Machine Learning on Sequential Data Using a Recurrent Weighted Average

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Abstract

Recurrent Neural Networks (RNN) are a type of statistical model designed to handle sequential data. The model reads a sequence one symbol at a time. Each symbol is processed based on information collected from the previous symbols. With existing RNN architectures, each symbol is processed using only information from the previous processing step. To overcome this limitation, we propose a new kind of RNN model that computes a recurrent weighted average (RWA) over every past processing step. Because the RWA can be computed as a running average, the computational overhead scales like that of any other RNN. The approach essentially reformulates the attention mechanism into a stand-alone model. When assessing a RWA model, it is found to train faster and generalize better than a standard LSTM model when performing the variable copy problem, the adding problem, classification of artificial grammar, classification of sequences by length, and classification of MNIST handwritten digits (where the pixels are read sequentially one at a time).

1. Introduction

Types of information as dissimilar as language, music, and genomes can be represented as sequential data. The essential property of sequential data is that the order of the information is important, which is why statistical algorithms designed to handle this kind of data must be able to process each symbol in the order that it appears. Recurrent neural network (RNN) models have been gaining interest as a statistical tool for dealing with the complexities of sequential data. The essential property of a RNN is the use of feedback connections. The sequence is read by the RNN one symbol at a time through the model’s inputs. The RNN starts by reading the first symbol and processing the information it contains. The processed information is then passed through a set of feedback connections. Every subsequent symbol read into the model is processed based on the information conveyed through the feedback connections. Each time another symbol is read, the processed information of that symbol is used to update the information conveyed in the feedback connections. The process continues until every symbol has been read into the model (Fig. 1a). The processed information is passed along each step like in the game telephone (a.k.a. Chinese whispers). With each

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step, the RNN produces an output that serves as the model’s prediction. The challenge of designing a working RNN is to make sure that processed information does not decay over the many steps. Error correcting information must also be able to backpropagate through the same pathways without degradation (Hochreiter, 1991; Bengio, Simard, and Frasconi, 1994). Hochreiter and Schmidhuber were the first to solve these issues by equipping a RNN with what they called long short-term memory (LSTM) (Hochreiter and Schmidhuber, 1997).

Since the introduction of the LSTM model, several improvements have been proposed. The attention mechanism is perhaps one of the most significant (Bahdanau, Cho, and Bengio, 2014). The attention mechanism is nothing more than a weighted average. At each step, the output from the RNN is weighted by a context model, creating a weighted output. The weighted outputs are then aggregated together by computing a weighted average (Fig. 1b). The outcome of the weighted average is used as the model’s result. The context model controls the relative contribution of each output, determining how much of each output is “seen” in the results. The attention mechanism has since been incorporated into several other neural network architectures leading to a variety of new models each specifically designed for a single task (partial reference list: Bahdanau et al. 2014; Vinyals et al. 2015b; Xu et al. 2015; Sønderby et al. 2015; Chan et al. 2015; Vinyals et al. 2015a). Unfortunately, the attention mechanism is not defined in a recurrent manner. The recurrent connections must come from a separate RNN model, restricting where the attention mechanism can be used.

Inspired by the attention mechanism used to modify existing neural network architectures, we propose a new kind of RNN that is created by reformulating the attention

Figure 1: Comparison of models for classifying sequential data. (a) Standard RNN architecture with LSTM requires that information contained in the first symbol $x_1$ pass through the feedback connections repeatedly to reach the output $h_t$, like in a game of telephone (a.k.a. Chinese whispers). (b) The attention mechanism aggregates the outputs into a single state by computing a weighted average. It is not recursively defined. (c) The proposed model incorporates pathways to every previous processing step using a recurrent weighted average (RWA). By maintaining a running average, the computational cost scales like that of other RNN models.
mechanism into a stand-alone model. The proposed model includes feedback connections from every past processing step, not just the preceding step (Fig. 1c). The feedback connections from each processing step are weighted by a context model. The weighted feedback connections are then aggregated together by computing a weighted average. The model is said to use a recurrent weighted average (RWA) because the context model also uses the feedback connections. At each step, the weighted average can be computed as a running average. By maintaining a running average, the model scales like any other type of RNN.

2. The RWA Model

2.1 Mathematical Description

The proposed model is defined recursively over the length of the sequence. Starting at the first processing step \( t = 1 \) a set of values \( h_0 \), representing the initial state of the model, is required. The distribution of values for \( h_0 \) must be carefully chosen to ensure that the initial state resembles subsequent processing steps. This is accomplished by defining \( h_0 \) in terms of \( s_0 \), a parameter that must be fitted to the data.

\[
h_0 = f(s_0)
\]  

The parameters \( s_0 \) are passed through the model’s activation function \( f \) to mimic the processes that generate the outputs of the later processing steps.

For every processing step that follows, a weighted average computed over every previous step is passed through the activation function \( f \) to generate a new output \( h_t \) for the model. The equation for the model is given below.

\[
h_t = f\left( \sum_{i=1}^{t} z(x_i, h_{i-1}) \circ e^{a(x_i, h_{i-1})} \right) / \sum_{j=1}^{t} e^{a(x_j, h_{j-1})}
\]  

The weighted average consists of two models: \( z \) and \( a \). The model \( z \) encodes the features \( x_i \) for each symbol in the sequence. Its recurrence relations, represented by \( h_{i-1} \), provide the context necessary to encode the features in a sequence dependent manner. The model \( a \) serves as the context model, determining the relative contribution of \( z \) at each processing step. The exponential terms of model \( a \) are normalized in the denominator to form a proper weighted average. The recurrent relations in \( a \), represented by \( h_{i-1} \), are required to compose the weighted average recursively. Because the weighted average is recursively defined, the model is said to use a RWA.

It is important that the recurrence relations in both \( z \) and \( a \) remain stable (i.e. the signal and gradients do not vanish or explode). The recurrence relations contained in \( a \) are stable because the exponential terms are normalized. The recurrence relations in \( z \) can be made stable by bounding \( h_{i-1} \), which must be done independently of \( x_i \) so that it can remain unbounded. This is accomplished by separating the model for \( z \) into a bounded component that includes the recurrent terms \( h_{i-1} \) and an unbounded component containing only \( x_i \).

\[
z(x_i, h_{i-1}) = u(x_i) \circ \tanh(g(x_i, h_{i-1}))
\]
The recurrent relations represented in the model for $g$ are confined between $[-1, 1]$ by the tanh function. The model for $g$ can control the sign of $z$ but cannot cause its absolute magnitude to increase. The model for $u$ contains only the features $x_i$ and is unbounded.

The terms $u$, $g$, and $a$ can be modelled as feed-forward linear networks.

\[
\begin{align*}
    u(x_i) &= W_u \cdot x_i + b_u \\
    g(x_i, h_{i-1}) &= W_g \cdot [x_i, h_{i-1}] + b_g \\
    a(x_i, h_{i-1}) &= W_a \cdot [x_i, h_{i-1}] + b_a
\end{align*}
\]

The matrices $W_u$, $W_g$, and $W_a$ represent the weights of the feed-forward networks, and the vectors $b_u$, $b_g$, and $b_a$ represent the bias terms.

While running through a sequence, the output $h_t$ from each processing step can be passed through a fully connected neural network layer to predict a label. Gradient descent based methods can then be used to fit the model parameters, minimizing the error between the true and predicted label.

### 2.2 Running Average

The RWA in equation (2) is recalculated from the beginning at each processing step. The first step to reformulate the model as a running average is to separate the RWA in equation (2) as a numerator term $n_t$ and denominator term $d_t$.

\[
\begin{align*}
    n_t &= \sum_{i=1}^{t} z(x_i, h_{i-1}) \circ e^a(x_i, h_{i-1}) \\
    d_t &= \sum_{j=1}^{t} e^a(x_j, h_{j-1})
\end{align*}
\]

Because any summation can be rewritten as a recurrence relation, the summations for $n_t$ and $d_t$ can be defined recurrently (see Appendix A). Let $n_0 = 0$ and $d_0 = 0$.

\[
\begin{align*}
    n_t &= n_{t-1} + z(x_t, h_{t-1}) \circ e^a(x_t, h_{t-1}) \\
    d_t &= d_{t-1} + e^a(x_t, h_{t-1})
\end{align*}
\]

By maintaining the previous state of the numerator $n_{t-1}$ and denominator $d_{t-1}$, the values for $n_t$ and $d_t$ can be efficiently computed using the work done during the previous processing step. The output $h_t$ from equation (2) can now be obtained from the relationship listed below.

\[
h_t = f\left(\frac{n_t}{d_t}\right)
\]

Using this formulation of the model, the RWA can efficiently be computed dynamically.
2.3 Equations for Implementation

The model can be implemented using equations (1) and (3)–(6), which are collected together and written below.

\[h_0 = f(s_0), \quad n_0 = 0, \quad s_0 = 0\]
\[u(x_t) = W_u \cdot x_t + b_u\]
\[g(x_t, h_{t-1}) = W_g \cdot [x_t, h_{t-1}] + b_g\]
\[a(x_t, h_{t-1}) = W_a \cdot [x_t, h_{t-1}] + b_a\]
\[z(x_t, h_{t-1}) = u(x_t) \circ \tanh g(x_t, h_{t-1})\]
\[n_t = n_{t-1} + z(x_t, h_{t-1}) \circ e^{a(x_t, h_{t-1})}\]
\[d_t = d_{t-1} + e^{a(x_t, h_{t-1})}\]
\[h_t = f\left(\frac{n_t}{d_t}\right)\] (7)

Starting from the initial conditions, the model is run recursively over an entire sequence. The features for every symbol in the sequence are contained in \(x_t\), and the parameters \(s_0\), \(W_u\), \(b_u\), \(W_g\), \(b_g\), \(W_a\), and \(b_a\) are determined by fitting the model to a set of training data. Appendix B provides a description of how to efficiently compute the gradients of the model required for parameter optimization.

3. Experiments

3.1 Implementations of the Models

A RWA model is implemented in TensorFlow using the equations in (7) (Abadi et al., 2016). The model is trained and tested on five different classification tasks each described separately in the following subsections.

The same configuration of the RWA model is used on each dataset. The activation function is \(f(x) = \tanh x\) and the model contains 250 units. Following general guidelines for initializing the parameters of any neural network, the initial weights in \(W_u\), \(W_g\), and \(W_a\) are drawn at random from the uniform distribution \([-\sqrt{3} \frac{(N_{in}+N_{out})}{2}, \sqrt{3} \frac{(N_{in}+N_{out})}{2}]\) and the bias terms \(b_u\), \(b_g\), and \(b_a\) are initialized to 0’s (Glorot and Bengio, 2010). The initial state \(s_0\) for the RWA model is drawn at random from a normal distribution according to \(s_0 \sim \mathcal{N}(\mu = 0, \sigma^2 = 1)\) \(^1\). To avoid overflow errors, the numerator and denominator terms in equations (7) are rescaled using equations (9) in Appendix C, which do not alter the model’s output.

The datasets are also scored on a LSTM model that contains 250 cells to match the number of units in the RWA model. Following the same guidelines used for the RWA model, the initial values for all the weights are drawn at random from the uniform distribution \([-\sqrt{3} \frac{(N_{in}+N_{out})}{2}, \sqrt{3} \frac{(N_{in}+N_{out})}{2}]\) and the bias terms are initialized to 0’s with the exception

\(^1\) On each task, the initial state for \(s_0\) was drawn from a normal distribution with a variance of \(\sigma^2 = 3\) instead of \(\sigma^2 = 1\). The performance of the RWA model is not expected to have been severely impacted.
of the forget gates (Glorot and Bengio, 2010). The bias terms of the forget gates are initialized to 1’s, since this has been shown to enhance the performance of LSTM models (Glorot and Bengio, 2010; Gers, Schmidhuber, and Cummins, 2000). All initial cell states of the LSTM model are 0.

A fully connected neural network layer transforms the output from the 250 units into a predicted label. The error between the true label and predicted label is then minimized by fitting the model’s parameters using ADAM optimization (Kingma and Ba, 2014). All values for the ADAM optimizer follow published recommended settings. A step size of 0.001 is used throughout this study, and the other optimizer settings are $\beta_1 = 0.9$, $\beta_2 = 0.999$, and $\epsilon = 10^{-8}$. Each parameter update consists of a batch of 100 training examples. Gradient clipping is not used.

Each model is immediately scored on the test set and no negative results are omitted. No hyperparameter search is done and no regularization is tried. At every 100 steps of training, a batch of 100 test samples are scored by each model to generate values that are plotted in the figures for each of the tasks described below. The code and results for each experiment may be found online (see: https://github.com/jostmey/rwa).

3.2 Classifying Artificial Grammar

It is important that a model designed to process sequential data exhibit a sensitivity to the order of the symbols. For this reason, the RWA model is tasked with proofreading the syntax of sentences generated by an artificial grammatical system. Whenever the sentences are valid with respect to the artificial grammar, the model must return a value of 1, and whenever a typo exists the model must return a value of 0. This type of task is considered especially easy for standard RNN models, and is included here to show that the RWA model also performs well at this task (Hochreiter and Schmidhuber, 1997).

The artificial grammar generator is shown in Figure 2a. The process starts with the arrow labeled $B$, which is always the first letter in the sentence. Whenever a node is encountered, the next arrow is chosen at random. Every time an arrow is used the associated

![Figure 2](https://github.com/jostmey/rwa)

Figure 2: (a) The generator function used to create each sentence. Examples of valid and invalid sentences are shown below. (b) A plot comparing the performance of the RWA and LSTM models. The traces show the accuracy of each model on the test data while the models are being fitted to the training data. The RWA model reaches 100% accuracy before the LSTM model.
letter is added to the sentence. The process continues until the last arrow $E$ is used. All valid sentences end with this letter. Invalid sentences are constructed by randomly inserting a typo along the sequence. A typo is created by an invalid jump between unconnected arrows. No more than one typo is inserted per sentence. Typos are inserted into approximately half the sentences. Each RNN model must perform with greater than 50% accuracy to demonstrate it has learned the task.

A training set of 100,000 samples are used to fit the model, and a test set of 10,000 samples are used to evaluate model performance. The RWA model does remarkably well, achieving 100% accuracy in just a few hundred training steps. The LSTM model also learns to identify valid sentences, but requires many times more training steps to achieve the same performance (Fig 2b.). This task demonstrates that the RWA model can classify patterns based on the order of information.

### 3.3 Classifying by Sequence Length

Classifying sequences by length requires that RNN models track the number of symbols contained in each sequence. For this task, the length of each sequence is randomly drawn from a uniform distribution over every possible length 0 to $T$, where $T$ is the maximum possible length of the sequence. Each step in the sequence is populated with a random number drawn from a unit normal distribution (i.e. $\mu = 0$ and $\sigma^2 = 1$). Sequences greater than length $T/2$ are labeled with 1 while shorter sequences are labeled with 0. The goal is to predict these labels, which indicates if a RNN model has the capacity to classify sequences by length. Because approximately half the sequences will have a length above $T/2$, each RNN model must perform with greater than 50% accuracy to demonstrate it has learned the task.

For this task, $T = 1,000$ (The task was found to be too easy for both models for $T = 100$). A training set of 100,000 samples are used to fit the model, and a test set of 10,000 samples are used to evaluate model performance. The RWA model does remarkably well, learning.

Figure 3: Plot comparing the performance of the RWA and LSTM models when classifying sequences by length. The models must determine if a sequence is longer than 500 symbols, which is half the maximum possible sequence length. Each sequence is populated with numbers drawn at random from a unit normal distribution. The traces show the accuracy of each model on the test data while the models are being fitted to the training data. The RWA model achieves a classification accuracy near 100% before the LSTM model.
to correctly classify sequences by their length in fewer than 100 training steps. The LSTM model also learns to correctly classify sequences by their length, but requires five times more training steps to achieve the same level of performance (Fig. 3).

### 3.4 Variable Copy Problem

The variable copy problem, proposed by Henaff et al. (2016) as a modification to the copy problem, requires that a RNN memorize a random sequence of symbols and recall it only when prompted. The input sequence starts with the recall sequence. The recall sequence for the RNN to memorize consists of \( S \) many symbols drawn at random from \( \{a_i\}_{i=1}^{K} \). After that, the input sequence contains a stretch of \( T \) blank spaces, denoted by the symbol \( a_{K+1} \). Somewhere along this stretch of blank spaces, one of the symbols is replaced by a delimiter \( a_{K+2} \). The delimiter indicates when the recall sequence should be repeated back. The input sequence is then padded with an additional stretch of \( S \) blank spaces, providing sufficient time for the model to repeat the recall sequence. The goal is to train the model so that its output is always blank except for the spaces immediately following the delimiter, in which case the output must match the recall sequence.

An example of the task is shown in Figure 4a. The recall pattern drawn at random from symbols \( A \) through \( H \) is \( DEAEEBHGBH \). Blank spaces represented by * fill the rest of the sequence. One blank space is chosen at random and replaced by \( X \), which denotes the delimiter. After \( X \) appears the model must repeat the recall pattern in the output. The naïve strategy is to always guess that the output is * because this is the most common

Figure 4: (a) An example of the variable copy problem. The input and target sequences represent a single sample of data. (b) A plot comparing the performance of the RWA and LSTM models when the sequences are 100 symbols in length. The traces show the error of each model on the test data while the models are being fitted to the training data. The RWA model beats the baseline score (dashed line) before the LSTM model. (c) The same problem as before except the sequences are 1,000 symbols in length. The RWA model beats the baseline score (dashed line) before the LSTM model.
symbol. Each RNN must perform better than the naïve strategy to demonstrate that it has learned the task. Using this naïve strategy, the expected cross-entropy error between the true output and predicted output is $\frac{S \ln k}{2S+T}$, represented by the dashed line in Figures 4b, c. This is the baseline to beat.

For this challenge, $K = 8$, $S = 10$, and models are trained and evaluated on two separate cases where $T = 100$ and $T = 1,000$. For both $T = 100$ and $T = 1,000$, the training set contains 100,000 examples and the test set contains 10,000 examples. For the case of $T = 100$, the RWA quickly beats the baseline error well before the LSTM model. At around 30,000 training steps, the RWA model appears to forget what it has learned, and its performance drops dramatically. This could be the result of using a minibatch of 100 samples on each training set instead of the full dataset or of not using gradient clipping. The LSTM model appears to sometimes forget too, but the drop in performance is never as severe (Fig. 4b). The RWA model scales well to $T = 1,000$, quickly beating the baseline score, whereas the LSTM model is only barely able to beat the baseline error after 50,000 training steps (Fig. 4c).

3.5 Adding Problem

The adding problem, first proposed by Hochreiter and Schmidhuber (1997), tests the ability of a RNN model to form long-range connections across a sequence. The task requires that the model learn to add two numbers randomly spaced apart on a sequence. The input sequence consists of two values at each step. The first value serves as an indicator marking the value to add while the second value is the actual number to be added and is drawn at random from a uniform distribution over $[0,1]$. Whenever the indicator has a value of 1, the randomly drawn number must be added to the sum, and whenever the indicator has a value of 0, the randomly drawn number must be ignored. Only two steps in the entire sequence will have an indicator of 1, leaving the indicator 0 everywhere else.

An example of the adding problem is shown in Figure 5a. The top row contains the indicator values and the bottom row contains randomly drawn numbers. The two numbers that have an indicator of 1 must be added together. The numbers in this example are 0.5 and 0.8, making the target output 1.3. Because the two numbers being added together are uniformly sampled over $[0,1]$, the naïve strategy is to always guess that the target output is 1. Each RNN must perform better than the naïve strategy to demonstrate that it has learned the task. Using this naïve strategy, the expected mean square error (MSE) between the true answer and the prediction is approximately 0.167, represented by the dashed line in Figures 5b,c. This is the baseline to beat.

The adding problem is repeated twice, first with sequences of length 100 and again with sequences of length 1,000. In both cases, a training set of 100,000 samples are used to fit the model, and a test set of 10,000 samples are used to evaluate the models performance. When the sequences are of length 100, the RWA model requires fewer than 1,000 training steps to beat the baseline score while the LSTM model requires around 3,000 steps (Fig. 5b). When the sequences are of length 1,000, the RWA model requires approximately 1,000 training steps, while the LSTM model requires almost 20,000 training steps on the same task (Fig. 5c). The RWA model appears to scale much better as the sequence length increases.
Figure 5: (a) An example of the adding problem. The input sequence and target output represent a single sample of data. (b) A plot comparing the performance of the RWA and LSTM models when the sequences are of length 100. The traces show the error of each model on the test data while the models are being fitted to the training data. The RWA model beats the baseline score (dashed line) before the LSTM model. (c) The same problem as before except the sequences are of length 1,000. The RWA model beats the baseline score (dashed line) before the LSTM model.

3.6 Classifying MNIST Images (Pixel by Pixel)

The MNIST dataset contains 28 × 28 pixel images of handwritten digits 0 through 9. The goal is to predict the digit represented in the image (LeCun, Bottou, Bengio, and Haffner, 1998). Using the same setup suggested by Le, Jaitly, and Hinton (2015), the images are arranged into a sequence of pixels. The length of each sequence is 28 × 28 = 784 pixels. Each RNN model reads the sequence one pixel at a time and must predict the digit being represented in the image from this sequence.

Examples of MNIST digits with the correct label are shown in Figure 6a. The pixels at the top and bottom of each image are empty. When the images are arranged into a sequence of pixels, all the important pixels will be located in the middle of the sequence. To utilize these pixels, each RNN model will need to form long-range dependencies that reach the middle of each sequence. The model will have formed the necessary long-range dependencies when it outperforms a naïve strategy of randomly guessing each digit. A naïve strategy will achieve an expected accuracy of 10%, represented by the dashed line in Figures 6b. This is the baseline to beat.

For this challenge, the standard training set of 60,000 images is used to fit the model, and the standard test set of 10,000 images is used to evaluate the models performance on unseen data. The RWA model fits the dataset in under 20,000 steps, considerably faster than the LSTM model (Fig. 6b). After a quarter million training steps, the RWA model achieves an accuracy of 98.8% with an error of 0.080 bits, while LSTM model achieves an accuracy of 99.0% with an error of 0.083 bits. The two models achieve approximately the same accuracy and error on the test set, but the RWA model requires considerably fewer
training steps. The performance of the RWA model exceeds that of fully connected neural network models and approaches the performance level of convolutional neural networks specifically tailored to the task of image classification.

A separate and more challenging task is to randomly permute the pixels, leaving the pixels out of order, as described in Le et al. (2015). The same permutation mapping must be used on each image to keep the data consistent between images.

The permutation task makes the classification of each digit more challenging, but the model no longer needs to form long-range dependencies. Before permuting the pixels, the pixels required to identify each image were located in the middle of each sequence. Permuting the pixels in the sequence disperses these important pixels everywhere along the sequence. Some of the important pixels will appear near the end of the sequence, requiring that the RNN form only short-range dependencies. As before, a naïve strategy of randomly guessing the answer will achieve an expected accuracy of 10%, represented by the dashed line in Figures 6c. This is the baseline to beat.

The classification task is repeated with the pixels randomly permuted. The LSTM model appears to fit the dataset using slightly fewer training steps (Fig. 6c). This is not surprising given that only short-range connections are required to reach some of the important pixels. After a quarter million training steps, the RWA model achieves an accuracy of 94.5% with an error of 0.444 bits, while LSTM model achieves an accuracy of 93.7% with an error of 0.705 bits. The performance of the RWA model is slightly better than that of the LSTM model, perhaps because the RWA model makes better use of every pixel in the sequence.

Figure 6: (a) Examples of the MNIST classification task. Each image of a handwritten digit must be classified by the value it represents. The images are feed into the RNNs as a sequence of pixels one at a time. (b) A plot comparing the performance of the RWA and LSTM models. The traces show the accuracy of each model on the test data while the models are being fitted to the training data. The RWA model beats the baseline score (dashed line) before the LSTM model. (c) Same task as before expect that the pixels have been randomly permuted. The performance of the RWA model appears to match that of the LSTM model.
4. Discussion

The RWA model reformulates the attention mechanism into a stand-alone model that can be optimized using gradient descent based methods. Given that the attention mechanism has been shown to work well on a wide range of problems, the robust performance of the RWA model on the five classification tasks in this study is not surprising (Bahdanau, Cho, and Bengio, 2014; Vinyals, Toshev, Bengio, and Erhan, 2015b; Xu, Ba, Kiros, Cho, Courville, Salakhutdinov, Zemel, and Bengio, 2015; Sønderby, Sønderby, Nielsen, and Winther, 2015; Chan, Jaitly, Le, and Vinyals, 2015; Vinyals, Kaiser, Koo, Petrov, Sutskever, and Hinton, 2015a). Moreover, the RWA model did not require a hyperparameter search to tailor the model to each task. The same configuration outperformed the LSTM model on every problem. Clearly, the RWA model can form long-range dependencies across the sequences in each task and does not suffer from the vanishing or exploding gradient problem that affects other RNN models (Hochreiter, 1991; Bengio, Simard, and Frasconi, 1994).

The RWA model requires less clock time and less computer memory than a LSTM model. On almost every task, the RWA model beat the baseline score using fewer training steps. The number of training steps could be further reduced using a larger step size for the parameter updates. It is unclear how large the step size can become before the convergence of the RWA model becomes unstable (a larger step size may or may not require gradient clipping, which was not used in this study). The RWA model requires less memory than a LSTM model because it uses over 25% fewer parameters per unit. Depending on the computer hardware, the RWA model can either run faster or contain more units than a LSTM model on the same computer.

Unlike previous implementations of the attention mechanism that read an entire sequence before generating a result, the RWA model computes an output in real time with each new input. With this flexibility, the RWA model can be deployed anywhere existing RNN models can be used. Several architectures are worth exploring. Bidirectional RWA models for interpreting genomic data could be created to simultaneously account for information that is both upstream and downstream in a sequence. Multi-layered versions could also be created to handle XOR classification problems at each processing step. In addition, RWA models could be used to autoencode sequences or to learn mappings from a fixed set of features to a sequence of labels. The RWA model offers a compelling framework for performing machine learning on sequential data.

5. Conclusion

The RWA model opens exciting new areas for research. Because the RWA model can form direct connections to any past processing step, it can detect patterns in a sequence that other models would miss. This could lead to dramatically different outcomes when applied to complex tasks like natural language processing, automated music composition, and the classification of genomic sequences. Given how easily the model can be inserted into existing RNN architectures, it is worth trying the RWA model on these tasks. The RWA model has the potential to solve problems that have been deemed too difficult until now.

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Appendix A.

Any summation of the form \( y_n = \sum_{i=1}^{n} r_i \) can be written as a recurrent relation. Let the initial values be \( y_0 = 0 \).

\[
y_n = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n-1} r_i + r_n = y_{n-1} + r_n
\]

The summation is now defined recursively.

Appendix B.

The parameters of the model can be fitted using gradient descent based optimization methods. Modern machine learning software such as TensorFlow can perform gradient optimization automatically (Abadi et al., 2016). The calculations are presented here only to demonstrate that it can be done efficiently.

The gradient calculations require computing \( \frac{\partial h_t}{\partial \theta} \), where \( h_t \) is the output from equation (7) and \( \theta \) represents one of the parameters \( s_0, W_u, b_u, W_g, b_g, W_a, \) or \( b_t \). To perform the calculation efficiently, the gradients need to be recursively defined. Starting from equations (7), the derivative of the initial conditions are \( \frac{\partial n_0}{\partial \theta} = 0 \), \( \frac{\partial d_0}{\partial \theta} = 0 \), and \( \frac{\partial h_0}{\partial \theta} = \frac{\partial \tanh s_0}{\partial \theta} \). The gradients for \( n_t, d_t, \) and \( h_t \) can then be computed recursively starting from these initial gradients.

\[
\begin{align*}
\frac{\partial n_t}{\partial \theta} &= \frac{\partial}{\partial \theta} \left( n_{t-1} + z(x_t, h_{t-1}) \circ e^{a(x_t, h_{t-1})} \right) \\
&= \frac{\partial n_{t-1}}{\partial \theta} + \left( \frac{\partial z(x_t, h_{t-1})}{\partial x_t} \circ \frac{\partial x_t}{\partial \theta} + \frac{\partial z(x_t, h_{t-1})}{\partial h_{t-1}} \circ \frac{\partial h_{t-1}}{\partial \theta} \right) \circ e^{a(x_t, h_{t-1})} \\
&\quad + z(x_t, h_{t-1}) \circ e^{a(x_t, h_{t-1})} \circ \left( \frac{\partial a(x_t, h_{t-1})}{\partial x_t} \circ \frac{\partial x_t}{\partial \theta} + \frac{\partial a(x_t, h_{t-1})}{\partial h_{t-1}} \circ \frac{\partial h_{t-1}}{\partial \theta} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial d_t}{\partial \theta} &= \frac{\partial}{\partial \theta} \left( d_{t-1} + e^{a(x_t, h_{t-1})} \right) \\
&= \frac{\partial d_{t-1}}{\partial \theta} + e^{a(x_t, h_{t-1})} \circ \left( \frac{\partial a(x_t, h_{t-1})}{\partial x_t} \circ \frac{\partial x_t}{\partial \theta} + \frac{\partial a(x_t, h_{t-1})}{\partial h_{t-1}} \circ \frac{\partial h_{t-1}}{\partial \theta} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial h_t}{\partial \theta} &= \frac{\partial}{\partial \theta} f_t \left( \frac{n_t}{d_t} \right) \circ \frac{\partial n_t}{\partial \theta} \circ d_t - n_t \circ \frac{\partial d_t}{\partial \theta}
\end{align*}
\]

Notice that the gradient term \( \frac{\partial n_t}{\partial \theta} \) is defined recursively using \( \frac{\partial n_{t-1}}{\partial \theta} \) and \( \frac{\partial h_{t-1}}{\partial \theta} \). Likewise, the gradient term \( \frac{\partial d_t}{\partial \theta} \) is defined recursively using \( \frac{\partial d_{t-1}}{\partial \theta} \) and \( \frac{\partial h_{t-1}}{\partial \theta} \). Once \( \frac{\partial n_t}{\partial \theta} \) and \( \frac{\partial d_t}{\partial \theta} \) have been computed, the gradient term \( \frac{\partial h_t}{\partial \theta} \) can be computed. With these equations, the gradient can be efficiently computed by maintaining a running calculation of the gradient as each input is feed into the model.

After completing the gradient computation \( \frac{\partial h_t}{\partial \theta} \), it can be used to fit the parameters of the model. To see how, suppose the output of the model at the last processing step is feed into a fully connected layer to predict a binary label.
\[ p = \sigma(W_o \cdot h_{t=\text{LAST}} + b_0) \]

The fully connected layer predicts a probability \( p \) that the label is 1, not 0. The matrix \( W_o \) represents the weights of the fully connected layer and the vector \( b_0 \) represents the bias terms. The activation function \( \sigma \) is the sigmoid function. The cross-entropy error between the true label \( y \) and predicted probability \( p \) is

\[ E = y \cdot \ln p + (1 - y) \cdot \ln (1 - p). \]

The model can be fitted by minimizing this error using gradient descent. Assuming that the parameter \( \theta \) is among \( s_0, W_u, b_u, W_g, b_g, W_a, \) or \( b_a \), the gradient of the error term is

\[ \frac{\partial E}{\partial \theta} = \frac{\partial}{\partial \theta} \left( y \cdot \ln p + (1 - y) \cdot \ln (1 - p) \right). \]

The result of the gradient calculation from equations (8) is inserted into \( \frac{\partial}{\partial \theta} h_{t=\text{LAST}}. \)

Appendix C.

The exponents \( a \) in equations (7) can become large enough to cause an overflow error. The overflow condition can be avoided with a simple modification to the model that does not affect its output.

To prevent the exponential terms from overflowing and to avoid the numerator and denominator terms from becoming larger with every step, \( n_t \) and \( d_t \) can be multiplied by \( e^{-a(x_t, h_{t-1})} \) whenever \( a(x_t, h_{t-1}) > 0. \)

\[
\text{IF: } a(x_t, h_{t-1}) > 0: \\
 n_t = (n_{t-1} + z(x_t, h_{t-1}) \circ e^{a(x_t, h_{t-1})}) \circ e^{-a(x_t, h_{t-1})} \\
 = n_{t-1} \circ e^{-a(x_t, h_{t-1})} + z(x_t, h_{t-1}) \\
 d_t = (d_{t-1} + e^{a(x_t, h_{t-1})}) \circ e^{-a(x_t, h_{t-1})} \\
 = d_{t-1} \circ e^{-a(x_t, h_{t-1})} + 1
\]

Notice that the largest exponent is still less than or equal to zero, avoiding the overflow condition. Because the numerator and denominator terms are scaled back by the same value, the fraction \( \frac{n_t}{d_t} \) remains unchanged and the output of \( h_t = f\left(\frac{n_t}{d_t}\right) \) is identical.

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