EXPLICIT FORMULAS FOR THE MOMENTS OF THE
SOJOURN TIME IN THE M/G/1 PROCESSOR
SHARING QUEUE WITH PERMANENT JOBS

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Abstract

We give some representation about recent achievements in analysis of the M/G/1 queue with egalitarian processor sharing discipline (EPS). The new formulas are derived for the $j$-th moments ($j \in \mathbb{N}$) of the (conditional) stationary sojourn time in the M/G/1—EPS queue with $K$ ($K \in 0 \cup \mathbb{N}$) permanent jobs of infinite size. We discuss also how to simplify the computations of the moments.

Keywords: processor sharing, sojourn time distribution, moments of sojourn time, permanent jobs, asymptotics

AMS subject classification: 60K25, 90B22

1 Introduction

Processor sharing queues, made very attractive models by the works of Kleinrock [1], [2] and Yashkov [3], [4], play a central role in queueing theory. These models were originally proposed to analyze the performance of scheduling algorithms in time-sharing computer systems, and continue to find new applications which pose interesting mathematical problems. Over the past few years, the processor sharing paradigm has emerged as a powerful concept for modeling of Web servers, in particular, for evaluating the flow-level performance of end-to-end flow control mechanisms like Transmission Control Protocol (TCP) in Internet.

The mathematical analysis of processor sharing queues has resulted in many insightful results. Yet, a number of challenging problems remains to be explored. The main goal of this paper is

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to gain understanding the problem of the moments of the stationary sojourn time in the M/G/1 queue with egalitarian processor sharing (EPS), and to derive the formulas for the j-th moments ($j \in \mathbb{N}$) of the (conditional) sojourn time in the M/G/1—EPS queue with $K$ ($K \in 0 \cup \mathbb{N}$) permanent jobs of infinite size. Our results complement and develop the corresponding sections of the paper by Yashkova and Yashkov [5].

An idea of the EPS discipline1 was introduced by Kleinrock [1] who studied only M/M/1 case as a limit of the round–robin queue. In particular, he first showed that the mean sojourn time conditioned on the initial job size (service requirement) of the job is linear function of the size of the job. For an overview of the literature on processor–sharing queueing systems we refer to Kleinrock [2] (1976), Koboyashi and Konheim [6] (1977), Jaiswal [7] (1982), Yashkov [4] (1987), [8] (1990) and Yashkova and Yashkov [5] (2003).

The exact determination of the stationary sojourn time distribution in the M/G/1—EPS queue was an open problem for a long time. After puzzling researchers for 15 years, Yashkov [9] (1981), [3] (1983) found an analytic solution of this problem in terms of double Laplace transforms (LT) (all details contains also his book [10] (1989)). Schassberger [11] (1984) provided another (completely different) approach to the exact solution by considering the EPS discipline as a limit of the round–robin model (in discrete time). Later similar solutions were also made by means of the variants of the methods from [3] and from [11] (or their combinations). See, for example, van den Berg [12] (1990) or Whitt [13] (1998) (here we do have no possibility to discuss the contributions of other authors (Brandt and Brandt [14] (1999), Asare and Foster (1983), Nunez–Queija (2000), Cheung et al. [15] (2005), et alii)to the closely related problems). We only mention that the EPS queue with permanent jobs has been studied in [12, 13, 5, 14, 15] from point of view which is different from our approach. A telecommunication system with CPU scheduling under SCO–UNIX can be considered as an example of using of the EPS model with permanent jobs for its description and predicting delays of the jobs.

In fact, our method has turned out to be a very fruitful to derive many further results, for example, the time-dependent queue-length and sojourn time distributions in this and related models (see, for example, [10, 8, 16, 5]). These results hold for any stability condition. Besides,

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1Under the EPS discipline, the processor (server) is shared equally by all jobs in the system. To put more concretely, when $1 \leq n < \infty$ jobs are present in the system, each job receives service at rate $1/n$. In other words, all these jobs receive $1/n$ times the rate of service which a solitary job in the processor would receive. Jumps of the service rate occur at the instants of arrivals and departures from the system. Therefore, the rate of service received by a specific job fluctuates with time and, importantly, its sojourn time depends not only on the jobs in the processor at its time of arrival there, but also on subsequent arrivals shorter of which can overtake a specific job. This makes the EPS system intrinsically harder to analyze than, say, the First Come — First Served (FCFS) queue.

2Indeed, the assumptions which required to use the steady–state (stationary) solutions of any queueing systems are rarely satisfied in real life. To be able really to apply queueing results in design and analysis of technical systems, in very many cases, the obtained results of steady state analysis are not sufficient. For example, it is often necessary to investigate the behaviour of the queue while it progresses towards a steady state (if and when a steady state exists). Even the average queue length at time $t$ gives us much more information in comparison with the stationary mean of the number of jobs.

However, few stochastic systems are known to have exact time–dependent (transient) solutions for the distributions of the processes. As a rule, such systems are the M/G/1 queues with simpler disciplines (for example, FCFS, see, for example, Takács [17]). Besides, all time–dependent solutions of the queues of the type M/G/1
the entire transient and equilibrium behaviour of the M/G/1—EPS queue is contained in the results mentioned, and the most (if not all) available at present analytic solutions (and also many new) can be derived from them as special cases via standard arguments (for example, by means of the Abel’s/Tauber’s theorems). However, we shall not consider the transient solutions in this paper.

The rest of the paper is organized as follows. In Section 2 we introduce some notations and describe our starting point represented by Theorem 2.1. In Section 3 we obtain some interesting consequences of Theorem 2.1, some of which were proved earlier as self-contained theorems but now ones are derived as special cases. The final section contains few closing remarks.

2 Preliminaries

In this section we give a short review of the M/G/1—EPS queue with \( K \) permanent jobs (only in steady state). For the time-dependent results we refer to [5] and also to [10, 8].

Jobs arrive to the single processor (server) according to a Poisson process with the rate \( \lambda > 0 \). Their sizes (required service times) are i.i.d. random variables with a general distribution function \( B(x) \) (\( B(0) = 0 \), \( B(\infty) = 1 \)) with the mean \( \beta_1 < \infty \) and the Laplace–Stieltjes transform (LST) \( \beta(s) \). Let \( \beta_j \) denote the \( j \)-th moment of \( B(x) \), \( j \in \mathbb{N} \). The service discipline is the EPS: every job is being served with rate \( 1/n \), when \( n > 0 \) jobs are present in the system. The EPS discipline is modified by having \( K \geq 0 \) extra permanent jobs with infinite sizes. The system works in steady state. In other words, \( \rho = \lambda \beta_1 < 1 \) and very long time went from the instant 0 that marks the start of the work of our system till current time.

It is well known, due to Sakata et al. [18], that the stationary distribution \( (P_n)_{n \geq 0} \) of the number of ordinary jobs in the M/G/1—EPS queue as \( K = 0 \) is geometrically distributed

\[
P_{0n} = (1 - \rho)\rho^n, \quad n \in 0 \cup \mathbb{N},
\]

where \( \rho = \lambda \int_0^\infty (1 - B(x))dx < 1 \). We note that \( (P_{0n})_{n \geq 0} \) depends on the service time only through its mean.

For \( K \geq 0 \) the equality (2.1) takes the form

\[
P_{Kn} = (1 - \rho)(n + K)^{K+1} \binom{n + K}{K} \rho^n, \quad n \in 0 \cup \mathbb{N}.
\]

We shall let that \( V_K(u) \) denotes the conditional sojourn time of a job of the size \( u \) upon its arrival. This job enter into the EPS system with \( K \geq 0 \) permanent jobs in steady state. Let \( V_{Kj} = \mathbb{E}[V_K(u)^j] \). (We shall omit the index \( K \) in these and similar notations when \( K = 0 \).)

are obtained in terms of double transforms (on space and time) from which it is very hard to extract necessary information concerning the behaviour of the system. (Moreover, much more advanced mathematical techniques become necessary for the time–dependent solutions in comparison with steady state analysis.) Some exceptions give variations of the M/M/1—FCFS queue for which closed–form transient solutions are known. As a rule, the exact transient analysis of the M/M/1—FCFS queue involves infinite sums of Bessel functions. In general, explicit exact solutions are highly unlikely for the time–dependent cases.
Define the LST of $V_K(u)$ by $v_K(r, u) = \mathbb{E}[e^{-rV_K(u)}]$ for $\Re r \geq 0$ and $u \geq 0$. Let $\pi(r)$ be the LST of the busy period distribution (due to ordinary, that is, non-permanent jobs). In other words, it is the positive root of the well-known Takács functional equation \cite{17}

$$
\pi(r) = \beta(r + \lambda - \lambda \pi(r))
$$

with the smallest absolutely value.

It is known from \cite{5} the following theorem

**Theorem 2.1** When $\rho < 1$,

$$
v_K(r, u) \doteq \mathbb{E}[e^{-rV_K(u)}] = v(r, u)^{K+1},
$$

where $v(r, u)$ is given by the the equality \cite{2.6}:

$$
v(r, u) = \mathbb{E}[e^{-rV(u)}] = \frac{(1 - \rho)e^{-u(r + \lambda)}}{\psi(r, u) - \tilde{a}(r, 0, u)},
$$

Here

$$
\tilde{a}(r, 0, u) = \lambda \psi(r, u) * [e^{-u(r + \lambda)}(1 - B(u))] + \lambda e^{-u(r + \lambda)} \int_{u}^{\infty} (1 - B(x))dx,
$$

where “$*$” is the Stieltjes convolution sign (on variable $u$), and $\psi(r, u)$ is the LST (with respect to $x$) of some function $\Psi(x, u)$ of two variables (possessing the probability density on variable $x$), which, in turn, has a Laplace transform (LT) with respect to $u$(argument $q$)

$$
\tilde{\psi}(r, q) = \frac{q + r + \lambda \beta(q + r + \lambda)}{(q + r + \lambda)(q + \lambda \beta(q + r + \lambda))} \quad (r \geq 0, q > -\lambda \pi(r)).
$$

In \cite{2.7}, $\beta(r) = \int_{0}^{\infty} e^{-rx} d\beta(x)$ and $\pi(r)$ (in the conditions imposed on \cite{2.7}) is understood as the minimal solution of the functional equation \cite{2.8}.

Thus, the function $\tilde{\psi}(r, q)$ is given in the form of the two-dimensional transform of the function $\Psi(x, u)$

$$
\tilde{\psi}(r, q) = \int_{0}^{\infty} \int_{0}^{\infty} e^{-rx - qu} d\Psi(x, u)du.
$$

In other words, $\psi(r, u)$ in equality \cite{2.9} is the Laplace transform inversion operator, $\psi(r, u) = \mathcal{L}^{-1}(\tilde{\psi}(r, q))(r, u)$, that is, the contour Bromvich integral

$$
\psi(r, u) = \frac{1}{2\pi i} \int_{-\infty+0}^{+\infty+0} \tilde{\psi}(r, q)e^{qu} dq.
$$

**Remark 2.1** Briefly, we have derived the expression for $\mathbb{E}[e^{-rV_K(u)}]$ by writing the sojourn time as some generalized functional on a branching process (like the processes by Crump–Mode–Jagers) by means of simple extensions of (non-trivial) arguments from \cite{9, 3}. Using the structure of the branching process, we found and solved a system of partial differential equations (of the first order) determining the components of a (non-trivial, too) decomposition of $V_K(u)$. It leads to $\mathbb{E}[e^{-rV_K(u)}]$ (see also Remark \cite{3.3}).
3 Results

We showed in the Section 2 that the determination of the steady–state sojourn time distribution in the queue $M/G/1$—EPS with $K$ permanent jobs is simple extension of the results from [3], [9]. However, the solution contains the Bromwich countour integrals. First we consider the case $K = 0$. Equivalent form of (2.5) (without contour integrals) is given in the following theorem.

**Theorem 3.1** Equivalent form of (2.5) (without the Bromwich countour integrals) is given by

$$
\frac{1}{v(r, u)} = \sum_{n=0}^{\infty} \frac{r^n}{n!} \xi_n(u),
$$

(3.1)

where

$$
\xi_0(u) = 1, \quad \xi_n(u) = \frac{n}{(1-\rho)^n} u^{n-1} * W^{(n-1)*}(u), \quad n = 1, 2, \ldots
$$

(3.2)

Here $W^{(n-1)*}(u)$ is $(n-1)$–fold convolution of the steady–state waiting time distribution $W(u)$ in the familiar $M/G/1$—FCFS system with itself ($W^0(u) = 1(u)$, $W^1(u) = W(u)$), the LST of $W(u)$ is given by the well–known Pollaczek–Khintchine formula as

$$
w(q) = \frac{1-\rho}{1-\rho f(q)},
$$

(3.3)

where $f(q) = (1-\beta(q))/(\beta_1)$ is the LST of the excess of $B(\cdot)$, that is, $F(x) = \beta_1^{-1} \int_{x}^{\infty} (1-B(y))dy$ ($F^0(x) = 1(x)$, the Heaviside function, $F^1(x) = F(x)$).

**Proof.** We rewrite (2.5) in the form of Theorem 3.2 from [11] (see also (5.5) in [4]), namely

$$
v(r, u) = \frac{(1-\rho)\delta(r, u)}{1-\rho \delta(r, u) \left[ \int_{0}^{u} \frac{dF(x)}{\delta(x)} + (1-F(u)) \right]} (\text{Re } r \geq 0),
$$

(3.4)

where

$$
\delta(r, u) = e^{-u(r+\lambda)}/\psi(r, u)
$$

(3.5)

and $F(x)$ is introduced in Theorem 3.1. To reach our aim, it is used the LT of $1/\delta(r, u)$ with respect to $u$ (argument $q$), which is found from (2.7) as $\psi(r, q-r-\lambda), \quad r \geq 0, \quad q > r + \lambda - \lambda \pi(r)$ (cf. also the third line on p.8 in [11]). Now we obtain after simple algebra the following power series expansion of the LT of the function $1/v(r, u), \quad r \geq 0, \quad u \geq 0$

$$
\int_{0}^{\infty} e^{-qu} \frac{1}{v(r, u)} du = \frac{1}{q} \left[ 1 + \frac{1}{1-\rho q} \frac{r}{1-\rho q} w(q) \right]
$$

$$
= \frac{1}{q} \left[ 1 + \sum_{n=1}^{\infty} \left( \frac{r}{1-\rho q} \right)^n w(q)^{n-1} \right]
$$

(3.6)

where $w(q)$ is given by (3.3). We note that $|r w(q)/(1-\rho q)| < 1$ as $q > r + \lambda - \lambda \pi(r), \quad \rho < 1$. Now it is easily to invert analytically (on argument $q$) each term of the power series in $r$ (3.6). The result
is given by (3.2) whence it follows (3.1), the right–hand side of which is the power series in $r$ with coefficients $\xi_n(u)/n!$.

The idea of such approach goes back to Heaviside. Similar results are obtained in [19], [20]. In fact, the form of $\text{Var}[V(u)]$ [19], [3] (see the equality (3.10) below) stimulates a guess about the possibility of such expansion.

**Remark 3.1** The formula for $W^{n*}(x)$ in (3.2) can be represented in the following form

$$W^{n*}(x) = (1 - \rho)^n \sum_{k=0}^{\infty} \left( \frac{k + n - 1}{n - 1} \right) \rho^k F^k(x).$$

It is done, for example, by inversion of $w(q)^n$, where $w(q)$ is given by (3.3).

**Remark 3.2** We note that the by–product of our analysis is the distribution function $W(x)$ whose LST is given by (3.3). However, the analysis of EPS queue gives the other quantity (corresponding to a non–probability measure) $W^o(x) = W(x)/(1 - \rho)$. The form of the LST of $W^o(x)$ is well–known: $w^o(q) = \sum_{n=0}^{\infty} \rho^n f^n(q)$. Unlike $W(x)$, $W^o(x)$ is well defined for all $\rho > 0$ and $x > 0$. It can be shown that $W^o(x) < \infty$ for all $\rho > 0$, $x > 0$ and for any $B(\cdot)$ (despite on the fact that, for $\rho \geq 1$, $W^o(x) \to \infty$ as $x \to \infty$).

**Theorem 3.2** Let $v_n(u) = \mathbb{E}[V(u)^n]$, $n = 1, 2, \ldots$. Then it holds the following recursive formula

$$v_n(u) = \sum_{i=1}^{n} \binom{n}{i} v_{n-i}(u) \xi_i(u)(-1)^{i+1}$$

**Proof.** Because $v(r, u)$ is analytical function in $r$ (in particular, in $r = 0$), we can use the Tailor series expansion of $v(r, u)$ for small $r > 0$

$$v(r, u) = 1 - \frac{r}{1!} v_1(u) + \frac{r^2}{2!} v_2(u) - \frac{r^3}{3!} v_3(u) + \ldots$$

The product of (3.8) and (3.1) gives

$$- \frac{r}{1!}[v_1(u) - \xi_1(u)] + \frac{r^2}{2!}[v_2(u) - 2v_1(u)\xi_1(u) + \xi_2(u)]
- \frac{r^3}{3!}[v_3(u) - 3v_2(u)\xi_1(u) + 3v_1(u)\xi_2(u) - \xi_3(u)] + \ldots = 0$$

and it leads to (3.7) after differentiating $n$ times with respect to $r$ and setting $r = 0$.

In particular, the expressions for the first two moments of $V(u)$ are:

$$v_1(u) = \mathbb{E}[V(u)] = u/(1 - \rho)$$

(3.9)
(this is well-known result due to Sakata et al. \[18\] (1969)),

\[
\text{Var}[V(u)] = v_2(u) - v_1^2(u) = \frac{2}{(1 - \rho)^2} \int_0^u (u - x)(1 - W(x)) \, dx,
\]

where \(W(x)\) is introduced in Theorem 3.1 and it is expressed as

\[
W(x) = (1 - \rho) \sum_{n=0}^{\infty} \rho^n F^{n*}(x)
\]

(3.11)

(other variables were introduced above).

The formula (3.2) implies that \(\xi_1(u) = \mathbb{E}[V(u)]\) in (3.9). The formula for the conditional variance (3.10) was first obtained by Yashkov [9]. The standard way for the computation of the moments is the following

\[
v_n(u) = \lim_{r \downarrow 0} (-1^n) \frac{\partial^n v(r, u)}{\partial r^n}, \quad n \in \mathbb{N}.
\]

(3.12)

However, the LST \(v(r, u)\) in Theorem 2.1 is very hard to differentiate in \(r\) more than once (practically almost impossible matter) since this LST has a rather complex form due to a highly complicated form of such constituents of (2.5) as \(\bar{a}\) and \(\bar{\psi}\). Therefore \(\text{Var}[V(u)]\) is first obtained by solving an alternative system of differential equations (see, for example, [10, Chapter 2]) which are derived by analogy with the equations of [9] Section 2 or with the equations in the proof of [3] Theorem 4. These equations are simpler forms of equations from [9, 3] because ones are composed not for the LST \(v(r, u)\) but only for the second and the first moments. Thus the formula (3.7) for the case \(n = 2\) in Theorem 3.2 was derived 20 years earlier than the same formula for arbitrary integer \(n\) (see [19, 20]).

It can be useful for asymptotic expansion of \(v_n(u)\) for small and large \(u\) in the spirit of such expansion for \(\text{Var}[V(u)]\) (see final section for some details). Such results for \(\text{Var}[V(u)]\) were obtained at first in [9] (1981) (see also [3]).

Now we show how to extend the formulas (3.9) and (3.10) to the case when the M/G/1—EPS queue is modified by having \(K \geq 0\) extra permanent jobs with infinite sizes.

**Theorem 3.3** In the setting above,

\[
\mathbb{E}[V_K(u)] = \frac{(K + 1)u}{1 - \rho},
\]

(3.13)

\[
\text{Var}[V_K(u)] = \frac{2(K + 1)}{(1 - \rho)^2} \int_0^u (u - x)(1 - W(x)) \, dx,
\]

(3.14)

where \(W(u)\) is the steady-state waiting time distribution in the M/G/1—FCFS queue, represented by the equality (3.11).

**Proof.** In our case, the equality (3.12) takes the form

\[
v_{Kn}(u) = \lim_{r \downarrow 0} (-1^n) \frac{\partial^n v_K(r, u)}{\partial r^n}, \quad n \in \mathbb{N}.
\]

(3.15)
The formula (3.13) follows directly from (2.4) by means of applying (3.15) as \( n = 1 \).

Taking into account (3.10), the formula (3.14) follows also from (2.4) by means of applying (3.15) as \( n = 2 \) after some simple algebra.

\( \Box \)

**Remark 3.3** An alternative way to obtain (3.14) is the following. We can compose and solve the system of the partial differential equations (of the first order) which satisfy the second and the first moments of \( V_K(u) \). The variant of such equations is known from [8, 10] as \( K = 0 \). We point out the following fact. These equations rely on a decomposition of the sojourn time of the (tagged) job with the size \( u \) that arrives to the EPS queue when \( n \) standard jobs are present with remaining service demands \( x_1, \ldots, x_n \) (a key ingredient of analysis). Denoting this conditional sojourn time by \( V_{Kn}(u; x_1, \ldots, x_n) \), it holds

\[
V_{Kn}(u; x_1, \ldots, x_n) \overset{d}{=} (K + 1)D(u) + \sum_{i=1}^{n} \Phi(x_i, u),
\]

(3.16)

where all components are independent random variables.

The random variable \( D(u) \) constitutes a “main” component of the sojourn time: it has the distribution of the sojourn time of a job with the size \( u \) that enters into a empty (from the standard jobs) system. By the way, its LST is given by (3.3). When the system is not empty, the \( i \)-th standard job (among the jobs which are sharing the capacity of the processor together with permanent jobs), having remaining size \( x_i \), “adds” a delay \( \Phi(x_i, u) = \Phi(x_i \land u, u) \) to the new job’s sojourn time. Note that \( D(u) = \Phi(x_i, u) \) for \( x_i \geq u \). Then the same chain of arguments as in [3] can be used to derive (2.4).

\section*{4 Conclusion}

Using Theorems 2.1 and 3.2, we can easily obtain all other moments of \( V_K(u) \) in \( \text{M/G/1—EPS} \) queue with \( K \geq 0 \) permanent jobs. However, the exact expression even for the variance of the sojourn time (see (3.14)) involves an integration term, making an exact computations difficult from practical point of view. The same holds for the third, fourth, etc. moments. This difficulty remains also in the case \( K = 0 \). To overcome the difficulty, it is possibly to obtain some simple approximations for the second moments, see, for example, Villela et al. [21] or van den Berg [12]. We note that there exist also an upper (and lower) bounds for \( \text{Var}[V(u)] \). These bounds only depends on \( \rho \) and the size of job \( u \). In addition, the bounds have the attractive property of intensivity to \( B(x) \), and the difference between the upper and lower bounds is small, particularly, for small and moderate values of \( \rho \). These second moments tight bounds can be easily generalized into higher moments of \( V(u) \) and also to the case \( K > 0 \).

It are also known the asymptotic estimates of \( \text{Var}[V(u)] \) as \( u \to 0 \) and \( u \to \infty \) [22]. For example,

\[
\text{Var}[V(u)] \sim \frac{u^2 \rho}{(1 - \rho)^2} \text{ as } u \to 0.
\]

This is some asymptotics of the sojourn time variance of a very small jobs, and it leads to intensive upper bounds with special structure requiring only knowledge of the traffic load and
the job size. Now our results may be easily extended to the higher moments and also to the case $K$ permanent jobs. Moreover, some preliminary analysis of asymptotics (see [9]) tells us about a high accuracy of such estimates in many typical cases.

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