A determinantal formula for the GOE Tracy-Widom distribution

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Abstract

Investigating the long time asymptotics of the totally asymmetric simple exclusion process, Sasamoto obtains rather indirectly a formula for the GOE Tracy-Widom distribution. We establish that his novel formula indeed agrees with more standard expressions.

1 Introduction

The Gaussian orthogonal ensemble (GOE) of random matrices is a probability distribution on the set of $N \times N$ real symmetric matrices defined through

$$ Z^{-1} e^{-\text{Tr}(H^2)/2N} dH. $$

$Z$ is the normalization constant and $dH = \prod_{1 \leq i \leq j \leq N} dH_{i,j}$. The induced statistics of eigenvalues can be studied through the method of Pfaffians. Of particular interest for us is the statistics of the largest eigenvalue, $E_1$. As proved by Tracy and Widom [8], the limit

$$ \lim_{N \to \infty} \mathbb{P}(E_1 \leq 2N + sN^{1/3}) = F_1(s) $$

exists, $\mathbb{P}$ being our generic symbol for probability of the event in parenthesis. $F_1$ is called the GOE Tracy-Widom distribution function. Following [3] it can be expressed in terms of a Fredholm determinant in the Hilbert space $L^2(\mathbb{R})$ as follows,

$$ F_1(s)^2 = \det \left( 1 - P_s(K + |g\rangle\langle f|)P_s \right), $$

1
where $K$ is the Airy kernel defined through

\[
K(x, y) = \int_{\mathbb{R}^+} d\lambda \, \mathrm{Ai}(x + \lambda) \, \mathrm{Ai}(y + \lambda),
\]

\[
g(x) = \mathrm{Ai}(x),
\]

\[
f(y) = 1 - \int_{\mathbb{R}^+} d\lambda \, \mathrm{Ai}(y + \lambda),
\]

and $P_s$ is the projection onto the interval $[s, \infty)$.

The GOE Tracy-Widom distribution $F_1(s)$ turns up also in the theory of one-dimensional growth process in the KPZ universality class, KPZ standing for Kardar-Parisi-Zhang [4]. Let us denote the height profile of the growth process at time $t$ by $h(x, t)$, either $x \in \mathbb{R}$ or $x \in \mathbb{Z}$. One then starts the growth process with flat initial conditions, meaning $h(x, 0) = 0$, and considers the height above the origin $x = 0$ at growth time $t$. For large $t$ it is expected that

\[
h(0, t) = c_1 t + c_2 t^{1/3} \xi_1. \tag{5}
\]

Here $c_1$ and $c_2$ are constants depending on the details of the model and $\xi_1$ is a random amplitude with

\[
\mathbb{P}(\xi_1 \leq s) = F_1(s). \tag{6}
\]

For the polynuclear growth (PNG) model the height $h(0, t)$ is related to the length of the longest increasing subsequence of symmetrized random permutations [5], for which Baik and Rains [1] indeed prove the asymptotics (5), (6), see [2] for further developments along this line. Very recently Sasamoto [6] succeeds in proving the corresponding result for the totally asymmetric simple exclusion process (TASEP). If $\eta_j(t)$ denotes the occupation variable at $j \in \mathbb{Z}$ at time $t$, then the TASEP height is given by

\[
h(j, t) = \begin{cases} 
2N_t + \sum_{i=1}^{j}(1 - 2\eta_i(t)) & \text{for } j \geq 1, \\
2N_t & \text{for } j = 0, \\
2N_t - \sum_{i=j+1}^{0}(1 - 2\eta_i(t)) & \text{for } j \leq -1,
\end{cases} \tag{7}
\]

with $N_t$ denoting the number of particles which passed through the bond $(0,1)$ up to time $t$. The flat initial condition for the TASEP is $\ldots 010101 \ldots$. For technical reasons Sasamoto takes instead $\ldots 010100000 \ldots$ and studies the asymptotics of $h(-3t/2, t)$ for large $t$ with the result

\[
h(-3t/2, t) = \frac{1}{2} t + \frac{1}{2} t^{1/3} \xi_{SA}. \tag{8}
\]
The distribution function of the random amplitude \(\xi_{SA}\) is

\[
P(\xi_{SA} \leq s) = F_{SA}(s)
\]

with

\[
F_{SA}(s) = \det(1 - P_s A P_s).
\]

Here \(A\) has the kernel \(A(x, y) = \frac{1}{2} \text{Ai}((x + y)/2)\) and, as before, the Fredholm determinant is in \(L^2(\mathbb{R})\).

The universality hypothesis for one-dimensional growth processes claims that in the scaling limit, up to model-dependent coefficients, the asymptotic distributions are identical. In particular, since (5) is proved for PNG, the TASEP with flat initial conditions should have the same limit distribution function, to say

\[
F_{SA}(s) = F_1(s).
\]

Our contribution provides a proof for (11).

2 The identity

As written above, the \(s\)-dependence sits in the projection \(P_s\). It will turn out to be more convenient to transfer the \(s\)-dependence into the integral kernel. From now on the determinants are understood as Fredholm determinants in \(L^2(\mathbb{R}^+)\) with scalar product \(\langle \cdot, \cdot \rangle\). Thus, whenever we write an integral kernel like \(A(x, y)\), the arguments are understood as \(x \geq 0\) and \(y \geq 0\).

Let us define the operator \(B(s)\) with kernel

\[
B(s)(x, y) = \text{Ai}(x + y + s).
\]

By \(\|B(s)^2\| < 1\) and clearly \(B(s)\) is symmetric. Thus also \(\|B(s)\| < 1\) for all \(s\). \(B(s)\) is trace class with both positive and negative eigenvalues. Shifting the arguments in (10) by \(s\), one notes that

\[
F_{SA}(s) = \det(1 - B(s)).
\]

Applying the same operation to (3) yields

\[
F_1(s)^2 = \det(1 - B(s)^2 - |g\rangle\langle f|)
\]

with

\[
\begin{align*}
g(x) &= \text{Ai}(x + s) = (B(s)\delta)(x), \\
f(y) &= 1 - \int_{\mathbb{R}^+} d\lambda \text{Ai}(y + \lambda + s) = ((1 - B(s))1)(y).
\end{align*}
\]
Here $\delta$ is the $\delta$-function at $x = 0$ and 1 denotes the function $1(x) = 1$ for all $x \geq 0$. $\delta$ and 1 are not in $L^2(\mathbb{R}_+)$. Since the kernel of $B(s)$ is continuous and has super-exponential decay, the action of $B(s)$ is unambiguous.

**Proposition 1.** With the above definitions we have

$$\det(1 - B(s)) = F_1(s). \quad (16)$$

**Proof.** For simplicity we suppress the explicit $s$-dependence of $B$. We rewrite

$$F_1(s)^2 = \det \left( (1 - B)(1 + B - |B\delta\rangle\langle 1|) \right) = \det(1 - B) \det(1 + B) \langle 1 - \langle \delta, B(1 + B)^{-1} \rangle \rangle = \det(1 - B) \det(1 + B) \langle \delta, (1 + B)^{-1} \rangle \quad (17)$$

since $1 = \langle \delta, 1 \rangle$. Thus we have to prove that

$$\det(1 - B) = \det(1 + B) \langle \delta, (1 + B)^{-1} \rangle. \quad (18)$$

Taking the logarithm on both sides,

$$\ln \det(1 - B) = \ln \det(1 + B) + \ln \langle \delta, (1 + B)^{-1} \rangle, \quad (19)$$

and differentiating it with respect to $s$ results in

$$- \text{Tr} \left( (1 - B)^{-1} \frac{\partial}{\partial s} B \right) = \text{Tr} \left( (1 + B)^{-1} \frac{\partial}{\partial s} B \right) + \frac{\partial}{\partial s} \langle \delta, (1 + B)^{-1} \rangle \quad (20)$$

where we used

$$\frac{d}{ds} \ln(\det(T)) = \text{Tr} \left( T^{-1} \frac{\partial}{\partial s} T \right). \quad (21)$$

Since $B(s) \to 0$ as $s \to \infty$, the integration constant for (20) vanishes and we have to establish that

$$-2 \text{Tr} \left( (1 - B^2)^{-1} \frac{\partial}{\partial s} B \right) = \frac{\partial}{\partial s} \langle \delta, (1 + B)^{-1} \rangle. \quad (22)$$

Define the operator $D = \frac{d}{dx}$. Then using the cyclicity of the trace and Lemma 2

$$-2 \text{Tr} \left( (1 - B^2)^{-1} \frac{\partial}{\partial s} B \right) = -2 \text{Tr} \left( (1 - B^2)^{-1} DB \right) = \langle \delta, (1 - B^2)^{-1} DB \rangle. \quad (23)$$

Using Lemma 3 and $D1 = 0$, one obtains

$$\langle \delta, \frac{\partial}{\partial s} (1 + B)^{-1} \rangle = \langle \delta, (1 - B^2)^{-1} B\delta \rangle \langle \delta, (1 + B)^{-1} \rangle. \quad (24)$$

Thus (22) follows from (23) and (24).
Lemma 2. Let $A$ be a symmetric, trace class operator with smooth kernel and let $D = \frac{\partial}{\partial x}$. Then
\[
2 \text{Tr}(DA) = -\langle \delta, A\delta \rangle
\]
where $DA$ is the operator with kernel $\frac{\partial}{\partial x}A(x, y)$.

Proof. The claim follows from spectral representation of $A$ and the identity
\[
\int_{\mathbb{R}^+} dx f'(x) f(x) = -f(0)f(0) - \int_{\mathbb{R}^+} dx f(x)f'(x).
\]
\[\square\]

Lemma 3. It holds
\[
\frac{\partial}{\partial s}(1 + B)^{-1} = (1 - B^2)^{-1}BD + (1 - B^2)^{-1}|B\delta\rangle\langle \delta| (1 + B)^{-1}.
\]

Proof. First notice that $\frac{\partial}{\partial s} B \equiv \dot{B} = DB$. For any test function $f$,
\[
(\dot{B}f)(x) = \int_{\mathbb{R}^+} dy \partial_y Ai(x + y + s)f(y)
\]
\[= -Ai(x + s)f(0) - \int_{\mathbb{R}^+} dy Ai(x + y + s)f'(y).
\]
Thus, using the notation $P = |B\delta\rangle\langle \delta|$, one has
\[
DB = -BD - P.
\]
Since $\|B\| < 1$, we can expand $\frac{\partial}{\partial s}(1 + B)^{-1}$ in a power series and get
\[
\frac{\partial}{\partial s}(1 + B)^{-1} = \sum_{n \geq 1} (-1)^n \frac{\partial}{\partial s} B^n = \sum_{n \geq 1} (-1)^n \sum_{k=0}^{n-1} B^k DB^{n-k}.
\]
Using recursively (29), we obtain
\[
\sum_{k=0}^{n-1} B^k DB^{n-k} = -\frac{1}{2}(-1)^n B^n D + \sum_{j=0}^{n-1} \sum_{k=j}^{n-1} (-1)^{j+1} B^k P B^{n-k-1}
\]
\[= -\frac{1}{2}(-1)^n B^n D + \sum_{k=0}^{n-1} \frac{1}{2}(-1)^k B^k P B^{n-k-1}.
\]
Inserting (31) into (30) and exchanging the sums results in
\[
\frac{\partial}{\partial s}(1 + B)^{-1} = \sum_{n \geq 1} B^{2n+1} D + \sum_{k \geq 0} \sum_{n \geq k+1} \frac{1}{2}(-1)^k B^k P(-B)^{n-(k+1)}
\]
\[= (1 - B^2)^{-1}BD + (1 - B^2)^{-1}P(1 + B)^{-1}.
\]
\[\square\]
3 Outlook

The asymptotic distribution of the largest eigenvalue is also known for Gaussian unitary ensemble of Hermitian matrices (\(\beta = 2\)) and Gaussian symplectic ensemble of quaternionic symmetric matrices (\(\beta = 4\)). As just established, for \(\beta = 1\),

\[
F_1(s) = \det(1 - B(s)),
\]

and, for \(\beta = 2\),

\[
F_2(s) = \det(1 - B(s)^2),
\]

which might indicate that \(F_4(s) = \det(1 - B(s)^4)\). This is however incorrect, since the decay of \(\det(1 - B(s)^4)\) for large \(s\) is too rapid. Rather one has

\[
F_4(s/\sqrt{2}) = \frac{1}{2} \left( \det(1 - B(s)) + \det(1 + B(s)) \right). \tag{35}
\]

This last identity is obtained as follows. Let

\[
U(s) = \frac{1}{2} \int_s^\infty q(x) \, dx
\]

with \(q\) the unique solution of the Painlevé II equation

\[
q'' = sq + 2q^3
\]

with \(q(s) \sim \text{Ai}(s)\) as \(s \to \infty\). Then the Tracy-Widom distributions for \(\beta = 1\) and \(\beta = 4\) are given by

\[
F_1(s) = \exp(-U(s))F_2(s)^{1/2}, \quad F_4(s/\sqrt{2}) = \cosh(U(s))F_2(s)^{1/2}, \tag{36}
\]

see [8]. Thus

\[
F_4(s/\sqrt{2}) = \frac{1}{2} (F_1(s) + F_2(s)/F_1(s)),
\]

from which (35) is deduced.

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