The energy density of an Ising half-plane lattice

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Abstract
We compute the energy density at arbitrary temperature of the half-plane Ising lattice with a boundary magnetic field $H_b$ at a distance $M$ rows from the boundary and compare limiting cases of the exact expression with recent calculations at $T = T_c$ done by means of discrete complex analysis methods.

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1. Introduction

Recently, Hongler and Smirnov [1] have studied the energy density of the Ising model at the critical temperature $T_c$ on discretizations by the square grid of Jordan domains a finite distance from the boundary with two special cases of boundary conditions: (1) free and (2) fixed with all spins up. The problem was considered on an isotropic lattice with a boundary of arbitrary shape. When specialized to the energy density a distance $M$ rows from the boundary of the half-plane lattice the results [2] as $M \to \infty$ are for free boundary conditions

$$\langle \sigma_M, 0 | \sigma_{M-1}, 0 \rangle - \langle E^v \rangle_{\text{bulk}} = -\frac{1}{2\pi M} + o(M^{-1})$$

and for fixed plus spin boundary conditions

$$\langle \sigma_M, 0 | \sigma_{M-1}, 0 \rangle - \langle E^v \rangle_{\text{bulk}} = \frac{1}{2\pi M} + o(M^{-1})$$

where in the isotropic lattice the vertical bulk energy density $\langle E^v \rangle_{\text{bulk}}$ is

$$\langle E^v \rangle_{\text{bulk}} \equiv \langle \sigma_M, 0 | \sigma_{M-1}, 0 \rangle_{\text{bulk}} = \frac{1}{\sqrt{2}}.$$ (3)

The computations of [1] are done by means of discrete complex analysis. It is the purpose of this paper to compute and study the energy density $\langle \sigma_M, 0 | \sigma_{M-1}, 0 \rangle$ for the anisotropic lattice at arbitrary temperature on a half-plane with a magnetic field $H_b$ applied to the boundary row. The energy operator is thus

$$\mathcal{E} = -\sum_{j=1}^\infty \sum_{k=-\infty}^\infty \{E_1 \sigma_{j,k} \sigma_{j,k+1} + E_2 \sigma_{j,k} \sigma_{j+1,k} \} - H_b \sum_{k=-\infty}^\infty \sigma_{1,k},$$ (4)
where we follow the notations of [3] and [4] and let \( \sigma_{i,k} \) specify the spin in row \( j \) and column \( k \). This reduces to the half-plane case of [1] when \( E_1 = E_2 \) and \( T = T_c \) with \( H_0 = 0 \) for free boundary conditions and \( H_b = \infty \) for plus spin boundary conditions. The exact result as calculated by Pfaffian methods is given in section 2. Limiting cases with \( M \to \infty \) for \( T < T_c \) and \( T \to T_c^- \) are obtained in section 3 and for \( T > T_c \) and \( T \to T_c^+ \) in section 4. We conclude in section 5 with a brief discussion.

2. The energy density \( \langle \sigma_{M,0,\sigma_{M-1,0}} \rangle \) for arbitrary \( T \) and \( H_b \)

The computation of \( \langle \sigma_{M,0,\sigma_{M-1,0}} \rangle \) is done straightforwardly by means of Pfaffian methods [5, 6]. The details are given in [3] and in chapter 7 of [4] and using the result (3.16d) on p 152 of [4] (or [3, (7.8d)]) we immediately find that

\[
\langle \sigma_{M,0,\sigma_{M-1,0}} \rangle - \langle E^v \rangle_{\text{bulk}} = I
\]  

(5)

with

\[
I = \frac{\alpha_2}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{1}{\alpha(\theta)^{2M-1}} \left[ \left( 1 - z_2^2 \right) - z_2 \alpha(\theta) \left( 1 + z_1^2 + 2z_1 \cos \theta \right) \right] \times \frac{\left[ (\cos \theta - 1)/(\cos \theta + 1) + iz_2 z_1^{-1} v/v' \right]}{\left[ (\cos \theta - 1)/(\cos \theta + 1) - iz_2 z_1^{-1} v/v' \right]},
\]

(6)

where we use the definitions

\[
z_j = \tanh E_j/k_B T, \quad z = \tanh H_0/k_B T
\]

(7)

\[
\alpha_1 = \frac{z_j(1 - z_2)}{(1 + z_2)}, \quad \alpha_2 = \frac{(1 - z_2)}{z_1(1 + z_2)}.
\]

(8)

The quantity \( v/v' \) is given by (3.7), (3.14) and (3.20) on pp 120–2 of [4], as

\[
v/v' = \frac{z_2^2(1 + z_1^2 + 2z_1 \cos \theta) - z_2(1 - z_1^2)\alpha(\theta)}{2z_1 z_2 \sin \theta}.
\]

(9)

Here, \( \alpha(\theta) \) is the largest root of (3.2) on p 86 of [4],

\[
(1 + z_1^2)(1 + z_2^2) - 2z_2(1 - z_1^2) \cos \theta - z_2(1 - z_1^2)\alpha(\theta) + \alpha(\theta)^{-1} = 0,
\]

(10)

which is explicitly given as

\[
\alpha(\theta) = \frac{1}{2z_2(1 - z_1^2)} \left\{ (1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \right\} \cos \theta + z_1(1 + z_2) \left\{ (1 + z_1^2)(1 + z_2^2) \right\}^{1/2},
\]

(11)

where the square root is defined to be positive for real \( \theta \). The vertical energy density \( \langle E^v \rangle_{\text{bulk}} \) is given in (4.1) of chapter VIII of [4] by the equivalent forms

\[
\langle E^v \rangle_{\text{bulk}} = z_2 + \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{(1 - z_2^2) - z_2 \alpha(\theta)(1 + z_1^2 + 2z_1 \cos \theta)}{z_2(1 - z_1^2)(1 - \alpha(\theta)^2)}
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} d\theta \frac{\left( 1 - \alpha_3 e^{i\theta} \right) \left( 1 - \alpha_4 e^{-i\theta} \right)}{\left( 1 - \alpha_3 e^{-i\theta} \right) \left( 1 - \alpha_4 e^{i\theta} \right)}^{1/2}
\]

(12)

with

\[
\alpha_3 = \frac{z_2(1 - z_1)}{1 + z_1}, \quad \alpha_4 = \frac{(1 - z_1)}{z_2(1 + z_1)}
\]

(13)

where to obtain the last line of (12) we have used identities of [5].
We make contact with the computations of [1] by computing the large $M$ behavior of $I$ as given by (6) in several cases.

3. Expansions for $M \to \infty$ for $T \leq T_c$

We make contact with the computations of [1] by computing the large $M$ behavior of $I$ as given by (6) in several cases.

3.1. $T < T_c$ and $H_0 = 0$

When $H_0 = 0$ and $T < T_c$ is fixed, we obtain the large $M$ behavior of $I$ by expanding (14) by steepest descents. The maximum of the integrand of (14) is at $\theta = 0$ and thus expanding for small $\theta$

\[
\ln \alpha(\theta) \simeq \ln \left( \frac{z_2(1 + z_1)}{(1 - z_1)} \right) + \frac{z_1 \alpha_2}{(1 - \alpha_1)(1 - \alpha_2)} \theta^2,
\]

\[(1 - z_1^2) - z_2 \alpha(1 + z_1^2 + 2z_1 \cos \theta) = -\frac{1 + z_1}{1 - z_1} \{z_2^2(1 + z_1)^2 - (1 - z_1)^2\} + O(\theta^2),
\]

we find

\[I \simeq \frac{1}{2\pi} \frac{z_2(1 - z_2^2)(1 + z_1)^2}{(1 - z_1)^2} \left[ \frac{z_2(1 + z_1)}{(1 - z_1)} \right]^{-2M} \int_{-\epsilon}^{\epsilon} d\theta \exp \left\{ -2M \theta^2 \frac{z_1 \alpha_2}{(1 - \alpha_1)(1 - \alpha_2)} \right\},
\]

where $0 < \epsilon \ll 1$. Then, in (18) setting

\[u^2 = 2M \theta^2 \frac{z_2 \alpha_2}{(1 - \alpha_1)(1 - \alpha_2)}
\]

and since $\epsilon \sqrt{M} \to \infty$, we obtain the result that as $M \to \infty$

\[
\langle \sigma_{M,0} \rangle \simeq \frac{z_2(1 - z_2^2)(1 + z_1)^2}{2(1 - z_1)^2} \left[ \frac{(1 - \alpha_1)(1 - \alpha_2)}{2\pi z_2 \alpha_2 M} \left( \frac{z_2(1 + z_1)}{(1 - z_1)} \right) \right]^{-2M}.
\]

3.2. $T < T_c$ and $H_0 > 0$

The large $M$ behavior of $I$ for $T < T_c$ and $H_0 > 0$ fixed is obtained from (6) also by steepest descents where, in addition to (15)–(17), we also need the expansion for $\theta \sim 0$

\[\frac{(e^{\theta_0} - 1)/(e^{\theta_0} + 1) + iz_2^2 z_2^{-1} v/v'}{(e^{\theta_0} - 1)/(e^{\theta_0} + 1) - iz_2^2 z_2^{-1} v/v} \simeq \frac{z_2^2 A[(1 - \alpha_2) - z^2 A]}{4z^2(1 - \alpha_2)^2} \theta^2,
\]

where

\[A = (1 - \alpha_2) + \frac{\alpha_2(1 + z_1)^2}{(1 - \alpha_1)} = \frac{4}{(1 + z_2)^2(1 - \alpha_1)}.
\]
Thus, using the same method as for (20), we find as \( M \rightarrow \infty \)
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle E \rangle_{\text{bulk}} \simeq -\frac{z_2^2(1 - z_2)(1 + z_1)}{8(1 + z_2)(1 - z_1)^2} 
\times \left[ \frac{(1 - \alpha_2) - z^2 A(1 - \alpha_1)^{1/2}}{z^2 \sqrt{2\pi}(1 - \alpha_2)(z_1 \alpha_2 M)^{3/2}} \right]^{2M} z_2(1 + z_1) \frac{1}{(1 - z_1)}. \tag{23}
\]
This is negative for
\[
z_2^2 < (1 - \alpha_2)/A = \frac{1}{4}(1 + z_2)^2(1 - \alpha_1)(1 - \alpha_2) \tag{24}
\]
and positive for
\[
z_2^2 > \frac{1}{4}(1 + z_2)^2(1 - \alpha_1)(1 - \alpha_2). \tag{25}
\]
We note that both (20) and (23) have the same exponential decay but that (23) decays faster by a factor of \( 1/M \) than does (20). We also note as \( z \rightarrow 0 \) that (23) diverges as \( z^{-2} \). Therefore, in order to connect together the regimes of \( H_b = 0 \) and \( H_b > 0 \), a crossover regime is required.

3.3. The crossover regime \( T < T_c \), \( H_b \rightarrow 0 \) with \( z^2 M \) fixed

The crossover from \( H_b = 0 \) to \( H_b > 0 \) for \( T < T_c \) is obtained by considering \( z \rightarrow 0 \) and \( M \rightarrow \infty \) with \( z^2 M = O(1) \). Then, when \( M \theta^2 = O(1) \) we have the expansion which replaces (21)
\[
\frac{(e^{i\theta} - 1)/(e^{i\theta} + 1) + iz_2^{-1} u'/v'}{(e^{i\theta} - 1)/(e^{i\theta} + 1) - iz_2^{-1} u'/v'} \simeq \frac{M \theta^2}{M \theta^2 + 4z^2 M(1 - \alpha_2)/z_2 A}. \tag{26}
\]
Then, using (15)–(17) and (19), and setting
\[
\zeta^2 = \frac{8z_2^2 Mz_1 \alpha_2}{(1 - \alpha_1)z_2^2 A} = 2Mz_2^2(1 + z_2)^2 \alpha_2 z_1 z_2^{-2} \tag{27}
\]
we find from (6) that the crossover function is
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle E \rangle_{\text{bulk}} \simeq -\frac{z_2^2(1 - z_2)(1 + z_1)^2}{2\pi(1 - z_1)^2} \left( \frac{(1 - \alpha_1)(1 - \alpha_2)}{2z_1 \alpha_2 M} \right)^{1/2} 
\times \left( \frac{z_2(1 + z_1)}{1 - z_1} \right)^{-2M} \int_{-\infty}^{\infty} du \frac{u^2}{u^2 + \zeta^2} e^{-u^2}. \tag{28}
\]
When \( \zeta \rightarrow 0 \), (28) reduces to (20) and when \( \zeta \rightarrow \infty \), (28) reduces to (23) with \( z \rightarrow 0 \).

3.4. \( T \rightarrow T_c \) and \( H_b = 0 \)

In order to obtain the results (1) and (2) of [1] at \( T = T_c \) where
\[
z_1z_2z_3 + z_1z_3 + z_2z_3 = 1, \quad \alpha_2 = 1 \tag{29}
\]
we consider \( \alpha_2 \rightarrow 1 \) in the asymptotic expansions for \( M \rightarrow \infty \) (20), (23), (28) and observe that the exponential factor
\[
\frac{z_2(1 + z_1)}{1 - z_1} \sim e^{ik\alpha_2((1 - \alpha_2)/(1 - z_1))} \tag{30}
\]
and that the coefficients either diverge or vanish. Therefore, when \( 1 - \alpha_2 \rightarrow 0 \), a separate expansion is needed. In the integral (6) for \( I \), we set
\[
\theta/(1 - \alpha_2) = x \tag{31}
\]
with $\theta \to 0$ and $\alpha_2 \to 1$ with $x$ fixed of order 1 and use the approximations
\[
\alpha(\theta) \sim 1 + \frac{2z_1 c}{1 - z_1 c} (1 - \alpha_2) \sqrt{1 + x^2}
\]  
(32)

and
\[
\frac{\alpha_2 \alpha \left(1 - z_1^2\right) - z_2 \alpha \left(1 + z_1^2 + 2z_1 \cos \theta\right)}{(1 + \alpha_1^2 - 2\alpha_1 \cos \theta)(1 + \alpha_2^2 - 2\cos \theta)} \sim -\frac{2z_1 c}{1 - z_1 c} \left[1 + \frac{1}{\sqrt{1 + x^2}}\right].
\]  
(33)

Then, defining for $M \to \infty$ and $1 - \alpha_2 \to 0$
\[
m = \frac{4z_1 c}{1 - z_1 c} M(1 - \alpha_2)
\]  
(34)

we obtain the result
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle E^v \rangle_{bulk} \simeq -\frac{z_1 c (1 - \alpha_2)}{\pi (1 - z_1 c)} \int_{-\infty}^{\infty} dx \left[1 + \frac{1}{\sqrt{1 + x^2}}\right] e^{-m \sqrt{1 + x^2}}
\]  
(35)

which, using $x = \sinh y$, is rewritten as
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle E^v \rangle_{bulk} \simeq -\frac{z_1 c (1 - \alpha_2)}{\pi (1 - z_1 c)} \int_{-\infty}^{\infty} dy [\cosh y + 1] e^{-m \cosh y}
\]
\[
= -\frac{2z_1 c (1 - \alpha_2)}{\pi (1 - z_1 c)} [K_1(m) + K_0(m)],
\]  
(36)

where $K_n(z)$ is the modified Bessel function of order $n$ [7].

When $m \to \infty$, we use the first term in the expansion
\[
K_n(m) = \sqrt{\frac{\pi}{2m}} e^{-m} \left(1 + \frac{4n^2 - 1}{8m} + O(m^{-2})\right)
\]  
(37)

to find that (36) reduces to
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle E^v \rangle_{bulk} \simeq -\frac{2z_1 c (1 - \alpha_2)}{\pi (1 - z_1 c)} M e^{-4z_1 c M(1 - \alpha_2)/(1 - z_1^2)}
\]  
(38)

which agrees with (20) in the limit $\alpha_2 \to 1$ when we use (30).

When $m \to 0$, we use
\[
K_1(m) \sim 1/m, \quad K_0(m) \sim -\ln m
\]  
(39)

to find that (36) reduces to
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle E^v \rangle_{bulk} \simeq -\frac{1}{2\pi M}
\]  
(40)

which agrees with the result of [1] for free boundary conditions (1).

3.5. $T \to T_c$ and $H_b > 0$

When $T \to T_c$ with $H_b > 0$, we need the further approximation that by using (31) in (9)
\[
v/v' \sim \frac{1 - \sqrt{1 + x^2}}{x}
\]  
(41)

and thus
\[
\frac{(e^{i\theta} - 1)/(e^{i\theta} + 1) + iz_2 z_1 c^{-1} v/v'}{(e^{i\theta} - 1)/(e^{i\theta} + 1) - iz_2 z_1 c^{-1} v'/v} \sim \left[\frac{\sqrt{1 + x^2} - 1}{x}\right]^{-2}.
\]  
(42)
Using (42) in (6) with (31)–(34), we obtain
\[
I \sim \frac{z_{1c}(1 - \alpha_2)}{\pi (1 - z_{1c}^2)} \int_{-\infty}^{\infty} \frac{\sqrt{1 + x^2} - 1}{x} \left[ 1 + \frac{1}{\sqrt{1 + x^2}} \right] e^{-m\sqrt{1+x^2}}
\]  
which, setting \( x = \sinh y \) gives the result
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \simeq \frac{z_{1c}(1 - \alpha_2)}{\pi (1 - z_{1c}^2)} \int_{-\infty}^{\infty} dy (\cosh y - 1) e^{-m\cosh y}
\]
which is independent of \( H_b \) and differs from the result (36) for \( H_b = 0 \) only in the sign of the term with \( K_1(m) \).

When \( m \to \infty \), we use (37) in (44) to find
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \simeq \frac{1}{8} \sqrt{\frac{1 - z_{1c}}{2\pi z_{1c}(1 - \alpha_2)}} \frac{1}{M^{3/2}} e^{-4z_{1c}M(1-\alpha_2)/(1-z_{1c}^2)}
\]  
which agrees with (23) with \( \alpha_2 \to 1 \).

When \( m \to 0 \), we use (39) to find
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \simeq \frac{1}{2\pi M}
\]  
which agrees with the result of [1] for fixed spin boundary conditions (2).

### 3.6. The crossover regime \( T \to T_c^- \), \( H_b \to 0 \) with \( z^2M \) fixed

It is of further interest to determine the crossover between the results (36) and (44) and the specialization to the crossover between the two results (1) and (2). When \( z \to 0 \) and \( M \to \infty \) with \( z^2M = O(1) \) and when \( M(1-\alpha_2) = O(1) \), we have the expansion which replaces (42)
\[
\frac{c(\theta - 1)(\theta + 1) + iz^2z_{1c}^{-1}v/v'}{c(\theta - 1)(\theta + 1) - iz^2z_{1c}^{-1}v/v'} \sim \frac{m x^2[\sqrt{1 + x^2} - 1] - \zeta^2 x^2[\sqrt{1 + x^2} - 1]^2}{m x^2[\sqrt{1 + x^2} - 1] + \zeta^2 x^2},
\]  
where \( \zeta^2 \) is obtained from (27) with \( \alpha_2 \to 1 \) as
\[
\zeta^2 = 2z^2M(z_{1c}^2 - 1).
\]  
Then, using (47) in (6) with (31)–(34), we obtain
\[
I \sim \frac{z_{1c}(1 - \alpha_2)}{\pi (1 - z_{1c}^2)} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{1 + x^2}} \frac{m x^2 - \zeta^2 x^2[\sqrt{1 + x^2} - 1]}{m[\sqrt{1 + x^2} - 1] + \zeta^2 x^2} e^{-m\sqrt{1+x^2}}
\]  
which setting \( x = \sinh y \) gives the result
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \simeq \frac{z_{1c}(1 - \alpha_2)}{\pi (1 - z_{1c}^2)} \int_{-\infty}^{\infty} dy \frac{m(cosh y + 1) - \zeta^2 (cosh y - 1)}{m(cosh y - 1) + \zeta^2} e^{-m\cosh y}.
\]  

When \( \zeta^2 \to 0 \), (36) is recovered, and when \( \zeta^2 \to \infty \), (44) is recovered.

Finally, in order to interpolate between the results (1) and (2), we let \( m \to 0 \) in (49) and set \( mx = t \) to obtain
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \simeq \frac{1}{2\pi M} \int_{0}^{\infty} dt \frac{t - \zeta^2}{t + \zeta^2} e^{-t}.
\]  

In figure 1, the integral in (51) is evaluated numerically for various \( \zeta^2 \). The integral vanishes at \( \zeta^2 = 0.610058 \cdot \cdot \cdot \).
4. Expansions for $M \to \infty$ for $T > T_c$

The fundamental result (6) holds for $T > T_c$ as well as $T < T_c$. The analysis of the various special limiting cases is parallel to $T < T_c$ where now $\alpha_2 > 1$ and (15)–(17) are replaced by

$$\ln \alpha(\theta) \sim \ln \left( \frac{1 - z_1}{z_2(1 + z_1)} \right) + \frac{z_1 \alpha_2}{(1 - \alpha_1)(\alpha_2 - 1)} \theta^2$$

(52)

$$1 - z_1^2 - z_2 \alpha (1 + z_1^2 + 2z_1 \cos \theta) \sim -\frac{4z_1(1 - z_1)}{(1 + z_2)^2(1 + z_1)(1 - \alpha_1)(\alpha_2 - 1)} \theta^2$$

(53)

$$\left\{(1 + \alpha_1^2 - 2\alpha_1 \cos \theta)(1 + \alpha_2^2 - 2\alpha_2 \cos \theta) \right\}^{1/2} = (1 - \alpha_1)(\alpha_2 - 1) + O(\theta^2).$$

(54)

4.1. $T > T_c$ and $H_b = 0$

Using (52)–(54) in (6), we find that $M \to \infty$,

$$\langle \sigma_{M,0} \sigma_{M-1,0} \rangle - \langle \mathcal{E}' \rangle_{\text{bulk}} \simeq -\frac{(1 - z_1)^2}{2\alpha_2(1 + z_1)^2(1 + z_2)^2 \sqrt{2\pi z_1 \alpha_2(1 - \alpha_1)(\alpha_2 - 1)} M^{3/2}}$$

$$\times \left[ \frac{(1 - z_1)}{z_2(1 + z_1)} \right]^{-2M}$$

(55)

which is to be compared with the corresponding result (20) for $T < T_c$.

4.2. $T > T_c$ and $H_b > 0$

When $T > T_c$ and $\theta \sim 0$

$$v/v' \sim -\frac{(1 + z_2)^2(1 - \alpha_1)(\alpha_2 - 1)}{2z_1 \theta}$$

(56)

and (21) is replaced by

$$\frac{(e^{i\theta} - 1)/(e^{i\theta} + 1) + iz_2^{-1} v/v'}{(e^{i\theta} - 1)/(e^{i\theta} + 1) - iz_2^{-1} v/v'} \simeq -\frac{z^2(1 + z_2)^2(1 - \alpha_1)(\alpha_2 - 1)^2}{z_2^2[(\alpha_2 - 1) + z^2 A] \theta^2}.$$

(57)
Thus, as $M \to \infty$,
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \simeq \frac{2(1-z_1^2)z_2^2}{z_1^2(1+z_1)^2[(\alpha_2 - 1) + z^2 A]}
\times \sqrt{\frac{z_1(\alpha_2 - 1)}{2\pi M(1-\alpha_1)}} \left[ \frac{(1-z_1)}{z_2(1+z_1)} \right]^{-2M}
\]
(58)
with $A$ given by (22). The result (58) is positive for all $z^2 > 0$ in contrast with the corresponding result (23) for $T < T_c$ which changes sign at $z^2 = (1 - \alpha_2)/A$.

4.3. The crossover regime $T > T_c$, $H_0 \to 0$ with $z^2 M$ fixed

In this case, we find that
\[
\frac{(e^{\theta} - 1)/(e^{\theta} + 1) + iz_2^{-1}v'/v'}{(e^{\theta} - 1)/(e^{\theta} + 1) - iz_2^{-1}v'/v'} \simeq 1 - \frac{\zeta^2}{\bar{\mu}^2}
\]
(59)
with $\zeta^2$ defined by (27) and
\[
\bar{\mu}^2 = 2M\theta^2 \frac{z_1\alpha_2}{(1-\alpha_1)(\alpha_2 - 1)}.
\]
(60)
Thus, we obtain the result
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \simeq -\frac{(1-z_1)^2(1-2\zeta^2)}{2z_2(1+z_1)^2(1+z_2)^2\sqrt{2\pi z_1\alpha_2(1-\alpha_1)(\alpha_2 - 1)M^{3/2}}}
\times \left[ \frac{(1-z_1)}{z_2(1+z_1)} \right]^{-2M}
\]
(61)
which agrees with (55) when $\zeta \to 0$ and agrees with the $z \to 0$ limit of (58) when $\zeta \to \infty$.

4.4. $T \to T_c^+$ and $H_0 = 0$

Approaching $T_c$ from above, (32) is replaced by
\[
\alpha(\theta) \sim 1 + \frac{2z_{1c}}{1 - z_{1c}}(\alpha_2 - 1)\sqrt{1 + x^2}
\]
(62)
where now,
\[
x = \theta/(\alpha_2 - 1)
\]
(63)
with $\theta \to 0$ and $\alpha_2 \to 1$ with $x$ fixed of order 1 and (33) is replaced by
\[
\frac{\alpha_2\alpha_1(1-z_1^2) - z_2\alpha_1(1+z_1^2 + 2z_1\cos \theta)}{\left[(1+\alpha_2^2 - 2\alpha_1\cos \theta)(1+\alpha_2^2 - 2\cos \theta)\right]^{1/2}} \sim \frac{2z_{1c}}{1 - z_{1c}} \left[ \frac{1}{\sqrt{1 + x^2}} \right].
\]
(64)
Defining
\[
\bar{m} = \frac{4z_{1c}}{1 - z_{1c}^2}M(\alpha_2 - 1)
\]
(65)
and setting $x = \sinh y$, we find
\[
\langle \sigma_{M,0}\sigma_{M-1,0} \rangle - \langle \mathcal{E}^v \rangle_{\text{bulk}} \simeq -\frac{z_{1c}(\alpha_2 - 1)}{\pi \left(1 - z_{1c}^2\right)} \int_{-\infty}^{\infty} dy [-1 + \cosh y] e^{-\bar{m}\cosh y}
\]
\[
= -\frac{2z_{1c}(\alpha_2 - 1)}{\pi \left(1 - z_{1c}^2\right)} [-K_0(\bar{m}) + K_1(\bar{m})]
\]
(66)
which is to be compared with (36).
Thus, setting
\[ x = \sinh y \]
we obtain the result
\[ \langle \sigma_{M,0} \rangle - \langle E \rangle_{\text{bulk}} \sim -\frac{2z_{lc}(\alpha_2 - 1)}{\pi(1 - z_{lc}^2)} \left[ K_1(m) + K_0(m) \right] \]
(71)
which is to be compared with (44).

When \( \tilde{m} \to \infty \), (71) reduces to
\[ \langle \sigma_{M,0} \rangle - \langle E \rangle_{\text{bulk}} \sim \sqrt{\frac{2z_{lc}(\alpha_2 - 1)}{\pi(1 - z_{lc}^2)}} \frac{m}{M} e^{-4z_{lc}M(\alpha_2 - 1)/(1 - z_{lc}^2)} \]
(72)
which agrees with (58) with \( \alpha_2 \to 1 \).

When \( \tilde{m} \to 0 \), (71) reduces to
\[ \langle \sigma_{M,0} \rangle - \langle E \rangle_{\text{bulk}} \sim \frac{1}{2\pi M} \]
(73)
which agrees with the result (2).

4.6. The crossover regime \( T \to T_c +, H_b \to 0 \) with \( z^2 M \) fixed

In this case, we have
\[ \frac{(e^{i\theta} - 1)/(e^{i\theta} + 1) + iz_{lc}^2 v'/v'}{(e^{i\theta} - 1)/(e^{i\theta} + 1) - iz_{lc}^2 v'/v} \sim \frac{m x^2 [\sqrt{1 + x^2} + 1 - \zeta^2_c [\sqrt{1 + x^2} + 1]^2}{m x^2 [\sqrt{1 + x^2} + 1] + \zeta^2_c x^2} \]
(74)
Thus, using (64) and setting \( x = \sinh y \), we obtain
\[ \langle \sigma_{M,0} \rangle - \langle E \rangle_{\text{bulk}} \sim \frac{z_{lc}(\alpha_2 - 1)}{\pi(1 - z_{lc}^2)} \int_{-\infty}^{\infty} dy \frac{\tilde{m}(\cosh y - 1) - \zeta^2_c (\cosh y + 1)}{\tilde{m}(\cosh y + 1) + \zeta^2_c (\cosh y + 1)} e^{-\tilde{m} \cosh y} \]
(75)
which is to be compared with (50). When \( \zeta^2_c \to 0 \), (66) is recovered, and when \( \zeta^2_c \to \infty \), (71) is recovered. When \( \tilde{m} \to 0 \) the result (51) is again obtained.
5. Discussion

In this paper, we have derived leading behavior of the energy-density operator of the Ising model on an anisotropic lattice $M$ rows from a half-plane boundary at criticality with a magnetic field $H_b$ on the boundary by use of Pfaffian methods. When the field is zero and infinity, we regain the results (1) and (2) obtained in [1] by means of discrete complex analysis. Furthermore, in (51) we have obtained the result in the more general situation, where $H^2 \alpha M$ is fixed with $H_b \to 0$ and $M \to \infty$. This result goes beyond the computations of [1] and we have obtained many results for $T \neq T_c$. It is of interest to obtain these results also by the methods of discrete complex analysis.

We would also like to take this opportunity to remark that it would be most useful to extend the results of [1] in several directions. One such direction is to consider discrete complex analysis on surfaces of higher genus and to derive and extend the results of [8] and [9].

A second direction is to consider inhomogeneous random lattices. One such case is the layered Ising model where the vertical interaction constants are the same in all columns but are chosen randomly from row to row [4, chapters 14 and 15] where it is known that there is an entire temperature region around $T_c$ where the correlation functions are algebraic. All of these problems can be considered as problems with free fermions and thus the methods of discrete complex analysis should apply.

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