Existence of an initial value problem for
time-fractional Oldroyd-B fluid equation using
Banach fixed point theorem

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Abstract

In this paper, we study the initial boundary value problem for time-fractional Oldroyd-B fluid equation. Our model contains two Riemann-Liouville fractional derivatives which have many applications, for example, in viscoelastic flows. For the linear case, we obtain regularity results under some different assumptions of the initial data and the source function. For the non-linear case, we obtain the existence of a unique solution using Banach’s fixed point theorem.

Keywords: Time-fractional Oldroyd-B fluid problem; Riemann-Liouville; Regularity; Banach fixed point theory.

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1. Introduction

In recent years, fractional calculus has been widely applied in many different fields science and technology. The generalized fractional Oldroyd-B fluid model is a special case of non-Newtonian fluids that is of paramount importance in a large number of industries and applied sciences. Therefore, the number of publications on this topic is very abundant with many different detailed research directions. There are currently several definitions for fraction derivatives and fraction integrals, such as Riemann-Liouville, Caputo, Hadamard, Riesz, etc. We can refer the reader to some papers [2,4,17,25]. The study of the exact analytical solutions have been found in some papers [6,8,11,12]. Numerical solutions for time fractional Oldroyd-B...
We investigate the following initial problem for the time-fractional Oldroyd-B fluid equation

\[
\begin{aligned}
(1 + a\partial^\alpha_t)u_t(x, t) &= \mu(1 + b\partial^\beta_t)\Delta u(x, t) + F(x, t, u(x, t)), \quad x \in \mathcal{D}, \quad 0 < t \leq T, \\
u &= 0, \quad (x, t) \in \partial \mathcal{D} \times (0, T), \\
u(0) &= u_0(x), \quad 1 - \alpha u_t(x, 0) = 0, \quad x \in \mathcal{D},
\end{aligned}
\]  

(1.1)

where \(\partial^\alpha_t\) is the Riemann-Liouville fractional derivative \([13]\)

\[
\partial^\alpha_t v(t) := \frac{\partial}{\partial t} \int_0^t \mu_{1-\alpha}(s)v(t - s, ds).
\]

Here \(\mu\) is the source functions which are defined later. Noting that if \(a = 0\) and \(b > 0\) then (1.1) describes a Rayleigh-Stokes problem for a generalized fractional second-grade fluid. The problem with \(b = 0\) and \(a > 0\) express in general fractional Maxwell model \([14, 16]\) and if \(a = b = 0\), we get immediately that classical Newtonian fluids.

The problem (1.1) was first mentioned by two authors E. Bazhlekovka and I. Bazhlekov \([1]\). However, the properties and existence of solutions were not investigated carefully in this paper. Continuation of work \([21, 26, 30]\), in recent paper \([5]\), M. Al-Maskari and S. Karaa considered the regularity result for homogeneous linear case, i.e, \(F = 0\). So far, there has not been any work related to the qualitative solution of the problem (1.1) for both cases \(F = F(x, t)\) and \(F = F(x, t, u)\). Motivated by this reason, we try to solve the above problem to consider the well-posedness of this problem. The two main results detailed in the paper are given below

- The first major result concerns the mild solution of the problem in the linear case. We investigate the regularity of the solution with two different cases of of the smoothness of input data.
- The second major result proves the global existence of a mild solution of the problem (1.1) in the nonlinear case. Using Banach’s fixed point theorem, we have proved the problem has only one solution. The difficulty that we face is choosing some suitable solution spaces.

2. Preliminaries

We recall the Hilbert scale space, which is given as follows

\[
\mathcal{H}^s(\mathcal{D}) = \left\{ f \in L^2(\mathcal{D}), \quad \sum_{j=1}^{\infty} \lambda_j^s (f, e_j)^2 _{L^2(\mathcal{D})} < \infty \right\},
\]

for any \(s \geq 0\). Here the symbol \(\langle \cdot, \cdot \rangle_{L^2(\mathcal{D})}\) denotes the inner product in \(L^2(\mathcal{D})\). It is well-known that \(\mathcal{H}^s(\mathcal{D})\) is a Hilbert space corresponding to the norm \(\| f \|_{\mathcal{H}^s(\mathcal{D})} = \sqrt{\sum_{j=1}^{\infty} \lambda_j^s (f, e_j)^2 _{L^2(\mathcal{D})}}\), \(f \in \mathcal{H}^s(\mathcal{D})\). In view of \(\mathcal{H}^\nu(\Omega) \equiv D((-\mathcal{L})^\nu)\) is a Hilbert space. Then \(D((-\mathcal{L})^{-\nu})\) is a Hilbert space with the norm

\[
\| v \|_{D((-\mathcal{L})^{-\nu})} = \left( \sum_{j=1}^{\infty} \| v, e_j \|^2 \lambda_j^{-2\nu} \right)^{\frac{1}{2}},
\]

where \(\langle \cdot, \cdot \rangle\) in the latter equality denotes the duality between \(D((-\mathcal{L})^{-\nu})\) and \(D((-\mathcal{L})^\nu)\).

**Lemma 2.1.** The following inclusions hold true:

\[
L^p(\Omega) \hookrightarrow D(A^\sigma), \quad \text{if} \quad \frac{N}{4} < \sigma \leq 0, \quad p \geq \frac{2N}{N - 4\sigma},
\]

\[
D(A^\sigma) \hookrightarrow L^p(\Omega), \quad \text{if} \quad 0 \leq \sigma < \frac{N}{4}, \quad p \leq \frac{2N}{N - 4\sigma}.
\]

(2.3)
3. Linear inhomogeneous source

In this section, we consider the \((1.1)\) problem in the linear case, that is, the source function has the simple form \(F = F(x, t)\). Applying eigenfunction decomposition, the solution \(u\) of Problem \((1.1)\) has the form of Fourier series \(u(x, t) = \sum_{j=1}^{\infty} u_j(t)e_j(x)\). Let us denote by \(u_j(t) = \langle u(x, t), e_j \rangle\). Then we get the following equation

\[
(1 + aD_t^\alpha) \frac{du_j(t)}{dt} = -\lambda_j \mu (1 + bD_t^\alpha) u_j(t) + F_j(t), \quad u_j(0) = \langle u_0(x), e_j \rangle.
\]

Our next step is to solving this equation. By applying the Laplace transform, we obtain the formal eigen expansion of solution \(u_j(t)\) as follows

\[
\begin{align*}
    u_j(t) &= K_j(t) \langle u_0, e_j \rangle + \int_0^t G_j(t - \tau) \langle F(\tau), e_j \rangle d\tau, \\
    u(x, t) &= \sum_{j=1}^{\infty} K_j(t) \langle u_0, e_j \rangle e_j(x) + \sum_{j=1}^{\infty} \left( \int_0^t G_j(t - \tau) \langle F(\tau), e_j \rangle d\tau \right) e_j(x).
\end{align*}
\]

Here two functions \(K_j\) and \(G_j\) have the following Laplace transform

\[
\mathcal{L}(K_j)(s) = \frac{1 + as^\alpha}{s(1 + as^\alpha) + \mu \lambda_j (1 + bs^\alpha)}, \quad \mathcal{L}(G_j)(s) = \frac{1}{s(1 + as^\alpha) + \mu \lambda_j (1 + bs^\alpha)}.
\]

Thanks for the results from the work of E. Bazhlekov and I. Bazhlekov [1], we have the following lemma right away

**Lemma 3.1.** Two expressions \(K_j\) and \(G_j\) have the following properties

\[
K_j(0) = 1, \quad G_j(0) = 0, \quad |K_j(t)| \leq C_1, \quad |G_j(t)| \leq \frac{C_2 (t^{\beta-1} + at^{\beta-\alpha-1})}{\lambda_j}, \quad \int_0^t |G_j(\tau)| d\tau \leq \frac{C_3}{\lambda_j},
\]

where the constants \(C_1, C_2, C_3\) are independent of \(n\) and \(t\).

**Theorem 3.1.** Let the source function \(F \in L^\infty(0, T; H^\theta(D))\).

a) If \(u_0 \in H^s(D)\) then

\[
\|u\|_{L^\infty(0, T; H^\theta(D))} \leq 2C_1 \|u_0\|_{H^s(D)} + 2\sqrt{C_1} C_2(s, \theta, N) C_3 \|F\|_{L^\infty(0, T; H^\theta(D))},
\]

Here \(s, \theta\) satisfies the condition \(4 + 4\theta - 4s > N\).

b) If \(u_0 \in H^{s-1}(D)\) then we get

\[
\|u(., t)\|_{H^\theta(D)} \leq 2C_2 \left( t^{\beta-1} + at^{\beta - \alpha - 1} \right) \|u_0\|_{H^{s-1}(D)} + \sqrt{2C_1} C_2(s, \beta, N) C_3 \|F\|_{L^\infty(0, T; H^\theta(D))},
\]

**Remark 3.1.** From part 2 of the above theorem, we notice that if \(u_0 \in H^{s-1}(D)\) then \(L^\infty(0, T; H^\theta(D))\) belongs to the space \(L^\infty(0, T; H^\theta(D))\) with \(\gamma \geq 1 + \alpha - \theta\).

**Remark 3.2.** Let us assume that \(u_0 \in L^p(D)\) for \(1 < p < 2\). Then using Lemma \((3.1)\), we find that \(u_0 \in H^\sigma(D)\) for \(-\frac{N}{4} < \sigma < \frac{(p-2)N}{4p}\). Let us choose \(\sigma = \frac{(p-2)N}{4p}\) then if \(F \in L^\infty(0, T; H^\theta(D))\) for \(\theta > \frac{1}{4} \left( N - \frac{2N}{p} - 3 \right)\) from Theorem \((3.1)\), we can deduce that \(u \in L^\infty(0, T; H^\sigma(D))\).
Proof. Using Parseval’s equality, we find that the following estimate

\[
\left\| u(\cdot, t) \right\|_{H^s(D)}^2 \leq 2 \sum_{j=1}^{\infty} \lambda_j^2 \left( \int_0^t G_j(t-\tau) \langle F(\tau), e_j \rangle d\tau \right)^2 \leq 2 \sum_{j=1}^{\infty} \lambda_j^2 \left( \int_0^t G_j(t-\tau) \langle F(\tau), e_j \rangle d\tau \right)^2.
\]

For the term \( \mathcal{J}_2 \), we first give the following bound by using Hölder inequality

\[
\lambda_j^{2s} \left( \int_0^t G_j(t-\tau) \langle F(\tau), e_j \rangle d\tau \right)^2 \leq \lambda_j^{2s} \left( \int_0^t G_j(t-\tau) d\tau \right)^2 \left( \int_0^t G_j(t-\tau) \langle F(\tau), e_j \rangle d\tau \right)^2 \leq C_3 \lambda_j^{2s-1} \left( \int_0^t G_j(t-\tau) \langle F(\tau), e_j \rangle d\tau \right)^2.
\]

By the definition of the function \( F \) on the space \( L^\infty(0,T;\mathcal{H}^{s-1}(D)) \), we find that

\[
\| F \|_{L^\infty(0,T;\mathcal{H}^s(D))}^2 = \sup_{0 \leq \tau \leq T} \| F(\tau) \|_{\mathcal{H}^0(D)}^2 \geq \lambda_j^{20} \langle F(\tau), e_j \rangle^2
\]

which allows us to obtain that

\[
\left( \int_0^t \lambda_j G_j(t-\tau) \lambda_j^{20} \langle F(\tau), e_j \rangle^2 d\tau \right) \leq \| F \|_{L^\infty(0,T;\mathcal{H}^s(D))}^2 \left( \int_0^t G_j(t-\tau) d\tau \right) \leq C_3 \| F \|_{L^\infty(0,T;\mathcal{H}^s(D))}^2.
\]

Combining (3.13), (3.14), and (3.16), we obtain that

\[
\mathcal{J}_2 \leq 2C_3^2 \| F \|_{L^\infty(0,T;\mathcal{H}^s(D))}^2 \sum_{j=1}^{\infty} \lambda_j^{2s-2-2\theta}.
\]

It is well-known that to recall \( \lambda_j \leq C_1 j^{2/N} \), for \( N \) is the dimensional number of the domain \( D \). Therefore, we arrive at the following estimate

\[
\sum_{j=1}^{\infty} \lambda_j^{2s-2-2\theta} \leq C \sum_{j=1}^{\infty} j^{4s-4-4\theta} \leq C \sum_{j=1}^{\infty} j^{4s-N}.
\]

Since the condition \( 4 + 4\theta - 4s > N \), we know that the infinite series \( \sum_{j=1}^{\infty} j^{4s-N} \) is convergent. Let us assume that \( \sum_{j=1}^{\infty} j^{4s-N} = C_2(s, \theta, N) \) then we follow from (3.17) that

\[
\mathcal{J}_2 \leq 2C_1 C_2(s, \theta, N) C_3^2 \| F \|_{L^\infty(0,T;\mathcal{H}^s(D))}^2.
\]

For considering the first term \( \mathcal{J}_1 \), we divide two cases.

Case 1. Let us assume that \( u_0 \in \mathcal{H}^s(D) \). Under this case, we can bound the quantity \( \mathcal{J}_1 \) as follows

\[
\mathcal{J}_1 = 2 \sum_{j=1}^{\infty} \lambda_j^2 |K_j(t)|^2 \langle u_0, e_j \rangle^2 \leq 2C_1 \sum_{j=1}^{\infty} \lambda_j^2 \langle u_0, e_j \rangle^2 = 2C_1 \| u_0 \|_{\mathcal{H}^s(D)}^2.
\]
Combining (3.18) and (3.22), we arrive at
\[ \|u(.,t)\|_{\mathcal{H}^s(D)}^2 \leq J_1 + J_2 \leq 2C_1\|u_0\|_{\mathcal{H}^s(D)}^2 + 2C_1C_2(\sigma,\beta,N)C_3^2\|F\|_{L^\infty(0,T;\mathcal{H}^s(D))}^2. \]  
(3.20)

The right hand side of the above expression is independent of \( t \), so we can deduce that \( u \in L^\infty(0,T;\mathcal{H}^s(D)) \). We also give the following regularity result
\[ \|u\|_{L^\infty(0,T;\mathcal{H}^s(D))}^2 \leq J_1 + J_2 \leq 2C_1\|u_0\|_{\mathcal{H}^s(D)}^2 + 2C_1C_2(\sigma,\beta,N)C_3^2\|F\|_{L^\infty(0,T;\mathcal{H}^s(D))}^2. \]  
(3.21)

Case 2. Let us assume that \( u_0 \in \mathcal{H}^{s-1}(D) \). Under this case, we give the following estimation for \( J_1 \) in the following
\[ J_1 = 2\sum_{j=1}^{\infty} \lambda^j_j |\mathbf{K}_j(t)|^2 \|u_0,e_j\|^2 \leq 2C_2^2 \left( t^{\beta-1} + at^{\beta-a-1} \right)^2 \sum_{j=1}^{\infty} \lambda^j_j 2^{2s-2} \|u_0,e_j\|^2 \]
\[ = 2C_2^2 \left( t^{\beta-1} + at^{\beta-a-1} \right)^2 \|u_0\|_{\mathcal{H}^{s-1}(D)}^2. \]  
(3.22)

Combining (3.18) and (3.22), we arrive at
\[ \|u(.,t)\|_{\mathcal{H}^s(D)} \leq \sqrt{J_1 + J_2} \leq 2C_2 \left( t^{\beta-1} + at^{\beta-a-1} \right)^2 \|u_0\|_{\mathcal{H}^{s-1}(D)} + \sqrt{2C_1C_2(\sigma,\beta,N)C_3}\|F\|_{L^\infty(0,T;\mathcal{H}^s(D))}. \]  
(3.23)

4. Nonlinear time-fractional Oldroyd-B fluid equation

In this section, we consider the following nonlinear problem
\[
\begin{cases}
(1 + a\partial_t^\sigma - \Delta u(x,t)) = \mu(1 + b\partial_t^\beta)\Delta u(x,t) + F(u(x,t)), \quad x \in D, \quad 0 < t \leq T, \\
u = 0, \quad (x,t) \in \partial D \times (0,T), \\
u(x,0) = u_0(x), \quad I^{1-\sigma}\nu(x,0) = 0, \quad x \in D.
\end{cases}
\]  
(4.24)

By using a similar explanation as in previous section, we derive that
\[ u(x,t) = \sum_{j=1}^{\infty} \mathbf{K}_j(t) \langle u_0,e_j \rangle e_j(x) + \sum_{j=1}^{\infty} \left( \int_0^t \mathbf{G}_j(t-\tau) \langle F(u(\tau)),e_j \rangle d\tau \right) e_j(x). \]  
(4.25)

**Theorem 4.1.** Let the initial datum \( u_0 \in \mathcal{H}^s(D) \). Let \( F \) satisfies that \( F(0) = 0 \) and
\[ \|F(w_1) - F(w_2)\|_{\mathcal{H}^s(D)} \leq K_f \|w_1 - w_2\|_{\mathcal{H}^s(D)}, \]  
(4.26)

for \( K_f \) is a positive constant. Then if \( K_f \) enough small then problem (4.24) has a unique solution \( u \in L^\infty(0,T;\mathcal{H}^s(D)) \).

**Remark 4.1.** In the above theorem, we need to assume the condition of the function \( F \) with a sufficiently small Lipschitz coefficient \( K_f \). We don’t have much information about \( \mathbf{G}_j \) so unbinding \( K_f \) is a thorny and challenging issue. There is only one information about \( \mathbf{G}_j \) then the best method in this case is Banach fixed point theorem applied to the solution space \( L^\infty(0,T;\mathcal{H}^s(D)) \). We will try to investigate it in another future article.
Proof. Set the following function

$$Q_w(t) = \sum_{j=1}^{\infty} K_j(t) (u_0, e_j) e_j(x) + \sum_{j=1}^{\infty} \left( \int_{0}^{t} G_j(t - \tau) \langle F(w(\tau)), e_j \rangle d\tau \right) e_j(x). \quad (4.27)$$

If $w = 0$ then since the condition $F(t) = 0$, we know that $Q_w(t) = \sum_{j=1}^{\infty} K_j(t) (u_0, e_j) e_j(x)$. Since the fact that $|K_j(t)| \leq C_1$ as in Lemma (4.1) and the initial datum $u_0 \in \mathcal{H}^\omega(D)$, we can easily to obtain that $Q_w \in L^\omega(0, T; \mathcal{H}^\omega(D))$. Take any functions $w_1, w_2 \in L^\omega(0, T; \mathcal{H}^\omega(D))$. It follows from (4.27) that

$$Q_{w_1}(t) - Q_{w_2}(t) = \sum_{j=1}^{\infty} \left( \int_{0}^{t} G_j(t - \tau) \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle d\tau \right) e_j(x). \quad (4.28)$$

By looking closely at the above expression and using Parseval’s equality and Hölder inequality, we get the following result by some calculations

$$\left\|Q_{w_1}(t) - Q_{w_2}(t)\right\|_{\mathcal{H}^\omega(D)}^2 = \sum_{j=1}^{\infty} \lambda_j^{2s} \left( \int_{0}^{t} G_j(t - \tau) \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle d\tau \right)^2$$

$$\leq \sum_{j=1}^{\infty} \lambda_j^{2s} \left( \int_{0}^{t} |G_j(t - \tau)| d\tau \right)^2 \left( \int_{0}^{t} |G_j(t - \tau)| \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle^2 d\tau \right)$$

$$\leq C_3 \sum_{j=1}^{\infty} \lambda_j^{2s-1} \left( \int_{0}^{t} |G_j(t - \tau)| \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle^2 d\tau \right). \quad (4.29)$$

It is easy to see that

$$\left\|Q_{w_1}(t) - Q_{w_2}(t)\right\|_{\mathcal{H}^\omega(D)}^2 \leq C_3 \sum_{j=1}^{\infty} \lambda_j^{2s-2-2\beta} \left( \int_{0}^{t} \lambda_j G_j(t - \tau) \lambda_j^{2\beta} \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle^2 d\tau \right). \quad (4.30)$$

Let us continue to deal with the integral term on the right hand side of the above expression. By looking at the globally Lipschitz condition of $F$ as in (4.26), we infer that

$$\lambda_j^{2\beta} \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle^2 \leq \left\|F(w_1(\tau)) - F(w_2(\tau))\right\|_{\mathcal{H}^\beta(D)}^2 \leq K_f \sup_{0 \leq \tau \leq T} \left\|w_1(\tau) - w_2(\tau)\right\|_{\mathcal{H}^\omega(D)}^2 \leq K_f \left\|w_1 - w_2\right\|_{L^\infty(0, T; \mathcal{H}^\omega(D))}^2. \quad (4.31)$$

By combining the two evaluations (4.30) and (4.31), we have immediately the result of the upper bound of the integral on the right hand side of (4.30)

$$\int_{0}^{t} \lambda_j G_j(t - \tau) \lambda_j^{2\beta} \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle^2 d\tau \leq K_f \left\|w_1 - w_2\right\|_{L^\infty(0, T; \mathcal{H}^\omega(D))}^2 \left( \int_{0}^{t} \lambda_j G_j(t - \tau) d\tau \right). \quad (4.32)$$

Hence, from some above observations, we can derive that

$$\left\|Q_{w_1}(t) - Q_{w_2}(t)\right\|_{\mathcal{H}^\omega(D)}^2 \leq K_f C_3 \left\|w_1 - w_2\right\|_{L^\infty(0, T; \mathcal{H}^\omega(D))}^2 \sum_{j=1}^{\infty} \lambda_j^{2s-2-2\beta} \left( \int_{0}^{t} \lambda_j G_j(t - \tau) d\tau \right)$$

$$\leq K_f C_3 \left\|w_1 - w_2\right\|_{L^\infty(0, T; \mathcal{H}^\omega(D))}^2 \sum_{j=1}^{\infty} \lambda_j^{2s-2-2\beta} \leq K_f C_4 \left\|w_1 - w_2\right\|_{L^\infty(0, T; \mathcal{H}^\omega(D))}^2. \quad (4.33)$$
where we note that the infinite series \(\sum_{j=1}^{\infty} \lambda_j^{2s-2} - \lambda_j^{-2} - \lambda_j^{-2} \theta \) is convergent. This latter inequality leads to
\[
\|Qw_1 - Qw_2\|_{L^\infty(0,T;H^s(D))} \leq \sqrt{K_f C_4} \|w_1 - w_2\|_{L^\infty(0,T;H^s(D))}.
\]
(4.34)

With the help of Banach Fixed Point Theorem and noting that \(K_f C_4 < 1\), if \(K_f\) is small enough, we immediately conclude that \(\Omega\) has a fixed point \(u \in L^\infty(0,T;H^s(D))\).

5. Conclusion

In this work, we focus on the time-fractional Oldroyd-B fluid equation with the initial boundary value problem. Here, the Riemann-Liouville fractional derivatives have many applications where we consider two cases. Firstly, we obtain regularity results under some different assumptions of the initial data and the source function for the linear problem. Secondly, for the non-linear problem, we obtain the existence of a unique solution using Banach’s fixed point theorem.

References

[1] E. Bazhlekov, I. Bazhlekov, Viscoelastic flows with fractional derivative models: computational approach by convolutional calculus of Dimovski, Fract. Calc. Appl. Anal. 17 (2014), no. 4, 954-976.

[2] J. Manimaran, L. Shangerganes, A. Debbouche, Finite element error analysis of a time-fractional nonlocal diffusion equation with the Dirichlet energy, J. Comput. Appl. Math. 382 (2021), 113066, 11 pp.

[3] J. Manimaran, L. Shangerganes, A. Debbouche, A time-fractional competition ecological model with cross-diffusion, Math. Methods Appl. Sci. 43 (2020), no. 8, 5197-5211.

[4] N.H. Tuan, A. Debbouche, T.B. Ngoc, Existence and regularity of final value problems for time fractional wave equations, Comput. Math. Appl. 78 (2019), no. 5, 1396-1414.

[5] M. Al-Maskari, S. Karaa, Galerkin FEM for a time-fractional Oldroyd-B fluid problem, Adv. Comput. Math. 45 (2019), no. 2, 1005-1029.

[6] E. Karapınar, H.D. Binh, N.H. Luc, & N.H. Can, Existence and regularity of final value problems for time fractional wave equations, Comput. Math. Appl. 78 (2019), no. 5, 1396-1414.

[7] Y. Zhang, J. Jiang, Y. Bai, MHD flow and heat transfer analysis of fractional Oldroyd-B nanofluid between two coaxial cylinders, Comput. Math. Appl. 78, 3408-3421 (2019).

[8] L. Feng, F. Liu, I. Turner, P. Zhuang, Numerical methods and analysis for simulating the flow of a generalized Oldroyd-B fluid between two infinite parallel rigid plates, Int. J. Heat Mass Transf. 115, 1309-1320 (2017).

[9] J. Zhang, F. Liu, V. Anh, Analytical and numerical solutions of a two-dimensional multi-term time fractional Oldroyd-B model, Numer. Methods Part. Differ. Equ. 35, 875-893 (2019).

[10] N.H. Tuan, Y. Zhou, T.N. Thach, N.H. Can, Initial inverse problem for the nonlinear fractional Rayleigh-Stokes equation with random discrete data, Commun. Nonlinear Sci. Numer. Simul. 78 (2019), 104873, 18 pp.

[11] T.B. Ngoc, N.H. Luc, V.V. Au, N.H. Tuan, Z. Yong (2020), Existence and regularity of inverse problem for the nonlinear fractional Rayleigh-Stokes equations, Mathematical Methods in the Applied Sciences, 1-27.

[12] Y. Zhou, J. N. Wang, The nonlinear Rayleigh-Stokes problem with Riemann-Liouville fractional derivative, Mathematical Methods in the Applied Sciences, https://doi.org/10.1002/mma.5926.

[13] M. Abdullah, A.R. Butt, N. Raza and E.U. Haque, Semi-analytical technique for solution of fractional Maxwell fluid, Can. J. Phys., 94 (2017), 472-478.

[14] E. Bazhlekov and I. Bazhlekov, Peristaltic transport of viscoelastic bio-fluids with fractional derivative models, Biomath, 5 (2016) 1605151.

[15] M. Jamil, A. Rauf, A.A. Zafar and N.A. Khan, New exact analytical solutions for Stokes first problem of Maxwell fluid with fractional derivative approach, Comput. Math. Appl., 62 (2011), 1013-1023.

[16] H. Afshari, E, Karapınar, A discussion on the existence of positive solutions of the boundary value problems via-Hilfer fractional derivative on b-metric spaces, Advances in Difference Equations, 2020(1), 1-24.

[17] H. Afshari, S. Kalantari, E. Karapınar, Solution of fractional differential equations via coupled fixed point, Electronic Journal of Differential Equations,Vol. 2015 (2015), No. 286, pp. 1-12

[18] B. Alqahtani, H. Aydi, E. Karapinar, V. Rakocevic, A Solution for Volterra Fractional Integral Equations by Hybrid Contractions, Mathematics 2019, 7, 694.

[19] E. Karapinar, A.Fulga, M. Rashid, L. Shahid, H. Aydi, Large Contractions on Quasi-Metric Spaces with an Application to Nonlinear Fractional Differential-Equations, Mathematics 2019, 7, 444.
[21] I.S. Kim, Semilinear problems involving nonlinear operators of monotone type. Results in Nonlinear Analysis, 2(1), 25-35.
[22] F.S. Bachir, S. Abbas, M. Benbachir, M. Benchohra, Hilfer-Hadamard Fractional Differential Equations, Existence and Attractivity, Advances in the Theory of Nonlinear Analysis and its Application, 2021, Vol 5, Issue 1, Pages 49-57.
[23] A. Salim, M. Benchohra, J. Lazreg, J. Henderson, Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations with Non-Instantaneous Impulses in Banach Spaces, Advances in the Theory of Nonlinear Analysis and its Application, Vol 4, Issue 4, Pages 332-348, 2020.
[24] Z. Baitiche, C. Derbazia, M. Benchohrab, $\psi$-Caputo Fractional Differential Equations with Multi-point Boundary Conditions by Topological Degree Theory, Results in Nonlinear Analysis 3 (2020) No. 4, 167-178
[25] T.N. Thach, N.H. Can, V.V. Tri, Identifying the initial state for a parabolic diffusion from their time averages with fractional derivative, Mathematical Methods in the Applied Sciences, (2021), pp. 1-16
[26] S. Muthaiah, Murugesan, N. Thangaraj, Existence of Solutions for Nonlocal Boundary Value Problem of Hadamard Fractional Differential Equations, Advances in the Theory of Nonlinear Analysis and its Application, 3 (3), 162-173.
[27] A. Ardjouni, A. Djoudi, Existence and uniqueness of solutions for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations, Results in Nonlinear Analysis, 2 (3), 136-142.
[28] J.E. Lazreg, S. Abbas, M. Benchohra, & E. Karapinar, Impulsive Caputo-Fabrizio fractional differential equations in $b$-metric spaces, Open Mathematics, 19(1), 363-372.
[29] S. Muthaiah, M. Murugesan, N. Thangaraj, Existence of Solutions for Nonlocal Boundary Value Problem of Hadamard Fractional Differential Equations, Advances in the Theory of Nonlinear Analysis and its Application, 3 (3), 162-173.
[30] N.D. Phuong, L.V.C. Hoan, E. Karapinar, J. Singh, H.D. Binh, & N.H. Can, Fractional order continuity of a time semilinear fractional diffusion-wave system, Alexandria Engineering Journal, 59(6), 4959-4968.