Long-lived Eccentric Modes in Circumbinary Disks

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Received 2020 August 1; revised 2020 November 2; accepted 2020 November 2; published 2020 December 18

Abstract

Hydrodynamical simulations show that circumbinary disks become eccentric, even when the binary is circular. Here we demonstrate that, in steady state, the disk’s eccentricity behaves as a long-lived free mode trapped by turning points that naturally arise from a continuously truncated density profile. Consequently, both the disk’s precession rate and eccentricity profile may be calculated via the simple linear theory for perturbed pressure-supported disks. By formulating and solving the linear theory, we find that (i) surprisingly, the precession rate is roughly determined by the binary’s quadrupole, even when the quadrupole is very weak relative to pressure; (ii) the eccentricity profile is largest near the inner edge of the disk and falls exponentially outward; and (iii) the results from linear theory indeed agree with what is found in simulations. Understanding the development of eccentric modes in circumbinary disks is a crucial first step for understanding the long-term (secular) exchange of eccentricity, angular momentum, and mass between the binary and the gas. Potential applications include the search for a characteristic kinematic signature in disks around candidate binaries and precession-induced modulation of accretion over long timescales.

Unified Astronomy Thesaurus concepts: Binary stars (154); Stellar accretion disks (1579); Protoplanetary disks (1300); Supermassive black holes (1663); Theoretical techniques (2093)

1. Introduction

Gas eccentricity is expected to grow in disks inside and around binaries (e.g., Lubow 1991; Whitehurst 1994; Paardekooper et al. 2008; Kley et al. 2008). In particular, 2D hydrodynamical simulations of circumbinary disks (CBDs) have consistently shown significant eccentricities (∼0.3), even when the binary is circular (MacFadyen & Milosavljević 2008; Miranda et al. 2017; Thun et al. 2017). Typically, the eccentricity is large near the circumbinary “cavity” and steeply declines outward. This overall behavior has been seen also in 3D magnetohydrodynamics simulations (Shi et al. 2012), which suggests that the growth of eccentricity is a robust property of disks around accreting binaries.

The mechanisms for eccentricity growth are thought to be either resonant excitation (Lubow 1991; Ogilvie 2007) or periodic “pumping” via oblique spiral shocks (Shi et al. 2012). The mechanisms for eccentricity damping are less constrained, although viscous damping (Goodchild & Ogilvie 2006), orbit crossing (Ogilvie 2001; Statler 2001), or combinations thereof are the likely processes behind eccentricity saturation. The nearly steady eccentricity profiles of some numerical simulations (Miranda et al. 2017) suggest that damping and excitation are in near equilibrium; if that is the case, eccentricity profiles can, in principle, be sustained through the lifetime of the disk.

In the past, the true longevity of eccentricity in simulations has been difficult to ascertain. CBD simulations are subject to slowly evolving transients that die out on the local viscous timescale. After that, and provided that a constant mass supply is available, a steady state can be achieved (Miranda et al. 2017; Muñoz et al. 2019; Moody et al. 2019), in which the CBD reaches a (quasi-)stationary density profile (see also Dempsey et al. 2020a). The transient phase exhibits a behavior that is not representative of that of the vastly longer disk lifetime, and it has been argued that many of the accepted outcomes of circumbinary accretion, such as binary migration being inward, have been inferred from simulations in the transient state (Muñoz et al. 2020). Getting past this transient phase is crucial for unveiling the true, long-term eccentricity profile of CBDs, since eccentric modes are sensitive to the background density profile (Teyssandier & Ogilvie 2016; Lee et al. 2019b).

Eccentricity longevity can be verified by comparing simulations to theoretical expectations. If disk eccentricities truly correspond to long-lived modes and endure over a large number of disk dynamical times, they will have a lasting impact on the binary–disk coupling, as the distribution of torques is bound to be different from that of circular disks, which is the standard assumption of the theory (Goldreich & Tremaine 1980; Artymowicz & Lubow 1994). Additional implications of long-lived eccentricities include the modification the processes taking place within the disk, such as planet formation (e.g., Silsbee & Rafikov 2015), and the use of kinematic observational signatures (e.g., Regály et al. 2011) as an independent diagnosis tool for disk structure.

In this work, we demonstrate that the CBD eccentricities seen in hydrodynamical simulations are consistent with long-lived normal modes. Through a comparison of linear analysis to hydrodynamical simulations in steady state, we confirm the agreement of the eigenfrequencies and eigenfunctions, finding that the circumbinary cavity size is crucial in determining the spatial extent and the precession rate of eccentricity profiles.

2. Linear Theory of Circumbinary Eccentricities

In simulations of CBDs, an initially circular disk becomes eccentric. Within a rather short time (∼1 viscous time at the CBD inner edge), the eccentricity profile saturates, and thereafter precesses uniformly (Miranda et al. 2017; Thun et al. 2017). The questions of how the eccentricity is excited and then saturates are difficult ones (e.g., Teyssandier & Ogilvie 2016). We hypothesize that the disk’s behavior in its saturated state is a normal mode of the disk and does not depend on how the eccentricity is excited or saturated. Such a
hypothesis is reasonable, provided that the mode’s precession rate is fast compared to excitation/saturation. In what follows, we calculate the disk’s eccentricity profile and precession rate in steady state. However, we do not address the amplitude of the mode, which requires one to consider excitation/saturation.

2.1. Basics of Linear Theory

The evolution of the complex eccentricity $E = ee^{i\omega}$, in a locally isothermal 2D disk of density profile $\Sigma$ and sound speed profile $c_s$, is governed by

$$2\Sigma R^2 \frac{\partial E}{\partial t} = \frac{i}{R} \frac{\partial}{\partial R} \left( \Sigma c_s^2 R \frac{\partial E}{\partial R} \right) + i R \frac{d}{dR} (\Sigma c_s^2) E$$

$$- \frac{i}{R} \frac{\partial}{\partial R} \left( \Sigma \frac{dc_s^2}{dR} R^2 E \right) + 2 \Sigma R^2 \frac{\partial E}{\partial t} \bigg|_{grav}$$

(Goodchild & Ogilvie 2006; Teyssandier & Ogilvie 2016; Lee et al. 2019a), with $\Omega(R)$ being the local orbital frequency, and where we have deliberately omitted terms due to excitation and damping (see Introduction). The first three terms on the right-hand side of Equation (1) are due to pressure (the third one being a consequence of a radially varying sound speed; Teyssandier & Ogilvie 2016). The last term is due to a non-Keplerian external potential (Goodchild & Ogilvie 2006). This term can be derived from the disturbing function (e.g., Mardling 2013), after ignoring high-frequency terms (i.e., under the secular approximation). For a circular binary of semimajor axis $a_b$ and mass ratio $q_b = m_b/m_1$, we have, to linear order in $E$ and second order in $a_b/R$,

$$\frac{\partial E}{\partial t} \bigg|_{grav} = i \Omega f_0(R) E,$$

where

$$f_0 = \frac{3}{4} \left( \frac{q_b}{1 + q_b} \right) \left( \frac{a_b}{R} \right)^2.$$

If the binary has a finite eccentricity $e_b$, then the right-hand side of Equation (2) includes an additional forcing term $i \Omega f_1(R) E_{b0}$, where $E_b = e_b e^{i\omega_b}$ is the binary’s complex eccentricity and $f_1$ is of third order in $a_b/R$. But throughout this paper, we consider only the case of circular binaries.

2.1.1. Disks with Central Cavities

We adopt a surface density profile representative of a CBD in viscous steady state (VSS; Muñoz & Lai 2016; Miranda et al. 2017; Muñoz et al. 2019)

$$\Sigma(R) = \left[ \Sigma_0 \left( \frac{R}{a_b} \right)^{-\gamma} \right]^{\frac{1}{2}} \left[ 1 - \frac{l_0}{\frac{3}{2} \left( \frac{a_b}{R} \right)^{\gamma}} \right] e^{-\left( \frac{R}{R_{cav}} \right)^{\gamma}}.$$

The first term in square brackets in Equation (4) is the steady-state solution of a zero-net torque disk with viscosity law $\nu \propto R^{1/2}$ (Lynden-Bell & Pringle 1974) and approximates the CBD solution far from the binary. The second term in square brackets is due to an inner “boundary effect” (Frank et al. 2002; Popham & Narayan 1991), which modifies the perfect power-law profile whenever the advection of angular momentum by the gas crossing the disk’s inner edge is not exactly balanced by outward viscous transport (see Equation (19) in Dempsey et al. 2020a). The quantity $l_0$, introduced by Miranda et al. (2017), is the net torque per unit accreted mass exerted by the CBD on the binary (see also Rafikov 2016), and $\Omega_b$ is the binary’s orbital frequency. The final exponential factor is the inner cutoff, with two adjustable parameters $\zeta$ and $R_{cav}$. In addition, we assume $\Omega = \Omega_b (R/a_b)^{-3/2}$ and that the disk’s aspect ratio ($h_0$) is constant, i.e., $c_s^2 = h_0^2 \Omega_b^2 (R/a_b)^{-1}$. The main adjustable parameters of the problem are $q_b$, $h_0$, and $R_{cav}$. The remaining parameters are fixed to $\zeta = 12$ (see Section 3.1 below) and, in the present section, to $l_0 = 0.75 h_0 a_b^2$, which is typical of simulations (Muñoz et al. 2020). A disk profile with fiducial parameters is depicted in the top panel of Figure 1.

2.2. Numerical Solution of the Boundary Value Problem

Inserting solutions of the form $E(t, R) = E(R)e^{i\omega t}$ into Equation (1) allows us to replace $\partial E/\partial t$ with $i\omega E$. Together with an appropriate boundary condition, the eccentricity equation defines a boundary value problem (BVP) with eigenvalue $\omega$. This BVP can be solved numerically (e.g., Lee et al. 2019a), and diverse numerical techniques exist for this purpose (e.g., Pryce 1993); in this work, we use a shooting method over a domain $R \in [R_{in}, R_{out}]$. At the boundaries, we impose the boundary condition $\frac{dE}{dR} = 0$, which is adequate for adiabatic perturbations. In general, the choice of boundary condition has an impact on the BVP eigenfunctions and eigenvalues, unless the modes are internally trapped in a resonant cavity away from the boundaries (see Section 2.3 below). For our particular disk model, the type of boundary condition is irrelevant. The computational domain extends from $R_{in} = 0.75 R_{cav}$ to $R_{out} = 450 a_b$. The location of the inner boundary is chosen to optimize convergence speed, but only after the results are

\[ \text{Figure 1. Top: surface density profile (Equation (4)) for the fiducial parameters used in this work. Bottom: numerically computed eccentricity eigenfunctions } |E| \text{ for different values of } q_b \text{ (increasing from lighter to darker blue). The dashed curve corresponds to the analytic estimate of } |E| \text{ (Equation (12)) evaluated at } q_b = 0.1. \]
checked to be independent of this choice. Robustness against the location of the computational boundary is possible only if the mode is trapped (see below), which is a distinct feature of our calculations. By contrast, if a density profile does not naturally trap modes (e.g., a power-law disk), then the eccentricity eigenfunctions and eigenfrequencies will depend sensitively on the location of the computational boundary (Miranda & Rafikov 2018).

In Figure 1 (bottom panel), we show the fundamental-mode eccentricity eigenfunctions $E$ for the values of $0.1 \leq q_b \leq 1$ typically explored in CBS simulations. These $E$ profiles peak within the cavity and extend out to several times the binary separation $a_b$. However, at distances $R \gtrsim 15a_b$, the eccentricity has decreased by 4 orders of magnitude, at which point the disk can be considered to be effectively circular.

In Figure 2, we show the eigenfrequencies $\omega_0$ of the fundamental mode as a function of $q_b$ for different values of $h_0$. These appear to roughly track the quadrupole precession frequency (e.g., Moriwaki & Nakagawa 2004) evaluated at $R_{\text{cav}}$:

$$\omega_0 \equiv \frac{3}{4} \frac{q_b}{(1 + q_b)^2} \left( \frac{R_{\text{cav}}}{a_b} \right)^2 = \frac{3}{4} \frac{q_b}{(1 + q_b)^2} \left( \frac{\Omega_{\text{cav}}}{a_b} \right)^2,$$

where $\Omega_{\text{cav}} = \Omega|_{R=R_{\text{cav}}}$ (MacFadyen & Milosavljević 2008). More precisely, $\omega_0$ is suppressed relative to $\omega_Q$ by a modest reduction factor that weakly depends on $h_0$. Surprisingly, $\omega_0$ continues to track $\omega_Q$ at very low $q_b$, where one might have naively expected pressure effects to dominate. We explore that behavior in more detail below.

Although there are three adjustable parameters ($q_b$, $h_0$, and $R_{\text{cav}}$), the nature of the solutions is determined by a single combination of them. Specifically, if we define the pressure-induced precession frequency evaluated at $R_{\text{cav}}$ via

$$\omega_P \equiv a_b^3 \Omega_{\text{cav}},$$

(Goldreich & Sari 2003; Lee et al. 2019a), then the determining parameter is $\omega_P/\omega_Q$, the ratio of the pressure-induced precession rate to the quadrupole rate, evaluated at $R_{\text{cav}}$. To show that, we solve the BVP for 1083 different sets of parameters, with $q_b \in [0.003, 1]$, $h_0 \in [0.01, 0.02, 0.05, 0.1, 0.2]$ and $R_{\text{cav}} \in \{2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0\}$ ($h_0 = 0.7a_b^3$ is held fixed). The resulting eigenfrequencies are depicted in Figure 3, where we show the normalized eigenfrequency $\omega_0/\omega_Q$ versus the ratio of characteristic frequencies $\omega_P/\omega_Q$ (colored squares). As can be seen from the figure, the solutions nearly collapse into a single line. The ratio $\omega_P/\omega_Q$ also dictates the shape of the $E$-eigenfunctions.

The eccentricity profiles associated with a subset of the 1083 BVPs of Figure 3 is shown in Figure 4 (left panels). When plotted as a function of $R/R_{\text{cav}}$, $E$-eigenfunctions with a given value of $\omega_P/\omega_Q$ line up with each other. When $\omega_P/\omega_Q \ll 1$ (top), the eccentricity profile is confined to the immediate vicinity of the circumbinary cavity; when $\omega_P/\omega_Q \gg 1$ (bottom), the eccentricity profile extends out to $R \gg R_{\text{cav}}$.

2.3. WKB Theory

2.3.1. Effective Potential

WKB theory provides further physical insight into the numerical results presented above. We introduce a rescaled eccentricity, $y$, defined via $E = y(\Sigma R^3)^{-1/2}$, which removes the first-order derivatives in the BVP (e.g., Gough 2007; Lanczos 2012) and results in

$$\frac{d^2y}{dR^2} + k^2y = 0,$$

where $\omega_{\text{pot}}(R)$ is an “effective potential” in units of frequency (e.g., Lee et al. 2019b). Except for the additional factor $2\Sigma c_s^2$, Equation (7) is the time-independent Schrödinger equation in 1D (Ogilvie 2008). For an axisymmetric density profile $\Sigma(R)$,

An exact collapse of the curves into a perfect line can be achieved by setting $l_0 = 0$ or by rescaling $l_0$ such that $l_0 R_{\text{cav}}^2 = \text{constant}$.\footnote{An exact collapse of the curves into a perfect line can be achieved by setting $l_0 = 0$ or by rescaling $l_0$ such that $l_0 R_{\text{cav}}^2 = \text{constant}$.}
The effective potential is

\[
\frac{\omega_p}{\omega_Q} \approx 0.05
\]

and

\[
\frac{\omega_p}{\omega_Q} \approx 50
\]

where primes denote radial derivatives. The first term on the right-hand side of Equation (8) is the “quadrupole contribution” (e.g., Moriwaki & Nakagawa 2004). The remaining terms are the “pressure contribution,” which depends on the profile but not its normalization.

The sign of \( \omega_{pot} \) helps us determine the sign of \( \omega \), i.e., whether modes are prograde or retrograde (e.g., Teyssandier & Ogilvie 2016). For instance, the quadrupole contribution is the precession rate of a test particle around a binary and is always positive. On the other hand, the pressure contribution is small and negative far from the cavity, as for pure power-law disks (e.g., Goldreich & Sari 2003; Miranda & Rafikov 2018), but it becomes positive near the cavity edge. As we show below, all of our modes are prograde, even when pressure dominates over the quadrupole.

When the circumbinary cavity is included, \( \omega_{pot} \) develops a local maximum, which is crucial for the existence of trapped modes. The function \( \omega_{pot} \) (depicted in Figure 5, left panel) exhibits a general shape that consists of a repulsive (\( \omega_{pot} \to -\infty \)) inner region and attractive (\( \omega_{pot} > 0 \)) “potential well,” which peaks at \( R_{peak} = gR_{cav} \) with \( g \) of order unity. The potential well can result in trapped modes (“bound states”) within that region provided that the peak is tall enough. The potential well is accompanied by two turning points, beyond which waves cannot propagate. In the “classically forbidden region” depicted in gray in Figure 5, the \( y \)-eigenfunctions are evanescent. Oscillatory solutions are allowed outside the “potential well,” but only in the form of traveling waves (“free states”) of negative frequency.

The trapping of the \( y \)-eigenfunctions can be seen in the right panels of Figure 4. These profiles are strongly peaked functions with exponential cutoffs due to the left and right turning points (when \( \omega = \omega_{pot} \)). The left exponential cutoff is largely independent of \( \omega_p/\omega_Q \); conversely, the right cutoff depends sensitively on \( \omega_p/\omega_Q \), with larger values producing more delocalized eigenfunctions.

2.3.2. Approximate Eigenfrequencies

The sharp local maximum in \( \omega_{pot} \) suggests that we can study eigenmodes trapped deeply into the potential well by expanding to high order in \( R \) and looking for known solutions of the time-independent Schrödinger equation. First, we write Equation (7) in “Liouville normal form” (Liouville 1837; Amrein et al. 2005) to eliminate the prefactor multiplying \( \omega \). With a change of variables...
\[ y = Y(R/R_{\text{cav}})^{1/8}, \quad \xi = (R/R_{\text{cav}})^{3/4}, \]

we have

\[ \frac{9\omega_p}{32} \frac{d^2 Y}{d\xi^2} + \left[ \omega_{\text{pot}}[R(\xi)] - \frac{7\omega_p}{128} \xi^2 \right] Y = \omega Y. \tag{9} \]

Second, we expand the pressure-dependent term in \( \omega_{\text{pot}} \) around the local maximum \( \xi_{\text{peak}} \) to second order in \( \xi \) and evaluate the remaining terms at \( \xi = \xi_{\text{peak}} \), i.e., the term in square brackets in Equation (9) is approximated by a quadratic potential. Therefore, the resulting expression can be cast into the standard Schrödinger equation for the quantum harmonic oscillator (QHO), and thus the eigenfrequencies in Equation (9) are given by

\[ \omega_n^{\text{QHO}} \approx 1.13\omega_Q + 26.1\omega_p - 130\omega_p \left( n + \frac{1}{2} \right) \tag{10} \]

(see Appendix B). Equation (10) is depicted in Figure 3 (thin black line) for \( n = 0 \). When \( \omega_p/\omega_Q \ll 1 \), the analytic \( \omega_n^{\text{QHO}} \) and the BVP frequencies show a moderate level of agreement, indicating that the modes are rarely deep enough into the potential well to be properly described with this local expansion.

2.3.3. WKB Eigenfrequencies from the Quantization Condition

The WKB method of elementary quantum mechanics can be directly applied to obtain the solutions to Equation (7), provided that \( k \) is sufficiently large compared to the length scale of variation of \( \omega_{\text{pot}} \). Trapped modes are those with discrete eigenvalues \( \omega_n \) that satisfy the Einstein–Brillouin–Keller quantization condition (Einstein 1917; Keller 1958), which is given by

\[ \oint k(\omega_n, R) dR = \left( 2n + \frac{\mu}{2} \right) \pi, \tag{11} \]

where \( k \) is from the dispersion relation (Equation (7)) and \( \mu \) is the amount of phase loss, sometimes called the Maslov index.\(^5\) In this case, \( \mu = 2 \), which is the number of classical turning points for a 1D potential like \( \omega_{\text{pot}} \) (Mark 1977; Shu et al. 1990; Lee et al. 2019a).

The behavior of a trapped mode is further illustrated in the dispersion relation map (DRM) of Figure 5 (right panel), which depicts constant-\( \omega \) contours of the dispersion relation (Equation (7)) for different values of \( R \) and \( kR \). Traveling waves propagate along open contours, while trapped modes (standing waves) trace closed loops. For the example of the figure, the quantization condition is satisfied by one and only one such closed loop (for \( n = 0 \), in red). The existence of just one trapped mode is a general property of CBDs with \( h_0 \sim 0.1 \), which is the typical aspect ratio of most CBD simulations (e.g., Farris et al. 2014; Miranda et al. 2017; Muñoz et al. 2019; Duffell et al. 2020). Colder disks \( (h_0 \lesssim 0.05) \), on the other hand, can support higher-\( n \) modes (Appendix A). This type of disk has been simulated recently (Thun et al. 2017; Tiede et al. 2020), but no multiharmonic disk eccentricity has been reported.

Figure 3 also includes the WKB frequencies as a function of \( \omega_p/\omega_Q \). The agreement with the BVP frequencies is excellent for \( \omega_p/\omega_Q < 1 \), but the two solutions quickly diverge when \( \omega_p/\omega_Q > 1 \), i.e., at low eigenfrequencies. To understand this discrepancy, we turn to studying the WKB eigenfunctions below.

2.3.4. Approximate Eccentricity Profiles

Far from the cavity, we can write the evanescent part of the \( y \)-eigenfunction using the WKB approximation

\(5\) The Maslov index corresponds to the number of turning points through a smooth potential (i.e., a soft reflection) plus twice the number of turning points under Dirichlet boundary conditions (i.e., a hard reflection).
y \sim \exp[-\int dR(-k^2)^{1/2}]. \quad \text{For } \omega \gg |\omega_{\text{post}}|, \text{ we obtain}

\[ E \sim R^{-5/4} \exp\left[-\frac{R^{3/4}}{\lambda}\right], \tag{12} \]

where

\[ \lambda = R_{\text{cav}} \left( \frac{9 \omega_{\text{P}}^2}{32 \omega_{0}} \right)^{2/3}, \tag{13} \]

i.e., the eccentricity profile is a tapered power law with tapering radius \( \lambda \) (see also Appendix C in Shi et al. 2012). A rough approximation of the quantization condition gives \( \lambda \sim R_{\text{cav}} (\omega_{\text{P}}/\omega_{0})^{2/3} [1 - 0.02(\omega_{0}/\omega_{\text{P}})] \) (Appendix C). Equation (12) is included in Figure 1 (for clarity, only for the \( q_{b} = 0.1 \) case). The tapering effect is a natural consequence of the \( \gamma \)-eigenfunction being a trapped mode. Note that, in Equation (13), \( \lambda \) is arbitrarily large for arbitrarily small \( \omega \), which is the usual behavior of marginally bound states in quantum mechanical potential wells. This “delocalization” of the eigenfunctions explains the radial extent of \( E \) and \( y \) in Figure 4, in turn explaining why WKB fails at low frequencies, since the spatial wavenumber is too small for the approximation to hold. As a consequence, when \( \omega_{\text{P}}/\omega_{0} \to \infty \) (i.e., when the quadrupole contribution is negligible), eigenfunctions are not trapped, even if the CBD is steeply truncated around the binary.

We may now also qualitatively understand the surprising result that \( \omega_{\text{P}} \sim \omega_{0} \) (Figures 2 and 3), even when pressure naively dominates over the quadrupole \( (\omega_{\text{P}}/\omega_{0} \gg 1) \). The reason behind this result is that, when \( \omega_{\text{P}}/\omega_{0} \) is large, the delocalized eigenfunction extends out to many times \( R_{\text{cav}} \), in which case the pressure-induced precession rate is much lower than the naive expectation \( \omega_{\text{P}} \sim h_{0}^{2}\Omega_{\text{cav}} \). Instead, the magnitude of the pressure-induced precession is set by the value of \( h_{0}^{2}\Omega_{\text{cav}} \) at \( R \gg R_{\text{cav}} \), where the eigenfunction is within a factor of a few of its peak value. At these distances, \( h_{0}^{2}\Omega_{\text{cav}} \) is much less than \( \omega_{\text{P}} \) and can never overcome quadrupole-induced precession.

3. Comparison to Hydrodynamical Simulations

3.1. Hydrodynamics of Circumbinary Accretion

We carry out 2D hydrodynamics simulations of steady-state CBDs using the moving-mesh code AREPO (Springel 2010; Pakmor et al. 2016) in its Navier–Stokes formulation (Muñoz et al. 2013), using a locally isothermal equation of state \( \epsilon_{\text{c}}^2 \propto R^{-1} \) and an \( \alpha \)-viscosity prescription. Once transients die out, these simulations are fully determined by four parameters: \( q_{b}, e_{b}, h_{0}, \) and the viscosity coefficient \( \alpha \). We focus on the case \( e_{b} = 0 \) and vary the other parameters, restricting ourselves to the regime with \( h_{0} \sim 0.1 \), which, from WKB analysis, is expected to develop a single trapped mode. We note that \( \alpha \) does not appear in the linear calculations (Section 2).

Details on the model setup and numerical scheme can be found in Muñoz & Lai (2016), Muñoz et al. (2019), and Muñoz et al. (2020). The general findings of the aforementioned works include the following:

1. A boundary condition with constant mass supply \( M_{0} \) enables CBDs to reach VSS (see also Miranda et al. 2017; Dempsey et al. 2020a). Once VSS is reached, the binary accretion rate \( \dot{M}_{0} \) equals (on average) the supply rate \( M_{0} \).
2. In VSS, the net angular momentum current in the CBD \( \dot{J}_{0} \) (including advective, viscous, and gravitational contributions) is statistically stationary and independent of radius (Miranda et al. 2017). In addition, the time average \( \langle \dot{J}_{0} \rangle \) equals the net angular momentum transfer rate from the disk onto the binary \( \langle J_{b} \rangle \) (which is composed of gravitational and accretional torques).
3. Most importantly, the quantity \( l_{0} \equiv \langle J_{b} \rangle / \langle \dot{M}_{0} \rangle \) is a positive constant. If \( l_{0} > l_{0,\text{crit}} \) (where \( l_{0,\text{crit}} > 0 \) depends on the binary properties; Muñoz et al. 2020), then the accretion process leads to binary expansion.

CBD simulations in VSS and positive \( l_{0} \) produce stationary \( \Sigma \) profiles that exhibit mass deficits in relation to the power-law profiles of equal accretion rate.\(^6\) Planetary-mass companions, by contrast, have \( l_{0} < 0 \) and exhibit mass pileups (e.g., Dempsey et al. 2020a). These stationary \( \Sigma \) profiles are well described by Equation (4), which is a function of three parameters: \( \Sigma_{0}, l_{0}, \) and \( R_{\text{cav}} \) (\( \epsilon_{1} = 12 \) is held fixed). Moreover, in VSS, \( \Sigma_{0} = M_{0}/(3\pi\alpha h_{0}^{3}\Omega_{\text{crit}}^{2}) \), and \( l_{0} \) can be computed from the torque balance in the simulations; hence, the only remaining parameter for the \( \Sigma \) profile is the cavity size \( R_{\text{cav}} \), which we fit from time-averaged density profiles.

In Figure 6 we show examples of the measured \( \Sigma \) (in blue) contrasted with the parametric model (Equation (4), in red) with best-fit values of \( R_{\text{cav}} \) for two different simulations with \( h_{0} = 0.1 (q_{b} = 0.2, \alpha = 0.1 \) on top, and \( q_{b} = 0.8, \alpha = 0.05 \) at the bottom). With a fully determined \( \Sigma \) profile and sound speed profile, we can compute the linear eigenfunctions and eigenfrequencies for a suite of hydrodynamical models (see Section 3.3).

3.2. Freely Precessing Eccentric Disks

CBD simulations are known to develop lopsided cavities that change orientation on timescales much longer than the binary’s orbital period (e.g., MacFadyen & Milosavljević 2008; Miranda et al. 2017; Thun et al. 2017). This slowly varying lopsidedness can be readily appreciated in Figure 7 (top panels), which shows the surface density every 60 binary orbits for a total time interval of 240 binary orbits.

One can compute a “dynamic” eccentricity from the velocity field (e.g., MacFadyen & Milosavljević 2008). As in Miranda et al. (2017), we measure disk eccentricity by first computing the Laplace-Runge-Lenz vector of the \( \text{rat} \) gas cell (see also Teyssandier & Ogilvie 2017)

\[ e_{b} = \frac{1}{\Omega_{b}^{2}d_{b}^{2}} \mathbf{r}_{b} \times (\mathbf{r}_{b} \times \mathbf{r}_{b}) - \frac{\mathbf{r}_{b}}{r_{b}} , \tag{14} \]

which can be combined into a global eccentricity vector \( \mathbf{e}_{b} \). By binning cells in semimajor axis, we can construct a set of co- focal Keplerian orbits, as shown in the bottom panels of Figure 7. These ellipses match the orientation and precession rate of the lopsided cavity. More importantly, the ellipses exhibit apsidal coherence (the longitudes of pericenter are

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\(^6\) This deficit is also a well-known consequence of the (misleadingly labeled) “zero-torque boundary condition” at some finite radius (Lynden-Bell & Pringle 1974; Frank et al. 2002; Dempsey et al. 2020b). Recently, however, Tiede et al. (2020) have reported pileups, rather than deficits, in simulations of cold \( (h_{0} \gtrsim 0.04) \) CBDs, which translate into negative values of \( l_{0} \).
aligned), meaning that they precess in tandem, which is suggestive of proper mode behavior.

The time evolution of the disk eccentricity can be better assessed from spacetime eccentricity maps. Following Miranda et al. (2017), we bin the gas eccentricity vector \( \mathbf{e}_d \), in barycentric radius to obtain \( \mathbf{e}_d \) as a function of \( R \) and \( t \). We visualize the eccentricity \( e_d = (e_{d,x}^2 + e_{d,y}^2)^{1/2} \) (top) and the longitude of pericenter \( \varpi_d = \tan^{-1}(e_{d,x}, e_{d,y}) \) (bottom) as intensity maps. The maps of Figure 8 correspond to a simulation with \( q_b = 0.8, h_0 = 0.1, \) and \( \alpha = 0.05 \) integrated for \( \approx 5000 \) binary orbits. Eccentricity growth is rapid: the \( e_d \) map saturates after \( \approx 200P_b \) and remains time-independent afterward. The \( \varpi_d \) map achieves a regularly repeating pattern after a few hundred binary orbits, showing that the disk precesses continuously, spanning the range \([0, 2\pi] \) at a fixed rate.

One can further define a global longitude of pericenter \( \varpi_d \equiv \tan^{-1}(\mathbf{e}_{d,x}, \mathbf{e}_{d,y}) \) (Figure 8, bottom panel), where \( \mathbf{e}_{d,x} \equiv \int \Sigma e_{d,x} R dR / \int \Sigma R dR \) denotes a mass-weighted radial average of the eccentricity vector over the entire disk. The global precession frequency,

\[
\varpi_d \equiv \frac{d}{dt} \varpi_d, \tag{15}
\]

in the example of the figure is \( \varpi_d \approx 2.77 \times 10^{-3} \Omega_b \). Hydrodynamical simulations typically exhibit precession rates \( \sim \mathcal{O}(10^{-3} \Omega_b) \) (MacFadyen & Milosavljević 2008; Miranda et al. 2017), which is consistent with the quadrupole-induced precession rate at the edge of the cavity.

### 3.3. Comparison of Theory and Simulations

As a test of the validity of the linear eccentricity Equation (1), we compare the hydrodynamic eccentricity profiles and precession rates to the eigenfunctions and eigenfrequencies, respectively, obtained from solving the BVP. The WKB approximation is not expected to hold, since our simulations have \( \omega_P/\omega_0 \approx 0.3-0.4 \), which is slightly outside the regime in which the WKB approximation is valid (see Figure 3).

Figure 9 shows the eccentricity profiles obtained from hydrodynamics (squares) and the linear BVP (lines) for \( q_b = 0.2, 0.4, \) and \( 1 \) and three different combinations of \( h_0 \) and \( \alpha \). In each case, the values \( R_{\text{cav}} \) and \( l_0 \) used in the BVP are extracted from simulations. The profiles show good agreement, revealing that the tapered power-law behavior of \( |E| \) (Equation (12)) is representative of the eccentricity behavior in hydrodynamical simulations. The difference between the panels is subtle, although the dependence on viscosity is clear: in general, a higher \( \alpha \) corresponds to a smaller \( R_{\text{cav}} \) (e.g., Artymowicz & Lubow 1994; Miranda & Lai 2015). Consequently, high viscosity shifts the eccentricity profile inward (see Equation (12)).

Similarly, in Figure 10 we compare the empirical precession rate \( \omega_d \) (Equation (15)) to the linear eigenfrequency \( \omega_d \). In general, the linear frequencies are within around 50% of the hydrodynamical ones, and the high-viscosity case (middle panel) shows a remarkable agreement between the two approaches.

### 4. Discussion

An exhaustive understanding of how disk and binary eccentricities couple to each other is still lacking. In this work, we have taken a first step toward a full picture of disk eccentricity by showing that CBDs naturally tend to trap eccentric modes in the vicinity of the central cavity. The turning points that allow for the trapping of waves are a consequence of the steeply but smoothly truncated CBD profile, which makes our results independent of the boundary conditions.

#### 4.1. Importance of Eccentric Disks

The inevitability of freely precessing disks around binaries can introduce an important change into our understanding of the physical processes in circumbinary environments. Long-lived disk eccentricities can (1) determine the dynamical coupling of binaries and disks; (2) modify the physical processes within protoplanetary disks, such as circumbinary planet formation; and (3) potentially leave observable imprints in the gas kinematics and disk morphologies.

A theoretical understanding of CBD eccentricity profiles might aid the interpretation of observational data. Recently, the “binary interpretation” of transitional disks (Ireland & Kraus 2008) has attracted renewed interest as a result of the rich morphological and kinematic signatures observed in astrophysical disks (see., e.g., Price et al. 2018, for the case of HD142527). Self-consistent eccentricity profiles will prove powerful tools as a diagnostic for hidden binaries and disk properties.
Moreover, eccentricity profiles inferred from observations can provide independent information on a disk’s density and temperature profiles. For example, Equation (12) is derived assuming that far from the cavity $\Sigma \propto R^{-1/2}$ (in turn a consequence of $\nu \propto R^{1/2}$) and that $c_5^2 \propto R^{-1}$, but a different set of assumptions would result in different eccentricity profiles.

### 4.2. Three-dimensional Effects

In this work, we have focused on gas eccentric modes in two dimensions, since most long-term hydrodynamical simulations are 2D. However, in three dimensions, the right-hand side of Equation (1) includes the additional term $\frac{2}{2} \Sigma^{-1} \frac{d}{r} (c_5^2 R^2) E$ (Ogilvie 2008; Teyssandier & Ogilvie 2016), which depends on $\Sigma$ but not on its derivative, and thus is always positive. However, the 2D pressure terms in the vicinity of $R_{\text{cav}}$—where the trapped mode lives—are already positive, thanks to the steep gradients in $\Sigma$. Thus, the 3D term can increase the eigenfrequency but cannot change its sign. The role of the additional term is made clear by transforming the 3D eccentricity equation into normal form, repeating the procedure of Section 2.3.1. The effective potential now becomes

$$\omega_{\text{pot}}^{(3D)} = \omega_{\text{pot}} \left( \frac{R}{R_{\text{cav}}} \right)^{-\frac{3}{2}} + \frac{\omega_{\text{f}}^2}{2} \left( \frac{R}{R_{\text{cav}}} \right)^{-\frac{3}{2}} \left[ \frac{R \Sigma'}{2 \Sigma} + \left( \frac{R \Sigma'}{2 \Sigma} \right)^2 - \frac{R^2 \Sigma''}{2 \Sigma} + \frac{3}{4} \right].$$

(16)

Remarkably, the only difference between $\omega_{\text{pot}}$ (Equation (8)) and $\omega_{\text{pot}}^{(3D)}$ (Equation (16)) is the numerical factor of $\pm \frac{3}{4}$ inside the square brackets, which is much smaller than the $\Sigma$-dependent
terms near the peak of the effective potential (Figure 5, left panel). Consequently, eccentric modes are essentially unaffected by 3D terms provided that they are confined by turning points. Note that the radial coordinate in $\omega_{\text{pot}}^{3D}$ scales with $R_{\text{cav}}$, which implies that larger cavities would simply displace the mode farther out and slow down the precession rate, but not change the sign of $\omega_0$. The robustness of eccentric mode trapping might explain why Moody et al. (2019) found no major differences between 2D and 3D (coplanar) simulations of circumbinary accretion.

4.3. Future Work: Eccentric Binaries

In this work, we have limited ourselves to the study of circular binaries. Evidently, a comprehensive study of binary–disk interaction must include eccentric binaries, since these binaries are known to exist in the T Tauri phase of stellar evolution (e.g., Tofflemire et al. 2017, 2019). Gas dynamics around moderate- to high-eccentricity binaries is rich and complex and can differ substantially from their circular counterparts (Muñoz & Lai 2016). Eccentric binaries accreting from VSS disks might see their eccentricity damped or excited (Muñoz et al. 2019), suggesting that there may exist an equilibrium eccentricity at $e_b \approx 0.2$–0.4 (see also Roedig et al. 2011), analogous to a similar phenomenon observed for migrating gas giants (Duffell & Chiang 2015). Precessing eccentric disks around binaries may be the culprits of the “alternating preferential accretion” phenomenon (Dunhill et al. 2015; Muñoz & Lai 2016), which could explain why primary accretion is dominant in the eccentric T Tauri binary TW A 3A (Tofflemire et al. 2019), despite most simulations suggesting that preferential accretion is invariably onto the secondary (e.g., Bate 2000; Farris et al. 2014; Muñoz et al. 2020).
For finite binary eccentricity $e_{b}$, the general solution $E$ will thus consist of a complementary (homogeneous) solution and a particular one, i.e., a “free” mode accompanied by a “forced” mode. In principle, for large enough forced eccentricity, the disk’s eccentricity vector will rotate around the tip of the forced eccentricity rather the origin, i.e., $\omega_{f}$ will librate instead of circulating. In upcoming work, we will simulate CBDs around eccentric binaries, probing the limitations of linear theory.

5. Summary and Conclusions

We have studied the properties of eccentricity modes in accretion disks around circular binaries using linear analysis and direct hydrodynamical simulations. Our findings are as follows:

(i) A linear eigenmode analysis shows that steeply truncated CBDs trap eccentricity modes between naturally arising turning points. Often, only one fundamental mode of low frequency $\omega_{0} \sim O(10^{-3} q_{h})$ is allowed. This mode precesses in a prograde way, and $\omega_{c}$ closely tracks the test-particle precession rate around a binary—even when pressure appears to dominate over the quadrupole.

(ii) The linear analysis shows that, when $q_{b} \gtrsim 1$, the eccentricity profiles are concentrated toward the edge of the circumbinary cavity (at a radius of $2a_{b}$) and decrease in a tapered power-law fashion, with an exponential drop-off at $\sim 10 a_{b}$. For $q_{b} \ll 1$ and/or $h_{0} \gtrsim 0.2$, the modes are poorly confined, effectively extending out to many times the binary separation $a_{b}$.

(iii) We have carried out nonlinear hydrodynamical simulations of CBDs in VSS for $0.2 \leq q_{b} \leq 1$ and different values of $h_{0}$ and $\alpha$. These simulations develop a steady-state, coherently precessing eccentricity profile. The precession rates and the radial dependence of the eccentricity are in good agreement with our analytical and numerical linear calculations, confirming that simulations around circular binaries develop free modes sustained by the close balance between excitation and damping.

We thank Wing-Kit Lee for helpful discussions and Tomoaki Matsumoto for comments on the manuscript. Y.L. acknowledges NSF grant AST1352369.

Figure 11. Left and middle panels: similar to Figure 5, but for $h_{0} = 0.03$, which allows for the existence of a fundamental mode ($n = 0$, red) and a first harmonic ($n = 1$, orange) to be trapped. Right panel: eigenfunctions associated with the $n = 0$ (red) and $n = 1$ (orange) modes; the first harmonic contains a node at $R \approx 3 a_{b}$, and its frequency $\omega_{1}$ is 10 times lower than that of the fundamental mode $\omega_{0}$.

Appendix A

Higher-order Eigenmodes

When $h_{0} \lesssim 0.05$, the quantization condition (11) can be satisfied by more than one frequency. This new behavior is illustrated in Figure 11, which shows that the function $\omega_{pot}$ (in units of $\omega_{0} / h_{0} q_{h}$, left panel) is taller and wider than that shown in Figure 5 for when $h_{0}$ is reduced from 0.1 to 0.03. This corresponds to changing $\omega_{p}/\omega_{q}$ from 0.33 to 0.03. This effective potential now allows for two trapped modes, as highlighted by the closed contours of the DRM (middle panel). A numerical solution of the BVP (right panel) shows that a lower-frequency one-node mode can accompany the fundamental mode.

Appendix B

Quantum Harmonic Oscillator

In the vicinity of $\xi = \xi_{peak}$, the term in square brackets in Equation (9) can be expanded to quadratic order in $\xi$, simplifying the eccentricity equation to

$$\frac{9 \omega_{p} d^{2}Y}{32 d^{2}\xi^{2}} + [25 \omega_{p} + 1.13 \omega_{q} + 260 \omega_{p}(\xi - \xi_{peak}) - 15000 \omega_{p}(\xi - \xi_{peak})^{2}] Y = \omega Y,$$

which is of the generic form

$$\hat{H} Y \equiv \left[ \frac{A d^{2}}{2 d^{2}\xi^{2}} + B + C(\xi - \xi_{0}) - \frac{D}{2}(\xi - \xi_{0})^{2} \right] \psi = \omega Y,$$

(B2)

If we transform the independent coordinate to $x = (D/A)^{1/4}(\xi - \xi_{0} - \frac{C}{D})$, Equation (B2) can be cast into a QHO form

$$\left[ -\frac{1}{2} d^{2} + \frac{x^{2}}{2} \right] Y = (AD)^{-1/2} \left[ \frac{C^{2}}{2D} + B - \omega \right] Y \equiv \omega Y,$$

(B3)

which has exact eigenfrequencies $\omega_{n} = n + 1/2$, and thus the operator $\hat{H}$ in Equation (B2) has eigenvalues

$$\omega_{n}^{QHO} = -\sqrt{AD} \left( n + \frac{1}{2} \right) + \frac{C^{2}}{2D} + B.$$
Since $A = (9/16)\omega_p$, $B \approx 25\omega_p + 1.13\omega_q$, $C \approx 260\omega_p$, and $D = 3 \times 10^4\omega_p$, the eccentricity eigenvalues can be approximated by

$$\omega_n^{OH} = 1.13\omega_q + 26.1\omega_p - 130\omega_p (n + \frac{1}{2}). \quad (B5)$$

**Appendix C**

**Approximated Quantization Condition**

The quantization condition (11) between two turning points, $R_{p1}$ (left) and $R_{p2}$ (right), cannot be solved analytically for $\omega_q$. However, an approximate expression can be obtained if we make three assumptions: (i) that the modes are shallow (i.e., that $\omega_q \ll \max|\omega_{pot}|$), (ii) that $R_{p1}$ is the same for all eigenmodes owing to the steep decline of $\omega_{pot}$, and (iii) that at the right turning point where $\omega_q = \omega_{pot}$, the disk is approximately a power law and $\omega_{pot} \approx \Omega_0$. Then, we approximate Equation (11) with

$$(kR)_{max} \Delta \ln R \sim \pi, \quad (C1)$$

where $\Delta \ln R \approx \ln(R_{p2}/R_{p1})$, and where assumption (iii) implies that the right turning point satisfies $R_{p2} \approx R_{\text{cav}}(\omega_0/\omega_q)^{-2/7}$. Since the modes are shallow, $kR \approx (R/R_{\text{cav}})^{3/4}(2\omega_{pot}/\omega_p)^{1/2}$ (Equation (7)), and thus, in order to solve Equation (C1) for $\omega_q$, we just need to know $\max[(R/R_{\text{cav}})^{3/4}(2\omega_{pot}/\omega_p)^{1/2}]$. If $C_1\omega_p R$ is the local maximum of the pressure contribution to $\omega_{pot}$, then we can rearrange the effective potential as $\omega_{pot} = C_1\omega_p \left[V_{\text{press}}(x) + \epsilon x^{-7/2}\right]$, where dimensionless function $V_{\text{press}}$ evaluates to 1 at $x = R/R_{\text{cav}} \approx 1$, and where we have defined $\epsilon \equiv \omega_q/(C_1\omega_p)$, which is small for all of the simulations carried out in this work. Thus, if we replace $(kR)_{max}$ with $\sqrt{2}C_1(1 + \epsilon/2)$ and $\Delta \ln R$ with $\ln((R_{\text{cav}}/R_{p1})^{1/4}(\omega_0/\omega_q)^{-2/7})$ in Equation (C1), then we can solve for $\omega_0$. We empirically find, for our assumed $\Sigma$ profile, that $2C_1 \approx 50$ and $R_{p1} \approx 0.92R_{\text{cav}}$, and therefore we have

$$\omega_0 \sim \epsilon^{-1.28 + \epsilon^2 \omega_q}. \quad (C2)$$

Therefore, the mode tapering radius (Equation (13)) is, to first order in $\epsilon$,

$$\lambda = R_{\text{cav}} \left(\frac{9 \omega_p}{32 \omega_0}\right)^{2/3} \sim R_{\text{cav}} \left(\frac{\omega_p}{\omega_q}\right)^{2/3} \left[1 - 0.02 \frac{\omega_q}{\omega_p}\right]. \quad (C3)$$

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