INVERSE SPECTRAL THEORY FOR PERTURBED TORUS

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Abstract. We consider an inverse problem for Laplacians on rotationally symmetric manifolds, which are finite for the transversal direction and periodic with respect to the axis of the manifold, i.e., Laplacians on tori. We construct an infinite dimensional analytic isomorphism between the space of profiles (the radius of the rotation) of the torus and the spectral data as well as the stability estimates: those for the spectral data in terms of the profile and conversely, for the profile in term of the spectral data.

1. Introduction

1.1. Geometry. In this paper we study inverse problems on the manifold $M = S^1 \times Y$, where $Y$ is a compact Riemannian manifold of dimension $m$ with or without boundary equipped with the metric $g_Y$, which gives the warped product metric

$$g_M = (dx)^2 + r^2(x)g_Y$$

(1.1)
on $M$. Here, we identify $S^1$ with $T^1 = \mathbb{R}^1/\mathbb{Z} = [0, 1]$ (see Fig. 1). We are mainly interested in the case in which $Y$ is diffeomorphic to (a part of) $S^1$. Therefore, we call $M$ a perturbed torus, and $Y$ a transversal manifold. The Laplacian $\Delta_M$ on $M$ has the form

$$-\Delta_M = -\frac{1}{r(x)^m} \partial_x \left( r(x)^m \partial_x \right) - \frac{\Delta_Y}{r(x)^2},$$

where $\Delta_Y$ is the Laplacian on $Y$, which has a discrete spectrum $0 \leq E_0 < E_1 \leq E_2 \leq ...$ and the associated complete orthonormal system of eigenfunctions $\Psi_\nu, \nu \geq 0$, in $L^2(Y)$. Introduce the Hilbert spaces

$$\mathcal{L}_\nu^2 = \left\{ f(x)\Psi_\nu(y); (x, y) \in S^1 \times Y, \int_0^1 |f(x)|^2 r^m(x)dx < \infty \right\}, \quad \nu \geq 0.$$ Then, the Laplacian on $(M, g)$ acting on $L^2(M) \simeq \bigoplus_{\nu \geq 0} \mathcal{L}_\nu^2$ is unitarily equivalent to a direct sum of one-dimensional operators $\Delta_\nu$, namely,

$$\Delta_M \simeq \bigoplus_{\nu = 0}^\infty \Delta_\nu.$$ Here the operator $\Delta_\nu$ acts in the space $L^2([0, 1], r^m(x)dx)$ and is given by

$$-\Delta_\nu = -\frac{1}{r(x)^m} \partial_x \left( r(x)^m \partial_x \right) + \frac{E_\nu}{r(x)^2}.$$
1.2. Example. Our motivating example is the following perturbed torus in $\mathbb{R}^3$:

\[
x = (a + R(\theta) \cos \phi) \cos \phi, \quad y = (a + R(\theta) \cos \theta) \sin \phi, \quad z = a + R(\theta) \sin \theta
\]

where

\[
a > 0, \quad R(\theta) \in C^2(S^1), \quad 0 < R(\theta) < a, \quad \theta, \phi \in [0, 2\pi].
\]

The induced metric on this surface is

\[
ds^2 = (R'(\theta)^2 + R(\theta)^2) (d\theta)^2 + (a + R(\theta) \cos \theta)^2 (d\phi)^2.
\]

We put

\[
b = \int_0^{2\pi} \sqrt{R'(\theta)^2 + R(\theta)^2} d\theta,
\]

and make the change of variable $t = t(\theta)$ by

\[
\frac{dt}{d\theta} = \sqrt{R'(\theta)^2 + R(\theta)^2}.
\]

Then we have

\[
ds^2 = (dt)^2 + r(t)^2 (d\phi)^2, \quad r(t) = a + R(\theta(t)) \cos \theta(t),
\]

where $0 \leq \phi \leq 2\pi, 0 \leq t \leq b$. We also have

\[
|r'(t)| \leq 1,
\]

since

\[
r'(t) = \frac{d}{d\theta} \left( \frac{R(\theta) \cos \theta d\theta}{dt} \right) = \frac{R'(\theta) \cos \theta - R(\theta) \sin \theta}{\sqrt{R'(\theta)^2 + R(\theta)^2}}.
\]

Putting $\tau = \frac{t}{b}$, we can rewrite it as

\[
ds^2 = b^2 \left( (d\tau)^2 + h(\tau)^2 (d\phi)^2 \right), \quad h(\tau) = \frac{r(t)}{b}, \quad |h'(\tau)| = |r'(t)| \leq 1, \quad \forall t \in [0,b].
\]

1.3. Problem. This example leads us to the manifold $M = S^1 \times Y$ equipped with metric

\[
y_M = b^2 \left( (dx)^2 + r(x)^2 g_Y \right),
\]

where $b > 0$ is a constant. We assume that

\[
r(x) \in C^\infty(S^1), \quad r(x) > 0, \quad |r'(x)| \leq 1.
\]

The Laplace-Beltrami operator on $M$ is

\[
\frac{1}{b^2} \left( \frac{1}{r^m(x)} \partial_x r^m(x) \partial_x - \frac{1}{r(x)^2} \Delta_Y \right).
\]
Letting $E_\nu$ be one of the eigenvalues of $-\Delta_Y$, we consider the operator $\Delta_{\nu,b}$ on $S^1$ given by
\[ -\Delta_{\nu,b} = \frac{1}{b^2} \left( -\frac{1}{r^m(x)} \partial_x r^m(x) \partial_x + \frac{E_\nu}{r^2(x)} \right). \] (1.5)

Let us note that one can compute the constant $b$ from the asymptotics of eigenvalues of the operator $-\Delta_{\nu,b}$. Therefore, below we take $b = 1$ for the sake of simplicity and let $\Delta_{\nu} = \Delta_{\nu,1}$.

Our problem is now stated as follows:

Determine $r(x)$ from the knowledge of the spectrum of $-\Delta_\nu$.

Assume that $r(x)$ is 1-periodic and is given by
\[ r(x) = r_0 e^{2\pi Q(x)}, \quad Q(x) = \int_0^x q(t) dt, \quad x \in [0,1], \] (1.6)
where $r_0 > 0$ is a constant. Our results are stated in terms of $q$ and summarized in Section 2.

1.4. Analytic approach to inverse problems. There are various methods for inverse problems. We employ here the analytic (or direct) approach due to [13], [17] based on nonlinear functional analysis, which we briefly explain below. Suppose that $\mathcal{H}, \mathcal{H}_1$ are real separable Hilbert spaces. The derivative of a map $f : \mathcal{H} \to \mathcal{H}_1$ at a point $y \in \mathcal{H}$ is a bounded linear map from $\mathcal{H}$ into $\mathcal{H}_1$, which we denote by $\frac{df}{dy} f$. A map $f : \mathcal{H} \to \mathcal{H}_1$ is compact on $\mathcal{H}$, if it maps any weakly convergent sequence in $\mathcal{H}$ to a strongly convergent sequence in $\mathcal{H}_1$.

A map $f : \mathcal{H} \to \mathcal{H}_1$ is a real analytic isomorphism between $\mathcal{H}$ and $\mathcal{H}_1$, if $f$ is bijective and both $f$ and $f^{-1}$ are real analytic. Let $\mathcal{H}_C$ be the complexification of $\mathcal{H}$. We recall the main result in [17], [18].

**Theorem 1.1.** Let $\mathcal{H}, \mathcal{H}_1$ be real separable Hilbert spaces equipped with norms $\| \cdot \|, \| \cdot \|_1$. Suppose that a map $f : \mathcal{H} \to \mathcal{H}_1$ satisfies the following conditions:
(A) $f$ is real analytic,
(B) $\frac{df}{dy} f$ has an inverse for each fixed $q \in \mathcal{H}$,
(C) there is a nondecreasing function $\eta : [0, \infty) \to [0, \infty)$ such that $\eta(0) = 0$, and
\[ \|q\| \leq \eta(\|f(q)\|_1), \quad \forall \ q \in \mathcal{H}. \]
(D) there exists a linear isomorphism $f_0 : \mathcal{H} \to \mathcal{H}_1$ such that the mapping $f - f_0 : \mathcal{H} \to \mathcal{H}_1$ is compact.

Then $f$ is a real analytic isomorphism between $\mathcal{H}$ and $\mathcal{H}_1$.

In our applications, $f(q)$ is supposed to be a spectral data. The main issue is to derive a priori estimates of $\|q\|$ in terms of spectral data $\eta(\|f(q)\|_1)$.

2. Main results

2.1. Inverse problem for perturbed torus. We assume that the profile $r(x), x \in [0,1]$, satisfies [17] with
\[ q \in \mathcal{H}_0 \cap \mathcal{H}_1, \quad r(0) = r(1) = r_0 > 0, \] (2.1)
for a fixed radius $r_0$. Here $\mathcal{H}_j$ is the Sobolev space of real functions defined by
\[ \mathcal{H}_j = \mathcal{H}_j(T) = \left\{ q \in L^2(T) : q^{(j)} \in L^2(T), \int_0^1 q(x) dx = 0 \right\}, \quad j \geq 0, \]
equipped with norm
\[ \|q\|_j^2 = \|q^{(j)}\|^2 = \int_0^1 |q^{(j)}(x)|^2 \, dx, \]
and
\[ \mathcal{H}_1 = \mathcal{H}_1(\mathbb{T}) = \{ f = g' \mid g \in \mathcal{H}_0 \} \]
equipped with norm
\[ \|f\|_{-1}^2 = \|g\|^2 = \int_0^1 |g(x)|^2 \, dx. \]

We also introduce a real Hilbert space \( \ell^2_\lambda \) equipped with norm
\[ \|f\|^2_{\ell^2_\lambda} = \sum_{n \geq 1} (2\pi n)^2 |f_n|^2, \quad j \in \mathbb{R}. \] (2.2)

We fix \( \nu \) arbitrarily and omit the subscript \( \nu \) of \( -\Delta_\nu \) for the sake of simplicity. In [26, 20], we have already developed the inverse spectral theory for the case \( E_\nu = 0 \). Below we consider the case \( E_\nu > 0 \). The spectrum of \( -\Delta_\nu \) on \( \mathbb{T} \) is discrete and consists of eigenvalues \( \lambda_{2n}^\pm = \lambda_{\nu,2n}^\pm, n \geq 0 \), for the equation
\[ -\frac{1}{\varrho^2}(\varrho^2 f')' + \frac{E_\nu}{\varrho^2} f = \lambda f \] (2.3)
with periodic boundary condition \( y(x + 1) = y(x), x \in \mathbb{R} \). They are labeled as
\[ \lambda_0^- < \lambda_1^- < \lambda_2^- < \lambda_3^- < \lambda_4^- < \cdots. \] (2.4)
The anti-periodic eigenvalues \( \lambda_{2n-1}^\pm, n \geq 1 \), are the eigenvalues of the equation (2.3) under antiperiodic boundary condition, i.e., \( y(x + 1) = -y(x), x \in \mathbb{R} \). Thus the eigenfunctions corresponding to \( \lambda_n^\pm \) are periodic with period 1 for even \( n \), and antiperiodic for odd \( n \). It is well-known that the periodic eigenvalues \( \lambda_0^+ = \lambda_{\nu,1}^+ > 0 \), \( \lambda_{2n}^+, n \geq 1 \), determine the anti-periodic eigenvalues \( \lambda_{2n-1}^+, n \geq 1 \), and that they satisfy
\[ \lambda_0^+ < \lambda_1^- < \lambda_1^+ < \lambda_2^- < \lambda_2^+ < \lambda_3^- < \lambda_3^+ < \lambda_4^- < \cdots. \] (2.5)
The gap \( \gamma_n \) for the operator \( -\Delta_\nu \) is defined by
\[ \gamma_n = [\lambda_n^-, \lambda_n^+], \quad n \geq 1. \] (2.6)

We also use the spectral data for the Sturm-Liouville problem in the impedance form under Dirichlet boundary condition:
\[ -\frac{1}{\varrho^2}(\varrho^2 f')' + \frac{E_\nu}{\varrho^2} f = \lambda f, \quad f(0) = f(1) = 0. \] (2.7)

Let \( \mu_n = \mu_n(q), n \geq 1 \), and \( f_n = f_n(x, q) \) be the eigenvalues and the associated eigenfunctions. It is well-known that all \( \mu_n \) are simple and are labeled as \( \mu_1 < \mu_2 < \mu_3 < \cdots \). Moreover, the Dirichlet eigenvalue \( \mu_n \) belongs to the interval \( \gamma_n \), i.e.,
\[ \mu_n \in [\lambda_n^-, \lambda_n^+], \quad \forall \ n \geq 1. \] (2.8)

As in [24, 28], we construct the gap length mapping \( \psi \) by
\[ \psi : q \rightarrow \psi(q) = (\psi_n)_1^\infty, \quad \psi_n = (\psi_{n1}, \psi_{n2}) \in \mathbb{R}^2, \] (2.9)
where $\psi_{n1}$ and $\psi_{n2}$ are given by

$$\psi_{n1} = \frac{\lambda_n^- + \lambda_n^+}{2} - \mu_n, \quad \psi_{n2} = \left| \frac{\gamma_n}{4} - \psi_{n1}^2 \right|^{\frac{1}{2}} \text{sign } \kappa_n,$$

the norming constants $\kappa_n = \kappa_{\nu,n}$ are defined by

$$\kappa_n = \log \left| \frac{f_n'(1)}{f_n'(0)} \right|, \quad n \geq 1,$$

and where $f_n$ is an eigenfunction for the Dirichlet problem (2.7) associated with the eigenvalue $\mu_n$. Such mapping was introduced in [24] for the Schrödinger operator with periodic potential on the circle. It is known that $\psi$ maps $\mathcal{H}$ into $\ell^2 \oplus \ell^2$ (see [27]). Note that the vector $\psi$ depends on the gap lengths $|\gamma_n|$, sign $\kappa_n$ and $\psi_{n1}$ for all $n \geq 1$, but is independent of the location of the gaps and the Dirichlet eigenvalues themselves.

The standard arguments (see (3.2)-(3.5)) show that the operator $\Delta_\nu$ is unitarily equivalent to a one-dimensional Schrödinger operators $S_{\nu,y} = -y'' + py$ on $L^2(\mathbb{T})$, where the potential $p$ is given by

$$p = q' + q^2 + u_\nu - c_{\nu,0}, \quad u_\nu = \frac{E_\nu}{r_0^2} e^{\frac{-1}{2\nu}} q, \quad c_{\nu,0} = \int_0^1 (q^2 + u_\nu) dx. \quad (2.12)$$

If $q \in \mathcal{H}_j, j \geq 1$, then we deduce that the potential $p \in \mathcal{H}_{j-1} \subset L^2(\mathbb{T})$. If $q \in \mathcal{H}_0$, then we deduce that the potential $p \in \mathcal{H}_{-1}$ is a distribution.

**Theorem 2.1.** Fix $\nu \geq 0$ and let $m \geq 1$. Then the mapping $\psi : \mathcal{H}_1 \to \ell^2 \oplus \ell^2$, given by (2.9), is a real analytic isomorphism between the Hilbert spaces $\mathcal{H}_1$ and $\ell^2 \oplus \ell^2$. Moreover, for all $q \in \mathcal{H}_1$ the following estimates hold true:

$$\|q'\| \leq \|p\| \leq 2\|q\| (1 + \|q\|^{\frac{1}{2}}),$$

$$\|q\| \leq w(1 + w^{\frac{1}{2}}), \quad (2.13)$$

where $w = \|q'\| + \|q\| (\|q'\| + C_* e^\beta |q|)$ and

$$C_* = A(\beta + \alpha)(2 + \beta A), \quad \beta = \frac{4}{m}, \quad A = \frac{E_\nu}{r_0^2}.$$  

We consider the inverse problem for more singular coefficients $q \in \mathcal{H}_0$. In this case, in general, the gap length $|\gamma_n|$ is increasing as $n \to \infty$.

**Theorem 2.2.** i) Fix $\nu \geq 0, m \geq 1$. Then the mapping $\psi : \mathcal{H}_0 \to \ell^2_{-1} \oplus \ell^2_{-1}$, given by (2.9), is real analytic, injective and satisfies

$$\|q\| \leq \|q'\| (1 + 2\|q\|^{3}),$$

$$\|p\| \leq \|q\| (3 + 2\|q\| + \beta A e^\beta |q|), \quad (2.15)$$

ii) Let, in addition, $m = 1$. Then the mapping $\psi : \mathcal{H}_0 \to \ell^2_{-1} \oplus \ell^2_{-1}$ is a real analytic isomorphism between the Hilbert spaces $\mathcal{H}_0$ and $\ell^2_{-1} \oplus \ell^2_{-1}$ and

$$\|q\|^2 \leq 2\|p\|^2 (1 + 2\|q\|^2),$$

$$\|p\| \leq 96\pi^2 \|q\|_{-1} (1 + 2\|q\|_{-1}^3). \quad (2.16)$$
Remark. 1) If we know the location of gaps $\gamma_n$, Dirichlet eigenvalues $\mu_n$ and sign $\varkappa_n$ for all $n \geq 1$, then we can obtain $q$ uniquely. Moreover, in order to determine $q$ we need to know only the vector $\psi(q)$. From the value $\psi(q)$, we can compute the gap length $|\gamma_n|$, the distance $\psi_n$ between the Dirichlet eigenvalue and the center of the gap for any $n \geq 1$.

2) The proof of Theorems 2.1 and 2.2 is based on non-linear functional analysis. It is crucial that the mapping $\psi(q)$ is a composition of two mappings $\Psi(p)$ and $p = P(q)$ given by (3.14) and (2.17). All necessary properties for $\Psi(q)$ are described in Theorem 3.1. The mapping $q \to P(q)$ is defined by Theorem 2.3. The a priori estimates of $\|q\|$ in terms of $\|P(q)\|$ play an important role in the proof.

2.2. Perturbed Riccati mappings. Recall that $q \to q' + \alpha q^2$ is the Riccati mapping. In order to prove our main theorems we use perturbed Riccati mappings $P : \mathcal{H}_j \to \mathcal{H}_{j-1}$ given by

$$q \to P, \quad P(q) = q' + \alpha q^2 + u - c_0,$$

$$u = Ar^{-\beta}Q, \quad Q(x) = \int_0^x q(t)dt, \quad c_0 = \int_0^1 (\alpha q^2 + u)dx,$$  \hspace{1cm} (2.17)

where $\alpha, \beta > 0$ and $A \geq 0$. We also consider the singular case $q \in \mathcal{H}_0$, which is more complicated.

Theorem 2.3. i) The mapping $P : \mathcal{H}_j \to \mathcal{H}_{j-1}, j \geq 1$, given by (2.17) is a real analytic isomorphism between the Hilbert spaces $\mathcal{H}_j$ and $\mathcal{H}_{j-1}$ and satisfies:

$$\|q'\|^2 \leq \|P(q)\|^2 \leq \|q'\|^2 + \alpha^2 2\|q\|^3\|q'\| + C_*\|q\|^2 e^{2\beta\|q\|},$$  \hspace{1cm} (2.18)

where the constant $C_* = A(\beta + \alpha)(2 + \beta A)$.

ii) Let $j = 0$. Then the mapping $P : \mathcal{H}_0 \to \mathcal{H}_{-1}$, given by (2.17) is real analytic and injective. Moreover,

$$\|P(q)\|_{-1} \leq \|q\|(3 + 2\|q\| + \beta A e^{\beta\|q\|}).$$  \hspace{1cm} (2.19)

If $m = 1$, then the mapping $P : \mathcal{H}_0 \to \mathcal{H}_{-1}$ is a real analytic isomorphism between the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_{-1}$ and satisfies:

$$\|q\|^2 \leq 2\|P(q)\|^2_{-1}(1 + 2\|P(q)\|^2_{-1}).$$  \hspace{1cm} (2.20)

Remark. 1) When $m \geq 2$, we do not have estimates of $\|q\|$ in terms of $\|P(q)\|$, which causes a difficulty in the inverse problem for the mapping $P : \mathcal{H}_0 \to \mathcal{H}_{-1}$.

2.3. Symmetric surfaces. Define the spaces of even functions $\mathcal{H}_j^{\text{even}}(\mathbb{T})$, and of odd functions $\mathcal{H}_j^{\text{odd}}(\mathbb{T})$ by

$$\mathcal{H}_j^{\text{even}} = \left\{ q \in \mathcal{H}_j : q(x) = q(1-x), \; \forall \; x \in (0,1) \right\},$$

$$\mathcal{H}_j^{\text{odd}} = \left\{ q \in \mathcal{H}_j : q(x) = -q(1-x), \; \forall \; x \in (0,1) \right\},$$  \hspace{1cm} (2.21)

Note that we have $\mathcal{H}_j = \mathcal{H}_j^{\text{even}} \oplus \mathcal{H}_j^{\text{odd}}$. It is well-known that if $P(q) \in \mathcal{H}_j^{\text{even}}$, we have either $\mu_n = \lambda_n^\ast$ or $\mu_n = \lambda_n^\ast$, see [12]. Then, $\psi_{n2} = 0$ and $|\psi_{n1}| = |\gamma_n|/2$ for all $n \geq 1$. 
Corollary 2.4. i) The mapping \( P : \mathcal{H}_j^{\text{odd}} \rightarrow \mathcal{H}_{j-1}^{\text{even}}, j \geq 1 \), given by (2.17) is a real analytic isomorphism between the Hilbert spaces \( \mathcal{H}_j^{\text{odd}} \) and \( \mathcal{H}_{j-1}^{\text{even}} \).

ii) Fix \( \nu \geq 0 \) and let \( m \geq 1 \). Then the mapping \( \psi^e : \mathcal{H}_1^{\text{odd}} \rightarrow \ell^2, \) given by
\[
q \rightarrow \psi^e = (\psi^e_n)_n, \quad \psi^e_n = \frac{\lambda_n^- + \lambda_n^+}{2} - \mu_n
\] (2.22)
is a real analytic isomorphism between the Hilbert spaces \( \mathcal{H}_1^{\text{odd}} \) and \( \ell^2 \).
Moreover, if \( m = 1 \) then the mapping \( \psi^e : \mathcal{H}_0^{\text{odd}} \rightarrow \ell^2_{-1}, \) given by (2.22) is a real analytic isomorphism between the Hilbert spaces \( \mathcal{H}_0^{\text{odd}} \) and \( \ell^2_{-1} \).

Remark. Similarly to (2.13) and (2.14), one can also derive the estimates of \( q \) in terms of gap-length.

2.4. Minkowski problem. The Minkowski problem deals with the existence of a convex surface with a prescribed Gaussian curvature. More precisely, for a given strictly positive real function \( F \) defined on a sphere, one seeks a strictly convex compact surface \( S \), whose Gaussian curvature at \( x \) is equal to \( F(\mathbf{n}(x)) \), where \( \mathbf{n}(x) \) denotes the outer unit normal to \( S \) at \( x \). The Minkowski problem was solved by Pogorelov \[36\] and by Cheng-Yau \[5\].

We consider only the case \( m = \dim Y = 1 \). Note that our surface is not convex, in general. We solve an analogue of the Minkowski problem in the case of the surface of revolution by showing the existence of a bijection between the Gaussian curvatures and the profiles of surfaces.

It is well-known that the Gaussian curvature \( G \) is given by
\[
G = -\frac{r''}{r} = -v' - v^2, \quad v = 2q, \quad r = r^\frac{1}{2}, \quad \text{for} \quad m = 1.
\] (2.23)

We define a new variable \( G_1 \in \mathcal{H}_0 \) by
\[
G = G_0 + G_1, \quad G_0 = \int_0^1 G(x)dx,
\]
\[
G_1 = -v' - v^2 + \int_0^1 v^2 dx.
\] (2.24)

Assuming that the function \( G_1 \in \mathcal{H}_0 \) is given, we determine the profile (radius) \( r = r(x), x \in [0, 1], \) and the constant \( G_0 \) by solving the equation (2.23).

Recall that \( m = \dim Y \). The eigenvalues \( \mathcal{E} \) and \( \epsilon_j, j = 1, \cdots, m, \) of the Ricci tensor \( \text{Ric}_{ij} \) are given by
\[
\mathcal{E} = -v' - \frac{v^2}{m}, \quad v = 2q,
\]
\[
\epsilon_j = \frac{\kappa_j}{r^2} - \frac{1}{m} (v' + v^2), \quad j = 1, \cdots, m,
\] (2.25)
where \( \kappa_j > 0 \) denote the eigenvalues of the Ricci tensor \( \text{Ric}_{ij} \) on \( Y \). In particular, if \( Y \) is a sphere, then all eigenvalues \( \kappa_j = \kappa > 0, j = 1, \cdots, m \) for some \( \kappa > 0 \). In this case, the Ricci tensor of the warped product has one simple eigenvalue \( E \) and an eigenvalue \( e_1 = \cdots = e_m \) of multiplicity \( m \) given by
\[
\mathcal{E} = -v' - \frac{1}{m} v^2, \quad e_1 = \frac{\kappa}{r^2} - \frac{1}{m} (v' + v^2).
\] (2.26)
Note that if \( m = 1 \), then the eigenvalues \( E \) of the Ricci tensor coincides with the Gaussian curvature \( G \), i.e.,

\[
E = G \quad \text{at} \quad m = 1.
\]

For the function \( q \in \mathcal{H}_j \) we define the constant \( E_0 \) and the function \( E_1 \in \mathcal{H}_{j-1} \) by

\[
E = E_0 + E_1, \quad E_0 = \int_0^1 E \, dx = -\frac{1}{m} \int_0^1 v^2(x) \, dx \leq 0,
\]

\[
E_1 = -v' - \frac{1}{m} v^2 + \frac{1}{m} \int_0^1 v^2(x) \, dx.
\]

If \( v = 2q \in \mathcal{H}_j \), then \((2.28)\) gives that \( E \in \mathcal{H}_{j-1} \). The function \( E_1(x), x \in [0,1] \), is the non-constant part of eigenvalue \( E(x), x \in [0,1] \) of the Ricci curvature tensor and \( E_0 = \text{const} \) is the constant part of eigenvalue \( E \).

**Corollary 2.5.** Let \( m \geq 1 \). Then the mapping \( q \to E_1 \) given by \((2.28)\) is a real analytic isomorphism between the Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_0 \) and satisfies:

\[
\|v'\|^2 \leq \|E_1\|^2 \leq \|v'\|^2 + \frac{\|v\|^2 \|v'\|^2}{m^2} - \frac{\|v\|^4}{m^2}.
\]

Moreover, the constant \( E_0 \) is uniquely defined by \( E_1 \).

**Remark.** 1) In the case \( m = 1 \), this corollary solves the Minkowski problem for the perturbed torus.

2) This theorem gives an isomorphism between \( E_1 \) and the function \( q \). Thus, in addition to Theorem \( 2.1 \) we can get a parametrization of the surface by \( E_1 \) or the function \( q \) or the vector \( \psi \):

\[
q \iff E_1 \iff \psi.
\]

2.5. **Brief overview.** There is an abundance of works devoted to the spectral theory and inverse problems for the surface of revolution from the view points of classical inverse Strum-Liouville theory, integrable systems, micro-local analysis, see \[1\], \[11\] and references therein. For integrable systems associated with surfaces of revolution, see e.g. \[21\], \[40\], \[39\] and references therein.

Bruning-Heintz \[4\] proved that the symmetric metric is determined from the spectrum by using the standard 1-dimensional Gel’fand-Levitan theory. We mention the work of Zelditch \[11\], which proved that the isospectral revolutionary surfaces of simple length spectrum, with some additional conditions, are isometric. Isozaki-Korotyaev \[15\], \[16\] solve the inverse spectral problem for rotationally symmetric manifolds (finite perturbed cylinders), which include a class of surfaces of revolution, by giving an analytic isomorphism from the space of spectral data onto the space of functions describing the radius of rotation. An analogue of the Minkowski problem is also solved.

As far as we know, there were no results about inverse problems for perturbed torus.

In this paper we use inverse spectral theory for Schrödinger operators with potentials on the circle. Let us briefly review the inverse spectral theory for Strum-Liouville operators on the circle, mostly focusing on the characterization problem, i.e., the complete description of spectral data that correspond to some fixed class of potentials. More information about different approaches to inverse spectral problems can be found in the monographs \[32\], \[31\] and the papers \[33\], \[22\], \[24\] and \[28\] and references therein.
Dubrovin [10], Its and Matveev [14], Novikov [35], Trubowitz [38] considered the inverse problem for finite band potentials (potentials were more general in [38]). Marchenko-Ostrovski [33] solved the inverse problem including the characterization in terms of spectral data associated with a "global quasimomentum". Note that their construction and the proof are complicated, for example, they used the inverse spectral theory for the scattering on the half-line.

Garnett–Trubowitz [13] solved the inverse problem for the gap-length mapping for the case of even potentials, under the conjecture that there exists an estimate of potentials in terms of their gap lengths. Kargaev–Korotyaev [17] gave a simplified proof of the result of Garnet and Trubowitz [13]. The Garnett–Trubowitz conjecture on the estimate of potentials in terms of their gap lengths was proved by Korotyaev [23], [26]. Korotyaev [22] reproved shortly Marchenko-Ostrovski results [33]. Korotyaev [24] constructed the gap-length mapping given by (3.14) for potentials from $H^0$ and from $H^{-1}$ and solved the corresponding inverse problems. Note that for the case $H^{-1}$ inverse problems for the operator in impedance form are important and were solved for the Dirichlet problem Coleman-McLaughlin [7] and for periodic case by Korotyaev [24].

We use also results on perturbed Riccati mappings from [27], [28], [2].

2.6. Plan of the paper. In Section 2, we prove Theorems 2.1 and 2.3, which are based on the Theorem 1.1 in non-linear functional analysis [17]. We do it after preparing a priori estimates for the perturbed Riccati mapping. In Section 3 we prove Theorems 2.2, where the main problem is a priori estimates for the perturbed Riccati mapping. For the moment we can do it only for $m = 1$. In Section 3, we consider the mapping associated with the Minkowski problem and the eigenvalue of the Ricci tensor.

3. Proof of Theorem 2.1

3.1. The unitary transformations. We begin by explaining the crucial role of the factorization and the non-linear mapping $P$ given by (2.17). For each $q \in H_j, j \geq 0$, we define the periodic weighted (model) operator $-\Delta_\nu$ given by

$$-\Delta_\nu f = -\frac{1}{r_m}(r^m f')' + u_\nu f, \quad u_\nu = \frac{E_\nu}{r^2}, \quad r = r_0 e^{2Q}, \quad Q = \int_0^x q(t) dt,$$

(3.1)

where $f \in L^2(\mathbb{T}, r^m dx)$. The property of the operator $-\Delta_\nu$ is well-known (see [30], [26], [28]). The spectrum of the operator $-\Delta_\nu$ is discrete and given by (2.5). We define the unitary transformation $\mathcal{U} : L^2(\mathbb{T}, g^2 dx) \to L^2(\mathbb{T}, dx)$ by

$$\mathcal{U} f = \varrho f, \quad \varrho = r_0^\frac{m}{2} = \varrho_0 e^{Q}, \quad \varrho_0 = r_0^\frac{m}{2}.$$

Using $\mathcal{U}$ and the equality $r = r_0 e^{\int_0^x q(t) dt}$, we transform the operator $-\Delta_\nu$ into the Schrödinger operator $S_\nu:$$$

\mathcal{U} (-\Delta_\nu) \mathcal{U}^{-1} = -\varrho^{-1} \partial_x \varrho \partial_x \varrho^{-1} + u_\nu = D^* D + u_\nu = S_\nu + c_{\nu,0},$$$

(3.2)

where

$$D = \varrho \partial_x \varrho^{-1} = \partial_x - q, \quad D^* = -\partial_x - q,$$

(3.3)

$$D^* D = - (\partial_x + q)(\partial_x - q) = - \partial_x^2 + q^2 + q^2,$$

(3.4)

$$S_\nu y = -y'' + P_\nu(q)y.$$
acting on $L^2(\mathbb{T}, dx)$, and where the periodic potential $P_\nu(q)$ has the form (2.17) with $A = \frac{E_\nu}{r_0^2}, \beta = \frac{4}{m}, \alpha = 1$, i.e.,

$$P_\nu(q) = q' + q^2 + u_\nu - c_{\nu,0}, \quad u_\nu = \frac{E_\nu}{r_0^2} e^{- \frac{1}{2} Q}, \quad c_{\nu,0} = \int_0^1 (q^2 + u_\nu) dx. \quad (3.5)$$

Thus we see that the Laplacian on $(M, g_M)$ is unitarily equivalent to a direct sum of one-dimensional Schrödinger operators $S_\nu$, namely,

$$\Delta_{(M,g)} \cong \oplus_{\nu \geq 1} (S_\nu + c_{\nu,0}), \quad (3.6)$$

where the direct sum acts on $\oplus_{\nu \geq 1} L^2(\mathbb{T}, dx)$.

3.2. Schrödinger operators on the circle. Using the global transformation $q \to P_\nu(q)$ we reduce our operators $-\Delta_\nu$ to the Schrödinger operators $S_\nu = -\frac{d^2}{dx^2} + P_\nu(q)$ with the potential $P_\nu(q)$ (including the singular case $P_\nu(q) \in \mathcal{H}_1^1$).

We recall some results about inverse spectral theory for Schrödinger operator with potential $p \in \mathcal{H}_j, j \geq -1$. We consider the operator $S = -\frac{d^2}{dx^2} + p(x)$, acting on $L^2(\mathbb{T})$, where the 1-periodic potential $p$ belongs to the Hilbert space $\mathcal{H}_j, j \geq -1$. The spectrum of $S$ on $\mathbb{T}$ is discrete and consists of eigenvalues $\lambda_n^+ + \lambda_n^-$, $n \geq 1$, which are eigenvalues of the equation

$$-f'' + pf = \lambda f. \quad (3.7)$$

with 1-periodic boundary conditions. They are labeled by

$$\lambda_0^+ < \lambda_2^- < \lambda_2^+ < \lambda_4^- < \ldots \quad (3.8)$$

It is well known that if we know all periodic eigenvalues $\lambda_0^+, \lambda_{2n}^+, n \geq 1$, then we can recover so-called anti-periodic eigenvalues $\lambda_{2n-1}^-, n \geq 1$. The anti-periodic eigenvalues $\lambda_{2n-1}^+, n \geq 1$ are eigenvalues of the equation (3.7) under antiperiodic boundary conditions, i.e., $y(x+1) = -y(x), \quad x \in \mathbb{R}$. Thus the eigenfunctions corresponding to $\lambda_n^\pm$ have period 1 when $n$ is even and they are antiperiodic, when $n$ is odd. It is important that they satisfy

$$\lambda_0^+ < \lambda_1^- < \lambda_1^+ < \lambda_3^- < \lambda_3^+ < \lambda_4^- < \lambda_4^+ < \ldots \quad (3.9)$$

Introduce so-called gaps $\gamma_n$ for the operator $S$ by

$$\gamma_n = [\lambda_n^-, \lambda_n^+], \quad n \geq 1. \quad (3.10)$$

It is important that each Dirichlet eigenvalue $\mu_n, n \geq 1$ belongs to the interval $[\lambda_n^-, \lambda_n^+]$, i.e.,

$$\mu_n \in [\lambda_n^-, \lambda_n^+] \quad \forall \ n \geq 1. \quad (3.11)$$

Introduce the fundamental solutions $\varphi(x, \lambda, p)$ and $\psi(x, \lambda, p)$ of the equation

$$-y'' + py = \lambda y, \quad \lambda \in \mathbb{C}, \quad (3.12)$$

under conditions

$$\varphi'(0, \lambda, p) = \psi(0, \lambda, p) = 1, \quad \varphi(0, \lambda, p) = \psi'(0, \lambda, p) = 0. \quad$$

We introduce the Lyapunov function (the discriminant) $\Lambda$ by

$$\Lambda(\lambda, p) = \frac{1}{2} (\varphi'(1, \lambda, p) + \psi(1, \lambda, p)).$$

Note that

$$\Lambda(\lambda_n^\pm, p) = (-1)^n, \quad \forall \ n \geq 1. \quad (3.13)$$

Here and in the sequel, $' = \partial/\partial x, \quad ^\cdot = \partial/\partial z, \quad \partial = \partial/\partial q.$
3.3. Gap length mapping. Using the gaps $\gamma_n = [\lambda_n^-, \lambda_n^+], n \geq 1$, the Dirichlet spectrum $\mu_n(p)$ and the sign of the norming constants $\text{sign} K_n$, we now construct the gap length mapping $\Psi : \mathcal{H}_1 \to \ell^2 \oplus \ell^2$ by

$$p \to \Psi = (\Psi_n)_1^\infty, \quad \Psi_n = (\Psi_{n1}, \Psi_{n2}) \in \mathbb{R}^2. \quad (3.14)$$

Here the coordinates $\Psi_{n1}, \Psi_{n2}$ are given by

$$\Psi_{n1} = \frac{\lambda_n^- + \lambda_n^+}{2} - \mu_n, \quad \Psi_{n2} = \frac{|\gamma_n|^2}{4} - \Psi_{n1}^2 \frac{1}{\nu} \text{sign} K_n, \quad (3.15)$$

where the norming constant $K_n = \log |\varphi'(1, \mu_n, p)|$ is defined by (2.11). Such mapping was introduced in [24] for the Schrödinger operator with periodic potential. It is known that $\Psi$ maps $\mathcal{H}_j$ into $\ell^2_j \oplus \ell^2_j$ (see [24]). Note that $\Psi$ is computed from the gaps lengths $|\gamma_n|$, $\text{sign} K_n$ and $\Psi_{n1}$ for any $n \geq 1$, however, we do not need the position of the gaps and the Dirichlet eigenvalues.

We introduce the Fourier transformation $\Phi : H_C \to \ell^2_C$ by $(\Phi q)_n = \sqrt{2} q_n, n \geq 1$, where $q_n = (q_{cn}, q_{sn})$ with

$$q_{cn} = \int_0^1 q(x) \cos 2\pi nx dx, \quad q_{sn} = \int_0^1 q(x) \sin 2\pi nx dx, \quad n \geq 1, \quad q \in H_C.$$

We now formulate the needed results from [24], [23] for $j = 0$ and from [28] for $j = -1$, where the inverse problem for the mapping $\Psi(\cdot)$ is solved by the direct method.

**Theorem 3.1.** Each mapping $\Psi : \mathcal{H}_j \to \ell^2_j \oplus \ell^2_j, j = -1, 0$ defined by (3.14), is a real analytic isomorphism between $\mathcal{H}_j$ and $\ell^2_j \oplus \ell^2_j$. Moreover the following estimates hold true:

$$\|p\| \leq 2\|\Psi\|(1 + \|\Psi\|^{\frac{1}{2}}), \quad \text{if } j = 0,$$

$$\|\Psi\| \leq \|p\|(1 + \|p\|^{\frac{1}{2}}), \quad \text{if } j = -1,$$

and

$$\|x\| \leq 96\pi^2\|\Psi\|_{-1}(1 + 2\|\Psi\|_{-1})^3, \quad \|\Psi\|_{-1} \leq \|x\|(1 + 2\|x\|)^3, \quad \text{if } j = -1, \quad p = x, x \in \mathcal{H}_0. \quad (3.17)$$

**Remark.** The proof is based on on nonlinear functional analysis see Theorem 1.1. The Gelfand-Levitan-Marchenko equation and a trace formula are not used in the proof.

3.4. **Proof of Theorem 2.1.** We assume that Theorem 2.3 i) holds true (in our case $\alpha = 1$) and then the mapping $q \to p = P_\nu(q)$ given by (3.5) is a real-analytic isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_0$. Recall that due to Theorem 3.1 from [24] the mapping

$$\Psi : p \mapsto \Psi(p) = (\Psi_n(p))_{n=1}^\infty \quad (3.18)$$

is a real-analytic isomorphism between $\mathcal{H}_1$ and $\ell^2 \oplus \ell^2$.

Thus due to (2.9)–(2.11), Theorem 3.1 and the relation between $q$ and $P$ given by (3.4), (3.5), we obtain the identity

$$\psi(q) = \Psi(P(q)), \quad \forall q \in \mathcal{H}_1.$$

The mapping $\psi(\cdot)$ is the composition of two mappings $\Psi$ and $P$, where each of them is the corresponding analytic isomorphism (see Theorem 3.1 and Theorem 2.3). Then the mapping

$$\psi : q \mapsto (\Psi_n(P(q)))_{n=1}^\infty,$$
is a real-analytic isomorphism between $\mathcal{K}_1$ and $\ell^2 \oplus \ell^2$. Using the factorization $\psi(q) = \Psi(P(q))$ and combining the estimates in Theorems 2.3 and 3.1, we obtain

$$
\|P\| \leq 2\|\psi\|(1 + \|\psi\|), \quad \|\Psi\| \leq \|P\|(1 + \|P\|),
$$

where $C_* = A(\beta + \alpha)(2 + \beta A)$ and $\beta = \frac{1}{m}$, $A = \frac{F_0}{F_0}$. This yields (2.13), (2.14).

4. Proof of Theorem 2.3 for smooth case

4.1. A priori estimates. Define the scalar product in $L^2(0, 1)$ by

$$(f, g) = \int_0^1 f\overline{g} dx.$$  

We recall the following simple estimates

$$
\|q\|_{L^\infty(0, 1)} \leq \|q\|, \quad \|q\| \leq \|q\|, \quad \forall \ q \in \mathcal{K}_1, \tag{4.19}
$$

and

$$
\|Q\|_{L^\infty(0, 1)} \leq \|q\|, \quad \|u(Q)\|_{L^\infty(0, 1)} \leq Ae^{\beta|q|}. \tag{4.20}
$$

We consider the mapping $q \to P(q)$ given by

$$
P(q) = q' + \alpha q^2 + u - c_0,
$$

$$
\alpha > 0, \quad u(Q) = Ae^{-\beta Q}, \quad Q(x) = \int_0^x q(t) dt, \quad c_0 = \int_0^1 f dx, \tag{4.21}
$$

where $f = \alpha q^2 + u$ and $A = \frac{F_0}{F_0}$ and $\beta = \frac{1}{m}$. We derive two-sided estimates of $q$ and $P(q)$.

Lemma 4.1. Let the mapping $q \to P(q)$ from $\mathcal{K}_j$ into $\mathcal{K}_{j-1}$, $j \geq 1$, be given by (4.21). Then the following estimates hold true:

$$
\|q''\| \leq \|P(q)\| \leq \|q''\| + \|\alpha q^2 + u - c_0\|^2 + 2\beta q^2, u, \tag{4.22}
$$

$$
\|P(q)\| = \left\|q''\right\| + \alpha q^2 + u - c_0 \leq \|q''\| + \|q^2\| + 2(\beta + \alpha)(q^2, u) - c_0^2, \tag{4.23}
$$

$$
\|P(q)\| \leq \|\alpha q^2 + \alpha^2 2\|q^2\| + \|u\|^2 + 2(\beta + \alpha)(q^2, u) - c_0^2 \tag{4.24}
$$

where the constant $C_* = A(\beta + \alpha)(2 + \beta A)$.

In particular, if $u = 0$, $\alpha = 1$, $A = 0$, then $c_0 = \|q\|^2$ and

$$
\|q''\| \leq \|P(q)\| \leq \|q''\| + \|q^2\|^2 + c_0^2 \leq \|q''\|^2 + \|q^2\|^2. \tag{4.25}
$$

Proof. Let $h = \alpha q^2 + u - c_0$. We have $p' = q' + h$, $h = f - c_0$ and

$$
\|P\|^2 = \|q''\|^2 + \|h\|^2 + 2(q', h),
$$

$$(q', h) = (q', \alpha q^2 + u) = (q', u) = \int_0^1 q'u(Q) dx = \beta \int_0^1 q^2 u(Q) dx = \beta(q^2, u),$$

where the integration by parts has been used. This yields (4.22).

By a direct computation

$$
\|h\|^2 = \|\alpha q^2 + u - c_0\| = \|\alpha q^2 + u\|^2 - 2(\alpha q^2 + u, c_0) + c_0^2 = \|\alpha q^2 + u\|^2 - c_0^2,
$$

$$
\|\alpha q^2 + u\|^2 = \alpha^2 \|q^2\|^2 + \|u\|^2 + 2\alpha q^2, u,$$

which, together with (4.22), yields (4.23).
4.2. Analyticity and invertibility.

We show (4.21). We have $c_0 = \alpha \|q\|^2 + u_0$ and $u = u_0 + u_1$, where $u_0 = (u, 1)$. Then

$$
\|u\|^2 - c_0^2 = \|u_1\|^2 - \alpha^2 \|q\|^4 - 2\alpha \|q\|^2 u_0 \leq \|u_1\|^2.
$$

We need to estimate $\|u_1\|^2$. Letting $Q_x = Q(x)$, we obtain

$$
\|u_1\|^2 = \int_0^1 (u(Q_x) - u_0)^2 \, dx, \quad u(Q_x) - u_0 = \int_0^1 (u(Q_x) - u(Q_t)) \, dt,
$$

and for some $y \in [x, t]$

$$
u(Q_x) - u(Q_t) = A(e^{-\beta Q_x} - e^{-\beta Q_t}) = -\beta A e^{-\beta Q_y} (Q_x - Q_t).
$$

These identities and the following estimates

$$
|u(Q_x) - u(Q_t)| = A|e^{-\beta Q_x} - e^{-\beta Q_t}| \leq \beta \|q\| A e^{\beta \|q\|}
$$

imply

$$
\|u\|^2 - c_0^2 \leq \|u_1\|^2 \leq \int_0^1 \beta^2 \|q\|^2 A^2 e^{2\beta \|q\|} \, dx = \beta^2 \|q\|^2 A^2 e^{2\beta \|q\|}.
$$

Moreover, we have

$$
\|P\|^2 = \|q'\|^2 + \alpha^2 \|q^2\|^2 + V,
$$

where,

$$
V = 2(\beta + \alpha) q^2, \quad u + \|u\|^2 - c_0^2,
$$

which is estimated as follows:

$$
V \leq 2(\beta + \alpha) \|q\|^2 A e^{\beta \|q\|} + \beta^2 \|q\|^2 A^2 e^{2\beta \|q\|} = A \left(2\beta + 2\alpha + \beta^2 A \right) \|q\|^2 e^{2\beta \|q\|} \leq A(\beta + \alpha)(2 + \beta A) \|q\|^2 e^{2\beta \|q\|} = C_* \|q\|^2 e^{2\beta \|q\|},
$$

with $C_* = A(\beta + \alpha)(2 + \beta A)$. In addition, we have

$$
\|q^2\|^2 = \int_0^1 q^4(y) \, dy = 2 \int_0^1 q^2(y) \, dy \int_0^y q(t) q'(t) \, dt \leq 2 \|q\|^3 \|q'\|,
$$

since $q(x_*) = 0$ for some $x_* \in [0, 1]$ and we have used the new variable $x = x_* + y$. Combining all these estimates, we obtain (4.21). If $u = 0$, (4.25) follows from (4.22)-(4.24). $\blacksquare$

4.2. Analyticity and invertibility. We show that that mapping $P : \mathcal{H}_j \to \mathcal{H}_j$, $j \geq 0$, given by (4.21), is real analytic and locally invertible. Here we have $P(q) = P(q, A, \beta, \alpha)$ for some fixed $A, \beta, \alpha$. First we discuss analyticity of $P$. Let

$$
Jf = \int_0^x f \, dx.
$$

Lemma 4.2. i) The map $P : \mathcal{H}_j \to \mathcal{H}_j$, $j \geq 0$, given by (4.21) is a real analytic and its gradient is given by

$$
\frac{\partial P(q)}{\partial q} f = f' + 2qf - \beta u(Q) Jf - \frac{\partial c_0(q)}{\partial q} f, \quad \forall \ q, f \in \mathcal{H}_j, \quad (4.26)
$$

$$
\frac{\partial c_0(q)}{\partial q} f = \int_0^1 \left[2qf - \beta u(Q) Jf \right] \, dx. \quad (4.27)
$$

ii) Moreover, for each $q \in \mathcal{H}_j$ the linear operator $\frac{\partial P(q)}{\partial q}$ acting in $\mathcal{H}_j$ is invertible.
Proof. The standard arguments (see [37]) give the proof of i).

ii) Due to (4.26) the linear operator \( \frac{\partial P(q)}{\partial q} : \mathcal{H}_j \to \mathcal{H}_j \) is a sum of the linear operator operator \( \frac{d}{dx} \) and the compact operator for all \( p \in \mathcal{H}_j \). Thus \( \frac{\partial P(q)}{\partial q} \) is a Fredholm operator. We prove that the operator \( \frac{\partial P(q)}{\partial q} \) is invertible by contradiction. Let \( f \in \mathcal{H}_j \) be a solution of the equation

\[
\frac{\partial P(q)}{\partial q} f = 0 \quad (4.28)
\]

for some fixed \( q \in \mathcal{H}_j \). Setting \( y = Jf \), we obtain the following equation for \( y \):

\[
-y'' - 2\alpha q y' + V y = B, \quad B = \int_0^1 (\alpha q y' + V y) dx, 
\]

\[
y \in \mathcal{H}_{j+1}, \quad V = \beta u(Q), \quad y(0) = y(1) = 0. \quad (4.29)
\]

We rewrite the equation (4.29) in the following form

\[
-\frac{1}{\varrho^2} (\varrho^2 y')' + V y = B, \quad \alpha q = \frac{\varrho'}{\varrho}, \quad y(0) = y(1) = 0. \quad (4.30)
\]

We have 2 cases. Firstly, let \( B = 0 \). Then multiplying (4.30) by \( \varrho^2 y \) we have

\[
0 = \int_0^1 \left[ - (\varrho^2 y') y + \varrho^2 V y^2 \right] dx = \int_0^1 \left[ (\varrho y')^2 + \varrho^2 V y^2 \right] dx,
\]

which shows \( y = 0 \).

Secondly, let \( B \neq 0 \). It is sufficiently to consider the case \( B = 1 \). The solution of the equation (4.30) with \( B = 1 \) has the form

\[
y = y_0 + y_1, \quad y_0 \in \mathcal{H}_{j+1}, \quad y \in \mathcal{H}_{j+2}, \\
-\frac{1}{\varrho^2} (\varrho^2 y_0')' + V y_0 = 0, \\
-\frac{1}{\varrho^2} (\varrho^2 y_1')' + V y_1 = 1, \quad y_1(0) = y_1(1) = 0. \quad (4.31)
\]

Arguing as above, we have \( y_0 = 0 \). We consider \( y_1 \), which has the form

\[
y_1(x) = \int_0^1 R(x, t) dt > 0,
\]

\( R(x, t) \) being the Green function for the last equation in (4.31). Note that \( R(x, t) \geq 0 \) for any \( x, t \in [0, 1] \times [0, 1] \), since the first eigenvalue of the Sturm-Liouville problem equation

\[
-\frac{1}{\varrho^2} (\varrho^2 \psi')' + V \psi = \lambda \psi, \quad \psi(0) = \psi(1) = 0, \quad (4.32)
\]

is positive. We have \( y_1'(0) \geq 0 \) and \( y_1'(1) \leq 0 \), which together with the equality

\[
\int_0^1 y_1'' dx = y_1'(1) - y_1'(0) = 0 \quad (4.33)
\]

yields \( y_1'(0) = y_1'(1) = 0 \). Let \( \varphi(x, t) \) be the solution of the equation

\[
-\varphi'' - 2\alpha q(x + t) \varphi' + V(x + t) \varphi = \lambda \varphi, \quad \varphi(0, t) = 0, \quad \varphi'(0, t) = 1, \quad t \in [0, 1].
\]
Since \( y_1(0) = y'_1(0) = 0 \), the solution of the last equation in (4.3) has the form
\[
y(x) = -\int_0^x \varphi(x-t, t) \cdot 1 \, dt < 0,
\]
which implies \( y_1 = 0 \), since \( \varphi(x, t) > 0 \), \( x, t \in [0, 1] \). ■

4.3. Proof of Theorem 2.3 ii). In order to prove Theorem 2.3 ii), we check all conditions
A)-D) in Theorem 1.1 for the mapping \( q \rightarrow P(q) \).

The statements A) and B) have been proved in Lemma 4.2.

C) The two-sided estimates (2.18) were proved in Lemma 4.1.

D) Let \( q'' \rightarrow q \) weakly in \( H_j \) as \( \nu \rightarrow \infty \). Then we deduce that \( q'' \rightarrow q \) strongly in \( H_{j-1} \)
as \( \nu \rightarrow \infty \) since the imbedding mappings \( H_j \rightarrow H_{j-1} \) are compact. Hence the mapping
\( q \rightarrow P(q) - q' \) is compact.

Hence all conditions in Theorem 1.1 hold true and the mapping \( P : H_j \rightarrow H_{j-1} \) is a real
analytic isomorphism between the Hilbert spaces \( H_j \) and \( H_{j-1} \). ■

5. The mapping \( P \) for singular case

5.1. Proof of Theorem 2.3 ii). We consider the mapping \( P \) given by
\[
z' = P(q) = q' + q^2 + h^2 - c_0,
\]
\[
h = h_0 e^{-\frac{2}{\nu} Q}, \quad h_0 \geq 0, \quad Q(x) = \int_0^x q(t) \, dt, \quad c_0 = \int_0^1 (q^2 + h^2) \, dx,
\]
for the singular case \( q \in H_0 \), where it is convenient to use \( z \in H_0 \).

Consider the case \( m \geq 1 \). By Lemma 1.2 the mapping \( P : H_j \rightarrow H_{j-1}, j \geq 0 \), is a real
analytic and locally invertible. We show that \( P \) is an injection.

Assume that \( P \) is not injective. Then there exist \( q, \tilde{q} \in H_0 \), \( q \neq \tilde{q} \), such that \( P(q) = P(\tilde{q}) \).
Then we have \( P(W) = B_{\varepsilon}(z) = P(\tilde{W}) \) where \( B_{\varepsilon}(z) = \{ e \in H_0 : \| z-e \| < \varepsilon \} \) for some domains
\( W, \tilde{W} \) satisfying \( q \in W \subset H_0 \), \( \tilde{q} \in \tilde{W} \subset H_0 \) and \( W \cap \tilde{W} = \emptyset \), since \( P \) is a real analytic and
locally invertible. If we take any \( q_1 \in W \cap H_1 \), then \( P(q_1) \in B_{\varepsilon}(z) \) and there exists \( \tilde{q}_1 \in \tilde{W} \cap H_1 \)
such that \( P(q_1) = P(\tilde{q}_1), q_1 \neq \tilde{q}_1 \). This yields contradiction since \( P : H_1 \rightarrow H_0 \) is a bijection.
We have thus proven that \( P \) is an injection.

We show the estimate (2.19). We prove that of \( z \) in terms of \( q \). Rewrite (5.1) in the form
\[
z = q + \int_0^x \left( f(t) - \int_0^t f(s) \, ds \right) \, dt + \int_0^1 \left( t - \frac{1}{2} \right) f(t) \, dt,
\]
where \( f = q^2 + u \). We further rewrite \( z \) in the form
\[
z = q + F - c_*, \quad c_* = \int_0^1 \left( t - \frac{1}{2} \right) f(t) \, dt,
\]
and
\[
F(x) = \int_0^x (f(t) - \int_0^t f(s) \, ds) \, dt = \int_0^x (q^2(t) - \| q \|^2) \, dt + F_0(x),
\]
\[
F_0(x) = \int_0^x (u(t) - \int_0^t u(s) \, ds) \, dt = A \int_0^1 \int_0^x (e^{-\beta Q_1} - e^{-\beta Q_2}) \, dt \, ds.
\]
Note here the following identity and estimate for some \( \tau \in [t, s] \):
\[
e^{-\beta Q_\tau} - e^{-\beta Q_s} = \beta(Q_s - Q_\tau)e^{-\beta Q_\tau},
\]
\[
|e^{-\beta Q_\tau} - e^{-\beta Q_s}| \leq \beta \|q\| e^{\beta \|q\|}.
\]

Then combining these estimates we obtain
\[
|F(x)| \leq 2\|q\|^2 + 2\|q\| + \beta A\|q\| e^{\beta \|q\|}.
\]

Then we get
\[
\|z\|^2 = (z, z) = (q + F - c_\tau, z) = (q + F, z) \leq \|q\|\|z\| + (F, z)
\]
\[
\leq \|q\|\|z\| + \|z\|\|q\|(2\|q\| + 2 + \beta A e^{\beta \|q\|}),
\]
which yields (2.19).

Let \( m = 1 \). We prove that the mapping \( P \) is a surjection.

We prove (2.20). We show by an explicit construction that for each \( z \in H_0 \), the equation (5.1), i.e., \( z' = P(q) = q' + q^2 + u(Q) - c_0(q) \), has a solution \( q \in H_0 \). For fixed \( z \in H_0 \) we consider the auxiliary equation
\[
-\phi'' + z' \phi = \lambda \phi, \quad x \in \mathbb{R}.
\]
We need the following results from [28]: there exists a unique solution \( \phi(x) \) of (5.6) with unique \( \lambda = \lambda_0(z) < \lambda_0^+(z) \) having the form
\[
\phi(x) = e^x \phi_1(x), \quad \phi_1(x + 1) = \phi_1(x), \quad \phi_1(x) > 0, \quad \phi_1 \in L^2(\mathbb{T}), \quad \|\phi_1\| = 1.
\]
Here \( \lambda_0^+(z) \) is the lowest eigenvalue of the periodic problem for (5.6). Note that the function \( \lambda_0(\cdot) \) is continuous on \( H_0 \).

Let \( z \in H_0 \). Then we have \( \phi \). We show that there exists \( q \in H_0 \) such that
\[
q + \frac{h_0}{v} = \frac{\phi'}{\phi}, \quad \text{where} \quad v(x) = e^{2\int_0^x q(s) \, ds}.
\]
The equation (5.8) is linear for \( v \), since \( v' = 2qv \) and
\[
\frac{v'}{2v} + \frac{h_0}{v} = \frac{\phi'}{\phi} \quad \iff \quad v' + 2h_0 = 2 \frac{\phi'}{\phi} v.
\]
This equation has a periodic solution of the form
\[
v(x) = C_1 \int_0^x \left( \frac{\phi(x)}{\phi(t)} \right)^2 \, dt + C_2 \int_x^{x+1} \left( \frac{\phi(x)}{\phi(t)} \right)^2 \, dt, \quad x \in \mathbb{R},
\]
where
\[
C_1 = \frac{2h_0}{e^2 - 1}, \quad C_2 = C_1 e^2.
\]
Thus we have \( v' \in L^2(0, 1) \) and \( v > 0 \). Then we obtain \( 2q = \frac{v'}{v} \in H_0 \) and \( h = h_0 e^{-2\int_0^x q(s) \, ds} \).

We show that \( z, q \) satisfy the equation (5.11). Differentiating (5.8) and using \( h = h_0/v \) we obtain
\[
(q + h)' = \frac{\phi''}{\phi} - (q + h)^2 = (q' - \lambda_0) - (q + h)^2,
\]
\[
q' - 2qh = (z' - \lambda_0) - (q^2 + 2qh + h^2),
\]
\[
2 + \lambda_0 = q' + q^2 + h^2 + \lambda_0.
\]
we obtain the following equality

\[ -\lambda_0(z) = \int_0^1 (q^2 + h^2)dx = \|q\|^2 + \|h\|^2. \]  

(5.12)

Thus \( q \) satisfies equation (5.11) and \( P \) is a surjection. Hence \( P : \mathcal{H}_0 \to \mathcal{H}_{-1} \) is a real analytic isomorphism between \( \mathcal{H}_0 \) and \( \mathcal{H}_{-1} \).

Now we prove the estimate (2.19). We show that it is sufficient to consider \( z \in \mathcal{H}_1 \). Substituting \( \phi = e^{\tau} \phi_1 \) into (5.6), we obtain the equation for \( \phi_1 \):

\[ -\phi_1'' - 2\phi_1' + z' \phi_1 = (\lambda_0(z) + 1)\phi_1, \quad (x,z) \in \mathbb{R} \times \mathcal{H}_1. \]  

(5.13)

Recall \( \|\phi_1\| = 1 \). Multiplying (5.13) in \( L^2(0,1) \) by \( \phi_1 \) we obtain the identity

\[ \|\phi_1\|^2 + (z' \phi_1, \phi_1) = \lambda_0(z) + 1. \]

Repeating the arguments from [28] we obtain the estimate

\[ -\lambda_0(z) - 1 \leq 2\|z\|^2 (1 + 2\|z\|^2), \quad z \in \mathcal{H}_1. \]  

(5.14)

By the continuity of \( \lambda_0(\cdot) \) on \( \mathcal{H}_0 \) the estimate (5.13) holds true for \( z \in \mathcal{H}_0 \). Substituting (5.13) into (5.14) we get

\[ \|q\|^2 + \|h\|^2 - 1 \leq 2\|z\|^2 (1 + 2\|z\|^2). \]  

(5.15)

Equation (5.9) and \( \phi = e^{\tau} \phi_1 \) give

\[ \int_0^1 h(x)dx = \int_0^1 \frac{\phi'(x)}{\phi(x)}dx = 1, \]

which implies \( \|h\|^2 \geq 1 \). Then (5.13) gives \( \|q\|^2 \leq 2\|z\|^2 (1 + 2\|z\|^2) \), which yields (2.19). □

**Proof of Theorem 2.2.** We recall that due to Theorem 3.1 in [28] the mapping

\[ \Psi : p \mapsto \Psi(p) = (\Psi_n(p))_{n=1}^\infty \]  

(5.16)

is a real-analytic isomorphism between \( \mathcal{H}_{-1} \) and \( \ell^2_{-1} \oplus \ell^2_{-1} \).

i) Fix \( m \geq 1 \). Then due to Theorem 2.3, the mapping \( P : \mathcal{H}_0 \to \mathcal{H}_{-1} \), given by (2.17), is real analytic and injective. Thus due to (2.9)–(2.11) and Theorem 3.1 and the relation between \( q \) and \( P \) given by (3.4), (3.5), we obtain the identity

\[ \psi(q) = \Psi(P(q)), \quad \forall q \in \mathcal{H}_0. \]  

(5.17)

The mapping \( \psi(\cdot) \) is the composition of two mappings \( \Psi \) and \( P \). The properties described above show that the mapping \( \psi : \mathcal{H}_0 \to \ell^2_{-1} \oplus \ell^2_{-1} \) is real analytic and injective. Moreover, combining (2.19) and (3.17), we obtain (2.13).

ii) Let \( m = 1 \). Recall that Theorem 2.3 (in our case \( j = 0 \)) proves that the mapping \( q \to P_P(q) \) defined by (3.5) is a real-analytic isomorphism between \( \mathcal{H}_0 \) and \( \mathcal{H}_{-1} \). Thus by virtue of (2.9)–(2.11), Theorem 3.1 and the relation between \( q \) and \( P \) given by (3.4), (3.5), we obtain the following equality

\[ \psi(q) = \Psi(P(q)), \quad \forall q \in \mathcal{H}_0. \]

The mapping \( \psi(\cdot) \) is the composition of two mappings \( \Psi \) and \( P \), each of which is a corresponding analytic isomorphism (see Theorem 3.1 and Theorem 2.3). Then the mapping \( \psi : q \mapsto (\Psi_n(P(q)))_{n=1}^\infty \), is a real-analytic isomorphism between \( \mathcal{H}_0 \) and \( \ell^2_{-1} \oplus \ell^2_{-1} \). Finally, combining (2.20) and (3.17), we obtain (2.16). □
Proof of Corollary 2.4. i) It is clear that the mapping $P$ given by (2.17) satisfies $P: \mathcal{H}_j^{\text{odd}} \to \mathcal{H}_{j-1}^{\text{even}}$, $j \geq 1$. Arguments in the proof of Theorem 2.3 prove that this mapping $P$ is a real analytic isomorphism between the Hilbert spaces $\mathcal{H}_j^{\text{odd}}$ and $\mathcal{H}_{j-1}^{\text{even}}$.

ii) In order to prove ii) we need the Garnett-Trubowitz’ results [12] or [13] on the gap-length mapping for Schrödinger operator $Sy = -y'' + py$ on the torus $\mathbb{T}$ with the potential $p \in \mathcal{H}_0^{\text{even}}$. Define the mapping $\Psi^e: \mathcal{H}_0^{\text{even}} \to \ell^2$ by

$$p \to \Psi^e(p) = (\Psi^e_n(p))_{n=1}^\infty, \quad \Psi^e_n(p) = \frac{\lambda^+_n(p) + \lambda^-_n(p)}{2} - \mu_n(p),$$

(5.18)

for all $n \geq 1$. Here $\lambda^\pm_n(p)$ is the periodic and antiperiodic eigenvalues for the equation $y'' + py = \lambda y$ and $\mu_n(p) \in [\lambda^+_n(p), \lambda^-_n(p)]$ are the corresponding Dirichlet eigenvalues for the problem $y'' + py = \lambda y, y(0) = y(1) = 0$. By Garnett-Trubowitz [12], we then see that the mapping $\Psi^e: \mathcal{H}_0^{\text{even}} \to \ell^2$ given by (5.18) is a real analytic isomorphism between the Hilbert spaces $\mathcal{H}_1^{\text{odd}}$ and $\ell^2$. Repeating the argument in the proof of Theorem 2.1 we prove Corollary 2.4 ii) for the case $\mathcal{H}_1^{\text{odd}}$.

We consider the singular case $\mathcal{H}_0^{\text{odd}}$ and deal with only the case $m = 1$. It is clear that the mapping $P$ given by (2.17) satisfies $P: \mathcal{H}_0^{\text{odd}} \to \mathcal{H}_1^{\text{even}}$. The results from Theorem 2.3 and arguments in the proof of Theorem 2.4 give that this mapping $P$ is a real analytic isomorphism between the Hilbert spaces $\mathcal{H}_0^{\text{odd}}$ and $\mathcal{H}_1^{\text{even}}$.

From Theorem 3.1 we deduce that the mapping $\Psi^e: \mathcal{H}_1^{\text{even}} \to \ell_{-1}^2$ given by (5.18), is a real analytic isomorphism between the Hilbert spaces $\mathcal{H}_0^{\text{odd}}$ and $\ell_{-1}^2$. Thus by the composition of the mapping $P: \mathcal{H}_0^{\text{odd}} \to \mathcal{H}_1^{\text{even}}$ and $\Psi^e: \mathcal{H}_1^{\text{even}} \to \ell_{-1}^2$, the mapping $\psi^e: \mathcal{H}_0^{\text{odd}} \to \ell_{-1}^2$, given by (2.22) is a real analytic isomorphism between the Hilbert spaces $\mathcal{H}_0^{\text{odd}}$ and $\ell_{-1}^2$. ■

5.2. Proof of Corollary 2.5. Due to Theorem 2.3 the mapping $v \to \mathcal{E}_1$ is a real analytic isomorphism between the spaces $\mathcal{H}_j$ and $\mathcal{H}_{j-1}$. Moreover, for each $\mathcal{E}_1 \in \mathcal{E}_0$ there exists a unique $v \in \mathcal{H}_j$ and the constant $\mathcal{E}_0$ in (2.28), which is uniquely defined by $\mathcal{E}_1$. We show estimates. Let $\mathcal{E}_1 = -v' - g$, where $g = \frac{a^2}{m} - \mathcal{E}_0$ and $\mathcal{E}_0 = -\frac{\|v\|^2}{m}$. We have

$$\|\mathcal{E}_1\|^2 = \|v' + g\|^2 = \|v'\|^2 + \|g\|^2 + 2(v', g)$$

$$= \|v'\|^2 + \|g\|^2 = \|v'\|^2 + \frac{1}{m^2}\|v\|^2 - \mathcal{E}_0^2;$$

(5.19)

since

$$(v', g) = \frac{1}{m}(v', v^2) = \frac{1}{m} \int_0^1 v'v^2 dx = 0,$$

$$\|g\|^2 = \frac{1}{m^2}\|v^2\|^2 + \mathcal{E}_0^2 - \frac{2}{m}\|v\|^2 \mathcal{E}_0 = \frac{1}{m^2}\|v\|^2 - \mathcal{E}_0^2.$$

This identity (5.19) and the following estimate

$$\|v^2\|^2 = \int_0^1 v^4 dx \leq \|v\|^2 \sup_{x \in \mathbb{T}} |v(x)|^2 \leq \|v\|^2 \|v'\|^2,$$

yield

$$\|v'\|^2 \leq \|\mathcal{E}_1\|^2 \leq \|v'\|^2 + \frac{1}{m^2}\|v\|^2\|v'\|^2 - \mathcal{E}_0^2.$$

(5.20)
Acknowledgments. We thank Andrei Badanin for the Fig. 1. Various parts of this paper were written during Evgeny Korotyaev’s stay in the Mathematical Institute of University of Tsukuba, Japan. He is grateful to the institute for the hospitality. H. Isozaki is supported by Grants-in-Aid for Scientific Research (S) 15H05740, and (B) 16H03944, Japan Society for the Promotion of Science. E. Korotyaev is supported by the RSF grant No. 18-11-00032.

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