LIE POWERS OF THE NATURAL MODULE FOR $\text{GL}(2, K)$

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Abstract. In recent work of R. M. Bryant and the second author a (partial) modular analogue of Klyachko’s 1974 result on Lie powers of the natural $\text{GL}(n, K)$ was presented. There is was shown that nearly all of the indecomposable summands of the $r$th tensor power also occur up to isomorphism as summands of the $r$th Lie power provided that $r \neq p^m$ and $r \neq 2p^m$, where $p$ is the characteristic of $K$. In the current paper we restrict attention to $\text{GL}(2, K)$ and consider the missing cases where $r = p^m$ and $r = 2p^m$. In particular, we prove that the indecomposable summand of the $r$th tensor power of the natural module with highest weight $(r - 1, 1)$ is a summand of the $r$th Lie power if and only if $r$ is a not power of $p$.

1. Introduction

Let $E$ denote the natural module for the general linear group $G = \text{GL}(n, K)$ over an infinite field $K$ of characteristic $p \geq 0$. The isomorphism types of the indecomposable summands of $E^\otimes r$ are parameterized by (row) $p$-regular partitions of $r$ into at most $n$ parts. We denote these summands by $T(\lambda)$. For each $p$-regular partition $\lambda$ of $r$ let $D^\lambda$ denote the simple $KS_r$-module labelled by $\lambda$. Then

$$E^\otimes r \cong \bigoplus d_\lambda T(\lambda),$$

(1.1)

where $d_\lambda = \dim D^\lambda > 0$ and the sum ranges over all $p$-regular partitions of $r$ into at most $n$ parts. (Here we use the notation $nV$ to denote $V \oplus \cdots \oplus V$.) Now let $L^r(E)$ denote the $r$th homogeneous component of the free Lie algebra $L(E)$. We take $L(E)$ to be the Lie subalgebra of the tensor algebra of $E$ generated by $E$; then $L^r(E)$ is a $KG$-submodule of $E^\otimes r$. Moreover, if $r$ is not divisible by $p$ then it is well known that $L^r(E)$ is a direct summand of $E^\otimes r$. Thus, for $p \nmid r$ we have

$$L^r(E) \cong \bigoplus l_\lambda T(\lambda),$$

(1.2)

where the sum ranges over all $p$-regular partitions of $r$ into at most $n$ parts and $0 \leq l_\lambda \leq d_\lambda$. Donkin and Erdmann [8] gave a formula describing the multiplicities $l_\lambda$ in terms of Brauer...
characters of the symmetric group $S_r$, as follows:

$$l_\lambda = \frac{1}{r} \sum_{d|r} \mu(d) \beta^\lambda(\sigma^{r/d}),$$

where $\mu$ is the Möbius function, $\beta^\lambda$ is the Brauer character of $D^\lambda$ and $\sigma = (12\cdots r) \in S_r$.

One would like to be able to calculate the multiplicities $l_\lambda$, however, the Brauer characters are not known in general. In particular, it is difficult to determine from this formula alone which multiplicities are non-zero.

In characteristic zero, Klyachko \[13\] has shown that almost all of the irreducible $KG$-submodules of the $r$th tensor power $E^{\otimes r}$ also occur up to isomorphism as submodules of the $r$th Lie power $L^r(E)$. Since $E^{\otimes r}$ is completely reducible in this case we obtain that the multiplicities $l_\lambda$ occurring on the right-hand side of (1.2) are almost always positive. In the spirit of this result we would like to know, for arbitrary characteristic, which indecomposable summands of the $r$th tensor power also occur up to isomorphism as summands of $L^r(E)$. For modules $U$ and $V$ we write $U \mid V$ to mean that $U$ is isomorphic to a direct summand of $V$. Thus, by (1.1), we would like to know for which $p$-regular partitions $\lambda$ of $r$ into at most $n$ parts we have $T(\lambda) \mid L^r(E)$.

When $K$ is an infinite field of prime characteristic $p$ Klyachko’s original argument can be modified to prove a similar result for Lie powers of certain degree, see \[1\]. Unfortunately the methods used there do not work well when the degree is a power of $p$ or twice a power of $p$.

Throughout this paper we shall restrict attention to the case where $K$ is an infinite field of prime characteristic $p$ and $G = \text{GL}(2, K)$. We shall prove the following theorems.

**Theorem A.** Let $K$ be an infinite field of characteristic 2, $G = \text{GL}(2, K)$ and let $E$ denote the natural $KG$-module. Let $r$ be a positive integer greater than 6 and $\lambda$ a 2-regular partition of $r$ into at most two parts.

(i) If $r$ is not a power of 2 then $T(\lambda) \mid L^r(E)$ if and only if $\lambda \neq (r)$.

(ii) If $r$ is a power of 2 then $T(\lambda) \mid L^r(E)$ if and only if $\lambda \neq (r), (r-1, 1)$.

(iii) Let $1 \leq t_1 < t_2 < \ldots < t_k$ be such that $r = 2s_i + 3t_i$ with $s_i \geq 1$. Then $\bigoplus_{i=1}^k E^{\otimes t_i} \mid L^r(E)$, considered as modules for $\text{SL}(2, K)$.

Part (i) of the theorem is a special case of \[1\] Theorem 6.8. We give an alternative proof here, using a result of Stöhr \[16\] Corollary 9.2 on free Lie algebras of rank two in characteristic 2. Part (ii) of the theorem deals with the cases not covered by \[1\] Theorem 6.8 when $p = n = 2$. Note that part (iii) can be used to give a fairly large lower bound for the multiplicity of a given indecomposable tilting module $T(\lambda)$ as a direct summand of $L^r(E)$. The precise statement is given in Corollary 3.8. In Section 3 we lay the groundwork for the proof of Theorem A by exploiting \[16\] Corollary 9.2. The remainder of the proof then follows from the following theorem (to be proved in Section 4) in the case where $p = 2$. 
Theorem B. Let \( K \) be an infinite field of prime characteristic \( p \), \( G = GL(2, K) \) and let \( E \) denote the natural \( KG \) module. Then \( T(r - 1, 1) \) is a summand of \( L^r(E) \) if and only if either \( r = p \) or \( r \) is not a power of \( p \).

Furthermore, in odd characteristic we prove the following:

Theorem C. Let \( K \) be an infinite field of odd characteristic \( p \), \( G = GL(2, K) \) and let \( E \) be the natural \( KG \) module. Then \( r > p \) and let \( \lambda \) be a partition of \( r \) into at most two parts.

(i) If \( r = p^m \) with \( p > 3 \) then \( T(\lambda) \mid L^r(E) \) if and only if \( \lambda \neq (r), (r - 1, 1) \).

(ii) Let \( r = p^m \) with \( p = 3 \) and suppose \( \lambda \neq (r), (r - 1, 1), ((r + 1)/2, (r - 1)/2) \). Then \( T(\lambda) \mid L^r(E) \).

(iii) Let \( r = 2p^m \) with \( p > 3 \) and suppose \( \lambda \neq (r), (p^m, p^m) \). Then \( T(\lambda) \mid L^r(E) \).

(iv) Let \( r = 2p^m \) with \( p = 3 \) and suppose \( \lambda \neq (r), (p^m, p^m), (p^m + 1, p^m - 1), (p^m + 2, p^m - 2) \). Then \( T(\lambda) \mid L^r(E) \).

This is proved by using a result of Stöhr and Vaughan-Lee [17, Theorem 1].

2. Polynomial representations of \( GL(2, K) \)

Let \( K \) be an infinite field of prime characteristic \( p \), \( n \) a fixed positive integer, and let \( G = GL(n, K) \). We begin by recalling a few facts about polynomial \( KG \)-modules (see [12] for further reference) and then quickly specialize to the case \( n = 2 \).

2.1 Let \( E \) denote the natural \( KG \)-module. Then \( E \) is a polynomial module of degree 1. If \( V \) is a polynomial module of degree \( r \) and \( W \) is a polynomial module of degree \( s \) then \( V \otimes W \) is a polynomial module (with diagonal action) of degree \( r + s \). Every submodule and every quotient of a polynomial module of degree \( r \) is polynomial of degree \( r \). Thus we see that \( E^\otimes r \) and \( L^r(E) \) are polynomial modules of degree \( r \).

Let \( \Lambda(n, r) \) denote the set of unordered partitions of \( r \) into at most \( n \) parts. Any polynomial module \( V \) of degree \( r \) can be written as a direct sum of weight spaces over \( K \),

\[
V = \bigoplus_{\alpha \in \Lambda(r)} V^\alpha,
\]

where \( V^\alpha \) is the \( K \)-vector space of all \( v \) such that \( tv = t_1^{\alpha_1}t_2^{\alpha_2} \ldots t_n^{\alpha_n}v \) with \( t \in G \) diagonal with \( i \)th diagonal entry \( t_i \). Let \( \Lambda^+(n, r) \) denote the set of (ordered) partitions of \( r \) into at most \( n \) parts. The simple polynomial modules of degree \( r \) are indexed by \( \Lambda^+(n, r) \) and we denote the simple module labelled by the partition \( \lambda \) by \( L(\lambda) \). To every partition \( \lambda \in \Lambda^+(n, r) \) let \( \Delta(\lambda) \) denote the Weyl module with unique simple quotient \( L(\lambda) \) (see [12] (5.3a),(5.3b) and (5.4b)]). All other composition factors of \( \Delta(\lambda) \) are isomorphic to modules of the form \( L(\mu) \) where \( \mu < \lambda \) with respect to the dominance ordering. Let \( \nabla(\lambda) \) denote the contravariant dual of \( \Delta(\lambda) \). We
say that a finite-dimensional $KG$-module admits a \textit{Weyl filtration} if it has a filtration in which every section is isomorphic to a Weyl module. Similarly we say that a finite-dimensional $KG$-module admits a \textit{dual Weyl filtration} if it has a filtration in which every section is isomorphic to a dual Weyl module.

The category of polynomial $KG$-modules of degree $r$ is equivalent to the module category of the Schur algebra $S(n,r)$ (for details see [12]). This has the advantage that there are finite-dimensional injective modules. This category is also a highest weight category in the sense of [4], with weight poset given by $\Lambda^+(n,r)$ with respect to the dominance ordering. The Weyl modules are the ‘standard modules’ and the dual Weyl modules the ‘costandard modules’. Equivalently, the Schur algebra $S(n,r)$ is quasi-hereditary. By a theorem of Ringel [15], and also Donkin [6] for the algebraic group setup, each highest weight category has a class of ‘canonical’ modules $T(\lambda)$ indexed by dominant weights. These are the indecomposable polynomial modules of degree $r$ admitting both a Weyl filtration and a dual Weyl filtration. We shall call the modules $T(\lambda)$ the \textit{indecomposable tilting modules} of degree $r$. Every Weyl filtration of $T(\lambda)$ contains exactly one section isomorphic to $\Delta(\lambda)$; all other sections are isomorphic to $\Delta(\mu)$, where $\mu < \lambda$. In fact, we have something stronger:

**Remark 2.1.** The $\lambda$-weight space of $T(\lambda)$ is one-dimensional, and the submodule generated by a weight vector of weight $\lambda$ is isomorphic to $\Delta(\lambda)$. This is the unique submodule of $T(\lambda)$ isomorphic to $\Delta(\lambda)$. We shall say that a polynomial module is a \textit{tilting module} if it is isomorphic to a direct sum of indecomposable tilting modules. An important property is that the tilting modules are closed under tensor products; this follows from the fact that tensor products of Weyl modules (respectively dual Weyl modules) have Weyl filtration (respectively dual Weyl filtration). Thus if $V$ and $W$ are tilting modules of degree $r$ and $s$ then $V \otimes W$ is a tilting module of degree $r + s$.

2.2 From now on we assume $n = 2$ so that $G = \text{GL}(2,K)$. We fix the basis $\{x,y\}$ of $E$ such that, when $E$ is identified with the column space, the vectors $x, y$ are identified with $\binom{1}{0}$ and $\binom{0}{1}$ respectively.

To simplify notation we shall write $\Lambda^+(r)$ to mean $\Lambda^+(2,r)$. Recall that a partition is (row) $p$-regular if no $p$ parts are equal. We write $\Lambda^+_p(r)$ for the set of $p$-regular partitions in $\Lambda^+(r)$. Thus for $p > 2$ we have $\Lambda^+_p(r) = \Lambda^+(r)$, whilst $\Lambda^+_2(r)$ consists of all partitions $\lambda = (\lambda_1, \lambda_2)$ of $r$ with $\lambda_1 > \lambda_2$. The isomorphism types of the indecomposable summands of $E^\otimes r$ are given by the $T(\lambda)$ where $\lambda \in \Lambda^+_2(r)$, as described in (1.1). Thus we would like to know for which $\lambda \in \Lambda^+_2(r)$ we have $T(\lambda) \mid L^r(E)$.

It will often be convenient to work with the subgroup $H = \text{SL}(2,K)$. Let $\mathcal{P}_r$ denote the category of $KH$-modules which are restrictions of polynomial $KG$-modules of degree $r$. Then
$\mathcal{P}_r$ is also a highest weight category. We identify the set of dominant weights with the set

$$W_r = \{m \in \mathbb{N}_0 : 0 \leq m \leq r, m \equiv r \mod 2\}.$$ 

The simple modules in $\mathcal{P}_r$ are then indexed by $W_r$, and we denote these by $L(m)$. Similarly, we denote the Weyl modules, dual Weyl modules and indecomposable tilting modules in $\mathcal{P}_r$ by $\Delta(m)$, $\nabla(m)$ and $T(m)$ respectively. We say that a module in $\mathcal{P}_r$ is a tilting module if it is isomorphic to a direct sum of indecomposable tilting modules.

If $\lambda = (\lambda_1, \lambda_2) \in \Lambda^+(r)$ then the simple module $L(\lambda)$ of $GL(2, K)$ restricts to $L(m)$, where $m = \lambda_1 - \lambda_2$. Similarly, $\Delta(\lambda), \nabla(\lambda)$ and $T(\lambda)$ restrict to $\Delta(m), \nabla(m)$ and $T(m)$, respectively. Suppose $M$ is any $S(2, r)$-module whose restriction to $SL(2, K)$ is isomorphic to $L(m)$, or $\Delta(m)$, or $\nabla(m)$, or $T(m)$. Then $m \leq r$ and $m \equiv r \mod 2$. Moreover there is a unique partition $\lambda(m) \in \Lambda^+(r)$ such that $M$ is isomorphic to $L(\lambda(m))$, or $\Delta(\lambda(m))$, or $\nabla(\lambda(m))$ or $T(\lambda(m))$ (see [5, 3.2.7]). We note that the restriction of $\Delta(1)$ to $SL(2, K)$ is given by $E^{\otimes r} \cong \bigoplus d_\lambda T(m)$, where the sum ranges over all $m$ in the set

$$(2.1) \quad A_r = \{k \in \mathbb{N} : 0 < k \leq r \text{ and } k \equiv r \mod 2\}.$$ 

We summarize some properties of the modules $T(m)$ and $\Delta(m)$ (see [6], [19] and [18] for further details). For a $KG$- or $KH$-module $M$, we denote its Frobenius twist by $M^F$.

(2.2a) For any $m \geq 0$, $\nabla(m)$ is isomorphic to the $m$-th symmetric power giving

$$\dim \Delta(m) = \dim \nabla(m) = m + 1.$$ 

(2.2b) Let $m$ be a non-negative integer. Then

$$T(m) \cong \Delta(m) \cong \nabla(m) \cong L(m)$$

if and only if either $m = 0$ or $m = ap^k - 1$, where $2 \leq a \leq p$ and $k \geq 0$ (see [18]). In particular, $E = \Delta(1) \cong \nabla(1) \cong L(1) \cong T(1)$ and $\Delta(0) = K$ as $SL(2, K)$-modules.

(2.2c) If $m = kp + i$ where $0 \leq i \leq p - 2$ and $k \geq 1$ then

$$T(m) \cong T(k - 1)^F \otimes T(p + i).$$

(2.2d) If $m = kp + (p - 1)$ where $k \geq 0$ then $T(m) \cong T(k)^F \otimes T(p - 1)$. If $T(k)$ is simple then so is $T(m)$.

(2.2e) For all $i, j \geq 0$ satisfying $i + j = p - 2$ there is a short exact sequence of $SL(2, K)$-modules given by

$$0 \to \Delta(p + j) \to T(p + j) \to \Delta(i) \to 0.$$ 

(2.2f) For all $m, n \geq 0$ with $m \geq n$ the tensor product $\Delta(m) \otimes \Delta(n)$ has a Weyl filtration with sections

$$\Delta(m + n), \Delta(m + n - 2), \ldots, \Delta(m - n).$$

(2.2g) For all $m, n \geq 0$, $T(n + m)$ is a direct summand of $T(n) \otimes T(m)$. 
(2.2h) If $0 \leq i, j \leq p - 2$ and $i + j = p - 2$ then for any $n \geq 1$ there is a short exact sequence

$$0 \to \Delta(n - 1)^F \otimes L(j) \to \Delta(pn + i) \to \Delta(n)^F \otimes L(i) \to 0.$$ 

3. Lie powers of the natural module in characteristic 2

In this section we lay the groundwork for the proof of Theorem A. The following result of Stöhr on free Lie algebras of rank two in characteristic two will be a key ingredient of our proof.

**Theorem 3.1.** [10 Corollary 9.2] Let $K$ be a field of characteristic 2, $G$ a group and let $V$ be a two-dimensional $KG$-module. For all $r \geq 4$ there is a direct sum decomposition of $L^r(V)$ as a $KG$-module:

$$L^r(V) = \bigoplus_{s,t \geq 1} m_{s,t}L^{r/(2s+3t)}(R^2(V)^{\otimes s} \otimes R^3(V)^{\otimes t}),$$

where $L^{r/d}(X) = 0$ if $d \nmid r$ and $m_{s,t} = \frac{1}{s+t} \sum d|s,t \mu(d) ((s+t)/d)! (s/d)(t/d)!$.

Here $R(V)$ is the free restricted Lie algebra on $V$. Notice that the multiplicity $m_{s,t}$ is equal to the dimension of the subspace of $L^{s+t}(V)$ spanned by the monomials of multidegree $(s,t)$, given by Witt’s dimension formula (see [14 Theorem 5.11] for example). In particular, these multiplicities are all positive.

We apply Theorem 3.1 in the case where $K$ is an infinite field of characteristic 2, $G = GL(2, K)$ and $V = E$ is the natural $KG$-module. Let $\{x, y\}$ be the basis of $E$ as defined in Section 2. We claim that the modules $R^2(E)$ and $R^3(E)$ occurring in this decomposition can be identified with certain Weyl modules. By definition $\Delta(2, 0)$ is the submodule of $E^{\otimes 2}$ generated by $x \otimes x$ (see [12 (5.3b)]). It has basis $\{x \otimes x, y \otimes y, x \otimes y + y \otimes x\}$. Since the characteristic of $K$ is two it then follows that $R^2(E) \cong \Delta(2, 0)$. Similarly, $\Delta(2, 1)$ is the submodule of $E^{\otimes 3}$ generated by $x \otimes x \otimes y + y \otimes x \otimes x$. It has basis $\{x \otimes x \otimes y + y \otimes x \otimes x, y \otimes y \otimes x + x \otimes y \otimes y\}$. Since $K$ has characteristic 2 we have $R^3(E) = L^3(E)$ with basis given by $\{[x, x, y], [x, y, x]\}$. Thus it is easy to see that $R^3(E) \cong \Delta(2, 1)$.

Let $D_{s,t} = \Delta(2, 0)^{\otimes s} \otimes \Delta(2, 1)^{\otimes t}$. Then Theorem 3.1 yields

$$L^r(E) \cong L^{r/2}(D_{1,0}) \oplus L^{r/3}(D_{0,1}) \oplus \bigoplus_{s,t \geq 1} m_{s,t}L^{r/(2s+3t)}(D_{s,t}),$$

for all $r \geq 4$. Since the multiplicities occurring on the right-hand side of (3.1) are all positive we see that $D_{s,t}$ is isomorphic to a direct summand of $L^{2s+3t}(E)$ for all $s, t \geq 1$. We shall show that each such summand $D_{s,t}$ is a tilting module for $G$. Let $\Delta_{s,t}$ denote the restriction of $D_{s,t}$ to $SL(2, K)$. Thus $\Delta_{s,t} = \Delta(2)^{\otimes s} \otimes \Delta(1)^{\otimes t}$. We shall soon see that it is enough to show that $\Delta_{s,t}$ is a tilting module for $SL(2, K)$.
Lemma 3.2. Let $K$ be an infinite field of characteristic 2, let $H = \text{SL}(2, K)$ and let $E$ denote the natural $KH$-module. For each $s, t \geq 1$ let $\Delta_{s,t}$ denote the $KH$-module defined by $\Delta_{s,t} = \Delta(2)^{\otimes s} \otimes \Delta(1)^{\otimes t}$. Then $\Delta_{s,t} \mid E^{\otimes 2s+t}$.

Proof. We first consider the case where $s = t = 1$. By (1.1) with $r = p = 2$ we obtain

$$E^{\otimes 2} \cong T(2).$$

Thus, by (2.2e), there is short exact sequence

$$0 \to \Delta(2) \to E^{\otimes 2} \to \Delta(0) \to 0$$

of $KH$-modules. Tensoring this with $E \cong \Delta(1)$ gives

$$0 \to \Delta(2) \otimes \Delta(1) \to E^{\otimes 3} \to \Delta(1) \to 0.$$  \hspace{1cm} (3.3)

Now, by (1.1) with $r = 3$ and $p = 2$, we have

$$E^{\otimes 3} \cong T(3) \oplus 2T(1),$$

so that, by (2.2b) and (2.2d), the middle term of (3.3) is semisimple. Thus the sequence (3.3) is split and we deduce that

$$\Delta_{1,1} = \Delta(2) \otimes \Delta(1) \cong T(3) \oplus T(1).$$

In particular, $\Delta_{1,1}$ is isomorphic to a direct summand of $E^{\otimes 3}$ and hence $\Delta_{1,t} = \Delta(2) \otimes E^{\otimes t} = \Delta_{1,1} \otimes E^{\otimes t-1}$ is isomorphic to a direct summand of $E^{\otimes t+2}$ for all $t \geq 1$. It follows that $\Delta(2) \otimes T \mid E^{\otimes t+2}$ for all $T \mid E^{\otimes t}$ and hence by induction that $\Delta_{s,t} \mid E^{\otimes 2s+t}$ for all $s, t \geq 1$. \hspace{1cm} $\square$

Corollary 3.3. Let $K$ be an infinite field of characteristic 2, let $G = \text{GL}(2, K)$ and let $E$ denote the natural $KG$-module. For each $s, t \geq 1$ let $D_{s,t}$ denote the $KG$-module defined by $D_{s,t} = \Delta(2, 0)^{\otimes s} \otimes \Delta(2, 1)^{\otimes t}$. Then $D_{s,t} \mid E^{\otimes 2s+3t}$.

Proof. Let $H = \text{SL}(2, K)$ and for all positive integers $m$ and $a$ let $g(m, a)$ denote the multiplicity of $T(m)$ in $E^{\otimes a}$ considered as a $KH$-module. By Lemma 3.2 we have that the restriction $\Delta_{s,t}$ of $D_{s,t}$ to $H$ is a summand of $E^{\otimes 2s+t}$. Thus we may write

$$\Delta_{s,t} = \bigoplus_{m \in A_{2s+t}} a_m T(m)$$

for some multiplicities $a_m$ satisfying $0 \leq a_m \leq g(m, 2s + t)$. Note that the sum ranges over all $m \in A_{2s+t}$ with $A_r$ as defined in (2.1). Since $A_{2s+t}$ is a subset of $A_{2s+3t}$, each indecomposable module $T(m)$ occurring on the right-hand side is a $KH$-summand of $E^{\otimes 2s+3t}$. We shall show that each such summand $T(m)$ occurs in $E^{\otimes 2s+3t}$ with multiplicity greater than or equal to $a_m$. Then by the unique lifting of $T(m)$ to a tilting module for $G$ (see Section 2) it will follow that $D_{s,t}$ is isomorphic to a $KG$-summand of $E^{\otimes 2s+3t}$. 

Thus we must show that \( a_m \leq g(m, 2s + 3t) \) for all \( m \in A_{2s+t} \). Since \( a_m \leq g(m, 2s + t) \), it is enough to show that \( g(m, 2s + t) \leq g(m, 2s + 3t) \). Now, by [10, Lemma 1.7.2 and Lemma 1.5(1)], it is known that for any positive integers \( a \) and \( m \) we have \( g(a, m) \leq g(a, m + 2) \). This completes the proof. \( \square \)

We shall now show that every indecomposable summand of \( E^\otimes 2s+t \) is isomorphic to a direct summand of \( \Delta_{s,t} \) as \( \text{SL}(2, K) \)-modules.

**Lemma 3.4.** Let \( K \) be an infinite field of characteristic 2, let \( H = \text{SL}(2, K) \) and let \( E \) denote the natural \( KH \)-module. Let \( s, t \geq 1 \). Then \( T(m) \mid \Delta_{s,t} \) if and only if \( T(m) \mid E^\otimes 2s+t \).

**Proof.** Let \( r = 2s + t \) and \( p = 2 \). As we have seen, the restriction of \( \Delta_{1,1} \) to \( H \) yields that \( T(m) \mid E^\otimes 2s+t \) if and only if \( m \in A_{2s+t} \), where \( A_{2s+t} \) is as in (2.1). Thus we must show that \( T(m) \mid \Delta_{s,t} \) if and only if \( m \in A_{2s+t} \). By Lemma 3.2 we may write

\[
\Delta_{s,t} \cong \bigoplus_{m \in A_{2s+t}} a_m T(m)
\]

for some multiplicities \( a_m \geq 0 \). So it is enough to show that each of the multiplicities \( a_m \) occurring on the right-hand side of (3.6) is non-zero.

We first note that this holds for \( \Delta_{1,1} \). Indeed \( A_{2(1)+1} = \{1, 3\} \) and by (3.5) \( \Delta_{1,1} \cong T(3) \oplus T(1) \). Thus we may suppose that the multiplicities \( a_m \) occurring on the right-hand side of (3.6) are all positive for some \( s, t \geq 1 \) and proceed by induction on \( s \) and \( t \). Since \( \Delta(1) = T(1) \), from (3.6) we obtain

\[
\Delta_{s,t+1} = \Delta_{s,t} \otimes T(1) \cong \bigoplus_{m \in A_{2s+t}} a_m T(m) \otimes T(1).
\]

Now by (2.2g) we find that \( T(m+1) \mid \Delta_{s,t+1} \) for all \( m \in A_{2s+t} \). We note that when \( t \) is odd \( A_{2s+(t+1)} = \{m+1 : m \in A_{2s+t}\} \), whilst when \( t \) is even, \( A_{2s+(t+1)} = \{m+1 : m \in A_{2s+t}\} \cup \{1\} \). Thus in order to show that the result holds for \( \Delta_{s,t+1} \) it remains to show that \( T(1) \mid \Delta_{s,t+1} \) whenever \( t \) is even. When \( t \) is even we have by induction that \( T(2) \mid \Delta_{s,t} \) and thus \( T(2) \otimes T(1) \mid \Delta_{s,t+1} \). By (3.2) and the fact that \( E \cong T(1) \) we see that \( T(2) \otimes T(1) \cong E^\otimes 3 \). Thus (3.4) yields that \( T(1) \mid \Delta_{s,t+1} \).

From (3.6) we also obtain

\[
\Delta_{s+1,t} = \Delta(2) \otimes \Delta_{s,t} \cong \bigoplus_{m \in A_{2s+t}} a_m \Delta(2) \otimes T(m).
\]

Since \( T(m) \) has Weyl filtration with sections \( \Delta(m), \Delta(m_1), \ldots, \Delta(m_k) \) where \( m > m_1, \ldots, m_k \) we deduce by (2.2f) that \( \Delta(2) \otimes T(m) \) has Weyl filtration with sections \( \Delta(m+2), \Delta(n_1), \ldots, \Delta(n_l) \) where \( m + 2 > n_1, \ldots, n_l \). By Lemma 3.2 we have \( \Delta_{s+1,t} \mid E^\otimes 2(s+1)+t \) and it follows that each of the summands \( \Delta(2) \otimes T(m) \) occurring on the right-hand side above must decompose as a direct sum of indecomposable tilting modules. By consideration of highest weights we deduce that
Thus in order to show that the result holds for \( \Delta \) whenever \( T \) that is denoted by \( \Delta \), we denote the natural \( KG \)-module \( \Delta(2) \) as at most two parts with highest weights yielding \( \Delta(2) \otimes \lambda \). By Corollary 3.3 we see that \( \Delta(2) \otimes T(1) \otimes \Delta(2)^{\otimes (2.2f)} \) we find that \( \Delta(2) \otimes \lambda \) only if \( \lambda \) denotes \( \Delta \) indecomposable tilting modules of the form \( \Delta(2) \otimes \lambda \) also have \( \lambda \) being a partition. The restriction of \( D_{s,t} \) to \( SL(2,K) \) is denoted by \( \Delta_{s,t} \) and \( \lambda \) is a direct summand of \( D_{s,t} \) if and only if \( \lambda \) is a direct summand of \( \Delta_{s,t} \). By Lemma 3.4, this happens if and only if \( \lambda_1 - \lambda_2 \in A_{2s+t} \). That is, if and only if \( 0 \leq \lambda_1 - \lambda_2 \leq 2s + t \) and \( \lambda_1 - \lambda_2 \equiv 2s + t \mod 2 \). Since \( \lambda \) is a partition of \( 2s + 3t \), we also have \( \lambda_1 + \lambda_2 = 2s + 3t \). Hence we deduce that \( T(\lambda) \mid D_{s,t} \) if and only if \( \lambda = (\lambda_1, \lambda_2) \) with \( \lambda_1 > \lambda_2 \geq t \).

**Proposition 3.6.** Let \( K \) be an infinite field of characteristic 2, let \( G = GL(2,K) \) and let \( E \) denote the natural \( KG \)-module. Let \( r \) be a positive integer greater than 6 and let \( \lambda \in \Lambda_s^+ (r) \).

(i) If \( r \) is odd and \( \lambda \neq (r) \) then \( T(\lambda) \mid L'(E) \).

(ii) If \( r \) is even and \( \lambda \neq (r), (r-1,1) \) then \( T(\lambda) \mid L'(E) \).

**Proof.** (i) Write \( r = 2k + 1 \) where \( k \geq 3 \). By equation (3.1), \( D_{k-1,1} \mid L'(E) \). Thus it is enough to show that \( T(\lambda) \mid D_{k-1,1} \) for all \( \lambda \neq (r) \). This follows immediately from Corollary 3.5.

(ii) Let \( r = 2k \) where \( k > 3 \). By equation (3.1), \( D_{k-3,2} \mid L'(E) \). Thus it is enough to show that \( T(\lambda) \mid D_{k-3,2} \) for all \( \lambda \neq (r), (r-1,1) \). This follows from Corollary 3.5.

The direct sum decomposition given in (3.1) also allows us to find lower bounds for the multiplicities of the indecomposable tilting modules occurring up to isomorphism as direct summands of \( L'(E) \). In fact we shall see that these multiplicities are large in general. Restricting (3.1) to \( SL(2,K) \) yields that \( \Delta_{s,t} \) is a summand of \( L'(E) \) whenever \( r = 2s + 3t \) and \( s,t \geq 1 \).

**Lemma 3.7.** Let \( K \) be an infinite field of characteristic 2, let \( H = SL(2,K) \) and let \( E \) denote the natural \( KH \)-module. Let \( r = 2s + 3t \) with \( s,t \geq 1 \). Then \( E^{\otimes t} \mid \Delta_{s,t} \).
Proof. By Lemma 3.4 and the fact that $E \cong T(1)$ we know that $E \mid \Delta_{s,1}$ as $\text{SL}(2, K)$-modules. Hence $E^{\otimes t} \mid \Delta_{s,1} \otimes E^{\otimes t-1} = \Delta_{s,t}$. □

Corollary 3.8. Let $K$ be an infinite field of characteristic 2, $G = \text{GL}(2, K)$ and let $E$ denote the natural $KG$-module. Let $r$ be a positive integer and let $\{(s_i, t_i) : i = 1, \ldots, k\}$ be a complete set of solutions to the equation $r = 2s + 3t$. For each partition $\lambda = (\lambda_1, \lambda_2)$ of $r$ with $0 < \lambda_1 - \lambda_2 \leq t_i$ define $\lambda(i) = (\lambda_1 - (s_i + t_i), \lambda_2 - (s_i + t_i))$. Then the multiplicity of $T(\lambda)$ in $L^r(E)$ is bounded below by

$$
\sum m_{s_i, t_i} \Delta_{\lambda(i)},
$$

where the sum ranges over all $i$ such that $0 < \lambda_1 - \lambda_2 \leq t_i$.

Proof. By (3.1) we have $\bigoplus_i m_{s_i, t_i} D_{s_i, t_i} \mid L^r(E)$. Thus the multiplicity of $T(\lambda)$ in $L^r(E)$ is greater than or equal to the multiplicity of $T(\lambda)$ in $\bigoplus_i m_{s_i, t_i} D_{s_i, t_i}$. Restriction to $\text{SL}(2, K)$ yields that the multiplicity of $T(\lambda)$ in $L^r(E)$ is greater than or equal to the multiplicity of $T(\lambda_1 - \lambda_2)$ in $\bigoplus_i m_{s_i, t_i} \Delta_{s_i, t_i}$. By Lemma 3.7 we see that $E^{\otimes t_i} \mid \Delta_{s_i, t_i}$. Thus the multiplicity of $T(\lambda)$ in $L^r(E)$ is greater than or equal to the multiplicity of $T(\lambda_1 - \lambda_2)$ in $\bigoplus_i m_{s_i, t_i} E^{\otimes t_i}$ and the result now follows from (1.1) restricted to $\text{SL}(2, K)$. □

Notice that Proposition 3.6 and Lemma 3.7 go most of the way to proving Theorem A. Indeed, it only remains to show that $T(r - 1, 1) \mid L^r(E)$ if and only if $r$ is not a power of 2. We shall prove this result in the following section.

In the remainder of this section we shall prove that, for $r > 6$, $L^r(E)$ is a tilting module if and only if $r$ is odd.

Lemma 3.9.  
(i) $L^2(T(2))$ contains a non-tilting summand.

(ii) $L^2(T(3))$ contains a non-tilting summand.

Proof. (i) By (3.2), $T(2) \cong E^{\otimes 2}$ and so $T(2)$ has basis $\{e_1, e_2, e_3, e_4\}$, where $e_1$ has weight 2, $e_2$ has weight $-2$ and $e_3, e_4$ have weight zero (for example, in terms of our basis for $E$ we may identify $e_1$ with $x \otimes x$, $e_2$ with $y \otimes y$ and so on). One checks that the only weights occurring in $L^2(T(2))$ are $0, \pm 2$ and that the composition factors are $L(2), L(2), L(0), L(0)$. Suppose for contradiction that $L^2(T(2))$ is a tilting module. Then, by consideration of highest weights, we must have that $T(2) \mid L^2(T(2))$. Since $T(2)$ has composition factors $L(2), L(0), L(0), L(0)$, this would leave only $L(2)$ in the complement which is non-tilting, thus contradicting the assumption that $L^2(T(2))$ is a tilting module.

(ii) By (2.2b) and (2.2a) $T(3) \cong \nabla^3(E) \cong S^3(E)$. Thus $T(3)$ has basis $\{e_1, e_2, e_3, e_4\}$, where $e_1$ has weight 3, $e_2$ has weight $-3$, $e_3$ has weight 1 and $e_4$ has weight -1. It is then easy to see that the only weights occurring in $L^2(T(3))$ are $0, \pm 2, \pm 4$ and that the composition factors are $L(4), L(2), L(0), L(0)$. Suppose for contradiction that $L^2(T(3))$ is a tilting module.
Then, by consideration of highest weights, we must have that $T(4) \mid L^2(T(3))$. However, $T(4)$ has composition factors $L(4), L(2), L(2), L(0), L(0)$, so it cannot be a direct summand of $L^2(T(3))$. □

**Theorem 3.10.** Let $r > 6$. Then $L^r(E)$ is a tilting module if and only if $r$ is odd.

*Proof.* If $r$ is odd then $L^r(E)$ is a summand of the tensor power, as explained in the introduction. To prove the converse, we consider the restriction to $H = SL(2, K)$. It suffices to prove that for all even $r$ with $r > 6$ we have that $L^r(E)$ contains a non-tilting summand. We will use the following argument. Suppose $W = U \oplus V$ as $KH$-modules, then $L^2(W)$ is a direct summand of $L^2(W)$. Namely, it is clear that there is a vector space decomposition

$$L^2(W) = L^2(U) \oplus L^2(V) \oplus [U, V]$$

and it is then easy to check that each summand is a $KH$-module.

Write $r = 2k$ and suppose first that $k$ is odd. Since $r > 6$ we have that $k > 3$ and hence $L^2(\Delta_{\frac{k-3}{2}, 1})$ is a direct summand of $L^r(E)$, by (3.1). By Lemma 3.4 we see that $T(3)$ is a direct summand of $\Delta_{s, 1}$ for all $s \geq 1$. Thus, by the above argument, $L^2(T(3))$ is a direct summand of $L^r(E)$ and hence $L^r(E)$ contains a non-tilting summand by Lemma 3.9 (ii).

Next suppose that $k$ is even. For $k > 6$ we have that $L^2(\Delta_{\frac{k-3}{2}, 2})$ is a direct summand of $L^r(E)$, by (3.1). Then by Lemma 3.4 we see that $T(2)$ is a direct summand of $\Delta_{s, 2}$ for all $s \geq 1$. Thus, again by the above argument, $L^2(T(2))$ is a direct summand of $L^r(E)$ and hence $L^r(E)$ contains a non-tilting summand by Lemma 3.9 (i).

It remains to deal with the cases $k = 4$ and $k = 6$. For $k = 4$ equation (3.1) gives

$$L^8(E) \cong L^4(\Delta(2)) \oplus \Delta_{1, 2}.$$  

By arguments similar to those in Lemma 3.9 it is easy to show that $L^4(\Delta(2))$ has composition factors

$$L(6), L(4), L(4), L(2), L(2), L(0), L(0), L(0), L(0), L(0).$$

Thus if $L^4(\Delta(2))$ is a tilting module, it must contain $T(6)$ as a direct summand. Since $T(6)$ has composition factors

$$L(6), L(4), L(4), L(2), L(2), L(0), L(0), L(0), L(0), L(0),$$

this would leave only $L(2)$ in the complement, which is non-tilting. Thus $L^8(E)$ contains a non-tilting summand. For $k = 6$, equation (3.1) gives that $L^4(E)$ is a direct summand of $L^{12}(E)$ and it is easy to check that $L^4(E)$ has composition factors $L(2), L(0)$, hence is not tilting, so that $L^{12}(E)$ contains a non-tilting summand. □

Note that it can be shown by direct computation that $L^2(E) \cong T(1, 1), L^4(E) \cong \nabla(3, 1)$ and $L^6(E) \cong T(5, 1) \oplus T(3, 3)$ as $GL(2, K)$-modules.
4. Lie powers of the natural module in arbitrary characteristic

We now return to the case where $K$ is an infinite field of arbitrary prime characteristic $p$. As before we let $G = \text{GL}(2, K)$, $H = \text{SL}(2, K)$ and let $E$ denote the natural $KG$-module with canonical basis $\{x, y\}$, as described in Section 2.

In this section we shall show that $T(r - 1, 1) \mid L^r(E)$ if and only if $r$ is not a power of $p$. To do so, we will exploit the fact that the highest weight in $L^r(E)$, namely $(r - 1, 1)$, has one-dimensional weight space.

**Remark 4.1.** If $r$ is not divisible by $p$ then, as we have seen in (2.2), $L^r(E)$ is a tilting module. Since the $(r - 1, 1)$ weight space is one-dimensional it follows immediately in this case that $L^r(E)$ has a unique summand isomorphic to $T(r - 1, 1)$.

Thus for the rest of this section we shall assume that $r$ is divisible by $p$. The $(r - 1, 1)$ weight space of $L^r(E)$ is spanned by the left-normed Lie monomial

$$\zeta := [\ldots[[y, x], x], \ldots, x].$$

We claim that if $T(r - 1, 1)$ is a submodule of $L^r(E)$ then the $KG$-submodule of $L^r(E)$ generated by $\zeta$, denoted $G\zeta$, is isomorphic to $\Delta(r - 1, 1)$. Indeed, suppose that $T(r - 1, 1)$ is a submodule of $L^r(E)$. Then $\zeta$ must be contained in $T(r - 1, 1)$ and it follows that $G\zeta$ is a submodule of $T(r - 1, 1)$. By Remark 2.1 this implies that $G\zeta \cong \Delta(r - 1, 1)$.

**Lemma 4.2.** Let $K$ be an infinite field of prime characteristic $p$ and let $r$ be a positive multiple of $p$.

(i) There is a homomorphism $\varphi : T(r - 1, 1) \rightarrow L^r(E)$, which maps the unique submodule of $T(r - 1, 1)$ isomorphic to $\Delta(r - 1, 1)$ onto the submodule $G\zeta$ of $L^r(E)$.

(ii) $T(r - 1, 1)$ is a submodule of $L^r(E)$ if and only if $G\zeta \cong \Delta(r - 1, 1)$.

**Proof.** (i) We have a short exact sequence of $KG$-modules

$$0 \rightarrow U \rightarrow L^{r-1}(E) \otimes E \xrightarrow{\psi} L^r(E) \rightarrow 0$$

(where $\psi$ maps $z_1 \otimes z_2$ to $[z_1, z_2]$). Since $p$ does not divide $r - 1$ we may apply Remark 4.1 to find that $L^{r-1}(E)$ has a unique summand isomorphic to the tilting module $T(r - 2, 1)$. By (2.2g), noting that $E \cong T(1)$, the module $T(r - 2, 1) \otimes E$ has $T(r - 1, 1)$ as a summand. The $(r - 1, 1)$ weight space of $L^{r-1}(E) \otimes E$ is one-dimensional, spanned by $\zeta' \otimes x$ where $\zeta' := [\ldots[[y, x], x], \ldots, x]$ spans the $(r - 2, 1)$ weight space of $L^{r-1}(E)$. Hence $\zeta' \otimes x$ must lie in the summand $T(r - 1, 1)$. By Remark 2.1, the submodule $G(\zeta' \otimes x)$ of $T(r - 1, 1)$ is the unique submodule isomorphic to the Weyl module $\Delta(r - 1, 1)$. Let $\varphi$ be the restriction of $\psi$ to $T(r - 1, 1)$. Then $\varphi(\zeta' \otimes x) = \zeta$ and so $\varphi$ maps $G(\zeta' \otimes x)$ onto $G\zeta$. This completes the proof of part (i).
(ii) By the remark preceding the lemma, we know that if \( T(r-1,1) \) is a submodule of \( L^r(E) \) (not necessarily via \( \varphi \)) then \( G\zeta \) must be isomorphic to \( \Delta(r-1,1) \). Conversely, if \( G\zeta \cong \Delta(r-1,1) \) then the restriction of \( \varphi \) to \( \Delta(r-1,1) \) is one-to-one. Since the socle of \( T(r-1,1) \) is simple (this follows from Lemmas 5 and 11 in [11]), and is contained in \( \Delta(r-1,1) \) it follows that \( \varphi \) is one-to-one and hence \( T(r-1,1) \) is isomorphic to a submodule of \( L^r(E) \).

In order to determine whether \( T(r-1,1) \mid L^r(E) \) we must study the \( KG \)-submodule \( G\zeta \) of \( L^r(E) \) generated by \( \zeta \). By Lemma 4.2 (i), this is a factor module of \( \Delta(r-1,1) \). In particular its (non-zero) weight spaces are one-dimensional. Our goal is to determine weight spaces of \( G\zeta \) for sufficiently many weights, so that we can identify its composition factors. Certainly \( L(r-1,1) \) occurs since \( \zeta \in G\zeta \).

Let \( g = \left( \begin{array}{cc} s & u \\ t & v \end{array} \right) \in G \), then \( g\zeta = (ux + vy)(ad(sx + ty))^{r-1} \). If \( s = 0 \) then this is just a scalar multiple of \([\ldots[[x,y],y],\ldots,y],\) which is in a one-dimensional weight space of a weight of \( L(r-1,1) \). So we assume now \( s \neq 0 \), and then without loss of generality, \( s = 1 \). Let \( \alpha = x + ty \). Since \((\alpha)(ad(\alpha))^{r-1} = 0\), the elements \((x)(ad\alpha)^{r-1}\) and \((y)(ad\alpha)^{r-1}\) are linearly dependent, so to identify weight spaces of \( G\zeta \), it is enough to consider the second of these two.

Let \( R_\alpha \) and \( L_\alpha \) be right- and left multiplication by \( \alpha \) in the associative tensor algebra on \( E \). These operations commute and

\[
(ad\alpha)^{r-1} = \sum_{k=0}^{r-1} \binom{r-1}{k} R_\alpha^{r-1-k}(-1)^k L_\alpha^k.
\]

Hence we have

\[
(4.1) \quad (y)(ad\alpha)^{r-1} = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k}(x + ty)^k y(x + ty)^{r-1-k}.
\]

If \( r = p^m \) we have \((p^{m-1}_k) \equiv (-1)^k \mod p\), and (4.1) specializes to

\[
(4.2) \quad (y)(ad\alpha)^{p^m-1} = \sum_{k=0}^{p^m-1} (x + ty)^k y(x + ty)^{p^m-1-k}.
\]

Note that (4.1) and (4.2) are expressions in the associative tensor algebra of \( E \). If we write either of these as a polynomial in \( t \), then for each \( i \) the coefficient of \( t^i \) is a weight vector, with \( i + 1 \) copies of \( y \) and \( r - (i + 1) \) copies of \( x \). Hence for different values of \( i \) the weights are distinct. Since the field is infinite, the module \( G\zeta \) has a basis consisting of the coefficients of the \( t^i \) which are non-zero.

**Lemma 4.3.** Assume \( r > 1 \) is a power of \( p \). Then \( G\zeta \) is a simple module isomorphic to \( L(r-1,1) \). Moreover \( G\zeta \cong \Delta(r-1,1) \) if and only if \( r = p \).
Proof. Let \( r = p^n \), where \( m \geq 1 \). The idea is to show that \( G \zeta \) is a module of the right dimension. Since \( L(r-1,1) \) is a composition factor of \( G \zeta \), it is enough to show that \( \dim G \zeta \leq \dim L(r-1,1) \). We use equation (4.2).

(i) First we show that for \( i = cp - 1 \), where \( 1 \leq c \leq p^m - 1 \), the coefficient of \( t^i \) in (4.2) is zero. Let \( \eta \) be a monomial in \( x \) and \( y \) of weight \( (p^m - cp, cp) \). We shall find the coefficient of \( \eta \) in (4.2). The monomial \( \eta \) has the form

\[
\eta = x^{a_0} y x^{a_1} y \ldots y x^{a_{cp}},
\]

where \( a_j \geq 0 \) for all \( j \) and \( \sum_j a_j = p^m - cp \). (Note that there are \( cp \) copies of \( y \) in total.) Then \( \eta \) occurs precisely \( cp \) times in (4.2), namely for the following values of \( k \),

\[
a_0, a_0 + a_1 + 1, \ldots, a_0 + a_1 + \ldots + a_{cp-1} + (cp - 1).
\]

Hence the coefficient of \( \eta \) in (4.2) is equal to

\[
cp t^{cp-1} \equiv 0 \mod p.
\]

(ii) Since \( G \zeta \) has basis consisting of the coefficients of the \( t^i \) which are non-zero, we compute an upper bound for the dimension of \( G \zeta \) as follows. The total number of possible weight vectors in \( G \zeta \) is \( p^m \), and we have shown that \( p^m - 1 \) of these are zero. Therefore \( \dim G \zeta \leq p^m - p^{m-1} \). Thus it suffices to show that \( \dim L(r-1,1) = p^m - p^{m-1} \). This is clear from (2.2b) and (2.2a) if \( m = 1 \). For \( m \geq 2 \) we have \( r - 2 = p - 2 + \sum_{j=1}^{m-1} (p - 1)p^j \). Thus, restricting to \( SL(2,K) \) yields

\[
L(r-2) \cong L(p-2) \otimes \bigotimes_{j=0}^{m-2} L(p-1)^{F^j},
\]

by Steinberg’s tensor product theorem. Applying (2.2b) and (2.2a), we see that the dimension of \( L(r-2) \) is \( (p - 1)p^{m-1} \), as required. Finally we note that \( \dim \Delta(r-1,1) = r - 1 = p^m - 1 \) and therefore \( \Delta(r-1,1) \cong L(r-1,1) \) if and only if \( m = 1 \). □

Lemma 4.4. Let \( r \) be a positive multiple of \( p \) that is not a power of \( p \). Then \( G \zeta \cong \Delta(r-1,1) \).

Proof. We show that \( G \zeta \) is a module of the right dimension. Since \( G \zeta \) is a factor module of \( \Delta(r-1,1) \), it is enough to show that \( \dim G \zeta \geq \dim \Delta(r-1,1) = r - 1 \). We use equation (4.1).

Consider the weight \( (r-v,v) \) with \( 1 \leq v \leq r/2 \). We will show that the coefficient of \( t^{v-1} \) in (4.1) is non-zero, so that the \( (r-v,v) \) weight space is non-zero. By applying a group element that interchanges \( x \) with \( y \) (up to a sign) it will follow that the \( (v,r-v) \) weight space is also non-zero. This will give in total \( r - 1 \) distinct non-zero weight spaces, which will prove the lemma.

Consider arbitrary monomials \( \eta = y x^{a_1} y x^{a_2} \ldots y x^{a_v} \) of weight \( (r-v,v) \) where \( v \) is fixed. It is enough to show that at least one of these monomials occurs with non-zero coefficient in (4.1).
Such $\eta$ occurs in (4.1) for $k = 0, a_1 + 1, (a_1 + 1) + (a_2 + 1), \ldots , \sum_{i=1}^{v-1}(a_i + 1)$.

The coefficient of $\eta$ in $(y)(\text{ad}\alpha)^{r-1}$ is therefore equal to $t^{v-1}\left(1 + (-1)^{a_1+1}(r - 1\atop a_1 + 1) + \ldots + (-1)^{\sum_{i=1}^{v-1}(a_i+1)}(r - 1\atop \sum_{i=1}^{v-1}(a_i + 1))\right)$.

Suppose, for contradiction, that this coefficient is equal to zero modulo $p$ for every such monomial $\eta$. Then taking $a_1 = \ldots = a_{v-1} = 0$ gives

$$1 + \sum_{i=1}^{v-1}(-1)^i\left(r - 1\atop i\right) \equiv 0 \pmod{p}. \quad (4.3)$$

Now take any $w$ with $v - 1 < w \leq r - 1$, and take the monomial $\eta$ with $a_1 = 1$, and $a_2 = \ldots = a_{v-2} = 0$ and $a_{v-1} = w - v$. (Note that $\sum_{i=1}^{v-1}a_i \leq r - v$, so such monomial is defined). Since the coefficient of $\eta$ is equal to zero modulo $p$ we obtain

$$1 + \sum_{i=2}^{v-1}(-1)^i\left(r - 1\atop i\right) + (-1)^w\left(r - 1\atop w\right) \equiv 0 \pmod{p}. \quad (4.4)$$

Subtracting (4.4) from (4.3) yields

$$(-1)^w\left(r - 1\atop w\right) \equiv 1 \pmod{p}, \quad (4.5)$$

since $r \equiv 0 \pmod{p}$. Note that if we let $w$ vary this tells us that all binomial coefficients $\left(r - 1\atop w\right)$ are all non-zero modulo $p$, since we are allowed to take $w$ to be any integer in the range $r/2 \leq w \leq r - 1$.

If $r - 1$ is odd and $p \neq 2$ then taking $w = r - 1$ gives $-1 \equiv 1 \pmod{p}$; a contradiction. If $r - 1$ is odd and $p = 2$, then all entries in the $(r - 1)$-th row of Pascal’s triangle are equal to 1 modulo 2 and it follows that $\left(r\atop k\right) \equiv 0 \pmod{2}$ for $1 \leq k \leq r - 1$. Thus $r$ must be a power of 2, contradicting the hypothesis.

Now assume $r - 1$ is even. By (4.5), the $(r - 1)$-th row of Pascal’s triangle modulo $p$ has entries 1 and $(-1)$ alternating, and we once more deduce that $\left(r\atop k\right) \equiv 0 \pmod{p}$ for $1 \leq k \leq r - 1$. Thus $r$ must be a power of $p$, contrary to the hypothesis. \hfill \Box

We shall show that whenever $T(r - 1, 1)$ is a submodule of $L^r(E)$, it is a direct summand. This will use the following:

**Proposition 4.5.** Let $s \geq 0$.

(i) The tilting module $T(s)$ is injective in degree $s$.

(ii) If $s + 2$ is not a power of $p$ then $T(s)$ is injective in degree $s + 2$. 

Proof. A tilting module is injective if and only if it is projective since it is self-dual. We show projectivity.

(i) For $0 \leq s \leq p - 1$ the Schur algebra $S(2, s)$ is semisimple, so any module is projective. For $p \leq s \leq 2p - 2$, $T(s)$ is projective, for example by [11] Lemma 20, Lemma 24 (with $u = 0$ and $s = 0$ respectively). Now let $s > 2p - 2$ and assume true for all weights $< s$.

Suppose first that $s = kp + (p - 1)$. Then $T(s) \cong T(p - 1) \otimes T(k)^F$, by (2.2d). By induction, $T(k)$ is projective in degree $k$. The functor $(-)^F \otimes T(p - 1)$ is an equivalence between the block containing $k$ and the block containing $s$ (see for example Lemma 1 in [11]), hence $T(s)$ is projective in degree $s$.

Now suppose that $s = kp + (p + i)$ where $0 \leq i \leq p - 2$. Then $T(s) \cong T(p + i) \otimes T(k)^F$ by (2.2c). By induction, $T(k)$ is projective in degree $k$, say $T(k) = P(u)$, the projective cover of the simple module $L(u)$. By Lemma 11 of [11], the module $T(s)$ has top $L(pu + j)$ where $i + j = p - 2$. Now the arguments in Lemma 20 and Lemma 24 of [11] show that $T(p+i) \otimes T(k)^F$ is projective in degree $s$.

(ii) Assume $s + 2$ is not a power of $p$. By part (i) we have that $T(s)$ is projective in degree $s$. Let $T(s) = P_s(u)$, the projective module in degree $s$ with simple top $L(u)$. In degree $s + 2$ there is then a surjective homomorphism

$$P_{s+2}(u) \twoheadrightarrow P_s(u),$$

where $P_{s+2}(u)$ is the projective cover of $L(u)$ in degree $s+2$, since $P_s(u)$ has simple top $L(u)$. To show this is an isomorphism, it suffices to show that both modules have the same $\Delta$-quotients. Let $[M : \Delta(t)]$ denote the number of quotients isomorphic to $\Delta(t)$ in a $\Delta$-filtration of $M$ (if $M$ has $\Delta$-filtration, this is well-defined). By ‘BGG reciprocity’ and duality, (see for example [3]), we have that for any $t \leq r$ and $t \equiv r \mod 2$,

$$[P_r(u) : \Delta(t)] = [\Delta(t) : L(u)]$$

where $[\Delta(t) : L(u)]$ is the multiplicity of $L(u)$ as a composition factor of $\Delta(t)$.

Since $s \equiv s + 2 \mod 2$, it follows that if $t \leq s$ then $[P_r(u) : \Delta(t)]$ is the same for $r = s$ and $r = s + 2$. It remains to show that $\Delta(s + 2)$ is not a $\Delta$-quotient of $P_{s+2}(u)$, or equivalently, that $L(u)$ does not occur as a composition factor of $\Delta(s + 2)$. We know that $L(u)$ is the socle of $T(s)$ and is therefore the socle of $\Delta(s)$. Since $s + 2$ is not a power of $p$ we can deduce using (2.2h) that $u \neq 0$. The claim follows now from the next lemma. \qed

Lemma 4.6. Suppose that $L(w) = \text{soc}\Delta(k)$ with $w \neq 0$. If $L(w)$ is a composition factor of $\Delta(k + 2)$ then $k + 2$ is a power of $p$.

Proof. Suppose first that $\Delta(k)$ is simple, so that $\Delta(k) = L(w)$. Then $L(w)$ takes up $k + 1$ dimensions from $\Delta(k + 2)$, which only has dimension $k + 3$. It follows that $L(k + 2)$ must be two-dimensional, which means that $k + 2 = p^a$ for some $a \geq 1$. 

Now suppose that $\Delta(k)$ is not simple. Thus $k \geq p$, by (2.2b). We proceed by induction on $k$. Since $k$ and $k+2$ are in the same block, we get that $k \equiv -2 \pmod{p}$. This follows from the Theorem in [1]. Thus we may write $k = pm + p - 2$ where $m \geq 1$, and by (2.2h), $L(w)$ is a submodule of $\Delta(m - 1)^F$. We see from this that $L(w) \cong L(v)^F$ where $w = pv$ with $v \neq 0$.

Then the end terms for the sequence (2.2h) of $\Delta(k+2)$ are $\Delta(m+1)^F$ and $\Delta(m)^F \otimes L(p-2)$. Since $\Delta(pm + p - 2)$ is multiplicity-free, $L(w)$ does not occur in $\Delta(m)^F \otimes L(p-2)$. So it must occur in $\Delta(m+1)^F$. Now we have $L(v) = \text{soc}\Delta(m - 1)$, and also $L(v)$ occurs in $\Delta(m + 1)$. By the inductive hypothesis, $m + 1 = p^a$ for some $a \geq 1$. Therefore $k + 2 = p^{a+1}$ and the lemma is proved.

\textbf{Theorem B.} Let $K$ be an infinite field of characteristic $p$, $G = \text{GL}(2,K)$ and let $E$ denote the natural $KG$ module. Then $T(r - 1, 1)$ is a summand of $L^r(E)$ if and only if either $r = p$ or $r$ is not a power of $p$.

\textbf{Proof.} By Remark 1.1 we have that $T(r-1,1) \mid L^r(E)$ whenever $r$ is not divisible by $p$. Thus we may assume that $r$ is a positive multiple of $p$. When $r = p$ it can also be shown that $T(p-1,1)$ is a summand of $L^p(E)$ using [3]. Indeed, it follows from [3] Corollary 3.2 and Lemma 4.2 that $L^p(E)$ has a direct summand isomorphic to $\nabla(p-1,1)$ and restriction to $\text{SL}(2,K)$ yields $T(p-2) \cong \nabla(p-2) \mid L^p(E)$, by (2.2b). By the unique lifting of $T(p-2)$ to a tilting module for $G$ (see Section 2) it then follows that $T(p-1,1)$ is isomorphic to a $KG$-summand of $L^r(E)$. Thus we may assume that $r = pk$ where $k > 1$.

Suppose first that $r$ is not a $p$-power. By Lemma 1.3 and Lemma 1.2 (ii) we get that $T(r-1,1)$ is isomorphic to a submodule of $L^r(E)$. By Proposition 1.5, $T(r - 1,1)$ is injective in degree $r$ and hence it is a summand of $L^r(E)$.

Next suppose that $T(r-1,1)$ is a summand of $L^r(E)$. By Lemma 1.2 (ii) we have $G_ζ \cong \Delta(r-1,1)$. Suppose for contradiction that $r = p^m$, then by Lemma 1.3 we have $G_ζ \cong L(r-1,1)$ and hence $m = 1$, contradicting our assumption that $r = pk$ where $k > 1$.

We note that [3] Corollary 3.2] used in proof of Theorem B concerns the $p$th metabelian Lie power. The $r$th metabelian Lie power of the natural module, denoted $M^r(E)$, is certain a quotient of the $r$th Lie power $L^r(E)$ (for details, see [16] section 1) or [3] for example). It was shown in [3] Corollary 3.2] that the $p$th metabelian Lie power occurs as a direct summand of the $p$th Lie power (see also [2] Section 2] for an explicit splitting map $M^p(E) \to L^p(E)$). One may wonder whether this quotient always occurs as a direct summand. We give a partial answer.

\textbf{Proposition 4.7.} Let $K$ be an infinite field of characteristic $p$, $G = \text{GL}(2,K)$ and let $E$ denote the natural $KG$ module. Let $r > 1$ be a positive integer that is not a power of $p$. Then $M^r(E)$ occurs as a direct summand of $L^r(E)$ if and only if $r = 2$ or $r = ap^k + 1$ for some $2 \leq a \leq p$ and $k \geq 0$. 

Proof. Assume that \( r \) is not a power of \( p \). Then by Remark 4.1, Lemma 4.2 (ii) and Lemma 4.4 we see that \( L^*(E) \) has a submodule isomorphic to \( T(r-1,1) \). Notice that \( L(r-1,1) \) occurs as a composition factor of \( T(r-1,1) \), namely as the top composition factor of the submodule \( \Delta(r-1,1) \) of \( T(r-1,1) \). Since the \( (r-1,1) \) weight space of \( L^*(E) \) is one-dimensional, this is the unique composition factor isomorphic to \( L(r-1,1) \) in \( L^*(E) \).

It was shown in [3, Lemma 4.2] that for \( r \geq 2 \) the \( r \)th metabelian Lie power \( M^r(E) \) is isomorphic to the dual Weyl module \( \nabla(r-1,1) \) as a module for the general linear group. Suppose that \( \nabla(r-1,1) \) occurs as a direct summand of \( L^*(E) \) as a module for \( GL(2,K) \), then in particular it is a submodule and hence its socle is a submodule of \( L^*(E) \). Since the socle of \( \nabla(r-1,1) \) is \( L(r-1,1) \) it follows that \( \Delta(r-1,1) \) must be simple, and by the results of section 2, so is the restriction to \( SL(2,K) \). Thus, by (2.2b), we see that this holds if and only if \( r - 2 = 0 \) or \( r - 2 = a p^k - 1 \) for some \( 2 \leq a \leq p \) and some \( k \geq 0 \).

For \( r = 1 \) we have, trivially, that \( M^r(E) = L^r(E) \). Consider the case where \( r > 1 \) is a power of \( p \). Then the situation is different, and by Lemma 4.3 the simple module \( L(r-1,1) \) is in the socle of \( L^*(E) \). We have seen that [3, Corollary 3.2] gives \( M^p(E) \) is a direct summand of \( L^p(E) \). We calculate some further examples for \( p = 2 \). If \( r = 4 \) then \( \nabla(3,1) \) is a summand of \( L^*(E) \); on the other hand \( \nabla(7,1) \) is not a summand of \( L^8(E) \). In general, for \( r = p^m \) with \( m \geq 2 \) we do not know whether \( M^r(E) \) occurs as a direct summand of \( L^*(E) \).

We conclude this section with the proof of Theorem A.

**Theorem A.** Let \( K \) be an infinite field of characteristic 2, \( G = GL(2,K) \) and let \( E \) denote the natural \( KG \)-module. Let \( r \) be a positive integer greater than 6 and \( \lambda \) a \( 2 \)-regular partition of \( r \) into at most two parts.

(i) If \( r \) is not a power of 2 then \( T(\lambda) \mid L^r(E) \) if and only if \( \lambda \neq (r) \).

(ii) If \( r \) is a power of 2 then \( T(\lambda) \mid L^r(E) \) if and only if \( \lambda \neq (r), (r-1,1) \).

(iii) Let \( 1 \leq t_1 < t_2 < \ldots < t_k \) be such that \( r = 2s_i + 3t_i \) with \( s_i \geq 1 \). Then \( \bigoplus_{i=1}^{k} E^{\otimes t_i} \mid L^r(E) \), considered as modules for \( SL(2,K) \).

**Proof.** We first note that the \((r)\)-weight space of \( L^r(E) \) is zero. Thus, by Remark 2.1, \( T(r) \) is not a summand of \( L^r(E) \) for any \( r > 1 \). Parts (i) and (ii) now follow from Proposition 3.6 and Theorem B, whilst part (iii) follows from [3,1] restricted to SL(2,K) and Lemma 3.7.

We note that Theorem A (i) can also be obtained as a special case of [1, Theorem 6.8], which is stated for \( GL(n,K) \) modules. However, the methods used in [1] do not apply when \( r = p^m \) or \( r = 2p^m \). Thus, for \( n = 2 \) and \( p = 2 \), we see that Theorem A (ii) deals with the cases not covered by [1, Theorem 6.8]. In the following section we give some partial results for \( n = 2 \) and \( p > 2 \) which are not covered by [1, Theorem 6.8]. Thus we look at Lie powers of degrees \( p^m \) and \( 2p^m \).
5. Lie powers of degree $p^m$ or $2p^m$ for $p > 2$

We return temporarily to the case $G = \text{GL}(n, K)$ and let $V$ denote the natural $KG$-module. Let $r > 1$. Then for all $a, b \geq 1$ satisfying $r = a + b$ we have that $[L^a(V), L^b(V)]$ is a submodule of $L^r(V)$. If $a$ and $b$ are coprime then it follows from [17, Theorem 1] that $[L^a(V), L^b(V)]$ is isomorphic to the tensor product $L^a(V) \otimes L^b(V)$ as a module for $\text{GL}(n, K)$. Namely, there is a surjective homomorphism from this tensor product onto $L^r(V)$, and in addition is injective in degree $r$. Hence whenever $T(\lambda)$ is a summand of this tensor product, and in addition is injective in degree $r$, it follows that $T(\lambda)$ is a summand of $L^r(V)$. Therefore it will be advantageous to take temporarily $n = r$ so that all summands of $V^{\otimes r}$ are injective (see for example 3.7 in [9], $V^{\otimes r} \cong S^{(1^r)}(V)$). The truncation functor, see [12, section 6.5], maps $V$ to $E$, $V^{\otimes r}$ to $E^{\otimes r}$ and $L^r(V)$ to $L^r(E)$. Moreover, by [9] given a partition $\lambda = (\lambda_1, \lambda_2)$, the truncation functor takes the tilting module $T(\lambda)$ for $\text{GL}(r, K)$ to the tilting module $T(\lambda)$ for $\text{GL}(2, K)$.

**Theorem C.** Let $K$ be an infinite field of odd characteristic $p$, $G = \text{GL}(2, K)$ and let $E$ be the natural $KG$ module. Let $r > p$ and let $\lambda$ be a partition of $r$ into at most two parts.

(i) If $r = p^m$ with $p > 3$ then then $T(\lambda) \mid L^r(E)$ if and only if $\lambda \neq (r), (r-1,1)$.

(ii) Let $r = p^m$ with $p = 3$ and suppose $\lambda \neq (r), (r-1,1), ((r+1)/2, (r-1)/2)$. Then $T(\lambda) \mid L^r(E)$.

(iii) Let $r = 2p^m$ with $p > 3$ and suppose $\lambda \neq (r), (p^m, p^m)$. Then $T(\lambda) \mid L^r(E)$.

(iv) Let $r = 2p^m$ with $p = 3$ and suppose $\lambda \neq (r), (p^m, p^m), (p^m + 1, p^m - 1), (p^m + 2, p^m - 2)$. Then $T(\lambda) \mid L^r(E)$.

**Proof.** Note that $r > p > 2$. Let $V$ be the natural $\text{GL}(r, K)$-module, which truncates to $E$ (see the remark above).

(i) and (ii) Assume first that $r = p^m$. Recall that the $(r)$-weight space of $L^r(E)$ is zero and hence $T(r)$ is not a summand of $L^r(E)$. By Theorem B we also have that $T(r-1,1)$ does not occur. Thus we must show that $T(\lambda)$ occurs when $\lambda_2 \geq 2$, with the exception of $\lambda = ((r+1)/2, (r-1)/2)$ when $p = 3$.

Consider the submodule $[L^2(V), L^{r-2}(V)]$ of $L^r(V)$. By the remarks preceding the theorem we have that $[L^2(V), L^{r-2}(V)] \cong L^2(V) \otimes L^{r-2}(V)$. We note that since $p \nmid 2$ and $p \nmid r - 2$, we may apply [11, Corollary 6.10] to find $T(1,1) \mid L^2(E)$ and $T(\mu) \mid L^{r-2}(V)$ for all partitions $\mu = (\mu_1, \mu_2)$ of $r - 2$ with $\mu_2 > 0$, except for $\mu = ((r-1)/2, (r-3)/2)$ when $p = 3$. Thus each such $T(1,1) \otimes T(\mu)$ is a submodule of $L^r(V)$ with highest weight $\lambda = (\mu_1 + 1, \mu_2 + 1)$. Since $T(1,1) \otimes T(\mu)$ it is a direct sum of tilting modules, it has $T(\lambda)$ as a direct summand. In fact, since $\lambda$ is a $p$-regular partition of $r$, we also have $T(\lambda) \mid V^{\otimes r}$, and since all summands of $V^{\otimes r}$ are injective in degree $r$ it follows that each such $T(\lambda)$ occurs as a direct summand of $L^r(V)$. 


Applying the truncation functor now yields that \( T(\lambda) \) occurs as a direct summand of \( L^r(E) \) for all \( \lambda = (\lambda_1, \lambda_2) \) with \( \lambda_2 \geq 2 \) except for \( \lambda = ((r+1)/2, (r-1)/2) \) when \( p = 3 \), as required.

(iii) Assume next that \( r = 2p^m \) and \( p > 3 \). Consider the submodule \([L^3(V), L^{r-3}(V)]\) of \( L^r(V) \). By the remarks preceding the theorem we have that \([L^3(V), L^{r-3}(V)] \cong L^3(V) \otimes L^{r-3}(V)\). We note that since \( p \nmid 3 \) and \( p \nmid r - 3 \), we may apply [9, Corollary 6.10] to find \( T(2,1) \mid L^3(V) \) and \( T(\mu) \mid L^{r-3}(V) \) for all partitions \( \mu = (\mu_1, \mu_2) \) of \( r - 3 \) with \( \mu_2 > 0 \). So each such \( T(2,1) \otimes T(\mu) \) is a submodule of \( L^r(V) \) with highest weight \( (\mu_1 + 2, \mu_2 + 1) \), giving that \( T(\lambda) \mid T(2,1) \otimes T(\mu) \) for all partitions \( \lambda = (\lambda_1, \lambda_2) \) of \( r \) with \( \lambda_2 \geq 2 \) and \( \lambda_1 - \lambda_2 > 0 \). Since each such \( \lambda \) is a \( p \)-regular partition of \( r \) we again conclude that \( T(\lambda) \) is injective in degree \( r \) and hence \( T(\lambda) \mid L^r(V) \) for all \( \lambda \neq (r), (r - 1, 1), (p^m, p^m) \). Applying the truncation functor, yields \( T(\lambda) \mid L^r(E) \) for all \( \lambda \neq (r), (r - 1, 1), (p^m, p^m) \) and by Theorem B we also know that \( T(r - 1, 1) \mid L^r(E) \).

(iv) Finally let \( r = 2p^m \) with \( p = 3 \). Consider the submodule \([L^5(V), L^{r-5}(V)]\) of \( L^r(V) \). By the remarks preceding the theorem we have that \([L^5(V), L^{r-5}(V)] \cong L^5(V) \otimes L^{r-5}(V)\). We note that since \( p \nmid 5 \) and \( p \nmid r - 5 \), we may apply [9, Corollary 6.10] to find \( T(4,1) \mid L^5(V) \) and \( T(\mu) \mid L^{r-5}(V) \) for all partitions \( \mu = (\mu_1, \mu_2) \) of \( r - 5 \) with \( \mu_2 > 0 \) and \( \mu_1 - \mu_2 > 1 \). So each such \( T(4,1) \otimes T(\mu) \) is a submodule of \( L^r(V) \) with highest weight \( (\mu_1 + 4, \mu_2 + 1) \), giving \( T(\lambda) \mid T(4,1) \otimes T(\mu) \) for all partitions \( \lambda \) of \( r \) with \( \lambda_2 \geq 2 \) and \( \lambda_1 - \lambda_2 > 4 \). Each such \( T(\lambda) \) is again injective in degree \( r \) giving \( T(\lambda) \mid L^r(V) \) for all \( \lambda \neq (r), (r - 1, 1), (p^m, p^m), (p^m + 1, p^m - 1), (p^m + 2, p^m - 2) \). Applying the truncation functor, yields \( T(\lambda) \mid L^r(E) \) for all \( \lambda \neq (r), (r - 1, 1), (p^m, p^m), (p^m + 1, p^m - 1), (p^m + 2, p^m - 2) \) and by Theorem B we also know that \( T(r - 1, 1) \mid L^r(E) \). \( \square \)

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