Lauricella hypergeometric function and its application to the solution of the Neumann problem for a multidimensional elliptic equation with several singular coefficients in an infinite domain

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Abstract. At present, the fundamental solutions of the multidimensional elliptic equation with the several singular coefficients are known and they are expressed in terms of the Lauricella hypergeometric function of many variables. In this paper we study the Neumann problem for a multidimensional elliptic equation with several singular coefficients in the infinite domain. Using the method of the integral energy the uniqueness of solution has been proved. In the course of proving the existence of the explicit solution of the Neumann problem, a differentiation formula, some adjacent and limiting relations for the Lauricella hypergeometric functions and the values of some multidimensional improper integrals are used.

Keywords: Neumann problem; multidimensional elliptic equations with several singular coefficients; Lauricella hypergeometric function of many variables; adjacent and limiting relations; multidimensional improper integrals.

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1 Introduction

It is known that the theory of boundary value problems for degenerate equations and equations with singular coefficients is one of the central sections of the modern theory of partial differential equations, which is encountered in solving many important problems of applied nature, for example, [1, 2]. Omitting the huge bibliography in which various local and nonlocal boundary value problems for equations of mixed type containing elliptic equations with singular coefficients are studied, we note the works that are most closely related to this work.

Let $\mathbb{R}_m$ be the $m$-dimensional Euclidean space ($m \geq 2$), $x := (x_1, ..., x_m)$ - arbitrary point in it and $n$ is a natural number, and $n \leq m$. The $2^n$-th part of the Euclidean space $\mathbb{R}_m$ is defined as follows:

$$
\Omega \equiv \Omega_m^{n^+} = \{x \in \mathbb{R}_m : x_i > 0, i = 1, ..., n; -\infty < x_j < +\infty, j = n + 1, ..., m\}. \quad (1)
$$

Fundamental solutions have an essential role in studying partial differential equations. The explicit form of the fundamental solution makes it possible to correctly formulate the problem statement and to study in detail the various properties of the solution of the equation under consideration. Fundamental solutions of singular elliptic equations are directly connected with multiple hypergeometric functions, the number of variables of which is determined by the number of singular coefficients. Indeed, all fundamental solutions of the
following elliptic equation with \( n \) singular coefficients

\[
E^{(m,n)}_{\alpha}(u) = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} + \sum_{j=1}^{n} 2\alpha_j \frac{\partial u}{\partial x_j} = 0 \quad (2)
\]

in the hyperoctant \( \Omega \) are expressed [3] by the Lauricella hypergeometric function \( F_A^{(n)} \) in \( n \) variables [4] where \( m \geq 2 \) is a dimension of the Euclidean space; \( n \geq 1 \) is a number of the singular coefficients; \( m \geq n \); \( \alpha_j \) are real constants and \( 0 < 2\alpha_j < 1 \) \((j = 1, n)\).

In a recent paper [5], the generalized Holmgren problem for equation (2) in some part of the first hyperoctant of the ball (in the finite domain) is written out in an explicit form and in case \( m = 3 \) and \( n = 1 \) the potential theory in the domain bounded in a half-space is constructed [6].

Relatively few works are devoted to the study of boundary value problems for spatial singular elliptic equations in infinite domains. We note the joint work of M. Salakhidinov and A. Hasanov [7], in which solutions of the Dirichlet and Holmgren problems for a multidimensional elliptic equation with one singular coefficient in the half-space were found in an explicit forms. For three- and four-dimensional elliptic equations with two [8], three [9] and four [10] singular coefficients boundary value problems in infinite domains are investigated. Recently [11], a solution to the external Dirichlet problem for an equation (2) in the hyperoctant \( \Omega \) is found explicitly.

In this paper, we study the Neumann problem for equation (2) in the infinite domain \( \Omega \). The outer Neumann problem for equation (2) on the plane has been studied in details by many authors (see, for example, [12]), therefore, for definiteness, we set \( m > 2 \) in this paper. Initially we introduce some formulas and definitions, then we proceed to solve boundary-value problem.

2 Preliminaries

Below we give some formulas for Euler gamma-function, Gauss hypergeometric function, Lauricella hypergeometric functions of three and more variables, which will be used in the next sections.

Let be \( N \) set of the natural numbers : \( N = \{1, 2, 3, \ldots\} \).

It is known that the Euler gamma-function \( \Gamma(a) \) has property [13, eq. 1.2(2)]

\[
\Gamma(a + m) = \Gamma(a)(a)_m.
\]

Here \((a)_m\) is a Pochhammer symbol, for which the equality \((a)_{m+n} = (a)_m(a + m)_n\) and its particular case \((a)_{2m} = (a)_m(a + m)_m\) is true.

For the Gamma function \( \Gamma(a) \) the Legendre’s duplication formula [13, eq. 1.2(15)] :

\[
\Gamma(2a) = \frac{2^{2a-1}}{\sqrt{\pi}} \Gamma(a) \Gamma\left( a + \frac{1}{2} \right) \quad (3)
\]

is valid.
A function
\[
F(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \quad c \neq 0, -1, -2, ...
\]
is known as the Gaussian hypergeometric function.

Multiple Lauricella hypergeometric function \(F^{(n)}_A\) in \(n \in \mathbb{N}\) (real or complex) variables is defined as following [14, eq. 1.4(1)]:

\[
\begin{align*}
F^{(n)}_A(a, b; c; x) &= \sum_{|\mathbf{k}|=0}^{\infty} \frac{(a)_{|\mathbf{k}|}(b_1)_{k_1} \cdots (b_n)_{k_n}}{(c_1)_{k_1} \cdots (c_n)_{k_n} k_1! \cdots k_n!} x^{k_1} \cdots x^{k_n} \\
&\quad \text{where}
\end{align*}
\]

\[
[c_k \neq 0, -1, -2, \ldots; i = 1, n; \ |x_1| + \ldots + |x_n| < 1]
\]

The function \(F^{(n)}_A\) satisfies the following differentiation formula [4]

\[
\frac{\partial}{\partial x_k} F^{(n)}_A(a, b; c; x) = \frac{ab_k}{c_k} F^{(n)}_A(a + 1, b_k; c_k; x)
\]
and adjacent relation [5]

\[
\sum_{k=1}^{n} \frac{a_k}{b_k} x_k F^{(n)}_A(a + 1, b_k; c_k; x) = F^{(n)}_A(a + 1, b; c; x) - F^{(n)}_A(a, b; c; x),
\]

where \(a_k\) and \(c_k\) are vectors obtained, respectively, from vectors \(b\) and \(c\) by increasing the \(k\)-th component by one (\(k = 1, n\)).

**Lemma 1** [11] [15] Let \(a, b_k\) and \(c_k\) are real numbers with \(c_k \neq 0, -1, -2, \ldots, a > b_1 + \ldots + b_n\) and \(c_k > b_k\) (\(k = 1, n\)). Then for \(n = 1, 2, \ldots\) the following limiting relation holds true

\[
\begin{align*}
\lim_{\varepsilon \to 0} & -b_1 - \ldots - b_n F^{(n)}_A(a, b; c; 1 - \frac{z_1(\varepsilon)}{\varepsilon}, \ldots, 1 - \frac{z_n(\varepsilon)}{\varepsilon}) \\
&= \frac{1}{\Gamma(a)} \Gamma \left( a - \sum_{k=1}^{n} b_k \right) \prod_{k=1}^{n} \left[ z_k(0) \right]^{b_k} \Gamma(c_k - b_k).
\end{align*}
\]

where \(z_k(\varepsilon)\) are arbitrary functions with \(z_k(0) \neq 0\).

**Lemma 2** If \(p_k, q_k, r_k, s, t,\) are real numbers and

\[
p_k > 0, \quad q_k > 0, \quad r_k > 0, \quad s > 0, \quad 0 < \frac{p_1}{q_1} + \ldots + \frac{p_n}{q_n} - t < s, \quad k = 1, n,
\]

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then the following equality holds true

\[
\int_0^\infty \ldots \int_0^\infty \frac{x_1^{p_1-1} \ldots x_n^{p_n-1} dx_1 \ldots dx_n}{[ (r_1 x_1)^{q_1} + \ldots + (r_n x_n)^{q_n} ]^t \left[ 1 + (r_1 x_1)^{q_1} + \ldots + (r_n x_n)^{q_n} \right]^s} = \Gamma \left( \frac{p_1}{q_1} \right) \ldots \Gamma \left( \frac{p_n}{q_n} \right) \Gamma \left( \frac{p_1}{q_1} + \ldots + \frac{p_n}{q_n} - t \right) \Gamma \left( s + t - \frac{p_1}{q_1} - \ldots - \frac{p_n}{q_n} \right) \frac{q_1 q_2 \ldots q_n \Gamma \left( \frac{p_1}{q_1} + \ldots + \frac{p_n}{q_n} \right) \Gamma \left( \frac{s}{q_1} \right)}{q_1 q_2 \ldots q_n \Gamma \left( \frac{p_1}{q_1} + \ldots + \frac{p_n}{q_n} \right) \Gamma \left( \frac{s}{q_1} \right)}.
\]

(7)

**Proof.** Into (7) we make a replacement of variables

\[
(r_1 x_1)^{q_1/2} = r \cos \phi_1,
\]

\[
(r_2 x_2)^{q_2/2} = r \sin \phi_1 \cos \phi_2,
\]

\[
(r_3 x_3)^{q_3/2} = r \sin \phi_1 \sin \phi_2 \cos \phi_3,
\]

\[
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ...)
3 Statement of the problem and an uniqueness theorem

Consider equation (2) in the infinite domain $\Omega$ defined in (1).

We introduce the following notation:

\[ x := (x_1, ..., x_m) \in \mathbb{R}^m; \quad R^2 := \sum_{i=1}^{m} x_i^2; \quad dx := \prod_{i=1}^{m} dx_i; \quad x^{(2\alpha)} := \prod_{j=1}^{n} x_j^{2\alpha_j}; \]

\[ \tilde{x}_k := (x_1, ..., x_{k-1}, x_{k+1}, ..., x_m) \in \mathbb{R}^{m-1}; \quad d\tilde{x}_k := \frac{dx}{dx_k}; \quad \tilde{x}_k^{(2\alpha)} := \frac{x^{(2\alpha)}}{x_k^{2\alpha_k}}; \]

\[ x^0_k := (x_1, ..., x_{k-1}, 0, x_{k+1}, ..., x_m) \in \mathbb{R}^m; \]

\[ S_k = \{ x : x_1 > 0, ..., x_{k-1} > 0, x_k = 0, x_{k+1} > 0, ..., x_n > 0, \]

\[ -\infty < x_{n+1} < +\infty, ..., -\infty < x_m < +\infty \}, \quad m \geq 2, \quad 1 \leq k \leq n \leq m. \]

The Neumann problem. Find a regular solution $u(x)$ of equation (2) from the class $C^1(\Omega) \cap C^2(\Omega)$, satisfying the conditions:

\[ \left( x_k^{2\alpha_k} \frac{\partial u}{\partial x_k} \right)_{x_k=0} = \nu_k(\tilde{x}_k), \quad \tilde{x}_k \in S_k, \]  

\[ \lim_{R \to \infty} u(x) = 0, \]  

where $\nu_k(\tilde{x}_k)$ are given continuous functions. The functions $\nu_k(\tilde{x}_k)$ can also turn to infinity of order less than $1 - 2\alpha_k$ and for sufficiently large values of $R$ the inequalities are valid

\[ |\nu_k(\tilde{x}_k)| \leq \frac{c_k}{(1 + x_1^2 + ... + x_{k-1}^2 + x_{k+1}^2 + ... + x_m^2)(1 - 2\alpha_k + \varepsilon_k)/2}, \]  

where $c_k = const > 0$, $0 < 2\alpha_k < 1$, and $\varepsilon_k$ are small enough positive numbers ($k = \overline{1,n}$).

Theorem 1. The Neumann problem has not more than one solution.

Proof. Suppose by contradiction. Let there be two solution $u_1$ and $u_2$ of the Neumann problem. We will denote via $u = u_1 - u_2$. Then it is clear that the function $u$ satisfies the equation (2) and homogeneous boundary conditions (8) and condition (9).

By $D_R$ we denote a bounded domain with a boundary $\partial D_R = \bigcup_{k=1}^{n} S_{Rk}$, where

\[ S_{Rk} = \{ x : 0 < x_1 < R, ..., 0 < x_{k-1} < R, x_k = 0, 0 < x_{k+1} < R, ..., 0 < x_n < R, \]

\[ -R < x_{n+1} < R, ..., -R < x_m < R \} \]

\[ \sigma_R := \{ x : x_1^2 + ... + x_m^2 = R^2, \quad x_1 > 0, ..., x_n > 0 - R < x_{n+1} < +R, ..., -R < x_m < R \}. \]
Choosing $R$ big enough, we integrate equation (2) over the domain $D_R$, having previously multiplied it by the function $u(x)$, we obtain

$$\int_{D_R} x^{(2\alpha)} u \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} dx = 0. \quad (11)$$

Taking into account in (11) the following equalities

$$u \frac{\partial^2 u}{\partial x_k^2} = \frac{\partial}{\partial x_k} \left( u \frac{\partial u}{\partial x_k} \right) - \left( \frac{\partial u}{\partial x_k} \right)^2, \quad k = 1, n,$$

after applying the Gauss-Ostrogradsky formula, we have

$$\int_{D_R} x^{(2\alpha)} m \sum_{i=1}^{m} \left( \frac{\partial u}{\partial x_i} \right)^2 dx = \int_{\sigma_R} x^{(2\alpha)} u \frac{\partial u}{\partial N} dS, \quad (12)$$

where

$$\frac{\partial u}{\partial N} = \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \cos (N, x_k); \quad \cos (N, x_k) dS = d\tilde{x}_k, \quad k = 1, n,$$

$N$ is outer normal to $\partial D_R$.

By virtue of condition (9), at $R \to \infty$, taking into account that

$$\lim_{R \to \infty} \int_{\sigma_R} x^{(2\alpha)} u \frac{\partial u}{\partial N} dS = 0$$

from (12), we have

$$\int_{D_R} x^{(2\alpha)} m \sum_{i=1}^{m} \left( \frac{\partial u}{\partial x_i} \right)^2 dx = 0. \quad (13)$$

From (13) we obtain $\frac{\partial u}{\partial x_i} = 0, \quad (i = 1, m)$ which means $u = \text{const}$. From the condition (9) follows that $u \equiv 0$. So, we have proved the uniqueness theorem for the Neumann problem.

### 4 Existence of a solution of the Neumann problem

Consider the function

$$u(\xi) := u(\xi_1, ..., \xi_m) = -\sum_{k=1}^{n} \int_{S_k} \tilde{x}_k^{(2\alpha)} \nu_k(\tilde{x}_k) q(x_0, \xi) dS_k, \quad (14)$$

where $\nu_k(\tilde{x}_k)$ are functions, defined in (8) and $q(x, \xi)$ is fundamental solution of equation (2) [3, 17]:
\[ q(x, \xi) = \gamma r^{-2\beta} F_A^{(n)} \left( \beta, \alpha_1, \ldots, \alpha_n; 2\alpha_1, \ldots, 2\alpha_n; -\frac{4x_1\xi_1}{r^2}, \ldots, -\frac{4x_n\xi_n}{r^2} \right), \]

\[ \beta := \frac{m - 2}{2} + \alpha, \quad \gamma = 2^{2\beta - m} \frac{\Gamma(\beta)}{\pi^{m/2}} \prod_{k=1}^{n} \frac{\Gamma(\alpha_k)}{\Gamma(2\alpha_k)}, \quad r^2 = \sum_{i=1}^{m} (x_i - \xi_i)^2. \]  

(15)

Here

\[ \int_{S_k} f(x, \xi) dS_k := \int_{0}^{+\infty} \ldots \int_{-\infty}^{0} \int_{-\infty}^{+\infty} \int_{-\infty}^{0} f(x, \xi) dx_1 \ldots dx_{k-1} dx_{k+1} \ldots dx_n dx_{n+1} \ldots dx_m. \]

It is easy to check that the fundamental solution \( q(x, \xi) \) has the properties

\[ \frac{\partial q(x, \xi)}{\partial x_k} \bigg|_{x_k=0} = 0, \quad k = 1, n. \]

Let us prove that the function (14) is a solution to the Neumann problem. It is clear that the function (14) in the arguments \( \xi \) satisfies the equation (2) (for details, see ). Let us show that the function (14) also satisfies the conditions of the Neumann problem. Due to the fact that \( \sigma_k|_{x_k=0} = 0 \), the expression (14) is converted to the form

\[ u(\xi) = \sum_{j=1}^{n} I_j(\xi), \]

(16)

where

\[ I_j(\xi) = -\gamma \int_{S_j} \nu_j(\tilde{x}_j) \tilde{x}_j^{(2\alpha)} r_j^{-2\beta} F_A^{(n-1)}(\beta, A_j; 2A_j; \Phi_j) dS_j, \]

(17)

\[ A_j := (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_n), \quad \Phi_j := (\phi_{1,j}, \ldots, \phi_{j-1,j}, \phi_{j+1,j}, \ldots, \phi_{n,j}), \]

\[ \phi_{i,j} := -\frac{4x_i\xi_i}{r_j^2}, \quad i \neq j, \quad r_j^2 := r_j^2|_{x_j=0}, \quad i, j = 1, n. \]

Let us show that the function (17) satisfies the conditions (8) of the Neumann problem as for \( j = k \) and for \( j \neq k \).

Let \( j = k \). Using the differentiation formula (4) and adjacent relation (5) for Lauricella hypergeometric function, we obtain

\[ \frac{\partial I_k}{\partial \xi_k} = 2\beta \gamma \xi_k \int_{S_k} \nu_k(\tilde{x}_k) r_k^{-2\beta-2} \tilde{x}_k^{(2\alpha)} F_A^{(n-1)}(\beta + 1, A_k; 2A_k; \Phi_k) dS_k. \]

(18)

On the right-hand side of the equality (18), we make the change of variables
then we have
\[
x_i = \xi_i + \xi_k t_i, \quad i = 1, m, \quad i \neq k,
\]

\[
\xi_k^{2\alpha_k} \frac{\partial I_k}{\partial \xi_k} = 2\beta \gamma \int_{T_k} \nu_k \left( \xi_1 + \xi_k t_1, \ldots, \xi_{k-1} + \xi_k t_{k-1}, \xi_{k+1} + \xi_k t_{k+1}, \ldots, \xi_m + \xi_k t_m \right) \frac{1 + t_1^2 + \ldots + t_{k-1}^2 + t_{k+1}^2 + \ldots + t_m^2)^{\beta+1}}{(1 + t_1^2 + \ldots + t_{k-1}^2 + t_{k+1}^2 + \ldots + t_m^2)^{\beta+1}} \times \prod_{s=1, s \neq k}^{n} \left( \frac{\xi_s + \xi_k t_s}{\xi_k} \right)^{2\alpha_s} F_A^{(n-1)} \left[ \frac{\beta + 1, \alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n; 1 - \frac{z_1(\tilde{t}_k, \xi_1, \xi_k)}{\xi_k^2}, \ldots, 1 - \frac{z_{k-1}(\tilde{t}_k, \xi_{k-1}, \xi_k)}{\xi_k^2}, 1 - \frac{z_{k+1}(\tilde{t}_k, \xi_{k+1}, \xi_k)}{\xi_k^2}, \ldots, 1 - \frac{z_n(\tilde{t}_k, \xi_n, \xi_k)}{\xi_k^2} \right] dT_k,
\]

where
\[
z_i(\tilde{t}_k, \xi_i, \xi_k) := \frac{4\xi_i (\xi_i + \xi_k t_i) + (1 + t_1^2 + \ldots + t_{k-1}^2 + t_{k+1}^2 + \ldots + t_m^2) \xi_k^2}{1 + t_1^2 + \ldots + t_{k-1}^2 + t_{k+1}^2 + \ldots + t_m^2},
\]

\[
\tilde{t}_k := (t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n, t_m),
\]

\[
\int_{T_k} \cdots dT_k := \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots dt_1 \cdots dt_{k-1} dt_{k+1} \cdots dt_n dt_{n+1} \cdots dt_m.
\]

In the both sides of the equality (19), we pass to the limit as \( \xi_k \to 0 \). Taking into account formulae (6) and (7), Legendre’s duplication formula (3) and the expression for the coefficient \( \gamma \) by (15), we have

\[
\lim_{\xi_k \to 0} \xi_k^{2\alpha_k} \frac{\partial I_k}{\partial \xi_k} = \nu_k \left( \hat{\xi}_k \right).
\]

Let now \( l \neq k(l, k = 1, n) \). For definiteness, we put \( l < k \). Repeating the reasoning for obtaining the formula (18), we have

\[
\frac{\partial I_k}{\partial \xi_l} = 2\beta \gamma \int_{S_k} \nu_k \left( \tilde{x}_k \right) (x_l - \xi_l)^{-2\beta - 2(2\alpha)} \tilde{a}_k^{(2\gamma)} F_A^{(n-1)} \left[ \beta + 1, \alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n; \sigma_k \right] dS_k.
\]

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\[-2\beta \gamma \int_{S_k} \nu_k (\tilde{x}_k) \, r_k^{2\beta - 2} \, x_k \, (2\alpha) \, F_\alpha^{(n-1)} \left[ \beta + 1, \alpha_1, \ldots, \alpha_l + 1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n; \sigma_k \right] dS_k.\]

Now using the obvious equality

\[(x_l - \xi_l) \, F_\alpha^{(n-1)} \left[ 1 + \beta, \alpha_1, \ldots, \alpha_l, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n; \Phi_k \right] \bigg|_{\xi_l=0} = x_l \, F_\alpha^{(n-1)} \left[ 1 + \beta, \alpha_1, \ldots, 1 + \alpha_l, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n; \Phi_k \right] \bigg|_{\xi_l=0}
\]

it is easy to prove that

\[
\lim_{\xi_l \to 0} \xi_l^{2\alpha_l} \frac{\partial I_k}{\partial \xi_l} = 0, \quad l < k. \tag{21}
\]

In this way,

\[
\lim_{\xi_l \to 0} \xi_l^{2\alpha_l} \frac{\partial I_k}{\partial \xi_l} = 0, \quad l > k. \tag{22}
\]

Therefore, by virtue of equalities (20), (21) and (22), we conclude that the function (14) satisfies the conditions (8) of the Neumann problem.

Next, we show that if the given functions \(\nu_k (\tilde{x}_k)\) for sufficiently large values of the arguments satisfy the inequalities (10), then the solution (14) of the Neumann problem also satisfies the condition (9).

Indeed, let the inequalities (10) hold, then in the equalities (17) we make the following changes of variables

\[y_i = \frac{1}{R_0} x_i, \quad \eta_i = \frac{1}{R_0} \xi_i, \quad i = 1, m,\]

where \(R_0^2 := \xi_1^2 + \ldots + \xi_m^2.\)

Then, by virtue of (17) we have the following inequalities for \(R_0 \to \infty\)

\[
\lim_{R_0 \to \infty} \left| I_k (\xi) \right| \leq \frac{2^{m-n} \gamma c_k}{R_0^{2\alpha}} \int_0^{+\infty} \ldots \int_0^{+\infty} \frac{dy_1 \ldots dy_{k-1} \ldots dy_m}{Y^{1-2\alpha_l + \epsilon_k} (1 + Y^2)^\beta}, \tag{23}
\]

where \(Y^2 := y_1^2 + \ldots + y_{k-1}^2 + y_{k+1}^2 + \ldots + y_m^2.\)

Now we will show that \((m - 1)\) - dimensional integrals occurring in inequalities (23), are limited.

Indeed, by virtue of the value of the integral (7), the inequalities (23) imply

\[
\lim_{R_0 \to \infty} \left| I_k (\xi) \right| \leq \frac{\tilde{c}_k}{R_0^{2\alpha}}, \tag{24}
\]

}\]
where $\tilde{c}_k$ are constants. Due to the inequalities (24) from the expression (16), we finally get

$$|I_k(\xi)| \leq \frac{c}{R^\varepsilon}, \quad c = \text{const}, \quad \varepsilon = \min_{1 \leq k \leq n} \{\varepsilon_k\}.$$  

The last inequality shows that the solution (14) vanishes for $R_0 \to \infty$. Therefore, condition (9) of the Neumann problem is satisfied. Thus, the solution (14) satisfies all the conditions of the Neumann problem.

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