Study of the influence factors on the accuracy of sound field reconstruction based on wave superposition method

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Abstract. Wave superposition method is a numerical method for sound field reconstruction, which can be applied in sound source identification and fault diagnosis. This study aims to obtain some guidelines to increase the reconstruction accuracy. A number of influence factors are analyzed through theoretical analysis and numerical simulation, including equivalent sources, measurement array, reconstruction distance and regularization methods. Results show that the distribution of the equivalent sources has a great impact on the reconstruction accuracy. The equivalent sources should be collocated interior to the radiator. It’s better that the equivalent source surface conforms to the radiator’s surface, while the measurement array doesn’t need to be conformal to the radiator’s surface. Relatively, the reconstruction distance has a little influence on the reconstruction accuracy. At last, four different regularization methods together with two regularization parameter selection approaches are combined with wave superposition method to solve the ill-posed problem. Comparative results indicate that Tikhonov and truncate singular value decomposition regularization methods work well with wave superposition method. In sum, these conclusions are useful to get higher reconstruction accuracy and consequently help wave superposition method become an effective fault diagnosis tool.

1. Introduction
As the result of structural vibration emission in air, acoustic signals can reflect the change of the mechanical working condition. Moreover, acoustic signals can be measured in a non-contact way, which can act as an alternative to the vibration signals under some special circumstances such as high temperature, high humidity, corroding environment and so on. So acoustic noise can be used for mechanical condition monitoring and fault diagnosis, and this technique is referred to as acoustical-based diagnosis (ABD) [1]. As one of the mature ABD techniques, acoustic emission (AE) technique has proven to be an important tool for condition monitoring, which can detect a defect even before it appears on the surface [2]. However, it needs special transducers to measure the AE signal released in the very high frequency range. Other ABD techniques are often based on acoustic pressure or acoustic

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intensity measurements, but the measurement results can be easily contaminated in industrial environment. Overall, in these traditional ABD techniques, acoustic signals are often measured by a few microphones and analyzed in the same way as vibration signals. As a result, even though the features in the time domain and frequency domain of the acoustic signals can be extracted, the number and locations of the noise sources can’t be obtained.

Actually, estimation of the number and locations of the sound sources is important for fault diagnosis. Near field acoustical holography (NAH) \[3,4\] has been widely applied in sound source localization and identification. Based on acoustic pressure measurements with a microphone array, NAH enables one to visualize the entire acoustic field of the sound source in three-dimensional space. Recently, a NAH-based fault diagnosis technique \[5, 6\] has been developed and applied in gearbox fault diagnosis. This technique combined NAH with image processing and pattern recognition techniques to diagnose different gearbox defects. Unlike traditional ABD techniques which is often based on single microphone measurement, it utilized the spatial distribution information of the acoustic signals for fault diagnosis. Anyway, the first step of this diagnosis technique is sound field reconstruction which has proven to be an effective way of identifying noise source and its transmission paths. So it’s important to enhance the accuracy of sound field reconstruction.

Wave superposition method (WSM) is proposed by Koopmann \[7\] in 1989 for sound field reconstruction. It is implemented based on the principle of equivalent sources: at first, a serial of equivalent sound sources are collocated on a source surface interior to the surface of a radiator; then, the strengths of these equivalent sources are evaluated in terms of a prescribed normal velocity distribution on the surface of the radiator; once evaluated, these equivalent sources allow the pressure distribution or normal velocity distribution exterior to the radiator to be reconstructed. WSM overcomes two intrinsic obstacles of boundary element method (BEM), and is shown to be a robust method at full wave-numbers \[8, 9\]. As an effective approach for sound field reconstruction for arbitrarily shaped radiators, WSM attracted many researchers’ attention \[10, 11, 12\]. Particularly, Some researchers employed it to identify and locate sound sources in an acoustical feature extraction technique \[13\].

However, WSM doesn't indicate definitely how to collocate the equivalent sources. As shown in references \[11\] and \[12\], engineers usually use an iterative or greedy approach to collocate equivalent sources. There’re also other doubts about the measurement array, reconstruction distance, etc. Because of these uncertainties, it is necessary to evaluate the reconstruction accuracy and analyze the factors that have influences on the reconstruction accuracy. In the present study, many influence factors such as equivalent sources, measurement array, reconstruction distance, and regularization methods are analyzed through theoretical analysis and numerical simulations. Firstly, the principle of sound field reconstruction based on wave superposition is described. Then, the influence factors are analyzed in detail. In the end, several conclusions are drawn.

2. Principle of wave superposition method
As shown in figure 1, an arbitrarily shaped vibrating structure with boundary $S'$ is immersed in a homogeneous, compressible, and inviscid fluid with density $\rho$ and sound speed $c$. The volume enclosed by $S'$ is denoted by $D$ and the volume exterior to it is denoted by $E$. $n$ is the outward normal vector of the surface $S'$. $P$ and $Q$ are two random points in the exterior and interior regions, respectively. $r$ represents the distance between them. Suppose a closed equivalent source surface $S$ locates interior to the surface $S'$. $\Omega$ is the interior region enclosed by $S$, containing a continuous distribution of sources. The exterior sound pressure field is acquired by a microphone array $M$. And sound field is reconstructed on a reconstruction surface $R$. 
Figure 1. Principle of wave superposition based sound field reconstruction.

Assuming a harmonic time dependence of the form $\exp(-i\omega t)$, the sound pressure $p$ in $E$ satisfies the Helmholtz equation under the Neumann boundary condition on $S'$ and a radiation condition at infinity, also termed as the Sommerfeld radiation condition (SRC).

$$\begin{cases}
\nabla^2 p + k^2 p = 0, & p \in E \\
v_n = \frac{\partial p}{\partial r} + ikp, & p \in S' \\
\lim_{r \to \infty} [r \left(\frac{\partial p}{\partial r} + ikp\right)] = 0 & p \in E
\end{cases} \tag{1}$$

where $\nabla^2$ is the Laplace operator, $k$ is the wave number, $v_n$ is the normal particle velocity, $\bar{v}_n$ is the given normal particle velocity, and $i = \sqrt{-1}$. Applying the principle of mass conservation to the volume $\Omega$ leads to a modified Helmholtz non-homogeneous equation. Combining the latter with equation (1) leads the sound pressure representation

$$p(P) = \int_S i\rho\omega q(Q)G(P,Q)dS(Q), \tag{2}$$

and the normal velocity representation

$$v_n(P) = \int_S q(Q)\frac{\partial G(P,Q)}{\partial n_p}dS(Q), \tag{3}$$

where $\omega$ is the angular frequency, $q$ is the source strength, and $G(P,Q)$ is the free-space Green function,

$$G(P,Q) = \frac{\exp(-ikr)}{4\pi r}. \tag{4}$$

To simplify the calculation formula, equivalent sources are laid on the surface $S$ instead of the volume $\Omega$. There is no reason to use a non-uniform distribution of sources on the surface $S$. So, the surface is evenly divided into grids; and each grid is collocated with a simple sound source (monopole or dipole). Assuming that the Green functions are constant over each grid, equations (2) and (3) can be rewritten as matrix forms

$$p = Hq, \tag{5}$$

$$v_n = Tq, \tag{6}$$

where $p$ and $v_n$ are the sound pressure vector and the normal velocity vector, respectively. $H$ and $T$ are the transfer matrices that relate the equivalent sources and the field points, also termed as the
radiation operators. \( q \) is equivalent sources’ strength vector, which is unknown. Once the source points’ and the field points’ locations are defined, the transfer function \( H \) and \( T \) can be easily calculated by equation (4).

Wave superposition based sound field reconstruction consists of two steps. Firstly, the unknown source strength vector \( q \) is solved with equation (5). It’s an inverse problem in the acoustic field, which is discussed in section 3.4. Let field point \( P \) represents any microphone on the measurement array \( M \); the transfer function \( H_{q,M} \) between the virtual sources and the measurement points can be evaluated. By substituting the measured sound pressure vector \( p_M \) into equation (5), the unknown source strength vector \( q \) can be solved as

\[
q = H_{q,M}^{-1} p_M,
\]

(7)

where \( H_{q,M}^{-1} \) is the pseudo-inverse of \( H_{q,M} \). Secondly, the exterior sound field quantities are reconstructed. Let field point \( P \) represent any point on the reconstruction surface \( R \); the transfer function \( H_{q,R} \) between the virtual sources and the reconstruction points can be evaluated. Again, by substituting the solved source strength vector \( q \) into equation (5), the exterior sound pressure field \( p_R \) can be reconstructed as

\[
p_R = H_{q,R} H_{q,M}^{-1} p_M.
\]

(8)

Similarly, the normal velocity field can also be evaluated with equation (6). Then, there is no difficulty in computing the sound intensity.

3. Analysis of the influence factors on reconstruction accuracy

It can be seen from the formulation in section 2 that the equivalent sources’ strengths \( q \) is an intermediate variable. So reconstruction accuracy is determined by measured sound pressure \( p_M \) or normal velocity \( v_n \) and the transfer function \( H \) or \( T \). The transfer functions are determined by the distribution of equivalent sources and field points. A suitable distribution must be found to get higher reconstruction accuracy. As for equivalent sources, they bridge the two steps of sound filed reconstruction and have great influence on the reconstruction accuracy. Since the first step of sound field reconstruction is an inverse problem, the measurement error will be amplified greatly in the solution. For such a problem, a special regularization method can be employed here to deal with it [7]. In consequence, the following factors are chosen to be discussed: equivalent sources, measurement array, reconstruction distance, and regularization methods.

A basic model is set up before the numerical analysis. Consider a typical pulse ball radiator with a spherical surface \( S' \), whose radius is \( r_0 \). The relationship between the radiator \( S' \), the equivalent sources’ surface \( S \), the measurement array and the reconstruction surface is depicted in figure 1. The surface \( S' \) pulses with a normal velocity \( v_0 \) and a frequency \( f_0 \). Let the origin of the coordinate system locate at the centre of the ball source. So the sound pressure at an exterior point \( r \) can be calculated as

\[
p(r) = \frac{\rho c k r^2 v_0}{r(1 + i k r_0)} \exp(\omega t - i k r).
\]

(9)

For all the following cases, \( \rho = 1.29 \text{kg/m}^3, c = 340 \text{m/s}, r_0 = 0.1 \text{m}, v_0 = 2.5 \text{m/s}, f_0 = 500 \text{Hz} \). An error equation can be defined as

\[
e(n) = \left| \frac{p_x(n) - p_y(n)}{p_y(n)} \right| \times 100\%,
\]

(10)
where $p_r$ is the reconstructed sound pressure, $p_t$ is the theoretical sound pressure, and $n$ represents the number of reconstructed points. In order to compare the overall performance of all kinds of conditions, a root mean square (RMS) error equation is also defined as

$$e_{rms} = \frac{\sum |p_r(n) - p_t(n)|^2}{N \sum |p_r(n)|^2} \times 100\% ,$$  

where $N$ represents the total number of points in the reconstruction surface.

### 3.1. Influence of equivalent sources

To determine how equivalent sources affect the reconstruction accuracy, three aspects are discussed: the shape of source surface, the distance between source surface and radiator’s surface, and the number of equivalent sources.

In the model described above, a grid style array consists of $11 \times 11$ microphones with spacing $\Delta = 0.2m$ between each grids is placed 1m away from the origin for sound pressure data acquisition. The reconstructed surface is a planar plane which is tangent to radiator’s surface. Also, it is evenly divided into $11 \times 11$ grids with spacing $\Delta = 0.2m$. The reconstruction distance is 0.1m. Totally, three kinds of source surface shape are chosen: a concentric spherical surface, an ellipsoid surface, and a planar surface, as shown in figure 2 (a), figure 3 (a) and figure 4 (a), respectively. They are divided into grids in every case, and a monopole is placed in each grid. The number of source in all these cases are 42, 42, and 41, respectively. Reconstruction errors are shown in figure 2 (b), figure 3 (b), and figure 4 (b), respectively. It can be found that errors in figure 2 (b) and figure 3 (b) are below 0.2%. While the errors in figure 4 (b) are relatively big, the peak value of which reaches 28%. The results in figure 2 (b) are slightly better than that in figure 3 (b). It reveals that concentric spherical source surface is slightly superior to concentric ellipsoid source surface. So, it’s better to let the source surface be conformal to the radiator’s surface.

Then discussion is focused on the number of equivalent source and the ratio of the radius of the inner equivalent source surface to that of the radiator. The simulation condition here is similar to that in figure 2 (a). Six concentric spherical surfaces with different numbers (14, 26, 42, 62, 86, 114) of equivalent sources are studied. Their reconstruction errors are compared in figure 5. The horizontal axis of figure 5 represents the ratio $r_v / r_o$, where $r_v$ is the radius of the equivalent source surface, and the vertical axis represents the relative reconstruction errors under different conditions. The errors shown here are RMS errors calculated with equation (11).

![Figure 2. Concentric spherical source surface and reconstruction errors.](image)
It can be seen from figure 5 that the errors increase sharply for all curves when the ratio approaches 1.0. It is because that there exists a singular integral problem when the source surface coincides with the radiator’s surface. When the ratio is above 1.0, the reconstruction errors increase greatly since they violate the principle of wave superposition which requires that the sources should be placed interior to the radiator’s surface. When the ratio is below 0.6, the errors of all curves decrease. But if the ratio is too small, the transfer matrices will become ill-conditioned since the matrix elements all approach nearly the same value. So a rule is made that the ratio is supposed to be between 0.1 and 0.6. In the following numerical simulation, the ratio is 0.5. It is also shown that the reconstruction errors decrease as the number of sources increase. The reason is obviously that the more the sources, the finer the radiator’s surface can be described. However, the unknown variables increase as the number of sources increases. So the number of microphone increases accordingly. Also, the transfer matrices will become larger, and the calculation will be more time-consuming consequently.

![Figure 3](image3.png)  ![Figure 4](image4.png)  ![Figure 5](image5.png)
3.2. Influence of measurement array

As for the measurement array, the shape of measurement array is taken into consideration. The focus is mainly concentrated on the problem whether a conformal shape is required or not in the wave superposition method.

The model is the same as that in figure 2 (a). A concentric spherical source surface is placed interior to the radiator with the ratio \( r_i / r_o = 0.5 \), and 42 equivalent sources are evenly collocated on the surface. The measurement distance is 1 m. Six typical array patterns which have the same microphone density are compared, as shown in table 1. Only the spherical cap style and the curve style are conformal to the radiator’s surface. The reconstruction RMS errors are listed in table 2. The underlined bold values represent the largest errors among six styles under each frequency.

**Table 1.** Comparison of six different array styles.

| Style name          | Grid | Spherical cap | Multiple circles | Cross | Curve | Star |
|---------------------|------|---------------|------------------|-------|-------|------|
| Number of microphones | 121  | 120           | 79               | 21    | 61    | 31   |

**Table 2.** Reconstruction errors with different microphone array styles.

| Style name          | \( f / \text{Hz} \) (error units: \( \times 10^{-4} \)) |
|---------------------|--------------------------------------------------------|
|                     | 100          | 200          | 400          | 600          | 800          | 1000          |
| Grid                | 2.0          | 1.7          | 1.8          | 1.9          | 2.0          | 2.0           |
| Spherical cap       | 2.0          | 1.8          | 8.1          | 8.5          | 9.1          | 9.0           |
| Multiple circles    | 11           | 1.7          | 7.8          | 4.3          | 9.1          | 10            |
| Cross               | 2.0          | 1.7          | 1.8          | 2.0          | 2.2          | 3.0           |
| Curve               | 4.0          | 3.9          | 4.5          | 4.7          | 4.8          | 5.0           |
| Star                | 8.0          | 8.0          | 2.9          | 3.1          | 3.5          | 2.0           |

It is seen that the spherical cap style has the largest errors at most frequencies. And the error of the curve style is not yet the smallest one. From the results of the two conformal shapes, it can be deduced that it is unnecessary to use a conformal microphone array. Instead, both grid style and the cross style have the smallest errors at all frequencies, and the grid style is preferred.

3.3. Influence of reconstruction distance

To investigate the influence of reconstruction distance on the reconstruction accuracy, a grid style array is used for sound pressure data acquisition, and simulation conditions are similar to the above. The measurement distance is 1 m, while the reconstruction distance ranges from 0.1 m to 1 m. Meanwhile, a theoretical condition without measurement noise and three practical conditions containing Gauss white noise with signal to noise ratios (SNRs) 80 dB, 40 dB, 20 dB are compared. Tikhonov regularization method together with L-curve regularization parameter selection technique is applied in the three practical conditions, which is discussed intensively in section 3.4. The reconstruction RMS errors are shown in figure 6.
In figure 6, as expected, measurement noise can decrease the reconstruction accuracy. For a given SNR, as the reconstruction distance increases, the reconstruction error decreases on the whole. As a result, if the reconstruction surface is near to the measurement surface, higher reconstruction accuracy will be achieved. Anyway, even under the worst condition that SNR is 20dB, relative RMS errors are below 0.02%. Furthermore, the errors at each individual reconstruction points are calculated with equation (10) under every condition. It is found that they’re all below 5%. Therefore, the reconstruction surface should be exterior to the radiator’s surface, and in that case, the reconstruction distance has a relatively little influence on the reconstruction accuracy.

3.4. Influence of regularization methods
Sound field reconstruction problem is an inverse problem, whose solution does not continuously depend on measurement data. So a small disturbance in the measurement data may cause a large oscillation in its solution. Such a problem is termed as a discrete ill-posed problem. For this problem, the straightforward solution is not satisfactory, whereas, a special regularization method can be employed to deal with it. Different regularization methods differ from each other for the definition of filter factors. Detailed definitions and formulations can be found in the reference [14].

Four well-known regularization methods, Tikhonov, truncate singular value decomposition (TSVD), damped singular value decomposition (DSVD) and conjugate gradients least square (CGLS), are compared here. Two different regularization parameter selection approaches, generalized cross validation (GCV) approach and L-curve criterion approach, are compared accompanied with Tikhonov regularization method. The numerical simulation condition is similar to the above except that the reconstruction distance is fixed at 0.1m, and SNRs are 40dB, 20dB and 10dB. The reconstruction RMS errors are shown in table 3.

Table 3. Reconstruction errors with different regularization methods.

| SNR | Direct | Tikhonov & L-curve | Tikhonov & GCV | TSVD | DSVD | CGLS |
|-----|--------|---------------------|----------------|------|------|------|
| ∞   | 1.8×10^{-4} | 1.8×10^{-4} | 1.8×10^{-4} | 1.8×10^{-4} | 1.8×10^{-4} | 7.9×10^{-4} |
| 40dB | 73.3   | 7.6×10^{-4} | 13×10^{-4} | 4.7×10^{-4} | 0.02 | 0.01 |
| 20dB | 159.2  | 0.01 | 0.01 | 0.01 | 0.04 | 0.01 |
| 10dB | 788.7  | 0.01 | 0.01 | 0.01 | 0.05 | 0.02 |

In table 3, the notation ∞ denotes the theoretical condition without measurement noise, and the underlined bold values represent the maximum values in each row. Obviously, errors increase as the SNR decreases. In the second row, the errors of six methods are approximately identical. Nonetheless, according to the second column, the direct solved solutions in the last three rows have very large errors. The largest error approaches 788% under the condition that SNR is 10dB. In contrast, small
errors are found in the same row with the regularized solutions. Hence, regularization methods play an important role in sound field reconstruction. In order to obtain high reconstruction accuracy, proper regularization method is necessary to be employed. Results also show that Tikhonov and TSVD are superior to the other two, while DSVD is the worst one among them. In addition, GCV and L-curve almost possess the same accuracy. Furthermore, errors at each individual reconstruction point are calculated with equation (10) for the remained three methods, Tikhonov, TSVD and CGLS. It’s found that Tikhonov and TSVD are a bit more accurate than CGLS. The largest errors of the former two under all conditions are below 3%, and they are also more robust than CGLS. In conclusion, Tikhonov and TSVD regularization methods are recommended for sound field reconstruction with wave superposition method.

4. Conclusions
Reconstructing acoustic field of a vibrating object has proven to be an effective way of identifying noise source and its transmission paths. So it is often the first step in noise diagnostic. WSM is an effective approach for sound field reconstruction, which can be applied in machinery condition monitoring and fault diagnosis. The ultimate goal of the present investigation is to provide some guidelines to enhance the reconstruction accuracy of WSM. Through theoretical and numerical analysis, a number of typical factors influencing the reconstruction accuracy are analyzed and some guidelines have been found.

1) The equivalent sources should be collocated to be conformal to the radiator’s surface and retracted inward with a scale between 0.1 and 0.6.

2) The measurement array doesn't need to be conformal to the radiator’s surface. Although the reconstruction distance doesn’t have an great effect on the reconstruction accuracy, it’s better that the reconstruction surface is placed close to the measurement surface to obtain higher accuracy.

3) Tikhonov and TSVD regularization methods perform well in combination with wave superposition method.

These conclusions may improve the reconstruction accuracy and thus be helpful for the application of WSM in fault diagnosis. Based on the accurate reconstructed acoustic quantities, the noise source can be identified correctly. Consequently, it’s easier to detect the locations of the defects. Moreover, combined with the acoustic image pattern recognition methods, WSM can also be further utilized in the NAH-based fault diagnosis technique.

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