Improved Convergence Proof of the Delta Expansion and Order Dependent Mappings

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ABSTRACT: We improve and generalize in several accounts the recent rigorous proof of convergence of delta expansion - order dependent mappings (variational perturbation expansion) for the energy eigenvalues of anharmonic oscillator. For the single-well anharmonic oscillator the uniformity of convergence in $g \in [0, \infty]$ is proven. The convergence proof is extended also to complex values of $g$ lying on a wide domain of the Riemann surface of $E(g)$. Via the scaling relation à la Symanzik, this proves the convergence of delta expansion for the double well in the strong coupling regime (where the standard perturbation series is non Borel summable), as well as for the complex “energy eigenvalues” in certain metastable potentials. Sufficient conditions for the convergence of delta expansion are summarized in the form of three theorems, which should apply to a wide class of quantum mechanical and higher dimensional field theoretic systems.
1 Introduction

A recent rigorous proof of the convergence of scaled delta expansion [1], (inspired by and developing further an earlier work by Duncan and Jones [2]), has provided for the first time a solid basis for understanding the success of the delta expansion - order dependent mappings (DE-ODM in the following).

However the proof given there has several unsatisfactory features. First, the proof does not apply as it stands to the double well case ($\omega^2 < 0$) while numerical evidence suggests that (optimized) delta expansion converges for coupling constant above some critical value (i.e., the central barrier lower than a critical height).

Also, delta expansion converges numerically well even in the $g \to \infty$ limit [2, 12], but the proof given in Ref. [1] is valid for finite $g$ only. (In other words, uniformity of the convergence in the whole $g (> 0)$ axis was not proven). Even for finite $g$ the absolute value of the remainder was numerically found to be smaller than our upper bound, suggesting that a tighter bound was possible to get, even if the bound found there was sufficient for proving the convergence for any finite $g$.

The proof of [1] furthermore does not apply if the coupling constant $g$ is taken to be complex. As the energy eigenvalues of the anharmonic oscillator and those of the double well are related by analytic continuation in $g$, it is interesting to know the domain in the complex $g$ plane in which delta expansion converges. Seznec and Zinn-Justin [3] analyzed these problems and argued in a non rigorous fashion that delta expansion or order dependent mappings converges in some complex domain of $g$.

These shortcomings of our previous proof have clearly to do with the fact that the detailed knowledge on the analytic property of the energy eigenvalue as a function of complex $g$, as explored by Bender and Wu [4], Loeffel and Martin [5], Simon [6], has not been fully exploited in the proof.

In the present paper, we improve on these points and generalize our convergence proof in several accounts. First we generalize the allowed form of the reparametrization (conformal transformation) of the coupling constant used to construct DE-ODM. Secondly, the DE-ODM for the anharmonic oscillator/double well system is proved to converge for complex coupling constant $g$ in a domain on the Riemann surface of $E(g)$, similar to the one suggested by Seznec and Zinn-Justin [3]. Also the proof is

We thank J. Zinn-Justin for stressing this point to us.
considerably simplified with respect to our previous work.

Furthermore, a general set of sufficient conditions for the convergence of DE-ODM will be given for a generic physical quantity considered as a function of the coupling constant and whose standard perturbative series is known.

To set the general background for the present work, some known facts about the scaling relation à la Symanzik, the relation between the energy eigenvalues in the single well and those in the double well oscillator, and the general analytic structure of $E(\omega^2, g)$ in $g$, will be briefly reviewed below.

Consider the anharmonic oscillator with $q^{2M}$ potential described by the Hamiltonian,

$$ H = \frac{p^2}{2} + \frac{\omega^2}{2} q^2 + \frac{g}{2M} q^{2M}. \quad (1.1) $$

By considering the scale transformation $q \rightarrow \xi q$, unitarily implemented on the Hilbert space of states (as first suggested by Symanzik), one finds [5, 6] a remarkable relation for the energy, considered as a function of $\omega$ and $g$,

$$ E(\omega^2, g) = \xi^{-2} E(\xi^4 \omega^2, \xi^{2M+2} g) \quad (1.2) $$

valid in the analyticity region for any complex number $\xi \neq 0$. This relation plays a powerful role in the study of analytic properties of $E(\omega^2, g)$.

An example of Eq. (1.2) is

$$ E_K(\omega^2, g) = g^{1/(M+1)} E_K(g^{-(2/(M+1))(M+1)}, 1) \quad (1.3) $$

valid for each level $K$. Combining this with the knowledge that the expansion in $\omega^2$ yields an analytic regular perturbation series, one proves [6] the (uniform) convergence of the large $g$ expansion,

$$ E_K(1, g^{1/(M+1)})/g^{1/(M+1)} = \sum_{n=0}^{\infty} d_n(g^{-(2/(M+1))} g^n), \quad (1.4) $$

for complex $g$ such that

$$ |g| \geq g_0 > 0 \quad (1.5) $$

(note that $g_0$ can always be increased so as to ensure the convergence at the border).

Also, the choice $\xi = e^{\pi i/2}$ leads to a periodicity relation for energy,

$$ E(\omega^2, e^{(M+1)\pi i} g) = -E(\omega^2, g). \quad (1.6) $$
Furthermore, by an appropriate choice of $\xi$ in Eq. (1.2) it is possible to relate the energy eigenvalues in the single well potential to those in the double well potential. Indeed, the choice $\xi = e^{i\pi/4}$ leads to

$$E^{(\text{DW})}_K(\omega^2, g) \equiv E_K(-\omega^2, g) = e^{-i\pi/2}E_K(\omega^2, e^{i(M+1)\pi/2}g).$$

Analytic continuation of $E(g) \equiv E(1, g)$ and the structure of its singularities in complex $g$ plane have been studied with different techniques and levels of rigor: Bender and Wu [4] and Bender et al. [7] used in their pioneering works the WKB approximation, together with some general arguments; Loeffel and Martin [5] studied analytic continuation of the exact solutions of differential Schrödinger equation, and Simon [6] used the general theory of linear operators in Hilbert spaces.

Some of their main results are:

i) $E(g)$ is analytic in the whole cut $g$ plane [5, 6];

ii) the Riemann surface has a global $M+1$-sheeted structure: a full rotation of angle $2(M+1)\pi$ at fixed $|g|$ brings back to the original value of $E(g)$ [4, 6];

iii) $E(g)$ has an infinite number of pairs of isolated square-root type branch points (Bender-Wu singularities) [4] on the symmetric positions with respect to the phase $\pm i(M+1)\pi/2$ that accumulate towards $g = 0$ (in some Riemann sheets other than the first), with asymptotic phase $e^{\pm i(M+1)\pi/2}$ [4, 6]; at these singularities in complex $g$ plane the crossing of energy levels occurs.

iv) A full closed tour of angle $2(M+1)\pi$ in $g$ plane starting from $E_K(g)$ but this time by crossing the Bender-Wu cuts several times, takes one back to the energy eigenvalue of a different level, $E_{K+n}(g)$, where $n$ depends on the number and the way the Bender-Wu cuts have been crossed [4].

v) For small enough $g$, $E(g)$ is analytic in a sectorial domain [6]: let $0 < \theta < (M+1)\pi/2$ then there is a $G$ such that $E(g)$ is analytic in the subset

$$\{ g \mid 0 < |g| < G, |\text{Arg } g| < \theta \}.$$  

\(^2\)A rigorous proof of the detailed structure of higher Riemann sheets is still lacking (to our knowledge): Simon [6] proved that the only isolated singularities of $E(g)$ are algebraic branch points with no negative powers, but he could only assume the non existence of non-isolated singularities other than $g = 0$ and of other pathologies.
of the Riemann surface. Also in any such sector the Rayleigh-Schrödinger perturbation theory is asymptotic uniformly in the angle \( \theta \).

vi) A unique strong coupling expansion Eq. (1.4) exists which converges uniformly for \( |g| \) large \( \xi \).

The rest of the paper is organized as follows. We first recall in Sec. 2 the relation between the delta expansion and the order dependent mappings, which serves also to review briefly these ideas and at the same time to define our convention. The convergence proofs are given in Sec. 3, which constitutes the main result of this paper. In Sec. 4 convergence of the optimized expansion is discussed. The case of general \( q^{2M} \) oscillators is discussed in Sec. 5. Sec. 6 discusses briefly the convergence of delta expansion for the Green’s functions of the anharmonic oscillator. Summary and Discussion is given in Sec. 7. Several technical issues and examples are collected in Appendices: Appendix A discusses some properties of the conformal transformation used in the proof, Appendix B the convergence proof of DE-ODM in a zero-dimensional analogue model. The condition on the trial parameter following from the optimization condition is discussed in Appendix C. In Appendix D we present the calculation of critical exponents in \( \phi^4_3 \) theory by use of DE-ODM. Finally a reconstruction of the strong coupling expansion coefficients from the standard perturbation series is discussed in Appendix E.

2 Delta expansion versus order dependent mapping

The delta expansion is a method to perform a systematic calculation, taking as the zeroth approximation an unperturbed action \( A_0(\{\Omega\}) \) which contains a set of trial “vacuum” parameters \( \{\Omega\} \) (e.g., trial frequency, mass). Perturbation with respect to the rest of the action \( \delta[A - A_0(\{\Omega\})] \) is calculated in the standard fashion, and the \( N \)th order result (with \( \delta = 1 \)) \( S_N^{(DE)} \) is computed by fixing \( \{\Omega\} \) order by order by some appropriate criterion, e.g., by an optimization procedure.

For the quartic quantum mechanical oscillator Eq. (1.1) with \( M = 2 \), the “unper-

\(^3\) In this respect the delta expansion is close in spirit to the Hartree or self consistent approximation. There are many works related to delta expansion. Standard earlier references can be found in [1, 2]. See [8] for more recent ones.
turbed" and "interaction" Hamiltonians can be taken as

\[ H_0(\Omega) = \frac{p^2}{2} + \frac{\Omega^2 q^2}{2}; \quad H_I(\Omega) = \frac{\omega^2 - \Omega^2}{2} q^2 + \frac{g}{4} q^4. \] (2.1)

In this case \( S_N^{(DE)} \) for the energy eigenvalues was proved [4] to converge to the exact answer as \( N \to \infty \) if the trial frequency is scaled with the order \( N \) as

\[ x_N \equiv \frac{\Omega_N}{\omega} = CN^\gamma, \] (2.2)

where either

\[ 1/3 < \gamma < 1/2, \quad C > 0, \] (2.3)

or

\[ \gamma = 1/3, \quad C \geq \alpha_c g^{1/3}, \quad \alpha_c = 0.5708751028937741 \cdots . \] (2.4)

Also the divergence of delta expansion with \( x_N \) lying outside the above range, was explicitly shown. The convergence of the "optimized" delta expansion (in which \( \Omega_N \) is determined for instance by the condition \( \partial S_N / \partial \Omega_N = 0 \), at each order), is a consequence of the above (see Sec. 4 below).

Formally, the delta expansion in this case can be generated by a substitution,

\[ \omega \to \Omega \left( 1 + \delta \frac{\omega^2 - \Omega^2}{\Omega^2} \right)^{1/2} = \omega x [1 - \delta \beta(x)]^{1/2}, \]

\[ g \to \delta \cdot g, \]

\[ x \equiv \frac{\Omega}{\omega}, \quad \beta(x) \equiv 1 - \frac{1}{x^2}, \] (2.5)

in the standard perturbation series

\[ E^{(pert)}(\omega^2, g) = \omega \sum_{n=0}^{\infty} c_n \left( \frac{g}{\omega^3} \right)^n, \]

\[ = \omega E^{(pert)}(1, g/\omega^3), \] (2.6)

followed by the expansion in \( \delta \) (and \( \delta = 1; \ \omega = 1 \) at the end).

In the order dependent mapping method of Seznec and Zinn-Justin [3], one considers a physical quantity \( S(g) \) regarded as the function of a (in general complex) coupling constant \( g \), and whose perturbative series in powers of \( g \) is known up to the \( N \)th order. One makes then a conformal transformation,

\[ g = \rho F(\beta) \] (2.7)
to the new (in general complex) variable $\beta$. The (order by order-) adjustable parameter of the transformation, $\rho$, is kept always real and positive. The transformation is such that $g = 0$ and $g = \infty$ are mapped to $\beta = 0$ and $\beta = 1$ respectively; the full range $[0, \infty]$ in $g$ is mapped to $[0, 1]$ in the space of $\beta$. For instance, in the case of the anharmonic oscillator the choice

$$F(\beta) \equiv \frac{\beta}{(1 - \beta)^{\alpha}},$$

(2.8)

with $\alpha > 0$ a fixed parameter, will work. The $N$th order approximation $S_N^{(ODM)}$ is obtained by re-expanding $S(g)$ in powers of $\beta$ up to $N$th order,

$$S(g) \rightarrow S_N^{(ODM)} = \sum_{n=0}^{N} P_n(\rho)\beta^n,$$

(2.9)

and $\rho$ is fixed by some criterion, order by order. In Ref. [3] Seznec and Zinn-Justin adopt the criterion of the fastest apparent convergence, i.e., they require $\rho$ at order $N$ to be a root of $P_{N+1}(\rho) = 0$.

In the case of the energy eigenvalues of the quartic anharmonic oscillator, the same authors considered the function $\Psi(\beta, \rho)$ related to the energy by

$$E(1, g) = \frac{1}{(1 - \beta)^{1/2}} \left[(1 - \beta)^{1/2}E(1, g(\beta))\right] \equiv \frac{1}{(1 - \beta)^{1/2}} \Psi(\beta, \rho);$$

(2.10)

with a particular conformal transformation

$$g(\beta) = \rho \frac{\beta}{(1 - \beta)^{3/2}}; \quad \beta \equiv 1 - \frac{1}{x^2}; \quad \rho = \frac{g}{x^3\beta(x)}.$$  
(2.11)

Only the function $\Psi(\beta, \rho)$ (but not the prefactor $1/(1 - \beta)^{1/2} = x$) is re-expanded in $\beta$.

Comparison between the set of equations (2.5) and (2.6), and the set Eqs. (2.10) and (2.11), shows that in this case the delta expansion and order dependent mapping (with the particular conformal transformation chosen above) coincide formally. $\beta$ in the two cases can be identified.

It is clear that the two approaches are closely related to each other. The original form of the delta expansion has a certain intuitive appeal, being physically motivated. However it is not obvious how to apply such a method to higher dimensional field theory models where mass renormalization is important. The order dependent mappings, on the other hand, being mainly mathematically motivated, appear to be more
flexible. In particular the conformal transformation in the latter can be of a more general form than suggested by the shift of the quadratic term. In the present paper these methods will be referred to indistinguishably as DE-ODM.

3 Convergence proof

In this section convergence proofs of DE-ODM are given. The convergence in the case of the quartic potential \((M = 2)\) for finite positive \(g\) is given in Sec. 3.1 for a wide class of conformal transformations, Eqs. (3.1)–(3.3). In Sec. 3.2, the convergence proof for quartic potential is generalized to complex values of \(g\) (including higher Riemann sheets of \(E(g)\)) by use of a particular conformal transformation \((\alpha = 3/2\) in Eq. (3.1)). All the (sufficient) conditions used for the convergence proof of DE-ODM are summarized in Sec. 3.3 in the form of three theorems to be applied for a generic function \(S(g)\) of \(g\).

The strategy for the proof of convergence is common in all cases, and can be summarized as follows (with reference to the energy of anharmonic oscillator).

1) Consider the energy as a function of the new variable \(\beta\), \(E(g(\beta))\), through the conformal transformation,

\[
g = \rho F(\beta) = \rho \frac{\beta}{(1 - \beta)^{\alpha}}; \quad (\rho \text{ real positive}).
\]

For consistency with delta expansion notation (see Sec. 2) it will be useful to define also the variable \(x\) by:

\[
\beta \equiv 1 - \frac{1}{x^2}; \quad \rho = \frac{g}{x^{2\alpha} \beta}.
\]

The parameter \(\alpha\) will be always chosen in the range

\[
1 < \alpha \leq 2
\]

to ensure some properties essential for the proof (see Appendix A). Since \(E(g)\) is analytic in the (open) cut \(g\) plane (see Sec. 1), the composed function \(E(g(\beta))\) is analytic (at least) in its (open) image \(D\) in \(\beta\) plane. It is shown in Appendix A that \(D\) is the interior of an apple like figure minus the negative real axis. (See Fig. 1 for the case \(\alpha = 3/2\).)
2) Introduce a function $\Psi(\beta, \rho)$ by

$$E(g(\beta)) = \frac{1}{(1-\beta)^\sigma} \Psi(\beta, \rho). \quad (3.4)$$

$E(g)$ in the case of the energy of $q^2M$-oscillator is known to behave as $g^{1/(M+1)}$ at $g \to \infty$ (see Sec. [1]); the extraction of the factor $1/(1-\beta)^\sigma$ is made so that the function $\Psi(\beta, \rho)$ is bounded at $\beta = 1$. The condition for that is:

$$\sigma \geq \frac{\alpha}{M+1}. \quad (3.5)$$

In cases we are interested in the convergence proof for values of $g$ in higher Riemann sheets, more special choices of $\sigma$ will be required; see below.

By construction $\Psi(\beta, \rho)$ is analytic in $\mathcal{D}$.

3) Write Cauchy’s theorem in the $\beta$ plane (See Fig. 1),

$$\Psi(\beta) = \oint_\Gamma \frac{d\beta'}{2\pi i} \frac{\Psi(\beta')}{\beta' - \beta}, \quad (3.6)$$

where $\Gamma$ is a closed contour which traces the boundary of the domain $\mathcal{D}$ (we assume that the energy is continuous on the two edges of the $g$ cut, see below). Note also that the contribution to the Cauchy integral from the portion of $\Gamma$ that turns around $\beta = 0$ is negligible because the uniform asymptoticity of the perturbative series for $E(g)$ implies that $E(g) \to 1/2$ for $|g| \to 0$, see Sec. [1]. In certain cases one can (and must) enlarge further the contour in the $\beta$ plane (depending on the analytic structure of $\Psi(\beta, \rho)$); such a deformation is crucial in the proof of convergence in higher sheets.

4) Taylor expand $\Psi(\beta, \rho)$ in $\beta$ up to the $N$th order

$$\Psi_N(\beta, \rho) = \sum_{n=0}^{N} \frac{1}{n!} \partial_\beta^n \Psi(0, \rho) \beta^n$$

$$= \sum_{n=0}^{N} \oint_\Gamma \frac{d\beta'}{2\pi i} \frac{\Psi(\beta')}{\beta'} \left( \frac{\beta}{\beta'} \right)^n. \quad (3.7)$$

$^4$Derivatives respect to $\beta$ in $\beta = 0$ are taken from the positive axis and can be exchanged with the integral in $\beta'$ of Cauchy formula essentially due to regularity of $\Psi$ in this region (in particular the behaviour of Disc $\Psi$ for $\beta' \to 0^-$, see Sec. [3.1]).
The remainder for the energy, \( R_N(\beta) = [\Psi(\beta, \rho) - \Psi_N(\beta, \rho)]/(1 - \beta)^\sigma \) is then given by

\[
R_N(\beta) = \frac{\beta^{N+1}}{(1 - \beta)\sigma} \int \frac{d\beta'}{2\pi i (\beta')^{N+1}(\beta' - \beta)} \Psi(\beta, \rho).
\]  \hspace{1cm} (3.8)

Study of various contributions (essentially from the two regions \( \beta' \sim 1 \) and \( \beta' \sim 0 \)) in Eq. (3.8) in the limit \( N \to \infty, \rho = g/(x^2 \beta) \to 0 \), leads to convergence proofs and the related conditions for various parameters.

3.1 Proof for the quartic oscillator \((M = 2)\) for real \( g \)

We first prove the convergence of DE-ODM to the energy eigenvalues of the quartic oscillator \((M = 2 \text{ in Eq. (1.1)})\) for any finite positive value of \( g \). Following the general steps outlined above, we make the conformal transformation

\[
g = \rho F(\beta) = \rho \frac{\beta}{(1 - \beta)^\alpha}; \quad (\rho \text{ real positive}).  \tag{3.9}
\]

with \( 1 < \alpha \leq 2 \). Finite positive real \( g \) will be mapped in \( \beta \in (0, 1) \) by this conformal transformation.

We introduce the function \( \Psi(\beta, \rho) \) by \( \Psi(\beta, \rho) = (1 - \beta)^\sigma E(g(\beta)) \) (with \( \sigma \) restricted as in Eq. (3.5)) and use the Cauchy’s theorem with the contour given in Fig. 1.

To analyze the remainder

\[
R_N(\beta) = \frac{\beta^{N+1}}{(1 - \beta)\sigma} \int \frac{d\beta'}{2\pi i (\beta')^{N+1}(\beta' - \beta)} \Psi(\beta, \rho), \tag{3.10}
\]

we divide the contour on \( \beta \) plane in two parts: \( \Gamma_A \) the part which wraps around the subset \([\beta_c \equiv -1/(\alpha - 1), 0] \) of the negative real axis, and \( \Gamma_B = \Gamma/\Gamma_A \) the rest of the contour, the upper and lower arms of the apple like curves in Fig. 1 (a complete study of \( \Gamma \) for generic \( \alpha \) is given in Appendix A).

Consider first the contribution from \( \Gamma_A \). Clearly the factor \((1 - \beta')^\sigma \) in

\[
\Psi(\beta, \rho) = (1 - \beta')^\sigma E(g(\beta')) \tag{3.11}
\]

takes the same value on the upper and lower sides of the cut along \( \beta \in [\beta_c, 0] \). Hence the discontinuity (imaginary part) of \( \Psi \) can be expressed in terms of that of \( E \) and this part of the remainder can be written as

\[
R_N^{(A)} = \frac{\beta^{N+1}}{(1 - \beta)\sigma} \int_{\beta_c}^0 \frac{d\beta'}{\pi} \frac{(1 - \beta')^\sigma \Im E(g(\beta'))}{(\beta')^{N+1}(\beta' - \beta)}. \tag{3.12}
\]
In the limit $N \to \infty, \rho \to 0$, we have

$$g(\beta') = \frac{\rho}{(1 - \beta')^\alpha} \to 0^-$$

for any value of $\beta' \in \Gamma_A$. Thus the known $g \to 0^-$ behavior of $\text{Im } E(g)$ [4],

$$\text{Im } E(g) \sim C E(-g)^{-c} \exp \left[ -\frac{a}{(-g)^{1/\alpha}} \right]$$

(3.14)

can be used in the integrand ($b \equiv M - 1 = 1, a = 4/3$ here and $c$ is a positive constant depending on the energy level considered). One then finds

$$|R_N^{(A)}| \leq \text{const.} \frac{|\beta|^N}{|1 - \beta|^\sigma} \rho^{-c} \int_0^{|\beta|} d\beta' \beta'^{(N+1+c)} \exp \left[ -\frac{a}{\rho \beta'} \right],$$

(3.15)

where the constant (const.) is independent of the order of DE-ODM and of $g$[4] and inequalities

$$|\beta - \beta'| \geq |\beta|$$

(3.16)

and

$$|1 - \beta_c| \geq |1 - \beta'| \geq 1, \rho \frac{|\beta'|}{|1 - \beta_c|^\alpha} \leq |g(\beta')| \leq \rho|\beta'|$$

(3.17)

valid for $\beta' \in \Gamma_A, \beta \in [0,1]$ have been used. Taking an upper bound by raising the upper limit of integration to infinity one gets:

$$|R_N^{(A)}| \leq \text{const.} \frac{|\beta|^N}{|1 - \beta|^\sigma} \rho^{-c} \left( \frac{\rho}{a} \right)^{N+c} \Gamma(N + c)$$

(3.18)

$$\sim (\rho N)^N \sim \left( \frac{N}{\pi^{2\sigma}} \right)^N,$$

(3.19)

where use was made of the asymptotic relation,

$$\beta = 1 - \frac{1}{x^2} \sim 1 - \left( \frac{\rho}{g} \right)^{1/\alpha}.$$  

(3.20)

Next consider the contribution from $\Gamma_B$. The range of $\alpha, 1 < \alpha \leq 2$, has been chosen so that the following inequalities

$$|\beta - \beta'| \geq |1 - \beta|;$$

(3.21)

$$|\beta'| \geq 1,$$

(3.22)

5 Any such constant will be indicated this way in the following.
hold for $\beta'$ on $\Gamma_B$ (see Appendix A) and $\beta \in [0, 1]$.

Furthermore we assume the continuity of $E(g)$ on the two edges of the negative $g$ cut. This physically reasonable assumption is motivated by the fact that all the known singularities of $E(g)$ in the nearby domain

$$g = |g|e^{i\theta}; \quad |g| > 0$$

$$\pi \leq \theta \leq \frac{3\pi}{2}, \quad \text{or} \quad -\frac{3\pi}{2} \leq \theta \leq -\pi$$

are simple square-root type branch points [4] (see footnote 2). Also, due to the fact that the energy is analytic on the cut plane, that $E(g) \sim g^{1/3}$ for large $g$, and that $E(g)$ is bounded in the limit $g \to 0$, the following bound holds on the cut plane including the two edges of the negative $g$ axis:

$$|E(g)| < (A + B|g|^{1/3}). \quad (3.24)$$

In terms of $\Psi$ we have (on the whole of $\Gamma \cup D$):

$$|\Psi(\beta, \rho)| < \left[A + B \left(\frac{\rho|\beta|}{|1 - \beta|^{1+\sigma}}\right)^{1/3}\right]|1 - \beta|^\sigma \quad (3.25)$$

from which (by use of Eq. (3.3)) one easily derives

$$|\Psi(\beta', \rho)| < \text{const.}; \quad \beta' \in \Gamma_B. \quad (3.26)$$

It follows from Eqs. (3.21), (3.22) and (3.26) that the contribution from $\Gamma_B$ to $R_N, R_N^{(B)}$, is bounded by

$$|R_N^{(B)}| \leq \text{const.} \frac{|\beta|^{(N+1)}}{|1 - \beta|^{1+\sigma}} \sim e^{-N/x^2} \sim \exp \left[-N \left(\frac{\rho}{g}\right)^{1/\alpha}\right] \quad (3.27)$$

(where Eq. (3.20) has been used).

It clear from Eqs. (3.19) and (3.27) that both pieces of the remainder, $R_N^{(A)}$ and $R_N^{(B)}$, vanish in the limit of large $N$ if $x$ or $\rho$ scale with $N$ as

$$x = CN^\gamma; \quad \frac{1}{2\alpha} < \gamma < \frac{1}{2} \quad (C > 0) \quad (3.28)$$

or

$$\rho = \frac{C'}{N^\gamma'}; \quad 1 < \gamma' < \alpha \quad (C' > 0). \quad (3.29)$$
Accordingly DE-ODM for an energy eigenvalue for the quartic oscillator, constructed with the class of conformal transformations in Eq. (3.9) with

\[ 1 < \alpha \leq 2; \]  

(3.30)

and

\[ \sigma \geq \frac{\alpha}{3}; \]  

(3.31)

converges pointwise to the exact answer for any positive \( g \), if \( x \) (or \( \rho \)) is scaled as above.

Finally, for completeness we state here that the convergence also holds if \( x \) (or \( \rho \)) is scaled precisely as

\[ x = CN^{1/(2\alpha)}; \quad \text{or} \quad \rho = \frac{C'}{N}; \]  

(3.32)

with

\[ C \geq \left( \frac{3u_*}{4} \right)^{1/(2\alpha)} g^{1/(2\alpha)}; \quad \text{or} \quad C' \leq \frac{4}{3u_*}. \]  

(3.33)

This is a special case of the Theorem 3 treated in Sec. 3.3 and \( u_* \) is a solution of the set of equations (3.77) and (3.80) (with \( b = 1 \)). In this case convergence is ensured by the factor \( |\beta|^N \) appearing in both pieces of the remainder. For a particular conformal transformation with \( \alpha = 3/2 \), it reduces to the condition with \( (3u_*/4)^{1/3} = 0.5708751089 \cdots \), found earlier [1].

**Remarks**

(i) The result of this subsection generalizes considerably the result found earlier (the case with \( \alpha = 3/2 \)) by the present authors [1] and argued for by others [3], allowing a class of conformal transformations to be used in the construction of DE-ODM. The convergence proof is also simpler than our original one.

(ii) As is suggested from the bounds Eq. (3.19) and Eq. (3.27), the convergence proof of this section can be actually extended to any complex \( g \) in the first Riemann sheet satisfying \( \text{Re} \ 1/g^{1/\alpha} > 0 \). See Theorem 1 in Sec. 3.3.

### 3.2 Proof for the anharmonic oscillator/double well system (\( M = 2 \)) with complex \( g \)

We consider the quartic oscillator (\( M = 2 \)) again but now the coupling constant \( g \) is allowed to lie anywhere in the Riemann surface of \( E(g) \). Through the scaling
relation Eqs. (1.2) and (1.7) this permits us to study simultaneously both the single and double well problems.

As before we make the conformal transformation,

$$ g = \rho F(\beta) = \rho \frac{\beta}{(1 - \beta)^\alpha}; \quad (\rho \text{ real positive}) \quad (3.34) $$

and introduce the function $\Psi$ by

$$ E(g(\beta)) = \frac{1}{(1 - \beta)^\sigma} \Psi(\beta, \rho). \quad (3.35) $$

The key observation which permits us to extend the proof to $g$ lying on higher Riemann sheets, is the following. The existence of the large $g$ expansion for $E(g)$,

$$ E_K(1, g) = g^{1/3} \sum_{n=0}^{\infty} d_n (g^{-2/3})^n, \quad (3.36) $$

with a convergence radius $g_0$, shows that for the following particular choice of parameters

$$ \sigma = \frac{\alpha}{3}; \quad \alpha = \frac{3}{2} k, \quad (k = 1, 2, 3, \ldots) \quad (3.37) $$

the function $\Psi(\beta, \rho)$ is analytic in $\beta$ inside a certain little circle $C$ of radius $r_0$ centered at $\beta = 1$, as well as in the domain $D$. The first of the conditions in Eq. (3.37) undoes the branch point at $g = \infty$ due to the overall $g^{1/(M+1)}$ factor; the second eliminates the same branch point due to the fractional powers of $g$ in the expansion parameter $\beta$.

Note that such elimination of the branch point at $\beta = 1$ is possible only for a particular choice of the conformal transformation. Indeed, the condition for analyticity in $C$, Eq. (3.37), together with the condition on the exponent $\alpha$ of the transformation, $1 < \alpha < 2$ (which is needed to keep $\beta'$ along $\Gamma_B$ outside the unit circle), implies that the only possible value for $k$ in Eq. (3.37) is $k = 1$. The only allowed conformal transformation is the one with

$$ \alpha = \frac{3}{2}. \quad (3.38) $$

From the first of Eq. (3.37) it also follows that $\sigma = 1/2$. These particular values are assumed hereafter in this subsection.

The radius $r_0$ of $C$ is easily found to be, in the limit $N \to \infty$,

$$ r_0 \sim \left( \frac{\rho}{g_0} \right)^{2/3}, \quad (3.39) $$

---

6 By use of Weierstrass’ theorem on series of uniformly convergent analytic functions.
by use of Eq. (3.20).

Since now the function $\Psi(\beta, \rho)$ is analytic in the joint region $D \cup C$, the contour $\Gamma$ can be enlarged to $\Gamma'$ so as to trace its boundary. See Fig. 2. As will be seen shortly, this apparently minor deformation of the integration contour is crucial. A further enlargement of the radius of $C$ is not possible because the images of the Bender-Wu branch points appear just outside $C$. The smallest distance from one of those to the point $\beta = 1$ shrinks to zero together with $r_0$ as $N \to \infty$.

Before proceeding further, some subtleties involved in analytic structure of the function $\Psi$ and in the consequent enlargement of the contour $\Gamma \to \Gamma'$ should be explained. First let us note that conformal transformation $g = \rho \beta/(1 - \beta)^{3/2}$ introduces the doubling of the number of the global Riemann sheets (six) in $E(g(\beta))$ considered as a function of $\beta$ as compared to the situation for $E(g)$ (three). As is clear from the form of the strong coupling expansion Eq. (3.36), there are a square-root branch point at $\beta = 1$ and a (global) cubic-root branch point at $\beta = 0$ ($\beta = \infty$ is a sixth root type branch point).

A detailed study of the conformal mapping from the $g$ Riemann surface to $\beta$ Riemann surface shows the following correspondence. Two values of $\beta$ with the same (geometrical) position on a pair of $\beta$ Riemann sheets, connected by going around $\beta = 1$ once (which may be called “mirror” Riemann sheets), are mapped to values of $g$ differing by the phase $e^{3\pi i}$. For example, the domain $D$ of Fig. 2 corresponding to the first and its mirror (second) sheet, is the image of the cut plane $-\pi < \text{Arg } g < \pi$ and the angular region $2\pi < \text{Arg } g < 4\pi$, respectively. Analogously, the part of the circle $C$ which lies outside $D$ (Fig. 2), represents the image of the region $\pi < \text{Arg } g < 2\pi$; $|g| > g_0$ and that of $4\pi < \text{Arg } g < 5\pi = -\pi$; $|g| > g_0$.

Now, the analyticity of $\Psi$ in $C$ around $\beta = 1$ implies, due to the uniqueness of analytic continuation, that the function $\Psi$ takes exactly the same value at corresponding points in each pair of mirror Riemann sheets.

In terms of $E = (1 - \beta)^{1/2}\Psi$ this implies a relation

$$E(e^{3\pi i}g) = -E(g)$$

(3.40)

for any $g$. (This relation of course follows immediately from the scaling relation,

$g_0$ is larger than the largest absolute value of the Bender-Wu branch points. For otherwise there would be values of $g$ such that $|g| > g_0$ and for which the strong coupling expansion for $E/g^{1/3}$ does not converge, which is a contradiction.
see Eq. (1.6). Nonetheless the above derivation, which essentially hinges upon the existence of a strong coupling expansion, can be used in a more general situation: See Sec. 3.3.

The true meaning of “enlarging the contour from $\Gamma$ to $\Gamma'$” can be appreciated now: As $\beta'$ moves along $\Gamma'$ twice, first in the first sheet and continuing in the second sheet and then back, its image $g'$ goes through the full $g$ Riemann surface once, as shown in Fig. 3.

Nevertheless, in applying Cauchy’s theorem for $\Psi(\beta, \rho)$ one may ignore the double looping of the $\Gamma'$ contour: on whichever sheet a given $\beta$ may lie, the integration over one of the loops vanishes, the factor $1/(\beta' - \beta)$ being analytic inside it.

Coming back to the convergence proof, for any $\beta$ lying inside $\mathcal{D} \cup \mathcal{C}$ we study the remainder

$$R_N(\beta) = \frac{\beta^{N+1}}{(1-\beta)^{1/2}} \int_{\Gamma'} \frac{d\beta'}{2\pi i} \frac{\Psi(\beta')}{(\beta')^{N+1}(\beta' - \beta)}. \quad (3.41)$$

By splitting again the contour in two parts, $\Gamma_A$ the part wrapping around the negative real axis (common to $\Gamma$ and $\Gamma'$), and $\Gamma'_B = \Gamma'/\Gamma_A$, the rest of the contour (see Fig. 2).

The contribution from $\Gamma_A$ can be treated as done in Sec. 3.1, except that $\alpha = 3/2$ and variables $g$ and $\beta$ are complex now. By use of Eq. (3.17) and of

$$|\beta - \beta'| \geq K_\beta \equiv \max(|\text{Im} \beta|, \text{Re} \beta), \quad (3.42)$$

one finds easily

$$|R_N^{(A)}| \leq \text{const.} \frac{|\beta|^{N+1}}{K_\beta |1-\beta|^{1/2}} \frac{\rho^{-c}}{(\rho/a)^{N+c}} \Gamma((N+c))$$

$$\sim \frac{(\rho N)^N}{|1-\beta|^{1/2}} \sim |g|^{1/3}(\rho N)^N, \quad (3.43)$$

where an asymptotic relation $\beta \sim 1 - (\rho/g)^{2/3}$ has been used.

The appearance of $K_\beta$ in the denominator of the first line of Eq. (3.43) means that the values of $\beta$ in $\mathcal{D} \cup \mathcal{C}$ which are arbitrarily close to the real negative axis, must be excluded. We shall see shortly that the remainder vanishes as $N \to \infty$ only for $g$ lying on a certain region of the Riemann surface (Eq. (3.49)). For any $g$ there and for any $N \geq N_0$ (with $N_0$ independent of $g$) the inequality $K_\beta \geq \text{const.} |\beta|$ holds, hence the convergence will indeed be uniform in $g$ in that region.
There is an important difference in the estimate of $R_N^{(B)}$ due to the enlargement of the contour from $\Gamma$ to $\Gamma'$. For $\beta'$ on $\Gamma_B'$, the following inequality holds:

$$|\beta'| \geq 1 + r_0 \cos(\phi_\alpha) + O(r_0^2)$$  \hspace{1cm} (3.44)

where $\phi_\alpha \equiv \pi(1 - 1/\alpha)$ is the angle between the apple-like curve and the positive $\beta$ axis in $\beta = 1$, see Appendix A. For the generic range $1 < \alpha < 2$ of $\alpha$, the angle $\phi_\alpha$ lies in the region $0 < \phi_\alpha < \pi/2$ so that $\cos(\phi_\alpha) > 0$. For the particular conformal transformation with $\alpha = 3/2$ we are considering here, $\phi_\alpha = \pi/3$; hence $\cos(\phi_\alpha) = 1/2$.

Also, to bound the factor $1/|\beta' - \beta|$, one must restrict $g$ to range in a complex domain such that the corresponding variable $\beta$ satisfies

$$|\beta' - \beta| \geq \epsilon r_0 \quad \forall \beta' \in \Gamma_B',$$  \hspace{1cm} (3.45)

where $\epsilon$ is a small positive fixed ($\beta$ independent) constant of order $O(1)$. Appearance of $r_0$ (radius of $C$) in the condition above is due to the fact that for any fixed finite $g$ its image $\beta$ approaches 1 in the limit $N \to \infty$ and that the nearest $\beta'$ are those on the boundary of $C$. In terms of $g$, it is sufficient that $g$ is restricted to be separated from the inverse image of the contour $\Gamma_B'$ (Fig. 3), by a fixed ($g$ independent) finite distance. It will be seen below that the condition for the (uniform) exponential vanishing of the remainder automatically ensures such a condition to be satisfied.

Furthermore, since $E(g) \sim g^{1/3}$ at large $g$ on the whole Riemann surface, $\Psi$ is bounded in the neighborhood of $\beta = 1$ (see Eq. (3.3)):  \hspace{1cm} (3.46)

Collecting these results one gets

$$|R_N^{(B)}| \leq \text{const} \frac{|\beta|^{N+1}}{|1 - \beta|^{1/2}r_0} \left( \frac{1}{1 + r_0/2} \right)^{N+1}$$

$$\sim |g|^{1/3} \exp \left[ -N \rho^{2/3} \left( \text{Re} \frac{1}{g^{2/3}} + \frac{1}{2g_0^{2/3}} \right) \right],$$  \hspace{1cm} (3.47)

where Eq. (3.39) has been used. Note the presence of the $g$-independent exponential factor in Eq. (3.47) which was absent in Eq. (3.27).

From Eqs. (3.43) and (3.47) one sees that the full remainder $R_N^{(A)} + R_N^{(B)}$ vanishes (uniformly) in the limit $N \to \infty$ with

$$\rho = \frac{C'}{N^{\gamma'}} \quad \text{with} \quad 1 < \gamma' < 3/2,$$  \hspace{1cm} (3.48)
if the complex values of \( g \) are restricted in (any closed subset of) the region
\[
\Re \left( \frac{1}{g^{2/3}} + \frac{1}{2g_0^{2/3}} \right) > 0.
\]
(3.49)

The regions excluded by such a condition are one heart-like closed domain \( \mathcal{H}_1 \)
\[
|g|^{2/3} \leq -2g_0^{2/3} \cos \frac{2\theta}{3}; \quad \frac{3\pi}{4} < \theta < \frac{9\pi}{4}; \quad \theta \equiv \text{Arg} \, g,
\]
and an analogous domain \( \mathcal{H}_2 \) with \( \theta \to -\theta \). See Fig. 3. The boundary curves of these domains cross the straight lines corresponding to \( \text{Arg} \, g = \pi, 2\pi, 4\pi (= -2\pi), \)
\( 5\pi (= -\pi) \) at modulus \( |g| = g_0 \).

As is clear from Fig. 3, the (inverse) image of the contour \( \Gamma' \) lies inside \( \mathcal{H}_1 \cup \mathcal{H}_2 \). It follows that the condition for the uniform vanishing of the remainder, Eq. (3.49), automatically guarantees Eq. (3.45).

We have thus proved that DE-ODM for an energy eigenvalue of the quartic oscillator, constructed with the conformal mapping with \( \alpha = 3/2 \) and with \( \sigma = 1/2 \), uniformly converges to the correct analytic continuation \( E(g) \) in every closed subset of the region
\[
\Re \left( \frac{1}{g^{2/3}} + \frac{1}{2g_0^{2/3}} \right) > 0,
\]
(3.51)
if \( \rho \) or \( x \) is scaled as
\[
\rho = \frac{C'}{N^{\gamma'}}; \quad 1 < \gamma' < \frac{3}{2},
\]
(3.52)
or
\[
x = cg^{1/3}N^\gamma; \quad \frac{1}{3} < \gamma < \frac{1}{2}
\]
(3.53)
\((C' \text{ or } c \text{ is real and independent of } g \text{ and } N)\).

As in the previous case, the convergence holds for the extreme case of scaling with
\[
\rho = \frac{C'}{N}; \quad \text{or} \quad x = cg^{1/3}N^{1/3},
\]
\( c > c_* = 0.5708751089 \cdots \) or \( C' < \frac{1}{c_*^3} \).

(3.54)

**Remarks**

(i) The \( g \)-independent factor in the exponent of Eq. (3.47) shows that even for real finite \( g \), the bound for the remainder is actually numerically smaller than suggested by Eq. (3.27), although the exponential behavior \((|R| < \exp(-\text{const. } N^{1-2\gamma'/3}))\) is the same. The convergence is fastest for the choice \( \gamma' = 1 \).
(ii) The same $g$ independent factor is especially important for the strong coupling limit, $g \to \infty$. By using Eqs. (3.43) and (3.47) we prove that DE-ODM converges to the correct answer for \( \lim_{g \to \infty} E(g)/g^{1/3} \), if $x$ or $\rho$ is scaled according to Eq. (3.53) or to Eq. (3.54), and if the limit $g \to \infty$ is taken at each order in $N$, before $N$ is sent to $\infty$. This remarkable fact was noted earlier empirically.

(iii) As is noted in Introduction, $E(g)$ continued analytically to $g = |g|e^{3\pi i/2}$ is (apart from a phase factor $-i$) equal to the energy eigenvalue of a double well. The result above proves that DE-ODM for the quantum double well converges to the correct answer as long as $|g| > 2^{3/2}g_0$. This explains the empirical fact that lower order (optimized) delta expansion for the double well gives good approximate results for coupling constant above some critical value (or the central barrier below a critical height) but fails below it. From our own numerical study of DE-ODM for the double well up to $N \simeq 50$, we find a numerical value $g_0 \approx 0.10 - 0.15$. Due to the fact that Eq. (3.51) refers to an upper bound of the remainder, this should be considered as an order of magnitude estimate for the convergence radius of the strong coupling expansion. See the following remark.

iv) Our result depends on the knowledge of analyticity of $E(g)$ in the cut $g$ plane: Bender-Wu singularities are confined in the angular region, $\pi < \text{Arg} g < 2\pi$; $|g| < g_0$ (and a similar region with $g \to -g$), but no further details on their location have been used. A tighter bound (and a wider convergence region in $g$) can be obtained by a further deformation of the contour near $\beta = 1$, if $E(g)$ turns out to be actually analytic in a wider region. Indeed, if $E(g)$ would be analytic in the region, $|\text{Arg} g| < \pi + \Delta$, $0 < \Delta < \pi/2$, then $1/(2g_0^{2/3})$ in Eqs. (3.47), (3.49), (3.51) would be replaced by $\cos[(\pi - 2\Delta)/3]/g_0^{2/3}$. The estimate of $g_0$ from our numerical study of delta expansion for the double well, would accordingly be changed to $0.10 \leq g_0 \leq 0.42$, which seems to be consistent with the known strong coupling expansion coefficients [2, 11].

---

8 The $N$th order delta expansion approximant for the double well is precisely \( \Phi \) given by the substitution $x^2 \to -x^2$ (that is, $\omega^2 \to -\omega^2$) made in the corresponding approximant of the single-well anharmonic oscillator. It follows that the remainder for the double well is correctly given by $R_N^{(A)} + R_N^{(B)}$ apart from an overall factor $i$.

9 This was noted independently by Okopinska [9], by Kleinert [10], by the present authors (unpublished), and by E. Hofstetter (private communication).
(v) $E(e^{i\pi}g)$ with real positive $g$ corresponds to the energy in an unstable potential well. According to our result, the complex energy “eigenvalues” of such a system can be correctly reconstructed by delta expansion/order dependent mapping, as long as $g > g_0$ (sliding regime, rather than tunnelling). This nicely explains the empirical success of a variational-perturbative approach discussed in Ref. [13].

### 3.3 General sufficient conditions for convergence

In this subsection the (sufficient) conditions for the convergence of DE-ODM are summarized in the form of three theorems.

**Theorem 1:** Let a function $S(g)$ be given such that:

1) $S(g)$ is analytic in the complex $g$ plane cut along negative axis, continuous on the two edges of the cut and bounded at $g = 0$ (on this sheet);

2) $S(g) \sim g^p$ with $p > 0$ for $g \to \infty$ on the cut plane;

3) The discontinuity of $S(g)$ along the cut behaves as

$$\text{Disc } S(g) \sim C_S(-g)^{-c} \exp \left[ -\frac{a}{(-g)^{1/b}} \right]$$

for $g \to 0^-$ with $a > 0$, $1 \leq b < 2$, $c$ real,

then the sequence $\{S_N(g)\}$ of approximants for $S(g)$, constructed with conformal transformation

$$g = \rho F(\beta) = \rho \frac{\beta}{(1 - \beta)^\alpha}; \quad (\rho \text{ real positive})$$

with

$$1 < \alpha \leq 2,$$

followed by a Taylor expansion in $\beta$ up to order $N$ of

$$\Psi(\beta, \rho) \equiv (1 - \beta)^\sigma S(g(\beta))$$

with

$$\sigma \geq \alpha p,$$
converges to $S(g)$ as $N \to \infty$ uniformly in every compact subset of the region

$$\text{Re} \frac{1}{g^{1/\alpha}} > 0,$$

(3.60)

if the real positive parameter $\rho$ is chosen to scale with $N$ as

$$\rho = \frac{C'}{N^{\gamma'}}; \quad \text{with} \quad b < \gamma' < \alpha. \quad (3.61)$$

($C'$ independent of $g$.)

Theorem 2: Let $S(g)$ be given which satisfies 1)–3) of Theorem 1 and moreover:

2') its large $g$ expansion

$$S(g) = g^p \sum_{n=0}^{\infty} d_n(g^{-q})^n$$

(3.62)

converges uniformly for $|g| \geq g_0$ and there exists a strictly positive integer $k$ such that

$$q < k < 2q. \quad (3.63)$$

Then the sequence $\{S_N(g)/g^p\}$ with $S_N(g)$ constructed as in Theorem 1 with the conformal transformation Eq. (3.56) with $\alpha = k/q$ (thus lying in the range $1 < \alpha < 2$), by use of the function $\Psi(\beta, \rho)$ with $\sigma = \alpha p$, converges to $S(g)/g^p$ as $N \to \infty$, uniformly in each closed subset of the domain (including $g = \infty$)

$$\text{Re} \frac{1}{g^{1/\alpha}} + \frac{1}{g_0^{1/\alpha}} \cos [\pi(1 - 1/\alpha)] > 0\quad (3.64)$$

for any choice of scaling Eq. (3.61) of the positive parameter $\rho = C'/N^{\gamma'}$. ($C'$ independent of $g$.)

Proof of Theorem 1: To prove Theorem 1 one proceeds as in Sec. 3.3.1: using 1) and 2) one can write a dispersion relation as in Eq. (3.6) for $\Psi$ from which the expression Eq. (3.8) for the remainder follows. In particular from 1) and 2) one derives $|S(g)| \leq A + B|g|^p$ in the cut plane including the edges of the cut along the negative real axis, hence (with the choice Eq. (3.59)) $\Psi$ is bounded by a constant on $\Gamma \cup \mathcal{D}$ ($\Gamma$ and $\mathcal{D}$ are defined in Sec. 3). The integration contour $\Gamma$ is then split in $\Gamma_A$ and $\Gamma_B$ as in Sec. 3.1.
As for $R_N^{(A)}$, one expresses the discontinuity of $\Psi$ along $\beta' \in [\beta_c, 0]$ by that of $S(g(\beta'))$, obtaining

$$R_N^{(A)} = \frac{\beta^{N+1}}{(1 - \beta)^\sigma} \int_{\beta_c}^{0} \frac{d\beta'}{2\pi i} (1 - \beta')^\sigma \text{Disc} S(g(\beta')).$$

(3.65)

Then by using Eq. (3.55) for Disc $S(g)$ (remember that $g(\beta') \to 0^-$ as $N \to \infty$ for $\beta' \in [\beta_c, 0]$), the inequality

$$|\beta - \beta'| \geq \text{const.} |\beta|,$$

(3.66)

(which is valid for $g$ in (3.60)) and the fact that the inequalities

$$1 \leq |1 - \beta'| \leq \text{const.},$$

$$\text{const.} \rho|\beta'| \leq |g(\beta')| \leq \rho|\beta'|$$

(3.67)

hold (uniformly) on $\Gamma_A$, one finds an upper bound as was done in Eq. (3.15). Extending the integration range up to $\beta' = \infty$ one gets (on the chosen compact):

$$|R_N^{(A)}| \leq \text{const.} \frac{|\beta|^N}{|1 - \beta|^\sigma} \rho^{-c} (\frac{\rho}{a^b})^{N+c} \Gamma(b(N + c)) \sim |g|^\sigma/\alpha (\rho a^b)^N.$$

(3.68)

As for $\Gamma_B$ contribution we observe that the choice of $\alpha$ Eq. (3.57) is essential to derive the inequality (see Appendix A),

$$|\beta'| \geq 1.$$

(3.69)

Also, for each compact subset of the domain Eq. (3.60) the inequality

$$|\beta - \beta'| \geq \text{const.} |1 - \beta|$$

(3.70)

holds (uniformly in $\beta$, $\beta'$). One immediately gets (exploiting the boundedness of $\Psi$)

$$|R_N^{(B)}| \leq \text{const.} \frac{|\beta|^{(N+1)}}{|1 - \beta|^{1+\sigma}} \sim \exp \left[ -NRe \left( \frac{\rho}{g} \right)^{1/\alpha} \right].$$

(3.71)

Theorem 1 easily follows from Eqs. (3.68), (3.71) and (3.61) (note that the upper limit on $\gamma'$ comes from $R_N^{(B)}$ while the lower limit from $R_N^{(A)}$). Q.E.D.

**Proof of Theorem 2** The new hypothesis $2')$ on the existence of a large $g$ expansion of $S(g)$, together with the choices $\alpha q = k$ and $\sigma = \alpha p$, ensures (see footnote [3]) the analyticity of $\Psi$ in the extended region $D \cup C$ enclosed by $\Gamma'$.
As in Sec. 3.2, the analyticity on the finite disk $C$ around $\beta = 1$ together with the uniqueness of analytic continuation implies that the function $\Psi(\beta, \rho)$ takes the same values on all the “mirror” Riemann sheets, i.e., those connected by going around the branch point at $\beta = 1$. As a result, on applying Cauchy’s theorem one can work on a single sheet corresponding to the given value of $g$; loops on all other Riemann sheets give vanishing contribution, enclosing no poles.

We proceed as in Sec. 3.2, by splitting $\Gamma'$ in $\Gamma_A$ and $\Gamma'_B$. The treatment of $\Gamma_A$ contribution is the same as in Theorem 1 and one finds Eq. (3.68).

As for the contribution of $\Gamma'_B$, we observe that the condition Eq. (3.63) (enforcing $1 < \alpha < 2$) and the chosen range of $g$, Eq. (3.64), lead to the inequalities

$$|\beta' - \beta| \geq \epsilon r_0;$$
$$|\beta'| \geq 1 + r_0 \cos(\phi_\alpha) + O(r_0^2)$$

for $\beta' \in \Gamma'_B$. ($\epsilon$ is a positive $\beta$, $\beta'$ independent fixed constant.) Together with the boundedness of $\Psi$ (which follows from $2'$), these inequalities give, as in Sec. 3.2,

$$|R_{N}^{(B)}| \leq \text{const.} \frac{|\beta|^{N+1}}{|1 - \beta|^\sigma r_0} \left( \frac{1}{1 + r_0 \cos \phi_\alpha} \right)^{N+1}$$

$$\sim |g|^p \exp \left[ -N \rho^{1/\alpha} \left( \Re \left( \frac{1}{g^{1/\alpha}} + \frac{1}{g_0^{1/\alpha} \cos \phi_\alpha} \right) \right) \right]. \quad (3.72)$$

The theorem follows from Eq. (3.68) and Eq. (3.72). Q.E.D.

The convergence proof expressed in the two precedent theorems actually holds if the trial parameter $\rho$ (or $x$) is scaled with the order $N$ precisely as $\rho \sim N^{-1}$ (or $x \sim N^{1/(2\alpha)}$), under certain conditions on the proportionality coefficient. Since this case was discussed earlier by several authors in connection with the optimized DE-ODM for the anharmonic oscillator, we state the result for this extreme case as a separate theorem.

**Theorem 3:** Theorem 1 and Theorem 2 hold if (maintaining the other respective

10 Note however that in general such a particular scaling of $\rho$ or $x$ does not follow from the optimization condition alone (either principle of minimum sensitivity or fastest apparent convergence) used frequently in the literature. It is only after imposing a further condition, e.g., of choosing the root of smallest modulus that optimization equation fixes the particular scaling of $x$. See the related discussion in Sec. 4.
hypotheses) we scale the parameter \( \rho \), instead of Eq. (3.61) as
\[
\rho = \frac{C'}{N^b}
\] (3.73)
and
\[
0 < C' < \left( \frac{a}{u_*} \right)^b
\] (3.74)
where \( u_* \) is a function of \( \alpha, b \) defined in Eq. (3.80) below (see also Eq. (3.82) for less stricter but \( \alpha, b \) independent range of convergence) and \( a, b \) are those of Eq. (3.55).

**Proof:** We proceed exactly as in Theorem 1 and Theorem 2, splitting the remainder in two pieces \( R_N^{(A)} \) and \( R_N^{(B)} \). With the scaling Eq. (3.73) the bounds on \( R_N^{(B)} \) Eqs. (3.71) and (3.72) still hold unmodified and enforce convergence of that contribution in the corresponding range of \( g \). It will be shown that with the assumed scaling the contribution \( R_N^{(A)} \) can be bounded by an exponentially vanishing term that is independent on \( g \); all the restrictions on the range of \( g \) come from \( R_N^{(B)} \), accordingly they are the same as in the previous theorems.

It is clear that with scaling Eq. (3.73) the previous bound Eq. (3.68) is too loose to prove the convergence: we thus step back to Eq. (3.65) that is common to both theorems and try to get a better bound. By use of asymptotic behavior Eq. (3.57) of Disc \( S(g) \), of bounds Eq. (3.60), Eq. (3.67), extending the integral range to \( \beta' \in [-\infty, 0] \) and changing the variable as \( \beta' \rightarrow -\beta' \), we find:
\[
|R_N^{(A)}| \leq \text{const.} \left| \frac{\beta^{N+1}}{1 - \beta^\sigma} \rho^{-c} \int_0^\infty d\beta' \exp \left[ -(N + 1 + c) \Phi \left( \beta', \frac{a}{(N + 1 + c)\rho^{1/b}} \right) \right] \right|
\] (3.75)
where
\[
\Phi(\beta', u) \equiv \log \beta' + u \frac{(1 + \beta')^{a/b}}{\beta^{1/b}}.
\] (3.76)
Eq. (3.75) can be estimated by the saddle point method. For fixed \( u > 0 \), \( \Phi \) behaves as \( u/\beta^{1/b} \) near \( \beta' = 0 \) and as \( u\beta^{(a-1)/b} \) for \( \beta' \rightarrow \infty \) thus at least one saddle point (depending on \( u \) and others parameters \( \alpha, b \)), \( \beta'_*(u) \) exists, defined by
\[
\partial_{\beta'} \Phi(\beta'_*, u) = 0.
\] (3.77)
If (depending on \( \alpha, b \) there are more than one such point, the one corresponding to smallest \( \Phi(\beta'_*, u) \) must be chosen. The bound for \( R_N^{(A)} \) is then
\[
|R_N^{(A)}| \leq \text{const.} \left| \frac{\beta^{N+1}}{1 - \beta^\sigma} \rho^{-c} \frac{1}{\sqrt{\partial^2_{\beta'} \Phi(\beta'_*, u)}} e^{-(N + 1 + c) \Phi(\beta'_*, u)} \right| e^{-N \Phi(\beta'_*, u)}
\] (3.78)
(in the last line powers of $N$ have been neglected). The function $\Phi_*(u) \equiv \Phi(\beta_*'(u), u)$ is an increasing function of $u$, as can be easily seen from (due to Eq. (3.77))

$$
\frac{d\Phi_*}{du} = \partial_u \Phi(\beta_*', u) + \partial_{\beta'} \Phi(\beta_*', u) \frac{\partial \beta_*'}{\partial u} = \partial_u \Phi(\beta_*', u) > 0 \quad (3.79)
$$

(last inequality comes from a direct inspection of Eq. (3.76)). Thus if we restrict ourselves to $u > u_*$, with $u_*$ defined by

$$
\Phi_*(u_*) = \Phi(\beta_*'(u_*), u_*) = 0, \quad (3.80)
$$

the bound on $R_N^{(A)}$ vanishes exponentially. Q.E.D.

The value of $u_*$ in Theorem 3 should be obtained (for fixed $\alpha, b$) by numerical solution of coupled equations (3.77) and (3.80). Nevertheless it is easy to get a (more conservative) $\alpha$, $b$ independent range of convergence in terms of $u$. Indeed, if we bound $\Phi$ in Eq. (3.76) with

$$
\Phi(\beta', u) \geq \tilde{\Phi}(\beta', u) \equiv \log \beta' + \frac{u}{\beta^{1/b}} \quad (3.81)
$$

and proceed as in Theorem 3, the equations analogous to Eq. (3.77) and Eq. (3.80) can be solved exactly, obtaining the exponential convergence ($\tilde{\Phi}(\beta_*', u) > 0$) for

$$
u > be^{-1} \geq e^{-1}. \quad (3.82)$$

4 Convergence of the Optimized (Variational) Expansion

Delta expansion - order dependent mappings diverges if the parameters $\rho$ or $x$ depends on $N$ improperly, i.e., if they lie outside the convergence range, Eqs. (3.28), (3.53), (3.32), (3.54). In the case where $x$ grows with $N$ faster than $N^{1/2}$, the behavior of the remainder $R_N$ can be estimated by observing that its main contribution comes from the integration region near $\beta' = 1$ on $\Gamma_B$. One finds

$$
S_N \sim \left(\frac{x^2}{N}\right)^{\sigma}, \quad (4.1)
$$

(note that $x/N^{1/2} \to \infty$; $\beta^N \to 1$). For the particular case $\sigma = 1/2$ it reduces to the result found earlier [1].
On the other hand, if \( x \) grows too slowly \( (x/N^{1/2\alpha} \to 0) \), the main contribution to \( R_N \) comes from the region \( \beta' \sim 0^- \). This contribution can be easily estimated by the saddle point approximation and gives:

\[
S_N \sim \left( -\frac{N}{x^{2\alpha}} \right)^N
\]

(4.2)

(this behavior is an offspring of the standard divergence of perturbative coefficients).

It is clear that there are no solutions of optimization condition (e.g., \( \partial S_N / \partial \Omega = 0 \), or \( S_{N+1} - S_N = 0 \)) in these ranges of \( x \equiv \Omega / \omega \). Optimization forces the parameter \( x \) (or \( \rho \)) to lie within the convergence range. A study of the roots of optimization equation (see Appendix C) with regard to the \( g \) dependence of \( \rho \) or \( x \), gives a further consistency check of this point.

In other words, we have proven that optimized DE-ODM for anharmonic oscillator/double well system converges as long as the coupling constant lies in the appropriate domain. This is important since in many (especially in higher dimensional) systems it is a hard task to do higher order calculations. Optimization is crucial to get a good approximation from lowest order calculations.

Of course, optimization condition alone does not usually yield a unique solution for \( x \) (or \( \rho \)), or a given optimization criterion may not have any real solution at all at a given order \( N \). Furthermore, as the number of the roots of optimization equation in general grows with \( N \), it is not very practical to focus our attention to a particular root and to try to follow it: it may disappear as \( N \) is varied and a new pair of roots may appear, etc. [2].

In the case of the energy eigenvalues of anharmonic oscillator or double well, a general criterion that follows from our analysis is that the convergence rate is fastest for the smallest possible \( \Omega \), in accordance with the earlier suggestions [3].

In any event, the importance of our results in the precedent sections lies in the fact that they do prove the convergence of optimized or variational expansion, without however letting ourselves involved into the analysis of complicated behavior of the solutions of a particular optimization scheme. We find it remarkable that a variational method is proven to converge, even if no variational principle is involved here.
5 $q^{2M}$ oscillators with $M \geq 3$

The convergence proofs presented in Sec. 3 do not apply directly to the case of higher anharmonic oscillators ($M \geq 3$ in Eq. (1.1)). For simplicity consider the case of real $g$. All the pre-requisites of Theorem 1 are satisfied: $E(g)$ is known to be analytic in the cut $g$ plane [3, 4], and the conditions 2) and 3) are satisfied with $p = 1/(M + 1)$ and $b = M - 1$. The convergence cannot however be proven in these cases because there are no values of the scaling index $\gamma'$ satisfying $b < \gamma' < \alpha$, since $b = M - 1 \geq 2$ on the one hand, and $1 < \alpha \leq 2$ on the other.

This does not yet mean that DE-ODM diverges in these cases, since Theorem 1 gives only sufficient conditions for convergence. Let us re-examine the last condition ($\alpha \leq 2$) imposed on the conformal transformation. It is needed so that the complex part $(\Gamma_B)$ of the image of the negative real $g$ axis lies outside of the unit circle, i.e., $|\beta'| > 1$, $\forall \beta' \in \Gamma_B$, $\beta' \neq 1$.

Now one knows that $E(g)$ is analytic on the cut $g$ plane, hence $E(g(\beta))$ analytic within $D$ surrounded by $\Gamma_A$, $\Gamma_B$. If the function $\Psi(\beta, \rho) = (1 - \beta)^{\sigma} E(g)$ would be analytic in a slightly (but sufficiently) larger region of $\beta$, one could consider a conformal transformation with $\alpha > 2$ (actually one needs $\alpha > M - 1$), deform (enlarge) the integration contour $\Gamma_A$, $\Gamma_B$ to $\Gamma'_A$, $\Gamma'_B$ until $|\beta'| > 1$, $\forall \beta' \in \Gamma'_B$. All the steps of the convergence proof would then be valid. Note however that such a deformation in the $\beta$ plane involves going onto the higher Riemann sheets in the original $g$ plane. It is here that Bender-Wu type singularities potentially make obstructions. Unfortunately, having no detailed knowledge on the positions of Bender-Wu type singularities in the general $q^{2M}$ oscillators, we are at present unable to answer whether or not DE-ODM for $q^{2M}$ ($M \geq 3$) oscillators converges with an appropriate scaling of $\rho$ (or by optimization).

It is interesting that Padé’s method does not converge [14] for $q^{2M}$ oscillators, for $M > 3$, either.

6 Green’s functions in the one dimensional $\phi^4$ theory

As an application of Theorem 1 of Sec. 3.3, consider the Euclidean Green’s functions of the anharmonic oscillator system:

$$\langle q(\tau_1)q(\tau_2)\cdots q(\tau_n) \rangle$$
\[
\frac{1}{Z} \int Dq \, q(\tau_1)q(\tau_2) \cdots q(\tau_n) \exp \left[ - \int_{-\infty}^{\infty} d\tau \left( \frac{q^2}{2} + \frac{\omega^2}{2} q^2 + \frac{g}{4} q^4 \right) \right] \equiv \frac{1}{\omega^{n/2}} G_n \left( \frac{g}{\omega^3}; \omega \tau_1, \omega \tau_2, \cdots \omega \tau_n \right).
\]

(6.1)

A simple-minded application of the delta-expansion based on the shift of the quadratic term of the Hamiltonian Eq. (2.5), seems to be problematic, due to the complicated dependence on \( \omega \tau_i \). One may, however, well apply Theorem 1 directly to \( G_n(\bar{g}, \{ \omega \tau_i \}) |_{\bar{g}=g/\omega^3} \) treating \( \{ \omega \tau_i \} \) as fixed parameters.

We first assume the analyticity of \( G_n(g, \{ \omega \tau_i \}) \) in the whole cut plane of \( g \). This is quite reasonable, since any given Green function can be expressed as an (infinite) sum of terms analytic in the cut plane, by expanding in the energy eigenstate basis. Such an assumption was also made by Strharsky [15] to compute successfully the large order behavior of the Euclidean Green’s functions by use of the dispersion relation.

The other conditions 2) and 3) of Theorem 1 are satisfied, with \( p = -n/4; a = 4/3; b = 1; c = n/2. \)

DE-ODM for \( G_n \) constructed from its standard perturbation series by the conformal transformation Eq. (3.56), followed by the Taylor expansion in \( \beta \), then converges to the correct answer, if \( \rho \) is scaled with the order \( N \) as in Eqs. (3.28), (3.29) or Eqs. (3.32), (3.33).

The convergence proof above holds pointwise for each fixed value of \( \{ \omega \tau_i \} \). In fact a uniform convergence over arbitrary value of \( \{ \omega \tau_i \} \) is desired, because the functional form of \( G_n, \{ \omega \tau_i \} \) as the argument, is what one wishes to reproduce. The following argument can be given on this point using the result of [1].

If we apply the same procedure as in [1] also for \( G_n \), we have an expression for the remainder similar to Eq. (2.53) of [1]. A part (corresponding to \( R_N^{(B)} \) above) of the remainder is bounded (essentially) by \( c_1 \beta^N \), where \( c_1 \) is the first order perturbative coefficients for \( G_n \). Another part (corresponding to \( R_N^{(A)} \) above) is estimated by the Euclidean bounce solution for \( g < 0 \). Now \( c_1 \) is computed by using the free propagator, \( \langle q(\tau)q(\tau') \rangle_0 = e^{-\omega|\tau-\tau'|}/(2\omega) \), thus obviously is bounded for the whole range of \( \{ \omega \tau_i \} \). On the other hand, \( \{ \omega \tau_i \} \) dependence of \( R_N^{(A)} \) is proportional to (for \( N \) large) \( \int_{-\infty}^{\infty} d\tau_0 \prod_i \sqrt{2sech} \omega(\tau_i - \tau_0) \), where \( \tau_0 \) is the position of the bounce center, and this is also bounded for arbitrary value of \( \{ \omega \tau_i \} \). This argues strongly for the uniform convergence with respect to \( \{ \omega \tau_i \} \).
7 Summary and discussion

Delta expansion - order dependent mappings, is a powerful resummation technique which converts a given divergent series, under certain conditions, into a convergent sequence of approximants. For its generality, the extreme simplicity of use (in particular, combined with an optimization procedure), and for its impressive success in simple quantum mechanical systems, it seems to be worthwhile to study carefully the mechanism underlying such a method.

In this paper, the convergence proof for the DE-ODM given previously by the present authors for the anharmonic oscillator has been refined and its domain of applicability considerably extended. The key step for the improvement was the use of Cauchy’s theorem, written directly in the complex plane of the conformally transformed variable, instead of making a conformal transformation in the standard dispersion relation. Use of the knowledge on the analyticity property of \( E(g) \) on the full Riemann surface was also crucial.

In particular, this allowed us to show the uniformity of convergence in a wide region of the Riemann surface (Fig. 3), including the strong coupling limit of anharmonic oscillator \((g = \infty)\), and the large \( g \) regime of the quantum mechanical double well.

It is interesting to note that in this last case the standard perturbation series is non Borel summable. Together with the example of the zero dimensional model discussed in Appendix B, our results convincingly demonstrate the fact that the non Borel summability of the standard perturbation series is no obstruction to the existence of a simple resummation procedure which converges to the exact answer. The possibility of reconstruction in non Borel summable cases here hinges upon the known existence of a wide region of analyticity of \( E(g) \), which is larger than normally required for the Borel summability \([16]\).

As for the variational aspect of the delta expansion - order dependent mappings, it is of considerable interest that a variational method can be proven to converge, without use of the variational principle.

A lesson to be learned from the results found here is the fact that, if only convergence for real finite \( g \) is required, quite a wide arbitrariness exits in the choice of the conformal transformation, to be used in the construction of DE-ODM. Our convergence proof can probably be generalized to conformal transformations of more general forms.
At the same time, we have clearly marked the limit of validity of the DE-ODM based on a simple conformal transformation of $g$. In particular, the perturbation series of the quantum mechanical double well in the regime of weak coupling (where tunnelling effects—or instantons—are clearly essential), cannot be resummed by the present method.

Also, in four dimensional field theory models such as Quantum Chromodynamics, where the analytic structure in the renormalized coupling constant is far more complicated \cite{17}, the validity of an analogous approach is not obvious.

Nevertheless, we believe that our general theorems apply to many quantum mechanical and field theory models in dimensions less than four.

**Note Added**

When this paper was being typed the authors were informed that an analogous technique was used by H. Kleinert \cite{18} to show the convergence of DE-ODM in the strong coupling limit.

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**Appendix A  Some properties of the conformal transformation**

We study here how the negative $g$ axis is mapped by the inverse of the conformal transformation

$$g = \rho \frac{\beta}{(1 - \beta)^{\alpha}} \quad (A.1)$$

for different values of $\alpha > 1$. The problem is equivalent to the study of solutions of the equation

$$H(\beta, s) \equiv (1 - \beta)^{\alpha} + s\beta = 0 \quad (A.2)$$
with $s = -\rho/g > 0$.

For $s > s_c \equiv \alpha^{\alpha}/(\alpha - 1)^{\alpha - 1}$ Eq. (A.2) has two real solutions that coalesce (one coming from $\beta = -\infty$, the other from $\beta = 0^-$) for $s = s_c$ at $\beta = \beta_c \equiv -1/(\alpha - 1)$. These values are found by imposing $H(\beta_c, s_c) = \partial_\beta H(\beta_c, s_c) = 0$. For $0 < s < s_c$ the equation (A.2) has two complex conjugate solutions that start from $\beta = \beta_c$ and join at $\beta = 1$.

Other complex conjugate solutions converging to $\beta = 1$ at $s = 0$ are in general also present, but the complex contour that wraps around the cut $g$ plane is mapped to the contour $\Gamma$ formed by the two above mentioned complex conjugate paths (called $\Gamma_B$ in Sec. 3) plus the path $[\beta_c, 0]$ ($\Gamma_A$) of the smaller negative root.

First of all, solving Eq. (A.2) in the limit $s \to 0$ (i.e., $\beta \sim 1$) one easily finds that $\Gamma_B$ leaves the point $\beta = 1$ with angle $\phi_\alpha = \pm (1 - 1/\alpha)\pi$.

To study better the shape of $\Gamma_B$, set $\beta = |\beta|e^{i\delta}$ and consider $\beta$ at a fixed phase $\delta$, $0 < \delta < \pi$, as a function of $\alpha$. One finds from Eq. (A.2), after eliminating $s$, 

$$|\beta| = \sin \frac{\pi - \delta}{\alpha} / \sin \frac{(\alpha - 1)(\pi - \delta)}{\alpha}.$$  \hfill (A.3)

It is then easy to see that 

$$|\beta| > 1, \quad \text{if} \quad 1 < \alpha < 2;$$

$$|\beta| < 1, \quad \text{if} \quad \alpha > 2,$$  \hfill (A.4)

for $\forall \delta, 0 < \delta < \pi$.

### Appendix B Zero dimensional model

In this appendix, we consider the delta expansion or, equivalently, the order dependent mapping with conformal transformation,

$$g = \rho \beta / (1 - \beta)^2,$$  \hfill (B.1)

of a simple integral,

$$Z(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dq e^{-q^2 - g q^4}.$$  \hfill (B.2)

---

11 It can be shown that there exist $2(k + 1)$ roots on the first Riemann sheet of the mapping Eq. (A.3), $-\pi < \text{Arg} (1 - \beta) < \pi$, in which $k$ is a nonnegative integer such that $2k + 1 \leq \alpha < 2k + 3$. For the case of our main interest, $1 < \alpha \leq 2$, therefore only two roots exist on this sheet.
(the partition function of $\phi^4_0$ theory). Due to the absence of Bender-Wu type singularities, DE-ODM in this case converges for each $g$ in the whole Riemann surface, in the limit, $N \to \infty$, $\rho = C'/N' \to 0$ with

$$1 < \gamma' < 2, \quad \text{or}$$

$$\gamma' = 1; \quad C' < C'_{\text{crit}} = 1.116711531873221 \cdots. \quad \text{(B.3)}$$

(In terms of $x$, $x \sim (g/\rho)^{1/4} \to \infty$.)

The proof for $g$ lying on higher Riemann sheets in this case requires a non-trivial extension of the treatment given in Sec. 3.

We start with the observation that the analytic continuation of $Z(g)$ is given in terms of the parabolic cylinder function by

$$Z(g) = \frac{1}{(2g)^{1/4}}e^{1/(8g)}D_{-1/2}(1/\sqrt{2g}) \quad \text{(B.4)}$$

with the strong coupling expansion

$$Z(g) = \frac{1}{2\sqrt{\pi}}g^{-1/4} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n/2 + 1/4)}{n!} (g^{-1/2})^n \quad \text{(B.5)}$$

(the latter can easily be derived directly from Eq. (B.2)).

Following the general procedure described in Sec. 3.2 we introduce the function $\Psi(\beta, \rho)$ by

$$Z(g) \equiv (1 - \beta)^{1/2}\Psi(\beta, \rho), \quad \text{(B.6)}$$

where the $N$th order approximant for $Z(g)$ is to be constructed as $Z_N(g) \equiv (1 - \beta)^{1/2} \Psi(\beta, \rho)$. $\{\Psi(\beta, \rho)\}_N; \{f(\beta)\}_k$ is the $k$th order truncation of the Taylor expansion of $f(\beta)$ in $\beta$.

In order to write Cauchy’s theorem in the $\beta$ variable, the analytic property of $\Psi$ must be studied first. A good knowledge of it can be deduced from the strong coupling expansion Eq. (B.3). Indeed, it is easily shown that this expansion converges for any $g \neq 0$, implying that the function $\Psi$ defined above is analytic everywhere, except at the quartic branch points at $\beta = 0$ and at $\beta = \infty$. ($\beta = 0$ is also an essential singularity, see Eq. (B.11) below.)

\[\text{12} \] Previously convergence was proved only pointwise for real $g$ \[3, 19, 1\] and for a particular phase of $g$, $\text{Arg} \ g = 2\pi$ (corresponding to the double well) \[4\].
The image of the cut $g$ plane is the interior of a unit circle (minus the negative real axis) in the first $\beta$ Riemann sheet and it is tempting to take its boundary as the integration contour $\Gamma = \Gamma_A + \Gamma_B$ and make a “small” enlargement of the contour around $\beta = 1$, keeping the rest of the contour unchanged, as was done in Sec. 3.2. It turns out that this procedure does not work here. (The convergence factor $1/(1 + r_0 \cos \phi_\alpha + O(\nu_0^2))^N$ in Eq. (3.72) is lost in the case $\alpha = 2$.) On the other hand, the analyticity domain of $\Psi$ is much wider here, so that the contour can be enlarged more substantially. Such enlargement is indeed possible but involves a delicate issue; it can be dealt with as follows.

We write the remainder $R_N(\beta) \equiv Z(g) - Z_N(g)$ as:

$$R_N(\beta) = (1 - \beta)^{1/2} \beta^{N+1} \int_{\Gamma_A' + \Gamma_B'} \frac{d\beta'}{2\pi i (\beta')^{N+1}(\beta' - \beta)}, \quad (B.7)$$

where $\Gamma_B'$ will be taken as a circle with the radius $R > 1$ and $\Gamma_A'$ as a contour which wraps around a part of the negative real axis $-R \leq \beta' \leq 0$ (see Fig. 4).

The radius $R$ will be taken dependent on the order $N$ as

$$R = 1 + dN^{-\kappa}, \quad (d > 0; \quad \gamma'/2 > \kappa > 0). \quad (B.8)$$

In deriving Eq. (B.7), it was implicitly assumed that $|\beta| < R$. This indeed holds for large $N$ with this choice of $R$. ($\beta \sim 1 - (\rho/g)^{1/2}$ as $\rho \to 0$.) For the subtleties involved in writing Cauchy’s theorem on $\beta$ Riemann surface and for its elucidation, see the comment before Eq. (3.40) in the main text.)

On $\Gamma_B'$, $|\beta'| = R$ and $|\beta' - \beta| \geq R - |\beta|$, hence the contribution to $R_N(\beta)$ is bounded by

$$|R_N^{(B)}(\beta)| \leq |1 - \beta|^{1/2} \left(\frac{|\beta|}{R}\right)^{N+1} \max \frac{|\Psi(\beta')|}{1 - |\beta|/R}. \quad (B.9)$$

When $\beta'$ is on the contour $\Gamma_B'$, the corresponding $g' = \rho \beta'/(1 - \beta')^2$ circles around the origin $g' = 0$ on the second Riemann sheet, with the radius,

$$|g'| = \rho \frac{R}{1 - \beta'^2} \leq \rho \frac{R}{(R - 1)^2}. \quad (B.10)$$

With $\rho$ scaling as $\rho = C'/N^{\gamma'}$, and $R$ as Eq. (B.8), $|g'|$ is small for $N$ large. We can therefore use the asymptotic expansion near the origin $^{33}$

$$Z(g) \sim \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \Gamma(2n + 1/2) \left[(-1)^n \pm i\sqrt{2}e^{1/(4g)}\right] g^n, \quad (B.11)$$

$^{13}$ The plus sign is for $-5\pi/2 < \text{Arg} \ g < -\pi/2$ and the minus sign is for $\pi/2 < \text{Arg} \ g < 5\pi/2$; the terms with the exponential factor is absent in other regions of $g$ $^{21}$ Eq. (9.246)].
to get the following bound on \( \Psi(\beta') \):

\[
|\Psi(\beta')| \leq \frac{\sqrt{2}}{(R - 1)^{1/2}} \exp \left[ \frac{(R - 1)^2}{4\rho R} \right] (1 + O(\rho R/(R - 1)^2)).
\]  

(B.12)

We get

\[
|R_N^{(B)}(\beta)| \leq \frac{\sqrt{2}|1 - \beta|^{1/2}}{(1 - |\beta|/R)(R - 1)^{1/2}} \left( \frac{|\beta|}{R} \right)^{N+1} \exp \left[ \frac{(R - 1)^2}{4\rho R} \right] 
\]

\[
\sim \exp \left( -dN^{1-\kappa} + \frac{d^2}{4C'} N^{\gamma'-2\kappa} \right). 
\]  

(B.13)

Note that for \( 0 < \gamma' < 2 \), it is always possible to choose \( \kappa \) in such a way that

\[
0 < \gamma' - 2\kappa < 1 - \kappa.
\]  

(B.14)

With the first factor dominating, one gets \( |R_N^{(B)}(\beta)| \to 0 \) as \( N \to \infty \).

On \( \Gamma_A, -R \leq \beta' \leq 0 \). The value of \( g' \) corresponding to it moves along \( -\rho/4 \leq g' \leq 0 \). Since \( \rho \ll 1 \) for \( N \) large, the asymptotic form

\[
|\Im \Psi(\beta')| = \frac{|\Im Z(g')|}{|1 - \beta'|^{1/2}} \sim \frac{1}{\sqrt{2}} \frac{e^{1/(4g')}}{|1 - \beta'|^{1/2}} (1 + O(g')), 
\]

(B.15)

can be used in

\[
R_N^{(A)}(\beta) = (1 - \beta)^{1/2} \beta^{N+1} \int_{-R}^{0} \frac{d\beta'}{\pi} \frac{\Im \Psi(\beta')}{(\beta')^{N+1}(\beta' - \beta)}. 
\]

(B.16)

Therefore, using \( |\beta' - \beta| \geq |\beta| \) and \( |1 - \beta'| \geq 1 \), we get

\[
|R_N^{(A)}(\beta)| \leq \frac{1}{\sqrt{2}} |1 - \beta|^{1/2} |\beta|^N \int_0^{\infty} \frac{d\beta'}{\pi} e^{-(N+1)\Phi(\beta')}(1 + O(1/N)),
\]

(B.17)

where \( \Phi(\beta') = \ln \beta' + u(1 + \beta')^2/\beta', \ u = 1/(4(N + 1)\rho) \) and in the last step we have changed the variable \( \beta' \to -\beta' \) and extended the upper limit of integration to \( \infty \).

The right hand side of Eq. (B.17) can be evaluated by the saddle point method for \( N \) large, as was done in Sec. 3.2, Sec. 3.3. Combining the bound for \( R_N^{(A)} \) thus obtained with that for \( R_N^{(B)} \), we get the announced result.

\footnote{We have taken the dominant term for \( \Re g' > 0 \). When \( \Re g' < 0 \), \( |Z(g')| \) is bounded by a power of \( |g'| \) and the contribution is harmless for the convergence. The existence of such a large factor near the origin on the second Riemann sheet is understood as \( g = e^{2\pi i |g|} \) corresponds to double well model.}
An interesting application of our result is the case of the “double well” model in zero dimension:

\begin{equation}
\tilde{Z}(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dq e^{q^2-gq^4} = \frac{1}{(2g)^{1/4}} e^{1/(8g)} D_{-1/2}(-1/\sqrt{2g}),
\end{equation}

with \( g \) real positive. From the known properties of the parabolic cylinder function one finds that \( \tilde{Z}(g) = (e^{2\pi i})^{1/4} Z(e^{2\pi i} g) \) (an analogue of the Symanzik’s scaling relation). Consequently, the (scaled) delta expansion of the double well model (whose standard perturbation series is non Borel summable) converges to the exact answer for any real value of \( g \neq 0 \), in accordance with an earlier result [19, 2].

As pointed out in [3], the case \( \text{Arg} \, g = \pi \) can also be related to \( \tilde{Z}(g) \). The crucial relation is

\begin{equation}
\tilde{Z}(g) = \sqrt{2} e^{1/(4g)} \text{Re} \, Z(e^{\pi i} g).
\end{equation}

Again according to our result, the exact answer for \( \tilde{Z}(g) \) can be correctly reconstructed systematically by using DE-ODM for \( Z(e^{\pi i} g) \), whose standard perturbation series \( Z(e^{\pi i} g) \sim (1/\sqrt{\pi}) \sum_{n=0}^{\infty} \Gamma(2n+1/2) g^n/n! \) is non Borel summable.

**Appendix C  Optimization condition on \( \rho \)**

In this appendix, we show that either of optimization conditions, the principle of minimal sensitivity (PMS) or the fastest apparent convergence (FAC), leads a simple equation for \( \rho \) which is independent on \( g \). This fact, for the particular case with \( \sigma = 1/2 \) and \( \alpha = (M + 1)/2 \), has recently been noted by [21].

By PMS, one imposes \( dS_N^{(ODM)}(dx)/dx = 0 \) (or equivalently \( dS_N^{(ODM)}/d\rho = 0 \)) order by order. From the definition of DE-ODM, we know

\begin{equation}
S_N^{(ODM)} = \frac{1}{(1-\beta)^\sigma} \sum_{n=0}^{N} c_n \rho^n \left\{ \beta^n (1-\beta)^{\alpha n + \sigma} \right\}_N \\
= \sum_{n=0}^{N} c_n g^n (1-\beta)^{\alpha n - \sigma} \left\{ (1-\beta)^{\alpha n + \sigma} \right\}_{N-n},
\end{equation}

where \( \{f(\beta)\}_k \) is a \( k \)th order truncation of the Taylor series on \( \beta \). With a use of a relation

\begin{equation}
\frac{d}{d\beta} \left[ (1-\beta)^\ell \left\{ (1-\beta)^{-\ell} \right\}_k \right] = -\frac{\Gamma(\ell + k + 1)}{k! \Gamma(\ell)} (1-\beta)^{\ell-1} \beta^k,
\end{equation}

34
we have (note the variation with respect to $x$ is taken while $g$ is kept fixed)

$$
\frac{d}{dx} S^{(ODM)}_N = - \frac{\beta^N}{d(1 - \beta)^{\sigma + 1}} \sum_{n=0}^{N} c_n \frac{\Gamma(N + (\alpha - 1)n - \sigma + 1)}{(N - n)!\Gamma(\alpha n - \sigma)} \rho^n.
$$

(C.3)

PMS therefore leads an equation

$$
\sum_{n=0}^{N} c_n \frac{\Gamma(N + (\alpha - 1)n - \sigma + 1)}{(N - n)!\Gamma(\alpha n - \sigma)} \rho^n = 0.
$$

(C.4)

On the other hand, it is easy to see that FAC, $S^{(ODM)}_N - S^{(ODM)}_{N-1} = 0$, leads a similar equation

$$
\sum_{n=0}^{N} c_n \frac{\Gamma(N + (\alpha - 1)n - \sigma)}{(N - n)!\Gamma(\alpha n - \sigma)} \rho^n = 0.
$$

(C.5)

Optimization condition thus leads to an order by order determination (albeit among all possible, at most $N$ple, roots) of $\rho$ which depends only on $N$ but independent of $g$. On the other hand, our convergence proof is valid if the parameter $\rho$ is appropriately scaled with $N$, but independently of $g$. See Eq. (3.52) or Eq. (3.53). These facts are consistent with each other: as noticed in Sec. 4, optimization forces $x$ or $\rho$ to lie within the range Eq. (3.52) or Eq. (3.53) for which DE-ODM converges, hence selects out $\rho$ which grows appropriately with $N$ but in a $g$ independent manner.

Appendix D  Critical exponents in $\phi^4_3$ theory

In this appendix, we apply DE-ODM to the renormalization group functions of $\phi^4_3$ theory and compute various critical exponents. This problem was already studied in [3], but our approach here differs in the following aspect: In [3], a renormalization group argument was used to relate first the analytic structure in $g_0$ around $g_0 = \infty$ ($g_0$ is the unrenormalized coupling constant) to the behavior of the beta function $W(g)$ near the infrared fixed point $g \simeq g^*$. The parameter of the conformal transformation is accordingly taken to be $\alpha = 1/\omega$ ($\omega$ being the slope of the beta function at the fixed point), in order to render $W(g(g_0(\beta)))$ analytic around $\beta = 1$. The input value of $\omega$ must somehow be guessed by e.g., other approximate methods for computing $W(g)$. Although quite encouraging results have been found [3], the fact that one must assume an input value of $\omega$ (which is one of the results being sought for) to start the calculation with, is a somewhat disturbing feature of the approach [22].
On the other hand, as is seen in Sec. 3.1, the requirement of analyticity near $\beta = 1$ is not really necessary, if one is only interested in the convergence for finite real positive $g$. Also, quite a wide range of conformal transformations can be used in such a case. In this spirit one may \textit{a priori} try a simpler conformal transformation, directly to the renormalized coupling constant $g$.

Here we choose $\alpha = 3/2$ and $\sigma = 1/2$, the same value as in the case of the anharmonic oscillator. Assuming the renormalization group functions to have the analyticity on the cut $g$ plane (which is not proven to our knowledge), one concludes that DE-ODM gives a convergent sequence for true functions.

We thus proceed as follows: take a perturbative series of, say, the beta function

$$W(g) \sim \sum_{n=0}^{\infty} c_n g^n,$$

for which the first six non-trivial coefficients have been calculated. Then ODM with $\alpha = 3/2$ and $\sigma = 1/2$ gives a new series

$$W_N(g, x) = \sum_{n=0}^{N} c_n g^n \frac{1}{x^{3n-1}} \sum_{k=0}^{N-n} \frac{\Gamma(3n/2 + k - 1/2)}{k! \Gamma(3n/2 - 1/2)} \left(1 - \frac{1}{x^2}\right)^k,$$

where $\rho$ and $\beta$ have been parameterized in terms of $x$ as in Eq. (3.2). At lower orders we expect that an optimized determination of $x_N$ gives a more accurate result than with a particular scaling of $x_N$ with $N$. We thus employ here the principle of minimal sensitivity, i.e., $dW_N/dx_N = 0$, for each value of $g$, by choosing the root closest to $x_N = 1$.

In Fig. 5, we plot the original beta function Eq. (D.1) up to $O(g^7)$ and the function $W_7(g, x_7(g))$ obtained in this way. A clear improvement and the appearance of an infrared fixed point can be seen.

We applied the same method for perturbative series of various renormalization group functions. The resulting values of the critical exponents, as a function of the order $N$, are summarized in Table 1. The first two columns of Table, $g^*$ and $\omega$, are the result of the method explained above. In the last two columns, we have used

$\text{15}$ It is known that the perturbation series for the renormalization group functions is asymptotic for $\text{Re } g \geq 0$, and the known large order behavior of those functions is consistent with the assumed form of the imaginary part Eq. (3.55) with $b = 1$.

$\text{16}$ For the definition of various critical exponents, see. 

36
\( g^* = 1.43645 \), the result of the seventh order resummation. The blank means there is no solution of the optimization condition near \( x_N \sim 1 \).

These results can be compared with those obtained by the Borel summation method \[26\]:

\[
\begin{align*}
g^* &= 1.416 \pm 0.005, \quad \omega = 0.79 \pm 0.03, \\
\eta &= 0.031 \pm 0.004, \quad \gamma = 1.241 \pm 0.0020.
\end{align*}
\] (D.3)

Considering the simplicity of our approach (in particular, the uniform convergence for \( g \in [0, \infty] \) is not guaranteed), the agreement is rather surprising and suggests the correctness of our assumptions.

**Appendix E  Strong coupling expansion coefficients from standard perturbation series**

As shown in Sec. 3.2 DE-ODM for the anharmonic oscillator converges even in the infinite coupling constant limit. This fact can be used to reconstruct the strong coupling expansion Eq. (1.4) from the perturbation series coefficients \( c_n \) \[12\]. Here we derive such a formula, which is more general than that given in \[12\] and whose derivation is also somewhat simpler.

It is interesting that a quite similar formula had been proposed earlier \[11\], following a completely different approach.

The \( N \)th order DE-ODM approximant is given by \[1\]

\[
S_N = \sum_{n=0}^{N} c_n g^n \frac{1}{x^{3n-1}} \sum_{k=0}^{N-n} \frac{\Gamma(3n/2 + k - 1/2)}{k! \Gamma(3n/2 - 1/2)} \left( 1 - \frac{1}{x^2} \right)^k.
\] (E.1)

Substituting \( x = g^{1/3} y_N \) (see Eqs. (3.53), (3.54)) into Eq. (E.1), we find

\[
S_N = g^{1/3} \sum_{n=0}^{N} \sum_{k=0}^{N-n} \sum_{l=0}^{k} c_n \frac{\Gamma(3n/2 + k - 1/2)}{\Gamma(3n/2 - 1/2)(k-l)!} \frac{1}{y_N^{3n+2l-1}} \frac{(-1)^l}{l!} (g^{-2/3})^l.
\] (E.2)

A comparison with Eq. (1.4) then gives for the \( l \)th strong coupling expansion coefficients \( d_l \),

\[
d_l = \lim_{N \to \infty} \frac{(-1)^l}{l!} \sum_{n=0}^{N} \sum_{k=0}^{N-n} c_n \frac{\Gamma(3n/2 + k + l - 1/2)}{k! \Gamma(3n/2 - 1/2)} \frac{1}{y_N^{3n+2l-1}}.
\] (E.3)
\[
\lim_{N \to \infty} \left( -1 \right)^l \frac{\Gamma(N + n/2 + 1/2)}{l! \left( N - l - n \right)! \Gamma(3n/2 - 1/2)(3n/2 + l - 1/2)} \frac{1}{y_N^{3n+2l-1}},
\]

where \( y_N \) must be scaled appropriately with \( N \) (\( y_N = cN^\gamma \); \( 1/3 < \gamma < 1/2 \) with arbitrary positive \( c \), or \( y_N = cN^{1/3} \); \( c \geq c_+ = 0.570875 \cdots \)), or simply determined order by order by optimization of \( S_N \) or \( d_l \).

Delta expansion for the double well potential is obtained simply by changing \( \beta \to 1 + 1/x^2 \) \[1\]. From the above derivation, it is obvious that \( d_l^{(DW)} = (-1)^l d_l^{(AHO)} \), which is of course a correct result, in view of Eq. (1.3).

Finally, as a further check, the known strong coupling expansion (B.5) in the zero dimensional model discussed in Appendix B can exactly be reproduced following the same procedure.

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Table Caption

1. Determination of various critical exponents in $\phi^4_3$ theory by DE-ODM.

Figure Captions

1. The domain of analyticity $\mathcal{D}$ in $\beta$ plane, corresponding to the cut $g$ plane. Its boundary is the curve $\Gamma_A$ which wraps around the negative real $\beta$ axis, and $\Gamma_B$, the upper and lower complex sections.

2. Enlargement of the contour $\Gamma_B$ to $\Gamma'_B$, so as to include part of the circle $\mathcal{C}$. The crosses represent the (schematic) position of Bender-Wu singularities.

3. The inverse image of the integration contour $\Gamma'_B$ (solid line) and the boundary of the region Eq. (3.49) (dashed line) on the $g$ Riemann surface. Straight sections of the contour image are actually on the axis, $\text{Arg} \ g = \pm \pi, \pm 2\pi, \pm 4\pi$.

4. The contour used in the convergence proof of the zero dimensional model.

5. Standard perturbation result for the beta function $W(g)$ (dotted line) and DE-ODM improved one (solid line).
Table 1: Determination of various critical exponents in $\phi^4$ theory by DE-ODM.

| $N$ | $g^*$ | $\omega$   | $\eta$   | $\gamma$ |
|-----|-------|------------|----------|----------|
| 2   | —     | —          | —        | 1.22847  |
| 3   | 1.77550 | 0.548573  | 0.025686 | —        |
| 4   | —     | —          | —        | 1.24137  |
| 5   | 1.48560 | 0.699459  | 0.030055 | —        |
| 6   | —     | —          | —        | 1.24366  |
| 7   | 1.43645 | 0.751379  | —        | —        |