REVIEW

Theory of decoherence of \( N \)-state quantum systems in the Born–Markov approximation

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Received 10 May 2010
Accepted for publication 7 June 2010
Published 3 August 2010
Online at stacks.iop.org/ANSN/1/023001

Abstract

We present a systematic formalism for the computation of the density matrix of an \( N \)-state quantum system in the presence of classical noise or a coupling to the environment. In this formalism, the density matrix of the system is given as an expansion in the generators of the SU(\( N \)) group with real coefficients. This leads to a system of master equations. The parameters in these equations may be approximately expressed in terms of the components of the Redfield tensor, when the Born and Markov approximations are valid. The general form of the solution of the system of master equations is established. All relaxation and dephasing rates are then very simply expressed as eigenvalues of a certain matrix. This gives the formulation its simplicity and makes it uniquely suitable for numerical computation. The spectral representation of the components of the Redfield tensor is derived in the case when the environment is a harmonic oscillator bath in thermal equilibrium. Beyond the Born approximation, the decoherence of the system is determined by the Lindblad formula for the Liouvillian superoperator. The Lindblad formulae of some models of multi-state quantum systems are also presented.

Keywords: decoherence, qubit, rate equation, relaxation, dephasing

Classification numbers: 2.00, 3.00, 3.01

1. Introduction

The prospect of quantum control has caused activity in several different subfields of physics. The goal is to maintain a quantum system with many degrees of freedom in a coherent, controllable many-body state. In the case of a quantum computer, this would enable the solution of certain very difficult computational problems. In particular, successful quantum computation (QC) would mean a revolution in the areas of cryptography \([1]\) and data-base searching \([2]\). But more generally, the control of a coherent quantum system would mean a great advance in general technical capabilities: it would open up a new field of science and technology.

One of the chief difficulties is that no quantum system can be entirely isolated from the environment, and the random perturbations arising from this coupling will ultimately destroy the necessary quantum coherence. In practice, one hopes (roughly speaking) to reduce the coupling to the point where the time scale \( s \) of the control operations and all quantum decoherence times \( \tau_c \) satisfy \( \tau_c \gg s \). The challenge for theory is to compute \( \tau_c \) for reasonable models of the environment and the coupling to the system. In general, there are many different coherence times for an \( N \)-state system, and part of the purpose of this paper is to properly count and categorize these times.
The chief tool for the computation of decoherence times goes by the name of Bloch–Redfield theory [3–5]. The main aim of this work is to give a compact and systematic formulation of Bloch–Redfield theory for an \( N \)-state quantum system using the properties of the SU(\( N \)) group of transformations that act on the system. We illustrate its application to the familiar case of a two-state system in order to connect with well-known physics. Bloch–Redfield theory relies on an approximation that is valid when the system–environment coupling is small in a sense that we will make precise, and when the correlation times of the environment are short compared with the decoherence times of the system. This approximation is usually called the Born–Markov approximation, and it has been summarized in numerous publications, but usually in less general formulations.

Our aim in this paper is to provide a general and systematic presentation of decoherence for an \( N \)-state quantum system. To do this, it is necessary to work in SU(\( N \)) Liouville space rather than the usual Hilbert state space, and to derive master equations for this rather general setting. The point is that master equations are almost always introduced either to describe a two-state system (the \( N = 2 \) case, in our language) or a harmonic oscillator (the \( N = \infty \) case), though sometimes special basis sets for other \( N \) are introduced in atomic physics problems. No systematic treatment for general \( N \) has been given.

In section 2, we describe how to formulate the density matrix description of the quantum mechanics of an isolated system in this general algebraic setting, since it is not familiar to most physicists. In section 3, we give the simplest generalization to an open system—a system subject to classical noise. In section 4, the more usual case of a system in contact with the environment is treated using the Markov approximation [6, 7]. Section 5 gives the results when perturbation theory is applicable to the system–environment coupling—the Bloch–Redfield formalism. Section 6 contains a presentation on the general solution of the system of rate equations in the Bloch–Redfield formalism. In section 7, we give the explicit solution for \( N = 2 \). Section 8 is devoted to three numerical examples of current interest. In section 9, we discuss the generalizations to some typical multi-state systems. Section 10 is the conclusion.

2. Quantum mechanics in SU(\( N \)) Liouville space

We now present quantum mechanics in Liouville space. This turns out to be very convenient when there is classical uncertainty about the quantum state of the system, i.e. we must work with density matrices rather than wave functions. The beauty of ordinary quantum mechanics in Hilbert space is that it provides a geometrical picture, so that unitary time evolution is a generalized rotation and orthogonality can be used to great effect. When we go to a density matrix formulation, this geometrical interpretation is usually lost. Working in Liouville space partially restores it, and this greatly simplifies and systematizes calculations.

Let the \( N \)-dimensional Hilbert state space of our system be denoted by \( V_S \). All traceless Hermitian operators \( O \) in \( V_S \) can be written as linear combinations of the \( N^2 - 1 \) generators \( \Gamma_A \) of SU(\( N \)) with real coefficients \( O_A \):

\[
O = \sum_{A=1}^{N^2-1} O_A \Gamma_A.
\]

where the index \( A \) runs from 1 to \( N^2 - 1 \) and the \( N \times N \) Hermitian matrices \( \Gamma_A \) satisfy

\[
\text{Tr} \Gamma_A = 0, \quad \text{Tr}(\Gamma_A \Gamma_B) = 2\delta_{AB},
\]

\[
[\Gamma_A, \Gamma_B] = i \sum_{C=1}^{N^2-1} f_{ABC} \Gamma_C, \quad f_{ABC} = f_{BCA}, \quad f_{BCA} = f_{ABC}.
\]

The \( f_{ABC} \) are the structure constants of the SU(\( N \)) group. For \( N = 2 \), the usual choice is the Pauli matrices: \( \Gamma_1 = \sigma_x = \sigma_1, \quad \Gamma_2 = \sigma_y = \sigma_2 \) and \( \Gamma_3 = \sigma_z = \sigma_3 \), while for \( N = 3 \), the usual choice is the Gell–Mann matrices [8]. For general \( N \), systematic ways of generating the \( \Gamma_A \) are known. \( d_{ABC} \) is a totally symmetric ‘tensor’ that also characterizes SU(\( N \)).

There are two key points. The first is that the density matrix \( \rho \) is a Hermitian matrix with unit trace and can therefore be written as

\[
\rho = \frac{1}{N} I + \sum_{A=1}^{N^2-1} \rho_A \Gamma_A,
\]

where \( I \) is the unit matrix and \( N^2 - 1 \) real numbers \( \rho_A \) can be thought of as the components of a vector. The second point is that the operation \( \{ \Gamma, \Gamma' \} = \frac{1}{2} \text{Tr}(\Gamma \Gamma') \) furnishes this vector space with an inner product (called the Hilbert–Schmidt inner product) so that the set of all density matrices is a real Hilbert space with \( N^2 - 1 \) dimensions, which has an orthonormal basis given by the \( \Gamma_A \). We call this the Liouville space \( L_S \).

The vector space structure means that we can easily find the coefficients \( \rho_A \):

\[
\rho_A = \frac{1}{2} \text{Tr}[\Gamma_A \rho] = \frac{1}{2} \sum_{\mu, \nu=1}^{N} (\Gamma_A)_{\mu \nu} \rho_{\mu \nu}.
\]

The second expression makes explicit reference to a basis in \( V_S \), often chosen as the basis of the energy eigenstates in \( V_S \).

The evolution of a quantum system is now viewed as a trajectory in \( L_S \) (or, equivalently, in the Lie algebra of SU(\( N \))). For an isolated system, this motion is governed by the Hamiltonian \( H \) and the von Neumann equation:

\[
\frac{i}{\hbar} \frac{d\rho(t)}{dt} = [H, \rho(t)].
\]

This is qualitatively similar to the evolution of a pure state, which is a trajectory in \( V_S \) governed by the Schrödinger equation.

To obtain the von Neumann equation in component notation, we write the Hamiltonian as

\[
H = \sum_{A=1}^{N^2-1} H_A \Gamma_A
\]
and we then get
\[ \frac{d\rho_A(t)}{dt} = \frac{i}{2} \text{Tr} \left[ H, \rho(t) \right] = \frac{i}{2} \sum_{B,C=1}^{N^2-1} H_B \rho_C \text{Tr} \left[ \Gamma_A \Gamma_D \right] \]
\[ = \frac{i}{2} \sum_{B,C=1}^{N^2-1} f_{BCD}^{} H_B \rho_C \text{Tr} \left[ \Gamma_A \Gamma_D \right] \]
\[ = \frac{i}{2} \sum_{B,C=1}^{N^2-1} f_{BCD}^{} H_B \rho_C \text{Tr} \]
\[ \times \left( \frac{1}{2} \sum_{E=1}^{N^2-1} f_{ADE}^{} \Gamma_E^{} + \frac{2}{N} \delta_{AB} I + \sum_{E=1}^{N^2-1} d_{ADE} \Gamma_E^{} \right), \]
which reduces to a set of \( N^2 - 1 \) real linear differential equations:
\[ \frac{d\rho_A(t)}{dt} = \sum_{B,C=1}^{N^2-1} f_{BCA}^{} H_B \rho_C = \sum_{B=1}^{N^2-1} M_{AB} \rho_B. \]

This equation has the formal solution
\[ \rho_A(t) = \exp(M_{AB}t) \rho_B(t = 0) = T_{AB} \rho_B(t = 0). \]

The matrix \( M_{AB} \) that acts in the space \( L_S \) is antisymmetric for the case of an isolated system, and the evolution matrix \( T_{AB} = \exp(M_{AB}t) \) is therefore orthogonal. The set \( \rho_A \) is a vector in \( L_S \) and its length is preserved in the evolution, which is simply a time-dependent rotation. The corresponding evolution of the wavefunction in \( V_S \) is unitary. Matrices such as \( M_{AC} \) and \( T_{AC} \) that act on the vector space \( L_S \) are known generally as ‘superoperators’. Those such as \( T_{AC} \) that guide the evolution of \( \rho \) are sometimes called ‘quantum dynamical maps’. Once we have the \( \rho_A \), we can always reconstruct the density matrix in any desired basis.

The eigenvalues of \( M_{AB} \) are purely imaginary, each corresponding to an oscillation frequency. The eigenvalues of \( T_{AB} \) are phases. Since \( T_{AB} \) is real, these eigenvalues come in complex conjugate pairs. This pattern of eigenvalues corresponds to purely oscillatory dynamics with no damping. These dynamics hold for isolated quantum systems, but that will, of course, cease to be the case when coupling to the environment is considered.

One virtue of this approach to quantum mechanics is that the real coefficients \( \rho_A \) have a direct physical interpretation. The expectation value of the Hermitian operator \( \Gamma_A \) is \( \text{Tr}(\rho_G \Gamma_A) = 2\rho_A \).

For \( N = 2 \), a simple example is \( \rho_1 = \text{Tr}(\rho \sigma_x) = \frac{1}{2} \), half the \( x \)-component of the spin. Therefore, the coefficients \( \rho_A \) themselves are already physical observables, in contrast to the usual formulation of quantum mechanics where the components of a vector in Hilbert space are probability amplitudes. The complex numbers that yield interference in quantum systems are present in the \( \Gamma_A \), but the equations of motion for the components are purely real.

Some rather less pleasant differences between the formulations in \( V_S \) and \( L_S \) now emerge. All vectors in \( V_S \) are associated with a pure state. However, not all vectors in \( L_S \) are associated with a physical density matrix. The probability interpretation for \( \rho \) implies that the eigenvalues of \( \rho \) lie between 0 and 1 inclusive. (More specifically, \( \rho \) is positive semi-definite and has unit trace.) This imposes restrictions on \( \rho_A \), the components of \( \rho \). For \( N = 2 \), a necessary and sufficient condition for positivity is that
\[ \sum_{A=1}^{3} \rho_A^2 \leq \frac{1}{4} \]
and the density matrix is a pure state when the equality holds. Thus, the set of admissible density matrices is a sphere in \( L_S \).

For \( N = 3 \), the situation is more complicated. The set of admissible density matrices is not a sphere. The necessary and sufficient conditions for positivity are
\[ \sum_{A=1}^{8} \rho_A^2 \leq \frac{1}{3} \]
and
\[ \sum_{B,A,C=1}^{8} d_{ABC} \rho_A \rho_B \rho_C \leq 1. \]
The subset of pure states satisfies the conditions
\[ \sum_{A=1}^{8} \rho_A^2 = \frac{1}{3} \]
and
\[ \frac{3}{2} \sum_{B,A,C=1}^{8} d_{ABC} \rho_A \rho_B = \rho_C. \]
The cubic nonlinearity in the second equation precludes a simple geometric description of the allowable ‘physical’ region of \( L_S \) for \( N = 3 \). For general \( N \), the situation is worse: there are \( N - 1 \) positivity conditions, one for each Casimir invariant. The allowable region of \( L_S \) is always convex, since for any two density matrices \( \rho_1 \) and \( \rho_2 \), \( \rho(\lambda) = \lambda \rho_1 + (1 - \lambda) \rho_2 \) is also a density matrix for any \( 0 \leq \lambda \leq 1 \). Unfortunately, this convex set has a complicated shape for \( N > 2 \).

3. Evolution in the presence of classical noise

For quantum systems that are not isolated but are subject to external influences from a bath that is not measured, the evolution is more general. Quantum information can leak out to external influences from a bath that is not measured, the evolution matrix \( T_{AB} \) is no longer orthogonal. This process cannot be described in \( V_S \).

Our first example of this will be a system subject to random classical noise [9]. The treatment is similar to that in the previous section, except that the Hamiltonian \( H \) is now a random variable with probability distribution \( P(H) \). Let \( \rho(H), T(H), \) etc, denote quantities for a fixed \( H \). The steps of section 2 are repeated and we once again obtain
\[ \rho_A(H, t) = \sum_{B=1}^{N^2-1} T_{AB}(H) \rho_B(t = 0). \]

\( T_{AB}(H) \) is orthogonal. Averaging this equation, we find the actual density matrix components
\[ \rho_A(t) = \int dH P(H) T_{AB}(H) \rho_B(t = 0) = T_{AB} \rho_B(t = 0). \]

\( T_{AB} \), which is the average of \( T_{AB}(H) \), is not orthogonal, but it must still map the allowable region into itself. A necessary condition that \( T_{AB} \) must satisfy in order that it may describe
a physical evolution is that its eigenvalues must lie on or in the interior of the unit circle in the complex plane. If this is so, then it will map the convex region of allowable density matrices into itself. It is easy to show that it satisfies this. Let $\psi$ be any vector in $L_2$. Then

$$
\langle\langle \psi | T | \psi \rangle\rangle = \left| \psi \right| \int dH P(H) T(H) | \psi \rangle
$$

$$
= \left| \int \left( dH P(H) \langle\langle \psi | T | \psi \rangle\rangle \right) \right|
$$

$$
\leq \left| \int dH P(H) \langle\langle \psi | T | \psi \rangle\rangle \right|
$$

$$
= \left| \int dH P(H) \langle\langle \psi | T | \psi \rangle\rangle = | \langle\langle \psi | \psi \rangle\rangle \right|
$$

So the eigenvalues of the real $(N^2 - 1) \times (N^2 - 1)$ matrix $T_{AB} = \exp(M_{AB})$ are complex numbers with modulus less than 1. They may be written as $\exp(-\lambda_n t)$, $|n| = 1, 2, \ldots, N^2 - 1$. Thus the positive number $\text{Re}(\lambda_n)$ is a rate, while $|\text{Im}(\lambda_n)|$ is an oscillation frequency. If $|\text{Im}(\lambda_n)| = 0$ for some $n$, then the mode is purely relaxational and $\text{Re}(\lambda_n)$ is a relaxation rate. If $|\text{Im}(\lambda_n)| \neq 0$, then $\text{Re}(\lambda_n)$ is a dephasing rate. $\rho(t = 0)$ may decompose into eigenvectors and thus the preparation of the system will determine which of the oscillation frequencies and rates are realized in an experiment.

Let us give a simple example of a system that undergoes non-unitary evolution in the presence of classical noise. A spin $1/2$ particle has its spin pointed in the $x$-direction at time $t = 0$. It finds itself in a steady magnetic field $B$ in the $z$-direction but the strength of this field is uncertain. The probability distribution for the field is $P(B)$. This is the case $N = 2$, and we take $\Gamma_1 = \sigma_x, \Gamma_2 = \sigma_y, \Gamma_3 = \sigma_z$ so the Hamiltonian is

$$
H = \frac{1}{2} B \sigma_x, \quad H_1 = 0, \quad H_2 = 0, \quad H_3 = -\frac{1}{2} B,
$$

$$
M_{AB} = \sum_{C=1}^{N^2-1} \delta_{ABC} H_C = \frac{B}{2} \delta_{AB}, \quad (12)
$$

$$
M_{12} = -\frac{B}{2}, \quad M_{21} = \frac{B}{2}.
$$

All other elements of the $3 \times 3$ matrix $M$ are zero. Exponentiating and averaging, we find

$$
T = \begin{pmatrix}
\cos B t / 2 & \sin B t / 2 & 0 \\
-\sin B t / 2 & \cos B t / 2 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (13)
$$

Here, the overbar indicates the average with respect to $P(B)$. The density matrix at $t = 0$ is

$$
\rho(t = 0) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} I + \frac{1}{2} \sigma_x.
$$

$$
\rho_1(t = 0) = \frac{1}{2}, \quad \rho_2(t = 0) = 0, \quad \rho_3(t = 0) = 0.
$$

Using $\rho(t) = T \rho(t = 0)$, we find

$$
\rho_1(t) = \frac{1}{2} \cos B t / 2, \quad \rho_2(t) = -\frac{1}{2} \sin B t / 2.
$$

$$
\rho(t) = \frac{1}{2} I + \frac{1}{2} \sigma_x \cos B t / 2 - \frac{1}{2} \sigma_y \sin B t / 2
$$

$$
= \frac{1}{2} \begin{pmatrix}
\cos B t / 2 & -\sin B t / 2 & 0 \\
\sin B t / 2 & \cos B t / 2 & 0 \\
0 & 0 & 1
\end{pmatrix} \cos B t / 2 + i \sin B t / 2.
$$

(14)

To make further progress, we must specify $P(B)$. We take $B(t) = B_0 + b(t)$, where $B_0$ is a steady field and $b(t)$ is a noise field with zero average. We also assume that $b(t)$ is Gaussian white noise with $\bar{b}(T)b(T) = B_0^2 \delta(T - t')$. Then the average is simple [10]: one has $\bar{e}^{i B t} = e^{i B_0 t} e^{-t / T}$; with $1 / T = B_0^2 / 2$. This is a standard result for the dephasing time $T_2$ [11]. The eigenvalues of $M$ are $\lambda_1 = -i B_0 + 1 / T_2$, $\lambda_2 = i B_0 + 1 / T_2$, $\lambda_3 = 0$ and the state of the system is $L_S$, which is given by $\rho(t = 0) = e^{-t / T_2} \cos B_0 t$, $\rho_2(t) = e^{-t / T_2} \sin B_0 t$. $\rho_3(t = 0) = 0$. Reconstructing $\rho$, we find

$$
\rho(t) = \frac{1}{2} \begin{pmatrix}
e^{i B_0 t} e^{-t / T_2} & 0 \\
e^{-i B_0 t} e^{-t / T_2} & 1
\end{pmatrix}.
$$

(15)

4. Master equation for reduced density matrix in Markov approximation

Consider an $N$-state quantum system interacting with the environment. The total physical system consists of two interacting subsystems: one of them is the given $N$-state subsystem and other one is the environment. Its total Hamiltonian $H_{tot}$ can be written in the form

$$
H_{tot} = H_S + H_E + H_{int}, \quad (16)
$$

where $H_S$ is Hamiltonian of the $N$-state quantum system without interaction with the environment, $H_E$ is that of the environment and $H_{int}$ is the Hamiltonian of the interaction between the two above subsystems.

It is natural and convenient to use each product of two basis vectors of the Hilbert spaces of the two above-mentioned subsystems as a basis vector of the Hilbert space of the total physical system. The trace of its density matrix $\rho_{tot}$ over the pair of indices labeling state vectors of the environment is an $N \times N$ matrix $\rho$ called reduced density matrix of a given $N$-state quantum system in the presence of its interaction with the environment. From the von Neumann equation for $\rho_{tot}$

$$
\frac{d\rho_{tot}(t)}{dt} = [H_{tot}, \rho_{tot}(t)], \quad (17)
$$

it follows a system of integro-differential equations called master equations, each of which expresses a time derivative of a matrix element of $\rho$ at a time moment $t > 0$ in the form of the sum of a linear combination of matrix element of $\rho$ at the same time moment $t$ and an integral of another linear combination of elements of $\rho$ with respect to the time variable in the interval from 0 to $t$: the quantum dynamical equations for the reduced density matrix is non-Markovian. However, if the interaction of an $N$-state quantum system with the environment is weak and the variation in the physical fields in the environment is slow, then one can assume the Markov approximation [6, 7]: replace values of matrix element of $\rho$ in

Adv. Nat. Sci.: Nanosci. Nanotechnol. 1 (2010) 020001
R. Joynt et al
the integral from 0 to $t$ by their values at $t$ and subsequently extend the integral to infinity. In this approximation, the system of master equations becomes that of linear differential equations for the elements of reduced density matrix $\rho$, which can be written in the form

$$\frac{d\rho}{dt} = -i[H, \rho] + L\rho$$

(18)

with some linear operator $L$ called the Liouvillian superoperator. This operator consists of two parts:

$$L\rho = L^{(1)}\rho + L^{(2)}\rho,$$

(19)

one of which, $L^{(1)}\rho$, is the effect of the renormalization of energy levels leading to the frequency shifts (Lamb shift), while the other part, $L^{(2)}\rho$, is a completely positive operator describing the dissipative and dephasing actions of the environment. $L^{(1)}\rho$ can be expressed as the addition of a new term $\delta H_S$ to the Hamiltonian

$$L^{(1)}\rho = -i[\delta H_S, \rho].$$

(20)

$\delta H_S$ has the following general form,

$$\delta H_S = \sum_{A=1}^{N^2-1} h_A \Gamma_A,$$

(21)

with real constants $h_A$. It was rigorously proved by Gorini et al. [12] that $L^{(2)}\rho$ has the general form

$$L^{(2)}\rho = \frac{1}{2} \sum_{A,B=1}^{N^2-1} \xi_{AB}[[\Gamma_A, \rho \Gamma_B] + [\Gamma_A \rho, \Gamma_B]],$$

(22)

where $\xi_{AB}$ are elements of a $(N^2 - 1) \times (N^2 - 1)$ positive matrix. Each constant $\xi_{AB}$ can be divided into the symmetric and antisymmetric parts,

$$\xi_{AB} = \xi_{AB}^{(s)} + i \xi_{AB}^{(a)}$$

(23)

with real constants

$$\xi_{AB}^{(s)} = -\xi_{BA}^{(s)} \quad \text{and} \quad \xi_{AB}^{(a)} = \xi_{BA}^{(a)}.$$

Matrix $L^{(2)}\rho$ can be represented in the form

$$L^{(2)}\rho = \sum_{A=1}^{N^2-1} (L^{(2)}\rho)_{A}\Gamma_A.$$

(24)

From its general formula (22) and expression (3) of $\rho$, it follows that

$$[L^{(2)}\rho(t)]_A = -\sum_{B=1}^{N^2-1} \lambda_{AB} \rho_B(t) - \varphi_A,$$

(25)

with following expressions for real constants $\lambda_{AB}$ and $\varphi_A$,

$$\lambda_{AB} = \frac{1}{2} \sum_{C,D,E=1}^{N^2-1} \left\{ f_{DCE} f_{BEC} f_{AEC} f_{BDC} \xi_{DE}^{(s)} + (f_{BEC} f_{DCA} - f_{BDC} f_{ECA} + 2 f_{DEC} f_{CBA}) \xi_{DE}^{(a)} \right\},$$

and

$$\varphi_A = \frac{1}{N} \sum_{D,E=1}^{N^2-1} f_{DEA} \xi_{DE}^{(a)}.$$
chosen so as to have vanishing statistical average over all states of the environment at a given temperature $T$, 
\[
(K_{\mu \nu})_\beta = 0,
\]
\[
\beta = 1/k_B T, \quad k_B \text{ being the Boltzmann constant. (Other ensembles for the environment can, of course, be considered, but we shall focus on the thermal equilibrium distribution.)}
\]
We work in the Markov approximation so that we have equation (18). We now make the additional approximation that the matrix elements of $K$ are small in the sense that $|K| \tau \ll 1$, where $\tau$ is the longest environmental correlation time. We employ second-order perturbation theory with respect to the interaction Hamiltonian. The action of $L$ on the reduced density matrix is expressed by the Redfield formula, 
\[
(L\rho)_{\mu \nu} = -\sum_{\sigma, \tau=1}^N R_{\mu \nu \sigma \tau} \rho_{\sigma \tau},
\]
(35)
with the Redfield tensor determined by the following relations:
\[
R_{\mu \nu \tau \sigma} = \delta_{\nu \tau} \sum_{\lambda=1}^N \kappa^{(e)}_{\mu \lambda \sigma} + \delta_{\sigma \tau} \sum_{\lambda=1}^N \kappa^{(-)}_{\mu \lambda \nu} - \kappa^{(-)}_{\lambda \tau \sigma},
\]
\[
\kappa^{(e)}_{\mu \sigma \nu \tau} = \int_0^\infty d\tau e^{-iE_{\mu \sigma}(\tau)} (K_{\mu \sigma}(\tau) K_{\nu \tau}(0))_\beta,
\]
(36)
\[
\kappa^{(-)}_{\mu \sigma \nu \tau} = \int_0^\infty d\tau e^{-iE_{\mu \sigma}(\tau)} (K_{\mu \sigma}(0) K_{\nu \tau}(\tau))_\beta,
\]
(37)
where $E_{\mu \sigma}$ is the difference in the energies of the levels $'\mu'$ and $'\sigma'$, $E_{\mu} = E_{\mu} - E_{\sigma}$, $K_{\mu \sigma}(t)$ is the operator $K_{\mu \sigma}$ in the interaction picture
\[
K_{\mu \sigma}(t) = e^{iH t} K_{\mu \sigma} e^{-iH t}.
\]
(38)
and $(K_{\mu \sigma}(t) K_{\nu \tau}(t'))_\beta$ denotes the statistical average of the product $K_{\mu \sigma}(t) K_{\nu \tau}(t')$ over all states of the environment at the given temperature. Since the interaction Hamiltonian is a Hermitian operator, the operators $K_{\mu \sigma}$ must satisfy the condition 
\[
(K_{\mu \sigma})^\dagger = K_{\mu \sigma}.
\]
(39)
From this condition, it is straightforward to derive the following general relations for the matrix elements $\kappa^{(e)}_{\mu \sigma \nu \tau}$ and $\kappa^{(-)}_{\mu \sigma \nu \tau}$
\[
(\kappa^{(e)}_{\mu \sigma \nu \tau})^\dagger = \kappa^{(-)}_{\mu \tau \sigma \nu},
\]
(40)
and the corresponding Hermiticity property of the Redfield tensor:
\[
(R_{\nu \tau \sigma \mu})^\dagger = R_{\mu \nu \sigma \tau}.
\]
(41)
Moreover, from expression (36), we derive also the relation
\[
\sum_{\mu=1}^N R_{\mu \mu \nu \tau} = 0,
\]
(42)
which implies that $\text{Tr}(\rho(t))$ is a constant,
\[
\frac{d}{dt} \text{Tr}(\rho(t)) = 0,
\]
(43)
as desired.

These formulae are general. For those familiar with thermal bosonic baths, it may be helpful to consider a simple example when the environment is a harmonic oscillator bath in thermal equilibrium and the Hamiltonian of the interaction of the $N$-level system with the environment has the expression
\[
H_{\text{int}} = \sum_{\mu, \nu = 1}^N \langle \mu | [g_{\mu \nu}^a b_\nu + (g_{\nu \mu}^a)^* b_\sigma^\dagger] | \nu \rangle,
\]
(44)
where $b_\nu$ and $b_\sigma^\dagger$ are the boson-like destruction and creation operators for the harmonic oscillators in the bath, $a$ denotes the set of quantum numbers of the harmonic oscillator with frequency $\omega_a$, and $g_{\mu \nu}^a$ are the corresponding coupling constants. In this case, the components of the Redfield tensor have the following spectral representation:
\[
R_{\nu \tau \sigma \mu} = i \int d\omega \left\{ \delta_{\nu \tau} \sum_{\lambda=1}^N \left[ I_{\lambda \lambda \lambda \lambda}(\omega) + J_{\lambda \lambda \lambda \lambda}(\omega) \right] \frac{\rho(\lambda \lambda \lambda \lambda)}{\omega_{\lambda \sigma} + \omega - i\epsilon} - \delta_{\sigma \tau} \sum_{\lambda=1}^N \left[ I_{\lambda \lambda \lambda \lambda}(\omega) + J_{\lambda \lambda \lambda \lambda}(\omega) \right] \frac{\rho(\lambda \lambda \lambda \lambda)}{\omega_{\lambda \sigma} + \omega + i\epsilon} \right\},
\]
(45)
and with the spectral densities
\[
I_{\lambda \lambda \lambda \lambda}(\omega) = \sum_{\lambda=1}^N (1 + n_{\omega_a}) g_{\mu \nu}^a g_{\mu \nu}^a g_{\nu \mu}^a \delta(\omega - \omega_a),
\]
(46)
\[
J_{\lambda \lambda \lambda \lambda}(\omega) = \sum_{\lambda=1}^N n_{\omega_a} g_{\mu \nu}^a g_{\mu \nu}^a g_{\nu \mu}^a \delta(\omega + \omega_a).
\]
Here, $n_{\omega_a} = [1 + \exp(\omega_a / k_B T)]^{-1}$ is the Bose distribution function.

We now recast these equations in component form in the Liouville space $L_\lambda$. Having chosen the $\Gamma_A$ coordinate system, we rewrite equations (18) with the Liouvillian superoperator (35) as
\[
\frac{d\rho_A(t)}{dt} = -i[H_S, \rho(t)]_A + [L\rho(t)]_A,
\]
(47)
where
\[
[H_S, \rho(t)]_A = \sum_{B, C=1}^{N^2-1} f_{BCA} E_B \rho_C
\]
(48)
and
\[
[L\rho(t)]_A = -\sum_{B=1}^{N^2-1} \alpha_{AB} \rho_B(t) - \frac{1}{2} \alpha_{A0},
\]
(49)
where
\[
\alpha_{AB} = \frac{1}{2} \sum_{\mu, \nu=1}^N \sum_{\lambda, \tau=1}^N (\Gamma_A)_{\mu \nu} R_{\mu \nu \tau \sigma} (\Gamma_B)_{\lambda \sigma \tau}
\]
(50)
and
\[
\alpha_{A0} = \frac{1}{N} \sum_{\mu, \nu=1}^N \sum_{\lambda, \tau=1}^N (\Gamma_A)_{\mu \nu} R_{\mu \nu \tau \sigma}.
\]
(51)
From the Hermiticity property (41) of the Redfield tensor, it follows that all coefficients $\alpha_{AB}$ are real constants. This leads finally to the set of inhomogeneous linear differential equations for $N^2 - 1$ functions $\rho_A(t)$:

$$\frac{d\rho_A(t)}{dt} = \sum_{B,C=1}^{N^2-1} f_{ABC} E_B \rho_C(t) - \sum_{B=1}^{N^2-1} \alpha_{AB} \rho_B(t) - \frac{1}{2} \alpha_{A0}. $$

(52)

These equations have the same form as those derived in the previous section for the Lindblad equation. Note that

$$H_A = E_A + h_A $$

and

$$\alpha_{AB} = \lambda_{AB} + \sum_{C=1}^{N^2-1} f_{ABC} h_C, \quad \alpha_{A0} = 2\varphi_A. $$

(53)

(54)

Formula (54) together with equations (26), (27) and (50), (51) mean that there exist relationships between constants $h_A$, $\xi_{AB}$ in formulae (21), (22) and components $R_{n\nu\sigma}$ of the Redfield tensor. In the sequel, we shall derive explicitly these relationships for the case $N = 2$.

**6. General solution and physical interpretation**

Denote by $\rho_A^\infty$ the asymptotic limit of $\rho_A(t)$ at $t \to \infty$, $A = 1, 2, \ldots, N^2 - 1$,

$$\rho_A^\infty = \lim_{t \to \infty} \rho_A(t). $$

(55)

They are determined by the system of algebraic equations

$$\sum_{B,C=1}^{N^2-1} f_{ABC} E_B \rho_C^\infty = \sum_{B=1}^{N^2-1} \alpha_{AB} \rho_B^\infty + \varphi_A. $$

(56)

$\rho_A^\infty$ is the equilibrium state of the system, determined by its own dynamics on the RHS of this equation, and the coupling to the environment on the RHS. As long as some of the constants $\varphi_A$ do not vanish, we have a non-trivial equilibrium state, i.e. $\rho_A^\infty \neq 0$ for some $A$.

Introducing new functions

$$p_A(t) = \rho_A(t) - \rho_A^\infty, \quad A = 1, 2, \ldots, N^2 - 1, $$

(57)

which tend to zero in the limit $t \to \infty$, we derive the master equations for these new functions:

$$\frac{dp_A(t)}{dt} = \sum_{B,C=1}^{N^2-1} f_{ABC} E_B p_C(t) - \sum_{B=1}^{N^2-1} \alpha_{AB} p_B(t). $$

(58)

The system of linear equations (58) is easily solved by the Laplace transform. Define

$$P_A(z) = \int_0^\infty dt e^{-zt} p_A(t). $$

(59)

These functions satisfy the following system of linear equations:

$$zP_A(z) - p_A(0) = \frac{1}{2} \sum_{B,C=1}^{N^2-1} f_{ABC} E_B P_C(z) - \sum_{B=1}^{N^2-1} \alpha_{AB} P_B(z) - \sum_{B=1}^{N^2-1} M_{AB} P_B(z), $$

(60)

where

$$M_{AB} = \alpha_{AB} - \frac{1}{2} \sum_{C=1}^{N^2-1} f_{ACB} E_C. $$

(61)

Let the matrix $M$ be diagonalized by a transformation $R : M = DR^{-1}$, where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{N^2-1})$. The solution of these equations is

$$P_A(z) = \sum_{n=1}^{N^2-1} \frac{K_A^{(n)}(\lambda_n)}{z - \lambda_n}. $$

(62)

where the coefficients $K_A^{(n)}$ are related to the initial conditions by

$$K_A^{(n)} = \sum_{B,C} R_{nB}(R^{-1})_{hC} p_C(0). $$

(63)

Transforming back to the time variable, we find

$$p_A(t) = \sum_{n=1}^{N^2-1} K_A^{(n)} e^{-\lambda_n t}. $$

(64)

In Markov approximation, the dynamics are entirely exponential. These equations give the full normal-mode decomposition.

**7. Two-state systems: explicit solution**

Let us apply the general theory presented in the preceding sections to the case of a two-state electron system. The basis vectors of the three-dimensional space $L_S$ are now the Pauli matrices and they are chosen so that the system Hamiltonian $H_S$ is

$$H_S = \frac{1}{2} E \sigma_3. $$

(65)

We can then express the reduced density matrix $\rho$ of the system in the form

$$\rho_S = \frac{1}{2} I + \sum_{A=1}^{3} \sigma_A \rho_A, $$

(66)

$\sigma_A$, $A = 1, 2, 3$, being the Pauli matrices. In terms of $\rho_A$, $A = 1, 2, 3$, the on Neumann equation for the reduced density matrix $\rho_S$ becomes a system of three equations,

$$\frac{d\rho_A}{dt} = \sum_{B,C=1}^{3} \epsilon_{ABC} E_B \rho_C - \sum_{B=1}^{3} \alpha_{AB} \rho_B - \frac{1}{2} \alpha_{A0}, $$

(67)

where

$$E_1 = E_2 = 0, \quad E_3 = E. $$

(68)
Using equations (50) and (51), the coefficients $\alpha_{AB}$ and $\alpha_A$ are expressed in terms of the components of the Redfield tensor:

$$\alpha_{10} = \frac{1}{2} [(R_{1211} + R_{2111}) + (R_{1222} + R_{2122})],$$

$$\alpha_{11} = \frac{1}{2} [(R_{1211} + R_{2111}) + (R_{1222} + R_{2122})],$$

$$\alpha_{12} = -\frac{i}{2} [(R_{1211} + R_{2111}) - (R_{1222} + R_{2122})],$$

$$\alpha_{13} = \frac{i}{2} [(R_{1211} + R_{2111}) - (R_{1222} + R_{2122})],$$

$$\alpha_{20} = \frac{i}{2} [(R_{1211} + R_{2111}) - (R_{1222} + R_{2122})],$$

$$\alpha_{21} = \frac{i}{2} [(R_{1212} + R_{2112}) + (R_{1221} + R_{2121})],$$

$$\alpha_{22} = \frac{i}{2} [(R_{1212} + R_{2112}) - (R_{1221} + R_{2121})],$$

$$\alpha_{23} = \frac{i}{2} [(R_{1211} + R_{2111}) - (R_{1222} + R_{2122})];$$

$$\alpha_{30} = R_{1111} + R_{1222},$$

$$\alpha_{31} = R_{1122} + R_{2121},$$

$$\alpha_{32} = -i(R_{1112} - R_{2122}),$$

$$\alpha_{33} = R_{1111} - R_{2122}.$$  

Define $\rho_A^\infty$ by the relations

$$\sum_{B,C=1}^3 \varepsilon_{ABC} E_B p_C - \sum_{B=1}^3 \alpha_{AB} p_B^\infty = \frac{1}{2} |\alpha_A| (70)$$

and rewrite equation (67) as the master equation for the functions $p_A = \rho_A - \rho_A^\infty$

$$\frac{dp_A}{dt} = \sum_{B,C=1}^3 \varepsilon_{ABC} E_B p_C - \sum_{B=1}^3 \alpha_{AB} p_B.$$  

(71)

It is easy to find the solution of this system of differential equations,

$$p_A(t) = \frac{N_A(\lambda_a)}{(\lambda_a - \lambda_b)(\lambda_c - \lambda_a)} e^{\lambda_a t}$$

$$+ \frac{N_A(\lambda_b)}{(\lambda_a - \lambda_b)(\lambda_c - \lambda_b)} e^{\lambda_b t}$$

$$+ \frac{N_A(\lambda_c)}{(\lambda_a - \lambda_c)(\lambda_b - \lambda_c)} e^{\lambda_c t},$$  

(72)

where $\lambda_a$, $\lambda_b$ and $\lambda_c$ are three roots of the third-order polynomial

$$D(x) = (x + \alpha_{11})(x + \alpha_{22})(x + \alpha_{33})$$

$$- \alpha_{23}(x + \alpha_{11}) - \alpha_{13}(x + \alpha_{22}) + (x + \alpha_{33})$$

$$+ \frac{\alpha_{23}(x + \alpha_{11}) + \alpha_{13}(x + \alpha_{22})}{6} (E - \alpha_{21})$$

$$= (x - \lambda_a)(x - \lambda_b)(x - \lambda_c)$$  

(73)

and

$$N_1(x) = [(x + \alpha_{11})(x + \alpha_{22}) - \alpha_{23}(x + \alpha_{11})] f_1(0)$$

$$- [(x + \alpha_{33})(x + \alpha_{11}) - \alpha_{13}(x + \alpha_{23})] f_2(0)$$

$$- [(x + \alpha_{13})(x + \alpha_{22}) - \alpha_{23}(x + \alpha_{12})] f_3(0),$$

$$N_2(x) = [(x + \alpha_{13})(x + \alpha_{11}) - \alpha_{13}(x + \alpha_{13})] f_1(0)$$

$$+ [(x + \alpha_{33})(x + \alpha_{11}) - \alpha_{13}(x + \alpha_{23})] f_2(0)$$

$$- [(x + \alpha_{23})(x + \alpha_{11}) + \alpha_{13}(x + \alpha_{23})] f_3(0);$$

$$N_3(x) = [(x + \alpha_{11})(x + \alpha_{22}) + (E - \alpha_{21})(E + \alpha_{12})] f_1(0)$$

$$- [\alpha_{12}(E - \alpha_{21}) + \alpha_{23}(x + \alpha_{22})] f_2(0)$$

$$- [\alpha_{23}(x + \alpha_{23}) - \alpha_{12}(E + \alpha_{23})] f_3(0).$$  

(74)

Since the coefficients of $D(x)$ are real numbers, it has either one real root $\lambda_A$ and two complex conjugate roots $\lambda_b$ and $\lambda_c$,  

$$\lambda_A = -\gamma_1, \quad \lambda_b = -\gamma_2 + i\omega, \quad \lambda_c = -\gamma_2 - i\omega,$$

or three real roots:

$$\lambda_A = -\gamma_1, \quad \lambda_b = -\gamma_2 = 0, \quad \lambda_c = \gamma_2 = -\gamma_2.$$  

(69)

In the former case, the real numbers $\gamma_1, \gamma_2$ are the relaxation and dephasing rates, respectively, and the imaginary parts $\pm i\omega$ of $\lambda_b$ and $\lambda_c$ generate the oscillation with the frequency $\omega$, while in the latter case there exist three relaxation rates but no oscillation. If all coefficients $\alpha_j$ vanish, then $\gamma_1 = \gamma_2 = 0$ and $\omega = 0$. The difference between $\omega$ and $E$ would mean the frequency shift due to the interaction between the two-state system and the environment. In the case where the environment is in thermal equilibrium, this would correspond to a Lamb shift, while if the environment includes a driving term, this would correspond to an ac Stark shift.

In the Markov approximation, without using perturbation theory, $L\rho$ in equation (18) is determined by the general formula

$$L\rho = -i \sum_{A=1}^3 [\sigma_A \rho] h_A$$

$$+ \frac{1}{2} \sum_{A,B=1}^3 \varepsilon_{AB} [\sigma_A \rho \sigma_B^*] + [\sigma_A \rho, \sigma_B].$$  

(75)

The constants $\alpha_{AB}$ and $\alpha_A$ in equation (67) are expressed in terms of the constants $h_A$ and $\xi_{AB}$ in formula (75) as follows:

$$\alpha_{AB} = 2\delta_{AB} \sum_{C=1}^3 \varepsilon_{ABC} (\xi_{AC} + \xi_{BC}^*) + \sum_{C=1}^3 \varepsilon_{ABC} h_C^*,$$  

(76)

$$\alpha_A = -i \sum_{B,C=1}^3 \varepsilon_{ABC} \xi_{BC}.$$  

(77)

From equations (69), (76) and (77), we derive expressions of the constants $h_A$ and $\xi_{AB}$ in terms of the components of the Redfield tensor:

$$h_1 = -\frac{1}{2} \text{Im}(2R_{1112} + R_{2111} - R_{1222}),$$

$$h_2 = \frac{1}{2} \text{Im}(2R_{1112} - R_{2111} + R_{1222}),$$

$$h_3 = \text{Im} R_{1212},$$

and

$$\xi_{12} = \frac{1}{2} \text{Im} R_{1221} + \frac{i}{4} (R_{1111} + R_{1122}),$$

$$\xi_{21} = \frac{1}{2} \text{Im} R_{1221} - \frac{i}{4} (R_{1111} + R_{1122}),$$

$$\xi_{23} = -\frac{1}{4} \text{Im}(2R_{1112} - R_{2111} + R_{1222}) + \frac{i}{4} \text{Re}(R_{1111} + R_{1122}).$$

(78)
\[ \xi_{32} = -\frac{1}{4} \text{Im}(2R_{1112} - R_{1211} + R_{1222}) \]
\[ -\frac{i}{4} \text{Re}(R_{1211} + R_{1222}), \]
\[ \xi_{31} = -\frac{1}{4} \text{Re}(2R_{1112} + R_{1211} - R_{1222}) \]
\[ -\frac{i}{4} \text{Im}(R_{1211} + R_{1222}), \]
\[ \xi_{13} = -\frac{1}{4} \text{Re}(2R_{1112} + R_{1211} - R_{1222}) \]
\[ +\frac{i}{4} \text{Im}(R_{1211} + R_{1222}). \] (79)

### 8. Examples of two-state systems

#### 8.1. Double quantum dot

As a simple first example, we investigate a non-magnetic double quantum dot (DQD) consisting of two single-level quantum dots (QDs) connected each with the other by the tunneling and containing only one electron. Because the electron spin projection plays no role, we can omit the electron spin index and denote \( e_0 \) and \( e_0^\dagger \), \( \lambda = 1, 2 \), the destruction and creation operators of the electron in the corresponding QD. Then we have the following Hamiltonian of the electron system:

\[ H_e = E_1^\dagger e_1^\dagger e_1 + E_2^\dagger e_2^\dagger e_2 + t(e_1^\dagger e_2 + e_2^\dagger e_1). \] (80)

with some effective tunneling coupling constant \( t \). We will consider both the electron–phonon interaction in each separated QD (intradot electron–phonon interaction) represented by an operator \( X \) and the phonon-assisted electron tunneling between two QDs (intradot electron–phonon interaction) represented by an operator \( Y \). Suppose that the interactions of the QDs with the environment are identical. Then the Hamiltonian of the interaction of the electron system with the environment has the symmetrical form

\[ H_{int} = (e_1^\dagger e_2^\dagger e_2 + e_2^\dagger e_1^\dagger e_1)X + (e_1^\dagger e_2^\dagger e_2 + e_2^\dagger e_1^\dagger e_1)Y. \] (81)

By means of the Bogolubov diagonalization transformation,

\[ e_1 = uc_1 + vc_2, \quad e_2 = -vc_1 + uc_2, \] (82)

with the real coefficients \( u \) and \( v \) determined by the relations

\[ u^2 = \frac{1}{2} \left[ 1 + \frac{E_1^\dagger - E_2^\dagger}{\sqrt{(E_1^\dagger - E_2^\dagger)^2 + 4t^2}} \right], \]
\[ v^2 = \frac{1}{2} \left[ 1 - \frac{E_1^\dagger - E_2^\dagger}{\sqrt{(E_1^\dagger - E_2^\dagger)^2 + 4t^2}} \right], \]
\[ uv = \frac{\lambda}{\sqrt{(E_1^\dagger - E_2^\dagger)^2 + 4t^2}}. \] (83)

we rewrite the Hamiltonian \( H_e \) in the form

\[ H_S = \sum_{i=1}^{3} E_i c_i^\dagger c_i, \] (84)

with

\[ E_1 = \frac{E_1^\dagger + E_2^\dagger}{2} + \frac{1}{2}\sqrt{(E_1^\dagger - E_2^\dagger)^2 + 4t^2}, \]
\[ E_2 = \frac{E_1^\dagger + E_2^\dagger}{2} - \frac{1}{2}\sqrt{(E_1^\dagger - E_2^\dagger)^2 + 4t^2}. \] (85)

The operators \( K_{\mu\nu} \) in the interaction Hamiltonian are

\[ K_{11} = X - 2uvY, \]
\[ K_{22} = X + 2uvY, \] (86)
\[ K_{12} = K_{21} = (u^2 - v^2)Y. \]

Introduce the notation

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{G}_{XY}(\omega) = \xi_1 + i\eta_1, \]
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{G}_{YY}(\omega) = \xi_2 + i\eta_2, \] (87)
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{G}_{XX}(\omega + \omega_1 - i\omega) = \xi_3 + i\eta_3, \]
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{G}_{YY}(\omega + \omega_2 - i\omega) = \xi_4 + i\eta_4, \]

with the real constants \( \xi_n \) and \( \eta_n \), \( n = 1, 2, 3, 4 \), where \( \tilde{G}_{XY}(\omega) \) and \( \tilde{G}_{YY}(\omega) \) are the Fourier transforms of the phonon Green functions

\[ G_{XY}(t) = -i[T[X(t)Y(0)]], \quad G_{YY}(t) = -i[T[Y(t)Y(0))]. \] (88)

In each concrete model of the environment and its interaction with the electron system in the DQD, these Green functions can be determined by means of some appropriate method. Using the expressions of the Fourier transforms of the Green functions, we can calculate the real constants \( \xi_n \) and \( \eta_n \), \( n = 1, 2, 3, 4 \). From the relations (69) and (86), we can now deduce expressions for the coefficients \( a_{AB} \) and \( d_{A0} \) with \( A, B = 1, 2, 3 \) in terms of the constants \( \xi_n \) and \( \eta_n \):

\[ a_{10} = 4uv(u^2 - v^2)(\xi_3 - \xi_4), \]
\[ a_{11} = 16(\mu v)^2 \xi_2, \]
\[ a_{12} = -8uv\eta_1, \]
\[ a_{13} = 4uv(u^2 - v^2)(\xi_3 + \xi_4), \]
\[ a_{120} = 4u^2v(u^2 - v^2)(\eta_3 + \eta_4 - 2\eta_2), \]
\[ a_{21} = 2(u^2 - v^2)^2(\eta_3 - \eta_4) + 8uv\eta_1, \]
\[ a_{22} = 2(u^2 - v^2)^2(\xi_3 + \xi_4) + 16(\mu v)^2 \xi_2, \]
\[ a_{23} = 4uv(u^2 - v^2)(\xi_3 - \xi_4) + 4(u^2 - v^2)\eta_1, \]
\[ a_{30} = 2(u^2 - v^2)^2(\xi_3 - \xi_4), \]
\[ a_{31} = 8uv(u^2 - v^2)\xi_2, \]
\[ a_{32} = -4(u^2 - v^2)\eta_1, \]
\[ a_{33} = 2(u^2 - v^2)^2(\xi_3 + \xi_4). \] (89)

It is worthwhile noting that, if the interdot electron–phonon interaction does not exist (i.e. \( Y = 0 \), then all coefficients...
\( \alpha_{AB} \) vanish. The experimental study of the relaxation and dephasing in the symmetrical DQD would give the information on the interdot electron–phonon interaction.

In order to estimate the order of magnitude of the parameters \( \xi_a \) and \( \eta_a \) determined by formulae (87), let us consider the simple case when the environment is a bath of non-interacting phonons and the operators \( X \) and \( Y \) in the interaction Hamiltonian (81) have the form

\[
X = \sum_a g_a (b_a + b_a^\dagger), \quad Y = \sum_a f_a (b_a + b_a^\dagger),
\]

where \( b_a \) and \( b_a^\dagger \) are the destruction and creation operators of the phonon with the energy \( \omega_a \). In this case we have

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega + \Omega - i\alpha} = i \sum_a g_a f_a \left[ -\frac{1 + n_a}{\omega_a + \Omega - i\alpha} + \frac{n_a}{\omega_a - \Omega + i\alpha} \right],
\]

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega + \Omega - i\alpha} = i \sum_a f_a^2 \left[ -\frac{1 + n_a}{\omega_a + \Omega - i\alpha} + \frac{n_a}{\omega_a - \Omega + i\alpha} \right],
\]

where \( n_a \) is the density of the phonons with the energy \( \omega_a \) in the equilibrium state at the given temperature \( n_a = \frac{1}{1 + e^{\beta \omega_a}} - 1 \). From the definitions (81) and the expressions (87), it follows that, if the difference \( \omega_{12} = E \) between the energy levels in the electron system is very large in comparison with the energies of the phonons, then \( \xi_1, \xi_2 \) and \( \eta_3, \eta_4 \) can be neglected in comparison with \( \xi_1, \xi_2 \) and \( \eta_1, \eta_2 \). Denote

\[
\xi = 8uv(u^2 - v^2)\xi_2, \quad \gamma = \frac{2uv}{u^2 - v^2} \xi, \quad \tau = 4(u^2 - v^2)\eta_1, \quad \varepsilon = \frac{2uv}{u^2 - v^2} \tau, \quad \tau' = 8uv(u^2 - v^2)\eta_2.
\]

These parameters have a magnitude of the same order as the linewidths of the energy levels in the QDs due to the electron–phonon interaction [14–16], which are typically smaller than 1 meV. Usually the energy level difference \( E = E_1' - E_2' \) due to the bias voltage applied to the gates of two QDs has the value of the order in the range of 10–100 meV. Therefore, the parameters (92) are very small in comparison with \( E \):

\[
\left| \frac{\xi}{E} \right| \ll 1, \quad \left| \frac{\tau}{E} \right| \ll 1.
\]

In the first order, with respect to the small parameters \( \xi/E \) and \( \tau/E \), we have the following expressions of the coefficients \( \alpha_{ij} \):

\[
\alpha_{00} = 0, \quad \alpha_{11} = \gamma, \quad \alpha_{12} = -\varepsilon, \quad \alpha_{13} = 0, \quad \alpha_{20} = -\tau', \quad \alpha_{21} = \varepsilon, \quad \alpha_{22} = \gamma, \quad \alpha_{23} = 0, \quad \alpha_{30} = -\tau, \quad \alpha_{31} = \xi, \quad \alpha_{32} = -\tau, \quad \alpha_{33} = 0.
\]

The polynomial \( D(x) \) becomes

\[
D(x) = x(x + \gamma)^2 + x(E - \varepsilon)^2 + (x + \gamma)\tau^2 + \tau \xi (E - \varepsilon).
\]

In the first order, with respect to the small parameters \( \xi/E \) and \( \tau/E \), the relaxation and dephasing rates \( \gamma_1 \) and \( \gamma_2 \) are

\[
\gamma_1 = \frac{\tau \xi}{E - \varepsilon} + \frac{\tau^2 \varepsilon}{E}, \quad \gamma_2 = \gamma = \frac{1}{2} \frac{\tau \xi}{E - \varepsilon}.
\]

Thus, we have shown that, if the environment is a bath of non-interacting phonons, the usual case is \( \gamma_1 \ll \gamma_2 \). However, if there is a strong interaction between the phonons and the electrons, and the width of the energy band of these electrons has the same order of the magnitude as the difference \( E \) between the energy levels in the DQD, then \( \gamma_1 \) and \( \gamma_2 \) may have the same order of magnitude.

### 8.2. Electron spin qubit with spin–phonon interactions

An electron in a QD or donor state whose orbital degrees of freedom are frozen out can act as a spin qubit. Besides the electron–phonon interaction, one may also have an interaction between the electron magnetic moment and the nuclear spins. In the general case Hamiltonian (34) of the interaction between the spin qubit and the environment contains four operators \( K_{\mu \nu} \) with \( \mu, \nu = 1, 2 \). For the study of the decoherence in the Born–Markov approximation, we can use the Green function technique to find Fourier transforms of Green functions and then apply formulae of the general theory presented above.

Consider the axially symmetric single-level QD with the following Hamiltonian of the interaction between the electron spin and the environment:

\[
H_{\text{int}} = e^\sigma_1 x + e^\sigma_2 y + e^\sigma_3 z,
\]

where \( e \) is the spinor with two components \( e_\lambda, \lambda = 1, 2 \), and \( X, Y, Z \) are three operators expressed in terms of the quantum operators of the quasi-particles in the environment. Due to the axial symmetry with the symmetry axis \( Oz \), there exist four relations between different Green functions:

\[
G_{XX}(t) = G_{YY}(t), \quad G_{YX}(t) = G_{XY}(t), \quad G_{XZ}(t) = G_{YZ}(t), \quad G_{ZX}(t) = G_{ZY}(t).
\]

In this case, the coefficients \( \alpha_{AB} \) and \( \alpha_{A0} \) with \( A, B = 1, 2, 3 \) are expressed in term of the following eight integrals:

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega + \omega_{12} - i\alpha} = \mu_1 + i\psi_1, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega + \omega_{21} - i\alpha} = \mu_2 + i\psi_2, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega + \omega_{12} - i\alpha} = \mu_3 + i\psi_3, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega + \omega_{21} - i\alpha} = \mu_4 + i\psi_4,
\]

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega + \omega_{21} - i\alpha} = \nu_1 + i\psi_1, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega + \omega_{12} - i\alpha} = \nu_2 + i\psi_2, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega + \omega_{21} - i\alpha} = \nu_3 + i\psi_3, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega + \omega_{12} - i\alpha} = \nu_4 + i\psi_4.
\]
We have
\[\begin{align*}
\alpha_{i0} &= -2(\mu_3 - \mu_4) - 4\varphi_2 + 2(\varphi_3 + \varphi_4), \\
\alpha_{i1} &= 4\mu_1 + 2(v_1 - v_2) + 2(\psi_1 - \psi_2), \\
\alpha_{i2} &= -2(v_1 - v_2) + 2(\psi_1 - \psi_2), \\
\alpha_{i3} &= -2(\mu_3 + \mu_4) + 2(\varphi_1 - \varphi_2), \\
\alpha_{i20} &= -2(\mu_3 - \mu_4) + 4\varphi_2 - 2(\varphi_3 + \varphi_4), \\
\alpha_{i21} &= 2(v_1 + v_2) - 2(\psi_1 - \psi_2), \\
\alpha_{i22} &= 4\mu_1 + 2(v_1 - v_2) - 2(\varphi_1 - \varphi_2), \\
\alpha_{i23} &= -2(\mu_3 + \mu_4) - 2(\psi_1 - \psi_2), \\
\alpha_{i30} &= 4(v_1 - v_2), \\
\alpha_{i31} &= -4\mu_2, \\
\alpha_{i32} &= 4(v_1 + v_2).
\end{align*}\]

The electron bound to a single level of the P donor in Si or Ge is a model of a spin qubit interacting with a bath of acoustic phonons [17]. In this case, operators Z and X (or Y) are proportional to two dimensionless constants A and B, depending on the strength H of the magnetic field and the temperature T. At \(H = 3 \times 10^4 G\) and \(T = 30\) mK, we have \(A^2 = 10^2\) and \(B^2 = 10^1\) for the P donor in Si and \(A^2 = 2^0\), \(B^2 = 10^6\) for the P donor in Ge. In the case of the P donor in Si, if we neglect the terms of the first and higher orders with respect to the small ratio \(B/A \approx 3 \times 10^{-2}\), then only two coefficients \(\alpha_{i1}\) and \(\alpha_{i2}\) are non-vanishing,
\[\alpha_{i1} = \alpha_{i22} = 4\mu_1,\]
and all other coefficients \(\alpha_{AB}\) and \(\alpha_{A0}\) vanish. The polynomial \(D(x)\) has the form
\[D(x) = x(x + 4\mu_1)^2 + x E^2\]
and therefore
\[\gamma_1 \approx 0, \quad \gamma_2 \approx 4\mu_1.\] 

It follows that the ratio \(\gamma_1/\gamma_2\) has the same order of magnitude as the ratio \(B/A \approx 3 \times 10^{-2}\). In the case of the P donor in Ge, because \(A = 0\), we have \(\mu_3 = \varphi_3 = 0\) and therefore there may exist only six non-vanishing coefficients \(\alpha_{i11}, \alpha_{i22}, \alpha_{i33}, \alpha_{i12}, -\alpha_{i21}, \alpha_{i30}\). The polynomial \(D(x)\) has the form
\[D(x) = (x + \alpha_{i11})(x + \alpha_{i22})(x + \alpha_{i33}) - (x + \alpha_{i12})(E + \alpha_{i12})^2.\]

It follows that
\[\gamma_1 = \alpha_{i33} = 4(v_1 + v_2), \quad \gamma_2 = \frac{\alpha_{i11} + \alpha_{i22}}{2} = 2v_1.\] 

The relaxation and dephasing rates may have the same order of magnitude, which is about 10^2 times the dephasing rate in the case of the P donor in Si.

For the electron at a single energy level of a QD in a constant magnetic field with only the (non-spin-flip) interaction of the electron charge with phonons and without the spin–lattice interaction, the interaction Hamiltonian has the simple form
\[H_{\text{int}} = e_1^*e_1X_1 + e_2^*e_2X_2,\] 
where the operators \(X_1\) and \(X_2\) are expressed in terms of the phonon creation and destruction operators. Due to the presence of the constant magnetic field, they may be different. In this case, from formulae (24) it follows that the Redfield tensor has only two complex conjugate non-vanishing components:
\[R_{1212} = \lambda + i\varphi, \quad R_{2121} = \lambda - i\varphi.\] 

The polynomial \(D(x)\) becomes
\[D(x) = x[(x + \lambda)^2 + (E + \varphi)^2].\]

The relaxation and dephasing rates are
\[\gamma_1 = 0, \quad \gamma_2 = \lambda.\]

8.3. Electron spin qubit with hyperfine interactions

The decoherence of a spin qubit due to the interaction with a spin bath has been studied in many works [18–29]. The origin of the interaction between the spin qubit and the spin bath is the hyperfine interaction between the electron spin and the magnetic moments of the magnetic ions in the QD. The interaction Hamiltonian has the general form (95), where \(X, Y, Z\) are expressed in terms of the magnetic moments \(I_n\) of the magnetic ions
\[X = \sum_n a_n I_n^x, \quad Y = \sum_n a_n I_n^y, \quad Z = \sum_n a_n I_n^z.\] 

In addition to the hyperfine interaction of the magnetic ions with the electron considered as the qubit, there also exist the interactions between the magnetic ions generated by different mechanisms: for example, the magnetic dipole–dipole interaction, the indirect interaction mediated by virtual excitation of electron-hole pairs, etc. [27]. The calculation of the relaxation and dephasing rates requires the study of the Green functions (96). In order to derive the differential equations for these Green functions by means of the equation of motion method, it is necessary to know the Hamiltonian of the interaction between the magnetic ions.

The hyperfine interaction Hamiltonian determined by formulae (95) and (104) consists of two parts: the longitudinal (or diagonal) part
\[H_{\text{int}}^l = e^*\sigma e \sum_n a_n I_n^z,\] 
and the transverse (or off-diagonal) one
\[H_{\text{int}}^t = e^*\sigma e \sum_n a_n I_n^x + e^*\sigma e \sum_n a_n I_n^y.\] 

In [27], it was shown that, at the limit of the very long longitudinal relaxation time \((T \to \infty)\), the off-diagonal part in the lowest order can be eliminated by means of a canonical transformation. Then the interaction Hamiltonian determined by formulae (95) and (104) is replaced by the effective interaction Hamiltonian of the form (101), with following operators \(X_1\) and \(X_2\):
\[X_1 = X_A + X_B + X_C + X_D, \quad X_2 = -X_A + X_B + X_C - X_D,\] 
where \(X_A, X_B, X_C\) and \(X_D\) have the forms
\[X_A = \sum_{n \neq m} A_{nm} I_n^{(x)} I_m^{(-x)}, \quad X_B = \sum_{n \neq m} B_{nm} I_n^{(y)} I_m^{(-y)}, \quad X_C = \sum_{n \neq m} C_{nm} I_n^+ I_m^-, \quad X_D = \sum_{n \neq m} D_{nm} I_n^-.\]
with the coefficients $A_{nm}$, $B_{nm}$, $C_{nm}$ and $D_{n}$ expressed in terms of the constants in the original Hamiltonian of the system. The use of this effective interaction significantly simplifies the calculation of the relaxation and dephasing rates because the Redfield tensor has only two non-vanishing complex conjugate components and therefore we have again the formulae of the form (102) and (103).

9. Examples of multi-state systems

As in the case of the two-state systems considered in two preceding sections, for the study of the time evaluation of any multi-state system in the Markov approximation, one starts from equation (29) with the term $L^{(2)}\rho$ determined by the Lindblad formula of the form (31). Since the Lindblad formula was established, from equation (29) follows the system of master equations. By solving this system of differential equations, one can derive explicit expressions of elements of reduced density matrix $\rho(t)$.

The concrete form of the Lindblad formula depends on the physical mechanism of the decoherence of the system. In this section, as examples, Lindblad formulae for some typical multi-state systems are presented.

9.1. Spin-qubit chains

Consider a chain of $n$ interacting spin-qubits. It is an $N$-state system with $N = 2^n$. For simplicity, we suppose that all spin-qubits are identical and denote $\sigma_i^{(j)}$, $j = 1, 2, 3$ the Pauli matrices in the two-dimensional vector space of the $j$th spin-qubit, $i = 1, 2, \ldots, n$.

The interaction of spin-qubits with the environment generates three different mechanisms of decoherence: the relaxation of spin-qubits with energy dissipation, the thermal excitation of spin-qubits due to the absorption of bosonic quasi-particles in the environment (phonons, for example), and the dephasing of qubits due to different spin–boson interaction processes. Therefore, $L^{(2)}\rho$ consists of three parts:

$$L^{(2)}\rho = L_{\text{re}}\rho + L_{\text{ex}}\rho + L_{\text{dp}}\rho,$$

(109)

where $L_{\text{re}}\rho$ is caused by the relaxation of spin-qubits, $L_{\text{ex}}\rho$ by their excitation and $L_{\text{dp}}\rho$ is the consequence of the dephasing.

If each spin-qubit is interacting only with its own environment and does not interact with those of others, then

$$L_{\text{re}}\rho = \frac{1}{2}\xi_{\text{re}}\sum_{i=1}^{n}[[\sigma_i^{(1)}\rho\sigma_i^{(1)}] + [\sigma_i^{(1)}\rho, \sigma_i^{(1)}]],$$

(110)

$$L_{\text{ex}}\rho = \frac{1}{2}\xi_{\text{ex}}\sum_{i=1}^{n}[[\sigma_i^{(1)}\rho\sigma_i^{(1)}] + [\sigma_i^{(1)}\rho, \sigma_i^{(1)}]],$$

(111)

$$L_{\text{dp}}\rho = \frac{1}{2}\xi_{\text{dp}}\sum_{i=1}^{n}[[\sigma_i^{(1)}\rho\sigma_i^{(1)}] + [\sigma_i^{(1)}\rho, \sigma_i^{(1)}]],$$

(112)

with non-negative real constant $\xi_{\text{re}}$, $\xi_{\text{ex}}$, $\xi_{\text{dp}}$. At zero temperature, there is no bosonic excitation in the environment and $\xi_{\text{ex}} = 0$.

If all spin-qubits are interacting with one and the same common environment, then

$$L_{\text{re}}\rho = \frac{1}{2}\xi_{\text{re}}[[\Sigma_+\rho\Sigma_+] + [\Sigma_+\rho, \Sigma_+]],$$

(113)

$$L_{\text{ex}}\rho = \frac{1}{2}\xi_{\text{ex}}[[\Sigma_+\rho\Sigma_+] + [\Sigma_+\rho, \Sigma_+]],$$

(114)

$$L_{\text{dp}}\rho = \frac{1}{2}\xi_{\text{dp}}[[\Sigma_+\rho\Sigma_+] + [\Sigma_+\rho, \Sigma_+]],$$

(115)

where

$$\Sigma_\pm \equiv \sum_{i=1}^{n} \sigma_i^{(i)} \quad \Sigma_3 = \sum_{i=1}^{n} \sigma_i^{(3)}.$$  

(116)

Consider the case of a system of two spin-qubits, each of them interacting only with its own environment, and denote $\rho(1,1,1)$ with $i_1, i_2, j_1, j_2 = 0, 1$ the element of reduced density matrix $\rho$ and similarly for other $4 \times 4$ matrices. $L_{\text{re}}\rho$ has following 16 matrix elements:

$$(L_{\text{re}}\rho)(00,00) = \xi_{\text{re}}[\rho(00,10) + \rho(01,01)],$$

$$(L_{\text{re}}\rho)(00,01) = \xi_{\text{re}}[\rho(01,10) - \frac{1}{2}\rho(00,01)],$$

$$(L_{\text{re}}\rho)(00,10) = \xi_{\text{re}}[\rho(10,01) + \frac{1}{2}\rho(00,10)],$$

$$(L_{\text{re}}\rho)(00,11) = -\xi_{\text{re}}[\rho(01,01)],$$

$$(L_{\text{re}}\rho)(10,00) = \xi_{\text{re}}[\rho(10,01) - \frac{1}{2}\rho(10,10)],$$

$$(L_{\text{re}}\rho)(10,01) = -\xi_{\text{re}}[\rho(01,10)],$$

$$(L_{\text{re}}\rho)(10,10) = -\xi_{\text{re}}[\rho(11,10)],$$

$$(L_{\text{re}}\rho)(10,11) = -\frac{1}{2}\xi_{\text{re}}[\rho(11,10)].$$

(117)

$L_{\text{ex}}\rho$ has the following 16 matrix elements:

$$(L_{\text{ex}}\rho)(00,00) = -2\xi_{\text{ex}}[\rho(00,00)],$$

$$(L_{\text{ex}}\rho)(00,01) = -\frac{3}{2}\xi_{\text{ex}}[\rho(00,01)],$$

$$(L_{\text{ex}}\rho)(00,10) = -\frac{3}{2}\xi_{\text{ex}}[\rho(00,10)],$$

$$(L_{\text{ex}}\rho)(00,11) = -\xi_{\text{ex}}[\rho(00,11)],$$

$$(L_{\text{ex}}\rho)(10,00) = -\frac{3}{2}\xi_{\text{ex}}[\rho(10,00)],$$

$$(L_{\text{ex}}\rho)(10,01) = -\xi_{\text{ex}}[\rho(10,01)],$$

$$(L_{\text{ex}}\rho)(10,10) = -\xi_{\text{ex}}[\rho(10,10)],$$

$$(L_{\text{ex}}\rho)(10,11) = -\xi_{\text{ex}}[\rho(10,11)].$$

(118)
$L_{dp}\rho$ has following 12 non-vanishing matrix elements:

\[
\begin{align*}
(L_{dp}\rho)_{\mathbf{00,01}} &= -\xi_{dp}\rho_{\mathbf{00,01}}, \\
(L_{dp}\rho)_{\mathbf{00,10}} &= -\xi_{dp}\rho_{\mathbf{00,10}}, \\
(L_{dp}\rho)_{\mathbf{01,11}} &= -\xi_{dp}\rho_{\mathbf{01,11}}, \\
(L_{dp}\rho)_{\mathbf{01,00}} &= -\xi_{dp}\rho_{\mathbf{01,00}}, \\
(L_{dp}\rho)_{\mathbf{10,11}} &= -\xi_{dp}\rho_{\mathbf{10,11}}, \\
(L_{dp}\rho)_{\mathbf{10,00}} &= -\xi_{dp}\rho_{\mathbf{10,00}}, \\
(L_{dp}\rho)_{\mathbf{11,10}} &= -\xi_{dp}\rho_{\mathbf{11,10}}, \\
(L_{dp}\rho)_{\mathbf{11,00}} &= -\xi_{dp}\rho_{\mathbf{11,00}}.
\end{align*}
\]

(119)

and the following excited eigenstates:

\[
|\psi_{\pm}^{(n)}\rangle = A_{\pm}^{(n)}|g, n\rangle + B_{\pm}^{(n)}|e, n - 1\rangle,
\]

(122)

with eigenvalues

\[
E_{\pm}^{(n)} = \frac{E - \omega}{2} + n\omega = \frac{1}{2} \Delta_n,
\]

(123)

\[
\Delta_n = \sqrt{(E - \omega)^2 + 4nf^2}.
\]

(124)

Up to a phase factor, coefficients $A_{\pm}^{(n)}$ and $B_{\pm}^{(n)}$ are determined as follows:

\[
A_{\pm}^{(n)} = \frac{1}{\sqrt{2}} \left[ 1 \mp \frac{E - \omega}{\Delta_n} \right],
\]

(125)

\[
B_{\pm}^{(n)} = \pm \frac{1}{\sqrt{2}} \left[ 1 \mp \frac{E - \omega}{\Delta_n} \right].
\]

There are an infinite number of eigenstates of the total Hamiltonian.

There are four different physical mechanisms for decoherence: relaxation, thermal excitation, dephasing of the electron in QD and photon leakage from the MC. The Liouvillian superoperator $L$ consists of four parts:

\[
L\rho = L_{re}\rho + L_{cx}\rho + L_{dp}\rho + L_i\rho,
\]

(126)

where

\[
L_{re}\rho = \frac{1}{2} \xi_{re} \left\{ [\sigma_- \rho \sigma_+] + [\sigma_- \rho, \sigma_+] \right\}
\]

(127)

is caused by the relaxation,

\[
L_{cx}\rho = \frac{1}{2} \xi_{cx} \left\{ [\sigma_+ \rho \sigma_-] + [\sigma_+ \rho, \sigma_-] \right\}
\]

(128)

is caused by the thermal excitation,

\[
L_{dp}\rho = \frac{1}{2} \xi_{dp} \left\{ [\sigma_3 \rho \sigma_3] + [\sigma_3 \rho, \sigma_3] \right\}
\]

(129)

is the consequence of the dephasing and

\[
L_i\rho = \frac{1}{2} \xi_{i} \left\{ [\gamma \rho \gamma^*] + [\gamma \rho, \gamma^*] \right\}
\]

(130)

is caused by the leakage of the photon from the MC.

Denote $\rho_{(g,m)}(g,n)$, $\rho_{(g,m)}(e,n)$, $\rho_{(e,m)}(g,n)$ and $\rho_{(e,m)}(e,n)$ elements of reduced density matrix $\rho$ in the basis with unit vectors $|g, n\rangle$ and $|e, n\rangle$, and similarly for other matrices. It is easy to check that $L_{re}\rho$ has the following matrix elements:

\[
(L_{re}\rho)_{(g,m)}(g,n) = \xi_{re}\rho_{(e,m)}(e,n),
\]

(131a)

\[
(L_{re}\rho)_{(g,m)}(e,n) = -\frac{1}{2} \xi_{re}\rho_{(g,m)}(g,n),
\]

(131b)

\[
(L_{re}\rho)_{(e,m)}(g,n) = -\frac{1}{2} \xi_{re}\rho_{(e,m)}(e,n),
\]

(131c)

\[
(L_{re}\rho)_{(e,m)}(e,n) = -\xi_{re}\rho_{(e,m)}(g,n).
\]

(131d)

$L_{cx}\rho$ has the following matrix elements:

\[
(L_{cx}\rho)_{(g,m)}(g,n) = -\xi_{cx}\rho_{(g,m)}(g,n),
\]

(132a)
The simplest system of this kind consists of two two-level QDs placed inside two single-mode MCs connected via a single-mode optical fiber (OF). In this system, there are also four types of decoherence physical mechanisms: relaxation, thermal excitation, dephasing and photon leakage. The Liouvillian superoperator consists of four parts, equation (126), with

\[ L_{re}\rho = \frac{1}{2} \sum_{k=1}^{2} \xi_{ex}^{(k)} \left[ \sigma_{-}^{(k)} \rho \sigma_{+}^{(k)} + \sigma_{+}^{(k)} \rho \sigma_{-}^{(k)} \right], \]

\[ L_{ex}\rho = \frac{1}{2} \sum_{k=1}^{2} \xi_{ex}^{(k)} \left[ \sigma_{+}^{(k)} \rho \sigma_{-}^{(k)} + \sigma_{-}^{(k)} \rho \sigma_{+}^{(k)} \right], \]

\[ L_{dp}\rho = \frac{1}{2} \sum_{k=1}^{2} \xi_{dp}^{(k)} \left[ \sigma_{3}^{(k)} \rho \sigma_{3}^{(k)} + \sigma_{3}^{(k)} \rho \sigma_{3}^{(k)} \right], \]

and

\[ L_{i}\rho = \frac{1}{2} \sum_{k=1}^{2} \xi_{c}^{(k)} \left[ \gamma_{k} \rho \gamma_{k}^{*} + \gamma_{k}^{*} \rho \gamma_{k} \right] + \frac{1}{2} \xi_{c} \left[ \gamma \rho \gamma^{*} + \gamma^{*} \rho \gamma \right], \]

where \( \sigma_{\pm}^{(k)} \) and \( \sigma_{3}^{(k)} \) are 2 × 2 matrices \( \sigma_{\pm} \) and \( \sigma_{3} \) in the vector space of state vectors of the \( k \)-th QD, \( k = 1, 2 \), \( \gamma_{k} \) and \( \gamma_{k}^{*} \) are the destruction and creation operators of the electron in the \( k \)-th MC, \( k = 1, 2 \), \( \gamma \) and \( \gamma^{*} \) are those in the OF.

Denote \( |i_1, n_1, i_2, n_2\rangle \) the state vector of the quantum system consisting of the first QD in the state \( |i_1\rangle (i_1 = g, e) \), \( n_1 \) photons in the first MC, second QD in the state \( |i_2\rangle (i_2 = g, e) \), \( n_2 \) photons in the second MC and \( n \) photons in the OF. In the basis with unit vectors of the form \( |i_1, n_1, i_2, n_2, n\rangle \) \( L_{re}\rho \) has the following matrix elements:

\[ \langle i_1, n_1, i_2, n_2, n| L_{re}\rho |i_1, n_1, i_2, n_2, n\rangle = \xi_{re}^{(1)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ + \xi_{re}^{(2)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ \langle i_1, n_1, i_2, n_2, n| L_{re}\rho |i_1, n_1, i_2, n_2, n\rangle = \xi_{re}^{(1)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ + \xi_{re}^{(2)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ \langle i_1, n_1, i_2, n_2, n| L_{re}\rho |i_1, n_1, i_2, n_2, n\rangle = \xi_{re}^{(1)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ + \xi_{re}^{(2)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ \langle i_1, n_1, i_2, n_2, n| L_{re}\rho |i_1, n_1, i_2, n_2, n\rangle = \xi_{re}^{(1)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ + \xi_{re}^{(2)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ \langle i_1, n_1, i_2, n_2, n| L_{re}\rho |i_1, n_1, i_2, n_2, n\rangle = \xi_{re}^{(1)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ + \xi_{re}^{(2)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ \langle i_1, n_1, i_2, n_2, n| L_{re}\rho |i_1, n_1, i_2, n_2, n\rangle = \xi_{re}^{(1)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ + \xi_{re}^{(2)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ \langle i_1, n_1, i_2, n_2, n| L_{re}\rho |i_1, n_1, i_2, n_2, n\rangle = \xi_{re}^{(1)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ + \xi_{re}^{(2)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ \langle i_1, n_1, i_2, n_2, n| L_{re}\rho |i_1, n_1, i_2, n_2, n\rangle = \xi_{re}^{(1)} \rho_{(i_1, n_1, i_2, n_2, n)} \]

\[ + \xi_{re}^{(2)} \rho_{(i_1, n_1, i_2, n_2, n)} \]
Expressions of elements of $L_{ee}$ are slightly different from those of $L_{dp}$. At zero temperature, they vanish. $L_{dp}$ has the following non-vanishing matrix elements:

\[
(L_{dp})_{(g,n; g,n; n)(g,m; e,m; m)} = -2\xi_{dp}^{(2)} \rho(g,n; g,n; n)(g,m; e,m; m), \\
(L_{dp})_{(g,n; e,n; n)(g,m; e,m; m)} = -2\xi_{dp}^{(2)} \rho(g,n; e,n; n)(g,m; e,m; m), \\
(L_{dp})_{(g,n; g,n; n)(e,m; e,m; m)} = -2\xi_{dp}^{(2)} \rho(g,n; g,n; n)(e,m; e,m; m), \\
(L_{dp})_{(e,n; e,n; n)(g,m; e,m; m)} = -2\xi_{dp}^{(2)} \rho(e,n; e,n; n)(g,m; e,m; m).
\]

$L_{ij}$ has the following matrix elements:

\[
(L_{ij})_{(i,n; j,n; n)(k,m; l,m; m)} = \xi_e^{(1)} \left[ \sqrt{(n_1 + 1)(m_1 + 1)} \rho(i,n_1+1; j,n_2; n)(k,m_1+1; l,m_2; m) \rho(i,n_1+1; j,n_2; n)(k,m_1+1; l,m_2; m) \right] \\
- \frac{1}{2} \left( n_1 + 1 \right) \rho(i,n_1; j,n_2; n)(k,m_1; l,m_2; m) \\
+ \xi_e^{(2)} \left[ \sqrt{(n_2 + 1)(m_2 + 1)} \rho(i,n_1+1; j,n_2; n)(k,m_1+1; l,m_2; m) \right] \\
- \frac{1}{2} \left( n_2 + 1 \right) \rho(i,n_1; j,n_2; n)(k,m_1; l,m_2; m) \\
+ \xi_f \left( \sqrt{(n + 1)(m + 1)} \rho(i,n_1; j,n_2; n)(k,m_1; l,m_2; m+1) \right).
\]

where $i, j, k, l = g, e$.

In [41], the master equations for the QC system consisting of two identical two-level QDs placed inside two single-mode identical MCs connected via a single mode FO were derived in the case where there is no decoherence due to the excitation, and then solved in the first order approximation with respect to the small decoherence constants.

10. Conclusion

We have presented a general method for the study of the decoherence of $N$-state systems. Expressing the $N \times N$ density matrix in terms of the $N^2 - 1$ generators of SU(N), we can present a completely systematic approach for the computation of the relaxation and dephasing times. In general, there may be as many as $N^2 - 1$ of these quantities, and they are obtained as the eigenvalues of a certain matrix. We have given explicit formulæ for this matrix in the case of a system subject to classical noise, one that evolves according to a generalized Lindblad equation, and one that satisfies the conditions for perturbation theory (Redfield theory) to be valid. Numerical examples for some two-state systems have been given. Lindblad formulæ for some typical multi-state systems have also been presented.

Acknowledgments

NVH and NBH express their gratitude to the Max-Planck Institute for the Physics of Complex Systems and the Vietnam Academy of Science and Technology for their support, and to Professor Peter Fulde for encouragement. Financial support for RJ was provided by the US National Science Foundation, grant nos. NSF-ECS-0524253, NSF-FRG-0805045 and OISE-0435632.

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