Emergent Elasticity in Amorphous Solids

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The mechanical response of naturally abundant amorphous solids such as gels, jammed grains, and biological tissues are not described by the conventional paradigm of broken symmetry that defines crystalline elasticity. In contrast, the response of such athermal solids are governed by local conditions of mechanical equilibrium, i.e., force and torque balance of its constituents. Here we show that these constraints have the mathematical structure of electromagnetism, where the electrostatic limit successfully captures the anisotropic elasticity of amorphous solids. The emergence of elasticity from constraints offers a new paradigm for systems with no broken symmetry, analogous to emergent gauge theories of quantum spin liquids. Specifically, our $U(1)$ rank-2 symmetric tensor gauge theory of elasticity translates to the electromagnetism of fractonic phases of matter with stress mapped to elec-
tric displacement and forces to vector charges. We present experimental evidence indicating that force chains in granular media are sub-dimensional excitations of amorphous elasticity similar to fractons in quantum spin liquids.

Solids that emerge in strongly nonequilibrium processes such as jamming or gelation, are characterized by strong stress heterogeneities, often referred to as force chains. They are rigid in that they can sustain external shear, yet they are often fragile. Analogous to classical elasticity theory, it is plausible to ask whether a long wavelength field theoretic description exists for the mechanical response of such athermal solids and if so, what are its characteristics and universal features, and what are the appropriate variables that can account for the underlying kinetic constraints in the emergent field theory? Any attempt to construct such a field theory must answer: (a) how to obtain the stress state, and (b) how to incorporate microscopic information about the structural disorder, accounting for the mechanical constraints into a continuum formulation. This second problem, in particular, has a close resemblance with kinetically constrained models such as hard-core dimer models on lattices where the hard-core constraint naturally allows for an emergent gauge theory description at low energy and long wavelengths.

In this work, we develop a theory of stress transmission in disordered granular solids, both with and without friction, where the local constraints of mechanical equilibrium are paramount—every grain satisfies the constraints of force and torque balance. These local constraints imply that the grain-level stress tensor \( \sigma_g \) is symmetric and satisfies

\[
(\nabla \cdot \sigma_g)_g = \sum_{c \in g} f_{g,c} = f_{g,ext},
\]

(1)
where $f_{g,c}$ are the contact forces acting on grain $g$ and $f_{g}^{\text{ext}}$ is the total external force acting on the grain. The divergence operator above is a discrete version of the differential operator, which is defined over the underlying contact network, as detailed in the SI. The condition encapsulating the local kinetic constraints is the exact analog of the celebrated Gauss’s law in the theory of electromagnetism. In granular solids, the external boundary forces give rise to the internal stress field; analogous to charges generating a non-zero electric field in electromagnetism.

The kinetic constraints in Eq. 1 on $\dot{\sigma}_g$ is the grain-level realization of the continuum equation of mechanical equilibrium: $\partial_i \sigma_{ij}(r) = f_j$. Following the electromagnetic analogy further, it is tempting to identify this condition with the Gauss’s law for a $U(1)$, symmetric rank-2 tensor gauge-theory with vector charges (VCT) \(^{21}\), with the mapping $E_{ij} = E_{ji} \leftrightarrow \sigma_{ij}$ and vector charges to unbalanced forces i.e. $\rho \leftrightarrow f$, satisfying

$$\partial_i E_{ij} = \rho_j,$$

Eq. 2 correctly incorporates, by construction, both the conservation of charge/force ($\int d\mathbf{r} \rho = 0$), and charge angular momentum/torque ($\int d\mathbf{r} (\mathbf{r} \times \rho) = 0$) \(^{21,22}\). Incidentally, as pointed out recently, the two conservation laws lead to sub-dimensional propagation of charges— a feature of recently discussed fractonic phases of matter \(^{21}\) as well as topological defects in elastic solids \(^{23}\). In the present context, it also provides a natural explanation for the visually striking “force chains” (see Fig. 3) observed in photoelastic images of granular solids \(^{24}\), as our analysis will demonstrate.

This reformulation of the stress equation in terms of the Gauss’s law for symmetric tensor electromagnetism gives us a natural starting point for understanding the mechanical response of
granular solids and a derivation of the correct continuum theory. The above formulation is similar to the problem of frustrated magnets and/or dimer models where due to non-trivial local energetic/kinetic constraints, the individual spins/dimers cease to be the right degrees of freedom and hence fail to describe the low energy theory, which in turn is often described by emergent gauge fields that naturally capture the constraints. Similarly, the displacement of the individual grains—the mainstay of the theory of elasticity of crystalline solids—cease to be the right variables in the absence of broken translation symmetry. However, the long-range stress correlations generated by Newton’s laws of force and torque balance are captured by the emergent electromagnetism.

Eq. 2 does not provide enough equations to solve for the field $E_{ij}$, since there are only $d$ equations in $d$ dimensions, for the $d(d+1)/2$ components of a symmetric tensor. This lack of equations is an issue that has plagued the solution of the granular stress problem. In VCT, the Gauss’s law constraint generates a gauge transformation for the symmetric tensor gauge potential, $A_{ij} \rightarrow A_{ij} + \frac{1}{2}(\partial_i \psi_j + \partial_j \psi_i)$. Since $E_{ij}$ is the momentum conjugate to $A_{ij}$, we have the analog of the Maxwell-Faraday condition: $\frac{\partial B_{ij}}{\partial t} = -\epsilon_{iab}\epsilon_{jcd}\partial_a \partial_c E_{bd}$, where $B_{ij}$ is the magnetic field: $B_{ij} = \epsilon_{iab}\epsilon_{jcd}\partial_a \partial_c A_{bd}$. Invoking the magnetostatic condition $\epsilon_{iab}\epsilon_{jcd}\partial_a \partial_c E_{bd} = 0$, leads to the potential formulation of electrostatics:

$$E_{ij} = \frac{1}{2}(\partial_i \phi_j + \partial_j \phi_i),$$

which can be used to obtain $E_{ij}$ for any charge configuration.

The theory described above, although self-consistent, does not capture the full complexity of granular mechanics. Crucial to their properties are the twin aspects of a granular solid: (1) it
is only defined under external pressure (as a packing of grains with purely repulsive interactions will fall apart in the absence of such boundary forces), and (2) it can support internal stresses. This translates, in terms of the VCT, to an assembly being subject to well defined boundary charges developing internal charge dipoles, akin to the response of a polarizable medium (dielectric). Stated differently, once under external pressure, although the granular solid is free of “charges” since every grain satisfies the constraints of force and torque balance, “bound charges” exist as pairs of equal and opposite forces at every contact of the disordered granular network. We need to include these force or charge dipoles in our theory through a tensorial dipole moment $P_{ij}$, whose divergence defines the bound charges. The structure of $P_{ij}$ is influenced by various microscopic details of the system such as the features of the underlying contact network and the nature of contact forces which, for example can be purely repulsive or attractive and frictionless or frictional. To construct a continuum theory, we assert that $P_{ij}$ is related to $E_{ij}$, through a fourth-rank polarizability tensor, as in linear dielectrics. For such a polarizable medium, Gauss’s law becomes:

$$\partial_i E_{ij} = \rho^\text{free}_j + \rho^\text{bound}_j,$$

$$\partial_i P_{ij} = -\rho^\text{bound}_j.$$

A complete derivation of these relations, and a detailed discussion of the structure of the theory will be presented in a future paper. The polarizability tensor, $\hat{\chi}$, defined via $P_{ij} = \chi_{ijkl} E_{kl}$, leads to the definition of a “Displacement” tensor, $D_{ij} = (\delta_{ik}\delta_{jl} + \chi_{ijkl}) E_{kl}$, satisfying $\partial_i D_{ij} = \rho^\text{free}_j$.

$$\partial_i D_{ij} = 0; \quad \epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c (\Lambda D)_{bd} = 0,$$

$$E_{ij} = (\delta_{ik}\delta_{jl} + \chi_{ijkl})^{-1} D_{kl} \equiv \Lambda_{ijkl}D_{kl}.$$  

Since both $\hat{D}$ and $\hat{E}$ are symmetric tensors, $\hat{\Lambda}$ has to satisfy $\Lambda_{ijkl} = \Lambda_{jikl} = \Lambda_{ijlk} = \Lambda_{jilk}$. Since the
inherent stresses in a granular solid satisfy the first of the equations above as a direct consequence of force balance, we need to interpret \( D_{ij} \) as the Cauchy stress tensor measured from contact forces and contact vectors inside the material: \( \sigma_{ij} \leftrightarrow D_{ij} \) in Eq. 2. In the presence of body forces such as gravity, the divergence of \( \hat{D} \) would be equated with the body force. We can compare this theory to anisotropic elasticity \(^8\):

\[
\partial_i \sigma_{ij} = 0; \quad \epsilon_{i(ab}\epsilon_{j(cd} \partial_a \partial_c U_{bd} = 0,
\]

\[
\sigma_{ij} = (\Lambda)^{-1}_{ijkl} U_{kl}, \quad (6)
\]

where \( U_{ij} \) is the macroscopic strain tensor. The tensor \( \hat{\Lambda} \) in Eq. 6 is the inverse of the elastic modulus tensor. Identifying \( E_{ij} \leftrightarrow U_{ij} \), and \( D_{ij} \leftrightarrow \sigma_{ij} \), the VCT-based framework is identical to the anisotropic elasticity theory with \( \phi \), a gauge potential, replacing the displacement field in elasticity theory, and providing the “missing” compatibility equations that allow us to solve the granular stress response problem without invoking a displacement field.

Eq. 5, our main theoretical result, demonstrates that an elasticity theory capturing the full range of stress responses in granular solids \(^8\) emerges from VCT. Although \( \hat{D} \) and \( \hat{E} \) are identified with \( \hat{\sigma} \) and \( \hat{U} \), the elasticity emerges from local constraints and not from broken symmetry: it is a \textit{stress-only} description that needs no reference to a stress-free state or displacement fields. Although \( \hat{\Lambda} \) is an effective elastic modulus, it depends on protocol, and there are no symmetry requirements imposed by a free-energy. Within the theory, protocols translate to stress-ensembles \(^3,4,19\) characterized by externally imposed stresses such as isotropic compression or pure shear.

Below, we compare the predictions for stress-stress correlations obtained from Eq. 5 to
experimental and numerical data. Through this analysis, we extract \( \hat{\Lambda} \) for frictionless and frictional granular solids prepared under different protocols. A hallmark of the VCT in free space, both in 2D and 3D, is the appearance of pinch point singularities in the Fourier transforms of \( E_{ij} \) correlators \(^{25}\):

\[
C^{\text{free}}_{ijkl}(q) \equiv \langle E_{ij}(q)E_{kl}(-q) \rangle 
= \frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2} + \frac{q_iq_jq_kq_l}{q^4} - \frac{1}{2} \left( \frac{\delta_{ik}q_jq_l}{q^2} + \frac{\delta_{jk}q_iq_l}{q^2} + \frac{\delta_{il}q_jq_k}{q^2} + \frac{\delta_{jl}q_iq_k}{q^2} \right). \tag{7}
\]

Eq. 7 is obtained by imposing the Gauss’s law constraint, \( \partial_iE_{ij} = 0 \), on Eq. 3, and assuming that all states are equiprobable \(^{25}\), i.e. the Edwards measure \(^{19}\). Earlier granular field theories \(^{3,4,26}\) based on this measure used a dual formulation of VCT \(^{27}\) where the emergence of elasticity is not evident. Since \( C_{ijkl}(q) \) is independent of \( |q| \), it is straightforward to show that the correlations in real-space decay as \( 1/r^d \). A stringent test of the theory, therefore is the pinch-point structure of the correlation functions.

For granular solids, we computed the correlators \( C_{ijkl}(q) = \langle D_{ij}(q)D_{kl}(-q) \rangle \) using Eq. 5, and tested the predictions in ensembles of 2D and 3D isotropically compressed soft particles (numerically), and in ensembles of 2D packings of frictional grains (experimentally). Pinch point singularities are clearly exhibited in both 2D (Figs. 1 and 3) and 3D (Fig. 2). We have determined \( \hat{\Lambda} \) through detailed comparisons between theory and measurements of \( C_{ijkl} \) (Figs. 1-3). Fig. 1 demonstrates that for packings created under isotropic compression, \( \hat{\Lambda} = \lambda 1 \), with \( 1 \) being the identity tensor. Additional tests of the theory are presented in the SI.
Figure 1: Comparisons in Fourier space between the theoretical predictions (solid black line) and the disorder-averaged angular dependent stress-stress correlations $C_{xxxy}(\theta)$ and $C_{xxyy}(\theta)$ in the numerical: a, and the experimental results (red symbols): b, for isotropically jammed systems. The range of pressure for the numerical data is $P \in [0.016, 0.017]$ and the results are displayed for five different system sizes $N = 512, 1024, 2048, 4096, 8192$. The experimental data is from frictional packings with a range of pressure $P \in [1.5 \times 10^{-4}, 2.9 \times 10^{-4}]$. All correlation functions are normalized by the peak value of $C_{xxxx}(\theta)$.

To illustrate the sensitivity of the $\hat{\Lambda}$ tensor to protocol (stress ensemble) we generated sheared packings of the same grains used in the isotropic compression. Under the experimental conditions of pure shear, with principle stress along $x$ and $y$, a diagonal form with different values of $\lambda_{ii}$ provides an excellent description of the experimental observations (Fig. 3). We find that $\lambda_{ii}$, satisfy
Figure 2: Comparisons in Fourier space between theoretical predictions (top panels) and numerical results (bottom panels) from jammed packings of frictionless spheres in three dimensions.

The figures display the radially averaged correlation functions \( a: C_{xxxx}(\theta, \phi) \), \( b: C_{xyxy}(\theta, \phi) \), \( c: C_{xxzz}(\theta, \phi) \) and \( d: C_{yxyz}(\theta, \phi) \) respectively. The coordinates \((H_x, H_y)\) represent a Hammer projection of the \((\theta, \phi)\) shell onto the plane. The results are presented for system size \( N = 27000 \), and have been averaged over 350 configurations. The range of packing fractions for these configurations is \( \phi \in [0.686, 0.689] \) and the range of pressure per grain is \( P \in [0.0136, 0.0147] \). Results for the \( \hat{\Lambda} \) tensor have not been presented due to the small effective system size: \( 30 \times 30 \times 30 \). The blank regions at the poles for the numerical results is due to the difficulty in sampling for distinct \( \phi \) at \( \theta = 0 \) and \( \theta = \pi \)
Figure 3: Comparisons in Fourier space between the experimental results (red symbols) in sheared frictional packings and the theoretical predictions (black line). a: Photoelastic images produced from a sheared packing. b: Contour plots of the stress-stress correlation functions: $C_{xxxx}(q, \theta)$ (Top), $C_{yyyy}(q, \theta)$ (Middle) and $C_{xyxy}(q, \theta)$ (Bottom). c: Corresponding angular plots. d: Angular plots of the correlation functions $C_{xxxy}(\theta)$ (Top), $C_{xxyy}(\theta)$ (Middle) and $C_{xyyy}(\theta)$ (Bottom). All these correlations are normalized by the peak value of $C_{xxxx}(\theta)$. The $\hat{\Lambda}$ tensor obtained from the fit is diagonal with $\lambda_{11} = 1$, $\lambda_{22} = 4.5$, and $\lambda_{33} = 1.46$. The ratio of the boundary stresses is $\Sigma_{xx}/\Sigma_{yy} = 1.94$, which satisfies the bound of positivity $\frac{\lambda_{22}}{\lambda_{11}} \geq (\frac{\Sigma_{xx}}{\Sigma_{yy}})^2$. 
a set of bounds imposed by the constraint of positivity of normal forces in granular media $^3, ^{26}$.

A consequence of the pinch point singularities is that in real-space, $\langle D_{ij}(q)D_{ij}(-q) \rangle$ is negative in transverse directions and positive along longitudinal directions, as shown in the SI. It is this property that is strikingly demonstrated in photoelastic images of force chains in Fig. 3 and Figs. S2, S3 and S4 in the SI.

To summarize, we demonstrate that elasticity of athermal amorphous solids emerges from the local constraints of force and torque balance, and not from broken symmetry. The theory is an exact analog of the electrostatics of a fractonic $U(1)$ gauge theory $^{21}$ in polarizable media. Although our analysis focused on granular solids, it marks a paradigm shift in our understanding of “amorphous elasticity” for a much broader class of solids such as jammed suspensions $^{16}$, and colloidal gels $^{18}$ where thermal fluctuations are completely irrelevant.

The current theoretical framework can be extended to thermal amorphous solids such as low-temperature glass formers $^{28-30}$. As in frustrated magnets $^9$, thermal fluctuations lead to a length-scale characterizing the distance between particles at which force and torque balance are violated, and wash out the pinch-point singularity, as shown in the SI. A fully dynamical theory of amorphous materials can be constructed by extending the “electrostatics” to “electrodynamics” through the identification of the analog of a magnetic field, and including unbalanced forces as charged excitations $^{22}$. 
Methods

The main quantity of interest in this study, for a given packing is the stress tensor field in Fourier space given by

\[ \hat{\sigma}^p (q) = \sum_{g=1}^{N_G^p} \hat{\sigma}^p_g \exp (iq \cdot r^p_g). \]  

(8)

Here, ‘p’ denotes a particular realization or packing of \( N_G^p \) grains, while \( g \) denotes a particular grain in the packing located at \( r^p_g \). \( \hat{\sigma}^p_g \) represents the force moment tensor for the grain \( g \), given by

\[ \hat{\sigma}^p_g = \sum_{c=1}^{n^g} r^g_c \otimes f^g_c. \]

(9)

Here \( r^g_c \) denotes the position of the contact \( c \) from the center of the grain \( g \) and \( f^g_c \) denotes the inter-particle force at the contact.

Numerical Methods: We generate jammed packings of frictionless spheres interacting through one-sided spring potentials in two and three dimensions. Our implementation follows the standard O’Hern protocol \(^{13,31,32}\), with energy minimization performed using two procedures (i) conjugate gradient minimization, and (ii) a FIRE \(^{33,34}\) minimization implementation in LAMMPS \(^{35}\). We have verified that these differences in protocol do not modify our results.

We simulate a 50:50 mixture of grains with diameter ratio 1:1.4. In our simulations, the system lengths are held fixed at \( L_x = L_y = 1 \) in 2D and \( L_x = L_y = L_z = 1 \) in 3D. We impose periodic boundary conditions in each direction, setting a lower cutoff between points in Fourier space \( q_{\text{min}} = 2\pi \). We choose an upper cutoff \( q_{\text{max}} = \pi/d_{\text{min}} \) so as to not consider stress fluctuations occurring at length scales shorter than \( d_{\text{min}} \), the diameter of the smallest grain in the packing.
We have presented data for system sizes $N = 512, 1024, 2048, 4096, 8192$ in 2D, averaged over at least 100 configurations for each system size. The results obtained for different system sizes have been collapsed (see Fig. 1) using the system size $N$ and $q_{\text{max}}$ as scaling parameters. This shows that the data presented is not significantly affected by finite size effects. All the 2D packings have a pressure per grain $P \in [0.016, 0.017]$ and packing fraction $\phi \in [0.878, 0.882]$. In 3D, the data for $N = 27000$ is presented in Fig. 2, the data have been averaged over 350 configurations. The range of packing fractions for these configurations is $\phi \in [0.686, 0.689]$, with a pressure per grain $P \in [0.0136, 0.0147]$.

**Experimental Methods:** The experimental results were produced from the analyses of isotropically jammed packings and pure-sheared packings, which were both prepared using a biaxial apparatus whose details can be found in Wang Et al. 2018. This apparatus mainly consists of a rectangular frame mounted on top of a powder-lubricated horizontal glass plate. Each pair of parallel walls of the rectangular frame can move symmetrically with a motion precision of 0.1 mm so that the center of mass of the frame remains fixed. To apply isotropic compression, the two pairs of walls are programmed to move inwards symmetrically. To apply pure shear, one pair of walls moves inwards, and the other pair of walls moves outwards, such that the area of the rectangle is kept fixed. The motion of walls is sufficiently slow to guarantee that the deformation is quasi-static. About 1.5 m above the apparatus, there is an array of $2 \times 2$ high-resolution (100 pixel/cm) cameras that are aligned and synchronized.

To prepare an isotropically jammed packing, we first filled the rectangular area with a 50:50 mixture of 2680 bi-dispersed photoelastic disks (Vishay PSM-4), with diameters of 1.4 cm and...
1.0 cm, to create the various unjammed random initial configurations. Next, we applied isotropic compression to the disks to achieve a definite packing fraction $\phi$, which is the ratio between the area of disks and that of the rectangle. To minimize the potential inhomogeneity of force chains in the jammed packing, we constantly applied mechanical vibrations before the $\phi$ exceeded the jamming point $\phi_J \approx 84.0\%$ of frictionless particles. The final isotropically jammed packing is confined in a square domain of $67.2 \text{ cm} \times 67.2 \text{ cm}$. Here, $\phi \approx 84.4\%$, the mean coordination number is around 4.1, the pressure is around 12 N/m, and the corresponding dimensionless pressure is $2 \times 10^{-4}$.

Once the isotropically jammed packing was prepared, we then applied pure shear of strain 1.5\% to the packing to produce the pure-sheared packing. For both types of packings, two different images were recorded. Disk positions were obtained using the normal image, recorded without polarizers. Contact forces were analyzed from the force-chain image, recorded with polarizers, using the force-inverse algorithm$^{24}$. 


Supplementary Information

In this supplementary document, we describe in detail several key aspects of the theoretical framework and analysis of numerical and experimental data. In Section 1, we outline the derivation of the Gauss’s law constraint on the Cauchy Stress tensor starting from the constraints of force and torque balance on every grain and discuss the mapping of grain-level properties to the continuum theory. In Section 2, we present results for the correlations of the electric displacement tensor $\hat{D}$, in a polarizable medium characterized by $\hat{\Lambda}$. Further in Section 3, we present experimental data for stress correlations from individual configurations. Finally, Section 4 describes the numerical results for the 2D system at finite temperature.

1 Mapping of Granular Media to Continuum VCT Theory

Figure S1: A section of a jammed configuration of soft frictionless disks in 2D. The centers of the grains are located at positions $r_g$. The contact points between grains are located at positions $r_c$. The triangle formed by the points $r_g, r_{g'}, r_c$ (shaded area) is uniquely assigned to the contact $c$ and has an associated area $a_{g,c}$. 
The VCT Gauss’s law (Eq. 2 in the main text), is widely accepted as the coarse-grained description of stresses in athermal solids in mechanical equilibrium\textsuperscript{5,7}. Here, we demonstrate the emergence of this Gauss’s law from local constraints of mechanical equilibrium, for the specific example of disordered granular solids. The arguments can be easily generalized to other amorphous packings at zero temperature. Granular materials consist of an assembly of grains that interact with each other via contact forces, as shown in Fig. S1. In a granular solid, each grain is in mechanical equilibrium and thus, satisfy the constraints of force and torque balance. The constraints of force and torque balance on a grain $g$, with no “body forces” can be written as:

$$\sum_{c \in g} f_{g,c} = 0,$$
$$\sum_{c \in g} r_{g,c} \times f_{g,c} = 0, \quad (S1)$$

respectively. Here, $f_{g,c}$ is the contact force, and $r_{g,c}$ the vector joining the center of grain $g$ to the contact $c$ (Fig. S1). This places $dN + d(d - 1)N$ nontrivial constraints on the $N$ grains that are part of the contact network. A grain is said to be a part of the contact network if it has more than one contact and grains which are not part of the contact network are defined to be “rattlers”. In our representation, the rattlers become part of voids. Given a set of $f_{g,c}$ and $r_{g,c}$, one can define a stress tensor for a grain with area $A_g$:

$$\hat{\sigma}_g = \frac{1}{A_g} \sum_{c \in g} r_{g,c} \otimes f_{g,c}. \quad (S2)$$

The coarse-grained stress tensor field, $\hat{D}(r)$ is obtained by summing $\hat{\sigma}_g$ over all grains included in a coarse-graining volume, $\Omega_r$, centered at $r$:

$$\hat{D}(r) = \frac{1}{\Omega_r} \sum_{g \in \Omega_r} A_g \hat{\sigma}_g. \quad (S3)$$
The symmetry of $\hat{\sigma}_g$ is easy to establish by writing every contact force as the sum of a normal force, which is along the contact vector $r_{g,c}$, and a tangential force perpendicular to it. The normal part leads to a symmetric contribution to $\hat{\sigma}_g$. Using the torque-balance equation, Eq. S1, the contribution from the tangential forces sum up to zero. To establish the divergence free condition, we follow the approach outlined in Degiuli, E. and McElwaine, J. 2011 by first subdividing $\hat{\sigma}_g$ into contributions from each contact. As seen from Fig. S1, we can associate a triangle of area $a_{g,c}$ with each contact, and $A_g = \sum_{c \in g} a_{g,c}$. Adopting a convention that we traverse around a grain in a counterclockwise direction, we associate with contact $c$, the triangle that is defined by $c$ and the contact $c'$ that follows it. We can then write: $A_g \hat{\sigma}_g = \sum_{c \in g} a_{g,c} \hat{\sigma}_c$, where $\hat{\sigma}_c$ is yet to be defined. Comparing to Eq. S2, we see that $a_{g,c} \hat{\sigma}_c = r_{g,c} \otimes f_{g,c}$, therefore $\hat{\sigma}_c = r_{g,c} \otimes f_{g,c}/a_{g,c}$. The signed area $a_{g,c}$ is given by $a_{g,c} = (1/2)r_{g,c} \times (r_{c'} - r_c)$. The divergence theorem is: $\int_V \nabla \cdot \sigma_{ij} = \int_{\partial V} n_i \sigma_{ij}$, where $\hat{n}$ is the unit normal to $\partial V$, which can be written as as $\int_V \nabla \cdot \hat{\sigma} = \int_{\partial V} (dr \times \hat{\sigma})_j$. We can apply the discrete version of this theorem to $\hat{\sigma}_g$ to get:

$$A_g(\nabla \cdot \hat{\sigma})_g = \sum_{c \in g} \hat{\sigma}_c \times (r_{c'} - r_c) = \sum_{c \in g} f_{g,c} = f_{ext}. \quad (S4)$$

In the absence of external forces, $\hat{\sigma}_g$ is divergence free. This grain-level condition leads to a similar condition on $\hat{D}(r)$: $\Omega \nabla \cdot \hat{D}(r) = \sum_{c \in \partial \Omega} f_c$, where the sum is over the contact forces on the boundary of $\Omega$, which is still discrete.

To map to the continuum theory, we posit that disorder averaging over all discrete networks that occur under given external conditions leads to
\[ \partial_i (\hat{D}(r))_{ij} = (f_{\text{ext}})_j. \]

We expect this mapping to be accurate if the coarse-graining volume \( \Omega \) is much larger than a typical grain volume. The excellent correspondence between disorder-averaged \( \hat{D} \) correlations measured in granular packings and theoretical predictions, shown in the main text justifies the above mapping. In Section 3 of this Supplementary Information, we present experimental measurements of \( \hat{D} \) correlations in individual configurations to show that self-averaging is a very good approximation for internal stresses in granular media.

2 Stress-Stress Correlations in Polarizable Media

In this section we present expressions for the correlations of the \( \hat{D} \) tensor, analogous to the expressions for the \( \hat{E} \) correlations in vacuum (Eq. 7 in main text). The starting point is Eq. 5 in the main text: Gauss’s law and the magnetostatic condition for a polarizable medium characterized by the rank-4 tensor, \( \hat{\Lambda} \). In the vacuum theory \(^{25}\), the strategy is to project out the divergence mode from the completely isotropic rank-4 tensor, using the magnetostatic condition. This condition in \( \mathbf{q} \)-space, for a polarizable medium is given by

\[ D_{ij}(\mathbf{q}) = (\hat{\Lambda}^{-1}\hat{\Lambda})_{ij}(\mathbf{q}); \ A_{ij}(\mathbf{q}) \equiv \mathbf{q} \otimes \phi, \quad (S5) \]

where \( \phi \) is the electrostatic gauge potential, as discussed in the main text. Since \( \hat{\Lambda} \) has to obey the symmetry \( ij \to ji \), it is simpler to write the components of \( \hat{D} \) as a vector of length 3 in 2D: \( (D_{xx}, D_{yy}, D_{xy}) \), and a vector of length 6 in 3D. The rank-4 tensor can be then expressed as a
3 × 3 (2D) and a 6 × 6 (3D) matrix \(^8\). Furthermore, if \(\hat{\Lambda}\) is a symmetric matrix in this representation, then the \(\hat{D} - \hat{D}\) correlations can be obtained from the \(\hat{E} - \hat{E}\) correlations by a transformation of the metric: \(q \rightarrow \bar{q}(\hat{\Lambda})\). Such a transformation is reminiscent of the emergence of birefringence in quantum spin ice in the presence of an applied electric field \(^{40}\). For the more general situation that can occur in granular media the matrix is not symmetric, and a cleaner approach is to use the dual formalism in which the potential is obtained by solving Gauss’s law \(^{38}\). In this dual formalism, potentials in 2D and 3D appear differently: a scalar in 2D and a second-rank symmetric tensor in 3D. The expression for the correlations of the potentials can be worked out explicitly, and from that the \(\hat{D} - \hat{D}\) correlations can be obtained in a straightforward manner. In 2D, \(\partial_i D_{ij} = 0\) is solved by introducing a potential \(^{3,26,37,38}\), \(\psi : D_{ij} = \epsilon_{ia}\epsilon_{jb}\partial_a\partial_b\psi\). The potential in 3D is a symmetric tensor, \(\psi_{ij} : D_{ij} = \epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c\psi_{bd}\)

Here, we present the explicit construction of the correlations of \(D_{ij}\) in 2D \(^{3,26,37}\). The magnetostatic condition implies that \(\hat{\Lambda}\) acts as a stiffness tensor in a Gaussian theory. Using the \(q\)-space representation: \(D_{ij}(q) = \epsilon_{ia}\epsilon_{jb}q_aq_b\psi(q)\). The correlations \(\langle \psi(q)\psi(-q) \rangle\) can be computed, and give:

\[
\langle \psi(q)\psi(-q) \rangle = [A_{ij}(q)\Lambda_{ijkl}A_{kl}(-q)]^{-1},
\]

\[
A_{ij} = q^2 \delta_{ij} - q_i q_j.
\]

(S6)

The correlations of \(D_{ij}\) then follow as:

\[
\langle D_{ij}(q)D_{kl}(-q) \rangle = \epsilon_{ia}\epsilon_{jb}\epsilon_{kc}\epsilon_{ld}q_aq_bq_cq_d\langle \psi(q)\psi(-q) \rangle.
\]
For the special case of $\hat{\Lambda}$ being a diagonal tensor with components $\lambda_i$, $i = xx, yy, xy$, the correlations simplify to:

\[
C_{xxxx}(q) = \langle D_{xx}(q) D_{xx}(-q) \rangle = \frac{q_y^4}{\lambda_{xx} q_y^2 + \lambda_{yy} q_x^2 + 2\lambda_{xy} q_x^2 q_y^2},
\]

\[
C_{xyxy}(q) = \langle D_{xy}(q) D_{xy}(-q) \rangle = \frac{q_x^4 q_y^2}{\lambda_{xx} q_y^2 + \lambda_{yy} q_x^2 + 2\lambda_{xy} q_x^2 q_y^2},
\]

\[
C_{yyyy}(q) = \langle D_{yy}(q) D_{yy}(-q) \rangle = \frac{q_x^4}{\lambda_{xx} q_y^2 + \lambda_{yy} q_x^2 + 2\lambda_{xy} q_x^2 q_y^2},
\]

\[
C_{xxxx}(q) = \langle D_{xx}(q) D_{xy}(-q) \rangle = -\frac{q_x^3 q_y^3}{\lambda_{xx} q_y^2 + \lambda_{yy} q_x^2 + 2\lambda_{xy} q_x^2 q_y^2},
\]

\[
C_{xyxy}(q) = \langle D_{xx}(q) D_{yy}(-q) \rangle = \frac{q_y^2 q_x^2}{\lambda_{xx} q_y^2 + \lambda_{yy} q_x^2 + 2\lambda_{xy} q_x^2 q_y^2},
\]

\[
C_{xyxy}(q) = \langle D_{xy}(q) D_{yy}(-q) \rangle = -\frac{q_y^3 q_x^3}{\lambda_{xx} q_y^2 + \lambda_{yy} q_x^2 + 2\lambda_{xy} q_x^2 q_y^2}.
\]

The experimental and numerical measurements of correlations in 2D, shown in Fig. 1 of the main text and in Fig. S2 of the supplementary, have been fit to the above forms. To analyze the correlations in isotropically compressed 3D packings, we assume that $\hat{\Lambda}$ is the identity tensor and use Eq. 7 of the main text, which gives the correlations in vacuum with an overall stiffness constant, $\lambda$.

3 Force Chains and Stress Correlations

A striking consequence of the anisotropic correlations in $q$-space is evident if we analyze the correlations of the normal stresses, $D_{xx}$ and $D_{yy}$ in real space. The Fourier Transform of $C_{xxxx}$ in
Figure S2: Comparisons in Fourier space between the theoretical predictions (black line) with $\Lambda = 1$, and the numerical and the experimental results (symbols) of the stress-stress correlations in 2D, isotropically jammed systems. 

a, Photo-elastic images, in which each grain is shaded according to the magnitude of its normal stress, exhibit clear filamentary structures that are normally referred to as force chains.

b, Theoretical predictions of $C_{xxxx}(q,\theta)$ and $C_{xyxy}(q,\theta)$, which are independent of $q$, and the corresponding angular functions $C_{xxxx}(\theta)$ and $C_{xyxy}(\theta)$.

c, Numerical data of the frictionless jammed packings within the range of pressure $P \in [0.016, 0.017]$. The results of the five different system sizes $N = 512, 1024, 2048, 4096, 8192$ are shown in the angular plots.

d, Experimental data from frictional packings within the range of pressure $P \in [1.5 \times 10^{-4}, 2.9 \times 10^{-4}]$. All correlation functions are normalized by their peak values of $C_{xxxx}(\theta)$. The units of $q$ are $2\pi/L$, where $L$ is the system size: $L \approx 100d_{\min}$ in simulations, $L = 40d_{\min}$ in experiments. Here $d_{\min}$ is the diameter of the small particle. Both the numerical and experimental data start to deviate from the theoretical predictions around $q \geq 2\pi/4d_{\min}$, indicating the breakdown of the continuum limit.
isotropic systems, with \( \hat{\Lambda} = \lambda \mathbb{I} \) illustrates the point:

\[
C_{xxxx}(r_x, r_y) = \frac{3}{2\lambda r_x^2} \quad \text{for} \quad r_x \gg r_y,
\]

\[
C_{xxxx}(r_x, r_y) = -\frac{1}{2\lambda r_y^2} \quad \text{for} \quad r_y \gg r_x.
\]

(S8)

The reverse is true for \( C_{yyyy} \). The consequence of this feature is that the transverse correlations become \textit{negatively} correlated. The photo-elastic images from 2D experiments, shown in the main text and in Figs. S3 and S4, are a striking visual representation of this stark difference between longitudinal and transverse correlations, which in turn is a manifestation of the conservation of “charge-angular-momentum”, and the resulting sub-dimensional propagation \(^{21}\). The \( U(1) \) gauge theory with vector charges, therefore, clarifies the meaning of force-chains within a continuum, disorder-averaged theory.

**Additional Analysis of Experiments** In this subsection, we present results of stress correlations from individual configurations in the sheared experimental packings to illustrate how well self-averaging works in these jammed packings. We note that our systems are deep in the jammed region: we do not address the possible breakdown of self-averaging close to the unjamming transition.
Figure S3: Experimental measurements of correlations in Fourier space, for a single packing in the ensemble of packings, used to generate the averaged correlations shown in the main text (Fig. 3). The features observed in these averaged correlations, are seen to emerge in a single packing, demonstrating the self-averaging property of the stress in these packings that are deep in the jammed regime.
Figure S4: Experimental measurements of correlations in Fourier space, for a second packing created under the same external conditions as in Fig. S3
4 Finite Temperature Results

Pinch point singularities are one of the salient features of the VCT correlation functions. These singularities originate from the strict constraints of mechanical equilibrium imposed on athermal systems. For a system at finite temperature however, these constraints can be violated and hence we expect the pinch point singularities to disappear at finite temperatures. Thus, the presence of a pinch point singularity is a hallmark of an athermal system. The numerically generated stress correlations from a 2D system at finite temperature is shown in Fig. S5 and it can be clearly seen that the pinch point singularity has vanished at this temperature \( \frac{E_{\text{thermal}}}{E_{\text{compression}}} = 3.9 \).

The numerical simulations were carried out in LAMMPS and the finite temperature was imposed through a Nosé-Hoover thermostat. The protocol is to start with a valid athermal \( T = 0 \) configuration, generated following the procedure described in the Numerical Methods Section of the main text and then perform finite temperature dynamics to compute the stress correlations at a non-zero temperature. This procedure is then repeated over multiple initial athermal configurations and ensemble averaged to obtain the finite temperature stress correlations. The results displayed are obtained for packings of 8192 disks with an average pressure per grain \( P \in [0.016, 0.017] \). The results shown have been averaged over 50 starting athermal configurations in 2D with 50 finite temperature configurations sampled during the finite temperature molecular dynamics run, for each of the 50 starting configurations.
Figure S5: Comparisons in Fourier space between stress correlations at zero (Top) and finite (Bottom) temperatures. The columns a, b, c, and d show the results for correlation functions $C_{xxxx}$, $C_{xyxy}$, $C_{xxyy}$ and $C_{xyxy}$ respectively. The packings used have an average compression energy per grain $E_{\text{compression}} \approx 10^{-4}$ and the finite temperature results have an average thermal energy per grain $E_{\text{thermal}} \approx 3.9 \times 10^{-4}$.

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