Hidden Positivity and a New Approach to Numerical Computation of Hausdorff Dimension

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Continued Fractions, Invariant Sets

For \( x \in (0, 1] \), define \( \lfloor \frac{1}{x} \rfloor = n_i \) = greatest integer \( n \) with \( n \leq \left( \frac{1}{x} \right) \)

Given \( x \in (0, 1] \) write \( x = n_1 + x_1 \), where \( n_1 = \lfloor \frac{1}{x} \rfloor \)

so

\[
x = \frac{1}{n_1 + x_1}, \text{ where } 0 \leq x_1 < 1
\]

If \( x_1 \neq 0 \), we can write

\[
x_1 = \frac{1}{n_2 + x_2}, \text{ where } n_2 = \lfloor \frac{1}{x_1} \rfloor \text{ and } 0 \leq x_2 < 1, \text{ so}
\]

\[
x = \frac{1}{n_1 + \frac{1}{n_2 + x_2}}
\]

If \( x_2 \neq 0 \), we can continue, etc.

The positive integers \( n_1, n_2, \ldots, n_k \) obtained in this way are the "terms" or "coefficients" of the continued fraction expansion of \( x \in (0, 1] \).

The continued fraction expansion of \( x \in (0, 1] \) has only finitely many terms iff \( x \) is rational.
The set $E[B]$, for $B \subset \mathbb{N}$

If $B$ is a finite set of positive integers, define

$$E[B] := \{ x \in [0,1] \setminus \mathbb{Q} \mid \text{all terms in the continued fraction expansion of } x \text{ lie in } B \}$$

**PROBLEM** Find high order **rigorous** approximations for $\dim_H (E[B])$
A Generalization of $E[B]$

$\forall b \in (0, \infty), \ \theta_b : [0, \infty) \to [0, \infty)$ is defined by

$$\theta_b(x) = \frac{1}{b + x}$$

$B$ = finite set of positive reals

$$\gamma := \min \{b \mid b \in B\}, \ \Gamma := \max \{b \mid b \in B\}$$

Theorem: There exists a unique, nonempty compact set $K[B]$ such that

$$K[B] = \bigcup_{b \in B} \theta_b(K[B])$$

Also, if (BCN),

$$K[B] = E[B]$$
Locating $K[\mathcal{B}]$

$\mathcal{B}$ is a finite set $\subset (0, \infty)$
$\gamma = \min \{ b | b \in \mathcal{B} \}$, $\Gamma = \max \{ b | b \in \mathcal{B} \}$.
$\mathcal{B}_m = \{ (b_1, b_2, \ldots, b_m) | b_j \in \mathcal{B}, 1 \leq j \leq m \}$

Given $\omega \in \mathcal{B}_m$, $\omega = (b_1, b_2, \ldots, b_m)$, $\Theta = \Theta_{b_1} \Theta_{b_2} \cdots \Theta_{b_m}$

where $\Theta_b(x) = \frac{1}{x + b}$

Define
$\alpha = -\frac{\gamma}{2} + \sqrt{\left(\frac{\gamma}{2}\right)^2 + \left(\frac{\gamma}{\Gamma}\right)^2}$, $\beta = -\frac{\Gamma}{2} + \sqrt{\left(\frac{\Gamma}{2}\right)^2 + \left(\frac{\Gamma}{\gamma}\right)^2}$ = ($\frac{\Gamma}{\gamma}$) $\alpha$

and $J = [\alpha, \beta]$.

Then
$\Theta_b(J) \subseteq J \quad \forall b \in \mathcal{B}$ and $K[\mathcal{B}] \subseteq J$
| $E[1, 2]$ | 14 | 0.0002 | 7  |
|-----------|----|--------|----|
|           |    | 0.531280506277205141624468647368471785493059109018398 |
| $E[1, 3]$ | 8  | 5.0e-05| 6  |
|           |    | 0.454489077661828743845777611651 |
| $E[1, 4]$ | 8  | 5.0e-05| 6  |
|           |    | 0.411182724774791776844805904696 |
| $E[1, 2, 3]$ | 5  | 0.0001 | 5  |
|           |    | 0.705660908028738 |
| $E[1, 3, 5]$ | 8  | 0.001  | 6  |
|           |    | 0.581366821182975 |
| $E[2, 3, 4, 5]$ | 16 | 0.005  | 4  |
|           |    | 0.559636450164776713312144913530 |
| $E[1, 2, 3, 4, 5]$ | 5  | 0.0005 | 5  |
|           |    | 0.836829443681209 |
| $E[1, 3, 5, \ldots, 33]$ | 10 | 0.01   | 1* |
|           |    | 0.770516008717163 |
\( \Theta_b(K[\mathcal{B}]) \cap \Theta_c(K[\mathcal{B}]) = \emptyset \) for \( b, c \in \mathcal{B} \), \( b \neq c \)?

\( \mathcal{B} \subset \mathbb{R}, (0, \infty) \) is a finite set; \( \forall j \in \mathbb{B} \) and \( \gamma \leq b \leq \Gamma \), \( \forall b \in \mathcal{B} \).

A sufficient condition that \( \Theta_b(K[\mathcal{B}]) \cap \Theta_c(K[\mathcal{B}]) = \emptyset \) \( \forall b, c \in \mathcal{B} \), \( b \neq c \) is

\( \# 1 \) \( |c-b| > \left( \frac{1}{3} - \frac{1}{5} \right) \left[ \frac{2}{1 + \sqrt{1 + \frac{4}{3}}} \right] \), \( \forall b, c \in \mathcal{B} \), \( b \neq c \)

\( \# 2 \) is satisfied if

\( \# 3 \) \( c - b \geq \frac{1}{5} \) whenever \( c, b \in \mathcal{B} \), \( c > b \)

\( \# 4 \) is satisfied if \( \mathcal{B} \subset \mathbb{N} \)
$\text{Dim}_H(K[B])$ and the Operator $L_\omega, \omega \geq 0$

Let $S$ be a compact interval, $S \subset [0, \infty)$.
Assume that $\Theta_b(S) < S \forall b \in B$.
For $s > 0$, define $L_\omega : X := C_0^\infty(S) \to X$ by

$$(L_\omega f)(x) = \sum_{b \in B} 1_{\Theta_b}(x) \lambda^\omega f(\Theta_b(x))$$

The map $\omega \mapsto \lambda_\omega := \text{R}(L_\omega)$ is spectral radius of $L_\omega$.

$C^\infty_0$ is strictly decreasing and log convex.

If $K := K[B]$ and $\Theta_b(K) \cap \Theta_c(K) = \emptyset \forall b \in B, b \neq c$,

$\text{dim}_H(K[B]) = \lambda_\omega$ where $\lambda_\omega = 1$.
The Spectrum of $\Lambda_\alpha$

Take $5 \subseteq \mathbb{C}$ to be a compact interval $\Theta_\alpha (S) \subseteq \mathbb{C}$ for all $\alpha \in \mathbb{B}$.

Let $Y$ denote one of $\text{Lip}_R (S)$ or $C^n_R (S)$ for some $n \in \mathbb{N}$.

Then $L_\alpha, Y$ defines a bounded linear operator $\Lambda_\alpha : Y \to Y$.

(i) $\rho (\Lambda_\alpha ) = \rho (L_\alpha ) = \lambda_\alpha > 0$, where $L_\alpha : C_R (X) \to C_R (X)$.

(ii) $\rho (L_\alpha ) = \sup \{ \frac{\| L_\alpha \|}{\lambda} : z \in \sigma (L_\alpha ) \}$.

(iii) $\lambda_\alpha$ is an algebraically simple eigenvalue of $\Lambda_\alpha$, with eigenvector $w_\alpha : \mathbb{Z} \to \mathbb{C}$ and $w_\alpha (x) > 0 \forall x \in S$ and $w_\alpha \in Y$ is unique (up to within positive scalar multiples).
Convergence of iterates of \( \left( \frac{1}{\lambda_n} \mathbf{A}_n \right) \)

If \( u \in Y \) (\( Y = \text{Lip}_{IR} (S) \) or \( Y = C^n_{IR} (S) \)) and \( u(x) > 0 \) \( \forall x \in S \), \( \exists \ a = a(u) > 0 \) such that

\[
\lim_{n \to \infty} \left( \frac{1}{\lambda_n} \mathbf{A}_n \right)^n (u) = a w_u.
\]

Convergence in \((\ast)\) is in the \( Y \) topology. Here \( w_u \) is the unique, normalized strictly positive eigenvector of \( \mathbf{A}_n \) and \( \lambda_n := R(\mathbf{A}_n) \).
Estimates for \( \frac{D^k v_\alpha(x)}{v_\alpha(x)} \), \( x \in [a,b] \), \( D = \frac{d}{dx} \)

There is a unique strictly positive eigenvector (to within scalar multiples) \( \psi \in C^n_{\mathbb{R}}([a,b]) \) for \( \lambda_\psi : Y = C^n_{\mathbb{R}}([a,b]) \to Y, \lambda_\psi(f) = L_\alpha(f) \) for \( f \in Y \).

The eigenvalue for \( \psi \) is \( R(L_\alpha) = R(\lambda_\psi) \)

(Here \( \alpha = -\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{8}\right)} \), \( \beta = \left(\frac{1}{\sqrt{8}}\right) \alpha \))

We have the following estimates for \( x \in [a,b] \):

\[
(-1)^k \left[ \frac{D^k v_\alpha(x)}{v_\alpha(x)} \right] \leq \frac{(2 \alpha)(2 \alpha + 1) \cdots (2 \alpha + k - 1)}{(\alpha \sqrt{1 + (\beta/n)})^k}
\]

\[
\frac{(\alpha \sqrt{1 + (\beta/n)})^k}{(2 \alpha)(2 \alpha + 1) \cdots (2 \alpha + k - 1)} \leq (-1)^k \left[ \frac{D^k v_\alpha(x)}{v_\alpha(x)} \right]
\]
Iterating $\Lambda^m$

For $f \in Y$,

$$(\Lambda^m f)(x) = \sum_{b \in B} 1_{\theta_b}(x) f(\theta_b(x))$$

One can check that

$$(\Lambda^m f)(x) = \sum_{\omega \in \Theta_m} 1_{\theta_\omega}(x) f(\theta_\omega(x))$$

(Recall that $\omega = (b_1, b_2, \ldots, b_m)$ and $\theta_\omega = \theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_m}$)

Remark. $(R(\Lambda^m))^m = R(\Lambda^m)$, so

$R(\Lambda^m) = 1 \iff R(\Lambda) = 1.$
Cônes and Positive Linear Operators

Let $X$ be a real Banach space. If $K \subseteq X$, $K$ is called a "cone" if

(i) $\alpha x + \beta y \in K$ whenever $x, y \in K$ and $\alpha, \beta \in \mathbb{R}$ are nonnegative

and

(ii) If $x \in C \setminus \{0\}$, then $-x \notin C$

If $K$ is closed in the topology on $X$, $K$ is a "closed cone".
An Important Example: The Cone $K_M(T)$

Let $T$ be a bounded complete metric space with metric $ρ$
Assume $T$ contains at least 2 points

$Y = \text{Lip}_1(T) := B$-space of Lipschitz maps $f: T \to \mathbb{R}$

$X = B$-space of bounded continuous maps $f: T \to \mathbb{R}$

$K_M(T) := \{ f \in X \mid f(t_1) \leq f(t_2) \exp(\rho(t_1, t_2)) \forall t_1, t_2 \in T \}$ in $Y$

$K_M(T)$ is a closed "normal cone" in $Y$ and
$\text{int}(K_M(T))$ in $Y$ is nonempty
\( \sigma(A) \) if \( A: Y \to Y \) and \( A(K_M(T)) \subset K_{M'}(T), \ M' < M \)?

\((T, \rho)\) is a bounded, complete metric space. \( Y := \text{Lip}_{\alpha}(T) \)

\( A: Y \to Y \) is a bounded linear map; \( A^2 \neq 0 \).

**Theorem** Assume also that \( A(K_M(T)) \subset K_{M'}(T), \ M' < M \).

Then

1. \( \exists v \in K_M(T) \setminus \{0\} \) with \( Av = R(A)v \), \( R(A) > 0 \) and \( R(A) \) has algebraic multiplicity equal to 1.

2. \( \sigma(A) \setminus \{R(A)\} \subset \{z \in \mathbb{C} \mid |z| \leq R'\} \), where \( R' < R(A) \).

3. \( u \in K_M(T) \) and \( A(u) \neq 0 \) \( \Rightarrow \exists a = a(u) > 0 \) with

\[
\lim_{j \to \infty} \left( \frac{1}{R(A)} A \right)^j(u) = a(u) v
\]
Piecewise Polynomials: the Collocation Method

Take \( N \in \mathbb{N} \) large, \( h = \frac{b - a}{N} \) and \( t_i = a + ih, \ 0 \leq i \leq N \).

Select \( r = \) positive integer and \( 0 \leq i \leq N, 0 \leq j \leq r, \) with

\[
0 = c_{i,0} < c_{i,1} < c_{i,2} < \ldots < c_{i,r} = t_i
\]

Define \( T = \{ c_{i,j} : 1 \leq i \leq N, 0 \leq j \leq r \} \) and \( |T| = N \cdot r + 1 \) and \( C_T(T) = \{ f : T \rightarrow \mathbb{R} \} = \mathbb{R}^{N \cdot r + 1} \) dimensional vector space.

If \( f : T \rightarrow \mathbb{R} \), there is a unique polynomial

\( p_i(t), t \in [t_{i-1}, t_i], \) such that \( \deg(p_i) \leq r \) and

\( p_i(c_{i,j}) = f(c_{i,j}), \ 0 \leq j \leq r \)

Define \( \mathcal{F} : [a, b] \rightarrow \mathbb{R} \)

\( \mathcal{F}(t) = p_i(t), \) for \( t \in [t_{i-1}, t_i], \ 1 \leq i \leq N \).
Extended Chebyshev Points on $[-1,1]$

Define $\hat{c}_k = -\left[ \cos\left(\frac{2k+1}{2n+2} \pi\right) / \cos\left(\frac{\pi}{2n+2}\right) \right]$

Important to define

$\hat{c}_{j+1} = t_{j+1} + \frac{\hat{c}_k}{1 + \hat{c}_k}, \quad 1 \leq j \leq N, \quad 0 \leq k \leq n$

If $x \in \left[ t_{j-1}, t_j \right]$, $x = t_{j-1} + \frac{\hat{c}_k}{1 + \hat{c}_k}$, $-1 \leq \hat{x} \leq 1$

If $f \in C^k_{[0,1]}$ and $f$ extends to a $C^{k+1}$ function on each interval $[t_{j-1}, t_j]$, $1 \leq j \leq N$, then

$\left| f(x) - F(x) \right| \leq \left[ \frac{1}{(k+1)!} \right] \left[ \max_{x \in [t_{j-1}, t_j]} \left| f^{(k+1)}(x) \right| \right] \left( \frac{\hat{c}_k}{2} \right)^{k+1} \prod_{\hat{x} = 0}^n \left| \frac{1}{\cos\left(\frac{\pi}{2n+2}\right)} \right|$

$\forall x \in \left[ t_{j-1}, t_j \right]$

$max_{-1 \leq \hat{x} \leq 1} \left| \prod_{\hat{x} = 0}^n \left( \hat{x} - \hat{c}_k \right) \right| = \left( \frac{1}{2^{2k}} \right) \left[ \frac{1}{\cos\left(\frac{\pi}{2n+2}\right)} \right]^{k+1}$
Defining the "Approximation" $A_{\ast n}$ to $L^m_{\ast 0}$

For $f \in C_{IR}([\alpha, \beta])$

$$(L^m_{\ast n} f)(x) = \sum_{\omega \in \mathcal{B}_n} \| \theta_\omega(x) \|^m f(\theta_\omega(x))$$

For $T = \{ c_{i,j} | 1 \leq i \leq N, 0 \leq j \leq n \}$ and $f \in C_{IR}(T)$

we have defined $F : [\alpha, \beta] \rightarrow IR$ by $F([t_{i-1}, t_i]) = p_i$

where $p_i(c_{i,j}) = f(c_{i,j})$, $0 \leq j \leq n$ and $p_i$ is a polynomial

of degree $\leq n$. We define $g \in C_{IR}(T)$ by

$g(c_{i,j}) = \sum_{\omega \in \mathcal{B}_n} 1_{\theta_\omega}(c_{i,j}) \| \theta_\omega(c_{i,j}) \|^m F(\theta_\omega(c_{i,j}))$

The map $f \in C_{IR}(T) \rightarrow g \in C_{IR}(T)$ defines a linear

map $A_{\ast n} : C_{IR}(T) \rightarrow C_{IR}(T)$
Approximating $R(L_n^m)$ by $R(A_{m,n})$

Assume that $r$ is a given positive integer and $0 < s < \frac{3}{2}$. There exists a positive integer $n$ and $\lambda_0 > 0$, such that if $h = \frac{\beta - d}{N} < \lambda_0$ then $\exists$ constants $M(r, s, \lambda_0) > 0$ and

$M' = M'(r, s, \lambda_0) < M$, $M' > 0$, with

$A_{m,n}(K_M(T)) \leq K_{M'}(T)$, $A_{m,n}^2 \neq 0$

There is a constant $H = H(r, s, \lambda_0) > 0$

such that for $x_n^m = R(L_n^m)$

$(\ast) \quad x_n^m(1 - Hh^r) \leq R(A_{m,n}) \leq x_n^m(1 + Hh^r)$