Theory of Josephson effects in iron-based multi-gap superconductor junctions

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Abstract. We revisit the theory for Josephson effects in Josephson junctions with multiple tunneling channels, which can be realized when multi-gap superconductors, e.g., iron-based superconductors or MgB\textsubscript{2} are employed as the electrodes. We mainly re-examine a hetero (multi-band)superconductor- insulator-(single-band)superconductor junction. Deriving an effective Lagrangian density based on the time-dependent Ginzburg-Landau model, we discuss how the relative fluctuations between multiple gaps (i.e., Leggett’s excitation modes) and the pairing symmetry modify the Shapiro steps.

1. Introduction

Josephson effect is one of the most drastic phenomena in superconductivity \cite{1, 2}. In addition to the remarkable effects applicable to devices, it is now well-known that the Josephson junction sensitively reflect fundamental superconducting properties like Cooper pair symmetry. Hence, one can utilize Josephson junctions to explore unsettled properties of novel superconducting materials and assess their application potential.

Multi-band superconductivity has increasingly attracted interests since the discovery of an iron-based superconductor \cite{3, 4, 5, 6, 7, 8}. Various experiments \cite{9, 10} revealed that multiple bands contribute to the superconductivity and multiple superconducting full (s-wave) gaps open below the transition temperature although exceptional nodal cases have been also reported \cite{11}. Several theoretical works \cite{12, 13, 14, 15, 16, 17, 18} have proposed that a sign change occurs between the multiple s-wave gaps when a strong repulsion works between the quasiparticles on the disconnected Fermi surfaces. The order-parameter symmetry with such a sign change has been called ±s-wave. Indeed, some experiments support the type of symmetry \cite{19, 20}, but the consensus of the pairing symmetry of the iron-based superconductors have not yet been attained.

Recently, various types of Josephson junctions with iron-based superconductors have been successfully fabricated and typical Josephson effects have been confirmed \cite{21, 22, 23, 24}. Among them, a hetero Josephson junction between an iron-based and a conventional s-wave single-gap superconductors has been regarded as a possible candidate to directly detect the pairing symmetry of iron-based superconductors. The hetero junction system is theoretically described based on multiple tunneling channels, some of which are π channels and the others are 0
ones \[25, 26\], depending on the pairing symmetry in multi-gap superconductors. It is now an urgent task to propose a definitive and reliable method to detect the pairing symmetry by using such Josephson junctions with iron-based superconductors. Therefore, we reveal how the multi-channels alter basic Josephson effects and how the pair symmetry are detectable.

In this paper, we review the theory of Josephson junctions whose electrodes are composed of multi-gap superconductors. We mainly focus on the hetero \(N\)-gap superconductor-insulator-superconductor junction. Deriving the effective Lagrangian density based on the time-dependent Ginzburg-Landau model, we investigate AC Josephson effects. In particular, we reveal how the relative fluctuations between multiple gaps (i.e., Leggett’s collective excitation modes) and the pairing symmetry modify Shapiro steps.

The paper is organized as follows. In the next section, we display the present target, i.e., the hetero Josephson junction between single-gap and two-gap superconductors. Then, we begin with the general theory for Josephson junctions between single-gap and \(N\)-gap superconductors in the subsequent two sections. Section 5 is the heart of this paper. A typical Josephson effect, i.e., Shapiro step is examined on the basis of the theory reviewed in the previous sections. We stress that the result offers a clear probe to detect the pairing symmetry of the iron-based superconductors. Section 6 is devoted to summary.

2. Model

In this paper, we focus on the hetero superconductor-insulator-superconductor (SIS) Josephson junction as shown in figure 1. The left (right) electrode is a single-(two-)gap superconductor with width \(s_L\) \((s_R)\). The insulator width and the dielectric constant are \(d\) and \(\epsilon\), respectively. The bias current and the magnetic field are applied along \(z\) and \(y\) direction, respectively. We have revealed that our theory for this system is applicable to other junction systems including multi-gap superconductors, e.g., grain-boundary junctions \[27\] and intrinsic Josephson junction stacks \[28\] with proper modification.

![Figure 1. A schematic view for a hetero SIS Josephson junction. The left superconducting electrode is a single-gap s-wave superconductor, while the right one a two-gap superconductor.](image)

Let us briefly summarize theoretical studies about such a two-tunneling-channel Josephson junction. A theoretical development for the present device was done by Brinkman \textit{et al.} \[29\] and Agterberg \textit{et al.} \[30\], as for MgB\(_2\). The modification in the conventional Ambegaokar-Baratoff relation \[31\] was shown in these literatures. Moreover, proximity effects were studied in another hetero structure composed of a normal metal and a multiugap superconductor \[32\]. The observation of collective modes in two-gap superconductors through Josephson junctions was also proposed \[33\]. After the discovery of iron-pnictide superconductors, the importance of examining effects of the sign change between the superconducting gaps has grown significantly. A large amount of studies on Josephson junctions or tunneling spectroscopy have been reported in terms of several issues, e.g., the Andreev bound states \[25, 34, 36, 37\], the DC Josephson
effect \cite{26, 27, 35, 38, 39}, the Riedel anomaly \cite{40}, the Josephson vortex \cite{41}, and the AC Josephson effect \cite{42}. For the present junction, Ng and Nagaosa \cite{43} studied the two tunneling channels from the viewpoint of a time-reversal symmetry breaking state.

In the subsequent section, we review the theory of the hetero SIS Josephson junction between single-gap and $N$-gap superconductors on the basis of the microscopic standard approach for Josephson junctions \cite{44, 45}.

3. Effective action

We begin with the microscopic description for the present junction system. In the present and the next sections, we give general formulation in which the number of the electron bands in the right electrode is an arbitrary integer $N$. Afterwards, the number will be reduced to two for simplicity. The system’s Hamiltonian is given by $H = H_L + H_R + H_T$, where

$$
\hat{H}_L = \int_{\mathcal{L}} d^3r \left[ \sum_{\sigma = \uparrow, \downarrow} \bar{\psi}_{\sigma} \epsilon_{\sigma} (-i \hbar \nabla) \psi_{\sigma} - g_{s} \sum_{\text{i}, \text{j}} (\bar{\psi}_{\text{i}}^{\text{\uparrow}} \psi_{\text{j}}^{\text{\downarrow}}) \right],
$$

$$
\hat{H}_R = \int_{\mathcal{R}} d^3r \left[ \sum_{\text{i} = 1}^{N} \sum_{\sigma = \uparrow, \downarrow} \bar{\psi}_{\text{i}}^{\sigma} \epsilon_{\text{i}} (-i \hbar \nabla) \psi_{\text{i}}^{\sigma} - \sum_{1 \leq \text{i}, \text{j} \leq N} g_{\text{ij}} \bar{\psi}_{\text{i}}^{\text{\uparrow}} \psi_{\text{j}}^{\text{\downarrow}} \right],
$$

$$
\hat{H}_T = \sum_{\text{i} = 1}^{N} \int_{\mathcal{L}} d^3r \int_{\mathcal{R}} d^3r' [T_{\text{rr'}}^{(\text{i})} \bar{\phi}_{\text{i}}^{\text{\uparrow}} (\text{r)} \phi_{\text{i}}^{(\text{\uparrow})} (\text{r'}) + \text{h.c.}],
$$

The Hamiltonian $\hat{H}_L$ is for the left superconducting electrode, while $\hat{H}_R$ for the right one. The Hamiltonian $\hat{H}_T$ describes the tunneling between the electrons in the left and the right electrodes, in which the tunneling matrix element $T_{\text{rr'}}^{(\text{i})}$ is real. We assume that $g_0$ is a real positive constant, $g_{\text{ij}}$ real constants, and $\mathcal{G}(\equiv (g_{\text{ij}}))$ is a positive definite matrix. Here, $g_{\text{ij}} = g_{\text{ji}}$ ($i \neq j$) since a positive definite matrix is a Hermitian one. We also note that the diagonal element $g_{\text{ii}}$ (i.e., intra-band pairing) is positive while the off-diagonal one $g_{\text{ij}}$ ($i \neq j$) (i.e., inter-band pairing) is either positive (i.e., attractive coupling) or negative (repulsive coupling). In the present paper, the right electrode is regarded as a $N$-band Bardeen-Cooper-Schrieffer type superconductor. The theoretical description for $N = 2$ is given in \cite{46, 47}. Although the momentum dependence in $g_{\text{ij}}$ should be taken into account in iron-based superconductors, basic physical properties in the present Josephson junction, i.e., Josephson effects ascribed to the multiple tunneling channels are fully illustrated through such a simplified model.

Introducing complex auxiliary fields $\Delta^{(\text{i})}, \Delta^{(\text{i})}^{\ast}$ and $\Delta^{\text{s}}, \Delta^{\text{s}}$ through the Hubbard-Stratonovich transformation \cite{44}, the partition function in imaginary time formalism is given by $Z = Z_0 \int \prod_{\text{i} = 1}^{N} \mathcal{D} \Delta^{(\text{i})} \mathcal{D} \Delta^{(\text{i})}^{\ast} \mathcal{D} \Delta^{\text{s}} \mathcal{D} \Delta^{\text{s}} e^{-S_{\text{eff}}/\hbar}$, where

$$
S_{\text{eff}} = \int_0^{\beta} d\tau \left( \int_{\mathcal{R}} d^3r \Delta^{(\text{i})}^{\ast} \mathcal{G}^{-1} \Delta^{(\text{j})} + \int_{\mathcal{L}} d^3r \frac{\Delta^{(\text{i})}^{\ast} (\text{r})}{g_0} \right) - \text{Tr} \ln \mathcal{G}_0 - \text{Tr} \ln \mathcal{G}^{-1}.
$$

$\beta$ is the inverse temperature, and $Z_0$ is the partition function for the non-interacting part. The condition $\mathcal{G} > 0$ ensures the presence of the inverse matrix $\mathcal{G}^{-1}$ and det $\mathcal{G} > 0$. The Green functions $\mathcal{G}_0$ and $\mathcal{G}$ are given by $\mathcal{G}_0^{-1}(x; x') = (-i \hbar \partial_x - \hat{h}_0) \delta(x - x')$ and $\mathcal{G}^{-1}(x; x') = [-i \hbar \partial_x - \hat{h}_0 - \hat{D}(x)] \delta(x - x') - \sum_{i = 1}^{N} \mathcal{T}^{(i)}_{\text{rr'}} \hbar \delta(x - x')$, respectively. Here, we use the notation as $x = (\text{r}, \tau)$ (e.g., $\delta(x - x') = \delta(\tau - \tau') \delta^{(3)}(\text{r} - \text{r'})$). $\hat{h}_0$ and $\hat{D}$ are $2(N + 1) \times 2(N + 1)$ matrices, $\hat{h}_0 = \tau_3 \xi_s \otimes (\oplus_{i = 1}^{N} \tau_i) \xi_i$ and $\hat{D} = \hat{D}_s \otimes (\oplus_{i = 1}^{N} \hat{D}_i)$, where $\xi_s = \xi_a - \mu_L, \xi_i = \epsilon_i - \mu_R, \hat{D}_s = \tau_1 \text{Re} \Delta^{\ast} - i \tau_2 \text{Im} \Delta^{\ast}$, and $\hat{D}_i = \tau_1 \text{Re} \Delta^{(i)} - i \tau_2 \text{Im} \Delta^{(i)}$ ($i = 1, \ldots, N$). The chemical potential on the left (right) electrode is denoted as $\mu_L$ ($\mu_R$), and $\tau_m$ is the $m$th component of the Pauli matrices ($m = 1, 2, 3$). The matrix $\hat{I}$ is the $2(N + 1) \times 2(N + 1)$ unit matrix, and $\hat{I}^{(i)}_{\text{rr'}}$ is
also $2(N + 1) \times 2(N + 1)$ matrix. One of the characteristic terms in the Josephson junctions with multi-gap superconductors is the inter-band Josephson coupling [48, 49, 50, 51, 52] given by the off-diagonal elements of $G^{-1}$. For the case of $N = 2$, this type of the coupling reads $-2(g_{12}/ \det G)|\Delta^{(1)}| |\Delta^{(2)}| \cos(\varphi(2) - \varphi(1))$, in which $\Delta^{(i)} = |\Delta^{(i)}| e^{i\psi^{(i)}}$.

4. Dynamical equations for gauge-invariant phase differences

4.1. Effective Lagrangian density

Using the standard procedure on (4) [44, 45], we obtain the effective Lagrangian density of the superconducting phases on $zx$ plane in the real time formalism

$$\mathcal{L}_{\text{eff}} = \frac{s_L}{8\pi\mu_5^2} q_s^2 + \sum_{i=1}^{N} \frac{s_R}{8\pi\lambda_5^2} q_i^2 - \sum_{i=1}^{N} \frac{s_R}{8\pi\lambda_5^2} v_i^2 + \sum_{i=1}^{N} \frac{\hbar j_i}{e^*} (1 - \cos \theta^{(i)}) + \sum_{i<j} \frac{\hbar J_{ij}}{e^*} [1 - \cos(\varphi^{(j)} - \varphi^{(i)})] + \frac{e d}{8\pi} \left( E_{RL}^z \right)^2 - \frac{d}{8\pi} \left( B_{RL}^z \right)^2. \quad (5)$$

We note that $q_s = (\hbar/e^*) \partial_t \varphi_s + A^0_L$, $v_s = (\hbar c/e^*) \partial_x \varphi_s - A^0_L$, $q_i = (\hbar/e^*) \partial_t \varphi^{(i)} + A^0_R$, $v_i = (\hbar c/e^*) \partial_x \varphi^{(i)} - A^0_R$, and $e^* = 2e$. The gauge invariant phase difference reads $\theta^{(i)} = \varphi^{(i)} - \varphi^s - (e^* d/\hbar c) A_{RL}^L$. The phase $\varphi^s$ is defined as $\Delta^s = |\Delta^s| e^{i\varphi^s}$. $j_i$ is the Josephson critical current between $i$th and single-band Cooper pairs. The charge screening length and the penetration depth on the left (right) electrode are $\mu_s (\mu_r)$ and $\lambda_s (\lambda_r)$, respectively. The last term in the gauge-invariant phase difference is the $z$ component of the spatial averaged vector potential in the insulator, defined as $A^z_{RL} = d^{-1} \int_{-d/2}^{d/2} A^z(z) \, dz$. The electric and the magnetic fields in the insulator are defined as $E^z_{RL} = -e^{-1} \partial_t A^z_{RL} - d^{-1} (A^0_R - A^0_L)$ and $B^z_{RL} = d^{-1} (A^0_R - A^0_L) - \partial_z A^z_{RL}$, respectively. The magnitude of the coupling constant $J_{ij}$ is proportional to $s_R |(G^{-1})_{ij}|/ \det G$. If $s_R$ is much larger than $d$ as the conventional Josephson junctions, then it allows us to regard $|J_{ij}| > j_i$. We assume that $J_{ij}$ is a real number in (5). The form of the inter-band Josephson coupling depends on the number of the bands in the right electrode. For example, $N = 2$, one can notice that $J_{ij}$ must be real [48, 49]. We remark that the sign of $J_{ij}$ is a matter of convention in bulk two-gap superconductors but not true in the present hetero Josephson junction due to the presence of the single-gap superconductor in the left electrode.

4.2. Josephson relations and Maxwell equation

From (5), we find the Euler-Lagrange equation with respect to $A^z_{RL}$ which gives the Maxwell equation

$$\frac{e^* d}{\hbar c} \partial_z B^y_{RL} = \sum_{i=1}^{N} \frac{1}{\lambda^2_{li}} \sin \theta^{(i)} + \frac{e^* d}{\hbar c} \partial_t E^z_{RL}, \quad (6)$$

where $\lambda^2_{li} = 4\pi e^* d j_l / \hbar c^2$. The first term on the right hand side of (6) is the summation of the Josephson current terms. Using the Euler-Lagrange equations with respect to $A^y_{L}$ and $A^y_{R}$ ($l = R, L$), we find that the Josephson relations are given by

$$\sum_{i=1}^{N} \frac{\bar{q_i}}{\alpha_i} \partial_t \theta^{(i)} = C \frac{e^* d}{\hbar c} E^z_{RL}, \quad \sum_{i=1}^{N} \frac{\bar{\eta_i}}{\eta_i} \partial_x \theta^{(i)} = L \frac{e^* d}{\hbar c} B^y_{RL}, \quad (7)$$

where the dimensionless parameters in each electrode are defined as $\alpha_i = \epsilon \mu^2_{\alpha_i} / s_i d$, $\alpha_i = \epsilon \mu^2_{\alpha_i} / s_i d$, $\bar{\alpha}^{-1} = \sum \alpha^{-1}_{li}$, $\eta_i = \lambda^2_{li} / s_i d$, $\eta_i = \lambda^2_{li} / s_i d$, and $\bar{\eta}^{-1} = \sum \eta^{-1}_{li}$. The magnitude of the electric (magnetic) field coupling is characterized by $\alpha$ and $\alpha_i$ ($\eta$ and $\eta_i$) [53]. The constants $C$ and $L$ are defined as $C = 1 + \alpha_s + \bar{\alpha}$ and $L = 1 + \eta_s + \bar{\eta}$, respectively.
4.3. In-phase motion
The present description is valid from very thin electrode junctions \((s_L \sim \mu_s \text{ and } s_R \sim \mu_i)\) to conventional thick ones \((s_L \gg \mu_s \text{ and } s_R \gg \mu_i)\). We stress that the junction electrode size considerably affects the dynamics of the gauge-invariant phase differences in the Josephson junction with multiple tunneling channels. Let us focus on the latter case. We find that the contributions from the charge compressibility (i.e., the first and the second terms in (5)) are negligible [54]. This case corresponds to the limit \(\alpha_s, \alpha_i \to 0\). Then, the temporal behaviors of \(\varphi_s\) and \(\varphi_i\) are rigidly pinned according to \(q_s = 0\) and \(q_i = 0\). It means that
\[
\partial_t \theta^{(1)} = \partial_t \theta^{(2)} = \cdots = \partial_t \theta^{(N)} = \frac{e^* d}{\hbar} E_{RL}^z .
\] (8)

Hence, in-phase motion with respect to \(\{\theta^{(i)}\}\) is only allowed, i.e., \(\theta^{(i)}(t) = \gamma(t) + \theta_{0,i}\), where \(\theta_{0,i}\) is a constant phase. In the absence of the magnetic field, the Maxwell equation leads to
\[
0 = \sum_{i=1}^{N} \lambda_{3i}^{-2} \sin(\gamma + \theta_{0i}) + \left(\epsilon/e^2\right) \left(e^*d/\hbar\right) \partial_t^2 \gamma .
\]
Thus, we find that the system is described by a single phase \(\gamma\). When the pairing symmetry of the right superconducting electrode is \(s\)-wave or \(\pm s\)-wave, the constant phases can have 0 (for \(s\)-wave) or \(\pi\) (for \(\pm s\)-wave). Therefore, the contribution from the multiple tunneling channels can be rewritten as \(\sum_i \lambda_{3i}^{-2} \cos(\theta_{0i} - \theta_{01}) \sin \gamma\).
We can find that a cancellation between the multiple Josephson currents occurs for the \(\pm s\)-wave symmetry while those are simply superimposed for the \(s\)-wave without the sign changes.

According to the above arguments, we can obtain a bound of the Josephson critical current in the present Josephson junction (i.e., the Ambegaokar-Baratoff relation) and the resultant expression gives an effective criterion for the \(\pm s\)-wave symmetry [39]. The out-of-phase motion between \(\theta^{(i)}\) and \(\theta^{(j)}\), whose basic mode is called Leggett’s mode [48], is excited as long as the contributions from the charge compressibility is not irrelevant to the dynamics of the present junction. We will examine this point in the subsequent section, in which we focus on the vicinity of the interface in the junction whose region is characterized by about \(\mu_i\) from the interface.

4.4. Phase equations
Now, let us derive the phase equations in the hetero SIS Josephson junction. First of all, using (6)-(7), we obtain
\[
\sum_{i=1}^{N} C_{\bar{i}} \eta_i \partial^2 x_{\theta^{(i)}} = \sum_{i=1}^{N} C L_{ij} \sin \theta^{(i)} + \sum_{i=1}^{N} L_{ij} \partial^2 t \theta^{(i)} .
\] (9)

Next, from the Euler-Lagrange equations about \(\varphi^{(i)}\), we have
\[
\frac{e^*}{e^2} \frac{e^*}{\hbar} \partial_t \left( \frac{q_i}{\alpha_i} - \frac{q_j}{\alpha_j} \right) = \frac{e^*}{\hbar} \partial_x \left( \frac{v_i}{\eta_i} - \frac{v_j}{\eta_j} \right) + \frac{1}{\lambda_{3i}^2} \sin \theta^{(i)} - \frac{1}{\lambda_{3j}^2} \sin \theta^{(j)} - (F_i^{IBJ} - F_j^{IBJ}) ,
\] (10)

where
\[
F_i^{IBJ} = \sum_{i<k} \kappa_{ik} \sin(\theta^{(i)} - \theta^{(k)}) - \sum_{k<i} \kappa_{ki} \sin(\theta^{(k)} - \theta^{(i)})
\] (11)

and
\[
\frac{1}{\lambda_{3ij}^2} = \frac{4\pi e^* d |J_{ij}|}{\hbar c^2} , \quad \kappa_{ij} = \text{sgn}(J_{ij}) .
\] (12)

Equation (10) describes the dynamics of electric and magnetic imbalance between the \(i\)th and the \(j\)th bands in the right superconducting electrode. The use of the kinematic relations \(q_s/\alpha_s = -dE_{RL}^z\) and \(v_s/\eta_s = -dB_{RL}^y\) given by the Euler-Lagrange equations with respect to
\(A_0^i\) and \(A_L^i\) make (10) a more convenient formula,

\[
\frac{e}{c^2} \partial_t^2 \left( \alpha_i^{-1} \theta^{(i)} - \alpha_j^{-1} \theta^{(j)} \right) - (\alpha_i^{-1} - \alpha_j^{-1})(1 + \alpha_s) \frac{e^* d}{\hbar} \partial_t E_{RL}^y \\
= \partial_x^2 \left( \eta_i^{-1} \theta^{(i)} - \eta_j^{-1} \theta^{(j)} \right) - (\eta_i^{-1} - \eta_j^{-1})(1 + \eta_s) \frac{e^* d}{\hbar c} \partial_x B_{RL}^y, \\
+ \frac{1}{\lambda_i^2} \sin \theta^{(i)} - \frac{1}{\lambda_j^2} \sin \theta^{(j)} - (F_{i}^{1BJ} - F_{j}^{1BJ}). \tag{13}
\]

The second terms in the both left and right hand sides represent the in-phase component of the dynamics. They are relevant to the system’s dynamics as far as \(\alpha_i \neq \alpha_j\) and \(\eta_i \neq \eta_j\). Using the Josephson relation (7), we find that (13) is described only by the gauge-invariant phase difference \(\theta^{(i)}\). Summarizing the above arguments, the dynamics of the hetero SIS Josephson junction is given by simultaneously solving (9) and (13).

5. Josephson effects in the presence of multiple tunneling channels

Here, let us examine basic Josephson effects in the hetero SIS Josephson junction between single-gap and two-gap superconductors. In this case, we have only an inter-band Josephson coupling constant, \(J_{12}\). We denote it as \(J_{in}\) in what follows. Associated with this notation, we write \(\lambda_{12}\) and \(\kappa_{12}\) as, respectively, \(\lambda_{in}\) and \(\kappa_{in}\). We focus on a typical AC Josephson effect, Shapiro steps in the hetero SIS Josephson junction betwee single- and two-gap superconductors [41].

To obtain the Shapiro step, we irradiate a microwave generating the AC voltage \(V_{mw} \cos \Omega_{mw} t\) under no external magnetic field. Thus, \(\theta^{(i)}\) does not depend on the spatial variables. To understand effects ascribed to the relative phase motion, we explicitly take the finite charge compressibility inside the superconducting electrodes, in which the compressible region is confined around the interface whose thickness is characterized by the charge screening length. Namely, the parameters \(\alpha_s\) and \(\alpha_i\) become, respectively, \(\alpha_s \approx \epsilon \mu_s / d\) and \(\alpha_i \approx \epsilon \mu_i / d\). The temporal evolution of \(\theta^{(i)}\) reads

\[
\theta^{(i)}(t) = \theta_0^{(i)} + \Lambda f(t) + (-1)^{i+1} \frac{\alpha_i}{\alpha_1 + \alpha_2} \tilde{\psi}(t), \quad f(t) = \Omega_0 t + u \sin \Omega_{mw} t
\]

where \(\Omega_0 = \epsilon^* V_0 / \hbar\), and \(u = \epsilon^* V_{mw} / \hbar \Omega_{mw}\). The phases \(\theta_0^{(1)}\) and \(\theta_0^{(2)}\) are constants satisfying the relation, \(\psi_0 = \theta_0^{(1)} - \theta_0^{(2)} = 0 \ (\pi)\) for \(J_{in} > 0 \ (< 0)\). The quantity \(\tilde{\psi}(t)\) describing the relative phase fluctuation, \(\tilde{\psi}(t) = \bar{\psi}(t) - \psi_0\). In this paper, \(\bar{\psi}\) is assumed to be small. Under this assumption, we find that a linerized equation for \(\bar{\psi}\) from (13) [26], which is similar to the one for a forced harmonic oscillator,

\[
\partial_t^2 \bar{\psi} = -\omega_{IL}^2 \bar{\psi} + F_C(t) + F_J(t), \tag{15}
\]

where

\[
F_C(t) = -\xi \left( \frac{\alpha_1 + \alpha_2}{2} \Omega_{mw}^2 u \sin \Omega_{mw} t \right), \quad F_J(t) = -\nu \left( \frac{\alpha_1 + \alpha_2}{2} \left( \omega_1^2 + \omega_2^2 \right) \sin \left[ \theta_0^{(i)} + \Lambda f(t) \right] \right), \tag{16}
\]

with \(\nu = (\omega_1^2 - \kappa_{in} \omega_2^2) / (\omega_1^2 + \omega_2^2)\), where the Josephson plasma frequency for the ith band is defined as \(\omega_i = c / \sqrt{\epsilon \lambda_{1i}}\). Similarly, we introduce a frequency associated with the inter-band Josephson coupling as \(\omega_{in} = c / \sqrt{\epsilon \lambda_{in}}\). The angular frequency \(\omega_{IL}\) is given as \(\omega_{IL} = \sqrt{\alpha_1 + \alpha_2 \omega_{in}}\), which is equal to the mass of the Josephson-Leggett mode [26]. The external force terms \(F_C\) and \(F_J\) are originated from, respectively, the charge imbalance and the Cooper pair tunneling. When \(F_C\) is predominant, we can obtain an interesting approximate solution for (15) [42], \(\bar{\psi}(t) \approx A \sin \Omega_{mw} t\),
in which where \( A = -\xi[(\alpha_1 + \alpha_2)/2][(\Omega_{mw}/2\Omega_{LL})(1 - \Omega_{mw}/\Omega_{LL})] \) with \( \Omega_k = \Lambda \Omega_0 + k \Omega_{nw} \) and \( J_k(\Lambda x) \) being the Bessel function of the \( k \)th order. It should be noted that \( \bar{\psi} \) has poles at \( \Omega_{mw}^2 = \omega_{LL}^2 \). Namely \( \bar{\psi} \) is resonantly enhanced at these frequencies. The resonant condition can be satisfied by tuning the frequency of the applied microwave. In a region far from the resonance we have \( \bar{\psi} \approx 0 \). By the use of the Fourier-Bessel expansion, we obtain a condition that DC component emerges in the superconducting current, i.e., \( \Lambda \Omega_0 + n \omega_{LL} = 0 \), in which \( n \) is an integer. Hence, the voltages at which the Shapiro steps appear can be determined by the relation \( V = -n \Lambda^{-1}(h \Omega_{nw}/e^*) \). The step height is obtained as \( I_{DC} = |j_1 J_p(x_1) + j_2 \cos \psi_0 J_p(x_2)| \), where \( x_1 = \Lambda u + (\alpha_1 A)/(\alpha_1 + \alpha_2) \) and \( x_2 = \Lambda u - (\alpha_2 A)/(\alpha_1 + \alpha_2) \). Since \( A \) is large near the resonance frequency, one may have an approximate relation, \( x_1 \approx -x_2 \). In this case, employing the relation \( J_n(-x) = (-1)^n J_n(x) \), we find that [42]

\[
I_{DC} = |j_1 + (-1)^p j_2 \cos \psi_0| |J_p(x_1)|. \tag{17}
\]

The coefficient \( |j_1 + (-1)^p j_2 \cos \psi_0| \) takes alternating values with respect to \( p \), that is, \( j_1 + (-1)^p j_2 \) in the \( s \)-wave case (\( \psi_0 = 0 \)), whereas \( j_1 + (-1)^p j_2 \) in the \( \pm s \)-wave case (\( \psi_0 = \pi \)), which leads to the fact that a larger DC current appears at even (odd) \( p \) in the \( s \)-wave (\( \pm s \)-wave) case. This remarkable feature of the Shapiro steps caused by the resonance with Leggett’s mode is quite clear-cut for the determination of the pairing symmetry in multi-gap superconductors.

6. Summary

We reviewed the theoretical description for the hetero SIS Josephson junction between single-gap and \( N \)-gap superconductors. The phase equations have the out-of-phase motion as well as the conventional in-phase motion. Then, we studied one of the basic Josephson effects, the Shapiro steps. We revealed that when an external microwave applied to the hetero SIS Josephson junction resonantly excites Leggett’s mode, the voltage dependence of the Shapiro steps is sensitive to the inter-band sign change, i.e., the pairing symmetry of the multi-based superconductor. We suggest that by tuning the microwave frequency close to the Leggett’s mode one has not only novel types of Shapiro steps but also a clear probe to the gap symmetry of a multi-gap superconductor.

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