta u_r(\xi) e^{\sigma \tau} Y^m_l(\theta, \varphi), \tag{24} \]

\[ u_\theta = \delta u_\theta(\xi) e^{\sigma \tau} \frac{\partial}{\ell + 1} Y^m_l(\theta, \varphi), \tag{25} \]

\[ u_\varphi = \delta u_\varphi(\xi) e^{\sigma \tau} \frac{\partial}{\ell + 1} \sin \theta \frac{\partial}{\partial \varphi} Y^m_l(\theta, \varphi), \tag{26} \]

\[ \phi = \phi_0 + \delta \phi(\xi) e^{\sigma \tau} Y^m_l(\theta, \varphi), \tag{27} \]

where the symbols with suffix, 0, denote the values in the similarity solution and the symbols with the symbol, \( \delta \), denote the perturbations. Substituting equations (23) throughout (27) into equations (12), (13), and (14), we obtain the following perturbation equations:

\[ (\sigma + 3\gamma - 4) \delta \rho + \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left[ \xi^2 (\rho_0 \delta u_r + u_{r0} \delta \rho) \right] \]

\[ - \ell \frac{\rho_0 \delta u_\theta}{r} = 0, \tag{28} \]

\[ (\sigma + 2\gamma - 3) \delta u_r + \frac{\partial}{\partial \xi} (u_{r0} \delta u_r) \]

\[ + \frac{\partial}{\partial \xi} \left( \frac{\delta \rho}{\rho_0^{2-\gamma}} \right) + \delta \Gamma = 0, \tag{29} \]
Fig. 2. Growth rate, $\sigma$, as a function of $\gamma$. The open squares denote numerically obtained data points.

\begin{equation}
(\sigma + 2\gamma - 3) \delta u_{\theta} + \frac{u_{r0}}{\xi} \frac{\partial}{\partial \xi} (\xi \delta u_{\theta})
+ \frac{\ell + 1}{\xi} \left( \frac{\delta \rho}{\rho \xi^{2-\gamma}} + \delta \phi \right) = 0,
\end{equation}
\begin{equation}
\frac{\partial}{\partial \xi} \delta \phi = \delta \Gamma,
\end{equation}
and
\begin{equation}
\frac{\partial}{\partial \xi} \delta \Gamma = - \frac{2\delta \Gamma}{\xi} + \frac{\ell (\ell + 1)}{\xi^{2}} \delta \phi + \delta \rho.
\end{equation}

These perturbation equations have singularities at the origin ($\xi = 0$), the sonic point ($[(u_{r0})^{2} - \rho^{\gamma-1}]$), and the infinity ($\xi = +\infty$). These perturbation equations reduce to those of Hanawa and Matsumoto (1999) when $\gamma = 1$. We solved these perturbation equations as an eigenvalue problem for the growth rate, $\sigma$, according to them. The details of the numerical procedures are given in Hanawa and Matsumoto (1999). The growth rate is obtained as a function of $\gamma$ and $\ell$. The growth is independent of $m$, since the unperturbed state is spherically symmetric (see, e.g., Hanawa, Matsumoto 1999).

Figure 2 shows the growth rate, $\sigma$, for $\ell = 2$ mode as a function of $\gamma$. The open squares denote the numerically obtained growth rates and the curve labeled “Bar” denotes a smooth fit to them. The growth rate is larger for smaller $\gamma$. A collapsing gas sphere is unstable only when $\gamma < 1.097$. We cannot find a damped mode since appropriate boundary condition are not given at $\xi = \infty$ for it (see appendix 2).

We searched for the $\ell = 3$ mode but could find none. Although the $\ell = 1$ mode exists, it is a trivial mode, having a growth rate of $\sigma = 1$. Its density perturbation is denoted as
\begin{equation}
\delta \rho = \frac{\partial \rho_{0}}{\partial \xi}.
\end{equation}

This $\ell = 1$ mode denotes only the misfit to the center of gravity, and is not relevant to any real instability (Hanawa, Matsumoto 1999). A higher $\ell$ mode is also unlikely to exist, since the wavelength is smaller for a higher $\ell$. Remember that the self-gravity induces an instability only when the wavelength of a perturbation is larger than the Jeans length. Thus, only the bar mode gives a non-spherical density perturbation growing during the collapse. Since the bar mode is stabilized for $\gamma > 1.097$, a similarity solution for $\gamma > 1.097$ is stable against any non-spherical density perturbation. This result answers a question posed by Goldreich, Lai, and Sahrling (1997). They asked whether a dynamically collapsing iron resultant type II-supernova is unstable against a large-scale non-spherical perturbation. Since a dynamically collapsing iron core is well approximated by a similarity solution for $\gamma \approx 4/3$, our analysis suggests that it is stable against this type of non-spherical density perturbation.

Figure 3 illustrates the variety of the bar mode. Each panel shows a dynamically collapsing isothermal gas cloud suffering a different bar-mode perturbation. It shows the density distribution perspectively by isodensity surfaces. Panel (a) denotes a bar mode of $m = 0$. In panel (a) the collapsing gas cloud is elongated in the $z$-direction. Panel (b) denotes the same bar mode of $m = 0$, but having the opposite sign. In panel (b) the collapsing gas cloud is oblate and compressed in the $z$-direction. Panels (c) and (d) denote the bar modes of $m = 1$ and 2, respectively. When the modes grow, the collapsing gas cloud becomes triaxial.

4. Spin-up Mode

In this section we consider the velocity perturbation, expressed as
\begin{equation}
\begin{aligned}
\left( \begin{array}{c}
\frac{u_{r}}{\xi} \\
\frac{u_{\theta}}{\xi} \\
\frac{u_{\phi}}{\xi}
\end{array} \right) &= \left[ \begin{array}{c}
\frac{A_{\ell, m}(\xi)}{\xi \sin \theta} \frac{\partial}{\partial \rho} Y_{\ell}^{m}(\theta, \phi) e^{\sigma r} \\
\frac{A_{\ell, m}(\xi)}{\xi} \frac{\partial}{\partial \rho} Y_{\ell}^{m}(\theta, \phi) e^{\sigma r}
\end{array} \right].
\end{aligned}
\end{equation}

Substituting equation (34) into equation (12) we obtain
\begin{equation}
\delta \rho = 0
\end{equation}
for this mode. Similarly we obtain
\begin{equation}
\delta \phi = 0
\end{equation}
Fig. 3. Collapsing isothermal cloud core suffering the bar mode. The density distribution is shown by the isodensity surfaces. The velocity in the $x-z$ plane is shown by the arrows. Each panel shows a different bar mode. Panel (a) denotes the bar mode of $m = 0$. Panel (b) denotes the same bar mode of $m = 0$ but having the opposite sign. Panel (c) denotes the bar mode of $m = 1$ while panel (d) does that of $m = 2$. In panels (a) and (b) the perturbation is added so that the radial velocity is $v_r \propto \left(-\frac{2}{3} \pm 0.10 (3 \cos^2 \theta - 1)\right) r$ near the center, respectively. In panel (c) the radial velocity is $v_r \propto \left(-\frac{2}{3} + 0.05 \sin 2\theta \cos \phi\right) r$ near the center. In panel (d) it is $v_r \propto \left(-\frac{2}{3} + 0.10 \sin^2 \theta \cos 2\phi\right) r$ near the center.

by substituting equation (35) into equation (14). Substituting equations (34), (35), and (36) into equation (13) we obtain

$$(\sigma - 7 + 5\gamma) \phi_0 \Lambda_{t,m} + \frac{1}{\xi^2} \frac{\partial}{\partial \xi} (\xi^2 \phi_0 u_0 \Lambda_{t,m}) = 0. \quad (37)$$

Substituting equation (15) into equation (37) we obtain

$$(\sigma - 3 + 2\gamma) \Lambda_{t,m} + u_0 \frac{\partial}{\partial \xi} (\Lambda_{t,m}) = 0. \quad (38)$$

Substituting equation (A2) into equation (38) we obtain

$$\sigma = 3 - 2\gamma + \left(\gamma - \frac{4}{3}\right) \left(\frac{d \ln \Lambda_{t,m}}{d \ln \xi}\right)_{\xi=0} - 2 \quad (39)$$

The growth rate, $\sigma$, should be smaller than $1/3$, since the velocity perturbation is regular at the origin only for $d \ln \Lambda_{t,m}/d \ln \xi \geq 2$ [see equation (34)]. When $d \ln A_{t,m}/d \ln \xi = 2$, the angular velocity is nearly constant around the origin. The growth rate of the spin-up mode, $\sigma = 1/3$, is also shown in figure 2 for a comparison with that of the bar mode. This spin-up mode is essentially the same as the spin-up mode shown in Appendix of Hanawa and Nakayama (1997).

5. Ori–Piran Mode

In this section we consider a perturbation emanating from the sonic point. According to Ori and Piran (1988) we now consider a spherical perturbation which vanishes...
inside the sonic point. Substituting \( \ell = 0 \) into equations (24) through (27) we obtain

\[
\frac{\partial}{\partial \tau} \delta \varrho + \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left[ \xi^2 (\delta \varrho u_{\,0} + \varrho_0 \delta u_r) \right] = (4 - 3\gamma) \delta \varrho ,
\]

\[
\frac{\partial}{\partial \tau} \delta u_r + \frac{\partial u_{\,0}}{\partial \xi} \delta u_r + u_{\,0} \frac{\partial}{\partial \xi} \delta u_r = \frac{\partial}{\partial \xi} \left[ (\varrho_0)^{\gamma - 1} \delta \varrho \right] + \delta \varrho = (3 - 2\gamma) \delta u_r ,
\]

and

\[
\delta \varrho = 0 ,
\]

\[
\delta \varrho = 0 ,
\]

at the sonic point. Equations (43) and (44) are equivalent and indicate that the sound wave traveling outwards with a phase speed of \( u_r + \varsigma_0 \) vanishes. In other words, the perturbation emanating from the sonic point is the other sound wave of which the phase velocity is zero at the sonic point, \( u_r - \varsigma_0 = 0 \). Taking a linear combination of the \( \xi \)-derivatives of equations (40) and (41) and evaluating it at the sonic point, we obtain

\[
\frac{\partial}{\partial \tau} \left( u_{\,0} \frac{\partial}{\partial \xi} \delta \varrho - \varrho_0 \frac{\partial}{\partial \xi} \delta u_r \right)
- \left[ -\frac{2}{\xi^2} \frac{\partial u_{\,0}}{\partial \xi} + (\gamma - 1) u_{\,0} \frac{\partial \ln \varrho_0}{\partial \xi} + \frac{7 - 5\gamma}{2} \right]
\times \left( u_{\,0} \frac{\partial}{\partial \xi} \delta \varrho - \varrho_0 \frac{\partial}{\partial \xi} \delta u_r \right) = 0 ,
\]

where equations (43) and (44) are substituted. Equation (46) means

\[
\sigma = -2 \frac{\partial u_{\,0}}{\partial \xi} + (\gamma - 1) u_{\,0} \frac{\partial \ln \varrho_0}{\partial \xi} + \frac{7 - 5\gamma}{2} .
\]

When \( \gamma = 1 \), equation (47) is equivalent to equation (15) of Ori and Piran (1988). To elucidate the growth (or damping) of this mode we rewrite equation (47) as

\[
\sigma = -2 \frac{\partial}{\partial \xi} (u_{\,0} - \varsigma_0) + \frac{7 - 5\gamma}{2} .
\]

The first term on the right-hand side of equation (48) denotes the dilution of the wave due to the spatial variation of the phase velocity. The phase speed vanishes at the sonic point and increases with the radial distance. Thus, the wave dilutes and the amplitude decreases. The second term in equation (48) denotes the self-reproduction. As shown on the right-hand side of equations (12) and (13), the density and momentum density reproduce and amplify themselves when measured in the zooming coordinates. When the self-reproduction overcomes the dilution, the collapsing gas sphere is unstable against the Ori-Piran mode.

The nonlinear growth of the Ori-Piran mode has not yet been studied. This is the first report showing that a similarity solution of a collapsing gas sphere can be unstable against the Ori-Piran mode. As far as we know, Ogino, Tomisaka, and Nakamura (1999) produced the first report on the collapse of a polytropic gas sphere with \( \gamma < 1 \). They reported a little on their numerical simulations of \( \gamma < 1 \), but mentioned nothing about the Ori-Piran instability. Since the similarity solution is unstable, their numerical solution may not be well approximated by it.

6. Summary

As shown in the previous sections, a gravitationally collapsing polytropic gas sphere can suffer from three types of instability. When \( \gamma < 4/3 \), it is unstable against the spin-up mode shown in section 5. The spin-up mode grows in proportion to \( |t - t_0|^{-1/3} \) and, accordingly, in proportion to \( \rho^{1/6} \). When \( \gamma < 1.097 \), it is also unstable against the bar mode instability. The growth rate of the bar mode is large for smaller \( \gamma \). When \( \gamma < 1.006 \), it is larger than that of the spin-up mode. When \( \gamma \leq 0.961 \), a gravitationally collapsing polytropic gas sphere is also unstable against the Ori-Piran mode.

It is particularly interesting that the bar mode has a larger growth rate when \( \gamma \) is smaller. This means that the bar mode grows faster when the sound speed decreases along with an increase in the density. This result may also be valid for a gravitationally collapsing gas sphere in which turbulence has an effective pressure; but the gas remains optically thin and thus can cool efficiently without suffering heating. Since the effective sound speed is lower in a denser part of a molecular cloud, a collapsing cloud core may be more unstable against the bar mode than a purely isothermal gas sphere. This enhancement in the bar mode instability may be relevant to the fragmentation of a collapsing cloud core and the formation of multiple stars during the collapse.

We thank Naoya Fukuda and Kazuya Saigo for useful discussions and their help in making figures. This research was financially supported in part by the Grant-in-Aid for Scientific Research (C) of the Ministry of Education, Science, Sports and Culture (No. 09640318) and by the Grant-in-Aid for Scientific Research on Priority
Appendix 1. Similarity Solution for Collapse of Polytropic Gas Sphere

In this appendix we summarize the main characteristics of the similarity solution for the collapse of a polytropic gas sphere. See Yahil (1983) and Suto and Silk (1988) for the derivation and more details.

When we expand the similarity solution around $\xi = 0$ by a Taylor series, they are expressed as

$$\varrho_0(\xi) = \varrho_0(0) - \frac{[\varrho_0(0)]^{2-\gamma}}{6} \left[ \varrho_0(0) - \frac{2}{3} \right] \xi^2 + O(\xi^3) \quad (A1)$$

and

$$u_r(\xi) = \left[ (2 - \gamma) - \frac{2}{3} \right] \xi + \frac{[\varrho_0(0)]^{1-\gamma}}{15} \left( \varrho_0 - \frac{2}{3} \right) \left( \frac{4}{3} - \gamma \right) \xi^3 + O(\xi^5) \quad (A2)$$

In the region $\xi \gg 1$ the similarity solution has the asymptotic form of

$$\varrho \propto \xi^{-2/(2-\gamma)} \quad (A3)$$

and

$$[u_r - (2 - \gamma) \xi] \propto \xi^{(1-\gamma)/(2-\gamma)} \quad (A4)$$

Appendix 2. Asymptotic Behavior of Bar-Mode Perturbation

We can derive the asymptotic forms for the bar-mode perturbation from the requirement that the perturbation is regular at $\xi = 0$. The density and velocity perturbations around $\xi = 0$ are expressed as

$$\delta \varrho = \alpha [\varrho_0(0)]^{2-\gamma} \xi^\ell \quad (A5)$$

$$\delta u_r = \beta \ell \xi^{\ell-1} \quad (A6)$$

$$\delta u_\theta = \beta (\ell + 1) \xi^{\ell-1} \quad (A7)$$

$$\delta \phi = - \left\{ \alpha + \left[ \frac{\sigma + 2\gamma - 3 + \frac{\ell(4 - 3\gamma)}{3}}{\beta} \right] \right\} \xi^\ell \quad (A8)$$

where $\alpha$ and $\beta$ are free parameters. When we derive the above Taylor series expansion, we use equation (A2).

The derivation is essentially the same as that shown in Hanawa and Matsumoto (1999).

As a boundary condition we assume that the relative density perturbation, $\delta \varrho / \varrho_0$, is vanishingly small at infinity, $\xi = \infty$. After some algebra we obtain the following asymptotic relations:

$$\frac{\delta \varrho}{\varrho_0} \propto \xi^{-\sigma/(2-\gamma)} \quad (A9)$$

$$\delta u_r \propto \xi^{-(\sigma+\gamma-1)/(2-\gamma)} \quad (A10)$$

$$\delta u_\theta \propto \xi^{-(\sigma+\gamma-4)/(2-\gamma)} \quad (A11)$$

$$\delta \phi \propto \xi^{-(\sigma-2\gamma+2)/(2-\gamma)} \quad (A12)$$

and

$$\delta \varrho = \left[ \frac{(\sigma - 2\gamma + 2)(\sigma - 3\gamma + 4) - \ell(\ell + 1)}{(2 - \gamma)^2} \right]^{-1} r^2 \delta \varrho \cdot \quad (A13)$$

When we derive the above relations, we use equations (A3) and (A4). See also Hanawa and Matsumoto (1999) for the derivation.

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