What is the right formalism to search for resonances?

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What is the right formalism to search for resonances?

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What is the right formalism to search for resonances? II. The pentaquark chain

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(Joint Physics Analysis Center)
Three body decays

Z(4430)⁻

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S-matrix principles

1. Something must happen
2. We can exchange particles and antiparticles
3. Causes precede effects

⇒

1. Unitarity
2. Crossing symmetry
3. Causality → Analyticity
Singularities

- We want to study scattering
- We need to build amplitudes according to S-matrix theory
  \[ S = I + 2i A \]
- That means to understand the singularities of the amplitude
  - **Kinematical** ⇒ From external momenta and spins
  - **Dynamical** ⇒ The physics we are after: resonances, QCD, BSM, etc.
2.3. CROSSING SYMMETRY

Figure 0E-1

Dalitz plot of s, t, and u-channels.

Example

We take a look at the Møller scattering Ce ⇤ e ⇤ e ⇤ e, which is the s-channel of the reaction depicted on Figure 0EA1. We get a u-channel reaction the Bhabha scattering C e + e ⇤ e + e ⇤ e, which is the reaction depicted on Figure 0EA1b3E.

The considerations of this chapter enable us to derive constraints on the possible dynamics but are not sufficient to decide on the dynamics. To "get" the dynamics we must calculate and compare to experimental decay rates and scattering cross sections.

\[
\begin{align*}
  s &= (p_a + p_b)^2 = (p_c + p_d)^2 \\
  t &= (p_a - p_b)^2 = (p_c - p_d)^2 \\
  u &= (p_a - p_d)^2 = (p_b - p_c)^2 \\
  s + t + u &= M_a + m_b + m_c + m_d
\end{align*}
\]

\[
M_a > m_b + m_c + m_d
\]
**B⁰ → ψπ⁻K⁺ amplitude**

- B⁰ decays weakly \(\Rightarrow\) PC and PV amplitudes
- We can use crossing symmetry to treat the decay channel
- The s channel is K* dominated
- Once we have the s channel, the t channel can be built similarly (ψπ resonances)

\[
s = (p_3 + p_4)^2, \\
t = (\bar{p}_1 + p_3)^2 \\
u = (\bar{p}_1 + p_4)^2 \\
s + t + u = \sum_i m_i^2
\]

\[
\langle \psi \pi K, \text{out} | B, \text{in} \rangle = (2\pi)^4 \delta^4(p_2 - \bar{p}_1 - p_3 - p_4)A_\lambda
\]
Non-PW expanded amplitude

\[ A_\lambda(s, t) = \epsilon_\mu(\lambda, p_1) \left[ \left( p_3 - p_4 \right)^\mu - \frac{m_3^2 - m_4^2}{s}(p_3 + p_4)^\mu \right] C(s, t) + \epsilon_\mu(\lambda, p_1)(p_3 + p_4)^\mu B(s, t) \]

This is a choice for the tensors, there are others and provide the same results

\( C(s, t) \) and \( B(s, t) \) are scalar functions that are kinematical singularity free

Fine, but if we are going to search for resonances we are going to need this \textbf{PW expanded}, and that is where the \textbf{headache starts}
To incorporate resonances in the πK system with certain spin \( j \), we expand the amplitude in partial waves

\[
\mathcal{A}_\lambda(s, t, u) = \frac{1}{4\pi} \sum_{j=|\lambda|}^{\infty} (2j + 1) A^j_\lambda(s) d^j_{\lambda \lambda_0}(z_s)
\]

The analysis of kinematical singularities has general validity, and may be applied to the original untruncated series.
Kinematical singularities

\[ d^{j}_{\lambda 0}(z_s) = \hat{d}^{j}_{\lambda 0}(z_s) \xi_{\lambda 0}(z_s), \]
where \[ \xi_{\lambda 0}(z_s) = \left( \sqrt{1 - z_{s}^2} \right)^{|\lambda|} = \sin|\lambda| \theta_s \]
is the so-called half angle factor that contains all the kinematical singularities in \( t \).

\[ \hat{d}^{j}_{\lambda 0}(z_s) \]
is a polynomial in \( s \) and \( t \) of order \( j - |\lambda| \) divided by the factor
\[ \frac{\lambda_{12} (j-|\lambda|)/2}{\lambda_{34} (j-|\lambda|)/2} \]

The helicity partial waves \( A^{j}_{\lambda}(s) \) have singularities in \( s \). These have both dynamical and kinematical origin.

First, the term \( (p q)^{j-|\lambda|} \) is factorized out from the helicity amplitude \( A^{j}_{\lambda}(s) \). This factor is there to cancel the threshold and pseudothreshold singularities in \( s \) that appear in \( \hat{d}^{j}_{\lambda 0}(z_s) \).

We introduce the kinematic factor \( K_{\lambda 0} \) (‘±’ is short for \( \lambda = \pm 1 \)), required to account for a mismatch between the \( j \) and \( L \) dependence in the angular momentum barrier factors in presence of particles with spin.
Kinematically singularity-free helicity partial waves

\[ A_j^0(s) = K_{00} \, (pq)^j \, \hat{A}_j^0(s) \quad \text{for } j \geq 1, \]
\[ A_{\pm}^j(s) = K_{\pm 0} \, (pq)^{j-1} \, \hat{A}_\pm^j(s) \quad \text{for } j \geq 1, \]
\[ A_0^0(s) = \frac{1}{K_{00}} \, \hat{A}_0^0(s) \quad \text{for } j = 0, \]

with \( K_{00} \) and \( K_{\pm 0} \) given by

\[ K_{00} = \frac{m_1}{p \sqrt{s}} = \frac{2m_1}{\lambda_{12}^{1/2}}, \]
\[ K_{\pm 0} = q = \frac{\lambda_{34}^{1/2}}{2 \sqrt{s}}. \]

\( A_j^j(s) \sim p^{L_1} q^{L_2} \) at threshold, where \( L_1 \) and \( L_2 \) are the lowest possible orbital angular momenta in the given helicity and parity combination.

The \( K \)-factors have powers of \( \sqrt{s} \) as required to ensure factorization of the vertices of Regge poles.
We match the PW and the non-PW expanded amplitudes

\[-C(s, t) \frac{n(s, t)(s + m_1^2 - m_2^2)}{4m_1^2 s} + B(s, t) \frac{\lambda_{12}}{4m_1^2} = \frac{A_0(s)}{K_{00} \xi_{00}(z_s)} = \frac{1}{4\pi} \left( \sum_{j>0} (2j + 1)(pq)^j \hat{A}^j_0(s) \hat{d}^j_{00}(z_s) + \frac{\lambda_{12}}{4m_1^2} \hat{A}^0_0(s) \right)\]

\[\pm \sqrt{2}C(s, t) = \frac{A_\pm(s)}{K_{\pm0} \xi_{10}(z_s)} = \pm \frac{1}{4\pi} \sum_{j>0} (2j + 1)(pq)^{j-1} \hat{A}^j_\pm(s) \hat{d}^j_{10}(z_s)\]

Combining them we obtain

\[4\pi B(s, t) = \hat{A}^0_0(s) + \frac{4m_1^2}{\lambda_{12}} \sum_{j>0} (2j + 1)(pq)^j \left[ \hat{A}^j_0(s) \hat{d}^j_{00}(z_s) + \frac{s + m_1^2 - m_2^2}{\sqrt{2}m_1^2} \hat{A}^j_\pm(s) z_s \hat{d}^j_{10}(z_s) \right]\]

\[\text{Two poles at } s_\pm = (m_1 \pm m_2)^2\]

Unless this guy cancel them out

The fact that $C(s, t)$ and $B(s, t)$ cannot have kinematical singularities imposes constrains in the PW expanded amplitudes
\section*{\textbf{s}_\pm \text{ poles}}

Consequence: the $\hat{A}^j_{\lambda}(s)$ for different $\lambda$ cannot be independent at pseudo(threshold)

In the $s \rightarrow s_\pm$ limit at fixed $t$, $z_s \rightarrow \infty$ so

\[
\begin{align*}
\hat{d}^j_{\lambda 0}(z_s) & \xrightarrow{z_s \rightarrow \infty} (-1)^{\frac{\lambda + |\lambda|}{2}} \frac{(2J)! \left[ J(2J - 1) \right]^{1/2}}{2^J J \left[ (1 + \lambda)! (1 - \lambda)! \right]^{1/2}} \frac{z_s^{J - |\lambda|}}{\langle j - 1, 0; 1, \lambda | j, \lambda \rangle} \\
\hat{A}^j_0(s) & \frac{(z_s)^j}{\langle j - 1, 0; 1, 0 | j, 0 \rangle} - \frac{s + m_1^2 - m_2^2}{\sqrt{2} m_1^2} \hat{A}^j_+(s) \frac{(z_s)^j}{\sqrt{2} \langle j - 1, 0; 1, 1 | j, 1 \rangle}
\end{align*}
\]

And the bracket in previous slide has to vanish, so

\[
\begin{align*}
\hat{A}^j_+(s) & = \langle j - 1, 0; 1, 1 | j, 1 \rangle g_j(s) + \lambda_{12} f_j(s) \\
\hat{A}^j_0(s) & = \langle j - 1, 0; 1, 0 | j, 0 \rangle \frac{s + m_1^2 - m_2^2}{2 m_1^2} g'_j(s) + \lambda_{12} f'_j(s)
\end{align*}
\]

where $g_j(s)$, $f_j(s)$, $g'_j(s)$, and $f'_j(s)$ are regular functions at $s = s_\pm$, and $g_j(s_\pm) = g'_j(s_\pm)$
Final PW amplitudes

\[ A^j_+ (s) = p^{j-1} q^j \left[ \langle j - 1, 0; 1, 1|j, 1 \rangle g_j (s) + \lambda_{12} f_j (s) \right] \]

\[ A^j_0 (s) = p^{j-1} q^j \left[ \langle j - 1, 0; 1, 0|j, 0 \rangle \frac{s + m_1^2 - m_2^2}{2m_1 \sqrt{s}} g'_j (s) + \frac{m_1}{\sqrt{s}} \lambda_{12} f'_j (s) \right] \]

and

\[ A^0_0 (s) = \lambda_{12}^{1/2} / (2m_1) \hat{A}^0_0 (s) \]
Comparison to LS and CPM
Helicity amplitudes for $B^+ \to X(3872)K^+$, $X(3872) \to J/\psi \rho$, $J/\psi \to \mu^+\mu^-$, $\rho \to \pi^+\pi^-$

$$M(X \to \psi \rho) = \sum_{\lambda_{\psi}=1,0,1} \sum_{\lambda_{\rho}=1,0,1} H^{J_{\psi}}_{\lambda_{\psi}, \lambda_{\rho}} D^{J_{\rho}}_{0, \lambda_{\rho}, -\lambda_{\rho}} (0, \theta_{\chi}, 0)^* D^{J_{\psi}}_{\lambda_{\psi}, -\lambda_{\rho}, -\lambda_{\rho}} (\alpha_{\psi}, \theta_{\psi}, 0)^* D^{J_{\rho}}_{0, 0, 0} (\alpha_{\rho}, 0)^*$$

$$H^{J_{\psi}}_{\lambda_{\psi}, \lambda_{\rho}} = \sum_{J_{\chi}} \sum_{L, S} B^{J_{\chi}}_{L, S} \left( \begin{array}{c} J_{\chi} \\ \lambda_{\chi} \\ \lambda_{\rho} \end{array} \right) \left( \begin{array}{ccc} J_{\chi} & J_{\rho} & S \\ \lambda_{\chi} & -\lambda_{\rho} & \lambda_{\chi} - \lambda_{\rho} \\ 0 & \lambda_{\chi} - \lambda_{\rho} & \lambda_{\chi} - \lambda_{\rho} \end{array} \right) \left( \begin{array}{c} J_{\chi} \\ \lambda_{\chi} \\ \lambda_{\rho} \end{array} \right)$$

Clebsch-Gordan coefficients

Number of $B_{LS}$ coupling equals number of independent $H_{\lambda_{\psi}, \lambda_{\rho}}$ couplings (1-5 depending on $J_{\chi}$)

- neglecting high $L$ values can reduce number of couplings to fit to the data

Stolen from Tomasz Skwarnicki (LHCb)
LS for $B^0 \rightarrow \psi \pi^- K^+$ amplitude

\[
|j\Lambda; LS> = \sqrt{\frac{2L + 1}{2j + 1}} \sum_{\lambda_1 \lambda_2} <L, 0; S, \lambda_1 - \lambda_2|j\Lambda><j_1, \lambda_1; j_2, -\lambda_2|S, \lambda_1 - \lambda_2>|j\Lambda; \lambda_1 \lambda_2>
\]

\[
G^j_L(s) = \sqrt{\frac{2L + 1}{2j + 1}} \sum_{\lambda} <L, 0; 1, \lambda|j\lambda> A^j_\lambda(s)
\]

We invert it

\[
A^j_\lambda(s) = p^{j-1} q^j \left( \sqrt{\frac{2j - 1}{2j + 1}} \langle j - 1, 0; 1, \lambda|j, \lambda\rangle \hat{G}^j_{j-1}(s) + \sqrt{\frac{2j + 3}{2j + 1}} \langle j + 1, 0; 1, \lambda|j, \lambda\rangle p^2 \hat{G}^j_{j+1}(s) \right)
\]

Note: relativistic but not covariant
Matching

\[ g_j(s) = \sqrt{\frac{2j - 1}{2j + 1}} \hat{G}_{j-1}^j(s) \]

\[ f_j(s) = \frac{1}{4s} \sqrt{\frac{2j + 3}{2j + 1}} \langle j + 1, 0; 1, 1|j, 1 \rangle \hat{G}_{j+1}^j(s) \]

\[ g'_j(s) = \frac{2m_1 \sqrt{s}}{s + m_1^2 - m_2^2} \sqrt{\frac{2j - 1}{2j + 1}} \hat{G}_{j-1}^j(s) \]

\[ f'_j(s) = \frac{1}{4m_1 \sqrt{s}} \sqrt{\frac{2j + 3}{2j + 1}} \langle j + 1, 0; 1, 0|j, 0 \rangle \hat{G}_{j+1}^j(s) \]
Covariant Projection Method

\[ \varepsilon_{\mu_1,\ldots,\mu_{j_0}}^0(p_0) \]

\[ X_{\mu_1,\ldots,\mu_L}(p_{1r}, P_{1r}) \]

\[ \varepsilon_{\mu_1,\ldots,\mu_{j_1}}^1(p_1) \]

\[ \varepsilon_{\mu_1,\ldots,\mu_{j_r}}^r(p_r) \]

\[ 0 \rightarrow 1 r (\rightarrow 23) \]

\[ X_{\mu_1,\ldots,\mu_L}(p_{23}, P_{23}) \]

\[ \varepsilon_{\mu_1,\ldots,\mu_{j_2}}^2(p_2) \]

\[ \varepsilon_{\mu_1,\ldots,\mu_{j_3}}^3(p_3) \]

Can be used both for scattering and decay
Scattering vs decay for CPM

CPM explicitly violates crossing symmetry

\[ B \rightarrow D\pi\pi \quad \text{and} \quad DB \rightarrow \pi\pi \]

in P wave

\[ A_{BD \rightarrow \pi\pi} = pq \cos \theta_s g_P(s), \quad A_{B \rightarrow D\pi\pi} = \gamma(s) \frac{\sqrt{s}}{m_2} pq \cos \theta_s g_P(s) \]

\[ \gamma(s)\sqrt{s}/m_2 = (s - m_1^2 + m_2^2)/(2m_2^2) \]
Simple model

\[ g_S(s) = \hat{G}_0^1(s) = F_0^1(s) = 0 \text{ and } g_D(s) = \hat{G}_2^1(s) = F_2^1(s) = T_{K^*}(s) B_1(q) B_2(p) \]

\[ T_{K^*}(s) \equiv \frac{0.1}{M_{K^*}(892)^2 - s - iM_{K^*}(892)\Gamma_{K^*}(892)} + \frac{1}{M_{K^*}(1410)^2 - s - iM_{K^*}(1410)\Gamma_{K^*}(1410)} \]

\[ B_1(q) = \sqrt{\frac{1}{1 + q^2 R^2}}; \quad B_2(q) = \sqrt{\frac{1}{9 + 3q^2 R^2 + q^4 R^4}} \]

\[ \frac{d\Gamma}{ds} = \sum_j N_j \left( |A_0^j(s)|^2 + 2 |A_+^j(s)|^2 \right) \rho(s) \]
Conclusions

❖ Kinematical singularities matter **A LOT**
❖ Kinematical and dynamical singularities are entangled
❖ **Careful** with the formalism, it introduces model dependencies
❖ **Careful** when you write your hadron model (BW?), you might be careful with the singularities
❖ Compare apples to apples
❖ Doing it properly is a **nightmare**
❖ **Growing spins** ⇒ **Growing pains**