Šapovalov elements and the Jantzen filtration for contragredient Lie superalgebras: a survey

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Abstract. This is a survey of some recent results on Šapovalov elements and the Jantzen filtration for contragredient Lie superalgebras. The topics covered include the existence and uniqueness of the Šapovalov elements, bounds on the degrees of their coefficients and the behavior of Šapovalov elements when the Borel subalgebra is changed. There is always a unique term whose coefficient has larger degree than any other term. This allows us to define some new highest weight modules. If $X$ is a set of orthogonal isotropic roots and $\lambda \in h^*$ is such that $\lambda + \rho$ is orthogonal to all roots in $X$, we construct highest weight modules with character $\epsilon^p_X$. Here $p_X$ is a partition function that counts partitions not involving roots in $X$. When $|X| = 1$, these modules are used to give a Jantzen sum formula for Verma modules in which all terms are characters of modules in the category $O$ with positive coefficients.

1. Introduction
Throughout this paper we work over an algebraically closed field $k$ of characteristic zero. If $g$ is a semisimple Lie algebra, necessary and sufficient conditions for the existence of a non-zero homomorphism between Verma modules can be obtained by combining work of Verma [15] with work of Bernstein, Gelfand and Gelfand [1], [2]. Such maps can be described explicitly in terms of certain elements introduced by Šapovalov in [14]. Necessary and sufficient conditions for a simple highest weight module to be a composition factor of a Verma module were also obtained in [1], [2]. A more elementary proof of the latter result can be given using the Jantzen filtration and sum formula [8]. However neither Šapovalov elements nor the Jantzen filtration have received much attention for classical simple Lie superalgebras. In this paper we review some recent results on Šapovalov elements, the Jantzen filtration and sum formula in the super case. New phenomena arise due to the presence of isotropic roots. Proofs not found here are given in [11] Chapter 9 and 10, [12] or [13].

Let $g = g(A, \tau)$ be a finite dimensional contragredient Lie superalgebra with Cartan subalgebra $h$, and set of simple roots $\Pi$, see [11] Chapter 5. Let $\Delta^+$ be the set of positive roots containing $\Pi$, and

$$g = n^- \oplus h \oplus n^+$$

1 We remark that the multiplicities of the composition factors in Verma modules for semisimple Lie algebras are given by the Kazdhan-Lusztig conjecture [10]. For type $A$ Lie superalgebras, they are given by Brundan’s analog of the Kazdhan-Lusztig conjecture, proven in [5], see also [4] Theorem A and [3] Theorem 3.6.
the corresponding triangular decomposition of $\mathfrak{g}$. We use the Borel subalgebras $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$. The Verma module $M(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$, and highest weight vector $v_\lambda$ is induced from $\mathfrak{h}$. Let $\rho_0$ (resp. $\rho_1$) be the half-sum of the positive even (resp. odd) positive roots and $\rho = \rho_0 - \rho_1$.

Fix a non-degenerate invariant symmetric bilinear form $(\ , \ )$ on $\mathfrak{h}^*$ as in [11] Theorem 5.4.1, and for all $\alpha \in \mathfrak{h}^*$, let $h_\alpha \in \mathfrak{h}$ be the unique element such that $(\alpha, \beta) = \beta(h_\alpha)$ for all $\beta \in \mathfrak{h}^*$. Then for all $\alpha \in \Delta^+$, choose elements $e_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ such that

$$[e_\alpha, e_{-\alpha}] = h_\alpha.$$  

We give bounds on the degrees of the coefficients of $H_\pi$ of the Šapovalov elements in Equation (1) below. The exact form of the coefficients depends on the way the positive roots are ordered. However there is always a unique coefficient of highest degree, and we determine the leading term of this coefficient up to a scalar multiple. These results appear to be new even for simple Lie algebras. The existence of a unique coefficient of highest degree is used to construct some new highest weight modules $M^\pi(\lambda)$, where $\gamma$ is an isotropic root and $(\lambda + \rho, \gamma) = 0$, [13]. There is a version of the Jantzen sum formula in [11] Theorem 10.3.1, see also [6], which involves infinite sums of Verma modules with alternating sign coefficients. The module $M^\pi(\lambda)$ has character $e^\lambda p_\gamma$, see (8) for notation. The infinite sums can be replaced by a finite sum of characters of modules of this form leading to a formula where both sides are sums of characters in the category $O$. In type A there are explicit expressions for Šapovalov elements, see [12] Section 7.

2. Preliminaries

To simplify the exposition, we assume in this survey that any non-isotropic root of $\mathfrak{g}$ is even. This should give the reader the flavor of the results in general, but the proofs and some of the statements should be modified if this is not the case.

If $\eta \in Q^+ = \sum_{\alpha \in \Pi} \mathbb{N}\alpha$, a partition of $\eta$ is a map $\pi : \Delta^+ \rightarrow \mathbb{N}$ such that $\pi(\alpha) = 0$ or 1 for all isotropic roots $\alpha$, and

$$\sum_{\alpha \in \Delta^+} \pi(\alpha)\alpha = \eta.$$  

Let $P(\eta)$ be the set of partitions of $\eta$. The degree of a partition $\pi$ is defined to be $|\pi| = \sum_{\alpha \in \Delta^+} \pi(\alpha)$. Partitions are useful because they can be used to index a basis for $U(\mathfrak{n}^-)$. Fix an ordering on the set $\Delta^+$, and for $\pi$ a partition, set

$$e_{-\pi} = \prod_{\alpha \in \Delta^+} e_{-\pi(\alpha)},$$  

the product being taken with respect to this order. Then the elements $e_{-\pi}$, with $\pi \in P(\eta)$ form a basis of $U(\mathfrak{n}^-)^{-\eta}$.

Suppose $\gamma$ is a positive root, $m$ is a positive integer and set

$$H_{\gamma,m} = \{\lambda \in \mathfrak{h}^* | (\lambda + \rho, \gamma) = m(\gamma, \gamma)/2\},$$  

and let $I(H_{\gamma,m})$ be the ideal of $S(\mathfrak{h})$ consisting of functions vanishing on $H_{\gamma,m}$. The Šapovalov element $\theta_{\gamma,m}$ corresponding to the pair $(\gamma, m)$ has the form

$$\theta_{\gamma,m} = \sum_{\pi \in P(m\gamma)} e_{-\pi} H_\pi,$$  

(1)
where \( H_\alpha \in U(\hbar) \), and satisfies

\[
e_\alpha \theta_{\gamma, m} \in U(\mathfrak{g})(h_\gamma + \rho(h_\gamma) - m(\gamma, \gamma)/2) + U(\mathfrak{g})n^+, \text{ for all } \alpha \in \Delta^+. \tag{2}
\]

This means that if \( \lambda \in \mathcal{H}_{\gamma, m} \), then \( \theta_{\gamma, m} v_\lambda \) is a highest weight vector in \( M(\lambda) \). If \( \gamma \) is simple, then \( \theta_{\gamma, m} = e_{m\gamma} \), so now assume that \( \gamma \) is not simple. Let \( m^0 \in \mathcal{P}(m\gamma) \) be the unique partition of \( m\gamma \) such that \( m^0(\alpha) = 0 \) if \( \alpha \in \Delta^+ \setminus \Pi \). The partition \( m\pi^\gamma \) of \( m\gamma \) is given by \( m\pi^\gamma(\gamma) = m \), and \( m\pi^\gamma(\alpha) = 0 \) for all positive roots \( \alpha \) different from \( \gamma \). We normalize \( \theta_{\gamma, m} \) so that the coefficient \( H_{\gamma, 0} \) is equal to 1. This guarantees that \( \theta_{\gamma, m} v_\lambda \) is never zero. For a semisimple Lie algebra, the existence of such elements was shown by Šapovalov, [14] Lemma 1.

For a non-isotropic root \( \alpha \), we set \( \alpha^\vee = 2\alpha/(\alpha, \alpha) \), and write \( s_\alpha \) for the reflection corresponding to \( \alpha \). As usual the Weyl group \( W \) is the subgroup of \( GL(\mathfrak{h}^*) \) generated by all such reflections. For \( u \in W \) set

\[
N(u) = \{ \alpha \in \Delta^+_0 | u\alpha < 0 \}, \quad \ell(u) = |N(u)|.
\]

Suppose \( \Pi_{\text{even}} \) is the set of even simple roots, and let \( W_{\text{even}} \) be the subgroup of \( W \) generated by the reflections \( s_\alpha \), where \( \alpha \in \Pi_{\text{even}} \). Note that \( W_{\text{even}} \) can be a proper subgroup of \( W \). For example this happens when \( \mathfrak{g} = \mathfrak{osp}(2m, 2n) \). If \( \Pi = \{ \alpha_i | i = 1, \ldots, t \} \) is the set of simple roots, and \( \gamma \) is a positive root such that \( \gamma = \sum_{i=1}^t \alpha_i \), then the height \( h_\gamma \) of \( \gamma \) is \( h_\gamma = \sum_{i=1}^t \alpha_i \).

**Theorem 2.1.** Let \( \mathfrak{g} \) be semisimple Lie algebra or a contragredient Lie superalgebra, and \( \gamma \) a positive root. If \( \gamma \) is isotropic assume that \( m = 1 \). Suppose \( \gamma = w\beta \) for a simple root \( \beta \) and \( w \in W_{\text{even}} \), and for \( \alpha \in N(w^{-1}) \), set \( q(w, \alpha) = (w\beta, \alpha^\vee) \). Then

(a) there exists a Šapovalov element \( \theta_{\gamma, m} \in U(\mathfrak{b}^-)^{-m\gamma} \), which is unique modulo the left ideal \( U(\mathfrak{b}^-)\mathcal{I}(\mathcal{H}_{\gamma, m}) \).

(b) the coefficients of \( \theta_{\gamma, m} \) satisfy

\[
|\pi| + \deg H_\pi \leq m h_\gamma, \tag{3}
\]

and

\[
H_{m\pi^\gamma} \text{ has leading term } \prod_{\alpha \in N(w^{-1})} h_\alpha^{m\pi(q(w, \alpha))}. \tag{4}
\]

Let \( d_{m\gamma} \) be the degree of \( H_{m\pi^\gamma} \).

**Corollary 2.2.** In Theorem 2.1 \( H_{m\pi^\gamma} \) is the unique term of degree \( d_{m\gamma} \) in \( \theta_{\gamma, m} \).

**Proof.** This follows easily from (3) and (4). \( \square \)

### 3. Šapovalov Elements and their Coefficients

A finite dimensional contragredient Lie superalgebra \( \mathfrak{g} \) has, in general several conjugacy classes of Borel subalgebras, and this both complicates and enriches the representation theory of \( \mathfrak{g} \). The complications are partially resolved by at first fixing a Borel subalgebra \( \mathfrak{b} \) (or equivalently a basis of simple roots for \( \mathfrak{g} \)) with special properties. The effect of changing the Borel subalgebra is studied in detail in [13], see also the next Section.

In [9] Table VI, Kac gave a particular diagram in each case that we will call distinguished. The corresponding set of simple roots and Borel subalgebra are also called distinguished.
The distinguished Borel subalgebra contains at most one simple isotropic root vector. Unless \( g = \mathfrak{osp}(1, 2n) \), or \( g = \mathfrak{osp}(2, 2n) \) there is exactly one other Borel subalgebra with this property up to conjugacy in \( \text{Aut} \ g \). A representative of this class (and its set of simple roots) will be called anti-distinguished. We assume that \( b \) is either distinguished or anti-distinguished.

Theorem 2.1 is proved by looking at the proofs given in [7] or [11] and keeping track of the coefficients. Given \( \lambda \in \mathfrak{h}^* \) and \( x = \sum_i a_i \otimes b_i \in U(b^-) = U(\mathfrak{n}^-) \otimes S(\mathfrak{h}) \), set \( x(\lambda) = \sum_i a_i b_i(\lambda) \in U(\mathfrak{n}^-) \). Let \((\gamma, m)\) be as in the statement of the Theorem and set \( H = H_{\gamma, m} \). The idea of the proof is to construct elements \( \theta^\lambda \in U(\mathfrak{n}^-)^{m\gamma} \) for all \( \lambda \) in a dense subset of \( \mathcal{H} \) such that \( \theta^\lambda v_\lambda \) is a highest weight vector in \( M(\lambda)^{\lambda - m\gamma} \), and that

\[
\theta^\lambda = \sum_{\pi \in \mathcal{P}(m\gamma)} a_{\pi, \lambda} e_{-\pi}
\]

where \( a_{\pi, \lambda} \) is a polynomial function of \( \lambda \in \Lambda \) satisfying suitable conditions. For \( \pi \in \mathcal{P}(m\gamma) \), the assignment \( \lambda \rightarrow a_{\pi, \lambda} \) for \( \lambda \in \Lambda \) determines a polynomial map from \( \mathcal{H} \) to \( U(\mathfrak{n}^-)^{m\gamma} \), so there exists an element \( H_\pi \in U(\mathfrak{h}) \) uniquely determined modulo \( I(\mathcal{H}) \) such that \( H_\pi(\lambda) = a_{\pi, \lambda} \) for all \( \lambda \in \Lambda \). We define the element \( \theta \in U(b^-) \) by setting

\[
\theta = \sum_{\pi \in \mathcal{P}(m\gamma)} e_{-\pi} H_\pi.
\]

Note that \( \theta(\lambda) = \theta^\lambda \). The Šapovalov element in Theorem 2.1 is constructed inductively using the next Lemma, see for example [7] Section 4.13 or [11] Theorem 9.4.3.

**Lemma 3.1.** Let \( \gamma \) be a positive root, and \( m \) a positive integer which is equal to 1 if \( \gamma \) is isotropic. Suppose that \( \alpha \in \Pi_{\text{even}} \), and set

\[
\mu = s_\alpha \cdot \lambda, \quad \gamma' = s_\alpha \gamma, \quad p = (\mu + \rho, \alpha') = (\gamma, \alpha').
\]

Assume that \( p, q \in \mathbb{N}\setminus\{0\} \) and \( \theta' \in U(\mathfrak{n}^-)^{-m\gamma'} \) is such that \( v = \theta' v_\mu \in M(\mu) \) is a highest weight vector. Then there is a unique \( \theta \in U(\mathfrak{n}^-)^{-m\gamma} \) such that

\[
e^{-\alpha} e_{-\alpha} \theta' = \theta e^p_{-\alpha}.
\]

Using Lemma 3.1 it follows that the coefficients of the Šapovalov elements \( \theta_{\gamma, m} \) are obtained by taking \( k \)-linear combinations of products of coefficients of the \( \theta_{\gamma', m} \), with binomial coefficients.

**4. Changing the Borel subalgebra**

Using adjacent Borel subalgebras (equivalently odd reflections), it is possible to give an alternative construction of Šapovalov elements corresponding to an isotropic root \( \gamma \), provided \( \gamma \) is a simple root for some Borel subalgebra. This condition always holds in type \( A \), but for other types, it is quite restrictive: if \( g = \mathfrak{osp}(2m, 2n + 1) \), (an algebra that does not satisfy the restrictions imposed at the start of section 2) the assumption only holds for roots of the form \( \pm(\epsilon_i - \delta_j) \), while if \( g = \mathfrak{osp}(2m, 2n) \) it holds only for these roots and the root \( \epsilon_m + \delta_n \).

Suppose that \( b \) is the distinguished Borel subalgebra, and let \( b' \) be another Borel subalgebra with the same even part as \( b \). Consider a sequence of Borel subalgebras

\[
b = b^{(0)}, b^{(1)}, \ldots, b^{(r)}.
\]

Assume there are isotropic roots \( \alpha_i \) such that \( g^{\alpha_i} \subset b^{(i-1)}, g^{-\alpha_i} \subset b^{(i)} \) for \( 1 \leq i \leq r \), and \( \alpha_1, \ldots, \alpha_r \) are distinct positive roots of \( b \).
The result follows since

\[ e_{a_1} \cdots e_{a_r} e_{-\gamma} e_{-a_r} \cdots e_{-a_1} v_\lambda = c \prod_{i \in F(\gamma)} (\lambda + \rho, a_i) \theta_\gamma v_\lambda. \]

5. The Square of a Šapovalov Element

When \( \gamma \) is an isotropic root we write \( H_\gamma \) and \( \theta_\gamma \) in place of \( H_{\gamma,1} \) and \( \theta_{\gamma,1} \) respectively. Here we record an elementary but important property of the Šapovalov element \( \theta_\gamma \) corresponding to such a root.

**Theorem 5.1.** If \( \lambda \in H_\gamma \), then \( \theta_\gamma^2 v_\lambda = 0 \). Equivalently, \( \theta_\gamma (\lambda - \gamma) \theta_\gamma (\lambda) = 0 \).

**Proof.** We assume \( \gamma = w \beta \) for \( \beta \in \Pi \) where \( \beta \) is isotropic, and \( w \in W_{\text{even}} \). For a suitable Zariski dense subset \( A \) of \( H_\beta \), suppose \( \lambda \in w \cdot A \). The proof is by induction on \( \ell(w) \). We can assume that \( w \neq 1 \). Suppose \( w = s_{\alpha_1} u \) with \( \ell(u) = \ell(w) - 1 \), and set \( \gamma' = s_{\alpha_1} \gamma \). Replace \( \mu \) with \( \mu - \gamma' \) and \( \lambda = s_{\alpha_1} \cdot \mu \) with \( s_{\alpha_1} \cdot (\mu - \gamma') = \lambda - \gamma \) in Equation (6). Then \( p \) is replaced by \( p + q \) and adopting the notation of (5),

\[ e_{-\alpha}^{p+2q} \theta_{\gamma'}^{\mu - \gamma'} \theta_{\gamma'}^\mu = \theta_{\gamma'}^{\lambda - \gamma} e_{-\alpha}^{p+q} \theta_{\gamma'}^\lambda e_{-\alpha}. \]

Combining this with Equation (6) and using induction we have

\[ 0 = e_{-\alpha}^{p+2q} \theta_{\gamma'}^{\mu - \gamma'} \theta_{\gamma'}^\mu = \theta_{\gamma'}^{\lambda - \gamma} \theta_{\gamma'}^\lambda e_{-\alpha}. \]

The result follows since \( e_{-\alpha} \) is not a zero divisor in \( U(n^-) \).

6. Modules with Prescribed Characters

We introduce some new highest weight modules whose characters are given by generating functions for certain kinds of partitions. If \( X \) is a set of pairwise orthogonal isotropic positive roots, set

\[ P_X(\eta) = \{ \pi \in P(\eta) | \pi(\alpha) = 0 \text{ for all } \alpha \in X \}. \]

and \( p_X(\eta) = |P_X(\eta)| \).

Set \( p_X = \sum p_X(\eta)e^{-\eta} \).

\[ p_X = \prod_{\alpha \in \Delta_1 \setminus X} (1 + e^{-\alpha})/\prod_{\alpha \in \Delta_0} (1 - e^{-\alpha}). \quad (7) \]

If \( X \) is empty, set \( p = p_X \), and if \( X = \{ \alpha \} \) is a singleton write

\[ P_\alpha(\eta), \quad p_\alpha(\eta), \quad \text{and} \quad p_\alpha \]

instead of \( P_X(\eta), \quad p_X(\eta), \quad \text{and} \quad p_X \).

For a module \( M \) in the BGG category \( O \), the character of \( M \) is defined by \( \chi M = \sum_{\eta \in P} \dim_k M^\eta e^\eta \). Recall that the Verma module \( M(\lambda) \) has character \( e^\lambda p \).

**Theorem 6.1.** Suppose that \( X \) is an isotropic set of positive roots and \( \lambda \in H_\gamma \) for all \( \gamma \in X \). Then there exists a factor module \( M_X(\lambda) \) of \( M(\lambda) \) such that

\[ \chi M_X(\lambda) = e^\lambda p_X. \]
If $X = \{\gamma\}$ we write $M^\gamma(\lambda)$ in place of $M^X(\lambda)$. The construction of the modules $M^X(\lambda)$ involves a process of deformation and specialization. First we extend scalars to $A = \mathbb{k}[T]$ and $B = \mathbb{k}(T)$. If $R$ is either of these algebras we set $U(g)_R = U(g) \otimes R$. Choose $\xi \in h^*$ such that $(\xi, \gamma) = 0$ for all $\gamma \in X$, and $(\xi, \alpha^\vee) \not\in \mathbb{Z}$ for all even roots $\alpha$. Next consider the $U(g)_B$-module $M(\lambda)_B$ with highest weight $\tilde{\lambda} = \lambda + T\xi$, and form the factor module of $M(\tilde{\lambda})_B$ obtained by setting $\theta_\gamma v_\lambda$ equal to zero for $\gamma \in X$. Then take a suitable $U(g)_A$-submodule of this factor module and reduce mod $T$ to obtain the module $M^X(\lambda)$. In more detail, we set

$$M^X(\tilde{\lambda})_B = M(\tilde{\lambda})_B / \sum_{\gamma \in X} U(g)_B \theta_\gamma v_\lambda.$$  

Then $M^X(\tilde{\lambda})_B$ is a $U(g)_B$-module generated by a highest weight vector $v_\tilde{\lambda}$ (the image of $v_\lambda$) with weight $\tilde{\lambda}$. Set $M^X(\tilde{\lambda})_A = U(g)_A v_\tilde{\lambda} \subset M^X(\tilde{\lambda})_B$, and

$$M^X(\lambda) = M^X(\tilde{\lambda})_A / TM^X(\tilde{\lambda})_A. \quad (9)$$  

Then

$$M^X(\tilde{\lambda})_A \otimes_A B = M^X(\tilde{\lambda})_B.$$  

Based on Corollary 2.2 we can show

**Lemma 6.2.** Let $M = U(g)_B v$ be a module with highest weight $\tilde{\lambda}$ and highest weight vector $v$. Suppose that $\theta_\gamma v = 0$ for all $\gamma \in X$. Then

(a) for all $\eta$ the weight space $M^{\tilde{\lambda} - \eta}$ is spanned over $B$ by all $e_{-\pi} v$ where $\pi \in \mathbb{P}(\lambda)$.

(b) $\dim_B M^{\tilde{\lambda} - \eta} \leq p_\lambda(\eta)$.

**Proof of Theorem 6.1 when $X = \{\gamma\}$.** We set

$$v = v_\tilde{\lambda} \in M_B = M(\tilde{\lambda})_B, \quad u = \theta_\gamma v, \quad N_B = U(g)_B u.$$  

For $R = A$ or $B$ we write $M^\gamma(\tilde{\lambda})_R$ in place of $M^X(\tilde{\lambda})_R$. Then the module $M^\gamma(\lambda)$ defined by (9) is generated by the image $\tilde{v}$ of $v$ which is a highest weight vector of weight $\lambda$. Also by Theorem 5.1, $\theta_\gamma u = 0$, so we can apply Lemma 6.2 to both $N_B$ and $M_B/N_B$. This gives

$$\dim(M_B/N_B)^{\tilde{\lambda} - \eta} \leq p_\gamma(\eta), \quad \dim N_B^{\tilde{\lambda} - \eta} \leq p_\gamma(\eta - \gamma). \quad (10)$$  

Since

$$p_\eta(\eta) = \dim M_B^{\tilde{\lambda} - \eta} = \dim(M_B/N_B)^{\tilde{\lambda} - \eta} + \dim N_B^{\tilde{\lambda} - \eta},$$  

and $p_\gamma(\eta - \gamma) = p_\gamma(\eta)$, it follows that equality holds in (10). Now we obtain the result from the following considerations applied to the weight spaces of the modules $M^\gamma(\tilde{\lambda})_R$ for $R = A, B$. If $K$ is an $A$-submodule of a finite dimensional $B$-module $L$ such that $K_A \otimes_A B = L$, then $\dim K/TK = \dim_B L$. \quad $\square$

**Remark 6.3.** If $M'$ is the kernel of the natural map $M(\lambda) \rightarrow M^\gamma(\lambda)$, then $U(g)_B \theta_\gamma v_\lambda \subseteq M'$, but the inclusion can be strict. Indeed this happens when $g = sl(2, 1)$ and $\lambda = -\rho$ [13], see also [11] Exercise 10.5.4.
7. The Jantzen Sum Formula

The Jantzen sum formula for a semisimple Lie algebra expresses the sum of the characters of the terms in the Jantzen filtration as a sum of characters of Verma modules. There is a version of the formula for contragredient Lie superalgebras in [11] Theorem 10.3.1, but it contains some terms that are not characters of Verma modules. Here we see that these extra terms are actually characters of the modules $M^\gamma(\lambda)$ introduced in Theorem 6.1.

For $\lambda \in \mathfrak{h}^*$ define

$$A(\lambda) = \{ \alpha \in \Delta_0^+ | (\lambda + \rho, \alpha^\vee) \in \mathbb{N} \setminus \{0\} \};$$

$$B(\lambda) = \{ \alpha \in \Delta_1^+ | (\lambda + \rho, \alpha) = 0 \}.$$

Now we state our improved version of the Jantzen sum formula. At the same time, rather than using characters as in [11], it is useful to rewrite the result using the Grothendieck group $K(O)$ of the category $O$. We define $K(O)$ to be the free abelian group generated by the symbols $[L(\lambda)]$ for $\lambda \in \mathfrak{h}^*$. If $M \in O$, the class of $M$ in $K(O)$ is defined as $[M] = \sum_{\lambda \in \mathfrak{h}^*} [M : L(\lambda)][L(\lambda)]$, where $[M : L(\lambda)]$ is the multiplicity of the composition factor $L(\lambda)$ in $M$.

**Theorem 7.1.** For all $\lambda \in \mathfrak{h}^*$

$$\sum_{i > 0} [M_i(\lambda)] = \sum_{\alpha \in A(\lambda)} [M(s_\alpha \cdot \lambda)] + \sum_{\gamma \in B(\lambda)} [M^\gamma(\lambda - \gamma)]. \quad (11)$$

**Proof.** Combine Theorem 6.1 with the result from [11] Theorem 10.3.1. \hfill \Box

The advantage of using this version of the formula is that $K(O)$ has a natural partial order. For $A, B \in O$ we write $A \geq B$ if $[A] - [B]$ is a linear combination of classes of simple modules with non-negative integer coefficients. Clearly if $B$ is a subquotient of $A$ we have $[A] \geq [B]$.

8. Orthogonal Isotropic Roots

We consider the structure of $M(\lambda)$ when $B(\lambda)$ consists of two orthogonal roots $\gamma, \gamma'$. We say that $\lambda \in \mathcal{H}_\gamma \cap \mathcal{H}_{\gamma'}$ is *weakly generic* if $A(\lambda) = \emptyset$, and $B(\lambda) = \{\gamma, \gamma'\}$, and *generic* if $\mu$ is weakly generic for all $\mu \in \lambda + \mathbb{Z} \gamma + \mathbb{Z} \gamma'$. Choose $\xi \in \mathfrak{h}^*$ such that $(\xi, \gamma) = 0$ for all $\gamma \in X$, and $(\xi, \alpha) \neq 0$ for all even roots $\alpha$.

**Lemma 8.1.** For $\lambda \in \mathcal{H}_\gamma \cap \mathcal{H}_{\gamma'}$ there are only finitely many $c \in k$ such that

$$\theta_\gamma(\lambda + c\xi - \gamma) \theta_{\gamma'}(\lambda + c\xi) = 0 \quad (12)$$

or

$$\theta_{\gamma'}(\lambda + c\xi - \gamma) \theta_\gamma(\lambda + c\xi) = 0. \quad (13)$$

**Proof.** Set $\tilde{\lambda} = \lambda + T\xi$. It follows from (4) and Corollary 2.2 that when $\theta_\gamma(\tilde{\lambda} - \gamma) \theta_{\gamma'}(\tilde{\lambda})v_{\tilde{\lambda}}$ is written as an $A$-linear combination of terms $e_{-\gamma}v_{\tilde{\lambda}}$, the coefficient of $e_{-\gamma}v_{\tilde{\lambda}}$ is a polynomial in $T$ of degree $d_\gamma + d_{\gamma'}$. Hence there are only finitely many $c$ such that Equation (12) holds, and a similar argument applies to Equation (13). \hfill \Box

It follows from Lemma 8.1 that the set

$$\Lambda = \{ \lambda \text{ generic in } \mathcal{H}_\gamma \cap \mathcal{H}_{\gamma'} | \theta_{\gamma'}(\lambda - \gamma) \theta_\gamma(\lambda) \neq 0 \neq \theta_\gamma(\lambda - \gamma') \theta_{\gamma'}(\lambda) \}$$

...
is Zariski dense in $\mathcal{H}_\gamma' \cap \mathcal{H}_\gamma$.

For $\lambda \in \Lambda$, the Jantzen sum formula (11) reads

\[ \sum_{i>0} [M_i(\lambda)] = [M^\gamma(\lambda - \gamma)] + [M^\gamma(\lambda - \gamma')]. \]  

(14)

Next we quote a result from [13] without proof.

**Lemma 8.2.** For $\lambda \in \Lambda$, 

\[ |M(\lambda) : L(\lambda - \gamma - \gamma')| = 1. \]

Now based on equation (14) we can show

**Lemma 8.3.** For $\lambda \in \Lambda$ we have

(a) If $|M(\lambda) : L(\mu)| > 0$, then $\mu \in \lambda - \mathbb{N}\gamma - \mathbb{N}\gamma'$.

(b) $|M^\gamma(\lambda) : L(\lambda)| = 1$.

(c) $|M^\gamma(\lambda - \gamma') : L(\lambda - r\gamma)| = 0$, for $r > 0$.

(d) $|M^\gamma(\lambda) : L(\lambda - \gamma)| = 0$.

(e) $\sum_{i>0} |M_i(\lambda) : L(\lambda - \gamma)| = 1$.

(f) $|M(\lambda) : L(\lambda - r\gamma)| = 0$ for $r \geq 2$.

(g) $|M(\lambda) : L(\lambda - \gamma - \gamma')| = |M_2(\lambda) : L(\lambda - \gamma - \gamma')| = 1$.

(h) $|M^\gamma(\lambda) : L(\lambda - \gamma - \gamma')| = 0$.

(i) $|M(\lambda) : L(\lambda - r\gamma - s\gamma')| = 0$ if $r + s \geq 3$.

**Proof.** (a) Follows from Equation (14) and induction on $\lambda - \mu$, while (b) holds since $M^\gamma(\lambda)$ is an image of $M(\lambda)$. We note that (c) follows from (a). To prove (d) set $L = L(\lambda - \gamma)$, and suppose for a contradiction, that $a = |M^\gamma(\lambda) : L| > 0$. Then

\[ a \leq |M(\lambda) : L| \leq |M^\gamma(\lambda - \gamma) : L| + |M^\gamma(\lambda - \gamma') : L| = 1, \]

by (b) and (c). Hence $a = |M(\lambda) : L| = 1$, but then

\[ |M(\lambda) : L| = |M^\gamma(\lambda) : L| + |M^\gamma(\lambda - \gamma) : L| = a + 1, \]

since in the Grothendieck group $K(O)$, we have

\[ [M(\lambda)] = [M^\gamma(\lambda)] + [M^\gamma(\lambda - \gamma)]. \]  

(15)

Now (e) follows from (b), (c) and (14), while (f) holds because for $r \geq 2$,

\[ |M(\lambda) : L(\lambda - r\gamma)| \leq |M^\gamma(\lambda - \gamma) : L(\lambda - r\gamma)| + |M^\gamma(\lambda - \gamma') : L(\lambda - r\gamma)| = 0, \]
by (c), (d) and induction. Next set $L = L(\lambda - \gamma - \gamma')$. To prove (g), it is enough, by Lemma 8.2 to show that $\sum_{i>0} |M_i(\lambda) : L| = 2$. First note that $|M(\lambda - 2\gamma) : L| = 0$ by (c), so $|M(\lambda - \gamma) : L| = |M(\lambda - \gamma) : L|$ and likewise $|M'(\lambda - \gamma') : L| = |M(\lambda - \gamma') : L|$. So by (e),

$$\sum_{i>0} |M_i(\lambda) : L| = |M(\lambda - \gamma) : L| + |M'\gamma(\lambda - \gamma') : L| = |M(\lambda - \gamma) : L| + |M(\lambda - \gamma') : L| = 2.$$

By the proof of (g) $|M(\lambda - \gamma) : L| = 1$, so (h) follows from (15). Finally (i) follows from (14) and induction on $r + s$. For example

$$|M(\lambda) : L(\lambda - \gamma - 2\gamma')| \leq |M(\lambda - \gamma) : L(\lambda - \gamma - 2\gamma')| + |M'\gamma(\lambda - \gamma') : L(\lambda - \gamma - 2\gamma')| = 0,$$

using (f) and (g).

Theorem 8.4. Suppose $\gamma$ and $\gamma'$ are orthogonal isotropic roots, and $\lambda \in \Lambda$. Define

$$V_1 = U(\mathbf{g})\theta_\gamma v_\lambda, \quad V_2 = U(\mathbf{g})\theta_{\gamma'} v_\lambda.$$

Then the Jantzen filtration $\{M_i = M_i(\lambda)\}_{i>0}$ on $M(\lambda)$ is given by

$$M_3 = 0, \quad M_2 = V_1 \cap V_2 \cong L(\lambda - \gamma - \gamma'), \quad M_1 = V_1 + V_2.$$

Moreover $M_1/M_2$ is the direct sum of $L(\lambda - \gamma)$ and $L(\lambda - \gamma')$.

Proof. By Lemma 8.3 the only composition factors of $M_1$ are $L(\lambda - \gamma), L(\lambda - \gamma')$, and $L(\lambda - \gamma - \gamma')$ each with multiplicity one. By (e) and (g) in the Lemma $L(\lambda - \gamma - \gamma') \subseteq M_2$ and

$$\sum_{i>0} |M_i| = |L(\lambda - \gamma)| + |L(\lambda - \gamma')| + 2|L(\lambda - \gamma - \gamma')|.$$

Note that $V_1$ and $V_2$ are generated by highest weight vectors of weights $\lambda - \gamma$ and $\lambda - \gamma'$. The result follows since for $\lambda \in \Lambda$, $\theta_{\gamma'}\theta_\gamma v_\lambda$ and $\theta_\gamma\theta_{\gamma'} v_\lambda$ are highest weight vectors with weight $\lambda - \gamma - \gamma'$.  

Corollary 8.5. There is a rational function $p$ of $\lambda \in \mathcal{H}_\gamma \cap \mathcal{H}_{\gamma'}$ such that

$$\theta_{\gamma'}(\lambda - \gamma)\theta_\gamma(\lambda) = p(\lambda)\theta_\gamma(\lambda - \gamma')\theta_{\gamma'}(\lambda).$$

Proof. The highest weight vectors referred to in the last sentence of the proof are necessarily proportional. Hence (16) holds since the coefficients of $e_{-\gamma} e_{-\gamma}$ in these highest weight vectors (when written as linear combinations of the $e_{-\pi}$ with $\pi \in P(\gamma + \gamma')$) are polynomials in $\lambda$. 

Corollary 8.5 is applied in the proof of Theorem 6.1 in the case $|X| \geq 2$. We sketch the main new ingredient when $X = \{\gamma, \gamma'\}$ as in the Corollary. In this case $M(\lambda)_B$ has a series of submodules

$$M(\tilde{\lambda})_B = W_0 \supset W_1 \supset W_2 \supset W_3 \supset W_4 = 0,$$

where

$$W_1 = U(\mathbf{g})B\theta_{\gamma'} v_\lambda + U(\mathbf{g})B\theta_\gamma v_\lambda, \quad W_2 = U(\mathbf{g})B\theta_\gamma v_\lambda, \quad W_3 = U(\mathbf{g})B\theta_{\gamma'} v_\lambda.$$

From the Corollary and Theorem 5.1 we deduce that $\theta_{\gamma'} v_\lambda \in W_2$, $\theta_{\gamma'} v_\lambda \in W_3$ and $\theta_{\gamma'} v_\lambda = 0$. Thus from Theorem 5.1 and Lemma 6.2 we obtain bounds on the dimensions of the weight spaces of the factors in the series (17) which are analogous to (10). The statement about characters follows as before.
9. Non-Orthogonal Isotropic Roots
If \( \gamma, \gamma' \) are non-orthogonal isotropic roots, then \( \gamma' = s_{\alpha} \gamma \) for some even root \( \alpha \). In this situation we relate the Šapovalov elements for \( \gamma, \gamma' \) and \( \alpha \). We set \( \theta_{\alpha,0} = 1 \) and \( Q^+ = \sum_{\alpha \in N} N \alpha \).

**Theorem 9.1.** Let \( \gamma \) be a positive isotropic root and \( \alpha \) a non-isotropic root contained in \( Q^+ \). Let \( v_\lambda \) be a highest weight vector in a Verma module with highest weight \( \lambda \), and set \( \gamma' = s_{\alpha} \gamma \). Suppose \( p = (\lambda + \rho, \alpha') \in N \setminus \{0\} \). Then

(a) If \( (\lambda + \rho, \gamma') = 0 \) we have

\[
\theta_\gamma \theta_{\alpha,p} v_\lambda = \theta_{\alpha,p+1} \theta_{\gamma'} v_\lambda. \tag{18}
\]

(b) If \( (\lambda + \rho, \gamma) = 0 \), and \( p - 1 \geq 0 \), we have

\[
\theta_\gamma \theta_{\alpha,p} v_\lambda = \theta_{\alpha,p-1} \theta_{\gamma'} v_\lambda. \tag{19}
\]

**Proof.** It suffices to prove (a) for all \( \lambda \) in the Zariski dense subset \( \Lambda \) of \( \mathcal{H}_{\gamma'} \cap \mathcal{H}_{\alpha,p} \) given by

\[
\Lambda = \{ \lambda \in \mathcal{H}_{\gamma'} \cap \mathcal{H}_{\alpha,p} \mid A(\lambda) = \{\alpha\}, \ B(\lambda) = \{\gamma'\} \}.
\]

However for \( \lambda \in \Lambda \), \( M(\lambda) \) contains a unique highest weight vector of weight \( s_{\alpha} \cdot \lambda - \gamma \). Thus, \( \theta_\gamma \theta_{\alpha,p} v_\lambda \) and \( \theta_{\alpha,p+1} \theta_{\gamma'} v_\lambda \) are equal up to a scalar multiple. If \( \pi^0 \) is the partition of \( \rho \alpha + \gamma \) with \( \pi^0(\sigma) = 0 \) for all non-simple roots \( \sigma \), it follows easily from the definition of Šapovalov elements, that \( e_{-\rho^0} v_\lambda \) occurs with coefficient equal to one in both \( \theta_\gamma \theta_{\alpha,p} v_\lambda \) and \( \theta_{\alpha,p+1} \theta_{\gamma'} v_\lambda \), and this gives the desired result. The proof of (b) is similar.

**Remark 9.2.** The most interesting case of Equation (18) arises when \( p = 0 \), since then we have an inclusion between submodules of a Verma module obtained by multiplying the highest weight vector \( v_\lambda \) by \( \theta_\gamma \) and \( \theta_{\gamma'} \). Similarly the most interesting case of Equation (19) is when \( p = 1 \).

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**References**
[1] Bernstein I N, Gelfand I M and Gelfand S I 1971 Structure of representations that are generated by vectors of higher weight Funkcional Anal. i Priložen. 5 1–9
[2] Bernstein I N, Gelfand I M and Gelfand S I 1975 Differential operators on the base affine space and a study of \( g \)-modules Lie groups and their representations (Proc. Summer School Bolyai János Math. Soc. Budapest 1971) (Halsted, New York) 21–64
[3] Brundan J 2014 Representations of the general linear Lie superalgebra in the BGG category \( \mathcal{O} \) Developments and Retrospectives in Lie Theory ed G Mason, I Penkov and J A Wolf (Berlin: Springer) 71–98
[4] Brundan J, Losev I and Webster B Tensor product categorifications and the super Kazhdan-Lusztig conjecture (arXiv:math/1310.0349)
[5] Cheng S-J, Wang W and Lam N 2008 Brundan-Kazhdan-Lusztig and super duality conjectures Publ. Res. Inst. Math. Sci. 44 1219–1272
[6] Gorelik M 2004 The Kac construction of the centre of \( \mathcal{U}(\mathfrak{g}) \) for Lie superalgebras J. Nonlinear Math. Phys. 11 325–349
[7] Humphreys J E 2008 Representations of Semisimple Lie Algebras in the BGG Category \( \mathcal{O} \) (Providence: American Mathematical Society)
[8] Jantzen J C 1979 Moduln mit einem höchsten Gewicht (Berlin: Springer)
[9] Kac V G 1977 Lie superalgebras Advances in Math. 26 8–96
[10] Kazhdan D and Lusztig G 1979 Representations of Coxeter groups and Hecke algebras Invent. Math. 53
[11] Musson I M 2012 Lie Superalgebras and Enveloping Algebras (Providence: American Mathematical Society)
[12] Musson I M Coefficients of Sapovalov elements for simple Lie algebras and contragredient Lie superalgebras (arXiv:math/1311.0570)
[13] Musson I M The Jantzen filtration and sum formula for basic classical Lie superalgebras (in preparation)
[14] Sapovalov N N 1972 A certain bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra Funkcional. Anal. i Priložen. 6 65–70
[15] Verma D-N 1968 Structure of certain induced representations of complex semisimple Lie algebras Bull. Amer. Math. Soc. 74 160–166