Statistical filtering algorithms based on the maximum cross section method for stochastic systems with regime switching

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Abstract. New statistical algorithms are formed to estimate the state vector of stochastic systems with regime switching. These algorithms are based on the particle method, i.e., on numerical methods for solving stochastic differential equations, methods for modeling conditional Markov processes with a finite set of states, and methods for calculating weights. An important part of proposed algorithms is the use of the maximum cross section method and its modification, which allows the exact simulation of the regime switching process. This paper continues the authors’ research in the field of statistical methods and algorithms of the analysis and filtering for continuous-time stochastic systems.

1. Introduction
In [1], the statistical algorithms for estimating trajectories of dynamic systems driven by the Wiener process and the compound Poisson process have been proposed. Observations have been defined by dynamic systems driven by the Wiener process only. Using observations we can estimate the state vector of a dynamic system at the current time (filtering), past (smoothing), and future (prediction). The proposed algorithms are based on numerical methods for solving stochastic differential equations (SDEs) and methods for modeling the Poisson flows. To simulate the Poisson flows, the maximum cross section method and its modification have been applied. This provides the lower complexity due to fewer random number generator calls [2,3].

The algorithms given in [2,3] are based on the particle method [4], where it is proposed to apply a similar approach to estimating trajectories of dynamic systems driven by the Wiener process and conditional Markov process with a finite set of states, i.e., we consider the estimation of trajectories of stochastic systems with regime switching (SSRS), or systems with random structure [5]. New statistical filtering algorithms are based on the particle method (particle filters). They include numerical methods for solving SDEs, the maximum cross section method and its modification for modeling a composition of the Poisson flows to simulate the regime switching process. Possible applications of proposed filtering algorithms for the SSRS are tracking [6–8], navigation [9–12] and others [13].

The solution of the optimal filtering problem for the SSRS in the most general case reduces to stochastic partial differential equations (generalized Stratonovich equations [5]), which are
similar to the Kushner–Stratonovich equation for dynamic systems driven by the Wiener process only [4]. Finding the posterior probability density function (pdf) as the solution of the Kushner–Stratonovich equation is a complicated problem. It is simplified for linear systems and some nonlinear systems that allow one to construct so-called finite-dimensional filters [4]. Finding conditionally posterior pdfs for the SSRS is more difficult, therefore, various suboptimal methods to solve the filtering problem in real time have been offered. Such methods are mainly related to the approximation of pdfs by partial sums of the function series using posterior moments and cumulant closed techniques [14]. The conditionally optimal filtering techniques and the construction of optimal structure filters are also applied [15,16].

An alternative approach consists of the application of the statistical modeling method (Monte Carlo method) and the construction of statistical filtering algorithms for the SSRS, i.e., particle filters. Their implementation requires the use of significant computational resources, but such algorithms can provide more accurate filtering results as compared to different suboptimal filters. Earlier, particle filters were applied to the SSRS but for simplified mathematical models, e.g., for the SSRS, in which the regime switching process does not use the information about the number, \(W\), of the regime switching intensities [5];\(\sigma\) switching intensities [5].

2. Optimal filtering problem for stochastic systems with regime switching

In the filtering problem it is assumed that a mathematical model of the dynamic system is known. Here we describe the dynamic system by the Itô SDE

\[
\begin{align*}
\text{d}X(t) &= f^{(l)}(t, X(t))\text{d}t + \sigma^{(l)}(t, X(t))\text{d}W(t), \quad X(0) = X_0, \\
\end{align*}
\]

where \(X\) is the \(\mathbb{R}^n\)-valued vector, \(t \in \mathbb{T} = [0, T]\), \(\mathbb{T}\) is the time interval, \(f^{(l)}(t, x): \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n\), \(\sigma^{(l)}(t, x): \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^{n \times s}\) are given functions for all \(l \in \mathbb{L} = \{1, 2, \ldots, L_{\text{max}}\}\), \(l\) is the regime number, \(W(t)\) is the \(s\)-dimensional standard Wiener process. The distribution law of the vector \(X_0\) is given.

The number \(l\) is the value of the regime switching process \(L(t)\) for a fixed \(t \in \mathbb{T}\). This random process is defined by functions \(\lambda_{lr}(t, x): \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}_+\) that determine the regime switching intensities [5]; \(l, r = 1, 2, \ldots, L_{\text{max}}, l \neq r\). This means that for small \(\Delta t\)

\[
\begin{align*}
\mathbb{P}(L(t + \Delta t) = r \mid L(t) = l, X(t) = x) &= \lambda_{lr}(t, x)\Delta t + o(\Delta t), \\
\mathbb{P}(L(t + \Delta t) = l \mid L(t) = l, X(t) = x) &= 1 - \sum_{r=1,r \neq l}^{L_{\text{max}}} \lambda_{lr}(t, x)\Delta t + o(\Delta t),
\end{align*}
\]

where \(\mathbb{P}\) is the probability.

The initial condition for the regime switching process \(L(t)\) is given, i.e., \(L(0) = L_0 \in \mathbb{L}\). Paths of the random process \(L(t)\) are piecewise constant functions (right continuous with left limits), their values are from the set \(\mathbb{L}\). It is assumed that \(X_0, L_0\) and \(W(t)\) are independent.

The state vector \(X\) of the SSRS consists of two parts. The first one is the continuous part \(X\) and the second one is the discrete part \(L\), i.e., the information about the current system state is \(\bar{X} = [X^T, L]^T\).

Observations are described by the following SDE

\[
\begin{align*}
\text{d}Y(t) &= c^{(l)}(t, X(t))\text{d}t + \zeta^{(l)}(t)\text{d}V(t), \quad Y(0) = Y_0 = 0,
\end{align*}
\]
in which \( Y \) is the \( \mathbb{R}^m \)-valued vector, \( c^{(l)}(t, x): \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^m \), \( \zeta^{(l)}(t): \mathbb{T} \to \mathbb{R}^{m \times d} \) are given functions for all \( l \in \mathbb{L} \), \( V(t) \) is the \( d \)-dimensional standard Wiener process \((W(t) \text{ and } V(t) \text{ are independent})\).

Note that in the general case, the regime indicator equation is needed for the SSRS [8, 15]. In this situation the observation vector as well as the state vector consists of continuous and discrete parts. However, in this paper we describe observations by equation (2) only and do not define the regime indicator equation.

All the functions \( f^{(l)}(t, x), \sigma^{(l)}(t, x), \lambda_{tr}(t, x), c^{(l)}(t, x) \), and \( \zeta^{(l)}(t) \) \((l = 1, 2, \ldots, L_{\text{max}})\) should satisfy the existence and uniqueness conditions for solutions of SDEs [21], and \( E[X_0]^2 < \infty \), where \( E \) is the mean. It is assumed that the matrix \( \eta^{(l)}(t) = \zeta^{(l)}(t)[\zeta^{(l)}(t)]^T \) is non-degenerate for all \( l \in \mathbb{L} \) and \( t \in \mathbb{T} \). Moreover, there exists the pdf of the vector \( X = X(t) \) for each time \( t \in \mathbb{T} \).

The optimal filtering problem for the SSRS is to find the estimate \( \hat{X}(t) \) of the state vector \( \hat{X} \) from observations \( Y^t_0 = \{Y(\tau), \tau \in [0, t]\} \) in the form \( \hat{X} = \Phi(t, Y^t_0), t \in \mathbb{T} \).

The function \( \Phi(t, \cdot) \) can be found from the estimate optimality condition, i.e., it is based on a given quality criterion that provides a minimum average risk [15]. So, \( E\Pi(\bar{x}, \hat{X}) \to \min \), where the minimum is taken over all admissible functions \( \Phi(t, \cdot) \), \( \Pi(\bar{x}, \hat{x}) \) is the loss function, \( \bar{x} = [x^T, l]^T \).

We consider the additive quadratic-simple loss function

\[
\Pi(\bar{x}, \hat{x}) = (x - \hat{x})^T C(x - \hat{x}) + (1 - \delta_{l,l}),
\]

where \( C \) is a symmetric positive definite matrix \( n \times n \), \( \delta_{l,l} \) is the Kronecker delta.

In order to write the solution of the optimal filtering problem, it is necessary to introduce notations for the posterior pdf of the vector \( X \) and for probabilities specifying the distribution of the regime number \( L \). Denote the function that describes the posterior distribution of the state vector \( \hat{X} \) by \( \varphi(t, x, l|Y^t_0)\): \( \mathbb{T} \times \mathbb{R}^n \times \mathbb{L} \to \mathbb{R}_+ \):

\[
\sum_{l=1}^{L_{\text{max}}} \int_{\mathbb{R}^n} \varphi(t, x, l|Y^t_0)dx = 1.
\]

For a fixed \( l \in \mathbb{L} \) this function is the unnormalized conditionally posterior pdf \( \varphi^{(l)}(t, x|Y^t_0) \) of the vector \( X \), and the function

\[
P^{(l)}(t|Y^t_0) = \int_{\mathbb{R}^n} \varphi^{(l)}(t, x|Y^t_0)dx
\]

(3)

determines the posterior probability of the event \( L(t) = l \) \((P^{(l)}(0) = P^{(l)}_0 \text{ is the distribution of the initial regime } L_0)\). Next, the function

\[
\hat{\varphi}^{(l)}(t, x|Y^t_0) = \frac{\varphi^{(l)}(t, x|Y^t_0)}{P^{(l)}(t|Y^t_0)}
\]

(4)

determines the normalized conditionally posterior pdf of the vector \( X \) for \( L(t) = l \), and the function

\[
\hat{\varphi}(t, x|Y^t_0) = \sum_{l=1}^{L_{\text{max}}} \varphi^{(l)}(t, x|Y^t_0)
\]

(5)

is the normalized posterior pdf of the vector \( X \).
According to [15], the optimal estimate $\hat{X}(t) = [\hat{X}^T(t), \hat{L}(t)]^T$ for the additive quadratic-

$$\hat{X}(t) = \sum_{l=1}^{L_{\text{max}}} P^{(l)}(t|Y_0^t) \hat{X}^{(l)}(t), \quad \hat{L}(t) = \arg\max_{l \in \mathbb{L}} P^{(l)}(t|Y_0^t),$$

where

$$\hat{X}^{(l)}(t) = \mathbb{E}[X(t)|L(t) = l, Y_0^t] = \int_{\mathbb{R}^n} x \tilde{p}^{(l)}_k(t, x|Y_0^t) dx, \quad l = 1, 2, \ldots, L_{\text{max}}.$$

3. Statistical filtering algorithms

To solve the optimal filtering problem, we apply the statistical modeling method (Monte Carlo

method). Its important advantage is the simplicity of the SSRS simulation by the use of the

known mathematical model including SDE (1) for the random process $X(t)$ and the definition

for the regime switching process $L(t)$. The estimate $\hat{X}$ can be found from the sample paths of

random processes $X(t)$ and $L(t)$, or realizations of the state vector $X$ at different time

moments (the time discretization nodes), using statistical methods. In fact, we can estimate the

posterior pdf of the vector $X$, conditional and unconditional moments of the vector $X$, and the

distribution for the regime number $L$. This requires the use of numerical methods for solving

SDEs and modeling Markov processes with a finite set of states.

It is proposed to apply the particle method with allowance for observations $Y_0^t$. So, we need to simulate the dynamic system (1) and the regime switching process as well as to calculate

weights according to a special rule. For stochastic continuous-time and discrete-time dynamic

systems, the rules for calculating weights are different [4, 22]. However, the use of numerical

methods for solving SDEs means the transition from continuous time to discrete time. It allows

the application of a simple rule for calculating weights.

We use the Euler–Maruyama method to construct a discrete approximation of the solution for equations (1) and (2) when $l \in \mathbb{L}$ is fixed. So,

$$X_{k+1} = X_k + hf^{(l)}(t_k, X_k) + \sqrt{h}\sigma^{(l)}(t_k, X_k)\Delta W_k, \quad (6)$$

$$Y_{k+1} = Y_k + hc^{(l)}(t_k, X_k) + \sqrt{h}\zeta^{(l)}(t_k)\Delta V_k, \quad (7)$$

where $\Delta W_k$ and $\Delta V_k$ are $s$-dimensional and $d$-dimensional random vectors, respectively, whose components are independent and have the standard normal distribution for all $k, h > 0$ is the integration step, i.e.,

$$t_{k+1} = t_k + h, \quad k = 0, 1, \ldots, N - 1, \quad t_0 = 0, \quad t_N = T.$$

The random vector $(Y_{k+1} - Y_k)/h - c^{(l)}(t_k, X_k)$ has the Gaussian distribution with zero mean and covariance matrix $\eta^{(l)}(t_k)/h$, i.e., the corresponding pdf is

$$p_k^{(l)}(x, h) = \frac{h^{n/2}}{(2\pi)^{n/2}\det \eta^{(l)}(t_k)} \exp\left\{ -\frac{h}{2} x^T[\eta^{(l)}(t_k)]^{-1} x \right\}.$$
The initial condition for the recurrence relation (8) should be positive. Its specific value does not matter, e.g., \( \omega_0 = 1 \). The weights are required to find the posterior moments for the vector \( X \). In particular, the posterior mean of the vector \( X \) at time \( t_k \) is estimated as follows

\[
\hat{X}_k = \frac{1}{\Omega_k} \sum_{i=1}^{M} \omega_i^k X_k^i, \quad \Omega_k = \sum_{i=1}^{M} \omega_i^k,
\]

where \( i \) is the number of the simulated path of the random process \( X(t) \), \( M \) is the sample size.

Similarly, posterior second-order moments can be estimated as

\[
\hat{\Psi}_k = \frac{1}{\Omega_k} \sum_{i=1}^{M} \omega_i^k (X_k^i)^T, \quad \hat{R}_k = \frac{1}{\Omega_k} \sum_{i=1}^{M} \omega_i^k (X_k^i - \hat{X}_k)(X_k^i - \hat{X}_k)^T,
\]

and the estimate \( \hat{R}_k \) (the posterior covariance matrix) can be used as the filtering accuracy characteristic.

Note that we can use the suitable numerical method for solving SDE (1) that provides a higher accuracy as compared to the Euler–Maruyama method [23–26]. A numerical solution of equation (2) is required for simulations only.

For the SSRS, we need to simulate paths of the random process \( X(t) \) and paths of the regime switching process \( L(t) \) in accordance with its definition [23, 27]. All the relations given above can be used if we let \( l = L_k = L(t_k) \).

For the simulation of the regime switching process, we can use the orderliness condition for the Poisson flows or can construct filtering algorithms based on the maximum cross section method and its modification. All necessary relations for filtering jump-diffusion random processes are described in [3].

Further, we form statistical algorithms to solve the optimal filtering problem using algorithms for the SSRS simulation [23, 24, 28] and algorithms for filtering jump-diffusion random processes [2, 3]. In new algorithms for the SSRS we use following notations and conditions: the intensity of a composition of the Poisson flows for the regime switching with the current number \( l \) is given by

\[
\lambda_l(t, x) = \sum_{r=1}^{m} \lambda_{lr}(t, x),
\]

values \( \lambda_{lr}^* \) \( (l, r = 1, 2, \ldots, L_{\text{max}}; l \neq r) \) satisfy the condition

\[
\lambda_{lr}(t, x) \leq \lambda_{lr}^*, \quad l, r = 1, 2, \ldots, L_{\text{max}}, \quad l \neq r,
\]

and

\[
\lambda_{l}^* = \sum_{r=1, r \neq l}^{m} \lambda_{lr}^*.
\]

**Algorithm 1 (based on the orderliness condition)**

(i) Specify \( M \) (the sample size), the number of paths for random processes \( X(t) \), \( L(t) \), and \( \omega(t) \) to be simulated; \( h \), the integration step. Simulate initial vectors \( X_0^i \) and regime numbers \( L_0^i \) according to given distributions and let \( k = 0, \omega_0^i = 1, i = 1, 2, \ldots, M \).

(ii) Estimate the posterior mean and the covariance matrix of the vector \( X \) as well as the regime number \( L \) at time \( t = t_k \) using the sample \( \{X_k^i, L_k^i, \omega_k^i\}_{i=1}^{M} \) as follows

\[
\hat{X}_k = \frac{1}{\Omega_k} \sum_{i=1}^{M} \omega_k^i X_k^i, \quad \hat{R}_k = \frac{1}{\Omega_k} \sum_{i=1}^{M} \omega_k^i (X_k^i - \hat{X}_k)(X_k^i - \hat{X}_k)^T, \quad \hat{L}_k = \arg \max_{l \in \mathbb{E}} \hat{P}_k^{(l)},
\]
where
\[ \Omega_k = \sum_{i=1}^{M} \omega^i_k, \quad \hat{P}^{(l)}_k = \frac{1}{\Omega_k} \sum_{i=1,2,\ldots,M} \omega^i_k. \]

If \( T - t_k = 0 \), then stop. Otherwise, let \( i = 1 \).

(iii) Obtain a realization of the vector \( X \) at time \( t = t_k + h \) by
\[ X^i_{k+1} = X^i_k + hf^{(l)}(t_k, X^i_k) + \sqrt{h} \sigma^{(l)}(t_k, X^i_k) \Delta W^i_k, \]
where \( l = L^i_k \) and \( \Delta W^i_k \) is a realization of the \( s \)-dimensional random vector with independent components having the standard normal distribution. Update the weight
\[ \omega^i_{k+1} = \omega^i_k \hat{P}^{(l)}_k \left( \frac{Y(t_{k+1}) - Y(t_k)}{h} - c^{(l)}(t_k, X^i_k, h) \right). \]

Verify the following regime switching condition: if \( \alpha = \lambda_l(t_k, X^i_k)h \), where \( \alpha \) is a realization of a random value with the uniform distribution on the interval \((0,1)\), then simulate a number \( r \) for a new regime with probability \( P_r = \lambda_{l^*}(t_{k+1}, X^i_{k+1})/\lambda_l(t_{k+1}, X^i_{k+1}) \) \( (r \in L \setminus \{l\}) \) and switch the current regime number by the \( r \)th number, i.e., let \( L^i_{k+1} := r \). Otherwise, let \( L^i_{k+1} := l \).

(iv) If \( i = M \), let \( k := k + 1 \), then go to Step (ii). Otherwise, let \( i := i + 1 \) and go to Step (iii).

**Algorithm 2 (Based on the Maximum Cross Section Method)**

(i) Specify \( M \) (the sample size), the number of paths for random processes \( X(t), L(t), \) and \( \omega(t) \) to be simulated; \( h \), the integration step. Simulate initial vectors \( X^i_0 \) and regime numbers \( L^i_0 \) according to given distributions and let \( k = 0, t^i_0 = 0, \omega^i_0 = 1, i = 1, 2, \ldots, M \). Obtain realizations \( \xi^i \) for a random value having the exponential distribution with the parameter \( \lambda^i_l \), where \( l = L^i_0, i = 1, 2, \ldots, M \).

(ii) See Step (ii) in Algorithm 1.

(iii) Let \( l = L^i_k \). If \( t^i_\ast + \xi^i \geq t_k + h \), then go to Step (iv). Otherwise, let \( \tau := t_k, \tilde{X} = X^i_k \) and go to Step (v).

(iv) Obtain a realization of the vector \( X \) at time \( t = t_k + h \) by
\[ X^i_{k+1} = X^i_k + hf^{(l)}(t_k, X^i_k) + \sqrt{h} \sigma^{(l)}(t_k, X^i_k) \Delta W^i_k, \]
where \( \Delta W^i_k \) is a realization of the \( s \)-dimensional random vector with independent components having the standard normal distribution. Let \( L^i_{k+1} := l \) and go to Step (vii).

(v) Obtain a realization of the vector \( X \) at time \( t = t^i_\ast + \xi^i \) by
\[ \tilde{X} = \tilde{X} + hf^{(l)}(\tau, \tilde{X}) + \sqrt{h} \sigma^{(l)}(\tau, \tilde{X}) \Delta \tilde{W}, \]
where \( \tilde{h} = t^i_\ast + \xi^i - \tau \) and \( \Delta \tilde{W} \) is a realization of the \( s \)-dimensional random vector with independent components having the standard normal distribution. Let \( \tau := \tau + \tilde{h} \) and simulate a number \( r \) for a new regime with probability \( P_r = \lambda_{l^*}(\tau, \tilde{X})/\lambda^i_l \) \( (r \in L \setminus \{l\}) \). Verify the following regime switching condition: if \( \alpha = \lambda_{l^*}(\tau, \tilde{X})/\lambda^i_l \), where \( \alpha \) is a realization of a random value having the uniform distribution on the interval \((0,1)\), then switch the current regime number by the \( r \)th number, i.e., let \( l := r \). Further, let \( t^i_\ast := t^i_\ast + \xi^i \) and obtain a new realization \( \xi^i \) for a random value having the exponential distribution with the parameter \( \lambda^i_l \). If \( t^i_\ast + \xi^i \geq t_k + h \), then go to Step (vi). Otherwise, go to Step (v).
(vii) Obtain a realization of the vector $X$ at time $t = t_k + h$ by

$$X'_{k+1} = \tilde{X} + \tilde{h}f^i(\tau, \tilde{X}) + \sqrt{\tilde{h}}\sigma^i(\tau, \tilde{X})\Delta\tilde{W},$$

where $\tilde{h} = t_k + h - \tau$ and $\Delta\tilde{W}$ is a realization of the $s$-dimensional random vector with independent components having the standard normal distribution. Let $L'_{k+1} := l$.

(viii) See Step (iv) in Algorithm 1.

Algorithm 3 (based on the modified maximum cross section method)

(i) Specify $M$ (the sample size), the number of paths for random processes $X(t)$, $L(t)$, and $\omega(t)$ to be simulated; $h$, the integration step. Simulate initial vectors $X_0^i$ and regime numbers $L_0^i$ according to given distributions and let $k = 0$, $t_k^i = 0$, $\omega_0^i = 1$, $i = 1, 2, \ldots, M$. Obtain realizations $\xi^i$ for a random value having the exponential distribution with the parameter $\lambda_r^i$, where $l = L_0^i$, and let $\Pi^i = \Phi^i = 1$, $i = 1, 2, \ldots, M$.

(ii) See Step (ii) in Algorithm 1.

(iii) Let $l = L_k^i$. If $t_k^i + \xi^i \geq t_k + h$, then go to Step (iv). Otherwise, let $\tau = t_k$, $\tilde{X} = X_k^i$. If $\Phi^i = 1$, obtain a realization of a random value $\alpha$ having the uniform distribution on the interval $(0, 1)$ and let $\Phi^i = 0$. Go to Step (v).

(iv) See Step (iv) in Algorithm 2.

(v) Obtain a realization of the vector $X$ at time $t = t_k^i + \xi^i$ by

$$\tilde{X} = \tilde{X} + \tilde{h}f^i(\tau, \tilde{X}) + \sqrt{\tilde{h}}\sigma^i(\tau, \tilde{X})\Delta\tilde{W},$$

where $\tilde{h} = t_k^i + \xi^i - \tau$ and $\Delta\tilde{W}$ is a realization of the $s$-dimensional random vector with independent components having the standard normal distribution. Let $\tau := \tau + \tilde{h}$ and simulate a number $r$ for a new regime with probability $P_r = \lambda_r^i/\lambda_r^i$ ($r \in \mathbb{L}\{l\}$). Let $\Pi^i := \Pi^i(1 - \lambda_r^i(\tau, \tilde{X})/\lambda_r^i)$. Verify the following regime switching condition: if $1 - \alpha > \Pi^i$, then switch the current regime number by the $r$th number, i.e., let $l := r$, and let $\Pi^i = \Phi^i = 1$. Further, let $t_k^i := t_k^i + \xi^i$ and obtain a new realization $\xi^i$ for a random value having the exponential distribution with the parameter $\lambda_r^i$. If $t_k^i + \xi^i \geq t_k + h$, then go to Step (vi). Otherwise, go to Step (v).

(vi) Obtain a realization of the vector $X$ at time $t = t_k + h$ by

$$X'_{k+1} = \tilde{X} + \tilde{h}f^i(\tau, \tilde{X}) + \sqrt{\tilde{h}}\sigma^i(\tau, \tilde{X})\Delta\tilde{W},$$

where $\tilde{h} = t_k + h - \tau$, $\Delta\tilde{W}$ is a realization of the $s$-dimensional random vector with independent components having the standard normal distribution. Let $L'_{k+1} := l$.

(vii) See Step (vii) in Algorithm 2.

(viii) See Step (iv) in Algorithm 1.
Let us discuss the advantages and limitations of statistical filtering algorithms given above.

Algorithm 1 (based on the orderliness condition) is the simplest to implement, it consists of four steps only. But we need to verify the regime switching condition at each time \( t_k \) by calling the random number generator. In addition, the regime switching time moments coincide with the time discretization nodes for the numerical solution of SDEs, therefore, the regime switching process is simulated approximately even in the trivial case of constant intensities \( \lambda_{lr}(t, x) = \lambda_l = \text{const} (l, r = 1, 2, \ldots, L_{\max}; l \neq r) \).

Algorithm 2 (based on the maximum cross section method) is more complicated, it consists of eight steps. In this algorithm we need to verify the regime switching condition at time moments corresponding to a composition of the Poisson flows. If \( \lambda^*_l h \ll 1 \ (l = 1, 2, \ldots, L_{\max}) \), then the random number generator calls are significantly reduced. However, this algorithm requires to save the additional information such as the previous regime switching time for all sample paths.

Finally, Algorithm 3 (based on the modified maximum cross section method) is an improved version of Algorithm 2. It also consists of eight steps. And this algorithm provides fewer random number generator calls as compared with Algorithm 2. Algorithm 3 needs to save even more additional information than Algorithm 2. In fact, the statistical efficiency in Algorithms 2 and 3 is achieved by saving the additional information.

4. Conclusion
New statistical algorithms based on the particle method to estimate the state vector of the SSRS have been proposed. These algorithms use numerical methods for solving SDEs, methods for modeling conditional Markov processes with a finite set of states, and methods for calculating weights. To simulate the Poisson flows for the regime switching process, the maximum cross section method and its modification have been applied. The important advantage of this method and its modification are the lower complexity due to fewer random number generator calls [2, 3, 23, 28].

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