Stochastic Drag Analysis via Polynomial Chaos Uncertainty Quantification*

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In this paper, uncertainty quantification approaches are applied to an aerodynamic uncertainty quantification problem with a far-field drag breakdown approach. Two uncertainty quantification approaches, a non-intrusive polynomial chaos approach and an inexpensive Monte-Carlo simulation approach on a response surface model are compared and investigated for more advanced aerodynamic uncertainty quantification. Using the drag breakdown approach, total drag of a body can be decomposed into three physical and one unphysical drag components: wave, viscous, induced and spurious drag components. The drag source distribution can also be visualized on the flowfield using this approach. An uncertainty quantification problem of two-dimensional airfoil with four uncertain input variables is analyzed with the drag breakdown approach to extract more aerodynamic design information about its uncertainty propagation.

Key Words: Uncertainty Quantification, Polynomial Chaos, Kriging Model, Drag Breakdown

Nomenclature

\( C_d \): drag coefficient
\( C_l \): lift coefficient
\( C_p \): pressure coefficient
\( f \): factor for number of sample (collocation) points
\( M_{\infty} \): freestream Mach number
\( m \): number of sample (collocation) points
\( n \): number of uncertain input variables
\( p \): order of polynomial chaos expansion
\( Q + 1 \): total number of terms of polynomial chaos expansion
\( W \): weight function for ensemble average
\( \omega \): arbitral stochastic function/variable
\( \xi \): vector of deterministic variables
\( \xi^* \): vector of uncertain input variables
\( \alpha \): angle of attack
\( \psi \): PDF of uncertain input variables
\( \mu \): mean
\( \sigma \): standard deviation
\( \xi \): vector of random variables
\( \Psi \): polynomial chaos basis

1. Introduction

Uncertainty quantification is being watched with keen interest these days\(^1\text{-}^\text{16}\) since high-fidelity computational fluid dynamics (CFD) simulations typically assume perfect knowledge of all parameters. In reality, however, there is much uncertainty due to manufacturing tolerances, in-service wear-and-tear, approximate modeling parameters and so on.\(^1\text{-}^\text{2}\) For more advanced engineering design, therefore, it is important to analyze mean, standard deviation, probability density function (PDF) and cumulative distribution function (CDF) of performance functions with respect to multiple uncertain input variables. The most straightforward method for uncertainty analysis is a full nonlinear Monte-Carlo simulation (MCS). Although this method is easy to implement, it is still prohibitively expensive for high-fidelity CFD computations. The moment method,\(^3\) based on Taylor series expansion, is an alternative approach to assess uncertainty. Applying surrogate model approaches is another way for uncertainty analysis which can reduce the computational cost dramatically. This approach is often referred to as an inexpensive Monte-Carlo simulation (IMCS).\(^7\text{-}^\text{10}\)

The polynomial chaos (PC) method\(^11\text{-}^\text{16}\) is another positive approach to model and propagate uncertainty in computational simulations. In the PC method, uncertain parameters or variables are expanded to include random variables by utilizing orthogonal polynomials. Therefore, all dependent variables, as well as uncertain input variables, are replaced with their polynomial chaos expansions. Although this expansion can be executed in a straightforward way, expanded governing equations are relatively complicated and computationally expensive. To overcome such difficulties, non-intrusive polynomial chaos (NIPC) approaches have been developed for efficient uncertainty quantification. There are two major methodologies in NIPC, sampling-based approaches\(^13\text{-}^\text{15}\) and quadrature-based approaches.\(^13\text{-}^\text{16}\) In both methodologies, the expanded polynomial coefficients are determined through sampling or numerical integration without any change in the deterministic high-fidelity CFD simulation code. The point-collocation NIPC approach\(^14\text{-}^\text{15}\) is one of the major sampling approaches that can efficiently analyze arbitral uncertainty quantification problems with multiple uncertain input variables.

The most important parameters for the aerodynamic performance of an aircraft are known to be the lift and drag forces when cruising. Traditionally, surface integration of the pressure and stress tensor on the surface of the aircraft body, which is referred to as “Surface Integration” or the “Near-Field method,” is used for drag prediction in CFD.
computations. Recently, advanced drag prediction methods based on the theory of momentum conservation around an aircraft have attracted much attention. These approaches are often referred to as the “Far-Field method.” In these advanced approaches, total drag can be decomposed into three physical components of wave, viscous and induced drag, and one spurious drag component. Furthermore, the drag source and its generated positions can be visualized in the flowfield using this approach. The ability to decompose and visualize drag would be beneficial to extract more design information from aerodynamic uncertainty quantification problems.

In this paper, the point-collocation non-intrusive polynomial chaos method and an inexpensive MCS method are applied to an aerodynamic uncertainty quantification problem in which aerodynamic performances are analyzed by the far-field drag breakdown approach. Firstly, the effect of the number of sample points and order of polynomial chaos expansion is examined in analytic uncertainty quantification problems. Then, these approaches are applied to an aerodynamic problem of a two-dimensional (2D) airfoil in which the freestream Mach number, angle of attack, and two airfoil configuration parameters are considered as uncertain input parameters. The drag breakdown approach is utilized to extract more aerodynamic design information about its uncertainty propagation.

2. Uncertainty Quantification Approaches

In this section, the uncertainty quantification approaches utilized in this research are introduced.

2.1. Monte-Carlo simulation

One of the most straightforward approaches for uncertainty analysis is Monte-Carlo simulation. The mean of a function \( w \) with respect to uncertain inputs \( \vec{z} \) is defined as follows:

\[
\mu(w_{\vec{z}}) = \int \cdots \int (w_{\vec{z}}\phi_{\vec{z}}) \, dz_1 \cdots dz_n
\]

where \( n \) and \( \phi \) are the number of uncertain inputs and PDF of \( \vec{z} \), respectively. Taking \( m \) sample points \( \vec{z}_i \) in the \( n \) dimensional uncertain inputs space by following the distribution of \( \phi \), the mean of \( w \) can be estimated by MCS as follows:

\[
\mu(w_{\vec{z}}) \approx \frac{1}{m} \sum_{i=1}^{m} w_{\vec{z}_i}
\]

The standard deviation of \( w \) can be estimated as follows:

\[
\sigma^2(w_{\vec{z}}) \approx \frac{1}{m} \sum_{i=1}^{m} (w_{\vec{z}_i} - \mu(w_{\vec{z}}))^2
\]

The PDF and CDF of \( w \) can also be estimated from the frequency distribution of \( w_{\vec{z}_i} \). In this research, the sample points are defined by utilizing a Latin Hypercube Sampling (LHS) method. Although the MCS approach is easy to implement, its computational cost is very high, especially with the increase in the number of uncertain inputs. The number of sample points for an accurate uncertainty analysis is usually more than 1,000, which means 1,000 or more CFD functional evaluations are required for an uncertainty analysis.

2.2. Inexpensive Monte-Carlo simulation (IMCS) on Kriging model

The concept of IMCS is identical to that of MCS. However, the huge number of exact functional evaluations is replaced by functional estimations on a response surface model, which results in a reduction in computational cost. In this research, an ordinary Kriging response surface model is utilized to construct an approximate model in the uncertain inputs space. A functional output of \( w_{\vec{z}} \) is estimated as:

\[
\hat{w}_{\vec{z}} = \hat{\beta} + r_{\vec{z}}^T \mathbf{R}^{-1} (Y - \hat{\beta} \mathbf{F})
\]

\[
\hat{\beta} = (\mathbf{F}^T \mathbf{R}^{-1} \mathbf{F})^{-1} \mathbf{F}^T \mathbf{R}^{-1} Y
\]

where \( Y \) and \( \mathbf{F} \) are the observed functional vector and regression vector, respectively. \( r \) and \( \mathbf{R} \) are the correlation vector and correlation matrix, respectively. \( \mathbf{R} \) expresses the correlations between all observed data locations and \( r \) stands for the correlations between the observed data locations and a location \( \vec{z} \). These correlations are calculated using the Wendland’s C4 radial basis function (RBF) in this study. Hyper-parameters to define the width of the RBF for each input variable are estimated by maximizing the likelihood (joint probability) function of the given samples. In this research, this maximization problem is solved numerically using a real-coded genetic algorithm.

2.3. Polynomial chaos method

PC is a stochastic method which is based on the spectral representation of uncertainty. Uncertain variables are represented by a series of orthogonal polynomials as follows:

\[
w(x, \vec{z}) = \sum_{i=0}^{Q} w_i(x) \Psi_i(\vec{z})
\]

where \( w_i \) is the polynomial coefficient and \( \Psi_i \) is the polynomial chaos basis of \( i \)-th mode. \( \vec{x} \) and \( \vec{z} \), \( Q + 1 \) is the total number of terms which is defined as follows:

\[
Q + 1 = \frac{(n + p)!}{n! \cdot p!}
\]

where \( n \) and \( p \) are the number of uncertain inputs and the order of PC expansion, respectively. The choice of the PC basis depends on the distribution of uncertain inputs. The optimal basis for a random distribution is found in the Askey scheme, e.g. Hermite polynomial for Gaussian random distributions and Legendre polynomial for uniform random distributions. The general form of the multivariate Hermite polynomials is defined as:

\[
H_k(\xi_1, \ldots, \xi_p) = e^{\frac{1}{2} \xi^2} (-1)^k \frac{d^k}{d\xi_1^k} \cdots \frac{d^k}{d\xi_p^k} (e^{-\frac{1}{2} \xi^2})
\]

\[
(k = 0, \ldots, p)
\]

There is a one-to-one correspondence between \( \Psi_i \) and \( H_k \).

The one-dimensional Hermite polynomials are given as follows:
\[ \Psi_0 = 1, \quad \Psi_1 = \xi, \quad \Psi_2 = \xi^2 - 1, \quad \Psi_3 = \xi^3 - 3\xi, \ldots \]  
(8)

In cases with multiple uncertain inputs, the PC basis is defined by the product of the one-dimensional polynomials. For example, the first several 2D Hermite polynomials are given as

\[ \Psi_0 = 1, \quad \Psi_1 = \xi_1, \quad \Psi_2 = \xi_2, \quad \Psi_3 = \xi_1^2 - 1, \quad \Psi_4 = \xi_1\xi_2, \quad \Psi_5 = \xi_2^2 - 1, \ldots \]  
(9)

The ensemble average of the basis functions can be defined as:

\[ \langle \Psi_i(\bar{\xi})\Psi_j(\bar{\xi}) \rangle = \int \Psi_i(\bar{\xi})\Psi_j(\bar{\xi}) W(\bar{\xi}) d\bar{\xi} \]  
(10)

where \( W(\bar{\xi}) \) is the weight function corresponding to the type of PC basis. For the Hermite polynomials, \( W(\bar{\xi}) \) is defined as follows:

\[ W(\bar{\xi}) = \frac{1}{\sqrt{2\pi}^n} e^{-\frac{1}{2}\bar{\xi}^2} \]  
(11)

The orthogonal property of the PC basis can be expressed as:

\[ \langle \Psi_i\Psi_j \rangle = \langle \Psi_i^2 \rangle \delta_{ij} \]  
(12)

where \( \delta_{ij} \) is the Kronecker delta. In the conventional PC approach, which is often referred to as an intrusive PC approach, all uncertain inputs, variables and outputs are expanded by Eq. (5) and then governing equations are expanded with a Galerkin projection using the relationships of Eqs. (10) and (12). Finally, the number of governing equations is increased from \( n_{eq} \) (original number) to \( n_{eq}(Q + 1) \), and then all polynomial coefficients can be calculated by solving the expanded governing equations.\(^{12}\)

2.4. Non-intrusive polynomial chaos method

In the NIPC approaches, the polynomial coefficients are estimated without modifying the governing equations. In quadrature approaches, the polynomial coefficient \( w_k \) is found by projecting Eq. (5) onto \( k\)-th basis:

\[ w(\bar{x}, \bar{\xi})\Psi_k(\bar{\xi}) = \sum_{i=0}^{Q} w_i(\bar{x})\Psi_i(\bar{\xi})\Psi_k(\bar{\xi}) \quad (k = 0, \ldots, Q) \]  
(13)

Taking the ensemble average and then considering Eq. (12) yields:

\[ w_k(\bar{x}) = \frac{\langle w(\bar{x}, \bar{\xi})\Psi_k(\bar{\xi}) \rangle}{\langle \Psi_k^2(\bar{\xi}) \rangle} \quad (k = 0, \ldots, Q) \]  
(14)

The most important point of the quadrature approaches is to compute the term of \( \langle w(\bar{x}, \bar{\xi})\Psi_k(\bar{\xi}) \rangle \) by numerical quadrature in the random space.\(^{13,16}\)

The point-collocation NIPC approach\(^{14,15}\) estimates the polynomial coefficients by solving a system of equations constructed from a number of deterministic computational results \( m \). The \( m \) vectors (samples) of \( \bar{\xi}_i \) \((i = 1, \ldots, m)\) are chosen in the random space, and then \( m \) deterministic computations are executed to obtain \( w(\bar{\xi}_i) \). Finally, a linear system of equations can be obtained from Eq. (5) as:

\[
\begin{bmatrix}
\Psi_0(\bar{\xi}_0) & \cdots & \Psi_Q(\bar{\xi}_0) \\
\Psi_0(\bar{\xi}_1) & \cdots & \Psi_Q(\bar{\xi}_1) \\
\vdots & \ddots & \vdots \\
\Psi_0(\bar{\xi}_m) & \cdots & \Psi_Q(\bar{\xi}_m)
\end{bmatrix}
\begin{bmatrix}
w_0 \\
w_1 \\
\vdots \\
w_Q
\end{bmatrix} =
\begin{bmatrix}
w(\bar{\xi}_0) \\
w(\bar{\xi}_1) \\
\vdots \\
w(\bar{\xi}_m)
\end{bmatrix}
\]  
(15)

The polynomial coefficients can be evaluated by solving the linear system of Eq. (15) as:

\[
\begin{bmatrix}
w_0 \\
w_1 \\
\vdots \\
w_Q
\end{bmatrix} =
\begin{bmatrix}
\Psi_0(\bar{\xi}_0) & \cdots & \Psi_Q(\bar{\xi}_0) \\
\Psi_0(\bar{\xi}_1) & \cdots & \Psi_Q(\bar{\xi}_1) \\
\vdots & \ddots & \vdots \\
\Psi_0(\bar{\xi}_m) & \cdots & \Psi_Q(\bar{\xi}_m)
\end{bmatrix}^{-1}
\begin{bmatrix}
w(\bar{\xi}_0) \\
w(\bar{\xi}_1) \\
\vdots \\
w(\bar{\xi}_m)
\end{bmatrix}
\]  
(16)

When the number of samples is greater than \( Q + 1 \), over-determined systems of equations are solved using the Moore-Penrose pseudo-inverse matrix approach in the least-squares sense. On the other hand, when the number of samples is less than \( Q + 1 \), there are usually an infinite number of solutions that satisfy Eq. (15). The pseudo-inverse approximate solution is the one which minimizes the norm of the vector of unknown coefficients. The effect of the variation of \( m \) has been investigated by Hosder et al.\(^{15}\) by introducing a factor \( f \) as:

\[ m = f \cdot (Q + 1) \]  
(17)

\( f = 2 \) is recommended by Hosder et al.,\(^{15}\) by taking into account the accuracy of uncertainty quantification and its computational expense. In cases with multiple uncertain parameters, the number of deterministic simulations is much smaller than the quadrature approaches. This is because \( m = 2(Q + 1) \) in the point-collocation approach while \( m = (\rho + 1)^n \) in an isotropic tensor-product quadrature approach. Once all the polynomial coefficients are estimated, the mean and standard deviation of an uncertain variable can be calculated as:

\[ \mu(w) = w_0 \]  
(18)

\[ \sigma^2(w) = \sum_{i=1}^{Q} w_i^2 \langle \Psi_i^2(\bar{\xi}) \rangle \]  
(19)

PDF and CDF of \( w \) can also be evaluated inexpensively using MCS for \( \bar{\xi} \) in Eq. (5).

3. Uncertainty Quantification in Analytic Problems

In this section, analytic uncertainty quantification problems are analyzed using the point-collocation NIPC approach as well as IMCS approach via the Kriging model.
Those results are compared with exact and MCS statistics. The following analytic function is utilized in this section:

\[ w(z_1, z_2, z_3, z_4) = \exp \left( \frac{3}{2}(z_1 + z_2 + z_3 + z_4) \right) \]  

where the four input parameters of \( z_1 \sim z_4 \) are considered as uniform or Gaussian uncertain inputs \( z_i(\xi_i) \) with the mean value of 0.4 and standard deviation of 0.16. In the uniform case, the Legendre polynomial basis is utilized with uniform distributions of \( \xi_i \) that are defined in the interval \([-1, 1]\). In the Gaussian case, the Hermite polynomial basis is utilized with Gaussian distributions of \( \xi_i \) with zero mean and unit variance. The performance of the point-collocation NIPC approach is evaluated with the variation of the number of sample points \( m \), as well as the PC order \( p \). To generate a set of sample points, Gaussian or uniform LHS methods are utilized. In the IMCS approach, the same set of sample points is utilized to construct a Kriging model. Conventional MCS computations are also performed using 50,000 functional evaluations. The accuracy of the uncertainty quantification results is evaluated by absolute errors from exact statistics as follows:

\[
\text{Error} = \begin{cases} 
|\mu(w)_{\text{exact}} - \mu(w)_{\text{NIPC/IMCS/MCS}}| \\
|\sigma(w)_{\text{exact}} - \sigma(w)_{\text{NIPC/IMCS/MCS}}| 
\end{cases}
\]  

In Fig. 1, the errors in the mean and standard deviation are compared between MCS, IMCS and NIPC in the case of the uniform uncertain inputs. In this case, various sets of sample points are generated by the uniform LHS in the range of \( 0.123 \leq z_{1-4} \leq 0.677 \). While the accuracy of the IMCS results stagnated at \( m > 100 \), the NIPC approaches provided much smaller errors at larger \( m \) with \( p = 3, 5 \) for both the mean and standard deviation. In Fig. 2, the effects of \( m \) and \( p \) in the NIPC computations are visualized on the parameter space. In this figure, the lines of \( f = 1, 2 \) (see Eq. (17)) are also shown for comparison purposes. It can be understood that more accurate uncertainty quantifications are achieved with larger \( m \) and \( p \). The accuracy of uncertainty quantifications clearly degenerated in the cases of under-determined systems of equations (lower-right zone under the line of \( f = 1 \)). With a fixed value of \( p \), the increase in \( m \) beyond the line of \( f = 2 \) doesn’t provide meaningful improvement in accuracy. This result is coincidental with that of Hosder et al.\(^{15}\) in which \( f = 2 \) is recommended. In Fig. 3, the distributions of PDF of the function are compared at \( m = 100 \) and 300 (top regions are respectively enlarged). The complicated shape of the exact PDF cannot be expressed in NIPC approaches with \( p = 1 \). The PDFs of the IMCS and NIPC approaches with larger \( p \) showed good agreement with that of MCS. \((m, p) = (300, 5)\) is the only case with an over-deter-
mined condition, which gives the most accurate PDF among the NIPC cases. The accuracy of PDF was worst at \( (m, p) = (100, 10) \) among the four NIPC cases with larger \( p \), since the value of \( f \) is the smallest \((\approx 0.1)\) in the four NIPC cases.

In Fig. 4, the errors in the mean and standard deviation are compared in the case of Gaussian uncertain inputs. In this case, the sets of sample points are generated by the Gaussian LHS in the range of \(-0.1 \leq z_1 \leq 0.9\). The effects of \( m \) and \( p \) in the NIPC approach are visualized in Fig. 5. Again, the NIPC approach realized more accurate uncertainty quantifications than the IMCS approach in the over-determined conditions.

In this section, uncertainty quantification problems with four uncertain inputs were analyzed using MCS, IMCS and NIPC approaches, and then compared with exact statistics. The uncertainty quantification using the IMCS and NIPC approaches can be performed reasonably with much fewer functional evaluations than in the MCS approach. The NIPC approach was the most accurate with appropriate sets of \( p \) and \( m \), especially at the conditions of \( f \geq 2 \). The NIPC approach is computationally less expensive than IMCS since the uncertainty quantification can be performed by calculating the inverse matrix only once (see Eqs. (15) and (16)).

The NIPC approach is considered to be promising for practical uncertainty quantifications during these discussions. Since the number of collocation points (CFD evaluations in our practical cases) \( m \) has to be determined from the viewpoint of its total computational cost, the order of PC \( p \) should be subsequently set to satisfy the requirement of \( f \geq 2 \).

4. Uncertainty Quantification in an Aerodynamic Problem

In this section, an aerodynamic uncertainty quantification problem is treated with a far-field drag breakdown approach. Firstly, the CFD methodologies as well as the far-field drag breakdown approach are briefly introduced. Then, an aerodynamic uncertainty quantification problem with four uncertain input variables is discussed.

4.1. CFD methodologies

In this research, steady viscous flows around 2D airfoils are considered. Two-dimensional flowfields are analyzed using a structured mesh CFD method. In the computation, compressible Reynolds Averaged Navier-Stokes (RANS) equations are solved on a C-type mesh. In space, the viscous diffusion terms are discretized by central difference approx-
For the convection terms, the third-order monotonic upstream-centered scheme for conservation laws (MUSCL) total variation diminishing (TVD) scheme is used. The implicit lower-upper symmetric-Gauss-Seidel (LU-SGS) method\textsuperscript{26} is used for time integration. The Baldwin-Lomax algebraic model\textsuperscript{27} is adopted to treat turbulent boundary layers, and fully turbulent flow is assumed in the RANS evaluation.

Traditionally, the surface integration of the pressure and stress tensor on the body surface of the aircraft is performed to predict drag force in CFD computations. In this near-field method, the drag force is computed as follows:

\[
D = \int \left[ -(P - P_\infty) n_x + \bar{\tau}_x \cdot \bar{n} \right] dS
\]  

where the integral area body indicates the surface of the aircraft. \( P, \bar{\tau} \) and \( \bar{n} \) are the pressure, stress tensor and outward unit normal vector to a surface, respectively. The first term corresponds to the pressure drag component and the second term to the skin friction drag component.

The drag breakdown approach is based on the theory of conservation of momentum around a body. The drag force is computed as follows:

\[
D = \int \left[ -\rho (u - U_\infty) \bar{u} \cdot \bar{n} - (P - P_\infty) n_x + \bar{\tau}_x \cdot \bar{n} \right] dS
\]  

where the integral area \( S_\infty \) indicates an arbitrary closed surface around the aircraft. \( \rho, \bar{u} = (u_x, u_y, u_z) \) and \( U_\infty \) are the density, velocity vector and freestream velocity, respectively. This equation can be modified using several formulas based on the first law of thermodynamics. Finally, the irreversible part of the aerodynamic drag can be calculated as follows\textsuperscript{17}:

\[
D_{ir} = \int \int \left[ \bar{F}_{(\Delta s, \Delta H)} \cdot \bar{n} \right] dS = \int \int \bar{V} \cdot \bar{F}_{(\Delta s, \Delta H)} dV
\]  

where \( \Delta s \) and \( \Delta H \) are the variation of entropy and stagnation enthalpy, respectively. The surface integration is transformed to the flowfield volume \( (V) \) integration using the divergence theorem. \( \bar{F}_{(\Delta s, \Delta H)} \) is the entropy/enthalpy drag vector of

\[
\bar{F}_{(\Delta s, \Delta H)} = -\rho \Delta \bar{u} \bar{u} - \frac{\Delta H}{U_\infty} \left[ 1 + \frac{2 \Delta H}{U_\infty^2} - 2 \left( \frac{\mu (\gamma - 1)}{\gamma} \Delta s/R \right) - \frac{1}{\gamma - 1} \right] - U_\infty
\]

where \( \gamma, R \) and \( M_\infty \) are the specific heat ratio, gas constant and freestream Mach number, respectively. \( \Delta \bar{u} \) of Eq. (25) can be expressed in Taylor’s series as:

\[
\Delta \bar{u} = \frac{\Delta s}{R} + \frac{\Delta s}{R} \left( \frac{\Delta H}{U_\infty^2} \right) + f_{H2} \frac{\Delta H}{U_\infty^2} \left( \frac{\Delta s}{R} \right) + f_{H3} \frac{\Delta H}{U_\infty^2} \left( \frac{\Delta s}{R} \right) + O(\Delta^3)
\]

\[
f_{H1} = -\frac{1}{\gamma M_\infty^2}, \quad f_{H2} = -\frac{1}{2 \gamma M_\infty^2}, \quad f_{H3} = \frac{1}{2 \gamma M_\infty^2}, \ldots
\]

In this research, only the first- and second-order terms are considered. The terms of \( \Delta H \) can be disregarded in cases where external work is not supplied in the flow. By decomposing the flowfield \( V \) into shock wave region \( V_{\text{wave}} \), wake/ boundary layer region \( V_{\text{viscous}} \) and the remaining region \( V_{\text{spurious}} \), as in Fig. 6, the irreversible entropy drag can be decomposed as:

\[
D_{ir} = \int \int \left[ \bar{V} \cdot \bar{F}_{(\Delta s, \Delta H)} \right] dV + \int \int \left[ \bar{V} \cdot \bar{F}_{(\Delta s, \Delta H)} \right] dV
\]

\[
= D_{\text{wave}} + D_{\text{viscous}} + D_{\text{spurious}}
\]
is outside of the boundary layer region\cite{19,21}. Therefore, the spurious drag generated in the shock and viscous regions is generally insignificant. It has already been demonstrated that accurate drag prediction can be achieved by eliminating the spurious drag component.\cite{19,21} The advantages of the far-field drag breakdown method are that it can divide the entropy drag into the wave, viscous and spurious drag components, and can visualize the generated positions and the strength of the entropy drag in the flowfield, since the integrand of Eq. (24), $\nabla \cdot \mathbf{F}_{(\Delta x, \Delta M_j)}$, indicates the entropy drag production rate per unit volume.\cite{21}

The domain decomposition of the flowfield is conducted based on the following shock and viscous detection functions. For detecting the shock region, the following function is used:\cite{28}:

$$f_{\text{wave}} = \frac{\bar{u} \cdot \nabla P}{a|\nabla P|}$$

(28)

where $a$ is the speed of sound. For detecting the viscous region, the following function is used:\cite{19}:

$$f_{\text{viscous}} = \frac{\mu_1 + \mu_2}{\mu_1}$$

(29)

where $\mu_1$ and $\mu_2$ are the laminar and eddy viscosity coefficients, respectively. The regions which satisfy $f_{\text{wave}} > 1$ and $f_{\text{viscous}} > C : \left(f_{\text{viscous}}\right)_\infty$ are recognized as the upstream region of shock waves and the viscous region, respectively. $C$ is a cutoff value for the viscous region, and $C = 1.1$ is commonly used in this research. The induced (vortex) drag component can be disregarded in this study because 2D flowfields are treated.

### 4.2. Aerodynamic uncertainty quantification problem with four uncertain inputs

In this study, viscous flowfields around the RAE2822 airfoil are considered. The computational grid has the C topology of $303 \times 171$ points. The flow conditions are given as $M_\infty$ of 0.729, Reynolds number of $6.5 \times 10^6$ and angle of attack ($\alpha$) of 2.31 degrees. The (deterministic) distribution of the pressure coefficient around the RAE2822 airfoil under these flow conditions is shown in Fig. 7 with experimental data.\cite{29}

For the stochastic aerodynamic problem, four uncertain input parameters are considered in this research. One is $M_\infty$, which is treated as a Gaussian variable with a mean value of 0.729 and a standard deviation of 0.02. $\alpha$ is also treated as a Gaussian variable with a mean value of 2.31 degrees and a standard deviation of 0.2. The other two parameters are used to fluctuate the geometry of the RAE2822 airfoil. For both upper and lower surfaces of the airfoil, the following Hicks-Henne function\cite{30} is considered:

$$h(x/c) = \sin^3 \left( \pi (1 - x/c) \log \frac{0.5}{0.5(1-(x/c)_{\text{peak}})} \right) \quad \left((x/c)_{\text{peak}} = 0.55\right)$$

(30)

where $x/c$ is the non-dimensional chordwise coordinate. The two parameters $\alpha_{\text{low}}$ and $\alpha_{\text{upp}}$ are used as the weights multiplying to the Hicks-Henne function, and then the lower and upper airfoil shapes are respectively modified as:

$$\begin{cases}
    y_{\text{low}}(x/c) = y_{\text{RAE}}(x/c) + \alpha_{\text{low}} h(x/c) \\
    y_{\text{upp}}(x/c) = y_{\text{RAE}}(x/c) + \alpha_{\text{upp}} h(x/c)
\end{cases}$$

(31)

These weights are also treated as Gaussian variables with a mean of 0 and a standard deviation of 0.004. The PDFs of $M_\infty$ and $\alpha$ and fluctuated airfoil shapes (geometrical uncertainties) using the other two uncertain inputs are shown in Fig. 8. In the end, the objective of the present stochastic aerodynamic problem is to evaluate the uncertainties of aerodynamic output functions with respect to the input uncertainties that are defined in Fig. 8.

For comparison purposes, the MCS of this stochastic aerodynamic problem is also executed by generating 1,000 sample points in the four-dimensional space of the uncertain input variables. For the 1,000 points, exact CFD computations and far-field drag analyses are executed. In Fig. 9, the balance between the near-field (NF) and far-field (FF) total drag is indicated for the 1,000 points. The validity and robustness of the far-field drag prediction can be observed from this comparison. In Fig. 10, the variations of the far-field drag components with respect to $M_\infty$ are indicated. Since the spurious drag amount is less than a few drag counts (1 count $= 1 \times 10^{-4}$) with the specified computational grid resolution, it is not shown in Fig. 10. As expected, $M_\infty$
has a significant influence on airfoil drag. It can be seen that the wave drag component is generated and increases as $M_{\infty}$ increases. There is no clear relationship between the other three uncertain inputs and the far-field drag components.

For the NIPC analysis, 100 sample points were generated using a Gaussian LHS method and then evaluated by deterministic CFD computations and far-field drag analyses. In Figs. 11 and 12, the variations in mean and standard deviation of $C_l$ and $C_d$ are analyzed with the variations in $m$ and $p$. All NIPC statistics showed a certain level of agreement with MCS/IMCS statistics at over-determined conditions. The statistics of $C_l$ with $p = 5$ don’t appear in Fig. 11 due to their less accurate prediction. The IMCS statistics showed better agreement with MCS with fewer sample points. All of the statistics obtained at $m = 50$ and 100 are directly compared with the MCS statistics in Fig. 13. In this figure, the MCS results obtained from the 100 sample points are also included. The MCS statistics obtained by the 100 points are less accurate than the others, especially in the standard deviation values. It is demonstrated that all NIPC statistics obtained at $(m, p) = (100, 3)$ fall in the range of 95–105% of the MCS statistics. In the NIPC case of $(m, p) = (50, 3)$, lower accuracy in the far-field physical drag components can also be observed. This is because $f = 1.43$ in the case of $(m, p) = (50, 3)$. Like the analytic problems in the previous section, the requirement of $f \geq 2$ is essential for accurate uncertainty quantification. (In this case, this requirement corresponds to $m \geq 70$.)

In Fig. 14, the PDFs of $C_l$, $C_d$ and two physical drag components are compared between the MCS and NIPC approaches. A certain level of agreement between the MCS and NIPC PDFs can be observed in all of the aerodynamic functional outputs. The PDF of $C_{d_{\text{viscous}}}$ has the largest kurtosis and skewness, which is considered to be the reason for the inaccurate estimation of $\sigma(C_{d_{\text{viscous}}})$ with $(m, p) = (50, 3)$ in Fig. 13. Higher orders of PC expansion and larger numbers of samples are essential for more accurate estimation of such complicated PDF. The PDF of $C_{d_{\text{wave}}}$ has a wider range of function value than $C_{d_{\text{viscous}}}$, which indicates more sensitive variation of the wave drag due to the fluctuation of the four uncertain inputs.

In Fig. 15, the stochastic $C_p$ distributions are compared between the MCS and NIPC approaches. The NIPC approaches showed good agreement with that of MCS in the case of $(m, p) = (100, 3)$. Larger uncertainty can be seen on the upper surface of the airfoil, especially around the lo-
cation of the shock wave. In Fig. 16, the stochastic entropy drag production rates are visualized by the MCS and NIPC approaches. For comparison purposes, the deterministic entropy drag distribution under the mean flow and shape conditions is also visualized in Fig. 16. In the deterministic case, it is clearly understood that entropy drag is generated at the shock, wake and boundary layer regions. Quantitative agreement of the stochastic entropy drag distributions between the three cases was confirmed. These visualizations provide more aerodynamic information about the uncertainty propagation. The distribution of the entropy drag at the shock location is diffused by the fluctuation in shock-wave location due to the input uncertainties. The variation (standard deviation) in entropy drag is observed mainly on the upper surface of the airfoil, especially at the location of the shock wave and downstream of the root of the shock wave. Although an uncertain input variable was set for the deformation of the lower surface of the airfoil, it had little effect in this problem since the standard deviation of the entropy drag production rate was very small on the lower side of the airfoil. Therefore, it can be understood that the variations in the wave and viscous drag components around the shock-wave location on the upper surface of the airfoil are the dominant physical phenomena in this aerodynamic uncertainty quantification problem.

For the uncertainty analyses of flowfield variables in Fig. 16, the uncertainty quantifications have to be performed on all computational grid points. The construction of a large number of Kriging response surface models for IMCS is relatively computationally expensive. The NIPC approach is, on the other hand, computationally inexpensive since the uncertainty quantification can be performed by calculating an inverse matrix only once (see Eqs. (15) and (16)).

5. Conclusion

In this paper, two uncertainty quantification approaches, non-intrusive polynomial chaos (NIPC) and inexpensive Monte-Carlo simulation (IMCS), were applied to an aerody-
Fig. 14. PDFs of $C_t$ and $C_d$ obtained by MCS and NIPC approaches.

Fig. 15. Stochastic $C_d$ distributions obtained by MCS and NIPC approaches (error bars indicate $\mu \pm \sigma$).

Fig. 16. Visualization of stochastic entropy drag production rate. Left: mean, Right: standard deviation.
namic uncertainty quantification problem with a far-field drag breakdown approach. In the drag breakdown approach, the total drag of a 2D airfoil can be decomposed into two physical components of wave and viscous drag, and one spurious drag component. In addition, the drag source and its generated positions can be visualized in the flowfield using this approach. The drag breakdown approach can extract more aerodynamic information from CFD results, which is also considered beneficial for aerodynamic uncertainty quantification problems.

Firstly, the uncertainty quantification approaches were investigated in analytic functional problems. In these problems, the NIPC approach was the most accurate, with appropriate sets of order for PC expansion (p) and number of sample points (m). The requirement for the appropriate sets can be summarized as $f \geq 2$, which gives over-determined systems of equations in the formulation of NIPC. The number of deterministic functional evaluations required by the NIPC and IMCS approaches was much less than that of MCS. Next, the drag breakdown approach was applied to an aerodynamic uncertainty quantification problem. In this problem, freestream Mach number, angle of attack and two airfoil configuration parameters were considered as uncertain input parameters. For comparison purposes, a full nonlinear MCS was also performed in which 1,000 exact CFD computations were required. The statistics obtained by the NIPC and IMCS approaches showed good agreement with that of MCS, while their computational costs were much lower than that of MCS. Thanks to the drag breakdown approach, we were able to conduct detailed stochastic drag analysis by investigating the statistics and PDFs of the physical drag components and the distribution of the stochastic entropy drag production rate in the flowfield. This approach can easily be extended to more practical 3D problems since the drag breakdown approach has already been developed to treat 3D flowfields. (In 3D cases, the induced drag component is also extracted). This detailed information on aerodynamic drag coefficients is considered beneficial for more advanced aerodynamic uncertainty quantification, as well as for robust and reliability design optimizations under uncertainty.

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