ON THE RIGIDITY FOR CONFORMALLY COMPACT EINSTEIN MANIFOLDS

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ABSTRACT. In this paper we prove that a conformally compact Einstein manifold with the round sphere as its conformal infinity has to be the hyperbolic space. We do not assume the manifolds to be spin, but our approach relies on the positive mass theorem for asymptotic flat manifolds. The proof is based on understanding of positive eigenfunctions and compactifications obtained by positive eigenfunctions.

0. Introduction

In this paper we study the rigidity problem for conformally compact Einstein manifolds with the round sphere as their conformal infinity. Quite recently there has been a great deal of interest in both physics and mathematics community in the so-called Anti-de-Sitter/Conformal Field Theory (in short AdS/CFT) correspondence. Conformally compact Einstein manifolds play an essential role in this correspondence. In mathematics it has been known for a long time that there are close connections between the geometry of Minkowski space-time, hyperbolic space and the round sphere. Notably in a seminal paper [FG] Fefferman and Graham showed this approach to be very fruitful in conformal geometry.

In establishing scalar curvature rigidity for asymptotically hyperbolic manifold as a generalization from the work of Witten [W] on positive mass theorem for asymptotically flat manifolds which are spin, in [AD], Andersson and Dahl proved that, if a conformally compact Einstein manifold with the round sphere as its conformal infinity is spin, then it has to be a hyperbolic space (please also see recent related works of X. Zhang [Z], Chruściel and Herzlich [CH], and X. Wang [Wa]). It opens an interesting question whether the spin structure is necessary to assure the rigidity. There is some progress made by Anderson in [An1].
Before we state our results, let us briefly introduce what is a conformally compact Einstein manifold. Let $X^{n+1}$ be a $n+1$ dimensional compact manifold with boundary $M^n = \partial X$. $(X, g)$ is said to be a conformally compact Einstein manifold if $\text{Ric}(g) = -ng$ and $(X, s^2 g)$ is a compact Riemannian manifold with boundary, where $s$ is a defining function of the boundary $M$. Clearly the restriction of $s^2 g$ to $TM$ is a metric $\hat{g}$ on the boundary and $\hat{g}$ rescales upon changing the defining function $s$. Thus $(M, [\hat{g}])$ is determined by $(X, g)$ and called the conformal infinity of $(X, g)$.

**Theorem 0.1.** Suppose that $(X^{n+1}, g)$ is a conformally compact Einstein manifold with the round sphere as its conformal infinity, and $3 \leq n \leq 6$. Then $(X, g)$ has to be the hyperbolic space.

One simple yet very interesting calculation leading to Theorem 0.1 is the following.

**Lemma 0.2.** Suppose that $(X, g)$ is a conformally compact Einstein manifold. And suppose that $u$ is a positive eigenfunction, i.e. $\Delta u = (n+1)u$. Then $(X, u^{-2} g)$ is with scalar curvature

\begin{equation}
R = n(n+1)(u^2 - |du|^2).
\end{equation}

Here $\Delta$ is the trace of the Hessian in metric $g$. Combining with the Bochner formula for eigenfunction $u$:

\begin{equation}
-\Delta(u^2 - |du|^2) = 2|Ddu - ug|^2,
\end{equation}

observed by Lee in [L], one may know the scalar curvature for the conformal compactification $(X, u^{-2} g)$ if one knows the asymptotic behavior of $u$ near the boundary. This turns out to be a very interesting construction for its own sake.

The paper is organized as follows. In Section 1 we will introduce notations and do some computations for the hyperbolic spaces. In Section 2, we will introduce conformally compact Einstein manifolds and relevant properties. And we will apply theory of uniformly degenerate elliptic linear PDE to solve for the eigenfunctions and their expansions. Finally in Section 3 we will introduce some conformal compactifications and prove Theorem 0.1.

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1. ON HYPERBOLIC SPACES

In this section let us recall some computations on hyperbolic spaces. We will present in the way that makes our choices of compactification in the Section 3 become natural. Meanwhile we will establish the notations for this paper in this section. First let us introduce the hyperbolic space as a submanifold in the Minkowski space-time. Namely,

\[ H^{n+1} = \{(x, t) \in \mathbb{R}^{n+1,1} : |x|^2 - t^2 = -1, t > 0\}, \]

and the hyperbolic metric \( g_H \) is induced from the standard Lorenz metric \( g_L \), i.e.

\[
(H^{n+1}, g_H) = (\mathbb{R}^{n+1,1}, \frac{(d|x|)^2}{1 + |x|^2} + |x|^2 h_0),
\]

where \( h_0 \) is the standard metric on the unit sphere. It is easily computed that the coordinate functions in \( \mathbb{R}^{n+1,1} \) are all eigenfunctions on \( H^{n+1} \) for \( g_H \), just as the coordinate functions in Euclidean space are all eigenfunctions on sphere for the round metric. Namely,

\[
\Delta x_i = (n + 1)x_i, \quad \Delta t = (n + 1)t, \quad \text{for } i = 1, \ldots, n+1
\]

where \( \Delta \) is the trace of the Hessian for the metric \( g_H \). One conformal compactification near the boundary is

\[
g_H = \rho^{-2}((dr)^2 + h_0),
\]

where

\[
\rho = 1/|x|, \quad \text{and } \rho = \sinh r.
\]

Another way to introduce hyperbolic space is the Poincaré ball

\[
(H^{n+1}, g_H) = (B^{n+1}, \frac{2}{1 - |y|^2}|dy|^2),
\]

where \( y \in B^{n+1} \) and \( B^{n+1} \) is the unit ball in \( \mathbb{R}^{n+1} \). We find that

\[
t = \frac{1 + |y|^2}{1 - |y|^2},
\]
and that \((B^{n+1}, (t+1)^{-2}g_H)\) is the Euclidean ball and \((B^{n+1}, t^{-2}g_H)\) is the round hemisphere. Finally let us introduce the hyperbolic space by the upper half space model

\[(H^{n+1}, g_H) = (R_+^{n+1}, \frac{|dz|^2}{z_{n+1}}),\]

where \(z \in R_+^{n+1}\) and \(R_+^{n+1} = \{(z_1, z_2, \ldots, z_{n+1}) \in R^{n+1} : z_{n+1} > 0\}\). We find that

\[\Delta \frac{1}{z_{n+1}} = (n + 1) \frac{1}{z_{n+1}},\]

and \((R_+^{n+1}, (\frac{1}{z_{n+1}})^{-2}g_H)\) is the flat half space, which can be considered as a partial compactification of the hyperbolic space.

Thus we observe here that the three compactifications: ball, hemisphere and half space (partial compactification), are all through conformal factors which come from eigenfunctions. Those are clearly better than the compactification (1.3) when we need to work not only near the boundary. Before we end this section let us recall the coordinate changes for the three different models for the hyperbolic space (cf. Chapter 4 in [R]).

\[x = \frac{2}{1 - |y|^2}y, \quad \left\{ \begin{array}{l} y_\alpha = \frac{2z_\alpha}{|z + e_{n+1}|^2} \\ y_{n+1} = \frac{|z|^2 - 1}{|z + e_{n+1}|^2} \end{array} \right\} \quad z_\alpha = \frac{2y_\alpha}{|y - e_{n+1}|^2}, \quad z_{n+1} = \frac{1 - |y|^2}{|y - e_{n+1}|^2},\]

where \(e_{n+1} = (0, \ldots, 0, 1)\), and derive

\[\frac{1}{z_{n+1}} = \frac{|y - e_{n+1}|^2}{1 - |y|^2} = t - x_{n+1}\]

and

\[\rho = \frac{1 - |y|^2}{2|y|}.

2. Conformally compact Einstein manifolds and the positive eigenfunctions.

Let us first define what is a conformally compact Einstein manifold. Suppose \(X^{n+1}\) is a compact \((n+1)\)-manifold with boundary \(\partial X = M^n\). A Riemannian metric
\( g \) in the interior of \( X \) is said to be \( C^{m,\alpha} \) conformally compact if \( \bar{g} = r^2 g \) extends as a \( C^{m,\alpha} \) metric on \( \bar{X} \), where \( r \) is a defining function of the boundary in the sense that: \( r > 0 \) in \( X \), \( r = 0 \) and \( dr \neq 0 \) on \( M \). Clearly defining functions are not unique. For a given defining function, the metric \( \bar{g} \) restricted to \( TM \) induces a metric \( \hat{g} \) on \( M \). \( \hat{g} \) rescales upon changing the defining function \( r \), therefore defines a conformal class \( [\hat{g}] \) on \( M \). We call \((M, [\hat{g}])\) the conformal infinity of the conformally compact manifold \((X, g)\). Conformally compact Einstein manifold \((X, g)\) is a conformally compact manifold which is Einstein, i.e. \( \text{Ric}(g) = -ng \).

The boundary regularity of the conformally compact Einstein metric is an important issue. Thanks to M. Anderson \([An]\), in 4 dimension, we know that \( C^{2,\alpha} \) would imply in Theorem 0.1 the full smoothness of the conformally compact Einstein metric. In other words, the conformally compact Einstein 4-manifold in Theorem 0.1 is automatically even analytic. Sufficient boundary regularity is assumed for our results in higher dimension to ensure the expansion in the following Lemma 2.1.

Solving a first order nonlinear PDE by the method of characteristics introduced in \([FG]\) \([GL]\), one at least has, near the boundary, as follows (cf. Lemma 5.4 in \([AD]\)).

**Lemma 2.1.** Given a conformally compact Einstein manifold with the round sphere as its conformal infinity. Then, taking the standard round metric \( h_0 \) and its associated special defining function, one has

\[
g = \rho^{-2}((dr)^2 + g_r)
\]

where

\[
g_r = h_0 + \rho^n h, \quad \text{Tr}_{h_0} h = O(\rho^n), \quad \rho = \sinh r.
\]

The choice of the defining function \( \rho \) in this lemma is different from the choice made by Fefferman and Graham in \([FG]\), particularly in Lemma 2.1 in \([Gr]\). Because of this choice the expansion of \( g_r \) becomes much nicer. And this choice comes from (1.3). One thing we learn from Section 1 is that the positive eigenfunctions seem to play a role in conformal compactifications. Therefore, let us turn our attention to study the positive eigenfunctions. We will use the theory of uniformly degenerate elliptic linear PDE on conformally compact manifolds developed in \([M]\), \([L]\) and \([GL]\). We first recall an analytic lemma with modifications for simplicity in the following Lemma 2.2 from \([M]\), \([L]\) and \([GL]\) (cf. Proposition 3.3 in \([L]\) for example).

Given a conformally compact manifold with a fixed defining function \( r \) which may be defined globally and is identical to \( r \) as in Lemma 2.1 near the boundary. We may define the space of functions

\[
C^{m,\alpha}_s(X) = \rho^s C^{m,\alpha}(X, g)
\]
where $C^{m,\alpha}(X,g)$ is defined as usual for a Riemannian manifold and $\alpha \in (0,1)$.

And

\[ (2.3) \quad \|f\|_{C^{m,\alpha}_s(X)} = \|\rho^{-s}f\|_{C^{m,\alpha}(X,g)}. \]

In our situation when the conformally compact manifold is Einstein with the round sphere as its conformal infinity, we have a coordinate near the boundary as the hyperbolic space, an annular region in $y$ coordinates (cf. please see notations in Section 1), for example.

**Lemma 2.2.** Suppose that $(X,g)$ is a conformally compact manifold. Then

\[ (2.4) \quad -\Delta + (n + 1) : C^{m+2,\alpha}_s \rightarrow C^{m,\alpha}_s \]

is an isomorphism whenever

\[ (2.5) \quad -1 < s < n + 1 \]

Next we are going to find eigenfunctions on the conformally compact Einstein manifold with the round sphere as its conformal infinity. It makes sense to consider functions $t, x_i$ for $i = 1, \ldots, n + 1$ near the boundary because we may adopt for $X$ near the boundary the same coordinate systems that we had in Section 1 for the hyperbolic space. To simplify the notation we will use $f = O(\rho^s)$ to stand for $f \in C^{2,\alpha}_s(X)$.

**Lemma 2.3.** Suppose that $(X,g)$ is a conformally compact Einstein manifold with the round sphere as its conformal infinity. Then there exist eigenfunctions $u > 0$, and $v_1, v_2, \ldots, v_{n+1}$ such that

\[ (2.6) \quad \Delta u = (n + 1)u, \quad \Delta v_i = (n + 1)v_i, \quad \text{for} \quad i = 1, 2, \ldots, n + 1 \]

and

\[ (2.7) \quad u = t + O(\rho^{n+\mu}) \quad \text{and} \quad v_i = x_i + O(\rho^{n+\mu}) \quad \text{for any} \quad \mu < 1. \]

**Proof.** First let us construct $u$ by the above Lemma 2.2. Since we are using the same coordinate systems as we did for the hyperbolic space, the only thing that is different from the hyperbolic space is the metric $g_r$. We calculate

\[ \Delta t = \frac{\rho^{n+1}}{\sqrt{\det g_r}} \partial_r (\rho^{1-n} \sqrt{\det g_r} \partial_r t) = (n + 1)t - \frac{1}{2} \Tr_{h_0} g'_r = (n + 1)t + O(\rho^{2n-1}), \]
where \( g_r = h_0 + \rho^n h \) according to Lemma 2.1, the prime means differentiating with respect to \( r \), and note also that \( t' = -1/\rho^2 \). Then, by Lemma 2.2, we know there exists a positive function \( u \) which satisfies \( \Delta u = (n + 1)u \) and
\[
u = t + O(\rho^{n+\mu}) \text{ for any } \mu < 1.
\]

Similarly, let us compute
\[
\Delta x_i = \frac{\rho^{n+1}}{\sqrt{\det g_r}} \partial_r (\rho^{1-n} \sqrt{\det g_r} \partial_r x_i) + \rho^2 \frac{1}{\sqrt{\det g_r}} \partial_\gamma (\sqrt{\det g_r g_\gamma^{\delta} \partial_\delta x_i}),
\]
where
\[
\frac{\rho^{n+1}}{\sqrt{\det g_r}} \partial_r (\rho^{1-n} \sqrt{\det g_r} \partial_r x_i) = \frac{\rho^{n+1}}{\sqrt{\det h_0}} \partial_r (\rho^{1-n} \sqrt{\det h_0} \partial_r x_i) + \frac{1}{2} \rho^2 x_i' \text{Tr} h_0 g'_r
\]
and
\[
\frac{\rho^2}{\sqrt{\det g_r}} \partial_\gamma (\sqrt{\det g_r g_\gamma^{\delta} \partial_\delta x_i})
\]
\[
= \frac{\rho^2}{\sqrt{\det g_r}} \partial_\gamma (\sqrt{\det g_r h_0^{\gamma\delta} \partial_\delta x_i}) + \frac{\rho^{n+1}}{\sqrt{\det g_r}} \partial_\gamma (\sqrt{\det g_r h^{\gamma\delta} \partial_\delta (\rho x_i)})
\]
\[
= \frac{\rho^2}{\sqrt{\det h_0}} \partial_\gamma (\sqrt{\det h_0 h_0^{\gamma\delta} \partial_\delta x_i}) + \frac{1}{2} \rho^2 h_0^{\gamma\delta} \text{Tr} h_0 \partial_\delta (g_r - h_0)
\]
\[
+ \rho^{n+1} \frac{1}{\sqrt{\det g_r}} \partial_\gamma (\sqrt{\det g_r h^{\gamma\delta} \partial_\delta (\rho x_i)}).
\]

Therefore
\[
(2.8) \quad \Delta x_i = (n + 1)x_i + O(\rho^{n+1}).
\]

Therefore, applying Lemma 2.2 again, we obtain functions \( v_i \) which solves \( \Delta v_i = (n + 1)v_i \) and
\[
v_i = x_i + O(\rho^{n+\mu}), \quad \text{for any } \mu < 1.
\]

3. Conformal compactifications.

In this section we will conformally compactify the manifolds with eigenfunctions obtained in Lemma 2.3. Before going to the proof of Theorem 0.1, let us mention that the compactification \((X, u^{-2}g)\) obtained by the eigenfunction \( u \) in Lemma 2.3 is a compact manifold with a totally geodesic standard sphere boundary and scalar
curvature $\geq n(n + 1)$. This can be easily verified similar to what we will do in the proof of following Lemma 3.2. It is worthwhile to note that the Bochner formula

$\Delta (u^2 - |\nabla u|^2) = 2|Ddu - ug|^2$

for the eigenfunction $u$ may very well invite people to show that $u^2 - |du|^2 = 1$ and $Ddu = ug$, which would quickly imply the rigidity. But, apparently, estimate (2.7) from Lemma 2.2 miserably just fails to provide the sufficient decay of $u^2 - |du|^2 - 1$.

To prove our Theorem 0.1 we consider the partial compactification corresponding to the half space. Following the notations in Section 1, we are changing into upper half space, $z$ coordinate. The metric in this coordinate becomes

$$g = \frac{1}{z^{n+1}}|dz|^2 + \rho^{n-2}h$$

in the light of (1.3) and (2.1). We first pay attention to the tail term $\rho^{n-2}h$ in $z$ coordinate.

**Lemma 3.1.** In $z$ coordinates,

$$\rho^{n-2}h = |z|^{-n-2}\tilde{h}_{ij}dz_idz_j$$

where $\tilde{h}_{ij}$ are well bounded in the sense that

$$|\tilde{h}_{ij}| + |z|\partial_z\tilde{h}_{ij} + |z|^2\partial^2_z\tilde{h}_{ij} < \infty,$$

at least for $|z|$ very large.

**Proof.** The proof is simply to perform the coordinate change. Again we follow notations used in Section 1. First, it is easily seen that $|y - e_{n+1}|$ is very small when $|z|$ is very large and vice versa. So let us restrict ourselves to the very small neighborhood of $e_{n+1}$. Then, according to (1.7),

$$\rho = \frac{1}{|x|} = \frac{1 - |y|^2}{2|y|} = \frac{z_{n+1}|y - e_{n+1}|^2}{2|y|} = \frac{2z_{n+1}}{|z + e_{n+1}|^2|y|}.$$ 

Meanwhile, in $y$ coordinates,

$$h = h_{\alpha\beta}d\phi_\alpha d\phi_\beta,$$

where $\{d\phi_\alpha\}_{\alpha=1}^n$ is an orthonormal co-frame on the unit sphere with the round metric and $h_{\alpha\beta} \in C^2$. We may write

$$d\phi_\alpha = \sum_{i=1}^{n+1} c_i^\alpha dy_i,$$
where \( \{c_i^\alpha\}_{i=1}^{n+1} \) are all well bounded. Therefore, by the transformation formula (1.7)

\[
\begin{align*}
  dy_\alpha &= \frac{2}{|z + e_{n+1}|^2} dz_\alpha - \frac{4}{|z + e_{n+1}|^4} z_\alpha (z + e_{n+1}) dz \\
  dy_{n+1} &= \frac{2}{|z + e_{n+1}|^2} \frac{2z_{n+1} + 2}{|z + e_{n+1}|^2} z dz - \frac{2(|z|^2 - 1)}{|z + e_{n+1}|^4} dz_{n+1}
\end{align*}
\]

This implies

(3.6) \[ \frac{\partial y_i}{\partial z_j} = O(|z|^{-2}). \]

Thus

\[ \rho^{n-2} h = |z|^{-n-2} \tilde{h}_{ij} dz_i dz_j, \]

where \( \tilde{h}_{ij} \) are well bounded as desired. So the lemma is proved.

We now consider the positive eigenfunction

(3.7) \[ \psi = u - v_{n+1} = t - x_{n+1} + O(\rho^{n+\mu}) = \frac{1}{z_{n+1}} + O(\rho^{n+\mu}) \text{ for any } \mu < 1 \]

in the light of (1.8) in Section 1. Again, positivity comes from a maximum principle and the boundary behavior of \( u - v_{n+1} \). More importantly we have

**Lemma 3.2.** The scalar curvature of the new metric \( g_h = \psi^{-2} g \) is nonnegative and integrable.

**Proof.** First we calculate the scalar curvature for the metric \( \psi^{-2} g \)

\[
\Delta \psi^{-\frac{n-1}{2}} = -\frac{n-1}{2} \psi^{-\frac{n+1}{2}} \Delta \psi + \frac{n^2 - 1}{4} \psi^{-\frac{n+3}{2}} |\nabla \psi|^2,
\]

that is

(3.8) \[ -\Delta \psi^{-\frac{n-1}{2}} - \frac{n^2 - 1}{4} \psi^{-\frac{n-1}{2}} = \frac{n^2 - 1}{4} (\psi^2 - |\nabla \psi|^2) \psi^{-\frac{n+3}{2}}. \]

Therefore

\[ R[\psi^{-2} g] = n(n+1)(\psi^2 - |\nabla \psi|^2). \]

Recall the Bochner formula for the eigenfunctions observed in [L] for \( \psi \)

(3.9) \[ -\Delta (\psi^2 - |\nabla \psi|^2) = 2|Dd\psi - \psi g|^2. \]
Thus to prove the scalar curvature $R[\psi^{-2}g] \geq 0$ one only needs to apply a maximum principle and to verify that $\psi^2 - |\nabla \psi|^2$ goes to zero towards the infinity. In fact we have

$$
(u - v_{n+1})^2 - |du - dv_{n+1}|^2 = O\left(\frac{\rho^{n+\mu}}{z_{n+1}}\right) = O(|y - e_{n+1}|^2 \rho^{n-1+\mu}) = O(|z|^{-n-1-\mu})
$$

in the light of (1.7), (2.7), (3.4) and (3.7). (3.10) also implies that the scalar curvature is integrable with respect to the metric $g_h$. It turns out that (3.10) is another key calculation in our approach regarding the remark made right after the formula (3.1).

For the convenience of readers we recall the definition of an asymptotically flat manifold. We simply use Definition 6.3 in [LP].

**Definition 3.3.** (Definition 6.3 [LP]) A Riemannian $(n+1)$-manifold $(M^{n+1}, g)$ is an asymptotically flat manifold of order $\tau$ if there exists a decomposition $M = M_0 \cup M_\infty$ (with $M_0$ compact) and a diffeomorphism $M_\infty \leftrightarrow \mathbb{R}^{n+1} \setminus B_R$ for some $R > 0$, satisfying, if $(M, g) = (\mathbb{R}^{n+1} \setminus B_R, g_{ij})$,

$$
g_{ij} = \delta_{ij} + O(|z|^{-\tau}), \quad \partial_k g_{ij} = O(|z|^{-\tau-1}), \quad \partial_k \partial_l g_{ij} = O(|z|^{-\tau-2})
$$

for all $i, j, k, l = 1, \ldots, n+1$, as $|z| \to \infty$ in the asymptotic coordinate chart $\mathbb{R}^{n+1} \setminus B_R$.

Also, given an asymptotically flat $(n+1)$-manifold and an asymptotic coordinate chart, one may define the mass for the asymptotically flat $(n+1)$-manifold as follows: (cf. Definition 8.2 in [LP])

$$
m(g) = \lim_{R \to \infty} \frac{1}{|S^n|} \int_{S_R} \sum_{i,j=1}^{n+1} (\partial_i g_{ij} - \partial_j g_{ii}) \frac{z_j}{|z|} d\sigma
$$

where $|S^n|$ is the volume of the unit n-sphere, if this limit exists.

**Lemma 3.4.** We may consider the doubling $(Y, G)$ of $(X, g_h)$ along the boundary in above partial compactification. Then $(Y, G)$ is at least $C^{n-1,1}$ (more precisely, $C^{n,1}$ if $n$ is even and $C^{n-1,1}$ is $n$ is odd), and is an asymptotically flat manifold of order $n - 1 + \mu$, therefore $m(G) = 0$, with integrable nonnegative scalar curvature.

**Proof.** By the above Lemma 3.2 one knows that the scalar curvature for $G$ is nonnegative. Note $G$ is $C^{2,1}$ at least, therefore its curvature tensor is well-defined.
To understand the metric $G$ better, we have

\begin{equation}
(3.11) \quad g_h = \psi^{-2} g = \psi^{-2} \frac{1}{z_{n+1}} |dz|^2 + \psi^{-2} \rho^{n-2} h.
\end{equation}

Let us first recall

$$\psi_{z_{n+1}} = 1 + z_{n+1} O(\rho^{n+\mu}),$$

then, plugging into (3.11),

\begin{align*}
g_h &= \frac{1}{(1 + z_{n+1} O(\rho^{n+\mu}))^2} |dz|^2 + \psi^{-2} \rho^{n-2} h \\
&= |dz|^2 + z_{n+1} O(\rho^{n+\mu}) |dz|^2 + z_{n+1}^2 \rho^{n-2} \frac{1}{|z + e_{n+1}|^4} \tilde{h}_{ij} dz_i dz_j.
\end{align*}

When $(z_1, \ldots, z_n)$ is fixed, from (3.4),

\begin{equation}
(3.12) \quad g_h = |dz|^2 + O(z_{n+1}^{n+1+\mu}) |dz|^2 + O(z_{n+1}^n) \tilde{h}_{ij} dz_i dz_j
\end{equation}

near the boundary, which tells us the smoothness of the metric $G$ at the boundary $z_{n+1} = 0$. Meanwhile, when $|z|$ is very large, we have, from (3.4),

\begin{equation}
(3.13) \quad g_h = |dz|^2 + O(|z|^{-(n+1+\mu)}) |dz|^2 + O(|z|^{-n}) \tilde{h}_{ij} dz_i dz_j.
\end{equation}

To check if $(Y, G)$ is an asymptotically flat metric one needs to verify that all terms $O(|z|^{-k})$ in (3.13) in fact satisfy $\partial_z O(|z|^{-k}) = O(|z|^{-k-1})$ and $\partial_z^2 O(|z|^{-k}) = O(|z|^{-k-2})$. Those indeed are true according to Lemma 2.2 for $m \geq 2$, Lemma 3.1, and (3.6). For example,

$$\partial_z f = \frac{\partial y}{\partial z} \partial_y f,$$

therefore

\begin{equation*}
\partial_z f = O(|z|^{-2}) \partial_y f = O(|z|^{-2}) O(\rho^{k-1}) = O(|z|^{-k-1})
\end{equation*}

if $f = O(\rho^k)$. Thus the lemma is proved.

**Theorem 3.5.** Suppose that $(X^{n+1}, g)$ is a conformally compact Einstein manifold with the round sphere as its conformal infinity, and $3 \leq n \leq 6$. Then $(X^{n+1}, g)$ must be the hyperbolic space.

**Proof.** This is a more or less straight consequence of the positive mass theorem of Schoen and Yau [Sc] except we need to make sure that their theorem applies to less smooth metrics as ours. By Theorem 4.2 in [Sc], for example, $(Y, G)$ has to be isometric to $R^{n+1}$, since $(Y, G)$ is an asymptotically flat manifold of order $n - 1 + \mu$...
with integrable nonnegative scalar curvature because of (3.10), and zero mass by Lemma 3.4. In the following we will point out that the positive mass theorem of Schoen and Yau indeed works for asymptotic manifolds which are at least $C^{2,1}$. First it is easily seen that Proposition 4.1 in [Sc] still holds in our situations without much modifications. Since the metric $\bar{g}$ in the proof of Proposition 4.1 in [Sc] was constructed from cutoff metric $g^{(\sigma)}$ which is Euclidean near the infinity, one may assume the asymptotically flat metrics are $C^\infty$ near the infinity to show that the mass have to be nonnegative following the minimal hypersurface argument in [Sc] (dimension is assumed to be less than and equal to seven in [Sc]). Then to show that the mass is zero implies that the asymptotically flat manifold has to be Euclidean one may follow the proof of Lemma 10.7 in [LP] (see also Lemma 3 and Proposition 3 in [Sc1]). The key is to show that the mass of $G$ is zero implies that $G$ is Ricci flat, in the light of Proposition 10.2 in [LP]. But in the argument given on page 84-85 in [LP] for Lemma 10.7 and in the argument given on page 80-81 in [LP] for Proposition 10.2 only derivatives of the metrics up to the second order were involved. And the minimality of the zero mass still holds since we just showed that the mass has to be nonnegative, for dimension less than 8. Moreover the variational formula (8.11) in [LP] certainly holds for metrics of $C^{2,1}$. Thus with little modifications Lemma 10.7 in [LP] holds in our cases.

**Remark 3.6.** The application of positive mass theorem to the doubling manifolds has been used by Escobar in [Es] in his proof of the Yamabe problem for manifolds with boundary. In the appendix of the paper [Es] he explained how one can apply the positive mass theorem to the doubling manifold which is asymptotically flat.

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