WHEN THE GROUP RING OF A FINITE GROUP OVER A FIELD IS SERIAL

ANDREI KUKHAREV AND GENA PUNINSKI

Abstract. We will describe finite simple groups whose group rings over a given field are serial.

1. Introduction

Let $F$ be a field and let $G$ be a finite group. In this paper we will attempt to make a list of pairs $(F,G)$ such that the group ring $FG$ is serial, i.e. each indecomposable projective right (equivalently left) $FG$-module has a unique composition series. Despite many important results has been proven, a complete description is still elusive. First of all, if the characteristic $p$ of $F$ does not divide the order of $G$, then, by Maschke’s theorem, $FG$ is a semisimple artinian ring, hence serial. Thus we may assume that $p$ divides the order of $G$, the modular case.

Since each artinian serial ring is of finite representation type, it follows from Higman [20] that seriality of $FG$ implies that each Sylow $p$-subgroup of $G$ is cyclic. If $p = 2$ then this condition is sufficient, because $G$ is 2-nilpotent in this case. This result can be extended to $p$-solvable groups. Namely, using Morita [36], one concludes that, if $G$ is a $p$-solvable group with a cyclic Sylow $p$-subgroup, then the ring $FG$ is serial. Despite Morita worked over an algebraically closed field, this is not a restriction, because Eisenbud and Griffith [12] showed that seriality of $FG$ depends only on characteristic of $F$.

However, if $G = SL_2(5)$, then each Sylow 5-subgroup of $G$ is cyclic, but the ring $FG$ is not serial for any field of characteristic 5. Nevertheless, every group $G$ with a cyclic Sylow $p$-subgroup is close to a $p$-solvable group. Namely, by Blau [5], if $G$ is not $p$-solvable, it admits a normal series $\{e\} \subset O_{p'} \subset K \subset G$, where $O_{p'}$ is the largest normal subgroups of $G$ do not containing elements of order $p$, and $K$ is the least normal subgroup of $G$ properly containing $O_{p'}$, in particular $P \subset K$ and $H = K/O_{p'}$ is a simple nonabelian group. Here is the main conjecture.

Conjecture 1.1. Suppose that $F$ is a field of characteristic $p$ dividing the order of $G$. Further assume that each Sylow $p$-subgroup of $G$ is cyclic. Then the group ring $FG$ is serial if and only if either $G$ is $p$-solvable, or the ring $FH$ is serial.

In this paper we will complete the description of finite simple groups with serial group rings.

Theorem 1.2. Let $H$ be a finite simple group and let $F$ be a field of characteristic $p$ dividing the order of $H$. Then the group ring $FH$ is serial if and only if one of the following holds.

1) $H = C_p$.

2) $H = PSL_2(q)$ and $p > 2$ divides $q - 1$.
3) \( H = \text{PSL}_2(q), \) \( q \neq 2 \) or \( H = \text{PSL}_3(q), \) where \( p = 3 \) and \( q \equiv 2, 5 \) (mod 9).

4) \( H = \text{PSU}_3(q^2) \) and \( p > 2 \) divides \( q - 1. \)

5) \( H = \text{Sz}(q), \) \( q = 2^{2n+1}, \) \( n \geq 1, \) where either \( p > 2 \) divides \( q - 1, \) or \( p = 5 \) divides \( q + r + 1, \) \( r = 2^{n+1}, \) but 25 does not divide this number.

6) \( H = 2^2 G_2(q^2), \) \( q = 3^{2n+1}, \) \( n \geq 1, \) where either \( p > 2 \) divides \( q^2 - 1, \) or \( p = 7 \) divides \( q^2 + \sqrt{3}q + 1, \) but 49 does not divide this number.

7) \( H = M_{11}, \) \( p = 5 \) or \( G = J_1, \) \( p = 3. \)

For instance, \( A_5 \) with \( p = 3 \) occurs twice in this list: first as \( \text{PSL}_2(4) \) in 2), and then as \( \text{PSL}_2(5) \) in 3).

Using this theorem we verify Conjecture 1.1 for groups of order \( \leq 10^4. \) However, we do not know whether it holds true in general even for \( p = 3; \) or if seriality of \( FG \) implies seriality of \( FH \) for each normal subgroup \( H \) of \( G. \)

The main tool in proving Theorem 1.2 is the following well known characterization of seriality. Namely, if \( F \) is sufficiently large, then the group ring \( FG \) is serial iff the Brauer tree of each \( p \)-block of \( G \) is a star whose exceptional vertex (if any) is located at its center. This allows us to use a well developed machinery of Brauer trees to complete the proof of this theorem. In fact, due to previously obtained results, the only remaining case is when \( H \) is a classical (in fact symplectic, unitary or orthogonal) group defined over a finite field of characteristic 2, and we address this case in the paper. We will try to make the paper self-contained by subsuming and elaborating existing knowledge on serial group rings.

Of course, there is a broader context for the problem we consider. For instance, Tuganbaev [45, part of Probl. 16.9] asks to characterize rings \( R \) and (finite or infinite) groups \( G \) such that the ring \( RG \) is serial. We will not comment on this general question, because it deserves a separate discussion, but some connections are obvious. Say, if \( R \) has a field as a factor, then all results of this paper are applicable.

The essential part of this research is included in the first named author PhD thesis [26], and otherwise is scattered in [27] [28] [46] [29] [30] [31] [32] [33]. The second named author participated in this thesis as a supervisor. He thanks Chris Gill for the initial discussion in Prague in Summer 2012, and for showing him how to check seriality using MAGMA. We are also indebted to Alexandre Zalesski, Alexej Kondratiev, David Craven and Meinhold Geck for their comments.

2. Basics

All rings in this paper will be associative with unity, and, by default, a module means a unital right module over a ring. A module \( M \) is said to be uniserial, if its lattice of submodules is a chain; and \( M \) is serial if it is a direct sum of uniserial modules. We say that a ring \( R \) is serial, if the right regular module \( RR \) is serial and the same holds true for the left module \( RR. \) In fact, \( R \) is serial iff there exists a set \( e_1, \ldots, e_n \) of pairwise orthogonal idempotents which is complete, i.e. \( e_1 + \cdots + e_n = 1, \) and each principal projective module \( e_iR \) is uniserial, so as each left module \( Re_i. \) Further, this collection of idempotents is unique up to conjugation by a unit. For
more on general theory of serial rings the reader is referred to [38], or to more recent [3]. We will downsize to a more restrictive setting.

Artinian serial rings were introduced by Nakayama (under the name of generalized uniserial rings), and nowadays are known (see [3]) as Nakayama rings, or Nakayama algebras, if they are algebras over a field; though [2] gives a broader meaning to the latter term. Recall that a serial ring is said to be basic, if different principal projective modules are not isomorphic. When classifying serial rings one can always assume (modulo Morita equivalence) that \( R \) is indecomposable and basic.

For instance, if \( R \) is an indecomposable basic artinian serial rings with simple modules \( S_i \), then (see [19 Cor. 12.4.2]) all skew fields \( D_i = \text{End}(S_i) \) are isomorphic. Suppose that \( R \) is an indecomposable basic finite dimensional Nakayama algebra over a perfect field and \( D \) is the common value of endomorphism rings of simple \( R \)-modules. It follows from Kupisch [34, Satz 1, Hilfsatz 2.1] that there exists an automorphism \( \alpha \) of \( D \) and a proper (hence uniserial) factor \( S = D[x, \alpha]/(x^k) \), with Jacobson radical \( J \), of the skew polynomial ring \( D[x, \alpha] \) such that \( R \) is a factor-ring of the following blow-up of \( S \):

\[
\begin{pmatrix}
S & S & \ldots & S \\
\vdots & \ddots & \ddots & \vdots \\
J & J & \ldots & S
\end{pmatrix}
\]

Furthermore, these factor rings are easily described, - see [34] for a complete list of invariants. Using the blow-up construction a classification of artinian serial rings can be reduced to the case of indecomposable basic rings with \( n \) simple modules, such that the length of all principal projectives is a constant equivalent to 1 modulo \( n \); for instance, this happens for serial group rings of finite groups. No reasonable classification is known in this case, - see [3, p. 275–276] for a list of open questions.

The very important property discovered by Nakayama is that any artinian serial ring \( R \) is of finite representation type: each \( R \)-module is a direct sum of modules \( e_i R / r R, \ r \in e_i R e_j \).

Suppose that \( F \) is a field and \( G \) is a finite group. We will be interested in when the group ring \( F G \) is serial. As we already mentioned, by Maschke’s theorem, we may (and will) assume that the characteristic \( p \) of \( F \) is finite and divides the order of \( G \).

An important necessary condition for seriality follows from Higman’s theorem [20] describing group rings of finite representation type.

**Fact 2.1.** Let \( F \) be a field of characteristic \( p \) dividing the order of \( G \). If the group ring \( FG \) is serial, then each Sylow \( p \)-subgroup of \( G \) is cyclic.

As we have mentioned above, the converse is not true. For instance, let \( F \) be an algebraically closed field and let \( G = \mathrm{SL}_2(p) \) for a prime \( p \geq 5 \). Then each Sylow \( p \)-subgroup of \( G \) consists of \( p \) elements, hence cyclic. Further, it follows from [11 p. 15] that \( FG \) has exactly \( p \) simple modules \( S_1, \ldots, S_p \). Also, for each \( 1 < i < p - 1 \), the projective cover \( P_i \) of \( S_i \) is not uniserial: \( \text{Jac}(P_i) / \text{Soc}(P_i) \) is the direct sum of two simple modules \( S_{p+1-i} \) and \( S_{p-1-i} \).
In fact, this conclusion holds for any field $F$ of characteristic $p$, due to the following result by Eisenbud and Griffith [12] (see [46] for another proof).

**Fact 2.2.** Let $F, F'$ be fields of characteristic $p$ dividing the order of $G$. Then the ring $FG$ is serial if and only if the ring $F'G$ is serial.

From this we derive a useful sufficient condition for seriality. Recall that a group $G$ is said to be $p$-**nilpotent**, if its Sylow $p$-subgroup admits a normal complement $H$, hence $G$ is a semidirect product $H : P$. The following result is contained in [27, Thm. 4.3].

**Fact 2.3.** Let $F$ be a field of characteristic $p$ dividing the order of $G$. Further assume that $G$ is $p$-nilpotent with a cyclic Sylow $p$-subgroup. Then $FG$ is a (left and right) principal ideal ring, in particular it is serial.

For instance, it is well known that each group $G$ with a cyclic 2-Sylow subgroup is 2-nilpotent, hence, for $p = 2$, the group ring $FG$ is serial iff each Sylow 2-subgroup of $G$ is cyclic. Thus from now on we will assume that $p > 2$ when investigating seriality.

This result can be extended to a larger class of groups. Recall that a group $G$ is said to be $p$-**solvable**, if it admits a composition series whose consecutive factors are either $p$-groups, or $p'$-groups. If $G$ has a cyclic Sylow $p$-subgroup (the case of our main interest), then, by an old results by Wielandt [17], we conclude that $G$ is $p$-solvable iff it possesses a 4-term series $\{e\} \subset \mathcal{O}_{p'} \subset K \subset G$ of completely characteristic subgroups, where $K$ is a semidirect product $\mathcal{O}_{p'} : P$ (hence $p$-nilpotent) and $G/K$ is a cyclic $p'$-group.

**Fact 2.4.** Let $F$ be a field of characteristic $p$ dividing the order of $G$. Further assume that $G$ is a $p$-solvable group with a cyclic Sylow $p$-subgroup. Then the group ring $FG$ is serial.

**Proof.** If $F$ is algebraically closed, it was proved by Morita [36] (and later by Srinivasan [43]). It remains to apply Fact 2.2. □

Furthermore, we conclude from Morita that, for a $p$-solvable group $G$, the Jacobson radical of $FG$ is principal as a left and right ideal, and the multiplicity of principal projective modules in a given block of $FG$ is constant. It follows from [28, Thm. 2.3] that both results hold over any field of characteristic $p$.

Note that each group ring $FG$ of a finite group is **quasi-Frobenius**, i.e. each projective module is injective and vice-versa, for instance $FG$ admits a self-duality. From this it easily follows that, when proving seriality, it suffices to verify that this ring is right serial, i.e. that each principal projective module $e_iFG$ is uniserial.

Furthermore, by Fact 2.2 when checking that the group ring of a given group is serial (say, by calculating radical series in MAGMA [6]), we may assume that $F$ is a prime field. When enlarging $F$, the idempotents may split, hence the block structure of $FG$ will change, but the result is still serial. Here is a typical example.

Let $2.S_4^-$ denote the **double covering** of $S_4$. This group is 3-solvable with a cyclic Sylow 3-subgroup, hence the group ring $FG$ is serial for any field $F$ of characteristic 3. By calculating in GAP [16], one recovers (see [26]) the above Kupisch structure of this group ring.
Example 2.5. Let $G = 2.S_4$ be the double covering of $S_4$.

1) If $F$ is the prime field $\mathbb{F}_3$, then $FG = M_3(F)^2 \oplus B \oplus M_2(W)$, where $B$ is the serial block

$$
\begin{pmatrix}
  F[x] & F[x] \\
  xF[x] & F[x]
\end{pmatrix} 
\bigg/ \begin{pmatrix}
  x^2F[x] & xF[x] \\
  x^2F[x] & x^2F[x]
\end{pmatrix}
$$

and $W$ is the factor $\mathbb{F}_9[y, \alpha]/(y^3)$ of the skew polynomial ring with the Frobenius automorphism $\lambda \mapsto \lambda^3$.

2) If $F$ is the Galois field $\mathbb{F}_9$, then $FG = M_3(F)^2 \oplus B \oplus M_2(B)$.

3. Brauer trees

In this section we will recall the main tool to investigate seriality of group rings - the Brauer trees. This graph is defined for each block $B$ with a cyclic defect group $D$, and is a tree whose one (exceptional) vertex is distinguished. For a definition and main properties of this object the reader is referred to the classical [13], or contemporary [35]; we will mention just few instances.

To calculate the Brauer tree one should fix a $p$-modular system $(K, R, F, \eta)$ (see [35, Def. 4.1.18]), where $R$ is a complete discrete valuation domain with the quotient field $K$ of characteristic zero, and $\eta : R \to F$ maps $R$ onto its residue field. We will consider this system to be fixed and large enough to split in $K$ some rational polynomials (called splitting system in [35]). The nature of this system will be of no importance for us.

Suppose that $G$ is a group with a cyclic Sylow $p$-subgroup $P$. Because the defect group $D$ of each block $B$ is a subgroup of $P$, it is cyclic, hence each block obtains its Brauer tree. For instance $P$ itself is the defect group of the principal block $B_0$, i.e. of the block which contains the trivial character.

The number of edges of the Brauer tree, and the multiplicity of the exceptional vertex admit a (more or less) straightforward calculation.

Fact 3.1. Let $G$ be a finite group with a (nontrivial) cyclic Sylow $p$-subgroup $P$.

1) The number of edges $e_0$ of the principal block of $G$ equals the index $|N_G(P)/C_G(P)|$ of the centralizer of $P$ in $G$ in its normalizer. Further, $e_0$ divides $p - 1$, and the multiplicity of the exceptional vertex (if any) equals $m_0 = (|P| - 1)/e_0$.

2) If $B$ is an arbitrary block with a defect group $D$, then the number of edges $e$ in its Brauer tree divides $e_0$. Furthermore the multiplicity of the exceptional vertex equals $|(D| - 1)/e$.

Proof. The first part follows from [5, p. 173] and [41 Thm. 6.5.5]. For the second claim one can argue as follows. Let $Q$ be the subgroup of order $p$ in $D$. Then (see [4, p. 212]) $e$ divides $|N_G(Q)/C_G(Q)|$. Now, by [40] Prop. 1, the latter divides $|N_G(P)/C_G(P)| = e_0$. □

Note that $N_G(P)$ acts on $P$ by conjugation, and $e_0$ is the order of this action. Thus, if $H$ is a normal subgroup of $G$ containing $P$, then $e_0(H) \leq e_0(G)$.

We say that a Brauer tree is a star, if at most one vertex of this graph (its center) has a valency exceeding 1. Here is the diagram of the star with 7 vertices, whose exceptional vertex is at its center.
Note that a line (or open polygon) with \(e\) edges is a star iff \(e = 1\), or \(e = 2\) with the exceptional vertex in the center. For instance the following line with 3 edges is not a star.

An irreducible character \(\chi\) is said to be real, if it is either real-valued, or \(\chi\) is an exceptional character taking real values on \(p\)-regular classes. By [22, p. 3] real characters of a given block form a real stem of its Brauer tree, which is a line.

For instance, let \(G\) be the special unitary group \(\text{SU}_3(4^2)\). If \(p = 13\), then each Sylow \(p\)-subgroup is cyclic. Further, from the character table we find that the Brauer tree of the principal block of \(G\) is the following line, where all non-exceptional characters are real valued. But each exceptional character \(\chi_{19-22}\) takes non-real values on each class of elements of order 13.

Thus the following is an instrumental criterion of seriality.

**Fact 3.2.** (see [14, Cor. 7.2.2]) Let \(G\) be a finite group with a nontrivial cyclic Sylow \(p\)-subgroup and let \(F\) be a field of characteristic \(p\). Then the ring \(FG\) is serial if and only if each block of \(G\) is a star whose exceptional vertex (if any) is located at its center.

For instance, let \(G = A_5\) and \(p = 5\), hence each Sylow 5-subgroup of \(G\) is cyclic. Then the Brauer tree of its principal block is the following:

Here the characters are indexed by degrees, for instance \(\chi_1\) denotes the trivial character. Further, the characters \(\chi_3, \chi'_3\) form the exceptional vertex of multiplicity 2. Because this vertex is not in the center, the group ring of \(A_5\) over any field of characteristic 5 is not serial.

The same results can be achieved (but with more efforts) by splitting idempotents in \(FG\) for \(F = \mathbb{F}_5\). Namely (see [3, p. 259]) the group ring \(FA_5\) decomposes as \(M_5(F) \oplus C\), where

\[
C = \begin{pmatrix} Q & Q & Q & X \\ Q & Q & Q & X \\ Q & Q & Q & X \\ Y & Y & Y & T \end{pmatrix}
\]

is a block of dimension 35, which is the blow-up of its \((3, 4)\)-minor \(\begin{pmatrix} Q & X \\ Y & T \end{pmatrix}\), where the diagonal ring \(Q\) has dimension 3, \(T\) is 2-dimensional, and bimodules \(X, Y\) are 1-dimensional. Further, (see [28]) the latter ring is isomorphic to the following factor-ring:
\[ R = \left( \frac{x^{3}F[x] x^{3}F[x]}{x^{3}F[x] x^{3}F[x]} \right) \].

Taking \( r = x^{1}Re_{1} \) and \( s = x^{1}Re_{2} \), it is easily seen that the module \( e_{1}R \) is not uniserial.

Clearly the above criterion (Fact 3.2) is equivalent to the following: each irreducible \( p \)-modular character lifts uniquely to an irreducible ordinary character via the \( p \)-modular system. For instance, for \( A_{5} \) and \( p = 5 \), each \( p \)-modular character can be lifted, but the uniqueness fails: the restrictions of ordinary characters \( \chi_{3}, \chi_{3}' \) to \( 5' \)-classes is the Brauer character \( \varphi_{3} \). This uniqueness requirement disappeared from the characterization of serial group rings in Janusz [25, Cor. 7.5].

Note that seriality is a Morita invariant property. Furthermore, each factor ring of a serial ring is serial. For instance, if \( H \) is a normal subgroup of a group \( G \), then the seriality of \( FG \) implies that the ring \( F(G/H) \) is serial.

We say that a block \( B \) of \( FG \) covers a block \( b \) of \( FH \), if there are irreducible characters \( \chi \in B \) and \( \xi \in b \) such that \( \xi \) is a component of \( \chi \) restricted to \( H \). For instance, the principal block of \( G \) covers the principal block of \( H \). We will put to use the following fact.

**Fact 3.3.** (see [13, Cor. 6.2.8]) Let \( F \) be a field of characteristic \( p \) dividing the order of \( G \) and let \( H \) be a normal subgroup of \( G \).

1) Let \( B \) be a block of \( FG \) which covers a block \( b \) of \( FH \). Assume that \( H \) contains a defect group of \( B \). Then \( B \) is a serial ring if and only if \( b \) is a serial ring.

2) For instance, if the index of \( H \) in \( G \) is coprime to \( p \), then the ring \( FG \) is serial if and only if the same holds true for \( FH \).

Note (see [13, L. 4.4.12]) that the annihilator of the principal block of any group \( G \) is its largest normal \( p' \)-subgroup \( O_{p'}(G) \). Thus, when investigating seriality of the principal block, we may factor out this subgroup.

We will also need the following result on groups with cyclic Sylow \( p \)-subgroups, which deserves a better publicity.

**Fact 3.4.** (see [5 L. 5.2] and [37 L. 6.1]) Let \( G \) be a non-\( p \)-solvable group with a nontrivial cyclic Sylow \( p \)-subgroup \( P \). Then there exists the least normal subgroup \( K \) of \( G \) properly containing \( O_{p'}(G) \). Furthermore, \( K \) contains \( P \) and the factor group \( H = K/O_{p'}(G) \) is simple non-abelian.

Note that this result does not extend to \( p \)-solvable groups. Namely, let \( G \) be the dihedral group \( D_{18} \) and \( p = 3 \). Then \( O_{p'}(G) = 1 \) and \( K = C_{3} \) does not contain \( P = C_{9} \).

Thus, each non-\( p \)-solvable group \( G \) with a cyclic Sylow \( p \)-subgroup admits the following 4-term normal series, similar to \( p \)-solvable groups: \( 0 \subset O_{p'} \subset K \subset G \). Further, the index of \( K \) in \( G \) is coprime to \( p \), hence, by Fact 3.3 when investigating seriality, we may assume that \( K = G \). Also, the group ring of the simple group \( H = K/O_{p'} \) must be serial and, if so, then the principal block of \( FG \) is serial.
In this section we will start proving Theorem 1.2, i.e. classifying finite simple groups $H$ with serial group rings. Because each abelian group is $p$-solvable, its group ring is serial, hence assume that $H$ is non-abelian.

Most cases has been already considered. We will just briefly comment on some instances.

1) The case of alternating groups was considered in [29] using Scope’s theorem [42] on Morita equivalence of blocks of symmetric groups.

2) The analysis of the PSL-series was completed in [26] using Burkhard’s paper [7].

3) The case of sporadic simple groups was settled in [30] by browsing known Brauer trees from Hiss–Lux [22]. In fact, few facts from Blau (like the parity of centralizers - see [5, Cor. 1]) simplifies matters greatly, and the cross-naught method of labeling Brauer trees (see [22, Def. 2.1.12]) was used for large groups, when the Brauer trees are not available.

As a kind of peculiarity note that the principal block of the Mathieu group $M_{23}$ for $p = 5$ is serial, because (see [22, p. 104]) its Brauer tree is a star with 4 edges without an exceptional vertex. However, there is a nonprincipal block whose Brauer tree is a line with 2 edges, whose exceptional vertex has multiplicity 2 and is attached to the end, hence this block is not serial.

4) The analysis of Suzuki groups in [30] is based on Burkhardt [8]. For instance, when $5 \mid q + r + 1$, then each Sylow 5-subgroup is cyclic, and the Brauer tree of the principal block has the following shape:

This block is serial iff the multiplicity of the exceptional vertex is 1, hence if 25 does not divide $q + r + 1$. For instance, this is the case, when $n = 3$, i.e. $q = 128$.

A similar situation occurs (see [33]), when $G$ is the Ree group $^2G_2(q^2)$. Namely, if $p$ divides $q^2 + \sqrt{3}q + 1$, then, according to [21, Thm. 4.3], each Sylow $p$-subgroup is cyclic and the Brauer tree of the principal block of $G$ is a star with 6 edges, whose exceptional vertex is on the boundary. Because the multiplicity equals $(|P|-1)/6$, if the ring $FG$ is serial, we conclude that $p = 7$ and $|P| = 7$. For instance, this is the case when $q^2 = 3^{13}$.

The remaining cases of exceptional Lie groups are ruled out in [33] using known information on Brauer trees, for instance see [23] and [24].

5) The symplectic, unitary and orthogonal groups defined over fields with odd number of elements were investigated in [32]. The main tool was an old result by Fong and Srinivasan [15]. It says that in this case (and when $p \neq 2$ does not divide $q$) the Brauer trees of all cyclic blocks of certain (not necessarily simple) groups in classical series are lines.

Thus, to complete the proof of Theorem 1.2 it remains to consider the case when $G$ is a symplectic, orthogonal or unitary simple group defined over a field with even number of elements. The proof proceeds similarly to [32], but first we will need a substitute for Fong and Srinivasan result.
Fact 4.1. Let \( G \) be one of groups \( \text{Sp}_{2m}(q) \), \( \text{GO}^\pm_{2m}(q) \), where \( q \) is even. Then the Brauer tree of each cyclic block of \( G \) is a line.

Proof. By [18, Thm.], each element of \( G \) is a product of two involutions. From this it follows easily that each element of \( G \) is conjugated to its inverse, hence each character of \( G \) is real-valued. Thus the Brauer tree of each cyclic block coincides with its real stem, hence is a line. \( \square \)

We copy from Stather [44, p. 549] the table (see Table 1) describing the sizes of Sylow \( p \)-subgroups of certain classical groups, when \( p > 2 \) does not divide \( q \). Here \( |H|_p \) denotes the order of a Sylow \( p \)-subgroup of a group \( H \), and \( \lfloor k/l \rfloor \) is the integer part of the fraction \( k/l \).

We will consider each case in turn. Since none of simple classical groups is 2-nilpotent, we may assume that \( p > 2 \). Furthermore (say, by [41, Prop. 5.1]), in the defining characteristic each Sylow \( p \)-subgroup is not cyclic, hence we will also assume that \( p \) does not divide \( q \).

Recall also a relevant information on Sylow subgroups of general linear groups. Suppose that \( d \) is the order of \( q \) modulo \( p \). Then \( G = \text{GL}_d(q) \) contains a copy of the multiplicative group \( F^{\ast}_{q^d} \) of the Galois field, the so-called Singer cycle, and a Sylow \( p \)-subgroup of \( G \) can be chosen within this cycle. Then the centralizer of \( P \) in \( G \) coincides with the cycle, and the normalizer of \( P \) is generated over the centralizer by an element \( y \), which acts by conjugation on a generator \( \alpha \) of \( P \) as \( \alpha y = \alpha q \), in particular the index \( |N_G(P)/C_G(P)| \) equals \( d \).

Note also that a Sylow \( p \)-subgroup of \( \text{GL}_m(q) \) is cyclic and nontrivial iff \( m/2 < d \leq m \). In this case \( P \) can be chosen within the copy of \( \text{GL}_d(q) \) embedded in the left upper corner.

The idea of forthcoming proofs is the following. If the ring \( FG \) is serial, then \( d \) cannot be too small comparing with the size of matrices, otherwise \( P \) is not cyclic. Further, \( d \) cannot be too large, because this would imply \( e_0 \geq 3 \). This effectively restricts the size of matrices for which serial rings occur, and the remaining cases are just few.

| Group          | Condition on \( d \)                  | Order of a Sylow \( p \)-subgroup | Sylow type |
|---------------|--------------------------------------|----------------------------------|-----------|
| \( \text{Sp}_{2m}(q) \) | \( d \) even                           | \( |\text{GL}_{2m}(q)|_p \) | A         |
|                | \( d \) odd                           | \( |\text{GL}_m(q)|_p \)          | B         |
| \( \text{GO}^+_m(q) \) | \( d \) odd                           | \( |\text{GL}_m(q)|_p \)          | B         |
|                | \( d \) even and \( \lfloor d/2m \rfloor \) odd | \( |\text{GL}_{2m-2}(q)|_p \)     | A         |
|                | otherwise                             | \( |\text{GL}_{2m}(q)|_p \)       | A         |
| \( \text{GO}^-_{2m}(q) \) | \( d \) odd                           | \( |\text{GL}_{m-1}(q)|_p \)      | B         |
|                | \( d \) even and \( \lfloor d/2m \rfloor \) even | | A        |
|                | otherwise                             | \( |\text{GL}_{2m}(q)|_p \)       | A         |
| \( \text{GU}_n(q^2) \) | \( d \equiv 2 \text{ (mod 4)} \)     | \( |\text{GL}_n(q^2)|_p \)       | B         |
|                | otherwise                             | \( |\text{GL}_{\lfloor n/2 \rfloor}(q^2)|_p \) | A         |
5. SYMPLECTIC GROUPS

Because $q$ is even, one may define symplectic groups using a non-singular symmetric bilinear form $f$ on a $2m$-dimensional vector space $V$ over the Galois field $\mathbb{F}_q$. This form is unique up to the choice of a basis in $V$, and is given by a matrix $W$ such that $W = W^t$, where $t$ means transpose.

A symplectic group, $\text{Sp}_{2m}(q)$, consists of invertible matrices $A$ of order $2m$ which preserve $f$, i.e. $AWA^t = W$. This group has the following order:

$$|\text{Sp}_{2m}(q)| = q^{m^2} \cdot (q^2 - 1) \cdot (q^4 - 1) \cdot \ldots \cdot (q^{2m} - 1).$$

For instance, one could take $W = \left( \begin{array}{cc} 0 & I_m \\ I_m & 0 \end{array} \right)$. Then the rule $A \mapsto \left( \begin{array}{cc} A & 0 \\ 0 & A^t \end{array} \right)$ defines an embedding from $\text{GL}_m(q)$ into $\text{Sp}_{2m}(q)$.

Since $q$ is even, $G = \text{Sp}_{2m}(q)$ coincides with its projective variant, and is simple with few exceptions: $\text{PSp}_2(2) \cong S_3$ and $\text{PSp}_2(4) \cong S_6$ (see Atlas [10, p. x]). Further, $\text{PSp}_2(q)$ is isomorphic to $\text{PSL}_2(q)$. This case has been already analyzed in [26], hence seriality occurs just in cases 2) and 3) of Theorem 1.2. Thus we may assume that $m > 1$, and we prove that no serial group rings occur in the case.

**Proposition 5.1.** Let $F$ be a field of characteristic $p$ dividing the order of a (simple) symplectic group $G = \text{Sp}_{2m}(q)$, where $m > 1$ and $q$ is even. Then the group ring $FG$ is not serial.

**Proof.** Let $d$ denote the order of $q$ modulo $p$. First we consider the case when $d = 1$, i.e. $p$ divides $q - 1$. Because $m \geq 2$, each Sylow $p$-subgroup of $\text{GL}_m(q)$ is not cyclic. The above mentioned diagonal embedding shows that the same holds true for $\text{Sp}_{2m}(q)$, contradicting seriality.

Otherwise $d \geq 2$ and we will show that the index $e = |N_G(P)/C_G(P)|$ is at least $d$. Because $e$ is the number of edges in the Brauer tree of the principal block of $G$, which is a line, non-seriality would follow.

The order and structure of $G$ depends on the parity of $d$ - see Table 1.

**Case A:** $d$ is even. In this case $P$ coincides with a Sylow $p$-subgroup of the ambient group $\text{GL}_{2m}(q)$. Because $P$ is nontrivial and cyclic, we conclude that $m < d \leq 2m$. Further, $m \geq 2$ yields $d > 2$, hence $d \geq 4$.

Under a suitable choice of the matrix $W$, the group $\text{Sp}_d(q) \times \text{Sp}_{2m-d}(q)$ is embedded into $\text{Sp}_{2m}(q)$ as a blocked diagonal, hence $P$ can be chosen in the left upper corner $\text{GL}_d(q)$. Thus it suffices to show that the index $e_d = |N_{G_d}(P_d)/C_{G_d}(P_d)|$ is at least $d$.

Let $\alpha$ be a generator of a Sylow $p$-subgroup $P_d$ of $\text{Sp}_d(q)$. From [44, L. 4.6] it follows that the factor $N_{G_d}(P_d)/C_{G_d}(P_d)$ is generated by an element $y$ acting by conjugation as $\alpha^y = \alpha^d$. Because this action has order $d$, we conclude that $e_d \geq d \geq 4$, as desired.

**Case B:** $d$ is odd. From Table 1 we derive that in this case the order of $P$ is the same as the order of a Sylow $p$-subgroup $P'$ of $\text{GL}_m(q)$. We may assume that $P$ is the image of $P'$, when $\text{GL}_m(q)$ is embedded into $\text{Sp}_{2m}(q)$ diagonally (as above). Because $P'$ is cyclic, hence $m/2 < d \leq m$. Since $m \geq 2$ and $d$ is odd, we derive $d \geq 3$, hence it suffices to check that $e \geq d$. 

10
As above we may assume that \( m = d \). Consider an element \( y' \in \text{GL}_d(q) \) which acts by conjugation on \( P' \) as an automorphism of order \( d \). Then the diagonal image of this element belongs to the normalizer of \( P \) and acts as an automorphism of order \( d \) on this subgroup. \( \square \)

### 6. Orthogonal groups

If \( q \) is even, then the odd-dimensional orthogonal group \( \text{O}_{2m+1}(q) \) is isomorphic to the symplectic group \( \text{PSp}_{2m}(q) \), and this case has been already considered. In the even-dimensional case we use [11] as a guide for definitions.

**Case +. A general orthogonal group of + type, \( \text{GO}_{2m}^+(q) \), consists of invertible matrices \( A \) of order \( 2m \) which preserve the nonsingular quadratic form given by the matrix \( W = \begin{pmatrix} 0 & I_m \\ 0 & 0 \end{pmatrix} \), i.e. \( AWA^t = W \). One can define a morphism from this group to \( \{ \pm 1 \} \) by sending an element to \((-1)^k\), where \( k \) is the dimension of its fixed subspace when acting on \( V \). The kernel of this map is the orthogonal group, \( \text{O}_{2m}^+(q) \), which is of index 2 in \( \text{GO}_{2m}^+(q) \). The order of this group is the following.

\[
| \text{O}_{2m}^+(q) | = q^{m(m-1)} \cdot (q^m - 1) \cdot \prod_{i=1}^{\frac{m-1}{2}} (q^{2i} - 1).
\]

For \( m \leq 3 \) these groups are isomorphic to groups from other series, namely \( \text{O}_{+}^+(q) \cong \text{PSL}_2(q) \times \text{PSL}_2(q) \) is not simple, and \( \text{O}_{+}^+(q) \cong \text{PSL}_4(q) \), hence only the case \( m \geq 3 \) is of potential interest.

**Proposition 6.1.** Let \( F \) be a field of characteristic \( p \) dividing the order of the orthogonal group \( G = \text{O}_{2m}^+(q) \), where \( q \) is even and \( m > 3 \). Then the group ring \( FG \) is not serial.

**Proof.** As usual we may assume that \( p > 2 \) does not divide \( q \), and that a Sylow \( p \)-subgroup \( P \) of \( G \) is nontrivial and cyclic. Since the index of \( G \) in \( H = \text{GO}_{2m}^+(q) \) equals 2, hence \( P \) is a Sylow \( p \)-subgroup of \( H \).

If \( p \) divides \( q - 1 \), then \( P \) is not cyclic, hence suppose that this is not the case. Thus, if \( d \) denotes the order of \( q \) modulo \( p \), then \( 2 \leq d \leq 2m - 2 \).

Recall (see Fact 4.1) that the Brauer tree of the principal block \( B_0' \) of \( H = \text{GO}_{2m}^+(q) \) is a line with \( e' \) edges. Further, Fact 4.3 implies that the principal block of \( G \) is serial iff the same holds true for \( H \), hence it suffices to prove that \( e' > 2 \).

We consider the possibilities for \( P \) given in Table 1

**Case A:** \( d \) is odd. Then the size of \( P \) equals \( | \text{GL}_d(q) |_p \). Since \( P \) is nontrivial and cyclic, we obtain \( m/2 < d \leq m \), hence \( m \geq 4 \) yields \( d > 2 \). Note that the image of \( \text{GL}_d(q) \) via the diagonal embedding \( A \rightarrow \left( \begin{array}{cc} A & 0 \\ 0 & A^{-1} \end{array} \right) \) is contained in \( \text{GO}_{2m}^+(q) \). Now, by considering the normalizer of a Sylow \( p \)-subgroup in \( \text{GL}_d(q) \) and this embedding, we derive \( e' \geq d > 2 \), as desired.

**Case B:** \( d \) is even. Then \( d \leq 2m - 2 \) implies that the integer part of the fraction \( d/2m \) is zero, hence \( P \) is a Sylow \( p \)-subgroup of \( \text{GL}_{2m}(q) \). Since \( P \) is cyclic and nontrivial, we derive that \( m < d \). Now, from [14] L. 4.6 it follows that \( e' > d \). Then \( d > 4 \) yields \( e' > 4 \), as desired. \( \square \)

**Case −.** Let \( \gamma \) be a primitive element of \( \mathbb{F}_{q^2} \) and set \( a = \gamma + \gamma q, b = \gamma^q + 1 \).

11
A general orthogonal group of $-$ type, $\text{GO}_{2m}^-(q)$, consists of invertible matrices $A$ of order $2m$ which preserve the nonsingular quadratic form given by the matrix $W = \begin{pmatrix} 0 & I_{m-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, i.e. $AWA^t = W$; and the orthogonal group, $\text{O}_{2m}^-(q)$, is defined as for the $+$ case. The order of this group is the following.

$$|\text{O}_{2m}^-(q)| = q^{m(m-1)} \cdot (q^m + 1) \cdot \prod_{i=1}^{m-1} (q^{2i} - 1).$$

For $m \leq 3$ these groups are isomorphic to groups from other series, namely $\text{O}_{4}^-(q) \cong \text{PSL}_2(q^2)$ (which we already know) and $\text{O}_{6}^+(q) \cong \text{PSU}_4(q)$ (which we consider later). Thus only the case $m > 3$ is of potential interest.

Proposition 6.2. Let $F$ be a field of characteristic $p$ dividing the order of the orthogonal group $G = \text{O}_{2m}^-(q)$, where $q$ is even and $m > 3$. Then the group ring $FG$ is not serial.

Proof. We use the same notations and assumptions as in the beginning of the proof of Proposition 6.1. The only difference is in the weaker conclusion $d \leq 2m$, because $d = 2m$ may occur, when $p$ divides $q^m + 1$. Again, it suffices to show that $e' > 2$. From Table I we obtain the following possibilities for $P$.

**Case B: $d$ is odd.** Then $P$ is the image, via the above skew diagonal embedding, of a Sylow $p$-subgroup $P'$ of $\text{GL}_{m-1}(q)$. Since $P'$ is cyclic, from $(m-1)/2 < d$, we conclude that $d \geq 3$. Applying [44, L. 4.6] we obtain $e' \geq d \geq 3$, as desired.

**Case A: $d$ is even.** This case splits in two subcases.

If the integer part of the fraction $d/2m$ is odd, then $d \leq 2m$ implies $d = 2m$. From Table I we see that $P$ is a Sylow $p$-subgroup of $\text{GL}_{2m}(q)$. Since $P$ is cyclic, from $m < d$ we derive $d \geq 6$. Again [44, L. 4.6] gives $e' \geq d \geq 6$, as desired.

Otherwise the integer part of $d/2m$ is zero, hence $P'$ is a Sylow $p$-subgroup of $\text{GL}_{2m-2}(q)$. Since $P'$ is cyclic, therefore $m-1 < d$ implies $d \geq 4$. One more application of [44, L. 4.6] yields $e' \geq d \geq 4$.

7. UNITARY GROUPS

It is a common knowledge that Fong–Srinivasan theorem also holds for $G = \text{GU}_n(q)$ when $q$ is even, i.e. the Brauer tree of each cyclic block of $G$ is a line. It would cut few lines from proofs, but is very difficult to find a proper reference. We will be content with the following.

Lemma 7.1. Let $G = \text{SU}_n(q)$ be a quasi-simple special unitary group with a nontrivial cyclic Sylow $p$-subgroup. Then the Brauer tree of the principal $p$-block of $G$ is a line.

Proof. It follows from [39, Sect. 6] that each non-exceptional character in the principal block of $G$ is rational-valued. By looking at the real stem we conclude the desired.

Now it follows from [14] that the Brauer trees of the principal blocks of $\text{SU}_n(q^2)$ and $\text{GU}_n(q^2)$ are obtained by unfolding the same line. Thus to prove that both trees are lines with the same
number of edges it sufficient to check that the indices $|N(P)/C(P)|$ in both groups are equal, but it is hard to find a reference.

Let $-\sigma$ denote the involution $\sigma = a^q$ of the Galois field $\mathbb{F}_q^2$ and let $V$ be an $n$-dimensional vector space over this field. There exists (an essentially unique) non-singular conjugate-symmetric sesquilinear form $f : V \times V \to \mathbb{F}_q^2$, i.e. $f(u, v) = \overline{f(v, u)}$ holds for all $u, v \in V$. If $f$ is given by a matrix $W$, then $W = \overline{W}$.

A general unitary group, $GU_n(q^2)$, consists of matrices $A \in GL_n(q^2)$ preserving $f$, i.e. $AW\overline{A}^t = W$. In particular, if $W = I_n$, then $GU_n(q^2)$ consists of matrices $A$ such that $A\cdot \overline{A}^t = I_n$. The order of this group is the following.

$$|GU_n(q^2)| = q^{n(n-1)/2} \cdot (q + 1) \cdot (q^2 - 1) \cdots \cdots \cdot (q^n - (-1)^n).$$

The unitary matrices of determinant 1 form a normal subgroup in $GU_n(q^2)$ of index $q + 1$, the special unitary group, $SU_n(q^2)$. The center $Z$ of this group consists of scalar matrices, and is of order $(n, q + 1)$. The factor group $PSU_n(q^2) = SU_n(q^2)/Z$ is the projective special unitary group. If $n \geq 3$, then this group is simple, with $PSU_3(2^2)$ being the only exception.

Note that $PSU_2(q^2) \cong PSL_2(q)$ hence, when investigating seriality, we may assume that $n \geq 3$. Thus it suffices to prove the following proposition.

**Proposition 7.2.** Let $F$ be a field of characteristic $p$ dividing the order of a simple unitary group $H = PSU_n(q^2)$, where $n \geq 3$ and $q$ is even. Then the group ring $FH$ is serial if and only if $n = 3$ and $p > 2$ divides $q - 1$.

**Proof.** First we consider the case $n = 3$. If $p \mid q - 1$, then it follows from Geck [17] Thm. 6.1 that the group ring of $SU_3(q^2)$ is serial. Since $H$ is a factor of this group, the group ring $FH$ is also serial. Furthermore, it is also derived from Geck that no other serial rings occur for $SU_3(q)$, because either $P$ is not cyclic, or the principal block is not serial. If $p$ does not divide $q + 1$ or 3, then both properties are inherited by $G$. In the remaining case, when $H = PSU_3(q^2)$ and $p = 3$ divides $q + 1$, it is easily checked that each Sylow 3-subgroup of $H$ is not cyclic.

Thus we may assume that $n \geq 4$. In this case, if $p$ divides $q + 1$, then each Sylow $p$-subgroup of $H$ is not cyclic. Otherwise, if $d$ is the order of $q$ modulo $p$, then $2 < d \leq 2n$.

Further, the principal block of $H$ coincides with the principal block $B_0$ of $G = SU_n(q^2)$, and the Brauer tree of $B_0$ is a line. Thus it suffices to show that the number of edges $e$ in this line exceeds 2. If $P$ denotes a Sylow $p$-subgroup of $G$ then it is a Sylow $p$-subgroup of $GU_n(q^2)$. We consider various possibilities for $P$ given in Table [1]

**Case B:** $d \equiv 2 \pmod{4}$. In particular $d > 2$ yields $d \geq 6$.

In this case $P$ is a Sylow $p$-subgroup of the ambient group $GL_n(q^2)$. Note that the order $f$ of $q^2$ modulo $p$ equals $d/2$. Because $P$ is cyclic, we conclude that $n/2 < f \leq n$, i.e. $n < d \leq 2n$.

If $n = f$ then the centralizer $C_G(P)$ is contained in the Singer cycle, hence is of odd order. Because $G$ has no involution in the annihilator of $B_0$ and at least two classes of involutions, as in Blau [5] proof of Thm. 1 and Cor. 1], we conclude that $B_0$ is not a star, hence $e \geq 3$ in this group. By considering the diagonal embedding from $SU_f(q^2) \times SU_{n-f}(q^2)$ into $SU_n(q^2)$, we obtain the same conclusion for $SU_n(q^2)$.
Case A: $d \equiv 0,1,3 \pmod{4}$. Let $n = 2m + \varepsilon$, where $\varepsilon = 0,1$. From Table 1 we see that the order of $P$ equals the order of a Sylow $p$-subgroup $P'$ of $GL_m(q^2)$. If $\varepsilon = 0$, then choosing $W = (\begin{smallmatrix} 0 & I_m \\ I_m & 0 \end{smallmatrix})$, we obtain the embedding from $GL_m(q^2)$ into $GU_n(q^2)$ which sends $A$ to $\left( \begin{array}{cc} A & 0 \\ 0 & \pi \end{array} \right)$, and a similar embedding takes place if $\varepsilon = 1$. Because $p$ does not divide $q^2 - 1$, the generator $\alpha'$ for $P'$ is in $SL_n(q^2)$, hence its image belongs to $SU_n(q^2)$.

Recall that $d$ is the order of $q$ modulo $p$, and $f$ is the order of $q^2$ modulo $p$. First we consider the case when $d$ is odd, hence $f = d > 2$.

Because $P'$ is cyclic and nontrivial, we conclude that $m/2 < f \leq m$. As above, to estimate $e$, we may assume that $n = 2d$. Choose an element in $GL_d(q^2)$ which acts by conjugation on $P'$ as an automorphism of order $f$. By multiplying by a constant we may assume that this element has determinant 1. Expanding diagonally, we conclude that $e \geq f > 2$, as desired.

Thus it remains to consider the case when $d$ is divisible by 4, in particular $d \geq 4$. Then $d = 2f$ yields $f \geq 2$. If $d > 4$, then, using the diagonal embedding, we conclude that $e \geq f > 2$.

Thus we may assume that $d = 4$ and $f = 2$, i.e. $G = SU_4(q^2)$ and $p | q^3 + 1$. In this case the generator $A = \alpha'$ for $P'$ can be chosen such that $A \cdot \overrightarrow{A^t} = I_2$. As above, using the diagonal embedding we obtain that $e \geq 2$. But also the matrix $W$ normalizers $P$, hence $e \geq 4$, as desired.

In fact, $e = 4$ in this case by looking at the generic character table for $GU_4(q)$. Say, the character degrees can be obtained formally extending Carter [9, p. 465] to the case of odd $q$.

$$
\begin{array}{cccc}
1 & q^3(q^2-q+1) & (q-1)(q+1)^3(q^2-q+1) & q^6 & q(q^2-q+1)
\end{array}
$$

\[\square\]

8. Small groups

In this section we will verify (modulo some calculations in MAGMA, which we omit) Conjecture 1 for small groups.

Proposition 8.1. Conjecture 1 holds true for all groups $G$ of order $\leq 10^4$.

Proof. Suppose otherwise. Thus, for some prime $p$, there exists a small non-$p$-solvable group $G$ with a cyclic Sylow $p$-subgroup $P$ and the normal series $\{e\} \subset O_{p'} \subset K \subset G$ such that the group ring of the simple group $H = K/O_{p'}$ over $\mathbb{F}_p$, but the ring $FG$ is not serial.

Suppose that $G$ is a minimal such counterexample. By Fact 3.3 it follows that $K = G$ and $H$ is a simple nonabelian finite group of order $\leq 10^4$, whose group ring is serial. Further, this fact implies that $G$ contains no proper normal subgroup containing $P$. Also, if $p = 3$ and $|P| = 3$, then the multiplicity of the exceptional vertex equals $(3 - 1)/2 = 1$. Thus in this case seriality of $H$ implies seriality of $G$, a contradiction. Thus we may assume that, if $p = 3$ divides the order $H$, then $|P| \geq 9$.

By these remarks, from the list of simple groups of size $\leq 5 \cdot 10^3$ whose group rings are serial (see Theorem 1.2) only the following are potential candidates for $H$ in the projected counterexample.

1) $H = PSL_2(8)$ of order 504, and $p = 7$, $|P| = 7$. 

14
This is the most difficult case: we have to check all extensions \( G = U.H \), where \( U = O_{p'} \) has order at most 19, and we use the MAGMA command `ExtensionsOfSolubleGroup` for this search. For instance, for \(|O_{p'}| = 24\) there are 15 extensions all of which contain \( H \) as a normal subgroup.

2) \( H = \text{PSL}_2(11) \) of order 660, and \( p = 5 \), \(|P| = 5\).

In this case we have to investigate extensions \( G = U.H \), where \( U = O_{p'} \) has order at most 14. For instance, the only extensions \( C_2.H \) are the direct product \( C_2 \times H \); and \( \text{SL}_2(11) \), where the normalizer of \( P \) coincides with \( G \). However the SL-series has been already considered: it follows from \( \text{[20]} \) that the ring \( FG \) is serial, a contradiction.

3) \( H = \text{PSL}_2(19) \) of order 3420, and \( p = 3 \), \(|P| = 9\).

Here the only case is \( O_{p'} = C_2 \). The only extension which is not a direct product is \( \text{SL}_2(19) \), for which seriality is known.

4) \( H = \text{PSL}_2(16) \) of order 4080, and \( p = 5 \), \(|P| = 5\).

Here the only extension \( C_2.H \) is the direct product \( C_2 \times H \). \( \square \)

By pushing harder, one probably could improve the above estimate, but not beyond \( 60 \cdot 504 = 30,240 \), because \(|A_5| = 60 \) and \( A_5 \) is not solvable.

9. Discussion

There are few open question on serial group rings of finite groups we would like to address.

**Question 9.1.** Is Conjecture \( \text{[1]} \) true for \( p = 3 \)?

Note that in this case \( e \) divides 2, hence only the case when \( e = 2 \) and \(|P| \geq 9\) is of interest. Further, under these restrictions, the list of simple groups in Theorem \( \text{[2]} \) contains only groups in \( \text{PSL}_2, \text{PSL}_3 \) and \( \text{PSU}_3 \)-series. Also, the principal block of \( G \) is serial, hence we should investigate non-principal blocks with large defect groups. If this block is non-serial, then \( e = 2 \) and the exceptional character occurs at the end of the line. Some information on values (hence degrees) of characters in this block can be extracted from \( \text{[13, Thm. 7.2.16]} \), but we were not able to draw a decisive conclusion. The feeling is that, even in this case, the main conjecture is based on a little empirical evidence.

The following simple question also shows our limits.

**Question 9.2.** Suppose that \( F \) is a field of characteristic \( p \) dividing the order of \( G \) and let \( H \) be a normal subgroup of \( G \). Is it true that, if \( FG \) is serial, then \( FH \) is serial?

Of course, modulo the main conjecture, this question has an affirmative answer.

Note that Blau \( \text{[5]} \) considered a similar question: when the Brauer tree of the principal block of a group \( G \) is a star (with no restriction of the position of the exceptional character). This is the same as each \( p \)-modular irreducible character in the principal block lifts to an ordinary character.

Comparing with seriality, there are some simplifications: from the very beginning one can factor out the subgroup \( O_{p'} \). The list of finite simple groups satisfying Blau’s condition will be
a bit larger than the one given in Theorem 1.2. For instance, the Mattieu group \( M_{23} \) will get there when \( p = 5 \), and there will be more varieties of Suzuki and Ree groups. However, the difficulty may lay when passing from a normal subgroup of coprime index to \( G \) and vice-versa.

By \( \mathbb{Z}(p) \) we will denote the localization of integers with respect to a prime ideal \( p \mathbb{Z} \).

**Question 9.3.** Describe finite groups \( G \) such that the group ring \( \mathbb{Z}(p)G \) is serial.

Of course, seriality of this ring implies seriality of the ring \( \mathbb{F}_pG \). Further, because the valuation domain \( \mathbb{Z}(p) \) is not complete, the indecomposable idempotents lift rarely, hence the list of simple groups in Theorem 1.2 will get scarce. However we will face the other difficulties, starting from the lack of Maschke’s theorem.

**References**

[1] J.L. Alperin, Local Representation Theory, Cambridge University Press, 1989.
[2] M. Auslander, I. Reiten, S. Smalo, Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics, Vol. 36, 1995.
[3] Y. Baba, K. Oshiro, Classical Artinian Rings and Related Topics, World Scientific Publ., 2009.
[4] Y. Benson, Representations and Cohomology, I, Cambridge Studies in Advanced Mathematics, Vol. 30, 1995.
[5] H.I. Blau, On Brauer stars, J. Algebra, 90 (1984), 169–188.
[6] W. Bosma, J. Cannon, C. Playoust, The MAGMA algebra system I: The user language, J. Symbolic Comput., 24 (1997), 235–265.
[7] R. Burkhardt, Die Zerlegungsmatrizen der Gruppen \( \text{PSL}(2,p^f) \), J. Algebra, 40 (1976), 75–96.
[8] R. Burkhardt, Über die Zerlegungszahlen der Suzukigruppen \( \text{Sz}(q) \), J. Algebra, 59(2) (1979), 421–433.
[9] R.W. Carter, Finite Groups of Lie Type: Conjugacy Classes and Complex Characters, John Wiley and Sons, 1985.
[10] J.H. Conway (et al.), Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups, Clarendon Press, 1985.
[11] H. Dietrich, C.R. Leedham-Green, F. Lübeck, E.A. O’Brien, Constructive recognition of classical groups in even characteristic, J. Algebra, 391 (2013), 227–255.
[12] D. Eisenbud, P. Griffith, Serial rings, J. Algebra, 17 (1971), 389–400.
[13] W. Feit, The Representation Theory of Finite Groups, North-Holland Mathematical Library, Vol. 25, 1982.
[14] W. Feit, Possible Brauer trees, Illinois J. Math., 28 (1984), 43–56.
[15] P. Fong, B. Srinivasan, Brauer trees in classical groups, J. Algebra, 131 (1990), 179–225.
[16] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.6, 2016, http://www.gap-system.org.
[17] M. Geck, Irreducible Brauer characters of the 3-dimensional unitary group in non-defining characteristic, Comm. Algebra, 18(2) (1990), 563–584.
[18] R. Gow, Products of two involutions in classical groups of characteristic 2, J. Algebra, 71 (1981), 583–591.
[19] M. Hazewinkel, N. Gubareni, V.V. Kirichenko, Algebras, Rings and Modules, Vol. 1, Kluwer, 2004.
[20] D.G. Higman, Indecomposable representations at characteristic \( p \), Duke Math. J., 21 (1954), 377–381.
[21] G. Hiss, The Brauer trees of the Ree groups, Comm. Algebra, 19(3) (1991), 871–888.
[22] G. Hiss, K. Lux, Brauer Trees of Sporadic Groups, Clarendon Press, Oxford, 1989.
[23] G. Hiss, F. Lübeck, The Brauer trees of the exceptional Chevalley groups of types \( F_4 \) and \( 2E_6 \), Arch. Math., 70(1) (1998), 16–21.
[24] G. Hiss, F. Lübeck, G. Malle, The Brauer trees of the exceptional Chevalley groups of type \( E_6 \), Manuscr. Math., 87(1) (1995), 131–144.
[25] G.J. Janusz, Indecomposable modules for finite groups, Annals of Math., 89 (1969), 209–241.
A. Kukharev, Serial group rings of finite groups, PhD Thesis, Belarusian State University, Minsk, 2016.

A. Kukharev, G. Puninski, Serial group rings of finite groups. $p$-nilpotency, Notes Research Semin. Steklov Institute Sanct-Petersb., 413 (2013), 134–152.

A. Kukharev, G. Puninski, Serial group rings of finite groups. $p$-solvability, Algebra Discr. Math., 16 (2013), 201–210.

A. Kukharev, G. Puninski, The seriality of group rings of alternating and symmetric groups, Vestnik BGU, math.-inform. ser., 2 (2014), 61–64.

A. Kukharev, G. Puninski, Serial group rings of finite groups. Simple sporadic groups and Suzuki groups, Notes Research Semin. Steklov Institute Sanct-Petersb., 435 (2015), 73–94.

A. Kukharev, G. Puninski, Serial group rings of finite groups. General linear and close groups, Algebra Discrete Math., 20(1) (2015), 259–269.

A. Kukharev, G. Puninski, Serial group rings of classical groups defined over fields with odd number of elements, Notes Research Semin. Steklov Institute Sanct-Petersb., 452 (2016), 158–176.

A. Kukharev, G. Puninski, Serial group rings of finite groups of Lie type, Fundam. Appl. Math., to appear.

H. Kupisch, Einreihige Algebren über eine perfekten Körper, J. Algebra, 33 (1975), 68–74.

K. Lux, H. Pahlings. Representations of Groups. A Computational Approach, Cambridge Studies in Advanced Mathematics, Vol. 124, 2010.

K. Morita, On group rings over a modular field which possess radicals expressible as principal ideals, Sci. Repts. Tokyo Daigaku, 4 (1951), 177–194.

N. Naerig, A construction of almost all Brauer trees, J. Group Theory, 11 (2008), 813–829.

G. Puninski, Serial Rings, Kluwer, 2001.

G.R. Robinson, Some uses of class algebra constants, J. Algebra, 91 (1984), 64–74.

M. Sawabe, A note on finite simple groups with abelian Sylow $p$-subgroups, Tokyo Math. J., 30 (2007), 293–304.

M. Sawabe, A. Watanabe, On the principal blocks of finite groups with abelian Sylow $p$-subgroups, J. Algebra, 237 (2001), 719–734.

J. Scopes, Cartan matrices and Morita equivalence for blocks of the symmetric groups, J. Algebra, 142 (1991), 441–455.

B. Srinivasan, On the indecomposable representations of a certain class of groups, Proc. Lond. Math. Soc., 10 (1960), 497–513.

M. Stather, Constructive Sylow theorems for the classical groups, J. Algebra, 316 (2007), 536–559.

A.A. Tuganbaev, Ring Theory, Arithmetical Rings and Modules, Moscow, Independent University, 2009.

Yu. Volkov, A. Kukharev, G. Puninski, The seriality of the group ring of a finite group depends only of characteristic of the field, Notes Research Semin. Steklov Institute Sanct-Petersb., 423 (2014), 57–66.

H. Wielandt, Sylowgruppen und Kompositions-Struktur, Abhand. Math. Sem. Hamburg, 22 (1958), 215–228.