DEFORMATIONS OF INFINITE PROJECTIONS

ETIENNE BLANCHARD

Abstract. Let $A = (A_x)$ be a (semi-)continuous field of $C^*$-algebras over a compact Hausdorff space $X$ and let $p = (p_x)$ be a projection in $A$ such that each $p_x \in A_x$ is properly infinite ($x \in X$). We prove that $p \oplus \ldots \oplus p$ ($l$ summands) is properly infinite in $M_l(A)$ for large enough $l \in \mathbb{N}$ if the $C(X)$-algebra $A$ is upper semi-continuous. But $p$ can be stably finite if $A$ is only lower semi-continuous.

1. Preliminaries

A powerful tool in the classification of $C^*$-algebras is the study of their projections. Two projections $p, q$ in a $C^*$-algebra $A$ are said to be Murray-von Neumann equivalent (respectively $p$ dominates $q$) if there exists a partial isometry $v \in A$ with $v^*v = p$ and $vv^* = q$ (resp. $v^*v \leq p$ and $vv^* = q$). For short we write $p \sim q$ (resp. $q \precsim p$). The non-zero projection $p$ is said to be infinite (resp. properly infinite) if $p$ is equivalent to a proper subprojection $q < p$ (resp. $p$ is equivalent to two mutually orthogonal projections $p_1, p_2$ with $p_1 + p_2 \leq p$) and $p$ is finite otherwise. J. Cuntz introduced the following generalization: A positive element $a$ in $A$ dominates another positive element $b$ in $A$ (written $b \precsim a$) if and only if (iff) there exists a sequence $\{d_n\}_n$ in $A$ such that $d_n^*ad_n \to b$ ([10]). Further $a \in A_x$ is called infinite (resp. properly infinite) iff there exists a non-zero positive element $b$ in $A$ such that $a \oplus b \precsim a \oplus 0$ in $M_2(A)$ (resp. $a \oplus a \precsim a \oplus 0$ in $M_2(A)$). And $a$ is said to be finite if $a$ is not infinite. Kirchberg and Rørdam proved that that these definitions coincide with the ones given in the previous paragraph in case $a$ is a projection ([10] Lemma 3.1)). Now a $C^*$-algebra $A$ is said to be infinite (resp. properly infinite) iff all strictly positive elements in $A$ are infinite (resp. properly infinite). It is said to be finite (resp. stably finite) if all strictly positive elements in $A$ are finite (resp. all strictly positive elements in $M_n(A)$ are finite for all positive integer $n$).

In order to study deformations of such algebras, let us recall a few notions from the theory of $C(X)$-algebras.

Let $X$ be a Hausdorff compact space and let $C(X)$ be the $C^*$-algebra of continuous functions on $X$ with values in the complex field $\mathbb{C}$.

Definition 1.1. A $C(X)$-algebra is a $C^*$-algebra $A$ endowed with a unital $*$-homomorphism from $C(X)$ to the centre of the multiplier $C^*$-algebra $M(A)$ of $A$.

For all $x \in X$, we denote by $C_x(X)$ the ideal of functions $f \in C(X)$ satisfying $f(x) = 0$, by $A_x$ the quotient of $A$ by the closed ideal $C_x(X)A$ and by $a_x$ the image of
an element $a \in A$ in the fibre $A_x$. Then the function

\begin{equation}
N(a) : x \mapsto \|a_x\| = \inf\{\|1 - f + f(x)\|a\| : f \in C(X)\}
\end{equation}

is upper semi-continuous by construction. The $C(X)$-algebra is said to be continuous (or to be a continuous $C^*$-bundle over $X$) if the function $x \mapsto \|a_x\|$ is actually continuous for all element $a \in A$.

**Definition 1.2.** ([12]) Given a continuous $C(X)$-algebra $B$, a $C(X)$-representation of a $C(X)$-algebra $A$ on $B$ is a $C(X)$-linear map $\pi$ from $A$ to the multiplier $C^*$-algebra $\mathcal{M}(B)$ of $B$. Further $\pi$ is said to be a continuous field of faithful representations if, for all $x \in X$, the induced representation $\pi_x$ of the fibre $A_x$ in $\mathcal{M}(B_x)$ is faithful.

Note that the existence of such a continuous field of faithful representations $\pi$ implies that the $C(X)$-algebra $A$ is continuous since the function

\begin{equation}
x \mapsto \|\pi_x(a_x)\| = \sup\{\|\pi(a)b_x\| : b \in B \text{ such that } \|b\| \leq 1\}
\end{equation}

is lower semi-continuous for all $a \in A$.

Conversely, any separable continuous $C(X)$-algebra $A$ admits a continuous field of faithful representations. More precisely, there always exists a unital positive $C(X)$-linear map $\varphi : A \to C(X)$ such that all the induced states $\varphi_x$ on the fibres $A_x$ are faithful ([3]). By the Gel’fand-Naimark-Segal (GNS) construction this gives a continuous field of faithful representations of $A$ on the continuous $C^*$-bundle of compact operators $\mathcal{K}(E)$ on the Hilbert $C(X)$-module $E = L^2(A, \varphi)$.

A simple $C^*$-algebra $A$ is purely infinite iff every non-zero hereditary $C^*$-subalgebra $B \subset A$ contains an infinite projection ([12]). Possible generalisations to the non-simple case are the following:

- A $C^*$-algebra $A$ is said to be purely infinite (p.i.) iff $A$ has no non-zero character and for all $a, b \in A_+, \varepsilon > 0$, with $b$ in the closed ideal of $A$ generated by $a$, there exists $d \in A$ with $\|b - d^*ad\| < \varepsilon$ ([11]).

- A $C^*$-algebra $A$ is said to be locally purely infinite (l.p.i.) iff for all $b \in A$ and all ideal $J \lhd A$ with $b \not\in J$, there exists a stable $C^*$-subalgebra $D_J \subset b^*Ab$ such that $D_J \not\subset J$.

Note that a $C^*$-algebra $A$ is p.i. iff for all $b \in A$, there exists a stable $C^*$-subalgebra $D \cong D \otimes \mathcal{K}$ contained in the hereditary $C^*$-subalgebra $b^*Ab$ such that for all (closed two sided) ideal $J \lhd A$ with $b \not\in J$, then $D \not\subset J$ ([12] prop. 5.4]). Hence, every p.i. $C^*$-algebra is l.p.i. ([3] prop. 4.11]). We shall study in this article a few problems linked to the converse implication.

The author is grateful to E. Kirchberg and M. Rørdam for helpful comments. He would also like to thank the Humboldt University for invitations during which part of that work was written.

**2. Continuous fields of properly infinite $C^*$-algebras**

In this section, we study the stability properties of proper infiniteness under (upper semi-)continuous deformations.

For all integer $n \geq 1$, $M_n(\mathbb{C})$ is the $C^*$-algebra linearly generated by $n^2$ operators.
\{e_{i,j}\} satisfying the relations \(e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}\) and \((e_{i,j})^* = e_{j,i}\) \((1 \leq i, j \leq n)\). The Cuntz \(C^*\)-algebra \(\mathcal{O}_n\) (resp. \(\mathcal{T}_n\)) is the unital \(C^*\)-algebra generated by \(n\) isometries \(s_1, \ldots, s_n\) satisfying the relation \(s_1s_1^* + \ldots + s_ns_n^* = 1\) (resp. \(s_1s_1^* + \ldots + s_ns_n^* \leq 1\)).

**Definition 2.1.** Given two \(C^*\)-algebra \(A\) and \(B\), a \(*\)-homomorphism \(\pi : A \to B\) is said to be unit full iff the closed two sided ideal generated by \(\pi(A)\) in \(B\) equals \(B\).

**Proposition 2.2.** Let \(X\) be a compact Hausdorff space and let \(D\) be a unital separable \(C(X)\)-algebra the fibres of which are properly infinite. Then \(M_l(D)\) is properly infinite for some integer \(l > 0\).

Let us first prove the following lemma which is essentially contained in [6].

**Lemma 2.3.** a) Let \(A, B\) be unital \(C^*\)-algebras, \(\pi : A \to B\) be a unital \(*\)-epimorphism, \(\theta : \mathcal{T}_2 \to A\) and \(\sigma : \mathcal{T}_2 \to B\) be unit full \(*\)-homomorphisms. Then there is a unit full \(*\)-homomorphism \(\theta' : \mathcal{T}_2 \to M_4(A)\) such that \((i \otimes \pi)\theta'(r) = e_{1,1} \otimes \sigma(r)\) for all \(r \in \mathcal{T}_2\).

b) Suppose that the \(C^*\)-algebra \(A\) is the pullback of the two unit \(C^*\)-algebras \(A_1\) and \(A_2\) along the \(*\)-epimorphisms \(\pi_k : A_k \to B\) \((k = 1, 2)\). Then there exists a unit full \(*\)-homomorphism \(\tilde{\theta} = (\theta_1, \theta_2) : \mathcal{T}_2 \to M_4(A)\) such that \(\tilde{\theta}(r) = e_{1,1} \otimes \theta(r)\) for all \(r \in \mathcal{T}_2\).

**Proof.** a) Let \(s_1, s_2\) be two isometries with orthogonal ranges generating the unital \(C^*\)-algebra \(\mathcal{T}_2\) and let \(p \in \mathcal{T}_2\) be the properly infinite projection \(p = s_1s_1^* + s_2s_2^*\). Then the two full projections \(\pi\theta(p)\) and \(\sigma(p)\) are Murray-von Neumann equivalent in \(B\) ([6] lemma 4.15]). Thus, there exists a unitary \(v \in M_2(B)\) with

\[
v^*(e_{1,1} \otimes \pi\theta(p))v = e_{1,1} \otimes \sigma(p).
\]

Define the the unitary \(u = 1_{M_2(B)} - e_{1,1} \otimes \sigma(p) + \sum_{k=1,2} (e_{1,1} \otimes \sigma(s_k))v^*(e_{1,1} \otimes \pi\theta(s_k^*))v\) in \(U(M_2(B))\). Then

\[e_{1,1} \otimes \sigma(s_k) = u v^* (e_{1,1} \otimes \pi\theta(s_k))v \quad \text{for} \quad k = 1, 2.
\]

Take unitary liftings \(\tilde{u}\) and \(\tilde{v}\) in \(U(M_4(A))\) of the unitaries \(u \oplus u^*\) and \(v \oplus v^*\) which are in the connected component of the identity. The formulae \(\theta'(s_k) = \tilde{u} \tilde{v}^* (e_{1,1} \otimes \theta(s_k))\tilde{v}\) \((k = 1, 2)\) define a relevant \(*\)-homomorphism \(\theta'\) from \(\mathcal{T}_2\) to \(M_4(A)\).

b) The \(C^*\)-algebra \(A\) is isomorphic to the \(C^*\)-subalgebra \(\{(a_1, a_2); a_j \in A_j\}_{a_j \in A_j, \pi_1(a_1) = \pi_2(a_2)}\) of \(A_1 \oplus A_2\). And \(e_{1,1} \otimes \pi\theta_1(p) = v^* (e_{1,1} \otimes \pi\theta_2(p))v\) for some \(v \in U(M_2(B))\).

Thus, by a), there exists an adequate \(*\)-morphism \(\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2)\) from \(\mathcal{T}_2\) to \(M_4(A)\).

**Proof of Proposition 2.2.** For all \(x \in X\) there exist a open neighbourhood \(U(x)\) of \(x\) in \(X\) with closure \(\overline{U(x)}\) and a unital \(*\)-homomorphism \(\mathcal{T}_2 \to D_{\overline{U(x)}} := D/C_0(X\setminus \overline{U(x)})\) since \(\mathcal{T}_2\) is semiprojective ([11] 4.7) and the fibre \(A_x\) is properly infinite. Thus, there exist a finite covering \(X = U_1 \cup \ldots \cup U_n\) by open subsets \(U_i\) and unital \(*\)-homomorphisms \(\sigma_i : \mathcal{T}_2 \to D_{\overline{U_i}} =: A_i\) \((1 \leq i \leq n)\). Now, step 2) of the above Lemma gives us a unit full \(*\)-homomorphism \(\tilde{\theta} : \mathcal{T}_2 \to M_l(D)\) for \(l = 4^{n-1}\), i.e. such that the closed two sided generated by the projection \(q = \pi(1)\) equals \(M_l(D)\).
If we embed each $M_k(D)$ in $M_{k+1}(D)$ by $d \mapsto d \oplus 0$ ($k \in \mathbb{N}$), then $1_{M_i(D)} \preceq q \oplus \ldots \oplus q$ in $M_{i+1}(D)$. But $q$ is properly infinite, i.e. $q \oplus q \preceq q$ (10), and so $1_{M_i(D)} \oplus 1_{M_i(D)} \preceq q \oplus \ldots \oplus q \preceq q \leq 1_{M_i(D)}$. □

Remark 2.4. Uffe Haagerup indicated me another way to prove Proposition 2.2: If the unital $C^*$-algebra $D$ is stably finite $C^*$-algebra, then there exists a bounded non-zero lower semi-continuous quasi-trace on $D$ (2). Now, if $D$ is also a $C(X)$-algebra for some compact Hausdorff space $X$, this implies that there is a bounded non-zero lower semi-continuous quasitrace $D_x \to \mathbb{C}$ for at least some point $x \in X$ ([2] Prop 3.7). But then, the fibre $D_x$ cannot be properly infinite.

Question 2.5. Does there exist a unital continuous $C(X)$-algebra $D$ the fibres of which are properly infinite and which is finite?

3. Lower semi-continuous fields of properly infinite $C^*$-algebras

Let us study whether the above results can be extended to lower semi-continuous (l.s.c.) $C^*$-bundles $(A, \{\sigma_x\})$ over a compact Hausdorff space $X$. Recall that any such separable l.s.c. $C^*$-bundle admits a faithful $C(X)$-linear representation on a Hilbert $C(X)$-module $E$ such that, for all $x \in X$, the fibre $\sigma_x(A)$ is isomorphic to the induced image of $A$ in $L(E_x)$ (5). Thus, the problem boils down to the following: Given a separable Hilbert $C(X)$-module $E$ with infinite dimensional fibres $E_x$, the unit $p$ of the $C^*$-algebra $L(E)$ of bounded adjointable $C(X)$-linear operators acting on $E$ has a properly infinite image in $L(E_x)$ for all $x \in X$. But is the projection $p$ itself (properly) infinite in $L(E)$?

Dixmier and Douady have proved that this is always the case if the space $X$ has finite topological dimension (8). But it does not hold anymore in the infinite dimensional case: Rørdam has constructed an explicit example where $L(E)$ is a finite $C^*$-algebra (13).

Question 3.1. What happens if the compact Hausdorff space $X$ is contractible?

References

1. B. Blackadar, *K-theory for Operator Algebras*, MSRI Publications 5, Cambridge Univ. Press (1998).
2. B. Blackadar, D. Handelman, *Dimension functions and traces on $C^*$-algebras*, J. Funct. Anal. 45 (1982), 297–340.
3. E. Blanchard, *Déformations de $C^*$-algèbres de Hopf*, Bull. Soc. Math. France 24 (1996), 141–215.
4. E. Blanchard, *Tensor products of $C(X)$-algebras over $C(X)$*, Astérisque 232 (1995), 81–92.
5. E. Blanchard, *A few remarks on $C(X)$-algebras*, Rev. Roumaine Math. Pures Appl. 45 (2001), 565–576.
6. E. Blanchard, E. Kirchberg, *Non-simple purely infinite $C^*$-algebras: the Hausdorff case*, J. Funct. Anal. 207 (2004), 461–513.
7. J. Cuntz, *K-theory for certain $C^*$-algebras*, Ann. of Math. 113 (1981), 181–197.
8. J. Dixmier, A. Douady, *Champs continus d’espaces hilbertiens et de $C^*$-algèbres*, Bull. Soc. Math. France 91 (1963), 227–284.
9. I. Hirshberg, M. Rørdam, W. Winter, *$C_0(X)$-algebras, stability and strongly self-absorbing $C^*$-algebras*, Preprint July 2006.
10. E. Kirchberg, M. Rørdam, *Non-simple purely infinite $C^*$-algebras*, Amer. J. Math. 122 (2000), 637–666.
[11] F. Larsen, N. Laustsen, M. Rørdam, *An Introduction to K-theory for C*-algebras*, London Mathematical Society Student Texts 49 (2000) CUP, Cambridge.

[12] M. Rørdam, *Stable C*-algebras*, Advanced Studies in Pure Mathematics 38 “Operator Algebras and Applications” (2004), 177-200.

[13] M. Rørdam, *A simple C*-algebra with a finite and an infinite projection*, Acta Math. 191 (2003), 109–142.

Etienne.Blanchard@math.jussieu.fr
IMJ, Projet Algèbres d’opérateurs, 175, rue du Chevaleret, F–75013 Paris