The KPZ fixed point for discrete time TASEPs

Yuta Arai *

Abstract

We consider two versions of discrete time totally asymmetric simple exclusion processes (TASEPs) with geometric and Bernoulli random hopping probabilities. For the process mixed with these and continuous time dynamics, we obtain a single Fredholm determinant representation for the joint distribution function of particle positions with arbitrary initial data. This formula is a generalization of the recent result by Mateski, Quastel and Remenik and allows us to take the KPZ scaling limit. For both the discrete time geometric and Bernoulli TASEPs, we show that the distribution function converges to the one describing the KPZ fixed point.

1 Introduction

The totally asymmetric simple exclusion process (TASEP) is a typical interacting stochastic particle system and can be interpreted as a stochastic growth model of an interface, which turns out to belong to the Kardar-Parisi-Zhang (KPZ) universality class introduced in [18]. In addition, the TASEP is one of the most basic model in the integrable probability [11]. Remarkable algebraic structures allow us to obtain exact explicit forms of distribution functions for some quantities.

On a macroscopic level, the particle density evolves deterministically according to the Burgers equation [30, 31]. Therefore, a natural question is to focus on fluctuation properties around the deterministic growth, which exhibit universal properties characterizing the KPZ class. There are many important results in the literature of the integrable probability. First, for the step initial condition, the one-point limiting distribution for the particle current in the TASEP has been obtained by Johansson [15] by converting the problem to the last passage percolation and then using the RSK correspondence. It turned out that the limiting distribution is the GUE Tracy-Widom distribution. In [21, 29], this result can also be obtained by using an explicit form of the the transition probability in the TASEP [33]. Similar results have been found for symmetrized last passage problems by Baik-Rains [2]. The results include the one-point limiting distribution of the particle current for the alternating initial condition in the language of the TASEP or equivalently, the height distribution for the flat initial condition in the language of the growth process called the polynuclear growth (PNG) model [24]. In this case, the limiting distribution is turned out to be the GOE Tracy-Widom distribution.

*Graduate School of Science and Engineering, Chiba University, Chiba-shi 263-8522, Japan. Email: yutaarai@chiba-u.jp
These results on the one-point fluctuations have been generalized to the case of the multi-point fluctuations. For the case corresponding to the step initial condition, a Fredholm determinant formula for the limiting multi-point distributions has been first obtained in the PNG model with space-time continuous setting [25] by using the technique related to the RSK correspondence. The same result has been obtained for the discretized PNG model [16]. The limiting process characterized by the multi-point distribution is called the Airy2 process. On the other hand, for the other conditions, the first important result has been given in [32]. Sasamoto has developed the technique for obtaining the multi-point function in terms of the transition probability in TASEP [33] not only for the step initial condition but also for the alternating one and has obtained a Fredholm determinant formula for the limiting functions in the alternating case. The process characterized by the multi-point distribution is now called the Airy1 process. This approach in [32] has been further studied and has been applied to the TASEP and the PNG model with different settings [5, 7, 8, 9].

We have been interested in the entire structure of the universal limiting process for more general initial data. Our understanding of this problem has been advanced by the recent result by Matetski, Quastel, and Remenik [20]. A Fredholm determinant formula for an arbitrary initial data has already been obtained in [8] based on the approach developed in [32]. The correlation kernel for the Fredholm determinant can be expressed in terms of the biorthogonal functions, say $\Phi_k(x)$ and $\Psi_k(x)$. The problem is that one of them, say $\Phi_k(x)$ does not have an explicit representation while $\Psi_k(x)$ does. Thus it had not been clear how to take the KPZ scaling limit of this kernel. [20] has overcome this situation. They represent the function in terms of a stopping time of the geometric random walk. This expression allows us to take the KPZ scaling limit since by Donsker’s invariance principle, we easily find this stopping time converges to the one for the Brownian motion in the KPZ scaling limit. Based on this technique, the limiting multi-point distribution functions for the particle positions in the arbitrary initial condition has been obtained. The process with this multi-point distribution is called the KPZ fixed point. Recently various interesting progresses on this problem have been made for example in [22, 23, 27].

In this paper, we show that the technique in [20] can be applicable to different versions of the TASEP beside the usual continuous time one. In particular, we focus on two versions of the discrete time TASEPs: the geometric random hopping with parallel update and the Bernoulli random hopping with sequential update. Furthermore, in both case, we consider the case where the hopping probabilities are time-dependent. For the step initial condition these dynamics has appear as a special case of the higher spin vertex model and has been recently studied in [19]. To the best of our knowledge, however, the analyses for the arbitrary initial condition has not been studied yet. We show Schütz’s type determinantal formulas for transition probabilities for both the geometric and Bernoulli TASEP with time dependent hopping probabilities. Combining these with Schütz’s formula for the continuous time TASEP, we get the determinantal transition probability for the system mixed with the three types of dynamics. Using this, we obtain a Fredholm determinant formula for the multi-point distribution for the particle positions, in which we can take the KPZ scaling limit. This is a generalized formula to the one [20]: When we vanish the whole parameters of the mixed dy-
dynamics except the part of the continuous time TASEP, the determinantal formula is reduced to the result in [20]. Finally taking the KPZ scaling limit for both discrete time geometric and Bernoulli TASEP, we see that the multi-point distribution functions converges to the one describing the KPZ fixed point.

The paper is organized as follows. In Sec. 2, we state the three versions of the TASEPs, continuous time and two types of discrete time version: geometric and Bernoulli random hopping. Their mixed version is also stated. We also give our main result: the Fredholm determinant formula for the mixed TASEP (Theorem 2.4) and the KPZ scaling limit in two cases of the geometric and Bernoulli TASEPs (Theorem 2.11, and Propositions 2.14 and 2.15). In Sec. 3, after giving the determinantal formulas for the transition probabilities for the above three types of TASEPs, we give the proof of Theorem 2.4 using the framework developed in [20]. In Sec. 4, we give the proofs of Theorem 2.11, and Propositions 2.14 and 2.15. The crucial step is the saddle point analysis for the kernels.

2 Models and results

In this section we define three versions of the TASEP and introduce our main results.

2.1 Models

In this paper we consider the TASEPs on \( \mathbb{Z} \). Each particle jumps only to the right independently and stochastically if the target site is empty. If the site is occupied by the other particle, it cannot move, which represents the exclusion interaction.

In the TASEPs we mainly focus on the position of each particle. Let \( X_t(i) \in \mathbb{Z} \) be a position of the \( i \)th particle at time \( t \). We set \( t \in \mathbb{Z} \) or \( t \in \mathbb{R} \) according to the version. Since the dynamics of the TASEPs preserves the order of the particles, we can always assume,

\[ \cdots < X_t(2) < X_t(1) < X_t(0) < X_t(-1) < X_t(-2) < \cdots. \]

The particles at \( \pm \infty \) are playing no role in the dynamics when adding \( \pm \infty \) into the state space.

2.1.1 Continuous time TASEP

The continuous time TASEP on \( \mathbb{Z} \) was introduced in [35] in the literature of mathematics. In this case \( t \in \mathbb{R}_{\geq 0} \) and each particle independently attempts to jump to the right neighboring site at rate \( \gamma \in \mathbb{R}_{\geq 0} \) provided this site is empty. It is a continuous time Markov process with the generator \( L \) defined as follows: Let \( \eta = (\eta_x)_{x \in \mathbb{Z}} \in \{0, 1\}^\mathbb{Z} \) be a particle configuration. \( \eta_x = 1 \) means the site \( x \) is occupied by a particle while \( \eta = 0 \) means it is empty. The generator \( L \) acting on cylinder functions \( f : \{0, 1\}^\mathbb{Z} \rightarrow \mathbb{R} \) is defined by

\[
(Lf)(\eta) = \gamma \sum_{x \in \mathbb{Z}} \eta_x (1 - \eta_{x+1})(f(\eta^{x,x+1}) - f(\eta))
\]
where
\[ \eta_x = \begin{cases} 
1, & \text{if the site is occupied by a particle}, \\
0, & \text{if the site } x \text{ is empty},
\end{cases} \]

and
\[ \eta_{y,x+1} = \begin{cases} 
\eta_{x+1}, & \text{for } y = x, \\
\eta_x, & \text{for } y = x + 1, \\
\eta_y, & \text{otherwise}.
\end{cases} \]

2.1.2 Discrete time Bernoulli TASEP with sequential update

We define the discrete time Bernoulli TASEP with sequential update on \( \mathbb{Z} \). This version was studied previously in [6] as a marginal of dynamics on Gelfand-Tsetlin patterns which preserve the class of Schur processes and more recently in [4, 19] in the studies of the integrable probability.

Let us assume the particle configurations at time \( t \in \mathbb{Z}_{\geq 0} \) as \( X_t(j) = a_j, j \in \mathbb{Z} \). The particle positions at time \( t+1 \) are determined by the following update rule: We update the position of the \( i \)th particle \( X_{t+1}(i) \) in increasing order. Suppose that we already updated the \( i-1 \)th particle and its position is \( b_{i-1} \) i.e. \( X_{t+1}(i-1) = b_{i-1} \). Then the update rule is given as follows:

- When \( X_{t+1}(i-1) - X_t(i) = b_{i-1} - a_i > 1 \),
  \[ \mathbb{P}(X_{t+1}(i) = a | X_t(i) = a_i, X_{t+1}(i-1) = b_{i-1}) = \begin{cases} 
1 - p_{t+1}, & \text{for } a = a_i, \\
p_{t+1}, & \text{for } a = a_i + 1, \\
0, & \text{otherwise}.
\end{cases} \]

- When \( X_{t+1}(i-1) - X_t(i) = b_{i-1} - a_i = 1 \),
  \[ \mathbb{P}(X_{t+1}(i) = a | X_t(i) = a_i, X_{t+1}(i-1) = b_{i-1}) = \begin{cases} 
1, & \text{for } a = a_i, \\
0, & \text{otherwise}.
\end{cases} \]

This dynamics mean that starting from right to left, for the time step \( t \to t+1 \), the \( i \)th particle jumps to the right neighboring site with probability \( p_{t+1} \in (0, 1) \) provided this site is empty. Since the update is sequential from right to left, during a time step, a block of consecutive particles can jump. For later use, we define \( \beta_t, t = 0, 1, 2, \ldots \) by
\[ p_t = \frac{\beta_t}{1 + \beta_t}, \quad \left( \beta_t = \frac{p_t}{1 - p_t} \right). \tag{2.1} \]

2.1.3 Discrete time geometric TASEP with parallel update

We define the discrete time geometric TASEP with parallel update on \( \mathbb{Z} \). This was studied previously in [36] as a marginal of dynamics on Gelfand-Tsetlin patterns which preserve the...
class of Schur processes. More recently it has been also investigated in [4, 19].

Let us assume that for $t \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}$, $X_t(j) = a_j$. The update rule of the positions at time $t + 1$ are given as follows: For each $1 \leq i \leq N$,

$$
\mathbb{P}(X_{t+1}(i) = a_i + a | X_t(i) = a_i, \ X_t(i-1) = a_{i-1})
= \begin{cases} 
\alpha_{t+1}^a (1 - \alpha_{t+1}) & \text{for } a = 0, 1, \cdots, a_{i-1} - a_i - 2, \\
\alpha_{t+1}^a & \text{for } a = a_{i-1} - a_i - 1, \\
0 & \text{otherwise},
\end{cases}
$$

where the update is independent for each $i$ and $t$.

Note that in this dynamics, the $j$th particle can jump with multiple cites according to the truncated geometric distribution defined in (2.2) with parameter $\alpha_t$.

### 2.1.4 TASEP$_{\alpha, \beta, \gamma}$: TASEP mixed with the continuous time TASEP and the discrete time TASEPs

In this paper, we consider the TASEP combined with the above three versions. First we take three time parameters $t_1$, $t_2 \in \mathbb{Z}_{\geq 0}$ and $t_3 \in \mathbb{R}_{\geq 0}$. Then particles evolve according to the discrete time geometric TASEP with parameter $\alpha := \{\alpha_1, \alpha_2, \cdots, \alpha_{t_1}\}$ (Sec. 2.1.3) from time 0 to $t_1$, the discrete time Bernoulli TASEP with parameter $\beta = \{\beta_{t_1+1}, \beta_{t_1+2}, \cdots, \beta_{t_1+t_2}\}$ (Sec. 2.1.2) from time $t_1$ to $t_1 + t_2$, and the continuous time TASEP with parameter $\gamma$ (Sec. 2.1.1) from $t_1 + t_2$ to $t_1 + t_2 + t_3$. In this paper we denote this mixed TASEP as TASEP$_{\alpha, \beta, \gamma}$.

This type of the mixed TASEP with $t_3 = 0$ has been introduced in [19]. We decided the order of the three dynamics as above. In fact the distribution of the particles’ positions remain unchanged if we freely exchange order of these dynamics since the semigroups of all the three dynamics are shown to be exchangeable thanks to the Yang-Baxter relations [10, 12].

### 2.2 Results

In this subsection, we give our main results.

#### 2.2.1 Joint distribution of the particle positions

Here we give a single Fredholm determinant formula for joint distribution of the particle position in TASEP$_{\alpha, \beta, \gamma}$ defined in Sec. 2.1.4. For the descriptions of the results below including the followin one, we state some definitions.

**Definition 2.1.** For a real single-valued function, $\hat{f} : \mathbb{A} \to (-\infty, \infty]$ with (in general an uncountable) domain $\mathbb{A}$, the epigraph $\text{epi}(\hat{f})$ and the hypograph $\text{hypo}(\hat{f})$ are defined as follows.

$$
\text{epi}(\hat{f}) = \{(x, y) : y \geq \hat{f}(x)\}, \quad \text{hypo}(\hat{f}) = \{(x, y) : y \leq \hat{f}(x)\}.
$$
Definition 2.2. Let $RW_m$, $m = 0, 1, 2 \cdots$ be the position of a random walker with $\text{Geom}[\frac{1}{2}]$ jumps strictly to the left starting at some fixed site $c$, i.e.,

$$RW_m = c - \chi_1 - \chi_2 - \cdots - \chi_m,$$

where $\chi_i$, $i = 1, 2, \cdots$ are the i.i.d. random variable with $\mathbb{P}(\chi_i = k) = 1/2^{k+1}$, $k = 0, 1, 2, \cdots$.

We also define the stopping time

$$\tau = \min\{m \geq 0 : RW_m > X_0(m + 1)\}, \quad (2.3)$$

where $\tau$ is the hitting time of the strict epigraph of the curve $(X_0(k + 1))_k=0\ldots,n-1$ by the random walk $RW_k$. When the number of particles is $N$, $X_0(m)$ is constant and defined only $m \leq N$.

At last we define the multiplication operators

Definition 2.3. For a fixed vector $a \in \mathbb{R}^m$ and indices $n_1 < \cdots < n_M$, we define $\chi_a$ and $\bar{\chi}_a$ by the multiplication operators acting on the space $\ell^2(\{n_1, \ldots, n_M\} \times \mathbb{Z})$ (or acting on the space $L^2(\{x_1, \ldots, x_m\} \times \mathbb{R})$) with

$$\chi_a(n_j, x) = 1_{x > a_j}, \quad \bar{\chi}_a(n_j, x) = 1_{x \leq a_j}. \quad (2.4)$$

We obtain the following result.

Theorem 2.4. We consider the TASEP$_{\alpha,\beta,\gamma}$ introduced in Sec. 2.1.4. Let $t = t_1 + t_2 + t_3$ be the final time, and $X_t(j)$, $j \in \mathbb{Z}$ be the the position of the particle labeled $j$ at $t = t_1 + t_2 + t_3$. Assume that the initial positions $X_0(j) \in \mathbb{Z}$ for $j = 1, 2, \cdots$ are arbitrary constants satisfying $X_0(1) > X_0(2) \cdots$ while $X_0(j) = \infty$ for $j \leq 0$.

For $n_j \in \mathbb{Z}_{\geq 1}$ $j = 1, 2, \cdots, M$ with $1 \leq n_1 < n_2 < \cdots < n_M$, and $a = (a_1, a_2, \cdots, a_M) \in \mathbb{Z}^M$ we have

$$\mathbb{P}(X_t(n_j) > a_j, j = 1, \ldots, M) = \det(I - \bar{\chi}_a K_t \chi_a)_{\ell^2(\{n_1, \ldots, n_M\} \times \mathbb{Z})}, \quad (2.5)$$
where $\bar{\chi}(n, x)$ is defined in (2.4) and the kernel $K_t$ is given by

$K_t(n, x; n, y) = -Q^{n-n}(x, y)1_{n_i<n_j} + (S_{-t, n})^\omega(X_t^\omega, 1_{n_i<n_j}, x, y), \quad (2.6)$

$Q_m(x, y) = \frac{1}{2^{x-y}} (x - y - 1)_{x \geq y + m}, \quad (2.7)$

$S_{-t, n}(z_1, z_2) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1 - w)^n}{2^{z_1+1}w^{n+1}2^{z_2-z_1}2^{z_2-1}S_{\alpha, \beta, \gamma}(w)}, \quad (2.8)$

$\bar{\delta}_{\alpha, \beta, \gamma}(w, t) = \prod_{j=1}^{t_1} \left(1 + \frac{2\alpha_j}{2 - \alpha_j} (w - \frac{1}{2}) \right) \prod_{j=t_1+1}^{t_1+t_2} \left(1 + \frac{2\beta_j}{2 + \beta_j} (w - \frac{1}{2}) \right) \cdot e^{\gamma t_3(w - \frac{1}{2})}, \quad (2.11)$

$\tilde{\delta}_{\alpha, \beta, \gamma}(w, t) = \prod_{j=1}^{t_1} \left(1 + \frac{2\alpha_j}{2 - \alpha_j} (w - \frac{1}{2}) \right) \prod_{j=t_1+1}^{t_1+t_2} \left(1 + \frac{2\beta_j}{2 + \beta_j} (w - \frac{1}{2}) \right) \cdot e^{\gamma t_3(w - \frac{1}{2})}, \quad (2.12)$

where $\Gamma_0$ is a simple counterclockwise loop around 0 not enclosing any other poles. The superscript epi($X_t$) in (2.10) refers to the fact that $\tau$ is the hitting time of the strict epigraph of the curve $(X_t, x, y)$ in 2.10 by the random walk $RW_k$ (see Def. 2.2).

Remark 2.5. In the case of continuous time TASEP, i.e. the special case $\alpha_i = \beta_j = 0$ with $1 \leq i \leq t_1$, $t_1 + 1 \leq t_1 + t_2$, this formula has been obtained in Theorem 2.6 in [20]. Theorem 2.4 above is the generalization of the result in [20] which involves the cases of the discrete time TASEPs as well as the continuous time one.

2.2.2 The Kardar-Parisi-Zhang (KPZ) scaling limit

Here we state our result on the scaling limit of the joint distribution function in Theorem 2.4. We focus on the following two cases:

- The discrete time Bernoulli TASEP (Sec. 2.1.2)

In the TASEP$_{\alpha, \beta, \gamma}$ introduced in Sec. 2.1.4, the case is realized by the specialization

$\alpha_1 = \alpha_2 = \alpha_{t_1} = \gamma = 0, \beta_{t_1+1} = \beta_{t_1+2} = \cdots = \beta_{t_1+t_2} = \beta = \frac{p}{1-p}.$

- The discrete time geometric TASEP (Sec. 2.1.3)

As above, it is realized by

$\alpha_1 = \alpha_2 = \alpha_{t_1} = \alpha, \gamma = \beta_{t_1+1} = \beta_{t_1+2} = \cdots = \beta_{t_1+t_2} = 0.$

To see the universal behavior of the fluctuations, we focus on the height function defined as follows.
Definition 2.6. For $z \in \mathbb{Z}$, the TASEP height function related to $X_t$ is given by

$$h_t(z) = -2(X_t^{-1}(z - 1) - X_0^{-1}(-1)) - z$$

where

$$X_t^{-1}(u) = \min\{k \in \mathbb{Z} : X_t(k) \leq u\}$$

denote the label of the rightmost particle which sits to the left of, or at, $u$ at time $t$ and we fix $h_0(0) = 0$.

Note that it can be represented as

$$h_t(z + 1) = h_t(z) + \hat{\eta}_t(z).$$

where

$$\hat{\eta}_t(z) = \begin{cases} 
1 & \text{if there is a particle at } z \text{ at time } t, \\
-1 & \text{if there is no particle at } z \text{ at time } t.
\end{cases}$$

We can extend the height function to a continuous function of $x \in \mathbb{R}$ by linearly interpolating between the integer points.

It is well known that the TASEP belongs to the Kardar-Parisi-Zhang (KPZ) universality class. Thus we expect that the proper scaling of the height function is

$$\frac{h_t(x) - At}{Ct^{1/3}}, \text{ with } x = Bt^{2/3}. \tag{2.16}$$

On average the height of the TASEP grows as $t^{1}$ with speed $A$, which is a constant. On the other hand the fluctuation of the height around the average is of order $t^{1/3}$ contrary to the $t^{1/2}$ of the usual scaling in the central limit theorem. The scaling exponent of the $x$-direction is 2/3, the twice of the one in $h$-direction 1/3, which suggest that the path of the height function becomes the Brownian motion like. The exponents (1/3, 2/3) are known to be universal and characterizing the KPZ universality class while the constants $A, B, C$ are not universal and depend on the models. As shown in Sec. 4, we have

- the discrete time Bernoulli TASEP case

$$A = \frac{p - 2}{2}, B = 2, C = 1, \tag{2.17}$$

- the discrete time geometric TASEP case

$$A = \frac{\alpha - 2}{2(1 - \alpha)}, B = 2, C = 1. \tag{2.18}$$

Based on the property of the height function, we define the scaled height, which is equivalent to (2.17) and (2.18) but a slightly different form appearing as the “1:2:3 scaling” in [28].
Definition 2.7. For \( t \in \mathbb{R}_{\geq 0} \) and \( x \in \mathbb{R} \), we define the scaling height function as the following.

- **The discrete time Bernoulli TASEP**
  \[
  \hat{h}^\varepsilon(t, x) = \varepsilon^\frac{1}{2} \left[ h_t(x) + \frac{2 - p}{2} \varepsilon^{-\frac{3}{2}} t \right],
  \]  
  where \( t \) and \( x \) are scaled as
  \[
  t = \frac{(2 - p)^3}{4p(1 - p)} \varepsilon^{-\frac{3}{2}}, \quad x = 2 \varepsilon^{-1} x.
  \]  

- **The discrete time geometric TASEP**
  \[
  \hat{h}^\varepsilon(t, x) = \varepsilon^\frac{1}{2} \left[ h_t(x) + \frac{2 - \alpha}{2(1 - \alpha)} \varepsilon^{-\frac{3}{2}} t \right],
  \]  
  where \( t \) and \( x \) are scaled as
  \[
  t = \frac{(2 - \alpha)^3}{4\alpha(1 - \alpha)} \varepsilon^{-\frac{3}{2}}, \quad x = 2 \varepsilon^{-1} x.
  \]

Our goal is to compute the \( \varepsilon \to 0 \) limit of the joint distribution function,

\[
\lim_{\varepsilon \to 0} \mathbb{P}_{\hat{h}^\varepsilon}(\hat{h}^\varepsilon(t, x_1) \leq a_1, \ldots, \hat{h}^\varepsilon(t, x_m) \leq a_m)
\]

for \( x_1 < x_2 < \cdots < x_m \in \mathbb{R} \) and \( a_1, \ldots, a_m \in \mathbb{R} \). Here \( \mathbb{P}_{\hat{h}^\varepsilon}() \) represents the probability measure in which the initial height profile is \( \hat{h}^\varepsilon(0, x) \). We will show that the limit converges to the joint distribution function characterizing the KPZ fixed point introduced in [20].

Here we introduce the KPZ fixed point. First we define UC and LC as follows.

**Definition 2.8.** (UC and LC [20]).
We define UC as the space of upper semicontinuous functions \( \hat{h} : \mathbb{R} \to [-\infty, \infty) \) with \( \hat{h}(x) \leq C_1 + C_2 |x| \) for some \( C_1, C_2 < \infty \) and \( \hat{h}(x) > -\infty \) for some \( x \) and LC as \( \text{LC} = \{ \hat{g} : -\hat{g} \in \text{UC} \} \).

Now we are ready to state the KPZ fixed point. For more detailed information, see [20].

**Definition 2.9** (The KPZ fixed point [20]). The KPZ fixed point is the unique Markov process on UC, \((\hat{h}(t, \cdot))_{t>0}\) with transition probabilities given by

\[
\mathbb{P}_{\hat{h}_0}(\hat{h}(t, x_1) \leq a_1, \ldots, \hat{h}(t, x_m) \leq a_m) = \det \left( I - \chi_a K_{t, \text{ext}}^{\text{hypo}(\hat{h}_0)} \chi_a \right)_{L^2(\{x_1, \ldots, x_m\} \times \mathbb{R})}.
\]  

Here in LHS, \( x_1 < x_2 < \cdots < x_m \in \mathbb{R} \) and \( a_1, \ldots, a_m \in \mathbb{R} \), \( \hat{h}_0 \in \text{UC} \) and \( \mathbb{P}_{\hat{h}_0} \) means the
measure on the process with initial data $\hat{h}_0$. In RHS, the kernel is given by

$$K_{t,\text{ext}}^{\text{hypo}(\hat{h}_0)}(x_i, v; x_j, u) = -\frac{1}{\sqrt{4\pi(x_j - x_i)}} \exp \left(-\frac{(u - v)^2}{4(x_j - x_i)}\right) \mathbf{1}_{x_i<x_j} + \left(S_{t,\text{ext}}^{\text{hypo}(\hat{h}_0)}\right)^* S_{t,x_j}(v, u),$$  \hspace{1cm} (2.25)

$$S_{t,x}(v, u) = t^{-\frac{3}{2}} e^{\frac{2(x^3 - v)}{t}} \text{Ai}(-t^{-\frac{1}{2}}(v - u) + t^{-\frac{4}{3}}x^2),$$ \hspace{1cm} (2.26)

$$S_{t,x}^{\text{hypo}(\hat{h})}(v, u) = \mathbb{E}_{B(0)=v}[S_{t,x-t'}(B(t'), u)1_{\tau<\infty}],$$ \hspace{1cm} (2.27)

where $(A)^*$ represents the adjoint of an integral operator $A$, and $B(x)$ is a Brownian motion with diffusion coefficient 2 and $\tau'$ is the hitting time of the hypograph of the function $\hat{h}$.

**Remark 2.10.** (2.25) and (2.26) can be written in terms of the differential operators $S_{t,x} = \exp\{x\partial^2 + t\partial^3/3\}$,

$$K_{t,\text{ext}}^{\text{hypo}(\hat{h}_0)}(x_i, \cdot; x_j, \cdot) = -e^{(x_j - x_i)\partial^2} \mathbf{1}_{x_i<x_j} + \left(S_{t,\text{ext}}^{\text{hypo}(\hat{h}_0)}\right)^* S_{t,x_j}.$$

In addition using the integral representation for the Airy function

$$\text{Ai}(z) = \frac{1}{2\pi i} \int_C dw \ e^{\frac{i}{3}w^3 - zw},$$

where $\int_C$ is the positively oriented contour going the straight lines from $e^{-\frac{i}{3}}\infty$ to $e^{\frac{i}{3}}\infty$ through 0, we find that $S_{t,x}(v, u)$ (2.26) can be expressed as

$$S_{t,x}(v, u) = \frac{1}{2\pi i} \int_C dw \ e^{\frac{i}{3}w^3 + xw^2 - (v - u)w}. \hspace{1cm} (2.28)$$

Now we assume that the limit

$$\hat{h}_0 = \lim_{\varepsilon \to 0} \hat{h}'(0, \cdot)$$ \hspace{1cm} (2.29)

exists. Note that by (2.13) and (2.29), (2.31), this assumption is rewritten as

$$\varepsilon^{\frac{1}{3}}[(X_0^{-1})^{-1}(x) + 2\varepsilon^{-1}x - 2] \xrightarrow{\varepsilon \to 0} -\hat{h}_0(-x),$$ \hspace{1cm} (2.30)

where $(X_0^{-1})^{-1}(x) := 2X_0^{-1}(-2\varepsilon^{-1}x - 1)$ and the left hand side is interpreted as a linear interpolation to make it a continuous function of $x \in \mathbb{R}$ and we chose the frame of reference by

$$X_0^{-1}(-1) = 1,$$ \hspace{1cm} (2.31)

i.e. the particle labeled 1 is initially the rightmost in $\mathbb{Z}_{<0}$.

Under this assumption, we have the following result for the limiting joint distribution function (2.23).

**Theorem 2.11.** *(One-sided fixed point formula).* Let $\hat{h}_0 \in \text{UC}$ with $\hat{h}_0(x) = -\infty$ for $x > 0$. ...
Then given \( x_1 < x_2 < \cdots < x_m \in \mathbb{R} \) and \( a_1, \ldots, a_m \in \mathbb{R} \), we have

\[
\lim_{\varepsilon \to 0} \mathbb{P}_{\tilde{h}_0} \left( \tilde{h}^\varepsilon(t, x_1) \leq a_1, \ldots, \tilde{h}^\varepsilon(t, x_m) \leq a_m \right) = \det \left( I - \chi \mathbb{K}_{t, \text{ext}}^{\text{hypo}(\tilde{h}_0)} \chi \right)_{L^2((x_1, \ldots, x_m) \times \mathbb{R})},
\]

where RHS is equivalent to that of (2.24).

**Remark 2.12.** The One-sided fixed point formula for the continuous time TASEP has been given in Proposition 3.6 in [20]. Our theorem 2.11 indicates that Bernoulli TASEP and geometric TASEP settle into the same class “KPZ fixed point. The KPZ fixed point is believed to be the universal process for the KPZ class with arbitrary fixed initial data. Our result supports this universality.

**Remark 2.13.** In fact we can remove the assumption \( \tilde{h}_0(x) = -\infty \) for \( x > 0 \) in the above theorem by using the similar argument in Theorem 3.8. in [20].

To prove Theorem 2.11, we use the following relationship between the particle positions \( X_j(t) \) and the height function \( h_t(z) \) (2.13). Let \( s_1, \ldots, s_k, m_1, \ldots, m_k \in \mathbb{R} \) and \( z_1, \ldots, z_k, n_1, \ldots, n_k \in \mathbb{Z} \). We have

\[
\mathbb{P}(h_t(z_1) \leq s_1, \ldots, h_t(z_k) \leq s_k) = \mathbb{P}(X_t(n_1) \geq m_1, \ldots, X_t(n_k) \geq m_k),
\]

which follows from the definitions of \( h_t(x) \) (2.13). By this relation, we see

\[
\lim_{\varepsilon \to 0} \mathbb{P}_{\tilde{h}_0} \left( \tilde{h}^\varepsilon(t, x_1) \leq a_1, \ldots, \tilde{h}^\varepsilon(t, x_m) \leq a_m \right) = \lim_{\varepsilon \to 0} \mathbb{P}_{\tilde{h}_0} \left( X_t^\varepsilon(n_1) > a_1, \ldots, X_t^\varepsilon(n_m) > a_m \right),
\]

where \( a_1, \ldots, a_m \in \mathbb{R} \) and \( t, n_j, x_j \) are scaled as

- the discrete time Bernoulli TASEP case

\[
t = \frac{(2 - p)^3}{4p(1 - p)} \varepsilon^{-3} t, \quad n_i = \frac{2 - p}{4} \varepsilon^{-3} t - \varepsilon^{-1} x_i - \frac{1}{2} \varepsilon^{-\frac{3}{2}} a_i + 1, \quad a_i = 2 \varepsilon^{-1} x_i - 2,
\]

- the discrete time geometric TASEP case

\[
t = \frac{(2 - \alpha)^3}{4\alpha(1 - \alpha)} \varepsilon^{-3} t, \quad n_i = \frac{2 - \alpha}{4(1 - \alpha)} \varepsilon^{-3} t - \varepsilon^{-1} x_i - \frac{1}{2} \varepsilon^{-\frac{3}{2}} a_i + 1, \quad a_i = 2 \varepsilon^{-1} x_i - 2.
\]

Thus we find that our goal, LHS of (2.34), can be obtained by taking the \( \varepsilon \to 0 \) limit of the expression (2.5) in Theorem 2.4 under the scaling (2.35) or (2.36). The critical step of this problem is the following propositions about the pointwise convergences. First, we state the result for the discrete time Bernoulli TASEP.

**Proposition 2.14.** (Pointwise convergence for the discrete time Bernoulli TASEP). Under the scaling (2.35),(dropping the \( i \) subscripts) and assuming that (2.30) holds, if we set \( z = \)
\[ \frac{p(2-p)}{4(1-p)} e^{-\beta t} + 2e^{-\beta x} + e^{-\beta (u+a)} - 2 \text{ and } y' = e^{-\beta v} \text{, then we have for } t > 0 \text{ as } \varepsilon \to 0, \]

\[ S^e_{t,x}(v, u) := e^{-\frac{\beta}{2} S^\text{Berg}_{t,n}(y', z)} \to S_{t,x}(v, u) \quad (2.37) \]

\[ S^e_{t,-x}(v, u) := e^{-\frac{\beta}{2} S^\text{Berg}_{t,n}(y', z)} \to S_{t,-x}(v, u) \quad (2.38) \]

\[ S^e_{t,-x}\epsilon(-h^-_0)(v, u) := e^{-\frac{\beta}{2} S^\text{Berg,\epsilon}_{t,n}(y', z)} \to S^\text{Berg,\epsilon}_{t,-x}\epsilon(-h^-_0)(v, u) \quad (2.39) \]

pointwise, where \( \hat{h}_0(x) = \hat{h}_0(-x) \) for \( x \geq 0 \), \( S_{t,x}(v, u) \) and \( S^\text{Berg,\epsilon}(v, u) \) are given by (2.26) and (2.27) respectively and

\[ S^\text{Berg}_{t,n}(z_1, z_2) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^n}{2z_2-w^n+1+z_2-z_1} \left( 1 + \frac{2p}{2-p} \left( \frac{w}{1} - 1 \right) \right)^t, \]

\[ \tilde{S}^\text{Berg}_{t,n}(z_1, z_2) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^n}{2z_2-w^n+1+z_2-z_1} \left( 1 - \frac{2p}{2-p} \left( \frac{w}{1} - 1 \right) \right)^{-t} \]

\[ \tilde{S}^\text{Berg,\epsilon}_{t,n}(z_1, z_2) = E_{RW_0=z_1} \left[ \tilde{R}^{\text{Berg,\epsilon}_{t-n}(RW_{t, \geq 1}1_{\tau<\infty})} \right] \]

with \( \Gamma_0 \) being a simple counterclockwise loop around 0 not enclosing 1, 1/p and \((1-p)/p\).

Next, we state that the point wise convergence for the discrete time geometric TASEP is obtained as the following.

**Proposition 2.15.** (Pointwise convergence for the discrete time geometric TASEP) Under the scaling (2.36), (dropping the i subscripts) and assuming that (2.30) holds in LC, if we set \( z = \frac{\alpha(2-\alpha)}{4(1-\alpha)^2} e^{-\beta t} + 2e^{-\beta x} + e^{-\beta (u+a)} - 2 \) and \( y' = e^{-\beta v} \), then we have for \( t > 0 \) as \( \varepsilon \to 0, \)

\[ S^e_{t,x}(v, u) := e^{-\frac{\beta}{2} S^\text{geo}_{t,n}(y', z)} \to S_{t,x}(v, u) \]

\[ S^e_{t,-x}(v, u) := e^{-\frac{\beta}{2} S^\text{geo}_{t,n}(y', z)} \to S_{t,-x}(v, u) \]

\[ S^e_{t,-x}\epsilon(-h^-_0)(v, u) := e^{-\frac{\beta}{2} S^\text{geo,\epsilon}_{t,n}(y', z)} \to S^\text{geo,\epsilon}_{t,-x}\epsilon(-h^-_0)(v, u) \]

pointwise, where \( \hat{h}_0(x) = \hat{h}_0(-x) \) for \( x \geq 0 \), \( S_{t,x}(v, u) \) and \( S^\text{geo,\epsilon}(v, u) \) are given by (2.26) and (2.27) respectively and

\[ S^\text{geo}_{t,-n}(z_1, z_2) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^n}{2z_2-w^n+1+z_2-z_1} \left( 1 - \frac{2\alpha}{2-\alpha} \left( \frac{w}{1} - 1 \right) \right)^{-t}, \]

\[ S^\text{geo}_{t,n}(z_1, z_2) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^n}{2z_2-w^n+1+z_2-z_1} \left( 1 + \frac{2\alpha}{2-\alpha} \left( \frac{w}{1} - 1 \right) \right)^{-t}, \]

\[ S^\text{geo,\epsilon}_{t,n}(z_1, z_2) = E_{RW_0=z_1} \left[ S^\text{geo,\epsilon}_{t-n}(RW_{t, \geq 1}1_{\tau<\infty}) \right] \]

with \( \Gamma_0 \) being a simple counterclockwise loop around 0 not enclosing 1, 1/\( \alpha \) and \((1-\alpha)/\alpha\).

**Remark 2.16.** Propositions 2.14 and (2.15) only give pointwise convergence of the kernels, but they can be upgraded to trace class convergence (see [20], [26] and [34]), which thus yields convergence of Fredholm determinants.
3 Distribution function of the TASEP

3.1 Transition probabilities

Let \( \Omega_N = \{ \vec{x} = (x_N, x_{N-1}, \cdots, x_1) \in \mathbb{Z}^N : x_N < \cdots < x_2 < x_1 \} \) be the Weyl chamber, whose elements express the particle positions of the TASEPs.

The main object of this subsection is the transition probability of the TASEP: For \( \vec{x}, \vec{y} \in \Omega_N \), we define

\[
G_t(x_N, \ldots, x_1) = \mathbb{P}(X_t = \vec{x} | X_0 = \vec{y}),
\]

which means the probability that at time \( t \) the particles are at positions \( x_N < \cdots < x_2 < x_1 \) provided that initially they are at positions \( y_N < \cdots < y_2 < y_1 \).

For all the three types of the TASEPs introduced in Sec. 2.1.1-2.1.3, the transition probabilities are obtained using Bethe ansatz (See [33]) and represented as determinants.

First, we give the result of the continuous time TASEP introduced in Sec. 2.1.1.

Lemma 3.1. ([33])

For the continuous time TASEP with \( N \in \{1, 2, 3, \cdots \} \) particles and rate \( \gamma \geq 0 \) introduced in Sec. 2.1.1, the transition probability has the following determinantal form

\[
G_t^{(\gamma)}(x_N, \cdots, x_1) = \det[F_n^{(\gamma)}(x_{N+1-i} - y_{N+1-j})]_{1 \leq i, j \leq N}
\]

with

\[
F_n^{(\gamma)}(x, t) = \frac{(-1)^n}{2\pi i} \oint_{\Gamma_{0,1}} dw \frac{(1 - w)^{-n}}{w^{x-n+1}} e^{\gamma t(w-1)}
\]

where \( \Gamma_{0,1} \) is any simple loop oriented anticlockwise which includes \( w = 0 \) and \( w = 1 \).

Next we introduce the result on the discrete time Bernoulli TASEP as follows.

Lemma 3.2. For the discrete time Bernoulli TASEP with \( N \in \{1, 2, \cdots \} \) particles and parameters \( \beta_i \geq 0, i = 1, 2, \cdots, t \) introduced in Sec. 2.1.2 , the transition probability has the following determinantal form

\[
G_t^{(\beta)}(\{x\}) = \det[F_n^{(\beta)}(x_{N+1-i} - y_{N+1-j}, t)]_{1 \leq i, j \leq N}
\]

with

\[
F_n^{(\beta)}(x, t) = \frac{(-1)^n}{2\pi i} \oint_{\Gamma_{0,1}} dw \frac{(1 - w)^{-n}}{w^{x-n+1}} \prod_{j=1}^{t} \frac{1 + \beta_j w}{1 + \beta_j}
\]

where \( \Gamma_{0,1} \) is any simple loop oriented anticlockwise which includes \( w = 0 \) and \( w = 1 \).

Proof. This determinantal formula has been obtained for the time homogeneous case \( \beta_1 = \beta_2 = \cdots = \frac{p}{1-p} \) in [7] and [29]. If we confirm that the following two equations hold, one
easily find that the result can be extended to the time inhomogeneous case:

\[
F_n^{(β)}(x, t+1) = \frac{1}{1 + β_{t+1}} F_n^{(β)}(x, t) + \frac{β_{t+1}}{1 + β_{t+1}} F_n^{(β)}(x - 1, t)
\]

and

\[
F_{n-1}^{(β)}(x, t) = F_n^{(β)}(x, t) - F_n^{(β)}(x + 1, t).
\]

It is easy to see that the above two equations hold.

We also give the result on the discrete time geometric TASEP introduced in Sec.2.1.3.

**Lemma 3.3.** For the discrete time geometric TASEP with \( N \in \{1, 2, \cdots \} \) particles and parameters \( 0 \leq α_i \leq 1, \ i = 1, 2, \cdots, t \) introduced in Sec. 2.1.3, the transition probability has the following determinantal form

\[
G_t^{(α)}(x_N, \cdots, x_1) = \det[F_{i-1}^{(α)}(x_{N+1-i} - y_{N+1-j}, t)]_{1 \leq i, j \leq N}
\]

with

\[
F_n^{(α)}(x, t) = \frac{(-1)^n}{2πi} \oint_{Γ_{0,1}} \frac{du}{w^{x-n+1}} \prod_{j=1}^{t} \left( 1 - \alpha_j \right) \prod_{i \in μ} \left( 1 - α_j \right) w
\]

where \( Γ_{0,1} \) is any simple loop oriented anticlockwise which includes \( w = 0 \) and \( w = 1 \).

**Proof.** We will check that the determinantal representation (3.8) satisfies the Kolmogorov forward equation.

\[
G_{t+1}^{(α)}(x_N, \cdots, x_1) = \sum_{μ \subset \{1, \cdots, N-1\}} (1 - α_{i+1})^{[μ]+1} \prod_{i \in μ} \alpha_i \prod_{j \in μ} \frac{k_j-1}{k_j} \cdot G_t^{(α)}(x(μ))
\]

where \( μ \) can take the empty set \( φ \), \( \bar{μ} := \{1, \cdots, N-1\} \setminus μ \), \( |\bar{μ}| \) means the number of elements in \( \bar{μ} \), and we define \( k_i \) and \( x(μ) := (x_N^μ, \cdots, x_1^μ) \) by

\[
k_i = \begin{cases} x_i - x_{i+1} & \text{for } i = 1, \cdots, N-1, \\ \infty & \text{for } i = N. \end{cases}
\]

\[
x^μ = \begin{cases} x_i + 1 & \text{for } i \in μ, \\ x_i - a_i & \text{for } i \in \bar{μ} \cup \{N\}. \end{cases}
\]

RHS in (3.9) consists of \( 2^{N-1} \) terms and each element \( j \) in the subset \( μ \) represents the label of the particle which is on \( x_{j+1} + 1 \) at time \( t \). Taking the hopping probability (2.2) in the geometric TASEP into account, we see that when \( j \in μ \), we should assign the weight \( α_j^{jump} \) without the factor \( 1 - α_{i+1} \) for the jump of the \( j + 1 \)th particle. Thus for the jumps of the \( N - |μ| = |\bar{μ}| + 1 \) particles, we put the factor \( (1 - α_{i+1})^{[μ]+1} \). In Appendix, we explain (3.9) in the case of \( N = 3 \).
We see that (3.9) is equivalent to the following two conditions

\[ G_{t+1}(x_N, \ldots, x_1) = \sum_{a_1, \ldots, a_n \in \{0, \ldots, \infty\}} (1 - \alpha t + 1)^N \alpha_t^{a_1 + \cdots + a_N} G_t^{(\alpha)}(x_N - a_N, \ldots, x_1 - a_1) \] (3.11)

\[ \sum_{m,n=0}^{\infty} (1 - \alpha t + 1) \alpha_t^{m+n} G_t^{(\alpha)}(x_N, \ldots, x_k - m - 1, x_k - n, x_{k-1}, \ldots, x_1) = \sum_{m=0}^{\infty} \alpha_t^m G_t^{(\alpha)}(x_N, \ldots, x_k - m - 1, x_k, x_{k-1}, \ldots, x_1) \] (3.12)

for \( k = 1, \ldots, N - 1 \). Now we show that (3.11) and (3.12) imply (3.9). First, we show below the equivalence

\[ \prod_{i=1}^{N} \sum_{a_i=0}^{\infty} (1 - \alpha t + 1) \alpha_t^{a_i} G_t^{(\alpha)}(x_N - a_N, \ldots, x_1 - a_1) = \sum_{\mu \subset \{1, \ldots, N\}} (1 - \alpha t + 1)^{\mu + 1} \prod_{i=\mu} \alpha_t^{a_i} \cdot \prod_{j \notin \mu} \alpha_t^{k_j - 1} \cdot G_t^{(\alpha)}(\vec{x}(\mu), x_1) \] (3.13)

by using the equation (3.12) with \( k = 1 \) and the version of \( N - 1 \) particles in (3.13),

\[ \prod_{i=1}^{N} \sum_{a_i=0}^{\infty} (1 - \alpha t + 1) \alpha_t^{a_i} G_t^{(\alpha)}(x_N - a_N, \ldots, x_1 - a_1) = \sum_{\nu \subset \{2, \ldots, N - 1\}} (1 - \alpha t + 1)^{\nu + 1} \prod_{i=\nu} \alpha_t^{a_i} \cdot \prod_{j \notin \nu} \alpha_t^{k_j - 1} \cdot G_t^{(\alpha)}(\vec{x}(\nu), x_1) \] (3.14)

where \( \bar{\nu} := \{2, \ldots, N - 1\} \setminus \nu \), \( \vec{x}(\nu) := (x_{2}^{\nu}, \ldots, x_{N}^{\nu}) \) with

\[ x_i^{\nu} = \begin{cases} x_i + 1 & \text{for } i \in \nu, \\ x_i - a_i & \text{for } i \in \bar{\nu} \cup \{N\}. \end{cases} \] (3.15)
In LHS of (3.13), we divide the sum of $a_1$ as follows.

\[
\text{LHS of (3.13)} = \left( \prod_{i=2}^{N} \sum_{a_i=0}^{\infty} (1 - \alpha_{t+1}) a_{t+1}^{a_i} \right) \left\{ \left( \sum_{a_1=0}^{k_1-2} (1 - \alpha_{t+1}) a_{t+1}^{a_1} \right) G_t^{(\alpha)}(x_N - a_N, \ldots, x_1 - a_1) \right. \\
+ \left. \left( \sum_{a_1=k_1-1}^{\infty} (1 - \alpha_{t+1}) a_{t+1}^{a_1} \right) G_t^{(\alpha)}(x_N - a_N, \ldots, x_1 - a_1) \right\} \\
= \left( \prod_{i=2}^{N} \sum_{a_i=0}^{\infty} (1 - \alpha_{t+1}) a_{t+1}^{a_i} \right) \left\{ \left( \sum_{a_1=0}^{k_1-2} (1 - \alpha_{t+1}) a_{t+1}^{a_1} \right) G_t^{(\alpha)}(x_N - a_N, \ldots, x_1 - a_1) \right. \\
+ \left. \left( \sum_{a_1=0}^{\infty} (1 - \alpha_{t+1}) a_{t+1}^{a_1+k_1-1} \right) G_t^{(\alpha)}(x_N - a_N, \ldots, x_2 - a_2, x_2 + 1 - a_1) \right\}.
\]

(3.16)

By using (3.12) with $k = 1$, we find

\[
(3.16) = \left( \prod_{i=2}^{N} \sum_{a_i=0}^{\infty} (1 - \alpha_{t+1}) a_{t+1}^{a_i} \right) \left\{ \left( \sum_{a_1=0}^{k_1-2} (1 - \alpha_{t+1}) a_{t+1}^{a_1} \right) G_t^{(\alpha)}(x_N - a_N, \ldots, x_1 - a_1) \right. \\
+ \left. \alpha_{t+1}^{k_1-1} G_t^{(\alpha)}(x_N - a_N, \ldots, x_2 - a_2, x_2 + 1) \right\}.
\]

(3.17)

By applying (3.14) to (3.17),

\[
(3.17) = \left\{ \sum_{\nu \in \{2, \ldots, N-1\}} (1 - \alpha_{t+1})^{|\nu|+1} \prod_{i \in \nu \cup \{N\}} \sum_{a_i=0}^{k_i-2} \prod_{j \in \nu} \alpha_{t+1}^{a_j} \right. \\
\times \left. \left( \sum_{a_1=0}^{k_1-2} (1 - \alpha_{t+1}) a_{t+1}^{a_1} \right) G_t^{(\alpha)}(\vec{x}(\nu), x_1 - a_1) + \alpha_{t+1}^{k_1-1} G_t^{(\alpha)}(\vec{x}(\nu), x_2 + 1) \right\} \\
= \text{RHS of (3.13)}.
\]

Thus we have shown (3.13) by using (3.14). Similarly, we can show (3.14) by using the equation (3.12) with $k = 2$ and

\[
\prod_{i=3}^{N} \sum_{a_i=0}^{\infty} (1 - \alpha_{t+1}) a_{t+1}^{a_i} G_t^{(\alpha)}(x_N - a_N, \ldots, x_3 - a_3, x_2, x_1) \\
= \sum_{\lambda \subseteq \{3, \ldots, N-1\}} (1 - \alpha_{t+1})^{|\lambda|+1} \prod_{i \in \lambda \cup \{N\}} \sum_{a_i=0}^{k_i-2} \prod_{j \in \lambda} \alpha_{t+1}^{a_j} G_t^{(\alpha)}(\vec{x}(\lambda), x_2, x_1)
\]

16
where \( \mathbf{x} := \{3, \ldots, N - 1\} \setminus \lambda \), \( \vec{x}^{(\lambda)} := (x_1^\lambda, \ldots, x_3^\lambda) \) and for \( i = 3, \ldots, N \)

\[
x_i^\lambda = \begin{cases} 
  x_{i+1} + 1 & \text{for } i \in \lambda, \\
  x_i - a_i & \text{for } i \in \mathbf{x} \cup \{N\}.
\end{cases}
\]

Therefore, by repeatedly using the similar calculation, we can show the equivalence (3.13) by using conditions can be obtained from (3.12) for \( k = 1, \ldots, N - 1 \), which leads to the equivalence between the Kolmogorov forward equation (3.9) and two conditions (3.11) and (3.12).

Now we will check (3.11) and (3.12). For convenience, we put \( F^\alpha_n(x, t) = F^{(\alpha)}_n(x - y_{N+1-j}, t) \).

Inserting (3.8) into RHS of (3.11) and using the multilinearity of the determinant, we find that RHS of (3.11) becomes

\[
\sum_{\alpha_1, \ldots, \alpha_N \in \{0, \ldots, \infty\}} (1 - \alpha_{t+1})^N \alpha_{t+1}^{\alpha_1 + \cdots + \alpha_N} \det[F^j_{i-j}(x_{N+1-i} - a_{N+1-j}, t)]_{1 \leq i, j \leq N}
\]

\[
= \det \left[ (1 - \alpha_{t+1}) \sum_{\alpha_{N+1-i} = 0}^\infty \alpha_{t+1}^{\alpha_{N+1-i}} F^j_{i-j}(x_{N+1-i} - a_{N+1-j}, t) \right]_{1 \leq i, j \leq N}.
\]

Thus we see that if the functions \( F^{(\alpha)}_n \) satisfies

\[
F^{(\alpha)}_n(x, t + 1) = \sum_{y=0}^\infty \alpha_{t+1}^y (1 - \alpha_{t+1}) F^{(\alpha)}_n(x - y, t),
\]

then (3.18) is equal to LHS of (3.11) \( G^{(\alpha)}_{t+1}(x_N, \ldots, x_1) = \det \left[ F^j_{i-j}(x_{N+1-i}, t + 1) \right]_{1 \leq i, j \leq N} \).

We also consider the condition (3.12). It can be written as

\[
0 = \det \begin{bmatrix}
  F^j_{N-k-j}(x_k, t + 1) - F^j_{N+1-k-j}(x_k, t) \\
  \vdots \\
  F^j_{N-k-j}(x_k, t + 1) - F^j_{N+1-k-j}(x_k, t)
\end{bmatrix}_{1 \leq j \leq N}.
\]

One easily sees that it holds if the functions \( F^{(\alpha)}_n \) satisfy

\[
F^{(\alpha)}_{n-1}(x - 1, t + 1) = c(F^{(\alpha)}_n(x, t + 1) - F^{(\alpha)}_n(x, t))
\]

for for arbitrary \( c \). Here we choose \( c = (1 - \alpha_{t+1})/\alpha_{t+1} \).

Therefore the function \( F^{(\alpha)}_n \) are determined by the two relations (3.19) and (3.21), as well as the initial condition

\[
G^{(\alpha)}_0(x_N, \ldots, x_1) = \delta_{y_N, x_N} \cdots \delta_{y_1, x_1}.
\]

\( F^{(\alpha)}_0(x, t) \) is already determined by one particle configurations. In fact, in this case, \( G^{(\alpha)}_t(x) = \)
\[ P(x(t) = x|x(0) = y) = F_0^{(\alpha)}(x - y, t). \] Therefore

\[ F_0^{(\alpha)}(x - y, t) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{1}{w^{x+y+1}} \prod_{j=0}^t \frac{1 - \alpha_j}{1 - \alpha_j w} \] \hspace{1cm} (3.23)

where \( \Gamma_0 \) is any simple loop around 0 oriented anticlockwise. This result is consistent with (3.19) and (3.22). Denote by \( \Delta \) the discrete derivative \( \Delta f(x, t) := \frac{1 - \alpha_t}{\alpha_t} (f(x + 1, t) - f(x + 1, t - 1)) \). Then by (3.21),

\[ F_n^{(\alpha)}(x, t) = (-1)^n (\Delta_n f_0^{(\alpha)})(x, t) \] \hspace{1cm} (3.24)

holds. Therefore to obtain \( F_n^{(\alpha)} \) we simply apply \( \Delta_n \alpha \)

\[ \Delta_n^{\alpha}, \frac{1}{w^{x+n+1}} \prod_{j=0}^t \frac{1 - \alpha_j}{1 - \alpha_j w} = (-1)^n (1 - w)^n \prod_{j=0}^t \frac{1 - \alpha_j}{1 - \alpha_j w}. \] \hspace{1cm} (3.25)

From the above, for \( n \geq 0 \),

\[ F_n^{(\alpha)}(x, t) = \frac{(-1)^n}{2\pi i} \oint_{\Gamma_0} dw \frac{(1 - w)^n}{w^{x+n+1}} \prod_{j=0}^t \frac{1 - \alpha_j}{1 - \alpha_j w}. \] \hspace{1cm} (3.26)

In this case, there is no pole at \( w = 1 \), and therefore replacing \( \Gamma_0 \) by \( \Gamma_{0,1} \) leaves the result unchanged.

For \( n > 0 \), \( F_n^{(\alpha)} \) is determined by the recurrence relation

\[ F_{n+1}^{(\alpha)}(x, t) = \sum_{y \geq x} F_n^{(\alpha)}(y, t) \] \hspace{1cm} (3.27)

together with the property that \( F_0^{(\alpha)}(x, t) = 0 \) for \( x \) large enough.

In order for (3.27) to be satisfied for all \( n \), we need to take the poles both at 0 and 1.

Finally, combining the above three formulas in Lemmas 3.1–3.3, we obtain the transition probability of the TASEP \( \alpha, \beta, \gamma \) introduced in Sec. 2.1.4.

**Proposition 3.4.** For the TASEP \( \alpha, \beta, \gamma \) with \( N \in \{1, 2, \cdots\} \) particles and parameters \( \alpha_t := (\alpha_1, \cdots, \alpha_t) \in [0, 1]^{t_1}, \beta_t := (\beta_{t_1+1}, \cdots, \beta_{t_1+t_2}) \in \mathbb{R}_{\geq 0}^{t_2}, \gamma > 0 \) and \( t_3 > 0 \) introduced in Sec. 2.1.4, the transition probability to \( t = t_1 + t_2 + t_3 \) has the following determinantal form

\[ G_t^{\alpha, \beta, \gamma}(x_N, \cdots, x_1) = \det [F_{i-j}^{\alpha, \beta, \gamma}(x_{N+1-i} - y_{N+1-j}, t)]_{1 \leq i,j \leq N} \] \hspace{1cm} (3.28)

with

\[ F_n^{\alpha, \beta, \gamma}(x, t) = \frac{(-1)^n}{2\pi i} \oint_{\Gamma_{0,1}} dw \frac{(1 - w)^{-n}}{w^{x+n+1}} f_{\alpha, \beta, \gamma}(w, t) \]
Remark 3.6. The fact that the joint distribution of particle positions can be expressed by the Fredholm determinant is proved by [7] for the discrete time Bernoulli TASEP and by [8] for the continuous time TASEP. The above result includes a generalization of the initial conditions for particle position in the results of distribution of particle position of [7] and [8], and is the result when the continuous time TASEP, the discrete time Bernoulli TASEP and the discrete time geometric TASEP are mixed.

Proof. We can obviously prove from Lemma 3.1, Lemma 3.2, and Lemma 3.3.

### 3.2 Biorthogonal ensembles for the joint distribution functions

In the following we consider the joint distribution function of the particle positions in the TASEP\(_{\alpha,\beta,\gamma}\) introduced in Sec. 2.1.4. We will give a formula in terms of a Fredholm determinant whose kernel can be written in an explicit form.

**Theorem 3.5.** We consider the TASEP\(_{\alpha,\beta,\gamma}\) introduced in Sec. 2.1.4. For \((n_1, n_2, \cdots, n_m) \in \mathbb{Z}^m\) with \(1 \leq n_1 < n_2 < \cdots < n_m \leq N\) and \((a_1, a_2, \cdots, a_m) \in \mathbb{Z}^m\), we have

\[
\mathbb{P}(X_i(n_j) > a_j, j = 1, \ldots, m) = \det(I - \chi_a K_i \chi_a)e^{(n_1, \ldots, n_m) \times \mathbb{Z}).
\]

Here the right hand side is a Fredholm determinant with the kernel

\[
K_i(n_1, x_i; n_j, x_j) = -Q^{n_j-n_i} (x_i, x_j) 1_{n_j < n_i} + \sum_{k=1}^{n_j} \Psi^{n_i}_{n_i-k}(x_i) \Phi^{n_j}_{n_j-k}(x_j)
\]

where \(Q^n(x, y)\) represents \(n\)-times convolution of \(Q(x, y) = 1/2^{x-y} \cdot 1_{x>y}\). The functions \(\Psi_k^n(x)\) and \(\Phi_k^n(x), k = 0, \ldots, n-1\) are defined as follows: For \(\Psi_k^n(x)\) with \(k \leq n-1\), we define

\[
\Psi_k^n(x) := \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^k}{2^{x-X_0(n-k)}w^{x+k+1-X_0(n-k)}} f_{\alpha,\beta,\gamma}(w, t)
\]

where \(\Gamma_0\) is any positively oriented simple loop including the pole at \(w = 0\) and \(f_{\alpha,\beta,\gamma}(w, t)\) is defined by (3.29). The functions \(\Phi_k^n(x), k = 0, \ldots, n-1\), are defined implicitly by

1. The biorthogonality relation

\[
\sum_{x \in \mathbb{Z}} \Psi_k^n(x) \Phi_l^n(x) = 1_{k=l};
\]

2. \(2^{-k}\Phi_k^n(x)\) is a polynomial of degree at most \(n-1\) in \(x\) for each \(k\).

**Remark 3.6.** The fact that the joint distribution of particle positions can be expressed by the Fredholm determinant is proved by [7] for the discrete time Bernoulli TASEP and by [8] for the continuous time TASEP. The above result includes a generalization of the initial conditions for particle position in the results of distribution of particle position of [7] and [8], and is the result when the continuous time TASEP, the discrete time Bernoulli TASEP and the discrete time geometric TASEP are mixed.

Proof. This proof can be proved in the same way as Theorem 4.3 in [28] by using the propositions and lemmas written in Chapter 4 of [28]. Therefore, only the outline of the proof is described below. (See [28] for more details.)
From the proof of Proposition 3.2 in [7] and the proof of Theorem 2.1 in [8], we can find that if the following three equations are satisfied, the proof can be done regardless of the form of \( f_{\alpha, \beta, \gamma}(w, t) \):

\[
F_{n+1}(x, t) = \sum_{y \geq x} F_n(y, t),
\]

\[
\Psi^n_k(x) = \frac{(-1)^k}{2^{x-X_0(n-k)}} F_{-k}(x - X_0(n-k), t), \quad k = 0, \ldots, n
\]

\[
Q^{n-m}\Psi^n_{n-k} = \Psi^m_{m-k}. \tag{3.33}
\]

Therefore, it is sufficient for us to check the above three equations, but it is not hard to confirm that the above three equations hold.

This completes the proof. \( \square \)

In the following, we will write \( \Phi^n_k(x) \) that was not explicitly written in previous research [7] in an explicit form.

First, we prepare the tools to use. \( Q^n \) can easily be taken from definition \( Q \); \n
\[
Q^n(x, y) = \frac{1}{2^{x-y}} \binom{x-y-1}{m-1} 1_{x \geq y+m}. \tag{3.34}
\]

As operators on \( \ell^2(\mathbb{Z}) \), \( Q \) and \( Q^n \) are invertible;

\[
Q^{-1}(x, y) = 2 \cdot 1_{x=y-1} - 1_{x=y}, \quad Q^{-m}(x, y) = (-1)^{y-x+m} 2^{y-x} \binom{m}{y-x}. \tag{3.35}
\]

Now we define

\[
R_{\alpha, \beta, \gamma, t}(x, y) := \frac{1}{2\pi i} \int_{\Gamma_0} dw \frac{f_{\alpha, \beta, \gamma}(w, t)}{2^{x-y} w^{x-y+1}}, \tag{3.36}
\]

where

\[
f_{\alpha, \beta, \gamma}(w, t) = \prod_{j=1}^{t_1} \frac{1 - \alpha_j}{1 - \alpha_j w} \cdot \prod_{j=t_1+1}^{t_1+t_2} \frac{1 + \beta_j w}{1 + \beta_j} \cdot e^{\gamma t_3 (w-1)}.
\]

Note that \( \Psi^n_0 = R_{\alpha, \beta, \gamma, t} \delta_{X_0(n)} \) with \( \delta_y(x) = 1_{x=y} \).

Then, the following lemma holds.

**Lemma 3.7.** For \( n \in \mathbb{Z} \),

\[
\Psi^n_k = R_{\alpha, \beta, \gamma, t} Q^{-k} \delta_{X_0(n-k)}. \tag{3.37}
\]

**Proof.** By (3.33) and (3.36),

\[
\Psi^n_k = Q^{-k} R_{\alpha, \beta, \gamma, t} \delta_{X_0(n-k)}
\]

holds.

Now, note that \( Q \) and \( R_{\alpha, \beta, \gamma, t} \) commute, because the kernels \( Q(x, y) \) and \( R_{\alpha, \beta, \gamma, t}(x, y) \) only depend on \( x - y \). Therefore, we obtain

\[
\Psi^n_k = R_{\alpha, \beta, \gamma, t} Q^{-k} \delta_{X_0(n-k)}.
\]

20
From the expression of $R_{\alpha,\beta,\gamma,t}$, we define
\[
R_{\alpha,\beta,\gamma,t}^{-1}(x,y) := \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{f_{\alpha,\beta,\gamma,t}^{-1}(w,t)}{2^{x-y}w^{x-y+1}},
\] (3.38)

It is not hard to check that $R_{\alpha,\beta,\gamma,t}R_{\alpha,\beta,\gamma,t}^{-1} = R_{\alpha,\beta,\gamma,t}^{-1}R_{\alpha,\beta,\gamma,t} = I$. At this time, the following theorem holds.

**Theorem 3.8.** Fix $0 \leq k < n$ and consider particles at $X_0(1) > X_0(2) > \cdots > X_0(n)$. Let $h^n_k(l, z)$ be the unique solution to the initial-boundary value problem for the backwards heat equation
\[
\begin{cases}
(Q^*)^{-1}h^n_k(l, z) = h^n_k(l + 1, z) & l < k, z \in \mathbb{Z}, \\
h^n_k(k, z) = 2^{z-X_0(n-k)} & z \in \mathbb{Z}, \\
h^n_k(l, X_0(n-l)) = 0 & l < k.
\end{cases}
\] (3.39)

Then the functions $\Phi^n_k$ from Theorem 3.5 are given by
\[
\Phi^n_k(z) = (R_{\alpha,\beta,\gamma,t}^*)^{-1}h^n_k(0, \cdot)(z) = \sum_{y \in \mathbb{Z}} h^n_k(0,y)R_{\alpha,\beta,\gamma,t}^{-1}(y,z).
\] (3.40)

Here $Q^*(x,y) = Q(y,x)$ is the kernel of the adjoint of $Q$ (and likewise for $R_{\alpha,\beta,\gamma,t}^*$).

**Remark 3.9.** It is not true that in general $Q^*h^n_k(l + 1, z) = h^n_k(l, z)$. In fact, $Q^*h^n_k(k, z)$ is divergent from the following.
\[
Q^*h^n_k(k, \cdot)(z) = \sum_{y \in \mathbb{Z}} h^n_k(k,y)Q(y,z) = \sum_{y \in \mathbb{Z}} 2^{y-X_0(n-k)} \frac{1}{2^{y-z}}1_{y \geq z} = \sum_{y \in \mathbb{Z}, y \geq z} 2^{z-X_0(n-k)} = \infty.
\]

**Proof.** This proof is almost the same as [20]. We show that the same can be proved in discrete time.

The existence and uniqueness of solutions of (3.39a)-(3.39c) is elementary consequence of the fact that $ker(Q^*)^{-1}$ has dimension 1 and it is spanned by the function $2^z$, which allows us to march forwards from the initial condition $h^n_k(k, z) = 2^{z-X_0(n-k)}$ uniquely solving the boundary value problem $h^n_k(l, X_0(n-k)) = 0$ at each step.

First, we prove that $2^{-x}h^n_k(0, x)$ is a polynomial of degree at most $k$. We use the mathematical
induction. By (3.39b),
\[ 2^{-x}h_k^n(k, x) = 2^{-x}2^{x-X_0(n-k)} = 2^{X_0(n-k)}. \] 

(3.41)

Therefore, \( 2^{-x}h_k^n(k, x) \) is polynomial of degree 0.

Now, assume that \( \hat{h}_k^n(l, x) := 2^{-x}h_k^n(l, x) \) is a polynomial of degree at most \( k-l \) for some \( 0 < l \leq k \). By (3.39a) and (3.35),
\[
\hat{h}_k^n(l, y) = 2^{-y}(Q^*)^{-1}h_k^n(l - 1, y) \\
= 2^{-y}(2 : h_k^n(l - 1, y - 1) - h_k^n(l - 1, y)) \\
= 2^{-(y+1)}h_k^n(l - 1, y - 1) - 2^{-y}h_k^n(l - 1, y) \\
= \hat{h}_k^n(l - 1, y - 1) - \hat{h}_k^n(l - 1, y).
\]

Taking the sum over \( x \geq X_0(n-l+1) \), one sees
\[
\sum_{y=X_0(n-l+1)+1}^{x} 2^{-y}h_k^n(l, y) = \sum_{y=X_0(n-l+1)+1}^{x} \hat{h}_k^n(l, y) \\
= \sum_{y=X_0(n-l+1)+1}^{x} (\hat{h}_k^n(l - 1, y - 1) - \hat{h}_k^n(l - 1, y)) \\
= \hat{h}_k^n(l - 1, X_0(n - l + 1)) - \hat{h}_k^n(l - 1, x).
\]

Therefore, using (3.39c), we have \( \hat{h}_k^n(l - 1, x) = \sum_{y=X_0(n-l+1)+1}^{x} 2^{-y}h_k^n(l, y) \).

By the induction hypothesis, \( \hat{h}_k^n(l - 1, x) \) is a polynomial of degree at most \( k-l+1 \) because \( \hat{h}_k^n(l, y) \) is a polynomial of degree at most \( k-l+1 \).

Similarly, taking the sum \( x < X_0(n-l+1) \), we get \( \hat{h}_k^n(l - 1, x) = \sum_{y=x+1}^{X_0(n-l+1)} \hat{h}_k^n(l, y) \), which is again a polynomial of degree at most \( k-l+1 \). From the above, it was shown that \( 2^{-x}h_k^n(0, x) \) is a polynomial of degree at most \( k \).

Now, we show that \( \sum_{y \in \mathbb{Z}} h_k^n(0, y) R_{a,\beta,\gamma,t}^{-1}(y, z) \), which is the rhs of (3.40), satisfies the condition

\[ \sum_{y \in \mathbb{Z}} h_k^n(0, y) R_{a,\beta,\gamma,t}^{-1}(y, z) = B_j n^{k+1-j} \] where \( B_j \) is Bernoulli number.

*1This can be understood from Faulhaber’s formula : \( \sum_{j=1}^{n} j^k = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j} \) where \( B_j \) is Bernoulli number.
(2) in Theorem 3.5. By (3.38), we have

\[
2^{-z} \sum_{y \in \mathbb{Z}} h_k^n(y, z) \mathcal{R}^{-1}_{\alpha, \beta, \gamma, t}(y, z) = 2^{-z} \sum_{y \geq z} \left( \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dw}{2y-\frac{z}{w} + 1} \right) h_k^n(0, y) \\
= \sum_{y \geq z} \left( \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dw}{2y-\frac{z}{w} + 1} \right) 2^{-y} h_k^n(0, y)
\]

(3.42)

Because $2^{-z} h_k^n(0, z)$ is a polynomial of degree at most $k$, it is enough to note that the sum is a polynomial of degree at most $k$ in $z$ as well. Next, we check the biorthogonality relation (1) of Theorem 3.5. Using (3.37), we get

\[
\sum_{z \in \mathbb{Z}} \mathcal{P}_n(z) \mathcal{P}_k(z) = \sum_{z_1, z_2 \in \mathbb{Z}} \sum_{z \in \mathbb{Z}} \mathcal{R}_{\alpha, \beta, \gamma, t}(z_1, z_2) Q^{-1}(z, X_0(n-l)) h_k^n(0, z_1) R_{\alpha, \beta, \gamma, t}(z_2, z)
\]

\[
= \sum_{z \in \mathbb{Z}} Q^{-1}(z, X_0(n-l)) h_k^n(0, z) = (Q^*)^{-l} h_k^n(0, X_0(n-l)),
\]

where in the first equality we have used the decay of $R_{\alpha, \beta, \gamma, t}$ and the fact that $2^{-z} h_k^n(0, x)$ is a polynomial together with the fact that the $z_1$ sum is finite to apply Fubini.

For $l \leq k$, from (3.39b) and (3.39c), we have the boundary condition

\[
h_k^n(l, X_0(n-l)) = 1_{l=k}.
\]

(3.43)

Thus, we get

\[
(Q^*)^{-l} h_k^n(0, X_0(n-l)) = h_k^n(l, X_0(n-l)) = 1_{l=k}.
\]

For $l > k$, we use (3.39a) and (3.39b), $2^z \in \ker(Q^*)^{-1}$,

\[
(Q^*)^{-l} h_k^n(0, X_0(n-l)) = (Q^*)^{-l-k} h_k^n(k, X_0(n-l)) = 0.
\]

This completes the proof.

\[\square\]

3.3 Representation of the TASEP kernel in terms of a hitting probability

Combining Theorem 3.5 with (3.37) and (3.40), we have obtained the following expression of the kernel $K_t$ (3.31),

\[
K_t(n_i, \cdots, n_j, \cdot) = -Q^n_{n_i, \cdots, n_j} 1_{n_i < n_j} + R_{\alpha, \beta, \gamma, t} Q^{-n_j} G_{0, n_j} R_{\alpha, \beta, \gamma, t}^{-1}.
\]

(3.44)
Here $Q$, $R_{\alpha,\beta,\gamma,t}$ and $R_{\alpha,\beta,\gamma,t}^{-1}$ are given by (3.34), (3.36) and (3.38) respectively and $G_{0,n_j}$ is defined by

$$G_{0,n}(z_1, z_2) = \sum_{k=0}^{n-1} Q^{n-k}(z_1, X_0(n - k)) h_k^n(0, z_2), \quad (3.45)$$

where $h_k^n$ is the solution of (3.39a)-(3.39c).

In this subsection, following the method in [20], we further rewrite the kernel in order to take the KPZ scaling limit. We use the fact that $h_k^n$ can be written as hitting probabilities of random walk. Let $RW_m^*$ with $RW_0^* = c$ be the position of the random walk with $\text{Geom}[\frac{1}{2}]$ jumps strictly to the right starting from $c \in \mathbb{Z}$, i.e.

$$RW_m^* = c + \chi_1 + \cdots + \chi_m,$$

where $\chi_j$, $j = 1, 2, \cdots, m$ are i.i.d. random variables with $\mathbb{P}(\chi_j = k) = 1/2^{k+1}$, $k \in \mathbb{Z}_{\geq 0}$. Note that $Q^*$ defined below (3.40) represents the transition kernel of the random walk: for $m = 0, 1, \cdots$, we have

$$Q^*(x, y) = \mathbb{P}(RW_{m+1}^* = x | RW_m^* = y). \quad (3.46)$$

For $0 \leq l \leq k \leq n - 1$ we define the stopping times

$$\tau_{l,n} = \min\{m \in \{l, \ldots, n-1\} : RW_m^* > X_0(n-m)\}, \quad (3.47)$$

where we set $\min \emptyset = \infty$.

Then, we have the following.

**Lemma 3.10.** ([20])

For $z \leq X_0(n-l)$, the function $h_k^n(l, z)$ can be written by

$$h_k^n(l, z) = \mathbb{P}_{RW_{l-1}^* = z} (\tau_{l,n} = k) \quad (3.48)$$

which is the probability of the walk starting at $z \in \mathbb{Z}$ at time $l-1 \in \mathbb{Z}$ and hitting $X_0(n-k)$ at time $k \in \mathbb{Z}$.

**Remark 3.11.** Eq. (3.48) is written in [20] and the proof is left to the readers as Exercise 5.17 in [28]. Here we give an answer.

**Proof.** By (3.39a)-(3.39c), it is enough to check $(Q^*)^{-1} \mathbb{P}_{RW_{l-1}^* = z}(\tau_{l,n} = k) = \mathbb{P}_{RW_l^* = z}(\tau_{l+1,n} = k)$. Now, we assume that $X_0(n-k) = x^{*2}$ for convenience. Then, by (3.35)

$$(Q^*)^{-1} \mathbb{P}_{RW_{l-1}^* = z}(\tau_{l,n} = k) = 2 \mathbb{P}_{RW_{l-1}^* = z-1}(\tau_{l,n} = k) - \mathbb{P}_{RW_{l-1}^* = z}(\tau_{l,n} = k) = 2 \left( \mathbb{P}_{RW_{l-1}^* = z-1}(\tau_{l,n} = k) - \frac{1}{2} \mathbb{P}_{RW_{l-1}^* = z}(\tau_{l,n} = k) \right). \quad (3.49)$$

*2Since we start from arbitrary fixed right finite initial configuration, we can write like this.
By the memoryless property of geometric distribution, for $\forall y > X_0(n-k)$,

$$
P_{R W^*_1 = z - 1}(\tau_{l,n} = k) = 2^{y - X_0(n-k)} P_{R W^*_1 = z - 1}(\tau_{l,n} = k, RW_k^* = y). \tag{3.50}
$$

Also, by (3.47),

$$
P_{R W^*_1 = z - 1}(\tau_{l,n} = k, RW_k^* = y) = \sum_{z-1 < y_1 < \cdots < y_{k-1} < x} \left( \frac{1}{2} \right)^{y-(z-1)} \left( \frac{1}{2} \right)^{y_1 - y} \cdots \left( \frac{1}{2} \right)^{y_{k-1} - y} \times 1_{y_1 \leq X_0(n-l)} \times \cdots \times 1_{y_{k-1} \leq X_0(n-k+1)}
= \left( \frac{1}{2} \right)^{y-(z-1)} \sum_{z-1 < y_1 < \cdots < y_{k-1} < x} 1_{y_1 \leq X_0(n-l)} \times \cdots \times 1_{y_{k-1} \leq X_0(n-k+1)}. \tag{3.51}
$$

Note that

$$
\sum_{z-1 < y_1 < \cdots < y_{k-1} < x} = \sum_{y_1 = z}^{x-1} \sum_{y_{k-1} = y_1 + 1}^{x-1} \cdots \sum_{y_{k-1} = y_{k-2} + 1}^{x-1}, \text{ by (3.50) and (3.51)}, \tag{3.49}
$$

$$
(3.49) = \left( \frac{1}{2} \right)^{x-z} \left\{ \sum_{y_1 = z}^{x-1} \sum_{y_{k-1} = y_1 + 1}^{x-1} \cdots \sum_{y_{k-1} = y_{k-2} + 1}^{x-1} 1_{y_1 \leq X_0(n-l)} \times \cdots \times 1_{y_{k-1} \leq X_0(n-k+1)} \\
- \sum_{y_1 = z+1}^{x-1} \sum_{y_{k-1} = y_1 + 1}^{x-1} \cdots \sum_{y_{k-1} = y_{k-2} + 1}^{x-1} 1_{y_1 \leq X_0(n-l)} \times \cdots \times 1_{y_{k-1} \leq X_0(n-k+1)} \right\}
= \left( \frac{1}{2} \right)^{x-z} \sum_{y_1 = z+1}^{x-1} \cdots \sum_{y_{k-1} = y_{k-2} + 1}^{x-1} 1_{z \leq X_0(n-l)} \times 1_{y_{k+1} \leq X_0(n-l-1)} \times \cdots \times 1_{y_{k-1} \leq X_0(n-k+1)}. \tag{3.52}
$$

Since $z \leq X_0(n-l)$ was assumed,

$$
(3.52) = \left( \frac{1}{2} \right)^{x-z} \sum_{z < y_1 < \cdots < y_{k-1} < x} 1_{y_{k+1} \leq X_0(n-l-1)} \times \cdots \times 1_{y_{k-1} \leq X_0(n-k+1)} \tag{3.53}
= P_{R W^*_1 = z}(\tau_{l+1,n} = k).
$$

This completes the proof. \qed

From the memoryless property of geometric distribution we get for $\forall y > X_0(n-k)$,

$$
P_{R W^*_1 = z}(\tau_{0,n} = k, RW_k^* = y) = 2^{X_0(n-k)-y} P_{R W^*_1 = z}(\tau_{0,n} = k) \tag{3.54}
$$
and as a consequence we get for \( z_2 \leq X_0(n) \), \( G_{0,n}(z_1, z_2) \) (3.3) can be expressed as

\[
G_{0,n}(z_1, z_2) = \sum_{k=0}^{n-1} \mathbb{P}_{\text{RW}_1^*} (\tau^{0,n} = k) (Q^*)^{n-k} (X_0(n-k), z_1) \\
= \sum_{k=0}^{n-1} \mathbb{P}_{\text{RW}_1^*} (\tau^{0,n} = k, \text{RW}_k^* = z) (Q^*)^{n-k-1} (z, z_1) \\
= \mathbb{P}_{\text{RW}_1^*} (\tau^{0,n} < n, \text{RW}_{n-1}^* = z_1),
\]

where in the second equality we used (3.34) and (3.54), while in the third one we used (3.46). Note that RHS of the above equation represents the probability for the walk starting at \( z_2 \in \mathbb{Z} \) at time \(-1\) to end up at \( z_1 \in \mathbb{Z} \) after \( n \) steps, having hit the curve \((X_0(n-m))_{m=0,\ldots,n-1}\) in between.

The next step is to extend the region \( z_2 \leq X_0(n) \) in (3.55) to \( z_2 \in \mathbb{Z} \). We begin by observing that for each fixed \( y_1 \) and \( n \geq 1 \), \( 2^{-y_2} Q^n(y_1, y_2) \) extends in \( y_2 \) to a polynomial \( 2^{-y_2} \bar{Q}^n(y_1, y_2) \) of degree \( n-1 \) with

\[
\bar{Q}^{(n)}(y_1, y_2) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1 + w)^{y_1-y_2-1}}{2^{y_2-w}w^n}. \tag{3.56}
\]

Now, for \( y_1 - y_2 \geq 1 \), we note that

\[
\bar{Q}^{(n)}(y_1, y_2) = Q^n(y_1, y_2). \tag{3.57}
\]

By (3.55) and (3.56), for \( n > 1 \), we get

\[
Q^{-1} \bar{Q}^{(n)} = \bar{Q}^{(n)} Q^{-1} = \bar{Q}^{(n-1)}. \tag{3.58}
\]

Also, we get

\[
Q^{-1} \bar{Q}^{(1)} = \bar{Q}^{(1)} Q^{-1} = 0. \tag{3.59}
\]

**Remark 3.12.** We note that

\[
\bar{Q}^{(n)} Q^{(m)}(x, y) = \sum_{z \in \mathbb{Z}} \bar{Q}^{(n)}(x, z) Q^{(m)}(z, y) \\
= \sum_{z \in \mathbb{Z}} \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1 + w)^{x-z-1}}{2^{x-z}z^n} \cdot \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1 + w)^{z-y-1}}{2^{z-y}w^n} \\
= \infty.
\]

Using the extension of \( Q^n \), we have the following lemma.

**Lemma 3.13.** ([20])

For all \( z_1, z_2 \in \mathbb{Z} \), we have

\[
G_{0,n}(z_1, z_2) = \mathbb{E}_{\text{RW}_0} \left[ \bar{Q}^{(n-\tau)}(\text{RW}_{\tau}, z_2) 1_{\tau < n} \right], \tag{3.60}
\]

where \( \text{RW}_m \) and \( \tau \) are defined by Definition. 2.2.
Proof. Although the proof is given in Lemma 2.4 of [20], we give its outline for self-containedness.

For \( z_2 \leq X_0(n) \), (3.55) can be written as

\[
G_{0,n}(z_1, z_2) = \mathbb{P}_{RW_{0}^{*}=z_2}(\tau_{0,n}^{0} \leq n-1, RW_{n-1}^{*} = z_1) = \mathbb{P}_{RW_0=z_1}(\tau \leq n-1, RW_n = z_2)
\]

\[
= \sum_{k=0}^{n-1} \sum_{\tau_k > X_0(k+1)} \mathbb{P}_{RW_0=z_1}(\tau = k, RW_k = z)Q^{n-k}(z, z_2)
\]

\[
= \mathbb{E}_{RW_0=z_1}[Q^{(n-\tau)}(RW_{\tau}, z_2)1_{\tau<n}],
\]

where in the second equality we used the fact \( Q^{n}(x, y) \) (3.34) represents the \( n \)-step transition probability of \( RW_n \). Let

\[
\tilde{G}_{0,n}(z_1, z_2) = \mathbb{E}_{RW_0=z_1}[\tilde{Q}^{(n-\tau)}(RW_{\tau}, z_2)1_{\tau<n}].
\]

From the relation \( z_2 < RW_{\tau} < z_1 \) for all \( \tau < n \) and (3.57) we see that for \( \tilde{G}_{0,n}(z_1, z_2) = \tilde{G}_{0,n}(z_1, z_2) \) for \( z_2 \leq X_0(n) \). Furthermore we find that \( 2^{-z_2}G_{0,n}(z_1, z_2) \) is polynomial in \( z_2 \) with degree at most \( k \), and similarly \( 2^{-z_2}G_{0,n}(z_1, z_2) \) is polynomial in \( z_2 \) with degree at most \( k \) since \( 2^{-y^2}b^n_k(0, y_2) \) is polynomial in \( y_2 \) with degree at most \( k \). From the above, we find that the equality \( \tilde{G}_{0,n}(z_1, z_2) = \tilde{G}_{0,n}(z_1, z_2) \) holds for the all \( z_2 \in \mathbb{Z} \).

Thus from (3.44) and (3.60), we see that the kernel \( K_t \) (3.31) can be expressed as

\[
K_t(n_1, x_1; n_2, x_2) = -Q^{n_2-n_1}(x_1, x_2)1_{n_1<n_2}
\]

\[
+ \sum_{x, y \in \mathbb{Z}} (R_{\alpha, \beta, \gamma, t}Q^{-n_1})(x_1, x)\mathbb{E}_{RW_0=x}[\tilde{Q}^{(n-\tau)}(RW_{\tau}, y)R_{\alpha, \beta, \gamma, t}^{-1}(y, x_2)1_{\tau<n_2}] .
\]

(3.62)

3.4 Formulas for the mixing time TASEP with right finite initial data:

Proof of Theorem 2.4

To show Theorem 2.4, we have the following relations.

Proposition 3.14.

\[
A_{\alpha, \beta, \gamma}^{-1}(t)(R_{\alpha, \beta, \gamma, t}Q^{-n})^{n}(z_1, z_2) = S_{-t, n}(z_1, z_2),
\]

(3.63)

And

\[
A_{\alpha, \beta, \gamma}(t)\tilde{Q}^{(n)}R_{\alpha, \beta, \gamma, t}^{-1}(z_1, z_2) = \tilde{S}_{-t, n}(z_1, z_2).
\]

(3.64)

Here \( S_{-t, n}(z_1, z_2) \) and \( \tilde{S}_{-t, n}(z_1, z_2) \) are defined by (2.8) and (2.9) respectively and \( A_{\alpha, \beta, \gamma}(t) \) is defined by

\[
A_{\alpha, \beta, \gamma}(t) := e^{-\frac{z_2}{2}} \prod_{j=1}^{t_1} \frac{1 - \alpha_j}{2 - \alpha_j} \prod_{j=t_1+1}^{t_1+t_2} \frac{2 + \beta_j}{1 + \beta_j}.
\]

(3.65)
Proof. By (3.37), the lhs of (3.63) becomes

\[ A_{\alpha,\beta,\gamma}^{-1}(t)(\Psi^n_\alpha)(z_1) \big|_{X_0(0)=z_2} = A_{\alpha,\beta,\gamma}^{-1}(t)(\Psi^n_\alpha)(z_2) \big|_{X_0(0)=z_1} \]

\[ = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^{n}}{2^{z_1-w_{n+1}}+z_2-z_1} \delta_{\alpha,\beta,\gamma}(w, t) = S_{-t,-n}(z_1, z_2). \]

By (3.38) and (3.56), the lhs of (3.64) is written as

\[ A_{\alpha,\beta,\gamma}(t) \sum_{z \in \mathbb{Z}} Q^{(n)}(z_1, z) R_{\alpha,\beta,\gamma, t}^{-1}(z, z_2) \]

\[ = A_{\alpha,\beta,\gamma}(t) \frac{1}{2^{z_1-z_2}} \sum_{z \in \mathbb{Z}} \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1+w)^{z_1-z-1}}{w^n} \cdot \frac{1}{2\pi i} \oint_{\Gamma_0} d\bar{w} f_{\alpha,\beta,\gamma}^{-1}(\bar{w}, t) \]

\[ = A_{\alpha,\beta,\gamma}(t) \frac{1}{2^{z_1-z_2}} \sum_{z \in \mathbb{Z}} \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^{z_1-z-1}}{w^n} (-1)^{n-1} \frac{1}{2\pi i} \oint_{\Gamma_0} d\bar{w} f_{\alpha,\beta,\gamma}^{-1}(\bar{w}, t) \]

\[ = A_{\alpha,\beta,\gamma}(t) \frac{1}{2^{z_1-z_2}} \sum_{z \in \mathbb{Z}} \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^{z_1-z_2-1}}{w^n} f_{\alpha,\beta,\gamma}^{-1} \left( \frac{1}{1-w}, t \right). \]

By changing variables \( w \mapsto \frac{-w}{1-w} \), we have

\[ \frac{A_{\alpha,\beta,\gamma}(t)}{2^{z_1-z_2}} \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^{z_2-z_1+n-1}}{w^n} f_{\alpha,\beta,\gamma}^{-1}(1-w, t) \]

\[ = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^{z_2-z_1+n-1}}{2^{z_1-z_2}w^n} \delta_{\alpha,\beta,\gamma}(w, t) = S_{-t,n}(z_1, z_2). \]

This completes the proof. \( \square \)

Proof of Theorem 2.4. First, we consider right finite initial data. If \( X_0(1) < \infty \) then we are in the setting of the above section. Formula (2.6) follow directly from above definition.

Now, we check (2.5). To check (2.5), it is enough to check

\[ Q^{n_j-n_i} K^{(n_j)}_t = (S_{-t,-n_i})^{\text{epi}(X_0)} \]

where \( K^{(n_j)}_t = RQ^{-n_j} G_{0,n_j} R_{\alpha,\beta,\gamma, t}^{-1}. \)

Because \( Q \) and \( R_{\alpha,\beta,\gamma, t} \) commute, by lemma 3.13,

\[ Q^{n_j-n_i} K^{(n_j)}_t = RQ^{-n_j} G_{0,n_j} R_{\alpha,\beta,\gamma, t}^{-1} = A_{\alpha,\beta,\gamma}^{-1}(t) RQ^{-n_j} G_{0,n_j} R_{\alpha,\beta,\gamma, t}^{-1} A_{\alpha,\beta,\gamma}(t) = (S_{-t,-n_i})^{\text{epi}(X_0)}. \]
If $X_0(j) = \infty$ for $j = 1, \ldots, l$ and $X_0(l + 1) < \infty$ then

$$\mathbb{P}_{X_0}(X_t(n_j) > a_j, j = 1, \ldots, M) = \det(I - \bar{\chi}_a^{(l)} \bar{\chi}_a)_{Z \times \{n_1, \ldots, n_M\}}$$

with the correlation kernel

$$K_t^{(l)}(n_i, \cdot; n_j, \cdot) = -Q_{n_j - n_i}^t 1_{n_i < n_j} + (S_{-t, -n_i}^\tau)^* \bar{S}_{-t, n_j}^\tau,$$

where $\theta_t X_0(j) = X_0(l + j)$. Now, using the fact that $Q_{S_{-t, n_j}^\tau}^\tau = S_{-t, n_j}^\tau$ and (3.63), we have that (2.6) still holds in this case.

4 Asymptotics

In this section we take the KPZ scaling limit for the discrete time Bernoulli and geometric TASEP and prove Proposition 2.14 and 2.15.

4.1 Proof of Proposition 2.14

First, we prove (2.37). By changing variables $w = \frac{1}{2}(1 - \varepsilon y)$, we have

$$\mathbb{P}_{X_0}(X_t(n_j) > a_j, j = 1, \ldots, M) = \det(I - \bar{\chi}_a^{(l)} \bar{\chi}_a)_{Z \times \{n_1, \ldots, n_M\}}$$

(4.1)

where $C_e$ is a circle of radius $\varepsilon^{-\frac{1}{2}}$ centred at $\varepsilon^{-\frac{1}{2}}$. In order to apply the saddle point method, we rewrite (4.1) as

$$\frac{1}{2\pi i} \int_{C_e} e^{f(\varepsilon y) + \varepsilon^{-1} \hat{F}_2(\varepsilon y) + \varepsilon^{-1} \hat{F}_1(\varepsilon y) + \hat{F}_0(\varepsilon y)} dy,$$

(4.2)

where the functions $f(x)$ and $F_i(x)$, $i = 0, 1, 2$ are defined by

$$f(x) = \frac{2 - p}{4} \log(1 + x) - \frac{2 - p}{4(1 - p)} \log(1 - x) + \frac{(2 - p)^3}{4p(1 - p)} \log \left(1 - \frac{p}{2 - p} x \right),$$

(4.3)

$$F_2(x) = -x \log(1 - x^2), \quad F_1(x) = (v - u - \frac{1}{2} \gamma) \log(1 - x) - \frac{1}{2} \gamma \log(1 + x), \quad F_0(x) := \log 2(1 + x).$$

(4.4)
Calculating the derivatives of $f(x)$ up to the third order, we have

$$f'(x) = \frac{x^2}{(1 - x^2)(1 - \frac{p}{2-p} x)}, \quad f''(x) = -\frac{2x - \frac{p}{2-p} x^2 - \frac{p}{2-p} x^4}{(1 - \frac{p}{2-p} x - x^2 + \frac{p}{2-p} x^3)^2},$$

$$f^{(3)}(x) = \frac{2}{1 + 3x^2 - 8\frac{p}{2-p} x^3 + 3\left(\frac{p}{2-p}\right)^2 x^4 + \left(\frac{p}{2-p}\right)^2 x^6}{(1 - \frac{p}{2-p} x - x^2 + \frac{p}{2-p} x^3)^3} \hat{t}.$$  \hspace{1cm} (4.5)

with $\hat{t} := \varepsilon^{-\frac{3}{2}}t$. Thus we see that $f(x)$ has the double saddle point at $x = 0$,

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 0 \text{ and } f^{(3)}(0) = 2\hat{t}. \hspace{1cm} (4.6)$$

Therefore, for small $\varepsilon$, $f(x)$ is expanded as

$$f(\varepsilon^{\frac{1}{2}} y) \approx \frac{t}{3} y^3. \hspace{1cm} (4.7)$$

For small $\varepsilon$, we also have

$$\varepsilon^{-1} F_2(\varepsilon^{\frac{1}{2}} y) \approx xy^2, \quad \varepsilon^{-\frac{1}{2}} F_1(\varepsilon^{\frac{1}{2}} y) \approx (u-v)y, \quad F_0(\varepsilon^{\frac{1}{2}} y) \approx \log 2. \hspace{1cm} (4.8)$$

Now, we see the convergence of the integration path. First, we deform $C_\varepsilon$ to the contour $\ell_\varepsilon \cup C_\varepsilon^\pi$, where $\ell_\varepsilon$ is the part of Airy contour $\ell$ within the ball of radius $\varepsilon^{-\frac{1}{2}}$ centred at $\varepsilon^{-\frac{1}{2}}$, and $C_\varepsilon^\pi$ is the part of $C_\varepsilon$ to the right of $\ell$. From (4.2), (4.7), and (4.8), we have

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\ell_\varepsilon} \varepsilon^{\frac{1}{2}} e^{f(\varepsilon^{\frac{1}{2}} y) + \varepsilon^{-1} F_2(\varepsilon^{\frac{1}{2}} y) + \varepsilon^{-\frac{1}{2}} F_1(\varepsilon^{\frac{1}{2}} y) + F_0(\varepsilon^{\frac{1}{2}} y)} dy = S_{t,x}(y), \hspace{1cm} (4.9)$$

where $S_{t,x}(y)$ is defined by (2.26).

Thus the remaining part is to show that the integral over $C_\varepsilon^\pi$ converges to 0. To see this note that the real part of the exponent of the integral over $C_\varepsilon$ in (4.1), parametrized as $y = \varepsilon^{-\frac{1}{2}}(1 - e^{i\theta})$, is given by

$$\varepsilon^{-\frac{3}{2}} t \left[ \frac{(2-p)^3}{4p(1-p)} \log \left( 1 + \frac{p}{2-p}(\cos \theta - 1) \right) + \left( \frac{2-p}{4(1-p)} + \mathcal{O}(\varepsilon^{\frac{1}{2}}) \right) \log(2 \cos \theta - 1) \right.$$  

$$\left. - \left( \frac{(2-p)p}{4(1-p)} + \mathcal{O}(\varepsilon^{\frac{1}{2}}) \right) \log(2 - \cos \theta) \right].$$

Note that we used an expression transform

$$\frac{2-p}{4} \log(2 - e^{i\theta}) - \frac{2-p}{4(1-p)} \log(e^{i\theta}) = \frac{2-p}{4(1-p)} [\log(2 - e^{i\theta}) - \log(e^{i\theta})] - \frac{(2-p)p}{4(1-p)} \log(2 - e^{i\theta})$$

$$= \frac{2-p}{4(1-p)} \log(2e^{-i\theta} - 1) - \frac{(2-p)p}{4(1-p)} \log(2 - e^{i\theta}).$$

30
Because the \( y \in C_\pi^\epsilon \) correspond to \( 0 < |\theta| < \frac{\pi}{3} \), using \( \log(1+x) < x^4 \) for \( x \in (-1, \infty) \setminus \{0\} \), we get

\[
\epsilon^{-\frac{3}{4}t} \left[ \frac{(2-p)^3}{4p(1-p)} \log \left( 1 + \frac{p}{2-p} (\cos \theta - 1) \right) \right] < \frac{(2-p)^2}{4(1-p)} \epsilon^{-\frac{3}{4}t} \left[ \cos \theta - 1 \right]
\]

(4.10)

and

\[
\epsilon^{-\frac{3}{4}t} \left[ \frac{2-p}{4(1-p)} \log (2 \cos -1) \right] < \frac{2-p}{2(1-p)} \epsilon^{-\frac{3}{4}t} \left[ \cos \theta - 1 \right]
\]

(4.11)

and

\[-\epsilon^{-\frac{3}{4}t} \left[ \frac{(2-p)p}{4(1-p)} \log (2 - \cos \theta) \right] < 0.\]

Therefore, for sufficiently small \( \epsilon \), the exponent there is less than \( -\epsilon^{-\frac{3}{4}t} \kappa t \) for some \( \kappa > 0 \). Hence we see that the part \( C_\pi^\epsilon \) of the integral vanishes and this completes the proof of (2.37). We can also prove (2.38) in the similar way to (2.37) thus omit the proof.

For the proof of (2.39), we define the scaled walk \( B^\epsilon(x) = \epsilon^\frac{1}{2} (RW_{\epsilon^{-1}x} + 2\epsilon^{-1}x - 1) \) for \( x \in \epsilon \mathbb{Z}_{\geq 0} \), interpolated linearly in between, and let \( \tau^\epsilon \) be the hitting time by \( B^\epsilon \) of \( \text{epi}(\bar{h}^\epsilon(0, \cdot)) \), where \( \bar{h}^\epsilon(t, x) \) is defined by (2.19) and \( \bar{h}^\epsilon(t, x) = \bar{h}^\epsilon(t, -x) \). By Donsker’s invariance principle [3], \( B^\epsilon(x) \) converges locally uniformly in distribution to a Brownian motion \( B(x) \) with diffusion coefficient 2. Combining this with (2.30), one finds the hitting time \( \tau^\epsilon \) converges to \( \tau \). (For more detailed proof, see Proposition 3.2 in [20].) This leads to (2.39).

### 4.2 Proof of Proposition 2.15

Proposition 2.15 can be shown in a similar manner to Proposition 2.14. Here we give only the proof of (2.43). (2.44) can be obtained in a parallel way to (2.43) whereas (2.45) follows from (2.44) and the Donsker’s invariance principle as in the case of (2.39) in Proposition 2.14.

As for (4.1) and (4.2), we rewrite (2.46) by changing variables \( w = (1 - \epsilon^\frac{1}{2} y)/2, \)

\[
(2.46) = \frac{1}{2\pi i} \oint_{C_{\theta}} \epsilon^{\frac{1}{2}y} dy \frac{(1 + \epsilon^\frac{1}{2} y)^n}{(1 - \epsilon^\frac{1}{2} y)^{n+1+z-y}} \left( 1 + \frac{\alpha}{2-\alpha} \epsilon^\frac{1}{2} y \right)^{-t}
\]

\[
= \frac{1}{2\pi i} \oint_{C_{\theta}} \epsilon^{\frac{1}{2}y} e^{\alpha(\epsilon^\frac{1}{2} y) + z(\epsilon^\frac{1}{2} y) + \epsilon^\frac{1}{2} G_2(\epsilon^\frac{1}{2} y) + \epsilon^\frac{1}{2} G_1(\epsilon^\frac{1}{2} y) + G_0(\epsilon^\frac{1}{2} y)} dy
\]

(4.12)

---

*3Since \( \theta = 0 \) corresponds to the origin \( 0 \in C_\pi^\epsilon \) and \( \) is the positively oriented contour going the straight lines from \( e^{-\frac{3}{4}t} \infty \) to \( e^{\frac{3}{4}t} \infty \) through 0, the domain of \( \theta \) can be written by this domain.

*4This inequality comes from \( \log(1+x) \leq x \) for \( x > -1 \), but since \( x = 0 \) corresponds to \( \theta = 0 \) in (4.10) and (4.11), we use this inequality to correspond to the calculations of (4.10) and (4.11).
where $C_\varepsilon$ is a circle of radius $\varepsilon^{-\frac{1}{2}}$ centred at $\varepsilon^{-\frac{1}{2}}$ and $g(x)$, $G_j(x)$, $j = 0, 1, 2$ are defined by

$$g(x) = \frac{2 - \alpha}{4(1 - \alpha)} \hat{t} \log(1 + x) - \frac{2 - \alpha}{4} \hat{t} \log(1 - x) - \frac{(2 - \alpha)^3}{4\alpha(1 - \alpha)} \hat{t} \log \left(1 + \frac{\alpha}{2} x \right).$$

$$G_2(x) = -x \log(1 - x^2), \ G_1(x) = (v - u - \frac{1}{2} a) \log(1 - x) - \frac{1}{2} a \log(1 + x), \ G_0(x) = \log 2(1 + x).$$

(4.13)

Here we apply the saddle point method to (4.12). Noting

$$g'(x) = \frac{x^2}{(1 - x^2)(1 + \frac{\alpha}{2 - \alpha} x)}, \ g''(x) = \frac{2x + \frac{\alpha}{2 - \alpha} x^2 + \frac{\alpha}{2 - \alpha} x^4}{(1 + \frac{\alpha}{2 - \alpha} x - x^2 - \frac{\alpha}{2 - \alpha} x^3)^2},$$

$$g^{(3)}(x) = \frac{2 \left[ 1 + 3x^2 + 8 \frac{\alpha}{2 - \alpha} x^3 + 3 \left( \frac{\alpha}{2 - \alpha} \right)^2 x^4 + \left( \frac{\alpha}{2 - \alpha} \right)^2 x^6 \right]}{(1 + \frac{\alpha}{2 - \alpha} x - x^2 - \frac{\alpha}{2 - \alpha} x^3)^3},$$

we find $g(x)$ has a double saddle point at $x = 0$,

$$g(0) = 0, \ g'(0) = 0, \ g''(0) = 0 \text{ and } g^{(3)}(0) = 2 \hat{t}.$$

(4.15)

Therefore, for small $\varepsilon$, we have

$$g(\varepsilon^\frac{1}{2} y) \approx \frac{t}{3} y^3.$$

(4.16)

For $G_i(x)$, $i = 0, 1, 2$, we easily see

$$\varepsilon^{-1} G_2(\varepsilon^\frac{1}{2} y) \approx xy^2, \ \varepsilon^{-\frac{1}{2}} G_1(\varepsilon^\frac{1}{2} y) \approx (u - v)y, \ G_0(\varepsilon^\frac{1}{2} y) \approx \log 2.$$  (4.17)

As discussed above (4.9), we divide the contour $C_\varepsilon$ in (4.12) into two parts $\left( \varepsilon \cup C^{\varepsilon} \right)$. From (4.12), (4.15), and (4.17), we have

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma} e^{\hat{t}(\varepsilon^\frac{1}{2} y) + \varepsilon^{-1} F_2(\varepsilon^\frac{1}{2} y) + \varepsilon^{-\frac{1}{2}} F_1(\varepsilon^\frac{1}{2} y) + F_0(\varepsilon^\frac{1}{2} y)} dy = S_{t,x}(y),$$

(4.18)

where $S_{t,x}(y)$ is defined by (2.26).

Finally we show that the part coming from $C^{\varepsilon}$ vanishes as $\varepsilon \to 0$. To see this note that the real part of the exponent of the integral over $C_\varepsilon$ in (4.12), parametrized as $y = \varepsilon^{-\frac{1}{2}} (1 - e^{i\theta})$, is given by

$$\varepsilon^{-\frac{3}{2}} \left[-\frac{(2 - \alpha)^3}{4\alpha(1 - \alpha)} \log \left(1 + \frac{\alpha}{2 - \alpha}(1 - \cos \theta)\right) + \left(\frac{2 - \alpha}{4(1 - \alpha)} + O(\varepsilon^{\frac{1}{2}})\right) \log(2 \cos \theta - 1)
+ \left(\frac{(2 - \alpha)\alpha}{4(1 - \alpha)} + O(\varepsilon^{\frac{1}{2}})\right) \log(\cos \theta)\right].$$
Note that we used an expression transform
\[
\frac{2 - \alpha}{4(1 - \alpha)} \log(2 - e^{i\theta}) - \frac{2 - \alpha}{4} \log(e^{i\theta}) = \frac{2 - \alpha}{4(1 - \alpha)} [\log(2 - e^{i\theta}) - \log(e^{i\theta})] + \frac{(2 - \alpha)\alpha}{4(1 - \alpha)} \log(e^{i\theta})
\]
\[
= \frac{2 - \alpha}{4(1 - \alpha)} \log(2e^{-i\theta} - 1) + \frac{(2 - \alpha)\alpha}{4(1 - \alpha)} \log(e^{i\theta}).
\]
Because the \( y \in C^\infty \) correspond to \( 0 < |\theta| < \frac{\pi}{2} \), using \( \log(1 + x) < x \) for \( x \in (-1, \infty) \setminus \{0\} \) (See *3 and *4), we get
\[
\varepsilon^{-\frac{3}{2} t} \left[ \frac{2 - \alpha}{4(1 - \alpha)} \log \left( 1 + \frac{\alpha}{2 - \alpha} (1 - \cos \theta) \right) \right] < 0
\]
and
\[
\varepsilon^{-\frac{3}{2} t} \left[ \frac{2 - \alpha}{4(1 - \alpha)} \log (2 \cos \theta - 1) \right] < \frac{2 - \alpha}{2(1 - \alpha)} \varepsilon^{-\frac{3}{2} t} [\cos \theta - 1]
\]
and
\[
\varepsilon^{-\frac{3}{2} t} \left[ \frac{(2 - \alpha)\alpha}{4(1 - \alpha)} \log \theta \right] < \frac{(2 - \alpha)\alpha}{4(1 - \alpha)} \varepsilon^{-\frac{3}{2} t} [\cos \theta - 1].
\]
Therefore, for sufficiently small \( \varepsilon \), the exponent there is less than \( -\varepsilon^{-\frac{3}{2}} \kappa t \) for some \( \kappa > 0 \). Hence this part of the integral vanishes.

4.3 Proof of Theorem 2.11

By using Propositions 2.14 or 2.15, we can prove Theorem 2.11 as following. This proof is almost the same as [20]. First, we change variables in the kernel as in Proposition 2.14 (resp. Proposition 2.15), so that for \( z_i = \frac{p(2-p)}{4(1-p)} \varepsilon^{-\frac{3}{2} t} + 2 \varepsilon^{-1} x_i + \varepsilon^{-\frac{1}{2} t}(u_i + a_i) - 2 \) (resp. \( z_i = -\frac{\alpha(2-\alpha)}{4(1-\alpha)} \varepsilon^{-\frac{3}{2} t} + 2 \varepsilon^{-1} x_i + \varepsilon^{-\frac{1}{2}} (u_i + a_i) - 2 \)) we need to compute the limit of \( \varepsilon^{-\frac{1}{2} t}(\chi_{2\varepsilon^{-1} x - 2 K_1(2\varepsilon^{-1} x - 2)}(z_i, z_j)) \). Note that the change of variables turns \( \chi_{2\varepsilon^{-1} x - 2}(z) \) into \( \chi_{-a}(u) \). We have \( n_i < n_j \) for small \( \varepsilon \) if and only if \( x_j < x_i \) and in this case we have, under our scaling,
\[
\varepsilon^{-\frac{1}{2} t} Q^{n_j-n_i}(z_i, z_j) \to e^{(u_i-x_j)\alpha^2}(u_i, u_j), \quad \text{(4.19)}
\]
as \( \varepsilon \to 0 \). For the second term in (2.6), by Proposition 2.14 we get
\[
\varepsilon^{-\frac{1}{2} t}(S_{-t,-n_j})^* S_{-t,x_j}^{\text{epi}(X_0)}(z_i, z_j) = \varepsilon^{-\frac{1}{2}} \int_{-\infty}^{\infty} d\nu(S_{-t,-n_j})^*(z_i, \nu) S_{-t,x_j}^{\text{epi}(X_0)}(\nu, z_j)
\]
\[
= \varepsilon^{-1} \int_{-\infty}^{\infty} d\nu(S_{-t,-n_j})^*(z_i, \varepsilon^{-\frac{1}{2}} \nu) S_{-t,x_j}^{\text{epi}(X_0)}(\varepsilon^{-\frac{1}{2}} \nu, z_j)
\]
\[
= \int_{-\infty}^{\infty} d\nu(S_{-t,x_j})^*(u_i, \nu) S_{-t,-x_j}^{\text{epi}(h_0^{\varepsilon})}(\nu, u_j)
\]
\[
= (S_{-t,x_j})^* S_{-t,-x_j}^{\text{epi}(h_0^{\varepsilon})}(u_i, u_j)
\]
\[
\xrightarrow{\varepsilon \to 0} (S_{-t,x_j})^* S_{-t,-x_j}^{\text{epi}(h_0^{\varepsilon})}(u_i, u_j).
\]
Therefore, we have a limiting kernel
\[ K_{\text{lim}}(x_i, u_i; x_j, u_j) = -e^{(x_i-x_j)\alpha_j^2} (u_i, u_j) \mathbf{1}_{x_i, x_j} + (S_{t-x_i}^* S_{t-x_j}^\text{hyp} \tilde{h}_0^{-1})(u_i, u_j) \] (4.21)
surrounded by projection \( \tilde{\chi}_{-a} \). It is nicer to have projection \( \chi_a \), so we change variables \( u_i \mapsto -u \) and replace the Fredholm determinant of the kernel by that of its adjoint to get
\[ \det \left( I - \chi_a K_{t, \text{ext}}^\text{hyp} \tilde{h}_0 \chi_a \right) \]
with \( K_{t, \text{ext}}^\text{hyp} \tilde{h}_0 (u_i, u_j) = K_{\text{lim}}(x_j, -u_j; x_i, -u_i) \).

By using \( (S_{t,x}^* S_{t-x}^\text{hyp} \tilde{h}_0)(v, u) = S_{t,x}^\text{hyp}(-\tilde{h})(-v, -u) \) (see [20] for more information on these equations), we get the following:
\[ K_{t, \text{ext}}^\text{hyp} \tilde{h}_0(x_i, \cdot, x_j, \cdot) = -e^{(x_j-x_i)\alpha_j^2} \mathbf{1}_{x_i, x_j} + \left( S_{t-x_i}^\text{hyp} \tilde{h}_0 \right)^* S_{t-x_j}. \]

### A The Kolmogorov forward equation for the discrete time geometric TASEP with \( N = 3 \)

Here, we explain (3.9) in more detail in the case of \( N = 3 \). In this case (3.9) can be decomposed into four terms,
\[ G_{t+1}(x_3, x_2, x_1) = G_t^{(\alpha,1)}(x_3, x_2, x_1) + G_t^{(\alpha,2)}(x_3, x_2, x_1) + G_t^{(\alpha,3)}(x_3, x_2, x_1) + G_t^{(\alpha,4)}(x_3, x_2, x_1) \] (A.1)

where for \( k_1 := x_1 - x_2, k_2 := x_2 - x_3 \), and
\[ G_t^{(\alpha,1)}(x_3, x_2, x_1) = \sum_{a_2=0}^{k_2-1} \sum_{a_1=0}^{k_1-1} (1 - \alpha_{t+1})^3 a_1 a_2 a_1 + a_1 G_t^{(\alpha)}(x_3 - a_3, x_2 - a_2, x_1 - a_1) \] (A.2)
\[ G_t^{(\alpha,2)}(x_3, x_2, x_1) = \sum_{a_2=0}^{k_2-1} \sum_{a_1=0}^{k_1-1} (1 - \alpha_{t+1})^2 a_1 a_2 + a_1 G_t^{(\alpha)}(x_3 - a_3, x_2 - a_2, x_1 + 1) \] (A.3)
\[ G_t^{(\alpha,3)}(x_3, x_2, x_1) = \sum_{a_2=0}^{k_2-1} \sum_{a_1=0}^{k_1-1} (1 - \alpha_{t+1}) a_1 a_2 + a_1 G_t^{(\alpha)}(x_3 - a_3, x_2 + 1, x_1 - a_1) \] (A.4)
\[ G_t^{(\alpha,4)}(x_3, x_2, x_1) = \sum_{a_2=0}^{k_2-1} a_1 a_2 + a_1 G_t^{(\alpha)}(x_3 - a_3, x_2 + 1, x_1 + 1) \] (A.5)

The four equations (A.2) through (A.5) correspond to the case \( \mu = \phi, \mu = \{1\}, \mu = \{2\}, \) and \( \mu = \{1, 2\} \) respectively and the situations for all the equations are illustrated in Fig. 1(a)-(d) below.
Fig. 1: The evolutions of the geometric TASEP with 3 particles. The circles correspond to the particles and they move to the directions of arrows during time step $t \rightarrow t + 1$. (a) The case $\mu = \phi$. Neither of the particles are blocked by each other. (b) The case $\mu = \{1\}$. At time $t$, the first particle (from the right) is at $x_2 + 1$ which leads to the blocking of the second particle. (c) The case $\mu = \{2\}$. At time $t$, the second particle is at $x_3 + 1$ which leads to the blocking of the third particle. (d) The case $\mu = \{1, 2\}$. At time $t$, the first and second particles are at $x_2 + 1$ and $x_3 + 1$ respectively which leads to the blockings of both the second and the third particles.

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