MABUCHI AND AUBIN-YAU FUNCTIONALS OVER COMPLEX MANIFOLDS

YI LI

Abstract. In the previous papers [2, 3] the author constructed Mabuchi and Aubin-Yau functionals over any complex surfaces and three-folds, respectively. Using the method in [3], we construct those functionals over any complex manifolds of the complex dimension bigger than or equal to 2.

Contents
1. Introduction 1
   1.1. Mabuchi and Aubin-Yau functionals on Kähler manifolds 2
   1.2. Mabuchi and Aubin-Yau functionals on complex surfaces 3
   1.3. Mabuchi and Aubin-Yau functionals on complex three-folds 3
   1.4. Mabuchi and Aubin-Yau functionals on complex manifolds 5
   1.5. Further questions 6
2. Mabuchi $L^M_\omega$ functional on complex manifolds 6
   2.1. The main idea 7
   2.2. The definitions of $c_1$ and $c_2$ 7
   2.3. The constructions of $I_5$ and $I_6$ 14
   2.4. The constructions of $I_7$ and $I_8$ 18
   2.5. Recursion formula 22
3. Aubin-Yau functionals on compact complex manifolds 32
   3.1. The main idea 32
   3.2. The construction of Aubin-Yau functionals 33
   Appendix A. Proof the identities (3.14), (3.21) and (3.25) 38
   Appendix B. Solve the system of the linear equations 41
   Appendix C. Explicit formulas of $I^A_Y(\varphi)$ and $J^A_Y(\varphi)$ 43
   References 47

1. Introduction

Mabuchi and Aubin-Yau functionals play a crucial role in studying Kähler-Einstein metrics and constant scalar curvatures (see [5]). How to generalize these functionals from Kähler geometry to complex geometry is an interesting problem:

Question 1.1. Can we define Mabuchi and Aubin-Yau functionals over compact complex manifolds so that these functionals coincide with the original definitions and satisfy the same basic properties?

In [2, 3], the author answered this question in the complex dimension two and three, respectively, and proved similar results in the Kähler setting. By carefully
Yi Li

checking and using a similar method in [3], we can construct those functionals in higher dimension cases. So, now, we give an affirmative answer to Question 1.1.

1.1. Mabuchi and Aubin-Yau functionals on Kähler manifolds. In this subsection we review Mabuchi and Aubin-Yau functionals on Kähler manifolds, and describe some basic properties of these functionals which also hold in any complex manifolds. Let $(X, \omega)$ be a compact Kähler manifold of the complex dimension $n$. Then the volume

$$V_\omega := \int_X \omega^n$$

depends only on the Kähler class of $\omega$. Let $P_{\text{Kähler}}^\omega$ denote the space of Kähler potentials and define the Mabuchi functional, for any smooth functions $\varphi', \varphi'' \in P_{\text{Kähler}}^\omega$, by

$$\mathcal{L}_{\omega}^{M, \text{Kähler}}(\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X \hat{\varphi} \omega^n_t dt$$

where $\varphi_t$ is any smooth path in $P_{\text{Kähler}}^\omega$ from $\varphi'$ to $\varphi''$. Mabuchi [4] showed that (1.2) is well-defined.

Using (1.2) we can define Aubin-Yau functionals, for any smooth function $\varphi \in P_{\text{Kähler}}^\omega$, as follows:

$$I_{\omega}^{A\text{Y}, \text{Kähler}}(\varphi) = \frac{1}{V_\omega} \int_X \varphi(\omega^n - \omega^\varphi),$$

$$J_{\omega}^{A\text{Y}, \text{Kähler}}(\varphi) = -\mathcal{L}_{\omega}^{M, \text{Kähler}}(0, \varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n.$$  

So Aubin-Yau functionals are also well-defined. The basic and often useful inequalities, for any smooth function $\varphi \in P_{\text{Kähler}}^\omega$, are

$$\frac{n}{n+1} I_{\omega}^{A\text{Y}, \text{Kähler}}(\varphi) - J_{\omega}^{A\text{Y}, \text{Kähler}}(\varphi) \geq 0,$$

$$(n+1) J_{\omega}^{A\text{Y}, \text{Kähler}}(\varphi) - I_{\omega}^{A\text{Y}, \text{Kähler}}(\varphi) \geq 0.$$  

An important consequence is that we will use the inequalities (1.5) and (1.6) to determine the extra terms on the definitions of $I_{\omega}^{A\text{Y}}(\varphi)$ and $J_{\omega}^{A\text{Y}}(\varphi)$, which are Aubin-Yau functionals over complex manifolds. However, if $\omega$ is not closed, then the above definitions (1.2), (1.3), and (1.4) do not make any sense. Hence we should add some extra terms on the definitions of those functionals; these extra terms should involve $\partial \omega$ and $\overline{\partial} \omega$, but, the essential question is to find the structure of the extra terms. Roughly speaking, $I_{\omega}^{A\text{Y}}(\varphi)$ and $J_{\omega}^{A\text{Y}}(\varphi)$ can be written as

$$I_{\omega}^{A\text{Y}}(\varphi) = I_{\omega}^{A\text{Y}, \text{Kähler}}(\varphi) + \text{terms involving } \partial \omega, \overline{\partial} \varphi + \text{terms involving } \overline{\partial} \omega, \partial \varphi,$$

$$J_{\omega}^{A\text{Y}}(\varphi) = J_{\omega}^{A\text{Y}, \text{Kähler}}(\varphi) + \text{terms involving } \partial \omega, \overline{\partial} \varphi + \text{terms involving } \overline{\partial} \omega, \partial \varphi.$$  

In the following sections, we will explicitly determine the extra terms.

Throughout the rest part of this paper, we denote by $(X, g)$ a compact complex manifold of the complex dimension $n \geq 2$, and $\omega$ be the associated real $(1, 1)$-form. Let

$$\mathcal{P}_\omega := \left\{ \varphi \in C^\infty(X) \big| \omega_\varphi := \omega + \sqrt{-1} \partial \overline{\partial} \varphi > 0 \right\}.$$  

2 YI LI
be the space of all real-valued smooth functions on $X$ whose associated real $(1,1)$-forms are positive.

1.2. Mabuchi and Aubin-Yau functionals on complex surfaces. In this subsection we recall the main result in [2]. Let $(X,g)$ be a compact complex manifold of the complex dimension 2 and $\omega$ be its associated real $(1,1)$-form. For any $\varphi', \varphi'' \in P_\omega$, we define

$$L^M_\omega(\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X \varphi_t \cdot \omega^2 dt$$

$$- \frac{1}{V_\omega} \int_0^1 \int_X \sqrt{-1} \partial \omega \wedge (\partial \varphi_t \cdot \varphi_t) dt$$

$$+ \frac{1}{V_\omega} \int_0^1 \int_X \sqrt{-1} \partial \omega \wedge (\partial \varphi_t \cdot \varphi_t) dt,$$

where $\{\varphi_t\}_{0 \leq t \leq 1}$ is any smooth path in $P_\omega$ from $\varphi'$ to $\varphi''$. Then in [2] we showed that the functional (1.8) is independent of the choice of the smooth path $\{\varphi_t\}_{0 \leq t \leq 1}$. If we set

$$L^M_\omega(\varphi) := L^M_\omega(0, \varphi),$$

then we have an explicit formula [2] of $L^M_\omega(\varphi)$:

$$L^M_\omega(\varphi) = \frac{1}{3V_\omega} \int_X \varphi (\omega^2 + \omega \wedge \omega \varphi + \omega^2)$$

$$+ \frac{1}{2V_\omega} \int_X \varphi \left( -\sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi + \sqrt{-1} \partial \omega \wedge \partial \varphi \right).$$

Now Aubin-Yau functionals are defined by

$$I^A_\omega(\varphi) := \frac{1}{V_\omega} \int_X \varphi (\omega^2 - \omega \varphi)$$

$$- \frac{1}{V_\omega} \int_X \varphi \cdot \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi + \frac{1}{V_\omega} \int_X \varphi \cdot \sqrt{-1} \partial \omega \wedge \partial \varphi,$$

$$J^A_\omega(\varphi) := -L^M_\omega(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^2$$

$$- \frac{1}{V_\omega} \int_X \varphi \cdot \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi + \frac{1}{V_\omega} \int_X \varphi \cdot \sqrt{-1} \partial \omega \wedge \partial \varphi.$$

Moreover they also satisfy the inequalities (1.5) and (1.6); that is

$$\frac{2}{3} I^A_\omega(\varphi) - J^A_\omega(\varphi) \geq 0,$$

$$3J^A_\omega - I^A_\omega(\varphi) \geq 0.$$

1.3. Mabuchi and Aubin-Yau functionals on complex three-folds. The functionals over complex three-folds are very different with these over complex
surfaces. For any \( \varphi', \varphi'' \in \mathcal{P}_\omega \), we define

\[
\mathcal{L}^M_\omega(\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X \varphi_t \omega_{\varphi_t}^3 \, dt \\
- \frac{3}{V_\omega} \int_0^1 \int_X \sqrt{-1} \partial \omega \wedge \omega_{\varphi_t} \wedge (\overline{\partial} \varphi_t \cdot \varphi_t) \, dt \\
+ \frac{3}{V_\omega} \int_0^1 \int_X \sqrt{-1} \partial \omega \wedge \omega_{\varphi_t} \wedge (\partial \varphi_t \cdot \varphi_t) \, dt \\
- \frac{1}{V_\omega} \int_0^1 \int_X \overline{\partial} \varphi_t \wedge \partial \varphi_t \wedge \partial \omega \wedge \partial \varphi_t \\
- \frac{1}{V_\omega} \int_0^1 \int_X \overline{\partial} \varphi_t \wedge \partial \varphi_t \wedge \overline{\partial} \omega \wedge \partial \varphi_t,
\]

where \( \{ \varphi_t \}_{0 \leq t \leq 1} \) is any smooth path in \( \mathcal{P}_\omega \) from \( \varphi' \) to \( \varphi'' \). In [3], we proved that (1.13) is well-defined.

For any \( \varphi \in \mathcal{P}_\omega \) we also set

\[
\mathcal{L}^M_\omega(\varphi) := \mathcal{L}^M_\omega(0, \varphi).
\]

If we chose \( \varphi_t = t \cdot \varphi \), then we have an explicit formula [3] of \( \mathcal{L}^M_\omega(\varphi) \):

\[
\mathcal{L}^M_\omega(\varphi) = \frac{1}{4V_\omega} \sum_{i=0}^{3} \int_X \varphi \omega^i \wedge \omega^{3-i} \\
- \frac{1}{2V_\omega} \sum_{i=0}^{1} \frac{i+1}{2} \int_X \varphi \omega^i \wedge \omega^{1-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi \\
+ \frac{1}{2V_\omega} \sum_{i=0}^{1} \frac{i+1}{2} \int_X \varphi \omega^i \wedge \omega^{1-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi.
\]

Now we define Aubin-Yau functionals \( \mathcal{I}^{AY}_\omega, \mathcal{J}^{AY}_\omega \) for any compact complex threefold \( (X, \omega) \):

\[
\mathcal{I}^{AY}_\omega(\varphi) := \frac{1}{V_\omega} \int_X \varphi (\omega^3 - \omega_{\varphi}^3) \\
- \frac{3}{2V_\omega} \int_X \varphi \omega^3 \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi - \frac{3}{2V_\omega} \int_X \varphi \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi \\
+ \frac{3}{2V_\omega} \int_X \varphi \omega^3 \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi + \frac{3}{2V_\omega} \int_X \varphi \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi,
\]

\[
\mathcal{J}^{AY}_\omega(\varphi) := -\mathcal{L}^M_\omega(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^3 \\
- \frac{3}{2V_\omega} \int_X \varphi \omega^3 \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi - \frac{3}{2V_\omega} \int_X \varphi \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi \\
+ \frac{3}{2V_\omega} \int_X \varphi \omega^3 \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi + \frac{3}{2V_\omega} \int_X \varphi \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi.
\]

Similarly, we have

\[
\frac{3}{4} \mathcal{I}^{AY}_\omega(\varphi) - \mathcal{J}^{AY}_\omega(\varphi) \geq 0,
\]

\[
4 \mathcal{J}^{AY}_\omega(\varphi) - \mathcal{I}^{AY}_\omega(\varphi) \geq 0.
\]
1.4. Mabuchi and Aubin-Yau functionals on complex manifolds. In this subsection we assume that the complex dimension \( n \) of the compact complex manifold \((X, \omega)\) is bigger than or equal to 3. For \( \varphi', \varphi'' \in \mathcal{P}_\omega \), define

\[
(1.18) \mathcal{L}_\omega^M(\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X \psi t \omega^n_{\varphi_t} dt
- \frac{1}{V_\omega} \int_0^1 \int_X \frac{n(n-1)\sqrt{-1}}{2} \partial \omega \wedge \omega_{\varphi_t}^{n-2} \wedge (\overline{\partial \varphi}_t \cdot \varphi_t) dt
+ \frac{1}{V_\omega} \int_0^1 \int_X \frac{n(n-1)\sqrt{-1}}{2} \overline{\partial} \omega \wedge \omega_{\varphi_t}^{n-2} \wedge (\partial \varphi_t \cdot \varphi_t) dt
+ \sum_{i=1}^{n-2} \frac{1}{\omega(V_\omega)} \int_X (-1)^i \left( \frac{n}{i+2} \right) \partial \varphi_t \wedge \partial \omega \wedge \overline{\partial} \varphi_t \wedge \varphi_t^{n-i-2} \wedge (\sqrt{-1} \overline{\partial} \varphi_t)^{i+1}
+ \sum_{i=1}^{n-2} \frac{1}{\omega(V_\omega)} \int_X (-1)^i \left( \frac{n}{i+2} \right) \overline{\partial} \varphi_t \wedge \overline{\partial} \omega \wedge \partial \varphi_t \wedge \varphi_t^{n-i-2} \wedge (\sqrt{-1} \overline{\partial} \varphi_t)^{i+1},
\]

where \( \{\varphi_t\}_{0 \leq t \leq 1} \) is any smooth path in \( \mathcal{P}_\omega \) from \( \varphi' \) to \( \varphi'' \). In Section 2, we prove

**Theorem 1.2.** For any \( n \geq 3 \), the functional \((1.18)\) is independent of the choice of the smooth path \( \{\varphi_t\}_{0 \leq t \leq 1} \) in \( \mathcal{P}_\omega \) from \( \varphi' \) to \( \varphi'' \).

As a consequence of Theorem 1.2, by taking \( \varphi_t = t \cdot \varphi \), we have, for any \( \varphi \in \mathcal{P}_\omega \),

\[
L^M(\varphi) := L^M(0, \varphi) = \frac{1}{V_\omega} \sum_{i=0}^n \int_X \frac{1}{n+1} \varphi \omega_{\varphi_t}^i \wedge \omega^{n-i}
- \sum_{i=0}^{n-2} \frac{i+1}{2V_\omega} \int_X \varphi \omega_{\varphi_t}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi
+ \sum_{i=0}^{n-2} \frac{i+1}{2V_\omega} \int_X \varphi \omega_{\varphi_t}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi.
\]

As before, we define Aubin-Yau functionals for any complex manifolds by

\[
I^{AY}(\varphi) := \frac{1}{V_\omega} \int_X \varphi(n_{\omega}^n - \omega_{\varphi}^n) - \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_{\varphi_t}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi
+ \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_{\varphi_t}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi
= \frac{1}{V_\omega} \sum_{i=0}^{n-1} \int_X \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega_{\varphi_t}^{i+1} \wedge \omega^{n-1-i},
\]

\[
J^{AY}(\varphi) := -L^M(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n - \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_{\varphi_t}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi
+ \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_{\varphi_t}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi
= \frac{1}{V_\omega} \sum_{i=0}^{n-1} \int_X \frac{n-i}{n+1} \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega_{\varphi_t}^i \wedge \omega^{n-1-i}.
\]
In Section 3, we shall show the following

**Theorem 1.3.** For any $\varphi \in \mathcal{P}_\omega$, one has

\begin{align}
\frac{n}{n+1} I_{\omega}^{AY}(\varphi) - J_{\omega}^{AY}(\varphi) &\geq 0, \\
(n+1) J_{\omega}^{AY}(\varphi) - I_{\omega}^{AY}(\varphi) &\geq 0.
\end{align}

In particular, $I_{\omega}^{AY}(\varphi), J_{\omega}^{AY}(\varphi)$ are nonnegative, and

\begin{align}
\frac{1}{n+1} I_{\omega}^{AY}(\varphi) &\leq J_{\omega}^{AY}(\varphi) \leq \frac{n}{n+1} I_{\omega}^{AY}(\varphi), \\
\frac{n+1}{n} J_{\omega}^{AY}(\varphi) &\leq I_{\omega}^{AY}(\varphi) \leq (n+1) J_{\omega}^{AY}(\varphi), \\
\frac{1}{n} J_{\omega}^{AY}(\varphi) &\leq \frac{1}{n+1} I_{\omega}^{AY}(\varphi) \leq I_{\omega}^{AY}(\varphi) - J_{\omega}^{AY}(\varphi) \\
&\leq \frac{n}{n+1} I_{\omega}^{AY}(\varphi) \leq n J_{\omega}^{AY}(\varphi).
\end{align}

1.5. **Further questions.** Up to now we consider functionals over compact complex manifolds without boundary, and we hope that the similar constructions can be achieved for compact complex manifolds with boundary.

**Question 1.4.** Can we define Mabuchi and Aubin-Yau functionals over compact complex manifolds with boundary so that these functionals coincide with the original definitions and satisfy the same basic properties?

There are other functionals, for example, Mabuchi $K_{\omega}^M$ functional, Chen-Tian functionals $\mathcal{H}$, etc. We can ask the following

**Question 1.5.** Can we define the analogy Mabuchi $K_{\omega}^M$ and Chen-Tian functionals over complex manifolds with(out) boundary?

In the future, we will study those two questions.

**Acknowledgements.** The author would like to thank Valentino Tosatti who read this note and pointed out a serious mistake in the first version.

2. **Mabuchi $L_{\omega}^M$ functional on complex manifolds**

In this section we assume that $(X, \omega)$ is a compact complex manifold of the complex dimension $n \geq 3$. For any two complex forms $\alpha$ and $\beta$, we frequently use the following formulas: if $|\alpha| + |\beta| = 2n - 1$, then

\begin{align}
\int_X \alpha \wedge \partial \beta &= (-1)^{|\beta|} \int_X \partial \alpha \wedge \beta = -(-1)^{|\alpha|} \int_X \partial \alpha \wedge \beta, \\
\int_X \alpha \wedge \bar{\partial} \beta &= (-1)^{|\beta|} \int_X \bar{\partial} \alpha \wedge \beta = -(-1)^{|\alpha|} \int_X \bar{\partial} \alpha \wedge \beta.
\end{align}

Another useful formula is

\begin{align}
\alpha \wedge \alpha = 0, \quad \text{if } |\alpha| \text{ is odd.}
\end{align}

By the definition of operators $\partial$ and $\bar{\partial}$, one has

\begin{align}
\partial \bar{\partial} = \bar{\partial} \partial = 0.
\end{align}

Hence the complex conjugate of the operator $\sqrt{-1} \partial \bar{\partial}$ is itself.
2.1. The main idea. Similarly in [3], we consider $\psi(s, t) = s \cdot \varphi_t$ and we can show that (see (2.49))

$$
\frac{2I^0}{n(n-1)\sqrt{-1}} = \frac{I^1}{a_1} - \frac{I^2}{a_2} + c_1, \\
(2.5)
$$

$$
\frac{I^3}{a_3} + \frac{I^4}{a_4} = \frac{3}{-(n-2)\sqrt{-1}}c_1 + c_2,
(2.6)
$$

where $I^i$ are functionals which can be determined \cite{1}, $c_j$ are also functionals but may not be determined, and $a_k$ are nonzero constants which can be determined later. We can use equation (2.6) to eliminate the undetermined term $c_1$, but there arises another undetermined term $c_2$. Our strategy is to find a determined expression for $c_2$. To achieve this we construct two sequences \{ $I^{2(i+1)}$ \}$_{2 \leq i \leq n-2}$, which can be determined, and \{ $c_i$ \}$_{2 \leq i \leq n-2}$, which may not be determined, satisfying

$$
\frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}} = \frac{i + 2}{n - (i + 1)}c_i + c_{i+1}, \\
(2.7)
$$

where $I^{2i+2} := \frac{I^{2i+1}}{a^{2i+1}}, a_{2i+2} := \frac{a_{2i+1}}{a_{2i+2}}$ and $a_{2i+1}$ can be determined later. By our construction we have $c_{n-1} = 0$ which gives us the determined and explicit formula for $c_2$ in terms of $I^{2i+1}$ and $a_{2i+1}$. More precisely, setting

$$
J_i := \frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}},
(2.8)
$$

yields

$$
c_{i+1} = -\frac{i + 2}{n - (i + 1)}c_i + J_i, \quad 2 \leq i \leq n - 2.
(2.9)
$$

By induction on $i$ we obtain

$$
c_i = (-1)^{i-2}(i + 1)!(n - i - 1)!c_2 + \sum_{k=2}^{i-1}(-1)^{i-1-k}(i + 1)!(n - i - 1)! \left( \frac{(k + 2)!(n - k - 2)!}{k!} \right) J_k.
(2.10)
$$

Since $c_{n-1} = 0$ it follows that

$$
c_2 = (-1)^{n-3}(n-3)! \left[ 0 - \sum_{k=2}^{n-2}(-1)^{n-2-k}\frac{n!}{(k + 2)!(n - k - 2)!} J_k \right]
(2.11)
$$

$$
= \sum_{k=2}^{n-2}(-1)^k\frac{3!(n-3)!}{(k + 2)!(n - k - 2)!} J_k, \quad n \geq 4.
$$

When $X$ is a three-fold, we knew that $c_2 = 0$ \cite{3}, but this can be seen from (2.11) if we take $n = 3$. Hence the formula (2.11) holds for any $n \geq 3$.

2.2. The definitions of $c_1$ and $c_2$. Firstly, we consider the "Kähler part" of Mabuchi functional. Let

$$
\mathcal{L}_\omega^0(\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X \varphi_t \omega_{\varphi_t}^n \, dt.
(2.12)
$$

\footnote{A functional $\mathcal{I}$ is said to be determined if $d\Psi = \mathcal{I} \cdot dt \wedge ds$ for some 1-form $\Psi$ on $[0, 1] \times [0, 1]$.}
As in [2, 3], we set \( \psi(s, t) := s \cdot \varphi_t, \) \( 0 \leq t, s \leq 1, \) and consider the corresponding 1-form on \([0, 1] \times [0, 1],\)

\[
\Psi^0 = \left( \int_X \frac{\partial \psi}{\partial s} \omega^\alpha_{\psi} \right) ds + \left( \int_X \frac{\partial \psi}{\partial t} \omega^\alpha_{\psi} \right) dt.
\]

(2.13)

Taking the differential on both sides implies

\[
d\Psi^0 = I^0 \cdot dt \wedge ds
\]

where

\[
I^0 := \int_X \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial s} \omega^\alpha_{\psi} \right) - \int_X \frac{\partial}{\partial s} \left( \frac{\partial \psi}{\partial t} \omega^\alpha_{\psi} \right)
\]

(2.14)

The explicit expression of \( I^0 \) can be determined as follows:

\[
I^0 = \int_X \left[ \frac{\partial \psi}{\partial s} n \omega^\alpha_{\psi} - \sqrt{-1} \partial n \omega^\alpha_{\psi} \right]
\]

\[
= \int_X n \frac{\partial \psi}{\partial s} \omega^\alpha_{\psi} - \sqrt{-1} \partial n \omega^\alpha_{\psi}
\]

\[
= \int_X n \frac{\partial \psi}{\partial t} \omega^\alpha_{\psi} + \sqrt{-1} \partial n \omega^\alpha_{\psi}
\]

\[
= - \int_X n \frac{\partial \psi}{\partial s} \omega^\alpha_{\psi} - \sqrt{-1} \partial n \omega^\alpha_{\psi}
\]

(2.15)

Here we have two slightly different expressions of \( I^0 \), and in the following we will use those expressions to find \( c_1 = A_1 + B_1 - (A_1 + B_1) \), where \( A_1, B_1, \overline{A}_1 \) and \( \overline{B}_1 \) are determined later. This technique will be frequently used in many places. Hence

\[
I^0 = \int_X \left[ -n \sqrt{-1} \partial \left( \frac{\partial \psi}{\partial s} \omega^\alpha_{\psi} \right) \right] - \partial \left( \frac{\partial \psi}{\partial s} \right)
\]

\[
+ \int_X \left[ n \left( \frac{\partial \psi}{\partial t} \omega^\alpha_{\psi} \right) \right] + \partial \left( \frac{\partial \psi}{\partial s} \right)
\]

\[
= \int_X \left[ n \left( \frac{\partial \psi}{\partial t} \omega^\alpha_{\psi} \right) \right] + \partial \left( \frac{\partial \psi}{\partial s} \right)
\]

Thus

\[
I^0 = \int_X \left[ n \left( \frac{\partial \psi}{\partial t} \omega^\alpha_{\psi} \right) \right] + \partial \left( \frac{\partial \psi}{\partial s} \right)
\]

(2.16)
Using another expression of $I^0$, or taking the complex conjugate on both sides of (2.16) since $I^0$ is real, one has

\[ I^0 = \int_X n(n-1) \sqrt{-1} \frac{\partial \psi}{\partial s} \omega_{\psi}^{n-2} \wedge \partial \omega \wedge \partial \left( \frac{\partial \psi}{\partial t} \right) \]

(2.17)

Hence, adding (2.17) to (2.16) and dividing by $n(n-1) \sqrt{-1}$ on both sides, we deduce

\[ \frac{2I^0}{n(n-1)\sqrt{-1}} = \int_X \overline{\psi} \left( \frac{\partial \psi}{\partial t} \right) \frac{\partial \psi}{\partial s} \wedge \omega_{\psi}^{n-2} \wedge \partial \omega + \int_X \overline{\psi} \left( \frac{\partial \psi}{\partial s} \right) \frac{\partial \psi}{\partial t} \wedge \omega_{\psi}^{n-2} \wedge \bar{\partial} \omega \]

(2.18)

According to the expression (2.18), we introduce two functionals

\[ L_1^\omega (\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X a_1 \partial \omega \wedge \omega_{\psi}^{n-2} \wedge (\overline{\partial} \varphi_t \cdot \varphi_t)dt, \]

(2.19)

\[ L_2^\omega (\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X a_2 \overline{\partial} \omega \wedge \omega_{\psi}^{n-2} \wedge (\partial \varphi_t \cdot \varphi_t)dt. \]

(2.20)

Here $a_1, a_2$ are non-zero constants determined later and we require $a_1 = a_2$. Satisfying this condition, $a_1$ and $a_2$ have lots of solutions. In the following we will see that we take only two special cases: $a_i$ are purely complex numbers; $a_i$ are real numbers. Consider the corresponding two 1-forms on $[0,1] \times [0,1]$,

\[ \Psi^1 = \left[ \int_X a_1 \partial \omega \wedge \omega_{\psi}^{n-2} \wedge \left( \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \cdot \psi \right) \right] ds \]

\[ + \left[ \int_X a_1 \partial \omega \wedge \omega_{\psi}^{n-2} \wedge \left( \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \cdot \psi \right) \right] dt, \]

(2.21)

\[ \Psi^2 = \left[ \int_X a_2 \overline{\partial} \omega \wedge \omega_{\psi}^{n-2} \wedge \left( \partial \left( \frac{\partial \psi}{\partial s} \right) \cdot \psi \right) \right] ds \]

\[ + \left[ \int_X a_2 \overline{\partial} \omega \wedge \omega_{\psi}^{n-2} \wedge \left( \partial \left( \frac{\partial \psi}{\partial t} \right) \cdot \psi \right) \right] dt. \]

(2.22)

Firstly, we compute the differential of $\Psi^1$, and the differential $d\Psi^2$ can be easily written down only by taking the complex conjugate on both sides. Calculate

\[ d\Psi^1 = I^1 \cdot dt \wedge ds \]

where

\[ I^1 = \int_X a_1 \frac{\partial}{\partial t} \left[ \partial \omega \wedge \omega_{\psi}^{n-2} \wedge \left( \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \cdot \psi \right) \right] \]

\[ - \int_X a_1 \frac{\partial}{\partial s} \left[ \partial \omega \wedge \omega_{\psi}^{n-2} \wedge \left( \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \cdot \psi \right) \right]. \]

(2.23)
Dividing by \(a_1\) on both sides of (2.24), we have
\[
\frac{I^1}{a_1} = \int_X -\frac{\partial}{\partial t} \left[ \left( \psi \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial s} \right) \right) \land \omega_{\psi}^{n-2} \land \partial \omega \right] \\
+ \int_X \frac{\partial}{\partial s} \left[ \left( \psi \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial t} \right) \right) \land \omega_{\psi}^{n-2} \land \partial \omega \right] \\
= \int_X -\frac{\partial \psi}{\partial t} \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial s} \right) + \psi \cdot \overrightarrow{\partial} \left( \frac{\partial^2 \psi}{\partial t \partial s} \right) \right] \land \omega_{\psi}^{n-2} \land \partial \omega \\
+ \int_X \left[ \frac{\partial \psi}{\partial s} \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial t} \right) + \psi \cdot \overrightarrow{\partial} \left( \frac{\partial^2 \psi}{\partial s \partial t} \right) \right] \land \omega_{\psi}^{n-2} \land \partial \omega \\
+ \int_X \psi \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land (n-2)\omega_{\psi}^{n-3} \land -\sqrt{-1}\overrightarrow{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \partial \omega \\
+ \int_X \psi \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land (n-2)\omega_{\psi}^{n-3} \land -\sqrt{-1}\overrightarrow{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \\
= \int_X \psi \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \omega_{\psi}^{n-2} \land \partial \omega + \int_X \frac{\partial \psi}{\partial s} \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \omega_{\psi}^{n-2} \land \partial \omega \\
+ \int_X \psi \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land (n-2)\omega_{\psi}^{n-3} \land -\sqrt{-1}\overrightarrow{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \\
+ \int_X \psi \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land (n-2)\omega_{\psi}^{n-3} \land \omega_{\psi}^{n-3} \land \sqrt{-1}\overrightarrow{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega.
\]

To simplify the notation, we set
\[
(2.25) \hspace{1cm} A_1 := \int_X \psi(n-2)\overrightarrow{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \omega_{\psi}^{n-3} \land \partial \omega \land -\sqrt{-1}\overrightarrow{\partial} \left( \frac{\partial \psi}{\partial t} \right), \\
(2.26) \hspace{1cm} B_1 := \int_X \psi(n-2)\overrightarrow{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \omega_{\psi}^{n-3} \land \partial \omega \land \sqrt{-1}\overrightarrow{\partial} \left( \frac{\partial \psi}{\partial s} \right).
\]

Consequently, \(I^1/a_1\) can be written as
\[
\frac{I^1}{a_1} = \int_X -\frac{\partial \psi}{\partial t} \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \omega_{\psi}^{n-2} \land \partial \omega + \int_X \frac{\partial \psi}{\partial s} \cdot \overrightarrow{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \omega_{\psi}^{n-2} \land \partial \omega \\
+ A_1 + B_1. \\
(2.27)
\]

Similarly, we can define \(I^2\) by
\[
(2.28) \hspace{1cm} d\psi^2 = I^2 \cdot dt \land ds,
\]
where
\[
\frac{I^2}{a_2} := \int_X -\frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \land \omega_{\psi}^{n-2} \land \bar{\partial} \omega + \int_X \frac{\partial \psi}{\partial s} \cdot \partial \left( \frac{\partial \psi}{\partial t} \right) \land \omega_{\psi}^{n-2} \land \bar{\partial} \omega \\
+ \frac{A_1}{A_1 + B_1}. \\
(2.29)
\]

Consequently, from (2.18), (2.27) and (2.29),
\[
(2.30) \hspace{1cm} \frac{2I^0}{n(n-1)\sqrt{-1}} = \frac{I^1}{a_1} - \frac{I^2}{a_2} + (A_1 + B_1) - (A_1 + B_1).
\]
By a direct computation and using (2.1) and (2.3), one can show that the sum $A_1 + B_1$ has a nice form:

$$A_1 = \int_X \sqrt{-1} \partial \left[ (n - 2)\psi \cdot \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-3} \wedge \partial \omega \right] \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right)$$

$$= \int_X \sqrt{-1} (n - 2) \left[ \partial \left( \psi \cdot \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \right) \wedge \omega_{\psi}^{n-3} \wedge \partial \omega \right] \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right)$$

$$= \int_X \sqrt{-1} (n - 2) \left[ \partial \psi \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) + \psi \cdot \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \right] \wedge \omega_{\psi}^{n-3} \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right)$$

$$= \int_X (n - 2) \psi \cdot \sqrt{-1} \partial \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-3} \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right)$$

$$+ \int_X (n - 2) \sqrt{-1} \partial \psi \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-3} \wedge \partial \omega \wedge \sqrt{-1} \partial \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right)$$

$$= \int_X -(n - 2) \psi \cdot \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-3} \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right)$$

$$+ \int_X -(n - 2) \sqrt{-1} \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-3}$$

$$= -B_1 - \int_X (n - 2) \sqrt{-1} \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-3}.$$ 

Adding the term $B_1$ on both sides gives

$$A_1 + B_1 = \int_X -(n - 2) \sqrt{-1} \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-3}. \tag{2.31}$$

According to (2.31), we define

$$L_\omega^3 (\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X a_3 \partial \varphi_1 \wedge \partial \omega \wedge \bar{\partial} \varphi_1 \wedge \bar{\partial} \varphi_1 \wedge \omega_{\varphi_1}^{n-3}, \tag{2.32}$$

$$L_\omega^4 (\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X a_4 \bar{\partial} \varphi_1 \wedge \bar{\partial} \omega \wedge \partial \varphi_1 \wedge \partial \varphi_1 \wedge \omega_{\varphi_1}^{n-3}, \tag{2.33}$$

where $a_3, a_4$ are nonzero constants determined later and we require $a_3 = a_4$. Consider

$$\Psi^3 = \left[ \int_X a_3 \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \bar{\partial} \psi \wedge \omega_{\psi}^{n-3} \right] ds$$

$$+ \left[ \int_X a_3 \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \bar{\partial} \psi \wedge \omega_{\psi}^{n-3} \right] dt, \tag{2.34}$$

$$\Psi^4 = \left[ \int_X a_4 \bar{\partial} \psi \wedge \bar{\partial} \omega \wedge \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \right] ds$$

$$+ \left[ \int_X a_4 \bar{\partial} \psi \wedge \bar{\partial} \omega \wedge \partial \left( \frac{\partial \psi}{\partial t} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \right] dt. \tag{2.35}$$

Calculate

$$d \Psi^3 = \Psi^3 \cdot dt \wedge ds, \tag{2.36}$$
where

\[ I^3 = \int_X a_3 \frac{\partial}{\partial t} \left[ \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \right] - \int_X a_3 \frac{\partial}{\partial s} \left[ \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \psi}{\partial t} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \right]. \]

Also, we can calculate

\[ d\Psi^4 = I^4 \cdot dt \wedge ds, \]

where

\[ I^4 = \int_X a_4 \frac{\partial}{\partial t} \left[ \partial \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \right] - \int_X a_4 \frac{\partial}{\partial s} \left[ \partial \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \psi}{\partial t} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \right]. \]

Hence

\[
\frac{I^3}{a_3} = \int_X \left[ \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial s} \right) \right] \wedge \partial \omega \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \\
+ \partial \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial^2 \psi}{\partial t \partial s} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} + \partial \psi \wedge \partial \omega \wedge \partial \psi \wedge \partial \psi \wedge \partial \psi \wedge \omega_{\psi}^{n-3} + \partial \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \\
- \int_X \left[ \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial s} \right) \right] \wedge \partial \omega \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \\
+ \partial \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial^2 \psi}{\partial s \partial t} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} + \partial \psi \wedge \partial \omega \wedge \partial \psi \wedge \partial \psi \wedge \partial \psi \wedge \omega_{\psi}^{n-3} + \partial \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \\
+ \partial \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \psi}{\partial t} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} + \partial \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} - \int_X \left[ \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial s} \right) \right] \wedge \partial \omega \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \right].
\]

The second term and the sixth term cancel with each other, and the third term and the seventh term are the same, so we have

\[
\frac{I^3}{a_3} = \int_X -\partial \left( \frac{\partial \psi}{\partial t} \right) \wedge \partial \omega \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \\
+ \int_X \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \\
+ 2 \int_X \partial \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \wedge \omega_{\psi}^{n-3} \\
+ \int_X (n-3) \partial \psi \wedge \partial \psi \wedge \partial \omega \wedge \partial \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{1-\partial t} \left( \frac{\partial \psi}{\partial t} \right) \\
- \int_X (n-3) \partial \psi \wedge \partial \psi \wedge \partial \omega \wedge \partial \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{1-\partial t} \left( \frac{\partial \psi}{\partial s} \right) \\
= H_1 + \frac{2}{-(n-2)\sqrt{-1}} (A_1 + B_1) + A_2 + B_2.
\]
where

\begin{align}
(2.40) \quad H_1 & := \int_X -\vartheta \left( \frac{\partial \psi}{\partial t} \right) \land \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \omega_\psi^{n-3} \\
& + \int_X \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \partial \omega \land \overline{\partial} \psi \land \omega_\psi^{n-3} \\
(2.41) \quad A_2 & := \int_X (n-3) \partial \psi \land \overline{\partial} \psi \land \partial \omega \land \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \\
& \land \omega_\psi^{n-4} \land \sqrt{-1} \partial \vartheta \left( \frac{\partial \psi}{\partial s} \right) \\
(2.42) \quad B_2 & := -\int_X (n-3) \partial \psi \land \overline{\partial} \psi \land \partial \omega \land \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \\
& \land \omega_\psi^{n-4} \land \sqrt{-1} \partial \vartheta \left( \frac{\partial \psi}{\partial t} \right).
\end{align}

Similarly,

\begin{equation}
(2.43) \quad \frac{\Gamma^4}{\alpha_4} = \overline{H}_1 + \frac{2}{(n-2)\sqrt{-1}}(\overline{A}_1 + \overline{B}_1) + \overline{A}_2 + \overline{B}_2.
\end{equation}

The hard part is to find some suitable expression of $H_1$. In the following we will see that $H_1 + \overline{H}_1$ has a nice form which contains only $A_1, B_1, \overline{A}_1$, and $\overline{B}_1$.

Now we compute $H_1$, using (2.40) and (2.41):

\begin{align*}
H_1 &= \int_X \vartheta \left( \frac{\partial \psi}{\partial t} \right) \land \partial \omega \land \overline{\partial} \psi \land \omega_\psi^{n-3} \left( \frac{\partial \psi}{\partial s} \right) \\
& + \int_X \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \omega_\psi^{n-3} \left( \frac{\partial \psi}{\partial t} \right) \\
& = \int_X \frac{\partial \psi}{\partial t} \left[ \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \omega_\psi^{n-3} - \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \partial \left( \partial \omega \land \overline{\partial} \psi \land \omega_\psi^{n-3} \right) \right] \\
& + \int_X \frac{\partial \psi}{\partial t} \left[ \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \omega_\psi^{n-3} - \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \partial \left( \partial \omega \land \overline{\partial} \psi \land \omega_\psi^{n-3} \right) \right] \\
& = \int_X \frac{\partial \psi}{\partial t} \cdot \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \left( \vartheta \overline{\partial} \psi \land \omega_\psi^{n-3} - \vartheta \partial \psi \land (n-3)\omega_\psi^{n-4} \land \partial \omega \right) \\
& + \int_X -\frac{\partial \psi}{\partial t} \cdot \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \partial \partial \omega \land \overline{\partial} \psi \land \omega_\psi^{n-3} \\
& + \int_X -\frac{\partial \psi}{\partial t} \cdot \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land (n-3)\omega_\psi^{n-4} \land \partial \omega \\
& = \int_X \vartheta \left( \frac{\partial \psi}{\partial t} \right) \land \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \omega_\psi^{n-3} \land \partial \psi \\
& + \int_X \frac{\partial \psi}{\partial t} \cdot \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \overline{\partial} \partial \omega \land \overline{\partial} \psi \land \omega_\psi^{n-3} \\
& - \int_X \frac{\partial \psi}{\partial t} \cdot \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \omega_\psi^{n-3} \\
& - \int_X \frac{\partial \psi}{\partial t} \cdot \vartheta \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land (n-3)\omega_\psi^{n-4} \land \partial \omega.
Therefore we obtain
\begin{equation}
(2.44) \quad H_1 = \int_X \partial \psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-3} \\
- \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge (n-3)\omega_{\psi}^{n-4} \wedge \overline{\partial} \omega \wedge \partial \psi \\
+ \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge (n-3)\omega_{\psi}^{n-4} \wedge \overline{\partial} \omega \wedge \overline{\partial} \psi,
\end{equation}
and, taking the complex conjugate yields, using (2.3) and (2.4)
\begin{equation}
(2.45) \quad \overline{H}_1 = \int_X \overline{\partial} \psi \wedge \overline{\partial} \omega \wedge \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-3} \\
- \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \overline{\partial} \omega \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-3} \\
- \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \partial \psi \wedge \omega_{\psi}^{n-3} \\
+ \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge (n-3)\omega_{\psi}^{n-4} \wedge \partial \omega \wedge \overline{\partial} \psi.
\end{equation}
Adding (2.45) to (2.45), it follows that
\begin{equation}
(2.46) \quad H_1 + \overline{H}_1 = \frac{A_1 + B_1}{-(n-2)\sqrt{-1}} + \frac{\overline{A}_1 + \overline{B}_1}{(n-2)\sqrt{-1}}.
\end{equation}
and, hence,
\begin{align}
\frac{I_3}{a_3} + \frac{I_4}{a_4} &= \frac{H_1 + \overline{H}_1 + \frac{2}{-(n-2)\sqrt{-1}}[(A_1 + B_1) - (\overline{A}_1 + \overline{B}_1)]}{A_2 + B_2 + \overline{A}_2 + \overline{B}_2} + \frac{3}{-(n-2)\sqrt{-1}}[(A_1 + B_1) - (\overline{A}_1 + \overline{B}_1)] + (A_2 + B_2) + (\overline{A}_2 + \overline{B}_2).
\end{align}
Set
\begin{equation}
(2.48) \quad c_1 := A_1 + B_1 - (\overline{A}_1 + \overline{B}_1), \quad c_2 := A_2 + B_2 + \overline{A}_2 + \overline{B}_2.
\end{equation}
we deduce
\begin{equation}
(2.49) \quad \frac{2I_0}{n(n-1)\sqrt{-1}} = \frac{I_1}{a_1} - \frac{I_2}{a_2} + c_1, \quad \frac{I_3}{a_3} + \frac{I_4}{a_4} = \frac{3}{-(n-2)\sqrt{-1}}c_1 + c_2.
\end{equation}

2.3. The constructions of $I^5$ and $I^6$. To give the general construction of $I^{2i+1}$ and $I^{2i+2}$, we firstly consider some special cases. In this subsection we give the construction of $I^5$ and $I^6$, and in the next subsection the construction of $I^7$ and $I^8$. Finally, we give the general construction.
From (2.41), it follows that

\[
A_2 = \int_X (n - 3)\partial \psi \wedge \overline{\partial} \psi \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-4} \wedge \partial \omega \wedge -\sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right)
\]

\[
= \int_X \sqrt{-1} \partial \left[ (n - 3)\partial \psi \wedge \overline{\partial} \psi \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-4} \wedge \partial \omega \right] \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right)
\]

\[
= \int_X \sqrt{-1}(n - 3) \left[ \partial \left( \partial \psi \wedge \overline{\partial} \psi \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-4} \wedge \partial \omega \right) \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \right]
\]

\[
= \int_X -\sqrt{-1}(n - 3) \left[ \partial \psi \wedge \left( \partial \overline{\partial} \psi \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) - \overline{\partial} \psi \wedge \partial \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \right) \right] \wedge \omega_{\psi}^{n-4} \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right)
\]

\[
= \int_X (n - 3)\partial \psi \wedge -\sqrt{-1} \partial \overline{\partial} \psi \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-4} \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right)
\]

\[
+ \int_X (n - 3)\partial \psi \wedge \overline{\partial} \psi \wedge \sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-4} \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right)
\]

\[
= \int_X (n - 3)\partial \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi - B_2;
\]

hence, by the definition (2.42), we have

\[
A_2 + B_2 = \int_X (n - 3)\partial \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi.
\]

Motivated by (2.50), we set

\[
(2.50) \quad A_2 + B_2 = \int_X (n - 3)\partial \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi.
\]

Consider again the 1-forms

\[
\Psi^5 = \left[ \int_X a_5 \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \right] ds
\]

\[
+ \left[ \int_X a_5 \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \right] dt,
\]

\[
\Psi^6 = \left[ \int_X a_6 \partial \psi \wedge \partial \omega \wedge \partial \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \right] ds
\]

\[
+ \left[ \int_X a_6 \partial \psi \wedge \partial \omega \wedge \partial \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \partial \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \right] dt.
\]

The differential of \( \Psi^5 \) is given by

\[
(2.55) \quad d\Psi^5 = I^5 \cdot dt \wedge ds,
\]
where
\[
\frac{I^5}{a_5} := \int_X \frac{\partial}{\partial t} \left[ \partial_\psi \wedge \partial_\omega \wedge \partial_\psi \left( \frac{\partial_\psi}{\partial s} \right) \wedge \partial_\psi \wedge \omega_\psi^{-4} \wedge \sqrt{-1} \partial_\psi \right] 
- \int_X \frac{\partial}{\partial s} \left[ \partial_\psi \wedge \partial_\omega \wedge \partial_\psi \left( \frac{\partial_\psi}{\partial t} \right) \wedge \partial_\psi \wedge \omega_\psi^{-4} \wedge \sqrt{-1} \partial_\psi \right].
\] (2.56)

Hence, using (2.56),
\[
\frac{I^5}{a_5} = \int_X \left[ \partial_\psi \wedge \partial_\omega \left( \frac{\partial_\psi}{\partial t} \right) \wedge \partial_\psi \wedge \omega_\psi^{-4} \wedge \sqrt{-1} \partial_\psi \right] 
+ \partial_\psi \wedge \partial_\omega \left( \frac{\partial^2_\psi}{\partial t \partial s} \right) \wedge \partial_\psi \wedge \omega_\psi^{-4} \wedge \sqrt{-1} \partial_\psi 
+ \partial_\psi \wedge \partial_\omega \left( \frac{\partial_\psi}{\partial t} \right) \wedge \partial_\psi \wedge \omega_\psi^{-4} \wedge \sqrt{-1} \partial_\psi 
+ \partial_\psi \wedge \partial_\omega \left( \frac{\partial_\psi}{\partial s} \right) \wedge \partial_\psi \wedge (n - 4) \omega_\psi^{-5} \wedge \sqrt{-1} \partial_\psi \left( \frac{\partial_\psi}{\partial t} \right) \wedge \sqrt{-1} \partial_\psi 
+ \partial_\psi \wedge \partial_\omega \left( \frac{\partial_\psi}{\partial t} \right) \wedge \partial_\psi \wedge \omega_\psi^{-4} \wedge \sqrt{-1} \partial_\psi 
= \int_X - \partial_\psi \left( \frac{\partial_\psi}{\partial t} \right) \wedge \partial_\psi \wedge \partial_\psi \wedge \omega_\psi^{-4} \wedge \sqrt{-1} \partial_\psi 
+ \int_X \partial_\psi \left( \frac{\partial_\psi}{\partial t} \right) \wedge \partial_\psi \wedge \partial_\psi \wedge \omega_\psi^{-4} \wedge \sqrt{-1} \partial_\psi 
+ 2 \int_X \partial_\psi \wedge \partial_\omega \left( \frac{\partial_\psi}{\partial s} \right) \wedge \partial_\psi \wedge \omega_\psi^{-4} \wedge \sqrt{-1} \partial_\psi 
+ \int_X \partial_\psi \wedge \partial_\omega \left( \frac{\partial_\psi}{\partial s} \right) \wedge \partial_\psi \wedge (n - 4) \omega_\psi^{-5} \wedge \sqrt{-1} \partial_\psi \left( \frac{\partial_\psi}{\partial t} \right) \wedge \sqrt{-1} \partial_\psi 
- \int_X \partial_\psi \wedge \partial_\omega \left( \frac{\partial_\psi}{\partial s} \right) \wedge \partial_\psi \wedge (n - 4) \omega_\psi^{-5} \wedge \sqrt{-1} \partial_\psi \left( \frac{\partial_\psi}{\partial t} \right) \wedge \sqrt{-1} \partial_\psi 
+ \frac{A_2}{n - 3} + \frac{B_2}{n - 3} 
= H_2 + \frac{3}{n - 3} (A_2 + B_2) + A_3 + B_3,
\]
where
\( H_2 := \int_X \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \) 
(2.57)
\[ + \int_X \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi, \]
\( A_3 := \int_X (n-4) \partial \psi \wedge \overline{\partial} \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-5} \wedge \sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \)
(2.58)
\[ \wedge \sqrt{-1} \partial \overline{\partial} \psi, \]
\( B_3 := \int_X (n-4) \partial \psi \wedge \overline{\partial} \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-5} \wedge -\sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \)
(2.59)
\[ \wedge \sqrt{-1} \partial \overline{\partial} \psi. \]

Similarly, for
\( d\Psi^6 := I^6 \cdot dt \wedge ds, \)
we have
\( \frac{I^6}{a_6} = H_2 + \frac{3}{n-3} (A_2 + B_2) + A_3 + B_3. \)

As (2.41), the hard part \( H_2 \) is calculated as follows:

\[ H_2 = \int_X \frac{\partial}{\partial t} \left[ \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \right] \frac{\partial \psi}{\partial t} \]
\[ + \int_X \overline{\partial} \left[ \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \right] \frac{\partial \psi}{\partial t} \]
\[ = \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \]
\[ - \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \left( \partial \omega \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \right) \]
\[ + \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \]
\[ - \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left( \partial \omega \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \right) \]
\[ = \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \]
\[ - \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \partial \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \]
\[ - \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge (n-4) \omega_{\psi}^{n-5} \wedge -\sqrt{-1} \partial \overline{\partial} \psi \]
\[ = \int_X \sqrt{-1} \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \partial \psi \]
\[ - \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \partial \overline{\partial} \psi \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \]
\[ - \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge (n-4) \omega_{\psi}^{n-5} \wedge -\sqrt{-1} \partial \overline{\partial} \psi \].
Hence

\[(2.62) \quad H_2 = \int_X \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \partial \psi \wedge \partial \omega \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \]

\[- \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \partial \omega \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \]

\[+ \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \partial \omega \wedge (n-4)\omega_{\psi}^{n-5} \wedge \sqrt{-1} \partial \overline{\partial} \psi \wedge \partial \psi \]

\[- \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \partial \omega \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \]

\[+ \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \partial \omega \wedge (n-4)\omega_{\psi}^{n-5} \wedge \sqrt{-1} \partial \overline{\partial} \psi \wedge \partial \psi \]

and, consequently, after taking the complex conjugate on both sides,

\[(2.63) \quad \overline{H_2} = \int_X \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \left( \frac{\partial \psi}{\partial t} \right) \wedge \overline{\partial} \psi \wedge \overline{\partial} \omega \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \]

\[- \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \psi \wedge \overline{\partial} \omega \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \]

\[+ \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \omega \wedge \partial \omega \wedge (n-4)\omega_{\psi}^{n-5} \wedge \sqrt{-1} \partial \overline{\partial} \psi \wedge \overline{\partial} \psi \]

\[- \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \partial \omega \wedge \omega_{\psi}^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \psi \]

\[+ \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \omega \wedge \partial \omega \wedge (n-4)\omega_{\psi}^{n-5} \wedge \sqrt{-1} \partial \overline{\partial} \psi \wedge \partial \psi \]

Therefore

\[(2.64) \quad H_2 + \overline{H_2} = \frac{A_2 + B_2}{n-3} + \frac{\overline{A_2} + \overline{B_2}}{n-3}, \]

\[(2.65) \quad \frac{I_5}{a_5} + \frac{I_6}{a_6} = \frac{4}{n-3} (A_2 + B_2 + \overline{A_2} + \overline{B_2}) + (A_3 + B_3 + \overline{A_3} + \overline{B_3}). \]

If we set

\[(2.66) \quad c_3 := A_3 + B_3 + \overline{A_3} + \overline{B_3} \]

we can rewrite (2.64) and (2.65) as

\[(2.67) \quad H_2 + \overline{H_2} = \frac{c_2}{n-3}, \quad \frac{I_5}{a_5} + \frac{I_6}{a_6} = \frac{4}{n-3} c_2 + c_3. \]

2.4. The constructions of $I^7$ and $I^8$. Recall

\[A_3 = \int_X \left[ (n-4)\partial \psi \wedge \overline{\partial} \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-5} \wedge \sqrt{-1} \partial \overline{\partial} \psi \right] \wedge \sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right). \]
By a direct computation, one deduces that

\[ A_3 = \int_X -\sqrt{-1}\partial \left[ (n - 4)\partial \psi \land \overline{\partial} \psi \land \partial \omega \land \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \right] \land \omega^{\psi n - 5} \land \sqrt{-1}d\overline{\partial} \psi \]

\[ = \int_X \sqrt{-1}(n - 4)\partial \psi \land \overline{\partial} \left[ \partial \overline{\partial} \psi \land \partial \omega \land \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \right] \land \omega^{\psi n - 5} \land \sqrt{-1}d\overline{\partial} \psi \]

Consequently, it follows that

\[ (2.68) \ A_3 + B_3 = \int_X (n - 4)\partial \psi \land \partial \omega \land \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \omega^{\psi n - 5} \land (\sqrt{-1}d\overline{\partial} \psi)^2. \]

Now we introduce the corresponding functionals

\[ (2.69) \ \mathcal{L}_\omega^7 (\varphi', \varphi'') := \frac{1}{V_{\omega}} \int_0^1 \int_X a_7 \partial \varphi_t \land \partial \omega \land \overline{\partial} \varphi_t \land \overline{\partial} \varphi_t \land \omega^{\varphi_t n - 5} \land (\sqrt{-1}d\overline{\partial} \varphi_t)^2, \]

\[ (2.70) \ \mathcal{L}_\omega^8 (\varphi', \varphi'') := \frac{1}{V_{\omega}} \int_0^1 \int_X a_8 \partial \varphi_t \land \partial \omega \land \partial \varphi_t \land \partial \varphi_t \land \omega^{\varphi_t n - 5} \land (\sqrt{-1}d\overline{\partial} \varphi_t)^2 \]

and consider the 1-forms

\[ \Psi^7 = \left[ \int_X a_7 \partial \psi \land \partial \omega \land \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \omega^{\psi n - 5} \land (\sqrt{-1}d\overline{\partial} \psi)^2 \right] ds \]

\[ + \left[ \int_X a_7 \partial \psi \land \partial \omega \land \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \omega^{\psi n - 5} \land (\sqrt{-1}d\overline{\partial} \psi)^2 \right] dt, \]

\[ \Psi^8 = \left[ \int_X a_8 \overline{\partial} \psi \land \overline{\partial} \omega \land \partial \left( \frac{\partial \psi}{\partial s} \right) \land \partial \psi \land \omega^{\psi n - 5} \land (\sqrt{-1}d\overline{\partial} \psi)^2 \right] ds \]

\[ + \left[ \int_X a_8 \overline{\partial} \psi \land \overline{\partial} \omega \land \partial \left( \frac{\partial \psi}{\partial t} \right) \land \partial \psi \land \omega^{\psi n - 5} \land (\sqrt{-1}d\overline{\partial} \psi)^2 \right] dt. \]
So, we have the expression,

\begin{align}
(2.73) \quad d\Psi^7 &= I^7 \cdot dt \wedge ds, \\
&\quad d\Psi^8 = I^8 \cdot dt \wedge ds,
\end{align}

where

\[
\frac{I^7}{a_7} := \int_X \frac{\partial}{\partial t} \left[ \partial \Psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \Psi}{\partial s} \right) \wedge \partial \Psi \wedge \omega^{-5}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2 \right]
\]

\[
- \int_X \frac{\partial}{\partial s} \left[ \partial \Psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \Psi}{\partial t} \right) \wedge \partial \Psi \wedge \omega^{-5}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2 \right]
\]

\[
= \int_X \left[ \partial \left( \frac{\partial \Psi}{\partial t} \right) \wedge \frac{\partial \Psi}{\partial t} \right] \wedge \partial \Psi \wedge \omega^{-5}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2
\]

\[
+ \partial \Psi \wedge \partial \omega \wedge \partial \left( \frac{\partial^2 \Psi}{\partial t \partial s} \right) \wedge \partial \Psi \wedge \omega^{-5}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2
\]

\[
+ \partial \Psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \Psi}{\partial s} \right) \wedge \partial \Psi \wedge \omega^{-5}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2
\]

\[
+ \partial \Psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \Psi}{\partial s} \right) \wedge \partial \Psi \wedge \omega^{-5}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2
\]

\[
+ \partial \Psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \Psi}{\partial s} \right) \wedge \partial \Psi \wedge \omega^{-5}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2
\]

\[
+ \partial \Psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \Psi}{\partial t} \right) \wedge \partial \Psi \wedge \omega^{-5}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2
\]

\[
= \int_X -\partial \left( \frac{\partial \Psi}{\partial t} \right) \wedge \partial \omega \wedge \partial \Psi \wedge \omega^{-5}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2
\]

\[
+ \int_X \partial \left( \frac{\partial \Psi}{\partial s} \right) \wedge \partial \omega \wedge \partial \Psi \wedge \omega^{-5}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2
\]

\[
+ 2 \int_X \partial \Psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \Psi}{\partial s} \right) \wedge \partial \Psi \wedge \omega^{-5}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2
\]

\[
+ \int_X \partial \Psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \Psi}{\partial t} \right) \wedge \partial \Psi \wedge (n-5) \omega^{-6}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2
\]

\[
- \int_X \partial \Psi \wedge \partial \omega \wedge \partial \left( \frac{\partial \Psi}{\partial t} \right) \wedge \partial \Psi \wedge (n-5) \omega^{-6}_\Psi \wedge (\sqrt{-1} \partial \Psi)^2
\]

\[
+ \frac{2}{n-4} (A_3 + B_3)
\]

\[
= H_3 + \frac{4}{n-4} (A_3 + B_3) + A_4 + B_4
\]
where

\[
H_3 := \int_X -\partial \left( \frac{\partial \psi}{\partial t} \right) \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \bar{\partial} \psi \wedge \omega_n^{-5} \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^2
\]

(2.74)

and

\[
A_4 := \int_X (n-5) \partial \psi \wedge \bar{\partial} \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_n^{-6}
\]

(2.75)

\[
B_4 := \int_X (n-5) \partial \psi \wedge \bar{\partial} \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_n^{-6}
\]

(2.76)

Hence

\[
\frac{r^7}{a_7} = H_3 + \frac{4}{n-4} (A_3 + B_3) + A_4 + B_4.
\]

(2.77)

Similarly

\[
\frac{r^8}{a_8} = H_3 + \frac{4}{n-4} (A_3 + B_3) + A_4 + B_4.
\]

(2.78)

Calculate

\[
H_3 = \int_X \bar{\partial} \left[ \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \bar{\partial} \psi \wedge \omega_n^{-5} \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^2 \right] \frac{\partial \psi}{\partial t}
\]

\[
+ \int_X \bar{\partial} \left[ \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \bar{\partial} \psi \wedge \omega_n^{-5} \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^2 \right] \frac{\partial \psi}{\partial t}
\]

(2.74)

\[
= \int_X \frac{\partial \psi}{\partial t} \left[ \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \bar{\partial} \psi \wedge \omega_n^{-5} \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^2 \right]
\]

\[
- \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \left( \partial \omega \wedge \bar{\partial} \psi \wedge \omega_n^{-5} \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^2 \right)
\]

\[
+ \int_X \frac{\partial \psi}{\partial t} \left[ \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \bar{\partial} \psi \wedge \omega_n^{-5} \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^2 \right]
\]

\[
- \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \left( \partial \omega \wedge \bar{\partial} \psi \wedge \omega_n^{-5} \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^2 \right)
\]

\[
= \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \bar{\partial} \psi \wedge \omega_n^{-5} \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^2
\]

\[
- \int_X \frac{\partial \psi}{\partial t} \cdot \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \bar{\partial} \omega \wedge \bar{\partial} \psi \wedge \omega_n^{-5} \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^2
\]

\[
- \int_X \frac{\partial \psi}{\partial t} \cdot \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \bar{\partial} \psi \wedge (n-5) \omega_n^{-6} \wedge \bar{\partial} \omega \wedge (\sqrt{-1} \partial \bar{\partial} \psi)^2.
\]
Since
\begin{align*}
(2.79) & \quad \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\sigma} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\sigma} \psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^2 \\
&= \int_X \left[ \frac{\partial \psi}{\partial t} \cdot \overline{\sigma} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\sigma} \partial \psi \wedge \omega_\psi^{n-5} \wedge \sqrt{-1} \partial \overline{\partial} \psi \right] \wedge -\sqrt{-1} \partial \overline{\partial} \psi \\
&= \int_X \sqrt{-1} \partial \left( \frac{\partial \psi}{\partial t} \right) \wedge \overline{\sigma} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\sigma} \partial \psi \wedge \omega_\psi^{n-5} \wedge \sqrt{-1} \partial \overline{\partial} \psi \wedge \partial \psi \\
&- \int_X \sqrt{-1} \frac{\partial \psi}{\partial t} \cdot \overline{\sigma} \left( \frac{\partial \psi}{\partial s} \right) \wedge \left( \overline{\sigma} \partial \omega \wedge \overline{\sigma} \partial \psi \wedge \omega_\psi^{n-5} \wedge \sqrt{-1} \partial \overline{\partial} \psi \right) \\
&- \partial \omega \wedge \overline{\sigma} \partial \psi \wedge (n-5) \omega_\psi^{n-6} \wedge \overline{\sigma} \omega \wedge \sqrt{-1} \partial \overline{\partial} \psi \rangle \wedge \partial \psi,
\end{align*}
\]

it follows that
\begin{align*}
H_3 &= \int_X \overline{\sigma} \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\sigma} \left( \frac{\partial \psi}{\partial t} \right) \wedge \partial \psi \wedge \partial \omega \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^2 \\
(2.80) &- \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\sigma} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \overline{\sigma} \partial \omega \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^2 \\
+ \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\sigma} \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\sigma} \partial \omega \wedge (n-5) \omega_\psi^{n-6} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^2 \wedge \partial \psi \\
- \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\sigma} \partial \psi \wedge \overline{\sigma} \partial \omega \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^2 \\
+ \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\sigma} \partial \omega \wedge (n-5) \omega_\psi^{n-6} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^2 \wedge \overline{\sigma} \psi.
\end{align*}

Hence
\begin{align*}
(2.81) & \quad H_3 + \overline{H}_3 = \frac{A_3 + B_3}{n-4} + \frac{\overline{A}_3 + \overline{B}_3}{n-4}, \\
(2.82) & \quad \frac{f^7}{a_7} + \frac{f^s}{a_s} = \frac{5}{n-4} \left( A_3 + B_3 + \overline{A}_3 + \overline{B}_3 \right) + \left( A_4 + B_4 + \overline{A}_4 + \overline{B}_4 \right).
\end{align*}

Set
\begin{equation}
(2.83) \quad c_4 := A_4 + B_4 + \overline{A}_4 + \overline{B}_4.
\end{equation}

Then
\begin{equation}
(2.84) \quad \frac{f^7}{a_7} + \frac{f^s}{a_s} = \frac{5}{n-4} c_3 + c_4.
\end{equation}

2.5. Recursion formula. Suppose now \( n \geq 4 \). We define, for \( 2 \leq i \leq n-2 \),
\begin{align*}
A_i &:= \int_X \left[ (n-i-1) \partial \psi \wedge \overline{\sigma} \psi \wedge \partial \omega \wedge \overline{\sigma} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^{i-2} \right] \\
(2.85) &\wedge \sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right), \\
B_i &:= \int_X \left[ (n-i-1) \partial \psi \wedge \overline{\sigma} \psi \wedge \partial \omega \wedge \overline{\sigma} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^{i-2} \right] \\
(2.86) &\wedge -\sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right).
\end{align*}

So
\[ A_i = \int_X \left[ (n - i - 1) \partial \psi \wedge \overline{\partial} \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-i-2} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^{i-2} \right] \]

\[ \wedge \sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \]

\[ = \int_X \sqrt{-1} (n - i - 1) \partial \psi \wedge \left[ \overline{\partial} \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-i-2} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^{i-2} \right] \]

\[ - \overline{\partial} \psi \wedge \partial \left( \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-i-2} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^{i-2} \right) \]

\[ = \int_X \sqrt{-1} (n - i - 1) \partial \psi \wedge \overline{\partial} \psi \wedge \partial \omega \wedge \partial \partial \left( \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-i-2} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^{i-2} \right) \]

\[ \wedge \sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \]

\[ + \int_X \sqrt{-1} (n - i - 1) \partial \psi \wedge \overline{\partial} \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \omega_{\psi}^{n-i-2} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^{i-2} \]

\[ \wedge \sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \]

thus

(2.87) \[ A_i + B_i = \int_X (n - i - 1) \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \omega_{\psi}^{n-i-2} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}. \]

Define, where \( a_{2i+1} \) and \( a_{2i+2} \) are nonzero constants and we require \( a_{2i+1} = a_{2i+2} \),

\[ L_{\omega}^{2i+1}(\phi', \phi'') := \frac{1}{V_\omega} \int_0^1 \int_X a_{2i+1} \partial \phi_t \wedge \partial \omega \wedge \overline{\partial} \phi_t \wedge \overline{\partial} \phi_t \wedge \omega_{\phi_t}^{n-i-2} \wedge (\sqrt{-1} \partial \overline{\partial} \phi_t)^{i-1}, \]

\[ L_{\omega}^{2i+2}(\phi', \phi'') := \frac{1}{V_\omega} \int_0^1 \int_X a_{2i+2} \overline{\partial} \phi_t \wedge \overline{\partial} \omega \wedge \partial \phi_t \wedge \partial \phi_t \wedge \omega_{\phi_t}^{n-i-2} \]

(2.88)

\[ \wedge (\sqrt{-1} \partial \overline{\partial} \phi_t)^{i-1}. \]

Consider

\[ \psi^{2i+1} = \left[ \int_X a_{2i+1} \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-i-2} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1} \right] ds \]

(2.89)

\[ + \left[ \int_X a_{2i+1} \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \overline{\partial} \psi \wedge \omega_{\psi}^{n-i-2} \wedge (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1} \right] dt, \]
and

\[ \Psi^{2i+2} = \left[ \int_X a_{2i+2} \frac{\partial \bar{\psi}}{\partial \omega} \frac{\partial \omega}{\partial \bar{\psi}} \left( \frac{\partial \omega}{\partial s} \right) \wedge \partial \psi \wedge \omega^{n-i-2} \wedge (\sqrt{-1}d\bar{\psi})^{i-1} \right] ds \]

(2.90)

\[ + \left[ \int_X a_{2i+2} \frac{\partial \bar{\psi}}{\partial \omega} \frac{\partial \omega}{\partial \bar{\psi}} \left( \frac{\partial \omega}{\partial t} \right) \wedge \partial \psi \wedge \omega^{n-i-2} \wedge (\sqrt{-1}d\bar{\psi})^{i-1} \right] dt. \]

So

(2.91)

\[ d\Psi^{2i+1} = I^{2i+1} \cdot dt \wedge ds, \quad d\Psi^{2i+2} = I^{2i+2} \cdot dt \wedge ds, \]

where

\[ I^{2i+1}_{a_{2i+1}} := \int_X \frac{\partial}{\partial t} \left[ \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \bar{\partial} \psi \wedge \omega^{n-i-2} \wedge (\sqrt{-1}d\bar{\psi})^{i-1} \right] \]

\[ - \int_X \frac{\partial}{\partial s} \left[ \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \bar{\partial} \psi \wedge \omega^{n-i-2} \wedge (\sqrt{-1}d\bar{\psi})^{i-1} \right] \]

\[ = \int_X \left[ \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \bar{\partial} \psi \wedge \omega^{n-i-2} \wedge (\sqrt{-1}d\bar{\psi})^{i-1} \right] \]

\[ - \int_X \left[ \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \bar{\partial} \psi \wedge \omega^{n-i-2} \wedge (\sqrt{-1}d\bar{\psi})^{i-1} \right] \]

\[ + \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial^2 \psi}{\partial s \partial t} \right) \wedge \bar{\partial} \psi \wedge \omega^{n-i-2} \wedge (\sqrt{-1}d\bar{\psi})^{i-1} \]

\[ + \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \bar{\partial} \psi \wedge \omega^{n-i-2} \wedge (\sqrt{-1}d\bar{\psi})^{i-1} \]

\[ + \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \bar{\partial} \psi \wedge (n-i-2) \omega^{n-i-3} \wedge \sqrt{-1}d\bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \]

\[ \wedge (\sqrt{-1}d\bar{\psi})^{i-1} \]

\[ + \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \bar{\partial} \psi \wedge \omega^{n-i-2} \wedge (i-1)(\sqrt{-1}d\bar{\psi})^{i-2} \]

\[ \wedge (\sqrt{-1}d\bar{\partial} \left( \frac{\partial \psi}{\partial t} \right)) \]

\[ - \int_X \left[ \partial \left( \frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \bar{\partial} \psi \wedge \omega^{n-i-2} \wedge (\sqrt{-1}d\bar{\partial} \psi)^{i-1} \right] \]

\[ + \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial^2 \psi}{\partial s \partial t} \right) \wedge \bar{\partial} \psi \wedge \omega^{n-i-2} \wedge (\sqrt{-1}d\bar{\partial} \psi)^{i-1} \]

\[ + \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \bar{\partial} \psi \wedge \omega^{n-i-2} \wedge (\sqrt{-1}d\bar{\partial} \psi)^{i-1} \]

\[ + \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \wedge \bar{\partial} \psi \wedge (n-i-2) \omega^{n-i-3} \wedge \sqrt{-1}d\bar{\partial} \left( \frac{\partial \psi}{\partial t} \right) \]

\[ \wedge (\sqrt{-1}d\bar{\partial} \psi)^{i-1} \]

\[ + \partial \psi \wedge \partial \omega \wedge \bar{\partial} \left( \frac{\partial \psi}{\partial s} \right) \wedge \bar{\partial} \psi \wedge \omega^{n-i-2} \wedge (i-1)(\sqrt{-1}d\bar{\partial} \psi)^{i-2} \]

\[ \wedge (\sqrt{-1}d\bar{\partial} \left( \frac{\partial \psi}{\partial t} \right)) \].
Therefore
\[
\frac{I_{a_{2i+1}}^{2i+1}}{a_{2i+1}} = \int_X -\partial \left( \frac{\partial \psi}{\partial t} \right) \land \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
+ \int_X \partial \left( \frac{\partial \psi}{\partial s} \right) \land \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \partial \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
+ 2 \int_X \partial \psi \land \partial \omega \land \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \omega_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
+ \int_X \partial \psi \land \partial \omega \land \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \overline{\partial} \psi \land (n-i-2)\omega_\psi^{n-i-3} \land \sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right)
\]
\[
\land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
- \int_X \partial \psi \land \partial \omega \land \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \overline{\partial} \psi \land (n-i-2)\omega_\psi^{n-i-3} \land \sqrt{-1} \partial \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right)
\]
\[
\land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
+ (i-1)(A_i + B_i)/(n-i-1)
\]
\[
(2.92) = H_i + \frac{i+1}{n-(i+1)}(A_i + B_i) + A_{i+1} + B_{i+1}.
\]
Here
\[
H_i := \int_X -\partial \left( \frac{\partial \psi}{\partial t} \right) \land \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
(2.93) + \int_X \partial \left( \frac{\partial \psi}{\partial s} \right) \land \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \partial \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}.
\]
Similarly
\[
(2.94) \quad \frac{I_{a_{2i+2}}}{a_{2i+2}} = \overline{\mathcal{I}}_i + \frac{i+1}{n-(i+1)}(\overline{A}_i + \overline{B}_i) + \overline{A}_{i+1} + \overline{B}_{i+1}.
\]
Calculate
\[
H_i = \int_X \partial \left[ \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1} \right] \frac{\partial \psi}{\partial t}
\]
\[
+ \int_X \overline{\partial} \left[ \partial \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1} \right] \frac{\partial \psi}{\partial t}
\]
\[
= \int_X \partial \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
- \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
+ \int_X \overline{\partial} \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
- \partial \left( \frac{\partial \psi}{\partial s} \right) \land \overline{\partial} \left( \frac{\partial \psi}{\partial t} \right) \land \partial \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
= \int_X \partial \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
- \int_X \partial \left( \frac{\partial \psi}{\partial s} \right) \land \overline{\partial} \omega \land \overline{\partial} \psi \land \overline{\omega}_\psi^{n-i-2} \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\]
\[
- \int_X \partial \left( \frac{\partial \psi}{\partial s} \right) \land \partial \omega \land \overline{\partial} \psi \land (n-i-2)\omega_\psi^{n-i-3} \land \overline{\partial} \omega \land (\sqrt{-1} \partial \overline{\partial} \psi)^{i-1}
\].
Since

\[
\int_x \frac{\partial \psi}{\partial t} \cdot \overline{\frac{\partial (\partial \psi)}{\partial s}} \wedge \partial \omega \wedge \partial \overline{\partial \delta \psi} \wedge \omega_n^{n-2} \wedge (\sqrt{-1} \partial \overline{\partial \delta \psi})^{i-1} \\
= \int_x \frac{\partial \psi}{\partial t} \cdot \overline{\frac{\partial (\partial \psi)}{\partial s}} \wedge \partial \omega \wedge \partial \overline{\partial \delta \psi} \wedge \omega_n^{n-2} \wedge (\sqrt{-1} \partial \overline{\partial \delta \psi})^{i-2} \wedge (-\sqrt{-1} \partial \overline{\delta \psi}) \\
= \int_x \sqrt{-1} \partial \left[ \frac{\partial \psi}{\partial t} \cdot \overline{\frac{\partial (\partial \psi)}{\partial s}} \wedge \partial \omega \wedge \partial \overline{\partial \delta \psi} \wedge \omega_n^{n-2} \wedge (\sqrt{-1} \partial \overline{\partial \delta \psi})^{i-2} \wedge \partial \psi \right] \\
= \int_x \sqrt{-1} \partial \left[ \frac{\partial \psi}{\partial t} \wedge \overline{\frac{\partial (\partial \psi)}{\partial s}} \wedge \partial \omega \wedge \partial \overline{\partial \delta \psi} \wedge \omega_n^{n-2} \wedge (\sqrt{-1} \partial \overline{\partial \delta \psi})^{i-2} \wedge \partial \psi \right] \\
- \frac{\partial \psi}{\partial t} \cdot \overline{\frac{\partial (\partial \psi)}{\partial s}} \wedge \overline{\frac{\partial (\partial \psi)}{\partial s}} \wedge \partial \omega \wedge \partial \overline{\partial \delta \psi} \wedge \omega_n^{n-2} \wedge (\sqrt{-1} \partial \overline{\partial \delta \psi})^{i-2} \wedge \partial \psi
\]

it follows that

\[
H_i = \int_x \frac{\partial (\partial \psi)}{\partial s} \wedge \overline{\frac{\partial (\partial \psi)}{\partial s}} \wedge \partial \psi \wedge \partial \omega \wedge \omega_n^{n-2} \wedge (\sqrt{-1} \partial \overline{\partial \delta \psi})^{i-1} \\
(2.95) - \int_x \frac{\partial \psi}{\partial t} \cdot \overline{\frac{\partial (\partial \psi)}{\partial s}} \wedge \partial \psi \wedge \overline{\partial \omega} \wedge \omega_n^{n-2} \wedge (\sqrt{-1} \partial \overline{\partial \psi})^{i-1} \\
+ \int_x \frac{\partial \psi}{\partial t} \cdot \overline{\frac{\partial (\partial \psi)}{\partial s}} \wedge \partial \omega \wedge \overline{\partial \omega} \wedge (n - i - 2)\omega_n^{n-3} \wedge (\sqrt{-1} \partial \overline{\partial \psi})^{i-1} \wedge \partial \psi \\
- \int_x \frac{\partial \psi}{\partial t} \cdot \overline{\frac{\partial (\partial \psi)}{\partial s}} \wedge \partial \omega \wedge \overline{\partial \psi} \wedge \omega_n^{n-2} \wedge (\sqrt{-1} \partial \overline{\partial \psi})^{i-1} \\
+ \int_x \frac{\partial \psi}{\partial t} \cdot \overline{\frac{\partial (\partial \psi)}{\partial s}} \wedge \partial \omega \wedge \overline{\partial \omega} \wedge (n - i - 2)\omega_n^{n-3} \wedge (\sqrt{-1} \partial \overline{\partial \psi})^{i-1} \wedge \overline{\partial \psi},
\]

and, similarly,

\[
\overline{H}_i = \int_x \frac{\partial (\partial \psi)}{\partial s} \wedge \partial \psi \wedge \overline{\partial \omega} \wedge \omega_n^{n-2} \wedge (\sqrt{-1} \partial \overline{\partial \delta \psi})^{i-1} \\
(2.96) - \int_x \frac{\partial \psi}{\partial t} \cdot \partial \frac{\partial (\partial \psi)}{\partial s} \wedge \overline{\partial \psi} \wedge \partial \overline{\partial \omega} \wedge \omega_n^{n-2} \wedge (\sqrt{-1} \partial \overline{\partial \psi})^{i-1} \\
+ \int_x \frac{\partial \psi}{\partial t} \cdot \partial \frac{\partial (\partial \psi)}{\partial s} \wedge \overline{\partial \omega} \wedge \partial \omega \wedge (n - i - 2)\omega_n^{n-3} \wedge (\sqrt{-1} \partial \overline{\partial \psi})^{i-1} \wedge \overline{\partial \psi} \\
- \int_x \frac{\partial \psi}{\partial t} \cdot \partial \frac{\partial (\partial \psi)}{\partial s} \wedge \partial \omega \wedge \partial \overline{\partial \omega} \wedge \omega_n^{n-2} \wedge (\sqrt{-1} \partial \overline{\partial \psi})^{i-1} \\
+ \int_x \frac{\partial \psi}{\partial t} \cdot \partial \frac{\partial (\partial \psi)}{\partial s} \wedge \partial \omega \wedge \partial \overline{\partial \omega} \wedge (n - i - 2)\omega_n^{n-3} \wedge (\sqrt{-1} \partial \overline{\partial \psi})^{i-1} \wedge \partial \psi.
So, for $2 \leq i \leq n - 2$,

\begin{align*}
(2.97) \\
H_i + \overline{H_i} &= \frac{A_i + B_i}{n - i - 1} + \frac{\overline{A_i} + \overline{B_i}}{n - i - 1}, \\
\frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}} &= H_i + \overline{H_i} + \frac{i + 1}{n - (i + 1)} (A_i + B_i + \overline{A_i} + \overline{B_i}) \\
&+ (A_{i+1} + B_{i+1} + \overline{A_{i+1}} + \overline{B_{i+1}}) \\
&= \frac{i + 2}{n - (i + 1)} (A_i + B_i + \overline{A_i} + \overline{B_i}) \\
&+ (A_{i+1} + B_{i+1} + \overline{A_{i+1}} + \overline{B_{i+1}}).
\end{align*}

Recall, see (2.49),

\begin{align*}
(2.98) \\
I^3 + I^4 &= \frac{3\sqrt{-1}}{n - 2} ((A_1 + B_1) - (\overline{A_1} + \overline{B_1})) + (A_2 + B_2 + \overline{A_2} + \overline{B_2}) \\
2I^0 &= \frac{I^1}{a_1} - \frac{I^2}{a_2} + (A_1 + B_1) - (\overline{A_1} + \overline{B_1}).
\end{align*}

Let

\begin{align*}
(2.99) \\
c_i &:= A_i + B_i + \overline{A_i} + \overline{B_i}, \quad 2 \leq i \leq n - 1, \\
(2.100) \\
c_1 &:= A_1 + B_1 - (\overline{A_1} + \overline{B_1}).
\end{align*}

Notice that $c_{n-1} = 0$. So

\begin{align*}
(2.101) \\
\frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}} &= \frac{i + 2}{n - (i + 1)} c_i + c_{i+1}, \quad 2 \leq i \leq n - 2, \\
(2.102) \\
\frac{I^3}{a_3} + \frac{I^4}{a_4} &= \frac{3}{(n - 2)\sqrt{-1}} c_1 + c_2,
\end{align*}

and

\begin{align*}
\frac{2I^0}{n(n - 1)\sqrt{-1}} &= \frac{I^1}{a_1} - \frac{I^2}{a_2} + c_1 \\
&= \frac{I^1}{a_1} - \frac{I^2}{a_2} + \left( \frac{I^3}{a_3} + \frac{I^4}{a_4} - c_2 \right) \frac{n - 2}{3\sqrt{-1}} \\
&= \left( \frac{I^1}{a_1} - \frac{I^2}{a_2} \right) + \frac{n - 2}{3\sqrt{-1}} \left( \frac{I^3}{a_3} + \frac{I^4}{a_4} \right) - \frac{n - 2}{3\sqrt{-1}} c_2.
\end{align*}

It is sufficient to determine $c_2$. Let

\begin{align*}
(2.103) \\
J_i &:= \frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}}.
\end{align*}

Then

\begin{align*}
(2.104) \\
c_{i+1} &= -\frac{i + 2}{n - (i + 1)} c_i + J_i, \quad 2 \leq i \leq n - 2.
\end{align*}
To completely determine \( c_2 \), it is sufficient to solve (2.104). A direct calculation shows

\[
c_3 = -\frac{4}{n-3} c_2 + J_2,
\]

\[
c_4 = -\frac{5}{n-4} c_3 + J_3 = -\frac{5}{n-4} \left( -\frac{4}{n-3} c_2 + J_2 \right) + J_3
\]

\[
= (-1)^2 \frac{5 \times 4}{(n-4)(n-3)} c_2 - \frac{5}{n-4} J_2 + J_3.
\]

\[
c_5 = -\frac{6}{n-5} c_4 + J_4
\]

\[
= -\frac{6}{n-5} \left[ (-1)^2 \frac{5 \times 4}{(n-4)(n-3)} c_2 - \frac{5}{n-4} J_2 + J_3 \right] + J_4
\]

\[
= (-1)^3 \frac{6 \times 5 \times 4}{(n-5)(n-4)(n-3)} c_2 + (-1)^2 \frac{6 \times 5}{(n-5)(n-4)} J_2 - \frac{6}{n-5} J_3 + J_4.
\]

Hence, we have

\[
(2.105) \quad c_i = (-1)^{i-2} \frac{(i+1)!(n-i-1)!}{3!(n-3)!} c_2 + \sum_{k=2}^{i-1} (-1)^{i-1-k} \frac{(i+1)!(n-i-1)!}{(k+2)!(n-k-2)!} J_k.
\]

By induction on \( i \), we have

\[
c_{i+1} = -\frac{i+2}{n-(i+1)} c_i + J_i
\]

\[
= -\frac{i+2}{n-(i+1)} \left[ (-1)^{i-2} \frac{(i+1)!(n-i-1)!}{3!(n-3)!} c_2
\right.
\]

\[
+ \sum_{k=2}^{i-1} (-1)^{i-1-k} \frac{(i+1)!(n-i-1)!}{(k+2)!(n-k-2)!} J_k \bigg] + J_i
\]

\[
= (-1)^{i+1-2} \frac{(i+2)!(n-i-2)!}{3!(n-3)!} c_2
\]

\[
+ \sum_{k=2}^{i-1} (-1)^{i-k} \frac{(i+2)!(n-i-2)!}{(k+2)!(n-k-2)!} J_k + J_i.
\]

So (2.105) holds for \( 2 \leq i \leq n-2 \) and we have

\[
c_2 = (-1)^{i-2} \frac{3!(n-3)!}{(i+1)!(n-i-1)!}
\]

\[
\cdot \left[ c_i - \sum_{k=2}^{i-1} (-1)^{i-1-k} \frac{(i+1)!(n-i-1)!}{(k+2)!(n-k-2)!} J_k \right].
\]

Setting \( i = n-1 \) yields

\[
c_2 = (-1)^{n-3} \frac{3!(n-3)!}{n!}
\]

\[
\cdot \left[ c_{n-1} - \sum_{k=2}^{n-2} (-1)^{n-2-k} \frac{n!}{(k+2)!(n-k-2)!} J_k \right], \quad n \geq 3.
\]
We deduce from (2.49) and (2.107) that

\[
\frac{2I^0}{n(n-1)\sqrt{-1}} = \left( \frac{I^0}{a_1} - \frac{I^2}{a_2} \right) + \frac{n-2}{3\sqrt{-1}} \left( \frac{I^3}{a_3} + \frac{I^4}{a_4} \right)
\]

\[
- \frac{n-2}{3\sqrt{-1}} (-1)^{n-3} \frac{3!(n-3)!}{n!}
\]

\[
\cdot \left[ c_{n-1} - \sum_{k=2}^{n-2} (-1)^{n-2-k} \frac{n!}{(k+2)!(n-k-2)!} J_k \right]
\]

\[
= \left( \frac{I^0}{a_1} - \frac{I^2}{a_2} \right) + \frac{n-2}{3\sqrt{-1}} \left( \frac{I^3}{a_3} + \frac{I^4}{a_4} \right)
\]

\[
+ \frac{2(n-2)!}{\sqrt{-1}} \sum_{k=2}^{n-2} \frac{(-1)^{k+1}}{k+2!(n-k-2)!} \left( \frac{I^{2k+1}}{a_{2k+1}} + \frac{I^{2k+2}}{a_{2k+2}} \right).
\]

Equivalently,

\[
(2.108) \quad \frac{2I^0}{n(n-1)\sqrt{-1}} = \left( \frac{I^1}{a_1} - \frac{I^2}{a_2} \right) + \sqrt{-1} \sum_{k=1}^{n-2} (-1)^k \frac{n!}{(n-k)!(k+2)!} \left( \frac{I^{2k+1}}{a_{2k+1}} + \frac{I^{2k+2}}{a_{2k+2}} \right).
\]

Set

\[
(2.109) \quad \frac{1}{a_1} = -\frac{2}{n(n-1)\sqrt{-1}}, \quad \frac{1}{a_2} = \frac{2}{n(n-1)\sqrt{-1}},
\]

\[
(2.110) \quad \sqrt{-1}(-1)^k \binom{n}{k+2} \frac{2}{a_{2k+1}} = -\frac{2}{n(n-1)\sqrt{-1}}, \quad a_{2i+1} = a_{2i+2},
\]

we obtain

\[
(2.111) \quad a_1 = -\frac{n(n-1)\sqrt{-1}}{2}, \quad a_2 = \frac{n(n-1)\sqrt{-1}}{2},
\]

\[
a_{2k+1} = a_{2k+2} = \frac{\sqrt{-1}(-1)^k \binom{n}{k+2} n(n-1)\sqrt{-1}}{-2\binom{n}{2}}
\]

\[
= \frac{(-1)^k n(n-1) \binom{n}{k+2}}{2\binom{n}{2}} = (-1)^k \binom{n}{k+2}.
\]

Consequently,

\[
(2.112) \quad I^0 + \sum_{k=0}^{n-2} (I^{2k+1} + I^{2k+2}) = 0.
\]
Theorem 2.1. The functional, for \( n \geq 2 \),

\[
L^M_\omega (\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X \varphi_t \omega_{\varphi_t}^n dt
\]

\[
- \frac{1}{V_\omega} \int_0^1 \int_X \frac{n(n-1)\sqrt{-1}}{2} \partial \omega \wedge \omega_{\varphi_t}^{n-2} \wedge (\bar{\partial} \varphi_t \cdot \varphi_t) dt
\]

\[
+ \frac{1}{V_\omega} \int_0^1 \int_X \frac{n(n-1)\sqrt{-1}}{2} \partial \omega \wedge \omega_{\varphi_t}^{n-2} \wedge (\partial \varphi_t \cdot \varphi_t) dt
\]

\[
+ \sum_{i=1}^{n-2} \frac{1}{V_\omega} \int_0^1 \int_X (-1)^i \left( \binom{n}{i+2} \partial \varphi_t \wedge \partial \omega \wedge \bar{\partial} \varphi_t \wedge \bar{\partial} \varphi_t \wedge \omega_{\varphi_t}^{n-i-2} \wedge (\sqrt{-1} \partial \omega \wedge \partial \varphi_t) \right) dt
\]

\[
+ \sum_{i=1}^{n-2} \frac{1}{V_\omega} \int_0^1 \int_X (-1)^i \left( \binom{n}{i+2} \bar{\partial} \varphi_t \wedge \partial \omega \wedge \partial \varphi_t \wedge \bar{\partial} \varphi_t \wedge \omega_{\varphi_t}^{n-i-2} \wedge (\sqrt{-1} \partial \omega \wedge \partial \varphi_t) \right) dt
\]

is independent of the choice of the smooth path \( \{ \varphi_t \}_{0 \leq t \leq 1} \) in \( \mathcal{P}_\omega \) from \( \varphi' \) to \( \varphi'' \).

Proof. Using (2.113), we can prove Theorem 2.1 in the similar way as \[2\].

Corollary 2.2. Suppose \( n \geq 2 \). For any \( \varphi \in \mathcal{P}_\omega \) one has

\[
L^M_\omega (\varphi) := L^M_\omega (0, \varphi) = \frac{1}{V_\omega} \sum_{i=0}^n \frac{1}{n+1} \int_X \varphi^n_i \wedge \omega^{n-i}
\]

\[
- \sum_{i=0}^{n-2} \frac{i+1}{2V_\omega} \int_X \varphi^n_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \bar{\partial}_\varphi
\]

\[
+ \sum_{i=0}^{n-2} \frac{i+1}{2V_\omega} \int_X \varphi^n_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi.
\]

Proof. Since \( L^M_\omega (\varphi) \) is independent of the choice of smooth path, we pick \( \varphi_t = t \varphi \), \( 0 \leq t \leq 1 \). Then \( \bar{\partial} \varphi_t \wedge \bar{\partial} \varphi_t = \partial \varphi_t \wedge \partial \varphi_t = 0 \), and

\[
L^M_\omega (\varphi) = \frac{1}{V_\omega} \int_0^1 \int_X \varphi t \omega_{\varphi_t}^n dt
\]

\[
- \frac{n(n-1)}{2V_\omega} \int_0^1 \int_X \sqrt{-1} \partial \omega \wedge \omega_{\varphi_t}^{n-2} \wedge (\bar{\partial} \varphi \cdot \varphi) dt
\]

\[
+ \frac{n(n-1)}{2V_\omega} \int_0^1 \int_X \sqrt{-1} \partial \omega \wedge \omega_{\varphi_t}^{n-2} \wedge (\partial \varphi \cdot \varphi) dt =: J_0 + J_1 + J_2.
\]

Now we compute \( J_0, J_1, J_2 \), respectively. Using

\[
\omega_{\varphi_t} = \omega + t \sqrt{-1} \partial \varphi = \omega + t(\varphi - \omega) = t \omega + (1 - t) \omega,
\]
it follows that
\[
J_0 = \frac{1}{V_\omega} \int_X \int_0^1 \varphi \sum_{i=0}^n \binom{n}{i} \omega^i \wedge \omega^{n-i} t^i (1-t)^{n-i} dt
\]
\[
= \sum_{i=0}^n \frac{1}{V_\omega} \int_X \varphi \binom{n}{i} \omega^i \wedge \omega^{n-i} \cdot \int_0^1 t^i (1-t)^{n-i} dt
\]
\[
= \sum_{i=0}^n \frac{1}{V_\omega} \int_X \left( \frac{n!}{i! (n-i)!} \right) \frac{\Gamma(i+1) \Gamma(n-i+1)}{\Gamma(n+2)} \omega_i \wedge \omega^{n-i}
\]
\[
= \sum_{i=0}^n \frac{1}{V_\omega} \int_X \frac{n!}{i! (n-i)!} \frac{\Gamma(n-i+1)}{\Gamma(n+2)} \omega_i \wedge \omega^{n-i} = \sum_{i=0}^n \frac{1}{V_\omega} \int_X \frac{1}{n+1} \omega_i \wedge \omega^{n-i},
\]
where \( \Gamma(x) \) is the Gamma function. Similarly, we have
\[
J_1 = -\frac{n(n-1)}{2V_\omega} \int_X \int_0^1 \sqrt{-1} \partial \omega \wedge [t \omega_i + (1-t)\omega]^{n-2} \wedge (\overline{\partial} \varphi \cdot t \varphi) dt
\]
\[
= -\frac{n(n-1)}{2V_\omega} \int_X \int_0^1 \sqrt{-1} \partial \omega \wedge \sum_{i=0}^{n-2} \binom{n-2}{i} \omega_i (1-t)^{n-2} \wedge (\overline{\partial} \varphi \cdot \varphi) dt
\]
\[
= -\frac{n(n-1)}{2V_\omega} \int_X \sqrt{-1} \partial \omega \wedge \sum_{i=0}^{n-2} \omega_i \wedge \omega^{n-2-i} \wedge (\overline{\partial} \varphi \cdot \varphi) \cdot \int_0^1 \binom{n-2}{i} (1-t)^{n-2-i} dt
\]
\[
= -\frac{n(n-1)}{2V_\omega} \int_X \sqrt{-1} \partial \omega \wedge \sum_{i=0}^{n-2} \omega_i \wedge \omega^{n-2-i} \wedge (\overline{\partial} \varphi \cdot \varphi) \cdot \frac{i+1}{n(n-1)}
\]
\[
= \sum_{i=0}^{n-2} -\frac{1}{2V_\omega} \int_X (i+1) \omega_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi.
\]
Taking the complex conjugate gives
\[
J_2 = \sum_{i=0}^{n-2} \frac{1}{2V_\omega} \int_X (i+1) \omega_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi.
\]
Together with the expressions of \(J_0, J_1\) and \(J_2\), we complete the proof. \(\square\)

**Remark 2.3.** When \((X, \omega)\) is a compact Kähler manifold, the functional \(\text{2.114}\) or \(\text{2.115}\) coincides with the original one.

Let \(S\) be a non-empty set and \(A\) an additive group. A mapping \(\mathcal{N} : S \times S \rightarrow A\) is said to satisfy the 1-cocycle condition if

(i) \(\mathcal{N}(\sigma_1, \sigma_2) + \mathcal{N}(\sigma_2, \sigma_1) = 0\);
(ii) \(\mathcal{N}(\sigma_1, \sigma_2) + \mathcal{N}(\sigma_2, \sigma_3) + \mathcal{N}(\sigma_3, \sigma_1) = 0\).
Corollary 2.4. (1) The functional $\mathcal{L}_\omega^M$ satisfies the 1-cocycle condition.
(2) For any $\varphi \in \mathcal{P}_\omega$ and any constant $C \in \mathbb{R}$, we have

$$\mathcal{L}_\omega^M(\varphi, \varphi + C) = C \cdot \left(1 + \frac{\text{Err}_\omega(\varphi)}{V_\omega}\right), \quad \text{Err}_\omega(\varphi) := \int_X \omega^n - \int_X \omega^n.$$  

In particular, if $\partial \bar{\partial} \omega = \partial \omega \wedge \bar{\partial} \omega = 0$, then $\mathcal{L}_\omega^M(\varphi, \varphi + C) = C$.
(3) For any $\varphi_1, \varphi_2 \in \mathcal{P}_\omega$ and any constant $C \in \mathbb{R}$, we have

$$\mathcal{L}_\omega^M(\varphi_1, \varphi_2 + C) = \mathcal{L}_\omega^M(\varphi_1, \varphi_2) + C \cdot \left(1 - \frac{\text{Err}_\omega(\varphi_2)}{V_\omega}\right).$$

In particular, if $\partial \bar{\partial} \omega = \partial \omega \wedge \bar{\partial} \omega = 0$, then $\mathcal{L}_\omega^M(\varphi_1, \varphi_2 + C) = \mathcal{L}_\omega^M(\varphi_1, \varphi_2) + C$.

Proof. The proof is similar to that given in [2] [3]. □

3. Aubin-Yau functionals on compact complex manifolds

3.1. The main idea. The strategy to construct Aubin-Yau functionals is to use the inequalities (1.5) and (1.6) to determine the extra terms. Firstly we can show that

$$\frac{n}{n+1} \mathcal{I}^{AY}_{\omega}(\varphi) - \mathcal{J}^{AY}_{\omega}(\varphi) = \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i}$$

$$- \frac{A_\omega(\varphi) + B_\omega(\varphi) + C_\omega(\varphi) + D_\omega(\varphi)}{2},$$

$$(n+1) \mathcal{I}^{AY}_{\omega}(\varphi) - \mathcal{J}^{AY}_{\omega}(\varphi) = \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n-1-i) \int_X \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i}$$

$$+ \frac{E_\omega(\varphi) + F_\omega(\varphi)}{2} + (n+1) \left[A_\omega^1(\varphi) + B_\omega^1(\varphi)\right]$$

$$- \frac{A_\omega^2(\varphi) + B_\omega^2(\varphi)}{n-1},$$

where $\mathcal{I}^{AY}_{\omega}(\varphi), \mathcal{J}^{AY}_{\omega}(\varphi), A_\omega(\varphi), B_\omega(\varphi), C_\omega(\varphi), D_\omega(\varphi), E_\omega(\varphi), F_\omega(\varphi), A_\omega^1(\varphi), B_\omega^1(\varphi), A_\omega^2(\varphi), B_\omega^2(\varphi)$ are functionals determined in next subsection. Inspired by (3.1) and (3.2), we define Aubin-Yau functionals as follows:

$$\mathcal{I}^{AY}_{\omega}(\varphi) := \mathcal{I}^{AY}_{\omega^\bullet}(\varphi)$$

$$+ a_1^1 A_\omega^1(\varphi) + a_2^1 A_\omega^2(\varphi) + b_1^1 B_\omega^1(\varphi) + b_2^1 B_\omega^2(\varphi)$$

$$+ c_1 C_\omega(\varphi) + d_1 D_\omega(\varphi) + e_1 E_\omega(\varphi) + f_1 F_\omega(\varphi),$$

$$\mathcal{J}^{AY}_{\omega}(\varphi) := \mathcal{J}^{AY}_{\omega^\bullet}(\varphi)$$

$$+ (a_1^2 - 1) A_\omega^1(\varphi) + (a_2^2 - 1) A_\omega^2(\varphi) + (b_1^2 - 1) B_\omega^1(\varphi) + (b_2^2 - 1) B_\omega^2(\varphi)$$

$$+ c_2 C_\omega(\varphi) + d_2 D_\omega(\varphi) + e_2 E_\omega(\varphi) + f_2 F_\omega(\varphi).$$

Here $a_j^i, b_j^i, c_k, d_k, e_k$, and $f_k$ are constants determined by the following two inequalities:

$$\frac{n}{n+1} \mathcal{I}^{AY}_{\omega}(\varphi) - \mathcal{J}^{AY}_{\omega}(\varphi) = \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i} \geq 0,$$

$$(n+1) \mathcal{I}^{AY}_{\omega}(\varphi) - \mathcal{J}^{AY}_{\omega}(\varphi) = \sum_{i=1}^{n-1} \frac{n-1-i}{V_\omega} \int_X \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i} \geq 0.$$
By a long computation we find an explicit and shorthed formulae for
these parameters:
\[ a_1^1 = b_1^1 = -\frac{n}{n-1}, \quad a_2^1 = b_2^1 = -\frac{n}{n^2-1}, \]
\[ a_1^2 = b_1^2 = \frac{n}{(n-1)^2}, \quad a_2^2 = b_2^2 = \frac{n}{n+1}\left(1 + \frac{n}{(n-1)^2}\right), \]
\[ c_1 = d_1 = -\frac{n+1}{2(n-1)}, \quad e_2 = f_2 = -\frac{n}{2(n^2-1)}, \]
\[ c_2 = d_2 = e_1 = f_1 = -\frac{1}{2(n-1)}. \]

By a long computation we find an explicit and shorthed formulae for \( I^A_Y(\varphi) \) and \( J^A_Y(\varphi) \):

\[ (3.5) \quad I^A_Y(\varphi) = \frac{1}{\omega} \int_X (\omega^n - \omega^n_\varphi) \]
\[ - \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\varphi} - \frac{n}{2V_\omega} \int_X \varphi \omega_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \varphi \wedge \overline{\varphi} \]
\[ + \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \varphi \wedge \overline{\varphi} + \frac{n}{2V_\omega} \int_X \varphi \omega_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \varphi \wedge \overline{\varphi}, \]

\[ (3.6) \quad J^A_Y(\varphi) = -L^M_\omega(\varphi) + \frac{1}{\omega} \int_X \varphi \omega^n \]
\[ - \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\varphi} - \frac{n}{2V_\omega} \int_X \varphi \omega_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \varphi \wedge \overline{\varphi} \]
\[ + \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \varphi \wedge \overline{\varphi} + \frac{n}{2V_\omega} \int_X \varphi \omega_i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \varphi \wedge \overline{\varphi}. \]

3.2. The construction of Aubin-Yau functionals. Let \((X, g)\) be a compact complex manifold of the complex dimension \(n \geq 3\) and \(\omega\) be its associated real \((1,1)\)-form. We recall some notation in [2]. For any \(\varphi \in P_\omega\) we set

\[ (3.7) \quad I^A_Y(\varphi) := \frac{1}{\omega} \int_X \varphi (\omega^n - \omega^n_\varphi), \]
\[ (3.8) \quad J^A_Y(\varphi) := \int_0^1 \frac{I^A_Y(s \cdot \varphi)}{s} ds = \frac{1}{\omega} \int_X \varphi (\omega^n - \omega^n_{s \cdot \varphi}) ds. \]

Two relations showed in [2] are

\[ (3.9) \quad \frac{n}{n+1} I^A_Y(\varphi) - J^A_Y(\varphi) \]
\[ = \frac{1}{\omega} \int_X \varphi \cdot (-\sqrt{-1} \partial \overline{\varphi}) \wedge \sum_{j=1}^{n-1} \frac{j}{n+1} \omega^{n-1-j} \wedge \omega_j, \]
\[ (3.10) \quad (n+1) I^A_Y(\varphi) - I^A_Y(\varphi) \]
\[ = \frac{1}{\omega} \int_X \varphi \cdot (-\sqrt{-1} \partial \overline{\varphi}) \wedge \sum_{j=0}^{n-1} (n-1-j) \omega^{n-1-j} \wedge \omega_j. \]
According to the expression of $\mathcal{L}_M^\omega(\varphi)$, we set

(3.11) $A_\omega(\varphi) := \sum_{i=0}^{n-2} \frac{i + 1}{2V_\omega} \int_X \varphi \omega^i_\varphi \wedge \omega^{n-2-i} \wedge -\sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi,$

(3.12) $B_\omega(\varphi) := \sum_{i=0}^{n-2} \frac{i + 1}{2V_\omega} \int_X \varphi \omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi.$

Using (3.9) we obtain

$$\frac{n}{n + 1} \mathcal{J}_{\omega|\bullet}(\varphi) - \mathcal{J}_{\omega|\bullet}^\lambda(\varphi) = \frac{1}{V_\omega} \int_X \sqrt{-1} \partial \left( \varphi \sum_{j=1}^{n-1} \frac{j}{n+1} \omega^{n-1-j} \wedge \omega^j_\varphi \right) \wedge \overline{\partial} \varphi$$

$$= \frac{1}{V_\omega} \int_X \sqrt{-1} \left( \partial \varphi \wedge \sum_{j=1}^{n-1} \frac{j}{n+1} \omega^{n-1-j} \wedge \omega^j_\varphi \right) \wedge \overline{\partial} \varphi$$

$$+ \frac{1}{V_\omega} \int_X \sqrt{-1} \varphi \sum_{j=1}^{n-1} \frac{j}{n+1} [(n - 1 - j)\omega^{n-2-j} \wedge \partial \omega \wedge \omega^j_\varphi$$

$$\omega^{n-1-j} \wedge j \omega^j_\varphi \wedge \partial \omega] \wedge \overline{\partial} \varphi;$$

from the identity $i(n - 1 - i) + (i + 1)^2 = (i + 1) + in$, it follows that

$$\frac{n}{n + 1} \mathcal{J}_{\omega|\bullet}(\varphi) - \mathcal{J}_{\omega|\bullet}^\lambda(\varphi) = \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^{n-1-i} \wedge \omega^i_\varphi$$

$$+ \frac{1}{V_\omega} \sum_{i=1}^{n-2} \frac{i(n - 1 - i)}{n+1} \int_X \varphi \omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi$$

$$+ \frac{1}{V_\omega} \sum_{i=1}^{n-2} \frac{(i + 1)^2}{n+1} \int_X \varphi \omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi$$

$$= \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^{n-1-i} \wedge \omega^i_\varphi$$

$$+ \frac{1}{V_\omega} \sum_{i=1}^{n-2} \frac{i + 1 + in}{n+1} \int_X \varphi \omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi$$

$$+ \frac{1}{(n+1)V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi.$$

To simplify the notation, we set

(3.13) $C_\omega(\varphi) := \frac{1}{V_\omega} \sum_{i=1}^{n-2} \frac{in}{n+1} \int_X \varphi \omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi.$
Since $n \geq 3$, the above expression is well defined. Therefore

\begin{align*}
(3.14) \quad \frac{n}{n+1} \mathcal{I}_{\omega}^{\text{AY}}(\varphi) - \mathcal{J}_{\omega}^{\text{AY}}(\varphi) \\
= \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^i_\varphi \wedge \omega^{n-1-i} - \frac{2}{n+1} A_\omega(\varphi) + C_\omega(\varphi).
\end{align*}

On the other hand, using the slightly different method, we obtain (see A.1)

\begin{align*}
(3.15) \quad \frac{n}{n+1} \mathcal{J}_{\omega}^{\text{AY}}(\varphi) - \mathcal{J}_{\omega}^{\text{AY}}(\varphi) \\
= \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^i_\varphi \wedge \omega^{n-1-i} - \frac{2}{n+1} B_\omega(\varphi) + D_\omega(\varphi)
\end{align*}

where

\begin{equation}
(3.16) \quad D_\omega(\varphi) := \frac{1}{V_\omega} \sum_{i=1}^{n-2} \frac{in}{n+1} \int_X \varphi \omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi.
\end{equation}

Equations (3.14) and (3.15) implies

\begin{equation}
(3.17) \quad \frac{n}{n+1} \mathcal{I}_{\omega}^{\text{AY}}(\varphi) - \mathcal{J}_{\omega}^{\text{AY}}(\varphi) = \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^i_\varphi \wedge \omega^{n-1-i} - \frac{A_\omega(\varphi) + B_\omega(\varphi) + C_\omega(\varphi)}{n+1} + D_\omega(\varphi).
\end{equation}

By the definition we have

\begin{align*}
\mathcal{J}_{\omega}^{\text{AY}}(\varphi) &= \frac{1}{V_\omega} \int_X (\varphi \omega^n - \varphi \omega^n_\varphi) ds = \frac{1}{V_\omega} \int_X \varphi \omega^n - \frac{1}{V_\omega} \int_0^1 \int_X \varphi \omega^n_\varphi dt \\
&= \frac{1}{V_\omega} \int_X \varphi \omega^n - (\mathcal{L}_\omega^M(\varphi) - A_\omega(\varphi) - B_\omega(\varphi)) \\
&= \frac{1}{V_\omega} \int_X \varphi \omega^n - \mathcal{L}_\omega^M(\varphi) + A_\omega(\varphi) + B_\omega(\varphi).
\end{align*}

If we define

\begin{align*}
(3.18) \quad \mathcal{E}_\omega(\varphi) &:= \sum_{i=0}^{n-3} \frac{n^2}{V_\omega} \int_X \varphi \omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi, \\
(3.19) \quad A_\omega^1(\varphi) &:= \sum_{i=0}^{n-3} \frac{i+1}{2V_\omega} \int_X \varphi \omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi \\
(3.20) \quad A_\omega^2(\varphi) &:= \frac{n-1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge -\sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi,
\end{align*}

then $A_\omega^1(\varphi) + A_\omega^2(\varphi) = A_\omega(\varphi)$ and it follows (see A.1)

\begin{align*}
(3.21) \quad (n+1) \mathcal{J}_{\omega}^{\text{AY}} - \mathcal{J}_{\omega}^{\text{AY}}(\varphi) &= \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n-1-i) \int_X \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^i_\varphi \wedge \omega^{n-1-i} \\
&+ \mathcal{E}_\omega(\varphi) + 2(n+1)A_\omega^1(\varphi) + \frac{2}{n-1} A_\omega^2(\varphi).
\end{align*}
Introduce the corresponding functionals

\[
\begin{align*}
\mathcal{F}_\omega(\varphi) &:= \sum_{i=0}^{n-3} \frac{n^2}{V_\omega} \int_X \varphi \omega_\omega^i \land \omega^{n-2-i} \land -\sqrt{-1} \overline{\partial} \omega \land \partial \varphi, \\
\mathcal{B}_\omega^1(\varphi) &:= \sum_{i=0}^{n-3} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\omega^i \land \omega^{n-2-i} \land \sqrt{-1} \partial \omega \land \partial \varphi, \\
\mathcal{B}_\omega^2(\varphi) &:= \frac{n-1}{2V_\omega} \int_X \varphi \omega_\omega^n \land \sqrt{-1} \overline{\partial} \omega \land \partial \varphi.
\end{align*}
\]

Then \( \mathcal{B}_\omega^1(\varphi) + \mathcal{B}_\omega^2(\varphi) = \mathcal{B}_\omega(\varphi) \) and hence (see A.2)

\[
(n+1)\mathcal{J}_{\omega^*}^{\Lambda Y}(\varphi) - \mathcal{I}_{\omega^*}^{\Lambda Y}(\varphi) = \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n-1-i) \int_X \sqrt{-1} \overline{\partial} \varphi \land \overline{\partial} \varphi \land \omega_\omega^i \land \omega^{n-1-i} + \mathcal{F}_\omega(\varphi) + 2(n+1)\mathcal{B}_\omega^1(\varphi) - \frac{2}{n-1} \mathcal{B}_\omega^2(\varphi).
\]

The equations (3.21) and (3.25) together gives

\[
(n+1)\mathcal{J}_{\omega^*}^{\Lambda Y}(\varphi) - \mathcal{I}_{\omega^*}^{\Lambda Y}(\varphi) = \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n-1-i) \int_X \sqrt{-1} \overline{\partial} \varphi \land \overline{\partial} \varphi \land \omega_\omega^i \land \omega^{n-1-i} + \mathcal{E}_\omega(\varphi) + \mathcal{F}_\omega(\varphi) + (n+1)(A_\omega^1(\varphi) + B_\omega^1(\varphi)) - \frac{A_\omega^2(\varphi) + B_\omega^2(\varphi)}{n-1}.
\]

Now, we define Aubin-Yau functionals over any compact complex manifolds as follows:

\[
\begin{align*}
\mathcal{I}_{\omega}^{\Lambda Y}(\varphi) &:= \mathcal{I}_{\omega^*}^{\Lambda Y}(\varphi) + a_1^1 A_\omega^1(\varphi) + a_2^1 A_\omega^2(\varphi) + b_1^1 B_\omega^1(\varphi) + b_2^1 B_\omega^2(\varphi) + c_1 C_\omega(\varphi) + d_1 D_\omega(\varphi) + e_1 E_\omega(\varphi) + f_1 F_\omega(\varphi), \\
\mathcal{J}_{\omega}^{\Lambda Y}(\varphi) &:= \mathcal{E}_\omega(\varphi) + \mathcal{F}_\omega(\varphi) + (n+1)(A_\omega^1(\varphi) + B_\omega^1(\varphi)) - \frac{A_\omega^2(\varphi) + B_\omega^2(\varphi)}{n-1}.
\end{align*}
\]

Plugging (3.27) and (3.28) into (3.26) and (3.17), we obtain

\[
\frac{n}{n+1} \mathcal{I}_{\omega}^{\Lambda Y}(\varphi) - \mathcal{J}_{\omega}^{\Lambda Y}(\varphi) = \frac{1}{V_\omega} \sum_{i=0}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1} \overline{\partial} \varphi \land \overline{\partial} \varphi \land \omega_\omega^i \land \omega^{n-1-i} \geq 0,
\]

and

\[
(n+1)\mathcal{J}_{\omega}^{\Lambda Y}(\varphi) - \mathcal{I}_{\omega}^{\Lambda Y}(\varphi) = \sum_{i=0}^{n-1} \frac{n-1-i}{V_\omega} \int_X \sqrt{-1} \overline{\partial} \varphi \land \overline{\partial} \varphi \land \omega_\omega^i \land \omega^{n-1-i} \geq 0.
\]
where we require that constants satisfy the following linear equations system

\begin{align}
(3.31) \quad \frac{n}{n+1} a_1 - (a_2 - 1) &= \frac{1}{n+1}, \quad \frac{n}{n+1} a_1^2 - (a_2^2 - 1) = \frac{1}{n+1}, \\
(3.32) \quad \frac{n}{n+1} b_1 - (b_2 - 1) &= \frac{1}{n+1}, \quad \frac{n}{n+1} b_1^2 - (b_2^2 - 1) = \frac{1}{n+1}, \\
(3.33) \quad \frac{n}{n+1} c_1 - c_2 &= -\frac{1}{2}, \quad \frac{n}{n+1} d_1 - d_2 = -\frac{1}{2}, \\
(3.34) \quad \frac{n}{n+1} e_1 - e_2 &= 0, \quad \frac{n}{n+1} f_1 - f_2 = 0, \\
(3.35) \quad (n+1)(a_1^2 - 1) - a_1 &= -(n+1), \quad (n+1)(a_2^2 - 1) - a_1^2 = \frac{1}{n-1}, \\
(3.36) \quad (n+1)(b_1^2 - 1) - b_1 &= -(n+1), \quad (n+1)(b_2^2 - 1) - b_1^2 = \frac{1}{n-1}, \\
(3.37) \quad (n+1)c_2 - c_1 &= 0, \quad (n+1)d_2 - d_1 = 0, \\
(3.38) \quad (n+1)e_2 - e_1 &= -\frac{1}{2}, \quad (n+1)f_2 - f_1 = -\frac{1}{2}.
\end{align}

The constants $a_1^j, b_1^j, c_1, d_1, e_1$ and $f_1$, calculated in Appendix B, are

\begin{align}
(3.39) \quad a_1^1 &= b_1^1 = -\frac{n}{n-1}, \quad a_2^1 = b_1^2 = -\frac{n}{n^2 - 1}, \\
(3.40) \quad a_1^2 &= b_1^2 = \frac{n}{(n-1)^2}, \quad a_2^2 = b_1^2 = \frac{n}{n+1} \left(1 + \frac{n}{(n-1)^2}\right), \\
(3.41) \quad c_1 &= d_1 = -\frac{n+1}{2(n-1)}, \quad e_2 = f_2 = -\frac{n}{2(n^2 - 1)}, \\
(3.42) \quad c_2 &= d_2 = e_1 = f_1 = -\frac{1}{2(n-1)}.
\end{align}

The explicit formulas for $I_\omega^{AY}(\varphi)$ and $J_\omega^{AY}(\varphi)$ are given in Proposition C.1 and C.2 respectively. Namely,

\begin{align}
I_\omega^{AY}(\varphi) &= \frac{1}{V_\omega} \int_X \varphi(\omega^n - \omega^n) - \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_i^\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi \\
&\quad + \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_i^\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi, \\
(3.43) &= \frac{1}{V_\omega} \sum_{i=0}^{n-1} \int_X \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega_i^\varphi \wedge \omega^{n-1-i},
\end{align}

\begin{align}
J_\omega^{AY}(\varphi) &= -\mathcal{L}_\omega^M(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n - \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_i^\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi \\
&\quad + \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_i^\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi \\
(3.44) &= \frac{1}{V_\omega} \sum_{i=0}^{n-1} \int_X \frac{n-i}{n+1} \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega_i^\varphi \wedge \omega^{n-1-i}.
\end{align}

Here the formulas (3.43) and (3.44) come from the solution of the system of linear equations (3.29) and (3.30).
From (3.29), (3.30), (3.37) and (3.44), we deduce the following

**Theorem 3.1.** For any $\varphi \in P_\omega$, one has

\[(3.45) \quad \frac{n}{n+1} I_{A^Y_\omega}(\varphi) - J_{A^Y_\omega}(\varphi) \geq 0,\]

\[(3.46) \quad (n+1)J_{A^Y_\omega}(\varphi) - I_{A^Y_\omega}(\varphi) \geq 0.\]

In particular

\[(3.47) \quad \frac{1}{n+1} I_{A^Y_\omega}(\varphi) \leq J_{A^Y_\omega}(\varphi) \leq \frac{n}{n+1} I_{A^Y_\omega}(\varphi),\]

\[(3.48) \quad \frac{n+1}{n} J_{A^Y_\omega}(\varphi) \leq I_{A^Y_\omega}(\varphi) \leq (n+1)J_{A^Y_\omega}(\varphi),\]

\[(3.49) \quad \frac{1}{n} J_{A^Y_\omega}(\varphi) \leq \frac{1}{n+1} J_{A^Y_\omega}(\varphi) \leq J_{A^Y_\omega}(\varphi) - J_{A^Y_\omega}(\varphi),\]

\[(3.50) \quad \leq \frac{n}{n+1} I_{A^Y_\omega}(\varphi) \leq nJ_{A^Y_\omega}(\varphi).\]

**Appendix A. Proof the identities (3.14), (3.21) and (3.25)**

In Appendix A we verify the identities (3.14), (3.21) and (3.25).
which gives (3.15). Calculate

\[(n + 1)J^\text{AY}_{\omega} - J^\text{AY}_{\phi}(\varphi)\]

\[= \frac{1}{V_\omega} \int_X \left( \varphi \sum_{j=0}^{n-1} (n - 1 - j)\omega^{n-1-j} \land \omega^j_\varphi \right) \land (-\sqrt{-i}\partial\bar{\partial}\varphi)\]

\[= \frac{1}{V_\omega} \int_X \sqrt{-1}\partial\varphi \sum_{j=0}^{n-1} (n - 1 - j)\omega^{n-1-j} \land \omega^j_\varphi \land \bar{\partial}\varphi\]

\[+ \frac{1}{V_\omega} \int_X \sqrt{-1}\varphi \sum_{j=0}^{n-1} [(n - 1 - j)^2\omega^{n-2-j} \land \partial\omega \land \omega^j_\varphi \land \bar{\partial}\varphi]\]

\[= \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n - 1 - j) \int_X \sqrt{-1}\partial\varphi \land \bar{\partial}\varphi \land \omega^1_\varphi \land \omega^{n-1-i}\]

\[+ \frac{1}{V_\omega} \sum_{j=0}^{n-1} (n - 1 - j)^2 \int_X \varphi\omega^{n-2-j} \land \omega^j_\varphi \land \sqrt{-1}\partial\omega \land \bar{\partial}\varphi\]

\[+ \frac{1}{V_\omega} \sum_{j=0}^{n-1} (n - 1 - j)j \int_X \varphi\omega^{n-1-j} \land \omega^j_\varphi \land \sqrt{-1}\partial\omega \land \bar{\partial}\varphi\]

where we use the elementary identity

\[(n - 1 - i)^2 + (i + 1)(n - i - 2)\]

\[= (n - 1)^2 + i^2 - 2(n - 1)i + (n - 2)(i + 1) - i(i + 1)\]

\[= n^2 - 2n + 1 + i^2 - 2ni + 2i + mi + n - 2i - 2 - i^2 - i\]

\[= n^2 - n - 1 - mi - i = -(n + 1)(i + 1) + n^2.\]
Using the definitions of $E_\omega(\phi), A_1^*(\phi), A_2^*(\phi)$, we have $A_1^*(\phi) + A_2^*(\phi) = A_\omega(\phi)$ and hence (3.21) holds. Similarly, we have $B_1^*(\phi) + B_2^*(\phi) = B_\omega(\phi)$ and

\[
\begin{align*}
(n + 1)\mathcal{J}_{\omega}^{\Lambda Y}(\phi) - \mathcal{T}_{\omega}^{\Lambda Y}(\phi) & = \frac{1}{V_\omega} \int_X -\sqrt{-1}\partial \left( \varphi \sum_{j=0}^{n-1} (n - 1 - j) \omega^{n-1-j} \wedge \omega_j^i \right) \wedge \partial \phi \\
 & = \frac{1}{V_\omega} \int_X -\sqrt{-1}\partial \phi \wedge \sum_{j=0}^{n-1} (n - 1 - j) \omega^{n-1-j} \wedge \omega_j^i \wedge \partial \phi \\
 & + \frac{1}{V_\omega} \int_X -\sqrt{-1}\partial \sum_{j=0}^{n-1} (n - 1 - j)(n - 1 - j) \omega^{n-2-j} \wedge \overline{\partial} \omega \wedge \omega_j^i \\
 & + j \omega^{n-1-j} \wedge \omega_j^{i-1} \wedge \overline{\partial} \omega) \wedge \partial \phi \\
 & = \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n - 1 - i) \int_X \sqrt{-1}\partial \phi \wedge \overline{\partial} \phi \wedge \omega^{n-1-i} \wedge \omega_j^i \\
 & + \frac{1}{V_\omega} \sum_{j=0}^{n-1} (n - 1 - j)^2 \int_X \varphi \omega^{n-2-j} \wedge \omega_j^i \wedge (-\sqrt{-1}\partial \omega \wedge \partial \phi) \\
 & + \frac{1}{V_\omega} \sum_{j=0}^{n-1} (n - 1 - j) j \int_X \varphi \omega^{n-1-j} \wedge \omega_j^{i-1} \wedge (-\sqrt{-1}\partial \omega \wedge \partial \phi) \\
 & = \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n - 1 - i) \int_X \sqrt{-1}\partial \phi \wedge \overline{\partial} \phi \wedge \omega^{n-1-i} \wedge \omega_j^i \\
 & + \frac{1}{V_\omega} \sum_{i=0}^{n-2} (n - 1 - i)^2 \int_X \varphi \omega^{n-2-i} \wedge \omega_j^i \wedge (-\sqrt{-1}\partial \omega \wedge \partial \phi) \\
 & + \frac{1}{V_\omega} \sum_{i=0}^{n-3} (n - 2)(i + 1) \int_X \varphi \omega^{n-2-i} \wedge \omega_j^i \wedge (-\sqrt{-1}\partial \omega \wedge \partial \phi) \\
 & = \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n - 1 - i) \int_X \sqrt{-1}\partial \phi \wedge \overline{\partial} \phi \wedge \omega^{n-1-i} \wedge \omega_j^i \\
 & + \frac{1}{V_\omega} \sum_{i=0}^{n-3} [n^2 - (n + 1)(i + 1)] \int_X \varphi \omega^{n-2-i} \wedge \omega_j^i \wedge (-\sqrt{-1}\partial \omega \wedge \partial \phi) \\
 & + \frac{1}{V_\omega} \int_X \varphi \omega^{n-2-i} \wedge (-\sqrt{-1}\partial \omega \wedge \partial \phi).
\end{align*}
\]

and hence

\[
\begin{align*}
(n + 1)\mathcal{J}_{\omega}^{\Lambda Y}(\phi) - \mathcal{T}_{\omega}^{\Lambda Y}(\phi) & = \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n - 1 - i) \int_X \sqrt{-1}\partial \phi \wedge \overline{\partial} \phi \wedge \omega_j^i \wedge \omega^{n-1-i} \\
& + \mathcal{F}_\omega(\phi) + 2(n + 1)B_1^*(\phi) - \frac{2}{n - 1} B_2^*(\phi).
\end{align*}
\]

(A.2)
Therefore (3.21) and (3.25) together gives
\[
(n + 1)J_{\omega|\bullet}^A Y (\varphi) - \mathcal{I}_{\omega|\bullet}^A Y (\varphi) = \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n - 1 - i) \int_X \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i}
\]
(A.3)
\[
+ \frac{\mathcal{E}_\omega(\varphi) + \mathcal{F}_\omega(\varphi)}{2} + (n + 1)(A^1_\omega(\varphi) + B^1_\omega(\varphi)) - \frac{A^2_\omega(\varphi) + B^2_\omega(\varphi)}{n - 1}.
\]

**Appendix B. Solve the system of the linear equations**

In this section we try to solve the system of the linear equations (3.31)-(3.38).
Firstly we solve (3.31) and (3.35) as follows: (3.31) and (3.35) gives us the following equations
\[
\frac{n}{n+1} a^1_1 - \frac{1}{n+1} = a^1_2 - 1,
\]
(B.1)
\[
(n + 1)(a^1_2 - 1) + (n + 1) = a^1_1,
\]
(B.2)
\[
\frac{n}{n+1} a^1_2 - \frac{1}{n+1} = a^2_2 - 1,
\]
\[
(n + 1)(a^2_2 - 1) - \frac{1}{n - 1} = a^2_1.
\]
Plugging the first equation into second equation in (B.1), we have
\[
(n + 1)\left(\frac{n}{n+1} a^1_1 - \frac{1}{n+1}\right) + (n + 1) = a^1_1
\]
which implies
\[
a^1_1 = -\frac{n}{n - 1}, \quad a^1_2 = -\frac{n}{n^2 - 1}.
\]
Similarly,
\[
(n + 1)\left(\frac{n}{n+1} a^2_2 - \frac{1}{n+1}\right) - \frac{1}{n + 1} = a^2_1,
\]
therefore
\[
a^2_1 = \frac{n}{(n - 1)^2}, \quad a^2_2 = \frac{n}{n + 1} \left(1 + \frac{n}{(n - 1)^2}\right) = \frac{n^3 - n^2 + n}{n^3 - n^2 - n + 1}.
\]
Secondly, (3.32) and (3.36) implies
\[
\frac{n}{n+1} b^1_1 - \frac{1}{n+1} = b^1_2 = 1 - (n + 1),
\]
(B.5)
\[
(n + 1)(b^1_2 - 1) = \frac{1}{n - 1}.
\]
(B.6)
\[
\frac{n}{n+1} b^2_2 - \frac{1}{n+1} = b^2_1 - 1,
\]
\[
(n + 1)(b^2_2 - 1) = b^2_1 + \frac{1}{n - 1}.
\]
The above linear equations system gives
\[
(n + 1)\left(\frac{n}{n+1} b^1_1 - \frac{1}{n+1}\right) = b^1_1 - (n + 1)
\]
and
\[
(n + 1) \left( \frac{n}{n+1} b_1^2 - \frac{1}{n+1} \right) = b_1^2 + \frac{1}{n-1},
\]
respectively. Hence
\[
\begin{align*}
(b.7) \\
b_1^1 &= -\frac{n}{n-1}, \\
b_2^1 &= -\frac{n}{n^2 - 1}, \\
(b.8) \\
b_1^2 &= \frac{n}{(n-1)^2}, \\
b_2^2 &= \frac{n}{n+1} \left( 1 + \frac{n}{(n-1)^2} \right).
\end{align*}
\]
Continuously, equations (3.33) and (3.37) shows that
\[
\begin{align*}
\frac{n}{n+1} c_1 - c_2 &= \frac{1}{2}, & (n+1)c_2 - c_1 &= 0, \\
\frac{n}{n+1} d_1 - d_2 &= -\frac{1}{2}, & (n+1)d_2 - d_1 &= 0.
\end{align*}
\]
Eliminating \(c_2\) and \(d_2\) respectively, we have
\[
\begin{align*}
(n+1) \left( \frac{n}{n+1} c_1 + \frac{1}{2} \right) - c_1 &= 0, \\
(n+1) \left( \frac{n}{n+1} d_1 + \frac{1}{2} \right) - d_1 &= 0.
\end{align*}
\]
Thus
\[
\begin{align*}
(b.9) \\
c_1 &= \frac{n+1}{2(n-1)}, \\
c_2 &= \frac{1}{2(n-1)}, \\
(b.10) \\
d_1 &= -\frac{n+1}{2(n-1)}, \\
d_2 &= -\frac{1}{2(n-1)}.
\end{align*}
\]
Similarly, from (3.34) and (3.38) we obtain
\[
\begin{align*}
\frac{n}{n+1} e_1 - e_2 &= 0, & (n+1)e_2 - e_1 &= -\frac{1}{2}, \\
\frac{n}{n+1} f_1 - f_2 &= 0, & (n+1)f_2 - f_1 &= -\frac{1}{2},
\end{align*}
\]
and hence
\[
\begin{align*}
(b.11) \\
e_1 &= f_1 = -\frac{1}{2(n-1)}, \\
(b.12) \\
e_2 &= f_2 = -\frac{n}{2(n^2 - 1)}.
\end{align*}
\]
APPENDIX C. Explicit formulas of $\mathcal{I}_\omega^{AY}(\varphi)$ and $\mathcal{J}_\omega^{AY}(\varphi)$

In this section we give the explicit formulas of $\mathcal{I}_\omega^{AY}(\varphi)$ and $\mathcal{J}_\omega^{AY}(\varphi)$. In what follows, we assume that $n \geq 3$. Using the constants determined in Appendix B, we have

$$
\mathcal{I}_\omega^{AY}(\varphi) = \frac{1}{V_{\omega}} \int_X \varphi(\omega^n - \omega^n_{\varphi})
$$

$$
= \frac{n}{n-1} \sum_{i=0}^{n-3} \frac{1}{2V_{\omega}} \int_X \varphi \omega_{\varphi}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial_{\omega} \wedge \overline{\partial}_{\varphi}
$$

$$
- \frac{n}{n-1} \frac{1}{2V_{\omega}} \int_X \varphi \omega_{\varphi}^{n-2} \wedge \sqrt{-1} \partial_{\omega} \wedge \overline{\partial}_{\varphi}
$$

$$
- \frac{n}{n-1} \sum_{i=0}^{n-3} \frac{1}{2V_{\omega}} \int_X \varphi \omega_{\varphi}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial_{\omega} \wedge \partial_{\varphi}
$$

$$
+ \frac{n}{n-1} \frac{1}{2V_{\omega}} \int_X \varphi \omega_{\varphi}^{n-2} \wedge \sqrt{-1} \partial_{\omega} \wedge \partial_{\varphi}
$$

$$
- \frac{n}{n-1} \sum_{i=1}^{n-2} \frac{1}{2V_{\omega}} \int_X \varphi \omega_{\varphi}^i \wedge \omega^{n-3-i} \wedge \sqrt{-1} \partial_{\omega} \wedge \overline{\partial}_{\varphi}
$$

$$
+ \frac{n}{n-1} \sum_{i=1}^{n-2} \frac{1}{2V_{\omega}} \int_X \varphi \omega_{\varphi}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial_{\omega} \wedge \partial_{\varphi}
$$

$$
- \frac{n^2}{n-1} \sum_{i=0}^{n-3} \frac{1}{2V_{\omega}} \int_X \varphi \omega_{\varphi}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial_{\omega} \wedge \overline{\partial}_{\varphi}
$$

$$
+ \frac{n^2}{n-1} \sum_{i=0}^{n-3} \frac{1}{2V_{\omega}} \int_X \varphi \omega_{\varphi}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial_{\omega} \wedge \partial_{\varphi}.
$$

When $n = 3$, it is easy to see that

$$
\mathcal{I}_\omega^{AY}(\varphi) = \frac{1}{V_{\omega}} \int_X \varphi(\omega^3 - \omega^3_{\varphi})
$$

$$
+ \frac{3}{4V_{\omega}} \int_X \varphi \omega \wedge \sqrt{-1} \partial_{\omega} \wedge \overline{\partial}_{\varphi} - \frac{3}{4V_{\omega}} \int_X \varphi \omega_{\varphi} \wedge \sqrt{-1} \partial_{\omega} \wedge \overline{\partial}_{\varphi}
$$

$$
- \frac{3}{4V_{\omega}} \int_X \varphi \omega \wedge \sqrt{-1} \partial_{\omega} \wedge \partial_{\varphi} + \frac{3}{4V_{\omega}} \int_X \varphi \omega_{\varphi} \wedge \sqrt{-1} \partial_{\omega} \wedge \partial_{\varphi}
$$

$$
- \frac{3}{4V_{\omega}} \int_X \varphi \omega_{\varphi} \wedge \sqrt{-1} \partial_{\omega} \wedge \overline{\partial}_{\varphi} + \frac{3}{4V_{\omega}} \int_X \varphi \omega_{\varphi} \wedge \sqrt{-1} \partial_{\omega} \wedge \partial_{\varphi}
$$

$$
- \frac{9}{4V_{\omega}} \int_X \varphi \omega \wedge \sqrt{-1} \partial_{\omega} \wedge \partial_{\varphi} + \frac{9}{4V_{\omega}} \int_X \varphi \omega \wedge \sqrt{-1} \partial_{\omega} \wedge \partial_{\varphi}
$$

$$
= \frac{1}{V_{\omega}} \int_X \varphi(\omega^3 - \omega^3_{\varphi})
$$

$$
- \frac{3}{2V_{\omega}} \int_X \varphi \omega \wedge \sqrt{-1} \partial_{\omega} \wedge \overline{\partial}_{\varphi} + \frac{3}{2V_{\omega}} \int_X \varphi \omega \wedge \sqrt{-1} \partial_{\omega} \wedge \partial_{\varphi}
$$

$$
- \frac{3}{2V_{\omega}} \int_X \varphi \omega_{\varphi} \wedge \sqrt{-1} \partial_{\omega} \wedge \overline{\partial}_{\varphi} + \frac{3}{2V_{\omega}} \int_X \varphi \omega_{\varphi} \wedge \sqrt{-1} \partial_{\omega} \wedge \partial_{\varphi}.$$
For general $n \geq 4$, a simple computation shows
\[
\mathcal{I}^\mathcal{A}_\omega(\varphi) = \frac{1}{V_\omega} \int_X \varphi(\omega^n - \omega_\varphi^n) + \sum_{i=1}^{n-3} \frac{1}{2V_\omega} \left[ \frac{n(i+1)}{n-1} - \frac{in}{n-1} - \frac{n^2}{n-1} \right] \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
+ \frac{n}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
- \frac{n(n-2)}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
- \frac{n}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
+ \sum_{i=1}^{n-3} \frac{1}{2V_\omega} \left[ -\frac{n(i+1)}{n-1} + \frac{in}{n-1} + \frac{n^2}{n-1} \right] \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
- \frac{n}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
+ \frac{n(n-2)}{n-1} \frac{2}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
+ \frac{n}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
- \frac{n^2}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi + \frac{n^2}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
= \frac{1}{V_\omega} \int_X \varphi(\omega^n - \omega_\varphi^n) \\
- \sum_{i=1}^{n-3} \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
+ \sum_{i=1}^{n-3} \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
- \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi + \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
- \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi + \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi.
\]

Thus

**Proposition C.1.** If $n \geq 3$, one has
\[
\mathcal{I}^\mathcal{A}_\omega(\varphi) = \frac{1}{V_\omega} \int_X \varphi(\omega^n - \omega_\varphi^n) \\
- \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}d\omega \wedge \partial \varphi + \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi \\
- \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}d\omega \wedge \partial \varphi + \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}d\omega \wedge \partial \varphi.
\]
Similarly, we have

$$J_{\omega}^{\text{AY}}(\varphi)$$

$$= -L_{\omega}^{M}(\varphi) + \frac{1}{\omega} \int_{X} \varphi \omega^{n}$$

$$+ \frac{n}{n^{2} - 1} \sum_{i=0}^{n-3} \frac{i + 1}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi}^{i} \wedge \omega^{n-2-i} \wedge \sqrt{-1} \omega \wedge \bar{\partial} \varphi$$

$$- \frac{n}{n + 1} \left( n - 1 + \frac{n}{n - 1} \right) \frac{1}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi}^{n-2} \wedge \sqrt{-1} \omega \wedge \bar{\partial} \varphi$$

$$+ \frac{n}{n^{2} - 1} \sum_{i=0}^{n-3} \frac{i + 1}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi}^{i} \wedge \omega^{n-2-i} \wedge \sqrt{-1} \omega \wedge \partial \varphi$$

$$+ \frac{n}{n + 1} \left( n - 1 + \frac{n}{n - 1} \right) \frac{2}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi}^{n-2} \wedge \sqrt{-1} \omega \wedge \partial \varphi$$

$$- \frac{n}{n^{2} - 1} \sum_{i=1}^{n-2} \frac{1}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi}^{i} \wedge \omega^{n-2-i} \wedge \sqrt{-1} \omega \wedge \bar{\partial} \varphi$$

$$+ \frac{n^{3}}{n^{2} - 1} \sum_{i=1}^{n-2} \frac{1}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi}^{i} \wedge \omega^{n-2-i} \wedge \sqrt{-1} \omega \wedge \partial \varphi$$

$$- \frac{n^{3}}{n^{2} - 1} \sum_{i=0}^{n-3} \frac{1}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi}^{i} \wedge \omega^{n-2-i} \wedge \sqrt{-1} \omega \wedge \bar{\partial} \varphi$$

$$+ \frac{n^{3}}{n^{2} - 1} \sum_{i=0}^{n-3} \frac{1}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi}^{i} \wedge \omega^{n-2-i} \wedge \sqrt{-1} \omega \wedge \partial \varphi$$

When $n = 3$, we have

$$J_{\omega}^{\text{AY}}(\varphi)$$

$$= -L_{\omega}^{M}(\varphi) + \frac{1}{\omega} \int_{X} \varphi \omega^{3}$$

$$+ 3 \frac{1}{8 \, 2V_{\omega}} \int_{X} \varphi \omega \wedge \sqrt{-1} \omega \wedge \bar{\partial} \varphi - \frac{3}{4} \left( 2 + \frac{3}{2} \right) \frac{1}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi} \wedge \sqrt{-1} \omega \wedge \bar{\partial} \varphi$$

$$- 3 \frac{1}{8 \, 2V_{\omega}} \int_{X} \varphi \omega \wedge \sqrt{-1} \omega \wedge \partial \varphi + \frac{3}{4} \left( 2 + \frac{3}{2} \right) \frac{1}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi} \wedge \sqrt{-1} \omega \wedge \partial \varphi$$

$$- 3 \frac{1}{8 \, 2V_{\omega}} \int_{X} \varphi \omega_{\varphi} \wedge \sqrt{-1} \omega \wedge \bar{\partial} \varphi + \frac{3}{8 \, 2V_{\omega}} \int_{X} \varphi \omega_{\varphi} \wedge \sqrt{-1} \omega \wedge \partial \varphi$$

$$- 27 \frac{1}{8 \, 2V_{\omega}} \int_{X} \varphi \omega \wedge \sqrt{-1} \omega \wedge \bar{\partial} \varphi + 27 \frac{1}{8 \, 2V_{\omega}} \int_{X} \varphi \omega \wedge \sqrt{-1} \omega \wedge \partial \varphi$$

$$= -L_{\omega}^{M}(\varphi) + \frac{1}{\omega} \int_{X} \varphi \omega^{3}$$

$$- 3 \frac{1}{2V_{\omega}} \int_{X} \varphi \omega \wedge \sqrt{-1} \omega \wedge \bar{\partial} \varphi + \frac{3}{2V_{\omega}} \int_{X} \varphi \omega \wedge \sqrt{-1} \omega \wedge \partial \varphi$$

$$- 3 \frac{2}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi} \wedge \sqrt{-1} \omega \wedge \bar{\partial} \varphi + \frac{3}{2V_{\omega}} \int_{X} \varphi \omega_{\varphi} \wedge \sqrt{-1} \omega \wedge \partial \varphi$$
When $n \geq 4$, we have

$$
J^{AV}_\omega(\varphi) = -L^M_\omega(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n
$$

$$
+ \frac{1}{2V_\omega} \sum_{i=1}^{n-3} \left[ \frac{n(i+1)}{n^2-1} - \frac{in}{n^2-1} - \frac{n^3}{n^2-1} \right] \int_X \varphi \omega^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi
$$

$$
+ \frac{n}{n^2-1} \frac{1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi
$$

$$
- \frac{n}{n+1} \left( n - 1 + \frac{n}{n-1} \right) \frac{1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi
$$

$$
+ \frac{1}{2V_\omega} \sum_{i=1}^{n-3} \left[ \frac{n(i+1)}{n^2-1} + \frac{in}{n^2-1} - \frac{n^3}{n^2-1} \right] \int_X \varphi \omega^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi
$$

$$
+ \frac{n}{n^3-1} \frac{1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi
$$

Using the identities

$$
\frac{n(i+1)}{n^2-1} - \frac{in}{n^2-1} - \frac{n^3}{n^2-1} = \frac{n-n^3}{n^2-1} = -n,
$$

$$
- \frac{n}{n+1} \left( n - 1 + \frac{n}{n-1} \right) - \frac{n(n-2)}{n^2-1} = \frac{-n(n-1)^2+n-n(n-2)}{n^2-1} = -n,
$$

the above expression can be simplified as

$$
J^{AV}_\omega(\varphi) = -L^M_\omega(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n
$$

$$
- \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi
$$

$$
+ \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi
$$

In summary,

**Proposition C.2.** If $n \geq 3$, one has

$$
J^{AV}_\omega(\varphi) = -L^M_\omega(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n
$$

$$
- \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi
$$

$$
+ \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \partial \varphi.$$
References

1. Chen, X. X., Tian, G., *Ricci flow on Kähler-Einstein surfaces*, Invent. Math. 147 (2002), no. 3, 487–544.
2. Li, Y., *Mabuchi and Aubin-Yau functionals over complex surfaces*, preprint, arXiv:1002.3411, 2010.
3. Li, Y., *Mabuchi and Aubin-Yau functionals over complex three-folds*, preprint, arXiv:1003.5307, 2010.
4. Mabuchi, T., *K-energy maps integrating Futaki invariants*, Tohoku Math. Journ., 38 (1986), 575–593.
5. Phong, D.H., Sturm, J., *Lectures on stability and constant scalar curvature*, preprint, arXiv:0801.4179v2, 2008.
6. Yau, S.-T., *Review of geometry and analysis*, Kodaira’s issue. Asian J. Math. 4 (2000), no. 1, 235–278

Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: yili@math.harvard.edu