A new proof of Grinberg Theorem
based on cycle bases

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Abstract
Grinberg Theorem, a necessary condition only for planar Hamilton graphs, was proved in 1968. In this paper, we use the cycles in a cycle basis of a simple connected graph to replace the faces in planar graphs and present a new proof of Grinberg Theorem, where $f'_i$ is the number of faces of degree $i$. This result implies that Grinberg Theorem can be extended into simple connected graphs.

Keywords Hamiltonian graphs Grinberg Theorem Cycle basis

1. Introduction
In 1968, took a Hamiltonian cycle on the plane as a Jordan curve, Emanuel Grinberg analyzed the relations between interior faces and outer faces along the curve and obtained an equality [1], called Grinberg formula or Grinberg criterion. This result was named Grinberg Theorem which is a well-known necessary condition for planar Hamilton graphs:

Grinberg Theorem Let $G$ be a planar graph of order $|V|$ with a Hamilton cycle $C$. Then $\sum_{i=3}^{|V|} (i - 2)(f'_i - f''_i) = 0$, where $f'_i$ and $f''_i$ are the numbers of faces of degree $i$ contained in inside $C$ and outside $C$, respectively.

In the original proof of the theorem [1], the given graph $G$ is a planar graph consisted of elementary cycles of being partitioned into interior and outer faces. So, it is easy, from the theorem, to derive the following the equality associated with interior faces,

$$\sum_{i=3}^{|V|} (i f'_i - 2 f''_i) = |V| - 2. \quad (1.1)$$
It is clear that the equality (1.1) holds only for planar graphs. Since then, some special graphs have been studied on the extension of Grinberg Theorem [2], [3], [4], but there has no the literature on simple connected graphs.

In this paper, using the cycles of a cycle basis, we investigate the combinatorial structure of the simple connected graphs according to the Inclusion-Exclusion Principle. By substituting interior faces with the cycles under the condition that the given graph is Hamiltonian, we derive the following theorem, which is the same as the equality (1.1). This result implies Grinberg Theorem can be extended into simple connected graphs.

**Theorem 1.1** Let $F$ be a cycle basis of a simple connected graph $G$. $f_i$ ($f_i \in F$) is referred to a cycle of order $i$ (interior faces $f'_i$ in Grinberg theorem). If $G$ is Hamiltonian, then $\sum_{i=3}^{|V|} (if_i - 2f_i) = |V| - 2$.

2. **Definitions**

The graphs considered in this paper are the finite, undirected, connected and simple graphs. A graph $G = (V,E)$ is a nonempty set $V$ of elements called vertices, together with a set $E$ of two element subsets of $V$ called edges. A walk is a finite alternating sequence of vertices and edges that begins and ends with two distinct vertices, such that each edge is incident with the vertices preceding and following it. No edge appears more than once in a walk. A closed walk in which no vertex (except for the beginning and ending vertex) appears more than once is called a cycle (that every vertex in a cycle is degree 2). A cycle that contains every vertex of a graph is called a Hamilton cycle. A graph is Hamiltonian if it contains a Hamiltonian cycle. In a given graph, a cycle basis is a set of cycles that forms a basis of the cycle space of the graph. Every closed walks in a graph can be expressed as a symmetric difference of the basis cycles. The dimension of the cycle space is equal to $|E| - |V| + 1$.

For notions and terminologies not defined in this paper, please refer to the Chapter 2 of [5], the Chapter 2-4 and Chapter 6 of [6].
3. The proof of Theorem 1.1

Proof Let G be a Hamilton graph and F a cycle basis of G. \( f_i \ (f_i \in F) \) is referred to a cycle (an interior face in Grinberg theorem) of order \( i \ (i \leq |V|) \). \(|f_i|\) denotes the number of \( f_i \). Therefore, \(|F| = |f_3| + |f_4| + \cdots + |f_{|V|}|\). Since the union of all cycles (both vertices and edges) in a cycle basis of G is the graph itself, then \(|V| = |V_3 \cup V_4 \cup \cdots \cup V_{|V|}|\).

By inclusion-exclusion principle, we have

\[
|V| = \sum_{i=3}^{|V|} |V_i| - \sum_{3 \leq i < j \leq |V|} |V_i \cap V_j |
+ \sum_{3 \leq i < j \leq |V|} |V_i \cap V_j \cap V_k| - \cdots + (-1)^{|V|-1} |V_3 \cap V_4 \cap V_5 \cdots \cap V_{|V|}|. \tag{3.2}
\]

Note that every Hamiltonian cycle in G can be represented by the symmetric difference of a set of cycles in a cycle basis of G and the cardinality of the union of every pair of disjoint cycles is null. So we reserve \( V_i \cap V_j \) to denote the item of that the cardinal number of the intersection of every pair of joint cycles is \( 2 \). Then, we rewrite the equality (3.2) as below,

\[
|V| = \sum_{i=3}^{|V|} |V_i| - \sum_{3 \leq i < j \leq |V|} |V_i \cap V_j |. \tag{3.3}
\]

Again, since \(|V_i \cap V_j| = 2\) and the number of pair of joint cycles is \(|F| - 1\), then \( \sum_{3 \leq i < j \leq |V|} |V_i \cap V_j| = 2(|F| - 1) \). Using \(|f_3| + |f_4| + \cdots + |f_{|V|}| \) to replace \(|F|\), we obtain,

\[
\sum_{3 \leq i < j \leq |V|} |V_i \cap V_j| = 2(|f_3| + |f_4| + \cdots + |f_{|V|}| - 1) = 2(\sum_{i=3}^{|V|} |f_i| - 1). \tag{3.4}
\]

In addition, since \( \sum_{i=3}^{|V|} |V_i| \) is the sum of all subsets of vertices, then

\[
\sum_{i=3}^{|V|} |V_i| = |V_3| + |V_4| + \cdots + |V_{|V|}|, \tag{3.5}
\]

where \(|V_3| = 3|f_3|\), \(|V_4| = 4|f_4|\), \(\cdots\), \(|V_{|V|}| = |V| \cdot |f_{|V|}|\); so the equality (3.5) can be rewritten as follows

\[
\sum_{i=3}^{|V|} |V_i| = 3|f_3| + 4|f_4| + \cdots + |V| \cdot |f_{|V|}| = \sum_{i=3}^{|V|} i|f_i|. \tag{3.6}
\]

Using the equality (3.4) and (3.6) to substitute the correspondent items in the equality (3.3), we derive
\[ \sum_{i=3}^{\mathcal{V}} i|f_i| - 2\left(\sum_{i=3}^{\mathcal{V}} |f_i| - 1\right) = |\mathcal{V}|. \]  

Rearranging the expression gives:

\[ \sum_{i=3}^{\mathcal{V}} (i|f_i| - 2f_i) = |\mathcal{V}| - 2. \]  

Particularly, in the case of a planar graph, replacing \( f_i \) with interior faces \( f'_i \), we obtain

\[ \sum_{i=3}^{\mathcal{V}} (i|f'_i| - 2f'_i) = |\mathcal{V}| - 2. \]  

The equality (3.9) is same to the equality (1.1) that completes the proof. \( \square \)

4. Conclusions and remarks

As the result of this paper, Theorem 1.1 is also a necessary condition, but it shows that we can extend Grinberg Theorem into simple connected graphs. In our proof, because the given graph is Hamiltonian, the cardinal number of the intersection of every pair of joint cycles is 2, denoted by \( |V_i \cap V_j| = 2 \). But when further investigating this cycle set, one could find that there exists another combinatorial structure satisfying \( |V_i \cap V_j| = 2 \) as well. Analyzing this structure will not only clarify the reason that Grinberg Theorem is to be a necessary condition but also, more importantly, will find some new properties of combinations of cycles. We will examine this structure in the next paper and propose some new properties of such graphs.

References

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