Probing Physics of Magnetohydrodynamic Turbulence Using Direct Numerical Simulation

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Abstract

The energy spectrum and the nonlinear cascade rates of MHD turbulence is not clearly understood. We have addressed this problem using direct numerical simulation and analytical calculations. Our numerical simulations indicate that Kolmogorov-like phenomenology with $k^{-5/3}$ energy spectrum, rather than Kraichnan’s $k^{-3/2}$, appears to be applicable in MHD turbulence. Here, we also construct a self-consistent renormalization group procedure in which the mean magnetic field gets renormalized, which in turns yields $k^{-5/3}$ energy spectrum. The numerical simulations also show that the fluid energy is transferred to magnetic energy. This result could shed light on the generation magnetic field as in dynamo mechanism.
1 Introduction

The fluid parcels in a turbulent flow have random motion. However, the random flow velocities obey certain properties. Since most of the talks in the conference dealt with chaos which also yields random signals, it is important to contrast the difference between chaos and turbulence. Chaos can occur in a nonlinear system with a few (3-10) degrees of freedom, but a turbulent system has many (order millions or more) degrees of freedom. Also, a turbulent system may be chaotic; may not be chaotic such as in structures (e.g., vortex street); or it may have both chaos and structure coexisting with each other.

Turbulence is ubiquitous and is of very practical importance. Some of the major applications are in mixing, aeroplane and high speed vehicle design, atmospheric flows and weather prediction, astrophysical objects like stars, jets, interplanetary medium etc. Even with its wide industrial, practical, and theoretical importance, the understanding of turbulence is very weak. There are many empirical laws from experiments and simulations. There are some phenomenologies, the most famous among these is by Kolmogorov. There only a few mathematically rigorous calculations, primarily Kraichnan’s direct interaction approximation, calculations based on renormalization group etc. In this paper we will focus on some of the statistical properties of velocity and magnetic fields in a turbulent magnetohydrodynamic (MHD) plasma. We have attempted to review in a concise manner some of the phenomenological, numerical, analytical work, and observations from the solar wind, with an emphasis on our work (with D. A. Roberts, M. L. Goldstein, J. K. Bhattacharjee, V. Eswaran).

Outline of the paper is as follows. Since the existing MHD turbulence phenomenologies are motivated by Kolmogorov’s fluid turbulence phenomenology, we review Kolmogorov's arguments in section 2. In section 3 we review the existing MHD turbulence phenomenologies. Section 4 contains the numerical results, which are compared with the predictions of the turbulence phenomenologies. In section 5 we briefly report the cascade rates of fluid and magnetic energies. Section 6 contains a renormalization group scheme which provides a self-consistent procedure to obtain the effective mean magnetic field. Section 7 contains conclusions.

2 Fluid Turbulence

The fluid flows are described by Navier-Stokes equation, which is

$$\left(\frac{\partial}{\partial t} + \mathbf{u}(x,t) \cdot \nabla\right)\mathbf{u}(x,t) = -\frac{1}{\rho} \nabla p(x,t) + \nu \nabla^2 \mathbf{u}(x,t)$$

where \(\mathbf{u}(x,t)\) is the velocity field, \(p(x,t)\) is the pressure field, \(\rho\) is the density, and \(\nu\) is the viscosity of the fluid. We assume that the fluid is incompressible, which implies that \(\nabla \cdot \mathbf{u}(x,t) = 0\). The above equation can be made non-dimensional by scaling the variables by large length, velocity, and time scales \((L, U, T)\). The ratio of the nonlinear terms (II and III term) and the viscous term (IV term) turns out to be \(UL/\nu\), which is called the Reynolds number. When the Reynolds number \(Re\) is large (greater than 10,000 or so), we say that the flow is turbulent. Please note that unlike chaos, there is no critical Reynolds number above which the flow becomes turbulent; experiments exhibit transitions at different \(Re\)’s.

Analytic solution to the above equation is not known in turbulent regime. The existence and uniqueness of the solution itself is not clear at this moment. From the experiments it is known
that the velocity field $u(x,t)$ is random. There is a range of scales present in the flow between the energy feeding scale ($L$) and the dissipation scale ($\eta$). This intermediate range is called the inertial range. Kolmogorov conjectured that the physics in this range does not depend on the energy feeding scale or the dissipation scale; the energy distribution is homogeneous and isotropic; the interaction between the velocity modes is local in the Fourier space; and the energy cascade rate $\Pi(k)$ is independent of $k$ under steady state. Using these assumptions, the energy spectrum can be deduced immediately using dimensional analysis. We find that one dimensional energy spectrum $E(k)$ ($\int E(k)dk = \text{total energy}$) is given by

$$E(k) = K_{Ko} \Pi^{2/3} k^{-5/3}$$

where $K_{Ko}$ is an universal constant called Kolmogorov’s constant, $k$ is the wavenumber. Note that the nonlinear energy cascade $\Pi(k)$ is equal to the dissipation rate and the energy supply rate of the fluid.

Experiments and simulations [1] are in reasonable agreement with the above phenomenology. There is a small deviation of the exponent from $5/3$ which is attributed to intermittency phenomenon. The Kolmogorov phenomenology described above is well supported by the calculations based on Direct interaction approximations [4], renormalization group techniques [2, 3], self-consistent mode coupling etc.

Regarding dependence on dimensionality, a brief remark is in order here. In two dimensions, there are two different powerlaw exponents, $5/3$ for small wavenumber and $3$ for large wavenumber regimes. The origin of the exponent $3$ is because of enstrophy conservation. We will not discuss this issue here.

After this brief introduction to fluid turbulence, now we turn to MHD turbulence.

3 MHD Turbulence

The incompressible MHD equations are

$$\left( \frac{\partial}{\partial t} + \mathbf{B}_0 \cdot \nabla + \mathbf{z}^\pm \cdot \nabla \right) \mathbf{z}^\pm = -\frac{1}{\rho} \nabla p_{\text{tot}} + \nu_+ \nabla^2 \mathbf{z}^+ + \nu_- \nabla^2 \mathbf{z}^-$$

$$\nabla \cdot \mathbf{z}^\pm = 0$$

where $\mathbf{z}^\pm = \mathbf{u} \pm \mathbf{b}$. Here $\mathbf{u}$ is the velocity fluctuation, $\mathbf{b}$ is the magnetic field fluctuation in velocity units (scaled by $(4\pi\rho)^{-1/2}$), $\mathbf{B}_0$ is the mean magnetic field in velocity units, and $\nu_\pm = (\nu \pm \lambda)/2$, where $\lambda$ is the resistivity. The field $p_{\text{tot}}$ is sum of thermal and magnetic pressure.

The Alfvén waves are the basic modes of incompressible MHD equations. In absence of the nonlinear term $(\mathbf{z}^\pm \cdot \nabla)\mathbf{z}^\pm$, $\mathbf{z}^\pm$ are the two independent modes travelling antiparallel and parallel to the mean magnetic field. However, when the nonlinear term is present, new modes are generated. These modes interact with each other, which results in a turbulent behaviour of the fluctuations.

Magnetohydrodynamic (MHD) turbulence is more complex than the fluid turbulence. There are several MHD turbulence phenomenologies. In the following discussion we will state the arguments of Dobrowolny et al. [5] from which we can obtain the existing MHD turbulence phenomenologies. The nonlinear interactions modify the amplitudes of the eddies. We assume that the interaction between the fluctuations are local in wavenumber space, and that in one interaction, the eddies $z^\pm_k$ interact with the other eddies of similar sizes for time interval $\tau^\pm_k$. Then from Eq. (3), the variation $\delta z^\pm_k$ in the amplitudes of these eddies during this interval is given by

$$\delta z^\pm_k \approx \tau^\pm_k z^\pm_k z^\mp_k.$$
In such interactions, because of their stochastic nature, the amplitude variation will be $\Delta z_k^\pm \approx \sqrt{N}\delta z_k^\pm$. Therefore, the number of interactions $N^\pm$ it takes to obtain a variation equal to its initial amplitude $z_k^\pm$ is

$$N^\pm \approx \frac{1}{k^2 \left( z_k^\pm \right)^2 \left( \tau_k^\pm \right)}$$

and the corresponding time $T^\pm = N\tau_k^\pm$ is

$$T^\pm \approx \frac{1}{k^2 \left( z_k^\pm \right)^2 \tau_k^\pm}.$$  

The time scale of the energy transfer at wavenumber $k$ is assumed to be $T^\pm$. Therefore, the fluxes $\Pi^\pm$ of the fluctuations $z_k^\pm$ can be estimated as

$$\Pi^\pm \approx \left( \frac{z_k^\pm}{T^\pm} \right)^2 \approx \tau_k^\pm \left( \frac{z_k^\pm}{z_k^\mp} \right)^2 k^2.$$  

By choosing different interaction time-scales, one can obtain different energy spectra. Kraichnan (1965) and Dobrowolny et al. (1980) argued that the interacting $z_k^+$ and $z_k^-$ modes will get separated because of the mean magnetic field in one Alfvén time-scale. Therefore, they chose Alfvén time scale $\tau_A = (kB_0)^{-1}$ as the relevant time-scale and found that

$$\Pi^+ \approx \Pi^- \approx \frac{1}{B_0} E^+(k)E^-(k)k^3 = \Pi.$$  

If we assume that $E^+(k) \approx E^-(k)$, then we immediately obtain

$$E^+(k) \approx E^-(k) \approx (B_0\Pi)^{1/2} k^{-3/2}.$$  

In absence of mean magnetic field, the magnetic field of the largest eddy was taken as $B_0$. Kraichnan (1965) also argued that the fluid and magnetic energies are equipartitioned. We refer the above phenomenology either as one due to Dobrowolny et al. or the generalized Kraichnan’s phenomenology.

If the nonlinear time-scale $\tau_{NL}^\pm \approx (kz_k^\pm)$ is chosen as the interaction time-scales for the eddies $z_k^\pm$, we obtain

$$\Pi^\pm \approx \left( \frac{z_k^\pm}{z_k^\mp} \right)^2 k,$$

which in turn leads to

$$E^\pm(k) = K^\pm (\Pi^\pm)^{4/3}(\Pi^\mp)^{-2/3}k^{-5/3},$$

where $K^\pm$ are constants, which we will refer to as Kolmogorov’s constants for MHD turbulence. Because of its similarity with Kolmogorov’s fluid turbulence phenomenology, we refer to this phenomenology as Kolmogorov-like MHD turbulence phenomenology. This phenomenology was first given by Marsch. It is also a limiting case of a more generalized phenomenology constructed by Matthaeus and Zhou, Zhou and Matthaeus, which is

$$\Pi^\pm = A^2 E^+(k)E^-(k)k^3$$

where $A$ is a constant. Here the small wavenumbers ($\sqrt{kE^\pm(k)} \gg B_0$) follow 5/3 spectrum, whereas the large wavenumbers ($\sqrt{kE^\pm(k)} \ll B_0$) follow 3/2 spectrum.
After the discussion on these phenomenologies one would naturally ask which of the phenomenologies is applicable in the solar wind. From the arguments by Matthaeus and Zhou, and Zhou and Matthaeus, we see that Kraichnan or Dobrowolny et al.’s phenomenology is expected to hold when $B_0 \gg \sqrt{kE^\pm(k)}$; on the other hand Kolmogorov-like phenomenology is expected to be applicable when $B_0 \ll \sqrt{kE^\pm(k)}$. We would like to test these scaling arguments from the solar wind observations and simulations. In the solar wind, which is a good test ground for MHD turbulence theories, we find that the exponent of the total energy is $1.69 \pm 0.08$, whereas the exponent of the magnetic energy is $1.73 \pm 0.08$, somewhat closer to $5/3$ than $3/2$. This is more surprising because $B_0 \gg \sqrt{kE^\pm(k)}$ for inertial range wavenumbers in the solar wind. Also, in the solar wind we do not find a break from $5/3$ to $3/2$ spectrum as predicted by Matthaeus and Zhou, Zhou and Matthaeus. Hence, from the comparison with the solar wind observations, it appears that there is some inconsistency in the phenomenological arguments given above. To explore these issues further, we performed numerical simulations and some analytic studies. As we will described below, the numerical simulations also tend to indicate that the Kolmogorov-like phenomenology, rather than Kraichnan’s or Dobrowolny et al.’s phenomenology, is probably applicable in MHD turbulence.

We have applied renormalization groups to analyse the above equations. We find that under certain assumptions, $B_0$ gets renormalized as we go from large length-scales to smaller length-scales. In other words, $B_0$ appearing in the Kraichnan’s or Dobrowolny et al.’s argument must be $k$ dependent. This leads to $k^{-5/3}$ energy spectra, which appears to be consistent with the solar wind observations and the simulation results. We will describe these ideas in more detail in the following sections.

Before we proceed further, we point out that the normalized cross helicity $\sigma_c$, defined as $(E^+ - E^-)/(E^+ + E^-)$, and the Alfvén ratio $r_A$, defined as the ratio of fluid energy and magnetic energy, play an important role in MHD turbulence.

4 Numerical Simulation

To probe numerically the physics at the inertial range, one usually solves the MHD equations using the pseudospectral method with introduction of hyperviscosity and hyperresistivity. For large resolution turbulence simulations, spectral method is preferred over finite difference methods because the derivatives can be calculated accurately using the Fourier transforms. Note that any error in these equations could propagate very fast, which could make the simulation unreliable. Most of our runs were performed in two-dimensions because of the expense associated with large three-dimensional runs. Fortunately, the scaling arguments described in the earlier section does not depend on the dimensionality because the absolute equilibrium theories predict a forward cascade of total energy in both two and three dimensions. The simulation method is as follows.

The equations are time advance in Fourier space. The nonlinear terms are calculated in real space, then its Fourier transform is calculated using FFT. For details of the simulation, refer to Biskamp and Welter and Verma et al. Earlier high resolution simulations were performed by Biskamp and Welter (1989), Pouquet et al. (1988), and Politano et al. (1989). In all these simulations there was no strong evidence for either 3/2 or 5/3 spectral evidence.

We have taken $\nu = \lambda$, (or $\nu_\perp = 0$) in our simulation. The viscous term has been modified by an addition of hyperviscous and hyperresistive term. The modified viscous term is

$$\nu \left( \nabla^2 + \frac{\nabla^4}{k_{eq}} \right) z^\pm.$$ (13)
Our $\nu = 5.0 \times 10^{-6}$, but $k_{eq} = 10.0$ which yields Reynolds number of approximately $2 \times 10^5$ at large scales and 500 at small scales. The maximum resolution of our simulation in 2D is $512^2$. The parameter $dt = 5.0 \times 10^{-4}$. The runs we consider have $(B_0, \sigma_c) = (0, 0), (0, 0.25), (0, 0.9), (1, 0), (1, 0.25), (1, 0.9), (5, 0.25), (5, 0.9), (5, 0.9)$. The $\sigma_c$ values are for $T = 0$. We have taken the initial Alfven ratio to be 1.0. The simulation were initialized with a box spectrum out to $k = 4$ for $B_0 = 0$, and a $k^{-1}$ spectrum out to $k = 15$ for $B_0 = 5.0$.

We find that after approximately $T = 10$ the system has reached a fully developed turbulent state, i.e., the energy spectrum shows approximate power laws in the inertial range. We obtained the spectral indices by fitting a straight line in the inertia l range, and found that it was difficult to distinguish between the indices 3/2 and 5/3. Therefore, we decided to study the energy cascade rates $\Pi^{\pm}$ leaving a sphere of radius $K$ in the wavenumber space which is given by

$$\Pi^{\pm}(K) = -\frac{\partial}{\partial t} \int_{0}^{K} E^{\pm}(k, t) dk - 2\nu \int_{0}^{K} [1 + (k/k_{eq})] k^2 E^{\pm}(k) dk. \quad (14)$$

The cascade rates $\Pi^{\pm}(K)$ were calculated numerically and compared with the predictions of the phenomenologies. The Kolmogorov-like phenomenology predicts that $(E_k^- / E_k^+) / (\Pi^- / \Pi^+) = K^- / K^+$, while the Dobrowolny et al.’s phenomenology predicts that $\Pi^+ = \Pi^-$. In our simulation we find that for initial $\sigma_c = 0$ and 0.25, $(E_k^- / E_k^+) / (\Pi^- / \Pi^+) \approx 1$. For large $\sigma_c$ this equality does not hold. Note that uncertainties is quite large in our simulations. However, we found in all our runs that the larger of $E_k^+$ and $E_k^-$ had larger cascade rate, thus negating Dobrowolny et al.’s predictions. In our simulations the mean magnetic field (or the magnetic field of the largest eddies) were at least $\approx 1$, whereas the amplitudes of the largest $z^{\pm}$ fluctuations in the inertial range ($k > 10$) were 0.2 to 0.5 (initial total energy = 1). Hence $z^{\pm} < B_0$, and generalized Kraichnan’s model should have been applicable, but we find otherwise. Hence we conclude that our numerical result are contrary to the predictions of the phenomenologies. The Kolmogorov-like phenomenology appears to hold for small $\sigma_c$ (with $K^+ = K^-$) in all $B_0 = 0 - 5$ regime. However, for large $\sigma_c$, there appears to be inconsistency. It is possible that the Kolmogorov-like phenomenology may still hold for large $\sigma_c$ with constants $K^+$ and $K^-$ being unequal and dependent on $\sigma_c$. This issue is under current investigation. (The details of the above simulation results are described in Verma et al. [8]).

5 Cascade rates of magnetic and fluid energies

The origin of magnetic field in the earth, sun, and in the universe is an important problem. It is generally believed that the fluid energy gets transferred to the magnetic energy due to nonlinear interactions, a mechanism know as dynamo. There are many models to get the desired magnetic field configuration in the astrophysical objects. Here we have attempted to investigate the cascade rates of magnetic and fluid energies. These studies will shed light on the energy transfer mechanisms involved in dynamo mechanism. Since this work is in progress, here we are reporting only the preliminary results.

The equations for fluid and magnetic energies are as follows:

$$\left( \frac{\partial}{\partial t} - 2\nu k^2 \right) u^2 = -u \cdot \nabla p - u \cdot [u \cdot \nabla u] + u \cdot [b \cdot \nabla b] \quad (15)$$

$$\left( \frac{\partial}{\partial t} - 2\lambda k^2 \right) b^2 = b \cdot [b \cdot \nabla u] - b \cdot [u \cdot \nabla b] \quad (16)$$

The cascade rates by evaluating $-\int_{0}^{K} T(k) dk$ where $T(k)$’s are the nonlinear terms appearing in the right hand side of the above equations. The pressure term does not yield any energy transfer.
The second term in the RHS of Eq. (15) yields the energy transfer from \( u \rightarrow u \) (\( \Pi_u \)), while the third term yields energy transfer \( u \rightarrow b(\Pi_{u \rightarrow b}) \). The terms in the RHS of Eq. (16) primarily yield the energy transfer from \( b \rightarrow u \) (\( \Pi_{b \rightarrow u}^{\text{total}} \)) (the transfer from \( b \rightarrow b \) has been ignored). The total energy transfer rates coming from \( u \) and \( b \) spheres of radius \( K \) are \( \Pi_u^{\text{total}} \) and \( \Pi_b^{\text{total}} \) respectively. We have obtained these cascade rates numerically.

We solve the MHD equations numerically one a 256 \( \times \) 256 grid using the method described in the earlier section. We choose \( \nu = \lambda = 10^{-5} \), \( k_{eq} = 10 \), and initial \( \sigma_c = 0, r_A = 1.0 \). Note that there is no forcing in our simulation. We have carried out our simulation till \( T = 7.5 \). On DEC 2000 4/233 the computer time required to reach \( T = 7.5 \) was approximately 4 hours. At \( T = 7.5 \) we have calculated the cascade rates by calculating the triple correlations. These cascade rates are shown in Figure 1. We find that \( \Pi_{u \rightarrow b} \) is positive, while \( \Pi_b \) negative. At \( k_{\max} \) we have \( \Pi_{u \rightarrow b} = -\Pi_b^{\text{total}} \). Hence, there is a net transfer of energy from the fluid energy to the magnetic energy. The energy evolution studies indicate that the total fluid energy \( (u^2/2) \) is decaying, while the magnetic energy \( (B^2/2) \) is almost constant in time. Hence, the dissipation rate of magnetic energy appears to be balanced by the energy gained from the transfer rate from the fluid energy. It is to be seen when the magnetic energy get enhanced. In any case, these simulations show that there is a net transfer of energy from velocity field to magnetic field. We also notice that the energy cascade rate \( \Pi_u \) is much smaller that both \( \Pi_{u \rightarrow b} \) and \( \Pi_b^{\text{total}} \). Careful analysis of the cascade rates for various parameter range is in progress.

We have attempted to obtain the perturbative solutions of MHD equations. This is described below.

6 Kolmogorov-like powerlaw using renormalization groups

There have been earlier attempts of applying renormalization groups [11] and other analytic technique, e.g. Eddy-Damped-Quasi-Normal-Markovian approximation, to MHD turbulence. In earlier renormalization group calculations, corrections to viscosity and diffusivity are evaluated by the nonlinear terms in presence of forcing. In our calculation presented here, we attempt to calculate corrections to the mean magnetic field due to the nonlinear terms.

The basic idea of our calculation is to obtain the effective \( B_0 \) at higher wavenumbers. In the phenomenology of Kraichnan and Dobrowolny et al. [3] the external field or the magnetic field of the largest eddy is taken to \( B_0 \). Here we construct a self-consistent scheme in which the effective \( B_0 \) is taken to be the magnetic field of the next-largest eddy. For example, for Alfvén waves of wavenumber \( k \), the effective magnetic field \( B_0 \) will be the magnetic field of the eddy of size \( k/10 \) or so. This argument is based on the physical intuition that for the scattering of the Alfvén waves at a wavenumber \( k \), the effects of the magnetic field of the next-largest eddy is much more than that of the external field. (For similarity, please note that in WKB method, local inhomogeneity of the medium determines the amplitude and phase evolution). We will find in the following discussion that the \( k \) dependent \( B_0 \) yields \( k^{-5/3} \) energy spectra. For simplicity we have taken \( E^+(k) = E^-(k) \) and \( r_A = 1 \).

The MHD equation in the Fourier space is [3]

\[
\frac{d}{dt} z_i^\pm (k, t) \mp i (B_0 \cdot k) z_i^\pm (k, t) = -i M_{ijm}(k) \int dp z_j^\mp (p, t) z_m^\pm (k - p, t)
\]  

(17)

where

\[ M_{ijm}(k) = k_j P_{im}(k); \quad P_{im}(k) = \delta_{im} - \frac{k_i k_m}{k^2}, \]

(18)
Here we have ignored the viscous terms. The above equation will, in principle, yield an anisotropic energy spectra (different spectra along and perpendicular to \( \mathbf{B}_0 \)). Here we modify the above equation in the following way to preserve isotropy

\[
\frac{d}{dt} z_i^\pm (k, t) \equiv i \left( B_0(k) \right) z_i^\pm (k, t) = -i M_{ijm}(k) \int dp z_j^\pm (p, t) z_m^\pm (k - p, t)
\]

(19)

This equation can be thought of as an effective equation with an isotropic random mean field.

The calculation of \( B_0(k) \) using the renormalization group is done by a procedure adopted by McComb, McComb and Watt, Zhou et al., and others \[3\]. The wavenumber range \((k_0..k_N)\), is divided logarithmically into \( N \) divisions. The \( n \)th shell is \((k_{n-1}..k_n)\) where \( k_n = s^n k_0 (s > 1) \). Brief summary of the RG operation used is given below (For details of the procedure, refer to \[3\] and \[12\]).

1. Decompose the modes into the modes to be eliminated \((k^<)\) and the modes to be retained \((k^>)\). In the first iteration \((k_0..k_1) = k^< \) and \((k_1..k_N) = k^> \).
2. We rewrite the Eq. (19) for \( k^< \) and \( k^> \). The equation for the retained modes are

\[
\frac{d}{dt} z_i^\pm (k, t) \equiv i \left( B_0(k) \right) z_i^\pm (k, t) = -i M_{ijm}(k) \int dp \left[ z_j^\pm (p, t) z_m^\pm (k - p, t) \right] + \left[ z_j^>(p, t) z_m^<(k - p, t) + z_j^<(p, t) z_m^>(k - p, t) \right]
\]

(20)

We get a similar equation for \( z_i^<(k, t) \) modes
3. The terms given in the second bracket in the RHS of Eq. (20) is calculated perturbatively and the \( z_i^<(k, t) \) modes are averaged out. The effective equation after this operation is

\[
\frac{d}{dt} z_i^\pm (k, t) \equiv i \left( B_0(k) + \delta B_0^\pm (k) \right) z_i^\pm (k, t) = -i M_{ijm}(k) \int dp \left[ z_j^\pm (p, t) z_m^\pm (k - p, t) \right]
\]

(21)

where

\[
\delta B_0^\pm (k) = -\frac{1}{2} \int_{p+q=k} dq b_2(k, p, q) \left( \frac{E(q)}{4\pi q^2} \right) \left( \frac{1}{p B_0(p) + q B_0(q)} \right)
\]

(22)

The term \( b_2(k, p, q) = kp(1 + z^2)(z + xy) \), where \( x, y, z \) are the cosine of angles between \((p, q), (q, k)\), and \((k, p)\) \[3\]. Note that \( \delta B_0^+ (k) = \delta B_0^- (k) \) (for \( E^+ = E^- \) and \( \tau_A = 1 \)). Let us denote

\[
B_1(k) = B_0(k) + \delta B_0(k)
\]

(23)

4. We keep eliminating the shells by the above procedure. After \( n + 1 \) iterations we obtain

\[
B_{n+1}(k) = B_n(k) + \delta B_n(k)
\]

(24)

where

\[
\delta B_n(k) = -\frac{1}{2} \int_{p+q=k} dq b_2(k, p, q) \left( \frac{E(q)}{4\pi q^2} \right) \left( \frac{1}{p B_n(p) + q B_n(q)} \right)
\]

(25)

5. We substitute the following forms for \( E(k) \) and \( B_n(k) \) in the Eqs. (24,25) and solve for \( B_n^*(k') \) numerically \[3,12\].

\[
E(k) = \alpha \Pi^{2/3} k^{-5/3}
\]

\[
B_n(k, k') = \alpha^{1/2} \Pi^{1/3} k_n^{-1/3} B_n^*(k')
\]
We start with initial $B_0(k_i)$ at each shell, and keep iterating till $B_{n+1}^*(k') \approx B_n^*(k')$, that is, till the solution converges. We find that $B_n^*(k')$ is approximately proportional to $k'^{-1/3}$. Hence, the mean magnetic field scales as $k^{-1/3}$, and the energy spectra scales as $k^{-5/3}$ in this self-consistent scheme.

Hence, we see that scaling of $B_0$ leads to $k^{-5/3}$ energy spectra. This is a self-consistent scheme that shows the plausibility of Kolmogorov-like energy spectra for MHD turbulence. Note that simulations and the solar wind observations appear to favour the Kolmogorov-like phenomenology. Therefore, the above analytical analysis is a promising step toward a proper understanding of the statistical theory of MHD turbulence.

7 Conclusions

In this paper we have reviewed some of the phenomenological, observational, numerical, and analytic work in the statistical theory of MHD turbulence, with emphasis on our studies. We find that the solar wind observations and the numerical results are inconsistent with the predictions of the existing MHD turbulence phenomenologies. The energy spectrum and the cascade rates appear to be closer the predictions of Kolmogorov-like phenomenology even when the mean magnetic field or the magnetic field of the largest eddies is large compared to the inertial range velocity and magnetic field fluctuations (a region where Kolmogorov-like theory is not expected to hold).

We have attempted to obtain Kolmogorov-like energy spectrum in MHD turbulence in presence of arbitrary $B_0$ by postulating that the effective $B_0$ is scale dependent, unlike what has been taken by Kraichnan and Dobrowolny et al. We have constructed a renormalization group scheme and shown that the self consistent $B_0(k) \propto k^{-1/3}$ and $E(k) \propto k^{-5/3}$. This analysis has been worked out when $E^+ = E^-$ and $r_A = 1$. The generalization to arbitrary parameters, or at least to the limiting cases, is also planned. We will be able to the get the Kolmogorov’s constants for MHD turbulence analytically using this procedure; these constants will be useful for the large-eddy-simulations (LES) of MHD turbulence.

We have also carried out a preliminary study of the cascade rates of fluid and magnetic energies. We find that there is a net transfer of fluid energy to magnetic energy. We are investigating the parameter regimes in which these cascade rates are large enough (more than dissipation rates) so that the total magnetic energy increases with time, what is found in dynamo theory.

Lastly, we would like to mention that the MHD turbulence theories are very important for modelling various astrophysical phenomena and plasma processes. We have estimated turbulent heating, nonclassical viscosity and resistivity of the solar wind using the MHD turbulence phenomenologies [13]. In this light, search for a satisfactory theory of MHD turbulence appears quite important.

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**FIGURE CAPTIONS**

Figure 1 This figure shows the fluxes for a simulation with initial $\sigma_c = 0.$ and $r_A = 1.0$. The fluxes are found at $t = 7.5$ when $\sigma_c = 0.015$ and $r_A = 0.4$. 
Figure 1