Novel Algorithms based on Majorization Minimization for Nonnegative Matrix Factorization

R. Jyothi, P. Babu and R. Bahl

Abstract

Matrix decomposition is ubiquitous and has applications in various fields like speech processing, data mining and image processing to name a few. Under matrix decomposition, nonnegative matrix factorization is used to decompose a nonnegative matrix into a product of two nonnegative matrices which gives some meaningful interpretation of the data. Thus, nonnegative matrix factorization has an edge over the other decomposition techniques. In this paper, we propose two novel iterative algorithms based on Majorization Minimization (MM)-in which we formulate a novel upper bound and minimize it to get a closed form solution at every iteration. Since the algorithms are based on MM, it is ensured that the proposed methods will be monotonic. The proposed algorithms differ in the updating approach of the two nonnegative matrices. The first algorithm Iterative Nonnegative Matrix Factorization (INOM) sequentially updates the two nonnegative matrices while the second algorithm Parallel Iterative Nonnegative Matrix Factorization (PARINOM) parallely updates them. We also prove that the proposed algorithms converge to the stationary point of the problem. Simulations were conducted to compare the proposed methods with the existing ones and was found that the proposed algorithms performs better than the existing ones in terms of computational speed and convergence.

Index terms— Nonnegative matrix factorization, Majorization Minimization, Big Data, Parallel, Multiplicative Update

The authors are with CARE, IIT Delhi, New Delhi, 110016, India.(email: jyothi.r@care.iitd.ac.in, prabhubabu@care.iitd.ac.in, rbahl@care.iitd.ac.in)
I. INTRODUCTION

Recent advancements in sensor technology and communications has created huge collection of data resulting in big data matrices, which when appropriately analyzed can give useful insights about the data. Many times, researchers reduce the dimension of the data matrix for easier visualization and to lessen the computational load \[1\]. There are various tools to reduce the dimension of the data, some of them are - Singular Value Decomposition (SVD) \[2\], Principal Component Analysis (PCA) \[3\] and Factor Analysis (FA) \[4\]. However, the major shortcoming of these techniques is that when the input data matrix is nonnegative as in the case of speech/image processing, the reduced data matrix can have negative entries, which makes the interpretation difficult. Nonnegative matrix factorization (NMF) is another dimension reduction technique, which as the name suggests decomposes the matrix such that the reduced dimension matrices are always nonnegative. NMF has found applications in music processing \[5\], data mining \[6\], image processing \[7\] and in neurobiology \[8\] to name a few. Mathematically, NMF problem can be written as:

\[
\text{NMF:} \quad \min_{W,H \geq 0} \{ f_{NMF}(W, H) \triangleq \| V - WH \|_F^2 \} \quad (1)
\]

where, \(\| X \|_F\) denotes the Frobenious norm of matrix \(X\), \(V\) is a data matrix made of real entries and of size \(n \times m\), \(W\) and \(H\) are decomposed matrices of size \(n \times r\) and \(r \times m\), respectively. Here, \(r\) is chosen to be lesser than \(m\) and \(n\). The constraint \((W, H \geq 0)\) is such that \(W\) and \(H\) matrices must not have any nonnegative element. The \(i^{th}\) column of \(V\) matrix, represented as \(v_i\), can be expressed as nonnegative weighted linear combination of columns of \(W\) matrix:

\[
v_i = \sum_{j=1}^{r} h_{ji} w_j \quad (2)
\]

where \(w_j\) is the \(j^{th}\) column of \(W\) matrix and \(h_{ji}\) is the \((j, i)^{th}\) element of \(H\) matrix. Geometrically, this means that the columns of \(W\) matrix generates a simplicial convex cone \(([9], [10])\) that contains the data matrix \(v_i\), where simplicial convex cone \(C\) is defined as:

\[
C = \left\{ \sum_{i=1}^{r} \theta_i u_i : \theta_i \geq 0 \right\} \quad (3)
\]
where $\theta_i \in \mathbb{R}^+$ and $u_i$ is a vector in $\mathbb{R}$ of arbitrary dimension. From (2), it can be seen that there exists many cones (whose vertices are defined by columns of $W$), containing the data $v_i$. In order to have a unique cone, one must normalize the columns of $W$ and $V$ matrix [10] and hence in this paper we do the same. The NMF problem in (1) is not jointly convex in $W$ and $H$. However, it is separately convex in either $W$ or $H$. Hence, alternatingly minimizing over $H$ and $W$ seems to be a favorable direction to solve the NMF problem in (1). This approach is traditionally named as **Alternating minimization**. In alternating minimization, constrained minimization is performed with respect to one matrix while keeping the other matrix fixed and vice-versa. The pseudo code of Alternating minimization is shown below:

| **Table 1: Pseudocode of Alternating Minimization ([11], [12] and [13])** |
|-----------------------------------------------|
| **Input:** Data sample $V$, with each column normalized, $r$, $m$ and $n$. |
| **Initialize:** Set $k = 0$. Initialize $W^0$ and $H^0$. Each column of $W^0$ normalized. |
| **Repeat:** |
| 1) Fix $W^k$ and find $H^{k+1} = \arg \min_{H \geq 0} f_{NMF}(W^k, H)$ |
| 2) Fix $H^{k+1}$ and find $W^{k+1} = \arg \min_{W \geq 0} f_{NMF}(W, H^{k+1})$ |
| 3) Normalize the columns of $W^{k+1}$ |
| $k \leftarrow k + 1$ |
| **until convergence** |

Many algorithms have been proposed to solve the NMF problem. Some of them are discussed as follows: The baseline algorithm used to solve the above problem is the **Multiplicative Update (MU)** algorithm proposed by Lee et. al. [14]. This algorithm which is iterative in nature, is based on **Block Majorization Minimization (MM)** (which we will introduce shortly in section [11]). The final update equations of both the matrices has only multiplication operation between the matrices and hence simple to implement, however, it was reported to have slow convergence [15]. Gradient descent algorithms [16], [17] are also employed to solve the NMF problem in (1); wherein the $W$ and $H$ matrices are updated by taking a step in a direction opposite to the direction of gradient of the function $f_{NMF}(W, H)$. The update equation of a gradient descent
algorithm for NMF problem is as follows:

$$W^{k+1} = W^k - \alpha \frac{\partial f_{\text{NMF}}(W^k, H^k)}{\partial W^k} = W^k - \alpha \left( (W^k H^k H^k T) - VH^k T \right)$$

$$H^{k+1} = H^k - \beta \frac{\partial f_{\text{NMF}}(W^{k+1}, H^k)}{\partial H^k} = H^k - \beta \left( (W^{k+1} H^k) - (W^{k+1} H^k)^T V \right)$$

where $\alpha$ and $\beta$ are the step sizes. MU algorithm can also be viewed as the gradient descent algorithm [18] with iteration dependent or adaptive step size $\alpha^k$ and $\beta^k$ defined for updating matrices $W$ and $H$ respectively as:

$$\alpha^k = W^k \odot (W^k H^k H^k T)$$

$$\beta^k = H^k \odot ((W^{k+1})^T W^{k+1} H^k)$$

where $\odot$ denotes element wise division. Hence, the update equation becomes:

$$W^{k+1} = W^k \odot \left( V H^k T \odot W^k H^k H^k T \right)$$

$$H^{k+1} = H^k \odot \left( W^k T V \odot (W^{k+1})^T (W^{k+1} H^k) \right)$$

where $\odot$ denotes element wise multiplication. Note that the above update equation for $H$ and $W$ have only multiplication operation between the matrices. This kind of algorithm falls under multiplicative update algorithms. If in contrast, if the step sizes are chosen such that the update equation have only addition operations between the matrices, then it is called an additive update algorithm [19]. Another group of algorithms use the fact that steps 1 and 2 of Alternating minimization (refer to pseudocode of alternating minimization in Table 1) fall under Nonnegative Least Squares problem (NLS) and solves each subproblem using active set methods ([20], [21], [22]), nevertheless, this approach can be computationally expensive [23]. Instead of alternatingly minimizing over the two blocks - $W$ and $H$ matrices, one can form $2r$ blocks by further partitioning the columns of $W$ matrix and rows of $H$ matrix as $W = [w_1, w_2 \cdots w_r]$, $H = [h_1, h_2 \cdots h_r]$ and solve the problem in (1) by updating the $w_j^{th}$ and $h_j^{th}$ block, while keeping the other blocks constant. Mathematically, this can be written as:

$$\text{minimize } \| V^{(j)} - w_j h_j^T \|_F^2, \text{ for } \{ j = 1, 2 \cdots r \}$$

May 14, 2019 DRAFT
where $V^{(j)} = V - W^kH^k - w_j^k h_j^{kT}$. Hierarchical Alternating Least Square (HALS) algorithm \cite{24}, \cite{25} follows this strategy and finds a closed form solution for the problem in (8) by alternatingly minimizing over $w_j$ and $h_j$. The advantage of this approach is that the computation of the solution for problem in (8) is computationally inexpensive compared to NLS. Fast-HALS is an efficient implementation of HALS algorithm wherein they do not explicitly compute $V^{(j)}$ and hence is computationally faster than HALS algorithm. Recently, Alternating Direction Method of Multipliers (ADMM) was also used to solve the NMF problem \cite{26}, \cite{27}. Vandaele et.al. \cite{28} proposed greedy randomized adaptive search procedure and simulated annealing to solve the NMF problem.

In this paper, we present two novel algorithms to solve the NMF problem-a sequential and a parallel algorithm. The major contributions of the paper are as follows:

1) A Block MM based sequential algorithm-Iterative Nonnegative Matrix Factorization (INOM) is proposed to solve the NMF problem. The update equation of this algorithm looks like that of the update equation of gradient descent algorithm with iteration dependent or adaptive step size. Typically, many algorithms use line search to estimate the step size to be taken at every iteration but our algorithm is developed in such a way that the MM procedure gives us the step size to be taken at every iteration and hence we don’t have to search for the optimal step size.

2) A parallel algorithm based on MM is presented, which parallely updates the $W$ and $H$ matrices. We address this algorithm as Parallel Iterative Nonnegative Matrix Factorization (PARINOM).

3) We discuss the convergence of the proposed algorithms and prove that they always converge to the stationary point.

4) Numerical simulations were conducted to compare the proposed algorithms with the existing algorithms and analyze the performance of the algorithms on the application of Blind Source Separation.

The paper is organized as follows. An overview of MM and block MM can be found in section II. In section III we first propose INOM to solve the NMF problem in (1). Next, we propose
PARINOM and in the same section, we also show that the proposed algorithms converge to stationary point and discuss the computational complexity of the algorithms. At the end of the section, we discuss an acceleration scheme to further accelerate the convergence of PARINOM algorithm. In section [IV] we compare the algorithms with the existing algorithms via computer simulations and evaluate the performance of our algorithm on the application of Blind Source Separation and conclude the paper in section [V].

Throughout the paper, bold capital and bold small letters are used to denote the matrix and vector, respectively. A scalar is denoted by a small letter. The $i^{th}$ entry of vector $v$ is denoted by $v_i$ and $(i, j)^{th}$ entry of the matrix $V$ is denoted by $v_{ij}$. $\odot$ and $\oslash$ denotes element wise multiplication and division operation respectively.

II. MAJORIZATION MINIMIZATION (MM): VANNILA MM AND BLOCK MM

A. Vanilla MM

Majorization Minimization (MM) is an iterative procedure which is mostly used to solve a non-convex, non-smooth or even a convex problem more efficiently. In MM, instead of minimizing the difficult optimization problem $f(x)$ over the constraint set $\chi$ directly, a “surrogate” function which majorizes the problem (at a given value of $x = x^k \in \chi$) is minimized at every iteration. The surrogate function $g(x)$ is the global upper bound of the objective function $f(x)$ i.e., it satisfies the following properties:

$$g(x^k|x^k) = f(x^k)$$  \hspace{1cm} (9)

$$g(x|x^k) \geq f(x) \quad \text{for any } x \in \chi$$ (10)

where, $x^k$ is the value taken by $x$ at the $k^{th}$ iteration. Hence, the MM algorithm generates a sequence of points $\{x^k\}$ according to the following rule:

$$x^{k+1} \in \arg\min_{x \in \chi} g(x|x^k)$$  \hspace{1cm} (11)
The MM procedure is depicted in Fig. 1, wherein \( g(x|x^k) \) is the surrogate function which majorizes \( f(x) \) around \( x^k \) at the \( k^{th} \) iteration. Pictorially, it can be seen that \( f(x^{k+2}) < f(x^{k+1}) < f(x^k) \). By using (9), (10) and (11) it can be shown mathematically that the objective function is monotonically decreased at every iteration:

\[
f(x^{k+1}) \leq g(x^{k+1}|x^k) \leq g(x^k|x^k) = f(x^k)
\]  

(12)

The first inequality and the last equality are by using (9) and (10), respectively. The second inequality is by (11). The convergence rate and computational complexity of the algorithm depends on how well one formulates the surrogate function. The convergence rate depends on how well the surrogate function follows the shape of the objective function \( f(x) \) and to have lower computational complexity, the surrogate function must be easy to minimize. Hence, the novelty of the algorithm based on MM lies in the design of the surrogate function. Moreover, in the case of multivariate optimization problem, if the surrogate function is separable in the optimization variables - then the minimization problem could be solved parallely, which gives the computational advantage. To design surrogate function there are no set steps to follow. However, there are few papers which give guidelines for designing various surrogate functions [30], [29].
B. Block MM

Suppose that the optimization variable $x$ can be split into $m$ blocks as $x = (x_1, x_2, \ldots, x_m)$, then one could apply Block MM to solve the optimization problem $f(x)$. Block MM, an extension of vanilla MM, is a combination of block coordinate descent and vanilla MM wherein the optimization variable is split into several blocks and each block is updated using vanilla MM while keeping the other blocks fixed. Hence, the $i^{th}$ block is updated by minimizing the surrogate function $g_i(x_i|x^k)$ which majorizes $f(x)$ on the $i^{th}$ block and has to satisfy the following properties:

$$g_i(x_i^k|x^k) = f(x^k)$$  \hspace{1cm} (13)

$$g_i(x_i|x^k) \geq f(x_1^k, \ldots, x_i, \ldots, x_m^k)$$  \hspace{1cm} (14)

where $x^k$ is the value taken by $x$ at the $k^{th}$ iteration. The $i^{th}$ block at $k + 1$ iteration is updated by solving the following problem:

$$x_i^{k+1} \in \arg\min_{x_i} g_i(x_i|x^k)$$  \hspace{1cm} (15)

Each block in Block MM is usually updated in a cyclic fashion. The Block MM procedure is as shown in Fig. 2.

![Figure 2: Block MM procedure](image-url)
The surrogate function for block MM must be chosen in such a way that the surrogate function is easy to minimize and must also approximate the objective function well to have faster convergence. Moreover, it is also reported that in some cases the surrogate function in case of block MM can approximate the objective function better than using a single block, leading to a faster convergence rate [29].

III. ALGORITHMS FOR NMF PROBLEM

In this section, we propose two iterative algorithms to solve the NMF problem in (1) based on block MM and vanilla MM. The first algorithm - INOM is based on block MM and sequentially updates both the matrices. The second algorithm - PARINOM is based on vanilla MM and parallely updates the $W$ and $H$ matrices. We then discuss the convergence of the two algorithms and show that they always converge to the stationary point. At the end of the section, we also show a way to accelerate the convergence of PARINOM algorithm, with a little additional computational cost.

A. INOM: Iterative NONnegative Matrix Factorization

This algorithm is based on Block Majorization Minimization principle wherein $W$ and $H$ are considered as blocks and each block is updated by vanilla MM scheme while keeping the other blocks fixed.

Before we derive the update equation for $W$ and $H$ matrices, we first point out an important observation that is subsequently used to form the surrogate functions $g_w (W|H^k, W^k)$ and $g_h (H|W^k, H^k)$, which is used to majorize the function $f_{NMF} (W, H)$ on the $W^{th}$ and $H^{th}$ block, respectively. To discuss the same, we re-write $f_{NMF} (W, H)$ as:

$$f_{NMF} (W, H) = \|V - WH\|^2_F = \sum_{j=1}^{m} \|v_j - Wh_j\|^2$$

(16)
where \( v_j \) and \( h_j \) is the \( j^{th} \) column of \( V \) and \( H \), respectively. From (16), it can be seen that given \( W = W^k \), the \( f_{\text{NMF}} (W^k, H) \) is separable on each column of \( H \). On taking the partial derivative of the above function with respect to \( h_j \), we get

\[
\nabla f_{\text{NMF}}(h_j|W^k) = -2(W^k)^T v_j + 2(W^k)^T W^k h_j
\]

(17)

\[
\nabla^2 f_{\text{NMF}}(h_j|W^k) = 2(W^k)^T W^k
\]

Note that the Hessian matrix \( 2(W^k)^T W^k \) consists only of nonnegative entries and is independent of \( h_j \). We now show that \( f_{\text{NMF}} (W, H^k) \) is separable on each row of \( W \) and the Hessian of \( f_{\text{NMF}} (W, H^k) \) with respect to \( j^{th} \) row of \( W \) is nonnegative and is equal to \( 2H^k(H^k)^T \):

\[
f_{\text{NMF}} (W, H^k) = \|V - WH^k\|_F^2 = \sum_{j=1}^{n} \|v'_j - w_j H^k\|^2_F
\]

(18)

where \( v'_j \) and \( w_j \) represent the \( j^{th} \) row of \( V \) and \( W \), respectively.

\[
\nabla f_{\text{NMF}}(w_j|H^k) = -2v'_j H^k^T + 2w_j H^k H^k^T
\]

(19)

\[
\nabla^2 f_{\text{NMF}}(w_j|H^k) = 2H^k H^k^T
\]

We smartly use the nonnegative Hessian matrix to design the upper bounds \( g_w (W|H^k, W^k) \) and \( g_H (H|W^k, H^k) \), based on the following lemmas.

**Lemma 3.1: Lower Quadratic Bound Principle** [30], [31]: Given \( x = x^k \), a twice differentiable function \( f(x) \), a square matrix \( \Lambda \) such that \( \Lambda \succeq \nabla^2 f(x) \), then \( f(x) \) can be upper bounded as

\[
f(x) \leq f(x^k) + \nabla f(x^k)^T (x - x^k) + \frac{1}{2} (x - x^k)^T \Lambda(x - x^k)
\]

(20)

The upper bound for \( f(x) \) is quadratic and differentiable in \( x \).

**Proof:** Suppose there exists a matrix \( \Lambda \), such that \( \Lambda \succeq \nabla^2 f(x) \), then we have the following equality by second order Taylor expansion:

\[
f(x) \leq f(x^k) + \nabla f(x^k)^T (x - x^k) + \frac{1}{2} (x - x^k)^T \Lambda(x - x^k)
\]
and equality is achieved at $x = x^k$.

**Lemma 3.2:** Let $A$ of size $n \times n$ denote a nonnegative, square and symmetric matrix and let $p \in \mathbb{R}^+$. We define $\Lambda$ as a diagonal matrix with its diagonal elements equal to $p$, the maximum row sum of $A$ matrix. Then, $\Lambda \gtrsim A$.

**Proof:** The above lemma is based on **Perron-Frobenius Theorem** [32], [33]. According to the theorem, there is a positive real number $p$, called the Perron root, such that any other eigenvalue $\lambda$ of $A$ in absolute value is strictly smaller than $p$ i.e. $|\lambda| < p$. To calculate $p$, an important corollary of the theorem is used:

$$p \leq \max_i \sum_{j=1}^{n} a_{ij}$$

(21)

where $a_{ij}$ is the $(i,j)^{th}$ element of $A$ matrix. Hence, we get the following inequality:

$$\max_i \sum_{j=1}^{n} a_{ij} \geq p > \lambda$$

(22)

By constructing a diagonal matrix $\Lambda$ with its diagonal elements as maximum row sum of $A$ and by using Eigen Value Decomposition, the inequality $\Lambda \gtrsim A$ is attained.

Using lemma 3.1 and lemma 3.2, we construct the surrogate function $g_H(H|H^k, W^k)$ for the problem in (16) to update the $H^{th}$ block with $W^k$ fixed.

$$g_H(H|H^k, W^k) = \sum_{j=1}^{m} g_{h_j}(h_j|h^k_j, W^k) = \sum_{j=1}^{m} f(h^k_j) + \nabla f(h^k_j)^T(h_j - h^k_j) + \frac{1}{2}(h_j - h^k_j)^T \Lambda^k(h_j - h^k_j)$$

(23)

where $\Lambda^k$ matrix is a diagonal matrix with its diagonal elements as maximum row sum of $2(W^k)^T W^k$ and $\nabla f(h^k_j)$ is equal to $-2(W^k)^T v_j + 2(W^k)^T W^k h^k_j$. The surrogate function is separable in each column of $H$. Hence at any iteration, given $W = W^k$ and $H = H^k$, the surrogate minimization problem is:

$$\min_{H \geq 0} g_H(H|H^k, W^k)$$

(24)
which has a closed form solution given by:

\[ h_{j}^{k+1} = h_{j}^{k} + \frac{1}{\mu_{k}} \left( 2(W^{k})^{T}v_{j} - 2(W^{k})^{T}W^{k}h_{j}^{k} \right) \]  

(25)

\[ h_{j}^{k+1} = h_{j}^{k} + \frac{1}{\mu_{k}} \left( \left( 2(W^{k})^{T}V \right)_{j} - \left( 2(W^{k})^{T}W^{k}h_{j}^{k} \right) \right) \]

where \( \mu_{k} \) = maximum row sum of \( 2(W^{k})^{T}W^{k} \). Since the update of \( j^{th} \) column of \( H \) does not depend on the update of \((j + 1)^{th}\) column of \( H \), we can re-write the above update equation as:

\[ H^{k+1} = H^{k} + \frac{1}{\mu_{k}} \left( 2(W^{k})^{T}V - 2(W^{k})^{T}W^{k}H^{k} \right) \]  

(26)

We now construct the surrogate function \( g_{w}(W|W^{k}, H^{k+1}) \) using lemma 3.1 and lemma 3.2 for the problem in (18) to update the \( W^{th} \) block with \( H^{k+1} \) fixed.

\[ g_{w}(W|W^{k}, H^{k+1}) = \sum_{j=1}^{n} g_{w_{j}}(w_{j}|w_{j}^{k}, H^{k}) = \sum_{j=1}^{n} f(w_{j}^{k}) + (w_{j} - w_{j}^{k})\nabla f(w_{j}^{k})^{T} + \frac{1}{2}(w_{j} - w_{j}^{k})\Lambda^{k}(w_{j} - w_{j}^{k}) \]  

(27)

where \( \Lambda^{k} \) matrix is constructed with its diagonal elements as maximum row sum of \( 2(H^{k+1})(H^{k+1})^{T} \) and \( \nabla f(w_{j}^{k}) = -2v_{j}'(H^{k+1})^{T} + 2w_{j}(H^{k+1})(H^{k+1})^{T} \). The surrogate function is separable in each row of \( W \). Hence at any iteration, given \( W = W^{k} \) and \( H = H^{k+1} \), the surrogate minimization problem is:

\[ \text{minimize} \ g_{w}(W|W^{k}, H^{k+1}) \]  

(28)

which has a closed form solution given by:

\[ W^{k+1} = W^{k} + \frac{1}{\nu_{k}} \left( 2VH^{k+1T} - 2W^{k}H^{k+1}H^{k+1T} \right) \]  

(29)

where \( \nu_{k} \) = maximum row sum of \( 2H^{k+1}H^{k+1T} \). Note that once \( H^{k+1} \) and \( W^{k+1} \) are computed using (26) and (29), the negative elements must be projected back to \( R^{+} \), to satisfy the nonnegative constraint. Also, observe that the update equations looks similar to gradient descent algorithm with “adaptive” step sizes equal to \( \mu_{k} \) and \( \nu_{k} \). Typically, line search algorithm is implemented to iteratively search for the optimal step sizes until the objective function decreases [34]. Nevertheless, we don’t have to iteratively search for the optimal step sizes; since the proposed
algorithm is based on block MM—it is guaranteed that the step sizes \( \nu^k \) and \( \mu^k \) will ensure the monotonic decrease of the respective cost functions. We now calculate the computational complexity of INOM algorithm. The complexity of computing \( \mu^k \) and \( \nu^k \) is \( O(r^2n) \) and \( O(r^2m) \), respectively. The complexity of computing \( H^{k+1} \) is \( O(rnm) + O(r^2(n + m)) \). The complexity of computing \( W^{k+1} \) is \( O(rnm) + O(r^2(n + m)) \). Since INOM is a sequential algorithm, the total complexity of the algorithm is \( O(2rnm + 2r^2(n + m)) \). Pseudo code of INOM is shown in Table 2.

| Table 2: INOM |
|--------------|
| **Input:** Data samples \( V \) with each column normalized; \( r, m \) and \( n \). |
| **Initialize:** Set \( k = 0 \). Initialize \( W^0 \) and \( H^0 \). Each column of \( W^0 \) is normalized. |
| **Repeat:** |
| Update \( H \) |
| 1) \( \mu^k \) = maximum row sum of \( 2W^kT W^k \) |
| 2) \( H^{k+1} = H^k + \frac{1}{\mu^k} \left( 2W^{kT}V - 2W^kT W^k H^k \right) \) |
| 3) \( H^{k+1} = \max(0, H^{k+1}) \) |
| Update \( W \) |
| 4) \( \nu^k \) = maximum row sum of \( 2H^{k+1}H^{k+1T} \) |
| 5) \( W^{k+1} = W^k + \frac{1}{\nu^k} \left( 2VH^{k+1T} - 2W^kH^{k+1}H^{k+1T} \right) \) |
| 6) \( W^{k+1} = \max(0, W^{k+1}) \) |
| 7) normalize each column of \( W \) |
| \( k \leftarrow k + 1 \), until \( \left| \frac{f_{\text{NMF}}(W^{k+1}, H^{k+1}) - f_{\text{NMF}}(W^k, H^k)}{f_{\text{NMF}}(W^k, H^k)} \right| \leq 10^{-6} \) |

B. PARINOM: Parallel Iterative Nonnegative Matrix Factorization

PARINOM solves the problem in (1) using Vanilla MM without alternatingly minimizing over \( W \) and \( H \) and hence at iteration \( i + 1 \), \( H^{i+1} \) matrix does not depend on \( W^{i+1} \) matrix. Therefore, \( H^{i+1} \) and \( W^{i+1} \) matrix can be parallely updated. On expanding \( f_{\text{NMF}}(W, H) \) we get:

\[
\|V - WH\|_F^2 = \text{Tr}(V^TV) + \text{Tr}(H^TW^TWH) - 2\text{Tr}(V^TWH)
\]
Ignoring the constant terms in (30), we will now show that the second and third term are sigmoidal functions, which is defined as:

**Sigmoidal function** [35], [36]: Let $c$ be a positive or a negative number and $x_1, x_2 \cdots x_n$ be the nonnegative components of a $n$-dimensional vector $x$. Let $\alpha_j$ be the $j^{th}$ component of $\alpha$, which is the fractional power (can be positive, negative or zero) of each component of $x$. Then, $c \prod_{j=1}^{n} x_j^{\alpha_j}$ is called a sigmoidal function.

The second term can be re-written as:

$$
\sum_{k=1}^{m} \sum_{j=1}^{n} \left( \sum_{l=1}^{r} \sum_{m=1}^{r} w_{jl} h_{lk} w_{jm} h_{mk} \right)
$$

where $w_{ab}$ and $h_{ab}$ represent the $(a, b)^{th}$ element of $W$ and $H$ matrix. The terms in (31) are sigmoidal functions with positive coefficient. Now, we show that the third term in (30) can also be written as sigmoidal function.

$$
-2 \text{Tr}(V^T WH) = -2 \sum_{j=1}^{n} \sum_{k=1}^{m} \left( v_{jk} \sum_{l=1}^{r} w_{jl} h_{lk} \right)
$$

This is a sigmoidal function with negative coefficient. Hence, the objective function $f_{\text{NMF}}(W, H)$ becomes:

$$
\|V - WH\|_F^2 = \sum_{k=1}^{m} \sum_{j=1}^{n} \left( \sum_{l=1}^{r} \sum_{m=1}^{r} w_{jl} h_{lk} w_{jm} h_{mk} \right) - 2 \sum_{j=1}^{n} \sum_{k=1}^{m} \left( v_{jk} \sum_{l=1}^{r} w_{jl} h_{lk} \right)
$$

(33)

Now, we propose the following lemma which is used to majorize the above objective function.

**Lemma 3.3:** When $c$ in $c \prod_{j=1}^{n} x_j^{\alpha_j}$ is positive, the following inequality holds:

$$
c \prod_{j=1}^{n} x_j^{\alpha_j} \leq c \prod_{j=1}^{n} (x_j^i)^{\alpha_j} \sum_{j=1}^{n} \frac{\alpha_j}{\|\alpha\|_1} \left( \frac{x_j}{x_j^i} \right) \|\alpha\|_1
$$

(34)

The above inequality is equal when $x$ is equal to $x^i$. When $c$ is negative, the inequality in (34) changes to:

$$
c \prod_{j=1}^{n} x_j^{\alpha_j} \leq c \left( \prod_{j=1}^{n} (x_j^i)^{\alpha_j} \right) \alpha_j \ln(x_j)
$$

(35)

**Proof:** See [Section 3, [36]].
Using the above lemma we get the following surrogate function $g(w_{jl}, h_{lk}|w_{jl}, h_{lk})$ for $f_{NMF}(W, H)$.

$$
g(w_{jl}, h_{lk}|w_{jl}, h_{lk}) = \sum_{k=1}^{m} \sum_{j=1}^{n} \left( \sum_{l=1}^{r} \sum_{m=1}^{r} w_{jl}^i h_{lk}^i w_{jm}^i h_{mk}^i \left( \frac{1}{4} \left( \frac{w_{jl}}{w_{jl}^i} \right)^4 + \frac{1}{4} \left( \frac{h_{lk}}{h_{lk}^i} \right)^4 + \frac{1}{4} \left( \frac{w_{jm}}{w_{jm}^i} \right)^4 + \frac{1}{4} \left( \frac{h_{mk}}{h_{mk}^i} \right)^4 \right) \right) - 2 \sum_{j=1}^{n} \sum_{k=1}^{m} \left( v_{jk} \sum_{l=1}^{r} \left( w_{jl}^i h_{lk}^i \right) \left( \ln w_{jl} + \ln h_{lk} \right) \right)
$$

where $w_{ab}^i$ and $h_{ab}^i$ represent the $(a, b)^{th}$ element of $W$ and $H$ matrix at the $i^{th}$ iteration. Note that the surrogate function $g(w_{jl}, h_{lk}|w_{jl}, h_{lk})$ is separable in the optimization variables: $w_{jl}$ and $h_{lk}$. Hence at any iteration, given $w_{jl} = w_{jl}^i$ and $h_{lk} = h_{lk}^i$, the surrogate minimization problem is:

$$
\text{minimize } g(w_{jl}, h_{lk}|w_{jl}, h_{lk})
$$

where $g(w_{jl}, h_{lk}|w_{jl}, h_{lk})$ is given by (36) which has a closed form solution is given by:

$$
w_{jl}^{i+1} = 4 \sqrt{p_{(jl)}^i (w_{jl}^i)^4} / z_{1(jl)}^i
$$

$$
h_{lk}^{i+1} = 4 \sqrt{q_{(lk)}^i (h_{lk}^i)^4} / z_{2(lk)}^i
$$

where

$$
z_{1(jl)}^i = \sum_{k=1}^{m} \sum_{M=1}^{r} h_{lk}^i w_{jm}^i h_{Mk}^i \quad p_{(jl)}^i = \sum_{k=1}^{m} v_{jk} h_{lk}^i
$$

$$
z_{2(lk)}^i = \sum_{j=1}^{n} \sum_{M=1}^{r} w_{jl}^i w_{jm}^i h_{Mk}^i \quad q_{(lk)}^i = \sum_{j=1}^{n} v_{jk} w_{jl}^i
$$

Taking the entire matrix into consideration, (38) can be re-written as:

$$
W_{i+1} = 4 \sqrt{\left( (V H^T) \circ W^i \right)^4 \otimes \left( W^i H^T H^T \right)} \quad H_{i+1} = 4 \sqrt{\left( (W^T V) \circ H^i \right)^4 \otimes \left( W^T W^i H \right)}
$$

In (40), the fourth power of the matrix $W$ and $H$ are done element wise. When compared to
MU and INOM algorithm, in PARINOM, update of $H^{i+1}$ does not depend on $W^{i+1}$ at $i + 1$ iteration. Hence, $W^{i+1}$ and $H^{i+1}$ matrices can be updated parallely. The complexity in computing $W^{i+1}$ is $O(nmr + r(m+n))$. The complexity to update $H^{i+1}$ is also $O(nmr + r(m+n))$. Since both the matrices can be updated parallely, the complexity of PARINOM is $O(nmr + r(m+n))$. The Pseudo code of PARINOM is shown in Table 3:

**Table 3: PARINOM**

**Input**: Data samples $V$, with each column normalized, $r$, $m$ and $n$.

**Initialize**: Set $i = 0$. Initialize $W^0$ and $H^0$. Each column of $W^0$ is normalized.

**Repeat**:

Update $H$ and $W$ parallely

1) $W^{i+1} = \sqrt{\left( (VH^T) \circ W^i \right) \circ (W^i H^i H^T)}$

2) $H^{i+1} = \sqrt{\left( (W^{i^T} V) \circ H^i \right) \circ (W^{i^T} W^i H)}$

normalize each column of $W^{i+1}$

$i \leftarrow i + 1$, until $\left| \frac{f_{\text{NMF}}(W^{i+1}, H^{i+1}) - f_{\text{NMF}}(W^i, H^i)}{f_{\text{NMF}}(W^i, H^i)} \right| \leq 10^{-6}$

C. Convergence Analysis

We first discuss the convergence of INOM, which is based on Block MM. To discuss the same, we first prove that the surrogate functions $g_H(H|H^k, W^k)$ in (23) and $g_W(W|W^k, H^{k+1})$ in (27) are quasi-convex and the problems in (24) and (28) have a unique minimum.

**Proof**: From (23) and (27), it can be seen that $g_H(H|H^k, W^k)$ and $g_W(W|W^k, H^{k+1})$ are separable in the columns and rows of $H$ and $W$, respectively i.e. $g_H(H|H^k, W^k) = \sum_{j=1}^{m} g_{h_j}(h_j|h_j^k, W^k)$ and $g_W(W|W^k, H^{k+1}) = \sum_{j=1}^{n} g_{w_j}(w_j|w_j^k, H^k)$. The Hessian of $g_{h_j}(h_j|h_j^k, W^k)$ and $g_{w_j}(w_j|w_j^k, H^k)$ for every $j$ is a matrix - which is a diagonal matrix made of non-negative elements and hence is positive semi-definite. This implies that $g_{h_j}(h_j|h_j^k, W^k)$ and $g_{w_j}(w_j|w_j^k, H^k)$ are convex functions. Since the sum of convex functions is convex, $g_H(H|H^k, W^k)$ and $g_W(W|W^k, H^{k+1})$ are also convex functions. Since every convex function has convex sublevel sets [37], $g_H(H|H^k, W^k)$ and $g_W(W|W^k, H^{k+1})$ are also quasi-convex functions.
Also, the problems in (24) and (28) has a unique minimum; since we are minimizing a convex function.

Razaviyayn et. al. in Theorem 2.(a) of [38] showed that the sequence of points generated by Block MM converge to the stationary point, provided the surrogate function is quasi-convex and the minimizer of the surrogate minimization problem is unique. Hence, INOM algorithm converges to the stationary point of the problem in (1) which is a direct application of Theorem 2. (a) in [38].

We now discuss the convergence of PARINOM; which is based on Vanilla MM. Note that $f_{NMF}(W, H)$ in (1) is bounded below by zero and the constraint set is closed and convex. Also from (13), the sequence of points $\{W^k, H^k\}$ monotonically decrease the NMF problem. Hence, the sequence $f_{NMF}(W^k, H^k)$ generated by PARINOM will at the least converge to the finite value.

We now show that the sequence $\{W^k, H^k\}$ will converge to the stationary point. To prove the same, we first group the variables $W, H$ into a single block $X$. From (13), we have:

$$f_{NMF}(X^0) \geq f_{NMF}(X^1) \geq f_{NMF}(X^2) \cdots$$

(41)

Assume that there is a subsequence $X^{r_j}$ converging to a limit point $Z$. Then, from (9), (10) and from (41) we obtain:

$$g(X^{r_{j+1}}|X^{r_j+1}) = f_{NMF}(X^{r_{j+1}}) \leq f_{NMF}(X^{r_j+1}) \leq g(X^{r_j+1}|X^{r_j}) \leq g(X|X^{r_j})$$

(42)

Letting $j \to \infty$, we get

$$g(Z|Z) \leq g(X|Z)$$

(43)

which implies $g'(Z|Z) \geq 0$. Since the first order behavior of surrogate function is same as function $f(\cdot)$, $g'(Z|Z) \geq 0$ implies $f'_{NMF}(Z) \geq 0$. Hence, $Z$ is the stationary point of $f_{NMF}(\cdot)$ and therefore the proposed algorithm converges to the stationary point.
D. Squarem Acceleration Scheme

We now describe a way to further accelerate the convergence of PARINOM algorithm based on Squared Iterative method (SQUAREM) [39] acceleration scheme. Originally, this scheme was proposed for fixed-point Expectation Maximization algorithm. However, since MM is a generalization of EM, this scheme could be used for accelerating MM based algorithms as well. SQUAREM is based on Cauchy-Brazilai-Browein method, which is a combination of Cauchy and BB (Brazilai and Browein) method to accelerate the convergence rate. This acceleration scheme is proven to be monotonic.

Let $f_1(W_k, H_k)$ and $f_2(W_k, H_k)$ denote fixed-point functions which update $W$ and $H$ respectively i.e

$$W^{k+1} = f_1(W^k, H^k) = \sqrt{V(H^k)^T \circ (W^k)^4 \odot (V^k((H^k)^T))}$$

$$H^{k+1} = f_2(W^k, H^k) = \sqrt{((W^k)^TV \circ (H^k)^4 \odot ((W^k)^TV)^k)}$$

The pseudo code of the acceleration scheme is shown in Table 4.

**Table 4: Acc-PARINOM**

**Input:** Data samples $V$, with each column normalized, $r$, $m$ and $n$.

**Initialize:** Set $k = 0$. Initialize $W^0$ and $H^0$. Each column of $W^0$ is normalized.

**Repeat:**

1. Parallely update: $W^1 = f_1(W^k, H^k), H^1 = f_2(W^k, H^k)$
2. Normalize each column of $W^1$
3. Parallely update: $W^2 = f_1(W^1, H^1), H^2 = f_2(W^1, H^1)$
4. Normalize each column of $W^2$
5. Compute: $r_h = H^1 - H^k, v_h = H^2 - H^1 - r_h$ and $\alpha_h = -\frac{||r_h||}{||v_h||}$
6. Parallely compute $r_w, v_w, \alpha_w$
7. $H = \max(0, H^k - 2\alpha_h r_h + \alpha_h^2 v_h)$
8. $W = \max(0, W^k - 2\alpha_w r_w + \alpha_w^2 v_w)$
9. Normalize each column of $W$
Table 4: Acc-PARINOM

10. while \( \|V - WH\|_F > \|V - W^kH^k\|_F \) do

11. \[ \alpha_h \leftarrow \frac{\alpha_h - 1}{2}, \quad \alpha_w \leftarrow \frac{\alpha_w - 1}{2} \]

12. \[ H = \max (0, H^k - 2\alpha_h r_h + \alpha_h^2 v_h) \]

13. \[ W = \max (0, W^k - 2\alpha_w r_w + \alpha_w^2 v_w) \]

14. end while

15. \( W^{k+1} = W, \quad H^{k+1} = H \)

16. normalize each column of \( W^{k+1} \)

17. \( k \leftarrow k + 1 \)

until convergence

SQUAREM acceleration scheme will sometimes violate the non-negative constraint. Hence, the negative values of the matrix must be made zero. To retain the descent property of the proposed algorithm; the value of \( \alpha_h \) and \( \alpha_w \) is found by backtracking which halves the distance between alpha and -1 until the descent property is satisfied. Note that when \( \alpha_h \) is equal to -1, \( \max (0, H^k - 2\alpha_h r_h + \alpha_h^2 v_h) \) becomes equal to \( H^2 \). Due to the monotonic behavior of MM, \( H^2 \) will be less than \( H^k \). Similarly when \( \alpha_w \) is equal to -1, \( \max (0, W^k - 2\alpha_w r_w + \alpha_w^2 v_w) \) is equal to \( W^2 \) - which is lesser than \( W^k \). Hence the descent property is guaranteed to hold as alpha is pushed towards -1.

IV. SIMULATIONS

In this section, we present numerical simulations to compare the proposed methods with the state of the art algorithms - Fast-HALS and MU. To have a fair comparison we accelerate the MU algorithm using the SQUAREM acceleration scheme. All the simulations were carried out on a PC with 2.40GHz Intel Xeon Processor with 64 GB RAM.

1. In the first simulation, we fix \( n = 100, \quad m = 200 \) and \( r = 1 \) and compare the convergence rate and per iteration cost of the proposed algorithms vs Fast-HALS, MU and Accelerated MU.
The elements of \( V \) matrix was randomly generated from a uniform distribution from \([100, 200]\). Initial values of \( W \) and \( H \) was also randomly generated from a uniform distribution from \([0, 1]\). The columns of \( V \) and \( W \) were normalized. The initial objective value \( f_{\text{NMF}}(W^0, H^0) \) for all the algorithms were kept same. Fig. 3(a) shows the run time vs the objective value in log scale of all the algorithms. Fig. 3(b) compares the run time vs the objective value in log scale of MU, Fast-HALS and INOM.

![Graphs showing comparison of run time and convergence](image)

(a) Objective value vs run time of the proposed and existing algorithms- Fast-HALS, MU and Accelerated MU

(b) Objective value vs run time of MU, Fast-HALS and INOM

Figure 3: Comparison of run time and convergence of proposed algorithm with existing algorithms when \( n = 100, m = 200 \) and \( r = 1 \).

From Fig. 3(a), it can be seen that the proposed algorithms are monotonic in nature. Accelerated PARINOM has faster convergence when compared to PARINOM algorithm. However, due to the additional steps involved in accelerated PARINOM, the per iteration cost is more in Accelerated PARINOM when compared to PARINOM. From Fig. 3(b), it can be seen that INOM algorithm takes lesser time to converge when compared to the rest of the algorithms.

2.a. In this simulation, we compare the proposed algorithms with Fast-HALS, MU and accelerated MU for a dense matrix \( V \) with \( m = 50000 \) and \( n = 10000 \). Dense matrix factorization has application in image processing and in video analysis. The elements of \( V \) matrix was randomly generated from a uniform distribution from \([100, 200]\). Initial values of \( W \) and \( H \) was randomly generated from a uniform distribution from \([0, 1]\).
generated from a uniform distribution from [0, 1]. The comparison is done based on how quickly the algorithms reduce the initial objective value $f_{\text{NMF}}(W^0, H^0)$ to about 70% of the initial objective value $f_{\text{NMF}}(W^0, H^0)$ for different values of $r$ - which was varied from 500 to 5000 in steps of 500. The columns of $V$ and $W$ were normalized. The initial objective value for all the algorithms were kept same. The run time was averaged over 50 trials. Fig. 4(a) compares the proposed algorithms with Fast-HALS, MU and accelerated MU algorithm for $m = 50000$ and $n = 10000$.

![Graph](image1)

(a) Comparison of run time of proposed algorithms with Fast-Hals, MU and Acc-MU.

![Graph](image2)

(b) Comparison of run time of MU, INOM and PARINOM.

Figure 4: Comparison of proposed algorithm with existing algorithms by varying $r$, $m = 50000$ and $n = 10000$ and when $V$ is a dense matrix.

From Fig. 4(a), it can be seen that as the size of $r$ increases, Fast-HALS takes the most time to reduce the initial objective value $f_{\text{NMF}}(W^0, H^0)$ to about 70% of the initial objective value $f_{\text{NMF}}(W^0, H^0)$. Fig. 4(b) compares MU, INOM and PARINOM. From this figure, it can be seen that INOM takes the least time.

2.b. We repeat the above simulation for a sparse matrix $V$. Sparse matrix factorization has application in text mining. The elements of a 70% sparse matrix was randomly generated from a normal distribution with negative elements pushed to zero. Fig. 5(a) compares the proposed algorithms with Fast-Hals, MU and accelerated MU algorithm for $m = 50000$ and $n = 10000$.
when the matrix $V$ is 70% sparse.

![Graph](image1.png)

(a) Comparison of run time of proposed algorithms with Fast-Hals, MU and Acc-MU.

![Graph](image2.png)

(b) Comparison of run time of MU, INOM and PARINOM

Figure 5: Comparison of proposed algorithm with existing algorithms by varying $r$, $m = 50000$ and $n = 10000$ and when $V$ is a 70% sparse matrix

From Fig.5(a), it can be seen that INOM and PARINOM takes the least time when compared to the state-of-the-art algorithms. Also, as the size of $r$ increases, Fast-HALS takes the most time to reduce the initial objective value $f_{NMF}(W^0, H^0)$ to 70% of the initial objective value $f_{NMF}(W^0, H^0)$. Fig.5(b) shows only the performance of INOM, PARINOM and MU for better readability.

3.a. In this simulation, we vary the size of $V$ and compare the performance of the algorithm. The comparison is done based on how quickly the algorithms reduce the initial objective value $f_{NMF}(W^0, H^0)$ to about 70% of the initial objective value $f_{NMF}(W^0, H^0)$. $m$ was varied from 100000 to 1000000 in steps of 100000 and $n$ and $r$ were equal to 1000 and 100, respectively. The elements of $V$ matrix was randomly generated from a uniform distribution from $[100, 200]$. Initial values of $W$ and $H$ was randomly generated from a uniform distribution from $[0, 1]$. The columns of $V$ and $W$ were normalized. The initial objective value for all the algorithms were kept same. The run time was averaged over 50 trials. Fig.6(a) compares the proposed algorithms with Fast-Hals, MU and accelerated MU algorithm for $r = 100$ and $n = 1000$. Fig.
(b) compares the performance of **INOM**, **MU** and **Fast-Hals**.

![Graph (a)](image1.png)  ![Graph (b)](image2.png)

(a) Comparison of run time of proposed algorithms with Fast-Hals, **MU** and Acc-MU.  
(b) Comparison of run time of **MU**, **INOM** and Fast-Hals

Figure 6: Comparison of proposed algorithm with existing algorithms for varying \( r, m = 50000 \) and \( n = 10000 \) when \( V \) is dense matrix.

From Fig.6 (b), it can be seen that **INOM** takes the least time when compared to the state-of-the-art algorithms.

3.b. The above simulation is repeated for a sparse matrix \( V \). The elements of a 70% sparse matrix was randomly generated from a normal distribution with negative elements pushed to zero. Fig.7 (a) compares the proposed algorithms with Fast-Hals, **MU** and accelerated **MU** algorithm for \( r = 100 \) and \( n = 1000 \). Fig.7 (b) compares the performance of **INOM**, **MU** and Fast-Hals. From Fig.7(b), it can be seen that **INOM** performs better than **MU** and Fast-Hals.
Figure 7: Comparison of proposed algorithm with existing algorithms for varying $r$, $m = 50000$ and $n = 10000$ when $V$ is 70% sparse matrix.

4. In this simulation, an application of Nonnegative Matrix Factorization on Blind Source Separation (BSS) is shown. In BSS, source signals have to be separated from a set of mixed signals without knowing the mixing process. It is commonly used in audio signal processing, image processing, biomedical signal processing and in digital communications. In the latter case, the $W$ matrix can be thought as the channel response and each row of $H$ consists of the source signal send by the transmitter array. The receiver sensor array then receives a linear mixture of the source signal. The task of BSS is to reconstruct the source signal from the received signal.

Five source signals for 10 seconds were simulated-a square wave, a rectangular wave, two sine waves of frequency 2 Hz and 20 Hz and a chirp signal which begins with 0 Hz at $t = 0$ sec and cross 30 Hz after $t = 5$ sec. The negative values of the source signals were made zero. $W$ matrix of size $200 \times 5$ was randomly generated. To evaluate the proposed algorithm in presence of noise, random noise with variance 0.01 was added to the five source signals. Source signals and some of the observed signals are shown in Fig. 8 and Fig. 9 respectively.
Figure 8: (a) Source signal without noise (b) Noisy source signal with variance = 0.1

Figure 9: (a) Observed signal when source signal has no noise. (b) Observed signal when source signal has noise variance = 0.1

\( W \) and \( H \) were initialized randomly from Uniform distribution from \([100, 500]\) and \([200, 400]\), respectively. The algorithms were made to run till 1000 iterations or unless the relative change in cost function was \(10^{-8}\). Figure 10 and 11 shows the reconstructed signal by INOM and Fast-Hals, respectively. MU algorithm was not able to reconstruct the signal in the presence of noise and the same is shown in Fig. 12.
Figure 10: Reconstructed signal by INOM (a) without noise. (b) with noise variance = 0.1

Figure 11: Reconstructed signal by Fast-Hals (a) without noise (b) with noise variance = 0.1
Figure 12: Reconstructed signal by MU (a) without noise (b) with noise variance = 0.1

V. POSSIBLE EXTENSION AND CONCLUSION

In this paper, we proposed two different MM based algorithms - INOM and PARINOM to solve the NMF problem. INOM sequentially updates the $W$ and $H$ matrices, while PARINOM parallely updates them. The update equations of $W$ and $H$ matrices in case of INOM resembles the update equation of Gradient Descent algorithms with adaptive step sizes. We also prove that the proposed algorithms are monotonic and converge to the stationary point of the NMF problem. Various computer simulations were performed to compare the proposed algorithms with the existing algorithms. It was found that INOM performed better than existing algorithms.

The effect of parallel update will be predominantly visible when one has to decompose a multidimensional array into matrices of varied dimensions. This problem is called NonNegative Tensor Factorization (NTF) [40], [41], [42], [43]. A brief overview of NTF is given below.

Suppose, a tensor $X$ of order $N \mathbb{R}^{m_1 \times m_2 \cdots \times m_N}$ is given. This can be represented as sum of rank one tensors i.e.

$$X = \sum_{k=1}^{K} a_k^{(1)} \otimes a_k^{(2)} \cdots \otimes a_k^{(N)} \quad (45)$$
where $a_k^{(1)}$ belongs to $\mathbb{R}^{m_1}$, $a_k^{(2)}$ belongs to $\mathbb{R}^{m_2}$ and so on. $K$ is rank of the tensor and $\otimes$ represents outer product. This can be written in a compact form as:

$$X = [A^{(1)}, A^{(2)} \ldots A^{(N)}]$$ (46)

where $A^{(1)} = [a_1^{(1)}, a_2^{(1)}, \ldots, a_K^{(1)}]$ is of size $m_1 \times K$. Similarly, $A^{(2)} = [a_1^{(2)}, a_2^{(2)}, \ldots, a_K^{(2)}]$ and is of size $m_2 \times K$. $[A^{(1)}, A^{(2)} \ldots A^{(N)}]$ represents $\sum_{k=1}^{K} a_k^{(1)} \otimes a_k^{(2)} \ldots a_k^{(N)}$. Hence, the NTF problem can be formulated as:

$$\text{NTF: } \minimize_{A^{(1)}, A^{(2)} \ldots A^{(N)} \geq 0} \left( \|X - [A^{(1)}, A^{(2)} \ldots A^{(N)}]\|_F^2 \right)$$ (47)

When compared to parallel update of matrices, alternating minimization will definitely take significantly more time to solve the problem. Since NTF is a generalization of NMF, the proposed algorithms can also be applied to solve NTF and the parallel update algorithm - PARINOM would be more handy for NTF.

REFERENCES

[1] S. Kaski and J. Peltonen, “Dimensionality reduction for data visualization [applications corner],” IEEE Signal Processing Magazine, vol. 28, no. 2, pp. 100–104, 2011.
[2] G. Strang, Linear Algebra and Its Applications. Thomson, Brooks/Cole, 2006.
[3] B. Moore, “Principal component analysis in linear systems: Controllability, observability, and model reduction,” IEEE Transactions on Automatic Control, vol. 26, no. 1, pp. 17–32, 1981.
[4] B. Thompson, “Factor analysis,” The Blackwell Encyclopedia of Sociology, 2007.
[5] P. Smaragdis and J. C. Brown, “Non-negative matrix factorization for polyphonic music transcription,” vol. 3, no. 3, pp. 177–180, 2003.
[6] V. P. Pauca, F. Shahnaz, M. W. Berry, and R. J. Plemmons, “Text mining using non-negative matrix factorizations,” pp. 452–456, 2004.
[7] J. Zhang, L. Wei, Q. Miao, and Y. Wang, “Image fusion based on nonnegative matrix factorization,” vol. 2, pp. 973–976, 2004.
[8] P. Fernsel and P. Maass, “A survey on surrogate approaches to non-negative matrix factorization,” arXiv preprint arXiv:1808.01975, 2018.
[9] D. Donoho and V. Stodden, “When does non-negative matrix factorization give a correct decomposition into parts?” pp. 1141–1148, 2004.
[10] S. Essid, “A single-class SVM based algorithm for computing an identifiable NMF,” pp. 2053–2056, 2012.
[11] J. Kim, Y. He, and H. Park, “Algorithms for nonnegative matrix and tensor factorizations: A unified view based on block coordinate descent framework,” *Journal of Global Optimization*, vol. 58, no. 2, pp. 285–319, 2014.
[12] Y. Xu and W. Yin, “A globally convergent algorithm for nonconvex optimization based on block coordinate update,” *Journal of Scientific Computing*, vol. 72, no. 2, pp. 700–734, 2017.
[13] S. Bonettini, “Inexact block coordinate descent methods with application to non-negative matrix factorization,” *IMA Journal of Numerical Analysis*, vol. 31, no. 4, pp. 1431–1452, 2011.
[14] D. D. Lee and H. S. Seung, “Algorithms for non-negative matrix factorization,” in *Advances in neural information processing systems*, 2001, pp. 556–562.
[15] E. F. Gonzalez and Y. Zhang, “Accelerating the Lee-Seung algorithm for nonnegative matrix factorization,” Tech. Rep., 2005.
[16] C.-J. Lin, “Projected gradient methods for nonnegative matrix factorization,” *Neural computation*, vol. 19, no. 10, pp. 2756–2779, 2007.
[17] N.-D. Ho, P. Van Dooren, and V. D. Blondel, “Descent methods for nonnegative matrix factorization,” pp. 251–293, 2011.
[18] T. D. Hien, D. Van Tuan, and P. Van At, “Additive update algorithm for nonnegative matrix factorization,” *arXiv preprint arXiv:1209.5647*, 2012.
[19] A. Cichocki, R. Zdunek, and S.-i. Amari, “Nonnegative matrix and tensor factorization [lecture notes],” *IEEE Signal Processing Magazine*, vol. 25, no. 1, pp. 142–145, 2008.
[20] H. Kim and H. Park, “Nonnegative matrix factorization based on alternating nonnegativity constrained least squares and active set method,” *SIAM journal on matrix analysis and applications*, vol. 30, no. 2, pp. 713–730, 2008.
[21] J. Kim and H. Park, “Toward faster nonnegative matrix factorization: A new algorithm and comparisons,” pp. 353–362, 2008.
[22] M. H. Van Benthem and M. R. Keenan, “Fast algorithm for the solution of large-scale non-negativity-constrained least squares problems,” *Journal of Chemometrics: A Journal of the Chemometrics Society*, vol. 18, no. 10, pp. 441–450, 2004.
[23] N. Gillis et al., “Nonnegative matrix factorization: Complexity, algorithms and applications,” *Unpublished doctoral dissertation, Université catholique de Louvain. Louvain-La-Neuve: CORE*, 2011.
[24] A. Cichocki, R. Zdunek, and S.-i. Amari, “Hierarchical als algorithms for nonnegative matrix and 3d tensor factorization,” *International Conference on Independent Component Analysis and Signal Separation*, pp. 169–176, 2007.
[25] A. Cichocki and A.-H. Phan, “Fast local algorithms for large scale nonnegative matrix and tensor factorizations,” *IEICE transactions on Fundamentals of Electronics, Communications and Computer sciences*, vol. 92, no. 3, pp. 708–721, 2009.
[26] D. Hajinezhad, T.-H. Chang, X. Wang, Q. Shi, and M. Hong, “Nonnegative matrix factorization using ADMM: Algorithm and convergence analysis,” pp. 4742–4746, 2016.
[27] S. Boyd, N. Parikh, E. Chu, B. Peleato, J. Eckstein et al., “Distributed optimization and statistical learning via the alternating direction method of multipliers,” *Foundations and Trends in Machine learning*, vol. 3, no. 1, pp. 1–122, 2011.
[28] A. Vandaele, N. Gillis, F. Glineur, and D. Tuytens, “Heuristics for exact nonnegative matrix factorization,” *Journal of Global Optimization*, vol. 65, no. 2, pp. 369–400, 2016.
[29] Y. Sun, P. Babu, and D. P. Palomar, “Majorization-minimization algorithms in signal processing, communications, and machine learning,” *IEEE Transactions on Signal Processing*, vol. 65, no. 3, pp. 794–816, 2017.
[30] D. R. Hunter and K. Lange, “A tutorial on MM algorithms,” *The American Statistician*, vol. 58, no. 1, pp. 30–37, 2004.

[31] D. Böhnning and B. G. Lindsay, “Monotonicity of quadratic-approximation algorithms,” *Annals of the Institute of Statistical Mathematics*, vol. 40, no. 4, pp. 641–663, 1988.

[32] S. U. Pillai, T. Suel, and S. Cha, “The Perron-Frobenius theorem: some of its applications,” *IEEE Signal Processing Magazine*, vol. 22, no. 2, pp. 62–75, 2005.

[33] S. Gaubert and J. Gunawardena, “The Perron-Frobenius theorem for homogeneous, monotone functions,” *Transactions of the American Mathematical Society*, vol. 356, no. 12, pp. 4931–4950, 2004.

[34] W. Sun and Y.-X. Yuan, *Optimization theory and methods: nonlinear programming*. Springer Science & Business Media, 2006, vol. 1.

[35] C. D. Maranas and C. A. Floudas, “Global optimization in generalized geometric programming,” *Computers & Chemical Engineering*, vol. 21, no. 4, pp. 351–369, 1997.

[36] K. Lange and H. Zhou, “MM algorithms for geometric and signomial programming,” *Mathematical programming*, vol. 143, no. 1-2, pp. 339–356, 2014.

[37] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[38] M. Razaviyayn, M. Hong, and Z.-Q. Luo, “A unified convergence analysis of block successive minimization methods for nonsmooth optimization,” *SIAM Journal on Optimization*, vol. 23, no. 2, pp. 1126–1153, 2013.

[39] R. Varadhan and C. Roland, “Simple and globally convergent methods for accelerating the convergence of any EM algorithm,” *Scandinavian Journal of Statistics*, vol. 35, no. 2, pp. 335–353, 2008.

[40] Y. Qian, F. Xiong, S. Zeng, J. Zhou, and Y. Y. Tang, “Matrix-vector nonnegative tensor factorization for blind unmixing of hyperspectral imagery,” *IEEE Transactions on Geoscience and Remote Sensing*, vol. 55, no. 3, pp. 1776–1792, 2017.

[41] N. B. Erichson, A. Mendible, S. Wihlborn, and J. N. Kutz, “Randomized nonnegative matrix factorization,” *Pattern Recognition Letters*, vol. 104, pp. 1–7, 2018.

[42] A. Sapienza, A. Bessi, and E. Ferrara, “Non-negative tensor factorization for human behavioral pattern mining in online games,” *Information*, vol. 9, no. 3, p. 66, 2018.

[43] N. D. Sidiropoulos, L. De Lathauwer, X. Fu, K. Huang, E. E. Papalexakis, and C. Faloutsos, “Tensor decomposition for signal processing and machine learning,” *IEEE Transactions on Signal Processing*, vol. 65, no. 13, pp. 3551–3582, 2017.