ON THE COHOMOLOGY OF THE CLASSIFYING SPACES OF PROJECTIVE UNITARY GROUPS

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Abstract. Let $BPU_n$ be the classifying space of $PU_n$, the projective unitary group of order $n$, for $n > 1$. We use the Serre spectral sequence associated to a fiber sequence $BU_n \to BPU_n \to K(Z, 3)$ to determine the ring structure of $H^*(BPU_n; \mathbb{Z})$ up to degree 10, as well as a family of distinguished elements of $H^{2k+2}(BPU_n; \mathbb{Z})$, for each prime divisor $p$ of $n$. We also study the primitive elements of $H^*(BU_n; \mathbb{Z})$ as a comodule over $H^*(K(Z, 2); \mathbb{Z})$, where the comodule structure is given by an action of $K(Z, 2) \cong BS^1$ on $BU_n$ corresponding to the action of taking the tensor product of a complex line bundle and an $n$ dimensional complex vector bundle.

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1. Introduction

Let $U_n$ be the unitary group of order $n$, and consider the unit circle group $S^1$ of complex numbers as the normal subgroup of scalars of $U_n$. The quotient group, denoted hereafter by $PU_n$, is called the projective unitary group of order $n$. Its classifying space $BPU_n$ is a topological space determined by $PU_n$ up to homotopy type, with a canonical base point, characterised by the fact that for a well behaved topological space $X$ with a base point, the set of pointed homotopy classes of maps, $[X, BPU_n]$, has a natural one-to-one correspondence with the isomorphism classes of $PU_n$ bundles, also known as topological Azumaya algebras of degree $n$, over $X$.

The space $BPU_n$ plays a fundamental role in the study of the topological period-index problem ([1], [2], Antieau and Williams, 2014, and Gu, The Topological Period-Index Problem over 8-Complexes, arXiv preprint, arXiv:1709.00787, 2017), a topological analog of the period-index problem concerning elements of Brauer groups and their representing Azumaya algebras ([7], Colliot-Thélène, 2002). The space $BPU_n$ also arises in the study of Dai-Freed anomalies in particle physics (Garca-Etxebarria and Montero, Dai-Freed Anomalies in Particle Physics, arXiv preprint, arXiv:1808.00009, 2018).

The cohomology rings of $BPU_n$ with coefficients in $\mathbb{Z}/p$ for various primes $p$, as well as its Brown-Peterson cohomology, for $n$ of various forms, have been considered by Kameko ([13], 2008), Kono and Mimura ([15], 1975), Kono, Mimura and Shimada ([16], 1975), Toda ([21], 1987), and Vavpetic and Viruel ([22], 2004). More results and their applications are discussed in [23] (Vistoli, 2007).

The subject of this paper is the integral cohomology $H^k(BPU_n; \mathbb{Z})$. This was known for $k \leq 5$, of which Section 3 of [1] (Antieau and Williams, 2014) is a good reference.

Our first theorem is as follows. The notations in the statement are made clear in the sequel.

Theorem 1.1. For an integer $n > 1$, $H^*(BPU_n; \mathbb{Z})$ in degrees $\leq 10$ is isomorphic to the following graded ring:

$$\mathbb{Z}[e_2, \cdots, e_j, x_1, y_{3,0}, y_{2,1}, z_1, z_2]/I_n.$$  

Here $e_i$ is of degree $2i$, $j_n = \min\{5, n\}$; the degrees of $x_1, y_{3,0}, y_{2,1}$ are $3, 8, 10$, respectively; and the degrees of $z_1, z_2$ are $9, 10$, respectively. $I_n$ is the ideal generated by

$$nx_1, \quad e_2(n)x_1^2, \quad e_3(n)y_{3,0}, \quad e_2(n)y_{2,1}, \quad e_3(n)z_1, \quad e_3(n)z_2,$$

$$\delta(n)e_2x_1, \quad (\delta(n) - 1)(y_{2,1} - e_2x_1^2), \quad e_3x_1,$$

where $e_p(n) = \gcd(p, n)$, and

$$\delta(n) = \begin{cases} 2, & \text{if } n = 4l + 2 \text{ for some integer } l, \\ 1, & \text{otherwise.} \end{cases}$$

We outline the strategy of the proof of Theorem 1.1 as follows. Consider the short exact sequence of Lie groups

$$1 \rightarrow S^1 \rightarrow U_n \rightarrow PU_n \rightarrow 1.$$
Applying the classifying space functor $B$, we obtain the following fiber sequence

$$BS^1 \to BU_n \to BPU_n.$$  

Notice that $BS^1$ is the Eilenberg-Mac Lane space $K(\mathbb{Z}, 2)$. Therefore we have the following fiber sequence:

$$BU_n \to BPU_n \xrightarrow{\chi} K(\mathbb{Z}, 3).$$

Let $\mathcal{U} E^*_{\ast}$ be the integral cohomological Serre spectral sequence induced by (1.1). This is the main object of interest in this paper.

Similarly, let $T^n$ and $PT^n$ be the maximal tori of $U_n$ and $PU_n$ respectively, and we have a fiber sequence

$$BT^n \to BPT^n \to K(\mathbb{Z}, 3).$$

Let $\mathcal{T} E^*_{\ast}$ be the integral cohomological Serre spectral sequence associated to it.

Finally we let $\mathcal{K} E^*_{\ast}$ be the integral cohomological Serre spectral sequence associated to the path fibration

$$K(\mathbb{Z}, 2) \to PK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3),$$

where $PK(\mathbb{Z}, 3)$ is the path space (with one end fixed) of $K(\mathbb{Z}, 3)$, which is contractible. We compare $\mathcal{U} E^*_{\ast}$ with the other two spectral sequences via homological algebra of differential graded algebras and the theory of Chern classes, in particular the splitting principle.

As we shall see in Section 3, the map $BU_n \xrightarrow{\chi} K(\mathbb{Z}, 3)$ in the fiber sequence (1.1) represents an additive generator of $H^3(BPU_n; \mathbb{Z}) \cong \mathbb{Z}/n$. It is well-known (and re-visited in Section 2) that for each prime $p$, the $p$-torsion element of $H^* (K(\mathbb{Z}, 3); \mathbb{Z})$ of the lowest dimension lives in dimension $2p + 2$, which we denote by $\gamma_{p, 0}$. In consistency with Theorem 1.1, we omit $X_{\ast}$, the notation for the induced homomorphism of cohomology rings, when there is no risk of ambiguity. In particular, in the following theorem we do not distinguish $\gamma_{p, 0}$ with $\chi^* (\gamma_{p, 0})$.

**Theorem 1.2.** Let $p$ be a prime. In $H^{2p+2}(BPU_n; \mathbb{Z})$, we have $\gamma_{p, 0} \neq 0$ of order $p$ when $p|n$, and $\gamma_{p, 0} = 0$ otherwise.

When $n = p$ is an odd prime, the essential content of this theorem is implied by work of Vistoli (Theorem 3.4, [23], 2009).

To state the next theorem, we recollect some generality on the topology of classifying spaces of compact Lie groups. Let $G$ be a compact Lie group, $\Gamma$ a closed subgroup of its center. Let $P : BG \to B(G/\Gamma)$ be the canonical map. Then $P$ fits in a fiber sequence

$$BG \xrightarrow{P} B(G/\Gamma) \to B^2 \Gamma,$$

which induces an action of $\Omega B^2 \Gamma \simeq \mathbf{B} \Gamma$ on $BG$:

$$\mu : \mathbf{B} \Gamma \times BG \to BG.$$  

(1.3)

Let $R$ be a discrete commutative unital ring such that the Künneth formula reduces to an isomorphism

$$H^* (\mathbf{B} \Gamma \times BG; R) \cong H^* (\mathbf{B} \Gamma; R) \otimes H^* (BG; R).$$

Then

$$\mu^* : H^* (BG; R) \to H^* (\mathbf{B} \Gamma; R) \otimes H^* (BG; R)$$
makes $H^*(BG; R)$ a comodule over $H^*(BΓ; R)$. We denote the subring of primitive elements of $H^*(BG; R)$ (the elements $x$ such that $μ(x) = 1 ⊗ x$) by $PH(BG; R)$.

It is well-known that we have $\text{Im} P^* \subseteq PH(BG; R)$ (Proposition 3.8 of [21], Toda, 1987). It is therefore natural to ask when the equality holds. Back to our case of $G = U_n$, $Γ = S^1$ and $G/Γ = PU_n$, the action

$$K(\mathbb{Z}, 2) \times BU_n \to BS^1 \times BU_n \to BU_n$$

corresponds to the action of taking the tensor product of a complex line bundle and an $n$ dimensional complex vector bundle. Toda ([21], 1987) showed $\text{Im} P^* = PH^*(BU_n; \mathbb{Z}/2)$ for $n \equiv 2 \pmod{4}$ and $n = 4$. Our second theorem concerns the case of integral cohomology:

**Theorem 1.3.** For $n > 1$ and $k \leq 12$, we have $\text{Im}(H^k(P; \mathbb{Z})) = PH^k(BU_n; \mathbb{Z})$.

In Section 2 we study the cohomology of $K(Z, 3)$ and the spectral sequence $^kE^{*, *}_\ast$. This is an example of the general theory of differential graded algebras and their bar constructions, developed in [6] (Cartan and Serre, 1954-1955) and reviewed in the appendix.

We set up the apparatus that computes the differentials of $^UE^{*, *}_\ast$ in Section 3. We compare $^U E^{*, *}_\ast$, $^TE^{*, *}_\ast$ with $^kE^{*, *}_\ast$ to give all the differentials of $^TE^{*, *}_\ast$ and show that they detect considerably many differentials of $^UE^{*, *}_\ast$ via the splitting principle of complex vector bundles (Chapter 16, [20], Switzer, 1975).

We collect some auxiliary results and present a proof of Theorem 1.2 in Section 4. Some of results in this section are devoted to simplify the proof of Theorem 1.1, which occupies Section 5, 6, and 7.

In Section 8 we discuss the primitive elements of $H^*(BU_n; \mathbb{Z})$ which enable us to prove Theorem 1.3. This also offers another way, in addition to the one discussed in Section 3, to study the differentials of $^UE^{*, *}_\ast$.

The appendix contains an outline and a summary of main results of exp. 2 to exp. 11 of [6] (Cartan and Serre, 1954-1955), as well as the theory of twisted tensor products ([4], Brown, 1959). They are essential to Section 2.

## 2. DG Algebras and the Path Fibration of $K(Z, 3)$

In [6] (Cartan and Serre, 1954-1955) a differential graded algebra is constructed to calculate the integral homology of $K(II, n)$ for any positive integer $n$ and a finitely generated abelian group $II$. See the appendix for an introduction to this topic.

This approach in particular gives us the cohomology of $K(Z, 3)$. However, for the purpose of this paper we need the differentials of the integral Serre spectral sequence of the fiber sequence

$$K(\mathbb{Z}, 2) \to PK(Z, 3) \to K(Z, 3)$$

which requires information on the level of chain complex rather then just cohomology. To obtain such information we consider an acyclic multiplicative construction (see Definition A.1 (A(2), A(3), M(3))) which models the fiber sequence above.

Throughout the rest of this section, we write $E_R(x; k)$, $P_R(y; l)$, $E(x; k)$, $P_R(y; l)$ for exterior algebras and divided power algebras with coefficients in a ring $R$, or $\mathbb{Z}$, and with one generator of a specified degree. Other than this, no attempt is made to keep notations in consistency with those in the appendix. All tensor products are over the base ring $\mathbb{Z}$ unless otherwise specified.
By Theorem [A.29] we should take \( A(n) = U(M^{(n)}) \), the universal algebra of \( M^{(n)} \) (Proposition [A.16]), a free graded abelian group defined as in Construction [A.28]. The differential of \( A(n) \) is defined in the same paragraph. We elaborate this construction for \( n = 2 \) and \( 3 \), fixing \( \Pi = \mathbb{Z} \). Notice that in these cases, since \( \Pi \) is a cyclic group of infinite order, \( M^{(n)} \) is identified to the free abelian group generated by admissible words of height \( n \).

In the case \( n = 2 \), the admissible words of height 2 are of the form \( \gamma_k(\sigma^2) \) for any nonnegative integer \( k \). Therefore we take \( A(2) = P(u, 2) \) where \( u \) corresponds to the operation \( \sigma^2 \). Since \( A(2) \) is 0 in odd degrees, the differential is 0.

In the case \( n = 3 \), the admissible words of height 3 are \( a_{p,k} = \varphi_p \gamma_p \sigma^2, b_{p,k} = \sigma\gamma_p \sigma^2, \) and \( b_1 = \sigma^3 \) for nonnegative integers \( k \) and prime numbers \( p \).

It follows that we have

\[
A(3) = \bigotimes_{p} \left[ \bigotimes_{k \geq 0} P(a_{p,k}; 2p^{k+1} + 2) \otimes \bigotimes_{k \geq 1} E(b_{p,k}; 2p^k + 1) \right] \otimes (b_1; 3)
\]

According to Construction [A.28], the differential is defined by

\[
(2.2) \quad \begin{cases}
\bar{d}(\gamma_1(a_{p,k})) = pb_{p, k+1} \gamma_{l-1}(a_{p,k}), \\
\bar{d}(b_{p,k}) = 0, \bar{d}(b_1) = 0
\end{cases}
\]

Notice that \( A(2) \) is torsion free, and that for each \( p \), the algebra

\[
\bigotimes_{k \geq 0} P(a_{p,k}; 2p^{k+1} + 2) \otimes \bigotimes_{k \geq 1} E(b_{p,k}; 2p^k + 1)
\]

has no \( p' \)-torsion for any prime \( p' \neq p \). Therefore we have

**Lemma 2.1**. The epimorphism of homology groups in Theorem [A.29] is an isomorphism of abelian groups, for \( n = 2, 3 \).

**Remark 2.2**. We may not assert, however, that this isomorphism is one of divided power algebras. Indeed, passing to the case where the base ring is \( \mathbb{Z}/2 \), the homology class represented by \( b_1 \) has a nontrivial divided power \( \gamma_2(b_1) \), whereas in \( A(3) \) we have \( \gamma_2(b_1) = 0 \).

We proceed to take \( M(3) = A(2) \otimes A(3) \) as a graded ring, and define its differential \( \bar{d} \) in such a way that it makes \( M(3) \) acyclic, and when passing to \( M(3) = \mathbb{Z} \otimes_{A(2)} M(3) \), it induces \( \bar{d} \), the differential of \( A(3) \). We will use the sketch of proof of Theorem [A.28] as our guide.

**Lemma 2.3** (Lucas’ Theorem). Let \( p \) be a prime number, \( k = \sum_{r=0}^{n} k_r p^r \), and \( l = \sum_{r=1}^{n} l_r p^r \) such that \( 0 \leq k_r, l_r \leq p - 1 \) are integers, and \( k_n \neq 0 \). Then

\[
\binom{k}{l} \equiv \prod_{r=0}^{n} \binom{k_r}{l_r} \mod p.
\]

Here \( \binom{i}{j} = 0 \) for \( i < j \).

**Proof.** For an independent variable \( w \) we have

\[
(1 + w)^k = \prod_{r=0}^{n} (1 + w)^{k_r p^r} = \prod_{r=0}^{n} (1 + w^{p^r})^{k_r} \mod p.
\]

The result is verified by comparing the coefficient of \( w^l \) on both sides of the equation above. \( \square \)
Lemma 2.4. For a prime $p$ and $1 \leq i \leq k + 1$ the greatest common divisor of
\[
\frac{1}{(p^{k+1} - 1)!p^i} = \lambda_i^{p,k+1} \frac{1}{(p^{k+1} - 1)!p^{i-1}} + \mu_i^{p,k+1} \frac{1}{(p^{k+1} - p^i)!p^i}.
\]

Proof. We have
\[
\frac{p^i}{p^i - 1} = \frac{p^i}{p^i - 1}[(p^i - 1)!] = p[(p^i - 1)!]; \quad \frac{(p^{k+1} - 1)!}{(p^{k+1} - p^i)!} = \left(\frac{p^{k+1} - 1}{p^i - 1}\right)[(p^i - 1)!]
\]
So
\[
(p^i - 1)! \mid \gcd\left\{\frac{p^i}{p^i - 1}, \frac{(p^{k+1} - 1)!}{(p^{k+1} - p^i)!}\right\}
\]
On the other hand, $p^{k+1} - 1 = (p-1)(1+p+\cdots+p^k)$, $p^i = (p-1)(1+p+\cdots+p^{i-1})$. By Lemma 2.3, we have
\[
\left(\frac{p^{k+1} - 1}{p^i - 1}\right) \equiv (p-1)^{i-1} \pmod{p},
\]
in particular
\[
\gcd\left\{\left(\frac{p^{k+1} - 1}{p^i - 1}\right), p\right\} = 1,
\]
Hence
\[
\gcd\left\{\frac{p^i}{p^i - 1}, \frac{(p^{k+1} - 1)!}{(p^{k+1} - p^i)!}\right\}|(p^i - 1)!,
\]
and the proof is completed. \(\square\)

Definition 2.5. We define $M(3)$ as follows:

1. Let $M(3)_k$ be the $k$th level of $A(2) \otimes A(3)$.

2. By Definition A.11 it is enough to define the differential of $M(3)$ by the following:
\[
d(u) = 0; \quad d(b_1) = u; \quad d(b_{p,k}) = \gamma_{p^k}(u);
\]
\[
d(a_{p,k}) = (pb_{p,k+1} - A_0^{p,k+1}b_1\gamma_{p^{k+1}}(u) - \sum_{i=1}^{k} A_i^{p,k+1}b_{p,i}\gamma_{p^{i+1}-p^i}(u)
\]
where $\{A_i^{p,k+1}\}_{i=0}^k \subset \mathbb{Z}$ are defined as follows.

3. Fix a set of integers $\{\lambda_i^{p,k+1}, \mu_i^{p,k+1}\}_{i=0}^k$ as in Lemma 2.4. Define
   (a) $A_0^{p,k+1} = \lambda_1^{p,k+1} \cdots \lambda_k^{p,k+1}$.
   (b) $A_i^{p,k+1} = \mu_i^{p,k+1}\lambda_{i+1}^{p,k+1} \cdots \lambda_k^{p,k+1}$, if $i = 1, \cdots, k - 1$.
   (c) $A_k^{p,k+1} = A_k^{p,k+1} = \mu_k^{p,k+1}$.

Lemma 2.4 ensures that $d$ is indeed a differential.
(4) We define a bi-degree on $M(3)$ as follows. Let $B \otimes C$ be a monomial in $M(3)$ such that

$$B \in A(3); \ C \in A(2).$$

Then the bi-degree of $B \otimes C$ is $(s, t) = (\deg(B), \deg(C))$. Clearly the total degree agrees with the usual degrees, and $s$ induces a filtration $F_K$ on $M(3)$.

The following result plays a central role in this section.

**Proposition 2.6.** $M(3)$ is acyclic.

**Remark 2.7.** It follows from Lemma 2.1, Proposition 2.6 and Corollary A.25 that the spectral sequence associated to the filtration $F$ on $M(3)$ is isomorphic to the Serre spectral sequence of the path fibration

$$K(\mathbb{Z}, 2) \to PK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3),$$

for cohomology, namely $^KE_\ast^\ast$ defined in Section 1, with coefficients in $\mathbb{Z}$. The case for the Serre spectral sequence for homology is similar.

**Remark 2.8.** The reason that we give the unpleasantly complicated formulae in Definition 2.5 is as follows. Remark 2.7 indicates that $^KE_\ast^\ast$ is the spectral sequence associated to the filtered chain complex $M(3)$. Therefore, at least in principle, all the differentials of $^KE_\ast^\ast$ can be determined mechanically, though the process might be tedious. This would be useful if one would like to explore the full potential of the spectral sequence $U_E_*^\ast$, perhaps with a computer program. The formulae is of little use though, if one is only interested in low dimensional computation.

Recall the following lemma from (Cartan and Serre, 1954-1955):

**Lemma 2.9.** For any integer $q$ and a prime number $p$ the following DGA over $\mathbb{Z}/p$ is acyclic:

$$P_p(a; 2pq + 2) \otimes E_p(b; 2q + 1) \otimes \mathbb{Z}/p[c; 2q]/c^p$$

with differential given by

$$d(c) = 0; \ d(b) = c; \ d(\gamma_l(a)) = c^{p-1}b^{l-1}(a), \ l \geq 1.$$  

Proof. In fact, a chain homotopy can be defined on linear generators as follows:

$$c \to b; \ c^{p-1}b^{l-1}(a) \to \gamma_l(a), \ l \geq 1$$

and all the other linear generators are sent to 0. □

Lemma 2.9 immediately generalize to the following

**Lemma 2.10.** Let $M(3)[p]$ be the DGA

$$\bigotimes_{k \geq 0} P(a_{p,k}; 2p^{k+1} + 2) \otimes \bigotimes_{k \geq 1} E(b_{p,k}; 2p^k + 1) \otimes E(b_1; 3) \otimes P(u; 2) \otimes \mathbb{Z}/p$$

with differential $d[p]$ defined as follows:

$$d[p](u) = 0; \ d[p](b_1) = u; \ d[p](b_{p,k}) = \gamma_p^k(u);$$

$$d[p](\gamma_l(a_{p,k})) = -\Lambda_k^{p,k+1}b_{p,k}^{p^{k+1}-}\gamma_p^k(u) \text{ if } k > 0,$$

$$d[p](\gamma_l(a_{p,0})) = -\Lambda_k^{p,1}b_1\gamma_p(u).$$

Then $M(3)[p]$ is acyclic.
Proof.

\[ M(3)[p] = \bigotimes_{k \geq 0} P(a,p,k; 2p^{k+1} + 2) \otimes E(b,p,k; 2p^k + 1) \otimes \mathbb{Z}[\gamma_p^k(u)]/(\gamma_p^k(u)^p) \otimes \mathbb{Z}/p, \]

where \( b_{p,0} = b_1 \). It is a tensor product of DGA’s of the form in Lemma 2.9 indexed by \( k \), and the result follows.

**Proof of Proposition 2.6.** We proceed to show that \( M(3) \otimes \mathbb{Z}/p \) is acyclic for all primes \( p \). Since \( M(3) \) is a degree-wise finitely generated free abelian group, this, together with the Künneth formula proves the theorem.

Let \( \mathcal{E}_s \) be the spectral sequence associated to the filtration \( \mathcal{F}_K \) on \( M(3) \otimes \mathbb{Z}/p \). Then obviously \( E_0^s = E_{s,t} \) and \( d_{s,t}^a \) is as follows:

\[
d_1^s(u) = 0; \quad d_1^s(b_1) = 0; \quad d_1^s(b_{p,k}) = 0;
\]

\[
d_1^s(\gamma_{a_{p',k}}) = \begin{cases} 
p' \gamma_{a_{p',k}} & \text{if } p' \neq p \\
0 & \text{if } p' = p \end{cases}
\]

for any prime \( p' \). Therefore we have

\[
\mathcal{E}_s \equiv \bigotimes_{k \geq 0} P_p(a,p,k; 2p^{k+1} + 2) \otimes E_p(b,p,k; 2p^k + 1) \otimes E_p(b; 3) \otimes P_p(u; 2) \otimes \mathbb{Z}/p \mathcal{E}_p[b_{p,k}^s; 2](p)
\]

where \( b_{p,0} = b_1 \). We proceed to consider all the higher differentials on \( \mathcal{E}_s \).

Notice that all the non-trivial differentials on \( P_p(a,p,k; 2p^{k+1} + 2) \otimes E_p(b,p,k; 2p^k + 1) \otimes \mathbb{Z}/p \mathcal{E}_p(b_{p,k}^s; 2)(p) \) are the following:

\[
d_2^{p,p+1}(a_{p,k}) = \Lambda_k^{p,k+1} b_{p,k}(u);
\]

\[
d_2^{p,p+1}(b_{p,k}) = \gamma_{p,k}(u);
\]

\[
d_2^{p,p+1}(\gamma_{p,k}(u)) = 0.
\]

By definition \( \Lambda_k^{p,k+1} = \mu_{p,k}^{p,k+1} \). By Lemma 2.4 \( p \Lambda_i^{p,k+1} + \mu_i^{p,k+1}(p^{k+1} - 1) = 1 \), which shows that \( \Lambda_k^{p,k+1} = \mu_{p,k}^{p,k+1} \) is invertible mod \( p \).

Notice that exact same statement of the definition of the filtration \( \mathcal{F}_K \) in (4) of Definition 2.3 can be applied to define a filtration on \( M(3)[p] \). A direct comparison shows that this filtration induces a spectral sequence which is identical to \( \mathcal{E}_s \) after the \( E_2 \)-page. The theorem then follows from Lemma 2.10.

**Remark 2.11.** The construction of \( M(3) \) and the proof of Proposition 2.6 are inspired by exp.9 and exp.11 of [2] (Cartan and Serre, 1954-1955).

**Corollary 2.12.** Let \( \mathcal{E}_s \) be the spectral sequence associated to the filtration \( \mathcal{F}_K \) on \( M(3) \). Then all the higher differentials are identified by

\[
d_3(b_1) = u;
\]

\[
d_{2p^k+1}(b_{p,k}) = \gamma_{p,k}(u), \quad k \geq 1;
\]

\[
d_r(\gamma_{i}(u)) = 0, \quad \text{for all } r, i
\]

together with the Leibniz rule.
Proof. This is straightforward computation, as the differential of $M(3)$ is described explicitly in Definition 2.6.

To study the cohomology, we will need a diagonal approximation

$$\Delta : A(n) \to A(n) \otimes A(n)$$

for $n = 2$ and 3. We take $\Delta$ to be the (unique) homomorphism of DGA’s with divided power operations such that for an individual admissible word $x$ it satisfies

$$\Delta(x) = \begin{cases} 
    a_{2,k} \otimes 1 + b_{2,k} \otimes b_{2,k} + 1 \otimes a_{2,k}, & \text{if } x = a_{2,k}, n = 3, k \geq 1, \\
    a_{2,0} \otimes 1 + b_1 \otimes b_1 + 1 \otimes a_{2,0}, & \text{if } x = a_{2,0}, n = 3, \\
    x \otimes 1 + 1 \otimes x, & \text{otherwise}.
\end{cases}$$

(2.5)

We proceed to consider the following homomorphism induced by $\Delta$:

$$\Delta_p : A_p(n) \to A_p(n) \otimes A_p(n),$$

where $A_p(n) = A(n) \otimes \mathbb{Z}/p$. As a morphism of $\mathbb{Z}/p$-modules this is simply $\Delta \otimes \mathbb{Z}/p$. However, for $n = 3$ and $p = 2$ there are some complications on the divided power algebra, as we are about to observe.

Lemma 2.13. For $n = 2, 3$ and any prime $p$, $\Delta_p$ is the (unique) homomorphism of DGA’s with divided power operations such that for an individual admissible word $x$ it satisfies

$$\Delta_p(x) = x \otimes 1 + 1 \otimes x.$$  

Proof. For $p \neq 2$ there is nothing to show. Let $\bar{a}_{2,k}, \bar{b}_{2,k}$ be the image of $a_{2,k}, b_{2,k}$ in $A_2(3)$ under the obvious projection. Then it suffices to verify that

$$\Delta_2(\bar{a}_{2,k}) = \bar{a}_{2,k} \otimes 1 + \bar{b}_{2,k} \otimes \bar{b}_{2,k} + 1 \otimes \bar{a}_{2,k}$$

can be derived from the characterization of $\Delta_2$ in the lemma. Indeed, we have the relation given in Proposition A.17

$$\varphi_2 = \gamma_2 \cdot \sigma,$$

which implies

$$\begin{cases} 
    \bar{a}_{2,k} = \gamma_2(\bar{b}_{2,k}), & k \geq 1, \\
    \bar{a}_{2,0} = \gamma_2(\bar{b}_1).
\end{cases}$$

(2.6)

The lemma then follows from the above, together with the product formula for divided power operations (4) of Definition A.11.

It then follows from exp.9, exp.10 of [1] (Cartan and Serre, 1954-1955) that for $n = 2, 3$ and each prime $p$, $\Delta_p$ induced the cup product over the cohomology ring $H^*(K(\mathbb{Z}, n); \mathbb{Z}/p)$. Therefore $\Delta$ is a diagonal approximation.

We proceed to study the dual complex of $A(2), A(3)$ and $M(3)$, which we denote by

$$\begin{align*}
    C(2)^i &= \text{Hom}(A(2),i, \mathbb{Z}), \\
    C(3)^i &= \text{Hom}(A(3),i, \mathbb{Z}), \\
    W(3)^i &= \text{Hom}(M(3),i, \mathbb{Z}).
\end{align*}$$
Let \( y_{p,k}, x_{p,k}, x_1, v \) be the dual of \( a_{p,k}, b_{p,k}, b_1, u \), respectively. Then it follows from (2.10) that we have
\[
(2.7) \quad \begin{cases} 
y_{2,k} = x_{2,k}^2, & k \geq 1, 
y_{2,0} = x_1^2.
\end{cases}
\]

The \( \mathbb{Z} \)-algebra structures of \( C(2) \) and \( C(3) \) induced by the diagonal approximations \( \Delta \) follows readily:
\[
C(2) = \mathbb{Z}[v; 2],
\]
\[
C(3) = \bigotimes_{p \neq 2} \left[ \bigotimes_{k \geq 0} \mathbb{Z}[y_{p,k}; 2p^{k+1} + 2] \right] \otimes \bigotimes_{k \geq 1} E(x_{p,k}; 2p^k + 1) \bigotimes_{k \geq 1} \mathbb{Z}[x_{2,k}; 2^{k+1} + 1] \otimes \mathbb{Z}[x_1; 3],
\]
\[
W(3) = C(2) \otimes C(3).
\]

We proceed to consider the differentials of the cochain complexes above. For \( C(2) \) it follows readily that the differential is trivial. For \( C(3) \) it follows from (2.5) and (2.7) that we have
\[
(2.9) \quad \begin{cases} 
d(x_{p,k+1}) = py_{p,k}, & d(y_{p,k}) = d(x_1) = 0, \text{for } p \text{ a prime and } k \geq 0, 
d(\alpha \beta) = d(\alpha) \beta + (-1)^{\deg(\alpha)} \alpha d(\beta),
\end{cases}
\]
where \( \alpha, \beta \) take the forms of monomials in \( x_1, x_{p,k+1} \) and \( y_{p,l} \) for various \( p, q, k, l \), such that the exponents of \( x_1 \) and \( x_{2,k} \) (\( k \geq 1 \)) in \( \alpha \beta \) are at most 1. The relation (2.9) ensures that the differential \( \bar{d} \) is uniquely determined by the above.

**Remark 2.14.** Notice that \( C(3) \) as defined above is not a DGA, though \( \bar{d} \) is so defined that its ring structure passes to cohomology.

The cohomology ring \( H^3(K(\mathbb{Z}, 3); \mathbb{Z}) \) can now be described in terms of generators and relations. It is easier to do so by describing its torsion free component, the \( p \)-primary components for all primes \( p \), and their relation. Whenever it is obvious from the context, we do not distinguish a cocycle in \( C(3) \) and the cohomology class it represents. Moreover, let \( pA \) denote the \( p \)-primary summand of a ring \( A \) (possibly without unit) of which the underlying abelian group is finitely generated. Finally, for \( R \) a (graded-commutative) ring without unit, let \( \hat{R} \) be the (graded anti-commutative) unital ring \( R \otimes \mathbb{Z} \), where the summand \( \mathbb{Z} \) lives in degree 0 and has a generator acting as the unit of the ring.

**Proposition 2.15.** The graded ring structure of \( H^*(K(\mathbb{Z}, 3); \mathbb{Z}) \) is described in terms of generators of and relations in \( C(3) \) as follows:
\[
H^*(K(\mathbb{Z}, 3); \mathbb{Z}) \cong \left[ \bigotimes_p pH^*(K(\mathbb{Z}, 3); \mathbb{Z}) \right] \otimes \mathbb{Z}[x_1]/(y_{2,0} - x_1^2),
\]
where \( p \) runs over all prime numbers, and \( pH^*(K(\mathbb{Z}, 3); \mathbb{Z}) \) is a free graded-commutative \( \mathbb{Z}/p \)-algebra without unit, generated by the elements of the form
\[
(2.10) \quad \frac{1}{p} \bar{d}(x_{p,i_1} \cdots x_{p,i_r}) = \sum_{j=1}^r (-1)^{k_j} x_{p,i_1} \cdots y_{p,i_j-1} \cdots x_{p,i_r},
\]
where \( k_j = \deg x_{p,i_1} + \cdots + \deg x_{p,i_j} \).
proof. It follows from (2.9) that the differential of \( C(3) \) localized at \( p \) is trivial modulo \( p \). Therefore the \( \mathbb{Z}/p \)-algebra \( H^*(K(\mathbb{Z}, 3); \mathbb{Z}/p) \) is isomorphic to (2.11)
\[
\left\{ \begin{array}{ll}
\bigotimes_{k \geq 0} \mathbb{Z}/p[y_p, k; 2p^{k+1} + 2] \otimes E(x_p, k; 2p^k + 1) & \text{if } p > 2, \\
\mathbb{Z}[x_2, k; 2^{k+1} + 1] \otimes \mathbb{Z}[x_1; 3] & \text{if } p = 2.
\end{array} \right.
\]
On the other hand, for any prime \( p \), a \( p \)-primary element of \( H^*(K(\mathbb{Z}, 3); \mathbb{Z}) \) is \( p \)-torsion. We proceed to prove the proposition by induction on dimension.

Let \( B : H^{s-1}(K(\mathbb{Z}, 3); \mathbb{Z}/p) \to pH^s(K(\mathbb{Z}, 3); \mathbb{Z}) \)
be the Bockstein homomorphism. Then the argument above indicates that \( B \) is surjective. The kernel of \( B \) consists of the modulo \( p \) reductions of integral cohomology classes, which, by (2.9) and the inductive hypothesis, is the \( \mathbb{Z}/2 \)-subalgebra of \( H^*(K(\mathbb{Z}, 3); \mathbb{Z}/p) \) generated by \( y_{p,k} \)'s. Notice that \( B \) is a homomorphism of \( \mathbb{Z}/p \)-vector spaces, it follows from a dimension counting argument that there is no nontrivial linear relation in \( \mathbb{Z}/pH^*(K(\mathbb{Z}, 3); \mathbb{Z}) \). Therefore, it is freely generated as a \( \mathbb{Z}/p \)-algebra by the elements given in (2.10).

To complete the proof, notice that the relation \( y_{2,0} - x_1^2 = 0 \) is given in (2.7). \( \square \)

It suffices in this paper to have the following

**Corollary 2.16.** In degree less then 15, \( H^*(K(\mathbb{Z}, 3); \mathbb{Z}) \) is isomorphic to the following ring:
\[
\mathbb{Z}[x_1; 3] \otimes \bigotimes_{k \geq 0; p} \mathbb{Z}/p[y_p, k; 2p^{k+1} + 2]/(x_1^2 - y_{2,0}),
\]
where \( p \) runs over all prime numbers.

We proceed to apply this setup to study the Serre spectral sequence \( \mathcal{K} \mathcal{E}^{s,*} \). Let \( \mathcal{F}_K \) be the filtration on \( W(3) \) obtained by dualizing the filtration \( \mathcal{F} \) on \( M(3) \). Then the spectral sequence associated to \( \mathcal{F}_K \) is isomorphic to \( \mathcal{K} \mathcal{E}^{s,*} \) starting from the \( E_2 \)-page (Remark 2.7).

In particular, we have
\[
\mathcal{K} E_2^{s,t} \cong H^s(K(\mathbb{Z}, 3); H^t(K(\mathbb{Z}, 2); \mathbb{Z})) \cong \begin{cases} 
H^s(K(\mathbb{Z}, 3); \mathbb{Z}), & t \text{ is even,} \\
0, & t \text{ is odd.}
\end{cases}
\]
The higher differentials of \( \mathcal{K} \mathcal{E}^{s,*} \) may be obtained by carefully writing down a formula of the dual of the differential of \( M(3) \) as given in Definition 2.6. Alternatively, we may consider the dual of the spectral sequence \( \mathcal{K} \mathcal{E}^{s,*} \). For example, the counterpart of Corollary 2.12 is given in the following

**Corollary 2.17.** The higher differentials of \( \mathcal{K} \mathcal{E}^{s,*} \) satisfy
\[
\begin{align*}
&d_3(v) = x_1 \\
&d_{2p^{k+1} - 1}(p^k x_1 v^{p^k - 1}) = v^{p^k - 1 - (p^{k+1} - 1)} y_{p,k}, \quad k \geq e, \gcd(l, p) = 1 \\
&d_r(x_1) = d_r(x_{p,k+1}) = d_r(y_{p,k}) = 0, \quad \text{for all } r, k > 0.
\end{align*}
\]
and the Leibniz rule. Here \( \mathcal{K} E_3^{s,2l(p^e - 1)} \) is generated by \( p^k v^{p^k - 1} x_1 \).

This corollary covers all the differentials we need from this spectral sequence. Figure 2 shows a low dimensional picture.
3. The Higher Differentials of $U^*_E$.

In this section we identify some higher differentials of the spectral sequence $U^*_E$, which make it possible to prove Theorem 1.1. To do so, we compare it to $K^*_E$, via the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{B}S^1 & \xrightarrow{\sim} & K(\mathbb{Z}, 2) \\
\Phi : & \Downarrow B \varphi & \Downarrow \\
\mathbb{B}T^n & \longrightarrow & \mathbb{B}PT^n \\
\Psi : & \Downarrow B \psi & \Downarrow B \psi' \\
\mathbb{B}U_n & \longrightarrow & \mathbb{B}PU_n \\
\end{array}
$$

Here $\varphi : S^1 \to T^n$ is the diagonal, and $\psi$ and $\psi'$ are the inclusions of the maximal tori. Then for any compact Lie group $G$ and its maximal torus $T$, $H^*(BG; \mathbb{Q})$ is the sub-$\mathbb{Q}$-algebra of $H^*(BT; \mathbb{Q})$ stable under the action of the Weyl group. In our case, we have

$$
H^*(\mathbb{B}PU_n; \mathbb{Q}) \xrightarrow{\cong} H^*(\mathbb{B}PT^n; \mathbb{Q})^W,
$$

where $W$ is the Weyl group. (See [12], Hsiang, 1975.)

**Remark 3.1.** Theorem 1.3 is equivalent to the assertion that the homomorphism (3.2) is a surjection in dimension $\leq 12$, with the coefficient ring $\mathbb{Q}$ replaced by $\mathbb{Z}$.

Let $v$ be the multiplicative generator of $H^*(\mathbb{B}S^1; \mathbb{Z})$, $v^i$ be the $i$th copy of $v$ in $H^*(\mathbb{B}(S^1)^n; \mathbb{Z}) \cong H^*((BT^n; \mathbb{Z})$, and $c_k \in H^*(\mathbb{B}U_n; \mathbb{Z})$ be the $k$th universal Chern class. Moreover let $\sigma_k \in H^*(BT^n; \mathbb{Z})$ be the $k$th elementary polynomial in the universal class $\sigma_k$. Then for each $k$, we have

$$
\sigma_k \in H^*(\mathbb{B}U_n; \mathbb{Z}) \xrightarrow{\cong} H^*(\mathbb{B}PU_n; \mathbb{Q})^W,
$$

where $W$ is the Weyl group.
v_1, \ldots, v_n$. In this case we have the splitting principal, which asserts that the above argument applies to integral cohomology \cite{20}, Switzer, 1975. More precisely, it says that $B\psi^*$ is the inclusion taking $c_k$ to $\sigma_k$, or in other words,

$$B\psi^*(\sum_{k=0}^n ckw^k) = \prod_{i=1}^n (1 + v_iw)$$

where $w$ is a polynomial generator.

Our first result is the following

Proposition 3.2. The differential $T_d^*\psi^*$, is partially determined as follows:

\begin{equation}
T_d^*2\psi^*(v^i\xi) = (B\pi)_v^*K(d_r^*2\psi^*(v^i\xi)),
\end{equation}

where $\xi \in T^{E_4}_{\ast,\ast}$, a quotient group of $H^\ast(K(\mathbb{Z}, 3); \mathbb{Z})$, and $\pi_v : T^n \rightarrow S^1$ is the projection of the $i$th diagonal entry. In plain words, $T_d^*2\psi^*(v^i\xi)$ is simply $K(d_r^*2\psi^*(v^i\xi))$ with $v$ replaced by $v_i$.

The proof is straightforward since the Serre spectral sequence is functorial. We proceed to study the differentials $T_d^\ast\psi^*$ with domain in the leftmost column.

Proposition 3.3. (1) The differential $T_d^3\psi^*$ is given by the “formal divergence”

$$\nabla = \sum_{i=1}^n (\partial/\partial v_i) : H^i(B\mathbb{T}^n; R) \rightarrow H^{i-2}(B\mathbb{T}^n; R),$$

in such a way that $T_d^3\psi^* = \nabla(-) \cdot x_1$. For any ground ring $R = \mathbb{Z}$ or $\mathbb{Z}/m$ for any integer $m$.

(2) The spectral sequence degenerates at $T^{E_4}_{\ast,\ast}$. Indeed, we have $T^{E_\infty}_{\ast,\ast} = T^{E_4}_{\ast,\ast} = \text{Ker} T_d^3\psi^* = \mathbb{Z}[v_1 - v_n, \ldots, v_{n-1} - v_n]$.

Proof. (1) is an immediate consequence of the Leibniz rule, chain rule, and Proposition 3.2. Given a polynomial $\theta(v_1, \ldots, v_n)$, we change variables to rewrite it as $\tilde{\theta}(v_1 - v_n, \ldots, v_{n-1} - v_n)$. \cite{20} then tells us that $H^\ast(B\mathbb{P}^n; \mathbb{Q})$ is the free commutative ring generated by $\{v_1 - v_n\}_{i=1}^{n-1}$. The equation

$$T_d^3(\theta(v_1, \ldots, v_n))$$

$$= \sum_{i=1}^n \frac{\partial}{\partial v_i} \tilde{\theta}(v_1 - v_n, \ldots, v_{n-1} - v_n, v_n)$$

$$= \frac{\partial}{\partial v_n} \tilde{\theta}(v_1 - v_n, \ldots, v_{n-1} - v_n, v_n)$$

then implies that $T_d^3(\theta(v_1, \ldots, v_n)) = 0$ if and only if $\tilde{\theta}(v_1 - v_n, \ldots, v_{n-1} - v_n, v_n)$ is independent of $v_n$. For obvious degree reasons this is the only nontrivial differential over $\mathbb{Q}$, which implies that $H^\ast(B\mathbb{T}^n; \mathbb{Q})$, as a $\mathbb{Q}$-subalgebra of $H^\ast(B\mathbb{T}^n; \mathbb{Q})$, is free commutative, generated by $\{v_i - v_n\}_{i=1}^{n-1}$. On the other hand, notice that the following homomorphism of Lie groups

$$T^n \rightarrow S^1, \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \mapsto \lambda_i \lambda_j^{-1}$$
factors through $PT^n$. This implies that $v_i - v_j \in H^*(BPT^n; \mathbb{Z})$, which proves (2).

**Corollary 3.4.** $Ud_3(c_k) = (n - k + 1)c_{k-1}x_1$.

**Proof.**

$$Ud_3(c_k) = Td_3(\sigma_k) = \sum_{i=1}^{n} \frac{\partial \sigma_k}{\partial v_i} = (n - k + 1)c_{k-1}x_1.$$

We will study the operator $\nabla$ in more details in Section 8. We proceed to study $T_d^*:*$ for greater $r$.

Consider a map $\kappa : T^n \to PT^n \times S^1$ defined as follows:

$$
\begin{pmatrix}
\lambda_1 & 0 & \ldots \\
0 & \lambda_2 & \ldots \\
\vdots & \vdots & \lambda_n
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\lambda_1 & 0 & \ldots \\
0 & \lambda_2 & \ldots \\
\vdots & \vdots & \lambda_n
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\lambda \lambda_1 / \lambda_n & 0 & \ldots \\
0 & \lambda \lambda_2 / \lambda_n & \ldots \\
\vdots & \vdots & \lambda
\end{pmatrix}.
$$

where the matrix in the square bracket denotes its class in $PT^n$. Then $\kappa$ is an isomorphism of Lie groups, its inverse being the following:

$$
\begin{pmatrix}
\lambda_1 & 0 & \ldots \\
0 & \lambda_2 & \ldots \\
\vdots & \vdots & \lambda_n
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\lambda \lambda_1 / \lambda_n & 0 & \ldots \\
0 & \lambda \lambda_2 / \lambda_n & \ldots \\
\vdots & \vdots & \lambda
\end{pmatrix}.
$$

Passing to classifying spaces, we have $B\kappa : BT^n \simeq BPT^n \times BS^1$.

**Proposition 3.5.** Let $\theta(v_1, \ldots , v_n) \in H^{2t}(BPT^n; \mathbb{Z})$ be an element in $TE^{0,2t}_2$. As in the proof of Proposition 3.3 we apply the change of variable $(v_1 - v_n, \ldots , v_{n-1} - v_n, v_n)$ and rewrite $\theta$ as

$$\theta = \sum_{i=0}^{t} \theta_i v_n^i$$

such that $\theta_i \in \text{Ker} \nabla$, $i = 0, \ldots , t$. Then

$$(B\kappa^{-1})^*(\theta) = \sum \theta_i \otimes v_n^i.$$

Consider the fiber sequence $BPT^n \to BPT^n \to *$ and let $^PE_0^{*,*}$ be its cohomological Serre Spectral sequence. We take the product of fiber sequences

$$(BPT^n \to BPT^n \to *) \times (BS^1 \to * \to K(\mathbb{Z}, 3))$$

or equivalently

$$BPT^n \times BS^1 \to BPT^n \to K(\mathbb{Z}, 3).$$

Then we have the following morphism between fiber sequences:

$$
\begin{array}{ccc}
BPT^n & \longrightarrow & BPT^n & \longrightarrow & K(\mathbb{Z}, 3) \\
\downarrow B\kappa & & \uparrow & & \uparrow \\
BPT^n \times BS^1 & \longrightarrow & BPT^n & \longrightarrow & K(\mathbb{Z}, 3)
\end{array}
$$

with the unspecified maps being the obvious ones. This diagram is easily seen to be commutative by the discussion above.
Remark

3.7

Proof. By Proposition 3.5 the commutative diagram above induces an isomorphism of spectral sequences $T^r E_r^{*,*} \cong P^r E_r^{*,*} \otimes K^r E_r^{*,*}$ for $r \geq 2$.

Proposition 3.6. Let $P^r E_r^{*,*}$ be the cohomological Serre Spectral sequence associated to $B P^n \rightarrow B P^n \rightarrow \ast$. Then the commutative diagram above induces an isomorphism of spectral sequences $E_r^{*,*} \cong P^r E_r^{*,*}$ for $r \geq 2$.

Proof. By Proposition 3.5 the commutative diagram above induces an isomorphism of $E_3$ pages, and the statement follows from Theorem 3.4 in \cite{McCleary} (McCleary, 2001).

Remark 3.7. Let $\theta(v_1, \cdots, v_n) \in H^{2t}(B T^n; \mathbb{Z})$ represent an element in $T E_2^{0,2t}$, $\xi$ be an element of $H^s(K(\mathbb{Z}, 3); \mathbb{Z})$ for some $s$. Suppose we have $\theta \xi \in T E_r^{*,*}$. Applying the change of variables $(v_1 - v_n, \ldots, v_{n-1} - v_n, v_n)$ we rewrite $\theta$ as

$$\theta = \sum_{i=0}^t m_i v_n^i \theta_i'$$

where $m_i v_n^i \xi \in T E_r^{*,*}$, $\theta_i' \in \text{Ker} \nabla$, $i = 0, \cdots, t$ for all $i$, such that $\theta_i'$ are primitive polynomials in the polynomial ring $\mathbb{Z}[v_1, v_n, \ldots, v_{n-1}, v_n]$, i.e., only $\pm 1$ divides all of its coefficients. In particular, this is just the change of variable considered in Proposition 3.5 where we have $\theta_i = m_i \theta_i'$. The differential $T d_r^{3,3}$ is hence determined by

$$T d_r(\theta \xi) = T d_r(\sum_{i=0}^t m_i v_n^i \theta_i' \xi) = \sum_{i=0}^t T d_r(m_i v_n^i \xi) \theta_i',$$

where $T d_r(m_i v_n^i \xi)$ is determined as in Proposition 3.5. This is how we compute the differentials in practice. This idea is explained more concretely in Example 3.9.

Finally we are at the place to state our main result of this section, of which the proof is already clear.

Proposition 3.8. Proposition 3.2, Proposition 3.3 and Proposition 3.6 determine all of the differentials of $T^r E_r^{*,*}$. Restricted to symmetric polynomials in $v_1, \cdots, v_n$, they determine all of the differentials $U d_r^{s-r+1, r}$ of $U^r E_r^{*,*}$ such that for any $r' < r$, $T d_{r'}^{s-r', r'-1} = 0$.

With enough patience one can apply this apparatus to calculate $H^k(B P U_n; \mathbb{Z})$ for many $k$ and $n$ up to group extension. The interested reader may take the following example as an exercise.

Example 3.9. Let $n = 3$. Show that $U d_3(2c_3 x_1) = 0$ but $2c_3 x_1$ is not a permanent cocycle.

Proof. By the splitting principle (16.13 Corollary, \cite{Switzer}, Switzer, 1975) the image of $c_3$ in $H^3(B T^3; \mathbb{Z})$ is $v_1 v_2 v_3$, and in $T^3 E_3^{*,*}$ we have

$$2v_1 v_2 v_3 x_1 = 2(v_1 - v_2)(v_2 - v_3)x_1 + 2[(v_1 - v_3)v_2^2 x_1 + 2v_3^3 x_1],$$

the first two terms on the right side being easily checked to be permanent cocycles. For the third term we have

$$T d_3(2v_3^3 x_1) = 6v_3^2 x_1^2 = 6v_3^2 y_2, 0 = 0$$

since $2y_2, 1 = 0$. This verifies the first statement. For the second one, we have

$$T d_5(2v_3^3 x_1) = 0$$
since $v_3^3$ has order 4 in $T E_4^{*,*}$ but $y_{3,1}$ is of order 3. However

$$Td_7(2v_3^3x_1) = y_{2,1} \neq 0,$$

whence the second statement. \hfill \Box

We end this section by several corollaries. Let $\nabla$, $\xi$ be the same as earlier, and let $\vartheta = \vartheta(c_1, \cdots, c_n) \in H^*(BU_n; \mathbb{Z})$.

**Corollary 3.10.** $U d_3(\vartheta \xi) = U d_3(\vartheta) \xi = \nabla(\vartheta)x_1\xi$.

This is immediate from the Leibniz rule and Proposition 3.3.

**Corollary 3.11.** If $\vartheta \xi \in U E_r^{a,r-1}$, or in other words, the image of $\rho_\xi$ of $U d_r$ lies in bottom row, then

$$U d_r(\vartheta \xi) = K d_r(B(\psi \cdot \varphi)\vartheta)\xi,$$

where $\varphi$ and $\psi$ are as in the diagram (3.7). In particular,

$$U d_r(c_k \xi) = \binom{n}{k} K d_r(v^k \xi).$$

**Proof.** Consider the commutative diagram (3.1) at the beginning of this section. The result is immediate upon passing to the diagram of Serre spectral sequences: $\Phi \cdot \Phi$ induces $c_k \mapsto (\binom{n}{k})v^k$ on the leftmost column and the identity on the rightmost one, the latter implying that the induced morphism of spectral sequences is the identity on $H^*(K(\mathbb{Z}, 3); \mathbb{Z})$ when restricted to the bottom rows of the $E_2$-page. \hfill \Box

4. **Auxiliary Results, Proof of Theorem 1.2**

In the statement of Theorem 1.1, the notations $x_1$, $y_{2,1}$ and $y_{3,0}$ are also used to denote elements in the cohomology group $H^*(K(\mathbb{Z}, 3); \mathbb{Z})$. This is on purpose, since as we shall see in the last three sections, $x_1$, $y_{2,1}$ and $y_{3,0} \in H^*(BU_n; \mathbb{Z})$ are the images of their namesakes in $H^*(K(\mathbb{Z}, 3); \mathbb{Z})$, under the homomorphism $H^*(K(\mathbb{Z}, 3); \mathbb{Z}) \to H^*(BU_n; \mathbb{Z})$, which is induced by the map $BU_n \to K(\mathbb{Z}, 3)$ as in the fiber sequence (1.1). We restate this fact as the following

**Corollary 4.1.**

1. When $n$ is even, the 2-torsion subgroup of $H^k(BPU_n; \mathbb{Z})$ for $k = 6, 9, 10$ are generated by the cohomology operations $y_{2,0}, y_{2,0}x_1$, and $y_{2,1}$ respectively, applied to a generator of $H^3(BPU_n; \mathbb{Z})$.

2. When $3|n$, and $k = 8$, the 3-torsion subgroup of $H^8(BPU_n; \mathbb{Z})$ is generated by the cohomology operation $y_{3,0}$ applied to a generator of $H^3(BPU_n; \mathbb{Z})$.

**Proof.** In the statement of Theorem 1.1 we see that for the k’s and n’s in the Corollary 4.1 the corresponding torsion subgroups of $H^k(BPU_n; \mathbb{Z})$ are generated by the images of elements $y_{2,0}, y_{2,0}x_1, y_{2,1}$ and $y_{3,0}$ respectively, under the homomorphism $H^k(K(\mathbb{Z}, 3); \mathbb{Z}) \xrightarrow{\cong} H^k(BPU_n; \mathbb{Z})$, and the proof is completed. \hfill \Box

We proceed to study the $p$-local components of $H^*(BPU_n; \mathbb{Z})$, for any prime number $p$.

**Lemma 4.2.** If $t + 1 < p$, then

$$U E_\infty^{3,2t} \otimes \mathbb{Z}_p \cong U E_3^{3,2t} / \text{Im}(d_3^{1,2t+2}) \otimes \mathbb{Z}_p.$$
Proof. This follows since
\[ U E^s,t_2 \cong H^s(K(\mathbb{Z}, 3); \mathbb{Z}) \otimes H^t(\mathbb{BU}_n; \mathbb{Z}) \]
has no \( p \)-torsion for \( s < 2p + 2 \). \qed

It follows from Lemma 4.2 that we have

**Proposition 4.3.** If \( 3 < k < 2p + 1 \) then \( H^k(\mathbb{BPU}_n; \mathbb{Z}(p)) \)
\[ \cong \begin{cases} 
\text{Ker} U d_3^*, k \text{ is even}, \\
H^{k-3}(\mathbb{BU}_n; \mathbb{Z}(p))/\nabla H^{k-1}(\mathbb{BU}_n; \mathbb{Z}(p)), k \text{ is odd}. 
\end{cases} \]

We proceed to prove a lemma that simplifies the computation of the torsion free summand of \( H^*(\mathbb{BPU}_n; \mathbb{Z}) \). We also present some immediate consequences of the computation apparatus developed in the previous section. At the end of this section we carefully decompose the statement of Theorem 1.1 to make its proof trackable.

**Lemma 4.4.** Let \( \bar{P} : \mathbb{BSU}_n \rightarrow \mathbb{BPU}_n \) be the map induced by the obvious quotient map \( \mathbb{SU}_n \rightarrow \mathbb{PU}_n \).

1. \( \bar{P} \) induces an isomorphism \( H^*(\mathbb{BPU}_n; \mathbb{Q}) \cong H^*(\mathbb{BSU}_n; \mathbb{Q}) \). In particular, \( \bar{P}^* : H^*(\mathbb{BPU}_n; \mathbb{Z}) \cong H^*(\mathbb{BSU}_n; \mathbb{Z}) \) is a monomorphism modulo torsion, such that \( \text{Im} \bar{P}^* \) has the same rank as \( H^*(\mathbb{BPU}_n; \mathbb{Z}) \) in each dimension.

2. Let \( m \) be an integer such that \( \text{gcd}(m, n) = 1 \), then \( H^*(\mathbb{BPU}_n; \mathbb{Z}) \) has no non-trivial \( m \)-torsion.

**Proof.** Let \( \mathbb{SU}_n \) be the special unitary group of degree \( n \). The short exact sequence of Lie groups
\[ 1 \rightarrow \mathbb{Z}/n \rightarrow \mathbb{SU}_n \xrightarrow{P} \mathbb{PU}_n \rightarrow 1 \]
induces a fiber sequence
\[ K(\mathbb{Z}/n, 1) \rightarrow \mathbb{BSU}_n \rightarrow \mathbb{BPU}_n. \]
We shift it to obtain another fiber sequence
\[ \mathbb{BSU}_n \rightarrow \mathbb{BPU}_n \rightarrow K(\mathbb{Z}/n, 2), \]
and consider the cohomological Serre spectral sequence with coefficients in \( \mathbb{Q} \) and \( \mathbb{Z}/m \). The first and second statement follows respectively from the vanishing of \( H^k(K(\mathbb{Z}/n, 2); \mathbb{Q}) \) and \( H^k(K(\mathbb{Z}/n, 2); \mathbb{Z}/m) \), for \( k > 0 \). \qed

**Remark 4.5.** The map \( \bar{P} \) factors as \( \mathbb{BSU}_n \rightarrow \mathbb{BU}_n \xrightarrow{P} \mathbb{BPU}_n \), where \( P \) is induced by the quotient map \( U_n \rightarrow PU_n \). By (1) of Lemma 4.4
\[ P^* : H^*(\mathbb{BPU}_n; \mathbb{Z}) \rightarrow H^*(\mathbb{BU}_n; \mathbb{Z}) \]
is a monomorphism modulo torsions. Furthermore, (1) of Lemma 4.4 shows that the torsion-free component of \( H^k(\mathbb{BPU}_n; \mathbb{Z}) \) is 0 if \( k \) is odd, and is a finitely generated free abelian group of the same rank as the abelian group of homogenous polynomials in \( \mathbb{Z}[c_2, c_3, \cdots, c_n] \) of degree \( k/2 \), if \( k \) is even. Therefore the group structure of the torsion-free component of \( H^*(\mathbb{BPU}_n; \mathbb{Z}) \) is determined, though it doesn’t seem to have a canonical choice of generators. The calculation of the torsion-free components is henceforth omitted unless we need some particular generators. An alternative method to calculate the torsion-free component is via representation theory. Examples can be found in [3] (Antieau, 2016) and [14] (Kameko, 2016).
Lemma 4.6. Let $p$ be a prime, $k, l \in \mathbb{N}$ satisfying $p|k$ and $l < p$. Then we have

$$p \binom{k}{l}.$$

Proof. It follows from direct computation, or set $k_0 = 0$ in Lemma 2.8 (Lucas’ Theorem).

We present the following

Proof of Theorem 1.2. For the absence of torsion-free or $p$-torsion in $U E_2^{s, \ast}$ for $3 < s < 2p + 2$ and obvious degree reasons, the only possibly nontrivial differential landing in $U E_2^{s, \ast}$ is

$$U d_{2p-1}^{3,2p-2} : U E_{2p-1}^{3,2p-2} \to U E_{2p-1}^{2p+2,0} \cong U E_2^{2p+2,0} \cong H^{2p+2}(K(\mathbb{Z}, 3); \mathbb{Z}),$$

where

$$U E_{2p-1}^{3,2p-2} \cong U E_2^{2p+2,0} / \text{Im } U d_{3}^{0,2p} = H^{2p-2}(BU_n; \mathbb{Z}) \otimes x_1 / \text{Im } U d_{3}^{0,2p}.$$

Let $c_1^{p-1} \cdots c_{2p-1}$ be a monomial in $H^{2p-2}(BU_n; \mathbb{Z})$. Then it follows from Corollary 2.17 and Corollary 3.11 that we have

$$U d_{2p-1}^{3,2p-2}(c_1^{p-1} \cdots c_{2p-1} x_1) = \prod_{j=1}^{p-1} \binom{n}{j} y_{p,j},$$

where $y_{p,j} \in H^{2p+2}(K(\mathbb{Z}, 3); \mathbb{Z})$ is of order $p$. It then follows from Lemma 4.6 that we have

Lemma 4.7. If $p|n$, then we have

$$U d_{2p-1}^{3,2p-2} = 0.$$

Therefore the case $p|n$ follows. The case $p \nmid n$ follows immediately from the second statement of Lemma 4.4.

We complete the proof of Theorem 1.2 in the sections 5, 6, and 7. The reader could refer to Figure 2 for the spectral sequence $U E_2^{s, \ast}$. For easier reference, we break the statement of the theorem down to the following list of assertions:

1. $H^k(BPU_n; \mathbb{Z}) = 0$ for $k = 1, 2$. $H^3(BPU_n; \mathbb{Z}) \cong \mathbb{Z}/n$ is generated by $x_1$ of order $n$. $x_1^n$ is of order 2 if $n$ is even and is 0 otherwise.
2. $H^4(BPU_n; \mathbb{Z}) \cong \mathbb{Z}$ is generated by $c_2$, such that $P^s(e_2) = 2nc_2 - (n-1)c_1^2$ when $n$ is even and $P^s(e_2) = nc_2 - \frac{n-1}{2}c_1^2$ when $n$ is odd.
3. $H^5(BPU_n; \mathbb{Z}) = 0$.
4. Let $n > 2$. If $n$ is even, $H^6(BPU_n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ is generated by $e_3$ of order infinity and $x_1^2$ of order 2. If $n$ is odd, $H^6(BPU_n; \mathbb{Z}) \cong \mathbb{Z}$ is generated by $e_3$, and $x_1^2 = 0$. In the exceptional case $n = 2$, the assertion holds with the absence of $e_3$ and its corresponding direct summand $\mathbb{Z}$.
5. If $n = 4l + 2$, $H^7(BPU_n; \mathbb{Z}) \cong \mathbb{Z}/2$ is generated by $e_2x_1$. Otherwise $H^7(BPU_n; \mathbb{Z}) = 0$ and in particular $e_2x_1 = 0$.
6. Let $n \geq 4$. If $n|n$, $H^8(BPU_n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3$ generated by $e_4$ and $e_2^2$ of order infinity and $y_{3,0}$ of order 3. Otherwise $H^8(BPU_n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $e_2^2$ and $e_4$. In the exceptional cases $n = 2, 3$, this assertion holds as well, with $e_4$ and its corresponding direct summand $\mathbb{Z}$ absent.
Figure 2. Some non-trivial differentials of $^U\text{E}_{s,t}^\ast$. A node with coordinate $(s,t)$ is unmarked if $^U\text{E}_{2,t}^s = 0$.

(7) $H^9(BPU_n; \mathbb{Z}) \cong \mathbb{Z}/\epsilon_2(n) \oplus \mathbb{Z}/\epsilon_3(n)$ is generated by $x_1^3$ of order $\epsilon_2(n)$ and $z_1$ of order $\epsilon_3(n)$.

(8) $e_3x_1 = 0$.

(9) If $n \geq 5$, $H^{10}(BPU_n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/\epsilon_2(n) \oplus \mathbb{Z}/\epsilon_3(n)$ is generated by $e_2^3, e_5$ of order infinity, $y_{2,1}$ of order $\epsilon_2(n)$ and $z_2$ of order $\epsilon_3(n)$. In the exceptional cases $n < 5$, the assertion holds, with the absence of monomials involving $e_i$ for $i > n$ and their corresponding direct summands $\mathbb{Z}$.

(10) $e_2x_1^2 = y_{2,1}$ when $n = 4l + 2$, and $e_2x_1^2 = 0$ otherwise.

5. $H^k(BPU_n; \mathbb{Z})$ for $1 \leq k \leq 6$

For $1 \leq r \leq 5$ the results are given by Antieau and Williams ([1], 2014). The interested readers can compare it to our computation.

Proof of (1) to (4). First notice $^U\text{E}_{2,t}^s = 0$ for $s = 1, 2, 4, 5, 7$ or $t$ odd. By Proposition 3.6, $^Ud_3(c_1) = Td_3(\sum_{i=1}^{n} v_i) = nx_1$. This immediately proves (1). See Figure 3 for a picture in low dimensions. By Corollary 5.4 we have

$$^Ud_3(c_2) = (n - 1)c_1x_1, \quad ^Ud_3(c_1^2) = 2nc_1x_1, \quad ^Ud_3(c_1x_1) = nx_1^2.$$

So $^U\text{E}_{4,1}^0 \cong ^U\text{E}_{\infty,1}^0 \cong H^4(BPU_n; \mathbb{Z})$ is easily verified as $\mathbb{Z}$. Interested readers can refer to Lemma 3.2 of [1] (Antieau and Williams, 2014) to identify $H^4(BPU_n; \mathbb{Z})$ as a subgroup of $H^4(BU_n; \mathbb{Z})$, or compute it directly. Either way, we proved (2).

Notice $^U\text{E}_{\infty,6}^0 \cong \mathbb{Z}$, by Remark 4.5.
(1) If $n$ is even, by (5.1), $Ud_3^{0,4}$ is a surjection and therefore $Ud_3^{3,2} = 0$. Hence $H^5(BPU_n; \mathbb{Z}) \cong U^{E_4^{6,0}} = 0$, and

$$H^6(BPU_n; \mathbb{Z}) \cong U^{E_\infty^{0,6}} \oplus U^{E_4^{6,0}} \cong U^{E_\infty^{0,6}} \oplus U^{E_3^{6,0}} \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2, & n > 2, \\ \mathbb{Z}/2, & n = 2. \end{cases}$$

with $U^{E_3^{6,0}}$ generated by $x_1^3$.

(2) If $n$ is odd, again by (5.1), the image of $Ud_3^{0,4}$ is $2c_1x_1$ which happens to be $\text{Ker} Ud_3^{3,2}$. By Proposition 3.3, $Ud_3^{3,2}(c_1x_1) = nx_1^3 = ny_2,0$ since $n$ is odd and $y_2,0$ is of order 2, which implies that $U^{E_4^{6,0}} = 0$. Hence $H^5(BPU_n; \mathbb{Z}) \cong U^{E_4^{3,2}} = 0$, and $H^6(BPU_n; \mathbb{Z}) \cong U^{E_\infty^{0,6}} \oplus U^{E_4^{6,0}} = \mathbb{Z}$.

In both cases we take $e_3 \in H^6(BPU_n; \mathbb{Z})$ generating $U^{E_\infty^{0,6}}$, for $n > 2$. Therefore, (3) and (4) follows. Notice that $e_3$ is determined by this argument modulo torsion. In fact, there is a unique choice of $e_3$ that fits the statement of Theorem 1.1.

This choice will be specified in Section 9, where we discuss the cup products in $H^*(BPU_n; \mathbb{Z})$.

6. $H^k(BPU_n; \mathbb{Z})$ for $k = 7, 8$ 

Proof of (5). Notice that the only bi-degree $(s, t)$ such that $s + t = 7$ and $Ud_3^{s,t}$ is nontrivial is $(3, 4)$. Therefore $H^7(BPU_n; \mathbb{Z}) \cong U^{E_\infty^{3,4}}$. We consider the $p$-local cohomology of $H^*(BPU_n; \mathbb{Z})_{(p)}$ for each prime $p$ separately. We consider the following relevant differentials:

1. $Ud_3^{0,6}: U^{E_3^{0,6}} \to U^{E_3^{3,4}}$

$$c_3 \mapsto (n - 2)c_2x_1, \quad c_1c_2 \mapsto [nc_2 + (n - 1)c_1^2]x_1, \quad c_1^3 \mapsto 3nc_1^2x_1.$$

2. When localized at 2, The only nontrivial differential from $U^{E_3^{3,4}}$ is $Ud_3^{3,4}: U^{E_3^{3,4}} \to U^{E_3^{6,2}}$

$$c_1^2x_1 \mapsto 2nc_1y_2,0 = 0, \quad c_2x_1 \mapsto (n - 1)c_1y_2,0.$$

3. When localized at 3, The only nontrivial differential from $U^{E_3^{3,4}}$ is $Ud_3^{3,4}: U^{E_3^{3,4}} \to U^{E_5^{8,0}}$

$$c_1^2x_1 \mapsto n^2y_3,0, \quad c_2x_1 \mapsto \frac{n(n - 1)}{2}y_3,0.$$

In what follows we compute the localization $H^7(BPU_n; \mathbb{Z})_{(p)}$ for each prime $p$ separately. Everything is implicitly assumed to be localized at the specified prime $p$ in each case.

Case 1. $p = 2$. In this case $U^{E_\infty^{3,4}} = \text{Ker} Ud_3^{3,4}/\text{Im} Ud_3^{0,6}$. By (2) of Lemma 4.4 we only need to consider the case that $n$ is even, in which, assuming $n > 2$ we have $\mathbb{Z}$-basis $\{c_1^2, c_1c_2, c_3\}$ and $\{c_1^2x_1, 2c_2x_1\}$ of $U^{E_3^{0,6}}$ and $\text{Ker} Ud_3^{3,4}$, respectively. The corresponding matrix for $Ud_3^{0,6}$ is

$$(6.1) \quad \begin{bmatrix} 3n & n - 1 & 0 \\ 0 & \frac{n}{2} & \frac{n-2}{2} \end{bmatrix}.$$
We apply invertible row and column operations to it and obtain
\[(6.2) \begin{bmatrix} 1 & -n/2 & 0 \\ -n/2 & 1 & n \end{bmatrix} \begin{bmatrix} 3n & n-1 & 0 & 0 \\ 0 & 1/2 & n/2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3n/2 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & n-1 & 0 \\ n & 0 & n-2 \end{bmatrix},
\]

with the row operation corresponding to the change of basis
\[
\begin{bmatrix} c_1^2 x_1 & 2c_2 x_1 \\ -1/n & 1 \end{bmatrix}^{-1} = \begin{bmatrix} c_1^2 x_1 + n \n-1 & 2c_2 x_1 \n0 & 1 \end{bmatrix}.
\]

By (6.2) and (6.3), \(\text{Im} U_{d_3}^{0,6} \) is generated by \(c_1^2 x_1 + n/2 - nc_2 x_1\) and \(\gcd\{n/2, n-2\} \cdot 2c_2 x_1\). Where \(\gcd\{2, n/2\} = 2\) if \(n = 4l + 2\) for some integer \(l\) and 1 otherwise. Hence \(U_{E_4}^{3,4} \cong U_{E_4}^{3,4} \) is isomorphic to \(\mathbb{Z}/2\) and is generated by \(2c_2 x_1\) if \(n = 4l + 2\), and is 0 otherwise.

In the exceptional case \(n = 2\), (6.1) is reduced to
\[
\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},
\]

and (6.2) is reduced to
\[
\begin{bmatrix} 6 & 1 \\ 0 & 1 \end{bmatrix},
\]

which yields the same argument as above.

The above discussion shows that \(H^7(BPU_n; \mathbb{Z}) \cong \mathbb{Z}/2\) is generated by \(2c_2 x_1\) if \(n = 4l + 2\) for some integer \(l\), and is 0 if \(4n/3\).

By (2), we know that for \(n\) even, \(H^4(BPU_n; \mathbb{Z}) \cong \mathbb{Z}\) is generated by \(e_2\), which is detected by \(2nc_2 - (n-1)c_1^2 \in U_{E_4}^{0,4}\). Furthermore, when \(n = 4l + 2\), in \(H^7(BPU_n; \mathbb{Z}) \cong \mathbb{Z}/2\) we have
\[
2c_2 x_1 = 3(4l + 2)c_2 x_1 = 3nc_2 x_1
\]
\[
= [(n-1)c_1^2 + nc_2]x_1 - [(n-1)c_1^2 - 2nc_2]x_1
\]
\[
= 2nc_2 - (n-1)c_1^2 x_1 \in U_{E_4}^{3,4},
\]

since by (6.2) and (6.3), \([(n-1)c_1^2 + nc_2]x_1\) is in the image of \(U_{d_3}^{0,6}\). Therefore, (6.5) shows that \(2c_2 x_1 = [2nc_2 - (n-1)c_1^2 x_1\) yields a non-trivial product in \(U_{E_4}^{3,4}\) that detects \(e_2 x_1\).

Case 2. \(p = 3\). In this case \(U_{E_3}^{3,4} = \text{Ker} U_{d_3}^{3,4}/\text{Im} U_{d_3}^{0,6}\) and by (2) of Lemma 4.4, we only care about the case \(3|n\). Notice that \(U_{d_3}^{3,4} = 0\) since its target is a 2-torsion group and we are working 3-locally. It follows from Lemma 4.7 that \(\text{Ker} U_{d_3}^{3,4} = U_{E_3}^{3,4}\) has a basis \(\{c_1^2 x_1, c_2 x_1\}\). Again taking the basis \(\{c_1^2, c_1 c_2, c_3\}\) for \(U_{E_3}^{0,6}\), we obtain the matrix
\[
\begin{bmatrix} 3n & n-1 & 0 \\ 0 & n & -n-2 \end{bmatrix}
\]
for $Ud_3^{0.6}$. We apply an invertible column operation to it and obtain

$$
\begin{pmatrix}
3n & n-1 & 0 \\
0 & n & n-2
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & \frac{-n}{n-2} & 1
\end{pmatrix}
= \begin{pmatrix}
3n & n-1 & 0 \\
0 & 0 & n-2
\end{pmatrix}.
$$

Therefore $Ud_3^{0.6}$ is surjective, and $H^7(BPU_n; \mathbb{Z})_{(3)} = 0$ when $3|n$.

Case 3. $p > 3$. If $n > 2$, then $U E_3^{3,4} = U E_3^{3,4}/\text{Im} Ud_3^{0.6}$. We take the basis $\{c_1^2x_1, c_2x_1\}$ for $U E_3^{3,4}$ and the matrix for $U d_3^{0.6}$ is again (6.1). Since $p > 3$, either $3n$ or $n-2$ is invertible, so it is easy to show that $U d_3^{0.6}$ is surjective. So $H^7(BPU_n; \mathbb{Z})_{(p)} = 0$ when $n > 2$. In the exceptional case $n = 2$, the same assertion follows easily from (2) of Lemma 4.4.

Summarizing the three cases above, (5) follows.

**Proof of (6).** It suffices to consider the torsion component. Therefore the relevant entries in $UE_3^{*,*}$ are $UE_2^{3,0}$ and $UE_2^{6,2}$. The relevant differentials are $U d_3^{3,4}$ and $U d_5^{3,4}$, as given in (2) and (3) in the proof for $k = 7$, together with the following one:

$$
U d_3^{6,2} : U E_3^{6,2} \to U E_3^{9,0}, \quad c_1y_{2,0} \mapsto nx_1y_{2,0} = nx_1^3.
$$

Again we consider $H^8(BPU_n; \mathbb{Z})_{(p)}$ for each prime $p$ separately. Remember that in each of the following cases we assume that $UE_3^{*,*}$ is localized at the specified prime $p$.

Case 1. $p = 2$. In this case the torsion component of $H^8(BPU_n; \mathbb{Z})_{(2)}$ is isomorphic to $UE_2^{6,2}/\text{Ker} Ud_3^{6,2}/\text{Im} Ud_3^{3,4}$. And by (2) of Lemma 4.4 we consider only the case that $n$ is even. By (6.8), $\text{Ker} Ud_3^{6,2} = \mathbb{Z}\{c_1y_{2,0}\}$. By the differentials given in (2) in the proof for (5), $\text{Im} Ud_3^{3,4} = \mathbb{Z}\{c_1y_{2,0}\}$. Therefore the torsion of $H^8(BPU_n; \mathbb{Z})_{(2)}$ is 0 when $n$ is even.

Case 2. $p = 3$. In this case the torsion component of $H^8(BPU_n; \mathbb{Z})_{(3)}$ is isomorphic to $UE_2^{8,0}/\text{Im} Ud_3^{3,4}$, where $UE_2^{8,0} \cong \mathbb{Z}/3$ is generated by a single element $y_{3,0}$. By the differentials given in (3) of the proof for $k = 7$, $\text{Im} Ud_3^{3,4} = 0$ when $3|n$. Therefore the torsion component of $H^8(BPU_n; \mathbb{Z})_{(3)}$ is isomorphic to $\mathbb{Z}/3$ and generated by $y_{3,0}$.

For $p > 3$ there is no $p$-torsion in the relevant range of $UE_3^{*,*}$. Hence (6) follows.

**7. $H^k(BPU_n; \mathbb{Z})$ for $k = 9, 10$**

The study of $H^9(BPU_n; \mathbb{Z})$ requires extra work when $n$ is even, since the differential $Ud_3^{0.8}$ cannot be determined by Proposition 3.3. Indeed, the differential $\tau d_3^{6,2} : \tau E_3^{6,2} \to \tau E_3^{9,0}$ is not trivial when $n$ is even. We consider the exceptional isomorphism of Lie groups $PU_2 \cong SO_3$. It is well-known that we have

$$
H^*(BP\mathbb{Z}U_2; \mathbb{Z}/2) \cong H^*(B\mathbb{S}O_3; \mathbb{Z}/2) \cong \mathbb{Z}/2[\omega_2, \omega_3],
$$

where $\omega_2, \omega_3$ are the universal Stiefel-Whitney classes. Furthermore, we have

$$
\text{Sq} \omega_3 = \omega_2 \omega_3.
$$

Indeed, the Adem relations indicate

$$
\text{Sq}^1 \text{Sq}^2(\omega_3) = \text{Sq}^3(\omega_3) = \omega_3^2 \neq 0.
$$
Therefore $\text{Sq}^2(\omega_3) \neq 0$. However, the only nonzero element in $H^5(\text{BPU}_2; \mathbb{Z}/2)$ is $\omega_2\omega_3$, and equation (7.2) follows.

As for the integral cohomology, we have the following result due to E. Brown. For more general cases see [5] (Brown, 1982).

**Proposition 7.1.** $H^*(\text{BPU}_2; \mathbb{Z}) \cong H^*(\text{BSO}_3; \mathbb{Z}) \cong \mathbb{Z}[e_2] \otimes_{\mathbb{Z}} \mathbb{Z}/2[x_1]$, where $e_2$ is of degree 4, and we abuse the notation $x_1$ to let it denote the image of $x_1 \in H^3(K(\mathbb{Z}, 3); \mathbb{Z})$ under the homomorphism induced by the second arrow of the fiber sequence $\text{BU}_2 \to \text{BPU}_2 \to K(\mathbb{Z}, 3)$.

Recall that in Theorem 1.1 we let $e_2$ denote the generator of $H^1(\text{BPU}_n; \mathbb{Z}) \cong \mathbb{Z}$ for all $n > 1$. Part (1) of the following lemma is due to A. Bousfield.

**Lemma 7.2.** For $n > 0$ even, let $\Delta : \text{BU}_2 \to \text{BU}_n$ denote the inclusion of block diagonal matrices, and $\Delta' : \text{BPU}_2 \to \text{BPU}_n$ be its induced homomorphism on quotients. Then we have the following assertions.

1. $(\Delta')^*(x_1) = x_1$. In particular, $x_1^3 \in H^9(\text{BPU}_n; \mathbb{Z})$ is non-zero.
2. $(\Delta')^*(e_2) = (\frac{3}{2})^2 e_2$. In particular, when $n = 4l + 2$ for some integer $l$, $e_2 x_1^2 \in H^{10}(\text{BPU}_n; \mathbb{Z})$ is non-zero.

**Proof.** Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{BS}^1 & \longrightarrow & \text{BU}_2 \\
\Delta & \downarrow & \Delta' \\
\text{BS}^1 & \longrightarrow & \text{BPU}_2 \\
\end{array}
\]

which induces another commutative diagram as follows:

\[
\begin{array}{ccc}
\text{BU}_2 & \longrightarrow & \text{BPU}_2 \\
\Delta & \downarrow & \Delta' \\
\text{BU}_n & \longrightarrow & \text{BPU}_n \\
\end{array}
\]

This diagram induces a homomorphism of Serre spectral sequences, such that its restriction on the bottom row of the $E_2$ pages is the identity. In particular it takes $x_1$ to itself. Moreover, it follows from Proposition 7.1 that $(\Delta')^*(x_1^3) = x_1^3 \in H^9(\text{BPU}_2; \mathbb{Z})$ is non-zero, which completes the proof of (1).

It is well known that the $U_n$-bundle over $\text{BU}_2$ induced by $\Delta$ is the Whitney sum of $\frac{n}{2}$ copies of the universal $U_2$-bundle, of which the total Chern class is

\[(1 + c_1 + c_2)^n = 1 + \frac{n}{2} c_1 + \left(\frac{n}{2} c_2 + \left(\frac{n/2}{2}\right) c_1^2\right) + (\text{terms of higher degrees}).\]

Therefore we have

\[
\Delta^* : H^2(\text{BU}_n; \mathbb{Z}) \to H^2(\text{BU}_2; \mathbb{Z}),
\]

\[
c_1 \mapsto \frac{n}{2} c_1, \quad c_2 \mapsto \frac{n}{2} c_2 + \left(\frac{n/2}{2}\right) c_1^2 = \frac{n}{2} c_2 + \frac{n(n-2)}{8} c_1^2.
\]

In particular,

\[
\Delta^*((n-1)c_1^2 - 2nc_2) = (n-1)\left(\frac{n}{2} c_1^2 - 2n\frac{n}{2} c_2 + \frac{n(n-2)}{8} c_1^2\right) = \left(\frac{n}{2}\right)^2 (c_1^2 - 4c_2).
\]
Notice that when $n = 2$, $P^*(e_2) = c_1^2 - 4c_2$, and the equation above implies $(\Delta')^*(e_2) = (\frac{7}{8})^2 e$. When $n = 4l + 2$, we have $(\Delta')^*(e_2 x_1^2) = (2l + 1)^2 e x_1^2 = e x_1^2 \in H^0(BP U_2; \mathbb{Z})$ which is non-zero by Proposition [1] and (2) follows. □

Proof of (7), (8). The relevant entries in $U E_{2}^{3,6}$ are $U E_{2}^{3,0}$ and $U E_{2}^{0,0}$. We study the localization of $U E_{2}^{*,*}$ at each prime $p$ separately.

Case 1. $p = 2$. Again we only consider the case that $n$ is even. When $n > 2$, with respect to the basis $\{c_1^2 x_1, c_1 c_2 x_1, c_3 x_1\}$ for $U E_{2}^{3,6}$ and $\{c_1^2 y_2, c_2 y_2\}$ for $U E_{2}^{6,4}$, the differential $U d_3^{3,6}: U E_{2}^{3,6} \to U E_{2}^{6,4}$ is represented by the following matrix:

\[(7.5) \begin{pmatrix} 3n & n - 1 & 0 \\ 0 & n & n - 2 \end{pmatrix} \]

Since $y_2$ is of order 2, and that $n$ is even, (7.5) implies that $\text{Ker} U d_3^{3,6}$ has a basis $\{c_1^2 x_1, 2c_1 c_2 x_1, c_3 x_1\}$. With this basis and $\{y_2, 1\}$ as a basis for $U E_{2}^{10,0}$, $U d_7^{3,6}$ is represented by the matrix

\[(7.6) \begin{pmatrix} \frac{n^3}{2} & \frac{n^2(n-1)}{2} & \frac{n(n-1)(n-2)}{12} \end{pmatrix} \]

A closer inspect shows that all three entries are even. Therefore $\text{Ker} U d_3^{3,6} = \text{Ker} U d_7^{3,6} = \{c_1^2 x_1, 2c_1 c_2 x_1, c_3 x_1\}$. With this basis and $\{c_1^2, c_1 c_2, c_1 c_3, c_2, c_4\}$ as a basis for $U E_{2}^{0,8}$, $U d_3^{0,8}: U E_{2}^{0,8} \to \text{Ker} U d_7^{3,6} \subset U E_{7}^{3,6}$ is represented by the following matrix:

\[(7.7) \begin{pmatrix} 4n & n - 1 & 0 & 0 & 0 \\ 0 & n & \frac{n-2}{2} & n - 1 & 0 \\ 0 & 0 & \frac{n}{2} & n & 0 \end{pmatrix} \]

We apply an invertible column operation on it as follows:

\[(7.8) \begin{pmatrix} 4n & n - 1 & 0 & 0 & 0 \\ 0 & n & \frac{n-2}{2} & n - 1 & 0 \\ 0 & 0 & \frac{n}{2} & n & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \]

\[= \begin{pmatrix} 4n & n - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & n - 1 & 0 \\ 0 & 0 & n & 0 & n - 3 \end{pmatrix} \]

Therefore $U d_3^{0,8}: U E_{2}^{0,8} \to \text{Ker} U d_7^{3,6}$ is onto. So $U E_{2}^{3,6} = 0$.

In the exceptional case $n = 2$, we have $c_3, c_4 = 0$, and (7.6), (7.7), are respectively reduced to

\[\begin{pmatrix} 4 & 2 \\ 8 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \]

Therefore, $U d_3^{0,8}: U E_{2}^{0,8} \to \text{Ker} U d_7^{3,6}$ is onto as well. It remains to consider $U d_9^{0,8}: U E_{9}^{0,8} \to U E_{9}^{9,0} \cong U E_{2}^{9,0} = \mathbb{Z}/2\{x_1^3\}$,
Since \( U^3 E_9 \cong \mathbb{Z}/2 \) is generated by \( x_1^3 \), \( U^0 d_9 \) is either surjective or 0, depending on whether \( x_1^3 \) is 0 or not. By Lemma (2), \( x_1^3 \neq 0 \) when \( n \) is even. Therefore \( U^0 d_9 = 0 \).

Summarising all above, when \( n \) is even, \( H^9(\text{BPU}_n; \mathbb{Z})_2 = \mathbb{Z}/2 \) is generated by \( x_1^3 \). We proceed to study the cup products in \( H^9(\text{BPU}_n; \mathbb{Z})_2 \). Since \( U^3 E_6 = 0 \), \( x_1 e_3 = x_1^3 \) or \( x_1 e_3 = 0 \). This merely depends on a choice of \( e_3 \); if the former case is true, then we simply replace \( e_3 \) by \( e_3 + x_1^2 \) to obtain the latter case. Therefore the 2-local case of (8) follows.

Case 2. \( p = 3 \). The only non-trivial target of differentials with domain \( U^3 E_2 \) is \( U^2 E_2 \). By Proposition 3.8 \( U^3 d_3 : U^3 E_2 \to U^2 E_2 \) is trivial since \( r d_3 \) is so. Hence \( U^3 E_\infty \cong U^2 E_3 / \text{Im} U^3 d_3 \). When \( n > 3 \), we take basis \( \{ c^3_1, c^2_1 c^2_2, c^1_3, c^2_2, c^4_3 \} \) for \( U^3 E_3 \) and \( \{ c^3_1 x_1, c^1_2 x_1, c^3_1 x_1 \} \) for \( U^2 E_2 \). Then the matrix representing \( U^3 d_3 \) is the following:

\[
(7.9) \begin{pmatrix}
4n & n - 1 & 0 & 0 & 0 \\
0 & 2n & n - 2 & 2(n - 1) & 0 \\
0 & 0 & n & 0 & n - 3
\end{pmatrix}
\]

We only consider the case \( 3 | n \), in which we apply an invertible column operation to (7.9) and obtain

\[
(7.10) \begin{pmatrix}
4n & n - 1 & 0 & 0 & 0 \\
0 & 2n & n - 2 & 2(n - 1) & 0 \\
0 & 0 & n & 0 & n - 3
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -n & -\frac{n - 2}{2(n - 1)} & 1 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

which shows that for \( n > 3 \) we have \( H^9(\text{BPU}_n; \mathbb{Z})_3 \cong U^3 E_6 \cong U^2 E_2 / \text{Im} U^3 d_3 \cong \mathbb{Z}/3 \), and that it is generated by \( c_3 x_1 \). We denote this cohomology by \( z_1 \).

In the exceptional case \( n = 3 \), the vanishing of \( c_4 \) makes the matrices (7.9) and (7.10) reduce to

\[
\begin{pmatrix}
4n & n - 1 & 0 & 0 \\
0 & 2n & n - 2 & 2(n - 1) \\
0 & 0 & n & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
4n & n - 1 & 0 & 0 \\
0 & 0 & 0 & 2(n - 1) \\
0 & 0 & n & 0
\end{pmatrix},
\]

respectively, from which the same assertion follows easily. Next, we consider the product \( e_3 x_1 \). The matrix (7.5) shows that for \( n > 2 \), \( U^3 E_\infty \) is generated by \( c_3 - \frac{3n}{n - 1} c_1 c_2 + \frac{3n^2}{(n - 1)(n - 2)} c_3 \). On the other hand \( U^3 E_\infty \cong \mathbb{Z}/3 \) is generated by \( c_3 x_1 \), as we just proved. Therefore,

\[
[c_3 - \frac{3n}{n - 1} c_1 c_2 + \frac{3n^2}{(n - 1)(n - 2)} c_3] x_1 = \frac{3n^2}{(n - 1)(n - 2)} c_3 x_1 = 0 \in U^3 E_\infty ,
\]

which proves the 3-local case of (8).

Case 3. \( p > 3 \). (7.7) clearly represents a surjection, even when the 3rd and 5th columns, corresponding to the image of \( c_1 c_3 \) and \( c_4 \), are removed. This completes the proof of (7) and (8). \( \square \)
Proposition 8.1 \cite{Toda87}.

The ring homomorphism
\[ \mu^* : H^* (BU_n; \mathbb{Z}) \to H^* (K(\mathbb{Z}, 2); \mathbb{Z}) \otimes H^* (BU_n; \mathbb{Z}) \]
is determined by
\[ \mu^* (c_k) = \sum_{i=0}^{k} v^i \otimes \binom{n-k+i}{i} c_{k-i}. \]
Recall from Section 1 that the primitive elements forms a subring of \( H^*(BU_n; \mathbb{Z}) \) denoted by \( PH^*(BU_n; \mathbb{Z}) \) which contains \( \text{Im} P^* \). Here \( P : BU_n \to BPU_n \) is the canonical map.

Recall from the Section \([\text{3}]\) that we have a homomorphism
\[
\nabla : H^*(BU_n; \mathbb{Z}) \to H^{*-2}(BU_n; \mathbb{Z})
\]
given by Corollary \([\text{3.3}]\)
\[
(8.1)
\nabla(c_k) = (n - k + 1)c_{k-1}
\]
satisfying the Leibniz rule. Then Proposition \([\text{8.1}]\) implies
\[
\mu^*(x) = 1 \otimes x + v \otimes \nabla(x) + \text{terms with higher orders of } v.
\]
Therefore we have
\[
(8.2) \quad \text{Im } P^* \subset PH^*(BU_n; \mathbb{Z}) \subset \text{Ker } \nabla = \text{Ker } U^*d_{3*}.
\]
This enables us to give the following

**Proof of Theorem \([\text{1.3}]\)** The computation in Section \([\text{5}]\) \([\text{6}]\) and \([\text{7}]\) shows that \( U^*d_{0,t}^r = 0 \) for all \( r > 3 \) and \( t \leq 8 \). It remains to show \( U^*d_{0,t}^r = 0 \) for \( r > 3 \) and \( t = 10, 12 \).

When \( t = 12 \), for obvious degree reasons we only need to check the cases \( r = 9, 11 \) and \( 13 \).

For \( r = 9, 11 \), a routine computation of \( U^*d_{3*}^r \) shows that the sequences
\[
\begin{align*}
U^*E_3^{12} & \xrightarrow{U^*d_{3*}^6} U^*E_3^{12,2} \\
U^*E_3^{14} & \xrightarrow{U^*d_{3*}^8} U^*E_3^{14,0}
\end{align*}
\]
are exact. Therefore we have \( U^*E_4^{9,4} = 0 \) and \( U^*E_4^{11,2} = 0 \). Therefore we have \( U^*d_{3*}^{9,12} = 0 \) and \( U^*d_{3*}^{11,12} = 0 \).

For \( r = 13 \), notice that the group
\[
U^*E_2^{13,0} = H^{13}(K(\mathbb{Z}, 3); \mathbb{Z}) = \mathbb{Z}/2
\]
is generated by \( x_1y_{2,1} \).

For \( n \) odd, \( x_1y_{2,1} \) is in the image of \( U^*d_{3*}^{10,2} \) and there is nothing else to prove.
For \( n \) even, recall the map \( \Delta' : BPU_2 \to BPU_n \) discussed in Section \([\text{7}]\) Let
\[
\rho_2 : H^*(-; \mathbb{Z}) \to H^*(-; \mathbb{Z}/2)
\]
be the canonical reduction. Then it follows from Lemma \([\text{7.11}]\) and equation \([\text{7.11}]\) that we have \( (\Delta')^*\rho_2(x_1) = \omega_3 \) and \( (\Delta')^*\rho_2(y_{2,1}) = (\text{Sq}^2(\omega_3))^2 = \omega_2^2\omega_3^2 \). Therefore we have
\[
(\Delta')^*\rho_2(x_1y_{2,1}) = \omega_2^2\omega_3^2 \neq 0.
\]
Hence there is no nontrivial differential into \( U^*E_*^{13,0} \). In particular, we have \( U^*d_{13}^{0,12} = 0 \).

When \( t = 10 \), the proof is very similar: we only need to check \( U^*d_{13}^{0,12} \) for \( r = 9 \) and \( r = 11 \).

For \( r = 9 \), again we have an exact sequence
\[
U^*E_3^{6,4} \xrightarrow{U^*d_{3*}^6} U^*E_3^{9,2} \xrightarrow{U^*d_{3*}^9} U^*E_3^{12,0}
\]
which yields \( U^*E_4^{9,2} = 0 \). Therefore \( U^*d_{3*}^{0,10} = 0 \).
For \( r = 11 \), the group
\[ U^\infty\infty \cong H^11(K(Z, 3); Z) \cong Z/3 \]
is generated by \( x_1y_{3,0} \).

For \( 3 \nmid n \), one readily verifies \( x_1y_{3,0} \in \text{Im} U_3^{8,2} \) and therefore \( U_4^{11,0} = 0 \). Hence there is nothing more to show.

For \( 3|n \), we have a “diagonal” map
\[ \Delta' : BPU_3 \to BPU_n \]
similar to the one considered in Lemma 7.1. In particular, we have
\[ (8.3) \Delta'^*(x_1) = x_1. \]

(Beware that the \( x_1 \)'s on both sides of the equation are not the same, as they live in different groups, unless \( n = 3 \).) Let
\[ \rho_3 : H^*(--; Z) \to H^*(--; Z/3) \]
be the canonical reduction map, and \( \mathcal{P}^i \) denote the \( i \)-th Steenrod power operation for \( p = 3 \). Finally let \( B \) be the Bockstein homomorphism. Then it is easy to deduce the following from Proposition 2.15:
\[ \rho_3(y_{3,0}) = B^1\mathcal{P}^1(\rho_3(x_1)). \]

It follows from Theorem 4.11 of [16] (Kono, Mimura and Shimada, 1975) that we have \( \rho_3(x_1)B\mathcal{P}^1(\rho_3(x_1)) \neq 0 \) for \( n = 3 \). Equation (8.3) shows that this is the case for all \( n \) such that \( 3|n \). In other words, we have \( U_3^{0,10} = 0 \). We have shown \( \text{Ker} U_3^{0,t} \subset \text{Im} P^* \) for \( t \leq 12 \). The theorem then follows from Proposition 8.2.

In fact, the assertion \( PH^*(BU_n; Z) \subset \text{Ker} \nabla \) may be improved:

**Proposition 8.2.** As subrings of \( H^*(BU_n; Z) \), the following holds:
\[ PH^*(BU_n; Z) = \text{Ker} \nabla. \]

**Proof.** It is straightforward to verify
\[ \mu^*(c_k) = \sum_{i=0}^k v^i \otimes \binom{n-k+i}{i} c_{k-i} = \sum_{i=0}^k v^i \otimes \frac{\nabla^i}{i!} c_k = \exp(\nabla)(1 \otimes c_k), \]
where
\[ \nabla : H^*(K(Z, 2); Z) \otimes H^*(BU_n; Z) \to H^*(K(Z, 2); Z) \otimes H^*(BU_n; Z) \]
is a homomorphism of abelian groups determined by
\[ \begin{cases} \nabla(1 \otimes c_k) = v \otimes \nabla(c_{k-1}), \\ \nabla(v \otimes 1) = 0, \end{cases} \]
together with the Leibniz rule. It is formal to verify that \( \exp(\nabla) \) is a ring homomorphism. Therefore we have \( \mu^* = \exp(\nabla) \). For any homogeneous element \( x \in H^*(BU_n; Z) \), clearly we have \( \nabla(1 \otimes x) = 0 \) if and only if \( \nabla(x) = 0 \), and the proposition follows. \( \square \)
We proceed to offer a way to study $U_{E}^{∗,∗}$ by using known results on primitive elements. It follows from Corollary 4.3 and Section 4.5 of [21] (Toda, 1987) that we have

$$\text{(8.4)} \quad \text{Im} P^{∗} = PH(BU_{n}; \mathbb{Z}/2)$$

for $n ≡ 2 \pmod{4}$ and $n = 4$.

Let $U_{E}^{∗,∗}(\mathbb{Z}/2)$ be the spectral sequence $U_{E}^{∗,∗}$ with coefficient ring $\mathbb{Z}$ replaced by $\mathbb{Z}/2$. Then we have the following

**Lemma 8.3.** For $t > 0$, the entry $U_{E}^{9,t}(\mathbb{Z}/2)$ is the target of a nontrivial differential if and only if $U_{E}^{9,t}$ is.

**Proof.** Recall that $H^{∗}(K(\mathbb{Z}, 3); \mathbb{Z}/2)$ is the polynomial algebra generated by the elements $Sq^{I}(\rho_{2}(x_{1}))$ where $x_{1}$ is the fundamental class and $I$ an admissible sequence of excess $< 3$. (See Chapter 3 and 9 of [19], Mosher and Tangora, 1968, for details.) Moreover, the subalgebra $\text{Im} \rho_{2}$ is generated by $\rho_{2}(x_{1})$ and $[Sq^{I}(\rho_{2}(x_{1}))]^{2}$ where $I \neq \phi$. Then in degree less than 9, the only elements of $H^{∗}(K(\mathbb{Z}, 3); \mathbb{Z}/2)$ not in $\text{Im} \rho_{2}$ are $Sq^{2}(\rho_{2}(x_{1}))$ and $\rho_{2}(x_{1}) Sq^{2}(\rho_{2}(x_{1}))$, of degree 5 and 8 respectively. The lemma then follows for obvious degree reasons. □

We have the following proposition regarding the differentials of $U_{E}^{∗,∗}$

**Proposition 8.4.** For $n ≡ 2 \pmod{4}$ and $n = 4$, $U_{E}^{9,t} = 0$ for all $t > 0$.

**Proof.** It follows from ((8.4), Toda, 1987) that this is true in the spectral sequence $U_{E}^{9,t}(\mathbb{Z}/2)$. The proposition then follows from Lemma 8.3. □

**Appendix A. Multiplicative Constructions and Twisted Tensor Products**

This appendix is a review of part Séminaire Henri Cartan required for Section 2 with twisted tensor products ([5], Brown, 1959) as an example. The author makes no claim of originality to materials in this appendix. As in the introduction, all DGA’s involved are graded-commutative and augmented over the base ring $R$, which is either $\mathbb{Z}$ or $\mathbb{Z}/p$ for some prime $p$.

**Definition A.1.** A construction is a triple $(A, N, M)$ where $A$ is a DGA and $N$ is a DG module, both with augmentations over $R$, and $M$ is a chain complex with a graded algebra structure, satisfying the following conditions:

1. As a graded algebra (not necessarily as a chain complex), we have $M = A \otimes_{R} N$.
2. The differential of $N$ is determined by that of $M$, via the relation $N = R \otimes_{A} M$ where $A$ acts on $R$ by augmentation.

An acyclic construction is one such that $M$ is acyclic. When $N$ is also a DGA, we say that $(A, N, M)$ is a multiplicative construction.

**Remark A.2.** $A$ (resp. $N$) is a sub-DGA of $M$ via $a \mapsto a \otimes 1$ (resp. $n \mapsto 1 \otimes n$). We will use this fact implicitly.

**Example A.3.** Let $A$ be a DGA, and $\epsilon$ be its augmentation. Let $\overline{A} = \text{Ker} \epsilon$. The bar construction of $A$ is a multiplicative construction $(A, \overline{A}, \mathcal{B}(A))$ where the DGA’s $\mathcal{B}(A)$ and $\overline{A}$ are defined as follows:
can iterate this procedure to construct inductively a chain homotopy between the identity of $\mathcal{B}(A)$ is denoted by

$$a[a_1] \cdots [a_k]$$

and

$$1 \cdot [a_1] \cdots [a_k] = [a_1] \cdots [a_k].$$

(2) The degree is defined by

$$\deg(a[a_1] \cdots [a_k]) = k + \deg(a) + \deg(a_1) + \cdots + \deg(k)$$

and the length of $a[a_1] \cdots [a_k]$ is defined to be $\deg([a_1] \cdots [a_k])$. We define a bi-degree on $\mathcal{B}(A)$ by saying that $a[a_1] \cdots [a_k]$ has bi-degree $(s,t)$ if $\deg(a) = s$, $\deg([a_1] \cdots [a_k]) = t$. The degree of $a[a_1] \cdots [a_k]$ is obviously $s + t$. Let $\mathcal{F}_B$ be the filtration induced by the first entry $s$ of this bi-degree.

(3) A chain map $s : \mathcal{B}(A) \to \mathcal{B}(A)$ of degree one is defined as follows:

$$s(a[a_1] \cdots [a_k]) = [a[a_1] \cdots [a_k]]s(1) = 0$$

(4) The differential of $\mathcal{B}(A)$ is defined by induction on $k$ as follows:

$$\begin{cases} d([a]) = a \cdot 1 - [d(a)] - \eta(a) \cdot 1 \\ d([a_1] \cdots [a_k]) = a_1[a_2] \cdots [a_k] - sd(a_2 \cdots [a_k]), k \geq 2 \end{cases}$$

and the differential of $\mathcal{B}(A)$ is induced by the above.

(5) $\mathcal{B}(A)$ has a product structure induced by that of $A$, sometimes called the shuffle product, which makes $\mathcal{B}(A)$ a DGA with respect to the differential defined above.

It is obvious and proved in Exp.3 of [6] (Cartan and Serre, 1954-1955) that $s$ is a chain homotopy between the identity of $\mathcal{B}(A)$ and the augmentation $\eta$. We can iterate this procedure to construct inductively $\mathcal{B}^{n+1}(A) = \mathcal{B}(\mathcal{B}^n(A))$ and $\mathcal{B}^{n+1}(A) = \mathcal{B}(\mathcal{B}^n(A))$. In the case $A = R[\Pi]$(concentrated in degree 0) where $\Pi$ is an abelian group, we have $H_*\mathcal{B}(\mathcal{B}(R[\Pi])) \cong H_*(K(\Pi, n); R)$. This is shown in, for example, [6] (Eilenberg and Mac Lane, 1953) and [13] (Milgram, 1967).

**Example A.4.** [5, Brown, 1959] Fix a base ring $R$. Let $K$ be a DG coalgebra and $A$ a DGA. In [5], Brown defined the notion of a twisted cochain, i.e., a cochain $\varphi \in C^\ast(K; A)$ satisfying certain axioms. Now given a differential graded $A$-module $L$, the twisted tensor product $K \varphi \otimes L$ is a chain complex, of which the underlying $R$-module is that of the graded algebra $K \otimes L$. Then $(A, K, K \varphi \otimes A)$ is a multiplicative construction.

The differential of $K \varphi \otimes L$ admits a filtration $G$ as follows:

$$G_s(K \varphi \otimes L) = \sum_{i=0}^s (K_s \otimes L),$$

which induces a spectral sequence $E^s_{s,t}$ such that

$$E^2_{s,t} \cong H_s(K; H_t(L)).$$
For a map of pointed spaces $\pi : X \to B$ admitting a weakly transitive function ($\pi$ a Serre fibration, for example), let $F = \pi^{-1}(b_0)$, where $b_0$ is the base point of $B$. Brown defined the loop space $\Omega B$ differently from the convention, such that it is homotopy equivalent to the conventional one, but the composition of loops yields an associative product, instead of that in the conventional case where the product is associative only up to homotopy. Therefore, the based singular complex $S_*(\Omega B)$ is a DGA, and $S_*(F)$ is a $S_*(\Omega B)$ module. Let $S_*^{(1)}(B)$ be the first Eilenberg sub-complex of $B$ (Eilenberg, 1944), i.e., the sub-complex of $S_*(B)$ of singular simplices sending all vertices of a standard simplex to $b_0$. Finally let $S_*^E(X)$ be the sub-complex of $S_*(X)$ of singular simplices sending all vertices of a standard simplex to $F$. Then we have the following

**Proposition A.5.** [E. H. Brown, Jr, (4.4) Corollary and (7.2) Corollary of [5], Brown, 1959] There is a twisted cochain $\Phi_B \in C^*(S_*^{(1)}(B), S_*(\Omega B))$ such that there is a chain homotopy equivalence

$$S_*^{(1)}(B)\Phi_B \otimes S_*(F) \to S_*^E(X).$$

Moreover, the spectral sequence induced by the filtration $G$ is isomorphic to the Serre spectral sequence of the fibration $\pi : X \to B$.

We proceed to introduce the operations on an acyclic multiplicative construction $(A,N,M)$ with base ring $R$, following Section 6 to 12 of [6] (Cartan and Serre, 1954-1955). Let $d_A, d_N, d_M$ be the differentials of $A, N$ and $M$ respectively. Furthermore, we assume that there are augmentations $\epsilon_A, \epsilon_N, \epsilon_M$ from $A, N, M$ respectively, to the base ring $R$. The subscripts will be omitted whenever there is no risk of ambiguity.

**Definition A.6.** Assume that the homomorphism $A \to M : a \mapsto a \otimes 1$ is injective. Let $\alpha \in H_k(A)$ be represented by $a \in \text{Ker } d_A$. Since $M$ is acyclic, there is some $x \in M$ such that $d_M(x) = a$. Passing to $N$, we obtain an element

$$\bar{x} = 1 \otimes x \in A \otimes_R M \cong N,$$

where $\bar{x}$ is easily verified to be a cycle in $N$. The homology class $\{\bar{x}\} \in H_{k+1}(N)$ is therefore called the suspension of the homology class $\alpha \in H_k(A)$, denoted by $\sigma(\alpha)$. It is easy to verify that $[\bar{x}]$ is independent of the choice of $a$ or $x$ and $\sigma : H_k(A) \to H_{k+1}(N)$ is a well defined homomorphism of graded abelian groups of degree 1.

**Example A.7.** If $(A,N,M) = (A,\overline{\mathcal{B}}(A),\mathcal{B}(A))$, then the suspension can be realized by the homomorphism $A \to \mathcal{B}(A), a \mapsto [a]$. Notice that the presence of the bracket lifts the degree by 1.

**Definition A.8.** Let the character of the base ring $R$ be a prime number $p$. Define the following $R$-submodule of of $A_{2q}$ by

$$pA_{2q} = \{a \in A_{2q} | d_A(a) = 0, a^p = (\epsilon(a))^p\}.$$

Take $x \in M_{2q+1}$ such that $d_M(x) = a - \epsilon(a)$. Then we have $d_M((a - \epsilon(a))^{p-1}x) = (a - \epsilon(a))^p = 0$. Now take $y \in M_{2pq+2}$ such that $d(y) = (a - \epsilon(a))^{p-1}x$. Passing to $N$ and taking homology we define the transposition $\psi_p(a) = y \in H_{2pq+2}(N)$.

Notice that $\psi_p : pA_{2q} \to H_{2pq+2}(N)$ is not necessarily a homomorphism, even of abelian groups, and does not necessarily pass to homology. However we have the following
Proposition A.9. \( (H.\ Cartan,\ Proposition\ 5,\ exp.\ 6,\ [6],\ Cartan\ and\ Serre,\ 1954-1955)\) Suppose we have

(1) \( a^p = 0, \) for all \( a \in A_{2q}, \) and
(2) \( b \cdot d_A(b^{p-1}) \) is in the image of \( d_A, \) for all \( b \in A_{2q+1}. \)

Then \( \psi_p \) passes to homology to define a map \( \varphi_p : H_{2q}(A) \to H_{2pq+2}(N). \)

This \( \psi_p \) is again not necessarily additive. But in the case that interests us, we have the following

Proposition A.10. \( (H.\ Cartan,\ Theorem\ 3,\ exp.\ 6,\ [6],\ Cartan\ and\ Serre,\ 1954-1955)\) Let \( A \) be a commutative \( R \)-algebra, regarded as a graded-commutative DGA concentrated in degree 0 over a base ring \( R \) of characteristic \( p \) where \( p \) is a prime. Then \( \psi_p : pA \to H_2(%(B(A)) \) is additive when \( p \) is odd. For all \( n \geq 1 \) and \( q \geq 1, \) the transposant \( \varphi_p : H_{2q}(%(B(A)) \to H_{2pq+2}(%(B(A)) \) induced by \( \psi_p \) is well defined, and if \( p \) is odd, it is additive with kernel containing all the decomposable elements of \( H_{2q}(%(B(A)) \).

Definition A.11. For \( A \) graded-commutative, a divided power operation on \( A \) is a collection of maps \( \gamma_k : A \to A \) for all integers \( k \geq 0, \) such that for any \( x, y \in A \) we have the following axioms:

(1) \( \gamma_0(a) = 1, \gamma_1(a) = a, \deg \gamma_k(a) = k \deg a. \)
(2) \( \gamma_k(x)\gamma_l(x) = \binom{k+l}{k}\gamma_{k+l}(x). \)
(3) (Leibniz rule) \( \gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y). \)
(4) \( \gamma_k(xy) = \begin{cases} 0, \text{deg}(x), \text{deg}(y) \text{ are odd, } k \geq 2, \\ x^k\gamma_k(y), \text{deg}(x), \text{deg}(y) \text{ are even, } \text{deg}(y) \geq 2. \end{cases} \)

When the characteristic of \( R \) is 2, we have in addition, for \( k \geq 2, \)

\( \gamma_k(xy) = \begin{cases} 0, \text{deg}(x), \text{deg}(y) > 0, \\ x^k\gamma_k(y), \text{deg}(x) = 0. \end{cases} \)

(5) \( \gamma_k(\gamma_l(x)) = \binom{2l-1}{l-1} \binom{2l-2}{l-2} \cdots \binom{l+1}{1} \gamma_{2k+l}(x). \) If in addition, \( A \) is a DGA with differential \( d \), then we require
(6) \( d(\gamma_k(x)) = \gamma_{k-1}(x)d(x) \) for \( k \geq 1. \)

A graded-commutative algebra with a divided power operation is called a divided power algebra. A map of divided power algebras is a homomorphism of graded algebras compatible with the divided power operation.

Remark A.12. We do not require \( A \) to have a differential, since we often wish to define a divided power operator on the homology of a DGA, rather than the DGA itself.

Example A.13. The prototype of a divided power algebra is \( P_R(y), \) which, as a graded \( R \)-algebra, is generated by element \( \gamma_k(y) \) for all \( k \geq 1 \) modulo the relations imposed by definition A.11. Here \( y \) is of degree 2. By (4) of definition A.11 we have \( k!\gamma_k(y) = y^k. \) In fact, when \( R \) is torsion-free, \( P_R(y) \) is isomorphic to the polynomial algebra \( R[y] \) adjoining all \( \gamma_k(y). \)

On the other hand, if \( R = \mathbb{Z}/p \) where \( p \) is a prime number, then (4) of definition A.11 implies that \( y^p = p!\gamma_p(y) = 0. \) Furthermore, for \( k = k_0 + k_1p + k_2p^2 + \cdots + k_rp^r, \)
where \(0 \leq k_i < p - 1, i = 0, 1, \cdots, r\), we have \(\gamma_k(y) = \prod_{0 \leq i < r} \gamma_i(\gamma_{p^i}(y)) = \gamma_{p^r}(y)/k_i!\). In fact, as a graded \(\mathbb{Z}/p\)-algebra

\[
(A.3) \quad P_{a/p}(y) \cong \bigotimes_{k \geq 0} \mathbb{Z}/p[\gamma_{p^k}(y)].
\]

A detailed discussion on divided power algebras over \(\mathbb{Z}/p\), including the proofs of the statements above, can be found in section 7, exp. 7 of \[6\] (Cartan and Serre, 1954-1955).

**Example A.14.**

1. Let \(M\) be a free graded \(R\)-module generated by elements in odd degrees. Then we can form the free exterior \(E(M)\) over a given set of generators and take the trivial divided power operations such that \(\gamma_0\) is the constant map on to \(R\), \(\gamma_1\) is the identity, and \(\gamma_k(x) = 0\) for \(k > 1\). This is considered usually when \(R\) is a field of character other than 2.

2. Let \(M\) be a free graded \(R\)-module generated by a set \(\{e_i\}\) of elements. If \(R\) has characteristic other than 2, we require \(e_i\) to have even degrees. Then we can form the the universal symmetric algebra \(S(M)\) which has the underlying graded \(R\)-module \(\sum_{i \geq 0} M^{\otimes i}\), with \(M^{\otimes 0} = R\) in degree 0, and with a structure of DGA such that for any divided power algebra \(A\), an \(R\)-linear map \(M \to A\) can be extended uniquely to a map of divided power algebra \(S(M) \to A\).

   The product on \(S(M)\) is shuffle product similar to that of the reduced bar construction. In particular \(e_i^k = k!e_i \otimes \cdots \otimes e_i\) with \(k\) copies of \(e_i\)'s on the righthand side of the equation. There is a power operation \(\{\gamma_k\}_k\) on \(S(M)\) determined by \(\gamma_k(e_i) = e_i \otimes \cdots \otimes e_i\) with \(k\) copies of \(e_i\) and the axioms in Definition A.11.

3. Let \(A\) be a DGA and \(a \in A\). Then \([a] \in \mathfrak{H}(A)\), and the shuffle product \([a]^k = k![a] \cdots [a]\) with \(k\) copies of \(a\) on the righthand side of the equation. We can define a divided power operation on \(\mathfrak{H}(A)\) similar to that of \(S(M)\), by requiring \(\gamma_k([a]) = [a] \cdots [a]\) with \(k\) copies of \(a\) in the bracket. For more details see section 4, exp. 8 of \[6\] (Cartan and Serre, 1954-1955).

**Example A.15.** Let \(A\) and \(A'\) be graded-commutative DGA’s with divided power operations. Then (3) and (4) of Definition A.11 gives a unique way to extend the divided power operations to \(A \otimes A'\), a DGA with differential determined by those of \(A\) and \(A'\) via the Leibniz rule. For details see Theorem 2, exp. 7 of \[6\] (Cartan and Serre, 1954-1955).

Let \(M\) be a free graded \(R\)-module and write \(M = M_+ \oplus M_-\), where \(M_+\) (resp. \(M_-\)) is the direct sum of \(M_k\)'s for even (resp. odd) \(k\). Let

\[
U(M) = \begin{cases} S(M), & R \text{ has character } 2, \\ S(M_+) \otimes E(M_-), & \text{otherwise}. \end{cases}
\]

Then in the sense of Example A.14 and Example A.15 \(U(M)\) is a graded-commutative algebra with a divided power operation. We call \(U(M)\) the universal algebra of \(M\) in the sense of the following
**Theorem A.16** (H. Cartan, Theorem 2, exp. 8, [6], Cartan and Serre, 1954-1955). Let $A$ be a graded-commutative algebra with a divided power operation as in Definition A.17, and $M$ a free graded module. Then any homomorphism of graded $R$-modules $f: M \to A$ can be extended uniquely to a map of divided power algebras $U(M) \to A$.

In the case $p = 2$ we have the following relation of the three operations:

**Proposition A.17** (Proposition 1, exp. 8, [6], Cartan and Serre, 1954-1955). For $q \geq 1$, and $A$ a graded-commutative DGA, we have

$$\varphi_2 = \gamma_2 \cdot \sigma : H_{2q}(A) \to H_{4q+2}(\mathcal{B}(A)).$$

The three operations, the suspension, the transpotence, and the divided power operation, as described above, are enough to describe the homology of $K(\Pi, n)$ with coefficients in $\mathbb{Z}$ or $\mathbb{Z}/p$, at least when $\Pi$ is a finitely generated abelian group. We start with the definitions of words and their heights and degrees.

**Definition A.18.** We give parallel definitions in the cases when $p$ is odd and $p = 2$.

1. By a word we mean a sequence consists of the three symbols $\sigma, \varphi_p,$ and $\gamma_p$ when $p$ is odd, or $\sigma, \varphi_2$ when $p = 2$, where repetition is allowed. The height of a word $\alpha$ is the total number of $\sigma$ and $\varphi_p$ in $\alpha$, counting repetition. We take the degree of the empty word to be 0 and inductively define the degree of the words $\sigma \alpha, \varphi_p \alpha$ and $\gamma_p \alpha$ as follows:

$$\begin{cases}
\deg(\sigma \alpha) = \deg(\alpha) + 2, \\
\deg(\varphi_p \alpha) = 2p \deg(\alpha) + 2, \\
\deg(\gamma_p \alpha) = p \deg(\alpha).
\end{cases}$$

(A.4)

2. When $p$ is odd, an admissible word, is a word $\alpha$ such that (i) $\alpha$ is non-empty and starts and ends with $\sigma$ or $\varphi_p$, and (ii) for every $\varphi_p$ and $\gamma_p$ appeared in $\alpha$, there are even number of copies of $\sigma$ on its righthand side. An admissible word is of type 1 if it ends with $\sigma$, and type 2 if it ends with $\varphi_p$.

3. When $p = 2$, an admissible word, is a word $\alpha$ that starts and ends with $\sigma$. $\alpha$ is of type 2 if it ends with $\gamma_2 \sigma$, and of type 1 otherwise.

The words consisting of $\sigma, \varphi_p,$ and $\gamma_p$ are also called $p$-words.

A word can be regarded as the compose of the sequence of operations $\sigma, \varphi_p$, and $\gamma_p$, in the obvious manner. Let $\mathbb{Z}[\Pi]$ be the group ring of a finitely generated abelian group $\Pi$, viewed as a DGA concentrated in degree 0 and with a trivial differential. Then $H_*(\mathbb{Z}[\Pi]) \cong \mathbb{Z}[\Pi]$. The alert reader will find that, an admissible word $\alpha$ is a well defined compose of operations on $\mathbb{Z}[\Pi]$, with image in $H_{\deg(\alpha)}(\mathcal{B}(\mathbb{Z}[\Pi])) \cong H_{\deg(\alpha)}(K(\Pi, n))$, where $n$ is the height of $\alpha$. In fact all homology classes are generated this way, as we will see soon.

**Definition A.19.** Let $\Pi$ be a finitely generated abelian group, and let $p\Pi$ be the subgroup of $\Pi$ of elements of order infinity or a power of $p$. We write $\Pi/(p\Pi) = \Pi_1, \Pi_2'$ and $p\Pi = \Pi_1^p, \Pi_2'$ as the decomposition of $\Pi/(p\Pi)$ and $p\Pi$ into direct products of cyclic groups of order infinity or a power of $p$.

Fix a positive integer $n$. Let $M^{(n)}$ be the free graded $R$-module generated by $\alpha_i$ of degree $\deg(\alpha)$ for every admissible word $\alpha$ of type 1 with height $n$ and $\Pi'$, and $\alpha'_j$ of degree $\deg(\alpha')$ for every admissible word $\alpha'$ with height $n$ of type 2. Let $U(M^{(n)})$ be as in Proposition A.16. Notice in particular that $U(M^{(0)}) \cong \mathbb{Z}/p[\Pi]$. 

In the following theorem we do not distinguish a word and the compose of operations that it represents.

**Theorem A.20** (H. Cartan, Théorème fondamental, exp. 9, [6], Cartan and Serre, 1954-1955). With the notations as above, let \( w'_j, w''_j \) be generators of \( \Pi'_i, \Pi''_i \) respectively. Take \( R = \mathbb{Z}/p \), where \( p \) is an odd prime number. Let \( f^{(n)} : M^{(n)} \to H_* (K(\Pi, n); \mathbb{Z}/p) \) be the homomorphism of \( \mathbb{Z}/p \)-modules taking \( \alpha_i \) (resp. \( \alpha'_j \)) to \( \alpha(w'_j) \) (resp. \( \alpha(w''_j) \)). Its unique extension to \( U(M^{(n)}) \), \( \tilde{f}^{(n)} \) given by Proposition A.16, is an isomorphism of divided power algebras.

We will give a sketch of proof of Theorem A.20 since the idea is relevant to our application. To do so we need the following theorems.

**Theorem A.21** (H. Cartan, Theorem 2, exp.2, [6], Cartan and Serre, 1954-1955). Let \( f : A \to A' \) be a morphism of DGA’s over \( R \). Let \( M \) (resp. \( M' \)) be an acyclic chain complex over \( R \) with a graded \( A \) (resp.\( A' \))-module structure. Let \( I \) (resp. \( I' \)) be the kernel of the augmentation of \( A \) (resp. \( A' \)). Then there is a morphism of chain complexes \( g : M/IM \to M'/IM' \) compatible with \( f \) in the obvious sense. The induced morphism \( H_* (g) \) is independent of the choice of \( g \). Moreover, if \( f \) is a weak equivalence, then so is \( g \).

**Theorem A.22** (H. Cartan, Theorem 5, exp.4, [6], Cartan and Serre, 1954-1955). Let \( (A, N, M) \) and \( (A', N', M') \) be two multiplicative constructions. Let \( N' \) be an \( R \)-subalgebra of \( M \) containing \( N \) such that \( d : N'_k \to \text{Ker}(d_k) \) is a degree-wise isomorphism of \( R \)-modules for all \( k \geq 0 \). Here \( d_0 = \epsilon \). In particular, \( M' \) is acyclic.

Let \( f : A \to A' \) be a map of DGA’s. Then there is a unique map \( g : M \to M' \) of DGA’s restricting to \( f \), such that \( g(N) \subset N' \).

**Sketch of proof of Theorem A.20** By Künneth formula it suffice to consider the case that \( \Pi \) is a cyclic group with a generator \( w \). We proceed to show that there is a multiplicative construction

\[(U(M^{(n)}), U(M^{(n+1)}), L)\]

with \( L \) acyclic.

One can easily show that an admissible word \( \alpha \) of height \( n + 1 \) is of the form

1. \( \alpha \alpha' \) where \( \alpha' \) is of height \( n \) and odd degree, or
2. \( \sigma \gamma \gamma' \alpha' \) where \( k \geq 0 \) and \( \alpha' \) is of height \( n \) and even degree.

The base case where \( n = 0 \) is easy. Let \( L = U(M^{(n)}) \otimes_{\mathbb{Z}/p} U(M^{(n+1)}) \) as a graded \( \mathbb{Z}/p \)-algebra, and consider \( U(M^{(n)}) \) and \( U(M^{(n+1)}) \) as its subalgebras in the obvious manner.

In the first case as above, let \( x = \alpha'(w) \in U(M^{(n)}) \). Then the free exterior algebra \( E_{\mathbb{Z}/p}(x) \) is a subalgebra of \( U(M^{(n)}) \). Let \( y = \alpha(w) = \sigma \alpha(w) \in U(M^{(n+1)}) \), and we have the subalgebra \( P_{\mathbb{Z}/p}(y) \) of \( U(M^{(n+1)}) \). Define the differentials of \( x \) and \( y \) in \( L \) by \( d_L(x) = 0 \) and \( d_L(y) = x \) together with the axioms in Definition A.11. Then the \( E_{\mathbb{Z}/p}(x) \otimes P_{\mathbb{Z}/p}(y) \) is acyclic.

In the second case, let \( x = \alpha'(w) \in U(M^{(n)}) \), \( y_k = \sigma \gamma \gamma_k \) and \( z_k = \phi \gamma_k \) for all \( k \geq 0 \). Define their differentials in \( L \) by \( d_L(x) = 0 \), \( d_L(y_k) = x \) and \( d_L(z_k) = x^{p-1} y_k \). Then one can show that

\[P_{\mathbb{Z}/p}(x) \otimes \bigotimes_{k \geq 0} E_{\mathbb{Z}/p}(y_k) \otimes \bigotimes_{k \geq 0} P_{\mathbb{Z}/p}(z_k)\]
is acyclic.

By Theorem A.21 and Theorem A.22 one can inductively prove the statement, the base case where \( n = 1 \) being standard homological algebra. \( \square \)

Remark A.23. This argument fails for \( p = 2 \), in which case \( \varphi \) is not additive.

For a free graded \( R \)-module \( M \), recall the graded \( R \)-algebra \( S(M) \) introduced in Example A.14 (2). We take \( M^{(n)} \) as in Definition A.19 and \( f^{(n)} \) as in Theorem A.20. Notice that the constructions apply to \( p = 2 \). The analog of Theorem A.20 in the case where \( p = 2 \) is the following:

**Theorem A.24** (H. Cartan, Théorème fondamental, exp. 9, [6], Cartan and Serre, 1954-1955). \( f^{(n)} : M^{(n)} \rightarrow H_*(K(\Pi, n);\mathbb{Z}/2) \) extends to \( f^{(n)} : S(M^{(n)}) \rightarrow H_*(K(\Pi, n);\mathbb{Z}/2) \) which is an isomorphism of divided power algebras.

The proof is similar to that of Theorem A.20.

For the following corollary, recall the notations from Theorem A.5. In particular, we may regard the based singular chain complex \( S_*(K(\Pi, n); R) \) as a DGA over \( R \). Also recall the multiplicative construction (A.5) and let \( F_U \) be the increasing filtration on \( L \) given by

\[
F_U(L) = \sum_{s \geq 0, k \leq \ell} U(M^{(n)})_s \otimes_{\mathbb{Z}/p} U(M^{(n+1)})_k.
\]

**Corollary A.25.** For \( n \geq 1 \) and each prime \( p \), the filtration \( F_U \) induces a spectral sequence isomorphic to the Serre spectral sequence associated to the path fibration

\[
K(\Pi, n) \rightarrow PK(\Pi, n + 1) \rightarrow K(\Pi, n + 1).
\]

**Proof.** In Section 10 of [4] (Brown, 1959), Brown defined a map of DG modules

\[
\varphi^B : S_*(\Omega Y) \rightarrow B_*(\Omega Y)
\]

for a paracompact space \( Y \), which is also a weak equivalence. The curious readers may easily check that when \( Y = \Omega X \) for some paracompact space \( X \), \( \varphi^B \) is a map of DGAs. Therefore, for \( Y = K(\Pi, n) \), we have a multiplicative construction \((\tilde{B}_*(K(\Pi, n - 1))_s \otimes \tilde{B}_*(K(\Pi, n))_k) \rightarrow \tilde{B}_*(K(\Pi, n + 1)) \) where the middle term is a twisted tensor product, quasi-isomorphic to \( S^F(PK(\Pi, n + 1)) \). Here \( F = K(\Pi, n) \).

It follows from Corollary 31.3 of [8] (Eilenberg, 1944) that \( S_*(K(\Pi, n + 1)) \) is chain equivalent to \( S_*(K(\Pi, n + 1)) \). Similarly one can show that \( S^F(PK(\Pi, n + 1)) \) is chain equivalent to \( S_*(PK(\Pi, n + 1)) \). In particular, this means that the twisted tensor product \( \tilde{B}_*(K(\Pi, n - 1))_s \otimes \tilde{B}_*(K(\Pi, n + 1)) \) is acyclic.

Therefore, it follows from Proposition A.5 that the bi-degree on the twisted tensor product induces the Serre spectral sequence of the path fibration in the corollary. It follows from Theorem A.22 that it suffices to construct a map of DGA’s with divided power operations

\[
U(M^{(n)}) \rightarrow \tilde{B}_*(\Omega K(\Pi, n + 1))
\]

which is a weak equivalence. It follows from Theorem A.20 and Theorem A.24 that such a chain equivalence exists, and in particular, sends each \( p \)-words \( \alpha \) to a cycle that represents the same homology class as \( \alpha \). Therefore we have a map of graded modules

\[
M^{(n)} \rightarrow \tilde{B}_*(\Omega K(\Pi, n + 1))
\]

for a prime \( p \).
By Proposition A.16 this extends to the desired DGA map. \qed

We proceed to consider the integral homology of \(K(\Pi,n)\). Recall the transposition \(\psi_p\) defined in Definition A.8. For an arbitrary integer \(l\), we have a similar operation \(\psi_l : \Pi \to H_2(K(\Pi;1);\mathbb{Z}/l)\) where \(\Pi\) is the subgroup of \(\Pi\) of \(l\)-torsion elements. \(\psi_l\) satisfies the following condition.

**Proposition A.26.** Let \(\delta_l : H_2(K(\Pi;1);\mathbb{Z}/l) \to H_1(K(\Pi;1);\mathbb{Z})\) be the Bockstein homomorphism and \(\sigma : \Pi \to H_1(K(\Pi;1);\mathbb{Z})\) be the suspension. Then \(\sigma = \delta_l\psi_l\).

For details see section 1, exp. 11 of [B] (Cartan and Serre, 1954-1955). In the case of integral cohomology, we extend the definition of an admissible \(p\)-word as follows.

**Definition A.27.** A \(p\)-word is a sequence consists of the symbols \(\sigma, \varphi_p, \gamma_p\), and \(\psi_{p^\lambda}\), for some positive integer \(\lambda\). An admissible \(p\)-word is a word satisfying (2) of Definition A.18 except that it can end with \(\psi_{p^\lambda}\). \(\psi_{p^\lambda}\), of height 1 and degree 2. The degree of a \(p\)-word is therefore given as in (1) of A.18. Notice we do not make the exception when \(p = 2\).

In what follows we abuse notations to let words denote elements of a DGA rather then homology classes, as we did earlier. Let \(\Pi = \prod_k \Pi_k\) be the decomposition of \(\Pi\) into cyclic groups of order infinity or a power of a prime, and let \(w_k\) be a generator of \(\Pi_k\). Also we recall the the decompositions \(\Pi/(p\Pi) = \prod_i \Pi_i\) and \(\Pi = \prod_i \Pi_i^p\) as well as the generators \(\{w'_i\}, \{w''_i\}\) as in Definition A.19 and Theorem A.20. We let \(E(a;k)\) or \(P(;k)\) denote exterior algebras or divided power algebras over \(\mathbb{Z}\) generated by a single element \(a\) of degree \(k\), suppressing the ring of coefficients, and consider them as graded \(\mathbb{Z}\)-algebras. We fix a positive integer \(n\), and construct a collection of DGA’s.

**Construction A.28.**

(1) For each \(w_k\) of order infinity, take the DGA \(E(\sigma^n(w_k);n)\) with trivial differential when \(n\) is odd, or \(P(\sigma^n(w_k);n)\) when \(n\) is even. We denote this DGA by \(A(n)_0\).

(2) For each \(w_k\) of order \(p^\lambda\) for some prime \(p\) and positive integer \(\lambda\), If \(n\) is odd take \(E(\sigma^n(w_k);n) \otimes P(\sigma^{n-1}(\psi_{p^\lambda})(w_k);n+1)\), or \(P(\sigma^n(w_k),n) \otimes E(\sigma^{n-1}(\psi_{p^\lambda})(w_k),n+1)\). In either case we define differential

\[
d(\sigma^{n-1}(\psi_{p^\lambda})(w_k)) = (-1)^{n-1}p^\lambda\sigma^n(k), d(\sigma^n(w_k)) = 0
\]

when \(n\) is even.

(3) Let \(\alpha'\) be an admissible \(p\)-word of height \(n-l-1\) and degree \(q\). Consider the pair of \(p\)-words \(\sigma^l\varphi_p\alpha'\) and \(\sigma^{l+1}\gamma_p\alpha'\). If \(n\) is odd, for each \(w'_i\) take \(E(\sigma^{l+1}\gamma_p\alpha'(w'_i);pq+l+1) \otimes P(\sigma^l\varphi_p\alpha'(w'_i);pq+l+2)\). If \(n\) is even, for each \(w'_i\) take \(P(\sigma^{l+1}\gamma_p\alpha'(w'_i);pq+l+1) \otimes E(\sigma^l\varphi_p\alpha'(w'_i),pq+l+2)\). In both cases we take differential

\[
d(\sigma^{l+1}\gamma_p\alpha'(w'_i)) = (-1)^{n-1}p(\sigma^{l+1}\gamma_p\alpha'(w'_i)), d(\sigma^{l+1}\gamma_p\alpha'(w'_i)) = 0.
\]

(4) Let \(\alpha',\sigma^l\varphi_p\alpha'\) and \(\sigma^{l+1}\gamma_p\alpha'\) be as above. If \(n\) is odd, for each \(w''_j\) take \(E(\sigma^{l+1}\gamma_p\alpha'(w''_j);pq+l+1) \otimes P(\sigma^l\varphi_p\alpha'(w''_j);pq+l+2)\). If \(n\) is even, for each \(w''_j\) take \(P(\sigma^{l+1}\gamma_p\alpha'(w''_j),pq+l+1) \otimes E(\sigma^l\varphi_p\alpha'(w''_j);pq+l+2)\). In both cases we take differential

\[
d(\sigma^{l+1}\gamma_p\alpha'(w''_j)) = (-1)^{n-1}p(\sigma^{l+1}\gamma_p\alpha'(w''_j)), d(\sigma^{l+1}\gamma_p\alpha'(w''_j)) = 0.
\]
We take all the DGA’s constructed in (2), (3), (4) and denote their tensor product by $A(n)_P$. Finally we take $A(n) = A(n)_0 \otimes_{\mathbb{Z}} A(n)_p$. It is easily seen that there is a homomorphism of divided power algebra $f : H_*(A(n)) \to H_*(K(\Pi, n); \mathbb{Z})$ taking the class represented by the word $\alpha(k)$ (resp. $\alpha(i), \alpha(j)$) to the homology class given by the operations $\alpha(w_k)$, (resp. $\alpha(w'_i, \alpha(w''_j)$). The following theorem from [10] (Cartan and Serre, 1954-1955) is stated in a more modern form than the original.

**Theorem A.29** (H. Cartan, Theorem 1, exp. 11, [3], Cartan and Serre, 1954-1955). $f : H_*(A(n)) \to H_*(K(\Pi, n); \mathbb{Z})$ is an epimorphism. For any prime $p$, the restriction $f : H_*(A(n)_0 \otimes A(n)_p) \to H_*(K(\Pi, n); \mathbb{Z})$ is a $p$-local isomorphism.

**Remark** A.30. The kernel of $f$ is not always trivial. Indeed, when $n = 2$ and $\Pi = \mathbb{Z}/2$ with a generator $w_1$, we take $x_k = \varphi_2 \gamma_2 \psi_2(w_1)$ for $k \geq 1$, $y_k = \sigma \gamma_2 \psi_2(w_1)$ for $k \geq 0$, and $z = \sigma^2(w_1)$. Consider the DGA over $\mathbb{Z}$

$$A = \bigotimes_{k \geq 1} P(x_k; 2^{k+1} + 2) \otimes \bigotimes_{k \geq 0} E(y_k; 2^{k+1} + 1) \otimes P(z; 2)$$

with $d(x_k) = -2y_k, d(y_k) = 0$ for $k \geq 1, d(y_0) = -2z$ and $d(z) = 0$. In particular we have

$$d(y_0 \gamma_2(z)) = -2z \gamma_2(z) = -6 \gamma_3(z),$$

which implies that the homology class $\gamma_2(z)$ is of order 6, but by Theorem A.20 $H_*(K(\mathbb{Z}/2, 2); \mathbb{Z})$ has only 2-primary elements. Hence $2 \gamma_3 z \neq 0$ is in the kernel of $f$ since it is a 3-torsion. However, as we see in the Section 2 when $\Pi = \mathbb{Z}$ we do have $H_*(A(n)) \cong H_*(K(\mathbb{Z}, n); \mathbb{Z})$.

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