Abstract

We use the boundary state formalism to study, from the closed string point of view, superpositions of branes and anti-branes which are relevant in some non-perturbative string dualities. Treating the tachyon instability of these systems as proposed by A. Sen, we show how to incorporate the effects of the tachyon condensation directly in the boundary state. In this way we manage to show explicitly that the D1 – anti-D1 pair of Type I is a stable non-BPS D-particle, and compute its mass. We also generalize this construction to describe other non-BPS D-branes of Type I. By requiring the absence of tachyons in the open string spectrum, we find which configurations are stable and compute their tensions. Our classification is in complete agreement with the results recently obtained using the K-theory of space-time.
1 Introduction

Among the various string dualities, the one between the SO(32) heterotic string and the Type I theory [1] is of special interest for many reasons. First of all, this weak/strong coupling duality relates two models with radically different perturbative expansions and spectra. Secondly, an analysis of the web of string dualities shows that all of them may be “derived” by combining it with T-duality [2].

The duality conjecture between the SO(32) heterotic string and the Type I theory basically relies on the uniqueness of the corresponding supergravity models. In fact, the two tree-level effective actions turn out to be related [1] through a field redefinition which inverts the coupling constant. This duality thus identifies the weak coupling regime of one theory with the strong coupling regime of the other. As a first check of the conjecture, it has been shown [3] that the world-volume theory on the D-string of Type I is identical to the world-sheet theory of the heterotic string. In particular this explains the appearance of the SO(32) spinor representation in the Type I theory.

More evidence of this conjectured duality has been given in the literature, mainly relying on the existence of BPS states. As is well known, BPS states form short (or ultra-short) multiplets of the supersymmetry algebra for which the mass is related to the charge. Because of this fact they are stable and protected from radiative corrections. Hence, their properties can be analysed and computed perturbatively at weak coupling and then reliably extrapolated at strong coupling. Such tests of the heterotic/Type I duality conjecture comprise comparison between BPS spectra in the compactified theories [4], but also between BPS saturated terms in the effective actions in both the uncompactified [5] and the compactified cases [6].

However, at the first massive level the heterotic string contains perturbative states which are stable but not BPS. Their stability follows from the fact that they are the lightest states carrying the quantum numbers of the spinor representation of SO(32). Since they cannot decay, these states should be present also in the strong coupling regime. Then, if the heterotic/Type I duality is correct, the Type I theory should support non-perturbative stable configurations that are spinors of SO(32). Finding them and checking their multiplicities is therefore a very non-trivial test on the heterotic/Type I duality.

It turns out [7] that a pair formed by a D1-brane and an anti D1-brane of Type I (wrapped on a circle and with a $Z_2$ Wilson line) describes a configuration with the quantum numbers of the spinor representation of SO(32) (for reviews see [8]). Thus this system is the right candidate to describe in the non-perturbative regime the stable non-BPS states of the heterotic string mentioned above. However, a superposition of a brane with an anti-brane is unstable due to the presence of
tachyons in the open strings stretching between the brane and the anti-brane \[4\]. Then the problem of defining properly this superposition and treating its tachyon instability arises. This has been addressed by A. Sen in a remarkable series of papers \[10, 11, 7, 12, 13\]. In particular in Ref. \[7\] he considered a D-string – anti D-string pair of Type IIB, and managed to prove that when the tachyon condenses to a kink along the compact direction of the D-string, the pair becomes tightly bound and, as a whole, behaves as a D-particle. He also computed its mass finding that it is a factor of $\sqrt{2}$ bigger than the one of the supersymmetric BPS D-particle of the Type IIA theory. The presence of a D-particle in the Type IIB spectrum looks surprising at first sight since one usually thinks that in Type IIB there are only D$p$-branes with $p$ odd. However, one should keep in mind that such a D-particle is a non-supersymmetric and non-BPS configuration. Furthermore it is unstable, due to the fact that there are tachyons in the spectrum of the open strings living on its world-line. These tachyons turn out to be odd under the world-sheet parity $\Omega$, and hence disappear if one performs the $\Omega$ projection to get the Type I string \[14\]. Therefore, the D-particle found by Sen is a stable non-perturbative configuration of Type I that transforms as a spinor of SO(32).

In this paper we analyze the Type I D-particle from the point of view of the closed string operator formalism, and construct its representation in terms of a boundary state \[15, 16\]. The boundary state approach is a very convenient and useful method to describe BPS D-branes, find their couplings with the various closed string fields \[17\] and also determine their interactions (for some applications of the boundary state see for example Refs. \[18, 19, 20\]). In Refs. \[11, 21\], it was shown that the boundary state formalism can also be applied successfully to describe non-BPS configurations in certain orbifold models.

Here, we use this approach to study the superposition of a D-string and an anti D-string in the Type IIB theory, and construct the corresponding boundary state with the proper normalization. Then, extending Sen’s arguments, we manage to incorporate the effects of the tachyon condensation directly in the boundary state, and show explicitly that at a particular value of the tachyon v.e.v. this boundary state describes a non-BPS D-particle. Moreover, we show that the same result can be achieved by means of a suitable discrete transformation which effectively accounts for the tachyon condensation on the initial D-string – anti D-string pair. Using this construction we can also compute directly the mass of the non-BPS D-particle finding agreement with Sen’s result.

In the closed string operator formalism, the $\Omega$ projection that reduces Type IIB to Type I is implemented by adding to the boundary state of Type II the crosscap state \[22, 16\] which is uniquely determined by the requirement of tadpole cancellation. In this way one obtains an effective boundary state for the Type I
D-branes which can be efficiently used to study their properties and interactions. Applying this procedure to the non-BPS D-particle, we construct its corresponding boundary state of Type I and use it to check its stability by showing that no tachyon poles develop in the open string vacuum amplitude at one loop.

This construction can be easily generalized to describe other non-BPS D-branes of Type I. In particular, by requiring the absence of tachyons in the open string spectrum, we determine which of these Type I configurations are stable and compute their tensions. Our classification is in complete agreement with the results recently obtained by E. Witten \[23\] and others \[24, 25\] using the K-theory of space-time. Other recent papers discussing related subjects are \[26, 27, 28\].

This paper is organized as follows: in Section 2 we briefly review the boundary state formalism and specify our notations; in Section 3 we derive the boundary state for the Type IIB D-particle as a bound state of D-strings. In Section 4 we perform the Ω projection on the boundary state to describe the D-particle of the Type I theory, and explicitly verify its stability. In Section 5 we generalize these results to construct the boundary state for non-BPS Dp-branes of Type I and discuss for which values of \( p \) the tachyon cancellation condition can be satisfied. Finally, a few more technical calculations are reported in two appendices.

2 The boundary state formalism

In the closed string operator formalism one describes the supersymmetric Dp-branes of Type II by means of boundary states \(|Dp\rangle\) \[15, 16\]. These are closed string states which insert a boundary on the world-sheet, enforce on it the appropriate boundary conditions and represent the source for the closed strings emitted by the branes \[17\]. In the fermionic string, both in the NS-NS and in the R-R sectors, there are two possible implementations for the boundary conditions of a Dp-brane which correspond to two boundary states \(|Dp, \pm\rangle\). However, only the combinations

\[
|Dp\rangle_{NS} = \frac{1}{2} \left[ |Dp, +\rangle_{NS} - |Dp, -\rangle_{NS} \right]
\]

(2.1)

and

\[
|Dp\rangle_{R} = \frac{1}{2} \left[ |Dp, +\rangle_{R} + |Dp, -\rangle_{R} \right]
\]

(2.2)

are selected by the GSO projection in the NS-NS and in the R-R sectors respectively. As discussed in Ref. \[13\], the boundary states \(|Dp, \pm\rangle\) can be written as the product of a matter part and a ghost part

\[
|Dp, \pm\rangle = \frac{T_{n}}{2} |Dp_{mat}, \pm\rangle |Dp_{g}, \pm\rangle ,
\]

(2.3)
where
\[ |D_p, \pm \rangle_{\text{mat}} = |D_pX, \pm \rangle |D_p\psi, \pm \rangle, \quad |D_p, \pm \rangle = |D_p_{gh}, \pm \rangle |D_p_{sgh}, \pm \rangle. \]  

(2.4)

The overall normalization \( T_p \) can be unambiguously fixed from the factorization of amplitudes of closed strings emitted from a disk \([29, 17]\) and is related to the brane tension \( \tau_p \) \([30]\) according to
\[ T_p = \tau_p \kappa_{10} = \sqrt{\pi} \left( \frac{2\pi}{\sqrt{\alpha'}} \right)^{3-p} \]  

(2.5)

where \( \kappa_{10} \) is the gravitational coupling constant in ten dimensions. The explicit expressions of the various components of \( |D_p, \pm \rangle \), together with their derivation, can be found for example in Ref. \([19]\). Here we just recall the structure of the matter part of the boundary state \([1]\), namely
\[ |D_pX \rangle = \exp \left\{ -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \cdot S^{(p)} \cdot \bar{\alpha}_{-n} \right\} |D_pX \rangle^{(0)}, \]  

(2.6)

and
\[ |D_p\psi, \pm \rangle_{\text{NS}} = \exp \left\{ \pm i \sum_{r=1/2}^{\infty} \psi_{-r} \cdot S^{(p)} \cdot \bar{\psi}_{-r} \right\} |0 \rangle, \]  

(2.7)

for the NS-NS sector, and
\[ |D_p\psi, \pm \rangle_{\text{R}} = \exp \left\{ \pm i \sum_{n=1}^{\infty} \psi_{-n} \cdot S^{(p)} \cdot \bar{\psi}_{-n} \right\} |D_p, \pm \rangle^{(0)}_{\text{R}}, \]  

(2.8)

for the R-R sector. The matrix \( S^{(p)}_{\mu\nu} \) encodes the boundary conditions which characterize the D-brane; in the case of a static brane without any external fields one simply has
\[ S^{(p)}_{\mu\nu} = (\eta_{\alpha\beta}, -\delta_{ij}) \]  

(2.9)

where the indices \( \alpha, \beta \) label the \((p + 1)\) longitudinal directions of the world-volume, and the indices \( i, j \) label the \((9 - p)\) transverse directions. Finally, the superscript \((0)\) in Eqs. \((2.4)\) and \((2.8)\) denotes the zero-mode contributions to the boundary state. These are
\[ |D_pX \rangle^{(0)} = \delta^{(9-p)}(q^i - y^i) \prod_{\mu=0}^{9} |k^\mu = 0 \rangle \]  

(2.10)

where \( y^i \) are the coordinates of the D-brane and \( |k^\mu \rangle \) is the Fock vacuum of momentum \( k^\mu \), and
\[ |D_p, \pm \rangle^{(0)}_{R} = \left( C \Gamma^0 \ldots \Gamma^p \frac{1 \pm i G_{11}}{1 \pm 1} \right)_{AB} |A \rangle |\bar{B} \rangle \]  

(2.11)

1The ghost and superghost parts will not play any significant role in our present analysis and thus we omit them to avoid clutter.
where $|A\rangle$ and $|\tilde{B}\rangle$ are the left and right spinor vacua of the R sector in the 32-dimensional Majorana representation.

Both in the NS-NS and in the R-R sectors, the two boundary states $|Dp, \pm\rangle$ are mapped into each other by the world-sheet fermion number operators $(-1)^F$ and $(-1)^{\tilde{F}}$. In particular, in the NS-NS sector one can show that

$$(-1)^F |Dp, +\rangle_{NS} = (-1)^{\tilde{F}} |Dp, +\rangle_{NS} = -|Dp, -\rangle_{NS} \tag{2.12}$$

where, as usual, we have taken the NS-NS vacuum to be odd under $(-1)^F$ and $(-1)^{\tilde{F}}$. Similarly, in the R-R sector, where the fermion number operators also measure the chirality of the spinor vacua, one has

$$(-1)^F |Dp, +\rangle_{R} = (-1)^p |Dp, -\rangle_{R}, \quad (-1)^{\tilde{F}} |Dp, +\rangle_{R} = |Dp, -\rangle_{R} \tag{2.13}$$

These relations become useful when one performs the GSO projection. For example, in the NS-NS sector where the GSO projector is

$$P_{GSO} = \frac{1 + (-1)^F}{2} \frac{1 + (-1)^{\tilde{F}}}{2}, \tag{2.14}$$

one simply has

$$|Dp\rangle_{NS} \equiv P_{GSO} |Dp, +\rangle_{NS} = \frac{1}{2} \left[ |Dp, +\rangle_{NS} - |Dp, -\rangle_{NS} \right], \tag{2.15}$$

i.e. Eq. (2.1). For the R-R sector, one has to remember that the boundary states are written in the left-right asymmetric superghost picture $(-1/2, -3/2)$ (see Ref. [19] for details) in which the appropriate GSO projector takes the form

$$P_{GSO} = \frac{1 + (-1)^p (-1)^F}{2} \frac{1 + (-1)^{\tilde{F}}}{2} \tag{2.16}$$

where $p$ is even for Type IIA and odd for Type IIB in accordance with the R-R charge carried by a D$p$-brane. Then, using Eq. (2.13), we easily see that the GSO projected boundary state in the R-R sector is

$$|Dp\rangle_{R} \equiv P_{GSO} |Dp, +\rangle_{R} = \frac{1}{2} \left[ |Dp, +\rangle_{R} + |Dp, -\rangle_{R} \right], \tag{2.17}$$

i.e. Eq. (2.2). This analysis shows that it is enough to consider the boundary states $|Dp, +\rangle_{NS}$ and $|Dp, +\rangle_{R}$ from which the complete boundary states are obtained by means of the GSO projection. Finally, we recall that an anti D$p$-brane is a brane with negative R-R charge, and in our formalism it is simply described by a boundary state with an overall minus sign in the R-R sector.

---

2We refer to Refs. [19] [17] for the conventions on spinors and $\Gamma$ matrices.
Let us now consider the case in which the $p$ longitudinal space directions are compactified on circles of radii $R$. The boundary state for a wrapped $D_p$-brane is still given by the previous expressions with only two changes. The first (and easiest) one is in $|D_pX⟩^{(0)}$ which becomes

$$|D_pX⟩^{(0)} = \delta^{(9-p)}(q^i - y^i) |k^0 = 0⟩ \prod_{\alpha=1}^{p} \sum_{w^\alpha} |n^\alpha = 0, w^\alpha⟩ \prod_{i=p+1}^{9} |k^i = 0⟩ \quad (2.18)$$

where $|n^\alpha, w^\alpha⟩$ is the bosonic vacuum with Kaluza-Klein index $n^\alpha$ and winding number $w^\alpha$. T-duality invariance requires that these vacuum states must be normalized as

$$⟨n', w' | n, w⟩ = \Phi \delta_{nn'} \delta_{ww'} \quad (2.19)$$

where $\Phi$ is the self-dual “volume” of a compact direction which satisfies the following properties

$$\Phi \sim 2\pi R \quad \text{for} \quad R \to \infty , \quad \Phi \sim \frac{2\pi \alpha'}{R} \quad \text{for} \quad R \to 0 . \quad (2.20)$$

As a consequence of this fact, a second (and less obvious) change occurs in the boundary state of a wrapped D-brane. Indeed, as shown in Ref. [29], the overall normalization factor $T_p$ in Eq. (2.3) must be replaced by

$$T_p \left(\frac{2\pi R}{\Phi}\right)^{p/2} . \quad (2.21)$$

Then, it is easy to check that the boundary state defined in this way correctly reproduces the vacuum amplitude for a wrapped $D_p$-brane, normalization factors included.

We conclude this brief summary by mentioning that in the boundary state formalism the incorporation of $U(1)$ Wilson lines for wrapped D-branes amounts simply to introduce in Eq. (2.18) appropriate phases in the sum over all winding numbers of the compact directions, so that $|D_pX⟩^{(0)}$ becomes

$$|D_pX⟩^{(0)} = \delta^{(9-p)}(q^i - y^i) |k^0 = 0⟩ \prod_{\alpha=1}^{p} \sum_{w^\alpha} e^{i\theta_\alpha w^\alpha} |n^\alpha = 0, w^\alpha⟩ \prod_{i=p+1}^{9} |k^i = 0⟩ \quad (2.22)$$

where the constants $\theta_\alpha$ parametrize the Wilson lines associated with the $U(1)$ gauge fields living on the D-brane world-volume [32, 29].

---

3The explicit expression of $\Phi$ as well as a detailed discussion of its properties can be found for example in Ref. [31].
3 The Type IIB D-particle as a bound state of D-strings

Following Ref. [7], we now study the Type IIB system formed by the superposition of a D-string and an anti D-string, both wrapped on a circle of radius $R$ and with a $\mathbb{Z}_2$ Wilson line on one of them, say for definiteness on the anti D-string. To describe this system we introduce the following boundary states

$$|B, +\rangle_{\text{NS}} \equiv |D1, +\rangle_{\text{NS}} + |D1', +\rangle_{\text{NS}}$$

$$|B, +\rangle_{\text{R}} \equiv |D1, +\rangle_{\text{R}} - |D1', +\rangle_{\text{R}}$$

where the $'$ indicates the presence of the $\mathbb{Z}_2$ Wilson line (i.e. $\theta = 1$ in the notation of Eq. (2.23)). Note that the minus sign in Eq. (3.2) accounts for the fact that one of the two members of the pair is an anti D-string. Using the explicit expressions reported in Section 2, we have

$$|B, +\rangle_{\text{NS}} = \frac{T_1}{2} \sqrt{\frac{2\pi R}{\Phi}} \exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_n \cdot \hat{S}^{(1)} \cdot \tilde{\alpha}_n \right] \exp \left[ +i \sum_{r=1/2}^{\infty} \psi_{-r} \cdot \hat{S}^{(1)} \cdot \tilde{\psi}_{-r} \right]$$

$$\exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_n \tilde{\alpha}_n \right] \exp \left[ +i \sum_{r=1/2}^{\infty} \psi_{-r} \tilde{\psi}_{-r} \right] |\Omega\rangle_{\text{NS}}$$

where we have denoted by $\hat{S}^{(1)}$ the D-string $S$-matrix for all non-compact directions and have separately indicated in the second line the contribution of the bosonic and fermionic non-zero modes of the compact direction (i.e. the modes of $X, \psi$ and $\tilde{\psi}$). Due to the presence of the $\mathbb{Z}_2$ Wilson line, the vacuum $|\Omega\rangle_{\text{NS}}$ is given by

$$|\Omega\rangle_{\text{NS}} = \delta^{(8)}(q^i)|k^0 = 0\rangle \left( \sum_w |0, w\rangle + \sum_w (-1)^w|0, w\rangle \right) \prod_{i=2}^{9} |k_i^0 = 0\rangle$$

where for simplicity we have set to zero the coordinates $y_i$ of the D-strings. Analogously, in the R-R sector we have

$$|B, +\rangle_{\text{R}} = \frac{T_1}{2} \sqrt{\frac{2\pi R}{\Phi}} \exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_n \cdot \hat{S}^{(1)} \cdot \tilde{\alpha}_n \right] \exp \left[ +i \sum_{r=1/2}^{\infty} \psi_{-r} \cdot \hat{S}^{(1)} \cdot \tilde{\psi}_{-r} \right]$$

$$\exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_n \tilde{\alpha}_n \right] \exp \left[ +i \sum_{r=1/2}^{\infty} \psi_{-r} \tilde{\psi}_{-r} \right] |D1, +\rangle^{(0)}_{\text{R}} |\Omega\rangle_{\text{R}}$$

where $|D1, +\rangle^{(0)}_{\text{R}}$ is defined in Eq. (2.11) with $p = 1$, and

$$|\Omega\rangle_{\text{R}} = \delta^{(8)}(q^i)|k^0 = 0\rangle \left( \sum_w |0, w\rangle - \sum_w (-1)^w|0, w\rangle \right) \prod_{i=2}^{9} |k_i^0 = 0\rangle$$
\[ 2 \delta^{(8)}(q^i | k^0 = 0) \sum_{w} |0, 2w + 1 \rangle \prod_{i=2}^{9} |k^i = 0 \rangle. \] (3.6)

Let us now suppose that the radius \( R \) of the compact direction \( X \) is given by
\[ R_c = \sqrt{\frac{\alpha'}{2}}. \] (3.7)

As shown in Ref. [7], at this particular value of the radius all excitations of the open strings stretched between the D-string and the anti D-string have non negative mass squared. Moreover, at \( R = R_c \) the bosonic coordinate \( X \) is equivalent to a pair of fermionic fields \( \xi \) and \( \eta \). Indeed, if we write
\[ X(z, \bar{z}) = \frac{1}{2} (X_L(z) + X_R(\bar{z})) \] (3.8)
where
\[ X_L(z) = q_L - i(2\alpha')p_L \ln z + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n} \]
\[ X_R(\bar{z}) = q_R - i(2\alpha')p_R \ln \bar{z} + i\sqrt{2\alpha'} \sum_{n \neq 0} \tilde{\alpha}_n \bar{z}^{-n} \] (3.9)

with
\[ p_L = \frac{1}{2} \left( \frac{n}{R_c} + \frac{wR_c}{\alpha'} \right), \quad p_R = \frac{1}{2} \left( \frac{n}{R_c} - \frac{wR_c}{\alpha'} \right) \] (3.10)
then, it is possible to prove that
\[ \exp \left[ \pm \frac{i}{\sqrt{2\alpha'}} X_L(z) \right] \approx \frac{1}{\sqrt{2}} (\eta(z) \pm i\xi(z)) \]
\[ \exp \left[ \pm \frac{i}{\sqrt{2\alpha'}} X_R(\bar{z}) \right] \approx \frac{1}{\sqrt{2}} (\tilde{\eta}(\bar{z}) \pm i\tilde{\xi}(\bar{z})). \] (3.11)

By recombining the fermions \( \eta, \xi \) and \( \psi \) (and \( \tilde{\eta}, \tilde{\xi} \) and \( \tilde{\psi} \)) in a different manner, we can obtain an equivalent representation of the same conformal field theory in terms of a new compact bosonic field
\[ \phi(z, \bar{z}) = \frac{1}{2} (\phi_L(z) + \phi_R(\bar{z})) \] (3.12)
defined through
\[ \frac{1}{\sqrt{2}} (\xi(z) \pm i\psi(z)) \approx \exp \left[ \pm \frac{i}{\sqrt{2\alpha'}} \phi_L(z) \right] \]
\[ \frac{1}{\sqrt{2}} (\tilde{\xi}(\bar{z}) \pm i\tilde{\psi}(\bar{z})) \approx \exp \left[ \pm \frac{i}{\sqrt{2\alpha'}} \phi_R(\bar{z}) \right]. \] (3.13)
The field $\phi(z, \bar{z})$ is a free boson of radius $R = R_c$ which has a mode expansion similar to the one of the original coordinate $X(z, \bar{z})$ (i.e. Eqs. (3.9) and (3.10)) with oscillators $\phi_n$ and $\tilde{\phi}_n$, and with Kaluza-Klein and winding numbers $n_\phi$ and $w_\phi$ respectively.

As emphasized in Ref. [7], if one uses the new fields $\phi$ and $\eta$ instead of the original $X$ and $\psi$, it is possible to study explicitly the effects of a non vanishing v.e.v. for tachyon of the open string stretched between the D-string and the anti D-string. In fact, in the 0 superghost picture such a tachyon is described by the following vertex operator

$$V(z) = \frac{i}{\sqrt{2\alpha'}} \partial \phi(z) \otimes \sigma^1$$  \hfill (3.14)

where we have denoted by $\phi(z)$ the open string field corresponding to Eq. (3.12) and by $\sigma^1$ the Chan-Paton factor appropriate for an open string stretched across the D-string – anti D-string pair [7]. As a matter of fact, the vertex operator (3.14) does not represent a true tachyon since it creates a state which, at the critical radius $R = R_c$, is massless. However, since such a state becomes tachyonic in the decompactification limit $R \to \infty$, the field associated to Eq. (3.14) is, nevertheless, called tachyon. From the explicit expression of $V(z)$, we easily see that giving a non vanishing v.e.v. to the tachyon field is equivalent to introducing a $U(1)$ Wilson line along $\phi$, which we can parametrize as follows

$$W(\theta) = \frac{1}{2} \text{Tr} \left[ \exp \left( \frac{\theta}{2} \oint dz \frac{i}{\sqrt{2\alpha'}} \partial \phi \otimes \sigma^1 \right) \right].$$  \hfill (3.15)

Since in a pure closed string amplitude there are no other sources of Chan-Paton factors, Eq. (3.15) simplifies to

$$W(\theta) = \cos \left( \frac{\pi \theta w_\phi}{2} \right)$$  \hfill (3.16)

where $w_\phi$ is the total winding number of the closed string state seen by the operator $W(\theta)$.

We are now in the position of writing the boundary state which describes the D-string – anti D-string pair in the presence of a non vanishing tachyon v.e.v. This is given by Eqs. (3.13) and (3.15) with the oscillators $\alpha_n, \tilde{\alpha}_n, \psi_r, \tilde{\psi}_r$ of the compact direction replaced by $\phi_n, \tilde{\phi}_n, \eta_r, \tilde{\eta}_r$, and with a vacuum that carries an explicit dependence on the parameter $\theta$ according to Eq. (3.16). In particular, in the NS-NS sector we have

$$|B(\theta), +\rangle_{\text{NS}} = \frac{T_1}{2} \sqrt{\frac{2\pi R_c}{\Phi}} \exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \cdot \tilde{S}^{(1)} \cdot \tilde{\alpha}_{-n} \right] \exp \left[ +i \sum_{r=1/2}^{\infty} \psi_{-r} \cdot \tilde{S}^{(1)} \cdot \tilde{\psi}_{-r} \right] \exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \phi_{-n} \tilde{\phi}_{-n} \right] \exp \left[ +i \sum_{r=1/2}^{\infty} \eta_{-r} \tilde{\eta}_{-r} \right] |\Omega(\theta)\rangle_{\text{NS}}$$  \hfill (3.17)
where
\[ |\Omega(\theta)\rangle_{NS} = 2 \delta^{(8)}(q^{i}) |k^{0} = 0 \rangle \sum_{w_{\phi}} \cos(\pi \theta w_{\phi}) \left| 0, 2w_{\phi} \right\rangle \prod_{i=2}^{9} |k^{i} = 0 \rangle. \]  

(3.18)

Analogously, in the R-R sector we have
\[ |B(\theta), +\rangle_{R} = \frac{T_{1}}{2} \sqrt{\frac{2\pi R_{c}}{\Phi}} \exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \cdot \hat{S}^{(1)} \cdot \alpha_{-n} \right] \exp \left[ +i \sum_{n=1}^{\infty} \psi_{-n} \cdot \hat{S}^{(1)} \cdot \bar{\psi}_{-n} \right] \exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \phi_{-n} \bar{\phi}_{-n} \right] \exp \left[ +i \sum_{n=1}^{\infty} \eta_{-n} \bar{\eta}_{-n} \right] |D_{1}, +\rangle_{R} \left| \Omega(\theta) \right\rangle_{R} \]  

(3.19)

Notice that at \( \theta = 0 \) the boundary states (3.17) and (3.19) are completely equivalent to the original ones written in Eqs. (3.3) and (3.5), as one can check for example by computing some correlation functions or the vacuum amplitude. In this respect, it is worth pointing out that in computing amplitudes with \( |B(\theta), +\rangle_{NS} \) and \( |B(\theta), +\rangle_{R} \), one has to be careful in performing correctly the GSO projection. In fact, this is not obtained by taking linear combinations as in Eqs. (2.1) and (2.2), since the operators \((-1)^{F} \hat{\Phi} \) and \((-1)^{F} \) are not related to the \( \eta \) and \( \bar{\eta} \) fermion numbers. Instead, as is clear from Eq. (3.13), in the NS-NS sector one has
\[ (-1)^{F} : \tilde{\alpha}_{n}^{\mu} \rightarrow \tilde{\alpha}_{n}^{\mu}, \; \tilde{\psi}_{r}^{\mu} \rightarrow -\tilde{\psi}_{r}^{\mu}, \; \tilde{\phi}_{n} \rightarrow -\tilde{\phi}_{n}, \; \tilde{\eta}_{r} \rightarrow \tilde{\eta}_{r} \]  

(3.21)

and similarly for \((-1)^{F} \) on the left moving oscillators. Using these rules, one can easily see, for example, that
\[ (-1)^{F} |B(\theta), +\rangle_{NS} = -T_{1} \sqrt{\frac{2\pi R_{c}}{\Phi}} \exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \cdot \hat{S}^{(1)} \cdot \alpha_{-n} \right] \exp \left[ -i \sum_{r=1/2}^{\infty} \psi_{-r} \cdot \hat{S}^{(1)} \cdot \bar{\psi}_{-r} \right] \exp \left[ +i \sum_{r=1/2}^{\infty} \phi_{-r} \bar{\phi}_{-r} \right] \exp \left[ +i \sum_{r=1/2}^{\infty} \eta_{-r} \eta_{-r} \right] \delta^{(8)}(q^{i}) |k^{0} = 0 \rangle \sum_{w_{\phi}} \cos(\pi \theta w_{\phi}) \left| w_{\phi}, 0 \right\rangle \prod_{i=2}^{9} |k^{i} = 0 \rangle. \]  

(3.22)

Notice that the action of the operator \((-1)^{F} \) on the bosonic field \( \phi \) given in Eq. (3.21) looks like a T-duality because it amounts to change the relative sign
between its left and right moving oscillators. However, since the compactification radius of $\phi$ is such that
\[ R_c = \frac{\alpha'}{2R_c}, \tag{3.23} \]
this change in sign implies that a state with Kaluza-Klein index $n_\phi$ and winding number $w_\phi$ is transformed into a state with Kaluza-Klein index $w_\phi/2$ (which is acceptable only when $w_\phi$ is even) and winding number $2n_\phi$, that is
\[ (-1)^{F} : |n_\phi, w_\phi\rangle \rightarrow \frac{|w_\phi}{2}, 2n_\phi\rangle. \tag{3.24} \]
This peculiar behavior explains the structure of the vacuum in Eq. (3.22). Of course, similar considerations apply also for the boundary states in the R-R sector.

Let us now turn to the vacuum amplitude of the theory defined on the world-volume of our D-string – anti D-string pair. In the boundary state formalism this amplitude is simply given by
\[ \mathcal{A}(\theta) = \langle B(\theta), + \mid P_{\text{GSO}} D \mid B(\theta), + \rangle \tag{3.25} \]
where the GSO projection operator is given in Eqs. (2.14) and (2.16) and $D$ is the closed string propagator
\[ D = \frac{\alpha'}{4\pi} \int \frac{d^2z}{|z|^2} z^{L_0-a} \bar{z}^{\bar{L}_0-a} \tag{3.26} \]
with intercept $a_{\text{NS}} = 1/2$ in the NS-NS sector, and $a_{\text{R}} = 0$ in the R-R sector. Using the explicit expressions of the boundary states written above, and performing standard manipulations, one finds
\[ \mathcal{A}_{\text{NS-NS}}(\theta) = \frac{VR_c}{2\pi^2\alpha'} \int_0^\infty dt \left( \frac{\pi}{t} \right)^4 \left[ \left( \sum_{n_\phi} \cos^2(\pi \theta w_\phi) q^{w_\phi^2} \right) \frac{f_3(q)}{f_3^2(q)} - \sqrt{2} \frac{f_2(q)}{f_1(q) f_2(q)} \right] \tag{3.27} \]
and
\[ \mathcal{A}_{\text{R-R}}(\theta) = -\frac{VR_c}{2\pi^2\alpha'} \int_0^\infty dt \left( \frac{\pi}{t} \right)^4 \left[ \sum_{n_\phi} \cos^2(\pi \theta (w_\phi + 1/2)) q^{(w_\phi+1/2)^2} \right] \frac{f_3(q)}{f_3^2(q)} \tag{3.28} \]
where $V$ is the (infinite) length of the time direction and
\[ f_1(q) = q^{\frac{1}{2}\phi} \prod_{n=1}^\infty (1 - q^{2n}) , \quad f_2(q) = \sqrt{2}q^{\frac{1}{2}\phi} \prod_{n=1}^\infty (1 + q^{2n}) \]
\[ f_3(q) = q^{-\frac{1}{2}\phi} \prod_{n=1}^\infty (1 + q^{2n-1}) , \quad f_4(q) = q^{-\frac{1}{2}\phi} \prod_{n=1}^\infty (1 - q^{2n-1}) \tag{3.29} \]
with \( q = e^{-t} \).

It is interesting to observe that the contribution of the NS-NS\((-1)F\) spin structure (i.e. the second term in Eq. (3.27)) does not depend on the tachyon v.e.v. \( \theta \). This is a direct consequence of the fact that this spin structure arises from the overlap between \( |B(\theta), +\rangle_{\text{NS}} \), whose vacuum contains states with only winding numbers, and \((-1)^F|B(\theta), +\rangle_{\text{NS}}\), whose vacuum instead contains states with only Kaluza-Klein numbers (see Eqs. (3.18) and (3.22)). Therefore, in the NS-NS\((-1)F\) spin structure there is no contribution from the bosonic zero modes of the compact direction \( \phi \), and hence no dependence on the tachyon v.e.v. \( \theta \).

If one performs the modular transformation \( t \rightarrow \pi/t \), the entire amplitude \( A(\theta) \) can be interpreted as the one-loop vacuum energy of the open strings living in the world-volume of the D-string – anti D-string pair. After this modular transformation, one can explicitly check that \( A(\theta) \) in Eq. (3.25) coincides with the annulus amplitude that follows from the rules described in Ref. [7] from the open string point of view (see Appendix A for some details). In particular, one sees that the \( \theta \)-independent NS-NS\((-1)F\) spin structure of the closed string channel goes into the R spin structure of the open string channel, that indeed has been shown in Ref. [7] to be independent of the tachyon v.e.v.

At \( \theta = 1 \) a remarkable simplification occurs: the R-R contribution to \( A \) vanishes, since, as is clear from Eq. (3.20), the R-R boundary state is null at \( \theta = 1 \). Thus, the entire vacuum amplitude becomes

\[
A(\theta = 1) = \frac{V R_c}{2 \pi^2 \alpha'} \int_0^\infty dt \left( \frac{\pi}{t} \right)^4 \left[ \left( \sum_{\phi} q^{w_\phi^2} \right) \frac{f_3^8(q)}{f_1^8(q)} - \sqrt{2} \frac{f_4^7(q)}{f_1^7(q)} \frac{f_3(q)}{f_2(q)} \right]
\]

\[
= \frac{V}{4 \pi^2 R_c} \int_0^\infty dt \left( \frac{\pi}{t} \right)^4 \left( \sum_{\phi} q^{w_\phi^2} \right) \left[ \frac{f_3^3(q)}{f_1^3(q)} - \frac{f_4^3(q)}{f_1^3(q)} \right]
\]

(3.30)

where in obtaining the last line we have used Eq. (3.23) to rewrite the prefactor, and exploited the identities

\[
f_2(q) f_3(q) f_4(q) = \sqrt{2} \quad , \quad f_1(q) f_3^2(q) = \sum_{n=-\infty}^{+\infty} q^{n^2}
\]

(3.31)

to transform the integrand. Notice that with these manipulations we have managed to reconstruct the typical combination of \( f \)-functions that is produced by the usual GSO projection of the NS-NS sector. Thus, we are lead to suspect that a simpler underlying structure may actually exist at \( \theta = 1 \), in accordance with the property, shown in Ref. [7], that at this value of the tachyon v.e.v. the D-string – anti D-string pair is most tightly bound and, as a whole, behaves like a D-particle. Indeed, our expectation is confirmed by the fact that the amplitude \( A(\theta = 1) \) can be factorized
in terms of a new boundary state according to

\[ \mathcal{A}(\theta = 1) = \langle \tilde{B}, + | P_{GSO} D | \tilde{B}, + \rangle \]  

(3.32)

where

\[ | \tilde{B}, \pm \rangle = \frac{T_1}{\sqrt{2}} \sqrt{\frac{\pi \alpha'}{R_c \Phi}} \exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \cdot \tilde{S}^{(1)} \cdot \tilde{\alpha}_{-n} \right] \exp \left[ \pm i \sum_{r=1/2}^{\infty} \psi_{-r} \cdot \tilde{S}^{(1)} \cdot \tilde{\psi}_{-r} \right] \]

\[ \exp \left[ + \sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \tilde{\alpha}_{-n} \right] \exp \left[ \mp i \sum_{r=1/2}^{\infty} \psi_{-r} \tilde{\psi}_{-r} \right] | \tilde{\Omega} \rangle \]  

(3.33)

with

\[ | \tilde{\Omega} \rangle = 2 \delta^{(8)}(q^i | k^0 = 0) \sum_w | w, 0 \rangle \prod_{i=2}^{9} | k^i = 0 \rangle \]  

(3.34)

Of course, the simple factorization of a vacuum amplitude does not allow to uniquely fix the structure of a boundary state, and thus one may think that there is some arbitrariness in Eq. (3.33). However, a detailed analysis of correlation functions shows that the new boundary state \( | \tilde{B}, + \rangle \), which is written in terms of the original degrees of freedom for the compact direction (i.e. \( X \) and \( \psi \)), is equivalent to the boundary state of Eq. (3.17) for \( \theta = 1 \). To see this, one considers closed string vertex operators written in terms of \( X \) and \( \psi \) and computes the corresponding correlation functions on a disk using the boundary state \( | \tilde{B}, + \rangle \); then, transforming \( (X, \psi) \) into \( (\phi, \eta) \) through the bosonization formulas, one checks that the same correlation functions are obtained with the boundary state \( | B(\theta = 1), + \rangle \) given in Eq. (3.17).

A few examples of such calculations are explicitly described in Appendix B.

Based on these results, we can conclude that in order to describe the D-string – anti D-string pair at \( R = R_c \) in terms of \( X \) and \( \psi \), we have to use the original boundary states of Eqs. (3.3) and (3.3) if \( \theta = 0 \), whereas we have to use the the new boundary state of Eq. (3.33) if \( \theta = 1 \). Of course, at this particular value of \( \theta \) there is no R-R sector, as we have explicitly shown. It is interesting to observe that \( | B, + \rangle_{NS} \) and \( | \tilde{B}, + \rangle \) can be related to each other by means of a discrete transformation \( T \). Indeed, as is clear from Eqs. (3.3) and (3.33), we may go from \( | B, + \rangle_{NS} \) to \( | \tilde{B}, + \rangle \) by changing the sign to the right moving oscillators of the compact direction, and consequently by changing the vacuum from \( | 0, 2w \rangle \) to \( | w, 0 \rangle \) since the radius of \( X \) satisfies Eq. (3.23) (cf also Eq. (3.24)). Like the usual T-duality, also \( T \) transforms a longitudinal direction into a transverse one, so that the new boundary state \( | \tilde{B}, + \rangle \) more properly describes a D0-brane with a compact transverse direction. However, unlike T-duality, \( T \) switches off the R-R sector. This fact suggests that, more than a symmetry of the theory, this transformation \( T \) has to be regarded simply as an effective way of implementing the change of the tachyon v.e.v. from \( \theta = 0 \) to \( \theta = 1 \).
on the original boundary states, which can be justified by introducing the new fields \( \phi \) and \( \eta \) through the bosonization procedure as we have done.

In Ref. [7] it was shown that the decompactification limit \( R \to \infty \) of the D-string – anti D-string pair is meaningful only at \( \theta = 1 \), where the system is most tightly bound and no tadpoles develop. In our formalism, this feature is revealed by the fact the limit \( R \to \infty \) is ill-defined on the original boundary states \(| B, + \rangle_{NS} \) and \(| B, + \rangle_{R} \) since their vacuum contains only a subset of all possible winding states, namely the states with only even or odd winding numbers respectively in the NS-NS and R-R sectors. On the contrary, there are no problems in taking the decompactification limit on the new boundary state \(| \tilde{B}, + \rangle \) which describes the theory at \( \theta = 1 \). In fact, when \( R \to \infty \) we can rewrite Eq. (3.34) as follows

\[
|\tilde{\Omega} \rangle = 2 \delta^{(8)}(q^i) |k^0 = 0\rangle 2\pi R \int \frac{dk^1}{2\pi} |k^1\rangle \prod_{i=2}^{9} |k^i = 0\rangle
\]

which resembles Eq. (2.10) for \( p = 0 \). Furthermore, combining the factor of \( 4\pi R \) from Eq. (3.35) with the prefactors of \( |\tilde{B}, + \rangle \), we see that the complete normalization of the boundary state becomes

\[
\frac{T_1}{2} \sqrt{\frac{\pi \alpha'}{R \Phi}} \frac{4\pi R}{\sqrt{2}} \to \frac{\sqrt{2} T_1}{2} \frac{(2\pi \sqrt{\alpha'})}{2} = \frac{\sqrt{2} T_0}{2}
\]

(3.36)

where we have used the asymptotic behavior of \( \Phi \) for \( R \to \infty \) (see Eq. (2.20)) and the explicit expression of the tensions \( T_p \) (see Eq. (2.5)). Thus, in the decompactification limit our system is described by

\[
|\tilde{B}, + \rangle = \frac{\sqrt{2} T_0}{2} \exp \left[ -\sum_{n=1}^{\infty} \frac{\alpha_{-n} \cdot S^{(0)} \cdot \tilde{\alpha}_{-n}}{n} \right] \exp \left[ +i \sum_{r=1/2} S^{(0)} \cdot \tilde{\psi}_{-r} \right] \delta^{(9)}(q^i) \prod_{\mu=0}^{9} |k^\mu = 0\rangle
\]

(3.37)

By performing the usual GSO projection, we then obtain the complete boundary state

\[
|\tilde{B} \rangle \equiv P_{GSO} |\tilde{B}, + \rangle = \frac{1}{2} \left[ |\tilde{B}, + \rangle - |\tilde{B}, - \rangle \right]
\]

(3.38)

which describes a D0-brane in the Type IIB theory. Since there is no R-R sector, this D-particle is non-supersymmetric and non-BPS. Moreover, from Eq. (3.37) we explicitly see that its tension (or mass) is a factor of \( \sqrt{2} \) bigger than the tension of the ordinary supersymmetric D-particle of the Type IIA theory. This fact has been proved and checked also in Ref. [7] using completely different arguments.
4 The boundary state for the Type I D-particle

The non-supersymmetric D-particle described in the previous section is an unstable configuration of the Type IIB theory. In fact, the absence of the R-R part in the D-particle boundary state $|\tilde{B}\rangle$ implies the absence of any GSO projection in the dual open string theory defined on its world-line. Thus, the spectrum of these open strings contains not only the states that a supersymmetric D-particle would sustain, but also all other states that are usually removed by the GSO projection.

In particular in the NS sector, one finds a tachyonic state which is responsible for the instability of the entire configuration. However, things might improve if there is a consistent truncation of the theory which is free of tachyons. One possibility is to study our system in the Type I theory, which can be viewed as an orbifold of the Type IIB theory with respect to the world-sheet parity $\Omega$ \footnote{Another possibility discussed in Refs. \cite{11,21} is to consider the orbifold IIB/(-1)^F_L I_4 where $F_L$ is the left space-time fermion number and $I_4$ is the space-time parity in four directions.}. To see whether or not the open string tachyon is removed by the $\Omega$ projection, we have to analyze the interaction amplitude of the D-particle with itself due to the exchange of unoriented strings. To this end, we follow the general procedure discussed in Ref. \cite{22,16}, and add to the boundary state $|\tilde{B}\rangle$ the so-called crosscap state $|C\rangle$ which inserts a boundary on the string world-sheet with opposite points being identified. Thus, we consider

$$|\tilde{D}0\rangle = \frac{1}{\sqrt{2}} [|\tilde{B}\rangle + |C\rangle] ,$$

where the factor of $1/\sqrt{2}$ has been introduced to have the proper normalization.

Like for a usual boundary, also for a crosscap there are two possible implementations of the overlap equations that correspond to two states $|C,\pm\rangle$. However, only a linear combination of them is selected by the GSO projection. Since for the case at hand the R-R sector does not play any role, we will concentrate only in the NS-NS sector where one finds

$$|C\rangle = P_{GSO} |C, +\rangle_{NS} = \frac{1}{2} [|C, +\rangle_{NS} - |C, -\rangle_{NS}] .$$

The explicit expression for the crosscap states is \footnote{Again we only consider the matter part. For the ghost and superghost contribution see for example Ref. \cite{13}.}

$$|C, \pm\rangle_{NS} = i 2^5 \frac{T_9}{2} \exp \left[ -\sum_{n=1}^{\infty} \frac{e^{i\pi n}}{n} \alpha_{-n} \cdot S^{(9)} \cdot \tilde{\alpha}_{-n} \right] \exp \left[ \pm i \sum_{r=1/2}^{\infty} e^{i\pi r} \psi_{-r} \cdot S^{(9)} \cdot \tilde{\psi}_{-r} \right] \prod_{\mu=0}^{9} |k^\mu = 0\rangle \quad (4.3)$$
where $S^{(9)}_{\alpha\beta} = \eta_{\alpha\beta}$, $T_9$ is given in Eq. (2.3) for $p = 9$ and the overall factor of $i$ has been introduced for later convenience. Like the D9-brane, also the crosscap is a source for the non-trivial 10-form potential of Type I theory \[^{16}\], and indeed the crosscap state $|C\rangle$ is strictly related to the boundary state of a D9-brane with a pure imaginary radius; in fact one has

$$|C\rangle \sim i^{L_0 + \tilde{L}_0}|D9\rangle \, . \quad (4.4)$$

This remark will prove to be useful for technical purposes. An important point we want to stress is that the normalization of the crosscap state in Eq. (4.3) is completely fixed by the requirement of tadpole cancellation of the Type I theory \[^{16}\].

We are now ready to study the interaction of a D-particle with itself. This is due to exchange of closed strings that propagate along a cylinder between two $|\tilde{D}0\rangle$ states. This amplitude comprises three types of contributions. The first one is with two boundaries and corresponds to a cylinder amplitude which is given by

$$A = \frac{1}{2}<\tilde{B}|D|\tilde{B}> \, .$$

Notice that this amplitude is half of the corresponding one in the Type IIB theory. The second type of contribution is with one boundary and one crosscap and corresponds to a Möbius strip amplitude which is given by

$$M = \frac{1}{2}<\tilde{C}|D|C> \, .$$

Of course, one has to consider also the conjugate expression $M^\ast = \frac{1}{2}<C|D|\tilde{B}>$ where the crosscap and the boundary have exchanged place. Finally, the third type of contribution is with two crosscaps and corresponds to the Klein bottle amplitude given by

$$K = \frac{1}{2}<C|P|C> \, .$$

This last contribution does not contain open string poles and refers only to the propagation of unoriented closed strings. For these reasons we shall not consider it in our analysis.

Using the explicit expressions for the boundary and the crosscap states, we find that the cylinder amplitude is

$$A = \frac{V}{2\pi} (8\pi^2 \alpha')^{-1/2} \int_0^\infty dt \left( \frac{\pi}{i} \right)^{1/2} \left[ \frac{f_3^8(q)}{f_1^8(q)} - \frac{f_4^8(q)}{f_1^8(q)} \right] \, . \quad (4.5)$$

where $q = e^{-t}$, while the Möbius strip amplitude is

$$M = 2^{9/2} \frac{V}{2\pi} (8\pi^2 \alpha')^{-1/2} \int_0^\infty dt \left[ \frac{f_3^8(iq) f_1(iq)}{f_2^8(iq) f_3(iq)} - \frac{f_4^8(iq) f_1(iq)}{f_2^8(iq) f_3(iq)} \right] \, . \quad (4.6)$$

A remark is in order here. Since the crosscap state is related to the boundary state of a D9-brane as we have seen in Eq. (4.3), the Möbius amplitude corresponds to a system with one NN direction and nine DN directions which cannot be studied in the light cone gauge. Hence in this case, the use of a covariant formalism is necessary.
To obtain the open string channel, we must perform the modular transformation $t \to 1/t$ in the previous expressions so that $\mathcal{A}$ and $\mathcal{M}$ become the planar and non-planar one-loop amplitudes of the open strings living on the D-particle world-line. The rules for the modular transformations of the functions $f_k$ with a real argument are well known (see for example Ref. [30]). The analogous rules for a pure imaginary argument are instead less known, and thus we list them below

$$f_1(\text{e}^{-\pi s}) = (2s)^{-1/2} f_1(\text{e}^{-\pi t}),$$
$$f_2(\text{e}^{-\pi s}) = f_2(\text{e}^{-\pi t}),$$
$$f_3(\text{e}^{-\pi s}) = \text{e}^{i\pi/8} f_4(\text{e}^{-\pi t}),$$
$$f_4(\text{e}^{-\pi s}) = \text{e}^{-i\pi/8} f_3(\text{e}^{-\pi t}).$$

These results for $f_1$ and $f_2$ may be obtained using the rules of modular transformation for $f_k(\text{e}^{-\pi s})$ and elementary algebraic manipulations. In order to fix the phases appearing in the transformation rules of $f_3$ and $f_4$ we have used Euler’s pentagonal identity.

Performing the modular transformation, we then obtain

$$\mathcal{A} = V (8\pi^2 \alpha')^{-1/2} \int_0^\infty \frac{ds}{2s} s^{-1/2} \left[ \frac{f_3^2(r)}{f_1^2(r)} - \frac{f_2^2(r)}{f_1^2(r)} \right]$$

and

$$\mathcal{M} = 2^3 V (8\pi^2 \alpha')^{-1/2} \int_0^\infty \frac{ds}{2s} s^{-1/2} \left[ \frac{e^{-i\pi/4} f_3^2(\text{i}r)f_1(\text{i}r)}{f_2^2(\text{i}r)f_4(\text{i}r)} - \frac{e^{i\pi/4} f_3^2(\text{i}r)f_1(\text{i}r)}{f_2^2(\text{i}r)f_3(\text{i}r)} \right]$$

where $r = e^{-\pi s}$. Notice that the crucial phases in Eq. (4.9) are a direct consequence of the phases appearing in the modular transformations (4.7). The total open string amplitude is then

$$\mathcal{A}_{\text{open}} = \mathcal{A} + \mathcal{M} + \mathcal{M}^*$$

which can be written as follows

$$\mathcal{A}_{\text{open}} = 2 V (8\pi^2 \alpha')^{-1/2} \int_0^\infty \frac{ds}{2s} s^{-1/2} \left( Z_{\text{NS-}}^I(r) + Z_{\text{NS+}}^I(r) - Z_{\text{R}}^I(r) \right)$$

where the various terms in the integrand are

$$Z_{\text{NS-}}^I(r) = \frac{f_3^2(r) + f_4^2(r)}{4f_1^2(r)} - i 2^\frac{3}{2} \left( \frac{f_3^2(\text{i}r)f_1(\text{i}r)}{4f_2^2(\text{i}r)f_4(\text{i}r)} + \frac{f_4^2(\text{i}r)f_1(\text{i}r)}{4f_2^2(\text{i}r)f_3(\text{i}r)} \right),$$

$$Z_{\text{NS+}}^I(r) = \frac{f_3^2(r) - f_4^2(r)}{4f_1^2(r)} + 2^\frac{3}{2} \left( \frac{f_3^2(\text{i}r)f_1(\text{i}r)}{4f_2^2(\text{i}r)f_4(\text{i}r)} - \frac{f_4^2(\text{i}r)f_1(\text{i}r)}{4f_2^2(\text{i}r)f_3(\text{i}r)} \right),$$

and

$$Z_{\text{R}}^I(r) = \frac{f_3^2(r)}{2f_1^2(r)}. $$
Notice that both $Z_{\text{NS}+}(r)$ and $Z_{\text{NS}-}(r)$ are real. The reason to introduce these quantities is that they can be interpreted as the partition functions in the various sectors of the open strings suspended between two Type I D-particles. To see this, we rely on the analysis of Ref. [12] where the rules for the interactions of these open strings are specified and shown to be consistent. Let us briefly recall the main points of this analysis that are relevant for our purposes. In the NS sector, the world-sheet fermion number $F$ acts in the usual manner both on the oscillators and on the Fock vacuum which is taken to be odd under $(-1)^F$. The world-sheet parity $\Omega$ has the following action on the various string oscillators (see for example Ref. [33])

$$\alpha_n^\mu \rightarrow \pm e^{i\pi n} \alpha_n^\mu, \quad \psi_r^\mu \rightarrow \pm e^{i\pi r} \psi_r^\mu$$

(4.15)

where the upper (lower) sign holds for NN (DD) directions. The action of $\Omega$ on the Fock vacuum in the odd $(-1)^F$ NS sector is taken to be

$$\Omega|0\rangle^{(odd)} = -|0\rangle^{(odd)}$$

(4.16)

while for the even $(-1)^F$ NS sector, where the states contain an odd number of fermionic oscillators under which $\Omega$ acts as $\pm i$, it is taken to be

$$\Omega|0\rangle^{(even)} = -i |0\rangle^{(even)}$$

(4.17)

According to these rules, it is not difficult to show that the expressions in Eqs. (4.12) and (4.13) may be reinterpreted as

$$Z_{\text{NS}+}^I(r) = \text{Tr}_{\text{NS}} \left( r^{(2L_0-1)} \frac{1 \pm (-1)^F}{2} \frac{1 + \Omega}{2} \right),$$

(4.18)

thus proving our previous statement. Similarly, in the R sector one can show that

$$Z_{\text{R}}^I(r) = \frac{1}{2} \text{Tr}_R \left( r^{2L_0} \frac{1 + \Omega}{2} \right),$$

(4.19)

where in the last step we have used the fact that $\Omega$ acts as a product of $\Gamma$ matrices on the spinor R vacuum, so that $\text{Tr}_R \left( r^{2L_0} \Omega \right)$ vanishes due to the presence of fermionic zero-modes.

Using these results, we can conclude that the total amplitude (4.11) is

$$A_{\text{open}} = 2 V \langle 8\pi^2 \alpha' \rangle^{1/2} \int_0^\infty \frac{ds}{2s} s^{-1/2} \left[ \text{Tr}_{\text{NS}} \left( r^{(2L_0-1)} \frac{1 + \Omega}{2} \right) - \text{Tr}_R \left( r^{2L_0} \frac{1 + \Omega}{2} \right) \right],$$

(4.20)

which is the one-loop vacuum energy of unoriented open strings without any GSO projection. Despite this fact, there is no instability in the system because the $\Omega$ projection removes the tachyon from the spectrum. This can be seen from the
explicit expression of $Z_{NS-}^I(r)$ corresponding to the odd $(-1)^F$ NS sector to which the tachyon belongs. Indeed, by expanding Eq. (4.12) in powers of $r$, one finds that

$$Z_{NS-}^I(r) = O(r)$$

which shows that the coefficient of the term $r^{-1}$ associated with the tachyon is vanishing. Notice that for this compensation to occur, it is crucial that the tension of the D-particle is fixed at the precise value $\sqrt{2} T_0$. Therefore, this analysis provides a further check of the extra factor of $\sqrt{2}$ found in Ref. \[7\].

The massless states in the spectrum of the open strings living on the Type I D-particle account for its degeneracy. To count such zero-modes, we can use the explicit expressions for the partition functions $Z_{NS}^I$ and $Z_{R}^I$ that we have previously derived. Eq. (4.21) shows that there are no massless states in the $(-1)^F$ odd part of the NS sector. On the contrary, the $(-1)^F$ even part of the NS sector contains nine bosonic zero-modes, as we can see by expanding Eq. (4.22) in powers of $r$:

$$Z_{NS+}^I(r) = 9 + O(r)$$

These nine massless modes correspond to the freedom of translating the D-particle along its nine transverse directions. Finally, from Eq. (4.14) we see that the R sector contains eight non-chiral fermionic zero-modes which upon quantization lead to a $2^8 = 256$ degeneracy for the D-particle, as it should be expected for a non-BPS multiplet of the $\mathcal{N}=1$ supersymmetry algebra in ten dimensions.

We conclude by mentioning that the D-particle is a completely stable configuration of the Type I theory because no tachyons develop in the spectrum of the open strings stretching between the D-particle and the thirty-two D9-branes which form the background. In the boundary state formalism, this can be checked by evaluating the amplitude $\langle \tilde{B} | D | D9 \rangle$ and performing the modular transformation to obtain the open string channel. The absence of these tachyons is also clear from the evaluation of the intercept for the open strings in the NS sector which is $a_{NS} = \frac{1}{2} - \frac{\nu}{8}$ where $\nu$ is the number of mixed Dirichlet-Neumann directions. In the case at hand, $\nu = 9$ and the intercept $a_{NS}$ is strictly negative so that the NS sector contains only massive modes. As usual, there are no problems from the R sector which contains both massless and massive modes, the former accounting for the $\text{SO}(32)$ spinor representation carried by the non-BPS D-particle.

5 Stable Dp-branes in the Type I theory

In the previous sections we have shown that in the Type IIB theory there is a non-supersymmetric D-particle whose boundary state $|\tilde{B}\rangle$ does not have any R-R
part. This unstable configuration of Type IIB becomes stable if one performs the $\Omega$ projection to obtain the Type I theory. In this section we want to see whether this construction can be generalized to investigate the possibility that other non-BPS but stable $D_p$-branes exist in Type I.

In order to cancel the R-R part in a boundary state the obvious thing to do is to consider a superposition of a $p$-brane with an anti $p$-brane (or $\bar{p}$-brane for short), and thus consider the boundary state

$$|\tilde{B}_p\rangle = |D_p\rangle + |D_{\bar{p}}\rangle$$

(5.1)

which clearly has only the NS-NS component. Since we are in a Type IIB context, $p$ has to be odd. As is well known \[9\], this configuration is unstable due to the presence of tachyons in the open strings stretching between the brane and the anti brane. To see whether or not these tachyons disappear in the Type I theory, we have to study their behavior under the world-sheet parity $\Omega$. To this end, we first recall that $\Omega$ does not change the R-R charge of a $D_p$-brane if $p = 1, 5, 9$, whereas it reverses its sign if $p = -1, 3, 7$. Thus we have the following rules

$$\Omega(p) = p \quad , \quad \Omega(\bar{p}) = \bar{p}$$

(5.2)

for $p = 1, 5, 9$, and

$$\Omega(p) = \bar{p} \quad , \quad \Omega(\bar{p}) = p$$

(5.3)

for $p = -1, 3, 7$. The tachyons we are concerned about, are $(-1)^F$ odd states in the $p-\bar{p}$ and $\bar{p}-p$ sectors of the open strings living on the brane world-volume. The vacuum states out of which they are constructed are therefore

$$|0\rangle_{p\bar{p}} = |0\rangle \otimes \Lambda_{p\bar{p}} \quad \text{and} \quad |0\rangle_{\bar{p}p} = |0\rangle \otimes \Lambda_{\bar{p}p}$$

(5.4)

where $|0\rangle$ is the NS Fock vacuum in the $-1$ superghost picture, and $\Lambda_{p\bar{p}}$ and $\Lambda_{\bar{p}p}$ are the Chan-Paton factors which label all states of the $p-\bar{p}$ and $\bar{p}-p$ sectors respectively. The operator $\Omega$ acts on the Fock vacuum in the usual manner, i.e. $\Omega |0\rangle = -i |0\rangle$. The action of $\Omega$ on the Chan-Paton factors is instead more subtle and is given by

$$\Omega \Lambda_{p\bar{p}} \Omega^{-1} = \omega_p \Lambda_{\Omega(\bar{p})\Omega(p)}$$

(5.5)

where $\omega_p$ is a suitable phase. From Eq. (5.2), we see that for $p = 1, 5, 9$, $\Omega$ maps $\Lambda_{p\bar{p}}$ into $\Lambda_{\bar{p}p}$ and vice versa. Since $\Omega$ relates the $p-\bar{p}$ and the $\bar{p}-p$ sectors, it cannot be used to remove the tachyons, and thus we conclude that for $p = 1, 5, 9$, the $p-\bar{p}$ system described by the boundary state (5.1) does not yield any stable configuration.

Things are different if $p = -1, 3, 7$. In this case, Eq. (5.3) shows that $\Omega$ does not exchange $\Lambda_{p\bar{p}}$ and $\Lambda_{\bar{p}p}$; furthermore using the arguments of Refs. \[33\], one can prove
that the intrinsic phase of Eq. (5.5) is \( \omega_p = (-i) \frac{p^2}{2} \), so that the states \(|0\rangle_{pp}\) and \(|0\rangle_{\bar{pp}}\) are eigenstates of \( \Omega \) with eigenvalues \((-i) \frac{11-p^2}{2}\). Since \( \Omega^2 = 1 \), it is conceivable that the \( \Omega \) projection can eliminate the tachyons and stabilize the \( p-\bar{p} \) systems when \( p = -1, 3, 7 \) [23].

As we have seen in Section 3, there is a less obvious way to remove the R-R part of a boundary state, namely by considering the superposition of a brane and an anti-brane in the presence of a \( \mathbb{Z}_2 \) Wilson line [7]. The boundary state corresponding to this configuration is

\[
|Dp\rangle + |D\bar{p}'\rangle
\]

(5.6)

where \( p \) is odd, and the \( ' \) denotes the Wilson line on the anti brane. As we have seen, the R-R part of this superposition does not trivially vanish, but after tachyon condensation it does. The resulting configuration is an unstable D\((p-1)\)-brane of the Type IIB theory whose boundary state is related to the combination Eq. (5.6) by the discrete transformation \( \mathcal{T} \) described in Section 3 and contains only a NS-NS component. The instability of these even D-branes is due to the presence of tachyons in the spectrum of the open strings living in their world-volume. For simplicity, we refer to these strings as the \( p-p \) strings (where now \( p \) is even!). The tachyons are then in the \((-1)^F \) odd part of the spectrum of the \( p-p \) strings, and are constructed from a vacuum which we denote by \(|0\rangle_{pp}^{(odd)}\). To decide about the fate of such tachyons in the Type I theory, we have to define the action of \( \Omega \) on the vacuum, and hence the operator \((1 + \Omega)/2\) cannot be used to truncate the spectrum and remove the tachyons. On the contrary, there are no obstructions in performing the \( \Omega \) projection in the other cases, i.e. \( p = 0, 4, 8 \).

This analysis leads us to consider only the Type IIB non-BPS \( Dp \)-branes with \( p = -1, 0, 3, 4, 7, 8 \), since these are the only ones that in principle can become stable in Type I. Correspondingly, for these values of \( p \) we consider GSO projected boundary states \(|\tilde{Bp}\rangle\) that contain only a NS-NS part and are given by

\[
|\tilde{Bp}\rangle = \frac{1}{2} \left( |\tilde{Bp}, +\rangle - |\tilde{Bp}, -\rangle \right)
\]

(5.8)

where

\[
|\tilde{Bp}, \pm\rangle = \frac{\mu_p T_p}{2} |D_{\text{mat}}, \pm\rangle_{\text{NS}}
\]

(5.9)

in the notation of Section 2. In this last equation we have introduced a positive parameter \( \mu_p \) to renormalize the brane tension, which will be fixed later by imposing
the stability condition, i.e. by requiring the cancellation of tachyons in the Type I theory.

We now proceed as in Section 4: we first add to the boundary state \(|\widetilde{Bp}\rangle\) the crosscap state \(|C\rangle\)

\[
|\widetilde{Dp}\rangle = \frac{1}{\sqrt{2}} \left[|\widetilde{Bp}\rangle + |C\rangle\right],
\]

and then study the interaction amplitude corresponding to the exchange of closed strings between two \(|\widetilde{Dp}\rangle\) states. For our purposes, we consider only the contributions with open string poles, namely the cylinder amplitude and the Möbius strip amplitude given by

\[
A = \frac{1}{2} \langle \widetilde{Bp}|D|\widetilde{Bp}\rangle \quad \text{and} \quad M = \frac{1}{2} \langle \widetilde{Bp}|D|C\rangle.
\]

Using the explicit expressions of the boundary and crosscap states, after standard manipulations, we find

\[
A = \frac{\mu_p^2}{2} \frac{V_{p+1}}{2\pi} (8\pi^2 \alpha')^{-\frac{p+1}{2}} \int_0^\infty dt \left(\frac{\pi}{t}\right)^\nu \frac{f_3^8(q) - f_4^8(q)}{f_1^8(q)}
\]

\[ (5.12) \]

and

\[
M = 2^4 \mu_p \frac{V_{p+1}}{2\pi} (8\pi^2 \alpha')^{-\frac{p+1}{2}} \int_0^\infty dt \left[ f_{3}^{8+\nu}(i q) f_{3}^{9}(i q) - f_{3}^{8-\nu}(i q) f_{4}^{9}(i q) \right]
\]

\[ (5.13) \]

where \(q = e^{-t}\) and \(\nu = 9 - p\). Note that if \(p = 0\) and \(\mu_0 = \sqrt{2}\) these expressions reduce to the amplitudes \((4.5)\) and \((4.6)\).

To obtain the open string channel we perform the modular transformations \(t \rightarrow \pi/s\) in \(A\) and \(t \rightarrow \pi/4s\) in \(M\), and after using the modular properties of the functions \(f_k\) (see e.g. Eq. \((1.7)\)), we find that Eqs. \((5.12)\) and \((5.13)\) become respectively

\[
A = \frac{\mu_p^2}{2} \frac{V_{p+1}}{2\pi} (8\pi^2 \alpha')^{-\frac{p+1}{2}} \int_0^\infty ds \left(\frac{\pi}{s}\right)^\nu \frac{f_3^8(r) - f_4^8(r)}{f_1^8(r)}
\]

\[ (5.14) \]

and

\[
M = 2^{\frac{\pi}{2}} \mu_p V_{p+1} (8\pi^2 \alpha')^{-\frac{p+1}{2}} \int_0^\infty ds \left(\frac{\pi}{s}\right)^\nu \left[ e^{-\nu \pi/4} \frac{f_{3}^{8+\nu}(i r) f_{4}^{9}(i r)}{f_{3}^{8-\nu}(i r) f_{2}^{9}(i r)} \right.
\]

\[
\left. -e^{\nu \pi/4} \frac{f_{3}^{8+\nu}(i r) f_{4}^{9}(i r)}{f_{1}^{8-\nu}(i r) f_{2}^{9}(i r)} \right]
\]

\[ (5.15) \]

where \(r = e^{-\pi s}\).
The spectrum of the open strings living on the world-volume of these Type I D-branes can be analyzed by expanding the total amplitude \( A_{\text{open}} = A + \mathcal{M} + \mathcal{M}^* \) in powers of \( r \). The leading term in this expansion is

\[
A_{\text{open}} \sim \int_0^\infty \frac{ds}{2s} s^{\frac{p+1}{2}} \left[ \mu_p^2 - 2\mu_p \sin\left(\frac{\pi}{4} \nu\right) \right] r^{-1} .
\] (5.16)

The \( r^{-1} \) behavior of the integrand signals the presence of tachyons in the spectrum; therefore, in order not to have them, we must require that

\[
\mu_p = 2 \sin\left(\frac{\pi}{4} \nu\right) .
\] (5.17)

Since \( \nu = 9 - p \) and \( \mu_p \) has to be positive, there is no solution if \( p = 3, 4 \); in the other cases we find

| \( p \) | \( \mu_p \) |
|---|---|
| -1 | 2 |
| 0 | \( \sqrt{2} \) |
| 7 | 2 |
| 8 | \( \sqrt{2} \) |

(5.18)

From this table we see that there exist two even non-BPS but stable Dp-branes: the D-particle and the D8-brane \[23\]. Both of them have a tension that is a factor of \( \sqrt{2} \) bigger than the corresponding branes of the Type IIA theory. This is in agreement with the result of the explicit construction presented in Sections 3 and 4. Moreover, there exist two odd non-BPS but stable Dp-branes: the D-instanton and the D7-brane \[23\]. Their tension is twice the one of the corresponding Type IIB branes, in accordance with the fact that they can be simply interpreted as the superposition of a brane with an anti brane, so that the R-R part of the boundary state cancels while the NS-NS part doubles.

It is interesting to see what happens in this construction if one performs the \( \Omega \) projection by using a symplectic crosscap state instead of the orthogonal one considered so far. This would correspond to quantize the thirty-two D9-branes of the background with symplectic rather than orthogonal Chan-Paton factors, a procedure which is known not to be fully consistent but which can nevertheless be considered up to some extent \[23\]. In this case everything proceeds as before, except that the Möbius strip contributions have a different sign. Thus the tachyon cancellation condition (5.17) gets replaced by

\[
\mu_p = -2 \sin\left(\frac{\pi}{4} \nu\right) .
\] (5.19)

Since \( \mu_p \) has to be positive, this equation has solution only if \( p = 3, 4 \), in which case
we get

\[
\begin{array}{cc}
    p & \mu_p \\
    3 & 2 \\
    4 & \sqrt{2}
\end{array}
\]  

(5.20)

The classification of the stable non-BPS D-branes of Type I based on tables (5.18) and (5.20) is in complete agreement with the results of Refs. [23, 24, 25] derived from the K-theory of space-time.

We end by observing that actually only the D-instanton and the D-particle are fully stable configurations of the Type I theory. In fact, in all other cases there are tachyons that develop in the spectrum of the open strings ending on the thirty-two D9-branes that form the Type I background. It would be very interesting to study and analyze in more detail the properties and the interactions of these non-BPS but stable D-branes. We hope that the boundary state formalism discussed in this paper might be useful for this purpose.

Acknowledgements

We would like to thank P. Di Vecchia, I. Pesando and R. Russo for many useful discussions.

Appendix A

In this appendix we compute the partition function of the open strings living on the world-volume of the D-string – anti D-string pair defined in Section 3 at the critical radius \( R_c = \sqrt{\alpha'}/2 \) and for a generic value \( \theta \) of the tachyon v.e.v. We present our calculation in the Type IIB theory. This open string partition function can be obtained by performing a modular transformation \( t \to 1/t \) on the amplitudes (3.27) and (3.28) that we computed using the boundary state formalism. As usual, in going from the open to the closed string channel we obtain the following identification of the various sectors: the NS sector of the open string comes from the NS-NS sector of the closed string, the NS(−)\( F \) from the R-R and the R from the NS-NS(−)\( F \).

Performing the modular transformation on the sum of Eqs. (3.27) and (3.28), after
standard manipulations we get

\[ \mathcal{A}(\theta) = 2V (8\pi^2 \alpha')^{-1/2} \int_0^\infty \frac{ds}{2s} s^{-1/2} \left( Z_{NS}(r; \theta) - Z_R(r; \theta) \right) \]  
(A.1)

where

\[ Z_{NS}(r; \theta) = \sum_n r^{4n^2} \frac{f_3^8(r) - f_3^8(r)}{2f_1^8(r)} + \sum_n r^{4(n+\frac{1}{2})^2} \frac{f_4^8(r) + f_4^8(r)}{2f_1^8(r)} \]  
(A.2)

and

\[ Z_R(r; \theta) = \sqrt{2} \frac{f_3^8(r) f_3(r)}{f_1^8(r) f_4(r)} \]  
(A.3)

with \( r = e^{-\pi s} \). Notice that \( Z_R(r; \theta) \) is actually independent of \( \theta \). Eqs. (A.2) and (A.3) are precisely the open string partition functions in the NS and R sectors that can be evaluated from the rules given in Ref. [7]. In particular, the R sector partition function (A.3) is obtained using as degrees of freedom for the compact direction the fields \((\phi, \eta)\) defined in Section 3. In this representation, the contribution of the compactified direction is that of a boson with mixed Neumann-Dirichlet boundary conditions (yielding the \( f_4 \) in the denominator of Eq. (A.3)) and of a NS fermion (yielding the \( f_3 \) in the numerator). Being independent of \( \theta \) this partition function can also be computed at \( \theta = 0 \) using the original \((X, \psi)\) degrees of freedom. In this representation we have the contribution of a compactified boson and of a Ramond fermion so that

\[ Z_R(r; \theta) = \sum_n r^{n^2} \frac{f_3^8(r)}{f_1^8(r)} \]  
(A.4)

The equality between Eqs. (A.3) and (A.4) is ensured by the identities (3.31).

If we set \( \theta = 1 \), we observe a remarkable simplification in the NS part of the partition function which becomes

\[ Z_{NS}(r; \theta = 1) = \sum_n r^{n^2} \frac{f_3^8(r)}{f_1^8(r)} \]  
(A.5)

Eqs. (A.5) and (A.4) are the partition functions of an open string in the NS and R sectors without any GSO projection.
Appendix B

In this appendix we show that the boundary state \( |\tilde{B},+\rangle \) given in Eq. (3.33) is equivalent to the boundary state \( |B(\theta),+\rangle_{\text{NS}} \) given in Eq. (3.17) for \( \theta = 1 \). To prove this equivalence we consider a few closed string states and check that they have the same overlap with both \( |\tilde{B},+\rangle \) and \( |B(\theta = 1),+\rangle_{\text{NS}} \). Since the difference between these two boundary states is only in the compact direction, in the following we will focus only on its contribution and understand all the rest.

We begin with the following state

\[
|\chi\rangle = |n = 1, w = 0\rangle + |n = -1, w = 0\rangle \quad (B.1)
\]

which is created by the vertex operator

\[
W_\chi(z, \bar{z}) = e^{i\sqrt{2\alpha'}[\phi_L(z)+\phi_R(\bar{z})]} + e^{-i\sqrt{2\alpha'}[\phi_L(z)+\phi_R(\bar{z})]} . \quad (B.2)
\]

When we saturate the boundary state \( |\tilde{B},+\rangle \) with \( |\chi\rangle \), we obtain

\[
\langle \chi | \tilde{B}, + \rangle = 2N \Phi \quad (B.3)
\]

where we have used Eq. (2.19) and absorbed in \( N \) all factors coming from the normalization of the boundary state and from the contributions of non-compact directions.

To compute the overlap between \( |\chi\rangle \) and \( |B(\theta),+\rangle_{\text{NS}} \), we have first to find the representation of \( |\chi\rangle \) in terms of the fields \((\phi, \eta)\). To this end, we use the bosonization formulas of Section 3 to write the vertex operator \( W_\chi \) as follows

\[
W_\chi(z, \bar{z}) = i \eta(z) \bar{\eta}(\bar{z}) - \frac{1}{2} \left\{ e^{\frac{i}{\sqrt{2\alpha'}}[\phi_L(z)+\phi_R(\bar{z})]} + e^{-\frac{i}{\sqrt{2\alpha'}}[\phi_L(z)+\phi_R(\bar{z})]} \right. \\
+ e^{\frac{i}{\sqrt{2\alpha'}}[\phi_L(z)-\phi_R(\bar{z})]} + e^{-\frac{i}{\sqrt{2\alpha'}}[\phi_L(z)-\phi_R(\bar{z})]} \right\} . \quad (B.4)
\]

From this expression we see that the \((\phi, \eta)\)-representation of \( |\chi\rangle \) is

\[
|\chi\rangle = i \eta_{-1/2} \bar{\eta}_{-1/2} |n_\phi = 0, w_\phi = 0\rangle - \frac{1}{2} \left[ |n_\phi = 1, w_\phi = 0\rangle + |n_\phi = 0, w_\phi = 2\rangle \\
+ |n_\phi = 0, w_\phi = -2\rangle + |n_\phi = -1, w_\phi = 0\rangle \right]. \quad (B.5)
\]

When we saturate \( |B(\theta),+\rangle_{\text{NS}} \) with this state, we get

\[
\langle \chi | B(\theta), + \rangle_{\text{NS}} = (1 - \cos(\pi\theta)) N \Phi \quad (B.6)
\]

which agrees with Eq. (B.3) if \( \theta = 1 \). Notice that for \( \theta = 0 \), this overlap vanishes; this is to be expected since \( |B(\theta = 0),+\rangle_{\text{NS}} \) is equivalent to the original boundary state \( |B, +\rangle_{\text{NS}} \) of Eq. (3.3) with which \( |\chi\rangle \) has zero overlap.
Let us now consider the following state
\[ |\rho\rangle = \alpha_{-1} \bar{\alpha}_{-1} |n = 0, w = 0 \rangle \] (B.7)
which corresponds to the vertex operator
\[ W_\rho(z, \bar{z}) = -\frac{1}{2\alpha'} \partial X_L(z) \bar{\partial} X_R(\bar{z}) \] . (B.8)
Saturating \(|\rho\rangle\) against the boundary state \(|\bar{B}, +\rangle\) we get
\[ \langle \rho | \bar{B}, + \rangle = N \Phi \] . (B.9)

After bosonization the vertex \(W_\rho\) becomes
\[ W_\sigma(z, \bar{z}) = -\frac{i}{2} \eta(z) \bar{\eta}(\bar{z}) \left\{ e^{\frac{i}{\sqrt{2\alpha'}}[\phi_L(z) + \phi_R(\bar{z})]} + e^{\frac{i}{\sqrt{2\alpha'}}[\phi_L(z) - \phi_R(\bar{z})]} \right\} , \] (B.10)
and the corresponding state is
\[ |\rho\rangle = -\frac{i}{2} \eta_{-1/2} \bar{\eta}_{-1/2} \left[ |n_\phi = 1, w_\phi = 0 \rangle + |n_\phi = 0, w_\phi = 2 \rangle + |n_\phi = 0, w_\phi = -2 \rangle + |n_\phi = -1, w_\phi = 0 \rangle \right] . \] (B.11)

Computing its overlap with the boundary state \(|B(\theta), +\rangle_{NS}\), we get
\[ \langle \rho | B(\theta), + \rangle_{NS} = -\cos(\pi \theta) N \Phi \] (B.12)
which again agrees with Eq. (B.9) for \(\theta = 1\).

Our last example concerns the state
\[ |\sigma\rangle = i \psi_{-1/2} \bar{\psi}_{-1/2} |n = 0, w = 0 \rangle \] (B.13)
which is created by the following vertex operator
\[ W_\sigma(z, \bar{z}) = i \psi(z) \bar{\psi}(\bar{z}) \] . (B.14)

Its overlap with the boundary state \(|\bar{B}, +\rangle\) is
\[ \langle \sigma | \bar{B}, + \rangle = -N \Phi \] (B.15)

Using the bosonization rules of Section 3, one easily sees that the vertex \(W_\sigma\) written in terms of \(\phi\) and \(\eta\) is
\[ W_\sigma(z, \bar{z}) = -\frac{1}{2} \left\{ e^{\frac{i}{\sqrt{2\alpha'}}[\phi_L(z) + \phi_R(\bar{z})]} - e^{\frac{i}{\sqrt{2\alpha'}}[\phi_L(z) - \phi_R(\bar{z})]} \right. \]
\[ -e^{-\frac{i}{\sqrt{2\alpha'}}[\phi_L(z) - \phi_R(\bar{z})]} + e^{-\frac{i}{\sqrt{2\alpha'}}[\phi_L(z) + \phi_R(\bar{z})]} \left\} , \] (B.16)
so that the corresponding state is

\[ |\sigma\rangle = -\frac{1}{2} \left[ |n_\phi = 1, w_\phi = 0\rangle - |n_\phi = 0, w_\phi = 2\rangle - |n_\phi = 0, w_\phi = -2\rangle + |n_\phi = -1, w_\phi = 0\rangle \right]. \tag{B.17} \]

When we compute its overlap with the boundary state \(|B(\theta), +\rangle_{NS}\) we get

\[ \langle \sigma | B(\theta), + \rangle_{NS} = \cos(\pi \theta) N \Phi \tag{B.18} \]

which once more coincides with Eq. (B.15) if \(\theta = 1\).

All these examples provide convincing evidence that indeed the boundary state \(|\tilde{B}, +\rangle\) defined in Eq. (3.33) is equivalent to the boundary state \(|B(\theta), +\rangle_{NS}\) defined in Eq. (3.17) for \(\theta = 1\). Furthermore, as a byproduct of these calculations, we can also verify that \(|B(\theta = 0), +\rangle_{NS}\) and the original boundary state \(|B, +\rangle_{NS}\) of Eq. (3.3) are equivalent as they should.

References

[1] J. Polchinski and E. Witten, Nucl. Phys. B460 (1996) 525, hep-th/9510169.
[2] A. Sen, Nucl. Phys. [Proc. Suppl.] 58 (1997) 5, hep-th/9609176.
[3] A. Dabholkar, Phys. Lett. B357 (1995) 307, hep-th/9511053.
[4] E. Gava, J.F. Morales, K.S. Narain and G. Thompson, Nucl. Phys. B528 (1998) 95, hep-th/9801128.
[5] A. Tseytlin, Phys. Lett. B367 (1996) 84, hep-th/9510173. A. Tseytlin, Nucl. Phys. B467 (1996) 383, hep-th/9512081.
[6] C. Bachas and E. Kiritsis, Nucl. Phys. [Proc. Suppl.] 55B (1997) 194, hep-th/9611205. C. Bachas, C. Fabre, E. Kiritsis, N. Obers and P. Vanhove, Nucl. Phys. B509 (1998) 33, hep-th/9707126. E. Kiritsis and N. Obers, JHEP 10 (1997) 004, hep-th/9709058.
[7] A. Sen, JHEP 9809 (1998) 023, hep-th/9808141.
[8] A. Sen, Non-BPS states and branes in string theory, hep-th/9904207. A. Lerda and R. Russo, Stable non-BPS states in string theory: a pedagogical review, hep-th/9905006.
[9] T. Banks and L. Susskind, *Brane - Anti-Brane Forces*, hep-th/9511194; M.B. Green and M. Gutperle, Nucl. Phys. B476 (1996) 484, hep-th/9604091.

[10] A. Sen, JHEP 9806 (1998) 007, hep-th/9803194; A. Sen, JHEP 9808 (1998) 012, hep-th/9805170.

[11] A. Sen, JHEP 9808 (1998) 010, hep-th/9805019.

[12] A. Sen, JHEP 9810 (1998) 021, hep-th/9809111.

[13] A. Sen, JHEP 9812 (1998) 021, hep-th/9812031.

[14] A. Sagnotti in *Non-Perturbative Quantum Field Theory*, eds G. Mack et al. (Pergamon Press, 1988) 521; N. Ishibashi and T. Onogi, Nucl. Phys. B310 (1989) 239; G. Pradisi and A. Sagnotti, Phys. Lett. 216B (1989) 59; P. Horava, Nucl. Phys. B327 (1989), 461.

[15] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, Nucl. Phys. B308 (1988) 221.

[16] J. Polchinski and Y. Cai, Nucl. Phys. B286 (1988) 91.

[17] P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda, R. Russo, Nucl. Phys. B507 (1997) 259, hep-th/9707068.

[18] M. Billó, D. Cangemi and P. Di Vecchia, Phys. Lett. B400 (1997) 63, hep-th/9701190.

[19] M. Billó, P. Di Vecchia, M. Frau, A. Lerda, I. Pesando, R. Russo and S. Sciuto, Nucl. Phys. B526 (1998) 199, hep-th/9802088.

[20] M. Billó, P. Di Vecchia, M. Frau, A. Lerda, R. Russo and S. Sciuto, Mod. Phys. Lett. A13 (1998) 2977, hep-th/9805091.

[21] O. Bergman and M.R. Gaberdiel, Phys. Lett. B441 (1998) 133, hep-th/9806155.

[22] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, Nucl. Phys. B293 (1987) 83.

[23] E. Witten, *D-branes and K-Theory*, hep-th/9810188.

[24] P. Horava, *Type IIA D-Branes, K-Theory, and Matrix Theory*, hep-th/9812139.
[25] S. Gukov, *K-Theory, Reality, and Orientifolds*, hep-th/9901042.

[26] O. Bergman and M.R. Gaberdiel, *Non-BPS States in Heterotic - Type IIA Duality*, hep-th/9901014.

[27] P. Yi, *Membranes from Five-Branes and Fundamental String from Dp-Branes*, hep-th/9901159.

[28] O. Bergman, E. Gimon and P. Horava, *Brane Transfer Operation and T-Duality of Non-BPS States*, hep-th/9902160.

[29] M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, Phys. Lett. **B400** (1997) 52, hep-th/9702037.

[30] J. Polchinski, *TASI lectures on D-branes*, hep-th/9611050.

[31] A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. **B322** (1988) 167.

[32] M.B. Green, Phys. Lett. **B266** (1991) 325.

[33] E. Gimon and J. Polchinski, Phys. Rev. **D54**(1996) 1667 hep-th/9601038.