Charged 4D Einstein-Gauss-Bonnet Black Hole: Vacuum solutions, Cauchy Horizon, Thermodynamics

M. Bousder\textsuperscript{1*}, K. El Bourakadi\textsuperscript{2†} and M. Bennai\textsuperscript{1,2‡}

\textsuperscript{1}LPHE-MS Laboratory, Department of physics, Faculty of Science, Mohammed V University in Rabat, Rabat, Morocco
\textsuperscript{2}Physics and Quantum Technology team, LPMC, Ben M’sik Faculty of Sciences, Hassan II de Casablanca University, Morocco

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Abstract

In this paper, we investigate the four-dimensional Einstein-Gauss-Bonnet black hole. The thermodynamic variables and equations of state of black holes are obtained in terms of a new parameterization. We discuss a formulation of the van der Waals equation by studying the effects of the temperature on $P-V$ isotherms. We show the influence of the Cauchy horizon on the thermodynamic parameters. We prove by different methods, that the black hole entropy obey area law (plus logarithmic term that depends on the Gauss-Bonnet coupling $\alpha$). We propose a physical meaning for the logarithmic correction to the area law. This work can be extended to the extremal EGB black hole, in that case, we study the relationship between compressibility factor, specific heat and the coupling $\alpha$.

Keywords: Black hole, Entropy, Einstein-Gauss-Bonnet gravity, van der Waals fluid.

1 Introduction

The study of Einstein-Gauss-Bonnet gravity (EGB) is very important as it offers a more comprehensive set up to explore many gravity-related conceptual problems. EGB gravity
is a major, higher dimensional generalization of Einstein gravity. Lovelock’s theorem \cite{1}, suggests that Einstein’s general relativity is a theory of gravity that respects various aspects such as spacetime is 4–dimensional. Recently, Glavan and Lin in Ref. \cite{2}, introduced a general covariant modified theory of gravity in 4–spacetime dimensions which propagates only the massless graviton and also bypasses Lovelock’s theorem. The case of 4–dimensional EGB gravity is noteworthy because the Euler-Gauss Bonnet term becomes a topological invariant, whereby the equations of motion and the gravitational dynamics are not affected. The intriguing idea of D. Glavan and C. Lin was to multiply the GB term by the factor \(1/(D-4)\) before taking the limit \(D \to 4\). This technique offers a new 4\(D\) gravitational theory with only two dynamical degrees of freedom \cite{3}. For example, it might solve the singularity problem of black holes \cite{4}. Indeed, by considering 4\(D\) EGB gravity in the static and spherically symmetric black holes, Boulware and Deser \cite{5,6} obtained the first black hole solution in the 5\(D\) EGB gravity, and since then steady attentions have been devoted to black hole solutions, including their formation, stability, and thermodynamics \cite{7}. The shadows of spherically symmetric \cite{8,9}, spinning \cite{10,11}, and charged EGB black holes in AdS space \cite{12,13}, and other works devoted to novel 4\(D\) EGB black holes have been published. However, it was shown in several papers that perhaps the idea of the limit \(D \to 4\) is not clearly defined, and several ideas have been proposed to remedy this inconsistency, as well as the absence of proper action \cite{14,15,16,17}. There are, indeed, some correct limits or procedures, that can lead to the same black hole solutions as naive 4\(D\) Gauss-Bonnet gravity, and different constructions were proposed, \cite{18,19}. In \(D \geq 5\) spacetime dimensions, BH-solutions were obtained for vacuum \cite{20}, anti-de Sitter (AdS) spaces \cite{21}, and attempts to build rotating solutions have taken place \cite{22,23} in the context of Einstein Gauss-Bonnet model \cite{24}. This paper aims to show the influence of the Cauchy horizon by presenting the 4\(D\) EGB black hole solution in terms of new parameterization. We use the unit system where \(c = G_N = 4\pi\epsilon_0 = \hbar = k_B = 1\).

This work is organized as follows. In Section 2, we study the solution for the charged EGB black hole in maximally symmetric vacuum and for a free photon orbiting around along a null geodesic. Section 3, discusses the black-hole equation of state and deals with the thermodynamic parameters, starting with the van der Waals equation to the black hole first law. In section 4, the discussion is extended to Extremal EGB black hole and we introduce the compressibility for the extremal case. We conclude in the final section.
2 Charged Einstein-Gauss-Bonnet black hole

2.1 Event horizon and Cauchy horizon

Consider now the charged Einstein-Gauss-Bonnet theory in D-dimensions with a negative cosmological constant \[ \Lambda = -\frac{(D-1)(D-2)}{2l^2}, \] (2.3)

by solving the field equation we use the following spherically symmetric,

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2, \] (2.4)

where \[ d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \] denotes the line element of the unit 3–sphere. Taking the limit \[ D \rightarrow 4, \] we obtain the exact solution in closed form

\[ f(r) = -g_{00} = 1 + \frac{r^2}{2\alpha} \left( 1 \pm \sqrt{1 + 4\alpha \left( \frac{2M}{r^3} - \frac{Q^2}{r^4} - \frac{1}{l^2} \right)} \right), \] (2.5)

where \[ 2M \] is the Schwarzschild radius. This last solution could be obtained directly from the derivation done in [26]. The extremal case correspond to \[ f(r_+) = 0. \] In the limit \[ \alpha \rightarrow 0, \] we can only recover the Reissner-Nordström-AdS solution. In the limit \[ r \rightarrow \infty \] with vanishing black hole charge, we asymptotically obtain the GR Schwarzschild solution. The event horizon in spacetime can be located by solving the metric equation: \[ f(r) = 0. \] The solutions show that the event horizon is located at:

\[ r_\pm = M \pm \sqrt{M^2 - Q^2 - \alpha}. \] (2.6)

We notice that the solution behaves like the Reissner-Nordström (RN) solution. The black hole event horizon is the largest root of the equation above, \[ r_+ \] is the black hole horizon radius [2]. However, the radius \[ r_- \] represents the Cauchy horizon radius. To explain the presence of two horizon \[ r_\pm, \] the mass of black holes can be rewritten as

\[ M^2 = \Gamma + Q^2 + \alpha, \] (2.7)
where \( \Gamma = \frac{(r_+ - r_-)^2}{2} \geq 0 \), when \( \Gamma = 0 \), we get an extremal EGB black hole. For a non-charged EGB black hole, we find \( M = \frac{(r_+ - r_-)}{\sqrt{2}} \). When a black hole has a horizon \( r_+ > r_- \), the black hole is locally stable. Most papers on EGB black hole do not indicate the importance of the Cauchy horizon \( r_- \). Our aim in this paper is show the influence of the Cauchy horizon radius \( r_- \) on the black hole thermodynamics, for that, it is necessary to study the term \( \Gamma \).

### 2.2 Maximally symmetric vacuum solutions

We choose the gauge field, the electric potentials \( \Phi_E \) arising from the black hole charge \( Q \), at the horizon \( r_+ \) given by

\[
A_\mu dx^\mu = \Phi_E(r) \, dt \quad \text{and} \quad \Phi_E(r) = Q/r.
\]

(2.8)

Note that while the scalar field possesses a harmonic time dependence, the gauge and metric fields are static and the energy-momentum tensor will be static and spherically symmetric. The event horizon satisfies the inequality \( r_+ \geq M \), which implies \( \Phi_E(r_+) \leq \Phi_E(M) \). The cosmological constant is considered to be dynamical, giving pressure [27].

We define the pressure [28] of the cosmological constant Eq.(2.3) for \( D \rightarrow 4 \)

\[
P = -M_p^2 \Lambda \quad \text{or} \quad 8\pi P = 3l^{-2} = -\Lambda,
\]

(2.9)

where \( M_p^2 = \frac{\alpha}{8\pi G} \approx 2 \times 10^{18} [\text{GeV}] \equiv 1 \). In the limit of vanishing mass and charge one obtains two AdS solutions with effective cosmological constants. The metric function can be rewritten as

\[
f(\chi) = 1 + \chi \pm \sqrt{(1 + \Phi_\Lambda) \chi^2 - (4\Phi_G + 2\Phi_E^2) \chi},
\]

(2.10)

where \( \Phi_G = -M/r \) is the gravitational potential at a distance. On this basis, we introduce a new order parameter

\[
\chi(r) = \frac{r^2}{2\alpha} \quad (\text{ex: } \Gamma = \alpha \chi(r_+ - r_-)),
\]

which makes it easier to calculate the thermodynamics of black holes. We introduce a new potential

\[
\Phi_\Lambda = \frac{4\alpha}{3} \Lambda.
\]

(2.11)

When \( \Phi_\Lambda = -1 \), the vacuum energy becomes zero. Indeed, we notice that \( \Phi_\Lambda = -2/\chi(l) \) describes the EGB vacuum. Later we will write all the formulas in term of \( (\chi, \Phi_\Lambda) \). For AdS space, when increasing \( \alpha \Lambda \) we expect that the corresponding \( \Phi_\Lambda \) should decrease.

We notice that, in the limit \( \alpha \rightarrow 0 \) or GR limit \( (\Phi_G = \Phi_E = 0) \), there exist two vacuum solutions:

\[
f_-(r) = 1 + \frac{\chi(r)}{\chi(l)},
\]

(2.12)
\[ f_+(r) = 1 - \frac{\chi(r)}{\chi(\ell)} + 2\chi(r). \] 
(2.13)
Thus, the \( f_- \) branch corresponds to the standard solution, whereas the \( f_+ \) branch reduces to the ds/AdS with an additional term \((2\chi(r))\). The metric reduce to the Reissner–Nordström black hole solution. In maximally symmetric vacuum solutions, there are two branches of solutions for the effective cosmological constant \[ \Lambda_{\text{eff}}^\pm = 2\Lambda_{\text{1}} \pm \sqrt{1 + \frac{4\alpha}{3} \Lambda}, \] 
(2.14)
If \( \alpha < 0 \) the solution is still an AdS space, if \( \alpha > 0 \) the solution is a de Sitter space, \[4\]. By evaluating the solution Eq.(2.14), we get
\[ \Phi_{\text{eff}}^\pm = \frac{2\Phi_{\Lambda}}{1 \pm \sqrt{1 + \Phi_{\Lambda}}}, \] 
(2.15)
these two solutions coincide when \( \Phi_{\Lambda} = -1 \). The potential of the cosmological constant must obey \( \Phi_{\Lambda} \geq -1 \). Therefore, the Gauss-Bonnet coupling develops a minimum \( \alpha_{\text{min}} = -3/4\Lambda \) at \( \Phi_{\Lambda_{\text{min}}} = -1 \). In the absence of a cosmological constant or \( \Phi_{\Lambda} = 0 \), one obtains the event horizon of EGB black hole Eq.(2.6).

### 2.3 Free photon and null geodesics solutions

Next, we analyze the evolution of a free photon orbiting around the equatorial orbit of the black hole along a null geodesic, by the affine parameter \( \lambda \). The photon Lagrangian for this system can be expressed as \[29\]
\[ \mathcal{L} = \frac{1}{2} \left[ -f(r)\dot{t}^2 + \frac{1}{f(r)}\dot{r}^2 + r^2\dot{\varphi}^2 \right], \] 
(2.16)
with \( \dot{r} = \partial r/\partial \lambda \). We get the generalized momenta \( p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu \), \( \mu = 0, 1, 2, 3 \). We obtain the energy \( E(=\text{constant}) \) and orbital angular momentum \( L(=\text{constant}) \) of the photon, which reads
\[ E = -p_t = f(r)\dot{t}^2 \text{ and } L = p_\varphi = r^2\dot{\varphi}, \] 
(2.17)
with \( p_r = \dot{r}/f(r) \). The Hamiltonian the moving free photon can be expressed as
\[ \mathcal{H} = 2(p_\mu\dot{x}^\mu - \mathcal{L}) = -E\dot{t} + \frac{\dot{r}^2}{f(r)} + L\dot{\varphi} = 0. \] 
(2.18)
Solving Eqs.(2.17), we easily get the equation of radial motion \( \dot{r}^2 = -V_{\text{eff}}, \) where the effective potential in terms of \( \chi \)-parameterization is
\[ V_{\text{eff}}^\pm/V_0 = f(\chi)/\chi - E^2/V_0, \] 
(2.19)
where \( V_0 = L^2/2\alpha \). We analyse the behaviour of the effective potential vs the parameter \( \chi \) plots shown in Fig.1

Figure 1: Figure showing how the effective potential is plotted as a function of \( \chi \) described by Eq.(2.19). (a) and (b) denotes the (+) and (-) part of \( f(\chi) \) Eq.(2.10), respectively. Parameters are chosen as \( \Phi_\Lambda = 0, 25 \) and \( k = (4\Phi_G + 2\Phi_E^2) \).

When the angular momentum of the free photon gets bigger in comparison to its energy, the effects of \( \alpha \) and \( Q \) become weak \[12\]. The circular unstable photon sphere satisfies the condition

\[
V_{\text{eff}} = \partial_\chi V_{\text{eff}} = 0 \quad \text{and} \quad \partial_\chi^2 V_{\text{eff}} < 0, \tag{2.20}
\]

where \( \partial_\chi = \partial/\partial \chi \). It is immediately clear that the radius of the photon sphere \( r_{ps} \) can be derived he above conditions by solving \( \partial_\chi V_{\text{eff}} = 0 \), one immediately obtains

\[
2f(\chi_{ps}) - \sqrt{2\alpha \chi_{ps}} \partial_\chi f(\chi_{ps}) = 0, \quad \text{where} \quad \chi_{ps} = r_{ps}^2/2\alpha.
\]

Secondly, solving \( V_{\text{eff}} = 0 \) for a spherically symmetric black hole, the shadow radius is \( R_S = \sqrt{2\alpha \chi_{ps}/f(\chi_{ps})} \). From Eq.(2.10), we can obtain the following expression

\[
R_S = \sqrt{2\alpha} \left( \frac{1 + \chi_{ps}}{\chi_{ps}} \pm \sqrt{1 + \Phi_\Lambda - \frac{2}{\chi_{ps}} \left( 2\Phi_G + \Phi_E^2 \right)} \right)^{-1/2}, \tag{2.21}
\]

As an example, if \( \Phi_\Lambda = -1 \) and \( 2\Phi_G = -\Phi_E^2 \), in the GR limit \((\alpha \to 0)\), the radius of the shadow can be very well approximated by \( R_S \sim \sqrt{2\alpha} \), that satisfies \( \chi(R_S) = 1 \). Within this limit, the surface of the shadow becomes

\[
A_S \sim 8\pi \alpha. \tag{2.22}
\]

To reach a maximum value of the shadow surface, using Eq.(2.6), we find \( A_{S_{\text{max}}} = 8\pi (M^2 - Q^2) \), which means that for \( M = Q \), we obtain a black hole without shadow. In the following, we would like to express logarithmic term of EGB entropy by \( A_S \).
3 Thermodynamics of 4D EGB Black hole

3.1 The van der Waals equation

We first compute the mass, temperature and entropy of the EGB black hole [31], in order to analyse the validity of the second law of black hole thermodynamics. We can express the ADM mass

\[ M = \frac{l^{-2}r_+^4 + r_+^2 + Q^2 + \alpha}{2r_+} \quad (3.1) \]

resulting in

\[ M = \frac{l^{-2}r_+^4 + r_+^2 + Q^2 + \alpha}{2r_+} \]

The temperature of the black hole corresponds to the tangent of the Newton’s potential at the event horizon [31]. The Hawking temperature

\[ T = \beta^{-1} \]

of the black hole can be calculated as [4]

\[ T = \frac{1}{4\pi} f'(r_+) = \frac{3l^{-2}r_+^4 + r_+^2 - Q^2 - \alpha}{8\pi \alpha r_+ + 4\pi r_+^3}. \quad (3.2) \]

One recovers the temperature of the Schwarzschild black hole in the limit of \( \alpha = 0, Q = 0 \) and \( l^{-2} = 0 \), in which case \( T = 1/4\pi r_+ \) as expected. For \( Q > 0, T > T_{Sch} \), where \( T_{Sch} \) is the temperature of Schwarzschild black hole. On the contrary, at \( Q < 0, T < T_{Sch} \). We can express the Eq.(3.2) as

\[ 1 + \chi + \frac{1}{\upsilon} T = \left( P + \frac{1}{16\pi \alpha \chi^+} \right) + \frac{\Gamma - M^2}{4\pi \alpha \chi^+ \upsilon^2}. \quad (3.3) \]

where \( \chi^+ = \chi(r_+) = A/8\pi \alpha \). We introduce the thermodynamic volume

\[ V = \frac{4\pi r_+^3}{3} \]

We use the specific volume

\[ \upsilon = 2r_+ l_P^2 \equiv 2r_+ = 6V/N > 0 \]

where \( l_P = \sqrt{\hbar G/c^3} \equiv 1 \) is the Planck length and \( N = A/l_P^2 \) is the number of states associated to the horizon [32]. Considering the transformations [33], the van der Waals equation can be obtained as

\[ P_{\chi} = P + \frac{1}{16\pi \alpha \chi^+}, \quad a_{\chi} = \frac{M^2 - \Gamma}{4\pi \alpha \chi^+} \quad \text{and} \quad b_{\chi} = \frac{\upsilon}{1 + \chi^+}. \quad (3.4) \]

The change of the black hole parameters are \( (P_{\chi}, a_{\chi}, b_{\chi}) \). In this case, the equation of state takes the form

\[ \left( P_{\chi} + \frac{a_{\chi}}{\upsilon^2} \right) (\upsilon - b_{\chi}) = T, \quad (3.5) \]

where the parameter \( a_{\chi} \) is the average attraction, which measures the attraction between particles. The parameter \( b_{\chi} \) corresponds to the volume of fluid of particles. From the last equation, by taking \( \Gamma = M^2 \) and \( \chi^+ \sim 0 \), i.e. non-charged EGB black hole, we can easily recover the ideal gas equation \( P_{\chi}^2 = T \). We notice that there is an appearance of term \( \Gamma \) in \( a_{\chi} \). If \( \Gamma \) is very low, the attraction between particles will be very large. One concludes that \( \Gamma \) maps the interacting gas into an ideal gas of point particles. The
attraction corresponds to $a_\chi > 0$ i.e. $\Gamma < M^2$. To show the effects of the temperature on $P_\chi - \upsilon$ isotherms of the van der Waals equation of state, we have plotted Fig.2.

For the choice of the transformations

$$P_\chi = P + \frac{1}{16\pi\alpha\chi_+}, \quad a_\chi = \frac{M^2 - \Gamma}{4\pi\alpha\chi_+} \quad \text{and} \quad T_\chi = \frac{1 + \chi_+}{\chi_+}T,$$

we get the equation of state

$$\left(P_\chi + \frac{a_\chi}{\upsilon^2}\right)\upsilon = T_\chi.$$  \hspace{1cm} (3.7)

It is interesting that the existence of a first-order phase transition between small and large black hole corresponding to a liquid and gas phase transition that automatically terminates at a critical point $(P_c, \upsilon_c, T_c)$ eventually satisfied $(\partial_\upsilon P_\chi)_{T\cdots} = (\partial^2_\upsilon P_\chi)_{T\cdots} = 0$.
Figure 3: $P_\chi - v$ isotherms of the van der Waals equation of state.

We shall now describe the Joule-Thomson effect, which is entirely related to the difference between a real gas and an ideal gas, especially the attraction and repulsion of the van der Waals forces. The Joule-Thomson coefficient $\mu$ is defined as

$$\mu = \left( \frac{\partial T}{\partial P} \right)_M .$$

For an ideal gas, $\mu$ is always equal to zero. The Joule-Thomson inversion temperature is the temperature for which the coefficient $\mu$ changes sign. It is easy to see that the system will experience a cooling (heating) process with $\mu > 0$ ($\mu < 0$), caused by the change in pressure is always negative during expansion. Let us consider the case Eq.(3.5):

$$\mu_\chi = \frac{\chi_+}{1 + \chi_+} v.$$  \hfill (3.9)

If $\mu_\chi > 0$, which means $\chi_+ > 0$ or $\chi_+ < -1$, then the gas temperature is below the inversion temperature. The interval $-1 < \chi_+ < 0$, implies that $\mu_\chi < 0$, then the gas temperature is above the inversion temperature. If we keep the specific volume fixed and vary the coefficient $\chi_+$ near $-1$ or $0$, then there is an inversion temperature similar to that of the phase transition. It can be explicitly verified that the inversion temperature corresponding to the evolution of the parameter $\chi_+$. Consider, for instance Eq.(3.7), it is immediately clear that $\mu_\chi = v > 0$. Then the gas temperature is above the inversion temperature. The important issue that requires attention is the role of $\chi_+$ appears in Eq.(3.5), and not in Eq.(3.7). For that, in what follows we will study only the case Eq.(3.5). To study the effect of $\chi_+$, we have plotted Fig.4.
To develop the average attraction $a_\chi$ of the new parametric expressions $(\chi_+, \Phi_\Lambda)$, it is helpful to first review the evolution of the thermodynamic variables, based on Ref.\[27\].

$$a_\chi/4 = \left( P_\chi + \frac{3}{4} \frac{\Phi_\Lambda}{8\pi\alpha} \right) (M^2 - \Gamma),$$  \hspace{1cm} (3.10)

This result is similar to that of the van der Waals equation. We imagine a scenario where the equation above represents a van der Waals equation. For this, one must have $8\pi\alpha \sim M^2$. Then we can impose a new temperature $T_a \propto a_\chi/4$, which leaves only.

Two-parameter equation of state induced by the transformations Eq.(3.4) of Ref.\[27\], which is the expression of van der Waals equation for the temperature $T_a$. The term $M^2$ can be shown to be a specific volume and $\Gamma$ can be a new volume excluded. Whereas, in asymptotic limit $4P_\chi \ll \frac{3\Phi_\Lambda}{8\pi\alpha}$, it is then clear that

$$\Gamma \sim M^2 - \frac{a_\chi}{4P_\chi}.$$  \hspace{1cm} (3.11)

**Solution without particles:** First, let us investigate the ADM mass Eq.(3.1) given by

$$\Gamma = \left( M - \frac{\nu^2}{2} \right)^2 - \frac{NP\nu^2}{6}.$$  \hspace{1cm} (3.12)

Indeed, if we compute $N = 0$ (black hole without particles), in the limit $\nu \ll 2M$, the ideal gas equation can be done by comparing Eqs.(3.11,3.12), at this point, we get $T_{\text{ideal}} \equiv a_\chi/4M \sim P_\chi\nu$. These results can be generalized to $N \neq 0$ and without asymptotic limit. Consequently, Eq.(3.10) take the form of the van der Waals equation. This analysis, leads us to find, $T_a \equiv a_\chi/4M$.  

Figure 4: $\mu_\chi/\sqrt{2\alpha}$ and $T_\chi/T$ vs $\chi_+$. 

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\[\text{Figure 4: $\mu_\chi/\sqrt{2\alpha}$ and $T_\chi/T$ vs $\chi_+$.}\]
3.2 The black hole first law

For EGB-AdS black hole, the mass of a black hole is more appropriately interpreted as enthalpy $H$. The black hole first law reads \[dM = \Phi_E dQ + A d\alpha + T dS + V dP,\] \hspace{1cm} (3.13)

where $A \equiv 4\pi r_+^2$ is the area of the event horizon of the black hole. The parameters $V$ and $A$ are the conjugate quantities of the pressure $P$ and GB coupling parameter $\alpha$, respectively \[34\]. If we fix $P$ and $\alpha$ Eq.(3.13), the Bekenstein-Hawking formula is then given by

$$S = \int \frac{dM}{T} = \frac{A}{4} + 2\pi \alpha \log \frac{A}{A_0},$$

where $A_0$ is some constant with units of area. An important justification for this new logarithmic term in Eq.(3.14), came from the Eq.(2.22), on using shadow surface, yields $2\pi \alpha = A_S/4$, which means that the logarithmic term (added by EGB gravity) depends on the shadow of black hole. We can express the EGB black hole entropy in terms of the shadow surface, one gets

$$S = \frac{A}{4} + \frac{A_S}{4} \log \frac{A}{A_0}.$$ \hspace{1cm} (3.15)

This investigation reveals a potential relationship between the entropy and the black hole shadow. These results can be generalized to the entropy of the 4D Hayward-EGB black hole \[35\]. Further in the GR limit ($\alpha \to 0$), we get the entropy obeying area law (ex: the Schwarzschild black hole) \[36\], the same if $A \to A_0$. Eq.(2.7) is based on the fact that, as thermodynamic systems, black holes must obey the first law of thermodynamics

$$dM = \frac{Q}{M} dQ + \frac{1}{2M} d\alpha + \frac{1}{2M} d\Gamma,$$ \hspace{1cm} (3.16)

where $\Phi_E (r_+) \leq \Phi_E (M)$. We now turn to determining the the entropy of the resulting black hole by another method. This can be done by comparing Eqs.(3.13,3.16), we find

$$dS = \frac{d\Gamma}{2MT},$$ \hspace{1cm} (3.17)

which results into the following relationship (3.11):

$$\frac{d\Gamma}{2MT} \sim \frac{dM}{T} - \frac{a_\alpha dP_\chi}{8P_\chi MT} + \frac{a_\alpha dP_\chi}{8P_\chi^2 MT},$$ \hspace{1cm} (3.18)

which turns into the expression of the entropy of 4D EGB black hole Eq.(3.14), when we fix $P_\chi$ and $a_\chi$ Eq.(3.4): $S \sim \int dM/T.$ This means that $\Gamma$ is a generator of the black hole entropy. Therefore, since $\Gamma$ depends on the Cauchy horizon radius, this shows that $r_-$ has a great effect on the thermodynamic parameters.
4 Extremal EGB black hole

We now describe as examples, the extremal EGB black hole. If we fix $P$ and $\alpha$ and by comparing Eq.(3.13) with Eq.(3.16), we indicate that

$$\Phi_E = \frac{Q}{M}, \quad \frac{dT}{2M} = T dS - P dV. \quad (4.1)$$

Next, we consider that $\alpha$ is fixed. As is well known that, the electric permittivity $\epsilon_S = (\partial Q/\partial \Phi_E)_S = M$ is relevant for stability in the grand canonical ensemble. Regarding this equation, we get an extremal black hole solution with degenerate horizon given by

$$r_+ = r_- = M = \sqrt{Q^2 + \alpha}. \quad (4.2)$$

In this case, the Hawking temperature reduces to

$$T = \frac{2P r_+^3}{2\alpha + r_+^2} \quad (4.3)$$

The entropy for this black hole can be obtained as

$$S = \int \frac{P}{T} dV = \frac{A}{4} + 2\pi \alpha \log \frac{A}{A_0} \sim 2\pi \alpha \log \left[\chi_+ \frac{1 + \chi_+}{\chi_0}\right], \quad (4.4)$$

where $A_0$ and $\chi_0$ are some constants with units of area. This expression coincides with the entropy formula in [34] (we have showed this entropy by a different method). $P$ and $\alpha$ are constant for an extremal black hole. This entropy led the study of black holes as a thermodynamical system. From Eq.(3.13), we get the equation of state for the extremal EGB black hole as follows

$$P_v = \frac{1 + \chi_+ T}{\chi_+}, \quad (4.5)$$

where $\chi_+ = \chi(r_+) \neq 0$. The extremal EGB black hole behaves like an ideal gas if $\chi_+ = c$. To determine the thermodynamic similarity class of different substances we use compressibility factor $Z = P_v/T > 0$ [37] in terms of the specific volume

$$Z = 1 + \frac{1}{\chi_+}, \quad (4.6)$$

we analyze in detail the thermodynamics and phase transitions for exact. The black holes with $\chi_+ < 0$, the compressibility factor is $Z < 1$, this shows that there is a great interaction between the gas particles. The pressures are lower, the particles are free to move. In this case, the attractive forces dominate. In GR limits ($\alpha = 0$): $\chi_+ \to \infty$, the compressibility factor is $Z = 1$, this shows that the gas behaves like an ideal gas. In this case, Eq.(4.5) represents the equation of state in the horizon. For $\chi_+ > 0$, the compressibility factor is $Z > 1$, the particles collide more often. This allows the repulsive
forces between molecules to have a noticeable effect. It is then clear that the relation between \( Z \) and \( \alpha \) is expressed as \( Z - 1 = 2\alpha/r_+^2 \), this means that the parameter \( \alpha \) increased the repulsion between the particles. Therefore, this \( \alpha \) maps the ideal gas into a noninteracting gas. From Eqs. (4.3), we calculate the specific heat of 4D extremal EGB black hole by

\[
C = \frac{\partial M}{\partial T} = \frac{3\Phi_\Lambda}{4\pi T} \frac{\chi_+^2}{3 + \chi_+}.
\]

The thermodynamical systems is the locally stable if \( C > 0 \) (or \( \chi_+ > -3 \)), and is unstable if \( C < 0 \) (or \( \chi_+ < -3 \)). Making use of the explicit case \( \chi_+ > -3 \), signifies that the extremal black holes are thermodynamically stable against perturbations in the region. For a thermodynamical systems, The behaviour of specific heat leads to find the regions of local and global stability of the 4D extremal EGB black hole. The diverging specific heat from \( C < 0 \) to \( C > 0 \), implies the existence of second-order phase transition.

5 Conclusion

First, we have considered the four-dimensional charged Einstein-Gauss-Bonnet theory. We also showed that the thermodynamic processes of black holes in AdS, can be modeled by the parameter \( \chi_+ \). The Joule-Thomson effect is entirely related to the difference between a real and an ideal gas, especially the attraction and repulsion of the van der Waals forces. To study the influence of the Cauchy horizon radius, we have introduced the parameter \( \Gamma \), which depends at the same time on the Cauchy horizon radius and the event horizon radius. In other words, the term \( \Gamma \) maps the interacting gas into an ideal gas of point particles. It makes the attraction decrease between particles. The results showed that all the thermodynamic quantities depend on the GB coupling parameter. The coexistence curve in the \( P_\chi - v \) diagram was shown, which is similar to the van der Waals fluid. After considering the GR limit for the evolution of a free photon orbiting along a null geodesic, we found the shadow surface. They satisfy the simple relation \( A_S \sim 8\pi\alpha \). Hence, we have obtained a relationship between the shadow surface and the black hole entropy. Finally, we investigated 4D extremal EGB black hole as a working substance and studied the entropy, the equation of state, compressibility and the diverging specific heat.
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