Criticality in the Quantum Kicked Rotor with a Smooth Potential

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We investigate the possibility of an Anderson type transition in the quantum kicked rotor with a smooth potential due to dynamical localization of the wavefunctions. Our results show the typical characteristics of a critical behavior i.e. multifractal eigenfunctions and a scale-invariant level-statistics at a critical kicking strength which classically corresponds to a mixed regime. This indicates the existence of a localization to delocalization transition in the quantum kicked rotor.

Our study also reveals the possibility of other type of transitions in the quantum kicked rotor, with a kicking strength well within strongly chaotic regime. These transitions, driven by the breaking of exact symmetries e.g. time-reversal and parity, are similar to weak-localization transitions in disordered metals.

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I. INTRODUCTION

The analogy of the statistical fluctuations of dynamical systems and disordered systems is well-known in the delocalized wave-regime (corresponding to the metallic limit in disordered systems and the classically chaotic limit in dynamical systems) and has been explained using random matrix theory as a tool [1, 2, 3, 4]. A similar analogy exists for fully localized regimes of the wavefunctions too (i.e. between the insulator limit of disordered systems and the integrable limit of dynamical systems) [1, 2, 3]. It is therefore natural to probe the presence/absence of the analogy in partially localized/critical regimes of these systems. Our analysis shows that, similar to the $d > 2$ Anderson Hamiltonian ($d$ as dimension), the $d = 1$ quantum kicked rotor (QKR) undergoes a localization-delocalization transition in the classically mixed regime. We also find quantum phase transitions in its chaotic regime due to breaking of the symmetries e.g. time-reversal and parity in the quantum system. Similar to disordered systems, the symmetry-breaking transitions in the QKR occur due to weak-localization effects. Similar phase transitions due to symmetry-breakings have been seen in a few other complex systems too e.g. the ensembles of distinguishable spins [5].

The connection of the kicked rotor to the $d = 1$ Anderson Hamiltonian has been known for several decades [2, 6, 7, 8]. A recent work [9] further explores the connection and shows that, for the non-analytic potentials in the QKR, the eigenstates show multifractality or power-law localization [10, 11, 12, 13], a behavior similar to the eigenstates of a $d > 2$ dimensional Anderson Hamiltonian [14] at its critical point. Our study shows existence of the multifractal eigenstates in the QKR with smooth potentials too e.g. $V(q) = K\cos(q)$ at specific parametric conditions. Furthermore, similar to a critical Anderson system, the multifractality in the QKR is accompanied by a critical level statistics (size-independent and different from the two ends of the transition), a necessary criteria for the critical behavior [10]. This indicates a much deeper connectivity of the kicked rotor to the Anderson Hamiltonian, not affected just by the nature of the potential or the dimension of system. As discussed here, the connection seems to be mainly governed by the ”degree of complexity” (measured by the complexity parameter discussed later) and may exist among a wider range of dynamical and disordered systems.

The present study is motivated by a recent analytical work [15, 16] leading to a common mathematical formulation for the statistical fluctuations of a wide range of complex systems. The work, based on the ensemble-averaging, shows that the fluctuations are governed by a single parameter $\Lambda$ besides global-constraints on the system [15, 16]. $\Lambda$ referred as the complexity parameter, turns out to be a function of the average accuracy of the matrix elements, measured in units of the mean-level spacing. The fluctuations in two different systems, subjected to similar global-constraints, are analogous if their complexity parameters are equal irrespective of other system-details.

The $\Lambda$-formulation was recently used by us to find the Gaussian Brownian ensemble (GBE) [17] analog and the power-law random banded matrix ensemble (PRBME) [18] analog of the Anderson system (for arbitrary $d$) [19]. However it can not directly be applied to find the QKR analog; this is, in principle, due to inapplicability of the ensemble averaging to dynamical systems. Fortunately it is possible to derive $\Lambda$ for dynamical systems by a semi-classical route, using the phase-space averages [20]. The ”semi-classical” $\Lambda$ was used by us [21, 22] to map the statistics
of the time-evolution operator $U$ of the QKR to the circular Brownian ensembles (CBE) \cite{17}. The lack of a suitable criteria for the critical statistical behavior prevented us earlier from a critical QKR-analysis. Our present work pursues the analysis by first analytically identifying the critical QKR-behavior using the "semi-classical" $\Lambda \equiv \Lambda_{kr}$; the limit $N \to \infty \Lambda \to \Lambda^*$, with $\Lambda^*$ independent of the size $N$, gives the critical points of transition. This is followed by a numerical analysis of an ensemble of the QKR which confirms critical behavior at the semi-classically predicted values. This in turn suggests a paradoxical validity of the ensemble averaging in dynamical systems at least of QKR type; (also indicated by the CBE-QKR mapping). A subsequent comparison of the "semi-classical" $\Lambda_{kr}$ to the "ensemble-based" $\Lambda_{ae}$ of the Anderson Hamiltonian (arbitrary $d$) gives us its QKR analog; the analogy is numerically confirmed too.

The statistical behavior in the bulk of the spectrum of a standard Gaussian ensemble is known to be analogous to a standard circular ensemble (for large matrix sizes) \cite{17, 22}. Our work extends this analogy to their Brownian ensemble counterparts too, that is, between the GBE and the CBE (by the mapping GBE $\to$ AE $\to$ QKR $\to$ CBE). This in turn indicates a connection among a wide-range of physical systems which are known to be well-modeled by the Anderson Hamiltonian, the kicked rotor and the Brownian ensembles of Gaussian and circular type.

The paper is organized as follows: The section II briefly reviews the basic features of the kicked rotor and the Anderson Hamiltonian required for our analysis; it also discusses the parametric conditions in the kicked rotor which can support critical points. The section III deals with the numerical confirmation of the critical level-statistics and the multifractality of the eigenfunctions at critical parametric conditions in the QKR and their comparison with a $d = 3$ dimensional Anderson Hamiltonian. We conclude in section IV with a brief discussion of our main results and open questions.

II. KICKED ROTOR AND ANDERSON HAMILTONIAN

The kicked rotor and the Anderson Hamiltonian have been subjects of intense study in past and many of their details can be found in several references \cite{11, 2, 4, 8, 23}. However, for self consistency of the paper, we present here a few details required for later discussion.

A. Kicked Rotor and Complexity Parameter

The kicked rotor can be described as a pendulum subjected to periodic kicks (of time-period $T$) with Hamiltonian $H$ given as

$$H = \frac{(p + \gamma)^2}{2} + K \cos(\theta + \theta_0) \sum_{n=-\infty}^{\infty} \delta(t - nT). \quad (1)$$

Here $K$ is the stochasticity parameter, $\gamma$ and $\theta_0$ are the time-reversal and parity symmetry breaking parameters in the quantum Hamiltonian while acting in a finite Hilbert space.

Integration of the equations of motion $\dot{\theta} = -\frac{\partial H}{\partial p}$, $\dot{p} = \frac{\partial H}{\partial \theta}$ between subsequent kicks e.g. $n$ and $n + 1$ gives the classical map,

$$p_{n+1} = p_n + K \sin(\theta_n + \theta_0) \pmod{2\pi},$$
$$\theta_{n+1} = \theta_n + p_{n+1} \pmod{2\pi} \quad \text{(2)}$$

The map is area-preserving and invariant under the discrete translation $\theta \to \theta + 2\pi, p \to p + 2\pi$. It also preserves the time-reversal symmetry $p \to 2\pi - p, \theta \to \theta, t \to -t$ and the parity $p \to 2\pi - p, \theta \to 2\pi - \theta$ for all values of $\gamma$ and $\theta_0$. Thus the classical dynamics depends only on $K$, changing from integrable ($K = 0$) to near integrable ($0 < K < 4.5$) to large scale chaos ($K > 4.5$).

The quantum dynamics can be described by a discrete time-evolution operator $U = G^{1/2} \cdot B \cdot G^{1/2}$ \cite{2, 5} where

$$B = \exp\left[-ik\cos(\theta + \theta_0)\right] \quad \text{(3)}$$
$$G = \exp\left[-iT(p + \gamma)^2/2\hbar\right] = \exp\left[i\tau \left(\frac{\partial}{\partial \theta} - i\frac{\gamma}{\hbar}\right)^2/2\right] \quad \text{(4)}$$
Here $k = K/\hbar$, $\tau = T\hbar$, $\theta$ and $p = i\hbar \partial/\partial \theta$ are the position and momentum operators respectively; $p$ has discrete eigenvalues, $p|m >= m\hbar|m$ $(m = 1 \to N)$ due to periodicity of $\theta$, $(\theta \to \theta + 2\pi)$. The choice of a rational value for $\tau/4\pi = M/N$ results in a periodicity also for momentum operator $p' = p + 4\pi M/T$, $(l' = l + N)$ and therefore in discrete eigenvalues for $\theta$, $(\theta | >= (2\pi l/N) | n >)$. The quantum dynamics can then be confined to a two dimensional torus (with a Hilbert space of finite size $N = 2\pi n_0/\tau$ with $n_0$ an integer). The classical analog of this model corresponds to the standard mapping on a torus of size $2\pi n_0/T$ in the momentum $p$; thus the classical limit is $\tau \to 0$, $k \to \infty$, $N \to \infty$ with $K = \text{constant}$ and $N\tau = \text{constant}$ [2, 7].

For the dynamical-localization analysis, it is useful to express the matrix $U$ in the momentum-basis [2, 2, 20]:

$$U_{mn} = \frac{1}{N} \exp [i\tau (m-\gamma/h)^2/4 + i\tau (n-\gamma/h)^2/4] \sum_{l=-N_1}^{N_1} \exp [-i k \cos (2\pi l/N + \theta_0)] \exp [-2\pi i l (m-n)/N]$$ (5)

where $n, m = -N_1, ..., N_1$ with $N_1 = (N-1)/2$ if $N$ is odd and $N_1 = N/2$ if $N$ is even. It is clear that the properties of $H$, eq.(4), are recovered in the infinite matrix size limit.

The quantum dynamics under exact symmetry conditions ($\gamma = 0$, $\theta_0 = 0$) can significantly be affected by relative values of $k, \tau, N$. It was first conjectured [2] and later on verified [20] that the statistical properties are governed by the ratio of the localization length $\zeta$ to the total number of states $N$ or, equivalently, $k^2/N$ (as $\zeta = D/2\tau^2$ with $D \approx K^2/2$ as the diffusion constant). However, to best of our knowledge, the critical behavior at $k^2 \approx N$ was not probed before. The other parameters playing a crucial role in the quantum dynamics are $\gamma$ and $\theta_0$, the measures of time-reversal and parity-symmetry breaking respectively (with $0 < \gamma < h$ and $-\pi/N < \theta_0 < \pi/N$). Note the change of $p \to p + \gamma$ or $\theta \to \theta + \theta_0$ is a canonical transformation, thus leaving the classical Hamiltonian unaffected. The corresponding quantum dynamics, however, is affected as the quantum Hamiltonian acting in a finite Hilbert space may not remain invariant under a unitary transformation.

Following eq.(3), the multi-parametric nature of $U$ is expected to manifest itself in the statistical behavior of its eigenvalues (quasi-energies) and the eigenfunctions. However, as shown in [20] using semi-classical techniques, the quasi-energy statistics of $U$ is sensitive to a single parameter $\Lambda_{kr}$ and the exact symmetry-conditions. Under exact time-reversal symmetry ($\gamma = 0$) and partially violated parity ($\theta_0 \neq 0$) (taking $T = 1$, equivalently, $\tau = h$, without loss of generality), we have [20]

$$\Lambda_{kr,t} = \frac{\theta_0^2 N k^2}{4\pi^2} = \frac{N^3\theta_0^2 K^2}{64\pi^4 M^2}$$ (6)

Note, for $\theta_0 = \pi/2N$, $\Lambda_{kr,t}$ is essentially the same as the one conjectured in [2] for scaling behavior of the spectral statistics. Similarly, for the strongly chaotic case ($k^2 > N$) with only parity symmetry ($\theta_0 = \pi/2N$) and no time-reversal ($\gamma \neq 0$),

$$\Lambda_{kr,nt} = \frac{\gamma_q^2 N^3}{48\pi^2}$$ (7)

with $\gamma_q = \gamma/h$. As eq.(6) indicates, $\Lambda_{kr,t} \to \infty$ in the strongly chaotic limit $k \to \infty$; similarly, from eq.(7), $\Lambda_{kr,t} \to \infty$ for $\gamma \approx h$; the statistics in these cases can be well-modeled [20, 24] by the standard random matrix ensembles of unitary matrices, known as the standard circular ensembles e.g. circular orthogonal ensemble (COE), circular unitary ensemble (CUE) etc [17]. The cases with a slow variation of $k$, $\gamma$ or $\theta_0$ (partial localization, partial time-reversal or parity-violation respectively) and finite size $N$ correspond to a smooth variation of $\Lambda_{kr,t}$ or $\Lambda_{kr,nt}$ between 0 and $\infty$. The intermediate statistics for these cases [20, 21] can be well-described by the circular Brownian ensembles. The latter are the ensemble of unitary matrices, described as $U_{w} = U_0^{1/2} \exp[iwV]U_0^{1/2}$ and characterized by a single parameter $\Lambda_{cbe} = w^2 \langle |V_{kl}|^2 \rangle / D^2$ ($D = 2\pi/N$) and the exact system-symmetries [17, 22]. The perturbation $V$ belongs to a standard Gaussian ensemble of the Hermitian matrices e.g Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE) etc [17]. The circular Brownian ensemble analog of a QKR is given by the condition

$$\Lambda_{kr} = \Lambda_{cbe}.$$ (8)

As mentioned above, the studies [20, 21] did not explore the infinite size limit of $\Lambda_{kr}$ and its application for the critical behavior analysis; we discuss it in next section.
B. Critical behavior of Quantum Kicked Rotor

For a critical point analysis of the QKR, we search for the system conditions leading to the critical eigenvalue-statistics and the multifractal eigenfunctions in the infinite size limit \((N \to \infty)\). The parametric conditions for the critical level statistics, characterized by a non-zero, finite \(\Lambda\) in the limit \(N \to \infty\), can be obtained from eqs. (6, 7). The analysis suggests the possibility of several continuous families of critical points, characterized by the complexity parameter and the exact symmetries; we mention here only the three main cases:

(i) \(k \propto \sqrt{N}, \gamma = 0, \theta_0 = \pi/2N\): Eq. (9) in this case leads to a size-independent \(\Lambda_{kr,t} = \chi^2/16\) with \(\chi = k/\sqrt{N}\). For small \(k\) (in the mixed regime), this corresponds to a localization \(\to\) delocalization transition under the time-reversal conditions (no parity symmetry) with a continuous family of critical points characterized by \(\chi\). The bulk-statistics here is analogous to a circular Brownian ensemble \(U_w\) (see eq. (8)) with \(w = \chi \pi/2\sqrt{N}\) (due to a GOE type perturbation \(V\) with \(|V_{kl}|^2 = (1 + \delta_{kl})\) of a Poisson matrix \([17]\), see eq. (5)). The two ends of the transition in this case are the Poisson \((\Lambda \to 0)\) and the COE ensemble \((\Lambda \to \infty)\).

(ii) \(\gamma \propto N^{-3/2}, \theta_0 = \pi/2N\): Eq. (7) in this case gives \(\Lambda_{kr,nt} = \lambda^2/48\pi^2\) with \(\lambda = \gamma q N^{3/2}\). For large \(k\) (in strongly chaotic regime), a finite \(\lambda\) gives the critical parameter for the transition from a time-reversible to a time-irreversible phase (both phases delocalized); it can be referred as the weak-localization critical point. The end-points of the transition are the COE \((\Lambda \to 0)\) and the CUE \((\Lambda \to \infty)\), with the critical statistics given by an intermediate circular Brownian ensemble with \(w = \lambda/(2N\sqrt{3})\) (due to a GUE type perturbation \(V\), with \(|V_{kl}|^2 = 1\), of a COE matrix \([17]\)).

(iii) \(\gamma = 0, \theta_0 \propto N^{-3/2}\): Here eq. (6) gives \(\Lambda_{kr,t} = \phi^2 K^2/(64\pi^4 M^2)\) with \(\phi = N^{3/2} \theta_0\) describing the critical point family for the transition from a parity-symmetric phase to a parity fully violated phase (both phases time-reversal). For \(K\) in the mixed regime \((K < Nh)\), a variation of \(\phi\) leads to the Poisson \(\to\) COE transition. For \(K\) in the strongly chaotic regime \((K > Nh)\), the transition end-points are the 2-COE \((\Lambda \to 0)\) and the COE \((\Lambda \to \infty)\). The critical statistics for a specific \(K\) is given by the intermediate Brownian ensemble with \(w = K\phi/(4\pi MN)\) (due to a GOE type perturbation \(V\), with \(|V_{kl}|^2 = 1\), of the Poisson ensemble if \(K << Nh\), or, the 2-COE ensemble if \(K >> Nh\)).

As eqs. (6, 7) indicate, a size-independent \(\Lambda_{kr}\) can be obtained by other combinations of \(k, \theta, \gamma\) too. This suggests critical behavior in the symmetry-spaces other than those mentioned above.

The critical nature of the system for specific parametric conditions can further be confirmed by an analysis of the eigenfunction fluctuations. The studies of a wide range of systems (see \([10, 11, 12, 13]\) and the references therein) reveal the presence of strong fluctuations in the eigenfunctions near a critical point. The fluctuations can be characterized through the set of generalized fractal dimension \(D_q\) or \(\tau_q = (q-1)D_q\), related to the scaling of the \(q^{th}\) moment of the wavefunction intensity \(|\phi(r)|^2\) with size \(N\) \([11]\): \(P_q = \int dr |\phi(r)|^2 = N^{-\tau(q)/d}\). The multifractality of the eigenfunctions can also be analyzed through the spectrum of singularity strengths \(f(\alpha)\) \([10]\), related to \(\tau(q)\) by a Legendre transformation: \(f(\alpha(q)) = \alpha q - \tau(q)\) (see \([11, 13]\) for details). In section III, we numerically analyze both \(\tau(q)\) and \(f(\alpha)\) to detect the multifractality of the QKR-eigenfunctions.

C. Anderson Hamiltonian and the Complexity Parameter

The Anderson model for a disordered system is described by a \(d\)-dimensional disordered lattice, of size \(L\), with a Hamiltonian \(H = \sum_n \epsilon_n a_n^\dagger a_n - \sum_{n \neq m} b_{mn}(a_m^\dagger a_n + a_n a_m^\dagger)\) in the tight-binding approximation \([14]\). In the site representation, \(H\) turns out to be a sparse matrix of size \(N = L^d\) with the diagonal elements as the site energies \(H_{kk} = \epsilon_k\) and the off-diagonals \(H_{mn} = b_{mn}\) given by the hopping conditions. For a Gaussian type on-site disorder (of variance \(\omega\) and zero mean) and a nearest neighbor (n.n.) isotropic hopping with both random (Gaussian) and/or non-random components, \(H\) can be modeled by an ensemble (later referred as the Anderson ensemble or AE) with following density

\[
\rho(H, v, b) = C \exp \left[ -\sum_k H_{kk}^2 / 2\omega - \sum_{(k,l)=n.n.} H_{kl}^2 / 2\eta \right] \prod_{(k,l)=n.n.} \delta(H_{kl} - t) \prod_{(k,l) \neq n.n.} \delta(H_{kl}) \tag{9}
\]
with $C$ as the normalization constant. As discussed in [19], the above ensemble can be rewritten as

$$\rho(H, h, b) = C \exp[-\sum_{k \leq l} (1/2h_{kl})(H_{kl} - b_{kl})^2]$$

(10)

where $b_{kl} = 0$, $h_{kk} = \omega$, $h_{kl} = \eta f_{kl}$ with $f(kl) = 1$ for $\{k, l\}$ pairs connected by the hopping, $f(kl) \rightarrow 0$ for all $\{k, l\}$ pairs representing the disconnected sites. The single parameter $\Lambda_{\text{ne}}$ governing the spectral statistics (see eq.(19) of [19]) can then be given as

$$\Lambda_{\alpha, e}(E, Y) = \left(\frac{|\alpha_w - \alpha_i| F^2}{\gamma_0}\right) \zeta^2 d L^{-d} \approx \left|\frac{\alpha_w - \alpha_i}{\gamma_0 N}\right| \left(\frac{F}{I_2^{\text{typ}}}\right)^2,$$

(11)

with

$$\alpha_w = \ln|1 - \gamma_0 \omega| + (z/2) \ln|1 - 2\gamma_0 \eta||t + \delta_0|$$

(12)

with $z$ as the number of the nearest-neighbors, $\alpha_i = -\ln 2$ and $\gamma_0$ as an arbitrary constant. Further $F(E)$ is the mean level density, $\zeta$ as the localization length and $I_2^{\text{typ}}$ as the typical inverse participation ratio: $I_2^{\text{typ}} \propto \zeta^{-d}$.

III. NUMERICAL ANALYSIS

The objectives of our numerical analysis are two fold: (i) a search for the critical points of the quantum kicked rotor, and, (ii) a comparison of its fluctuation measures with those of a 3-dimensional Anderson ensemble. For this purpose, we analyze the following cases:

(i) **QKR1: quantum dynamics time-reversal but parity broken**; $k^2 = \chi^2 N, \chi \approx 1.5, \gamma = 0, \theta_0 = \pi/2N, T = 1, \tau = \hbar = 40\pi/N$ which gives $K \approx 189/\sqrt{N}$. This case corresponds to the critical set (i) in section II.B and is analyzed for many sizes ($N = 213 \rightarrow 1013$) to verify the critical behavior.

(ii) **QKR2: quantum dynamics with both time-reversal and parity broken**; $k \approx 20000, \gamma = \lambda_q N^{-3/2}, \lambda_q = 6, T = 1, \theta_0 = \pi/2N, \tau = \hbar = 40\pi/N$. This case belongs to the set (ii) in section II.B and its critical nature is also confirmed by analyzing many $N$ values.

(iii) **QKR3**; same as QKR1 but with $\chi = 0.8$ which gives $K \approx 100/\sqrt{N}$. We consider this case to verify the analogy with a $d = 3$ Anderson ensemble.

(iv) **QKR4: quantum dynamics time-reversal but parity broken**; $k \approx 4.5/h, \gamma = 0, \theta_0 = \phi N^{-3/2}, \phi = 0.84\pi^2, T = 1, \tau = \hbar = 8\pi/N$ (case (iii) in section II.B). This is also analogous to the Anderson system mentioned above in the QKR3 case, notwithstanding the crucial changes in $k$ and $\theta_0$ for the two QKR cases.

To explore critical behavior, we analyze large ensembles of the matrices $U$ for both QKR1 and QKR2 for various matrix sizes $N$; the ensemble in each case is obtained by varying $k$ in a small neighborhood while keeping $N$ fixed. The chosen $N$ range give $K$ in the mixed regime for QKR1 ($6.25 < K < 13$) and in the chaotic regime for QKR2 ($1000 < K < 12000$). Prior to the analysis, the quasi-energies (the eigenvalues of $U$) are unfolded by the local mean level density $D^{-1} (= N/2\pi$, a constant due to repulsion and a unit-circle confinement of the quasi-energies). The figures 1.2 display the nearest-neighbor spacing distribution $P(s)$ and the number variance $\Sigma^2(r)$, the measures of the short and long-range spectral correlations respectively, for the QKR1 and the QKR2. Note the curves in figure 1 are intermediate to the Poisson and the COE limits; the size independence implies their survival in the infinite size limit too. This indicates the QKR1 as the critical point of transition from a localized phase to the delocalized phase. Similarly the curves in figure 2, intermediate to the COE and the CUE limits, suggest the QKR2 as the critical point of transition from the time-reversed phase to the time-irreversible phase (both phases in the chaotic regime).

To reconfirm the critical nature of the QKR1 and the QKR2, we numerical analyze the moments of their local eigenfunctions intensity for various sizes. The results shown for $\tau_q$ in figures 3a,4a indicate the multifractal nature
of the eigenfunctions; for small-$q$ ranges, $\tau_q$ shows a behavior $\tau_q = (1 + c) q - d - c q^2$ (or $D_q = 1 - c q$) with $d = 1$ and $c = 0.06, 0.075$ for the QKR1 and the QKR2 respectively (see table 1 for the first few values of $D_q$). These results are reconfirmed by a numerical study of the $f(\alpha)$ spectrum displayed in figures 3b, 4b. For this purpose, we use the procedure based on the evaluation of moments, described in \cite{13} (using eq.(4) and eq.(10) of \cite{13}) which has the advantage of full control over the finite-size corrections. The numerical results for $\alpha_0, \alpha_1$ and $\alpha_1/2$ for the QKR (see table 1) indicate a parabolic form of $f(\alpha)$ (also confirmed by the fits shown in figure 3b, 4b) and therefore a log-normal behavior of the local eigenfunction intensity $u = |\psi(r)|^2$ for large $u$-regions (see \cite{10, 11, 13} for details). As shown in figures 3c, 4c, the tail-behavior of $P_\alpha(u')$ can be well-approximated by the function

$$f(u') = \sqrt{a/2\pi e^{b+a u'} e^{-a(u'+c)^2/2}}.$$  

(13)

with $u' = [\ln u - \langle u \rangle]/\langle \ln^2 u \rangle$ (see figures 3c, 4c for numerical values of $a, b, c$).

The bulk statistical behavior of the Hermitian matrices is known to be analogous to the unitary matrices \cite{17, 22}. This, along with the single parametric formulation of the statistics of the Hermitian matrix ensemble \cite{15}, suggests the analogy of the QKR-Anderson ensemble statistics if their $\Lambda$ parameters are equal (besides similar global constraints e.g. global symmetries) \cite{16}. Our next step is the comparison of the fluctuation measures of a time-reversal Anderson case with the QKR3, a time-reversal system with partially localized wavefunction in the momentum space (dynamical localization). For this purpose, we analyze a cubic ($d = 3$) Anderson lattice of linear size $L$ ($N = L^d$) with a Gaussian site disorder (of variance $\omega = W^2/12$, $W = 4.05$ and mean zero), same for each site, an isotropic Gaussian hopping (of variance $\eta = 1/12$ and mean zero) between the nearest-neighbors with hard wall boundary conditions; these condition correspond to the critical point for a disorder driven metal-insulator transition \cite{19}). A substitution of the above values (with $t = 0$) in eq. (12) gives $\alpha_{\text{ae}} - \alpha_i = 1.36$. As shown by the numerical analysis in \cite{10, 22, 23}, $F(E) \approx 0.26 e^{-E^2/5}$ and $f_2^{\text{pp}} \approx 0.04$ which on substitution in eq. (11) gives $\Lambda_{\text{ae}} = 0.056$ (with $\gamma_i = 2$).

For AE-QKR comparison, we analyze the ensembles of 2000 matrices with matrix size $N = L^3 = 512$ for the 3d AE case and $N = 513$ for the QKR case. The energy dependence of $\Lambda$ (see \cite{19, 20}) forces us to confine our analysis to only 10% of the levels near the band-center from each such local matrix. The levels are unfolded by respective local mean-level density in each case (so as to compare the level-density fluctuations on a same density-scale) \cite{17}. Figure 5a shows the AE-QKR3-QKR4 comparison of $P(s)$; the good agreement among the three curves verifies the $\Lambda$-dependence of the spectral correlations. This is reconfirmed by figure 5b showing a comparison of the number variance. Note that $\phi \approx 0.84\pi^2$ for QKR4 is same as the theoretical analog given by the condition $\Lambda_{\text{kr},t} = \Lambda_{\text{ae}} = 0.056$ (see eqs. 11, 11). However $\chi = 0.8$ for QKR3 show a small deviation from the theoretically prediction ($\chi \approx 0.95$). This may be due to $\phi$ being a poor approximation at the integrable-nonintegrable boundary $K = 0$. Note that the classical limits of QKR3 and QKR4 are different ($K = 0$ for QKR3, $K = 4.5$ for QKR4), although their "semi-classical" $K$ are same ($K = 4.5$ with $N = 513$ for QKR3, $K = 4.5$ for any $N$ for QKR4). As clear from eq. (2), $K = 0$ marks the boundary between the integrable and mixed dynamics; $K = 4.5$ corresponds to the mixed nature of the dynamics (see figure 5c).

As discussed in \cite{23}, the eigenfunction fluctuations of finite systems are influenced by two parameters, namely, system size $N$ as well as $\Lambda_{\text{measure}}$. To compare $\Lambda_{\text{measure}}$ dependence of an eigenfunction measure, therefore, same system size should be taken for each system. Figure 6a shows the distribution $P_\alpha(u')$ of the local eigenfunction intensity, $u' = [\ln u - \langle u \rangle]/\langle \ln^2 u \rangle$, for the AE and the QKR3. The close proximity of the two curves suggests $\Lambda_{\text{kr},t}$ as the parameter governing the local eigenfunctions intensity too; (we have verified the analogy also with QKR4). A comparison of the AE-QKR3 multifractality spectrum, shown in figure 6b, reconfirms their close similarity at least on the statistical grounds.

The numerical confirmation of the statistical analogy of QKR4-AE-QKR3 systems supports our claim regarding single parametric ($\Lambda$)-dependence of the statistics besides global constraints. Note the latter are same for both QKR3 and QKR4 (i.e parity violated, time-reversal preserved and mixed dynamics) which results in their statistics intermediate to same universality classes, namely, Poisson and COE although the transition parameters are different in the two cases.

The QKR-AE analogy can be utilized to connect them to other complex systems too. In \cite{19, 23}, we studied the AE-connection with the PRBME (described by $\langle H_{kl} \rangle = 0$, $\langle H_{kl}^2 \rangle \propto [1 + |k - l|^2/p^2]^{-1}$ \cite{18}) and the GBE ($\langle H_{kl} \rangle = 0$, $\langle H_{kl}^2 \rangle \propto [1 + cN^2]^{-1}$ \cite{19}). For the AE case considered above, the PRBME and the GBE analogs for the spectral statistics turned out to be $p = 0.4$ and $c = 0.1$; these systems are therefore the spectral statistical analogs of QKR3, QKR4 as well as of a $N \times N$ circular Brownian ensemble with $w \approx 0.4\pi N^{-1/2}$ (see case(i) of section II.B).
IV. CONCLUSION

To summarize, we studied the statistical analogy of two paradigmatic models of dynamical and disordered systems, namely, the quantum kicked rotor and the Anderson Hamiltonian, in the partially localized regimes. Our results indicate the existence of critical behavior in the classically mixed regime of the QKR with a smooth potential. This is qualitatively analogous to a disorder driven metal-insulator transition in the Anderson system; the quantitative analogy for their statistical behavior follows if their complexity parameters are equal. Our study also reveals the possibility of other transitions in the QKR e.g from a symmetry preserving phase to a symmetry fully violated phase. These transitions are analogous to similar symmetry breaking transitions in disordered metals e.g the Anderson Hamiltonian in the weak disorder limit in the presence of a slowly varying magnetic field.

As with the Anderson transition, the QKR transitions are governed by the complexity parameter \( \Lambda \) too. However, contrary to the \( \Lambda \)-derivation for the Anderson case by an ensemble route, \( \Lambda \) for the QKR is derived by a semi-classical method. The semi-classical \( \Lambda \)-formulation is also numerically verified for the QKR-ensembles. This indicates an equivalence of the ensemble-averaging and the phase-space averaging for the statistical analysis. This further lends credence to the single parametric formulation of the statistical behavior of complex systems, irrespective of the origin of their complexity. However it needs to be examined for other dynamical systems.

Research has indicated a multi-parametric dependence of the spectral-statistics at long energy scales of the dynamical systems [26], originating in the level-density oscillations due to short periodic orbits. However, these studies are not at variance with our work. This is because the ”semi-classical” \( \Lambda \)-derivation in [20] is based on the assumed equivalence of the traces of the operators with their phase-space averages. The assumption may not be valid on short time-scales of the dynamics. One should also understand the exact role of the ensemble-averaging for the statistical analysis of dynamical systems.

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VI. FIGURE CAPTION

Fig. 1 [Color online]. Spectral Measures of QKR1:
(a) the nearest-neighbor spacing distribution $P(S)$ of the eigenvalues for various sizes $N$, with inset showing behavior on the linear scale, (b) Number variance $\Sigma^2(r)$ for various sizes.

The convergence of the curves for different sizes indicates scale-invariance of the statistics. The behavior is critical, being different from the two end-points, namely, Poisson and CUE statistics even in infinite size limit.

Fig. 2 [Color online]. Spectral Measures of QKR2: (a) Distribution $P(S)$ of the nearest-neighbor eigenvalue spacings $S$ for various sizes, with inset showing behavior on the linear scale, (b) Number variance $\Sigma^2(r)$ for various sizes.

Again the statistics being intermediate between COE and CUE, and, convergence of the curves for different sizes indicates its critical behavior.

Fig. 3 [Color online]. Multifractality of QKR1:
(a) Fractal Dimension $\tau_q$ along with the fit $y(q) = (1+c)q - 1 - cq^2$ with $c = 0.06$ (good only for $q \leq 3$). A fit for the large $q$ regime suggest following behavior: $\tau_q = q - 1 + 0.02q^2$. (b) Multifractal spectrum $f(\alpha)$ for various sizes along with the parabolic fit $f(\alpha) = d - \frac{(\alpha - \alpha_0)^2}{4(c_0 - 2)}$ with $\alpha_0 = 1.09$ and $d = 1$. (c) Distribution $P_u(u')$ with $u' = [\ln u - \langle u \rangle]/\langle \ln^2 u \rangle$ of the local intensity of an eigenfunction for QKR1. The solid line represent the function $f(u')$ given by eq. (13) with $a = 5.2$, $b = 1.2$ and $c = 0.78$ (corresponding to an approximate log-normal behavior of $P_u(u)$), a good approximation in tail-region as expected. The inset shows the behavior on linear scale.

Fig. 4 [Color online]. Multifractality of QKR2: (a) Fractal Dimension $\tau_q$ along with the fit $y(q) = (1+c)q - 1 - cq^2$ with $c = 0.075$ (good only for $q < 4$). (b) Multifractal spectrum $f(\alpha)$ for various sizes along with a parabolic fit, of the same form as in figure 3b with $\alpha_0 = 1.045$ and $d = 1$. (c) Distribution $P_u(u')$ of the local intensity of an eigenfunction for QKR2, with $u'$ same as in figure 3c. The solid line represent the function $f(u')$ with $a = 5.3$, $b = 0.95$ and $c = 0.73$ (corresponding to a log-normal behavior of $P_u(u)$) which fits well in the tail region of $P_u(u')$. The inset shows the behavior on the linear scale.

Fig. 5 [Color online]. Comparison of spectral statistics of the Anderson ensemble with QKR3 and QKR4: (a) $P(S)$, with inset showing the linear behavior, (b) $\Sigma^2(r)$. Note QKR3 and QKR4 turns out to be close to the one suggested by the relation $\Lambda_n = \Lambda_{kr,t}$ (giving $\chi = 0.95$ for QKR3 and $\phi = 0.84\pi^2$ for QKR4). (c) Phase-space behavior of the classical kicked rotor at $K = 4.5$ (see eq. (2))

Fig. 6 [Color online]. Comparison of the multifractality of the eigenfunctions of QKR3 with the Anderson ensemble: (a) local intensity distribution $P_u(u')$ of an eigenfunction with $u'$ same as in figure 3,4. Also shown is the function $f(u')$ with $a = 15.9$, $b = 7.7$, $c = 0.99$, a good fit in the tail region of $P_u(u')$. The inset shows the behavior on linear scale. (b) Multifractal spectrum $f(\alpha)$: Note the QKR3 analog of the Anderson ensemble here is same as the one in figure 5. Again the fit has the same parabolic form as in figures 3,4, with $\alpha_0 = 1.1$ and $d = 1$. 

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TABLE I: Multifractality Analysis of QKR: \( \alpha \) values here are obtained by an \( L \to \infty \) extrapolation of \( \alpha_L \) \[13\]. The \( 0 < D_2 < 1 \)-behavior indicates a multifractal nature of the three QKR cases.

| Case | QKR1 | QKR2 | QKR3 |
|------|------|------|------|
| \( \alpha_0 \) | 1.034 | 1.008 | 2.466 |
| \( \alpha_{1/2} \) | 1.007 | 1.001 | 0.812 |
| \( \alpha_1 \) | 0.934 | 0.991 | 0.543 |
| \( D_0 \) | 1     | 1     | 1     |
| \( D_1 \) | 0     | 0     | 0     |
| \( D_2 \) | 0.825 | 0.811 | 0.89  |
\[ \sum^2(r)/r \]

- \( N=213 \)
- \( N=313 \)
- \( N=413 \)
- \( N=513 \)
- \( N=613 \)
- \( N=713 \)
- \( N=813 \)
- \( N=913 \)
- \( N=1013 \)
Parabolic fit
$f(\alpha)$

Parabolic fit

$N=213$

$N=313$

$N=413$

$N=513$

$N=613$

$N=713$

$N=813$

$N=913$

$N=1013$
$P(u')$ vs $u'$ for different $N$: $N=213$, $N=313$, $N=413$, $N=513$, $N=613$, $N=713$, $N=813$, $N=913$, $N=1013$. The inset shows $f(u')$. The data points are represented by various markers corresponding to different $N$ values.
$\Sigma^2(r)/r$ vs $r$ for AE, QKR3, and QKR4.
$P(u')$

$u'$

$\text{AE}$

$\text{QKR3}$

$f(u')$
