ORDER STATISTICS FROM INDEPENDENT NON-IDENTICALLY DISTRIBUTED DISCRETE RAYLEIGH DISTRIBUTION

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ABSTRACT

This study deals with the subject of Order Statistics (OS) and its moments for independent non-identically distributed discrete random variables rv’s drawn from one parameter discrete Rayleigh distribution or dRayleigh (θ). Distributions of single order statistics and their moments for dRayleigh (θ) are studied. The kth moments of single os and the mean of the largest and smallest os are obtained for n = 3.

Keywords: Order Statistics, Discrete Random Variable, Discrete Rayleigh Distribution, Permanents

1. INTRODUCTION

The subject of order statistics for Independent Identically Distributed (IID) discrete random variates is referred to the work published by (Raider, 1951; Abdel-Aty, 1954; Siotani, 1956; Kabe, 1969; Arnold et al., 1992; Nagaraga, 1992). Not much attention has been paid to the case when assuming that the random variables are discrete and Independent Non-Identically Distributed (INID). Basic distribution theory of order statistics for this case was first discussed by (Gungor et al., 2009). The appearance of new discrete distributions in the literature such as: discrete gamma distribution (Chakraborty, 2012; Alhazzani, 2012), discrete burr and pareto distribution (Krishna and Pundir, 2009), discrete Weibull distribution (Nakagawa and Osaki, 1975; Stein and Dattero, 1984; Khan et al., 1989), discrete normal distribution (Roy, 2003) and discrete Rayleigh distribution (Roy, 2004) opens the door for lots of scientific papers in order statistics for both iid and inid cases.

Let X1,X2,...,Xn be Independent Non-Identically Distributed (INID) random variables having dRayleigh distributions Equation 3:

In this study we will study os from discrete Rayleigh distribution dRayleigh (θ) when the Xi’s are iid random variables.

1.1. Discrete Rayleigh Distribution

The discrete Rayleigh distribution was obtained and studied by (Roy, 2004), by using the general approach of discretizing a continuous distribution. The probability mass function (pmf) and the cumulative distribution function (cdf) of the dRayleigh (θ) are given by Equation 1 and 2:

\[
p(x) = \text{Pr}(X = x) = \theta^x - \theta^{x+1}, \quad x = 0,1,2,..., \quad 0 < \theta < 1
\]

\[
P(x) = \text{Pr}(X \leq x) = 1 - \theta^x
\]

Let X1, X2,..., Xn be Independent Non-Identically Distributed (INID) random variables having dRayleigh distributions Equation 3:

\[
p(r) = \text{Pr}(X = x) = \theta^x - \theta^{x+1}, \quad x = 0,1,2,..., \quad 0 < \theta_i < 1
\]

1.2. Distribution of single os from inid dRayleigh (θ) rv’s

The general form of the pmf of rth os; Xr:n from iid discrete distribution is given by (Gungor et al., 2009) Equation 4:
\[ p_{x:n}(x) = P_r(X_{x:n} = x) = \sum_{k=m}^{n-r} \sum_{j=0}^{m-k} (r-k-1)!(j+m)!/(n-r-m)! \sum_{b=0}^{j} \prod_{i=1}^{k} (\theta_b^x - \theta_b^{(x+1-i)}) \prod_{i=1}^{j} (\theta_b^{(x-i+1)}) \]  

(4)

\[ \Pi_{x:n}^{-1}(x) = \Pi_{n+1}^{-1}(x) = \Pi_{kk+1}^{-1}(x) \Pi_{kk+2}^{-1}(1-P_{x:n}(x)) \]

where, \( i = 1,2,\ldots,n \) and \( \Sigma_p \) denotes the summation over all \( n! \) permutations \((i_1,i_2,\ldots,i_n)\) of \((1,2,\ldots,n)\). The pmf of \( r^0:os; X_{x:n} \) from iniddRayleigh \((\theta)\) distribution is given by Equation 5:

\[ p_{x:n}(x) = \sum_{k=0}^{n-r} \sum_{j=m}^{n-k} (r-k-1)!(j+m)!/(n-r-m)! \prod_{i=1}^{k} (1 - \theta_i^{(x+1-i)}) \prod_{i=1}^{j} (\theta_i^{(x-i+1)}) \]  

(5)

Using permanent notation the above pmf can be written as (Vaughan and Venables, 1972) Equation 6:

\[ p_{x:n}(x) = \sum_{k=m}^{n-r} \sum_{j=0}^{m-k} (r-k-1)!(j+m)!/(n-r-m)! \frac{\prod_{i=1}^{k} (1 - \theta_i^{(x+1-i)}) \prod_{i=1}^{j} (\theta_i^{(x-i+1)})}{\{r-k-1\}(k+m+1)!(n-r-m)!} \]  

(6)

where, \( k = m = 0 \) i.e., the chance of ties is zero and all the observations are distinct, an easier form of the pmf of \( X_{x:n} \) is now obtained Equation 7 and 8:

\[ p_{x:n}(x) = \frac{\sum_{j=0}^{m-k} \prod_{i=1}^{k} (1 - \theta_i^{(x+1-i)}) \prod_{i=1}^{j} (\theta_i^{(x-i+1)})}{\{r-k-1\}(n-r)!} \frac{1}{(n-r-m)!} \]  

(7)

\[ p_{x:n}(x) = \frac{1}{(n-r)!} \sum_{j=0}^{m-k} \prod_{i=1}^{k} (1 - \theta_i^{(x+1-i)}) \prod_{i=1}^{j} (\theta_i^{(x-i+1)}) \]  

(8)

1.3. Distribution of the Smallest and Largest os Following inidd Rayleigh \((\theta) CVs\)

The pmf of \( X_{1:n} \) and \( X_{n:n} \) from inidd Rayleigh \((\theta)\) are given by Equation 9 and 10:

\[ p_{x:n}(x) = \frac{1}{(n-r)!} \sum_{j=0}^{m-k} \prod_{i=1}^{k} (1 - \theta_i^{(x+1-i)}) \prod_{i=1}^{j} (\theta_i^{(x-i+1)}) \]  

(9)

\[ p_{x:n}(x) = \frac{1}{(n-r)!} \sum_{j=0}^{m-k} \prod_{i=1}^{k} (1 - \theta_i^{(x+1-i)}) \prod_{i=1}^{j} (\theta_i^{(x-i+1)}) \]  

(10)

Example 1

For \( n = 3 \) the pmf of \( X_{1:3} \) and \( X_{3:3} \) from iniddRayleigh \((\theta)\) are given by Equation 11:

\[ p_{x:1}(x) = \frac{1}{2!} \sum_{j=0}^{1} \prod_{i=1}^{1} (1 - \theta_i^{(x+1-i)}) \prod_{i=1}^{j} (\theta_i^{(x-i+1)}) \]  

(11)

The cdf of \( X_{x:n} \) from iniddRayleigh \((\theta)\) is given by Equation 12:

\[ p_{x:n}(x) = \sum_{j=0}^{m-k} \prod_{i=1}^{k} (1 - \theta_i^{(x+1-i)}) \prod_{i=1}^{j} (\theta_i^{(x-i+1)}) \]  

(12)

where, \( \Sigma_p \) denotes the summation over all permutations \((i_1,i_2,\ldots,i_n)\) of \((1,2,3,\ldots,n)\) for which \( i_1 < i_2 < \ldots < i_n \) and \( i_{j+1} < i_{j+2} \). Equation 13:

\[ p_{x:n}(x) = \sum_{j=0}^{m-k} \prod_{i=1}^{k} (1 - \theta_i^{(x+1-i)}) \prod_{i=1}^{j} (\theta_i^{(x-i+1)}) \]  

(13)

Then the cdf of \( X_{1:n} \) and \( X_{n:n} \) are Equation 14-16:
Table 1. The mean of the smallest os from inid dRayleigh ($\theta$)

|   | 2   | 3   | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|-----|-----|-----|-----|-----|-----|-----|
| 0.1 | 0.1 | 1.001 | 1.025 | 1.05001 | 1.07503 | 1.10010 |
|    | 0.2 | 1.002 | 1.002 | 1.00300 | 1.00400 | 1.00500 |
|    | 0.3 | 1.003 | 1.004 | 1.00600 | 1.00800 | 1.01000 |
|    | 0.4 | 1.004 | 1.006 | 1.00900 | 1.01200 | 1.01500 |
|    | 0.5 | 1.005 | 1.008 | 1.01200 | 1.01600 | 1.02000 |
| 0.2 | 0.1 | 1.002 | 1.010 | 1.01500 | 1.02000 | 1.02500 |
|    | 0.2 | 1.004 | 1.004 | 1.00600 | 1.00800 | 1.01000 |
|    | 0.3 | 1.006 | 1.008 | 1.01200 | 1.01600 | 1.02000 |
|    | 0.4 | 1.008 | 1.012 | 1.01800 | 1.02400 | 1.03000 |
|    | 0.5 | 1.010 | 1.016 | 1.02400 | 1.03200 | 1.04000 |
| 0.3 | 0.1 | 1.010 | 1.020 | 1.03000 | 1.04000 | 1.05001 |
|    | 0.2 | 1.003 | 1.006 | 1.00900 | 1.01200 | 1.01500 |
|    | 0.3 | 1.006 | 1.012 | 1.01800 | 1.02400 | 1.03000 |
|    | 0.4 | 1.009 | 1.018 | 1.02700 | 1.03600 | 1.04500 |
|    | 0.5 | 1.012 | 1.024 | 1.03600 | 1.04801 | 1.06001 |
| 0.4 | 0.1 | 1.015 | 1.030 | 1.04500 | 1.06001 | 1.09001 |
|    | 0.2 | 1.004 | 1.008 | 1.01200 | 1.01600 | 1.04000 |
|    | 0.3 | 1.008 | 1.016 | 1.02400 | 1.03200 | 1.06001 |
|    | 0.4 | 1.012 | 1.024 | 1.03600 | 1.04801 | 1.08004 |
|    | 0.5 | 1.016 | 1.032 | 1.04801 | 1.06402 | 1.10010 |
| 0.5 | 0.1 | 1.020 | 1.040 | 1.06001 | 1.08004 | 1.12524 |
|    | 0.2 | 1.005 | 1.010 | 1.01500 | 1.02000 | 1.05001 |
|    | 0.3 | 1.010 | 1.020 | 1.03000 | 1.04000 | 1.07503 |
|    | 0.4 | 1.015 | 1.030 | 1.04500 | 1.06001 | 1.08004 |
|    | 0.5 | 1.020 | 1.040 | 1.06001 | 1.08004 | 1.10010 |

Table 2. The mean of the largest os from inid dRayleigh ($\theta$)

|   | 2   | 3   | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|-----|-----|-----|-----|-----|-----|-----|
| 0.1 | 0.1 | 1.27130 | 1.35380 | 1.44130 | 1.53979 | 1.65769 |
|    | 0.2 | 1.35380 | 1.42730 | 1.50579 | 1.59526 | 1.70409 |
|    | 0.3 | 1.44130 | 1.50579 | 1.57523 | 1.65559 | 1.75519 |
|    | 0.4 | 1.53979 | 1.59526 | 1.65559 | 1.72664 | 1.81659 |
|    | 0.5 | 1.65769 | 1.70409 | 1.75519 | 1.81659 | 1.89618 |
| 0.2 | 0.1 | 1.35380 | 1.42730 | 1.50579 | 1.59526 | 1.70409 |
|    | 0.2 | 1.42730 | 1.49279 | 1.56327 | 1.64472 | 1.74550 |
|    | 0.3 | 1.50579 | 1.56327 | 1.62571 | 1.69904 | 1.79158 |
|    | 0.4 | 1.59526 | 1.64472 | 1.69904 | 1.76406 | 1.84796 |
|    | 0.5 | 1.70409 | 1.74550 | 1.79158 | 1.84796 | 1.92250 |
| 0.3 | 0.1 | 1.44130 | 1.50579 | 1.57523 | 1.65559 | 1.75519 |
|    | 0.2 | 1.50579 | 1.56327 | 1.62571 | 1.69904 | 1.79158 |
|    | 0.3 | 1.57523 | 1.62571 | 1.68110 | 1.74732 | 1.83263 |
|    | 0.4 | 1.65559 | 1.69904 | 1.74732 | 1.80624 | 1.88390 |
|    | 0.5 | 1.75519 | 1.79158 | 1.83263 | 1.88390 | 1.95321 |
| 0.4 | 0.1 | 1.53979 | 1.59526 | 1.65559 | 1.72664 | 1.81659 |
|    | 0.2 | 1.59526 | 1.64472 | 1.69904 | 1.76406 | 1.84796 |
|    | 0.3 | 1.65559 | 1.69904 | 1.74732 | 1.80624 | 1.88390 |
|    | 0.4 | 1.72664 | 1.76406 | 1.80624 | 1.85885 | 1.92989 |
|    | 0.5 | 1.81659 | 1.84796 | 1.88390 | 1.92989 | 1.99359 |
| 0.5 | 0.1 | 1.65769 | 1.70409 | 1.75519 | 1.81659 | 1.89618 |
|    | 0.2 | 1.70409 | 1.74550 | 1.79158 | 1.84796 | 1.92250 |
|    | 0.3 | 1.75519 | 1.79158 | 1.83263 | 1.88390 | 1.95321 |
|    | 0.4 | 1.81659 | 1.84796 | 1.88390 | 1.92989 | 1.99359 |
|    | 0.5 | 1.89618 | 1.92250 | 1.95321 | 1.99359 | 2.05103 |
For \( n = 3 \) the cdf of \( X \) is given by Equation 18:

\[
P_{x}(x) = \sum_{j=3}^{n} \frac{1}{j!} \prod_{j=3}^{n} \left[ \frac{1 - \theta^{2j}}{j} \right]
\]

Example 2

For \( n = 3 \) the cdf of \( X_{1,3} \) from \text{inidRayleigh (Θ)} is given by Equation 17:

\[
P_{x_{1,3}}(x) = \frac{1}{3!} \left[ 1 - \frac{\theta^{2}}{2} \right] + \frac{1}{2} \frac{1}{2!} \left[ 1 - \frac{\theta^{2}}{3} \right] + \frac{1}{3!} \left[ 1 - \theta^{2} \right] + \frac{1}{4!} \left[ 1 - \theta^{2} \right] + \frac{1}{5!} \left[ 1 - \theta^{2} \right] + \frac{1}{6!} \left[ 1 - \theta^{2} \right] = 1 - \theta^{2} \theta^{2} \theta^{2}
\]

For \( n = 3 \) the cdf of \( X_{3,3} \) from \text{inidRayleigh (Θ)} is given by Equation 18:

\[
P_{x_{3,3}}(x) = \frac{1}{6!} \left[ 1 - \theta^{2} \right] + \frac{1}{2} \left[ 1 - \theta^{2} \right] + \frac{1}{3!} \left[ 1 - \theta^{2} \right] + \frac{1}{4!} \left[ 1 - \theta^{2} \right] + \frac{1}{5!} \left[ 1 - \theta^{2} \right] + \frac{1}{6!} \left[ 1 - \theta^{2} \right] = 1 - \theta^{2} \theta^{2} \theta^{2} \theta^{2}
\]

1.4. Moments of os from \text{inid Rayleigh (Θ)} rv’s

The \( k \)th moment of \( X_{n} \) from \text{inid Rayleigh (Θ)} is given by Equation 19:

\[
\mu_{x_{n}}^{(k)} = M^{k} - \sum_{j=0}^{n} \sum_{j=0}^{n} \prod_{j=0}^{n} \left[ 1 - \theta^{2j} \right] \prod_{j=1}^{n} \left[ 1 - \theta^{2j} \right]
\]

The mean of the \( r \)th os is Equation 20:

\[
\mu_{r,n} = \sum_{j=0}^{n} \sum_{j=0}^{n} \prod_{j=1}^{n} \left[ 1 - \theta^{2j} \right] \prod_{j=1}^{n} \left[ 1 - \theta^{2j} \right]
\]

and the mean for the smallest and largest os from \text{inid Rayleigh (Θ)} are given by Equation 21 and 22:

\[
\mu_{r,n} = \sum_{j=0}^{n} \left[ 1 - \prod_{j=1}^{n} \left[ 1 - \theta^{2j} \right] \right]
\]

Example 3

For \( M = 3, n = 3 \), the mean of the smallest and largest os are then Equation 23 and 24:

\[
\mu_{r,1} = \sum_{j=0}^{n} \prod_{j=1}^{n} \left[ \theta^{2j} \right]
\]

\[
\mu_{r,1} = \sum_{j=0}^{n} \left[ 1 - \prod_{j=1}^{n} \left[ 1 - \theta^{2j} \right] \right]
\]

Calculations of the mean of the smallest and largest os from \text{dRayleigh (Θ)} are given in Table 1 and 2 for several values of 1, 2, 3.

2. CONCLUSION

Discrete Rayleigh distribution under contaminations (INID) is introduced. Exact forms of the pmf and cdf of order statistics and single moment are derived. Further studies are suggested for other discrete distributions.

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