DUALITY ON GENERALIZED CUSPIDAL EDGES PRESERVING SINGULAR SET IMAGES AND FIRST FUNDAMENTAL FORMS

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Dedicated to Professor Toshizumi Fukui for his sixtieth birthday.

Abstract. In the second, fourth and fifth authors’ previous work, a duality on generic real analytic cuspidal edges in the Euclidean 3-space $\mathbb{R}^3$ preserving their singular set images and first fundamental forms, was given. In this paper, we show that this duality extends to generalized cuspidal edges in $\mathbb{R}^3$, including cuspidal cross caps, and $5/2$-cuspidal edges. When the singular set image has no symmetries and does not lie in a plane, the dual generalized cuspidal edge is not congruent to the original one. We construct concrete examples of these duals. Moreover, we give several new geometric insights on this duality.

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1. Introduction

By the terminology ‘$C^r$’-differentiable we mean $C^\infty$-differentiability if $r = \infty$ and real analyticity if $r = \omega$. We denote by $\mathbb{R}^3$ the Euclidean 3-space. Let $U$ be a neighborhood of the origin

\begin{equation}
    o := (0,0)
\end{equation}
in the uv-plane $\mathbb{R}^2$, and let $f : U \to \mathbb{R}^3$ be a $C^r$-map. Without loss of generality, we may assume $f(o) = 0$, where

$$ (1.2) \quad 0 := (0, 0, 0). $$

A point $p \in U$ is called a singular point if $f$ is not an immersion at $p$. A singular point $o \in U$ is called a cuspidal edge (resp. a generalized cuspidal edge) if there exist a local $C^r$-diffeomorphism $\varphi$ on $\mathbb{R}^2$ and a local $C^r$-diffeomorphism $\Phi$ on $\mathbb{R}^3$ such that

$$ (1.3) \quad (f_{3/2} :=)(u, v^2, v^3) = \Phi \circ f \circ \varphi(u, v) \quad (\text{resp. } (u, v^2, v^3) = \Phi \circ f \circ \varphi(u, v)), $$

where $h(u, v)$ is a $C^r$-function. Similarly, a singular point $o \in U$ is called a 5/2-cuspidal edge (resp. a fold singularity) if there exist a local $C^r$-diffeomorphism $\varphi$ on $\mathbb{R}^2$ and a local $C^r$-diffeomorphism $\Phi$ on $\mathbb{R}^3$ such that

$$ (1.4) \quad (f_{3/2} :=)(u, v^2, v^3) = \Phi \circ f \circ \varphi(u, v) \quad (\text{resp. } (u, v^2, 0) = \Phi \circ f \circ \varphi(u, v)). $$

Also, a singular point $o \in U$ is called a cuspidal cross cap if there exist a local $C^r$-diffeomorphism $\varphi$ on $\mathbb{R}^2$ and a local $C^r$-diffeomorphism $\Phi$ on $\mathbb{R}^3$ such that

$$ (1.5) \quad (f_{ccr} :=)(u, v^2, w^3) = \Phi \circ f \circ \varphi(u, v). $$

Cuspidal edges, 5/2-cuspidal edges and cuspidal cross caps are all generalized cuspidal edges.

Let $G^c_{3/2}(\mathbb{R}^2, \mathbb{R}^3)$ (resp. $G^c(\mathbb{R}^2, \mathbb{R}^3)$) be the set of germs of $C^r$-cuspidal edges (resp. generalized $C^r$-cuspidal edges) $f(u, v)$ such that $f(o) = 0$. We fix an embedding (i.e. a simple regular space curve) $\Gamma : (-1, 1) \to \mathbb{R}^3$ such that $\Gamma(0) = 0$, and denote by $C$ the image of $\Gamma$. Here, we ignore the orientation of $C$ and think of $C$ as the singular set image (i.e. the image of the singular set) of $f$. We let $G^c_{3/2}(\mathbb{R}^2, \mathbb{R}^3, C)$ (resp. $G^c(\mathbb{R}^2, \mathbb{R}^3, C)$) be the subset of $G^c_{3/2}(\mathbb{R}^2, \mathbb{R}^3)$ (resp. $G^c(\mathbb{R}^2, \mathbb{R}^3)$) such that $f \in G^c_{3/2}(\mathbb{R}^2, \mathbb{R}^3, C)$ (resp. $f \in G^c(\mathbb{R}^2, \mathbb{R}^3, C)$) if and only if the singular set image of $f$ is contained in $C$ (we call the space curve $C$ the edge of $f$). Similarly, a subset of $G^c(\mathbb{R}^2, \mathbb{R}^3, C)$ denoted by

$$ G^c_{ccr}(\mathbb{R}^2, \mathbb{R}^3, C), \quad (\text{resp. } G^c_{3/2}(\mathbb{R}^2, \mathbb{R}^3, C)), $$

which consists of germs of cuspidal cross caps (resp. 5/2-cuspidal edges) is also defined.

We denote by $\kappa(P)$ the curvature function of $C$ at a point $P \in C$. Throughout this paper, we assume

$$ (1.6) \quad \kappa(P) > 0 \quad (P \in C). $$

We fix $f \in G^c(\mathbb{R}^2, \mathbb{R}^3, C)$ arbitrarily. For each point $P$ on the edge $C$, the section of the image of $f$ by the normal plane $\Pi(P)$ of $C$ at $P$ is a planar curve with a singular point at $P$. We call this the sectional cusp of $f$ at $P$. Moreover, we can find a unit vector, which points in the tangential direction of the sectional cusp at $P$. We call this vector the cuspidal direction (cf. [21]). The angle $\theta$ of the cuspidal direction from the principal normal vector of $C$ at $P$ is called the cuspidal angle.

We can take the arc-length parametrization $\hat{\gamma}(u)$ of $C$ such that $\hat{\gamma}(0) = 0$. If we normalize the initial value $\theta(0) \in (-\pi, \pi]$ at $f(o) = 0$, then the cuspidal angle $\theta(u)$ at $f(u)$ can be uniquely determined as a continuous function. In [13] [16], the
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Figure 1. A cuspidal edge and its sectional cusp

singular curvature $\kappa_s(u)$ and the limiting normal curvature $\kappa_\nu(u)$ along the edge $\hat{\gamma}(u)$ are defined. In our present situation, they can be expressed as

$$\kappa_s := \kappa \cos \theta, \quad \kappa_\nu := \kappa \sin \theta.$$

By definition, the following identity holds:

$$\kappa(\hat{\gamma}(u)) = \sqrt{\kappa_s(u)^2 + \kappa_\nu(u)^2} \quad (|u| < \varepsilon).$$

We say that $f$ is generic if

$$|\kappa_\nu(0)| > 0, \quad \text{(i.e. } \kappa(\hat{\gamma}(0)) > |\kappa_s(0)|).$$

We denote by $G^*_r(R^2_0, R^3, C)$ the set of germs of generic generalized $C^r$-cuspidal edges in $G^r(R^2_0, R^3, C)$, and set

$$G^*_{s,3/2}(R^2_0, R^3, C) := G^*_r(R^2_0, R^3, C) \cap G^*_{3/2}(R^2_0, R^3, C),$$

$$G^*_r,R,cc(R^2_0, R^3, C) := G^*_r(R^2_0, R^3, C) \cap G^*_{cc}(R^2_0, R^3, C),$$

$$G^*_{s,5/2}(R^2_0, R^3, C) := G^*_r(R^2_0, R^3, C) \cap G^*_{5/2}(R^2_0, R^3, C).$$

Let $O(3)$ (resp. $SO(3)$) be the isometry group (resp. the orientation preserving isometry group) of $R^3$ fixing the origin $0$. We denote by $\text{Diff}^r(R^2_0)$ the set of germs of $C^r$-diffeomorphisms on $R^2$ fixing the origin $o$.

**Definition 1.1.** For a given generalized cuspidal edge $f \in G^r(R^2_0, R^3_0)$, a generalized cuspidal edge $g \in G^r(R^2_0, R^3_0)$ is said to be congruent to $f$ if there exists an orthogonal matrix $T \in O(3)$ such that

$$T(\text{Im}(g)) = \text{Im}(f)$$

holds, where $\text{Im}(g)$ and $\text{Im}(f)$ are the images of $g$ and $f$, respectively.

We then define the following:
Definition 1.2. For a given generalized cuspidal edge \( f \in \mathcal{G}^{r}(R^3_0, R^3_0) \), a generalized cuspidal edge \( g \in \mathcal{G}^{r}(R^2_0, R^3_0) \) is called an isomer of \( f \) (cf. [14]) if it satisfies the following conditions:

1. the first fundamental form of \( g \) coincides with that of \( f \),
2. the image of \( g \) does not coincide with that of \( f \), but
3. \( f \) and \( g \) have the same singular set image.

By (1), \( f \) and \( g \) share a common singular set \( \Sigma \) in \( R^2 \). We can parametrize it as a regular curve \( \gamma(u) \) in \( R^2 \). We say that \( g \) is a faithful isomer of \( f \) if the orientation of \( g \circ \gamma(u) \) is compatible with that of \( f \circ \gamma(u) \). This definition is independent of the choice of the initial orientation of \( \gamma \) parametrizing \( \Sigma \). Moreover, if a faithful isomer is not congruent to \( f \), it is called a proper isomer.

Remark 1.3. If \( g \) is a faithful isomer of \( f \), it is said to be strongly isometric to \( f \) in [14]. A faithful isomer might not be a proper isomer, in general, see the statements of Corollary V, Proposition VI and Theorem VII.

We give here the following terminologies:

Definition 1.4. The curve \( C \) admits a non-trivial symmetry if there exists \( T \in O(3) \) such that \( T(C) = C \), which is non-trivial, that is, if \( P \in C \) satisfies \( T(P) = P \) then \( P = 0 \).

If \( C \) lies in a plane \( \Pi \), then there exists a reflection \( S \in O(3) \) with respect to \( \Pi \). Then \( S \) is a symmetry of \( C \) which is not non-trivial, but is not the identity map. In [14], it was shown the existence of an involution

\[
I_0 : \mathcal{G}^w_{s,3/2}(R^3_0, R^3_0, C) \ni f \mapsto \hat{f} \in \mathcal{G}^w_{s,3/2}(R^2_0, R^3_0, C).
\]

The map \( \hat{f} := I_0(f) \) is called the dual of \( f \), which satisfies the following properties:

(i) The dual \( \hat{f} \) is a faithful isomer of \( f \).
(ii) If \( \{f_t\}_{t \in R} \) is a continuous family with respect to \( t \), then so is \( \{\hat{f}_t\}_{t \in R} \).
(iii) If \( C \) lies in a plane, then \( \hat{f} \) is congruent to \( f \). On the other hand, if \( C \) does not lie in any plane and does not admit any non-trivial symmetry at the origin 0 of \( R^3 \), then \( \hat{f} \) is not congruent to \( f \).

In [14], a necessary and sufficient condition for a given positive semi-definite metric to be realized as a generic cuspidal edge, is given. Moreover, a generalization of this result for swallowtails and cuspidal cross caps is given in [6] Theorem B. The following assertion is a special case of [6] Theorem B].

Theorem I. There exists an involution

\[
I_C : \mathcal{G}^w_s(R^3_0, R^3_0, C) \ni f \mapsto \hat{f} \in \mathcal{G}^w_s(R^3_0, R^3_0, C)
\]

satisfying the properties (i), (ii). If \( \theta(P) \) is the cuspidal angle of \( f \) at \( P(\in C) \), then \( -\theta(P) \) is the cuspidal angle of \( \hat{f} \) at \( P \). Moreover, the restriction of the map \( I_C \) to \( \mathcal{G}^w_{s,3/2}(R^3_0, R^3_0, C) \) coincides with the map \( I_0 \) as in (1.10).

In this paper, we prove Theorem I as a modification of the proof of [14].

Since \( \text{Diff}^r(R^3_0) \) acts on \( \mathcal{G}^r(R^3_0, R^3_0) \) and \( \mathcal{G}^r(R^2_0, R^3_0) \) from the right we can define the quotient space by

\[
\mathcal{G}^{r}(R^3) := \mathcal{G}^{r}(R^3_0, R^3_0)/\text{Diff}^r(R^3_0), \quad \mathcal{G}^{r}(R^3, C) := \mathcal{G}^{r}(R^3_0, R^3, C)/\text{Diff}^r(R^3_0).
\]

\(^{1}\)We discuss the property (iii) later, see Theorem IV.
Similarly,

\[ \mathcal{G}_{3,2}^r(R^3, C), \quad \mathcal{G}_{cc}^r(R^3, C), \quad \mathcal{G}_{5,2}^r(R^3, C) \]

and

\[ \mathcal{G}_{c}^r(R^3), \quad \mathcal{G}_{s}^r(R^3, C), \quad \mathcal{G}_{s,3/2}^r(R^3, C), \quad \mathcal{G}_{s,cc}^r(R^3, C), \quad \mathcal{G}_{s,5/2}^r(R^3, C) \]

are also defined. Since SO(3) canonically acts on \( \mathcal{G}^r(R^3_0, R^3_0) \) from the left, we can define the quotient space

\[ \pi : \mathcal{G}^r(R^3_0, R^3_0) \rightarrow \mathcal{G}^r(R^3_0, R^3_0)/\text{SO}(3). \]

We then set

\[ \mathcal{G}^r(R^3_0, [C]) := \pi(\mathcal{G}^r(R^3_0, R^3_0), C), \]

where \([C]\) denotes the set of curves belonging to the orbit of \( C \) by the action of SO(3).

Kossowski [9] defined a certain kind of positive semi-definite metrics called ‘Kossowski metrics’ (cf. Section 2). A singular point (or a semi-definite point) of a Kossowski metric is a point where the metric is not positive definite. The first fundamental forms (i.e. the induced metrics) of germs of generalized cuspidal edges are Kossowski metrics of type I (cf. Lemma [2.9]). We denote by \( K^r_o(R^3_0) \) the set of germs of \( C^r \)-Kossowski metrics of type I. We fix \( ds^2 \in K^r_o(R^3_0) \), and let \( U \) be the domain of definition of \( ds^2 \). Then the metric is expressed as

\[ ds^2 = Edu^2 + 2Fdu dv + Gdv^2, \]

and there exists a \( C^r \)-function \( \lambda \) such that \( EG - F^2 = \lambda^2 \). Let \( K \) be the Gaussian curvature of \( ds^2 \) defined at points where \( ds^2 \) is positive definite. Then the function

\[ (1.11) \quad \hat{K} = \lambda K \]

is \( C^r \)-differentiable even at the singular points of \( ds^2 \). If \( \hat{K} \) vanishes (resp. does not vanish) at the singular point \( o \) of \( ds^2 \), then \( ds^2 \) is said to be parabolic (resp. non-parabolic) (see Definition [2.0]). We denote by \( K^r_o(R^3_0) \) (resp. \( K^r_p(R^3_0) \)) the set of germs of non-parabolic (resp. parabolic) \( C^r \)-Kossowski metrics of type I. The subset of \( K^r_o(R^3_0) \) defined by

\[ K^r_o(R^3_0) = \{ ds^2 \in K^r_o(R^3_0) : \hat{K}_u(o) \neq 0 \} \]

plays an important role in this paper. Metrics belonging to \( K^r_p(R^3_0) \) are called \( p \)-generic. On the other hand, if \( \hat{K} \) vanishes identically along the singular curve of \( ds^2 \in K^r_o(R^3_0) \), we call \( ds^2 \) an asymptotic Kossowski metric of type I. This terminology comes from the following two facts:

- for a regular surface, a direction where the normal curvature vanishes is called an asymptotic direction, and
- the induced metric of a cuspidal edge whose limiting normal curvature \( \kappa_u \) vanishes along its singular set belongs to \( K^r_o(R^3_0) \). (Such a cuspidal edge is called an asymptotic cuspidal edge, see Definition [3.11] and Proposition [4.12].)
We let $K^2_\eta(R_0^2)$ be the set of germs of such metrics. By definition, we have

$$K^2_\eta(R_0^2) \cap K^3_\eta(R_0^3) = \emptyset, \quad K^2_\eta(R_0^2) \cup K^3_\eta(R_0^3) = K^2_\eta(R_0^2),$$

$$K^3_\eta(R_0^3) \subset K^3_\eta(R_0^3) \subset K^2_\eta(R_0^2).$$

For an asymptotic $C^r$-Kossowski metric $ds^2$ of type I, the Gaussian curvature $K$ can be extended on $U$ as a $C^r$-differentiable function. Let $\eta \in T_oR^2$ be the null vector at the singular point $o$ of the asymptotic Kossowski metric $ds^2$. If

$$dK(\eta)(o) \neq 0,$$

then $ds^2$ is said to be $a$-generic, and we denote by $K^2_\eta(R_0^2)$ the set of germs of a-generic asymptotic $C^r$-Kossowski metrics. Considering the first fundamental form $ds^2_f$ of $f$, we can define a map

$$J_C : G^2_\eta(R_0^2, \mathbb{R}^3, C) \ni f \mapsto ds^2_f \in K^2_\eta(R_0^2).$$

**Theorem II.** The map $J_C : G^2_\eta(R_0^2, \mathbb{R}^3, C) \rightarrow K^2_\eta(R_0^2)$ has the following properties:

1. $J_C$ is surjective and $J_C \circ I_C = J_C$, where $I_C$ is the involution defined on $G^2_\eta(R_0^2, \mathbb{R}^3, C)$ as in Theorem I.
2. For each $f \in G^2_\eta(R_0^2, \mathbb{R}^3, C)$, there exists $g \in G^2_\eta(R_0^2, \mathbb{R}^3, C)$ such that

$$J_C^{-1}(J_C(f)) = \{f, I_C(f), g, I_C(g)\}.$$

Here $I_C(f)$ is a faithful isomer of $f$, but $g$ and $I_C(g)$ are not.

It should be remarked that to prove Theorems I and II, we use the existence and uniqueness result for partial differential equations of real analytic category so called the Cauchy-Kowalevski theorem. Recently, Fukui [3] gave a representation formula for generalized cuspidal edges along their edges in $\mathbb{R}^3$. (In [3], a somewhat similar formula for swallowtails is also given, although it is not applied in this paper.)

We let $\hat{\gamma}(u)$ be the arc-length parametrization of $C$ such that $\hat{\gamma}(0) = 0$. We fix a $a$-generic generalization for $f \in G^2_\eta(R_0^2, \mathbb{R}^3, C)$ arbitrary. The curvature function $\kappa(u)$ and the torsion function $\tau(u)$ along the edge $\hat{\gamma}$ can be considered as invariants of $f$, which we call the edge-curvature and the edge-torsion, respectively.

We denote by $C^r(R_0^3)$ (resp. $C^r(R_0^3)$) the set of $C^r$-function germs defined on a neighborhood of the origin $o$ of $R$ (resp. $R^3$). The sectional cusp of $f$ at $\hat{\gamma}(u)$ induces a function $\mu(u, t) \in C^r(R_0^3)$ which gives the normalized curvature function (see the appendix), where $t$ is the normalized half-arc-length parameter of the sectional cusp. We call $\mu(u, t)$ the slice function of the generalized cuspidal edge $f$. The value

$$\kappa_c(u) := \mu(u, 0)$$

coincides with the cuspidal curvature at the singular point of the sectional cusp, and is called the cuspidal curvature function of $f$ (cf. [13]). As a consequence, the map

$$F : G^r(R^3) \ni f \mapsto (\kappa, \gamma, \theta, \mu) \in C^r(R_0^3) \times C^r(R_0^3) \times C^r(R_0^3) \times C^r(R_0^3)$$

is canonically induced. (Here $\kappa$, $\tau$ and $\theta$ are the curvature, torsion and cuspidal angle along the edge of $f$, respectively, and $\mu$ is the slice function of $f$.) The data $(\kappa, \tau, \theta, \mu)$ is called the fundamental data or the modified Fukui data (see [6, 3] for the definition of Fukui data) of $f$. It can be easily checked that two given generalized cuspidal edges are congruent if and only if the associated fundamental
data coincide. In Section 4, we give a Bj"orling-type representation formula as the inverse map of $\mathcal{F}$ (cf. Proposition 4.2), which is a modification of the formula given in Fukui \[3\]. (In fact, Fukui \[3\] expressed the sectional cusp as a pair of functions, but did not use the function $\mu$.) Fukui \[3\] explained several geometric invariants of cuspidal edges in terms of $\kappa_\alpha, \kappa_\beta$ and $\theta$. For example, the \textit{cusp-directional torsion} $\kappa_t$ is defined in \[12\] and has the expression (cf. \[3\] Page 7) as follows:

(1.14)

$$\kappa_t = \tau - \theta'.$$

As shown in Section 4, the map $\mathcal{F}$ is a bijection. Using several properties of the inverse map $\mathcal{F}$ together with the proof of Theorem I, we can determine the images of the maps $I_C$ and $J_C$ as follows:

**Theorem III.** The maps $I_C$ and $J_C$ satisfy the following properties:

1. $I_C$ is an involution on $\mathcal{G}_{\omega,3/2}(\mathbb{R}^2_0, \mathbb{R}^3, C)$, and $J_C$ maps $\mathcal{G}_{\omega,3/2}(\mathbb{R}^2_0, \mathbb{R}^3, C)$ onto $\mathcal{K}_{\omega}(\mathbb{R}^2_0)$ (cf. \[14\] Theorem 12)),

2. $I_C$ is an involution on $\mathcal{G}_{\omega,5/2}(\mathbb{R}^2_0, \mathbb{R}^3, C)$ and $J_C$ maps $\mathcal{G}_{\omega,5/2}(\mathbb{R}^2_0, \mathbb{R}^3, C)$ onto $\mathcal{K}_{\omega}^*(\mathbb{R}^2_0)$ (cf. \[7\] Theorem A),

3. $I_C$ is an involution on $\mathcal{G}_{\omega,5/2}(\mathbb{R}^2_0, \mathbb{R}^3, C)$ and $J_C$ maps $\mathcal{G}_{\omega,5/2}(\mathbb{R}^2_0, \mathbb{R}^3, C)$ onto $\mathcal{K}_{\omega}^{**}(\mathbb{R}^2_0)$ (cf. \[7\] Theorem 5.6).

In \[7\], the singular points of Kossowski metrics belonging $\mathcal{K}_{\omega}^*(\mathbb{R}^2_0)$ are called \textit{intrinsic} 5/2-cuspidal edges. This terminology comes from the latter assertion in (2).

**Definition 1.5.** A Kossowski metric $ds^2 \in \mathcal{K}_{\omega}^*(\mathbb{R}^2_0)$ is called \textit{symmetric} (at the origin $o$) if there exists a local diffeomorphism $\varphi \in \text{Diff}(\mathbb{R}^2)$ which is not an identity map such that $\varphi^* ds^2 = ds^2$. If $ds^2$ is not symmetric, then $ds^2$ is called \textit{non-symmetric}.

**Definition 1.6.** Two germs $f, g \in \mathcal{G}^*(\mathbb{R}^2_0, \mathbb{R}^3_0)$ are said to be \textit{distinct} if the image of $g$ is different from that of $f$, that is, if $U$ is a neighborhood of $o$, then $g(V)$ is not contained in $f(U)$ for any neighborhood $V(\subset U)$ of $o$.

We can prove the following assertion.

**Theorem IV.** Let $f \in \mathcal{G}_{\omega,3/2}(\mathbb{R}^2_0, \mathbb{R}^3, C)$ be an arbitrarily given real analytic generic cuspidal edge. Suppose that

(a) $C$ is not planar and does not admit any non-trivial symmetry at the origin,

(b) $ds^2$ is not symmetric.

Then any two of the generalized cuspidal edges belonging to $J_C^{-1}(J_C(f))$ are not congruent each other. In particular, the images of the four maps in $J_C^{-1}(J_C(f))$ are distinct.

Surprisingly, as a consequence, each non-parabolic cuspidal edge has three non-congruent isomers, in general. Moreover, we can prove:

**Corollary V.** Let $f \in \mathcal{G}_{\omega,3/2}(\mathbb{R}^2_0, \mathbb{R}^3, C)$ be an arbitrarily given real analytic generic cuspidal edge. Then the images of the four maps belonging to $J_C^{-1}(J_C(f))$ consist of four (resp. two) sets if and only if the induced metric $ds^2(\equiv J_C(f))$ is non-symmetric (resp. symmetric).
The existence of faithful isomers does not follow from Theorem I when \( f \) is \( C^\infty \)-differentiable but not real analytic. However, in the following two cases, we can find them even for \( C^\infty \)-differentiable case as follows:

**Proposition VI.** Let \( C \) be a space curve lying in a plane \( \Pi \). If \( f \in \mathcal{G}^\infty_{4,3/2}(\mathbb{R}^3_0, \mathbb{R}^3, C) \), then \( S \circ f \) is a faithful isomer of \( f \) (which is not proper), where \( S \) is the reflection with respect to the plane \( \Pi \).

**Theorem VII.** Let \( f \in \mathcal{G}^\infty(\mathbb{R}^3_0, \mathbb{R}^3, C) \) be a generalized cuspidal edge whose fundamental data is \((\kappa, \tau, \theta, \mu)\). Suppose that \( \kappa, \tau \) and \( \theta \) are constant, and the slice function \( \mu(u,t) \) does not depend on \( u \). Then there exists an orientation preserving isometry \( T \in SO(3) \) and a local involution \( \varphi \in \text{Diff}^\infty(\mathbb{R}^3_0) \) such that \( T \circ f \circ \varphi \) is a faithful isomer of \( f \). (Since \( T \circ f \circ \varphi \) is congruent to \( f \), it is not a proper isomer.)

Theorem IV, Corollary V, Proposition VI and Theorem VII are new results, and in particular, the last two assertions imply that the faithful isomers are constructed explicitly for a certain subclass of \( C^\infty \)-differentiable generalized cuspidal edges, which suggests that the duality might be described without applying the Cauchy-Kowalevski theorem, in general.

The paper is organized as follows: In Section 2, we review the definition and properties of Kossowski metrics, and prove that the first fundamental forms of generalized cuspidal edges are all Kossowski metrics of type I (cf. Lemma 2.9). In Sections 3, we prove Theorems I and II. In Section 4, we give the representation formula representing generalized cuspidal edges and prove Theorem III. In Section 5, we investigate symmetric properties of generalized cuspidal edges. In Section 6, we prove Theorem IV, Corollary V, Proposition VI and Theorem VII. Several examples are given in Section 7. Finally, in the appendix, a representation formula for generalized cusps in the Euclidean plane is given.

### 2. Kossowski metrics and generalized cuspidal edges

Let \( ds^2 \) be a \( C^r \)-differentiable positive semi-definite metric on a \( C^r \)-differentiable 2-manifold \( M^2 \). A point \( p \in M^2 \) is called a regular point of \( ds^2 \) if it is positive definite at \( p \), and is called a singular point (or a semi-definite point) if \( ds^2 \) is not positive definite at \( p \).

**Definition 2.1.** Let \( p \) be a singular point of the metric \( ds^2 \) on \( M^2 \). Then a non-zero tangent vector \( v \in T_p M^2 \) is called a null vector if

\[
(2.1) \quad ds^2(v, v) = 0.
\]

Moreover, a local coordinate neighborhood \((U; u, v)\) is called adjusted at \( p \in U \) if \( \partial_e := \partial/\partial v \) gives a null vector of \( ds^2 \) at \( p \).

It can be easily checked that (2.1) implies that \( ds^2(v, w) = 0 \) for all \( w \in T_p M^2 \). If \((U; u, v)\) is a local coordinate neighborhood adjusted at \( p = (0,0) \), then

\[
F(0,0) = G(0,0) = 0
\]

holds, where

\[
(2.2) \quad ds^2 = E \, du^2 + 2F \, du \, dv + G \, dv^2.
\]
Definition 2.2. A singular point \( p \in M^2 \) of a \( C^r \)-differentiable positive semi-definite metric \( ds^2 \) on \( M^2 \) is called admissible if there exists a local coordinate neighborhood \((U; u, v)\) adjusted at \( p \) satisfying
\[
E_v(p) = 2F_u(p), \quad G_u(p) = G_v(p) = 0,
\]
where \( E, F, G \) are the \( C^r \)-functions on \( U \) given in (2.2).

The property (2.3) does not depend on the choice of a local coordinate system adjusted at \( p \), as shown in [9] and [5, Proposition 2.7]. In fact, a coordinate-free treatment for the admissibility of singular points is given in [5, Definition 2.3].

Definition 2.3. A positive semi-definite metric \( ds^2 \) is called a Kossowski metric if each singular point \( p \in M^2 \) of \( ds^2 \) is admissible and there exists a \( C^r \)-function \( \lambda(u, v) \) defined on a local coordinate neighborhood \((U; u, v)\) of \( p \) such that
\[
EG - F^2 = \lambda^2 \quad \text{(on } U\text{)},
\]
\[
(\lambda_u(p), \lambda_v(p)) \neq (0, 0),
\]
where \( E, F, G \) are \( C^r \)-functions on \( U \) given in (2.2).

The above function \( \lambda \) is determined up to \( \pm \)-ambiguity (see [6]). We call such a \( \lambda \) the signed area density function of \( ds^2 \) with respect to the local coordinate neighborhood \((U; u, v)\). The following fact is known (cf. [9, 16]).

Fact 2.4. Let \( ds^2 \) be a \( C^r \)-differentiable Kossowski metric defined on a domain \( U \) of the \( uv \)-plane. Then the 2-form \( d\hat{A} := \lambda du \wedge dv \) on \( U \) is defined independently of the choice of admissible local coordinates \((u, v)\).

We call \( d\hat{A} \) the signed area form of \( ds^2 \).

Fact 2.5. Let \( K \) be the Gaussian curvature defined on the complement of the singular set of \( ds^2 \). Then \( \Omega := K d\hat{A} \) can be extended to a \( C^r \)-differentiable 2-form on \( U \).

Definition 2.6. We call \( \Omega \) the Euler form of \( ds^2 \). If \( \Omega \) vanishes (resp. does not vanish) at a singular point \( p \in U \) of \( ds^2 \), then \( p \) is called a parabolic point (resp. non-parabolic point).

The following fact is also known (cf. [9, 5, 6]).

Fact 2.7. Let \( p \) be a singular point of the Kossowski metric \( ds^2 \). Then the null space (i.e. the subspace consisting of null vectors at \( p \)) of \( ds^2 \) is 1-dimensional.

Applying the implicit function theorem for \( \lambda \) as in (2.5), we can conclude that there exists a regular curve \( \gamma(t) \) (\( |t| < \varepsilon \)) in the uv-plane (called the singular curve) parametrizing the singular set of \( ds^2 \) such that \( \gamma(0) = p \). Then there exists a \( C^\infty \)-differentiable non-zero vector field \( \eta(t) \) along \( \gamma(t) \) which points in the null direction of the metric \( ds^2 \). We call \( \eta(t) \) a null vector field along the singular curve \( \gamma(t) \).

Definition 2.8 ([5]). A singular point \( p \in M^2 \) of a Kossowski metric \( ds^2 \) is said to be of type I or an \( A_2 \) point if the derivative \( \gamma'(0) \) of the singular curve at \( p \) (called the singular direction at \( \gamma(t) \)) is linearly independent of the null vector \( \eta(0) \). A singular point \( p \) which is not of type I is called a singular point of type II.

A Kossowski metric \( ds^2 \) is called of type I if all of the singular points of \( ds^2 \) are of type I.
We now fix a $C^r$-generalized cuspidal edge $f \in \mathcal{G}^r(\mathbb{R}^3_0, \mathbb{R}^3, C)$. By the definition of generalized cuspidal edges, we can take a local coordinate system $(u, v)$ such that
$$\hat{\gamma}(u) := f \circ \gamma(u), \quad (\gamma(u) := (u, 0))$$
parametrizes $C$. Then we can write $f(u, v) = \hat{\gamma}(u) + v\xi(u, v)/2$. Since $v = 0$ is a singular set, we have $\xi(u, 0) = 0$ for each $u$. So there exists an $\mathbb{R}^3$-valued $C^r$-function $\hat{\xi}(u, v)$ such that $\xi(u, v) = v\hat{\xi}(u, v)$ and
$$f(u, v) = \hat{\gamma}(u) + \frac{v^2}{2} \hat{\xi}(u, v)$$
hold. Since $f$ is a generalized cuspidal edge, $\hat{\xi}(u, 0)$ is linearly independent of $\hat{\gamma}'(u)$.

**Lemma 2.9.** The induced metrics of $C^r$-differentiable generalized cuspidal edges are $C^r$-differentiable Kossowski metrics whose singular points are of type I.

**Proof.** Let $f$ be a generalized cuspidal edge as in (2.6), and let $ds^2 = Edu^2 + 2Fdu dv + Gdv^2$ be the first fundamental form of $f$. Then
$$E = f_u \cdot f_u, \quad F = f_u \cdot f_v, \quad G := f_v \cdot f_v$$
hold, where $\cdot$ is the inner product of $\mathbb{R}^3$. Since $f_v(u, 0) = 0$, one can easily check (2.4). By (2.6), we have
$$EG - F^2 = |f_u \times f_v|^2 = v^2 \left| \left( \frac{\hat{\gamma}' + \frac{v^2}{2} \hat{\xi}_u}{2} \times \left( \frac{\hat{\xi} + \frac{v}{2} \hat{\xi}_v}{2} \right) \right) \right|^2,$$
where $\times$ denotes the cross product in $\mathbb{R}^3$. Since two vectors $\hat{\gamma}'(u)$, $\hat{\xi}(u, 0)$ are linearly independent, the function given by
$$\lambda := v\lambda_0, \quad \lambda_0 := \left| \left( \frac{\hat{\gamma}' + \frac{v^2}{2} \hat{\xi}_u}{2} \times \left( \frac{\hat{\xi} + \frac{v}{2} \hat{\xi}_v}{2} \right) \right) \right|$$
is $C^r$-differentiable and $\lambda_0(u, 0) \neq 0$. Moreover, $\lambda^2$ coincides with $EG - F^2$. Since $\lambda_u \neq 0$, $ds^2$ is a Kossowski metric. Since $f_v(u, 0) = 0$, $\partial_u := \partial/\partial u$ gives the null-direction, which is linearly independent of the singular direction $\partial_u$. So all singular points of $ds^2$ are of type I. $\square$

**Remark 2.10.** For a generalized cuspidal edge $f$,
$$\nu := \frac{(2\hat{\gamma}' + v^2 \hat{\xi}_u) \times (2\hat{\xi} + \frac{v}{2} \hat{\xi}_v)}{|(2\hat{\gamma}' + v^2 \hat{\xi}_u) \times (2\hat{\xi} + \frac{v}{2} \hat{\xi}_v)|}$$
gives a $C^r$-differentiable unit normal vector field. So $f$ is a frontal map.

**Corollary 2.11.** Let $\hat{K}$ be the function (cf. [111]) associated with a Kossowski metric $ds^2$ of type I such that the $u$-axis is the singular set of $ds^2$. By setting $K := v\hat{K}$, the following assertion hold (where $K$ is the Gaussian curvature of $ds^2$):

1. $\hat{K}(o) \neq 0$ if and only if $K(o) \neq 0$.
2. under the assumption $K(o) = 0$, $\hat{K}_u(o) \neq 0$ if and only if $\hat{K}_u(o) \neq 0$.

**Proof.** By [111], we have the expression $\lambda = v\lambda_0$, where $\lambda_0(o) \neq 0$. So if we set $\hat{K} = vK$, then
$$\hat{K} = \lambda_0 \hat{K},$$
and \( \hat{K}(o) = \lambda_0(o)\hat{K}(o) \) hold, and so the first assertion is obvious. Differentiating \([2,3]\), we have

\[
\hat{K}_u = (\lambda_0)_u\hat{K} + \lambda_0\hat{K}_u
\]

and using the fact \( \hat{K}(o) = 0 \), we have \( \hat{K}_u(o) = \lambda_0(o)\hat{K}_u(o) \), proving the second assertion.

3. Generalized cuspidal edges and the proofs of Theorems I, II

In this section, we fix a \( C^\infty \)-differentiable generalized cuspidal edge \( f : U \to \mathbb{R}^3 \), where \( U \) is a domain of \( \mathbb{R}^2 \) containing the origin \( o \). We assume \( o \) is a singular point of \( f \).

**Definition 3.1.** A parametrization \((u, v)\) of \( f \) is called an adapted coordinate system (cf. \([13\) Definition 3.7]) if

1. \( f_v(u, 0) = 0 \) and \(|f_u(u, 0)| = |f_{vv}(u, 0)| = 1 \) along the \( u \)-axis,
2. \( f_{vv}(u, 0) \) is perpendicular to \( f_u(u, 0) \).

The existence of an adapted coordinate system is proved in \([16\) Lemma 3.2]. Then \( u \) is the arc-length parameter of the edge \( \hat{\gamma}(u) := f(u, 0) \). We denote by \( C \) the image of the curve \( \hat{\gamma} \). In this section, we assume that the curvature function \( \kappa(u) \) of \( \hat{\gamma}(u) \) is positive for each \( u \). Then the torsion function \( \tau(u) \) is well-defined. We can take the unit tangent vector \( e(u) := \hat{\gamma}'(u) \) (\( \prime = d/du \)), and the unit principal normal vector \( n(u) \) satisfying \( \hat{\gamma}''(u) = \kappa(u)n(u) \). We set

\[
b(u) := e(u) \times n(u),
\]

which is the binormal vector of \( \hat{\gamma}(u) \). Since \( f_{uv}(u, 0) \) is perpendicular to \( e(u) \), we can write

\[
f_{uv}(u, 0) = \cos \theta(u)n(u) - \sin \theta(u)b(u),
\]

which is called the cuspidal direction. Here, the plane \( \Pi(\hat{\gamma}(u)) \) passing through \( \hat{\gamma}(u) \) spanned by \( n(u) \) and \( b(u) \) is the normal plane of the space curve \( \hat{\gamma}(u) \). The section of the image of \( f \) by \( \Pi(\hat{\gamma}(u)) \) is a plane curve, which is called the sectional cusp at \( u \). The vector \( f_{uv}(u, 0) \) points in the tangential direction of the sectional cusp at \( \hat{\gamma}(u) \). So we call \( \theta(u) \) the cuspidal angle function as defined in the introduction. Then, we define two functions \( \kappa_s(u) \) and \( \kappa_v(u) \) by \([17\] \( (1) \)), which are called the singular curvature and the limiting normal curvature along the edge of \( f \) (cf. \([16\) ]).

By Lemma \([23\]) the first fundamental form \( ds^2 \) of \( f \) is a Kossowski metric of type I. To prove Theorem I, we need to show the existence of a special local coordinate system for an arbitrarily given Kossowski metric of type I as follows (see \([6\) Theorem 2.6]).

**Fact 3.2.** Let \( ds^2 \) be a \( C^r \)-differentiable \((r = \infty \text{ or } \omega) \) Kossowski metric of type I belonging to \( K\alpha^r(\mathbb{R}^3) \). Then there exists a \( C^r \)-differentiable local coordinate system \((u, v)\) centered at \( o \) such that the coefficients \( E, F, G \) of the first fundamental form \( ds^2 = Edu^2 + 2Fdudv + Gdv^2 \) satisfying:

1. \( E = 1 \) holds along the \( u \)-axis,
2. \( F = 0 \) on \( U \),
3. \( E_u = 0 \) along the \( u \)-axis, and
4. there exists a \( C^r \)-function \( G_0 \) defined on a neighborhood of the origin \( o \) such that \( G = v^2G_0 \) and \( G_0(u, 0) = 1 \).
Proof. In fact, in [6, Lemma 2.3], the existence of such a coordinate system satisfying (1)-(3) and $G_{vv}(u,0) > 0$, was shown. Then the adjustment of $G_0(u,0) = 1$ is done by the coordinate change $v \mapsto a(u)v$ with a suitable function $a(u)$ of one variable. □

Proposition 3.3. If $f \in G^r(R^2_0, R^3, C)$ and $ds^2 = J_C(f)$, then the above orthogonal coordinate system is an adapted coordinate system of $f$ at $o$ (cf. [11,1]).

Proof. Since the $u$-axis is the singular set of $ds^2$, we have $f_v(u,0) = 0$. On the other hand, $f_u(u,0) = f_u(u,0) = 1$

and

(3.2) $f_{uv}(u,0) = f_{uv}(u,0) = \frac{\partial F(u,v)}{\partial v} \bigg|_{v=0} = 0$.

Finally, we have

$f_{vv}(u,0) = \frac{\partial^2 G(u,v)}{\partial v^2} \bigg|_{v=0} = G_0(u,0) = 1$,

proving the assertion. □

The following fact is important:

Lemma 3.4 ([16]). The singular curvature is intrinsic. More precisely,

(3.3) $\kappa_s(u) = \frac{-E_{vv}(u,0)}{2}$ holds, where $(u,v)$ is the coordinate system as in Fact 3.2.

Proof. In fact, we have

$\kappa_s = \kappa \cos \theta = f_{uu}(u,0) \cdot f_{vv}(u,0)$

$= (f_u(u,0) \cdot f_{vv}(u,0))_u - f_u(u,0) \cdot f_{vu}(u,0) = f_u(u,0) \cdot f_{vv}(u,0)$

$= -(f_u(u,v) \cdot f_{uv}(u,v))_v \big|_{v=0} + f_{uv}(u,0) \cdot f_{uv}(u,0)$.

Since $f_v(u,0) = 0$, we have $f_{uv}(u,0) = 0$. So it holds that

$\kappa_s = -(f_u(u,v) \cdot f_{uv}(u,v))_v \big|_{v=0} = \frac{1}{2} (f_u(u,v) \cdot f_{uv}(u,v))_v \big|_{v=0} = \frac{-E_{vv}}{2}$,

proving the assertion. □

Remark 3.5. As shown in [16, Prop. 1.8], $\kappa_s$ is expressed as

$\kappa_s = -F \eta + 2EF_{uv} - EE_{vv}$

under the assumption that $\lambda_v > 0$, where $(u,v)$ is a local coordinate system such that the $u$-axis is the singular set and $\partial/\partial v$ points in the null direction at each point on the $u$-axis. If $(u,v)$ is the local coordinates as in Fact 3.2 then $F = 0$, $\lambda = \sqrt{EG_0}$ and $E(u,0) = 1$. So we can reprove 3.3.

We now prove the following theorem:
Theorem 3.6. Let \( C \) be a regular space curve as in the introduction whose curvature function does not vanish at 0. We let \( ds^2 \in K^\omega_1(R^2_0) \) be a non-parabolic Kossowski metric of type I defined on a neighborhood of the origin in the \( uv \)-plane \( R^2 \) satisfying (1)–(4) of Fact 3.2. Suppose that the absolute value of the singular curvature \( \kappa_s \) of \( ds^2 \) at 0 is less than \( \kappa(0) \). Then there exist two generalized cuspidal edges \( g_+, g_- \in G^\omega(R^2_0, R^3, C) \) satisfying the following properties:

1. \( ds^2 \) is the common first fundamental form of \( g_+ \) and \( g_- \).
2. \( g_- \) is a faithful isomer of \( g_+ \).

This theorem is obtained as a corollary of \([6, \text{Theorem B}]\). In fact, Theorem B is more general assertion which can be applied not only to generalized cuspidal edges but also swallowtail singularities. However, the proof is rather complicated. So we prove Theorem 3.6 from here on out, as a modification of the proof given in \([14]\).

Let \( \hat{\gamma}(u) \) be the parametrization of the space curve \( C \) with arc-length parameter such that \( \hat{\gamma}(0) = 0 \). We now show the existence of a generalized cuspidal edge \( g(u,v) \) such that

\[
\begin{align*}
g_u & = \hat{\gamma}(u), \\
g_v & = \hat{\gamma}_v(u), \\
g_{uv} & = \hat{\gamma}_{uv}(u)
\end{align*}
\]

on \( U \) using the Cauchy-Kowalevski theorem. The following lemma holds:

Lemma 3.7. If there exists a generalized cuspidal edges \( g = (g_\pm) \) as in Theorem 3.6, then it is a solution of the following partial equation

\[
\begin{align*}
g_v & = v\zeta, \\
\xi_v & = g_{uv} = v\zeta_u, \\
\zeta_v & = \frac{1}{2} ( (\zeta, g_u, \xi_u)^T )^{-1} \left( (G_0)_v, -v(G_0)_u, r - v(G_0)_{uu} + 2v\zeta_u \cdot \zeta_u \right)
\end{align*}
\]

of unknown \( R^3 \)-valued functions \( g, \xi, \zeta \) with the initial data

\[
\begin{align*}
g(u,0) & = \hat{\gamma}(u), \\
\xi(u,0) & = g_u(u,0), \\
\zeta(u,0) & = x(u),
\end{align*}
\]

where

\[
x(u) = \cos \theta(u)n(u) \pm \sin \theta(u)b(u),
\]

and

\[
\cos \theta(u) := \frac{\kappa_s(u)}{\kappa(u)}.
\]

Remark 3.8. Since \( g_v = v\zeta \) and \( \xi_v = v\zeta_u \), we have \( \xi_v = v\zeta_u = g_{uv} \). Thus, the initial condition \( \xi(u,0) = g_u(u,0) \) yields \( \xi(u,v) = g_u(u,v) \).

Proof. Since \( ds^2 \) is real analytic, \( E \) and \( G \) are real analytic functions. Since \( g_v(u,0) = 0 \), we can write

\[
g_v(u,v) = v\zeta(u,v),
\]

where \( \zeta(u,v) \) is a real analytic function defined near the origin \( o \in R^2 \). Then

\[
\zeta_v \cdot \zeta = \frac{\zeta \cdot \zeta_v}{2} - \frac{(G_0)_v}{2}.
\]

On the other hand, since

\[
v g_v \cdot \zeta = g_u \cdot g_v = F = 0,
\]
we have \( g_u \cdot \zeta = 0 \). Differentiating this, we have

\[
0 = v(\zeta \cdot g_u)_v = v\zeta_v \cdot g_u + v\zeta \cdot g_{uv} = v\zeta_v \cdot g_u + g_v \cdot g_{uv} = v\zeta_v \cdot g_u + \frac{G_u}{2}.
\]

Since \( G = v^2G_0 \), we have

\[
(3.9) \quad \zeta_v \cdot g_u = -\frac{v}{2}(G_0)_u.
\]

We now obtain information on \( \zeta_v \cdot g_{uu} \). It holds that

\[
v\zeta_v \cdot g_{uu} = g_v \cdot g_{uu} = (g_v \cdot g_u)_u - g_{uv} \cdot g_u = -g_{uv} \cdot g_u = -\frac{E_v}{2},
\]

that is, we obtain

\[
(3.10) \quad \zeta \cdot g_{uu} = -\frac{E_v}{2v}.
\]

On the other hand, we have that

\[
\zeta \cdot g_{uu} + v\zeta_v \cdot g_{uu} = g_{uv} \cdot g_{uu} = (g_{uv} \cdot g_u)_u - g_{uvu} \cdot g_u
\]

\[
= \{ (g_v \cdot g_u)_v - (g_v \cdot g_{uv}) \}_u - (g_{uv} \cdot g_{uv})_v + g_{uv} \cdot g_{uv}
\]

\[
= (-G_u/2)_u - (E_v/2)_v + g_{uv} \cdot g_{uv},
\]

where we have used the fact that \( g_v \cdot g_u = F \) vanishes identically. This, together with \( (3.10) \), gives the following identity

\[
v\zeta_v \cdot g_{uu} = \frac{E_v}{2v} - \frac{E_{uv}}{2} - \frac{G_{uu}}{2} + g_{uv} \cdot g_{uv},
\]

that is,

\[
(3.11) \quad \zeta_v \cdot g_{uu} = \frac{E_v - vE_{uv}}{2v^2} - \frac{(G_0)_{uu}}{2} + v\zeta_u \cdot \zeta_u
\]

is obtained.

Since \( E_v(u,0) = 0 \), the function \( E_v/v \) is a real analytic function, and the function

\[
r(u,v) := \frac{E_v - vE_{uv}}{v^2} = \left( \frac{E_u}{v} \right)_v
\]

is real analytic along the \( u \)-axis. By \( (3.8), (3.9) \) and \( (3.11) \), we have

\[
(3.13) \quad (\zeta, g_u, g_{uu})^T \zeta_v = \frac{1}{2} \left( (G_0)_v, -v(G_0)_u, r - v(G_0)_{uu} + 2v\zeta_u \cdot \zeta_u \right),
\]

where \( (\zeta, g_u, g_{uu})^T \) is the transposition of the \( 3 \times 3 \)-matrix \( M := (\zeta, g_u, g_{uu}) \). If matrix \( M \) is regular, then \( g \) satisfies \( (3.4) \) by setting \( \xi := g_u \) The map \( g \) must have the initial data \( (3.3) \), where

\[
x(u) = \zeta(u,0) = \lim_{v \to 0} \frac{g_u(u,v)}{v} = g_{uv}(u,0).
\]

By \( (3.4) \), \( x(u) \) can be written in the form

\[
x_+(u) = \cos \theta(u)n(u) - \sin \theta(u)b(u),
\]

where \( \theta(u) \) is the function defined by \( (3.6) \) and \( \kappa(u) \) (resp. \( \kappa_s(u) \)) is the curvature function of \( \gamma(u) \) (resp. the singular curvature function defined by \( (3.3) \)). Such a function \( \theta(u) \) is determined up to a \( \pm \)-ambiguity and so

\[
x_-(u) = \cos \theta(u)n(u) + \sin \theta(u)b(u)
\]

is the other possibility. \( \square \)
We show that there are generalized cuspidal edges \( g_\pm(u, v) \) and \( g_\mp(u, v) \) corresponding to the initial data \( x_\pm(u) \) and \( x_- (u) \), respectively. After constructing such \( g_\pm \), the functions \( \pm \theta \) coincide with the cuspidal angles of \( g_\pm \). Without loss of generality, we consider the case \( g = g_+ \) with initial condition \( x(u) := x_+(u) \). Then we have

\[
(M_0 :=) \quad (\zeta(u, 0), g_u(u, 0), g_{uu}(u, 0)) = (\cos \theta(u)n(u) - \sin \theta(u)b(u), e(u), \kappa(u)n(u)).
\]

Since the singular curvature of \( ds^2 \) satisfies \( |\kappa_s(0)| < \kappa(0) \), we may assume \( |\kappa_s(u)| < \kappa(u) \), that is, \( \sin \theta(u) \neq 0 \) for sufficiently small \( |u| \). Thus the matrix \( M_0 \) is regular for each \( u \). We can then apply the Cauchy-Kowalevski theorem (cf. [10]) for the system of partial differential equations (3.4) with initial data (3.5) and get the desired real analytic solution \( g \). We next show that the first fundamental form of \( g \) coincides with \( ds^2 \). Since

\[
(g_v \cdot g_v)_v = 2v\zeta \cdot \zeta + 2v^2 \zeta_u \cdot \zeta = 2vG_0 + v^2(G_0)_v = G_v
\]

and

\[
g_v(u, 0) \cdot g_v(u, 0) = 0 = G(u, 0),
\]

the Cauchy-Kowalevski theorem yields that \( g_v \cdot g_v = G \), which is equivalent to the condition

(3.16) \[ \zeta \cdot \zeta = G_0. \]

On the other hand, using (3.4),

\[
(\xi - g_u)_v = \xi_v - g_{uv} = v\zeta_u - g_{uv} = (g_v)_u - g_{uv} = 0
\]

holds. The initial condition \( \xi(u, 0) = g_u(u, 0) \) yields that

(3.17) \[ g_u = \xi. \]

Then \( g_{uv} = \xi_v = v\zeta_u \) and

\[
g_{uv} \cdot \zeta = v\zeta_u \cdot \zeta = v\frac{(\zeta \cdot \zeta)_u}{2} = \frac{vG_0}{2}
\]

holds. Using this, we have

\[
(g_u \cdot \zeta)_v = g_{uv} \cdot \zeta + g_u \cdot \zeta_v = \frac{vG_0}{2} - \frac{vG_0}{2} = 0.
\]

Since \( g_u(u, 0) \cdot \zeta(u, 0) = 0 \), we can conclude that \( g_u \cdot \zeta = 0 \), that is,

(3.18) \[ g_u \cdot g_v = 0 (= F) \]

is obtained. We now prepare the following:

Lemma 3.9. Suppose that (which is one of the conditions in (3.13))

\[
\zeta_u \cdot \xi_v (= \zeta \cdot g_{uu}) = r - v(G_0)_{uu} + 2v\zeta_u \cdot \zeta_u.
\]

Then the initial condition (3.14) implies the following identity

(3.19) \[ \frac{E_v}{2} + v\zeta \cdot \xi_u = 0. \]
whose first fundamental forms coincide with
\[ ds_f \]
and the first fundamental form of
\[ f_{u,v} \]
parameters holds. On the other hand, we have

**Proof.** Using (3.16), we have that

\[ \left( \zeta \cdot \xi_u \right)_v = \zeta_v \cdot \xi_u + \zeta \cdot \xi_{uv} = \zeta_v \cdot \xi_u + \zeta \cdot (v^2 \xi_{uu}) \]

\[ = \frac{1}{2} \left( r - v(G_0)_{uu} + 2v \zeta_u \cdot \xi_u \right) + \zeta \cdot (v^2 \xi_{uu}) \]

\[ = \frac{r}{2} - \frac{v}{2} (G_0)_{uu} + v(\zeta_u \cdot \xi_u + \zeta \cdot \zeta_{uu}) \]

\[ = \frac{r}{2} - \frac{v}{2} (\zeta \cdot \zeta)_{uu} + \frac{v}{2} (\zeta \cdot \zeta)_{uu} = \frac{r}{2} . \]

By (3.12),

\[ \left( \zeta \cdot \xi_u - E_u \right)_{2v} = 0 \]

holds. On the other hand, we have

\[ \zeta(u,0) \cdot \xi_u(u,0) = x(u) \cdot f_{uu}(u,0) = (\cos \theta(u) n(u) - \sin \theta(u) b(u)) \cdot \hat{\gamma}'(u) \]

\[ = (\cos \theta(u) n(u) - \sin \theta(u) b(u)) \cdot (\kappa(u) n(u)) = \kappa(u) \cos \theta(u) \]

\[ = \kappa(u) \kappa'(u) = \kappa(u) = -E_{uv}(u,0) = \lim_{v \to 0} -E_u(u, v) . \]

So we obtain (3.19). \( \square \)

We return to the proof of Theorem 3.6. By (3.19), we have that

\[ \frac{1}{2} \left( g_u \cdot g_u \right)_v = g_{uv} \cdot g_u = (g_v \cdot g_u)_{u} - g_v \cdot g_{uu} = -v \zeta \cdot \xi_u = \frac{E_v}{2} . \]

This, with the initial condition

\[ g_u(u,0) \cdot g_u(u,0) = \hat{\gamma}'(u) \cdot \hat{\gamma}'(u) = 1 , \]

implies

\[ (3.20) \quad g_u \cdot g_u = E . \]

By (3.20), (2.6) and (3.18), the existence of \( g = g_+ \) is shown. Replacing \( \theta \) by \( -\theta \), we also obtain the existence of \( g = g_- \). Since the cuspidal angles of \( g_{\pm} \) are distinct, the image of \( g_- \) does not coincide with \( g_+ \). Since the orientation of \( u \mapsto g_-(u,0) \) is compatible with that of the curve \( u \mapsto g_+(u,0) \), the map \( g_- \) is a faithful isomer of \( g_+ \). This proves Theorem 3.6. \( \square \)

By the proof of Theorem 3.6 we obtain the following:

**Corollary 3.10.** The cuspidal angle of \( g_- \) is \( -\theta \), where \( \theta \) is the cuspidal angle of \( g_+ \). In particular, \( g_- \) is a faithful isomer of \( g_+ \) whenever \( \theta \neq 0 \).

Now we give the proof of Theorem I.

**Proof of Theorem I.** We take a generalized cuspidal edge \( f \in G_c^w (R^3, R^3, C) \). We let \( ds^2 \) be the first fundamental form of \( f \). Then \( ds^2 \) is a Kossowski metric of type I, by Lemma 2.9. By Fact 3.2, there exists \( \varphi \in \text{Diff}^w (R^3) \) such that the parameters \( u, v \) of \( f \circ \varphi (u, v) \) satisfy the four properties of Fact 3.2. Let \( ds^2 \) be the first fundamental form of \( f \circ \varphi \). Then we can apply Theorem 3.6 for this Kossowski metric \( ds^2 \). Since \( f \) is generic (cf. (1.9)), the singular curvature \( \kappa_s \) of \( ds^2 \) is less than \( \kappa \). By Theorem 3.6 there exist two generalized cuspidal edges \( g_+ , g_- \in G_c^w (R^3, [C]) \) whose first fundamental forms coincide with \( ds^2 \). Since \( ds^2 \) is the first fundamental form of \( f \circ \varphi \), either \( f \circ \varphi = g_+ \) or \( f \circ \varphi = g_- \) holds. Without loss of generality,
we may set $f \circ \varphi = g_+$, then $g_- \circ \varphi^{-1}$ gives the desired dual of $f$. The Cauchy-Kowalevski theorem is proved by showing the convergence of formal power series. So the convergence is uniform, and so the uniqueness of the solution yields the assertion (ii). \hfill \Box

Proof of Theorem II. The surjectivity of the map $J_C$ and the property (1) follow from Theorem 3.6. So, it is sufficient to show the property (2). We fix $f \in G^\infty_c(R^3_0, R^3)$ arbitrarily. Since the first fundamental form is determined independently of a choice of local coordinate system, we have

$$I_C(f \circ \varphi) = I_C(f) \circ \varphi \quad (\varphi \in \text{Diff}^\infty(R^2_0)).$$

So, to prove the assertion, we may assume that the parameters $u, v$ of $f \circ \varphi(u, v)$ satisfy the four properties of Fact 3.4. By Theorem 3.6, there exist two distinct generalized cuspidal edges $f_{\pm} \in G^\infty_c(R^3_0, R^3, C)$ such that $f_{\pm} = f$, and $u \mapsto f_{\pm}(u, 0)$ has the same orientation as that of $u \mapsto f(u, 0)$. On the other hand, by replacing $u$ with $-u$ and applying Theorem 3.6 again, there exist two distinct generalized cuspidal edges $g_{\pm} \in G^\infty_c(R^3_0, R^3, C)$ such that $u \mapsto g_{\pm}(u, 0)$ have the same orientation as that of $u \mapsto f(-u, 0)$. Then $ds^2$ gives the common first fundamental form of these four generalized cuspidal edges. The cuspidal angles $\theta_{\pm}(u)$ of $g_{\pm}$ satisfy $\theta_{\pm}(u) = \theta_{\pm}(-u)$, where $\theta_{\pm}(u)$ is the cuspidal angle of $f_{\pm}$. Obviously $f_{-}$ is a faithful isomer of $f$. On the other hand, since the images of the singular curves of $g_{\pm}$ have opposite orientation of the image of the singular curve of $f$, the two maps $g_{\pm}$ are non-faithful isomers of $f$.

Finally, if there exists a generalized cuspidal edge $h$ whose first fundamental form coincides with $ds^2$, then $h$ must coincide with one of $\{f_+, f_-, g_+, g_-\}$, because of the uniqueness of the partial differential equation (3.3) with initial condition (3.3). Thus, we obtain the second assertion of Theorem II. \hfill \Box

4. A REPRESENTATION FORMULA FOR GENERALIZED CUSPIDAL EDGES AND THE PROOF OF THEOREM III

We now let $f : (U; u, v) \rightarrow R^3$ be a generalized cuspidal edge such that $f(o) = 0$, where $o := (0, 0) \in U$. As mentioned just after Definition 3.1, there exists an adapted coordinate system centered at $o$, that is,

$$e(u) := f_u(u, 0), \quad v_2(u) := f_{uv}(u, 0), \quad v_3(u) := e(u) \times v_2(u)$$

give an orthonormal frame field along the singular set (i.e. the $u$-axis). In particular, $u$ is the arc-length parameter of the edge

$$\hat{\gamma}(u) := f(u, 0)$$

of $f$. We denote by $C$ the image of $\hat{\gamma}$. We consider the normal plane $\Pi(\hat{\gamma}(u))$ of the curve $\hat{\gamma}$, which is passing through $\hat{\gamma}(u)$ and is perpendicular to the tangential direction $\hat{\gamma}'(u)$. Then $\Pi(\hat{\gamma}(u))$ is spanned by $v_2(u)$ and $v_3(u)$, and the intersection of the image of $f$ with the plane $\Pi(\hat{\gamma}(u))$ is a generalized cusp (see the appendix for the definition) as a planar curve. So $f$ can be written (as in Fukui [3]) as follows:

$$f(u, v) = \hat{\gamma}(u) + \alpha(u, v)v_2(u) + \beta(u, v)v_3(u).$$
Since \( f_{uv}(u, 0) = \mathbf{v}_2(u) \), we can write
\[
\alpha(u, v) = \frac{v^2}{2} + v^3 \alpha_0(u, v),
\]
\[
\beta(u, v) = v^3 \beta_0(u, v)
\]
where \( \alpha_0(u, v) \) and \( \beta_0(u, v) \) are \( C^\infty \)-functions. If we fix \( u \), then the curve on \( \Pi(\hat{\gamma}(u)) \)
given by
\[
\hat{\sigma}_u(v) := f(u, v)
\]
is called the sectional cusp at \( u \) as mentioned in the introduction. In this situation, \( \mathbf{v}_2(u) \) points in the same direction as \( d^2\hat{\sigma}_u(v)/dv^2 \) at \( v = 0 \). So we call \( \mathbf{v}_2(u) \) the cuspidal direction. As a plane curve, \( \hat{\sigma}_u \) is congruent to the plane curve
\[
\sigma_u(v) := (\alpha(u, v), \beta(u, v)).
\]
We will take a new parameter \( t \) instead of \( v \) as the normalized half-arc-length parameter of \( \sigma_u \) (see the appendix). Then \( s := t^2/2 \) gives the arc-length parameter of the generalized cusp \( \sigma_u \), and
\[
\hat{\sigma}_u(0) := \left. \frac{d^2\sigma_u(t)}{dt^2} \right|_{t=0}
\]
is a unit vector (cf. Lemma 3.1), and by the replacement of parameter \( (u, v) \) with \( (u, t) \), we have this expression for the cuspidal direction:
\[
\mathbf{v}_2(u)(= f_{uv}(u, 0)) = f_{tt}(u, 0).
\]
So \( (u, t) \) is also an adapted coordinate system along the edge of \( f \). From here, we will introduce this new parameter \( (u, t) \) of \( f \) as follows: Let \( \mu(u, t) \) be the normalized curvature function of \( \sigma_u(t) = (\alpha(u, t), \beta(u, t)) \) (cf. the appendix), which is a \( C^\infty \)-function of \( u \) and \( t \), called the slice function of \( f \) (as defined in the introduction). By Shiba-Umehara’s formula (A.2) in the appendix, the curve \( \sigma_u(t) \) satisfies the following formula
\[
(\alpha(u, t), \beta(u, t)) = \int_0^t v \left( \cos \lambda(u, v), \sin \lambda(u, v) \right) dv,
\]
\[
\lambda(u, t) := 2 \int_0^t \mu(u, v) dv.
\]
We now suppose that the curvature function \( \kappa(u) := |\hat{\gamma}''(u)| \) of \( \hat{\gamma}(u) \) never vanishes for each \( u \), where the prime denotes the differential \( d/du \). We set
\[
\mathbf{e}(u) := \hat{\gamma}'(u).
\]
We can take the unit principal normal vector field \( \mathbf{n}(u) \) and the unit binormal vector field \( \mathbf{b}(u) \) along \( \hat{\gamma}(u) \). Since \( (\hat{\gamma}', \mathbf{n}, \mathbf{b}) \) gives an orthonormal frame for each \( u \), we can write
\[
\begin{pmatrix}
\mathbf{v}_2(u) \\
\mathbf{v}_3(u)
\end{pmatrix}
= \begin{pmatrix}
\cos \theta(u) & -\sin \theta(u) \\
\sin \theta(u) & \cos \theta(u)
\end{pmatrix}
\begin{pmatrix}
\mathbf{n}(u) \\
\mathbf{b}(u)
\end{pmatrix}.
\]
In particular, we have
\[
\begin{pmatrix}
\mathbf{n}(u) \\
\mathbf{b}(u)
\end{pmatrix}
= \begin{pmatrix}
\cos \theta(u) & \sin \theta(u) \\
-\sin \theta(u) & \cos \theta(u)
\end{pmatrix}
\begin{pmatrix}
\mathbf{v}_2(u) \\
\mathbf{v}_3(u)
\end{pmatrix},
\]
where \( \theta(u) \) is the cuspidal angle along the edge of \( f \) defined in the introduction.
Definition 4.1. Let \((a, b)\) \((a < b)\) be an interval on \(\mathbb{R}\), and \(\delta \in (0, \infty)\) a positive number. A \(C^r\)-differentiable \((r = \infty \text{ or } r = \omega)\) quadruple \((\kappa, \tau, \theta, \mu)\) is called a fundamental data (or a modified Fukui-data) if

- \(\kappa : (a, b) \to \mathbb{R}\) is a \(C^r\)-function such that \(\kappa > 0\),
- \(\tau, \theta : (a, b) \to \mathbb{R}\) and \(\mu : (a, b) \times (-\delta, \delta) \to \mathbb{R}\) are \(C^r\)-functions.

Summarizing the above discussions, one can easily show the following representation formula for generalized cuspidal edges, which is a mixture of Fukui’s representation formula as in [3, (1.1)] for generalized cuspidal edges and Shib a-Umehara’s formula for cusps in the appendix:

Proposition 4.2. Let \((\kappa, \tau, \theta, \mu)\) be a given fundamental data and \(\hat{\gamma}(u)\) the space curve with arc-length parameter whose curvature function and torsion function are \(\kappa(u)\) and \(\tau(u)\). Then,

\[
\hat{\gamma}(u) + (\alpha(u, t), \beta(u, t)) = \left(\frac{\cos \theta(u)}{\sin \theta(u)}, -\frac{\sin \theta(u)}{\cos \theta(u)}\right) \left(n(u) + b(u)\right) \quad (u \in (a, b), \ |t| < \delta)
\]

is a generalized cuspidal edge, where \((\alpha, \beta)\) is given by

\[
(\alpha(u, t), \beta(u, t)) = \int_0^t v(\cos \lambda(u, v), \sin \lambda(u, v)) dv, \quad \lambda(u, t) := 2 \int_0^t \mu(u, v) dv.
\]

Moreover,

1. \(\theta\) gives the cuspidal angle of \(f\) along \(\hat{\gamma}\),
2. \(t \mapsto \mu(u, t)\) is the normalized curvature function of the sectional cusp of \(f\) at \(u\),

where \(n(u)\) and \(b(u)\) are the principal normal vector and the binormal vector of \(\hat{\gamma}(u)\), respectively. Conversely, any generalized cuspidal edge is congruent to such a \(f\) constructed in this manner.

Remark 4.3. The fundamental data \((\kappa, \tau, -\theta, -\mu)\) produces a generalized cuspidal edge which is congruent to the one associated to \((\kappa, \tau, \theta, \mu)\). In fact, we can prove this fact as follows: Let \(\hat{\gamma}_0(u)\) be a space curve parametrizing the arc-length parameter \(u\) whose curvature function and torsion function are \(\kappa(u)\) and \(\tau(u)\) respectively. We may assume that \(\hat{\gamma}(0) = 0\). Let \(T \in O(3) \setminus SO(3)\) be an involution. Then \(\hat{\gamma}_1(u) := T \hat{\gamma}_0(u)\) is a space curve whose curvature function and torsion function are \(\kappa(u)\) and \(-\tau(u)\) respectively. We denote by \(e_i(u), n_i(u), b_i(u)\) \((i = 0, 1)\) the unit tangent vector, the unit principal normal vector and the unit binormal vector of \(\hat{\gamma}_i(u)\), respectively. Differentiating \(T \circ \hat{\gamma}_0(u) = \hat{\gamma}_1(u)\), we have

\[
Te_0 = T \circ \hat{\gamma}_0' = \hat{\gamma}_1' = e_1, \quad \kappa T n_0 = T \circ \hat{\gamma}_0'' = \hat{\gamma}_1'' = \kappa n_1.
\]

In particular, \(Te_0 = e_1\) and \(Tn_0 = n_1\) hold. Since \(T \in O(3) \setminus SO(3)\), we have

\[
b_0 = e_0 \times n_0 = (Te_1) \times (Tn_1) = -T(e_1 \times n_1) = -Tb_1.
\]

We set

\[
f_i := \hat{\gamma}_i + (\alpha_i, \beta_i) \left(\frac{\cos \theta_i}{\sin \theta_i}, -\frac{\sin \theta_i}{\cos \theta_i}\right) \left(n_i + b_i\right) \quad (i = 0, 1).
\]

If \(f_0\) (resp. \(f_1\)) is associated to \((\kappa, \tau, \theta, \mu)\) (resp. \((\kappa, -\tau, -\theta, -\mu)\)), then we have

\[
\alpha_0 = \alpha_1, \quad \beta_0 = -\beta_1, \quad \theta_0 = -\theta_1,
\]
and
\[
T \circ f_0 = T \gamma_0 + (\alpha_0, \beta_0) \begin{pmatrix} \cos \theta_0 & -\sin \theta_0 \\ \sin \theta_0 & \cos \theta_0 \end{pmatrix} \begin{pmatrix} T n_0 \\ T b_0 \end{pmatrix} = \gamma_1 + (\alpha_1, \beta_1) \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix} \begin{pmatrix} n_1 \\ b_1 \end{pmatrix} = f_1,
\]
proving the assertion.

The following assertion holds:

**Proposition 4.4.** Let \( f \) be the generalized cuspidal edge associated to a fundamental data \( (\kappa, \tau, \theta, \mu) \). Then

1. \( f \) gives a cuspidal edge along the \( u \)-axis if \( \mu(u, 0) \neq 0 \),
2. \( f \) gives a cuspidal cross cap at \( o \) if \( \mu(0, 0) = 0 \) and \( \mu_u(0, 0) \neq 0 \),
3. \( f \) gives a \( \frac{5}{2} \)-cuspidal edge cap along the \( u \)-axis if \( \mu(u, 0) = 0, \mu_{uu}(u, 0) \neq 0 \).

The first and the second assertions have been proved in [3, Prop. 1.6].

**Proof.** The first assertion follows from (1) of Proposition A.2. The second assertion follows from the criterion for cuspidal cross caps given in [2], but can be proved much easier using (2) of [3, Prop. 4.4]. The third assertion is a consequence of (2) of Proposition A.2. \( \square \)

To compute the first and the second fundamental forms of \( f \) in terms of fundamental data, the following Frenet-type formula for singular curves is convenient.

**Lemma 4.5** (Izumiya-Saji-Takeuchi [8] and Fukui [3]). The following formula holds:

\[
\begin{pmatrix} e' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa \cos \theta & \kappa \sin \theta \\ -\kappa \cos \theta & 0 & \tau - \theta' \\ -\kappa \sin \theta & -(\tau - \theta') & 0 \end{pmatrix} \begin{pmatrix} e \\ v_2 \\ v_3 \end{pmatrix}.
\]

**Proof.** For the sake of the readers’ convenience, we give here a proof (which is the same as in [8 Lemma 1.3]). Since \( e' = \kappa n \), [17] yields the first row of (4.10). On the other hand, using the well-known Frenet-equation

\[
\begin{pmatrix} e' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e \\ n \\ b \end{pmatrix},
\]

we have that

\[
\begin{pmatrix} v_2' \\ v_3' \end{pmatrix} = \theta' \begin{pmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix} \begin{pmatrix} n \\ b \end{pmatrix} + \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} n' \\ b' \end{pmatrix}.
\]

Since

\[
\begin{pmatrix} n' \\ b' \end{pmatrix} = \begin{pmatrix} -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e \\ n \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e \\ v_2 \\ v_3 \end{pmatrix}
\]
and
\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix}
= \begin{pmatrix}
-\kappa \cos \theta & 0 & \tau \\
-\kappa \sin \theta & -\tau & 0
\end{pmatrix},
\]

it holds that
\[
\begin{pmatrix}
\nu_2 \\
\nu_3
\end{pmatrix}
= \theta' \begin{pmatrix}
-\sin \theta & -\cos \theta \\
\cos \theta & -\sin \theta
\end{pmatrix}
\begin{pmatrix}
\kappa_s & \kappa_v \\
-\kappa_s & 0
\end{pmatrix}
+ \begin{pmatrix}
-\kappa \cos \theta & 0 & \tau \\
-\kappa \sin \theta & -\tau & 0
\end{pmatrix}
\begin{pmatrix}
e \\
\nu_2 \\
\nu_3
\end{pmatrix}
= -\kappa \begin{pmatrix}
\cos \theta \\
\sin \theta
\end{pmatrix}
\begin{pmatrix}
u_2 + (\tau - \theta') \\
0 -1
\end{pmatrix}
\begin{pmatrix}
\nu_2 \\
\nu_3
\end{pmatrix},
\]

proving the second and third rows of (4.10). □

Remark 4.6. The above formula can be rewritten as
\[
\begin{pmatrix}
e' \\
\nu_2' \\
\nu_3'
\end{pmatrix}
= \begin{pmatrix}0 & \kappa_s & \kappa_v \\
-\kappa_s & 0 & \kappa_v \\
-\kappa_v & -\kappa_l & 0
\end{pmatrix}
\begin{pmatrix}
e \\
\nu_2 \\
\nu_3
\end{pmatrix},
\]

which is the one given in Izumiya-Saji-Takeuchi [8, Prop. 3.1].

Using this, one can easily obtain the following by a straightforward computation:

Proposition 4.7 (Fukui [3]). The first fundamental form
\[
ds^2 = Edu^2 + 2F du dt + G dt^2
\]
of \( f \) as in (4.8) is given by
\[
E = (1 - \kappa (\alpha \cos \theta + \beta \sin \theta))^2 + (\alpha_u + \beta (\theta' - \tau))^2 + (\beta_u - \alpha (\theta' - \tau))^2,
\]
\[
F = \alpha_t (\alpha_u + \beta (\theta' - \tau)) + \beta_t (\beta_u - \alpha (\theta' - \tau)),
\]
\[
G = t^2,
\]

where \( \alpha \) and \( \beta \) are the functions given by (4.5).

Proof. Differentiating \( f = \gamma + \alpha \nu_2 + \beta \nu_3 \), we have
\[
f_u = (1 - \kappa (\alpha \cos \theta + \beta \sin \theta)) e + (\alpha_u + \beta (\theta' - \tau)) \nu_2 + (\beta_u - \alpha (\theta' - \tau)) \nu_3,
\]
\[
f_t = \alpha_t \nu_2 + \beta_t \nu_3.
\]

Since \( E = f_u \cdot f_u, \ F = f_u \cdot f_t \) and \( G = f_t \cdot f_t \), we obtain the assertion. □

We can write
\[
\mu(u,t) = \mu_0(u) + \mu_1(u)t + \mu_2(u)t^2 + \mu_3(u,t)t^3,
\]
and then Lemma A.1 yields that
\[
\alpha = \frac{t^2}{2} - \frac{\mu_0(u)^2}{2} t^4 - \frac{2 \mu_0(u) \mu_1(u)}{5} t^5 + t^6 a_6(t,u),
\]
\[
\beta = \frac{2 \mu_0(u)}{3} t^3 + \frac{\mu_1(u)}{4} t^4 + \frac{2}{15} \left( 2 \mu_0(u)^3 + \mu_2(u) \right) t^5 + t^6 b_6(t,u),
\]

where \( a_6(t,u) \) and \( b_6(t,u) \) denote \( C^r \)-functions. Using this, we can show the following:
Corollary 4.8. The Gaussian curvature $K$ of $ds^2$ satisfies
\[ K(u, t) = \frac{K_0(u)}{t} + K_1(u) + K_2(u) t + K_3(u, t) t^2, \]
where
\[ K_0 := 2\mu_0 \kappa_\nu, \quad K_1 := -4\kappa_\nu \mu_0^2 - \kappa_t^2 + 2\kappa_\nu \mu_1, \]
\[ K_2 := -4\kappa_\nu \mu_0^3 + \kappa_\nu \kappa_\mu_0 - 6\kappa_\nu \mu_0 \mu_1 + 2\kappa_\nu \mu_2 - 4\mu_0' \kappa_t + \mu_0 \kappa_t', \]
and $K_3(u, t)$ is a $C^r$-function. Here $\kappa_\nu, \kappa_\mu$ and $\kappa_t$ are defined in (1.8) and (1.14). Moreover, $\mu_0 = \kappa_c$ (cf. (1.13)) and $\kappa_t' = d\kappa_t(u)/du$.

Remark 4.9. Fukui [3, Theorem 1.8] has already determined the first two terms $K_0$ and $K_1$. So the essential part of the above corollary is the statement for $K_2$.

Proof. One can obtain this formula by computing the sectional curvature of $ds^2$, or alternatively, one can get it by computing the second fundamental form of $f$ using Proposition 4.10 as Fukui did in [3].

This implies that the first term
\[ K_0 := 2\mu_0 \kappa_\nu = 2\kappa_c \kappa_\nu \]
defined in [3] is an intrinsic invariant, which is called the product curvature. The second term $K_1$ is an intrinsic invariant. Also
\[ K_2 = -4\kappa_\nu \mu_0^3 + \kappa_\nu \kappa_\mu_0 - 6\kappa_\nu \mu_0 \mu_1 + 2\kappa_\nu \mu_2 - 4\mu_0' \kappa_t + \mu_0 \kappa_t' \]
is intrinsic. Since $K_0 = 2\kappa_c \kappa_\nu$, and since $\mu_0$ is equal to the cuspidal curvature $\kappa_c$, the fact that $\kappa_\nu$ and $\kappa_c \kappa_\nu$ are intrinsic yields that
\[ K_2 := -4\kappa_\nu \kappa_c^3 - 6\kappa_\nu \kappa_c \mu_1 + 2\kappa_\nu \mu_2 - 4\mu_0' \kappa_t + \kappa_c \kappa_t' \]
is an intrinsic invariant. Using this, we can prove the following assertion:

Proposition 4.10. Let $f$ be the generalized cuspidal edge associated to a fundamental data $(\kappa, \tau, \theta, \mu)$ satisfying $\sin \theta \neq 0$. Then
1. $f$ gives a cuspidal edge along the $u$-axis if $K_0(u) \neq 0$,
2. $f$ gives a cuspidal cross cap at $u = 0$ if $K_0(0) = 0$ and $dK_0(0)/du = 0$, and
3. $f$ gives a $5/2$-cuspidal edge along the $u$-axis if $K_1(u) = 0$ and $K_2(0) \neq 0$.

In particular, these conditions depend only on the first fundamental form of $f$.

Proof. Since $\sin \theta(u) \neq 0$, we have $\kappa_\nu(u) \neq 0$. Since $K_0 = 2\mu_0 \kappa_\nu$, $K_0(u) = 0$ if and only if $\mu_0(u) = 0$. Since $\mu_0(u) = \mu(u, 0)(= \kappa_c(u))$, the first and second assertions follow from (1) and (2) of Proposition 4.4 respectively. On the other hand, if $\mu_0(= \kappa_c) = 0$, then $K_2 = 2\mu_2 \kappa_\nu$. So $K_2(u) \neq 0$ if and only if $\mu_2(u) \neq 0$. Thus, the third assertion immediately follows from (3) of Proposition 4.4.

We now prove Theorem III.

Proof of Theorem III. Since $\sin \theta \neq 0$ if and only if $\kappa_\nu \neq 0$, the assertions (1) and (2) follow from Theorem 4.6. We next prove (3), (4) and (5). We remark that
\[ K_{\nu, \tau}^s(R_0^2) = \{ ds^2 \in K_f(R_0^2) ; K_0(0) \neq 0 \}, \]
\[ K_{\nu, s}^p(R_0^2) = \{ ds^2 \in K_f(R_0^2) ; K_0(0) = 0, dK_0(0)/du \neq 0 \}, \]
\[ K_{\nu, s}^w(R_0^2) = \{ ds^2 \in K_f(R_0^2) ; K_0(0) = 0, K_2(0) \neq 0 \} \]
hold in terms of our coordinates \((u, t)\). We have shown the following (cf. Propositions 4.4 and 4.10).

- \(K_0(0) \neq 0\) if and only if \(\mu_0(0)(= \kappa_c(0)) \neq 0\).
- \(K_0(0) = 0\) and \(dK_0(0)/du \neq 0\) if and only if \(\mu_0(0)(= \kappa_c(0)) = 0\) and \(d\mu_0(0)/du \neq 0\).
- \(K_0(u) = 0\) and \(K_2(0) \neq 0\) if and only if \(\mu_0(u) = 0\) and \(\mu_2(0) \neq 0\).

By Corollary 2.11, the following assertions hold:

- \(\hat{K}(0) \neq 0\) if and only if \(K_0(0) \neq 0\).
- \(\hat{K}(0) = 0\) and \(\hat{K_u}(0) \neq 0\) if and only if \(K_0(0) = 0\) and \(dK_0(0)/du \neq 0\).

So the first fundamental form \(ds^2\) of \(f\) belongs to \(K_{\omega^*}^{\omega, 3/2}(R^2_0)\) (resp. \(K_{\omega^*}^{p, \omega}(R^2_0)\)) if and only if \(\mu_0(0)(= \kappa_c(0)) \neq 0\) (resp. \(\mu_0(0)(= \kappa_c(0)) = 0\) and \(d\mu_0(0)/du \neq 0\)). On the other hand, \(ds^2\) belongs to \(K_{\omega^*}^{\omega, 3/2}(R^2_0)\) if and only if \(\mu_0(u) = 0\) and \(\mu_1(0) \neq 0\). In fact, \(\eta := \partial/\partial t\) gives the null direction of \(f\) along the \(u\)-axis (as the singular curve of \(ds^2\)), and we have (cf. (1.12)) \(dK(\eta) = K_t(u, 0) = K_2(u)\).

Finally, we consider the generalized cuspidal edges with vanishing limiting normal curvature:

**Definition 4.11.** A cuspidal edge is called *asymptotic* if its cuspidal angle \(\theta(u)\) is constantly equal to 0 or \(\pi\) along its edge.

If \(f\) is an asymptotic cuspidal edge, the singular curvature \(\kappa_s\), the limiting normal curvature \(\kappa_\nu\) and the cusp-directional torsion \(\kappa_t\) satisfies

\[
(4.12) \quad \kappa_s = \varepsilon \kappa, \quad \kappa_\nu = 0, \quad \kappa_t = \tau,
\]

where \(\varepsilon := \cos \theta\). So we get the following:

**Proposition 4.12.** Let \(f\) be the cuspidal edge associated to a fundamental data \((\kappa, \tau, \theta, \mu)\) satisfying \(\sin \theta = 0\). Then

1. the limiting normal curvature \(\kappa_\nu\) vanishes identically,
2. the Gaussian curvature \(K\) can be extended across its singular set as a \(C^r\)-function,
3. the first fundamental form of \(f\) is an asymptotic Kossowski metric.

Moreover, the sign of \(K\) along the singular set coincides with the sign of

\[
(K_1 =) 4\varepsilon \kappa \mu_0^2 - \tau^2
\]

whenever \(K_1 \neq 0\), where \(\varepsilon := \cos \theta\).

If \(\theta = \pi\) and \(\mu_0\) is sufficiently large, then the Gaussian curvature \(K\) near the singular set can be positive. So we can construct cuspidal edges with \(K > 0\). Conversely, the following assertion is an immediate consequence of Proposition 4.12.

**Corollary 4.13.** Let \(f\) be a cuspidal edge whose Gaussian curvature \(K\) is bounded near singular set and positive, then it is asymptotic satisfying \(\theta = \pi\) and \(\kappa_s < 0\).

The negativity of \(\kappa_s\) has been pointed out in [16]. Although Theorem 3.6 does not cover the case \(\kappa_\nu = 0\), Brander [11] showed the existence of cuspidal edges in the case of \(K = 1\) along a given space curve \(C\) of \(\kappa_\nu > 0\) using the loop group theory.
5. Relationships among Isomers

In this section, we show several properties of isomers, prove the last four statements in the introduction.

Let $ds^2 \in K_{1}(R^3_0)$ ($r = \infty$ or $\omega$) be a germ of a Kossowski metric of type I. Recall that (cf. Definition 1.3) $ds^2$ is said to have a symmetry if there exists a non-identical map $\varphi \in \text{Diff}^+(R^3_0)$ satisfying $\varphi^*ds^2 = ds^2$. On the other hand, if such a $\varphi$ does not exist, we say that $ds^2$ has no symmetry.

We first remark the following assertion can be shown:

**Theorem 5.1.** Let $f \in G^{r,1/2}_{\ast}([R^3_{0}], [R^3], C)$ be a non-parabolic cuspidal edge, where $r = \infty$ or $\omega$. Suppose that there exists an orthogonal matrix $T \in O(3)$ satisfying $T(\text{Im}(f)) = \text{Im}(f)$ and there exists at least one point $P \in \text{Im}(f)$ such that $T(P) \neq P$. Then $T$ is an involution, and there exists an involution $\varphi \in \text{Diff}^+(R^3_0)$ which is not the identity map such that $T \circ f \circ \varphi = f$. Moreover, $T$ (resp. $\varphi$) reverses the orientation of the image of the singular curve (resp. the orientation of the singular curve). In particular, the first fundamental form $ds^2$ of $f$ admits a symmetry.

**Proof.** We set $g := T \circ f$. Since $f$ is symmetric at $p$, the image of $g$ coincides with that of $f$ as map germs. Then by Zakalyukin’s lemma [11, Appendix], there exists a local diffeomorphism $\varphi$ such that

\[ (T(f \circ \varphi) = g \circ \varphi = f. \]

In particular, we have $\varphi^*ds^2 = ds^2$, where $ds^2$ is the first fundamental form of $f$. We let $dr^2$ be the pull-back metric of the immersion $L_0 := (f, \nu)$, where $\nu$ is the unit normal vector field along $f$. Since $\tilde{\nu} := T\nu$ is a unit normal vector field along $g$, we have

\[ d\nu = d(T(\nu \circ \varphi)) = |d(T(\nu \circ \varphi))|^2 = |d(\nu \circ \varphi)|^2 = \varphi^*|d\nu|^2, \]

and

\[ \varphi^*dr^2 = \varphi^*|d\nu|^2 + \varphi^*ds^2 = |d\nu|^2 + ds^2 = |d\nu|^2 + |df|^2 = dr^2. \]

Since this new metric $dr^2$ is positive definite, the fact that $\varphi$ is not the identity implies that the differential $d\varphi$ at $o$ is also a non-identical linear involution. We let $\gamma(t)$ be the characteristic curve of $ds^2$ ($|t| < \varepsilon$) such that $\gamma(0) = o$. We may assume that $t$ is the arc-length parameter with respect to $dr^2$. Then, either $\varphi \circ \gamma(s) = \gamma(s)$ or $\varphi \circ \gamma(s) = \gamma(-s)$ holds. However, the former case $\varphi \circ \gamma(s) = \gamma(s)$ never happens: In fact, suppose that $\varphi \circ \gamma(s) = \gamma(s)$. If $\varphi$ is the identity map, then it contradicts the existence of a point $P \in \text{Im}(f)$ such that $T(P) \neq P$. So $\varphi$ is not the identity map. Then the fact that $(d\varphi)_o$ is not a non-identical linear involution implies that $\varphi$ must map one side of $\gamma$ to the other side. However, this contradicts the fact that the Gaussian curvature of $ds^2$ takes different signs on these two sides. So we have $\varphi \circ \gamma(s) = \gamma(-s)$, and $\varphi$ reverses the orientation $\gamma$. Thus,

\[ T \circ f \circ \gamma(-s) = T \circ f \circ \varphi \circ \gamma(s) = f \circ \gamma(s). \]

So we can conclude that $T$ reverses the orientation of the image of $\dot{\gamma} := f \circ \gamma$. By differentiating it, $T \circ \dot{\gamma}'(s) = -\dot{\gamma}'(-s)$ holds. Since $o$ is a cuspidal edge, $e := \dot{\gamma}'(0)(\neq 0)$ gives a $(-1)$-eigenvector of $T$. On the other hand, since the normal direction is invariant under $T$, we have

\[ (5.1) \quad T \circ \nu(o) = \pm \nu(o). \]
Thus, $T$ has two linearly independent $\pm 1$-eigenvectors $\nu(o)$ and $e$. Since the determinant $\det(T)$ is $\pm 1$, all of the eigenvalues of $T$ are $\pm 1$. So $T^2$ is the identity matrix. For each $g \in U$, we have
\[
f(q) = T^2 \circ f(q) = T \circ (f \circ \varphi^{-1})(q) = (T \circ f) \circ \varphi^{-1}(q) = f((\varphi^{-1})^2(q)).
\]
Since $p$ is a cuspidal edge, we may assume that $f$ is injective on $U$. So $\varphi^2$ implies that $\varphi^2$ is the identity map on $U$. □

We now prove Theorem IV:

**Proof of Theorem IV.** Let $g \in G_{e,3/2}(R^2_0, R^3_0)$ be an isomer of $f$. Suppose that $g$ is congruent to $f$, then (cf. Fact 5.1) there exist a non-trivial involution $T \in O(3)$ and an involution $\varphi \in \text{Diff}^\omega(R^3_0)$ such that $T \circ g \circ \varphi = f$. However, this contradicts that $ds^2(=: J_C(f))$ is not symmetric at the origin $o$. Thus, any pair of the four maps in $J_C^{-1} \circ J_C(f)$ is not congruent. In particular, any pair of the images of isomers cannot coincide. □

Using a similar argument, the following is obtained:

**Corollary 5.2.** Let $f \in G_{e,3/2}(R^2_0, R^3_0)$. Suppose that

1. $C$ is planar and does not admit any non-trivial symmetry at the origin $0$, and
2. $ds^2(=: J_C(f))$ admits no symmetry.

- Then $S \circ f$ is a faithful isomer, where $S \in O(3)$ is the reflection with respect to the plane containing $C$,
- if $f$ is real analytic, then there exists $g \in G_{e,3/2}(R^2_0, R^3_0)$ such that $J_C^{-1}(J_C(f)) = \{f, S \circ f, g, S \circ g\}$ holds, and
- $g$ is not congruent to $f$.

In particular, the images of the four maps are distinct.

**Proof.** We know that

(a) the cuspidal angle $\theta$ of $f$ is not equal to an integer multiple of $\pi$, since $f$ is generic, and

(b) the images of two cuspidal edges $f_1, f_2 \in G_{e,3/2}(R^2_0, R^3_0)$ whose cuspidal angle functions are different do not coincide.

So the first assertion follows. We next suppose $f$ is real analytic. Since $C$ lies in a plane, $I_C(f) = S \circ f$ holds. By applying Theorem II, there exists $g \in G_{e,3/2}(R^2_0, R^3_0)$ such that
\[
J_C^{-1}(J_C(f)) = \{f, S \circ f, g, S \circ g\}.
\]
It is sufficient to show that $g$ is not congruent to $f$. If not, then there exist $T \in O(3)$ and non-identity involution $\varphi \in \text{Diff}^\omega(R^3_0)$ such that $T \circ g \circ \varphi = f$ (cf. Fact 5.1). In particular, $\varphi^* ds^2 = ds^2$ holds, contradicting the assumption (2). □

**Remark 5.3.** If $C$ is planar but $f \in G_{3/2,3/2}(R^2_0, R^3_0)$ is not generic, then $S \circ f$ is not a faithful isomer of $f$ in general. For example, the standard cuspidal edge $f_{3/2}$ in the introduction is not generic. In fact, its first fundamental form belongs to $K_{a}(R^3_0)$). The singular set image is a line on the $xy$-plane, and $f(R^2)$ is symmetric with respect to the $xy$-plane.

We next consider the case that $ds^2$ has a symmetry.
Proposition 5.4. Let $f \in G_{s,3/2}(R_0^2, R^3, C)$. Suppose that

1. $C$ is non-planar and does not admit any non-trivial symmetry at the origin $0$.
2. $ds^2 := J_C(f)$ admits a symmetry $\varphi$.

Then

- $\hat{f} := I_C(f)$ is not congruent to $f$, and
- $J^{-1}_C(J_C(f)) = \{f, \hat{f}, f \circ \varphi, \hat{f} \circ \varphi\}$.

In particular, the images of the four maps consist of two sets.

Proof. Suppose that $\hat{f}$ is congruent to $f$, then there exist $T \in O(3)$ and $\psi \in \text{Diff}^\omega(R_0^2)$ such that $T \circ \hat{f} \circ \psi = f$. By (1), $T$ must be the identity matrix. So $\hat{f} \circ \psi = f$ holds. If $\hat{f}$ is the faithful isomer of $f$, then $\psi$ must preserve the orientation of the singular curve. But this contradicts the assertion of Fact 5.1.

Corollary 5.5. Let $f \in G_{s,3/2}(R_0^2, R^3, C)$. Suppose that

1. $C$ is planar and does not admit any non-trivial symmetry at the origin $0$,
2. $ds^2 := J_C(f)$ admits a symmetry $\varphi$.

Then

- $S \circ f$ is a faithful isomer, where $S \in O(3)$ is the reflection with respect to the plane containing $C$,
- if $f$ is real analytic, then $J^{-1}_C(J_C(f)) = \{f, S \circ f, f \circ \varphi, S \circ f \circ \varphi\}$, holds.

As a consequence, the images of these four maps consist of two sets.

Proof. By (a) and (b) in the proof of Corollary 5.2, the first assertion is obtained. We now suppose that $f$ is real analytic. Since $C$ is planar, $I_C(f) = S \circ f$ holds. So the assertions are obvious.

We then consider the case that $C$ has a non-trivial symmetry.

Proposition 5.6. Let $f \in G_{s,3/2}(R_0^2, R^3, C)$. Suppose that

1. $C$ is non-planar and admits a non-trivial symmetry $T \in O(3)$ at $0$ (cf. Definition 1.4),
2. $ds^2 := J_C(f)$ admits no symmetry.

Then

- $\hat{f} := I_C(f)$ is not congruent to $f$, and
- $J^{-1}_C(J_C(f)) = \{f, \hat{f}, T \circ f, T \circ \hat{f}\}$.

Moreover, these four maps are distinct (cf. Definition 1.6).

Proof. If $\hat{f}$ is congruent to $f$, then there exist $T' \in O(3)$ and $\varphi \in \text{Diff}^\omega(R_0^2)$ such that $T' \circ \hat{f} \circ \varphi = f$. Then $T'$ must be the isometry preserving the orientation of $C$, and so it is the identity matrix, because of condition (2). Then $\hat{f} \circ \varphi = f$ holds, but this contradicts that the image of $\hat{f}$ is different from $f$. We next suppose that the four maps are not distinct. Since $f$ is not congruent to $\hat{f}$, either $\text{Im}(f) = T(\text{Im}(f))$ or $\text{Im}(\hat{f}) = T(\text{Im}(f))$ holds. By Fact 5.1, the induced metric $ds^2$ admits a symmetry, a contradiction. So we obtain the conclusion.
Corollary 5.7. Let \( f \in G^\infty_{3/2}(\mathbb{R}^3_0, \mathbb{R}^3, C) \). Suppose that \( C \) is planar and admits a non-trivial symmetry \( T \) at the origin \( 0 \). Then \( S \circ f \) gives a faithful isomer, and \( S \circ f, TS \circ f \) are non-faithful isomers of \( f \), where \( S \) is the reflection fixing the plane containing \( C \). When \( f \) is real analytic, then \( J_{C}^{-1}(J_{C}(f)) = \{ f, S \circ f, T \circ f, TS \circ f \} \) holds. Moreover, the images of these four maps consist of four (resp. two) sets if and only if the induced metric \( ds^2(= J_{C}(f)) \) admits no symmetry (resp. a symmetry).

Proof. Obviously, \( g := S \circ f \) gives an isomer, which is not proper. By (a) and (b) in the proof of Corollary 5.2, the assertion is obtained. \( \square \)

Proposition VI in the introduction is an immediate consequence of Corollaries 5.2, 5.5 and 5.7. We give here an application of Corollary 5.7:

Example 5.8. We set

\[
    f(u,v) := \left( -v^3u-2v^3-v^2+1 \right) \cos u-1, \left( -v^3u-2v^3-v^2+1 \right) \sin u, \quad v^3u+2v^3-v^2
\]

Then, it has cuspidal edge singularities along

\[
    \hat{\gamma}(u) := f(u,0) = (\cos u-1, \sin u, 0).
\]

By setting,

\[
    S := \begin{pmatrix}
        1 & 0 & 0 \\
        0 & 1 & 0 \\
        0 & 0 & -1
    \end{pmatrix}, \quad T := \begin{pmatrix}
        1 & 0 & 0 \\
        0 & -1 & 0 \\
        0 & 0 & 1
    \end{pmatrix},
\]

\( S \circ f \) is the faithful isomer, and \( S \circ f, TS \circ f \) are non-faithful isomers. We remark that \( f \) is associated to the Fukui data \( (\hat{\gamma}, \theta, A, B) \) in (6.2) given by

\[
    \theta = \frac{\pi}{4}, \quad A(u,v) := \sqrt{2}v^3, \quad B(u,v) := \sqrt{2}v^3(u+2).
\]

Finally, we consider the case that \( C \) and \( ds^2 \) both admit a non-trivial symmetry.

Proposition 5.9. Let \( f \in G^r(\mathbb{R}^3_0, \mathbb{R}^3, C) \) for \( r = \infty \) or \( \omega \). Suppose that

1. \( C \) is non-planar and admits a non-trivial symmetry \( T \in \text{O}(3) \) at the origin \( 0 \),
2. \( ds^2(= J_{C}(f)) \) admits a symmetry \( \varphi \).


If $T \in SO(3)$, then $T \circ f \circ \varphi$ is the faithful isomer of $f$, and $T \circ f$, $f \circ \varphi$ are non-faithful isomers. On the other hand, if $f \in G_{\mathrm{c},3/2}(\mathbf{R}^2, \mathbf{R}^3, C)$ and $T \in O(3) \setminus SO(3)$, then $\hat{f} := I_C(f)$ satisfies
\begin{equation}
J_C^{-1}(J_C(f)) = \{ f, \hat{f}, T \circ f, T \circ \hat{f} \}.
\end{equation}
Moreover, $\hat{f}$ is a proper isomer of $f$.

**Proof.** We set $g := T \circ f \circ \varphi$. If $T$ is orientation preserving (resp. orientation reversing), then the cuspidal angle of $T \circ f \circ \varphi$ takes the opposite sign of (resp. the same sign as) that of $f$. So $g$ is a faithful isomer of $f$ if $T \in SO(3)$, and $g = f$ if $T \in O(3)$. On the other hand, $T \circ f$ and $f \circ \varphi$ are non-faithful isomers of $f$.

We next prove the second assertion. We let $f \in G_{\mathrm{c},3/2}(\mathbf{R}^2, \mathbf{R}^3, C)$. Suppose that $T \in O(3) \setminus SO(3)$. We have already seen that $f = T \circ f \circ \varphi$. In particular, $f \circ \varphi$ and $\hat{f}$ are both non-faithful isomers of $f$. It is sufficient to show that $\hat{f} = I_C(f)$ is not congruent to $f$. If not, there exist a non-trivial $S \in O(3)$ and $\varphi \in \text{Diff}^\infty(\mathbf{R}^3)$ such that $S \circ f \circ \varphi = f$. Since $\hat{f}$ is a faithful isomer of $f$, the transformation $S$ preserves the given orientation of $C$, and it must be the identity, a contradiction. \hfill $\Box$

**Proof of Corollary V.** Summarizing the assertions of Theorem IV, Corollary 5.2, Proposition 5.3, Corollary 5.5, Proposition 5.6, Corollary 5.7, and Proposition 5.9 we get the assertion. \hfill $\Box$

As an application, we prove Theorem VII in the introduction.

**Proof of Theorem VII.** By the assumption of the theorem, $C$ has the constant curvature $\kappa$ and the constant torsion $\tau$. If $\tau = 0$, then the assertion reduces to Proposition VI. So we may assume that $\tau \neq 0$. Then, $C$ is a helix in $\mathbf{R}^3$ and there exists a $180^\circ$-rotation $T \in SO(3)$ with respect the the principal normal line at $0 \in C$ such that $T(C) = C$. By the first part of Proposition 5.9 it is sufficient to show that the first fundamental form $ds^2 = E du^2 + 2Fdudt + Gdt^2$ of $f$ admits a symmetry $\varphi$ as an involution. In fact, if such a $\varphi \in \text{Diff}^\infty(\mathbf{R}^3)$ exists, then $T \circ f \circ \varphi$ gives a faithful isomer of $f$. Since $\mu = \mu(t)$ does not depend on $u$, we can write (cf. (5.6))
\begin{align*}
\alpha(t) := & \int_0^t \alpha_0(u) du, \\
\beta(t) := & \int_0^t \beta_0(u) du,
\end{align*}
where $\alpha_0(t)$ and $\beta_0(t)$ are $C^*$-functions. By Proposition 4.7
- $E(t) \neq 0$ for each $t$,
- there exists a $C^*$-function $F_0(t)$ such that $F(t) = t^4 F_0(t)$, and
- $G(t) = t^2$.

Setting
\begin{align*}
\omega_1 = & \sqrt{E(t)} \left( du + \frac{F(t)}{E(t)} dt \right), \\
\omega_2 = & \int E(t) - t^6 F_0(t)^2 \sqrt{E(t)} dt,
\end{align*}
we have $ds^2 = (\omega_1)^2 + (\omega_2)^2$. Moreover, if we set
\begin{align*}
x(u, t) := & u + \int_0^t \frac{F(v)}{E(v)} dv, \\
y(t) := & \int_0^t \sqrt{E(v) - v^6 F_0(v)^2} dv.
\end{align*}
Then we can take $(x, y)$ as a new local coordinate system centered at $(0, 0)$, and $t$ can be considered as a function of $y$. So we can write $t = t(y)$, and
\begin{align*}
ds^2 = & E(y) dx^2 + t(y)^2 dy^2.
\end{align*}
So the local diffeomorphism \( \varphi : (x, y) \mapsto (-x, y) \) preserves the metric and reverses the orientation of the singular curve.

6. Examples

One method to give a numerical approximation of a dual \( g \) of a real analytic cuspidal edge \( f \) is to determine the Taylor expansion of \( g(u, v) \) at \( v = 0 \) along the \( u \)-axis as a singular set so that \( g = I_C(f) \). In [14, Page 85], we give a numerical approximation of the dual of \( f_0(u, v) = (u, -v^2/2 + u^3/6, u^2 + v^3/6) \).

We denote by \( C \) its image. In the figure of the dual \( g_0 = I_C(f_0) \) given in [14, Figure 2], the dual surface \( g_0 \) seems like it is lying on the almost opposite side of \( f_0 \). This is the reason why the cuspidal angle \( \theta(u) \) of \( f_0(u, v) \) is \( \pi/2 \) at \( u = 0 \). The red lines of Figure 3 (left) indicates the section of \( f_0, g_0 \) at \( u = -1/4 \). The orange (resp. blue) surface corresponds to \( f_0 \) (resp. \( g_0 \)). We can recognize that the cuspidal angle takes value less than \( \pi/2 \), that is, the normal direction of \( g_0 \) is linearly independent of that of \( f_0 \) at \( (u, v) = (-1/4, 0) \). On the other hand, Figure 3 (right) indicates the images of the numerical approximations of the two non-faithful isomers \( f_1, g_1 \) of \( f_0 \).

![Figure 3](image_url)

**Figure 3.** The images of \( f_0, g_0 \) (left), and the images of \( f_0, f_1, g_1 \) (right), where \( f_0 \) is indicated as orange surfaces.

To give several examples of generalized cuspidal edges, we give here a modified version of the representation formula given in Section 4 as follows: Let \( \hat{\gamma}(u) \ (|u| < \delta) \) be a space curve parametrized with arc-length parameter whose curvature function is \( \kappa(u) > 0 \) and torsion function is \( \tau \), where \( \delta \) is a positive constant. Consider functions of two variables \( A(u, v) \) and \( B(u, v) \) satisfying

\[
(6.1) \quad A(u, 0) = A_v(u, 0) = B(u, 0) = B_v(u, 0) = B_{vv}(u, 0) = 0, \quad A_{vv}(u, 0) > 0.
\]

Then we set

\[
(6.2) \quad f_\theta := \hat{\gamma}(u) + (A(u, v), B(u, v)) \begin{pmatrix} \cos \theta(u) & -\sin \theta(u) \\ \sin \theta(u) & \cos \theta(u) \end{pmatrix} \begin{pmatrix} n(u) \\ b(u) \end{pmatrix} \quad (|u| < \delta, \ v \in R).
\]
When $u$ is the arc-length parameter and $v$ is the normalized $1/2$-arc-length parameter, then this formula coincides with the formula given in Fukui [3]. So we call (6.2) the Fukui-formula. In our situation, we do not assume such a restriction for $u$ and $v$. This formula is useful for constructing several examples of isomers, and we call the quadruple

\[(\hat{\gamma}, \theta, A, B)\]

the Fukui-data associated with $f$. By a straightforward calculation, the coefficients of the first fundamental form of $f$ are expressed as

\[
E = |\hat{\gamma}'|^2 \left(1 - \kappa(A \cos \theta + B \sin \theta)\right)^2 + \left((B\Delta - A_n) \sin \theta + (A\Delta + B_n) \cos \theta\right)^2
\]

\[+ \left((A\Delta + B_n) \sin \theta + (B\Delta + A_n) \cos \theta\right)^2,
\]

(6.5)

\[F = -(BA_v + AB_v)\Delta + A_uA_v + B_uB_v, \quad G = A_u^2 + B_u^2,
\]

where $\Delta := |\hat{\gamma}'(u)|\tau(u) - \theta'(u) = |\hat{\gamma}'(u)|\kappa_t(u)$ and $\kappa_t$ is the function defined in (1.14).

The following assertion gives a criterion when $f_{-\theta}$ is congruent to $f_{\theta}$.

**Proposition 6.1.** Let $C$ be a space curve which admits a non-trivial symmetry $T \in \text{SO}(3)$ at $0$, and let $f := f_{\theta} \in G^\infty(SO_3, R^3, C)$ be a generalized cuspidal edge as in the formula (6.2) such that

- $T \circ f(-u,0) = f(u,0)$, and
- the cuspidal angle $\theta$ satisfies $\theta(u) = \pm \theta(-u)$.

Suppose that $A(u,v)$ and $B(u,v)$ satisfy one of the following two conditions:

1. $A(-u,v) = A(u,v)$ and $B(-u,v) = -B(u,v)$ or
2. $A(-u,v) = A(u,v)$ and $B(-u,v) = -B(v,u)$.

Then $f_{\theta} = T \circ f_{\varepsilon_{\theta}} \circ \varphi$ holds, where $\varphi(u,v) = (-u,-v)$ (resp. $\varphi(u,v) = (-u,u)$) in the case of (1) (resp. (2)). In particular, $f_{-\theta} \circ \varphi$ gives a non-faithful isomer of $f(= f_{\theta})$ if $\theta(u) = \theta(-u)$, and $f$ has a symmetry if $\theta(u) = -\theta(-u)$.

**Proof.** We consider the case $\theta(u) = \theta(-u)$. Differentiating $T \circ \hat{\gamma}(-u) = \hat{\gamma}(u)$, we have $-Te(-u) = e(u)$ and $Tn(-u) = n(u)$. Since $T \in \text{SO}(3)$, we have

\[b(u) = e(u) \times n(u) = -(Te(-u)) \times Tn(-u) = -T(e(-u) \times n(-u)) = -Tb(-u).
\]

In the case of (1) (resp. (2)), we set $\varphi(u,v) := (-u,-v)$ (resp. $\varphi(u,v) := (-u,u)$), and then we have $A \circ \varphi(u,v) = A(u,v)$ and $B \circ \varphi(u,v) = -B(u,v)$. So it holds that

\[T \circ f_{\theta} \circ \varphi(u,v) = \hat{\gamma}(u) + (A(u,v), -B(u,v))\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}\begin{pmatrix} n(u) \\ -b(u) \end{pmatrix}
\]

\[= \hat{\gamma}(u) + (A(u,v), B(u,v))\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}\begin{pmatrix} n(u) \\ b(u) \end{pmatrix} = f_{-\theta}(u,v),
\]

proving the assertion. The case $\theta(u) = -\theta(-u)$ is proved in the same way. \(\square\)
Remark 6.2. When $\theta(u) = \theta(-u)$, the image of $f_{-\theta}$ might coincide with the image of the faithful isomer of $f$. By Proposition 5.9, this happens when the induced metric $ds^2$ admits a symmetry. In fact, if $f(= f_{\theta})$ is as in Theorem VII, the image of $f_{-\theta}$ gives the image of faithful isomer and the image of of non-faithful isomer at the same time.

Similarly, the following assertion holds.

Corollary 6.3. Let $C$ be a space curve which admits a non-trivial symmetry $T \in O(3) \setminus SO(3)$ at $0$, and let $f := f_{\theta} \in G^{\infty}(R_0, R^3, C)$ be a generalized cuspidal edge as in the formula (6.2) such that

- $T \circ f(-u,0) = f(u,0)$, and
- the cuspidal angle $\theta$ satisfies $\theta(u) = \pm \theta(-u)$.

Suppose that $A(u,v) \neq B(u,v)$ satisfy one of the following two conditions:

1. $A(-u,-v) = A(u,v)$ and $B(-u,-v) = B(u,v)$.
2. $A(-u,v) = A(u,v)$ and $B(-u,v) = B(u,v)$.

Then $f_{\theta} = T \circ f_{\pm \theta} \circ \varphi$ holds, where $\varphi(u,v) = (-u,-v)$ (resp. $\varphi(u,v) = (-u,v)$) in the case of (1) (resp. (2)). In particular, $f_{-\theta} \circ \varphi$ gives a non-faithful isomer of $f(= f_{\theta})$ if $\theta(u) = -\theta(-u)$, and $f$ has a symmetry if $\theta(u) = \theta(-u)$.

Proof. Like as in the case of the proof of Theorem 6.1, $-Te(-u) = e(u)$ and $Tn(-u) = n(u)$ hold. Since $\det(T) = -1$, we have $Tb(-u) = b(u)$. In the case of (1) (resp. (2)), we set $\varphi(u,v) := (-u,-v)$ (resp. $\varphi(u,v) := (-u,v)$), then we can prove the assertions. \[\square\]

Figure 4. The images of cuspidal edges $g_{1,\pm}$ (left), 5/2-cuspidal edges $g_{2,\pm}$ (center) and cuspidal cross caps $g_{3,\pm}$ (right) given in Example 6.4. (The orange surfaces corresponds to $g_{i,+}$ and the blue surfaces corresponds to $g_{i,-}$ for $i = 1, 2, 3$.)

Example 6.4. Let $a, b$ be real numbers so that $a > 0$ and $b \neq 0$. Then the space curve $\check{\gamma}(u) := \left( a \cos \left( \frac{u}{c} \right) - a, \ a \sin \left( \frac{u}{c} \right), \ \frac{bu}{c} \right)$ gives a helix of constant curvature $\kappa := a/c^2$ and constant torsion $\tau := b/c^2$, where $c := \sqrt{a^2 + b^2}$. At the point $0 := \check{\gamma}(0)$ on the helix, $\check{\gamma}$ satisfies $T(\check{\gamma}(R)) = \check{\gamma}(R)$.
where $T \in SO(3)$ is the $180^\circ$-rotation with respect to the line passing through the origin $0$ which is parallel to the principal normal vector $\mathbf{n}(0)$. We set $a = b = 1$, $\theta = \pi/4$. By setting

- $A_1(u,v) := v^2$, $B_1(u,v) := v^3$,
- $A_2(u,v) := v^2$, $B_2(u,v) := v^5$,
- $A_3(u,v) := v^2$, $B_3(u,v) := uv^3$.

The surfaces $g_{i,\pm} := f_{\pm\pi/4} (i = 1, 2, 3)$ associated to the Fukui data $(\hat{\gamma}, \pm\pi/4, A_i, B_i)$ correspond to cuspidal edges, $5/2$-cuspidal edges, and cuspidal cross caps, respectively. The first two cases satisfy (1) of Theorem 6.1 and the third case satisfies (2) Theorem 6.1. So $h$ is a non-faithful isomer of $g$ (see Figure 4). The image of $g$ coincides with the image of $T \circ h$, where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. $$

Moreover, by Theorem VII, the image of $g$ in the cases of (1) and (2) coincides with the image of the faithful isomer of $f$.

Finally, we consider the case of fold singularities:

**Example 6.5.** We let $\hat{\gamma}(u)$ be a $C^\infty$-regular space curve with $\kappa(u) > 0$ and $\tau(u) > 0$. If we set

$$g(u,v) = \hat{\gamma}(u) + \frac{v^2}{2} (\cos \theta \mathbf{n}(u) - \sin \theta \mathbf{b}(u))$$

and

$$h(u,v) = \hat{\gamma}(u) + \frac{v^2}{2} (\cos \theta \mathbf{n}(u) + \sin \theta \mathbf{b}(u)),$$

then it can be easily checked that $h$ is a faithful isomer of $g$, where $\theta$ is a constant. These two surfaces can be extended to the following regular ruled surfaces:

$$\tilde{g} = \hat{\gamma}(u) + \frac{v}{2} (\cos \theta \mathbf{n}(u) - \sin \theta \mathbf{b}(u)), \quad \tilde{h} = \hat{\gamma}(u) + \frac{v}{2} (\cos \theta \mathbf{n}(u) + \sin \theta \mathbf{b}(u)).$$

**APPENDIX A. A REPRESENTATION FORMULA FOR GENERALIZED CUSPS**

A plane curve $\sigma : I \to \mathbb{R}^2$ is said to have a singular point at $t = t_0$ if $\dot{\sigma}(t_0) = 0$ (the dot means $d/dt$). The singular point $t = t_0$ is called a **generalized cusp** if $\ddot{\sigma}(t_0) \neq 0$. In this situation, it is well-known that

(i) $t = t_0$ is a cusp if and only if $\ddot{\sigma}(t_0)$, $\dddot{\sigma}(t_0)$ are linearly independent,

(ii) $t = t_0$ is a $5/2$-cusp if and only if $\ddot{\sigma}(t_0)$, $\dddot{\sigma}(t_0)$ are linearly dependent and

$$3\det(\ddot{\sigma}(t_0), \sigma^{(5)}(t_0))\ddot{\sigma}(t_0) - 10\det(\ddot{\sigma}(t_0), \sigma^{(4)}(t_0)) \dddot{\sigma}(t_0) \neq 0$$

holds (cf. [15]).

From now on, we set $t_0 = 0$. The arc-length parameter $s(t)$ of $\sigma$ given by

$$s(t) := \int_0^t |\dot{\sigma}(u)| \text{d}u$$

is not smooth at $t = 0$, but if we set

$$w := \text{sgn}(t) \sqrt{|s(t)|}$$

then this gives a parametrization of $\sigma$ at $t = 0$, which is called the half-arc-length parameter of $\sigma$ at $t = 0$ in [17]. However, for our purpose, as Fukui [3] did, the parameter

$$v := \frac{w}{\sqrt{2}} = \frac{\text{sgn}(t)}{\sqrt{2}} \left( \int_0^t |\dot{\sigma}(u)| du \right)^{1/2}$$

called the normalized half-arc-length parameter is convenient, since it is compatible with the property $|f_{uv}| = 1$ for adapted coordinate systems (cf. Definition 3.1) of generalized cuspidal edges. This normalized half-arc-length parameter can be characterized by the property that $v^2 / 2$ gives the arc-length parameter of $\sigma$. Then by [17, Theorem 1.1], we can write

$$\sigma(v) = \int_0^v u(\cos \theta(u), \sin \theta(u))du, \quad \theta(v) = 2 \int_0^v \mu(u)du.$$  

We call this Shiba-Umehara’s formula, where $\mu(v)$ is called the normalized curvature function of the generalized cusp at $t = 0$, and the value $\mu(0) = \det(\ddot{\sigma}(0), \ldots, \sigma(0)) / |\ddot{\sigma}(0)|^{5/2}$ is called the cuspidal curvature, defined in [18]. If $\mu(0) \neq 0$, there exists a best-approximated cycloid of $\sigma$ at $v = 0$. This cycloid is generating by rolling a circle of radius $r$. Then the value $|\mu(0)|$ coincides with $1 / \sqrt{r}$. We need the following lemma, which can be proved by a straightforward computation.

**Lemma A.1.** Let $v$ be the normalized half-arc-length parameter of the generalized cusp $\sigma(w)$ at $w = 0$. Then there exists an orientation preserving isometry $T$ of $\mathbb{R}^2$ such that the following formula holds

$$T \circ \sigma(v) = \left( \frac{v^2}{2} - \frac{\mu_0 v^4}{2}, -\frac{2\mu_0 v^3}{5} + \frac{2\mu_0 v^3}{3} + \frac{\mu_1 v^4}{4} + \frac{2(-2\mu_0^3 + \mu_2 v^5)}{15} \right) + o(v^5),$$

where $\mu(v) = \sum_{j=0}^3 \mu_j v^j + o(v^3)$, and $o(v^5)$ (resp. $o(v^3)$) is a term higher than $v^5$ (resp. $v^3$).

Using this, one can easily obtain the following assertion:

**Proposition A.2.** Let $v$ be the normalized half-arc-length parameter of the generalized cusp $\sigma(w)$ at $w = 0$. Then

1. $w = 0$ is a cusp of $\sigma$ if and only if $\mu_0 \neq 0$, and
2. $w = 0$ is a 5/2-cusp of $\sigma$ if and only if $\mu_0 = 0$ and $\mu_2 \neq 0$.

It is remarkable that the coefficient $\mu_1$ does not affect the criterion of the 5/2-cusp. If $\mu_0 = 0$, then (A.3) reduces to

$$T \circ \sigma(v) = \left( \frac{v^2}{2}, \frac{\mu_1 v^4}{4} + \frac{2\mu_2 v^5}{15} \right) + o(v^5),$$

and $6\mu_1$ and $48\mu_2$ are called the secondary cuspidal curvature and the bias of $\sigma(t)$ at $t = 0$. Geometric meanings for these two invariants for 5/2-cusps can be found in [17, Prop 2.2].

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