ASYMMETRIC DIRECTED POLYMERS IN RANDOM ENvironments

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Abstract. The model of Brownian Percolation has been introduced as an approximation of discrete last-passage percolation models close to the axis. It allowed to compute some explicit limits and prove fluctuation theorems for these, based on the relations between the Brownian percolation and random matrices.

Here, we present two approaches that allow to treat discrete asymmetric models of directed polymers. In both cases, the behaviour is universal, meaning that the results do not depend on the precise law of the environment as long as it satisfies some natural moment assumptions.

First, we establish an approximation of asymmetric discrete directed polymers in random environments at very high temperature by a continuous-time directed polymers model in a Brownian environment, much in the same way than the last passage percolation case. The key ingredient is a strong embedding argument developed by Komlos, Major and Tusnady.

Then, we study the partition function of a 1 + 1-dimensional directed polymer in a random environment with a drift tending to infinity. We give an explicit expression for the free energy based on known asymptotics for last-passage percolation and compute the order of the fluctuations of the partition function. We conjecture that the law of the properly rescaled fluctuations converges to the GUE Tracy-Widom distribution.

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1. Introduction

The Brownian Percolation model was introduced by Glynn and Whitt in [10], where the authors studied the asymptotic of passage times for customers in an infinite network of M/M/1 queues in tandem. This continuous model was easier to handle than the original discrete problem, mostly because of the scaling properties of the Brownian motion.

Let state the problem more precisely in its original setting: Let $\Omega_{N,M}$ be the set of directed paths from $(0,0)$ to $(N,M)$, i.e., the paths with steps equal to $(0,1)$ or $(1,0)$. Let $\{\eta(x): x \in \mathbb{Z}^2\}$ be a collection of (centered) i.i.d. random variables with finite exponential moments $e^{\lambda(\beta)} = Q(e^{\beta \eta}) < +\infty$, which will be referred as the environment variables, or just as the environment.

Define

\[ T(N,M) = \max_{S \in \Omega_{N,M}} H(S). \]  \hspace{1cm} (1.1)

where $H(S) = \sum_{(t,x) \in S} \eta(t,x)$ will be called the energy of the path $S$. This is usually referred to as a last-passage percolation problem (LPP). It can be interpreted as the departure time of the $M$-th customer from the $N$-th queue in a series of queues in tandem. The variable $\eta(k,n)$ has then to be understood as the service time of the $k$-th customer in the $n$-th queue.

A regime of special interest occurs when

\[ M = O(N^a) \]

for some $a \in (0,1)$. Glynn and Whitt [10] proved that

\[ \lim_{N \to +\infty} \frac{T(N, \lfloor xN^a \rfloor)}{N^{(1+a)/2}} = c\sqrt{x}, \]  \hspace{1cm} (1.2)

where the constant is independent of $a$ and of the distribution of the service times, given that they satisfy some mild integrability conditions. The proof used a strong approximation of sums of i.i.d. random variables by Brownian motions (see [18, 19]) in order to approximate $T(N, \lfloor xN^a \rfloor)$ by the corresponding maximal energy along continuous-time paths in a Brownian environment (see below for precise definitions). Then, scaling arguments lead to (1.2). Based on simulations, they conjectured that $c = 2$.

The proof of this conjecture was first given by Seppäläinen in [31]. It uses a coupling between queues in tandem and TASEP. Later proofs used an interesting relation between the Brownian model and the eigenvalues of random matrices. For a shorter proof using ideas from queueing theory and Gaussian concentration, see [12]. A complete review of the ideas of these proofs can be found in [26].

Let us now state the following as a summary of the previous discussion:

**Theorem 1.1.**

\[ \lim_{N \to +\infty} \frac{T(N, \lfloor xN^a \rfloor)}{N^{(1+a)/2}} = 2\sqrt{x}, \]  \hspace{1cm} (1.3)
Some fluctuation results are also available (see [2, 5]). The limiting law is identified as the Tracy-Widom distribution. This is closely related to the link between Brownian percolation and random matrices we have mentioned. See also [13] for large deviations results at the Tracy-Widom scale. As usual in this type of models, the upper deviations are much larger than the lower ones (see [16] for last-passage percolation, [9] and [32] for the related model of increasing subsequences in the plane and [4] for directed polymers. See also [21] for a general discussion on the subject, including random matrices). This can be explained heuristically by noticing that, in order to increase the values of the max, it is enough to increase the values of the environment along a single path. Decreasing the value of the max requires to decrease the values of the whole environment.

We will be mostly concerned with non-zero temperature analogs to the LPP problem, namely directed polymers in random environment. Let \( P_{N,M} \) be the uniform probability measure on \( \Omega_{N,M} \). For a given realization of the environment, we define on \( \Omega_{N,M} \) the polymer measure at inverse temperature \( \beta \) as

\[
\mu_{N,M}^{\beta}(\omega = S) = \frac{1}{Z_{\beta}(N,M)} e^{\beta H(S)} P_{N,M}(\omega = S), \quad \forall S \in \Omega_{N,M},
\]

where \( Z_{\beta}(N,M) \) is a normalizing constant called the (point-to-point) partition function, given by

\[
Z_{\beta}(N,M) = P_{N,M} \left( e^{\beta H(\omega)} \right).
\]

It is easy to show the existence of the limit of the free energy in the regime considered above for the LPP. Indeed, for \( M = O(N^a) \) for some \( a \in (0, 1) \), the following limit holds for almost every realization of environment:

\[
\lim_{N \to +\infty} \frac{1}{N^{(1+a)/2}} \log Z_{\beta}(N, N^a) = 2\beta.
\]

The proof is straightforward as it applies directly the corresponding result for last-passage percolation. Just note that

\[
- \log |\Omega_{N,N^a}| + \beta T(N, N^a) \leq \log Z_{\beta}(N, N^a) \leq \beta T(N, N^a),
\]

observe that \( \log |\Omega_{N,N^a}| = O(N^a \log N) \), divide by \( N^{(1+a)/2} \) and let \( N \) goes to +\( \infty \).

To obtain a non trivial regime, we have to ensure that the normalizing term is of the same order than \( |\Omega_{N,N^a}| \). This will be done by increasing the temperature with \( N \) (equivalently, decreasing \( \beta \)). Although this is not the usual situation in statistical mechanics, it allows us to recover a well known model of continuous-time directed polymer in a Brownian environment (see below for a precise definition). Until now, no precise relation between discrete models and this Brownian model has been given in the literature.
Let us introduce more precisely the Brownian setting: let \((B_{\cdot}^{(i)})_i\) be an i.i.d. sequence of one-dimensional Brownian motions. Let \(\Omega^c_{N,M}\) be the set of increasing sequences \(0 = u_0 < u_1 < \cdots < u_M < u_{M+1} = N\). This can be identified as the set of piecewise constant paths with \(M\) positive jumps of size 1 in the interval \([0, N]\). Note that \(|\Omega^c_{N,M}| = N^M/M!\), where \(|\cdot|\) stands here for the Lebesgue measure. Denote by \(P^c_{N,M}\) the uniform probability measure on \(\Omega^c_{N,M}\). For \(u \in \Omega^c_{M,N}\), define

\[
Br(N, M)(u) = Br(u) = \sum_{i=0}^{M} (B_{u_{i+1}}^{(i)} - B_{u_i}^{(i)}),
\]

\[(1.8)\]

\[
L(N, M) = \max_{u \in \Omega^c_{N,M}} Br(u),
\]

\[(1.9)\]

\[
Z_{\beta}^{Br}(N, M) = P^c_{N,M} (e^{\beta Br(u)}).
\]

The functional \((1.9)\) is the aforementioned Brownian percolation problem from queueing theory. Observe that it has the interesting property that

\[
L(N, M) = \sqrt{N}L(1, M),
\]
in law. This is due to the scaling properties of Brownian motions. It is now a well known fact that \(L(1, M)\) has the same law as the larger eigenvalue of a Gaussian Unitary random matrix (GUE, see [3, 29] among other proofs). As a consequence,

\[(1.10)\]

\[
N^{1/6} (L(1, N) - 2N^{1/2}) \longrightarrow F_2,
\]

where \(F_2\) denotes the Tracy-Widom distribution [34]. It describes the fluctuations of the top eigenvalue of the GUE and its distribution function can be expressed as

\[
F_2(s) = \exp \left\{ - \int_s^{+\infty} (x - s)u(s)^2 dx \right\},
\]

where \(u\) is the unique solution of the Painlevé II equation

\[
u'' = 2u^3 + xu,
\]

with asymptotics

\[
u(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}} \exp \left\{ -\frac{2}{3}x^{3/2} \right\}.
\]

The distribution function \(F_2\) is non-centered and its asymptotics behavior is as follows:

\[
F_2(s) \sim e^{\frac{1}{2} s^3}, \text{ as } s \to -\infty, \quad 1 - F_2(s) \sim e^{-\frac{4}{3} t^{3/2}} \text{ as } t \to +\infty.
\]

See [1] for more details about the Tracy-Widom distribution and random matrices in general. In the discrete setting, it is shown in [5] that, for \(M = N^a\) with \(0 < a < 3/7\),
\[
\frac{T(N, N^a) - 2N^{(1+a)/2}}{N^{1/2-a/6}} \longrightarrow F_2.
\]

The proof uses similar approximations than the seminal work of Glynn and Whitt. See also [2] for similar results.

The third display (1.10) is the partition function of the continuous-time directed polymers in Brownian environment. The free energy of this polymer model is explicit. Its exact value was first conjectured in [29] based on a generalized version of the Burke’s Theorem and detailed heuristics. The proof was then completed in [27]:

**Theorem 1.2** (Moriarty-O’Connell). [27]

\[
(1.12) \quad \lim_{N \to +\infty} \frac{1}{N} \log Z_{\beta}^B(N, N) = f(\beta),
\]

where

\[
(1.13) \quad f(\beta) = \begin{cases} 
-(-\Psi)^*(-\beta^2) - 2\log |\beta| & : \beta \neq 0 \\
0 & : \beta = 0
\end{cases}
\]

where \( \Psi(m) \equiv \Gamma'(m)/\Gamma(m) \) is the restriction of the digamma function to \((0, +\infty)\), \( \Gamma \) is the Gamma function

\[
\Gamma(m) = \int_0^{+\infty} t^{m-1} e^{-t} dt,
\]

and \(-\Psi^*\) is the convex dual of the function \(-\Psi:\)

\[
(-\Psi)^*(u) = \inf_{m \geq 0} \{mu + \Psi(m)\}.
\]

We now search for a ‘regime’ in which the limiting free energy of the discrete model is the same as the Brownian one. It turns out that a way to achieve this is to increase the temperature in the asymmetric discrete model, as \( N \) tends to \(+\infty\). So the Moriarty-O’Connell polymer can be viewed as an approximation of a discrete polymer close to an axis at a very high temperature.

**Theorem 1.3** (The Moriarty-O’Connell regime). Let \( \beta_{N,a} = \beta N^{(a-1)/2} \),

\[
(1.14) \quad \lim_{N \to +\infty} \frac{1}{\beta_{N,a} N^{(1+a)/2}} \log Z_{\beta_{N,a}}(N, N^a) = f(\beta)/\beta
\]

In Section 4, we will give a proof of a \( d \) dimensional version of this fact. Unfortunately, we are no longer able to compute explicitly the free energy for the Moriarty-O’Connell model when \( d \geq 2 \). We can even treat more asymmetric cases, where the additional asymmetry translates in a lost of dimensions in the limit (Section 4.2). The proof of this fact is closely related to the continuity of the free energy of point-to-point directed polymers at fixed temperature at the border of an octant which is discussed in Section 2.
We then turn to the problem of computing the free energy of a directed polymers model with a drift that grows with $N$. Let

$$Z^{(h)}_{\beta,N} = \sum_{1 \leq n \leq N} \overline{Z}_\beta(n, N - n) e^{-h \times (N - n)},$$

where, for each $n$, $\overline{Z}_\beta(n, N - n)$ is the (non-normalized) point-to-point partition function

$$\overline{Z}_\beta(n, N - n) = \sum_{\omega \in \Omega_{n,N-n}} e^{\beta H_N(\omega)}.$$

This can also be seen as a generating function or a Poissonization of the point-to-point partition function. Recall that, when $N - n = O(N^a)$, Theorem 1.1 implies that

$$\lim_{N \to +\infty} \frac{1}{\sqrt{n(N - n)}} \log \overline{Z}_\beta(n, N - n) = 2\beta,$$

as, in this regime, $\log |\Omega_{n,N-n}|$ is of much smaller order than $\sqrt{n(N - n)}$ (see also (1.6)). The role of the drift $h$ in (1.15) is to penalize the paths for which the final point is far from the horizontal axis. It has to be calibrated in order to favor final points such that $N - n = O(N^a)$.

Our first result for this model concerns the value of the free energy.

**Theorem 1.4.** Take $h = h_N = \gamma N^{(1-a)/2}$. Then,

$$\lim_{N \to +\infty} \frac{1}{N^{(1+a)/2}} \log Z^{(h)}_{\beta,N} = \beta^2 \gamma,$$

for all environment laws such that $Q(e^{\beta \eta}) < +\infty$ for all $\beta > 0$.

We can even give the correct order of the fluctuations of the free energy. The bounds we obtain have a certain flavor of variance bounds without being exactly such.

**Theorem 1.5.** For all $a < 1/3$ ($a < 3/7$ for a Gaussian environment), there exists a constant $C > 0$ such that, for all $N \geq 1$,

$$\frac{1}{C} N^{1-a/3} \leq Q\left\{ \left( \log Z^{h_N}_{\beta,N} - \frac{\beta^2}{\gamma} N^{(1+a)/2} \right)^2 \right\} \leq C N^{1-a/3}.$$

The proof is based on non-asymptotic deviation inequalities for the partition function. These are reminiscent of similar bounds for random matrices proved by Ledoux and Rider ([23]). Similar bounds were obtained in the context of LPP in [13] for Gaussian or Bounded environments. We strongly believe that the properly rescaled fluctuations should converge to the Tracy-Widom distribution. However, we would need a more precise analysis to prove this affirmation (see Remark 5.9). Note that the recent article [33] includes fluctuation bounds for a (symmetric) one-dimensional model of directed polymers in a log-Gamma environment.
The rest of this work is organized as follows:

- We prove the continuity of the point-to-point partition function for discrete models in Section 2.
- In Section 3, we discuss the existence of the free energy for the directed polymers in Brownian environments.
- In Section 4, we discuss the links between asymmetric directed polymers and directed polymers in a Brownian environment. We give the proof of a multidimensional version of Theorem 1.3 in Section 4 and discuss a more asymmetric situation in Section 4.2.
- Finally, we study the model of directed polymers with a huge drift in Section 5. Theorem 1.4 is proved in Subsection 5.1, while the fluctuation bounds (Theorem 1.5) are proved in Subsection 5.3.

An extended version of this article can be found in the Thesis [26]. It includes a complete review of the literature about Brownian percolation and one-dimensional directed polymers in a Brownian environment.

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2. Continuity of the Point-to-Point Partition Function for the Discrete Model

We prove here the continuity of the point-to-point free energy seen as a function from the octant \( \{ x \in \mathbb{R}^d : x_i \geq 0 \} \) to \( \mathbb{R} \). Only the continuity at the boundary of the octant requires a proof, as the continuity in the interior is an easy consequence of the concavity properties of the free energy (which itself follows from sub-additivity).

For \( y \in \mathbb{R}^d_+ \), define

\[
Z_N^\beta(y) = \sum_{s \in \Omega_{Ny}} \exp \beta H(s),
\]

where \( \Omega_{Ny} \) is the set of directed paths from the origin to \( Ny \), which, by notational abuse, denotes the point in \( \mathbb{Z}^d \) which \( i \)-th coordinate is \( \lfloor Ny_i \rfloor \). Note that the dimension here is \( d \) and not \( d + 1 \) as usual. We will be interested in directions of the form \( y_h = (h, x) \) with \( x \in \mathbb{R}^{d-1}_+ \) (i.e. \( x \in \mathbb{R}^{d-1}, x_i > 0 \)), and \( h \geq 0 \). In this case, we just denote the partition function by \( Z_N(h, x) \). We also define the point-to-point free energy:

\[
\psi^\beta(y) = \lim_{N \to +\infty} \frac{1}{N} \log Z_N^\beta(y),
\]

and we adopt the convenient notation \( \psi(h, x) \) for \( \psi(y_h) \) (we also dropped the dependence in \( \beta \)). \( \psi \) is a function from the octant \( \{ x \in \mathbb{R}^d : x_i \geq 0 \} \) to \( \mathbb{R} \).
Proposition 2.1.

\[ \lim_{{h \to 0}} \psi(h, x) = \psi(0, x). \]

Proof. Each path from the origin to \(N(h, x)\) can be decomposed into \(Nh\) segments with constant first coordinate: for each path, there is a collection of points \((m_i)_{i \leq Nh}\) with \(m_i \in \mathbb{Z}^{d-1}_+\) and such that for each \(0 \leq i < Nh\), there is a segment of the path linking \((i, m_i)\) and \((i, m_{i+1})\). So the partition function can be decomposed itself as

\[ Z_N(h, x) = \sum_{(m_i)} \Pi_i Z(i; m_i, m_{i+1}), \tag{2.1} \]

where, for each \(i\), \(Z(i; m_i, m_{i+1})\) is a sum over directed paths linking \((i, m_i)\) and \((i, m_{i+1})\). The collection of possible points \((m_i)\) runs over a set \(J_{Nh}^N\) which cardinality satisfies \(\log |J_{Nh}^N| = N\phi(h, x) + o(N)\) for some \(\phi(h, x) \to 0\) as \(h \to 0\) (see remark below). We will analyze each summand of the right hand side of (2.1) separately:

\[ Q(\log \Pi_i Z(i; m_i, m_{i+1})) = \sum_i Q(\log Z(i; m_i, m_{i+1})) \]

\[ = \sum_i Q(\log Z(0; m_i, m_{i+1})) \]

\[ \leq Q \left( \log Z_N(0, x) + \beta \sum \eta(i, m_{i+1}) \right) \]

\[ \leq N \psi(0, x). \tag{2.2} \]

The second equality follows by translation invariance; in the third line, we use the fact that the partition functions do not consider the environment at the starting point; the last inequality follows by subadditivity, as

\[ \psi(y) = \sup_N \frac{1}{N} Q \log Z_N(y), \]

and the fact that \(Q\eta = 0\). Now, the concentration inequality implies that

\[ Q \left( \left| \log \Pi_i Z(i; m_i, m_{i+1}) - Q \log \Pi_i Z(i; m_i, m_{i+1}) \right| \geq \epsilon N \right) \leq e^{-\epsilon^2 N}. \tag{2.3} \]

for \(\epsilon\) small enough (see [22] and [8] Proposition 3.2.1-b). Using (2.1), we can see that, if

\[ \log Z_N(h, x) \geq N \psi(0, x) + \epsilon N, \]

for some \(\epsilon > 0\), then, for some \((m_i)_{i \in J_{Nh}^N}\), it must happen that

\[ \log \Pi_i Z(i; m_i, m_{i+1}) \geq N \psi(0, x) + \epsilon N - \log |J_{Nh}^N|. \]
By (2.2), this means that the quantity in the left hand side deviates more than $\epsilon N - \log |J_{h,x}^N|$ from its mean. By the asymptotics on $|J_{h,x}^N|$, for $h$ small enough, we will have that $\log |J_{h,x}^N| < \epsilon N/2$, and then the inequality (2.3) applies. Then,

$$Q(\log Z_N(h,x) \geq Np(0,x) + \epsilon N) \leq \exp \left\{ N\phi(h,x) - c\epsilon^2 N + o(N) \right\}.$$ 

By taking $h$ even smaller if necessary, the right hand side of this inequality becomes summable. By Borel-Cantelli we will then have that

$$\log Z_N(h,x) \leq \psi(0,x) + \epsilon,$$

for $h$ small enough. We now have to check the reverse inequality. But it follows easily that

$$\psi(h,x) \leq \psi(0,x) + \epsilon,$$

for $h$ small enough. We now have to check the reverse inequality. But it follows easily that

$$\log Z_N(h,x) \geq \log Z_N(0,x) + \beta \sum_{i=0}^{hN} \eta(Nx,i).$$

Recalling that the $\eta$'s are centered, dividing by $N$ and taking the limit $N \to +\infty$ give that $\psi(h,x) \geq \psi(0,x)$. \qed

Remark 2.2. The function $\phi$ can be made explicit: as

$$\log |J_{h,x}^N| = \prod_{i=2}^d \left( \frac{\lfloor Nx_i \lfloor + \lfloor Nh \rfloor}{\lfloor Nh \rfloor} \right),$$

by Stirling formula, we have $\log |J_{h,x}^N| = N\phi(h,x) + o(N)$, with

$$\phi(h,x) = \sum_{2 \leq i \leq d} \left( h \log \frac{x_i + h}{h} + x_i \log \frac{x_i + h}{x_i} \right).$$

Corollary 2.3. The point-to-point free energy is continuous on $\mathbb{R}_+^d$.

Proof. The continuity in the interior of $\mathbb{Z}_+^d$ is a consequence of the concavity properties arising from the subadditivity. See the proof of Theorem 3.3 where this is explained in the continuous setting. The continuity at the boundary follows from repeated use of the preceding Proposition. \qed

Remark 2.4. In the one-dimensional case, a very precise asymptotic for the last-passage percolation is available. It implies that $\psi(1,h) = 2\sqrt{h} + o(\sqrt{h})$ as $h \downarrow 0$ (see [25], Theorem 2.3).
Remark 2.5. This scheme of proof will reappear later in the proof of a certain continuity at the borders property for very asymmetric directed polymers, in the regime where the limit is the Brownian free energy.

3. Directed Polymers in a Brownian Environment

We will now generalize the Brownian setting introduced before to larger dimensions. Let \( x \in \mathbb{Z}^d \) such that \( x_i \geq 1 \) for all \( i = 1, \ldots, d \). Let \( M = \sum_{i=1}^d x_i \). This is basically the length of a nearest-neighbor path from the origin \( 0 \) to \( x \). Let \( \Omega_{t,x}^c \) be the set of right-continuous paths \( s \) such that:

(i) \( s_0 = 0 \) and \( s_t = x \).

(ii) \( s \) performs exactly \( M \) jumps, according to the coordinate vectors.

So the skeleton of \( s \) can be thought of as a discrete nearest-neighbor path from the origin to \( x \). Let \( \Omega_{c,t,x} \) be the set of right-continuous paths \( s \) such that:

(i) \( s_0 = 0 \) and \( s_t = x \).

(ii) \( s \) performs exactly \( M \) jumps, according to the coordinate vectors.

So the skeleton of \( s \) can be thought of as a discrete nearest-neighbor path from the origin to \( x \). Let \( \Omega_{c,t,x} \) be the uniform measure on \( \Omega_{c,t,x}^c \).

Now consider a family \( \{ B(y) : y \in \Lambda \} \) of independent Brownian motions, where \( \Lambda = \{ y \in \mathbb{Z}^d : 0 \leq y_i \leq x_i, \forall i = 1, \ldots, d \} \). Define the energy of a path \( s \) in the following way: let \( 0 = t_0 < t_1 < \cdots < t_M < t \) be the jumps times of \( s \) and put \( t_{M+1} = t \), then

\[
\text{Br}(s) = \text{Br}(t, x)(s) = \sum_{k=1}^{M+1} (B_{t_k}(s_{t_k}) - B_{t_{k-1}}(s_{t_k})).
\]

The partition function of the directed polymers in Brownian environment at inverse temperature \( \beta \) is

\[
Z^\text{Br}_\beta(t, x) = P_{c,t,x}^c(\exp \beta \text{Br}(s)).
\]

We first prove the existence of the free energy in the linear regime. Take \( \alpha \in \mathbb{R}^d \) with strictly positive entries.

**Theorem 3.1.** Let \( \alpha N \) be the point of \( \mathbb{Z}^d \) whose \( i \)-th coordinate is equal to \( \lfloor \alpha_i N \rfloor \). Then the following deterministic limit

\[
p(\beta, \alpha, d) = \lim_{N \to +\infty} \frac{1}{N} \log Z^\text{Br}_\beta(N, \alpha N)
\]

exists \( Q \)-a.s.. Moreover, the function \( \alpha \mapsto p(\beta, \alpha, d) \) is continuous on its domain.

**Proof.** First, fix \( \alpha \). The proof uses subadditivity. To lighten notation, denote \( |\Omega_N| \) for \( |\Omega_{N, \alpha N}| \). We consider unnormalized versions of the partition function:

\[
\int_{\Omega_{N+M}} e^{\beta \text{Br}(N+M, x_{N+M})(s)} \geq \int_{\Omega_{N+M}} e^{\beta \text{Br}(N+M, x_{N+M})(s)} 1_{s_N = \alpha N}
\]

\[
= \int_{\Omega_N} e^{\beta \text{Br}(N, \alpha N)(s)} \times \left( \int_{\Omega_M} e^{\beta \text{Br}(M, \alpha M)(s)} \circ \theta_{N, \alpha N} \right)
\]
where the shift $\theta_{k,x}$ means that we use the Brownian motions

$$B^{(y)}(\cdot) = B^{(y+x)}(\cdot + k),$$

to define $B_{\cdot}$. By subadditivity, it follows that there exists a deterministic function $p(\beta, \alpha, d)$ such that

$$p(\beta, \alpha, d) = \lim_{N \to +\infty} \frac{1}{N} \log \int_{\Omega_N} e^{\beta B_{N,\alpha N}(s)}.$$

$Q$-almost surely. Apply this with $\beta = 0$ and the theorem follows with $p(\beta, \alpha, d) = p(\beta, \alpha, d) / p(0, \alpha, d)$. Now, take $\alpha_1$ and $\alpha_2$ in $\mathbb{R}^d$ with strictly positive coordinates, and $\lambda \in (0, 1)$. Then,

$$Z_{\beta}^{B_{\cdot}}(N, N(\lambda \alpha_1 + (1 - \lambda) \alpha_2)) \geq Z_{\beta}^{B_{\cdot}}(\lambda N, \lambda \alpha_1 N) \times Z_{\beta}^{B_{\cdot}}((1 - \lambda)N, (1 - \lambda)\alpha_2 N) \circ \theta_{\lambda N, \lambda \alpha_1 N}.$$

Taking logarithms in both sides, dividing by $N$ and taking limits, leads to,

$$p(\beta, \lambda \alpha_1 + (1 - \lambda) \alpha_2, d) \geq \lambda p(\beta, \alpha_1, d) + (1 - \lambda) p(\beta, \alpha_2, d)$$

So $\alpha \mapsto p(\beta, \alpha, d)$ is concave, and then continuous. As $p(\beta, \alpha, d) = \overline{p}(\beta, \alpha, d)/\overline{p}(0, \alpha, d)$, it is also continuous. □

**Remark 3.2.** Note that as we have true subadditivity, we can avoid the use of concentration. However, we can state the following result:

$$Q \left( |\log Z_{\beta}^{B_{\cdot}}(N, \alpha N) - Q \log Z_{\beta}^{B_{\cdot}}(N, \alpha N) | > uN \right) \leq C \exp \left\{ - \frac{Nu^2}{C\beta^2} \right\}.$$

(3.4)

This can be proved as Formula (9) in [30], using ideas from Malliavin Calculus.

4. **Asymmetric Directed Polymers in a Random Environment**

The central part of this Section is the proof of a multidimensional version of Theorem 1.3.

Let $x \in \mathbb{Z}^d$ such $x_i \geq 1$ for all $i = 1, \ldots, d$ and $N \geq 1$. Let $M = \sum_{i=1}^{d} x_i$ be the distance between the origin and $x$ in $\mathbb{Z}^d$. Let $\Omega_{N,x}$ be the set of directed paths from the origin in $\mathbb{Z}^{d+1}$ to $(N, x)$ that is

$$\Omega_{N,x} = \{ S : \{0, \ldots, N + M \} \to \mathbb{Z}^{d+1} : S_0 = 0, S_{N+M} = (N, x), \forall t, S_{t+1} - S_t \in \{ e_i : i = 1, \ldots, d \} \}.$$

Consider a collection of i.i.d. random variables $\{ \eta(k, x) : k \in \mathbb{Z}, x \in \mathbb{Z}^d \}$. We will assume that $Q(e^{\beta \eta}) < +\infty$ for all $\beta \geq 0$. For a fixed realization of the environment, define the energy of a path $S \in \Omega_{N,x}$ as
The polymer measure at inverse temperature $\beta$ is now defined as the measure on $\Omega_{N,x}$ such that

$$d\mu_{N,x}(S) = \frac{1}{Z_\beta(N, x)} \exp \beta H(S),$$

where $Z_\beta(N, x)$ is the point-to-point partition function

$$Z_\beta(N, x) = P_{N,x} (\exp \beta H(S)).$$

We will be interested in the limit as $N$ grows to infinity and $x = x_N$, with $|x_N| \to +\infty$ with $N$ in an appropriate way. Take $\alpha \in \mathbb{R}^d$ with strictly positive coordinates. Let $\alpha^N$ be the point in $\mathbb{Z}^d$ which $i$-th coordinate is equal to $[\alpha_i^N]$. The following theorem is the generalization to $\mathbb{Z}^d$ of Theorem 1.3.

**Theorem 4.1.** Let $\beta_{N,a} = \beta N^{(a-1)/2}$. Then,

$$\lim_{N \to +\infty} \frac{1}{\beta_{N,a} N^{(1+a)/2}} \log Z_{\beta_{N,a}}(N, \alpha N^a) = p(\beta, \alpha, d)/\beta,$$

$Q$-almost surely, where $p(\beta, \alpha, d)$ is the free energy of the continuous-time directed polymer in a Brownian environment as in (3.3).

**Remark 4.2.** To lighten notation, in the following, $C$ will denote a generic constant whose value can vary from line to line. Also, we can consider $\alpha = (1, \cdots, 1)$ for simplicity and introduce the notations $\Omega_{N,a} = \Omega_{N,N^a}$ and $P_{N,a} = P_{N,N^a}$, and similarly for their continuous counterparts.

4.1. **Proof of Theorem 4.1.** The proof is carried on in 4 Steps. Much of the computations in Steps 1 and 2 are inspired by [5, 13], while the scaling argument in Step 3 is already present in [10].

4.1.1. **First Step: approximation by a Gaussian environment.** The central ingredient of this part of the proof is a strong approximation technique by Komlós, Major and Tusnády: let $\{\eta_t : t \geq 0\}$ denote an i.i.d. family of random variables, with $Q(\eta_0) = 0$, $Q(\eta_0^2) = 1$ and $Q(e^{\beta_0}) < +\infty$ for all $0 \leq \beta \leq \beta_0$ for some $\beta_0 > 0$. Let $\{g_t : t \geq 0\}$ denote an i.i.d. family of standard normal variables. Denote

$$S_N = \sum_{t=0}^{N} \eta_t, \quad T_N = \sum_{t=0}^{N} g_t.$$
Theorem 4.3 (KMT approximation). \[19\] The sequences \( \{\eta_t : t \geq 0\} \) and \( \{g_t : t \geq 0\} \) can be constructed in such a way that, for all \( x > 0 \) and every \( N \),

\[
Q \left\{ \max_{k \leq N} |S_k - T_k| > K_1 \log N + x \right\} \leq K_2 e^{-K_3 x},
\]

where \( K_1, K_2 \) and \( K_3 \) depend only on the distribution of \( \eta \), and \( K_3 \) can be taken as large as desired by choosing \( K_1 \) large enough. Consequently, \( |S_N - T_N| = O(\log N) \), Q-a.s..

Now consider our environment variables \( \{\eta(t, x) : t \in \mathbb{Z}, x \in \mathbb{Z}^d\} \). Use Theorem 4.3 to couple each ‘row’ \( \eta(\cdot, x) \) with standard normal variables \( g(\cdot, x) \) such that

\[
Q \left\{ \max_{k \leq N} |S(k, x) - T(k, x)| > C \log N + \theta \right\} \leq K_2 e^{-K_3 \theta}, \quad \forall \theta > 0,
\]

where \( S(k, x) = \sum_{t=0}^k \eta(t, x) \) and \( T(k, x) = \sum_{t=0}^k g(t, x) \).

Now, we need to decompose each path \( S \in \Omega_{N,a} \) into its ‘jump’ times \( T = (T_i)_i \) and its position between jump times \( L = (L_i)_i \). We say that \( T \) is a jump time if one of the coordinates of \( S \) other than the first changes between instants \( T_i \) and \( T_{i+1} \). We can order the jump times of \( S \): \( T_0 = 0 < \cdots < T_{dN^a} < T_{dN^a+1} = N \). We can then define \( L_i \) as the point \( y \in \mathbb{Z}^d \) such that \( S_{T_i} = (T_i, y) \). We can rewrite the Hamiltonian (4.1) as

\[
H(S) = \sum_{i=0}^{dN^a} \Delta H(S, i),
\]

where

\[
\Delta H(S, i) = \sum_{k=T_i}^{T_{i+1}-1} \eta(k, L_i).
\]

Define \( g(S) \) and \( \Delta g(S, i) \) just in the same way by replacing the variables \( \eta \) by the Gaussians \( g \). Then,

\[
|H(S) - g(S)| \leq \sum_{i=0}^{dN^a} |
\Delta H(S, i) - \Delta g(S, i)|.
\]

Let \( \theta_N \) be an increasing function to be determined later and \( \Lambda_{N,a} = \{y \in \mathbb{Z}^d : 0 \leq y_i \leq \lfloor N^a \rfloor\} \):

\[
Q \left\{ |H(S) - g(S)| > 2dN^a \theta_N, \text{ for some } S \in \Omega_{N,a} \right\} \leq Q \left\{ \sum_{i=1}^{dN^a} |\Delta H(S, i) - \Delta g(S, i)| > 2dN^a \theta_N, \text{ for some } S \in \Omega_{N,a} \right\}
\]
In order to apply Theorem 4.3, we have to take \( \theta_N = K_1 \log N + \epsilon_N \), and to apply Borel-Cantelli, as \( |\Lambda_{N,a}| \leq N^{da} \), it is enough to take \( \epsilon_N = c \log N \) with \( c \) large enough to make \( N^a e^{-K_3 \epsilon_N} \) summable. Then, \( Q \)-a.s., \( |H(S) - g(S)| \leq CN^a \log N \) for all \( S \in \Omega_{N,a} \), for \( N \) large enough. This shows that

\[
P_{N,a}(e^{\beta N,a H(S)}) = P_{N,a}(e^{\beta N,a g(S)}) O(e^{C\beta N,a N^a \log N}).
\]

Recall that \( \beta_{N,a} = \beta N^{(a-1)/2} \), so that \( \beta_{N,a} N^a \log N = O(N^{(3a-1)/2} \log N) \). As \( 0 < a < 1 \), we have \( \beta_{N,a} N^a \log N \ll \beta_{N,a} N^{(a+1)/2} = \beta N^a \), and then

\[
\log Z_{\beta_{N,a}}(N, \alpha N^a) = \log Z_{\beta_{N,a}}^g(N, \alpha N^a) + o(\beta_{N,a} N^{(1+a)/2}),
\]

where the superscript \( g \) means that the environment is Gaussian.

We can conclude that, if the limit free energy exists \( Q \)-a.s. for Gaussian environment variables, it exists for all environment variables having some finite exponential moments, and the limit is the same.

### 4.1.2. Second Step: approximation by continuous-time polymers in Brownian environment.

Having replaced our original disorder variables by Gaussians, we can take them as unitary increments of independent one-dimensional Brownian motions. We then just have to control their fluctuations to replace the discrete paths by continuous paths in a Brownian environment. This is what will be done in the following paragraphs.

We first need to establish a correspondence between continuous paths and discrete ones.

Take \( s \in \Omega_{N,a}^c \), and recall the definition (3.1) for the Brownian Hamiltonian \( Br(s) \) and that \( 0 = t_0 < t_1 < \cdots < t_{dN^a+1} = N \) denote the jump times of \( s \). Let \( l_i = s_{t_i} \). The path \( s \) can be discretized by defining the following Gaussian Hamiltonian:

\[
H^g(s) = \sum_{i=0}^{dN^a} \left( B_{[t_{i+1}]}^{(l_i)} - B_{[t_i]}^{(l_i)} \right).
\]

This is equivalent to consider \( g(S) \) where \( S \in \Omega_{N,a} \) is defined through its jump times \( T_i \) and successive positions \( L_i \) by
\[ T_i = \lfloor t_i \rfloor, \]
\[ L_k = l_i, \quad \forall T_i \leq k < T_{i+1}, \]

(Recall that the Gaussian variables obtained in the previous step are now embedded in the Brownian motions). In this way,

\[ P_{N,a}^c (\exp \beta H^{Br}(s)) = P_{N,a} (\exp \beta g(S)). \]

We have now to approximate the previous expression by \( Z^{Br}_\beta (N, N^a) \). Take \( s \in \Omega_{N,a}^c \):

\[ |H^g(s) - Br(s)| = \left| \sum_{i=0}^{dN^a} (B_1^{(l_i)} - B_1^{(r)}) - \sum_{i=0}^{dN^a} (B_{i+1}^{(l_i)} - B_{i+1}^{(r)}) \right| \]
\[ \leq \sum_{i=0}^{dN^a} |B_1^{(l_i)} - B_1^{(r)}| + \sum_{i=0}^{dN^a} |B_{i+1}^{(l_i)} - B_{i+1}^{(r)}| \]
\[ \leq 2 \sum_{i=0}^{dN^a} \sup_{0 \leq s, t \leq N+1, |s-t| < 2} |B_1^{(l_i)} - B_1^{(r)}|. \]

This can be handled with basic properties of Brownian motion: denote by \( x_N \) an increasing function to be determined,

\[ Q \left( \sum_{i=0}^{dN^a} \sup_{0 \leq s, t \leq N+1, |s-t| < 2} |B_1^{(l_i)} - B_1^{(r)}| > dN^a x_N \text{ for some } s \in \Omega_{N,a} \right) \]
\[ \leq Q \left( \max_{1 \leq i \leq dN^a} \sup_{0 \leq s, t \leq N+1, |s-t| < 2} |B_1^{(l_i)} - B_1^{(r)}| > x_N \text{ for some } s \in \Omega_{N,a} \right) \]
\[ \leq Q \left( \max_{x \in \Lambda_{N,a}} \sup_{0 \leq s, t \leq N+1, |s-t| < 2} |B_1^{(x)} - B_1^{(x)}| > x_N \right) \]
\[ \leq |\Lambda_{N,a}| Q \left( \sup_{0 \leq s, t \leq N+1, |s-t| < 2} |B_1 - B_1| > x_N \right) \]
\[ \leq CN^{d a} \sum_{i=0}^{N-2} Q \left( \sup_{i \leq t \leq i+3} B_t - \inf_{i \leq t \leq i+3} B_t > x_N \right) \]
\[ \leq CN^{d a+1} Q \left( \sup_{0 \leq t \leq 3} |B_t| > \frac{x_N}{2} \right) \]
\[ \leq CN^{d a+1} Q \left( B_3 > \frac{x_N}{2} \right) \]
\[ \leq CN^{d a+1} e^{-C x_N^2}. \]

With \( x_N = \log N \) and recalling (4.4) from Step 1, we see that \( Q \)-a.s., for \( N \) large enough,
\[ P_{N,a} \left( e^{\beta N,a H(N,\alpha N^a)} \right) = P_{N,a} \left( e^{\beta N,a Br(N,\alpha N^a)} \right) \times O(e^{\beta N,a N^a \log N}). \]

Again, this will imply that
\[ \log Z_{\beta N,a} (N, \alpha N^a) = \log Z_{\beta N,a}^{Br}(N, \alpha N^a) + o(\beta N,a N^{(1+a)/2}), \tag{4.5} \]

4.1.3. Third Step: scaling. Observe that, for a fixed path \( s \in \Omega_{N,a} \),
\[ Br(N,\alpha N^a)(s) = \sqrt{N} Br(1,\alpha N^a)(s/N) = N^{(1-a)/2} Br(N^a,\alpha N^a)(s_{\times N^{a-1}}), \]
where the equalities hold in law. Note also that \( s_{\times N^{a-1}} \in \Omega_{N^a,\alpha N^a} \). It follows that
\[ Z_{\beta N,a}^{Br}(N, \alpha N^a) = P_{N,a} \left( \exp \beta N,a Br(N, \alpha N^a)(s) \right) \]
\[ = P_{N^a,\alpha N^a} \left( \exp \beta N,a N^{(1-a)/2} Br(N^a, \alpha N^a)(s_{\times N^{a-1}}) \right) \]
\[ = P_{N^a,\alpha N^a} \left( \exp \beta Br(N^a, \alpha N^a)(s_{\times N^{a-1}}) \right). \]

But the last expression is simply \( Z_{\beta}^{Br}(N^a, \alpha N^a) \) so that, by Theorem 3.1
\[ \lim_{N \to +\infty} \frac{1}{N^a} \log Z_{\beta N,a}^{Br}(N, \alpha N^a) = p(\beta, \alpha, d). \tag{4.6} \]

From (4.5) and (4.6), we can deduce that the limit (4.2) holds in law.

4.1.4. Final Step: concentration. So far, we proved convergence in law for the original problem. But we can write a convenient concentration inequality for the free energy with respect to his average, in the Gaussian case. So, a.s. convergence holds for Gaussian, and, according to step 1, for any environment.

The classical concentration inequality for Gaussian random variables can be stated as follows:

**Theorem 4.4.** Consider the standard normal distribution \( \mu \) on \( \mathbb{R}^K \). If \( f : \mathbb{R}^K \to \mathbb{R} \) is Lipschitz continuous with Lipschitz constant \( L \), then
\[ \mu \left( x : |f(x) - \int f d\mu| \geq u \right) \leq 2 \exp\{-\frac{u^2}{2L^2}\}. \]
For a detailed exposition of concentration of measures, see for example, the lecture notes of Ledoux [20]. Define

\[ F(z) = \frac{1}{N^a} \log P_{N,a} \left( e^{\beta N,a \sum_{i=1}^{N+d^N} z(S_i)} \right). \]

It is easy to prove that \( F \) is a Lipschitz continuous function with Lipschitz constant \( CN^{-a/2} \). By Gaussian concentration, this yields

\[ Q \left\{ \left| \frac{1}{N^a} \log Z_{\beta N,a} (N, \alpha N^a) - \frac{1}{N^a} Q \log Z_{\beta N,a} (N, \alpha N^a) \right| > u \right\} \leq 2 \exp \left( - \frac{N^a u^2}{2C^2} \right). \]

(4.7)

This ends the proof of the theorem. \( \square \)

4.2. Very asymmetric cases. We now consider an even more asymmetric case: let \( a = (a_1, \ldots, a_d) \) with \( 0 \leq a_i \leq a \) for all \( i \) but \( a_i = a \) for exactly \( d - l \) values of \( i \), \( 1 \leq l < d \), and consider paths from the origin to points of type \( \alpha N^a \) with coordinates \( \alpha_i N^{a_i} \), \( \alpha_i > 0 \).

**Theorem 4.5.** Let \( \alpha' \) be the vector of \( \mathbb{R}^{d-l} \) which coordinates are those of \( \alpha \) for the indexes \( i \) such that \( a_i = a \). Then,

\[ \lim_{N \to +\infty} \frac{1}{\beta_{N,a} N^{(1+\alpha)/2}} \log Z_{\beta N,a} (N, \alpha N^a) = p(\beta, \alpha', d-l)/\beta. \]

**Proof.** The proof is very similar to the proof of Proposition 2.1. We will consider the simple case \( d = 2 \) and a final point of type \( (N^a, N^b) \) with \( b < a \). We then have to prove convergence to \( p(\beta, 1, 1)/\beta \). The general case follows easily. We can think of \( h \) as \( h = h_N = N^{(b-a)} \).

From the proof of Theorem 4.4, we have to remember that

\[ \log Z_{\beta N,a} (N, \alpha N^a) = \log Z_{\beta}^{Br} (N^a, \alpha N^a) + o(N^a) \]

Denote by \( Z(N, M, L) \) (resp. \( Z(N, M, L) \)) the normalized (resp. non-normalized) partition function over discrete paths from the origin to \( (N, M, L) \). We perform the same decomposition than before:

\[ Z_{\beta N,a} (N, N^a, N^b) = \frac{Z_{\beta N,a} (N, N^a, N^b)}{Z_0 (N, N^a, N^b)} \]

\[ = \frac{1}{Z_0 (N, N^a, N^b)} \sum_{(m_i)} \Pi_i Z(i; m_i, m_{i+1}) \]
Here, $0 \leq i \leq N^b - 1$ and $m_{N^b} = N^a$. Recalling Remark 2.2, the cardinality of the set $J_N$ of the possible configurations of $(m_i)$ satisfies $|J_N| \sim \exp\{cN^{(a+b)/2} \log N\}$. For a fixed $m_i$, recalling that the environment variables are centered,

\[
Q \left( \log \prod_i Z_{\beta_{N,a}} (i; m_i, m_{i+1}) \right) = Q \left( \log \prod_i Z_{\beta_{N,a}} (0; m_i, m_{i+1}) \right)
\]

\[
= \log Z_0(N, N^a, 0) + Q \left( \frac{\prod_i Z_{\beta_{N,a}} (0; m_i, m_{i+1})}{Z_0(N, N^a, 0)} \right)
\]

\[
\leq \log Z_0(N, N^a, 0) + Q \left( \log Z_{\beta_{N,a}} (N, N^a, 0) \right)
\]

\[
\leq \log Z_0(N, N^a, 0) + Q \left( \log Z_{\beta_{N,a}} (N^a, N^a) \right) + o(N^a)
\]

\[
\leq \log Z_0(N, N^a, 0) + N^a p(\beta, 1, 1) + o(N^a).
\]

Now, if

\[
\log Z_{\beta_{N,a}} (N, N^a, N^b) > N^a p(\beta, 1, 1) + \epsilon,
\]

there must exist some $(m_i)_i$ such that

\[
\log \prod_i Z_{\beta_{N,a}} (i; m_i, m_{i+1}) \succ \log Z_0(N, N^a, N^b) + N^a p(\beta, 1, 1) + \epsilon - \log |J_N|,
\]

Using the fact that $Z_0(N, N^a, N^b) > Z_0(N, N^a, 0)$, (4.7) and the union bound, we find that

\[
Q \left( Z_{\beta_{N,a}} (N, N^a, N^b) > N^a p(\beta, 1, 1) + \epsilon \right) \leq |J_N| \exp \{-c\epsilon^2 N^a\},
\]

for $\epsilon$ small enough. As $\log |J_N| = o(N^a)$, the RHS of the last display is summable. The result follows by Borel-Cantelli.

\[
\square
\]

5. One-dimensional directed polymers with a huge drift

We now turn to the study of directed polymers with a drift growing with $N$. This section contains the proofs of Theorems 1.4 and 1.5.

5.1. The free energy. Let us first sketch the proof of Theorem 1.4: we parametrize the terminal points conveniently:

\[
N = n(1 + u).
\]

Thus $n = N/(1 + u)$ and $N - n = Nu/(1 + u)$. We can then rewrite (1.15) as

\[
Z_N^{(h)} = \sum_u Z_{\beta} \left( \frac{N}{1 + u}, \frac{Nu}{1 + u} \right) \exp \left\{ -\gamma N^{(1-a)/2} \times \frac{Nu}{(1 + u)} \right\}.
\]

Now, for $u$ in an interval $I_N = [N^{\kappa_0}, N^{\kappa_1}]$, we will have
\[
\mathcal{Z}_\beta \left( \frac{N}{1+u}, \frac{Nu}{1+u} \right) = \exp \left\{ 2\beta N \sqrt{\frac{u}{1+u}} + o(1) \right\}
\]
uniformly in \( u \). Then,

\[
Z^{(h)}_{\beta,N} \sim \sum_{u \in I_N} \exp N \left\{ 2\beta \frac{\sqrt{u}}{1+u} - \gamma N^{(1-a)/2} \times \frac{u}{(1+u)} \right\}.
\]

Define the function

\[
f_N(u) = 2\beta \frac{\sqrt{u}}{1+u} - \gamma N^{(1-a)/2} \times \frac{u}{(1+u)}.
\]

It attains its global maximum at a point \( u^*_N \sim \left( \frac{\beta^2}{\gamma^2} \right)^N \), with \( f_N(u^*_N) \sim \frac{\beta^2}{\gamma} N^{(a-1)/2} \). So, by Laplace method, we will have

\[
Z^{(h)}_{\beta,N} = \exp \{ Nf(u^*_N) + o(1) \} = \exp \{ N^{(1+a)/2} \beta^2/\gamma + o(1) \},
\]
which would finish the proof.

**Remark 5.1.** The proof is split in three steps. The first one gives the lower bound in the Theorem, minoring the whole sum by one term, given by a \( u \) very close to the minimizer. This is the easy part.

The second step will consist mainly in proving the uniformity in (5.2) (but replacing \( = \) by \( \leq \)). This will be done by applying uniformly the KMT approximation in the whole interval \( I_N \), and then applying some deviation inequality for the Brownian percolation. The third step will be to prove that the \( u \)'s outside \( I_N \) do not contribute to the sum.

**Proof of Theorem 1.4:**

**First step:** We will now provide the lower bound: recall the notation in (5.1) and observe that for the value \( u^* \), the asymptotics of \( n \) and \( N - n \) fit the situation studied in (1.6). An easy computation yields:

\[
\liminf_{N \to +\infty} \frac{1}{N^{(1+a)/2}} \log \mathcal{Z}^{(h)}_{\beta,N} \\
\geq \lim_{N \to +\infty} \frac{1}{N^{(1+a)/2}} \log \mathcal{Z}_\beta \left( \frac{N}{1+u^*}, \frac{Nu^*}{1+u^*} \right) \exp \left\{ -\gamma N^{(1-a)/2} \times \frac{Nu^*}{(1+u^*)} \right\} \\
= \frac{\beta^2}{\gamma}.
\]

**Second step:** Let \( \delta > 0 \) and take \( \kappa_1 = (a - 1)/2 - \delta \) in order to define \( I_N = [N^{\kappa_0}, N^{\kappa_1}] \). Here, \( \kappa_0 > -1 \) is introduced to discard small values of \( u \) that have to be treated separately. Note that, in this interval, \( N - n \sim Nu \leq N^{1+\kappa_1} = o(N^{(a+1)/2}) \).

We first couple the environment variables \{\( \eta(t,x) : 1 \leq t \leq N, 1 \leq x \leq N^{\kappa_1} \)\} row by row with Brownian motions as in the proof of Theorem 1.2. This yields
\[
Z_\beta \left( \frac{N}{1 + u}, \frac{Nu}{1 + u} \right) = Z_{\beta}^{Br} \left( \frac{N}{1 + u}, \frac{Nu}{1 + u} \right) \times O(e^{N^{1+\kappa_1} \log N}) \\
\leq \exp \left\{ \beta L \left( \frac{N}{1 + u}, \frac{Nu}{1 + u} \right) \right\} \times O(e^{N^{1+\kappa_1} \log N}),
\]
uniformly for \( u \in I_N \), where \( Z_{\beta}^{Br} (N, M) \) denotes the unnormalized partition function of the Brownian model (In the following, we only make use of the domination by \( L(\cdot, \cdot) \) which can also be guessed directly from the results in [5]). Note that (1.6) holds for \( Z_{\beta}^{Br} (N, M) \) with \( M = O(N^a) \), as \( |\Omega_{N,M}^c| \) is small compared to \( \sqrt{MN} \):

(5.4) \[
\log |\Omega_{N,M}^c| \sim \log \frac{N^M}{(M)!} = O(N^a \log N).
\]

We now search for a convenient upper bound for the (normalized) Brownian partition function:

\[
Q \left\{ Z_N^{Br} \left( \frac{N}{1 + u}, \frac{Nu}{1 + u} \right) > \exp \frac{\beta N \sqrt{u}}{1 + u} (2 + \epsilon_N) \right\} \\
\leq Q \left\{ \max_\omega Br \left( \frac{N}{1 + u}, \frac{Nu}{1 + u} \right) > \frac{N \sqrt{u}}{1 + u} (2 + \epsilon_N) \right\} \\
\leq Q \left\{ \max_\omega Br \left( 1, \frac{Nu}{1 + u} \right) > \frac{N \sqrt{u}}{1 + u} (2 + \epsilon_N) \right\} \\
\leq C \exp \left\{ -\frac{1}{C} \frac{N \sqrt{u}}{1 + u} \epsilon_N^{3/2} \right\}.
\]

The last inequality follows from Ledoux [21], Section 2.1 (see also Proposition [5.3]). Taking \( \epsilon_N = N^{-\theta} \) with \( \theta > 0 \) small enough, and applying Borel-Cantelli, we conclude that, for \( N \) large enough,

\[
Z_N^{Br} \left( \frac{N}{1 + u}, \frac{Nu}{1 + u} \right) \leq \exp \left\{ 2 \beta \frac{N \sqrt{u}}{1 + u} + o(1) \right\},
\]
for all \( u \in I_N \). Now, thanks to (5.4), this is still true with \( Z_{Br}^{\ast} \) instead of \( Z_{Br}^{\ast} \). We then get

\[
Z_N \left( \frac{N}{1 + u}, \frac{Nu}{1 + u} \right) \leq \exp \left\{ 2 \beta \frac{N \sqrt{u}}{1 + u} + o(1) \right\},
\]
uniformly for \( u \in I_N \). Once the Third Step is achieved, this uniform bound and Laplace Method will finish the proof.

**Third Step:** We are now interested in values \( u \leq N^{\kappa_0} \) and \( u \geq N^{\kappa_1} \). Again we have to split the proof in three.
Let us first focus on small values of $u$. Recall that, in this region, by the KMT coupling, we can work directly with Gaussians. Take $\theta' > 0$.

$$
Q \left\{ \mathcal{Z}^g \left( \frac{N}{1+u} \cdot \frac{Nu}{1+u} \right) > e^{\beta N^{\theta'}} \right\} \leq Q \left\{ T^g \left( \frac{N}{1+u} \cdot \frac{Nu}{1+u} > N^{\theta'} \right) \right\}
$$

$$
\leq Q \left\{ \exists s \in \Omega \frac{N}{1+u} \cdot \frac{Nu}{1+u} : H(s) > N^{\theta'} \right\}
$$

$$
\leq |\Omega \frac{N}{1+u} \cdot \frac{Nu}{1+u}| \exp\{-N^{2\theta'-1}\}
$$

$$
\leq \exp\{cN(1+\kappa_0)\log N - N^{2\theta'-1}\}.
$$

So, choosing $\kappa_0$ small enough and $1 + \kappa_0/2 < \theta' < (1 + a)/2$, we get, by Borel-Cantelli and by a computation analogous to (5.4), that for $N$ large enough,

$$
\mathcal{Z}^g \left( \frac{N}{1+u} \cdot \frac{Nu}{1+u} = o \left( e^{N(1+a)/2} \right) \right),
$$

for all $u \leq N^{\kappa_0}$.

For $N^{(a-1)/2-\delta} \leq u \leq N^{(a-1)/2+\delta}$, we have to couple the environment row by row with Gaussians until $N - n = N(1+a)/2 + \delta$ (just conserve the coupling already done in Step 2 and add the missing rows). This will yield an error uniformly of order $N^{(1+a)/2+\delta} \log N$

The point is that for $\delta$ small enough, the drift will be large compared with the point-to-point partition functions and the error in the approximation. In fact,

$$
h \times (N - n) \geq \gamma N^{1-\delta}.
$$

Recall that we are working with Gaussians, denote $\Omega_{N,u} = \Omega \frac{N}{1+u} \cdot \frac{Nu}{1+u}$,

$$
Q \left\{ \max_{\omega \in \Omega_{N,u}} H_N(\omega) > N^{\theta'} \right\} \leq \exp(1 + a)N^{(1+a)/2+\delta} \log N - N^{2\theta'-1},
$$

and, by Borel-Cantelli (taking, of course, $(1 + a)/2 + \delta < 2\theta' - 1$),

$$
\mathcal{Z}^\beta \left( \frac{N}{1+u} \cdot \frac{Nu}{1+u} \right) e^{h \times (N-n)} \leq \exp \left\{ (1 + a)N^{(1+a)/2+\delta} \log N + \beta N^{\theta'} - \gamma N^{1-\delta} \right\},
$$

where, as usual, the overline denotes that the partition function is unnormalized and the superscript $g$ stands for Gaussian environment. To insure that the drift is larger than the other terms, we have to take $\theta' < 1 - \delta$ and $(1 + a)/2 + \delta < 1 - \delta$, both holding for $\delta < (1 - a)/4$ and $\theta$ small enough. Now, this is also enough to neglect the error in the approximation as it is of order $N^{(1+a)/2+\delta}$ too. The first condition we have encountered, namely $(1 + a)/2 + \delta < 2\theta' - 1$ is satisfied for $\delta < (1 - a)/6$ and $\theta' < 1 - \delta$, so that, choosing $\delta$ and $\theta$ according to these last restrictions gives that

$$
(5.5) \quad \mathcal{Z}_\beta \left( \frac{N}{1+u} \cdot \frac{Nu}{1+u} \right) e^{-h \times (N-n)} \rightarrow 0,
$$
as \( N \to +\infty \) uniformly for \( N^{(1-a)/2-\delta} \leq u \leq N^{(1-a)/2+\delta} \).

We are then left with the values \( u > N^{(a-1)/2+\delta} \). This is an easy task: we can dominate each point-to-point partition function by the whole partition function (without drift!):

\[
Z_N = Z_{\beta,N} = \sum_{\omega \in \Omega_N} e^{\beta H(\omega)},
\]

where \( \Omega_N \) is the set of directed nearest-neighbor paths of length \( N \). \( Z_N \) grows at most as \( e^{CN} \) for some constant \( C > \lambda(\beta) + \log 2d \), as we can see from

\[
Q(Z_N \geq e^{CN}) \leq e^{-CN} QZ_N = e^{(\lambda(\beta)+\log 2d-C)N}
\]

and Borel-Cantelli. Now, for the range of \( u \)’s we are considering, the drift satisfies,

\[
h(N - n) > N^{1+\delta'},
\]

for large \( N \), whenever \( \delta' < \delta \), and then (5.5) holds in this interval as well.

\[\square\]

5.2. Moderate deviations for the partition function. We now discuss the fluctuation of \( \log Z_{\beta,N}^{(h_N)} \). For technical reasons, we have to restrict to \( a < 1/3 \) for variable with finite exponential moments, and to \( a < 3/7 \) for Gaussian variables (see Remark 5.8 at the end of this section).

We start proving the two following deviation inequalities:

**Theorem 5.2.** For all \( a < 3/7 \), there exists a constant \( C > 0 \) such that, for all \( N \geq 1 \) and \( 0 \leq \epsilon \leq N^{1-a} \),

\[
Q \left\{ \log Z_{\beta,N}^{(h_N)} \geq \frac{\beta^2}{\gamma} N^{(1+a)/2} (1 + \epsilon) \right\} \leq C \exp \left\{ -\frac{N^a}{C} \epsilon^{3/2} \right\},
\]

and for \( a < 1/3 \), \( a < 3/7 \) for Gaussian disorder), \( 0 \leq \epsilon \leq 1 \),

\[
Q \left\{ \log Z_{\beta,N}^{(h_N)} \leq \frac{\beta^2}{\gamma} N^{(1+a)/2} (1 - \epsilon) \right\} \leq C \exp \left\{ -\frac{N^{2a}}{C} \epsilon^3 \right\}.
\]

These are consequences of similar non-asymptotics deviation inequalities for the top eigenvalue of GUE random matrices that we recall here in the context of Brownian percolation (see [23], Theorem 1 and [21], Chapter 2, for a complete discussion of this topic):

**Proposition 5.3.** There exists a constant \( C_1 > 0 \) such that, for all \( N \geq 1 \) and \( \epsilon \geq 0 \),

\[
Q \left\{ L(1, N) \geq 2\sqrt{N} (1 + \epsilon) \right\} \leq C_1 \exp \left\{ -\frac{N}{C_1} \epsilon^{3/2} \right\},
\]

and for \( 0 \leq \epsilon \leq 1 \),
\begin{equation}
Q \left\{ L(1, N) \leq 2\sqrt{N}(1 - \epsilon) \right\} \leq C \exp \left\{ -\frac{N^2}{C} \epsilon^3 \right\}.
\end{equation}

We first transfer these inequalities to the LPP context:

**Proposition 5.4.** For $M = O(N^a)$ with $a < 3/7$, there exists a constant $C_a > 0$ such that, for all $\epsilon \geq 0$,

\begin{equation}
Q \left\{ T(N, M) \geq 2\sqrt{NM}(1 + \epsilon) \right\} \leq C_a \exp\{-Me^{3/2}/C_a}\).
\end{equation}

and, for $a < 1/5$ (a $< 1/2$ if the environment is Gaussian) and $0 \leq \epsilon \leq 1$,

\begin{equation}
Q \left\{ T(N, M) \leq 2\sqrt{NM}(1 - \epsilon) \right\} \leq C_a \exp\{-M^2\epsilon^3/C_a\}.
\end{equation}

**Remark 5.5.** The result for Gaussian variables can be found in [13]. The exponential case is covered in the Thesis of the same author. We present here a rather complete proof of the exponential case using KMT, both for completeness and to state some inequalities that will be used in the following.

The core of the proof of Proposition 5.4 consists in the following Lemma, whose proof is a simple application of the KMT coupling (Theorem 4.3):

**Lemma 5.6.** There exist positive constants $C_2, C_3$ such that,

\[ Q \left\{ \exp\{|T(N, M) - T^g(N, M)|\} \right\} \leq e^{C_2 M \log N}, \]

\[ Q \left\{ \exp\{M^{-1}|L(N, M) - T^g(N, M)|^2\} \right\} \leq e^{C_3 M \log N}, \]

where $T^g(N, M)$ is the discrete Gaussian last passage functional given by the KMT approximation.

**Proof.** By the KMT approximation,

\[ Q \left\{ e^{(|T(N, M) - T^g(N, M)|)} \right\} \leq Q \left\{ \exp\{2 \sum_{i=0}^{M} \sup_{j \leq N} \sum_{k=0}^{j} \eta(k, i) - B_j^{(i)}(1) \} \right\} \]

\[ \leq \left( Q \left\{ \exp\{2 \sup_{j \leq N} \sum_{k=0}^{j} \eta(k, 1) - B_j^{(1)}(1) \} \right\} \right)^M \]

\[ \leq e^{C_1 M \log N}. \]

For the second affirmation, remember that in the second step of the proof of Theorem 4.3 we noticed that
\[(5.12) \quad Q \left\{ \sup_{0 \leq s, t \leq N+1, |s-t| \leq 2} |B_{s} - B_{t}| > x \right\} \leq C_{4} e^{-x^2/C_{4}}, \]

for some \( C_{4} > 0 \). Then,

\[ Q \left\{ \exp \left\{ M^{-1} |L(N, M) - T^{g}(N, M)|^2 \right\} \right\} \leq \left( \exp \left\{ 2 \left( \sup_{1 \leq s, t \leq N, |s-t| \leq 2} |B_{s}^{(1)} - B_{t}^{(1)}| \right)^2 \right\} \right)^{M} \leq e^{C_{3} M \log N} \]

where we use (5.12) in the last step. \( \Box \)

The following Corollary is now straightforward:

**Corollary 5.7.** For any sequence \((\theta_{N})_{N}\) with \( \theta_{N} > 0 \), we have

\[(5.13) \quad Q \{ |T(N, M) - T^{g}(N, M)| \geq \theta_{N} \} \leq e^{-\theta_{N} + C_{2} M \log N}, \]

\[(5.14) \quad Q \{ |L(N, M) - T^{g}(N, M)| \geq \theta_{N} \} \leq e^{-M^{-1} \theta_{N}^{2} + C_{3} M \log N}. \]

**Proof of Proposition 5.4**

Let us prove (5.10) ((5.11) is proved following the same lines). Remember \( T^{g}(N, M) \) denotes the last passage percolation functional for the Gaussian environment given by the KMT coupling. Then,

\[ Q \left\{ T(N, M) \geq 2 \sqrt{NM}(1 - \epsilon) \right\} \leq Q \left\{ L(N, M) \geq 2 \sqrt{NM}(1 - \epsilon/2) \right\} + Q \left\{ |T(N, M) - T^{g}(N, M)| \geq \frac{\epsilon}{2} \sqrt{NM} \right\} + Q \left\{ |L(N, M) - T^{g}(N, M)| \geq \frac{\epsilon}{2} \sqrt{NM} \right\} \]

The first term can be treated using (5.8) and Brownian scaling:

\[ Q \left\{ L(N, M) \geq 2 \sqrt{NM}(1 - \epsilon/2) \right\} \leq C_{1} e^{-N^{a} \epsilon^{3/2}/C_{1}}. \]

The remaining term can be treated with Corollary 5.7 taking \( \theta_{N} = \epsilon/2 \sqrt{NM} \).

The careful analysis performed in [13], Section 5, allows us to choose the uniform constant in (5.10). \( \Box \)

We turn now to the proof of Theorem 5.2.

**Proof of the Inequality 5.7.** This follows by lowering the partition function by one term: recall that \( u^{*} \sim \beta^{2}/\gamma N^{a-1} \), and define \( n^{*} = N/(1 + u^{*}) \). Then,
\[
Q \left\{ \log Z_{\beta,N}^{(h_N)} \leq \frac{\beta^2}{\gamma} N^{(1+a)/2} (1 - \epsilon) \right\}
\]
\[
\leq Q \left\{ \beta T(n^*, N - n^*) - h_N \times (N - n^*) \leq \frac{\beta^2}{\gamma} N^{(1+a)/2} (1 - \epsilon) \right\},
\]

Observe that
\[
h_N \times (N - n^*) = \frac{\beta^2}{\gamma} N^{(1+a)/2}.
\]

We are then reduced to estimate the quantity
\[
(5.15) \quad Q \left\{ T(n^*, N - n^*) \leq \frac{2\beta}{\gamma} N^{(1+a)/2} (1 - \epsilon/2) \right\}.
\]

which can be handled with (5.11). \(\square\)

**Proof of the inequality 5.6** This proof is more involved as it requires to control all the terms in the sum defining \(Z_{\beta,N}^{(h_N)}\). Again, we need to give a special treatment to the terms for which \(N - n\) is not of the relevant order (namely \(O(N^a)\)). We use the convenient parametrization \(N - n = vN^a\) for some \(v \geq 0\), i.e., \(u = vN^{a-1}\). To lighten notation, let us denote
\[
q(\epsilon, v) = Q \left\{ \beta T(N, vN^a) - \gamma v N^{(1+a)/2} \geq \frac{\beta^2}{\gamma} N^{(1+a)/2} (1 + \epsilon) \right\}.
\]

Several cases have to be analyzed separately:

**Case** \(v \leq \beta^2/(2\gamma)^2\): We use the fact that, for these values of \(v\), \(T(N, vN^a)\) is stochastically dominated by \(T(N, \beta^2/(2\gamma)^2 N^a)\). Then, neglecting the term \(\gamma v\),
\[
q(v, \epsilon) \leq Q \left\{ T(N, \beta^2/(2\gamma)^2 N^a) \geq \frac{\beta}{\gamma} N^{(1+a)/2} (1 + \epsilon) \right\}
\]
\[
\leq C_a \exp \left\{ -\frac{\beta^2 N^a}{4\gamma^2 C_a^2} \epsilon^{3/2} \right\}.
\]

**Case** \(\beta^2/(2\gamma)^2 \leq v \leq 16\beta^2/\gamma^2\): We make use of the fact that \(2\beta\sqrt{v} - \gamma v \leq \beta^2/\gamma\) for all \(v \geq 0\) and that, for these values of \(v\), we have \(1/\sqrt{v} \geq \gamma/(16\beta)\). Then,
\[ q(\epsilon, v) \leq Q \left\{ \beta T(N, vN^a) \geq 2\beta \sqrt{vN^{(1+a)/2}} + \frac{\beta^2 \epsilon}{\gamma} N^{(1+a)/2} \right\} \]
\[ \leq Q \left\{ \beta T(N, vN^a) \geq 2\beta \sqrt{vN^{(1+a)/2}} \left( 1 + \frac{\beta \epsilon}{2\gamma \sqrt{v}} \right) \right\} \]
\[ \leq Q \left\{ \beta T(N, vN^a) \geq 2\beta \sqrt{vN^{(1+a)/2}} \left( 1 + \frac{\epsilon}{32} \right) \right\} \]
\[ \leq C_a \exp \left\{ -\sqrt{\frac{N^a}{C_a}} \epsilon^{3/2} \right\} \]
\[ \leq C_a \exp \left\{ -\frac{\beta^2 N^a}{\gamma^2 C_a} \epsilon^{3/2} \right\} , \]

thanks to the lower bound we assumed on \( v \). So far, we made no additional assumption on \( \epsilon \).

**Case 16**: \( \beta^2 / \gamma^2 \leq v \):

\[ q(\epsilon, v) \leq Q \left\{ \beta L(N, vN^a) - \frac{\gamma v}{2} N^{(1+a)/2} \geq \frac{\beta^2}{2\gamma} N^{(1+a)/2} (1 + \epsilon) \right\} \]
\[ + Q \left\{ \beta |T(N, M) - T^g(N, M)| - \frac{\gamma v}{4} N^{(1+a)/2} \geq \frac{\beta^2}{4\gamma} N^{(1+a)/2} (1 + \epsilon) \right\} \]
(5.16)
\[ + Q \left\{ \beta |L(N, M) - T^g(N, M)| - \frac{\gamma v}{4} N^{(1+a)/2} \geq \frac{\beta^2}{4\gamma} N^{(1+a)/2} (1 + \epsilon) \right\} \]
(5.17)

Let us treat the first summand: assume that \( K \beta^2 / \gamma^2 < v \leq (K + 1) \beta^2 / \gamma \), for some \( K \geq 16 \). Then,

\[ \frac{\beta^2}{2\gamma} (1 + \epsilon) + K \frac{\beta^2}{2\gamma} \geq \frac{2\beta^2}{\gamma} \sqrt{K + 1} \left( 1 + \frac{\epsilon}{4\sqrt{K + 1}} \right) . \]

So, recalling that \( L(N, (K + 1) \beta^2 N^a / \gamma^2) \) stochastically dominates \( L(N, vN^a) \) for these values of \( v \),

\[ Q \left\{ \beta L(N, vN^a) \geq \left( \frac{\beta^2}{2\gamma} (1 + \epsilon) + K \frac{\beta^2}{2\gamma} \right) N^{(1+a)/2} \right\} \]
\[ \leq Q \left\{ L \left( N, (K + 1) \frac{2\beta}{\gamma^2} N^a \right) \geq \frac{2\beta}{\gamma} \sqrt{K + 1} N^{(1+a)/2} \left( 1 + \frac{\epsilon}{4\sqrt{K + 1}} \right) \right\} \]
\[ \leq C_1 \exp \left\{ -\frac{1}{C_1} (K + 1) \frac{\beta^2}{\gamma^2} N^a \left( \frac{\epsilon}{4\sqrt{K + 1}} \right)^{3/2} \right\} \]
\[ \leq C_5 \exp \left\{ -\frac{1}{C_5} N^a \epsilon^{3/2} \right\} . \]
The remaining terms (5.16) and (5.17) can be handle with (5.13) and (5.14) respectively. We then get

\[ q(\epsilon, v) \leq C_5 \exp \left\{ -\frac{1}{C_5} N^a \epsilon^{3/2} \right\} + \exp \left\{ -c_1 v N^{(1+a)/2} - c_2 N^{(1+a)/2} (1 + \epsilon) \right\} + \exp \left\{ -c_3 v N - c_4 v^{-1} (1 + \epsilon)^2 \right\} \leq C_5 \exp \left\{ -\frac{1}{C_5} N^a \epsilon^{3/2} \right\} + \exp \left\{ -c'_2 \epsilon N^{(1+a)/2} \right\} + \exp \left\{ -c_2 v N - c'_4 v^{-1} N \right\} \leq C_5 \exp \left\{ -\frac{1}{C_5} N^a \epsilon^{3/2} \right\} + c_5 \exp \left\{ -\frac{1}{c_5} \epsilon N^{(1+a)/2} \right\}. \]

Observe that, for \( 0 \leq \epsilon \leq N^{1-a} \), there is a constant \( C_6 > 0 \) such that

\[ q(\epsilon, v) \leq C_6 \exp \left\{ -\frac{1}{C_6} N^a \epsilon^{3/2} \right\}, \]

which ends the proof. \( \square \)

Let us observe that, for \( \epsilon > N^{1-a} \), there is a constant \( C_7 > 0 \) such that

\[ (5.18) \quad q(\epsilon, v) \leq C_7 \exp \left\{ -\frac{1}{C_7} \epsilon N^{(1+a)/2} \right\}. \]

5.3. **Fluctuation bounds.** We can now complete the proof of Theorem 1.5. The argument to deduce the fluctuation bounds from our moderate deviations is very general and can be found in [21] in the context of random matrices. To lighten notations, let us denote

\[ X_N = \log Z_{B,N}^{(h_N)}, \quad x_N = \frac{\beta^2}{\gamma} N^{(1+a)/2}. \]

The upper bound follows from the previous deviation inequalities by a direct computation:
\[ Q(X_N - x_N)^2 = \int_0^{+\infty} Q \left\{ (X_N - x_N)^2 \geq t \right\} dt \]
\[ \leq \int_0^{+\infty} Q \left\{ X_N - x_N \geq \sqrt{t} \right\} dt + \int_0^{X_N} Q \left\{ X_N - x_N \leq -\sqrt{t} \right\} dt \]
\[ = 2N^{1+a} \int_0^{+\infty} u Q \left\{ X_N \geq \frac{\beta^2}{\gamma} (1 + u) \right\} du \]
\[ + 2N^{1+a} \int_0^{\beta^2/\gamma} u Q \left\{ X_N \leq \frac{\beta^2}{\gamma} (1 - u) \right\} du \]

Let us bound the first integral. The second one can be treated in the same way. We apply (5.6) from Theorem 5.2 and split the interval of integration:

\[ \int_0^{+\infty} u Q \left\{ X_N \geq \frac{\beta^2}{\gamma} (1 + u) \right\} du = C \int_0^1 u e^{-\frac{1}{N^a}u^{3/2}} du + C_7 \int_1^{+\infty} u e^{-\frac{1}{N^a}u^{1+a/2}} du. \]

(5.19)

The second integral in this last display is easily seen to decrease as \( \exp \{-N^{(1+a)/2}\} \). For the first integral, observe that the integrand is \( O(N^{-2a/3}) \) in \([0, N^{-2a/3}]\) and decreases exponentially fast outside this interval. Then,

\[ \int_0^{1} u e^{-\frac{1}{N^a}u^{3/2}} du \leq C' e^{-N^{-a/3}}, \]

for some \( C' > 0 \). Putting this back into (5.19), we found

\[ \int_0^{+\infty} u Q \left\{ X_N \geq \frac{\beta^2}{\gamma} (1 + u) \right\} du \leq C e^{N^{1-a/3}}. \]

As we already mentioned, the deviations on the left of the mean can be treated similarly. This gives the upper bound. For the lower bound, observe that

\[ \beta T(n^*, N - n^*) - h_N \times (N - n^*) \sim \beta T(N, \frac{\beta^2}{\gamma^2} N^a) - \frac{2\beta^2}{\gamma} N^{(1+a)/2}. \]

Then, applying Jensen’s inequality,

\[ Q \left\{ \left( \log Z_{\beta,N}^{(h_N)} - \frac{\beta^2}{\gamma} N^{(1+a)/2} \right)^2 \right\} \geq \left( Q \left\{ \log Z_{\beta,N}^{(h_N)} - \frac{\beta^2}{\gamma} N^{(1+a)/2} \right\} \right)^2 \]
\[ \geq \left( Q \left\{ \beta T(N, \frac{\beta^2}{\gamma^2} N^a) - \frac{2\beta^2}{\gamma} N^{(1+a)/2} \right\} \right)^2 \]
Now, recall [5] that
\[
T(N, \beta^2/\gamma^2 N^a) - 2\beta^2/\gamma N^{(1+a)/2} \over N^{(\frac{1}{2} - \frac{a}{2})}
\]
converges in law to a Tracy-Widom. Then, recalling that the Tracy-Widom law has a strictly positive expected value,
\[
\left( Q \left\{ \beta T(N, \beta^2/\gamma^2 N^a) - 2\beta^2/\gamma N^{(1+a)/2} \right\} \right)^2 \geq cN^{1-a/3},
\]
for some \( c > 0 \). This ends the proof.

\[\square\]

**Remark 5.8.** Again, the condition \( a < 1/5 \) seems to be a technical limitation due to our use of the KMT approximation. For a more extensive discussion on asymptotics and non-asymptotics small deviations for asymmetric last-passage percolation, see [13].

**Remark 5.9.** The limit law of the properly centered and rescaled partition function should be the GUE Tracy-Widom law from random matrix theory. A proof of this fact would need to refine the analysis performed in the first section of this chapter to reduce the relevant values of \( u \)'s to an interval \([cN^{1-a} - \epsilon_N, cN^{1-a} - \epsilon_N]\) with \( c = \beta^2/\gamma^2 \) and \( \epsilon_N \to 0 \) fast enough. This can be done without much effort, but, in order to identify the limit law as the Tracy-Widom, we also need a joint control of expressions of the form
\[
L(N - cN^a - sN^{2a/3}, N + sN^{2a/3})
\]
for \( s \) ranging over a large interval. The result we are searching for can be expressed as follows: for \( s \in \mathbb{R} \)
\[
N^{1/6} \left\{ L(N - sN^{2/3}, N + sN^{2/3}) - 2N \right\} \to \text{Ai}(s) - s^2
\]
where \( \text{Ai}(\cdot) \) is a continuous version the Airy process. This is a stationary process which marginals are the Tracy-Widom law. See [17] for a related result and a precise description of the Airy process.

**References**

[1] Anderson, D., Guionnet, A., Zeitouni, O. (2009) *An Introduction to Random Matrices*, Cambridge Studies in Advanced Mathematics, Cambridge University Press.

[2] Baik, J., Suidan, T. (2005) *A GUE central limit theorem and universality of directed first and last passage percolation site percolation*, Int. Math. Res. Not., 6, 325-337.

[3] Baryshnikov, Yu. (2001) *GUEs and Queues*, Probab Theory Relat. Fields 119 2, 256-274.

[4] Ben-Ari, I. (2009) *Large deviations for partition functions of directed polymers in an IID field*, Ann. Inst. H. Poincaré Probab. Statist. 45, 3, 770-792.

[5] Bodineau, T., Martin, J. (2005) *Universality for last passage percolation close to an axis*, Elec. Comm. Prob., 105-112, June 9.

[6] Burke, P. (1956) *The output of a queuing system*, Oper. Res. 4 6, 699-704.

[7] F. Comets, T. Shiga, and N. Yoshida *Probabilistic Analysis of Directed Polymers in a Random Environment: a Review*, Adv. Stud. Pure Math. 39 (2004), 115–142.

[8] Comets, F., Yoshida, N. (2009) *Branching random walks in a random environment: survival probability, global and local growth rates*, preprint.

[9] Dembo, A., Zeitouni, O. (1999) *On increasing subsequences of i.i.d. samples*, Combin. Comput. Probab. 8, 247-263.

[10] Glynn, P., Whitt, W. (1991) *Departures from many queues in serie* Annals of Applied Prob. 1, 4, 546-572.
[11] Henley, C., Huse, D. (1985) Pinning and roughening of domain wall in Ising systems due to random impurities, Phys. Rev. Lett. **54**, 2708-2711.

[12] Hambly, B., Martin, J., O’Connell, N. (2002) Concentration results for a Brownian directed percolation problem, Stoch. Proc. and App. **102**, 207-220.

[13] Ibrahim, J-P (2007) Large deviations for directed last passage percolation on a thin box, to appear in ESAIM: probability and statistics.

[14] Imbrie, J.Z., Spencer, T. (1988) Diffusion of directed polymers in random environment, J. Stat. Ph. **52** 3/4, 609-626.

[15] Johansson, K. (2000) Transversal fluctuations for increasing subsequences on the plane, Probab. Theor. Rel. Fields **116**, 445-456.

[16] Johansson, K. (2000) Shape fluctuation and random matrices, Comm. Math. Phys. **209**, 2, 437-476.

[17] Johansson, K. (2003) Discrete Polynuclear Growth and Determinantal Processes, Comm. Math. Phys. **242**, 277-329.

[18] Komlos, Major, Tusnady (1975) An approximation of partial sums of RV’s, and the sample DF, I, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **32**, 111–131.

[19] Komlos, Major, Tusnady (1976) An approximation of partial sums of RV’s, and the sample DF, II, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **34**, 33–58.

[20] Ledoux, M. (2007) Isoperimetry and Gaussian analysis, Lecture Notes in Math. **1648**, 167-219.

[21] Ledoux, M. (2007) Deviation inequalities on largest eigenvalues, Lecture Notes in Math. **1910**, 165-294.

[22] Liu, Q., Watbled, F. (2009) Large deviation inequalities for supermartingales and applications to directed polymers in a random environment C. R. Acad. Sci. Paris, Ser. I 347.

[23] Ledoux, M., Rider, B. (2009) Small deviations for beta ensembles, preprint.

[24] Licea, C., Newman, C., Piza, M. (1996) Superdiffusivity in first-passage percolation, Probab. Theory Relat. Fields **106**, 559–591.

[25] Martin, J. (2004) Limiting shape for directed percolation models Ann. Probab. **32**, 4, 2908-2937.

[26] Moreno, G. (2010) Modèles de polymères dirigés en milieux aléatoires, PhD Thesis, Université Paris 7.

[27] Moriarty, O’Connell (2007) On the free energy of a directed polymer in a brownian environment, Markov Processes and Related Fields **13**, 251-266

[28] O’Connell, N. (2009) Directed polymers and the quantum Toda lattice, preprint.

[29] O’Connell, N., Yor, M. (2001) Brownian analogues of Burke’s theorem, Stoch. Proc. and Appl., **96**, 285-304

[30] Rovira, C., Tindel, S. (2005) On the Brownian directed polymer in a Gaussian random environment, Journal of Functional Analysis **222**, 178-201.

[31] Seppalainen, T. (1997) A scaling limit for queues in series, The Annals of Applied Probability **7**, 4, 855-872.

[32] Seppalainen, T. (1998) Large deviations for increasing sequences on the plane, Probab. Theory Relat. Fields **112**, 221-244.

[33] Seppalainen, T. (2009) Scaling for a one-dimensional directed polymer with boundary conditions, preprint. Fund. Math. **147**, 173-180. J. Statist. Phys., **1/2**, 277-289.

[34] Tracy, C., Widom, H. (1993) Level spacing distributions and the Airy kernel, Phys. Lett. **305**, 1-2, 115-118.

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