A gauge-invariant Hamiltonian analysis for non-Abelian
gauge theories in (2+1) dimensions

DIMITRA KARABALI
Physics Department*
Rockefeller University
New York, New York 10021
and
The Institute for Advanced Study
Princeton, New Jersey 08540

V.P. NAIR
Physics Department
City College of the City University of New York
New York, New York 10031.

Abstract

The Hamiltonian formulation for a non-Abelian gauge theory in two spatial dimensions is carried out in terms of a gauge-invariant matrix parametrization of the fields. The Jacobian for the relevant transformation of variables is given in terms of the WZW-action for a hermitian matrix field. Some gauge-invariant eigenstates of the kinetic term of the Hamiltonian are given; these have zero charge and exhibit a mass gap.

* Present address
E-mail addresses: karabali@theory.rockefeller.edu, vpn@ajanta.sci.ccny.cuny.edu
1. Introduction

The importance of non-Abelian gauge theories to our understanding of physical phenomena is well recognized and needs no reiteration. While the perturbative aspects of such theories were understood many years ago, nonperturbative phenomena have been difficult to analyze. Detailed analyses and calculational techniques are still lacking, even though most of the qualitative features are more or less clear. Gauge theories in two spatial dimensions, which are simpler than their analogues in three spatial dimensions, are interesting as a guide to the more realistic case, and not surprisingly, there have been many analyses of such theories starting from the early days \[1,2\]. For a non-Abelian gauge theory spontaneously broken to an Abelian subgroup, one can show the existence of a mass gap or the absence of massless gauge particles due to the condensation of monopoles \[1\]. It is very likely that this feature holds true for the unbroken theory as well. The subject has recently been receiving renewed attention. New gauge-invariant parametrizations, variables similar to gravitational degrees of freedom as well as extensions of old techniques are being explored \[3,4\]. In this paper, we introduce a fairly simple matrix parametrization of the gauge fields which amounts to a particular choice of coordinates on the space of gauge-invariant configurations and work out the Hamiltonian formulation. This approach is closest in spirit to the work of Feynman (and also Singer) who emphasized the importance of the geometry of the configuration space \[5\]. The kinetic energy term for the Hamiltonian is essentially a Laplace operator on the configuration space. Now, for a finite-dimensional compact manifold the Laplacian has a discrete spectrum, the gap between the first and second eigenvalues being the analogue of the mass gap. Feynman argued that this picture should remain qualitatively correct for the infinite-dimensional case of the configuration space of the \((2+1)\)-dimensional non-Abelian gauge theory. Since the Hamiltonian is a real operator, the ground state wave function can be chosen real and positive everywhere on the configuration space. The excited states, since they have to be orthogonal to the ground state, must involve nodes. The low-lying excited states can thus be analyzed by considering the minimization of the expectation value of the Hamiltonian for wave functions with nodes. The potential energy can be minimized by making the
magnitude of the wave function small. The kinetic energy is proportional to the square of the gradient (over the configuration space) of the wave function. One can try to minimize the magnitude of the gradient by choosing a function for which the regions where it is positive and where it is negative are, roughly speaking, as far separated as possible. However, if the gauge-invariant distance between configurations cannot be made arbitrarily large, the gradient of the wave function and hence the kinetic energy cannot be made arbitrarily small; in other words, one has a mass gap for the theory. This is the essence of Feynman’s arguments, although, of course, distances, expectation values, etc. have to be understood with regularizations and all that. By choosing a simple enough parametrization of the fields, which makes the evaluation of the relevant Jacobian factors and the transformation of the Hamiltonian quite elementary, we expect to make these intuitive and qualitative arguments more explicit. This is also equivalent to making a special choice of ‘collective coordinates’ and so our analysis is similar to some previous work along these lines [6].

For the matrix parametrization we use, the relevant Jacobian factor is related to the action $S$ for a hermitian Wess-Zumino-Witten (WZW) model [7], i.e., a $G^C/G$-coset model [8,9], where $G$ is the gauge group and $G^C$ is its complexification. We obtain the Hamiltonian and other generators of the Poincaré algebra in this parametrization. In view of Feynman’s arguments, it is interesting to consider the spectrum of the kinetic energy term by itself. We obtain some (gauge-invariant) eigenstates of the kinetic energy; these correspond to massive states. The construction of the states involve nonlocal composite operators (in terms of the gauge potentials) and there are arguments which suggest that the norms should exist in a suitably regularized sense. We also give arguments regarding the completeness of these states.

In the next section, we introduce the matrix parametrization of the gauge fields. Section 3 deals with the reduction of the metric on the space of configurations and the Hamiltonian. In section 4, we discuss the vacuum and other eigenstates of the kinetic energy. A short discussion and recapitulation of results is given in section 5 and the paper concludes with an appendix where some estimate of the effect of coordinate singularities is given.
2. The matrix parametrization of fields

We consider a $G = SU(N)$-gauge theory in the $A_0 = 0$ gauge. The gauge potential can be written as $A_i = -it^a A^a_i$, $i = 1, 2$, where $t^a$ are hermitian $N \times N$-matrices which form a basis of the Lie algebra of $SU(N)$ with $[t^a, t^b] = i f^{abc} t^c$, $\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$. The Hamiltonian can be written as

$$H = T + V$$

$$T = \frac{e^2}{2} \int d^2 x \, E^a_i E^a_i$$

$$V = \frac{1}{2e^2} \int d^2 x \, B^a B^a$$

where $e$ is the coupling constant; $e^2$ has the dimension of mass. Further, $B^a = \frac{1}{2} \epsilon_{jk} (\partial_j A^a_k - \partial_k A^a_j + f^{abc} A^b_j A^c_k)$ and we have the standard commutation rules

$$[A^a_i(x), A^b_j(y)] = [E^a_i(x), E^b_j(y)] = 0$$

$$[A^a_i(x), E^b_j(y)] = i \delta_{ij} \delta^{ab} \delta^{(2)}(x - y)$$

These commutation rules or the corresponding Poisson brackets are given by the symplectic two-form

$$\omega = \delta \Theta = \int d^2 x \, \delta E^a_i \delta A^a_i$$

$$\Theta = \int d^2 x \, E^a_i \delta A^a_i$$

We shall use $\delta$ to denote exterior differentiation on the space of gauge potentials $A = \{A^a_i(x)\}$. For fields which fall off at spatial infinity, the spatial manifold can be taken as the Riemann sphere $\mathbb{C} \cup \infty$ with complex coordinates $z = x_1 - ix_2$, $\bar{z} = x_1 + ix_2$. The complex components of the potential, viz., $A_z = \frac{1}{2} (A_1 + iA_2)$, $A_{\bar{z}} = \frac{1}{2} (A_1 - iA_2) = -(A_z)\dagger$, can be parametrized in terms of a complex $G^\mathbb{C}$-, specifically $SL(N, \mathbb{C})$-, matrix $M$ as

$$A_z = -\partial_z M M^{-1}, \quad A_{\bar{z}} = M^{\dagger - 1} \partial_{\bar{z}} M^{\dagger}$$

For a given potential, the matrix $M$ is not uniquely defined; $M$ and $M\bar{V}$, where $\bar{V}$ is antiholomorphic, lead to the same potential. For Riemann sphere, the only (nonsingular or globally defined) antiholomorphic function being a constant, $\bar{V}$ has to be constant. This
ambiguity can be eliminated by requiring $M \to 1$ as $|x| \to \infty$. (If we allow singularities in $M$, more general choices of $\tilde{V}$ are possible. These can be related to the coordinate singularities of our parametrization of the configuration space; we shall comment on them in the appendix.) The Green's functions for the $\partial_z$ and $\bar{\partial}_{\bar{z}}$ can be defined as follows.

\[
\bar{\partial}_z G(x, y) = \partial_z G(x, y) = \delta^{(2)}(x - y) \\
\bar{G}(x, x') = \frac{1}{\pi (z - z')}, \quad G(x, x') = \frac{1}{\pi (\bar{z} - \bar{z}')} \quad (2.5)
\]

Using these Green's functions, eqs.(2.4) can be inverted to obtain $M$ and $M^\dagger$ in terms of $A_z$ and $A_{\bar{z}}$ respectively, at least as a power series in the gauge potentials.

Using the above parametrization, we find

\[
\Theta = 2 \int \text{Tr} \left( \bar{p} \delta M^\dagger M^\dagger - \bar{1} + \bar{p} M - M^\dagger \delta M \right) \quad (2.6)
\]

where

\[
\bar{p} = \bar{p}^at^a = 2 \partial_{\bar{z}}(M^\dagger E_z M^\dagger - 1) \\
p = p^at^a = -2 \partial_z(M^{-1} E_{\bar{z}} M) = (\bar{p})^\dagger \quad (2.7)
\]

The symplectic two-form is obtained from (2.6) as

\[
\omega = \delta \Theta = 2 \int \text{Tr} \left[ \delta \bar{p} \delta M^\dagger M^\dagger - 1 + \bar{p} (\delta M^\dagger M^\dagger - 1)^2 + \delta p M^{-1} \delta M - p (M^{-1} \delta M)^2 \right] \quad (2.8)
\]

We want to rewrite the theory in terms of $M$'s. The transformed version of the commutation rules (2.2) can be obtained from (2.8) as

\[
[F, G] = i(V_F[V_G] \omega) = -i(V_F[\delta G]) \quad (2.9)
\]

where $V_F$ is the Hamiltonian vector field corresponding to $F$, defined by $V_F[\omega] = -\delta F$ and the contraction $|$ is defined by $V | \omega = V^I \omega_{IJ} \delta \xi^J$ for $V = V^I \delta \xi^J$, $\omega = \frac{1}{2} \omega_{IJ} \delta \xi^I \delta \xi^J$.

The vector fields $V(\phi)$, $\bar{V}(\phi)$ corresponding to $\int p_a \phi^a$ and $\int \bar{p}_a \phi^a$ are easily identified as

\[
V(\phi) = v(\phi) - i \int f^{abc} \phi_a p_b \frac{\delta}{\delta p_c} \\
\bar{V}(\phi) = \bar{v}(\phi) + i \int f^{abc} \phi_a \bar{p}_b \frac{\delta}{\delta \bar{p}_c} \quad (2.10)
\]
where $v(\phi) \delta M = M(\phi a t^a)$, $\bar{v}(\phi) \delta M^\dagger = (\phi a t^a)M^\dagger$. Using (2.9) and (2.10), we immediately find the commutation rules

$$
[p_a(x), M(y)] = -iM(x)t_a \delta^{(2)}(x - y)
$$

$$
[p_a(x), p_b(y)] = f_{abc}p_c(x) \delta^{(2)}(x - y)
$$

$$
[\bar{p}_a(x), M^\dagger(y)] = -it_aM^\dagger(x) \delta^{(2)}(x - y)
$$

$$
[\bar{p}_a(x), \bar{p}_b(y)] = -f_{abc}\bar{p}_c(x) \delta^{(2)}(x - y)
$$

$$
[M(x), M(y)] = [M^\dagger(x), M^\dagger(y)] = [M(x), M^\dagger(y)] = 0
$$

$$
[p_a(x), M^\dagger] = [\bar{p}_a(x), M(y)] = [p_a(x), \bar{p}_b(y)] = 0
$$

The $SL(N, \mathbb{C})$-matrix $M$ can be parametrized in terms of a complex field $\theta_a(x)$. We write

$$
M^{-1}\delta M = \delta \theta^a R_{ab} b
$$

$$
\delta M^\dagger M^{\dagger -1} = \delta \bar{\theta}^a R_{ab}^* b
$$

(2.12)

The definition of the potentials leads to

$$
\delta A_{z,k} = -i\mathcal{E}_{ka} \partial_z (\delta \theta^b R_{ba})
$$

$$
\delta A_{\bar{z},k} = i\mathcal{E}_{ak} \partial_{\bar{z}} (\delta \bar{\theta}^b R_{ba}^*)
$$

(2.13)

where

$$
\mathcal{E}_{ka} = 2 \text{Tr}(t_k M a M^{-1}), \quad \mathcal{E}_{ak}^\dagger = 2 \text{Tr}(t_a M^\dagger t_k M^{\dagger -1})
$$

(2.14)

The commutation rules (2.11) can be realized by the following differential operator representation for $p_a, \bar{p}_a$.

$$
p_a(x) = -iR_{ab}^{-1}(x) \frac{\delta}{\delta \theta^b(x)}
$$

$$
\bar{p}_a(x) = -iR_{ab}^{-1}(x) \frac{\delta}{\delta \bar{\theta}^b(x)}
$$

(2.15)

The components of the electric field can be written as

$$
E_{z,k} = \frac{i}{2} \mathcal{E}_{ak} \int_y \bar{G}(x,y)\bar{p}_a(y)
$$

$$
E_{\bar{z},k} = \frac{i}{2} \mathcal{E}_{ka} \int_y G(x,y)p_a(y)
$$

(2.16)
These expressions have the correct ordering of the operators; this can be seen by deriving them from a change of variables. Thus

\[
E_{z \ k}(x)\Psi(M, M^\dagger) = -\frac{i}{2} \frac{\delta \Psi}{\delta A_{z \ k}(x)} = -\frac{i}{2} \int_y \frac{\delta \Psi}{\delta \theta^a(y)} \frac{\delta \theta^a(y)}{\delta A_{z \ k}(x)}
\]

(2.17)

where we have used (2.13).

The new variables \( M \) and \( M^\dagger \) are not gauge-invariant. Gauge transformations act on \( M, M^\dagger \) as

\[
M \rightarrow gM \simeq (1 + i\phi^a t_a)M, \quad M^\dagger \rightarrow M^\dagger g^{-1} \simeq M^\dagger(1 - i\phi^a t_a), \quad g(x) \in SU(N).
\]

The theory can be written in terms of the gauge-invariant combination \( H = M^\dagger M \), which can be considered as the fundamental dynamical variable. From the commutation rules (2.11), we obtain

\[
[p_a(x), H(y)] = -iH(x) t_a \delta^{(2)}(x - y)
\]

(2.18)

When acting on functions of \( H, p_a \) and \( \bar{p}_a \) can be represented as

\[
p_a(x) = -ir_{ab}^{-1}(x) \frac{\delta}{\delta \varphi^b(x)}
\]

(2.19)

\[
\bar{p}_a(x) = -ir_{ab}^{-1}(x) \frac{\delta}{\delta \varphi^b(x)} = K_{ab}(x)p_b(x)
\]

where we have defined \( H \) in terms of the real parameters \( \varphi^a \) and

\[
H^{-1} \delta H = \delta \varphi^a r_{ab} t^b
\]

(2.20)

The Gauss law operator or the generator of gauge transformations can be obtained as

\[
G(\phi) = i \int \phi_k \ (E_{ka} p_a - \mathcal{E}_{ak}^\dagger \bar{p}_a)
\]

(2.21)

From the commutation rules, we see that \( p_a, \bar{p}_a \) are gauge-invariant operators, i.e.,

\[
[G_k, p_a] = [G_k, \bar{p}_a] = 0
\]

(2.22)

We have already used this fact in representing \( p_a, \bar{p}_a \) as derivatives with respect to \( \varphi^a \) in (2.19). The differential operator representations (2.15) which are appropriate when acting
on functions of $M$ and $M^\dagger$ naturally go over to the representation (2.19) for action on functions of $H$.

The magnetic field can be written as

$$B = B^a t_a = -2 M[\partial(H^{-1}\bar{\partial}H)]M^{-1}$$

where $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$. The potential term in the Hamiltonian can thus be written as

$$V = \frac{2}{e^2} \int d^2 x \left[ \partial(H^{-1}\bar{\partial}H)_a \partial(H^{-1}\bar{\partial}H)_a \right]$$

Using the expressions (2.16) for the electric fields, the kinetic term $T$ can be written as

$$T = \frac{e^2}{2} \int d^2 x \int_{y,z} G(x,y)\bar{p}_a(y)K_{ab}(x)G(x,z)p_b(z)$$

where

$$K_{ab} = 2 \text{Tr}(t_a H t_b H^{-1}) = \mathcal{E}_{ak}^T \mathcal{E}_{kb}$$

$K_{ab}$ is the adjoint representation of the matrix $H$; we also have $K^T K = 1$, $\mathcal{E}^T \mathcal{E} = 1$, $\mathcal{E}^T \mathcal{E}^\dagger = 1$, where the superscript $T$ denotes the transpose.

The kinetic term of the Hamiltonian as given in (2.25) is still essentially classical since we have not included the corrections from the Jacobians which arise in writing the inner product of wavefunctions in terms of $M$, and eventually $H$. The kinetic energy $T$, as given by (2.1), involves the Laplacian on $A$, since $E_i^a(x) = -i \delta A_i^a(x)$. Upon changing to $M, M^\dagger$, this should be the Laplacian expressed in terms of $M, M^\dagger$ (or $H$). Expression (2.25) gives only one of the terms in the Laplacian; the remaining terms, which involve the determinant of the metric, will arise from the Jacobian in the inner product for wave functions.

The physical states are gauge-invariant and obey $G|\text{phys} \rangle = 0$. They can be represented as functions of $H$. Further, from the discussion following eq.(2.4), $H$ and $V H \bar{V}$, where $V$ is a constant $SL(N, \mathbb{C})$-matrix, describe the same physical configuration; thus we must require that physical states, and correspondingly observables, should be invariant under $H \rightarrow VH\bar{V}$.

3. The reduction of the metric, the Hamiltonian and Poincaré invariance
The metric on the space of gauge potentials $A$ is given by

$$ds^2 = \int \delta A_a^a \delta A_{\bar{a}}^a = -8 \int Tr(\delta A_A \delta A_{\bar{A}})$$  \hspace{1cm} (3.1)$$

The wavefunctions have the inner product

$$\langle 1|2 \rangle = \int d\mu(C) \Psi_1^\dagger \Psi_2$$  \hspace{1cm} (3.2)$$

where $d\mu(A)$ is defined by the metric (3.1) and $d\mu(C)$, the measure on the space of physical configurations $C$, is obtained from $d\mu(A)$ by factoring out the volume of gauge transformations. Formally

$$d\mu(A) = \prod_x dA_z(x) dA_{\bar{z}}(x)$$  \hspace{1cm} (3.3)$$

More properly (3.3) should be understood with appropriate regularization.

We first express the measure $d\mu(C)$ in the inner product (3.2) in terms of $H$. Using the definition of $A_z$, $A_{\bar{z}}$, we have

$$ds^2 = 8 \int Tr[D_z(\delta MM^{-1})D_{\bar{z}}(M^{\dagger -1}\delta M^{\dagger})]$$  \hspace{1cm} (3.4)$$

where $D_i \phi = \partial_i \phi + [A_i, \phi]$. The metric for $SL(N, \mathbb{C})$-matrices can be written as

$$ds^2 = 8 \int Tr[(\delta MM^{-1})(M^{\dagger -1}\delta M^{\dagger})]$$

$$= 2 \int \delta \bar{\theta}(R^*KR^T)\delta \theta + \delta \theta(RK^TR^{*T})\delta \bar{\theta}$$  \hspace{1cm} (3.5)$$

The Haar measure $d\mu(M, M^{\dagger})$ is the volume associated with this metric. Explicitly, $d\mu(M, M^{\dagger}) = (\det RR^*)[\delta \theta \delta \bar{\theta}]$. From eqs.(3.4, 3.5) we see immediately that [8,9,10]

$$d\mu(A) = [dA_z dA_{\bar{z}}] = (\det D_z D_{\bar{z}})d\mu(M, M^{\dagger})$$  \hspace{1cm} (3.6)$$

$D_z$ and $D_{\bar{z}}$ are chiral covariant derivatives in two dimensions and their determinants are well known. The gauge-invariant evaluation of these determinants gives [8-11]

$$\det(D_z D_{\bar{z}}) = \exp[2c_A S(H)] \sigma$$

$$\sigma = \left[ \frac{\det'(\partial \bar{\theta})}{\int d^2x} \right]^{\dim G}$$  \hspace{1cm} (3.7)$$
where $c_A$ is the quadratic Casimir of the adjoint representation, viz., $c_A \delta_{ab} = f_{amn} f_{bmn}$; $\dim G = (N^2 - 1)$ is the dimension of the gauge group $SU(N)$. The prime on the determinant of $(\partial \bar{\partial})$ and the division by the volume of space denote the factoring out of the constant or zero mode of $(\partial \bar{\partial})$. $S(H)$ is the Wess-Zumino-Witten (WZW) action for $H$.

$$S(H) = \frac{1}{2\pi} \int \text{Tr}(\partial H \bar{\partial} H^{-1}) + \frac{i}{12\pi} \int \epsilon^{\mu \nu \alpha} \text{Tr}(H^{-1} \partial_\mu H \bar{\partial}_\nu H^{-1} \partial_\alpha H)$$

As is usual for the WZW action, we write the second term in $S$ in terms of an extension of $H$ into a three-dimensional space with space as the boundary. Actually, for hermitian matrices such as $H$, this is unnecessary and the second term can be integrated and written as an integral over space only [9,12].

Consider now the reduction of $d\mu(M, M^\dagger)$ by factoring out the volume of gauge transformations. Since gauge transformations act as $M \rightarrow gM$, $g \in G$, the physical degrees of freedom belong to $G^C/G$ and we need to construct the volume element on $G^C/G$. The metric obtained from (3.5) by factoring out gauge transformations is

$$ds^2 = 2 \int \text{Tr}(H^{-1} \delta H)^2$$

The volume corresponding to this is given by

$$d\mu(H) = \det r[\delta \varphi]$$

A simple way to see this result is as follows. $\rho \equiv ((M^\dagger)^{-1}dM^\dagger + dMM^{-1})$ is a differential form on $G^C$ which transforms as $\rho \rightarrow g\rho g^{-1}$ under $M \rightarrow gM$. Thus $\text{Tr}(\rho^n) = \text{Tr}(H^{-1}dH)^n$ are differential forms on $G^C/G$. The volume element is given by the differential form of maximal degree, i.e., for $n = (N^2 - 1)$. This is easily seen to be $\det r[\delta \varphi]$. For matrices which are functions of the spatial coordinates, as in our case, we have the product over the spatial points as well [9,10]; this is understood in eq.(3.10).

Using eqs.(3.6-3.10), we can write the inner product as

$$\langle 1|2 \rangle = \int d\mu(C) \Psi_1^\dagger \Psi_2 = \int \sigma \ d\mu(H) e^{2c_A S(H)} \Psi_1^\dagger \Psi_2$$
With the inner product (3.11) and the representation (2.19), the expression for the kinetic energy \( T \) can now be obtained as follows. We can write

\[
\langle 1|T|2 \rangle = e^2 \int d\mu(C) \left[ (E^a \Psi_1)^*(E^a \Psi_2) + (E^a \Psi_1)^*(E^a \Psi_2) \right]
\]

(3.12)

\[
= e^2 \int \sigma d\mu(H) e^{2cA(T)} \left[ (Gp_a \Psi_1)^* K_{ab} (Gp_b \Psi_2) + (\bar{G} \bar{p}_a \Psi_1)^* K^T_{ab} (\bar{G} \bar{p}_b \Psi_2) \right]
\]

where we have used the change of variables procedure as in (2.17) and \( (Gp_a)(x) = \int_y G(x,y)p_a(y) \). We do a partial integration using the easily verified property

\[
\frac{\delta}{\delta \varphi_b} [(r^{*^{-1}} \det r^*) f] = r^{*^{-1}} (\det r^*) \frac{\delta}{\delta \varphi_b} f.
\]

(3.13)

This gives

\[
\langle 1|T|2 \rangle = e^2 \int \sigma d\mu(H) \Psi_1^* [(\bar{G} \bar{p}_a)(K_{ab} e^{2cA S} Gp_b) + (Gp_a) K^T_{ab} e^{2cA S} (\bar{G} \bar{p}_b)] \Psi_2
\]

(3.14)

We can thus identify the operator \( T \) as

\[
T = e^2 \int \sigma d\mu(H) \left[ (\bar{G} \bar{p}_a)(K_{ab} e^{2cA S} Gp_b) + (Gp_a) K^T_{ab} e^{2cA S} (\bar{G} \bar{p}_b) \right]
\]

(3.15)

\( T \) is of course self-adjoint by construction. Notice that if we write \( T = -\frac{\xi^2}{2} \Delta \), then \( \Delta \) has the form of a Laplacian. The above method is standard for working out the form of the kinetic energy under change of variables. It has been used for collective coordinates and also in the context of \( A_0 = 0 \) gauge quantization of a gauge theory (with parametrizations of the fields different from ours) [6]. The form of the metric is really not explicitly used in this derivation; given the Jacobian and the change of variables formula (2.17), eq.(3.15) follows. If required, the inverse of the metric can actually be read off from this equation by comparison with the standard form of the Laplacian.

Given the inner product (3.11), we see that it is convenient to define the wavefunctions

\[
\Phi = e^{cA S} \Psi
\]

(3.16)

The inner product is now

\[
\langle 1|2 \rangle = \int \sigma d\mu(H) \Phi_1^* \Phi_2
\]

(3.17)
and the Hamiltonian can be written as

\[ H = \frac{e^2}{4} \int (P_a^\dagger K_{ab} P_b + Q_a^\dagger K_{ab}^T Q_b) + V \]  

(3.18)

where

\[ P_a = \int G(p_a - c_A p_a S) \]

\[ P_a^\dagger = \int \bar{G}(\bar{p}_a + c_A \bar{p}_a S) \]

\[ Q_a = \int \bar{G}(\bar{p}_a - c_A \bar{p}_a S) \]

\[ Q_a^\dagger = \int G(p_a + c_A p_a S) \]

Explicitly

\[ p_a S = -\frac{i}{\pi} \text{Tr}[t_a \partial (H^{-1} \bar{\partial} H)] , \quad \bar{p}_a S = -\frac{i}{\pi} \text{Tr}[t_a \bar{\partial} (\partial H H^{-1})] \]  

(3.20)

One can expand \( H \) in terms of \( p_a, \bar{p}_a \) and reorder terms bringing the derivatives to the right end. In doing so we encounter the commutator \([ \bar{G} \bar{p}_a(x), K_{ab}(x) ]\) which is singular, since the operators are at the same point. This can be evaluated as follows. Keeping in mind the relation \( \bar{p}_a = K_{ab} p_b \) which is required by Gauss law,

\[ \int z [\bar{G}(x, z) \bar{p}_a(z), K_{ab}(x)] = \int z \bar{G}(x, z) [K_{ac}(z) p_c(z), K_{ab}(x)] \]

\[ = \int z \bar{G}(x, z) f_{cbm} K_{ac}(z) K_{am}(x) \Delta(x, z) \]

\[ = \int f_{cbm} \bar{G}(x, z) \{ \delta_{cm} - (K^T \partial K)_{mc}(x - z) + ... \} \Delta(x, z) \]

\[ = \frac{i c_A}{\pi} (H^{-1} \partial H)_b(x) \]

(3.21)

where \( \Delta(x, z) \) is a regularization of \( \delta^{(2)}(z - x) \); it can be considered as a narrow Gaussian function of \( (z - x) \). Since it has support only for \( z \approx x \), we have expanded \( K_{ac}(z) \) around \( x \). The result (3.21) then follows in the limit \( \Delta(z, x) \to \delta^{(2)}(z - x) \). This evaluation of the commutator can also be checked with a Pauli-Villars regulator for the Green’s functions in (3.18) or (3.21) as well as against the operator equations of motion. Using (3.20), we may write this result as

\[ \int \bar{G}(x, z) \bar{p}_a(z) K_{ab}(x) = \int K_{ab}(x) \bar{G}(x, z) \bar{p}_a(z) - 2 c_A K_{ab}(x) (G \bar{p}_a) S \]

(3.22)

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With this result and a similar one for \((Gp)_a K^T_{ab}\) we get for the kinetic energy operator

\[
T = \frac{e^2}{2} \int K_{ab} Q_a P_b
\]  

(3.23)

We have made transformations and reductions of variables in the Hamiltonian framework and since some of these involve singular products, a useful consistency check is the Poincaré invariance of the theory. We have explicitly checked that the generators

\[
\mathcal{P}^\mu = \int d^2 x \, T^{0\mu}
\]

\[
\mathcal{J}^{\mu\nu} = \int d^2 x \, (x^\mu T^{0\nu} - x^\nu T^{0\mu})
\]

(3.24)
do satisfy the Poincaré algebra, i.e.,

\[
[\mathcal{P}^\mu, \mathcal{P}^\nu] = 0
\]

\[
i[\mathcal{P}^\mu, \mathcal{J}^{\nu\lambda}] = g^{\mu\lambda} \mathcal{P}^\nu - g^{\mu\nu} \mathcal{P}^\lambda
\]

\[
i[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\kappa\lambda}] = g^{\nu\lambda} \mathcal{J}^{\mu\kappa} - g^{\mu\lambda} \mathcal{J}^{\nu\kappa} + g^{\mu\kappa} \mathcal{J}^{\nu\lambda} - g^{\nu\kappa} \mathcal{J}^{\mu\lambda}
\]

(3.25)

In our case, the densities for the energy and momentum are given by

\[
T^{00}(x) = \frac{e^2}{4} [P^\dagger_a(x) K_{ab}(x) P_b(x) + Q^\dagger_a(x) K_{ba}(x) Q_b(x)] + \frac{2}{e^2} \partial(H^{-1} \bar{\partial} H) a \partial(H^{-1} \bar{\partial} H) a
\]

\[
T^{0z}(x) = -\frac{1}{4} [\bar{\partial}(\partial H H^{-1}) a (P^\dagger_a + Q_a) + (P^\dagger_a + Q_a) \bar{\partial}(\partial H H^{-1}) a]
\]

\[
T^{0z}(x) = -\frac{1}{4} [\partial(H^{-1} \bar{\partial} H) a (P_a + Q^\dagger_a) + (P_a + Q^\dagger_a) \partial(H^{-1} \bar{\partial} H) a]
\]

(3.26)

It is straightforward to verify that \(\mathcal{P}^i\) and \(\mathcal{J}^{12}\) are the generators of translations and rotation respectively, namely,

\[
[\Phi(x), \mathcal{P}_i] = i \partial_i \Phi(x)
\]

\[
[\Phi(x), \mathcal{J}^{12}] = [S_\Phi + i(x_1 \partial_2 - x_2 \partial_1)] \Phi(x)
\]

(3.27)

where \(\Phi(x)\) denotes any function of the fields of the theory. \(S_\Phi\) is the spin carried by \(\Phi(x)\); for example \(S_{P^\dagger} = S_Q = 1\), \(S_P = S_{Q^\dagger} = -1\), \(S_{K_{ab}} = 0\).
Given eq.(3.27), along with the fact that $T^{00}$ has no explicit coordinate dependence, the other necessary and sufficient condition for Poincaré invariance is the Dirac-Schwinger condition

$$[T^{00}(x), T^{00}(y)] = i[T^{0i}(x) + T^{0i}(y)]\partial_i \delta(x - y)$$  (3.28)

We have explicitly checked eqs.(3.27,3.28) for the operators (3.24,3.26) of the reduced theory. The commutation rules for the operators $P_a, Q_a, P^\dagger_a, Q^\dagger_a$, useful for checking these equations, can be worked out from (2.11) and are the following.

$$[P_a(x), P_b(y)] = -f_{abc}G(x,y)\left(P_c(x) - P_c(y)\right)$$

$$[P^\dagger_a(x), P^\dagger_b(y)] = f_{abc}G(x,y)\left(P^\dagger_c(x) - P^\dagger_c(y)\right)$$

$$[P_a(x), P^\dagger_b(y)] = \frac{cA}{\pi}K_{ba}(x)\delta^{(2)}(x - y)$$

$$[Q_a(x), Q_b(y)] = f_{abc}G(x,y)\left(Q_c(x) - Q_c(y)\right)$$

$$[Q^\dagger_a(x), Q^\dagger_b(y)] = -f_{abc}G(x,y)\left(Q^\dagger_c(x) - Q^\dagger_c(y)\right)$$

$$[Q_a(x), Q^\dagger_b(y)] = \frac{cA}{\pi}K_{ab}(x)\delta^{(2)}(x - y)$$

$$[P_a(x), Q_b(y)] = [P^\dagger_a(x), Q^\dagger_b(y)] = 0$$

$$[P_a(x), Q^\dagger_b(y)] = -\frac{cA}{\pi}\delta_{ab}\partial_y G(x,y) + f_{abc}G(x,y)Q^\dagger_c(y) - f_{abc}G(x,y)P_c(x)$$

$$[P^\dagger_a(x), Q_b(y)] = \frac{cA}{\pi}\delta_{ab}\partial_y G(x,y) - f_{abc}\bar{G}(x,y)Q_c(y) + f_{abc}G(x,y)P^\dagger_c(x)$$  (3.29)

4. Eigenstates of the kinetic energy

The total volume of the space $C$ of gauge-invariant configurations is given by $V = \sigma Z$, where

$$Z = \int d\mu(H)e^{2c_A S}$$  (4.1)

$Z$ is the partition function on the Riemann sphere of a $G^C/G$-WZW model. This partition function is finite, modulo regularization of the Laplacian $\partial\bar{\partial}$, and can be written as [10]

$$Z = \left[(\det'\partial\bar{\partial})^{-\dim G}\right]$$  (4.2)
which gives \( V = 1 \). For an Abelian theory with \( c_A = 0 \), \( Z \) is infinite. (If we make a mode decomposition of \( H \) or \( \varphi^a \) over the eigenmodes of the Laplacian \( \partial \bar{\partial} \), the integration over the amplitude of each mode is finite for the non-Abelian case because of the exponential; the divergence arises from the infinity of modes and can be regularized by truncation to a finite number of modes. For the Abelian case, the integration for each mode is divergent.)

The finiteness of \( Z \), albeit modulo regularization, suggests that the Laplace operator \( \Delta \) on \( \mathcal{C} \) should have a mass gap. Thus the diagonalization of the kinetic energy operator \( T \) by itself could give us some insight into the origin of the mass gap. The potential term \( V \) involves more derivatives and inverse powers of \( e^2 \) and for states for which the momenta are small compared to \( e^2 c_A \), we may expect that it can be treated as a perturbation. Of course, neglecting gradient terms and making such an ‘ultralocal’ approximation [3] will be consistent only if we already have a mass term or, in our case, if the kinetic term by itself leads to a mass gap of order \( e^2 c_A \).

For the purpose of looking for eigenstates, it is actually convenient to go back to the wavefunctions \( \Psi \) rather than the redefined wavefunctions \( \Phi \). As an operator on \( \Psi \)'s, the kinetic energy can be written, taking account of (3.22), as

\[
T = \frac{e^2}{2} \int e^{-2c_A S} \bar{G} \bar{p}_a K_{ab} e^{2c_A S} (Gp_b) \quad (4.3)
\]

\[
= \frac{e^2}{2} \int K_{ab} (\bar{G} \bar{p}_a) (Gp_b) \quad (4.4)
\]

The inner product is given by (3.11). \( T \) is a manifestly positive operator and its ground state or vacuum state is defined by \( T |0\rangle = 0 \). The solution of this condition is \( \Psi_0 = \) constant, say \( \mathcal{N} \). This may seem somewhat trivial; the key point, however, is that this function is normalizable with appropriate regularizations. The normalization integral is given by \( \mathcal{V} \) in (4.1). We can choose \( \mathcal{N} \) to obtain \( \langle 0 | 0 \rangle = 1 \); with the regularization as in (4.1,4.2), \( \mathcal{N} = 1 \).

The matrix elements of the operators and the normalization integrals for wavefunctions are given by expectation values or correlation functions of the hermitian \( G^C/G \)-WZW model. These can be obtained by analytic continuation of the correlation functions of the corresponding unitary WZW model [9]. The \( G^C/G \)-analogues of the level number \( k \) and the renormalized level \( \kappa = k + c_A \) of the unitary WZW model are \(- (k + 2c_A)\) and
\(-(k + c_A) = -\kappa\). The correlators can be obtained by the continuation \(\kappa \to -\kappa\). In our case, since we have only \(e^{2c_A S}\), rather than \(e^{(k + 2c_A) S}\), we must also take \(k \to 0\). In the unitary case, correlators involving primary fields belonging to the nonintegrable representations of the current algebra vanish. The corresponding statement for the hermitian model is that such correlators become infinite or undefined [9,10]. In our case, since \(k \to 0\), only the identity and its current algebra descendents have well-defined correlators. (The divergence of correlators of other operators has to do with \(k \to 0\), not coincidence of arguments.) We must thus conclude that the wavefunctions for the higher states can be taken as functions of the current

\[
J_a = \frac{c_A}{\pi} (\partial H \ H^{-1})_a
\]  

(4.5)

The arguments given above are the conformal field-theoretic reason for the currents being the quantities of interest. The result, however, is not surprising since the Wilson loop operator can be written in terms of \(H\) as

\[
W(C) = \text{Tr} \ P \exp \left( - \oint_C dz \ \partial HH^{-1} \right)
\]  

(4.6)

The Wilson loop operators form a complete set and hence the currents \(J_a\) should suffice to generate the gauge-invariant states.

Consider now a wavefunction

\[
\Psi_a = \int J_a(x) f(x)
\]  

(4.7)

for some c-number function \(f(x)\). From (4.3) we get

\[
T \Psi_a = \frac{e^2}{2} \int e^{-2c_A S(G\bar{p}_m)K_{mn}e^{2c_A S(Gp_n)}} \Psi_a
\]

\[
= i \frac{e^2 c_A}{2\pi} \int_{z,x} e^{-2c_A S(G\bar{p}_m)(z)K_{mn}(z)e^{2c_A S}K_{an}(x)\delta(z-x)f(x)}
\]

\[
= i \frac{e^2 c_A}{2\pi} \int e^{-2c_A S(G\bar{p}_a)(x)e^{2c_A S}f(x)}
\]

\[
= \left( \frac{e^2 c_A}{2\pi} \right) \Psi_a = m \Psi_a
\]  

(4.8)

where \(m = e^2 c_A / 2\pi\). This shows that \(J_a\) effectively behaves as an eigenfunction of eigenvalue \(m\). The result (4.8) can also be obtained from the version (4.4) of \(T\) using the
relation

\[ [K_{mn}(z)\tilde{G}\tilde{p}_m)(z), K_{an}(x)]|_{z \to x} = -iJ_a(x) \quad (4.9) \]

Of course, \( J_a \) by itself is not an acceptable eigenfunction since it does not have invariance under \( H \to V \tilde{H} \tilde{V} \). We can construct eigenfunctions with this invariance by taking products of \( J \)'s. The state

\[ \Psi_2(J) = \int [J_a(x)J_a(y)f(x, y) + \frac{c_A}{\pi^2} \frac{\dim G}{(x-y)^2} f(x, y)] \quad (4.10) \]

is orthogonal to the ground state and obeys

\[ T \Psi_2(J) = 2m \Psi_2(J) \quad (4.11) \]

This state has invariance under \( H \to V \tilde{H} \tilde{V} \) for constant \( V, \tilde{V} \). Wavefunctions which have invariance for all (local) \( V(z), \tilde{V}(\bar{z}) \) can be obtained by taking appropriate limit; for example \( f(x, y) = \bar{\partial}_x \partial_y \delta(x-y)f(x) \) would lead to \( \bar{\partial}J_a(x) \partial J_a(y) \) which is invariant under \( H \to V(z)H \tilde{V}(\bar{z}) \). Given the result (4.8), it is clear that one can construct higher eigenfunctions by proper orthogonalizations. Parity transforms of these states can also be obtained by using \( \bar{J}_a = (c_A/\pi)(H^{-1} \bar{\partial}H)_a \). Each \( J_a \) carries a spin of +1, while \( \bar{J}_a \) has a spin of −1. The normalization integral for these states will involve the current correlators of the \( G^C/G \)-WZW model and hence we expect the states to be normalizable (with suitable regularization, as is standard for states in field theory).

The requirement that we have eigenstates does not constrain the functions \( f(x), f(x, y) \), apart from normalization; we have an eigenstate for any \( f \). Thus the states are infinitely degenerate. This degeneracy will be lifted by the inclusion of the potential term; the effects of the potential should be calculated using degenerate-state perturbation theory.

5. Discussion

We have used the parametrization of the gauge potential \( A_z = -\partial M M^{-1} \), where \( M \) is an \( SL(N, \mathbb{C}) \)-matrix, to work out the Hamiltonian formulation of an \( SU(N) \) gauge theory in two spatial dimensions. The matrix field \( M \) is not gauge-invariant, but the theory can be written in terms of \( H = M^\dagger M \) which is gauge-invariant. The hermitian matrix field \( H \) can thus be taken as the basic field variable of the theory. The simplicity of the
parametrization we use is in the fact that the relevant Jacobian for the change of variables is easily evaluated in terms of the WZW-action for the matrix field $H$. The volume of the configuration space is then given by the partition function for the two-dimensional WZW-theory for $H$. Within the standard requirements of regularization (such as limiting to a large but finite number of modes), the volume is then finite; this is in contrast to an Abelian theory where the volume so defined would be infinite. This ‘finiteness’ of the non-Abelian case is very suggestive and is presumably related to the mass gap, especially in view of intuitive arguments outlined in the introduction.

We have also obtained the Poincaré generators and done a direct check of Poincaré invariance within our parametrization. The kinetic energy term of the Hamiltonian is related to the Laplace operator on the configuration space. If the ‘finiteness’ of the volume of the configuration space is the reason for the mass gap, one would expect to see it already at the level of the spectrum of the kinetic term. The kinetic energy operator is especially simple in our parametrization of fields (see eq. (3.18) or (4.6)). The ground state wave function for the kinetic operator is given by a constant and is normalizable with the gauge-invariant measure. We have obtained some excited states which show a discrete spectrum, with a gap $m = (e^2 c_A / 2\pi)$. The excited states are infinitely degenerate, as should be expected, since they are eigenstates of only the kinetic operator; the degeneracy will be lifted by the inclusion of the potential term. Even though the Hamiltonian has a simple structure, clearly we do not yet have a systematic calculational scheme. The proper inclusion of the effects of the potential term and the full construction of the eigenstates are questions which have to be addressed before meaningful calculations can be attempted. These issues are under investigation.

We thank R. Jackiw, B. Sakita and E. Witten for useful discussions. Discussions with G. Alexanian, A. Kavalov, Chanju Kim and D. Minic are also gratefully acknowledged. Special thanks are due to S. Samuel for pointing out a regularization ambiguity in a previous version of this paper. This work was supported in part by the Department of Energy, grant numbers DE-FG02-90ER40542 and DE-FG02-91ER40651-Task B and the National Science Foundation, grant number PHY-9322591.
Appendix

In parametrizing the fields as in eq.(2.4) and using the gauge-invariant variable $H$, one of the potential difficulties that one may worry about is the question of coordinate singularities. The physical configuration space $C$ is the space of gauge potentials $A$ modulo $G_*$, the latter being the set of gauge transformations which go to the identity at spatial infinity. $A$, considered as a $G_*$-bundle over $C$ is nontrivial. Thus one cannot choose global sections; this is the well known Gribov problem [13]. The space $C$ is topologically and geometrically nontrivial.

The simplest way to see the nontriviality of $A$ as a $G_*$-bundle is to consider the homotopy groups. Since $A$ is homotopically trivial, if we show that $C$ has nontrivial homotopy groups, it is clear that $A$ cannot be written as a product $C \times G_*$. Since $\Pi_1(G_*) \approx \Pi_3(G) = \mathbb{Z}$, it is easily checked that $\Pi_2(C) = \mathbb{Z}$. This implies that there are noncontractible closed two-surfaces in $C$. This is the simplest obstruction to the triviality of the bundle [13]. The existence of topologically nontrivial structures in the configuration space implies that any gauge-invariant parametrization of the fields or choice of coordinates on $C$ will necessarily have coordinate singularities [14]. We may hope to gain some understanding of the importance of the coordinate singularities by studying configurations which form a noncontractible two-surface.

It is not too difficult to construct a set of configurations which form a noncontractible two-surface since they are related to the instanton of the four-dimensional gauge theory. This can be seen as follows. In addition to $\Pi_2(C)$ being nontrivial, the second cohomology group of $C$ is nontrivial as well. In other words, there is a closed but not exact two-form on $C$. In terms of the potentials, a representative of this can be written as

$$\Omega = \frac{1}{4\pi} \int \text{Tr}(\delta A \delta A) \quad (A.1)$$

The integral of $\Omega$ over the closed noncontractible two-surface in $C$ gives a winding number $\nu$ by $\int \Omega = 2\pi \nu$. The two-surface in $C$ along with the two-dimensional spatial manifold gives a four-dimensional space and $\nu$ is the instanton number on this space [15]. Specifically,

$$\nu = \frac{1}{8\pi^2} \int \text{Tr}(\tilde{F} \tilde{F}) \quad (A.2)$$
where $\tilde{F} = (d+\delta)\tilde{A} + \tilde{A}\tilde{A}$. Here $\tilde{A}$ is the four-dimensional gauge potential; it is constructed from the two-dimensional potential $A$ as $\tilde{A} = A + c_w dw + c_\bar{w} d\bar{w}$ and $\delta = dw \partial_w + d\bar{w} \partial_{\bar{w}}$. $c_w$, $c_\bar{w}$ are given in terms of $M$ and $M^\dagger$ by

$$c_w = -\partial_w M M^{-1}, \quad c_\bar{w} = (M^\dagger)^{-1} \partial_{\bar{w}} M^\dagger$$ \hspace{1cm} (A.3)

In terms of these variables

$$\Omega = \frac{1}{2\pi} \int \text{Tr} \left[ \partial (H^{-1}\bar{\partial} H) \delta (H^{-1}\bar{\partial} H) + \partial (H^{-1}\bar{\partial} H) \delta (H^{-1}\bar{\partial} H) \right]$$ \hspace{1cm} (A.4)

We can exploit the connection outlined above between the two-form $\Omega$ on $\mathbb{C}$ and the instanton number to construct an example of the noncontractible two-surface of configurations. The standard instanton in $\mathbb{R}^4$ [16] can be rewritten using complex coordinates and interpreting one pair of complex coordinates as internal coordinates, viz., as parametrizing the two-surface in $\mathbb{C}$, we can get a set of configurations of interest. Explicitly we have

$$H = \exp(2fJ^3) = \cosh 2f + J^3 \sinh 2f$$ \hspace{1cm} (A.5)

Here $J^3 = \sigma \cdot n$; $\sigma^a$, $a = 1, 2, 3$, are the Pauli matrices and the unit vector $n^a$ is given by

$$n^a = \frac{1}{(\bar{z}z + \bar{w}w)} (\bar{z}w + \bar{w}z, i(\bar{w}z - \bar{z}w), \bar{z}z - \bar{w}w)$$ \hspace{1cm} (A.6)

Also

$$f = \frac{1}{2} \log \left( \frac{\bar{z}z + \bar{w}w + \mu^2}{\bar{z}z + \bar{w}w} \right)$$ \hspace{1cm} (A.7)

$\mu$ is a scale parameter and $(w, \bar{w})$ parametrize the two-surface in $\mathbb{C}$. It is easily verified that $\nu = \int \Omega/2\pi$ is equal to 3 for this set of configurations and hence eq.(A.5) gives a noncontractible two-surface in $\mathbb{C}$. As $\bar{z}z \to \infty$, $H \to 1$. For almost all $w, \bar{w}$, $H$ is nonsingular; however, the configuration at $w = 0$ has a singularity at the spatial point $z = 0$. One can shift the position of this singularity by transformations of the type $H \to VH\bar{V}$, where $V$ is holomorphic in $z$. Nonsingular configurations are given by nonsingular formulae for $H$ in different coordinate patches with transition relations given by transformations of this type. Since the singularity in our example is at a point, viz., at $w = 0$, even if we simply use the formulae (A.5-7) with the coordinate singularity, the effect on the quantum wave
functions is minimal; one can see this explicitly by constructing wave functions $\psi(w, \bar{w})$ for the reduced set of configurations (A.5). We can also consider the effect on the vacuum wave function $\Phi_0$. The WZW-action is invariant under transformations of the type $H \to VH\bar{V}$ and we therefore do not expect any pathology for the wave function. Explicitly, for the set of configurations (A.5), the WZW-action is given by

$$S(H) = 5\mu^2 + 4w\bar{w} + \frac{3}{2} \mu^2 + 4w\bar{w} + \mu^2 \log \left[ \frac{\mu^2 + w\bar{w}}{w\bar{w}} \right]$$  \hspace{1cm} (A.8)

When $w \to 0$, $\Phi_0 = \exp(c_A S)$ vanishes as $(w\bar{w})^{3c_A}$. The coordinate singularity does not lead to difficulties, at least for this case, and a nonsingular description is not essential.

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