CALCULUS OF FUNCTORS AND MODEL CATEGORIES II

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Abstract. This is a continuation, completion, and generalization of our previous joint work [2] with B. Chorny. We supply model structures and Quillen equivalences underlying Goodwillie’s constructions on the homotopy level for functors between certain simplicial model categories.

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1. Introduction

In a series of three papers, Tom Goodwillie developed a method of analyzing functors from spaces to spaces, with concrete applications towards Waldhausen’s algebraic $K$-theory of spaces. The third paper provides the setup and the basic
theorems for the resulting calculus of functors, with many reminiscences to the classical calculus of real functions. Whereas Goodwillie’s original proofs of the basic theorems usually involve certain connectivity assumptions, his more recent proofs in [12] consist of clever diagram manipulations which apply in great generality. In fact, this generality is one reason “for reworking this whole theory in the context of closed model categories”, which is the present goal. This goal has been addressed already in [17], [2], [18], and most recently in [21].

One aspect of our “reworking” is a lift of Goodwillie’s tower as a tower of model structures on the category \( \mathcal{F} \) of pointed simplicial functors from \( \mathcal{C} \) to \( \mathcal{D} \). Here \( \mathcal{D} \) is a pointed simplicial model category and \( \mathcal{C} \) is a small full subcategory of a pointed simplicial model category \( \mathcal{B} \), and both \( \mathcal{D} \) and \( \mathcal{B} \) are required to satisfy further, but not too restrictive, conditions, which are given in Conventions [3.8, 4.7, 5.2, and 6.1].

To be more precise, we explicitly describe a sequence

\[
\mathcal{F}_{hf} \rightarrow \cdots \rightarrow \mathcal{F}_{(n+1)-\text{exc}} \rightarrow \mathcal{F}_{n-\text{exc}} \rightarrow \cdots \rightarrow \mathcal{F}_{1-\text{exc}} \rightarrow \mathcal{F}_{0-\text{exc}} \simeq *
\]

of left Bousfield localizations of a homotopy functor (hf) model structure on \( \mathcal{F} \), such that the respective fibrant replacements are \( n \)-excisive approximations. These model structures are also constructed for unpointed simplicial functors. To go further we use the pointed setting. Here, right Bousfield localization supplies “fiber sequences”

\[
\mathcal{F}_{n-\text{hom}} \rightarrow \mathcal{F}_{n-\text{exc}} \rightarrow \mathcal{F}_{(n-1)-\text{exc}}
\]

of model structures for every \( n \), such that cofibrant replacement in \( \mathcal{F}_{n-\text{hom}} \) yields an \( n \)-reduced approximation. Moreover, we supply model structures that promotes Goodwillie’s commuting diagram

\[
\begin{array}{c}
\Omega^\infty \downarrow \\
\text{Ho(symmetric multilinear functors to } \text{Sp}(\mathcal{D})) \overset{\text{hocr}}{\longrightarrow} \text{Ho(}\ell n\text{-homog. functors to } \text{Sp}(\mathcal{D}))
\end{array}
\]

\[
\begin{array}{c}
\Omega^\infty \downarrow \\
\text{Ho(symmetric multilinear functors to } \mathcal{D}) \overset{\text{hocr}}{\longrightarrow} \text{Ho(}\ell n\text{-homog. functors to } \mathcal{D})
\end{array}
\]

of equivalences of homotopy categories to a commuting diagram of Quillen equivalences. The horizontal right adjoints are given by a strict version of the \( n \)-th cross effect whose total right derived functor is then Goodwillie’s \( n \)-th homotopy cross effect.

This article applies for example to the target \( \mathcal{D} = \mathcal{S} \), the category of all pointed simplicial sets, with the full subcategory of finite pointed simplicial sets \( \mathcal{C} = \mathcal{S}^{\text{fin}} \) as source. An important variation is the relative setting where one considers objects retractive over a fixed simplicial set \( K \). The category of finite pointed CW complexes does not satisfy the assumptions we use on the source category for the construction of the homotopy functor model structure. However, all finite CW complexes are both fibrant and cofibrant and, thus, all simplicial functors defined on finite CW complexes are automatically homotopy functors. In this case, the homotopy functor model structure is not needed and one can make all the constructions in this article by using the projective model structure directly. Shortly, finite pointed CW complexes as source category \( \mathcal{C} \) and all pointed topological spaces as target category \( \mathcal{D} \) qualify as further examples.

In the special case, where \( \mathcal{C} \) is the category of finite pointed simplicial sets, the model structure for symmetric multilinear functors is shown to be Quillen equivalent.
to the stable model category of (naive) $\Sigma_n$-spectra in $\mathcal{D}$, thus lifting Goodwillie’s derivative to the level of model categories.

**Notation 1.1.** The category of simplicial sets (unpointed spaces) is denoted $\mathcal{U}$, and the category of pointed simplicial sets (spaces) is denoted $\mathcal{S}$. The corresponding full subcategories of finite (pointed) simplicial sets are denoted $\mathcal{U}_\text{fin}$ and $\mathcal{S}_\text{fin}$, respectively. Left adjoints are always on top. A terminal object in a category $\mathcal{C}$ will be denoted $\ast_\mathcal{C}$ or simply $\ast$.

**Acknowledgements:** We would like to thank Bill Dwyer and André Joyal for many encouraging and enlightening discussions. We thank Luís Alexandre Pereira for his detailed and helpful comments.

2. **Enriched functors**

2.1. **Preliminaries on enriched functors.** References for enriched category theory are [3], [8], and [16]. This section mainly presents notation.

**Convention 2.1.** Let $((\mathcal{V}, \otimes, I))$ be a closed symmetric monoidal category. The two main examples of closed symmetric monoidal categories are mentioned in Notation 1.1: $((\mathcal{U}, \times, \ast))$, the category of simplicial sets equipped with the categorical product, and $((\mathcal{S}, \wedge, S^0))$ the category of pointed simplicial sets equipped with the smash product.

**Notation 2.2.** The $\mathcal{V}$-object of morphisms from $A$ to $B$ in any given $\mathcal{V}$-category $\mathcal{C}$ is denoted $\mathcal{V}_\mathcal{C}(A, B)$. The category of $\mathcal{V}$-functors from a small $\mathcal{V}$-category $\mathcal{C}$ to another $\mathcal{V}$-category $\mathcal{D}$ is again a $\mathcal{V}$-category, denoted by $\text{Fun}(\mathcal{C}, \mathcal{D})$ or simply $\text{Fun}(\mathcal{C}, \mathcal{D}) = \mathcal{D}_\mathcal{C}$ if no confusion may arise. One example is the functor category $\mathcal{D}_G$, where $G$ is a monoid, considered as a (discrete) $\mathcal{V}$-category with a single object.

**Definition 2.3.** Given $\mathcal{V}$-categories $\mathcal{C}_i$ for $i = 1, \ldots, n$, the monoidal product category $\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n$ has as objects ordered $n$-tuples $(K_1, \ldots, K_n)$ of objects $K_i$ in $\mathcal{C}_i$, and as $\mathcal{V}$-object of morphisms from $\underline{K} = (K_1, \ldots, K_n)$ to $\underline{L} = (L_1, \ldots, L_n)$ the $n$-fold monoidal product

$$\mathcal{V}_{\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n}(\underline{K}, \underline{L}) := \bigotimes_{i=1}^n \mathcal{V}_{\mathcal{C}_i}(K_i, L_i).$$

Composition and units are readily introduced, giving $\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n$ a $\mathcal{V}$-category structure.

Of course it suffices to give Definition 2.3 for two factors. The general case is presented in view of discussing enriched functors in several variables.

**Definition 2.4.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories, with $\mathcal{C}$ small. Recall that if $\mathcal{D}$ is tensored over $\mathcal{V}$, there is a $\mathcal{V}$-functor

$$\text{Fun}(\mathcal{C}, \mathcal{V}) \otimes \mathcal{D} \to \text{Fun}(\mathcal{C}, \mathcal{D})$$

sending $(D, X)$ to the $\mathcal{V}$-functor

$$X \otimes D : C \mapsto X(C) \otimes D.$$

For fixed $D$ in $\mathcal{D}$ the $\mathcal{V}$-functor $X \mapsto X \otimes D$ has as a right adjoint $Y \mapsto Y^D$, where $(Y^D)(C) = \mathcal{V}_\mathcal{D}(D, Y(C))$. For fixed $X \in \text{Fun}(\mathcal{C}, \mathcal{V})$ the functor $D \mapsto X \otimes D$ has a
right adjoint
\[ Y \mapsto \text{hom}(X, Y) = \int_C \left(Y(C) \right)^{X(C)} \in D \]
if \( D \) is also cotensored over \( V \). Slightly adapting these definitions supplies \( V \)-functors
\[ \text{Fun}(C, D) \otimes V \to \text{Fun}(C, D) \]
and
\[ Y^{\text{op}} \otimes \text{Fun}(C, D) \to \text{Fun}(C, D) \]
giving the functor category \( \text{Fun}(C, D) \) the usual structure of a tensored and coten-
sored \( V \)-category.

**Notation 2.5.** The covariant \( V \)-functor represented by the object \( C \in C \) is denoted
\[ R^C = R^C_C = V_C(C, -) : C \to V. \]

**Lemma 2.6 (Yoneda).** Let \( C \) be an object in \( C \) and \( Y \) in \( \text{Fun}(C, D) \). The \( V \)-natural transformation
\[ \left\{ Y(C) \to Y(D)^{R^C(D)} \right\}_{D \in C} \]
induces an isomorphism:
\[ Y(C) \cong \text{hom}(R^C, Y). \]

**Definition 2.7.** Let \( C \) be a small \( V \)-category. The objectwise tensor product of \( X \) and \( Y \) in \( \text{Fun}(C, V) \), denoted by \( X \otimes Y \), is given by the equation
\[ (X \otimes Y)(C) := X(C) \otimes Y(C). \]

2.2. The projective model structure. For terminology concerning model cate-
gories, consider Hirschhorn [13] or Hovey [14]. A (co)fibration which is also a weak
equivalence will be called acyclic (co)fibration.

**Convention 2.8.** Let \( V \) be a symmetric monoidal model category.

Again the main examples are (pointed) simplicial sets, with monomorphisms as
cofibrations and maps inducing homotopy equivalences after geometric realization
as weak equivalences. As explained in the references mentioned, a \( V \)-model category
is tensored and cotensored over \( V \), and the compatibility of the model structures
with the enrichment is expressed using the following definition.

**Definition 2.9.** Let \( f : A \to B \) and \( g : C \to D \) be two maps in \( V \). The map
\[ f \square g : (A \otimes D) \cup_{(A \otimes C)} B \otimes C \to B \otimes D \]
induced by \( f \otimes D \) and \( B \otimes g \) is called the pushout product of \( f \) and \( g \). The analogous
construction where \( g \) is a map in a tensored \( V \)-category \( D \) with pushouts yields a
map \( f \square g \) is a map in \( D \).

Hirschhorn in [13, 9.3.5.(2)] calls it the pushout corner map. In order to equip
the category \( \text{Fun}(C, D) \) of \( V \)-functors from a small \( V \)-category \( C \) to a \( V \)-model category
\( D \) with the projective model structure whose weak equivalences and fibrations are
objectwise, certain assumptions on \( D \) are necessary.
Definition 2.10. Let $\mathcal{V}$ be a monoidal model category, and let $\mathcal{D}$ be a $\mathcal{V}$-model category. The $\mathcal{V}$-model category $\mathcal{D}$ satisfies the $\mathcal{V}$-monoid axiom if the following property holds: Let acof$_{\mathcal{D}}$ be the class of acyclic cofibrations in $\mathcal{D}$. Let $\mathcal{E}_D$ be the class of relative cell complexes in $\mathcal{D}$ generated by the class of morphisms

$$\{ j \otimes A \mid j \in \text{acof}_D, A \in \text{Ob}\mathcal{V}\}.$$

Then every morphism in $\mathcal{E}_D$ is a weak equivalence.

Definition 2.11. Let $\mathcal{V}$ be a monoidal model category, and let $\mathcal{D}$ be a $\mathcal{V}$-model category. Let $\mathcal{F}_D$ be the class of relative cell complexes in $\mathcal{D}$ generated by the class of morphisms

$$\{ i \otimes A \mid i \in \text{cof}_D, A \in \text{Ob}\mathcal{V}\}.$$

The $\mathcal{V}$-model category $\mathcal{D}$ is $\mathcal{V}$-left proper if weak equivalences in $\mathcal{D}$ are closed under cobase change along morphisms in $\mathcal{F}_D$.

Remark 2.12. If all objects in $\mathcal{V}$ are cofibrant, the $\mathcal{V}$-monoid axiom holds automatically in any $\mathcal{V}$-model category $\mathcal{D}$. Furthermore, in that case, $\mathcal{V}$-left properness is equivalent to left properness. This holds in particular for the cases $\mathcal{V} = S$ or $\mathcal{U}$.

Theorem 2.13. Let $\mathcal{D}$ be a bicomplete $\mathcal{V}$-model category which is cofibrantly generated. If the $\mathcal{V}$-monoid axiom holds in $\mathcal{D}$, the category

$$\text{Fun}_\mathcal{V}(\mathcal{C}, \mathcal{D})$$

of $\mathcal{V}$-functors from a small $\mathcal{V}$-category $\mathcal{C}$ to $\mathcal{D}$ carries a cofibrantly generated model structure, where the weak equivalences and fibrations are defined objectwise. If the model structure on $\mathcal{D}$ is right proper, so is the projective model structure. If the model structure on $\mathcal{D}$ is $\mathcal{V}$-left proper, the projective model structure is left proper.

Proof. This follows by adapting the proof of [7, Theorem 4.4]. Left properness is shown as in the proof of [7, Cor. 4.8]. Following standard terminology, this model structure will be referred to as the projective model structure. For future reference, generating sets for cofibrations and acyclic cofibrations in the projective model structure are constructed as follows: Tensoring the functor $R^K$ with an object $E \in \text{Ob}(\mathcal{D})$ yields a $\mathcal{V}$-functor

$$R^K \otimes E : \mathcal{C} \rightarrow \mathcal{D}, \quad L \mapsto \mathcal{V}_\mathcal{C}(K, L) \otimes E.$$

The $\mathcal{V}$-Yoneda lemma [2,0] implies that any $\mathcal{V}$-functor $X : \mathcal{C} \rightarrow \mathcal{D}$ is naturally isomorphic to the coend

$$\int_{K \in \mathcal{C}} R^K \otimes X(K).$$

Given generating sets $I_D$ and $J_D$ for the model structure on $\mathcal{D}$, the sets

$$I_{\text{Fun}_\mathcal{V}(\mathcal{C}, \mathcal{D})}^{\text{proj}} := \{ R^K \otimes i \mid K \in \text{Ob}(\mathcal{C}), i \in I_D \}$$

$$J_{\text{Fun}_\mathcal{V}(\mathcal{C}, \mathcal{D})}^{\text{proj}} := \{ R^K \otimes j \mid K \in \text{Ob}(\mathcal{C}), j \in J_D \}$$

are generating (acyclic) cofibrations for the projective model structure. \qed
2.3. **Enriched functors in several variables.** A \( \mathcal{V} \)-functor in several variables is simply a \( \mathcal{V} \)-functor

\[
C_1 \otimes \cdots \otimes C_n \to \mathcal{D}
\]

where \( \mathcal{D} \) and \( C_i \) for \( i = 1, \ldots, n \) are \( \mathcal{V} \)-categories. In order to translate between \( \mathcal{V} \)-functors in several variables and in a single variable, let \( \mathcal{I}_V \) denote the \( \mathcal{V} \)-category given by the full subcategory of \( \mathcal{V} \) containing as its single object the unit \( I \). In other words, it is a \( \mathcal{V} \)-category with one object, also denoted \( I \), and endomorphism object \( I \). For every \( \mathcal{V} \)-category \( \mathcal{A} \) there are canonical unit isomorphisms

\[
\mathcal{I}_V \otimes \mathcal{A} \xrightarrow{\cong} \mathcal{A} \xrightarrow{\cong} \mathcal{A} \otimes \mathcal{I}_V
\]

of \( \mathcal{V} \)-categories.

For any object \( B \) in a \( \mathcal{V} \)-category \( \mathcal{B} \), a \( \mathcal{V} \)-functor

\[
i_B : \mathcal{A} \otimes \mathcal{I}_V \to \mathcal{A} \otimes \mathcal{B}
\]

is given on objects by \((A, I) \mapsto (A, B)\) and on morphisms by

\[
\mathcal{V}_A(A_1, A_2) \otimes I \to \mathcal{V}_A(A_1, A_2) \otimes \mathcal{V}_B(B, B),
\]

where \( I \to \mathcal{V}_B(B, B) \) is the canonical unit map. Of course, there is an analogous functor \( i_A \) for every \( A \) in \( \mathcal{A} \). Given a \( \mathcal{V} \)-functor \( G : \mathcal{A} \otimes \mathcal{B} \to \mathcal{D} \), every object \( B \) in \( \mathcal{B} \) defines the partial functor \( G_B : \mathcal{A} \to \mathcal{D} \) by composition:

\[
\mathcal{A} \xrightarrow{\gamma_A} \mathcal{A} \otimes \mathcal{I}_V \xrightarrow{i_B} \mathcal{A} \otimes \mathcal{B} \xrightarrow{G} \mathcal{D}
\]

The functor \( G \) is uniquely determined by all its partial functors \( G_A \) and \( G_B \):

**Proposition 2.14** (Prop. 4.2, [8]). Given for all objects \( A \) and \( B \) \( \mathcal{V} \)-functors

\[
G_A : \mathcal{B} \to \mathcal{D} \quad \text{and} \quad G_B : \mathcal{A} \to \mathcal{D}
\]

with the property \( G_A(B) = G_B(A) =: G(A, B) \), then there exists a unique \( \mathcal{V} \)-functor \( G : \mathcal{A} \otimes \mathcal{B} \to \mathcal{D} \) with \( \{G_A\} \) and \( \{G_B\} \) as partial functors if and only if the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{V}_A(A, A') \otimes \mathcal{V}_B(B, B') & \xrightarrow{G_B \otimes G_A} & \mathcal{V}_D(G(A, B'), G(A', B')) \otimes \mathcal{V}_D(G(A, B), G(A, B')) \\
\text{switch} & & \downarrow \text{composition} \\
\mathcal{V}_B(B, B') \otimes \mathcal{V}_A(A, A') & \xrightarrow{G_B \otimes G_A} & \mathcal{V}_D(G(A', B), G(A', B')) \otimes \mathcal{V}_D(G(A, B), G(A', B)) \\
\end{array}
\]

In other words, a \( \mathcal{V} \)-functor from a monoidal product category is essentially a functor in \( n \) variables which is componentwise enriched over \( \mathcal{V} \).

**Example 2.15.** In the case where the closed symmetric monoidal base category is \((\mathcal{U}, \times, \ast)\), the underlying category of a monoidal product category coincides with the ordinary product category. This is different in the case of \((\mathcal{S}, \land, S^0)\). For example, any object of the form \((K, \ast)\) or \((\ast, L)\) in \( \mathcal{S} \land \mathcal{S} \) is a zero object.

The analogous result for \( \mathcal{V} \)-natural transformations will be used as well.

**Proposition 2.16** (Prop. 4.12, [8]). Let \( \mathcal{V} \) be a symmetric monoidal category and \( T, S : \mathcal{A} \otimes \mathcal{B} \to \mathcal{D} \) be two \( \mathcal{V} \)-functors. For all objects \( A \) in \( \mathcal{A} \) and \( B \) in \( \mathcal{B} \) let

\[
\alpha_{A,B} : S(A, B) \to T(A, B)
\]
be a map in the underlying category of \( D \). The maps \( \alpha_{A,B} \) are the components of a \( \mathcal{V} \)-natural transformation \( \alpha : S \to T \) if and only if, for each \( A \), the map \( \alpha_{A,B} \) is the \( B \)-component of a \( \mathcal{V} \)-natural transformation \( \alpha_A : S_A \to T_A \) and, for each \( B \), the map \( \alpha_{A,B} \) is the \( A \)-component of a \( \mathcal{V} \)-natural transformation \( \alpha_B : S_B \to T_B \).

Recall that a terminal object in a category \( C \) is denoted \( \ast_C \) or simply \( \ast \).

**Definition 2.17.** Suppose that the categories \( C \) and \( D \) admit a terminal object. A functor \( F : C \to D \) is called reduced if \( F(\ast_C) \cong \ast_D \). A functor \( F \) to \( D \) in \( n \) variables is called multireduced if

\[
F(K_1, \ldots, K_n) \cong \ast_D
\]

whenever \( K_i \) is a terminal object for at least one \( i \in \{1, \ldots, n\} \).

**Remark 2.18.** Suppose that \( \mathcal{V} \) and the categories \( C_i \) are pointed categories, and that \( D \) is a cocomplete \( \mathcal{V} \)-category. Then every object in \( \text{Fun}(C_1 \otimes \ldots \otimes C_n, D) \) is multireduced. In fact, every representable functor is multireduced. By the \( \mathcal{V} \)-Yoneda lemma 2.6, every \( \mathcal{V} \)-functor is a colimit of representable functors.

### 2.4. Smash product and product categories.

This section treats monoidal product categories in the special cases of unpointed and pointed simplicial sets. Since the functor \( u : S \to U \) forgetting the base point is lax symmetric monoidal, every \( S \)-category \( C \) is a \( U \)-category \( uC \) by simply forgetting base points in all morphism objects.

**Lemma 2.19.** Let \( C \) and \( D \) be \( S \)-categories. Then \( uC \times uD \) is an \( S \)-category in a natural way.

**Proof.** Given objects \( K = (K_1, K_2), L = (L_1, L_2), M = (M_1, M_2) \in uC \times uD \), the simplicial set

\[
\mathcal{U}_{uC \times uD}((K_1, K_2), (L_1, L_2)) = uSC(K_1, L_1) \times uSD(K_2, L_2)
\]

is naturally a pointed simplicial set. The \( S \)-composition is induced by the \( U \)-composition map

\[
S(L, M) \times S(K, L) \to S(K, L)
\]

since if \( f = \ast \) or \( g = \ast \), then \( f_i \circ g_i = \ast \) for all \( 1 \leq i \leq n \). The unit

\[
S^0 \to uSC \times uD(K, K)
\]

is induced by the diagonal. Associativity and unitality of the \( U \)-composition imply associativity and unitality for the \( S \)-composition. \( \square \)

**Notation 2.20.** Let \( C \) and \( D \) be \( S \)-categories. The \( S \)-category from Lemma 2.19 is denoted \( C \times D \).

**Lemma 2.21.** Let \( C \) and \( D \) be \( S \)-model categories. Then \( C \times D \) is an \( S \)-model category with the componentwise model structure.

**Proof.** The category underlying the \( S \)-category \( C \times D \) is simply the product category. Hence, the existence of the componentwise model structure follows from [14, Ex. 1.1.6.]. It remains to prove that \( C \times D \) is tensored and cotensored over \( S \), and to verify the pushout product axiom. Tensor and cotensor are defined componentwise. It is straightforward to check that they constitute \( S \)-functors which are part of \( S \)-adjunctions. The pushout product axiom follows immediately. \( \square \)
Definition 2.22. Let \( C_1, \ldots, C_n \) be \( S \)-categories. The canonical functor
\[
p: C_1 \times \cdots \times C_n \to C_1 \land \cdots \land C_n
\]
being the identity on objects and the quotient map from the Cartesian product to the smash product on morphisms is an \( S \)-functor. If the \( C_i \)'s are small and \( D \) is another \( S \)-category, \( p \) induces an \( S \)-adjoint pair
\[
p_*: \text{Fun}(C_1 \times \cdots \times C_n, D) \rightleftharpoons \text{Fun}(C_1 \land \cdots \land C_n, D) : p^*.
\]

Lemma 2.23. Let \( D \) be an \( S \)-model category. The adjoint pair \((p_*, p^*)\) is a Quillen pair of projective model structures. The functor \( p^* \) preserves and detects objectwise weak equivalences and objectwise fibrations.

Proof. This is immediate, since \( p^* \) is precomposition with a functor being the identity on objects. \( \square \)

We denote by \( \{\ast\} = I_U \) the unpointed simplicial category with one object and no non-identity morphisms and by \( \{S^0\} = I_S \) the corresponding \( S \)-category. There is exactly one \( U \)-functor
\[
I: \{\ast\} \to \{S^0\},
\]
and it is given on underlying simplicial sets by sending \( \ast \) to the non-basepoint. There is a canonical isomorphism of unpointed simplicial categories
\[
\pi_A: A \rightarrow A \times \{\ast\}
\]
and analogous ones with entries switched. The functors \( I \) and \( \pi_A \) are unpointed but not pointed simplicial. In particular, the functor
\[
J: A \times \{\ast\} \overset{(id,I)}\longrightarrow A \times \{S^0\} \overset{p}\longrightarrow A \land \{S^0\}
\]
is unpointed simplicial. For any \( B \) in \( B \) we obtain an unpointed simplicial functor
\[
i_B: A \times \{\ast\} \to A \times B
\]
given on objects by \( (A, \ast) \mapsto (A, B) \) and on morphisms by \( (f, \ast) \mapsto (f, id_B) \). Again, there are also functors \( i_A \). We hope no confusion with the analogous definition in the pointed case will arise from the indiscriminate notation.

For every object \( B \) in \( B \) the following diagram commutes
\[
\text{Diagram (2.2)}
\]
where the upper row consists of \( U \)-functors and the lower row consists of \( S \)-functors. Obviously, there is an analogous commutative diagram for every object \( A \) in \( A \). Given a \( U \)-functor \( F \) and an object \( B \) as above we define the partial functor \( F_B \) by composing the upper row. Given an \( S \)-functor \( G \) and \( B \) as above we define the partial functor \( G_B \) by composing the lower row. Similarly, we define partial functors \( F_A \) and \( G_A \) for every \( A \) in \( A \). The functor \( F \) in (2.2) is \( S \)-enriched if and only if \( F(*_A,*_B) = *_D \). The latter does not imply that \( F \) is multireduced.

Lemma 2.24. Let \( A, B \) and \( D \) be \( S \)-categories. For a \( U \)-functor \( F: A \times B \to D \), the following are equivalent:
(1) The functor $F$ is multireduced.
(2) All partial functors of $F$ are reduced.
(3) The functor $F$ is isomorphic to $p^*G$ for some $S$-functor $G: A \wedge B \to D$.

If these conditions hold, $F$ is in particular an $S$-functor.

Proof. Since all $S$-functors $G: A \wedge B \to D$ are multireduced and $p^*$ is the identity on objects, (3) implies (1). Obviously, (1) implies (2). Now, assume (2). Then, for all objects $A$ in $A$ and $B$ in $B$ the partial functors $F_A: B \to D$ and $F_B: A \to D$ are $S$-functors. The partial functors $G_A := F_A$ and $G_B := F_B$ assemble by proposition 2.14 to an $S$-functor $G: A \wedge B \to D$. Because the diagram (2.2) commutes, there are canonical isomorphisms $(p^*G)_A \cong G_A = F_A$ and $(p^*G)_B \cong G_B = F_B$ for all $A$ and $B$. This gives (3). □

3. Symmetric functors

The purpose of this section is to interpret Goodwillie’s cross effect construction as a Quillen functor between appropriate model categories.

3.1. Symmetric functors.

Definition 3.1. Let $C$ and $D$ be $\mathcal{V}$-categories. A $\mathcal{V}$-functor $X: C \otimes \cdots \otimes C \to D$ in $n$ variables with values in $D$ is symmetric if it is equipped with a $\mathcal{V}$-natural isomorphism
$$\sigma_X: X(K_1, \ldots, K_n) \cong X(K_{\sigma(1)}, \ldots, K_{\sigma(n)})$$
for every $\sigma \in \Sigma_n$, such that the equalities $id_X = id$ and $(\tau \sigma)_X = \tau_X \sigma_X$ hold. A symmetric $\mathcal{V}$-natural transformation is a $\mathcal{V}$-natural transformation between two symmetric functors that respects the symmetry in the obvious way. Let $\text{Fun}_{\text{sym}}(C^\otimes n, D)$ denote the corresponding $\mathcal{V}$-category.

The standard example of a symmetric $\mathcal{V}$-functor is the $n$-fold monoidal product
$$\mathcal{V} \otimes \cdots \otimes \mathcal{V} \to \mathcal{V}.$$  

The main example of interest here are cross effect functors, to be described in the next section. In order to introduce model structures for symmetric functors, it will be convenient to describe them as a genuine functor category instead of just a proper subcategory of a functor category.

Convention 3.2. Suppose that the closed symmetric monoidal category $(\mathcal{V}, \otimes, I)$ has finite coproducts, denoted as $\vee$.

Definition 3.3. Let $C$ be a $\mathcal{V}$-category. The wreath product category $\Sigma_n \wr C^\otimes n$ has as its objects $n$-tuples $(K_1, \ldots, K_n)$ of objects in $C$. The morphisms from $K = (K_1, \ldots, K_n)$ to $L = (L_1, \ldots, L_n)$ are given by
$$\mathcal{V}_{\Sigma_n \wr C^\otimes n}(K, L) := \bigvee_{\sigma \in \Sigma_n} \mathcal{V}_C(K_{\sigma^{-1}(i)}).$$

Composition is defined as it is in the wreath product of groups or, more generally, in a semi-direct product by the following formula:
$$(3.1) \ (\tau, (g_1 \otimes \cdots \otimes g_n)) \circ (\sigma, (f_1 \otimes \cdots \otimes f_n)) = (\sigma \tau, (g_1 f_{\sigma^{-1}(1)} \otimes \cdots \otimes g_n f_{\sigma^{-1}(n)}))$$
More formally, composition is a map as follows:

\[ \mathcal{V}_{\Sigma_n \mathbb{I} \otimes \mathbb{I}}(L, M) \otimes \mathcal{V}_{\Sigma_n \mathbb{I} \otimes \mathbb{I}}(K, L) \to \mathcal{V}_{\Sigma_n \mathbb{I} \otimes \mathbb{I}}(K, M). \]

Observe that the source of the composition map is canonically isomorphic to the term

\[ \bigvee_{(r, \sigma) \in \Sigma_n \times \Sigma_n} \left[ \bigotimes_{j=1}^n V_C(L_{\tau^{-1}(j)}) \otimes \bigotimes_{i=1}^n V_C(K_{\sigma^{-1}(i)}) \right] \]

while the target is given by

\[ \bigvee_{\omega \in \Sigma_n} \bigotimes_{k=1}^n V_C(K_k, M_{\omega^{-1}(k)}). \]

Given \( \sigma \) and \( \tau \), the corresponding summand in the first term is mapped to the summand corresponding to \( \omega = \sigma \tau \), with the map being the \( n \)-fold monoidal product of the \( V \)-composition

\[ \mathcal{V}_C(L_{\sigma^{-1}(k)}, M_{\tau^{-1}(\sigma^{-1}(k))}) \otimes \mathcal{V}_C(K_{\sigma^{-1}(k)}, L_{\sigma^{-1}(k)}), \]

up to a permutation of monoidal factors. This amounts to the formula (3.1). Associativity and identity conditions are checked readily, implying that \( \Sigma_n \mathbb{I} \otimes \mathbb{I} \) is indeed a \( V \)-category.

**Remark 3.4.** Here are two interpretations of this construction.

1. The \( V \)-category \( \Sigma_n \mathbb{I} \otimes \mathbb{I} \) is the \( V \)-category of unordered \( n \)-tuples. More precisely, in \( \Sigma_n \mathbb{I} \otimes \mathbb{I} \) there is for every \( \sigma \in \Sigma_n \) and every \( n \)-tuple \( K \) a canonical map

\[ (\sigma, f_1 \otimes \cdots \otimes f_n) \mapsto (\sigma, \text{id} \otimes \cdots \otimes \text{id}) \]

that is an isomorphism with inverse \( (\sigma, \text{id} \otimes \cdots \otimes \text{id}) \). Moreover, any map in \( \Sigma_n \mathbb{I} \otimes \mathbb{I} \) can be written as the composition of such an isomorphism with a map from \( \mathbb{I} \otimes \mathbb{I} \):

\[ (\sigma, f_1 \otimes \cdots \otimes f_n) = (\sigma, \text{id} \otimes \cdots \otimes \text{id}) \circ (\text{id}, f_1 \otimes \cdots \otimes f_n) = (\text{id}, f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(n)}) \circ (\sigma, \text{id} \otimes \cdots \otimes \text{id}) \]

2. The \( V \)-category \( \Sigma_n \mathbb{I} \mathbb{I} \) is obtained as the \( V \)-Grothendieck construction or \( V \)-category of elements of the functor \( \Sigma_n \to V \text{-} \text{Cat} \) sending the unique object to \( \mathbb{I} \mathbb{I} \) with the permutation action.

**Definition 3.5.** For every \( V \)-category \( C \) there is a functor

\[ \varepsilon: \mathbb{I} \mathbb{I} \to \Sigma_n \mathbb{I} \mathbb{I} \]

which is the identity on objects and the inclusion of the summand indexed by the identity in \( \Sigma_n \) on morphisms: \( f_1 \otimes \cdots \otimes f_n \mapsto (\text{id}, f_1 \otimes \cdots \otimes f_n) \).

**Lemma 3.6.** Let \( C \) and \( D \) be \( V \)-categories. Suppose also that \( C \) is small. Precomposition with \( \varepsilon \) induces an equivalence

\[ \varepsilon^*: \text{Fun}(\Sigma_n \mathbb{I} \mathbb{I}, D) \to \text{Fun}_{\text{sym}}(\mathbb{I} \mathbb{I}, D) \]

of \( V \)-categories.
Proof. Precomposition with \( \varepsilon \) defines a \( \mathcal{V} \)-functor

\[
\varepsilon^*: \text{Fun}(\Sigma_n \wr \mathcal{C} \otimes n, D) \to \text{Fun}(\mathcal{C} \otimes n, D)
\]

By construction, every \( \mathcal{V} \)-functor (\( \mathcal{V} \)-natural transformation) in the image of \( \varepsilon^* \) is symmetric. The \( \mathcal{V} \)-functor with target restricted to the category of symmetric functors will also be denoted \( \varepsilon^* \). Unravelling the definitions shows that a \( \mathcal{V} \)-functor \( C \otimes n \to D \) is symmetric precisely if its domain extends (via the extra data) to the wreath product category \( \Sigma_n \wr \mathcal{C} \otimes n \), which essentially completes the proof. \( \Box \)

**Definition 3.7.** Let \( D \) be a tensored and cotensored \( \mathcal{V} \)-category. For an object \( L \) in \( \mathcal{V}^\Sigma \) and a functor \( X \) in \( \text{Fun}(\Sigma_n \wr \mathcal{C} \otimes n, D) \) set

\[
(X \otimes_{\Sigma_n} L)(K) := X(K) \otimes L
\]

using the tensor \( D \otimes \mathcal{V} \to D \). This is a functor in \( K \). The symmetry automorphisms

\[
\sigma_{X \otimes L}: X(K_1, \ldots, K_n) \otimes L \to X(K_{\sigma^{-1}(1)}, \ldots, K_{\sigma^{-1}(n)}) \otimes L
\]

defined by \( \sigma_{X \otimes L} := \sigma_X \otimes \sigma_L \) for every permutation \( \sigma \in \Sigma_n \) turn \( X \otimes_{\Sigma_n} L \) into a symmetric functor. For fixed \( L \) in \( \mathcal{V}^\Sigma \), the functor \( X \mapsto X \otimes_{\Sigma_n} L \) has as a \( \mathcal{V} \)-right adjoint \( Y \mapsto \text{hom}_{\Sigma_n}(L, Y) \), where

\[
\text{hom}_{\Sigma_n}(L, Y)(K) := \text{hom}_D(L, Y(K))
\]

with symmetric structure obtained by the diagonal action.

### 3.2. The cross effect.

**Convention 3.8.** Suppose that \( \mathcal{V} = \mathcal{S} \), so that Convention 3.2 is satisfied. Suppose further that \( D \) is a bicomplete \( \mathcal{S} \)-category, and that \( C \) is a small \( \mathcal{S} \)-category with finite coproducts and terminal object \( * \).

**Notation 3.9.** The \( \mathcal{S} \)-functor \( \text{tr}: D \to D^{\Sigma_n} \), sending an object to itself equipped with the trivial \( \Sigma_n \)-action, has a \( \mathcal{S} \)-left adjoint given by the orbit functor \( (\_)^{\Sigma_n} \).

**Definition 3.10.** Let \( \underline{n} = \{1, \ldots, n\} \), with associated power set \( P(\underline{n}) \), and let

\[
P_0(\underline{n}) := P(\underline{n}) - \{\emptyset\}
\]

be the partially ordered set of non-empty subsets of \( \underline{n} \). For every \( n \)-tuple \( K = \{K_1, \ldots, K_n\} \) of objects in \( \mathcal{C} \) and every \( S \in P_0(\underline{n}) \) there is a map

\[
\bigvee_{i=1}^{n} K_i \to \bigvee_{i \in \underline{n} - S} K_i
\]

induced by the canonical inclusion \( K_i \to \bigvee_{i \in \underline{n} - S} K_i \) if \( i \notin S \) and the trivial map \( K_i \to * \) if \( i \in S \).

**Definition 3.11.** The \( n \)-th cross effect \( \text{cr}_n: \mathcal{C}^{\wedge n} \to \mathcal{D} \) of a functor \( X: \mathcal{C} \to \mathcal{D} \) is given by the formula

\[
\text{cr}_n X(K_1, \ldots, K_n) := \text{fib}
\]

\[
\left[\bigvee_{i=1}^{n} K_i \to \lim_{S \in P_0(\underline{n})} X(\bigvee_{i \in \underline{n} - S} K_i)\right]
\]

where the map is induced by the maps described in Definition 3.10.
Remark 3.12. The $n$-th cross effect, as defined above, does not coincide with the construction (denoted by the same symbol) $\text{cr}_n$ introduced in [12]. Goodwillie’s $\text{cr}_n$ refers to the functor

$$\text{hocr}_n X(K_1, \ldots, K_n) := \text{hofib} \left[ \bigvee_{i=1}^n X(\bigvee_{i \in I} K_i) \to \text{holim}_{S \in P_0} \bigvee_{i \in I} X(\bigvee_{i \in I} K_i) \right]$$

that we call the $n$-th homotopy cross effect. Section 3.3 supplies a model structure on $\text{Fun}(\mathcal{C}, \mathcal{D})$ such that the homotopy cross effect becomes the right derived functor of the strict cross effect.

A permutation $\sigma \in \Sigma_n$ defines an automorphism of an $n$-fold coproduct, whence $\text{cr}_n$ produces symmetric functors:

$$\text{cr}_n : \text{Fun}(\mathcal{C}, \mathcal{D}) \to \text{Fun}(\Sigma_n \wr \mathcal{C} \wedge n, \mathcal{D}).$$

The main goal of this section is to construct a left adjoint to this functor.

Definition 3.13. For every object $K = \{K_1, \ldots, K_n\}$ in $\Sigma_n \wr \mathcal{C} \wedge n$, the maps given in Definition 3.10 induce a map

$$\phi_K : \text{colim}_{S \in P_0} R^{\bigvee_{i \in I} K_i} \to R^{\bigwedge_{i=1}^n K_i}$$

in $\text{Fun}(\mathcal{C}, \mathcal{S})$, functorial in $K$.

Lemma 3.14. For $K$ in $\Sigma_n \wr \mathcal{C} \wedge n$ and $X$ in $\text{Fun}(\mathcal{C}, \mathcal{D})$ there is a canonical isomorphism:

$$\text{hom}(\bigwedge_{i=1}^n R^{K_i}, X) \cong \text{cr}_n X(K_1, \ldots, K_n)$$

Proof. The map $\phi_K$ given in Definition 3.13 induces an objectwise cofiber sequence

$$\text{colim}_{S \in P_0} R^{\bigvee_{i \in I} K_i} \xrightarrow{\phi_K} R^{\bigwedge_{i=1}^n K_i} \to \bigwedge_{i=1}^n R^{K_i}$$

in $\text{Fun}(\mathcal{C}, \mathcal{S})$, where $\bigwedge_{i=1}^n R^{K_i} : \mathcal{C} \to \mathcal{S}$ is the objectwise smash product described in Definition 2.7. The result then follows from the enriched Yoneda lemma 2.6.

Lemma 3.15. The functor

$$\text{cr}_n : \text{Fun}(\mathcal{C}, \mathcal{D}) \to \text{Fun}(\Sigma_n \wr \mathcal{C} \wedge n, \mathcal{D})$$

has a left $\mathcal{S}$-adjoint $L_n$, sending the symmetric functor $X$ to the functor

$$K \mapsto \left( X(K_1, \ldots, K) \right)_{\Sigma_n}$$

where $\Sigma_n$ operates by permuting the entries in $(K_1, \ldots, K)$.

Lemma 3.14 implies for the case $\mathcal{D} = \mathcal{S}$ that a left $\mathcal{S}$-adjoint of $\text{cr}_n$ (if it exists) sends the functor represented by $K \in \Sigma_n \wr \mathcal{C} \wedge n$ to the objectwise smash product $\bigwedge_{i=1}^n R^{K_i}$. This supplies a candidate for the definition of the left adjoint by the enriched Yoneda lemma 2.6.

Proof. Let $X$ in $\text{Fun}(\Sigma_n \wr \mathcal{C} \wedge n, \mathcal{D})$ be written as a colimit of representable functors:

$$X \cong \int^{K} X(K) \wedge R^{K}.$$
Then the functor $\mathcal{L}_n : \text{Fun}(\Sigma_n \mathcal{C}, \mathcal{D}) \to \text{Fun}(\mathcal{C}, \mathcal{D})$ is defined by the following coend:

$$\mathcal{L}_n(X) \cong \int^K X(K) \land \mathcal{L}_n(R^K) \cong \int^K \left( X(K) \land \bigwedge_{i=1}^n R^{K_i} \right).$$

For every $\mathcal{S}$-functor $Y : \mathcal{C} \to \mathcal{D}$, one obtains natural isomorphisms:

$$\mathcal{S}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(\mathcal{L}_n(X), Y) \cong \mathcal{S}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(\int^K (X(K) \land \bigwedge_{i=1}^n R^{K_i}), Y)$$

$$\cong \int^K \mathcal{S}_{\mathcal{D}}(X(K), \text{hom}(\bigwedge_{i=1}^n R^{K_i}, Y))$$

$$\cong \int^K \mathcal{S}_{\mathcal{D}}(X(K), (\text{cr}_n Y)(K))$$

$$\cong \mathcal{S}_{\text{Fun}(\Sigma_n \mathcal{C}, \mathcal{D})}(X, \text{cr}_n Y)$$

Lemma 3.14 is used for the third isomorphism. Hence, the functor $\mathcal{L}_n$ is $\mathcal{S}$-left adjoint to $\text{cr}_n$. One identifies the functor $\mathcal{L}_n$ explicitly by the following formula:

$$\mathcal{L}_n \cong (\Delta^*_n(-))_{\Sigma_n} = \text{LKan} \circ \Delta^*_n$$

Here $\text{pr}_c : \mathcal{C} \times \Sigma_n \to \mathcal{C}$ is the projection onto the first factor, and $\Delta^*_n$ is precomposition with the symmetric diagonal $\Delta_n : \Sigma_n \times \mathcal{C} \to \Sigma_n \mathcal{C}^\land$ sending an object $K$ to the $n$-tuple $\Delta_n(K) = (K, \ldots, K)$ and a morphism $(\sigma, f)$ to $(\sigma, (f, \ldots, f))$. A straightforward computation shows that the right hand side of (3.2) sends the functor represented by $K \in \Sigma_n \mathcal{C}^\land$ to the objectwise smash product $\bigwedge_{i=1}^n R^{K_i}$. Since it also commutes with colimits, the isomorphism (3.2) holds by the universal property of the left Kan extension. This supplies the formula stated in the Lemma.

3.3. The cross effect model structure. Suppose that $\mathcal{D}$ is a cofibrantly generated $\mathcal{S}$-model category, so that Theorem 2.13 is applicable. In all interesting cases, the cross effect

$$\text{cr}_n : \text{Fun}(\mathcal{C}, \mathcal{D})_{\text{proj}} \to \text{Fun}(\Sigma_n \mathcal{C}, \mathcal{D})_{\text{proj}}$$

is not a right Quillen functor of projective model structures because it does not preserve fibrant objects. The purpose of this section is to supply a model structure on $\text{Fun}(\mathcal{C}, \mathcal{D})$ with objectwise weak equivalences, such that the $n$-th cross effect is a right Quillen functor. The task will be accomplished by introducing more cofibrations. The right derived functor of the cross effect turns out to be Goodwillie’s homotopy cross effect, as promised in Remark 3.12.

**Definition 3.16.** Let $n \geq 1$. If $K = \{K_1, \ldots, K_n\}$ is an $n$-tuple of objects in $\mathcal{C}$, one says $|K| = n$. Definition 3.13 supplies a map

$$\phi_{|K|} : \colim_{S \in P_{|K|}(\mathcal{C})} R^{V_{\geq |K|} K_i} \to R^{V_{\geq 1} K_i}$$

in $\text{Fun}(\mathcal{C}, \mathcal{S})$ for every $K$ with $|K| = n$. Let $\Phi_0 := \emptyset$, $\Phi_n := \{\phi_{|K|} | |K| \leq n\}$, and $\Phi := \Phi_\infty := \bigcup_{n \geq 1} \Phi_n$.

For $n \geq 2$, most of the maps in $\Phi_n$ are not projective cofibrations.
Definition 3.17. The tensor \( S \times D \to D \) induces a tensor \( \text{Fun}(C, S) \times D \to \text{Fun}(C, D) \). Hence, the pushout product \( \Box \) of Definition 2.3 of two maps in \( \text{Fun}(C, D) \) is defined. For \( 0 \leq n \leq \infty \), we define two sets of maps in \( \text{Fun}(C, D) \):

\[
I^\text{cr}_n = (\Phi_n \Box I_D) \quad J^\text{cr}_n = (\Phi_n \Box J_D)
\]

Set \( I^\text{cr} := I^\text{cr}_\infty \) and \( J^\text{cr} := J^\text{cr}_\infty \). A map in \( \text{Fun}(C, D) \) is

1. a \textit{cross effect fibration} or \textit{cr fibration} if it belongs to the class \( J^\text{cr}_{\infty} \)-inj, i.e. it has the right lifting property with respect to all maps in \( J^\text{cr} \).

2. a \textit{cross effect cofibration} or \textit{cr cofibration} if it belongs to the class \( I^\text{cr}_{\infty} \)-cof, i.e. it has the left lifting property with respect to all cr fibrations.

Weak equivalences are still given by objectwise weak equivalences. The model structure is called the \textit{cross effect} or simply \textit{cr model structure} and denoted by \( \text{Fun}(C, D)_{\text{cr}} \).

Remark 3.18. Note that \( I^\text{cr}_1 = I^\text{proj}_{\text{Fun}(C, D)} \) and \( J^\text{cr}_1 = J^\text{proj}_{\text{Fun}(C, D)} \). In particular, every projective cofibration is a cr cofibration and every acyclic projective cofibration is an acyclic cr cofibration.

Theorem 3.19. The classes of objectwise weak equivalences, cross effect fibrations and cross effect cofibrations form a cofibrantly generated \( S \)-model structure on the category \( \text{Fun}(C, D) \), which is as proper as \( D \).

The idea for the proof of the theorem can be found in [15]. It is an application of the recognition principle for cofibrantly generated model structures [13, 11.3.1]. Replacing \( \Phi \) with \( \Phi_n \) for some \( 1 \leq n \leq \infty \) leads to a model structure where only the \( k \)-th cross effects for \( 1 \leq k \leq n \) become right Quillen functors.

Proof. Remark 3.18 implies that every projective cofibration is a cr cofibration. Sources and targets of maps in \( I^\text{cr} \) and \( J^\text{cr} \) are as small as the sources and targets of the maps in \( I_D \) and \( I_P \), thus allowing the small object argument. Lemmata 3.21, 3.22 and 3.22 below conclude the proof of the existence of the cofibrantly generated model structure. The model structure is an \( S \)-model structure, as one checks on the generators \( I^\text{cr} \) and \( J^\text{cr} \). The statement regarding properness is proved in 3.23.

Lemma 3.20. A map in \( \text{Fun}(C, D) \) is in \( I^\text{cr}_{\infty} \)-inj if and only if it is an objectwise weak equivalence and a cross effect fibration.

Proof. Let \( f \colon X \to Y \) be in \( I^\text{cr}_{\infty} \)-inj. Then \( f \) is in \( I^\text{proj}_{\text{Fun}(C, D)} \)-inj, and so an acyclic projective fibration. In particular, \( f \) is an objectwise weak equivalence. The inclusion \( I_D^{-\inj} \subset J_D^{-\inj} \) implies

\[
(\Phi \Box I_D)^{-\inj} \subset (\Phi \Box J_D)^{-\inj},
\]

whence \( f \) is a cr fibration as well.

Conversely, let \( f \) be an objectwise equivalence and a cr fibration. Then it is in particular an acyclic projective fibration, hence in \( I^\text{proj}_{\text{Fun}(C, D)} \)-inj. By assumption, it is also in \( (\Phi \Box J_D)^{-\inj} \), which means exactly that the map

\[
X\left( \bigvee_{i=1}^n K_i \right) \to \lim_{S \in P_0(\omega)} \left( \bigvee_{i \in S} K_i \right) \times \lim_{S \in P_0(\omega)} \left( \bigvee_{i \in S} K_i \right)
\]

(3.3)
is a fibration for every possible choice of $n$ and $K$. It remains to show that $f$ is in $(\Phi \Box I_D)$-inj. This is equivalent to the map in (3.3) being an acyclic fibration. By assumption, the map

$$X^n \bigvee_{i=1}^n K_i \rightarrow Y^n \bigvee_{i=1}^n K_i$$

is a weak equivalence. Thus it suffices to show that the map

$$\lim_{S \in P_0(\underline{n})} X_{\bigvee_{i \in n-S} K_i} \rightarrow \lim_{S \in P_0(\underline{n})} Y_{\bigvee_{i \in n-S} K_i}$$

is an acyclic fibration. To conclude this, recall that $P_0(\underline{n})$ is an inverse category by the functor $\text{deg}: P_0(\underline{n})^{\text{op}} \rightarrow \mathbb{N}$ which sends $S$ to the number of elements in $\underline{n} \setminus S$. By [13, Thm. 5.1.3], there is a model structure on the category of functors from $P_0(\underline{n})$ to any model category. It has objectwise weak equivalences and cofibrations, and the fibrations are characterized by an appropriate matching space condition. The limit is thus a right Quillen functor on this functor category, and in particular preserves acyclic fibrations. Since $f: X \rightarrow Y$ is a cr fibration, the induced natural transformation $f'$ of functors on $P_0(\underline{n})$ is a fibration. And since $f$ is an objectwise equivalence, $f'$ is an objectwise weak equivalence. The result follows. □

Lemma 3.21. A map in $J^r$-cof is a cr cofibration.

Proof. If a map is in $J^r$-cof, it has the left lifting property with respect to all cr fibrations. In particular, it has the left lifting property with respect to all cr fibrations that are also objectwise weak equivalences. So, by 3.20 it is an cr cofibration. □

Lemma 3.22. A map in $J^r$-cof is an objectwise weak equivalence.

Proof. By the small object argument, every map in $J^r$-cof is a retract of a map in $J^r$-cell. Since $\mathcal{V} = \mathcal{S}$, every map in $\Phi$ is an objectwise cofibration in Fun($\mathcal{C}, \mathcal{S}$). This implies that every map in $J^r$-cell is an objectwise weak equivalence. □

Lemma 3.23. If $\mathcal{D}$ is right or left proper, then the cross effect model structure is right or left proper, respectively.

Proof. Any cr fibration is an objectwise fibration and any cr cofibration is an objectwise cofibration, again using $\mathcal{V} = \mathcal{S}$ for the latter statement. Since pullbacks and pushouts are formed objectwise, the statement follows. □

Lemma 3.24. If the functor $X$ is cross effect fibrant, the canonical map

$$\text{cr}_n X \rightarrow \text{hocr}_n X$$

is an objectwise weak equivalence.

Proof. If $X$ is cr fibrant, then – as in the proof of Lemma 3.20 – the $P_0(\underline{n})$-diagram

$$S \mapsto X_{\bigvee_{i \in n-S} K_i}$$

is Reedy (or, what is the same here, injectively) fibrant. This follows from the right lifting property of the map $X \rightarrow *$ with respect to $J^r_{m}$ for $1 \leq m < n$. Thus, the map

$$\lim_{S \in P_0(\underline{n})} X_{\bigvee_{i \in n-S} K_i} \rightarrow \holim_{S \in P_0(\underline{n})} X_{\bigvee_{i \in n-S} K_i}$$
is a weak equivalence. The right lifting property with respect to $J^{cr}$ implies that

$$X \left( \bigvee_{i=1}^n K_i \right) \to \lim_{S \in P(E)} X \left( \bigvee_{i \in S} K_i \right)$$

is a fibration. The claim follows. □

Proposition 3.25. The functor

$$cr_n : \text{Fun}(\mathcal{C}, \mathcal{D})_{cr} \to \text{Fun}(\Sigma_n \wr \mathcal{C}, \mathcal{D})_{proj}$$

is a right Quillen functor and $hocr_n$ is its right derived functor.

Proof. Let $\underline{K} = (K_1, \ldots, K_n)$ be an $n$-tuple of objects in $\mathcal{C}$. The cofiber sequence

$$\text{colim}_{S \in P(E)} R^n_{\in S} K_i \xrightarrow{\phi_K} R^n_{\in i=1} K_i \to \bigwedge_{i=1}^n R^K_i$$

in $\text{Fun}(\mathcal{C}, \mathcal{S})$ implies that the functor $\bigwedge_{i=1}^n R^K_i$ is $\text{cr}$ cofibrant, because the map $\phi_K$ is a $\text{cr}$ cofibration. By the formula 3.14

$$\text{hom}(\bigwedge_{i=1}^n R^K_i, X) \cong cr_n X(K_1, \ldots, K_n),$$

the strict cross effect is a right Quillen functor. Its right derived functor $hocr_n$ is identified by Lemma 3.24. □

4. Homotopy functors

Definition 4.1. Suppose $\mathcal{B}$ and $\mathcal{D}$ are model categories and $\mathcal{C}$ is a small full subcategory of $\mathcal{B}$. A functor in $\text{Fun}(\mathcal{C}, \mathcal{D})$ is called a homotopy functor if for every weak equivalence $A \to B$ in $\mathcal{C}$ the image $F(A) \to F(B)$ is a weak equivalence in $\mathcal{D}$.

Homotopy functors are the main object of study in Goodwillie’s calculus of functors. From the point of view of model categories, the full subcategory of homotopy functors is usually inadequate. The aim of this section is to construct a model structure in which every functor is a homotopy functor, up to weak equivalence.

4.1. Homotopy functors and simplicial functors. A preliminary goal is to show that every homotopy functor of reasonable categories is objectwise weakly equivalent to a simplicial functor. The following statement is a slight generalization of a Lemma by Waldhausen [25, Lemma 3.1.2, 3.1.3] to certain $\mathcal{U}$-model categories.

Lemma 4.2. Let $\mathcal{C}$ be a small subcategory of a simplicial model category, closed under cotensoring with finite simplicial sets, and let $\mathcal{D}$ be a $\mathcal{U}$-model category. Suppose that $X : \mathcal{C} \to \mathcal{D}$ is a homotopy functor. If the simplicial object $n \mapsto X(A^{\Delta^n})$ is Reedy cofibrant for every object $A \in \mathcal{C}$, then there exists a $\mathcal{U}$-functor $\underline{X} : \mathcal{C} \to \mathcal{D}$ and a natural objectwise weak equivalence $f : X \to \underline{X}$.

Proof. The value of the functor $\underline{X}$ at an object $A$ of $\mathcal{C}$ is defined as the coend

$$\underline{X}(A) := \int^n X(A^{\Delta^n}) \times \Delta^n,$$
which is in fact the standard realization of a simplicial object in $D$. Expressing $X(A)$ as the standard realization of a constant simplicial object, one obtains a natural transformation $f: X \to X$ via $\Delta^n \to \Delta^0$:

$$f(A): X(A) \cong \int^n X(A^{\Delta^0}) \times \Delta^n \to \int^n X(A^{\Delta^0}) \times \Delta^n = X(A)$$

The map $A = A^{\Delta^0} \to A^{\Delta^n}$ is a simplicial homotopy equivalence, since $\Delta^n \to \Delta^0$ is one. It follows that it is a weak equivalence in $C$. Hence, so is its image under $X$ by assumption. The constant simplicial object $X(A)$ is cofibrant in the Reedy model structure. Since by assumption the target of $f(A)$ is Reedy cofibrant as well, $f$ is a natural weak equivalence. The reason is that realization is a left Quillen functor on the Reedy model structure.

It remains to prove that $X$ is a $U$-functor. A map of simplicial sets

$$U_C(A, B) \to U_D(X(A), X(B))$$

will be given in simplicial degree $m$ as follows. An $m$-simplex $A \times \Delta^m \to B$ can equivalently be described as a map $\alpha: A \to B^{\Delta^m}$. Consider the simplicial objects $[n] \mapsto F_n = X(A^{\Delta^n})$, $[n] \mapsto G_n = X(B^{\Delta^n})$. An $m$-simplex $F_\bullet \to G_\bullet$ is the same as a natural transformation

$$(\gamma: [n] \to [m]) \mapsto (t_\gamma: F_n \to G_n).$$

Set $t_\gamma$ to be the composition

$$X(A^{\Delta^n}) \xrightarrow{X(\alpha^{\Delta^n})} X(B^{\Delta^m \times \Delta^n}) \xrightarrow{X(B^\gamma \cdot \text{id})} X(B^{\Delta^n})$$

which induces the desired map

$$X(A) \times \Delta^m \to X(B).$$

The verification of the relevant axioms this map has to fulfill is left to the reader. □

Note that the condition on Reedy cofibrancy is fulfilled automatically in many cases, for example in the category of simplicial presheaves with the injective model structure.

**Remark 4.3.** Recall from Definition 2.17 that a functor $X$ between pointed categories is reduced if $X(\ast) \cong \ast$. A $U$-functor is an $S$-functor if and only it is reduced. Hence, the analog of Lemma 4.2 for $S$-model categories and reduced homotopy functors holds as well. In fact, one can replace a homotopy functor with $X(\ast) \simeq \ast$ by a weakly equivalent $S$-functor.

### 4.2. A model structure for simplicial homotopy functors.

The purpose of this section is to construct a model structure on a category of enriched functors in which every enriched functor is weakly equivalent to an enriched homotopy functor. For specific categories of enriched functors this has been obtained in [19], [7] and [2]. Although more general results are possible, a restriction to simplicial functors seems adequate, as Lemma 4.2 suggests. None of this is necessary if all functors are already homotopy functors, as it is the case if the source category is the category of finite CW complexes. This section is written in the pointed setting. All statements in this section and their proofs have unpointed variants, whose formulation is left to the reader.
Definition 4.4. An $\mathcal{S}$-model category $\mathcal{B}$ has a decent fibrant replacement functor if there exists a $\mathcal{S}$-natural transformation

$$\phi_{\text{Fibr}} : \text{Id}_\mathcal{B} \to \text{Fibr}$$

of $\mathcal{S}$-functors satisfying the following conditions:

1. For every object $A \in \mathcal{B}$ the object $\text{Fibr}(A)$ is fibrant and $\phi_{\text{Fibr}}(A)$ is an acyclic cofibration.
2. The functor $\text{Fibr}$ sends weak equivalences of cofibrant objects to simplicial homotopy equivalences.
3. The functor $\text{Fibr}$ commutes with filtered colimits.

Example 4.5. In the case $\mathcal{B} = \mathcal{S}$ or $\mathcal{U}$ one can use Kan’s $\text{Ex}_\infty$, as well as the composition of the geometric realization and the singular complex, as a decent fibrant replacement functor.

Lemma 4.6. Let $\mathcal{B}$ be an $\mathcal{S}$-model category. Suppose there exists a set $\{j : s_j \to t_j\}_{j \in J}$ of acyclic cofibrations in $\mathcal{B}$ with the following properties:

1. An object $A \in \mathcal{B}$ is fibrant if $A \to *$ has the right lifting property with respect to $J$.
2. The functor $\mathcal{S}_\mathcal{B}(s_j, -) = R^{s_j} : \mathcal{B} \to \mathcal{S}$ commutes with filtered colimits for every $j \in J$.

Then $\mathcal{B}$ has a decent fibrant replacement functor.

Proof. This is an enriched version of Quillen’s small object argument, as constructed in [7]. Let $\text{Fibr}_1$ be the $\mathcal{S}$-functor defined as the pushout of

$$\bigvee_{j \in J} R^{s_j} \wedge t_j \longrightarrow \bigvee_{j \in J} R^{s_j} \wedge s_j \longrightarrow \text{Id}_\mathcal{B}.$$ 

It comes together with an $\mathcal{S}$-natural transformation $\phi_1 : \text{Id} \to \text{Fibr}_1$. For $n \geq 1$ set $\text{Fibr}_{n+1} = \text{Fibr}_1 \circ \text{Fibr}_n$ and let

$$\text{Fibr} = \text{colim}(\text{Id} \xrightarrow{\phi_1} \text{Fibr}_1 \xrightarrow{\phi_1 \circ \text{Fibr}_1} \text{Fibr}_2 \to \cdots).$$

The natural transformation $A \to \text{Fibr}(A)$ is then an acyclic cofibration. Since every $s_j$ is in particular finitely presentable, a morphism $\alpha : s_j \to \text{Fibr}(A)$ factors over $\text{Fibr}_n(A)$. The composition

$$t_j \equiv S^0 \wedge t_j \to \bigvee_{j \in J} \mathcal{S}_\mathcal{B}(s_j, \text{Fibr}_n(A)) \wedge t_j \to \text{Fibr}_{n+1}(A) \to \text{Fibr}(A)$$

solves the lifting problem given by $\alpha$. Thus $\text{Fibr}(A)$ is fibrant. Weak equivalences of bifibrant objects in an $\mathcal{S}$-model category are simplicial homotopy equivalences. Thus it remains to prove the third condition. It follows because $\text{Fibr}_1$ is a colimit of functors preserving filtered colimits by definition. \qed

Convention 4.7. In addition to Convention 3.8 the following statements are assumed to be true:

1. The category $\mathcal{C}$ is a small full sub-$\mathcal{S}$-category of an $\mathcal{S}$-model category $\mathcal{B}$, containing only $\mathcal{S}$-finitely presentable cofibrant objects.
2. There exists a decent fibrant replacement functor $\text{Id}_\mathcal{B} \to \text{Fibr}$ such that, for every $A \in \mathcal{C}$, the object $\text{Fibr}(A) \in \mathcal{B}$ is a filtered colimit of objects in $\mathcal{C}$. 

(3) The category $D$ is a right proper cofibrantly generated $\mathcal{S}$-model category.
(4) In $D$, weak equivalences, fibrations with fibrant codomain, and pullbacks are preserved under filtered colimits.

A sufficient condition on the model category $D$ to satisfy 4.7(4) is essentially due to Voevodsky.

**Definition 4.8.** [7, 3.4] A cofibrantly generated model category $D$ is called weakly finitely generated if we can choose a set of generating cofibrations $I$ and of generating acyclic cofibrations $J$ such that the following conditions hold:

1. The domains and codomains of the maps in $I$ are finitely presentable.
2. The domains of the maps in $J$ are small.
3. There exists a subset $J'$ of $J$ of maps with finitely presentable domains and codomains such that a map in $D$ with fibrant codomain is a fibration if and only if it is in $J'$-inj.

If $D$ is a locally finitely presentable category, then pullbacks are preserved under filtered colimits in $D$. The rest of the requirements from 4.7 are met, as proved in [7, Lemma 3.5], if the model structure on $D$ is weakly finitely generated. The class of weakly finitely generated model structures is closed under left Bousfield localization with respect to a set of morphisms with finitely presentable (co)domains.

Suppose from now on that Convention 4.7 holds.

**Lemma 4.9.** Every $\mathcal{S}$-functor $X: C \to D$ preserves filtered colimits.

**Proof.** Condition 4.7(1) implies that every representable $\mathcal{S}$-functor commutes with filtered colimits. The $\mathcal{S}$-Yoneda lemma implies that any $\mathcal{S}$-functor is a colimit of representables, which gives the result. □

**Definition 4.10.** Let $i: C \to B$ be the inclusion functor. Every $\mathcal{S}$-functor $X: C \to D$ admits a left $\mathcal{S}$-Kan extension $i_*(X): B \to D$. Note that if $X$ commutes with filtered colimits, then so does $i_*(X)$. Let

$$
\phi_X: X \to X^{hf} := i_*(X) \circ \text{Fibr} \circ i
$$

denote the canonical map to the composition. The composition $X^{hf}: C \to D$ is again an $\mathcal{S}$-functor.

**Definition 4.11.** A map $f: X \to Y$ is

1. an $hf$ equivalence if the map $f^{hf}: X^{hf} \to Y^{hf}$ is an objectwise weak equivalence.
2. an $hf$ fibration if it is an objectwise fibration $X \to Y$ such that the square

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X^{hf} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f'^{hf}} & Y^{hf}
\end{array}
$$

is an objectwise homotopy pullback square.

The $hf$ cofibrations are the projective cofibrations. Theorem 4.14 states that these classes form a model structure on $\text{Fun}(C, D)$. It is called the homotopy functor model structure or $hf$ model structure for short, and denoted $\text{Fun}(C, D)^{hf}$. Analogous
definitions can be given starting from the cross effect model structure instead of the projective model structure.

**Remark 4.12.** In an \(\mathcal{S}\)-model category, any simplicial homotopy equivalence is in particular a weak equivalence. Simplicial homotopy equivalences – unlike weak equivalences – are preserved by any \(\mathcal{S}\)-functor.

**Remark 4.13.** A few words about hf fibrant \(\mathcal{S}\)-functors:

1. A (cr) fibrant functor \(X\) is (cr) hf fibrant if and only if the map (4.11) is an objectwise weak equivalence.
2. The functor \(X^{hf}\) preserves weak equivalences. Thus a (cr) hf fibrant functor preserves weak equivalences.
3. A (cr) fibrant functor preserving weak equivalences is (cr) hf fibrant.
4. The hf fibrant functors are exactly the objectwise fibrant homotopy functors. The cr hf fibrant functors are exactly the cr fibrant homotopy functors.

**Theorem 4.14.** Assume Convention 4.7. The classes of maps given in Definition 4.11, starting from the projective or the cross effect model structure, constitute a right proper cofibrantly generated \(\mathcal{S}\)-model structure. It is left proper if \(D\) is left proper.

**Proof.** As in our previous article [2], it suffices to check that the natural transformation \(\phi_X: X \to X^{hf}\) satisfies the axioms (A1), (A2), and (A3) given by Bousfield in [5, 9.2]. Cofibrant generation is delegated to Lemma 4.16.

**Axiom (A1):** Let \(f: X \to Y\) be an objectwise weak equivalence. To prove that \(f^{hf}\) is an objectwise weak equivalence, let \(A \in \mathcal{C}\) and express \(\text{Fibr}(A)\) as a filtered colimit of objects \(B_i\) in \(\mathcal{C}\) by condition 4.7.2. Lemma 4.9 implies that \(f^{hf}(A)\) is the morphism induced on filtered colimits by the morphisms \(f(B_i)\), which are weak equivalences. Condition 4.7.3 implies that \(f^{hf}(A)\) is itself a weak equivalence.

**Axiom (A2):** The task is to identify the two natural transformations

\[
\phi_X^{hf}, \phi_X^{hf}: X^{hf} \to (X^{hf})^{hf}
\]

as weak equivalences. The triangular identities and the natural isomorphism

\[X \overset{\cong}{\to} i_*(X) \circ i\]

reduce the problem to the value of the two natural maps

\[\text{Fibr}(A) \to \text{Fibr}(\text{Fibr}(A))\]

under the \(\mathcal{S}\)-functor \(X\). Both maps are simplicial homotopy equivalences, since \(\text{Fibr}\) is a decent fibrant replacement functor and all objects in \(\mathcal{C}\) are cofibrant by Condition 4.7.4. Remark 4.12 then implies that the maps in question are objectwise weak equivalences.

**Axiom (A3):** Let \(f: X \to Y\) be an hf weak equivalence and let \(p: Z \to Y\) be an objectwise fibration with \(Y\) objectwise fibrant; cr fibrations are not necessary. Consider the pullback diagram

\[
\begin{array}{ccc}
X \times_Y Z & \rightarrow & Z \\
\downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array}
\]
The goal is to prove that \( g^{hf} \) is an objectwise weak equivalence. Pullbacks are computed objectwise. Lemma 4.9 and Condition 4.7 imply that the diagram

\[
\begin{array}{ccc}
(X \times Y Z)^{hf} & \xrightarrow{g^{hf}} & Z^{hf} \\
\downarrow & & \downarrow p^{hf} \\
X^{hf} & \xrightarrow{f^{hf}} & Y^{hf}
\end{array}
\]

is a pullback diagram. Moreover, since fibrations in \( D \) with fibrant target are closed under filtered colimits, \( p^{hf} \) is still an objectwise fibration. Now \( f^{hf} \) is an objectwise weak equivalence by assumption and \( D \) is right proper. Thus, \( g^{hf} \) is an objectwise weak equivalence, which finishes the proof. □

**Lemma 4.15.** Let \( p: X \to Y \) be an objectwise fibration. Then the following statements are equivalent:

(i) The map \( p \) is an hf fibration.

(ii) The induced square

\[
\begin{array}{ccc}
X & \longrightarrow & X^{hf} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y^{hf}
\end{array}
\]

is an objectwise homotopy pullback.

(iii) For each weak equivalence \( A \to B \) in \( C \) the induced square

\[
\begin{array}{ccc}
X(A) & \longrightarrow & X(B) \\
\downarrow & & \downarrow \\
Y(A) & \longrightarrow & Y(B)
\end{array}
\]

is a homotopy pullback square.

The corresponding statement for cr fibrations holds as well.

**Proof.** The equivalence of (i) and (ii) follows from the characterization \([5, \text{Thm. 9.3}]\) of fibrations in the localized model structure. The equivalence of (ii) and (iii) will be shown now. Let \( X \to Y \) satisfy (ii). Since the functors \( X^{hf} \) and \( Y^{hf} \) are homotopy functors, the induced diagram

\[
\begin{array}{ccc}
X^{hf}(A) & \longrightarrow & X^{hf}(B) \\
\downarrow & & \downarrow \\
Y^{hf}(A) & \longrightarrow & Y^{hf}(B)
\end{array}
\]

is trivially a homotopy pullback diagram for any weak equivalence \( A \to B \) in \( C \). This means that the composed outer square and the right hand square in the following
diagram are homotopy pullbacks:

\[
\begin{array}{ccc}
X(A) & \rightarrow & X(B) \\
\downarrow & & \downarrow \\
Y(A) & \rightarrow & Y(B)
\end{array}
\rightarrow
\begin{array}{ccc}
X^\text{hf}(B) & \rightarrow & X^\text{hf}(B) \\
\downarrow & & \downarrow \\
Y^\text{hf}(B) & \rightarrow & Y^\text{hf}(B)
\end{array}
\]

It follows that the left hand square is a homotopy pullback.

Now let \(X \rightarrow Y\) satisfy property (iii), and let \(A \rightarrow B\) be a weak equivalence in \(C\). Consider a decent fibrant replacement \(\phi_A: A \rightarrow \text{Fibr}(A)\). By Convention 4.7.2, this map factors through a colimit

\[
A \rightarrow \cdots \rightarrow B_i \rightarrow B_{i+1} \rightarrow \cdots \rightarrow \text{Fibr}(A)
\]

where all objects \(B_i\) are in \(C\). By (iii) there are homotopy pullback diagrams:

\[
\begin{array}{ccc}
X(A) & \rightarrow & X(B_i) \\
\downarrow & & \downarrow \\
Y(A) & \rightarrow & Y(B_i)
\end{array}
\]

Their colimit yields the desired homotopy pullback in (ii) since homotopy pullbacks commute with filtered colimits in \(D\) by Convention 4.7.4. □

**Lemma 4.16.** Assume Convention 4.7. Then the homotopy functor model structure is cofibrantly generated.

**Proof.** In order to enlarge the set of generating acyclic cofibrations of the cr or projective model structure, respectively, take an arbitrary weak equivalence \(w: A \xrightarrow{\sim} B\) in \(C\). It induces the map

\[w^*: R^B \rightarrow R^A\]

which, using the simplicial mapping cylinder construction, can be factored as a projective cofibration \(w\|^\circ\), followed by a simplicial homotopy equivalence. The additional set of generating acyclic cofibrations is

\[\{w^\circ i\}\]

where \(i\) runs through a set \(I_D\) of generating cofibrations of \(D\) and \(w\) runs through the set of weak equivalences in \(C\). The fact that hf fibrations are exactly those objectwise fibrations with the right lifting property with respect to this set follows from Lemma 4.15. □

**Remark 4.17.** It is now clear that the (cr) hf model structure on \(\text{Fun}(C, D)\) can also be viewed as the left Bousfield localization of the (cr) projective model structure with respect to the set

\[\{R^B \rightarrow R^A | A \rightarrow B\ \text{weak equivalence in} \ \mathcal{C}\}\].

**4.3. Homotopy functors in several variables.** Recall that in the product of \(S\)-model categories \(B_1 \times \cdots \times B_n\), a morphism \(f: K \rightarrow L\) is a weak equivalence (or fibration, or cofibration) if every component \(K_i \rightarrow L_i\) is so in \(C_i\). This defines an \(S\)-model structure. In particular, Definition 4.1.4 is applicable in \(\text{Fun}(C_1 \times \cdots \times C_n, D)\), where \(C_k \subset B_k\) is a full subcategory for every \(k \in n\). Note that a functor in the product category is a homotopy functor if and only if all its partial functors are homotopy functors. The results of Section 4.2 apply.
Corollary 4.18. Assume that for all \( k \in \mathbb{N} \) the categories \( C_i \subset B_i \) and \( D \) satisfy Convention [4.7]. Then the homotopy functor model structure obtained from the projective model structure on the category \( \text{Fun}(C_1 \times \cdots \times C_n, D) \) exists. Furthermore, it is a right proper cofibrantly generated \( S \)-model structure, and it is left proper if \( D \) is left proper.

We want to obtain an hf model structure on the category \( \text{Fun}(C_1 \wedge \cdots \wedge C_n, D) \).

Unfortunately, the ambient category \( B_1 \wedge \cdots \wedge B_n \) is usually neither cocomplete, nor complete. In particular, we cannot directly apply the results from section 4.2.

We first define homotopy functors.

Definition 4.19. A functor \( X \) in \( \text{Fun}(C_1 \wedge \cdots \wedge C_n, D) \) is called a homotopy functor if for any object \((K_1, \ldots, \hat{K}_i, \ldots, K_n)\) in \( C_1 \wedge \cdots \wedge \hat{C}_i \wedge \cdots \wedge C_n \) the associated partial functor

\[
X_{(K_1, \ldots, \hat{K}_i, \ldots, K_n)} : C_i \to D
\]

is a homotopy functor. The hat indicates that the corresponding entry is left out.

A coaugmented \( S \)-functor \( \text{Fibr} \) from \( B_1 \wedge \cdots \wedge B_n \) to itself is defined by

\[
\text{Fibr}(K) := (\text{Fibr}_{B_1}(K_1), \ldots, \text{Fibr}_{B_n}(K_n)),
\]

using the decent fibrant replacement in each category \( B_i \). Analogous to (4.1), a coaugmented \( S \)-functor \((\_)^{hf} : \text{Fun}(C_1 \wedge \cdots \wedge C_n, D) \to \text{Fun}(C_1 \wedge \cdots \wedge C_n, D)\) is defined by

\[
\phi_X : X \to (X)^{hf}(K) := ((i^\wedge n)_*X) \circ \text{Fibr} \circ i^\wedge n
\]

where \( i : C_1 \wedge \cdots \wedge C_n \to B_1 \wedge \cdots \wedge B_n \) is the inclusion. These enriched functors are well defined by proposition [2.16].

Definition 4.20. A functor \( X \) in \( \text{Fun}(\Sigma_n \wr C \wedge n, D) \) is called a homotopy functor if \( \varepsilon^*X \) is a homotopy functor, where \( \varepsilon : C^\wedge n \to \Sigma_n \wr C^\wedge n \) was defined in [3.5].

We observe that the previous constructions extend to the wreath product category. The inclusion \( C \to B \) induces a symmetric inclusion \( \Sigma_n \wr C^\wedge n \to \Sigma_n \wr B^\wedge n \) and a decent fibrant replacement functor of \( B \) extends to a symmetric functor

\[
\text{Fibr} = (\text{Fibr}_B, \ldots, \text{Fibr}_B) : \Sigma_n \wr B^\wedge n \to \Sigma_n \wr B^\wedge n;
\]

which is a decent fibrant replacement functor for \( \Sigma_n \wr B^\wedge n \). There is then a coaugmented \( S \)-functor

\[
(\_)^{hf} : \text{Fun}(\Sigma_n \wr C^\wedge n, D) \to \text{Fun}(\Sigma_n \wr C^\wedge n, D)
\]

as above and an associated \( S \)-natural transformation \( \phi_X \).

Remark 4.21. For any \( X \) in \( \text{Fun}(C_1 \wedge \cdots \wedge C_n, D) \) or \( \text{Fun}(\Sigma_n \wr C^\wedge n, D) \) the functor \( X^{hf} \) is a homotopy functor and analogues of remark [4.13] hold.

In order to treat functors out of the smash product category and the wreath product category simultaneously, the construction \((\_)^{hf}\) has to be interpreted appropriately in the following statements.

Definition 4.22. A map \( f : X \to Y \) in \( \text{Fun}(C_1 \wedge \cdots \wedge C_n, D) \) or \( \text{Fun}(\Sigma_n \wr C^\wedge n, D) \) is called
(1) an hf equivalence if the map $f^{hf}: X^{hf} \rightarrow Y^{hf}$ is an objectwise weak equivalence, and
(2) an hf fibration if it is an objectwise fibration such that the square

\[
\begin{array}{ccc}
X & \longrightarrow & X^{hf} \\
\downarrow f & & \downarrow f^{hf} \\
Y & \longrightarrow & Y^{hf}
\end{array}
\]

is a homotopy pullback square in the objectwise model structure.

The hf cofibrations are the projective cofibrations.

**Theorem 4.23.** Assume Convention 4.7. The classes given in Definition 4.22 constitute a right proper cofibrantly generated $S$-model structure on the categories $\text{Fun}(C_1 \wedge \cdots \wedge C_n, D)$ and $\text{Fun}(\Sigma_n \wr C \wedge \wedge n, D)$, respectively. It is left proper if $D$ is left proper.

*Proof.* The arguments of the proof of Theorem 4.14 showing the existence of the hf model structure on $\text{Fun}(C, D)$ apply componentwise. □

The model structure from Theorem 4.23 is called the homotopy functor model structure and is denoted by $\text{Fun}(C_1 \wedge \cdots \wedge C_n, D)^{hf}$ and $\text{Fun}(\Sigma_n \wr C \wedge \wedge n, D)^{hf}$, respectively.

**Proposition 4.24.** Assume Convention 4.7. The adjoint pair
\[
p_*: \text{Fun}(C_1 \times \cdots \times C_n, D)^{hf} \rightleftharpoons \text{Fun}(C_1 \wedge \cdots \wedge C_n, D)^{hf}: p^*
\]
is a Quillen pair of homotopy functor model structures. The functor $p^*$ preserves and detects weak equivalences and fibrations. Also, the adjoint pair
\[
\varepsilon_*: \text{Fun}(C^n, D)^{hf} \rightleftharpoons \text{Fun}(\Sigma_n \wr C \wedge \wedge n, D)^{hf}: \varepsilon^*
\]
is a Quillen pair of homotopy functor model structures. The functor $\varepsilon^*$ preserves and detects weak equivalences and fibrations.

*Proof.* The pair $(p_*, p^*)$ is a Quillen pair for the respective projective model structures by Lemma 2.23. The same holds for the pair $(\varepsilon_*, \varepsilon^*)$. The claims above then follow from the canonical natural isomorphisms $p^* \circ (\varepsilon^*)^{hf} \cong (\varepsilon^*)^{hf} \circ p^*$ and $\varepsilon^* \circ (\varepsilon_*)^{hf} \cong (\varepsilon_*)^{hf} \circ \varepsilon^*$. □

**Lemma 4.25.** Consider the homotopy functor model structure on $\text{Fun}(C, D)$ obtained from the cross effect model structure. Then the $n$-th cross effect
\[
cr_n: \text{Fun}(C, D)^{hf} \rightarrow \text{Fun}(\Sigma_n \wr C \wedge \wedge n, D)^{hf}
\]
is a right Quillen functor.

*Proof.* Let $f: X \rightarrow Y$ be an hf fibration, that is, a cr fibration such that the diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X^{hf} \\
\downarrow f & & \downarrow f^{hf} \\
Y & \longrightarrow & Y^{hf}
\end{array}
\]

is a homotopy pullback square in the objectwise model structure.
is a homotopy pullback square. The construction of \((-\)hf and Convention 4.7 imply that \(f\) hf is a cr fibration as well. By Proposition 3.25, the map \(\text{cr}_n f\) is an objectwise fibration and the diagram

\[
\begin{array}{c}
\text{cr}_n X \\
\downarrow \text{cr}_n f \\
\text{cr}_n Y
\end{array} \quad \begin{array}{c}
\text{cr}_n (X\text{hf}) \\
\downarrow \text{cr}_n (f\text{hf}) \\
\text{cr}_n (Y\text{hf})
\end{array}
\]

is a homotopy pullback square. One checks directly that there is a canonical isomorphism

\[
\text{cr}_n (X\text{hf}) \cong (\text{cr}_n X)^{\text{hf}},
\]

which shows that \(\text{cr}_n f\) is an hf fibration in \(\text{Fun}(\Sigma_n \wr C \land n, D)\).

\[\Box\]

5. Excisive functors

The goal of this section is to localize the homotopy functor model structures on the various functor categories further, such that every functor is weakly equivalent to an \(n\)-excisive functor. Recall that a homotopy functor is \(n\)-excisive if it maps strongly homotopy cocartesian \((n + 1)\)-cubes to homotopy Cartesian ones.

5.1. The excisive model structures. Set \(\underline{n} := \{1, \ldots, n\}\), let \(P(\underline{n})\) be its power set, and let \(P_0(\underline{n}) := P(\underline{n}) - \{\emptyset\}\).

**Definition 5.1.** For an object \(A\) in \(\mathcal{C}\), let \(CA\) be the simplicial cone over \(A\). This is the reduced or unreduced cone depending on whether \(\mathcal{C}\) is a \(U\)- or \(S\)-category. For a finite set \(U\), the join \(A \star U\) is defined as

\[
CA \sqcup |U|^n A,
\]

where \(|U|^n A\) gluing \(|U|\) many copies of \(CA\) along their base \(A\).

**Convention 5.2.** In addition to Convention 4.7, suppose that for any object \(A\) in \(\mathcal{C}\) and any finite set \(U\) the object \(A \star U\) is also in \(\mathcal{C}\).

**Remark 5.3.** There are other models for \(A \star U\). In an ambient model category \(\mathcal{B}\), the join \(A \star U\) is weakly equivalent to a homotopy colimit of the asterisk-shaped diagram given by \(|U|\) copies of the map \(A \rightarrow \ast\) out of a single copy of \(A\). For instance, \(A \star U\) is weakly equivalent to \(|U| - 1\) wedge summands of \(\Sigma A\). Hence, the assumption that \(\mathcal{C}\) is closed under suspensions and finite coproducts is an equally good convention. Because \(\mathcal{C}\) is a full subcategory of \(\mathcal{B}\), a reasonable sufficient condition is to assume that \(\mathcal{C}\) is closed under finite pushouts along cofibrations.

The join is an enriched bifunctor and comes with a natural map \(A \rightarrow A \star U\) induced by the inclusion \(\emptyset \subset U\). The \(P_0(\underline{n} + 1)\)-diagram \(U \mapsto R^U \star A\) of representable functors yields the functor hocolim \(\lim_{U \in P_0(\underline{n} + 1)} R^U \star A\). Using repeated factorization by suitable simplicial mapping cylinder constructions supplies a cr cofibrant model, denoted as

\[
A_n \xrightarrow{\xi A, n} \text{Cyl}(\xi A, n) \xrightarrow{\cong} R^A.
\]

The induced natural transformation is factored via a simplicial mapping cylinder as a cr cofibration, followed by a simplicial homotopy equivalence:
Definition 5.4. Goodwillie’s construction $T_n$ [12, p. 657] on the category of objectwise fibrant homotopy functors may be rewritten as

$$T_nX(A) := \text{hom}(A_n, X).$$

Let $P_n$ be the colimit of the following sequence:

$$\text{Id} \to (-)^{\text{hf}} \to T_n(-)^{\text{hf}} \to T_n^2(-)^{\text{hf}} \to \cdots \to \text{colim}_k T_n^k(-)^{\text{hf}}$$

Convention 4.7 implies that filtered colimits and filtered homotopy colimits are weakly equivalent in $D$. Thus, Goodwillie’s $n$-excisive approximation is weakly equivalent to $P_n$ as defined above. The canonical inclusion $\mathbb{Z} \to \mathbb{Z} + 1$ induces a map $P_0(n) \to P_0(n + 1)$ of posets. This induces natural transformations $T_n \to T_{n-1}$ and $q_n: P_n \to P_{n-1}$ which commute with the coaugmentations from the identity. The map $q_n$ is constructed for categories of spaces or spectra in [12, p. 664] and generalizes to this setup.

Lemma 5.5. The functor $T_n$ commutes up to natural weak equivalence with all homotopy limits. The functor $P_n$ commutes up to natural weak equivalence with finite homotopy limits. Both functors commute up to natural weak equivalence with filtered homotopy colimits.

Proof. Since $A_n$ is cr cofibrant, $T_n$ is a right Quillen functor and commutes with all homotopy limits. Convention 4.7 ensures that $T_n$ commutes with filtered colimits. It then follows from its definition that $P_n$ commutes with filtered colimits. Again Convention 4.7 implies that $P_n$ commutes at least with finite homotopy limits. □

Lemma 5.6. Let $X$ be an objectwise fibrant homotopy functor.

1. The functor $P_nX$ is $n$-excisive.
2. The map $X \to P_nX$ is initial among maps in the objectwise homotopy category out of $X$ into an $n$-excisive functor.
3. Both maps $P_n(\text{pr}_n X)$ and $\text{pr}_n(P_nX)$ are objectwise weak equivalences and homotopic to each other.

Proof. Part (1) follows by adapting Goodwillie’s opaque [12, pp. 662] or Rezk’s slightly less opaque proof [23]. Part (2) and (3) are as in [12, pp. 661]. □

Definition 5.7. A map $X \to Y$ in $\text{Fun}(C, D)$ is called

1. an $n$-excisive equivalence if the induced map $P_nX \to P_nY$ is an objectwise weak equivalence.
2. an $n$-excisive fibration if it is an hf fibration and the following diagram

$$\begin{array}{ccc}
X & \longrightarrow & P_nX \\
f \downarrow & & \downarrow \text{pr}_n(f) \\
Y & \longrightarrow & P_nY
\end{array}$$

is a homotopy pullback square in the homotopy functor model structure.

The $n$-excisive cofibrations are projective cofibrations. Analogous definitions can be given starting from the cr model structure.

In the case of $D = S$ and $C = S^{\text{fin}}$, the following theorem was already obtained in [2].
**Theorem 5.8.** Assume Convention 5.2. The classes described in Definition 5.7 form a right proper cofibrantly generated S-model structure on $\text{Fun}(\mathcal{C}, \mathcal{D})$, which is left proper if $\mathcal{D}$ is left proper.

**Proof.** It suffices to show that the coaugmented functor $P_n$ satisfies the axioms (A1), (A2), and (A3) given in 5.9.2. The functors $T_n$ and $(-)^{hf}$ preserve objectwise weak equivalences by construction. Filtered colimits in $\mathcal{D}$ preserve weak equivalences by Convention 4.7. whence $P_n$ maps weak equivalences in $\mathcal{C}$ to weak equivalences in $\mathcal{D}$. This implies (A1). Property (A2) is verified in Lemma 5.6(3). Property (A3) follows directly from Lemma 5.5. The remaining task is to add further generating acyclic cofibrations for the $n$-excisive model structure. With $I_D$ being a set of generating cofibrations of $\mathcal{D}$, let

$$J_n := \{ \xi_{A,n} \square_i \}_{A \in \mathcal{C}, i \in I_D}.$$ 

An objectwise fibration $X \to Y$ has the right lifting property with respect to the set $J_n$ if and only if the morphism of pointed simplicial sets

(5.1) $$S_D(\text{Cyl}(\xi_{A,n}), X) \to S_D(A_n, X) \times_{S_D(A_n, Y)} S_D(\text{Cyl}(\xi_{A,n}), Y)$$

has the right lifting property with respect to $I_D$. Since the map in (5.1) is a fibration in $\mathcal{D}$ anyway, $X \to Y$ has the right lifting property with respect to $J_n$ if and only if (5.1) is a weak equivalence. The simplicial homotopy equivalence

$$\text{Cyl}(\xi_{A,n}) \cong R^A$$

induces a simplicial homotopy equivalence on $\mathcal{D}$-mapping objects. As $S_D(-, Z)$ transforms homotopy colimits to homotopy limits for $Z$ objectwise fibrant, and $X \to Y$ is an objectwise fibration, the map $X \to Y$ has the right lifting property with respect to $J_n$ if and only if the square

$$\begin{array}{ccc}
X(A) & \longrightarrow & \text{holim} X(A \ast U) \\
\downarrow & & \downarrow \\
Y(A) & \longrightarrow & \text{holim} Y(A \ast U)
\end{array}$$

is a homotopy pullback square. Thus adding $J_n$ to a suitable set of generating acyclic cofibrations for the hf model structure yields a set of generating acyclic cofibrations for the $n$-excisive model structure; analogously for their cr versions. □

The model structures provided by Theorem 5.8 are the $n$-excisive model structures. They are denoted $\text{Fun}(\mathcal{C}, \mathcal{D})_{n-\text{exc}}$ and $\text{Fun}(\mathcal{C}, \mathcal{D})_{n-\text{exc-cr}}$, respectively. The following statement is analogous to Lemma 4.15.

**Lemma 5.9.** A map $f : X \to Y$ is an $n$-excisive fibration if and only if it is an hf fibration and the following diagram

$$\begin{array}{ccc}
X & \longrightarrow & T_n X \\
f \downarrow & & \downarrow \quad T_n(f) \\
Y & \longrightarrow & T_n Y
\end{array}$$

is a homotopy pullback square in the hf model structure.

**Proof.** This is straightforward using the fact that in $\mathcal{D}$ filtered colimits preserve homotopy pullbacks. □
Remark 5.10. The proof of theorem 5.8 shows that the \( n \)-excisive \((cr)\) model structure is a left Bousfield localization of the \( hf \) \((cr)\) model structure with respect to the set \( \{ \xi_{A,n} \mid A \in C \} \) or equivalently the set
\[
\left\{ \text{hocolim}_{U \in \mathcal{P}_{0(n+1)}} R^{U \times A} \to R^A \mid A \in C \right\}.
\]

5.2. Excisive functors in several variables.

Definition 5.11. A functor \( X : C_1 \times \cdots \times C_n \to \mathcal{D} \) is \textit{multi-excisive} if for every object \( (K_1, \ldots, \hat{K}_i, \ldots, K_n) \) in \( C_1 \times \cdots \times \hat{C}_i \times \cdots \times C_n \) the associated partial functor
\[ X : (K_1, \ldots, \hat{K}_i, \ldots, K_n) : C_i \to \mathcal{D} \]
is excisive. More generally, the functor \( X \) is \((d_1, \ldots, d_n)\)-excisive if, for every \( 1 \leq i \leq n \), the associated partial functor \( X : (K_1, \ldots, \hat{K}_i, \ldots, K_n) \) is \( d_i \)-excisive.

The augmented functor \( P_1 \) from Definition 5.4 can be applied componentwise to functors in \( \text{Fun}(C_1 \times \cdots \times C_n, \mathcal{D}) \). More precisely, for each object \( K \) in \( C_1 \times \cdots \times C_n \) and each \( 1 \leq i < j \leq n \), the diagram
\[
\begin{array}{ccc}
X(K_1, \ldots, K_n) & \rightarrow & (P_1^{C_i} X(K_1, \ldots, K_n))(K_i) \\
\downarrow & & \downarrow \\
(P_1^{C_j} X(K_1, \ldots, \hat{K}_j, \ldots, K_n))(K_j) & \rightarrow & (P_1^{C_j} P_1^{C_i} X(K_1, \ldots, \hat{K}_i, \ldots, \hat{K}_j, \ldots, K_n))(K_i, K_j)
\end{array}
\]
in \( \mathcal{D} \) is natural in \( K \) and \( X \). By definition, \( P_1^{C_k} \) and \( P_1^{C_j} \) commute with each other up to canonical isomorphism, and thus the order \( i < j \) is purely cosmetic. These diagrams assemble into a \( U \)-natural transformation
\[
(5.2) \quad p_{1, \ldots, 1}(X) : X \to P_{1, \ldots, 1} X := P_1^{C_1} \cdots P_1^{C_n} X
\]
of functors from \( C_1 \times \cdots \times C_n \) by Propositions 2.14 and 2.16. If the functor \( X \) is multireduced, then this natural transformation is in fact enriched over \( S \).

A functor \( X \) is multi-excisive if and only if the natural transformation \( p_{1, \ldots, 1}(X) \) is an objectwise weak equivalence.

Definition 5.12. A map \( f : X \to Y \) in \( \text{Fun}(C_1 \times \cdots \times C_n, \mathcal{D}) \) is

1. a \textit{multi-excisive equivalence} if the map \( P_{1, \ldots, 1}(f) \) is an \( hf \) equivalence.
2. a \textit{multi-excisive fibration} if it is an \( hf \) fibration such that the square
\[
\begin{array}{ccc}
X & \rightarrow & P_{1, \ldots, 1} X \\
\downarrow f & & \downarrow P_{1, \ldots, 1}(f) \\
Y & \rightarrow & P_{1, \ldots, 1} Y
\end{array}
\]
is a homotopy pullback square in the \( hf \) model structure.

The multi-excisive cofibrations are the projective cofibrations.

Theorem 5.13. The classes given in Definition 5.12 are a right proper cofibrantly generated \( S \)-model structure on \( \text{Fun}(C_1 \times \cdots \times C_n, \mathcal{D}) \), which is left proper if \( \mathcal{D} \) is left proper.
Proof. The fact that each functor $P^k_\alpha$ above satisfies properties (A1), (A2) and (A3) by the proof of theorem 5.8 implies that the composite functor $P_{1,\ldots,1}$ satisfies them. Generating acyclic cofibrations are constructed as in the proof of Theorem 5.8. □

The corresponding model structure on $Fun(C_1 \times \cdots \times C_n, D)$ is called the multexcisive model structure and is denoted $Fun(C_1 \times \cdots \times C_n, D)_{\text{mexc}}$.

5.3. Multilinear and symmetric multilinear functors. The natural transformation $p_{1,\ldots,1}$ from $\langle 5.2 \rangle$ for functors $X: C_1 \times \cdots \times C_n \rightarrow D$ will be constructed differently for functors in $Fun(C_1 \wedge \cdots \wedge C_n, D)$ and $\Sigma_n \downarrow C \wedge n$. Let $S^n$ denote the $n$-fold smash product of $S^1 = \Delta^1/\partial \Delta^1$. Adapting Convention $\langle 5.2 \rangle$ slightly, if necessary, each of the categories $C_\alpha$ is closed under tensoring with $S^1$ as a sub-$S$-category of the ambient model category $B_k$. In particular, every object $K$ has a functorial suspension $K \wedge S^1 = (K_1 \wedge S^1, \ldots, K_n \wedge S^1)$, allowing the following observation.

Definition 5.14. For every object $K$ in $C_1 \wedge \cdots \wedge C_n$ there is a natural map:

$$
\mathcal{S}_{C_1 \wedge \cdots \wedge C_n}(\bigwedge_{i=1}^n K_i \wedge S^1, K \wedge S^1) = \bigwedge_{i=1}^n \mathcal{S}_{C_i}(K_i \wedge S^1, K_i \wedge S^1)
$$

$$
\cong \bigwedge_{i=1}^n \mathcal{S}(S^1, \mathcal{S}_{C_i}(K_i, K_i \wedge S^1))
$$

$$
\rightarrow \bigwedge_{i=1}^n \mathcal{S}(S^1, \mathcal{S}_{C_i}(K_i, K_i \wedge S^1))
$$

$$
= \mathcal{S}(S^n, \mathcal{S}_{C_1 \wedge \cdots \wedge C_n}(K \wedge S^1, K \wedge S^1)).
$$

The isomorphism uses that the ambient categories $B_\alpha$ are tensored over $S$. The map labelled $f$ is an $n$-fold version of the canonical map

$$
\mathcal{S}(S^1, L) \wedge \mathcal{S}(S^1, M) \rightarrow \mathcal{S}(S^1 \wedge S^1, L \wedge M).
$$

If $X$ is a functor in $Fun(C_1 \wedge \cdots \wedge C_n, D)$, the composition above induces a map

$$
S^n \rightarrow \mathcal{S}_{C_1 \wedge \cdots \wedge C_n}(\bigwedge_{i=1}^n K_i \wedge S^1) \rightarrow \mathcal{S}_D(X(K), X(K \wedge S^1)).
$$

that has an adjoint map

$$
X(K) \rightarrow (X(K \wedge S^1))^{S^n}
$$

which is natural in $K$. The result is a composite natural transformation

$$
(t_X : X \rightarrow T_{(1,\ldots,1)}(X) := \Omega^n(X(- \wedge S^1))
$$

of $S$-functors. Let $p_{1,\ldots,1}(X): X \rightarrow P_{1,\ldots,1}(X)$ denote the canonical map to the colimit

$$
X \rightarrow X^{\text{hf}} \rightarrow T_{(1,\ldots,1)}(X^{\text{hf}}) \rightarrow \cdots \rightarrow \text{colim}_n T_{(1,\ldots,1)}^{n}(X^{\text{hf}}).
$$

One can check that $p^*(p_{1,\ldots,1}(X))$ is canonically weakly equivalent to the natural transformation obtained in $\langle 5.2 \rangle$. Here, $p$ is the functor defined in $\langle 5.2 \rangle$.

Remark 5.15. In order to repeat this for symmetric functors, observe that the $n$-fold smash product $S^n$ of the unit circle carries a natural $\Sigma_n$-action given by permutation.
Definition 5.16. For every object $K$ in $\Sigma_n \& C^{\land n}$ there is a natural map

$$S_{\Sigma_n \& C^{\land n}}(K \land S^1, K \land S^1) \cong \bigvee_{\sigma=1}^{n} S(S^1, S_C(K_i, K_{\sigma^{-1}(i)} \land S^1))$$

$$\xrightarrow{f} \bigvee_{\sigma=1}^{n} S(S^n, S_C(K_i, K_{\sigma^{-1}(i)} \land S^1))$$

$$\xrightarrow{g} S(S^n, S_{\Sigma_n \& C^{\land n}}(K, K \land S^1)),$$

where the first two maps are as in Definition 5.14 and the map $g$ is the natural map relating the compositions of coproduct and the functor $S(S^n, -)$. The induced map

$$S^n \to S_{\Sigma_n \& C^{\land n}}(K, K \land S^1) \to S_P(X(K), X(K \land S^1))$$

for any functor $X$ in $\text{Fun}(\Sigma_n \& C^{\land n}, \mathcal{D})$. This has as adjoint the map

$$X(K) \to (X(K \land S^1))^S^n,$$

where the target has the correct $\Sigma_n$-action from Definition 5.7 and Remark 5.15.

The resulting map of symmetric $S$-functors is denoted

$$t_X : X \to T_{(1,\ldots,1)}(X) = (X(\_ \land S^1))^S^n.$$  

Let $p_X : X \to P_{(1,\ldots,1)}(X)$ denote the canonical map to the colimit of the sequence

$$X \to X^{hf} \xrightarrow{t_X^{hf}} T_{(1,\ldots,1)}(X^{hf}) \xrightarrow{t_{(1,\ldots,1)}(X^{hf})} \cdots \to \text{colim} T^n(X^{hf}) =: P_{(1,\ldots,1)}X.$$  

Definition 5.17. A functor $X : C_1 \land \cdots \land C_n \to \mathcal{D}$ is called multilinear if it is multi-excisive in the sense of Definition 5.11; that is, if all partial functors are excisive. A functor $\hat{X} : \Sigma_n \& C^{\land n} \to \mathcal{D}$ is called multilinear if $\varepsilon^* \hat{X} : C^{\land n} \to \mathcal{D}$ is multilinear.

In order to treat the categories $\text{Fun}(C_1 \land \cdots \land C_n, \mathcal{D})$ and $\text{Fun}(\Sigma_n \& C^{\land n}, \mathcal{D})$ simultaneously, the notation in what follows has to be interpreted appropriately.

Remark 5.18. Let $X$ be a functor in $\text{Fun}(C_1 \land \cdots \land C_n, \mathcal{D})$ or $\text{Fun}(\Sigma_n \& C^{\land n}, \mathcal{D})$. In particular, $X$ is multi-reduced. Hence, being multi-excisive is equivalent to being multilinear. In any case, the functor $P_{(1,\ldots,1)}X$ is componentwise excisive by construction and the map $X \to P_{(1,\ldots,1)}X$ is initial among maps to multilinear functors in the objectwise homotopy category.

Definition 5.19. A map $f : X \to Y$ in $\text{Fun}(C_1 \land \cdots \land C_n, \mathcal{D})$ or $\text{Fun}(\Sigma_n \& C^{\land n}, \mathcal{D})$ is

1. a multilinear equivalence if the map $P_{(1,\ldots,1)}(f)$ is an hf equivalence, and
2. a multilinear fibration if it is an hf fibration such that the square

$$\begin{array}{ccc}
X & \xrightarrow{f} & P_{(1,\ldots,1)}X \\
\downarrow & & \downarrow P_{(1,\ldots,1)}(f) \\
Y & \xrightarrow{P_{(1,\ldots,1)}(f)} & P_{(1,\ldots,1)}Y
\end{array}$$

is a homotopy pullback square in the hf model structure.

The multilinear cofibrations are the projective ones.
Theorem 5.20. Assume Convention 4.7. The classes described in Definition 5.19 constitute right proper cofibrantly generated $S$-model structures on $\text{Fun}(\Sigma_n \wr C_n, D)$ and $\text{Fun}(C_1 \times \cdots \times C_n, D)$, respectively. They are left proper if $D$ is left proper. The adjoint pairs

$$p_* : \text{Fun}(C_1 \times \cdots \times C_n, D)_{\text{mexc}} \rightleftarrows \text{Fun}(\Sigma_n \wr C_n, D)_{\text{ml}} : p^*$$

and

$$\varepsilon_* : \text{Fun}(C^\wedge, D)_{\text{mexc}} \rightleftarrows \text{Fun}(\Sigma_n \wr C^\wedge, D)_{\text{ml}} : \varepsilon^*$$

are Quillen pairs. The functors $p^*$ and $\varepsilon^*$ preserve and detect multilinear equivalences and multilinear fibrations.

Proof. The assertions about $p^*$ and $\varepsilon^*$ are straightforward observations. Further, it is tedious but true that $\varepsilon^*$ maps the symmetric version (5.6) of the natural transformation $p(1, \ldots, 1)$ to the non-symmetric version (5.4), and that $p^*$ maps the $S$-enriched version (5.4) to the unpointed version (5.2). The proof then proceeds as the proof of Theorem 5.8. In order to describe additional generating acyclic cofibrations, let $K$ be an object in $C$. By Lemma 5.21 below, the map $f$ is a multilinear fibration if and only if the diagram

\[
\begin{array}{ccc}
X(K) & \longrightarrow & T_{(1, \ldots, 1)}(X)(K) \\
\downarrow & & \downarrow \\
Y(K) & \longrightarrow & T_{(1, \ldots, 1)}(Y)(K)
\end{array}
\]

is a homotopy pullback diagram in $D$ for all $K$. The horizontal maps in this square are from (5.3). Evaluated at $K$, the upper horizontal map may be written as follows:

$$\text{hom}(R^K, X) \cong X(K) \rightarrow (X(K \wedge S^1))^S \cong \text{hom}(R^K \wedge S^1 \wedge S^n, X)$$

By the $S$-Yoneda lemma 2.6, it is thus induced by a map

$$\tau_K : R^K \wedge S^1 \wedge S^n \rightarrow R^K$$

of simplicial functors. Hence, the square above is isomorphic to the following square:

$$\begin{array}{ccc}
\text{hom}(R^K, X) & \longrightarrow & \text{hom}((R^K \wedge S^1 \wedge S^n), X) \\
\downarrow & & \downarrow \\
\text{hom}(R^K, Y) & \longrightarrow & \text{hom}((R^K \wedge S^1 \wedge S^n), Y)
\end{array}$$

Since $\tau_K$ is a map between projectively cofibrant objects, the simplicial mapping cylinder yields a factorization as

$$R^K \wedge S^1 \wedge S^n \xrightarrow{j(K)} \text{Cyl}(\tau_K) \xrightarrow{q(K)} R^K$$
where \(j(K)\) is a projective cofibration, \(q(K)\) is a simplicial homotopy equivalence, and all objects in this factorization are finitely presentable and projectively cofibrant. The square above factors accordingly as follows:

\[
\begin{array}{cccc}
\text{hom}(\mathcal{R}\mathcal{K}, X) & \rightarrow & \text{hom}((\mathcal{R}\mathcal{K} \wedge S^1 \wedge S^n), X) \\
\downarrow & & \downarrow \\
\text{hom}(\mathcal{R}\mathcal{K}, Y) & \rightarrow & \text{hom}((\mathcal{R}\mathcal{K} \wedge S^1 \wedge S^n), Y)
\end{array}
\]

Since the map \(q(K)\) is a simplicial homotopy equivalence, then so are the horizontal maps on the left hand square. Since the map \(j(K)\) is a projective cofibration and \(f\) is at least an objectwise fibration, the map (5.8)

\[
\text{hom}(\text{Cyl}(\tau K), X) \rightarrow \text{hom}(\text{Cyl}(\tau K), Y) \times \text{hom}((\mathcal{R}\mathcal{K} \wedge S^1 \wedge S^n), X)
\]

is a fibration in \(\mathcal{D}\). Hence, this fibration is acyclic if and only if the square above is a homotopy pullback square. Because \(\mathcal{D}\) is cofibrantly generated, the map in question is an acyclic fibration if and only if it has the right lifting property with respect to \(I_\mathcal{D}\). By adjointness, the map (5.8) is thus a weak equivalence if and only if \(f\) has the right lifting property with respect to the set of maps \(\{i \Box j(K)\}_{i \in I_\mathcal{D}}\).

The model structures provided by Theorem 5.20 are referred to as the \textit{multilinear model structures}, and denoted \text{Fun}(C_1 \wedge \cdots \wedge C_n, \mathcal{D})_{ml} \text{ and Fun}(\Sigma_n \wr C \wedge S^n, \mathcal{D})_{ml}, respectively. The proof of Theorem 5.20 shows that they can be seen as left Bousfield localizations.

\textbf{Lemma 5.21.} A map \(f: X \rightarrow Y\) is a multilinear fibration if and only if it is an \(hf\) fibration and the following diagram

\[
\begin{array}{ccc}
X & \rightarrow & T_{(1,\ldots,1)}X \\
\downarrow f & & \downarrow T_{(1,\ldots,1)}(f) \\
Y & \rightarrow & T_{(1,\ldots,1)}Y
\end{array}
\]

is an objectwise homotopy pullback square.

\textit{Proof.} This is straightforward using the fact that in \(\mathcal{D}\) filtered colimits preserve homotopy pullbacks. \(\square\)

\textbf{5.4. Coefficient spectra.} The aim of this section is to connect the multilinear model category of symmetric functors with the model category of spectra with an action of a symmetric group.

\textbf{Definition 5.22.} Let \(\text{Sp}(\mathcal{D})\) denote the category of \textit{Bousfield-Kan spectra} in the \(\mathcal{S}\)-model category \(\mathcal{D}\) as defined by Schwede [23]. An object in \(\text{Sp}(\mathcal{D})\) is a sequence \((E_0, E_1, \ldots)\) of objects in \(\mathcal{D}\), together with structure maps \(\sigma_n^E: \Sigma E_n \rightarrow E_{n+1}\). A morphism of such objects is a sequence of morphisms in \(\mathcal{D}\) commuting strictly with the structure maps. A map \(f: E \rightarrow F\) in \(\text{Sp}(\mathcal{D})\) is called

\(1\) a stable equivalence of spectra in \(\mathcal{D}\) if \(QE \rightarrow QF\) is a levelwise equivalence where \(Q\) is a certain model for “Ω-spectrum” given in [24, p. 90].
(2) a projective cofibration if the map $E_0 \to F_0$ and for all $n \geq 1$ the maps

$$E_n \lor_{E_{n-1}} F_{n-1} \to F_n$$

are cofibrations in $\mathcal{D}$.

**Theorem 5.23** (Schwede). Suppose that $\mathcal{D}$ satisfies Convention 4.7. Then there is an $S$-model structure on $\text{Sp}(\mathcal{D})$ with stable equivalences as weak equivalences and projective cofibrations as cofibrations. It satisfies its elf Convention 4.7. The $S$-functor $\text{Ev}_0: \text{Sp}(\mathcal{D}) \to \mathcal{D}$, $E \mapsto E_0$ is a right Quillen functor.

The proof can be found in [24], but some aspects are relevant later: If $\{i\}_{i \in I_D}$ is a set of generating cofibrations for $\mathcal{D}$, then $\{\text{Fr}_k(i)\}_{k \in \mathbb{N}, i \in I_D}$ is a set of generating cofibrations for $\text{Sp}(\mathcal{D})$. Here the functor $\text{Fr}_k: \mathcal{D} \to \text{Sp}(\mathcal{D})$ is left adjoint to evaluating at the $k$-th level, explicitly

$$(\text{Fr}_k(D))_\ell = \begin{cases} * & \text{for } 0 \leq \ell < k \\ \Sigma^{\ell-k} D & \text{for } \ell \geq k \end{cases}$$

The functor $\text{Ev}_0$ commutes with all limits and colimits.

**Theorem 5.24.** Composing with $\text{Fr}_0$ and $\text{Ev}_0$ induces a Quillen equivalence:

$$F: \text{Fun}(\Sigma_n \wr C^n, \mathcal{D})_{ml} \rightleftarrows \text{Fun}(\Sigma_n \wr C^n, \text{Sp}(\mathcal{D}))_{ml}: G$$

**Proof.** Since the functor $\text{Ev}_0: \text{Sp}(\mathcal{D}) \to \mathcal{D}$ preserves objectwise fibrations and objectwise acyclic fibrations, the same is true for $G$. Hence, $G$ is a right Quillen functor for the projective model structures. The functor $\text{Ev}_0$ commutes with all limits, colimits, and is a right Quillen functor. Hence $\text{Ev}_0: \mathcal{S} \to \mathcal{S}$ commutes up to natural weak equivalence with $X \to X^{h\text{f}}$ and $T_{(1,...,1)}$. In particular, the induced functor $G$ is a right Quillen functor on homotopy functor and multilinear model structures. A right Quillen functor is a Quillen equivalence if and only if its total right derived functor is an equivalence. The proof of [12, Prop. 3.7], which states that $\text{Ev}_0$ induces an equivalence on the (naive) homotopy categories of multilinear functors, extends to the setup here, which concludes the proof. □

**Corollary 5.25.** The multilinear model structure on symmetric functors is stable.

A suitable evaluation functor connects symmetric functors directly with spectra having a symmetric group action. In order to describe it, recall from Notation 2.2 that the category $\text{Sp}(\mathcal{D})^{\Sigma_n}$ is the category of functors $\Sigma_n \to \text{Sp}(\mathcal{D})$, where $\Sigma_n$ is viewed as a category with one object. In other words, an object in $\text{Sp}(\mathcal{D})^{\Sigma_n}$ is a spectrum with a right $\Sigma_n$-action. Since the stable model structure on $\text{Sp}(\mathcal{D})$ is cofibrantly generated, the category $\text{Sp}(\mathcal{D})^{\Sigma_n}$ carries a cofibrantly generated model structure with fibrations and weak equivalences defined on underlying spectra. This is sometimes called the model structure for “naive $\Sigma_n$-spectra”.

**Definition 5.26.** Precomposition with the symmetric diagonal $S$-functor

$$\Delta_n: \Sigma_n \times C \to \Sigma_n \wr C^{\land n}$$

as introduced in the proof of Lemma 3.15 defines a functor

$$\Delta_n^*: \text{Fun}(\Sigma_n \wr C^{\land n}, \text{Sp}(\mathcal{D})) \to \text{Fun}(\Sigma_n \times C, \text{Sp}(\mathcal{D})) \cong \text{Fun}(C, \text{Sp}(\mathcal{D})^{\Sigma_n})$$

Evaluating at an object $C$ in $\mathcal{C}$ induces the functor

$$\text{ev}_C: \text{Fun}(C, \text{Sp}(\mathcal{D})^{\Sigma_n}) \to \text{Sp}(\mathcal{D})^{\Sigma_n}.$$
The composition is denoted
\[ \text{Ev}_C = \text{ev}_C \circ \Delta^* : \text{Fun}(\Sigma_n \wr \mathcal{C}, \text{Sp}(\mathcal{D})) \to \text{Sp}(\mathcal{D})^{\Sigma_n}. \]

As a composition of two right adjoint functors, the functor \( \text{Ev}_C \) has a left \( S \)-adjoint denoted by
\[ \mathcal{L}	ext{Ev}_C : \text{Sp}(\mathcal{D})^{\Sigma_n} \to \text{Fun}(\Sigma_n \wr (\mathcal{C} \wedge n), \text{Sp}(\mathcal{D})). \]

**Theorem 5.27.** Suppose that \( \mathcal{C} \) is the category \( S_{\text{fin}} \) of finite pointed simplicial sets. The functor
\[ \mathcal{L}	ext{Ev}_{S^0} : \text{Sp}(\mathcal{D})^{\Sigma_n} \to \text{Fun}(\Sigma_n \wr (S_{\text{fin}} \wedge n), \text{Sp}(\mathcal{D}))_{\text{ml}} \]
is a left Quillen equivalence.

**Proof.** Choosing \( S^0 \in \mathcal{C} = S_{\text{fin}} \) yields the functor
\[ \text{Ev}_{S^0} : \text{Fun}(\Sigma_n \wr (S_{\text{fin}} \wedge n), \text{Sp}(\mathcal{D})) \to \text{Sp}(\Sigma_n). \]

Explicitly, it is given by \( X \mapsto X(S^0, \ldots, S^0) \) with \( \Sigma_n \)-action induced by permuting the \( n \)-tuple \( (S^0, \ldots, S^0) \). Its left adjoint \( \mathcal{L}	ext{Ev}_{S^0} \) sends a \( \Sigma_n \)-spectrum \( E \) to the symmetric functor
\[ K = (K_1, \ldots, K_n) \mapsto \mathcal{L}	ext{Ev}_{S^0}(K) = E \wedge K_1 \wedge \cdots \wedge K_n \]
having the following effect on morphism spaces:
\[ \bigvee_{\sigma \in \Sigma_n} \bigwedge_{i=1}^n S(K_i, L_{\sigma^{-1}(i)}) \longrightarrow S(E \wedge K_1 \wedge \cdots \wedge K_n, E \wedge L_1 \wedge \cdots \wedge L_n) \]

\[ (\sigma, f = (f_1, \ldots, f_n)) \longrightarrow \sigma E \wedge (\sigma \circ (f_1 \wedge \cdots \wedge f_n)) \]

Here \( \sigma \) denotes the permutation \( L_{\sigma^{-1}(1)} \wedge \cdots \wedge L_{\sigma^{-1}(n)} \to L_1 \wedge \cdots \wedge L_n \) induced by \( \sigma \).

The unit \( E \to \text{Ev}_{S^0}(\mathcal{L}	ext{Ev}_{S^0}(E)) \) is the canonical isomorphism identifying \( E \) with \( E \wedge S^0 \wedge \cdots \wedge S^0 \). The counit \( \mathcal{L}	ext{Ev}_{S^0}(\text{Ev}_{S^0}(X)) \to X \) is the natural transformation
\[ X(S^0, \ldots, S^0) \wedge K_1 \wedge \cdots \wedge K_n \to X(K_1, \ldots, K_n) \]
which is a special case of the assembly map
\[ X(L_1, \ldots, L_n) \wedge K_1 \wedge \cdots \wedge K_n \to X((K_1 \wedge L_1), \ldots, (K_n \wedge L_n)). \]

The latter is adjoint to the natural map
\[ \Sigma_n \wr (S_{\text{fin}} \wedge n)((L_1, \ldots, L_n), (K_1 \wedge L_1, \ldots, K_n \wedge L_n)) \]
\[ \longrightarrow \text{Sp}(X(L), X(K_1 \wedge L_1, \ldots, K_n \wedge L_n)). \]
Since $\Delta^*$ and $ev_{S^0}$ are right Quillen functors for projective model structures on functor categories, so is their composition. Hence, $\mathcal{L}ev_{S^0}$ is a left Quillen functor to the projective model structure, and to the multilinear model structure as well.

To show that the derived unit $E \to ev_{S^0}(P(1,\ldots,1)(\mathcal{L}ev_{S^0}(E)))$ is a weak equivalence for $E$ cofibrant, recall that the unit is an isomorphism. Further, the functor $\mathcal{L}ev_{S^0}(E)$ preserves weak equivalences and the canonical map

$$E \wedge S^k \wedge \cdots \wedge S^k \to \Omega^n \left((E \wedge S^{k+1} \wedge \cdots \wedge S^{k+1})_{\text{fib}}\right)$$

is a weak equivalence in the stable model structure, where $(\_)$ fib denotes fibrant replacement in $Sp_{/X}$. It follows that $E \to P(1,\ldots,1)(\mathcal{L}ev_{S^0}(E))(\underline{S^0})$ is a weak equivalence. It remains to prove that $Ev_{S^0}$ detects weak equivalences of multilinear functors. As in the proof of [12, Prop. 5.8], the symmetry is irrelevant, and the case $n = 1$ is sufficient. If $f: X \to Y$ is a map of linear functors with $f(S^0)$ a weak equivalence, then $f(S^k)$ is a weak equivalence for every $k$, as one deduces from the natural weak equivalence [5.3]. It then follows that $f(K)$ is a weak equivalence for every $K$ in $S$ fib by induction on the cells in $K$, using that $X$ and $Y$ are linear. □

For $X: \Sigma_n (\underline{S^{fin}})^{\wedge n} \to Sp(D)$, the $\Sigma_n$-spectrum $Ev_{S^0}(X) = X(S^0,\ldots,S^0)$ in $D$ is called the coefficient spectrum of $X$. It has the correct homotopy type if $X$ is multilinear. Given a functor $Y: S^{fin} \to S$, Goodwillie calls the coefficient spectrum of the multilinear functor $\text{hocr}_nP_nY \simeq \text{hocr}_nD_nY$ the $n$-th derivative of $Y$.

5.5. Goodwillie’s theorem on multilinearized homotopy cross effects.

**Proposition 5.28** (Prop. 3.3 [12]). Let $0 \leq m \leq n$. For any $n$-excisive functor $X$, the functor $\text{hocr}_{m+1}X$ is $(n-m)$-excisive in each variable. In particular, the $n$-th homotopy cross effect is multilinear if $X$ is $n$-excisive, and it is contractible if $X$ is $(n-1)$-excisive.

**Proof.** The proof by Goodwillie is again a variation on opaqueness and applies to the setup here. □

**Definition 5.29.** A functor $X$ is $n$-reduced if $P_{n-1}X \simeq \ast$, and $n$-homogeneous if it is $n$-excisive and $n$-reduced.

**Definition 5.30.** In order to distinguish a functor $X: C_1 \times \cdots \times C_n \to D$ in $n$ variables $K_1,\ldots,K_n$ notationally from the same functor $\tilde{X}$ when viewed as a functor in one variable $\tilde{K} = (K_1,\ldots,K_n)$, the latter is denoted $\lambda X$.

The $n$-excisive approximation functor $P_n$ applies to the functor $\lambda X$. There is a commutative diagram:

$$\begin{array}{ccc}
\lambda X & \xrightarrow{\beta} & P_n\lambda X \\
\downarrow & & \downarrow \\
\lambda P_1,\ldots,1X & \xrightarrow{\alpha} & P_n\lambda(P_1,\ldots,1X)
\end{array}$$

**Lemma 5.31.** If a functor $X: C_1 \times \cdots \times C_n \to D$ is $(d_1,\ldots,d_n)$-excisive (as defined in [5.1]), then $\lambda X$ is $(d_1 + \cdots + d_n)$-excisive.

**Proof.** This is Goodwillie’s Lemma 6.6 in [12] whose proof refers to [11, 6.6]. The proof applies here. □

**Lemma 5.32.** If a functor $X: C_1 \times \cdots \times C_n \to D$ is multireduced, $\lambda X$ is $n$-reduced.
Proof. The proofs of Lemma 3.2 and Lemma 6.7 in [12] apply. □

Lemma 5.33. If \(X: C_1 \times \cdots \times C_n \to D\) is multireduced and \(\lambda X\) is \(n\)-excisive, then \(X\) is multilinear.

Proof. The proof of [12, Lemma 6.9] applies here as well. □

Corollary 5.34. The maps \(\alpha\) and \(\beta\) in diagram (5.9) are objectwise weak equivalences for every functor in \(\text{Fun}(C_1 \land \cdots \land C_n, D)\).

Proof. In order to apply the Lemmata above, the functor in question is pulled back via \(p: C_1 \times \cdots \times C_n \to C_1 \land \cdots \land C_n\). Then the map \(\alpha\) is an objectwise weak equivalence by Lemma 5.31. The analogous map \(\gamma: P_n \lambda X \to P_1,\ldots,1(P_n \lambda X)\) is one by Lemma 5.33. The maps \(\beta\) and \(\gamma\) are related via a natural weak equivalence commuting the constructions \(P_n\) and \(P_1,\ldots,1\), since homotopy limits, as well as joins, commute among themselves. Hence, \(\beta\) is an objectwise weak equivalence as well. The proof finishes by noting that the functor \(p^*\) detects objectwise weak equivalences, see Lemma 2.23. □

Theorem 5.35 (Theorem 6.1 [12]). For all \(S\)-functors \(X: C \to D\) there is a natural objectwise weak equivalence under \(\text{hocr}_n X\):
\[
\text{hocr}_n P_n X \simeq P_{(1,\ldots,1)} \text{hocr}_n X.
\]

Proof. Let \(X\) be a functor in \(\text{Fun}(C, D)\). Substituting \(\text{hocr}_n X\) in diagram (5.9) supplies a natural zig-zag of objectwise weak equivalence
\[
P_{(1,\ldots,1)} \text{hocr}_n X \simeq P_n \lambda (\text{hocr}_n X)
\]
under \(\text{hocr}_n X\) by Corollary 5.34. In order to prove that the functors \(P_n \lambda (\text{hocr}_n X)\) and \(\text{hocr}_n P_n X\) are naturally weakly equivalent under \(\text{hocr}_n X\), denote by \(J_U(X)\) the functor
\[
K \mapsto J_U(X)(K) = X(K \ast U).
\]
for each finite set \(U\). The join with \(U\) commutes with coproducts in \(C\). Thus, for each \(K\) in \(C^{\land n}\) there is a weak equivalence
\[
J_U \lambda (\text{hocr}_n X)(K) \simeq \text{hofib} \left[ X \left( \bigvee_{i=1}^n K_i \ast U \right) \to \lim_{S \in P_0 \Delta} X \left( \bigvee_{i \not\in S} K_i \ast U \right) \right]
\]
\[
\simeq \text{hofib} \left[ X \left( \bigvee_{i=1}^n (K_i \ast U) \right) \to \lim_{S \in P_0 \Delta} X \left( \bigvee_{i \not\in S} (K_i \ast U) \right) \right]
\]
\[
\simeq \text{hocr}_n J_U X(K)
\]
and therefore an objectwise weak equivalence \(T_n \lambda (\text{hocr}_n X) \simeq \text{hocr}_n T_n X\). It induces the desired objectwise weak equivalence \(P_n \lambda (\text{hocr}_n X) \simeq \text{hocr}_n P_n X\). □

Corollary 5.36. The \(n\)-th cross effect
\[
\text{cr}_n: \text{Fun}(C, D)_{n-\text{exc-cr}} \to \text{Fun}(\Sigma_n \land C^{\land n}, D)_{\text{ml}}
\]
is a right Quillen functor.
Proof. The left adjoint preserves cofibrations by Lemma 4.25. It suffices to show that \( \text{cr}_n \) preserves fibrations. Let \( f : X \rightarrow Y \) be an \( n \)-excisive cr fibration, hence an hf cr fibration such that the diagram

\[
\begin{array}{ccc}
X & \longrightarrow & P_n X \\
\downarrow f & & \downarrow P_n f \\
Y & \longrightarrow & P_n Y
\end{array}
\]

is a homotopy pullback square. Lemma 4.25 implies that the map \( \text{cr}_n f \) is an hf fibration. It remains to check that the diagram

\[
\begin{array}{ccc}
\text{cr}_n X & \longrightarrow & P_{1, \ldots, 1} \text{cr}_n X \\
\downarrow \text{cr}_n f & & \downarrow P_{1, \ldots, 1} \text{cr}_n f \\
\text{cr}_n Y & \longrightarrow & P_{1, \ldots, 1} \text{cr}_n Y
\end{array}
\]

is a homotopy pullback. This square is the front of a commutative cube, whose sides are induced by the natural map \( \text{cr}_n X \rightarrow \text{hocr}_n X \). The back of the cube is the following diagram:

\[
\begin{array}{ccc}
\text{hocr}_n X & \longrightarrow & \text{hocr}_n P_n X \\
\downarrow & & \downarrow \\
\text{hocr}_n Y & \longrightarrow & \text{hocr}_n P_n Y
\end{array} \cong \begin{array}{ccc}
P_{1, \ldots, 1} \text{hocr}_n X \\
P_{1, \ldots, 1} \text{hocr}_n Y
\end{array}
\]

The horizontal maps on the right are objectwise weak equivalences by Goodwillie’s Theorem 5.35. The square on the left hand side is the image of a homotopy pullback square under \( \text{hocr}_n \). Thus, \( \text{cr}_n f \) is a multilinear fibration, once the sides of the commutative cube are proven to be homotopy pullback squares as well. In fact, it suffices to check that the square

\[
(5.10)
\begin{array}{ccc}
\text{cr}_n X & \longrightarrow & \text{hocr}_n X \\
\downarrow & & \downarrow \\
\text{cr}_n Y & \longrightarrow & \text{hocr}_n Y
\end{array}
\]

is a homotopy pullback square, because the opposite side of the cube is obtained by applying \( P_{1, \ldots, 1} \) and inherits the homotopy pullback property. Let \( F \) be the fiber of \( f \). It is cr fibrant, thus by Lemma 3.24 the canonical map

\[
\text{cr}_n F \rightarrow \text{hocr}_n F
\]

of vertical (homotopy) fibers in diagram (5.10) is an objectwise weak equivalence. As the multilinear model structure is stable by Corollary 5.25 diagram (5.10) is a homotopy pullback square, which completes the proof. \( \square \)

6. Homogeneous Functors

As recalled in Definition 5.29 a functor \( X : \mathcal{C} \rightarrow \mathcal{D} \) is \( n \)-homogeneous if it is \( n \)-excisive and \( P_{n-1} X \) is contractible. In this section, \( \text{Fun}_S(\mathcal{C}, \mathcal{D}) \) will be equipped via right Bousfield localization with a model structure in which every functor is weakly equivalent to an \( n \)-homogeneous functor. It is Quillen equivalent to the
multilinear model structure, and hence by Theorem 5.27 to the model category of \( \Sigma_n \)-spectra in \( \mathcal{D} \), provided \( \mathcal{C} \) is the category of finite pointed simplicial sets.

**Convention 6.1.** Suppose in addition to Convention 5.2 that \( \mathcal{D} \) admits a set of generating cofibrations with cofibrant domains.

### 6.1. The homogeneous model structure.

**Definition 6.2.** Consider the following set of objects of \( \text{Fun}(\mathcal{C}, \mathcal{S}) \):

\[
\Lambda_n = \left\{ \bigwedge_{i=1}^n R^{K_i} \mid K_1, \ldots, K_n \in \text{Ob}(\mathcal{C}) \right\}.
\]

More generally, let \( I_D \) be a set of generating cofibrations with cofibrant domains in \( \mathcal{D} \), which exists by Convention 6.1. Let \( \text{cd}(I_D) \) denote the set of domains and codomains of all morphisms \( i \in I \), and set

\[
\Lambda_{n,I_D} = \left\{ \left( \bigwedge_{i=1}^n R^{K_i} \right) \wedge D \mid K_1, \ldots, K_n \in \text{Ob}(\mathcal{C}), D \in \text{cd}(I_D) \right\}
\]

A map \( f \) in \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) is

1. an \textit{n-homogeneous equivalence} if it is a \( \Lambda_{n,I_D} \)-colocal equivalence.
2. an \textit{n-homogeneous cofibration} if it has the left lifting property with respect to all \( n \)-excisive \( \text{cr} \) fibrations that are also \( n \)-homogeneous equivalences.

The \( n \)-homogeneous fibrations are the \( n \)-excisive \( \text{cr} \) fibrations.

Choosing a different set of generating cofibrations with cofibrant domains in \( \mathcal{D} \) yields the same classes, due to the following well-known lemma, whence the choice of generating cofibrations of \( \mathcal{D} \) will be omitted from the notation.

**Lemma 6.3.** Let \( \mathcal{D} \) be an \( \mathcal{S} \)-model category and \( f \) a morphism of fibrant objects. Suppose \( \mathcal{D} \) admits a set \( I_D \) of generating cofibrations with cofibrant domains. The morphism \( f \) is a weak equivalence if and only if for every domain and every codomain \( D \) appearing in \( \text{cd}(I_D) \), the map \( \mathcal{D}(D, f) \) is a weak equivalence of simplicial sets.

**Theorem 6.4.** Assume Convention 6.1. The classes described in Definition 6.2 form a right proper \( \mathcal{S} \)-model structure on \( \text{Fun}(\mathcal{C}, \mathcal{D}) \).

**Proof.** This follows from [6, Thm. 2.6], which applies to any cofibrantly generated right proper model category. In our case this is the \( n \)-excisive model structure on \( \text{Fun}(\mathcal{C}, \mathcal{D}) \). \( \square \)

The right Bousfield localization of the \( n \)-excisive cross effect model structure on the category \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) with respect to the set \( \Lambda_{n,I_D} \) is the \textit{n-homogeneous model structure} and is denoted by \( \text{Fun}(\mathcal{C}, \mathcal{D})_{n-\text{hom}} \). Theorem 5.23 implies that the \( n \)-homogeneous model structure on \( \text{Fun}(\mathcal{C}, \text{Sp}(\mathcal{D})) \) exists.

**Lemma 6.5.** A map \( f \) in \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) is an \textit{n-homogeneous equivalence} if and only if the induced map \( \text{hocr}_n(f) \) is a multilinear equivalence.

**Proof.** An \( n \)-homogeneous equivalence is a map that induces weak equivalences on the derived mapping spaces out of all objects in \( \Lambda_n \). Hence, suppose that domain and codomain of \( f \) are \text{cr} fibrant \( n \)-excisive homotopy functors. For \( X \) \( n \)-excisive \text{cr}
fibrant, there is a natural weak equivalence \( \text{cr}_n X \simeq \text{hocr}_n X \simeq \text{hocr}_n P_n X \) as well as a natural isomorphism

\[
S_{\text{Fun}(C, D)} \left( \bigotimes_{i=1}^{n} R^{K_i} \right) \cong S_{D} \left( D, \text{cr}_n X(K_1, \ldots, K_n) \right)
\]

by Lemma 3.14 where \( D \) is in \( \text{cd}(I_D) \). Lemma 6.3 implies that \( f \) is an \( n \)-homogeneous equivalence if and only if the induced morphism \( \text{hocr}_n(f) \simeq \text{hocr}_n(P_n f) \) is an objectwise weak equivalence. The equivalence \( \text{hocr}_n P_n(f) \simeq P_{(1, \ldots, 1)} \text{hocr}_n(f) \) from Theorem 5.35 finishes the argument.

**Definition 6.6.** The \( n \)-homogeneous part of a functor \( X \) is defined as

\[
D_n X := \text{hofib}[g_n : P_n X \to P_{n-1} X].
\]

By construction, \( D_n X \) is indeed \( n \)-homogeneous.

**Corollary 6.7.** For every functor \( X \) the map \( D_n X \to P_n X \) induces an objectwise equivalence \( \text{hocr}_n D_n X \to \text{hocr}_n P_n X \).

*Proof.* Proposition 5.28 implies that \( \text{hocr}_n P_{n-1} X \simeq * \). The chain

\[
\text{hocr}_n D_n X \simeq \text{hocr}_n \text{hofib}[P_n X \to P_{n-1} X] \simeq \text{hofib}[\text{hocr}_n P_n X \to \text{hocr}_n P_{n-1} X] \simeq \text{hocr}_n P_n X
\]

of natural objectwise equivalences completes the proof. \( \square \)

**Corollary 6.8.** Every functor \( X \) is \( n \)-homogeneously equivalent to \( D_n X \).

*Proof.* The map \( X \to P_n X \) is an \( n \)-excisive equivalence, hence an \( n \)-homogeneous equivalence. By Lemma 6.5 and Corollary 6.7 the map \( D_n X \to P_n X \) is an \( n \)-homogeneous equivalence.

The next statement also holds for functors to an unstable category \( D \), but it will be shown later in 6.19 after some auxiliary statements.

**Corollary 6.9.** A map \( f \) in \( \text{Fun}(C, \text{Sp}(D)) \) is an \( n \)-homogeneous equivalence if and only if the induced map \( D_n(f) \) is an objectwise equivalence.

*Proof.* By Corollary 6.8 it remains to show that \( D_n(f) \) is an objectwise equivalence if and only if \( \text{hocr}_n D_n(f) \) is an objectwise equivalence. This is the content of [12 Prop. 3.4] where Goodwillie actually shows that a functor to \( \text{Sp}(D) \) is \((n-1)\)-excisive if it is \( n \)-excisive with contractible \( n \)-th homotopy cross effect. In the proof, the following two properties are used:

1. Every strongly homotopy cocartesian cube of cofibrant objects admits a weak equivalence from a pushout cube [11] Prop. 2.2.\footnote{For a proof, see [11] Prop. 2.2.}
2. If a map \( X \to Y \) of \( n \)-cubes is homotopy Cartesian as an \((n+1)\)-cube and \( Y \) is homotopy Cartesian, then \( X \) is homotopy Cartesian. This holds in every model category.

This implies the statement for stable model categories, because in such a map is a weak equivalence if and only if its homotopy fiber is contractible. \( \square \)

**Theorem 6.10.** Assume Convention 6.1. The functors

\[
L_n : \text{Fun}(\Sigma_n \times C^n, \text{Sp}(D))_{\text{ml}} \cong \text{Fun}(C, \text{Sp}(D))_{\text{n-hom}} : \text{cr}_n
\]

form a Quillen equivalence.
Proof. The $n$-th cross effect is a right Quillen functor from the $n$-excisive cr model structure by Corollary 5.36. In particular, $cr_n$ preserves fibrations. Lemma 3.24 and Lemma 6.5 show that $cr_n$ preserves and detects $n$-homogeneous equivalences on cr fibrant objects. Hence if $p$ is an acyclic fibration in the $n$-homogeneous cr model structure with fiber $F$, the map $cr_n(p)$ is a fibration with contractible fiber $cr_n(F)$. As the multilinear model structure is stable by Corollary 5.25, $cr_n(p)$ is an acyclic fibration. Thus $cr_n$ is a right Quillen functor on the $n$-homogeneous model structure. The argument from [12, pp. 678] extends to show that the derived unit map $X \xrightarrow{\sim} hocr_nL_nX$ is an equivalence. The already mentioned fact that $cr_n$ detects $n$-homogeneous equivalences on cr fibrant objects implies it is a Quillen equivalence. □

6.2. Goodwillie’s delooping theorem.

Theorem 6.11. Suppose that $C$ and $D$ satisfy Convention 6.1. The pair of adjoint functors obtained by composing with the functors

$$F_0: D \to \text{Sp}(D) \quad \text{and} \quad \text{Ev}_0: \text{Sp}(D) \to D$$

is a Quillen equivalence:

$$F: \text{Fun}(C,D)_{n-\text{hom}} \rightleftharpoons \text{Fun}(C,\text{Sp}(D))_{n-\text{hom}}: G$$

Proof. The proof is divided into several steps. The pair is a Quillen adjunction by Lemma 6.12. The total right derived functor of $G$ is faithful by Lemma 6.13 and essentially surjective and full by Lemma 6.16. □

Lemma 6.12. The functors $(F,G)$ form a Quillen pair for the $n$-homogeneous model structures on both sides.

Proof. The functor $F$ maps the generating sets $I^\text{cr}$ and $J^\text{cr}$ on the left hand side into the corresponding ones on the right hand side. This implies that $F$ is a left Quillen functor on cr model structures. The functor $\text{Ev}_0$ commutes with all colimits, limits and homotopy limits. In particular, it commutes with the functors $(-)^{hf}$ and $P_n$ up to natural weak equivalence. Thus, the characterization of fibrations in the hf model structure 4.15 and in the $n$-excisive model structure 5.7 yields that $G$ is a right Quillen functor on hf model structures and the $n$-excisive model structure. The fibrations agree in all these model structures.

The fibrations in the $n$-homogeneous and the $n$-excisive model structure agree. Suppose that $f$ is an $n$-homogeneous acyclic fibration. Since $\text{Ev}_0$ commutes with $P_n$ and with homotopy fibers, we have $hocr_nP_nG(f) \simeq G(hocr_nP_nf)$. It follows from Lemma 6.15 that $G(f)$ is an $n$-homogeneous acyclic fibration, whence $G$ is a right Quillen functor also for the $n$-homogeneous model structure. Alternatively, the left adjoint $F$ preserves the set of objects that define the right Bousfield localization. □

Lemma 6.13. The functor $G$ preserves and detects $n$-homogeneous equivalences of bifibrant functors, and its total right derived functor is faithful on morphisms.

Proof. General localization theory of model categories implies that a map between $n$-homogeneously bifibrant functors is an $n$-homogeneous equivalence if and only if it is an objectwise equivalence. The functor $\text{Ev}_0$ preserves and detects weak equivalences of stably fibrant spectra. Thus, the functor $G$ preserves and detects $n$-homogeneous equivalences of bifibrant functors.
Moreover, the structure maps \( X \to \Omega X \) of an \( n \)-homogeneously bifibrant functor \( X : C \to \text{Sp}(D) \) are objectwise weak equivalences. Hence, if \( f \) and \( g \) are maps of bifibrant objects in the \( n \)-homogeneous model structure on \( \text{Sp}(D) \)-valued functors such that \( f_0 \) and \( g_0 \) are homotopic, then \( f \) and \( g \) are homotopic. In particular, the total right derived functor of \( G \) is faithful on morphisms. □

For the next lemma recall that all \( S \)-functors are reduced.

**Lemma 6.14.** Let \( n > 0 \) and \( X \) a functor. Then there is a natural commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{K_n X} & \tilde{P}_n X \\
\downarrow & & \downarrow \\
R_n X & \xrightarrow{P_{n-1} X} & P_{n-1} X
\end{array}
\]

in which the left hand square is a homotopy pullback square, the functor \( K_n X \) is contractible and the functor \( R_n X \) is \( n \)-homogeneous.

**Proof.** The diagram above is diagram (2.3) in [12] and Goodwillie’s proof of its existence in section 2 of that article applies. □

**Lemma 6.15.** Let \( X : C \to D \) be cofibrant in the \( n \)-homogeneous model structure. Then there exists a spectrum-valued functor \( R_n X : C \to \text{Sp}(D) \) and a natural objectwise equivalence \( G(R_n X) \simeq X \).

**Proof.** If a functor \( X : C \to D \) is \( n \)-homogeneous, the upper left corner \( K_n X \) and the lower right corner \( P_{n-1} X \) of the homotopy pullback square from Lemma 6.14 are contractible. Let \( U_n X \) denote the homotopy pullback of that square. The natural objectwise weak equivalences

\[
\begin{align*}
X & \xrightarrow{\sim} \tilde{P}_n X \xrightarrow{\sim} U_n X \xrightarrow{\sim} \Omega R_n X,
\end{align*}
\]

do not form a direct map which can be iterated to obtain a spectrum-valued functor \( R_n X \). However, a trick by Goodwillie [10] given at the end of the introduction works. For \( j \geq 0 \), let \( (R_n X)_j \) denote the homotopy limit (not colimit) of the diagram

\[
\begin{align*}
R_n^j X & \xrightarrow{\sim} U(R_n^j X) \xrightarrow{\sim} \Omega R_n^{j+1} X \xrightarrow{\sim} \Omega U(R_n^{j+1} X) \xrightarrow{\sim} \Omega^2 R_n^{j+2} X \xrightarrow{\sim} \cdots,
\end{align*}
\]

starting with \( R_n^0 X = X \). Then \( (R_n X)_j \xrightarrow{\sim} R_n^j X \) for all \( j \geq 0 \), and there are structure maps \( (R_n X)_j \xrightarrow{\sim} \Omega(R_n X)_{j+1} \) defining a spectrum-valued functor \( R_n X \) such that \( G(R_n X) \simeq X \). □

**Lemma 6.16.** The total right derived functor of \( G \) is essentially surjective and full.

**Proof.** By Lemma 6.8, any functor in \( \text{Fun}(C, D) \) is \( n \)-homogeneously equivalent to an \( n \)-homogeneous functor \( X \). Lemma 6.15 supplies a functor \( R_n X \) with \( G(R_n X) \simeq X \). The statement follows. □
6.3. The Quillen equivalences. There is a commutative diagram of Quillen pairs:

\[
\begin{array}{ccc}
\Fun(\Sigma \wr \mathcal{C} \wedge, \mathcal{D})_{\text{ml}} & \xrightarrow{\mathcal{L}_n} & \Fun(\mathcal{C}, \mathcal{D})_{n-\text{hom}} \\
F & \downarrow G & \downarrow F \\
\Fun(\Sigma \wr (\mathcal{C} \wedge, \Sp(\mathcal{D}))_{\text{ml}} & \xrightarrow{\mathcal{L}_n} & \Fun(\mathcal{C}, \Sp(\mathcal{D}))_{n-\text{hom}}
\end{array}
\]

The left vertical Quillen pair was shown to be a Quillen equivalence in Theorem 5.24, the lower horizontal one in Theorem 6.10, and the right vertical pair in Theorem 6.11. The 2-out-of-3 property of Quillen equivalences yields the following statement.

**Corollary 6.17.** Suppose Convention 6.1 holds. Then the pair $\mathcal{L}_n: \Fun(\Sigma \wr \mathcal{C} \wedge, \mathcal{D})_{\text{ml}} \rightleftarrows \Fun(\mathcal{C}, \mathcal{D})_{n-\text{hom}} : \mathcal{C} \mathcal{r}_n$ is a Quillen equivalence.

If $\mathcal{C} = \mathcal{S}^{\text{fin}}$, evaluation at $\mathcal{S}_0$ prolongs this Quillen equivalence to the category of $\Sigma_n$-spectra in $\mathcal{D}$, by Theorem 5.27. The associated diagram of total derived functors on homotopy categories yields Goodwillie’s diagram of equivalences of homotopy categories (as displayed in the introduction) if $\mathcal{D} = \mathcal{S}$.

**Corollary 6.18.** The $n$-homogeneous model structure on $\Fun(\mathcal{C}, \mathcal{D})$ is stable, if Convention 6.1 holds.

6.4. More on the homogeneous model structures. The following assertion was proved for $\Sp(\mathcal{D})$ as target category already in Corollary 6.9.

**Lemma 6.19.** Suppose Convention 6.1 holds. A map $f$ in $\Fun(\mathcal{C}, \mathcal{D})$ is an $n$-homogeneous equivalence if and only if the induced map $D_n(f)$ is an objectwise weak equivalence.

**Proof.** By Lemma 6.5 and Theorem 6.35, the map $f$ is an $n$-homogeneous equivalence if and only if $\text{hocr}_n P_n(f) \simeq \text{hocr}_n D_n(f)$ is a multilinear equivalence, and this holds for any target category. This shows that if $D_n(f)$ is an objectwise equivalence, then $f$ is an $n$-homogeneous equivalence.

Conversely, note that any $n$-homogeneous equivalence $f$ in $\Fun(\mathcal{C}, \mathcal{D})$ extends to an $n$-homogeneous equivalence $f' \in \Fun(\mathcal{C}, \Sp(\mathcal{D}))$ by Theorem 6.14. Now $D_n(f) \simeq D_n(G(f')) \simeq G(D_n(f'))$ is an objectwise equivalence by the commutativity of the right adjoints in diagram 6.2 and Corollary 6.9 for the target category $\Sp(\mathcal{D})$. $\square$

**Corollary 6.20.** For any functor $X: \mathcal{C} \to \mathcal{D}$ and $n > 0$ there exists a functor $R_n X: \mathcal{C} \to \mathcal{D}$ and a natural weak equivalence $D_n X \simeq \Omega R_n X$.

**Proof.** This follows from Lemma 6.14. $\square$

**Corollary 6.21.** Let $n > 0$. A map $f: X \to Y$ in $\Fun(\mathcal{C}, \mathcal{D})$ induces an objectwise weak equivalence $D_n(f)$ if and only if the diagram

\[
\begin{array}{ccc}
P_n X & \xrightarrow{P_n} & P_n^{-1} X \\
\downarrow & & \downarrow \\
P_n Y & \xrightarrow{P_n^{-1}} & P_n^{-1} Y
\end{array}
\]

is an objectwise homotopy pullback square.
Proof. If the diagram is a homotopy pullback square, then its homotopy fibers are weakly equivalent via $D_n(f)$. For the converse note that $R_n(f)$ is a weak equivalence if $D_n(f)$ is by Corollary 6.20. The horizontal maps are part of homotopy fiber sequences

$$P_nX \to P_{n-1}X \to R_nX,$$

by Lemma 6.14. So the square is a homotopy pullback. □

Lemma 6.22. A map is an $n$-homogeneous acyclic fibration if and only if it is an $(n - 1)$-excisive fibration and an $n$-homogeneous equivalence.

Proof. By Definition 5.7, a map is an $(n - 1)$-excisive fibration if and only if it is a fibration $f : X \to Y$ in the hf model structure that induces the following objectwise homotopy pullback diagram:

$$
\begin{array}{ccc}
X & \longrightarrow & P_{n-1}X \\
\downarrow f & & \downarrow P_{n-1}(f) \\
Y & \longrightarrow & P_{n-1}Y
\end{array}
$$

So it follows that in the following diagram

(6.4)

$$
\begin{array}{ccc}
X & \longrightarrow & P_nX \longrightarrow P_{n-1}X \\
\downarrow f & & \downarrow P_n(f) \downarrow P_{n-1}(f) \\
Y & \longrightarrow & P_nY \longrightarrow P_{n-1}Y
\end{array}
$$

the outer square and the right hand square are homotopy pullbacks. Therefore, the left hand square is a homotopy pullback, and $f$ is an $n$-excisive fibration which is the same as an $n$-homogeneous fibration. Because it is an $n$-homogeneous equivalence by assumption, it is an $n$-homogeneous acyclic fibration.

Suppose now that $f$ is an $n$-homogeneous acyclic fibration. An $n$-homogeneous fibration is the same as an $n$-excisive fibration, hence the left hand square of diagram (6.4) is a homotopy pullback. Because (6.3) is also a homotopy pullback, the combined outer square is so. Hence, $p$ is an $(n - 1)$-excisive fibration. It is an $n$-homogeneous equivalence by assumption. □

Lemma 6.23. Let $X$ and $Y$ be $n$-excisively fibrant. Then a map $p : X \to Y$ is an $n$-homogeneous acyclic fibration if and only if it is an $(n - 1)$-excisive fibration.

Proof. According to Lemma 6.22 it suffices to show that an $(n - 1)$-excisive fibration between $n$-excisively fibrant objects is already an $n$-homogeneous equivalence. In this case the outer square of diagram (6.4) is a homotopy pullback and the horizontal maps in the left hand square are weak equivalences. Thus, the square on the right hand side is a homotopy pullback square, and the induced map of fibers $D_nX \to D_nY$ is a weak equivalence. □

Lemma 6.24. A cr cofibration is an $n$-homogeneous cofibration if and only if it is an $(n - 1)$-excisive equivalence.

Proof. Right properness implies that a map is an acyclic cofibration in the $n$-homogeneous model structure if and only if it has the left lifting property with
respect to all \(n\)-homogeneous fibrations between fibrant objects. The fibrant objects are exactly the \(n\)-excisively fibrant functors. Thus, the stated equivalence follows from Lemma 6.23. □

**Corollary 6.25.** A functor is cofibrant in the \(n\)-homogeneous model structure if and only if it is \(cr\) cofibrant and \(n\)-reduced.

**Proof.** This follows from Lemma 6.24. □

In particular, one can view the \(n\)-homogeneous model structure as the “fiber” model structure of the left Quillen functor

\[
\text{Id} : \text{Fun}(C, \mathcal{D})_{n-\text{exc} -\text{cr}} \to \text{Fun}(C, \mathcal{D})_{(n-1)-\text{exc} -\text{cr}}
\]

of excisive model structures.

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