Polar Codes for the Deletion Channel: Weak and Strong Polarization

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Abstract—This paper presents the first proof of polarization for the deletion channel with a constant deletion rate and a regular hidden-Markov input distribution. A key part of this work involves representing the deletion channel using a trellis and describing the plus and minus polar-decoding operations on this trellis. In particular, the plus and minus operations can be seen as combining adjacent trellis stages to yield a new trellis with half as many stages. Using this viewpoint, we prove a weak polarization theorem for standard polar codes on the deletion channel. To achieve strong polarization, we modify this scheme by adding guard bands of repeated zeros between various parts of the codeword. Using this approach, we obtain a scheme whose rate approaches the mutual information and whose probability of error decays exponentially in the cube-root of the block length.

I. INTRODUCTION

In many communications systems, symbol-timing errors may result in insertion and deletion errors. For example, the deletion channel maps a length-$N$ input string to a substring using an i.i.d. process that deletes each input symbol with probability $\delta$. These types of channels were first studied in the 1960s [1], [2] and modern coding techniques were first applied to them in [3]. Over the past 15 years, numerical bounds on the capacity of the deletion channel have been significantly improved but a closed-form expression for the capacity remains elusive [4]–[7]. Recently, polar codes were applied to the deletion channel in a series of papers but the question of polarization for non-vanishing deletion rates remained open [8]–[11]. In this work, we show that polar codes can be used to efficiently approach the mutual-information rate between a regular hidden-Markov input process and the output of the deletion channel with constant deletion rate.

In [8], a polar code is designed for the binary erasure channel (BEC) and evaluated on a BEC that also introduces a single deletion. An inner cyclic-redundancy check (CRC) code is used and decoding is performed by running the successive cancellation list (SCL) decoder [12] exhaustively over all compatible erasure locations. The results show one can recover a single deletion in this setting. Extensions to a finite number of deletions are also discussed but the decoding complexity grows faster than $N^{d+1}$.

In [9] a low-complexity decoder is proposed for the same setup. Its complexity, for a length-$N$ polar code, is roughly $d^2N\log N$ when $d$ deletions occur. The paper also presents simulation results for polar codes with lengths ranging from 256 to 2048 on two deletion channels. The first channel has a fixed deletion rate of 0.002 and the second introduces exactly 4 deletions. Based on their results, they conjecture that polarization occurs when $N \to \infty$ while the total number of deletions, $d$, is fixed.

The final papers in this series [10], [11] extend the previous results by proving that weak polarization occurs when $N \to \infty$ and $d = o(N)$. While this result is quite interesting, its proof does not extend to the case of constant deletion rate. Strong polarization is shown for the case where $N \to \infty$ with $d$ fixed.

In this paper, we combine the well-known trellis representation for channels with synchronization errors [3] with low-complexity joint successive-cancellation decoding for channels with memory [13], [14]. In particular, [3] describes how the input-output mapping of the deletion channel (and other synchronization-error channels) can be represented using a trellis. The main advantage of the trellis perspective is that it naturally generalizes to other channels with synchronization errors (e.g., with insertions, deletions, and errors). The papers [13], [14] describe how the plus and minus polar-decoding operations can be efficiently applied to a channel whose input-output mapping is represented by a trellis. Putting these ideas together defines a low-complexity successive-cancellation decoder for polar codes on the deletion channel that is essentially equivalent to the decoder defined in [9].

Building on previous proofs of polarization for channels with memory [15], [16], this paper also proves weak and strong polarization for the deletion channel. In order to prove strong polarization, guard bands of ‘0’ symbols are embedded in the codewords of Arıkan’s standard polar codes. These guard bands allow the decoder to effectively work on independent blocks and enable the proof of strong polarization.

The following theorem describes the main result of this research. We note that the family of allowed input distributions will be defined in Subsection II-D whereas the structure of the codeword will be defined in Section VII-A. The theorem will be proved in Section VII-B.

Theorem 1: Fix a regular hidden-Markov input process. For any fixed $\gamma \in (0, 1/3)$, the rate of our coding scheme approaches the mutual-information rate between the input process and the deletion channel output. For large enough blocklength $\Lambda$, the probability of error is at most $2^{-\Lambda^\gamma}$.

Here is an outline of the structure of this paper. Section II sets up the basic notation and definitions used in this paper.
Section III defines the concept of a trellis and shows how it can be used to compactly represent various deletion patterns and their corresponding probabilities. In Section IV we show how minus and plus polarization operations are applied to trellises to yield new trellises. This allows one to easily and efficiently adapt the successive cancellation decoding to channels with memory. It is our hope that all sections up to and including Section V will be accessible to practitioners who are more interested in the implementation of codes than proving theorems. Section VI discusses information rates and Section VII proves that, in our setting, weak polarization occurs. Section VII focuses on fast polarization. The practitioner is recommended to read Section VII-A which defines the structure and operation of an encoder with guard bands. The proof of the main theorem is presented in Section VII-B.

II. BACKGROUND

A. Notation

The natural numbers are denoted by \( \mathbb{N} = \{1, 2, \ldots \} \). We also define \( [m] = \{1, 2, \ldots, m\} \) for \( m \in \mathbb{N} \). Let \( \mathcal{X} \) denote a finite set (e.g., the input alphabet of a channel). In this paper, we fix \( \mathcal{X} = \{0, 1\} \) as the binary alphabet. Extensions to non-binary alphabets are straightforward, see for example [17] Chapter 3. Let \( x = (x_1, \ldots, x_N) \in \mathcal{X}^N \) be a vector of length \( N = 2^n \). We use [statement] to denote the Iverson bracket which evaluates to 1 if statement is true and 0 otherwise. The concatenation of vectors \( y \in \mathcal{X}^{N_1} \) and \( y' \in \mathcal{X}^{N_2} \) lives in \( \mathcal{X}^{N_1+N_2} \) and is denoted by \( y \circledast y' \). The length of a vector \( y \) is denoted by \( |y| \).

In this paper, we use the standard Arıkan transform presented in the seminal paper [18]. Generalization to other kernels [19] is straightforward. The Arıkan transform of \( x \in \mathcal{X}^N, N = 2^n \), is defined recursively using length-N/2 binary vectors, \( x^{[0]} \) and \( x^{[1]} \):

\[
\begin{align*}
x^{[0]} & \equiv (x_1 \oplus x_2, x_3 \oplus x_4, \ldots, x_{N-1} \oplus x_N), \\
x^{[1]} & \equiv (x_2, x_4, \ldots, x_N),
\end{align*}
\]

where \( \oplus \) denotes modulo-2 addition. Then, for any sequence \( b_1, b_2, \ldots, b_\lambda \in \{0, 1\} \) with \( \lambda \leq n \), we extend this notation to define the vector \( x^{[b_1, b_2, \ldots, b_\lambda]} \in \mathcal{X}^{2^n-\lambda} \) recursively via

\[
\begin{align*}
z = x^{[b_1, b_1, \ldots, b_{\lambda-1}]} \quad x^{[b_1, b_2, \ldots, b_{\lambda}]} = z^{[b_{\lambda}]}.
\end{align*}
\]

Specifically, if \( \lambda = n \), then the vector \( x^{[b_1, b_2, \ldots, b_n]} \) is a scalar. This scalar is denoted \( u_{i(b)} \), where \( b \) defines the index

\[
\begin{align*}
i(b) & \equiv 1 + \sum_{j=1}^{n} b_j 2^{n-j}.\end{align*}
\]

The transformed length-\( N \) vector is given by

\[
\begin{align*}
u = (u_1, \ldots, u_N) = A_n(x),
\end{align*}
\]

where \( A_n: \mathcal{X}^{2^n} \to \mathcal{X}^{2^n} \) is called the Arıkan transform of order \( n \). Its inverse is denoted \( A_n^{-1} \) and satisfies \( A_n^{-1} = A_n \).

B. Deletion Channel

Let \( W(y|x) \) denote the transition probability of \( N \) uses of the deletion channel with constant deletion rate \( \delta \). The input is denoted by \( x \in \mathcal{X}^N \) and the output \( y \) has a random length \( M = |y| \) supported on \( \{0, 1, \ldots, N\} \). This channel is equivalent to a BEC with erasure probability \( \delta \) followed by a device that removes all erasures from the output. Thus, \( W(y|x) \) equals the probability that \( N - M \) deletions have occurred, which is \( (1-\delta)^M, f^N-M \), times the number of distinct deletion patterns that produce \( y \) from \( x \), see [4] Section 2.

We will also consider a trimmed deletion channel whose output is given by removing all leading and trailing zeros from the output of the standard deletion channel. See Section VII for details.

C. Trellis Definition

An \( N \)-segment trellis \( T \) is a labeled weighted directed graph \((\mathcal{V}, \mathcal{E})\). Let \( \mathcal{V}_n \) denote the set of possible channel states after \( n \) steps. We assume that \( \mathcal{V} \) can be partitioned into \( \mathcal{V}_0, \ldots, \mathcal{V}_N \) so that \( \mathcal{V} \) is the union of \( N + 1 \) disjoint sets:

\[
\begin{align*}
\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{N-1} \cup \mathcal{V}_N,
\end{align*}
\]

where \( \cup \) denotes a disjoint union. Similarly, the edge set \( \mathcal{E} \) is arranged into a sequence of \( N \) disjoint sets:

\[
\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_{N-1} \cup \mathcal{E}_N.
\]

An edge in \( \mathcal{E}_j \) connects a vertex in \( \mathcal{V}_{j-1} \) to a vertex in \( \mathcal{V}_j \). We define \( \sigma(e) \) and \( \tau(e) \) to be the starting and ending vertices of edge \( e \). Thus, for \( e = u \rightarrow v \), we have \( \sigma(e) = u \) and \( \tau(e) = v \). Then,

\[
e \in \mathcal{E}_j \text{ implies } \sigma(e) \in \mathcal{V}_{j-1} \text{ and } \tau(e) \in \mathcal{V}_j.
\]

A trellis section comprises two adjacent sets of vertices along with the edges that connect them. That is, for \( 1 \leq j \leq N \), section \( j \) comprises vertex sets \( \mathcal{V}_{j-1} \) and \( \mathcal{V}_j \), as well as edge set \( \mathcal{E}_j \). See Fig. 1 for an example of a trellis with 4 sections.

Each edge \( e \in \mathcal{E} \) has a weight \( w(e) \in [0, 1] \) and a label \( \ell(e) \in \mathcal{X} \). We also assume that \( \mathcal{V}_0 \) and \( \mathcal{V}_N \) have weight functions,

\[
q: \mathcal{V}_0 \to [0, 1] \text{ and } r: \mathcal{V}_N \to [0, 1],
\]

that are associated with the initial and final states.

A path through a trellis is a sequence of \( N \) edges, \( e_1, e_2, \ldots, e_N \), which starts at a vertex in \( \mathcal{V}_0 \) and ends at a vertex in \( \mathcal{V}_N \). Namely, \( \sigma(e_1) \in \mathcal{V}_0, \tau(e_N) \in \mathcal{V}_N \), and for each \( 1 \leq j \leq N-1 \), we have \( \tau(e_j) = \sigma(e_{j+1}) \). The weight of a path through the trellis is defined as the product of the weights on each edge in the path times the weights of the initial and final vertices. Namely, the weight of the above path is

\[
q(\sigma(e_1)) \cdot r(\tau(e_N)) \times \prod_{j=1}^{N} w(e_j).
\]

Thus, an \( N \)-section trellis naturally defines a path-sum function \( T:\mathcal{X}^N \to \mathbb{R} \), where \( T(x) \) equals the sum of the path weights over all paths whose length-\( N \) label sequences match \( x \). That is,
We now define the corresponding trellis, having
\[ T(x) = \sum_{e_1 \in E_1} \sum_{e_2 \in E_2} \cdots \sum_{e_N \in E_N} q(\sigma(e_1)) r(\tau(e_N)) \]
\[ \times \prod_{j=1}^{N} w(e_j) \times \prod_{j=1}^{N-1} [\tau(e_j) = \sigma(e_{j+1})] . \]  

\[ (6) \]

D. FAIM processes

In latter parts of this paper, for simplicity, we will often introduce key ideas by first framing them in the context of the uniform input distribution. That is, by first considering the case in which the input distribution is i.i.d. Bernoulli 1/2. However, the uniform input distribution, or indeed any i.i.d. input distribution, is known to generally be sub-optimal with respect to the information rate between input and output, when transmitting over a deletion channel. Thus, we stand to benefit by considering a larger class of input distributions.

Towards this end, let \( S \) be a given finite set. Each element of \( S \) is a state of an input process. In the following definition, we have for all \( j \in Z \) that \( S_j \in S \) and \( X_j \in X \).

Definition 1 (FAIM process): A strictly stationary process \((S_j, X_j)\), \( j \in Z \) is called a finite-state, aperiodic, irreducible, Markov (FAIM) process if, for all \( j \),

\[ P_{S_j, X_j|S_{j-1}^{-1}, X_{j-1}^{-1}} = P_{S_j, X_j|S_{j-1}}, \]

is independent of \( j \) and the sequence \((S_j), j \in Z \) is a finite-state Markov chain that is stationary, irreducible, and aperiodic.

For a FAIM process, consider the sequence \( X_j \), for \( j \in Z \). In principle, the distribution of this sequence can be computed by marginalizing the states of the FAIM process \((S_j, X_j)\). Such a sequence is typically called a hidden-Markov process. In this paper, we sometimes add the term regular to emphasize that the hidden state process is a regular finite-state Markov chain.

Let us now tie the concept of a FAIM process to that of a trellis. Let a FAIM process \((X_j, S_j)\) be given, and fix \( N \geq 1 \). We define the corresponding trellis, having \( N \) stages. The vertex set is \( V = V_0 \cup V_1 \cup \cdots \cup V_N \), where we define

\[ V_j = \{ s_j : s \in S \} \]

for \( 0 \leq j \leq N \) so that each \( V_j \) contains a distinct copy of \( S \). For each \( x \in X \), \( 1 \leq j \leq N \), \( \alpha_j \in V_j \), and \( \beta_j \in V_j \), define an edge \( e \) from \( \alpha_j \) to \( \beta_j \) with label \( \ell(e) = x \) and weight \( w(e) = P_{S_j, X_j|S_{j-1}}(\beta, x|\alpha) \). Lastly, for all \( \alpha_0 \in V_0 \) define \( q(\alpha_0) = \pi(\alpha) \), where \( \pi(\alpha) \) is the stationary probability of state \( \alpha \) in the Markov process \((S_j)_{j \in Z} \), and define \( r(\beta_N) = 1 \) for all \( \beta_N \in V_N \). It follows that the probability of \((X_1, X_2, \ldots, X_N) = (x_1, x_2, \ldots, x_n)\) equals \( T(x) \), where \( T \) was defined in (6).

III. TRELLIS REPRESENTATION OF JOINT PROBABILITY

We have just seen that a trellis is instrumental in compactly representing a hidden-Markov input distribution. In fact, it is much more versatile than this. Namely, we will now show how a trellis can be used to represent the joint distribution of a hidden-Markov input process, as well as its corresponding deleted output.

A. Trellis for uniform input

This trellis representation for the deletion channel can also be found in [3].

As previously explained, it is generally beneficial to use an input distribution with memory. However, for the sake of an easy exposition, we will first consider the simplest possible input distribution, a uniform input distribution (i.e., i.i.d. and Bernoulli 1/2).

The trellis representation will be used on the decoder side. Thus, when building the trellis we will have already received the output vector \( y \). Hence, the primary role of the trellis is to evaluate the probabilities associated with possible input vectors \( x \), of length \( N \). That is, the trellis will be used to calculate the joint probability of \( x \) and \( y \), denoted \( P_X(x) \cdot W(y|x) \), for \( y \) fixed. Recall that \( W(y|x) \) is the deletion channel law, and in this subsection \( P_X \) is the uniform input distribution.

We will shortly define the concept of a valid path in the trellis. Each valid path will correspond to a specific transmitted \( x \) and a specific deletion pattern that is compatible with the received \( y \) (see Fig. 1). We term this trellis the base trellis, as we will ultimately construct other trellises derived from it.

Recalling our notation, we have \( x \) as the unknown input vector, of known length \( N \). The vector \( y \) is the known output, having known length \( M = |y| \). The deletion probability is \( \delta \). The base trellis is defined as follows.

Definition 2 (Base Trellis for Uniform Input): For \( N, \delta, M \), and \( y \in X^M \):

1) The vertex set \( V \) equals the disjoint union

\[ V = V_0 \cup V_1 \cup \cdots \cup V_N \]

where, for \( 0 \leq j \leq N \),

\[ V_j = \{ v_{i,j} : 0 \leq i \leq M \} . \]

Fig. 1. A trellis for the binary deletion channel corresponding to a codeword \( v \) of length \( N = 4 \) and received word \( y = (011) \) of length \( M = 3 \). Vertices are denoted \( v_{i,j} \) with \( 0 \leq i \leq M \) and \( 0 \leq j \leq N \). All blue edges have label ‘0’ while all red edges have label ‘1’. The horizontal edges are weighted by the probability \( \delta/2 \). Diagonal edges are weighted by the probability \((1-\delta)/2 \). The two circled vertices have \( q(v_{0,0}) = r(v_{0,1},N) = 1 \), while all other vertices in \( V_0 \) and \( V_N \) have \( q \) and \( r \) values equal to 0, respectively. Edges that can be pruned without changing \( T(x) \) are dashed.
2) A path passing through vertex \(v_{i,j}\) corresponds to the event where only \(i\) of the first \(j\) transmitted symbols were received. That is, from \(x_1, x_2, \ldots, x_j\), the channel has deleted \(j-i\) symbols.\(^3\)

3) Vertices \(v_{i,j}\) with \(0 \leq i \leq M\) and \(0 \leq j < N\) each have up to three outgoing edges: two ‘horizontal’ edges, each corresponding to a deletion, and one ‘diagonal’ edge, corresponding to a non-deletion.

4) For \(0 \leq i \leq M\) and \(0 \leq j < N\), there are two edges \(e, e'\) from \(v_{i,j}\) to \(v_{i,j+1}\). From \(^2\) above, we deduce that these two ‘horizontal’ edges are associated with \(x_{j+1}\) being deleted by the channel. The first is associated with \(x_{j+1} = 0\) and has \(\ell(e) = 0\), while the second is associated with \(x_{j+1} = 1\) and has \(\ell(e') = 1\). Since the probability of deletion is \(\delta\), and in the uniform distribution \(x_{j+1} = 0\) and \(x_{j+1} = 1\) each occur with probability 1/2, we set \(w(e) = w(e') = \delta/2\).

5) For \(0 \leq i < M\) and \(0 \leq j < N\), there is a single edge \(e\) from \(v_{i,j}\) to \(v_{i+1,j}\). Recalling \(^2\) above, we deduce that this ‘diagonal’ edge represents \(x_{j+1}\) not being deleted, and being observed as \(y_{i+1}\). Thus, \(\ell(e) = y_{i+1}\). Since the probability of sending \(x_{j+1}\) in the uniform case is 1/2, regardless of its value, and the probability of a non-deletion is \(1 - \delta\), we set \(w(e) = (1 - \delta)/2\).

6) For all \(v \in \mathcal{V}_0\), we set \(q(v) = [v = v_{0,0}]\). Thus, with respect to \(^6\), we effectively force all paths to start at \(v_{0,0}\). Namely, when starting a path, no symbols have yet been transmitted, and hence no symbols have yet been received.

7) For all \(v \in \mathcal{V}_N\), we set \(r(v) = [v = v_{M,N}]\). Thus, with respect to \(^6\), we effectively force all paths to end at \(v_{M,N}\). That is, at the end of a path, \(N\) symbols have been transmitted, and of these, \(M\) have been received.

In line with the definitions above, let us call a path valid if it starts at \(v_{0,0}\) and ends at \(v_{M,N}\). For example, in Figure \(^1\) valid paths are those that start at the circled vertex on the top left, end at the circled vertex on the bottom right, and hence contain only solid edges. Clearly, such a path is comprised of \(N\) edges, \(e_1, e_2, \ldots, e_N\). Denote by \(x = x_1, x_2, \ldots, x_N\) the input vector corresponding to the above path, where \(x_i = \ell(e_i)\). Each such \(x\) is consistent with our received \(y\). Indeed, tracing the path, the type of the corresponding edge (horizontal or diagonal) shows exactly which \(x_i\)’s to delete and which to keep in order to arrive at \(y\). Also, the probability of the input sequence \(x\) being transmitted and experiencing the above chain of deletion/no-deletion events is exactly equal to the product of the \(w(e_i)\), times \(q(v_{0,0}) \cdot r(v_{M,N}) = 1\).

From the above discussion, one has the following key lemma.

**Lemma 2:** Let \(T\) be a trellis as described in Definition \(^2\). Then, for \(x \in \mathcal{X}^N\) and \(T(x)\) as defined in \(^6\), we have

\[
T(x) = P_X(x) \cdot W(y|x), \tag{9}
\]

\(^2\) Note that we could have optimized our definition of \(V_j\). Namely, only \(i\) in the range \(\max(0, M - N + j) \leq i \leq \min(j, M)\) are actually consistent with the described event (i.e., only the solid edges in Figure \(^3\)). We leave such optimization to the practitioner, and settle for the simpler description in \(^6\). where \(P_X\) is the uniform input distribution and \(W\) is the deletion channel law.

**Proof:** First, we observe that the weight of a trellis path equals the joint probability of \((x, y)\) and the deletion pattern. Then, the claim follows from the fact that \(T(x)\) sums the path weight over all paths through the trellis (i.e., all deletion patterns) consistent with the given \((x, y)\) pair. \(\blacksquare\)

### B. Trellises for hidden-Markov inputs

As explained earlier, a trellis is used on the decoding side, in order to capture the joint probability of \(x\) and \(y\). We now show how such a trellis is built for the more general case in which \(x\) is drawn from a regular hidden-Markov input process. Intuitively, this is done by simply “multiplying” the trellis corresponding to the input distribution, as described at the end of Section \(^2\) with the trellis defined for the uniform case (with the correction that the edge weights \(\delta/2\) and \((1 - \delta)/2\) are replaced by \(\delta\) and \((1 - \delta)\), respectively). A formal definition follows.

**Definition 3 (Base Trellis for Hidden-Markov Input):** For \(N\), \(\delta\), \(M, S, P_{S_j, x_j} | S_{j-1}, \pi\), and \(y \in \mathcal{X}^M\):

1) The vertex set \(\mathcal{V}\) equals the disjoint union

\[
\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_N,
\]

where, for \(0 \leq j \leq N\),

\[
\mathcal{V}_j = \{s_{i,j} : 0 \leq i \leq M, s \in S\}. \tag{10}
\]

2) A path passing through vertex \(s_{i,j}\) corresponds to the event in which only \(i\) symbols were received, out of the first \(j\) symbols transmitted. Such a path further implies that \(S_j = s\). That is, the hidden state at stage \(j\) equals \(s\), the state by which \(s_{i,j}\) was indexed, as per \(^10\).

3) Vertices \(s_{i,j}\) with \(0 \leq i \leq M\), \(0 \leq j < N\), and \(s \in S\) each have up to \(3 \cdot |S|\) outgoing edges.

4) For \(0 \leq i \leq M\), \(0 \leq j < N\), and \(s, \alpha, \beta \in S\), there are two edges \(e, e'\) from \(s_{i,j}\) to \(s_{i+1,j+1}\). From item \(^2\) we deduce that these two ‘horizontal’ edges are associated with \(x_{j+1}\) being deleted by the channel. The first is associated with \(x_{j+1} = 0\) and has \(\ell(e) = 0\), while the second is associated with \(x_{j+1} = 1\) and has \(\ell(e') = 1\). We set

\[
w(e) = \delta \cdot P_{S_j, x_j} | S_{j-1}, (\beta, 0|\alpha) \tag{11}
\]

and

\[
w(e') = \delta \cdot P_{S_j, x_j} | S_{j-1}, (\beta, 1|\alpha). \tag{12}
\]

That is, the probability of a deletion, times the probability implied by the underlying FAIM distribution.

5) For \(0 \leq i < M\), \(0 \leq j < N\), and \(s, \alpha, \beta \in S\), there is a single edge \(e\) from \(s_{i,j}\) to \(s_{i+1,j+1}\). Recalling item \(^2\) above, we deduce that this ‘diagonal’ edge represents \(x_{j+1}\) not being deleted, and being observed as \(y_{i+1}\). Thus, \(\ell(e) = y_{i+1}\). We set

\[
w(e) = (1 - \delta) \cdot P_{S_j, x_j} | S_{j-1}, (\beta, y_{i+1}|\alpha). \tag{13}
\]

That is, the probability of a non-deletion, times the probability implied by the underlying FAIM distribution.\(^4\)

\(^3\) As in the uniform case, we have opted for simplicity of exposition over reduced algorithmic complexity. That is, in the uniform case, we can take the index \(i\) in \(^10\) to have range \(\max(0, M - N + j) \leq i \leq \min(j, M)\). Also, edges \(e\) with probability \(w(e) = 0\) can be removed from the trellis.
6) For all \( s_{0,0} \in V_0 \), where \( s \in S \), we set \( q(s_{0,0}) = \pi(s) \). All other vertices \( v \in V_0 \) have \( q(v) = 0 \). Thus, with respect to (6), we effectively force all paths to start at a vertex \( s_{0,0} \), where \( s \in S \). Namely, when starting a path, no symbols have yet been transmitted, and hence no symbols have yet been received. Moreover, the probability of starting the path at \( s_{0,0} \) is \( \pi(s) \), the stationary probability of \( s \) in the FAIM input process.

7) For all \( s_{M,N} \in V_N \), we set \( r(s_{M,N}) = 1 \). All other vertices \( v \in V_N \) have \( r(v) = 0 \). Thus, with respect to (6), we effectively force all paths to end at a vertex \( s_{M,N} \). That is, at the end of a path, \( N \) symbols have been transmitted, and of these, \( M \) have been received.

As in the uniform case, we have the following lemma, which is easily proved.

**Lemma 3:** Let \( T \) be a trellis as per Definition 3. Then, for \( x \in \mathcal{X}^N \) and \( T(x) \) as defined in (6),

\[
T(x) = P_x(x) \cdot W(y|x),
\]

where \( P_x \) is the hidden-Markov input distribution and \( W \) is the deletion channel law.

**Proof:** First, we observe that the weight of a trellis path equals the joint probability of \((x,y)\) and the deletion pattern. Then, the claim follows from the fact that \( T(x) \) sums the path weight over all paths through the trellis (i.e., all deletion patterns) consistent with the given \((x,y)\) pair. \( \blacksquare \)

### C. Trellises for trimmed deletion channels

For reasons that will shortly become clear, we will now consider a slight variation of the deletion channel. Namely, we now define the trimmed deletion channel (TDC). A TDC is a deletion channel that, after the deletion process, trims its output of leading and trailing '0' symbols. Thus, by definition, the output of a TDC is either an empty string, or a string that starts and ends with a '1' symbol.

We now show how to alter Definition 3 in order to account for this variation. The change turns out to be minimal.

**Definition 4 (Base Trellis for Hidden-Markov Input and TDC):** For \( N, \delta, M, S, P_{S, X | S_{j-1}}, \pi, \) and \( y \in \mathcal{X}^M \), define the trellis \( T \) as in Definition 3 but with the following changes.

- The probability of an edge \( e \) from \( \alpha_{0,j} \) to \( \beta_{0,j+1} \) with \( \ell(e) = 0 \) must be changed to \( w(e) = P_{S, X | S_{j-1}}(\beta, 0 | \alpha) \). Namely, the factor in (11) is removed. In short, if the path is currently at vertex \( \alpha_{0,j} \), then none of the \( j \) symbols \( x_1, x_2, \ldots, x_j \) have made it to the output of the channel (they have either been deleted or trimmed). Thus, if \( x_{j+1} = 0 \), it will surely be either deleted, or else trimmed.

- The probability of an edge \( e \) from \( \alpha_{M,j} \) to \( \beta_{M,j+1} \) with \( \ell(e) = 0 \) must be changed to \( w(e) = P_{S, X | S_{j-1}}(\beta, 0 | \alpha) \). Namely, the factor in (11) is removed. Note that the exact same reasoning from the previous point applies; the only difference is that now we are correcting for the trimming of the trailing '0' symbols.

The result of the above altered trellis definition is the following lemma.

**Lemma 4:** Let \( T \) be a trellis as described in Definition 4. Then, for \( x \in \mathcal{X}^N \) and \( T(x) \) as defined in (6),

\[
T(x) = P_x(x) \cdot W^*(y^*|x),
\]

where \( P_x \) is the hidden-Markov input distribution and \( W^* \) is the law of the TDC.

**Proof:** First, we observe that the weight of a trellis path equals the joint probability of \((x,y^*)\) and the deletion/trimming event associated with that path. Then, the claim follows from the fact that \( T(x) \) sums the path weight over all paths through the trellis (i.e., all deletion/trimming events) consistent with the given \((x,y^*)\) pair. \( \blacksquare \)

### IV. Polarization operations on a trellis

Polar plus and minus transforms for channels with memory were first presented in [13], [14]. Let an input distribution on \( \mathcal{X}^N \) be given, for \( N \) even. For this input distribution and a vector channel with input \( x \in \mathcal{X}^N \) and output \( y \), let \( T \) be a trellis with \( N \) sections whose path-sum function satisfies

\[
T(x) = \Pr(Y = y, X = x).
\]

**A. Minus transform**

The polar minus transform of the path-sum function \( T(x) \) given in (13) rewrites this function in terms of \( z = x^{[0]} = (x_1 \oplus x_2, \ldots, x_{N-1} \oplus x_N) \).

The path-sum function becomes

\[
T^{[0]}(z) = \Pr(Y = y, X^{[0]} = z) = \sum_{x \in \mathcal{X}^N} T(x) \prod_{j=1}^{N/2} [x_{2j-1} \oplus x_{2j} = z_j].
\]

Due to the local nature of this reparameterization, there is a modified trellis \( T^{[0]} \) with \( N/2 \) sections that represents the new path-sum function.

**Definition 5 (Minus Transform):** Let \( T = T(V, E, w, \ell, q, r) \) be a length-\( N \) trellis, where \( N \) is even. The trellis \( \tilde{T} = \tilde{T}(\tilde{V}, \tilde{E}, \tilde{w}, \tilde{\ell}, \tilde{q}, \tilde{r}) = T^{[0]} \) is defined as follows.

- The vertex set of \( \tilde{T} \) is \( \tilde{V} = \tilde{V}_0 \cup \tilde{V}_1 \cup \ldots \cup \tilde{V}_{N/2} \),

where

\[
\tilde{V}_j = V_{2j}.
\]

- We next define the edge set \( \tilde{E} \) implicitly. Consider an edge \( \tilde{e} = e \rightarrow (\alpha, \beta, \gamma) \in \tilde{E} \) in section \( j \) of \( \tilde{T} \) with label \( \ell(\tilde{e}) = z \).

Then,

\[
\alpha \in \tilde{V}_{j-1} = V_{2j-2} \quad \text{and} \quad \gamma \in \tilde{V}_j = V_{2j}.
\]

The weight \( \tilde{w}(\tilde{e}) \) of this edge equals the sum of the product of the edge weights along each two-step path \( \alpha \circ \beta \rightarrow \gamma \) in \( T \) with \( \ell(e_1) \oplus \ell(e_2) = z \). That is,

\[
\tilde{w}(\tilde{e}) = \sum_{e_1 e_2 e_{2j-1}: \sigma(e_1) \circ \sigma(e_2) = z} \sum_{e_{2j}: \tau(e_{2j}) = \gamma} w(e_1) w(e_2)
\]

\[
\times [\tau(e_1) = \sigma(e_2)] : [\ell(e_1) \oplus \ell(e_2) = z].
\]
Edges with weight 0 may be removed from \( \tilde{T} \).

- The minus operation does not affect initial and final vertices and this implies that \( \tilde{q}(s) = q(s) \) and \( \tilde{r}(s) = r(s) \). This lemma states the key property of the minus transform.

**Lemma 5:** For a length-\( N \) trellis \( T \) and \( z \in \mathcal{X}^{N/2} \), we have
\[
T^{[0]}(z) = \sum_{x \in \mathcal{X}^N \mid x^{[0]} = z} T(x) .
\]

**Proof:** This follows from the fact that the minus trellis is constructed by merging adjacent trellis stages and then combining paths according to their \( x^{[0]} \) values. Finally, the new paths are relabeled by their \( x^{[0]} \) values. \( \square \)

**B. Plus transform**

The polar plus transform rewrites \( T(x) \), given in (15), as a function of
\[
x' = x^{[1]} = (x_2, x_4, \ldots, x_N) .
\]
This is done by using a previously calculated vector \( z \in \mathcal{X}^{N/2} \) and setting \( x_2j-1 = x_2j \oplus z_j \) for \( j \in \{N/2\} \). That is, we have \( z_j = x_2j-1 \oplus x_2j \). The implied path-sum function for \( x' \in \mathcal{X}^{N/2} \) is
\[
T^{[1]}(x') = \Pr(Y = y, X^{[1]} = x', X^{[0]} = z) = \sum_{x \in \mathcal{X}^N} T(x) \prod_{j=1}^{N/2} [z_j = x_2j-1 \oplus x_2j] \cdot [x_2j = x'_j] .
\]

Below, the transformed trellis \( T^{[1]} \) is defined in detail for a fixed vector \( z \). Sometimes \( z \) is not specified when its value is clear from the context.

**Definition 6 (Plus Transform):** Let \( T = (\mathcal{V}, \mathcal{E}, w, \ell, q, r) \) be a length-\( N \) trellis, where \( N \) is even and let \( z \in \mathcal{X}^{N/2} \) be given. The trellis \( T = (\mathcal{V}, \dot{\mathcal{E}}, \dot{w}, \dot{\ell}, \dot{q}, \dot{r}) = T^{[1]} \) is defined as follows.

- The vertex set of \( \tilde{T} \) is the same as the minus trellis \( T^{[0]} \).
- This is also the case for the functions \( \dot{q} \) and \( \dot{r} \).
- We next define the edge set \( \dot{E} \) implicitly. Consider an edge \( \tilde{e} = \alpha \rightarrow \gamma \in \dot{E} \) in section \( j \) of \( \tilde{T} \) with label \( \ell(\tilde{e}) = x' \).

Then,
\[
\alpha \in \dot{V}_{j-1} = V_{2j-2} \quad \text{and} \quad \gamma \in \dot{V}_j = V_{2j} .
\]

The weight \( \dot{w}(\tilde{e}) \) of this edge equals the sum of the product of the edge weights along each two-step path \( \alpha \rightleftharpoons \beta \rightarrow \gamma \) in \( T \) with \( \ell(e_1) \oplus \ell(e_2) = z_j \) and \( \ell(e_2) = x' \).

That is,
\[
\dot{w}(\tilde{e}) = \sum_{e_1 \in E_{2j-1}} \sum_{\sigma(\tilde{e}) = \alpha} \sum_{\tau(e_2) = \gamma} w(e_1) w(e_2) \cdot \sigma(e_1) \cdot \tau(e_2) \cdot [\ell(e_1) \oplus x' = z_j] \cdot [\ell(e_2) = x'_j] .
\]

Edges with weight 0 may be removed from \( \tilde{T} \).

This lemma states the key property of plus transform.

**Lemma 6:** Let \( T \) be a length \( N \) trellis with \( N \) even and let \( z \in \mathcal{X}^{N/2} \) be given. Construct \( T^{[1]} \) with respect to fixed vector \( z \), then for any \( x' \in \mathcal{X}^{N/2} \), we have
\[
T^{[1]}(x') = T(x) , \quad \text{where} \ x^{[0]} = z \quad \text{and} \ x^{[1]} = x' .
\]

Note that the vector \( x \in \mathcal{X}^N \) is uniquely defined by \( x' \) and \( z \).

**Proof:** This follows from the fact that the plus trellis is constructed by merging adjacent trellis stages and then pruning paths that do not satisfy \( x^{[0]} = z \). Finally, the remaining paths are relabeled with \( x^{[1]} \) values. \( \square \)

**C. Successive cancellation decoding**

As in Arkan’s seminal paper [18], the transform defined above leads to a successive cancellation decoding algorithm. In brief, given \( y \) we first construct a base trellis \( \tilde{T} \). Then, there is a recursive decoder that, given \( T^{[1]} \), constructs \( T^{[b_1,b_2,\ldots,b_k,0]} \) and calls itself with that argument. When this returns the decoded vector \( x^{[b_1,b_2,\ldots,b_k,0]} \), it then builds \( T^{[b_1,b_2,\ldots,b_k,1]} \) with respect to those hard decisions and calls itself to decode \( x^{[b_1,b_2,\ldots,b_k,1]} \). Then, the two decoded vectors are combined to form \( x^{[b_1,b_2,\ldots,b_k]} \) and the function returns. The following lemma makes this precise.

**Lemma 7:** Let \( T \) be a base trellis with \( N = 2^n \) sections corresponding to a received word \( y \). For each \( i \in [N] \) in order, let \( u_i^{-1} \) be a vector of past decisions and \( b_1, b_2, \ldots, b_n \in \{0, 1\} \) satisfy \( i(b) = i \). Construct \( T^{[b_1,b_2,\ldots,b_k]} \) iteratively as follows. For \( \lambda = 1, 2, \ldots, n \), let us define
\[
T^{[b_1,b_2,\ldots,b_k,\lambda]} = \begin{cases} T^{[b_1,b_2,\ldots,b_k,\lambda-1]} & \text{if } \lambda \geq 2 , \\ T^{[b_1]} & \text{if } \lambda = 1 . \end{cases}
\]

If \( b_\lambda = 1 \), then we apply the plus transform with respect to the fixed vector
\[
\tilde{x}^{[b_1,b_2,\ldots,b_k,\lambda-1]} = A^{-1}(\tilde{u}_\lambda) ,
\]
where \( \tilde{u}_\tau^{\theta} = (\tilde{u}_\tau, \tilde{u}_{\tau+1}, \ldots, \tilde{u}_\theta) \) and
\[
\theta = \sum_{j=1}^{b_\lambda} 2^{n-j} , \quad \tau = 2^{n-\lambda} + 1 .
\]

Then, for \( U = \mathcal{A}_u(X) \in \mathcal{X}^N \) we have
\[
T^{[b_1,b_2,\ldots,b_k]}(u) = \Pr(U_i = u, U_{i-1} = \tilde{u}_i^{-1}, Y = y) .
\]

**Sketch of Proof:** The workings of this decoder are a natural generalization of those in the SC decoder presented in the seminal paper [13]. Namely, at step \( i \), we make a decision as to the value of \( \tilde{u}_i \). Then, this decision is propagated to previous levels in the decoder. \( \square \)

Actually, the above lemma is not unique to the deletion channel and it applies to any base trellises for which \( \mathcal{L} \) holds. The above lemma also gives an efficient method for deciding the value of \( \tilde{u}_i \) at stage \( i \), since
\[
\Pr(U_i = u | U_{i-1} = \tilde{u}_i^{-1}, Y = y) = \frac{T^{[b_1,b_2,\ldots,b_k]}(u)}{\sum_{u' \in \mathcal{X}} T^{[b_1,b_2,\ldots,b_k]}(u')} .
\]

**V. INFORMATION RATES**

In this section, we will introduce and analyze various information rates related to polar codes on the deletion channel. For a given regular hidden-Markov input distribution, let \( X \) be an input vector of length \( N \) and let \( Y \) be the corresponding output vector (i.e., the observation of \( X \) through the deletion
channel). The main goal of this paper is to show that our polar coding scheme achieves the information rate

$$\mathcal{I} = \lim_{N \to \infty} \frac{I(X; Y)}{N}, \quad (16)$$

where $X$ and $Y$ depend implicitly on $N$. This existence of this limit is well-known [2] but we revisit it here because the same argument will be used later with slight variations.

**Lemma 8**: Fix a hidden-Markov input distribution. For a given $N$, let $X = (X_1, X_2, \ldots, X_N)$ be a random vector with the above distribution. Let $Y$ be the result of passing $X$ through a deletion channel with deletion probability $\delta$. Then, the following two limits exist,

$$\lim_{N \to \infty} \frac{H(X)}{N} \quad \text{and} \quad \lim_{N \to \infty} \frac{H(X|Y)}{N}. \quad (17)$$

**Proof**: The proof of this lemma is detailed below for uniform inputs in Section V-A and hidden-Markov inputs in Section V-B.

Once the limits in (17) are established, the limit in (16) follows because

$$\frac{I(X; Y)}{N} = \frac{H(X)}{N} - \frac{H(X|Y)}{N}. \quad (18)$$

**A. Uniform input**

In this subsection, we prove Lemma 8 for the restricted case in which the input distribution is i.i.d. and uniform.

**Proof of Lemma 8 for Uniform Inputs**: In such a setting, the first limit in (17) clearly exists and equals 1. To prove the second limit in (17), let us first define

$$\mathcal{H}_N = H(X|Y), \quad |X| = N. \quad (19)$$

Our plan is to show that the sequence $\mathcal{H}_N$ is superadditive, implying [20] Lemma 1.2.1, page 3 the existence of the second limit in (17). Indeed, let $N_1$ and $N_2$ be given, and let $X$ and $X'$ be distributed according to the input distribution, and having lengths $N_1$ and $N_2$, respectively. Denote the outputs corresponding to $X$ and $X'$ by $Y$ and $Y'$, respectively. We have

$$\mathcal{H}_{N_1+N_2} = H(X \odot X'|Y \odot Y') = H(X, X'|Y \odot Y') \quad (a)$$

$$\geq H(X, X'|Y, Y') \quad (b)$$

$$\geq H(X|Y, Y') + H(X'|Y, Y') \quad (c)$$

$$= H(X|Y) + H(X'|Y') \quad = \mathcal{H}_{N_1} + \mathcal{H}_{N_2},$$

where (a) holds because $Y \odot Y'$ is a function of $Y$ and $Y'$; (b) follows by the chain rule; (c) holds because, for the i.i.d. uniform input distribution, the pair $(X, Y)$ is independent of the pair $(X', Y')$. Hence, the sequence $\mathcal{H}_N$ is indeed superadditive.

**B. Hidden-Markov input**

We now prove Lemma 8 for the case where the input distribution is a regular hidden-Markov process. Since now $\mathcal{H}_N$ is not generally superadditive, we will take an indirect route to prove Lemma 8. Indeed, the following lemma is proved by defining a related quantity, $\hat{\mathcal{H}}_N$, which is superadditive.

**Lemma 9**: Fix a regular hidden-Markov input distribution. For a given $N$, let $X = (X_1, X_2, \ldots, X_N)$ be a random vector with the above distribution. Let $Y$ be the result of passing $X$ through a deletion channel with deletion probability $\delta$. Then, the following limit exist,

$$\lim_{N \to \infty} \frac{H(Y|S_0, S_N)}{N}. \quad (20)$$

**Proof**: Define

$$\hat{\mathcal{H}}_N = H(X|Y, S_0, S_N), \quad |X| = N. \quad (21)$$

To borrow the terminology of [15], the above defines the boundary-state-aware entropy. Note that $S_0$ and $S_N$ are the states just before transmission has started, and just after transmission has ended, respectively.

We now show that $\hat{\mathcal{H}}_N$ is superadditive. Indeed, let $X$ and $X'$ be consecutive input vectors of length $N_1$ and $N_2$, respectively. That is, $X \odot X'$ is a vector of length $N_1 + N_2$ drawn from the input distribution. Denote by $Y$ and $Y'$ the output vectors corresponding to $X$ and $X'$, respectively. Then,

$$\hat{\mathcal{H}}_{N_1+N_2} = H(X \odot X'|Y \odot Y', S_0, S_{N_1+N_2})$$

$$\geq H(X, X'|Y, Y', S_0, S_{N_1+N_2}) \quad (a)$$

$$\geq H(X, X'|Y, Y', S_0, S_N, S_{N_1+N_2}) \quad (b)$$

$$= H(X, X'|Y, Y', S_0, S_N, S_{N_1+N_2})$$

$$+ H(X|Y, Y', S_0, S_N, S_{N_1+N_2}) \quad (c)$$

$$+ H(X|Y, S_0, S_N) + H(Y'|Y', S_N, S_{N_1+N_2})$$

$$= \hat{\mathcal{H}}_{N_1} + \hat{\mathcal{H}}_{N_2},$$

where (a) holds because $Y \odot Y'$ is a function of $Y$ and $Y'$; (b) follows by the chain rule; (c) holds because of conditional independence: given $S_{N_1}$, $(X, Y, S_0)$ is independent of $(X', Y', S_{N_1+N_2})$. Hence, the sequence $\hat{\mathcal{H}}_N$ is indeed superadditive, and the following limit exists by [20] Lemma 1.2.1, page 3,

$$\lim_{N \to \infty} \frac{\hat{\mathcal{H}}_N}{N}. \quad (22)$$

All that remains now is to account for the difference in the entropies of $\mathcal{H}_N$ and $\hat{\mathcal{H}}_N$, incurred by conditioning on $S_0$ and $S_N$. As will be made clear in the following proof, this difference can be bounded by a constant, and hence vanishes when we divide by $N$.

**Proof of Lemma 8 for hidden-Markov inputs**: We first note that the existence of the second limit in (17) implies the existence of the first limit. Indeed, taking the deletion probability $\delta$ equal to 1 makes the second limit equal the first. Hence, all that remains is to prove the existence of the second limit.
To show that the second limit in (17) exists, note that, for $|X| = N$, we have on the one hand that

\[ H(X, S_0, S_N | Y) = H(X | Y) + H(S_0, S_N | X, Y) \]

\[ \geq H(X | Y) = \mathcal{H}_N, \]

and on the other hand that

\[ H(X, S_0, S_N | Y) = H(S_0, S_N | Y) + H(X | S_0, S_N, Y) \]

\[ \leq 2 \log_2 |S| + H(X | S_0, S_N, Y) \]

\[ = 2 \log_2 |S| + \mathcal{H}_N. \]

Thus, \[ \mathcal{H}_N \leq \mathcal{H}_N + 2 \log_2 |S|. \]

Since it is easily seen that \( \mathcal{H}_N \leq \mathcal{H}_N \), we have that

\[ \frac{\mathcal{H}_N}{N} \leq \frac{\mathcal{H}_N}{N} + \frac{2 \log_2 |S|}{N}. \]

(21)

We have already proved that the limit of the LHS of (21) exists, in Lemma 9. Since the limit of \((2 \log_2 |S|)/N\) is 0, the limit of the RHS of (21) exists, and equals that of the LHS. By the sandwich property, the limit of the middle term exists as well, which is the desired result.

We finish by restating the last part of the proof as a lemma.

**Lemma 10:** Fix a hidden-Markov input distribution. For a given $N$, let $X = (X_1, X_2, \ldots, X_N)$ be a random vector with the above distribution. Let $Y$ be the result of passing $X$ through a deletion channel with deletion probability $\delta$. Then,

\[ \lim_{N \to \infty} \frac{H(X | Y, S_0, S_N)}{N} = \lim_{N \to \infty} \frac{H(X | Y)}{N}. \]

(22)

**VI. WEAK POLARIZATION**

In this section, we prove weak polarization for both the deletion channel and the trimmed deletion channel, as defined in Subsection III-C. As in [13], we will first prove that a certain process is submartingale, and then prove that it either converges to 0 or to 1.

As a first step, we will shortly define three entropies. These are defined with respect to an input $X$ of length $N = 2^n$, which has a regular hidden-Markov input distribution, and $U = A_n(X)$. The corresponding output is denoted $Y$. Recall that $S_0$ and $S_N$ are the (hidden) states of the input process, just before $X$ is transmitted and right after $X$ is transmitted, respectively. Lastly, denote by $Y^*$ the result of trimming all leading and trailing ‘0’ symbols from $Y$. Then, for a given $n$ and $1 \leq i \leq N = 2^n$, define the following (deterministic) entropies:

\[ h_i = H(U_i | U_{i-1}^{|2^{i-1}|}, Y), \]

(23)

\[ \hat{h}_i = H(U_i | U_{i-1}^{|2^{i-1}|}, S_0, S_N, Y), \]

(24)

\[ h_i^* = H(U_i | U_{i-1}^{|2^{i-1}|}, Y^*). \]

(25)

Clearly,

\[ h_i^* \geq \hat{h}_i \geq h_i. \]

Note that in the case of a uniform input distribution, there is only one state, and hence $h_i$ and $\hat{h}_i$ are equal.

Following [13], we show weak polarization by considering a sequence $B_1, B_2, \ldots$ of i.i.d. Ber(1/2) random variables. For any $n \in \mathbb{N}$, let $J_n = i(B_1, B_2, \ldots, B_n)$ be the random index defined by [4], with $b_i$ in place of $b_n$. We will study the three related random processes defined for $n \in \mathbb{N}$ by

\[ H_n = h_{J_n}, \]

(26)

\[ \hat{H}_n = \hat{h}_{J_n}, \]

(27)

\[ H_n^* = h_{J_n}^*. \]

(28)

The arguments below will show that that $\hat{H}_n$ is a submartingale, converging to either 0 or 1. From this we will infer that $H_n$ and $H_n^*$ must converge to either 0 or 1 as well. Though neither $H_n$ nor $H_n^*$ are necessarily submartingales.

**Theorem 11:** The sequence $\hat{H}_n$ converges (almost surely and in $L^1$) to a well-defined random variable $\hat{H}_\infty \in \{0, 1\}$ and, for any $\epsilon > 0$, it follows that

\[ \frac{1}{N} \left| \left\{ i \in [N] \mid H(U_i | U_1^{i-1}, S_0, S_N, Y) \in [\epsilon, 1 - \epsilon] \right\} \right| \to 0. \]

(29)

**Proof:** Lemma 12 below shows that $H_1, \hat{H}_2, \hat{H}_3, \ldots \in [0, 1]$ is a bounded submartingale with respect to $J_n$. This implies that the sequence $\hat{H}_n$ converges (almost surely and in $L^1$) to a limit that is denoted by $\hat{H}_\infty$ [21] p. 236. Lemma 17 shows that, for any $\epsilon > 0$, there is a $\Delta > 0$ such that $\hat{H}_n \in [\epsilon, 1 - \epsilon]$ implies $H_{n+1} > H_n + \Delta$ with probability $\frac{1}{2}$. Thus, the sequence $H_n$ cannot converge to the set (0,1) and hence $\hat{H}_\infty \in \{0, 1\}$.

From (23) and (25), we see that $\Pr(\hat{H}_n \in [\epsilon, 1 - \epsilon])$ equals

\[ \frac{1}{N} \left| \left\{ i \in [N] \mid H(U_i | U_1^{i-1}, S_0, S_N, Y) \in [\epsilon, 1 - \epsilon] \right\} \right|. \]

Since $\hat{H}_n$ converges almost surely to $\hat{H}_\infty$ and $\epsilon, 1 - \epsilon$ are continuity points of $\Pr(\hat{H}_\infty \leq x)$ [21] Ch. 4, it follows that

\[ \lim_{n \to \infty} \Pr(\hat{H}_n \in [\epsilon, 1 - \epsilon]) = \Pr(\hat{H}_\infty \in [\epsilon, 1 - \epsilon]) = 0. \]

This completes the proof.

**Lemma 12:** For a hidden-Markov input distribution and a deletion channel with deletion probability $\delta$, let $H_n$ and $J_n$ be as defined above. Then, the sequence $H_1, H_2, H_3, \ldots$ is a bounded submartingale with respect to the $J_1, J_2, J_3, \ldots$ sequence.

**Proof:** Since $\hat{H}_n$ is clearly bounded between 0 and 1, it remains to show that $E(\hat{H}_{n+1} | J_1, J_2, \ldots, J_n) \geq \hat{H}_n$. Let $X \otimes X'$ be a length-2N input to the channel. Denote by $Y \otimes Y'$ the corresponding output, where $Y$ only contains inputs from $X$ and $Y'$ only contains inputs from $X'$. Recall that $U = A_n(X)$ and define $V = A_n(X')$ and

\[ F = (U_1 \oplus V_1, U_2 \oplus V_2, \ldots, U_N \oplus V_N, V_N). \]

Then,

\[ E(\hat{H}_{n+1} | J_n^*) = E\left(H \left( F_{J_{n+1}} | F_{J_{n+1}}^* \oplus Y \otimes Y', S_0, S_{2N} \right) \right) \]

\[ = \frac{1}{2} H \left( F_{J_2, J_{n+1}} | F_{J_2, J_{n+1}}^*, Y \otimes Y', S_0, S_{2N} \right) \]

\[ + \frac{1}{2} H \left( F_{J_2, J_{n+1}} | F_{J_2, J_{n+1}}^*, Y \otimes Y', S_0, S_{2N} \right) \]
\[ = \frac{1}{2} H(F_{2Jn} - F_{2Jn}, F_{2Jn} - F_{1Jn} - \beta - Y \otimes Y', S_0, S_{2N}) \]
\[ = \frac{1}{2} H(U_{Jn} \oplus V_n, V_n, F_{2Jn} - Y \otimes Y', S_0, S_{2N}) \]
\[ = \frac{1}{2} H(U_{Jn} \oplus V_n, V_n, F_{2Jn} - Y \otimes Y', S_0, S_{2N}) \]
\[ = \frac{1}{2} H(U_{Jn} \oplus V_n, V_n, F_{2Jn} - Y \otimes Y', S_0, S_{2N}) \]
\[ = \frac{1}{2} H(U_{Jn} \oplus V_n, V_n, F_{2Jn} - Y \otimes Y', S_0, S_{2N}) \]
\[ = \frac{1}{2} H(U_{Jn} \oplus V_n, V_n, F_{2Jn} - Y \otimes Y', S_0, S_{2N}) \]
\[ = \frac{1}{2} H(U_{Jn} \oplus V_n, V_n, F_{2Jn} - Y \otimes Y', S_0, S_{2N}) \]
\[ = \frac{1}{2} H(U_{Jn} \oplus V_n, V_n, F_{2Jn} - Y \otimes Y', S_0, S_{2N}) \]
\[ = \frac{1}{2} H(U_{Jn} \oplus V_n, V_n, F_{2Jn} - Y \otimes Y', S_0, S_{2N}) \]

The inequality (a) follows from the fact that \( Y \otimes Y' \) is a deterministic function of \( Y, Y' \). Inequality (b) follows since conditioning reduces entropy. Step (c) holds by the Markov property. Finally, (d) is due to stationarity, \( H_n = H(U_{Jn}, V_n, Y, S_0, S_{2N}) = H(V_n, Y, S_0, S_{2N}) \), \[ H_n = H(U_{Jn}, V_n, Y, S_0, S_{2N}) = H(V_n, Y, S_0, S_{2N}) \).

Since the sequence \( H_n \) is a bounded submartingale, it converges almost surely and in \( L_1 \) to a random variable \( \hat{H}_n \in [0, 1] \). To show that \( \hat{H}_n \) is a deterministic function of \( Y, Y' \). Inequality (b) follows since conditioning reduces entropy. Step (c) holds by the Markov property. Finally, (d) is due to stationarity, \( H_n = H(U_{Jn}, V_n, Y, S_0, S_{2N}) = \frac{1}{2} H(U_{Jn} | F_{2Jn} - Y \otimes Y', S_0, S_{2N}) \).

The following lemma states how \( \alpha \) and \( \beta \) are related to our quantities of interest, \( H_n \) and \( \hat{H}_n \).

**Lemma 15:** Let \( N = 2^m > 2^{n_0} \), where \( n_0 \) was promised in Lemma 14. Then, for \( \alpha \) and \( \beta \) as defined above, we have that
\[ \hat{H}_n \geq \sum_{s_0,s_n,s_{2N} \in S} p(s_0, s_n, s_{2N}) \cdot \alpha(s_0, s_n, s_{2N}) \],
\[ \hat{H}_n \geq \sum_{s_0,s_n,s_{2N} \in S} p(s_0, s_n, s_{2N}) \cdot \alpha(s_0, s_n, s_{2N}) \]

Furthermore, for all \( s_0, s_n, s_{2N} \in S \),
\[ \alpha(s_0, s_n, s_{2N}) \geq \beta(s_0, s_n, s_{2N}) \]

**Proof:** Define \( i = J_n \). In order to prove (36), we proceed similarly to the proof in Lemma 12 and deduce that
\[ \hat{H}_n = H(U_i \oplus V_i, V_i, Y \otimes Y', S_0, S_{2N}) \]
\[ H(U_i \oplus V_i | U_i^{i-1}, V_i^{i-1}, Y, Y', S_0, S_{2N}) \geq H(U_i | U_i^{i-1}, V_i^{i-1}, Y, Y', S_0, S_{2N}) \]

\[ = \sum_{s_0, s_N, s_{2N} \in S} p(s_0, s_N, s_{2N}) \cdot \alpha(s_0, s_N, s_{2N}) \]

The proof of (37) follows by stationarity. That is,

\[ \hat{H}_n = H(U_i | U_i^{i-1}, Y, S_0, S_N) \]

\[ = H(U_i | U_i^{i-1}, Y, S_0, S_N) + H(V_i | V_i^{i-1}, Y', S_N, S_{2N}) \]

\[ = \sum_{s_0, s_N, s_{2N} \in S} p(s_0, s_N, s_{2N}) \times \frac{\gamma(s_0, s_{2N}) + \gamma(s_N, s_{2N})}{2} \]

\[ = \sum_{s_0, s_N, s_{2N} \in S} p(s_0, s_N, s_{2N}) \times \beta(s_0, s_N, s_{2N}) \].

By (34), we deduce that (38) will follow from proving that

\[ \alpha(s_0, s_N, s_{2N}) \geq \gamma(s_0, s_N) \quad (39) \]

and

\[ \alpha(s_0, s_N, s_{2N}) \geq \gamma(s_N, s_{2N}) \quad (40) \]

W.l.o.g., we prove (39). Indeed, given that $S_N = s_N$, we have by Markovity that $(S_0, U_i^{i-1}, U_i, Y)$ and $(V_i^{i-1}, V_i, Y', S_2N)$ are independent. Hence, for any $s_{2N}$ we may also write $\gamma$, defined in (35), as

\[ \gamma(s_0, s_N) = H(U_i | U_i^{i-1}, V_i, Y, Y', S_0 = s_0, S_N = s_N, S_{2N} = s_{2N}) \].

Lastly, note that in the above expression for $\gamma$, since we condition on $V_i$, we could have written $U_i \oplus V_i$ in place of $U_i$. This would give us the expression for $\alpha$ in (33), up to a further conditioning on $V_i$. Since conditioning reduces entropy, (39) follows. As noted, the proof of (40) is similar. Hence, we deduce (38).

In light of Lemma 16 our plan is to show the existence of a triplet $(s_0, s_N, s_{2N})$ for which $\alpha(s_0, s_N, s_{2N})$ is substantially greater than $\beta(s_0, s_N, s_{2N})$. The next lemma assures us such a triplet indeed exists.

**Lemma 17:** For every $\epsilon > 0$ there exists a $\Delta' = \Delta'(\epsilon)$ for which the following holds. Let $N = 2^n > 2^{n_0}$, where $n_0$ was promised in Lemma 14. Then, if $\epsilon \leq \hat{H}_n \leq 1 - \epsilon$, there exists a triplet $(s_0, s_N, s_{2N})$ such that

\[ \alpha(s_0, s_N, s_{2N}) > \beta(s_0, s_N, s_{2N}) + \Delta' \quad (41) \]

Proof: By definition of $\gamma$ in (35), we have that

\[ \hat{H}_n = \sum_{s_0, s_N, s_{2N} \in S} \text{Pr}(S_0 = s_0, S_N = s_N) \cdot \gamma(s_0, s_N) \quad (42) \]

A crucial point will be to show the existence of a triplet $(s_0, s_N, s_{2N})$ for which either

\[ \gamma(s_0, s_N) \leq \hat{H}_n \quad \text{and} \quad \gamma(s_N, s_{2N}) \geq \hat{H}_n \quad (43) \]

or

\[ \gamma(s_0, s_N) \geq \hat{H}_n \quad \text{and} \quad \gamma(s_N, s_{2N}) \leq \hat{H}_n \quad (44) \]

Assume to the contrary that this is not the case. Fix some state $\rho \in S$, and note that one of the following two assertions must hold:

1) For all $s_0, s_{2N} \in S$, we have $\gamma(s_0, \rho) < \hat{H}_n$ and $\gamma(\rho, s_{2N}) < \hat{H}_n$.

2) For all $s_0, s_{2N} \in S$, we have $\gamma(s_0, \rho) > \hat{H}_n$ and $\gamma(\rho, s_{2N}) > \hat{H}_n$.

Indeed, if this were not the case, then either (43) or (44) would hold, with $S_N = \rho$.

Assume w.l.o.g. that assertion 1 holds for our fixed $\rho$. Now consider some arbitrary $\rho' \in S$ such that $\rho' \neq \rho$. Again, by assumption, either assertion 1 or assertion 2 must hold, with $\rho'$ in place of $\rho$. In fact, assertion 1 must hold. Indeed, we have previously established that $\gamma(\rho, s_{2N}) < \hat{H}_n$ for all $s_{2N}$. In particular, we can set $s_{2N} = \rho'$, and deduce that $\gamma(\rho, \rho') < \hat{H}_n$. Thus, if assertion 2 were to hold for $\rho'$, we would have $\gamma(s_0, \rho') > \hat{H}_n$ for all $s_0$. In particular, we could set $s_0 = \rho$, and deduce that $\gamma(\rho, \rho') > \hat{H}_n$, contradicting our previous conclusion.

From the above paragraph, we conclude that for all $s_0, s_N \in S$, we must have that $\gamma(s_0, s_N) < \hat{H}_n$. However, recalling from (42) that $\hat{H}_n$ is a weighted average of such $\gamma$ terms, we arrive at a contradiction. Hence, there exists a triplet $(s_0, s_N, s_{2N})$ for which either (43) or (44) holds. This is the triplet we are searching for.

Fix the above defined triplet, $s_0, s_N, s_{2N}$, and assume w.l.o.g. that (43) holds. Our result now follows by combining part (i) of (22) Lemma 2.2 with (23) Lemma 11.

Combining Lemmas 15 and 16 gives the following key result.

**Lemma 18:** For every $n \in \mathbb{N}$, let $A_n$ and $B_n$ be real random variables defined on a common probability space. Suppose $B_n$ converges in $L^1$ to $B_\infty$ and $E(A_n)$ converges to $E(B_\infty)$. If $A_n \geq B_n$ for all $n \in \mathbb{N}$, then $A_n$ converges in $L^1$ to $B_\infty$.

Proof: By definition, $B_n$ converges to $B_\infty$ in $L^1$ if and only if $E(|B_n - B_\infty|) \to 0$. Thus, we can write

\[ E(|A_n - B_\infty|) \leq E(|A_n - B_n|) + E(|B_n - B_\infty|) \]

\[ = E(A_n - B_n) + E(|B_n - B_\infty|) \]

In the limit as $n \to \infty$, the first two terms converge to $E(B_\infty)$ and the last term converges to 0. Thus, $E(|A_n - B_\infty|) \to 0$. The following theorem claims weak polarization for the three cases discussed earlier.

**Theorem 19:** Fix $\epsilon \in (0, 1)$ and let $N = 2^n$. Then,

\[ \lim_{n \to \infty} \left\lfloor \frac{\lfloor i : H(U_i | U_i^{i-1}, Y, S_0, S_N) < \epsilon \rfloor}{N} \right\rfloor \]

The first two strict inequalities in the statement of (23) Lemma 11] are essentially typos: they should both be replaced by weak inequalities, as is evident from reading the beginning of the proof.
\[
\text{Proof: For simplicity, the proof is split into 4 parts.}
\]

\textbf{Part I:} \((45a)\) and \((46d)\) are well defined: First, recall from Lemma 8 that \(\lim_{n \to \infty} E(X|Y) / N\) exists. Thus, the right hand sides of both \((45a)\) and \((46d)\) are well defined.

\textbf{Part II:} \((45a) \Rightarrow (45d)\) and \((46a) \Rightarrow (46d)\): Since the Ankan transform is invertible, it follows that \(\mathcal{H}_N = H(X|Y, S_0, S_N) = H(U|Y, S_0, S_N)\), where \(\mathcal{H}_N\) is defined in \((20)\). Thus, from the chain rule for entropy, we observe that
\[
E(\hat{H}_n) = \frac{1}{N} \sum_{i=1}^{N} H(U_i|U_i^{i-1}, Y, S_0, S_N)
= \frac{1}{N} H(U|Y, S_0, S_N)
= \frac{1}{N} \mathcal{H}_N.
\]

From Theorem 11 we see that \(\hat{H}_n\) converges in \(L^1\) to \(\hat{H}_\infty\) \(\in\{0,1\}\). This implies that \(E(\hat{H}_\infty) = \lim_{n \to \infty} E(\hat{H}_n)\) which exists and equals \(\lim_{N \to \infty} \mathcal{H}_N / N\) by Lemma 9. Since \(\hat{H}_\infty \in \{0,1\}\), observing that \(E(\hat{H}_\infty) = \Pr(\hat{H}_\infty = 1)\) shows that
\[
(45a) = \lim_{n \to \infty} \Pr(\hat{H}_n > 1 - \epsilon) = \Pr(\hat{H}_\infty = 1) = \lim_{n \to \infty} \frac{1}{N} \mathcal{H}_N,
\]
where the second equality holds because convergence in \(L^1\) implies convergence in distribution and \(1 - \epsilon\) is a continuity point of \(\Pr(\hat{H}_\infty \leq x)\) [21] Ch. 4. Since Lemma 10 shows that \(\lim_{N \to \infty} \mathcal{H}_N / N\) equals \((46d)\), it follows that \((46a)\) equals \((46d)\). The last step is observing that
\[
(45a) = \lim_{n \to \infty} \Pr(\hat{H}_n < \epsilon) = \Pr(\hat{H}_\infty = 0) = 1 - \Pr(\hat{H}_\infty = 1)
\]
holds because convergence in \(L^1\) implies convergence in distribution and \(\epsilon\) is a continuity point of \(\Pr(\hat{H}_\infty \leq x)\). Thus, \((45a)\) equals \((45d)\).

\textbf{Part III:} \((45c) \Rightarrow (45d)\) and \((46c) \Rightarrow (46d)\): To prove these equalities, we will apply Lemma 18 to the sequences \(A_n = H_n^*\) and \(B_n = H_n\). Theorem 11 shows that \(\hat{H}_n\) converges in \(L^1\) to \(\hat{H}_\infty\) and we established in the previous part that \(E(\hat{H}_\infty)\) equals \((46d)\). From the definitions in \((27)\) and \((28)\), it follows that \(H_n^* \geq H_n\) for all \(n \in \mathbb{N}\). The only other element required for Lemma 18 is that \(E(H_n^*) \to E(H_\infty)\) and this will be shown below. Assuming this for now, we observe Lemma 18 implies that \(H_n^*\) converges in \(L^1\) to \(\hat{H}_\infty\) and gives the desired result
\[
\text{where the second equality on each line holds because convergence in \(L^1\) implies convergence in distribution and \(1 - \epsilon\) are continuity points of \(\Pr(\hat{H}_\infty \leq x)\) [21] Ch. 4.}
\]
To show that \(E(H_n^*) \to E(\hat{H}_\infty)\), we will use the fact that
\[
H(U|Y, S_0, S_N) \leq H(U|Y^*) \leq H(U|Y, S_0, S_N) + 2 \log_2 |S| + 2 \log_2 (N + 1).
\]
Indeed, the first inequality holds because \(Y^*\) is a function of \(Y\). The second inequality follows from first noting that
\[
H(U, S_0, S_N, Y|Y^*) \geq H(U|Y^*).
\]
And then observing that
\[
H(U, S_0, S_N, Y|Y^*) = H(Y|Y^*) + H(S_0, S_N|Y, Y^*) + H(U|S_0, S_N, Y, Y^*)
\]
holds because \(Y^*\) is a function of \(Y\), \(Y\) follows from \(S_0\) and \(S_N\) each having a support of size \(|S|\), and \(Y\) follows since in order to construct \(Y\) from \(Y^*\), it suffices to be told how many ‘0’ symbols have been trimmed from each side of \(Y\), and both numbers are always between 0 and \(N\). Combining the above two displayed equations yields the RHS of \((47)\).

Finally, we divide both sides of \((47)\) by \(N\) and take the limit as \(N \to \infty\). Since the left-most and right-most terms converge to \(E(\hat{H}_\infty)\), the sandwich property implies that the center term, \(E(H_n^*)\) also converges to this quantity.

\textbf{Part IV:} \((45a) \Rightarrow (45b)\) and \((46a) \Rightarrow (46b)\): Note that, for \(1 \leq i \leq N\), we have
\[
H(U_i|U_i^{i-1}, Y, S_0, S_N) \leq H(U_i|U_i^{i-1}, Y) \leq H(U_i|U_i^{i-1}, Y^*)
\]
We have already proved that \((45a) \Rightarrow (45c)\) and \((46a) \Rightarrow (46c)\). Thus, by the sandwich property, \((45a) \Rightarrow (45b)\) and \((46a) \Rightarrow (46b)\).

\[\]

\textbf{VII. Strong polarization}\n
To rigorously claim a coding scheme for the deletion channel, one must also show strong polarization. For this, Theorem 19 is not sufficient and, so far, we have been unable to prove strong polarization for the standard polar code construction. Thus, we will modify the standard coding scheme to proceed.
A. Coding Scheme for Strong Polarization

The basic idea is to use standard polar encoding for the first \( n_0 \) stages, and then to add a guard band in the middle of the codeword during each additional encoding stage. That is, we will have independent blocks of length \( N_0 = 2^{n_0} \) bits distributed according to our input distribution, and between each two consecutive blocks we will have a string of ‘0’ symbols, which we term a guard band. The real trick is to remove these guard bands in a controlled fashion.

Assume for a moment that this can be done perfectly. If that were the case, the effect of the guard bands would be to add commas between blocks of length \( N_0 \). The received sequence would then become \( Y_1, Y_2, \ldots, Y_{\phi} \), where \( \Phi = 2^{n-n_0} \) and \( Y_{\phi} \) is the output of the channel corresponding to the input segment \( X_i = X_{\phi-i}^{\phi-i} \). In this case, the blocks are statistically independent and hence strong polarization occurs after stage \( n_0 \). The claim just made about strong polarization is a bit subtle: we carry out one process for the first \( n_0 \) stages, and another for the rest. Hence, we are in a different setting than that considered in the seminal paper on strong polarization, [24]. However, by [25, Lemma 40], we indeed have strong polarization (see also [26]). The scheme by Honda and Yamamoto [27] can be used to encode the information bits into length-2\(^n_0\) codeword blocks that are consistent with the hidden-Markov input distribution.

Our procedure to remove the above guard bands will be imperfect. Let the transmitted word be \( G_1 \odot G_\Delta \odot G_{II} \), where \( G_\Delta \) is a string of ‘0’ symbols termed the guard band, and \( G_1 \) and \( G_{II} \) are of equal length. Denote the corresponding parts of the received word by \( Y_1, Y_\Delta, \) and \( Y_{II} \). As a preliminary step, we will remove from the received word \( Y \) all leading and trailing ‘0’ symbols. Then, we will assume that the middle index (rounding down) in the resulting word originated from a guard band symbol. We will partition the word into two words according to this middle index, and remove all leading and trailing ‘0’ symbols from these two words. A moment’s thought reveals that if our assumption is correct (the middle index corresponds to a guard band symbol), then the two resulting words are simply \( Y^*_1 \) and \( Y^*_{II} \). That is, \( Y_1 \) and \( Y_{II} \), with leading and trailing ‘0’ symbols removed. That is, in effect, we have transmitted \( G_1 \) and \( G_{II} \) over a deletion channel, but over the trimmed deletion channel (TDC) defined earlier. We will apply this procedure recursively for \( n-n_0 \) stages. If during all the recursive steps the middle index does indeed belong to the corresponding guard band, we will have produced \( Y^*_1, Y^*_2, \ldots, Y^*_\phi \).

Recall that we have proved in Theorem 19 that weak polarization occurs also for the TDC channel, with the same proportion of high-entropy and low-entropy indices as in the deletion channel. Hence, our plan for this section is as follows. We first define exactly how the guard bands are added. That is, what is the length of a guard band at stage \( n \). Then, we show that this added length is negligible in terms of the rate penalty incurred. Finally, we show that our assumption of constantly hitting guard band symbols in our recursive partitioning is correct, with very high probability.

We start by defining how guard bands are added. For \( x = x_1 \odot x_{II} \in \mathcal{X}^{2^n} \) with

\[
\begin{align*}
x_1 &= x_1^{2^{n-1}} \in \mathcal{X}^{2^{n-1}}, \\
x_{II} &= x_{2^{n+1}}^{2^n} \in \mathcal{X}^{2^{n-1}}
\end{align*}
\]

being the first and second halves of \( x \), respectively, we define

\[
g(x) = \begin{cases} 
\phi & \text{if } n \leq n_0 \\
(\phi \odot 00 \ldots 0 \odot g(x_{II})) & \text{if } n > n_0,
\end{cases}
\]

where \( \phi \in \{0,1/2\} \) is a ‘small’ constant, to be specified later. Then, the channel input with added guard bands is given by \( g(x) \).

B. Proof of Strong Polarization

In this section, we begin by stating and proving a number of lemmas before combining them into the proof of Theorem 1.

Since \( \ell_n \) is defined by a fixed \( \epsilon > 0 \), the following lemma shows that the rate-penalty for transmitting \( g(x) \) in place of \( x \) becomes negligible as \( n_0 \) increases. In the sequel, we choose \( n_0 \) to be roughly \( n/3 \), so that the rate penalty is also negligible as \( n \to \infty \).

**Lemma 20:** Let \( x \) be a vector of length \( |x| = 2^n \). Then,

\[
|x| \leq |g(x)| < \left(1 + \frac{2^{-c\cdot n_0 + 1}}{2^{-\epsilon} - 1}\right) |x|.
\]

**Proof:** From the definition of \( g(x) \), induction shows

\[
|g(x)| = \begin{cases} 
2^n & \text{if } n \leq n_0 \\
2^n + \sum_{t=n_0+1}^{2^n} \ell_t & \text{otherwise.}
\end{cases}
\]

Thus, the lower bound in (50) is trivial, since \( |x| = 2^n \), and every term in the sum in (51) is non-negative, by (49). The upper bound in (50) is trivially true for \( n \leq n_0 \). For the case \( n > n_0 \), we have we have that

\[
|g(x)|/|x| \overset{(a)}{\leq} 1 + \sum_{t=n_0+1}^{2^n} \ell_t
\]

\[
\overset{(b)}{\leq} 1 + \sum_{t=n_0+1}^{2^n} 2^{-t} \cdot 2^{(-\epsilon)(t-1)}
\]

\[
= 1 + \sum_{t=n_0+1}^{2^n} 2^{-c(t-1)-1}
\]

\[
< 1 + \sum_{t=n_0+1}^{\infty} 2^{-c(t-1)-1}
\]

\[
= 1 + \frac{2^{-c\cdot n_0 + 1}}{2^{-\epsilon} - 1},
\]

where (a) follows from \( |x| = 2^n \) and (51); (b) follows from (49); (c) is simply the sum of geometric series.

**Lemma 21:** Let the guard-band lengths \( \ell_n \) in (49) use a fixed \( \epsilon \in \{0,1/2\} \). Fix the channel deletion probability \( \delta \) and a regular hidden-Markov input distribution.

Let \( n > n_0 \) and let \( X \) be a random vector of length \( 2^n \) distributed according to the modified input distribution described above: i.i.d. blocks of length \( N_0 = 2^{n_0} \), each distributed according to the specified input distribution. Denote by \( Y \) the result of transmitting \( g(X) \) through the deletion channel. Then, there exists a constant \( \theta > 0 \), dependent only

\[
\begin{align*}
x_1 \odot x_{II} \in \mathcal{X}^{2^n} & \\
x_1 = x_1^{2^{n-1}} \in \mathcal{X}^{2^{n-1}}, & \\
x_{II} = x_{2^{n+1}}^{2^n} \in \mathcal{X}^{2^{n-1}}
\end{align*}
\]
on the input distribution and the deletion probability such that, for \( n_0 \) large enough, the probability that the middle symbol of \( Y^* \) (rounding down) is not a ‘0’ from the guard band is at most \( 2^{-\frac{1}{2}(1-2)n_0} \).

**Proof:** Let \( G = g(X) \) (see Fig. 2). Denote the first and second halves of \( X \) by \( X_I \) and \( X_{II} \), respectively. Let \( G_I = g(X_I) \) and \( G_{II} = g(X_{II}) \), and denote by \( G_\Delta \) the length \( \ell_n \) guard band of ‘0’ corresponding to \( G_I \) and \( G_{II} \). Hence, by \(|G| = G_I \odot G_\Delta \odot G_{II} \).

Denote by \( Y \) the (untrimmed) result of passing \( G \) through the deletion channel. Let \( Y_I, Y_{II}, \) and \( Y_\Delta \) be the parts of \( Y \) corresponding to \( G_I, G_{II}, \) and \( G_\Delta \), respectively. Let \( Z = Y^* \) be the trimmed \( Y \). Define \( Z_I, Z_{II}, \) and \( Z_\Delta \), as the parts of \( Z \) corresponding to \( G_I, G_{II}, \) and \( G_\Delta \), respectively.

For \( Z = (Z_1, Z_2, \ldots, Z_t) \) with \( t \geq 1 \), the middle index of \( Z \) (rounding down) is \( s = \lfloor (t + 1)/2 \rfloor \). Thus, if \( Z \) is a guard band symbol, then the split and trim operation will succeed with

\[
(Z_1, \ldots, Z_t)^* = Y_I^* \quad \text{and} \quad (Z_{t+1}, \ldots, Z_r)^* = Y_{II}^*.
\]

A sufficient condition for success, which we will use, is

\[
|Z_I| < |Z_\Delta| + |Z_{II}|, \quad |Z_{II}| < |Z_I| + |Z_\Delta|.
\] (52)

To see that this is sufficient, we observe that \( |Z_I| < |Z_\Delta| + |Z_{II}| \) implies that the middle index does not fall in \( Z_I \) because then

\[
\lfloor (|Z| + 1)/2 \rfloor = \lfloor (|Z_I| + |Z_\Delta| + |Z_{II}| + 1)/2 \rfloor \geq \lfloor (|Z_I| + |Z_{II}| + 2)/2 \rfloor = |Z_I| + 1.
\]

Similarly, if \( |Z_{II}| < |Z_I| + |Z_\Delta| \), then the middle index does not fall in \( Z_{II} \) because then

\[
\lfloor (|Z| + 1)/2 \rfloor = \lfloor (|Z_I| + |Z_\Delta| + |Z_{II}| + 1)/2 \rfloor \leq \lfloor (|Z_I| + |Z_{II}| + |Z_\Delta|)/2 \rfloor = |Z_I| + |Z_\Delta|.
\]

Now, we will analyze the probability of a successful split. Denote by \( \alpha, \beta, \) and \( \gamma \) the following length differences between the three parts of \( G \) and the three corresponding parts of \( Y \),

\[
\alpha = |G_I| - |Y_I|,
\beta = |G_\Delta| - |Y_\Delta|,
\gamma = |G_{II}| - |Y_{II}|.
\]

Also, denote by \( \alpha', \beta', \) and \( \gamma' \) the length differences resulting from trimming,

\[
\alpha' = |Y_I| - |Z_I|,
\beta' = |Y_\Delta| - |Z_\Delta|,
\gamma' = |Y_{II}| - |Z_{II}|.
\]

Consider the case in which \( \beta' = 0 \). That is, the trimming on both sides has stopped short of the guard band. Since \(|G_I| = |G_{II}| \) and \(|G_\Delta| = \ell_n \), condition (52) reduces to

\[
\alpha + \alpha' < \gamma + \gamma' + \ell_n - \beta', \quad (53)
\]
\[
\gamma + \gamma' < \alpha + \alpha' + \ell_n - \beta'. \quad (54)
\]

Recall that \( \delta \) is the channel deletion probability and let

\[
\hat{\ell} = \ell_n \cdot (1 - \delta)/2. \quad (55)
\]

We define the following ‘good’ events on the random variables \( \alpha, \alpha', \beta, \beta', \gamma, \) and \( \gamma' \):

\[
A : \delta|G_I| - \hat{\ell}/4 < \alpha < \delta|G_I| + \hat{\ell}/4
\]
\[
A' : 0 \leq \alpha' < \hat{\ell}/4
\]
\[
B : 0 \leq \beta \leq \ell_n (1 + \delta)/2
\]
\[
B' : \beta' = 0
\]
\[
C : \delta|G_{II}| - \hat{\ell}/4 < \gamma < \delta|G_{II}| + \hat{\ell}/4
\]
\[
C' : 0 \leq \gamma' < \hat{\ell}/4
\]

First, we note that the total number of symbols deleted or trimmed from \( G_I \) is given by \(|G_I| - |Z_I| = \alpha + \alpha'\). If \( A \) and \( A' \) hold, then this is bounded by

\[
\alpha + \alpha' < \delta|G_I| + \hat{\ell}/4 + \hat{\ell}/4 = \delta|G_I| + \hat{\ell}/2. \]

Since \(|G_I| = 2^{n-1}\), this quantity is always less than \(|G_I|\) because

\[
\delta|G_I| + \hat{\ell}/2 < \delta 2^{n-1} + 2^{-1} (1 - \delta) 2^{(1-\epsilon)(n-1)}
\]
\[
\leq 2^{n-1} + (1 - \delta) 2^{n-2}
\]
\[
\leq 2^{n-1}.
\]

The analogous claim also holds for \( C, C', \) and \( G_{II} \). Thus, if \( A, A', C, \) and \( C' \) hold, then some parts of \( G_I \) and \( G_{II} \) must remain in \( Z_I \) and \( Z_{II} \) after deletion and trimming. Hence, this also implies that \( B' \) must hold because trimming does not affect \( G_\Delta \).

If, in addition, \( B \) occurs, then both (53) and (54) must also hold. To verify that (53) holds, we simply combine (62) with

\[
\gamma + \gamma' + \ell_n - \beta > \delta|G_{II}| - \hat{\ell}/4 + \ell_n - \ell_n (1 + \delta)/2
\]
\[
= \delta|G_{II}| - \hat{\ell}/4 + \ell_n (1 - \delta)/2
\]
\[
= \delta|G_{II}| - \hat{\ell}/4 + \hat{\ell}/12
\]
\[
= \delta|G_{II}| + 3\hat{\ell}/4
\]
\[
> \delta|G_{II}| + \hat{\ell}/2
\]

and observe that \(|G_I| = |G_{II}|\). The proof of (54) is the same except that the upper and lower bounds are swapped for \( \alpha + \alpha' \) and \( \gamma + \gamma' \).

To recap, the occurrence of all the ‘good’ events in (56)–(61) implies that the middle index falls inside \( Z_\Delta \). Hence, the
The next step is to show that each of the above events occurs with very high probability.

We now recall Hoeffding’s bound \([28]\) proof of Lemma 4.13 and apply it to the deletion channel with deletion probability \(\delta\). It says that the probability of \(D\) deletions during \(N\) channel uses can be bounded with

\[
\Pr(D \geq \delta N + t) \leq e^{-2t^2/N}, \quad (62)
\]

and \(\Pr(D \leq \delta N - t) \leq e^{-2t^2/N}. \quad (63)
\]

Recalling that \(\epsilon > 0\), we now require that \(n_0\) be large enough that the bracketed term in \((50)\) is at most 2. That is, we assume that \(n_0\) is large enough such that for \(n > n_0\),

\[
|G_1| \leq 2 \cdot 2^{n-1}. \quad (64)
\]

Applying both \((62)\) and \((63)\), we deduce that

\[
1 - \Pr(A) \leq 2e^{-2(\ell/4)^2/|G_1|} = 2e^{-2(\ell_n(1-\delta)/8)^2/|G_1|} \leq 2e^{-2(\ell_n(1-\delta)/8)^2/|G_1|} \leq 2e^{-2(\ell_n(1-\delta)/8)^2+(2n-1)} = 2e^{-2(\ell_n(1-\delta)/8)^2+2n-1}. \quad (65)
\]

where (a) follows from \((49)\); (b) holds by combining \((50)\) with our assumption on \(n_0\) leading to \((64)\), and (c) follows from \(n > n_0\). Exactly the same bound applies to \(1 - \Pr(C)\). For \(Pr(B)\), we again deduce that

\[
1 - \Pr(B) \leq e^{-2(\ell_n(1-\delta)/4)^2/|G_n|} = e^{-2(\ell_n(1-\delta)/4)^2/|G_n|} \leq e^{-2(\ell_n(1-\delta)/4)^2+(2n-1)} = e^{-2(\ell_n(1-\delta)/4)^2+2n-1}. \quad (66)
\]

where (a) follows from \((49)\) and (b) holds because \(n > n_0\).

We now bound \(1 - \Pr(A')\) from above. First, recall that by the recursive definition of \(g\) in \((48)\), the prefix of length \(N_0 = 2^{n_0}\) of \(G_1\) is distributed according to the underlying regular Markov input distribution (it does not contain a guard band). Denote this prefix as \(X_1, X_2, \ldots, X_{N_0}\), and denote the state of the process at time 0 as \(S_0\). Since our input distribution is not degenerate, there exists an integer \(t > 0\) and a probability \(0 < p < 1\) such that for any \(s \in S\),

\[
\Pr((X_1, X_2, \ldots, X_t) = (0, 0, \ldots, 0)|S_0 = s) < p. \quad (67)
\]

Let

\[
\tilde{\ell} = \ell_{n_0+1} \cdot (1-\delta)/2. \quad (68)
\]

Since \(n > n_0\), we have by \((49)\) and \((55)\) that \(\tilde{\ell} \leq \tilde{\ell}\) and that

\[
\tilde{\ell}/4 < 2^{n_0}. \quad (69)
\]

Let

\[
\rho = t \cdot \left[1 + \frac{\tilde{\ell}/4}{\ell} \right]. \quad (70)
\]

and partition \(X_1, X_2, \ldots, X_t\) into consecutive segments of length \(t\). Then, we define event \(A''\) to occur if there exists a segment which is not an all-zero vector of length \(t\), and its first non-zero entry has not been deleted. By construction, the event \(A'\) contains the event \(A''\).

Also, if event \(A''\) occurs, then the number of symbols trimmed from \(G_1\) is strictly less than \(\ell\), since the above non-zero non-deleted symbol is not trimmed, and this assures that the “trimming from the left” stops before it. Thus, \(1 - \Pr(A') \leq 1 - \Pr(A'')\).

Since \((67)\) holds for all \(s \in S\), we have by Markovity that

\[
1 - \Pr(A'') < (1 - (1 - p)(1 - \delta))^{\rho/2}. \quad (69)
\]

Indeed, if \(A''\) does not hold, this means that we have “failed” on each of the \(\rho/t\) blocks, in the sense that each such block was either all-zero, or its first non-zero symbol was deleted. Since the probability of “success” conditioned on any given string of past failures is always greater than \((1 - p)(1 - \delta)\), the above follows.

Define

\[
\zeta = \log_e \left(1 - (1 - p)(1 - \delta)\right), \quad (70)
\]

and note that \(\zeta > 0\). Next, we bound \(\rho\) as

\[
\rho > \frac{\tilde{\ell} - 4 - t}{t - 4} = \frac{\tilde{\ell}_0 + 1 \cdot (1 - \delta)/8 - t}{t - 4} > \frac{(2^{n_0+1} - 1)(1 - \delta)/8 - t}{t - 4}.
\]

Recalling that \(1 - \Pr(A') \leq 1 - \Pr(A'')\), we have that

\[
1 - \Pr(A') < e^{-\zeta/(2^{n_0+1} - 1)(1 - \delta)/8 - t)} \quad (71)
\]

Of course, exactly the same bound holds for \(1 - \Pr(C')\).

Putting \((65)\), \((66)\), and \((70)\) together, and applying the union bound proves the lemma.

We conclude this section with the proof of our main theorem.

**Proof of Theorem 1** We let \(n_o = [n/3]\), choose \(\gamma' \in (\gamma, \tilde{\gamma})\), and set the \(\epsilon\) by which the guard band length \(\ell_n\) is defined to \(\epsilon = \frac{\tilde{\gamma} - 2\gamma'}{2}\). Also, let \(X\) and \(Y\) be defined as in Lemma \((21)\). Recall that Lemma \((20)\) already shows that the rate penalty incurred by adding guard bands becomes negligible as \(n_0 \to \infty\), which occurs because \(n \to \infty\). Thus, the claim will follow once we prove two points:

- First, we must show that the probability of making a mistake during the partitioning of \(Y\) into the \(\Phi = 2^{n-n_0}\) trimmed blocks \(Y_1, Y_2, \ldots, Y_\Phi\) is at most \(2^{-2\gamma}\), for \(N = 2^n\) large enough.
- Second, for \(U = A(X)\), we must show that the fraction of indices \(i\), for which the total variation parameter (see \((13)\) Definition 3) satisfies

\[
K(U_i|U_i^{i-1}) < 2^{-2\gamma}, \quad (72)
\]

approaches the first limit in \((17)\), while the fraction of indices \(i\) for which the Bhattacharyya parameter satisfies

\[
Z(U_i|U_i^{i-1}, Y_1^*, Y_2^*, \ldots, Y_\Phi^*) < 2^{-2\gamma} \quad (73)
\]

approaches one minus the second limit in \((17)\).
From these two points the main claim follows, exactly along the lines of [27].

By Lemma [21] and the union bound, the probability of ‘missing’ any of the $\Phi - 1$ guard bands when partitioning $Y$ is at most

$$ (\Phi - 1) \cdot 2^{-\Phi^2n(1-2\gamma)} = (2^{2n/3} - 1) \cdot 2^{-2\gamma 3^{3/5}} , $$

for $n$ large enough. Thus, for $n$ large enough, the above probability is indeed at most $2^{-2\gamma}$. To prove the second point, consider first a block $X_i$ of length $N_0$, along with the corresponding TDC output $Y_i$, for $1 \leq i \leq \Phi$. Denote $V(t) = V = \mathcal{A}(X_i)$. Then, recall from Theorem [19] that for a fixed $\xi > 0$, the fraction of indices $1 \leq i_0 \leq N_0$ for which $H(V_{i_0}, Y_{i_0}^t) < \xi$ holds approaches one minus the second limit in (17). By specializing $\delta$ to 1 for a moment, we further deduce from Theorem [19] that the fraction of indices $i_0$ for which $H(V_{i_0}, Y_{i_0}^t) > 1 - \xi$ holds approaches the first limit in (17).

For $b = (b_1, b_2, \ldots, b_n)$, recall from (4) the definition of $i(b)$, and denote

$$ i_0(b) \triangleq 1 + \sum_{j=1}^{n_0} b_j 2^{n_0-j} . $$

Thus, we may think of the random process by which $i(B_1, B_2, \ldots, B_n)$ is chosen as first selecting $i_0$, which is in fact a function of $B_1, B_2, \ldots, B_{n_0}$, and then completing the choice of $i$ according to a new process $\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_{n-n_0}$, where

$$ \tilde{B}_1 = B_{n_0+1}, \tilde{B}_2 = B_{n_0+2}, \ldots, \tilde{B}_{n-n_0} = B_n . $$

This second process can be thought of as applying

$$ n - n_0 = [2n/3] $$

polar transforms to $\Phi = 2n-n_0$ i.i.d. input/output pairs $(X_t, Y_{i_0})_{t=1}^n$, where $X_t = V_{i_0}(t)$ and $Y_t = (V_{i_0}^{-1}(t), Y_{i_0}^{-1}(t))$. For $i_0$ fixed, we conclude that the Bhattacharyya inequalities in [29] Proposition 1, as well as the total-variation inequalities in [15] Equation 13 hold for this second process. Extremal values of $H$ imply extremal values of $K$ and $Z$ [25] Equation 4, and we may set the above $\xi$ to be as small as we like. Thus, we conclude the second point from the above and [25] Lemma 40, where there we choose $\beta = 3\gamma/2$ and replace $n$ by $[2n/3]$. ■

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