Resolving Small-scale Structures in Two-dimensional Boussinesq Convection by Spectral Methods with High Resolutions

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Two-dimensional Boussinesq convection is studied numerically with very fine spatial resolutions up to $4096^2$. Our numerical study starts with a smooth asymmetric initial condition, which is chosen to make the flow field well confined in the computational domain until the blow-up time ($T_c$). Our study shows that the vorticity will blow up at a finite time with $|\omega|_{\text{max}} \sim (T_c - t)^{-1.61}$ and $|\nabla \theta|_{\text{max}} \sim (T_c - t)^{-3.58}$.

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Understanding whether smooth initial conditions in three-dimensional (3D) Euler equations can develop singularities in a finite time is an important step in understanding high-Reynolds-number hydrodynamics. Two-dimensional (2D) Boussinesq convection is a simplified model of the 3D axisymmetric flows with swirl if main flow structures are well away from the symmetry axis. The computing requirements of the 2D Boussinesq simulations are significantly less than those of the 3D Euler equations. However, the spatial resolutions adopted have still been not fine enough to resolve the small-scale structure. Therefore, the spectral methods and the adaptive mesh methods are mostly adopted. One central issue of this study is on whether there exist singularities in the 2D Boussinesq flows. If we denote $T_c$ the time of blowup, the minimum criterion for the breakdown of smooth solutions in the 2D Boussinesq equations is: $|\omega|_{\text{max}} \sim (T_c - t)^{-\alpha}$ and $|\nabla \theta|_{\text{max}} \sim (T_c - t)^{-\beta}$, where $\alpha \geq 1$ and $\beta \geq 2$. Adaptive mesh techniques and the spectral method are adopted to investigate this problem, which yield various conclusions, e.g., observe no vorticity blow-up, while only provide the marginal values ($\alpha = 1$ and $\beta = 2$) although these studies predict vorticity singularities.

Since the fast developed adaptive mesh methods are limited by the finite-order accuracy and mesh equality, a further “spectral” effort seems necessary to investigate this challenging problem. The spectral methods have been used intensively in 3D Euler simulations, but they are limited by the available computing capability: the finest resolutions used so far have been $2048^3$. The conclusions are affected by certain kind of symmetric assumptions introduced to increase the effective resolutions. We can argue that the conclusion drawn by the 3D studies is not necessarily more convincing than the axisymmetric assumption where a resolution of 1500$^2$ is adopted. In this study, spectral computations with extremely high resolution (from $2048^2$ to $4096^2$) are carried out to investigate the blow-up issue for the 2D Boussinesq convection problem. Moreover, we try to maintain the flow structure well away from the axis at the time when solutions begin to blow up. This is done by choosing appropriate initial data. The governing equations are solved by a fully de-alias pseudospectral Fourier methods with 8/9 phase-shifted scheme. The digital filter is adopted to modify the Fourier coefficient to increase the stability of the numerical codes. The machine accuracy of our computer with double precision is $\epsilon = 10^{-16} \approx e^{-37}$, and the modifying factor in the filter is $\varphi(k) = e^{-37(k/N)^{16}}$ for $k < N$. With these efforts, the vorticity blow-up is observed, which is in contrast to the conclusion drawn by the earlier spectral computations.

The non-dimensionized 2D inviscid Boussinesq convection equations can be written in the $\omega$-$\psi$ formulation:

\begin{align}
\theta_1 + u \cdot \nabla \theta &= 0, \\
\omega_1 + u \cdot \nabla \omega &= -\theta_2, \\
\Delta \psi &= -\omega,
\end{align}

where the gravitational constant is normalized to $g = (0,-1)$, $\theta$ is the temperature, $u = (u,v)$ the velocity, $v$ the kinematic viscosity, $\omega = (0,0,\omega) = \nabla \times u$ vorticity, and $\psi$ stream function. The simulation is carried out in the $[0,2\pi]^3$ doubly-periodic domain. At first, we take the initial condition with unified zero vorticity and a cap-like contour of temperature with the following expression:

\begin{equation}
\theta(x,y,0) = \left( \frac{4x - 3\pi}{\pi} \right) \theta_1(x,y) \theta_2(x,y) [1 - \theta_1(x,y)], \tag{4}
\end{equation}

where if $S(x,y) := \pi^2 - y^2 - (x - \pi)^2$ is positive, then $\theta_1 = \exp \left( 1 - \pi^2/S(x,y) \right)$, and zero otherwise; if $s(y) := |y - 2\pi|/1.95\pi$ is less than 1, then $\theta_2 = \exp \left( 1 - \pi^2/S(x,y) \right)$, and zero otherwise.

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FIG. 1: Contour plots of temperature and vorticity at different times with the resolution of 4096².

FIG. 2: Three-dimensional perspective plot of the computed vorticity and temperature versus \(x\) and \(y\) at \(t = 0.88\) in 4096² run. The vorticity peak near the “+” on (a) is the location of the maximum \(|\omega|\).

\[
\exp\left(1 - (1 - s(y)^2)^{-1}\right), \text{ and zero otherwise. This initial condition}\]

\[
\text{(4) is similar to}\]

\[
\text{except that a factor}\]

\[
(4x - 3\pi)/\pi\]

\[
s\text{used to break down the field symmetry with respect to}\]

\[
x = \pi.\]

\[
The main flow structure looks like a rising bubble which touches the\]

\[
y = 0\] and \(y = 2\pi\)

\[
\text{boundaries at } t \approx 2.0.\]

\[
\text{This will cause the singularity at the axis if we transform the 2D Boussinesq results back}
\]

\[
to the 3D axisymmetric flow}\]

\[
\text{(4) the initial condition}\]

\[
\text{to form a new initial data. More precisely, we let }\]

\[
\omega(x, y, 0) = \omega'(x, 2y - 0.4\pi, 1.2)\]

\[
\text{and}\]

\[
\theta(x, y, 0) = \theta'(x, 2y - 0.4\pi, 1.2), \text{ for } (x, y) \in [0, 2\pi] \times [0, \pi]\]

\[
(\text{where } \theta' \text{ and } \omega' \text{ are obtained by solving (4) with a}\]

\[
2048² \text{ grid, and zero otherwise. The new initial condition}\]

\[
\text{only needs about } 1/4 \text{ simulation time to reach } T_c\]

\[
\text{comparing with the run in}\]

\[
\text{Figs.}\]

\[
\text{and advantages of higher resolutions show up at early stages of simulations (Figs.}\]

\[
\text{The time steps for the resolutions}\]

\[
1024², 2048² \text{ and}\]
FIG. 3: The evolution of the $T_2$, $T_4$ and maximum $\theta$ errors for three different resolutions ($1024^2$: solid line, $2048^2$: dashline, $4096^2$: circle line). The errors are respectively defined as $(T_2(0) - T_2(t))/T_2(0)$, $(T_4(0) - T_4(t))/T_4(0)$ and $|\theta_{max}(0) - \theta_{max}(t)|/|\theta_{max}(0)|$.

4096$^2$ are 0.0004, 0.0002 and 0.0001, respectively, given by the CFL condition.

The compressed bubble (Figs. 1(a)(d)) will continue to rise, and the edge of the cap will roll up at later times (Figs. 1(c)(f)). At $t \approx 0.9$, the density and vorticity contours develop into the shape of “two asymmetric eyes.” The contour evolutions on three different resolutions reveal similar phenomena at this point.

On the other hand, combining the divergence-free condition, the doubly-periodic condition and Eq. 1, we can verify that $T_2(t) = \int_0^{2\pi} \theta^2(x,y,t)dxdy$ and $T_4(t) = \int_0^{2\pi} \theta^4(x,y,t)dxdy$ are time independent. Figs. 3(a) and (b) demonstrate that the global average quantities are well conserved within 1% error for all the three resolutions used. The flow field develops into many small vortices after $t = 0.8$ and more and more energy is removed by the filter in the time evolution. Hence, it seems that simulations with higher resolution do not have more advantages here. However, the time evolution of the maximum values of $\theta$, which is also time-independent, shows much better improvement for the $4096^2$ run by comparing with lower resolution runs (Fig. 3(c)): the relative error for $|\theta|_{max}$ with the $4096^2$ run is always lower than $10^{-4}$, while the corresponding error with the $1024^2$ run is about $10^{-2}$. Because the global maximum values of $|\omega|$ and $|\nabla \theta|$ are normally used as the indicators in the singularity analysis, the $4096^2$ run may lead to a more accurate conclusion than that of the $1024^2$ and $2048^2$ runs. It should be pointed out that if resolutions higher than $4096^2$ are employed then the quadruple precision ($\epsilon = 10^{-32}$) instead of the double precision ($\epsilon = 10^{-16}$) may have to be used; otherwise the results may be spoiled by the round-off errors (see Fig. 4 for more details). The maximum vorticity is initially located on the left edge of the rising bubble (Fig. 1(a)), and later moved to the outer layer of the “eyes”, (see, Figs. 1(c) and 2(a)). This observation is in good agreement to the prediction in Fig. 4. At $t = 0.88$, the $|\omega|_{max}$ locates at $(5.31, 2.72)$ (see the “+” in Fig. 2(a)).

Around $t \approx 0.88$, some very small vorticity structures begin to appear at the lower part of the smooth outer layer where the maximum $|\omega|$ appears. The filter we adopted in the code will remove more and more energy from the system, so that the global average values like $T_2$ and $T_4$ are greatly affected (Figs. 3(a)(b)). Consequently, after this critical time the numerical results in the $4096^2$ run may not be accurate and reliable. Actually, it is observed that the $|\omega|_{max}$ and $|\nabla \theta|_{max}$ experience a drop-down after $t \approx 0.9$. This indicates that the filter prevents the blowup of the maximum vorticity.
FIG. 5: The time evolutions of re-scaled $|\omega|_{\text{max}}$ (square line), $|\nabla \theta|_{\text{max}}$ (star line) and their least square fit to a straight line (dash line) for the (a) 2048$^2$ and (b) 4096$^2$ run. The blow-up times ($T_c$) are the points where x-axes are intersected by dash lines. The 1024$^2$ run has not been plotted because there is no blow-up signal until $t = 1.0$. 

from happening, which is similar to the viscosity effect in high Reynolds number simulations (e.g. [11], [12]). Therefore, the sample maximum values after the dropping down should not be used in the singularity analysis. For the 1024$^2$ and 2048$^2$ runs, however, the drop-downs appear later than $t \simeq 0.88$ and the blow-up time $T_c$ is also delayed. In Figs. 3 we only provide time evolutions for $|\omega|_{\text{max}}$ and $|\nabla \theta|_{\text{max}}$ in different resolutions before they drop down. We re-plot these maximum value evolutions in Fig. 4. It seems that the growth of $|\omega|_{\text{max}}$ and $|\nabla \theta|_{\text{max}}$ in the 4096$^2$ run (Fig. 4(b)) plausibly terminates in a finite time with $|\omega|_{\text{max}} \sim (T_c - t)^{-1.61}$, and crudely $|\nabla \theta|_{\text{max}} \sim (T_c - t)^{-3.58}$ (in the later case we only adopt the values of $|\nabla \theta|_{\text{max}}$ after $t = 0.75$). It is noticed that our values ($\alpha, \beta = (1.61, 3.58)$ are larger than the minimum blow-up values [4, 7] and the existing results [3, 4, 5]. We run our simulations until $t = 2.0$ and find that the main structure of the flow field stays away from $y = 0$ and $y = 2\pi$. The axisymmetric assumption adopted in the original 3D Euler equation model does not cause any axis singularity difficulty as discussed earlier with our new initial condition.

We have not performed the simulation on grid finer than 4096$^2$, but we can predict some results from the present computations. It is well known that when a delta function is approximated by a finite Fourier series, each doubling of the resolutions will cause doubling of the maximum value and $2^n + 1$ times the maximum values of the $n^{th}$ order space derivative (see the analysis in Appendix A of [11]). In the final state of our simulation (Fig. 2(a)), the cut-line at $y = 2.72$ through the outer layer of the “eye” looks very similar to a delta function. Therefore, when finer and finer resolutions are used the peak vorticity and temperature gradient are getting larger and larger. Further 2D Boussinesq simulations with even higher resolution will support our singularity prediction although it seems impossible to accurately predict the singularity time with the current computational scheme. In Fig. 4(b), the gradients of the curves become larger when the grids are refined. The 1024$^2$ result indicates no blow-up, and the singularity analysis for the 2048$^2$ run (Fig. 4(a)) shows $\alpha = 1.28, \beta = 2.32$ and $T_c = 1.47$. The results obtained by the three resolutions reveal that the values of $T_c$ may be smaller than 1.34, and the values of $\alpha$ and $\beta$ may be larger than 1.61 and 3.58 if much fine resolutions (> 4096$^2$) are used. Although our simulations can not provide accurate value for $T_c$, the existence of singularity in the 2D Boussinesq simulations is strongly supported by our numerical computations.

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