A model for next-to-leading order resummed form factors

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Abstract

We present a model for next-to-leading order resummed threshold form factors based on a time-like coupling recently introduced in the framework of small $x$ physics. Improved expressions for the form factors in $N$-space are obtained which are not plagued by Landau-pole singularities, as the included absorptive effects — usually neglected — act as regulators. The physical reason is that, because of faster decay of gluon jets, there is not enough resolution time to observe the Landau pole. Our form factors reduce to the standard ones when the absorptive parts related to the coupling are neglected. The inverse transform from $N$-space to $x$-space can be done directly without any prescription and we obtain analytical expressions for the form factors, which are well defined in all $x$-space.

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1 Introduction

Resummation of large infrared logarithms in form factors and shape variables is essential in order to predict accurate cross sections in many phenomenologically relevant processes \cite{1 2 3 4 5}. In this paper we present a model for next-to-leading order (NLO) resummed form factors based on the time-like coupling recently introduced by B. Ermolaev, M. Greco and S. Troyan in \cite{6} in the framework of small $x$ physics (see also \cite{6}). The usual expression for resummed threshold form factors in $N$-space is \cite{3 4 5}:

\[
  f_N(\alpha_S) = \exp \left\{ \int_0^1 \frac{dz}{z} \left[ \sum_{n=1}^{N-1} \left( \int_{Q^2(1-z^2)}^{Q^2} \frac{dk_t^2}{k_t^2} \right) \left[ A_1 \alpha_S(k_t^2) + A_2 \alpha_S(k_t^2)^2 + \cdots \right] + B_1 \alpha_S(Q^2(1-z)) + \cdots + D_1 \alpha_S(Q^2(1-z)^2) + \cdots \right] \right\}. \tag{1}
\]

$Q$ is the hard scale of the process. $A_1, A_2, \cdots B_1, \cdots D_1, \cdots$ are the first coefficients of the functions $A(\alpha_S)$, $B(\alpha_S)$ and $D(\alpha_S)$:

\[
  A(\alpha_S) = A_1 \alpha_S + A_2 \alpha_S^2 + \cdots, \quad B(\alpha_S) = B_1 \alpha_S + B_2 \alpha_S^2 + \cdots, \quad D(\alpha_S) = D_1 \alpha_S + D_2 \alpha_S^2 + \cdots. \tag{2}
\]

$A(\alpha_S)$ describes the emission of partons which are both soft and collinear, $B(\alpha_S)$ describes hard and collinear partons while $D(\alpha_S)$ describe partons which are soft and at large angles. The knowledge of the quantities $A_1, A_2, B_1$ and $D_1$ is needed for resummation at next-to-leading order (see later for definition). For instance, in the case of the thrust distribution we have \cite{8}:

\[
  A_1 = \frac{2C_F}{\pi}, \quad A_2 = \frac{2C_F}{\pi^2} \left[ C_A \left( \frac{67}{36} - \frac{\pi^2}{12} \right) - \frac{5}{9} n_F T_R \right], \quad B_1 = -\frac{3C_F}{2\pi}, \quad D_1 = 0, \tag{3}
\]

where $C_A = N_C, C_F = (N_C^2 - 1)/(2N_C), T_R = 1/2, N_C = 3$ is the number of colors and $n_F$ is the number of active quark flavors\footnote{The value of $A_2$ is given in the $\overline{\text{MS}}$ scheme for the coupling constant.}. For heavy flavor decays we have instead \cite{9} \cite{10}:

\[
  A_1 = \frac{C_F}{\pi}, \quad A_2 = \frac{C_F}{\pi^2} \left[ C_A \left( \frac{67}{36} - \frac{\pi^2}{12} \right) - \frac{5}{9} n_F T_R \right], \quad B_1 = -\frac{3C_F}{4\pi}, \quad D_1 = -\frac{C_F}{\pi}. \tag{4}
\]

Generally, one assumes $\alpha_S \ll 1$ and uses truncated $\alpha_S$ expansions for the functions $A(\alpha_S)$, $B(\alpha_S)$ and $D(\alpha_S)$ in eq. \cite{6}. Since the running coupling $\alpha_S(k_t^2)$ is integrated over all gluon transverse momenta $k_t$ from the hard scale $Q$ down to zero, the Landau pole is hit. Therefore $\alpha_S$ diverges inside the integration region — it is certainly not small there — indicating that a truncated expansion for the $A$, $B$ and $D$ functions is not correct. We have conceptually a breakdown of the scheme: resummed perturbation theory assumes $\alpha_S$ small at the beginning, while it ends up with a large $\alpha_S$. A prescription has to be assigned to give a meaning to the formal expression \cite{6}.

The standard solution is to expand the exponent of eq. \cite{6} in a function series of the form \cite{12}:

\[
  f_N(\alpha_S) = \exp \left[ \sum_{n=0}^{\infty} \frac{\alpha_S^n}{n!} g_n(\lambda) \right] = \exp \left[ \frac{L \left( g_1(\lambda) + g_2(\lambda) + \alpha_S g_3(\lambda) + \cdots \right)}{\beta_0} \right], \tag{5}
\]

where

\[
  \lambda = \beta_0 \alpha_S(Q^2) L, \quad L = \log N \tag{6}
\]

and $\beta_0 = (11/3 N_C - 2/3 n_F)/(4\pi)$. The functions $g_i(\lambda)$ have a power-series expansion:

\[
  g_i(\lambda) = \sum_{n=1}^{\infty} b_{i,n} \lambda^n. \tag{7}
\]

The effects of the Landau pole in this framework are the following:
1. the series in eq. (5) is divergent as the higher order functions have factorially growing coefficients \([10, 11]\);

2. the functions \(g_i(\lambda)\) have branch cuts starting at \(\lambda = 1/2\) and going up to infinity. The form factors are then formally well defined up to a critical value \(N_{\text{crit}} \sim Q/\Lambda\) (\(\Lambda\) is the QCD scale), above which they acquire a (completely unphysical) imaginary part.

The first problem is solved by means of a truncation of the series to its first few terms — typically two or three — which is effectively a prescription of the Landau pole. This is the so-called fixed logarithmic accuracy, which we will describe later. The resummation to leading order requires the knowledge of the function \(g_1\), the resummation to next-to-leading order also requires the knowledge of \(g_2\), and so on. As far as the second problem is concerned, one simply restricts himself to a fiducial region in \(N\)-space below \(N_{\text{crit}}\). The inverse Mellin transform from \(N\)-space to \(x\)-space, at next-to-leading \(\log(1-x)\) accuracy, is well defined up to \(x_{\text{crit}} \sim 1-\Lambda/Q\), above which singularities occur. However, a form factor formally well defined in the whole \(x\)-space and containing all the requested \(\log(1-x)\) terms can be obtained by means of an additional prescription for the contour integration in \(N\)-space, the so-called minimal-prescription \([12]\).

Another common solution to this problem involves the renormalon calculus \([11]\). The latter uses the inconsistency discussed above to get information about non-perturbative effects, in the form of power-suppressed corrections to the cross sections:

\[
\delta \sigma \sim c \left( \frac{\Lambda}{Q} \right)^b.
\]

The exponent \(b\) of the power correction can be computed, but not its coefficient \(c\). The resulting information is therefore of a qualitative kind and, ultimately, perturbation theory is deprived of predictive power. In fact, one can substantially modify the spectra coming out of a renormalon calculation by changing the prescriptions in the Borel plane, which are completely arbitrary.

Our perspective is different: we want to use resummed perturbation theory in a predictive way, by curing the "disease" of usual resummation formula. Therefore we begin by re-analyzing the derivation of the standard formula. \(\alpha_S(k_t^2)\) occurs in eq. \([11]\) because we are computing a so-called inclusive-gluon-decay quantity, in which one does not observe the development of the jets originating from the gluons emitted by the hard partons. One then sums over all possible final states, i.e. formally over all cuts of dressed gluon propagators, reconstructing their discontinuity. The radiated gluons have a positive virtuality \(k_t^2 > 0\), as the development of jets is intrinsically a time-like process. Absorptive effects, i.e. the \(-i\pi\) terms in gluon polarization functions, are usually neglected in literature: by taking them into account, one enforces that all gluons participating to the cascade are unstable particles \([6]\). They are analogous to the quasi-particles of statistical physics, possessing in lowest order — non interacting quasi-particles — a non-zero width \([13]\). The outcome of the inclusion of this physical effect is that \(\alpha_S(k_t^2)\) does not occur any longer in the resummation formula, and it is replaced by a different function of \(k_t^2\),

\[
\alpha_S(k_t^2) \rightarrow \tilde{\alpha}_S(k_t^2),
\]

which does not possess the Landau pole. We call this new function "effective coupling", as it specifies the effective strength of the interaction in the gluon cascade — not in a general QCD process. The main point is that the effective coupling never becomes large, as the typical expansion parameter is

\[
\frac{\tilde{\alpha}_S(k_t^2)}{\pi} < \frac{1}{\beta_0 \pi} < 1 \quad \text{for any} \quad k_t^2
\]

and one never leaves the perturbative domain. By including absorptive effects related to the coupling constant — usually neglected — we derive an improved expression for the resummation formula of the form factors. The resummation formula is free of Landau pole pathologies, it does not involve any new free parameter and is strictly predictive.
2 The improved resummation formula

By taking into account renormalization effects in the form factors, the tree-level, momentum-independent coupling is replaced by the integral on the discontinuity of the gluon propagator [12, 13]:

$$\alpha_s \rightarrow \hat{\alpha}_S(k_t^2) = \frac{1}{2\pi i} \int_0^{k_t^2} ds \text{ Disc} \frac{1}{s \beta_0 \log \frac{s}{\Lambda^2}}. \quad (11)$$

The discontinuity is defined as usual: Disc$_F(s) = F(s + i\epsilon) - F(s - i\epsilon)$, with $\epsilon$ being a positive infinitesimal number. We now close the integration contour with a circle of radius $k_t^2$ and a circle of infinitesimal radius. By using Cauchy’s theorem and neglecting the residue of the pole in $s = -\Lambda^2$ in order to preserve asymptotic freedom $^4$, we obtain:

$$\hat{\alpha}_S(k_t^2) = \int_{-\pi}^{+\pi} d\varphi \frac{1}{2\pi \beta_0} \left[ \log \frac{k_t^2}{\Lambda^2} + i\varphi \right]. \quad (12)$$

The standard resummation formula (1) is obtained by neglecting the imaginary term $i\varphi$ in the denominator. In hard processes $Q \gg \Lambda$ and one expects on physical grounds that $k_t \sim Q$, implying:

$$\log \frac{k_t^2}{\Lambda^2} \gg \pi. \quad (13)$$

As a consequence, one obtains the coupling evaluated at the gluon transverse momentum squared:

$$\hat{\alpha}_S(k_t^2) \rightarrow \frac{1}{\beta_0 \log \frac{k_t^2}{\Lambda^2}} = \alpha_S(k_t^2). \quad (14)$$

However, the assumption (13) is not correct, because the transverse momentum $k_t$ is integrated from $Q$ down to very small scales. We propose to modify the resummation formula (1) according to this criticism: approximation (13) disregards absorptive effects related to the gluon jet decay, which are important in the infrared region, and therefore has to be avoided. By performing the integration exactly, we obtain:

$$\hat{\alpha}_S(k_t^2) = \frac{1}{\beta_0} \left[ \frac{1}{2} - \frac{1}{\pi} \arctan \frac{\log \frac{k_t^2}{\Lambda^2}}{\pi} \right]. \quad (15)$$

$^4$The integral on the discontinuity of the gluon propagator is transformed into the integral over a circle of radius $k_t^2$ plus the residue of the pole in $s = -\Lambda^2$. In our pragmatic approach, we have just omitted the latter.
Figure 2: Form factor in $N$-space $f_N(\alpha_S)$ for the beauty case. Solid line: our model; dashed line: standard form factor, i.e. $r = 0$ in our model.

The effective coupling \(^{[14]}\) approaches the standard one in the asymptotic region $k^2_t \gg \Lambda^2$, but it does not contain the infrared pole in $k^2_t = \Lambda^2$. \(^{[5]}\) Furthermore, $\tilde{\alpha}_S(k^2_t)$ is positive definite, monotonically decreasing in all the $k_t$ range and has a finite limit at zero momentum (see fig. 1):

$$\lim_{k^2_t \to 0} \tilde{\alpha}_S(k^2_t) = 1.$$

According to eq. \(^{[15]}\), the effective coupling deviates from the standard one and saturates at a scale of order

$$k_t \sim \Lambda e^{\pi/2} \sim 1 \text{ GeV},$$

for $\Lambda \sim 200$ MeV. Our improved expression for the form factor at one-loop approximation reads:

$$f_N(\alpha_S) = \exp{\int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \left\{ \int_{Q^2(1-z)^2}^{Q^2} \frac{dk^2_t}{k^2_t} A_1 \tilde{\alpha}_S(k^2_t) + B_1 \tilde{\alpha}_S(Q^2(1-z)) + D_1 \tilde{\alpha}_S(Q^2(1-z)^2) \right\}},$$

where the expression for the effective coupling $\tilde{\alpha}_S$ is given by eq. \(^{[15]}\). By comparing with the standard resummation formula \(^{[1]}\), we see that the only difference is the appearance of $\tilde{\alpha}_S$ in place of $\alpha_S$. Let us now consider two-loop effects. Note that eq. \(^{[11]}\) can be written as:

$$\tilde{\alpha}_S(k^2_t) = \frac{1}{2\pi i} \int_{k^2_t}^{\infty} ds \text{ Disc}_s \frac{\alpha_S(-s)^L}{s},$$

where

$$\alpha_S(\mu^2)^{1L} = \frac{1}{\beta_0 \log \frac{\mu^2}{\Lambda^2}},$$

is the one-loop coupling. We now assume that eq. \(^{[10]}\) generalizes to the two-loop coupling

$$\alpha_S(\mu^2) = \frac{1}{\beta_0 \log \frac{\mu^2}{\Lambda^2}} - \frac{\beta_1}{\beta_0^3} \log \log \frac{\mu^2}{\Lambda^2}.$$

\(^{[5]}\) We define the arctan function as the one being continuous in zero and discontinuous at infinity, with image in the interval $(-\pi/2, +\pi/2)$. 

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4
where $\beta_1 = (51 - 19/3 \cdot n_F)/(8\pi^2)$. Therefore, the subleading contribution to the effective coupling $\tilde{\alpha}_S(k_t^2)$ is:

$$
\frac{1}{2\pi i} \int_0^{k_t^2} ds \text{Disc_s} \frac{1}{s} \left\{ S \log(-s/A^2) \right\} \left[ \frac{1}{2} + \frac{1}{\pi} \log \frac{k_t^2}{A^2} \right] = \frac{1}{\beta_0^3 \pi^2 + \log^2 k_t^2 / A^2}.
$$

(22)

It is not easy to derive the terms proportional to $A_2$. Let us make the following guess:

$$
\alpha_2^2(k_t^2) \rightarrow \tilde{\alpha}_2^2(k_t^2) = \frac{1}{2\pi i} \int_0^{k_t^2} ds \text{Disc_s} \frac{\alpha_2^2(-s)^{1L}}{s} = \frac{1}{\beta_0^3 \pi^2 + \log^2 k_t^2 / A^2}.
$$

(23)

An explicit verification of the validity of eq. (23) would require a three loops computation, which is beyond the purpose of the present work. That is because $A_2$ first occurs at order $\alpha_S^2$ and running coupling effects involve at least one additional power of $\alpha_S$, i.e. $\alpha_3^2$ in total. Let us stress that also the standard replacement $\alpha_S^2 \rightarrow \alpha_S(k_t^2)^2$ has never been explicitly checked, as its verification requires a three loop computation as well. In eq. (23) we just assumed that the $s$-discontinuity had to be taken after squaring the gluon polarization function; it is remarkable that our choice gives a simple analytic result. Other choices are possible at this level, such as squaring the effective coupling: $\alpha_2^2(k_t^2) \rightarrow \tilde{\alpha}_S(k_t^2)^2$. Anyway, since two-loop terms are rather small, our choice seems acceptable in a model of soft gluon dynamics.

Figure 3: $f_N(\alpha_S)$ for the top case. Solid line: our model; dashed line: standard form factor, i.e. $r = 0$ in our model.

In conclusion, our expression for the resummed form factor at next-to-leading order reads:

$$
f_N(\alpha_S) = \exp \int_0^1 \frac{dz}{1-z} \sum_{N-1}^{z} \left\{ \int_{Q^2(1-z)^2}^{Q^2(1-z)} \frac{dk_t^2}{k_t^2} \left[A_1 \tilde{\alpha}_S(k_t^2) + A_2 \tilde{\alpha}_S^2(k_t^2) + \cdots \right] + B_1 \tilde{\alpha}_S(Q^2(1-z)) + \cdots + D_1 \tilde{\alpha}_S(Q^2(1-z)^2) + \cdots \right\},
$$

(24)

With this choice there is only one effective coupling, while with our choice there are two effective couplings.
where:

\[
\tilde{\alpha}_S(k_t^2) = \frac{1}{\beta_0} \left[ \frac{1}{2} - \frac{1}{\pi} \arctan \frac{\log k_t^2 / \Lambda^2}{\pi} \right] + \frac{\beta_1}{\beta_0^3 \pi^2 + \log^2 k_t^2 / \Lambda^2} \left\{ \left[ -\frac{1}{2} + \frac{1}{\pi} \arctan \frac{\log k_t^2 / \Lambda^2}{\pi} \right] \log \frac{k_t^2}{\Lambda^2} + \frac{1}{2} \log \left[ \pi^2 + \log^2 \frac{k_t^2}{\Lambda^2} \right] + 1 \right\}
\] (25)

and \(\tilde{\alpha}_S^2(k_t^2)\) is given by eq. (23).

Eq. (24) is the main result of this work and replaces eq. (1). The main problem of eq. (1), discussed before, does not occur anymore in our improved expression: the effective couplings \(\tilde{\alpha}_S\) and \(\tilde{\alpha}_S^2\) do not diverge for small values of the argument and actually remain acceptably small.

Our treatment of the running coupling constant resembles the exponentiation of the \(\pi^2\) terms in the \(K\)-factor of Drell-Yan coefficient function [16]. It is well known that, going from the space-like kinematics of Deep-Inelastic-Scattering \((q^2 < 0)\) to the time-like kinematics of Drell-Yan \((q^2 > 0)\), double logarithms in the vertex correction generate a large constant term proportional to \(\pi^2\), because:

\[
\log^2(-q^2 - i0) \rightarrow \log^2(q^2) - \pi^2 + \cdots.
\] (26)

Since double logs exponentiate, it has been suggested that also the \(\pi^2\) terms exponentiate:

\[
e^{-c\alpha_S \log^2(-q^2 - i0)} \rightarrow e^{c\alpha_S \pi^2} e^{-c\alpha_S \log^2(q^2)},
\] (27)

where \(c\) is a positive constant. It has been however established that the exponentiation of \(\pi^2\) is not exact, being violated at two loops by terms of the form \(\zeta(2) \log^2(-q^2 - i0)\), where \(\zeta\) is the Riemann zeta function \((\zeta(2) = \pi^2/6)\) [17]. Analogously, in our case, we cannot expect to have a complete resummation; we resum, however, a set of constants which are certainly there, and this seems to be a reasonable approximation.

### 3 The functions \(g_1\) and \(g_2\)

Let us begin this section by defining the logarithmic accuracy in our model. We have powers of the coupling at the hard scale,

\[
\alpha \equiv \alpha_S(Q^2),
\] (28)

multiplied by powers of the infrared logarithm, \(L = \log N\), i.e. monomials of the form

\[
\alpha^n L^k.
\] (29)

Within the usual resummation scheme,

- by leading logarithms \((LL)\) we mean all the terms in the exponent of the form factor of the form \(\alpha^n L^{n+1}\) with \(n = 0, 1, 2, \ldots \infty\);
- by next-to-leading logarithms \((NLL)\) we mean all the terms of the form \(\alpha^n L^n\) with \(n = 0, 1, 2, \ldots \infty\).

\(\text{In our work we are dealing with a geometric series (cfr. eq. (12)), while in the } K\text{-factor an exponential series is resummed.}
The coupling $\alpha_S(k_t^2)$ at one-loop, when expanded in powers of $\alpha$, produces a series of leading logarithms:

$$\alpha_S(k_t^2) = \alpha - \beta_0 \alpha^2 \log \frac{k_t^2}{Q^2} + \beta_0^3 \alpha^3 \log^2 \frac{k_t^2}{Q^2} + \cdots$$  \hfill (32)

The powers of $\log k_t^2/Q^2$ give indeed powers of $L = \log N' \theta$ after the integration over $k_t$ and $z$ in eq. (1).

According to eq. (13), the relation between the standard and the effective coupling reads $^8$:

$$\tilde{\alpha}_S(k_t^2) = \frac{1}{\pi \beta_0} \arctan \left[ \pi \beta_0 \alpha_S(k_t^2) \right] \quad \text{for} \quad \alpha_S(k_t^2) > 0.$$  \hfill (33)

Therefore, the expansion of our effective coupling in the parameter $\alpha$ is:

$$\tilde{\alpha}_S(k_t^2) = \alpha - \beta_0 \alpha^2 \log \frac{k_t^2}{Q^2} + \beta_0^3 \alpha^3 \log^2 \frac{k_t^2}{Q^2} - \frac{(\beta_0 \pi)^2}{3} \alpha^3 + \cdots$$  \hfill (34)

Comparing eq. (32) with eq. (34), we see that the expansion of the effective coupling involves subleading logarithms compensated by powers of $\beta_0 \pi$, coming from the absorptive effects discussed before. The main point is that the $\beta_0 \pi$ terms have a fundamental regulating effect over the logarithmic ones; therefore one has to consider them on equal footing in the counting. The hierarchy in our model can therefore be defined as follows:

- by $LL$ we mean all the terms of the form\n\n\[ \alpha^n (\beta_0 L)^{n+1-k} (\beta_0 \pi)^k \quad \text{with} \quad k = 0, 1, \cdots n + 1; \]  \hfill (35)\n
- by $NL$ we mean all the terms of the form\n\n\[ \alpha^n (\beta_0 L)^{n-k} (\beta_0 \pi)^k \quad \text{with} \quad k = 0, 1, \cdots n. \]  \hfill (36)\n
The usual scheme is “minimal” in the sense that it deals only with powers of the logarithms and of the coupling; the related problem is that of the Landau pole discussed above. Instead, our scheme, in order to solve the Landau problem, has to be non-minimal.

The next step is to perform the integrations within the above defined next-to-leading accuracy. In general, we believe that fixed logarithmic accuracy is a consistent approximation in our scheme because series of logarithms of higher order are suppressed by powers of $\tilde{\alpha}_S$, which is always small. The integrations over transverse and longitudinal gluon momenta are easily done within the standard approximation $^{13}$: $z^{N-1}-1 \simeq -\theta (1-z-1/n)$, where $\theta$ is the step function and $n = N e^x$, with $\gamma_E$ being the Euler constant. This approximation, as shown in our previous paper $^{10}$, misses for instance terms of the form: $\pi^2 \alpha^n L^{n-1}$, which are leading according to our power counting. However, in our model, we are interested in picking up only those terms proportional to $\beta_0 \pi$, which come from the analytic continuation of the coupling from space-like to time-like region. These are the terms coming from the effective coupling, having the regulating effect on the Landau pole.

In order to obtain the functions $g_1$ and $g_2$, we express the logarithm of the hard scale $\log Q^2/\Lambda^2$, coming from the integrations, in terms of the coupling evaluated at a general renormalization scale $\mu$ by means of the formulas:

$$\log \frac{Q^2}{\Lambda^2} = \log \frac{\mu^2}{\Lambda^2} - \log \frac{\mu^2}{Q^2},$$  \hfill (37)

and

$$\log \frac{\mu^2}{\Lambda^2} = \frac{1}{\beta_0 \alpha_S(\mu^2)} + \frac{\beta_1}{\beta_0} \log \left( \beta_0 \alpha_S(\mu^2) \right).$$  \hfill (38)

$^8$One has to use the relation $\arctan(1/x) = \pi/2 - \arctan(x)$, which is valid for $x \geq 0$ with our definition of the arctan function.
We finally Taylor expand the exponent on the r.h.s of eq. (18) in powers of $\gamma_E$, $\beta_1$ and $\log \mu^2/Q^2$ up to first order included. Our result for the leading function $g_1$ reads:

$$g_1(\lambda; r) = \frac{A_1}{4\beta_0} \left\{ -(1 - 2\lambda) \log[(1 - 2\lambda)^2 + r^2] + 2 (1 - \lambda) \log[(1 - \lambda)^2 + r^2] - \log[1 + r^2] + \frac{1}{r} \left[ -\pi \lambda^2 + (1 - r^2) \arctan \frac{1}{r} + \left( (1 - 2\lambda)^2 - r^2 \right) \arctan \frac{1 - 2\lambda}{r} - 2 \left( (1 - \lambda)^2 - r^2 \right) \arctan \frac{1 - \lambda}{r} \right] \right\}. \tag{39}$$

The subleading function $g_2$ reads:

$$g_2(\lambda; r) = \frac{D_1}{4\beta_0} \left\{ \log[(1 - 2\lambda)^2 + r^2] - \log[1 + r^2] + \frac{2}{r} \left[ -\pi \lambda + \arctan \frac{1}{r} - (1 - 2\lambda) \arctan \frac{1 - 2\lambda}{r} \right] \right\} + \frac{B_3}{2\beta_0} \left\{ \log[(1 - \lambda)^2 + r^2] - \log[1 + r^2] + \frac{1}{r} \left[ -\pi \lambda + 2 \arctan \frac{1}{r} - 2 (1 - \lambda) \arctan \frac{1 - \lambda}{r} \right] \right\} + \frac{A_2}{4\beta_0} \left\{ \log[(1 - 2\lambda)^2 + r^2] - 2 \log[(1 - \lambda)^2 + r^2] + \log[1 + r^2] + \frac{2}{r} \left[ -1 - \frac{2}{\pi} \arctan \frac{1 - 2\lambda}{r} \right] \log[(1 - 2\lambda)^2 + r^2] + \frac{4\pi}{r} \left[ (1 - \lambda) \left[ 1 - \frac{2}{\pi} \arctan \frac{1 - \lambda}{r} \right] \log[(1 - \lambda)^2 + r^2] + \frac{2\pi}{r} \left[ -1 - \frac{2}{\pi} \arctan \frac{1}{r} \right] \log[1 + r^2] + 4 \arctan \frac{1 - 2\lambda}{r} \left[ -\pi + \arctan \frac{1 - 2\lambda}{r} \right] + 8 \arctan \frac{1 - \lambda}{r} \left[ \pi - \arctan \frac{1 - \lambda}{r} \right] \right\} + \frac{A_1}{4\beta_0} \log \mu^2 \left\{ \log[(1 - 2\lambda)^2 + r^2] - 2 \log[(1 - \lambda)^2 + r^2] + \log[1 + r^2] + \frac{2}{r} \left[ -1 - \frac{2}{\pi} \arctan \frac{1}{r} \right] \log[(1 - 2\lambda)^2 + r^2] + \frac{A_1 \gamma_E}{2\beta_0} \left\{ \log[(1 - 2\lambda)^2 + r^2] - \log[(1 - \lambda)^2 + r^2] + \frac{2}{r} \left[ (1 - \lambda) \arctan \frac{1 - 2\lambda}{r} + (1 - \lambda) \arctan \frac{1 - \lambda}{r} - \frac{\pi \lambda}{2} \right] \right\}. \tag{40}$$

We have defined — note the general renormalization scale $\mu$ in the coupling:

$$\lambda \equiv \beta_0 \alpha_S(\mu^2) L \quad \text{and} \quad r \equiv \pi \beta_0 \alpha_S(\mu^2). \tag{41}$$

Note that $\lambda$ and $r$ are obtained from each other by interchanging $L$ with $\pi$. All this is in line with the logarithmic accuracy defined above.

A few remarks are in order. The first one is that, unlike the standard case, $g_1$ and $g_2$ do not depend only on $\lambda$ but also on $r$, i.e. on $\alpha_S$ alone. Our $g_1$ and $g_2$ are then “non-minimal”, as a consequence of the inclusion of absorptive constants in the coupling. The parameter $r \neq 0$ acts as a regulator of Landau-pole singularities: $g_1$
Figure 4: Renormalization scale dependence of form factor in $N$-space $f_N(\alpha_S)$ for the beauty case. Solid line: $\mu^2 = Q^2$; dashed line: $\mu^2 = Q^2/4$; dotted line: $\mu^2 = 4Q^2$.

and $g_2$ are regular for any value of $\lambda$. As expected, in the limit $r \to 0$ we recover the standard $g_1$ and $g_2$ [4]:

$$g_1(\lambda; r = 0) = \frac{A_1}{2 \beta_0} \{ - (1 - 2 \lambda) \log(1 - 2 \lambda) + 2 (1 - \lambda) \log(1 - \lambda) \}$$  \hspace{1cm} (42)

and

$$g_2(\lambda; r = 0) = \frac{D_1 \log(1 - 2 \lambda) + B_1 \log(1 - \lambda)}{2 \beta_0} + \frac{A_2 [\log(1 - 2 \lambda) - 2 \log(1 - \lambda)]}{2 \beta_0^2} + \frac{A_1 \beta_1 [2 \log(1 - 2 \lambda) + \log^2(1 - 2 \lambda) - 4 \log(1 - \lambda) - 2 \log^2(1 - \lambda)]}{4 \beta_0^3}$$

$$+ \frac{A_1 \log(\mu^2/Q^2) [\log(1 - 2 \lambda) - 2 \log(1 - \lambda)]}{2 \beta_0} + \frac{A_1 \gamma_E [\log(1 - 2 \lambda) - \log(1 - \lambda)]}{\beta_0}.$$  \hspace{1cm} (43)

If we expand our $g_1$ and $g_2$ for small $r$, we find that the leading corrections are of order $r^2\lambda^n$: this means corrections of next-to-next-to-leading order (NNLO) $\alpha_S^{n+1} L^n$ in the standard counting, which had been overlooked in our previous evaluation of the $g_3$ [10]. Our $g_1$ and $g_2$ contain some terms from the standard $g_3$, $g_4$, $g_5$, . . . or from the coefficient function $C(\alpha_S)$ multiplying the form factor. It is remarkable that eqs. (39) and (40) involve only powers of the log and arctan functions.

In fig. 2 we have plotted the form factor $f_N(\alpha_S)$ as a function of $N$ for the decay of a beauty quark, i.e. for $Q = m_b$ [9], for which $\alpha_S = 0.21$ and $n_F = 3$. The dashed line is the plot of the standard form factor, i.e. of the same form factor in the limit $r \to 0$. We have plotted moments up to $N \sim 10$, the latter being steeper for larger $N$. For $N = 20$ there is a difference of a factor 2 circa. Fig. 3 shows similar plots for the top case, i.e. for $Q = m_t$ for which $\alpha_S = 0.11$ and $n_F = 5$. Due to the increased hard scale by more than one order of magnitude:

- differences between the two models are much smaller;
- the peak above one, related to the subleading, single-logarithmic effects, is barely visible, while it is rather pronounced in the beauty case.

[9] In general, for heavy flavor decays $Q = 2E_X$, where $E_X$ is the energy of the hadronic final state. One can set $2E_X = m_b$ in $b \to s\gamma$, while this is not true in semi-leptonic $b \to u$ decays [18].
Figure 5: Form factor in $x$-space $f(x; \alpha_S)$ for the beauty case. Solid line: our model; dashed line: standard form factor.

Figure 6: Form factor in $x$-space $f(x; \alpha_S)$ for the top case. Solid line: our model; dashed line: standard form factor.

The scale dependence is shown in fig. 4, where we have plotted $f_N(\alpha_S)$ for the beauty case for $\mu^2 = Q^2$, $\mu^2 = Q^2/4$ and $\mu^2 = 4Q^2$. The scale ambiguity is smaller than the difference between the models shown in fig. 2.

Let us now discuss the extension of our model to NNLO, i.e. the computation of the function $g_3$. It is clear that one has to make similar guesses to the ones needed at NLO to evaluate the terms proportional to $A_2$. Apart from that, the computation, although technically rather cumbersome, does not seem to present any specific difficulty. In general, our fixed logarithmic accuracy for the exponent in the form factor,

$$\Sigma = L g_1(\lambda; r) + g_2(\lambda; r) + \alpha g_3(\lambda; r) + \alpha^2 g_4(\lambda; r) + \cdots,$$

is an expansion with better convergence properties than the usual one, because the effective coupling becomes at the most of order one. In other words, our expansion is certainly better than an expansion in $1/(\beta_0 \pi)^n$, which is already convergent as $1/(\beta_0 \pi) < 1$. We expect the extension of our model to NNLO to be relevant also for beauty physics. In a previous work [10] we have found instead that usual NNLO effects could not be included in the case of beauty decay. In fact, the hard scale $Q = m_b$ was not large enough to avoid effects of the divergence of the perturbative series even at relatively low values of $x$. Such divergence was related to the integration over the Landau pole and is therefore absent in the present scheme.
4 Form Factors in $x$ Space

Up to now we have considered form factors in $N$-space. The $N$-moments are indeed physical quantities, but in practice a measure of the moments for large $N$ is difficult. Therefore, let us transform back to $x$-space. The inverse transform of $f_N(\alpha_S)$ is defined as:

$$f(x; \alpha_S) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} dN x^{-N} f_N(\alpha_S),$$  \hspace{1cm} (45)

where $C$ is a constant such that the integration contour lies to the right of all the singularities of $f_N$. A standard computation gives to next-to-leading log($1-x$) accuracy \[8\]:

$$f(x; \alpha_S) = -\frac{d}{dx} \left\{ \theta(1-x-0) \exp \left[ \frac{g_1(\tau;r) + g_2(\tau;r)}{\Gamma[1 - \frac{d}{d\tau}(\tau g_1(\tau;r))] \right] \right\},$$ \hspace{1cm} (46)

where we have defined:

$$\tau \equiv \beta_0 \alpha_S(\mu^2) l \quad \text{and} \quad l \equiv -\log(1-x).$$ \hspace{1cm} (47)

In eq. (46) we have neglected terms $O(1-x)$, which are small in the threshold region. The form factor $f(x; \alpha_S)$ in eq. (46) is plotted in fig. 5 for the beauty case. Also shown is the standard form factor, i.e. the same formula for $r = 0$. Our form factor is formally well defined up to $x = 1$, while the standard one presents a singularity at $x_{\text{crit}} \sim 1 - \Lambda/Q$, so it is plotted only below this point. There are substantial differences between the models: our distribution is broader, the peak is smaller and occurs at much larger $x$ than in the standard case. The reason is the following. Our model contains an effective coupling which approaches a constant value for $N \to \infty$ or, equivalently, $x \to 1$ (see eq. (16)). Therefore, for $x \sim 1$ our model resembles a frozen coupling scheme, while the standard form factor contain a coupling divergent at the non-zero momentum $\Lambda$ — the well-known Landau pole. In fig. 6 we plot $f(x; \alpha_S)$ for the top case in both models; as expected, differences are much smaller with respect to the beauty case. The dependence of our $f(x; \alpha_S)$ from the renormalization scale $\mu$ is shown in fig. 7 for the beauty case and turns out to be moderate.

5 Phenomenology

Let us now discuss the phenomenological relevance of our model. A natural application concerns beauty physics, i.e. a case with a moderately large hard scale — when the hard scale is very large, as in top decays, there are
basically no differences between our model and the standard one. In the beauty case, large non-perturbative corrections are expected. The main problem, in our opinion, is that of combining perturbative and non-perturbative effects in the most efficient way. In many phenomenological studies, one assumes a shape for the non-perturbative effects and convolutes it with a frozen-coupling distribution. Since $\alpha_S(m_b) \sim 0.21$ (that is, rather small) and the coupling does not increase at small $k_t$, the perturbative effects turn out to be extremely small. Since the theoretical prediction is a convolution of perturbative and non-perturbative effects, it is completely dominated by the assumed form of the latter. The standard resummation scheme — running coupling — includes much more perturbative dynamics, but it is technically more difficult to implement in a phenomenological analysis and it has been barely used. That is because the perturbative form factors in $x$-space are generally computed with a numerical contour integration, by using the so-called minimal prescription \[12\]. This is a consistent prescription based again on ”minimality”: the correct logarithms of $1 - x$ are included and the distribution is formally well defined up to $x = 1$, but no physically motivated effects are included, only logarithms. Our approach includes instead a (perturbative) physical effect, namely parton decay, giving rise to analytic distributions well-defined in the whole $x$-space. One can then easily convolve our spectra with an assumed form of non-perturbative effects:

$$f_{xx}(x) = \int_0^1 \int_0^1 dx' dx'' \delta(x - x' x'') f_{np}(x'; c_k) f(x''; \alpha_S)$$

$$= \int_x^1 \frac{dx'}{x'} f_{np}(x'; c_k) f(x/x'; \alpha_S),$$

where $f_{np}$ is a non-perturbative function and $f_{xx}$ is the spectrum to be compared with the data. $c_k$ denotes schematically non-perturbative parameters entering $f_{np}$.

It is well known that non-perturbative effects broaden the peaks of perturbative spectra and shift them to smaller $x$’s. They have, as intuitively expected, a smearing effect, representing the final stage of parton evolution, not described by perturbation theory. We have not done yet any detailed phenomenological analysis, but the following points can already be made. Our model gives a broader spectrum compared to the usual one, while it gives a peak at larger values of $x$. The peak we find corresponds indeed to a jet invariant mass below $1 \text{ GeV}$. We then expect that our model has to be supplemented with non-perturbative effects shifting the peak towards smaller $x$.

6 Conclusions

Our main result is the improved threshold resummation formula given by eq. (24). The latter involves — in place of the usual running coupling possessing the Landau pole — an effective coupling which is not singular in the infrared region and remains rather small in all the integration domain. It approaches the standard coupling for large transverse momenta, while it deviates from the latter at a scale of order

$$k_t \sim \Lambda e^{\pi/2} \sim 1 \text{ GeV}. \quad (49)$$

The effective coupling, unlike the usual one, includes absorptive effects related to the decay of gluon jets. The physical explanation of the absence of the Landau pole in our resummation formula is the following. The inclusive gluon branching, inducing the running coupling in resummation formula, is described by taking into account absorptive effects in addition to the usual dispersive ones. The gluon cascade is then treated, more realistically, as an intrinsically unstable process and there is not enough resolution time to see the effects of the Landau pole.

By performing the integrations over longitudinal and transverse momenta with next-to-leading accuracy, we have obtained the functions $g_1$ and $g_2$ in the exponent of the form factor given in eqs. (39) and (40) respectively. These functions define next-to-leading order resummed threshold distributions which are well defined in the
whole moment or physical space. There are large differences between our next-to-leading order form factor and the standard one for beauty decays. We believe that our model can be used to describe the variety of spectra in inclusive or rare $b$ decays. Natural extensions of our approach involve resummation of shape variables or the inclusion of mass effects in semi-leptonic $b \to c$ transitions.

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References

[1] G. Parisi and R. Petronzio, Nucl. Phys. B 154, 427 (1979); G. Curci and M. Greco, Phys. Lett. B92, 175 (1980).

[2] J. Kodaira and L. Trentadue, SLAC-PUB-2934 (1982); Phys. Lett. B 112, 66 (1982); S. Catani, E. D’Emilio and L. Trentadue, Phys. Lett. B 211, 335 (1988).

[3] G. Sterman, Nucl. Phys. B 281, 310 (1987).

[4] S. Catani and L. Trentadue, Nucl. Phys. B 327, 323 (1989).

[5] S. Catani and L. Trentadue, Nucl. Phys. B 353, 183 (1991).

[6] B. Ermolaev, M. Greco, S. Troyan, Phys. Lett. B 522, 57 (2001).

[7] M. Pennington, R. Roberts and G. Ross, Nucl. Phys. B 242, 69 (1984).

[8] S. Catani, L. Trentadue, G. Turnock and B. Webber, Nucl. Phys. B 407, 3 (1993).

[9] G. Altarelli, N. Cabibbo, G. Corbó, L. Maiani and G. Martinelli, Nucl. Phys. B 208, 365 (1982); G. Korchemsky and G. Sterman, Phys. Lett. B 340, 96 (1994); R. Akhoury and I. Rothstein, Phys. Rev. D 54, 2349 (1996); M. Cacciari, G. Corcella and A. Mitov, JHEP 0212, 015 (2002).

[10] U. Aglietti and G. Ricciardi, Phys. Rev. D 66, 074003 (2002).

[11] E. Gardi, Nucl. Phys. B 622, 365 (2002).

[12] S. Catani, M. Mangano, P. Nason and L. Trentadue, Nucl. Phys. B 478, 273 (1996).

[13] See for instance, *Methods of Quantum Field Theory in Statistical Physics*, by A. Abrikosov, L. Gorkov and I. Dzyaloshinskii, Dover Publications, 1975.

[14] Y. Dokshitzer, D. Dyakonov and S. Troyan, Phys. Rept. 58, 269 (1980).

[15] D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini and G. Veneziano, Nucl. Phys. B 173, 429 (1980).

[16] G. Parisi, Phys. Lett. B 90, 295 (1980); J. Kripfganz, Nucl. Phys. B 177, 509 (1980).

[17] W. Van Neerven, Nucl. Phys. B 268, 453 (1986).

[18] U. Aglietti, Nucl. Phys. B 610, 293 (2001).