Loop conditions

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Abstract

We prove that the existence of a term $s$ satisfying $s(x, y, y, z, z, x) = s(y, x, z, y, x, z)$ is the weakest non-trivial strong Maltsev condition given by a single identity.

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1 Introduction

One of the most interesting by-products of the research on the fixed-template constraint satisfaction problems is the result of Mark Siggers [11], which says that there exists a weakest non-trivial strong Maltsev condition for idempotent locally finite varieties.

Let us recall that a strong Maltsev condition (see [5, 9, 4]) is a condition for a variety (or an algebra) postulating the existence of finitely many terms satisfying a given finite set of identities. Such conditions can be compared by their strength: a condition $C$ is weaker than $D$ if each variety satisfying $D$ also satisfies $C$. The weakest conditions are those satisfied in every variety, we call them trivial. The concept of strength can be naturally relativized to special types of varieties, such as idempotent locally finite varieties in the above mentioned result of Siggers.

His weakest non-trivial condition has an especially simple form: it is given by a single linear identity in one operation symbol appearing on both sides, namely

$s(x, y, y, z, z, x) = s(y, x, z, y, x, z)$.

A well known fact, stated here as Proposition [1] is that strong Maltsev conditions of this form can be characterized by the existence of a loop in certain binary relations compatible with algebras in the variety. This structural property of compatible relations proved useful (see [1, 2]) and inspired the following terminology.

Definition 1. A loop condition is a strong Maltsev condition given by a single identity of the form

$t(x_1, \ldots, x_n) = t(y_1, \ldots, y_n),$

where $x_1, \ldots, x_n, y_1, \ldots, y_n$ are variables.
In idempotent locally finite varieties, the 6-ary loop condition by Siggers is equivalent to many other loop conditions, such as the existence of a 4-ary term $t$ satisfying $t(r,a,r,e) = t(a,r,e,a)$ [7].

What happens when we drop the local finiteness assumption? While there is no weakest non-trivial strong Maltsev condition for general varieties [12, 8], it turned out that there is one for idempotent varieties [10]; for instance, the existence of a 6-ary $t$ satisfying $t(x,y,y,y,x,x) = t(y,x,y,x,y) = t(y,y,x,x,x,y)$. Such a condition cannot be a loop condition: A. Kazda [6] proved that the free idempotent algebra over $\{x, y\}$ in the signature consisting of one ternary operation symbol $w$ modulo the weak near unanimity identities $w(x,y,y) = w(y,x,y) = w(y,y,x)$ does not satisfy any non-trivial loop condition.

On the other hand, a consequence of the main result of this paper, Theorem 1, is that the existence of a 6-ary Siggers term is a weakest non-trivial loop condition in general, that is, for varieties that are not necessarily idempotent or locally finite. To describe the main result, it will be convenient to assign a directed graph (digraph) to each loop condition in a natural way:

**Definition 2.** Let $C$ be a loop condition given by an identity $t(x_1,\ldots,x_n) = t(y_1,\ldots,y_n)$, where $x_1,\ldots,x_n, y_1,\ldots,y_n$ are variables from a set $V$. The digraph of $C$, denoted $G_C$, is the digraph $(V,E)$ with vertex set $V$ equals to the variable set and edge set $E = \{(x_i, y_i) : i = 1,\ldots,n\}$.

For example, the digraph of the loop condition $s(x,y) = s(y,x)$ is $K_2$ and the digraph of the 6-ary loop condition by Siggers is $K_3$, where $K_i$ denotes the complete loopless digraph on $i$ vertices.

It is easy to see that two loop conditions, whose associated digraphs are isomorphic, are equivalent. Therefore, we may talk about “loop condition $G$” (or sometimes “$G$ loop condition”) instead of “a loop condition whose associated digraph is $G$”. Also note that $G_C$ contains a loop if and only if $C$ is trivial, i.e., satisfied in every algebra.

Our main result, Theorem 1, fully classifies the strength of undirected loop conditions: Each such nontrivial condition is either equivalent to the existence of a commutative term (condition $K_2$) or the existence of a Siggers 6-ary term (condition $K_3$).

![Figure 1: The digraph of the Siggers term.](image)

2 Preliminaries

An $n$-ary operation $f$ on a set $A$ is *compatible* with an $m$-ary relation $R \subseteq A^m$, or $R$ is *compatible* with $f$, if $f(r_1,\ldots,r_n) \in R$ for any $r_1,\ldots,r_n \in R$. Here (and later as well) we abuse the notation and use $f$ also for the $n$-ary operation on $A^m$ defined from $f$ coordinate-wise.
An algebra $A = (A, f_1, f_2, \ldots)$ is said to be compatible with a relational structure $A = (A, R_1, R_2, \ldots)$ if all the operations $f_1, f_2, \ldots$ are compatible with all the relations $R_1, R_2, \ldots$.

**Digraph** is a relational structure $G = (G, E)$ with one binary relation $E$. If $E$ is symmetric, then the digraph $G$ is called undirected. A loop in $G$ is a pair of the form $(x, x) \in E$.

We will extensively use a standard method for building compatible relations from existing ones — primitive positive (pp, for short) definitions. A relation $R$ is pp-definable from relations $R_1, \ldots, R_n$ if it can be defined by a first order formula using variables, existential quantifiers, conjunctions, the equality relations, and predicates $R_1, \ldots, R_n$. Clauses in pp-definitions are also referred to as constraints. Recall that $R_1, \ldots, R_n$ are compatible with an algebra, then so is $R$.

It is sometimes helpful to visualize a pp-definition of $k$-ary relation $R$ from a digraph $G$ as a digraph $H$ with $k$ distinguished vertices $u_1, \ldots, u_k$: the vertices of $H$ are the variables, its edges correspond to the constraints, and the distinguished vertices are the free variables. Observe that $(v_1,\ldots,v_k)$ is in $R$ if and only if there is a digraph homomorphism $H \to G$ which maps $u_i$ to $v_i$ for each $i = 1, \ldots, k$. In a similar way, we can visualize pp-definitions from a relational structure consisting of more than one binary relation.

A **pp-power** of a relational structure $A$ on $A$ is a relational structure $B$ on $A^l$ whose relations are pp-definable from $A$ in the sense that a $k$-ary relation from $B$, regarded as a $(k \cdot l)$-ary relation on $A$, is pp-definable from $A$. Observe that if $A$ is compatible with an algebra $A$, then the pp-power $B$ is compatible with the algebraic power $A^l$.

For more background on pp-definitions and its relevance for constraint satisfaction problems we refer the reader to [3].

Finally, we state the promised correspondence between loop conditions and loops in digraphs.

**Proposition 1.** Let $V$ be a variety and $C$ a loop condition. The following conditions are equivalent.

(i) $V$ satisfies $C$.

(ii) For every $A \in V$ and for every digraph $G = (A, G)$ compatible with $A$, the following holds: If there is a digraph homomorphism $G_C \to G$ then $G$ contains a loop.

(iii) For every $A \in V$ and for every digraph $G = (A, G)$ compatible with $A$, the following holds: If a subdigraph of $G$ is isomorphic to $G_C$ then $G$ contains a loop.

**Proof.** (i)⇒(ii) Let $C$ be of the form $t(x_1, \ldots, x_n) = t(y_1, \ldots, y_n)$ and $f : G_C \to G$ be a digraph homomorphism. Then $(f(x_i), f(y_i)) \in G$ for all $i = 1, \ldots, n$. By compatibility of $G$ with $A$ also

$$(t(f(x_1), \ldots, f(x_n)), t(f(y_1), \ldots, f(y_n))) \in G,$$

Since equality

$$t(f(x_1), \ldots, f(x_n)) = t(f(y_1), \ldots, f(y_n))$$

is ensured by $C$, we get the desired loop in $G$. 

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(ii)$\Rightarrow$(iii) Trivial.

(iii)$\Rightarrow$(i) Let $F$ be the free algebra in $V$ generated by the vertices of $G_C$ and let $E$ be the subuniverse of $F^2$ generated by the edges of $G_C$. The set $E$ is the edge–set a digraph compatible with $F$ containing $G_C$ as a subgraph. Therefore, by (iii), there is loop $(a, a)$ in $E$. The pair $(a, a)$ is generated from edges of $G_C$, so there is a term operation $t$ of $F^2$ taking edges of $G_C$ as arguments and returning $(a, a)$. Thus, in $F$, the term $t$ satisfies the loop condition $C$, at least when we plug in the generators. By the universality of free algebra, the term $t$ satisfies $C$ in general. 

**Corollary 1.** Let $C$, $D$ be loop conditions. If there is a digraph homomorphism $G_C \rightarrow G_D$, then $C$ implies $D$. In particular, $C$ implies $D$ whenever $G_C$ is a subdigraph of $G_D$.

**Corollary 2.** Loop conditions with homomorphically equivalent digraphs are equivalent. In particular, loop conditions with isomorphic digraphs are equivalent.

### 3 Undirected case

In this section, we will focus on loop conditions with undirected digraphs, briefly graphs.

We start with several simple consequences of Corollary 1.

**Proposition 2.**

1. Any loop condition of a bipartite graph is equivalent to the edge loop condition (commutativity).
2. The $(l + 2)$-cycle loop condition implies the $l$-cycle loop condition for any odd length $l \geq 3$.
3. The $n$-clique loop condition implies the $(n + 1)$-clique loop condition for any size $n \geq 3$.
4. For any non-bipartite graph $G$ there is an odd length $l \geq 3$ such that the $l$-cycle loop condition implies the loop condition given by $G$.
5. For any loopless digraph $G$ there is a size $n \geq 3$ such that the loop condition given by $G$ implies the $n$-clique loop condition.

![Figure 2: Scheme of the easy implications between undirected loop conditions given by Proposition 2](image-url)
Our aim in the rest of the chapter is to reverse the second and the third implication, that is, $(2n+3)$-cycle to $(2n+1)$-cycle and $(n+2)$-clique to $(n+3)$-clique, where $n \geq 1$. It will follow that all the loopless non-bipartite graphs are equivalent as loop conditions and that they are weakest among all non-trivial loop conditions.

**Proposition 3.** The $l$-cycle loop condition implies the $(l+2)$-cycle loop condition for any odd length $l \geq 3$.

**Proof.** There is a graph homomorphism from the $l^2$-cycle to the $(l+2)$-cycle because both cycles are odd and $l^2 \geq l+2$. It is thus sufficient to show that the $l$-cycle loop condition implies the $l^2$-cycle one.

We use Proposition 1. Let $A$ be an algebra on a set $A$ such that every digraph compatible with $A$ containing a homomorphic image of the $l$-cycle has a loop. We need to prove that every compatible digraph $G = (A, G)$ containing a homomorphic image of the $l^2$-cycle has a loop. Take such a digraph $G$ where the cycle is formed by vertices $v_0, v_1, \ldots, v_{l^2-1}$. We may assume that $G$ is symmetric, since the set of symmetric edges of $G$ is clearly pp-definable from $G$. 

![Figure 3: Getting a loop from 9-cycle using the triangle loop condition.](image)

We construct a binary relation $H$ on $A$: vertices $x$ and $y$ are by definition $H$-adjacent if there is a $G$-walk from $x$ to $y$ of length $l$. Since $H$ is pp-defined from a compatible relation, it is compatible with $A$ too. Vertices $v_0, v_1, v_2, \ldots, v_{l(l-1)}$ form a cycle in $H$ of length $l$. By the assumption on algebra $A$, there is a loop in $H$. A loop in $H$ is a homomorphic image of an $l$-cycle in $G$. Using the assumption on $A$ again we get a loop in $G$. \hfill $\square$

**Proposition 4.** The $(n+1)$-clique loop condition implies the $n$-clique loop condition for any $n \geq 3$.

**Proof.** As in the previous proposition, we use Proposition 1. Let $A$ on a set $A$ be an algebra satisfying the $(n+1)$-clique loop condition. Let $G = (A, G)$ be a digraph compatible with $A$ containing an $n$-clique $a_1, a_2, \ldots, a_n$ as a subgraph. It suffices to prove that $G$ has to have a loop. As before we may assume that $G$ is a graph.

Let us pp-define a 4-ary relation $R$ on $A$ as follows. $R(u, v, x, y)$ if and only if there are elements $x_1, \ldots, x_{n-2}, w \in A$ such that all the vertices $x_i$ are pairwise $G$-adjacent to each other, they are also $G$-adjacent to vertices $x, y, v, w$ and moreover $G(u, w), G(u, x), G(v, y)$. From $R$ we pp-define a binary relation $F(x, y) \iff \exists u \in A: R(u, u, x, y)$. 

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Observe that the pp-definition of $F$ is symmetric, so $F$ itself is symmetric. We regard $F$ as the set of edges of a graph $F = (A, F)$ compatible with $A$.

**Claim 1.** Let $u, v, x, y \in \{a_1, \ldots, a_n\}$. Then $R(u, v, x, y)$ whenever one of the following condition are met:

(a) $u \neq v$ and $x = y \neq v$.

(b) $u = v$ and $x = u = v$, $y \neq x$.

To prove the claim, we need to find correct values of variables $x_1, \ldots, x_{n-2}$, $w$ in the definition of $R$ to meet the constraints of $R$. We do it separately for the two cases.

(a) We set $w = v$ and variables $x_i$ to vertices $a_i$ different from $v$, $x$.

(b) We set $w = y$ and variables $x_i$ to vertices $a_i$ different from $x$, $y$.

The next claim immediately follows from the case (b) of the previous one.

**Claim 2.** If $x, y \in \{a_1, \ldots, a_n\}$ and $x \neq y$, then $F(x, y)$.

Finally we define a digraph $Q = (A^2, Q)$, where the binary relation $Q$ is defined as follows: $Q((u_1, u_2), (v_1, v_2))$ if and only if there are $x_1, x_2, \ldots, x_{n+1}$ such that every pair of different indices $i_1, i_2$, with the possible exception of pairs $\{1, 2\}$ and $\{3, 4\}$, satisfies $F(x_{i_1}, x_{i_2})$ and, moreover, $R(u_1, v_1, x_1, x_2)$ and $R(u_2, v_2, x_3, x_4)$.

**Claim 3.** If $u_1, u_2, v_1, v_2 \in \{a_1, \ldots, a_{n+1}\}$ and $(u_1, u_1) \neq (v_2, v_2)$, then $Q((u_1, u_1), (v_2, v_2))$.

To prove the claim, let variables $x_1, \ldots, x_{n+1}$ be as in the definition of $Q$. We satisfy the constraints of $Q$ by means of Claims [1][2]. Because of $F$ clauses in the definition of $Q$, all pairs of variables from $\{x_1, \ldots, x_{n+1}\}$ should differ, with two possible exceptions $x_1 = x_2$ and $x_3 = x_4$. We analyse cases according to equalities among $u_1, u_2, v_1, v_2$. In each case we use Claim [3] and assign suitable values to $x_1, x_2, x_3, x_4$.

(a) $u_1 \neq v_1$ and $u_2 \neq v_2$: We want to use (a), that is, we want to have $x_1 = x_2 \neq v_1$ and $x_3 = x_4 \neq v_2$. First we choose $x_1 = x_2$, then $x_3 = x_4$ different from $v_2$ and $x_1$. This is possible since $n \geq 3$. 

![Figure 4: A visualization of the definitions of $R$ and $F$.](image-url)
Figure 5: A visualization of the definitions of $Q$.

(b) $u_1 = v_1$ and $u_2 \neq v_2$: We want to satisfy $x_1 = u_1$, $y_1 \neq x_1$ (and use (b)), and $x_3 = x_4 \neq v_2$ (and use (a)). So we put $x_1 = u_1$, then choose a value for $x_3 = x_4$ different from $x_1, v_2$ and, finally, we choose a value for $x_2$ different from $x_1, x_3$.

(c) $u_1 \neq v_1$ and $u_2 = v_2$ is analogous to the previous case.

(d) $u_1 = v_1$ and $u_2 = v_2$ can not happen since $(u_1, u_1) \neq (v_2, v_2)$.

In every case $x_1 = x_2$ or $x_3 = x_4$, therefore the remaining variables $x_i$ can be completed so that all the constraints of $Q$ are satisfied.

Let us finish the proof. The digraph $Q$ is compatible with the algebra $A^2$ since it is a pp-power of $G$. Moreover, $Q$ contains a clique of size $n^2 \geq n + 1$.

The $n$-clique loop condition holds also for $A^2$, so there is a loop in $Q$.

The loop in $Q$ is represented by elements $x_1, \ldots, x_{n+1}$ in $A$ such that $F(x_i, x_j)$ whenever $i \neq j$. Since $F$ is pp-defined from $G$, it is compatible with $A$. Therefore, there is a loop in $F$.

Finally, a loop in $F$ yields a $(n+1)$-clique in $G$ and, consequently, the sought after loop in $G$.

**Theorem 1.** There are exactly three equivalence classes of loop conditions given by undirected digraphs $G$:

1. loop conditions $G$, where $G$ is bipartite,
2. loop conditions $G$, where $G$ is non-bipartite and loopless,
3. loop conditions $G$, where $G$ contains a loop (trivial).

Conditions (1) imply (2) imply (3). Conditions (2) are the weakest non-trivial loop conditions.

**Proof.** If a graph $G$ is bipartite (contains a loop, respectively), then the $G$ loop condition is equivalent to the edge loop condition by item 1 of Proposition 2 (is trivial, respectively). If $G$ is non-bipartite and loopless, then the $G$ loop condition implies a clique loop condition (item 5 of Proposition 2), which implies the triangle loop condition (by Proposition 4), which implies the $l$-cycle loop condition.
condition for any odd \( l \) (by Proposition 3), which, finally, implies the \( G \) loop condition (by item 4 of Proposition 2).

Clearly, conditions (1) imply (2) imply (3), and (3) do not imply (2). The implication from (2) to (1) cannot be reversed even for idempotent finitely generated varieties: an example of an algebra satisfying (2) but not (1) is the algebra \( (\{0,1,2\}, m) \) where \( m(x,y,z) = x + y - z \) modulo 3.

Theorem 1 also provides an alternative proof to the fact [10] that the existence of a near unanimity term implies the existence of a Siggers term. Recall that a near unanimity (NU) term is an \( n \)-ary term \( t \) satisfying the identity 
\[
t(x, \ldots, x, y, x, \ldots, x) = x
\]
for all positions of \( y \). 

**Theorem 2.** If an algebra (or a variety) has an \( n \)-ary NU term, then it also has the Siggers term.

**Proof.** Let \( t \) be the NU term. The value \( t(x_1, x_2, \ldots, x_n) \) may be expressed in the following two ways.

\[
t(t(x_2, x_1, x_1, \ldots, x_1), t(x_2, x_3, x_2, \ldots, x_2), \ldots, t(x_n, \ldots, x_n, x_n, x_1))
\]

This identity may be interpreted as a loop identity (whose graph is the \( n \)-clique) for a composed \( n^2 \)-ary term. Therefore the \( n \)-clique loop condition is satisfied and then also the triangle loop condition is satisfied since all the loop conditions given by non-bipartite loopless graphs are equivalent.

4 Directed case

A digraph \( G \) is said to be smooth if every vertex has an incoming and an outgoing edge; \( G \) is said to have algebraic length 1 if there is no graph homomorphism from \( G \) to a directed cycle of length greater than one. The following theorem holds for finite algebras [7].

**Theorem 3.** Let \( A \) be a finite algebra. Let \( G, H \) be weakly connected smooth digraphs of algebraic length one. Then \( A \) satisfies loop condition \( G \) if and only if it satisfies loop condition \( H \).

Less formally, all connected smooth loop conditions with algebraic length 1 are equivalent for finite algebras. This gives us a simpler weakest loop condition than the triangle for finite algebras: 
\[
s(a, r, e, a) = s(r, a, r, e).
\]

For general varieties it is no longer true that all such loop conditions are equivalent. However, very recently, we have shown that all loop conditions given by strongly connected graphs of algebraic length 1 are. These results will appear in a forthcoming paper.

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