Necessary and sufficient condition for the comparison theorem of multidimensional anticipated backward stochastic differential equations

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Abstract

Anticipated backward stochastic differential equations, studied the first time in 2007, are equations of the following type:

\[ -dY_t = f(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\zeta(t)})dt - Z_t dB_t, \quad t \in [0, T]; \]
\[ Y_t = \xi_t, \quad t \in [T, T+K]; \]
\[ Z_t = \eta_t, \quad t \in [T, T+K]. \]

In this paper, we give a necessary and sufficient condition under which the comparison theorem holds for multidimensional anticipated backward stochastic differential equations with generators independent of the anticipated term of \( Z \).

Keywords: comparison theorem, multidimensional anticipated backward stochastic differential equation, necessary and sufficient condition

1 Introduction

Backward stochastic differential equation (BSDE) of the general form

\[ Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s \quad (1) \]

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was considered the first time by Pardoux-Peng \cite{7}. Since then, the theory of BS-DEs has been studied with great interest. One of the achievements of this theory is the comparison theorem. It is due to Peng \cite{9} and then generalized by Pardoux-Peng \cite{8} and El Karoui-Peng-Quenez \cite{4}. It allows to compare the solutions of two BSDEs whenever we can compare the terminal conditions and the generators. The converse comparison theorem for BSDEs has also been studied (see \cite{1, 3, 6}). Besides, a necessary and sufficient condition for the comparison theorem in the multidimensional case was given by Hu-Peng \cite{5}, and their main method consists in translating the comparison principle into an equivalent viability property for BSDEs, studied by Buckdahn-Quincampoix-Răşcanu \cite{2}.

Recently, a new type of BSDE, called anticipated BSDE (ABSDE), was introduced by Peng-Yang \cite{10} (see also Yang \cite{11}). The ABSDE is of the following form:

\[
\begin{cases}
-dY_t = f(t,Y_t,Z_t,Y_{t+\delta(t)},Z_{t+\zeta(t)})dt - Z_t dB_t, & t \in [0,T]; \\
Y_t = \xi_t, & t \in [T,T+K]; \\
Z_t = \eta_t, & t \in [T,T+K],
\end{cases}
\]

(2)

where $\delta(\cdot): [0,T] \to \mathbb{R}^+ \setminus \{0\}$ and $\zeta(\cdot): [0,T] \to \mathbb{R}^+ \setminus \{0\}$ are continuous functions satisfying

(a1) there exists a constant $K \geq 0$ such that for each $t \in [0,T]$,

$$t + \delta(t) \leq T + K, \quad t + \zeta(t) \leq T + K;$$

(a2) there exists a constant $M \geq 0$ such that for each $t \in [0,T]$ and each nonnegative integrable function $g(\cdot)$,

$$\int_t^T g(s+\delta(s))ds \leq M \int_t^{T+K} g(s)ds, \quad \int_t^T g(s+\zeta(s))ds \leq M \int_t^{T+K} g(s)ds.$$

\cite{10} tells us that \cite{2} has a unique solution under proper assumptions. Furthermore, for 1-dimensional ABSDEs there is a comparison theorem, which requires that the generators of the ABSDEs cannot depend on the anticipated term of $Z$ and one of them must be increasing in the anticipated term of $Y$.

The aim of this paper is to give a comparison theorem for multidimensional ABSDEs with generators independent of the anticipated term of $Z$ and possibly not increasing in the anticipated term of $Y$. Moreover, the condition under which the comparison theorem holds is necessary and sufficient. The main approach we adopt is to consider an ABSDE as a series of BSDEs and then apply the results in \cite{5}. It should be mentioned here that the reason why the generators are still required to be independent of the anticipated term of $Z$ is that the continuity property of $f(\cdot,y,z,Y_{+\delta(\cdot)},Z_{+\zeta(\cdot)})$, where $(Y,Z)$ is the unique solution to a BSDE, is hard to depict.
The paper is organized as follows: in Section 2, we list some notations and some existing results which will be used in the text. In Section 3, we mainly study the comparison theorem for multidimensional ABSDEs, besides, we also discuss a lot about that for 1-dimensional ABSDEs.

2 Preliminaries

Let \( \{B_t; t \geq 0\} \) be a \( d \)-dimensional standard Brownian motion on a probability space \((\Omega, \mathcal{F}, P)\) and \( \{\mathcal{F}_t; t \geq 0\} \) be its natural filtration. Denote by \( |\cdot| \) the norm in \( \mathbb{R}^m \).

Given \( T > 0 \), we will use the following notations:

- \( L_2^2(\mathcal{F}_T; \mathbb{R}^m) := \{\xi \in \mathbb{R}^m \mid \xi \) is an \( \mathcal{F}_T \)-measurable random variable such that \( E|\xi|^2 < +\infty\};\)
- \( L_2^2(0, T; \mathbb{R}^m) := \{\varphi: \Omega \times [0, T] \to \mathbb{R}^m \mid \varphi \) is progressively measurable and \( E \int_0^T |\varphi_t|^2 dt < +\infty\};\)
- \( S_2^2(0, T; \mathbb{R}^m) := \{\psi: \Omega \times [0, T] \to \mathbb{R}^m \mid \psi \) is progressively measurable and \( E[\sup_{0 \leq t \leq T}|\psi_t|^2] < +\infty\}.\)

2.1 Comparison theorem for multidimensional BSDEs

Consider the BSDE (1). For the generator \( g: \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m \), we make the following assumptions, which are essential for [2] as well as [5]:

(A1) \( g(\cdot, \cdot, y, z) \) is progressively measurable, and for each \( (y, z) \), \( g(\omega, \cdot, y, z) \) is continuous, a.s.;

(A2) there exists a constant \( L_g \geq 0 \) such that for each \( s \in [0, T] \), \( y, y' \in \mathbb{R}^m \), \( z, z' \in \mathbb{R}^{m \times d} \), the following holds:

\[
|g(s, y, z) - g(s, y', z')| \leq L_g(|y - y'| + |z - z'|);
\]

(A3) \( \sup_{0 \leq s \leq T} |g(s, 0, 0)| \in L^2(\mathcal{F}_T; \mathbb{R}). \)

Then according to [7], for each \( \xi \in L^2(\mathcal{F}_T; \mathbb{R}^m) \), BSDE (1) has a unique solution.

We recall here the comparison theorem for multidimensional BSDEs from [3]. For \( j = 1, 2 \), let \( (Y^{(j)}, Z^{(j)}) \) be the unique solution to the following BSDE:

\[
Y^j_t = \xi^j + \int_t^T g_j(s, Y^j_s, Z^j_s) ds - \int_t^T Z^j_s dB_s, \tag{3}
\]

where \( \xi^j \in L^2(\mathcal{F}_T; \mathbb{R}^m) \) and \( g_j \) satisfies (A1)–(A3).
Theorem 2.1 The following are equivalent:

(i) for all $\tau \in [0,T]$, $\xi^j \in L^2(\mathcal{F}_\tau; \mathbb{R}^m)$ $(j = 1, 2)$ such that $\xi^1 \geq \xi^2$, the unique solutions $(Y^j, Z^j) \in S^2_\mathfrak{F}(0, \tau; \mathbb{R}^m) \times L^2_\mathfrak{F}(0, \tau; \mathbb{R}^{m \times d})$ $(j = 1, 2)$ to the BSDE \[3\] over time interval $[0, \tau]$:

$$Y^j_t = \xi^j + \int_t^\tau g_j(s, Y^j_s, Z^j_s)ds - \int_t^\tau Z^j_s dB_s,$$

satisfy

$$Y^1_t \geq Y^2_t, \text{ for all } t \in [0, \tau], \text{ a.s.};$$

(ii) for all $t \in [0, T]$, $(y, z), (y', z') \in \mathbb{R}^m \times \mathbb{R}^{m \times d},$

$$-4\langle y^-, g_1(t, y^+ + y', z) - g_2(t, y', z') \rangle \leq 2 \sum_{k=1}^m 1_{\{y_k < 0\}}|z_k - z'_k|^2 + C|y^-|^2, \text{ a.s., (4)}$$

where $C > 0$ is a constant.

Remark 2.1 In fact, the constant $C$ in (4) only depends on the Lipschitz coefficients $L_{g_j}$ $(j = 1, 2)$ of the generators $g_j$ $(j = 1, 2)$, which can be easily got from the detailed proofs in [2] and [3].

Remark 2.2 Let $m = 1$. Then (4) is equivalent to

$$g_1(t, y, z) \geq g_2(t, y, z).$$

This has been stated already in [2].

2.2 Multidimensional anticipated BSDEs

Now let us consider the ABSDE [2]. First for the generator $f(\omega, s, y, z, \theta, \phi) : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times S^2_\mathfrak{F}(s, T + K; \mathbb{R}^m) \times L^2_\mathfrak{F}(s, T + K; \mathbb{R}^{m \times d}) \to L^2(\mathcal{F}_s; \mathbb{R}^m)$, we introduce two hypotheses:

(H1) there exists a constant $L_f > 0$ such that for each $s \in [0, T]$, $y, y' \in \mathbb{R}^m$, $z, z' \in \mathbb{R}^{m \times d}$, $\theta, \theta' \in L^2_\mathfrak{F}(s, T + K; \mathbb{R}^m)$, $\phi, \phi' \in L^2_\mathfrak{F}(s, T + K; \mathbb{R}^{m \times d})$, $r, r' \in [s, T + K]$, the following holds:

$$|f(s, y, z, \theta_r, \phi_r) - f(s, y', z', \theta'_r, \phi'_r)| \leq L_f(|y - y'| + |z - z'| + E^{\mathcal{F}_r}[|\theta_r - \theta'_r| + |\phi_r - \phi'_r|]);$$

(H2) $E[\int_0^T |f(s, 0, 0, 0, 0)|^2 ds] < +\infty$.

Let us review the existence and uniqueness theorem for ABSDEs from [10]:
Theorem 2.2 Assume that $f$ satisfies (H1) and (H2), $\delta$, $\zeta$ satisfy (a1) and (a2), then for arbitrary given terminal conditions $(\xi, \eta) \in S^2_\mathcal{F}(T, T + K; \mathbb{R}^m) \times L^2_\mathcal{F}(T, T + K; \mathbb{R}^{m \times d})$, the ABSDE (2) has a unique solution, i.e., there exists a unique pair of processes $(Y, Z) \in S^2_\mathcal{F}(0, T + K; \mathbb{R}^m) \times L^2_\mathcal{F}(0, T + K; \mathbb{R}^{m \times d})$ satisfying (2).

Next we will recall the comparison theorem from Peng-Yang [10]. For $j = 1, 2$, let $(Y^{(j)}, Z^{(j)})$ be the unique solution to the following 1-dimensional ABSDE:

\[
\begin{cases}
-dY_t^{(j)} = f_j(t, Y_t^{(j)}, Z_t^{(j)}, Y_{t+\delta(t)})dt - Z_t^{(j)}dB_t, & t \in [0, T] ; \\
Y_0^{(j)} = \xi^{(j)}, & t \in [T, T + K].
\end{cases}
\]

Theorem 2.3 Assume that $f_1$, $f_2$ satisfy (H1) and (H2), $\xi^{(1)}, \xi^{(2)} \in S^2_\mathcal{F}(T, T + K; \mathbb{R})$, $\delta$ satisfies (a1), (a2), and for each $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f_2(t, y, z, \cdot)$ is increasing, i.e., $f_2(t, y, z, \theta_t) \geq f_2(t, y, z, \theta_t')$, if $\theta_t \geq \theta_t'$, $\theta, \theta' \in L^2_\mathcal{F}(t, T + K; \mathbb{R}), r \in [t, T + K]$. If $\xi^{(1)} \geq \xi^{(2)}$, $s \in [T, T + K]$ and $f_1(t, y, z, \theta_t) \geq f_2(t, y, z, \theta_t)$, $t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d, \theta \in L^2_\mathcal{F}(t, T + K; \mathbb{R}), r \in [t, T + K]$, then $Y_t^{(1)} \geq Y_t^{(2)}$, a.e., a.s.

At the end of this subsection, for $f(\omega, s, y, z, \theta) : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times S^2_\mathcal{F}(s, T + K; \mathbb{R}^m) \to L^2(\mathcal{F}_s; \mathbb{R}^m)$ particularly, let us introduce three more hypotheses:

(H3) for each $(y, z, \theta) \in \mathbb{R}^m \times \mathbb{R}^{m \times d} \times S^2_\mathcal{F}(\cdot, T + K; \mathbb{R}^m)$, $f(\omega, \cdot, y, z, \theta_{s+\delta(s)})$ is continuous, a.s.;

(H4) $\sup_{0 \leq s \leq T} |f(s, 0, 0, 0)| \in L^2(\mathcal{F}_s; \mathbb{R})$;

(H4') $\sup_{0 \leq s \leq T} |f(s, 0, 0, \theta_{s+\delta(s)})| \in L^2(\mathcal{F}_s; \mathbb{R})$, for all $\theta \in S^2_\mathcal{F}(s, T + K; \mathbb{R}^m)$.

Remark 2.3 $(H4') \Rightarrow (H4) \Rightarrow (H2); (H1) + (H4) \Rightarrow (H4')$. Indeed, we have

\[
E \left[ \sup_{0 \leq s \leq T} |f(s, 0, 0, \theta_{s+\delta(s)})|^2 \right] \\
\leq 2E \left[ \sup_{0 \leq s \leq T} |f(s, 0, 0, \theta_{s+\delta(s)}) - f(s, 0, 0, 0)|^2 + \sup_{0 \leq s \leq T} |f(s, 0, 0, 0)|^2 \right] \\
\leq 2L_j^2 E \left[ \sup_{0 \leq s \leq T} E^{\mathcal{F}_s} |\theta_{s+\delta(s)}|^2 \right] + 2E \left[ \sup_{0 \leq s \leq T} |f(s, 0, 0, 0)|^2 \right] \\
\leq 2L_j^2 E \left[ \sup_{0 \leq s \leq T} |\theta_{s+\delta(s)}|^2 \right] + 2E \left[ \sup_{0 \leq s \leq T} |f(s, 0, 0, 0)|^2 \right] \\
< +\infty.
\]
3 Comparison theorem for anticipated BSDEs

Consider the following multidimensional ABSDE:

\[
\begin{aligned}
-dY_t^{(j)} &= f_j(t, Y_t^{(j)}, Z_t^{(j)}, Y_{t+\delta^{(j)}(t)}^{(j)})dt - Z_t^{(j)}dB_t, \\ Y_t^{(j)} &= \xi_t^{(j)},
\end{aligned}
\]

where \( j = 1, 2, f_j \) satisfies (H1), (H3) and (H4), \( \xi_t^{(j)} \in S_2^2(T, T + K; \mathbb{R}^m) \), \( \delta^{(j)} \) satisfies (a1) and (a2). Then by Theorem 2.2, \( \xi \) has a unique solution.

Proposition 3.1 Putting \( t_0 = T \), we define by iteration

\[
t_i := \min\{t \in [0, T] : \min\{s + \delta^{(1)}(s), s + \delta^{(2)}(s)\} \geq t_{i-1}, \text{ for all } s \in [t, T]\}, \quad i \geq 1.
\]

Set \( N := \max\{i : t_{i-1} > 0\} \). Then \( N \) is finite, \( t_N = 0 \) and

\[ [0, T] = [0, t_{N-1}] \cup [t_{N-1}, t_{N-2}] \cup \cdots \cup [t_2, t_1] \cup [t_1, T]. \]

Proof. Let us first prove that \( N \) is finite. For this purpose, we apply the method of reduction to absurdity. Suppose that \( N \) is infinite. From the definition of \( \{t_i\}_{i=1}^{+\infty} \), we know

\[
\min\{t_i + \delta^{(1)}(t_i), t_i + \delta^{(2)}(t_i)\} = t_{i-1}, \quad i = 1, 2, \ldots. \tag{7}
\]

Since \( \delta^{(j)}(\cdot) (j = 1, 2) \) are continuous and positive, thus obviously we have

\[
t_i < t_{i-1}, \quad i = 1, 2, \ldots.
\]

Therefore \( \{t_i\}_{i=1}^{+\infty} \) converges as a strictly monotone and bounded sequence. Denote its limit by \( \bar{t} \). Letting \( i \to +\infty \) on both sides of (7), we get

\[
\min\{\bar{t} + \delta^{(1)}(\bar{t}), \bar{t} + \delta^{(2)}(\bar{t})\} = \bar{t}.
\]

Hence \( \delta^{(1)}(\bar{t}) = 0 \) or \( \delta^{(2)}(\bar{t}) = 0 \), which is just a contradiction since both \( \delta^{(1)} \) and \( \delta^{(2)} \) are positive. Consequently, \( N \) is finite.

Next we will show that \( t_N = 0 \). In fact, the following holds obviously:

\[
\min\{t_N + \delta^{(1)}(t_N), t_N + \delta^{(2)}(t_N)\} > t_N,
\]

which implies \( t_N = 0 \), or else we can find a \( \tilde{t} \in [0, t_N) \) due to the continuity of \( \delta^{(j)}(\cdot) (j = 1, 2) \) such that

\[
\min\{s + \delta^{(1)}(s), s + \delta^{(2)}(s)\} \geq t_N, \quad \text{for all } s \in [\tilde{t}, T],
\]

from which we know that \( \tilde{t} \) is an element of the sequence as well. \( \Box \)
Proposition 3.2 For \( j = 1, 2 \), suppose that \((Y^{(j)}, Z^{(j)})\) is the unique solution to the ABSDE (8). Then for fixed \( i \in \{1, 2, \ldots, N\} \), \( \tau \in [t_i, t_{i-1}] \), over time interval \([t_i, \tau]\), ABSDE (8) is equivalent to the following ABSDE:

\[
\begin{cases}
-d\tilde{Y}^{(j)}_t = f_j(t, \tilde{Y}^{(j)}_t, \tilde{Z}^{(j)}_t, \tilde{Y}^{(j)}_{t+\delta(j)(t)})dt - \tilde{Z}^{(j)}_tdB_t, & t \in [t_i, \tau]; \\
Y^{(j)}_t = Y^{(j)}_{t_i}, & t \in [\tau, T + K],
\end{cases}
\]

which is also equivalent to the following BSDE with terminal condition \( Y^{(j)}_\tau \):

\[
\tilde{Y}^{(j)}_t = Y^{(j)}_\tau + \int_t^\tau f_j(s, \tilde{Y}^{(j)}_s, \tilde{Z}^{(j)}_s, Y^{(j)}_{s+\delta(j)(s)})ds - \int_t^\tau \tilde{Z}^{(j)}_s dB_s.
\]

That is to say,

\[
Y^{(j)}_t = \tilde{Y}^{(j)}_t = \tilde{Y}^{(j)}_{t_i}, \ Z^{(j)}_t = \tilde{Z}^{(j)}_t = \tilde{Z}^{(j)}_{t_i}, \ t \in [t_i, \tau], \ j = 1, 2.
\]

Proof. The conclusion immediately follows from

(i) for each \( s \in [t_i, \tau] \), \( s + \delta(j)(s) \geq t_{i-1} \);

(ii) write \( f_j^Y(s, y, z) = f_j(s, y, z, Y^{(j)}_{s+\delta(j)(s)}) \), then \( f_j^Y \) satisfies (A1)–(A3);

(iii) \((Y^{(j)}_t, Z^{(j)}_t)_{t \in [t_i, \tau]} \) satisfies both ABSDE (8) and BSDE (9). \( \square \)

3.1 Comparison theorem in \( \mathbb{R}^m \)

Next we will study the following problem: under which condition the comparison theorem for multidimensional ABSDEs holds?

Lemma 3.1 For fixed \( i \in \{1, 2, \ldots, N\} \), \( s \in (t_i, t_{i-1}) \), the following are equivalent:

(i) for all \( \tau \in [t_i, s] \), \( \xi^{(j)} \in S^2_{\bar{F}}(\tau, T + K; \mathbb{R}^m) \) \((j = 1, 2)\) such that \( \xi^{(1)}(\tau) \geq \xi^{(2)}(\tau) \) and \( (\xi^{(j)}_t)_{t \in [t_i, t_i+\tau]} \) \((j = 1, 2)\) are fixed, the unique solutions \((Y^{(j)}_i, Z^{(j)}_i) \in S^2_{\bar{F}}(t_i, T + K; \mathbb{R}^m) \times L^2_\mathbb{F}(t_i, \tau; \mathbb{R}^{m\times d}) \) \((j = 1, 2)\) to the following ABSDE over time interval \([t_i, \tau]\):

\[
\begin{cases}
-dY^{(j)}_t = f_j(t, Y^{(j)}_t, Z^{(j)}_t, Y^{(j)}_{t+\delta(j)(t)})dt - Z^{(j)}_tdB_t, & t \in [t_i, \tau]; \\
Y^{(j)}_t = \xi^{(j)}_t, & t \in [\tau, T + K],
\end{cases}
\]

satisfy

\[ Y^{(1)}_t \geq Y^{(2)}_t, \ \text{for all} \ t \in [t_i, \tau], \ \text{a.s.}; \]

(ii) for all \( t \in [t_i, s] \), \( (y, z, (y', z')) \in \mathbb{R}^m \times \mathbb{R}^{m\times d} \),

\[
-4\langle y^{-}, f_1(t, y^+ + y', z, \xi^{(1)}_{t+\delta(1)(t)})\rangle - f_2(t, y', z', \xi^{(2)}_{t+\delta(2)(t)}) \leq 2 \sum_{k=1}^{m} \mathbf{1}_{\{y_k < 0\}} \|z_k - z'_k\|^2 + C|y^-|^2, \ \text{a.s.},
\]

where \( C > 0 \) is a constant.
\textbf{Proof.} According to Proposition 3.2, we can equivalently consider the following BSDE over time interval \([t_i, \tau]\) instead of (10):

\begin{equation}
\tilde{Y}_t^{(j)} = \xi_t^{(j)} + \int_t^\tau f_j(r, \tilde{Y}_r^{(j)}, \tilde{Z}_r^{(j)}, \xi_{r+s(1)(s)}^{(j)})dr - \int_t^\tau \tilde{Z}_r^{(j)}dB_r.
\end{equation}

Write \(f_j(s, y, z) = f_j(s, y, z, \xi_{s+\delta(1)(s)}^{(j)})\), then \(f_j\) satisfies (A1)-(A3).

On the other hand, it is obvious that (i) is equivalent to

(iii) for all \(\tau \in [t_i, s]\), \(\xi_t^{(j)} \in L^2(\mathcal{F}_\tau; \mathbb{R}^m) (j = 1, 2)\) such that \(\xi_t^{(1)} \geq \xi_t^{(2)}\), the unique solutions \((Y_t^{(j)}, Z_t^{(j)})\) \(\in S_2^2(t_i, \tau; \mathbb{R}^m) \times L_2^2(t_i, \tau; \mathbb{R}^{m \times d})\) \(j = 1, 2\) to the BSDE (\[11\]) satisfy

\(\tilde{Y}_t^{(2)} \geq \tilde{Y}_t^{(1)}\), for all \(t \in [t_i, \tau]\), a.s.

By Theorem 2.1, (iii) is equivalent to (ii). \(\square\)

\textbf{Remark 3.1} From the proof of Lemma 3.1, we can find that the result holds true for arbitrary values of \((\xi_t^{(j)})_{t \in [t_i, \tau-1]}\) such that \(\xi_t^{(j)} \in S_2^2(\tau, T + K; \mathbb{R}^m)\). In fact we even can choose them according to a fixed formula, for example, all \(\tau \in [t_i, s]\), for all \(\xi_t^{(j)} \in L^2(\mathcal{F}_\tau; \mathbb{R}^m)\) \(j = 1, 2\) such that \(\xi_t^{(1)} \geq \xi_t^{(2)}\), and the fixed processes \((\xi_t^{(j)})_{t \in [t_i, \tau+K]}\) \(j = 1, 2\) such that \(\xi_t^{(1)} \geq \xi_t^{(2)}\), we can construct \(\xi_t^{(j)} \in S_2^2(\tau, t_i-1; \mathbb{R}^m)\) \(j = 1, 2\), thanks to the strict inequality \(\tau < t_i-1\), such that \(\xi_t^{(1)} \geq \xi_t^{(2)}\) as follows:

\(\xi_t^{(j)} = \frac{t_i-1 - t}{t_i-1 - \tau} \xi_t^{(j)} + \frac{t - \tau}{t_i-1 - \tau} E_{\mathcal{F}_t}[\xi_{t_i-1}^{(j)}], \ t \in [\tau, t_i-1].\)

\textbf{Theorem 3.1} The following are equivalent:

(i) for all \(\tau \in [0, T]\), \(\xi_t^{(j)} \in S_2^2(\tau, T + K; \mathbb{R}^m)\) \(j = 1, 2\) such that \(\xi_t^{(1)} \geq \xi_t^{(2)}\), the unique solutions \((Y_t^{(j)}, Z_t^{(j)})\) \(\in S_2^2(0, T + K; \mathbb{R}^m) \times L_2^2(0, T; \mathbb{R}^{m \times d})\) \(j = 1, 2\) to the following ABSDE:

\begin{align*}
-\text{d} Y_t^{(j)} &= f_j(t, Y_t^{(j)}, Z_t^{(j)}, Y_{t+\delta(1)(t)}^{(j)})dt - Z_t^{(j)}dB_t, \ t \in [0, \tau]; \\
Y_t^{(j)} &= \xi_t^{(j)}, \ t \in [\tau, T + K],
\end{align*}

satisfy

\(\tilde{Y}_t^{(1)} \geq \tilde{Y}_t^{(2)}\), for all \(t \in [0, \tau]\), a.s.;

(ii) for all \(s \in [0, T]\), \((y, z), (y', z') \in \mathbb{R}^m \times \mathbb{R}^{m \times d}\) and all \(\theta^{(j)} \in S_2^2(s, T + K; \mathbb{R}^m)\) \(j = 1, 2\) such that \(\theta^{(1)} \geq \theta^{(2)}\),

\[-4\langle y^\top, f_1(s, y'^\top + y', z, \theta_{s+\delta(1)(s)}^{(1)}) - f_2(s, y', z', \theta_{s+\delta(2)(s)}^{(2)}) \rangle \leq 2 \sum_{k=1}^m 1_{\{y_k < 0\}}|z_k - z'_k|^2 + C|y^\top|^2, \ \text{a.s.},\]

where \(C > 0\) is a constant.
Proof. (a) (i) ⇒ (ii): without loss of generality, we may assume that \( s \in [t_i, t_{i-1}] \) \((i \in \{1, 2, \ldots, N\})\). For some convenience of techniques, we first consider the case when \( s \in (t_i, t_{i-1})\).

According to Proposition 3.2, (i) implies

(iii) for all \( \tau \in [t_i, s] \), the unique solutions \((Y^{(j)}, Z^{(j)}) \in S^2_F(t_i, T + K; \mathbb{R}^m) \times L^2_F(t_i, \tau; \mathbb{R}^{m \times d})\) \((j = 1, 2)\) to the following BSDE over time interval \([t_i, T + K]\):

\[
\left\{
\begin{align*}
-dY^{(j)}_t &= f_j(t, Y^{(j)}_t, Z^{(j)}_t, Y^{(j)}_{t+\delta(t)}(t))dt - Z^{(j)}_tdB_t, & t \in [t_i, \tau]; \\
Y^{(j)}_t &= \xi^{(j)}_t, & t \in [\tau, T + K],
\end{align*}
\right.
\]

satisfy

\[Y^{(1)}_t \geq Y^{(2)}_t, \text{ for all } t \in [t_i, \tau], \text{ a.s.}\]

In the above BSDE, let

\[(\xi^{(j)}_t)_{t \in [t_i-1, T+K]} = (\tilde{\theta}^{(j)}_t)_{t \in [t_i-1, T+K]}\].

Then by Lemma 3.1, (iii) is equivalent to

(iv) for all \( t \in [t_i, s] \), \((y, z), (y', z') \in \mathbb{R}^m \times \mathbb{R}^{m \times d},\)

\[-4(y^-, f_1(t, y^+ + y', z, \tilde{\theta}^{(1)}_{t+\delta(t)}(t)) - f_2(t, y', z', \tilde{\theta}^{(2)}_{t+\delta(t)}(t))) \leq 2 \sum_{k=1}^m \sum_{j=1}^N 1_{\{y_k < 0\}} |z_k - z'_k|^2 + C|y^-|^2, \text{ a.s.} \tag{14}\]

Setting \( t = s \) in (14), we can get (13) for \( s \in (t_i, t_{i-1})\). Note the continuity property of \( f_j \) \((j = 1, 2)\), then (13) holds for each \( s \in [t_i, t_{i-1}]\).

(b) (ii) ⇒ (i): we only need to consider the nontrivial case \( \tau \neq 0 \). Without loss of generality, we may assume that \( \tau \in (t_i, t_{i-1}) \) \((i \in \{1, 2, \ldots, N\})\). Our aim is to show that

\[Y^{(1)}_t \geq Y^{(2)}_t, \text{ for all } t \in [0, \tau], \text{ a.s.},\]

where \( Y^{(j)} \) \((j = 1, 2)\) are the unique solutions of ABSDE (12).

Consider the ABSDE (12) one time interval by one time interval.

For the first step, we consider the case when \( t \in [t_i, \tau] \). According to Proposition 3.2, we can equivalently consider the following BSDE instead of ABSDE (12):

\[\tilde{Y}^{(j)}_t = \xi^{(j)}_\tau + \int_t^\tau f_j(s, \tilde{Y}^{(j)}_s, \tilde{Z}^{(j)}_s, \xi^{(j)}_{s+\delta(s)}(s))ds - \int_t^\tau \tilde{Z}^{(j)}_sdB_s, \]

Noticing that \( \xi^{(j)} \in S^2_F(\tau, T + K; \mathbb{R}^m) \((j = 1, 2)\) and \( \xi^{(1)} \geq \xi^{(2)} \) from (ii), we get

(v) for all \( s \in [t_i, \tau] \), \((y, z), (y', z') \in \mathbb{R}^m \times \mathbb{R}^{m \times d},\)

\[-4(y^-, f_1(s, y^+ + y', z, \xi^{(1)}_{s+\delta(s)}(s)) - f_2(s, y', z', \xi^{(2)}_{s+\delta(s)}(s))) \leq 2 \sum_{k=1}^m 1_{\{y_k < 0\}} |z_k - z'_k|^2 + C|y^-|^2, \text{ a.s.}\]
Then thanks to Theorem 2.1, (v) implies
\[
\tilde{Y}_t^{(1)} \geq \tilde{Y}_t^{(2)}, \text{ for all } t \in [t_i, \tau], \text{ a.s.}
\]
i.e.,
\[
Y_t^{(1)} \geq Y_t^{(2)}, \text{ for all } t \in [t_i, \tau], \text{ a.s.}
\]
Consequently,
\[
Y_t^{(1)} \geq Y_t^{(2)}, \text{ for all } t \in [t_i, T + K], \text{ a.s.}
\]
For the second step, we consider the case when \( t \in [t_{i+1}, t_i] \). Similarly, according to Proposition 3.2, we can consider the following BSDE equivalently:
\[
\tilde{Y}_t^{(j)} = Y_t^{(j)} + \int_t^{t_i} f_j(s, \tilde{Y}_s^{(j)}, \tilde{Z}_s^{(j)}, Y_{s+\delta(s)}^{(j)})ds - \int_t^{t_i} \tilde{Z}_s^{(j)}dB_s.
\]
Noticing (15), according to (ii), we have
\[
\text{for all } s \in [t_{i+1}, t_i], (y, z), (y', z') \in \mathbb{R}^m \times \mathbb{R}^{m \times d},
\]
\[
-4(y^-, f_1(s, y^+ + y', z, Y_s^{(1)} - f_2(s, y', z', Y_s^{(2)})) \leq 2 \sum_{k=1}^m 1_{\{y_k < 0\}}|z_k - z_k'|^2 + C|y^-|^2, \text{ a.s.}
\]
Applying Theorem 2.1 again, from (vi), we can finally get
\[
Y_t^{(1)} \geq Y_t^{(2)}, \text{ for all } t \in [t_{i+1}, t_i], \text{ a.s.}
\]
Similarly to the above steps, we can give the proofs for the other cases when \( t \in [t_{i+2}, t_{i+1}], [t_{i+3}, t_{i+2}], \ldots, [t_N, t_{N-1}] \).

**Remark 3.2** By Remark 2.1, we can deduce from the fact
\[
L_f = L_{f'} \text{, for all } \theta, \theta' \in S^2(s, T + K; \mathbb{R}^m)
\]
where \( f^\theta(s, y, z) = f(s, y, z, \theta_{s+\delta(s)}) \) and \( f'^\theta(s, y, z) = f(s, y, z, \theta'_{s+\delta(s)}) \), that the constant \( C \) in (13) is independent of \( \theta^{(j)} \) \((j = 1, 2)\) and only depends on the Lipschitz coefficients of \( f_j \) \((j = 1, 2)\).

**Remark 3.3** If \( \delta^{(1)} = \delta^{(2)} =: \delta \), then (13) is reduced to
\[
-4(y^-, f_1(s, y^+ + y', z, \theta^{(1)}_{s+\delta(s)}) - f_2(s, y', z', \theta^{(2)}_{s+\delta(s)})) \leq 2 \sum_{k=1}^m 1_{\{y_k < 0\}}|z_k - z_k'|^2 + C|y^-|^2, \text{ a.s.}
\]
Note that this conclusion is just with respect to the ABSDE (2) in the multidimensional case.
For the special case when $f_1 = f_2 =: f$ and $\delta^{(1)} = \delta^{(2)} =: \delta$, we have the following result:

**Theorem 3.2** The following are equivalent:

(i) for all $\tau \in [0, T]$, $\xi^{(j)} \in S^2_F(\tau, T + K; \mathbb{R}^m)$ $(j = 1, 2)$ such that $\xi^{(1)} \geq \xi^{(2)}$, the unique solutions $(Y^{(j)}, Z^{(j)}) \in S^2_F(0, T + K; \mathbb{R}^m) \times L^2_F(0, T; \mathbb{R}^{m \times d})$ $(j = 1, 2)$ to the following ABSDE

\[
\begin{aligned}
-dY_t^{(j)} &= f(t, Y_t^{(j)}, Z_t^{(j)}, Y_{t+\delta(t)}^{(j)}) dt - Z_t^{(j)} dB_t, \quad t \in [0, \tau]; \\
Y_t^{(j)} &= \xi_t^{(j)}, \quad t \in [\tau, T + K],
\end{aligned}
\]

satisfy

$Y_t^{(1)} \geq Y_t^{(2)}$, for all $t \in [0, \tau]$, a.s.

(ii) for any $k = 1, 2, \ldots, m$, for all $s \in [0, T]$, $y \in \mathbb{R}^m$, $\theta \in S^2_F(s, T + K; \mathbb{R}^m)$, $\theta^{(j)} \in S^2_F(s, T + K; \mathbb{R}^m)$ $(j = 1, 2)$ such that $\theta^{(1)} \geq \theta^{(2)}$, $f_k(s, y, \cdot, \theta)$ depends only on $z_k$, and

$f_k(s, \delta^ky + y', z_k, \theta^{(1)}_{s+\delta(s)}) \geq f_k(s, y', z_k, \theta^{(2)}_{s+\delta(s)})$, for any $\delta^ky \in \mathbb{R}^m$ such that $\delta^ky \geq 0, (\delta^ky)_k = 0$.

**Proof.** According to Theorem 3.1, (i) is equivalent to

(iii) for all $s \in [0, T]$, $(y, z), (y', z') \in \mathbb{R}^m \times \mathbb{R}^{m \times d}$ and all $\theta^{(j)} \in S^2_F(s, T + K; \mathbb{R}^m)$ $(j = 1, 2)$ such that $\theta^{(1)} \geq \theta^{(2)}$,

\[
-4\langle y^-, f(s, y^+ + y', z, \theta^{(1)}_{s+\delta(s)}) - f(s, y', z', \theta^{(2)}_{s+\delta(s)}) \rangle \leq 2 \sum_{k=1}^m \mathbf{1}_{\{y_k < 0\}} |z_k - z'_k|^2 + C|y^-|^2, \quad \text{a.s.}
\]

(17)

On the one hand, suppose that (17) holds. Let us pick $y_k < 0$, and $y = y_k e_k$, $\theta^{(1)} = \theta^{(2)} =: \theta$. Then we get

\[
4y_k (f_k(s, y', z, \theta^{(1)}_{s+\delta(s)}) - f_k(s, y', z', \theta^{(2)}_{s+\delta(s)})) \leq 2 |z_k - z'_k|^2 + C|y_k|^2, \quad \text{a.s.,}
\]

which implies that $f_k$ depends only on $z_k$. Furthermore, for $\delta^ky \in \mathbb{R}^m$ such that $\delta^ky \geq 0, (\delta^ky)_k = 0$, putting in (17) $y = \delta^ky - \varepsilon e_k, \varepsilon > 0, z' = z$, dividing by $-\varepsilon$ and letting $\varepsilon \to 0^+$, we can deduce that

$f_k(s, \delta^ky + y', z_k, \theta^{(1)}_{s+\delta(s)}) \geq f_k(s, y', z_k, \theta^{(2)}_{s+\delta(s)})$.

On the other hand, it is easy to check that if (ii) holds, then (17) holds. \qed
3.2 Comparison theorem in \( \mathbb{R} \)

From now on, we will mainly consider the special case when \( m = 1 \). The following is immediate according to Remark 2.2.

**Theorem 3.3** Assume that \( m = 1 \). Then the following are equivalent

(i) for all \( \tau \in [0, T] \), \( \xi^{(j)} \in S^{2}_{F}(\tau, T + K; \mathbb{R}) \) (\( j = 1, 2 \)) such that \( \xi^{(1)} \geq \xi^{(2)} \),
the unique solutions \( (Y^{(j)}; Z^{(j)}) \in S^{2}_{F}(0, T + K; \mathbb{R}) \times L^{2}_{F}(0, \tau; \mathbb{R}^{d}) \) (\( j = 1, 2 \)) to the ABSDE (12) with terminal conditions \( \xi^{(j)} \) (\( j = 1, 2 \)) satisfy

\[
Y^{(1)}_t \geq Y^{(2)}_t, \quad \text{for all} \ t \in [0, \tau], \ a.s.;
\]

(ii) for all \( s \in [0, T] \), \( (y, z, (y', z')) \in \mathbb{R} \times \mathbb{R}^{d} \) and all \( \theta^{(j)} \in S^{2}_{F}(s, T + K; \mathbb{R}) \) (\( j = 1, 2 \)) such that \( \theta^{(1)} \geq \theta^{(2)} \),

\[
f_1(s, y, z, \theta^{(1)}_{s+d(s)}) \geq f_2(s, y, z, \theta^{(2)}_{s+d(s)}).
\]

**Remark 3.4** \( \text{[17]} \) is equivalent to

\[
f_1(s, y, z, \theta^{(1)}_{s+d(s)}) \geq f_2(s, y, z, \theta^{(2)}_{s+d(s)}). \tag{18}
\]

**Remark 3.5** The generators \( f_1 \) and \( f_2 \) will satisfy (18), if for all \( s \in [0, T] \), \( y \in \mathbb{R} \), \( z \in \mathbb{R}^{d} \), \( \theta \in L^{2}_{F}(s, T + K; \mathbb{R}) \), \( r \in [s, T + K] \), \( f_1(s, y, z, \theta_{r}) \geq f_2(s, y, z, \theta_{r}) \),
together with one of the following:

(i) for all \( s \in [0, T] \), \( y \in \mathbb{R} \), \( z \in \mathbb{R}^{d} \), \( f_1(s, y, z, \cdot) \) is increasing, \( \text{i.e.,} \ f_1(s, y, z, \theta_{r}) \geq f_1(s, y, z, \theta_{s}) \), \( \theta \geq \theta' \), \( \theta, \theta' \in L^{2}_{F}(s, T + K; \mathbb{R}) \), \( r \in [s, T + K] \);

(ii) for all \( s \in [0, T] \), \( y \in \mathbb{R} \), \( z \in \mathbb{R}^{d} \), \( f_2(s, y, z, \cdot) \) is increasing, \( \text{i.e.,} \ f_2(s, y, z, \theta_{r}) \geq f_2(s, y, z, \theta_{s}) \), \( \theta \geq \theta' \), \( \theta, \theta' \in L^{2}_{F}(s, T + K; \mathbb{R}) \), \( r \in [s, T + K] \).

Note that the latter is just the case that Peng-Yang \( \text{[10]} \) discussed (see Theorem 2.3).

**Remark 3.6** The generators \( f_1 \) and \( f_2 \) will satisfy (18), if there exists a function \( \tilde{f} \) such that

\[
f_1(s, y, z, \theta_{r}) \geq \tilde{f}(s, y, z, \theta_{r}) \geq f_2(s, y, z, \theta_{r}),
\]

for all \( s \in [0, T] \), \( y \in \mathbb{R} \), \( z \in \mathbb{R}^{d} \), \( \theta \in L^{2}_{F}(s, T + K; \mathbb{R}) \), \( r \in [s, T + K] \). Here the function \( \tilde{f}(s, y, z, \cdot) \) is increasing, \( \text{for all} \ s \in [0, T] \), \( y \in \mathbb{R} \), \( z \in \mathbb{R}^{d} \), \( \text{i.e.,} \ \tilde{f}(s, y, z, \theta_{r}) \geq \tilde{f}(s, y, z, \theta_{s}) \), \( \theta \geq \theta' \), \( \theta, \theta' \in L^{2}_{F}(s, T + K; \mathbb{R}) \), \( r \in [s, T + K] \).
Example 3.1 Now suppose that we are facing with the following two ABSDEs:

\[
\begin{cases}
- dY^{(1)}_t = E^{F_t} [Y^{(1)}_{t+\delta(t)} + 2 \sin Y^{(1)}_{t+\delta(t)} + 1] dt - Z^{(1)}_t dB_t, & t \in [0, T]; \\
Y^{(1)}_t = \xi^{(1)}_t, & t \in [T, T+K],
\end{cases}
\]

\[
\begin{cases}
- dY^{(2)}_t = E^{F_t} [Y^{(2)}_{t+\delta(t)} + \cos(2Y^{(2)}_{t+\delta(t)}) - 2] dt - Z^{(2)}_t dB_t, & t \in [0, T]; \\
Y^{(2)}_t = \xi^{(2)}_t, & t \in [T, T+K],
\end{cases}
\]

where \(\xi^{(1)}_t \geq \xi^{(2)}_t, t \in [T, T+K]\).

As neither of the generators is increasing in the anticipated term of \(Y\), we cannot apply Peng, Yang’s comparison theorem to compare \(Y^{(1)}\) and \(Y^{(2)}\).

While noting that \(f_1(s, y, z, \theta) = E^{F_s}[\theta + 2 \sin \theta + 1], f_2(s, y, z, \theta) = E^{F_s}[\theta + \cos(2\theta) - 2]\), we can choose \(\tilde{f}(s, y, z, \theta) = E^{F_s}[\theta + \sin \theta]\), due to the following facts:

\[
x + 2 \sin x + 1 \geq x + \sin x \geq x + \cos(2x) - 2, x + \sin x \geq y + \sin y, \text{ for all } x \geq y, x, y \in \mathbb{R}.
\]

Then obviously, \(f_1, \tilde{f}\) and \(f_2\) satisfy the conditions in Remark 3.6. Thus according to Theorem 3.3 (together with Remark 3.6), we get \(Y^{(1)}_t \geq Y^{(2)}_t\), for all \(t \in [0, T+K]\), a.s.

Remark 3.7 In [10], Peng and Yang gave a counterexample which indicates that if \(f_2\) is not increasing in the anticipated term of \(Y\) then their comparison theorem (see Theorem 2.3) will not hold. In fact the main reason is that the generators appearing in that example do not satisfy the necessary and sufficient condition listed in Remark 3.4.

That is to say, there is no contradiction between our result and Peng and Yang’s.

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