Paramagnetism, zero modes and mass singularities in

QED in 1 + 1, 2 + 1 and 3 + 1 dimensions

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Abstract

The interplay of paramagnetism, zero modes of the Dirac operator and fermionic mass
singularities on the fermionic determinants in quantum electrodynamics in two, three and
four dimensions is discussed.

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I. INTRODUCTION

One of the oldest outstanding problems of nonperturbative QED is the calculation of its fermionic determinant. This determinant is obtained by integrating over the fermionic degrees of freedom in the presence of a background potential $A_\mu$, thereby producing the one-loop, gauge invariant effective action $\ln \det(F_{\mu\nu})$ that appears in the calculation of all physical processes. The problem is that since the determinant is itself part of a functional integral over $A_\mu$, it has to be known for all fields. If the gauge field is given an infrared cutoff then $A_\mu$ is concentrated on $\mathcal{S}'$, the Schwartz space of real tempered distributions. Consequently, $\det$ is needed for fields that can be rough and that have no particular symmetry. Nevertheless, modulo certain technical assumptions, bounds can be placed on the Euclidean determinants in QED$_2$, QED$_3$ and QED$_4$ for particular classes of fields. These bounds are reviewed in [1]. Here we wish to discuss what additional insight can be gained by paying particular attention to the interplay of the paramagnetic tendency of charged fermions in external magnetic and electric fields, zero modes of the Dirac operator $\mathcal{D}=\mathcal{P}-\mathcal{A}$, and fermionic mass singularities. Recall that in Euclidean space $\mathcal{E}$ and $\mathcal{B}$ are on the same footing, so that one may speak of paramagnetism in the presence of an electric field as well.

The effect of paramagnetism is particularly transparent in Schwinger’s proper time definition of the fermionic determinant [2–4],

$$\ln \det_{\text{ren}} = \frac{1}{2} \int_0^\infty \frac{dt}{t} \left\{ \text{Tr} \left( e^{-D^2 t} - \exp[-(D^2 + \frac{1}{2}\sigma^{\mu\nu} F_{\mu\nu})t] + \frac{\|F\|^2}{24\pi^2} \right)e^{-tm^2} \right\}. \quad (1.1)$$

Here $D^2 = (\mathcal{P}-\mathcal{A})^2$; $m$ is the unrenormalized fermion mass; $\sigma^{\mu\nu} = (1/2i)[\gamma^\mu, \gamma^\nu], \gamma^{\mu\dagger} = -\gamma^\mu$. 

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and $\|F\|^2 = \int d^4 x F_{\mu \nu}^2(x)$. The procedure for smoothing the rough fields and for introducing a volume cutoff is described in refs. [1,5]. Here we will just assume that $A_\mu$ and $F_{\mu \nu}$ are smooth and that $F_{\mu \nu}$ is square-integrable. The last item in (1.1) is the second-order mass shell charge renormalization subtraction required for the small $t$ limit of the integral to converge. The same definition may be used in lower dimensions by omitting the charge renormalization subtraction and dropping the subscript “ren” on the determinant.

Paramagnetism is immediately manifest through the square-integrable zero modes and zero energy threshold resonance states of the nonnegative Pauli operator $D^2 + \frac{1}{2} \sigma F$ for suitable $A_\mu$. By inspection of definition (1.1), these tend to drive $\ln \det_{\text{ren}}$ toward negative values and will introduce nonperturbative mass singularities.

In two and three dimensions paramagnetism extends on up the spectrum of the Pauli operator by lowering its eigenvalues on average relative to those of the free Hamiltonian $P^2$, as manifested by the bound $\ln \det \leq 0$ [3,6,7]. There is no such bound in four dimensions [3] due to the positive sign of the charge renormalization counterterm and, hence, ultimately due to the nonasymptotic freedom of QED$_4$. It is of interest to quantify this interplay between nonasymptotic freedom and paramagnetism as this bears on the stability of QED$_4$. For if $\ln \det_{\text{ren}}$ grows faster than a quadratic in the field strength then it is doubtful that it is integrable with respect to the potential’s gauge-fixed Gaussian measure. These introductory remarks will be developed in Sec. III. In Sec. II we take up the case of QED$_2$ and conclude with a discussion of QED$_3$ in Sec. IV.

II. ZERO MODES IN QED$_2$

For Euclidean QED in two-dimensional space-time, otherwise known as the massive Schwinger
model, Eq. (1.1) gives

\[ \frac{\partial}{\partial m^2} \ln \det_{QED2} = \frac{1}{2} \operatorname{Tr} \left[ (D^2 - \sigma_3 B + m^2)^{-1} - (P^2 + m^2)^{-1} \right], \]

(2.1)

where \( B = F_{01} = \partial_0 A_1 - \partial_1 A_0 \), and the coupling \( e \) has been absorbed into \( A_\mu \) so that the flux \( \Phi = \int d^2r B(r) \) is dimensionless. Thus, \( \ln \det_{QED2} \) is determined by the quantum mechanics of a charged fermion confined to a plane in the presence of a static magnetic field perpendicular to the plane. As noted in Sec. I, this field and its associated potential are assumed sufficiently smooth with enough fall off so that everything done here makes mathematical sense. It was conjectured in [8] (see Eq. (5.67) and below) that

\[ \lim_{m^2 \to 0} m^2 \frac{\partial}{\partial m^2} \ln \det_{QED2} = \frac{|\Phi|}{4\pi}. \]

(2.2)

Referring to (2.1), this would indicate that the leading mass singularity of \( \ln \det_{QED2} \) is determined by the zero modes of the Pauli operator \((P - A)^2 - \sigma_3 B\). The number of square-integrable zero modes is \([|\Phi|/2\pi]\), all with \( \sigma_3 = 1 \) (\( \sigma_3 = -1 \)) if \( \Phi > 0 \) (\( \Phi < 0 \)), where \([x]\) stands for the nearest integer less than \( x \) and \([0] = 0 \) [9]. The remaining fractional part of \( \Phi \) is related to the zero energy scattering phase shifts for \(-\Phi^2\) [10]. Here we will prove (2.2) for the case \( B(r) \geq 0 \) or \( B(r) \leq 0 \). But first note that (2.2) contradicts the perturbative result

\[ \lim_{m^2 \to 0} m^2 \frac{\partial}{\partial m^2} \ln \det_{QED2} = \frac{\Phi^2}{8\pi^2} + O(\Phi^4), \]

(2.3)

obtained by expanding (1.1) in the field strength. In fact, the series in (2.3) is not even asymptotic.
The reason is clear: square-integrable zero modes and zero-energy threshold resonances cannot be captured by a perturbative expansion. Eqs. (2.2) and (2.3) stand as a salutary lesson on how badly perturbation theory can go wrong in QED.

Now for the proof. Suppose \( B(r) \geq 0 \). From Eqs. (5) and (6) of [11] one has

\[
m^2 \frac{\partial}{\partial m^2} \ln \det \text{QED}^2 = \frac{\Phi}{4\pi} + m^2 \text{Tr}[(D^2 + B + M^2)^{-1} - (P^2 + m^2)^{-1}]
\leq \frac{\Phi}{4\pi} - \frac{m^2}{4\pi} \int d^2 r \ln \left(1 + \frac{B(r)}{m^2}\right),
\]

(2.4)

where the trace in the first line of (2.4) is over space indices only. Since

\[
m^2 \int d^2 r \ln \left(1 + \frac{B(r)}{m^2}\right) \leq \int d^2 B(r) = \Phi,
\]

(2.5)

for \( m^2 \geq 0 \), the integral in (2.4) converges uniformly for \( m^2 \geq 0 \). Therefore the limit \( m^2 = 0 \) may be taken inside the integral, giving

\[
\lim_{m^2=0} m^2 \frac{\partial}{\partial m^2} \ln \det \text{QED}^2 \leq \frac{\Phi}{4\pi}.
\]

(2.6)
Next, note that for $M^2 \geq m^2$, 

\[
\begin{align*}
\text{Tr}[(D^2 + B + m^2)^{-1} - (P^2 + m^2)^{-1}] &
\geq \text{Tr}[(D^2 + B + M^2)^{-1} - (P^2 + M^2)^{-1}] \\
&= \text{Tr}[(D^2 + B + M^2)^{-1} - (P^2 + M^2)^{-1}] \\
&\quad + \text{Tr}[(P^2 + M^2)^{-1} - (P^2 + m^2)^{-1}] \\
&= \text{Tr}[(D^2 + B + M^2)^{-1} - (P^2 + M^2)^{-1}] + \pi \ln(m^2/M^2). \quad (2.7)
\end{align*}
\]

Thus, (2.7) and the first line of (2.4) imply

\[
\lim_{m^2\to 0} m^2 \frac{\partial}{\partial m^2} \ln \det_{QED2} \geq \frac{\Phi}{4\pi}, \quad (2.8)
\]

and hence (2.6) and (2.8) together imply (2.2). The case $B(r) \leq 0$ is dealt with along the same lines, the only change being that $B \to -B \geq 0$ and $\Phi \to -\Phi \geq 0$ in (2.4) following ref. [11], thereby establishing (2.2) for both cases.

From (2.2) one gets the small mass limit

\[
\ln \det_{QED2} = \frac{|\Phi|}{4\pi} \ln m^2 + R(m^2), \quad (2.9)
\]

where $\lim_{m^2=0} (R(m^2)/\ln m^2) = 0$. Letting $A_\mu(x) \to \lambda A_\mu(\lambda x)$, $F_{\mu\nu}(x) \to \lambda^2 F_{\mu\nu}(\lambda x)$, $\lambda > 0$, in (1.1) there results the exact scaling relation

\[
\ln \det(\lambda^2 F_{\mu\nu}(\lambda x), m^2) = \ln \det(F_{\mu\nu}(x), m^2/\lambda^2), \quad (2.10)
\]
which holds in two, three and four dimensions. Applying (2.10) to (2.9) gives

$$\ln \det_{QED2}(\lambda^2 B(\lambda r), m^2) \sim \frac{|\Phi|}{4\pi} \ln \lambda + \mathcal{A}(\lambda),$$

(2.11)

where \( \lim_{\lambda \to \infty} (\mathcal{A}(\lambda)/\ln \lambda) = 0 \). Note the minus sign, indicating paramagnetism. Also note that the coefficient of \( \ln \lambda \) is the absolute value of the anomaly. However, this still does not answer the main question: how does \( \ln \det_{QED2} \) grow for large field strength obtained by the scaling \( B(r) \to \lambda B(r), \lambda > 0 \)? In [8], Eq. (5.67), we conjected

$$\lim_{\lambda \to \infty} \left( \ln \det_{QED2}(\lambda B, m^2)/\lambda \ln \lambda \right) = -\frac{|\Phi|}{2\pi}.$$  

(2.12)

III. PARAMAGNETISM, ZERO MODES AND MASS SINGULARITIES IN \( QED_4 \)

From (1.1),

$$m^2 \frac{\partial}{\partial m^2} \ln \det_{ren} = \frac{m^2}{2} \text{Tr} \left[ (D^2 + \frac{1}{2} \sigma F + m^2)^{-1} - (P^2 + m^2)^{-1} \right] - \frac{||F||^2}{48\pi^2}. $$

(3.1)

The main question here is: What is the \( m^2 = 0 \) limit of (3.1)? The answer is not so straightforward as in the case of \( QED_2 \). Since \( \mathcal{D} \) and \( \gamma_5 \) anti-commute, then in a basis in which \( \gamma_5 \) is diagonal, \( \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( \mathcal{D} \) takes the supersymmetric form [12]

$$\mathcal{D} = \begin{pmatrix} 0 & \Delta_- \\ \Delta_+ & 0 \end{pmatrix},$$

(3.2)
with $\Delta^\dagger = -\Delta_+$, and hence

$$ - \mathcal{D}^2 = D^2 + \frac{1}{2} \sigma F = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, $$

(3.3)

where

$$ H_\pm = (P - A)^2 - \sigma \cdot (B \pm E). $$

(3.4)

The global anomaly for $\mathcal{D}$ may be put in the form [10]

$$ \frac{1}{4\pi^2} \int d^4x \, \mathbf{E} \cdot \mathbf{B}(x) = \lim_{m^2 \to 0} \text{Tr} \left( \frac{m^2}{H_+ + m^2} - \frac{m^2}{H_- + m^2} \right) = n_+ - n_- + \frac{1}{\pi} \sum_l \mu(l) \left( \delta^l_+(0) - \delta^l_-(0) \right), $$

(3.5)

where $n_\pm$ are the number of square integrable zero modes of $H_\pm$; $\delta^l_\pm(0)$ are the scattering phase shifts for $H_\pm$ as the energy tends to zero; $l$ is a degeneracy parameter such as angular momentum, and $\mu(l)$ is a weight factor. The last line in (3.5) is a generalization of the Atiyah-Singer index theorem [13] to non-compact manifolds.

We will now use (3.1) and (3.5) to rederive the asymptotic form of the constant field result for $\ln \det_{\text{ren}}$. Although it is a rederivation, a new physical insight is gained that goes well beyond the constant field case.

By making two rotations (corresponding to a Lorentz boost and a rotation in Minkowski space) a frame can be found in which $\mathbf{E}$ and $\mathbf{B}$ are parallel. Select the frame where $\mathbf{B}' = (0, 0, B')$,
and of course $\mathbf{B}' \cdot \mathbf{E}' = \mathbf{B} \cdot \mathbf{E}$. Suppose $\mathbf{B} \cdot \mathbf{E} > 0$. Then $B', E' > 0$ and $H_{\pm}$ take the form

$$H_{\pm} = (P - A)^2 - \sigma_3 (B' \pm E').$$

Choose $A_x = -yB'$, $A_z = -tE'$, $A_t = A_y = 0$ so that

$$(P - A)^2 = H_{E'} + H_{B'},$$

where $H_{E'}, H_{B'}$, describe two oscillators with energies $(2n + 1)E'$, $(2m + 1)B'$ with $m, n = 0, 1, ...$

By inspection of (3.8), all zero modes have positive chirality. Likewise, when $\mathbf{B} \cdot \mathbf{E} < 0$ then $B' > 0$ and $E' < 0$, in which case we conclude that all zero modes have negative chirality. Because there are no scattering states, the phase shifts are zero in (3.5).

Referring back to (3.1), suppose $\mathbf{E} \cdot \mathbf{B} > 0$. Then the fact that there are no negative chirality zero modes combined with (3.5) gives

$$\lim_{m^2 = 0} m^2 \frac{\partial}{\partial m^2} \ln \text{det}_{\text{ren}} = \frac{1}{2} \lim_{m^2 = 0} m^2 \text{Tr}[(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}]$$

$$+ \lim_{m^2 = 0} m^2 \text{Tr}[(H_- + m^2)^{-1} - (P^2 + m^2)^{-1}] - \frac{\|F\|^2}{48\pi^2}$$

$$= \frac{\mathbf{E} \cdot \mathbf{B} V}{8\pi^2} - \frac{(E^2 + B^2)V}{24\pi^2},$$

(3.10)
where \( V \) is the volume of the space-time box, and the trace operation includes a trace over a \( 2 \times 2 \) spin space. Similarly, when \( \mathbf{E} \cdot \mathbf{B} < 0 \) there are no positive chirality zero modes and so (3.5) gives

\[
\lim_{m^2=0} m^2 \frac{\partial}{\partial m^2} \ln \det_{\text{ren}} = -\frac{\mathbf{E} \cdot \mathbf{B} V}{8\pi^2} - \frac{(E^2 + B^2)V}{24\pi^2}.
\]

(3.11)

Hence we can conclude that for small \( m^2 \),

\[
\ln \det_{\text{ren}} = -\frac{V}{24\pi^2} (B^2 + E^2 - 3|\mathbf{E} \cdot \mathbf{B}|) \ln m^2 + R(m^2),
\]

(3.12)

using the same definition of \( R \) as in Sec. II. From the scaling relation (2.10) we obtain for \( \lambda >> 1 \),

\[
\ln \det_{\text{ren}}(\lambda^2 F_{\mu\nu}(\lambda x), m^2) = \frac{V}{24\pi^2} (B^2 + E^2 - 3|\mathbf{E} \cdot \mathbf{B}|) \ln \lambda^2 + \mathcal{A}(\lambda),
\]

(3.13)

where \( \mathcal{A}(\lambda) \) is defined in Sec. II. This is the same result that is obtained by scaling the exact constant field expression for \( \ln \det_{\text{ren}} \) \([2,4]\).

On the basis of (3.13) it is tempting to make the general field conjecture

\[
\lim_{\lambda \to \infty} \ln \det_{\text{ren}}(\lambda^2 F_{\mu\nu}(\lambda x), m^2)/\ln \lambda^2 = \frac{1}{8}\beta_1 \|F\|^2 - \frac{1}{2}|A|,
\]

(3.14)

where \( \beta_1 \) is obtained from the Callan-Symanzik function \( \beta = 2\alpha/3\pi + O(\alpha^2) \), i.e. \( \beta_1 = 1/6\pi^2 \), and \( A \) is the anomaly, \( \frac{1}{16\pi} \int d^4x F_{\mu\nu}^* F_{\mu\nu} \), with \( F_{\mu\nu}^* = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \). Recall that \( \|F\|^2 \geq 16\pi^2|A| \) so that the right-hand side of (3.14) is greater than or equal to \(-|A|/6\).

Whether or not (3.14) is correct, this much is now clear. Paramagnetism results in zero modes.
of $B^2$ driving $\ln \det_{\text{ren}}$ toward negative values that tend to offset the nonasymptotic freedom of QED$_4$ that enters through the positive sign of the charge renormalization subtraction. The zero modes contribute to the leading mass singularity of $\ln \det_{\text{ren}}$ which can be related to its strong field behavior by scaling. As a consequence of paramagnetism and associated zero modes the sign of $\ln \det_{\text{ren}}$ for large field strength is not definite, and so the question of QED$_4$’s stability becomes all the more interesting.

In order to make further progress one has to go beyond constant fields. For example, self-dual (anti-self-dual) fields $B = -E$ ($B = E$) will, by inspection of (3.4), have only negative (positive) chirality square-integrable zero modes. It would be helpful if one knew the rules, if any, that allow one to predict such results in more general cases.

**IV. ZERO MODES IN QED$_3$**

Making the continuation of QED$_{2+1}$ to Euclidean space, the planar electric field combines with the normal magnetic field to form a three dimensional static magnetic field $B(r)$ on $\mathbb{R}^3$. Here we will use the $2 \times 2$ Dirac matrices $\gamma^\mu = (i\sigma_1, i\sigma_2, i\sigma_3)$. Then (1.1) gives

$$m^2 \frac{\partial}{\partial m^2} \ln \det_{\text{QED}_3} = \frac{m^2}{2} \text{Tr}[[((P - A)^2 - \sigma \cdot B + m^2)^{-1} - (P^2 + m^2)^{-1}].$$

(4.1)

Again we ask the question: What is the $m^2 = 0$ limit of (4.1)? Of course the answer is zero for a constant magnetic field [14], or indeed any unidirectional magnetic field, since the momentum parallel to the field serves as an extra infrared cutoff. Remarkably, it was not known until 1986
whether the equation

\[ [(\mathbf{P} - \mathbf{A})^2 - \mathbf{\sigma} \cdot \mathbf{B}]\psi = 0, \quad (4.2) \]

had any bound state solutions for \( \mathbf{B} \in L^2(\mathbb{R}^3) \), and hence for \( \mathbf{A} \in L^6(\mathbb{R}^3) \) when \( \mathbf{A} \) is in the Coulomb gauge \( \nabla \cdot \mathbf{A} = 0 [15,16] \). Matters are not helped by the absence of an index theorem in three dimensions. It is now known that bound states exist and that their degeneracy can be large, with an upper bound growing like \( \int d^3r |\mathbf{B}(r)|^{3/2} [16] \). This will be derived below.

Since it is now known that bound state solutions to (4.2) exist, (4.1) tells us that \( \ln \det_{QED} \) has a logarithmic mass singularity whose coefficient can grow as fast as \( \int |\mathbf{B}|^{3/2} \). By scaling, Eq. (2.10), we expect

\[ \ln \det_{QED}(\lambda^2 \mathbf{B}(\lambda \mathbf{r}), m^2) \sim -c \int d^3r |\mathbf{B}(r)|^{3/2} \ln \lambda^2, \quad (4.3) \]

where \( c \) is a positive constant, again reflecting paramagnetism. Therefore, the strong field “folk theorem” that \( \ln \det_{QED} \sim -\int |\mathbf{B}|^{3/2} \) is probably wrong. One wonders if the coefficient of the logarithm in (4.3) has a topological origin as in two dimensions.

Since the upper bound, \( c \int |\mathbf{B}|^{3/2} \), on the number of bound state solutions of (4.2) is not derived in [16] it might be helpful to provide a derivation here. It is suggested in [16] that a variational principle be used together with the inequality \( |(\psi, \mathbf{\sigma} \cdot \mathbf{B} \psi)| \leq (\psi, |\mathbf{B}| \psi) \). Accordingly,
the ground state energy is given by

\[ E(= 0) = \inf_{\|\psi\|=1} \langle \psi, [(P - A)^2 - \sigma \cdot B] \psi \rangle \]
\[ \geq \inf_{\|\psi\|=1} \langle \psi, [(P - A)^2 - |B|] \psi \rangle \]
\[ \geq \inf_{\|\psi\|=1} \langle |\psi|, [(P^2 - |B|)|\psi| \rangle \]
\[ \geq \inf_{\|\phi\|=1} \langle \phi, [(P^2 - |B|)|\phi| \rangle. \quad (4.4) \]

The third line in (4.4) follows from Simon’s variant of Kato’s inequality expressing the diamagnetism of spinless bosons [17]. By (4.4), the number of bound states of the Hamiltonian \( P^2 - |B| \) provides an upper bound on the number of bound states of the Pauli Hamiltonian \( (P - A)^2 - \sigma \cdot B \). The number \( N \) of bound states for the Hamiltonian \( P^2 - |B| \) can be estimated by the Cwikel-Lieb-Rozenblum bound [18]

\[ N \leq 0.1156 \int d^3r |B(r)|^{3/2}, \quad (4.5) \]

which is the result of [16] with the addition of a precise constant.

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