Hydrogen-like Spectrum of Spontaneously Created Brane Universes with deSitter Ground State

Aharon Davidson

Physics Department, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

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Unification of Randall-Sundrum and Regge-Teitelboim brane cosmologies gives birth to a serendipitous Higgs-deSitter interplay. A localized Dvali-Gabadadze-Porrati scalar field, governed by a particular (analytically derived) double-well quartic potential, becomes a mandatory ingredient for supporting a deSitter brane universe. When upgraded to a general Higgs potential, the brane surface tension gets quantized, resembling a Hydrogen atom spectrum, with deSitter universe serving as the ground state. This reflects the local/global structure of the Euclidean manifold: From finite energy density no-boundary initial conditions, via a novel acceleration divide filter, to exact matching conditions at the exclusive nucleation point. Imaginary time periodicity comes as a bonus, with the associated Hawking temperature vanishing at the continuum limit. Upon spontaneous creation, while a finite number of levels describe universes dominated by a residual dark energy combined with damped matter oscillations, an infinite tower of excited levels undergo a Big Crunch.

Introduction

The no-boundary proposal [1] invokes basic quantum mechanics to avoid the classically unavoidable Big Bang singularity. Creation in this language is a smooth Euclidean to Lorentzian transition, with the emerging (finitely scaled) universe resembling alpha decay. The simplest model of this kind is constructed at the level of the mini superspace, requires a positive cosmological constant \( \Lambda > 0 \), and can only be implemented for a closed \( k > 0 \) space. A variant which introduces a supplementary embryonic era can be realized, ad-hoc [2] by including a radiation energy density term, field theoretically by invoking the embedding approach [3], or via the landscape of string theory [4]. Brane extensions have also been discussed [5]. The theoretical highlight of the no-boundary proposal is the wave function of the universe, the solution of the Schrodinger Wheeler-deWitt (WdW) equation [6].

The two Randall-Sundrum (RS) models [7], followed by their Dvali-Gabadadze-Porrati (DGP) and Collins-Holdom (CH) extensions [8] which supplement a 4-dim Einstein-Hilbert part to the underlying 5-dim action, are presumably the prototype brane models. The first rights are reserved, however, to the Regge-Teitelboim (RT) model [9] where the universe is treated as a 4-dim extended test object floating geodesically [10] in a 5-dim non-dynamical background. Moreover, the first field theoretically consistent brane variation, albeit in a flat spacetime, was formulated by Dirac [11]. Exporting the Dirac prescription to the gravitational regime [12] allows us to treat the variety of models as special limits of a single unified brane cosmology. This Letter attempts to take the no-boundary proposal one step further to expose the Hydrogen-like spectrum (with deSitter as the ground state) of spontaneously created unified brane universes.

Unified brane cosmology in a nutshell

Let the 4-dim FLRW cosmological line element

\[
  ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)
\]

be isometrically embedded within a \( Z_2 \)-symmetric (L and R branches, respectively) 5-dim AdS background characterized by a negative cosmological constant \( \Lambda_5 < 0 \). This can be done for any scale factor \( a(t) \) and without imposing any geometrical constraints. The associated extrinsic curvatures are given explicitly by

\[
  K_{L,R}^{\mu\nu} = \begin{bmatrix}
  \frac{1}{\xi} (\frac{a}{a}) - \frac{1}{5} \Lambda_5 & 0 \\
  0 & -\xi a^2 \gamma_{ij}(r, \theta)
  \end{bmatrix}.
\]

It is \( \xi(a) \) which governs the cosmic evolution equation, with the latter cast into the familiar FLRW format

\[
  \frac{\dot{a}^2 + k}{a^2} = \frac{\Lambda_5}{6} + \xi^2(a).
\]

Within the framework of unified brane cosmology [14], in a nutshell, \( \xi(a) \) is the root of the cubic equation

\[
  \rho = \frac{3\xi^2}{8\pi G_4} + \frac{3\xi}{4\pi G_5} + \frac{\Lambda_5}{16\pi G_4} + \frac{\omega}{\sqrt{3\xi} a^4}.
\]

\( G_{4,5} \) denote the 4,5-dim gravitational constants, respectively, and \( \rho(a) \) stands for the localized DGP energy density on the brane. No specific equation of state \( P = P(\rho) \) has been assumed. The \( \omega \)-term (\( \omega \) is a conserved charge), resembles (but not to be confused with) the dark radiation term which is known to accompany RS cosmology, is the fingerprint of the underlying RT model. It owes its existence to the built-in integrability of the brane’s geodesic equations of motion. The special limits include:

- DGP limit (\( \omega = 0 \)): The now quadratic eq. (4) admits two branches \( \xi_\pm(a) \).
- RS limit (\( \omega = 0, G_4 \to \infty \)): \( \xi_+(a) \) becomes proportional to \( \rho(a) \), so that the FLRW equation is unconventionally sourced [13] by \( \rho_{\text{total}} = \frac{\Lambda_5}{2} + \frac{1}{3} (4\pi G_5 \rho)^2 \).
- GR limit (\( \omega = 0, G_5 \to \infty \)): \( \Lambda_5 \) simply decouples.
- RT limit (\( \omega \neq 0, G_5 \to \infty \)): The bulk is kept nondynamical, \( \Lambda_5 \neq 0 \) is optional. Sticking to the original FLRW format, one formally replaces \( \rho \) by \( \rho_{\text{total}} = \rho + \rho_4 \).
compactely squeezing the entire deviation from GR into an effective ‘dark’ component $\rho_d(\rho)$. The latter must of course vanish for $\omega = 0$, obeying
\[ \rho_d^2 \left( 8\pi G_4 (\rho + \rho_d) - \frac{\Lambda_5}{2} \right) = \frac{\omega^2}{a^2} . \]  
\[ (5) \]

- In the general case [12], one may follow the formalism specified by eq. [5], only with modified $\{\rho^*, \rho_d^*\}$ replacing $\{\rho, \rho_d\}$, where $\rho^* = \rho - 3\xi / 4\pi G_5$. 

**Higgs ↔ deSitter interplay**

We start with a deceptively naive question: What are the field theoretical ingredients necessary for supporting a deSitter brane? It is well known that, within the framework of GR, introducing a positive cosmological constant $\Lambda_4 > 0$ will do. However, once a non-trivial $\rho_d(\rho)$ enters the game, the answer is not straight forward any more. Our goal is to end up with a constant $\xi$ charge, and (iii) Bypassing fine tuning. This can be achieved by introducing a DGP brane localized real scalar field $\phi(x)$, subject to a particular uniquely prescribed (analytically derived by means of reverse engineering [15]) scalar potential $V(\phi)$.

The scalar potential and its derivative enter the game via the intrinsic energy density $V(\phi) = \rho - \frac{a^2}{2} \dot{\phi}^2$, and via the Klein-Gordon (KG) equation $V'(\phi) = -\dot{\phi} - 3\xi \dot{\phi}$, respectively. The constant value of $\xi(a)^2 = \frac{1}{3}(\Lambda_4 - \frac{a^2}{2} \Lambda_5)$ then allows to express both of them as explicit functions of $a$. Self consistency of these two expressions is achieved only provided a certain differential equation is satisfied
\[ \frac{1}{4} W''(\phi)^2 = -k \sqrt{\frac{3}{\omega} \Lambda_4 - \frac{1}{2} \Lambda_5} W(\phi)^3 + \frac{A_4}{3} W(\phi) , \]
\[ (7) \]
where $W(\phi) = V(\phi) - \sigma$. Counter intuitively, the exact analytic solution is surprisingly familiar
\[ V(\phi) = \sigma + \lambda^2 \left( \phi^2 - v^2 \right)^2 \]
\[ (8) \]
A restricted Higgs potential has made its appearance
\[ \lambda^2 = \frac{3k^2}{16\omega} \sqrt{\Lambda_4 - \frac{1}{2} \Lambda_5} , \quad \lambda^2 v^2 = \frac{\Lambda_4}{12} , \]
\[ (9) \]
\[ \sigma(\lambda, v) = \frac{\Lambda_4}{8\pi G_4} + \sqrt{\frac{3}{4\pi G_5}} \Lambda_4 - \frac{1}{2} \Lambda_5 \equiv \sigma_0 . \]
\[ (10) \]
Note that the Higgs potential is a necessary but not a sufficient ingredient for supporting a de-Sitter brane. Since $3a^4 \sqrt{\Lambda_4 - \frac{1}{2} \Lambda_5} \dot{\phi}^2 = 4\omega$, the initial value $\dot{\phi}_c$, required by the 2nd order differential KG equation, gets fixed by the initial scale factor value $a_c$. The classical solution of the field equations is given by
\[ a(t) = \sqrt{\frac{3k}{\Lambda_4}} \cos \sqrt{\frac{\Lambda_4}{3}}(\tau - \tau_c) , \]
\[ (11) \]
\[ \phi(t) = i\chi(\tau) = iv \tan \sqrt{\frac{\Lambda_4}{3}}(\tau - \tau_c) , \] 
\[ (13) \]
normalized such that $a(0) = 0$, constituting a circle in the Euclidean phase plane $\alpha(\tau)^2 + \beta(\tau)^2 = \frac{3k}{\Lambda_4}$ (see Fig.3), where $b(t) \rightarrow i\beta(\tau)$. Globally, the de-Sitter imaginary time periodicity $\Delta \tau = 2\pi \sqrt{3/\Lambda_4}$ is now clearly manifest.

**Variant Euclidization**

Performing a Wick rotation $t \rightarrow i(\tau - \tau_c)$ implies
\[ a(t) \rightarrow a(\tau) = \sqrt{\frac{3k}{\Lambda_4}} \cos \sqrt{\frac{\Lambda_4}{3}}(\tau - \tau_c) , \]
\[ (12) \]
\[ \phi(t) \rightarrow i\chi(\tau) \equiv iv \tan \sqrt{\frac{\Lambda_4}{3}}(\tau - \tau_c) , \]
\[ (13) \]
Had $a(t) \rightarrow a(\tau)$ been conventionally accompanied by $\phi(t) \rightarrow \chi(\tau)$, the KG $\tau$-evolution in the Euclidean regime would have been governed [16] by $V_E(\chi) = -V(\chi)$. This in turn would give rise to the well known upside-down potential $V_E(\chi) = -\sigma - \lambda^2(\chi^2 - v^2)^2$. However, in the present case $a(t) \rightarrow a(\tau)$ is unconventionally accompanied by $\phi(t) \rightarrow i\chi(\tau)$ and hence $b(t) \rightarrow i\beta(\tau)$, so the rules of the game are changed dramatically. A closer inspection reveals that the KG equation, while keeping its generic form, is actually being governed by $V_E(\chi) = +V(i\chi)$, translated in our case into
\[ V_E(\chi) = \sigma + \lambda^2(\chi^2 + v^2)^2 \]
\[ (14) \]
The dimensionless function involved a such that finite. This physical
exactly as they do for the deSitter brane
definite). The two zeroes must then cancel each other,
acceleration, being of the form
advances, we arrive at a critical crossroad where the cosmic
parameter space stays unconstrained at this stage.

\[ \phi(0) = \chi(\tau_e) = 0 . \]  

(15)

From no-boundary to spontaneous creation
Allowing now for the most general (arbitrary \( \sigma, \lambda, v \))
Higgs potential, the forthcoming physics heavily depends
on the local/global structure of the Euclidean manifold.
We show that only a discrete class of smooth Euclidean
manifolds is in fact creative (= supports Hubble-Hawking
creation). Our analysis stands on three legs:

(i) **No-boundary:** The essence of the no-boundary
proposal is keeping the Euclidean origin perfectly smooth
(singularity free). The corresponding expansion reads

\[
\alpha(\tau) \simeq \sqrt{k} \tau \left[ 1 - \frac{\tau^2}{18} \left( \frac{256\lambda^4 \omega^2}{9k^4} + \frac{\Lambda_5}{2} \right) \right], \\
\beta(\tau) \simeq \sqrt{k} \frac{2}{2\lambda v} \left[ 1 - \frac{\tau^2}{18} \left( 12\lambda^2 \omega^2 + \frac{512\lambda^4 \omega^2}{9k^4} + \Lambda_5 \right) \right].
\]

Consistently, as \( \tau \to 0 \), in spite of the \( \chi(\tau) \simeq 1/\lambda \tau \)
behavior, the total energy density approaches a finite value

\[ \rho_{\text{total}} \simeq \frac{\Lambda_5}{2} + \frac{256}{9k^4} \omega^2 \lambda^4 . \]  

(18)

This regularity is rooted in the quartic term of the Higgs
potential. The point to notice is that the entire \( \{ \sigma, \omega \} \)
parameter space stays unconstrained at this stage.

(ii) **Acceleration divide:** As the imaginary time ad-
advances, we arrive at a critical crossroad where the cosmic
acceleration, being of the form \( a''(\tau) = u(\tau)/d(\tau) \) gives
rise to a novel \( \omega \)-selector which resembles the Wien
filter. A zero of the \( d(\tau) \) is potentially harmful, and unless
protected, is translated into an unacceptable curvature
singularity. A zero of \( u(\tau) \) humbly signals a velocity
minimum. Ironically, this is problematic from a differ-
ent point of view, as the creation point, characterized by
\( a'(\tau) = 0 \), does not really have a chance to be reached
(since once \( a'(\tau) \) starts rising again, \( d(\tau) \) turns positive
definite). The two zeroes must then cancel each other,

\[ s(\tau_f) = 1 , \quad s'(\tau_f) = 0 , \]

such that finite \( a''(\tau_f) \neq 0 \) (subscript \( f \) stands for 'filter').
The dimensionless function involved

\[
s(\tau) = \frac{1}{3} \left[ 8\pi G_4 \rho - \frac{\Lambda_5}{2} + \frac{4G^3}{G_5^3} \right]^{\frac{1}{4}} \\
\left[ \frac{4\pi G \omega}{3\sqrt{3}a^4} + \frac{G_4}{9G_5} \left( 8\pi G_4 \rho - \frac{\Lambda_5}{2} \right) + \frac{8G^3}{27G_5^3} \right] \frac{1}{4}.
\]

(20)

constitutes the Vieta solution of the cubic eq.\( (14) \). In turn,
for any value of the surface tension \( \sigma \), only a particular
\( \omega(\sigma) \), to be referred to as the **acceleration divide line**,
remains physically permissible. The numerical solution
is depicted in Fig.\( (1) \) for the RT special case.

(iii) **Creation:** Equipped with the no-boundary initial
conditions, smoothly passing the acceleration divide
filter with a proper \( \omega(\sigma) \) charge, the coupled differential
field equations are prematurely expected to lead their sol-
solutions directly to the nucleation point. This is, however,
not necessarily the case as no freedom is left to make ev-
ery solution arrive at

\[ a'(\tau_c) = 0 , \quad \phi(\tau_c) = 0 , \]

at some finite \( \tau_c \) (subscript \( c \) stands for 'creation'). In
other words, not every Euclidean configuration can un-
dergo a quantum mechanical transition into a Lorentzian
universe. Only a discrete spectrum of surface tensions
\( \sigma_n(n = 0, 1, 2, \ldots) \) will do. Furthermore, associated with
the surviving configurations are already fixed \( a(\tau_c) \) and
\( \phi(\tau_c) \) values. No room for creation initial conditions.

**Hydrogen-like spectrum**
At this stage, we can offer exact analytic \( \{ \sigma_n, \omega_n \} \) pair
formulas for the extreme levels \( n = 0 \) and \( n \to \infty \), and
semi-analytically extract (numerically verified) the lead-
ing \( n \)-dependence at the near continuum (large \( n \))
approximation. The \( n = 0 \) ground state corresponds to the
deSitter brane. Hence, eqs.\( (10) \) apply. It is by far
the most isolated level. For \( n > 0 \), the scalar field \( \chi(\tau) \)
oscillates \( n \) times around the bottom of the Euclidean
potential before finally reaching there the exact vanish-
ing velocity \( \alpha'(\tau_c) = 0 \) mandatory for creation.

![FIG. 2: Hydrogen-like (large-\( n \)) surface tension spectrum (blue), demonstrated for the RT special case, is depicted along the acceleration divide curve \( \omega(\sigma) \). deSitter ground state and the 1st excited state are located farther away on the rhs.](image-url)
For $\sigma = \sigma_{\infty} + \epsilon^2$ (arbitrarily tiny $\epsilon$), the velocity $\alpha(\tau)$ gradually decreases, while the frustrated $\chi(\tau)$ spends an infinite amount of imaginary time damped oscillating around the bottom of the potential. For $\sigma = \sigma_{\infty} - \epsilon^2$, on the other hand, $\alpha(\tau)$ becomes a monotonically increasing function of $\tau$, with $\chi(\tau)$ still damped oscillating, and consequently, the solution totally loses its chance to eventually hit the creation point. At the continuum limit, the velocity gets stuck $\alpha'(\tau) \to \sqrt{\lambda}$, and $\chi(\tau)$ settles down at the minimum of the potential, so that $\rho(\tau) \to \sigma + \lambda^2 \nu^4$.

Altogether, the condition $\rho' = 0$ (generalizing $\rho = 0$ in the presence of a finite $G_5$) is translated into

$$\sigma_{\infty} = \frac{3}{4\pi G_5} \sqrt{-\frac{\Lambda_5}{6} - \lambda^2 \nu^4} .$$

(22)

The full $\sigma$-spectrum, Hydrogen atom like for large-$n$, numerically verified excellent approximation, is given by

$$\sqrt{\frac{\sigma_0 - \sigma_{\infty}}{\sigma_n - \sigma_{\infty}}} \approx 2(n + 1) - \frac{1}{n + 1} .$$

(23)

This formula exhibits an empirical universal structure, with the variety of parameters involved entering only implicitly via $\sigma_0$ and $\sigma_{\infty}$. The full $\{\sigma_n, \omega_n\}$ list, a collection of points along the acceleration divide, is depicted in Fig 3 (for $\Lambda_5 = 0, G_5 \to \infty$). Note the linear large-$n$ approximation, and notice the fact that $\omega_{\infty} \neq 0$.

**Imaginary time periodicity**

The three-leg foundation of the Euclidean manifold, achieved by means $\sigma$-quantization, comes with a bonus. Namely, associated with every $n$-level there is a loop in the $\{\alpha(\tau), \beta(\tau)\}$ phase plane. These loops, see Fig. 3, form closed Lissajous like trajectories, and differ from each other by means of their total number $2n + 1$ of holes. Most importantly, they dictate $n$-dependent imaginary time periodicities $\Delta_n = 4\tau$. And since $\Delta_n$ and $\sqrt{\sigma_n - \sigma_{\infty}}$ have been numerically observed to share exactly the same functional behavior, the corresponding Hawking temperatures are consequently given by

$$T_n = \frac{1}{\Delta_n} = \frac{\lambda \nu}{\pi} \sqrt{\frac{\sigma_n - \sigma_{\infty}}{\sigma_0 - \sigma_{\infty}}} .$$

(24)

The temperature ranges from $T_0 = \lambda \nu / \pi$ for the de-Sitter ground state down to $T_{\infty} \to 0$ at the continuum limit.

At the large-$n$ approximation, the Euclidean FLRW $\tau$-evolution near the nucleation point is characterized by an asymptotically constant $\rho_{\text{total}}(\tau)$, see Fig. 4. On continuity grounds, the created Lorentzian universe is then governed by the inflationary cosmological constant

$$\Lambda_{\text{inf}} \simeq 12 \lambda^2 \nu^2 \frac{\sigma_n - \sigma_{\infty}}{\sigma_0 - \sigma_{\infty}} > 0 .$$

(25)

As the time keeps ticking, the system is aiming toward the local minimum of the Higgs potential associated with the residual DGP/RS/GR cosmological constant

$$\Lambda_{\text{res}} = \frac{\Lambda_5}{2} + 3 \left[ \frac{G_4}{G_5} - \sqrt{\frac{G_4^2}{G_5^2} + \frac{8\pi G_4}{3} \sigma_n - \frac{\Lambda_5}{6}} \right] .$$

(26)

Be aware, however, that in the present theory the Higgs VEV is not always accessible.

**FIG. 3:** $\tau$-periodicity: Associated with the $n$-th level is a closed contour ($2n+1$ holes) in phase plane. The arrow leads from no-boundary (initial dot) to creation (final dot).

**FIG. 4:** The total energy density evolution is depicted for the lowest $n$-states (here $\Lambda_5 < 0, G_5$ finite). The black dots mark the Creation point. $n = 0$ universe is deSitter. $n = 1$ universe transforms from an inflationary $\Lambda_{inf}$ into a residual $\Lambda_{res}CDM$. $n \geq 2$ universes already suffer a Big Crunch.

The various options are demonstrated in Fig 4

- $n = 0$ stands for the deSitter universe.
- $n = 1$ resembles $\Lambda_{res}CDM$, and is somewhat closer to the universe that we observe. The residual cosmological constant $0 < \Lambda_{res} \ll \Lambda_{inf}$ is accompanied by dark matter (in average). The damped oscillations of $\phi(t)$ near the VEV are simply translated into $(p, q$ constants)

$$\rho_{\text{total}}(t) \simeq \Lambda_{res} + \frac{p^2 + q^2 \cos^2 \sqrt{8\lambda^2 \nu^4 - \frac{4\Lambda_{res}}{\lambda v^2}}} {\left( e^{\sqrt{\frac{\Lambda_{res}}{\nu^4}} t} \right)^3} .$$

(27)

- $n \geq 2$ eventually undergo a Big Crunch.

In the most general case, a finite number of states, characterized by $4\pi G_5 \sigma \geq -3\Lambda_5/2$, is associated with eternally expanding $\Lambda_{res} > 0$ universes. This leaves an infinite tower of excited states associated with eventually collapsing universes (and hence $\Lambda_{res} = 0$, no fine tuning).
Epilogue

We have shown that associated with spontaneous creation of a unified brane universe is a Hydrogen atom like spectrum, with deSitter universe playing the role of the ground state. Our discussion has been carried out at the semi classical level, with the focus on the local/global structure of the 4-dim Euclidean manifold. Smoothness and regularity of the no-boundary Euclidean manifold combined with the exclusiveness of the spontaneous creation mechanism is bonused by imaginary time periodicity. In the near future we would like to (i) Solve the WdW equation for the Hartle-Hawking wave eigenfunctions, (ii) Replace the present Bohr-like approach by a path integral calculation, (iii) Calculate the finite amplitudes, (ii) Replace the present Bohr-like approach by a path integral calculation, (iii) Calculate the finite amplitudes, (ii) Replace the present Bohr-like approach by a path integral calculation, (iii) Calculate the finite amplitudes, (ii) Replace the present Bohr-like approach by a path integral calculation, (iii) Calculate the finite amplitudes, (ii) Replace the present Bohr-like approach by a path integral calculation, (iii) Calculate the finite amplitudes, (ii) Replace the present Bohr-like approach by a path integral calculation, (iii) Calculate the finite amplitudes, (ii) Replace the present Bohr-like approach by a path integral calculation, (iii) Calculate the finite amplitudes.