WEAK MATRIX ELEMENTS AND K-MESON PHYSICS

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An overview is presented about old and recent methods to compute the $K \to \pi\pi$ decay amplitude.

1 Introduction

Kaon Physics is a very complicated blend of Ultraviolet and Infrared effects which still defies complete physical understanding.

The problem consists in the large enhancement ($\approx 20$) of the $\Delta I = \frac{1}{2}$ amplitude with respect to the $\Delta I = \frac{3}{2}$ one.

Being a process involving hadrons, $K$-decay must be treated non-perturbatively, so that lattice discretization is the ideal tool to deal with this problem. In fact lattice regularization is the only convergent (as $a \to 0$) approximation scheme to QCD.

Due to the difficulties of putting the Standard Model on the lattice, one can use weak interaction perturbation theory, which, together with Asymptotic Freedom of Strong Interactions, allows the definition of an effective low energy actions for non-leptonic decays:

$$H_{\Delta S=1}^\text{eff} = \lambda_u \frac{G_F}{\sqrt{2}} \left[ C_+ (\mu) O^+(\mu) + C_- (\mu) O^-(\mu) \right]$$

where $\lambda_u = V_{ud} V_{us}^*$ and:

$$O^{(\pm)} = [ (\bar{s} \gamma^\mu d) (\bar{u} \gamma^\mu u) \pm (\bar{s} \gamma^\mu u) (\bar{u} \gamma^\mu d)] - [u \leftrightarrow c]$$

$O^(-)$ is a pure $I = \frac{1}{2}$, while $O^+$ is a mixture of $I = \frac{1}{2}$ and $I = \frac{3}{2}$.

The $O^{(\pm)}$'s transform as $(8,1) \oplus (1,8)$ and $(27,1) \oplus (1,27)$ under the $SU(3) \otimes SU(3)$ chiral group and some discrete symmetries. The coefficients $C_\pm (\mu)$ reliably computed in Perturbation Theory, show a slight octet enhancement:

$$\frac{|C_- (\mu \approx 2 \text{ GeV})|}{|C_+ (\mu \approx 2 \text{ GeV})|} \approx 2$$

The rest of the enhancement ($\approx 10$) should, then, be provided by the matrix elements of $O^{(\pm)}$ and is a non perturbative, infrared effect.

The difficulty with Lattice regularization lies in the fact that naive discretization of Dirac fermions entails a multiplication of low energy degrees of freedom (Doublers) whose elimination complicates the scheme.

There are, essentially, two possibilities:

- Wilson Fermions

A term is added to the Lagrangian, breaking explicitly the chiral symmetry, which can be restored, as $a \to 0$, by the inclusion of appropriate counterterms. This formulation is ultra-local (at the lagrangian level only near neighbors interactions are involved) and it is very convenient for numerical purposes.

- Ginsparg-Wilson Fermions

This discretization is much more respectful of the chiral properties of the (continuum) QCD lagrangian, at the expense of being non local at the lattice level, which makes it, at the moment, numerically very demanding.

My remarks on renormalization will, therefore, be addressed to Wilson fermions formulation.
The difficulty of the problem consists, first of all, in giving the correct definition of the operators $O_i\pm$.

In order to construct finite composite operator of dimension 6, $O_6(\mu)$, we must mix the original bare operator, $O_6(0)$, with bare operators of equal ($O_6^{(0)}(a)$) or smaller ($O_6(a)$) dimension, in general with different naive chiralities.

A general non perturbative technique to construct composite operators is based on the systematic exploitation of Chiral Ward Identities.

It turns out that, in order to minimize the renormalization procedure, the best strategy is to compute $\pi\pi \to \pi\pi$ in the world in which $m_K = 2m_\pi$ or $m_K = m_\pi$, with pions at rest (see section 2.1) and then extrapolate to the real world through chiral perturbation theory. In these cases the ultraviolet subtractions are limited to an overall renormalization which could be determined non perturbatively.

2 Infrared Problems

Approaches requiring the construction of an asymptotic two pion state face the problems due to the fact that the theory is defined, through the functional integral, in the euclidean region. In the next two subsections we will briefly discuss the nature of the problem and possible proposals to solve it.

2.1 Infinite volume

In order to compute the $K \to \pi\pi$ width we have to evaluate the matrix element $(\pi(0)|\pi(t_2)|\mathcal{H}_W|K)\otimes$ with two interacting hadrons in the final state. This is not easy to do in the euclidean region, but it can be shown that:

$$\langle \phi_{\pi}(t_1)\phi_{\pi}(t_2)\mathcal{H}_W(0)K(t_K) \rangle \approx e^{m_Kt_K-E_{t_1}t_1-E_{t_2}t_2} \sqrt{\frac{Z_{\pi}}{2m_K}} \frac{Z_{K}}{2m_\pi}$$

One sees from eq.(7) that for $\pi\pi$ at rest and weakly interacting it is possible to extract a meaningful matrix element.

2.2 Finite volume

Lellouch and Lüscher have recently formulated a strategy based on the exploitation of the finiteness of volume in lattice simulations. Their proposal is based on the following relation between finite and infinite volume matrix elements:

$$|\langle \pi\pi, E = m_K |\mathcal{H}_W(0) |K\rangle|^2 = V^2 |\langle \pi\pi, E |\mathcal{H}_W(0) |K\rangle|^2 \left(\frac{m_K}{k}\right)^3 \times 8\pi[\xi\delta'(k)]$$

where:

$$P(t_2) = -\sum \exp[-(E_n - 2E_0)t_2]2\pi^3\delta^3(P_n) \times$$

$$N_n [M(q, -q; n)]^* (n, out) |\mathcal{H}_W(0) |K\rangle$$
In eq.\([8]\) \(|\pi\pi, E\rangle\) denotes a finite volume two pion state with zero total momentum and "angular momentum" and energy \(E\), while \(|K\rangle\) denotes a single finite volume kaon state with zero momentum. Both states are normalized to 1. \(|\pi\pi, E\rangle\) and \(|K\rangle\) denote the corresponding infinite volume states covariantly normalized according to the usual convention which, for single particle states reads as:

\[
\langle p | q \rangle = (2\pi)^3 2\omega_\pi \delta^{(3)}(p - q) \tag{9}
\]

In a finite volume the allowed values, \(k\), of the 'radial' relative momentum of a zero total momentum s-wave two particle state obey the relation:\([4]\):

\[
n\pi - \delta_0(k) = \phi(q) \tag{10}
\]

where \(\delta_0(k)\) is the s-wave phase-shift, \(q \equiv \frac{h_k}{2\pi}\) \(k\) is related to the center of mass energy, \(E\) as:

\[
E = 2\sqrt{m_\pi^2 + k^2} \tag{11}
\]

and:

\[
\tan \phi(q) = -\frac{3}{2}\pi q \quad Z_{00}(1; q^2) \tag{12}
\]

\[
Z_{00}(s;q^2) = \frac{1}{\sqrt{4\pi}} \sum_{n\in\mathbb{Z}} (n^2 - q^2)^{-s} \tag{13}
\]

Eqs.\([10]-[13]\) completely define the quantities appearing in eq.\([8]\).

I will now present a different approach\([7]\) to the relation between finite and infinite volume matrix elements, which may lead to a better understanding of the nature of eq.\([8]\).

The argument goes as follows.

In order to relate the states at finite volume with those at infinite volume we take the two-point Green function of a scalar operator \(\sigma(x), \int d^3x \langle \sigma(x, t)\sigma(0)\rangle\), and consider its behavior as the space volume \(V\) becomes large. We have:

\[
\int d^3x \langle \sigma(x, t)\sigma(0)\rangle_V \rightarrow \int d^3x \langle \sigma(x, t)\sigma(0)\rangle \quad \text{as} \quad V \rightarrow \infty \tag{14}
\]

\[
\frac{(2\pi)^3}{2(2\pi)^3} \int \frac{dp_1}{2\omega_1} \frac{dp_2}{2\omega_2} \delta(p_1 + p_2) e^{-(\omega_1 + \omega_2)t} \times \langle 0 | \sigma(0) | p_1, p_2 \rangle \bigg|^2 = \frac{1}{2(2\pi)^3} \int dE e^{-Et} |\langle 0 | \sigma(0) | \pi\pi, E \rangle|^2 \times \int \frac{dp_1}{2\omega_1} \frac{dp_2}{2\omega_2} \delta(p_1 + p_2) \delta(E - \omega_1 - \omega_2) = \frac{\pi}{2(2\pi)^3} \int dE e^{-Et} |\langle 0 | \sigma(0) | \pi\pi, E \rangle|^2 k(E) \tag{15}
\]

where:

\[
k(E) = \sqrt{E^2 - m_\pi^2} \tag{16}
\]

On the other hand:

\[
\int d^3x \langle \sigma(x, t)\sigma(0)\rangle = \int d^3x \langle \pi\pi, E\rangle |\langle \sigma(0) | \pi\pi, n\rangle|^2 e^{-Et} \rightarrow \int dE \rho(E) |\langle \sigma(0) | \pi\pi, E\rangle|^2 e^{-Et} \tag{17}
\]

where \(|\pi\pi, n\rangle\) denote the finite volume two pion states classified according to the quantum number \(n\) defined in eq.\([10]\) and:

\[
\rho(E) \equiv \frac{\Delta n}{\Delta E} = \frac{q\phi(q) + k\delta_0(k)}{4\pi k^2} E \tag{18}
\]

denotes the density of states of energy \(E\).

Comparing eqs.\([14]\) and \([18]\), we get the correspondence:

\[
|\pi\pi, E\rangle \leftrightarrow 4\pi \sqrt{VE} \rho(E) k(E) \tag{19}
\]

In a similar way it is easy to show:

\[
|\pi = 0\rangle \leftrightarrow \sqrt{2mV} |\sigma(0)\rangle \quad \text{as} \quad V \rightarrow \infty \tag{20}
\]

From eqs.\([18]\) and \([19]\) we get:

\[
|\langle \pi\pi, E = m_k | H_W(0) | K \rangle|^2 = 32\pi^2 V^2 \rho(m_k) m_k^2 \frac{1}{k_\pi^2} |\langle \pi\pi, E | H_W(0) | K \rangle|^2 \tag{21}
\]

where:

\[
k_\pi \equiv \sqrt{\frac{m_k^2}{4} - m_\pi^2} \tag{22}
\]
Using the expression of $\rho(E)$ given by eq.(17), eq.(20) looks the same as eq.(8). There is, however an important difference. In fact the derivation of eq.(8) requires to work at a fixed volume $V$ and at a fixed value of $n$, defined in eq.(10). Eq.(20), on the contrary, is valid at fixed energy $E$, asymptotically in $V$, so that, while we let $V \to \infty$, we must allow simultaneously $n \to \infty$. A possible relation between the two approaches will be discussed in a forthcoming paper.

The strategy proposed by Lellouch and Lüscher consists in tuning the volume $V$ so that the first excited two-pion state ($n=1$) is degenerate in energy with the kaon state ($L \approx \frac{5}{6}$ $Fm$) and compute the finite volume Green’s function:

$$\int d^3x d^3y \langle \sigma(\mathbf{r},t)H_{W}(0)K(y,t') \rangle_{V} \approx e^{m_{K}t'} \langle K(0) |0\rangle V^2 \times$$

$$\times \sum_{n} \langle 0 |\sigma(0)|\pi\pi,n \rangle_{V} \times$$

$$\times \langle \pi\pi,n |H_{W}(0)|K \rangle_{V} e^{-E_{n}t} =$$

$$= e^{m_{K}t'} \langle K(0) |0\rangle V^2 \times$$

$$\times \sum_{n} \langle 0 |\sigma(0)|\pi\pi,n \rangle_{V} \times$$

$$\times \langle \pi\pi,n |H_{W}(0)|K \rangle_{V} e^{-E_{n}t}$$

(22)

The last equality in eq.(22) is justified by the cancellation of the final state interactions phases in $\langle 0 |\sigma(0)|\pi\pi,n \rangle_{V}$ and $\langle \pi\pi,n |H_{W}(0)|K \rangle_{V}$.

Then, from

$$\int d^3x \langle \sigma(\mathbf{r},t)\sigma(0) \rangle =$$

(23)

$$V \sum_{n} \langle 0 |\sigma(0)|\pi\pi,n \rangle_{V}^2 e^{-E_{n}t}$$

we compute $\langle 0 |\sigma(0)|\pi\pi,1 \rangle_{V}$ and, finally, $\langle \pi\pi,1 |H_{W}(0)|K \rangle_{V}$.

In the case of a $\Delta I=\pm 1$ transition we face a further complication: independently of the chosen procedure, a subtraction has to be performed, due to the fact that the relevant correlator $\langle \sigma(t)H_{W}(0)K(t') \rangle$ is dominated, for large $t$, by the vacuum insertion between $\sigma(t)$ and $H_{W}(0)K(t')$. As a consequence, the relevant physical information about the $K$ decay is contained in the connected correlator:

$$\langle \sigma(t)H_{W}(0)K(t') \rangle_{\text{conn}} \equiv \langle \sigma(t)H_{W}(0)K(t') \rangle - \langle \sigma(0) \rangle \langle H_{W}(0)K(t') \rangle$$

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