On the $D_s^+ \to \pi^+\pi^+\pi^-$-nonresonant decay in the effective quark model with chiral symmetry

A. N. Ivanov* and N.I. Troitskaya†

Abstract

The partial widths of the decays $D_s^+ \to \phi \pi^+$ and $D_s^+ \to \pi^+\pi^+\pi^-$-nonresonant are computed within the effective quark model with chiral symmetry involving Heavy quark effective theory (HQET) and Chiral perturbation theory at the quark level (CHPT)$_q$ with linear realization of chiral $U(3) \times U(3)$ symmetry. It is shown that for the explanation of the experimental probability of the $D_s^+ \to \pi^+\pi^+\pi^-$-nonresonant decay one does not need to use unnaturally heavy light current quarks as has been suggested in Ref.[1].

PACS number(s): 13.30.Ce, 12.39.Ki, 14.20.Lq, 14.20.Mr

*E–mail: ivanov@kph.tuwien.ac.at, Tel.: +43–1–58801–5598, Fax: +43–1–5864203
†Permanent Address: State Technical University, Department of Theoretical Physics, 195251 St. Petersburg, Russian Federation
1 Introduction

The problem of the $D_s^+ \rightarrow (\pi^+ \pi^+ \pi^-)_{\text{NR}}$-nonresonant (NR) decay has been recently discussed in Ref.[1]. According to the conclusion suggested in Ref.[1] the theoretical explanation of the experimental probability of this decay can be reached via the adoption of unnaturally heavy light current quark masses $\bar{m} = \frac{1}{2} (m_{0u} + m_{0d}) = 22 \div 38 \text{ MeV}$ [1] instead of the widely accepted value $\bar{m} = \frac{1}{2} (m_{0u} + m_{0d}) = 5.5 \text{ MeV}$ [2].

The application of the effective quark model with chiral symmetry involving Heavy quark effective theory (HQET) [3-5] supplemented by Chiral perturbation theory at the quark level (CHPT) [6-8] with linear realization of chiral $U(3) \times U(3)$ symmetry, to the calculation of chiral corrections to the mass spectra and leptonic constants of charmed mesons [9], and to the form factors of semileptonic $D$-meson decays [10] has shown that the widely accepted values of the current quark masses $m_{0u} = 4 \text{ MeV}, m_{0d} = 7 \text{ MeV}$ and $m_{0s} = 135 \text{ MeV}$ [2,6-8] describe well the experimental data.

This paper is to apply the effective quark model with chiral $U(3) \times U(3)$ symmetry incorporating HQET and (CHPT) to the computation of the partial width and the probability of the $D_s^+ \rightarrow (\pi^+ \pi^+ \pi^-)_{\text{NR}}$ decay. Since experimentally the probability of this decay has been defined compared the probability of the $D_s^+ \rightarrow \phi \pi^+$ decay [11], i.e. $\frac{\Gamma(D_s^+ \rightarrow (\pi^+ \pi^+ \pi^-)_{\text{NR}})}{\Gamma(D_s^+ \rightarrow \phi \pi^+)} = 0.29 \pm 0.09 \pm 0.06$, (1.1) we also compute the partial width of the $D_s^+ \rightarrow \phi \pi^+$ decay.

The effective low–energy Lagrangian responsible for non–leptonic decays of the $D_s^+$–meson reads [12]

$$\mathcal{L}_{\text{eff}}(x) = -\frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} C_1 [\bar{s} \gamma^\mu (1 - \gamma^5) c(x)] [\bar{u} \gamma_\mu (1 - \gamma^5) d(x)],$$ (1.2)

where $G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2}$ is the Fermi weak constant, $V_{cs}^*$ and $V_{ud}$ are the elements of the CKM–mixing matrix and $C_1 \simeq 1.3$ is the Wilson coefficient caused by the strong quark–gluon interactions at scales $p > \mu$, where $\mu$ is a normalization scale. In (CHPT)$_q$ we should identify $\mu$ with the scale of spontaneous breaking of chiral symmetry (SB$\chi$S) $\Lambda_\chi = 0.94 \text{ GeV}$ [7], i.e. $\mu = \Lambda_\chi = 0.94 \text{ GeV}$ (see also [6,8]). The contribution of strong interactions at scales $p \leq \mu = \Lambda_\chi$ is described by (CHPT)$_q$ in terms of constituent quark loop diagrams, where the momenta of virtual quarks are restricted from above by the SB$\chi$S scale $\Lambda_\chi$ [6–10,13–16].

The effective quark model incorporating HQET and (CHPT)$_q$ resembles that suggested by Bardeen and Hill [17] that is also based on the Nambu–Jona–Lasinio model. There is only distinction that in the Bardeen–Hill model heavy–light mesons are considered like partners of light mesons, whereas in our effective quark model, heavy mesons are external states with respect to the light ones. This distinction influences only the redefinition of phenomenological parameters that are introduced in the model. Nevertheless, all results obtained within our effective quark model should be valid too in the Bardeen–Hill model.
2 The $D_s^+ \to \phi \pi^+$ decay

The amplitude of the non–leptonic decay $D_s^+ \to \phi \pi^+$ can be defined as follows

$$M(D_s^+ \to \phi \pi^+) = < \pi^+(q)\phi(Q)|L_{\text{eff}}(0)|D_s^+(p)>.$$  

(2.1)

In (CHPT)$_q$ at the tree–meson approximation the computation of the matrix element of the non–leptonic decay $D_s^+ \to \phi \pi^+$ Eq.(2.1) agrees with the vacuum insertion approximation leading to a factorized amplitude [7]. This gives

$$M(D_s^+ \to \phi \pi^+) = < \pi^+(q)\phi(Q)|L_{\text{eff}}(0)|D_s^+(p)> = -\frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} C_1 < \pi^+(q)|[\bar{u} \gamma^\mu(1 - \gamma^5) d(x)]|0 > \times < \phi(Q)|[\bar{s} \gamma_\mu(1 - \gamma^5) c(x)]|D_s^+(p)> = -i G_F V_{cs}^* V_{ud} C_1 F_\pi q^\mu < \phi(Q)|[\bar{s} \gamma_\mu(1 - \gamma^5) c(x)]|D_s^+(p)>.$$  

(2.2)

The computation of the matrix element $< \pi^+(q)|[\bar{u} \gamma^\mu(1 - \gamma^5) d(x)]|0 >$ has been carried out in Ref.[7]

$$< \pi^+(q)|[\bar{u} \gamma^\mu(1 - \gamma^5) d(x)]|0 > = i \sqrt{2} F_\pi q^\mu,$$  

(2.3)

where $F_\pi \approx 92$ MeV is a lepton constant of the $\pi^+$–meson [11].

The matrix element $< \phi(Q)|[\bar{s} \gamma_\mu(1 - \gamma^5) c(x)]|D_s^+(p)>$ of the $D_s^+ \to \phi$ transition can be expressed in terms of the form factors of the semileptonic $D_s^+ \to \phi \ell \nu_\ell$ decay [10,16]

$$< \phi(Q)|[\bar{s} \gamma_\mu(1 - \gamma^5) c(x)]|D_s^+(p)> = i a_1(q^2) e_\mu^*(Q) - i a_2(q^2) (e^*(Q) \cdot p) (p + Q)_\mu - i a_3(q^2) (e^*(Q) \cdot p) (p - Q)_\mu - 2 b(q^2) \varepsilon_{\mu \nu \alpha \beta} e^{*\nu}(Q) p^\alpha Q^\beta, \ (\varepsilon^{0123} = 1),$$  

(2.4)

where $e_\mu^*(Q)$ is the polarization 4–vector of the $\phi$–meson. As a result the amplitude of the $D_s^+ \to \phi \pi^+$ decay reads

$$M(D_s^+ \to \phi \pi^+) = G_F V_{cs}^* V_{ud} C_1 F_\pi [a_1(M_\pi^2) - (M_{D_s^+}^2 - M_\phi^2) a_2(M_\pi^2) - M_\pi^2 a_3(M_\pi^2)] q \cdot e^*(Q),$$  

(2.5)

where we have set $p^2 = M_{D_s^+}^2$, $Q^2 = M_\phi^2$ and $q^2 = M_\pi^2$. Setting $M_\pi = 0$ we reduce the r.h.s. of Eq.(2.3) to the form

$$M(D_s^+ \to \phi \pi^+) = G_F V_{cs}^* V_{ud} C_1 F_\pi [a_1(0) - (M_{D_s^+}^2 - M_\phi^2) a_2(0)] q \cdot e^*(Q).$$  

(2.6)

In HQET supplemented by (CHPT)$_q$ the form factors $a_i(0) \ (i = 1, 2)$ have been computed in Ref.[16] and read

$$a_1(0) = 1.47 \text{ GeV}, \quad a_2(0) = 0.21 \text{ GeV}^{-1}.$$  

(2.7)
The partial width of the $D_s^+ \to \phi \pi^+$ decay is given by

$$\Gamma(D_s^+ \to \phi \pi^+) = |C_1|^2 |G_F V_{cs} V_{ud}|^2 F_\pi^2 \left[ a_1(0) - (M_{D_s^+}^2 - M_\phi^2) a_2(0) \right] \frac{|\vec{q}|^2}{8\pi M_{D_s^+}^2} =$$

$$= |C_1|^2 |G_F V_{cs} V_{ud}|^2 F_\pi^2 \left[ a_1(0) - (M_{D_s^+}^2 - M_\phi^2) a_2(0) \right] \left( 1 - \frac{M_\phi^2}{M_{D_s^+}^2} \right)^3 =$$

$$= |C_1|^2 \times 0.36 \times 10^{11} \text{ s}^{-1}, \quad (2.8)$$

where $|\vec{q}| = (M_{D_s^+}^2 - M_\phi^2)/2M_\phi$ is a relative momentum of a $\phi$–meson and a massless $\pi^+$–meson. The numerical value has been obtained at $|V_{cs}| = 1.01$, $|V_{ud}| = 0.975$, $M_{D_s^+} = 1.97 \text{ GeV}$, $M_\phi = 1.02 \text{ GeV}$ [11]. We have dropped the dependence on the $\pi$–meson mass and set $M_\tau = 0$.

When using the partial width Eq. (2.8) we can estimate $\tau_{D_s^+}$ the time life of the $D_s^+$–meson:

$$\tau_{D_s^+} = \frac{B(D_s^+ \to \phi \pi^+)}{|C_1|^2 \times 0.36 \times 10^{11}} = \frac{(3.6 \pm 0.9) \times 10^{-2}}{|C_1|^2 \times 0.36 \times 10^{11}} = (0.59 \pm 0.15) \times 10^{-12} \text{ s}, \quad (2.9)$$

where we have applied the experimental value $B(D_s^+ \to \phi \pi^+) = (3.6 \pm 0.9)\%$ [11]. Our estimate $\tau_{D_s^+} = (0.59 \pm 0.15) \times 10^{-12} \text{ s}$ agrees well with the experimental value $\tau_{D_s^+} = (0.467 \pm 0.017) \times 10^{-12} \text{ s}$.

### 3 The $D_s^+ \to (\pi^+ \pi^+ \pi^-)_{\text{NR}}$ decay

The amplitude of the $D_s^+ \to (\pi^+ \pi^+ \pi^-)_{\text{NR}}$ decay is defined

$$M(D_s^+ \to (\pi^+ \pi^+ \pi^-)_{\text{NR}}) = < \pi^+(q_+)\pi^+(p_+)\pi^-(p_-) >_{\text{NR}} |\mathcal{L}_{\text{eff}}(0)| D_s^+(p) >=$$

$$= - \frac{G_F}{\sqrt{2}} V_{cs} V_{ud} C_1 < \pi^+(q_+)\pi^+(p_+)\pi^-(p_-) >_{\text{NR}} |\bar{u}(0)\gamma_\mu(1 - \gamma^5) d(0)|0 > \times < 0|\bar{s}(0)\gamma_\mu(1 - \gamma^5) c(0)|D_s^+(p) >. \quad (3.1)$$

The matrix element $< 0|\bar{s}(0)\gamma_\mu(1 - \gamma^5) c(0)|D_s^+(p) >$ is determined by the $D_s^+$ leptonic constant $F_{D_s^+}$:

$$< 0|\bar{s}(0)\gamma_\mu(1 - \gamma^5) c(0)|D_s^+(p) >= - i \sqrt{2} F_{D_s^+} p^\mu, \quad (3.2)$$

where $F_{D_s^+} = 1.17 F_{D^0} = 1.41 F_\pi$ [9,13]. Thus the amplitude of the $D_s^+ \to (\pi^+ \pi^+ \pi^-)_{\text{NR}}$ decay reads

$$M(D_s^+ \to (\pi^+ \pi^+ \pi^-)_{\text{NR}}) = i G_F V_{cs} V_{ud} C_1 F_{D_s^+} \times p_\mu < \pi^+(q_+)\pi^+(p_+)\pi^-(p_-) >_{\text{NR}} |\bar{u}(0)\gamma_\mu(1 - \gamma^5) d(0)|0 >. \quad (3.3)$$

In (CHPT) the matrix element $< \pi^+(q_+)\pi^+(p_+)\pi^-(p_-) >_{\text{NR}} |\bar{u}(0)\gamma_\mu(1 - \gamma^5) d(0)|0 >$ is defined by the box–constituent quark diagrams [8] and can be described by the momentum
integral
\begin{align}
< (\pi^+(q_+))\pi^+(p_+)\pi^-(-p_-)|\bar{u}(0)\gamma^\mu(1-\gamma^5)d(0)|0> &= i \frac{Ng_\pi^3}{16\pi^2} \times \\
\int \frac{d^4k}{\pi^2i} \text{tr}\left\{\gamma^\mu\gamma^5 \frac{1}{m-k+p^-} \gamma^5 \frac{1}{m-k+p^-} \gamma^5 \frac{1}{m-k+q^-} \gamma^5 \frac{1}{m-k}\right\} \\
+ (q_+ \leftrightarrow p_+) + \ldots,
\end{align}

(3.4)

where the ellipses denote the contribution of the matrix element of the vector current \(\bar{u}(0)\gamma^\mu d(0)\) not contributing to the \(D^+_s \to (\pi^+\pi^+\pi^-)\) decay amplitude, then \(m = 0.33\) GeV is the constituent light quark mass calculated in the chiral limit [6–8], and \(g_\pi = \sqrt{2} m/F_\pi\), the coupling constant of quark–pion interaction, satisfies the constraint \(Ng_\pi^3/8\pi^2 = 1\) [13]. The former becomes rather obvious, if one would take into account that \(F_\pi = O(\sqrt{N})\) at \(N \to \infty\) and make a change \(F_\pi \to F_\pi\sqrt{N/3}\), where in the r.h.s. of this relation \(F_\pi = 92\) MeV. Then, one finds substituting the numerical data that \(Ng_\pi^3/8\pi^2 = 3m^2/4\pi^2F_\pi^2 \simeq 1\). We should emphasize that we have ignored the contribution of the \(\pi^+\)–meson pole contribution to the r.h.s. of Eq.(3.4). As it is shown in Appendix the contribution of the \(\pi^+\)–meson pole is of order \(O(1/M^4_{D_s})\) compared with the contribution of the momentum integral left in Eq.(3.4).

Since in (CHPT)\(_q\) [8] virtual momenta of constituent quark loops are restricted by the SB\(_{\chi}\)S scale, i.e. \(k \leq \Lambda_\chi\), and HQET assumes that \(M_{D_s}^+ \simeq M_c\), so in a heavy–quark mass limit we following the Appelquist–Carazzone theorem [22] reduce the r.h.s. of Eq.(3.4) to the form
\begin{align}
< (\pi^+(q_+)\pi^+(p_+)\pi^-(-p_-))_{\text{NR}}|\bar{u}(0)\gamma^\mu(1-\gamma^5)d(0)|0> &= \\
= - \frac{i m^2}{\sqrt{2}F_\pi M^2_{D_s}} \int \frac{d^4k}{\pi^2i} \text{tr}\left\{\gamma^\mu\gamma^5 \frac{1}{m-k+p^-} \gamma^5 \frac{1}{m-k} \right\} \\
+ (q_+ \leftrightarrow p_+),
\end{align}

(3.5)

where we have used the relations \(g_\pi = \sqrt{2} m/F_\pi\), \(Ng_\pi^2/8\pi^2 = 1\) and \(p^2 = M^2_{D_s}\). The calculation of the integral over \(k\) gives one
\begin{align}
\int \frac{d^4k}{\pi^2i} \text{tr}\left\{\gamma^\mu\gamma^5 \frac{1}{m-k+p^-} \gamma^5 \frac{1}{m-k} \right\} &= 4 m q_+^4 \frac{16\pi^2}{Ng_\pi^2} g^2_\pi I_2(m).
\end{align}

(3.6)

In the r.h.s. of Eq.(3.6) we encounter a logarithmically divergent integral which is well defined in (CHPT)\(_q\) by the compositeness condition \(g^2_\pi I_2(m) = 1\) [6–9] (see also Refs.[25,26]), where we have used the definition of \(I_2(m)\) [6–9]
\begin{align}
I_2(m) &= \frac{N}{16\pi^2} \int \frac{d^4k}{\pi^2i} \frac{1}{(m^2-k^2)^2},
\end{align}

(3.7)

and by applying the relation \(Ng_\pi^2/8\pi^2 = 1\) [13]. The compositeness condition \(g^2_\pi I_2(m) = 1\) realizes correct kinetic terms of low–lying mesons, \(\bar{q}q\)–collective excitations, in effective chiral Lagrangians derived in effective quark models based on the Nambu–Jona–Lasinio approach [6–9,17,23,24].

Formally, the calculation of the momentum integrals representing one–loop constituent quark diagrams should be carried out keeping only the divergent parts and dropping the
contributions of the parts finite in the infinite limit of the cut–off. Such a prescription realizes a naive description of confinement of quarks. Indeed, dropping the finite parts of quark diagrams we remove the imaginary parts of them and suppress by this the appearance of quarks in the intermediate states of low–energy hadron interactions. This naive description of confinement has turned out to be rather useful for the derivation of effective Chiral Lagrangians \[6–9,23,24\]. As has been shown in Ref. \[21\] this prescription can be justified in the multicolour QCD \((N \rightarrow \infty)\) with a linearly rising interquark confinement potential. Thereby, within such a naive approach to the quark confinement mechanism one can bridge quantatively the quark and the hadron level of the description of strong low–energy interactions of hadrons. The cut–off \(\Lambda_\chi = 0.94 \text{GeV}\) and the constituent quark mass \(m = 0.33 \text{GeV}\) can be considered in such an approach as input parameters and fixed at one–loop approximation via the experimental values of the \(\rho \pi \pi\) coupling constant \(g_\rho\) and the leptonic pion constant \(F_\pi\) \[6\]. Thus, we should accentuate that in the effective quark models based on the Nambu–Jona–Lasinio approach quark diagrams lose the meaning of quantum field theory objects and only display how quark flavours can be transferred form an initial hadron state to a final hadron state in hadron–hadron low–energy transitions. The coupling constants of such transitions described in terms of divergent parts of quark diagrams and depending of the cut–off \(\Lambda_\chi = 0.94 \text{GeV}\) and the constituent quark mass \(m = 0.33 \text{GeV}\) can be expressed in terms of effective phenomenological coupling constants of low–energy hadron interactions given by effective chiral Lagrangians \[6–9,17,23,24\]. Hence, such a description of strong low–energy interaction of hadrons can be valued as good established as the effective chiral Lagrangian approach. As has been shown in Refs.\[6–10,14–16\] (CHPT) allows to develop consistent expansions in powers of both current quark masses and external momenta of interacting hadrons, i.e. chiral perturbation theory at both tree–meson and one–loop meson level, in good agreement with experimental data.

For the matrix element \(< (\pi^+(q_+)\pi^+(p_+)\pi^-(p_-))_{\text{NR}}|\bar{u}(0)\gamma^\mu(1 - \gamma^5)d(0)|0 \rangle\) we obtain the following

\[
< (\pi^+(q_+)\pi^+(p_+)\pi^-(p_-))_{\text{NR}}|\bar{u}(0)\gamma^\mu(1 - \gamma^5)d(0)|0 \rangle = -i\frac{4\sqrt{2}}{F_\pi}\frac{m^2}{M_{D^+}^2}(p_+ + q_+)^\mu. \quad (3.8)
\]

This gives one

\[
p_\mu < (\pi^+(q_+)\pi^+(p_+)\pi^-(p_-))_{\text{NR}}|\bar{u}(0)\gamma^\mu(1 - \gamma^5)d(0)|0 \rangle = -i\frac{4\sqrt{2}}{F_\pi}\frac{m^2}{M_{D^+}^2}(p_+ + q_+)^\mu. \quad (3.9)
\]

where we have set \(q_+^2 = p_+^2 = p_-^2 = 0\) and \(q^2 = (p_+ + q_+)^2\).

Thus, in our approach the r.h.s. of Eq.(3.9) is not proportional to the sum of current quark masses

\[
p_\mu < (\pi^+(q_+)\pi^+(p_+)\pi^-(p_-))_{\text{NR}}|\bar{u}(0)\gamma^\mu(1 - \gamma^5)d(0)|0 \rangle = -(m_{0u} + m_{0d}) < (\pi^+(q_+)\pi^+(p_+)\pi^-(p_-))_{\text{NR}}|\bar{u}(0)\gamma^5d(0)|0 \rangle, \quad (3.10)
\]

that can be expected when assuming the validity of the application of the equations of motion for the free current quark fields, i.e. \(\bar{u}(0)\hat{p}(1 - \gamma^5)d(0) = -(m_{0u} + m_{0d})\bar{u}(0)\gamma^5d(0)\). This entails a vanishing of the matrix element \(p_\mu < (\pi^+(q_+)\pi^+(p_+)\pi^-(p_-))_{\text{NR}}|\bar{u}(0)\gamma^\mu(1 - \gamma^5)d(0)|0 \rangle\).
$\gamma^5 d(0)|0 >$ in the chiral limit. Let us show now that at $p^2 = M_{D_s^+}^2 \neq M_\pi^2$ the matrix element $p_\mu < (q^+(q_+)\pi^+(p_+)\pi^-(p_-))_{NR}|\bar{u}(0)\gamma^\mu(1 - \gamma^5) d(0)|0 >$ does not vanish in the chiral limit.

Let us consider a more general matrix element $< \pi^+(q_+)\pi^+(p_+)\pi^-(p_-)|\bar{u}(0)\gamma^\mu(1 - \gamma^5) d(0)|0 >$ and represent it in terms of invariant amplitudes

$$-i < \pi^+(q_+)\pi^+(p_+)\pi^-(p_-)|\bar{u}(0)\gamma^\mu(1 - \gamma^5) d(0)|0 >= F_1 (p_\mu + q_+)^\mu + F_2 p_\mu + iF_3 \epsilon_{\mu\nu\alpha\beta} p_\nu p_{\alpha} q_+^\beta, \quad (3.11)$$

where $F_i (i = 1, 2, 3)$ are invariant amplitudes free of kinematical singularities. Multiplying both sides of Eq. (3.11) by a momentum $p_\mu$ we arrive at the expression

$$-ip_\mu < \pi^+(q_+)\pi^+(p_+)\pi^-(p_-)|\bar{u}(0)\gamma^\mu(1 - \gamma^5) d(0)|0 >= \frac{1}{2} F_1 (p^2 - p_\mu^2 + q^2) + F_2 p_\mu^2. \quad (3.12)$$

Imposing then the Adler condition, i.e. demanding a vanishing of the amplitude at $p_\mu \rightarrow 0$, we obtain a relation between invariant amplitudes $F_2 = -F_1$, which ensues the equation

$$-i \lim_{p_\mu \rightarrow 0} p_\mu < \pi^+(q_+)\pi^+(p_+)\pi^-(p_-)|\bar{u}(0)\gamma^\mu(1 - \gamma^5) d(0)|0 >= \frac{1}{2} (F_1 + F_2)p^2 = 0, \quad (3.13)$$

where we have used the relation $q^2 \rightarrow p^2$ valid in the limit $p_\mu \rightarrow 0$ due to a conservation of 4–momentum $p = q_+ + p_+ - p_-$. Using the relation $F_2 = -F_1$ we get

$$-ip_\mu < \pi^+(q_+)\pi^+(p_+)\pi^-(p_-)|\bar{u}(0)\gamma^\mu(1 - \gamma^5) d(0)|0 >= \frac{1}{2} F_1 (q^2 - p^2 - p_-^2). \quad (3.14)$$

The correctness of this expression can be verified by matching Eq. (3.14) with the amplitude of the elastic $\pi^+\pi^+$–scattering ($\pi^+ + \pi^+ \rightarrow \pi^+ + \pi^+$). Setting $p^2 = M_\pi^2$ and denoting $q^2 = (p_+ + q_+)^2 = s$ we reduce the r.h.s. of Eq. (3.14) to the form

$$-ip_\mu < \pi^+(q_+)\pi^+(p_+)\pi^-(p_-)|\bar{u}(0)\gamma^\mu(1 - \gamma^5) d(0)|0 >= \frac{1}{2} F_1 (s - 2 M_\pi^2), \quad (3.15)$$

which up to the common factor agrees well with the amplitude $\mathcal{M}(\pi^+ + \pi^+ \rightarrow \pi^+ + \pi^+) = (2 M_\pi^2 - s)/F_\pi^2$ given by Weinberg.

In the chiral limit $p_+^2 = q_+^2 = p_-^2 = M_\pi^2 \rightarrow 0$ we find

$$-ip_\mu < \pi^+(q_+)\pi^+(p_+)\pi^-(p_-)|\bar{u}(0)\gamma^\mu(1 - \gamma^5) d(0)|0 >= \frac{1}{2} F_1 s. \quad (3.16)$$

It is correct, since nothing suppresses the elastic $\pi^+\pi^+$–scattering for massless $\pi^+$–mesons. This confirms the validity of Eq. (3.14) and non–vanishing behaviour of the matrix element $p_\mu < (\pi^+(q_+)\pi^+(p_+)\pi^-(p_-))_{NR}|\bar{u}(0)\gamma^\mu(1 - \gamma^5) d(0)|0 >$ in the chiral limit at $p^2 = M_{D_s^+}^2 \neq M_\pi^2$.

Substituting Eq. (3.9) in Eq. (3.3) we get the amplitude of the $D_s^+ \rightarrow (\pi^+\pi^+\pi^-)_{NR}$ decay amplitude

$$M(D_s^+ \rightarrow (\pi^+\pi^+\pi^-)_{NR}) = C_1 2\sqrt{2} m_2 G_F V_{cs}^* V_{ud} \frac{F_{D_s^+}}{M_{D_s^+}} \frac{M_{D_s^+}^2 + q^2}{M_{D_s^+}^2}. \quad (3.17)$$
The partial width of the $D^+_s \rightarrow (\pi^+\pi^+\pi^-)_{\text{NR}}$ decay computed at $M_\pi = 0$ is given by

$$
\Gamma(D^+_s \rightarrow (\pi^+\pi^+\pi^-)_{\text{NR}}) = \frac{|C_1|^2 |G_F V_{cs}^* V_{ud}|^2}{384\pi^3} \frac{11m^4}{F^2} \frac{F^2_D}{F^2_\pi} M_{D^+_s} = |C_1|^2 \times 0.87 \times 10^{10} \text{ s}^{-1},
$$

where the numerical value has been obtained at $m = 0.33 \text{ GeV}$, $|V_{cs}| = 1.01$, $|V_{ud}| = 0.975$ and $M_{D^+_s} = 1.97 \text{ GeV}$.

Matching this value with the value of the partial width of the $D^+_s \rightarrow \phi \pi^+$ decay we arrive at the ratio

$$
\frac{\Gamma(D^+_s \rightarrow (\pi^+\pi^+\pi^-)_{\text{NR}})}{\Gamma(D^+_s \rightarrow \phi \pi^+)} = 0.24 \pm \Delta,
$$

which agrees well with the experimental data Eq.(3.1).}

### 4 Conclusion

The application of the effective quark model with chiral $U(3) \times U(3)$ symmetry incorporating HQET and (CHPT)$_q$ to the computation of the partial widths of the $D^+_s \rightarrow \phi \pi^+$ and $D^+_s \rightarrow (\pi^+\pi^+\pi^-)_{\text{NR}}$ decays has shown the agreement of the theoretical values with the experimental data. We have found $\Gamma(D^+_s \rightarrow (\pi^+\pi^+\pi^-)_{\text{NR}})/\Gamma(D^+_s \rightarrow \phi \pi^+) = 0.24 \pm 0.12$, which agrees well with the experimental value $\Gamma(D^+_s \rightarrow (\pi^+\pi^+\pi^-)_{\text{NR}})/\Gamma(D^+_s \rightarrow \phi \pi^+) = 0.29 \pm 0.09 \pm 0.06$.

Thus, one can conclude that such an effective quark model describes reasonably well a low–energy dynamics of heavy–light meson interactions, and one does not need to include unnaturally heavy light current quarks in order to explain the experimental data on the $D^+_s \rightarrow (\pi^+\pi^+\pi^-)_{\text{NR}}$ decay [1].

Using the experimental probability of the $D^+_s \rightarrow \phi \pi^+$ decay, i.e. $Br(D^+_s \rightarrow \phi \pi^+) = (3.6 \pm 0.9)\%$, we have estimated the time life of the $D^*_s$–meson: $\tau_{D^*_s} = (0.59 \pm 0.15) \times 10^{-12} \text{ s}$. This theoretical value agrees well with the experimental one: $(\tau_{D^*_s})_{\text{exp}} = (0.467 \pm 0.017) \times 10^{-12} \text{ s}$ [11].

As has been discussed in Ref.[14] the theoretical uncertainty of our effective quark model is about 50%. However, for all cases of the application of this model, the resultant agreement between theoretical and experimental data turns out as usually much better.

Recall that our results have been obtained for the factorized amplitudes. The step beyond the factorization approximation is to take into account one–meson loop contributions. In (CHPT)$_q$ the procedure of the computation of one–meson loop corrections has been considered in Ref.[6] (Ivanov). We are planning to investigate such contributions in our forthcoming publications.

### 5 Appendix. The $\pi^+$–meson pole contribution

In (CHPT)$_q$ the contribution of the $\pi^+$–meson pole to the matrix element Eq.(3.4) is given by [8]

$$
i < (\pi^+ (q_+) \pi^+ (p_+) \pi^- (p_-))_{\text{NR}} | \bar{u}(0) \gamma^\mu (1 - \gamma^5) d(0) | 0 >_{\pi^+\text{pole}=}$$

8
\[ \frac{N g_\pi}{16 \pi^2} \int \frac{d^4 k}{\pi^2 i} \text{tr} \left\{ \gamma^\mu \gamma^5 \frac{1}{m - k + \hat{p}} \gamma^5 \frac{1}{m - \hat{k}} \right\} \frac{1}{p^2 - M_\pi^2} \left( - \frac{N g_\pi^4}{16 \pi^2} \int \frac{d^4 k}{\pi^2 i} \text{tr} \left\{ \gamma^5 \frac{1}{m - k + \hat{p}} \gamma^5 \frac{1}{m - k} \right\} \right) \]

(5.1)

Taking into account that \( M_{D^+_s} \gg \Lambda \gg k \) we reduce the r.h.s. of Eq.(5.1) to the form

\[ i < \left( \pi^+(q_+) \pi^+(p_+) \pi^-(p_-) \right)_{\text{NR}} |\bar{u}(0) \gamma^\mu (1 - \gamma^5) d(0)|0 >_{\pi^+ - \text{pole}} = \frac{1}{2 \sqrt{2}} \frac{m}{F_\pi} \frac{1}{M_{D^+_s}^2} \]

(5.2)

where we have used the relations \( g_\pi = \sqrt{2} m / F_\pi, \quad N g_\pi^2 / 8 \pi^2 = 1 \) and \( p^2 = M_{D^+_s}^2 \).

The momentum integrals entering the r.h.s. of Eq.(5.2) equal \[6–9]\n
\[ \int \frac{d^4 k}{\pi^2 i} \text{tr} \left\{ \gamma^\mu \gamma^5 \frac{1}{m - k} \right\} \frac{1}{p^2 - M_\pi^2} \int \frac{d^4 k}{\pi^2 i} \text{tr} \left\{ \gamma^5 \frac{1}{m - k + \hat{p}} \frac{\gamma^5}{m - k} \right\} + (q_+ \leftrightarrow p_+) \]

\[ i < \left( \pi^+(q_+) \pi^+(p_+) \pi^-(p_-) \right)_{\text{NR}} |\bar{u}(0) \gamma^\mu (1 - \gamma^5) d(0)|0 >_{\pi^+ - \text{pole}} = \frac{1}{2 \sqrt{2}} \frac{m}{F_\pi} \frac{1}{M_{D^+_s}^2} \]

(5.3)

where \( \bar{v} = - \langle 0 | \bar{q} q | 0 > / F_\pi^2 = 1.92 \text{ GeV} \ [6–9] \). This gives the following contribution of the \( \pi^+ - \text{pole} \)

\[ i < \left( \pi^+(q_+) \pi^+(p_+) \pi^-(p_-) \right)_{\text{NR}} |\bar{u}(0) \gamma^\mu (1 - \gamma^5) d(0)|0 >_{\pi^+ - \text{pole}} =

= 8 \sqrt{2} \frac{m^4}{F_\pi} \frac{\bar{v}^2}{M_{D^+_s}^6} p^\mu. \]

(5.4)

The contribution of the \( \pi^+ \)-meson pole to the amplitude Eq.(3.3) is given by

\[ M(D_s^+ \rightarrow (\pi^+ \pi^+ \pi^-)_{\text{NR}})_{\pi^+ - \text{pole}} = C_1 \frac{2 \sqrt{2} m^2 G_F V_{us}^* V_{ud}}{F_\pi} \frac{F_{D_s^+}}{F_\pi} \frac{2 m^2 \bar{v}^2}{M_{D_s^+}^4}. \]

(5.5)

Thus, we have shown that the contribution of the \( \pi^+ \)-meson pole is of order \( O(1/M_{D_s^+}^4) \) compared with the contribution of the momentum integral Eq.(3.5) and can be ignored, correspondingly.
References

[1] Hoang N. L., Nguyen A. V. and Pham X. Y., Phys. Lett. B, 357 (1995) 177 and references therein.

[2] Gasser J. and H. Leutwyler H., Phys. Rep., 87 (1982) 77.

[3] Eichten E. and Feinberg F. L., Phys. Rev. D, 23 (1981) 2724; Eichten E., Nucl. Phys. B, 4 (Proc.Suppl.) (1988) 70; Voloshin M. B. and Shifman M. A., Sov. J. Nucl. Phys., 45 (1987) 292;

[4] Politzer H. D. and Wise M., Phys. Lett. B, 206 (1988); ibid. B, 208 (1988) 504.

[5] Georgi H., Phys. Lett. B, 240 (1990) 447.

[6] Ivanov A. N., Nagy M. and Troitskaya N. I., Int. J. Mod. Phys. A, 7 (1992) 7305; Ivanov A. N., Int. J. Mod. Phys. A, 8 (1993) 853.

[7] Ivanov A. N., Phys. Lett. B, 275 (1992) 450; Ivanov A. N., Troitskaya N. I. and Nagy M., Int. J. Mod. Phys. A, 8 (1993) 2027, 3425.

[8] Ivanov A. N. and Troitskaya N. I., Nuovo Cim. A, 108 (1995) 555.

[9] Ivanov A. N. and Troitskaya N. I., Phys. Lett. B, 342 (1995) 323.

[10] Hussain F., Ivanov A. N. and Troitskaya N. I., Phys. Lett. B, 348 (1995) 609; ibid. B, 369 (1996) 351.

[11] Partical Data Group, Phys. Rev. D, 54(1996) 1, Part1.

[12] Lee B. W. and Gaillard M. K., Phys. Rev. Lett., 33 (1974) 108; Altarelli G., Curci G., Martinelli G. and Petrarca S., Nucl. Phys. B, 187 (1981) 461; Buras A., Gérard J.-M. and Rückl R., Nucl. Phys. B, 268 (1986) 16; Bauez M., Stech B. and Wizbel M., Z. Phys. C, 34 (1987) 103.

[13] Ivanov A. N., Troitskaya N. I. and Nagy M., Phys. Lett. B, 339 (1994) 167; Hussain F., Ivanov A. N. and Troitskaya N. I., Phys. Lett. B, 329 (1994) 98; ibid. B, 334 (1994) E450;

[14] Ivanov A. N. and Troitskaya N. I., Phys. Lett. B, 345 (1995) 175; ibid. B, 390 (1997) 341.

[15] Ivanov A. N. and Troitskaya N. I., Nuovo Cim. A, 110 (1997) 65.

[16] Ivanov A. N. and Troitskaya N. I., Phys. Lett. B, 394 (1997) 195.

[17] Bardeen W. A. and Hill C. T., Phys. Rev.D, 49 (1994) 409.

[18] Nambu Y. and Jona–Lasinio G., Phys. Rev., 122 (1961) 345; ibid. 124 (1961) 246.

[19] Eguchi T., Phys. Rev. D, 14 (1976) 2755; Kikkawa K., Progr. Theor. Phys., 56 (1976) 947; Kleinert H., Proc. of Int. Summer School of Subnuclear Physics, Erice 1976, Ed. A.Zichichi, p.289.
[20] Klint S., Lutz M., Vogl V. and Weise W., Nucl. Phys. A, 516 (1990) 429, 469 and references therein.

[21] Ivanov A. N., Troitskaya N. I., Faber M., Schaler M. and Nagy M., Nuovo Cim. A, 107 (1994) 1667; Phys. Lett. B, 336 (1994) 555.

[22] Appelquist T. and Carazzone J., Phys. Rev. D, 11 (1975) 2856.

[23] Dhar A. and Wadia S. R., Phys. Rev. Lett., 52 (1984) 959; Dhar A., Shankar R. and Wadia S. R., Phys. Rev. D, 31 (1985) 3256; Ebert D. and Reinhardt H., Nucl. Phys. B, 271 (1986) 188; Wakamatsu M., Ann. Phys., 193 (1989) 287; Ruiz Arriola E., Phys. Lett. B, 264 (1991) 178.

[24] Bijnens J., Bruno C. and de Rafael E., Nucl. Phys. B, 390 (1993) 501 and references therein; Bijnens J., de Rafael E. and Zheng H., Z. Phys. C, 62 (1994) 437 and references therein.

[25] Hayashi K., Hirayama M., Muta T., Seto N. and Shirafuji T., Fortschritte der Physik, 15 (1967) 625.

[26] Efimov G. V. and Ivanov M. A., in Quark confinement model of hadrons, Institute of Physics Publishing Bristol and Philadelphia, London 1993, p.21; Efimov G. V., On bound states in quantum field theory, Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia, hep-ph/9607425 25 July 1996, 33p.