GROTHENDIECK–LEFSCHETZ FOR AMPLE SUBVARIETIES AND EXTENSION OF FIBER STRUCTURES

TOMMASO DE FERNEX AND CHUNG CHING LAU

In memory of Mauro Beltrametti

Abstract. In this paper, we establish a Grothendieck–Lefschetz theorem for smooth ample subvarieties of smooth projective varieties over an algebraically closed field of characteristic zero. We then address a conjecture of Sommese on the extension of fiber structures from a smooth ample subvariety to its ambient variety. Using cohomological methods, we propose a solution to the conjecture which relies on strengthening the positivity assumption in a suitable arithmetic sense. The same methods are applied to verify the conjecture in special cases, including when the ambient variety is abelian, when the subvariety is abelian or toric, or when the morphism is a smooth fibration in abelian or toric varieties. Using a different approach based on deformation theory of rational curves, we settle the conjecture for smooth fibrations with rationally connected fibers and prove a classification theorem for projective bundles and quadric fibrations.

1. Introduction

The notion of ample subscheme, whose origins can be traced back to the work of Hartshorne [Har70], was only recently formalized by Ottem in [Ott12]: a closed subscheme $Y$ of codimension $r$ of a projective variety $X$ of characteristic zero is said to be ample if the exceptional divisor of the blow-up of $X$ along $Y$ is $(r-1)$-ample in the sense of [Tot13]. If $r = 1$ and $X$ is normal, this is equivalent to $Y$ being an effective ample Cartier divisor; in higher codimensions, examples are given by subschemes defined by the vanishing of regular sections of ample vector bundles. The assumption on the characteristic is relevant, and in fact it remains unclear at the moment what the correct definition of ample subscheme should be in positive characteristics.

Several properties of ample subschemes are established in [Ott12]. The connection between Ottem’s definition of ampleness and the treatment in [Har70] is manifest in the property stating that, when $X$ smooth, a locally complete intersection subscheme $Y \subset X$ of codimension $r$ is ample if and only if the normal bundle $N_{Y/X}$ is ample and the complement has cohomological dimension $\text{cd}(X \setminus Y) = r - 1$. Ottem deduces from this property a version of the Lefschetz hyperplane theorem which states that if $X$ is a smooth complex projective variety and $Y$ is an ample locally complete intersection subscheme, then $H^i(X, \mathbb{Q}) \rightarrow H^i(Y, \mathbb{Q})$ is an isomorphism for $i < \dim Y$ and injective for $i = \dim Y$. It is important to remark that, differently from the case of ample divisors or, more generally, of schemes defined by regular sections of ample vector bundles, the Lefschetz hyperplane theorem with integral coefficients can fail, even assuming that $Y$ is smooth.

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The first part of the paper is devoted to the following Grothendieck–Lefschetz-type theorem for ample subvarieties, a more general version of which is given in the main body of the paper (see Theorem 3.6).

**Theorem A.** Let $X$ be a smooth projective variety defined over an algebraically closed field of characteristic zero, and let $Y \subset X$ be a smooth ample subvariety. If $\dim Y \geq 2$, then the restriction map $\text{Pic}(X) \to \text{Pic}(Y)$ is injective and induces an isomorphism $\text{Pic}^0(X) \cong \text{Pic}^0(Y)$, and if $\dim Y \geq 3$ then the restriction map has finite cokernel.

Related results are also obtained in arbitrary characteristics. Theorem A implies by duality that if $\dim Y \geq 2$ then the natural map on Albanese varieties $\text{Alb}(Y) \to \text{Alb}(X)$ is an isomorphism, so that the Albanese morphism of $X$ extends the one of $Y$. Veronese embeddings show that the cokernel of $\text{Pic}(X) \to \text{Pic}(Y)$ can be nontrivial, hence Theorem A is optimal.

Before discussing the second part of the paper, we mention here two applications of Theorem A and the results from Section 3 leading to it. The first is that if $X$ is a normal projective variety and $Y \subset X$ is a positive dimensional connected closed subscheme that is $G_2$ in $X$, then the induced map on Albanese varieties $\text{Alb}(Y) \to \text{Alb}(X)$ is surjective, and if $Y$ is $G_3$ then this map has connected fibers. This improves upon a theorem of Matsumura where the surjectivity of the map established under the more restrictive assumptions that both $X$ and $Y$ are smooth and $Y$ has ample normal bundle. The second application is that with the exception of elliptic curves, abelian varieties cannot be realized as ample subvarieties of any smooth projective variety. This extends a result of Sommese stating that they cannot be realized as ample divisors.

The second part of the paper focuses on the question of extendability of fiber structures from a smooth ample subvariety to a smooth ambient variety. The question originates from a conjecture of Sommese [Som76] predicting that if $X$ is a smooth complex projective variety and $Y \subset X$ is defined by the vanishing of a regular section of an ample vector bundle $E$ of rank $r$, then any morphism $\pi: Y \to Z$ with $\dim Y - \dim Z > r$ extends to a morphism $\tilde{\pi}: X \to Z$.

The condition that the dimensions of the fibers of $\pi$ be strictly larger than the codimension of $Y$ is easily seen to be sharp. When $Y$ is an ample divisor, the conjecture is proved in [Som76] and the case of relative dimension one (the first case beyond the bound imposed in the conjecture) has been studied in the literature, especially the case of $\mathbb{P}^1$-bundles, see, e.g., [BS95, BI09, Lit17a, Liu19] and the references therein. Apart from some special cases where Sommese’s conjecture has been verified, to which we will come back at the end of the introduction, not much is known about these questions when $r > 1$.

In this paper, we consider Sommese’s conjecture in the more general context of ample subvarieties discussed above, a notion that was not available at the time of the writing of [Som76] but nonetheless fits very naturally.

**Conjecture B.** Let $X$ be a smooth projective variety over an algebraically closed field of characteristic zero, and let $Y \subset X$ be a smooth ample subvariety of codimension $r$. Then any morphism $\pi: Y \to Z$ with $\dim Y - \dim Z > r$ extends uniquely to a morphism $\tilde{\pi}: X \to Z$.

Theorem A provides the first step towards this conjecture, implying unicity of the extension and setting up the argument for existence. The same cohomological arguments used in the proof in the divisor case in [Som76] lead to a general sufficient condition (a
Kodaira-type vanishing) for extendability. While we are unable to establish this condition in full generality, we propose a solution of the problem which relies on strengthening the positivity assumption on $Y$ in a suitable arithmetic sense. To this end, we introduce the notion for a subvariety to be \textit{arithmetically ample}. Roughly speaking, this means that the reductions of $Y$ modulo $p$ along some arithmetic thickening of $(X,Y)$ over a $\mathbb{Z}$-algebra of finite type remain (naively) ample; we refer to Section 2 for the precise definition.

Our result is that $\pi$ extends to $X$ if $Y$ is arithmetically ample. Going back to the original framework of Sommese’s conjecture where $Y$ is defined by a regular section of an ample vector bundle $E$ on $X$, it follows that $\pi$ extends if $E$ is \textit{arithmetically $\Gamma$-ample}, which essentially means that its reductions modulo $p$ over a suitable arithmetic thickening remain $\Gamma$-ample. We also prove that analogous results hold in positive characteristic.

It remains unclear at the moment how restrictive the conditions of arithmetic ampleness and arithmetic $\Gamma$-ampleness actually are; this is something that seems worthwhile to explore.

We apply the same Kodaira-type vanishing condition for extendability to verify Conjecture B without requiring additional positivity on $Y$ but rather assuming that either $X$ or $Y$ or $\pi$ are special. Our results provide affirmative answers to the conjecture when $X$ is an abelian variety, when $Y$ is a toric variety, and, under some additional assumptions, when $\pi$ is either a family of abelian varieties or toric varieties. The first case follows from a vanishing theorem due to Debarre [Deb95], and the next two cases rely on Manivel’s vanishing theorems [Man96].

In the last two sections of the paper, we address the case where $\pi$ is a morphism with rationally connected fibers. The methods applied in this section are different from previous sections, and use deformation theory of rational curves. A special case of the main theorem from this part of the paper gives the following result.

**Theorem C.** Conjecture B holds when $\pi$ is a smooth morphism with rationally connected fibers.

The condition that $\pi$ is smooth can be relaxed; we refer to Theorem 7.3 for a more general result. The proof builds upon the main result of [BdFL08] which can be viewed as a ‘birational’ solution of the conjecture in the context of rationally connected fibrations.

The theorem applies, for instance, to smooth Mori contractions; as before, the smoothness assumption can be relaxed. Various results generally related to the problem of extending Mori contractions were previously obtained in [LM96, AO99, dFL99, dF00, LM01, ANO06, Occ06, BdFL08]. As an application of our theorem, we settle Conjecture B for all fibrations in Fano complete intersections of index larger than the condimension of complete intersection. In particular, this proves the conjecture, and leads to a classification theorem, when $\pi$ is a projective bundle or a quadric fibration, two cases that were investigated under more restrictive conditions in several of the above references.

Unless otherwise specified, we work over an algebraically closed field of characteristic zero.

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2. Positivity conditions

We start by recalling the definitions of q-ampleness and ample subscheme from [Tot13, Ott12]. Let q be a nonnegative integer. A line bundle \( L \) on a projective variety \( X \) over a field is said to be, respectively:

1. **q-T-ample** if, for a given ample line bundle \( O_X(1) \) on \( X \) and for some positive integer \( N \), we have \( H^{q+i}(X, L \otimes N(-n-i)) = 0 \) for \( 1 \leq i \leq \dim X - q \);
2. **naively q-ample** if for every coherent sheaf \( F \) on \( X \) we have \( H^i(X, F \otimes L^{\otimes m}) = 0 \) for all \( i > q \) and all \( m \) sufficiently large depending on \( F \);
3. **uniformly q-ample** if there exists a constant \( \lambda > 0 \) such that for every coherent sheaf \( F \) on \( X \) we have \( H^i(X, L^{\otimes m}(-j)) = 0 \) for all \( i > q, j > 0 \), and \( m \geq \lambda j \).

In general, we have (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1), and the three properties are equivalent in characteristic zero [Tot13, Theorem 6.2]. In characteristic zero, we say that \( L \) is **q-ample** if it satisfies any of these equivalent conditions. The same terminology is used for a Cartier divisor \( D \) if the condition is satisfied by \( O_X(D) \).

The notion of q-ampleness presents some subtleties when passing to positive characteristic. For instance, q-T-ampleness is an open property, but naively q-ampleness is not open in mixed characteristic: if it were, then it would follow by [Ara04, Corollary 8.5] that q-ample line bundles on smooth varieties of characteristic zero satisfy the Kodaira vanishing \( H^i(X, \omega_X \otimes L) = 0 \) for \( i > q \), but there are examples where such vanishing property does not hold, see [Ott12, Section 9] or [Lau19, Section 5].

In characteristic zero, a closed subscheme \( Y \) of a projective variety \( X \) is said to be **ample** if the exceptional divisor \( E \) of the blow-up of \( X \) along \( Y \) is \((r-1)\)-ample where \( r \) is the codimension of \( Y \). Examples are given by subschemes defined (scheme theoretically) by regular sections of ample vector bundles [Ott12, Proposition 4.5], and ample subschemes can be thought of as a generalization of this notion. In the smooth case, they have several similar properties such as having an ample normal bundle and satisfying a Lefschetz hyperplane theorem for rational cohomology [Ott12, Corollary 5.6], though some differences occur: for instance, the Lefschetz hyperplane theorem for integral cohomology does not hold in general (e.g., see [Ott12, Example 7.3]).

It is not clear what should be the correct definition of ‘ample subscheme’ in positive characteristic. When working in arbitrary characteristic, we will say that a closed subscheme \( Y \) of codimension \( r \) of a projective variety \( X \) is **naively ample** if the exceptional divisor \( E \) of the blow-up of \( X \) along \( Y \) is naively \((r-1)\)-ample.

The subtleties occurring when passing from characteristic zero to positive characteristic are similar, and in fact related, to the difference between ampleness and \( \Gamma \)-ampleness of vector bundles. Recall that a vector bundle \( E \) on a projective variety \( X \) is **ample** if for every coherent sheaf \( F \) on \( X \) we have \( H^i(X, F \otimes \text{Sym}^t E) = 0 \) for \( i > 0 \) and \( t \gg 1 \), and is **\( \Gamma \)-ample** if for every coherent sheaf \( F \) on \( X \) we have \( H^i(X, F \otimes (\text{Sym}^t E)^*) = 0 \) for \( i > 0 \) and \( t \gg 1 \). These two notions are equivalent in characteristic zero but not in positive characteristics (see [Har70, Example III.4.10]). Now, the normal bundle of a (naively) ample locally complete intersection subscheme is ample in characteristic zero, but in positive characteristic is only known to be \( \Gamma \)-ample. Similarly, in characteristic zero a subscheme defined by a regular section of an ample vector bundle is ample, but this is not known in positive characteristic, where the correct assumption is \( \Gamma \)-ampleness. For a discussion, see [Ott12, Remark 4.2].

In characteristic zero, a useful characterization of ample subschemes is given in [Ott12, Theorem 5.4], which says that a locally complete intersection subscheme of codimension \( r \) of a smooth projective variety is ample if and only if the normal bundle is ample.
and the complement has cohomological dimension \( r - 1 \). This characterization can be extended to arbitrary characteristic as follows.

**Proposition 2.1.** A locally complete intersection subscheme \( Y \) of codimension \( r \) of a smooth projective variety \( X \) is naively ample if and only if the normal bundle \( N_{Y/X} \) is \( \Gamma \)-ample and the complement \( X \setminus Y \) has cohomological dimension \( \text{cd}(X \setminus Y) = r - 1 \).

**Proof.** The proof is the same as the proof of [Ott12, Theorem 5.4] once the ampleness of \( N_{Y/X} \) is replaced with \( \Gamma \)-ampleness. The fact that the complement of a \( q \)-ample effective divisor has cohomological dimension at most \( q \) is proved in characteristic zero in [Ott12, Theorem 5.4], but the proof extends to positive characteristics by combining the proof of [Ott12, Proposition 5.1] with [Tot13, Theorem 5.1]. In particular, combining this with [Har70, Corollary III.3.6], we see that, in all characteristics, the complement of a naively ample subscheme \( Y \subset X \) of codimension \( r \) has cohomological dimension \( r - 1 \). \( \square \)

Given a scheme \( X \) over a field \( k \) of characteristic zero, an arithmetic thickening of \( X \) is a choice of a finitely generated \( \mathbb{Z} \)-subalgebra \( A \subset k \) and a scheme \( X_A \) over \( \text{Spec} \ A \) such that \( X \cong X_A \times_{\text{Spec} \ A} \text{Spec} \ k \). Similar definitions are given for morphisms, sheaves, etc (see [Ara04]).

We are interested in situations where positivity properties that are not open in general are assumed to be preserved when spreading out to positive characteristics over some arithmetic thickening. One such example occurs when dealing with nefness, which is known not to be an open property (see [Les14,Lan15]): in the terminology of [Ara04], a line bundle \( L \) on \( X \) is said to be arithmetically nef if there exists an arithmetic thickening \((X_A, L_A)\) of \((X, L)\) such that \((L_A)_p\) is a nef line bundle on \((X_A)_p\) for every closed point \( p \in \text{Spec} \ A \).

In a similar spirit, we give the following definitions. Let \( X \) be a projective variety over an algebraically closed field \( k \) of characteristic zero. We say that a vector bundle \( E \) on \( X \) is arithmetically \( \Gamma \)-ample if there exists an arithmetic thickening \((X_A, E_A)\) of \((X, E)\) such that \((E_A)_p\) is a \( \Gamma \)-ample vector bundle on \((X_A)_p\) for every closed point \( p \in \text{Spec} \ A \). We say that a line bundle \( L \) on \( X \) is arithmetically \( q \)-ample if there exists an arithmetic thickening \((X_A, L_A)\) of \((X, L)\) such that \((L_A)_p\) is a naive \( q \)-ample divisor of \((X_A)_p\) for every closed point \( p \in \text{Spec} \ A \); analogous definition is given for Cartier divisors. Finally, we say that a closed subscheme \( Y \) of codimension \( r \) of \( X \) is arithmetically ample if the exceptional divisor \( E \) of the blow-up of \( X \) along \( Y \) is arithmetically \((r - 1)\)-ample.

The argument of the proof of [Ott12, Proposition 4.5] extends to give the following property.

**Proposition 2.2.** Any subscheme of a projective scheme defined by a regular section of a \( \Gamma \)-ample (resp., an arithmetically \( \Gamma \)-ample) vector bundle is naively ample (resp., arithmetically ample).

**Remark 2.3.** In [Gie71, Theorem 7.1] (see also [Har70, Exercise III.4.10]), Gieseker gives an example of an ample vector bundle on \( \mathbb{P}^2 \) that fails to be \( \Gamma \)-ample in all positive characteristics and hence is not arithmetically \( \Gamma \)-ample.

**Remark 2.4.** Over an algebraically closed field of characteristic zero, if a smooth subvariety \( Y \) of codimension \( r \) of a smooth projective variety \( X \) is arithmetically ample, then the normal bundle \( N_{Y/X} \) is arithmetically \( \Gamma \)-ample and \( \text{cd}(X \setminus Y) = r - 1 \). It is not known whether, in analogy with Proposition 2.1, the converse is true.
Remark 2.5. The conditions in above definitions of arithmetic $q$-ampleness, arithmetic ample subvariety, and arithmetically $\Gamma$-ampleness can be relaxed by only requiring that, over the arithmetic thickening, the reductions modulo $p$ are naively $q$-ample (respectively, naively ample, $\Gamma$-ample) for a dense subset of points $p \in \text{Spec} \, A$. In fact, an even more lax definition arithmetic ample subvariety that would suffice for the purpose of this paper would be to only require that the exceptional divisor $E$ of the blow-up $\widetilde{X} = \text{Bl}_Y X$ satisfying the following condition: for any arithmetic thickening $(\widetilde{X}_A, E_A)$ of $(\widetilde{X}, E)$ there exists a dense set of closed points $p \in \text{Spec} \, A$ with residue fields $k(p_i)$ of arbitrarily large characteristics such that $(E_A)_p$ has $F$-amplitude $\phi((E_A)_p) \leq r - 1$ (cf. [Ara04, Section 1]; the definition of $F$-amplitude is also recalled below in Section 5). The term arithmetically naively ample may be a more precise way to referring to subvarieties that are called here arithmetically ample.

3. Grothendieck–Lefschetz for ample subvarieties

The purpose of this section is to establish a Grothendieck–Lefschetz type theorem in the context of ample subvarieties. The case where the subvariety is defined by the vanishing of a regular section of an ample vector bundle is well understood, thanks to the Lefschetz–Sommese theorem [Som76] (see also [Laz04]). To deal with the more general setting, we will rely on the Lefschetz hyperplane theorem with rational coefficients obtained in [Ott12] and some properties related cohomological dimension and formal rational functions discussed below.

We start with some results about torsion in the kernel of the restriction map on Picard groups and the surjectivity of the map between Albanese varieties which hold in arbitrary characteristics. We will later focus on the case of characteristic zero.

We denote by $\widehat{X}$ the formal completion of a variety $X$ along a closed subscheme $Y$ and by $K(\widehat{X})$ the ring of formal rational functions on $\widehat{X}$. Note that there is a natural inclusion $K(X) \subset K(\widehat{X})$, where $K(X)$ is the function field of $X$. The study of $K(\widehat{X})$ was first initiated by Hironaka and Matsumura in [HM68]. If $K(\widehat{X})$ is a finite module over $K(X)$, then $Y$ is said to be $G2$ in $X$. If the natural inclusion is an isomorphism, then $Y$ is said to be $G3$ in $X$. These properties can be seen as properties that measure the positivity of the embedding $Y \subset X$. For instance, over an algebraically closed field of characteristic zero, a locally complete intersection subscheme with ample normal bundle is $G2$ [Har68, Corollary 6.8], and in fact the same is true if if the normal bundle is just $(\dim Y - 1)$-ample [Hal19, Corollary 2.6]. Similarly, a smooth ample subvariety is $G3$ [Ott12, Corollary 5.6].

The next proposition and corollary relate the aforementioned properties on the ring of formal rational functions and the induced map on the Albanese varieties of the subvariety and the ambient variety.

Proposition 3.1. Let $X$ be a projective variety over an algebraically closed field of arbitrary characteristic, and let $Y \subset X$ be a positive dimensional closed subscheme. Assume that $K(\widehat{X})$ is a field and $K(X)$ is algebraically closed in $K(\widehat{X})$ (e.g., $Y$ is $G3$ in $X$). Then:

1. The kernel of the map $\text{Pic}(X) \rightarrow \text{Pic}(Y)$ is torsion free.
2. The map $\text{Pic}^0(X) \rightarrow \text{Pic}^0(Y)$ is injective.
3. The map $\text{Alb}(Y) \rightarrow \text{Alb}(X)$ is surjective with connected fibers.
Proof. By [BS02, Theorem 2.7], the condition that $K(\tilde{X})$ is a field and $K(X)$ is algebraically closed in it is equivalent to saying that for any proper surjective morphism of projective varieties $g: V \to X$, the set $g^{-1}(Y)$ is connected.

To prove (1), we argue by contradiction and suppose there is a non-trivial, torsion line bundle $L \in \text{Pic}(X)$ such that $L|_Y \cong \mathcal{O}_Y$. Let $m$ be the smallest integer such that $L^\otimes m \cong \mathcal{O}_X$. We may take the étale cyclic Galois cover $\pi: \tilde{X} \to X$ associated with $L$ and the section $1 \in \Gamma(\mathcal{O}_X) \cong \Gamma(L^\otimes m)$. Since

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^0(X, \pi_*\mathcal{O}_X) \cong H^0(X, \mathcal{O}_X \oplus L^{-1} \oplus \cdots \oplus L^{-m+1}) \cong H^0(X, \mathcal{O}_X),$$

$\tilde{X}$ is connected. Therefore, $\tilde{X}$ is irreducible and reduced. By contrast, the fact that $L|_Y \cong \mathcal{O}_Y$ implies that $\pi^{-1}(Y)$ consists of exactly $m$-copies of $Y$, and since $m \geq 2$, this contradicts what was just observed at the beginning of the proof.

Since the kernel of a non-injective morphism between abelian varieties always contains torsion, (2) follows from (1).

(2) and (3) are equivalent statements. We show how to deduce (3) from (2). The proof of the converse statement is similar. Denote for short $A := \text{Pic}^0(Y)$ and $B := \text{Pic}^0(X)$, so that $A^\vee = \text{Alb}(Y)$ and $B^\vee = \text{Alb}(X)$. By (2), we have an inclusion $B \subset A$, and Poincaré’s reducibility theorem says that there is an abelian subvariety $C \subset A$ such that the map $B \times C \to A$ given by $(b, c) \mapsto b + c$ is an isogeny. Dualizing, we obtain an isogeny $A^\vee \to (B \times C)^\vee$, and since the inclusion $B \subset B \times C$ gives a surjection $(B \times C)^\vee \to B^\vee$, the resulting map $A^\vee \to B^\vee$ is surjective. Let $K$ be the kernel of this map, and let $K_0 \subset K$ be the connected component through the origin. If $A^\vee \to B^\vee$ is not a morphism with connected fibers, then $K_0 \neq K$, and hence the map factors as $A^\vee \to A^\vee/K_0 \to B^\vee$ where the last arrow is an isogeny of degree $> 1$. Dualizing, this contradicts the fact that the map $B \to A$ is injective.

A theorem attributed by Hartshorne to Matsumura (cf. [Har70, Exercise III.4.15]) states that if $X$ is a positive dimensional smooth proper variety and $Y \subset X$ is a smooth closed subvariety with ample normal bundle then the induced map $\text{Alb}(Y) \to \text{Alb}(X)$ is surjective. As we already mentioned, over an algebraically closed field of characteristic zero, a smooth subvariety with ample normal bundle is $G2$. The following result can be seen as a strengthened version of Matsumura’s result.

Corollary 3.2. Let $X$ be a normal projective variety over an algebraically closed field of arbitrary characteristic, and let $Y \subset X$ be a positive dimensional connected closed subscheme. Assume that $Y$ is $G2$ in $X$. Then the induced map $\text{Alb}(Y) \to \text{Alb}(X)$ is surjective.

Proof. By the proof of [Gie77, Theorem 4.3], there is a finite surjective map $f: X' \to X$ and a closed subscheme $Y' \subset X'$, such that $Y'$ is $G3$ in $X'$, $f$ is étale at each point of $Y'$, and the restriction of $f$ to $Y'$ gives an isomorphism $Y' \to Y$. By Proposition 3.1, the map $\text{Alb}(Y') \to \text{Alb}(X')$ is surjective with connected fibers. Since $f$ is surjective, $\text{Alb}(X') \to \text{Alb}(X)$ is surjective, and therefore $\text{Alb}(Y) \to \text{Alb}(X)$ is also surjective. □

In [BS96, Theorem 2], Bădescu and Schneider proved that for any rational homogeneous space $X$ over $\mathbb{C}$, the diagonal $\Delta_X$ is $G3$ in $X \times X$. In [Hal19, Proposition 3.10], Halic showed that for a smooth projective variety $X$ with non-pseudoeffective cotangent bundle over an algebraically closed field, the diagonal is $G2$. In the opposite direction, we obtain the following property.
Corollary 3.3. Let X be a projective variety with nontrivial Albanese variety, and let $\hat{X} \times X$ denote the completion of $X \times X$ along the diagonal $\Delta_X$. Then $K(X \times X)$ is not algebraically closed in $K(\hat{X} \times \hat{X})$ and $\Delta_X$ is not $G2$ in $X \times X$.

Proof. As $\text{Alb}(X \times X) \cong \text{Alb}(X) \times \text{Alb}(X)$, the induced map $\text{Alb}(\Delta_X) \to \text{Alb}(X \times X)$ cannot be surjective, hence the corollary follows from Proposition 3.1 and Corollary 3.2.

Under some stronger conditions on $X$ and $Y$, the next theorem, due to Speiser, can be used to reformulate Proposition 3.1 by replacing the local condition on the field of formal rational functions with a (more global) condition on the cohomological dimension of the complement.

Theorem 3.4 ([Spe80, Theorem 3.1]). Let $X$ be a Gorenstein projective variety over an algebraically closed field of arbitrary characteristic and $Y \subset X$ be a positive dimensional closed subscheme, such that $X$ is smooth at all points of $Y$. Then $\text{cd}(X - Y) < n - 1$ if and only if $Y$ is $G3$ in $X$ and $Y$ meets every divisor of $X$.

With the next proposition, which is independent of Speiser’s result, we provide such a reformulation of Proposition 3.1 without imposing any additional conditions.

Proposition 3.5. Let $X$ be a projective variety over an algebraically closed field of arbitrary characteristic, and let $Y \subset X$ be a positive dimensional closed subscheme. Assume that $\text{cd}(X \setminus Y) \leq \dim X - 2$. Then $K(\hat{X})$ is a field and $K(X)$ is algebraically closed in $K(\hat{X})$. In particular, the same conclusions of Proposition 3.5 hold.

Proof. By [Bäd04, Theorem 7.6], the condition on cohomological dimension implies that $Y$ is connected.

In order to prove that $K(\hat{X})$ is a field and $K(X)$ is algebraically closed in $K(\hat{X})$, we may assume without loss of generality that $X$ is normal. Indeed, if $\pi: W \to X$ is the normalization map then we see by [Har68, Proposition 1.1] that $\text{cd}(W \setminus \pi^{-1}(Y)) = \text{cd}(X \setminus Y) \leq \dim X - 2$, so the hypothesis holds on the normalization, and if $\hat{W}$ is the formal completion along $\pi^{-1}(Y)$ then the canonical map $K(\hat{X}) \to K(\hat{W})$ is an isomorphism by [BS02, Theorem 2.2]. Thus the claimed property can be deduced from the normalization.

Assume therefore that $X$ is normal. By [HM68, Lemma (1.4)], the connectedness of $Y$ implies that $K(\hat{X})$ is a field. Suppose by contradiction that $K(X)$ is not algebraically closed in $K(\hat{X})$. Then there is an element $\eta \in K(\hat{X}) \setminus K(X)$ that is algebraic over $K(X)$. By [BS02, Theorem 2.5], there is a finite surjective morphism $f: X' \to X$ from a variety $X'$, and a closed subscheme $Y' \subset X'$, such that $f$ is étale at each point of $Y'$ and its restriction to $Y'$ gives an isomorphism $Y' \to Y$; moreover, if $\hat{X}'$ is the formal neighborhood of $Y'$ in $X'$ and $\hat{f}: \hat{X}' \to \hat{X}$ is the morphism of formal schemes induced by $f$, then $\hat{f}^*\eta \in K(X')$. Note that $X'$ is projective, since $X$ is. By Stein factorization and the fact that we are assuming that $X$ is normal, we have $\deg(f) > 1$. The induced map $(X' \setminus f^{-1}(Y)) \to (X \setminus Y)$ is finite and surjective, and therefore $\text{cd}(X' \setminus f^{-1}(Y)) = \text{cd}(X \setminus Y) \leq \dim X - 2$ by [Har68, Proposition 1.1]. Thus $f^{-1}(Y)$ has to be connected, again by [Bäd04, Theorem 7.6]. As $f^{-1}(Y)$ contains $Y'$ and $f$ is étale at all points of $Y'$, it follows that $Y'$ must be a connected component in $f^{-1}(Y)$, but this contradict the connectedness of the latter since $f$ has degree $> 1$ and its restriction to $Y'$ has degree $1$. □
We now restrict ourselves to the case of characteristic zero. We begin by recalling the following extension of the Lefschetz hyperplane theorem, due to Ottem.

**Theorem 3.6** ([Ott12, Section 5]). Let $X$ be a smooth complex projective variety, and let $Y \subset X$ be a smooth subvariety such that $\text{cd}(X \setminus Y) = q$. Then the map $H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$ is an isomorphism for $i < \dim X - 1 - q$ and is injective for $i = \dim X - 1 - q$.

As in the classical setting, one deduces restriction properties at the level of Picard groups. Because of the rational coefficients, torsion is possible. We will use Proposition 3.5 to control the torsion in the kernel, but simple examples show that torsion can indeed appear in the cokernel.

**Theorem 3.7.** Let $X$ be a smooth projective variety defined over an algebraically closed field of characteristic zero, and let $Y \subset X$ be a smooth closed subvariety. Assume that $\text{cd}(X \setminus Y) \leq \dim X - 3$ (e.g., $Y$ is an ample subvariety of dimension $\geq 2$). Then:

1. The canonical map $\text{Pic}^0(X) \to \text{Pic}^0(Y)$ is an isomorphism.
2. The restriction map $\text{NS}(X) \to \text{NS}(Y)$ is injective.
3. The restriction map $\text{Pic}(X) \to \text{Pic}(Y)$ is injective.

Moreover, if $\text{cd}(X \setminus Y) \leq \dim X - 4$ (e.g., $Y$ is an ample subvariety of dimension $\geq 3$), then the maps in (2) and (3) have finite cokernel.

**Proof.** We can reduce to the case where the ground field is the field of complex numbers, as follows. First, we find a finite field extension $k_0$ of $\mathbb{Q}$ such that both $X$ and $Y$ are defined over $k_0$, and embed $k_0$ into $\mathbb{C}$. Then, we can find an algebraically closed field $K$ that contains both the original ground field and $\mathbb{C}$. Letting $k$ denote either the original ground field or $\mathbb{C}$, and considering $X$ and $Y$ as being defined over $k$, we only need to check that both hypothesis and conclusions of the theorem hold true over $k$ if and only if they hold true over $K$. Regarding the hypothesis, we use [Har70, Proposition 3.1] and the fact that for an ample line bundle $\mathcal{O}(1)$ on $X$ we have that $\mathcal{O}(1)_K$ is ample and $H^i(X_K \setminus Y_K, \mathcal{O}(m)_K) = H^i(X \setminus Y, \mathcal{O}(m)) \otimes_k K$ for all $i$ and $m$ to check that $\text{cd}(X_K \setminus Y_K) = \text{cd}(X \setminus Y)$. We denote $\text{Pic}(X)$ and $\text{Pic}^0(X)$ to be the Picard scheme and the Picard variety of $X$ respectively. From the fact that $\text{Pic}(X_K) \cong \text{Pic}(X)_K$ and $\text{Pic}^0(X_K) \cong \text{Pic}^0(X)_K$, we see that (1) and (3) hold over $k$ if and only if they hold over $K$, and this shows that the same equivalence holds for (2). As for the last assertion, it suffices to look at the cokernel of $\text{NS}(X) \to \text{NS}(Y)$. Since $k$ is algebraically closed, the irreducible components of $\text{Pic}(X)$ correspond to that of $\text{Pic}(X_K)$ via base change. This correspondence gives a natural identification $\text{NS}(X) = \text{NS}(X_K)$ and similarly we have $\text{NS}(Y) = \text{NS}(Y_K)$. These identifications are compatible with the restriction maps, and therefore they induce an identification between their cokernels.

Consider the compatible exponential sequences on the associated complex analytic varieties $X^\text{an}$ and $Y^\text{an}$:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}_{X^\text{an}} & \longrightarrow & \mathcal{O}_{X^\text{an}} & \overset{\exp}{\longrightarrow} & \mathcal{O}^*_X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}_{Y^\text{an}} & \longrightarrow & \mathcal{O}_{Y^\text{an}} & \overset{\exp}{\longrightarrow} & \mathcal{O}^*_Y & \longrightarrow & 0
\end{array}
$$

We shall apply the GAGA principle liberally in the following and, as usual, drop the superscript in the notation of complex analytic varieties when writing singular cohomology. If $\text{cd}(X \setminus Y) \leq \dim X - 4$, then Theorem 3.6 implies that for $i = 1, 2$ the map $H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$ has finite kernel and cokernel, and by Hodge theory this implies
that $H^i(X,\mathcal{O}_X) \to H^i(Y,\mathcal{O}_Y)$ is an isomorphism for $i = 1, 2$. If $\text{cd}(X \setminus Y) \leq \dim X - 3$, then these conclusions hold for $i = 1$.

For (1), we take the long exact sequence in cohomology associated with (3.1). Observing that $H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X^*)$ is nothing but the exponential function $\mathbb{C} \to \mathbb{C}^*$ and is surjective, we have the diagram with exact rows

$$
\begin{array}{cccccc}
0 & \to & H^1(X,\mathbb{Z}) & \to & H^1(X,\mathcal{O}_X) & \to & \text{Pic}^0(X) & \to & 0 \\
& & \downarrow & & \cong & & \downarrow & & \\
0 & \to & H^1(Y,\mathbb{Z}) & \to & H^1(Y,\mathcal{O}_Y) & \to & \text{Pic}^0(Y) & \to & 0
\end{array}
$$

By inverting the middle vertical arrow, we see that $\text{Pic}^0(X) \to \text{Pic}^0(Y)$ is surjective, and the snake lemma implies that $\ker(\text{Pic}^0(X) \to \text{Pic}^0(Y)) \cong \text{coker}(H^1(X,\mathbb{Z}) \to H^1(Y,\mathbb{Z}))$, which, as we already noted, is a finite abelian group. We then apply Proposition 3.5 to conclude that this last group is trivial.

For (2) and (3), note that we have commutative diagrams with exact rows

$$
\begin{array}{cccccc}
0 & \to & \text{NS}(X) & \to & H^2(X,\mathbb{Z}_X) & \to & C_1 & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{NS}(Y) & \to & H^2(Y,\mathbb{Z}_Y) & \to & C_2 & \to & 0
\end{array}
$$

and

$$
\begin{array}{cccccc}
0 & \to & C_1 & \to & H^2(X,\mathcal{O}_X) & \to & D_1 & \to & 0 \\
& & \downarrow & & \cong & & \downarrow & & \\
0 & \to & C_2 & \to & H^2(Y,\mathcal{O}_Y) & \to & D_2 & \to & 0
\end{array}
$$

If $\text{cd}(X \setminus Y) \leq \dim X - 3$, then we see from the second diagram that $C_1 \to C_2$ is injective. Going back to the first diagram, we see that $\ker(\text{NS}(X) \to \text{NS}(Y)) \cong \ker(H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z}))$, which is finite. Moreover, if $\text{cd}(X \setminus Y) \leq \dim X - 4$, then we also have that $\text{coker}(\text{NS}(X) \to \text{NS}(Y)) \subset \text{coker}(H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z}))$, which is finite.

We see by the diagram

$$
\begin{array}{cccccc}
0 & \to & \text{Pic}^0(X) & \to & \text{Pic}(X) & \to & \text{NS}(X) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Pic}^0(Y) & \to & \text{Pic}(Y) & \to & \text{NS}(Y) & \to & 0
\end{array}
$$

that $\ker(\text{Pic}(X) \to \text{Pic}(Y)) \cong \ker(\text{NS}(X) \to \text{NS}(Y))$ and $\text{coker}(\text{Pic}(X) \to \text{Pic}(Y)) \subset \text{coker}(\text{NS}(X) \to \text{NS}(Y))$, hence both kernel and cokernel of $\text{Pic}(X) \to \text{Pic}(Y)$ are finite. We then apply again Proposition 3.5 to conclude that the kernel is trivial. \qed

The next example shows that Theorem 3.7 is optimal.

**Example 3.8.** With the same assumptions as in Theorem 3.7, the map $\text{Pic}(X) \to \text{Pic}(Y)$ can have nontrivial torsion in the cokernel. For example, consider the $d$-uple embedding $\mathbb{P}^n \to \mathbb{P}^N$ over the complex numbers. Since in this case the Lefschetz hyperplane theorem with rational coefficient holds, it follows by a theorem of Oguš that $\mathbb{P}^n$ is an ample subvariety of $\mathbb{P}^N$ (cf. [Ott12, Theorem 7.1]). Nonetheless, the induced map on Picard groups has cokernel isomorphic to $\mathbb{Z}/d\mathbb{Z}$. This also gives an example where the effective Lefschetz condition $\text{Leff}(X,Y)$ defined in [Har70] fails for an ample subvariety. Note, by
contrast, that the Lefschetz condition Lef(X, Y) always holds for ample locally complete intersection subschemes of smooth projective varieties by [Har70, Proposition IV.1.1].

**Lemma 3.9.** Let X be a projective variety defined over a field k of arbitrary characteristic and Y ⊂ X be a subvariety. Let π : X → Z be a morphism to a projective variety Z such that its restriction to Y, π|Y : Y → Z, is an algebraic fiber space. Then π is also an algebraic fiber space.

*Proof.* Let X → Specπ∗OX → Z be the Stein factorization of π. The composition OZ → π∗OX → (π|Y)∗OY is an isomorphism, since π|Y is an algebraic fiber space. This implies OZ → π∗OX split-injects, giving a section of Specπ∗OX → Z. Since X is connected, Specπ∗OX is also connected. Thus, OZ → π∗OX is an isomorphism. □

**Lemma 3.10.** Let X be a smooth projective variety defined over an algebraically closed field k of characteristic 0 and Y ⊂ X be a smooth ample subvariety. Let π : X → Z be a dominant morphism to a projective variety Z such that the general fiber dimension of π|Y is positive. Then π and π|Y have the same Stein factorization.

*Proof.* By [Lau18, Theorem B], π|Y : Y → Z is dominant as well. Let Y → Z′ → Z be the Stein factorization of π|Y and X → W → Z be the Stein factorization of π. Then the morphism Z′ → Z factors through W → Z and the induced map g : Z′ → W is finite surjective. We claim that g is an isomorphism. To see this, fix a general closed point w ∈ W. Note that Xw is a smooth connected variety and Xw ∩ Y is positive dimensional. In fact, Xw ∩ Y is the union of the fibers of Y → Z′ over g−1(w), which is a finite set. In particular, Xw ∩ Y is connected if and only if g is birational. By [Lau19, Proposition 4.8], Xw ∩ Y is an ample subscheme in Xw. Applying Theorem 3.6, we see that Xw ∩ Y must be connected, hence g is birational. Since W is normal, it follows that g is an isomorphism. This completes the proof. □

**Corollary 3.11.** Let X be a smooth variety defined over an algebraically closed field of characteristic zero and Y ⊂ X a smooth ample subvariety of dimension ≥ 2.

1. The induced map on Albanese varieties Alb(Y) → Alb(X) is an isomorphism, hence we have the commutative diagram

   \[
   \begin{array}{ccc}
   Y & \xrightarrow{\text{alb}_Y} & X \\
   \| \downarrow & & \| \downarrow \\
   \text{Alb}(Y) & \cong & \text{Alb}(X)
   \end{array}
   \]

2. If alb_Y is an algebraic fiber space, then so is alb_X.

3. If Y is not of maximal Albanese dimension, then neither is X, and furthermore alb_X and alb_Y have the same image and share the same Stein factorization.

*Proof.* By Theorem 3.7, Pic^0(X) → Pic^0(Y) is an isomorphism, hence so is the dual map Alb(Y) → Alb(X). This gives (1). (2) and (3) follow from Lemma 3.9 and Lemma 3.10 respectively. □

We close this section with some applications to abelian varieties.

**Lemma 3.12.** Let f : X → Z be a morphism of projective varieties. If f∗ : Pic(Z) → Pic(X) has torsion cokernel, then f is a finite morphism.

*Proof.* Take an ample line bundle L on X. By the hypothesis, after replacing L by a suitable multiple, L can be expressed as the pull-back of a line bundle A on Z. If there
is any irreducible curve \( C \) contracted by \( f \), then \( \mathcal{L} \cdot C = f^*A \cdot C = 0 \), contradicting the assumption that \( \mathcal{L} \) is ample.

**Corollary 3.13.** Let \( A \) be an abelian variety. If \( \dim A \geq 2 \), then \( A \) cannot be realized as an ample subvariety of any smooth projective variety.

**Proof.** Suppose by contradiction that \( A \) is an ample subvariety of a smooth projective variety \( X \). By Corollary 3.11, the Albanese morphism on \( X \) gives a retraction from \( X \) to \( A \). Since the map Pic(\( X \)) → Pic(\( A \)), induced by the inclusion \( A \subset X \), is injective by Theorem 3.7, and the composition Pic(\( A \)) → Pic(\( X \)) → Pic(\( A \)) is the identity map, the map Pic(\( A \)) → Pic(\( X \)) induced by the retraction map is an isomorphism. This implies, by Lemma 3.12, that the retraction map is finite, which is impossible. \( \square \)

**Corollary 3.14.** Let \( Y \) be a smooth subvariety of an abelian variety \( A \). Assume that
1. \( Y \) is ample in \( A \) and \( \dim Y \geq 3 \), or
2. \( Y \) has ample normal bundle in \( A \) and \( 2 \dim Y \geq \dim A + 2 \).

If \( Y \) admits a closed embedding as an ample subvariety of a smooth projective variety \( X \), then the Albanese morphism of \( X \) is finite.

**Proof.** We reduce to the case where the ground field is the field of complex numbers. In both cases, the map Pic(\( A \)) → Pic(\( Y \)) is injective and has finite cokernel and Pic\( ^0 \)(\( A \)) → Pic\( ^0 \)(\( Y \)) is an isomorphism. In the first case, this follow by Theorem 3.7. In the second case, it follows from the fact that \( \pi_i(A, Y; \mathbb{Z}) = 0 \) for \( i \leq 3 \) [Som82], since the latter implies, by the Hurewicz theorem, that \( H_i(A, Y; \mathbb{Z}) = 0 \), and hence \( H^i(A, Y; \mathbb{Z}) = 0 \), for \( i \leq 3 \). In fact, we have in this case that Pic(\( A \)) → Pic(\( Y \)) is an isomorphism. In particular, the Albanese morphism of \( Y \) coincides with the inclusion \( Y \subset A \). By Corollary 3.11(1), we now have the following commutative diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{alb}_Y} & X \\
\downarrow & & \downarrow \text{alb}_X \\
A & \xrightarrow{\cong} & \text{Alb}(X)
\end{array}
\]

Since Pic(\( X \)) → Pic(\( Y \)) is injective, the map Pic(\( A \)) → Pic(\( X \)) has finite cokernel. It follows by Lemma 3.12 that the map \( X \rightarrow A \) is finite. \( \square \)

The example of a smooth cubic in \( \mathbb{P}^2 \) shows that the condition on dimension in Corollary 3.13 is sharp. Corollary 3.13 generalizes the first assertion in [Som76, Corollary I-A] to the case of ample subvarieties of arbitrary codimension. In fact, one can see that the second assertion of Corollary I-A as well as Proposition I and Corollary I-B of [Som76] can also be extended in a similar fashion by combining the same arguments given there with Corollary 3.11. While the settings are different, it is interesting to compare the previous result to [Som76, Conjecture IV-A].

## 4. Sommese’s extendability conjecture

In [Som76, Page 71], Sommese conjectures that if \( X \) is a smooth projective variety and \( Y \subset X \) is a smooth subvariety of codimension \( r \) defined by the vanishing of a regular section of an ample vector bundle \( \mathcal{E} \) on \( X \), then any morphism \( \pi: Y \rightarrow Z \) with \( \dim Y - \dim Z > r \) extends to a morphism \( \tilde{\pi}: X \rightarrow Z \).

**Remark 4.1.** In the original formulation of the conjecture, the section of \( \mathcal{E} \) is allowed to vanish to higher order along \( Y \). That is, \( Y \) is only assumed to be defined set-theoretically.
as the zero locus of the section. Note that while some properties of the complement of $Y$ may not be sensitive to the order of vanishing, local properties along $Y$ such as the ampleness of the normal bundle are. In this paper we will exclusively consider the original conjecture assuming that the section vanishes scheme-theoretically (i.e., to the first order) along $Y$. The difference is only relevant when $Y$ is not a divisor, and we do not see enough evidence in higher codimensions if the section is allowed to define an arbitrary scheme structure along $Y$.

A natural way of generalizing the setting is to just assume that $Y$ is an ample subvariety in the sense of [Ott12]. As we discussed already, subschemes defined by regular sections of ample vector bundles are ample in the ambient variety. We can therefore restate Sommese’s conjecture in the context of ample subvarieties, as follows.

**Conjecture 4.2.** Let $X$ be a smooth projective variety and $Y \subset X$ a smooth ample subvariety of codimension $r$. Then any morphism $\pi: Y \to Z$ with $\dim Y - \dim Z > r$ extends uniquely to a morphism $\tilde{\pi}: X \to Z$.

It is easy to see that the condition that $\dim Y - \dim Z > r$ is sharp. When $r = 1$, this is discussed in [BI09, Section 3], and the construction given there can be extended to include the following example in arbitrary codimension $r$.

**Example 4.3.** Let $r, s$ be two positive integers. Denoting by $u_0, \ldots, u_r \in H^0(\mathbb{P}^r, O_{\mathbb{P}^r}(1))$ a set of generators, consider the exact sequence

$$0 \to O_{\mathbb{P}^r}^\oplus r \to O_{\mathbb{P}^r}(r + 1)^{\oplus r + 1} \to O_{\mathbb{P}^r}(2r + 1) \to 0$$

where $\beta$ is given on global sections by

$$\beta: (s_0, \ldots, s_r) \mapsto \sum_{i=0}^r s_i u_i^r,$$

and $\alpha$ is given on global sections by

$$\alpha: (t_1, \ldots, t_r) \mapsto \left(-\sum_{i=1}^r t_i u_i^{r+1}, t_1 u_0 u_1, \ldots, t_r u_0 u_r\right).$$

Adding a new summand $O_{\mathbb{P}^r}(2r + 1)^{\oplus s}$ to the middle and right terms, with the identity map in between, we obtain the exact sequence

$$0 \to O_{\mathbb{P}^r}^{\oplus r} \to O_{\mathbb{P}^r}(r + 1)^{\oplus r + 1} \oplus O_{\mathbb{P}^r}(2r + 1)^{\oplus s} \to O_{\mathbb{P}^r}(2r + 1)^{\oplus s + 1} \to 0.$$

Let $X = \mathbb{P}(O_{\mathbb{P}^r}(r + 1)^{\oplus r + 1} \oplus O_{\mathbb{P}^r}(2r + 1)^{\oplus s})$ and $Y = \mathbb{P}(O_{\mathbb{P}^r}(2r + 1)^{\oplus s + 1})$. We have a fiberwise embedding $Y \subset X$ of scrolls over $\mathbb{P}^r$. By construction, $Y$ is defined, scheme theoretically, by a regular section of $O_X(1)^{\oplus r}$, where $O_X(1)$ is the tautological line bundle. Note that this is an ample vector bundle; in particular, $Y$ is an ample subvariety of $X$. Now, we have $Y \cong \mathbb{P}^r \times \mathbb{P}^s$, and the second projection $Y \to \mathbb{P}^s$ does not extend to $X$. Note that this projection has relative dimension $r$.

Extending the proof of [Som76, Proposition III] (see in particular the proof given in [BI09, Theorem 3.1]), we obtain the following general condition for a morphism to extend as predicted by Conjecture 4.2.

**Proposition 4.4.** Let $X$ be a smooth projective variety and $Y \subset X$ a smooth ample subvariety of codimension $r$. Assume that for any globally generated line bundle $\mathcal{M}$ on $Y$ we have

$$H^i(Y, \omega_Y \otimes \text{Sym}^t N_{Y/X} \otimes \mathcal{M}) = 0 \quad \text{for } i > r, \ t \geq 1.$$  

(4.1)
Then any morphism \( \pi : Y \to Z \) with \( \dim Y - \dim Z > r \) extends to a morphism \( \tilde{\pi} : X \to Z \).

**Proof.** As the statement is trivial if \( Z \) is a point, we may assume that \( \dim Z \geq 1 \) and hence that \( \dim Y \geq 3 \). After taking Stein factorization, we may also assume that the natural map \( O_Z \to \pi_*O_Y \) is an isomorphism. Let \( A \) be a very ample line bundle on \( Z \). By projection formula, we have \( H^0(Y, \pi^*A) \cong H^0(Z, A) \). The composition \( Y \to Z \to \mathbb{P}(H^0(Z, A)) \) is just given by the isomorphism \( H^0(Z, A) \cong H^0(Y, \pi^*A) \).

By Theorem 3.7, after possibly replacing \( A \) by a multiple, there exists a line bundle \( L \) on \( X \) such that \( L|_Y \cong \pi^*A \). The plan is to show that

\[
H^0(X, L) \to H^0(Y, L|_Y)
\]

is bijective. This will then imply that \( |H^0(X, L)| \) is basepoint-free and gives a morphism that extends \( \pi \).

Let \( \sigma : \tilde{X} \to X \) be the blow-up of \( X \) along \( Y \) and \( E \) the exceptional divisor. By definition of ampleness of \( \pi \) in \( X \), \( E \) is an \((r-1)\)-ample divisor on \( \tilde{X} \). Note that \( E \cong \mathbb{P}(N^*_Y/X) \). Let \( \tilde{L} := \pi^*L \) be the pull-back of \( L \) to \( X \) via the blow-up morphism. Setting \( n = \dim X \), we claim that

\[
H^{n-1-k}(E, \omega_{\tilde{E}}(tE) \otimes \tilde{L}^*) = 0 \quad \text{for } t \geq 1 \text{ and } k = 0, 1. \tag{4.2}
\]

Note that these cohomology groups are isomorphic to \( H^{n-r-k}(Y, \omega_Y \otimes \text{Sym}^i N^*_Y/X \otimes L^*) \).

Here, we implicitly used the fact that working over a field of characteristic zero, we have \( (\text{Sym}^i N^*_Y/X)^* \cong \text{Sym}^i N^*_Y/X \).

In order to prove (4.2), first observe that our hypothesis (4.1) imply, by the Leray spectral sequence, that

\[
R^j\pi_*(\omega_Y \otimes \text{Sym}^i N^*_Y/X) = 0 \quad \text{for } j \geq r \text{ and } t \geq 1 \tag{4.3}
\]

by [Laz04, Lemma 4.3.10]. Here we use the fact that the pull-back to \( Y \) of any ample line bundle on \( Z \) is globally generated. Then the Leray spectral sequence gives

\[
H^i(Z, R^j\pi_*(\omega_Y \otimes \text{Sym}^i N^*_Y/X) \otimes A^*) \Rightarrow H^{n-r-k}(Y, \omega_Y \otimes \text{Sym}^i N^*_Y/X \otimes L^*) \tag{4.4}
\]

for \( k = 0, 1 \), where \( i + j = n - r - k \). The relative vanishing (4.3) kills the terms with \( j \geq r \) on the left-hand side. For those terms with \( j < r \) and \( i \geq n - 2r \), they vanish since \( \dim Z < n - 2r \). Thus, we have \( H^{n-r-k}(Y, \omega_Y \otimes \text{Sym}^i N^*_Y/X \otimes L^*) = 0 \), and (4.2) follows.

Consider the following exact sequence

\[
0 \to \omega_{\tilde{X}}(tE) \otimes \tilde{L}^* \to \omega_{\tilde{X}}((t+1)E) \otimes \tilde{L}^* \to \omega_E(tE) \otimes \tilde{L}^*|_E \to 0.
\]

By (4.2), we have the bijectivity of

\[
H^{n-k}(\tilde{X}, \omega_{\tilde{X}}(tE) \otimes \tilde{L}^*) \to H^{n-k}(\tilde{X}, \omega_{\tilde{X}}((t+1)E) \otimes \tilde{L}^*) \quad \text{for } t \geq 1, k = 0, 1.
\]

By the \((r-1)\)-ampleness of \( E \), these groups vanish for \( t \gg 1 \), hence they are zero for all \( t \geq 1 \). It then follows by Serre duality that \( H^k(\tilde{X}, \mathcal{O}_{\tilde{X}}(E) \otimes \tilde{L}) = 0 \) for \( k = 0, 1 \), and hence \( H^0(\tilde{X}, \tilde{L}) \to H^0(\mathcal{E}, \mathcal{E}|_E) \) is bijective. This proves that

\[
H^0(X, L) \to H^0(Y, L|_Y) \tag{4.5}
\]

is bijective.

We shall now show that \( L \) is generated by global sections. We see by (4.5) that the base locus \( B \) of the linear system \( |H^0(X, L)| \) is disjoint from \( Y \). Suppose that \( B \neq \emptyset \). Note that \((\dim Z + 1)\)-many general sections in \( H^0(Y, L|_Y) \) have no common zeroes. Lifting
these sections to $H^0(X, \mathcal{L})$, they cut out a closed sub scheme $B'$ of $X$ that contains $B$, is disjoint from $Y$, and has dimension $\geq \dim X - \dim Z - 1$. Since, by [Ott12, Theorem 5.4], $X \setminus Y$ has co homological dimension $r - 1$, it cannot contain a projective sub scheme of dimension greater than $r - 1$. Since by our hypothesis we have $\dim B' > r - 1$, this gives the desired contradiction.

We have the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & Z' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\tilde{\pi}} & Z'' = \mathbb{P}(H^0(Z, \mathcal{A}))
\end{array}
\]

where $Z'$ is the image of $\tilde{\pi}$. If $Z' = Z$, then we are done. Suppose otherwise that $Z \subsetneq Z'$. After replacing $\mathcal{A}$ in the beginning with a sufficiently large multiple, we may assume that $\mathcal{O}_{Z'} \cong \tilde{\pi}_* \mathcal{O}_X$ [Laz04, Theorem 2.1.27] and that $H^0(Z', \mathcal{I}_Z \otimes \mathcal{O}_P(1)|_{Z'}) \neq 0$. Now, $H^0(Z', \mathcal{O}_P(1)|_{Z'})$ contains a section that vanishes on $Z$. Since $H^0(X, \mathcal{L}) \cong H^0(Z', \mathcal{O}_P(1)|_{Z'})$, this means that $H^0(X, \mathcal{L})$ contains a section on $X$ that vanishes on $Y$, contradicting the bijectivity in (4.5). \hfill \square

Remark 4.5. The vanishing condition in (4.1) is only needed to ensure the vanishing of higher direct images in (4.3), so the hypothesis of the proposition can be relaxed by only assuming the latter. In fact, all really needed is to guarantee the vanishing of the left-hand side of (4.4). Therefore, the vanishing hypothesis in Proposition 4.4 can be replaced with the weaker condition that

\[
\dim \text{Supp}(R^j \pi_*(\omega_Y \otimes \text{Sym}^t \mathcal{N}_{Y/X})) < n - r - j - 1 \quad \text{for} \quad j \geq r, \ t \geq 1. \tag{4.6}
\]

The following proposition addresses the problem of uniqueness of extension from a subvariety. It will be tacitly applied in the proofs of Propositions 5.3, 6.1, 6.5 and 6.6 and Theorem 7.3.

**Proposition 4.6.** Let $X$ be a projective variety over an arbitrary field and $Y \subset X$ a closed subvariety. Let $\pi: Y \to Z$ be a dominant morphism. Assume that either

1. $\text{Pic}(X) \to \text{Pic}(Y)$ is injective and $\pi: Y \to Z$ is an algebraic fiber space, or
2. the ground field is algebraically closed of characteristic zero, $X$ and $Y$ are smooth, $Y$ is ample in $X$, and $\dim Y = \max\{1, \dim Z\}$.

Then any extension $\tilde{\pi}: X \to Z$ of $\pi$, if it exists, is unique.

**Proof.** Assume first that the hypotheses in (1) hold. By Lemma 3.9, $\tilde{\pi}$ is an algebraic fiber space. Let $\mathcal{A}$ be a very ample line bundle on $Z$ and $\mathcal{L} := \tilde{\pi}^* \mathcal{A}$ be its pull-back to $X$. Since $\text{Pic}(X) \to \text{Pic}(Y)$ is injective, the line bundle $\mathcal{L}$ is independent of the choice of extension $\tilde{\pi}$ of $\pi$. Thanks to the fact that $\tilde{\pi}$ is an algebraic fiber space, we have the isomorphism $H^0(X, \mathcal{L}) \cong H^0(Z, \mathcal{A})$. Thus, the morphism $\tilde{\pi}: X \to Z$ is determined by the complete linear system $|H^0(X, \mathcal{L})|$ and therefore has to be unique.

If the hypotheses in (2) hold, then we apply Lemma 3.10 and Theorem 3.7 to reduce to the first case. \hfill \square

Remark 4.7. If we remove the assumption that $\text{Pic}(X) \to \text{Pic}(Y)$ is injective, the conclusion of Proposition 4.6 can fail. For instance, let $Y$ be a projective variety diagonally embedded in the product $X := Y \times Y$, and let $\pi: Y \to Z$ be any morphism. Composing either projection $X \to Y$ with $\pi$ gives an extension $X \to Z$ of $\pi$, and these two extensions do not agree if $\dim Z \geq 1$. 

Proposition 4.8. Let $X$ be a projective Cohen–Macaulay variety and $Y \subset X$ a regularly embedded ample subvariety. Let $\phi: X \to Z$ be a morphism with $Z$ smooth. If $\phi|_Y: Y \to Z$ is flat, then so is $\phi$.

Proof. By [Sta19, Lemma 00R4], it suffices to show that $\phi$ is equidimensional. The flatness of $\phi|_Y$ implies that the map is surjective and equidimensional. Each irreducible component of any fiber of $\phi$ must intersect $Y$, by ampleness of $Y$, and the fact that $Y$ is regularly embedded in $X$ implies that such intersection will be of codimension $\leq \text{codim}(Y, X)$ in the given component. This forces $\phi$ to be equidimensional. \qed

We close this section with the following positive characteristic version of Proposition 4.4.

Proposition 4.9. Let $X$ be a smooth projective variety over an algebraically closed field of characteristic $p > 0$, and let $Y \subset X$ be a smooth naively ample subvariety of codimension $r$. Suppose that $\pi: Y \to Z$ is a morphism such that $\dim Y - \dim Z > r$. Letting $\tilde{X}$ denote the formal completion of $X$ along $Y$, assume that the cokernel of the restriction map $\text{Pic}(X) \to \text{Pic}(\tilde{X})$ is torsion. Furthermore, assume that for any globally generated line bundle $\mathcal{M}$ on $Y$ we have

$$H^i(Y, \omega_Y \otimes \text{Sym}^t(\mathcal{N}^e_Y)^* \otimes \mathcal{M}) = 0 \quad \text{for } i \geq r, t \geq 1.$$  \hfill (4.7)

Then any morphism $\pi: Y \to Z$ with $\dim Y - \dim Z > r$ extends to a morphism $\tilde{\pi}: X \to Z$.

Proof. The proof is similar to the one of Proposition 4.4 but requires some adjustments. The statement is trivial if $Z$ is a point, so we may assume that $\dim Z \geq 1$ and $\dim Y \geq 3$. Let $I \subset O_X$ denote the ideal sheaf of $Y$. By duality, the vanishing condition (4.7) implies that

$$H^i(Y, I^t/I^{t+1}) = 0 \quad \text{for } i \leq 2 \text{ and } t \geq 1.$$  \hfill (4.8)

For every $t \geq 1$, we have an exact sequence

$$0 \to I^t/I^{t+1} \to O_{X_{t+1}}^* \to O_{X_t}^* \to 0$$

where $Y_t \subset X$ is the $t$-th neighborhood of $Y$ (defined by $I^t$) and the map $I^t/I^{t+1} \to O_{X_{t+1}}^*$ is defined by $x \mapsto 1 + x$. By (4.8), we see that the restriction map $\text{Pic}(\tilde{X}) \to \text{Pic}(Y)$ is an isomorphism. Combining this with the assumption that the cokernel of $\text{Pic}(X) \to \text{Pic}(\tilde{X})$ is torsion, we conclude that the cokernel of $\text{Pic}(X) \to \text{Pic}(Y)$ is also torsion. We can therefore fix a line bundle $\mathcal{L}$ on $X$ such that its restriction $\mathcal{L}|_Y$ is the pull-back of a very ample line bundle $\mathcal{A}$ on $Z$. Using the vanishing assumption (4.7) as in the proof of Proposition 4.4, we see that the restriction map $H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L}|_Y)$ is bijective. The closing argument of the proof of Proposition 4.4 applies here as well, thanks to Proposition 2.1. \qed

5. Arithmetic positivity

Let us recall the following key result of Deligne and Illusie.

Theorem 5.1 ([DI87, Theorem 2.1]). If $X$ a smooth variety of dimension $n$ defined over a perfect field $k$ of characteristic $p > n$, and $X$ can be lifted to the ring of second Witt vectors $W_2(k)$, then $F_{X/k} \Omega^i_{X/k} \equiv \bigoplus_{i=0}^n \Omega^i_{X/k}[-i]$ as objects in the derived category. Here $F_{X/k}: X \to X'$ denotes the relative Frobenius morphism.
The relative Frobenius $F_{X/k}$ is defined using the absolute Frobenius $F_X: X \to X$ and the universal property of the Cartesian diagram:

![Diagram of Frobenius and absolute Frobenius](image)

As a corollary, Raynaud made the observation that the Kodaira–Akizuki–Nakano vanishing theorem holds in such setting \cite[Corollary 2.9]{DI87}. Using a standard reduction modulo $p$ argument, this also gives a new proof of the Kodaira–Akizuki–Nakano vanishing theorem in the characteristic zero setting.

Using a similar argument, Arapura deduced from Deligne–Illusie’s result the following vanishing theorem over a field of characteristic zero. Before stating Arapura’s theorem, we recall the definition of $F$-amplitude. If $X$ is a projective variety over a field of characteristic $p > 0$, then the $F$-amplitude $\phi(E)$ of a coherent sheaf $E$ on $X$ is defined to be the least nonnegative integer $l$ such that for every locally free sheaf $F$ there exists an integer $N$ such that $H^i(X, E^{p^m} \otimes F) = 0$ for all $i > l$ and $m > N$. Here $E^{p^m} := F^{m \ast}E$ is the pull-back of $E$ by the $m$-th iterated absolute Frobenius. If $X$ is a projective variety over a field of characteristic zero, then the $F$-amplitude $\phi(E)$ of a coherent sheaf $E$ on $X$ is defined to be the least nonnegative integer $l$ for which there exists an arithmetic thickening $(X_A, E_A)$ such that for every closed point $p \in \text{Spec } A$ we have $\phi((E_A)_p) \leq l$.

**Theorem 5.2** ([Ara04, Corollary 8.5]). Let $X$ be a smooth projective variety over a field of characteristic zero and $E$ be a vector bundle over $X$. Then

$$H^i(X, \Omega^j_X \otimes E) = 0 \quad \text{for } i + j > n + \phi(E).$$

Combining this vanishing theorem with Proposition 4.4, we obtain the following extension result.

**Proposition 5.3.** Let $X$ be a smooth projective variety defined over an algebraically closed field of characteristic zero. Let $Y \subset X$ be a smooth arithmetically ample subvariety of $X$ of codimension $r$. Then any morphism $\pi: Y \to Z$ with $\dim Y - \dim Z > r$ extends uniquely to a morphism $\tilde{\pi}: X \to Z$.

**Proof.** Using the same notation as in the proof of Proposition 4.4, let $\sigma: \tilde{X} \to X$ be the blow-up of $X$ along $Y$ and $E$ the exceptional divisor. By assumption, $\phi(\mathcal{O}(tE)) \leq r - 1$ for all $t \geq 1$ and therefore $\phi(\mathcal{O}_E(tE)) \leq r - 1$ for all $t \geq 1$.

By Theorem 5.2, for any globally generated line bundle $\tilde{\mathcal{M}}$ on $E$ we have

$$H^i(E, \omega_E(tE) \otimes \tilde{\mathcal{M}}) = 0 \quad \text{for } i \geq r, \ t \geq 1.$$ 

Indeed, since globally generated line bundles are arithmetically nef, we have $\phi(\mathcal{O}_E(tE) \otimes \tilde{\mathcal{M}}) \leq \phi(\mathcal{O}_E(tE)) = r - 1$ by [Ara04, Theorem 3].

Now, if $\mathcal{M}$ is a globally generated line bundle on $Y$, then its pull-back $\tilde{\mathcal{M}}$ to $E$ is also globally generated, hence the vanishing condition in Proposition 4.4 is satisfied. The existence of the extension follows from there. \qed

In view of Proposition 2.2, the next corollary follows immediately from Proposition 5.3.
Corollary 5.4. Let $X$ be a smooth projective variety, let $E$ be an arithmetically $\Gamma$-ample vector bundle on $X$ of rank $r$, and let $Y \subset X$ be a smooth subvariety defined (scheme theoretically) by the vanishing of a regular section of $E$. Then any morphism $\pi: Y \to Z$ with $\dim Y - \dim Z > r$ extends uniquely to a morphism $\tilde{\pi}: X \to Z$.

Using Proposition 4.9, one obtains analogous results in positive characteristics. We state without proof the analogue of Proposition 5.3.

Proposition 5.5. Let $X$ be a smooth projective variety defined over an algebraically closed field $k$ of characteristic $p > 0$, and let $Y \subset X$ be a smooth naively ample subvariety of $X$ of codimension $r$. Suppose $\dim Y < p$ and that $Y$ can be lifted to $\text{Spec} W_2(k)$.

Denoting by $\tilde{X}$ the formal completion of $X$ along $Y$, assume that the cokernel of the restriction map $\text{Pic}(X) \to \text{Pic}(\tilde{X})$ is torsion. Then any morphism $\pi: Y \to Z$ with $\dim Y - \dim Z > r$ extends to a morphism $\tilde{\pi}: X \to Z$.

6. Special varieties

Here we apply Proposition 4.4 to verify Conjecture 4.2 in a number of special cases. We first study the case where the ambient variety is an abelian variety.

Proposition 6.1. Let $X$ be an abelian variety and $Y \subset X$ a smooth subvariety of codimension $r$ with ample normal bundle (e.g., $Y$ is an ample subvariety). Then any dominant morphism $\pi: Y \to Z$ with $\dim Y - \dim Z > r$ extends uniquely to a morphism $\tilde{\pi}: X \to Z$. Moreover, the Stein factorization of $\tilde{\pi}$ is, up to translation, a quotient morphism of abelian varieties $X \to X/B$ composed with a finite surjective map $X/B \to Z$.

Proof. Note that here we do not assume $Y$ to be ample in $X$. Nonetheless, we shall show that the proof of Proposition 4.4 goes through. We follow the notation used in Proposition 4.4.

We may reduce to the case where the ground field is $\mathbb{C}$. To see this, assume that $X$ and $Y$ are defined over an algebraically closed field $k$, and let $K$ be an algebraically closed field extension. Since $N_{Y/K} = (N_Y/X)_K$, we see that $N_{Y/X}$ is ample if and only if $N_{Y/K/X_K}$ is ample. Let $\text{Hom}_k(X, Z)$ be the scheme whose closed points parametrize $k$-morphisms from $X \to Z$. Note that $\pi$ corresponds to a closed point on $\text{Hom}_k(Y, Z)$ Let $\text{Hom}_k(X, Z; \pi)$ be the fiber of the natural restriction morphism $\text{Hom}_k(X, Z) \to \text{Hom}_k(Y, Z)$ over this point. We have $\text{Hom}_k(X, Z, \pi) \times_{\text{Spec} K} \text{Spec} K \cong \text{Hom}_K(X_K, Z_K, \pi_K)$, where $\pi_K: Y_K \to Z_K$ is the base change of $\pi$ over $K$. Thus, $\text{Hom}_k(X, Z, \pi)$ is nonempty if and only if $\text{Hom}_k(X_K, Z_K, \pi_K)$ is nonempty. Note also that the proof that an extension $\tilde{\pi}$ of $\pi$ satisfies the properties listed in the last part of the statement uses Lemma 6.2, which does not require working over the complex numbers.

Therefore both hypothesis and conclusions hold over $k$ if and only if they hold over $K$, and this allows us to reduce to the case where the ground field is $\mathbb{C}$.

We may assume $\dim Z \geq 1$, as otherwise there is nothing to prove, and hence $\dim Y \geq r + 2$. By the Barth–Lefschetz theorem on abelian varieties [Deb95, Theorem 4.5], $H^q(X, Y; \mathcal{C}) = 0$ for $q \leq \dim Y - r + 1$. Since $\dim Y - r + 1 \geq 3$, it follows from the proof of Theorem 3.7 that $\text{Pic}(X) \to \text{Pic}(Y)$ has finite kernel and cokernel. In fact, since $2 \dim Y \geq \dim X$, we see by [BS02, Proposition 4.8] that $Y$ is $G3$ in $X$, that is, $K(X) \cong K(\tilde{X})$ where $\tilde{X}$ is the formal completion along $Y$, and therefore $\text{Pic}(X) \to \text{Pic}(Y)$ is injective by Proposition 3.1. Thus, after replacing $\mathcal{A}$ be a positive multiple, there exists a line bundle $\mathcal{L}$ on $X$ such that $\mathcal{L}|_Y \cong \pi^* \mathcal{A}$. 

[Note: The document contains mathematical content and is structured as a series of propositions and corollaries, with a focus on special varieties and their properties. The text is precise and technical, typical of advanced mathematical research papers.]
Let \( n = \dim X \). By [Deb95, Theorem 4.4], for any nef line bundle \( \mathcal{M} \) on \( Y \) we have 
\[
H^i(Y, \omega_Y \otimes \text{Sym}^i \mathcal{N}_Y/X \otimes \mathcal{M}) = 0 \quad \text{for } i \geq r, \ t \geq 1.
\]
This implies the maps
\[
H^{n-j}(\tilde{X}, \omega_{\tilde{X}}(tE) \otimes \tilde{\mathcal{L}}^*) \to H^{n-j}(\tilde{X}, \omega_{\tilde{X}}((t+1)E) \otimes \tilde{\mathcal{L}}^*)
\]
are bijective for \( j = 0, 1 \) and all \( t \geq 1 \). It follows then by [Ott12, Equation (5.1)] that
\[
H^{n-j}(\tilde{X}, \omega_{\tilde{X}}(tE) \otimes \tilde{\mathcal{L}}^*) \cong H^{n-j}(\tilde{X} \setminus E, (\omega_{\tilde{X}} \otimes \tilde{\mathcal{L}}^*)|_{\tilde{X} \setminus E}) \quad \text{for } j = 0, 1 \text{ and } t \geq 1.
\]

On the one hand, since \( Y \) has ample normal bundle, it is geometrically non-degenerate in \( X \), and therefore for any subvariety \( W \subset X \) with \( \dim W + \dim Y \geq \dim X \) we have \( W \cap Y \neq \emptyset \), see [Deb, Corollary 2.5]. In particular, the complement \( X \setminus Y \) supports no divisors. On the other hand, we have by [BS02, Proposition 4.8] and the fact that \( \dim Y \geq \frac{n}{2} \) that \( Y \) is G3 in \( X \), that is, \( K(X) \cong K(\tilde{X}) \) where \( \tilde{X} \) is the formal completion along \( Y \). We may then apply [Spe73, Theorem 3], which implies that \( \text{cd}(\tilde{X} \setminus E) = \text{cd}(X \setminus Y) \leq n - 2 \).

Therefore, we have \( H^{n-j}(\tilde{X} \setminus E, (\omega_{\tilde{X}} \otimes \tilde{\mathcal{L}}^*)|_{\tilde{X} \setminus E}) = 0 \) for \( j = 0, 1 \), hence
\[
H^{n-j}(\tilde{X}, \omega_{\tilde{X}}(tE) \otimes \tilde{\mathcal{L}}^*) = 0 \quad \text{for } j = 0, 1 \text{ and } t \geq 1.
\]

This implies that the map \( H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L}|_Y) \) is bijective.

Now take \( (\dim Z + 1) \)-many general sections in \( H^0(X, \mathcal{L}) \). They cut out a closed subscheme \( B' \subset X \), which is either empty or has dimension \( \geq n - \dim Z \). By what we observed before, if \( B' \) is nonempty then it must intersect \( Y \), which is impossible. Thus, \( |H^0(X, \mathcal{L})| \) is base-point free. The last part of the proof of Proposition 4.4 can be repeated here verbatim.

The last statement of the proposition follows from the next lemma. \( \square \)

**Lemma 6.2.** Let \( A \) be an abelian variety and \( \pi: A \to W \) be an algebraic fiber space (i.e., a morphism such that \( O_W \cong \pi_\ast O_A \)). Then \( W \) is an abelian variety and, up to a translation, \( \pi \) is a morphism of abelian varieties.

**Proof.** First, let us show that if \( \pi \) is birational then it is an isomorphism. Suppose \( C \) is an irreducible curve contracted by \( \pi \). Let \( A \) be an ample line bundle on \( W \) and \( \mathcal{L} := \pi_\ast A \) be its pull-back to \( A \). Then \( \mathcal{L}|_C \) is trivial. Let \( a \) be an arbitrary closed point of \( A \) and \( \pi_\ast a : A \to A \) be the translation by \( \pi \). Identifying \( C \) and \( a + C \) via the translation map \( t^a \), we have \( \mathcal{L}|_{a + C} \cong t^\ast a \mathcal{L}|_C \). Note that \( t^\ast a \mathcal{L} \) is algebraically equivalent to \( \mathcal{L} \), so \( \mathcal{L}|_{a + C} \) is numerically trivial. Therefore, \( a + C \) is contracted by \( \pi \) as well. Since \( a \) is arbitrary, this shows that if \( \pi \) is birational then \( \pi \) has to be an isomorphism.

Now assume that \( \dim A > \dim W \). Take the fiber \( A_w \) over a general closed point \( w \in W \). Note that \( A_w \) is smooth and has trivial normal bundle. Therefore the tangent bundle of \( A_w \) is trivial as well. It is a fact that a smooth projective variety over an algebraically closed field of characteristic 0 with trivial tangent bundle has to be isomorphic to an abelian variety [MS87, p.191]. Therefore, \( A_w \) is isomorphic to an abelian variety \( B \). We may identify \( B \) with an abelian subvariety of \( A \) so that \( A_w = a + B \) for some closed point \( a \in A \). We claim that \( \pi \) factors through the quotient morphism \( A \to A/B \). To see this, note that the pull-back of any ample line bundle on \( W \) is numerically trivial on \( a' + B \) for any closed point \( a' \in A \). This implies that \( a' + B \) is contracted by \( \pi \), thus \( \pi \) factors through \( A \to A/B \). The induced map \( A/B \to W \) is birational and therefore is an isomorphism by the above argument. \( \square \)
We fix an algebraically closed field $k$ of characteristic zero, and define of two classes $\mathcal{U}$ and $\mathcal{V}$ of varieties over $k$ whose union contains abelian varieties and toric varieties.

The first class, $\mathcal{U}$, was introduced by Manivel in [Man96]. The key notion is that of \textit{uniformly nef vector bundle}, for which we refer to [Man96, Section 2.2]; roughly speaking, the category $\mathcal{C}$ of uniformly nef vector bundles is the smallest subcategory of the category of vector bundles on $k$-varieties which contains all bundles of the form $E \otimes \mathcal{L}$ where $E$ is a Hermitian flat vector bundle and $\mathcal{L}$ is a nef line bundle, that is closed under quotient, extension, and direct sum decomposition, and that satisfies the condition that for a finite morphism $f : Y \to X$ and a vector bundle $E$ on $X$, one has $E \in \mathcal{C}$ if and only if $f^* E \in \mathcal{C}$. One then defines $\mathcal{U}$ to be the class of smooth projective varieties over $k$ with uniformly nef tangent bundle.

Example 6.3. Prototypes of smooth projective varieties with uniformly nef tangent bundle are projective spaces and abelian varieties. More examples can be constructed starting from these using the fact that the class $\mathcal{U}$ is closed under products, finite étale covers, and the following construction: given $X \in \mathcal{U}$ and $E_1, \ldots, E_m$ are numerically flat vector bundles on $X$, we have $\mathbb{P}(E_1) \times_X \cdots \times_X \mathbb{P}(E_m) \in \mathcal{U}$; see [Man96, Section 2.3].

The second class of varieties, which we denote by $\mathcal{V}$, consists of those smooth projective varieties $X$ over $k$ which admit an arithmetic thickening $X_A \to \text{Spec} A$ with a dense set of fibers with the $F$-liftability property, by which we mean that for a dense set of closed points $p \in \text{Spec} A$ there exists a lift of the absolute Frobenius of $(X_A)_p$ to the second Witt vectors $W_2(k(p))$.

Example 6.4. Examples of varieties in the class $\mathcal{V}$ are toric varieties, abelian varieties whose arithmetic thickenings contain a dense set of ordinary abelian varieties as fibers, étale quotients of abelian varieties with the above property, and toric fibrations over such abelian varieties [AWZ17, Examples 3.1.2–3.1.5]. It is expected that all abelian varieties satisfy the above property, see [MS11, Conjecture 1.1 and Example 5.4].

Proposition 6.5. Let $X$ be a smooth projective variety and $Y \subset X$ a smooth ample subvariety of codimension $r$. Suppose that $Y$ belongs to the union $\mathcal{U} \cup \mathcal{V}$ of the two classes defined above. Then any morphism $\pi : Y \to Z$ with $\dim Y - \dim Z > r$ extends uniquely to a morphism $\tilde{\pi} : X \to Z$.

Proof. The result follows from Proposition 4.4. If $Y$ is toric or is a variety with uniformly nef tangent bundle, then the necessary vanishing follows from [Man96, Théorème 2.5 and Théorème 5.3]; the statement of [Man96, Théorème 5.3] does not include the twist by a nef line bundle, but the proof extends to cover that case. For the remaining cases, we apply [Lit17b, Theorem 2.2.1] (with $j = 1$) along a dense set of fibers over an arithmetic thickening and use upper semicontinuity of cohomology, relying on [Ara04, Theorem 3] as in the proof of Proposition 5.3 to allow for the twist of a globally generated line bundle.

For the following statement, we say that a surjective morphism of varieties $f : V \to W$ does not contract divisors if there are no prime divisors $D$ in $V$ such that $f(D)$ has codimension greater than one in $W$.

Proposition 6.6. Let $X$ be a smooth projective variety, $Y \subset X$ a smooth ample subvariety of $X$ of codimension $r$, and $\pi : Y \to Z$ a surjective morphism with $\dim Y - \dim Z > r$. Suppose that $\pi$ does not contract divisors and that there exists an open set $Z^* \subset Z$ with complement of codimension $\geq 2$ such that $\pi$ restricts to a smooth family $Y^* \to Z^*$ of varieties belonging to $\mathcal{U} \cup \mathcal{V}$. Then $\pi$ extends uniquely to a morphism $\tilde{\pi} : X \to Z$. 


Proof. Here we apply [Man96, Théorème 2.5 and Théorème 5.3] and [Lit17b, Théorème 2.2.1] along the fibers of $\pi$ over $Z^*$ as in the proof of Proposition 6.5, and use the hypothesis that $\pi$ does not contract divisors to check the condition in (4.6).

Corollary 6.7. Let $Y = A \times B$ where $A$ and $B$ are two varieties in $\mathcal{U} \cup V$. If $Y$ can be embedded as an ample subvariety of codimension $r$ of a smooth projective variety $X$, then $\min \{\dim A, \dim B\} \leq r$.

Proof. If $\min \{\dim A, \dim B\} > r$, then both projections of $A \times B$ extend to $X$ by Proposition 6.6. Starting from a sufficiently divisible line bundle $\mathcal{L}$ on $X$, we have $\rho^*(\mathcal{L}|_Y) \cong \mathcal{L}$ by Theorem 3.7, which is impossible if $\mathcal{L}$ is ample.

Corollary 6.8. Let $Y$ be a smooth projective variety. Suppose that $\pi: Y \to Z$ is a smooth family of abelian varieties of dimension $d \geq 2$. Then $Y$ cannot be realized as an ample subvariety of codimension $r < d$ in any smooth projective variety.

Proof. Suppose by contradiction that $Y$ is an ample subvariety of codimension $r < d$ of a smooth projective variety $X$. By Proposition 6.6, $\pi$ extends to a morphism $\tilde{\pi}: X \to Z$. Let $z \in Z$ be a general closed point and $Y_z$ and $X_z$ the respective fibers over $z$. By [Lau19, Proposition 4.8], $Y_z$ is an ample subvariety of $X_z$. Since $Y_z$ is an abelian variety of dimension $d \geq 2$, this contradicts Corollary 3.13.

7. Rationally connected fibrations

The following theorem from [BdFL08] provides a ‘rational’ solution to Conjecture 4.2 in the context of rationally connected fibrations. A related result which applies to maximal rationally connected fibrations over bases of positive geometric genus was also obtained in [Occ06]. In [BdFL08], the ground field is assumed to be $\mathbb{C}$, but the results of the paper hold more generally over any algebraically closed field of characteristic zero, which is the setting adopted here.

Theorem 7.1 ([BdFL08, Theorem 3.6]). Let $X$ be a smooth projective variety and $Y \subset X$ a smooth ample subvariety of codimension $r$. Denote by $\iota: Y \hookrightarrow X$ the inclusion map. Let $V \subset \text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ be a family of rational curve, and assume that the restriction to $Y$ of every irreducible component of $V$ is a covering family of rational curves on $Y$. Let $V_Y := \iota^{-1}_Y(V)$ be the restriction of $V$ to $Y$, and let $\alpha: X \dashrightarrow X//V$ and $\beta: Y \dashrightarrow Y//V_Y$ denote the respective rationally connected fibrations. Assume that $\dim Y - \dim Y//V_Y > r$. Then there is a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\iota} & X \\
\downarrow{\beta} & & \downarrow{\alpha} \\
Y//V_Y & \xrightarrow{\delta} & X//V
\end{array}
$$

where $\delta$ is a birational map.

The statement above, as well as the proof of Theorem 7.3 below, uses some terminology and notation from [Kol96, BdFL08] which we now recall.

Let $X$ be a smooth variety. We denote by $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ the scheme parameterizing morphisms from $\mathbb{P}^1$ to $X$ that are birational to their images. An element $[f] \in \text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ is said to be a free rational curve (resp., a very free rational curve) if $f^*TX$ is nef (resp., ample). A family of rational curves $V$ on $X$ is, by definition, an
arbitrary union of irreducible components of $\text{Hom}_\text{bir}(\mathbb{P}^1, X)$. If $0 \in \mathbb{P}^1$ is a fixed point and $Z \subset X$ is a closed subscheme, then $V(\{0\} \to Z)$ denotes the closed subscheme of $V$ defined by the condition that, for $[f] \in V$, we have $[f] \in V(\{0\} \to Z)$ if and only if $f(0) \in Z$. The image of the evaluation map $\mathbb{P}^1 \times V \to X$ is denoted by $\text{Locus}(V)$, and $\text{Locus}(V; \{0\} \to Z)$ is defined similarly.

Assume now that $X$ is projective. A family $V \subset \text{Hom}_\text{bir}(\mathbb{P}^1, X)$ is a covering family if $\text{Locus}(V_i)$ is dense in $X$ for every irreducible component $V_i$ of $V$. Associated to every covering family $V$, there is a model $X//V$ (only defined up to birational equivalence), called the $\text{RC}_V$-quotient, and a dominant rational map $\phi: X \dasharrow X//V$, called the $\text{RC}_V$-fibration, such that $\phi$ restricts to a proper morphism over a nonempty open set of $X//V$ and a very general fiber is an equivalence class of the equivalence relation defined by $V$. The variety $X$ is said to be $\text{RC}_V$-connected if $X//V$ is a point.

Given a closed embedding $\iota: Y \subset X$ of a smooth subvariety, there is a natural map $\iota_*: \text{Hom}_\text{bir}(\mathbb{P}^1, Y) \to \text{Hom}_\text{bir}(\mathbb{P}^1, X)$ given by composition. For a set $S \subset \text{Hom}_\text{bir}(\mathbb{P}^1, X)$ we denote by $\iota_*^{-1}(S) \subset \text{Hom}_\text{bir}(\mathbb{P}^1, Y)$ its inverse image via $\iota_*$, and for a set $T \subset \text{Hom}_\text{bir}(\mathbb{P}^1, Y)$ we denote by $\iota_*(T) \subset \text{Hom}_\text{bir}(\mathbb{P}^1, Y)$ its image via $\iota_*$. Given a family of rational curves $V$ on $X$, its restriction $\iota_*^{-1}(V)$ to $Y$ is defined to be the largest family of rational curves on $Y$ that is contained in $\iota_*^{-1}(V)$; equivalently, $\iota_*^{-1}(V)$ is the union of all irreducible components of $\iota_*^{-1}(V)$ that are also irreducible components of $\text{Hom}_\text{bir}(\mathbb{P}^1, Y)$. Similarly, given a family of rational curves $W$ on $Y$, its extension $\langle \iota_*(W) \rangle$ to $X$ is the union of all irreducible components of $\text{Hom}_\text{bir}(\mathbb{P}^1, X)$ that contains at least one irreducible component of $\iota_*(W)$.

Remark 7.2. The statement of Theorem 7.1 is actually a slight variation of [BdFL08, Theorem 3.6]. The original statement in [BdFL08] imposes a weaker condition on $Y$, only requiring that the normal bundle $\mathcal{N}_{Y/X}$ is ample and the induced map on Néron–Severi spaces $N^1(X) \to N^1(Y)$ is surjective (here, $N^1(X) = \text{NS}(X)_{\mathbb{R}}$); however, the conclusion is also weaker, namely, that the map $\delta$ is dominant and generically finite. Assuming that $Y$ is an ample subvariety, which implies the ampleness of $\mathcal{N}_{Y/X}$ plus the surjectivity of $N^1(X) \to N^1(Y)$ but is in general a stronger condition, allows us to immediately conclude that $\delta$ is birational. To see this, let $s \in X//V$ be a general point, and let $X_s$ and $Y_s$ be the fibers over $s$. Here we assume in particular that the fiber $\delta^{-1}(s)$ is a finite set of cardinality equal to the degree of $\delta$. Note that $X_s$ is smooth and connected. By [Lau19, Proposition 4.8], $Y_s$ is a positive dimensional ample subvariety of $X_s$, and therefore it is connected since, by Theorem 3.6, the map $H^0(X_s, \mathbb{Q}) \to H^0(Y_s, \mathbb{Q})$ is an isomorphism. This implies that $\delta$ is birational.

As an application of the above theorem, Conjecture 4.2 was verified in [BdFL08] when $\pi: Y \to Z$ is a projective bundle or a quadric fibration with integral fibers and relative Picard number 1, assuming that either $Y$ is defined by a regular section of an ample vector bundle on $X$ (as in the original conjecture of Sommese), or that $Z$ is a curve. Other special cases and related result were previously obtained in [LM96, AO99, dF09, dF00, LM01, AN06, Oc06]. We refer to the introduction of [BdFL08] and Section 8 for quick overviews of some of these results.

Using Theorem 7.1, we prove Conjecture 4.2 for Mori contractions and, more generally, fibrations with rationally connected fibers under some conditions on the fibers.

Recall that, as we defined in Section 6, a surjective morphism of varieties $\phi: X \to Y$ is said not to contract divisors if there are no prime divisors $D$ in $X$ such that $f(D)$ has codimension greater than one in $Y$. 


Recall also that a Mori contraction is of fiber-type if it has positive dimensional fibers. We say that a Mori contraction $\phi: X \to Y$ is of pure fiber-type if every extremal ray of the face of the Mori cone $\overline{\text{NE}}(X) \subset N_1(X)$ contracted by $\phi$ defines an extremal Mori contraction of fiber-type. For example, any extremal Mori contraction of fiber-type is of pure fiber-type, but a conic bundle over a curve admitting reducible fibers is not of pure fiber-type.

While the next result falls short from settling the conjecture for all Mori contractions of pure fiber-type and all rationally connected fibrations not contracting divisors, it is possible that with a more delicate analysis of deformation theory of 1-cycles, using Chow varieties in place of the Hom scheme, one may be able to further push this approach and prove the general case.

**Theorem 7.3.** Let $X$ be a smooth projective variety and $Y \subset X$ a smooth ample subvariety of codimension $r$. Let $\pi: Y \to Z$ be a surjective morphism with $\dim Y - \dim Z > r$, and assume that either

1. $\pi$ is a Mori contraction of pure fiber-type, or
2. $\pi$ does not contract divisors.

Assume furthermore that there exists an open set $Z^* \subset Z$ with complement of codimension $\geq 2$ such that for every $z \in Z^*$ the fiber $Y_z$ is irreducible and contains in its smooth locus a very free rational curve. Then $\pi$ extends uniquely to a morphism $\tilde{\pi}: X \to Z$.

**Proof.** Since the statement is trivial if $Z$ is a point, we can assume that $\dim Z \geq 1$, and hence $\dim Y \geq 3$. By Theorem 3.7, the inclusion $\iota: Y \xhookrightarrow{} X$ induces an isomorphism $\iota^*: N^1(X) \to N^1(Y)$ and, by duality, an isomorphism $\iota_*: N_1(Y) \to N_1(X)$.

Let $F$ be a fiber of $\pi$ of dimension $\dim F = \dim Y - \dim Z$ such that the smooth locus $F_{\text{sm}}$ of $F$ contains a very free rational curve $h: \mathbb{P}^1 \to F_{\text{sm}}$. Note that any fiber over $Z^*$ will satisfy this condition. Let $U \subset Y$ be an open set containing the image of $h$ and such that $F \cap U \subset F_{\text{sm}}$. We see by the splitting of the exact sequence

$$0 \to h^*T_{F\cap U} \to h^*T_U|_{F\cap U} \to h^*N_{F\cap U/U} \to 0$$

that $h$ defines, by composition with the inclusion of $F$ in $Y$, a free rational curve $f: \mathbb{P}^1 \to Y$. If $W$ is the irreducible component of $\text{Hom}_{\text{bir}}(\mathbb{P}^1, Y)$ containing $[f]$, then $W$ is a covering family of rational curves on $Y$. Note that $\mathbb{R}_{\geq 0}[W] \subset N_1(Y)$ is contained in the extremal face of $\overline{\text{NE}}(Y)$ contracted by $\pi$, and this means that the latter, viewed as a rational map, factors through the $\text{RC}_W$-fibration $Y \to Y//W$. As $F$ is $\text{RC}_W$-connected, we conclude that these two maps have the same very general fibers and hence $\pi$ agrees, as rational maps, with the $\text{RC}_W$-fibration.

Let $V := \langle i_*(W) \rangle \subseteq \text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ be the extension of $W$ to $X$, and consider the restriction $V_Y := i_Y^*V(\subseteq \text{Hom}_{\text{bir}}(\mathbb{P}^1, Y)$ of $V$ to $Y$. By [BdFL08, Proposition 3.11], the $\text{RC}_{V_Y}$-fibration $Y \to Y//V_Y$ agrees with the $\text{RC}_W$-fibration of $Y$ and hence with the contraction $\pi$. Note also that

$$\mathbb{R}_{\geq 0}[V_Y] = \mathbb{R}_{\geq 0}[V] = \mathbb{R}_{\geq 0}[W]$$

via the identification $\iota_*: N_1(Y) \cong N_1(X)$.

Let $\tilde{\pi}: X \to X//V$ be the $\text{RC}_{V_Y}$-fibration. The models $X//V$ and $Y//W$ are defined up to birational equivalence, but $Z$, which is a model for $Y//W$, is uniquely determined, up to isomorphism, by the contraction $\pi$. By Theorem 7.1, $X//V$ is birational to $Z$, thus
we have a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\iota} & X \\
\downarrow{\pi} & & \downarrow{\tilde{\pi}} \\
Z & \xrightarrow{\pi} & \tilde{\pi} 
\end{array}
\]

Fix an embedding \( Z \subset \mathbb{P}^m \). Let \( \mathcal{A} = \mathcal{O}_{\mathbb{P}^m}(1)|_Z \), and let \( \mathcal{L} \) be a line bundle whose global sections define the rational map \( X \dashrightarrow \mathbb{P}^m \). If \( p: X' \rightarrow X \) is a proper birational morphism such that \( q := \tilde{\pi} \circ p: X' \rightarrow Z \) is a morphism, then we have \( \mathcal{L} \cong \mathcal{O}_X(p_*q^*A) \) for any \( A \in |\mathcal{A}| \). Our goal is to show that \( \mathcal{L}|_Y \cong \pi^*\mathcal{A} \).

By construction, \( \tilde{\pi} \) is defined by a linear subsystem \( |\Lambda| \) of \( |\mathcal{L}| \), where \( \Lambda \subset H^0(X, \mathcal{L}) \) is a subspace. Let \( B \subset X \) denote the base scheme of \( |\Lambda| \). Note that the support of \( B \) is the indeterminacy locus of \( \tilde{\pi} \). To prove the theorem, we need to show that \( B = \emptyset \). This will show that \( \mathcal{L}|_Y \cong \pi^*\mathcal{A} \), hence that \( \phi \) is a morphism giving the desired extension of \( \pi \).

Suppose by contradiction that \( B \neq \emptyset \). Then

\[
\dim B \geq \dim X - \dim Z - 1.
\]

This is proved (over the complex numbers) in [Ste68]. Alternatively, one can see this directly by taking a general linear projection \( \mathbb{P}^m \dashrightarrow \mathbb{P}^{\dim Z} \). Since the induced map \( Z \to \mathbb{P}^{\dim Z} \) is a morphism, it follows that the indeterminacy locus of \( \tilde{\pi} \) is the same as the one of its composition with the projection to \( \mathbb{P}^{\dim Z} \), and hence \( B \) is cut out, set theoretically, by \( \dim Z + 1 \) divisors. This implies the lower-bound on \( \dim B \) stated above.

Since \( Y \) is ample in \( X \) and \( \dim Y + \dim B \geq \dim X \), it follows that

\[
B \cap Y \neq \emptyset.
\]

Let \( \Lambda_Y \subset H^0(Y, \mathcal{L}|_Y) \) be the image of \( \Lambda \) under restriction map \( H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L}|_Y) \). The commutativity of the above diagram implies that \( B \) cuts, scheme theoretically, a nonempty effective Cartier divisor \( E \) on \( Y \) such that

\[
|\Lambda_Y| = |\pi^*\mathcal{A}| + E.
\]

Note that, in particular, \( \mathcal{L}|_Y \cong \pi^*\mathcal{A} \otimes \mathcal{O}_Y(E) \).

We claim that \( \pi(\text{Supp}(E)) \) has codimension one in \( Z \). This is clear if \( \pi \) satisfies the condition given in (2) in the statement of the theorem. Suppose then that \( \pi \) satisfies (1). In this case, every irreducible curve \( C \) in \( Y \) that is contracted by \( \pi \) is numerically equivalent to a multiple of a curve \( C' \) that is contained in a general fiber of \( \pi \). Since \( \tilde{\pi} \) restricts to a proper morphism \( X^\circ \to Z^\circ \) for some nonempty open subset \( Z^\circ \subset Z \), it follows that \( \mathcal{L}|_Y \cdot C = 0 \). As clearly \( \pi^*\mathcal{A} \cdot C \), this gives \( E_Y \cdot C = 0 \), hence it follows by the cone theorem that \( \mathcal{O}_Y(E) \) is the pull-back of a line bundle on \( Z \). This means that \( E \) is the pull-back of a Cartier divisor on \( Z \), and therefore \( \pi(\text{Supp}(E)) \) has codimension one in \( Z \), as claimed.

Recall the assumption stated in the theorem on the fibers of \( \pi \) over the open set \( Z^* \subset Z \). Since the complement of \( Z^* \) has codimension \( \geq 2 \), it follows by the above claim that

\[
\pi(\text{Supp}(E)) \cap Z^* \neq \emptyset.
\]

Since the fibers of \( \pi \) over \( Z^* \) are irreducible, \( E \) must contain in its support a fiber \( F := Y_z \) over a point \( z \in Z^* \), and such fiber contains a very free rational curve \( h: \mathbb{P}^1 \to F_{\text{sm}} \) within its smooth locus. By composing with the inclusion of \( F \) in \( Y \), this yields a free rational curve \( f: \mathbb{P}^1 \to Y \) supported in \( F_{\text{sm}} \). We may assume without loss of generality that this
fiber $F$ is the same as the fiber picked at the beginning of the proof, and that the maps $h$ and $f$ are also the same. Let $g := t \circ f : \mathbb{P}^1 \to X$. Note that $[g] \in V$.

Pick an irreducible component $V'$ of $V$ that contains $[g]$. Note that $[g] \in V'(\{0\} \to Y)$ and, in fact, $[g] \in V'(\{0\} \to E)$. The argument of [BdFL08, Lemma 3.4] shows that $V'(\{0\} \to Y)$ is smooth at $[g]$ and that the evaluation map

$$\mathbb{P}^1 \times V'(\{0\} \to Y) \to X$$

has full rank, equal to $\dim X$, at $(q, [g])$ where $q$ is any point in $\mathbb{P}^1 \setminus \{0\}$. Its restriction to $V'(\{0\} \to E) \subset V'(\{0\} \to Y)$, namely, the evaluation map

$$\mathbb{P}^1 \times V'(\{0\} \to E) \to X,$$

has rank $\geq \dim X - 1$ at $(q, [g])$. This follows from the fact that the subscheme $V'(\{0\} \to E) \subset V'(\{0\} \to Y)$ is cut out by one equation, locally at $[g]$. Indeed, we have the following fiber diagram

$$
\begin{array}{ccc}
V'(\{0\} \to E) & \to & V'(\{0\} \to Y) \\
\downarrow & & \downarrow \\
E' & \to & Y
\end{array}
$$

where the vertical arrows send any element $[g']$ to $g'(0)$, and $E$ is locally cut out by one equation in a neighborhood of $g(0)$. Therefore $\text{Locus}(V'(\{0\} \to E))$ has dimension at least $\dim X - 1$.

To conclude, it suffices to show that under our assumption that $B \neq \emptyset$, we have

$$\text{Locus}(V'; \{0\} \to E) \subset B.$$ 

This will contradict the fact that the indeterminacy locus of a rational map on a normal variety must have codimension $\geq 2$, thus finishing the proof.

The above inclusion follows from the following observation. Let $C$ be an irreducible curve on $X$ with numerical class in $\mathbb{R}_{\geq 0}[V]$. Recall that this cone is the image of $\mathbb{R}_{\geq 0}[W]$ under the isomorphism $\iota_* : N_1(Y) \cong N_1(X)$. Using again that $\bar{\pi}$ restricts to a proper morphism $X^o \to Z^o$ and $W$ is a covering family, we see that $\mathcal{L} \cdot C = 0$. This implies that for any such curve $C$ we have that either $C \cap B = \emptyset$ or $C \subset B$. Now, since every curve parameterized by an element of $V'(\{0\} \to E)$ meets $E$ and hence $B$, it follows that $\text{Locus}(V'; \{0\} \to E)$ must be fully contained in $B$. \hfill $\square$

8. Fano fibrations

Theorem 7.3 can be used to settle Conjecture 4.2 for fibrations in Fano complete intersections of index larger than the codimension of complete intersection.

Recall that a morphism of varieties $\pi : Y \to Z$ is a projective bundle (or $\mathbb{P}^n$-bundle, if $n$ is the relative dimension) if it is locally of the form $U \times \mathbb{P}^n \to U$, with $U \subset Z$ open, and the transition functions are linear. If $Z$ is smooth, then every projective bundle over $Z$ is isomorphic to the projectivization of a locally free sheaf on $Z$ [Har77, Exercise II.7.10] and therefore admits a polarization $\mathcal{H}$ inducing a linear polarization on the fibers.

We say that a flat morphism $\pi : Y \to Z$ of relative dimension $n \geq 1$ is a fibration in Fano complete intersections if there exists a $\mathbb{P}^{n+c}$-bundle $\pi' : Y' \to Z$ and a fiberwise embedding $Y \hookrightarrow Y'$ over $Z$ such that the general fiber of $\pi$ is a Fano variety and every fiber of $\pi$ is embedded as a (possibly singular) nondegenerate complete intersection of codimension $c$ in the corresponding fiber of $\pi'$. The number $c$ is called the codimension of $\pi$, and the index of $\pi$ is the Fano index of a general fiber. If $n$ is the relative dimension...
of $\pi$ and $d_1, \ldots, d_c$ are the degrees of the equations cutting the fibers of $\pi$ in the fibers of $\pi'$, then the index is given by $n + c + 1 - \sum d_i$, with the only exception when $\pi$ is a conic bundle, which has index 2 and not 1.

Special cases of fibrations in Fano complete intersections include projective bundles, which correspond to the case $c = 0$, and quadric fibrations, which correspond to the case $c = 1$ and $d_1 = 2$. The following result implies, in particular, that Conjecture 4.2 holds for all projective bundles and quadric fibrations.

**Corollary 8.1.** Let $X$ be a smooth projective variety and $Y \subset X$ a smooth ample subvariety of codimension $r$. Assume that $\pi: Y \to Z$ is a fibration in Fano complete intersections of codimension $c \geq 0$ and index $c$, with $\dim Y - \dim Z > r$. Then $\pi$ extends uniquely to a morphism $\tilde{\pi}: X \to Z$.

**Proof.** The statement is trivial if $Z$ is a point, so we can assume that $\dim Z \geq 1$. Hence $\pi$ has relative dimension $n \geq 2$. Note that $\pi$ satisfies the condition of Theorem 7.3 given in (2). Then the corollary follows from theorem once we verify the condition on the fibers of $\pi$ on a suitable open set $Z' \subset Z$ stated in the theorem.

We fix a fiberwise embedding of $Y$ into a $\mathbb{P}^{n+c}$. bundle $\pi': Y' \to Z$ as in the definition.

By taking $c$ general hyperplane sections, one sees that every Fano complete intersection $V \subset \mathbb{P}^{n+c}$ of dimension $n \geq 2$, codimension $c \geq 0$, and index $c$ contains a very free rational curve in its smooth locus, provided the singular locus of $V$ has dimension $< c$. So, all we need to check is that, away from a set of codimension $\geq 2$ in the base $Z$, the fibers of $\pi$ are smooth at a fiber of dimension $< c$.

This can be checked by restricting $\pi$ over a general complete intersection curve $B \subset Z$. Set $W := \pi^{-1}(B)$ and $W' := (\pi')^{-1}(B)$, and let $\pi|_W: W \to B$ and $\pi'|_{W'}: W' \to B$ be the restrictions of $\pi$ and $\pi'$. By Bertini, we can assume that $W, W'$ and $B$ are all smooth.

As the fibers of $\pi|_W$ have dimension $\geq 2$, a local computation of the equations of $W$ in $W'$ then shows that the presence of fibers of $\pi|_W$ with singular locus of dimension $\geq c$ would confute the smoothness of $W$.

To see this, assume by contradiction that $\pi|_W$ has a fiber $F$ with singular locus of dimension $\geq c$. Let $t$ be a local parameter on $B$ centered at the base $p$ of the fiber, and let $(x_0 : \ldots : x_{n+c})$ be homogeneous coordinates of $\mathbb{P}^{n+c}$, where $n$ is the relative dimension of $\pi$. We can assume that $W$ is defined in a local trivialization $U \times \mathbb{P}^{n+c}$ of $W'$ by the equations $f_i + tg_i = 0$, for $1 \leq i \leq c$, where $f_i \in k[x_0, \ldots, x_{d+1}]$ are the forms defining $F$ in $\mathbb{P}^{n+c}$ and $g_i \in \mathcal{O}_C(U)[x_0, \ldots, x_{d+1}]$ are forms of the same degree in the variables $x_i$. Let $g_i^0 \in k[x_0, \ldots, x_{d+1}]$ be the specialization of $g_i$ at the point $p \in B$. By computing the Jacobian ideal, we see that $W$ is singular along the set $\text{Sing}(F) \cap \{g_0^0 = \cdots = g_c^0 = 0\} \times \{p\}$, and this set is non-empty if $\dim \text{Sing}(F) > c$. □

By imposing an additional condition on the restriction map on Picard groups, we obtain the following classification result for projective bundles and quadric fibrations.

**Theorem 8.2.** Let $X$ be a smooth projective variety and $Y \subset X$ a smooth ample subvariety of codimension $r$. Assume that the restriction map $\text{Pic}(X) \to \text{Pic}(Y)$ is surjective. Let $\pi: Y \to Z$ be either

1. a projective bundle or
2. a quadric fibration with integral fibers,

and assume that $\dim Y - \dim Z > r$. Then $\pi$ extends uniquely to a morphism $\tilde{\pi}: X \to Z$ which is a projective bundle in case (1), and either a projective bundle or a quadric fibration with integral fibers in case (2). In both cases, the fibers of $\pi$ are linearally embedded in the fibers of $\tilde{\pi}$. 
When \( r = 1 \), the case where \( \pi \) is a projective bundle follows from [Som76, Proposition III] and [BI09, Theorem 5.5]. When \( Z \) is a curve and \( \pi \) has relative Picard number 1, the theorem follows from case (a) of [BdFL08, Theorem 5.8].

By the Lefschetz–Sommese theorem [Som76, Lemma A] (see also [Laz04, Example 7.1.5]), the hypothesis that \( \text{Pic}(X) \to \text{Pic}(Y) \) is surjective is automatic, when \( \dim Y \geq 3 \), if \( Y \) is assumed to be defined by a regular section of an ample vector bundle \( \mathcal{E} \) on \( X \). In this more restrictive setting, Theorem 8.2 was proved (assuming that \( \pi \) has relative Picard number 1) in case (b) of [BdFL08, Theorem 5.8], and fits within a general study aiming to understand how much constraint a subvariety defined by a regular section of an ample vector bundle puts on the ambient variety, when the subvariety is special from the point of view of adjunction theory. This study, which is inspired by analogous studies related to hyperplane sections, started with Lanteri and Maeda’s paper [LM95], where the case where \( Y \) is a projective space or a smooth quadric was first settled (this case corresponds to the special case of Theorem 8.2 where \( Z \) is a point). Still restricting to the ample vector bundle setting, the case where \( Z \) is a curve of positive genus was first obtained in [LM96], and the theorem was proved in [AO99, Theorems 4.1 and 5.1] with no restrictions on \( Z \) but assuming \textit{a priori} the existence of a global polarization of \( X \) inducing a relatively linear polarization on \( Y \to Z \).

\textbf{Proof of Theorem 8.2.} By Corollary 8.1, \( \pi \) extends to a morphism \( \tilde{\pi} : X \to Z \).

We claim that \( \pi \) has relative Picard number 1. Otherwise \( \pi \) is necessarily a \( \mathbb{P}^1 \times \mathbb{P}^1 \)-bundle with trivial monodromy on the cohomology of the fibers, given by the contraction of a 2-dimensional face of the Mori cone \( \overline{\text{NE}}(Y) \). In this case, by contracting the two extremal rays of this face independently, we obtain two \( \mathbb{P}^1 \)-bundles \( \sigma_i : Y \to W_i, \ i = 1, 2 \), where each \( W_i \) is a \( \mathbb{P}^1 \)-bundle over \( Z \). Since in this case \( \pi \) has relative dimension 2, we have \( r = 1 \) and hence \( Y \) is an ample divisor on \( X \). We can therefore apply [Liu19, Theorem 1.3]. The surjectivity of \( \text{Pic}(X) \to \text{Pic}(Y) \) implies that the cases (iii) and (iv) of the quoted theorem cannot occur. By the remaining cases (i) and (ii), we see that both \( \mathbb{P}^1 \)-bundles \( \sigma_i \) extend to \( \mathbb{P}^2 \)-bundles \( \overline{\sigma}_i : X \to W_i \). Restricting to a general fiber \( G \) of \( \overline{\pi} \), which is 3-dimensional, this gives two distinct \( \mathbb{P}^2 \)-bundle structures \( G \to \mathbb{P}^1 \), which is clearly impossible.

We see by [Sta19, Lemma 02K4] that \( Z \) is smooth, since \( Y \) is smooth and the fibers of \( \pi \) are reduced. Therefore there exists a line bundle \( \mathcal{H} \) on \( Y \) inducing a linear polarization on the fibers of \( \pi \). By our hypothesis on the Picard groups, we can pick a line bundle \( \mathcal{L} \) on \( X \) such that \( \mathcal{L}|_Y \cong \mathcal{H} \). The same condition on the Picard groups implies that \( \overline{\pi} \), like \( \pi \), has relative Picard number 1, and therefore \( \mathcal{L} \) is relatively ample. After twisting by the pull-back of a sufficiently ample line bundle on \( Z \), we can assume that \( \mathcal{L} \) is an ample line bundle.

Arguing as in the proof of [BdFL08, Theorem 5.8], we see that \( \overline{\pi} \) is equidimensional with integral fibers. Since the setting here is slightly different, we sketch the argument. Note that \( \pi \) is equidimensional, say of relative dimension \( n \), and the general fiber of \( \pi \) has dimension \( n + r \). If \( G_i \) is any irreducible component of \( G \), then \( Y \cap G_i \neq \emptyset \) by the ampleness of \( Y \) in \( X \), and since \( Y \) is locally complete intersection of codimension \( r \) in \( X \), it follows that \( \dim(Y \cap G_i) \geq \dim G_i - r \). Therefore \( G_i \) has dimension \( n + r \) and \( F \subset G_i \). Note, in particular, that \( G \) is regularly embedded in \( X \) since \( Z \) is smooth and \( \text{codim}(G, X) = \dim Z \), and therefore it has no embedded components by [Mat69, Theorem 17.6]. Since \( \mathcal{O}_{X,F} \) is a regular local ring with a regular sequence locally defining \( G \) forming part of a regular system of parameters, \( \mathcal{O}_{G,F} \) is a regular local ring. As every irreducible component of \( G \) contains \( F \), it follows that \( G \) is integral.
Let $m = n + r$ denote the dimension of the fibers of $\tilde{\pi}$. Let $G$ be a smooth fiber of $\tilde{\pi}$, let $F \subseteq G$ be the corresponding fiber of $\pi$, and let $C \subseteq F$ be a line. By adjunction formula, we have
\[(K_G + a c_1(\mathcal{L}|G)) \cdot C = (K_F + a c_1(\mathcal{H}|F) - c_1(N_{F/G})|C) \cdot C\]
for any integer $a$. Since $N_{F/G} = N_{Y/X}|F$ is an ample vector bundle of rank $r$, we see that the nef value of $(G, \mathcal{L}|G)$ is at least $m + 1$ in case (1), and at least $m$ in case (2).

We can therefore apply the main result of [Fuj86] (see also [Fuj92]). In case (1), this implies that $(G, \mathcal{L}|G) \cong (\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$. In case (2), we see that $(G, \mathcal{L}|G)$ can either be $(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)), (Q, \mathcal{O}_{Q}(1))$ where $Q \subseteq \mathbb{P}^{m+1}$ is a smooth quadric hypersurface, or a scroll over a curve.

The last case can be excluded, as follows. Assume that $G$ is a scroll over $\mathbb{P}^1$. First, note that $n \geq 2$, and since $F$ is ample in $G$, the map Pic($G$) $\rightarrow$ Pic($F$) is injective by Theorem 3.7. Therefore $F \cong \mathbb{P}^1 \times \mathbb{P}^1$. Since $\pi$ has relative Picard number 1, $Z$ cannot be a point. Let $B \subseteq Z$ be a general complete intersection curve, and let $W = \pi^{-1}(B)$. If $\pi|W : W \rightarrow B$ is a smooth fibration, then, arguing as at the beginning of the proof, we see that the monodromy action on $N_1(F)$ must swap the two rulings in the fibers of $\pi|W$. We claim that the same happens even if $\pi$ has some singular fibers. Suppose this is not the case. Let $C \subseteq F$ be a line. By taking a general one-parameter deformation of $C$ in $W$, we construct a divisor $D$ on $W$ which is Cartier since, by Bertini, we can assume that $W$ is smooth. If the monodromy acts trivially on $N_1(F)$, then $D$ intersects $F$ into a finite union of lines in the same ruling of $C$. This implies that $D \cdot C = 0$, and hence $D$ cannot be relatively ample (or antieample) over $B$. Since on the other hand $D$ is not numerically trivial over $B$, as it intersect positively any line in the other ruling of $F$, this contradicts the fact that $\pi|W$, having singular fibers, has relative Picard number 1. Therefore the monodromy action on $N_1(F)$ cannot be trivial and must swap the two rulings. Now, the map $N_1(F) \rightarrow N_1(G)$ sends one of the extremal rays of the Mori cone $\overline{NE}(F)$ to the extremal ray $R$ of $\overline{NE}(G)$ defining the projective bundle fibration $G \rightarrow \mathbb{P}^1$. The contradiction follows by observing that the monodromy action on $N_1(G)$ must stabilize the ray $R$ since, for dimensional reasons, $G$ cannot have two distinct fibrations to $\mathbb{P}^1$. Therefore this case cannot occur, hence we conclude that $(G, \mathcal{L}|G)$ can only be either $(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$ or $(Q, \mathcal{O}_Q(1))$.

Note that, by Proposition 4.8, $\pi$ is flat. To finish the proof, we apply semi-continuity of the $\Delta$-genus along the fibers of $\tilde{\pi}$ [Fuj75, Theorem 5.2] and the classification of polarized varieties with $\Delta$-genus zero [Fuj75, Theorems 2.1 and 2.2], as in the proof of [BdFL08, Theorem 5.8]. This allows us to conclude that all fibers of $\tilde{\pi}$ are projective spaces or quadric hypersurfaces, depending of the situation. The sheaf $\pi_*\mathcal{L}$ is locally free on $Z$, the surjection $\pi^*\pi_*\mathcal{L} \rightarrow \mathcal{L}$ gives the desired linear embedding $X \hookrightarrow \mathbb{P}(\pi_*\mathcal{L})$ that gives $X$ the projective bundle or quadric fibration structure, and the surjection $\pi_*\mathcal{L} \rightarrow \pi_*\mathcal{H}$ gives the fiberwise linear embedding of $Y$ into $X$. \qed

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(T. de Fernex) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 SOUTH 1400 EAST, SALT LAKE CITY, UT 84112 (USA)

E-mail address: defernex@math.utah.edu

(C. C. Lau) SHANGHAI CENTER FOR MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, 2005 SONGHU ROAD, SHANGHAI 200438 (CHINA)

E-mail address: malccad@gmail.com