A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo-Fabrizio derivative

Dumitru Baleanu1,2*, Asef Mousalou3 and Shahram Rezapour3

Abstract
We present a new method to investigate some fractional integro-differential equations involving the Caputo-Fabrizio derivation and we prove the existence of approximate solutions for these problems. We provide three examples to illustrate our main results. By checking those, one gets the possibility of using some discontinuous mappings as coefficients in the fractional integro-differential equations.

Keywords: approximate solution; Caputo-Fabrizio derivative; fractional integro-differential equation; generalized α-contractive map

1 Introduction
The fractional calculus has an old history and several fractional derivations where defined but the most utilized are Caputo and Riemann–Liouville derivations [1–5]. In 2015, Caputo and Fabrizio defined a new fractional derivation without singular kernel [6]. Immediately, Losada and Nieto wrote a paper about properties of the new fractional derivative [7] and several researchers tried to utilize it for solving different equations (see [8–14] and the references therein).

Let \( b > 0, \ u \in H^1(0, b) \) and \( \alpha \in (0, 1) \). The Caputo-Fabrizio fractional derivative of order \( \alpha \) for the function u is defined by

\[
\text{CF} D^\alpha u(t) = \frac{(2–\alpha)M(\alpha)}{2\alpha M(\alpha)} \int_0^t \exp\left(\frac{–\alpha}{1–\alpha}(t–s)\right)u'(s) \, ds,
\]

where \( t \geq 0 \) and \( M(\alpha) \) is normalization constant depending on \( \alpha \) such that \( M(0) = M(1) = 1 \) [6]. Also, Losada and Nieto showed that the fractional integral of order \( \alpha \) for the function u is given by

\[
\text{CF} I^\alpha u(t) = \frac{M(\alpha)}{2\alpha M(\alpha)} \int_0^t \exp\left(\frac{–\alpha}{1–\alpha}(t–s)\right)u(s) \, ds
\]

whenever \( 0 < \alpha < 1 \). They showed that \( M(\alpha) = \frac{2}{\alpha} \) for all \( 0 \leq \alpha \leq 1 \) [7]. Thus, the fractional Caputo-Fabrizio derivative of order \( \alpha \) for the function u is given by

\[
\text{CF} D^\alpha u(t) = \frac{1}{(2–\alpha)M(\alpha)} \int_0^t \exp\left(\frac{–\alpha}{1–\alpha}(t–s)\right)u'(s) \, ds
\]

where \( t \geq 0 \) and \( 0 < \alpha < 1 \). If \( n \geq 1 \) and \( \alpha \in [0, 1] \), then the fractional derivative \( \text{CF} D^{n+\alpha} \) of order \( n + \alpha \) is defined by

\[
\text{CF} D^{n+\alpha} u := \text{CF} D^\alpha (D^n u(t))
\]

We need the following results.

Theorem 1.1 ([6]) Let \( u, v \in H^1(0, 1) \) and \( \alpha \in (0, 1) \). If \( u' = 0 \), then \( \text{CF} D^\alpha D^\beta u(t) = \text{CF} D^{\beta+\alpha} u(t) \), \( \lim_{\alpha \to 0} \text{CF} D^\alpha u(t) = u(t) – u(0) \), \( \lim_{\lambda \to 0} \text{CF} D^\alpha u(t) = u'(t) \) and \( \text{CF} D^\alpha (\lambda u(t) + \gamma v(t)) = \lambda \text{CF} D^\alpha u(t) + \gamma \text{CF} D^\alpha v(t) \).

© The Author(s) 2017. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.
Lemma 1.2 ([7]) Let $0 < \alpha < 1$. Then the unique solution for the problem $\alpha D^\alpha u(t) = v(t)$ with boundary condition $u(0) = c$ is given by $u(t) = c + a_\alpha (v(t) - v(0)) + b_\alpha \int_0^t v(s) \, ds$, where $a_\alpha = \frac{2(1-\alpha)}{[2(1-\alpha)M(a)]} = 1 - \alpha$ and $b_\alpha = \frac{2\alpha}{[2(1-\alpha)M(a)]} = \alpha$. Note that $v(0) = 0$ whenever $u(0) = 0$.

To discuss the existence of solutions for most fractional differential equations in analytic methods, the well-known fixed point results such as the Banach contraction principle is used. In fact, the existence of solutions and the existence of fixed points are equivalent. As is well known, there are many fractional differential equations which have no exact solutions. Thus, the researchers utilize numerical methods usually for obtaining an approximation of the exact solutions. We say that $u$ is an approximate solution for fractional integro-differential equation whenever we could obtain a sequence of functions $\{u_n\}_{n \geq 1}$ with $u_n \to u$. We use this notion when we could not obtain the exact solution $u$. This appears usually when you want to investigate the fractional integro-differential equation in a non-complete metric space.

In this manuscript, we prove the existence of approximate solutions analytically for some fractional integro-differential equations involving the Caputo-Fabrizio derivative. In fact, the approximate solution of an equation is equivalent to the approximate fixed point of an appropriate operator. This says that by using numerical methods, one can obtain approximations of the unknown exact solution. We will not check the estimates of the exact solution in our examples because our aim is to show the existence of approximate solutions within the analytical method.

Here, we provide some basic needed notions.

Let $(X, d)$ be a metric space, $F$ a selfmap on $X$, $\alpha : X \times X \to [0, \infty)$ a mapping and $\varepsilon$ a positive number. We say that $F$ is $\alpha$-admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha(Fx, Fy) \geq 1$ [15]. An element $x_0 \in X$ is called $\varepsilon$-fixed point of $F$ whenever $d(Fx_0, x_0) \leq \varepsilon$. We say that $F$ has the approximate fixed point property whenever $F$ has an $\varepsilon$-fixed point for all $\varepsilon > 0$ [15]. Some mappings have approximate fixed points, while they have no fixed points [15]. Denote by $\mathcal{R}$ the set of all continuous mappings $g : [0, \infty)^2 \to [0, \infty)$ satisfying $g(1, 1, 1, 0) = g(1, 1, 1, 0, 2) = h \in (0, 1), g(\mu x_1, \mu x_2, \mu x_3, \mu x_4, \mu x_5) \leq \mu g(x_1, x_2, x_3, x_4, x_5)$ for all $(x_1, x_2, x_3, x_4, x_5) \in [0, \infty)^5$ and $\mu \geq 0$ and also $g(x_1, x_2, x_3, x_4, 0) \leq g(y_1, y_2, y_3, y_4, 0)$ whenever $x_1, \ldots, x_4, y_1, \ldots, y_4 \in [0, \infty)$ with $x_i < y_i$ for $i = 1, 2, 3, 4$ [15]. We say that $F$ is a generalized $\alpha$-contractive mapping whenever there exists $g \in \mathcal{R}$ such that $\alpha(x, y) d(Fx, Fy) \leq g(d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx))$ for all $x, y \in X$ ([15]).

Theorem 1.3 ([15]) Let $(X, d)$ be a metric space, $\alpha : X \times X \to [0, \infty)$ a mapping and $F$ a generalized $\alpha$-contractive and $\alpha$-admissible selfmap on $X$. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geq 1$. Then $F$ has an approximate fixed point.

2 Main results

Now, we are ready to state and prove our main results.

Lemma 2.1 Suppose that $u, v \in H^1(0, 1)$ and there exists a real number $K$ such that

$$|u(t) - v(t)| \leq K$$

for all $t \in [0, 1]$. Then $|\alpha D^\alpha u(t) - \alpha D^\alpha v(t)| \leq \frac{2-\alpha}{(1-\alpha)^2} K$ for all $t \in [0, 1]$. 
Proof. Note that
\[
\begin{align*}
\mathbf{CD}^\alpha u(t) &= \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) u'(s) \, ds \\
&= \frac{1}{1-\alpha} \exp\left(-\frac{\alpha}{1-\alpha}(t)\right) u(0) - \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) u(s) \, ds
\end{align*}
\]
and so
\[
\begin{align*}
\mathbf{CD}^\alpha u(t) - \mathbf{CD}^\alpha v(t) &\leq \frac{1}{1-\alpha} \left| (u(t) - v(t)) \right| + \frac{1}{1-\alpha} \left| \exp\left(-\frac{\alpha}{1-\alpha}(t)\right) \right| \left| u(0) - v(0) \right| \\
&\quad + \frac{\alpha}{(1-\alpha)^2} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) \left| (u(s) - v(s)) \right| \, ds \\
&\leq \frac{2}{1-\alpha} K + \frac{\alpha}{(1-\alpha)^2} K = \frac{2-\alpha}{(1-\alpha)^2} K
\end{align*}
\]
for all \( t \in [0,1] \). Hence, \( \left| \mathbf{CD}^\alpha u(t) - \mathbf{CD}^\alpha v(t) \right| \leq \left( \frac{2-\alpha}{(1-\alpha)^2} \right) K \) for all \( t \in [0,1] \). \qed

If \( u \in H^1(0,1) \) and there exists \( K \geq 0 \) such that \( |u(t)| \leq K \) for all \( t \in [0,1] \), then by using last result we get \( |\mathbf{CD}^\alpha u(t)| \leq \left( \frac{2-\alpha}{(1-\alpha)^2} \right) K \) for all \( t \in [0,1] \). Also by checking the proof of the last result, one can prove the next lemma.

Lemma 2.2 Suppose that \( u, v \in H^1(0,1) \) with \( u(0) = v(0) \) and there exists a real number \( K \) such that \( |u(t) - v(t)| \leq K \) for all \( t \in [0,1] \). Then \( |\mathbf{CD}^\alpha u(t) - \mathbf{CD}^\alpha v(t)| \leq \left( \frac{2-\alpha}{(1-\alpha)^2} \right) K \) for all \( t \in [0,1] \).

Lemma 2.3 Suppose that \( u, v \in C[0,1] \) and there is \( K \geq 0 \) such that \( |u(t) - v(t)| \leq K \) for all \( t \in [0,1] \). Then \( |\mathbf{DF}^\alpha u(t) - \mathbf{DF}^\alpha v(t)| \leq K \) for all \( t \in [0,1] \).

Proof. Note that for each \( t \in [0,1] \) we have
\[
\begin{align*}
\mathbf{DF}^\alpha u(t) - \mathbf{DF}^\alpha v(t) &= a_u (u(t) - v(t)) + b_u \int_0^t (u(s) - v(s)) \, ds \\
&\leq a_u K + b_u K = K,
\end{align*}
\]
where \( a_u \) and \( b_u \) are given in Lemma 1.2. This completes the proof. \qed

If \( u \) is an element of \( C[0,1] \) such that \( |u(t)| \leq K \) for some \( K \geq 0 \) and all \( t \in [0,1] \), then the last result implies that \( |\mathbf{DF}^\alpha u(t)| \leq K \) for all \( t \in [0,1] \).

Lemma 2.4 Let \( b > 0 \) be given and \( 0 \leq \alpha \leq 1 \). If \( u \) is an element of \( H^1(0,b) \) such that \( u(0) = 0, \ u'(0) = 0, \ u' \in H^1(0,b) \) and \( \mathbf{CD}^\alpha u \in H^1(0,b) \), then \( \mathbf{CD}^\alpha (\mathbf{DF}^\alpha u(t)) = \mathbf{CD}^\alpha (\mathbf{DF}^\alpha u(t)) = a_u u'(t) + b_u u(t) \) and \( \mathbf{CD}^\alpha u'(t) = \mathbf{CD}^\alpha u'(t) \) for all \( t \geq 0 \). If \( u'(t) \geq 0 \) for all \( t \geq 0 \), then \( \mathbf{CD}^\alpha u \) is increasing on \([0,b]\). Also, \( \mathbf{DF}^\alpha u \) is decreasing on \([0,b]\) whenever \( u''(t) \leq 0 \) for all \( t \geq 0 \).
Proof. Note that \( \text{CF} D^\alpha \text{CF} D^1 u(t) \) = \( a_\alpha u'(t) + b_\alpha \int_0^t u'(s) \, ds \) = \( a_\alpha u'(t) + b_\alpha u(t) \) and

\[
\text{CF} D^1 \text{CF} D^\alpha u(t) = \text{CF} D^1 \left( a_\alpha u(t) + b_\alpha \int_0^t u(s) \, ds \right)' = a_\alpha u'(t) + b_\alpha \left( \int_0^t u(s) \, ds \right)'
\]

for all \( t \geq 0 \). Also, \( \text{CF} D^\alpha \text{CF} D^\beta u(t) \) = \( \text{CF} D^\beta \text{CF} D^\alpha u(t) \) = \( \text{CF} D^\gamma \text{CF} D^\delta u(t) \) = \( \text{CF} D^\delta \text{CF} D^\gamma u(t) \) for all \( t \geq 0 \). Since \( \text{CF} \lambda(t) = \frac{1}{t} \int_0^t \exp(-\frac{\lambda}{t^2}(t-s))u'(s) \, ds \) for all \( t \geq 0 \), we see that \( \text{CF} \lambda \) is increasing on \([0, b]\) whenever \( u''(t) \geq 0 \) for all \( t \in [0, b] \). Also, \( \text{CF} \lambda \) is decreasing on \([0, b]\) whenever \( u''(t) \leq 0 \) for all \( t \in [0, b] \).

Note that the conditions \( u' \in L^1(0, b) \) and \( \text{CF} \lambda \in H^1(0, b) \) in Lemma 2.4 just impose a unique condition on \( u \). Let \( \gamma, \lambda : [0, 1] \times [0, 1] \to [0, \infty) \) be two continuous maps such that \( \sup_{t \in [0, 1]} \int_0^t \gamma(t, s) \, ds < \infty \) and \( \sup_{t \in [0, 1]} \int_0^t \lambda(t, s) \, ds < \infty \). Consider the maps \( \phi \) and \( \varphi \) defined by \( \phi u(t) = \int_0^t \gamma(t, s)u(s) \, ds \) and \( \varphi u(t) = \int_0^t \lambda(t, s)u(s) \, ds \). Throughout this paper, we put \( \gamma_0 = \sup_{t \in [0, 1]} \int_0^t \gamma(t, s) \, ds \), \( \lambda_0 = \sup_{t \in [0, 1]} \int_0^t \lambda(t, s) \, ds \) and \( \eta(t) \in L^\infty(I) \) with \( \eta^* = \sup_{t \in I} |\eta(t)| \). Here, we investigate the fractional integro-differential problem

\[
\text{CF} D^\alpha \eta(t) = f(t, u(t), (\phi u)(t), (\varphi u)(t))
\]

with boundary condition \( u(0) = 0 \), where \( \alpha \in (0, 1) \).

Theorem 2.5 Let \( \eta(t) \in L^\infty(I) \) and \( f : I \times \mathbb{R}^3 \to \mathbb{R} \) be a continuous function such that

\[
|f(t,x,y,w) - f(t,x',y',w')| \leq \eta(t) \left( |x-x'| + |y-y'| + |w-w'| \right)
\]

for all \( t \in I \) and \( x, y, w, x', y', w' \in \mathbb{R} \). Then the problem (1) with the boundary condition has an approximate solution whenever \( \Delta_1 = \eta^*(1 + \gamma_0 + \lambda_0) < 1 \).

Proof. Consider the space \( H^1 \) endowed with the metric \( d(u, v) = \|u - v\| \), where \( \|u\| = \sup_{t \in I} |u(t)| \). Now, define the selfmap \( F : H^1 \to H^1 \) by

\[
(Fu)(t) = a_\alpha f(t, u(t), (\phi u)(t), (\varphi u)(t)) + b_\alpha \int_0^t f(s, u(s), (\phi u)(s), (\varphi u)(s)) \, ds,
\]

where \( a_\alpha \) and \( b_\alpha \) are given in Lemma 1.2. Note that

\[
(Fu)(t) = a_\alpha \text{CF} D^\alpha \eta(t) + b_\alpha \int_0^t f(s, u(s), (\phi u)(s), (\varphi u)(s)) \, ds
\]

\[
= a_\alpha \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) u'(s) \, ds
\]

\[
+ b_\alpha \int_0^t f(s, u(s), (\phi u)(s), (\varphi u)(s)) \, ds
\]
for all $t$. This shows that $F$ maps $H^1$ into $H^1$. Thus, we have

$$
|F(u)(t) - (Fv)(t)| \\
\leq a_o \left| f(t, u(t), (\phi u)(t), (\varphi u)(t)) - f(t, v(t), (\phi v)(t), (\varphi v)(t)) \right| \\
+ b_o \int_0^t \left| f(s, u(s), (\phi u)(s), (\varphi u)(s)) - f(s, v(s), (\phi v)(s), (\varphi v)(s)) \right| \, ds \\
\leq a_o |\eta(t)| |(u(t) - v(t))| + |(\phi u)(t) - (\phi v)(t)| + |(\varphi u)(t) - (\varphi v)(t)|| \\
+ b_o \int_0^t \left| (u(s) - v(s)) + |(\phi u)(s) - (\phi v)(s)| + |(\varphi u)(s) - (\varphi v)(s))| \right| \eta(s) \, ds \\
\leq \eta^* (1 + \gamma_0 + \lambda_0)[a_o + b_o] \|u - v\| = \eta^* (1 + \gamma_0 + \lambda_0) \|u - v\|
$$

for all $t \in I$ and $u, v \in H^1$. Now, consider $g : [0, \infty)^5 \to [0, \infty)$ and $\alpha : H^1 \times H^1 \to [0, \infty)$ defined by $g(t_1, t_2, t_3, t_4, t_5) = \Delta_1 t_1$ and $\alpha(x, y) = 1$ for all $x, y \in H^1$. One can easily check that $g \in \mathcal{R}$ and $F$ is a generalized $\alpha$-contraction. By using Theorem 1.3, $F$ has an approximate fixed point which is an approximate solution for the problem (1).

Note that $H^1$ with the sup norm is not Banach. Thus, we used a new method for investigation of the problem. Now, we investigate the fractional integro-differential problem

$$
\text{CF} D^\mu u(t) = \mu(\text{CF} D^\beta u(t) + \text{CF} D^\gamma u(t)) \\
+ f(t, u(t), (\phi u)(t), (\varphi u)(t), \text{CF} D^\rho u(t), \text{CF} D^\delta u(t)) \\
(2)
$$

with boundary condition $u(0) = c$, where $\mu \geq 0$ and $\alpha, \beta, \gamma, \theta, \delta \in (0, 1)$ and $c \in \mathbb{R}$.

**Theorem 2.6** Let $\eta(t) \in L^\infty(I)$ and $f : [0, 1] \times \mathbb{R}^5 \to \mathbb{R}$ be a continuous function such that

$$
|f(t, x, y, w, u_1, u_2) - f(t, x', y', w', u_1, u_2)| \\
\leq \eta(t)(|x - x'| + |y - y'| + |w - w'| + |u_1 - u_1| + |u_2 - u_2|)
$$

for all $t \in I$ and $x, y, w, x', y', w', u_1, u_2, v_1, v_2 \in \mathbb{R}$. Then the problem (2) with the boundary condition has an approximate solution whenever $\Delta_2 < 1$, where

$$
\Delta_2 = \eta^* \left( 2 + \gamma_0 + \lambda_0 + \frac{1}{(1 - \delta)^2} \right) + \mu \left( \frac{1}{(1 - \beta)^2} + \frac{1}{(1 - \gamma)^2} \right).
$$

**Proof** Consider the space $H^1$ endowed with the metric $d(u, v) = \|u - v\|$, where $\|u\| = \sup_{t \in I} |u(t)|$. Define the map $F : H^1 \to H^1$ by

$$
(Fu)(t) = u(0) + a_o \left[ \mu(\text{CF} D^\rho u(t) + \text{CF} D^\gamma u(t)) \\
+ f(t, u(t), (\phi u)(t), (\varphi u)(t), \text{CF} D^\rho u(t), \text{CF} D^\gamma u(t)) - \mu(\text{CF} D^\beta u(0) + \text{CF} D^\gamma u(0)) \right] \\
- f(0, u(0), (\phi u)(0), (\varphi u)(0), \text{CF} D^\rho u(0), \text{CF} D^\gamma u(0)) \\
+ b_o \int_0^t \left[ \mu(\text{CF} D^\rho u(s) + \text{CF} D^\gamma u(s)) \\
+ f(s, u(s), (\phi u)(s), (\varphi u)(s), \text{CF} D^\rho u(s), \text{CF} D^\gamma u(s)) \right] \, ds,
$$

where $\Delta_2 < 1$. Then $F$ is a generalized $\alpha$-contraction. By using Theorem 1.3, $F$ has an approximate fixed point which is an approximate solution for the problem (2).
where $a_u$ and $b_u$ are given in Lemma 1.2. By using Lemmas 2.2 and 2.3, we obtain

$$
|Fu(t) - Fv(t)| \leq |u(0) - v(0)| + a_u \left[ \mu |CF D^\delta u(t) - CF D^\delta v(t)| + \frac{1}{(1 - \delta)^2} \left( \frac{1}{(1 - \beta)^2} + \frac{1}{(1 - \gamma)^2} \right) \left( |Au(t)| + |Bu(t)| \right) \right]
$$

for all $u, v \in H^1$ and $t \in I$. Hence,

$$
\|Fu - Fv\| \leq \eta^* \left( 2 + \gamma_0 + \lambda_0 + \frac{1}{(1 - \delta)^2} \right) + \mu \left( \frac{1}{(1 - \beta)^2} + \frac{1}{(1 - \gamma)^2} \right) \|u - v\| + \Delta_2 \|u - v\|
$$

for $u, v \in H^1$. Now, consider the maps $g : [0, \infty)^5 \to [0, \infty)$ and $\alpha : H^1 \times H^1 \to [0, \infty)$ defined by $g(t_1, t_2, t_3, t_4, t_5) = \Delta_2(t_1 + 2t_2)$ and $\alpha(x, y) = 1$ for all $x, y \in H^1$. One can easily see that $g \in \mathcal{R}$ and $F$ is a generalized $\alpha$-contractive map. By using Theorem 1.3, $F$ has an approximate fixed point which is an approximate solution for the problem (2). $\square$

Let $k$ and $h$ be bounded functions on $I = [0, 1]$ and $s$ an integrable bounded function on $I$ with $M_1 = \sup_{t \in I} |k(t)|$, $M_2 = \sup_{t \in I} |s(t)| < \infty$ and $M_3 = \sup_{t \in I} |h(t)| < \infty$. Now, we
investigate the fractional integro-differential problem

\[
\text{CF}^\alpha u(t) = \mu_1 k(t)\text{CF}^\theta u(t) + \mu_2 (\phi s)(t)\text{CF}^\rho (\text{CF}^\alpha u(t)) + f(t, u(t), (\phi u)(t), h(t)\text{CF}^\nu u(t))
\]

with boundary condition \(u(0) = 0\), where \(\mu_1, \mu_1 \geq 0\) and \(\alpha, \beta, \theta, \rho, \nu \in (0, 1)\). Note that the functions \(k, s\) and \(h\) are not necessarily continuous. Since the left side of equation (3) is continuous, the right side so is as the problem (3) is a well-defined equation. For this reason, we supposed continuity of the function \(f\) in Theorems 2.5 and 2.6 where equations (1) and (2) are well defined.

**Theorem 2.7** Let \(\eta(t) \in L^\infty(I)\) and \(f : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}\) be a function such that

\[
|f(t, x, y, w) - f(t, x', y', w')| \leq \eta(t)\left(|x - x'| + |y - y'| + |w - w'|\right)
\]

for all \(t \in I\) and \(x, y, w, v, x', y', w' \in \mathbb{R}\). Then the problem (3) has an approximate solution whenever \(\Delta_3 < 1\), where

\[
\Delta_3 = \left(1 + \frac{1}{(1 - \beta)^2} + \frac{1}{(1 - \theta)^2(1 - \rho)^2} + \frac{1}{(1 - \nu)^2}\right)\left[\mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3)\right].
\]

**Proof** Consider the space \(H^1\) endowed with the metric \(d(u, v) = \|u - v\|\), where

\[
\|u\| = \max_{t \in I}\|u(t)\| + \max_{t \in I}\|\text{CF}^\alpha u(t)\| + \max_{t \in I}\|\text{CF}^\beta (\text{CF}^\alpha u(t))\| + \max_{t \in I}\|\text{CF}^\nu u(t)\|.
\]

Define the map \(F : H^1 \rightarrow H^1\) by

\[
(Fu)(t) = a_u\left[\mu_1 k(t)\text{CF}^\theta (\text{CF}^\alpha u(t)) + \mu_2 (\phi s)(t)\text{CF}^\rho (\text{CF}^\alpha u(t))ight] + f(t, u(t), (\phi u)(t), h(t)\text{CF}^\nu u(t))
+ \int_0^t \left[\mu_1 k(s)\text{CF}^\theta (\text{CF}^\alpha u(s)) + \mu_2 (\phi s)(t)\text{CF}^\rho (\text{CF}^\alpha u(s))ight] + f(t, u(s), (\phi u)(s), h(s)\text{CF}^\nu u(s))] ds
\]

for all \(t \in I\), where \(a_u\) and \(b_u\) introduced in Lemma 1.2. By using Lemma 2.2, we get

\[
\left[\mu_1 k(t)\text{CF}^\theta (\text{CF}^\alpha u(t)) + \mu_2 (\phi s)(t)\text{CF}^\rho (\text{CF}^\alpha u(t)) + f(t, u(t), (\phi u)(t), h(t)\text{CF}^\nu u(t)) - f(t, v(t), (\phi v)(t), h(t)\text{CF}^\nu v(t))\right]
\]

\[
\leq \mu_1 k(t)\|\text{CF}^\theta (\text{CF}^\alpha u(t)) - \text{CF}^\theta (\text{CF}^\alpha v(t))\| + \mu_2 (\phi s)(t)\|\text{CF}^\rho (\text{CF}^\alpha u(t)) - \text{CF}^\rho (\text{CF}^\alpha v(t))\|
\]

\[
+ f(t, u(t), (\phi u)(t), h(t)\text{CF}^\nu u(t)) - f(t, v(t), (\phi v)(t), h(t)\text{CF}^\nu v(t))\right]
\]

\[
\leq \mu_1 M_1 \|u - v\| + \mu_2 \lambda_0 M_2 \|u - v\| + \eta^*(\|u - v\| + \gamma_0 \|u - v\| + M_3 \|u - v\|)
\]

\[
= \left[\mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3)\right]\|u - v\|
\]
for all \( t \in I \) and \( u, v \in H^1 \). Hence

\[
|F(u)(t) - (Fv)(t)| \leq \left[ \mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3) \right] ||u - v|| = \Delta_3 ||u - v||
\]

for all \( t \in I \) and \( u, v \in H^1 \). Also, we have

\[
\begin{align*}
|CF D^\beta (Fu - Fv)(t)| &\leq \frac{1}{(1 - \beta)^2} \left[ \mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3) \right] ||u - v||, \\
|CF D^\beta (CF D^\gamma (Fu - Fv)(t))| &\leq \frac{1}{(1 - \gamma)^2(1 - \rho)^2} \left[ \mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3) \right] ||u - v||,
\end{align*}
\]

and \( |CF D^\gamma (Fu - Fv)(t)| \leq \frac{1}{|\gamma|^2} \left[ \mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3) \right] ||u - v|| \) for all \( u, v \in H^1 \) and \( t \in I \). Hence, \( || Fu - Fv || \leq \Delta ||u - v|| \) for all \( u, v \in H^1 \). Now, consider the maps \( g : [0, \infty)^2 \rightarrow [0, \infty) \) and \( \alpha : H^1 \times H^1 \rightarrow [0, \infty) \) defined by \( g(t_1, t_2, t_3, t_4, t_5) = \Delta_3 \max(t_1, t_2, t_3, t_4, t_5) \) and \( \alpha(x, y) = 1 \) for all \( x, y \in H^1 \). One can check that \( g \in \mathcal{R} \) and \( F \) is a generalized \( \alpha \)-contraction. By using Theorem 1.3, \( F \) has an approximate fixed point which is an approximate solution for the problem (3).

Let \( k, s, h, g \) and \( q \) be bounded functions on \([0,1]\) with \( M_1 = \sup_{t \in I} |k(t)| < \infty \), \( M_2 = \sup_{t \in I} |s(t)| < \infty \), \( M_3 = \sup_{t \in I} |h(t)| < \infty \), \( M_4 = \sup_{t \in I} |g(t)| < \infty \), and \( M_5 = \sup_{t \in I} |q(t)| < \infty \). Here, we investigate the fractional integro-differential problem

\[
\begin{align*}
CF D^\rho u(t) = \lambda k(t) CF D^\rho u(t) + \mu s(t) CF I^\rho u(t) \\
+ f_1(t, u(t), (\phi u)(t), (h(t) CF I^\rho u(t), g(t) CF D^\gamma u(t)) \\
+ \int_0^t f_2(s, u(s), (\phi u)(s), (q(t) CF D^\gamma u(s)) ds
\end{align*}
\]

(4)

with boundary condition \( u(0) = 0 \), where \( \lambda, \mu \geq 0 \) and \( \alpha, \beta, \rho, \nu, \delta, \gamma \in (0, 1) \). Note that the maps \( k, s, h, g \) and \( q \) should be chosen such that the right side of equation (4) is continuous.

**Theorem 2.8** Let \( \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6 \) and \( \xi_7 \) be nonnegative real numbers. Suppose that \( f_1 : [0,1] \times \mathbb{R}^4 \rightarrow \mathbb{R} \) and \( f_2 : [0,1] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) are integrable functions such that

\[
|f_1(t, x, y, w, v) - f_1(t, x', y', w', v')| \leq \xi_1 |x - x'| + \xi_2 |y - y'| + \xi_3 |w - w'| + \xi_4 |v - v'|
\]

and \( |f_2(t, x, y, w) - f_2(t, x', y', w')| \leq \xi_5 |x - x'| + \xi_6 |y - y'| + \xi_7 |w - w'| \) for all real numbers \( x, y, w, v, x', y' \) and \( w' \) and \( t \in I \). If \( \Delta_4 < 1 \), then the problem (4) has an approximate solution, where \( \Delta_4 := \max \left\{ \frac{1}{(1 - \beta)^2}, \frac{1}{(1 - \beta)^2}, \frac{1}{(1 - \beta)^2}, \frac{1}{(1 - \beta)^2}, \frac{1}{(1 - \beta)^2}, \frac{1}{(1 - \beta)^2} \right\} \).

**Proof** Consider the space \( H^1 \) endowed with the metric \( d(u, v) = ||u - v|| \), where

\[
||u|| = \max \left\{ \sup_{t \in I} |u(t)|, \sup_{t \in I} |CF D^\rho u(t)|, \sup_{t \in I} |CF I^\rho u(t)|, \sup_{t \in I} |CF D^\gamma u(t)| \right\}
\]

(5)
Define the map \( F : H^1 \to H^1 \) by

\[
(Fu)(t) = a_u \left[ \lambda k(t)^C D^\beta u(t) + \mu s(t)^C I_0^\alpha u(t) \\
+ f_1(t, u(t), (\phi u)(t), h(t)^C I^v u(t), g(t)^C D^\beta u(t)) \right. \\
+ \left. \int_0^t f_2(s, u(s), (\phi u)(s), q(t)^C D^\gamma u(s)) \, ds \right] \\
+ b_u \left[ \lambda k(s)^C D^\beta u(s) + \mu s(s)^C I_0^\alpha u(s) \\
+ f_1(s, u(s), (\phi u)(s), h(s)^C I^v u(s), g(s)^C D^\beta u(s)) \right. \\
+ \left. \int_0^t f_2(r, u(r), (\phi u)(r), q(r)^C D^\gamma u(r)) \, dr \, ds \right],
\]

where \( a_u \) and \( b_u \) are introduced in Lemma 1.2. By using Lemmas 2.2 and 2.3, we obtain

\[
\left| \lambda k(t)^C D^\beta (u(t) - v(t)) + \mu s(t)^C I_0^\alpha (u(t) - v(t)) \right| \\
+ \left| f_1(t, u(t), (\phi u)(t), h(t)^C I^v u(t), g(t)^C D^\beta u(t)) \right| \\
- \left| f_1(t, v(t), (\phi v)(t), h(t)^C I^v v(t), g(t)^C D^\beta v(t)) \right| \\
+ \left| \int_0^t f_2(s, u(s), (\phi u)(s), q(s)^C D^\gamma u(s)) \, ds \right| \\
\leq \left[ \lambda \frac{M_1}{(1 - \beta)^2} \| u - v \| + \mu M_2 \| u - v \| + \xi_1 \| u - v \| + \xi_2 \gamma_0 \| u - v \| + \xi_3 M_3 \| u - v \| \\
+ \xi_4 \frac{M_4}{(1 - \delta)^2} \| u - v \| + \xi_5 \| u - v \| + \xi_6 \lambda_0 \| u - v \| + \xi_7 \frac{M_5}{(1 - \gamma)^2} \| u - v \| \\
\times \| u - v \| \right]
\]

for all \( u, v \in H^1 \) and \( t \in I \). Hence,

\[
\left| (Fu)(t) - (Fv)(t) \right| \\
\leq a_u \left| \lambda k(t)^C D^\beta (u(t) - v(t)) + \mu s(t)^C I_0^\alpha (u(t) - v(t)) \right| \\
+ f_1(t, u(t), (\phi u)(t), h(t)^C I^v u(t), g(t)^C D^\beta u(t))
\]
\[-f_1(t, v(t), (\phi v)(t), h(t)^CT D^\nu v(t), g(t)^CT D^\nu v(t))
+ \int_0^t \left( f_2(s, u(s), (\phi u)(s), q(s)^CT D^\nu u(s)) - f_2(s, v(s), (\phi v)(s), q(s)^CT D^\nu v(s)) \right) ds \right| \\
+ b_a \int_0^t \left| \lambda k(s)^CT D^\beta \left( u(s) - v(s) \right) + \mu s(s)^CT D^\nu \left( u(s) - v(s) \right) \right| ds \\
+ f_1(s, u(s), (\phi u)(s), h(s)^CT D^\nu u(s), g(s)^CT D^\nu u(s)) \\
- f_1(s, v(s), (\phi v)(s), h(s)^CT D^\nu v(s), g(s)^CT D^\nu v(s)) \\
+ \int_0^t \left( f_2(r, u(r), (\phi u)(r), q(r)^CT D^\nu u(r)) - f_2(r, v(r), (\phi v)(r), q(r)^CT D^\nu v(r)) dr \right) ds \right| \\
\leq a_a \left[ \lambda \frac{M_1}{(1 - \beta)^2} + \mu M_2 + \xi_1 + \xi_2 \gamma_0 + \xi_3 M_3 + \xi_4 \frac{M_4}{(1 - \delta)^2} + \xi_5 \lambda_0 + \xi_6 \frac{M_5}{(1 - \gamma)^2} \right] \\
x \parallel u - v \parallel \\
+ b_a \int_0^t \left[ \lambda \frac{M_1}{(1 - \beta)^2} + \mu M_2 + \xi_1 + \xi_2 \gamma_0 + \xi_3 M_3 + \xi_4 \frac{M_4}{(1 - \delta)^2} + \xi_5 \lambda_0 + \xi_6 \frac{M_5}{(1 - \gamma)^2} \right] \parallel u - v \parallel ds \\
\leq \left[ a_a + b_a \right] \left[ \lambda \frac{M_1}{(1 - \beta)^2} + \mu M_2 + \xi_1 + \xi_2 \gamma_0 + \xi_3 M_3 + \xi_4 \frac{M_4}{(1 - \delta)^2} + \xi_5 \lambda_0 + \xi_6 \frac{M_5}{(1 - \gamma)^2} \right] \parallel u - v \parallel := \Delta \parallel u - v \parallel \\
\text{for all } u, v \in H^1. \text{ Also by using Lemmas 2.2 and 2.3, we get} \\
\left| ^{CT} D^\delta F u(t) - ^{CT} D^\delta F v(t) \right| \leq \frac{1}{(1 - \beta)^2} \Delta \parallel u - v \parallel, \\
\left| ^{CT} D^\nu F u(t) - ^{CT} D^\nu F v(t) \right| \leq \frac{1}{(1 - \beta)^2} \Delta \parallel u - v \parallel, \\
\left| ^{CT} I^\nu F u(t) - ^{CT} I^\nu F v(t) \right| \leq \Delta \parallel u - v \parallel \text{ and } \left| ^{CT} I^\nu F u(t) - ^{CT} I^\nu F v(t) \right| \leq \Delta \parallel u - v \parallel \text{ for all } u, v \in H^1 \\
\text{and } t \in I. \text{ Hence, we obtain} \\
\parallel F u - F v \parallel \leq \max \left\{ \frac{1}{(1 - \beta)^2}, \frac{1}{(1 - \delta)^2}, \frac{1}{(1 - \gamma)^2} \right\} \Delta \parallel u - v \parallel = \Delta_4 \parallel u - v \parallel \\
\text{for all } u, v \in H^1. \text{ Consider the maps } g : [0, \infty)^2 \to [0, \infty) \text{ and } \alpha : H^1 \times H^1 \to [0, \infty) \text{ defined by } g(t_1, t_2, t_3, t_4) = \frac{\lambda_3}{3} (3t_1 + 2t_2 + 4t_3) \text{ and } \alpha(x, y) = 1 \text{ for all } x, y \in H^1. \text{ One can check that} \\
g \in R \text{ and } F \text{ is a generalized } \alpha \text{-contraction. By using Theorem 1.3, } F \text{ has an approximate} \\
\text{fixed point which is an approximate solution for the problem (4).} \quad \square \\

Here, we provide three examples to illustrate our some main results.

Example 2.1 Define the functions \eta \in L^\infty([0, 1]) \text{ and } \gamma, \lambda : [0, 1] \times [0, 1] \to [0, \infty) \text{ by } \eta(t) = e^{-2t}, \gamma(t, s) = \sin(1) \text{ and } \lambda(t, s) = e^{-s}. \text{ Then } \eta^* = \frac{1}{2}, \gamma_0 = \sin(1) \text{ and } \lambda_0 \leq e. \text{ Put } \alpha = \frac{3}{2}. \\
Consider the problem \\
^{CT} D^{\frac{1}{2}} u(t) = e^{-(t+6)} \left[ 2t + u(t) + \frac{1}{20} \int_0^t \sin(1)u(s)ds + \frac{1}{6} \int_0^t e^s u(s)ds \right] \\
(5)
with boundary condition \( u(0) = 0 \) and the function \( f(t,x,y,w) = e^{-(\pi t+\delta)}(2t + x + \frac{1}{20}y + \frac{1}{6}w) \). Note that \( \Delta_1 = \eta^*(1 + \gamma_0 + \lambda_0) < 0.00926 < 1 \). By utilizing Theorem 2.5, (5) has an approximate solution.

**Example 2.2** Define the functions \( \eta \in L^\infty([0,1]) \) and \( \gamma, \lambda : [0,1] \times [0,1] \rightarrow [0, \infty) \) by \( \eta(t) = \frac{\pi}{\sqrt{1 - t^2}}, \gamma(t,s) = e^{\eta(t,s)} \) and \( \lambda(t,s) = \log e^{\sin(\pi t s^{-1})} \). Then \( \eta^* = \frac{\pi}{\sqrt{17}}, \gamma_0 \leq e \) and \( \lambda_0 \leq \log e \). Put \( \alpha = \frac{1}{2}, \mu_1 = \frac{1}{120}, \mu_2 = \frac{1}{28}, \beta = \frac{3}{2}, \theta = \frac{1}{2}, \rho = \frac{1}{2} \) and \( \nu = \frac{1}{2} \). Consider the functions \( k(t) = \sin t, h(t) = \tan^{-1}(t) \), and \( s(t) = \frac{1}{7} \) whenever \( x = \frac{x}{n} \in Q \cap [0,1] \) with \( (m,n) = 1 \) and \( s(t) = 0 \) whenever \( x \in Q' \cap [0,1] \) or \( x = 0 \). Then \( M_1 = \sup_{t \in [0,1]} |k(t)| = 1, M_2 = \sup_{t \in [0,1]} |s(t)| = 1 \) and \( M_3 = \sup_{t \in [0,1]} |h(t)| = \frac{\pi}{2} \). Consider the fractional integro-differential problem

\[
\begin{align*}
\text{CF}D^\frac{1}{2}u(t) &= \frac{1}{120} \sin(t)\text{CF}D^{\frac{3}{2}}u(t) \\
+ \frac{1}{28} \left( \int_0^t \log(e^{\sin(\pi t s^{-1})})s(s)ds \right) \text{CF}D^\frac{1}{2}(\text{CF}D^{\frac{3}{2}}u(t)) \\
+ \frac{\pi}{\varepsilon^{(1+\varepsilon)}} \left[ t + u(t) \right] + \int_0^t e^{-\eta(t,s)}ds + \tan^{-1}(t)\text{CF}D^{\frac{1}{2}}u(t)
\end{align*}
\]

with boundary condition \( u(0) = 0 \). Put \( f(t,x,y,w,v) = \frac{\pi}{\sqrt{1 - t^2}}(t + x + y + w) \). Note that \( \Delta_3 = \left(1 + \frac{1}{(1-\rho^2)^{\frac{1}{2}}} + \frac{1}{(1-\nu^2)^{\frac{1}{2}}} + \frac{1}{(1-\beta^2)^{\frac{1}{2}}} \right) \mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3) < 0.5485 < 1 \). Then by using Theorem 2.7, it implies that (6) has an approximate solution.

**Example 2.3** Define the functions \( \gamma, \lambda : [0,1] \times [0,1] \rightarrow [0, \infty) \) by \( \nu(t,s) = \frac{\pi}{\sqrt{1 - t^2}} \) and \( \gamma(t,s) = 0 \). Then \( \gamma_0 = 0 \) and \( \lambda_0 \leq e \). Put \( \alpha = \frac{1}{2}, \beta = \frac{1}{4}, \nu = \frac{1}{7}, \delta = \frac{1}{4}, \gamma = \frac{1}{7}, \lambda = \frac{1}{177}, \mu = 0 \), \( \xi_1 = \frac{2}{17}, \xi_2 = \frac{1}{30}, \xi_3 = \frac{1}{30}, \xi_4 = \frac{1}{30}, \xi_1' = \frac{1}{30}, \xi_2' = \frac{1}{30}, \xi_3' = \frac{1}{177} \). Let \( s \) be an arbitrary bounded map, \( q(t) = \tan^{-1}(t) \), \( h(t) = \sin(t) \) for all \( t \in I \), \( k(t) = 1 \) whenever \( x \in Q \cap [0,1] \) and \( k(t) = 0 \) whenever \( x \in Q' \cap [0,1] \) and \( g(t) = 0 \) whenever \( x \in Q \cap [0,1] \) and \( g(t) = 2 \) whenever \( x \in Q' \cap [0,1] \). Then \( M_1 = \sup_{t \in [0,1]} |k(t)| = 1, M_2 \) is a real number, \( M_3 = \sup_{t \in [0,1]} |h(t)| = 1, M_4 = \sup_{t \in [0,1]} |g(t)| = 2 \) and \( M_5 = \sup_{t \in [0,1]} |q(t)| = \frac{\pi}{4} \). Now, consider the well-defined fractional integro-differential problem

\[
\begin{align*}
\text{CF}D^{\frac{1}{2}}u(t) &= \frac{1}{200} k(t)\text{CF}D^\frac{3}{2}u(t) + \frac{3}{40} t + \frac{2}{41} u(t) + \frac{1}{48} \sin(t)\text{CF}D^\frac{1}{2}u(t) + \frac{1}{400} g(t)\text{CF}D^\frac{3}{2}u(t) \\
+ \int_0^t \left[ \frac{2}{56} s + \frac{1}{320} u(s) + \frac{1}{40} \int_0^s \frac{e^{2v-\rho}}{e^{u(r)}}dr + \frac{1}{119} \tan^{-1}(t)\text{CF}D^\frac{1}{2}u(s) \right]ds
\end{align*}
\]

with boundary condition \( u(0) = 0 \). Put \( f_1(t,x,y,w,v) = \frac{3}{30} t + \xi_1 x + \xi_2 y + \xi_3 w + \xi_4 v \) and \( f_2(t,x,y,w) = \frac{2}{30} t + \xi_1' x + \xi_2' y + \xi_3' w \) for all \( t \in I \) and \( x, y, w, v \in \mathbb{R} \). Note that

\[
\Delta_4 = \max \left\{ \frac{1}{(1-\beta)^2}, \frac{1}{(1-\delta)^2}, \frac{1}{(1-\gamma)^2} \right\} \\
\times \left[ \lambda \frac{M_1}{(1-\beta)^2} + \mu M_2 + \xi_1 + \xi_2\gamma_0 + \xi_3 M_3 + \xi_4 \frac{M_4}{(1-\delta)^2} + \xi_1' + \xi_2'\lambda_0 + \xi_3' M_5 \right]
\]

\(< 0.8451 < 1 \).
Thus, taking into account Theorem 2.8 we conclude that the problem (7) has an approximate solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of this study equally and approved the final version of the manuscript.

Author details
1Department of Mathematics, Cankaya University, Balgat, Ankara, 06530, Turkey. 2Institute of Space Sciences, Magurele-Bucharest, Romania. 3Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.

Acknowledgements
The authors express their gratitude to the Professor JJ Nieto and an unknown referee for their helpful suggestions, which improved the final version of this paper. Research of the second and third authors was supported by Azarbaijan Shahid Madani University.

Received: 9 September 2016 Accepted: 17 January 2017 Published online: 09 February 2017

References
1. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego, CA (1999)
2. Samko, G, Kilbas, AA, Marichev, S: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon (1993)
3. Kilbas, AA, Srivastava, MH, Trujillo, JJ: Theory and Application of Fractional Differential Equations. North Holland Mathematics Studies, vol. 204 (2006)
4. Magin, RL: Fractional Calculus in Bioengineering. Begell House Publishers, Redding (2006)
5. Baleanu, D, Diethelm, K, Scalas, E, Trujillo, JJ: Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos. World Scientific, Singapore (2012)
6. Caputo, M, Fabrizo, M: A new definition of fractional derivative without singular kernel. Prog. Fract. Differ. Appl. 1(2), 73-85 (2015)
7. Losada, J, Nieto, JJ: Properties of a new fractional derivative without singular kernel. Prog. Fract. Differ. Appl. 1(2), 87-92 (2015)
8. Alsaedi, A, Baleanu, D, Etemad, S, Rezapour, Sh: On coupled systems of time-fractional differential problems by using a new fractional derivative. J. Funct. Spaces 2016, Article ID 4626940 (2016)
9. Atangana, A: On the new fractional derivative and application to nonlinear Fisher’s reaction-diffusion equation. Appl. Math. Comput. 273(6), 948-956 (2016)
10. Atangana, A, Alkahtani, BT: Analysis of the Keller-Segel model with a fractional derivative without singular kernel. Entropy 17(6), 4439-4453 (2015)
11. Atangana, A, Nieto, JJ: Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel. Adv. Mech. Eng. 7, 1-7 (2015)
12. Gomez-Aguilar, JF, Yépez-Martínez, H, Calderon-Ramón, C, Cruz-Orduña, I, Escobar-Jimenez, RF, Olivares-Peregrino, VH: Modeling of a mass-spring-damper system by fractional derivatives with and without a singular kernel. Entropy 17(9), 6289-6303 (2015)
13. Doungmo G, Emile, F, Pene, MK, Mwambakana, JN: Duplication in a model of rock fracture with fractional derivative without singular kernel. Open Math. 13, 839-846 (2015)
14. Al-Salti, N, Karimov, ET, Sadarangani, K: On a differential equation with Caputo-Fabrizio fractional derivative of order $1 \leq \beta \leq 2$ and application to mass-spring-damper system. Prog. Fract. Differ. Appl. 2(4), 257-263 (2016)
15. Mirdadagh, MA, Postolache, M, Rezapour, Sh: Some approximate fixed point results for generalized $\alpha$-contractive mappings. Sci. Bull. “Politeh.” Univ. Buchar., Ser. A, Appl. Math. Phys. 75(2), 3-10 (2013)