A Class of Weighted Point Schemes for the Grünwald Implicit Finite Difference Solution of Time-Fractional Parabolic Equations Using KSOR method

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Abstract. In this study, system of Grünwald implicit approximation equations has been developed through the discretization of one-dimensional linear time-fractional parabolic equations using the Grünwald fractional derivative operator and second-order implicit finite difference scheme. The aim of this paper is to examine the effectiveness of Kaudd Successive Over-Relaxation (KSOR) iterative method, which is one of the weighted point iterative schemes for solving the proposed time-fractional parabolic equations by considering the Grünwald implicit approximation equation. To investigate the effectiveness of the proposed iterative method, numerical experiments and comparison are made in terms of number of iterations, execution time, and maximum absolute error. Based on numerical results, the accuracy of Grünwald implicit solution obtained by proposed iterative method is in excellent agreement, and it can be concluded that the proposed KSOR iterative method requires less number of iterations and execution time as compared to the existing point iterative method.

1. Introduction

For the past few decades, the interest on the application of fractional order partial differential equations (FPDEs) in diverse fields such as biomedics, engineering, finance and natural sciences fields [1-3] have grown among researchers for its potential in describing the non-Markovian random walks problems [4] which is also stated by [5] as an effective tool to describe real world phenomena such as diffusion process. In this paper, we consider the one-dimensional inhomogeneous time-fractional parabolic equations (TFPE) of order $0 < \alpha < 1$ as

$$\frac{\partial^\alpha U(x,t)}{\partial x^\alpha} + p(x) \frac{\partial U(x,t)}{\partial x} + q(x) \frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t), \quad t > 0, \quad x \in \Omega, \quad 0 < \alpha \leq 1,$$

subject to the initial condition

$$U(x,0) = f(x), \quad a \leq x \leq b,$$

and the boundary conditions

$$U(a,t) = g_1(t); \quad U(b,t) = g_2(t), \quad 0 \leq t \leq T,$$

where $p(x)$ and $q(x)$ are real parameters and $f(x,t)$ is the source term.
To construct the approximation equation of problem (1), we used the implicit finite difference scheme with Grünwald fractional derivative order \( \alpha \), which defined as [6,7]

\[
D^\alpha_{\text{G}} f(t) = \lim_{N \to \infty} \frac{1}{(\Delta t)^\alpha} \sum_{k=0}^{l} g_{\alpha,k} f(t - k\Delta t), \quad 0 < \alpha < 1
\]

where the Grünwald weights are \( g_{\alpha,k} = -\frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} \).

Numerous numerical approaches which dealing with TFPE could be found in the literatures. Several recent researches have used the reduced spline (RS) method based on a proper orthogonal decomposition (POD) technique [8], Crank-Nicholson scheme with finite element method [9] and alternating segment explicit-implicit/ implicit-explicit parallel difference method [10]. Meanwhile, some earlier researchers have applied the finite difference method based on implicit scheme [11-17] and compact difference scheme [18], which are mostly described based on Caputo sense.

Similar to the integer order PDEs, the discretization of TFPE will also generate a large sparse linear systems. Where in this case, iterative method is famously used by researchers for its reputation in reducing the time consumption of the direct method. Gauss-Seidel (GS) iterative method is one of the most used classical iterative methods. Yet, the convergence rate of this method is still relatively slow. Hence, previous researchers such as [19-20] have devoted their studies to improve the convergence rate of large sparse linear systems. Prior to that, [21-22] has introduced and discussed another approach called the Successive Over-Relaxation (SOR) iterative method.

Just recently, a new concept known as Kaudd Successive Over-Relaxation (KSOR) iterative method have been introduced by [23-24], which have sparked the interest of this paper. So far, only some researches [15-17] could be found in solving TFPE iteratively and none of the work related to solving TFPE using KSOR iterative method could be found in the literatures. Figure 1 shows the finite grid network that is used to formulate the approximate equation, where the GS and KSOR iterative methods are applied onto each interior node point until the convergence test can be reached.

![Figure 1](image_url)

**Figure 1.** The distribution of uniform node points for the solution domain at \( m=8 \).

### 2. Grünwald finite difference approximation

In this section, the time-fractional parabolic equation (1) is discretized using the Grünwald fractional derivative as stated in equation (2) and implicit difference scheme. To derive numerical approximations based on Grünwald derivative, first let the mesh point \( x_i = \alpha + ih \), where \( i = 0, 1, ..., N \) and \( h = \Delta x = (b - a) / N \) denotes the uniform step-size. Hence, the discrete equation of problem (1) is given by

\[
\frac{1}{(\Delta t)^\alpha} \sum_{k=0}^{l} g_{\alpha,k} U_{i,j-k} + \frac{p(x) (U_{i+1,j} - U_{i-1,j})}{2\Delta x} + \frac{q(x)}{(\Delta x)^2} (U_{i+1,j} - 2U_{i,j} + U_{i-1,j}) = f_{i,j}
\]

Then, by letting

\[
G_k = \frac{g_{\alpha,k}}{(\Delta t)^\alpha}, \quad a_i = \frac{p(x)}{2\Delta x}, \quad b_j = \frac{q(x)}{(\Delta x)^2},
\]

the simplified approximation equation can be stated in the form
\[ \alpha_i U_{i-1,j} + \beta_i U_{i,j} + \gamma_i U_{i+1,j} = F_{i,j} \]  

(4)

where,

\[ \alpha_i = b_i - a_i, \quad \beta_i = G_0 - 2b_i, \quad \gamma_i = a_i + b_i \]

and where, at time level \( j = 1 \), \( F_{i,j} = f_{i,j} - G_0 \) and at time level \( j = 2, 3, \ldots, M \),

\[ F_{i,j} = f_{i,j} - G_k \sum_{k=1}^{j} g_{a,k} U_{i,j-k} \]

Meanwhile, the illustration of their respective computational molecules when time level \( j = 1 \) and \( j = 2 \) as shown in Figures 2 and 3.

**Figure 2.** The computational molecule for Grünwald time-fractional parabolic approximation equation at time level \( j = 1 \).

**Figure 3.** The computational molecule for Grünwald time-fractional parabolic approximation equation at time level \( j = 2 \).

Based on Figure 2, it shows that the computational molecules for Grünwald time-fractional parabolic approximation equation is similar to the regular computational molecules of implicit finite difference approximation at time level \( j = 1 \). Meanwhile, at time level \( j = 2, 3, \ldots, M \), a series of points will appear as a tail-like as shown in Figure 3.

Hence, equation (4) will lead to large sparse linear system in matrix form as

\[ AU_{-j} = F_{-j} \]  

(5)

where

\[ A = \begin{bmatrix} \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ & \alpha_3 & \beta_3 & \gamma_3 \\ & & \ddots & \ddots & \ddots \\ & & & \alpha_{m-2} & \beta_{m-2} & \gamma_{m-2} \\ & & & & \alpha_{m-1} & \beta_{m-1} \end{bmatrix} \quad U = \begin{bmatrix} U_{1,j} \\ U_{2,j} \\ \vdots \\ U_{m-2,j} \\ U_{m-1,j} \end{bmatrix} \quad F = \begin{bmatrix} F_{1,j} - \alpha U_{0,j} \\ F_{2,j} \\ \vdots \\ F_{m-2,j} \\ F_{m-1,j} - \gamma_{m-1} U_{m,j} \end{bmatrix} \]

and

\[ \alpha_i = b_i - a_i, \quad \beta_i = G_0 - 2b_i, \quad \gamma_i = a_i + b_i \]

3. **Derivation of KSOR iteration scheme**

Due to the performance of KSOR iterative method as discussed in [23-24], we investigate the effectiveness of KSOR iterative method in solving problem (1) based on Grünwald derivative and implicit finite difference scheme. To do so, the linear system in equation (5) are solved iteratively by using GS iterative method as a control method. First, let the coefficient matrix \( A \) be decomposed as
where \( D, V \) and \( L \) are diagonal, lower triangular and upper triangular matrices respectively.

To formulate the KSOR iterative method, let us recall the SOR iterative method stated in general form as [21-22,25]

\[
U_j^{(k+1)} = (1 - \omega)U_j^{(k)} + \omega(D + L)^{-1}(F_j - VL_j^{(k)})
\]

noted that, the SOR iterative method in equation (7) can be reduced to GS iterative method when \( \omega = 1 \).

By using the decomposition matrix in equation (6), the KSOR method can be written in general form as [23-24]

\[
U_j^{(k+1)} = (((1 + \omega)D - \omega L)^{-1}(D + \omega V))U_j^{(k)} + (((1 + \omega)D - \omega L)^{-1}V \omega F_j)
\]

where the relaxation parameter \( \omega \) is extended to \( R - [-2,0] \) for KSOR iterative method, compared to \( \omega \in (1,2) \) for SOR iterative method.

The implementation of KSOR iterative method to solve problem (1) is as described in Algorithm 1.

**Algorithm 1: KSOR scheme**

1. Initialize \( U_0 \) and \( \varepsilon \leftarrow 10^{-10} \)
2. Assign the optimal value of \( \omega \),
3. For \( j = 1, 2, \ldots, M \), perform:
   a. Set initial value, \( U_j^{(0)} \leftarrow 0 \)
   b. Calculate the approximate value of \( U_j^{(k+1)} \) iteratively
      \[
      U_j^{(k+1)} \leftarrow (((1 + \omega)D - \omega L)^{-1}(D + \omega V))U_j^{(k)} + (((1 + \omega)D - \omega L)^{-1}V \omega F_j)
      \]
   c. Perform the convergence test, \( |U_{i,j}^{(k+1)} - U_{i,j}^{(k)}| \leq \varepsilon = 10^{-10} \). If yes, go to step (d).
      Otherwise repeat step (b).
   d. Check time level, \( j = M \). If yes, go to step (iv). Otherwise, repeat step (a).
4. Display approximate solutions.

4. Numerical experiments

In this section, we tested one-dimensional inhomogeneous TFPEs for \( \alpha = 0.333, \alpha = 0.666 \) and \( \alpha = 0.999 \) in order to verify the performance of the KSOR iterative method over the GS iterative method. Comparison was made based on three criteria, which are the iteration numbers, time of execution and maximum absolute error. In the implementation of the iterative methods, the convergence test considered the tolerance error \( \varepsilon = 10^{-10} \).

**Example 1:** Consider the following one-dimensional linear inhomogeneous fractional Burger’s equation [26]

\[
\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} + \frac{\partial U(x,t)}{\partial x} + \frac{\partial^2 U(x,t)}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2.
\]

with \( p(x) = 1 \) and \( q(x) = -1 \), and subject to the initial condition \( U(x,0) = x^2 \).

The exact solution is \( U(x,t) = x^2 + t^2 \).

**Example 2:** Consider the following one-dimensional inhomogeneous time-fractional parabolic equation [27]
\[
\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} - \frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t),
\]

with \( p(x) = 0 \) and \( q(x) = -1 \), while \( f(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}t^{2-\alpha}\sin(2\pi x) + 4\pi^2x^2\sin(2\pi x), \)

and subject to the initial condition \( U(x,0) = 0 \). The exact solution is given by \( U(x,t) = t^2\sin(2\pi x) \).

**Example 3:** Consider the following one-dimensional linear inhomogeneous time-fractional parabolic equation [28]

\[
\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} - \frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t), \quad t > 0, x \in R, 0 < \alpha \leq 1,
\]

with \( p(x) = 0 \) and \( q(x) = -1 \), while \( f(x,t) = \frac{2e^x t^{2-\alpha}}{\Gamma(3-\alpha)}-t^\alpha e^x \),

subject to the initial condition \( U(x,0) = 0 \). The exact solution is given by \( U(x,t) = t^\alpha e^x \).

Then, the numerical results of both GS and KSOR iterative methods for examples (1) to (3) are recorded as in Tables 1 to 3. Overall, the percentage of reductions of all examples at \( \alpha=0.333, 0.666, 0.999 \) are respectively between 93.89-99.80% for the iteration numbers and 13.45-99.35% for the time of execution.

**Table 1.** Comparison of Number Iterations (k), execution time (t) (in seconds) and maximum absolute error for the iterative methods using example 1 at \( \alpha=0.333, 0.666, 0.999 \).

| M Method | \( \alpha=0.333 \) | k | t (seconds) | Max Error | \( \alpha=0.666 \) | k | t (seconds) | Max Error | \( \alpha=0.999 \) | k | t (seconds) | Max Error |
|----------|-------------------|---|----------------|------------|-------------------|---|----------------|------------|-------------------|---|----------------|------------|
| GS 128 | 18325 | 7.89 | 2.5972e-02 | 9897 | 5.36 | 1.3065e-02 | 2772 | 3.69 | 1.2478e-03 |
| KSOR 256 | 404 | 3.2 | 2.5972e-02 | 283 | 3.11 | 1.3065e-02 | 166 | 3.14 | 1.2480e-03 |
| GS 512 | 67139 | 41.9 | 2.5973e-02 | 32944 | 23.62 | 1.3066e-02 | 10244 | 11.42 | 1.2473e-03 |
| KSOR 1024 | 877165 | 1914.74 | 2.5981e-02 | 435083 | 965.24 | 1.3074e-02 | 137338 | 23.49 | 1.2454e-03 |
| GS 2048 | 314564 | 13356.94 | 2.6008e-02 | 1556326 | 6730.67 | 1.3102e-02 | 496352 | 2216.78 | 1.2066e-03 |
| KSOR | 6365 | 87.08 | 2.5972e-02 | 4519 | 75.36 | 1.3065e-02 | 2618 | 63.69 | 1.2480e-03 |

**Table 2.** Comparison of Number Iterations (k), execution time (t) (in seconds) and maximum absolute error for the iterative methods using example 2 at \( \alpha=0.333, 0.666, 0.999 \).

| M Method | \( \alpha=0.333 \) | k | t (seconds) | Max Error | \( \alpha=0.666 \) | k | t (seconds) | Max Error | \( \alpha=0.999 \) | k | t (seconds) | Max Error |
|----------|-------------------|---|----------------|------------|-------------------|---|----------------|------------|-------------------|---|----------------|------------|
| GS 128 | 14148 | 6.4 | 3.4745e-04 | 7120 | 4.72 | 5.2947e-04 | 2358 | 75.36 | 1.3065e-02 |
| KSOR 256 | 47194 | 28 | 2.0203e-04 | 24125 | 17.5 | 3.8529e-04 | 8332 | 10.12 | 3.1294e-04 |
| GS 512 | 764 | 6.66 | 2.0160e-04 | 513 | 6.4 | 3.8483e-04 | 291 | 6.2 | 3.1267e-04 |
| KSOR 1024 | 151187 | 149.13 | 1.6691e-04 | 79109 | 86.99 | 3.5049e-04 | 29799 | 42.17 | 2.7708e-04 |
| GS 2048 | 454367 | 812.72 | 1.6304e-04 | 251077 | 509.06 | 3.4511e-04 | 107228 | 240.64 | 2.6674e-04 |
| KSOR | 5708 | 80.73 | 1.5380e-04 | 4056 | 71.23 | 3.3742e-04 | 2155 | 60.45 | 2.6568e-04 |

**Table 3.** Comparison of Number Iterations (k), execution time (t) (in seconds) and maximum absolute error for the iterative methods using example 3 at \( \alpha=0.333, 0.666, 0.999 \).
5. Conclusion
In this paper, we proposed the Grünwald implicit finite difference scheme and KSOR iterative method to solve inhomogeneous time-fractional parabolic equations. By taking GS iterative method as the control method, the numerical results of three examples show that the KSOR iterative method is able to substantially improve the iteration numbers and time of execution of the GS iterative method in solving one-dimensional inhomogeneous time-fractional parabolic equations. This concluded that, the KSOR iterative method is superior than the GS iterative method.

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