New similarity solutions for the generalized variable-coefficients KdV equation by using symmetry group method

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\textbf{ABSTRACT}

In this paper, a generalized variable-coefficients KdV equation (gvcKdV) arising in fluid mechanics, plasma physics and ocean dynamics is investigated by using symmetry group analysis. Two basic generators are determined, and for every generator, the admissible forms of the variable coefficients and the corresponding reduced ordinary differential equations are obtained. Finally, by searching for solutions to those reduced ordinary differential equations, many new exact solutions for the gvcKdV equation have been found.

\textbf{1. Introduction}

This paper is devoted to studying the generalized variable-coefficients KdV equation (gvcKdV), which is given by (Wang, 2006)

\begin{equation}
    u_t + g_1 u_{xxx} + (g_2 u^2 + g_3 u^3 + g_4 u + g_5) u_x + g_6 u + g_7 = 0,
\end{equation}

where, \(g_i(t)\) with \(i = 1, 2, \ldots, 7\) are arbitrary functions of \(t\). When \(g_2 = 0\), Equation (1) is derived by considering the time-dependent basic flow and boundary conditions from the well-known Euler equation with an earth rotation see (Tang, Huang, & Lou, 2006) and analytical solitonic solution is obtained for it by considering \(g_3 = g_5 = g_7 = 0\), which means that Equation (1) not really solved but only the known famous variable coefficients KdV equation. Furthermore, many physical and mechanical situations governed by Equation (1) like pressure pulses in fluid-filled tubes of special value in arterial dynamics, trapped quasi-one-dimensional Bose–Einstein condensates, ion-acoustic solitary waves in plasmas and the effect of a bump on wave propagation in a fluid-filled elastic tube; moreover, a model for strongly nonlinear internal waves in the ocean (Alam & Ali Akbar, 2015; Bibi & Mohyud-Din, 2014; El-Shiekh, 2012; Hu, Tan, & Al-Nowehy, 2013; Moussa & El-Shiekh, 2010, 2012).

Recently, Wang used the semi-inverse method to obtain the variational principle for Equation (1) but no solutions obtained. Therefore, the most important target for this paper is obtaining new exact solutions for Equation (1) under some constraints among the variable coefficients by using the symmetry group analysis.

\textbf{2. Symmetry method}

Recently, many methods have been investigated to deal with nonlinear partial differential equations like bilinear representation, Bäcklund transformation methods (El-Shiekh, 2015; Lü & Peng, 2013; Lü, Lin, & Qi, 2015a; Lü & Lin, 2016), tanh function and sine-cosine methods (Bibi & Mohyud-Din, 2014; El-Shiekh, 2012, 2015; El-Wakil, Abulwafa, El-hanbaly, El-shewy, & Abd El-Hamid, 2016; Hu et al., 2016; Moussa & El-Shiekh, 2011), direct reduction method (El-Shiekh, 2012, 2015, 2017) and the symmetry group analysis (El-Sayed, Moatimid, Moussa, El-shiekh, & El-Satar, 2014; El-Sayed et al., 2015; Moatimid, El-Shiekh, & Al-Nowehy, 2013; Moussa & El-Shiekh, 2010, 2012).

Symmetry method is one of the new modification of Lie group analysis; it is more easy and simple in calculations than Lie method (Moatimid et al., 2013; Wang, Liu, & Zhang, 2013; Wang, Kara, & Fakhar, 2015; Wang, 2016) and can be briefly described in the following steps:

Suppose that the differential operator \(L\) can be written in the form

\begin{equation}
    L(u) = \frac{\partial^p u}{\partial t^p} - H(u),
\end{equation}

where, \(u = u(t, x)\) and \(H\) may depend on \(t, x, u\) and any derivative of \(u\) as long the derivative of \(u\) does not really solved but only the known famous variable coefficients KdV equation. Furthermore, many physical and mechanical situations governed by Equation (1) like pressure pulses in fluid-filled tubes of special value in arterial dynamics, trapped quasi-one-dimensional Bose–Einstein condensates, ion-acoustic solitary waves in plasmas and the effect of a bump on wave propagation in a fluid-filled elastic tube; moreover, a model for strongly nonlinear internal waves in the ocean (Alam & Ali Akbar, 2015; Bibi & Mohyud-Din, 2014; El-Shiekh, 2012; Hu, Tan, & Al-Nowehy, 2013; Moussa & El-Shiekh, 2010, 2012).

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where, \(u = u(t, x)\) and \(H\) may depend on \(t, x, u\) and any derivative of \(u\) as long the derivative of \(u\) does
not contain more than \((p - 1), t\) derivatives. We will consider the symmetry operator (called infinitesimal symmetry) in the form
\[
S(u) = A(t, x, u) \frac{\partial u}{\partial t} + \sum_{i=1}^{n} B_i(t, x, u) \frac{\partial u}{\partial x_i} + C(t, x, u),
\]
and the Fréchet derivative of \(L(u)\) is given by
\[
F(L, u, v) = \frac{d}{dt}L(u + \varepsilon v)|_{\varepsilon = 0}
\]
With these definitions, we will compute the following:

i. \(F(L, u, v)\);

ii. \(F(L, u, S(u))\);

iii. Substitute \(H(u)\) for \(\frac{\partial S}{\partial x_i}\) in \(F(L, u, S(u))\);

iv. Set this expression to zero and perform a polynomial expansion;

v. Solve the resulting partial differential equations. Once this system of partial differential equations is solved for the coefficients of \(S(u)\), Equation (2) can be used to obtain the functional form of the solutions.

3. Determination of symmetries

In order to find the symmetries of Equation (1), we set the following symmetry operator
\[
S(u) = A(x, t, u)u_t + B(x, t, u)u_x + C(x, t, u).
\]
Calculating the Fréchet derivative \(F(L, u, v)\) of \(L(u)\) in the direction of \(v\), given by Equation (4), and replacing \(v\) by \(S(u)\) in \(F\), we get
\[
F(L, u, S(u)) = S_t + g_1 S_{xx} + 3g_2 u^2 S + g_2 u^2 S_x + g_2 u^2 S_x + 2g_3 u u_x S + g_4 u S + g_4 u S + g_5 u S_x + g_5 u S_x + g_6 S + g_7.
\]
Substituting the values of different derivatives of \(S(u)\) in \(F\) with the aid of Maple program, we get a polynomial expansion in \(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots\). On making use of Equation (1) in the polynomial expression for \(F\), rearranging terms of various powers of derivatives of \(u\) and equating them to zero, we obtain
\[
A_x = A_u = B_u = C_u = C_x = 0,
\]
\[
3g_1 B_x - (A_1)_t = 0,
\]
\[
C_t - (A_7)_t - (A_6)_t + g_1 C_{xx} + g_2 C_x u^3 + g_3 C_x u^2 + g_4 C u + g_5 C u - g_6 C = 0,
\]
\[
g_2 B_x u^3 + g_3 B_x - (A_7)_t - (A_6)_t - (A_3)_t u^2 + 2g_3 u C + B_t + g_4 C + 3g_3 u^2 C - (A_4)_u + g_4 B_u + g_3 B_u u^2 = 0.
\]
On solving system (7), the infinitesimal \(A, B\) and \(C\) in the above equations are:
\[
A = \frac{1}{\Gamma(t)} [3c_1 \Gamma(t) + c_2],
\]
\[
B = c_x + c_3,
\]
\[
C = c_4 u + c_5,
\]
where, \(c, i = 1, 2, \ldots, 5\), are arbitrary constants. The functions \(g_i = g_i(t), i = 1, 2, \ldots, 7\), are governed by the following equations:
\[
\begin{align*}
(Ag_2)_t - (c_1 + 3c_4) g_2 &= 0, \\
(Ag_3)_t - (c_1 + 2c_4) g_3 - 3c_5 g_2 &= 0, \\
(Ag_4)_t - (c_1 + c_4) g_4 - 2c_5 g_3 &= 0, \\
(Ag_5)_t - c_1 g_5 + c_4 g_2 &= 0, \\
(Ag_6)_t &= 0, \\
(Ag_7)_t + c_4 g_7 + c_5 g_6 &= 0.
\end{align*}
\]
The symmetry Lie algebra of Equation (1) is generated by the operators
\[
\begin{align*}
V_1 &= \frac{3\Gamma(t)}{\Gamma(t)} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \\
V_2 &= \frac{\Gamma(t)}{\Gamma(t)} \frac{\partial}{\partial t}, \\
V_3 &= \frac{\partial}{\partial u}, \\
V_4 &= \frac{\partial}{\partial u}, \\
V_5 &= \frac{\partial}{\partial u}, \\
V_6 &= \frac{\partial}{\partial u},
\end{align*}
\]
and the commutator table of it is given by

\[
\begin{array}{cccccc}
V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \\
V_1 & 0 & -5V_1 & -5V_1 & 0 & 0 \\
V_2 & 0 & 0 & 0 & 0 & 0 \\
V_3 & 0 & 0 & 0 & 0 & 0 \\
V_4 & 0 & 0 & 0 & 0 & 0 \\
V_5 & 0 & 0 & 0 & 0 & 0 \\
V_6 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Now, we are going to search for a one-dimensional optimal system of the Lie algebra generated by the operators (10) as follows:
Consider a general element of \(V = \sum_{i=1}^{5} a_i V_i\) and checking whether \(V\) can be mapped to a new element \(V'\) under the general adjoint transformation \(Ad(\exp(cV')) V_j = V_j - c_i V_i, V_j + \frac{c_j}{2} [V_i, [V_i, V_j]]\), to simplify it as much as possible.

The adjoint table

\[
\begin{array}{cccccc}
V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \\
V_1 & \exp(3c)V_2 & \exp(c)V_1 & V_3 & V_4 & V_5 \\
V_2 & V_1 - 3cV_2 & V_2 & V_3 & V_4 & V_5 \\
V_3 & V_1 - 3cV_3 & V_2 & V_3 & V_4 & V_5 \\
V_4 & V_1 - 3cV_4 & V_2 & V_3 & V_4 & V_5 \\
V_5 & V_1 - 3cV_5 & V_2 & V_3 & V_4 & V_5 \\
V_6 & V_1 - 3cV_6 & V_2 & V_3 & V_4 & V_5 \\
\end{array}
\]
Following Olver (1986), we can deduce the following basic fields which form an optimal system for the gvckDv,

i. \( V_1 + k_1 V_4 \),
ii. \( V_2 + k_2 V_3 + k_3 V_4 \),
iii. \( V_3 + k_4 V_5 \),
iv. \( V_4 \),
v. \( V_5 \)

where, \( k_i, i = 1, \ldots, 4 \) are arbitrary constants. The cases (iii), (iv) and (v) give trivial reductions. Therefore, we will discuss the first and second case by only using the following characteristic equation:

\[
\frac{dt}{A(x, t, u)} = \frac{dx}{B(x, t, u)} = -\frac{du}{C(x, t, u)}. \tag{11}
\]

By solving Equation (11) for both generators (I) and (II)

| The similarity | Similarity | Integrability conditions |
|---------------|------------|--------------------------|
| variable \( \xi \) | solution \( u(x, t) \) | (x + k_1)\( \Gamma(t)^r \), \( F(\xi)\Gamma(t)^{r/2} \), \( g_1(t) = \frac{3}{2} \Gamma(t)\Gamma(t)^{r/2} + k_1 \), \( g_2(t) = \frac{3}{2} \Gamma(t)\Gamma(t)^{r/2} - k_1 \), \( g_3(t) = \frac{3}{2} \Gamma(t)\Gamma(t)^{r/2} - k_1 \), \( g_4(t) = \frac{3}{2} \Gamma(t)\Gamma(t)^{r/2} - k_1 \), \( g_5(t) = m_1\Gamma(t)e^{-\kappa_1}, \) \( g_6(t) = m_1\Gamma(t)e^{-\kappa_1} \), \( g_7(t) = m_1\Gamma(t)\), \( g_8(t) = m_1\Gamma(t)e^{-\kappa_1} \). |
| \( (x + k_1)\Gamma(t) - x \) | \( F(\xi)e^{-\kappa_1} \) |

where, \( n_1, \ldots, n_6 \) and \( m_1, \ldots, m_6 \) are arbitrary constants.

4. Reductions and exact solutions

In this section, the primary focus is on the reductions associated with the two generators (i) and (ii), and their solutions.

**Generator (I)**

Corresponding to this generator, the gvckDv is reduced to the following ordinary differential equation.

\[
3F'' + n_1F^3F' + n_2F^2F' + n_3FF' + (n_4 - \zeta)F' + (n_5 - k_1)F + n_6 = 0. \tag{12}
\]

To solve Equation (12), we seek a special solution in the form

\[
F = A_0 + A_1 \zeta^{-\frac{1}{2}}, \tag{13}
\]

where, \( A_0 \) and \( A_1 \) are arbitrary constants to be determined. Substituting Equation (13) into Equation (12) and equating the coefficients of different powers of \( \zeta \) to zero, we get a system of algebraic equations, solutions of which give rise to the relations on the constants as

\[
A_0 = 3n_6, \quad A_1 = \frac{3n_6}{n_2}, \quad n_1 = -\frac{2n_2}{9n_6}, \quad n_3 = -\frac{3}{2}n_2n_6, \quad n_4 = \frac{3}{2}n_2n_5, \quad k_1 = \frac{2}{3} + n_5. \tag{14}
\]

Finally, we get the following exact solution for the gvckDv

\[
u_1(x, t) = \frac{\Gamma(t)^{(\frac{1}{1-n_6})}}{\Gamma(t)^{(\frac{1}{1-n_6})}} \left( \frac{3n_6}{2} + \frac{3n_6}{n_2} \left( \left[ x + \frac{2}{3} + n_5 \right] \Gamma(t)^{\frac{1}{2}} \right) \right)^{-\frac{1}{2}}. \tag{15}
\]

**Generator (II)**

The reduced nonlinear ordinary differential equation corresponding to this case is

\[
F'' + m_1F^3F' + m_2F^2F' + m_3FF' + (m_4 - k_3)F' + (k_4 - m_5)F + m_6 = 0. \tag{16}
\]

Herein, we apply the modified extended tanh function method (El-Shiekh, 2015) to obtain rational exact solitary wave solutions to Equation (16). Let us assume that Equation (16) has a solution in the form

\[
F(\zeta) = A_0 + \sum_{i=1}^{N} A_i \phi^i + B_i \phi^{-i}, \tag{17}
\]

where, \( \phi(\zeta) \) is a solution of the following Riccati equation

\[
\phi' = r + \phi^2, \tag{18}
\]

which has the following solutions

\[
\phi(\zeta) = -\sqrt{-r} \tanh(\sqrt{-r} \zeta), \quad r < 0, \quad \phi(\zeta) = -\sqrt{-r} \coth(\sqrt{-r} \zeta), \quad r < 0, \quad \phi(\zeta) = \sqrt{r} \tan(\sqrt{r} \zeta), \quad r > 0, \quad \phi(\zeta) = -\frac{1}{\sqrt{r}}, \quad r = 0. \tag{19}
\]

Substituting Equation (17) into Equation (16) and by balancing the linear term with the greatest nonlinear term, we get

\[
N = \frac{2}{3} \tag{20}
\]

Therefore,

\[
F(\zeta) = A_0 + A_1 \phi^\frac{2}{3} + B_1 \phi^{-\frac{2}{3}}. \tag{21}
\]
Substituting Equation (21) into Equation (16) and equating the powers of \( \phi, j = 0, -\frac{1}{3}, -3, -\frac{7}{3}, \ldots \) to zero, we obtain a system of algebraic equations. By solving that system with maple program yields the following solution

\[
A_1 = -2 \left( \frac{5}{9m_1} \right)^{\frac{1}{3}}, B_1 = \left( \frac{15}{m_1} \right)^{\frac{2}{3}} \left( 3m_1m_3 - m_2^2 \right), \\
r = \pm \frac{1}{42} \sqrt{\frac{5}{14m_1}} \left( m_2^2 - 3m_1m_3 \right) \frac{1}{m_1}, \\
A_0 = -\frac{m_2}{3m_1}, m_6 = 0, k_3 = m_5, \\
k_2 = m_4 + \frac{1}{1323m_1^2} \times \left( 98m_2^2 - 441m_1m_2m_3 \pm 2\sqrt{70}(m_2^2 - 3m_1m_3) \right) \frac{1}{2}.
\]

Substituting Equation (22) into Equation (21), we get

\[
F(\zeta) = -\frac{m_2}{3m_1} - 2 \left( \frac{5}{9m_1} \right)^{\frac{1}{3}} \phi^3 + \left( \frac{15}{m_1} \right)^{\frac{2}{3}} \left( 3m_1m_3 - m_2^2 \right) \phi^2, 
\]

where, \( \phi \) is given by Equation (18). Substituting Equation (23) into the similarity solution corresponding to this case, we obtain

\[
u_2(x, t) = e^{-k_2 \Gamma(t)} \left[ -\frac{m_2}{3m_1} + 2 \left( \frac{5}{9m_1} \right)^{\frac{1}{3}} r \tanh \left( \sqrt{-r} (k_2 \Gamma(t) - x) \right) - \frac{15}{m_1} \left( 3m_1m_3 - m_2^2 \right) \left( 63m_1 \right)^{\frac{2}{3}} \right] \\
\]

\[
u_3(x, t) = e^{-k_2 \Gamma(t)} \left[ -\frac{m_2}{3m_1} + 2 \left( \frac{5}{9m_1} \right)^{\frac{1}{3}} r \coth \left( \sqrt{-r} (k_2 \Gamma(t) - x) \right) + \frac{15}{m_1} \left( 3m_1m_3 - m_2^2 \right) \left( 63m_1 \right)^{\frac{2}{3}} \right] \\
\]

where

\[
r = \frac{1}{42} \sqrt{\frac{5}{14m_1}} \left( m_2^2 - 3m_1m_3 \right) \frac{1}{m_1} \text{ and} \\
k_2 = m_4 + \frac{1}{1323m_1^2} \left( 98m_2^2 - 441m_1m_2m_3 \pm 2\sqrt{70}(m_2^2 - 3m_1m_3) \right) \frac{1}{2}.
\]

5. Conclusion

In this paper, we have applied the symmetry group analysis to the gvcKdV. This application leads to two nonequivalent generators; for every generator in the optimal system, the admissible forms of the coefficients and the corresponding reduced ordinary differential equation are obtained. The search for solutions to those reduced ordinary differential equations using tanh function method has yielded many exact new solutions that were not obtained before.

Disclosure statement

No potential conflict of interest was reported by the author.

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