On cubic hypersurfaces of dimensions 7 and 8

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Abstract

In this paper, we study cubic sevenfolds and cubic eightfolds in relation with the Cartan cubic, the $E_6$-invariant cubic in $\mathbb{P}^{26}$. We prove that a generic cubic sevenfold $X$ can be represented as a linear section of the Cartan cubic by finitely many ways. This allows one to define auto-equivalences of the non-commutative Calabi–Yau threefold associated to the generic cubic sevenfold $X$. Finally, we show that the generic eight-dimensional section of the Cartan cubic is rational.

1. Introduction

Manifolds of Calabi–Yau type were defined in [25] as compact complex manifolds of odd dimension whose middle-dimensional Hodge structure is similar to that of a Calabi–Yau threefold. Certain manifolds of Calabi–Yau type were used in order to construct mirrors of rigid Calabi–Yau threefolds, and in several respects they behave very much like Calabi–Yau threefolds. This applies in particular to the cubic sevenfold, which happens to be the mirror of the so-called $Z$-variety, a rigid Calabi–Yau threefold obtained as a desingularization of the quotient of a product of three Fermat cubic curves by the group $\mathbb{Z}_3 \times \mathbb{Z}_3$; see, for example, [6, 10, 37].

We observed in [25] that one can also construct manifolds of Calabi–Yau type as linear sections of the Cayley plane, and surprisingly, this construction turns out to be again related to cubic sevenfolds. The Cayley plane, which is homogeneous under $E_6$, is the simplest of the homogeneous spaces of exceptional type. It has many beautiful geometric properties, in particular it supports a plane projective geometry, whose ‘lines’ are eight-dimensional quadrics. Its minimal equivariant embedding is inside a projective space of dimension 26, in which its secant variety is an $E_6$-invariant cubic hypersurface that we call the Cartan cubic. The Cartan cubic is singular exactly along the Cayley plane, so that its general linear section of dimension at most 8 is smooth.

We prove in Proposition 2.2 that a general cubic sevenfold $X$ can always be described as a linear section of the Cartan cubic. Moreover, there is, up to isomorphism, only finitely many such Cartan representations. Using the plane projective geometry supported by the Cayley plane, we show that such a representation induces on $X$ a rank 9 vector bundle $\mathcal{E}_X$ with special properties: in particular, it is simple, infinitesimally rigid (Theorem 2.7), and arithmetically Cohen–Macaulay (aCM) (Proposition 2.5). Kuznetsov observed in [28] that the derived category of coherent sheaves on a smooth Fano hypersurface contains, under some hypothesis on the dimension and the degree, a full subcategory that is a Calabi–Yau category, of smaller dimension than the hypersurface. In particular, the derived category $D^b(X)$ of the smooth cubic sevenfold contains a full subcategory $\mathcal{A}_X$ which is Calabi–Yau of dimension 3 (a ‘non-commutative Calabi–Yau threefold’). It turns out that $\mathcal{E}_X(-1)$ and $\mathcal{E}_X(-2)$ both define objects in $\mathcal{A}_X$, and these objects are spherical. As shown by Seidel and Thomas, the

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corresponding spherical twists define auto-equivalences of \( A_X \) (Corollary 4.3). Recall that \( D^b(X) \) itself is poor in symmetries, \( X \) being Fano. Since \( A_X \) is Calabi–Yau, its structure and its symmetries are potentially much richer.

Cartan representations of cubic sevenfolds are very similar to Pfaffian representations of cubic fourfolds, which are linear sections of the secant cubic to the Grassmannian \( G(2,6) \). However, while a general cubic fourfold is not Pfaffian, we show that a general cubic sevenfold admits a Cartan representation. Starting from a Pfaffian representation, a classical construction consists in considering the orthogonal linear section of the dual Grassmannian: this is a K3 surface of genus 8, whose derived category can be embedded inside the derived category of the fourfold \([30]\). We expect something similar for a Cartan representation of a general cubic sevenfold \( X \). The orthogonal linear section to \( X \) of the dual Cayley plane is a Fano sevenfold \( Y \). What about cubics of higher dimensions? The determinantal cubics of dimensions at least 3 are singular, but one can nevertheless proceed by replacing determinantal by Pfaffian representations, that is, representations as Pfaffians of skew-symmetric matrices of order 6 with linear entries. It has been found essentially in \([15, 26, 35]\), and restated in this form in \([4]\), that the general cubic threefold \( Y \) has a family of Pfaffian representations which is five-dimensional and birational to the intermediate Jacobian of \( Y \). As for cubic fourfolds, it is well known that the general cubic fourfold is not Pfaffian. For cubics of higher dimensions, the situation is even worse, while the general Pfaffian cubic fourfold is still smooth (and rational!) the Pfaffian cubics of dimension greater than 5 are singular, and the question is which generalization of the Pfaffian, if any, can represent cubics of higher dimensions.

Looking back, we can observe that the spaces of symmetric \( 3 \times 3 \) matrices, matrices of order 3, and skew-symmetric matrices of order 6 are in fact the first three of a series of four Jordan algebras, the algebras \( J_3(\mathbb{A}) \) of \( 3 \times 3 \) Hermitian matrices over the composition algebras \( \mathbb{A} = \mathbb{R}, \mathbb{H}, \mathbb{O}, \mathbb{C} \).
C and H. Then, writing down a cubic form $F$ as a symmetric determinant, as a determinant, or as a Pfaffian is the same as giving a presentation of the cubic $F = 0$ as a linear section of the determinant cubic hypersurfaces in the first three Jordan algebras over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We refer the reader to the paper [4] for an excellent description of this connection: Corollary 6.5 for determinantal representations of cubic surfaces, Proposition 8.5 for Pfaffian representations of cubic threefolds and §9 for a discussion on the case of Pfaffian cubic foufolds.

The fourth Jordan algebra $J_3(\mathbb{O})$ is the space of $3 \times 3$ Hermitian matrices over the Cayley algebra $\mathbb{O}$ of octonions. The octonionic determinant is a special $E_6$-invariant cubic in $\mathbb{P}^{26}$: the Cartan cubic which is singular in codimension 9. Therefore, we can ask about representations of the smooth cubics $X$ as linear sections of the Cartan cubic up to dimension 8. We call them the Cartan representations of $X$.

As we show below, the general cubic sevenfold admits a Cartan representation, and in fact a finite number of inequivalent such representations. By this property, it looks very similar to the cubic curve and the cubic surface. But while the numbers of determinantal representations as above of cubic curves and of cubic surfaces are known since Hesse and Clebsch, finding the number of Cartan representations of the general cubic sevenfold remains an unsolved problem.

2.1. The Cayley plane and the Cartan cubic

Let $\mathbb{O}$ denote the normed algebra of real octonions (see, for example, [3]). This is an eight-dimensional real vector space with a non-commutative, non-associative product. Let $\mathbb{O}$ be its complexification. The space

$$J_3(\mathbb{O}) = \left\{ x = \begin{pmatrix} r & w & \bar{v} \\ \bar{w} & s & u \\ v & \bar{u} & t \end{pmatrix} : r, s, t \in \mathbb{C}, u, v, w \in \mathbb{O} \right\} \cong \mathbb{C}^{27}$$

of $\mathbb{O}$-Hermitian matrices of order 3 is traditionally known as the exceptional simple Jordan algebra for the Jordan multiplication $A \circ B = \frac{1}{2}(AB + BA)$. The automorphism group of this Jordan algebra is an algebraic group of type $F_4$, and preserves the determinant

$$\text{Det}(x) = rst - r|u|^2 - s|v|^2 - t|w|^2 + 2 \text{Re}(uvw),$$

where the term $\text{Re}(uvw)$ makes sense uniquely, despite the lack of associativity of $\mathbb{O}$. The subgroup of $\text{GL}(J_3(\mathbb{O}))$ consisting of automorphisms preserving this cubic polynomial is the simply connected group of type $E_6$. The Jordan algebra $J_3(\mathbb{O})$ and its dual are the minimal (non-trivial and irreducible) representations of this group. An equation of the invariant cubic hypersurface was once written down by Elie Cartan in terms of the configuration of the 27 lines on a smooth cubic surface. We will call it the Cartan cubic.

Orbits. The action of $E_6$ on the projectivization $\mathbb{P}J_3(\mathbb{O})$ has exactly three orbits: the complement of the determinantal hypersurface ($\text{Det} = 0$), the regular part of this hypersurface, and its singular part which is the closed $E_6$-orbit. These three orbits can be considered as the (projectivizations of the) sets of matrices of ranks 3, 2, and 1, respectively.

The closed orbit, corresponding to rank 1 matrices, is called the Cayley plane and denoted by $\mathbb{O}^{\mathbb{P}^2}$ (see [24]) (the reasons for this notation and terminology will soon be explained). It can be defined by the quadratic equation

$$x^2 = \text{trace}(x)x, \quad x \in J_3(\mathbb{O}).$$

It will be useful to note that a dense open subset of $\mathbb{O}^{\mathbb{P}^2}$ can be parameterized explicitly as the set of matrices of the form

$$\begin{pmatrix} 1 & w & \bar{v} \\ \bar{w} & |w|^2 & \bar{w}\bar{v} \\ v & v\bar{w} & |v|^2 \end{pmatrix}, \quad v, w \in \mathbb{O}. \quad (2.1)$$
The Cayley plane can also be identified with the quotient of $E_6$ by the maximal parabolic subgroup $P_3$ defined by the simple root $\alpha_3$ corresponding to the end of one of the long arms of the Dynkin diagram (we follow the notation of [9]). The semi-simple part of this maximal parabolic is isomorphic to Spin$_{10}$, as can be seen from the fact that, suppressing the node corresponding to the simple root $\alpha_1$, we get a Dynkin diagram of type $D_5$. (See [1] for generalized flag manifolds and homogeneous vector bundles.)

\[ \begin{array}{cccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\alpha_2 
\end{array} \]

The node $\alpha_6$ symmetric to $\alpha_1$ in the Dynkin diagram corresponds to the dual representation $J_3(\mathbb{O})^\vee$, which is exchanged with $J_3(\mathbb{O})$ by an outer automorphism of $E_6$. In particular, the orbits inside $\mathbb{P}J_3(\mathbb{O})$ and $\mathbb{P}J_3(\mathbb{O})^\vee$ are isomorphic (although non-equivariantly), and the closed orbit $E_6/P_6$ inside $\mathbb{P}J_3(\mathbb{O})^\vee$ is again a copy of the Cayley plane. We denote it by $\overline{\mathbb{P}^2}$ and call it the dual Cayley plane. There are two different dualities involved here.

Projective and octonionic dualities. The Cartan cubic inside $\mathbb{P}J_3(\mathbb{O})^\vee$ is the projectively dual variety to the Cayley plane. It can also be described as the secant variety to the dual Cayley plane. As a secant variety, it is degenerate, and therefore any point $p$ in the open orbit of the Cartan cubic defines a positive-dimensional entry locus in the dual Cayley plane (the trace on $\overline{\mathbb{P}^2}$ of the secants through $p$; see, for example, [33, 42] for a background and more details on Severi varieties). One can check that these entry-loci are eight-dimensional quadrics (see [33, § 2]), and that their family is parameterized by the Cayley plane itself. Explicitly, if $x$ is in $\mathbb{O}P^2$, then the orthogonal in $\mathbb{P}J_3(\mathbb{O})^\vee$ to its projective tangent space cuts the dual Cayley plane along one of these quadrics $Q_x$; see, for example, [23, Lemma 3.9].

The quadrics $Q_x \subset \overline{\mathbb{P}^2}$ for $x \in \mathbb{O}P^2$ can be considered as $\mathbb{O}$-lines (that is copies of $\mathbb{O}P^1$) inside the dual Cayley plane $\overline{\mathbb{P}^2}$. What we really get is a plane projective geometry: the two Cayley planes parameterize points and octonionic lines (respectively, lines and points) on them; see [20, 33, 40]. More precisely, the following two properties hold, which by symmetry take place also for the dual Cayley plane.

1. Let $x, x' \in \mathbb{O}P^2$ be two distinct points, such that the usual projective line $xx'$ is not contained in $\mathbb{O}P^2$. Then $Q_x \cap Q_{x'}$ is a unique reduced point on $\overline{\mathbb{P}^2}$; see, for example, [33, § 2, Corollaries 2.6 and 2.8, p. 22].

2. Let $x, x' \in \mathbb{O}P^2$ be such that the usual projective line $xx'$ is not contained in $\mathbb{O}P^2$. Then there is a unique $y$ in $\overline{\mathbb{P}^2}$ such that the $\mathbb{O}$-line $Q_y \subset \mathbb{O}P^2$ passes through $x$ and $x'$.

Remark 2.1. In coordinates $\left( \begin{array}{cccc} r & w & v & t \\ w & u & v & t \\ v & u & u & t \\ t & 0 & 0 & 0 \end{array} \right)$ on $\mathbb{O}P^2 \subset J_3(\mathbb{O})$ as above, one can suppose that $x = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$. Then the second point $x' \in \mathbb{O}P^2$ lies on a complex projective (a usual) line $L$ on the complex projective sixteenfold $\mathbb{O}P^2 \subset J_3(\mathbb{O}) = \mathbb{P}^{26}$, such that $L$ passes through $x$, if and only if $x' \in \mathbb{O}P^2$ lies on the tangent projective space to $\mathbb{O}P^2$ at $x$, that is, when $x'$ lies on $\mathbb{O}P^2$ and has coordinates $\left( \begin{array}{cccc} r & w & v & t \\ w & u & v & t \\ v & u & u & t \\ t & 0 & 0 & 0 \end{array} \right)$. One can see that the set of such points $x'$ is a cone over the Spinor tenfold $S_{10} = OG(5, 10)$ swept out by the lines $L$ on $\mathbb{O}P^2$ which pass through $x$: see [32]. If this is not the case, then one can choose coordinates such that $x$ is as above and $x' = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$. Then the unique eight-dimensional quadric on $\mathbb{O}P^2$ which passes through $x$ and $x'$ is $Q = \left\{ \left( \begin{array}{cccc} r & w & v & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) : rt - |v|^2 = 0 \right\} \subset \mathbb{P}^9(r : v : t)$. As discussed above, $Q = Q_y$ for a unique point $y \in \overline{\mathbb{P}^2}$.
The incidence variety

\[ \Sigma = \{(x, y) \subset \mathbb{OP}^2 \times \mathbb{OP}^2, \ x \in Q_y\} \]
equals \( E_6/P_{1,6} \), where \( P_{1,6} \subset E_6 \) is the non-maximal parabolic subgroup \( P_1 \cap P_6 \).

A birational map. The Cartan cubic is a homoloidal polynomial, in the sense that its derivatives define a birational transformation of the projective space [16]. More intrinsically, denote by \( \text{Det}(\ast, \ast, \ast) \) the polarization of the determinant (the unique symmetric trilinear map such that \( \text{Det}(x, x, x) = \text{Det}(x) \)). Then we have a birational quadratic map

\[ \text{PDet} : \mathbb{PJ}_3(\mathcal{O}) \to \mathbb{PJ}_3(\mathcal{O})^\vee, \]

\[ x \mapsto \text{Det}(x, x, \ast), \]

compatible with the \( E_6 \)-action. The polarized determinant \( \text{PDet} \) defines an isomorphism between the open orbits on both sides, but it contracts the Cartan cubic to the dual Cayley plane. Moreover, it blows up the Cayley plane itself, which is its indeterminacy locus, to the dual Cartan cubic in such a way that a point \( x \in \mathbb{OP}^2 \) is blown up into the linear span of the corresponding quadric \( Q_x \). Of course, the picture being symmetric, the inverse of \( \text{PDet} \) must be the quadratic map defined by the derivatives of the dual Cartan cubic.

In order to make explicit computations, it is convenient to identify \( J_3(\mathcal{O}) \) and \( J_3(\mathcal{O})^\vee \) (as vector spaces) through the following scalar product:

\[ \langle \begin{pmatrix} r & w & \bar{v} \\ \bar{w} & s & u \\ v & \bar{u} & t \end{pmatrix}, \begin{pmatrix} r' & u' & \bar{v}' \\ \bar{w}' & s' & u' \\ v' & \bar{u}' & t' \end{pmatrix} \rangle = rr' + ss' + tt' - 2\text{Re}(\bar{u}u' + \bar{v}v' + \bar{w}w'). \]

With this convention, the map \( \text{PDet} \) is given explicitly by

\[ x = \begin{pmatrix} r & w & \bar{v} \\ \bar{w} & s & u \\ v & \bar{u} & t \end{pmatrix} \mapsto \begin{pmatrix} st - |u|^2 & tw - \bar{v}u & uw - s\bar{v} \\ tw - \bar{v}u & rt - |v|^2 & ru - \bar{w}v \\ \bar{u}w - sv & r\bar{u} - vw & rs - |w|^2 \end{pmatrix}. \]

(2.2)

The main interest of this presentation is that the equation of the dual Cartan cubic is given exactly by the same formula as \( \text{Det} \). In particular, the above expression of \( \text{PDet} \), which is a kind of comatrix map, defines an involutive quadratic Cremona transformation of \( \mathbb{P}^{26} \).

2.2. Cartan representations

Given a cubic hypersurface \( X \) of dimension \( d \), we define a Cartan representation of \( X \) as a linear section of the Cartan cubic isomorphic to \( X \):

\[ X \simeq \mathcal{C} \cap \mathbb{PL} \quad \text{for} \quad \mathbb{P}^{d+1} \simeq \mathbb{PL} \subset \mathbb{PJ}_3(\mathcal{O}). \]

Note that the Cartan cubic being singular in codimension 9, a smooth cubic hypersurface of dimension bigger than 8 cannot have any Cartan representation. The main result of this section is that, on the contrary, a general cubic hypersurface of dimension smaller than 8 always admits a Cartan representation. (The boundary case of eight-dimensional sections will be considered later.) The following proposition is a more precise statement.

**Proposition 2.2.** A generic cubic sevenfold admits a Cartan representation. Moreover, the number of Cartan representations is finite.

**Proof.** Let \( \mathcal{M}_3^7 \) denote the moduli space of cubic sevenfolds. What we need to prove is that the rational map

\[ \Psi : G(9, J_3(\mathcal{O}))/E_6 \dashrightarrow \mathcal{M}_3^7, \]
obtained by sending $L \in G(9, J_3(\mathbb{O}))$ to the isomorphism class of the cubic sevenfold $X_L = C \cap P^L$, is generically finite. The two moduli spaces have the same dimension (respectively, $\dim G(9, J_3(\mathbb{O})) - \dim E_6 = 9 \times 18 - 78 = 84$ and $\dim |\mathcal{O}_{P^2}(3)| - \dim \text{PGL}_9 = 164 - 80 = 84$; the coincidence was first noted in [25]). So we just need to check that $\Psi$ is generically étale. We will find a special vector space $L$ at which the differential of $\Psi$ is surjective, as will follow from the surjectivity of the map

$$\varphi \in \text{Hom}(L, J_3(\mathbb{O})) \mapsto P_\varphi(x) = \text{Det}(x, x, \varphi(x)) \in S^3L^\vee.$$ 

We will choose $L$ as the image of a map

$$(r, y) \mapsto \begin{pmatrix} r & \alpha(y) & \bar{y} \\ \bar{\alpha}(y) & r & \beta(y) \\ y & \bar{\beta}(y) & r \end{pmatrix},$$

where $\alpha, \beta : \mathbb{O} \to \mathbb{O}$ are linear endomorphisms (of $\mathbb{O}$ considered as an eight-dimensional vector space).

Consider the morphism $\varphi$ from $L$ to $J_3(\mathbb{O})$ defined by

$$(r, y) \mapsto \begin{pmatrix} a & w & \bar{v} \\ \bar{w} & b & u \\ v & \bar{u} & c \end{pmatrix},$$

for some linear forms $a, b, c$ and linear functional $u, v, w$. The corresponding polynomial $P_\varphi(x)$ is

$$P_\varphi(x) = r^2(a + b + c) - a|\alpha(y)|^2 - b|y|^2 - c|\beta(y)|^2 - 2r\text{Re}(u\beta(y) + vy + w\alpha(y)) + 2\text{Re}(w\beta(y)y + \beta(y)v\alpha(y) + u\alpha(y)).$$

So what we need to prove is that the 27 following quadrics in $(r, y)$ generate $H^0(L, \mathcal{O}_L(3))$; the three expressions in the second column have values in $\mathbb{O}$; therefore, they count each for eight quadrics:

$$r^2 - |\alpha(y)|^2, \quad y\alpha(y) - r\beta(y),$$
$$r^2 - |y|^2, \quad \alpha(y)\beta(y) - ry,$$
$$r^2 - |\beta(y)|^2, \quad \beta(y)y - r\alpha(y).$$

It is easy to see that a sufficient condition for this to be true is that the sixteen quadrics $y\alpha(y)$ and $\beta(y)y$ generate $H^0(\mathbb{O}P^2, \mathcal{O}_{\mathbb{O}P^2}(3))$.

In order to complete the proof, we thus only need to exhibit $\alpha(y)$ and $\beta(y)$ with this property. We use the traditional basis $(e_0 = 1, e_1, \ldots, e_7)$ of $\mathbb{O}$ in which the multiplication table is given by the Fano plane (see [3]). We will choose $\alpha(y)$ and $\beta(y)$ of the form

$$\alpha(y) = \sum_i s_i y_{\sigma(i)}e_i, \quad \beta(y) = \sum_i t_i y_{\tau(i)}e_i,$$

for some coefficients $s_i, t_i$ and permutations $\sigma, \tau$. Let $\sigma(i) = i + 1$ and $\beta(i) = i + 3$ (using the cyclic order on $0, \ldots, 7$), and $s = t = (1, 2, -1, 3, 1, -1, 2, 1)$. A computation with MACAULAY2 shows that the lemma does indeed hold in that case.

**Question 2.3.** What is the number of Cartan representations of a generic cubic sevenfold? (In other words, what is the degree of $\Psi$?) Is it equal to 1, in which case we would get a canonical form for cubic sevenfolds? Or bigger, in which case the derived category would have many interesting symmetries, as we will see later on?
Remark 2.4. In [25], we have shown that the Cayley plane belongs to a series of four homogeneous spaces for which we expect very similar phenomena. One of these is the Grassmannian \( G(2, 10) \), and the statement analogous to Proposition 2.2 in that case is the fact that a general quintic threefold (the archetypal Calabi–Yau) admits a finite number of Pfaffian representations. This was proved by Beauville and Schreyer but the precise number of such representations is not known (although it may be computable, being a Donaldson–Thomas invariant; see [4, (8,10)]). The two other cases are spinor varieties, for which the corresponding statements have not been checked yet.

2.3. A special vector bundle

Pfaffian representations of quintic threefolds are in correspondence with certain special rank 2 vector bundles, obtained as the restriction of the kernel bundle on the regular part of the Pfaffian cubic which parameterizes skew-symmetric forms of corank 2 [4]. These bundles belong to the extensively studied class of aCM, a bundle \( \mathcal{F} \) on an irreducible variety \( Z \) being aCM if
\[
H^i(Z, \mathcal{F}(\ell)) = 0 \quad \text{for} \quad 0 < i < \dim Z, \ \ell \in \mathbb{Z}.
\]

In this section, we will attach to any Cartan representation of a cubic sevenfold a very special vector bundle on the cubic that we believe to be the correct generalization of these rank 2 bundles attached to Pfaffian representations. The construction is the following. On the regular part of the Cartan cubic, there is a natural quadric bundle \( \mathcal{Q} \) of relative dimension 8, whose fiber at \( p \) is the corresponding entry locus \( \mathcal{Q}_p \). The linear span \( \langle \mathcal{Q}_p \rangle \) contains \( p \), and since \( \mathcal{Q}_p \) is always smooth, the polar to \( p \) with respect to \( \mathcal{Q}_p \) is a hyperplane \( \mathbb{P}(\mathcal{E}_p^\vee) \subset \langle \mathcal{Q}_p \rangle \).

This defines a vector bundle \( \mathcal{E}^\vee \) of rank 9, which is a sub-bundle of the trivial vector bundle with fiber \( J_3(\mathcal{O}) \). We will denote by \( \mathcal{K} \) the quotient bundle, of rank 18.

For a Cartan representation \( X = \mathcal{C} \cap \mathbb{P}L \) of a cubic sevenfold \( X \), we simply denote by \( \mathcal{E}_X \) and \( \mathcal{K}_X \) the restrictions of \( \mathcal{E} \) and \( \mathcal{K} \) to \( X \), respectively. The bundle \( \mathcal{E}_X \) is our main object of interest.

Proposition 2.5. The bundles \( \mathcal{E}_X \) and \( \mathcal{K} \) have the following properties:

1. \( \mathcal{E}_X \) is aCM;
2. its Hilbert polynomial is \( P_{\mathcal{E}_X}(k) = 27 \binom{k+7}{7} \);
3. the space of its global sections is \( H^0(X, \mathcal{E}_X) = J_3(\mathcal{O}) \);
4. \( \mathcal{E}_X^\vee = \mathcal{E}_X(-2) \) and \( \mathcal{K}^\vee_X = \mathcal{K}_X(-1) \);
5. \( \chi(\text{End}(\mathcal{E}_X)) = 0 \).

Proof. The polarization of Det defines over \( \mathbb{P}J_3(\mathcal{O})^\vee \) a map
\[
J_3(\mathcal{O})^\vee \otimes \mathcal{O}_{\mathbb{P}J_3(\mathcal{O})^\vee}(-1) \overset{\delta}{\longrightarrow} J_3(\mathcal{O}) \otimes \mathcal{O}_{\mathbb{P}J_3(\mathcal{O})^\vee}.
\]
This map is invertible outside \( \mathcal{C} \); in particular it is everywhere injective as a map of sheaves. We claim that its cokernel \( \mathcal{F} \) has constant rank 9 over the smooth part \( \mathcal{C}_0 \) of \( \mathcal{C} \) (that is, outside the Cayley plane). By homogeneity, we only need to check it at one point \( p \) of \( \mathcal{C}_0 \); we choose
\[
p = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
The entry locus of \( p \) is the quadric
\[
Q_p = \left\{ \begin{pmatrix} r & z & 0 \\ z & s & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad r, s \in \mathbb{C}, z \in \mathbb{O}, \quad rs - |z|^2 = 0 \right\}.
\]

An easy computation shows that the image of \( \delta_p \) is generated by the linear forms \( r + s, t, x, y \). The dual of its cokernel can then be identified with the kernel of these linear forms, that is,
\[
\mathcal{F}^\vee_p = \left\{ \begin{pmatrix} r & z & 0 \\ z & -r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad r \in \mathbb{C}, z \in \mathbb{O} \right\}.
\]

But this is clearly the linear space polar to \( p \) with respect to \( Q_p \), which yields an identification \( \mathcal{F}^\vee \simeq \mathcal{E}^\vee \) over \( \mathcal{O}_0 \). In particular, we deduce over the linear span \( \mathbb{P}_X \) of \( X \) an exact sequence of sheaves
\[
0 \longrightarrow J_3(\mathcal{O})^\vee \otimes \mathcal{O}_{\mathbb{P}_X}(-1) \longrightarrow J_3(\mathcal{O}) \otimes \mathcal{O}_{\mathbb{P}_X} \longrightarrow \mathcal{E}_X \longrightarrow 0.
\]

This implies that \( \mathcal{E}_X \) is aCM, and that its space of global sections is \( J_3(\mathcal{O}) \). Also, by the above exact sequence, one directly computes the Hilbert polynomial of \( \mathcal{E}_X \). The equality \( \mathcal{E}_X^\vee = \mathcal{E}_X(-2) \) follows from the fact that the quadric bundle \( Q \) is everywhere non-degenerate over \( \mathcal{O}_0 \). Similarly, we get \( \mathcal{K}_X^\vee = \mathcal{K}_X(1) \).

Now we can restrict the previous sequence to \( X \). A standard computation yields
\[
\text{Tor}_1^{\mathcal{O}_X}(\mathcal{E}_X, \mathcal{O}_X) = \mathcal{E}_X^\vee(-1) = \mathcal{E}_X(-3). \quad \text{Therefore, we get}
\]
\[
0 \longrightarrow \mathcal{E}_X(-3) \longrightarrow J_3(\mathcal{O})^\vee \otimes \mathcal{O}_X(-1) \longrightarrow J_3(\mathcal{O}) \otimes \mathcal{O}_X \longrightarrow \mathcal{E}_X \longrightarrow 0. \quad (2.3)
\]

This exact sequence is self-dual up to a twist. We can tensor it by \( \mathcal{E}_X^\vee \) to deduce the Hilbert polynomial of \( \text{End}(\mathcal{E}_X) \). Indeed, this polynomial verifies the two equations
\[
P_{\text{End}(\mathcal{E}_X)}(k) - P_{\text{End}(\mathcal{E}_X)}(k - 3) = 27(P_{\mathcal{E}_X}(k - 2) - P_{\mathcal{E}_X}(k - 3)),
\]
\[
P_{\text{End}(\mathcal{E}_X)}(k) = -P_{\text{End}(\mathcal{E}_X)}(-k - 6).
\]

The first equation follows from the exact sequence above, and the second one from Serre duality, since \( \omega_X = \mathcal{O}_X(-6) \). There is a unique solution to these two equations, given by the following formula:
\[
P_{\text{End}(\mathcal{E}_X)}(k) = 3^5 \binom{k + 6}{7} - 3^4 \binom{k + 5}{5} + 3^3 \binom{k + 4}{3} - 3^2 \binom{k + 3}{1}.
\]

In particular, \( \chi(\text{End}(\mathcal{E}_X)) = P_{\text{End}(\mathcal{E}_X)}(0) = 0. \)

**Remark 2.6.** The characteristic classes of \( \mathcal{E}_X \) can easily be computed from the sequence (2.3), for example, the Chern character
\[
\text{ch}(\mathcal{E}_X) = 27 \frac{1 - \exp(-h)}{1 - \exp(-3h)},
\]
where \( h \) is the hyperplane class on \( X \). One deduces the Chern classes of the self-dual bundle \( \mathcal{E}_X(-1) \):
\[
c(\mathcal{E}_X(-1)) = 1 - 3h^2 + 9h^4 - 39h^6.
\]

### 2.4. Rigidity

Recall that, with our conventions, the Cayley plane \( \mathbb{O} \mathbb{P}^2 \) is \( E_6/P_1 \). In particular, the category of \( E_6 \)-homogeneous vector bundles on \( \mathbb{O} \mathbb{P}^2 \) is equivalent to the category of \( P_1 \)-modules [1, Chapter 4]. But recall that the semi-simple part of \( P_1 \) is a copy of \( \text{Spin}_{10} \), whose minimal non-trivial
representations are the ten-dimensional vector representation and the two sixteen-dimensional half-spin representations. The corresponding homogeneous vector bundles are, up to twists, the tangent bundle $T$ and cotangent bundle $\Omega$ for the latter two, and the normal bundle for the former. We will denote by $S$ the rank 10 irreducible homogeneous vector bundle on $\mathbb{OP}^2$ such that $H^0(\mathbb{OP}^2, S) = J_3(\zeta)$, and call $S$ the spinor bundle on $\mathbb{OP}^2$.

The normal bundle is then $N = S(1)$. This implies that the projectivization of the bundle $S^\vee$ over $\mathbb{OP}^2$ is a desingularization of the dual Cayley cubic: the line bundle $O_S(1)$ is generated by its global sections, whose space is isomorphic with $H^0(\mathbb{OP}^2, S) = J_3(\zeta)$. We have a diagram

\[
\begin{array}{ccc}
\mathbb{OP}^2 & \xrightarrow{\gamma} & C \supset \mathbb{OP}^2 \\
\pi \downarrow & & \downarrow \pi \\
\mathbb{P}(S^\vee) & & \\
\end{array}
\]

where the morphism $\gamma$, which is defined by the linear system $|O_S(1)|$, is an isomorphism outside the dual Cayley plane $\overline{\mathbb{OP}^2}$. (We use the same notation for the Cartan cubic in the dual space.) Moreover, the pre-image of $\overline{\mathbb{OP}^2}$ is the incidence variety $\Sigma = E_6/F_{1,6}$, embedded as a divisor in $\mathbb{P}(S^\vee)$. Note that the rational map $\pi \circ \gamma^{-1} : C \to \overline{\mathbb{OP}^2}$ is the restriction of the comatrix map $\text{PDet}$. Since $X$ does not meet $\overline{\mathbb{OP}^2}$, $\gamma$ is an isomorphism over $X$, and $\gamma^{-1}(X)$ is the complete intersection, in $\mathbb{P}(S^\vee)$, of eighteen general sections of $O_S(1)$. Note that the restriction of $O_S(1)$ coincides with the pull-back of $O_X(1)$ by $\gamma$. Let us denote by $\pi^*S_X$ the restriction of $\pi^*S$ to $\gamma^{-1}(X) \simeq X$. The tautological bundle $O_X(-1)$ is a sub-bundle of $\pi^*S_X$. Moreover, $E_X^\vee$ is defined as the orthogonal to $O_X(-1)$ in $S_X^\vee$, with respect to its natural quadratic form, corresponding to the family of $O$-lines parameterized by $X$. Since $X$ does not meet the Cayley plane, $O_X(-1)$ does not meet these $O$-lines, and we therefore get the direct sum decomposition

$$\pi^*S_X^\vee = E_X^\vee \oplus O_X(-1).$$

This relation is important because it will allow us to compute the cohomology groups of $\text{End}(E_X)$, essentially by reducing to that of $\text{End}(S_X)$. The point is that $S_X$ is the restriction of the homogeneous vector bundle $S$, while $X \simeq \gamma^{-1}(X)$ is a complete intersection in $\mathbb{P}(S^\vee)$. We will therefore be able to use Koszul complexes in order to reduce to the cohomology of $\text{End}(S)$, which we control owing to Bott’s theorem. We can prove the following theorem.

**Theorem 2.7.** The vector bundle $E_X$ is simple and infinitesimally rigid. More precisely,

$$h^i(X, \text{End}(E_X)) = \delta_{i,0} + \delta_{i,3}.$$  

The proof of this statement, being rather technical, is deferred to the Appendix. Note that it is not true that $h^i(X, \text{End}(E_X)) = 0$ for $i > 0$. The theorem nicely reflects the fact that $E_X(-1)$ belongs to a three-dimensional Calabi–Yau category $\mathcal{A}_X$; see Proposition 4.2. Moreover, it shows that is defines a spherical object in this category.

The theorem is another indication that the bundle $E_X$ is really special: it cannot be deformed, at least infinitesimally. What we expect is that, a cubic sevenfold being given, each of its Cartan representation should correspond to such a very special vector bundle. We would like to be able to solve the following reconstruction problem: given a smooth cubic sevenfold $X$, and a rank
9 vector bundle $\mathcal{E}$ on $X$, having all the properties we have just established for the bundle $\mathcal{E}_X$ (and maybe some others), does it necessarily come from a Cartan representation of $X$? And if this is the case, how can we reconstruct effectively this representation? The global sections of $\mathcal{E}$ provide what should be a copy of $J_3(\mathcal{O})$, and one would like to reconstruct the cubic determinant just from the bundle. Unfortunately, we have not been able to do that starting from $\mathcal{E}_X$.

3. The Tregub–Takeuchi birationality for cubic sevenfolds

By the results of the previous section, the general cubic sevenfold $X$ has a finite number of Cartan representations, where a Cartan representation is the same as a presentation of $X$ as a linear section $X = \mathcal{C} \cap \mathbb{P}L$ of the Cartan cubic $\mathcal{C} \subset \mathbb{P}J_3(\mathcal{O})$ with a linear subspace $\mathbb{P}L \cong \mathbb{P}^8$. In the dual space $\mathbb{P}J_3(\mathcal{O})^\vee$, the linear forms defining $\mathbb{P}L$ span a subspace $\mathbb{P}L^\perp$ which intersects the dual Cayley plane $\mathbb{O}\mathbb{P}^2 \subset \mathbb{P}J_3(\mathcal{O})^\vee$ along a sevenfold $Y$, the orthogonal linear section of $X$ with respect to its Cartan representation. In this section, we prove that $X$ and $Y$ are birational to each other by constructing a birationality that is an analog of the Tregub–Takeuchi birationality between the cubic threefold and its general orthogonal linear section in $G(2,6)$.

The analogy is the following. A Pfaffian representation of the cubic threefold $X'$ is the same as a presentation of $X'$ as a linear section of the Pfaffian cubic $Pf \subset \mathbb{P}J_3(\mathbb{H})$ with a subspace $\mathbb{P}L \cong \mathbb{P}^4$, the projective Jordan algebra $\mathbb{J}_3(\mathbb{H})$ being identified with the projective space $\mathbb{P}(\wedge^2 V)$, where $V = \mathbb{C}^6$. The linear forms defining $\mathbb{P}L$ span a subspace $\mathbb{P}L^\perp$ in $\mathbb{P}(\wedge^2 V^\vee)$ which intersects the quaternionic plane $\mathbb{H}\mathbb{P}^2 = G(2,6) \subset \mathbb{P}(\wedge^2 V^\vee)$ along a Fano threefold $Y'$ of degree 14, the orthogonal section of $X'$. It is known since Fano that $Y'$ are $X'$ are birational. The two types of birationalities between $X'$ and $Y'$, of Fano–Iskovskikh and of Tregub–Takeuchi, are related to curves on the cubic threefold $X'$ that are hyperplane sections of surfaces with OADP, correspondingly del Pezzo surfaces of degree 5 and rational quartic scrolls; see [2, 11, 39, 41]. Especially, the Tregub–Takeuchi birationality, or more precisely its inverse $Y' \to X'$, starts from a point $p \in Y'$, and after a blow-up of $p$ and a flop, ends with a contraction of a divisor to a rational normal quartic curve in $Y'$.

We will describe the analog of the inverse Tregub–Takeuchi birationality between the cubic sevenfold $X$ and its orthogonal sevenfold $Y$ defined by a Cartan representation of $X$. It is an analog because the construction described in the proof of Proposition 3.5 applied to the cubic threefold $X'$ and its dual $Y'$ as above and, after replacing $J_3(\mathcal{O})$ by $J_3(\mathbb{H})$, gives the Tregub–Takeuchi birationality as described, for example, in [39, 41]. Just as in the three-dimensional case, the inverse birationality $Y \to X$ starts with a blow-up of a point $p \in Y$. In the seven-dimensional case, the birationality ends with a contraction of a divisor onto a prime Fano threefold $Z \subset X$ of degree 12, which is a hyperplane section of a fourfold with OADP. The connection with varieties with OADP inside the Pfaffian cubic fourfolds and Cartan cubic eightfolds is commented in §5. One geometric explanation why the rational quartic curve $C \subset X$ and the Fano threefold $Z \subset X$ of degree 12 appear as indeterminacy loci of the birationalities related to points on their dual varieties is that the projective tangent space to a point $p$ on $\mathbb{H}\mathbb{P}^2 = G(2,6)$ intersects on $G(2,6)$ a cone over $\mathbb{P}^1 \times \mathbb{P}^3$, whose general linear section of dimension is a rational normal quartic, while the projective tangent space to $\mathbb{O}\mathbb{P}^2$ at $p$ intersects on $\mathbb{O}\mathbb{P}^2$ a cone over the spinor variety $S_{10} = OG(5,10)$, whose general three-dimensional linear section is a prime Fano threefold of degree 12; see [31].

3.1. The Tregub–Takeuchi birationality

First, we recall the classical description of the Tregub–Takeuchi birationality; see [39]. Let $Y = Y_{14}$ be a smooth prime Fano threefold of degree 14 and let $y_0 \in Y$ be a point through
which do not pass any of the one-dimensional family of lines on $Y$. The linear system $|I^6_0(2)|$ defines a birationality $\psi_T : Y \to X$ with a cubic threefold $X$; this is the Tregub–Takeuchi birationality. It can be decomposed as

$$
\begin{array}{c}
Y' \xrightarrow{\sigma} \ Y^+ \\
Y \xrightarrow{p} \ Y \\
\end{array}
\quad
\begin{array}{c}
\sigma \\
p \\
\end{array}
\quad
\begin{array}{c}
Y^+ \\
X \\
\end{array}
$$

where $p : Y \dashrightarrow \ Y$ is the double projection through $y_0$, $\sigma : Y' \to Y$ is the blow-up of $y_0$, $Y' \dashrightarrow Y^+$ is the flop over $\ Y$ of the finitely many conics through $y_0$, and $Y^+ \to X$ is the contraction of an irreducible divisor $N^+$ of $Y^+$ onto a rational normal quartic curve $\Gamma$ on $X$ (the blow-up of $\Gamma$). There is on $Y$ a $\mathbb{P}^1$-family of 1-cycles whose general member is a rational quintic curve $C_t$ with a double point at $y_0$, and the proper image on $Y^+$ of the general $C_t$ is the general fiber of the $\mathbb{P}^1$-bundle $N^+ \to \Gamma$.

We have already mentioned the fact that

$$G(2, 6) \simeq \mathbb{H}P^2 \subset \mathbb{P}J_3(\mathbb{H}) \simeq \mathbb{P}(\wedge^2 \mathbb{C}^6).$$

The corresponding ‘plane projective geometry’ is defined by the family $Q_z$ of $\mathbb{H}$-lines, which are copies of $G(2, 4) \simeq \mathbb{Q}^4$ parameterized by $z \in G(4, 6)$, the dual Grassmannian. In fact, $z$ defines a four-dimensional subspace $V_z$ of $V = \mathbb{C}^6$, and $Q_z \subset G(2, 6)$ is just the sub-Grassmannian of planes $P_y$ contained in $V_z$.

As Mukai has shown in [36], the general prime Fano threefold $Y$ can be realized as a linear section $Y = G(2, 6) \cap \mathbb{P}L^\perp$, with $\mathbb{P}L^\perp \simeq \mathbb{P}^9$. The intersection of the Pfaffian cubic in the dual projective space, with $\mathbb{P}L \simeq \mathbb{P}^4$, is a smooth cubic threefold, and it follows from the results of [26] that it must coincide with the target $X$ of the Tregub–Takeuchi birationality.

On the other hand, we can construct a birational map $\psi : Y \dashrightarrow X$ as follows. Take a general point $y$ on $Y$. Then $Q_y \cap Q_{y_0}$ is a single point $z$ given by $V_z = P_y + P_{y_0}$ (provided these planes are transverse). Moreover, the tangent space $T_z$ to the Grassmannian $G(4, 6)$ at $z$ meets $\mathbb{P}L$ at a unique point: this is $\psi(y)$.

**Proposition 3.1.** The birational maps $\psi$ and $\psi_T$ are the same.

**Proof.** First observe that the rational map $y \mapsto z$ defined by $V_z = P_y + P_y$, is the double projection from $y_0$.

Then we study the $\mathbb{P}^1$-family of quintic rational curves $C_t \subset Y$, $t \in \mathbb{P}^1$, passing doubly through $y_0$. Let $T_{y_0} = \mathbb{P}(P_{y_0} \cap \mathbb{C}^6)$ be the projective tangent space to $G(2, 6)$ at $y_0$ and let $\mathbb{P}^1_{y_0} = \mathbb{P}U_4 = \mathbb{P}((P_{y_0} \cap \mathbb{C}^6) \cap L^\perp)$ be the projective tangent space to $Y$ at $y_0$. These quintics are obtained as follows. For any hyperplane $U_t \subset \mathbb{C}^6$, the intersection $C = Y \cap G(2, U_5)$ is a quintic curve of arithmetic genus $p_a(C) = 1$, since $G(2, U_5)$ has degree 5 and index 5. In this $\mathbb{P}^5$-family, those having a double point at $y_0$ are the rational curves of the $\mathbb{P}^1$-family $C_t$. They correspond to those 5-spaces $U_t$, such that the projective tangent spaces to $G(2, U_t)$ and to $Y$ at $y_0$ intersect each other along a plane (the tangent space to $C_t$ at the node $y_0$), that is,

$$\dim(U_4 \cap \wedge^2 U_t) = 3.$$

**Lemma 3.2.** Let $C_t$ be a rational quintic on $Y$ passing doubly through $y_0$. Then the birationality $\psi : Y \dashrightarrow X$ contracts $C_t$. 

\qed
Proof. Since the quintic curve $C_t$ has a double point at $y_0$, the double projection $\bar{p}$ sends $C_t$ to a line $\ell_t \subset \bar{Y}$. A point $y_s \in C_t$, $y_s \neq y_0$ is mapped by $\bar{p}$ to a point $z_s = z(y_s, y_0)$ on the line $\ell_t \subset \bar{Y} \subset Q_{y_0} \subset G(4, 6)$. We will then obtain $\psi(y_s)$ as the intersection point of the projective tangent space $T_{z_s}$ to $G(4, 6)$ at $z_s$, with $PL \simeq \mathbb{P}^5$, and we must check that this point does not depend on $s$. For this, we observe that

$$\bigcap_{z \in \ell_t} T_z := T_{\ell_t}$$

is five-dimensional. Moreover, we claim that $T_{\ell_t}$ is orthogonal to the linear span of $C_t$. This will imply the claim, since $(C_t)_{\perp}$ is a $\mathbb{P}^9$ inside which $PL \simeq \mathbb{P}^5$ and $T_{\ell_t} \simeq \mathbb{P}^5$ will meet, generically, at a unique point $x_t$. But then $\{\psi(y_s)\} = T_{z_s} \cap PL \subset T_{\ell_t} \cap PL = \{x_t\}$, hence $\psi$ contracts $C_t$ to the point $x_t$.

There remains to prove the previous claim. It is enough to check that any $y_s \in C_t$ is orthogonal to $T_{z_s}$. Recall that if $y_s$ represents a plane $P_{y_s} \subset \mathbb{C}^6$, transverse to $P_{y_0}$, then $z_s$ represents the four-space $P_{y_s} + P_{y_0}$. But then the affine tangent space $T_{z_s} = \wedge^3(P_{y_s} + P_{y_0}) \wedge \mathbb{C}^6 \subset P_{y_s} \wedge (\wedge^3 \mathbb{C}^6)$, and therefore $\wedge^2 P_{y_s} \wedge T_{z_s} = 0$, which is the required orthogonality condition.

We can finally conclude the proof of the proposition. The two birational maps $\psi$ and $\psi_t$ induce two birational morphisms $\psi^+$ and $\psi^+_t$ from $Y^+$ to $X$. These two morphisms contract the divisor $N^+$, and the fibers of their restrictions to $N^+$ are the same. Since the relative Picard number is 1, $\psi^+$ and $\psi^+_t$ cannot contract any other divisor. Then $\psi^+ \circ (\psi^+_t)^{-1}$ is an isomorphism between $X - \psi^+_t(N^+)$ and $X - \psi^+(N^+)$, and extends to $X$, since it is normal and $\psi^+_t(N^+)$, $\psi^+(N^+)$ have codimension greater than 1. Hence, it must extend to an automorphism of $X$.

3.2. The orthogonal linear sections of the cubic sevenfold on the Cayley plane

In [25], we considered the Cayley plane and its linear sections as candidates for being Fano manifolds of Calabi–Yau type. Starting from a Cartan representation of a general cubic sevenfold, $X = C \cap PL$, we consider the orthogonal section $Y = \overline{CP^2} \cap PL^\perp$ of the dual Cayley plane.

Proposition 3.3. The variety $Y$ is a Fano manifold of Calabi–Yau type, of dimension 7 and index 3.

Proof. The Cayley plane has dimension 16 and index 12, hence a general section of codimension 9 has dimension 7 and index 3. The fact that it is of Calabi–Yau type follows from [25, Proposition 4.5] and Proposition 2.2.

Remark 3.4. The situation here is very similar to what happens for cubic fourfolds. Recall that the Pfaffian cubic fourfolds are those that can be obtained as linear sections of the Pfaffian cubic in $\mathbb{P}^{14}$. The Pfaffian cubic can be defined as the secant variety to $G(2, 6)$ (or the projective dual variety to the dual Grassmannian $G(4, 6)$), and that $G(2, 6)$ is nothing else but the projective plane $\mathbb{H}P^2$ over the quaternions.

Given such a smooth four-dimensional section of the Pfaffian cubic, the orthogonal section of the dual Grassmannian is now a smooth K3 surface, and the Hodge structure of the cubic fourfold is indeed ‘of K3 type’. The main difference with cubic sevenfolds is that the general cubic fourfold is not Pfaffian, the Pfaffian ones form a codimension 1 family in the moduli
space. Nevertheless, the Hodge structure of a cubic fourfold is always of K3 type, and this allows one in particular to associate to them complete families of symplectic fourfolds [5, 27].

3.3. The Tregub–Takeuchi birationalities for the cubic sevenfold

We will use the basic properties of the incidence geometry on the Cayley plane in order to prove the following proposition.

**Proposition 3.5.** The general cubic sevenfold \( X = \mathcal{C} \cap \mathbb{P}L \) and the dual section \( Y = \mathbb{P}^2 \cap \mathbb{P}L^\perp \) are birationally equivalent.

**Remark 3.6.** Since \( X \) has index 6 but \( Y \) has index 3, they are not isomorphic. Their Hodge numbers coincide in middle dimension, and in fact they do coincide in general: this follows from the Lefschetz hyperplane theorem since the Cayley plane and the projective space have the same Betti and Hodge numbers up to degree 6.

**Proof.** We start by fixing a general point \( y_0 \in Y \subset \mathbb{OP}^2 \), and let \( Q_{y_0} \subset \mathbb{OP}^2 \) be its corresponding quadric.

Let \( y \in Y \) be general. The corresponding quadric \( Q_y \subset \mathbb{OP}^2 \) meets \( Q_{y_0} \) at a unique point \( z = z(y) \in \mathbb{OP}^2 \). The tangent space \( T_z = T_zQ_y \subset \mathbb{OP}^2 \subset \mathbb{OP}J(\mathcal{O}) \) is contained in the Cartan cubic \( \mathcal{C} \), and we claim that inside the projective 26-space \( \mathbb{OP}J(\mathcal{O}) = \mathbb{P}^{26} \), the projective 16-space \( T_z = T_z(y) \) meets the projective 8-space \( \mathbb{P}L \) at a unique point \( \psi(y) \).

First, because \( y \) and \( y_0 \) belong to \( \mathbb{P}L^\perp \), the projective 8-space \( \mathbb{P}L \) is contained in the codimension 2-space \( \mathbb{P}^{24} \subset \mathbb{OP}L \). Second, the projective 16-space \( T_z \) is also contained in \( \mathbb{P}^{24} \). Indeed \( z = Q_y \cap Q_{y_0} \), and the condition that \( z \in Q_y \) (respectively, \( z \in Q_{y_0} \)) is equivalent to the condition that \( y \in Q_z \) (respectively, \( y_0 \in Q_z \)), where \( Q_z \subset \mathbb{OP}^2 \) is the dual quadric to \( z \in \mathbb{OP}^2 \). Therefore, since \( T_z \) is the orthogonal projective space to the linear span of the quadric \( Q_z \subset \mathbb{OP}^2 \), it follows that \( T_z \subset \mathbb{P}^{24} \).

Therefore, the intersection \( \psi(y) = \mathbb{P}L^\perp \cap T_z \) is non-empty, and now it is not hard to see that \( \psi(y) \) is a point for the general \( y \in Y \).

Conversely, let \( X = \mathcal{C} \cap \mathbb{P}L \) be general (in particular, \( X \subset \mathcal{C} - \mathbb{OP}^2 \)), and let \( x \) be a general point of \( X = \mathcal{C} \cap \mathbb{P}L \).

Then the entry locus \( Q_x \subset \mathbb{OP}^2 \) of the point \( x \in \mathcal{C} - \mathbb{OP}^2 \) meets the dual quadric \( Q_{y_0} \subset \mathbb{OP}^2 \) of the point \( y_0 \in \mathbb{OP}^2 \) (where \( y_0 \) is as above) at a unique point \( z \). The point \( z \in \mathbb{OP}^2 \) defines its dual quadric \( Q_z \subset \mathbb{OP}^2 \) that we will cut out with \( \mathbb{P}L^\perp \). The latter has codimension 9 but it contains \( x \), and we claim that the orthogonal hyperplane \( x^\perp \subset \mathbb{OP}^2 \) to \( x \) contains \( Q_z \). We shall check this by a direct computation. We may suppose that \( y_0 \) and \( \text{PDet}(x) = y_1 \) are two general points on \( \mathbb{OP}^2 \). The group \( E_6 \) acts transitively on pairs of points in the Cayley plane which are not joined by a line contained in \( \mathbb{OP}^2 \). (Indeed, suppose that we have two such pairs. Each of them is contained in a unique \( \mathcal{O} \)-line, and since \( E_6 \) acts transitively on the family of \( \mathcal{O} \)-lines, we can suppose that the two pairs are contained in the same \( \mathcal{O} \)-line. The stabilizer of this \( \mathcal{O} \)-line is a parabolic subgroup of \( E_6 \) with a semisimple part of type \( \text{Spin}_{10} \), acting on the \( \mathcal{O} \)-line through the usual action of \( \text{SO}_{10} \) on an eight-dimensional quadric. Then the statement follows from the fact that \( \text{SO}_{10} \) acts transitively on pairs of points in the quadric that are not joined by a line contained in that quadric.) We can therefore suppose that

\[
y_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]
This implies that
\[
Q_{y_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad Q_{y_1} = \begin{pmatrix} * & 0 & * \\ * & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.
\]
(We have omitted the quadratic conditions expressing that the determinants of the non-trivial 2 \times 2 blocs in the matrices above must vanish.) Hence, the intersection point
\[
z = Q_{y_0} \cap Q_{y_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_z = \begin{pmatrix} * & 0 & * \\ * & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.
\]
Finally, recall that a point \(x\), such that \(\text{PDet}(x) = y_1\), must be contained in the linear span of \(Q_{y_1}\). If we take into account the condition that \(x\) is orthogonal to \(y_0\), then we deduce that it must be of the form
\[
x = \begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & 0 & 0 \end{pmatrix},
\]
and therefore \(Q_z \subset x^\perp\), as claimed.

We conclude that the intersection of \(Q_z\) with \(\mathbb{P}L^\perp\) is in fact given by eight general linear conditions, and we end up with two points: one must be \(y_0\), and we denote the other one by \(\phi(x)\).

It is then straightforward to check that the rational maps \(\psi\) and \(\phi\) between \(X\) and \(Y\) are inverse to one another. This concludes the proof.

**Question 3.7.** Note that \(\psi\) and \(\phi\) factorize through a map to \(Q_{y_0}\), which implies that the general cubic sevenfold is birational to a complete intersection of bidegree \((2, d)\) in \(\mathbb{P}^9\). What is \(d\)?

Analyzing in detail the structure of the birational maps \(\phi\) and \(\psi\) seems to be rather complicated. The very first steps in this direction are the following two statements. 

**Proposition 3.8.** The rational map from \(\mathbb{OP}^2\) to \(\overline{\mathbb{OP}^2}\), mapping a general point \(y\) to \(Q_y \cap Q_{y_0}\), is the double projection \(p\) from \(y_0\).

**Proof.** We may suppose that \(y_0\) is the point that we have chosen in the proof of Proposition 3.5. Then we can use the local parameterization of \(\mathbb{OP}^2\) around \(y_0\) given by (2.1):
\[
y = \begin{pmatrix} 1 \\ \bar{w} \\ w \end{pmatrix} \quad \text{and} \quad p(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & |w|^2 & \bar{w}\bar{v} \\ 0 & vw & |v|^2 \end{pmatrix}.
\]
The point \(p(y)\) certainly belongs to \(Q_{y_0}\). There remains to prove that it also belongs to \(Q_y\), hence that it is orthogonal to the tangent space to the Cayley plane at \(y\). This is a straightforward explicit computation.

**Proposition 3.9.** Suppose that \(X, Y\) and \(y_0\) are general. The indeterminacy locus of \(\psi\) is a cone over a smooth canonical curve of genus 7. The indeterminacy locus of \(\phi\) is a smooth prime Fano threefold of degree 12.
Proof. There are two possible accidents that could prevent \( \psi(y) \) from being defined. The first one is that \( Q_y \) meets \( Q_{y_0} \) in more than one point. The second one is that \( Q_y \cap Q_{y_0} \) is a single point \( z \) but such that the tangent space \( T_z \) meets \( P L \) in more than one point. A dimension count shows that for \( y_0 \) generic, the second situation cannot happen. The first one happens when \( y \) belongs to the cone \( C_{y_0} \) spanned by the (usual) projective lines through \( y_0 \) in the Cayley plane. By Landsberg and Manivel [32], this is a cone over the spinor variety \( S_{10} = OG(5, 10) \), inside a sixteen-dimensional projectivized half-spin representation. Cutting this cone by \( P L^\perp \), we get a cone over a codimension 9 generic linear section of \( S_{10} \), which is a canonical curve of genus 7.

Now let \( x \) be a point in \( X \). To prevent \( \phi(x) \) from being defined, the only possible accident is that \( Q_x \) meets \( Q_{y_0} \) in more than one point. If we let \( y = \text{PDet}(x) \), so that \( Q_x = Q_y \), this means that \( y \) belongs the cone \( C_{y_0} \). Let us determine \( \text{PDet}^{-1}(C_{y_0}) \). We may suppose that

\[
y_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{hence } C_{y_0} = \left\{ \begin{pmatrix} 1 & a & b \\ \bar{a} & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, \quad |a|^2 = |b|^2 = ab = 0 \right\},
\]

as we can see from the explicit parameterization of a neighborhood of \( y_0 \) given in (2.1). This implies in particular that the conditions \( |a|^2 = |b|^2 = ab = 0 \) on the pair \((a, b) \in \mathbb{O} \oplus \mathbb{O}\) define a copy of the spinor variety \( S_{10} \) in a half-spin representation. Now, (2.2) shows that \( x \) belongs to \( \text{PDet}^{-1}(C_{y_0}) \) if and only if \( \text{Det}(x) = 0 \) and

\[
|v|^2 = rt, \quad |w|^2 = rs, \quad vw = r\bar{u}.
\]

If \( r \neq 0 \), then these conditions imply that \( wu = s\bar{v} \) and \( uv = t\bar{w} \), so that in fact \( \text{PDet}(x) = y_0 \). Generically, this cannot happen in our situation since this would imply that \( x \) belongs to the linear span of \( Q_{y_0} \), which is too small to meet \( P L \). So we must let \( r = 0 \), in which case we are left with the conditions

\[
|v|^2 = 0, \quad |w|^2 = 0, \quad vw = 0.
\]

As we have just seen, this defines a copy of \( S_{10} \). Since the parameters \( s, t, u \) remain free, we conclude that \( x \) must belong to the join of \( S_{10} \) with \( \mathbb{P}^9 \). Cutting this join with \( P L \) amounts to cutting \( S_{10} \) along a generic three-dimensional linear section, which is a prime Fano threefold of degree 12.

\( \square \)

4. Derived categories

4.1. The Calabi–Yau subcategory

Let again \( X \) be a smooth cubic sevenfold. Since it is Fano of index 6, the collection \( \mathcal{O}_X, \ldots, \mathcal{O}_X(5) \) is exceptional. Denote by \( \mathcal{A}_X \) the full subcategory of \( D^b(X) \), the derived category of coherent sheaves on \( X \), defined as the left semi-orthogonal to this exceptional collection. The following statement is a special case of [28, Corollary 4.3] (to which we also refer for the basic terminology we use about derived categories).

Proposition 4.1. \( \mathcal{A}_X \) is a three-dimensional Calabi–Yau category.

The terminology non-commutative Calabi–Yau is sometimes used. Kuznetsov has given in [30] interesting examples of non-commutative K3’s, which are deformations of commutative ones (that is, of derived categories of coherent sheaves on genuine K3 surfaces). Here the situation is a bit different, since the non-commutative Calabi–Yau cannot be the derived
category of any Calabi–Yau threefold $Z$, or even a deformation of such a derived category. Indeed, computing the Hochschild cohomology of $A_X$ with the help of [29, Corollary 7.5], we would deduce from the HKR isomorphism theorem that the Hodge numbers of $Z$ should be such that

$$\sum_p h^{p,p}(Z) = \sum_p h^{p,p}(X) - 6 = 2!.$$  

4.2. Spherical twists

Suppose that $X$ is given with a Cartan representation $X = C \cap \mathbb{P}L$. Recall from §2.3 that this Cartan representation induces a rank 9 vector bundle $E_X$ on $X$, such that $E_X(-1)$ is self-dual.

**Proposition 4.2.** $E_X(-1)$ and $E_X(-2)$ are two objects of $A_X$.

**Proof.** The proof is immediate consequence of Proposition 2.5 and Serre duality. 

As a consequence, Theorem 2.7 can be interpreted as the assertion that $E_X(-1)$ and $E_X(-2)$ are spherical objects in $A_X$, in the sense of Seidel and Thomas [38, Definition 1.1]. In particular, with the notation and terminology of [38] one can state the following corollary.

**Corollary 4.3.** The spherical twists $T_{E_X(-1)}$ and $T_{E_X(-2)}$ are auto-equivalences of $A_X$.

These auto-equivalences have infinite order. This can be seen at the level of K-theory: the K-theory of $A_X$ is free of rank 2, hence has a basis given by the classes of $E_X(-1)$ and $E_X(-2)$. The matrices $\Phi_{E_X(-1)}$ and $\Phi_{E_X(-2)}$ of the induced automorphisms of $K(A_X)$, expressed in this basis, are easy to compute. Since $P_{\text{End}(E_X)}(-1) = 9$, they are given by

$$\Phi_{E_X(-1)} = \begin{pmatrix} 1 & 0 \\ -9 & 1 \end{pmatrix}, \quad \Phi_{E_X(-2)} = \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}.$$  

**Remark 4.4.** The category $A_X$ should have a very interesting group of auto-equivalences, including the auto-equivalences from Corollary 4.3 coming from the Cartan representations of the cubic sevenfold. This group could be big if the number of Cartan representations of the cubic is big (see Question 2.3 at the end of §2.2). This is of course related to the fact that $A_X$ is Calabi–Yau, contrary to $D^b(X)$ which, $X$ being Fano, has no interesting auto-equivalence [7].

4.3. Homological projective duality

We expect for cubic sevenfolds certain phenomena that would illustrate the principles of homological projective duality [28]. Starting once again from a Cartan representation $X = C \cap \mathbb{P}L$ of a smooth cubic sevenfold, we would like to compare the derived category of $X$ with that of the orthogonal section $Y = \mathbb{O}\mathbb{P}^2 \cap \mathbb{P}L^\perp$, which, for the general choice of $X$, is a smooth Fano sevenfold of index 3 birational to $X$; see Propositions 3.3 and 3.5.

In order to describe $D^b(Y)$, we will start from the Cayley plane. Recall that on $\mathbb{O}\mathbb{P}^2$ we have denoted by $S$ the rank 10 spin bundle. It is endowed with the non-degenerate quadratic form

$$\text{Sym}^2 S \to \mathcal{O}_{\mathbb{O}\mathbb{P}^2}(1),$$
which defines the quadrics on the dual Cayley plane parameterized by $\mathbb{OP}^2$. Its kernel is an irreducible homogeneous bundle that we denote by $S_2$. The following result was proved in [17, 34].

**Theorem 4.5.** In the derived category of coherent sheaves on $\mathbb{OP}^2$, let

$$A_0 = A_1 = A_2 = \langle S_2^\vee, S_2, O_{\mathbb{OP}^2} \rangle,$$

$$A_3 = \cdots = A_{11} = \langle S_2^\vee, O_{\mathbb{OP}^2} \rangle.$$

Then there is a semi-orthogonal decomposition

$$D^b(\mathbb{OP}^2) = \langle A_0, A_1(1), \ldots, A_{11}(11) \rangle.$$

If we denote by $S_Y$ the restriction of $S$ to the linear section $Y$, then we deduce the following proposition.

**Proposition 4.6.** The sequence $O_Y, S_Y, O_Y(1), S_Y(1), O_Y(2), S_Y(2)$ is exceptional.

**Proof.** Use the Koszul complex describing $O_Y$ and apply Bott’s theorem on the Cayley plane. Then use [34, Lemmas 1 and 3].

It is then tempting to consider the right orthogonal $A_Y$ in $D^b(Y)$ to this exceptional collection and ask.

(i) Is $A_Y$ a three-dimensional Calabi-Yau category?

(ii) Is $A_Y$ equivalent to $A_X$?

Note that if $X$ has $d > 1$ Cartan representations, then the corresponding orthogonal sections $Y_1, \ldots, Y_d$ would define non-commutative Fourier partners to the non-commutative Calabi-Yau threefold $A_X$.

**Remark 4.7.** It follows from the results of [34, Lemma 4] that $S_{2,Y}(-1)$ belongs to $A_Y$. Moreover, completing the computations made for proving this lemma, one gets the following statement: if $0 \leq i \leq 13$,

$$h^q(\mathbb{OP}^2, \text{End}(S_2)(-i)) = \delta_{i,0} \delta_{q,0} + \delta_{i,3} \delta_{q,4} + \delta_{i,6} \delta_{q,8} + \delta_{i,9} \delta_{q,12}.$$

From this, one deduces that the values of $h^q(Y, \text{End}(S_{2,Y}))$ are equal to 1, 84, 84, 1, 0, 0, 0, 0. This is remarkably coherent with the expected Calabi-Yau property of $A_Y$.

5. **The rationality of the Cartan cubic eightfolds**

Contrary to cubic sevenfolds, an eight-dimensional linear section of the Cartan cubic cannot be a general cubic eightfold. Indeed, the dimension of this family of cubics is $\dim G(10, 27) - \dim E_6 = 170 - 78 = 92$, while the moduli space of cubic eightfolds has dimension $\dim |O_{\mathbb{OP}^9}(3)| - \dim \text{PGL}_{10} = 219 - 99 = 120$.

In the spirit of the study in § 2, the Cartan cubic eightfold is the octonionic analog of the Pfaffian cubic fourfold, the last corresponding to the quaternions. The most important property of the Pfaffian cubic fourfold, known since G. Fano is that it is rational; see [18]. In this section, we prove that the Cartan cubic eightfold is also rational.
The basic observation, originating from Fano, is that the Pfaffian cubic fourfold contains surfaces with OADP: quintic del Pezzo surfaces and rational quartic scrolls; see [2, §4] for an up-to-date treatment of the results of Fano. More generally, a cubic 2n-fold containing an n-fold with OADP is rational, and as far as we know, all known rational smooth cubics of dimension 2n \( \geq 6 \) contain n-folds with OADP (see, for example, [2, §6] for a study of smooth cubic sixfolds containing OADP threefolds). One delicate part in these examples is the proof that a general cubic 2n-fold through a particular kind of an OADP n-fold is smooth. Moreover, the examples of n-folds with OADP are not too many (see [11]), which seem to be one more reason why there are very few known examples of rational smooth cubic 2n-folds of dimension bigger than four. Note, however, that interesting series of examples of cubic fourfolds \( X \) whose rationality does not come from the existence of OADP surfaces were discovered by Hassett [21].

We shall see below that the general Cartan cubic eightfold \( X \) contains a fourfold \( Y \) with OADP, the and hence \( X \) is rational. This special OADP fourfold \( Y \) is a transversal section of the spinor variety \( S_{10} \); see [11, Example 2.8]. By the comments to § 3, the rationality construction coming from the degree 12 OADP fourfold \( Y \) on the Cartan cubic eightfold \( X \) is the analog of the rationality construction coming from the quartic rational normal scroll on the Pfaffian cubic fourfold, as described in [2, Theorem 4.3; 19].

**Theorem 5.1.** The general Cartan cubic eightfold is rational.

**Proof.** The following lemma is the key observation of this proof.

**Lemma 5.2.** Any general \( \mathbb{P}^{15} \subset \mathbb{P} \dot{H} \) is the linear span of a copy of the spinor variety \( S_{10} \), contained in the Cayley cubic.

As we have already mentioned, this implies the theorem as follows. Let \( X \) be the cubic eightfold defined as the linear section of the Cayley cubic by a general \( \mathbb{P}^{9}_X \subset \mathbb{P} \dot{H} \). By the lemma, there exists a copy \( S_{10} \) of the spinor variety, contained in the Cayley cubic, whose linear span contains \( \mathbb{P}^9_X \) as a general linear subspace. Then the intersection \( Y = S_{10} \cap \mathbb{P}^9_X \) is smooth of dimension 4, and is a variety with OADP as we mentioned above.

Now, any cubic \( X \) of even dimension 2m containing an m-dimensional variety \( Y \) with OADP is rational. Indeed, consider a general hyperplane \( H \) in \( \mathbb{P}^{2m+1} \) and \( x \) a general point of \( X \). There is a unique secant line \( \ell_x \) to \( Y \) passing through \( x \) and we can let \( \pi(x) = \ell_x \cap H \). Then \( \pi \) is a birationality whose inverse is described in much the same way: if \( h \) is a general point of \( H \), then there is a unique secant line \( d_h \) to \( Y \) passing through \( h \); this line \( d_h \) cuts \( X \) in two points of \( Y \), and a third point, which is \( \pi^{-1}(h) \).

**Proof of the lemma.** Recall that the spinor variety \( S_{10} \) parameterizes the space of lines in the Cayley plane passing through a given point. We have deduced that

\[
S_{10} \simeq \mathbb{P}\{u + v \in \mathfrak{O} \oplus \mathfrak{O}, |u|^2 = |v|^2 = u\bar{v} = 0\} \subset \mathbb{P}(\mathfrak{O} \oplus \mathfrak{O}) \simeq \mathbb{P}^{15}.
\]

Let \( \rho, \tau \) be linear forms on \( \mathfrak{O} \oplus \mathfrak{O} \) and let \( \alpha, \beta \) be endomorphisms of \( \mathfrak{O} \). Then, letting \( y = \alpha(u) + \tau(v) \), the set of matrices of the form

\[
\begin{pmatrix}
\rho(u, v) & u & \bar{y} \\
\bar{u} & 0 & \bar{v} \\
y & v & \sigma(u, v)
\end{pmatrix}, \quad |u|^2 = |v|^2 = u\bar{v} = 0,
\]

defines a copy of \( S_{10} \) contained in the Cayley cubic. The corresponding family \( F_0 \) of linear spaces is a smooth subset of \( G(16, H) \); to be precise, consider the decomposition \( H = L_0 \oplus L_1 \),
The central coefficient defines the restriction to $H$. Then the set of points in $G(16, H)$ defined by subspaces transverse to $y_1$ forms an affine neighborhood of $L_0$ isomorphic to $\text{Hom}(L_0, L_1)$, inside which $F_0$ is the linear subspace $\text{Hom}(L_0, L_1')$ if $L_1' \subset L_1$ denotes the hyperplane defined by $s = 0$. Consider the map

$$\psi : E_6 \times F_0 \longrightarrow G(16, H),$$

$$(g, L) \mapsto g(L).$$

The image of $\psi$ consists of linear spaces spanned by a copy of the spinor variety contained in the Cayley cubic, since this is the case for $F_0$ and this property is preserved by the action of $E_6$. We claim that $\psi$ is dominant, which implies the lemma. We will check that the differential of $\psi$ at $(1, L_0)$ is surjective. To see this, first observe that $\text{Im}(\psi_*) \subset T_{L_0}G(16, H) \simeq \text{Hom}(L_0, L_1)$ contains $T_{L_0}F_0 \simeq \text{Hom}(L_0, L_1')$. It also contains the image $\psi_*(\epsilon_6)$ of the Lie algebra $\epsilon_6$, obtained by differentiating the restriction of $\psi$ to $E_6 \times \{L_0\}$. So we just need to prove that $\text{Hom}(L_0, L_1')$ and $\psi_*(\epsilon_6)$ do span $\text{Hom}(L_0, L_1')$, which is equivalent to the fact that the quotient map $\epsilon_6 \twoheadrightarrow \text{Hom}(L_0, L_1/L_1') \simeq L_0^\vee$ is surjective.

To check this, recall that $\epsilon_6 \simeq f_4 \oplus H_0$, where $H_0 \subset H$ is the hyperplane of traceless matrices for the action on $H$ by Jordan multiplication:

$$M \in H_0 \mapsto T_M \in \text{End}(H), \quad T_M(X) = \frac{1}{2}(XM + MX).$$

In particular, for a traceless matrix

$$M = \begin{pmatrix} a & c & \bar{b} \\ \bar{c} & \beta & a \\ \bar{b} & \bar{a} & \gamma \end{pmatrix},$$

a short computation yields

$$T_M(L_0) = \left\{ \begin{pmatrix} \ast & \ast & \ast \\ \ast & (a, u) + (c, v) & \ast \\ \ast & \ast & \ast \end{pmatrix}, \quad u, v \in \mathcal{O} \right\}.$$  

The central coefficient defines the restriction to $H_0 \subset \epsilon_6$ of the map to $L_0^\vee$ we are interested in. It is obviously surjective, hence we are done. 

\bigskip

Appendix: Proof of Theorem 2.7

We start with a simple observation showing that the higher cohomology groups of $\text{End}(\mathcal{E}_X)$ cannot all be trivial.

**Lemma A.1.** One has a natural duality

$$H^i(X, \text{End}(\mathcal{E}_X)) \simeq H^{3-i}(X, \text{End}(\mathcal{E}_X))^\vee.$$  

In particular, $H^i(X, \text{End}(\mathcal{E}_X)) = 0$ for $i \geq 4$.

**Proof.** Consider the exact sequence (2.3). Observe that $\mathcal{E}_X(-3)$ is acyclic. Indeed, $\mathcal{E}_X$ is aCM, $H^0(\mathcal{E}_X(-3)) = 0$, and by Serre duality, $H^7(\mathcal{E}_X(-3)) = H^0(\mathcal{E}_X^\vee(3)) = H^0(\mathcal{E}_X(-5)) = 0$. We deduce that, for any $i$,

$$H^i(\text{End}(\mathcal{E}_X)) = H^{i+2}(\text{End}(\mathcal{E}_X)(-3)).$$
But the latter is Serre dual to $H^{5-i}(\text{End}(\mathcal{E}_X)(-3))$ which, by the previous assertion, is isomorphic to $H^{3-i}(\text{End}(\mathcal{E}_X))$. 

Now we use the fact that $\mathcal{E}_X^\vee = \mathcal{E}_X(2)$ to decompose $\text{End}(\mathcal{E}_X)$ as the sum of its symmetric and skew-symmetric parts, $S^2\mathcal{E}_X^\vee(-2)$ and $\wedge^2\mathcal{E}_X^\vee(-2)$, which we treat separately.

(A) We start with the latter. To determine its cohomology, we will use the fact that $\wedge^2(\pi^*\mathcal{S}_X) = \wedge^2\mathcal{E}_X \oplus \mathcal{E}_X(1)$. Moreover, we observe that we can use the Koszul complex

$$0 \longrightarrow \wedge^1 L \otimes \mathcal{O}_S(-18) \longrightarrow \cdots \longrightarrow L \otimes \mathcal{O}_S(-1) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_X \longrightarrow 0 \quad (A.1)$$

to deduce the cohomology of $\wedge^2(\pi^*\mathcal{S}_X)$ from that of $\wedge^2(\pi^*\mathcal{S})$ and its twists, which we can derive from Bott's theorem.

Let $H_X$ denote the restriction to $\gamma^{-1}(X)$ of $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. We have $H_X = \mathcal{O}_X(2)$ since the rational map $C \to \mathbb{P}^2$ is defined by quadrics. We twist the Koszul complex by $\pi^*(\wedge^2 \mathcal{S} \otimes H^{-1})$. This bundle is acyclic since $\wedge^2 \mathcal{S} \otimes H^{-1}$ is acyclic on $\mathbb{P}^2$. Its twists by $\mathcal{O}_S(-k)$ are also acyclic for $k \leq 9$, since they are acyclic on the fibers of $\pi$. For $k \geq 10$ and any bundle $F$ on $\mathbb{P}^2$, we have an isomorphism

$$H^3(\mathbb{P}(S^\vee), \pi^* F \otimes \mathcal{O}_S(-k)) = H^{q-9}(\mathbb{P}^2, F \otimes \text{Sym}^{k-10} S^\vee \otimes \det(S^\vee))$$

$$= H^{q-9}(\mathbb{P}^2, F \otimes \text{Sym}^{k-10} S \otimes H^{5-k}).$$

We will need a few decomposition formulas. Denote by $\mathcal{S}_m$ the homogeneous vector bundle on $\mathbb{P}^2$ defined by the weight $m\omega_6$. In particular, $\mathcal{S}_1 = \mathcal{S}$.

**Lemma A.2.** For any positive integer $m$,

$$\text{Sym}^n \mathcal{S} = \bigoplus_{\ell \geq 0} \mathcal{S}_{m-2\ell} \otimes H^\ell.$$

**Proof.** The proof follows from the decomposition formulas of the symmetric powers of the natural representations of special orthogonal groups, which are classical. 

Applying the formula above to $F = \wedge^2 \mathcal{S} \otimes H^{-1}$, we thus get

$$H^3(\mathbb{P}(S^\vee), \pi^*(\wedge^2 \mathcal{S} \otimes H^{-1}) \otimes \mathcal{O}_S(-k)) = \bigoplus_{\ell \geq 0} H^{q-9}(\mathbb{P}^2, \wedge^2 \mathcal{S} \otimes \mathcal{S}_{k-2\ell-10} \otimes H^{4-k+\ell}).$$

Then we need to decompose the tensor products $\wedge^2 \mathcal{S} \otimes \mathcal{S}_m$ into irreducible components. This decomposition is given by the next lemma, where for future use we include a few more similar formulas. We shall denote by $\mathcal{S}_{p,q,r}$ the homogeneous vector bundle on $\mathbb{P}^2$ defined by the weight $p\omega_6 + q\omega_5 + r\omega_4$, and we let $\mathcal{S}_{p,q} = \mathcal{S}_{p,q,0}$.

**Lemma A.3.** (1) $\mathcal{S}_m \otimes \mathcal{S} = \mathcal{S}_{m+1} \oplus \mathcal{S}_{m-1,1} \oplus \mathcal{S}_{m-1} \otimes H$.

(2) $\mathcal{S}_m \otimes \wedge^2 \mathcal{S} = \mathcal{S}_{m,1} \oplus \mathcal{S}_{m-1,0,1} \oplus \mathcal{S}_m \otimes H \oplus \mathcal{S}_{m-2,1} \otimes H$.

(3) $\mathcal{S}_m \otimes \mathcal{S}_2 = \mathcal{S}_{m+2} \oplus \mathcal{S}_{m,1} \oplus \mathcal{S}_{m-2,2} \oplus \mathcal{S}_m \otimes H \oplus \mathcal{S}_{m-2,1} \otimes H \oplus \mathcal{S}_{m-2} \otimes H^2$.

**Proof.** The proof is obtained from routine representation theory. 

Now we can decompose each term $\wedge^2 \mathcal{S} \otimes \mathcal{S}_{k-2\ell-10}$ into irreducible components, and compute the cohomology of each term. The special cases of Bott’s theorem we will need are covered by the following lemma.
Lemma A.4. Let \( p \geq 0 \) and \( m \geq 1 \). Then

1. \( H^i(\mathbb{P}^2, S_p \otimes H^{-m}) = 0 \) for \( i < 16 \), except
   \[ H^8(\mathbb{P}^2, S_p \otimes H^{-m}) = V_{(p+4-m)\omega_1 + (m-8)\omega_6} \] if \( 8 \leq m \leq p + 4 \);
2. \( H^i(\mathbb{P}^2, S_{p,1} \otimes H^{-m}) = 0 \) for \( i < 16 \), except
   \[ H^8(\mathbb{P}^2, S_{p,1} \otimes H^{-m}) = V_{(p+5-m)\omega_1 + (m-9)\omega_6} \] if \( 9 \leq m \leq p + 5 \);
3. \( H^i(\mathbb{P}^2, S_{p,2} \otimes H^{-m}) = 0 \) for \( i < 16 \), except
   \[ H^4(\mathbb{P}^2, S_{p,2} \otimes H^{-5}) = V_{p\omega_1} \],
   \[ H^8(\mathbb{P}^2, S_{p,2} \otimes H^{-m}) = V_{(p+6-m)\omega_1 + (m-10)\omega_6} \] if \( 10 \leq m \leq p + 6 \),
   \[ H^{12}(\mathbb{P}^2, S_{p,2} \otimes H^{-p-11}) = V_{p\omega_6} \];
4. \( H^i(\mathbb{P}^2, S_{p,0,1} \otimes H^{-m}) = 0 \) for \( i < 16 \), except
   \[ H^2(\mathbb{P}^2, S_{p,0,1} \otimes H^{-3}) = V_{p\omega_1} \],
   \[ H^6(\mathbb{P}^2, S_{p,0,1} \otimes H^{-7}) = V_{(p-2)\omega_1} \],
   \[ H^8(\mathbb{P}^2, S_{p,0,1} \otimes H^{-m}) = V_{(p+5-m)\omega_1 + (m-10)\omega_6} \] if \( 10 \leq m \leq p + 5 \),
   \[ H^{10}(\mathbb{P}^2, S_{p,0,1} \otimes H^{-p-8}) = V_{(p-2)\omega_6} \],
   \[ H^{14}(\mathbb{P}^2, S_{p,0,1} \otimes H^{-p-12}) = V_{p\omega_6} \].

Proof. Apply Bott’s theorem as in [34].

We deduce the following statement: for \( q < 25 \),
\[ H^q(\mathbb{P}(S^\vee), \pi^*(\wedge^2 S \otimes H^{-1}) \otimes O_S(-k)) = \delta_{q,19}V_{(k-15)\omega_6}. \]

This implies that, for \( q < 25 - 18 = 7 \), \( H^q(X, \wedge^2 S_X^\vee(-2)) \) is the \( q \)th cohomology group of the complex
\[ 0 \longrightarrow V_{3\omega_6} \longrightarrow L \otimes V_{2\omega_6} \longrightarrow \wedge^2 L \otimes V_{\omega_6} \longrightarrow \wedge^3 L \longrightarrow 0, \]
going from degree 0 to degree 5. In particular, \( H^0(X, \wedge^2 S_X^\vee(-2)) = 0 \).

Proposition A.5. For \( L \) generic, the map \( L \otimes V_{2\omega_1} \rightarrow V_{3\omega_1} \) is surjective.

Therefore, \( H^1(X, \wedge^2 S_X^\vee(-2)) = 0 \).

Proof. We start with the commutative diagram
\[
\begin{array}{cccccc}
V_{\omega_1} \otimes V_{\omega_6} & \longrightarrow & V_{\omega_1} \otimes S^2 V_{\omega_1} & \longrightarrow & S^3 V_{\omega_1} \\
\wedge^2 V_{\omega_1} \otimes V_{\omega_1} & \longrightarrow & V_{\omega_1} \otimes S^2 V_{\omega_1} & \longrightarrow & S^3 V_{\omega_1} \\
\wedge^2 V_{\omega_1} \otimes V_{\omega_1} & \longrightarrow & V_{\omega_1} \otimes V_{2\omega_1} & \longrightarrow & V_{3\omega_1} \\
\end{array}
\]

The horizontal complex in the middle is exact, being part of a Koszul complex. It surjects to the horizontal complex of the bottom, which is again exact, as one can check from the decompositions into irreducible components, as given by LiE [13]. The vertical complex in the middle is also exact, the map \( V_{\omega_1} = V_{\omega_1}^\vee \rightarrow S^2 V_{\omega_1} \) being given by the differential of the invariant cubic Det.

We deduce from this diagram that the kernel \( \tilde{K} \) of the map \( L \otimes V_{2\omega_1} \rightarrow V_{3\omega_1} \) is simply the image of the kernel \( K \) of the map \( L \otimes S^2 V_{\omega_1} \rightarrow S^3 V_{\omega_1} \).
Lemma A.6. Let $L \subset V$ be any subspace of dimension $\ell$ and codimension $c$. Then the kernel $K$ of the map $L \otimes S^2V \to S^3V$ is the image of $\wedge^2L \otimes V$. In particular,
\[
\dim(K) = \frac{\ell(\ell^2 - 1)}{3} + c \frac{\ell(\ell - 1)}{2}.
\]

Proof. The proof is straightforward. \qed

In our case, letting $\ell = 18$ and $c = 9$, we get $\dim(K) = 3315$.

Lemma A.7. The map $K \to \bar{K}$ is an isomorphism.

Proof. We need to check that $K$ does not meet the kernel $L \otimes V_{\omega_6}$ of the projection to $L \otimes V_{2\omega_1}$. In other words, we need to check that the map to $\text{Sym}^3 V_{\omega_1}$ is injective on $L \otimes V_{\omega_6} \subset V_{\omega_1} \otimes S^2V_{\omega_1}$. Let $\ell_i$ be a basis of $L$ that we complete into a basis of $V_{\omega_1}$ by some vectors $m_j$ generating a supplement $M$ to $L$. Let $(y_k) = (\ell_i^\vee, m_j^\vee)$ denote the dual basis. Any element of $L \otimes V_{\omega_6}$ can be written as $\sum_i \ell_i \otimes x_i$ for some vectors $v_i \in V_{\omega_6} = V_{\omega_1}^\vee$. Its image in $L \otimes S^2V_{\omega_1}$ is
\[
\sum_i \ell_i \otimes \text{Det}(x_i, y_j, y_k) y_j^\vee y_k^\vee \mapsto \sum_i \text{Det}(x_i, y_j, y_k) \ell_i^\vee y_j^\vee y_k^\vee \in S^3V_{\omega_1}.
\]

The decomposition $V_{\omega_1} = L + M$ induces a direct sum decomposition $S^3V_{\omega_1} = S^3L \oplus S^2LM \oplus L^2M \oplus S^3M$. The component of the previous tensor on $L^2M$ is
\[
\sum_{i,j,k} \text{Det}(x_i, m_j^\vee, m_k^\vee) \ell_i m_j m_k.
\]

For it to vanish, we need $\text{Det}(x_i, m_j^\vee, m_k^\vee) = 0$ for any $i, j, k$. But this implies that $x_i = 0$ for all $i$ since, for a generic $L$, one can check that the map
\[
S^2L^\perp \hookrightarrow S^2V_{\omega_6} \longrightarrow V_{\omega_6}^\vee
\]
is surjective.

We can now complete the proof of the surjectivity of $\phi : L \otimes V_{2\omega_1} \to V_{3\omega_1}$. We know that $\dim V_{2\omega_1} = 351$ and $\dim V_{3\omega_1} = 3003$. By the two previous lemmas, the kernel of $\phi$ has dimension 3315, and therefore the dimension of its image is $18 \times 351 - 3315 = 3003$. Hence, the surjectivity. This completes the proof of the proposition. \qed

(B) Now we treat $\text{Sym}^2 \mathcal{E}_X(-2)$, essentially in the same way. We first observe that $\text{Sym}^2(\pi^*S_X) = \text{Sym}^2 \mathcal{E}_X \oplus \pi^*S_X(1)$. Recall that $\text{Sym}^2(\mathcal{S}) = S_2 \oplus H$, where the second factor will contribute to the homotheties in $\text{End}(\mathcal{E}_X)$. Therefore, what we need to compute is the cohomology of $\pi^*(S_2 \otimes H^{-1})$ restricted to $X$. Proceeding as before, we check that, for $q < 25$,
\[
H^q(\mathbb{P}(\mathcal{E}^\vee), \pi^*(S_2 \otimes H^{-1}) \otimes \mathcal{O}_X(-k)) = \delta_{q,17} V(k-12)_{\omega_6} + \delta_{q,21} V(k-18)_{\omega_6}.
\]

This implies that, for $q < 21 - 18 = 3$, the cohomology group $H^q(X, \pi^*(S_2 \otimes H^{-1})|_X)$ can be computed as the $q$th cohomology group of the complex
\[
0 \longrightarrow V_{6\omega_6} \longrightarrow L \otimes V_{5\omega_6} \longrightarrow \wedge^2 L \otimes V_{4\omega_6} \longrightarrow \wedge^2 L \otimes V_{3\omega_6} \longrightarrow \wedge^3 L,
\]
where the rightmost term is in degree 2. On the other hand, recall that $\text{Sym}^2 \mathcal{E}_X(-2)$ is only a direct factor of $\pi^*(S_2 \otimes H^{-1})|_X = \pi^*(S_2)|_X(-2)$, with complementary factor $S_X(-1) = S_X \otimes H_X^{-1}(1)$. After a computation similar to (but easier than) the previous one, we obtain that, for $q < 25$,
\[
H^q(\mathbb{P}(\mathcal{E}^\vee), \pi^*(S \otimes H^{-1}) \otimes \mathcal{O}_X(-1 - k)) = \delta_{q,17} V(k-12)_{\omega_6},
\]
and we can conclude that, for $q < 25 - 18 = 7$, $H^q(X, S_X(-1))$ is the $q$th cohomology group of the complex
\[ 0 \rightarrow V_{8\omega_6} \rightarrow L \otimes V_{8\omega_6} \rightarrow \wedge^2 L \otimes V_{4\omega_6} \rightarrow \wedge^2 L \otimes V_{3\omega_6} \rightarrow \wedge^3 L \rightarrow \cdots, \tag{A.3} \]
with the same grading as in the complex (A.2). In particular, $H^q(X, S_X(-1))$ coincides with $H^q(X, \pi^*(S_2 \otimes H^{-1})|_X)$ for $q < 3$, and therefore
\[ H^q(X, \Sym^2 \mathcal{E}_X(-2)) = \delta_{q,0} \mathbb{C} \quad \text{if } q < 3. \]

The proof of the theorem is now complete. Indeed, we have proved that
\[ H^0(X, \End(\mathcal{E}_X)) = H^0(X, \wedge^2 \mathcal{E}_X(-2)) \oplus H^3(X, \Sym^2 \mathcal{E}_X(-2)) = 0 \oplus \mathbb{C} = \mathbb{C}, \]
so that $\mathcal{E}_X$ is simple, and moreover that $H^1(X, \End(\mathcal{E}_X)) = 0$. By Lemma A.1, we deduce that $H^2(X, \End(\mathcal{E}_X)) = 0$ and $H^3(X, \End(\mathcal{E}_X)) = \mathbb{C}$, and we already know that $H^q(X, \End(\mathcal{E}_X)) = 0$ for $q > 3$. \hfill\(\square\)

References

1. D. Akhiezer, *Lie group actions in complex analysis*, Aspects of Mathematics E27 (Vieweg & Sohn, Braunschweig, 1995).
2. A. Alzati and F. Russo, ‘Some extremal contractions between smooth varieties arising from projective geometry’, *Proc. London Math. Soc.* 89 (2004) 25–53.
3. J. Baez, ‘The octonions’, *Bull. Amer. Math. Soc.* 39 (2002) 145–205.
4. A. Beauville, ‘Determinantal hypersurfaces’, *Michigan Math. J.* 48 (2000) 39–64.
5. A. Beauville and R. Donagi, ‘La variété des droites d’une hypersurface cubique de dimension 4’, *C. R. Acad. Sci. Paris Sér. I Math.* 301 (1985) 703–706.
6. K. Becker, M. Becker, C. Vafa and J. Walcher, ‘Moduli stabilization in non-geometric backgrounds’, *Nucl. Phys. B* 770 (2007) 1–46.
7. A. Bondal and D. Orlov, ‘Reconstruction of a variety from the derived category and groups of autoequivalences’, *Compos. Math.* 125 (2001) 327–344.
8. L. Borisov and A. Caldararu, ‘The Pfaffian–Grassmannian derived equivalence’, *J. Algebraic Geom.* 18 (2009) 201–222.
9. N. Bourbaki, *Groupes et algèbres de Lie* (Hermann, Paris, 1968).
10. P. Candelas, E. Derrick and L. Parkes, ‘Generalized Calabi–Yau manifolds and the mirror of a Rigid Manifold’, *Nucl. Phys. B* 407 (1993) 115–154.
11. C. Ciliberto, M. Mella and F. Russo, ‘Varieties with one apparent double point’, *J. Algebraic Geom.* 13 (2004) 475–512.
12. A. Clebsch, ‘Die Geometrie auf den Flächen dritter Ordnung’, *J. reine angew. Math.* 65 (1866) 359–380.
13. M. A. Cohen, M. A. van Leeuwen and B. Lisser, ‘LiE, a package for Lie group computations’, http://www.mat.univie.ac.at/~slc/105/lie.html.
14. A. C. Dixon, ‘Note on the reduction of a ternary quantic to a symmetric determinant’, *Proc. Cambridge Philos. Soc.* 11 (1902) 350–351.
15. S. Druel, ‘Espace des modules des faisceaux semi-stables de rang 2 et de classes de Chern $c_1 = 0$, $c_2 = 2$ et $c_3 = 0$ sur une hypersurface cubique lisse de $P^4$’, *Int. Math. Res. Not.* 19 (2000) 985–1004.
16. L. Ein and N. Shepherd-Barron, ‘Some special Cremona transformations’, *Amer. J. Math.* 111 (1989) 783–800.
17. D. Faenzi and L. Manivel, ‘On the derived category of the Cayley plane II’, Preprint, 2012, arXiv:1201.0420v2.
18. G. Fano, ‘Sulle sezioni spaziali della varietà grassmanniana delle rette dello spazio a cinque dimensioni’, *Rend. Accad. d. L. Roma* (6) 11 (1930) 329–335.
19. G. Fano, ‘Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del 4° ordine’, *Comm. Math. Helv.* 15 (1942) 71–80.
20. H. Freudenthal, ‘Lie groups in the foundations of geometry’, *Adv. Math.* 1 (1964) 145–190.
21. B. Hassett, ‘Some rational cubic fourfolds’, *J. Algebraic Geom.* 8 (1999) 103–114.
22. L. Hesse, ‘Über die Elimination der Variablen aus drei algebraischen Gleichungen von zweiten Grad mit zwei Variablen’, *J. reine angew. Math.* 28 (1844) 68–96.
23. A. Iliev and L. Manivel, ‘Severi varieties and their varieties of reductions’, *J. reine angew. Math.* 585 (2005) 93–139.
24. A. Iliev and L. Manivel, ‘The Chow ring of the Cayley plane’, *Compos. Math.* 141 (2005) 146–160.
25. A. Iliev and L. Manivel, ‘Fano manifolds of Calabi–Yau type’, Preprint, 2011, arXiv:1102.3623.
26. A. Iliev and D. Markushevich, ‘The Abel–Jacobi Map for a cubic threefold and periods of Fano threefolds of degree 14’, *Doc. Math.* 5 (2000) 23–47.
27. A. ILIEV and K. RANESTAD, ‘K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds’, Trans. Amer. Math. Soc. 353 (2001) 1455–1468.
28. A. KUZNETSOV, ‘Derived categories of cubic and $V_{14}$ threefolds’, Proc. Steklov Inst. Math. 246 (2004) 171–194.
29. A. KUZNETSOV, ‘Hochschild homology and semiorthogonal decompositions’, Preprint, 2009, arXiv:0904.4330.
30. A. KUZNETSOV, ‘Derived categories of cubic fourfolds’, Cohomological and geometric approaches to rationality problems, Progress in Mathematics 282 (Birkhäuser, Boston, 2010) 219–243.
31. J. M. LANDSBERG and L. MANIVEL, ‘The projective geometry of Freudenthal’s magic square’, J. Algebra 239 (2001) 447–512.
32. J. M. LANDSBERG and L. MANIVEL, ‘On the projective geometry of rational homogeneous varieties’, Comment. Math. Helv. 78 (2003) 65–109.
33. R. LAZARSFELD and A. VAN DE VEN, Topics in the geometry of projective space. Recent work of F.L.Zak. With an addendum by Zak, DMV Seminar. 4 (Birkhäuser Verlag, Boston, 1984).
34. L. MANIVEL, ‘On the derived category of the Cayley plane’, J. Algebra 330 (2011) 177–187.
35. D. MARKUSHEVICH and A. TIKHOMIROV, ‘The Abel–Jacobi map of a moduli component of vector bundles on the cubic threefold’, J. Algebraic Geom. 10 (2001) 37–62.
36. S. MUKAI, Curves, K3 surfaces and Fano 3-folds of genus $\leq 10$, Algebraic geometry and commutative algebra, vol. I (Kinokuniya, Tokyo, 1988) 357–377.
37. R. SCHIMMICH, ‘Mirror symmetry and string vacua from a special class of Fano varieties’, Int. J. Mod. Phys. A 11 (1996) 3049–3096.
38. P. SEIDEL and R. THOMAS, ‘Braid group actions on derived categories of coherent sheaves’, Duke Math. J. 108 (2001) 37–108.
39. K. TAKEUCHI, ‘Some birational maps of Fano 3-folds’, Compos. Math. 71 (1989) 265–283.
40. J. TITS, ‘Les groupes de Lie exceptionnels et leur interprétation géométrique’, Bull. Soc. Math. Belg. 8 (1956) 48–81.
41. S. TREGUB, ‘Construction of a birational isomorphism of a cubic threefold and Fano variety of the first kind with $g = 8$, associated with a normal rational curve of degree 4’, Moscow Univ. Math. Bull. 40 (1985) 78–80.
42. F. L. ZAK, Tangents and secants of algebraic varieties, Translations of Mathematical Monographs 127 (American Mathematical Society, Providence, RI, 1993).

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