Research Article
Numerical Solutions of Time Fractional Zakharov-Kuznetsov Equation via Natural Transform Decomposition Method with Nonsingular Kernel Derivatives

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In this paper, we have studied the time-fractional Zakharov-Kuznetsov equation (TFZKE) via natural transform decomposition method (NTDM) with nonsingular kernel derivatives. The fractional derivative considered in Caputo-Fabrizio (CF) and Atangana-Baleanu derivative in Caputo sense (ABC). We employed natural transform (NT) on TFZKE followed by inverse natural transform, to obtain the solution of the equation. To validate the method, we have considered a few examples and compared with the actual results. Numerical results are in accordance with the existing results.

1. Introduction

Fractional calculus is an emerging field in various branches of engineering science. Fractional differential equations attracted researchers as they used to model a variety of diverse applications such as visco elasticity, heat conduction, biology, and dynamical systems [1–7]. Due to its importance in diverse fields, considerable methods developed to study the exact and computational solutions of fractional differential equations. Other than the modelling, divergence and convergence of the solutions are also equally important. A suitable definition is essential for a fractional generalization of a physical model. Several fractional derivative definitions developed in the last few decades. Some of the popular definitions in the literature are Riemann-Liouville (R-L), Caputo, CF, ABC, Grunwald-Letnikov, and Riesz fractional derivatives. For more details, we refer to [8, 9] and the references therein. R-L and Caputo fractional derivatives have a singular kernel. Recently, two nonsingular kernel fractional derivative definitions are developed by Atangana-Baleanu and Caputo-Fabrizio. Several methods are being investigated for the analysis of fractional differential equations for accuracy and reliable solutions. Some of the popular semi analytical and numerical methods are variational iteration method (VIM) [10], fractional differential transform method [11–14], homotopy perturbation transform method (HPTM) [15], homotopy analysis transform method [16, 17], residual power series method (RPS) [18], q-homotopy analysis transform method (q-HATM) [19–21], operational matrix method [22], tension spline method [23], parametric cubic
Able in the literature; for more details, we refer [8, 36–38]. In this section, we give the definitions of R-L, Caputo, CF, and ABC fractional derivatives for the benefit of the readers.

**Definition 1** (see [36]). The R-L left-sided fractional integral operator of a function \( f \in C_{\nu}, \nu \geq -1 \) is given as

\[
I^\nu f(\omega) = \frac{1}{\Gamma(\mu)} \int_0^\omega (\omega - \zeta)^{\mu-1} f(\zeta) d\zeta, \mu > 0, \omega > 0,
\]

and \( I^0 f(\omega) = f(\omega) \).

**Definition 2** (see [1]). The Caputo sense fractional derivative of \( f(\omega) \) is defined by

\[
C^\mu_0 D^\mu f(\omega) = I^{m-\mu} D^m f(\omega) = \frac{1}{m-\mu} \int_0^\omega (\omega - \zeta)^{m-\mu-1} f^m(\zeta) d\zeta,
\]

for \( m-1 < \mu \leq m, m \in \mathbb{N}, \omega > 0, f \in C^m_\nu, \) and \( \nu \geq -1 \).

**Definition 3** (see [39]). The CF fractional derivative of \( f(\omega) \) is given by

\[
C^\mu_0 D^\mu f(\omega) = B(\mu) \int_0^\omega \exp \left( \frac{-\mu(\omega - \zeta)}{1-\mu} \right) D(f(\zeta)) d\zeta,
\]

where \( 0 < \mu < 1 \) and \( B(\mu) \) is a normalization function, where \( B(0) = B(1) = 1 \).

**Definition 4** (see [40]). The ABC fractional derivative of \( f(\omega) \) is presented as

\[
A^\mu_0 D^\mu f(\omega) = B(\mu) \int_0^\omega E_\mu \left( -\mu(\omega - \zeta) \right) D(f(\zeta)) d\zeta,
\]

where \( 0 < \mu < 1 \). Normalization function is \( B(\mu) \), and the Mittag-Leffler function is \( E_\mu(z) = \sum_{n=0}^{\infty} (z/\Gamma(\mu l + 1))^n \). These definitions widely used to study the fractional differential equation solutions using numerous integral transform techniques such as Sumudu transform, Shehu transform, and Laplace transform. Recently, natural transform of these definitions applied to study various differential equations; for more details, we refer [33, 41, 42].

### 2. Basic Definitions

There are various fractional derivative definitions that are available in the literature; for more details, we refer [8, 36–38]. In the present study, we consider the TFZKE. Rawashdeh and Maitama [33] introduced NTDM for nonsingular differential equations; for more details, we refer [33, 41, 42]. Computational time can

| \( x \) | \( \zeta \) | \( t \) | \( \text{NTDM}_{\text{CF}} \) | \( \text{NTDM}_{\text{ABC}} \) | \( \text{PIA} [31] \) | \( \text{RPSM} [31] \) |
|---|---|---|---|---|---|---|
| 0.1 | 0.1 | 0.2 | 3.8519 E-07 | 3.8519 E-07 | 3.8521 E-07 | 3.8521 E-07 |
| 0.6 | 0.6 | 0.2 | 4.6474 E-05 | 4.6474 E-05 | 4.6634 E-05 | 4.6639 E-05 |
| 0.9 | 0.9 | 0.2 | 4.9249 E-04 | 4.9249 E-04 | 5.1213 E-04 | 5.1424 E-04 |
| 0.9 | 0.9 | 0.3 | 6.7503 E-04 | 6.7503 E-04 | 7.3819 E-04 | 7.4845 E-04 |
| 0.9 | 0.9 | 0.4 | 8.1510 E-04 | 8.1510 E-04 | 9.5794 E-04 | 9.8914 E-04 |
be reduced in this transform than other traditional methods while preserving the efficiency. When \( v = 1 \) and \( s = 1 \), the NT reduced to the Laplace transform and Sumudu integral transform, respectively.

**Definition 5.** The natural transform of \( u(t) \) is defined by

\[
\text{NT}(u(t)) = \mathcal{H}(s,v) = \int_{-\infty}^{\infty} e^{-st} u(\nu t) dt, \ s, v \in (-\infty, \infty). \tag{7}
\]

For \( t \in (0,\infty) \), natural transform of \( u(t) \) is defined by

\[
\text{NT}(u(t)H(t)) = \text{NT}^+ \mathcal{H}(s,v) = \mathcal{H}^+(s,v) = \int_{0}^{\infty} e^{-st} u(\nu t) dt, \ s, v \in (0,\infty), \tag{8}
\]

where \( H(t) \) is the Heaviside function.

**Definition 6.** The inverse natural transform of \( \mathcal{H}(s,v) \) is given by

\[
\text{NT}^{-1} \mathcal{H}(s,v) = u(t), \forall t \geq 0. \tag{9}
\]

**Lemma 7** (linearity property). If natural transform of \( u_1(t) \) is \( \mathcal{H}_1(s,v) \) and \( u_2(t) \) is \( \mathcal{H}_2(s,v) \), then

\[
\text{NT}[c_1 u_1(t) + c_2 u_2(t)] = c_1 \text{NT}[u_1(t)] + c_2 \text{NT}[u_2(t)] \tag{10}
\]

where \( c_1 \) and \( c_2 \) are constants.

**Lemma 8** (inverse linearity property). If inverse natural transform of \( u_1(s,v) \) and \( u_2(s,v) \) is \( u_1(t) \) and \( u_2(t) \), respectively, then

\[
\text{NT}^{-1}[c_1 u_1(s,v) + c_2 u_2(s,v)] = c_1 \text{NT}^{-1}[u_1(s,v)] + c_2 \text{NT}^{-1}[u_2(s,v)] = c_1 u_1(t) + c_2 u_2(t), \tag{11}
\]

where \( c_1 \) and \( c_2 \) are constants.
Figure 1: Continued.
Definition 9 (see [43]). Natural transform of $D_t^\mu u(t)$ by means of Caputo sense is given as

$$\mathrm{NT}[D_t^\mu u(t)] = \left(\frac{s}{v}\right)^\mu \left(\mathrm{NT}[u(t)] - \frac{1}{s} u(0)\right).$$

(12)

Definition 10 (see [44]). Natural transform of $D_t^\mu u(t)$ by means of CF is defined as

$$\mathrm{NT}[D_t^\mu u(t)] = \frac{1}{1-\mu + \mu \left(\frac{v}{s}\right)} \left(\mathrm{NT}[u(t)] - \frac{1}{s} u(0)\right).$$

(13)

With this motivation, we defined natural transform of ABC derivative as follows.

Definition 11. Natural transform of $D_t^\mu u(t)$ by means of ABC derivative is defined as

$$\mathrm{NT}[D_t^\mu u(t)] = \frac{M[\mu]}{1-\mu + \mu \left(\frac{v}{s}\right)} \left(\mathrm{NT}[u(t)] - \frac{1}{s} u(0)\right).$$

(14)

3. Methodology

In this section, we present a novel approximate analytical procedure based on natural transform [42] to the following equation

$$D_t^\mu u(\zeta, t) = \mathcal{D}(u(\zeta, t)) + \mathcal{N}(u(\zeta, t)) + h(\zeta, t),$$

(15)

with the initial condition

$$u(\zeta, 0) = \phi(\zeta),$$

(16)

where $\mathcal{N}$, $\mathcal{D}$, and $h(\zeta, t)$ are nonlinear, linear, and source terms, respectively. Now we employing NT on equation (15) by considering fractional derivative by means of three fractional definitions.

Case 1. (NTDM$_{CF}$). By taking natural transform of equation (15), by means of CF fractional derivative, we obtain

$$\frac{1}{p(\mu, v, s)} \left(\mathrm{NT}[u(\zeta, t)] - \frac{\phi(\zeta)}{s}\right) = \mathrm{NT}[M(\zeta, t)],$$

(17)

where

$$p(\mu, v, s) = 1 - \mu + \mu \left(\frac{v}{s}\right).$$

(18)

By taking inverse natural transform using (8), we rewrite (17) as

$$u(\zeta, t) = \mathrm{NT}^{-1} \left[\frac{\phi(\zeta)}{s} + p(\mu, v, s)\mathrm{NT}[M(\zeta, t)]\right].$$

(19)

$\mathcal{N}(u(\zeta, t))$ can be decomposed into

$$\mathcal{N}(u(\zeta, t)) = \sum_{l=0}^{\infty} A_l,$$

(20)

where $A_l$ is the Adomian polynomials [45, 46]. We assume that equation (15) has the analytical expansion

$$u(\zeta, t) = \sum_{l=0}^{\infty} u_l(\zeta, t).$$

(21)

By substituting equations (20) and (21) into (19), we obtain

\begin{align*}
\end{align*}
Figure 2: Continued.
\[\sum_{l=0}^{\infty} u_l(\zeta, t) = N T^{-1} \left[ \frac{\phi(\zeta)}{s} + p(\mu, v, s) N T[h(\zeta, t)] \right] + N T^{-1} \left[ p(\mu, v, s) N T \left[ \sum_{l=0}^{\infty} \mathcal{L}[u_l(\zeta, t)] + A_l \right] \right].\]

From (22), we get
\[u_0^{CF}(\zeta, t) = N T^{-1} \left[ \frac{\phi(\zeta)}{s} + p(\mu, v, s) N T[h(\zeta, t)] \right] u_1^{CF}(\zeta, t) + N T^{-1} \left[ p(\mu, v, s) N T \left[ \sum_{l=0}^{\infty} \mathcal{L}[u_l(\zeta, t)] + A_l \right] \right].\]

By substituting (23) into (21), we get the NTDM solution of (15) and (16) as
\[u^{CF}(\zeta, t) = u_0^{CF}(\zeta, t) + u_1^{CF}(\zeta, t) + u_2^{CF}(\zeta, t) + \cdots.\]

**Case 2. (NTDM_{ABC}).** By taking natural transform of equation (15), by means of ABC derivative, we acquire
\[\frac{1}{q(\mu, v, s)} \left( N T[u(\zeta, t)] - \frac{\phi(\zeta)}{s} \right) = N T[M(\zeta, t)],\]
where
\[q(\mu, v, s) = \frac{1 - \mu + \mu(\nu/s)^\mu}{B(\mu)}.\]
By taking inverse natural transform using (8), we rewrite (25), as
\[ u(\zeta, t) = NT^{-1} \left[ \frac{\phi(\zeta)}{s} + q(\mu, v, s) NT[h(\zeta, t)] \right]. \] (27)

\[ \mathcal{N}(u(\zeta, t)) \text{ can be decomposed into} \]
\[ \mathcal{N}(u(\zeta, t)) = \sum_{i=0}^{\infty} A_i, \] (28)

where \( A_i \) is the Adomian polynomials. We assume that equation (15) has the analytical expansion
\[ u(\zeta, t) = \sum_{i=0}^{\infty} u_i(\zeta, t). \] (29)

By substituting equations (28) and (29) into (27), we obtain
\[ \sum_{i=0}^{\infty} u_i(\zeta, t) = NT^{-1} \left[ \frac{\phi(\zeta)}{s} + q(\mu, v, s) NT[h(\zeta, t)] \right] + NT^{-1} \left[ q(\mu, v, s) NT \left( \sum_{i=0}^{\infty} \mathcal{D}(u_i(\zeta, t)) + A_1 \right) \right]. \] (30)

From (30), we get
\[ u_{ABC}(\zeta, t) = NT^{-1} \left[ \frac{\phi(\zeta)}{s} \right] + NT^{-1} \left[ q(\mu, v, s) NT[h(\zeta, t)] \right] u_{ABC}^{ABC}(\zeta, t) = NT^{-1} \left[ q(\mu, v, s) NT[\mathcal{D}(u_i(\zeta, t)) + A_1] \right], \] (31)

By substituting (31) into (29), we get the NTDM_{ABC} solution of (15)–(16) as
\[ u_{ABC}(\zeta, t) = u_0^{ABC}(\zeta, t) + u_1^{ABC}(\zeta, t) + u_2^{ABC}(\zeta, t) + \cdots \] (32)

4. Convergence Analysis

We have presented uniqueness and convergence of the NTDM_{CF} and NTDM_{ABC} in this section.

Theorem 12. The NTDM_{CF} solution of (15) is unique when \( 0 < (\delta_1 + \delta_2)(1-\mu + \mu t) < 1 \).

Proof. Let \( F = (C[\mathcal{M}], \|\cdot\|) \) be the Banach space with the norm \( \|\phi(t)\| = \max_{t \in [0,t]} |\phi(t)| \), \( \forall \) continuous functions on \( J \). Let \( G : F \rightarrow F \) is a nonlinear mapping, where
\[ u_{i+1}(\zeta, t) = u_i^C + NT^{-1}[p(\mu, v, s) NT[\mathcal{D}(u_i(\zeta, t)) + \mathcal{N}(u_i(\zeta, t))]], i \geq 0. \] (33)

Suppose that \( |\mathcal{D}(u) - \mathcal{D}(u^*)| < \delta_1 |u - u^*| \) and \( |\mathcal{N}(u) - \mathcal{N}(u^*)| < \delta_2 |u - u^*| \), where \( \delta_1 \) and \( \delta_2 \) are Lipschitz constants and \( u = u(\zeta, t) \) and \( u^* = u^*(\zeta, t) \) are two different function values.
\[ ||Gu - Gu^*|| \leq \max_{\mathcal{D}[\mathcal{M}]} [NT^{-1}[p(\mu, v, s) NT[\mathcal{D}(u) - \mathcal{D}(u^*)] + p(\mu, v, s) NT[\mathcal{N}(u) - \mathcal{N}(u^*)]] \]
\[ \leq \max_{\mathcal{D}[\mathcal{M}]} [\delta_1 NT^{-1}[p(\mu, v, s) NT[u - u^*]] \]
\[ + \delta_2 NT^{-1}[p(\mu, v, s) NT[u - u^*]] \]
\[ \leq \max_{\mathcal{D}[\mathcal{M}]} [\delta_1 + \delta_2] [NT^{-1}[p(\mu, v, s) NT[u - u^*]] \]
\[ \leq (\delta_1 + \delta_2) [NT^{-1}[p(\mu, v, s) NT[u - u^*]] \]
\[ = (\delta_1 + \delta_2)(1 - \mu + \mu t) ||u - u^*||. \] (34)

\( G \) is contraction as \( 0 < (\delta_1 + \delta_2)(1 - \mu + \mu t) < 1 \). The solution of (15) is unique from Banach fixed point theorem. 

\( \square \)
Theorem 13. The $\text{NTDM}_{ABC}$ solution of (15) is unique when $0 < (\delta_1 + \delta_2)(1 - \mu + \mu(t^n/\Gamma(\mu + 1))) < 1$.

Proof. Let $F = (C/J, ||||)$ be the Banach space with the norm $\|\phi(t)\| = \max_{t<\tau}||\phi(t)||$, $\forall$ continuous functions on $J$. Let $G: F \to F$ be a nonlinear mapping, where

$$u_{\ell+1}(t, \tau) = u_{\ell}(t, \tau) + N^T[q(\mu, \nu, s)\mathcal{DC}(u_{\ell}(t, \tau)) + \mathcal{M}(u_{\ell}(t, \tau))]$$

Suppose that $|\mathcal{D}(u) - \mathcal{D}(u^*)| < \delta_1|u - u^*|$ and $|\mathcal{M}(u) - \mathcal{M}(u^*)| < \delta_2|u - u^*|$, where $\delta_1$ and $\delta_2$ are Lipschitz constants and $u = u(t, \tau)$ and $u^* = u^*(t, \tau)$ are two different function values.

$$\|Gu - Gu^*\| \leq \max_{t, s} |N^{-1}q(\mu, \nu, s)\mathcal{D}(u) - \mathcal{D}(u^*)|$$
$$+ |\max_{t, s} \delta_1|u - u^*||$$
$$\leq \max_{t, s} | \delta_2|\mathcal{D}(u) - \mathcal{D}(u^*)|$$
$$\leq \max_{t, s} \delta_2|u - u^*||$$
$$\leq \max_{t, s} \delta_1 + \delta_2\|u - u^*\|$$

(36)

$$= (\delta_1 + \delta_2)\left[1 - \mu + \mu(t^n/\Gamma(\mu + 1))\right]|u - u^*|.$$

$G$ is contraction as $0 < (\delta_1 + \delta_2)(1 - \mu + \mu(t^n/\Gamma(\mu + 1))) < 1$. The solution of (15) is unique from Banach fixed point theorem.

(37)

Theorem 14. $\text{NTDM}_{CF}$ solution of (15) is convergent.

Proof. Let $u_m = \sum_{r=0}^{m} u_{r}(t, \tau)$. To prove that $u_m$ is a Cauchy sequence in $F$. Consider

$$\|u_{m} - u_{n}\| = \max_{t, s} \|u_{m} - u_{n}\| = \max_{t, s} \sum_{r=0}^{m} u_{r}||, n = 1, 2, 3, \ldots, \leq \max_{t, s} |N^{-1}$$

$$\left[ p(\mu, \nu, s)\mathcal{D}(u_{r-1}) + \mathcal{M}(u_{r-1}) \right] ||$$
$$\leq \max_{t, s} |N^{-1}p(\mu, \nu, s)\mathcal{D}(u_{r-1}) + \mathcal{M}(u_{r-1}) ||$$
$$\leq \max_{t, s} |N^{-1}p(\mu, \nu, s)\mathcal{D}(u_{r-1}) + \mathcal{M}(u_{r-1}) ||$$

$$= (\delta_1 + \delta_2)(1 - \mu + \mu(t^n/\Gamma(\mu + 1))|u_{m-1} - u_{n-1}|.$$

Table 3: Absolute errors of $\text{NTDM}_{CF}$ and $\text{NTDM}_{ABC}$ with existing methods of Example 2 when $\lambda = 0.001$.

| $x, \xi$ | $t$ | $\text{NTDM}_{CF}$ | $\text{NTDM}_{ABC}$ | $\text{FNDM}_{[28]}$ | $\eta - \text{HATM}_{[28]}$ |
|----------|-----|----------------|----------------|----------------|----------------|
| 0.02     | 0.02| 4.9926E-09    | 4.9926E-09    | 4.9926E-09    | 4.9926E-09    |
| 0.04     | 0.06| 9.9852E-09    | 9.9852E-09    | 9.9852E-09    | 9.9852E-09    |
| 0.08     | 0.08| 1.9970E-08    | 1.9970E-08    | 1.9970E-08    | 1.9970E-08    |
| 0.10     | 0.10| 2.4963E-08    | 2.4963E-08    | 2.4963E-08    | 2.4963E-08    |
| 0.02     | 0.06| 1.4979E-08    | 1.4979E-08    | 1.4979E-08    | 1.4979E-08    |
| 0.08     | 0.08| 1.9972E-08    | 1.9972E-08    | 1.9972E-08    | 1.9972E-08    |
| 0.10     | 0.10| 2.4965E-08    | 2.4965E-08    | 2.4965E-08    | 2.4965E-08    |
| 0.02     | 0.04| 4.9934E-09    | 4.9934E-09    | 4.9934E-09    | 4.9934E-09    |
| 0.04     | 0.04| 4.9934E-09    | 4.9934E-09    | 4.9934E-09    | 4.9934E-09    |
| 0.06     | 0.06| 1.4980E-08    | 1.4980E-08    | 1.4980E-08    | 1.4980E-08    |
| 0.08     | 0.08| 1.9974E-08    | 1.9974E-08    | 1.9974E-08    | 1.9974E-08    |
| 0.10     | 0.10| 2.4967E-08    | 2.4967E-08    | 2.4967E-08    | 2.4967E-08    |
| 0.02     | 0.06| 1.4983E-08    | 1.4983E-08    | 1.4983E-08    | 1.4983E-08    |
| 0.08     | 0.08| 1.9977E-08    | 1.9977E-08    | 1.9977E-08    | 1.9977E-08    |
| 0.10     | 0.10| 2.4971E-08    | 2.4971E-08    | 2.4971E-08    | 2.4971E-08    |
| 0.02     | 0.06| 1.4986E-08    | 1.4986E-08    | 1.4986E-08    | 1.4986E-08    |
| 0.08     | 0.08| 1.9981E-08    | 1.9981E-08    | 1.9981E-08    | 1.9981E-08    |
| 0.10     | 0.10| 2.4976E-08    | 2.4976E-08    | 2.4976E-08    | 2.4976E-08    |
Let $m = n + 1$, then

$$
\| u_{n+1} - u_n \| \leq \delta \| u_n - u_{n-1} \| \leq \delta^2 \| u_{n-1} - u_{n-2} \| \\
\leq \cdots \leq \delta^n \| u_1 - u_0 \|,
$$

where $\delta = (\delta_1 + \delta_2)(1 - \mu + \mu t)$. Similarly, we have

$$
\| u_m - u_n \| \leq \delta^n \| u_{n+1} - u_n \| + \| u_{n+2} - u_{n+1} \| + \cdots + \| u_m - u_{m-1} \|,
$$

$$
\leq \delta^n \left( \frac{1 - \delta^{m-n}}{1 - \delta} \right) \| u_1 - u_0 \|,
$$

(38)

As $0 < \delta < 1$, we get $1 - \delta^{m-n} < 1$. Therefore

$$
\| u_m - u_n \| \leq \delta^n \frac{\max \| u_1 \|}{1 - \delta} \| u_1 \|.
$$

(39)

Since $\| u_1 \| < \infty$, $\| u_m - u_n \| \to 0$ when $n \to \infty$. Hence, $u_m$ is a Cauchy sequence in $F$; therefore, the series $u_m$ is convergent.

**Theorem 15.** $NTDM_{ABC}$ solution of (15) is convergent.

**Proof.** Let $u_m = \sum_{r=0}^{m} u_r(\xi, t)$. To prove that $u_m$ is a Cauchy sequence in $F$, consider

$$
\| u_m - u_n \| = \max_{t \in [0, T]} \| u_m - u_n \| = \max_{t \in [0, T]} \left| \sum_{r=-1}^{m} u_r \right|, n = 1, 2, 3, \ldots \leq \max_{t \in [0, T]} |NT^{-1} q(\mu, v, s)F(u_{m-1}, u_{n-1})| \leq \delta_1 \max_{t \in [0, T]} \left| q(\mu, v, s)F(u_{m-1}, u_{n-1}) \right|
$$

$$
\leq \max_{t \in [0, T]} |NT^{-1} \left[ q(\mu, v, s)F(u_{m-1}, u_{n-1}) \right] + \| F(u_{m-1}, u_{n-1}) \| |
$$

$$
= \delta_1 \max_{t \in [0, T]} \left| q(\mu, v, s)F(u_{m-1}, u_{n-1}) \right|
$$

(40)
Figure 4: (a) Exact solution, (b) absolute error of NTDM_{CF}, and (c) absolute error of NTDM_{ABC} for $\mu = 1, t = 0.5$, and $\lambda = 0.001$ of Example 2.
Figure 5: Continued.
Let \( m = n + 1 \), then

\[
\|u_{n+1} - u_n\| \leq \delta \|u_n - u_{n-1}\| \leq \delta^2 \|u_{n-1} - u_{n-2}\| \leq \cdots \leq \delta^n \|u_1 - u_0\|, 
\]

where \( \delta = (\delta_1 + \delta_2)(1 - \mu + \mu(t^\mu / \Gamma(\mu + 1))) \). Similarly, we have

\[
\|u_m - u_n\| \leq \|u_{n+1} - u_n\| + \|u_{n+2} - u_{n+1}\| + \cdots + \|u_m - u_{m-1}\| \\
\leq (\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}) \|u_1 - u_0\| \\
\leq \delta^n \left( \frac{1 - \delta^{m-n}}{1 - \delta} \right) \|u_1\|, 
\]

As \( 0 < \delta < 1 \), we get \( 1 - \delta^{m-n} < 1 \). Therefore

\[
\|u_m - u_n\| \leq \frac{\delta^n}{1 - \delta} \max \|u_1\|. 
\]

Since \( \|u_1\| < \infty \), \( \|u_m - u_n\| \to 0 \) when \( n \to \infty \). Hence, \( u_m \) is a Cauchy sequence in \( F \); therefore, the series \( u_m \) is convergent.

\section*{5. Numerical Examples}

This section includes the approximate analytical solutions for a few examples of TFZKE. We have chosen these equations as the closed form solutions are available and also well-known methods employed to study the solutions in the literature.
**Example 1.** TFZKE (1) is considered with the following parameters. Let $\xi = \eta = \delta = 2, a = 1, b = c = 1/8$, and $u(x, \zeta, 0) = (4/3)\lambda \sinh^2(x + \zeta)$ \cite{47, 48}. When $\mu = 1$, exact solution \cite{49} is $u(x, \zeta, t) = (4/3)\lambda \sinh^2(x + \zeta - \lambda t)$.

**NTDM$_{CF}$:** By employing NTDM$_{CF}$, we get
\begin{align*}
u_0^{CF}(x, \zeta, t) &= \frac{2}{3} \lambda \sinh^2(x + \zeta), \\
u_1^{CF}(x, \zeta, t) &= -\frac{8}{9} \lambda^2 (\mu(t-1) + 1)(5 \sinh(4(x + \zeta)) - 4 \sinh(2(x + \zeta))), \\
u_2^{CF}(x, \zeta, t) &= \frac{32}{27} \lambda^3 (2 \mu^2 - 4 \mu + (4 \mu - 4 \mu^2) t + \mu^2 t^2) \\
&\times (13 \cosh(2(x + \zeta)) + 75 \cosh(6(x + \zeta))) \\
&\quad - 70 \cosh(4(x + \zeta))).
\end{align*}

Substituting $u_0^{CF}(x, \zeta, t)$, $u_1^{CF}(x, \zeta, t)$, in (24), we obtain the NTDM$_{CF}$ solution as
\begin{align*}
u_0^{CF}(x, \zeta, t) &= \frac{2}{3} \lambda \sinh^2(x + \zeta) \\
&\quad - \frac{8}{9} \lambda^2 \mu(t-1) + 1) \sinh(2(x + \zeta)) \\
&\quad + \frac{32}{27} \lambda^3 (2 \mu^2 - 4 \mu + (4 \mu - 4 \mu^2) t + \mu^2 t^2) \\
&\quad \times (13 \cosh(2(x + \zeta)) + 75 \cosh(6(x + \zeta))) \\
&\quad - 70 \cosh(4(x + \zeta))).
\end{align*}

**NTDM$_{ABC}$:** By employing the NTDM$_{ABC}$, we get
\begin{align*}
u_0^{ABC}(x, \zeta, t) &= \frac{2}{3} \lambda \sinh^2(x + \zeta), \\
u_1^{ABC}(x, \zeta, t) &= \frac{8}{9} \lambda^2 \mu(t-1) + 1) \sinh(2(x + \zeta)) \\
&\quad - \frac{32}{27} \lambda^3 (2 \mu^2 - 4 \mu + (4 \mu - 4 \mu^2) t + \mu^2 t^2) \\
&\quad \times (13 \cosh(2(x + \zeta)) + 75 \cosh(6(x + \zeta))) \\
&\quad - 70 \cosh(4(x + \zeta))).
\end{align*}

Substituting $u_0^{CF}(x, \zeta, t)$, $u_1^{CF}(x, \zeta, t)$, in (24), we obtain the NTDM$_{CF}$ solution as
\begin{align*}
u_0^{ABC}(x, \zeta, t) &= \frac{2}{3} \lambda \sinh^2(x + \zeta) \\
&\quad - \frac{8}{9} \lambda^2 \mu(t-1) + 1) \sinh(2(x + \zeta)) \\
&\quad + \frac{32}{27} \lambda^3 (2 \mu^2 - 4 \mu + (4 \mu - 4 \mu^2) t + \mu^2 t^2) \\
&\quad \times (13 \cosh(2(x + \zeta)) + 75 \cosh(6(x + \zeta))) \\
&\quad - 70 \cosh(4(x + \zeta))).
\end{align*}

**Example 2.** TFZKE (1) is considered with the following parameters. Let $\xi = \eta = \delta = 3, a = 1, b = c = 2$, and $u(x, \zeta, 0) = (3/2)\lambda \sinh((x + \zeta)/6)$ \cite{47, 48}. When $\mu = 1$, exact solution \cite{49} is given by $u(x, \zeta, t) = (3/2)\lambda \sinh((x + \zeta - \lambda t)/6)$.

**NTDM$_{CF}$:** By employing the NTDM$_{CF}$, we get
\begin{align*}
u_0^{CF}(x, \zeta, t) &= \frac{2}{3} \lambda \sinh^2(x + \zeta), \\
u_1^{CF}(x, \zeta, t) &= \frac{3}{32} \lambda^2 \mu(t-1) + 1) \sinh(2(x + \zeta)) \\
&\quad - \frac{32}{27} \lambda^3 (2 \mu^2 - 4 \mu + (4 \mu - 4 \mu^2) t + \mu^2 t^2) \\
&\quad \times (5 \sinh(2(x + \zeta)) - 9 \cosh(2(x + \zeta))) \\
&\quad - 70 \cosh(4(x + \zeta))).
\end{align*}

**NTDM$_{ABC}$:** By employing the NTDM$_{ABC}$, we get
\begin{align*}
u_0^{ABC}(x, \zeta, t) &= \frac{2}{3} \lambda \sinh^2(x + \zeta), \\
u_1^{ABC}(x, \zeta, t) &= \frac{8}{9} \lambda^2 \mu(t-1) + 1) \sinh(2(x + \zeta)) \\
&\quad - \frac{32}{27} \lambda^3 (2 \mu^2 - 4 \mu + (4 \mu - 4 \mu^2) t + \mu^2 t^2) \\
&\quad \times (5 \sinh(2(x + \zeta)) - 9 \cosh(2(x + \zeta))) \\
&\quad - 70 \cosh(4(x + \zeta))).
\end{align*}

Substituting $u_0^{CF}(x, \zeta, t)$, $u_1^{CF}(x, \zeta, t)$, in (24), we obtain the NTDM$_{CF}$ solution as
\begin{align*}
u_0^{ABC}(x, \zeta, t) &= \frac{2}{3} \lambda \sinh^2(x + \zeta) \\
&\quad - \frac{8}{9} \lambda^2 \mu(t-1) + 1) \sinh(2(x + \zeta)) \\
&\quad + \frac{32}{27} \lambda^3 (2 \mu^2 - 4 \mu + (4 \mu - 4 \mu^2) t + \mu^2 t^2) \\
&\quad \times (5 \sinh(2(x + \zeta)) - 9 \cosh(2(x + \zeta))) \\
&\quad - 70 \cosh(4(x + \zeta))).
\end{align*}
\[ u_{\text{CF}}(x, \zeta, t) = \frac{3}{2} \lambda \sinh \left( \frac{x + \zeta}{6} \right) + \frac{3}{32} \lambda^3 (\mu(t-1)+1) \left( 5 \cosh \left( \frac{x + \zeta}{6} \right) - 9 \cosh \left( \frac{x + \zeta}{2} \right) \right) \]
\[ + \frac{3 \lambda^5}{1024} \left( -621 \sinh \left( \frac{x + \zeta}{2} \right) + 70 \sinh \left( \frac{x + \zeta}{6} \right) \right) + 765 \sinh \left( \frac{5(x + \zeta)}{6} \right) \]
\[ \times (\mu^2((t-4)t+2)+4\mu(t-1)+2) + \text{NTDMABC} \]

By employing the NTDMABC, we get
\[ u_{0}^{\text{ABC}}(x, \zeta, t) = \frac{3}{2} \lambda \sinh \left( \frac{x + \zeta}{6} \right), \]
\[ u_{1}^{\text{ABC}}(x, \zeta, t) = \frac{3 \lambda^5}{1024} \left( -621 \sinh \left( \frac{x + \zeta}{2} \right) + 70 \sinh \left( \frac{x + \zeta}{6} \right) \right) + 765 \sinh \left( \frac{5(x + \zeta)}{6} \right) \]
\[ \times (\mu^2((t-4)t+2)+4\mu(t-1)+2) + \text{NTDMABC} \]

Substituting \( u_{0}^{\text{ABC}}(x, \zeta, t) \) and \( u_{1}^{\text{ABC}}(x, \zeta, t) \), in (32), we obtain the NTDMABC solution as
\[ u_{2}^{\text{ABC}}(x, \zeta, t) = \frac{3 \lambda^5}{1024} \left( -621 \sinh \left( \frac{x + \zeta}{2} \right) + 70 \sinh \left( \frac{x + \zeta}{6} \right) \right) + 765 \sinh \left( \frac{5(x + \zeta)}{6} \right) \]
\[ \times (\mu^2((t-4)t+2)+4\mu(t-1)+2) + \text{NTDMABC} \]

6. Numerical Results and Discussion

Tables 1 and 2 demonstrate the comparison of absolute errors with the existing methods and approximate solutions for different fractional orders with different fractional derivatives, respectively, of Example 1. Absolute errors of Example 1 graphically represented in Figure 1 for fixed \( t \) when \( \mu = 1 \). In Figure 2, we plotted approximate solutions for different values of \( \mu \) for fixed \( t \) of Example 1. Figure 3 presents the comparison of NTDMABC solutions of Example 1 with exact solution for different values of fractional order \( \mu \) for fixed \( x \) and \( \zeta \). In Table 3, we presented absolute errors of two fractional derivative solutions and existing results of Example 2. We have tabulated the approximate solution of Example 2 for noninteger fractional values in

| \mu | Absolute Error |
|-----|---------------|
| 0.5 | 0.001         |
| 0.7 | 0.002         |
| 1   | 0.003         |

7. Conclusions

In this paper, we have studied the TFZKE through natural transformation by means of CF and ABC derivatives. We compared numerical results with the existing results. It is observed that the present method results are in accordance with existing methods. The NTDM is simple in its principles; also, NTDM is effective in solving nonlinear fractional differential equations, and promising method for a large varieties of such equations arises in mathematical physics.

Data Availability

There is no any data available.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors’ Contributions

All authors contributed equally to this work. And all the authors have read and approved the final version manuscript.

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