Deconstruction, 2d lattice Yang-Mills, and the dynamical lattice spacing

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Abstract

We study expectation values related to the dynamical lattice spacing that occurs in the recent supersymmetric 2d lattice Yang-Mills constructions of Cohen et al. [hep-lat/0307012]. For the purposes of this preliminary analysis, we restrict our attention to the bosonic part of that theory. That is, we compute observables in the fully quenched ensemble, equivalent to non-supersymmetric 2d lattice Yang-Mills with a dynamical lattice spacing and adjoint scalars. Our numerical simulations indicate difficulties with the proposed continuum limit. We find that expectation values tend to those of the undeformed “daughter theory,” in spite of the deformation suggested by Cohen et al. In an effort to understand these results, we examine the zero action configurations, with and without the deformation. Based on these considerations, we are able to interpret the simulation results in terms of entropic effects.
1 Introduction

In [1], a lattice action has been proposed by Cohen, Kaplan, Katz and Unsal (CKKU) for the \( (4,4) \) 2d \( U(k) \) super-Yang-Mills.\(^1\) The Euclidean target theory action is:

\[
S_{(4,4)} = \int d^2x \frac{1}{g_2^2} \Tr \left[ (Ds_\mu) \cdot (Ds_\mu) + \bar{\psi}_i D\psi_i + \frac{1}{4} F \cdot F + \bar{\psi}_i \left( s_0 \delta_{ij} + i\gamma_3 s \cdot \sigma_{ij} \right) \psi_j - \frac{1}{2} [s_\mu, s_\nu]^2 \right]
\] (1.1)

where \( s_\mu (\mu = 0, 1, 2, 3) \) are hermitian scalars, \( F \) is the 2d Yang-Mills field strength, and \( \psi_i (i = 1, 2) \) are 2d Dirac fermions, all in the adjoint representation of \( U(k) \). In [2], some aspects of the fermion determinant were examined, with results very similar to those in [3]. Here we will study a rather fundamental aspect of the construction: the proposed emergence of a “dynamical” lattice spacing. CKKU make use of an \( N \times N \) lattice that contains link and site fields that are \( k \times k \) matrices. The classical lattice action contains many zero action configurations. CKKU expand the classical lattice action about a particular class of configurations (discussed below) that are characterized by a parameter \( a \). For this reason we will refer to such a configuration as an \( a \)-configuration. In the limit \( a \to 0 \), \( N \to \infty \), the classical action tends to the continuum action of \( (4,4) \) 2d \( U(k) \) super-Yang-Mills. Thus \( a \) is interpreted as a lattice spacing. However, it is dynamical as it has to do with a particular background configuration for the lattice fields. This strategy is based on the ideas of deconstruction [4, 5]. Studies where a partial latticization of 4d supersymmetric theories has been obtained by this approach include [6, 7].

It is obvious that to arrive at the proposed continuum theory requires that a semi-classical expansion about the \( a \)-configuration gives a good approximation to the behavior of the full lattice theory. But all \( a \)-configurations are energetically equivalent for any value of \( a \). Furthermore, there exist other zero action configurations that do not fall into the \( a \)-configuration class (shown below). For this reason, CKKU deform the action by adding an \( a \)-dependent potential that favors the \( a \)-configuration. The “continuum limit” then includes sending \( a \to 0 \) in this potential. Although this deformation breaks the exact lattice supersymmetry, it is rendered harmless by scaling the relative strength of the deformation potential to zero in the thermodynamic limit. For this reason it has been argued by CKKU that the quantum continuum limit is nothing but the target theory, without the need for fine-tuning. For further details, we refer the reader to [1], as well as the articles leading up to it [8, 9].

The deconstruction method for latticizing a 2d continuum target theory does not require fermions or supersymmetry. The bosonic system (i.e., setting all lattice fermions to zero) contained in the model of CKKU already has the interesting feature of a deconstructed lattice Yang-Mills. The same semi-classical arguments yield \( U(k) \) Yang-Mills with 4 adjoint scalars; that is, the bosonic part of (1.1). In the present article we will investigate the validity

\(^1\) The \( (4,4) \) 2d \( U(k) \) super-Yang-Mills is best defined as the dimensional reduction of \( \mathcal{N} = 1 \) 6d \( U(k) \) super-Yang-Mills; the notation “(4,4)” denotes the number of left and right 2d chirality supercharges.
of the semi-classical argument. We will take into account quantum effects, estimating key expectation values by means of Monte Carlo simulation. Of course we expect such results to differ from those that would be obtained in the related supersymmetric theory of CKKU; for in that case one would have dynamical fermions, and in the limit of vanishing deformation, supersymmetric nonrenormalization theorems. However, we find it sufficiently interesting to analyze the bosonic system as a preliminary step. Indeed, for the expectation values that we study, we find that semi-classical arguments are not a good indicator of the full quantum behavior.

In the next section we introduce the essential features of the bosonic part of the CKKU construction that will be needed for the subsequent discussion. In Section 3 we study the classical minima of the undeformed and deformed actions. In Section 4 we outline the methods and results of our lattice simulation. In Section 5 we suggest an interpretation of these results in terms of the analysis of Section 3. In Section 6 we make some concluding remarks. In the Appendix, we provide a brief review of the $\mathcal{N} = 4$ 4d super-Yang-Mills moduli space, which arises in the discussion of Section 3.

2 Quiver lattice Yang-Mills

For CKKU, the starting point is the Euclidean $\mathcal{N} = 1$ 6d $U(kN^2)$ super-Yang-Mills. In our case, the starting point will instead be Euclidean 6d $U(kN^2)$ Yang-Mills. This is dimensionally reduced to 0d to obtain a $U(kN^2)$ matrix model. The matrix model naturally possesses $SO(6)$ Euclidean invariance. Next we note $SO(6) \supset SO(2) \times SO(2) \supset Z_N \times Z_N$. CKKU have identified a homomorphic embedding of $Z_N \times Z_N$ into the $U(kN^2)$ gauge symmetry group. Using this, a $Z_N \times Z_N$ orbifold projection is performed to obtain a $U(k)$ 0d quiver, or, product group theory.

In every respect we follow CKKU, except that we have set all fermions to zero. For details of the matrix model and orbifold projection, we refer the reader to [1]; in the interests of brevity, we will only give the final result, the undeformed lattice action:

$$S_0 = \frac{1}{g^2} Tr \sum_{\mathbf{n}} \left[ \frac{1}{2} (x_{\mathbf{n}-\mathbf{i} \mathbf{n}-\mathbf{i}}^{\dagger} x_{\mathbf{n}-\mathbf{i} \mathbf{n}-\mathbf{i}} - x_{\mathbf{n} \mathbf{n}}^{\dagger} x_{\mathbf{n} \mathbf{n}} + y_{\mathbf{n}-\mathbf{j} \mathbf{n}-\mathbf{j}}^{\dagger} y_{\mathbf{n}-\mathbf{j} \mathbf{n}-\mathbf{j}} - y_{\mathbf{n} \mathbf{n}}^{\dagger} y_{\mathbf{n} \mathbf{n}} + z_{\mathbf{n} \mathbf{n}}^{\dagger} z_{\mathbf{n} \mathbf{n}} - z_{\mathbf{n} \mathbf{n}}^{\dagger} z_{\mathbf{n} \mathbf{n}})^2 
+ 2(x_{\mathbf{n} \mathbf{n}+\mathbf{i}} y_{\mathbf{n} \mathbf{n}+\mathbf{j}} - y_{\mathbf{n} \mathbf{n}+\mathbf{j}} x_{\mathbf{n} \mathbf{n}+\mathbf{j}}^{\dagger} x_{\mathbf{n} \mathbf{n}+\mathbf{j}}^{\dagger} y_{\mathbf{n} \mathbf{n}+\mathbf{j}}^{\dagger}) 
+ 2(y_{\mathbf{n} \mathbf{n}+\mathbf{j}} z_{\mathbf{n} \mathbf{n}} - z_{\mathbf{n} \mathbf{n}} y_{\mathbf{n} \mathbf{n}}^{\dagger} z_{\mathbf{n} \mathbf{n}}^{\dagger} y_{\mathbf{n} \mathbf{n}}^{\dagger} z_{\mathbf{n} \mathbf{n}}^{\dagger}) 
+ 2(z_{\mathbf{n} \mathbf{n}} x_{\mathbf{n} \mathbf{n}+\mathbf{i}} - x_{\mathbf{n} \mathbf{n}+\mathbf{i}} x_{\mathbf{n} \mathbf{n}+\mathbf{i}}^{\dagger} z_{\mathbf{n} \mathbf{n}}^{\dagger} x_{\mathbf{n} \mathbf{n}+\mathbf{i}}^{\dagger} z_{\mathbf{n} \mathbf{n}}^{\dagger}) \right] \tag{2.1}$$

Here, $x_{\mathbf{m}}, y_{\mathbf{m}}, z_{\mathbf{m}}$ are bosonic lattice fields that are $k \times k$ unconstrained complex matrices; $\mathbf{m} = (m_1, m_2)$ labels points on an $N \times N$ lattice, and $\mathbf{i} = (1, 0), \mathbf{j} = (0, 1)$ are unit vectors. The $U(k)^{N^2}$ symmetry is nothing but the local $U(k)$ symmetry of the lattice action $S_0$, with link bosons $x_{\mathbf{m}}$ in the $\mathbf{i}$ direction, link bosons $y_{\mathbf{m}}$ in the $\mathbf{j}$ direction, and sites bosons $z_{\mathbf{m}}$, all

\[\text{Quiver theories were originally studied many years ago in other contexts [10, 11].}\]
transforming in the usual manner:

\[ x_m \rightarrow \alpha_m x_m \alpha_m^{\dagger} + i, \quad y_m \rightarrow \alpha_m y_m \alpha_m^{\dagger} + i, \quad z_m \rightarrow \alpha_m z_m \alpha_m^{\dagger} \] (2.2)

Canonical mass dimension 1 is assigned to \( x_m, y_m, z_m \), whereas \( g \) has mass dimension 2.

Although \( S_0 \) is a lattice action that describes a statistical system with interesting features, it is not in any obvious way related to a 2d continuum field theory. As will be explained below, \( S_0 \geq 0 \) and a vast number of nontrivial solutions to \( S_0 = 0 \) exist, not all of which are gauge equivalent. In fact, the space of minimum action configurations, or moduli space, is a multi-dimensional noncompact manifold with various branches (classes of configurations). For this reason it is difficult to say what a “continuum limit” might be; for there exists an infinite number of energetically equivalent configurations about which to expand, not all of which are gauge equivalent.

A surprising result—pointed out by CKKU, and based on ideas from deconstruction—is obtained if one expands about the \( a \)-configuration

\[ x_m = \frac{1}{a\sqrt{2}} 1, \quad y_m = \frac{1}{a\sqrt{2}} 1, \quad z_m = 0, \quad \forall m \] (2.3)

keeping \( g_2 = ga \) and \( L = Na \) fixed, treating \( a \) as small. (It is easy to see that \( S_0 = 0 \) for this configuration.) That is, we associate \( a \) with a lattice spacing (mass dimensions -1), even though it arises originally from a specific background field configuration. In this case, one finds that the classical continuum limit is nothing but the bosonic part of (1.1), which is a variety of 2d \( U(k) \) Yang-Mills with adjoint scalars. In the case of the supersymmetric quiver theory of CKKU, where fermions are present, one obtains (1.1) in full; i.e., (4,4) 2d \( U(k) \) super-Yang-Mills.

The trick is how to make the configuration (2.3) energetically preferred without destroying all of the pleasing symmetry properties of the theory. (This is particularly true in the supersymmetric case.) In the quantum analysis, we must address the more delicate complication of entropy as well. CKKU suggest a deformation of the bosonic action in an effort to stabilize the theory near the \( a \)-configuration (2.3):

\[ S = S_0 + S_{SB} \]

\[ S_{SB} = \frac{a^2 \mu^2}{2g^2} \sum_n \text{Tr} \left[ \left( x_n x_n^\dagger - \frac{1}{2a^2} \right)^2 + \left( y_n y_n^\dagger - \frac{1}{2a^2} \right)^2 + \frac{2}{a^2} z_n z_n^\dagger \right] \] (2.5)

Here the strength of the deformation is determined by the quantity \( \mu \), which has mass dimension 1. It is clear that the configuration (2.3) minimizes \( S_{SB} \). (Other configurations that minimize \( S_0 \) and \( S_{SB} \) will be discussed below.) Unfortunately, in the supersymmetric version of CKKU, the deformation \( S_{SB} \) breaks the exact supersymmetry of their original lattice action (hence the subscript “SB”). For this reason they demand that the strength of \( S_{SB} \) relative to \( S_0 \), conveyed by \( \mu^2 \), be scaled to zero in the thermodynamic limit. Thus we are mostly interested in the effects of the deformation subject to this scaling.

In much of what follows we will specialize to the case of \( U(2) \). This is merely because it is the simplest case and the most efficient to simulate. In this special case, \( x_m, y_m, z_m \) will be unconstrained \( 2 \times 2 \) complex matrices.
The classical analysis

Here details are given of the classical analysis of the minima of the action. In Section 3.1, we consider the undeformed action $S_0$; then in Section 3.2 we consider the modifications induced by the deformation $S_{SB}$, which has the effect of lifting some directions in moduli space. Through understanding this classical picture, naive expectations of what will occur in the quantum theory, based on energetics, can be formulated.

3.1 Undeformed theory

Here we neglect $S_{SB}$ and examine the minima of $S_0$. Note that (2.1) is a sum of terms of the form $\text{Tr} AA^\dagger$ (the first line involves squares of hermitian matrices). Thus $S_0 \geq 0$ with $S_0 = 0$ iff the following equations hold true:

$$x_n^{\dagger} x_{n-i} - x_n x_{n-i} + y_{n-j} y_{n-j} - y_n y_n + [z_n^{\dagger}, z_n] = 0 \quad (3.1)$$

$$x_n y_{n+i} - y_n x_{n+i} = y_n z_{n+j} - z_n y_n = z_n x_n - x_n z_{n+i} = 0 \quad (3.2)$$

together with the h.c. of (3.2). The set of solutions is the moduli space of the undeformed theory.

3.1.1 Zeromode branch

To begin a study of the moduli space, we isolate the zero momentum modes: $x_n \equiv x \forall n$, etc. Then Eqs. (3.1) and (3.2) reduce to

$$[x^{\dagger}, x] + [y^{\dagger}, y] + [z^{\dagger}, z] = 0$$
$$[x, y] = [y, z] = [z, x] = 0 \quad (3.3)$$

together with the h.c. of the second line of (3.3). Eqs. (3.3) may be recognized as nothing but the $D$-flatness and $F$-flatness constraints that describe the moduli space associated with the classical scalar vacuum of $N = 4$ 4d super-Yang-Mills. The equations are invariant with respect to the global gauge transformation

$$x \to \alpha x \alpha^{\dagger}, \quad y \to \alpha y \alpha^{\dagger}, \quad z \to \alpha z \alpha^{\dagger} \quad (3.4)$$

Then it is well-known that solutions to (3.3) consist of $x, y, z$ that lie in a Cartan subalgebra of $U(k)$; the proof is reviewed in Appendix A. The global gauge transformations (3.4) allow one to change to a basis where this Cartan subalgebra has a diagonal realization. Thus one can think of the moduli space as the set of all possible diagonal matrices $x, y, z$, and all global gauge transformations (3.4) of this set.

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We thank Erich Poppitz for pointing this out to us, as well as the branch of moduli space given below.
In particular, the zeromode moduli space of the undeformed $U(2)$ theory is completely described by

$$x = x^0 + x^3 \sigma^3, \quad y = y^0 + y^3 \sigma^3, \quad z = z^0 + z^3 \sigma^3,$$  \hspace{1cm} (3.5)

with arbitrary complex numbers $x^0, x^3, y^0, y^3, z^0, z^3$, together with $U(2)$ transformations of these solutions.

Eqs. (3.1) and (3.2) also have non-zeromode solutions. We do not attempt to present an exhaustive account of them. We will merely point out a few such branches in order to illustrate that the undeformed theory has a very complicated and large set of $S_0 = 0$ configurations. This observation will be relevant to our interpretation of the simulation results in Section 5.

3.1.2 $x_m = y_m = 0$ non-zeromode branch

We have the very “large” branch of moduli space described by

$$x_m = y_m = 0, \quad z_m = z^0_m + z^3_m \sigma^3, \quad \forall m$$  \hspace{1cm} (3.6)

Again, $z^0_m, z^3_m$ are arbitrary complex numbers. Furthermore, $z_m$ is a site variable and thus transforms independently at each site as

$$z_m \rightarrow \alpha_m z_m \alpha_m^\dagger$$  \hspace{1cm} (3.7)

It can be seen that this branch affords a vast number of solutions to (3.1) and (3.2); there are $N^2$ such solutions, modulo choices for $z^0_m, z^3_m \in \mathbb{C}$ and gauge equivalences. We will argue below that an understanding of the entropic effects that result from this branch is necessary to understanding expectation values in the quantum theory—even when a potential is introduced which gives these configurations nonvanishing action.

3.1.3 $z_n = 0$ non-zeromode branch

Another branch in moduli space is the following. First we set $z_n = 0, \forall n$, and introduce Fourier space variables

$$x_n = \frac{1}{N} \sum_k \omega^{k_n} f_k, \quad y_n = \frac{1}{N} \sum_k \omega^{k_n} g_k, \quad \omega = \exp(2\pi i/N)$$  \hspace{1cm} (3.8)

where $k = (k_1, k_2)$ and $k_1, k_2 \in [0, 1, \ldots, N - 1]$. Then taking into account $z_n = 0$, the conditions (3.1) and (3.2) are equivalent to:

$$0 = \sum_k \left( \omega^{k_\ell} f_k \tilde{f}_{k-\ell} + \omega^{3k_\ell} f_k \tilde{f}_{k+\ell} + \omega^{k_\ell} g_k \tilde{g}_{k-\ell} - g_k \tilde{g}_{k+\ell} \right)$$

$$0 = \sum_k \left( \omega^{-k_{(\ell+k)}} f_k g_{k-\ell} - \omega^{3k} f_k g_{k-\ell} \right)$$  \hspace{1cm} (3.9)
for all $\ell = (\ell_1, \ell_2)$ and $\ell_1, \ell_2 \in [0, 1, \ldots, N - 1]$. Next we turn off all modes except one for both $f_k$ and $g_k$:

$$f_k = \delta_{k,k'} f_{k'}, \quad g_k = \delta_{k,-k'} g_{-k'}$$

(3.10)

Here and below, no sum over $k'$ is implied. When substituted into (3.9), only 2 nontrivial conditions survive:

$$0 = \left[ f_{k'}, f^\dagger_{k'} \right] + \left[ g^\dagger_{-k'} g_{-k'} \right], \quad 0 = f_{k'} g_{-k'} - \omega^{(i+j)k'} g_{-k'} f_{k'}$$

(3.11)

For the $U(2)$ case, we find that solutions exist if $\omega^{(i+j)k'} = \pm 1$. We already know from the zeromode considerations that for $\omega^{(i+j)k'} = 1$ we have solutions for $f_{k'}, g_{-k'}$ diagonal matrices. In the case of $\omega^{(i+j)k'} = -1$ it is easy to see that there are solutions, say, of the form

$$f_{k'} = z_f \sigma^3, \quad g_{-k'} = z_g (\sigma^1 + b \sigma^2), \quad z_f, z_g \in \mathbf{C}, \quad b \in \mathbf{R}$$

(3.12)

There are many values of $k'$ for which $\omega^{(i+j)k'} = \pm 1$. For $N$ even these are

$$k'_1 + k'_2 = 0, \frac{N}{2}, N, \frac{3N}{2}$$

(3.13)

For $N$ odd, $k'_1 + k'_2 = 0, N$ are allowed and in the cases where

$$k'_1 + k'_2 = \frac{N \pm 1}{2}, \frac{3(N \pm 1)}{2}$$

(3.14)

(3.12) yield approximate solutions to (3.11), with an error of order $1/N$. Thus in the $N \to \infty$ limit the number of $S_0 = 0$ configurations in this class is vast; in fact, it is easy to check that the number of such configurations is approximately $2N$, modulo gauge equivalences and various choices for the constants in (3.12).

### 3.2 Deformed theory

Now we consider the supersymmetry breaking deformation $S_{SB}$ introduced by CKKU. To see its effect it is handy to rewrite the quantities that appear in it. Recall that $x_m$ is a complex $2 \times 2$ matrix. Dropping the subscript, we can always define

$$x = x^0 + x^a \sigma^a, \quad x^\dagger = \overline{x}^0 + \overline{x}^a \sigma^a$$

(3.15)

Then it is straightforward to work out ($\mu = 0, \ldots, 3$)

$$\begin{align*}
xx^\dagger &= x^\mu \overline{x}^\mu + (x^0 \overline{x}^c + \overline{x}^0 x^c + ix^a \overline{x}^b \epsilon^{abc}) \sigma^c \equiv \phi^{x,0} + \phi^{x,c} \sigma^c \equiv \phi^x
\end{align*}$$

(3.16)
Note that $\phi^{x,\mu}$ are real, and that $\phi^{x,0}$ is positive definite. With similar definitions for $\phi^{y,0}$, $\phi^{z}$, the CKKU deformation is

$$S_{SB} = \frac{a^2 \mu^2}{2g^2} \sum_m \text{Tr} \left[ \left( \phi_m^x - \frac{1}{2a^2} \right)^2 + \left( \phi_m^y - \frac{1}{2a^2} \right)^2 + \frac{2}{a^2} \phi_m^z \right]$$

$$= \frac{a^2 \mu^2}{g^2} \sum_m \left[ \left( \phi_m^{x,0} - \frac{1}{2a^2} \right)^2 + \left( \phi_m^{y,0} - \frac{1}{2a^2} \right)^2 + \frac{2}{a^2} \phi_m^{z,0} \right]$$

$$+ \sum_a \left[ (\phi_m^{x,a})^2 + (\phi_m^{y,a})^2 \right]$$

(3.17)

It can be seen that the deformation drives $\phi_m^{x,a}, \phi_m^{y,a}, \phi_m^{z,0}$ toward the origin, and $\phi_m^{x,0}, \phi_m^{y,0}$ toward $1/2a^2$. When $\phi_m^{z,0} = 0$, it is easy to see that $\phi_m^{x,a} = 0$ identically.

To continue the analysis, it is convenient to rescale to dimensionless quantities using the parameter $\tilde{a}$:

$$\tilde{g} = ga^2, \quad \tilde{\mu} = \mu a, \quad \tilde{\phi}_m^x = a^2 \phi_m^x, \quad \tilde{x}_m = ax_m, \quad \text{etc.}$$

(3.18)

Then

$$S_{SB} = \frac{\tilde{g}^2}{\tilde{g}^2} \sum_m \left[ \left( \tilde{\phi}_m^{x,0} - \frac{1}{2} \right)^2 + \left( \tilde{\phi}_m^{y,0} - \frac{1}{2} \right)^2 + 2\tilde{\phi}_m^{z,0} + \sum_a \left[ (\tilde{\phi}_m^{x,a})^2 + (\tilde{\phi}_m^{y,a})^2 \right] \right]$$

(3.19)

For any value of the lattice spacing $a$, the minimum of $S_{SB}$ is obtained iff

$$\tilde{\phi}_m^{x,0} = \tilde{\phi}_m^{y,0} = \frac{1}{2}, \quad \tilde{\phi}_m^{z,0} = \tilde{\phi}_m^{x,a} = \tilde{\phi}_m^{y,a} = 0, \quad \forall m$$

(3.20)

The conditions involving $x_m$ are just

$$\tilde{x}_m^\mu \tilde{x}_m^\mu = \frac{1}{2}, \quad \tilde{x}_m^0 \tilde{x}_m^0 + \tilde{x}_m^0 \tilde{x}_m^c + i\tilde{x}_m^a \tilde{x}_m^b \epsilon^{abc} = 0$$

(3.21)

Let us examine what additional constraint this places on classical solutions to $S = 0$, beyond the restrictions of the undeformed theory.

First we note that neither of the non-zeromode branches discussed in Section 3.1 above are minima of $S_{SB}$. Thus we pass on to the zeromode configurations (3.5). Eqs. (3.21) then imply that (3.5) is restricted to the form

$$\tilde{x} = \frac{e^{i\gamma x}}{\sqrt{2}} \text{diag} \left( e^{i\varphi_x}, e^{-i\varphi_x} \right)$$

and global gauge transformations of this. That is, $\tilde{x}$ is restricted to be an element of the maximal abelian subgroup $U(1)^2$ of $U(2)$, up to an overall factor of $1/\sqrt{2}$. Similarly, we have for $\tilde{y}$,

$$\tilde{y} = \frac{e^{i\gamma y}}{\sqrt{2}} \text{diag} \left( e^{i\varphi_y}, e^{-i\varphi_y} \right)$$

(3.22)
Finally, $\hat{\varphi}_{z,0}^0 = 0$ implies $z_m^0 = 0$, which has the unique solution $z_m = 0$, $\forall m$.

Apart from global obstructions that are essentially Polyakov loops in the $i$ or $j$ directions, the configuration (3.22) and (3.23) can be gauged away. It is straightforward to verify that the required gauge transformation is (2.2) with

$$
\alpha_{m_1,m_2} = e^{i(m_1 \gamma_x + m_2 \gamma_y)} \times \text{diag} \left( e^{i(m_1 \varphi_x + m_2 \varphi_y)}, e^{-i(m_1 \varphi_x + m_2 \varphi_y)} \right)
$$

This sets all $\hat{x}_m, \hat{y}_m$ to unity except at the “boundaries”:

$$
\hat{x}_{N,m_2} = \frac{e^{iN\gamma_x}}{\sqrt{2}} \text{diag} \left( e^{iN\varphi_x}, e^{-iN\varphi_x} \right), \forall m_2
$$

$$
\hat{y}_{m_1,N} = \frac{e^{iN\gamma_y}}{\sqrt{2}} \text{diag} \left( e^{iN\varphi_y}, e^{-iN\varphi_y} \right), \forall m_1
$$

For most purposes, we do not expect such vacua to distinguish themselves from the trivial vacua in the thermodynamic limit. In any case, global features such as these are typical of classical vacua of other lattice Yang-Mills formulations, such as the Wilson action. In our simulation study we will avoid this issue by restricting our attention to the expectation value of a quantity that is independent of these angles.

### 4 Simulation

The emergence of the effective lattice theory relies upon the assumption that fluctuations about this classical minimum are small, and that the equations (3.20) are a good approximation to the corresponding expectation values in the quantum theory. For this reason we study

$$
\langle \hat{\varphi}_{z,0} \rangle = \langle \hat{x}_m^\mu \hat{y}_m^\mu \rangle = \left( \frac{1}{2} \text{Tr} (\hat{x}_m \hat{x}_m^\dagger) \right)
$$

in our simulations, and compare the expectation values to the classical prediction (3.20).

#### 4.1 Scaling

We study (4.1) along a naive scaling trajectory:

$$
g_2 = a^{-1} \hat{g}(a) = \text{fixed}
$$

That is, we hold the bare coupling in physical units, $g_2$, fixed; this is equivalent to neglecting its anomalous dimension. The dimensionless bare coupling $\hat{g}$ is then a function of $a$ that vanishes linearly with $a$ as the UV cutoff is removed.

With regard to $\hat{\mu}$ we follow the instructions of CKKU: we send the dimensionless coefficient $\hat{\mu}$ of the deformation $S_{SB}$ to zero as $1/N$ while increasing $N$.

$$
\hat{\mu}^{-1} = cN, \quad c = \mathcal{O}(1)
$$
This is equivalent to scaling $\mu = 1/cL$, where $c$ is a constant and $L = Na$ is the extent of the system.

In the rescaled variables (5.18), the coefficient of the undeformed action is $1/\hat{g}^2$, whereas the coefficient of the deformation is $\hat{\mu}^2/\hat{g}^2$, as can be seen from (3.19). Thus it is that the relative strength of the deformation vanishes in the thermodynamic limit, when (4.3) is imposed.

We perform these scalings for a sequence of decreasing values of $a$. That is, we study the thermodynamic limit for fixed values of $a$. We then extrapolate toward $a = 0$ to obtain the continuum limit.

The physical length scales are set by $g_{2}^{-1}$ and the system size $L = Na$. To keep discretization effects to a minimum we would like to take $g_{2}^{-1} \gg a$. Equivalently, $\hat{g}^{-1} \gg 1$. On the other hand we are most interested in what happens at large or infinite volume. To render finite volume effects negligible would require $g_{2}^{-1} \ll L$. Equivalently, $\hat{g}^{-1} \ll N$. In the simulations we study the system for various choices of $\hat{g}^{-1}$ and $N$. We extrapolate to the regime $1 \ll \hat{g}^{-1} \ll N$, but often violate the bounds $1 \leq \hat{g}^{-1} \leq N$ for specific points where measurements are taken. The reason for this is that data outside the optimal window $1 \ll \hat{g}^{-1} \ll N$ is informative to the extrapolation.

4.2 Sampling procedure

We update the system using a multi-hit Metropolis algorithm. We attempt to update a single site or link field 10 times before moving to the next, with an acceptance rate of approximately 50 percent for a single hit. We find that this minimizes autocorrelations while maintaining program efficiency. We have examined autocorrelations and the dependence of our observables on initial conditions. These studies have led us to make 500 thermalization sweeps after a random initialization, and 100 updating sweeps between each sample. 1/2 to 2 percent standard errors result from accumulating 1000 samples at each data point.

4.3 Results

In Fig. 1 we show $\langle \hat{\phi}^{x,0}_{m} \rangle$ [cf. (4.1)] as a function of $N$ for various values of $\hat{g}^{-1}$, having set $c = 1$ in (4.3). Doubling $\hat{g}^{-1}$ is equated with halving the lattice spacing, according to (4.2). It can be seen that $\langle \hat{\phi}^{x,0}_{m} \rangle$ appears to tend toward smaller values as the lattice regulator is removed, contrary to the classical expectations (3.20). At large enough values of $N$ the curves flatten out to a constant value.

To understand this behavior, we first note that we are computing an expectation value that is already nonvanishing in the undeformed theory ($\mu \equiv 0$). The undeformed theory expectation values

\[ \langle g^{-1}\hat{\phi}^{x,0}_{m} \rangle = \langle \hat{g}^{-1}\hat{\phi}^{x,0}_{m} \rangle \]  

\[ (4.4) \]

for various values of $N$ are shown in Fig. 2. The rescaling by $g^{-1} = a^{2}\hat{g}^{-1}$ is useful because it corresponds to removing $g$ from the undeformed action $S_{0}$ by a rescaling of the lattice
Figure 1: Trajectories of fixed lattice spacing, increasing volume, with data connected by lines to guide the eye. Each line is marked by the corresponding value of $\hat{g}^{-1}$. A doubling of $\hat{g}^{-1}$ corresponds to a halving of the lattice spacing.
variables [cf. (2.1)]; this amounts to studying the undeformed theory in units of $\sqrt{g}$. It can be seen from Fig. 2 that this quantity is far from zero, and is rather insensitive to $N$. The undeformed theory only contains two length scales, $1/\sqrt{g}$ and $N/\sqrt{g}$. For large $N$, it is not surprising that the local expectation value (4.4) is insensitive to this long distance scale. Rather, it is determined by the short distance scale $1/\sqrt{g}$. The deformed theory retains this short distance scale. Once the system size is much larger than this scale, the local expectation value becomes independent of the system volume; this is particularly true because the relative strength of the deformation is being scaled away to zero [cf. (4.3)].

In Table 1 we show the large $N$ expectation values (4.4) in the deformed theory as well as those of the undeformed theory. Here, the values for $N = 16, 18, 20$ were averaged for each $\hat{g}^{-1}$, which should provide a good estimate of the asymptotic value, as can be seen in Fig. 1. The error was estimated based on the maximum deviation from this mean, among the three data points, taking into account the $1\sigma$ error estimates that have been represented in the figure by error bars. Table 1 shows that the large $N$ expectation values (4.4) in the deformed theory are (up to statistical errors) the same as those of the undeformed theory.

5 Interpretation

In our simulations, we have observed that under the scaling (4.3), the deformation becomes ineffective at changing the expectation value (4.4), or equivalently (4.1), away from the value that would be obtained in the undeformed theory. What has happened is that the flat directions that were lifted by the deformation are becoming flat all over again as $N \rightarrow \infty$.

More precisely, the deformation is proportional to $1/g_s^2 L^2$, and in the thermodynamic limit this quantity vanishes. Whereas the configurations of the moduli space of the undeformed theory that were lifted by $S_{SB}$ cost energy at finite $N$, the number of such configurations
\[
\hat{g}^{-1} \langle \hat{\phi}^{x,0} \rangle \langle \hat{g}^{-1} \hat{\phi}^{x,0} \rangle
\]

| \( \hat{g}^{-1} \) | \( \langle \hat{\phi}^{x,0} \rangle \) | \( \langle \hat{g}^{-1} \hat{\phi}^{x,0} \rangle \) |
|---|---|---|
| 2 | 0.2469(94) | 0.494(19) |
| 4 | 0.1249(27) | 0.500(11) |
| 8 | 0.0618(22) | 0.494(18) |
| 16 | 0.03064(75) | 0.490(12) |
| 32 | 0.01586(64) | 0.507(21) |
| \( 1 (\mu \equiv 0) \) | | 0.496(22) |

Table 1: Large \( N \) asymptotic values for the \( \hat{\mu}^{-1} = N \) trajectories. For comparison, the result for the undeformed \( (\mu \equiv 0) \) expectation value is shown in the bottom line. Estimated errors in the last 2 digits are shown in parentheses.

is becoming vast due to the approximate flatness in those directions. For \( N \gg \hat{g}^{-1/2} \), the entropy of these configurations wins out over the energy arguments that prefer (3.20).

One lifted region of the \( S_0 \) moduli space where this is particularly clear is the branch (3.6). The action for such configurations is (setting \( c = 1 \))

\[
S = \frac{1}{2g^2} + \frac{2}{\hat{g}^2 N^2} \sum_m (|\hat{z}_0^0|^2 + |\hat{z}_3^0|^2) = \frac{N^2}{2g^2 L^2} + \frac{2}{g^2 L^2} \sum_m (|\hat{z}_0^0|^2 + |\hat{z}_3^0|^2)
\]

(5.1)

Integrating \( \exp(-S) \) over all \( \hat{z}_{0,3}^m \) we obtain

\[
\left( \frac{\pi}{2} \right)^{2N^2} \exp \left[ \frac{N^2}{2g^2 L^2} (-1 + 4g^2 L^2 \ln(g^2 L^2)) \right]
\]

(5.2)

Note that for large system size \( g_2 L \gg 1 \). Thus, the positive (entropic) term under the exponential wins out over the negative (energetic) term by a large margin. It would seem that the weight of these configurations increases exponentially as we increase \( N \) while holding \( g_2 L \) fixed; that is, in the continuum limit.

Indeed the observed behavior summarized in Table 1 is that in the deformed theory

\[
\left\langle \frac{1}{2} \text{Tr} (\hat{x}_m \hat{x}_m^\dagger) \right\rangle \approx \frac{g_2 L}{2N} = \frac{1}{2} \hat{g} a
\]

(5.3)

Thus the simulations likewise indicate that configurations with \( \hat{x}_m = 0 \) are dominating as we increase \( N \) while holding \( g_2 L \) fixed, which is nothing but the continuum limit. By symmetry, the same also holds for \( \hat{y}_m \).

6 Conclusions

Naturally we find our results disappointing. Attempts to formulate supersymmetric field theories on the lattice have a long and troubled history (see for example [12, 13, 14, 15, 16].

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and references therein). Recent success in non-gauge models is very encouraging [17, 18, 19]. Of particular importance is the understanding of these models as “topological” or $Q$-exact, where $Q$ is an exact supercharge of the lattice system [20, 21, 22]. Indeed, the CKKU undeformed action can be written in a $Q$-exact form. An exciting development has been Sugino’s exploitation of this $Q$-exact idea to construct lattice super-Yang-Mills with compact gauge fields and an ordinary (non-dynamical) lattice spacing [23]. While the Sugino construction has its benefits, he notes (based on a remark by Y. Shamir) that these systems also suffer from a vacuum degeneracy problem that renders the classical continuum limit ambiguous. Consequently, Sugino has introduced a non-supersymmetric deformation in these models that lifts the unwanted vacua; he demands that the relative strength of this deformation be scaled to zero in the thermodynamic limit. Thus in some respects the actions of Sugino suffer from a problem that is similar to that of CKKU. However, he argues that entropic effects do not destroy the vacuum selection imposed by his deformation. We are currently investigating this matter [24].

However, we continue to find the constructions of CKKU intriguing. We would like to explore other possibilities for the deformation. In the non-supersymmetric case, we do not see any compelling reason to scale the relative strength of the deformation to zero in the thermodynamic limit. We are currently investigating whether or not a well-defined continuum limit can be obtained for the present system if this is not done. Furthermore, we would like to better understand the phase structure of the deformed theory. Finally, it would be interesting to extend the present analysis to the supersymmetric system of CKKU, as well as higher dimensional systems. In the former case we expect supersymmetric nonrenormalization theorems to play a role in the behavior of expectation values as the deformation is scaled away. In the latter case, spontaneous symmetry breaking is allowed—since we do not face the theorems special to 2d [25, 26]; so a nontrivial phase structure may exist that would allow for a well-defined continuum limit.

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Appendix

A $\mathcal{N} = 4$ moduli space

Here we establish the well-known solution to (3.3). One way to see this is as follows [27]. First we note that $S_0$ reduced to the zero modes, which we write as $S_z$, takes the form

$$S_z = \frac{N^2}{g^2} \text{Tr} \left( \frac{1}{2} \left( [x^\dagger, x] + [y^\dagger, y] + [z^\dagger, z] \right)^2 + 2[x, y][y^\dagger, x^\dagger] + 2[y, z][z^\dagger, y^\dagger] + 2[z, x][x^\dagger, z^\dagger] \right)$$

(A.1)

Now note that the $U(1)$ parts of $x, y, z$ do not appear and can take any value. Thus we can restrict our attention the the $SU(k)$ parts, which we choose to express in terms of Hermitian matrices $a_p, b_p, \ p = 1, 2, 3$:

$$x^c T^c = (a_1^c + i b_1^c) T^c = a_1 + i b_1$$
$$y^c T^c = (a_2^c + i b_2^c) T^c = a_2 + i b_2$$
$$z^c T^c = (a_3^c + i b_3^c) T^c = a_3 + i b_3$$

(A.2)

Substitution into (A.1) and a bit of algebra yields

$$S_z = -\frac{N^2}{g^2} \text{Tr} \left[ 2 \left( \sum_p (a_p, b_p) \right)^2 + \sum_{p,q} (a_p, b_q)^2 \right]$$

$$= -\frac{N^2}{g^2} \sum_{p,q} \text{Tr} \left( [a_p, a_q]^2 + [b_p, b_q]^2 + 2[a_p, b_q]^2 \right)$$

(A.3)

Using positivity arguments quite similar to those above, one finds that $S_z \geq 0$ and that $S_z = 0$ iff

$$[a_p, a_q] = [b_p, b_q] = [a_p, b_q] = 0, \ \forall p, q$$

(A.4)

which is nothing other than Eq. (53) of [27]. Since the matrices are all hermitian and they all commute, it is obviously possible to choose a basis which simultaneously diagonalizes them. This basis will be related to the one used in (A.2) according to $T^c \rightarrow T^c' = \alpha T^c \alpha^\dagger$, which is nothing other than the global gauge transformations (3.4).

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