ON GLOBAL SOLUTIONS TO THE VLASOV-POISSON SYSTEM WITH RADIATION DAMPING

MEIXIA XIAO AND XIANWEN ZHANG

School of Mathematics and Statistics
Huazhong University of Science and Technology
Wuhan, Hubei 430074, China

Abstract. In this paper, the dynamics of three dimensional Vlasov-Poisson system with radiation damping is investigated. We prove global existence of a classical as well as weak solution that propagates boundedness of velocity-space support or velocity-space moment of order two respectively. This kind of solutions possess finite mass but need not necessarily have finite kinetic energy. Moreover, uniqueness of the classical solution is also shown.

1. Introduction. As is known to all, the Vlasov-Poisson system is a typical non-linear kinetic equation modeling the time evolution of a plasma or a galaxy at the mesoscopic level [13]. The research of the system culminated in the 1990s when two different proofs for global existence of classical solutions with general data were obtained almost simultaneously but independently by P. L. Lions and B. Perthame [23] and by K. Pfaffelmoser [33]. Before this, the first global existence result of a modified system was proved [2] and global existence of spherically or cylindrically symmetric solutions was established in [3, 16, 17]. In recent years a significant interest has been focused on the research of this system. Many research papers on the mathematical aspects of this theory have been published about uniqueness, propagation of moments and large time behavior of compactly supported classical solution (see e.g., [6, 10, 13, 19, 20, 24, 26, 27, 28, 29, 30, 31, 34, 35, 36, 37, 38] and the references therein).

Recently, based on the Vlasov-Poisson theory a new model for plasma physics by considering radiation damping effects has been introduced by M. Kunze and A. D. Rendall in [21], namely,

\[
\begin{aligned}
\frac{\partial f^\pm}{\partial t} + (v \pm \varepsilon D^{[2]}(t)) \cdot \nabla_x f^\pm \pm \nabla_x U \cdot \nabla_v f^\pm &= 0, \\
\Delta U &= 4\pi \rho, \\
\lim_{x \to \infty} U(t,x) &= 0, \\
D^{[2]}(t) &= \int_{\mathbb{R}^3} \nabla_x U(t,x)(\rho^+(t,x) + \rho^-(t,x))dx, \\
f^\pm(0,x,v) &= f_0^\pm(x,v),
\end{aligned}
\]

(1)

where \(\varepsilon > 0\) is a given small constant, \(f^\pm(t,x,v)\) are microscopic densities of two species of charged particles (i.e., ions ("+") and electrons ("-")) at time \(t \geq 0\) and
position \( x \in \mathbb{R}^3 \), moving with velocity \( v \in \mathbb{R}^3 \), and \( U(t, x) \) is the self-induced electrostatic potential of the plasma. As usual, the spatial densities \( \rho^{\pm}(t, x) \) corresponding to \( f^{\pm}(t, x, v) \) are defined by

\[
\rho^{\pm}(t, x) = \int_{\mathbb{R}^3} f^{\pm}(t, x, v) dv,
\]

respectively. So, the net charge density of the plasma can be calculated by

\[
\rho(t, x) = \rho^{+}(t, x) - \rho^{-}(t, x).
\]

Due to the vanishing condition at infinity of the potential \( U(t, x) \), we know that the Poisson equation \( \Delta U = 4\pi \rho \) has a unique solution \( U(t, x) = -\int_{\mathbb{R}^3} \frac{\rho(t, y)}{|x-y|} dy \). So, the electrostatic force is defined by

\[
E(t, x) = \nabla_x U(t, x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho(t, y) dy.
\]

The radiation reaction force in (1) is described by \( D^{(2)}(t) \). Actually, effect of radiation damping should is characterized by the second derivative \( \ddot{D}(t) = D^{(2)}(t) + \varepsilon \dot{D}(t) \int (\rho^{+} + \rho^{-}) \) of the dipole moment \( D(t) = \int_{\mathbb{R}^3} x \rho(t, x) dx \). Nevertheless, the corresponding mathematical model becomes more intricate and there has been no any progress up to now (see [21, 4] for detailed discussion). For another simplified model proposed in [4] which is slightly more complex than (1), S. Bauer established local existence of classical solutions for general smooth initial data \( f_0^{\pm}(x, v) \), at present existence and asymptotic behavior of global classical solutions are obtained only for small initial data [7].

For the more simplified model (1), assuming that the initial data satisfy \( f_0^{\pm} \geq 0 \) and \( f_0^{\pm} \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \), M. Kunze and A. D. Rendall established global existence and large time behavior of compactly supported classical solutions [21]. More precisely, they showed that there is a unique nonnegative solution \( f^{\pm} \in C([0, \infty); C_c(\mathbb{R}^3 \times \mathbb{R}^3)) \cap C^1([0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3) \) to the initial value problem (1) such that

\[
\dot{E}(t) = -\varepsilon |D^{(2)}(t)|^2, \quad t \geq 0,
\]

where \( E(t) = E_{\text{kin}}(t) + E_{\text{pot}}(t) \) represents the total energy of the system with \( E_{\text{kin}}(t) \) and \( E_{\text{pot}}(t) \) being the kinetic and potential energy respectively, i.e.,

\[
E_{\text{kin}}(t) = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 (f^+ + f^-) dx dv, \quad E_{\text{pot}}(t) = \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla_x U(t, x)|^2 dx.
\]

Moreover, the following asymptotic behaviors were also proved in that paper:

\[
\|\rho^{\pm}(t)\|_p \leq C(1+t)^{-3(p-1)/2p}, \quad p \in [1, 5/3], \quad t > 0,
\]

(2)

\[
\|\nabla_x U(t)\|_p \leq C(1+t)^{-5(p-3)/7p}, \quad p \in [2, 15/4], \quad t > 0,
\]

(3)

and

\[
|D^{(2)}(t)| \leq C(1+t)^{-8/7}, \quad t > 0,
\]

(4)

where \( C \) is a positive constant depending only upon the initial data.

As for weak solutions in distributional sense, in [8] the authors considered the following class of initial data:

\[
f_0^{\pm} \geq 0, \quad (1 + |v|^2) f_0^{\pm} \in L^1(\mathbb{R}^3 \times \mathbb{R}^3), \quad f_0^{\pm} \in L^p(\mathbb{R}^3 \times \mathbb{R}^3).
\]

They showed that for \( p > 3 \) there is a global weak solution \( f^{\pm}(t, x, v) \in C([0, \infty); (w) \sim L^1(\mathbb{R}^6)) \) to system (1) such that

\[
\|f^{\pm}(t)\|_1 = \|f_0^{\pm}\|_1, \quad \|f^{\pm}(t)\|_p \leq \|f_0^{\pm}\|_p, \quad t \geq 0,
\]
\[ \mathcal{E}_{\text{kin}}(t) + \mathcal{E}_{\text{pot}}(t) \leq \mathcal{E}_{\text{kin}}(0) + \mathcal{E}_{\text{pot}}(0) - \int_0^t \varepsilon |D^{[2]}(s)|^2 ds. \]

Here, \( (w) \sim L^1(\mathbb{R}^6) \) denotes the topological vector space \( L^1(\mathbb{R}^6) \) equipped with its weak topology. Furthermore, large time behaviors for weak solutions like estimates (2)–(4) were also proved to hold true.

All the above discussions and results concern for solutions with finite kinetic energy. In this paper, inspired by a recent work [9] we are aimed at constructing classical as well as weak solutions to system (1) which allow for infinite kinetic energy and establishing their large time behaviors. Throughout the paper \( \alpha \) is a given positive number, and for the sake of convenience we denote by \( C_0^1(\mathbb{R}^3 \times \mathbb{R}^3) \) the function class consisting of continuously differentiable functions \( f(x,v) \) defined on \( \mathbb{R}^3 \times \mathbb{R}^3 \) vanishing at infinity.

Now, we are in a position to give the first result concerning classical solutions with unbounded supports.

**Theorem 1.1.** If the nonnegative initial data \( f_0^\pm(x,v) \in C_0^1 \cap L^1(\mathbb{R}^3 \times \mathbb{R}^3) \) verify
\[
\sup \{|x - \alpha v| : (x,v) \in \text{supp} f_0^\pm \} < \infty, \tag{5}
\]
then there exists a unique global classical solution \( f^\pm(t,x,v) \in C^1([0,\infty);C_0^1 \cap L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \) to system (1) such that
\[
\sup \{|x - (t + \alpha)v| : (x,v) \in \text{supp} f^\pm(t), \ t \in [0,T]\} < \infty \tag{6}
\]
for any \( T > 0 \).

Moreover, there is positive constant \( C \) depending only upon \( \|f_0^\pm\|_1, \|f_0^\pm\|_\infty \) and \( \|v - \frac{x}{2}\|_1^2 f_0^\pm \|_1 \) such that for \( t > 0 \)
\[
\|\rho^\pm(t)\|_2 \leq C(t + \alpha)^{-\frac{3}{2}}, \quad \|E(t)\|_{1/2} \leq C(t + \alpha)^{-\frac{7}{2}},
\]
\[
|D^{[2]}(t)| \leq C(t + \alpha)^{-\frac{5}{2}}.
\]

**Remark 1.** Compared with [21], we use weaker initial data, but obtain the global existence and uniqueness of classical solutions. The proof of the existence of global classical solutions does not depend on an a-priori bound of the kinetic energy. Actually, the solution constructed in Theorem 1.1 need not necessarily have finite kinetic energy. To overcome this difficulty we explore dispersive effect associated with potential energy and velocity-space moment of order two and deduce a-priori estimates for various macroscopic quantities.

The second result in this paper concerns for global weak solutions to system (1). A pair of nonnegative functions \( f^\pm(t,x,v) \) are said to be a global weak solution to system (1) if \( f^\pm(t,x,v) \in L^\infty_{loc}(0,\infty);L^1(\mathbb{R}^3 \times \mathbb{R}^3) \) such that \( \nabla_x U(t,x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho(t,y)dy \) and \( D^{[2]}(t) = \int_{\mathbb{R}^3} \nabla_x U(t,x)(\rho^+(t,x) + \rho^-(t,x))dx \) are well-defined, and for any test function \( \varphi(t,x,v) \in C^\infty_c(0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \)
\[
\int_0^{+\infty} dt \int_{\mathbb{R}^3} f^\pm[\partial_t \varphi + (v \pm \varepsilon D^{[2]}(t)) \cdot \nabla_x \varphi \pm \nabla_x U(t,x) \cdot \nabla_v \varphi] dx dv
\]
\[+ \int_{\mathbb{R}^3} f_0^\pm \varphi|_{t=0} dx dv = 0. \]

Then, we have
Theorem 1.2. Suppose that the nonnegative initial data $f_0^\pm(x,v)$ satisfy
\[ f_0^\pm(x,v) \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3), \quad (|x| + |x - \alpha v|^2) f_0^\pm(x,v) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3), \]
then there exists a global weak solution $f^\pm(t,x,v)$ of system (1) such that
\[ \|f^\pm(t)\|_s \leq \|f_0^\pm\|_s, \quad s \in [1, \infty], \quad t > 0. \]
Moreover, there exists a positive constant $C$ depending only upon the initial data such that
\[ \int_{\mathbb{R}^6} \left| v - \frac{x}{t + \alpha} \right|^2 (f^+ + f^-)(t,x,v)dvdx \leq \frac{C}{t + \alpha}, \quad t > 0. \]

Throughout the paper, $C$ denotes a generic positive constant which may depend on the initial data, but not on $T$. If a constant depends on $T$, we will give an explanation. $C_0, C_1, \cdots$ denote numerical positive constants and $\| \cdot \|_p$ represents the norm of the usual Lebesgue space $L^p$ over $\mathbb{R}^n$ with $n = 3$ or $n = 6$ as the case may be. Finally, $B_r = \{ x \in \mathbb{R}^3 : |x| < M \}$ denotes the ball in $\mathbb{R}^3$ with radius $M$ and centered at the origin.

The outline of the remainder of this paper is as follows. In section 2, we address the estimates of field and its derivative, meanwhile, establish a new conservation law which is different from [21]. Section 3 is devoted to establishing global existence and uniqueness of classical solutions to system (1). In section 4, we use the results in section 3 to investigate global existence of weak solutions to system (1). In appendix, we prove Proposition 2.

2. Preliminaries. In this section, we collect some useful tools. The first one is the following interpolation inequality established in [9].

Lemma 2.1. Let $f(x,v) \in L^\infty_r(\mathbb{R}^3 \times \mathbb{R}^3)$, and $m \geq l \geq -3$, $\alpha > 0$. There exists a constant $C$ depending only on $m, l$ and $\|f\|_\infty$ such that for all $t \geq 0$
\[ \int_{\mathbb{R}^3} |x - (t + \alpha)v|^l f dv \leq C(t + \alpha)^{\frac{3(l-m)}{m+3}} \left( \int_{\mathbb{R}^3} |x - (t + \alpha)v|^m f dv \right)^{\frac{l+3}{m+3}}, \quad (7) \]
and
\[ \left\| \int_{\mathbb{R}^3} |x - (t + \alpha)v|^l f dv \right\|_\frac{m+3}{l+3} \leq C(t + \alpha)^{\frac{3(l-m)}{m+3}} \left( \int_{\mathbb{R}^6} |x - (t + \alpha)v|^m f dv dx \right)^{\frac{l+3}{m+3}}. \quad (8) \]

In particular,
\[ \left\| \int_{\mathbb{R}^3} f dv \right\|_\frac{3q}{q+3} \leq C(t + \alpha)^{-\frac{3m}{m+3}} \left( \int_{\mathbb{R}^6} |x - (t + \alpha)v|^m f dv dx \right)^{\frac{3}{m+3}}. \quad (9) \]

From Hardy-Littlewood-Sobolev inequality and Calderón-Zygmund inequality, we can show (see [21] for details):

Lemma 2.2. We have
\[ \left\| (\cdot/|\cdot|^3) \ast \rho \right\|_q \leq C\|\rho\|_p, \quad q \in \left( \frac{3}{2}, \infty \right), \quad p = \frac{3q}{q+3}. \]
In addition,
\[ \left\| (\cdot/|\cdot|^3) \ast \text{div}\Gamma \right\|_q \leq C\|\Gamma\|_q, \quad q \in (1, \infty), \]
for smooth and compactly supported vector fields $\Gamma : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. 

Lemma 2.3. Let $\rho^\pm(x) \in L^p \cap L^\infty(\mathbb{R}^3)$ with $p \in [1,3)$, and define

$$E(x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3}(\rho^+(y) - \rho^-(y))dy.$$ 

Then we have

$$\|E\|_\infty \leq C_0 \left( \sum_{j=\pm} \|\rho^j\|_\infty \right)^{1-\frac{2}{p}} \left( \sum_{j=\pm} \|\rho^j\|_p \right)^{\frac{2}{p}}, \quad (10)$$

where the constant $C_0 > 0$ depends only on $p$. If we further assume $\rho^\pm(x) \in W^{1,\infty}(\mathbb{R}^3)$, then

$$\|\partial_x E\|_\infty \leq C_1 [(1 + \sum_{j=\pm} \|\rho^j\|_\infty)(1 + \ln(1 + \sum_{j=\pm} \text{Lip}_x(\rho^j))) + \sum_{j=\pm} \|\rho^j\|_1]. \quad (11)$$

Proof. The proof of (10) is the same as that in [21], we only sketch the proof of (11). Combining the definition of $E(x)$ with classical differentiability give that for any $x \in \mathbb{R}^3$, $d > 0$ and $i,k = 1,2,3$, we have

$$\partial_{x_i} E_k(x) = - \int_{|x-y| \geq d} \frac{3(x_i - y_i)(x_k - y_k)}{|x-y|^5} - \frac{\delta_{ik}}{|x-y|^3} \left( \rho^+(x) - \rho^-(x) \right) dy$$

$$- \int_{|x-y| < d} \frac{3(x_i - y_i)(x_k - y_k)}{|x-y|^5} - \frac{\delta_{ik}}{|x-y|^3} \left( \rho^+(y) - \rho^+(x) - \rho^-(y) + \rho^-(x) \right) dy + \frac{4\pi}{3} \delta_{ik} (\rho^+(x) - \rho^-(x)).$$

With $0 < d_1 < d_2 < \infty$, the above identity implies that

$$|\partial_{x_i} E_k(x)| \leq \frac{4\pi}{3} \sum_{j=\pm} |\rho^j(x)| + \sum_{j=\pm} \text{Lip}_x(\rho^j) \int_{|x-y| \leq d_1} \frac{4}{|x-y|^7} dy$$

$$+ \int_{d_1 < |x-y| < d_2} \frac{4}{|x-y|^3} \sum_{j=\pm} |\rho^j(y)| dy + \int_{|x-y| \geq d_2} \frac{4}{|x-y|^3} \sum_{j=\pm} |\rho^j(y)| dy$$

$$\leq C_1 \left[ \sum_{j=\pm} |\rho^j(x)| + d_1 \sum_{j=\pm} \text{Lip}_x(\rho^j) + \ln(d_2/d_1) \sum_{j=\pm} \|\rho^j\|_\infty + d_2^3 \sum_{j=\pm} \|\rho^j\|_1 \right]$$

Choosing $d_2 = 1$ and $d_1 = 1/(1 + \sum_{j=\pm} \text{Lip}_x(\rho^j))$, we get (11). \qed

Under the framework of classical solutions, it is convenient to use characteristic curves of the Vlasov equation. Suppose $E(t,x) \in C(\mathbb{R}_+; C^1_0(\mathbb{R}^3))$, then the characteristics generated by the field $E(t,x)$ is defined by the ODE system

$$\begin{aligned}
\frac{dX^\pm(s,t,x,v)}{ds} &= V^\pm(s,t,x,v) = \pm E(s, X^\pm(s,t,x,v)), \\
\frac{dv^\pm(s,t,x,v)}{ds} &= \pm E(s, X^\pm(s,t,x,v)).
\end{aligned} \quad (12)$$

By the Cauchy-Lipschitz theorem we know that there exists a unique global solution $(X^\pm(s,t,x,v), V^\pm(s,t,x,v))$ to (12), in the following we shall use the shorthand:

$$(X^\pm(s), V^\pm(s)) = (X^\pm(s,t,x,v), V^\pm(s,t,x,v)).$$

It follows from Liouville theorem that the characteristics $(X^\pm(s), V^\pm(s))$ defines a $C^1$ homeomorphism from $\mathbb{R}^3 \times \mathbb{R}^3$ onto itself which preserves Lebesgue measure. As
Assume that Proposition 1. Then, we are going to show solution to system (1) usual, we can rewrite the Vlasov-Poisson system (1) along the characteristics and obtain

\[ f^\pm(t, x, v) = f_0^\pm(X^\pm(0), V^\pm(0)). \] (13)

By (12), it is easy to find that

\[ \frac{d}{ds}((s + \alpha)V^\pm(s) - X^\pm(s)) = \pm(s + \alpha)E(s, X^\pm(s)) \mp \varepsilon D[2](s). \]

Integrating it against \( s \) we have

\[ \left| V^\pm(s) - \frac{X^\pm(s)}{s + \alpha} \right| \leq \left| v - \frac{x}{t + \alpha} \right| + \int_t^s |E(\tau, X^\pm(\tau))|d\tau + \frac{\varepsilon}{s + \alpha} \int_t^s |D[2](\tau)|d\tau \] (14)

for \( 0 \leq t < s < \infty \).

Suppose that \( f^\pm(t, x, v) \) is a classical solution to (1) and verifies (6), its kinetic energy \( E_{\text{kin}}(t) \) is not necessary finite although its potential energy \( E_{\text{pot}}(t) \) remains bounded. Therefore, the energy dissipation equation

\[ \dot{E}(t) = \dot{E}_{\text{kin}}(t) + \dot{E}_{\text{pot}}(t) = -\varepsilon|D[2](t)|^2 \] (15)

obtained in [21] for compactly supported solutions is not available in the present situation. On the other hand, it seems to be reasonable to choose the inertia

\[ \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |x - (t + \alpha)v|^2(f^+ + f^-)dxdv \]

as an alternative for the kinetic energy in (15) due to (6). In fact, inspired by [9], we can really extend (15) to some extent. To this end, we need to introduce some notation. For any \( k > 0 \), we define

\[ H_k^+(t) = \int_{\mathbb{R}^6} |x - (t + \alpha)v|^k f^+(t, x, v)dvdx, \]

\[ M_k^+(t) = \int_{\mathbb{R}^6} |v - \frac{x}{t + \alpha}|^k f^+(t, x, v)dvdx, \]

and

\[ H_k(t) = \sup \{ H_k^+(s) + H_k^-(s) \}. \]

Then, we are going to show

**Proposition 1.** Assume that \( f^\pm \in C^1(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3) \) is a nonnegative classical solution to system (1) as stated in Theorem 1.1. Then for any \( 1 \leq p \leq \infty \)

\[ ||f^\pm(t)||_p = ||f_0^\pm||_p, \quad t \geq 0, \] (16)

and

\[ \frac{d}{dt} \left( H_2^+(t) + H_2^-(t) + 2(t + \alpha)^2 E_{\text{pot}}(t) \right) = 2(t + \alpha)E_{\text{pot}}(t) - 2\varepsilon(t + \alpha)^2|D[2](t)|^2 + 2\varepsilon|D[2](t)| \cdot \int_{\mathbb{R}^6} (x - (t + \alpha)v)(f^+ - f^-)dvdx, \quad t \geq 0. \] (17)

Moreover, there exists a constant \( C > 0 \) depending only on \( ||f_0^\pm||_1, ||f_0^\pm||_\infty \) and \( M_2^+(0) \) such that

\[ E_{\text{pot}}(t) \leq C(\alpha + t)^{-1} \quad \text{and} \quad H_2(t) \leq C(\alpha + t) \] (18)

for any \( t > 0 \).
Proof. (16) follows from (13) and measure preserving of the characteristics. Now we are in a position to prove (17). Using the Vlasov equation in (1) and integration by parts, we have that

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^6} |x - (t + \alpha)v|^2 (f^+ (t, x, v) + f^- (t, x, v)) dvdx \right)
\]

\[= -\int_{\mathbb{R}^6} |x - (t + \alpha)v|^2 [\text{div}_x (v f^+ + v f^-) + \text{div}_v (Ef^+ - Ef^-)] dvdx
\]

\[-\varepsilon D^{[2]}(t) \cdot \int_{\mathbb{R}^6} |x - (t + \alpha)v|^2 \nabla_x (f^+ - f^-) dvdx
\]

\[-2 \int_{\mathbb{R}^6} (x - (t + \alpha)v) \cdot v (f^+ + f^-) dvdx
\]

\[= -2(t + \alpha) \int_{\mathbb{R}^3} x \cdot E dx + 2(t + \alpha)^2 \int_{\mathbb{R}^3} E \cdot j dx
\]

\[+ 2\varepsilon D^{[2]}(t) \cdot \int_{\mathbb{R}^6} (x - (t + \alpha)v) (f^+ - f^-) dvdx,
\]

where

\[j(t, x) = j^+(t, x) - j^-(t, x), \quad j^\pm (t, x) = \int_{\mathbb{R}^3} vf^\pm dv.
\]

On the other hand, we know from [21] that

\[\int_{\mathbb{R}^3} x \cdot E(t, x) \rho(t, x) dx = \mathcal{E}_{\text{pot}}(t)
\]

(20)

and

\[\dot{\mathcal{E}}_{\text{pot}}(t) = - \int_{\mathbb{R}^3} E \cdot j dx - \varepsilon |D^{[2]}(t)|^2.
\]

(21)

Inserting (20), (21) into (19) we then get (17).

Finally, we show (18). If \((t + \alpha)^2 |D^{[2]}(t)| \leq |\int_{\mathbb{R}^6} (x - (t + \alpha)v) (f^+ - f^-) dvdx|\), by (17) we can directly verify

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^6} |x - (t + \alpha)v|^2 (f^+ + f^-) dvdx + 2(t + \alpha)^2 \mathcal{E}_{\text{pot}}(t) \right)
\]

\[\leq 2(t + \alpha) \mathcal{E}_{\text{pot}}(t) + 2\varepsilon (t + \alpha)^{-2} \int_{\mathbb{R}^6} (x - (t + \alpha)v) (f^+ - f^-) dvdx \right)^2.
\]

(22)

Otherwise, we have \((t + \alpha)^2 |D^{[2]}(t)| > |\int_{\mathbb{R}^6} (x - (t + \alpha)v) (f^+ - f^-) dvdx|\), it is obvious that

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^6} |x - (t + \alpha)v|^2 (f^+ + f^-) dvdx + 2(t + \alpha)^2 \mathcal{E}_{\text{pot}}(t) \right) \leq 2(t + \alpha) \mathcal{E}_{\text{pot}}(t).
\]

In both cases we obtain (22). Note that \(\varepsilon > 0\) is a small constant (might as well let \(0 \leq \varepsilon \leq 1\)), from (22) we get by Hölder’s inequality that

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^6} |x - (t + \alpha)v|^2 (f^+ + f^-) dvdx + 2(t + \alpha)^2 \mathcal{E}_{\text{pot}}(t) \right)
\]

\[\leq 2(t + \alpha) \mathcal{E}_{\text{pot}}(t) + C(t + \alpha)^{-2} \int_{\mathbb{R}^6} |x - (t + \alpha)v|^2 (f^+ + f^-) dvdx.
\]
Integration of this inequality from 0 to $t \geq 0$ yields
\[
\int_{\mathbb{R}^3} |x - (t + \alpha)v|^2(f^+ + f^-)dvdx
\]
\[
\leq \int_{\mathbb{R}^3} |x - (t + \alpha)v|^2(f^+ + f^-)dvdx + 2(t + \alpha)^2E_{\text{pot}}(t)
\]
\[
\leq C + 2\int_0^t (\tau + \alpha)E_{\text{pot}}(\tau)d\tau
\]
\[
+ C\int_0^t \frac{1}{(\tau + \alpha)^2} \int_{\mathbb{R}^3} |x - (\tau + \alpha)v|^2(f^+ + f^-)dvdxdr. \tag{23}
\]
Consequently, the Gronwall’s inequality deduce that
\[
\int_{\mathbb{R}^3} |x - (t + \alpha)v|^2(f^+ + f^-)dvdx \leq C \left(1 + \int_0^t (\tau + \alpha)E_{\text{pot}}(\tau)d\tau\right). \tag{24}
\]
Inserting this inequality into (23), we get
\[
2(t + \alpha)^2E_{\text{pot}}(t)
\]
\[
\leq C + 2\int_0^t (\tau + \alpha)E_{\text{pot}}(\tau)d\tau + C\int_0^t \left(1 + \int_0^\tau (s + \alpha)E_{\text{pot}}(s)ds\right)\frac{d\tau}{(\tau + \alpha)^2}
\]
\[
\leq C + 2\int_0^t (\tau + \alpha + 1)E_{\text{pot}}(\tau)d\tau + C\int_0^t \left(\frac{1}{\alpha + \tau} - \frac{1}{\alpha + t}\right) (\tau + \alpha)E_{\text{pot}}(\tau)d\tau.
\]
Applying Gronwall’s inequality, we have $E_{\text{pot}}(t) \leq C(\alpha + t)^{-1}$ for $t \geq 0$. Insertion of the estimate for $E_{\text{pot}}(t)$ into (24) results in (18). \hfill \Box

3. Proof of Theorem 1.1. To prove Theorem 1.1, we start with constructing local solution and establishing its continuation criterion. Actually, we have the following result.

Proposition 2. Assume that the initial data $f_0^\pm(x,v)$ satisfy the conditions as stated in Theorem 1.1. Then $f_0^\pm(x,v)$ launch a unique classical solution $f^\pm(t,x,v)$ to system (1) such that $f^\pm(t,x,v) \in C^1([0,T_{\text{max}}) \times \mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty([0,T_{\text{max}}); L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ and
\[
\sup\{|x - (t + \alpha)v| : (x,v) \in \text{supp}f^\pm(t)\} < \infty \tag{25}
\]
for any given $t \in [0,T_{\text{max}})$. Furthermore, there is positive constant $C$ depending only upon $\|f_0^\pm\|_1$, $\|f_0^\pm\|_\infty$ and $M_0^2(0)$ such that $t > 0$
\[
\|\rho^\pm(t)\|_2 \leq C(t + \alpha)^{-\frac{\beta}{2}}, \quad \|E(t)\|_{\frac{1}{2}} \leq C(t + \alpha)^{-\frac{\beta}{2}}, \tag{26}
\]
\[
|\bar{D}[1](t)| \leq C(t + \alpha)^{-\frac{\beta}{4}}. \tag{27}
\]
Moreover, if $T_{\text{max}}$ is the maximal life span of the solution $f^\pm(t,x,v)$ and if
\[
\sup\{|x - (t + \alpha)v| : (x,v) \in \text{supp}f^\pm(t), \ t \in [0,T_{\text{max}})\} < \infty, \tag{28}
\]
then $T_{\text{max}} = \infty$, i.e., the solution is global in time.

The proof of this Proposition is postponed to the Appendix. Based on Proposition 2, in order to prove global existence of classical solution as stated in Theorem 1.1 it is sufficient to show that condition (28) holds true. To this end, we shall adopt the method developed in [9, 23].
Proposition 3. Let \( f^\pm(t, x, v) \in C^1([0, T_{\text{max}}) \times \mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty([0, T_{\text{max}}); L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)) \) be the solution constructed in Proposition 2 with \( T_{\text{max}} \) being its maximal life span. Then for any \( k > 3 \)

\[
\sup_{t \in [0, T_{\text{max}})} H_k^\pm(t) < \infty. \tag{29}
\]

**Proof.** Due to estimate (25), we know that

\[
H_k^\pm(t) = \int_{\mathbb{R}^6} |x - (t + \alpha)v|^k f^\pm(t, x, v)dv < \infty
\]

for any fixed \( t \in [0, T_{\text{max}}) \) and \( k \geq 0 \). Therefore, what we need to show is to prove that \( H_k^\pm(t) \) does not blow up at \( T_{\text{max}} \). Similar to [9, 23], the proof is divided into several steps.

**Step 1.** An inequality of \( H_k^\pm(t) \). In view of the Vlasov equation in (1), integration by parts, Hölder’s inequality, and Lemma 2.1 we obtain

\[
\frac{d}{dt} H_k^\pm(t) = \int_{\mathbb{R}^6} |x - (t + \alpha)v|^k [-E \cdot \nabla_v f^\pm - (v + \varepsilon D^{[2]}(t)) \cdot \nabla_x f^\pm]dvdx
\]

\[
- k \int_{\mathbb{R}^6} |x - (t + \alpha)v|^{k-2} (x - (t + \alpha)v) v f^\pm dvdx
\]

\[
\leq k(t + \alpha) \int_{\mathbb{R}^6} |E(t, x)||x - (t + \alpha)v|^{k-1} f^\pm dvdx
\]

\[
+k\varepsilon |D^{[2]}(t)| \int_{\mathbb{R}^6} |x - (t + \alpha)v|^{k-1} f^\pm dvdx
\]

\[
\leq C(k + \alpha)^{\frac{k}{k+2}} ||E(t)||_{k+3} H_k^\pm(t)^{\frac{k+2}{k+3}} + k\varepsilon |D^{[2]}(t)| H_{k-1}^\pm(t).
\]

Note that \( H_{k-1}^\pm(t) \leq ||f^\pm(t)||_{1/k} H_k^\pm(t)^{(k-1)/k} \) and without loss of generality we may assume that \( H_k^\pm(t) > 1 \), then we deduce that

\[
\frac{d}{dt} H_k^\pm(t) \leq C \left[ (t + \alpha)^{\frac{k}{k+2}} ||E(t)||_{k+3} + |D^{[2]}(t)| \right] H_k^\pm(t)^{\frac{k+2}{k+3}}. \tag{30}
\]

Thus, we get that

\[
H_k^\pm(t) \leq C \left[ H_k^\pm(0) + \left( \int_0^t ((t + \alpha)^{\frac{k}{k+2}} ||E(\tau)||_{k+3} + |D^{[2]}(\tau)|) d\tau \right)^{k+3} \right]. \tag{31}
\]

It is obvious that the last two inequalities also hold true if we replace \( H_k^\pm \) by \( H_k^- \). Consequently, they hold true with \( H_k^\pm \) replaced by \( H_k \).

**Step 2.** The estimates of \( ||E(t)||_{k+3} \) and the term of \( D^{[2]}(t) \). Based on references [9, 21, 23], we have

\[
||E(t)||_{k+3} \leq C||\rho_0(t)||_{\frac{2k+3}{k+3}} + C||\sigma(t)||_{k+3},
\]

where \( \rho_0(t, x) = \int_{\mathbb{R}^3} f_0^+(x - tv, v) - f_0^-(x - tv, v)dv \) and \( \sigma(t, x) = \int_0^t (s-t) \int (Ef^+ - Ef^-)(s, x + (s-t)v, v)dvds \). Since \( k > 3 \) implies \( 1 < \frac{3(k+3)}{k+6} < \frac{k+3}{k+2} \), we get by Hölder’s inequality that

\[
||\rho_0(t)||_{\frac{2k+3}{k+3}} \leq ||\rho_0(t)||_{\frac{k+3}{k+6}}||\rho_0(t)||_{\frac{2k+3}{k+2}} \leq C||\rho_0(t)||_{\frac{2k+3}{k+2}}.
\]
where $C > 0$ depends only on $\|f_0^\pm\|_1$. By Lemma 2.1, we get
\[
\left\| \int_{\mathbb{R}^3} f_0^+(x-tv,v)dv \right\|_{k+3} \leq C(t+\alpha)^{-\frac{2k}{k+3}} H_k^+(0)^{\frac{3}{k+3}}.
\]
Combining the above two estimates with Minkowski’s inequality, we then have
\[
\|\rho_0(t)\|_{\frac{2(k+3)}{k+4}} \leq C(t+\alpha)^{-\frac{2k+3}{k+4}} \left( H_k^+(0) + H_k^-(0) \right)^{\frac{2k+3}{k+4}}
\]
\[
\leq C(t+\alpha)^{-\frac{2k+3}{k+4}}.
\] (33)

Next, we estimate $\|\sigma(t)\|_{k+3}$. For some $t_0 \in (0,T_{max})$, which will be specified later, we have by the definition of $\sigma(t)$,
\[
\|\sigma(t)\|_{k+3} = \left\| \int_0^t s \int_{\mathbb{R}^3} (Ef^+ - Ef^-)(t-s,x-sv,v)dvds \right\|_{k+3}
\]
\[
\leq \left\| \int_0^{t_0} s \int_{\mathbb{R}^3} (Ef^+ - Ef^-)(t-s,x-sv,v)dvds \right\|_{k+3}
\]
\[
+ \left\| \int_{t_0}^t s \int_{\mathbb{R}^3} (Ef^+ - Ef^-)(t-s,x-sv,v)dvds \right\|_{k+3}
\]
\[
=: J_1 + J_2.
\] (34)

We firstly compute $\langle Ef^+\rangle(t-s,x-sv,v)dv$, with $\frac{1}{r} + \frac{1}{r'} = 1$
\[
\int_{\mathbb{R}^3} |Ef^+|(t-s,x-sv,v)dv
\]
\[
\leq C \|f^+(t-s)\|_{r}^{\frac{1}{r}-1} \left( \int_{\mathbb{R}^3} |E(t-s,x-sv)|^r dv \right)^{1/r} \left( \int_{\mathbb{R}^3} f^+(t-s,x-sv)dv \right)^{1/r'}
\]
\[
\leq C s^{-\frac{m}{2}} \sup_{r \in [0,t]} \|E(\tau)\|_r \left( \int_{\mathbb{R}^3} f^+(t-s,x-sv)dv \right)^{1/r'}.
\] (35)

To estimate $J_1$, we choose $r'$ close enough to 3 such that $\max\left\{ \frac{3(k+3)}{k+4}, 6-k \right\} \leq r' < 3$, then by (35) and (26) we have that
\[
\left\| \int_0^{t_0} s \int_{\mathbb{R}^3} (Ef^+)(t-s,x-sv,v)dvds \right\|_{k+3}
\]
\[
\leq C \left\| \int_0^{t_0} s^{-\frac{m}{2}} \left( \sup_{r \in [0,t]} \|E(\tau)\|_r \right) \left( \int_{\mathbb{R}^3} f^+(t-s,x-sv,v)dv \right)^{1/r'} \right\|_{k+3}
\]
\[
\leq C t_0^{-\frac{m}{2}} \left( \sup_{r \in [0,t]} \|E(\tau)\|_r \right) \sup_{s \in [0,t]} \left\| \int_{\mathbb{R}^3} f^+(t-s,x-sv,v)dv \right\|_{k+3}^{\frac{1}{r'}}
\]
\[
\leq C t_0^{-\frac{m}{2}} \sup_{s \in [0,t]} \left\| \int_{\mathbb{R}^3} f^+(t-s,x-sv,v)dv \right\|_{k+3}^{\frac{1}{r'}}.
\] (36)

In the above, we have used the fact $\|E(\tau)\|_r \leq C$. Actually, from weak Young’s inequality (see [35]) we have $\|E(\tau)\|_r \leq C\|\rho(t)\|_q$ for $1 + \frac{1}{r} = \frac{1}{q} + \frac{2}{q}$ and $r, q \in (1, \infty)$, then by (26) we also have $\|\rho(t)\|_q \leq C$ for any $q \in [1, \frac{6}{5}]$. 

Note that $\frac{3(k+3)}{k+4} \leq r' < 3$, there exists $m \in (k, k+1]$ such that $\frac{m+3}{3} = \frac{k+3}{r'}$.

Using (9) and (31) in succession we find

$$\left\| \int_{\mathbb{R}^3} f^+(t-s, x - sv, v) dv \right\|_{\frac{k+3}{r'}} = \left\| \int_{\mathbb{R}^3} f^+(t-s, x - sv, v) dv \right\|_{\frac{m+3}{3}}$$

$$\leq C(t + \alpha)^{-\frac{3m}{(m+3)r'}} \left( \int_{\mathbb{R}^3} |x - (t + \alpha)v|^m f^+(t-s, x - sv, v) dv dx \right)^{\frac{1}{r'}}$$

$$= C(t + \alpha)^{-\frac{m}{m+3}} H^+_{m1}(t-s)^{\frac{1}{s+3}}$$

$$\leq C(t + \alpha)^{-\frac{m}{m+3}} \left[ 1 + \int_0^{t-s} \left( (\tau + \alpha)^{\frac{3m}{3+1+\gamma}} \frac{\|E(\tau)\|_{m+3} + |D^{[2]}(\tau)|}{d\tau} \right) d\tau \right]^{\frac{m+3}{3+1+\gamma}}. \quad (37)$$

On the other hand, by the definition of $m$ and $r' \geq 6 - k$, there exists $m_1 \in (3, k]$ such that $\frac{3(m+3)}{m+6} = \frac{m_1+3}{3}$. Thanks to (9), Lemma 2.2, Hölder’s inequality and the boundedness of $\|f^\pm(\tau)\|_{1}$, we get

$$\|E(\tau)\|_{m+3} \leq C\|\rho(\tau)\|_{\frac{3(m+3)}{3+1+\gamma}} = C\|\rho(\tau)\|_{\frac{m_1+3}{3+1+\gamma}}$$

$$\leq C(t + \alpha)^{-\frac{m_1}{m+1+\gamma}} H^+_{m1}(\tau)^{\frac{3}{m_1+1+\gamma}}$$

$$\leq C(t + \alpha)^{-\frac{m_1}{m+1+\gamma}} H^+_{k}(\tau)^{\frac{3m}{m+1+\gamma}}. \quad (38)$$

Due to (27), we can find that

$$\int_0^{t-s} |D^{[2]}(\tau)| d\tau \leq C \int_0^{t-s} (\tau + \alpha)^{-\frac{1}{s+3}} d\tau \leq C. \quad (39)$$

Inserting (38) and (39) into (37) and noticing that $\frac{m}{m+3} - \frac{3m_1}{3+1+\gamma} = -1$, we then obtain

$$\left\| \int_{\mathbb{R}^3} f^+(t-s, x - sv, v) dv \right\|_{\frac{k+3}{r'}}$$

$$\leq C(t + \alpha)^{-\frac{m}{m+3}} \left[ 1 + \sup_{\tau \in [0, t-s]} H^+_{k}(\tau)^{\frac{3m_1}{3+1+\gamma}} \int_0^{t-s} (\tau + \alpha)^{-1} d\tau \right]^{\frac{m+3}{3+1+\gamma}}$$

$$\leq C(t + \alpha)^{-\frac{m}{m+3}} \left[ 1 + \left( \ln(1 + \frac{t-s}{\alpha}) \right)^{\frac{m+3}{3+1+\gamma}} \sup_{\tau \in [0, t-s]} H^+_{k}(\tau)^{\frac{2m+3}{3+1+\gamma}} \right]. \quad (40)$$

Then inserting (40) into (36), we finally obtain that

$$\left\| \int_0^t \int_{\mathbb{R}^3} (Ef^+)(t-s, x - sv, v) dv ds \right\|_{\frac{k+3}{r'}}$$

$$\leq C_0^{2-3/r} (t+\alpha)^{-\frac{m}{m+3}} \left( 1 + \ln(1 + \frac{t}{\alpha}) \right)^{\frac{m+3}{3+1+\gamma}} \sup_{s \in [0, t]} H^+_{k}(\tau)^{\frac{2m+3}{3+1+\gamma}}. \quad (41)$$

Obviously, replacing $f^+$ and $H^+$ by $f^-$ and $H^-$ respectively, the above inequality remains true. Thus, we deduce that

$$J_1 \leq C_1^{2-3/r} (t+\alpha)^{-\frac{m}{m+3}} \left( 1 + \ln(1 + \frac{t}{\alpha}) \right)^{\frac{m+3}{3+1+\gamma}} H_k(t)^{\frac{2m+3}{3+1+\gamma}}. \quad (41)$$

To estimate $J_2$, we need the following result (Lemma 1.13 in [35]).
Lemma 3.1. For all functions $u(x) \in L^1 \cap L^\infty(\mathbb{R}^3)$ and $z(x) \in L^{3/2}_w(\mathbb{R}^3)$, 
\[
\int_{\mathbb{R}^3} |u_\tau| dx \leq 3(\frac{3}{2})^{2/3} \|u\|_{1/3}^{1/3} \|u\|_{\infty}^{2/3} \|z\|_{2,w},
\]
where $z(x) \in L^{3/2}_w(\mathbb{R}^3)$ if and only if $z$ is measurable and $\|z\|_{2,w} := \sup_{\tau > 0} (\text{vol}\{x \in \mathbb{R}^3 : |z(x)| > \tau\})^{\frac{2}{3}} < \infty$.

We take advantage of Lemma 3.1, (9) and $\|E(t)\|_{2,w} \leq C$ (see e.g. (1.32) in [35]), then
\[
\left\| \int_0^t s \int_{\mathbb{R}^3} (Ef^+)(t-s, x-sv, v) dv ds \right\|_{k+3} \leq C \left\| \int_0^t s \sup_{\tau \in [0,t]} \|E(\tau)\|_{2,w} \left( \int_{\mathbb{R}^3} f^+ (t-s, x-sv, v) dv \right)^{1/3} ds \right\|_{k+3} \leq C \ln \left( \frac{t}{t_0} \right) \sup_{s \in [0,t]} \left\| \int_{\mathbb{R}^3} f^+ (t-s, x-sv, v) dv \right\|_{k+3} \leq C \ln \left( \frac{t}{t_0} \right) \sup_{s \in [0,t]} H_k^+(s) \frac{1}{t^{\frac{m}{k+3}}}. \]

Similarly, replacing $f^+$ and $H_k^+$ by $f^-$ and $H_k^-$ respectively, the above inequality remains true. Then we obtain that
\[
J_2 \leq C (t + \alpha)^{-\frac{m}{k+3}} \ln \left( \frac{t}{t_0} \right) H_k(t)^{\frac{1}{k+3}}. \] (42)

Inserting (41), (42), (34) and (33) into (32) we then get that
\[
\|E(t)\|_{m+3} \leq C \left[ t_0^{2-\frac{2}{\alpha}} (t + \alpha)^{-\frac{m}{k+3}} \left( 1 + \ln \left( 1 + \frac{t}{t_0} \right) \right)^{\frac{m+2}{k+3}} H_k(t)^{\frac{2m+3}{k+3}} + (t + \alpha)^{-\frac{2m}{k+3}} \right]. \] (43)

**Step 3.** Gronwall estimate of $H_k(t)$. Combining (43) and (30), we get that
\[
\frac{d}{dt} H_k(t) \leq C \left[ t_0^{2-\frac{2}{\alpha}} (t + \alpha)^{-\frac{m}{k+3}} \left( 1 + \ln \left( 1 + \frac{t}{t_0} \right) \right)^{\frac{m+2}{k+3}} H_k(t)^{\frac{2m+3}{k+3}} + \ln \left( \frac{t}{t_0} \right) H_k(t)^{\frac{1}{k+3}} + (t + \alpha)^{-\frac{2m}{k+3}} H_k(t)^{\frac{2m+3}{k+3}} \right] \quad H_k(t) H_k(t)^{\frac{1}{k+3}} + C(t + \alpha)^{-\frac{2m}{k+3}} H_k(t)^{\frac{2m+3}{k+3}}. \]

By the definition of $m$ we have $2-\frac{2}{\alpha} = \frac{m-k}{k+3}$, in consideration of $\beta = \frac{2m+3}{k+3} - \frac{1}{k+3} > 0$ and $H_k(t) > 1$, we find
\[
\frac{d}{dt} H_k(t) \leq C \left( \frac{t_0}{t + \alpha} \right)^{2-\frac{2}{\alpha}} \left( 1 + \ln \left( 1 + \frac{t}{t_0} \right) \right)^{\frac{m+2}{k+3}} H_k(t)^{\beta+1} + C \ln \left( \frac{t}{t_0} \right) H_k(t) + CH_k(t). \] (44)

Now, assume that $\sup \{ H_k(t) : t \in [0, T_{\max}) \} = \infty$. By monotonicity there exists a unique time $t^* \in (0, T)$ such that
\[
H_k(t)^\beta > \left( \frac{t}{t + \alpha} \right)^{\frac{2}{\alpha} - 2}, \quad \forall \ t > t^*. \]
For $t \geq t^*$ we choose $t_0 \in (0, t)$ such that
$$t_0^{\frac{3}{2} - 2} = (t + \alpha)^{\frac{3}{2} - 2} H_k(t)^{\beta}.$$ Then putting this $t_0$ into (44), we have
$$\frac{d}{dt} H_k(t) \leq C \left[ \left( 1 + \ln(1 + \frac{t}{\alpha}) \right)^{\frac{m+3}{m+2}} + \ln H_k(t) \right] H_k(t)$$ for any $t \geq t^*$. Integrating it against $t$ we get
$$H_k(t) \leq C + C \int_{t_0}^{t} \left( 1 + \ln(1 + \frac{\tau}{\alpha}) \right)^{\frac{m+3}{m+2}} + \ln H_k(\tau) \ d\tau, \quad t \in [t^*, T_{\text{max}}).$$ Thus, Gronwall’s inequality implies that $H_k(t)$ is bounded on $[0, T_{\text{max}})$, which is a contradiction to the assumption. This concludes the proof of (29).

Proof of Theorem 1.1. In consideration of (9), we know that $\|\rho^\pm(t)\|_2 \leq CH_3^+/2(t)^{\frac{1}{2}}$. By (10) with $p = 2$ and (16), we deduce that
$$\|E(t)\|_{\infty} \leq C_0 \left( \sum_{j=\pm} \|\rho^j(t)\|_{\infty} \right)^{\frac{1}{3}} \left( \sum_{j=\pm} \|\rho^j(t)\|_{2} \right)^{\frac{2}{3}} \leq CH_3^+(t)^{1/3} R(t),$$ where $R(t) = R^+(t) + R^-(t)$ and $R^\pm(t)$ are defined by
$$R^\pm(t) = \sup \{|v - \frac{x}{t + \alpha}| : (x, v) \in \text{supp} f^\pm(t)\}.$$ For any $s \in [0, T]$, $(x, v) \in \text{supp} f^\pm_0$, combining (14) and (27) we get
$$\left| V^\pm(t, 0, x, v) - \frac{X^\pm(t, 0, x, v)}{s + \alpha} \right| \leq \left| v - \frac{x}{\alpha} \right| + \int_0^t \|E(\tau)\|_{\infty} d\tau + \frac{\varepsilon}{t + \alpha} \int_0^t |D[\mathcal{E}](\tau)| d\tau, \quad t \geq t^*$$ Note that $f^\pm(t, X^\pm(t, 0, x, v), V^\pm(t, 0, x, v)) = f^\pm_0(x, v)$, we know for any $0 \leq t < T_{\text{max}}$
$$R(t) \leq C + R(0) + C \int_0^t H_3^+(\tau)^{1/3} R(\tau) d\tau, \quad t \geq t^*$$ i.e.,
$$R(t) \leq (C + R(0)) e^{C \int_0^t H_3(s)^{1/3} ds}.$$ Thanks to Proposition 3 and conservation of mass we can directly verify sup $\{H_3^+(t) : t \in [0, T_{\text{max}})\} < \infty$, consequently sup $\{R(t) : t \in [0, T_{\text{max}})\} < \infty$. By Proposition 2 this completes the proof of Theorem 1.1.
4. Proof of Theorem 1.2. Taking a sequence of smooth function \( f_{0,n}^\pm \) \((n = 1, 2, \ldots)\), each of which satisfies the assumption (5) and (6) of Theorem 1.1 such that
\[
\|f_{0,n}^+\|_s \leq \|f_0^+\|_s, \quad \|(|x| + (x - \alpha v)^2)f_{0,n}^+\|_1 \leq \|(|x| + (x - \alpha v)^2)f_0^+\|_1, \tag{45}
\]
\[
\lim_{n \to \infty} \|f_{0,n}^+ - f_0^+\|_s = 0, \quad \lim_{n \to \infty} \int_{\mathbb{R}^6} [1 + (x - \alpha v)^2]\|f_{0,n}^+ - f_0^+\| (x, v) dx dv = 0, \tag{46}
\]
where \( s \in [1, \infty] \). Using these regularized initial data, we construct approximate equations of system (1) as follows
\[
\left\{ \begin{array}{l}
\partial_t f_n^\pm + (v \pm \varepsilon D_n^{|v|^2}(t)) \cdot \nabla_x f_n^\pm \pm E_n \cdot \nabla_v f_n^\pm = 0, \\
E_n(t, x) = \nabla_x U_n = \int_{\mathbb{R}^3} \frac{v - y}{|x - y|^2} \rho_n(t, y) dy, \\
D_n^{|v|^2}(t) = \int_{\mathbb{R}^3} E_n(t, x)(\rho_n^+(t, x) + \rho_n^-(t, x)) dx, \\
\rho_n(t, x) = \rho_n^+(t, x) - \rho_n^-(t, x), \quad \rho_n^+(t, x) = \int f_n^+(t, x, v) dv, \\
f_n^+(t)|_{t=0} = f_{0,n}^+(x, v).
\end{array} \right. \tag{47}
\]
According to Theorem 1.1, system (47) has a unique nonnegative solution \( f_n^+ \in C^1([0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3) \) and satisfies
\[
\|f_n^+(t)\|_s = \|f_{0,n}^+\|_s \leq \|f_0^+\|_s \quad \text{with} \quad s \in [1, \infty], \quad t \geq 0. \tag{48}
\]
From Proposition 1 and Theorem 1.1, we know that for \( t \geq 0 \)
\[
\int_{\mathbb{R}^6} (v - \frac{x}{t + \alpha})^2(f_n^+ + f_n^-) dx dv \leq C(\alpha + t)^{-1}, \quad \|E_n(t)\|_{\mathbb{R}^6} \leq C(\alpha + t)^{-\frac{2}{4}} \tag{49}
\]
\[
\|\rho_n^+(t)\|_2 \leq C(\alpha + t)^{-\frac{1}{2}}, \quad |D_n^{|v|^2}(t)| \leq C(\alpha + t)^{-\frac{1}{4}}, \quad \|\rho_n^+(t)\|_\infty \leq C(\alpha + t)^{-1}, \tag{50}
\]
where the \( C \) is a constant depending only on \( \|f_0^+\|_1, \|f_0^+\|_\infty \) and \( M_2^2(0) \).

Proof of Theorem 1.2. By virtue of the reflexivity of \( L^p \) space \((1 < p < \infty)\), the Banach-Alauglu theorem for \( L^\infty \) and the Dunford-Pettis theorem for \( L^1 \), as well as estimate (48) and (49), there exists a pair of nonnegative functions \( f^+(t, x, v) \) defined on \([0, \infty) \times \mathbb{R}^6\) such that as \( n \to \infty \),
\[
f_n^+ \to f^+ \quad \text{weakly in} \ L^p([0, T] \times \mathbb{R}^6) \quad \text{for} \ 1 < p < \infty, \\
f_n^+ \to f^+ \quad \text{weakly}^* \ \text{in} \ L^\infty([0, T] \times \mathbb{R}^6), \\
f_n^+ \to f^+ \quad \text{weakly} \ \text{in} \ L^{1}_{loc}([0, T] \times \mathbb{R}^6)
\]
for \( T < \infty \) and up to a subsequence. Next, we show that \( f_n^+ \) is equicontinuous. As \( f_n^+ \) is a classical solution, for any \( \phi(x, v) \in C_c^\infty(\mathbb{R}^6) \), due to (50) we get
\[
\left| \frac{d}{dt} \int_{\mathbb{R}^6} \phi(x, v)f_n^+ dv \right| dx \leq C\|\nabla_x \phi\|_{\infty} \int_{\mathbb{R}^6} f_n^+ dv + \int_{\mathbb{R}^6} (v \cdot \nabla_x \phi \pm E_n \cdot \nabla_v \phi) f_n^+ dv \right| dx.
\]
The second term on the right hand side can be handled according to the discussion in [18], thus, we get
\[
\left| \frac{d}{dt} \int_{\mathbb{R}^6} \phi(x, v)f_n^+ dv \right| \leq C, \quad t > 0,
\]
where \( C \) is a constant and only depends on \( \|f_0^+\|_1, \|f_0^+\|_\infty, \|\nabla x, or \ \|\nabla_v \phi\|_\infty \) and supp. \( \phi \). Then we obtain the desired equicontinuity. Thus, following from [18], there exists
a nonnegative function $\tilde{f}^\pm(t, x, v)$ defined on $\mathbb{R}^6$ such that for any $t \in [0, \infty)$ as $n \to \infty$

$$
\begin{align*}
&f_n^\pm(t) \to \tilde{f}^\pm(t) \text{ weakly in } L^s(\mathbb{R}^3 \times \mathbb{R}^3) \text{ for } 1 < s < \infty, \\
&f_n^\pm(t) \to \tilde{f}^\pm(t) \text{ weakly in } L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3), \\
&f_n^\pm(t) \to \tilde{f}^\pm(t) \text{ weakly}^* \text{ in } L^\infty(\mathbb{R}^3 \times \mathbb{R}^3).
\end{align*}
$$

Then, a further computation shows that $f^\pm = \tilde{f}^\pm$. Thus, the weakly (weakly*) lower semicontinuous of norms implies

$$
\|f^\pm(t)\|_s = \|\tilde{f}^\pm(t)\|_s \leq \|f_0^\pm\|_s, \quad s \in [1, \infty], \quad t > 0.
$$

To proceed further, especially to prove the continuity of nonlinear terms in (1), we need the following velocity averaging lemma.

**Lemma 4.1.** Let $g \in L^2((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$, $h \in (L^2((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3))^3$ and $f \in C([0, T]; L^2(\mathbb{R}^3 \times \mathbb{R}^3))$ satisfy

$$
\partial_t f + (v + a(t)) \cdot \nabla_x f = g + \text{div}_h, \quad \text{in } D'(0, T) \times \mathbb{R}^3 \times \mathbb{R}^3
$$

where $a(t) : (0, T) \mapsto \mathbb{R}^3$ is integrable on $(0, T)$. Then, for any $\phi \in C^\infty_c(\mathbb{R}^3)$, we have

$$
\int_{\mathbb{R}^3} f(t, x, v)\phi(v)dv \in L^2((0, T); H^{-\frac{1}{2}}(\mathbb{R}^3_x))
$$

and

$$
\|F^\phi\|_{L^2((0, T); H^{-\frac{1}{2}}(\mathbb{R}^3_x))} \leq C(\|f(0)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} + \|f\|_{L^2((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)} + \|g\|_{L^2((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)} + \|h\|_{L^2((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)}),
$$

where the positive constant $C$ depends only on $\phi$ and its support.

Velocity averaging lemmas was discovered at the middle of 1980s and have been proved to be a greatly useful tool in kinetic theory (see e.g.: [11, 12, 14, 15, 22, 32]). The above one is a variant among others which was deduced in [8] by the method of [5].

Now, let $R > 0$, $\psi(v) \in C^\infty_c(\mathbb{R}^3)$ with $\text{supp} \psi \subset B_R$, then

$$
\partial_t (f_n^+ \psi(v)) + (v + cD_n^2(t)) \cdot \nabla_x (f_n^+ \psi(v)) = (E_n \cdot \nabla_v \psi(v))f_n^+ - \nabla_v \cdot (E_n f_n^+ \psi(v)).
$$

In view of (48)-(50), we get

$$
\|f_n^+(t)\psi\|_2 \leq \|\psi\|_\infty \|f_n^+(t)\|_2 \leq C\|f_0^+\|_2
$$

and

$$
\|E_n f_n^+ \psi(v)\|_2^2 + \|E_n \cdot \nabla_v \psi(v)f_n^+\|_2^2 \\
\leq (\|\psi\|_\infty^2 + \|\nabla_v \psi\|_\infty^2) \int_{\mathbb{R}^3} |E_n(t, x)|^2 \int_{|v| \leq R} |f_n^+|^2 dxdv \leq CR^3,
$$

where $C$ is a constant depending only on $\|f_0^+\|_1$, $\|f_0^+\|_\infty$, $M^2_\frac{3}{2}(0)$, $\|\psi\|_\infty$ and $\|\nabla_v \psi\|_\infty$. By Lemma 4.1, we then have that

$$
\left\{ \int_{\mathbb{R}^3} f_n^+(t) \psi(v)dv \right\} \text{ is bounded in } L^2((0, T); H^{-\frac{1}{2}}(\mathbb{R}^3_x)).
$$

On the other hand, we further have that

$$
\left\{ \frac{d}{dt} \int_{\mathbb{R}^3} f^+_n(t, x)\psi(v)dv \right\} \text{ is bounded in } L^2((0, T); H^{-1}(\mathbb{R}^3_x)).
Actually, from the Vlasov equation in (1), we deduce

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \psi(v) f_n^+(t, x, v) dv = \int_{\mathbb{R}^3} E_n \cdot \nabla_v \psi(v) f_n^+(t, x, v) dv - \nabla_x \cdot \int_{\mathbb{R}^3} (v + \varepsilon D_n^{[2]}(t)) \psi(v) f_n^+(t, x, v) dv.
\]

We estimate the two terms on the right hand side separately:

\[
\left\| \int_{\mathbb{R}^3} E_n \cdot \nabla_v \psi(v) f_n^+(t, x, v) dv \right\|^2_{L^2(\mathbb{R}_t^3)} \leq \|f_n^+(t)\|_{L^2(\mathbb{R}_t^3)}^2 \int_{\mathbb{R}^3} |E_n(t, x)|^2 dx \left\| \nabla_v \psi(v) dv \right\|^2 \leq C,
\]

and

\[
\left\| \int_{\mathbb{R}^3} (v + \varepsilon D_n^{[2]}(t)) \psi(v) f_n^+(t, x, v) dv \right\|^2_{L^2(\mathbb{R}_t^3)} \leq 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \psi(v) f_n^+(t, x, v) dv dx + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varepsilon D_n^{[2]}(t) \psi(v) f_n^+(t, x, v) dv dx \leq C,
\]

where \(C\) is a constant depending only on \(\|f_0^+\|_1, \|f_0^+\|_{L^\infty}, M_2^+(0), \|\psi\|_{L^\infty}, \|\psi\|_2, \|\nabla_v \psi\|_{L^\infty}\) and \(\text{supp} \psi\).

In order to finish the proof, we also need the following lemma (see e.g. [1]).

**Lemma 4.2.** Assume \(U_1, U_2\) and \(U_3\) are Banach spaces and satisfy \(\U_1 \hookrightarrow \hookrightarrow \U_2 \hookrightarrow \U_3\) (\(\hookrightarrow\) denote continuously embedding and \(\hookrightarrow \hookrightarrow\) denote compact embedding). If \(0 < T < \infty, 1 \leq p, q < \infty,\) and

\[
W = W(0, T) = \{ \mu \in L^p(0, T; U_1) : \mu_t \in L^q(0, T; U_3) \},
\]

then \(W \hookrightarrow \hookrightarrow L^p(0, T; U_2)\).

Now we turn to the proof of Theorem 1.2. For all \(R' > 0,\) let \(U_1 = H^\frac{1}{4}(B_{R'})\), \(U_2 = L^2(B_{R'})\), \(U_3 = H^{-1}(B_{R'})\) (obviously \(U_1 \hookrightarrow \hookrightarrow U_2 \hookrightarrow U_3\)) and \(p = q = 2\) in Lemma 4.2, then \(\int_{\mathbb{R}^3} f_n^+(t, x, v) dv\) is compact in \(L^1((0, T); L^2(B_{R'})).\) Note that \(L^2((0, T); L^2(B_{R'})) \hookrightarrow L^\theta((0, T) \times B_{R'})\) for \(\theta \in [1, 2]\), then \(\int_{\mathbb{R}^3} f_n^+(t, x, v) dv\) is compact in \(L^\theta((0, T) \times B_{R'}).\) In particular, we have extracting a subsequence if necessary

\[
\int_{B_{R'}} f_n^+(\cdot, \cdot, v) dv \to \int_{B_{R'}} f^+(\cdot, \cdot, v) dv \text{ strongly in } L^1((0, T) \times B_{R'}). \tag{51}
\]

Obviously, the above conclusion also holds true for \(f_n^-\). Now, we show

\[
\int_{\mathbb{R}^3} f_n^+(\cdot, \cdot, v) dv \to \int_{\mathbb{R}^3} f^+(\cdot, \cdot, v) dv \text{ strongly in } L^1((0, T) \times B_{R'}). \tag{52}
\]

In fact

\[
\int_0^T dt \int_{B_{R'}} \left| \int_{\mathbb{R}^3} f_n^+(t, x, v) dv - f^+(t, x, v) dv \right| dx \leq \int_0^T dt \int_{B_{R'}} \left| f_n^+(t, x, v) - f^+(t, x, v) dv \right| dx
\]
Using Gronwall’s inequality and (45), we get for \(0 \leq t \leq T\), shows that (52) holds true. Consequently, we have (up to a subsequence)

\[
Q_1 + Q_2.
\]

It follows from (49) that there exists a positive constant \(C\) such that \(Q_2 \leq C/R_0^2\). For any \(\epsilon > 0\), we choose \(R'_0\) so large such that \(Q_2 < \epsilon/2\). For such fixed \(R'_0\), we get

\[
Q_1 = \int_0^T dt \int_{B_{R_0}} \left| f_{n}^+(t, x, v) - f^+(t, x, v) \right| dv dx
\]

Due to this estimate and (51), we have

\[
\rho_{n}^+ \to \rho^+ \quad \text{in} \quad L^1_\text{loc}([0, T] \times \mathbb{R}^3).
\]

Next, we shall prove

\[
\rho_{n}^\pm \to \rho^\pm \quad \text{in} \quad L^1((0, T) \times \mathbb{R}^3).
\]

By (50) and (49), we have for \(0 \leq t \leq T\)

\[
\frac{d}{dt} \int_{\mathbb{R}^6} |x| f_{n}^+ (t, x, v) dv dx = \int_{\mathbb{R}^6} (|V_{n}(t, 0, x, v)| + \epsilon |D_{n}^{[2]}(t)|) f_{n}^+ (x, v) dv dx
\]

Using Gronwall’s inequality and (45), we get for \(0 \leq t \leq T\),

\[
\int_{\mathbb{R}^6} |x| f_{n}^+ (t, x, v) dv dx \leq e^{t/\alpha} \int_{\mathbb{R}^6} |x| f_{0,n}^+ (x, v) dv dx + C\epsilon e^{t/\alpha}
\]

Furthermore, by (53) it comes

\[
\int_0^T \int_{|x| \geq R_1} \rho_{n}^\pm (t, x) dx dt \leq \frac{\int_0^T \int_{|x| \geq R'_1} |x| \rho_{n}^\pm (t, x) dx dt}{R_1^2} \leq C(1 + \alpha) e^{T/\alpha}.
\]
Taking $R'_1 \to \infty$, then we obtain $\rho_n^\pm \to \rho^\pm$ in $L^1((0,T) \times \mathbb{R}^3)$. According to $\|\rho_n^\pm(t)\|_2 \leq C(t + \alpha)^{-\frac{\theta}{4}}$ and interpolation inequality, we have

$$\rho_n^\pm \to \rho^\pm \quad \text{in} \quad L^s((0,T) \times \mathbb{R}^3), \quad 1 \leq s < \frac{5}{3}.$$  

By Lemma 2.2, we deduce

$$E_n \to E \quad \text{in} \quad L^\theta((0,T) \times \mathbb{R}^3), \quad \frac{3}{2} < \theta < \frac{15}{4},$$

$$E_n f_n^\pm \to Ef^\pm \quad \text{in} \quad D'(((0,T) \times \mathbb{R}^3 \times \mathbb{R}^3).$$

It is clear that

$$\int_0^T |D_n^{[2]}(\tau) - D^{[2]}(\tau)|d\tau$$

$$\leq \int_0^T \left| \int_{\mathbb{R}^3} (E_n(\tau,x) - E(\tau,x))(\rho_n^+ + \rho_n^-)(\tau,x)dx \right|d\tau$$

$$+ \int_0^T \left| \int_{\mathbb{R}^3} E(\tau,x)[(\rho_n^+ - \rho^+ + (\rho_n^- - \rho^-)](\tau,x)dx \right|d\tau. \quad (54)$$

Applying Hölder’s inequality and Lemma 2.2, we have

$$\int_0^T \left| \int_{\mathbb{R}^3} (E_n(\tau,x) - E(\tau,x))(\rho_n^+ - \rho^+)(\tau,x)dx \right|d\tau$$

$$\leq \int_0^T \|E_n(\tau) - E(\tau)\|_2 \cdot \|\rho_n^+ + \rho_n^-\|_2 d\tau$$

$$\leq C \int_0^T \|\rho_n^+ - \rho^+(\tau)\|_1 \cdot \|\rho_n^+ + \rho_n^-\|_2 d\tau$$

$$\leq C \int_0^T \|\rho_n^+ - \rho^+(\tau)\|_1^{1/75} \cdot \|\rho_n^+ + \rho_n^-\|_2^{64/75} \cdot \|\rho_n^+ + \rho_n^-\|_2 d\tau$$

$$\to 0 \quad (n \to \infty).$$

Similarly, we can show the second term on the right hand side in (54) tends to 0 as $n \to \infty$. Therefore, it comes

$$\int_0^T |D_n^{[2]}(\tau) - D^{[2]}(\tau)|d\tau \to 0 \quad \text{as} \quad n \to \infty.$$  

Thus, for $\varphi(t,x,v) \in C^\infty_\circ((0,T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ we have

$$\left| \int_0^T dt \int_{\mathbb{R}^3} f_n^\pm D_n^{[2]}(t) \nabla_x \varphi dx dv - \int_0^T dt \int_{\mathbb{R}^3} f^\pm D^{[2]}(t) \nabla_x \varphi dx dv \right|$$

$$\leq \left| \int_0^T dt \int_{\mathbb{R}^3} (f_n^\pm - f^\pm) D_n^{[2]}(t) \nabla_x \varphi dx dv \right|$$

$$+ \left| \int_0^T dt \int_{\mathbb{R}^3} f_n^\pm (D_n^{[2]}(t) - D^{[2]}(t)) \nabla_x \varphi dx dv \right|$$

$$\to 0 \quad (n \to \infty).$$

Since for any $\varphi(t,x,v) \in C^\infty_\circ((0,T) \times \mathbb{R}^3 \times \mathbb{R}^3)$, we have

$$\int_0^T dt \int_{\mathbb{R}^3} f_n^\pm [\partial_t \varphi + (v \pm \varepsilon D_n^{[2]}(t)) \cdot \nabla_x \varphi \pm E_n(t,x) \cdot \nabla_v \varphi] dx dv$$
Proof of Proposition 2. Let $S^\pm$ be subsets of $C_b([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ consisting of nonnegative functions $g^\pm(t, x, v)$ which satisfy the following conditions:

$$
\|g^\pm(t)\|_1 = \|f_0^\pm\|_1, \quad \|g^\pm(t)\|_\infty \leq \|f_0^\pm\|_\infty, \quad (55)
$$

$$
\sum_{j = \pm} \|\text{Lip}_x g^j(t)\|_{L^\infty(\mathbb{R}^3)} \leq L(t) \quad \text{and} \quad g^\pm(t, x, v) = 0 \quad \text{if} \quad |v - \frac{x}{t + \alpha}| \geq R(t), \quad (56)
$$

where $t \in [0, T]$ and $T > 0$. $R(t) = R^+(t) + R^-(t)$ and $R^\pm(t)$, $L(t) > 0$ are continuous functions, which will be fixed later. It is clear that $S^\pm$ is a closed and bounded convex subset of $C_b([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$. For any $g^\pm(t, x, v) \in S^\pm$, let $g = g^+ - g^-$. Then by virtue of the definition of $E(t, x)$ and $D[2](t)$, we have

$$
E_g(t, x) = \int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \rho_g(t, y) dy, \\
\rho_g^\pm(t, x) = \int_{\mathbb{R}^3} g^\pm(t, x, v) dv \quad \rho_g(t, x) = \rho_g^+ - \rho_g^-, \quad (57)
$$

Let $(X_g^\pm(s), V_g^\pm(s)) = (X_g^\pm(s, t, x, v), V_g^\pm(s, t, x, v))$ be the solution to the ordinary differential system

$$
\begin{cases}
X_g^\pm(s, t, x, v) = V_g^\pm(s, t, x, v) \pm \varepsilon D_g[2](s), \quad X_g^\pm(t, t, x, v) = x, \\
V_g^\pm(s, t, x, v) = \pm E_g(s, X_g^\pm(s, t, x, v)), \quad V_g^\pm(t, t, x, v) = v,
\end{cases}
$$

then the solution to the linear partial differential equation

$$
\partial_t f^\pm + (v \pm \varepsilon D_g[2](t)) \cdot \nabla_x f^\pm \pm E_g \cdot \nabla_v f^\pm = 0, \quad f^\pm(0, 0, x, v) = f_0^\pm(x, v)
$$

is given formally by $f^\pm(t, x, v) = f_0^\pm(X_g^\pm(0, s), V_g^\pm(0, s))$. In this way we shall assign a pair of functions $(f^+, f^-)$ to the given a pair of functions $(g^+, g^-)$, which are denoted by $(f^+, f^-) = \mathcal{F}(g^+, g^-)$. Therefore, $\mathcal{F}$ is a map defined on $S^+ \times S^-$. Clearly, if we can show that $\mathcal{F}$ maps $S^+ \times S^-$ into itself and is continuous and compact, then the Schauder’s fixed point theorem ensures that $\mathcal{F}$ has a fixed point $(f^+, f^-)$ in $S^+ \times S^-$.\hfill\Box
First, we display $\mathcal{F}$ maps $S^+ \times S^-$ into itself. It is only to verify (56). Note that sup\{|x - \alpha v| : (x, v) \in \text{supp} f_0^\pm\} < \infty, we have

$$f_0^\pm(x, v) = 0, \quad |v - \frac{x}{\alpha}| \geq R_0,$$

where $R_0 = R_0^+ + R_0^-$ and $R_0^\pm = \sup\{|x - \frac{x}{\alpha}| : (x, v) \in \text{supp} f_0^\pm\}$. We have

$$\|\rho_\tau^\pm(t)\|_\infty \leq \frac{2\pi}{3} \|f_0^\pm\|_\infty R^3(t),$$

since vol\{v \in \mathbb{R}^3 : |v - \frac{x}{t + \alpha}| \leq R(t)\} = \frac{4\pi}{3} R^3(t). Combining (10) with the definition of $S^\pm$, we obtain

$$\|E_\tau(t)\|_\infty \leq C_0 \left( \sum_{j=\pm} \|\rho_j^\tau(t)\|_\infty \right)^{\frac{3}{2}} \left( \sum_{j=\pm} \|\rho_j^\tau(t)\|_1 \right)^{\frac{1}{3}} \leq C_2 R^2(t),$$

(58)

where $C_2 = \frac{4\pi C_0}{3} \left( \sum_{j=\pm} \|f_0^\tau(t)\|_\infty \right)^{2/3} \left( \sum_{j=\pm} \|f_0^\tau(t)\|_1 \right)^{1/3}$. Let $R : (0, \delta) \to (0, \infty)$ denote the maximal solution of the integral equation

$$R(t) = R_0 + C_2 \int_0^t R^2(s)ds,$$

i.e., $R(t) = R_0(1 - C_2 R_0 t)^{-1}$, $0 \leq t < \delta = (C_2 R_0)^{-1}$. Combining the definition of $D^{[2]}_g(t)$, (14) with (58), we get that

$$\left| V_g^\pm(s, 0, x, v) - X_g^\pm(s, 0, x, v) \right| \leq \left| v - \frac{x}{\alpha} \right| + \int_0^s \|E_\tau(t)\|_\infty dt + \frac{\varepsilon}{\alpha} \int_0^s \|D^{[2]}_g(t)\|dt$$

$$\leq \left| v - \frac{x}{\alpha} \right| + \int_0^t \|E_\tau(t)\|_\infty dt + \frac{\varepsilon}{\alpha} \int_0^t \|E_\tau(t)\|_\infty \sum_{j=\pm} \|\rho_j^\tau(t)\|_1 dt$$

$$\leq R_0 + C \int_0^t R^2(\tau)d\tau = R(t),$$

(59)

for any $0 \leq s \leq t < \delta$ and $(x, v) \in \text{supp} f_0^\pm$. By virtue of $f^\pm(t, X_g^\pm(t, 0, x, v), V_g^\pm(t, 0, x, v)) = f_0^\pm(x, v)$ and (59), we finally get that

$$f^\pm(t, x, v) = 0 \quad \text{if} \quad |v - \frac{x}{t + \alpha}| \geq R(t).$$

If $\delta_0 \in (0, \delta)$, we have that for any $t \in [0, \delta_0]$, $R(t) \leq C$ and $\|E_\tau(t)\|_\infty \leq C$, where the constants $C$ only dependent on $\|f_0^\pm\|_1, \|f_0^\pm\|_\infty, R_0^\pm$ and $\delta_0$.

We note that

$$\text{Lip}_x f^\pm(t, \cdot, v) \leq \sup_{x, v} |\partial_x (f_0^\pm(X_g^\pm(0, t, x, v), V_g^\pm(0, t, x, v)))|$$

$$\leq C(\|\partial_x X_g^\pm(0, t, \cdot, \cdot)\|_\infty + \|\partial_x V_g^\pm(0, t, \cdot, \cdot)\|_\infty).$$

We fix $x, v \in \mathbb{R}^3$ and write $(X_g^\pm, V_g^\pm)(s)$ instead of $(X_g^\pm, V_g^\pm)(s, t, x, v)$. If we differentiate the characteristic system (57) with respect to $x$ and integrate with respect to time, we get

$$|\partial_x X_g^\pm(s)| + |\partial_x V_g^\pm(s)| \leq 1 + \int_s^t (1 + \|E_\tau(t)\|_\infty)(|\partial_x X_g^\pm(\tau)| + |\partial_x V_g^\pm(\tau)|)d\tau.$$
By Gronwall’s inequality,
\[
|\partial_x X_g^+(s)| + |\partial_x V_g^+(s)| \leq \exp \left\{ \int_0^t (1 + \|\partial_x E_g(\tau)\|_{\infty}) d\tau \right\},
\]
and hence
\[
\sum_{j = \pm} \|\text{Lip}_x f^j(t)\|_{L^\infty(\mathbb{R}^3)} \leq C \exp \left\{ \int_0^t (1 + \|\partial_x E_g(\tau)\|_{\infty}) d\tau \right\}, \quad 0 \leq s \leq t \leq \delta_0. \tag{60}
\]
Inserting the uniform bound on \([0, \delta_0]\) of \(\|\rho_g^\pm(t)\|_{\infty}\) and the above estimate into (11), we find that
\[
\|\partial_x E_g(t)\|_{\infty} \leq C \left[ 1 + \ln \left( 1 + \sum_{j = \pm} \text{Lip}_x \rho_g^j(t) \right) \right]
\leq C \left[ 1 + \ln \left( 1 + \sum_{j = \pm} \|\text{Lip}_x g^j(t)\|_{L^\infty(\mathbb{R}^3)} \right) \right].
\]
By (60), it comes
\[
\ln \left( 1 + \sum_{j = \pm} \|\text{Lip}_x f^j(t)\|_{L^\infty(\mathbb{R}^3)} \right) \leq C \left[ 1 + \int_0^t \ln \left( 1 + \sum_{j = \pm} \|\text{Lip}_x g^j(\tau)\|_{L^\infty(\mathbb{R}^3)} \right) d\tau \right],
\]
which implies that there exists positive and continuous function \(L(t)\) on \([0, \delta_0]\) such that
\[
\sum_{j = \pm} \|\text{Lip}_x f^j(t)\|_{L^\infty(\mathbb{R}^3)} \leq L(t) \text{ if } \sum_{j = \pm} \|\text{Lip}_x g^j(t)\|_{L^\infty(\mathbb{R}^3)} \leq L(t).
\]
Hence, the operator \(\mathcal{F}\) maps \(S^+ \times S^-\) into itself.

Second, we show \(\mathcal{F}\) maps \(S^+ \times S^-\) into \(Q^+ \times Q^-\), where \(Q^\pm\) is defined by
\[
\{|f^\pm| \in C([0, \delta_0]; C^0_b(\mathbb{R}^3 \times \mathbb{R}^3)) \cap C^1([0, \delta_0]; C^0_b(\mathbb{R}^3 \times \mathbb{R}^3)) : \forall t \in [0, \delta_0], \|f^\pm(t)\|_1 = \|f_0^\pm\|_1, \|f^\pm(t)\|_\infty \leq \|f^\pm_0\|_\infty, \|\partial_{x,v} f^\pm(t)\|_\infty \leq C, \text{ and } f^\pm(t, x, v) = 0 \text{ if } |v - \frac{x}{t + \alpha}| \geq R(t)\}
\]
with the constant \(C\) depending only on \(\|f^\pm_0\|_1, \|f^\pm_0\|_\infty, \|\partial_{x,v} f^\pm_0\|_\infty, R^\pm_0\) and \(\delta_0\). It is obvious that \(Q^\pm \subseteq S^\pm\). Now we prove \(\|\partial_{x,v} f^\pm(t)\|_\infty \leq C, f^\pm \in C^1([0, \delta_0] \times \mathbb{R}^3 \times \mathbb{R}^3)\) and \(f^\pm \in C([0, \delta_0]; C^0(\mathbb{R}^3 \times \mathbb{R}^3))\).

For any \(g^\pm \in S^\pm\), we have that \(p^\pm_g(t, x) \in C_b([0, \delta_0] \times \mathbb{R}^3)\) is Lipschitz on \(x\). Combining the theory of Poisson equation with Lemma 2.3 we get \(E_g(t, x) \in C([0, \delta_0]; C^0_b(\mathbb{R}^3))\). By the characteristic (57), we obtain \((X_g^\pm(s), V_g^\pm(s)) \in C^1([0, \delta_0] \times [0, \delta_0] \times \mathbb{R}^3 \times \mathbb{R}^3), \text{ which implies that } f^\pm \in C^1([0, \delta_0] \times \mathbb{R}^3 \times \mathbb{R}^3)\). The analogues of the estimates for \(\sum_{j = \pm} \|\text{Lip}_x f^j(t)\|_{L^\infty(\mathbb{R}^3)} \) read
\[
\sum_{j = \pm} \|\text{Lip}_x f^j(t)\|_{L^\infty(\mathbb{R}^3)} \leq L(t) \text{ for } t \in [0, \delta_0].
\]
If \(\sup\{|x - \alpha v| : (x, v) \in \text{supp} f^\pm_0\} < \infty\), then the assertions \(\lim_{|x| + |v| \to \infty} f^\pm_0(x, v) = 0\) and \(\lim_{|v| \to \infty} f^\pm_0(x, v) = 0\) are equivalent. For any \(f^\pm \in Q^\pm\), we have \(f^\pm(t, x, v) = f^\pm_0(X_g^\pm(0), V_g^\pm(0))\) and \(|V^\pm(0)| \geq |v| - |V^\pm(0) - v| \geq |v| - C\) for \(t \in [0, \delta_0]\), which imply \(\lim_{|v| \to \infty} f^\pm(t, x, v) = 0\). In consideration of \(\sup\{|x - (t + \alpha)v| : (x, v) \in \text{supp} f^\pm(t), t \in [0, \delta_0]\} < \infty\), we then deduce \(f^\pm(t, x, v) \in C([0, \delta_0]; C^0(\mathbb{R}^3 \times \mathbb{R}^3))\).

Third, we show that \(\mathcal{F}(S^+ \times S^-)\) is relatively compact and the operator \(\mathcal{F}\) is continuous. It is obvious that for any \(\epsilon_0 > 0\) there exists a constant \(M > 0\) such that
\[
|f^\pm(t, x, v)| < \epsilon_0, \text{ if } (t, x, v) \in [0, \delta_0] \times (B_M \times B_M)^c.
\]
For \((t, x, v) \in [0, \delta_0] \times B_M \times B_M\), by Vlasov equation, boundedness of \(\|\partial_x v f^\pm(t)\|_\infty\) and \(|D_g^{[2]}(t)| \leq \|E_g(t)\|_\infty \sum_{j=\pm} \|p_j(t)\|_1\), we get that
\[
|\partial_t f^\pm(t, x, v)| \leq |(v \pm \varepsilon D_g^{[2]}(t)) \cdot \nabla_x f^\pm| + |E_g(t, x) \cdot \nabla_v f^\pm| \leq C(M + 1).
\]
Hence, \(\mathcal{F}(S^+, S^-)\) is equicontinuous in \([0, \delta_0] \times B_M \times B_M\) and then \(\mathcal{F}(S^+, S^-)\) is relatively compact.

To prove the continuity of \(\mathcal{F}\), let \(\mathcal{F}(\tilde{g}^+, \tilde{g}^-) = (\tilde{f}^+, \tilde{f}^-)\). The electrostatic force and the characteristics corresponding to \((\tilde{g}^+, \tilde{g}^-)\) are respectively denoted by \(E_{\tilde{g}}\) and \((\tilde{X}_{\tilde{g}}^\pm(s), \tilde{V}_{\tilde{g}}^\pm(s))\). Then
\[
\begin{cases}
\tilde{X}_{\tilde{g}}^\pm(s, t, x, v) = \tilde{V}_{\tilde{g}}^\pm(s, t, x, v) \pm \varepsilon D_g^{[2]}(s), & \tilde{X}_{\tilde{g}}(t, t, x, v) = x, \\
\tilde{V}_{\tilde{g}}^\pm(s, t, x, v) = \pm E_{\tilde{g}}(s, \tilde{X}_{\tilde{g}}^\pm(s, t, x, v)), & \tilde{V}_{\tilde{g}}(t, t, x, v) = v.
\end{cases}
\]
For \(t \in [0, \delta_0]\), \(s, v \in \mathbb{R}^3\)
\[
|\tilde{f}^\pm(t, x, v) - f^\pm(t, x, v)| = |f_{\tilde{g}}^\pm(\tilde{X}_{\tilde{g}}^\pm(0), \tilde{V}_{\tilde{g}}^\pm(0)) - f_0^\pm(X_g^\pm(0), V_g^\pm(0))| \\
\leq C(|\tilde{X}_{\tilde{g}}^\pm(0) - X_g^\pm(0)| + |\tilde{V}_{\tilde{g}}^\pm(0) - V_g^\pm(0)|). \ (61)
\]
For any \(0 \leq s < t \leq \delta_0\), we get
\[
\int_s^t |D_g^{[2]}(\tau) - D_g^{[2]}(\tau)|d\tau \\
= \int_s^t \int_{\mathbb{R}^3} E_{\tilde{g}}(\tau, x) \int_{\mathbb{R}^3} [(\tilde{g}^+ + \tilde{g}^-) - (g^+ + g^-)](\tau, x, v)dvdx \\
+ \int_{\mathbb{R}^3} (E_{\tilde{g}} - E_g)(\tau, x) \int_{\mathbb{R}^3} (g^+ + g^-)(\tau, x, v)dvdx |d\tau \\
\leq C \int_s^t \sum_{j=\pm} \|\tilde{g}^j(\tau) - g^j(\tau)\|_1d\tau + C \int_s^t \|E_{\tilde{g}}(\tau) - E_g(\tau)\|_\infty d\tau,
\]
and
\[
\int_s^t |E_{\tilde{g}}(\tau, \tilde{X}_{\tilde{g}}^\pm(\tau)) - E_g(\tau, X_g^\pm(\tau))|d\tau \\
\leq \int_s^t |E_{\tilde{g}}(\tau, \tilde{X}_{\tilde{g}}^\pm(\tau)) - E_g(\tau, X_g^\pm(\tau))|d\tau + \int_s^t |E_{\tilde{g}}(\tau, \tilde{X}_{\tilde{g}}^\pm(\tau)) - E_g(\tau, X_g^\pm(\tau))|d\tau \\
\leq C \int_s^t |\tilde{X}_{\tilde{g}}^\pm(\tau) - X_g^\pm(\tau)|d\tau + \int_s^t \|E_{\tilde{g}}(\tau) - E_g(\tau)\|_\infty d\tau,
\]
where \(C\) depends upon \(\|\tilde{g}^\pm\|_1, \|g^\pm\|_1, \|\tilde{g}^\pm\|_\infty\) and \(\delta_0\). Thus
\[
|\tilde{X}_{\tilde{g}}^\pm(s) - X_g^\pm(s)| + |\tilde{V}_{\tilde{g}}^\pm(s) - V_g^\pm(s)| \\
\leq \int_s^t |\tilde{V}_{\tilde{g}}^\pm(\tau) - V_g^\pm(\tau)|d\tau + \varepsilon \int_s^t |D_g^{[2]}(\tau) - D_g^{[2]}(\tau)|d\tau \\
+ \int_s^t |E_{\tilde{g}}(\tau, \tilde{X}_{\tilde{g}}^\pm(\tau)) - E_g(\tau, X_g^\pm(\tau))|d\tau \\
\leq C \int_s^t |\tilde{X}_{\tilde{g}}^\pm(\tau) - X_g^\pm(\tau)| + |\tilde{V}_{\tilde{g}}^\pm(\tau) - V_g^\pm(\tau)|d\tau \\
+ C \int_s^t \sum_{j=\pm} \|\tilde{g}^j(\tau) - g^j(\tau)\|_1d\tau + C \int_s^t \|E_{\tilde{g}}(\tau) - E_g(\tau)\|_\infty d\tau,
which implies
\[
|\tilde{X}^\pm_g(s) - X^\pm_g(s)| + |\tilde{V}^\pm_g(s) - V^\pm_g(s)| \\
\leq C \int_0^t \left[ \left( \sum_{j=\pm} \|\tilde{g}^j_g - g^j\|_1 + \|\tilde{g}^j_g - g^j\|_\infty \right) \right]^2 \left( \sum_{j=\pm} \|\tilde{g}^j_g - g^j\|_1 \right) \, d\tau,
\]
since (10). Hence, by (61) we obtain for any \( t \in [0, \delta_0] \)
\[
|f^\pm(t, x, v) - f^\pm(t, x, v)| \leq C \sup_{\tau \in [0, \delta_0]} \left[ \left( \sum_{j=\pm} \|\tilde{g}^j_g - g^j\|_1 + \|\tilde{g}^j_g - g^j\|_\infty \right) \right]^2 \left( \sum_{j=\pm} \|\tilde{g}^j_g - g^j\|_1 \right) \, d\tau,
\]
(62)
i.e., \( F \) is continuous. Combining the above results with Schauder’s fixed point theorem, we read that \( F \) has a fixed point \((f^+, f^-)\) in \((S^+, S^-)\). Furthermore, we get that \((f^+, f^-)\) is a classical solution on \([0, \delta_0]\) to system (1).

Then, we prove the uniqueness of local solutions by the method of [25]. Let \( f_1^\pm \) and \( f_2^\pm \) be two solutions with the same initial datum which exist on some common time interval \([0, T_0]\). We denote the distance between the Lagrangian flows
\[
\Upsilon(t) = \int_{\mathbb{R}^6} [X_1^\pm(t) - X_2^\pm(t)] f_0^\pm(x, v) + |X_1^\pm(t) - X_2^\pm(t)| f_0^\pm(x, v) \, dvdx,
\]
where \( X_1^\pm(t) = X_1^\pm(t, 0, x, v) \) and \( X_2^\pm(t) = X_2^\pm(t, 0, x, v) \). By the characteristics (12), we have for any \( t \in [0, T_0] \)
\[
|X_1^\pm(t) - X_2^\pm(t)| \\
\leq \int_0^t ds \int_{\mathbb{R}^6} |E_1(\tau, X_1^\pm(\tau)) - E_2(\tau, X_2^\pm(\tau))| \, d\tau + \varepsilon \int_0^t |D_1^{[2]}(\tau) - D_2^{[2]}(\tau)| \, d\tau.
\]
Then, we get
\[
\Upsilon(t) \leq \int_0^t ds \int_0^s d\tau \int_{\mathbb{R}^6} |E_1(\tau, X_1^\pm(\tau)) - E_2(\tau, X_2^\pm(\tau))| f_0^\pm(x, v) \, dvdx \\
+ \int_0^t ds \int_0^s d\tau \int_{\mathbb{R}^6} |E_1(\tau, X_1^\pm(\tau)) - E_2(\tau, X_2^\pm(\tau))| f_0^\pm(x, v) \, dvdx \\
+ \varepsilon (\|f_0^\pm\|_1 + \|f_0^\pm\|_1) \int_0^t |D_1^{[2]}(\tau) - D_2^{[2]}(\tau)| \, d\tau =: J_1 + J_2 + J_3.
\]
Next, we estimate \( J_3 \) and we have
\[
J_3 \leq C \int_0^t \left( \int_{\mathbb{R}^3} E_1(\tau, x) \int_{\mathbb{R}^3} |(f_1^+ + f_1^-) - (f_2^+ + f_2^-)(\tau, x, v)| \, dvdx \right) \, d\tau \\
+ C \int_0^t \left( \int_{\mathbb{R}^3} (E_1 - E_2)(\tau, x) \int_{\mathbb{R}^3} (f_2^+ + f_2^-)(\tau, x, v) \, dvdx \right) \, d\tau \\
\leq C \int_0^t \left( \int_{\mathbb{R}^3} E_1(\tau, x) \int_{\mathbb{R}^3} (f_1^+ - f_1^-)(\tau, x, v) \, dvdx \right) \, d\tau \\
+ C \int_0^t \left( \int_{\mathbb{R}^3} E_1(\tau, x) \int_{\mathbb{R}^3} (f_2^+ - f_2^-)(\tau, x, v) \, dvdx \right) \, d\tau.
Similarly, it comes

\[ + C \int_0^t \left| \int_{\mathbb{R}^3} (E_1 - E_2)(\tau, x) \int_{\mathbb{R}^3} f_2^+(\tau, x, v) dv \right| d\tau \]

\[ + C \int_0^t \left| \int_{\mathbb{R}^3} (E_1 - E_2)(\tau, x) \int_{\mathbb{R}^3} f_2^-(\tau, x, v) dv \right| d\tau \]

\[ = J_{11} + J_{12} + J_{13} + J_{14}, \]

where the constant \( C > 0 \) depends only on \( \| f_0^\pm \|_1 \). In order to proceed we need another auxiliary result (see [25] for details).

**Lemma 5.1.** There exists \( C > 0 \) such that for all \( r > n \) and \( g \in L^1 \cap L^r(\mathbb{R}^n) \),

\[ \int_{\mathbb{R}^n} \left| \frac{x - z}{|x - z|^n} - \frac{y - z}{|y - z|^n} \right| g(z) dz \leq C r (\| g \|_1 + \| g \|_r) |x - y|^{-\frac{n}{2}}. \]

Since \( \| \rho_i^+(t) \|_1 = \| f_0^+ \|_1 \), \( \{ i = 1, 2 \} \) and \( \| \rho_i^+(t) \|_\infty \leq C(T_0, \| f_0^+ \|_\infty) \), we get \( \| \rho_i^+(t) \|_p \leq C(T_0, f_0^+) \quad (p \in [1, \infty]) \) for \( t \in [0, T_0] \). Combining measure preserving of the characteristics, Lemma 5.1 and Jensen’s inequality, we get for any \( r > 3 \) and \( t \in [0, T_0] \)

\[ J_{11} = C \int_0^t \left| \int_{\mathbb{R}^3} [E_1(\tau, X_1^+(\tau)) - E_1(\tau, X_2^+(\tau))] f_0^+(x, v) dv dx \right| d\tau \]

\[ \leq C r \int_0^t (\| \rho_1(\tau) \|_1 + \| \rho_1(\tau) \|_r) \int_{\mathbb{R}^6} |X_1^+(\tau) - X_2^+(\tau)|^{1-\frac{r}{2}} f_0^+(x, v) dv dx d\tau \]

\[ \leq C(T_0, f_0^+) r \int_0^t \Upsilon(\tau)^{1-\frac{r}{2}} d\tau. \]

Similarly, it comes

\[ J_{12} \leq C(T_0, f_0^+) r \int_0^t \Upsilon(\tau)^{1-\frac{r}{2}} d\tau. \]

Note that

\[ (E_1 - E_2)(\tau, x) \]

\[ = \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \int_{\mathbb{R}^3} (f_1^+ - f_1^-)(\tau, y, v) dv dy \]

\[ - \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \int_{\mathbb{R}^3} (f_2^+ - f_2^-)(\tau, y, v) dv dy \]

\[ = \int_{\mathbb{R}^6} \left( \frac{x - X_1^+(\tau, 0, y, w)}{|x - X_1^+(\tau, 0, y, w)|^3} - \frac{x - X_2^+(\tau, 0, y, w)}{|x - X_2^+(\tau, 0, y, w)|^3} \right) f_0^+(y, w) dv dy \]

\[ - \int_{\mathbb{R}^6} \left( \frac{x - X_1^-(\tau, 0, y, w)}{|x - X_1^-(\tau, 0, y, w)|^3} - \frac{x - X_2^-(\tau, 0, y, w)}{|x - X_2^-(\tau, 0, y, w)|^3} \right) f_0^-(y, w) dv dy, \quad (63) \]

then by Lemma 5.1 we get

\[ \int_{\mathbb{R}^3} (E_1 - E_2)(\tau, x) \int_{\mathbb{R}^3} f_2^+(\tau, x, v) dv dx \]

\[ \leq \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^6} \left( \frac{x - X_1^+(\tau, 0, y, w)}{|x - X_1^+(\tau, 0, y, w)|^3} - \frac{x - X_2^+(\tau, 0, y, w)}{|x - X_2^+(\tau, 0, y, w)|^3} \right) f_0^+(y, w) dv dy \right| d\tau \]

\[ + \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^6} \left( \frac{x - X_1^-(\tau, 0, y, w)}{|x - X_1^-(\tau, 0, y, w)|^3} - \frac{x - X_2^-(\tau, 0, y, w)}{|x - X_2^-(\tau, 0, y, w)|^3} \right) f_0^-(y, w) dv dy \right| d\tau \]

\[ \leq C \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^6} \left( \frac{x - X_1^+(\tau, 0, y, w)}{|x - X_1^+(\tau, 0, y, w)|^3} - \frac{x - X_2^+(\tau, 0, y, w)}{|x - X_2^+(\tau, 0, y, w)|^3} \right) \rho_2^+(\tau, x) dx \right| f_0^+(y, w) dv dy \]
By Jensen’s inequality, it comes

\[ J_{13} \leq C(T_0, f_0^+) r \int_0^t \Upsilon(\tau)^{1 - \frac{3}{2}} d\tau. \]

Similarly, we get

\[ J_{14} \leq C(T_0, f_0^+) r \int_0^t \Upsilon(\tau)^{1 - \frac{3}{2}} d\tau. \]

Hence

\[ J_3 \leq C(T_0, f_0^+) r \int_0^t \Upsilon(\tau)^{1 - \frac{3}{2}} d\tau. \] (64)

By measure preserving of the characteristics, the estimates of \( J_{11} \) and \( J_{13} \), we get

\[ J_1 \leq \int_0^t ds \int_0^s ds' \int_{\mathbb{R}^6} |E_1(\tau, X_1^+(\tau)) - E_1(\tau, X_1^-(\tau))| f_0^+(x, v) dv dx \]

\[ + \int_0^t ds \int_0^s ds' \int_{\mathbb{R}^6} |E_1(\tau, X_2^+(\tau)) - E_2(\tau, X_2^-(\tau))| f_0^+(x, v) dv dx \]

\[ = \int_0^t ds \int_0^s ds' \int_{\mathbb{R}^6} |E_1(\tau, X_1^+(\tau)) - E_1(\tau, X_1^-(\tau))| f_0^+(x, v) dv dx \]

\[ + \int_0^t ds \int_0^s ds' \int_{\mathbb{R}^6} |E_1(\tau, x) - E_2(\tau, x)| f_0^+(x, v) dv dx \]

\[ \leq C(T_0, f_0^+) r \int_0^t \Upsilon(\tau)^{1 - \frac{3}{2}} d\tau. \] (65)

Similarly, it comes

\[ J_2 \leq C(T_0, f_0^+) r \int_0^t \Upsilon(\tau)^{1 - \frac{3}{2}} d\tau. \] (66)

Therefore, combining (64), (65) and (66), we obtain

\[ \Upsilon(t) \leq C(T_0, f_0^+) r \int_0^t \Upsilon(\tau)^{1 - \frac{3}{2}} d\tau. \]

Applying Gronwall’s inequality, we have for any \( r > 3 \) and \( t \in [0, T_0] \)

\[ \Upsilon(t) \leq [C(T_0, f_0^+) r]^\frac{3}{2}. \]

Taking \( r \to \infty \), we have \( \Upsilon(t) = 0 \) for \( t \in [0, (C(T_0, f_0^+))^{-1}] \). Repeating the argument of intervals of length \( (C(T_0, f_0^+))^{-1} \), we prove \( \Upsilon(t) = 0 \) for \( t \in [0, T_0] \). Then, \( X_1^\pm(t) = X_2^\pm(t) \) a.e. on \( \mathbb{R}^3 \times \mathbb{R}^3 \). By (63) and the characteristics (12), we know that \( V_1^\pm(t) = V_2^\pm(t) \). Thus, we obtain \( f_1^\pm(t, x, v) = f_2^\pm(t, x, v) \) a.e. on \( \mathbb{R}^3 \times \mathbb{R}^3 \) and prove the uniqueness.

Finally, let \( f^\pm(t, x, v) \in C^1([0, T_{max}) \times \mathbb{R}^3 \times \mathbb{R}^3) \) be the maximally extended classical solution obtained above, and suppose that \( R^* = R_+^* + R_-^* \),

\[ R_+^* = \sup\{|x - (t + \alpha)v| : (x, v) \in \text{supp} f^\pm(t), t \in [0, T_{max})\} < \infty, \]
but $T_{\text{max}} < \infty$. We can choose a new initial time $\hat{t} \in [0, T_{\text{max}})$. From the proof of local existence result, there exists $\delta_1 = 1/4\varepsilon T^4$, which is independent of $t$ such that the solution can be extended to $[\hat{t}, \hat{t} + \delta_1]$. Hence, if we take $\hat{t}$ sufficiently close to $T_{\text{max}}$, then $\hat{t} + \delta_1 > T_{\text{max}}$, which is a contradiction.

From Lemma 2.1 and Lemma 2.2, the proof of (26) is obvious. Now we prove (27). Note that $\|E(t)\|_2 = \sqrt{8\pi} \varepsilon_{\text{pot}}^{\frac{1}{4}} \leq (\alpha + t)^{-\frac{1}{4}}$, hence by interpolation estimate we have $\|E(t)\|_2 \leq C(t + \alpha)^{-\frac{19}{40}}$. By the definition of $D^{[2]}(t)$ and Hölder inequality, we get
\[
\|D^{[2]}(t)\| \leq \|\rho(t)\|_2 \|E(t)\|_2 \leq C(t + \alpha)^{-\frac{8}{5}},
\]
which gives (27).

\[\square\]

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E-mail address: xiao_meixia@163.com

E-mail address: xwzhang@hust.edu.cn