**MMSE Approximation For Sparse Coding Algorithms Using Stochastic Resonance**

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**Abstract**—Sparse coding refers to the pursuit of the sparsest representation of a signal in a typically overcomplete dictionary. From a Bayesian perspective, sparse coding provides a Maximum a Posteriori (MAP) estimate of the unknown vector under a sparse prior. Various nonlinear algorithms are available to approximate the solution of such problems.

In this work, we suggest enhancing the performance of sparse coding algorithms by a deliberate and controlled contamination of the input with random noise, a phenomenon known as stochastic resonance. This not only allows for increased performance, but also provides a computationally efficient approximation to the Minimum Mean Square Error (MMSE) estimator, which is ordinarily intractable to compute. We demonstrate our findings empirically and provide a theoretical analysis of our method under several different cases.

**Index Terms**—Sparse coding, stochastic resonance, basis pursuit, orthogonal matching pursuit, noise-enhanced pursuit

**I. INTRODUCTION**

In signal processing, often times we have access to a corrupted signal and we wish to estimate its clean version. This process includes a wide variety of problems, such as denoising, where we wish to remove noise from a noisy signal; deblurring where we look to sharpen an image that has been blurred or was taken out of focus; and inpainting in which we fill-in missing data that have been removed from the image. All the aforementioned tasks, and many others, include a linear degradation operator and a stochastic corruption. The forward model can be described by \( y = Hx + \nu \), where \( x \) is the clean signal, \( H \) is the linear degradation operator, \( \nu \) denotes additive noise, and \( y \) stands for the noisy measurements.

In order to successfully restore \( x \), it is useful to incorporate both the statistical properties of the corruption and a prior knowledge on the signal. In image processing many such priors have been developed over the years, such as total-variation, self-similarity, sparsity, and many others [1–3].

In this work we focus our attention to the sparse model prior, which assumes that the clean signal \( x \in \mathbb{R}^n \) is a linear combination of a small number of columns from an overcomplete dictionary \( D \in \mathbb{R}^{n \times m} \), where \( n < m \), referred to as atoms. In this case, we can write \( x = D\alpha \), where the representation vector \( \alpha \in \mathbb{R}^m \) is sparse. One of the most fundamental problems in this sparse model is termed “sparse coding”: given \( x \), find the sparsest \( \alpha \) such that \( x = D\alpha \).

Essentially, this calls for solving the following optimization problem:

\[
(P_0) : \quad \hat{\alpha} = \arg\min_{\alpha} ||\alpha||_0 \quad \text{s.t.} \quad D\alpha = x.
\]

where \( || \cdot ||_0 \) stands for the \( l_0 \) pseudo-norm that counts the number of non-zero elements in the vector.

Typically, one does not have access to \( x \), but rather to a degraded version of it, \( y \). When addressing a denoising task, the degradation is usually formed by an additive noise \( \nu \) with a bounded energy \( ||\nu||_2 \leq \epsilon \), and therefore \( y = x + \nu \). Hence, the above problem needs to be modified in order to accommodate model deviations, leading to:

\[
(P_\epsilon) : \quad \hat{\alpha} = \arg\min_{\alpha} ||\alpha||_0 \quad \text{s.t.} \quad ||D\alpha - y||_2 \leq \epsilon.
\]

The solution of \((P_\epsilon)\) can be used to provide an estimate of \( x \), in the form of \( \hat{x} = D\hat{\alpha} \).

Without further assumptions, the \((P_\epsilon)\) problem is non-convex and NP-Hard in general [4], as it requires iterating through all the possible supports of \( \alpha \). That being said, approximation algorithms, also referred to as pursuit algorithms, have been developed in order to manage this task effectively. These include Basis-Pursuit (BP) [5] and Orthogonal Matching Pursuit (OMP) [6]. Pursuit algorithms recover a highly likely support (and in some cases, the most probable one) and they can be seen as providing an approximation to the Maximum A Posteriori (MAP) estimator under a Bayesian framework. While their use is widespread, the performance of these pursuit algorithms is limited by that of the MAP estimator. In principle, one should use the Minimum Mean Squared Error (MMSE) estimator, but its computational complexity makes this optimal operator intractable.

In this paper, we show how to improve sparse coding algorithms’ performance by a deliberate and controlled contamination of the input signal by noise. The proposed strategy not only allows for improved performance but also provides a computationally efficient approximation to the optimal MMSE estimator. The proposed method leans on the phenomenon termed Stochastic Resonance (SR). In SR, one adds noise to a weak sub-threshold signal to increase its output Signal-to-Noise Ratio (SNR), or other statistical measure, when going through a threshold quantizer. This field has been further developed and has shown the ability to improve the performance of sub-optimal detectors [7], non-linear parametric estimators [8] and image processing algorithms [9]. Moreover, Suprathreshold SR (SSR) is a phenomenon in which statistical measures improve even when the signal’s amplitude is greater than the thresholding quantizer.

In our work we will seek answers to the following questions:
1) Can noise be employed to improve the performance of sparse coding algorithms?
2) How significant can this improvement be?
3) Can we use noise enhancement to achieve MMSE approximation?

We address these and closely related questions. We show how SR could be of great benefit for pursuit algorithms, and indeed provide an MMSE approximation. The rest of the paper is structured as follows. Section II reviews Bayesian estimation under the sparse prior, in Section III we introduce our proposed algorithm and in Section IV we explore related work in the field. Then, sections V and VI we analyze the proposed method in the unitary case and the general case respectively. Section VII reviews numerous SR noise priors and in Section VIII we demonstrate a practical usage of our proposed algorithm for image denoising. Finally, in Section IX we conclude this work.

II. BAYESIAN ESTIMATION UNDER THE SPARSE PRIOR

We now formalize our model in detail. $D \in \mathbb{R}^{n \times m}$ is an over-complete dictionary, with $n < m$. $\alpha \in \mathbb{R}^m$ is a sparse vector with either fixed cardinality $\|\alpha\|_0 = M$ or with a prior probability $P_i$ for each entry to be non-zero (we will variate $i$). As surprising as it might seem, the MMSE is actually a dense representation, and $\alpha$ is known to be a constant, $\|\alpha\|_0 = M$, and all are equally likely, we can omit the last two sums since they indicate the prior of the support $P(S)$ and they are uniformly distributed. Once the most probable support is recovered, we use the Oracle estimator to recover the clean sparse vector:

$$\hat{\alpha}_{y}^{\text{Oracle}} = \alpha^{\text{Oracle}}_{\hat{S}, y}. $$

The last estimator is the MMSE. A well known result from estimation theory is that the MMSE estimator is given by the conditional expectation:

$$\hat{\alpha}_{y}^{\text{MMSE}} = \mathbb{E}[\alpha|y] = \sum_S P(S|y)\mathbb{E}[\alpha|y, S]$$

This is a weighted sum of all the possible supports. The probability $P(S|y)$ is given in [10]:

$$P(S|y) = \frac{t_S}{t}, \quad t \triangleq \sum_S t_S$$

$$t_S \triangleq \frac{1}{\sqrt{\det(C_S)}} \exp \left\{ -\frac{1}{2} y^T C_S^{-1} y \right\} \prod_{i \in S} P_i \prod_{i \notin S} (1 - P_i).$$

As surprising as it might seem, the MMSE is actually a dense vector. Both estimators, the MMSE and the MAP, are generally NP hard to obtain, since they require either to sum all the possible supports or to compute all the posterior probabilities and pick the highest one. Though in principle these estimators could be computed, this is prohibitive as the number of possible supports is exponentially large. Therefore, approximation methods are needed. As shown in (3), the posterior probabilities $P(S|y)$ have an exponential nature, which means that the sum in (2) is practically dominated by only a small number of elements:

$$\hat{\alpha}(y) = \sum_{S \in \Omega} P(S|y)\hat{\alpha}(y, S) \approx \sum_{S \in \omega \in \Omega} P(S|y)\hat{\alpha}(y, S),$$

for a proper choice of the subset $\omega$. If one could obtain the significant elements and their proper weights, then an MMSE approximation would be attainable. This is the motive behind our proposed method.

III. THE PROPOSED ALGORITHM

We present the proposed approach in Algorithm 1. The algorithm adds a small amount of noise $n_k$ to the already

1) In fact, this is the MAP of the support. We use it to avoid the probable case where the recovered signal is the 0 vector as described in [10].
noisy signal $y$ and then applies any given pursuit which may be considered as a MAP approximation. Finally all the evaluations are averaged to a single final estimation. Note that two forms of estimations are returned by the algorithm. The first one, the “non-subtractive”, uses the extra-noisy evaluations as the basic estimators and averages them. The second one, the “subtractive”, uses only the supports recovered by the noisy evaluations in order to provide an estimation that is not effected by the SR added noise, before returning their average.

Before we go through the analytic arguments provided in the coming sections, we present some intuition behind our proposed method. Assume the SR noise added has a small variance which will lead to a small perturbation in the signal $y + n$, before returning their average.

As we will see in the following sections, our approach does not suffer from such limitation.

Similar to SR, SSR is an observation of noise enhanced behaviour in signal processing systems. The phenomenon was first described in [13] in which a signal is passed through a set of thresholds, or other non-linearities as suggested in [14], each contaminated by noise. The outputs of all the non-linearities are then averaged to obtain a final estimator. Indeed, like SR, there exists some noise properties that lead to optimal performance w.r.t some statistical measure (SNR, mutual information, MSE, etc...). But, unlike regular SR, improvement exists even when the signal is stronger than the thresholding operator. Even though our proposed method resembles the described structure of a set of thresholds, our work is different: our approach considers not only nonlinear closed form functions but also more general and complex pursuit algorithms. In addition, we present and analyze the case of subtracting the projected SR noise, improving the estimation further. Finally, in Section VI-B2, we propose a different approach to use SR noise which follows a Monte Carlo framework and assures asymptotic convergence to the MMSE.

### V. MMSE Approximation in the Unitary Case

#### A. The Unitary Sparse Estimators

When the dictionary is a unitary $n \times n$ matrix, we can simplify the expressions associated with the oracle, MAP and MMSE estimators as suggested in [10]. The MAP estimator is reduced to the elementwise hard thresholding operator applied on the projected measurements $\beta = D^T y$, given by:

$$\hat{\alpha}_{\text{MAP}}(\beta) = H_{\lambda_{\text{MAP}}} (\beta) = \begin{cases} c^2 \beta & \text{if } |\beta| \geq \lambda_{\text{MAP}}, \\ 0 & \text{otherwise} \end{cases}$$

where $c^2 = \frac{\sigma_n^2}{\sigma_0^2 + \sigma_n^2}$, $\lambda_{\text{MAP}} = \frac{\sqrt{2} \sigma_0}{c} \sqrt{\log \left( \frac{1 - p_1}{p_1 \sqrt{1 - c^2}} \right)}$, and $\alpha$ and $\beta$ are the elements of the vectors $\alpha$ and $\beta$.

The MMSE estimator in the unitary case is a simple elementwise shrinkage operator of the following form:

$$\hat{\alpha}_{\text{MMSE}} = \psi(\beta) = \frac{\exp \left( \frac{c^2}{2 \sigma_n^2} \beta^2 \right) \frac{p_1}{1 - p_1} \sqrt{1 - c^2} }{1 + \exp \left( \frac{c^2}{2 \sigma_n^2} \beta^2 \right) \frac{p_1}{1 - p_1} \sqrt{1 - c^2}} c^2 \beta.$$
B. The Unitary SR Estimators

In Section II we mentioned that the MMSE is a weighted sum of all the probable solutions, where the weights are the posterior probabilities of each support. Similarly, we propose to approximate the MMSE by summing many probable solutions, by the following procedure: First, we add white zero mean Gaussian noise \( n_k \) to the signal \( y \). Then, we apply hard thresholding \( H \) resulting from a MAP estimation \( \hat{\alpha}_k \) of the original signal. This process is repeated \( K \) times, each time with a different noise realization. The final step includes an arithmetic mean over all the estimations\(^2\)

\[
\hat{\alpha}_{\text{stochastic}} = \frac{1}{K} \sum_{k=1}^{K} \hat{\alpha}_k = \frac{1}{K} \sum_{k=1}^{K} H(\beta + n_k).
\]

As \( K \to \infty \), the described process asymptotically converges to the following expectation:

\[
\mathbb{E}_n \left[ H_{\lambda} (\beta + n) \right] = \int_{-\infty}^{\infty} H_{\lambda} (\beta + n) p(n) \, dn
\]

\[
= e^2 \left[ \beta Q \left( \frac{\lambda + \beta}{\sigma_n} \right) + \beta Q \left( \frac{\lambda - \beta}{\sigma_n} \right) + e^2 \frac{\sigma_n}{2\sqrt{2\pi}} \left( e^{-\frac{(\lambda+\beta)^2}{2\sigma_n^2}} - e^{-\frac{(\lambda-\beta)^2}{2\sigma_n^2}} \right) \right],
\]

where we have denoted \( Q(\cdot) \) as the tail probability of the standard normal distribution, i.e. \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-u^2/2} \, du \). The full derivation can be found in Appendix A. Note that the estimator \( \hat{\alpha}_k \) is still contaminated with the SR noise \( n_k \) that has been added to it. We refer to it as the non-subtractive estimator.

Conversely, one might consider the subtractive estimator, in which we remove the projection of the added noise, resulting in the following shrinkage operator:

\[
\hat{\alpha}_k (\beta, n_k) = H^{-1} (\beta, n_k)
\]

\[
H^{-1} (\beta, n_k) = \left\{\begin{array}{ll}
e^2 (\beta + n_k) - e^2 n_k & \text{if } |\beta + n_k| \geq \lambda_{\text{MAP}}, \\
0 & \text{otherwise}
\end{array}\right.
\]

Using this shrinkage operator, and following the same procedure described previously, results in the following estimator:

\[
\mathbb{E}_n \left[ H^{-1} (\beta, n) \right] = \int_{-\infty}^{\infty} H^{-1} (\beta + n) p(n) \, dn
\]

\[
= e^2 \beta \left[ Q \left( \frac{\lambda + \beta}{\sigma_n} \right) + Q \left( \frac{\lambda - \beta}{\sigma_n} \right) \right].
\]

The full derivation can be found in Appendix A as well.

Note that these estimators have two parameters yet to be set: \( \sigma_n \) and \( \lambda \). The former tunes the magnitude of the added noise, while the latter controls the value of the thresholding operation. The original MAP threshold might be sub-optimal due to the added noise and therefore we leave \( \lambda \) as a parameter needed to be set. We will explore how to set these parameters later in this section.

\(^2\)Notice that the written equation operates on the the vectors elementwise.
In the unitary case this is further simplified to an element-wise sum:

\[ \mu(\mathcal{H}(\beta, \lambda, \sigma_n), \beta) = \sum_i \mu(\mathcal{H}(\beta_i, \lambda, \sigma_n), \beta_i) \]

\[ = \sum_i \mathcal{H}(\beta_i, \lambda, \sigma_n)^2 - 2\mathcal{H}(\beta_i, \lambda, \sigma_n)\beta_i + 2\sigma^2_\nu \frac{d}{d\beta_i} \mathcal{H}(\beta_i, \lambda, \sigma_n), \]

(4)

and we wish to optimize \( \sigma_n \) and \( \lambda \):

\[ \sigma_n, \lambda = \arg \min_{\sigma_n, \lambda} \mu(\mathcal{H}(\beta, \lambda, \sigma_n), \beta). \]

Plugging in the subtractive estimator \( \mathcal{H}^- \) results in a closed form expression as can be seen in Appendix B. Also, in Appendix B we show the surface of the MAP estimator described in Section II boils down to the following form:

\[ \hat{x}(y) = c^2 y_S, \quad y_S = d_S d_S^T y, \quad c^2 = \frac{\sigma^2_\alpha}{\sigma^2_\alpha + \sigma^2_\nu}, \]

where the chosen atom \( d_S \) is:

\[ d_S = \arg \min_{d_S} \|y_S - y\|^2_2 \]

\[ = \arg \min_{d_S} \|d_S d_S^T y - y\|^2_2 = \arg \max_{d_S} y^T d_S d_S^T y. \]

(5)

Following the subtractive concept we propose for the unitary case, we introduce the following SR estimator:

\[ \hat{x}(y) = c^2 y_S, \quad y_S = d_S d_S^T y, \]

where this time the choice of the atom \( d_S \) is affected by an additive SR noise:

\[ d_S = \arg \min_{d_S} \|y_S(n) - y(n)\|^2_2 \]

\[ = \arg \max_{d_S} (y + n)^T d_S d_S^T (y + n). \]

Employing this as a pursuit to be used in Algorithm 1 we now analyze the proposed asymptotic estimator:

\[ \mathbb{E}_n [\hat{x}(y, n)] = \mathbb{E}_n [c^2 y_S(n)] = \mathbb{E}_S [\mathbb{E}_n|S| c^2 y_S|S]] \]

\[ = c^2 \sum_{i=1}^m \mathbb{E}_n|S| d_i d_i^T y P(\hat{S} = i) \]

\[ = c^2 \sum_{i=1}^m d_i d_i^T y P(\hat{S} = i). \]

Just like the MMSE in the general case, the asymptotic estimator is a weighted sum of all the possible solutions (all supports with a single atom in this case). Each support is weighted by its probability to be chosen in the SR process, which we now analyze. As stated in (5), the chosen atom \( i \) is the most correlated atom with the input signal:

\[ P(\hat{S} = i) = P \left( |d_i^T (y + n)| > \max_{j \neq i} |d_j^T (y + n)| \right) \]

\[ = P \left( |\tilde{n}_i| > \max_{j \neq i} |\tilde{n}_j| \right), \]

(6)

where we defined \( \tilde{n} \) as a random Gaussian vector such that

\[ \tilde{n} = \begin{bmatrix} \tilde{n}_1 \\ \vdots \\ \tilde{n}_m \end{bmatrix} \sim \mathcal{N} \left( D^T y, \sigma_n^2 D^T D \right). \]

This means that the probability of choosing the \( i \)th atom is distributed as the probability of the maximum value of a random Gaussian vector with correlated variables. The vector’s elements are correlated since in the non-unitary case, \( D^T D \) is not a diagonal matrix.

Facing this difficulty, we propose to tackle it in several directions:

1) Instead of adding the SR noise to \( y \), we can add it to the projected signal \( D^T y \), thus avoiding the variables \( \{\tilde{n}_i\} \) being correlated.

2) We can add some assumptions regarding the dictionary \( D \), thus deriving average case conclusions.

3) We can change the pursuit to use a constant threshold \( \lambda \) for the choice of the support instead of comparing the correlated variables. This means that when applying the algorithm, a cardinality of \( |S| = 0 \) or \( |S| \geq 2 \) might enter the averaging process. This is clearly a sub-optimal choice and we leave the study of this option for future work.

We shall now analyze the first two proposed alternatives.

1) Adding Noise to the Representation: Under this assumption, we continue from (6), only this time the noise \( \tilde{n}_i \) is white and has the following properties:

\[ \tilde{n} \sim \mathcal{N} \left( D^T y, \sigma_n^2 I_{m \times m} \right) \]

Plugging this into (6):

\[ P(\hat{S} = i) = P \left( |d_i^T (y + n_i)| > \max_{j \neq i} |d_j^T (y + n_j)| \right) \]

\[ = P \left( |\tilde{n}_i| > \max_{j \neq i} |\tilde{n}_j| \right) \]

\[ = \int_0^\infty P \left( \max_{j \neq i} |\tilde{n}_j| < t, |\tilde{n}_i| = t \right) P \left( |\tilde{n}_i| = t \right) dt \]

\[ = \int_0^\infty P \left( \max_{j \neq i} |\tilde{n}_j| < t \right) P \left( |\tilde{n}_i| = t \right) dt. \]

(7)
For the first term, the elements of $\tilde{n}$ are independent, and therefore
\[
P \left( \max_{j \neq i} |\tilde{n}_j| < t \right)
= \prod_{j \neq i} P (|\tilde{n}_j| < t) = \prod_{j \neq i} \left[ 1 - P (|\tilde{n}_j| > t) \right]
= \prod_{j \neq i} \left[ 1 - \left( Q \left( \frac{t + \beta_j}{\sigma_n} \right) + Q \left( \frac{t - \beta_j}{\sigma_n} \right) \right) \right],
\]
where the last equality follows similar steps as in Appendix A with $\beta_j \triangleq d_i^T y$ and $t = \lambda$. The second term in (7) is simply an absolute value of a Gaussian, therefore
\[
P (|\tilde{n}_i| = t) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \left( e^{-\frac{(t-\beta_i)^2}{2\sigma_n^2}} + e^{-\frac{(t+\beta_i)^2}{2\sigma_n^2}} \right).
\]
Putting the two terms back into (7)
\[
P(\tilde{s} = i) = \int_0^\infty \prod_{j \neq i} \left[ 1 - \left( Q \left( \frac{t + \beta_j}{\sigma_n} \right) + Q \left( \frac{t - \beta_j}{\sigma_n} \right) \right) \right] dt.
\]
(8)

The obtained expression cannot be solved analytically but can be computed numerically. In Figure 3 we present a simulation for the single atom case where the noise is added to the projection $\beta_i = d_i^T y$. In Figure 3a we can observe that indeed, even in the non-unitary case, SR for the optimal choice of $\sigma_n$ applied on the representation $D^T y$ approximates the MMSE pretty well.

In Figure 3b we show the probability of recovering the true support $P_{\text{success}}$ as a function of $\sigma_n$, both from (8) and from simulations that pick the most correlated atom. We also compare it to the MMSE weight from (3). The optimal $\sigma_n$ in terms of MSE is drawn as a black line. Notice that for the optimal choice of $\sigma_n$, the probability of the true support to be chosen is similar to that given in the MMSE solution. In other words, the optimal $\sigma_n$ is the one that approximates the weight of the support to the weight given by the MMSE expression. The trend in (8) shows, unsurprisingly, that as we add noise, the probability of successfully recovering the true support decreases. In the limit, when $\sigma_n \to \infty$ the signal will be dominated by the noise and the success probability will be uniform among all the atoms, i.e., equal to $P_{\text{success}} = \frac{1}{m}$. Due to these findings, and since the optimal MSE is comparable to that of the MMSE estimator, one might expect a similar behavior for most of the other possible supports. To examine this idea, we carry out the following experiment. We randomize many representations $\alpha$, all containing a non-zero coefficient in the same index. Then, we plot the histogram of the average empirical probability of each element in the vector $\alpha$ to be non-zero (obtained by pursuit). Finally, we compare these probabilities to the weights of the MMSE from (3). This experiment will run for different $\sigma_n$ values and each time we can compare the entire support histogram. We expect the two histograms (SR and MMSE) to fit for the right choice of added noise parameter $\sigma_n$. In Figure 4 we see the results of the described experiment.
the expression is a weighted sum of $n$ iid random variables $\{n_k\}_{k=1}^n$. Therefore, using the central limit theorem, and the fact that $\|d_i\|_2 = 1$, for a large dimension $n$, $\sum_{k=1}^n d_i \cdot n_k$ is a Gaussian variable. Clearly, its mean value is 0, and its standard deviation is $\sigma_n$, hence $\hat{n}_i[d_i \sim \mathcal{N}(d_i^T y, \sigma_n)$ for $n \to \infty$.

Now we turn to analyze the properties of the entire vector $\hat{n}$. From the previous analysis we know that given the dictionary $D$, it is a random Gaussian vector with the mean vector $\mu_{\hat{n}} D = D^T y$. Using the properties of the noise $\mathbb{E}[n n^T] = \sigma_n^2 I$, the auto-correlation matrix of $\hat{n}_i[D$ is by definition:

$$\Sigma D = \mathbb{E} [D^T n n^T D] = D^T \mathbb{E}[nn^T] D = \sigma_n^2 D^T D.$$ 

Analyzing the average case, the mean vector is of the form:

$$\mu_{\hat{n}} = \mathbb{E}_D [D^T y],$$

and the auto-correlation matrix is simply diagonal:

$$\Sigma = \mathbb{E}_D [\Sigma D] = \mathbb{E} [\sigma_n^2 D^T D] = \sigma_n^2 I,$$

where we used the assumptions in (9) and (10). This means that the uncorrelated atom assumption leads $\hat{n}$ to have the same properties as seen in the previous section, and therefore their analysis is the same.

To demonstrate that indeed the two are the same, we performed the following experiment. We sampled a random dictionary $D$ and random sparse representations $\alpha$ with cardinality of 1 as the generative model described earlier suggests. In this experiment we used a dictionary $D$ of size $200 \times 400$ and 2000 random sparse representations. Using the generated vectors and dictionary we created signals $y$ simply by multiplying and adding noise $y = D \alpha + \nu$. To denoise the signals, we once ran the stochastic resonance algorithm with noise $n_1 \sim \mathcal{N}(0, \sigma_n I_{n \times n})$ added to the signal vectors $y + n_1$, and once with noise $n_2 \sim \mathcal{N}(0, \sigma_n I_{n \times n})$ added to the representation domain $D^T y + n_2$. Observe that due to the cardinality of the sparse representation, a simple Hard Thresholding is the MAP estimator, thus we shall use that as our non-linear estimator. In Figure 6 we see that the MSE of the two cases result in an almost identical curve. Small differences might exist due to the finite dimensions used in the experiment.

Note that the noise energy added in the representation domain is much larger than that of the noise added to the signal, i.e. $E[\|n_2\|^2] = n \sigma_n^2 > n \sigma_n^2 = E[\|n_1\|^2]$, but the results remain the same due to the unit norm of the dictionary $\|d_i\|_2 = 1$, and of course the uncorrelated assumption.
model are known and slightly modify Algorithm 1 so that Algorithm 1 provides a significant improvement over standard pursuits. Then, we assume that the parameters of the generative model are known and slightly modify Algorithm 1 so that it will be equivalent to a Monte Carlo simulation in which MMSE convergence is asymptotically guaranteed.

1) **The Generative Model is Not Known**: In the general case, the \( P_0 \) optimization is non-convex, posing a hard problem to tackle and has been shown to be NP-Hard [4]. Nevertheless, approximation algorithms have been developed in order to manage this task effectively. One of these approximations, known as the Basis-Pursuit (BP) [5] is a relaxation method where the \( l_0 \) norm is replaced by an \( l_1 \). An alternative approach adopts a greedy strategy, such as in the case of the Orthogonal Matching Pursuit (OMP) [6], where the non-zeros in \( \hat{\alpha} \) are found one-by-one by minimizing the residual energy at each step.

These approximation algorithms have been accompanied by theoretical guarantees for finding a sparse representation \( \hat{\alpha} \) leading to a bounded error \( ||\hat{\alpha} - \alpha||_2 \) [16]. These results rely on the cardinality of \( \alpha \), the range of the non-zero values and properties of the dictionary \( D \). In practice, under non-adversarial assumptions, these algorithms succeed with high probability even when the theoretical conditions of the worst case analysis are not met [17, 18].

We shall now show that applying Algorithm 1 in the general case of an overcomplete dictionary with many non-zero coefficients in \( \alpha \) leads to superior denoising performance over the aforementioned standard pursuit algorithms. We also show that this algorithm can be used with not only the \( l_0 \) pseudo-norm (OMP) but also with \( l_1 \) norm algorithms (Basis Pursuit) as well. Note that the literature regarding an MMSE approximation under the sparse model, such as the Random OMP, modified the greedy components of \( l_0 \) pursuits by a stochastic one. This work, however, is the first to present an approximation applicable when using the \( l_1 \) norm, making it accessible to problems with large dimensions.

In order to recover the clean signal, we use Algorithm 1 as follows. In each SR iteration, as previously, SR noise is added to \( y \). Then, we apply an entire pursuit \( (l_0 \text{ or } l_1) \) and recover a support containing multiple non-zeros. Using the recovered support we compute a clean signal estimator by multiplying the sparse vector by the dictionary \( D \). This process is repeated \( K \) times and finally all the estimators are averaged in order to imitate the MMSE estimator structure. As before we use a random Gaussian dictionary and generate random Gaussian coefficients, this time with a random number of non zero coefficients and a prior probability \( P_1 = 0.05 \) for each atom. We then add Gaussian noise to the signals and use Algorithm 1 with BP or OMP. Since the cardinality of the sparse vector is unknown, we use the bounded noise formulation of the pursuit algorithms, i.e.

\[
\text{OMP: } \min_{\alpha} ||\alpha||_0 \quad \text{s.t.} \quad ||y - D\alpha||_2 \leq \epsilon,
\]

\[
\text{BP: } \min_{\alpha} ||\alpha||_1 \quad \text{s.t.} \quad ||y - D\alpha||_2 \leq \epsilon.
\]

The results of the described experiment can be seen in Figure 7. We note that although BP is in general inferior to OMP in terms of MSE, the SR method improves both algorithms’ performance, showcasing the flexibility of our proposed approach. We should mention that even though OMP achieved better performance in terms of MSE, as the number of non-zeros increases, BP converges much faster.

Without any further knowledge regarding the generative model, proving MMSE convergence using this method is a challenging task and we defer it to future work. That being said, in cases where the generative model is assumed to be known, MMSE convergence can be proved as we show next.

2) **The Generative Model is Known**: In the case where the generative model is fully known, the MMSE can, in principle, be computed. The problem is that due to the large number of possible supports, it is impractical. In order to solve this problem, we can turn to Monte Carlo simulations in order to achieve an asymptotically converging approximation. As discussed in Section II, even though the MMSE is a weighted sum of all possible supports, it is practically dominated by...
only a few number of elements. If one can find the significant elements and weight them properly, then an approximation is at hand. To do so, we turn to a Monte Carlo simulation named Importance Sampling.

In the literature, there are numerous ways of approximating non-analytic integrals [19]. Specifically for the posterior expectation case, the Monte Carlo Importance Sampling approach [20] is known to work well. Generally the integral we wish to approximate is:

\[ \mathbb{E}_x \left[ h(x) \right] = \int_{\mathcal{X}} h(x) f(x) dx. \]

Importance Sampling essentially calculates the above integral by sampling \( S \) using the importance sampling fundamental identity:

\[ \mathbb{E}_x \left[ h(x) \right] = \int_{\mathcal{X}} h(x) f(x) dx = \int_{\mathcal{X}} h(x) \frac{f(x)}{g(x)} g(x) dx, \]

\[ \{g(x) \neq 0| x \in \mathcal{X}, f(x) \neq 0\}. \]  \hspace{1cm} (11)

Note that the condition given in brackets means that the \( \text{supp}(g) \supseteq \text{supp}(f) \). The above integral can be approximated by sampling \( J \) samples from the distribution \( g \), i.e. \( X_j \sim g \), \( 1 \leq j \leq J \) and then average:

\[ \mathbb{E}_x \left[ h(x) \right] \approx \frac{1}{m} \sum_{X_j} f(x) h(x). \]  \hspace{1cm} (12)

This sum asymptotically converges to the integral in (11) by the strong law of large numbers. A well known alternative to the above sum is:

\[ \mathbb{E}_x \left[ h(x) \right] \approx \frac{\sum_{X_j} f(x) h(x)}{\sum_{X_j} g(x) h(x)}. \]  \hspace{1cm} (13)

This formulation addresses some stability issues regarding the tail of \( f \) and \( g \) that are present in (12) and therefore is commonly used. As the previous sum, it also converges to (11) by the strong law of large numbers.

We propose to use stochastic resonance in order to retrieve the potentially important supports that have the dominant weights in the MMSE, and weight them properly by using importance sampling. Formally, we use SR as a support generator PDF \( S_j | y \sim g(S | y) \), use the oracle estimators \( h(\alpha | S, y) = \hat{\alpha}_{S,y}^{\text{Oracle}} \) and their MMSE unnormalized weights \( f(S | y) \). Plugging this into (13) we get:

\[ \hat{\alpha}(y) = \mathbb{E} \left[ \alpha | y \right] \approx \frac{\sum_{S_j} f(S_j | y) \hat{\alpha}_{S_j,y}^{\text{Oracle}}}{\sum_{S_j} f(S_j | y)}, \quad S_j | y \sim g(S | y). \]

We now write the explicit expressions for each of the components described above:

- \( \hat{\alpha}_{S,y}^{\text{Oracle}} \) – This is the same as stated in (1).
- \( f(S | y) \) – This is the un-normalized probability for a support \( S \) given the noisy measurements vector \( y \). Again, given the described generative model and from (3)

\[ P(S | y) \propto f(S | y) = t_S. \]

- \( g(S | y) \) – This is the support generator probability function given the noisy measurements \( y \). As previously mentioned we would like to have a way of generating probable supports. Using the SR concept we shall create likely supports simply by adding SR noise \( n \) to the measurements \( y \) and run a pursuit algorithm. Clearly, as we add more noise, each of the iterations of the algorithm might retrieve a different support. If we add too much noise then the supports recovered are not necessarily likely and we might miss the “preferred supports” with the highest MMSE weights. This means that another parameter for this generating function \( g \) is also the amount of noise to be added, \( \sigma_n \). That being said, as long as \( \sigma_n > 0 \), after sufficient iterations the algorithm will converge to the MMSE.

In order to quantify \( g_{\sigma_n}(S | y) \) we simply use the empirical distribution. In other words, if we run \( K \) iterations and a specific support occurred \( k \) times, its probability is simply \( g_{\sigma_n}(S | y) = \frac{k}{K} \).

We now approximate the NP-Hard MMSE computation simply by recovering the likely supports using any pursuit algorithm on the SR-noisy measurements and use (13). Note that this approximation is guaranteed to asymptotically converge to the MMSE with probability 1. As a result, one might wonder about the rate of convergence. We explore this question numerically, and describe the following experiment.

We start by repeating the previously described experiments for comparison. Note that in this experiment there are only 100 possibilities for the support and therefore when running 100 iterations we are not surprised that the algorithm has successfully converged. These results can be seen in Figure 8a. To show the efficiency of this technique we compare it with the same settings, only this time the cardinality is \( ||\alpha||_0 = 3 \) giving it \( \binom{100}{3} = 161,700 \) possibilities, all apriori equally likely. These results can be seen in Figure 8b. Note that the MAP and the MMSE are missing in this figure due to the impractical amount of computation required. Compared to the SR methods described earlier, this method is less sensitive to the variance of the added noise. As the noise increases we observe a minor degradation since the most likely supports are not recovered anymore. That being said, it seems that as the number of possible supports increases, more iterations will be required for convergence, or at least for outperforming the previous SR method. In Figure 8c we see that after 200 iterations the two perform roughly the same, but importance sampling is much less sensitive to \( \sigma_n \). Note that with 200 iterations we recover at most 200 different supports and that is only \( \approx 0.12\% \) of all the possible supports.

To summarize, as we add more iterations we are guaranteed to asymptotically converge to the MMSE, but a major improvement can be achieved even with a small amount of iterations. The biggest advantage of this method is its robustness to the SR noise energy, or variance.

VII. WHAT NOISE SHOULD BE USED?

Throughout this work we naturally used a white Gaussian noise by default. In this section we question this decision and wonder whether we can use noise models with different distributions and whether it affects the performance of the stochastic resonance estimator.
As mentioned briefly in VI-A1, the result of the additive noise multiplied by the dictionary $D^T$ is Gaussian under mild conditions. Denoting $n \triangleq D^T n$, each element $n_i$ is:

$$n_i = d_i^T n = \sum_{j=1}^{m_0} d_{ij}n_j.$$ 

Without loss of generality, we assume that the atoms are normalized, i.e., $\|d_i\|_2 = 1$. The above expression is a weighted average of $m_0$ iid variables $\{n_j\}^{m_0}_{j=1}$. If the noise $n_j$ has bounded mean and variance and, of course, assumed to have iid elements, then the central limit theorem holds. In this case, for large enough signal dimensions, $n_i$ is asymptotically Gaussian regardless of the distribution of the original additive noise $n_j$.

Following the previous statement, we experiment with a different distribution for a random noise vector. We will employ an element-wise iid uniform noise with 0 mean $n \sim U[-r,r]$. In order to compare with a Gaussian noise $n \sim N(0,\sigma_n^2)$ we choose $r = \sqrt{3}\sigma_n$ thus assuring the same standard deviation for the two cases. In Figure 9a we compare random Gaussian noise with a uniform noise distribution as described. By using uniform noise with a zero mean and the same standard deviation as the Gaussian noise, we see a perfect match between the two curves. Due to the central limit theorem, the noise distribution will practically not change much as long as they uphold the described conditions, and the signal’s dimensions are large enough.

Now that we know that the choice of the noise’s distribution will not effect the performance, is there a distribution from which we can benefit more than others in terms of computational efficiency? To explore this question, consider the following. Given the signal $y$, we define the subsampling noise $r_{\text{subsample}}$ in the following way:

$$n_i(y_i) = \begin{cases} 0 & \text{w.p. } p \\ -y_i & \text{w.p. } 1 - p \end{cases}.$$ 

This means that each of the elements of the SR-noisy measurements will be zero with probability $1 - p$ and only $p$ measurements will remain in the signal. This distribution is interesting because of the following reason. When zeroing out an element in the vector $y$, the matching row in the dictionary $D$ will always be multiplied by the zero element when calculating the correlations $D^T y_{\text{SR}}$ as done in most pursuits. This multiplication obviously has no contribution to the inner product and we might as well omit the zero elements from $y_{\text{SR}}$ and the corresponding rows from $D$, remaining with only a subsampled version of the signal $y$ and the dictionary $D$. In other words, in each of the SR iterations we simply subsample a random $np$ portion from the signal $y$ and the matching $np$ portion of rows from the dictionary $D$, which leads to $y_{\text{subsample}}$ of size $pn \times 1$ and a dictionary $D_{\text{subsample}}$ of size $pn \times m$. Finally we use apply a pursuit algorithm on the subsampled vectors. Recall that once the pursuit is done,
its result contains a noisy estimation of the signal due to the added SR noise \( n_{\text{subsample}} \). Just like in the previous cases we should now use the pursuit’s result only as a support estimator in order to calculate the subtractive SR estimator. To do so, once the pursuit is done, we should turn back to the full sized signal \( y \) and dictionary \( D \) and calculate the oracle estimator using the support recovered by the subsampled pursuit.

The described process can be equally formulated by sampling a random mask \( M_{pn \times n} \) which is a random subsample of \( pn \) rows from the diagonal identity matrix \( I_{n \times n} \). In this formulation, we create many base estimators by applying a pursuit on the sub-sampled signal \( y_{\text{sub}} = My \), using the sub-sampled dictionary \( D_{\text{sub}} = MD \). Note that when applying this method, each pursuit has a computational benefit over the previous methods due to the decreased size of the signal’s dimension. In Figure 9b we show the results for the same experiment described in Figure 9a, this time using the subsampling approach. In this figure the \( x \) axis represents the probability \( p \), the percentage of elements kept in each pursuit. We see that the two methods have similar MSE performance for the optimal choice of \( \sigma_v \) and \( p \), but this method is faster due to the decreased dimension of the problem.

VIII. IMAGE DENOISING

In this section we demonstrate the benefits of using SR for facial image denoising. In this experiment we use the Trainlets [21] dictionary trained on facial images from the Chinese Passport dataset as described in [22]. In the dataset, each image is of size \( 100 \times 100 \) pixels and contains a grayscale aligned face. The application we address is denoising, that is approximating the following optimization problem:

\[
\hat{\alpha} = \arg \min_{\alpha} \| D\alpha - y \|_2 \quad \text{s.t.} \quad \| \alpha \|_0 = L.
\]

The approximation is achieved using the Subspace Pursuit (SP) algorithm [23] which provides a fast converging algorithm for a fixed number of non-zeros \( L \). The number of non-zeros \( L \) was empirically set to maximize denoising performance. For Stochastic Resonance we used Algorithm 1 in its subtractive form with 200 iterations and the same SP settings. Note that we do not seek state of the art denoising results but rather to show that SR can improve real image processing tasks.

We corrupt an image from the dataset with additive white Gaussian noise, each time with different standard deviation \( \sigma_v \). After that, for each image we applied SP using the Trainlets dictionary and for each \( \sigma_v \) choose \( L \) such that the denoised results would be optimal under the PSNR measure. Next we took the noisy images and used Algorithm 1 to denoise the image for varying SR noise standard deviation while fixing the parameter \( L \) to the optimal value found earlier.

In Figure 10 the results for \( \sigma_v = 40 \) can be seen. Importantly, the SR result yield a clearer image with much less artifacts. Figure 11a presents the effectiveness of SR under varying SR noise \( \sigma_n \). We see that a gain of almost 2 dB is achieved by using SR with a proper \( \sigma_n \) over the regular pursuit. Figure 11b presents a comparison of SP vs. SR for varying values of the noise’s standard deviation \( \sigma_v \). In all of the described experiments, SR improved the denoising results. Generally we observe that as the noise is increased, the improvement is more significant.

IX. CONCLUSION

In this work we suggested algorithms leveraging the idea of stochastic resonance under the context of sparse coding algorithms. We analyzed their theoretical properties and showed that one can efficiently deploy pursuit algorithms while boosting their performance with SR, which provides an approximation to the MMSE estimator. The unitary dictionary case was fully analyzed, while we provided different theoretical insights for the general dictionary case. We have demonstrated stochastic resonance as a practical and effective MMSE approximation that has the ability to use any pursuit algorithm as a “black box”, thus opening the door for MMSE approximation in large dimension regimes for the first time.
APPENDIX A
UNITARY SR ESTIMATOR EXPECTATION

For the non-subtractive case:
\[ E_n [\mathcal{H}_\lambda (\beta + n)] = \int_{-\infty}^{\infty} \mathcal{H}_\lambda (\beta + n) p(n) \, dn \]
\[ = \int_{|\beta + n| \geq \lambda} c^2 (\beta + n) p(n) \, dn \]
\[ = c^2 \left[ \int_{-\infty}^{-\lambda - \beta} (\beta + n) p(n) \, dn + \int_{\lambda - \beta}^{\infty} (\beta + n) p(n) \, dn \right] \]
\[ = c^2 \left[ \int_{-\infty}^{-\lambda - \beta} 1 \frac{1}{\sqrt{2\pi} \sigma_n^2} e^{-\frac{n^2}{2\sigma_n^2}} \, dn + \int_{\lambda - \beta}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_n^2} e^{-\frac{n^2}{2\sigma_n^2}} \, dn \right] \]
\[ = c^2 \beta Q \left( \frac{\lambda + \beta}{\sigma_n} \right) + \beta Q \left( \frac{\lambda - \beta}{\sigma_n} \right) + \frac{\sigma_n}{\sqrt{2\pi}} \left( e^{\frac{(\lambda - \beta)^2}{2\sigma_n^2}} - e^{\frac{(\lambda + \beta)^2}{2\sigma_n^2}} \right) . \]

Similarly, for the subtractive case:
\[ E_n [\mathcal{H}_\lambda^- (\beta, n)] = \int_{-\infty}^{\infty} \mathcal{H}_\lambda^- (\beta + n) p(n) \, dn \]
\[ = \int_{|\beta + n| \geq \lambda} c^2 \beta p(n) \, dn \]
\[ = c^2 \left[ \int_{-\infty}^{-\lambda - \beta} \beta p(n) \, dn + \int_{\lambda - \beta}^{\infty} \beta p(n) \, dn \right] \]
\[ = c^2 \beta \left[ \int_{-\infty}^{-\lambda - \beta} \frac{1}{\sqrt{2\pi} \sigma_n^2} e^{-\frac{n^2}{2\sigma_n^2}} \, dn + \int_{\lambda - \beta}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_n^2} e^{-\frac{n^2}{2\sigma_n^2}} \, dn \right] \]
\[ = c^2 \beta \left[ Q \left( \frac{\lambda + \beta}{\sigma_n} \right) + Q \left( \frac{\lambda - \beta}{\sigma_n} \right) \right] . \]

APPENDIX B
THE SURE SURFACE FOR THE UNITARY CASE

Plugging the subtractive estimator \( \mathcal{H}_\lambda^- \) into (4) leads to the following expression:
\[ E_n [\mu (\mathcal{H}_\lambda^-)] = \sum_i \left( c^2 \beta_i \left[ Q \left( \frac{\lambda + \beta_i}{\sigma_n} \right) + Q \left( \frac{\lambda - \beta_i}{\sigma_n} \right) \right] \right) + \sum_i 2c^2 \beta_i^2 \left[ Q \left( \frac{\lambda + \beta_i}{\sigma_n} \right) + Q \left( \frac{\lambda - \beta_i}{\sigma_n} \right) \right] + \sum_i 2\sigma_n^2 c^2 \beta_i \left[ Q \left( \frac{\lambda + \beta_i}{\sigma_n} \right) + Q \left( \frac{\lambda - \beta_i}{\sigma_n} \right) \right] + \sum_i 2\sigma_n^2 c^2 \beta_i \left[ \frac{1}{\sqrt{2\pi} \sigma_n} e^{-\frac{(\lambda - \beta_i)^2}{2\sigma_n^2}} - \frac{1}{\sqrt{2\pi} \sigma_n} e^{-\frac{(\lambda + \beta_i)^2}{2\sigma_n^2}} \right] . \]

In order to show that it is indeed easy to optimize \( \lambda \) and \( \sigma \) on the SURE surface, we demonstrate it by the following experiment. We generated a sparse signal with probability of \( P_i = 0.01 \) for any coefficient to be non-zero. The coefficients of the non-zero entries are drawn from a Gaussian distribution \( \mathcal{N} (0, 1) \). We project the signal by a unitary dictionary and added random Gaussian noise \( \mathcal{N} (0, 0.2) \). Each signal has been estimated using the described subtractive estimator. Figure 12 shows the SURE surface over different \( \lambda \) and \( \sigma_n \) values, and Figure 13 shows the MSE results respectively. We can see that the SURE surface behaves just like the true MSE up to an additive constant and that it is smooth and rather easy to optimize. Also, in terms of MSE we see the obvious superiority of the proposed estimator over the MAP estimator, and that it is quite close to the MMSE.

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