Deterministic Completion of Rectangular Matrices
Using Ramanujan Bigraphs – II:
Explicit Constructions and Phase Transitions

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Abstract

Matrix completion is a part of compressed sensing, and refers to determining an unknown
low-rank matrix from a relatively small number of samples of the elements of the matrix. The
problem has applications in recommendation engines, sensor localization, quantum tomography
etc. In a companion paper (Part-1), the first and second author showed that it is possible to
guarantee exact completion of an unknown low rank matrix, if the sample set corresponds to the
edge set of a Ramanujan bigraph. In this paper, we present for the first time an infinite family of
unbalanced Ramanujan bigraphs with explicitly constructed biadjacency matrices. In addition,
we also show how to construct the adjacency matrices for the currently available families of
Ramanujan graphs.

In an attempt to determine how close the sufficient condition presented in Part-1 is to being
necessary, we carried out numerical simulations of nuclear norm minimization on randomly gen-
erated low-rank matrices. The results revealed several noteworthy points, the most interesting
of which is the existence of a phase transition. For square matrices, the maximum rank \( \bar{r} \)
for which nuclear norm minimization correctly completes all low-rank matrices is approximately
\( \bar{r} \approx d/3 \), where \( d \) is the degree of the Ramanujan graph. This upper limit appears to be inde-
pendent of the specific family of Ramanujan graphs. The percentage of low-rank matrices that
are recovered changes from 100% to 0% if the rank is increased by just two beyond \( \bar{r} \). Again,
this phenomenon appears to be independent of the specific family of Ramanujan graphs.

1 Introduction

Compressed sensing refers to the recovery of high-dimensional but low-complexity entities from a
limited number of measurements. Two of the most popular applications of compressed sensing are
the recovery of high-dimensional but sparse vectors from a small number of linear measurements,
and the completion of high-dimensional but low rank matrices from measurements of a few of its
entries. The latter problem, known as the matrix completion problem, is the object of study here.
Matrix completion has application in recommendation engines (e.g., the “Netflix problem”), sensor
localization, quantum tomography, etc.

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1.1 Matrix Completion Problem

The matrix completion problem can be stated precisely as follows: Suppose $X \in \mathbb{R}^{n_r \times n_c}$ is an unknown matrix that we wish to recover whose rank is bounded above by a known integer $r$. Let $[n]$ denote the set $\{1, \ldots, n\}$ for each integer $n$. In the matrix completion problem, a set $\Omega \subseteq [n_r] \times [n_c]$ is specified, known as the sample set. The measurements consist of $X_{i,j}$ for all $(i, j) \in \Omega$. If we define $E_{\Omega} \in \{0, 1\}^{n_r \times n_c}$ by

$$(E_{\Omega})_{ij} = \begin{cases} 1, & \text{if } (i, j) \in \Omega, \\ 0, & \text{if } (i, j) \not\in \Omega, \end{cases}$$

then the measurements can be expressed as the Hadamard product $E_{\Omega}X$.¹ The objective is to use these measurements, together with the bound $r$ on the rank of $X$, to recover $X$ fully, that is, to “complete” the matrix. A common approach to achieve this is via nuclear norm minimization: Define

$$\hat{X} = \arg\min_Z \|Z\|_N \text{ s.t. } Z_{ij} = X_{ij}, \forall (i, j) \in \Omega,$$  

where $\| \cdot \|_N$ refers to the nuclear norm of a matrix, that is, the sum of its singular values. This approach is suggested in [1, 2], and is by now one of the most popular approaches to matrix completion. Part I of this paper [3] contains a literature review on when this approach can lead to $X$ equalling the unknown matrix $X$; therefore the literature review is not repeated here. Note too that an alternate approach consists of minimizing the so-called “max-norm” instead of the nuclear norm. It is proposed in [4] and many papers have followed up on this approach. Max-norm minimization is analyzed in detail in Part I of this paper [3] and is not further discussed here.

1.2 Some Terminology from Graph Theory

The measurement matrix $E_{\Omega}$ can be interpreted as the biadjacency matrix of a bipartite graph with $n_r$ vertices on one side and $n_c$ vertices on the other, while the sample set $\Omega$ can be interpreted as its edge set. If $n_r = n_c$, then the bipartite graph defined by $E_{\Omega}$ is said to be balanced or symmetric, and is said to be unbalanced or asymmetric if $n_r \neq n_c$. The prevailing convention is to refer to the side with the larger ($n_c$) vertices as the “left” side and the other as the “right” side. A bipartite graph is said to be left-regular with degree $d_r$ if every left vertex has degree $d_r$, and right-regular with degree $d_c$ if every right vertex has degree $d_c$. It is said to be $(d_r, d_c)$-biregular if it is both left- and right-regular with row-degree $d_r$ and column-degree $d_c$. Obviously, in this case we must have that $n_r d_r = n_c d_c$. It is convenient to say that a matrix $B \in \{0, 1\}^{n_r \times n_c}$ is “$(d_r, d_c)$-biregular” to mean that the associated bipartite graph is $(d_r, d_c)$-biregular.

1.3 Exact Matrix Completion Using Biregular Graphs

Theorem 7 of [3] presents a sufficient condition under which nuclear norm minimization as in (1) and sampling matrix from a biregular graph leads to exact recovery of the unknown matrix. For the convenience of the reader, that theorem is repeated here. The theorem is stated in terms of the coherence of the unknown matrix, so we begin with that definition.

**Definition 1.** Suppose $X \in \mathbb{R}^{n_r \times n_c}$ has rank $r$ and the reduced singular value decomposition $X = U \Sigma V^\top$, where $U \in \mathbb{R}^{n_r \times r}$, $V \in \mathbb{R}^{n_c \times r}$, and $\Sigma \in \mathbb{R}^{r \times r}$ is the diagonal matrix of the nonzero

¹Recall that if $A, B$ have the same dimensions, then their Hadamard product $C = A \cdot B$ is defined by $C_{ij} = A_{ij}B_{ij}$.
singular values of $X$. Let $P_U = UU^\top \in \mathbb{R}^{n_r \times n_r}$ denote the orthogonal projection of $\mathbb{R}^{n_r}$ onto $U \mathbb{R}^{n_r}$. Finally, let $e_i \in \mathbb{R}^{n_r}$ denote the $i$-th canonical basis vector. Then we define

$$
\mu_0(U) := \frac{n_r}{r} \max_{i \in [n_r]} \|P_U e_i\|_2^2 = \max_{i \in [n_r]} \|u_i^i\|_2^2,
$$

where $u_i^i$ is the $i$-th row of $U$. The quantity $\mu_0(V)$ is defined analogously, and

$$
\mu_0(X) := \max\{\mu_0(U), \mu_0(V)\}.
$$

To facilitate the statement of the sufficient condition, we introduce two assumptions.

(A1). There is a known upper bound $\mu_0$ on $\mu_0(X)$.

(A2). There is a constant $\theta$ such that

$$
\| \sum_{k \in S} \frac{n_r}{d_c} (U^k^\top U^k) - I_r \|_S \leq \theta, \ \forall S \subseteq [n_r], |S| = d_c,
$$

$$
\| \sum_{k \in S} \frac{n_c}{d_r} (V^k^\top V^k) - I_r \|_S \leq \theta, \ \forall S \subseteq [n_c], |S| = d_r,
$$

where $U^k^\top$ is shorthand for $(U^k)^\top$.

Now we repeat [3, Theorem 7] for the convenience of the reader.

Theorem 1. Suppose $X \in \mathbb{R}^{n_r \times n_c}$ is a matrix of rank $r$ or less. Suppose $E_\Omega \in \{0, 1\}^{n_r \times n_c}$ is the biadjacency matrix of a $(d_r, d_c)$ biregular graph $\Omega$, and let $\sigma_2$ denote the second largest singular value of matrix $E_\Omega$. (Of course $\sigma_1 = \sqrt{d_r d_c}$.) Define

$$
\phi = \frac{\sigma_2}{\sigma_1} \mu_0 r,
$$

and suppose that

$$
\theta + \phi < 1/2,
$$

$$
\left(1 + \frac{4}{3} \sqrt{\frac{r}{2}}\right) \phi + \theta < 1.
$$

Then $X$ is the unique solution of (1).

1.4 Contributions of the Present Paper

Theorem 1 shows that the smaller the ratio $\sigma_2/\sigma_1$ is for the biregular measurement matrix $E_\Omega$, the weaker are the sufficient conditions in (7) and (8). As we shall see in Section 2, Ramanujan graphs and Ramanujan bigraphs are those for which this ratio is as small as possible. Please see Section 2 for the relevant definitions. This means that if the biregular graph defining the measurement set is chosen to be a Ramanujan graph in the case of square matrices, and a Ramanujan bigraph in the case of rectangular matrices, then we optimize the bound in (6). At present, there are relatively few methods for the explicit construction of Ramanujan graphs, and none at all for the explicit construction of unbalanced Ramanujan bigraphs. In this paper, we present for the first time explicit constructions of an infinite family of Ramanujan bigraphs. This is the first contribution of the paper.
Along the way, we revisit two earlier known constructions of Ramanujan bipartite graphs, due to Lubotzky-Phillips-Sarnak (LPS) [5] and Gunnells [6], and show that each can be converted to a non-bipartite graph. Note that every graph can be associated with a bipartite graph, but the converse is not true in general. Also, the research community prefers non-bipartite Ramanujan graphs over bipartite graphs. Thus our proof that the LPS and Gunnells constructions can be converted to non-bipartite graphs is of some interest.

The second contribution of the paper is to carry out a simulation study of how close the sufficient conditions of Theorem 1 are to being necessary. Not surprisingly perhaps, the sufficient conditions are quite far from being necessary. Our simulations show a few interesting aspects.

1. When Ramanujan graphs are used to complete square matrices, the limit \( \hat{r} \) for 100\% recovery of randomly-generated low-rank matrices is approximately \( \hat{r} \approx d/3 \), where \( d \) is the degree of the Ramanujan graph.

2. This limit is apparently independent of the specific family of Ramanujan graphs used to carry out nuclear norm minimization.

3. There is a very sharp “phase transition” whereby the recovery percentage drops abruptly from 100\% to 0\% as the rank \( r \) is increased by just two or three above this limit.

Such phase transition phenomena are reported in \( \ell_1 \)-norm minimization with randomly generated Gaussian measurement matrices. However, to date there have been very few phase transition studies for matrix completion. This topic is discussed in greater detail in Section 8.

1.5 Organization of the Paper

The paper is organized as follows: In Section 2, we present a brief review of Ramanujan graphs and Ramanujan bigraphs. In Section 3, we review most of the known methods for constructing Ramanujan graphs, based on the original publications. In Section 4, we shed further light on these construction procedures, by proving some new relationships. In Section 5, we present, for the first time, an explicit construction of an infinite family of Ramanujan bigraphs. In Section 6, we present some procedures for actually implementing the various abstract constructions of Ramanujan graphs discussed in Section 3. In Section 7, we carry out various numerical studies on matrix completion using nuclear norm minimization and the various classes of Ramanujan graphs as well as the new class of Ramanujan bigraphs introduced here for the first time. Finally, in Section 8, we discuss the results presented in the present paper, as well as some important problems that merit further research.

2 Review of Ramanujan Graphs and Bigraphs

In this subsection we review the basics of Ramanujan graphs and Ramanujan bigraphs. Further details about Ramanujan graphs can be found in [7, 8].

Recall that a graph consists of a vertex set \( \mathcal{V} \) and an edge set \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \). If \((v_i, v_j) \in \mathcal{E}\) implies that \((v_j, v_i) \in \mathcal{E}\), then the graph is said to be undirected. A graph is said to be bipartite if \( \mathcal{V} \) can be partitioned into two sets \( \mathcal{V}_r, \mathcal{V}_c \) such that \( \mathcal{E} \cap (\mathcal{V}_r \times \mathcal{V}_r) = \emptyset, \mathcal{E} \cap (\mathcal{V}_c \times \mathcal{V}_c) = \emptyset \). Thus, in a bipartite graph, all edges connect one vertex in \( \mathcal{V}_r \) with another vertex in \( \mathcal{V}_c \). A bipartite graph is said to be balanced if \(|\mathcal{V}_r| = |\mathcal{V}_c|\), and unbalanced otherwise.
A graph is said to be \(d\)-regular if every vertex has the same degree \(d\). A bipartite graph is said to be \((d_r, d_c)\)-biregular if every vertex in \(V_r\) has degree \(d_r\), and every vertex in \(V_c\) has degree \(d_c\). Clearly this implies that \(d_c|V_c| = d_r|V_r|\).

Suppose \((V, E)\) is a graph. Then its adjacency matrix \(A \in \{0, 1\}^{|V| \times |V|}\) is defined by setting \(A_{ij} = 1\) if there is an edge \((v_i, v_j) \in E\), and \(A_{ij} = 0\) otherwise. In an undirected graph (which are the only kind we deal with in the paper), \(A\) is symmetric and therefore has only real eigenvalues. If the graph is \(d\)-regular, then \(d\) is an eigenvalue of \(A\) and is also its spectral radius. The multiplicity of \(d\) as an eigenvalue of \(A\) equals the number of connected components of the graph. Thus the graph is connected if and only if \(d\) is a simple eigenvalue of \(A\). The graph is bipartite if and only if \(-d\) is an eigenvalue of \(A\). If the graph is bipartite, then its adjacency matrix \(A\) looks like

\[
A = \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix},
\]

where \(B \in \{0, 1\}^{|V_r| \times |V_c|}\) is called the biadjacency matrix. The eigenvalues of \(A\) equal \(\pm \sigma_1, \ldots, \pm \sigma_l\) together with a suitable number of zeros, where \(l = \min\{|V_r|, |V_c|\}\), and \(\sigma_1, \ldots, \sigma_l\) are the singular values of \(B\). In particular, in a \((d_r, d_c)\)-biregular graph, \(\sqrt{d_r d_c}\) is the largest singular value of \(B\). These and other elementary facts about graphs can be found in [7].

**Definition 2.** A \(d\)-regular graph is said to be a Ramanujan graph if the second largest eigenvalue by magnitude of its adjacency matrix, call it \(\lambda_2\), satisfies

\[
|\lambda_2| \leq 2\sqrt{d - 1}. \tag{9}
\]

- A \(d\)-regular bipartite graph\(^2\) is said to be a bipartite Ramanujan graph if the second largest singular value of its biadjacency matrix, call it \(\sigma_2\), satisfies

\[
\sigma_2 \leq 2\sqrt{d - 1}. \tag{10}
\]

Note the distinction being made between the two cases. If a graph is \(d\)-regular and bipartite, then it cannot be a Ramanujan graph, because in that case \(\lambda_2 = -d\), which violates (9). On the other hand, if it satisfies (10), then it is called a bipartite Ramanujan graph. Observe too that not all authors make this distinction.

**Definition 3.** A \((d_r, d_c)\)-biregular bipartite graph is said to be a Ramanujan bigraph if the second largest singular value of its biadjacency matrix, call it \(\sigma_2\), satisfies

\[
\sigma_2 \leq \sqrt{d_r - 1} + \sqrt{d_c - 1}. \tag{11}
\]

It is easy to see that Definition 3 contains the second case of Definition 2 as a special case when \(d_r = d_c = d\). A Ramanujan bigraph with \(d_r \neq d_c\) is called an unbalanced Ramanujan bigraph.

The rationale behind the bounds in these definitions is given the following results. In the interests of brevity, the results are paraphrased and the reader should consult the original sources for precise statements.

**Theorem 2.** (Alon-Boppana bound; see [9].) Fix \(d\) and let \(n \to \infty\) in a \(d\)-regular graph with \(n\) vertices. Then

\[
\lim \inf_{n \to \infty} |\lambda_2| \geq 2\sqrt{d - 1}. \tag{12}
\]

\(^2\)Note that such a bipartite graph must perforce be balanced with \(|V_r| = |V_c|\) and \(d_r = d_c = d\).
Theorem 3. (See [10].) Fix $d_r, d_c$ and let $n_r, n_c$ approach infinity subject to $d_r n_r = d_c n_c$. Then
\[
\liminf_{n_r \to \infty, n_c \to \infty} \sigma_2 \geq \sqrt{d_r - 1 + \sqrt{d_c - 1}}.
\] (13)

Given that a $d$-regular graph has $d$ as its largest eigenvalue $\lambda_1$, a Ramanujan graph is one for which the ratio $\lambda_2/\lambda_1$ is as small as possible, in view of the Alon-Boppana bound of Theorem 2. Similarly, given that a $(d_r, d_c)$-regular bipartite graph has $\sigma_1 = \sqrt{d_r d_c}$, a Ramanujan bigraph is one for which the ratio $\sigma_2/\sigma_1$ is as small as possible, in view of Theorem 3. Therefore, in order to apply the sufficient conditions of Theorem 1 using a biregular graph to generate the samples, it makes sense to use a Ramanujan graph for square matrices and an unbalanced Ramanujan bigraph in the case of rectangular matrices. In a certain sense, Ramanujan graphs and Ramanujan bigraphs are pervasive. To be precise, if $d$ is kept fixed and $n \to \infty$, then the fraction of $d$-regular, $n$-vertex graphs that satisfy the Ramanujan property approaches one; see [11, 12]. Similarly, if $d_r, d_c$ are kept fixed and $n_r, n_c \to \infty$ (subject of course to the condition that $d_r n_r = d_c n_c$, then the fraction of $(d_r, d_c)$-biregular graphs that are Ramanujan bigraphs approaches one; see [13]. However, despite their prevalence, there are relatively few explicit methods for constructing Ramanujan graphs. Many of the currently known techniques are reprinted in Section 3. In the case of Ramanujan bigraphs, there are a couple of abstract constructions in [14, 15], but these are not explicit. In Section 5, we present the first such construction.

3 Explicit Constructions of Ramanujan Graphs

At present there are not too many methods for explicitly constructing Ramanujan graphs. In this section we reprise most of the known methods. These techniques are used in the phase transition studies of Section 7. Note that the authors have written Matlab codes for all of the constructions in this section; these codes are available upon request.

The available construction methods can be divided into two categories, which for want of a better terminology we call “one-parameter” and “two-parameter” constructions. One-parameter constructions are those for which, once the degree $d$ is specified, the number $n$ of vertices is also fixed by the procedure. In contrast, two-parameter constructions are those in which it is possible to specify $d$ and $n$ independently. The methods of Lubotzky-Phillips-Sarnak (LPS) and of Gunnells are two-parameter, while those of Winnie Li and Bibak et al. are one-parameter. Of course, not all combinations of $d$ and $n$ are permissible.

In this connection it is worth mentioning the results of [16, 17, 18], which show that there exist bipartite Ramanujan graphs of all degrees and all sizes. However these results do not imply the existence of Ramanujan graphs of all sizes and degrees. Moreover, the ideas in [16, 17, 18] do not lead to an explicit construction. There is a preprint [19] that claims to give a polynomial time algorithm for implementing the construction of [16, 17]. However, no code for the claimed implementation is available.

All but one of the constructions described below are Cayley graphs. So we begin by describing that concept. Suppose $G$ is a group, and that $S \subseteq G$ is “symmetric” in that $a \in S$ implies that $a^{-1} \in S$. Then the Cayley graph $C(G, S)$ has the elements of $G$ as the vertex set, and the edge set is of the form $(x, xa), x \in G, a \in S$. Due to the symmetry of $S$, the graph is undirected even if $G$ is noncommutative.

There is however a paper under preparation by C. Ballantine, S. Evra, B. Feigon, K. Maurischat, and O. Parzanchevski that will present an explicit construction.
3.1 Lubotzky-Phillips-Sarnak Construction

The Lubotzky-Phillips-Sarnak (referred to as LPS hereafter) construction [5] makes use of two unequal primes \( p, q \), each of which is \( \equiv 1 \mod 4 \). As is customary, let \( \mathbb{F}_q \) denote the finite field with \( q \) elements, and let \( \mathbb{F}_q^* \) denote the set of nonzero elements in \( \mathbb{F}_q \). The general linear group \( GL(2, \mathbb{F}_q) \) consists of all \( 2 \times 2 \) matrices with elements in \( \mathbb{F}_q \) whose determinant is nonzero. If we define an equivalence relation \( \sim \) on \( GL(2, \mathbb{F}_q) \) via \( A \sim B \) whenever \( A = \alpha B \) for some \( \alpha \in \mathbb{F}_q^* \), then the resulting set of equivalence classes \( GL(2, \mathbb{F}_q)/\sim \) is the projective general linear group \( PGL(2, \mathbb{F}_q) \). Next, it is shown in [5] that there are exactly \( p + 1 \) solutions of the equation

\[
\begin{aligned}
p &= a_0^2 + a_1^2 + a_2^2 + a_3^2,
\end{aligned}
\]

(14)

where \( a_0 \) is odd and positive, and \( a_1, a_2, a_3 \) are even (positive or negative). Choose an integer \( i \) such that \( i^2 \equiv -1 \mod q \). Thus \( i \) is a proxy for \( \sqrt{-1} \) in the field \( \mathbb{F}_q \). Such an integer always exists.

The LPS construction is a Cayley graph where the group \( G \) is the projective general linear group \( PGL(2, \mathbb{F}_q) \). The generator set \( S \) consists of the \( p + 1 \) matrices

\[
M_j = \begin{bmatrix}
a_{0j} + ia_{1j} & a_{2j} + ia_{3j} \\
-a_{2j} + ia_{3j} & a_{0j} - ia_{1j}
\end{bmatrix} \mod q,
\]

(15)

as \((a_{0j}, a_{1j}, a_{2j}, a_{3j})\) range over all solutions of (14). Note that each matrix \( M_j \) has determinant \( p \mod q \). It is clear that the LPS graph is \((p + 1)\)-regular, and the number of vertices equals the cardinality of \( PGL(2, \mathbb{F}_q) \), which is \((q(q^2 - 1))/2\).

To proceed further, it is necessary to introduce the Legendre symbol. If \( q \) is an odd prime and \( a \) is not a multiple of \( q \), define

\[
\left( \frac{a}{q} \right) = \begin{cases} 
1 & \text{if } \exists x \in \mathbb{Z} \text{ such that } x^2 \equiv a \mod q, \\
-1 & \text{otherwise}
\end{cases}
\]

Partition \([q - 1]\) into two subsets, according to

\[
S_{1,q} := \left\{ a \in [q - 1] : \left( \frac{a}{q} \right) = 1 \right\},
\]

(16)

\[
S_{-1,q} := \left\{ a \in [q - 1] : \left( \frac{a}{q} \right) = -1 \right\}.
\]

(17)

Then it can be shown that each set \( S_{1,q} \) and \( S_{-1,q} \) consists of \((q - 1)/2\) elements of \([q - 1]\). One of the many useful properties of the Legendre symbol is that, for integers \( a, b \in \mathbb{Z} \), neither of which is a multiple of \( q \), we have

\[
\left( \frac{ab}{q} \right) = \left( \frac{a}{q} \right) \left( \frac{b}{q} \right).
\]

Consequently, for a fixed odd prime number \( q \), the map

\[
a \mapsto \left( \frac{a}{q} \right) : \mathbb{Z} \setminus q\mathbb{Z} \to \{-1, 1\}
\]

is multiplicative. Further details about the Legendre symbol can be found in any elementary text on number theory; see for example [20, Section 6.2], or [21, Section 5.2.3].

The LPS construction gives two distinct kinds of graphs, depending on whether \( \left( \frac{p}{q} \right) = 1 \) or \(-1\). To describe the situation, let us partition \( PGL(2, \mathbb{F}_q) \) into two disjoint sets \( PSL(2, \mathbb{F}_q) \) and
$PSL^c(2, \mathbb{F}_q)$, that are defined next. Partition $GL(2, \mathbb{F}_q)$ into two sets $GL_1(2, \mathbb{F}_q)$ and $GL_2(2, \mathbb{F}_q)$ as follows:

$$GL_1(2, \mathbb{F}_q) = \{ A \in GL(2, \mathbb{F}_q) : \det(A) \in S_{1,q} \},$$

$$GL_2(2, \mathbb{F}_q) = \{ A \in GL(2, \mathbb{F}_q) : \det(A) \in S_{-1,q} \},$$

Because of the multiplicativity of the Legendre symbol, it follows that $GL_1(2, \mathbb{F}_q)$ is a subgroup of $GL(2, \mathbb{F}_q)$. Next, define

$$PSL(2, \mathbb{F}_q) := GL_1(2, \mathbb{F}_q)/\sim,$$

$$PSL^c(2, \mathbb{F}_q) := GL_2(2, \mathbb{F}_q)/\sim.$$  

Then $PSL(2, \mathbb{F}_q)$ and $PSL^c(2, \mathbb{F}_q)$ form a partition of $PGL(2, \mathbb{F}_q)$, and each set contains $(q(q^2 - 1))/2$ elements.

Now we come to the nature of the Cayley graph that is generated by the LPS construction.

- If $\left( \frac{p}{q} \right) = 1$, then each $M_j$ maps $PSL(2, \mathbb{F}_q)$ onto itself, and $PSL^c(2, \mathbb{F}_q)$ into itself. Thus the Cayley graph consists of two disconnected components, each with $(q(q^2 - 1))/2$ elements. It is shown in [5] that each component is a Ramanujan graph.

- If $\left( \frac{p}{q} \right) = -1$, then each $M_j$ maps $PSL(2, \mathbb{F}_q)$ onto $PSL^c(2, \mathbb{F}_q)$, and vice versa. In this case the Cayley graph of the LPS construction is a balanced bipartite graph, with $(q(q^2 - 1))/2$ elements in each component. It is shown in [5] that the graph is a bipartite Ramanujan graph.

### 3.2 Gunnells’ Construction

Next we review the construction in [6]. Suppose $q$ is a prime or a prime power, and as usual, let $\mathbb{F}_q$ denote the finite field with $q$ elements. Let $l$ be any integer, and view $\mathbb{F}_{ql}$ as a linear vector space over the base field $\mathbb{F}_q$. Then the number of one-dimensional subspaces of $\mathbb{F}_{ql}$ equals

$$\nu(l, q) := \frac{q^l - 1}{q - 1} = \sum_{i=0}^{l-1} q^i.$$ 

Let us denote the set of one-dimensional subspaces by $\mathcal{V}_1$. Correspondingly, the number of subspaces of $\mathbb{F}_{ql}$ of codimension one also equals $\nu(l, q)$. Let us denote this set by $\mathcal{V}_{l-1}$ because (obviously) every subspace of codimension one has dimension $l - 1$. The Gunnells construction is bipartite with $\mathcal{V}_1$ and $\mathcal{V}_{l-1}$ as the two sets of vertices. There is an edge between $S_a \in \mathcal{V}_1$ and $T_b \in \mathcal{V}_{l-1}$ if and only if $S_a$ is a subspace of $T_b$. The graph is bipartite and balanced with

$$n = |\mathcal{V}_1| = |\mathcal{V}_{l-1}| = \nu(l, q) = \sum_{i=0}^{l-1} q^i.$$ 

The graph is also biregular with

$$d = \nu(l - 1, q) = \sum_{i=0}^{l-2} q^i.$$ 

It is shown in [6] that the graph is a bipartite Ramanujan graph.
3.3 Winnie Li’s Construction

Suppose $G$ is an Abelian group and let $n$ denote $|G|$. Then a character $\chi$ on $G$ is a homomorphism $\chi : G \rightarrow S^1$, where $S^1$ is the set of complex numbers of magnitude one. Thus $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in G$. There are precisely $n$ characters on $G$, and it can be shown that $[\chi(a)]^n = 1$ for each character $\chi$ and each $a \in G$. Therefore each character maps an element of $G$ into an $n$-th root of one. The character defined by $\chi_0(a) = 1$ for all $a \in G$ is called the trivial character. Let us number the remaining characters on $G$ as $\chi_i, i \in [n - 1]$ in some manner.

Suppose that $S$ is a symmetric subset of $G$, and consider the associated Cayley graph. Thus the vertices of the graph are the elements of $G$, and the edges are $(x, xa), x \in G, a \in S$. Clearly the Cayley graph has $n$ vertices and is $d$-regular where $d = |S|$. Now a key result [22, Proposition 1.(1)] states that the eigenvalues of the adjacency matrix are

$$\lambda_i = \sum_{s \in S} \chi_i(s), i = 0, 1, \ldots, n - 1.$$

If $i = 0$, then $\lambda_0 = d$, which we know (due to regularity). Therefore, if the set $S$ can somehow be chosen in a manner that

$$\left| \sum_{s \in S} \chi_i(s) \right| \leq 2\sqrt{d - 1}, i \in [n - 1],$$

then the Cayley graph would have the Ramanujan property. Several Ramanujan graphs can be constructed using the above approach. In particular, the following construction is described in [22, Section 2].

Let $F_q$ be a finite field, so that $q$ is a prime or prime power. Let $F_{q^2}$ be a degree 2 extension of $F_q$ and choose $S$ to be the set of all primitive elements of $F_{q^2}$, that is, the set of elements of multiplicative norm 1. Here, the multiplicative norm of $\alpha \in F_{q^2}$ is defined as

$$N(\alpha) := \alpha \cdot \alpha^q = \alpha^{q + 1}.$$

Note that $S$ is symmetric and contains $q + 1$ elements. A deep and classical theorem due to Deligne [23] states that for every nontrivial character $\chi_i, i \geq 1$ on the additive group of $F_{q^2}$, we have

$$\left| \sum_{s \in S} \chi_i(s) \right| \leq 2\sqrt{q}, i \in [q^2 - 1].$$

Therefore the Cayley graph with $F_{q^2}$ as the group and $S$ as the generator set has the Ramanujan property.

The above result presents only an abstract construction, and it is still a challenge to construct this graph explicitly. The main difficulty is that as yet there is no polynomial-time algorithm to find all elements of the generator set $S$. However, for relatively small values of $q$ (say 101 or less), it is still feasible to execute “nonpolynomial” algorithms. The approach adopted by us to implement this construction is described in Section 6.

3.4 Bibak et al. Construction

The next construction is found in [24]. Suppose $q$ is a prime $\equiv 3 \mod 4$. Let $G$ be the additive group $F_{q^2}$, consisting of pairs $z = (x, y)$ where $x, y \in F_q$. Clearly $n = |G| = q^2$. The “norm” of an element $z$ is defined as $(x^2 + y^2) \mod q$, and the set $S$ consists of all $z$ such that the norm equals one. It can be shown that $|S| = q + 1$ and that $S$ is symmetric. It is shown in [24] that the associated Cayley graph has the Ramanujan property.
4 Further Analysis of Earlier Constructions

It is seen from Section 3 that the LPS construction leads to a disconnected graph if \( \left( \frac{q}{p} \right) = 1 \). It would be of interest to know whether the two connected components are isomorphic to each other. It turns out to be really easy to demonstrate that this is indeed so. Next, the LPS construction leads to a bipartite Ramanujan graph if the Legendre symbol \( \left( \frac{q}{p} \right) = -1 \). The Gunnells construction always leads to a bipartite Ramanujan graph. Graph theorists prefer non-bipartite (or “real”) Ramanujan graphs to bipartite Ramanujan graphs. It is easy to show that every Ramanujan graph leads to a bipartite Ramanujan graph, but the converse is not necessarily true. Specifically, suppose \((V, E)\) is a Ramanujan graph with adjacency matrix \( A \). Define two sets \( V_r \) and \( V_c \) to be copies of \( V \), and define an edge \((v_i, v_j)\) in the bipartite graph if and only if there is an edge \((v_i, v_j)\) in the original graph. Then it is obvious that the biadjacency matrix of this bipartite graph is also a Ramanujan graph if the Legendre symbol \( \left( \frac{q}{p} \right) \) is a Ramanujan graph with adjacency matrix \( A \). Moreover, it is easy to show that, if \( B \) is the biadjacency matrix of the original graph, and \( A \) is the adjacency matrix of the nonbipartite graph, then \( A = B \Pi \) where \( \Pi \) is the matrix representation of \( \pi \). Thus the eigenvalues of \( A \) are the singular values of \( B \), which implies that if the bipartite graph has the Ramanujan property, so does the nonbipartite graph.

However, the passage in the opposite direction is not guaranteed. Suppose that there is a balanced bipartite graph with vertex sets \( V_r \) and \( V_c \) and edge set \( E \subseteq V_r \times V_c \). In order to convert this bipartite graph into a nonbipartite graph, it is necessary and sufficient to find a one-to-one and onto map \( \pi : V_c \to V_r \) (which is basically a permutation), such that whenever there is an edge \((v_i, v_j)\) in the bipartite graph, there is also an edge \((\pi^{-1}(v_j), \pi(v_i))\). In this way, the “right” vertex set can be identified with its image under \( \pi \) and the result would be an undirected nonbipartite graph. Moreover, it is easy to show that, if \( B \) is the biadjacency matrix of the original graph, and \( A \) is the adjacency matrix of the nonbipartite graph, then \( A = B \Pi \) where \( \Pi \) is the matrix representation of \( \pi \). Thus the eigenvalues of \( A \) are the singular values of \( B \), which implies that if the bipartite graph has the Ramanujan property, so does the nonbipartite graph.

With this background, in the present section we first remark that, when \( \left( \frac{q}{p} \right) = 1 \), the two connected components of the LPS construction are isomorphic. Indeed, choose any \( A \in PGL(2, \mathbb{F}_q) \) such that \( \text{det}(A) \in S_{-1,q} \), and define the map \( \pi : PSL(2, \mathbb{F}_q) \to PSL^c(2, \mathbb{F}_q) \) via \( X \mapsto AX \). Then the edge incidence is preserved.

Next we show that, when \( \left( \frac{q}{p} \right) = -1 \), the resulting bipartite graph can be mapped into a (non-bipartite) Ramanujan graph. For this purpose we establish a preliminary result.

**Lemma 1.** There exists a matrix \( A \in GL(2, \mathbb{F}_q) \) such that \( A^2 \sim I_{2 \times 2} \).

**Proof.** Choose elements \( \alpha, \beta, \gamma \in \mathbb{F}_q \) such that \(-\alpha^2 + \beta \gamma \in S_{-1,q}\). This is easy: Choose an arbitrary \( \delta \in S_{-1,q} \), arbitrary \( \alpha \in \mathbb{F}_q \), \( \gamma = -1 \), and \( \beta = \alpha^2 + \delta \). Now define

\[
A = \begin{bmatrix}
\alpha & \beta \\
\gamma & -\alpha
\end{bmatrix}.
\] (18)

Then

\[
A^2 = \begin{bmatrix}
\alpha & \beta \\
\gamma & -\alpha
\end{bmatrix} \begin{bmatrix}
\alpha & \beta \\
\gamma & -\alpha
\end{bmatrix} = \begin{bmatrix}
\alpha^2 + \beta \gamma & 0 \\
0 & \alpha^2 + \beta \gamma
\end{bmatrix}.
\]

Hence \( A^2 \sim I \). Clearly \( \text{det}(A) = -\alpha^2 + \beta \gamma \in S_{-1,q} \). \( \square \)

**Theorem 4.** Suppose \( \left( \frac{q}{p} \right) = -1 \), and consider the bipartite Ramanujan graph \( C(PGL(2, \mathbb{F}_q), S) \).

Then there exists a map \( \pi : PSL(2, \mathbb{F}_q) \to PSL^c(2, \mathbb{F}_q) \) that is one-to-one and onto such that, whenever there is an edge \((X, Z)\) with \( X \in PSL(2, \mathbb{F}_q) \) and \( Z \in PSL^c(2, \mathbb{F}_q) \), there is also an edge \((\pi^{-1}(Z), \pi(X))\).
Proof. For a matrix $\tilde{X} \in GL(2, \mathbb{F}_q)$, let $[\tilde{X}] \in PGL(2, \mathbb{F}_q)$ denote its equivalence class under $\sim$. Construct the matrix as in Lemma 1. Suppose there exists an edge $(X, Z)$ with $X \in PSL(2, \mathbb{F}_q)$ and $Z \in PSL^c(2, \mathbb{F}_q)$. Thus there exist representatives $\tilde{X} \in GL1(2, \mathbb{F}_q), \tilde{Z} \in GL2(2, \mathbb{F}_q)$ and an index $i \in [p+1]$ such that $\tilde{Z} \sim \tilde{X}M_i$. Now observe that if $(a_0, a_1, a_2, a_3)$ solves (14), then $(a_0, -a_1, -a_2, -a_3)$ also solves (14). By examining the definition of the matrices $M_j$ in (15), it is clear that for every index $i \in [p+1]$, there exists another index $j \in [p+1]$ such that $M_iM_j \sim I$. Now define $\tilde{Y} = A\tilde{Z}$ and note that $\tilde{Y} \sim A^{-1}\tilde{Z}$ because $A^2 \sim I$. Also, $\tilde{Y} \in GL1(2, \mathbb{F}_q)$ because $\tilde{Z} \in GL2(2, \mathbb{F}_q)$ and $\det(A) \in S_{-1,q}$. By assumption $\tilde{Z} \sim \tilde{X}M_i$. So $A\tilde{Z} \sim A\tilde{X}M_i$. Now choose the index $j \in [p+1]$ such that $M_iM_j \sim I$. Then

$$\tilde{Y}M_j = A\tilde{Z}M_j \sim A\tilde{X}M_iM_j \sim A\tilde{X}.$$ 

Hence there is an edge from $[\tilde{Y}] = A^{-1}(Z)$ to $[A\tilde{X}] = A(X)$. 

\[\square\]

**Theorem 5.** The Gunnells construction can be converted into a nonbipartite Ramanujan graph of degree $d = \nu(l - 1, q)$ and $n = \nu(l, q)$ vertices.

Proof. As before, it suffices to find a one-to-one and onto map $\pi$ from $\mathcal{V}_{l-1}$ to $\mathcal{V}_1$ such that, if there is an edge $(S, T)$ where $S \in \mathcal{V}_1$ and $T \in \mathcal{V}_{l-1}$, then there is also an edge $(\pi(T), \pi^{-1}(S))$. Accordingly, if $T \in \mathcal{V}_{l-1}$, so that $T$ is a subspace of codimension one, define $\pi(T)$ to be an “annihilator” $T^\perp$ of $T$, consisting of all vectors $v \in \mathbb{F}_q^l$ such that $v^\top u = 0$ for every $u \in T$. Then $T^\perp$ is a one-dimensional subspace of $\mathbb{F}_q^l$ and thus belongs to $\mathcal{V}_1$. Moreover, for $S \in \mathcal{V}_1$, we have that $\pi^{-1}(S) = S^\perp$. It is obvious that this map is one-to-one and onto. Now suppose there is an edge $(S, T)$ where $S \in \mathcal{V}_1$ and $T \in \mathcal{V}_{l-1}$. This is the case if and only if $S \subseteq T$. But this implies that $T^\perp \subseteq S^\perp$. Hence there is an edge from $\pi(T)$ to $\pi^{-1}(S)$. 

\[\square\]

5 Explicit Constructions of Ramanujan Bigraphs

According to a recent result [13], randomly generated $(d_r, d_c)$-biregular bipartite graphs with $(n_r, n_c)$ vertices satisfy the Ramanujan property with probability approaching one as $n_r, n_c$ simultaneously approach infinity (of course, while satisfying the constraint that $d_r n_r = d_c n_c$). This result generalizes an earlier result due to [11, 12] which states that randomly generated $d$-regular, $n$-vertex graph satisfies the Ramanujan property as $n \rightarrow \infty$ with probability approaching one. A “road map” for constructing Ramanujan bigraphs is given in [25], and some abstract constructions of Ramanujan bigraphs are given in [14, 15]. These bigraphs have degrees $(p+1, p^2+1)$ for various values of $p$, such as $p \equiv 5 \mod 12, p \equiv 11 \mod 12$ [14], and $p \equiv 3 \mod 4$ [15]. For each suitable choice of $p$, these papers lead to an infinite family of Ramanujan bigraphs. At present, these constructions are not explicit in terms of resulting in a biadjacency matrix of 0s and 1s. There is a paper under preparation by these authors to make these constructions explicit.

In contrast, in this section, we state and prove two such explicit constructions, namely $(lq, q^2)$-biregular graphs where $q$ is any prime and $l$ is any integer that satisfies $2 \leq l < q$, and $(q^2, lq)$-biregular graphs where $q$ is any prime and $l$ is any integer greater than $q$. Thus we can construct Ramanujan bigraphs for a broader range of degree-pairs compared to [14, 15]. However, for a given pair of integers $l, q$, we can construct only one Ramanujan bigraph.

Our construction is based on so-called “array code” matrices from LDPC (low density parity check) coding theory, first introduced in [26, 27]. Suppose $q$ is a prime number, and let $P \in \{0, 1\}^{q \times q}$

\[\text{The authors thank Prof. Cristina Ballantine for aiding us to interpret these papers.}\]
be a cyclic shift permutation matrix on \( q \) numbers. Then the entries of \( P \) can be expressed as
\[
P_{ij} = \begin{cases} 
1, & j = i - 1 \mod q \\
0, & \text{otherwise}
\end{cases}
\]

Now let \( q \) be a prime number, and define
\[
B(q, l) = \begin{bmatrix}
I_q & I_q & I_q & \cdots & I_q \\
I_q & P & P^2 & \cdots & P^{(l-1)} \\
I_q & P^2 & P^4 & \cdots & P^{2(l-1)} \\
I_q & P^3 & P^6 & \cdots & P^{3(l-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
I_q & P^{(q-1)} & P^{3(q-1)} & \cdots & P^{(l-1)(q-1)}
\end{bmatrix}
\] (19)

where \( P^j \) represents \( P \) raised to the power \( j \). Now \( B = B(q, l) \) is binary with \( q^2 \) rows and \( lq \) columns, row degree of \( l \) and column degree of \( q \). If \( l < q \), we study the matrix \( B^\top \), whereas if \( l \geq q \), we study \( B \). In either case, the largest singular value of \( B \) is \( \sqrt{ql} \). The fact that these bipartite graphs satisfy the Ramanujan property is now established.

**Theorem 6.**

1. Suppose \( 2 \leq l \leq q \). Then the matrix \( B^\top \) has a simple singular value of \( \sqrt{ql} \), \( l(q-1) \) singular values of \( \sqrt{q} \), and \( l-1 \) singular values of zero. Therefore \( B^\top \) represents a Ramanujan bigraph.

2. Suppose \( l \geq q \). The matrix \( B \) has a simple singular value of \( \sqrt{ql} \). Now two subcases need to be considered:
   
   (a) When \( l \mod q = 0 \), in addition \( B \) has \( (q-1)q \) singular values of \( \sqrt{l} \) and \( q-1 \) singular values of \( 0 \).
   
   (b) When \( l \mod q \neq 0 \), let \( k = l \mod q \). Then \( B \) has, in addition, \( (q-1)k \) singular values of \( \sqrt{l + q - k} \), \( (q-1)(q-k) \) singular values of \( \sqrt{l - k} \), and \( q-1 \) singular values of \( 0 \).

   Therefore, whenever \( l \geq q \), \( B(q, l) \) represents a Ramanujan bigraph.

**Proof.** Let \( B \) be a shorthand for \( B(q, l) \). Note that \( P \) is a cyclic shift permutation; therefore \( P^\top = P^{-1} \). The proof consists of computing \( B^\top B \) and determining its eigenvalues. Throughout we make use of the fact that \( P^\top = P^{-1} \).

We begin with the case \( l \leq q \). Use block-partition notation to divide \( BB^\top \) into \( l \) blocks of size \( q \times q \). Then
\[
(BB^\top)_{ij} = \sum_{s=1}^{q} P^{(i-1)(s-1)}(P^\top)^{(s-1)(j-1)}
\]
\[
= \sum_{s=1}^{q} P^{(i-j)(s-1)} = \sum_{s=0}^{q-1} P^{(i-j)s}
\]

It readily follows that
\[
(BB^\top)_{ii} = qI_q, i = 1, \ldots, q.
\]
Now observe that, for any nonzero integer \( k \), the set of numbers \( ks \mod q \) as \( s \) varies over \{0, \ldots, q-1\} equals \{0, \ldots, q-1\}. (This is where we use the fact that \( q \) is a prime number.) Therefore, whenever \( i \neq j \), we have that
\[
(B^\top B)_{ij} = \sum_{s=0}^{q-1} P^s = 1_{q \times q},
\]
where \( 1_{q \times q} \) denotes the \( q \times q \) matrix whose entries are all equal to one. Observe that the largest eigenvalue of \( B^\top B \) is \( qI \), with normalized eigenvector \((1/\sqrt{q})1_{qI}\). Therefore if we define \( M_l = B^\top B - 1_{qI} \) and partition it commensurately with \( B \), we see that the off-diagonal blocks of \( M_l \) are all equal to zero, while the diagonal blocks are all identical and equal to \( qI - 1_{qI} \). This is the Laplacian matrix of a fully connected graph with \( q \) vertices, and thus has \( q-1 \) eigenvalues of \( q \) and one eigenvalue of 0. Therefore \( M_l = BB^\top - 1_{qI} \) has \( l(q-1) \) eigenvalues of \( q \) and \( l \) eigenvalues of 0. Moreover, \( 1_{qI} \) is an eigenvector of \( M \) corresponding to the eigenvalue zero. Therefore \( BB^\top = M_l + 1_{qI} \) has a single eigenvalue of \( qI \), \( l(q-1) \) eigenvalues of \( q \), and \( l-1 \) eigenvalues of 0. This is equivalent to the claim about singular values of \( B^\top \).

Now we study the case where \( l \geq q \). Let \( M_q \in \{0,1\}^{q^2 \times q^2} \) denote the matrix in the previous case with \( l = q \). This matrix can be block-partitioned into \( q \times q \) blocks, with
\[
M_{ij} = \begin{cases} 
qI_q - 1_{q \times q}, & \text{if } j - i \equiv 0 \mod q, \\
0, & \text{otherwise}.
\end{cases}
\]

Now consider the subcase that \( l \equiv 0 \mod q \). Then \( B^\top B \) consists of \( l/q \) repetitions of \( M_q \) on each block row and block column of size \( q^2 \times q^2 \). Therefore each such row and column block gives \((q-1)q\) eigenvalues of magnitude \( l \) and rest of the eigenvalues of will be zero. Next, if \( l \mod q =: k \neq 0 \), then \( B^\top B \) consists of \((l-k)/q\) repetitions of \( M_q \) on each block row and column of size \( q^2 \times q^2 \). In addition it contains first \( k \) column blocks of \( M_q \) concatenated \( l \) times column-wise as the last column blocks. It contains first \( k \) row blocks of \( M_q \) concatenated \( l - k \) times row-wise as the last block of rows. The extra \( k \) rows and columns of \( B^\top B \) give \((q-1)k\) eigenvalues of magnitude \( l + (q-k) \). Another set of row and column blocks give \((q-1)(q-k)\) eigenvalues of magnitude \( l - k \). The remaining eigenvalues are 0.

Note that, when \( l = q \), the construction in (19) leads to a new class of Ramanujan graphs of degree \( q \) and \( q^2 \) vertices. In our terminology, it is a “one-parameter” family. So far as we are able to determine, this family is new and is not contained any other known family.

6 Construction of Ramanujan Graphs: Implementation Details

In this section we discuss some of the implementation details of constructing Ramanujan graphs using the various methods discussed in Section 3. The authors have written MATLAB codes for all of these implementations, which can be made available upon request. Note that the biadjacency matrix of the Ramanujan bigraphs presented in Section 5 is quite explicit, and does not require further elaboration.

We begin with the LPS construction. In order to implement this construction, it is desirable to have systematic enumerations of the projective groups \( PGL(2, \mathbb{F}_q) \), \( PSL(2, \mathbb{F}_q) \), and \( PSL^c(2, \mathbb{F}_q) \). The approach used by us is given next. Define
\[
\mathcal{M}_1 = \left\{ \begin{bmatrix} 0 & 1 \\ g & h \end{bmatrix} : g \in \mathbb{F}_q^*, h \in \mathbb{F}_q \right\},
\]
\[ M_2 = \left\{ \begin{bmatrix} 1 & f \\ g & h \end{bmatrix} : f, g \in \mathbb{F}_q^*, h - fg \in \mathbb{F}_q^* \right\}. \]

Then every matrix in \( PGL(2, \mathbb{F}_q) \) is equivalent under \( \sim \) to exactly one element of \( PGL(2, \mathbb{F}_q) \). Specifically, let \( A \in GL(2, \mathbb{F}_q) \) be arbitrary. If \( a_{11} = 0 \), then \( A \sim (a_{12})^{-1}A \in M_1 \), whereas if \( a_{11} \neq 0 \), then \( A \sim (a_{11})^{-1}A \in M_2 \). The rest of the details are easy and left to the reader. This provides an enumeration of \( PGL(2, \mathbb{F}_q) \). To provide an enumeration of \( PSL(2, \mathbb{F}_q) \), we modify the sets as follows:

\[
M_{1,1} = \left\{ \begin{bmatrix} 0 & 1 \\ g & h \end{bmatrix} : g \in S_{1,q}, h \in \mathbb{F}_q \right\}, \\
M_{2,1} = \left\{ \begin{bmatrix} 1 & f \\ g & h \end{bmatrix} : f, g \in \mathbb{F}_q^*, h - fg \in S_{1,q} \right\}, \\
m_{1,-1} = \left\{ \begin{bmatrix} 0 & 1 \\ g & h \end{bmatrix} : g \in S_{-1,q}, h \in \mathbb{F}_q \right\}, \\
m_{2,-1} = \left\{ \begin{bmatrix} 1 & f \\ g & h \end{bmatrix} : f, g \in \mathbb{F}_q^*, h - fg \in S_{-1,q} \right\}.
\]

Then the set of matrices \( M_{1,1} \cup M_{2,1} \) provides an enumeration of \( PSL(2, \mathbb{F}_q) \), while the set of matrices \( M_{1,-1} \cup M_{2,-1} \) provides an enumeration of \( PSL^c(2, \mathbb{F}_q) \). Once the vertex sets are enumerated, each representative matrix of an element in \( PGL(2, \mathbb{F}_q) \) is multiplied by each generator matrix \( M_1 \) through \( M_{p+1} \), and then converted to one of the above representations, depending on whether the \((1,1)\)-element is zero or nonzero.

In the original LPS construction, it is assumed that \( p < q \). However, the LPS construction can still be used with \( p > q \),\(^5\) provided that the \( p+1 \) generating matrices \( M_1 \) through \( M_{p+1} \) are distinct elements of \( \mathbb{F}_q^{2 \times 2} \). In our implementation, we handle the case \( p > q \) by verifying whether this is indeed the case.

For the Gunnells construction, the vertex set is the set of one-dimensional subspaces of \( \mathbb{F}_{q^l} \).

For this purpose we identify \( \mathbb{F}_{q^l} \) with \( \mathbb{F}_q^l \), the set of \( l \)-dimensional vectors over \( \mathbb{F}_q \). To enumerate these, observe that there are \( q^l - 1 \) nonzero vectors in \( \mathbb{F}_q^l \). For any nonzero vector, there are \( q - 1 \) nonzero multiples of it, but all these multiples generate the same subspace. Therefore the number of one-dimensional subspaces is

\[
\frac{q^l - 1}{q - 1} = \sum_{i=0}^{l-1} q^i.
\]

To enumerate these subspaces without duplications, we proceed as follows: In step 1, fix the first element of a vector \( x \in \mathbb{F}_q^l \) to 1, and let the elements \( x_2 \) through \( x_l \) be arbitrary. This generates \( q^{l-1} \) nonzero vectors that generate distinct subspaces. In step 2, fix \( x_1 = 0 \), \( x_2 = 1 \) and \( x_3 \) through \( x_l \) be arbitrary. This generates \( q^{l-2} \) nonzero vectors that generate distinct subspaces. And so on. To construct the edge set, suppose \( x, y \) are two nonzero generating vectors defined as above (which could be equal). Then the one-dimensional subspace generated by \( x \) is contained in the one-dimensional subspace annihilated by \( y \) if and only if \( y^\top x \equiv 0 \mod q \).

The Winnie Li construction requires more elaborate computation. The main source of difficulty is that as of now there is no polynomial-time algorithm for finding a primitive element in a finite field. The best known algorithm to date is due to [28], which finds a primitive element of \( \mathbb{F}_{q^l} \) in time \( O(q^{(l/4)+\epsilon}) \). Thus, for really large primes \( q \), the Li construction would be difficult to implement. At

\(^5\)We are grateful to Prof. Alex Lubotzky for clarifying this point.
1. We identify an irreducible polynomial \( \phi(x) \) of degree 2 in \( \mathbb{F}_q[x] \). For instance, if \( q \equiv 5 \mod 8 \), then \( x^2 + 2 \) is irreducible. In particular, \( x^2 + 2 \) is an irreducible polynomial in \( \mathbb{F}_{13}[x] \). Other irreducible polynomials are known for other classes of prime numbers. In the worst case, an irreducible polynomial can be found by enumerating all polynomials of degree 2 in \( \mathbb{F}_q[x] \), and then deleting all products of the form \( (x - a)(x - b) \) where \( a, b \in \mathbb{F}_q \). This would leave \( (q(q - 1))/2 \) irreducible polynomials, but we can use any one of them.

2. Once this is done, \( \mathbb{F}_{q^2} \) is isomorphic to the quotient field \( \mathbb{F}_q[x]/(\phi(x)) \). Let us denote \( x = x + (\phi(x)) \). Then the elements of \( \mathbb{F}_{q^2} \) can be expressed as \( a\bar{x} + b, a, b \in \mathbb{F}_q \).

3. To determine the set of units \( S \) in the field \( \mathbb{F}_{q^2} \), we have to find all elements \( \alpha \) in \( \mathbb{F}_{q^2}^* \) such that \( \alpha^{q+1} = 1 \). Now, if \( \beta \) is a primitive element of the \( \mathbb{F}_q^* \), then
   \[
   \mathbb{F}_{q^2}^* = \{ \beta^m, 1 \leq m \leq q^2 - 1 \}.
   \]
   Among the elements of \( \mathbb{F}_{q^2}^* \), the units are precisely the roots of \( \alpha^{q+1} = 1 \). With the above enumeration, we get
   \[
   S = \{ \beta^k(q-1), 1 \leq k \leq q + 1 \}.
   \]

4. We now enumerate all elements in the multiplicative group \( \mathbb{F}_{q^2}^* \) and test each element to see if it is a primitive element. To ensure that \( g \in \mathbb{F}_{q^2}^* \) is a primitive element, it is sufficient to check that \( g^{q^2-1} \neq 1 \) for any prime divisor \( l \) of \( q^2 - 1 \). Because \( q^2 - 1 = (q - 1)(q + 1) \), the prime divisors of \( q^2 - 1 \) are 2 and the other prime divisors of \( q - 1 \) and \( q + 1 \). For example, with \( q = 13 \), the prime divisors of \( q^2 - 1 \) are \( l = 2, 3, 7 \), with the corresponding values \( k = (q^2 - 1)/l = 84, 56, 24 \). Therefore if \( g^k \neq 1 \) for these values of \( k \), then \( g \) is a primitive element. There are also efficient ways to raise a field element to large powers, for example, by expanding the power to the base 2. One can check that \( g = \bar{x} + 1 \) is a primitive element of the field \( \mathbb{F}_{13^2} \).

5. Once a primitive element \( g \in \mathbb{F}_{q^2} \) is determined, the generator set \( S \) equals \( \{ g^{(q-1)k}, k \in [q + 1] \} \). Thus we can construct the Cayley graph \( C(\mathbb{F}_{q^2}, S) \) as follows: Each vertex \( a\bar{x} + b \) is connected to the vertices
   \[
   \{ a\bar{x} + b + g^{(q-1)k}, k \in [q + 1] \}.
   \]
   In this way we can construct a \( (q + 1) \)-regular Ramanujan graph with \( q^2 \) vertices.

Finally, for the Bibak construction, we simply enumerate all vectors in \( \mathbb{Z}_{q^2}^2 \), compute all of their norms, and identify the elements of norm one. Again, for \( q \leq 103 \), this works quite well.
7 Phase Transition Studies

7.1 Background

The bounds in (7) and (8) provide sufficient conditions for matrix completion using nuclear norm minimization when the sample set is chosen as the edge set of a Ramanujan graph or a Ramanujan bigraph. This is in contrast to earlier papers such as [1, 29] in which the sample set is chosen at random. If one were to work through how the numbers $\theta$ and $\phi$ depend on the degree $d$ (for a balanced graph) and the rank $r$, it turns out that the sufficient conditions imply that $r = O(d^{1/3})$. Thus, in order to guarantee exact matrix completion using Theorem 1, either the rank of the unknown matrix has to be really small, or the degree of the Ramanujan graph has to be really large, or both. However, these are only sufficient conditions. It is possible to determine how close these sufficient conditions are to being necessary via numerical simulations.

So far as the authors could determine, the only paper that studies the behavior of nuclear norm minimization for the completion of randomly generated low rank matrices is [1]. A related paper is [30], which studies matrix recovery, and not matrix completion. In this paper, the measurements consist of taking the Frobenius inner product of the unknown matrix with randomly generated Gaussian matrices. Finally, a very general theory of the behavior of convex optimization on randomly generated data is given in [31]. A concept called the “statistical dimension” is introduced, and it is established that convex optimization exhibits “phase transition,” whereby the success rate of the optimization algorithm goes from 100% to 0% very quickly as the input to the algorithm is changed. The width of the transition region is linear in the statistical dimension. The theory in this paper applies also to matrix completion using nuclear norm minimization. However, it is not easy to work out the appropriate statistical dimension in this case.

Against this background, we carried out several numerical experiments on the behavior of nuclear norm minimization for matrix completion, on randomly generated low rank matrices. We used all the various construction techniques for Ramanujan graphs described in previous sections. In particular, we also studied the completion of randomly generated rectangular matrices.

7.2 Comparison Across Ramanujan Graphs

The first study we carried out was to construct Ramanujan graphs using the five methods described in this paper, and test their performance on the same set of randomly generated low rank matrices.

Now we describe how the parameters of our study were chosen. Obviously, each method for constructing a Ramanujan graph permits only certain combinations of graph size and degree. In particular, the LPS graphs have degree $p + 1$ where $p$ is a prime $\equiv 1 \pmod{4}$, while the Bibak constructions have degree $p + 1$ where $p$ is a prime $\equiv 3 \pmod{4}$. Of course, the Winnie Li, Gunnells, and Array code constructions work with any prime number. To ensure that the LPS and Bibak constructions have comparable degrees, it is desirable to use “twin primes” such as 41 and 43, 59 and 61, 71 and 73, 101 and 103 etc. In the constructions of Li, Bibak, and our array code method for square matrices, the size $n$ approximately equals $d^2$. In order to be able to run a large number of nuclear norm minimizations on a standard laptop, we chose to limit the size of the graph $n$ to around 1,500. With these considerations in mind, we choose $p = 41$ for the LPS construction and $q = 43$ for the rest. In the case of the LPS construction, we chose $q = 13$ which led to Ramanujan graphs with $n = 1092$ vertices. The other constructions had roughly $43^2 \sim 1,700$ vertices.

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6In this connection we again remind the reader that while Ramanujan graphs of all sizes and degrees are known to exist as per [18], there are as yet no efficient procedures for constructing them. As and when such procedures become available, we would be in a position to undertake a more thorough study.
Figure 1: Comparison across various classes of Ramanujan graphs

Once the Ramanujan graph of degree 41 or 43 and a corresponding number of vertices was constructed using each method, the corresponding edge set was used as the measurement set. For purpose of comparison, we also used random sampling. A total of $dn$ elements were selected at random from among the $n^2$ elements, so that the average number of elements sampled in each row and each column was equal to $d$. We carried out nuclear norm minimization using each of these sampling sets. For each value of the rank $r$, we generated 100 $n \times n$ square matrices at random, by choosing $X = UV^\top$ where $U, V \in \mathbb{R}^{n \times r}$ were random Gaussian matrices. Then we solved the nuclear norm minimization problem of (1) for each of these 100 random matrices. If $\hat{X}$ denotes the solution of (1), then we computed the relative error $\|\hat{X} - X\|_F / \|X\|_F$, and deemed that the matrix $X$ was accurately completed if the relative error is less than 0.005. Obviously this threshold is arbitrary and can be varied. By way of comparison, the threshold used in [1] is 0.001. Further, for each choice of the rank $r$ and each family of Ramanujan graphs, we computed the fraction of the 100 random graphs that were accurately completed.

Figure 1 shows a comparison between the various methods as well as random sampling. As we increased the rank, naturally the performance deteriorated irrespective of the Ramanujan graph used to generate the sample set. But in each case, the success ratio went from 100% to 0% with an increase of just two in $r$. This behavior is usually referred to as a “phase transition.” In the case of $\ell_1$-norm minimization with randomly generated Gaussian measurement matrices, a similar phase transition is reported in [32]. The performance of the array code matrix and the Gunnells construction degrades less sharply compared to other methods. However, we believe that not too much should be read into this, because the critical value which is the maximum rank at which 100% success is achieved is the same as for the other methods.

7.3 Study on LPS Ramanujan Graphs

As mentioned above, the LPS and Gunnells constructions are the only “two-parameter” construction methods, in the sense that the degree $d$ and size $n$ can be chosen independently. In this study, we focused on the LPS construction. We chose $q = 13$, which leads to $n = 1,092$. We varied $p$ from
Figure 2: Critical value of rank versus the degree of the LPS Ramanujan graph with 1,092 vertices

| $d$  | $\bar{r}$ | $\bar{r}/d$ |
|-------|-----------|-------------|
| 197   | 62        | 0.3147      |
| 229   | 75        | 0.3275      |
| 293   | 102       | 0.3481      |

Table 1: Degree vs. critical rank in high degree LPS Ramanujan graphs

5 to 157, which is the largest number such that *every* prime number equal to 1 mod 4 results in a Ramanujan graph using the LPS construction. Still higher values of $p$ (e.g., 197) can also lead to a Ramanujan graph. However, for some intermediate values of $p$ between 157 and 197, the resulting LPS construction would have multiple edges between some pairs of vertices. The objective was to determine how the critical value of the rank $r$ (beyond which the success ratio falls off drastically) depends on $d = p + 1$. For each choice of $p$, we noted the maximum value of $r$, call it $\bar{r}$, such that when $r \leq \bar{r}$, the recovery percentage was 100%. The results are shown in Figure 2. It can be observed from this figure that $\bar{r} \approx d/3 = (p + 1)/3$ over the entire range of $p$.

7.4 Phase Transition in High Degree LPS Ramanujan Graphs

As mentioned earlier, the LPS construction can be used with $p > q$, provided that the $p+1$ generator matrices defined in (15) are distinct elements of $\mathbb{F}_q^{2 \times 2}$. With $q = 13$, $p = 157$ is the largest value of $p$ for which *every* choice of $p$ is permissible. However, still larger choices of $p$ are permissible, such as 197, 229 and 293. The previous experiment was carried out for the LPS Ramanujan graph for each of these values of $p$, and the results are shown in Figures 3, 4, and 5. Two observations can be made from this study. First, in each case, there is still a phase transition, and second, the critical value $\bar{r} \approx d/3$. 
Figure 3: Phase transition behavior in LPS graph with 1,092 vertices with $p = 197$

Figure 4: Phase transition behavior in LPS graph with 1,092 vertices with $p = 229$
Finally, to illustrate the use of a Ramanujan bigraph for sampling, we chose \( q = 43 \), and constructed array code matrices with \( l = 26, 60, 100 \). Figures 6, 7 and 8 show the results of the various studies. It can be seen that there is phase transition behavior in the rectangular case as well.

### 8 Discussion and Conclusions

In a companion paper (Part-1), we showed that it is possible to guarantee exact completion of an unknown low rank matrix, if the sampling set corresponds to the edge set of a Ramanujan bigraph. While that set of results is interesting in itself, it has left open the question of just how Ramanujan bigraphs are to be constructed. In the literature to date, there are relatively few explicit constructions of Ramanujan graphs, and no explicit constructions of a Ramanujan bigraph. In this paper, we presented for the first time an infinite family of unbalanced Ramanujan bigraphs with explicitly constructed biadjacency matrices. In addition, we have also shown how to construct the adjacency matrices for the currently available families of Ramanujan graphs. These explicit constructions, as well as forthcoming ones based on [14, 15], are available for only a few combinations of degree and size. In contrast, it is known from [17] and [18] that Ramanujan graphs are known to exist for all degrees and all sizes. The main limiting factor is that these are only existence proofs and do not lead to explicit constructions. A supposedly polynomial-time algorithm for constructing Ramanujan graphs of all degrees and sizes is proposed in [19]. But it is still a conceptual algorithm and no code has been made available. Therefore it is imperative to develop efficient implementations of the ideas proposed in [18], and/or to develop other methods to construct Ramanujan graphs of most degrees and sizes. In this connection, it is worth pointing out that efficient solutions of the matrix completion problem do not really require the existence of Ramanujan graphs of all sizes and degrees. It is enough if the “gaps” in the permissible values for the degrees and the sizes are very small. If this extra freedom leads to substantial simplification in the construction procedures, then it would be a worthwhile tradeoff. However, research on this
Figure 6: Phase transition behavior in the recovery of rectangular matrices using the array code matrix of size $1118 \times 1849$

Figure 7: Phase transition behavior in the recovery of rectangular matrices using the array code matrix of size $1849 \times 2580$
The sufficient condition for matrix completion derived in Part-1 is rather restrictive. In an attempt to determine how close it is to being necessary, we carried out numerical simulations of nuclear norm minimization on randomly generated low-rank matrices. The results revealed several interesting points:

1. For square matrices, the maximum rank $\tilde{r}$ for which nuclear norm minimization correctly completes all low-rank matrices is approximately $\tilde{r} \approx d/3$, where $d$ is the degree of the Ramanujan graph.

2. This upper limit appears to be independent of the specific family of Ramanujan graph.

3. There is a very noticeable phase transition, whereby the percentage of low-rank matrices that are recovered changes from 100% to 0% if the rank is increased by just two beyond $\tilde{r}$. Again, this phenomenon appears to be independent of the specific family of Ramanujan graph.

Phase transition behavior is already known to occur in the case of $\ell_1$-norm minimization with randomly generated Gaussian measurement matrices. See for example [32]. A theoretical analysis is carried out in several papers, including [33, 34] and others. Yet another paper [35] reports purely numerical results which show that, when the random measurement matrix is replaced by any of several deterministic matrices, the phase transition behavior continues to manifest itself. This conclusion is further reinforced in [36], through additional numerical simulations. In the case of matrix completion, the phase transition phenomenon is reported in [1], but in that paper the sampling set is chosen at random, and not in a deterministic fashion as in the present paper. In [30], phase transition behavior is reported for the matrix recovery problem, in which the measurements consist of Frobenius inner products $\langle A_i, X \rangle_F$ where $A_i$ is a random Gaussian matrix and $X$ is the unknown matrix to be recovered. There is no theoretical analysis, and only numerical results are reported. Our observations regarding phase transition for deterministic sampling sets are along similar lines – only simulations but no theory. Therefore it would be worthwhile to develop a
theoretical foundation for phase transition in matrix completion, as was done earlier for $\ell_1$-norm minimization in [32]. However, this appears to be a difficult task.

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