ON THE INDEPENDENT SUBSETS OF POWERS OF PATHS AND CYCLES

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Abstract. In the first part of this work we provide a formula for the number of edges of the Hasse diagram of the independent subsets of the $h^{th}$ power of a path ordered by inclusion. For $h = 1$ such a value is the number of edges of a Fibonacci cube. We show that, in general, the number of edges of the diagram is obtained by convolution of a Fibonacci-like sequence with itself.

In the second part we consider the case of cycles. We evaluate the number of edges of the Hasse diagram of the independent subsets of the $h^{th}$ power of a cycle ordered by inclusion. For $h = 1$, and $n > 1$, such a value is the number of edges of a Lucas cube.

1. Introduction

For a graph $G$ we denote by $V(G)$ the set of its vertices, and by $E(G)$ the set of its edges.

Definition 1.1. For $n, h \geq 0$,

(i) the $h$-power of a path, denoted by $P_n^{(h)}$, is a graph with $n$ vertices $v_1, v_2$, ..., $v_n$ such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(P_n^{(h)})$ if and only if $|j - i| \leq h$;

(ii) the $h$-power of a cycle, denoted by $Q_n^{(h)}$, is a graph with $n$ vertices $v_1, v_2$, ..., $v_n$ such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(Q_n^{(h)})$ if and only if $|j - i| \leq h$ or $|j - i| \geq n - h$.

Thus, for instance, $P_n^{(0)}$ and $Q_n^{(0)}$ are the graphs made of $n$ isolated nodes, $P_n^{(1)}$ is the path with $n$ vertices, and $Q_n^{(1)}$ is the cycle with $n$ vertices. Figures 1(a) and 1(b) show some powers of paths and cycles, respectively.

Definition 1.2. An independent subset of a graph $G$ is a subset of $V(G)$ not containing adjacent vertices.

Let $H_n^{(h)}$, and $M_n^{(h)}$ be the Hasse diagrams of the posets of independent subsets of $P_n^{(h)}$, and $Q_n^{(h)}$, respectively, ordered by inclusion. Clearly, $H_n^{(0)} \cong M_n^{(0)}$ is a Boolean lattice with $n$ atoms ($n$-cube, for short).

Every independent subset $S$ of $P_n^{(h)}$ can be represented by a binary string $b_1b_2\cdots b_n$, where, for $i = 1, \ldots, n$, $b_i = 1$ if and only if $v_i \in S$. Specifically, each independent subset of $P_n^{(h)}$ is associated with a binary string of length $n$ such
that the distance between any two 1’s of the string is greater than $h$. Following [MS02] (see also [Kla11]), a Fibonacci string of order $n$ is a binary string of length $n$ without (two) consecutive 1’s. Recalling that the Hamming distance between two binary strings $\alpha$ and $\beta$ is the number $H(\alpha, \beta)$ of bits where $\alpha$ and $\beta$ differ, we can define the Fibonacci cube of order $n$, denoted $\Gamma_n$, as the graph $(V,E)$, where $V$ is the set of all Fibonacci strings of order $n$ and, for all $\alpha, \beta \in V$, $(\alpha, \beta) \in E$ if and only if $H(\alpha, \beta) = 1$. One can observe that for $h = 1$ the binary strings associated with independent subsets of $P_n$ are Fibonacci strings of order $n$, and the Hasse diagram of the set of all such strings ordered bitwise is $\Gamma_n$. Fibonacci cubes were introduced as an interconnection scheme for multicomputers in [Hsu93], and their combinatorial structure has been further investigated, e.g. in [KP07, MS02]. Several generalizations of the notion of Fibonacci cubes has been proposed (see, e.g., [IKR12a, Kla11]).

Remark. Consider the generalized Fibonacci cubes described in [IKR12a], i.e., the graphs $B_n(\alpha)$ obtained from the $n$-cube $B_n$ of all binary strings of length $n$ by removing all vertices that contain the binary string $\alpha$ as a substring. In this notation the Fibonacci cube is $B_n(11)$. It is not difficult to see that $H_n^{(h)}$ cannot be expressed, in general, in terms of $B_n(\alpha)$. Instead we have:

$$H_n^{(2)} = B_n(11) \cap B_n(101), \quad H_n^{(3)} = B_n(11) \cap B_n(101) \cap B_n(1001), \ldots,$$

where $B_n(\alpha) \cap B_n(\beta)$ is the subgraph of $B_n$ obtaining by removing all strings that contain either $\alpha$ or $\beta$.

A similar argument can be carried out in the case of cycles. Indeed, every independent subset $S$ of $Q_n^{(h)}$ can be represented by a circular binary string (i.e., a sequence of 0’s and 1’s with the first and last bits considered to be adjacent) $b_1b_2\cdots b_n$, where, for $i = 1, \ldots, n$, $b_i = 1$ if and only if $v_i \in S$. Thus, each independent subset of $Q_n^{(h)}$ is associated with a circular binary string of length $n$ such that the distance between any two 1’s of the string is greater than $h$. A Lucas cube of order $n$, denoted $\Lambda_n$, is defined as the graph whose vertices are the binary strings of length $n$ without either two consecutive 1’s or a 1 in the first and in the
last position, and in which the vertices are adjacent when their Hamming distance is exactly 1 (see [MPCZS01]). For $h = 1$ the Hasse diagram of the set of all circular binary strings associated with independent subsets of $Q_n^{(h)}$ ordered bitwise is $\Lambda_n$. A generalization of the notion of Lucas cubes has been proposed in [IKR12b].

Remark. Consider the generalized Lucas cubes described in [IKR12b], that is, the graphs $B_n(\hat{\alpha})$ obtained from the $n$-cube $B_n$ of all binary strings of length $n$ by removing all vertices that have a circular containing $\alpha$ as a substring (i.e., such that $\alpha$ is contained in the circular binary strings obtained by connecting first and last bits of the string). In this notation the Lucas cube is $B_n(\hat{1})$. It is not difficult to see that $M_n^{(h)}$ cannot be expressed, in general, in terms of $B_n(\hat{\alpha})$. Instead we have:

$$M_n^{(2)} = B_n(\hat{1}) \cap B_n(\hat{0}1), \quad M_n^{(3)} = B_n(\hat{1}) \cap B_n(\hat{1}0) \cap B_n(\hat{0}01), \ldots$$

As far as we now, our $H_n^{(h)}$, and $M_n^{(h)}$ are new generalizations of Fibonacci and Lucas cubes, respectively.

In the first part of the paper we evaluate $p_n^{(h)}$, i.e., the number of independent subsets of $P_n^{(h)}$, and $H_n^{(h)}$, i.e., the number of edges of $H_n^{(h)}$. Our main result (Theorem 3.4) is that, for $n, h \geq 0$, the sequence $H_n^{(h)}$ is obtained by convolving the sequence $1, 1, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \ldots$ with itself.

In the second part of the paper we derive similar results for $q_n^{(h)}$, i.e., the number of independent subsets of $Q_n^{(h)}$, and $M_n^{(h)}$, i.e., the number of edges of $M_n^{(h)}$.

2. The independent subsets of powers of paths

For $n, h, k \geq 0$, we denote by $p_n^{(h)}$ the number of independent $k$-subsets of $P_n^{(h)}$.

Remark. For $h = 1$, $p_n^{(h)}$ counts the number of binary strings $\alpha \in \Gamma_n$ such that $H(\alpha, 00 \cdots 0) = k$.

Lemma 2.1. For $n, h, k \geq 0$,

$$p_{n,k}^{(h)} = \binom{n - hk + h}{k}.$$ 

This result is Theorem 1 of [Hog70]. Below we write down a different proof.

Proof. By Definition 1.1, any two elements $v_i, v_j$ of an independent subset of $P_n^{(h)}$ must satisfy $|j - i| > h$. It is straightforward to check that whenever $n - hk - h < 0$, $p_{n,k}^{(h)} = 0 = \binom{n - hk + h}{k}$. It is also immediate to see that when $n = h = 0$ our lemma holds true.

Suppose now $n - hk - h \geq 0$. We can complete the proof of our lemma by establishing a bijection between independent $k$-subset of $P_n^{(h)}$ and $k$-subsets of a set with $(n - hk + h)$ elements. Let $X$ be the set of all $k$-subsets of a set $B = \{b_1, b_2, \ldots, b_{n-hk+h}\}$, and $I_k$ the set of all independent $k$-subsets of $P_n^{(h)}$. Consider the map $f : X \to I_k$ such that, for any $S = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\} \in X$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq n - hk + h$,

$$f(\{b_{i_1}, b_{i_2}, \ldots, b_{i_j}, \ldots, b_{i_k}\}) = \{v_{i_1}, v_{i_2+h}, \ldots, v_{i_j+(j-1)h}, \ldots, v_{i_k+(k-1)h}\}.$$
Claim 1. The map $f$ associates with each $k$-subset $S = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\} \in \mathcal{S}$ an independent $k$-subset of $\mathbf{P}_n^{(h)}$.

To see this we first remark that $f(S)$ is a $k$-subset of $V(\mathbf{P}_n^{(h)})$. Furthermore, for each pair $b_{i_j}, b_{i_{j+t}} \in S$, with $t > 0$, we have

\[ i_{j+t} + (j + t - 1)h - (i_j + (j - 1)h) = i_{j+t} - i_j + th > h. \]

Hence, by Definition 1.1, $(f(b_{i_1}), f(b_{i_{j+t}})) = (v_{i_{j+t} + (j+t-1)h}, v_{i_j + (j-1)h}) \notin E(\mathbf{P}_n^{(h)})$. Thus, $f(S)$ is an independent subset of $\mathbf{P}_n^{(h)}$.

Claim 2. The map $f$ is bijective.

It is easy to see that $f$ is injective. Then, we consider the map $f^{-1} : \mathcal{I}_k \to \mathcal{S}$ such that, for any $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in \mathcal{I}$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq n,

f^{-1}(\{v_{i_1}, v_{i_2}, \ldots, v_{i_1}, \ldots, v_{i_k}\}) = \{b_{i_1}, b_{i_2-h}, \ldots, b_{i_j-(j-1)h}, \ldots, b_{i_k-(k-1)h}\}.

Following the same steps as for $f$, one checks that $f^{-1}$ is injective. Thus, $f$ is surjective.

By Claims 1 and 2 we have established a bijection between independent $k$-subsets of $\mathbf{P}_n^{(h)}$ and $k$-subsets of a set with $(n - hk + h) \geq 0$ elements. The lemma is proved.

The coefficients $p_{n,k}^{(h)}$ also enjoy the following property: $p_{n,k}^{(h)} = p_{n-k+1,k}^{(h-1)}$.

For $n, h \geq 0$, the number of all independent subsets of $\mathbf{P}_n^{(h)}$ is

\[ p_n^{(h)} = \sum_{k \geq 0} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} \binom{n - hk + h}{k}. \]

Remark. Denote by $F_n$ the $n$th element of the Fibonacci sequence $F_1 = 1, F_2 = 1,$ and $F_i = F_{i-1} + F_{i-2}$, for $i > 2$. Then, $p_n^{(1)} = F_{n+2}$ is the number of elements of the Fibonacci cube of order $n$.

The following, simple fact is crucial for our work.

Lemma 2.2. For $n, h \geq 0$,

\[ p_n^{(h)} = \begin{cases} n + 1 & \text{if } n \leq h + 1, \\ p_{n-1}^{(h)} + p_{n-h-1}^{(h)} & \text{if } n > h + 1. \end{cases} \]

A proof of this Lemma can be also obtained using the first part of [Hog70] Proof of Theorem 1].

Proof. For $n \leq h + 1$, by Definition 1.2 the independent subsets of $\mathbf{P}_n^{(h)}$ have no more than 1 element. Thus, there are $n + 1$ independent subsets of $\mathbf{P}_n^{(h)}$.

Consider the case $n > h + 1$. Let $\mathcal{I}$ be the set of all independent subsets of $\mathbf{P}_n^{(h)}$, let $\mathcal{I}_{in}$ be the set of the independent subsets of $\mathbf{P}_n^{(h)}$ that contain $v_n$, and let $\mathcal{I}_{out} = \mathcal{I} \setminus \mathcal{I}_{in}$. The elements of $\mathcal{I}_{out}$ are in one-to-one correspondence with the $p_{n-1}^{(h)}$ independent subsets of $\mathbf{P}_n^{(h)-1}$, and those of $\mathcal{I}_{in}$ are in one-to-one correspondence with the $p_{n-h-1}^{(h)}$ independent subsets of $\mathbf{P}_{n-h-1}^{(h)}$. \qed
3. The poset of independent subsets of powers of paths

Figure 2 shows a few Hasse diagrams \( H^{(h)}_n \). Notice that, as stated in the introduction, for each \( n \), \( H^{(1)}_n \) is the Fibonacci cube \( \Gamma_n \).

![Hasse diagrams](image)

**Figure 2.** Some \( H^{(h)}_n \).

Let \( H^{(h)}_n \) be the number of edges of \( H^{(h)}_n \). Noting that in \( H^{(h)}_n \) each non-empty independent \( k \)-subset covers exactly \( k \) independent \((k-1)\)-subsets, we can write

\[
H^{(h)}_n = \sum_{k=1}^{\lfloor n/(h+1) \rfloor} k p^{(h)}_{n,k} = \sum_{k=1}^{\lfloor n/(h+1) \rfloor} k \left( \frac{n - hk + h}{k} \right).
\]

**Remark.** For \( h = 1 \), \( H^{(1)}_n \) counts the number of edges of \( \Gamma_n \).

Let now \( T^{(n,h)}_{k,i} \) be the number of independent \( k \)-subsets of \( P^{(h)}_n \) containing the vertex \( v_i \), and let, for \( h,k \geq 0, n \in \mathbb{Z} \),

\[
p^{(h)}_{n,k} = \begin{cases} 
 p^{(h)}_{0,k} & \text{if } n < 0, \\
 p^{(h)}_{n,k} & \text{if } n \geq 0.
\end{cases}
\]

**Lemma 3.1.** For \( n,h,k \geq 0, \) and \( 1 \leq i \leq n \),

\[
T^{(n,h)}_{k,i} = \sum_{r=0}^{k-1} p^{(h)}_{i-r-1,h-1-r} p^{(h)}_{n-i-h,k-1-r}.
\]

**Proof.** No independent subset of \( P^{(h)}_n \) containing \( v_i \) contains any of the elements \( v_{i-h}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+h} \). Let \( r \) and \( s \) be non-negative integers whose sum is \( k-1 \). Each independent \( k \)-subset of \( P^{(h)}_n \) containing \( v_i \) can be obtained by adding \( v_i \) to a \((k-1)\)-subset \( R \cup S \) such that

(a) \( R \subseteq \{v_1, \ldots, v_{i-h-1}\} \) is an independent \( r \)-subset of \( P^{(h)}_n \);

(b) \( S \subseteq \{v_{i+h+1}, \ldots, v_n\} \) is an independent \( s \)-subset of \( P^{(h)}_n \).

Viceversa, one can obtain each of this pairs of subsets by removing \( v_i \) from an independent \( k \)-subset of \( P^{(h)}_n \) containing \( v_i \). Thus, \( T^{(n,h)}_{k,i} \) is obtained by counting independently the subsets of type (a) and (b). Noting that the subsets of type (b) are in bijection with the independent \( s \)-subsets of \( P^{(h)}_{n-i-h} \), the lemma is proved.

**Remark.** \( T^{(n,1)}_{k,i} \) counts the number of strings \( \alpha = b_1 b_2 \cdots b_n \in \Gamma_n \) such that: (i) \( H(\alpha,00 \cdots 0) = k \), and (ii) \( b_1 = 1 \).

In order to obtain our main result, we prepare a lemma.
Lemma 3.2. For positive \( n \),
\[
\sum_{k=1}^{\lfloor n/(h+1) \rfloor} \sum_{i=1}^{n} T_{k,i}^{(n,h)} = H_n^{(h)}.
\]

Proof. The inner sum counts the number of \( k \)-subsets exactly \( k \) times, for each element of the subset. That is, \( \sum_{i=1}^{n} T_{k,i}^{(n,h)} = k p_{n,k}^{(h)} \). The lemma follows directly from Equation (1).

Next we introduce a family of Fibonacci-like sequences.

Definition 3.3. For \( h \geq 0 \), and \( n \geq 1 \), we define the \( h \)-Fibonacci sequence \( \mathcal{F}^{(h)} = \{ F_n^{(h)} \}_{n \geq 1} \) whose elements are
\[
F_n^{(h)} = \begin{cases} 
1 & \text{if } n \leq h + 1, \\
F_{n-1}^{(h)} + F_{n-h-1}^{(h)} & \text{if } n > h + 1.
\end{cases}
\]

From Lemma 2.2 and setting for \( h \geq 0 \) and \( n \in \mathbb{Z}, p_n^{(h)} = \begin{cases} p_0^{(h)} & \text{if } n < 0, \\
p_n^{(h)} & \text{if } n \geq 0,
\end{cases} \) we have that,
\[
F_i^{(h)} = \bar{p}_{i-h-1}, \text{ for each } i \geq 1.
\]

Thus, our Fibonacci-like sequences are obtained by adding a prefix of \( h \) ones to the sequence \( p_0^{(h)}, p_1^{(h)}, \ldots \). Therefore, we have:
- \( \mathcal{F}^{(0)} = 1, 2, 4, \ldots, 2^n, \ldots \);
- \( \mathcal{F}^{(1)} \) is the Fibonacci sequence;
- more generally, \( \mathcal{F}^{(h)} = 1, \ldots, 1, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \ldots \).

In the following, we use the discrete convolution operation \( * \), as follows.
\[
\left( \mathcal{F}^{(h)} * \mathcal{F}^{(h)} \right)(n) = \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)}
\]

Theorem 3.4. For \( n, h \geq 0 \), the following holds.
\[
H_n^{(h)} = \left( \mathcal{F}^{(h)} * \mathcal{F}^{(h)} \right)(n).
\]

Proof. The sum \( \sum_{k=1}^{\lfloor n/(h+1) \rfloor} T_{k,i}^{(n,h)} \) counts the number of independent subsets of \( P_i^{(h)} \) containing \( v_i \). We can also obtain such a value by counting the independent subsets of both \( \{v_1, \ldots, v_{i-h-1}\} \) and \( \{v_{i+h+1}, \ldots, v_n\} \). Thus, we have:
\[
\sum_{k=1}^{\lfloor n/(h+1) \rfloor} T_{k,i}^{(n,h)} = \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}.
\]

Using Lemma 3.2 we can write
\[
H_n^{(h)} = \sum_{k=1}^{\lfloor n/(h+1) \rfloor} \sum_{i=1}^{n} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \sum_{k=1}^{\lfloor n/(h+1) \rfloor} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}.
\]

By Equation (2) we have \( \sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)} = \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)} \). By (3), the theorem is proved. □
Remark. For \( h = 1 \), we obtain the number of edges of \( \Gamma_n \) by using Fibonacci numbers:
\[
H_n^{(h)} = \sum_{i=1}^{n} F_i F_{n-i+1}.
\]
The latter result is \cite{Kla05} Proposition 3].

4. THE INDEPENDENT SUBSETS OF POWERS OF CYCLES

For \( n, h, k \geq 0 \), we denote by \( q_{n,k}^{(h)} \) the number of independent \( k \)-subsets of \( Q_n^{(h)} \).

Remark. For \( h = 1 \), \( n > 1 \), \( q_{n,k}^{(h)} \) counts the number of binary strings \( \alpha \in \Lambda_n \) such that \( H(\alpha,00\cdots0) = k \).

**Lemma 4.1.** For \( n, h \geq 0 \), and \( k > 1 \),
\[
q_{n,k}^{(h)} = \frac{n}{k} \binom{n-hk-1}{k-1}.
\]
Moreover, \( q_{n,0}^{(h)} = 1 \), and \( q_{n,1}^{(h)} = n \), for each \( n, h \geq 0 \).

**Proof.** Fix an element \( v_i \in V(Q_n^{(h)}) \), and let \( n > 2h \). Any independent subset of \( Q_n^{(h)} \) containing \( v_i \) does not contain the \( h \) elements preceding \( v_i \) and the \( h \) elements following \( v_i \). Thus, the number of independent \( k \)-subsets of \( Q_n^{(h)} \) containing \( v_i \) equals
\[
p_{n-2h-1,k-1}^{(h)} = \binom{n-hk-1}{k-1}.
\]
The total number of independent \( k \)-subsets of \( Q_n^{(h)} \) is obtained by multiplying \( p_{n-2h-1,k-1}^{(h)} \) by \( n \), then dividing it by \( k \) (each subset is counted \( k \) times by the previous proceeding). The case \( n \leq 2h \), as well as the cases \( k = 0, 1 \), can be easily verified. \( \square \)

For \( n, h \geq 0 \), the number of all independent subsets of \( Q_n^{(h)} \) is
\[
q_n^{(h)} = \sum_{k \geq 0} q_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} q_{n,k}^{(h)}.
\]

**Remark.** Denote by \( L_n \) the \( n \)th element of the Lucas sequence \( L_1 = 1, L_2 = 3 \), and \( L_i = L_{i-1} + L_{i-2}, \) for \( i > 2 \). Then, for \( n > 1 \), \( q_1^{(1)} = L_n \) is the number of elements of the Lucas cube of order \( n \).

The coefficients \( q_n^{(h)} \) satisfy a recursion that closely resemble that of Lemma 2.2.

**Lemma 4.2.** For \( n, h \geq 0 \),
\[
q_n^{(h)} = \begin{cases} 
\frac{n+1}{h+1} & \text{if } n \leq 2h+1, \\
q_{n-1}^{(h)} + q_{n-h-1}^{(h)} & \text{if } n > 2h+1.
\end{cases}
\]

**Proof.** The case \( n \leq 2h+1 \) can be easily checked. Let \( n > 3h+2 \), and let \( \mathcal{I} \) be the set of the independent subsets of \( Q_n^{(h)} \). Let \( \mathcal{I}_{in} \) be the subset of these subsets that (i) do not contain \( v_n \), and that (ii) contain no one of the following pairs: \((v_1,v_{n-h}),(v_2,v_{n-h+1}),\ldots,(v_h,v_{n-1})\). Furthermore let \( \mathcal{I}_{out} \) be the subset of the remaining independent subsets of \( Q_n^{(h)} \).
It is easy to see that the elements of $I_{in}$ are exactly the independent subsets of $Q_n^{(h)}$. Indeed, $v_n$ is not a vertex of $Q_n^{(h)}$ and the vertices of pairs $(v_1, v_{n-h}), (v_2, v_{n-h+1}), \ldots, (v_h, v_{n-1})$ are connected in $Q_n^{(h)}$. On the other hand, to show that

$$|I_{out}| = q_{n-h-1}^{(h)}$$

we argue as follows. First we recall (see the proof of Lemma 4.1) that the number of independent $k$-subsets of $Q_n^{(h)}$ that contain $v_n$ is $p_{n-2h-1,k-1}^{(h)}$. Secondly we obtain that the number of independent $k$-subsets of $Q_n^{(h)}$ containing one of the pairs $(v_1, v_{n-h}), (v_2, v_{n-h+1}), \ldots, (v_h, v_{n-1})$ is $h p_{n-3h-2,k-2}^{(h)}$. To see this, consider the pair $(v_1, v_{n-h})$. The independent subsets containing such a pair do not contain the $h$ vertices from $v_n - h + 1$ to $v_n$, do not contain the $h$ vertices from $v_2$ to $v_{h+1}$, and do not contain the $h$ vertices from $v_{n-2h}$ to $v_{n-h-1}$. Thus, the removal of such vertices and of the vertices $v_1$ and $v_{n-h}$ turns $Q_n^{(h)}$ into $P_{n-3h-2}^{(h)}$. Hence we can obtain all the independent $k$-subsets of $Q_n^{(h)}$ that contain the pair $(v_1, v_{n-h})$ by simply adding these two vertices to one of the $p_{n-3h-2,k-2}^{(h)}$ independent $k-2$-subsets of $P_{n-3h-2}^{(h)}$. Same reasoning can be carried out for any other one of the pairs: $(v_2, v_{n-h+1}), \ldots, (v_h, v_{n-1})$.

Using Lemmas 2.1 and 4.1 one can easily derive that

$$p_{n-2h-1,k-1}^{(h)} + h p_{n-3h-2,k-2}^{(h)} = q_{n-h-1,k-1}^{(h)}.$$

Hence, we derive the size of $I_{out}$:

$$|I_{out}| = q_{n-h-1}^{(h)} = \sum_{k \geq 1} p_{n-2h-1,k-1}^{(h)} + h \sum_{k \geq 2} p_{n-3h-2,k-2}^{(h)}.$$

Summing up we have shown that $|I| = |I_{in}| + |I_{out}|$, that is

$$q_n^{(h)} = q_{n-1}^{(h)} + q_{n-h-1}^{(h)}.$$

The proof of the case $2h + 1 < n \leq 3h + 2$ is obtained in a similar way, observing that $|I_{out}| = n - h$, and that $n - h - 1 \leq 2h + 1$. \hfill \Box

5. The poset of independent subsets of powers of cycles

Figure 3 shows a few Hasse diagrams $M_n^{(h)}$. Notice that, as stated in the introduction, for each $n$, $M_n^{(1)}$ is the Lucas cube $\Lambda_n$. 

![Figure 3. Some $M_n^{(h)}$.](image-url)
Let $M_n^{(h)}$ be the number of edges of $M_n^{(h)}$. As done in Section 3 for the case of paths, we immediately provide a formula for $M_n^{(h)}$.

\begin{equation}
M_n^{(h)} = \sum_{k=0}^{\left\lceil \frac{n}{h+1} \right\rceil} kq_n^{(h)} = n \sum_{k=0}^{\left\lceil \frac{n}{h+1} \right\rceil} \left( \binom{n-hk-1}{k-1} \right).
\end{equation}

**Remark.** For $h = 1$, $n > 1$, $M_n^{(h)}$ counts the number of edges of $\Lambda_n$. As shown in [MPCZS01, Proposition 4(ii)], $M_n^{(h)} = nF_{n-1}$.

As shown in the proof of Lemma 4.1, the value $p_{n-2h-1,k-1}^{(h)} = \left( \binom{n-hk-1}{k-1} \right)$ is the analogue of the coefficient $T_{k,i}^{(n,h)}$: in the case of cycles we have no dependencies on $i$, because each choice of vertex is equivalent. We can obtain $M_n^{(h)}$ in terms of a fibonacci-like sequence, as follows.

**Proposition 5.1.** For $n > h \geq 0$, the following holds.

$$M_n^{(h)} = nF_{n-h}^{(h)}.$$

**Proof.** Using Equation (2) we obtain:

$$M_n^{(h)} = n \sum_{k=1}^{\left\lceil \frac{n}{h+1} \right\rceil} p_{n-2h-1,k-1}^{(h)} = nF_{n-h}^{(h)}.$$ 

\[\square\]

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