FACTORIAL THREEFOLDS AND SHOKUROV VANISHING

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ABSTRACT. We apply the Shokurov vanishing theorem to prove the factoriality of the following nodal threefolds: a complete intersection of hypersurfaces $F$ and $G \subset \mathbb{P}^5$ of degree $n$ and $k$ respectively, where $G$ is smooth, $|\text{Sing}(F \cap G)| \leq (n+k-2)(n-1)/5$, $n \geq k$; a double cover of a smooth hypersurface $F \subset \mathbb{P}^4$ of degree $n$ branched over a surface that is cut out on $F$ by a hypersurface $G$ of degree $2r \geq n$, and $|\text{Sing}(F \cap G)| \leq (2r+n-2)r/4$.

1. Introduction.

A Weil divisor is a $\mathbb{Q}$-Cartier divisor if some its nonzero multiple is a Cartier divisor, a variety has $\mathbb{Q}$-factorial singularities if every Weil divisor on it is a $\mathbb{Q}$-Cartier divisor, a variety is $\mathbb{Q}$-factorial if its singularities are $\mathbb{Q}$-factorial. Smooth varieties are $\mathbb{Q}$-factorial.

Birational geometry of many singular varieties crucially depend on the $\mathbb{Q}$-factoriality condition. For example, all $\mathbb{Q}$-factorial nodal quartic threefolds are known to be nonrational (see [36], [48], [45]) and all $\mathbb{Q}$-factorial double covers of $\mathbb{P}^3$ branched over nodal sextic surfaces are nonrational (see [31], [49], [12]), which is false in the non-$\mathbb{Q}$-factorial case.

Example 1. Every nodal quartic threefold in $\mathbb{P}^4$ does not have more than 45 singular point (see [55], [26]). There are nodal quartic threefolds having any number of singular points up to 45, and there is a unique (see [38]) nodal quartic threefold $B_4$ with 45 singular points, which is called a Burkhardt quartic (see [54], [47]) and can be given as

$$w^4 - w(x^3 + y^3 + z^3 + t^3) + 3xyzt = 0 \subset \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x,y,z,t,w]),$$

which implies that $B_4$ is determinantal and rational. The quartic $B_4$ is a unique invariant having degree 4 of the simple group $\text{PSp}(4,\mathbb{Z}_3)$ of order 25920 (see [27], [31], [29], [30]), and singular points of the quartic $B_4$ corresponds to 45 tritangents of a smooth cubic surface, which is related to the fact that the Weil group $E_6$ is a nontrivial extension of the group $\text{PSp}(4,\mathbb{Z}_3)$ by $\mathbb{Z}_2$. The quartic $B_4$ contains a plane, which is not a Cartier divisor, because the plane is not cut out on $B_4$ by any hypersurface. On the other hand, the local class group of an ordinary double point is $\mathbb{Z}$, which implies that every nonzero multiple of a plane contained in $B_4$ is not a Cartier divisor. So, the quartic $B_4$ is not $\mathbb{Q}$-factorial, one can show that $\text{Cl}(B_4) \cong \mathbb{Z}^{16}$ (see [29]), but the Lefschetz implies that $\text{Pic}(B_4) \cong \mathbb{Z}$.

Example 2. Let $\pi : X \to \mathbb{P}^3$ be a double cover ramified in a Barth sextic surface

$$4(\tau^2x^2-y^2)(\tau^2y^2-z^2)(\tau^2z^2-x^2) = w^2(1+2\tau)(x^2+y^2+z^2-w^2)^2 \subset \mathbb{P}^3 \cong \text{Proj}(\mathbb{C}[x,y,z,w]),$$

where $\tau = (1 + \sqrt{5})/2$. Then $X$ is nodal and has 65 singular points (see [33]), but every nodal sextic surface has at most 65 singular points (see [37], [57]). There is a determinantal

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All varieties are assumed to be projective, normal and defined over $\mathbb{C}$.

A variety is called nodal if all its singular points are isolated ordinary double points.
quartic threefold $V \subset \mathbb{P}^4$ with 42 ordinary double points such that the diagram

$$
\begin{array}{ccc}
Y & \stackrel{\psi}{\longrightarrow} & \mathbb{P}^4 \\
\downarrow \rho & & \downarrow \gamma \\
X & \stackrel{\pi}{\longrightarrow} & \mathbb{P}^3
\end{array}
$$

commutes (see [23], [47]), where $\rho$ is a birational map and $\gamma$ is the projection from an ordinary double point of the quartic $Y$, which implies the rationality of $X$, because determinantal quartics are rational. The rational map $\rho$ can be decomposed as a composition of a blow up of a singular point of the quartic $Y$ and a consecutive blow down of the proper transforms of 24 lines on the quartic $Y$ that pass through the blown up singular point. Every nonzero multiple of the image of the exceptional divisor of the blow up of the singular point of $Y$ is not a Cartier divisor, which implies that $X$ is not $\mathbb{Q}$-factorial, but one can show that $\text{Pic}(X) \cong \mathbb{Z}$ and $\text{Cl}(X) \cong \mathbb{Z}^{14}$ (see Example 3.7 in [23]).

It is natural to ask how a global topological condition of being $\mathbb{Q}$-factorial depends on the number of singular points of a nodal threefold. To illustrate a general picture let us consider nodal hypersurfaces in $\mathbb{P}^4$. Let $V$ be a hypersurface in $\mathbb{P}^4$ of degree $n$ that has at most ordinary double points. Then the variety $V$ is $\mathbb{Q}$-factorial if and only if

$$\text{rk } H^2(V, \mathbb{Z}) = \text{rk } H_4(V, \mathbb{Z}),$$

which is true in the smooth case due to Poincare duality. Moreover, the following important result holds (see [15], [58], [22], [19]).

**Proposition 3.** The hypersurface $V$ is $\mathbb{Q}$-factorial if and only if its singular points impose independent linear conditions on global sections of the sheaf $\mathcal{O}_{\mathbb{P}^4}(2n - 5)$.

In particular, the hypersurface $V$ is always $\mathbb{Q}$-factorial when $|\text{Sing}(V)| \leq 2n - 4$.

**Remark 4.** Let $X$ be either a nodal complete intersection of two hypersurface in $\mathbb{P}^5$, or a nodal double cover of a smooth hypersurface in $\mathbb{P}^4$. Then the group $\text{Pic}(X)$ is generated either by the class of a hyperplane section or by a pull back of the class of a hyperplane section. The threefold $X$ is usually called factorial in the case when a similar statement holds for the group $\text{Cl}(X)$. However, the local class group of an isolated ordinary double point is $\mathbb{Z}$ (see [16]), which implies that the following conditions are equivalent:

- the variety $V$ is $\mathbb{Q}$-factorial;
- the variety $V$ is factorial;
- the isomorphism $\text{Cl}(V) \cong \text{Pic}(V)$ holds;
- the isomorphism $\text{Cl}(V) \cong \mathbb{Z}$ holds;
- the equality $\text{rk Cl}(V) = 1$ holds.

Let us consider the simplest example of the hypersurface $V$ that is not $\mathbb{Q}$-factorial.

**Example 5.** Suppose that the hypersurface $V$ is given by the equation

$$xg(x, y, z, t, w) + yf(x, y, z, t, w) = 0 \subset \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $g$ and $f$ are general polynomials of degree $n - 1$. Then $V$ is indeed nodal, it contains the plane $x = y = 0$ and $|\text{Sing}(V)| = (n - 1)^2$, which implies that $V$ is not $\mathbb{Q}$-factorial.

The problem of $\mathbb{Q}$-factoriality of nodal threefolds is related to the Shokurov vanishing theorem (see [52], [42], [43], [1]). Let us illustrate this relation on the following example.
Proposition 6. Let \( \mathcal{H} \) be a linear system of hypersurface in \( \mathbb{P}^4 \) of degree \( k < n/2 \) that pass through the points of the set \( \text{Sing}(V) \), and let \( \mathcal{H}' = \mathcal{H}|_V \). Suppose that the base locus of the linear system \( \mathcal{H} \) is zero-dimensional. Then the hypersurface \( V \) is \( \mathbb{Q} \)-factorial.

Proof. Let \( P \) be an arbitrary singular point of \( V \). Then it follows from Proposition 3 that in order to conclude the proof we must find a hypersurface in \( \mathbb{P}^4 \) of degree \( 2n - 5 \) that pass through all points of the set \( \text{Sing}(V) \setminus P \) and does not pass through the point \( P \).

Suppose that the base locus of the linear system \( \mathcal{H} \) is zero-dimensional. Let \( \Lambda \) be a base locus of \( \mathcal{H} \). Then \( \text{Sing}(V) \subseteq \Lambda \). Take sufficiently general divisors \( H_1, \ldots, H_s \) in the linear system \( \mathcal{H} \) for \( s \gg 0 \). Put \( X = \mathbb{P}^4 \), \( B_X = 4 \sum_{i=1}^{s} H_i \) and

\[
\text{Sing}(V) \setminus P = \{ P_1, \ldots, P_r \},
\]

where \( P_i \) is a point. Let \( f : V \to X \) be a blow up of all points in \( \text{Sing}(V) \setminus P \). Then

\[
K_V + B_V + \sum_{i=1}^{r} \left( \text{mult}_{P_i}(B_X) - 4 \right) E_i + f^*(H) = f^*\left( (4k - 4)H \right) - \sum_{i=1}^{r} E_i,
\]

where \( E_i \) is the \( f \)-exceptional divisor such that \( f(E_i) = P_i \), the divisor \( B_V \) is a proper transform of the divisor \( B_X \) on the variety \( V \), and \( H \) is a hyperplane in \( \mathbb{P}^4 \). Let

\[
\hat{B}_V = B_V + \sum_{i=1}^{r} \left( \text{mult}_{P_i}(B_X) - 4 \right) E_i,
\]

and \( \hat{P} \) be a point of the variety \( V \) such that \( f(\hat{P}) = (P) \). Then the divisor \( \hat{B}_V \) is effective, because \( \text{mult}_{P_i}(B_X) \geq 4 \) for every \( i \). We have \( \text{mult}_{P}(B_V) \geq 4 \), which implies that \( \hat{P} \) is an isolated center of log canonical singularities of the log pair \( (V, \hat{B}_V) \). Hence, the map

\[
H^0\left( \mathcal{O}_V\left( f^*\left( (4k - 4)H \right) - \sum_{i=1}^{r} E_i \right) \right) \to H^0\left( \mathcal{O}_{\mathcal{L}(V, \hat{B}_V)} \otimes \mathcal{O}_V\left( f^*\left( (4k - 4)H \right) - \sum_{i=1}^{r} E_i \right) \right)
\]

is surjective by the Shokurov vanishing theorem (see Theorem 23), where \( \mathcal{L}(V, \hat{B}_V) \) is a subscheme of log canonical singularities of the log pair \( (V, \hat{B}_V) \).

In the neighborhood of the point \( \hat{P} \) the support of the subscheme \( \mathcal{L}(V, \hat{B}_V) \) consists of the point \( \hat{P} \), which implies the existence of an effective divisor

\[
D \in \left| f^*\left( (4k - 4)H \right) - \sum_{i=1}^{r} E_i \right|
\]

that does not pass through the point \( \hat{P} \). Therefore, the divisor \( f(D) \) is a hypersurface of degree \( 4k - 4 \) that passes through all points of the set \( \text{Sing}(V) \setminus P \) but does not pass through the point \( P \). We have \( 4k - 4 \leq 2n - 5 \), which implies the existence of a hypersurface of degree \( 2n - 5 \) that contains the set \( \text{Sing}(V) \setminus P \) and does not pass through the point \( P \).

In general case we can apply the previous arguments to the linear system \( \mathcal{H}' \) instead of the linear system \( \mathcal{H} \), put \( X = V \), and use the projective normality of \( V \subset \mathbb{P}^4 \). \( \square \)

Corollary 7. Suppose that the subset \( \text{Sing}(V) \subset \mathbb{P}^4 \) is a set-theoretical intersection hypersurfaces of degree \( k < n/2 \). Then the hypersurface \( V \) is \( \mathbb{Q} \)-factorial.

Every smooth surface on \( V \) is a Cartier divisor when \( \text{Sing}(V) < (n-1)^2 \) due to [13], and it is natural to expect that \( V \) is \( \mathbb{Q} \)-factorial in the case when \( |\text{Sing}(V)| < (n-1)^2 \), which is proved only for \( n \leq 4 \) (see [25], [11]). The arguments of the proof of Proposition 3 and
known properties of linear systems on rational surfaces are used in [10] to prove that the hypersurface $V$ is $\mathbb{Q}$-factorial in the case when $|\text{Sing}(V)| \leq (n - 1)^2/4$.

The main result of the given paper is the following theorem.

**Theorem 8.** The following nodal threefolds are $\mathbb{Q}$-factorial:

- a complete intersection of the hypersurface $F$ and $G$ in $\mathbb{P}^5$ of degree $n$ and $k$ respectively such that $G$ smooth, $|\text{Sing}(X)| \leq (n + k - 2)(n - 1)/5$, and $n \geq k$;
- a double cover of a smooth hypersurface $F \subset \mathbb{P}^4$ of degree $n \geq 2$ branched in a surface $S \subset F$ that is cut out on $F$ by a hypersurface of degree $2r \geq n$ such that the number of singular points of the surface $S$ does not exceed $(2r + n - 2)r/4$.

Nodal threefolds arise naturally in many problems of algebraic geometry.

**Example 9.** Let $Y$ be a general divisor of bi-degree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^3$ given by

$$f_3(x, y, z, w)s^2 + g_3(x, y, z, w)st + h_3(x, y, z, w)t^2 = 0,$$

where $(s : t; x : y : z : w)$ are bihomogeneous coordinates, and $f_3, g_3, h_3$ are homogeneous polynomials of degree 3. Let $\xi : Y \to \mathbb{P}^3$ be a natural projection. Then $Y$ has 27 rational curves $C_1, C_2, \ldots, C_{27}$ such that $-K_Y \cdot C_i = 0$, because the system of equations

$$f_3(x, y, z, w) = g_3(x, y, z, w) = h_3(x, y, z, w) = 0$$

has exactly 27 solutions. The projection $\xi$ has degree 2 outside of $C_1, C_2, \ldots, C_{27}$, but

$$X = \text{Proj} \left( \bigoplus_{n \geq 0} H^0(Y, \mathcal{O}_Y(-nK_Y)) \right)$$

is a double cover of $\mathbb{P}^3$ branched over a nodal surface

$$g_3^2(x, y, z, w) - 4f_3(x, y, z, w)h_3(x, y, z, w) = 0,$$

which implies that the threefold $X$ is nodal and has exactly 27 ordinary double points that are images of the smooth rational curves $C_1, C_2, \ldots, C_{27}$, which are contracted by the morphism $\phi_{-nK_Y} : Y \to X$ for some natural $n \gg 0$. The threefold $X$ is not $\mathbb{Q}$-factorial, and it is well known, that the threefold $X$ is not rational (see [2], [23], [9]).

The geometry of nodal threefolds is more complicated than of smooth ones:

- every surface on smooth hypersurface in $\mathbb{P}^4$ is a complete intersection due to Lefschetz theorem, which is no longer true in the nodal case (see Example 2);
- the group of birational automorphisms of a smooth quartic threefold is a finite group (see [36]), which is no longer true in the nodal case (see [48], [45]);
- smooth cubic threefolds are not rational (see [16]), but singular ones are rational.

Isolated ordinary double point has two small resolutions, which are birational via an ordinary flop (see [58], [11]). Therefore, every nodal threefold having $k$ singular points has exactly $2^k$ small resolutions, which all must be non-projective in the $\mathbb{Q}$-factorial case, because every exceptional curves must be homological to zero. Thus, it is quite natural to expect that a singular nodal threefold is $\mathbb{Q}$-factorial if and only if all its small resolutions are not projective. The following example of L. Wotzlaw shows that the latter is not true.

**Example 10.** Let $\mathcal{I}_5$ be a quintic hypersurface

$$x_5 - 6x_5^3 \sum_{i=0}^{6} x_i - 27x_5 \left( \sum_{i=0}^{5} x_i \right)^2 - 4 \sum_{i=0}^{5} \sum_{j=i+1}^{5} x_ix_j \right) - 648x_0x_1x_2x_3x_4 = 0$$

be a quintic hypersurface $F \subset \mathbb{P}^5$. The geometry of nodal threefolds is more complicated than of smooth ones:

- every surface on smooth hypersurface in $\mathbb{P}^4$ is a complete intersection due to Lefschetz theorem, which is no longer true in the nodal case (see Example 2);
- the group of birational automorphisms of a smooth quartic threefold is a finite group (see [36]), which is no longer true in the nodal case (see [48], [45]);
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in \( \mathbb{P}^5 \cong \text{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]) \). Then the quintic \( \mathcal{I}_5 \) is invariant under the standard action of the Weil group \( E_6 \) on \( \mathbb{P}^5 \) by reflection. Moreover, the quintic \( \mathcal{I}_5 \) is the only invariant of degree 5 of the Weil group \( E_6 \) under such action (see §6 in [31, 32]).

The singularities of the quintic \( \mathcal{I}_5 \) consist of lines \( L_1, \ldots, L_{120} \), which intersect each other in points \( O_1, \ldots, O_{36} \), the projectivization of a tangent cone to \( \mathcal{I}_5 \) in \( O_k \) is isomorphic to a so-called Segre cubic (see [24], [31], [32]), but in every point of the set

\[
\bigcup_{i=1}^{120} L_i \setminus \bigcup_{k=1}^{36} O_k
\]

the quintic \( \mathcal{I}_5 \) is locally isomorphic to a product \( \mathbb{C} \times \mathbb{A}_1 \), where \( \mathbb{A}_1 \) is a neighborhood of a three-dimensional ordinary double point.

Let \( H_\alpha \) be a hyperplane section of the quintic \( \mathcal{I}_5 \) that corresponds to a general point \( \alpha \) of the dual space \( (\mathbb{P}^5)^* \), and \( T_\beta \) be a hyperplane section of \( \mathcal{I}_5 \) that corresponds to a general point \( \beta \in (\mathcal{I}_5)^* \subset (\mathbb{P}^5)^* \) and tangents \( \mathcal{I}_5 \) in a point \( P \in \mathcal{I}_5 \). Therefore, there is a five-dimensional family of hyperplane sections \( H_\alpha \), and four-dimensional family of tangent hyperplane sections \( H_\beta \). It follows from [31] that both families are versal.

The variety \( H_\alpha \) is a nodal hypersurface in \( \mathbb{P}^4 \) of degree 5 that has 120 ordinary double points \( Q_1, \ldots, Q_{120} \) such that \( Q_i = L_i \cap H_\alpha \). The variety \( T_\beta \) is a nodal hypersurface of degree 5 that has 121 ordinary double points \( P_1, \ldots, P_{120} \) and \( P \) such that \( P_i = L_i \cap T_\beta \).

It follows from the Lefschetz theorem that \( \text{rk} \text{Pic}(H_\alpha) = \text{rk} \text{Pic}(T_\beta) = 1 \), but it follows from [3] that \( \text{rk} \text{Cl}(H_\alpha) = \text{rk} \text{Cl}(T_\beta) = 25 \), and \( H_\alpha \) and \( T_\beta \) are not \( \mathbb{Q} \)-factorial.

Let \( \pi : \hat{T}_\beta \to T_\beta \) be a small resolution, and \( C_i \) and \( C \) be the curves on \( \hat{T}_\beta \) that are contracted to the points \( P_i \) and \( P \) respectively. Then

\[
N_{C/\hat{T}_\beta} \cong N_{C_i/\hat{T}_\beta} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1),
\]

where \( C \cong C_i \cong \mathbb{P}^1 \).

Let \( \psi : \hat{H}_\alpha \to H_\alpha \) be a small resolution, and \( \tau : \hat{T}_\beta \to \hat{T}_\beta \) be a small contraction of a smooth rational curve \( C \) to an ordinary double point \( \hat{P} \in \hat{T}_\beta \). Then \( \hat{P} \) is the only singular point of the variety \( \hat{T}_\beta \), and five-dimensional family of smooth threefolds \( H_\alpha \) is a smooth deformation of the threefold \( \hat{T}_\beta \). Therefore, there is an exact sequence (see [58])

\[
0 \to H_3(\hat{T}_\beta, \mathbb{Z}) \to H_3(T_\beta, \mathbb{Z}) \to H_2(C, \mathbb{Z}) \to H_2(\hat{T}_\beta, \mathbb{Z}) \to H_2(T_\beta, \mathbb{Z}) \to 0
\]

and an isomorphism \( H_2(T_\beta, \mathbb{Z}) \cong H_2(\hat{T}_\beta, \mathbb{Z}) \), but

\[
h_2(\hat{T}_\beta, \mathbb{Z}) = \text{rk} \text{Cl}(T_\beta) = \text{rk} \text{Cl}(H_\alpha) = h_2(H_\alpha, \mathbb{Z}),
\]

which implies that the natural map \( H_2(C, \mathbb{Z}) \to H_2(\hat{T}_\beta, \mathbb{Z}) \) maps the whole homology group \( H_2(C, \mathbb{Z}) \) to the zero. Hence, the curve \( C \) is homological to the zero on the smooth threefold \( \hat{T}_\beta \), which implies that \( \hat{T}_\beta \) is not projective.

Let us consider two examples, which are inspired by the papers [10] and [15].

**Example 11.** Let \( \pi : X \to \mathbb{P}^3 \) be the double cover ramified along a surface \( S \) given by

\[
u^2 + g_3^2(x, y, z, w) = h_1(x, y, z, w) f_5(x, y, z, w) \subset \mathbb{P}(1^4, 3) \cong \text{Proj}(\mathbb{C}[x, y, z, w, u]),
\]

where \( g_3, h_1, \) and \( f_5 \) are general polynomials defined over \( \mathbb{R} \) of degree 3, 1, and 5, respectively. Then the double cover \( X \) is not \( \mathbb{Q} \)-factorial over \( \mathbb{C} \) because the divisor \( h_1 = 0 \) splits into two non-\( \mathbb{Q} \)-Cartier divisors conjugated by \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) and given by the equation

\[
\left(u + \sqrt{-1}g_3(x, y, z, w)\right)\left(u - \sqrt{-1}g_3(x, y, z, w)\right) = 0.
\]
The sextic surface $S \subset \Proj(\mathbb{C}[x, y, z, w])$ has 15 ordinary double points at the intersection points of the three surfaces
\[
\{ h_1(x, y, z, w) = 0 \} \cap \{ g_3(x, y, z) = 0 \} \cap \{ f_5(x, y, z, w) = 0 \},
\]
which gives 15 simple double points of $X$. Introducing variables $s$ and $t$ defined by
\[
\begin{align*}
s &= \frac{u + \sqrt{-1}g_3(x, y, z, w)}{h_1(x, y, z, w)} = \frac{f_5(x, y, z, w)}{u - \sqrt{-1}g_3(x, y, z, w)} \\
t &= \frac{u - \sqrt{-1}g_3(x, y, z, w)}{h_1(x, y, z, w)} = \frac{f_5(x, y, z, w)}{u + \sqrt{-1}g_3(x, y, z, w)}
\end{align*}
\]
we can unproject $X$ in the sense of [50] into two complete intersections
\[
\begin{align*}
V_s &= \left\{ sh_1(x, y, z, w) = u + \sqrt{-1}g_3(x, y, z, w) \right\} \subset \mathbb{P}(1^4, 3, 2) \\
V_t &= \left\{ th_1(x, y, z, w) = u - \sqrt{-1}g_3(x, y, z, w) \right\} \subset \mathbb{P}(1^4, 3, 2),
\end{align*}
\]
respectively, which are not defined over $\mathbb{R}$. Eliminating variable $u$, we get
\[
\begin{align*}
V_s &= \left\{ s^2h_1 - 2\sqrt{-1}sg_3 - f_5 = 0 \right\} \subset \mathbb{P}(1^4, 2) \\
V_t &= \left\{ t^2h_1 + 2\sqrt{-1}tg_3 - f_5 = 0 \right\} \subset \mathbb{P}(1^4, 2)
\end{align*}
\]
and for the maps $\rho_s : X \dashrightarrow V_s$ and $\rho_t : X \dashrightarrow V_t$ we obtain a commutative diagram
\[
\begin{array}{ccc}
Y_s & \overset{\psi_s}{\leftarrow} & X & \overset{\psi_t}{\rightarrow} & Y_t \\
\phi_s & & & & \phi_t \\
V_s & \leftarrow & X & \rightarrow & V_t
\end{array}
\]
with birational morphisms $\phi_s$, $\psi_s$, $\phi_t$, and $\psi_t$ such that $\psi_s$ and $\psi_t$ are extremal contractions in the sense of [17], while $\phi_s$ and $\phi_t$ are flopping contractions.

The weighted hypersurfaces $V_s$ and $V_t$ are quasi-smooth (see [33]), which implies that they are $\mathbb{Q}$-factorial and have Picard group $\mathbb{Z}$ (see [4]). The hypersurfaces $V_s$ and $V_t$ are projectively isomorphic in $\mathbb{P}(1^4, 2)$ by the action of $\text{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}_2$. We have
\[
\text{Pic}(Y_s) \cong \text{Pic}(Y_t) \cong \mathbb{Z} \oplus \mathbb{Z},
\]
which gives $\text{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}$. The $\text{Gal}(\mathbb{C}/\mathbb{R})$-invariant part of the group $\text{Cl}(X)$ is $\mathbb{Z}$, which implies that $X$ is $\mathbb{Q}$-factorial over $\mathbb{R}$. The threefold $X$ is not rational over $\mathbb{R}$ due to [12], but $X$ is also not rational over $\mathbb{C}$ due to [13]. Moreover, the involution of $X$ interchanging fibers of $\pi$ induces a non-biregular involution of $V_s$ which is regularized by $\rho_s$ (see [3]).

Example 12. Let $V \subset \mathbb{P}^4$ be a general hypersurface of degree 4 such that $V$ has exactly one ordinary double point $O$. Then $V$ is $\mathbb{Q}$-factorial and can be given by the equation
\[
t^2f_2(x, y, z, w) + tf_3(x, y, z, w) + f_4(x, y, z, w) = 0 \subset \mathbb{P}^4 \cong \Proj(\mathbb{C}[x, y, z, w, t]),
\]
where $O = (0 : 0 : 0 : 0 : 1)$. The threefold $V$ is known to be non-rational (see [45], [48]), but the projection $\phi : V \dashrightarrow \mathbb{P}^3$ from the singular point $O$ has degree 2 at a generic point of the threefold $V$ and induces a non-biregular involution $\tau \in \text{Bir}(V)$. 


Let $f: Y \to V$ be the blow up of the point $O$. Then the linear system $| - nK_Y |$ does not have base points for $n \gg 0$ and gives a birational morphism $g: Y \to X$ contracting every curve $C_i \subset Y$ such that $f(C_i)$ is a line on the quartic threefold $V$ passing through the singular point $O$. We then obtain the double cover $\pi: X \to \mathbb{P}^3$ ramified along the nodal sextic surface $S \subset \mathbb{P}^3$ given by the equation

$$f_3^2(x, y, z, w) - 4f_2(x, y, z, w)f_4(x, y, z, w) = 0.$$  

Each line $f(C_i)$ corresponds to an intersection point of three surfaces

$$\{f_2(x, y, z, w) = 0\} \cap \{f_3(x, y, z, w) = 0\} \cap \{f_4(x, y, z, w) = 0\} \subset \mathbb{P}^3 \cong \text{Proj}\left( \mathbb{C}[x, y, z, w] \right),$$

which gives 24 smooth rational curves $C_1, C_2, \ldots, C_{24}$ such that

$$N_{\pi/C_i} \cong \mathcal{O}_{C_i}(-1) \oplus \mathcal{O}_{C_i}(-1)$$

and $g$ is a standard flopping contraction which maps every curve $C_i$ to an ordinary double point of the threefold $X$. In particular, the sextic $S \subset \mathbb{P}^3$ has exactly 24 simple double points. However, the threefold $X$ is not $\mathbb{Q}$-factorial and $\text{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}$.

Put $\rho = g \circ f^{-1}$. Then the involution $\gamma = \rho \circ \tau \circ \rho^{-1}$ is birational on $X$ and interchanges the fibers of the double cover $\pi$. Thus the map $\rho$ is a regularization of the birational non-biregular involution $\tau$ in the sense of [8], while the commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{\rho} & X \\
\downarrow{g} \quad \downarrow{\gamma} & & \quad \downarrow{\gamma} \quad \downarrow{g} \\
Y & \xrightarrow{\rho} & V \\
\end{array}$$

is a decomposition of $\tau$ in a sequence of so-called Sarkisov links (see [17], [18], [35]).

Suppose that $f_2(x, y, z, w)$ and $f_4(x, y, z, w)$ are defined over $\mathbb{Q}$ and

$$f_3(x, y, z, w) = \sqrt{2}g_3(x, y, z, w),$$

where $g_3(x, y, z, t)$ is defined over $\mathbb{Q}$. Then the threefold $V$ is defined over $\mathbb{Q}(\sqrt{2})$, but the hypersurface $V$ is not defined over $\mathbb{Q}$, because the threefold $V$ is not invariant under the action of $\text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$. However, the sextic surface $S \subset \mathbb{P}^3$ is given by the equation

$$2g_3^2(x, y, z, w) - 4f_2(x, y, z, w)f_4(x, y, z, w) = 0 \subset \mathbb{P}^3 \cong \text{Proj}\left( \mathbb{Q}[x, y, z, w] \right),$$

which implies that $X$ is defined over $\mathbb{Q}$ as well. Moreover, the $\text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$-invariant part of the group $\text{Cl}(X)$ is $\mathbb{Z}$, which implies that $X$ is $\mathbb{Q}$-factorial over $\mathbb{Q}$.

Thus, the condition of $\mathbb{Q}$-factoriality depends also on the field of definition.

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2. Preliminaries.

The following result is well known (see [13], [51], [58], [22], [19]).

**Theorem 13.** Let $W$ be a smooth fourfold, $Y$ be an ample reduced and irreducible divisor on the fourfold $W$ such that the only singularities of the threefold $Y$ are nodal and

$$h^2(\Omega^1_W) = h^3(\Omega^1_W \otimes \mathcal{O}_W(-Y)) = h^1(\mathcal{O}_W) = h^2(\mathcal{O}_W) = 0,$$
and let \( \tilde{Y} \) be a small resolution of the threefold \( Y \). Then
\[
\begin{align*}
    h^1(\mathcal{O}_Y) &= h^2(\mathcal{O}_Y) = 0, \quad h^1(\Omega^1_Y) = h^1(\Omega^1_W) + \delta, \\
    h^2(\Omega^1_Y) &= h^0\left(K_W \otimes \mathcal{O}_W(2Y)\right) + h^3(\mathcal{O}_W) - h^0\left(K_W \otimes \mathcal{O}_W(Y)\right) - \\
    &\quad - h^3(\Omega^1_W) - h^4(\Omega^1_W \otimes \mathcal{O}_W(-Y)) - |\text{Sing}(Y)| + \delta,
\end{align*}
\]
where \( \delta \) is the number of dependent conditions that vanishing at the nodes of the threefold \( Y \) imposes on the global sections of the line bundle \( K_W \otimes \mathcal{O}_W(2Y) \).

The proof of Theorem 13 in [19] implies the following result.

**Corollary 14.** Let \( W \) be a smooth fourfold, and \( Y \) be an ample reduced and irreducible divisor on the fourfold \( W \) such that the threefold \( Y \) is nodal. Suppose that
\[
h^2(\Omega^1_W) = h^1(\mathcal{O}_W) = h^2(\mathcal{O}_W) = 0,
\]
but singular points of the threefold \( Y \) impose independent linear conditions on the global sections of the line bundle \( K_W \otimes \mathcal{O}_W(2Y) \). Then \( Y \) is \( \mathbb{Q} \)-factorial.

The following result is due to [4].

**Theorem 15.** Let \( \pi : Y \to \mathbb{P}^2 \) be a blow up of points \( P_1, \ldots, P_s \) such that
\[
s \leq \frac{d^2 + 9d + 10}{6}
\]
and at most \( k(d + 3 - k) - 2 \) points among the points \( P_1, \ldots, P_s \) are contained in a curve of degree \( k \leq (d + 3)/2 \) for some natural number \( d \geq 3 \). Then the linear system
\[
\left| \pi^*\left(\mathcal{O}_{\mathbb{P}^2}(d)\right) - \sum_{i=1}^s E_i \right|
\]
does not have base points, where \( E_i \) is the \( \pi \)-exceptional divisor such that \( \pi(E_i) = P_i \).

In the case \( d = 3 \) the claim of Theorem 15 is a base point freeness of the anticanonical linear system of a weak del Pezzo surface of degree \( 9 - s \geq 2 \) (see [21], [28], [44]).

**Corollary 16.** Let \( \Sigma \) be a finite subset of \( \mathbb{P}^2 \) and \( d \geq 3 \) be a natural number such that
\[
|\Sigma| \leq \frac{d^2 + 9d + 16}{6}
\]
and at most \( k(d + 3 - k) - 2 \) points of the set \( \Sigma \) lie on a possibly reducible plane curve of degree \( k \leq (d + 3)/2 \). Then for every point \( P \in \Sigma \) there is a curve on \( \mathbb{P}^2 \) of degree \( d \) that passes through all points of the set \( \Sigma \setminus P \) and does not pass through the point \( P \).

The claim of Theorem 15 is strengthen in [20] in the following way.

**Theorem 17.** Let \( \pi : Y \to \mathbb{P}^2 \) be a blow up of points \( P_1, \ldots, P_s \) such that
\[
s \leq \max\left\{ \left\lfloor \frac{(d + 3)}{2} \right\rfloor \left( d + 3 - \left\lfloor \frac{(d + 3)}{2} \right\rfloor \right) - 1, \left\lfloor \frac{(d + 3)}{2} \right\rfloor^2 \right\},
\]
and at most \( k(d + 3 - k) - 2 \) points among the points \( P_1, \ldots, P_s \) are contained in a curve of degree \( k \leq (d + 3)/2 \) for some natural number \( d \geq 3 \). Then the linear system
\[
\left| \pi^*\left(\mathcal{O}_{\mathbb{P}^2}(d)\right) - \sum_{i=1}^s E_i \right|
\]
does not have base points, where \( E_i \) is the \( \pi \)-exceptional divisor such that \( \pi(E_i) = P_i \).
3. Connectedness principle.

Let $X$ be a smooth variety, and $B_X = \sum_{i} a_i B_i$ be a $\mathbb{Q}$-divisor, where $B_i$ is a prime divisor and $a_i$ is a positive rational number. Let $\pi : Y \to X$ be a birational morphism such that $Y$ is smooth, and the union of all the proper transforms of the divisors $B_i$ and all the $\pi$-exceptional divisors form a divisor with simple normal crossing. Let $B_Y$ be the proper transform of $B_X$ on the variety $Y$, and put

$$B^Y = B_Y - \sum_{i} c_i E_i,$$

where $E_i$ is an $\pi$-exceptional divisor and $c_i$ is a rational number such that the equivalence

$$K_Y + B^Y \sim_{\mathbb{Q}} \pi^* (K_X + B_X)$$

holds. Then the log pair $(Y, B^Y)$ is called the log pull back of the log pair $(X, B_X)$ with respect to the birational morphism $\pi$, while the number $c_i$ is called the discrepancy of the log pair $(X, B_X)$ in the $\pi$-exceptional divisor $E_i$.

**Definition 18.** A proper irreducible subvariety $Z \subset X$ is called a center of log canonical singularities of the log pair $(X, B_X)$ if there is a divisor $E$ on $Y$ contained in the support of the effective part of the divisor $\lceil B^Y \rceil$ such that $\pi(E) = Z$.

In particular, the proper irreducible subvariety $\pi(E_i) \subset X$ is a center of log canonical singularities of the log pair $(X, B_X)$ if $c_i \leq -1$. Similarly, the prime divisor $B_i$ is center of log canonical singularities of the log pair $(X, B_X)$ if $a_i \geq 1$.

The set of all centers of log canonical singularities of the log pair $(X, B_X)$ are usually denoted as $\mathbb{LCS}(X, B_X)$. Similarly, the union of all centers of log canonical singularities of the log pair $(X, B_X)$ considered as a proper subset of the variety $X$ are called the locus of log canonical singularities of the log pair $(X, B_X)$ and denoted as $\mathcal{LCS}(X, B_X)$.

**Example 19.** Let $O$ be a smooth point on $X$. Then the inequality $\text{mult}_O(B_X) \geq \dim(X)$ implies that $O \in \mathbb{LCS}(X, B_X)$. Moreover, the inequality $\text{mult}_O(B_X) \geq 1$ holds in the case when $O \in \mathbb{LCS}(X, B_X)$ and the boundary $B_X$ is effective.

**Remark 20.** Let $H$ be a general hyperplane section of the variety $X$, and $Z$ be subvariety of the variety $X$ that is an element of the set $\mathbb{LCS}(X, B_X)$. Then every component of the intersection $Z \cap H$ is contained in $\mathbb{LCS}(H, B_X|_H)$.

**Example 21.** Let $O$ be a smooth point of the variety $X$. Suppose that $O$ is a center of log canonical singularities of the log pair $(X, B_X)$. Let $f : V \to X$ be the blow up of the point $O$, and $E$ be the $f$-exceptional divisor. Then either $E \in \mathbb{LCS}(V, B^V)$, or there is a proper irreducible subvariety $Z \subset E$ that is a center of log canonical singularities of the log pair $(V, B^V)$. Moreover, the exceptional divisor $E$ is a center of log canonical singularities of the log pair $(V, B^V)$ if and only if $\text{mult}_O(B_X) \geq \dim(X)$.

**Definition 22.** The subscheme associated to the ideal sheaf

$$\mathcal{I}(X, B_X) = f_*(\mathcal{O}_Y(\lceil -B^Y \rceil))$$

is called the log canonical singularity subscheme of $(X, B_X)$ and denoted as $\mathcal{L}(X, B_X)$.

The support of the subscheme $\mathcal{L}(X, B_X)$ consists of the set-theoretic union of all centers of log canonical singularities of the log pair $(X, B_X)$, which implies that

$$\text{Supp}\left(\mathcal{L}(X, B_X)\right) = \mathcal{LCS}(X, B_X) \subset X.$$
The following result is the Shokurov vanishing theorem (see \cite{52, 42, 43, 1}).

**Theorem 23.** Suppose that $K_X + B_X + H$ is numerically equivalent to a Cartier divisor, where $H$ a $\mathbb{Q}$-divisor on the variety $X$ that is nef and big\(^2\). Then for every $i > 0$ we have

$$H^i \left( X, \mathcal{I}(X, B_X) \otimes \mathcal{O}_X(K_X + B_X + H) \right) = 0.$$  

**Proof.** It follows from the Kawamata-Viehweg vanishing theorem (see \cite{39, 56}) that

$$R^i f_* \left( f^*(K_X + B_X + H) + [-B_Y] \right) = 0$$

for all $i > 0$. The degeneration of the local–to–global spectral sequence and

$$R^0 f_* \left( f^*(K_X + B_X + H) + [-B_Y] \right) = \mathcal{I}(X, B_X) \otimes \mathcal{O}_X(K_X + B_X + H)$$

imply that

$$H^i \left( X, \mathcal{I}(X, B_X) \otimes \mathcal{O}_X(K_X + B_X + H) \right) = H^i \left( Y, f^*(K_X + B_X + H) + [-B_Y] \right)$$

for $i \geq 0$. On the other hand, we have

$$H^i \left( Y, f^*(K_X + B_X + H) + [-B_Y] \right) = 0$$

for $i > 0$ by the Kawamata-Viehweg vanishing theorem. \hfill \Box

The claim of Theorem 23 implies the following result.

**Lemma 24.** Let $\mathcal{M}$ be a linear subsystem in $|\mathcal{O}_{\mathbb{P}^n}(k)|$ such that the base locus of the linear system $\mathcal{M}$ is zero-dimensional. Then the points of the base locus of $\mathcal{M}$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^n$ of degree $n(k - 1)$.

**Proof.** Let $\Lambda$ be the base locus of the linear system $\mathcal{M}$, and $H_1, \ldots, H_r$ be general divisors in the linear system $\mathcal{M}$, where $r$ is sufficiently big. Put $B_{\mathbb{P}^n} = \frac{n}{r} \sum_{i=1}^r H_i$. Then the singularities of the log pair $(\mathbb{P}^n, B_{\mathbb{P}^n})$ are log terminal (see \cite{43}) outside of the set $\Lambda$, but

$$\text{mult}_P \left( B_{\mathbb{P}^n} \right) = n \sum_{i=1}^r \frac{\text{mult}_P (H_i)}{r} \geq n$$

for every point $P \in \Lambda$. Thus, we have $\text{Supp}(\mathcal{L}(\mathbb{P}^n, B_{\mathbb{P}^n})) = \Lambda$.

Since $K_{\mathbb{P}^n} + B_{\mathbb{P}^n} + H \sim_{\mathbb{Q}} n(k - 1)H$, where $H$ is a hyperplane in $\mathbb{P}^n$, we see that

$$H^1 \left( \mathbb{P}^n, \mathcal{I}(\mathbb{P}^n, B_{\mathbb{P}^n}) \otimes \mathcal{O}_{\mathbb{P}^n} \left( n(k - 1) \right) \right) = 0$$

by Theorem 23. Hence, the points of $\Lambda$ impose independent linear conditions on homogeneous forms of degree $n(k - 1)$, because $\text{Supp}(\mathcal{L}(\mathbb{P}^n, B_{\mathbb{P}^n})) = \Lambda$. \hfill \Box

---

\(^2\)It should be pointed out that a $\mathbb{Q}$-Cartier divisor $H \in \text{Div}(X) \otimes \mathbb{Q}$ is called numerically effective or nef if for every curve $C \subset X$ the inequality $H \cdot C \geq 0$ holds. A numerically effective divisor $H$ is called big if the inequality $H^n > 0$ holds, where $n = \dim(X)$. 

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4. Complete intersections.

Let $X$ be a complete intersection of hypersurfaces $F$ and $G$ in $\mathbb{P}^5$ such that the singularities of $X$ are nodal. Put $n = \deg(F)$ and $k = \deg(G)$. Suppose that $n \geq k$.

**Example 25.** Let $F$ and $G$ be general hypersurfaces containing a plane. Then $X$ is nodal and not $\mathbb{Q}$-factorial, both $F$ and $G$ are smooth, and $|\text{Sing}(X)| = (n + k - 2)^2$.

The following result is proved in [13].

**Theorem 26.** Suppose that $G$ is smooth and $|\text{Sing}(X)| \leq 3n/8$. Then $X$ is $\mathbb{Q}$-factorial.

In this section we prove the following result.

**Theorem 27.** Suppose that $G$ is smooth. Then $X$ is $\mathbb{Q}$-factorial in the case when

$$|\text{Sing}(X)| \leq \frac{(n + k - 2)(n - 1)}{5}.$$ 

The claim of Theorem 27 is not true in the case when the hypersurface $G$ is singular.

**Example 28.** Let $Q \subset \mathbb{P}^5$ be a smooth quadric surface, and $G$ be a cone over the quadric surface $Q$ whose vertex is a general line $L \subset \mathbb{P}^5$. Take a general hypersurface $F \subset \mathbb{P}^5$ of degree $n$. Let $X$ be the complete intersection of the hypersurfaces $G$ and $F$. Then $X$ is a nodal threefold of degree $2n$ and $|\text{Sing}(X)| = n$. Let $\Omega$ be a linear subspace in $\mathbb{P}^5$ spanned by a line contained in $Q$ and a line $L$. Then $\Omega \subset G$, the surface $\Omega \cap F$ has degree $n$ and is not a $\mathbb{Q}$-Cartier divisor on the threefold $X$.

In the case $k = 1$ the claim of Theorem 27 follows from [10].

**Conjecture 29.** Suppose that $G$ is smooth. Then $X$ is $\mathbb{Q}$-factorial in the case when

$$|\text{Sing}(X)| \leq (n + k - 2)^2.$$ 

Suppose that $G$ is smooth. Then the following result follows from Corollary 14.

**Proposition 30.** The threefold $X$ is $\mathbb{Q}$-factorial in the case when its singular points impose independent linear conditions on the sections in $H^0(\mathcal{O}_{\mathbb{P}^5}(2n + k - 6)|_G)$.

**Corollary 31.** Suppose that $|\text{Sing}(X)| \leq 2n + k - 5$. Then $X$ is $\mathbb{Q}$-factorial.

The variety $X$ is $\mathbb{Q}$-factorial if and only if the group $\text{Cl}(X)$ is generated by the class of a hyperplane section (see Remark 11). Every surface contained in the threefold $X$ is a complete intersection in $\mathbb{P}^5$ in the case when $X$ is $\mathbb{Q}$-factorial.

Now we prove Theorem 27. Suppose that $a |\text{Sing}(X)| \leq (n + k - 2)(n - 1)/5$, but the hypersurfaces $G$ is smooth. We have $n = \deg(F) \geq k = \deg(G)$. Let us show that the singular points of the complete intersection $X \subset \mathbb{P}^5$ impose independent linear conditions on the hypersurface in $\mathbb{P}^5$ of degree $2n + k - 6$, which implies the claim of Theorem 27.

The claim of Theorem 27 follows from [10] in the case $k = 1$, and in the case $4 \geq n$ the claim of Theorem 27 follows Corollary 31. Thus, we assume that $k \geq 2$ and $n \geq 5$.

**Lemma 32.** There is a hypersurface $\hat{F} \subset \mathbb{P}^5$ of degree $n$ such that the threefold $X$ is a complete intersection of the hypersurfaces $\hat{F}$ and $G$, but $\text{Sing}(\hat{F}) \subset \text{Sing}(X)$.

**Proof.** The threefold $X$ is given by the system of equations

\[
\begin{align*}
  f(x_0, x_1, x_2, x_3, x_4, x_5) &= 0 \\
  g(x_0, x_1, x_2, x_3, x_4, x_5) &= 0 \\
  &\subset \mathbb{P}^5 \cong \text{Proj}\left(\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]\right),
\end{align*}
\]

Therefore, $\text{Sing}(\hat{F}) \subset \text{Sing}(X)$.
where \( f \) and \( g \) be are homogeneous polynomials of degree \( n \) and \( k \) that define the hypersurface \( F \) and \( G \) respectively. Consider linear system

\[
\mathcal{L} = | \lambda f + h(x_0, x_1, x_2, x_3, x_4, x_5) g | \subset | \mathcal{O}_{\mathbb{P}^5}(n) |,
\]

where \( \lambda \in \mathbb{C} \), and \( h \) is a homogeneous polynomial of degree \( n - k \). Then the base locus of the linear system \( \mathcal{L} \) is the variety \( X \). The Bertini theorem implies the existence of a hypersurface \( \hat{F} \subset \mathcal{L} \) such that \( X = \hat{F} \cap G \), but \( \text{Sing}(\hat{F}) \subset \text{Sing}(X) \).

We may assume that \( \text{Sing}(F) \subset \text{Sing}(X) \).

**Definition 33.** We say that the points of a subset \( \Gamma \subset \mathbb{P}^r \) have property \( \star \) in the case when at most \( t(n + k - 2) \) points of the set \( \Gamma \) lie on a curve in \( \mathbb{P}^r \) of degree \( t \in \mathbb{N} \).

Let \( \Sigma = \text{Sing}(X) \subset \mathbb{P}^5 \).

**Proposition 34.** The points of the subset \( \Sigma \subset \mathbb{P}^5 \) have property \( \star \).

**Proof.** The hypersurface \( F \subset \mathbb{P}^5 \) can be given by the equation

\[
f(x_0, x_1, x_2, x_3, x_4, x_5) = 0 \subset \mathbb{P}^5 \cong \text{Proj} \left( \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5] \right),
\]

where \( f \) is a homogeneous polynomial of degree \( n \), and \( G \) can be given by the equation

\[
g(x_0, x_1, x_2, x_3, x_4, x_5) = 0 \subset \mathbb{P}^5 \cong \text{Proj} \left( \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5] \right),
\]

where \( g \) is a homogeneous polynomial of degree \( k \). Then the set \( \Sigma \) is given by the vanishing of polynomials \( f \) and \( g \), and by vanishing of all minors of size 1 of the matrix

\[
\begin{pmatrix}
\frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & \frac{\partial f}{\partial x_4} & \frac{\partial f}{\partial x_5} \\
\frac{\partial g}{\partial x_0} & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} & \frac{\partial g}{\partial x_4} & \frac{\partial g}{\partial x_5}
\end{pmatrix},
\]

which implies that \( \Sigma \) is a set-theoretical intersection of hypersurfaces of degree \( n + k - 2 \), which concludes the proof.

Take an arbitrary point \( P \in \Sigma \). Then we must show that there is a hypersurface of degree \( 2n + k - 6 \) that contains the set \( \Sigma \setminus P \) and does not contain the point \( P \).

**Lemma 35.** Suppose that there is a plane \( \Pi \subset \mathbb{P}^5 \) such that \( \Sigma \subset \Pi \subset \mathbb{P}^5 \). Then there is a hypersurface of degree \( 2n + k - 6 \) that contains \( \Sigma \setminus P \) and does not contain \( P \).

**Proof.** We want to apply Corollary 16 to \( \Sigma \subset \Pi \) and \( d = 2n + k - 6 \geq 6 \). Let us check that all conditions of Corollary 16 are satisfied.

Suppose that \( \left| \Sigma \right| > (d^2 + 9d + 16)/6 \). Then

\[
\frac{(n + k - 2)(n - 1)}{5} > \frac{(2n + k - 6)^2 + 9(2n + k - 6) + 16}{6},
\]

where \( n \geq 5 \) and \( k \geq 2 \). Put \( A = n + k \geq 7 \). Then

\[
0 > (A + n - 6)^2 + 9(A + n - 6) + 16 - 6An = 5A^2 - 3A - 10 + 5n^2 - 3n + 4An \geq 464,
\]

which is a contradiction.

Now must show that at most \( t(2n + k - 3 - t) - 2 \) points of the set \( \Sigma \) lie on a curve of degree \( t \leq (2n + k - 3)/2 \). However, at most \( t(n + k - 2) \) points of the set \( \Sigma \) lie on a curve of degree \( t \) by Proposition 34. In particular, in the case \( t = 1 \) we have

\[
t(2n + k - 3 - t) - 2 = 2n + k - 6 \geq n + k - 2 = t(n - 1),
\]
because $n \geq 5$. In the case when $t > 1$ it is enough to show that
\[
t(2n + k - 3 - t) - 2 \geq t(n + k - 2)
\]
for every $t \leq (2n + k - 3)/2$ such that $t(2n + k - 3 - t) - 2 < |\Sigma|$. We have
\[
t(2n + k - 3 - t) - 2 \geq t(n + k - 2) \iff n - 1 > t
\]
in the case when $t > 1$. Therefore, we may assume that $t \geq n - 1$, which implies that
\[
t(2n + k - 3 - t) - 2 \geq (n - 1)(n + k - 2) > |\Sigma|.
\]

Therefore, it follows from Corollary 16 that there is a curve $C \subset \Pi$ of degree $2n + k - 6$ that contains the set $\Sigma \setminus P$ and does not contain $P$. Let $Y$ be a general four-dimensional cone in $\mathbb{P}^5$ over the curve $C$. Then $Y$ is the required hypersurface. □

Let $\Pi$ and $\Gamma$ be sufficiently general planes in $\mathbb{P}^5$, and $\psi : \mathbb{P}^5 \to \Pi$ be a projection from the plane $\Gamma$. Put $\Sigma' = \psi(\Sigma) \subset \Pi \cong \mathbb{P}^2$ and $\tilde{P} = \psi(P) \in \Sigma'$.

**Lemma 36.** Suppose that the points of $\Sigma' \subset \Pi$ have property $\star$. Then there is a hypersurface of degree $2n + k - 6$ that contains $\Sigma \setminus P$ and does not contain $P$.

**Proof.** The proof of Lemma 35 implies the existence of a curve $C \subset \Pi$ of degree $2n + k - 6$ that contains $\Sigma' \setminus \tilde{P}$ but does not pass through the point $\tilde{P}$. Let $Y \subset \mathbb{P}^5$ be the cone over the curve $C$ whose vertex is $\Gamma$. Then $Y$ is a hypersurface in $\mathbb{P}^5$ of degree $2n + k - 6$ that passes through all points of the set $\Sigma \setminus P$ and does not contain the point $P \in \Sigma$. □

Therefore, we may assume that the points of the set $\Sigma' \subset \Pi \cong \mathbb{P}^2$ does not have the property $\star$. There is subset $\Lambda^1_r \subset \Sigma$ such that $|\Lambda^1_r| > r(n + k - 2)$, but the subset
\[
\tilde{\Lambda}^1_r = \psi(\Lambda^1_r) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2
\]
is contained in a curve $C \subset \Pi$ of degree $r$. Moreover, we may assume that $r$ is the smallest natural number having such property, which implies that the curve $C$ is irreducible and reduced. We can iterate the construction of $\Lambda^1_r$ to get the disjoint union of subsets
\[
\bigcup_{j=r}^{t} \bigcup_{i=1}^{c_j} \Lambda^j_i \subset \Sigma
\]
such that $|\Lambda^j_i| > j(n + k - 2)$, the points of the set
\[
\tilde{\Lambda}^j_i = \psi(\Lambda^j_i) \subset \Sigma'
\]
lie on an irreducible curve in $\Pi \cong \mathbb{P}^2$ of degree $j$, and the points of the subset
\[
\Sigma = \Sigma' \setminus \bigcup_{j=r}^{t} \bigcup_{i=1}^{c_j} \tilde{\Lambda}^j_i \subset \Sigma' \subset \Pi \cong \mathbb{P}^2
\]
have property $\star$, where $c_j \geq 0$. Then $c_r > 0$ and
\[
|\Sigma| < \frac{(n + k - 2)(n - 1)}{5} - \sum_{i=r}^{t} c_i(n - 1)i = \frac{n + k - 2}{5}(n - 1 - \sum_{i=r}^{t} 5ic_i).
\]

**Corollary 38.** The inequality $\sum_{i=r}^{t} ic_i < (n - 1)/5$ holds.

In particular, we have $j < (n - 1)/5$ in the case when $\Lambda^j_i \neq \emptyset$. 13
Lemma 39. Suppose that $\Lambda^j_i \neq \emptyset$. Let $\mathcal{M}$ be a linear system of hypersurfaces in $\mathbb{P}^5$ of degree $j$ that contains $\Lambda^j_i$. Then the base locus of the linear system $\mathcal{M}$ is zero-dimensional.

Proof. The construction of the set $\Lambda^j_i$ implies that all points of the subset
$$\tilde{\Lambda}^j_i = \psi(\Lambda^j_i) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$
are contained in an irreducible curve $C \subset \Pi$ of degree $j$. Let $Y$ be a cone in $\mathbb{P}^5$ over the curve $C$ whose vertex is some plane $\Upsilon \subset \mathbb{P}^5$. Then $Y$ is a hypersurface in $\mathbb{P}^5$ of degree $j$ that contains all points of the set $\Lambda^j_i$, which implies that $Y \in \mathcal{M}$.

Suppose that the base locus of $\mathcal{M}$ contains an irreducible curve $Z \subset \mathbb{P}^5$. Then $Z \subset Y$, but the generality of $\psi$ and the irreducibility of $Z$ and $C$ imply that $\psi(Z) = C$ and $\Lambda^j_i \subset Z$.

but $\psi|_Z : Z \rightarrow C$ is a birational morphism. In particular, the equality $\deg(Z) = j$ holds, but $Z$ contains at least $|\Lambda^j_i|$ points of $\Sigma \subset \mathbb{P}^4$, which is impossible by Proposition 34. □

Corollary 40. The inequality $r \geq 2$ holds.

Let $\Xi^i_j$ be a base locus of the linear system of hypersurfaces in $\mathbb{P}^4$ of degree $j$ that contains the set $\Lambda^j_i$. Then $\Xi^i_j$ is a finite subset in $\mathbb{P}^5$ by Lemma 39 while we have $\Lambda^j_i \subseteq \Xi^i_j$.

Lemma 41. Suppose that $\Xi^i_j \neq \emptyset$. Then the points of the set $\Xi^i_j$ impose independent linear conditions on hypersurfaces in $\mathbb{P}^5$ of degree $5(j - 1)$.

Proof. The claim follows from Lemma 24. □

In particular, the points of the set $\Lambda^j_i$ impose independent linear conditions on hypersurfaces in $\mathbb{P}^5$ of degree $5(j - 1)$ in the case when $\Lambda^j_i \neq \emptyset$.

Lemma 42. Suppose that $\bar{\Sigma} = \emptyset$. Then there is a hypersurface in $\mathbb{P}^5$ of degree $2n + k - 6$ that contains all points of the set $\Sigma \setminus P$ and does not contain the point $P \in \Sigma$.

Proof. We have a disjoint union of the subsets
$$\Sigma = \bigcup_{j=r}^{l} \bigcup_{i=1}^{c_j} \Lambda^j_i,$$
which implies that there is a unique set $\Lambda^b_a$ that contains the point $P$. In particular, the point $P$ is contained in the set $\Xi^b_a$.

It follows from Lemma 41 that for every non-empty set $\Xi^i_j$ containing $P$ there is a hypersurface of degree $5(j - 1)$ that passes through all points of the set $\Xi^i_j \setminus P$ and does not contain the point $P$. On the other hand, the construction of the set $\Xi^i_j$ implies that for every non-empty set $\Xi^i_j$ not containing $P$ there is a hypersurface of degree $j$ that passes through all points of the set $\Xi^i_j$ and does not contain the point $P$.

We have $j < 5(j - 1)$, because $j \geq r \geq 2$ (see Corollary 40).

Thus, for every $\Xi^i_j$ containing $P$ there is hypersurface $F^i_j \subset \mathbb{P}^5$ of degree $5(j - 1)$ that contains the set $\Xi^i_j \setminus P$ and does not contain the point $P$. Consider a hypersurface
$$F = \bigcup_{j=r}^{l} \bigcup_{i=1}^{c_j} F^i_j \subset \mathbb{P}^5.$$
of degree \( \sum_{i=r}^l 5(i-1)c_i \). Then \( F \) contains \( \Sigma \setminus P \) and does not contain \( P \), but
\[
\text{deg}(F) = \sum_{i=r}^l 5(i-1)c_i < \sum_{i=r}^l 5ic_i \leq n-1 \leq 2n+k-6
\]
by Corollary 38, because \( n \geq 5 \).

Put \( \tilde{\Sigma} = \bigcup_{j=r}^l \cup_{i=1}^j A_j \) and \( \Sigma = \Sigma \setminus \tilde{\Sigma} \). Then \( \Sigma = \Sigma \cup \Sigma \) and \( \psi(\Sigma) = \Sigma \subset \Pi \).

**Remark 43.** The proof of Lemma 42 implies the existence of a hypersurface \( F \subset \mathbb{P}^5 \) of degree \( \sum_{i=r}^l 5(i-1)c_i \) such that \( F \) passes through all points of the subset \( \tilde{\Sigma} \setminus P \subset \Sigma \) and does not contain the point \( P \in \Sigma \).

Put \( d = 2n+k-6 - \sum_{i=r}^l 5(i-1)c_i \). Let us check that the subset \( \tilde{\Sigma} \subset \Pi \cong \mathbb{P}^2 \) and the number \( d \) satisfy all conditions of Theorem 15. We may assume that \( \tilde{\Sigma} \neq \emptyset \) and \( \tilde{\Sigma} \neq \emptyset \).

**Lemma 44.** The inequality \( d \geq 5 \) holds.

**Proof.** The claim follows from Corollary 38 because \( c_r \geq 1 \).

**Lemma 45.** The inequality \( |\tilde{\Sigma}| \leq (d^2 + 9d + 10)/6 \) holds.

**Proof.** Let us show that \( 6(n+k-2)(n-1-\sum_{i=r}^l 5ic_i) \) does not exceed
\[
5\left(2n+k-6 - \sum_{i=r}^l 5(i-1)c_i \right)^2 + 45\left(2n+k-6 - \sum_{i=r}^l 5(i-1)c_i \right) + 50,
\]
which implies \( |\tilde{\Sigma}| \leq (d^2 + 9d + 10)/6 \), because
\[
|\tilde{\Sigma}| < \frac{(n+k-2)}{5} \left(n-1-\sum_{i=r}^l ic_i \right)
\]
due to the inequality 37. Suppose that the inequality that we want to prove is not true, and put \( A = n-1-\sum_{i=r}^l 5ic_i \) and \( B = \sum_{i=r}^l 5c_i \). Then
\[
6A(n+k-2) > 5(A+n+k-5+B)^2 + 45(A+n+k-5+B) + 50,
\]
which is impossible, because \( A > 0 \) by Corollary 38 and \( n \geq 5 \).

**Lemma 46.** At most \( t(d+3-t) - 2 \) points of the set \( \tilde{\Sigma} \) lie on a curve of degree \( t \) for every \( t \leq (d+3)/2 \).

**Proof.** Suppose that \( t = 1 \). Then
\[
t(d+3-t) - 2 = d = 2n+k-6 - \sum_{i=r}^l 5(i-1)c_i \geq n+k-5 + \sum_{i=r}^l 5c_i \geq n+k-5 + 5c_r \geq n+k-2
\]
by Corollary 38, which implies that at most \( d \) points of \( \tilde{\Sigma} \) lie on a line by Proposition 34.

Suppose that \( t > 1 \). The points of the subset \( \tilde{\Sigma} \subset \mathbb{P}^2 \) have property \( \star \), which implies that at most \( (n+k-2)t \) points of \( \tilde{\Sigma} \) lie on a curve of degree \( t \). Therefore, we it is enough to show that
\[
t(d+3-t) - 2 \geq (n+k-2)t
\]
for all \( t > 1 \) such that \( t \leq (d+3)/2 \) and \( t(d+3-t) - 2 < |\tilde{\Sigma}| \).
It is easy to see that

\[ t(d + 3 - t) - 2 \geq t(n + k - 2) \iff n - 1 - \sum_{i=r}^{l} 5(i - 1)c_i > t, \]

because \( t > 1 \). Suppose that the inequalities

\[ n - 1 - \sum_{i=r}^{l} 5(i - 1)c_i \leq t \leq \frac{d + 3}{2} \]

and \( t(d + 3 - t) - 2 < |\Sigma| \) hold. Let us show that our assumptions lead to a contradiction. Put \( g(x) = x(d + 3 - x) - 2 \). Then \( g(x) \) is increasing for \( x \leq \frac{d + 3}{2} \). Hence, we have

\[ g(t) \geq g\left(n - 1 - \sum_{i=r}^{l} 5(i - 1)c_i\right), \]

which implies the inequalities

\[ \frac{n + k - 2}{5}(n - 1 - \sum_{i=r}^{l} 5ic_i) > |\Sigma| > g(t) \geq g\left(n - 1 - \sum_{i=r}^{l} 5(i - 1)c_i\right). \]

Let \( A = n - 1 - \sum_{i=r}^{l} 5ic_i \) and \( B = \sum_{i=r}^{l} 5c_i \). Then

\[ A\frac{n + k - 2}{5} > g(A + B), \]

where \( A > 0 \) by Corollary \( 38 \). Hence, we have

\[ 0 > 4(n + k - 2)(A + B) + 5(A + B) - 2 \geq 118, \]

which is a contradiction. \( \square \)

There is a curve \( C \subset \Pi \) of degree \( 2n + k - 6 - \sum_{i=r}^{l} 5(i - 1)c_i \) that contains \( \hat{\Sigma} \setminus \hat{P} \) and does not contain \( \hat{P} \) by Theorem \( 13 \) and there is a hypersurface \( F \) of degree \( \sum_{i=r}^{l} 5(i - 1)c_i \) that contains \( \Sigma \setminus P \) and does not contain \( P \). Let \( G \) be a cone over the curve \( C \) whose vertex is \( \Gamma \). Then \( F \cup G \) is a hypersurface of degree \( 2n + k - 6 \) that contains \( \Sigma \setminus P \) and does not contain \( P \), which concludes the proof of Theorem \( 27 \).

5. Double hypersurfaces.

Let \( \eta : X \to F \) be a double cover such that \( F \) is a smooth hypersurface of degree \( n \geq 2 \), and \( \eta \) is branched in a nodal surface \( S \subset F \) that is cut out on the hypersurface \( F \) by a hypersurface \( G \subset \mathbb{P}^4 \) of degree \( 2r \geq n \). In this section we prove the following result.

**Theorem 47.** Suppose that \( |\text{Sing}(X)| \leq (2r + n - 2)r/4 \). Then \( X \) is \( \mathbb{Q} \)-factorial.

The following result follows from Corollary \( 14 \).

**Proposition 48.** The threefold \( X \) is \( \mathbb{Q} \)-factorial if and only if the singular points of the surface \( S \) impose independent linear conditions on the sections in \( H^0(\mathcal{O}_{\mathbb{P}^4}(3r + n - 5)|_F) \).

**Corollary 49.** Suppose \( |\text{Sing}(X)| \leq 3r + n - 4 \). Then \( X \) is \( \mathbb{Q} \)-factorial.

Let us prove Theorem \( 47 \). Suppose that \( |\text{Sing}(X)| \leq (2r + n - 2)r/4 \). We are about to show that the singular points of the surface \( S \subset \mathbb{P}^4 \) impose independent linear conditions on hypersurfaces of degree \( 3r - n - 5 \). We may assume that \( r \geq 3 \) and \( n \geq 2 \), because the claim of Theorem \( 47 \) follows from Corollary \( 49 \) and \( 10 \) otherwise.
Lemma 50. There is a hypersurface \( \hat{G} \subset \mathbb{P}^4 \) of degree \( 2r \) such that the surface \( S \) is a complete intersection of \( \hat{G} \) and \( F \), but \( \text{Sing}(\hat{G}) \subseteq \text{Sing}(S) \).

Proof. See the proof of Lemma 32 \( \square \)

We may assume that \( \text{Sing}(G) \subseteq \text{Sing}(S) \). Let \( \Sigma = \text{Sing}(S) \), and \( P \) be an arbitrary point of the set \( \Sigma \). We must show the existence of a hypersurface of degree \( 3r + n - 5 \) that contains \( \Sigma \setminus P \) and does not contain \( P \). The proof of Proposition 44 implies that at most \( t(2r + n - 2) \) points of the set \( \Sigma \) lie on a curve in \( \mathbb{P}^4 \) of degree \( t \in \mathbb{N} \).

Lemma 51. Suppose that there is a plane \( \Pi \subset \mathbb{P}^4 \) such that \( \Sigma \subset \Pi \). Then there is hypersurface of degree \( 3r + n - 5 \) that contains \( \Sigma \setminus P \) and does not contain \( P \).

Proof. It follows from the proof of Lemma 35 that to conclude the proof it is enough to check that we can apply Corollary 16 to \( \Sigma \subset \Pi \) and the number \( d = 3r + n - 5 \geq 6 \).

The inequality

\[
|\Sigma| \leq \frac{d^2 + 9d + 16}{6}
\]

is obvious, because \( r \geq 3 \), \( 2r \geq n \) and \( |\Sigma| \leq (2r + n - 2)r/4 \). Therefore, we must show that at most \( t(3r + n - 2 - t) - 2 \) points of \( \Sigma \) lie on a curve of degree \( t \leq (3r + n - 2)/2 \), which implies that it is enough to show that

\[
t(3r + n - 2 - t) - 2 \geq t(2r + n - 2)
\]

for all \( t \) such that \( t \leq (3r + n - 2)/2 \) and \( t(3r + n - 2 - t) - 2 < |\Sigma| \).

We may assume that \( t \geq 2 \), because \( 3r + n - 5 \geq 2r + n - 2 \). Then

\[
t(3r + n - 2 - t) - 2 \geq t(2r + n - 2) \iff r > t.
\]

Suppose that \( r \leq t \) for some natural \( t \) such that

\[
t \leq \frac{3r + n - 2}{2}
\]

and \( t(3r + n - 2 - t) - 2 < |\Sigma| \). Put \( g(x) = x(3r + n - 2 - x) - 2 \). Then \( g(x) \) is increasing for all \( x < (3r + n - 2)/2 \), which implies that \( g(t) \geq g(r) \). Therefore, we have

\[
\frac{(2r + n - 1)r}{4} \geq |\Sigma| > g(t) \geq g(r) = r(2r + n - 2) - 2,
\]

which is impossible for \( r \geq 3 \) \( \square \).

Let \( \Pi \) and \( \Gamma \) be general plane and a line in \( \mathbb{P}^4 \) respectively, and \( \psi : \mathbb{P}^4 \rightarrow \Pi \) be a projection from the line \( \Gamma \). Put \( \Sigma' = \psi(\Sigma) \subset \Pi \cong \mathbb{P}^2 \) and \( \hat{P} = \psi(P) \in \Sigma' \).

Lemma 52. Suppose that at most \( t(2r + n - 2) \) points of the set \( \Sigma' \) lie on a possibly reducible curve of degree \( t \in \mathbb{N} \). Then there is hypersurface in \( \mathbb{P}^4 \) of degree \( 3r + n - 5 \) that contains the set \( \Sigma \setminus P \) and does not contain the point \( P \).

Proof. The proof of Lemma 51 implies the existence of a curve \( C \subset \Pi \) of degree \( 3r + n - 5 \) that contains the set \( \Sigma' \setminus \hat{P} \) and does not contain the point \( \hat{P} \). Let \( Y \) be a cone in \( \mathbb{P}^4 \) over the curve \( C \) whose vertex is a line \( \Gamma \). Then \( Y \) is a hypersurface in \( \mathbb{P}^4 \) of degree \( 3r + n - 5 \) that contains the set \( \Sigma \setminus P \) and does not contain the point \( P \in \Sigma \). \( \square \)
Therefore, to conclude the proof of Theorem 47 we may assume that the points of the set \( \Sigma' \subset \Pi \cong \mathbb{P}^2 \) do not satisfy the conditions of Lemma 52, which implies that there is a subset \( \Lambda_1^k \subset \Sigma \) such that \( |\Lambda_1^k| > k(2r + n - 2) \), but the points of the set
\[
\tilde{\Lambda}_1^j = \psi(\Lambda_1^j) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2
\]
are contained in a curve \( C \subset \Pi \) of degree \( k \). Moreover, we may assume that \( k \) is the minimal natural number of such property, which implies that \( C \) is irreducible and reduced.

We can iterate the construction of the subset \( \Lambda_1^k \subset \Sigma \) to get the disjoint union of subsets
\[
\bigcup_{j=k}^{t} \bigcup_{i=1}^{c_j} \Lambda_i^j \subset \Sigma
\]
such that \( |\Lambda_i^j| > j(2r + n - 2) \), the points of the set \( \tilde{\Lambda}_i^j = \psi(\Lambda_i^j) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2 \) are contained in a curve of degree \( t \), where \( c_j \geq 0 \) and \( c_k > 0 \). Hence, we have
\[
|\Sigma| < \frac{(2r + n - 2)r}{4} - \sum_{i=k}^{t} c_i(2r + n - 2)i = \frac{2r + n - 2}{4} \left( r - \sum_{i=k}^{t} 4ic_i \right).
\]

**Corollary 54.** The inequality \( \sum_{i=k}^{t} ic_i < r/4 \) holds.

**Lemma 55.** Let \( M \) be linear system of hypersurfaces in \( \mathbb{P}^4 \) of degree \( j \) that contains all points of the set \( \Lambda_i^j \). Then the base locus of the linear system \( M \) is zero-dimensional.

**Proof.** See the proof of Lemma 39. \( \square \)

**Corollary 56.** The inequality \( k \geq 2 \) holds.

Let \( \Xi_j^i \) be a base locus of the linear system of hypersurfaces in \( \mathbb{P}^4 \) of degree \( j \) that contains the set \( \Lambda_j^i \). Then \( \Xi_j^i \) is a finite subset in \( \mathbb{P}^4 \) by Lemma 55 such that \( \Lambda_j^i \subset \Xi_j^i \).

**Lemma 57.** The points of the set \( \Xi_j^i \) impose independent linear conditions on the hypersurface of degree \( 4(j - 1) \).

**Proof.** The required claim follows from Lemma 24. \( \square \)

The points of \( \Lambda_j^i \) impose independent linear conditions on hypersurface of degree \( 4(j - 1) \).

**Lemma 58.** Suppose that \( \bar{\Sigma} = \emptyset \). Then there is hypersurface in \( \mathbb{P}^4 \) degree \( 3r + n - 5 \) that contains set \( \Sigma \setminus P \), but does not contain the point \( P \in \Sigma \).

**Proof.** See the proof of Lemma 12. \( \square \)

Put \( \tilde{\Sigma} = \bigcup_{j=k}^{t} \bigcup_{i=1}^{c_j} \Lambda_i^j \), \( \bar{\Sigma} = \Sigma \setminus \tilde{\Sigma} \) and \( d = 3r + n - 5 - \sum_{i=k}^{t} 4(i - 1)c_i \). Then it immediately follows from the proof of Theorem 27 that to conclude the proof of Theorem 47 it is enough to check that we can apply Theorem 15 to the subset \( \bar{\Sigma} \subset \Pi \) and the number \( d \).

**Lemma 59.** The inequality \( d \geq 3 \) holds.

**Proof.** The required claim follows from Corollary 53 because \( r \geq 3 \) and \( c_k \geq 1 \). \( \square \)

**Lemma 60.** The inequality \( |\bar{\Sigma}| \leq (d^2 + 9d + 10)/6 \) holds.
Proof. Suppose that $|\Sigma| > (d^2 + 9d + 10)/6$. Then

$$6(2r + n - 2)\left(r - \sum_{i=k}^{l} 4ic_i\right) > 4(d^2 + 9d + 10),$$

and putting $A = r - \sum_{i=k}^{l} 4ic_i$ and $B = \sum_{i=k}^{l} c_i$ we see that

$$6A(2r + n - 2) > 4(2r + n - 5 + A + 4B)^2 + 36(2r + n - 5 + A + 4B) + 40,$$

where $r \geq 3$, but $A > 0$ by Corollary 54, which is a contradiction. □

Lemma 61. At most $t(d + 3 - t) - 2$ points of the set $\Sigma$ lie on a possibly reducible curve of degree $t$ for every $t \leq (d + 3)/2$.

Proof. Let us consider the case $t = 1$. Then it follows from Corollary 54 that

$$t(d + 3 - t) - 2 = d = 3r + n - 5 - \sum_{i=k}^{l} 4(i - 1)c_i \geq 2r + n - 5 + 4c_k \geq 2r + n - 2.$$

Now we consider the case $t > 1$. Then at most $(2r + n - 2)t$ points of the set $\Sigma$ lie on a curve in $\mathbb{P}^2$ of degree $t$. Therefore, to conclude the proof it is enough the show that

$$t(d + 3 - t) - 2 \geq (2r + n - 2)t$$

for every $t > 1$ such that $t \leq (d + 3)/2$ and $t(d + 3 - t) - 2 < |\Sigma|$. However, we have

$$t(d + 3 - t) - 2 \geq t(2r + n - 2) \iff r - \sum_{i=k}^{l} 4(i - 1)c_i > t,$$

because $t > 1$. Thus, we may assume that

$$r - \sum_{i=k}^{l} 4(i - 1)c_i \leq t \leq \frac{d + 3}{2}$$

and $t(d + 3 - t) - 2 < |\Sigma|$. Let us deduce a contradiction, which concludes the proof.

Put $g(x) = x(d + 3 - x) - 2$. Then $g(x)$ is increasing for $x \leq (d + 3)/2$. Hence, we have

$$g(t) \geq g\left(r - \sum_{i=k}^{l} 4(i - 1)c_i\right).$$

Put $A = r - \sum_{i=k}^{l} 4ic_i$ and $B = \sum_{i=k}^{l} c_i$. Then

$$A\frac{2r + n - 2}{4} > g(A + 4B) = (A + 4B)(2r + n - 5) - 2,$$

which is impossible, because $A > 0$ by Corollary 54. □

Thus, we proved that we can apply Theorem 15 to the subset $\Sigma \subset \Pi$ and the natural number $d$, which concludes the proof of Theorems 47 and 8.
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