A UNIVERSAL CLOSED FORM FOR SQUARE MATRIX POWERS

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ABSTRACT.

This note presents a simple, universal closed form for the powers of any square matrix. A diligent search of the internet gave no indication that the form is known.

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1. THE CLOSED FORM

Let $M = [m_{i,j}]$ be an $n \times n$ square matrix, real or complex, and let $n m_{i,j}$ be the $(i, j)^{th}$ element of $M^n$. Then $n m_{i,j}$ can be expressed as the sum of $n$ terms, each of which is the product of a constant not depending on $n$, a binomial coefficient, and a power of an eigenvalue. In particular, for each unique eigenvalue, $\lambda$, of $M$, with multiplicity $m_{\lambda}$, the portion of $n m_{i,j}$ arising from $\lambda$ is

$$c_{i,j,1}(\binom{n-1}{0})\lambda^{n-1} + c_{i,j,2}(\binom{n-1}{1})\lambda^{n-2} + \cdots + c_{i,j,m_{\lambda}}(\binom{n-1}{m_{\lambda}-1})\lambda^{n-m_{\lambda}}.$$

If the matrix is not singular, the formula works for any power of $M$, negative, zero or positive; if the matrix is singular, for positive powers only. In the case of zero eigenvalues, $0^n$ is taken as 0 if $n < 0$, and 1 if $n = 0$. 

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2. ILLUSTRATION

Let $M$ be the matrix

$$
M = \begin{pmatrix}
66 & 83 & 95 & 31 & -50 & -63 \\
-71 & -79 & -86 & -22 & 59 & 72 \\
83 & 74 & 69 & 4 & -77 & -90 \\
-74 & -50 & -34 & 16 & 77 & 86 \\
-7 & -31 & -47 & -32 & -12 & -5 \\
65 & 89 & 105 & 41 & -40 & -56
\end{pmatrix}
$$

The 6 eigenvalues of $M$ are $\{-3, 2, 2, 1, 1, 1\}$. Hence,

$$
n_{m_{i,j}} = c_{i,j,1}(-3)^{n-1} + c_{i,j,2}2^{n-1} + c_{i,j,3}(n-1)^{2n-2} + c_{i,j,4}1^{n-1} + c_{i,j,5}(n-1)^{1n-2} + c_{i,j,6}(n-1)^{1n-3}.
$$

To determine the values of the $c$ constants for say, $n_{m_{2,5}}$, we first obtain by direct numerical calculation the values of $n_{m_{2,5}}$ for $n = 1$ to 6, namely, 59, 229, 764, 1915, 5270, and 11377, respectively.

We then solve the 6 simultaneous equations

$$
in_{m_{2,5}} = 59, 2n_{m_{2,5}} = 229, 3n_{m_{2,5}} = 764, 4n_{m_{2,5}} = 1915, 5n_{m_{2,5}} = 5270, 6n_{m_{2,5}} = 11377,
$$

obtaining

$$
c_{2,5,1} = 3, c_{2,5,2} = 65, c_{2,5,3} = 126, c_{2,5,4} = -9, c_{2,5,5} = -9, c_{2,5,6} = 0.
$$

Hence, we have the closed form

$$
n_{m_{2,5}} = 3(-3)^{n-1} + 652^{n-1} + 126\binom{n-1}{1}2^{n-2} - 91^{n-1} - 9\binom{n-1}{1}1^{n-2}.
$$

3. DERIVATION OF THE CLOSED FORM

Any square matrix $A$ can be written as $A = QTQ^{-1}$, where $Q$ is a unitary matrix, and $T$ is an upper triangular matrix (Schur decomposition). Hence, $A^n = QT^nQ^{-1}$. The eigenvalues of $A$ and $T$ are the same, and are the main diagonal elements of $T$. Theorem 3 in [2] shows that every element of $T^n$ can be expressed by the linear combination described above in the first section of this note. Therefore, every

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1 this matrix appeared in [3]
4. CONCLUSION

The purpose of this note was not to present a new method of calculating the numerical values of the elements of a matrix raised to the \(n^{th}\) power. Nor does it present the only way to obtain a closed form for matrix powers. For example, the Mathematica\(^2\) function `MatrixPower[m,n][[2,5]]` gives

\[ n m_{2,5} = \frac{1}{2} (-2(-3)^n + 2^{1+n} - 18n + 63 2^n n) \]

This is identical to the closed form for \(n m_{2,5}\) given above, but with the beauty and symmetry absent.

Rather, the primary purpose of this note was to share with others the aesthetic beauty of a simple universal form, which represents some very complicated calculations.

References

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\(^2\)Version 10.3.0.0