Notes on the arithmetic of Hecke \( L \)-functions

A RAGHURAM

Department of Mathematics, Fordham University Lincoln Center, 113 West 60th Street, New York, NY 10023, USA  
E-mail: araghuram@fordham.edu

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Abstract. This is an expository article that concerns the various related notions of algebraic idèle-class characters, the \( \text{Größencharaktere} \) of Hecke, and cohomological automorphic representations of \( \text{GL}(1) \), all under the general title of algebraic Hecke characters. The first part of the article systematically lays the foundations of algebraic Hecke characters. The only pre-requisites are: basic algebraic number theory, familiarity with the adelic language, and basic sheaf theory. Observations that play a crucial role in the arithmetic of automorphic \( L \)-functions are also discussed. The second part of the article, on the ratios of successive critical values of the Hecke \( L \)-function attached to an algebraic Hecke character, concerns certain variations on a theorem of Harder \[6\], especially drawing attention to a delicate signature that apparently has not been noticed before.

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1. Introduction

In \[17\], Weil introduced a special class of characters on the group of idèle-classes of an algebraic number field \( F \) which he called characters of type \( (A_0) \). The importance of these characters are best captured in Weil’s words:

“As to the non-trivial characters of type \( (A_0) \), some of them arise with the theory of abelian varieties with complex multiplication ... Taniyama has proved that the \( L \)-series attached to the characters of type \( (A_0) \) belonging to abelian varieties with complex multiplication are precisely those which occur in the zeta-functions of such varieties ... all that can be said here is that they tend to emphasize the importance of the characters we have discussed and of their remarkable properties.”

With the hindsight afforded by decades of development, we can now say that the modern arithmetic theory of automorphic forms has Weil’s landmark 1955 paper as one of its points of origin.

In this article, we are primarily concerned with the arithmetic aspects of automorphic forms and \( L \)-functions on \( \text{GL}(1) \). While working on the special values of automorphic \( L \)-functions, especially in the context of \[11,12\], the author was naturally confronted with
the various notions connecting Weil’s characters of type \((A_0)\), the \textit{Größencharaktere} of Hecke, and cohomological automorphic representations of \(GL(1)\), all under the general title of algebraic Hecke characters. The idea of writing these notes is so that anyone with a similar interest in the arithmetic theory of \(L\)-functions has a ready reference at hand with all the necessary details. There is nothing new in the first five sections other than a reorganization of one’s thoughts to wade through the various dictionaries; this part of the article is based upon the references: Deligne [3,4], Harder [6], Hida [8], Neukirch [10], Serre [14], Schappacher [13], Waldspurger [16] and Weil [17]. Also included are observations that play a crucial role in the study of special values of automorphic \(L\)-functions. The only pre-requisites to read the first part are basic algebraic number theory, the language of adèles and idèles (for example, the first seven chapters of Weil’s book [18] are sufficient), and some basic theory of sheaf cohomology.

The last section (Section 6), which concerns the special values of Hecke \(L\)-functions, is conceptually and technically much deeper. In a pioneering paper, Harder [6] laid the foundations of Eisenstein cohomology for \(GL(2)\) and as a consequence proved a rationality result on the ratios of critical values of Hecke \(L\)-functions. To give an idea of the shape of this result, let \(E\) and \(F\) be number fields and \(\chi\) an algebraic Hecke character of \(F\) with coefficients in \(E\). For an embedding of fields \(\iota : E \to \mathbb{C}\), consider the \(\mathbb{C}\)-valued Hecke \(L\)-function \(L(s, \iota\chi)\). Suppose \(m \in \mathbb{Z}\) is such that both \(m\) and \(m+1\) are critical for \(L(s, \iota\chi)\). This condition restricts \(F\) to be a totally imaginary field that contains a maximal CM subfield. Let \(\delta_{F/\mathbb{Q}}\) denote the absolute discriminant of \(F\). Harder [6] proved that the complex number \(\left|\delta_{F/\mathbb{Q}}\right|^{1/2} \frac{L(m, \iota\chi)}{L(m+1, \iota\chi)}\) is algebraic (see (25)) and satisfies a reciprocity law that it is equivariant under the Galois group \(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\) of \(\mathbb{Q}\) (see (26)); it follows then that the algebraicity statement can be strengthened to assert that the above quantity is in \(\iota(E)\) (see (27)). The \textit{raison d’être} for writing this article is that whereas the preliminary algebraicity result (25) is correct, the reciprocity law (26) is not correct as it stands by producing an example (see Section 6.2) when \(F\) is a totally imaginary field that is not of CM type that gives a counterexample to (27). The real aim of this article is to state the correct reciprocity law which is stated in Theorem 19. The missing signature, which is the term \(\varepsilon_{n,\iota}(\xi) \cdot \varepsilon_{\bar{n},\iota}(\xi)\) in Theorem 19, needed to rectify the reciprocity law is rather complicated, and turns out to be trivial when \(F\) is a CM field, and can be nontrivial for a general totally imaginary field. The presence of this subtle signature can be finessed using, interestingly enough, some other passages in [6]. Section 6 is an exposition of some results in my paper [12] for Rankin–Selberg \(L\)-functions for \(GL_n \times GL_{n'}\) over a totally imaginary base field in the simple situation of \(n = n' = 1\).

That there is a signature missing was suggested by calculations due to Deligne (see Proposition 20), in the context of motivic \(L\)-functions, carried out from the point of view afforded by his celebrated conjecture – which is recalled in Conjecture 6.4.1 in the context at hand. After relating the missing signature to the signature predicted by Deligne, the main theorem is restated as a much cleaner looking Theorem 25. As a final comment to end the Introduction, the results of this article show that whereas the analytic theory of \(L\)-functions of automorphic forms on \(GL(1)\) is not sensitive to the inner structure of the base field, the arithmetic theory of \(L\)-functions concerning the special values of \(L\)-functions crucially depends on the arithmetic structure of the base field.
2. Hecke characters

We set up some general notation for this article:

- \( \mathbb{Q} \) is the rational number field with closure \( \overline{\mathbb{Q}} \) inside complex numbers \( \mathbb{C} \);
- \( F \) is a number field, i.e., a finite extension of \( \mathbb{Q} \); we do not assume that \( F \) is contained in \( \overline{\mathbb{Q}} \subset \mathbb{C} \);
- \( d_F = d = [F : \mathbb{Q}] \);
- \( \mathcal{O}_F \) is the ring of integers of \( F \);
- \( U_F = \mathcal{O}_F^\times \) is the group of units of \( \mathcal{O}_F \);
- \( \Sigma_F \) = Hom(\( F, \mathbb{C} \)), all embeddings of \( F \) into \( \mathbb{C} \); note: Hom(\( F, \overline{\mathbb{Q}} \)) = Hom(\( F, \mathbb{C} \));
- \( S_\infty = \) set of all archimedean places of \( F \);
- \( S_\infty = S_r \cup S_c \) the real and complex places, respectively;
- \( r_1 = \# S_r, r_2 = \# S_c ; d_F = r_1 + 2r_2 \);
- \( p \) is a finite prime ideal of \( \mathcal{O}_F \) or an infinite place;
- \( F_p \) is the completion of \( F \) at \( p \);
- \( \mathcal{O}_p \) is the ring of integers of \( F_p \) at a finite prime \( p \);
- \( \mathcal{O}_p^\times \) is the group of units of \( \mathcal{O}_p \) for a finite prime \( p \).

2.1 Class groups

We start with some preliminaries on class groups, towards which we need some more notation:

- \( J_F \) is the group of all fractional ideals of \( F \);
- \( \mathbb{P}_F \) is the group of all principal fractional ideals \( (x) = x \mathcal{O}_F \) for \( x \in F^\times \);
- \( x \gg 0 \) means that \( x \in F \) is totally positive, i.e., \( \rho(x) > 0 \) for all \( \rho \in \text{Hom}(F, \mathbb{R}) \);
- \( \mathbb{P}_F^+ = \{ (x) \in \mathbb{P}_F : x \gg 0 \} \);
- \( \text{Cl}_F = J_F / \mathbb{P}_F \) is the class group of \( F \);
- \( \text{Cl}_F^+ = J_F / \mathbb{P}_F^+ \) is the narrow class group of \( F \).

The adjective ‘narrow’ is to suggest narrow conditions in \( \mathbb{P}_F^+ \) that we divide by to get \( \text{Cl}_F^+ \); the narrow class group surjects onto the class group; we have the exact sequence:

\[
0 \xrightarrow{} \mathbb{P}_F / \mathbb{P}_F^+ \xrightarrow{} \mathbb{J}_F / \mathbb{P}_F^+ \xrightarrow{} \mathbb{J}_F / \mathbb{P}_F \xrightarrow{} 0.
\] (1)

Observe that

\[
\frac{\mathbb{P}_F}{\mathbb{P}_F^+} \simeq \frac{F^\times / U_F}{F_+^\times / U_F} \simeq \frac{F_+^\times / U_F}{F_+^\times / U_F},
\]

where \( F_+^\times = \{ x \in F^\times : x \gg 0 \} \) and \( U_F^+ = U_F \cap F_+^\times \) is the group of totally positive units.

We have an exact sequence:

\[
0 \xrightarrow{} U_F / U_F^+ \xrightarrow{} F^\times / F_+^\times \xrightarrow{} F_+^\times / U_F \xrightarrow{} 0.
\] (2)

Splicing the two exact sequences, we get a four-term exact sequence relating the narrow class group to the class group:

\[
0 \xrightarrow{} U_F / U_F^+ \xrightarrow{} F^\times / F_+^\times \xrightarrow{} \text{Cl}_F^+ \xrightarrow{} \text{Cl}_F \xrightarrow{} 0.
\] (3)
The narrow class group surjects onto the class group and the ratio of their orders is a power of 2 since \( [F^\times : F_\infty^\times] = 2^{t_1}. \) (This suggests an interesting exercise in basic number theory: given \( r_1, t_1 \in \mathbb{Z} \) with \( r_1 \geq 1 \) and \( 0 \leq t_1 \leq r_1 \), construct a number field \( F \) with exactly \( r_1 \) real embeddings and such that the order of the group \( U_F / U_F^+ \) is \( 2^{t_1} \), equivalently, that the order of the kernel of \( \text{Cl}_F^+ \rightarrow \text{Cl}_F \) is \( 2^{t_1-t_1}. \))

To discuss the class groups in terms of idèles, here are some more notations:

- \( \mathbb{A}_F \) is the adèle ring of \( F; \)
- \( \mathbb{I}_F \) is the group of idèles of \( F; \)
- \( C_F = \mathbb{I}_F / F^\times \) is the group of idèle-classes of \( F; \)
- \( U_p := \mathbb{O}_F^\times \) for finite \( p \) as before, but moreover
- \( U_p := \mathbb{R}_+^\times \) for \( p \in S_r \), and \( U_p := \mathbb{C}^\times \) for \( p \in S_c; \)
- \( \mathbb{I}_F^S = \) group of \( S \)-idèles for any finite set \( S \) of places;
- \( F_\infty := F \otimes \mathbb{R} \cong \prod_{v \in S_r} F_v \times \prod_{w \in S_c} F_w \cong \prod_{v \in S_r} \mathbb{R} \times \prod_{w \in S_c} \mathbb{C}; \)
- \( F_{\infty +} = \{ x = (x_v) \in F_\infty: x_v > 0, \forall v \in S_r \}. \)

There is a canonical map \( \iota: \mathbb{I}_F \rightarrow \mathbb{I}_F \) which sends an idèle \( a = (a_p) \) to the ideal \( \iota(a) := \prod_{p \notin S_\infty} \mathfrak{p}^{\text{ord}_p(a_p)}. \) It is clear that \( \text{Kernel}(\iota) = F_\infty^\times \prod_{p \notin S_\infty} U_p. \) The map \( \iota \) induces an isomorphism which gives an idèle-theoretic description of the class group of \( F: \)

\[
\mathbb{I}_F / F^\times (\prod_{p \notin S_\infty} U_p) \cong \frac{\mathbb{I}_F}{\mathbb{P}_F} = \text{Cl}_F.
\]

Similarly, one has for the narrow class group:

\[
\mathbb{I}_F^+ / F_\infty^\times (\prod_{p \notin S_\infty} U_p) \cong \frac{\mathbb{I}_F^+}{\mathbb{P}_F^+} = \text{Cl}_F^+.
\]

The weak-approximation gives that \( F^\times \) is dense in \( F_\infty^\times; \) hence \( F_\infty^\times F_\infty^\times = F^\times F_\infty^\times; \) whence

\[
\frac{\mathbb{I}_F}{F^\times (\prod_{p \notin S_\infty} U_p)} \cong \frac{\mathbb{I}_F^+}{\mathbb{P}_F^+} = \text{Cl}_F^+.
\]

The four-term long exact sequence relating the class group and the narrow class group described using idèles takes the form

\[
0 \rightarrow U_F^+ \rightarrow F_\infty^\times \rightarrow F_\infty^\times (\prod_{p \notin S_\infty} U_p) \rightarrow \mathbb{I}_F \rightarrow \mathbb{I}_F^+ \rightarrow 0.
\]

To compare the second term from the left in (3) with that in (6), we note that weak-approximation says that the canonical map \( F^\times / F_+^\times \rightarrow F_\infty^\times / F_\infty^\times + \) induced by the diagonal inclusion \( F^\times \hookrightarrow F_\infty^\times \) is an isomorphism.

2.2 Basics of Hecke characters

A Hecke character is a continuous homomorphism

\[
\chi : \mathbb{I}_F / F^\times \rightarrow \mathbb{C}^\times.
\]

We do not ask \( \chi \) to be unitary. Some people prefer calling a unitary Hecke character as a Hecke character, and what we call a Hecke character, as a Hecke quasi-character. It is painful to keep writing ‘quasi-character’ all the time, and so, with a possible abuse of
terminology, we will work with the above definition. This is not so serious as we now explain: Consider the norm of an idèle \( \alpha \in \mathbb{I}_F \) defined as \( ||\alpha|| := \prod_p |\alpha|_p \), where \( p \) runs over all valuations (finite and archimedean), and each valuation is normalized. Then \( || : \mathbb{I}_F \rightarrow \mathbb{R}_+^\times \) is a surjective homomorphism and let \( 0_{\mathbb{I}_F} \) denote its kernel. The product formula says that \( ||a|| = 1 \) for all \( a \in F^\times \), i.e., \( F^\times \subset 0_{\mathbb{I}_F} \). We have the following exact sequence

\[
0 \rightarrow 0_{\mathbb{I}_F} \rightarrow \mathbb{I}_F \rightarrow \mathbb{R}_+^\times \rightarrow 0.
\]

This sequence splits. There are several splittings; for example, map \( t \in \mathbb{R}_+^\times \) into the idèle which is \( t \) at a particular real infinite place and \( 1 \) elsewhere, or map it to \( t^{1/2} \) at a particular complex infinite place and \( 1 \) elsewhere. But this depends on a choice of an infinite place. Instead, using \( d = d_F = [F : \mathbb{Q}] \), we will always consider the following splitting:

\[
t \mapsto (t^{1/d}, \ldots, t^{1/d}; 1, 1, \ldots),
\]

where the idèle has the positive \( d \)-th root of \( t \) at every infinite place and \( 1 \) at every finite place. (Note that it is indeed a splitting, because for any complex place, the normalized valuation satisfies \( |x|_C = |x|^2_{\mathbb{R}} \) for \( x \in \mathbb{R} \).) This splitting gives

\[
\mathbb{I}_F/F^\times \cong 0_{\mathbb{I}_F}/F^\times \times \mathbb{R}_+^\times.
\]

It is a fundamental fact that \( 0_{\mathbb{I}_F}/F^\times \) is compact [10, Theorem VI.1.6]. A continuous homomorphism of \( 0_{\mathbb{I}_F}/F^\times \) into \( \mathbb{C}^\times \) has a compact image and so lands in \( S^1 \). Further, any homomorphism \( \mathbb{R}_+^\times \rightarrow \mathbb{C}^\times \) is of the form \( x \mapsto |x|^w \) for a complex number \( w = \sigma + i\varphi \). (see Section 2.5.1 below). Putting these remarks together, any Hecke character \( \chi \) can be uniquely factored as

\[
\chi = 0_{\mathbb{I}_F} \otimes ||^\sigma \quad (7)
\]

for a unitary Hecke character \( 0_{\mathbb{I}_F} : \mathbb{I}_F/F^\times \rightarrow S^1 \) and \( \sigma \in \mathbb{R} \); sometimes \( 0_{\mathbb{I}_F} \) is denoted also as \( \chi^u \).

### 2.3 Dirichlet characters: Hecke characters of finite order

A continuous homomorphism \( \chi : \mathbb{I}_F/F^\times \rightarrow \mathbb{C}^\times \) with finite image and which is unramified everywhere (i.e., \( \chi_p := \chi|_{F_p^\times} \) is trivial on the units \( U_p \) for all finite \( p \)) gives a character of \( \mathbb{I}_F/F^\times \big/(F^\times(\mathbb{Z}_p, F_p^\times) \cap \mathbb{Q}_p) \), and by (5), is a character of the narrow class group \( \text{Cl}_F^+ \). This generalizes by introducing some level structure giving us a description of Hecke characters of finite order in terms of characters of narrow ray class groups with level structure. Here are some more notations:

- \( m \) will be an integral ideal with prime-factorization \( m = \prod_{p \notin S_\infty} p^{m_p} \);
- \( U_p(m_p) \) will be defined as

\[
U_p(m_p) = \begin{cases} 
1 + p^{m_p}, & \text{if } p \notin S_\infty, \ p \mid m, \\
U_p, & \text{if } p \notin S_\infty, \ p \nmid m, \\
\mathbb{R}_+^\times, & \text{if } p \in S_r, \\
\mathbb{C}^\times, & \text{if } p \in S_c.
\end{cases}
\]

- \( \mathbb{U}_F(m) := \prod_p U_p(m_p) \);
\[ U_{F, f}(m) := \prod_{p \in S_{\infty}} U_p(m_p); \]
\[ C_F(m) := U_F(m) F^\times / F^\times \] is called the congruence subgroup mod \( m \) in \( C_F; \)
\[ C_F / C_F(m) := I_F / U_F(m) F^\times \] is the idèle-theoretic narrow ray class group mod \( m; \)
\[ J_F(m) := I_F / F^\times \] is the group of all fractional ideals relatively prime to \( m; \)
\[ \mathbb{P}^+_F(m) \] is the group of all principal fractional ideals \( (x) \) with \( x \equiv 1 \pmod{m}; \)
\[ C_F(m) := J_F(m) / \mathbb{P}_F(m) \] is the ray class group mod \( m; \)
\[ C_F^+(m) := J_F(m) / \mathbb{P}_F^+(m) \] is the narrow ray class group mod \( m. \)

**Proposition 1**

The canonical homomorphism \( \iota : I_F \rightarrow J_F \) induces an isomorphism

\[ \frac{C_F}{C_F(m)} \cong \frac{J_F(m)}{\mathbb{P}_F^+(m)}. \]

**Proof.** Define

\[ I_F(m) := \{ \alpha \in I_F : \alpha_p \in U_p(m_p), \forall p | m \cdot \infty \}. \]

The weak approximation implies that \( I_F = F^\times \cdot I_F(m). \) Hence

\[ C_F = I_F / F^\times = I_F(m) / (F^\times \cap I_F(m)). \]

Note that \( \iota \) maps \( I_F(m) \) onto \( J_F(m) \) and similarly, maps \( F^\times \cap I_F(m) \) onto \( \mathbb{P}_F^+(m). \) Further, \( \text{Ker} (\iota : I_F(m) \rightarrow J_F(m)) = U_F(m) \). The rest is easy. \( \square \)

A fundamental property of these congruence subgroups \( C_F(m) \) is captured by the following proposition.

**Proposition 2**

For any integral ideal \( m, \) the congruence subgroup \( C_F(m) \) is a subgroup of the idèle class group \( C_F \) of finite index. Conversely, any subgroup of finite index of \( C_F \) contains \( C_F(m) \) for some \( m. \)

**Proof.** See [10, Proposition VI.1.8]. \( \square \)

The moral of the above discussion is that Hecke characters of finite order are exactly the characters of narrow ray class groups.

**Corollary 3** (Finite order Hecke characters)

Consider a continuous homomorphism

\[ \chi : I_F / F^\times \rightarrow \mathbb{C}^\times \]
whose image is of finite order. Then there is an integral ideal \( m \) such that \( \chi \) factors as

\[
\mathbb{I}_F/F^\times = C_F \xrightarrow{\chi} \mathbb{C}^\times \xrightarrow{\chi} C_F/C_F(m)
\]

The smallest such \( m \) is called the conductor of \( \chi \) and will be denoted as \( \mathfrak{f}_\chi \).

A Dirichlet character of \( F \) is a (necessarily unitary) character of a narrow ray class group of \( F \) with level structure, i.e., it is a homomorphism

\[
\chi : \mathbb{I}_F(m)/\mathbb{I}_F^+(m) \rightarrow S^1,
\]

for some integral ideal \( m \). By Proposition 1, Proposition 2 and Corollary 3, it follows that Dirichlet characters are exactly the Hecke characters of finite order. The algebraic Hecke characters that we want to understand are in general not of finite order.

An easy exercise is to take \( F = \mathbb{Q}, m = N \mathbb{Z} \) to see that a Dirichlet character for \( \mathbb{Q} \) modulo \( N \mathbb{Z} \) is exactly a homomorphism \((\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times\); such characters were considered by Dirichlet in his proof of infinitude of primes in arithmetic progressions; this exercise justifies the terminology.

### 2.4 Local components at finite places and conductor

Fix any place \( p \) of \( F \). Then \( F_p^\times \) embeds into \( \mathbb{I}_F \) as \( x_p \mapsto (1, \ldots, 1, x_p, 1, \ldots) \), the idele having \( x_p \) at the place corresponding to \( p \) and 1 elsewhere. The restriction \( \chi_p \) of \( \chi \) to this copy of \( F_p^\times \) is called the local character at \( p \). Sometimes we use \( v \) for a place instead of \( p \). For almost all finite \( p \), the character \( \chi_p \) is unramified, i.e., it is trivial on \( U_p \). This may be seen by considering the restriction of \( \chi \) to \( \prod_{p \notin S_\infty} U_p \), a compact and totally disconnected group the image of which under a continuous homomorphism into \( \mathbb{C}^\times \) must be finite by the so-called ‘no small-subgroups argument’; see, for example, [2, Exercise 3.1.1]. One can write \( \chi = \otimes_v \chi_v \) as a restricted tensor product. By the same token, \( \chi \) is trivial on \( \bigcup_{F, f}(m) \) for some integral ideal \( m \). The smallest ideal \( m \) such that \( \chi \) is trivial on \( \bigcup_{F, f}(m) \) is called conductor of \( \chi \) and will be denoted by \( \mathfrak{f}_\chi \).

### 2.5 The character at infinity of a Hecke character

#### 2.5.1 Characters of \( \mathbb{R}^\times \).

Any continuous homomorphism \( \chi : \mathbb{R}^\times \rightarrow \mathbb{C}^\times \) is of the form

\[
\chi(x) = \text{sgn}(x)^n |x|^w = \left(\frac{x}{|x|}\right)^n |x|^w,
\]

with \( n \in \{0, 1\} \) and \( w \in \mathbb{C} \). Such a character \( \chi \) is unitary if and only if \( w = i\varphi \in i\mathbb{R} \).
2.5.2 Characters of $\mathbb{C}^\times$.

For $z = x + iy \in \mathbb{C}$, define $|z| := |z|_\mathbb{R} := \sqrt{x^2 + y^2}$, and the normalised valuation on $\mathbb{C}$ as $|z|_\mathbb{C} := |z|_\mathbb{R}^2 = x^2 + y^2$. A continuous homomorphism $\chi : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ is of the form

$$
\chi(z) = \left(\frac{z}{|z|}\right)^n |z|_C^w,
$$

with $n \in \mathbb{Z}$ and $w \in \mathbb{C}$. If $z = re^{i\theta}$ with $r \in \mathbb{R}_+^\times$ and $\theta \in \mathbb{R}$, then $\chi(z) = r^{2w}e^{in\theta}$. Such a character $\chi$ is unitary if and only if $w = i\varphi \in i\mathbb{R}$.

2.5.3 Description of $\chi_\infty$.

For a Hecke character $\chi$, let $\chi_\infty = \chi|_{F_\infty^\times}$ where $F_\infty^\times \hookrightarrow \mathbb{I}_F$. To describe this character at infinity explicitly, we introduce some more notations:

- $\lambda$ is any infinite place; $\lambda \in S_\infty$;
- $v$ is any real place; $v \in S_\mathbb{R}$ and $F_v \simeq \mathbb{R}$ canonically;
- $w$ is any complex place; $w \in S_\mathbb{C}$ and $F_w \simeq \mathbb{C}$ non-canonically;
- $|x_\infty|_\mathbb{C} = \prod_{\lambda} |x_\lambda|_\mathbb{C}$ for $x_\infty \in F_\infty$ is the product of normalized valuations.

Keeping in mind that a Hecke character factorizes as $\chi = ^0\chi \otimes | |^\sigma$ (see (7)), one can write the character at infinity $\chi_\infty$ on $x_\infty \in F_\infty^\times$ as

$$
\chi_\infty(x_\infty) = \left(\prod_{\lambda \in S_\infty} \frac{x_\lambda}{|x_\lambda|} \right)^{n_\lambda} |x_\lambda|_\mathbb{C}^{i\varphi_\lambda} |x_\infty|_\mathbb{C}^\sigma,
$$

where $n_\lambda \in \{0, 1\}$, $n_w \in \mathbb{Z}$, $\varphi_\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{R}$.

3. Größencharaktere mod $m$

3.1 Definition

We follow the treatment in [10, Section VII.6], except that for us these characters can take values in $\mathbb{C}^\times$, i.e., they need not be unitary. Before getting started, the following notations will be useful:

$$
U_F^1(m) := \{u \in U_F \mid u \equiv 1 \pmod{m}\} \text{ are the units congruent to 1 mod } m;
$$

$$
U_F^1(m)^+ := \{u \in U_F^1(m) \mid u > 0\};
$$

$$
O_F(m) := \{a \in O_F \mid (a, m) = 1\} \text{; nonzero integers in } F \text{ relatively prime to } m;
$$

$$
F(m) := \{x \in F \mid (x, m) = 1\} \text{; nonzero elements of } F \text{ relatively prime to } m;
$$

$$
F^1(m) := \{x \in F \mid x \equiv 1 \pmod{m}\} \text{; elements of } F \text{ congruent to 1 mod } m.
$$

By a Größencharakter mod $m$, we mean a homomorphism $\psi : \mathbb{J}_F(m) \rightarrow \mathbb{C}^\times$ for which there exists a pair of characters $(\psi_f, \psi_\infty)$ with

$$
\psi_f : (O_F/m)^\times \rightarrow \mathbb{C}^\times \quad \text{and} \quad \psi_\infty : F_\infty^\times \rightarrow \mathbb{C}^\times
$$

such that for all $a \in O_F(m)$, we have

$$
\psi((a)) = \psi_f(a \mod m) \psi_\infty(a),
$$

(9)
where in the left-hand side \((a) := a\mathcal{O}_F\) is the principal ideal generated by \(a\). We call \(\psi_f\) the finite part and \(\psi_\infty\) the infinite part of \(\psi\); they satisfy the compatibility condition: if \(u \in U_F = \mathcal{O}_F^\times\) is a unit, then \(\psi((u)) = 1\) and so we require
\[
\psi_f(u \mod m) \psi_\infty(u) = 1.
\]
Further, if \(\epsilon \in U_F^1 := \{u \in U_F \mid u \equiv 1 \mod m\}\), then the above requirement implies that \(\psi_\infty(\epsilon) = 1\). The following proposition explains the relation between \(\psi\) and \((\psi_f, \psi_\infty)\).

**PROPOSITION 4**

A Größencharakter \(\psi\) \(\mod m\) uniquely determines \(\psi_f\) and \(\psi_\infty\); indeed, the restriction of \(\psi\) to the group \(\mathbb{P}_F(m)\) uniquely determines \(\psi_f\) and \(\psi_\infty\), and these satisfy the above compatibility conditions.

Conversely, given homomorphisms \(\psi_f' : (\mathcal{O}_F/\mathbb{m})^\times \to \mathbb{C}^\times\) and \(\psi_\infty' : F_\infty^\times \to \mathbb{C}^\times\) satisfying the compatibility condition \(\psi_f'(u \mod m) \psi_\infty'(u) = 1\) for all \(u \in U_F\) (and so necessarily \(\psi_\infty'(\epsilon) = 1\) for all \(\epsilon \in U_F^1(m)\)), there exists a Größencharakter \(\psi\) \(\mod m\) such that \(\psi_f = \psi_f'\) and \(\psi_\infty = \psi_\infty'\).

Finally, if \(\xi\) is another Größencharakter \(\mod m\) such that \(\xi_f = \psi_f\) and \(\xi_\infty = \psi_\infty\), then \(\psi\xi^{-1}\) is a character of \(\text{Cl}_F(m)\)-the ray class group \(\mod m\).

**Proof.** Using \(F^1(m) \rightarrow \mathbb{P}_F(m) \subset \mathbb{J}_F(m)\), we see that a Größencharakter \(\psi\) \(\mod m\) determines via restriction a character of \(F^1(m)\). By weak approximation, \(F^1(m)\) is dense in \(F_\infty^\times\); thus \(\psi\) restricted to \(\mathbb{P}_F(m)\) uniquely determines \(\psi_\infty\). Clearly, \(\psi\) and \(\psi_\infty\) uniquely determine \(\psi_f\) by (9). For the converse, given \(\psi_f'\) and \(\psi_\infty'\), define \(\bar{\psi}'\) on \(\mathcal{O}_F(m)\) by
\[
\bar{\psi}'((a)) := \psi_f'(a \mod m) \psi_\infty'(a).
\]

The compatibility conditions say that \(\bar{\psi}'(a)\) is well-defined. This \(\bar{\psi}'\) extends uniquely to \(F(m)\), and being trivial on \(U_F^1(m)\), we get a character \(\psi' : \mathbb{P}_F(m) \to \mathbb{C}^\times\) as \(\psi'((a)) = \bar{\psi}'(a)\). Now induce \(\psi'\) to \(\mathbb{J}_F(m)\), i.e., consider the induced representation
\[
\text{Ind}_{\mathbb{P}_F(m)}^{\mathbb{J}_F(m)}(\psi').
\]

An easy exercise with Mackey theory says that this representation is a multiplicity free direct sum of characters \(\psi\) of \(\mathbb{J}_F(m)\) such that \(\psi\mid_{\mathbb{P}_F(m)} = \psi'\). Moreover, fix such a \(\psi\), then
\[
\text{Ind}_{\mathbb{P}_F(m)}^{\mathbb{J}_F(m)}(\psi') = \psi \otimes \text{Ind}_{\mathbb{P}_F(m)}^{\mathbb{J}_F(m)}(\mathbb{1}) = \bigoplus \psi \otimes \chi
\]
with \(\chi\) running over the characters of \(\mathbb{J}_F(m)/\mathbb{P}_F(m) = \text{Cl}_F(m)\) the ray class group modulo \(m\). Hence, if \(\xi\) is a Größencharakter with same finite and infinite parts as \(\psi\), then \(\xi = \psi \otimes \chi\).

Following the German language, the plural of Größencharakter will be Größencharaktere.
3.2 The correspondence between Hecke characters and Größencharaktere

The domain of definition of a Größencharakter surjects onto the domain of definition of a Hecke character as in the following exact sequence.

**PROPOSITION 5**

We have an exact sequence

\[
1 \rightarrow \frac{F(m)}{U_F^1(m)} \xrightarrow{\times} \mathbb{J}_F(m) \times (\mathcal{O}_F/m)_{\times} \times \frac{F_{\infty}^\times}{U_F^1(m)} \rightarrow \mathbb{J}_F \times \mathbb{I}_F \rightarrow 1,
\]

where

1. \(\times(a) = ((a)^{-1} \mod m, a \mod U_F^1(m))\) for all \(a \in F(m)\);
2. \(\varrho = \alpha \otimes \beta \otimes \gamma^{-1}\),

(a) \(\alpha : \mathbb{J}_F(m) \rightarrow \mathbb{I}_F/(F^\times : \mathbb{I}_F,F_f(m))\) is induced by the map \(\mathbb{J}_F \rightarrow \mathbb{I}_F\) that sends a prime ideal \(q\) to the idèle that has \(\sigma_q\) at the place corresponding to \(q\) and \(1\)'s elsewhere;
(b) \(\beta : (\mathcal{O}_F/m)_{\times} \rightarrow \mathbb{I}_F/(F^\times : \mathbb{I}_F,F_f(m))\) is induced by the map that sends \(a \in \mathcal{O}_F(m)\) to the idèle that has \(a\) at all infinite places and also at all finite places \(p\) not dividing \(m\), and is \(1\) at all \(p|m\); and
(c) \(\gamma : F_{\infty}^\times/U_F^1(m) \rightarrow \mathbb{I}_F/(F^\times : \mathbb{I}_F,F_f(m))\) is induced by the inclusion \(F_{\infty}^\times \subset \mathbb{I}_F\); check that \(U_F^1(m)\) is mapped into \(F^\times : \mathbb{I}_F,F_f(m)\).

See [10, Proposition VII.6.13] for a proof but it is an instructive exercise to fix a proof for oneself. Let us denote an \(a \in \mathbb{I}_F\) as \((a_\infty; a_1, a_2, \ldots, a_q, \ldots)\) for \(p|m\) and \(q \not\mid m\), and \([a]\) the class of the idèle \(a\) in \(\mathbb{I}_F/(F^\times : \mathbb{I}_F,F_f(m))\). Note that

1. \(\alpha(q) = [(\ldots 1 \ldots; 1 \ldots 1, \ldots, \sigma_q, 1, \ldots)]\) for any prime ideal \(q \not\mid m\).
2. \(\beta(a) = [(\ldots 1 \ldots; a^{-1}, a^{-1}, \ldots, 1 \ldots)]\).
3. \(\gamma(x_\infty) = [(x_\infty; 1 \ldots 1 \ldots)]\).

The rest of the details are left to the reader.

A Hecke character \(\chi\) trivial on \(\mathbb{I}_F,F_f(m)\) will be called a Hecke character mod \(m\); necessarily the conductor of \(\chi\) divides \(m\). From the exact sequence, the dictionary between Größencharaktere mod \(m\) and Hecke characters mod \(m\) is clear: Given a Größencharakter \(\psi\) mod \(m\), with \(\psi_f\) and \(\psi_\infty\), its finite and infinite parts, we get a character \(\psi_f \otimes \psi_f \otimes \psi_\infty\) on \(\mathbb{J}_F(m) \times (\mathcal{O}_F/m)_{\times} \times (F_{\infty}^\times/U_F^1(m))\) and (9) says that this is trivial on the image of \(\chi\) and hence we get a character \(\chi = \chi_\psi\) on \(\mathbb{J}_F/(F^\times : \mathbb{I}_F,F_f(m))\), i.e., \(\chi\) is a Hecke character mod \(m\). Conversely, given such a \(\chi\), we inflate it via \(\varrho\) to get \(\psi_f \otimes \psi_f \otimes \psi_\infty = \chi_\varrho\).

3.3 The infinity type of a Größencharakter

Given a Größencharakter \(\psi = (\psi_f, \psi_\infty)\) mod \(m\), by using the above dictionary and (8), one sees that the character at infinity, which is a continuous character \(\psi_\infty : F_{\infty}^\times \rightarrow \mathbb{C}^\times\), is of the form

\[
\psi_\infty(x_\infty) = \left( \prod_{\lambda \in \mathcal{S}_\infty} \left( \frac{x_\lambda}{|x_\lambda|} \right)^n \right) |x_\lambda|^{i\psi_\lambda} |x_\infty|_\infty^\rho,
\]
where $n_v \in \{0, 1\}$, $n_w \in \mathbb{Z}$, $\varphi_\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{R}$. Furthermore, $\psi_\infty$ being trivial on $U^1_F(m)$ imposes certain restrictions on the infinity type, i.e., the collection $((n_\lambda, \varphi_\lambda)_{\lambda}, \sigma)$ of exponents, that can occur.

**Lemma 6.** Let $\psi$ be a Größencharakter mod $m$ with the above notations for $\psi_\infty$. Then

(i) there are no restrictions on $n_\lambda$ for any $\lambda \in S_\infty$ and on $\sigma$; however,
(ii) $2\varphi_v = \varphi_w = \varphi$ (say), for all $v \in S_r$ and all $w \in S_c$.

Hence, the character at infinity takes the shape

$$\psi_\infty(x_\infty) = \left( \prod_{\lambda \in S_\infty} \left( \frac{x_\lambda}{|x_\lambda|} \right)^{n_\lambda} \right) |x_\infty|^{\sigma+i\varphi}.$$

We adumbrate how one proves this lemma. Write the character at infinity as

$$\psi_\infty(x_\infty) = \prod_{v \in S_r} \text{sgn}(x_v)^{n_v} |x_v|^\sigma |\varphi_v| \cdot \prod_{w \in S_c} x_w^{\sigma+n_w/2+i\varphi_w/2} \bar{x}_w^{-\sigma-n_w/2+i\varphi_w/2}.$$

For brevity, denote $f_v := \sigma + i\varphi_v$, $f_w := \sigma + n_w/2 + i\varphi_w/2$, $f_w^\sim := \sigma - n_w/2 + i\varphi_w/2$. As $\psi_\infty$ is trivial on any $u \in U^1_F(m)$ (totally positive global units that are $1 \mod m$), we get

$$\prod_{v \in S_r} u_f v^f \prod_{w \in S_c} w(u)^{f_w} \bar{w}(u)^{f_w^\sim} = 1.$$

We will go through the rest of the proof, which uses the ingredients of the proof in Dirichlet’s unit theorem in detail, in the proof of Lemma 7 below (the main focus of this article is the notion of an algebraic Hecke character to which that lemma particularly applies) giving us the constraint $2f_v = f_w$, from which the above lemma follows.

### 4. Algebraic Hecke characters

#### 4.1 Definition and purity

There is a canonical map $\Sigma_F \to S_\infty$. Let $\lambda \in S_\infty$. If $\lambda = v \in S_r$, i.e., $\lambda$ is a real place $v$, then it corresponds to a unique real embedding $\tau_v : F \to \mathbb{R}$, and if $\lambda = w \in S_c$, i.e., $\lambda$ is a complex place $w$, then it corresponds to a conjugate pair of complex embeddings $\{\tau_w, \bar{\tau}_w\}$ with the understanding that the choice of $\tau_w : F \to \mathbb{C}$ is not canonical; we use $\tau_w$ to identify $F_w \simeq \mathbb{C}$. Recall, as a reminder of our notational artifice, that $F_\infty = F \otimes \mathbb{R} \simeq \prod_{\lambda \in S_\infty} F_\lambda \simeq \prod_{v \in S_r} \mathbb{R} \times \prod_{w \in S_c} \mathbb{C}$. One may write $x_\infty \in F_\infty$ simply as $x_\infty = (x_\lambda)_{\lambda \in S_\infty}$, or more elaborately as $x_\infty = (x_v)_{v \in S_r}, (z_w)_{w \in S_c}$; both notations have their intrinsic virtue. Let $\chi$ be a Hecke character of $F$, and $\chi_\infty$ its character at infinity; see (8). We say that $\chi$ is an *algebraic Hecke character* if for every $\tau \in \Sigma_F$, there exists an integer $n_\tau$ such that for $x_\infty \in F_\infty^\times$, we have

$$\chi_\infty(x_\infty) = \prod_{v \in S_r} x_v^{n_{\tau_v}} \prod_{w \in S_c} z_w^{n_{\tau_w}} \bar{z}_w^{-n_{\tau_w}}.$$  

(10)
By separating the right-hand side into unitary and non-unitary parts, while noting that 
\( z^{p \bar{z}^q} = \left( \frac{z}{|z|} \right)^{p-q} |z|^{(p+q)/2} \), we have

\[
\chi_\infty(x_\infty) = \left( \prod_{v \in S_r} \left( \frac{x_v}{|x_v|} \right)^{n_{\tau_v}} \prod_{w \in S_c} \left( \frac{z_w}{|z_w|} \right)^{n_{\tau_w} - n_{\bar{\tau_w}}} \right) \left( \prod_{v \in S_r} |x_v|^{n_{\tau_v}} \prod_{w \in S_c} |z_w|^{(n_{\tau_w} + n_{\bar{\tau_w}})/2} \right).
\]

If we compare this with (8), we see \( v \in S_r, w \in S_c \) and \( \lambda \in S_\infty \) such that 
\[
n_v \equiv n_{\tau_v} \mod 2, \quad n_w = n_{\tau_w} - n_{\bar{\tau}_w}, \quad \varphi_\lambda = 0, \quad 2\sigma = 2n_{\tau_v} = n_{\tau_w} + n_{\bar{\tau}_w}.
\]

Especially note the constraint \( 2n_{\tau_v} = n_{\tau_w} + n_{\bar{\tau}_w} \) for all \( v \in S_r, w \in S_c \). Now, let us suppose the algebraic Hecke character has modulus \( m \). Then this constraint gets significantly tighter by the following.

**Lemma 7 (Purity).** For each \( \tau \in \Sigma_F \), suppose we are given \( n_\tau \in \mathbb{Z} \). Suppose for some integral ideal \( m \) of \( F \) we have

\[
\prod_{\tau \in \Sigma_F} \tau(u)^{n_\tau} = 1, \quad \forall u \in U^1_F(m).
\]

Then, there exists \( w \in \mathbb{Z} \) such that

(i) if \( S_r \neq \emptyset \), then \( n_\tau = w \) for all \( \tau \in \tau_F \), and

(ii) if \( S_r = \emptyset \), then \( n_{\gamma \tau} + n_{\gamma \bar{\tau}} = w \) for all \( \tau \in \Sigma_F \) and \( \gamma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \).

**Proof.** The proof uses the ingredients of the proof in Dirichlet’s unit theorem. Note that \( U^1_F(m) \) has finite index in \( U_F \). Enumerate the real embeddings of \( F \) as \( \{v_1, \ldots, v_{r_1}\} \) and the complex embeddings as \( \{w_1, \bar{w}_1, \ldots, w_{r_2}, \bar{w}_{r_2}\} \). With a minor abuse of notation, for brevity, write \( \tau_{v_i} = v_i \) and \( \tau_{w_j} = w_j \). Let \( H \subset \mathbb{R}^{r_1+r_2} \) be the hyperplane given by the sum of all coordinates being 0. The Minkowski map \( l : U_F \rightarrow H \) is given by

\[
l(u) = (\log|v_1(u)|_\mathbb{R}, \ldots, \log|v_{r_1}(u)|_\mathbb{R}, \log|w_1(u)|_\mathbb{C}, \ldots, \log|w_{r_2}(u)|_\mathbb{C}).
\]

In (11), apply \( \log|\cdot|_\mathbb{C} \) to get

\[
2n_{v_1}\log|v_1(u)|_\mathbb{R} + \cdots + 2n_{v_{r_1}}\log|v_{r_1}(u)|_\mathbb{R} + \\
(n_{w_1} + n_{\bar{w}_1})\log|w_1(u)|_\mathbb{C} + \cdots + (n_{w_{r_2}} + n_{\bar{w}_{r_2}})\log|w_{r_2}(u)|_\mathbb{C} = 0,
\]

for all \( u \in U^1_F(m)^+ \). The proof of Dirichlet’s unit theorem gives that \( \Gamma = \text{im}(l(U_F)) \) is a lattice in \( H \); hence \( \Gamma^1_F(m) : = l(U^1_F(m)) \) is also a lattice in \( H \); whence, there exists \( u_1, \ldots, u_{t-1} \in U^1_F(m) \) such that \( \{v_1 := l(u_1), \ldots, v_{t-1} := l(u_{t-1})\} \) is an \( \mathbb{R} \)-basis for \( H \), where, for brevity, we denote \( t = r_1 + r_2 \). Let us write \( v_i = (a_{i1}, \ldots, a_{it}) \) as a vector in \( \mathbb{R}^t \), and consider the \((t-1) \times t\)-sized matrix \( A = [a_{ij}] \). The rank of \( A \) is \( t-1 \) and hence its nullity is 1. Since each \( v_i \in H \), we know \( \sum_{j} a_{ij} = 0 \). We deduce that for any \( X \in \mathbb{R}^{t \times 1} \), if \( AX = 0 \), then all the coordinates of \( X \) are equal. Now (12) applied to \( u_1, \ldots, u_{t-1} \) gives a solution to the system of equations \( AX = 0 \) from which we get \( 2n_{v_1} = \cdots = 2n_{v_{r_1}} = n_{w_1} + n_{\bar{w}_1} = \cdots = n_{w_{r_2}} + n_{\bar{w}_{r_2}} \). (The reader may pause here to
note that the above argument proves Lemma 6.) Let us denote the common value by $w'$. Now, take any $\gamma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, apply $\gamma^{-1}$ to (11) to get
\[
1 = \prod_{\tau \in \tau_F} \gamma^{-1}(\tau(u))^{n_\tau} = \prod_{\tau \in \tau_F} \tau(u)^{n_{\gamma \tau}},
\]
and we go through the same argument to get $n_{\gamma \tau} + n_{\gamma \tau} = w'$ for all $\tau \in \tau_F$ and $\gamma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. If $S_r \neq \emptyset$, then since $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts transitively on $\Sigma_F$, we see that $n_\tau = n_{v_1}$ for all $\tau$; then take $w = n_{v_1} = w'/2$. If $S_r = \emptyset$, then take $w = w'$. □

Let $\chi$ be an algebraic Hecke character of $F$ mod $m$ and with infinity type $(n_\tau)_{\tau \in \Sigma_F}$ and purity weight $w$. Then the exponents $n_\tau$ satisfy the conditions of Lemma 7. The integer $w$ is called the purity weight of $\chi$. It is worth noting that when $F$ is totally real or totally imaginary then there is no constraint on the parity of the purity weight $w$, but when $F$ has both real and complex embeddings, then the purity weight $w$ is necessarily an even integer. Note the following easy consequence of purity.

COROLLARY 8

Let $\chi$ be an algebraic Hecke character of $F$ mod $m$ and with infinity type $(n_\tau)_{\tau \in \Sigma_F}$ and purity weight $w$.

(i) If $S_r \neq \emptyset$, then $\chi = \chi \circ \|w\|$ for a Dirichlet character $\chi$ of $F$ mod $m$.

(ii) If $S_r = \emptyset$, then $\chi = \chi \circ \|w/2\|$ for a unitary Hecke character $\chi$ of $F$ mod $m$.

In case (ii), let us observe that the character at infinity of $\chi$ has the following shape:
\[
\chi^u_\infty(x_\infty) = \prod_{w \in S_c = S_\infty} (x_w/\bar{x}_w)^{n_w-w/2}.
\]

4.2 The $L$-function of an algebraic Hecke character

4.2.1 Definition(s) of Hecke $L$-function.

Hecke considered the class of $\text{Größencharaktere}$ mod integral ideals as the optimal class of characters for which one may study the analytic properties of $L$-functions, called Hecke $L$-functions, that we now define. Let $\chi$ be an algebraic Hecke character, and $\psi = (\psi_f, \psi_\infty)$ the associated $\text{Großencharakter}$. Suppose that $m$ is the conductor of $\chi$, and so also $m$ is the conductor of $\psi$. The (finite part of the) Hecke $L$-function of $\chi$ may be defined in two possible ways:

(1) As defined by Hecke in the form of a Dirichlet series:
\[
L_f(s, \chi) = \sum_a \frac{\psi(a)}{N_{F/Q}(a)^s},
\]
the summation running over integral ideals $a$ of $\mathcal{O}_F$ that are relatively prime to $m$.

(2) As defined by Tate in the form of an Euler product
\[
L_f(s, \chi) = \prod_p \left(1 - \chi_p(\omega_p)N_{F/Q}(p)^{-s}\right)^{-1},
\]
the product is over prime ideals $p$ of $\mathcal{O}_F$ that are relatively prime to $m$. 
Here, \( N_{F/Q}(p) \) is defined as \(|O_F/p|\), and is multiplicatively extended to define \( N_{F/Q}(a) \). One has the usual analytic properties: absolute convergence in a half-plane, analytic continuation, and functional equation. In the form of Hecke’s definition, we refer the reader to [10]; in the form of Tate’s definition, we refer the reader to Tate’s thesis [15] and Weil’s book [18].

4.2.2 The coefficients of the \( L \)-function of an algebraic Hecke character.

The main reason to consider algebraic Hecke characters (called characters of type \( A_0 \) by Weil [17]) is the following proposition which puts us in the right context to study the arithmetic properties of Hecke \( L \)-functions.

**PROPOSITION 9**

Let \( \chi \) be an algebraic Hecke character, and \( \psi = (\psi_f, \psi_\infty) \) the associated Großencharakter of modulus, say, \( m \). Then the coefficients of the Dirichlet series giving the finite part of the Hecke \( L \)-function \( L_f(s, \chi) \) are contained in a number field, i.e., there is a finite extension \( E \) of \( \mathbb{Q} \) such that \( \psi(a) \in E \) (resp., \( \chi_p(\sigma_p) \in E \)) for all integral ideals \( a \) (resp., prime ideals \( p \)) relatively prime to \( m \).

**Proof.** By (9), it follows that for \( a \in O_F(m) \), and hence also for \( a \in F(m) \), \( \psi((a)) \) takes values in a finite extension \( E_1 \) of \( \mathbb{Q} \) inside \( \mathbb{Q} \), since \( (O_F/m)^{\times} \) is a finite group, and since \( \psi_\infty \) is algebraic (10), we see that \( \psi_\infty(a) \) takes values in the compositum of all the conjugates of \( F \) inside \( \mathbb{Q} \). Hence the values of \( \psi \) on \( \mathbb{P}_F(m) \) lie in \( E_1 \). Since \( \mathbb{P}_F(m) \) has finite index in \( J_F(m) \), we see that the values of \( \psi \) are in a finite extension \( E \) of \( E_1 \) contained inside \( \mathbb{Q} \). By Proposition 5, \( \psi(p) = \chi_p(\sigma_p) \) for all prime ideals \( p \) not dividing \( m \). \( \square \)

4.2.3 The rationality field of an algebraic Hecke character

The smallest subfield of \( \mathbb{Q} \) that contains all the ‘values’ of an algebraic Hecke character \( \chi \), denoted as \( \mathbb{Q}(\chi) \), is called its rationality field. More precisely, with notations as in the above proposition, suppose \( \chi \) has conductor \( m \), we may define

\[
\mathbb{Q}(\chi) = \mathbb{Q}([\chi(p) : \text{for all prime ideals } p \text{ relatively prime to } m])
\]

\[
= \mathbb{Q}([\psi(a) : \text{for all integral ideals } a \text{ relatively prime to } m]).
\] (13)

As Weil comments in [17], \( \mathbb{Q}(\chi) \) need not contain \( F \); consider, for example, a quadratic character of any number field \( F \).

4.2.4 The critical values of the \( L \)-function of an algebraic Hecke character

Let us recall the shape of the functional equation satisfied by \( L_f(s, \chi) \), in as much as we recall the precise recipe for the \( \Gamma \)-factors \( L_\infty(s, \chi) \) necessary to define the completed \( L \)-function

\[
L(s, \chi) := L_\infty(s, \chi)L_f(s, \chi),
\]

where, if \( \chi \) has infinity type \( (n_\tau)_{\tau \in \Sigma_F}, n_\tau \in \mathbb{Z}, \) satisfying the purity condition of Lemma 7, then we define

\[
L_\infty(s, \chi) = \prod_{v \in S_F} \Gamma_{\mathbb{R}} \left( s + n_v + \epsilon_v \right) \cdot \prod_{w \in S_c} \Gamma_{\mathbb{C}} \left( s + \frac{n_w + n_{\bar{w}}}{2} + \frac{|n_w - n_{\bar{w}}|}{2} \right),
\]
where $\Gamma_R(s) = \pi^{-s/2} \Gamma(s/2)$, $\Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s)$ and $\epsilon_v \in \{0, 1\}$ with $\epsilon_v \equiv n_v \mod 2$. When $\chi$ is nontrivial, the completed $L$-function admits an analytic continuation to an entire function of $s \in \mathbb{C}$, and a beautiful aspect of $L$-functions is the functional equation

$$L(s, \chi) = \epsilon(s, \chi) L(1 - s, \chi^{-1}),$$

where $\epsilon(s, \chi)$ is an exponential function (see [10] or [18]). An integer $m$ is said to be critical for the Hecke $L$-function $L(s, \chi)$ if both $L_{\infty}(s, \chi)$ and $L_{\infty}(1 - s, \chi^{-1})$ are regular (i.e., finite) at $s = m$. We may say that the $\Gamma$-factors on either side of the functional equation do not have poles at $s = m$. The set of all critical integers for $L(s, \chi)$, denoted as $\text{Crit}(L(s, \chi))$, is given by the following.

**PROPOSITION 10**

Let $\chi$ be an algebraic Hecke character of a number field $F$ with infinity type $(n_{\tau})_{\tau \in \Sigma_F}$, $n_{\tau} \in \mathbb{Z}$, satisfying the purity condition of Lemma 7 with purity weight $w$. Then the critical set of integers for $L(s, \chi)$ is given by

(i) $F$ is totally real ($S_c = \emptyset$): Recall that $n_{\tau} = w$ for all $\tau$.

(a) If there exists $v_1, v_2 \in S_r$ such that $\epsilon_{v_1} \neq \epsilon_{v_2}$, then $\text{Crit}(L(s, \chi)) = \emptyset$.

(b) If $\epsilon_v = 0$ for all $v \in S_r$, then $\text{Crit}(L(s, \chi))$ is given by

$$\{\ldots, -w + 1 - 2k, \ldots, -w - 3, -w - 1; -w + 2, -w + 4, \ldots, -w + 2k, \ldots\}_{k \in \mathbb{Z}_{\geq 0}}.$$

The critical set is centered at $\frac{1}{2} - w$.

(c) If $\epsilon_v = 1$ for all $v \in S_r$, then $\text{Crit}(L(s, \chi))$ is given by

$$\{\ldots, -w - 2k, \ldots, -w - 2, -w, -w + 1, -w + 3, \ldots, -w + 1 + 2k, \ldots\}_{k \in \mathbb{Z}_{\geq 1}}.$$

The critical set is centered at $\frac{1}{2} - w$.

(ii) $F$ is totally imaginary ($S_r = \emptyset$): Define the width of $\chi$ as the non-negative integer:

$$\ell = \ell(\chi) = \min\{|n_w - n_{\bar{w}}| : w \in S_c\}.$$ Then $w \equiv \ell \mod 2$. We have

$$\text{Crit}(L(s, \chi)) = \left\{ m \in \mathbb{Z} : 1 - \frac{w}{2} - \frac{\ell}{2} \leq m \leq -\frac{w}{2} + \frac{\ell}{2} \right\}.$$

The critical set, centered at $\frac{1}{2} - w$, is a finite set of cardinality $\ell$.

(iii) If $F$ has real and complex places ($S_r \neq \emptyset \neq S_c$), then $\text{Crit}(L(s, \chi)) = \emptyset$.

We omit the tedious proof of the above proposition and just add a few comments after which it is an extended exercise that is left to the reader. Recall that the poles of $\Gamma(s)$ are simple and located at non-positive integers, and furthermore that $\Gamma(s) \neq 0$ for all $s$. In case (i)(a), one sees the presence of a product $\Gamma(s/2)\Gamma((1 + s)/2)$ because of which no integer can be critical. In case (i)(b), up to shifting by $-w$, it is the same as $\text{Crit}(\xi(s))$, the critical set of the Riemann zeta function. In case (i)(c), up to shifting by $-w$, it is the same as the critical set of an odd classical Dirichlet character. For case (ii), the reader is referred to [12, Section 3.1] where the details are explicated. Case (iii) is like the intersection of cases (i) and (ii); by Lemma 7, $n_{\tau} = w$ for all $\tau; \ell = 0$, where $\ell$ is defined using only $S_c$, forcing no critical points if there are real and complex places.
There has been a huge amount of work on the nature of the critical values of $L(m, \chi)$. Deligne’s conjecture [4] on the special values of motivic $L$-functions sets the stage for the sort of theorem one seeks. The interested reader is referred to [1,6] and [13], and all the references therein. My reason for writing this article is that while I was working on the critical values of automorphic $L$-functions ([11] and [12]), I had to work through the results of this article frequently; and these results, although well-known to experts, did not seem to be readily available in the literature in the way I needed them. There are some very deep sign problems, apparently hitherto not well-appreciated, that appear in these works which have their roots – to paraphrase Weil [17] – in the remarkable properties of such algebraic Hecke characters.

The above proposition begets the terminology of a critical algebraic Hecke character by which we mean an algebraic Hecke character $\chi$ such that $\text{Crit}(L(s, \chi)) \neq \emptyset$. Existence of a critical algebraic Hecke character $\chi$ of a number field $F$ implies first of all that $F$ is totally real or totally imaginary, and furthermore, when $F$ is totally real then the parities $\epsilon_v$ of all the local archimedean characters are equal. We will now discuss the existence of critical algebraic Hecke characters with prescribed infinity type subject to the purity condition.

4.3 Existence of (critical) algebraic Hecke characters

For a given number field $F$, we discuss the problem of existence of algebraic Hecke characters that may or may not be critical. The purity lemma naturally leads us to consider two cases: (i) $S_r \neq \emptyset$ and (ii) $S_r = \emptyset$; we will see that case (ii) when $F$ is totally imaginary is in some sense far more interesting.

4.3.1. When $F$ has a real place $(S_r \neq \emptyset)$. From Corollary 8(i), we know that an algebraic Hecke character $\chi$ is necessarily of the form $\chi = \chi^0 \| w$, for a character $\chi^0$ of finite order and for an integer $w$. We may fine-tune our question a little further to note that we can arrange for $\chi^0$ to have any prescribed signature at infinite places: for any (possibly empty) $S \subseteq S_r$, by weak approximation, we can find $x \in F^\times$ such that $v(x) > 0$ for $v \in S$, and $v(x) < 0$ for $v \in S_r \setminus S$. Take $E = F(\sqrt{x})$, and $\chi = \omega_{E/F}$ the quadratic character of $F$ corresponding to $E$ by class field theory. Then $\chi_v(1) = 1$ for $v \in S$ and $\chi_v(-1) = -1$ for $v \in S_r \setminus S$. In particular, when $F$ is a totally real field, by taking $S = \emptyset$ or $S = S_r = S_\infty$, we see that there exists critical algebraic Hecke characters of any prescribed parity (totally even or totally odd) and any prescribed purity weight $w$.

4.3.2. When $F$ is totally imaginary $(S_r = \emptyset)$. Consider an infinity type $n := (n_\tau)_{\tau \in \Sigma_F}$, $n_\tau \in \mathbb{Z}$, satisfying the purity condition $n_{\gamma \sigma} + n_{\gamma \sigma \bar{\tau}} = w$ for all $\tau \in \Sigma_F$ and all $\gamma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. The question we wish to address is whether there exists an algebraic Hecke character $\chi$ of $F$ with infinity type $n$. As Weil says in [17], Artin pointed out to him that this is essentially an exercise in Galois theory.

First, consider the case when $F$ is a CM field, by which one means that $F$ is a totally imaginary quadratic extension of a totally real field $F^+$. Let us note an easy lemma.
Lemma 11. Let $F$ be a totally imaginary quadratic extension of a totally real field $F^+$. Let $\tau : F \to \bar{Q}$ and $\gamma \in \text{Gal}(\bar{Q}/Q)$ and let $\epsilon$ stand for complex conjugation as an element of $\text{Gal}(\bar{Q}/Q)$. Then $\gamma \circ \epsilon \circ \tau = \epsilon \circ \gamma \circ \tau$, i.e., complex conjugation and any automorphism of $\bar{Q}$ commute on the image of a CM field. This also holds for a totally real field.

**Proof.** Let $\tau_1 = \gamma \circ \epsilon \circ \tau$ and $\tau_2 = \epsilon \circ \gamma \circ \tau$. Then it is easy to see that $\tau_1|_{F^+} = \tau_2|_{F^+}$ (since $F^+$ is totally real). This means that $\tau_1 = \tau_2$ or $\tau_1 = \epsilon \circ \tau_2$; if it is the latter, then $\gamma \circ \epsilon \circ \tau = \epsilon \circ \tau$, whence $\epsilon \circ \tau = \tau$ or that $\tau$ lands inside $\mathbb{R}$ which is not possible. Hence $\tau_1 = \tau_2$. The last assertion for a totally real field is obvious. □

For a CM field $F$, consider an infinity type $n := (n_{\tau})_{\tau \in \Sigma_F}$, $n_{\tau} \in \mathbb{Z}$, satisfying the purity condition $n_{\gamma \circ \epsilon \circ \tau} + n_{\gamma \circ \tau} = w$. The above lemma gives that $n_{\gamma \circ \tau} = n_{\gamma \circ \epsilon \circ \tau}$. Hence, purity for a CM field is the same as asking $n_{\tau} + n_{\bar{\tau}} = w$ for all $\tau \in \Sigma_F$.

**PROPOSITION 12**

Let $F$ be a totally imaginary quadratic extension of a totally real field $F^+$. Let $n := (n_{\tau})_{\tau \in \Sigma_F}$, $n_{\tau} \in \mathbb{Z}$ satisfying the purity condition $n_{\tau} + n_{\bar{\tau}} = w$ for all $\tau \in \Sigma_F$. Then there exists an algebraic Hecke character (of some modulus) with that infinity type.

**Proof.** If such a character exists, then the character at infinity would have the shape

$$
\chi_{\infty}(x_{\infty}) = \left( \prod_{w \in \Sigma_c} \frac{x_w^{n_{\tau_w} - n_{\bar{\tau}_w}}}{x_w^{n_{\tau_w}}} \right)^{w/2},
$$

and also for some integral ideal $m$, $\chi_{\infty}$ needs to be trivial on $U_F^1(m)$. Put $p_w = n_{\tau_w} - n_{\bar{\tau}_w}$, and write $x_w$ in polar form as $r_w e^{i\theta_w}$. Then we need to construct a unitary Hecke character $\chi^u : \mathbb{F}/\mathbb{F}^\times \to S^1$ such that $\chi^u(x_{\infty}) = \prod_w r_w e^{ip_w \theta_w}$, and $\chi^u_{\infty}$ needs to be trivial on $U_F^1(m)$. We can then take $\chi = x_u \|^{w/2}$. Note that $F_F^\infty = (S^1 \times \cdots \times S^1) \times (\mathbb{R}^\times_+ \times \cdots \times \mathbb{R}^\times_+)$. The units $U_F$ (via the proof of Dirichlet’s unit theorem) intersects with the compact part in roots of unity $\mu_F$ in $F^\times$. Take an integral ideal $m$ such that $U_F^1(m) \cap \mu_F = \{1\}$; hence, $U_F^1(m) \cap (S^1 \times \cdots \times S^1) = \{1\}$. Define a unitary character $\psi^u$ that maps $x_{\infty}$ to $\prod_w e^{ipa_w \theta_w}$. Then $\psi^u_{\infty}$ defines a character of $F_F^\infty/U_F^1(m)$. Take any character $\psi^u_F$ of $(O_F/m)^\times$, and by Proposition 4, we get a Größencharakter $\psi^u$ mod $m$ with the necessary infinity type. Take $\chi^u$ to be the corresponding Hecke character as in the paragraph after Proposition 5. □

Next, consider the case of a general totally imaginary field $F$. There is a maximal CM or totally real subfield $F_1$ of $F$ that plays an important role. Following Weil [17], let $F_0$ be the largest totally real subfield of $F$. Then $F_0$ admits at most one totally imaginary quadratic extension contained inside $F_0$. If $F$ does admit such a CM subfield, denote it as $F_1$; and if there is no such extension inside $F$, then put $F_1 = F_0$. A basic property of a pure infinity type of a totally imaginary field is that it is the base-change of a pure infinity type of $F_1$ as explained by the following.
PROPOSITION 13

Let $F$ be a totally imaginary field. Consider an infinity type $\mathbf{n} := (n_\tau)_{\tau \in \Sigma_F}$, $n_\tau \in \mathbb{Z}$, satisfying the purity condition $n_{\gamma \tau} + n_{\gamma \bar{\tau}} = \mathbf{w}$ for all $\tau \in \Sigma_F$ and all $\gamma \in \text{Gal}(\bar{Q}/Q)$. Then $n_\tau$ as a function of $\tau$ depends only on the restriction of $\tau$ to $F_1$, i.e., there exists a necessarily pure infinity type $\mathbf{m} := (m_{\tau_1})_{\tau_1 \in \Sigma_{F_1}}$ for the field $F_1$, such that if $\tau \in \Sigma_F$ with $\tau_1 = \tau|_{F_1}$, then $n_\tau = m_{\tau_1}$. (In this situation, we may say that $\mathbf{n}$ is the base-change of $\mathbf{m}$.)

Proof. (I learnt of the following proof from Deligne; it appears in a different context in [5].) The purity condition $n_{\gamma \tau} + n_{\gamma \bar{\tau}} = \mathbf{w}$, for all $\tau \in \Sigma_F$ and all $\gamma \in \text{Gal}(\bar{Q}/Q)$, also implies that $n_{\gamma \tau} + n_{\bar{\gamma} \bar{\tau}} = \mathbf{w}$, from which we deduce

$$n_{\gamma \tau} = n_{\bar{\gamma} \bar{\tau}}, \quad \forall \tau, \forall \gamma.'$$

Consider the function $n : \text{Gal}(\bar{Q}/Q) \times \Sigma_F \to \mathbb{Z}$, defined as $n(\gamma, \tau) = n_{\gamma \tau}$. For any $x \in \text{Gal}(\bar{Q}/Q)$, we have $n(\gamma x, \tau) = n(\gamma, x\tau)$. The above displayed equation reads

$$n(\gamma, c\tau) = n(c, \gamma \tau).$$

Replacing $\tau$ by $\gamma^{-1}\tau$, we also have

$$n(\gamma, \tau) = n(\gamma, c\gamma^{-1}\tau) = n(\gamma c\gamma^{-1}, \tau).$$

Replacing $\gamma$ by $c\tau$, we get

$$n(1, \tau) = n(\gamma c\gamma^{-1}c, \tau).$$

Now, take any $x \in \text{Gal}(\bar{Q}/Q)$, and we see that by using the above relations (for all $\gamma$ and $\tau$),

$$n(x\gamma c\gamma^{-1}cx^{-1}, \tau) = n(x\gamma c\gamma^{-1}x^{-1}xcx^{-1}, \tau) = n((x\gamma c)(x\gamma)^{-1}, xcx^{-1}\tau)$$

$$n(x, xcx^{-1}\tau) = n(xcx^{-1}, c\tau) = n(xcx^{-1}c, \tau) = n(1, \tau).$$

In other words, $n_{h\tau} = n_\tau$ for all $h$ in the normal subgroup $\mathcal{N}$ of $\mathcal{G} := \text{Gal}(\bar{Q}/Q)$ generated by $\{\gamma c\gamma^{-1}c : \gamma \in \text{Gal}(\bar{Q}/Q)\}$. Note that $\mathcal{G}/\mathcal{N}$ is the largest quotient in which $c$ is central. We see that $\mathcal{G}$ acts transitively on $\Sigma_F$; the transitive action of $\mathcal{G}$ on $\Sigma_{F_1}$ is via $\mathcal{G}/\mathcal{N}$ (Lemma 11); and $\mathcal{N}$ acts transitively on the fiber over $\tau_1 \in \Sigma_{F_1}$ in the restriction map $\Sigma_F \to \Sigma_{F_1}$. In particular, if $\tau, \tau' \in \Sigma_F$ with $\tau_1 = \tau|_{F_1} = \tau'|_{F_1}$, and suppose $h \in \mathcal{N}$ is such that $h\tau = \tau'$, then $n_{\tau'} = n_{h\tau} = n_\tau$ and we call this common value as $m_{\tau_1}$. \qed

We are now ready to state the basic fact about the existence of algebraic Hecke characters for a totally imaginary field in the following.

PROPOSITION 14

Let $F$ be a totally imaginary field. Consider an infinity type $\mathbf{n} := (n_\tau)_{\tau \in \Sigma_F}$, $n_\tau \in \mathbb{Z}$, satisfying the purity condition $n_{\gamma \tau} + n_{\gamma \bar{\tau}} = \mathbf{w}$ for all $\tau \in \Sigma_F$ and all $\gamma \in \text{Gal}(\bar{Q}/Q)$. Then there exists an algebraic Hecke character of infinity type $\mathbf{n}$.
(i) If $F_0 = F_1$ (this is the case when $F$ has no CM subfield), then there exists $n \in \mathbb{Z}$ such that $n_{\tau} = n$ for all $\tau \in \Sigma_F$. Any algebraic Hecke character $\chi$ of infinity type $\mathbf{n}$ is of the form $\chi = \chi^0 \parallel n$ for a Dirichlet character $\chi^0$ of $F$.

(ii) If $F_1$ is a CM field, then $\mathbf{n}$ is the base-change of an infinity type $\mathbf{m}$ for $F_1$ (Proposition 13); any algebraic Hecke character $\chi$ of $F$ of infinity type $\mathbf{n}$ is of the form

$$\chi = \chi_1 \circ N_{F/F_1} \otimes \chi^0,$$

for some algebraic Hecke character $\chi_1$ of $F_1$ with infinity type $\mathbf{m}$ and some Dirichlet character $\chi^0$ of $F$.

Proof. In both cases, from Proposition 13, there exists an infinity type $\mathbf{m}$ over $F_1$ whose base-change to $F$ is $\mathbf{n}$. In Case (i), since $F_1 = F_0$ is totally real, all the $m_{\tau}$ are equal, and hence $n_{\tau}$ are all equal, to say, $n$. If there is an algebraic Hecke character $\chi$ of this infinity type, then necessarily we have $\chi \parallel n$ is a finite-order character, say, $\chi^0$. To construct $\chi$, take any finite-order Hecke character $\chi^0$ of $F$, and put $\chi = \chi^0 \parallel n$. In Case (ii), from Proposition 12, there exists an algebraic Hecke character $\chi_1$ of $F_1$ with infinity type $\mathbf{m}$. Consider its base-change $\chi_2 := \chi_1 \circ N_{F/F_1}$ to $F$; it is clear that this base-change has the same infinity type as $\chi$, if one knew the existence of $\chi$, and in which case $\chi \chi_2^{-1}$ would be a character of finite-order, say, $\chi^0$. Going backwards, construct the required $\chi$ via $\chi_1 \circ N_{F/F_1} \otimes \chi^0$. \hfill \Box

I have found it helpful to keep some examples in mind while thinking of a totally imaginary field $F$, with various possible scenarios for $F_0$ and $F_1$. Some of these examples were suggested to me by Haruzo Hida.

Example 14. When $F$ is totally imaginary but not CM, then one has to be careful about what constitutes a pure infinity type. The following example is instructive: take $F = \mathbb{Q}(2^{1/3}, \omega)$, where $2^{1/3}$ is the real cube root of 2 and $\omega = e^{2\pi i/3}$. Then $\Sigma_F = \text{Gal}(F/\mathbb{Q}) \simeq S_3$ the permutation group in 3 letters taken to be $\{2^{1/3}, 2^{1/3}\omega, 2^{1/3}\omega^2\}$. Let $s \in S_3$ correspond to $\tau_s : F \to \mathbb{C}$. Consider the infinity types $\mathbf{n} = (n_{\tau_s})_{s \in S_3}$ and $\mathbf{n}' = (n'_{\tau_s})_{s \in S_3}$:

| $s$ | $e$ | (12) | (23) | (13) | (123) | (132) |
|-----|-----|------|------|------|-------|-------|
| $n_{\tau_s}$ | $a$ | $b$ | $w - a$ | $c$ | $w - c$ | $w - b$ |
| $n'_{\tau_s}$ | $a$ | $w - a$ | $w - a$ | $w - a$ | $a$ | $a$ |

where $a, b, c, w \in \mathbb{Z}$. For the tautological embedding $F \subset \hat{\mathbb{Q}}$, the set $\Sigma_F$ is paired into complex conjugates as $\{(\tau_e, \tau_{(23)}), (\tau_{(12)}, \tau_{(132)}), (\tau_{(13)}, \tau_{(123)})\}$. Then, $n_{\tau_e} + n'_{\tau_e} = w$. However, $\mathbf{n}'$ is not a pure infinity type in general. All other possible pairings of $\Sigma_F$ into conjugates via automorphisms of $\hat{\mathbb{Q}}$ ($F$ being Galois this simply boils down to composing these embeddings $\tau_s$ by a fixed one $\tau_{\sigma} = \tau_{s_0}$, and using $\tau_{s_0} \circ \tau_s = \tau_{s_0 s}$) are given by $\{(\tau_e, \tau_{(12)}), (\tau_{(23)}, \tau_{(132)}), (\tau_{(13)}, \tau_{(123)})\}$ and $\{(\tau_e, \tau_{(13)}), (\tau_{(23)}, \tau_{(132)}), (\tau_{(12)}, \tau_{(123)})\}$, from which we see that $\mathbf{n}'$ is not a pure infinity type unless $w - a, b$ and $c$ are all equal; $\mathbf{n}$ is pure and has purity weight $w$. Also, $\mathbf{n}$ is the base-change of the infinity type $\mathbf{m}$ of $F_1 = \mathbb{Q}(\omega)$, where $m_{\tau_e} = a$ and $m_{\tau_{(12)}} = w - a$.

Example 15. Take a CM field $F_{\text{cm}}$ and a number field $L$ that is linearly disjoint from $F_{\text{cm}}$, and put $F = L \cdot F_{\text{cm}}$. This is a particularly pleasant example of a totally imaginary
field. If $L$ is totally real, then $F$ is also of CM type. For concreteness, take $n \geq 3$, then $F = \mathbb{Q}(2^{1/n}, \zeta_n)$, where $\zeta_n = e^{2\pi i/n}$ is a totally imaginary field of this form; $F^{\text{cm}} = \mathbb{Q}(\zeta_n)$ and $L = \mathbb{Q}(2^{1/n})$. This example is slightly more general than the field considered in the previous example.

**Example 16.** For a totally imaginary field not of the above form, take $F_0 = \mathbb{Q}$, and put $F_1 = \mathbb{Q}(i)$ and $F = \mathbb{Q}(\sqrt{4 + i}, i)$. (This example is used in Section 6.2.)

**Example 17.** For a totally imaginary field with no CM subfield, take $F_0$ to be a cubic totally real field; for example, $F_0 = \mathbb{Q}(\sqrt[3]{3} + \sqrt[3]{3}^{-1})$. Choose non-squares $a$ and $b$ in $F_0$ with conjugates $a, a', a''$ and $b, b', b''$, respectively, and chosen so that $a > 0, a' < 0, a'' < 0$ and $b < 0, b' < 0, b'' > 0$. Such a choice is possible by weak approximation. Then the field $F = F_0[\sqrt{a}, \sqrt{b}]$ is totally imaginary with no CM subfield. Such fields do not support critical algebraic Hecke characters since the width of a pure infinity type is 0 (see Proposition 10).

**Remark 18.** There is a slightly different way to organise one’s thoughts about the subfields $F_0$ and $F_1$ of a totally imaginary field $F$, and which at times is useful to keep in mind. Recall that $c$ denotes the complex conjugation on $\mathbb{C}$. Let $F_1$ be maximal among all subfields $K$ of $F$ that have an involution $\iota_K : K \to K$ such that $c \circ \sigma = \sigma \circ \iota_K$ for all embeddings $\sigma : K \to \mathbb{C}$. In particular, $F_1$ has its involution $\iota_{F_1}$, and let $F_0$ denote the subfield of $F_1$ that is fixed by $\iota_{F_1}$. There are two cases: (i) $\iota_{F_1}$ is non-trivial; then $F_1$ is a totally imaginary quadratic extension of the totally real $F_0$; (ii) $\iota_{F_1}$ is trivial; then $F_1 = F_0$ is a totally real field.

5. Cohomological automorphic representations of $GL_1$

So far, we have treated Hecke characters as complex-valued functions on adèlic spaces, and one might say that this is from the point of view of automorphic forms on $GL(1)$. We may also treat them as algebro-geometric objects; see, for example, [3] or [14]. Of course, these points of view are inter-related for which the reader is recommended to [13, Chapter 0] for a lucid summary. In this section, we consider Hecke characters from the point of view often taken in the cohomology of arithmetic groups. We begin by setting up the context, adopting the treatment in [9, Chapter 2].

5.1 Locally symmetric spaces for $GL_1/F$

Let $G = \text{Res}_{F/\mathbb{Q}}(GL_1/F)$. Then $G(\mathbb{Q}) = GL_1(F) = \mathbb{F}^\times$;

$G(\hat{\mathbb{A}}) = GL_1(\hat{\mathbb{A}}_F) = \mathbb{I}_F$;

$G(\mathbb{A}_f) = \hat{\mathbb{A}}_f^\times$ the group of finite idèles;

$G(\mathbb{R}) = GL_1(F \otimes \mathbb{R}) = \mathbb{F}_\infty^\times = \prod_{v \in S_F} \mathbb{R}_v^\times \times \prod_{w \in S_c} \mathbb{C}_w^\times$;

$G(\mathbb{R})^0 = \mathbb{F}_\infty^+$, the connected component of the identity in $G(\mathbb{R})$;

$K_\infty = \prod_{v \in S_F} \{\pm 1\} \times \prod_{w \in S_c} S_1^1$, the maximal compact subgroup of $G(\mathbb{R})$;

$K_\infty^0 \simeq \prod_{w \in S_c} S_1^1$, the connected component of the identity in $K_\infty$;

$A = GL_1/\mathbb{Q} = G/\mathbb{Q}$ is the canonical inclusion in $\text{Hom}(GL_1/\mathbb{Q}, \text{Res}_{F/\mathbb{Q}}(GL_1/F))$;

$A(\mathbb{R})$ is the diagonal copy of $\mathbb{R}^\times$ inside $G(\mathbb{R}) = \prod_{v \in S_F} \mathbb{R}_v^\times \times \prod_{w \in S_c} \mathbb{C}_w^\times$;
There are various quotients of $I_F$ that one may consider depending on how much to divide by inside $G(\mathbb{R})$. Here are five possible situations:

\begin{align}
S^G_m &:= G(\mathbb{Q})\backslash G(\mathbb{A})/K_m. \\
X^G_m &:= G(\mathbb{Q})\backslash G(\mathbb{A})/A(\mathbb{R})^o K_m. \\
Y^G_m &:= G(\mathbb{Q})\backslash G(\mathbb{A})/K^\infty_m. \\
T^G_m &:= G(\mathbb{Q})\backslash G(\mathbb{A})/A(\mathbb{R})^o K^\infty_m K_m. \\
C^G_m &:= G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})^o K_m.
\end{align}

Of course, $G$ being abelian, the double-coset-spaces are all quotients of $I_F$; for example,

$$C^G_m = G(\mathbb{A})/(G(\mathbb{Q})G(\mathbb{R})^o K_m) = I_F/(F^\times \cup F_F(m)) \approx \text{Cl}_F^+(m)$$

the narrow ray class group modulo $m$. Writing as a double-coset space is simply to suggest how things would generalize if $G$ is taken to be a general reductive group over $\mathbb{Q}$. There are the canonical projection maps:

\begin{equation}
\begin{array}{c}
S^G_m \\
X^G_m \\
\downarrow \\
T^G_m \\
\downarrow \\
C^G_m \approx \text{Cl}_F^+(m)
\end{array}
\end{equation}

The bottom-most is a finite-abelian group being a narrow ray class group. To get a feel for the space $S^G_m$ at the top, think of

$$G(\mathbb{A})/K_m = G(\mathbb{R}) \times (G(\mathbb{A}_f)/K_m),$$

as the product of $G(\mathbb{R})$ with the totally disconnected space $G(\mathbb{A}_f)/K_m$; on which the discrete subgroup $G(\mathbb{Q})$ – it is discrete in $G(\mathbb{A})$ – acts diagonally, and taking the quotient by that action, we have $G(\mathbb{Q})\backslash (G(\mathbb{R}) \times (G(\mathbb{A}_f)/K_m))$. Note that $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_m$ is a finite set

$$G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_m = \mathbb{A}^\times_{F,f}/(F^\times \cup F_F,m) \hookrightarrow I_F/(F^\times (F^\times_{\infty} \cup F_F,m)) = \text{Cl}_F^+(m).$$

There exists a finite set $\{x_{1,f}, \ldots, x_{k,f}\}$ of finite idèles such that $\mathbb{A}^\times_{F,f} = \bigcup_{i=1}^k x_{i,f} F^\times \cup F_F,m$. Using these representatives, a bijection

$$S^G_m \simeq \prod_{i=1}^k (F^\times/ U^1_F(m)) x_{i,f}$$
can be described as follows: for \( x = (x_\infty, x_f) \in \mathbb{I}_F = F^\infty_\infty \times A^\infty_{F,f} \), there is a unique \( i \) such that \( x_f = x_i, f \gamma u_f \) for some \( \gamma \in F^\infty \) and \( u_f \in \mathbb{U}_{F,f}(m) \); the bijection is given by
\[
S^G_m \ni x(F^\infty \cup F_{f,f}(m)) \mapsto (\gamma^{-1} x_\infty U^1_{f}(m), x_i, f) \in (F^\infty / U^1_{f}(m)) x_i, f.
\]

Let us describe the connected components of \( S^G_m \), which are all copies of
\[
\frac{F^\infty_{\infty+}}{U^1_{f}(m)} \simeq \frac{(\mathbb{R}^\infty_+)^1 \times (\mathbb{C}^\infty)^2}{U^1_{f}(m)}.
\]
Let \( 0 F^\infty_{\infty+} \) denote the kernel of norm at infinity \( \| \infty : F^\infty_{\infty+} \rightarrow \mathbb{R}^\infty_+ \). Then
\[
F^\infty_{\infty+} = 0 F^\infty_{\infty+} \times A(\mathbb{R})^0.
\]

By the product formula, \( F^\infty_{\infty+} \) contains \( U^1_{f}(m) \). Note that if \( u \in U^1_{f}(m) \cap A(\mathbb{R})^0 \), then \( \sigma(u) = \sigma'(u) \) for all \( \sigma, \sigma' \in \Sigma_F \), hence \( u \in \mathbb{Q}^\infty \); whence \( u \in \{ \pm 1 \} \). Assume the mild condition that \( U^1_{f}(m) \cap \{ \pm 1 \} = \{ 1 \} \). Hence we get
\[
\frac{F^\infty_{\infty+}}{U^1_{f}(m)} = A(\mathbb{R})^0 \times \frac{0 F^\infty_{\infty+}}{U^1_{f}(m)}.
\]

Next, under the identification \( F^\infty_{\infty+} \simeq (\mathbb{R}^\infty_+)^1 \times (\mathbb{C}^\infty)^2 \), after writing \( \mathbb{C}^\infty = \mathbb{R}^\infty_+ \times S^1 \), and noting that \( K(\mathbb{R})^0 = (S^1)^2 \) which is the product of the \( S^1 \)'s coming from the complex places, we have
\[
0 F^\infty_{\infty+} \simeq K(\mathbb{R})^0 \times \frac{0 (\mathbb{R}^\infty_+)^1 + r_2}{U^1_{f}(m)},
\]
where \( 0 (\mathbb{R}^\infty_+)^1 + r_2 \) consists of tuples \((r_v)_{v \in S_\infty}\) such that \( \sum_v |r_v|_v = 0 \). Assume furthermore that \( m \) is deep enough so that \( U^1_{f}(m) \cap K(\mathbb{R})^0 \), which \textit{a priori} is finite being discrete and compact in \( F^\infty_{\infty+} \), is in fact trivial, i.e., it amounts to assuming that \( U^1_{f}(m) \) has no roots of unity. Then, by the proof of Dirichlet’s unit theorem using the Minkowski map (see the notations in the proof of Lemma 7), one has
\[
\frac{0 (\mathbb{R}^\infty_+)^1 + r_2}{U^1_{f}(m)} \approx \mathcal{H} \approx \mathbb{R}^{r_1 + r_2 - 1} \Gamma^1_{F}(m) \approx (S^1)^{r_1 + r_2 - 1}.
\]

To summarise, any connected component of \( S^G_m \), for \( m \) deep enough, is homeomorphic to
\[
A(\mathbb{R})^0 \times K(\mathbb{R})^0 \times (S^1)^{r_1 + r_2 - 1},
\]
which in turn is isomorphic to \( \mathbb{R}^\infty_+ \times (S^1)^{r_1 + 2r_2 - 1} \). Note that
\[
\dim(S^G_m) = r_1 + 2r_2 = d_F = [F : \mathbb{Q}].
\]
Similarly, the other spaces in (19) can be described. We can grant ourselves the license to call any of these as a ‘locally symmetric space’ for \( \text{GL}_1 \) over \( F \) with level \( m \).

5.2 Sheaves on locally symmetric spaces

Take a field \( E \), which is assumed to be a finite Galois extension of \( \mathbb{Q} \) that takes a copy of \( F \); hence \( \text{Hom}(F, E) = \text{Hom}(F, \bar{E}) \). This field \( E \) may be called the field of coefficients. Consider an infinity type \( n = (n_\infty)_\times : F \rightarrow E \) with \( n_\infty \in \mathbb{Z} \); one may also denote this as \( n \in \mathbb{Z}[\text{Hom}(F, E)] \). This is like an infinity type from before, except that it is now parametrised over \( \text{Hom}(F, E) \) instead of \( \text{Hom}(F, \bar{Q}) \) as before. An embedding \( \iota : E \rightarrow \bar{Q} \) gives an
identification \( \tau_n : \text{Hom}(F, E) \to \text{Hom}(\widehat{F}, \widehat{Q}) \) as \( \tau \mapsto \tau_n \tau = \iota \circ \tau \). Such an \( n \) gives us an algebraic representation \( \vartheta_n \) on a one-dimensional \( E \)-vector space \( \mathcal{M}_{n,E} \) denoted by

\[
\vartheta_n : G \times E \longrightarrow \text{GL}_1(\mathcal{M}_{n,E}).
\]

Since \( G \times E = \text{Res}_{F/\mathbb{Q}}(\text{GL}_1/F) \times E = \prod_{\tau: F \to E} \text{GL}_1 \times_{F, \tau} E \), and on the component indexed by \( \tau \), the representation is \( x \mapsto \vartheta_n(\tau) = \tau^n \tau \) for \( x \in \text{GL}_1(E) \). For \( m \in G(\mathbb{Q}) = \text{F} \), we have

\[
\vartheta_n(a) = \prod_{\tau: F \to E} \vartheta_n(\tau(a)) = \prod_{\tau: F \to E} \tau(\tau(a)^n).
\]

Given an infinity type \( n \in \mathbb{Z}[\text{Hom}(F, E)] \), and the associated representation \( (\vartheta_n, \mathcal{M}_{n,E}) \), there is a sheaf \( \widetilde{M}_{n,E} \) of \( E \)-vector spaces on \( S_m^G \) constructed as follows. Consider the canonical projection map of dividing by \( F^\times \):

\[
\pi : \mathbb{I}_F / \mathbb{I}_F(f)(m) \longrightarrow F^\times / \mathbb{I}_F(f)(m) = S_m^G.
\]

For any open subset \( V \subset S_m^G \), define the sections of \( \widetilde{M}_{n,E} \) over \( V \) as the \( E \)-vector space:

\[
\widetilde{M}_{n,E}(V) = \{ s : \pi^{-1}(V) \to \mathcal{M}_{n,E} : (ax) = \vartheta_n(a)s(x), \forall a \in F^\times, \forall x \in \pi^{-1}(V) \}.
\]

Checking the sheaf axioms is a routine exercise. The purity lemma (Lemma 7) can be recast as the following lemma.

**Lemma 15 (Purity).** Given \( n \in \mathbb{Z}[\text{Hom}(F, E)] \), the sheaf \( \widetilde{M}_{n,E} \) of \( E \)-vector spaces on \( S_m^G \) is non-zero if and only if there exists \( w \in \mathbb{Z} \) such that

(i) if \( S_r \neq \emptyset \), then \( n_r = w \) for all \( \tau \in \text{Hom}(F, E) \), and

(ii) if \( S_r = \emptyset \), then \( n_{\iota \tau} = w \) for all \( \iota \in \text{Hom}(F, E) \) and for all \( \iota \in \text{Hom}(E, \widehat{Q}) \).

**Proof.** A sheaf is nonzero if and only if it admits a nonzero stalk. A stalk over a point \( y \in V \) consists of all \( s : \pi^{-1}(y) \to \mathcal{M}_{n,E} \), such that \( s(ax) = \vartheta_n(a)s(x) \), for all \( a \in F^\times \) and all \( x \in \pi^{-1}(y) \). This is possible if and only if \( \vartheta_n(a) = 1 \) for all \( a \in F^\times / \mathbb{I}_F(f)(m) = U^1_F(m) \).

After passing to \( \widehat{Q} \) via \( \iota \), the rest is basically the same as the proof of Lemma 7. \( \square \)

Analogously, one may construct sheaves on the other spaces in (19); one may pull-back or push-forward the so-constructed sheaves, and explicate the mutual relations between them.

### 5.3 Cohomology of \( \text{GL}_1 \) and algebraic Hecke characters of \( F \) with coefficients in \( E \)

Given an integral ideal \( m \), and an infinity type \( n \in \mathbb{Z}[\text{Hom}(F, E)] \) satisfying the purity conditions of Lemma 15, a fundamental object of interest is

\[
H^\bullet(S_m^G, \widetilde{M}_{n,E}),
\]

the sheaf-cohomology of a locally symmetric space for \( \text{GL}_1/F \) with coefficients in the sheaf \( \widetilde{M}_{n,E} \). One may also consider \( H^\bullet_\iota(S_m^G, \widetilde{M}_{n,E}) \), which is the cohomology with compact supports. It will take us way too much of an effort without substantial dividends for the purposes of this article to develop this systematically. Any reader who might feel this is brewing heavy weather is recommended to look into Harder [6, §2.5] on the
cohomology of tori, where the torus at hand is the diagonal torus in GL$_2$, which, up to the easily controlled centre, is basically the above kind of cohomology group. This context naturally arises in the theory of Eisenstein cohomology where the cohomology of the Levi subgroups (in particular, tori) play a crucial role. The reader is referred to Harder’s forthcoming book [7] where this point of view is treated in great detail.

Let us suffice here to note that an algebraic Hecke character $\chi$ of $F$ modulo $m$, taking values in $\mathbb{Q}$ inside $\mathbb{C}$, and with infinite type $n$, contributes to $H^0(S^G_m, \mathcal{M}_n, \mathbb{Q})$, and by duality also to the top-degree cohomology with compact supports $H^d_{cFG}(S^G_m, \mathcal{M}_n, \mathbb{Q})$. For arithmetic applications, it is better to work with an arbitrary coefficient field $E$, from which one can then map into $\mathbb{Q}$ or $\mathbb{C}$; this necessitates clarifying the meaning of an algebraic Hecke character of $F$ with coefficients in $E$.

The definition below follows [3, §5]; see also [13, Chapter 0]. Whereas in [3] and [13], $E$ is any number field with no relation to $F$. We will continue with our restriction that $E$ is a finite Galois extension of $\mathbb{Q}$ that contains a copy of $F$. Given an integral ideal $m$ of $F$, and an infinity type $n \in \mathbb{Z}[\text{Hom}(F, E)]$, define an algebraic Hecke character of $F$ with values in $E$ of modulus $m$ and infinity type $n$ as a homomorphism

$$\chi : \mathbb{I}_F(m) \to E^\times$$

such that for any principal ideal $(a) \in \mathbb{P}_F^+(m)$ (recall $a \gg 0$ and $a \equiv 1 \pmod{m}$), one has

$$\chi((a)) = \prod_{\tau \in \text{Hom}(F, E)} \tau(a)^{n_\tau}.$$

Note the similarity to Hecke’s definition of a Größencharakter mod $m$, but with values in $E^\times$ instead of $\mathbb{C}^\times$, and while asking for the character at infinity to be of algebraic type. To see the relation with our previous treatment of algebraic Hecke characters, following [13, §0.5], ‘localise’ over any place of $E$. We will work with archimedean places, but note that as in loc. cit., we may also go through the discussion below for any finite place which would give us $\ell$-adic versions of algebraic Hecke characters.

Let $\chi$ be an algebraic Hecke character of $F$ with values in $E$ of modulus $m = \prod_{p|m} p^{n_p}$ and infinity type $n$. Then there is unique continuous homomorphism

$$\chi_{\mathbb{A}} : \mathbb{I}_F \to E^\times,$$

with the discrete topology on $E^\times$, such that

1. $\chi_{\mathbb{A}}^{-1}(1) = \prod_{v \in \mathcal{S}_\infty} F_v^\times \prod_{p|m} (1 + p^{n_p} \mathcal{O}_p) \prod_{p | m} \mathcal{O}_p^\times$, which is open in $\mathbb{I}_F$.
2. For $p \nmid m$, the local component $\chi_p$ of $\chi_{\mathbb{A}}$ satisfies $\chi_p(\mathcal{O}_p) = \chi(p)$.
3. $\chi_{\mathbb{A}}|_{F^\times} = \vartheta_n$; (recall: $\vartheta_n(a) = \prod_{\tau : F \to E} \tau(a)^{n_\tau}$).

By weak approximation, $\mathbb{I}_F = F^\times \cdot (\prod_{v \in \mathcal{S}_\infty} F_v^\times \prod_{p|m} (1 + p^{n_p} \mathcal{O}_p) \prod_{p | m} \mathcal{O}_p^\times)$, giving us the uniqueness of $\chi_{\mathbb{A}}$. Let us note en passant that such a $\chi_{\mathbb{A}}$ is an element of $H^0(S^G_m, \mathcal{M}_n, \mathbb{E})$. Fix an embedding $i : E \hookrightarrow \mathbb{C}$, and suppose it corresponds to the archimedean place $v_i$ of $E$. The character $\vartheta_n$ is an element of $\text{Hom}_{\text{alg}}(\text{Res}_{F/Q}(\text{GL}_1/F), \text{Res}_{E/Q}(\text{GL}_1/E))$, and taking the $\mathbb{A}$-points, one gets a continuous homomorphism $\vartheta_n, \mathbb{A} : \mathbb{I}_F \to \mathbb{I}_E$. Composing with the projection $\mathbb{I}_E^\times \to E_{v_i}^\times \simeq \mathbb{C}^\times$ gives

$$\vartheta_{n,i} : \mathbb{I}_F \to \mathbb{C}^\times.$$

Now, define a homomorphism $\mathbb{I}_F \to \mathbb{C}^\times$ as

$$i \chi : = (i \circ \chi_{\mathbb{A}}) \cdot \vartheta_{n,i}^{-1},$$

(24)
We are interested in the special values of the Hecke \( L \)-function; in particular, there is a mapping of infinity types \( \mathbb{Z}[\text{Hom}(E,F)] \rightarrow \mathbb{Z}[\text{Hom}(F,\mathbb{C})] \) as \( n \mapsto 'n \); for \( \eta \in \text{Hom}(E,\mathbb{C}) \), the \( \eta \)-component of \( 'n \) is given by \( 'n_\eta = n_{1-\eta} \). Hence, \( \iota \circ \vartheta_n = \vartheta_{'n} \), as homomorphisms from \( F^\times \) to \( \mathbb{C}^\times \) since

\[
(t \circ \vartheta_n)(a) = t \left( \prod_{\tau \in \text{Hom}(F,E)} \tau(a)^{\eta(\tau)} \right) = \prod_{\tau \in \text{Hom}(F,E)} (t_\tau \tau(a))^{\eta(\tau)} = \prod_{\eta \in \text{Hom}(F,\mathbb{C})} (\eta(a))^{n_{1-\eta}}.
\]

The properties of \( '\chi \) are summarised in the following proposition, the proof of which is now a routine check.

**Proposition 16**

Let \( \chi \) be an algebraic Hecke character of \( F \) with values in \( E \) of modulus \( m \) and infinity type \( n \). For any embedding \( \iota : E \to \mathbb{C} \), the homomorphism \( '\chi \) satisfies the following:

1. \( '\chi : \mathbb{I}_F/F^\times \to \mathbb{C}^\times \) is a continuous homomorphism;
2. For all \( p \not\in S_\infty \), \( '\chi_p = \iota \circ \chi_p \); recall that \( \chi_p \) is the local character of \( \chi_{\mathbb{A}} \);
3. \( '\chi_{\infty} = \vartheta_{n,1}^{-1}|_{F^\times_{\infty}} \) is completely determined by its values on \( F^\times_{\infty} \) embedded diagonally in \( F^\times_{\infty} \), on which we have \( \vartheta_{n,1}^{-1}|_{F^\times} = \iota \circ \vartheta_{n,1}^{-1} = \vartheta_{-n} \); i.e., \( '\chi \) is a Hecke character of \( F \) of modulus \( m \) and infinity type \( -'n \).

Furthermore, for \( k \in \mathbb{Z} \), the Tate-twist \( '\chi(k) := '\chi \otimes \mathbb{I}^k \) has infinity type \( -'n + k \).

We may say that \( '\chi \) is an automorphic representation of \( \text{GL}_1/F \) of cohomological type. We are interested in the special values of the Hecke \( L \)-function attached to \( '\chi \).

### 6. Ratios of critical values of Hecke \( L \)-functions

Let \( \chi \) be an algebraic Hecke character of \( F \) with values in \( E \) of modulus \( m \) and infinity type \( n = \sum_{\tau:F \to E} n_\tau \tau \). For any embedding \( \iota : E \to \mathbb{C} \), consider the algebraic Hecke character \( '\chi \) as in Proposition 16. The rest of this article concerns the special values of the \( \mathbb{C} \)-valued \( L \)-function

\[
L(s, \iota, \chi) := L(s, '\chi),
\]

the right-hand side is the usual Hecke \( L \)-function; see Subsection 4.2.1. It is sometimes convenient to consider the \( E \otimes \mathbb{C} \)-valued \( L \)-function

\[
\mathbb{L}(s, \chi) := \{ L(s, \iota, \chi) \}_{\iota : E \to \mathbb{C}},
\]

where \( E \otimes \mathbb{C} \) is identified with \( \prod_{l : E \to \mathbb{C}} \mathbb{C} \). From Proposition 10 and Lemma 17 below, it follows that the critical set of \( L(s, '\chi) \) is independent of \( \iota \); one may therefore allude to the critical set for \( \mathbb{L}(s, \chi) \). Furthermore, we are interested in the rationality results for the ratios of successive critical \( L \)-values that arise naturally in the theory of Eisenstein cohomology as in Harder [6]. Henceforth, we assume that \( F \) is a totally imaginary number field, in which case it follows from Proposition 10 that if the width \( \ell \) of \( '\chi \) satisfies \( \ell \geq 2 \), then we are guaranteed the existence of two consecutive integers that are critical for \( \mathbb{L}(s, \chi) \).
6.1 A result of Harder on ratios of critical $L$-values

The relative discriminant $\delta_{F/K}$ of any extension $F/K$ of number fields is defined as: suppose $\text{Hom}(F, \bar{K}) = \{\sigma_1, \ldots, \sigma_r\}$, and $\{\omega_1, \ldots, \omega_r\}$ is a $K$-basis of $F$, then define $\delta_{F/K} := \det([\sigma_i(\omega_j)])^2$; taking the square makes it invariant under $\text{Gal}(\bar{K}/K)$, and hence $\delta_{F/K}$ is in $K^\times$, and as an element of $K^\times/K^{\times 2}$ it is independent of the choice of the basis $\{\omega_j\}$. The absolute discriminant of $F$ is $\delta_{F/Q}$. Harder proved in [6, Corollary (4.2.2)] that if $m$ and $m + 1$ are critical for $L(s, \chi)$, then the number

$$C(\iota\chi) := |\delta_{F/Q}|^{1/2} \frac{L(m, \iota\chi)}{L(m + 1, \iota\chi)} \quad (25)$$

is in $\bar{\mathbb{Q}}$, and for any $\varsigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, one has

$$\varsigma(C(\iota\chi)) = C(\varsigma^{\text{col}}\chi). \quad (26)$$

Let us note that we work with the completed $L$-function, and not with just the finite part of the $L$-function as in loc. cit. In particular, the power of $\pi$ in [6, Corollary (4.2.2)] is subsumed by the ratio of archimedean local-factors. Also, the result is stated therein for $m = -1$; for a general critical integer $m$ (with $m + 1$ also critical), the result follows by taking Tate-twists of $\chi$.

Note that (25) is asserting that the ratio of $L$-values therein is algebraic, since $|\delta_{F/Q}|^{1/2}$ is algebraic; but if we only look at the ratios of $L$-values then the reciprocity law becomes obscured. At any rate, the algebraicity result as stated in (25) together with the reciprocity law in (26) gives that

$$|\delta_{F/Q}|^{1/2} \frac{L(m, \iota\chi)}{L(m + 1, \iota\chi)} \in \iota(E). \quad (27)$$

We contend that whereas just the algebraicity result in (25) is correct, but restated as the stronger statement in (27) via the reciprocity law, it is not correct as it stands by producing an example when $F$ is a totally imaginary field that is not of CM type. The reason turns out to be, and this is the *raison d’être* for writing this article, that the reciprocity law in (26) needs to be modified by a sign; this sign is rather complicated, and turns out to be trivial when $F$ is a CM field, and can be nontrivial for a general totally imaginary field. We will first discuss this example in Section 6.2 that says that (27) is not stable under base-change, making it incorrect as it stands, in general. In Section 6.3, we will see how to rectify the reciprocity law in (26) using, interestingly enough, some other passages in [6] itself.

6.2 An example: Stability under base-change

Consider the fields in Example 16: $F_0 = \mathbb{Q}$, and put $F_1 = \mathbb{Q}(i)$ and $F = \mathbb{Q}(i, \sqrt{4 + i})$. To emphasize, this $F$ is totally imaginary but not CM, and $F_1$ is a CM subfield. Changing notations mildly to make it more suggestive, we denote

$$F = \mathbb{Q}(i, \sqrt{4 + i})$$

$$F^{\text{cm}} = \mathbb{Q}(i)$$

$$F^+ = \mathbb{Q}$$
Let us compute $\delta_{F/Q}$. First of all,

$$\delta_{F/F^{\text{cm}}} = \delta_{Q(i, \sqrt{4+i})/Q(i)} = 4(4 + i),$$

computed with respect to the basis $\{1, \sqrt{4+i}\}$; hence

$$N_{F^{\text{cm}}/Q}(\delta_{F/F^{\text{cm}}}) = N_{Q(i)/Q}(4(4 + i)) = 16 \cdot 17.$$  

Also, using the basis $\{1, i\}$ for $Q(i)$, we have

$$\delta_{F^{\text{cm}}/Q} = \delta_{Q(i)/Q} = -4.$$  

Using the well-known formula for the discriminant of a tower of fields, we get

$$\delta_{F/Q} = \delta_{F^{\text{cm}}/Q} \cdot N_{F^{\text{cm}}/Q}(\delta_{F/F^{\text{cm}}}) = (-4)^2 \cdot 16 \cdot 17 = 2^8 \cdot 17.$$ 

Let $\psi$ be an algebraic Hecke character of $F^{\text{cm}} = Q(i)$; as a continuous homomorphism $\psi : A_{Q(i)/Q} \to \C^\times$, with the character at infinity, $\psi_{\infty} : \C^\times \to \C^\times$, being of the form $\psi_{\infty}(z) = z^a z^b$ for $a, b \in \Z$. We may take (without loss of much generality) that $a \geq b$. The critical set for $L(s, \psi)$ is given by

$$\text{Crit}(L(s, \psi)) = \{1 - a, 2 - a, \ldots, -b\}.$$  

Let us further assume that $a - b \geq 2$ so that there are at least two critical points; let $m, m + 1 \in \text{Crit}(L(s, \psi))$. Then, (27) will remark that

$$|\delta_{F^{\text{cm}}/Q}|^{1/2} \frac{L(m, \psi)}{L(m + 1, \psi)} \in \Q(\psi),$$

where $\Q(\psi)$ is the number field generated by the values of the finite part $\psi_f$ of $\psi$. (We could have set up the context so that $\psi$ takes values in a coefficient field $E$, and then used an $\iota : E \to \C$ and to assert that the left-hand side, for $L(s, \iota, \psi)$ in place of $L(s, \psi)$, lies in $\iota(E)$.) Since $|\delta_{F^{\text{cm}}/Q}|^{1/2} = | -4 |^{1/2} \in \Q^\times$, we have

$$\frac{L(m, \psi)}{L(m + 1, \psi)} \in \Q(\psi).$$  

Suppose $\omega$ is a quadratic Dirichlet character of $F^{\text{cm}}$, then $\omega_{\infty}$ is trivial, and (28) applied to $\psi \omega$ will give

$$\frac{L(m, \psi \omega)}{L(m + 1, \psi \omega)} \in \Q(\psi \omega) = \Q(\psi).$$  

Now let $\chi$ be the base-change of $\psi$ to give a Hecke character of $F$:

$$\chi := \psi \circ N_{F/F^{\text{cm}}}. $$

Note that $F_{\infty}^{\text{cm}} = \C \times \C$, $F_{\infty}^{\text{cm}} = \C$ and $N_{F_{\infty}/F_{\infty}^{\text{cm}}}(z_1, z_2) = z_1 z_2$. Hence $\chi_{\infty}(z_1, z_2) = (z_1 z_2)^a z_1^b z_2^b = z_1^a z_2^b$, which is to say $\chi_{\infty} = \psi_{\infty} \otimes \psi_{\infty}$, whence $L_{\infty}(s, \chi) = L_{\infty}(s, \psi)^2$, from which we deduce

$$\text{Crit}(L(s, \chi)) = \text{Crit}(L(s, \psi)).$$  

Also, a basic property of base-change is that

$$L(s, \chi) = L(s, \psi) L(s, \psi \omega),$$  

where $\omega = \omega_{F/F^{\text{cm}}}$ is the quadratic character of $F^{\text{cm}}$ attached by class field theory to the quadratic extension $F/F^{\text{cm}}$. From (30), (28) and (29), we deduce

$$\frac{L(m, \chi)}{L(m + 1, \chi)} \in \Q(\psi) = \Q(\chi).$$
But, on the other hand, one may directly apply (27) to \( L(s, \chi) \) to get
\[
|\delta_{F/Q}|^{1/2} \frac{L(m, \chi)}{L(m + 1, \chi)} \in \mathbb{Q}(\chi).
\]
For the discriminant term, \( |\delta_{F/Q}|^{1/2} = (2^8 \cdot 17)^{1/2} = \sqrt{17} \pmod{\mathbb{Q}^\times} \). Hence
\[
\sqrt{17} \frac{L(m, \chi)}{L(m + 1, \chi)} \in \mathbb{Q}(\chi).
\] (32)
Of course, (31) and (32) are contradictory, because in general \( \sqrt{17} \not\in \mathbb{Q}(\chi) \). We may say that the formal statement in (27) is not stable under base-change. The reader can readily appreciate that the above example can be generalized.

### 6.3 Variations on a result of Harder

In this subsection, we state the main theorem of this article; see Theorem 19. Before doing so, we need to introduce notions and notations that lead to a delicate signature that appears in that theorem. For the rest of this subsection, we will let \( F \) be a totally imaginary number field, and \( E \) a number field that is Galois over \( \mathbb{Q} \) and contains a copy of \( F \). Let \( n = \sum \tau n_\tau \in \mathbb{Z}[\text{Hom}(F, E)] \) be an infinity type satisfying the purity condition of Lemma 15, with purity weight \( w \). Suppose \( \chi \) is an algebraic Hecke character of \( F \) with values in \( E \) of infinity type \( n \). Take any embedding \( \iota : E \to \mathbb{C} \), and let \( \iota \chi \) be the character as in Proposition 16.

Recall from Proposition 10 that the width of \( n \), for a given \( \iota \), is defined as
\[
\ell(\iota \chi) = \ell(\iota n) := \min_{\tau : F \to E} \{|n_{\text{tot}} - n_{\text{tot}}|\}.
\]
The Hecke \( L \)-function \( L(s, \iota \chi) \) has \( \ell(\iota \chi) \) many critical points. Note the following simple but important lemma.

**Lemma 17.** \( \ell(\iota n) \) is independent of \( \iota \), and depends only on the pure infinity type \( n \).

**Proof.** If \( F \) has no CM subfield (\( F_1 = F_0 \) is totally real), then from Proposition 14 it follows that \( \ell(\iota n) = 0 \), which proves the lemma; but this case is not interesting because we would like to have critical points. Assume henceforth that \( F \) is a totally imaginary field that contains a (maximal) CM subfield \( F_1 \), in which case \( n \) is the base-change from a pure infinity type \( m \) over \( F_1 \). It is clear that \( \ell(\iota n) = \ell(\iota m) \). Suppose \( \text{Hom}(F_1, E) = \{\tau_1, \tau'_1, \ldots, \tau_k, \tau'_k\} \), where \( 2k = [F_1 : \mathbb{Q}] \), and \( \tau_j \) and \( \tau'_j \) have the same restriction to the maximal totally real subfield \( F_0 \) of \( F_1 \). Then it follows from (the proof of) Lemma 11 that \( \ell(\iota m) = \min_j \{|m_{\tau_j} - m_{\tau'_j}|\} \).

The above lemma justifies the notation \( \ell = \ell(n) \).

#### 6.3.1 A combinatorial lemma

The following is a special case of a general ‘combinatorial lemma’ that is at the heart of what makes the theory of Eisenstein cohomology work so well for studying the special values to automorphic \( L \)-functions.
Lemma 18. The following three conditions are equivalent:

(1) \( s = -1 \) and \( s = 0 \) are critical for \( L(s, \chi) \).

(2) \(-\ell \leq w \leq -4 + \ell\).

(3) For each \( \tau : F \to E \) and \( \iota : E \to \mathbb{C} \), there exists \( w_{10\tau} \) in the Weyl group of \( GL(2) \) such that

(a) \( l(w_{10\tau}) + l(w_{\tau\tau}) = 1 \), and

(b) \( w_{10\tau} \cdot (n_{10\tau}) \) is dominant.

The reader is referred to [12, Section 3.2] for a proof; to compare statement (2) above to the corresponding statement in [12, Lemma 3.16], note that \( N, \ell, \mu, \mu' \), and \( a(\mu, \mu') \) of [12] take values 2, \( \ell \), and \( w/2 \), respectively in this article; Statement (3) and its consequences need to be elaborated upon. Given \( n_1, n_2 \in \mathbb{Z} \), by \( (n_1/n_2) \), one means the algebraic character of the diagonal torus \( GL(2) \) given by \( t_1, t_2) \mapsto t_1^{n_1} t_2^{n_2} \). Such a character is said to be dominant if \( n_1 \geq n_2 \) (where the choice of the Borel subgroup \( B \) is taken to be the upper-triangular matrices in \( GL(2) \)). The Weyl group \( W \) of \( GL(2) \), which has order 2, acts on this torus and hence on its characters. The twisted action of the nontrivial element \( w_0 \in W \) on such a character is explicitly given by \( w_0 \cdot (n_1/n_2) = (n_2-1/n_1+1) \). Of course, the length \( l(w) = 1 \) if \( w = w_0 \), and \( l(w) = 0 \) if \( w \) is the trivial element. Now the statement in (3) makes sense. Furthermore, in each pair \( \{\iota \circ \tau, \tau \circ \iota\} \) of conjugate embeddings \( \Sigma_F \), by (a), exactly one of the Weyl group elements is nontrivial and so having length 1, and the other is trivial; this defines a CM-type for \( F \) (recall that by a CM-type \( \Phi \) for \( F \) one means \( \Phi \subset \Sigma_F \) such that \( \Sigma_F = \Phi \sqcup \bar{\Phi} \)):

\[
\Phi(n, \iota) := \{ \eta \in \Sigma_F : l(w_\eta) = 1 \},
\]

and a bijection \( \beta_{n,\iota} : \Phi(n, \iota) \to S_\infty \) with the set of archimedean places of \( F \). Statement (3) also implies that

\[
\eta \in \Phi(n, \iota) \implies n_\eta \leq -2 \quad \text{and} \quad \eta \notin \Phi(n, \iota) \implies n_\eta \geq 0.
\]

For each \( \eta \in \Sigma_F \), we have \( n_\eta \leq -2 \) and \( n_{\bar{\eta}} \geq 0 \) or \( n_\eta \geq 0 \) and \( n_{\bar{\eta}} \leq -2 \).

Let us digress for a moment: what condition (3) really says is that the automorphic representation \( a \text{Ind}_{B(\mathbb{A}_F)}^{GL_2(\mathbb{A}_F)}(\chi, 1) \), that is algebraically and parabolically induced from the Borel subgroup, contributes to the cohomology in degree \( [F : \mathbb{Q}] \)/2 of the Borel–Serre boundary \( \partial_B S^G \) of a locally symmetric space \( S^G \) for \( G = GL(2)/F \). This is the starting point of using Eisenstein cohomology for \( GL(2)/F \) to deduce results on the special values of \( L \)-functions attached to algebraic Hecke characters that was pioneered by Harder in [6]. In that paper, Harder proved, under the conditions of Lemma 18, a rationality result for the ratio of critical values \( L(-1, \chi)/L(0, \chi) \), from which an analogous rationality result follows for the ratio of other successive pair of critical values by allowing ourselves Tate-twists of \( \chi \); statement (2) bounds the possible twists we can make, and it is a striking aspect of the lemma that one so obtains all successive pairs of critical values, \textit{no more and no less}! Let me note that in [6] such a lemma and its consequences are implicit; the explicit version of the above lemma and its philosophical content came out in my work with Harder [9]; that was further developed in my sequel [12] – especially, see paragraph 5.3.2.2 in [12].
6.3.2 A tale of two signatures. Fix an ordering on the set \( S_{\infty} \) of archimedean places of \( F \); say \( S_{\infty} = \{ w_1, \ldots, w_r \} \), where \( r = d_F/2 = [F : \mathbb{Q}] / 2 \). For each \( \zeta \in \text{Gal}(\bar{\mathbb{Q}} / \mathbb{Q}) \), define a permutation \( \pi_{n, i}(\zeta) \) of \( S_{\infty} \) by the diagram

\[
\begin{array}{ccc}
\Phi(n, t) & \xrightarrow{\beta_{n, i}} & S_{\infty} \\
\zeta \circ - & \downarrow & \pi_{n, i}(\zeta) \\
\Phi(n, \zeta \circ t) & \xrightarrow{\beta_{n, \zeta \circ t}} & S_{\infty},
\end{array}
\]

where the left vertical arrow needs an explanation: the \( t \)-independence of (1) and/or (2) of Lemma 18 translates to \( t \)-independence of (3) also, whence if \( t \circ \tau \in \Phi(n, t) \), then \( \zeta \circ (t \circ \tau) \in \Phi(n, \zeta \circ t) \). Put in other words, in general, a Galois conjugate of a CM-type of a totally imaginary field need not be a CM-type; however, since \( n \) is a base-change of an infinity-type \( m \) from \( F_1 \) (the key ingredient that went into \( t \)-independence), we see that the CM-type \( \Phi(n, t) \) for \( F \) is deduced from the CM-type \( \Phi(m, t) \) for \( F_1 \) under the canonical map \( \Sigma_F \rightarrow \Sigma_{F_1} \); and for a CM-field such as \( F_1 \), a Galois conjugate of a CM-type is a CM-type; hence we deduce that \( (\zeta \circ -)(\Phi(n, t)) = \Phi(n, \zeta \circ t) \) is a CM-type. The sign of the permutation \( \pi_{n, i}(\zeta) \) defines the signature

\[
\varepsilon_{n, i}(\zeta) := \text{sgn}(\pi_{n, i}(\zeta)).
\]

Let us digress for another moment: the key idea in the theory of Eisenstein cohomology is to see the map induced in cohomology by the standard intertwining operator

\[
T_{st}(s)|_{s=1} : \text{Ind}^\text{GL}_2(\mathbb{A}_F)_{t}^1(\chi, \mathbb{I}) \rightarrow \text{Ind}^\text{GL}_2(\mathbb{A}_F)_{t}^1(\mathbb{I}, 1, t\chi(-1))
\]

is realised in terms of the restriction map from total cohomology of \( S^G \) to the cohomology of the isotypic component cut out by these induced representations in the cohomology of the Borel–Serre boundary \( \partial_B S^G \) (see [9, pp. 82–83].) The infinity type of the inducing representation \( t\chi \otimes \mathbb{I} \) in the domain of \( T_{st}(s)|_{s=1} \) corresponds to the weight \( \text{'}n := (\begin{smallmatrix} n \\ 0 \end{smallmatrix}) \); similarly the infinity type of the inducing representation \( \mathbb{I}(1) \otimes t\chi(-1) \) for the codomain is the weight \( \text{'}n := (\begin{smallmatrix} n+1 \end{smallmatrix}) \).

Complementary to \( \Phi(n, i) \), we define the CM-type

\[
\Phi(\text{'}n, i) := \{ \eta \in \Sigma_F : l(w_\eta) = 0 \} = \{ \eta \in \Sigma_F : n_\eta \geq 0 \}.
\]

Note that \( \Phi(n, i) = \bar{\Phi}(\text{'}n, i) \), and the bijection \( \beta_{\text{'}n, i} : \Phi(\text{'}n, i) \rightarrow \Sigma_{\infty} \) is the complex-conjugate of \( \beta_{n, i} \). For each \( \zeta \in \text{Gal}(\bar{\mathbb{Q}} / \mathbb{Q}) \), define a permutation \( \pi_{\text{'}n, i}(\zeta) \) of \( \Sigma_{\infty} \) by the diagram

\[
\begin{array}{ccc}
\Phi(\text{'}n, i) & \xrightarrow{\beta_{\text{'}n, i}} & S_{\infty} \\
\zeta \circ - & \downarrow & \pi_{\text{'}n, i}(\zeta) \\
\Phi(\text{'}n, \zeta \circ t) & \xrightarrow{\beta_{\text{'}n, \zeta \circ t}} & S_{\infty},
\end{array}
\]

the sign of which defines the signature

\[
\varepsilon_{\text{'}n, i}(\zeta) := \text{sgn}(\pi_{\text{'}n, i}(\zeta)).
\]
6.3.3 The main theorem on $L$-values. We are now ready to state the main theorem which is a variation of a result due to Harder.

**Theorem 19.** Let $F$ be a totally imaginary number field, and $E$ a number field that is Galois over $\mathbb{Q}$ and containing a copy of $F$. Let $n = \sum \tau n_\tau \in \mathbb{Z}[\text{Hom}(F, E)]$ be an infinity type satisfying the purity condition of Lemma 15, with purity weight $w$. Suppose $\chi$ is an algebraic Hecke character of $F$ with values in $E$ of infinity type $n$. Take any embedding $\iota : E \rightarrow \mathbb{C}$, and let $\iota \chi$ be the character as in Proposition 16. Assume the condition on $n$ from Lemma 18:

$$-\ell \leq w \leq -4 + \ell.$$ 

Hence $\ell \geq 2$ and $n$ is the base-change to $F$ from an infinity-type of a maximal CM-subfield of $F$. Suppose $m, m + 1 \in \text{Crit}(\mathbb{L}(s, \chi))$; see Proposition 10. Then

$$|\delta_{F/\mathbb{Q}}|^{1/2} \frac{L(m, \iota \chi)}{L(m + 1, \iota \chi)} \in \iota(E),$$

and, furthermore, for every $\zeta \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have the reciprocity law

$$\zeta \left(|\delta_{F/\mathbb{Q}}|^{1/2} \frac{L(m, \iota \chi)}{L(m + 1, \iota \chi)}\right) = \varepsilon_{n, \iota}(\zeta) \cdot \varepsilon_{\bar{n}, \iota}(\zeta) \cdot |\delta_{F/\mathbb{Q}}|^{1/2} \frac{L(m, \iota \chi)}{L(m + 1, \iota \chi)},$$

(39)

6.3.4 Remarks on the proof. The above theorem is a special case of a more general theorem on Rankin–Selberg $L$-functions for $\text{GL}(n) \times \text{GL}(n')$ over a totally imaginary field; see [12, Theorem 5.16, (1), (2)]. When $F$ is a CM field, one has $\zeta \circ c \circ \eta = c \circ \zeta \circ \eta$ (see Lemma 11), hence, $\pi_{n, \iota}(\zeta) = \pi_{\bar{n}, \iota}(\zeta)$ is an equality of the permutations of $S_\infty$; whence, $\varepsilon_{n, \iota}(\zeta) = \varepsilon_{\bar{n}, \iota}(\zeta)$. In particular, the reciprocity law in (39) is the same as (26). As will be seen below, this is no longer the case for a general total imaginary field; even though one has an equality of sets $c(\Phi(n, \zeta) \circ i) = \zeta(c(\Phi(n, i)))$, the permutations $\pi_{n, \iota}(\zeta)$ and $\pi_{\bar{n}, \iota}(\zeta)$ can be distinct, and the signature $\varepsilon_{n, \iota}(\zeta) \cdot \varepsilon_{\bar{n}, \iota}(\zeta)$ can be nontrivial. (This is in essence the explanation of the counterexample in Section 6.2.) Even though Theorem 19 and especially (39), for a general totally imaginary field, are not stated like this in [6], its proof follows by putting together the proof of [6, Corollary 4.2.2] amplified by the discussion in [6, Section 2.4] of a rational system of generators for the unipotent cohomology of the coefficient system on $\text{GL}(2)$. The signature $\varepsilon_{n, \iota}(\zeta)$ is described in [6, (2.4.1)].

6.4 Compatibility with Deligne’s conjecture

In this subsection, we recall Deligne’s celebrated conjecture on the special values of motivic $L$-functions, as applied to ratios of successive critical values, and also recall a result of Deligne on the ratios of motivic periods that gives a motivic explanation to the ratio of successive critical $L$-values for a Hecke $L$-function.

6.4.1 Statement of Deligne’s conjecture. We will freely use the notation of [4]; a motive $M$ over $\mathbb{Q}$ with coefficients in a field $E$ will be thought in terms of its Betti, de Rham, and $\ell$-adic realizations. For a critical motive $M$, we have its periods $c^{±}(M) \in (E \otimes \mathbb{C})^\times$ as in loc. cit., that are well-defined in $(E \otimes \mathbb{C})^\times / E^\times$. We begin with a relation between the
two periods over a totally imaginary base field $F$; recall that $F_0$ is the largest totally real subfield of $F$; then $F_0$ admits at most one totally imaginary quadratic extension contained inside $F$; if $F$ does admit such a CM subfield, we denote it as $F_1$; and if there is no such extension inside $F$, then we put $F_1 = F_0$. If $F_1$ is indeed a CM field, and suppose $F_1 = F_0(\sqrt{D})$ for a totally negative $D \in F_0$, then define

$$\Delta_{F_1} := \sqrt{N_{F_0/Q}(D)}, \quad \Delta_F := \Delta_{F_1}^{[F:F_1]}.$$  

If $F_1 = F_0$ is totally real, then define

$$\Delta_{F_1} := 1, \quad \Delta_F := \Delta_{F_1}^{[F:F_1]} = 1.$$  

The following proposition is proved in [5].

**PROPOSITION 20**

Let $M_0$ be a pure motive of rank 1 over a totally imaginary number field $F$ with coefficients in a number field $E$. Put $M = \text{Res}_{F/Q}(M_0)$, and suppose that $M$ has no middle Hodge type. Let $c^\pm(M)$ be the periods defined in [4]. Then

$$\frac{c^+(M)}{c^-(M)} = 1 \otimes \Delta_F, \quad \text{in } (E \otimes \mathbb{C})^\times / E^\times.$$  

Let us also note that $1 \otimes \Delta_F$ is $\pm 1$ in each component of $(E \otimes \mathbb{C})^\times / E^\times$, since its square is trivial. Based on Proposition 20, Deligne’s conjecture [4] for the ratios of successive critical values of the completed $L$-function of a rank-one motive may be stated as follows:

**Conjecture** [4]. Let $M_0$ be a pure motive of rank 1 over a totally imaginary $F$ with coefficients in $E$. Put $M = \text{Res}_{F/Q}(M_0)$, and suppose that $M$ has no middle Hodge type. For $\iota : E \to \mathbb{C}$, let $L(s, \iota, M)$ denote the completed $L$-function attached to $(M, \iota)$. Put $L(s, M) = \{L(s, \iota, M)\}_{\iota : E \to \mathbb{C}}$ for the array of $L$-functions taking values in $E \otimes \mathbb{C}$. Suppose $m$ and $m + 1$ are critical integers for $L(s, M)$, and assuming that $L(m + 1, M) \neq 0$, we have

$$\frac{L(m, M)}{L(m + 1, M)} = 1 \otimes i^{d_F/2} \Delta_F,$$

where the equality is in $(E \otimes \mathbb{C})^\times / E^\times$.

A word of explanation is in order, since, in [4], Deligne formulated his conjecture for critical values of $L_f(s, M)$ – the finite-part of the $L$-function attached to $M$; which takes the shape

$$\frac{L_f(m, M)}{L_f(m + 1, M)} = (1 \otimes (2\pi i)^{-d^\pm(M)}) \frac{c^\pm(M)}{c^\pm(M)}, \quad \text{in } E \otimes \mathbb{C}.$$  

The assumption on $M$ that there is a middle Hodge type gives $d^\pm(M) = d(M)/2 = d_F/2$. Knowing the $L$-factor at infinity, we have

$$L_\infty(m, M) / L_\infty(m + 1, M) = 1 \otimes (2\pi)^{d_F/2}.$$
For the completed $L$-function, we get
\[
\frac{L(m, M)}{L(m+1, M)} = (1 \otimes i^{d_F/2}) \frac{c^\pm(M)}{c^\mp(M)}.
\]
(40)

It is clear now that (40) and Proposition 20 give Conjecture 6.4.1.

Using the conjectural correspondence between an algebraic Hecke character of $F$ with values in $E$, and a rank-one motive over $F$ with coefficients in $E$, Conjecture 6.4.1 may be restated as follows.

**Conjecture (Deligne’s conjecture for ratios of critical values of Hecke $L$-functions).** Let the notations and hypothesis be as in Theorem 19. Then
\[
i^{d_F/2} \Delta_F \frac{L(m, i^\chi)}{L(m+1, i^\chi)} \in \iota(E),
\]
(41)

and, furthermore, for every $\xi \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have the reciprocity law
\[
\xi \left(i^{d_F/2} \Delta_F \frac{L(m, i^\chi)}{L(m+1, i^\chi)}\right) = i^{d_F/2} \Delta_F \frac{L(m, \xi i^\chi)}{L(m+1, \xi i^\chi)}.
\]
(42)

6.4.2 **Theorem 19 $\implies$ Conjecture 6.4.1.** This follows from the following proposition.

PROPOSITION 21

Let the notations and hypothesis be as in Theorem 19. Then
\[
\frac{\xi((i^{d_F/2} \Delta_F)}{i^{d_F/2} \Delta_F} = \frac{\xi(|\delta_F/\mathbb{Q}|^{1/2})}{|\delta_F/\mathbb{Q}|^{1/2}} \cdot \epsilon_{\mathfrak{n}, i}(\xi) \cdot \epsilon_{\mathfrak{n}, i}(\xi).
\]

**Proof.** When $F$ is a CM-field, we have $\epsilon_{\mathfrak{n}, i}(\xi) \cdot \epsilon_{\mathfrak{n}, i}(\xi) = 1$ from 6.3.4; the proof will follow from the following lemma.

**Lemma 22.** Let $F$ be a CM field. Then, as elements of $\mathbb{C}^\times/\mathbb{Q}^\times$, we have
\[
|\delta_F/\mathbb{Q}|^{1/2} = i^{d_F/2} \cdot \Delta_F.
\]

**Proof of Lemma 22.** Let $F_0$ be the maximal totally real subfield of $F$; suppose $F = F_0(\sqrt{D})$, for a totally negative $D \in F_0$; then $\Delta_F = \sqrt{N_{F_0/\mathbb{Q}}(D)}$. For the absolute discriminant of $F$, using the well-known formula for the discriminant of a tower of fields, we have
\[
\delta_F/\mathbb{Q} = \delta_{F_0/\mathbb{Q}}^{[F:F_0]} \cdot N_{F_0/\mathbb{Q}}(\delta_F/F_0) = \delta_{F_0/\mathbb{Q}} \cdot N_{F_0/\mathbb{Q}}(\delta_F/F_0).
\]

Using the basis $\{1, \sqrt{D}\}$, we get the relative discriminant of $F/F_0$ as $\delta_{F/F_0} = 4D$. Hence, taking square-root of the absolute value of $\delta_F/\mathbb{Q}$, we get
\[
\sqrt{|\delta_F/\mathbb{Q}|} = \sqrt{|N_{F_0/\mathbb{Q}}(D)|} \pmod{\mathbb{Q}^\times}.
\]
Since \( D \) is totally negative, the sign of \( N_{F_0/Q}(D) \) is the same as \((-1)^{[F_0:Q]} = (-1)^{d_F/2}\), i.e., \( |N_{F_0/Q}(D)| = (-1)^{d_F/2}N_{F_0/Q}(D) \). Hence

\[
|\delta_{F/Q}|^{1/2} = \sqrt{(-1)^{d_F/2}N_{F_0/Q}(D)} = i^{d_F/2}\sqrt{N_{F_0/Q}(D)} = i^{d_F/2}\Delta_F \pmod{Q^\times}.
\]

\[\square\]

Suppose now that \( F \) is a totally imaginary field; the hypothesis of Theorem 19 guarantees that \( F \) contains a CM subfield \( F_1 \) and its maximal totally real subfield \( F_0 \). Lemma 22 generalizes to the following lemma.

**Lemma 23.** Let \( F \) be a totally imaginary field, and \( F_1 \) its maximal CM subfield. Then, as elements of \( C^\times/Q^\times \), we have

\[
|\delta_{F/Q}|^{1/2} = i^{d_F/2} \cdot \Delta_F \cdot (N_{F_1/Q}(\delta_{F/F_1}))^{1/2}.
\]

**Proof of Lemma 23.** Begin by noting that \( \delta_{F/Q} = \delta_{F_0/Q} \cdot N_{F_0/Q}(\delta_{F/F_0}) \), and since \([F:F_0] = 2[F:F_1] \) is even, we get

\[
|\delta_{F/Q}|^{1/2} = |N_{F_0/Q}(\delta_{F/F_0})|^{1/2} \pmod{Q^\times}.
\]

Next, since \( \delta_{F/F_0} = \delta_{F_1/F_0} \cdot N_{F_1/F_0}(\delta_{F/F_1}) \), using the \( F_0 \)-basis \( \{1, \sqrt{D}\} \) for \( F_1 \), one has \( \delta_{F_1/F_0} = 4D \); therefore,

\[
N_{F_0/Q}(\delta_{F/F_0}) = N_{F_0/Q}(4D)^{[F:F_1]} \cdot N_{F_0/Q}(N_{F_1/F_0}(\delta_{F/F_1})) = N_{F_0/Q}(D)^{[F:F_1]} \cdot N_{F_1/Q}(\delta_{F/F_1}) \pmod{Q^\times}.
\]

Since \( F_1/Q \) is a CM-extension, \( N_{F_1/Q}(\delta_{F/F_1}) > 0 \); hence, we get

\[
|\delta_{F/Q}|^{1/2} = |N_{F_0/Q}(\delta_{F/F_0})|^{1/2} = |N_{F_0/Q}(D)|^{[F:F_1]/2} \cdot (N_{F_1/Q}(\delta_{F/F_1}))^{1/2} \pmod{Q^\times}.
\]

Since \( D \ll 0 \) in \( F_0 \), we see that \((-1)^{[F_0:Q]}N_{F_0/Q}(D) > 0 \). Hence

\[
|N_{F_0/Q}(D)|^{[F:F_1]/2} = ((-1)^{[F_0:Q]}N_{F_0/Q}(D))^{[F:F_1]/2} = (i^{[F_0:Q]}\Delta_{F_1})^{[F:F_1]} = i^{[F_0:Q][F:F_1]}\Delta_F = i^{d_F/2}\Delta_F.
\]

\[\square\]

**Lemma 24.** With notations as in Lemma 23 and Theorem 19, we have the equality of signatures

\[
\varepsilon_{n,i}(\zeta) \cdot \varepsilon_{\tilde{n},i}(\zeta) = \frac{\zeta(N_{F_1/Q}(\delta_{F/F_1})^{1/2})}{N_{F_1/Q}(\delta_{F/F_1})^{1/2}}.
\]
Proof of Lemma 24. For the left-hand side, recall that \(\varepsilon_{n,t}(\zeta)\) (resp., \(\overline{\varepsilon}_{n,t}(\zeta)\)) is the sign of the permutation \(\pi_{n,t}(\zeta)\) (resp., \(\overline{\pi}_{n,t}(\zeta)\)) induced by the map \(\zeta \circ - : \Phi_{n,t} \to \Phi_{n,\zeta|t}\) (resp., \(\overline{\Phi}_{n,t} \to \overline{\Phi}_{n,\zeta|t}\)). For brevity, let \(\Phi = \Phi_{n,t}\), then \(\Phi = \overline{\Phi}_{n,t}\). Recall from Proposition 13 that \(n\) is the base-change of \(m\), hence the CM-type \(\Psi = \Phi_{m,t}\) for the CM-field \(F_1\) has the property that \(\Phi \to \Psi\) under the restriction map; also \(\Phi \to \Psi\). One has the following relation with respect to conjugating by a Galois element \(\zeta\):

\[
\begin{array}{c}
\Phi \cup \overline{\Phi} \xrightarrow{\zeta \circ -} \zeta(\Phi) \cup \zeta(\overline{\Phi}) \\
\Psi \cup \overline{\Psi} \xrightarrow{\zeta \circ -} \zeta(\Psi) \cup \zeta(\overline{\Psi})
\end{array}
\]

Fixing an ordering on \(S_{\infty}(F_1)\), the bottom row involves two permutations corresponding to \(\Psi \to \zeta(\Psi)\) and \(\overline{\Psi} \to \zeta(\overline{\Psi})\), but from Lemma 11, it follows that they are equal, hence \(\varepsilon_{m,t}(\zeta) = \overline{\varepsilon}_{m,t}(\zeta)\). However, the two permutations and their signatures corresponding to \(\Phi \to \zeta(\Phi)\) and \(\Phi \to \zeta(\overline{\Phi})\) need not be equal.

For the right-hand side, let \(\Sigma_{F_1} = \{\theta_1, \ldots, \theta_{d_1}\}; d_1 = [F_1 : \mathbb{Q}]\); then for any \(x \in F_1^\times\) one has \(N_{F_1/\mathbb{Q}} = \prod_{j=1}^{d_1} \theta_j(x) > 0\). Also, suppose \(\{\rho_1, \ldots, \rho_k\}\) denotes all embeddings of \(F\) into \(\tilde{F}_1\) over \(F_1\), and \(\omega_1, \ldots, \omega_k\) is an \(F_1\)-basis for \(F\), then \(\delta_{F_1/F_1} = \det[\rho_1(\omega_j)]^2\). Putting together, we get

\[
N_{F_1/\mathbb{Q}}(\delta_{F_1/F_1}) = \prod_{\theta \in \Sigma_{F_1}} \theta(\det[\rho_1(\omega_j)])^2 = \prod_{\theta \in \Sigma_{F_1}} \det[\rho_{\theta}(\omega_j)]^2,
\]

where \(\{\rho_{\theta}, \ldots, \rho_{\theta}\}\) is the set of all embeddings of \(F\) into \(\tilde{Q}\) that restrict to \(\theta : F_1 \to \tilde{Q}\). Hence,

\[
N_{F_1/\mathbb{Q}}(\delta_{F_1/F_1})^{1/2} = \pm \det\begin{bmatrix}
[\rho_{\theta_1}(\omega_j)] \\
[\rho_{\theta_2}(\omega_j)] \\
\vdots \\
[\rho_{\theta}(\omega_j)]
\end{bmatrix}, \tag{43}
\]

where the appropriate sign \(\pm\) is chosen to make the right-hand side positive. Denote by \(A^\theta\) the \(k \times k\) block \([\rho_{\theta}(\omega_j)]\). To the above equation, apply a Galois element \(\zeta\). To see how the determinant changes, note that the blocks along the diagonal are permuted according to how \(\zeta\) permutes \(\Sigma_{F_1}\), which is the same as \(\Psi \to \zeta(\Psi)\) and \(\overline{\Psi} \to \zeta(\overline{\Psi})\), since \(\varepsilon_{m,t}(\zeta) = \varepsilon_{\overline{m},t}(\zeta)\). Just a permutation of the blocks of a block diagonal matrix does not change the determinant. Next, if \(\theta \mapsto \zeta \theta\) then the rows of the block \(A^\theta\) are permuted to get the rows of \(A^{\zeta \theta}\). Fix an ordering on \(S_{\infty}(F)\) and correspondingly suppose \(\Phi = \{\theta_1, \ldots, \theta_{d_1}\}\), where \(r_1 = d_1/2\). The blocks of the block-diagonal matrix \(\text{diag}(A^\theta_1, \ldots, A^\theta_{r_1})\) (which is ‘half’ the matrix in (43)) get permuted to give \(\text{diag}(A^{\zeta \theta_1}, \ldots, A^{\zeta \theta_{r_1}})\), and within the blocks the net effect of the permutation is exactly the permutation \(\pi_{n,t}(\zeta)\) from Section 6.3.2, and so the determinant changes by the signature \(\varepsilon_{n,t}(\zeta)\). Similarly, working with the ‘other-half’ \(\text{diag}(A^\theta_1, \ldots, A^\theta_{r_1})\), we get the signature \(\overline{\varepsilon}_{n,t}(\zeta)\). Hence, the effect of applying \(\zeta\) to the matrix in (43) changes its determinant by \(\varepsilon_{n,t}(\zeta)\overline{\varepsilon}_{n,t}(\zeta)\).

Lemma 23 and Lemma 24 prove Proposition 21. \(\square\)
6.4.3 Some final remarks

Using Proposition 21, the result in Theorem 19 may be restated as the following theorem:

**Theorem 25.** Let the notations and hypothesis be as in Theorem 19. Then

\[ i^{d_F/2} \Delta_F \frac{L(m, \chi)}{L(m + 1, \chi)} \in \iota(E), \]

and, furthermore, for every \( \zeta \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \), we have the reciprocity law

\[ \zeta \left( i^{d_F/2} \Delta_F \frac{L(m, \chi)}{L(m + 1, \chi)} \right) = i^{d_F/2} \Delta_F \frac{L(m, \chi)}{L(m + 1, \chi)}. \]

We conclude this article by drawing attention to following piquant detail in the special case when \( F \) is a CM field. As before, \( F_0 \) is its maximal totally real subfield. Suppose \( \omega_{F/F_0} \) is the quadratic Hecke character of \( F_0 \) associated to \( F/F_0 \) by class field theory, and let \( G(\omega_{F/F_0}) \) denote the associated quadratic Gauss sum. Then

\[ |\delta_{F/Q}|^{1/2} = i^{d_F/2} \Delta_F \equiv i^{d_F/2} G(\omega_{F/F_0}) \mod \mathbb{Q}^\times; \quad (44) \]

the first congruence is exactly Lemma 22, and the second congruence is well-known (see Proposition 33 of [11]). Such a Gauss sum naturally appears if we were to use automorphic induction and study the \( L \)-function of a Hecke character of the CM field \( F \) as the \( L \)-function of a Hilbert modular form over \( F_0 \); see [12, Section 5.5]. More generally, as shown in [11], the Gauss sum naturally appears when we study the standard \( L \)-functions for \( \text{GL}(n)/F \) via automorphic induction as the standard \( L \)-functions of \( \text{GL}(2n)/F_0 \).

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