Hitting and return times in ergodic dynamical systems

N. Haydn, Y. Lacroix, & S. Vaienti
Los Angeles, Toulon, & Marseille

Abstract
Given an ergodic dynamical system $(X, T, \mu)$, and $U \subset X$ measurable with $\mu(U) > 0$, let $\mu(U)\tau_U(x)$ denote the normalized hitting time of $x \in X$ to $U$. We prove that given a sequence $(U_n)$ with $\mu(U_n) \to 0$, the distribution function of the normalized hitting times to $U_n$ converges weakly to some sub-probability distribution $F$ if and only if the distribution function of the normalized return time converges weakly to some distribution function $\tilde{F}$, and that in the converging case,

$$F(t) = \int_0^t (1 - \tilde{F}(s))ds, \ t \geq 0.$$  

This in particular characterizes asymptotics for hitting times, and shows that the asymptotics for return times is exponential if and only if the one for hitting times is too.

1. Introduction.
In the recent years there has been an interest in the statistics of hitting and return times. Typically a neighbourhood of a point is considered which can be either a metric ball or a ‘cylinder set’ associated with a measurable partition. In accordance with a theorem due to Kac one then looks at the return times which are normalized by the measure of the return set. A number of recent papers (e.g. [A1], [A2], [AG], [BV1], [BV2], [Co], [CG1], [CO], [GS], [H], [H1], [H2], [HSV], [HV], [P], [PI]) have provided conditions under which this distribution converges to the exponential distribution if the set is shrunk so that its measure converges to zero. On a different note, Lacroix and Kupsa have shown that with a suitable choice of return set one can realize any arbitrarily chosen limiting return time distribution [L] and hitting time distribution [K-L] within some class (see Theorem 1).

2000 Mathematics Subject Classification 37A05, 37A50, 60F05, 28D05.
Key words and phrases. asymptotic distribution, hitting, return times, Kac.
The purpose of this note is to show that limiting distributions for hitting and return times are intimately linked by the transformation \( \Diamond \).

Let \((X, \mathcal{B}, \mu)\) be a probability space, \(T: X \to X\) a measurable transformation that preserves \(\mu\), i.e. \(T^*\mu = \mu\). We also assume the dynamical system \((X, T, \mu)\) to be ergodic.

For (measurable) \(U \subset X\) with \(\mu(U) > 0\) we define the return/hitting time function \(\tau_U\) by

\[
\tau_U(x) = \inf\{k \geq 1: T^k x \in U\}.
\]

For \(x \in U\), \(\tau_U(x)\) denotes the return time. On the other hand if we refer to \(\tau_U(x)\) as a function on all of \(X\) then we call it the hitting time function. Poincaré’s recurrence theorem [K, Theorem 1'], together with ergodicity, then shows that \(\tau_U\) is a.s. finite. We also have Kac’s theorem [K, Theorem 2'] according to which

\[
\int_U \tau_U(x) \, d\mu(x) = \sum_{k=1}^{\infty} k \mu(U \cap \{\tau_U = k\}) = 1.
\]

Finer statistical properties of the variable \(\mu(U)\) have been investigated, in a rather large number of recent papers, where particular attention was given to the study of weak convergence of \(\mu(U_n)\) as \(\mu(U_n) \to 0\). See [AG] for a recent survey in the mixing case.

We say a sequence of distribution functions \(F_n, n = 1, 2, \ldots,\) converges weakly to a function \(F\) (which might not be a distribution itself) if \(F\) is (non-strictly) increasing right continuous and satisfies \(\lim_{n \to \infty} F_n(t) = F(t)\) at every point \(t\) of continuity of \(F\). We will write \(F_n \Rightarrow F\) if \(F_n\) converges weakly to \(F\).

Given a \(U \subset X\) measurable with \(\mu(U) > 0\), we define

\[
\tilde{F}_U(t) = \frac{1}{\mu(U)} \mu(U \cap \{\tau_U \mu(U) \leq t\})
\]

\[
F_U(t) = \mu(\{\mu(U) \tau_U \leq t\}).
\]

Define

\[
\mathcal{F} = \left\{ F: \mathbb{R} \to [0, 1], \ F \equiv 0 \text{ on } ]-\infty, 0], \ F \text{ (non-strictly) increasing, continuous, concave on } [0, +\infty[, \ F(t) \leq t \text{ for } t \geq 0 \right\};
\]

\[
\tilde{\mathcal{F}} = \left\{ \tilde{F}: \mathbb{R} \to [0, 1], \ \tilde{F} \text{ (non strictly) increasing, right-continuous,} \right\}
\]

\[
\tilde{F} \equiv 0 \text{ on } ]-\infty, 0], \ \int_0^{+\infty} (1 - \tilde{F}(s))ds \leq 1 \right\}.
\]

These functional classes appear in the following (\(U_n\) is always assumed to be measurable):
Theorem 1. Let $(X, T, \mu)$ be an ergodic and aperiodic dynamical system. Then:

(a) [L] for any $\tilde{F} \in \tilde{F}$ there exists a sequence $\{U_n \subset X : n = 1, 2, \ldots\}$ such that $\mu(U_n) \to 0$ and $\tilde{F}_{U_n} \Rightarrow \tilde{F}$.

(b) [L] if $\mu(U_n) \to 0$ and $\tilde{F}_{U_n} \Rightarrow \tilde{F}$ then $\tilde{F} \in \tilde{F}$.

(c) [K-L] for any $F \in F$, there exists $\{U_n \subset X : n = 1, 2, \ldots\}$ such that $\mu(U_n) \to 0$ and $F_{U_n} \Rightarrow F$.

(d) [K-L] if $\mu(U_n) \to 0$ and $F_{U_n} \Rightarrow F$ then $F \in F$.

In this note we prove the following rather unexpected and surprisingly unknown result:

Main Theorem. Let $(X, T, \mu)$ be ergodic, and $\{U_n \subset X : n \geq 1\}$ a sequence of positive measure measurable subsets. Then the sequence of functions $\tilde{F}_{U_n}$ converges weakly if and only if the functions $F_{U_n}$ converge weakly.

Moreover, if the convergence holds, then

$$\langle \diamond \rangle \quad F(t) = \int_0^t (1 - \tilde{F}(s))ds, \quad t \geq 0,$$

where $\tilde{F}$ and $F$ are the corresponding limiting (sub-probability) distributions.

The only previous result in this direction was obtained in [HSV] where it is shown that $\tilde{F}_{U_n} \to \tilde{F}$ and $\tilde{F}(t) = 1 - e^{-t}$ for $t \geq 0$ if and only if $F_{U_n} - \tilde{F}_{U_n} \to 0$ in the supremum norm on the real line. Our Main Theorem shows that the exponential distribution is the only distribution which can be asymptotic to both return and hitting times, as it is clearly the only fixed point of $\langle \diamond \rangle$. The equation $\langle \diamond \rangle$ also gives an equivalence between parts (a) and (b) of Theorem 1. We state a Corollary in this direction:

Corollary 2. (i) The asymptotic distribution for hitting times, if it exists, is positive exponential with parameter 1 if and only if the one for return times is too.

(ii) Parts (a) and (b) of Theorem 1 are equivalent.

Before we prove the Main Theorem in Section 3, and Corollary 2, we present an application to irrational rotations on the torus.

2. Homeomorphisms of the circle.

In the paper [CF] (see also [C] for further developments), Coelho and De Faria were able to characterize the limiting laws for the distribution of hitting time for orientation preserving homeomorphisms of the circle without periodic points, provided the set $U$ is chosen in a descending chain of renormalisation intervals. We could therefore apply our theorem to compute the corresponding distribution for the first return time.
Let \( f : S^1 \to S^1 \) be an orientation preserving homeomorphism of the circle without periodic points and let \( \alpha = \alpha(f) \in [0, 1) \) be its irrational rotation number; we consider here its continued fraction expansion: \( \alpha = [a_0, a_1, \ldots, a_n, \ldots] \). It is well known that \( a_n = \lfloor H^n(\alpha) \rfloor \), where \( \lfloor . \rfloor \) denotes now the integer part and \( H : [0, 1] \to [0, 1] \) is the Gauss transformation defined by \( H(0) = 0 \) and \( H(\alpha) = \{1/\alpha\} \) for \( \alpha \neq 0 \), denoting \( \{.\} \) the fractional part. The truncated expansion of order \( n \) of \( \alpha \) is given by \( p_n/q_n = [a_0, a_1, \ldots, a_{n-1}] \), where \( p_n \) and \( q_n \) verify the recursive relations

\[
q_{k+1} = a_k q_k + q_{k-1} \\
p_{k+1} = a_k p_k + p_{k-1}
\]

with \( p_0 = 1 \), \( p_1 = 1 \) and \( q_0 = 0 \), \( q_1 = 1 \). Then, for any number \( \beta \in [0, 1] \) and by setting \( b_j = \lfloor H^j(\beta) \rfloor \) for \( j \geq 0 \), we can construct the following quantities for \( n \geq 1 \) (natural extension of \( H \))

\[
\Gamma^n(\alpha, \beta) = (H^n(\alpha), [a_{n-1}, a_{n-2}, \ldots, a_0, b_0, b_1, \ldots]).
\]

Notice that the convergent subsequences of \( \Gamma^n(\alpha, \beta) \) for \( n \geq 0 \) do not depend on \( \beta \). The sets \( U \), where the points of the circle enter, are constructed in the following way. Let us take any point \( z \) on the circle and define \( J_n \) as the closed interval of endpoints \( f^{q_n-1}(z) \) and \( f^{q_n}(z) \): the sequence of sets \( U \) shrinking to \( z \) is taken as the descending chain of intervals \( J_n \). We call \( F_n(t) \) the distribution function of the hitting time into the sets \( U = \{J_n\} \). Coelho and De Faria proved the following theorem:

**Theorem [CF].** For each subsequence \( \sigma = \{n_i\} \) of \( \mathbb{N} \), the corresponding distribution functions \( F_{n_i} \) converge (pointwise or uniformly) if and only if either

(a) \( H^{n_i}(\alpha) \to 0 \), in which case the limit distribution is the uniform distribution on the unit interval; or

(b) \( \Gamma(\alpha, \cdot) \to (\theta, \omega) \) for some \( \theta > 0 \) and \( \omega < 1 \), in which case the limit distribution is the continuous piecewise linear function \( F_\sigma \) given by

\[
F_\sigma(t) = \begin{cases} 
\frac{1}{1+\theta} t + \frac{\omega^2(1+\theta)}{(1+\theta\omega)(1+\omega)} & \text{if } 0 \leq t < \frac{(1+\theta)\omega}{1+\theta\omega} \\
\frac{(1+\theta)\omega}{1+\theta\omega} & \text{if } \frac{(1+\theta)\omega}{1+\theta\omega} \leq t < \frac{1+\theta}{1+\theta\omega} \\
1 & \text{if } t \geq \frac{1+\theta}{1+\theta\omega}
\end{cases}
\]

For the second, more interesting case, we get immediately the distribution of the first return time by applying our result. It reads:

\[
\tilde{F}_\sigma(t) = \begin{cases} 
0 & \text{if } 0 \leq t < \frac{(1+\theta)\omega}{1+\theta\omega} \\
\frac{\omega}{1+\omega} & \text{if } \frac{(1+\theta)\omega}{1+\theta\omega} \leq t < \frac{1+\theta}{1+\theta\omega} \\
1 & \text{if } t \geq \frac{1+\theta}{1+\theta\omega}
\end{cases}
\]
As a concrete example we could take \( \alpha = \sqrt{\frac{5}{2}} - 1 \), the golden number, which exhibits the continued fraction expansion \([1, 1, 1, ...]\). In this case it is easy to check that \( \Gamma^n(\alpha, .) \) converges, when \( n \to \infty \), to \((\theta = \alpha, \omega = \alpha)\).

3. Proof of the Main Theorem.

For \( k \geq 1 \), let us denote \( V_k = \{ x \in U : \tau_U = k \} \). Then up to a zero measure set \( X \) is the disjoint union of the sets \( \bigcup_{j=0}^{k-1} T^j V_k \), \( k = 1, 2, \ldots \) (this is Kac’s Theorem 1’, no invertibility of the transformation is needed).

Let us also introduce \( U_k = \{ x : \tau_U(x) = k \} \). Then

\[
U_k = \bigcup_{j=0}^{\infty} T^j V_{k+j}, \quad \text{and} \quad \mu(U_k) = \sum_{j \geq k} \mu(V_j).
\]

Also for \( t \geq 0 \), \( F_U(t) = \mu(\{ \tau_U \leq t/\mu(U) \}) = \sum_{k \leq t/\mu(U)} \mu(U_k) \) : notice that \( F_U \) is constant on intervals \([k\mu(U), (k+1)\mu(U)]\), with a jump \( \mu(U_k) \) at \( k\mu(U) \).

We next define \( \bar{F}_U \) by

\[
\begin{align*}
(a) & : \bar{F}_U(k\mu(U)) = F_U(k\mu(U)); \\
(b) & : \bar{F}_U \text{ is linear on } [k\mu(U), (k+1)\mu(U)].
\end{align*}
\]

Then because the discontinuity jumps of \( F_U \) decrease, the function \( \bar{F}_U \) is (non-strictly) increasing, concave on the positive axis, piecewise linear, continuous, and the right hand side derivative satisfies

\[
\bar{F}_U'(t) = \frac{\mu(U_{k+1})}{\mu(U)} \quad \text{for } t \in [k\mu(U), (k+1)\mu(U)].
\]

On the other hand, \( \tilde{F}_U(t) = \frac{1}{\mu(U)} \sum_{k \leq t/\mu(U)} \mu(V_k) \), whence

\[
\tilde{F}_U \text{ is constant on } [k\mu(U), (k+1)\mu(U)] \text{ and has jump } \frac{\mu(V_k)}{\mu(U)} \text{ at } k\mu(U).
\]

Putting the above together yields

\[
(\star) \quad \bar{F}_U'(t) = 1 - \tilde{F}_U(t), \quad t \geq 0.
\]

Notice also that

\[
(\star\star) \quad \| F_U - \bar{F}_U \|_\infty \leq \mu(U)
\]

since \( F_U \) has its discontinuities located at points \( \mu(U), 2\mu(U), \ldots \).

We continue with the proof of the Main Theorem:
(I) Let us assume there is a sequence of subsets $U_n \subset X$ so that $\mu(U_n) \to 0$ and $\tilde{F}_{U_n} \Rightarrow \tilde{F}$ (then $\tilde{F} \in \tilde{F}$). Since $\tilde{F}$ is (non-strictly) increasing, this implies that $\tilde{F}_{U_n} \to \tilde{F}$ Lebesgue almost surely on $[0, +\infty[$. Whence, for given $t \geq 0$, by the Lebesgue dominated convergence theorem on $[0, t]$ ($\tilde{F} \in [0, 1]$), combining with $(\star)$ one has

$$\bar{F}_{U_n}(t) = \int_0^t (1 - \tilde{F}_{U_n}(s))ds \to \int_0^t (1 - \tilde{F}(s))ds =: F(t).$$

We put $F(t) = 0$ for $t < 0$. Since $\tilde{F} \in \tilde{F}$, it follows that $F \in \mathcal{F}$.

Moreover, by $(\ast \ast)$, $F_{U_n}(t) \to F(t)$ for all $t \in \mathbb{R}$ (the convergence is in fact uniform on compact subsets of $\mathbb{R}$ by [R, Theorem 10.8]).

Hence if $\tilde{F}_{U_n} \Rightarrow \tilde{F}$, then $(F_{U_n})$ converges weakly to the $F$ associated to $\tilde{F}$ by formula $(\Diamond)$.

Proving the converse for the Main Theorem, we need the following:

**Lemma 3.** Let $f_n$, $n = 1, 2, \ldots$, be a sequence of concave functions defined on a non-empty open interval $I$ and assume that $f_n$ converges pointwise to a limit function $f$. Then off an at most countable subset of $I$ the sequence of derivatives $f'_n$ converges pointwise to the derivative $f'$ of $f$.

**Proof of Lemma 3.** By [R, Theorem 25.3], off an at most countable subset of $I$, the functions $f_n$, and $f$, are differentiable, as concave functions.

Next, using the argument for the proof of [R, Theorem 25.7], but for a fixed $x \in I$ rather than along a sequence of point $x_i$ or points $x_i$ in a closed bounded subset of $I$, the convergence of the derivatives, when all defined, follows at once. □

(II) Let us now assume that $F_{U_n} \Rightarrow F$. Then [KL] implies that $F \in \mathcal{F}$. Whence by $(\star)$ and $(\ast \ast)$, we have, for $t \geq 0$,

$$\tilde{F}_{U_n}(t) = \int_0^t \tilde{F}'_{U_n}(s)ds = \int_0^t (1 - \tilde{F}_{U_n}(s))ds \to F(t) (= \int_0^t F'(s)ds).$$

It now follows from Lemma 3 that off an at most countable subset $\Omega$ of $]0, +\infty[$, the functions $1 - \tilde{F}_{U_n}(s)$ converge pointwise to $F'+(s)$. We a priori have $F'+(s)$ non-strictly decreasing. We then set $\tilde{F}(s) = 1 - F'(s)$ where $F'$ is continuous, else we ask $\tilde{F}$ to be right-continuous which sets automatically its values at eventual discontinuity points of $F'$.

It remains to show that $\bar{F}_{U_n}(s) \to \bar{F}(s)$ at points $s$ of continuity of $F'$. Clearly if $s \notin \Omega$ or $s < 0$ there is nothing to do. Else, for any $s_1$ and $s_2$ not in $\Omega$ such that $s_1 < s < s_2$, we have

$$\bar{F}(s_1) \leq \liminf_n \bar{F}_{U_n}(s) \leq \limsup_n \bar{F}_{U_n}(s) \leq \bar{F}(s_2),$$
and since $\Omega$ is dense in $[0, +\infty]$, the conclusion follows. So $\tilde{F}_{U_n} \Rightarrow \tilde{F} = 1 - F^{'+}$, which ends the proof. □

**Proof of Corollary 2.**

(i) : the proof is standard.

(ii) : we have a map $\tilde{F} \mapsto F$ defined by $\langle \diamond \rangle$ which clearly maps $\tilde{F} \to F$ in a one to one way.

To prove this map is surjective we may proceed as follows : given $F \in \mathcal{F}$ we have for any $s \geq 0$, $F(s) = \int_0^s F^{'+}(t)dt$, where $F^{'+}$ is the righthand side derivative of $F$. Since $F \in \mathcal{F}$, it follows quite easily that $F^{'+}$ is positive (non-strictly) decreasing and takes on values in $[0, 1]$. By integrability ($F(s) \to 1$ when $s \to +\infty$), $\lim_{s \to +\infty} F^{'+} = 0$.

We can set $\tilde{F}$ as to be equal to $1 - F^{'+}$ on $]0, +\infty[$, at continuity points of $F^{'+}$. We else set $\tilde{F} \equiv 0$ on $]-\infty, 0]$, and we eventually complete the definition of $\tilde{F}$ on $]0, +\infty[$ by requiring that it is right-continuous.

Then one has $\tilde{F} \mapsto F$ by $\langle \Diamond \rangle$, and $\tilde{F} \in \tilde{\mathcal{F}}$. □

**References**

[A1] M. Abadi, *Exponential approximation for hitting times in mixing processes*, Math. Phys. Elec. J. 7, 2, (2001).

[A2] M. Abadi, *Sharp error terms and necessary conditions for exponential hitting times in mixing processes*, to appear in Ann. Probability.

[AG] M. Abadi & A. Galves, *Inequalities for the occurrence times of rare events in mixing processes. The state of the art*, Markov Process. Related Fields 7 (2001), 97–112.

[BV1] H. Bruin, B. Saussol, S. Troubetzkoy & S. Vaienti, *Return time statistics via inducing*, Ergodic theory and dynamical systems 23 (2003), 991–1013.

[BV2] H. Bruin & S. Vaienti, *Return times for unimodal maps*, Forum Math. 176 (2003), 77–94.

[C] Z. Coelho, *The loss of tightness of time distribution for homeomorphisms of the circle*, to appear in Trans. Math. Soc. (2004).

[CF] Z. Coelho & E. De Faria, *Limit laws of entrance times for homeomorphisms of the circle*, Israel J. Math. 93 (1996), 93–112.

[Co] P. Collet, *Statistics of closest return times for some non uniformly hyperbolic systems*, Ergodic Theory and Dynamical Systems 21 (2001), 401–420.

[CG] P. Collet & A. Galves, *Statistics of close visits to the indifferent fixed point of an interval map*, J. Stat. Phys. 72 (1993), 459–478.

[CO] Z. Coelho, *Asymptotic laws for symbolic dynamical systems*, LMS Lectures Notes 279 (2000), 123–165.

[GS] A. Galves & B. Schmitt, *Inequalities for hitting time in mixing dynamical systems*, Random Comput. Dynam. 5 (1997), 337–347.

[H] M. Hirata, *Poisson law for Axiom-A diffeomorphisms*, Ergodic Theory and Dynamical Systems 13 (1993), 533–556.

[H1] N. Haydn, *Statistical properties of equilibrium states for rational maps*, Ergodic Theory and Dynamical Systems 201 (2000), 1371–1390.

[H2] N. Haydn, *The distribution of the first return time for rational maps*, J. Stat. Phys. 94 (1999), 1027–1036.
[HSV] M. Hirata, B. Saussol & S. Vaienti, Statistics of return times: a general framework and new applications, Comm. Math. Phys. 206 (1999), 33–55.

[HV] N. Haydn & S. Vaienti, The limiting distribution and error terms for return time of hyperbolic maps, Discrete and Continuous Dynamical Systems 10 (2004), 584–616.

[K] M. Kac, On the notion of recurrence in discrete stochastic processes, Bull. A.M.S. 53 (1947), 1002–1010.

[KL] M. Kupsa & Y. Lacroix, Asymptotics for hitting times, To appear, Annals of Probability (2005).

[L] Y. Lacroix, Possible limit laws for entrance times of an ergodic aperiodic dynamical system, Israel J. Math. 132 (2002), 253–264.

[P] F. Paccaut, Statistics of return time for weighted maps of the interval, Ann. Inst. H. Poincaré Probab. Statist. 36 (2000), 339–366.

[PI] B. Pitskel, Poisson limit law for Markov chains, Ergodic Theory and Dynamical Systems 11 (1991), 501–513.

[R] R. Rockafellar, Convex analysis, Princeton University Press, Princeton, New Jersey, 1970.

Nicolai Haydn
University of Southern California
Department of Mathematics
Denney Research Building
DRB 155
Los Angeles
California 90089-1113
USA

Yves Lacroix
Université du Sud Toulon Var
ISITV
Avenue Georges Pompidou
BP 56
83162 La Valette Cedex
France

Sandro Vaienti
UMR-6207 Centre de Physique Théorique, CNRS,
Universités d’Aix-Marseille I, II, et du Sud Toulon Var
FRUMAM (Fédération de Recherche des
Unités de Mathématiques de Marseille)
CPT
Campus of Luminy
Case 907
13288 Marseille Cedex 9
France

nhaydn@usc.edu
ylacroix@univ-tln.fr

vaienti@cpt.univ-mrs.fr