ASYMPTOTIC STABILITY OF PLANAR RAREFACTION WAVES FOR 3-D ISENTROPIC NAVIER-STOKES EQUATIONS UNDER PERIODIC PERTURBATIONS

FEIMIN HUANG$^{1,2}$, LINGDA XU$^3$, AND QIAN YUAN$^1$*

ABSTRACT. We study the asymptotic stability of a planar rarefaction wave (in the $x_1$-direction) for the 3-d isentropic Navier-Stokes equations, where the initial perturbation is periodic on the torus $\mathbb{T}^3$ with zero average. To solve this Cauchy problem in which the initial data is periodic with respect to only $x_2$ and $x_3$ but not to $x_1$, we construct a suitable ansatz carrying the oscillations of the solution in the $x_1$-direction, but remaining to be periodic in the transverse $x_2$- and $x_3$-directions. In such a way, the difference between the ansatz and the solution can be integrable on the region $\mathbb{R} \times \mathbb{T}^2$, which allows us to utilize the energy method with the aid of a Gagliardo-Nirenberg type inequality on $\mathbb{R} \times \mathbb{T}^2$ to prove the result.

CONTENTS

1. Introduction 1
2. Preliminaries 6
3. Reformulation of the problem and proof 9
4. Appendix 24
References 28

1. Introduction

We consider a Cauchy problem of the three-dimensional (3-d) isentropic compressible Navier-Stokes (CNS) equations, which read in $\mathbb{R}^3$ as

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) &= \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \text{div} \mathbf{u},
\end{align*}
\]

where $t > 0, x = (x_1, x_2, x_3) \in \mathbb{R}^3, \rho(x, t) > 0$ is the density, $\mathbf{u}(x, t) = (u_1, u_2, u_3)(x, t) \in \mathbb{R}^3$ is the velocity, the pressure $p(\rho)$ satisfies $p(\rho) = \rho^\gamma$ with $\gamma \geq 1$, and $\mu > 0$ and $\lambda + \frac{2}{3} \mu \geq 0$ are the viscous coefficients.

When $\mu = \lambda = 0$, (1.1) turns to the 3-d isentropic compressible Euler equations. A planar centered rarefaction wave $(\rho^r, \mathbf{u}^r)(x, t) = (\rho^r, u_1^r, 0, 0)(x_1, t)$ is a weak entropy

* Corresponding author.

Feimin Huang is partially supported by NSFC Grant No. 11371349 and 11688101.

Qian Yuan is supported by the China Postdoctoral Science Foundation funded projects 2019M660831 and 2020TQ0345.
solution to this hyperbolic system, where \((\rho^r, u^r_1)\) solves the following Riemann problem in one dimension,

\[
\begin{align*}
\hat{c}_t \rho + \hat{c}_1 (\rho u_1) &= 0, \\
\hat{c}_t (\rho u_1) + \hat{c}_1 (\rho u_1^2) + \hat{c}_1 p(\rho) &= 0, \\
(\rho, u_1)(x_1, 0) &= \begin{cases} 
(p^-, u^-_1), & x_1 < 0, \\
(p^+, u^+_1), & x_1 > 0.
\end{cases}
\end{align*}
\]  

(1.2)

In this paper, we consider only the 2-rarefaction wave, i.e., the constants of the initial data in (1.2) satisfy the relation

\[
\bar{u}_1^+ = \bar{u}_1^- + \int_{\bar{p}^-}^{\bar{p}^+} \frac{\sqrt{p'(s)}}{s} ds \quad \text{with} \quad \bar{p}^- < \bar{p}^+.
\]  

(1.3)

We remark that the cases for the 1-rarefaction wave and a combination of two families of rarefaction waves can be proved in a similar way.

The 2-rarefaction wave \((\rho^r, u^r_1)(x_1, t)\) can be solved as follows. For \(\rho > 0\), the system (1.2) is strictly hyperbolic with two distinct eigenvalues

\[
\lambda_1(\rho, u_1) = u_1 - \sqrt{p'(\rho)}, \quad \lambda_2(\rho, u_1) = u_1 + \sqrt{p'(\rho)}.
\]

One can normalize the corresponding right eigenvectors \(r_i\) as \(\nabla \lambda_i \cdot r_i \equiv 1 \ (i = 1, 2)\). And the i-Riemann invariant \((i = 1, 2)\) is given by

\[
Z_i(\rho, u_1) = u_1 + (-1)^{i+1} \int_1^\rho \frac{\sqrt{p'(s)}}{s} ds,
\]

which satisfies \(\nabla Z_i \cdot r_i \equiv 0\). Denote \(\lambda_2^\pm := \lambda_2\left(\bar{p}^\pm, \bar{u}_1^\pm\right)\) and \(Z_2^\pm := Z_2\left(\bar{p}^\pm, \bar{u}_1^\pm\right)\). Then the 2-centered rarefaction wave \((\rho^r, u^r_1)\) can be solved exactly by

\[
\begin{align*}
\lambda_2 (\rho^r(x_1, t), u^r_1(x_1, t)) &= u^r_1(x_1, t) + \sqrt{p'(\rho^r)}(x_1, t) = \omega \left(\frac{x_1}{t}\right), \\
Z_2 (\rho^r(x_1, t), u^r_1(x_1, t)) &= u^r_1(x_1, t) - \int_1^{\rho^r(x_1, t)} \frac{\sqrt{p'(s)}}{s} ds = Z_2^- (= Z_2^+),
\end{align*}
\]  

(1.4)

where

\[
\omega (\xi) = \begin{cases} 
\lambda_2^-, & \xi < \lambda_2^-, \\
\xi, & \lambda_2^- \leq \xi < \lambda_2^+, \\
\lambda_2^+, & \xi \geq \lambda_2^+.
\end{cases}
\]

It is well-known that the compressible Euler equations have three important entropy solutions, the shock, the rarefaction wave and the contact discontinuity, which are called the Riemann solutions. For the 1-d case, it has been shown in many literatures that these Riemann solutions characterize the large time behaviors of the solutions, as long as the initial data tend to constant states at far field. In other words, the Riemann solutions are time-asymptotically stable if the initial perturbations are integrable on \(\mathbb{R}\) (at least \(L^1 \cap L^2\)-integrable). For the CNS equations, due to the effect of viscosity, the large time behaviors of the solutions are governed by the viscous versions of these three basic waves, i.e. the viscous shock wave, the rarefaction wave and the viscous contact discontinuity. For instance, Goodman [1] and Matsumura-Nishihara [19] independently
proved the stability of a single viscous shock wave with a zero-mass condition by using the anti-derivative method. And then there have been a lot of efforts [16, 17, 24] to remove the zero-mass condition, where the key is to introduce some diffusion waves propagating along the directions of other characteristic families in order to carry the excessive masses. For the rarefaction wave, Matsumura-Nishihara [21] was the first to show the stability for the isentropic CNS system, where the initial perturbations can be large but the density is away from the vacuum, and we also refer to [23] for the stability result of the full CNS system. For the contact discontinuity, Huang-Matsumura-Xin [6] and Huang-Xin-Yang [7] proved the stability with an algebraic decay rate by constructing a suitable ansatz and using the anti-derivative method.

For the multi-dimensional (m-d) wave patterns, Xin [25] showed in 1990 that the planar rarefaction waves are stable for the scalar viscous conservation laws through an $L^2$-energy method. Since then, [12, 22, 13] and the reference therein further improved the results of Xin [25], where both the strength of the wave and the initial perturbation can be large and an optimal decay rate $t^{-\frac{1}{2}}$ is also obtained. However, for the case of the m-d Navier-Stokes equations, the stability of the planar rarefaction waves is still a challenging open problem. For a $2 \times 2$ system with an artificial viscosity matrix, Hokari-Matsumura [2] proved the stability of the planar rarefaction wave in two dimensions, which crucially depends on the strict positivity of the viscosity matrix. Recently, Li-Wang [14] and Li-Wang-Wang [15] showed the stability of the planar rarefaction waves for the Navier-Stokes equations on the domain $\mathbb{R} \times \mathbb{T}$ and on an infinitely long nozzle domain $\mathbb{R} \times \mathbb{T}^2$, respectively, where the periodic boundary conditions are imposed in their settings.

In this paper, we consider a Cauchy problem of the 3-d isentropic CNS system, concerning the stability of the planar rarefaction waves under space-periodic perturbations. It is important and interesting to study the stability of the Riemann solutions under periodic perturbations, where the initial data tend to different periodic functions at far fields. It was shown in [26, 27, 28] that for the 1-d scalar conservation laws, the periodic oscillations around the shocks and rarefaction waves can be canceled as time increases due to the genuine nonlinearity of the flux. Recently, Huang-Yuan [10] further studied the m-d scalar viscous conservation laws to show the stability of the scalar planar rarefaction waves under m-d periodic perturbations. In particular, a Gagliardo-Nirenberg type inequality on $\mathbb{R} \times \mathbb{T}^{n-1}$ was introduced, which plays an important role when doing energy estimates on this unbounded domain without zero boundary conditions. We also refer to [9, 29] for the viscous shocks under periodic perturbations for the 1-d CNS equations.

Now we formulate the main result of this paper. Since the centered rarefaction wave $(\rho^r, u_1^r) (x_1, t)$ is only Lipschitz continuous, we need to construct a smooth approximation as in [20]. Let $\tilde{\omega}(x_1, t)$ be the unique smooth solution to the problem,

\[
\begin{aligned}
&\tilde{\omega}_t + \tilde{\omega}_1 \left( \frac{\tilde{\omega}^2}{2} \right) = 0, \\
&\tilde{\omega}(x_1, 0) = \frac{\lambda^2_2 + \lambda^2_1}{2} + \frac{\lambda^2_2 - \lambda^2_1}{2} \tanh(x_1).
\end{aligned}
\]
Same as (1.4), we let \((\tilde{\rho}_r, \tilde{u}_r)\) be the smooth functions solved uniquely by
\[
\tilde{u}_r(x_1, t) + \sqrt{\rho'(\tilde{\rho}_r)}(x_1, t) = \tilde{\omega}(x_1, t),
\]
\[
\tilde{u}_r(x_1, t) - \int_1^{\tilde{\rho}(x_1, t)} \frac{\sqrt{\rho'(s)}}{s} ds = Z_2^- ( = Z_2^+ ).
\]
From this, one has that
\[
\tilde{u}_r(x_1, t) = \tilde{u}_1 + \int_{\tilde{\rho}}^{\tilde{\rho}(x_1, t)} \frac{\sqrt{\rho'(s)}}{s} ds.
\]

To study the stability of the planar rarefaction waves under 3-d periodic perturbations, we prescribe the initial data for (1.1) as
\[
(\rho, \rho u)(x, 0) = (\tilde{\rho}_r, \tilde{\rho}_r \tilde{u}_r, 0, 0) (x_1) + (v_0, w_0)(x), \quad x \in \mathbb{R}^3,
\]
where \(v_0(x) \in \mathbb{R}\) and \(w_0(x) = (w_{1,0}, w_{2,0}, w_{3,0})(x) \in \mathbb{R}^3\) are periodic functions defined on the 3-d torus \(T^3 := [0, 1]^3\), satisfying
\[
\int_{T^3} (v_0, w_0)(x) dx = 0.
\]

**Remark 1.1.** The condition (1.8) indicates that the periodic perturbations of the conservative quantities, the density and the momentum, should have zero averages. Otherwise, if the condition (1.8) does not hold, the problem (1.1), (1.7) turns to be connected with other kinds of Riemann solutions, such like a shock or a combination of multiple waves, which is not the topic of this paper and will be studied in future works.

**Remark 1.2.** The solution \((\rho, u)\) to the problem (1.1), (1.7) is periodic with respect to only \(x_2\) and \(x_3\), but not to \(x_1\). Thus, the problem cannot be studied on the bounded torus \(T^3\), but on the unbounded domain \(\Omega := \mathbb{R} \times T^2\) instead. However, the solution keeps oscillating as \(|x_1| \to +\infty\), thus a suitable ansatz is needed if we want to use the energy method.
tends to the periodic solutions (\(\rho^\pm, u^\pm\)) as \(x_1 \to \pm \infty\), respectively. It is noted that due to the conservative form of (1.1) with the condition (1.8), the periodic perturbations

\[ (v^\pm, w^\pm) = (\rho^\pm, \rho^\pm u^\pm) - (\bar{\rho}, \bar{\rho} u^\pm, 0, 0) \]  

(1.10)

satisfy that \(\int_{\Omega^3} (v^\pm, w^\pm) (x, t) dx \equiv 0\), i.e. the periodic solutions \((\rho^\pm, \rho^\pm u^\pm)(x, t)\) still have the same averages \((\bar{\rho}^\pm, \bar{\rho}^\pm u^\pm)\) as the initial data \((1.9) \pm\) for all \(t \geq 0\). And it is well-known (see Lemma 2.3) that the periodic solutions \((\rho^\pm, \rho^\pm u^\pm)(x, t)\) tend to their averages as \(t \to +\infty\). And this is the reason why the condition (1.8) is necessary to ensure the stability of the background rarefaction wave, as stated in Remark 1.1. The ansatz is constructed as follows, which is similar to [10].

**Ansatz.** Inspired by the formulas of the background smooth rarefaction wave,

\[
\begin{align*}
\bar{\rho}'(x_1, t) &= \bar{\rho}^- (1 - \sigma(x_1, t)) + \bar{\rho}^+ \sigma(x_1, t), \\
\bar{u}'_1(x_1, t) &= \bar{u}^- (1 - \eta(x_1, t)) + \bar{u}^+ \eta(x_1, t),
\end{align*}
\]

(1.11)

where

\[
\begin{align*}
\sigma(x_1, t) &:= \frac{\bar{\rho}'(x_1, t) - \bar{\rho}^-}{\bar{\rho}^+ - \bar{\rho}^-}, & \eta(x_1, t) &:= \frac{\bar{u}'_1(x_1, t) - \bar{u}^-}{\bar{u}^+ - \bar{u}^-},
\end{align*}
\]

(1.12)

we set the ansatz \((\rho, \bar{u}) = (\rho, \bar{u}_1, \bar{u}_2, \bar{u}_3)\) as

\[
\begin{align*}
\rho(x, t) &:= \rho^-(x_1, t) (1 - \sigma(x_1, t)) + \rho^+(x_1, t) \sigma(x_1, t), \\
\rho^-(x_1, t) + v^-(x_1, t) (1 - \sigma(x_1, t)) + v^+(x_1, t) \sigma(x_1, t), \\
\bar{u}(x, t) &:= u^-(x_1, t) (1 - \eta(x_1, t)) + u^+(x_1, t) \eta(x_1, t), \\
\bar{u}^-(x_1, t) + z^-(x_1, t) (1 - \eta(x_1, t)) + z^+(x_1, t) \eta(x_1, t),
\end{align*}
\]

(1.13)

where \(e_1 := (1, 0, 0)\) is a unit vector and \(z^\pm(x, t) := u^\pm(x, t) - \bar{u}^\pm\). Note that if \(\|v_0\|_{L^\infty(\Omega^3)}\) is small enough, it follows form (1.10) that

\[
z^\pm(x, t) = \frac{w^\pm(x, t) - v^\pm(x, t) \bar{u}^\pm}{\rho^\pm(x, t)}.
\]

(1.14)

Same as the solution to (1.1), (1.7), the ansatz (1.13) is also periodic with respect to only \(x_2\) and \(x_3\), but not to \(x_1\). Recall the domain \(\Omega = \mathbb{R} \times \mathbb{T}^2\). We are ready to state the main theorem.

**Theorem 1.3.** Assume in (1.7) that the periodic perturbation \((v_0, w_0) \in H^5(\mathbb{T}^3)\) and satisfies (1.8). Then there exist small \(\delta_0 > 0\) and \(\varepsilon_0 > 0\) such that, if

\[
|\overline{\rho}^+ - \overline{\rho}^-| \leq \delta_0 \quad \text{and} \quad \|(v_0, w_0)\|_{H^5(\mathbb{T}^3)} \leq \varepsilon_0,
\]

(1.15)
the problem (1.1), (1.7) admits a unique global solution \((\rho, u) (x, t)\), which is periodic with respect to \(x_2\) and \(x_3\), and satisfies that
\[
(\rho - \bar{\rho}, u - \bar{u}) \in C \left( 0, +\infty; H^2(\Omega) \right),
\]
\[
\nabla (\rho - \bar{\rho}) \in L^2 \left( 0, +\infty; H^1(\Omega) \right),
\]
\[
\nabla (u - \bar{u}) \in L^2 \left( 0, +\infty; H^2(\Omega) \right),
\]
with the large time behavior
\[
\sup_{x \in \mathbb{R}^3} \| (\rho, u) (x, t) - (\rho^*, u^*_1, 0, 0) (x_1, t) \| \to 0 \quad \text{as } t \to +\infty.
\] (1.17)

The rest of this paper is organized as follows. In the next section, we will first introduce some useful lemmas and notations. In Section 3, we will prove the a priori estimates and then complete the proof of the main result. In the last appendix, the exponential decay rates of both the periodic solutions and the error terms produced by the ansatz are obtained.

2. Preliminaries

Notations. For convenience, we denote
\[
\delta := |\bar{\rho}^+ - \bar{\rho}^-| \quad \text{and} \quad \varepsilon := \| v_0, w_0 \|_{H^5(\mathbb{T}^2)},
\]
and we use the notations
\[
\| \cdot \| := \| \cdot \|_{L^2(\Omega)}, \quad \| \cdot \|_l := \| \cdot \|_{H^l(\Omega)} \quad \text{for } l \geq 1.
\]

Lemma 2.1 ([20], Lemma 2.1). The smooth rarefaction wave \((\bar{\rho}^*, \bar{u}_1^*)\) solving (1.5) satisfies the following properties.

i) \((\bar{\rho}^*, \bar{u}_1^*)\) solves the 1-d isentropic Euler equations;

ii) \(\hat{\partial}_1 \bar{\rho}^* > 0, \hat{\partial}_1 \bar{u}_1^* > 0\) and there exists a constant \(C > 0\), independent of either \(\delta\) or \(t\), such that \(|\bar{\rho}_1^2 \bar{u}_1^*| \leq C \bar{\rho}_1^2 \bar{u}_1^*\) for all \(t \geq 0\) and \(x_1 \in \mathbb{R}\);

iii) \(\sup_{x_1 \in \mathbb{R}} \| (\bar{\rho}^*, \bar{u}_1^*) - (\rho^*, u_1^*) \| (x_1, t) \to 0 \quad \text{as } t \to +\infty\);

iv) For any \(p \in [1, +\infty]\) and \(t \geq 0\), it holds that
\[
\| \nabla_{t,x_1} (\bar{\rho}^*, \bar{u}_1^*) \|_{L^p(\mathbb{R})} \leq C \min \left\{ \delta, \delta^{1/p} (1 + t)^{-1 + 1/p} \right\},
\]
\[
\| \nabla_{t,x_1}^m (\bar{\rho}^*, \bar{u}_1^*) \|_{L^p(\mathbb{R})} \leq C \min \left\{ \delta, (1 + t)^{-1} \right\} \quad \text{for } m = 2, 3,
\]
where \(C > 0\) is independent of either \(\delta\) or \(t\).

Lemma 2.2. The functions \(\sigma\) and \(\eta\) defined in (1.12) are smooth and satisfy the following properties.

i) \(0 < \sigma(x_1, t), \eta(x_1, t) < 1\) and \(\hat{\partial}_1 \sigma(x_1, t), \hat{\partial}_1 \eta(x_1, t) > 0\) for any \((x_1, t) \in \mathbb{R} \times [0, +\infty)\).

ii) For any \(p \in [1, +\infty]\), it holds that
\[
\| \sigma(1 - \sigma), \sigma(1 - \eta), \eta(1 - \sigma), \eta(1 - \eta) \|_{L^p(\mathbb{R})} \leq C(1 + t)^{1/p},
\]
\[
\| \sigma - \eta \|_{L^p(\mathbb{R})} \leq C\delta (1 + t)^{1/p},
\]
\[
\sup_{t > 0} \left\| \nabla_{t,x_1}^m (\sigma, \eta) \right\|_{L^p(\mathbb{R})} \leq C \quad \text{for } m = 1, 2, 3,
\]
where the constant \( C > 0 \) is independent of either \( \delta \) or \( t \).

**Proof.** Let \( C \) denote a constant independent of either \( \delta \) or \( t \). From Lemma 2.1, it is direct to prove i). And for ii), we prove only \( \sigma(1 - \eta) \) and \( \sigma - \eta \), since the proof of others is similar and the proof of the derivatives is straightforward.

It follows from (1.3) that \( C^{-1} \delta \leq |\overline{u}_1^+ - \overline{u}_1^-| \leq C\delta \). Then by (1.5), one has that

\[
\hat{\mathcal{c}}_1 \hat{u}_1^+ = \frac{\sqrt{p'(\rho^+)}}{\rho'} \hat{\mathcal{c}}_1 \hat{\rho}^+, \quad \hat{\mathcal{c}}_1 \hat{u}_1^- = \left(1 + \frac{\rho''(\rho^+ \rho^+)}{2p'(\rho^+)}\right)^{-1} \hat{\mathcal{c}}_1 \hat{\omega} \leq C \hat{\mathcal{c}}_1 \hat{\omega}.
\]

which yields that

\[
0 < \hat{\rho}^- - \hat{\rho}^- = \int_{-\infty}^{x_1} \hat{\gamma}_y \hat{\rho}^+(y,t) \, dy \leq C \int_{-\infty}^{x_1} \hat{\gamma}_y \hat{\omega}(y,t) \, dy \leq C \left( \hat{\omega} - \lambda_2^- \right),
\]

and similarly,

\[
0 < \hat{\rho}^+ - \hat{\rho}^- \leq C \left( \lambda_2^+ - \hat{\omega} \right).
\]

Thus, one has that

\[
\left\| \sigma(1 - \eta) \right\|_{\text{L}^p(\mathbb{R})} \leq C \delta^{-2} \left\| \left( \hat{\rho}^- - \hat{\rho}^- \right) \left( \hat{u}_1^+ - \overline{u}_1^+ \right) \right\|_{\text{L}^p(\mathbb{R})} \leq \left\| \left( \hat{\omega} - \lambda_2^- \right) \left( \hat{\omega} - \lambda_2^- \right) \right\|_{\text{L}^p(\mathbb{R})} \leq C(1 + t)^{1/p},
\]

where the last inequality can be derived from the characteristic curve method and the fact that \( |\lambda_2^+ - \lambda_2^-| \leq C\delta \).

Then we show the proof of \( |\sigma - \eta| \). Denote \( A(\rho) := \int_1^\rho \frac{\sqrt{p'(s)}}{s} \, ds \). It can follow from (1.3) and (1.6) that

\[
\sigma - \eta = \sigma \left[ 1 - \int_0^1 A'(\overline{\rho}^- + s(\rho^+ - \rho^-)) \, ds \right] - \int_0^1 A'(\overline{\rho}^- + s(\rho^+ - \rho^-)) \, ds 
\]

where

\[
b(\overline{\rho}) = \left( \overline{\rho}^- - \overline{\rho}^- \right) \int_0^1 A''(\overline{\rho}^- + s(\rho^+ - \rho^-)) + rs(\overline{\rho}^- + \rho^-) drds \]

which satisfies that \( \left\| \frac{\sqrt{p'(s)}}{s} \right\|_{\text{L}^p(\mathbb{R} \times [0, +\infty])} \leq C\delta \). Thus, one has that

\[
\left\| \sigma - \eta \right\|_{\text{L}^p(\mathbb{R})} \leq C\delta \left\| \sigma(1 - \sigma) \right\|_{\text{L}^p(\mathbb{R})} \leq C\delta(1 + t)^{1/p},
\]

which finishes the proof. \( \square \)
Moreover, the periodic perturbations (1.10) and (1.14) satisfy that

Moreover, it holds that

Lemma 2.5. Without any additional boundary conditions do not satisfy the 3-d Gagliardo-Nirenberg inequality, either $\delta, \varepsilon$ or $\alpha$.

By using the energy method with the aid of the Poincaré inequality on $\mathbb{T}^3$, the proof of Lemma 2.3 is standard and is placed in the appendix.

Although the ansatz (1.13) is not a solution to (1.1), its error terms

$$h_0 := \partial_t \tilde{\rho} + \text{div}(\tilde{\rho} \tilde{u}),$$

$$h = (h_1, h_2, h_3) := \partial_t (\tilde{\rho} \tilde{u}) + \text{div}(\tilde{\rho} \tilde{u} \otimes \tilde{u}) + \nabla p(\tilde{\rho}) - \mu \Delta \tilde{u} - (\mu + \lambda) \nabla \text{div} \tilde{u}$$

(2.4)

decay exponentially fast with respect to time. More precisely, it holds that

Lemma 2.4. Under the assumptions of Theorem 1.3, the error terms (2.4) satisfy that

$$\|h_0\|_{W^{2,p}(\Omega)} + \|h + (2\mu + \lambda) \tilde{u}^t \tilde{e}_1\|_{W^{1,p}(\Omega)} \leq C \varepsilon e^{-\alpha t}, \quad p \in [1, +\infty],$$

(2.5)

where $\alpha > 0$ is the constant in Lemma 2.3 and the constant $C > 0$ is independent of either $\delta, \varepsilon$ or $t$.

The proof of Lemma 2.4 is similar to [10, Lemma 2.3], which is based on Lemmas 2.2 and 2.3. We still place it in the appendix for brevity.

As indicated in [10], the functions which are integrable on the domain $\Omega = \mathbb{R} \times \mathbb{T}^2$ without any additional boundary conditions do not satisfy the 3-d Gagliardo-Nirenberg (G-N) inequalities in general (any 1-d function $f(x_1) \in C_{c}^{\infty}(\mathbb{R})$, which is periodic with respect to $x_2$ and $x_3$, is a counterexample). To solve the problem (1.1), (1.7) on the domain $\Omega = \mathbb{R} \times \mathbb{T}^2$, we need the following G-N type inequality.

Lemma 2.5 ([10], Theorem 1.4). Assume that $u(x)$ is in the $L^q(\Omega)$ space with $\nabla^m u \in L^p(\Omega)$, where $1 \leq q, r \leq +\infty$ and $m \geq 1$, and $u$ is periodic with respect to $x_2$ and $x_3$.

Then there exists a decomposition $u(x) = \sum_{k=1}^{3} u^{(k)}(x)$ such that each $u^{(k)}$ satisfies the $k$-dimensional G-N inequality,

$$\|\nabla^j u^{(k)}\|_{L^p(\Omega)} \leq C \|\nabla^m u\|_{L^r(\Omega)}^{\theta_k} \|u\|_{L^q(\Omega)}^{1-\theta_k},$$

(2.6)

where $0 \leq j < m$ is any integer and $1 \leq p \leq +\infty$ is any number, satisfying

$$\frac{1}{p} = \frac{j}{k} + \left(\frac{1}{r} - \frac{m}{k}\right) \theta_k + \frac{1}{q} (1 - \theta_k) \quad \text{with} \quad \frac{j}{m} \leq \theta_k \leq 1.$$

Moreover, it holds that

$$\|\nabla^j u\|_{L^p(\Omega)} \leq C \sum_{k=1}^{3} \|\nabla^m u\|_{L^r(\Omega)}^{\theta_k} \|u\|_{L^q(\Omega)}^{1-\theta_k},$$

(2.7)
The constants $C > 0$ in (2.6) and (2.7) are independent of $u$.

3. Reformulation of the Problem and Proof

We first show the equations satisfied by the perturbation terms,

$$
\phi := \rho - \bar{\rho} \quad \text{and} \quad \psi = (\psi_1, \psi_2, \psi_3) := u - \bar{u}.
$$

(3.1)

It follows from (1.1) and (2.4) that

$$
\begin{align*}
\partial_t \phi + \rho \partial_t \psi &= \nabla \phi + \phi \partial_t \bar{u} + \nabla \bar{\rho} \cdot \psi = -h_0, \\
\rho \partial_t \psi + \rho u \cdot \nabla \psi + \rho \psi \cdot \nabla \bar{u} + \partial_t (\rho) \nabla \phi + \left( \rho' \rho - \rho' \bar{\rho} \right) \nabla \bar{\rho} \\
&\quad - \mu \Delta \psi - (\mu + \lambda) \nabla \psi = f - \phi g + (2\mu + \lambda) \bar{\partial}_t^2 \bar{u}_t e_1,
\end{align*}
$$

(3.3)

where

$$
\begin{align*}
f &= (f_1, f_2, f_3)^T = h_0 \bar{u} - h - (2\mu + \lambda) \bar{\partial}_t^2 \bar{u}_t e_1, \\
g &= (g_1, g_2, g_3)^T = \frac{1}{\rho} \left[ \mu \Delta \bar{u} + (\mu + \lambda) \nabla \partial_t \psi \bar{u} + h - h_0 \bar{u} \right], \\
&\quad = \frac{1}{\rho} \left[ \mu \Delta (\bar{u} - \bar{u}_t e_1) + (\mu + \lambda) \nabla \partial_t (\bar{u} - \bar{u}_t e_1) - f \right],
\end{align*}
$$

which satisfy from Lemma 2.4 that

$$
\begin{align*}
\|f\|_{W^{1,p}(\Omega)} &\leq C \varepsilon e^{-a t} \quad \forall p \in [1, +\infty], \\
\|g\|_{W^{1,\infty}(\Omega)} &\leq C \varepsilon e^{-a t}.
\end{align*}
$$

(3.4)

From (1.7), (1.13) and (1.14), one has that

$$
\begin{align*}
\phi(x, 0) &= \rho(x, 0) - \bar{\rho} \cdot (x_1, 0) - v_0(x) (1 - \sigma(x_1, 0)) - v_0(x) \sigma(x_1, 0) = 0, \\
\psi(x, 0) &= \frac{\bar{\rho} \cdot (x_1, 0) \bar{u}_1 \cdot (x_1, 0) e_1 + w_0(x)}{\bar{\rho} \cdot (x_1, 0) + v_0(x)} - \bar{u}_1 \cdot (x_1, 0) e_1 \\
&\quad - \frac{w_0(x) - \bar{u}_1 \cdot v_0(x) e_1}{\bar{\rho} + v_0(x)} (1 - \eta(x_1, 0)) - \frac{w_0(x) - \bar{u}_1 \cdot v_0(x) e_1}{\bar{\rho} + v_0(x)} \eta(x_1, 0) \\
&= \left( \frac{1}{\bar{\rho} \cdot (x_1, 0) + v_0(x)} - \frac{1 - \eta(x_1, 0)}{\bar{\rho} + v_0(x)} - \frac{\eta(x_1, 0)}{\bar{\rho} + v_0(x)} \right) w_0(x) \\
&\quad - \left( \frac{\bar{u}_1 \cdot (x_1, 0)}{\bar{\rho} \cdot (x_1, 0) + v_0(x)} - \frac{1 - \eta(x_1, 0)}{\bar{\rho} + v_0(x)} - \frac{\eta(x_1, 0)}{\bar{\rho} + v_0(x)} \right) v_0(x) e_1 \\
&\quad = \frac{\bar{\rho} + v_0(x)}{\bar{\rho} \cdot (x_1, 0) + v_0(x)} \left[ \frac{(\eta(1 - \sigma))(x_1, 0)}{\bar{\rho} + v_0(x)} - \frac{(\sigma(1 - \eta))(x_1, 0)}{\bar{\rho} + v_0(x)} \right] w_0(x) \\
&\quad - \frac{\bar{\rho} + v_0(x)}{\bar{\rho} \cdot (x_1, 0) + v_0(x)} \left[ \frac{\bar{\rho} \cdot (\eta(1 - \sigma))(x_1, 0)}{\bar{\rho} + v_0(x)} - \frac{\bar{\rho} \cdot (\sigma(1 - \eta))(x_1, 0)}{\bar{\rho} + v_0(x)} \right] v_0(x) e_1 \\
&\quad := q_1(x) w_0(x) - q_2(x) v_0(x) e_1.
\end{align*}
$$

If $\delta_0 \leq 1$ and $\varepsilon_0$ is small enough, it follows from Lemma 2.2 that

$$
\|\psi(\cdot, 0)\|_2 \leq \|w_0, v_0\|_{W^{2,2}} \|q_1, q_2\|_2 \leq C_0 \delta \leq C_0 \varepsilon,
$$

(3.5)
where \( C_0 > 0 \) is independent of \( \varepsilon \) or \( \delta \).

Now we aim to solve the problem (3.2) and (3.3) with the initial data
\[
\begin{align*}
\phi(x, 0) &= \phi_0(x) = 0, \\
\psi(x, 0) &= \psi_0(x) = q_1(x)w_0(x) - q_2(x)v_0(x)e_1,
\end{align*}
\tag{3.6}
\]
where \( \psi_0 \) satisfies (3.5). The proof consists of the a priori estimates (Theorem 3.1) and the local existence (Proposition 3.7). For \( T > 0 \), denote
\[
N(T) := \left\{ \sup_{t \in (0, T)} \|\phi, \psi\|_2^2 + \int_0^T \left( \|\nabla \phi\|_1^2 + \|\nabla \psi\|_2^2 \right) dt \right\}^{\frac{1}{2}}.
\tag{3.7}
\]

**Theorem 3.1.** Under the assumptions of Theorem 1.3, let \( T > 0 \) and
\[
(\phi, \psi) \in C \left( 0, T; H^2(\Omega) \right) \quad \text{with} \quad \nabla \phi \in L^2 \left( 0, T; H^1(\Omega) \right), \quad \nabla \psi \in L^2 \left( 0, T; H^2(\Omega) \right),
\]
solve the problem (3.2), (3.3) with initial data \((\phi_0, \psi_0)(x) \in H^2(\Omega)\). Then there exist \( \varepsilon_0 > 0, \delta_0 > 0 \) and \( \nu_0 > 0 \) such that if \( \varepsilon < \varepsilon_0, \delta < \delta_0 \) and \( N(T) < \nu_0 \), then
\[
N(T)^2 \leq C \|\phi_0, \psi_0\|_2^2 + C(\varepsilon + \delta^\frac{1}{2}),
\tag{3.8}
\]
where the constant \( C > 0 \) is independent of \( \varepsilon, \delta \) or \( N(T) \).

For convenience, in the remaining part of this paper, we let \( C > 0 \) denote a generic constant which is independent of \( \varepsilon, \delta \) or \( N(T) \). Under the assumptions of Theorem 3.1, it follows from Lemma 2.5 that
\[
\sup_{t \in (0, T)} \|\phi, \psi\|_{L^\infty(\Omega)} \leq C \sup_{t \in (0, T)} \left\{ \|\nabla (\phi, \psi)\|_2^\frac{1}{2} \|\phi, \psi\|_2^\frac{1}{2} + \|\nabla (\phi, \psi)\| + \|\nabla^2 (\phi, \psi)\|_2^{\frac{3}{2}} \|\phi, \psi\|_2^{\frac{1}{2}} \right\}
\leq C \sup_{t \in (0, T)} \|\phi, \psi\|_2 \leq CN(T).
\]
Thus, one can first choose \( 0 < \nu_0 < 1 \) small enough such that
\[
\frac{1}{2\bar{p}^+} \leq \inf_{x \in \Omega} \rho(x, \bar{t}) \leq \sup_{x \in \Omega} \rho(x, \bar{t}) \leq 2\bar{p}^+ \quad \text{and} \quad \sup_{x \in \Omega} |u(x, t)| \leq C.
\tag{3.9}
\]

**Lemma 3.2.** Under the assumptions of Theorem 3.1, if \( \varepsilon_0 > 0 \) and \( \nu_0 > 0 \) are small enough, then
\[
\sup_{t \in (0, T)} \|\phi, \psi\|_2^2 + \int_0^T \left( (\tilde{\partial}_1 \tilde{u}_1)^\frac{1}{2} (\phi, \psi_1) \right)^2 dt + \int_0^T \|
abla \psi\|_2^2 dt \leq C \|\phi_0, \psi_0\|_2^2 + C(\varepsilon + \delta)^{\frac{1}{2}}.
\tag{3.10}
\]

**Proof.** Define
\[
\Phi(\rho, \tilde{\rho}) = \int_{\rho}^{\tilde{\rho}} \frac{p(s) - p(\tilde{\rho})}{s^2} ds = \frac{\gamma}{(\gamma - 1)\tilde{\rho}} \left[ p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho})(\rho - \tilde{\rho}) \right],
\]
which satisfies \( C^{-1} |\phi|^2 \leq \Phi(\rho, \tilde{\rho}) \leq C |\phi|^2 \) for some constant \( C > 0 \), and
\[
\partial_\rho \Phi = \frac{p(\rho) - p(\tilde{\rho})}{\rho^2}, \quad \partial_{\tilde{\rho}} \Phi = p'(\tilde{\rho}) \left( \frac{1}{\rho} - \frac{1}{\tilde{\rho}} \right).
\]
Then multiplying $\Phi$ on the first equation of (1.1) yields that
\[
\dot{c}_t (\rho \Phi) + \text{div} (\rho \Phi u) = \rho (\dot{c}_t \Phi + u \cdot \nabla \Phi)
\]
\[
= \rho \dot{c}_t \rho (\dot{c}_t \rho + u \cdot \nabla \rho) + \rho \dot{c}_t \rho (\dot{c}_t \rho + u \cdot \nabla \rho)
\]
\[
= - [p(\rho) - p(\bar{\rho})] \text{div} u - p'(\bar{\rho}) \dot{\rho} \left[ \frac{1}{\bar{\rho}} (h_0 + \psi \cdot \nabla \bar{\rho} - \text{div} \bar{u}) \right]
\]
\[
= - [p(\rho) - p(\bar{\rho}) - p'(\bar{\rho}) \dot{\rho}] \text{div} \bar{u} - \text{div} [(p(\rho) - p(\bar{\rho})) \psi]
\]
\[
+ \psi \cdot \nabla (p(\rho) - p(\bar{\rho})) - \frac{p'(\bar{\rho})}{\bar{\rho}} \dot{\rho} (h_0 + \psi \cdot \nabla \bar{\rho}) ,
\]
which gives that
\[
\dot{c}_t (\rho \Phi) + [p(\rho) - p(\bar{\rho}) - p'(\bar{\rho}) \dot{\rho}] \dot{c}_t \bar{u}_1^r - p'(\rho) \psi \cdot \nabla \phi
\]
\[
= - \text{div} Q_1 + \left( p'(\rho) - \frac{\rho}{\bar{\rho}} p'(\bar{\rho}) \right) \psi \cdot \nabla \bar{\rho}
\]
\[
- [p(\rho) - p(\bar{\rho}) - p'(\bar{\rho}) \dot{\rho}] (\text{div} \bar{u} - \dot{c}_t \bar{u}_1^r) - \frac{p'(\bar{\rho})}{\bar{\rho}} h_0 \phi ,
\]
where $Q_1 = \rho \Phi u + (p(\rho) - p(\bar{\rho})) \psi$, and it follows from Lemma 2.3 that
\[
\| \text{div} \bar{u} - \dot{c}_t \bar{u}_1^r \|_{L^2(\Omega)} \leq C \| z^\pm \|_{W^{1,\infty}(\Omega)} \leq C \varepsilon e^{-\alpha t}.
\]
Multiplying $\cdot \psi$ on both sides of (3.3) yields that
\[
\dot{c}_t \left( \frac{1}{2} \rho |\psi|^2 \right) - \frac{1}{2} \dot{c}_t \rho |\psi|^2 + \rho (u \cdot \nabla \psi) \cdot \psi + \rho (\psi \cdot \nabla \bar{u}) \cdot \psi + p'(\rho) \nabla \phi \cdot \psi
\]
\[
+ \left( p'(\rho) - \frac{\rho}{\bar{\rho}} p'(\bar{\rho}) \right) \nabla \rho \cdot \psi + \mu |\nabla \psi|^2 - \frac{\mu}{2} \text{div} (\nabla |\psi|^2) + (\mu + \lambda) |\text{div} \psi|^2
\]
\[
- (\mu + \lambda) \text{div} (\psi \text{div} \psi) = f \cdot \psi - \phi g \cdot \psi + (2\mu + \lambda) \dot{c}_t \bar{u}_1^r \psi_1 .
\]
Note that
\[
- \frac{1}{2} \dot{c}_t \rho |\psi|^2 + \rho (u \cdot \nabla) \psi \cdot \psi = - \frac{1}{2} \left[ \dot{c}_t \rho + \text{div} (\rho u) \right] |\psi|^2 + \text{div} \left( \frac{1}{2} \rho |\psi|^2 u \right)
\]
\[
= \text{div} \left( \frac{1}{2} \rho |\psi|^2 u \right) ,
\]
and
\[
\rho (\psi \cdot \nabla \bar{u}) \cdot \psi = (\rho (\psi \cdot \nabla (\bar{u} e_1)) \cdot \psi + \rho (\psi \cdot \nabla (\bar{u} - \bar{u}_1 e_1)) \cdot \psi
\]
\[
= \rho \dot{c}_t \bar{u}_1^r \psi_1^2 + \rho \left( \psi \cdot \nabla (\bar{u} - \bar{u}_1 e_1) \right) \cdot \psi .
\]
By (1.13) and Lemma 2.3, one has that
\[
| I_1 | \leq C \| z^\pm \|_{W^{1,\infty}(\Omega)} |\psi|^2 \leq C \varepsilon e^{-\alpha t} |\psi|^2 .
\]
Then one has
\[
\dot{c}_t \left( \frac{1}{2} \rho |\psi|^2 \right) + \mu |\nabla \psi|^2 + (\mu + \lambda) |\text{div} \psi|^2 + p'(\rho) \psi \cdot \nabla \phi + \rho \dot{c}_t \bar{u}_1^r \psi_1^2
\]
\[
= f \cdot \psi - \phi g \cdot \psi + (2\mu + \lambda) \dot{c}_t \bar{u}_1^r \psi_1 + \text{div} Q_2 - I_1 - \left( p'(\rho) - \frac{\rho}{\bar{\rho}} p'(\bar{\rho}) \right) \nabla \bar{\rho} \cdot \psi ,
\]
(3.12)
where \( Q_2 = \frac{\mu}{2} \nabla |\psi|^2 + (\mu + \lambda) \psi \text{div} \psi - \frac{1}{2} \rho |\psi|^2 \mathbf{u} \). Summing (3.11) and (3.12) together, integrating the resulting equation over \( \Omega \times (0, T) \) and combining (3.4), one has that

\[
\sup_{t \in (0,T)} \|\phi, \psi\|^2 + \int_0^T \|\nabla \psi\|^2 \, dt + \int_0^T \left\| (\partial_t \tilde{u}_1)^2 (\phi, \psi) \right\|^2 \, dt \\
\leq C \|\phi_0, \psi_0\|^2 + C \int_0^T \left\{ \varepsilon e^{-\alpha t} \left( \|\phi\|^2 + \|\psi\|^2 \right) + \|h_0\| \|\phi\| + \left( \|f\| + \|g\|_{L^\infty(\Omega)} \|\phi\| \right) \|\psi\| + \int_\Omega \left| \bar{c}_1^2 \tilde{u}_1 \right| |\psi_1| \, dx \right\} \, dt \\
\leq C \|\phi_0, \psi_0\|^2 + C \varepsilon \sup_{t \in (0,T)} \|\phi, \psi\|^2 + C \varepsilon + C \int_0^T \int_\Omega \left| \bar{c}_1^2 \tilde{u}_1 \right| |\psi| \, dx \, dt.
\] (3.13)

Decompose \( \psi = \sum_{k=1}^3 \psi^{(k)} \) as in Lemma 2.5 such that \( \psi^{(k)} \) satisfies the \( k \)-dimensional G-N inequalities. Then it follows from Lemma 2.1 that the last term in (3.13) satisfies that

\[
C \int_0^T \int_\Omega \left| \bar{c}_1^2 \tilde{u}_1 \right| |\psi| \, dx \, dt \leq C \int_0^T \int_\Omega \sum_{k=1}^3 \left| \bar{c}_1^2 \tilde{u}_1 \right| |\psi^{(k)}| \, dx \, dt \\
\leq C \int_0^T \left[ \left\| \bar{c}_1^2 \tilde{u}_1 \right\|_{L^1(\Omega)} \left( \|\psi^{(1)}\|_{L^\infty(\Omega)} + \|\psi^{(2)}\|_{L^\infty(\Omega)} \right) + \left\| \bar{c}_1^2 \tilde{u}_1 \right\|_{L^2(\Omega)} \|\psi^{(3)}\|_{L^2(\Omega)} \right] \, dt \\
\leq C \int_0^T \min\{\delta, (1+t)^{-1}\} \left( \|\nabla \psi^{(1)}\|_{L^\infty(\Omega)} + \|\psi^{(2)}\|_{L^\infty(\Omega)} \right) + \|\nabla \psi^{(3)}\|_{L^\infty(\Omega)} \) \, dt \\
\leq C \int_0^T \min\{\delta, (1+t)^{-1}\} \|\nabla \psi\|^2 \, dt + \frac{1}{2} \int_0^T \|\nabla \psi\|^2 \, dt + C \int_0^T \min\{\delta, (1+t)^{-1}\} \frac{1}{2} \|\psi\|^2 \, dt \\
\leq C \delta + \frac{1}{2} \int_0^T \|\nabla \psi\|^2 \, dt + \frac{1}{4} \int_0^T (1+t)^{-\frac{3}{2}} \|\psi\|^2 \, dt + C \int_0^T (1+t)^{\frac{3}{4}} \min\{\delta, (1+t)^{-1}\} \|\psi\|^2 \, dt \\
\leq \frac{1}{2} \int_0^T \|\nabla \psi\|^2 \, dt + \sup_{t \in (0,T)} \|\psi\|^2 + C \delta^{\frac{1}{2}}.
\] (3.14)

Collecting (3.13) and (3.14), one can obtain (3.10) if \( \varepsilon > 0 \) is small enough.

\[ \Box \]

**Lemma 3.3.** Under the assumptions of Theorem 3.1, if \( \delta_0 > 0, \varepsilon_0 > 0 \) and \( \nu_0 > 0 \) are small enough, then one has that

\[
\sup_{t \in (0,T)} \|\nabla \phi\|^2 + \int_0^T \|\nabla \phi\|^2 \, dt \leq C \|\phi_0\|^2 + C \|\psi_0\|^2 + C (\varepsilon + \delta^{\frac{1}{2}}) + C \nu_0 \int_0^T \|\nabla^3 \psi\|^2 \, dt.
\] (3.15)

**Proof.** Taking the gradient \( \nabla \) on (3.2) and then multiplying the result by \( \cdot \frac{\nabla \phi}{\rho^2} \), one has that

\[
\partial_t \left( \frac{|\nabla \phi|^2}{2\rho^2} \right) + \frac{|\nabla \phi|^2}{\rho^2} \partial_t \rho + \frac{\text{div} \psi}{\rho^2} (\nabla \tilde{\rho} + \nabla \phi) \cdot \nabla \phi + \frac{1}{\rho} \nabla \text{div} \psi \cdot \nabla \phi
\]
\[ + \frac{1}{2\rho^2} (u \cdot \nabla) |\nabla \phi|^2 + \frac{1}{\rho^2} \nabla \phi \cdot \nabla u \nabla \phi + \frac{\text{div} \bar{u}}{\rho^2} |\nabla \phi|^2 + \frac{\phi}{\rho^2} \nabla \text{div} \bar{u} \cdot \nabla \phi \]  
\[ + \frac{1}{\rho^2} \nabla \phi \cdot \nabla^2 \rho \psi + \frac{1}{\rho^2} \nabla \phi \cdot \nabla \psi \nabla \rho = -\frac{1}{\rho^2} \nabla h_0 \cdot \nabla \phi. \]  

Note that in (3.16), the sum of the second term on the first line and the first term on the second line satisfies that
\[
\frac{|\nabla \phi|^2}{\rho^3} \partial_t \rho + \frac{1}{2\rho^2} (u \cdot \nabla) |\nabla \phi|^2 = \frac{|\nabla \phi|^2}{\rho^3} \partial_t \rho + \text{div} \left( \frac{|\nabla \phi|^2}{2\rho^2} \bar{u} \right) - \text{div} \left( \frac{u}{2\rho^2} \right) |\nabla \phi|^2 
\]
\[
= \text{div} \left( \frac{|\nabla \phi|^2}{2\rho^2} \bar{u} \right) - \frac{3}{2\rho^2} \text{div} |\nabla \phi|^2 - \frac{3}{2\rho^2} \text{div} \psi |\nabla \phi|^2.
\]

Then (3.16) yields that
\[
\partial_t \left( \frac{|\nabla \phi|^2}{2\rho^2} \right) - \text{div} \bar{u} \frac{|\nabla \phi|^2}{2\rho^2} - \text{div} \psi \frac{|\nabla \phi|^2}{2\rho^2} + \text{div} \frac{\psi}{\rho^2} \nabla \rho \cdot \nabla \phi + \frac{1}{\rho} \nabla \text{div} \psi \cdot \nabla \phi 
\]
\[
+ \frac{1}{\rho^2} \left[ \nabla \phi \cdot \nabla (\bar{u} + \psi) \nabla \phi + \phi \nabla \text{div} \bar{u} \cdot \nabla \phi + \nabla \phi \cdot \nabla^2 \rho \psi + \nabla \phi \cdot \nabla \psi \nabla \rho \right] 
\]
\[
= -\text{div} \left( \frac{|\nabla \phi|^2}{2\rho^2} \bar{u} \right) - \frac{1}{\rho^2} \nabla h_0 \cdot \nabla \phi.
\]

Multiplying \( \frac{\nabla \phi}{\rho} \) on (3.3) yields that
\[
\partial_t \psi \cdot \nabla \phi + (u \cdot \nabla \psi) \cdot \nabla \phi + (\psi \cdot \nabla \bar{u}) \cdot \nabla \phi + \frac{\rho'(\rho)}{\rho} |\nabla \phi|^2 
\]
\[
+ \left( \frac{\rho'(\rho)}{\rho} - \frac{\rho'(\bar{\rho})}{\bar{\rho}} \right) \nabla \bar{\rho} \cdot \nabla \phi - \frac{1}{\rho} \left[ \mu \Delta \psi + (\mu + \lambda) \nabla \text{div} \psi \right] \cdot \nabla \phi 
\]
\[
= \frac{1}{\rho} \mathbf{f} \cdot \nabla \phi - \frac{\phi}{\rho} \mathbf{g} \cdot \nabla \phi + \frac{2\mu + \lambda}{\rho} \bar{c}_1 \bar{u}_1 \bar{c}_1 \phi.
\]

Note that
\[
\partial_t \psi \cdot \nabla \phi = \partial_t (\psi \cdot \nabla \phi) - \psi \cdot \nabla \partial_t \phi 
\]
\[
= \partial_t (\psi \cdot \nabla \phi) - \text{div} \left( \psi \partial_t \phi \right) 
\]
\[
- \text{div} \psi (h_0 + \rho \text{div} \psi + \bar{u} \cdot \nabla \phi + \phi \text{div} \bar{u} + \nabla \bar{\rho} \cdot \psi), \tag{3.18}
\]

and
\[
\frac{1}{\rho} \left[ \mu \Delta \psi + (\mu + \lambda) \nabla \text{div} \psi \right] \cdot \nabla \phi 
\]
\[
= \frac{2\mu + \lambda}{\rho} \nabla \text{div} \psi \cdot \nabla \phi + \frac{\mu}{\rho} \left( \Delta \psi - \nabla \text{div} \psi \right) \cdot \nabla \phi 
\]
\[
= \frac{2\mu + \lambda}{\rho} \nabla \text{div} \psi \cdot \nabla \phi + \frac{\mu}{\rho} \nabla \text{div} (\nabla \phi \times \text{curl} \psi) 
\]
\[
= \frac{2\mu + \lambda}{\rho} \nabla \text{div} \psi \cdot \nabla \phi + \mu \text{div} \left( \frac{\nabla \phi \times \text{curl} \psi}{\rho} \right) + \frac{\mu \nabla \rho \cdot (\nabla \phi \times \text{curl} \psi)}{\rho^2} \tag{3.19}
\]
where

\[ \frac{\mu + \lambda}{\rho} \nabla \text{div} \psi \cdot \nabla \phi + \mu \text{div} \left( \frac{\nabla \phi \times \text{curl} \psi}{\rho} \right) + \frac{\mu \nabla \bar{\rho} \cdot (\nabla \phi \times \text{curl} \psi)}{\rho^2}, \]

here and hereafter “\( \times \)” denotes the external product of vectors. Thus it holds that

\[
\partial_t (\psi \cdot \nabla \phi) + \frac{p'(\rho)}{\rho} |\nabla \phi|^2 - \frac{2\mu + \lambda}{\rho} \nabla \text{div} \psi \cdot \nabla \phi
\]

\[= \text{div} \left( \psi \partial_t \phi + \frac{\mu}{\rho} \nabla \phi \times \text{curl} \psi \right) + \psi \partial_t (h_0 + \rho \text{div} \psi + \mathbf{u} \cdot \nabla \phi + \phi \text{div} \tilde{\mathbf{u}} + \nabla \bar{\rho} \cdot \psi)
\]

\[- (\mathbf{u} \cdot \nabla \psi) \cdot \nabla \phi - (\psi \cdot \nabla \tilde{\mathbf{u}}) \cdot \nabla \phi - \left( \frac{p'(\rho)}{\rho} - \frac{p'(|\rho|)}{\rho} \right) \nabla \bar{\rho} \cdot \nabla \phi + \frac{\mu \nabla \bar{\rho} \cdot (\nabla \phi \times \text{curl} \psi)}{\rho^2}
\]

\[+ \frac{1}{\rho} f \cdot \nabla \phi - \frac{\phi}{\rho} g \cdot \nabla \phi + \frac{2\mu + \lambda}{\rho} c_1^2 \bar{u}_i \partial_i \phi. \tag{3.20}\]

Then by multiplying the constant \(2\mu + \lambda\) on (3.17) and then adding the result onto (3.20), one can get that

\[
\partial_t \left( \frac{2\mu + \lambda}{2\rho^2} |\nabla \phi|^2 + \psi \cdot \nabla \phi \right) + \frac{p'(\rho)}{\rho} |\nabla \phi|^2 = \text{div} Q_3 + \sum_{j=2}^{5} I_j. \tag{3.21}\]

where \(Q_3 = -\frac{|\nabla \phi|^2}{2\rho^2} \mathbf{u} + \psi \partial_t \phi + \frac{\mu}{\rho} \nabla \phi \times \text{curl} \psi\) and

\[
I_2 = \frac{2\mu + \lambda}{\rho^2} \left( \frac{1}{2} \text{div} \tilde{\mathbf{u}} \nabla \phi \cdot \nabla \phi \right) - \text{div} \psi \nabla \bar{\rho} \cdot \nabla \phi - \nabla \phi \cdot \nabla \tilde{\mathbf{u}} \nabla \phi - \nabla \phi \cdot \nabla \psi \nabla \bar{\rho}
\]

\[+ \rho (\text{div} \psi)^2 + \text{div} \psi \nabla \phi - (\mathbf{u} \cdot \nabla \psi) \cdot \nabla \phi - \frac{\mu}{\rho^2} \nabla \bar{\rho} \cdot (\nabla \phi \times \text{curl} \psi),
\]

\[
I_3 = - \frac{2\mu + \lambda}{\rho^2} \left( \phi \text{div} \tilde{\mathbf{u}} \cdot \nabla \phi + \nabla \phi \cdot \nabla \psi \nabla \bar{\rho} \right) + \text{div} \psi \phi \text{div} \tilde{\mathbf{u}} + \nabla \bar{\rho} \cdot \psi
\]

\[+ (\psi \cdot \nabla \tilde{\mathbf{u}}) \cdot \nabla \phi - \left( \frac{p'(\rho)}{\rho} - \frac{p'(|\rho|)}{\rho} \right) \nabla \bar{\rho} \cdot \nabla \phi - \frac{\phi}{\rho} g \cdot \nabla \phi,
\]

\[
I_4 = \frac{2\mu + \lambda}{\rho^2} \left( \frac{1}{2} \text{div} \psi |\nabla \phi|^2 - \nabla \phi \cdot \nabla \psi \nabla \phi \right),
\]

\[
I_5 = - \frac{2\mu + \lambda}{\rho^2} \nabla h_0 \cdot \nabla \phi + \text{div} \psi h_0 + \frac{1}{\rho} f \cdot \nabla \phi + \frac{2\mu + \lambda}{\rho} c_1^2 \bar{u}_i \partial_i \phi.
\]

For \(I_2\), first note that

\[
\|\nabla \tilde{\mathbf{u}}\|_{L^\infty(\Omega)} \leqslant \|\partial_1 \tilde{u}_i^r\|_{L^\infty(\mathbb{R})} + C \|v^\perp\|_{W^{1,+\infty}(\Omega)} \leqslant C(\delta + \varepsilon),
\]

\[
\|\nabla \bar{\rho}\|_{L^\infty(\Omega)} \leqslant \|\partial_1 \bar{\rho}^r\|_{L^\infty(\mathbb{R})} + C \|v^\perp\|_{W^{1,+\infty}(\Omega)} \leqslant C(\delta + \varepsilon).
\]

Then it follows from (3.9) that

\[
\int_0^T \|I_2\|_{L^1(\Omega)} \, dt \leqslant \left( \frac{1}{8} + C(\delta + \varepsilon) \right) \int_0^T \int_\Omega \frac{p'(\rho)}{\rho} |\nabla \phi|^2 \, dx \, dt + C \int_0^T \|\nabla \psi\|^2 \, dt. \tag{3.22}\]
Thus, by integrating (3.21) over one can finish the proof. And combining the fact that \( p \parallel \cdot \psi - \partial_1 \tilde{u}_1 \partial_1 \phi \leq C\varepsilon e^{-a_t} \parallel \psi \parallel \nabla \phi \), 
\[ |\psi \cdot \nabla \tilde{u}_1 \cdot \nabla \phi - \partial_1 \tilde{u}_1 \partial_1 \phi| \leq C\varepsilon e^{-a_t} \parallel \psi \parallel \nabla \phi, \]
\[ |\nabla \tilde{p} \cdot \nabla \phi - \partial_1 \tilde{u}_1 \partial_1 \phi| \leq C\varepsilon e^{-a_t} \parallel \nabla \phi \parallel. \]
And combining the fact that \( \|g\|_{L^\infty(\Omega)} \leq C\varepsilon e^{-a_t}, \partial_1 \tilde{u}_1 \leq C\partial_1 \tilde{u}_1 \parallel \|g\|_{L^\infty(\Omega)} \leq N(T) \leq \nu_0 < 1 \), it holds that
\[ \int_0^T \|I_3\|_{L^1(\Omega)} \leq CN(T) \int_0^T \left( \|\partial_2^2 \tilde{u}_1\| + \|\partial_2^1 \tilde{p}\| \right) \|\nabla \phi\| dt + C\varepsilon \int_0^T e^{-a_t} \|\phi, \psi\| \|\nabla (\phi, \psi)\| dt \]
\[ + C \int_0^T \int_{\Omega} \partial_1 \tilde{u}_1 (|\phi| + |\psi_1|) (|\nabla \phi| + |\nabla \psi|) dx dt \]
\[ \leq C\delta^{\frac{1}{2}} + C(\delta^{\frac{1}{2}} + \varepsilon) \int_0^T \||\nabla (\phi, \psi)||^2 dt + C\varepsilon \sup_{t \in (0, T)} \||\phi, \psi||^2 \]
\[ + C \int_0^T \left( \|\partial_1 \tilde{u}_1\| \right) \||\phi, \psi||^2 \] (3.23)
For \( I_5 \), it can follow easily from Lemma 2.4 and (3.4) that
\[ \int_0^T \|I_5\|_{L^1(\Omega)} dt \leq C \int_0^T \varepsilon e^{-a_t} \||\nabla (\phi, \psi)|| dt + C \varepsilon \int_0^T \min\{\delta, (1 + t)^{-1}\} \||\nabla \phi|| dt \]
\[ \leq C(\varepsilon + \delta^{\frac{1}{2}}) \int_0^T \||\nabla \phi||^2 dt + C\varepsilon \int_0^T \||\nabla \psi||^2 dt + C(\varepsilon + \delta^{\frac{1}{2}}). \] (3.24)
And for the most difficult term \( I_4 \), decompose \( \psi = \sum_{k=1}^3 \psi^{(k)} \) as in Lemma 2.5 and the a priori assumption (3.7) yield that
\[ \int_0^T \|I_4\|_{L^1(\Omega)} dt \leq C \int_0^T \int_{\Omega} |\nabla \psi| |\nabla \phi|^2 dx \]
\[ \leq C \sum_{k=1}^3 \int_0^T \||\nabla \psi^{(k)}||_{L^\infty(\Omega)} \||\nabla \phi||^2 dt \]
\[ \leq C \int_0^T \left( \||\nabla^2 \psi||^{\frac{1}{2}} \||\nabla \psi||^{\frac{1}{2}} + \||\nabla^2 \psi||^{\frac{3}{2}} \||\nabla \psi||^{\frac{1}{2}} \right) \||\nabla \phi||^2 dt \]
\[ \leq C \int_0^T \left( \||\nabla \psi||_1 + \||\nabla^2 \psi|| \right) \||\nabla \phi||^2 dt \]
\[ \leq CN(T) \int_0^T \||\nabla \phi||^2 dt + CN(T) \int_0^T \||\nabla^3 \psi|| \||\nabla \phi|| dt \]
\[ \leq C\nu_0 \int_0^T \||\nabla \phi||^2 dt + C\nu_0 \int_0^T \||\nabla^3 \psi||^2 dt. \] (3.25)
Thus, by integrating (3.21) over \( \Omega \times (0, T) \), using (3.22) to (3.25) and applying Lemma 3.2, one can finish the proof.
Lemma 3.4. Under the assumptions of Theorem 3.1, if $\delta_0 > 0, \varepsilon_0 > 0$ and $\nu_0 > 0$ are small enough, then
\[
\sup_{t \in (0, T)} \| \nabla \psi \|^2 + \int_0^T \| \nabla^2 \psi \|^2 \, dt \leq C \| \phi_0, \psi_0 \|_1^2 + C(\varepsilon + \delta^2) + C \nu_0 \int_0^T \| \nabla^3 \psi \|^2 \, dt. \tag{3.26}
\]

Proof. Multiplying $-\frac{\Delta \psi}{\rho}$ on (3.3) yields that
\[
- \partial_t \psi \Delta \psi - (\mathbf{u} \cdot \nabla \psi) \Delta \psi - (\psi \cdot \nabla \hat{\mathbf{u}}) \cdot \Delta \psi - \frac{p'(\rho)}{\rho} \nabla \phi \cdot \Delta \psi \\
- \left( \frac{p'(\rho)}{\rho} - \frac{p'(\bar{\rho})}{\bar{\rho}} \right) \nabla \bar{\rho} \cdot \Delta \psi + \frac{\mu}{\rho} |\Delta \psi|^2 + \frac{\mu + \lambda}{\rho} \nabla \div \psi \cdot \Delta \psi \\
= -\frac{1}{\rho} \mathbf{f} \cdot \Delta \psi + \frac{\phi}{\rho} \div - \frac{2\mu + \lambda}{\rho} \hat{c}_1^2 \bar{u}_1^2 \Delta \psi_1.
\]
Note that $-\partial_t \psi \cdot \Delta \psi = \frac{1}{2} \partial_t (|\nabla \psi|^2) - \div (\nabla \psi \partial_t \psi)$ and
\[
\frac{1}{\rho} \div \psi \cdot \Delta \psi = \frac{1}{\rho} |\div \psi|^2 + \frac{1}{\rho} \div \psi \cdot (\Delta \psi - \div \psi) \\
= \frac{1}{\rho} |\div \psi|^2 + \frac{1}{\rho} \div \left[ \div \psi (\Delta \psi - \div \psi) \right] \\
= \frac{1}{\rho} |\div \psi|^2 + \div \left[ \frac{\div \psi}{\rho} (\Delta \psi - \div \psi) \right] \\
+ \frac{\div \psi}{\rho^2} (\nabla \bar{\rho} + \nabla \phi) \cdot (\Delta \psi - \div \psi). \tag{3.27}
\]

Then one has that
\[
\frac{1}{2} \partial_t |\nabla \psi|^2 + \frac{\mu}{\rho} |\Delta \psi|^2 + \frac{\mu + \lambda}{\rho} |\div \psi|^2 = \div Q_4 + \sum_{j=6}^{9} I_j, \tag{3.28}
\]
where $Q_4 = \nabla \psi \partial_t \psi - \frac{\mu + \lambda}{\rho} \div (\Delta \psi - \div \psi)$ and
\[
I_6 = (\mathbf{u} \cdot \nabla \psi) \cdot \Delta \psi + \frac{p'(\rho)}{\rho} \nabla \phi \cdot \Delta \psi - \frac{\mu + \lambda}{\rho^2} \div \nabla \bar{\rho} \cdot (\Delta \psi - \div \psi), \\
I_7 = (\psi \cdot \nabla \hat{\mathbf{u}}) \cdot \Delta \psi + \left( \frac{p'(\rho)}{\rho} - \frac{p'(\bar{\rho})}{\bar{\rho}} \right) \nabla \bar{\rho} \cdot \Delta \psi + \frac{\phi}{\rho} \div \psi, \\
I_8 = -\frac{1}{\rho} \mathbf{f} \cdot \Delta \psi - \frac{2\mu + \lambda}{\rho} \hat{c}_1^2 \bar{u}_1^2 \Delta \psi_1, \\
I_9 = -\frac{\mu + \lambda}{\rho^2} \div \psi \nabla \phi \cdot (\Delta \psi - \div \psi).
Since $\int_\Omega \frac{2}{\rho} |\nabla \psi|^2 \, dx \geq C \|\nabla \psi\|^2 \geq a_0 \|\nabla^2 \psi\|^2$ for some $C > 0$ and $a_0 > 0$, which depend only on $\mu\left(\inf_{x,t} \rho(x,t)\right)^{-1}$, then similar to the proof of Lemma 3.3, one can get that

$$\int_0^T \|I_6\|_{L^1(\Omega)} \, dt \leq \frac{a_0}{8} \int_0^T \|\nabla^2 \phi\|^2 \, dt + C \int_0^T \|
abla (\phi, \psi)\|^2 \, dt,$$

$$\int_0^T \|I_7\|_{L^1(\Omega)} \, dt \leq C\varepsilon \sup_{t \in (0,T)} \|\phi, \psi\|^2 + C(\varepsilon + \delta) \int_0^T \|\nabla^2 \psi\|^2 \, dt + C \int_0^T \|\nabla (\phi, \psi)\|^2 \, dt,$$

$$\int_0^T \|I_8\|_{L^1(\Omega)} \, dt \leq \frac{a_0}{8} \int_0^T \|\nabla^2 \psi\|^2 \, dt + C(\varepsilon + \delta),$$

$$\int_0^T \|I_9\|_{L^1(\Omega)} \, dt \leq C \int_0^T \|\nabla \psi\| \|\nabla^2 \psi\| \, dt \leq C \int_0^T \left(\|\nabla \psi\|_1 + \|\nabla^3 \psi\| \right) \|\nabla \phi\| \|\nabla^2 \psi\| \, dt \leq C N(T) \int_0^T \left(\|\nabla \psi\|_1 \|\nabla^2 \psi\| + \|\nabla^2 \psi\| \|\nabla^3 \psi\| \right) \, dt \leq CN_0 \int_0^T \|\nabla \psi\|_1^2 \, dt + C\nu_0 \int_0^T \|\nabla^3 \psi\|^2 \, dt.$$

Then by integrating (3.28) over $\Omega \times (0, T)$ and applying Lemmas 3.2 and 3.3, one can finish the proof. 

\[ \square \]

**Lemma 3.5.** Under the assumptions of Theorem 3.1, if $\delta_0 > 0$, $\varepsilon_0 > 0$ and $\nu_0 > 0$ are small enough, then

$$\sup_{t \in (0,T)} \|\nabla \phi\|^2 + \int_0^T \|\nabla^2 \phi\|^2 \, dt \leq C \|\phi_0\|_2^2 + \|\psi_0\|^2 + C(\varepsilon + \delta^2) + C \nu_0 \int_0^T \|\nabla^3 \psi\|^2 \, dt. \quad (3.29)$$

**Proof.** Let $i \in \{1, 2, 3\}$ be fixed. Taking the second derivative $\nabla \hat{c}_i$ on (3.2), then multiplying the result by $\frac{\nabla \hat{c}_i}{\rho}$ and using the fact

$$-\frac{\hat{c}_i \rho}{\rho^3} |\nabla \hat{c}_i\phi|^2 + \frac{1}{2\rho^2} (\mathbf{u} \cdot \nabla) |\nabla \hat{c}_i\phi|^2 = \text{div} \left(\frac{\nabla \hat{c}_i\phi}{2\rho^2} \mathbf{u} \right) - \frac{3}{2\rho^2} \text{div} \mathbf{u} |\nabla \hat{c}_i\phi|^2,$$

one can get that

$$\hat{c}_i \left(\frac{\nabla \hat{c}_i\phi}{2\rho^2} \right) + \frac{1}{\rho} \nabla \hat{c}_i \text{div} \psi \cdot \nabla \hat{c}_i \phi = -\text{div} \left(\frac{\nabla \hat{c}_i\phi}{2\rho^2} \mathbf{u} \right) + I_{10} + I_{11}, \quad (3.30)$$

where

$$I_{10} = \frac{3}{2\rho^2} \text{div} \mathbf{u} |\nabla \hat{c}_i\phi|^2 - \frac{1}{\rho^2} \left[\nabla \hat{c}_i (\rho \text{div} \psi) - \rho \nabla \hat{c}_i \text{div} \psi \right] \cdot \nabla \hat{c}_i \phi$$

$$- \frac{1}{\rho^2} \left[\nabla \hat{c}_i (\mathbf{u} \cdot \nabla \phi) - (\mathbf{u} \cdot \nabla) \nabla \hat{c}_i \phi \right] \cdot \nabla \hat{c}_i \phi$$
Then combining the a priori assumption (3.7), one has that

$$I_{11} = -\frac{\phi}{\rho^2} \nabla \chi_i \text{div} \psi \cdot \nabla \chi_i \phi - \frac{1}{\rho^2} \left[ \nabla \chi_i (\nabla \tilde{\rho} \cdot \psi) - \nabla^2 \tilde{\rho} \tilde{\rho} \psi \right] \cdot \nabla \chi_i \phi.$$

Note that $\frac{1}{\rho^2} \nabla \chi_i \text{div} \psi \cdot \nabla \chi_i \phi$ is a high-order term in (3.30), which can be canceled by the equation (3.3). And similar to the proof of Lemmas 3.2 to 3.4, the other lower-order terms satisfy that

$$I_{10} \leq C |\nabla^2 \phi| \left[ \left( |\nabla \tilde{\mathbf{u}}| + |\nabla \psi| \right) |\nabla \phi| + \left( |\nabla^2 \tilde{\rho}| + |\nabla^2 \phi| \right) |\nabla \psi| + \left( |\nabla \tilde{\rho}| + |\nabla \phi| \right) |\nabla^2 \psi| \right]$$

$$\leq C(\varepsilon + \delta) |\nabla^2 \phi| \left( |\nabla^2 \phi| + |\nabla \psi| + |\nabla^2 \psi| + |\nabla \phi| \right) + C |\nabla \psi| |\nabla^2 \phi|^2$$

$$+ C |\nabla \phi| |\nabla^2 \psi| |\nabla \phi|,$$

$$|I_{11}| \leq C \| \phi, \psi \|_{L^\infty} \left( \| \nabla^2 \chi_i \tilde{u}_i |^2 + \| \nabla^2 \chi_i \tilde{u}_1 | \right) |\nabla^2 \phi| + C \varepsilon e^{-\alpha t} (|\phi| + |\psi|) |\nabla^2 \phi| + C |\nabla^2 h_0| |\nabla \phi|.$$

For $I_{10}$, first note that by Lemma 2.5, one has

$$\| \nabla \phi \|_{L^4(\Omega)} \leq C \sum_{k=1}^3 \| \nabla^2 \phi \|^k \| \nabla \phi \|^{1-k/4} \leq C N(T),$$

$$\| \nabla^2 \psi \|_{L^4(\Omega)} \leq C \sum_{k=1}^3 \| \nabla^3 \psi \|^k \| \nabla^2 \psi \|^{1-k/4}.$$  \hspace{1cm} (3.31)

Then combining the a priori assumption (3.7), one has that

$$\int_0^T \int_\Omega |\nabla \phi| |\nabla^2 \psi| |\nabla^2 \phi| \, dx \, dt \leq \int_0^T \| \nabla \phi \|_{L^4(\Omega)} \| \nabla^2 \psi \|_{L^4(\Omega)} \| \nabla^2 \phi \| \, dt$$

$$\leq C N(T) \int_0^T \sum_{k=1}^3 \| \nabla^3 \psi \|^k \| \nabla^2 \psi \|^{1-k/4} \| \nabla \phi \| \, dt$$

$$\leq C \nu_0 \int_0^T \left( \| \nabla^3 \psi \|^2 + \| \nabla^2 \psi \|^2 + \| \nabla \phi \|^2 \right) \, dt$$

$$\leq C \nu_0 \int_0^T \| \nabla^2 (\phi, \psi) \|^2 \, dt + C \nu_0 \int_0^T \| \nabla^3 \psi \|^2 \, dt, \hspace{1cm} (3.32)$$

where the Holder inequality for $1 = \frac{k}{8} + \frac{4-k}{8} + \frac{1}{2}$ is used. Thus, it holds that

$$\int_0^T \| I_{10} \|_{L^1(\Omega)} \, dt \leq C(\varepsilon + \delta) \int_0^T \| \nabla (\phi, \psi) \|_1^2 \, dt + C \int_0^T \| \nabla \psi \|_{L^\infty(\Omega)} \| \nabla^2 \phi \|^2 \, dt$$

$$+ C \nu_0 \int_0^T \| \nabla^2 (\phi, \psi) \|^2 \, dt + C \nu_0 \int_0^T \| \nabla^3 \psi \|^2 \, dt$$

$$\leq C(\varepsilon + \delta + \nu_0) \int_0^T \| \nabla (\phi, \psi) \|_1^2 \, dt + C \nu_0 \int_0^T \| \nabla^3 \psi \|^2 \, dt.$$
\[ + CN(T) \int_0^T \left( \| \nabla \psi \|_1 + \| \nabla^2 \psi \| \right) \| \nabla^2 \phi \| \, dt \]
\[ \leq C(\varepsilon + \delta + \nu_0) \int_0^T \| \nabla (\phi, \psi) \|_1^2 \, dt + C \nu_0 \int_0^T \| \nabla^2 \psi \|^2 \, dt. \quad (3.33) \]

For \( I_{11} \), one has that
\[
\int_0^T \| I_{11} \|_{L^1(\Omega)} \, dt \leq CN(T) \int_0^T \min\{\delta, (1 + t)^{-1}\} \| \nabla^2 \phi \| \, dt \\
+ C\varepsilon \int_0^T e^{-\alpha t} \| \phi, \psi \| \| \nabla^2 \phi \| \, dt + C \varepsilon \int_0^T e^{-\alpha t} \| \nabla^2 \phi \| \, dt \\
\leq C(\varepsilon + \nu_0) \int_0^T \| \nabla^2 \phi \|^2 \, dt + C(\varepsilon + \delta) + C \varepsilon \sup_{t \in (0, T)} \| \phi, \psi \|^2. \quad (3.34) \]

For fixed \( i \in \{1, 2, 3\} \), taking the derivative \( \partial_i \) on \( \frac{1}{\rho} \) (3.3), then multiplying the result by \( \partial_i \), and using the fact that
\[
\partial_i \partial_j \psi \cdot \nabla \partial_i \phi = \partial_i (\partial_i \psi \cdot \nabla \partial_i \phi) - \text{div} (\partial_i \partial_i \phi \partial_i \psi) + \partial_i \partial_i \phi \text{div} \partial_i \psi \\
= \partial_i (\partial_i \psi \cdot \nabla \partial_i \phi) - \text{div} (\partial_i \partial_i \phi \partial_i \psi) \\
- \partial_i (h_0 + \rho \text{div} \psi + \mathbf{u} \cdot \nabla \phi + \text{div} \mathbf{u} \phi + \nabla \rho \cdot \psi) \text{div} \partial_i \psi,
\]
and
\[
\frac{1}{\rho} \left[ \mu \Delta \partial_i \psi + (\mu + \lambda) \text{div} \partial_i \psi \phi \right] \cdot \nabla \partial_i \phi \\
= \frac{2\mu + \lambda}{\rho} \nabla \text{div} \partial_i \psi \cdot \nabla \partial_i \phi + \mu \nabla \phi \times \nabla \partial_i \psi \\
\quad + \mu \phi \cdot (\nabla \partial_i \phi \times \nabla \partial_i \psi)
\]
(which is similar to (3.19)), one can get that
\[
\partial_i (\partial_i \psi \cdot \nabla \partial_i \phi) + \frac{p'(\rho)}{\rho} |\nabla \partial_i \phi|^2 - \frac{2\mu + \lambda}{\rho} \nabla \text{div} \partial_i \psi \cdot \nabla \partial_i \phi = \text{div} Q + \sum_{j=12}^{14} I_j, \quad (3.35) \]
where \( Q = \partial_i \partial_i \phi \partial_i \psi - \frac{\mu}{\rho} \nabla \partial_i \phi \times \nabla \partial_i \psi \) and
\[
I_{12} = \partial_i \left( \frac{f - \phi \mathbf{g}}{\rho} \right) \cdot \nabla \partial_i \phi + \partial_i \left( \frac{2\mu + \lambda}{\rho} \partial_i \tilde{u}_1 \right) \partial_i \phi + \partial_i (h_0 \text{div} \partial_i \psi), \\
I_{13} = \partial_i \left( \text{div} \mathbf{u} \partial_i \psi + \nabla \rho \cdot \psi \right) \text{div} \partial_i \psi + \partial_i (\psi \cdot \nabla \mathbf{u}) \cdot \nabla \partial_i \phi \\
+ \partial_i \left[ \left( \frac{p'(\rho)}{\rho} - \frac{p'_{\rho}(\tilde{\rho})}{\tilde{\rho}} \right) \nabla \tilde{\rho} \right] \cdot \nabla \partial_i \phi, \\
I_{14} = \partial_i (\rho \text{div} \psi + \mathbf{u} \cdot \nabla \phi) \text{div} \partial_i \psi + \partial_i (\mathbf{u} \cdot \nabla \psi) \cdot \nabla \partial_i \phi + \partial_i \left( \frac{p'(\rho)}{\rho} \right) \nabla \phi \cdot \nabla \partial_i \phi \\
- \partial_i \left( \frac{\mu}{\rho} \right) \Delta \psi \cdot \nabla \partial_i \phi - \partial_i \left( \frac{\mu + \lambda}{\rho} \right) \text{div} \psi \cdot \nabla \partial_i \phi + \frac{\mu \nabla \rho \cdot (\nabla \partial_i \phi \times \nabla \partial_i \psi)}{\rho^2}.
\]
Similar to the estimates of (3.33) and (3.34), one can get that

\[
\int_0^T \|I_{12}\|_{L^1(\Omega)} dt \leq C \int_0^T \left( \|\nabla (f - \phi g)\| + \|\partial_1^2 \tilde{u}_1\| \right) \|\nabla^2 \phi\| dt \\
+ C \int_0^T \int_\Omega \left( |f - \phi g| + |\partial_1^2 \tilde{u}_1| \right) (\varepsilon + \delta + |\nabla \phi|) |\nabla^2 \phi| dx dt \\
+ \int_0^T \|\nabla h_0\| \|\nabla^2 \psi\| dt \\
\leq C \int_0^T \left( \|f - \phi g\|_1 + \|\partial_1^2 \tilde{u}_1\|_1 \right) \|\nabla^2 \phi\| dt \\
+ C \int_0^T (\|f - \phi g\|_{L^\infty} + \delta) \|\nabla \phi\| \|\nabla^2 \phi\| dt + C\varepsilon \int_0^T e^{-\alpha t} \|\nabla^2 \psi\| dt \\
\leq C \int_0^T (\varepsilon e^{-\alpha t} + \varepsilon e^{-\alpha t} N(T) + \min\{\delta, (1+t)^{-1}\}) \|\nabla^2 \phi\| dt \\
+ C \int_0^T (\varepsilon + \varepsilon N(T) + \delta) \|\nabla \phi\| \|\nabla^2 \phi\| dt + C\varepsilon + C\varepsilon \int_0^T \|\nabla^2 \psi\|^2 dt \\
\leq C(\varepsilon + \delta^2) \int_0^T \|\nabla^2 \phi\|^2 dt + C(\varepsilon + \delta^2) + C \int_0^T \|\nabla \phi, \nabla^2 \psi\|^2 dt;
\]

\[
\int_0^T \|I_{13}\|_{L^1(\Omega)} dt \leq C \int_0^T \int_\Omega \left( |\partial_1^2 \tilde{u}_1| + |\partial_1^2 \tilde{\rho}^r| + |\tilde{c}_1 \tilde{\rho}^r|^2 + \varepsilon e^{-\alpha t} \right) \\
\times (|\phi| + |\psi|)(|\nabla^2 \psi| + |\nabla^2 \phi|) dx dt \\
+ C(\varepsilon + \delta) \int_0^T \|\nabla (\phi, \psi)\| \|\nabla^2 (\psi, \phi)\| dt \\
\leq CN(T) \int_0^T \left( \min\{\delta, (1+t)^{-1}\} + \min\{\delta^2, \delta^2 + (1+t)^{-\frac{3}{2}}\} \right) \|\nabla^2 (\phi, \psi)\| dt \\
+ C\varepsilon \sup_{t \in (0,T)} \|\phi, \psi\|^2 + C(\varepsilon + \delta) \int_0^T \|\nabla^2 (\psi, \phi)\|^2 dt + C \int_0^T \|\nabla (\phi, \psi)\|^2 dt \\
\leq C(\varepsilon + \delta + \nu_0) \int_0^T \|\nabla^2 \phi\|^2 dt + C\delta + C\varepsilon \sup_{t \in (0,T)} \|\phi, \psi\|^2 \\
+ C \int_0^T (\|\nabla \phi\|^2 + \|\nabla \psi\|^2) dt;
\]

\[
\int_0^T \|I_{14}\|_{L^1(\Omega)} dt \leq C \int_0^T \int_\Omega \left[ |\nabla^2 \psi| (|\nabla^2 \psi| + |\nabla^2 \phi| + |\nabla \psi| + |\nabla \phi|) \\
+ |\nabla^2 \phi| (|\nabla \psi| + |\nabla \phi|) \right] dx dt \\
+ C \int_0^T \int_\Omega \left[ |\nabla^2 \psi| (|\nabla \phi| |\nabla^2 \phi| + |\nabla \phi| |\nabla \phi|) \right] dx dt
\]
+ |\nabla^2 \phi| \left( |\nabla \psi|^2 + |\nabla \phi|^2 \right) dx dt.

Besides $|\nabla \phi| |\nabla^2 \psi| |\nabla^2 \phi|$ which has been estimated in (3.32), the other triple terms in $I_{14}$ can be estimated as follows.

\[
\int_0^T \int_{\Omega} |\nabla \psi| \left( |\nabla^2 \psi| |\nabla \phi| + |\nabla^2 \phi| |\nabla \psi| \right) dx dt + \int_0^T \int_{\Omega} |\nabla \phi|^2 |\nabla^2 \phi| dx dt \leq C \int_0^T \|\nabla \psi\|_{L^\infty} \left( \|\nabla \phi\| \|\nabla^2 \psi\| + \|\nabla \psi\| \|\nabla^2 \phi\| \right) dt + C \int_0^T \|\nabla \phi\|_{L^4(\Omega)}^2 \|\nabla^2 \phi\| dt
\]

\[
\leq C N(T) \int_0^T \left( \|\nabla \psi\|_1 + \|\nabla^3 \psi\| \right) \left( \|\nabla^2 \psi\| + \|\nabla^2 \phi\| \right) dt + C \int_0^T \sum_{k=1}^3 \|\nabla^2 \phi\|^{1+k/2} \|\nabla \phi\|^{2-k/2} dt
\]

\[
\leq C \nu_0 \int_0^T \|\nabla \psi\|_1^2 + \|\nabla^2 \phi\|_1^2 dt + C \nu_0 \int_0^T \|\nabla^3 \psi\|_1^2 dt + C N(T) \sum_{k=1}^3 \|\nabla^2 \phi\|^{k/2} \|\nabla \phi\|^{2-k/2} dt;
\]

\[
\leq C \nu_0 \int_0^T \|\nabla (\psi, \phi)\|_1^2 dt + C \nu_0 \int_0^T \|\nabla^3 \psi\|_1^2 dt.
\]

Thus, it holds that

\[
\int_0^T \|I_{14}\|_{L^1(\Omega)} dt \leq \left( \frac{1}{8} + C \nu_0 \right) \int_0^T \int_{\Omega} \frac{p'(\rho)}{\rho} |\nabla^2 \phi|^2 dx dt + C \int_0^T \left( \|\nabla \psi\|_1^2 + \|\nabla \phi\|^2 \right) dt
\]

\[
+ C \nu_0 \int_0^T \|\nabla^3 \psi\|_1^2 dt.
\]

Thus, collecting the estimates of $I_{10}$ to $I_{14}$ above, by adding the two equations $(2\mu + \lambda)(3.30)$ and $(3.35)$ together, and summing the results with respect to $i$ from 1 to 3, one has that

\[
\sup_{t \in (0, T)} \|\nabla^2 \phi\|^2 + \int_0^T \left( \|\nabla^2 \phi\|^2 + \|\nabla \phi\|^2 \right) dt \leq C \|\phi_0\|_2^2 + C \|\psi_0\|_1^2 + C \|\nabla \psi\|^2 + C(\varepsilon + \delta^1)
\]

\[
+ C \int_0^T \left( \|\nabla \psi\|_1^2 + \|\nabla \phi\|^2 \right) dt + C \nu_0 \int_0^T \|\nabla^3 \psi\|_1^2 dt.
\]

Thus (3.29) follows from Lemmas 3.2–3.4. \[\square\]

**Lemma 3.6.** Under the assumptions of Theorem 3.1, if $\delta_0 > 0, \varepsilon_0 > 0$ and $\nu_0 > 0$ are small enough, then

\[
\sup_{t \in (0, T)} \|\nabla^2 \psi\|^2 + \int_0^T \|\nabla^3 \psi\|^2 dt \leq C \|\phi_0, \psi_0\|_2^2 + C(\varepsilon + \delta^1).
\] (3.36)
Proof. For fixed $i \in \{1, 2, 3\}$, taking the derivative $\partial_i$ on $\frac{1}{\rho} \cdot (3.3)$ and then multiplying the result by $\cdot (-\partial_i \Delta \psi)$, one has that

$$
- \partial_i \partial_i \psi \cdot \partial_i \Delta \psi + \partial_i \left( \frac{\mu}{\rho} \Delta \psi \right) \cdot \partial_i \Delta \psi + \partial_i \left( \frac{\mu + \lambda}{\rho} \nabla \text{div} \psi \right) \cdot \partial_i \Delta \psi
$$

$$
= \partial_i \Delta \psi \cdot \partial_i \left[ \mathbf{u} \cdot \nabla \psi + \psi \cdot \nabla \tilde{\mathbf{u}} + \frac{p'(\rho)}{\rho} \nabla \phi + \left( \frac{p'(\rho)}{\rho} - \frac{p'(\tilde{\rho})}{\tilde{\rho}} \right) \nabla \tilde{\rho} \right] - \frac{f - \phi g}{\rho} - \frac{2\mu}{\rho} \partial_i^2 \tilde{u}_i^1 \mathbf{e}_1 := I_{15}.
$$

For the left-hand side of (3.37), similar to (3.27), one can get that

$$
- \partial_i \partial_i \psi \cdot \partial_i \Delta \psi = \frac{1}{2} \partial_t \left( |\nabla \partial_i \psi|^2 \right) - \text{div} \left( \nabla \partial_i \psi \partial_i \partial_i \psi \right),
$$

$$
\frac{1}{\rho} \partial_i \nabla \text{div} \psi \cdot \partial_i \Delta \psi = \frac{1}{\rho} |\nabla \text{div} \partial_i \psi|^2 + \text{div} \left( \frac{\text{div} \partial_i \psi}{\rho} \Delta \partial_i \psi - \frac{\nabla (\text{div} \partial_i \psi)^2}{2\rho} \right)
$$

$$
+ \frac{\text{div} \partial_i \psi}{\rho} \nabla \rho \cdot (\Delta \partial_i \psi - \nabla \partial_i \text{div} \psi).
$$

Then one has that

$$
\frac{1}{2} \partial_t \left( |\nabla \partial_i \psi|^2 \right) + \frac{\mu}{\rho} |\partial_i \Delta \psi|^2 + \frac{1}{\rho} |\nabla \text{div} \partial_i \psi|^2 = \text{div} Q_6 + I_{15} + I_{16},
$$

where

$$
Q_6 = \nabla \partial_i \psi \partial_i \partial_i \psi - \frac{\text{div} \partial_i \psi}{\rho} \Delta \partial_i \psi + \frac{\nabla (\text{div} \partial_i \psi)^2}{2\rho},
$$

$$
I_{16} = \frac{\partial_i \rho}{\rho^2} \left[ \mu \Delta \psi + (\mu + \lambda) \nabla \text{div} \psi \right] \cdot \partial_i \Delta \psi - \frac{\text{div} \partial_i \psi}{\rho} \nabla \rho \cdot (\Delta \partial_i \psi - \nabla \partial_i \text{div} \psi),
$$

and there holds that $\int_\Omega \frac{\mu}{\rho} |\partial_i \Delta \psi|^2 \, dx \geq a_0 \left\| \nabla^2 \partial_i \psi \right\|^2$ for some constant $a_0 > 0$.

From (3.37), it holds that

$$
|I_{15}| \leq C \left| \partial_i \Delta \psi \right| \left[ |\nabla^2 \psi| + |\nabla \tilde{\mathbf{u}}| \left| \nabla \psi \right| + |\nabla \psi|^2 + |\nabla^2 \tilde{\mathbf{u}}| \left| \nabla \psi \right| + |\nabla^2 \phi| + |\nabla \tilde{\rho}| \left| \nabla \phi \right| + |\nabla \phi|^2 + (|\nabla \tilde{\rho}|^2 + |\nabla^2 \tilde{\rho}|) |\phi| + |\nabla \phi| \left| \nabla \phi \right| + |\nabla \phi| \left| \nabla \phi \right| + |\nabla \phi| \left| \nabla \phi \right| + |\nabla \phi| \left| \nabla \phi \right|
$$

$$
+ |\mathbf{g}| \left| \nabla \phi \right| + |\mathbf{g}| \left| \nabla \phi \right| \left( |\nabla \tilde{\rho}| + |\nabla \phi| \right) + |\partial_i^2 \partial_i \tilde{u}_i^1| + |\partial_i^2 \partial_i \tilde{u}_i^1| \left( |\nabla \tilde{\rho}| + |\nabla \phi| \right)
$$

Similar to the estimates before, e.g.,

$$
\int_0^T \int_\Omega |\partial_i \Delta \psi| \left( |\nabla \tilde{\rho}|^2 + |\nabla^2 \tilde{\rho}| \right) |\phi| \, dx dt
$$

$$
\leq CN(T) \int_0^T \left( \min\{\delta^2, \delta^2 (1 + t)^{-\frac{3}{2}} \} + \min\{\delta, (1 + t)^{-1} \} + C \varepsilon e^{-\alpha t} \right) \left\| \nabla^2 \psi \right\| \, dt
$$

$$
\leq (\delta + \varepsilon) + C \nu \int_0^T \left\| \nabla^3 \psi \right\|^2 \, dt,
$$

one can get that

$$
\int_0^T \left\| I_{15} \right\|_{L^1(\Omega)} \, dt \leq C \int_0^T \left\| \nabla^3 \psi \right\| \left\| \nabla (\psi, \phi) \right\|_1 \, dx dt + C \int_0^T \left\| \nabla^3 \psi \right\| \left\| \nabla (\psi, \phi) \right\|^2_{L^4(\Omega)} \, dt
$$

$$
\leq C \int_0^T \left\| \nabla^3 \psi \right\|^2 \, dx dt + C \int_0^T \left\| \nabla^3 \psi \right\| \left\| \nabla (\psi, \phi) \right\|^2_{L^4(\Omega)} \, dt
$$
+ \int_0^T \| \nabla^3 \psi \| \left( \| f \|_1 + \| \tilde{\phi}_t \|_1 + \| g \|_{W^{1,\infty}} \| \phi \| \right) \, dt \\
+ C(\delta + \varepsilon) + C\nu_0 \int_0^T \| \nabla^3 \psi \|^2 \, dt \\
\leq \left( \frac{C_0}{4} + C(\varepsilon + \delta + \nu_0) \right) \int_0^T \| \nabla^3 \psi \|^2 \, dt + C \int_0^T \| \nabla (\psi, \phi) \|^2 \, dt \\
+ C\varepsilon \sup_{t \in (0,T)} \| \phi \|^2 + C(\delta + \varepsilon), \tag{3.40}
\end{align*}

where the $L^4$-estimate (3.31) and the Hölder inequality for $1 = \frac{1}{2} + \frac{k}{8} + \frac{4-k}{8}$ are used when dealing with the integral involving $\| \psi, \phi \|_{L^4(\Omega)}$.

Similarly, $I_{16}$ satisfies that

\begin{align*}
\int_0^T \| I_{16} \|_{L^1(\Omega)} \, dt & \leq C \int_0^T \| \nabla^2 \psi \| \| \nabla^3 \psi \| \, dt + C \int_0^T \| \nabla \phi \|_{L^\infty} \| \nabla^2 \psi \| \| \nabla^3 \psi \| \, dt \\
& \leq \left( \frac{C_0}{4} + C\nu_0 \right) \int_0^T \| \nabla^3 \psi \|^2 \, dt + C \int_0^T \| \nabla^2 \psi \|^2 \, dt. \tag{3.41}
\end{align*}

Then integrating (3.38) over $\Omega \times (0,T)$ and summing the results with respect to $i$ from 1 to 3 can finish the proof of Lemma 3.6.

Thus, Theorem 3.1 can follow from Lemmas 3.2–3.6 easily.

**Proof of the main result, Theorem 1.3.**

*Proof.* We first introduce the local existence theorem.

**Proposition 3.7** ([15]). **Local existence theorem.** *There exists a positive constant $b$, such that if $\| \phi_0, \psi_0 \|_2 \leq M$ and $\inf_x (\bar{\rho}(x,0) + \phi_0(x)) \geq m > 0$, then there exists a $T_0 = T_0(m, M)$, such that, the problem (3.2), (3.3) admits a unique solution $(\phi, \psi) \in X_{\frac{1}{2}, bM}(0, T_0)$ where

\[ X_{m, M}(0, T) := \{ (\phi, \psi) : (\phi, \psi) \in C \left( 0, T; H^2(\Omega) \right) \text{ with } \inf_{x, t} \phi \geq m, \sup_t \| \phi, \psi \|_2 \leq M, \]

\[ \nabla \phi \in L^2 \left( 0, T; H^1(\Omega) \right) \text{ and } \nabla \psi \in L^2 \left( 0, T; H^2(\Omega) \right) \}.
\]

Under the assumptions of Theorem 1.3, we let $m = \frac{1}{2} \bar{\rho}^-$ and $M = C_0 \varepsilon_0 + C(\varepsilon_0 + \delta_0^\frac{1}{2})$, where $C_0$ is the constant in (3.5) and $C, \varepsilon_0, \delta_0$ are the constants in Theorem 3.1. And we can let $\varepsilon_0$ and $\delta_0$ small enough such that

\[ \inf_{x, t} \bar{\rho}(x, t) \geq \bar{\rho}^- - C \varepsilon_0 \geq \frac{1}{2} \bar{\rho}^- + M. \tag{3.42}
\]

Then by Proposition 3.7, the solution $(\phi, \psi)$ to (3.2) and (3.3) exists on $[0, T_0]$, satisfying $(\phi, \psi) \in X_{\frac{1}{2}, bM}(0, T_0)$, i.e.

\[ \sup_{0 \leq t \leq T_0} \| (\phi, \psi)(t) \|_2 \leq bM. \]
Since either \( b \) or \( M \) is independent of the constant \( \nu_0 > 0 \) in Theorem 3.1, one can choose \( \varepsilon_0 \) and \( \delta_0 \) small enough such that \( bM < \nu_0 \). Thus, it follows from the priori estimates, Theorem 3.1, that

\[
\sup_{0 \leq t \leq T_0} \| (\phi, \psi)(t) \|_2 \leq C \| \phi_0, \psi_0 \|_2 + C(\varepsilon + \delta_1^{1/2}) \leq M,
\]

and hence,

\[
\inf_{x \in \Omega} (\bar{\rho}(x, t) + \phi(x, t)) \geq \frac{1}{2} \bar{\rho}^- + M - M = \frac{1}{2} \bar{\rho}^-.
\]

Using Proposition 3.7 once more, the solution to (3.2) and (3.3) also exists on \([T_0, 2T_0]\). By induction, we obtain the global in time solution to the Cauchy problem of (3.2) and (3.3), if \( \varepsilon_0 > 0 \) and \( \delta_0 \) are small enough.

Hence, to complete the proof of Theorem 1.3, it remains to prove the large time behavior of the solution.

We give only the proof of \( \phi \), since it is similar to prove \( \psi \). It follows from Lemma 2.5 that

\[
\| \phi \|_{L^\infty(\mathbb{R}^3)} = \| \phi \|_{L^\infty(\Omega)} \leq C \| \nabla \phi \|^{1/2} \| \phi \|^{1/2} + \| \nabla^2 \phi \|^{1/2} \| \nabla \phi \|^{1/2} \leq C \delta_1^{1/2} \| \nabla \phi \|^{1/2}.
\]

Thus if suffices to prove that \( \| \nabla \phi \| (t) \to 0 \) as \( t \to +\infty \), which can be easily derived from

\[
\int_0^{+\infty} \left( \| \nabla \phi \|^2 + \left| \frac{d}{dt} \| \nabla \phi \|^2 \right| \right) dt < +\infty.
\]

The proof of Theorem 1.3 is completed.

\[\square\]
Now we prove the exponential decay rate of the solution by using the Poincaré inequality. Denote the perturbations 

\[ v := \rho - \bar{\rho}, \quad z := u - \bar{u} = u \quad \text{and} \quad w := \rho u - \bar{\rho} \bar{u} = \rho z. \]

Due to the conservative form of (1.1), it holds that 

\[
\int_{\mathbb{T}^3} v(x, t) = 0, \quad \int_{\mathbb{T}^3} w(x, t)dx = 0, \quad t \geq 0.
\]

Thus, it follows from the Poincaré inequality that 

\[
\|v\| \leq C_0 \|\nabla v\|, \\
\|w\| \leq C_0 \|\nabla w\| \leq C_0 \|z\|_{L^\infty} \|\nabla v\| + C_1 \|\nabla z\| \leq C_0 \varepsilon \|\nabla v\| + C_1 \|\nabla z\|, \hspace{1cm} (4.1)
\]

where the positive constants \(C_0\) and \(C_1\) are independent of \(\varepsilon\) or \(t\). For the derivatives, it also follows from the Poincaré inequality that 

\[
\|\nabla v\| \leq C_0 \|\nabla^2 v\|, \quad \|\nabla z\| \leq C_0 \|\nabla^2 z\| \quad \text{and} \quad \|\nabla^2 z\| \leq C_0 \|\nabla^3 z\|. \hspace{1cm} (4.2)
\]

It is noted that the perturbations \(v\) and \(z\) here satisfy the same equations as (3.2) and (3.3), where \(\bar{\rho}\) and \(\bar{u}\) become constants and \(h_0 = f = g = 0\). Thus, similar estimates as Lemmas 3.2–3.6 yield that 

\[
E'(t) + C_2 \|\nabla v\|^2_1 + C_2 \|\nabla z\|^2_2 \leq 0, \hspace{1cm} (4.3)
\]

for some energy functional \(E(t)\) satisfying 

\[
C^{-1} \|v, z\|^2_2 \leq E(t) \leq C \|v, z\|^2_2,
\]

for some constant \(C > 0\). Thus, it follows from (4.1) to (4.3) that 

\[
E'(t) + \frac{C_2}{2} \|\nabla v\|^2_1 + \frac{C_2 C_0}{2} \|v\|^2_1 + \frac{C_2}{2} \|\nabla z\|^2_2 + C_3 \|z\|^2_2 - C_4 \varepsilon^2 \|\nabla v\|^2 \leq 0.
\]

Thus, if \(\varepsilon > 0\) is small enough, one can get that 

\[
E'(t) + 2C_2 E(t) \leq 0,
\]

which implies that 

\[
\|v, z\|^2_2(t) \leq Ce^{-C_2 t}.
\]

And the higher order estimates 

\[
\|v, z\|^5_5(t) \leq Ce^{-C_6 t}.
\]

can be proved similarly, which is omitted for brevity.

\[ \square \]

**Proof of Lemma 2.4.** The idea is to extract the “well-decay terms” \(R_i (i = 1, 2, \cdots)\) from the equations (2.4), where all the \(R_i\) are products of space-periodic functions decaying exponentially fast with respect to \(t\) (e.g. \(v^\pm, z^\pm, \rho - \bar{\rho}, \partial_i u^\pm, \nabla p^\pm, \cdots\)) and integrable functions with respect to \(x_1 \in \mathbb{R}\) (e.g. \(\eta(1 - \sigma), \sigma - \eta, \partial_i \sigma, \partial_i \eta, \cdots\)).

Denote \(\varepsilon = \|v_0, z_0\|_{H^5(\mathbb{T}^3)}\). Then it follows from Lemma 2.3 that 

\[
\|v^\pm, z^\pm\|_{W^{3, +\infty}(\mathbb{T}^3)} \leq C\varepsilon e^{-2\alpha t}.
\]
i) For $h_0$ given in (2.4), first note that
\[
\begin{align*}
\partial_t \tilde{\rho} &= \partial_t \tilde{\rho}^+ (1 - \sigma) + \partial_t \tilde{\rho}^+ \sigma + (\rho^+ - \rho^-) \partial_t \sigma, \\
= &\tilde{\partial}_t \tilde{\rho}^- (1 - \sigma) + \tilde{\partial}_t \tilde{\rho}^+ \sigma + \tilde{\partial}_t \tilde{\rho}^r + R_1,
\end{align*}
\]
where the remainder $R_1 = (v^+ - v^-) \partial_t \sigma$. It follows from Lemmas 2.2 and 2.3 that
\[
\|R_1(\cdot, t)\|_{W^2, p(\Omega)} \leq C\varepsilon e^{-2\alpha t} \|\tilde{\partial}_t \sigma(\cdot, t)\|_{W^2, p(\mathbb{R})} \leq C\varepsilon e^{-2\alpha t}.
\]
And
\[
\begin{align*}
\text{div} (\tilde{\rho} \tilde{u}) &= \left[ \rho^-(1 - \sigma) + \rho^+ \sigma \right] \left[ \text{div} u^- (1 - \eta) + \text{div} u^+ \eta \right] + \rho \left[ \partial_t \tilde{u}_i^r + (z_1^+ - z_1^-) \partial_t \eta \right] \\
&+ \rho \left( \partial_1 \tilde{u}_i^r + (z_1^+ - z_1^-) \partial_1 \eta \right)
\end{align*}
\]
where
\[
R_2 = \text{div} \left( \rho^- u^- \right) (1 - \sigma) \eta + \text{div} \left( \rho^+ u^- \right) \sigma (1 - \eta) + (\tilde{\rho} - \tilde{\rho}^r) \partial_t \tilde{u}_i^r + (\tilde{u}_1^r - \tilde{u}_1^\gamma) \partial_t \tilde{\rho}^r
\]
which satisfy that
\[
\sum_{i=2}^3 \|R_i\|_{W^2, p(\Omega)} \leq C\varepsilon e^{-\alpha t}.
\]
Collecting (4.4) and (4.5), one can get that
\[
\begin{align*}
\tilde{h}_0 &= \left[ \partial_t \tilde{\rho}^- + \text{div} (\rho^- u^-) \right] (1 - \sigma) + \left[ \partial_t \tilde{\rho}^+ + \text{div} (\rho^+ u^+) \right] \sigma + \tilde{\partial}_t \tilde{\rho}^r + \tilde{\partial}_1 (\tilde{\rho}^r \tilde{u}_i^r) + \sum_{i=1}^3 \|R_i\|_{W^2, p(\Omega)}
\end{align*}
\]
which satisfies
\[
\|\tilde{h}_0\|_{W^2, p(\Omega)} \leq C\varepsilon e^{-\alpha t}.
\]
ii) Now we prove the source term $h$ deduced from the momentum equations (2.4). Note that
\[
\begin{align*}
\tilde{\rho} \partial_t \tilde{u} &= \tilde{\rho} \left[ \partial_t \tilde{u}^- (1 - \eta) + \partial_t \tilde{u}^+ \eta + \partial_t \tilde{u}_i^r \right] e_1 + \left( z_1^+ - z_1^- \right) \partial_t \eta
\end{align*}
\]
where
\[
\begin{align*}
R_4 &= \rho^- \partial_t \tilde{u}^- (1 - \sigma) \eta + \rho^+ \partial_t \tilde{u}^+ \sigma (1 - \eta) + (\tilde{\rho} - \tilde{\rho}^r) \partial_t \tilde{\rho}^r \\
R_5 &= -\rho^- \partial_t \tilde{u}^- (1 - \sigma) \eta - \rho^+ \partial_t \tilde{u}^+ \sigma (1 - \eta) + (\tilde{\rho} - \tilde{\rho}^r) \partial_t \tilde{\rho}^r
\end{align*}
\]
And
\[
\tilde{\rho}\hat{u} \cdot \nabla \hat{u} = \tilde{\rho}\hat{u} \cdot [\nabla u^-(1-\eta) + \nabla u^+\eta] + \tilde{\rho}\hat{u}_1 \left[ \hat{c}_1 \hat{u}_1^r e_1 + \hat{c}_1 \eta (z^+ - z^-) \right]
\]
\[
= \rho^- u^- \cdot \nabla u^- (1-\sigma)(1-\eta)^2 + \rho^+ u^+ \cdot \nabla u^+ \eta^2 + \tilde{\rho}\hat{u}_1 \hat{c}_1 \hat{u}_1^r e_1 + R_6
\]
\[
= \rho^- u^- \cdot \nabla u^- (1-\sigma) + \rho^+ u^+ \cdot \nabla u^+ \sigma + \tilde{\rho}\hat{u}_1 \hat{c}_1 \hat{u}_1^r e_1 + R_6 + R_7,
\]  
(4.7)
where
\[
R_6 = \nabla u^- \cdot \left[ \tilde{\rho}\hat{u} - \rho^- u^- (1-\sigma)(1-\eta) \right] (1-\eta) + \nabla u^+ \cdot \left( \tilde{\rho}\hat{u} - \rho^+ u^+ \sigma \right) \eta
\]
\[+ \tilde{\rho}\hat{u}_1 \hat{c}_1 \eta (z^+ - z^-),
\]
\[
R_7 = \rho^- u^- \cdot \nabla u^- (1-\sigma) \eta (\eta - 2) + \rho^+ u^+ \cdot \nabla u^+ (\eta^2 - 1)
\]
\[+ (\tilde{\rho}\hat{u}_1 - \tilde{\rho}\hat{u}_1^r) \hat{c}_1 \hat{u}_1^r e_1.
\]
Moreover,
\[
\nabla p(\tilde{\rho}) = p'(\tilde{\rho}) \left[ \nabla \rho^- (1-\sigma) + \nabla \rho^+ \sigma + \hat{c}_1 \tilde{\rho}^r e_1 + (v^+ - v^-) \hat{c}_1 \sigma e_1 \right]
\]
\[
= (p'(\tilde{\rho}) - p'(\tilde{\rho}^r)) \nabla \rho^- (1-\sigma) + \nabla p(\rho^-)(1-\sigma) + (p'(\tilde{\rho}) - p'(\tilde{\rho}^r)) \nabla \rho^+ \sigma
\]
\[+ \nabla p(\rho^+) \sigma + (p'(\tilde{\rho}) - p'(\tilde{\rho}^r)) \hat{c}_1 \tilde{\rho}^r e_1 + \nabla p(\tilde{\rho})^r + p'(\tilde{\rho}) (v^+ - v^-) \hat{c}_1 \sigma e_1
\]
\[
= \nabla p(\rho^-)(1-\sigma) + \nabla p(\rho^+) \sigma + \hat{c}_1 p(\tilde{\rho}^r) e_1 + R_8,
\]  
(4.8)
where the remainder \( R_8 \) satisfies that
\[
R_8 = (\rho^+ - \rho^-) \left[ a(\rho, \tilde{\rho}) \nabla \rho^- - a(\rho^+, \tilde{\rho}) \nabla \rho^+ \right] \sigma(1-\sigma)
\]
\[+ a(\tilde{\rho}, \tilde{\rho}^r)(\rho - \rho^-) \hat{c}_1 \tilde{\rho}^r e_1 + p'(\tilde{\rho}) (v^+ - v^-) \hat{c}_1 \sigma e_1,
\]
where \( a(u, v) := \int_0^1 p''(u + \theta (v - u)) d\theta \).

The remaining second-order terms satisfy that
\[
\Delta \hat{u} = \Delta (\hat{u} - \hat{u}_1^r e_1) + \hat{c}_1 \tilde{u}_1^r e_1
\]
\[
= \Delta \left[ z^- (1-\sigma) + z^+ \sigma \right] + \Delta \left[ (z^+ - z^-)(\eta - \sigma) \right] + \hat{c}_1 \tilde{u}_1^r e_1
\]
\[
= \Delta u^- (1-\sigma) + \Delta u^+ \sigma + \hat{c}_1 \tilde{u}_1^r e_1 + R_9,
\]
\[
\nabla \text{div} \hat{u} = \nabla \text{div} \left[ z^- (1-\sigma) + z^+ \sigma \right] + \nabla \text{div} \left[ (z^+ - z^-)(\eta - \sigma) \right] + \hat{c}_1 \tilde{u}_1^r e_1
\]
\[
= \nabla \text{div} u^- (1-\sigma) + \nabla \text{div} u^+ \sigma + \hat{c}_1 \tilde{u}_1^r e_1 + R_{10},
\]
where
\[
R_9 = \Delta \left[ (z^+ - z^-)(\eta - \sigma) \right] + (z^+ - z^-) \hat{c}_1^2 \sigma + 2 \hat{c}_1 (z^+ - z^-) \hat{c}_1 \sigma,
\]
\[
R_{10} = \nabla \text{div} \left[ (z^+ - z^-)(\eta - \sigma) \right] + \nabla \left[ (z_1^+ - z_1^-) \hat{c}_1 \sigma \right] + (\text{div} u^+ - \text{div} u^-) \hat{c}_1 \sigma e_1.
\]
Collecting (4.6) to (4.9), one can get that
\[
h = \left[ \rho^- \hat{c}_1 u^- + \rho^+ u^+ \cdot \nabla u^- + \nabla p(\rho^-) - \mu \Delta u^- - (\mu + \lambda) \nabla \text{div} u^- \right] (1-\sigma)
\]
\[
+ \left[ \rho^+ \hat{c}_1 u^+ + \rho^+ u^+ \cdot \nabla u^+ + \nabla p(\rho^+) - \mu \Delta u^+ - (\mu + \lambda) \nabla \text{div} u^+ \right] \sigma
\]
\[+ \left[ \tilde{\rho} \hat{c}_1 \tilde{u}_1^r + \tilde{\rho} \tilde{u}_1^r \hat{c}_1 \tilde{u}_1^r + \hat{c}_1 p(\tilde{\rho}^r) \right] e_1 + h_0 \hat{u}
\]
\[+ \sum_{i=4}^{8} R_i - \mu R_9 - (\mu + \lambda) R_{10} - (2\mu + \lambda) \hat{c}_1^2 \tilde{u}_1^r e_1,
\]  
(4.9)
\[
\begin{align*}
= h_0 \tilde{u} + \sum_{i=4}^{8} R_i - \mu R_9 - (\mu + \lambda) R_{10} - (2\mu + \lambda) \partial_1^2 \tilde{u} R^i e_1,
\end{align*}
\]

which satisfies

\[
\| h + (2\mu + \lambda) \partial_1^2 \tilde{u} R^i e_1 \|_{W^{1,p}(\Omega)} \leq C \| h_0 \|_{W^{1,p}(\Omega)} + C \sum_{i=4}^{10} \| R_i \|_{W^{1,p}(\Omega)} \leq C \varepsilon e^{-at}.
\]

\[\square\]

REFERENCES

[1] J. Goodman, Nonlinear asymptotic stability of viscous shock profiles for conservation laws, Arch. Ration. Mech. Anal., 95 (1986), 325-344.

[2] H. Hokari, A. Matsumura, Asymptotics toward one-dimensional rarefaction wave for the solution of two-dimensional compressible Euler equation with an artificial viscosity. Asymptot. Anal. 15:283?298 (1997)

[3] F. Huang, J. Li and A. Matsumura, Asymptotic stability of combination of viscous contact discontinuity with rarefaction waves for one-dimensional compressible Navier-Stokes system, Arch. Ration. Mech. Anal., 197 (2010), 89-116.

[4] F. Huang and A. Matsumura, Stability of a composite wave of two viscous shock waves for the full compressible Navier-Stokes equation, Comm. Math. Phys., 289 (2009), 841-861.28

[5] F. Huang, A. Matsumura and X. D. Shi, On the stability of contact discontinuity for compressible Navier-Stokes equations with free boundary, Osaka J. Math., 41 (2004), 193-210.

[6] F. Huang, A. Matsumura and Z. Xin, Stability of contact discontinuities for the 1-d compressible Navier-Stokes equations, Arch. Ration. Mech. Anal., 179 (2006), 55-77.

[7] F. Huang, Z. Xin and T. Yang, Contact discontinuities with general perturbation for gas motion, Adv. Math., 219 (2008), 1246-1297.

[8] F. Huang and T. Yang, Stability of contact discontinuity for the Boltzmann equation, J. Differential Equations, 229 (2006), 698–742.

[9] F. Huang and Q. Yuan, A viscous shock under periodic perturbations for 1-D isentropic Navier-Stokes equations, preprint, (2020), 1–25.

[10] F. Huang and Q. Yuan, Stability of planar rarefaction waves for scalar viscous conservation laws under periodic perturbations, preprint, (2020), 1–17.

[11] F. Huang and H. J. Zhao, On the global stability of contact discontinuity for compressible Navier-Stokes equations, Rend. Sem. Mat. Univ. Padova., 109 (2003), 283–305.

[12] K. Ito, Asymptotic decay toward the planar rarefaction waves of solutions for viscous conservation laws in several space dimensions, Math. Models Methods Appl. Sci., 6 (1996), pp. 315–338.

[13] S. Kawashima, S. Nishibata and M. Nishikawa, Lp energy method for multi-dimensional viscous conservation laws and application to the stability of planar waves, J. Hyperbolic Differ. Equations 01 (2004), no. 03, 581–603.

[14] L. Li, Y. Wang, Stability of the planar rarefaction wave to the two-dimensional compressible Navier-Stokes equations, SIAM Journal on Mathematical Analysis 50(5), 2018

[15] L. Li, T. Wang, Y. Wang, Stability of Planar Rarefaction Wave to 3D Full Compressible Navier-Stokes Equations, Archive for Rational Mechanics and Analysis. NO.05 2018.

[16] T.-P. Liu, Nonlinear stability of shock waves for viscous conservation laws, Mem. Amer. Math. Soc. 56 (1985), no. 328, v+108.

[17] T.-P. Liu, Pointwise convergence to shock waves for viscous conservation laws, Comm. Pure Appl. Math. 50 (1997), no. 11, 1113–1182.

[18] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ. (1980) 67–104.

[19] A. Matsumura and K. Nishihara, On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas, Japan J. Appl. Math. 2 (1985), no. 1, 17-25.
[20] A. Matsumura and K. Nishihara, *Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas*, Japan Journal of Applied Mathematics 3 (1986), no. 1, 1–13.

[21] A. Matsumura and K. Nishihara, Global stability of the rarefaction wave of a onedimensional model system for compressible viscous gas, Comm. Math. Phys., 144 (1992), 325-335.

[22] M. Nishikawa and K. Nishihara, *Asymptotics towards the planar rarefaction wave for viscous conservation law in two space dimensions*, Amer. Math. Soc. Transl. Vol. 352(2000), 1203-1215.

[23] K. Nishihara, T. Yang and H. J. Zhao, Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations, SIAM J. Math. Anal., 35 (2004), 1561-1597.

[24] A. Szepessy and Z. Xin, Nonlinear stability of viscous shock waves, Arch. Ration. Mech. Anal. 122 (1993), 53-103.

[25] Z. Xin, *Asymptotic stability of planar rarefaction waves for viscous conservation laws in several dimensions*, Trans. Amer. Math. Soc. 319 (1990), no. 2, 805–820.

[26] Z. Xin, Q. Yuan, and Y. Yuan, *Asymptotic stability of shock waves and rarefaction waves under periodic perturbations for 1-d convex scalar conservation laws*, SIAM Journal on Mathematical Analysis, 51 (2019), no. 4, 2971–2994.

[27] Z. Xin, Q. Yuan, and Y. Yuan, *Asymptotic stability of shock profiles and rarefaction waves under periodic perturbations for 1-d convex scalar viscous conservation laws*, arxiv:1902.09772 (2019), 1–42.

[28] Q. Yuan, and Y. Yuan, *On Riemann solutions under different initial periodic perturbations at two infinities for 1-d scalar convex conservation laws*, J. Differential Equations, 1 (2019), 1–16.

[29] Q. Yuan, and Y. Yuan, *Periodic perturbations of a composite wave of two viscous shocks for 1-D full Navier-Stokes equations*, Preprint, (2020), 1–29.

---

1. Academy of Mathematics and Systems Science, CAS, Beijing 100190, China
2. School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
3. Department of Mathematics, Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China

Email address: fhuangamt.ac.cn

Email address: xulingda@tsinghua.edu.cn

Email address: qyuan103@link.cuhk.edu.hk