On tournaments’ combinatorics

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Abstract

In this paper we present a new proof of a proposition presented in the article: Complexes of tournaments, directionality filtrations and persistent homology [1]. This paper is part of Joaquín Castañeda’s undergraduate thesis.

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Definition 1.1 [1] For a non-negative integer n, an n-tournament is a digraph with no reciprocal edges whose underlying undirected graph is an n-clique. An n–tournament is said to be

1. transitive, if its edge orientation defines a total ordering on its vertex set,
2. regular, if for each vertex v ∈ σ its in-degree and out-degree are equal.

Definition 1.2 [1] Let G = (V, E) be a digraph. For a vertex v ∈ V, the following is defined:

1. In-degree of v in G, iG(v), as the number of edges coming into v.
2. Out-degree of v in G, oG(v), as the number of edges going out of v.
3. Signed degree of v in G by sdG(v) = iG(v) − oG(v).

Let U ⊆ V, a subset of vertices.

1. The signed degree of U relative to G is defined as sdG(U) = \sum_{v \in U} sdG(v).
2. The directionality of U relative to G is defined as DrG(U) = \sum_{v \in U} sdG(v)^2.
**Definition 1.3** [1] For any \( n \)-tournament \( \sigma \) in a graph \( G \), let \( V_{\sigma} \) denote the vertex set of \( \sigma \) and define:

1. **Local directionality**: \( Dr(\sigma) = Dr_{\sigma}(V_{\sigma}) \)
2. Let \( c_3(\sigma) \) denote the number of regular 3-sub-tournaments in \( \sigma \).

**Proposition 1** Let \( \sigma \) a tournament and \( v, w \in \sigma \) two vertices such that the following requirement are met:

1. \( sd_{\sigma}(v) \geq sd_{\sigma}(w) \)
2. \( i_{\sigma}(w) - i_{\sigma}(v) = k \)

Let \( \tau \) the tournament formed by changing the direction of the edge between \( v \) and \( w \).

**First case**: \((v, w) \in \sigma\), then \( Dr(\tau) = Dr(\sigma) - 8(k-1) \).

**Second case**: \((w, v) \in \sigma\), then \( Dr(\tau) = Dr(\sigma) + 8(k+1) \).

**Proof:** For any vertex \( u \in \sigma \) it is true that if \( i_{\sigma}(u) + o_{\sigma}(u) = n - 1 \), then \( o_{\sigma}(u) = n - 1 - i_{\sigma}(u) \).

Compute:

\[
sd_{\sigma}(v) - sd_{\sigma}(w) = \left[i_{\sigma}(v) - o_{\sigma}(v)\right] - \left[i_{\sigma}(w) - o_{\sigma}(w)\right] = 2i_{\sigma}(v) - 2i_{\sigma}(w) = -2[i_{\sigma}(w) - i_{\sigma}(v)] = -2k.
\]

Let \( sd_{\sigma}(v) = a \) and \( sd_{\sigma}(w) = b \), thus \( a - b = -2k \).

**First case:** If the direction of the edge \((v, w)\) is changed, then for a vertex \( u \in V_{\sigma} \setminus \{v, w\}\), \( sd_{\tau}(u) = sd_{\tau}(u) \). Moreover, \( sd_{\tau}(v) = a + 2 \) and \( sd_{\tau}(w) = b - 2 \).

Thus,

\[
Dr(\tau) = Dr(\sigma) - a^2 - b^2 + (a + 2)^2 + (b - 2)^2 = Dr(\sigma) + 4(a - b) + 8 = Dr(\sigma) + 4(-2k) + 8 = Dr(\sigma) - 8(k - 1).
\]

**Second case:** As in the previous case, for a vertex \( u \in V_{\sigma} \setminus \{v, w\}\), \( sd_{\tau}(u) = sd_{\tau}(u) \) and for \( v, w \) \( sd_{\tau}(v) = a - 2 \) y \( sd_{\tau}(w) = b + 2 \).
Thus,

\[
Dr(\tau) = Dr(\sigma) - a^2 - b^2 + (a - 2)^2 + (b + 2)^2 = Dr(\sigma) + 4(b - a) + 8 = Dr(\sigma) + 8(k + 1).
\]

**Proposition 2** Let \( \sigma \) be a tournament that satisfies the same hypothesis as in the previous proposition.

Let \( \tau \) the tournament formed by changing the direction of the edge between \( v \) and \( w \).

**First case:** \( (v, w) \in \sigma \), then \( \tau \) has \( k - 1 \) regular 3-cycles more than \( \sigma \).

**Second case:** \( (w, v) \in \sigma \), then \( \tau \) has \( k + 1 \) regular 3-cycles less than \( \sigma \).

**Proof:** **First case:** Given \( v, w \) and \( u \) another vertex, then the 3-cycle \( \{u, v, w\} = \rho \) lies in one and only one of the following:

1. It is regular in \( \sigma \) but not in \( \tau \).
2. It is not regular in \( \sigma \) and becomes regular in \( \tau \).
3. It is not regular in \( \sigma \) and still not regular in \( \tau \).

If \( \rho \) is such that \( (u, v), (v, w) \in \rho \), then \( \rho \) is type 1 or 2. The maximum number \( D \) of regular 3-cycles that could be destroyed is equal to the number of 3-cycles of those two types, i.e the number of 3-cycles in \( \sigma \) with an edge incident on \( v \). Such number \( D \) is equal to \( i_\sigma(v) = i_\tau(v) - 1 \).

Now, the 3-cycles \( \rho \) in that could become regular are those formed by edges
\((u, w)\) and \((v, w)\), but the edge between vertices \(u\) and \(v\) can go in any direction. So, the maximum number \(C\) of 3-cycles that could become regular is equal to the number of 3-cycles of Types 2 and 3, i.e. the number of 3-cycles in \(\tau\) with an edge incident on \(w\). Such number \(C\) is equal to \(i_\tau(w) = i_\sigma(w) - 1\).

Thus, the number \(C - D = i_\sigma(w) - i_\sigma(v) - 1 = k - 1\) represents the difference \(c_3(\tau) - c_3(\sigma)\).

Therefore, there are \((k - 1)\) regular 3-cycles more in \(\tau\) than in \(\sigma\).

**Second case:** The proof for this case is similar to the previous case.

However, now \(D = i_\tau(w) - 1 = i_\sigma(w)\) and \(C = e_\tau(v) = i_\sigma(v) - 1\). Then, the number \(D - C = i_\sigma(w) - (i_\sigma(v) - 1) = k + 1\) represents the difference \(c_3(\sigma) - c_3(\tau)\).

Therefore, there are \((k + 1)\) regular 3-cycles less in \(\tau\) than in \(\sigma\).

**Proposition 3** [1] Let \(\sigma\) be an \(n\)-tournament. Then

\[
Dr(\sigma) = 2\binom{n+1}{3} - 8c_3(\sigma).
\]

**Proof:** Proceed by induction on \(n\).

For \(n = 3\), \(\sigma\) is regular or transitive. In the first case each vertex has signed degree equal to zero, then \(Dr(\sigma) = 0\). If \(\sigma\) is transitive, the signed degree of its vertices are 2, -2 and 0. Therefore \(Dr(\sigma) = 8\). Thus the claim follows. Assuming the formula holds for all \(n\)-tournaments and prove it holds for
\[ n \text{-tournaments.} \]

**Particular case:** Let \( \sigma \) an \( n \)-tournament. Add a new vertex \( v \) and form a \((n+1)\)-tournament \( \tau \) where all the new edges are of the form \((v, u)\) or \((u, v)\).

**Observation:** All the 3-cycles which contain the vertex \( v \) are transitive. Thus, \( c_3(\sigma) = c_3(\tau) \). Therefore, if the proposition is true

\[ Dr(\tau) - Dr(\sigma) = 2\binom{n+2}{3} - 2\binom{n+1}{3}. \]

Note that \( \binom{n+2}{3} \) is the number of 3-cycles in an \((n+2)\)-tournament \( \rho \) and \( \binom{n+1}{3} \) is the number of 3-cycles in \( \eta = \rho \setminus \{w\} \), where \( w \) is any vertex. Then

\[ \binom{n+2}{3} - \binom{n+1}{3} \]

is the number of 3-cycles in \( \rho \setminus \eta \). That is equal to the number of edges in \( \eta \). Thus \( 2\left[\binom{n+2}{3} - \binom{n+1}{3}\right] = 2\left[\binom{n}{2}\right] = n^2 + n. \)

On the other hand,

\[ Dr(\tau) - Dr(\sigma) = \sum_{u \in \tau}[i_{\tau}(u) - o_{\tau}(u)]^2 - \sum_{u \in \sigma}[i_{\sigma}(u) - o_{\sigma}(u)]^2. \]

Regardless of the direction of the edges in \( \tau \setminus \sigma \), \( [i_{\tau}(v) - o_{\tau}(v)]^2 = n^2 \). Then

\[ Dr(\tau) = \sum_{u \in \sigma}[i_{\tau}(u) - o_{\tau}(u)]^2 + n^2 \] and

\[ Dr(\tau) - Dr(\sigma) = \sum_{u \in \sigma}[i_{\tau}(u) - o_{\tau}(u)]^2 + n^2 - \sum_{u \in \sigma}[i_{\sigma}(u) - o_{\sigma}(u)]^2 \]

For the formula to be true, it is sufficient to prove that

\[ \frac{\sum_{u \in \sigma}[i_{\tau}(u) - o_{\tau}(u)]^2 - \sum_{u \in \sigma}[i_{\sigma}(u) - o_{\sigma}(u)]^2}{\sum_{u \in \sigma}[i_{\tau}(u) - o_{\tau}(u)]^2 - [i_{\sigma}(u) - o_{\sigma}(u)]^2} = n. \]

If the edges in \( \tau \setminus \sigma \) come out of \( v \) then, for each vertex \( u \in \sigma \), \( o_{\tau}(u) = o_{\sigma}(u) \) and \( i_{\tau}(u) = i_{\sigma}(u) + 1 \). For the case where all the edges in \( \tau \setminus \sigma \) incident on \( v \) then, \( o_{\tau}(u) = o_{\sigma}(u) + 1 \) and \( i_{\tau}(u) = i_{\sigma}(u) \). In either case

\[ \frac{\sum_{u \in \sigma}[i_{\tau}(u) - o_{\tau}(u)]^2 - [i_{\sigma}(u) - o_{\sigma}(u)]^2}{\sum_{u \in \sigma}[1 + (i_{\sigma}(u) - o_{\sigma}(u))]^2 - [i_{\sigma}(u) - o_{\sigma}(u)]^2} = \]

\[ = \sum_{u \in \sigma} 1 + 2[\varepsilon_{\sigma}(u) - s_{\sigma}(u)] = n + 2\left(\sum_{u \in \sigma} \varepsilon_{\sigma}(u) - \sum_{u \in \sigma} s_{\sigma}(u)\right) = \]

\[ n + 2(0) = n. \]

Therefore, the formula is valid for this particular case.
**General case:** Suppose that $\tau$ has $m$ edges that are incident on the vertex $v$. Consider the tournament $\rho$ where all edges go out from $v$ and changing, one by one, the direction of the appropriate edges to form $\tau$. With the help of the two previous propositions we have that:

$$Dr(\tau) = Dr(\rho) - 8K.$$ 

Where $K = c_3(\tau) - c_3(\rho)$. Therefore,

$$Dr(\tau) = 2\binom{n+2}{3} - 8c_3(\rho) - 8[c_3(\tau) - c_3(\rho)] = Dr(\tau) = 2\binom{n+2}{3} - 8c_3(\tau).$$

**Bibliography**

[1] D. Govc, R. Levi, J. P. Smith, Complexes of tournaments, directionality filtrations and persistent homology, 2021.