Non-Hermitian adiabatic transport in the space of exceptional points

J. H"oller\textsuperscript{1}, N. Read\textsuperscript{1,2}, and J.G.E. Harris\textsuperscript{1,2}

\textsuperscript{1}Department of Physics, Yale University, P.O. Box 208120, New Haven, CT 06520-8120
\textsuperscript{2}Department of Applied Physics, Yale University, P.O. Box 208284, New Haven, CT 06520-8284

(Dated: September 20, 2018)

We consider adiabatic transport \textit{within} the space of non-Hermitian Hamiltonians that are equivalent to a single $n \times n$ Jordan block, with $T = 1/\varepsilon \to \infty$ as the time-scale of the transport around a closed path (a loop). We show that, for a certain class of loops and a choice of initial state, the complex phase acquired is $\varepsilon^{-1}$ times an expansion in powers of $\varepsilon^{1/n}$ (a Puiseux expansion). The exponential of the term of $n$th order (equivalent to the “geometric” or Berry phase modulo $2\pi$), which is thus independent of $\varepsilon$ as $\varepsilon \to 0$, depends only on the homotopy class of the loop and is an integer power of $e^{2\pi i/n}$. One of the conditions for these results is that the state being transported is, for all points on the loop, that of slowest decay.

The introduction of weak linear dissipation or amplification into a system of $n$ classical harmonic oscillators results in time evolution that can be described using an $n \times n$ Hamiltonian matrix $H$ that is non-Hermitian. The non-Hermiticity of $H$ gives rise to the familiar decay (or growth) of such a system’s eigenstates (normal modes). It also opens the possibility of “exceptional points” (EPs) in parameter space, at which $H$ is not fully diagonalizable. In the neighborhood of an EP, the eigenvalues exhibit branch-point behavior as functions of the parameters, and so encircling it permutes the eigenvalues and eigenspaces \cite{1–3}, an effect referred to as flipping, monodromy, or spectral flow.

In recent years, EPs have been studied experimentally in a wide range of settings, including microwave \cite{4, 5}, electrical \cite{6}, optical \cite{7}, cavity QED \cite{8}, exciton \cite{9}, acoustic \cite{10}, and mechanical \cite{11} systems. While each of these realizations has offered some degree of control over $H$, in most experiments the number $m$ of independent control parameters is insufficient to specify an arbitrary $H$. As a result, EPs are typically observed to occur at isolated points within the $m$-dimensional space of control parameters. In contrast, if we consider the space $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ of all $n \times n$ complex matrices $H$, then EPs are not isolated, but in fact form subspaces of $M_n(\mathbb{C})$ of dimension larger than zero.

The evolution of a system under an asymptotically slow (“adiabatic”) smooth variation of some parameters in time has been the subject of much study in both Hermitian and non-Hermitian cases. In what follows, we concentrate on evolution along a closed path (a loop) in parameter space. In Hermitian systems, the adiabatic theorem \cite{12, 13} guarantees, in terms of the eigenstates of the “instantaneous” Hamiltonian at any point of the loop, that if the system is initially in an eigenstate (or subspace of degenerate eigenvalues) and if the degeneracy of that eigenvalue does not change at any point during the evolution, then at the end of the adiabatic evolution the system will be found in the same eigenspace in which it started. Moreover, the phase of the state vector changes by an amount whose asymptotic form as the time $T$ taken for the loop tends to infinity has two leading contributions: the integral of the eigenvalue along the loop (the dynamical phase), which typically is linear in $T$; and the geometric or Berry phase, which is independent of $T$ \cite{14}. The Berry phase modulo $2\pi$, or phase factor, is the holonomy of a natural connection, and may have further topological significance \cite{14, 15}. (For evolution of a degenerate eigenspace, the phase factor becomes a unitary map \cite{16}.)

In contrast, in non-Hermitian systems there exists a mode or modes that are “dominant”, meaning they have the largest rate of exponential growth (or slowest decay). During adiabatic evolution along a generic loop the system tends to transition into this subspace, which then dominates at long times, even when the eigenvalue (or its real or imaginary parts) of the chosen mode does not coincide with that of another mode at any point on the loop. In this situation the adiabatic theorem does not hold for all the modes, but only for the dominant one \cite{17}. If a different mode becomes dominant somewhere along the loop, then even this statement breaks down. This occurs generically when the loop encircles an EP (because of monodromy); as a result, no adiabatic theorem applies for strict adiabatic transport around such a loop \cite{18, 19}.

In this Letter, we consider adiabatic transport around a loop \textit{within} a space of EPs. Specifically, we consider the space $\mathcal{E}P_n \subset M_n(\mathbb{C})$ of Hamiltonians similar to a single $n \times n$ Jordan block, and describe its geometry. Then, as for the usual adiabatic theorems, we consider a loop in $\mathcal{E}P_n$, parametrized by Hamiltonians $H(s)$ for $0 \leq s \leq 1$, where $H(1) = H(0)$, and evolve the system in time $t$ with $s = t/T$; finally the asymptotics as $T \to \infty$ are studied, with the loop $H(s)$ fixed (independent of $T$). We find that the result of adiabatic transport in $\mathcal{E}P_n$ has features in common with the case of a non-degenerate eigenstate in a Hermitian system, but also substantial differences. For a class of loops and a choice of initial state (to be described in a moment), we find that as $T \to \infty$ the state returns to its initial subspace, multiplied by a complex number whose logarithm has the Puiseux series form

\begin{equation}
\sum_{r=0}^{n} T^{1-r/n} \int_{0}^{1} ds \, a_r(s)
\end{equation}

(plus terms higher order in $1/T^{1/n}$), where $a_r(s)$ are com-
plex functions that can be calculated from $H(s)$. Thus the (complex) dynamical phase (the terms with $r < n$) includes fractional powers of $T^{-1}$ (of which the $r = 1$ term at least has been noticed previously \[21\]). The term of order $T^0$ (the $r = n$ term) is the geometric or Berry phase, and is only well-defined modulo $2\pi r$. Remarkably, the exponential of this term is again the holonomy of a connection, and is an integer power of $e^{2\pi in/\hbar}$ that depends only on the homotopy class \[21\] of the loop in $\mathcal{EP}_n$ (specifically, the power is the winding number that characterizes the homotopy class).

To describe the conditions under which this result holds, it is useful to change basis in the evolution equation to the basis of instantaneous generalized eigenvectors of $H(s)$. In this basis, the effective Hamiltonian describing evolution is $H' = J + T^{-1}A(s)$, where $J$ is the Jordan block and $A(s)$ is the adiabatic (Berry) connection matrix along the loop at $s$ (details appear below). The most generic case is that in which the element $A_{n1}$ of $A$ is nonzero, and then $H'$ is diagonalizable. For adiabatic transport on $\mathcal{EP}_n$, it is $H'$ (rather than $H$) that determines the dominant mode. We assume that the same mode remains dominant everywhere on the loop, which is ensured if $A_{n1}$ does not touch the negative real axis or zero. Our result holds for such loops and when the initial state is this dominant mode.

In addition, we identify an important subclass of such loops for which adiabatic transport of any eigenstate of $J + T^{-1}A$ stays in that state for all time. This consists of loops for which in $H(s)$, $A(s) = A$ is independent of $s$. As long as $A_{n1}$ is not zero, such a “straight” loop in $\mathcal{EP}_n$ produces the form \[1\] for each eigenstate, not only the dominant one (the coefficients $a_r$ for $r \neq n$ differ for each eigenstate, however). Topologically non-trivial loops of this form exist in $\mathcal{EP}_n$, and the holonomy of such a loop is independent of which eigenstate is transported.

To obtain these results, it is convenient to use the evolution (Schrödinger) equation in the form $\partial|\psi\rangle = H|\psi\rangle$, so $i\hbar H$ would be the usual Hamiltonian of quantum mechanics. We treat $H$ as a matrix, and $|\psi\rangle$ stands for a column vector. Adding a multiple of the identity $I$ to a Hamiltonian merely shifts all the eigenvalues by the same amount, so we can assume that the trace of $H$ is zero. An $n \times n$ complex Hamiltonian that is similar to a single Jordan block, meaning that $H = \Lambda J \Lambda^{-1}$ where

\[
J = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\] (2)

and $\Lambda$ is invertible, is said to lie at an EP of type $EP_n$. (Note that the columns of $\Lambda$ are generalized eigenvectors of $H$; we denote them by $|u_i\rangle$ for $i = 1, \ldots, n$.) To describe the geometry of the space $\mathcal{EP}_n$ of all such $H$, first notice that the matrices that commute with $J$ have the form $aI + b_1 J + \cdots + b_{n-1} J^{n-1}$, where $I$ is the identity, and $a, b_i$ are complex numbers. Such matrices with $a \neq 0$ form a Lie group $\mathbb{C}^\times \times J$, which is a subgroup of $GL_n(\mathbb{C})$, the group of invertible $n \times n$ complex matrices. Here $\mathbb{C}^\times$ is the group of nonzero complex numbers (under multiplication), and $J$ is the group of matrices of the form $I + b_1 J + \cdots + b_{n-1} J^{n-1}$. Then $\mathcal{EP}_n$ can be identified \[22\] as $\mathcal{EP}_n \cong GL_n(\mathbb{C})/(\mathbb{C}^\times \times J)$. It is a non-compact space of complex dimension $n(n - 1)$, but has a deformation retract \[21\] onto $SU(n)/\mathbb{Z}_n$, the group of unitary matrices of determinant 1, modulo its center $\mathbb{Z}_n$, the cyclic group of order $n$. The fundamental group of this space is $\pi_1(\mathcal{EP}_n) \cong \mathbb{Z}_n \times \mathbb{Z}_2$.

As illustration, for $n = 2$, we can more explicitly describe traceless Hamiltonians as

\[
H = \begin{pmatrix}
Z & X - iY \\
X + iY & -Z \\
\end{pmatrix},
\] (3)

where $X, Y, Z$ are complex numbers. With our conventions, $i\hbar$ would be Hermitian (with respect to the standard inner product) if $X, Y, Z$ were all imaginary. However, for general non-Hermitian $H$, an inner product plays no essential role, and we avoid using one on $\mathbb{C}^n$ anywhere. If $X = (X, Y, Z)^T$, then $H$ is in $\mathcal{EP}_2$ if and only if $|\text{Re} X| = |\text{Im} X| > 0$ and $\text{Re} X \cdot \text{Im} X = 0$ \[22\] (here the standard inner product and norm on $\mathbb{R}^3$ were used). If we fix $|\text{Re} X|$ to 1, then because an ordered pair of orthogonal unit vectors in $\mathbb{R}^3$ determines an orthonormal basis with positive orientation in $\mathbb{R}^3$, the space of such pairs forms the special orthogonal group in three dimensions, or real projective 3-space, $SO(3) \cong \mathbb{R}P^3 \cong SU(2)/\mathbb{Z}_2$. It is well-known that the fundamental group of this space is $\pi_1 \cong \mathbb{Z}_2$. Then $\mathcal{EP}_2 \cong SO(3) \times \mathbb{R}$, where the second factor represents $\text{ln}|\text{Re} X|$ and is contractible.

Now consider adiabatic transport in $\mathcal{EP}_n$ for general $n$. We choose a smooth loop in $\mathcal{EP}_n$, so we have $H = H(s)$, a function of $s \in [0,1]$ with $H(1) = H(0)$ and $H \in \mathcal{EP}_n$ for all $s$. We evolve the system in time $t$ from 0 to $T > 0$ with the time-dependent Hamiltonian $H = H(s = t/T)$ as in the usual adiabatic evolution. If we express the evolution equation $\partial|\psi\rangle = H|\psi\rangle$ in a smooth basis of generalized eigenvectors $|u_i(s)\rangle$ of $H(s)$ for each $s$ then it takes the form

\[
\varepsilon \partial s |u_i\rangle = (J + \varepsilon A)|u_i\rangle,
\] (4)

where $\varepsilon = 1/T$ and $A_{ij} = -\langle u_i|\partial_s u_j\rangle$ (i.e. $A = -\Lambda^{-1}\partial_s \Lambda$) is the Berry connection evaluated on the tangent vector to the loop. Here the bras $\langle u_i(s)\rangle$ are a basis set of row vectors that is dual to the set of kets $|u_i(s)\rangle$, so $\langle u_i(s)|u_j(s)\rangle = \delta_{ij}$ for each $s$ (this is not a use of an inner product); they are rows of $\Lambda^{-1}$, and $|u_i\rangle = \Lambda^{-1}|\psi\rangle$.

To keep later arguments simpler, we assume without loss of generality that $\Lambda(s)$ is periodic $[\Lambda(1) = \Lambda(0)]$, and so also $A(1) = A(0)$. There is a residual gauge freedom when we obtain eq. \[4\]: the form is preserved under a further differentiable periodic $s$-dependent change of basis by $\tilde{\Lambda}(s) \in \mathbb{C}^\times \times J$ for all $s \in [0,1]$. The key point now is that while $J$ is not diagonalizable, $J + \varepsilon A$ often is. As $\varepsilon \to 0$, det$(J + \varepsilon A) = \varepsilon$. \[22\]
\((-1)^{n-1} \varepsilon A_0 + O(\varepsilon^2)\), where we write \(A_0 = A_{n+1}(s)\) and we assume henceforth that \(A_0\) is nonzero for all \(s\). From the characteristic equation, we find that the \(s\)-dependent eigenvalues are \(\lambda_\mu = \zeta^{\mu}(\varepsilon A_0)^{1/n}(\mu = 0, 1, \ldots, n - 1)\) to leading order, where \(\zeta = e^{2\pi i/n}\). Here we choose one root of \(A_0\), which we take to be the principal branch, for which \(\arg A_0^{1/n} \in (-\pi/n, \pi/n]\), and denote it \(A_0^{1/n}\), and \((\varepsilon A_0)^{1/n} = \varepsilon^{1/n} A_0^{1/n}(\varepsilon^{1/n} > 0)\). Solving iteratively for each eigenvalue, we can obtain series expansions

\[
\lambda_\mu = \sum_{r=1}^{\infty} a_r \varepsilon^{r/n} \zeta^{\mu r}
\]

with nonzero radius of convergence; such an expansion is called a Puiseux expansion. Note that here the coefficients \(a_r = a_r(s)\) \((a_1 = A_0^{1/n})\) are independent of \(\mu\) (once \(\zeta^{\mu r}\) has been extracted), because if such an expansion satisfies the characteristic equation for one value \(\mu\), then it does so for all \(\mu\). Then as \(\sum_\mu \lambda_\mu = \text{tr}(J + \varepsilon A) = \varepsilon \text{tr} A\), we find that

\[
a_n = \frac{1}{n} \text{tr} A
\]

and \(a_2n = a_3n = \cdots = 0\). If \(A_0 = 0\) for some \(s\) (contrary to our assumption), then the remaining elements of \(A\) become important as \(\varepsilon \to 0\), and there are different cases to study; we do not consider these in this paper.

If \(A_0 \neq 0\) makes a circuit around the origin (say, as \(s\) varies), then following the eigenvalues continuously along the circuit produces a net cyclic permutation of \(\mu \pmod{n}\). This monodromy of the eigenvalues has the same form as that we mentioned in the first paragraph of this Letter. The situations are related because \(\varepsilon A\) can be considered as a perturbation from the exceptional point \(H = J\) to an “effective” Hamiltonian \(H' = J + \varepsilon A\), and we are considering a case in which the degeneracy of eigenvalues is fully lifted. Henceforth we assume that \(A_0\) does not encircle the origin as \(s\) varies from 0 to 1.

If, in the adiabatic limit \(\varepsilon \to 0\) and under our assumptions, a state prepared in the \(\mu\)th instantaneous eigenvector of \(J + \varepsilon A\) stayed in the corresponding eigenstate until \(s = 1\), then there would be a change in its “phase” (which here is complex) of

\[
\sum_{r=1}^{n} \varepsilon^{r/n-1} \zeta^{\mu r} \int_{0}^{1} ds a_r(s)
\]

plus order \(\varepsilon^{1/n}\), plus possibly a further contribution to the geometric phase, which we discuss in the following. Apart from the \(r = n\) term, the terms displayed in expression (7) are dynamical phases. In addition to the usual one that is of order \(T\) (omitted here because we subtracted off the trace of \(H\)), there are also fractional powers of \(T\).

In view of our choice that \(\Lambda(1) = \Lambda(0)\), the final \(r = n\) term in (7) is a geometric phase, like the usual Berry phase in the case of non-degenerate eigenvalues, but given by the average of the diagonal elements of \(A\). It can change by a multiple of \(2\pi i\) under a “large" residual gauge transformation that winds in \(C^X\) as a function of \(s\); thus it is well-defined only modulo \(2\pi i\).

In other words, it is the Berry phase factor or holonomy \(\exp -\frac{1}{n} \int_{0}^{1} \text{tr} A\) that is well-defined if we do not keep track of the choice of basis along the path (i.e. is fully gauge invariant). The dynamical phase terms \(r < n\) are gauge invariant; all of this is similar to the usual Hermitian case.

An important special case of adiabatic transport in \(\mathcal{EP}_n\) is that in which \(A\) (and hence \(a_r\)) is independent of \(s\) (for all \(r\)). In that case, the coefficients in eq. (4) are constant, so the system does stay in an initial eigenstate for all \(s\), and the preceding remarks conclude the calculation. We note that the corresponding path is “straight” in \(GL_n(C)\), with \(\Lambda(s) = \Lambda(0) \exp(-s A)\), and that such paths can return to the starting point, so \(\Lambda(1) = \Lambda(0)\). Moreover, these loops can be non-trivial in both \(\pi_1(GL_n(C))\) and \(\pi_1(\mathcal{EP}_n)\), that is, they can have non-zero winding number (modulo \(n\)) when projected to \(\mathcal{EP}_n\). In the general case, when \(A\) is not constant, there could be further contributions to the geometric phase of the same order, which we now discuss.

In order to examine the general scenario, we apply the adiabatic theorem to \(J + \varepsilon A\). At leading order, the \(\mu\)th eigenvector of \(J + \varepsilon A\) can be chosen to be

\[
|v_\mu\rangle = \begin{pmatrix}
1 \\
(\varepsilon A_0)^{1/n} \zeta^\mu \\
(\varepsilon A_0)^{(n-1)/n} \zeta^\mu
\end{pmatrix} [1 + O(\varepsilon^{1/n})],
\]

which is periodic in \(s\) under our assumptions. If we use these instantaneous eigenvectors as a basis set (together with a dual basis as before), then in this basis the evolution equation becomes

\[
\varepsilon \partial_s |v\rangle = (D + \varepsilon A') |v\rangle
\]

where \(D = \text{diag}(\lambda_0, \ldots, \lambda_{n-1})\), and \(A'_{\mu\nu} = -\langle v_\mu | \partial_s v_\nu \rangle\). Now we use the adiabatic theorem for this non-Hermitian non-degenerate situation [17]. As mentioned already, in this case, with eigenvalues \(\lambda_\mu\) with differences much larger than \(\varepsilon\) as \(\varepsilon \to 0\), the adiabatic theorem in general does not hold for all the eigenspaces of \(D\), but only for the dominant mode (the one with the largest real part of its eigenvalue). (In the less general case in which a permutation of the \(|v_\mu\rangle\)’s makes \(A'\) block diagonal with the same block structure for all \(s\), then the adiabatic theorem holds for the dominant mode in each block.) If we make the stronger assumption that \(\arg A_0(s) < \pi\) for all \(s\) (i.e. \(A_0\) does not touch or cross the negative real axis) then, for all \(s\) and as \(\varepsilon \to 0\), Re \(\lambda_\mu\) is largest when \(\mu = 0\), and is non-degenerate. [Technically, we assume that \(A_0(s)\) does not approach the negative real axis or zero closer than some bound, say \(d > 0\).] It is then not difficult to show that if the initial state is purely the dominant mode, then it remains in it for all \(s\) with sufficient
accuracy as $\varepsilon \to 0$ \cite{22}. Moreover, the additional contribution to the geometric or Berry “phase” is found by integrating the diagonal element $A'_{00}$ for the dominant mode.

In the present case, we find that

\[ A'_{\mu\nu} = -\sum_{r=1}^{n-1} \frac{r}{n^2} \varepsilon^{(r-\mu)} \frac{\partial A_0}{A_0} + O(\varepsilon^{1/n}), \]

so the diagonal elements are given by $\frac{1}{2n} \partial_s \ln A_0$ to leading order. Integrating from $s = 0$ to $1$, the change in the complex amplitude is $\varepsilon A$. Because $\Lambda(1) = \Lambda(0)$, the non-Abelian holonomy is $1$, but for a non-contractible loop it is a power of $\zeta$. This concludes the derivation of the general results stated at the beginning of the Letter.

As an illustration of the general results, we solve the $n = 2$ model explicitly for $A$ constant and $A_0 \neq 0$. An example of a non-contractible loop in $EP_2$ is parametrized by $X(s) = (i \cos \phi(s), i \sin \phi(s), 1)^T$ with $\phi(s) = 2\pi s$. Then with $|u_1| = (1, i e^{i\phi})^T$ and $|u_2| = (1, 0)^T$, we find

\[ A = \begin{pmatrix} -2\pi i & 0 \\ 2\pi i & 0 \end{pmatrix}, \]

so $A_0 = 2\pi i$, and the eigenvalues of $J + \varepsilon A$ are

\[ \lambda_{0,1} = -i\pi \varepsilon \pm \sqrt{2\pi i \varepsilon - \pi^2 \varepsilon^2} \]

\[ = \pm \pi^{1/2} (1 + i) \varepsilon^{1/2} - i\pi \varepsilon + O(\varepsilon^{3/2}). \]

The two leading terms agree with $a_1 = A_0^{1/2}$ and $a_2 = \frac{1}{2} \text{tr} A = -i\pi$, and further $a_{2k} = 0$ for $k > 1$; note that the Berry phase is $\pi$ (modulo $2\pi$). Our general theory tells us that these results are independent of the choice of gauge, and that the Berry phase is invariant under sufficiently slowly varying smooth changes in the loop. More generally, if we choose $\phi(s) = 2\pi m s$ for an integer $m$, then the holonomy is $(-1)^m$.

Clearly, we could carry out an analysis by similar methods in other cases, such as when $A_0 = 0$ all along the loop. Alternatively, we can also consider, for example, spaces of EPs of types $EP_n$ for $n' < n$ within $M_n(C)$, or spaces of Hamiltonians for which the Jordan canonical form is a direct sum of several Jordan blocks with the same eigenvalue. We expect these cases to involve the non-Abelian connection for transporting a proper subspace \cite{10}, as well as effects similar to those we found for a Jordan block of size $n$. We leave these cases for later study.

To conclude, we studied adiabatic transport around a loop in a space of exceptional points of type $EP_n$ and found that the dynamical phase is given by a Puiseux series of fractional powers of $T$, and that the Berry phase (modulo $2\pi$) depends only on the homotopy class of the loop. The results hold for a choice of initial state that depends on both the loop (which must be in a certain class of loops), and on $T$.

We are grateful for discussions with A. Alexandrini and G. Moore. We acknowledge support from Yale University (JH) and from AFOSR grant FA9550-15-1-0270 and ONR MURI N00014-15-1-2761 (JGEH).

[1] T. Kato, *Perturbation theory for linear operators* (Springer, New York, 1980), Ch. 2.
[2] C. Miniatura, C. Sire, J. Baudon, and J. Bellissard, Europhys. Lett. 13, 199 (1990).
[3] W.D. Heiss, Phys. Rev. E 61, 929 (2000).
[4] C. Dembowski, H.-D. Gräf, H.L. Harney, A. Heine, W.D. Heiss, H. Rehfeld, and A. Richter, Phys. Rev. Lett. 86, 787 (2001).
[5] J. Doppler, A.A. Mailybaev, J. Böhm, U. Kuhl, A. Girschik, F. Libisch, T.J. Milburn, P. Rabl, N. Moiseyev, and S. Rotter, Nature 537, 76 (2016).
[6] T. Stehmann, W.D. Heiss and F.G. Scholtz, J. Phys. A 37, 7813 (2004).
[7] S.-B. Lee, J. Yang, S. Moon, S.-Y. Lee, J.-B. Shim, S.W.
C\(^x\) is embedded in GL\(_n\)(C) as the subgroup of non-zero complex multiples of the identity] is also known as the projective linear group PGL\(_n\)(C). It has a deformation retract onto U(n)/U(1) \(\cong SU(n)/\mathbb{Z}_n\).

Finally, we must also mod out by \( J \). It is a contractible subgroup of GL\(_n\)(C), and intersects C\(^x\)I only at I, so its image is still a contractible subgroup in PGL\(_n\)(C). As for any space that is the quotient space G/H of a group G by a subgroup H \( \subseteq G \), PGL\(_n\)(C) can be viewed as a fibre bundle over the quotient space EP\(_n\) with fibre \( \cong J \). Because the fibre is contractible, EP\(_n\) has the same homotopy type as PGL\(_n\)(C), or as SU\((n)/\mathbb{Z}_n\). That is, these spaces are homotopy equivalent, due to the existence of deformation retracts from PGL\(_n\)(C) to EP\(_n\), and from either of these to SU\((n)/\mathbb{Z}_n\).

The homotopy type of a space can also be studied by using the homotopy groups. We can apply the homotopy long exact sequence of a fibration to the fibre bundle G over H with fibre H (where H \( \subseteq G \) are groups) [21],

\[ \cdots \rightarrow \pi_{i+1}(G) \rightarrow \pi_{i+1}(G/H) \rightarrow \pi_i(H) \rightarrow \pi_i(G) \rightarrow \cdots, \]

which holds down to \( \pi_0(G/H) \) for \( i = 0 \), \( \pi_0 \) of a space is in general a set without a group structure, however for the present case the \( \pi_0 \)s are groups, except for \( \pi_0(G/H) \) in the case that H is not a normal subgroup of G. If we apply this with \( G = PGL_n(C) \) and \( H = J \), then as all homotopy groups (or sets) of \( J \) are zero (it is contractible), we of course find again that \( \pi_i(EP_n) = \pi_i(PGL_n(C)) \approx \pi_i(SU(n)/\mathbb{Z}_n) \) for all \( i \geq 0 \). Applying the sequence again with \( G = SU(n) \), \( H = \mathbb{Z}_n \), and using \( SU(n) \) (with \( n \geq 2 \) from here on) \( \pi_1(SU(n)) = \mathbb{Z} \), \( \pi_3(SU(n)) = \mathbb{Z} \), and of course \( \pi_i(\mathbb{Z}_n) = 0 \) for \( i > 0 \), \( \pi_0(\mathbb{Z}_n) = \mathbb{Z} \), we obtain \( \pi_1(EP_n) = \mathbb{Z} \), \( \pi_2(EP_n) = 0 \), \( \pi_3(EP_n) = \mathbb{Z} \).

Appendix A: Supplemental material

1. Geometry and topology of EP\(_n\)

With \( H = \Lambda J A^{-1} \) and \( A \) an invertible complex matrix [an element of the general linear group GL\(_n\)(C)], we see that multiplying \( A \) on the right by any invertible matrix that commutes with \( J \) produces the same \( H \). The latter matrices form the subgroup \( \mathbb{C}^x \times J \). It follows that the space of such \( H \) is GL\(_n\)(C)/[\( \mathbb{C}^x \times J \)]. The division by the product group can be investigated one factor at a time.

To study the geometry and topology of this space, a useful first step is to analyze that of GL\(_n\)(C). If we arbitrarily choose an inner product on \( \mathbb{C}^n \), say the standard one, then any invertible matrix \( g \in GL_n(C) \) can be expressed in the polar decomposition \( g = Uh \), where \( U \) is unitary and \( h \) is a positive-definite Hermitian matrix (i.e. all its eigenvalues are strictly positive). This can be obtained from \( h = (g^*g)^{1/2} \), where the positive square root is taken for each positive eigenvalue of \( g^*g \), and \( U = gh^{-1} \). The space of such \( h \) is contractible; \( h \) can be deformed to the identity \( I \). Hence GL\(_n\)(C) has a "deformation retract" [21] to the space \( U(n) \) of unitary matrices.

For the second step, the group GL\(_n\)(C)/\( \mathbb{C}^x \) [where
The characteristic equation of $J + \varepsilon A$ has the form
\[ \lambda^n + \sum_{r=0}^{n-1} c_r \lambda^r = 0 \] (A2)
where the $c_r$ are similarity invariants of $J + \varepsilon A$, and all are of order $O(\varepsilon)$ as $\varepsilon \to 0$. Indeed, the terms of first order in $\varepsilon$ in $c_r$ are $c_r = -\varepsilon \sum_{j=0}^{r} A_{n+j-r,1+j} + O(\varepsilon^2)$ for all $r = 0, \ldots, n-1$.

Because any root of the equation must tend to zero as $\varepsilon \to 0$, it is not difficult to see that the $c_0$ term is the most important of the terms containing a $c_i$, provided that $\lim_{\varepsilon \to 0} c_0/\varepsilon \neq 0$. Then $\lambda \sim (-c_0)^{1/n} \sim \varepsilon^{1/n}$, and the other $c_i$ ($i \neq 0$) do not contribute at leading order in this limit.

3. Gauge transformations and invariance of results

The evolution equation (4) is covariant under $s$-dependent transformations in the subgroup $C^\times \times J$. Precisely, if the column vector $|u\rangle$ is replaced by $|v\rangle = g|u\rangle$, where $g \in C^\times \times J$, then the evolution equation becomes $\varepsilon \partial_s |v\rangle = (J + \varepsilon A)|v\rangle$, where
\[ A^g = g A g^{-1} + \partial_s g g^{-1}. \] (A3)
As $g$ is upper triangular, the inhomogeneous term $\partial_s g g^{-1}$ is also (it is in the Lie algebra of $C^\times \times J$).

Now we recall from Sec. A.2 that the terms $\alpha \varepsilon$ in the coefficients $c_r$ in the characteristic equation contain elements of $A$ or below the diagonal. It follows that the inhomogeneous terms in $A^g$ have no effect on the coefficients $c_r$ in order $\varepsilon$ except for the diagonal of $A^g$ which affects $c_{n-1} \sim \varepsilon \text{tr} A$. Moreover, $J + \varepsilon A g^{-1} = g(J + \varepsilon A) g^{-1}$ has the same eigenvalues as $J + \varepsilon A$. Hence the terms displayed in expression (7) are gauge invariant to the order shown, except for the $r = n$ term; the latter transforms as a $C^\times$ connection. Then the $s$ integrals of these terms are also gauge invariant, except that the Berry phase $(r = n)$ term changes by a multiple of $2\pi i$, and thus is invariant except under a “large” gauge transformation.

4. Generalized adiabatic theorem

Here for completeness we prove the generalized version of the adiabatic theorem (including the Berry phase) in the context of the main text. The result is contained in Ref. [17], but our proof is different. We consider the evolution equation
\[ \partial_s |v\rangle = (\varepsilon^{-1} D + A')|v\rangle, \] (A4)
where $D = \text{diag}(\lambda_0, \ldots, \lambda_{n-1})$ and $A'$ is the Berry connection. In our case, $D$ consists of eigenvalues proportional to $\varepsilon^{1/n}$ that are never equal, and $A'$ has entries independent of $\varepsilon$. We assume that $\lambda_0$ has largest real part (it is dominant), and that the differences of real parts are bounded away from zero (this holds under the assumptions in the text). We also assume that all elements of $A'$ are bounded in magnitude by the same constant $B > 0$ uniformly for all $s$. First, we suppose that the system is prepared in the $\mu = 0$ (dominant) eigenstate at $s = 0$, and consider the amplitude for it to return to that eigenstate at $s = 1$. If the possible transitions to other modes ($\mu = 0$) are neglected, then the change in the complex amplitude of the dominant state will be a factor $\exp\{\int_0^1 ds [\varepsilon^{-1} \lambda_0(s) + A'_{00}(s)]\}$. We will show that corrections to this due to transitions, and the amplitude for ending in a different state, are of relative size $O(\varepsilon^{n-1}/n)$ at most. We emphasize that our general argument applies whenever $D$ is diagonal and the differences of the real parts of the diagonal entries from the dominant one are bounded below by a non-zero constant times $\varepsilon$ to any power $< 1$.

We first extract the factor $\exp\{\int_0^1 ds [\varepsilon^{-1} \lambda_0(s) + A'_{00}(s)]\}$, to calculate relative to this expected factor; this has the effect of taking $\lambda_0 = 0$ (and $A'_{00} = 0$) without loss of generality, by subtracting these from the diagonal elements of $D$ (resp., $A'$). Now we begin by considering $n = 2$, and set $A'_{11} = 0$ for now. The change in amplitude is
\[ \langle v_0| \mathcal{P} \exp \int_0^1 ds [\varepsilon^{-1} D + A']|v_0\rangle, \] (A5)
where the initial $|v_0\rangle = (1,0)^T$ is the dominant eigenstate in this basis, (and the dual basis vector $\langle v_0|$ corresponds). This transition amplitude is
\[ = \sum_{r=0}^{\infty} \int_D \prod_{j=1}^{2r} ds_j A'_{01}(s_{2r}) A'_{10}(s_{2r-1}) \cdots A'_{10}(s_1) \times e^{\int_{s_1}^{s_{2r}} \varepsilon^{-1} \lambda_1(s') ds'} + \cdots + e^{\int_{s_{2r-1}}^{s_{2r}} \varepsilon^{-1} \lambda_1(s') ds'}, \] (A6)
where $D$ is the region defined by $s_1 < s_2 < \cdots < s_{2r}$ and all $s_j$ in $[0,1]$. Subtracting the initial 1 and taking the absolute value, we have the bound
\[ \leq \sum_{r=1}^{\infty} \int_D \prod_{j=1}^{2r} ds_j B^{2r} e^{-\varepsilon^{-1/2} L \sum_{j=1}^{r} (s_{2j} - s_{2j-1})}, \] (A7)
where $\varepsilon^{1/2}L > 0$ is a lower bound on $-\text{Re}\lambda_1 > 0$ for all $s \in [0, 1]$, which exists by our assumptions. [This expression, with 1 added, can be viewed as the partition function of a statistical mechanics problem of domain walls at positions $s_j$, where at $s = 0, 1$, the state is fixed at 0, and transitioning to the other state 1 carries a fugacity $B$ and energy penalty $\varepsilon^{-1/2}L > 0$ per unit length. As this penalty is large, domain walls are bound in pairs, and it is unlikely that state 1 is found.] This is in turn less than or equal to

$$\leq \sum_{r=1}^{\infty} \frac{1}{r!} \int_{D'} \prod_{j=1}^{2r} ds_j B^{2r} e^{-\varepsilon^{-1/2}L \sum_{j=1}^{r} (s_j - s_{2j-1})}$$  \hspace{1cm} (A8)

(where $D'$ is the domain $s_1 < s_2, s_3 < s_4, \ldots, s_{2r-1} < s_{2r}$ and all $s_j$ in $[0, 1]$), because the parts where some of the intervals $[s_{2j-1}, s_{2j}]$ overlap give positive contributions, and discarding these leaves a region that covers $D$ $r!$ times. This multiple integral is a product, and gives rise to an exponential series with the initial term 1 omitted.

Each two-dimensional integral factor can be evaluated to give $\varepsilon^{1/2}/L + \varepsilon(e^{-\varepsilon^{-1/2}L} - 1)/L^2$, where the subleading terms are introduced by the integration limits at $s = 0, 1$. Hence we have found the upper bound

$$= \exp[\varepsilon^{1/2}B^2/L + \varepsilon B^2(e^{-\varepsilon^{-1/2}L} - 1)/L^2] - 1.$$  \hspace{1cm} (A9)

As $\varepsilon \to 0$, this gives $\sim \varepsilon^{1/2}B^2/L$, which is simply the first term in the series.

In general, one should include $A'_{11}$. This can be added to $\varepsilon^{-1/2}\lambda_1$, and can be absorbed into a change in the bound $L$ when $\varepsilon$ is sufficiently small. Hence the full result is

$$\langle v_0|P \exp \int_0^1 ds [\varepsilon^{-1}D + A']|v_0\rangle = $$

$$\exp \left\{ \int_0^1 ds [\varepsilon^{-1}\lambda_0(s) + A'_{00}(s)] \right\} \cdot [1 + \mathcal{O}(\varepsilon^{1/2})]$$

for the $n = 2$ case. Similarly, we can show that the amplitude for making a transition to the state 1 [i.e. $|v_1\rangle = (0, 1)^T$] during the evolution is

$$\langle v_1|P \exp \int_0^1 ds [\varepsilon^{-1}D + A']|v_0\rangle = $$

$$\mathcal{O}(\varepsilon^{1/2}B/L) \cdot \exp \left\{ \int_0^1 ds [\varepsilon^{-1}\lambda_0(s) + A'_{00}(s)] \right\}$$

as $\varepsilon \to 0$, and the same bound also applies to the amplitude for starting in 1 and ending in 0.

Finally, for $n > 2$ we can absorb $A'_{\mu\nu}$ ($\mu > 0$) into a change in the lower bound on all $-\text{Re}\lambda_\mu$, $\mu \neq 0$. The remaining elements of $A'_{\mu\nu}$ with $\mu, \nu \neq 0$, which produce transitions among those modes, are bounded in magnitude by $B$, and in the above argument simply lead to another order 1 contribution that can also be absorbed into $\varepsilon^{(1-n)/n}L$. Hence the result is similar, with corrections of relative order $\sim \varepsilon^{(n-1)/n}B^2/L$, and the amplitude for a transition to any state $\mu \neq 0$ is of this order also.