Operator normalized quantum arrival times in the presence of interactions

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We model ideal arrival-time measurements for free quantum particles and for particles subject to an external interaction by means of a narrow and weak absorbing potential. This approach is related to the operational approach of measuring the first photon emitted from a two-level atom illuminated by a laser. By operator-normalizing the resulting time-of-arrival distribution, a distribution is obtained which for freely moving particles not only recovers the axiomatically derived distribution of Kijowski for states with purely positive momenta but is also applicable to general momentum components. For particles interacting with a square barrier the mean arrival time and corresponding “tunneling time” obtained at the transmission side of the barrier becomes independent of the barrier width (Hartman effect) for arbitrarily wide barriers, i.e., without the transition to the ultra-opaque, classical-like regime dominated by wave packet components above the barrier.

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I. INTRODUCTION

A major open question in quantum theory is how to include time observables and time measurements in a satisfactory way into the formalism. Pauli pointed out, in a famous footnote [1], that it is not possible to define self-adjoint time operators because of the semi-boundedness of the canonically conjugate Hamiltonian $H$ (see [2, 3] for recent discussions of Pauli’s argument, its domain of applicability and implications). A great deal of effort and ingenuity has been devoted to pinpoint and overcome this difficulty, in particular for describing the time of arrival of a quantum particle [3–39]. For an extensive review up to 2000 see [40].

In a physically motivated, axiomatic approach, Kijowski [4] derived an arrival-time distribution for free particles coming in from the left (or right), i.e., with only one sign of the momentum. For a one dimensional wave packet coming in from the left, with momentum (wave number) representation $\tilde{\psi}(k)$, $k > 0$, at time $t = 0$, and the arrival point at $x = 0$, Kijowski’s distribution is given by

$$\Pi_K(t) = \frac{\hbar}{2\pi m} \left| \int_0^\infty dk \tilde{\psi}(k) \sqrt{K} e^{-ihk^2t/2m} \right|^2.$$  \hspace{1cm} (1)

Much more recently, this distribution has been related to the positive operator-valued measure (POVM) generated by the eigenstates of the Aharonov-Bohm (maximally symmetric) time-of-arrival operator [41], and generalized for systems with interaction [28, 31] and multi-particle states [33]. As pointed out in several reviews [2, 36, 40], one of the important pending questions of Kijowski’s distribution and its generalizations is the absence of an operational interpretation, in terms of some measuring procedure. This problem becomes particularly acute when interpreting the puzzling results obtained for complicated cases such as (i) freely moving particles with wave components coming from both sides to the arrival position, and (ii) particles subject to some interaction potential [34, 36].

Another research thread on quantum arrival times is based on modeling the irreversible detection by means of effective absorbing complex potentials. They were first used by Alcock in this context [5]. He chose an imaginary potential step and found that a strong potential (therefore a short detection time) lead to important reflection, i.e., many particles of the quantum ensemble associated with the incident wave packet could not be detected; this affected more intensely the slow end of the momentum distribution and distorted the measured signal. In the opposite limit, i.e., for a very weak absorbing potential, the reflection was avoided but at the price of inducing a very large detection delay. By deconvolving the absorption-time probability density with the absorption-time distribution for a particle immersed at rest in the complex potential, in the limit of vanishing absorption, he could define an “ideal distribution” which turned out to be equal to the flux and therefore not positive definite, even for states formed only with positive momentum components. Kijowski’s distribution for positive-momentum states emerged as the best positively defined fit to this “ideal distribution” [5].

Twenty five years later, further investigations of the quantum arrival time using complex potentials revealed that perfect and fast detection were compatible by means of properly designed complex potential profiles [9, 22]. However, an ambiguity in the detailed form of these potentials remained, so it was not clear how to define, within the complex potential approach, an ideal time-of-arrival distribution in an arbitrary case. The distributions obtained with potentials constructed to absorb perfectly in the energy range of a given incident wave packet were in general close but not equal to Kijowski’s distribution, and the relation with actual measurement procedures remained vague. A model proposed by Halli-
well of time-of-arrival detection by means of an unstable two-level detector made such a relation more plausible \cite{19} (he derived a complex potential, again with a step form, from the original full set of equations for the two-level system), but the model lacked a precise physical content since the nature of the actual system or coupling mechanism were not provided.

An important step forward to relate “ideal” and “operational” distributions has been the development of a model, using the “quantum jump” technique \cite{42}, for the detection of first photons emitted by atoms illuminated by a laser beam localized in a semi-infinite \cite{35, 43} or finite region of space \cite{37}. Closely resembling Allcock’s results, it was found that strong driving by the laser caused reflection, whereas the weak driving regime could be used to get the flux by deconvolution from the first-photon distribution, but Kijowski’s distribution was not obtained in any limit \cite{35}. The analogy with Allcock’s work was not accidental since in the limit where the inverse life-time $\gamma$ and the Rabi frequency $\Omega$ become very large, but keeping the ratio $\Omega^2/\gamma$ constant, and $\gamma/\Omega \gg 1$, it is possible to find a closed equation for the atomic ground-state amplitude in terms of an effective complex potential \cite{38, 43}. For the semi-infinite laser model, which is physically valid for slow enough atoms \cite{37}, and the laser tuned at resonance with the atomic transition, it takes the simple form $-i\hbar\Omega\Theta(x)^2/(2\gamma)$, where $\Theta(x)$ is the Heaviside function, and the laser illuminates the region $x \geq 0$. In other words, once again, a purely imaginary absorbing step potential arises, but now with a clearcut physical content in terms of laser and atomic parameters.

As pointed out before, though, the atom-laser model alone did not lead to Kijowski’s distribution. In a very recent article, Kijowski’s distribution has been obtained for freely moving states with positive momentum components \cite{38}, by applying the “operator normalization” proposed by Brunetti and Fredenhagen \cite{44} to the model that describes the interaction between the atom and a semi-infinite laser in the intuitive limit of strong laser field and fast decay (large $\Omega$ and $\gamma$ respectively).

From the previous discussion it should be clear that the problem of undetected atoms, i.e., atoms reflected (or transmitted in the finite beam case) without emitting any photon has to be faced in some way. For example in \cite{35} “normalization by hand” for the fraction of detected atoms was used for strong driving conditions. Instead, for states with incident positive momenta the strategy of the “operator normalization” amounts to enhance the amplitude of each incident momentum component by an amount inversely proportional to the square root of the corresponding detection probability. Operator normalization may be applied whether or not the laser-atom parameters allow for a simplified complex-potential description, but the advantage of such a parameter domain is that a much simpler one-channel calculation may be carried out; this is particularly useful to tackle the more complicated physical situations examined in the present work.

One obvious limitation of the semi-infinite models (with the laser in explicit form or with a complex potential) is that it is not possible to study with them the arrival at a point, say $x = 0$, corresponding to a state incident from both sides, $x > 0$ and $x > 0$, with negative and positive momenta respectively; similarly, if one is interested in the arrivals in the midst of a potential interaction region, the semi-infinitely extended measurement will severely affect the dynamics of the unperturbed system on one side. The aim of this paper is to apply, instead of the semi-infinite interaction, a weak and narrow, minimally perturbing absorbing complex potential combined with “operator normalization” to these two more elusive arrival-time problems.

The motivation for looking at the first case (free motion) is a question originally posed by Leavens \cite{16, 40}: When positive and negative momenta are present in the wavefunction, Kijowski’s original axiomatic approach does not apply and he obtained, in a heuristic way, a distribution which in one dimension is given by a sum of independent contributions. The immediate consequence of this structure is that symmetrical and antisymmetrical states with respect to the arrival point $x = 0$, formed by adding or substracting two waves with opposite momenta, but identical otherwise, have the same arrival distribution, i.e., the distribution is “blind” to the interference between the positive and negative momentum components and predicts arrivals for the antisymmetric case even where the wave function vanishes at all times. This result may be understood formally from the resolution of the identity in terms of eigenstates of the time-of-arrival operator, and corresponds to a projection of the state onto the positive or negative momentum subspaces \cite{40}, but a hint on how that process could be implemented was missing \cite{36}. We shall obtain this distribution by using the weak and narrow absorbing potential, i.e., an appropriate laser interaction, combined with “operator normalization”.

The analysis of arrival times when the particle interacts with a potential has been problematic, too. It took a long time to generalize Kijowski’s distribution in this case because the original axiomatic method is not applicable \cite{13, 27, 28, 31, 45}. The proposed generalizations, though, are purely formal in character. In principle, a narrow and weak laser may be used to excite and detect the atom with minimal disturbance, even when the system’s motion is affected by an additional interaction potential; by compensating the detection losses by “operator normalization” a physically motivated generalization is obtained. As before, for simplicity, a complex potential will play the role of the laser. In this paper we apply the method to tunneling across the paradigmatic square barrier potential, and obtain an arrival time distribution at the transmission side. It is similar to Kijowski’s expression, but modified by the phase of the transmission amplitude. The effect of operator normalization is that the mean arrival time becomes the average of the “phase times” with respect to the initial quantum state, instead
of the transmitted wave packet. As a consequence, the Hartman effect \cite{46, 48}, namely, the essential independence of the average arrival time with respect to the barrier width, is obtained for arbitrarily large barriers, without the transition to an ultra-opaque, classical-like regime dominated by over-the-barrier momentum components \cite{46, 48}.

II. ABSORBING POTENTIALS AND QUANTUM ARRIVAL TIME DISTRIBUTION

Let us consider an asymptotically free, moving wave packet impinging on an imaginary potential \(-iV_c\) which is located in \(-\epsilon \leq x \leq \epsilon\) and a real potential \(U\) localized between \(a\) and \(b\). The Hamiltonian is given by

\[
H_c = \frac{p^2}{2m} - iV_c \chi_c(x) + U \chi_{[a,b]}(x)
\]

where \(\chi_{[a,b]}\) is one in \([a, b]\) and zero elsewhere, and

\[
\chi_c(x) = \begin{cases} 
1 & -\epsilon \leq x \leq \epsilon \\
0 & \text{elsewhere}
\end{cases}
\]

The scaling of the potential \(V_c\) in the limit \(\epsilon \to 0\) is given by a function \(c(\epsilon)\) and

\[
V_c = \frac{V_0 L_0}{2c(\epsilon)}.
\]

Here \(V_0\) and \(L_0\) are some arbitrary initial values for the potential height and potential width. Depending on the structure of \(c(\epsilon)\), we may investigate two cases:

(a) \(c(\epsilon) = \epsilon\), \(0 < \alpha < 1\).

(b) \(c(\epsilon) \sim e^{\alpha \epsilon}\), \(0 < \alpha < 1\).

The case (a) yields in the limit \(\epsilon \to 0\) a delta-function potential with “strength” \(V_0 L_0\), \(V_c \chi_c(x) \to V_0 L_0 \delta(x)\), whereas case (b) implies a weaker and less perturbing measurement. We shall eventually prefer this later case but include an analysis of the first one for completion and comparison. In the following we shall always apply the limit \(\epsilon \to 0\) and denote these two cases by \(\epsilon \to 0\), \(\epsilon \to 0\), respectively.

To obtain the time development under \(H_c\) of a wave packet which is asymptotically free, one first solves the stationary equation

\[
H_c \phi_k = E_k \phi_k
\]

for scattering states with real energy

\[
E_k = \frac{\hbar^2 k^2}{2m}.
\]

By decomposing an initial state as a superposition of eigenfunctions of \(H_c\), its time development is obtained. This is easy for an initial free wave packet coming in from \(x = -\infty\) in the remote past. Indeed,

\[
\psi(x, t) = \int_0^\infty dk \tilde{\psi}(k) \phi_k(x) e^{-i\hbar^2 k^2 t/2m}
\]

describes the time development of a state which in the remote past behaves like a free wave packet with \(\tilde{\psi}(k), k > 0\), the momentum amplitude it would have at \(t = 0\), \(\phi_k\) would correspond in that case to the scattering states for left incidence. In the absence of a real potential \((U = 0)\) we shall also be interested in states with both positive and negative momenta. The formal treatment for symmetrical and antisymmetrical wave components is analogous, as shown later with more detail.

It is convenient to use the interaction picture with respect to \(H_0 = \hat{p}^2/(2m)\),

\[
H_c^I = e^{iH_0 t/\hbar} (H_c - H_0) e^{-iH_0 t/\hbar}
\]

\[
U_c^I(t, t_0) = e^{iH_0 (t-t_0)/\hbar} e^{-iH_0 t_0/\hbar},
\]

where \(U_c^I\) is the conditional time development corresponding to \(H_c^I\). Then Eq. (7) can be written as

\[
\tilde{\psi}_t = e^{-iH_0 t/\hbar} U_c^I(t, -\infty) |\psi\rangle,
\]

where \(|\psi\rangle \equiv \int dk |k\rangle \tilde{\psi}(k)\). An arrival time distribution in the region \(-\epsilon \leq x \leq \epsilon\) is given by the absorption rate \cite{38}

\[
\Pi(t) = \frac{2V_c}{\hbar} \int_{-\epsilon}^{\epsilon} dx |\psi(x, t)|^2
\]

\[
= \int dk dk' \tilde{\psi}(k) \tilde{\psi}(k') e^{i\hbar(k^2 - k'^2)t/2m} f_c(k, k')
\]

with the kernel function

\[
f_c(k, k') = \frac{2V_c}{\hbar} \int_{-\epsilon}^{\epsilon} dx \tilde{\phi}(x) \tilde{\phi}(x).
\]

Now \(\Pi(t)\) can be written as an expectation value on incoming states in the form

\[
\Pi(t) = \langle \psi|\hat{\Pi}_t|\psi\rangle
\]

and

\[
\hat{\Pi}_t = \frac{2V_c}{\hbar} U_c^I(t, -\infty) \chi_c(x) U_c^I(t, -\infty).
\]

To normalize on the level of operators we define \cite{38, 54}

\[
\hat{B} = \int_{-\infty}^{\infty} dt \hat{\Pi}_t = I - U_c^I(\infty, -\infty) U_c^I(\infty, -\infty)
\]

and

\[
\Pi^{\text{ON}}_t = \hat{B}^{-1/2} \hat{\Pi}_t \hat{B}^{-1/2}.
\]

In Ref. \cite{38} it has been shown that \(\hat{B}\) is diagonal in \(k\) space, \(|k|\hat{B}|k'\rangle = b(k, k') \delta(k - k')\) (remember that the
integrals over \( k \) run from 0 to \( \infty \), and this leads to the normalized distribution

\[
\Pi^{ON}(t) = \langle \psi | \Pi^{ON}_{t} | \psi \rangle
\]

\[
= \int dk dk' \overline{\psi}(k) \psi(k') b(k, k')^{-1/2} b(k', k')^{-1/2} e^{i(k^2-k'^2)t/2m f_{k}(k, k')}.
\]

The kernel function \( b(k, k') \) can be calculated as in Ref. [38] or by the following simple argument. Because of \( \int dt \Pi^{ON}(t) = 1 \), \( b(k, k')^{-1/2} \) has to cancel the factors which arise from integrating \( \Pi(\sqrt{\epsilon}) \) in the arrival time distribution, modified by a model dependent term

\[
\Pi^{ON}(t) = \frac{\hbar}{2\pi m} \int dk dk' \overline{\psi}(k) \psi(k') e^{i(k^2-k'^2)t/2m \sqrt{k k'}} f_{k}(k, k') \sqrt{f_{k}(k, k) f_{k'}(k', k')}.
\]

The normalization leads immediately to Kijowski’s kernel \( \sqrt{kk'} \) in the arrival time distribution, modified by a model dependent term

\[
F_{\epsilon}(k, k') = \frac{f_{k}(k, k')}{\sqrt{f_{k}(k, k) f_{k'}(k', k')}}
\]

which has to be investigated for specific situations in the limit \( \epsilon \to 0 \).

### III. EXAMPLES: FROM THE FREE CASE TO TUNNELING PARTICLES

#### A. Arrival time distribution for free particles

The easiest case to evaluate Eq. (19) is a free incoming wave packet of positive momenta. Its Hamiltonian is given by Eq. (2) with \( U = 0 \) and the general solution of Eq. (5) in the region \( i, i = 0, 1, 2 \) associated with \( x \leq -\epsilon, -\epsilon \leq x \leq \epsilon, \epsilon \leq x \), respectively, is given by

\[
\phi^{(i)}_{\epsilon}(x) = \frac{1}{\sqrt{2\pi}} (A^{+}_{\epsilon} e^{ikx} + A^{-}_{\epsilon} e^{-ikx})
\]

where \( k_{0} = k_{2} \equiv k \) and \( k_{1} \equiv q_{\epsilon} = \sqrt{k^{2} + \frac{imV_{0}L_{0}}{\hbar^{2}q_{\epsilon}}})^{1/2} \) with \( \text{Im} q_{\epsilon} > 0 \) (Fig. 1a). The amplitudes \( A^{+}_{\epsilon} \) and \( A^{-}_{\epsilon} \) are determined by the matching conditions at \( x = -\epsilon \) and \( x = \epsilon \) and they are explicitly derived in Appendix A.

(i) **Left incoming states**: The appropriate eigenstates have boundary conditions \( A^{+}_{0} = 1 \) and \( A^{-}_{2} = 0 \). Then one can solve for the other amplitudes to obtain

\[
A^{+}_{\epsilon} = e^{-2i\epsilon k}/D
\]

\[
A^{-}_{0} = -\frac{i}{2} \left( \frac{k}{q_{\epsilon}} - \frac{q_{\epsilon}}{k} \right) \sin(2q_{\epsilon} \epsilon) e^{-2i\epsilon k}/D
\]

\[
A^{+}_{1} = \frac{1}{2} \left( 1 + \frac{k}{q_{\epsilon}} \right) e^{-i(k+q_{\epsilon}) \epsilon}/D
\]

\[
A^{-}_{1} = \frac{1}{2} \left( 1 - \frac{k}{q_{\epsilon}} \right) e^{-i(k-q_{\epsilon}) \epsilon}/D
\]

with the common denominator

\[
D = \cos(2q_{\epsilon} \epsilon) - \frac{i}{2} \left( \frac{k}{q_{\epsilon}} + \frac{q_{\epsilon}}{k} \right) \sin(2q_{\epsilon} \epsilon).
\]

In the limit \( \epsilon \to 0 \) one has

\[
A^{+}_{0} \to -\frac{mV_{0}L_{0}}{\hbar^{2}k + mV_{0}L_{0}}
\]

\[
A^{-}_{2} \to \frac{\hbar^{2}k}{\hbar^{2}k + mV_{0}L_{0}}
\]

\[
A^{+}_{1}, A^{-}_{1} \to \frac{1}{2} \frac{\hbar^{2}k}{\hbar^{2}k + mV_{0}L_{0}}
\]

These results may be checked by considering a \( \delta \)-potential in Eq. (2) from the start.

The limit \( \epsilon \to 0 \) yields

\[
A^{0} \to 0
\]

\[
A^{+}_{2} \to 1
\]

\[
A^{+}_{1}, A^{-}_{1} \to \frac{1}{2}
\]

In this limit the measurement region has no effect on the motion of the atoms, as seen by Eq. (A10). However, by operator normalization a finite distribution is obtained in both limits. Inserting \( \phi^{(1)}_{\epsilon}(x) \) from Eq. (21) into Eq. (12) and calculating Eq. (20) leads to

\[
F_{\epsilon}(k, k') \to 1, \quad \epsilon \to 0, \epsilon \to 0
\]
and the arrival time distribution, $\Pi^{ON}_a(t)$, for a left incident free wave packet equals Kijowski's distribution,

$$\Pi^{ON}_a(t) = \Pi_K(t). \quad (34)$$

(ii) The case of general states: The Hamiltonian $H_c$ in Eq. (2) commutes with the parity operator and, as a consequence, so does the operator $\Pi_l$ in Eq. (14). Therefore its matrix elements between symmetric and antisymmetric states vanish so that the operator normalization of $\Pi_l$ can be performed in the subspaces of symmetric and antisymmetric states independently.

Let us first consider an antisymmetric incident state $|\psi_a\rangle$ composed of two identical wave packets with opposite momenta coming from the right and from the left [16, 40]. We first consider the antisymmetric subspace. The antisymmetric eigenstates $|k; a\rangle$ of $H_0$ and $\phi^a_k$ of $H_c$ ($k > 0$) are given by

$$|k; a\rangle \equiv \frac{1}{\sqrt{2}}(|k\rangle - |−k\rangle) \quad (35)$$

and an antisymmetric wavefunction $\psi_a$ can be decomposed as

$$\psi_a = \int_{0}^{\infty} dk |k; a\rangle \langle k; a| \psi_a. \quad (36)$$

The antisymmetric eigenstates $\phi^a_k$ of $H_c$ are obtained with the boundary conditions $A_0^+ = 1$ and $A_2^- = -1$, which gives

$$A_2^- = -A_0^+ = \frac{2 + i(k\frac{\hbar}{\kappa} - \frac{\kappa}{2}) \sin(2q\epsilon)}{D} e^{-2i\kappa} \quad (37)$$

$$A_1^+ = -A_1^- = \frac{\hbar}{2k} \cos(q\epsilon) - i \sin(q\epsilon). \quad (38)$$

The wave function in the presence of the imaginary potential can now be written as

$$\psi_a(x, t) = \int_{0}^{\infty} dk \langle k; a| \phi^a_k(x) e^{-i\hbar k^2 t/2m}. \quad (39)$$

In spite of the fact that the wave function vanishes at $x = 0$ the operator normalization preserves a finite arrival distribution even when the width of the measurement region contracts to 0. With Eq. (20), Eq. (21) and Eq. (12) one has, using Eq. (20),

$$F_\epsilon(k, k') \to 1, \quad \epsilon \to 0, \quad (a) \to 0. \quad (40)$$

which yields

$$\Pi^{ON}_a(t) = \frac{\hbar}{2\pi m} \left| \int_{0}^{\infty} dk \langle k; a| \psi_a \rangle \sqrt{k} e^{-i\hbar k^2 t/2m} \right|^2 \quad (41)$$

$$= \frac{\hbar}{\pi m} \left| \int_{0}^{\infty} dk \tilde{\psi}_a(k) \sqrt{k} e^{-i\hbar k^2 t/2m} \right|^2. \quad (42)$$

In the last line we have changed to the ordinary momentum representation and have taken the antisymmetry into account.

A similar treatment may be applied to a symmetric wavefunction $\psi_s$ by using symmetric eigenfunctions and the result is again of the form of Eq. (42) with $\psi_a$ replaced by $\psi_s$.

An arbitrary state $\psi(k)$ can be written in terms of its symmetric and antisymmetric part as $\psi(k) = \psi_s(k) + \psi_a(k)$. By parity, $\Pi^{ON}_a(t)$ is the sum of the corresponding symmetric and antisymmetric contribution since the cross terms vanish. By means of a trivial calculation the sum can be written as

$$\Pi^{ON}_\psi(t) = \frac{\hbar}{2\pi m} \sum_x \left| \int_{0}^{\infty} dk \tilde{\psi}(\pm k) \sqrt{k} e^{-i\hbar k^2 t/2m} \right|^2. \quad (43)$$

This is the operator-normalized arrival-time distribution for a general free wavefunction. The expression has been proposed in Refs. [4, 40] on more heuristic grounds as a generalization of the distribution in Eq. (1) from left (or right) incoming states to general free states.

B. Arrival time distribution for tunneling particles

A major advantage of the operational fluorescence model for the determination of arrival time distributions is that, in contrast to the approach of Kijowski, it is not restricted to free particles. This means that arbitrary potentials with bounded support can be considered and the arrival time distribution in the presence of these interactions can be calculated.

For simplicity, we consider here the case of a rectangular potential barrier. The Hamiltonian is given by Eq. (2) where, as before, the arrival at $x = 0$ is measured and the additional real potential $U$ is located in $a \leq x \leq b$. We here investigate the case with $a \leq b \leq 0$ (Fig. 1b).

For solving the stationary Schrödinger equation, Eq. (5), the $x$-axis has to be divided into five regions $i, i = 0, \ldots, 4$, corresponding to $x \leq a, a \leq x \leq b, b \leq x \leq -\epsilon, -\epsilon \leq x \leq \epsilon, \epsilon \leq x$, respectively. The general solution in region $i$ is given by Eq. (21) with $k_0 = k_2 = k_4 \equiv k, k_1 \equiv \infty = [k^2 - 2mU/\hbar^2]^{1/2}$ and $k_3 \equiv q \equiv [k^2 + 2mV_0L_0/\hbar^2]^{1/2}$. In Appendix A we present the derivation of the $A_i^\pm$ using transfer matrices.

In the case of an initial wave packet coming from the left and crossing the potential region the eigenstates required have boundary conditions $A_0^+ = 1$ and $A_4^- = 0$. Then one can solve for $A_0^+$ and $A_4^-$ and obtain the amplitudes $A_i^\pm$, i.e., the solution in the measurement region.

In the limit $\epsilon \to 0$ one has with $l = b - a$ and $s = a + b$

$$A_3^+ \to \left[ e^{ikl} \left( 2 \cos(\kappa l) - i \left( \frac{\kappa}{k} + \frac{s}{l} \right) \sin(\kappa l) \right) \right. \times \left( 1 + \frac{mV_0L_0}{\hbar^2k} \right)$$

$$\left. + e^{iks} \left( \frac{s}{k} - \frac{k}{l} \right) \sin(\kappa l) \frac{imV_0L_0}{4\hbar^2k} \right]^{-1}. \quad (44)$$
whereas in the weak case the limit $\epsilon^{(b)} \to 0$ yields

$$A_3^+ \rightarrow \frac{e^{-ikl}}{2\cos(\epsilon l) - i\left(\frac{\epsilon}{2} + \frac{k}{m}\right)\sin(\epsilon l)}, \quad (45)$$

which may also be obtained from Eq. (44) in the limit $V_0 \to 0$. The last expression is independent of $V_0L_0$ and

of the position of the potential. It is half the transmission amplitude $T(k)$ of a rectangular potential barrier without

any arrival time measurement, see Fig. 2. Inserting $A_3^+$

$$\frac{e^{ikx}}{R(k)} e^{-ikx}$$

real potential

$U$

$T(k) e^{ikx}$

Fig. 2: The stationary scattering function for a wave impinging from the left on a real potential barrier. $T$ and $R$ are the transmission and reflection amplitudes.

into $\phi_k^{(3)}$ given by Eq. (21) and calculating $F_i(k, k')$ by

Eq. (12) and Eq. (20) we obtain

$$F_i(k, k') = \frac{T(k)T(k')}{|T(k)||T(k')|}, \quad \epsilon^{(b)} \to 0, \quad (46)$$

and the operator-normalized arrival time distribution of a tunneled particle takes, with Eq. (19), the form

$$\Pi^{ON}_{\text{pot}}(t) = \frac{\hbar}{2\pi m} \left|\int dk \tilde{\psi}(k)e^{-ik^2t/2m} \sqrt{k} \frac{T(k)}{|T(k)|}\right|^2. \quad (47)$$

The effect of the additional potential is the introduction of a phase factor, which is the phase of the transmission amplitude for the real potential in the weak limit $\epsilon^{(b)} \to 0$.

The dependence of the arrival time distribution $\Pi^{ON}_{\text{pot}}(t)$ on the potential height $U$ with potential width $l = 10$ is shown in Fig. 3. With decreasing $U$ the arrival time at $x = 0$ of a particle starting at $\langle x \rangle = x_0 < 0$ with mean velocity $\langle v \rangle = v_0$ is first delayed but, for increasing potential strength, an asymptotic distribution is reached with a mean arrival time (“Hartman time”) $t_H = (|x_0| - l)\langle v^{-1} \rangle$ which is smaller than the free arrival time $|x_0|\langle v^{-1} \rangle$. This is related to the Hartman effect [46], which is discussed in detail in Section IV. The effect can indeed be observed [49] and it is complete in the limit $U \to \infty$, corresponding to an arrival point located in a “forbidden” region, in which case one has

$$F_i(k, k') \to e^{i(k-k')l}, \quad U \to \infty. \quad (48)$$

In the above setup an incoming free particle was prepared far to the left and then interacted with an external real potential. A different situation arises when one includes the real potential as part of the preparation procedure through which the particle passes far away on the left and then continues to propagate freely. In this case the incident state for operator normalization purposes would be the normalized transmitted wave packet. Formally, this packet may be formed by a projection onto a large region to the right of the potential. For the positive results, i.e. transmissions, the normalized incoming free state is then characterized by

$$T(k)\tilde{\psi}(k)/(\int dk |T(k)\tilde{\psi}(k)|^2)^{1/2} \text{ instead of } \tilde{\psi}(k).$$

Applying Kijowski’s distribution to the incoming free state thus prepared gives

$$\Pi^N_k(t) = \frac{\hbar}{2\pi m} \left|\int dk \tilde{\psi}(k)e^{-ik^2t/2m} \sqrt{k} T(k)\right|^2 \times \left(\int dk |T(k)\tilde{\psi}(k)|^2\right)^{-1}. \quad (49)$$

This expression coincides with the proposal of Refs. [13, 27, 31] for a generalized Kijowski distribution in the presence of an external potential and it is thus seen to be related to our result by a state preparation procedure that selects the transmitted particles.

Remark: The transmission probability through a potential barrier as in Fig. 2 is given by $\int dk |\tilde{\psi}(k)T(k)|^2$ and one may argue that in this case the total arrival-time probability should equal this transmission probability. Instead of $\Pi^{ON}_{\text{pot}}(t)$ one would then have a modified distribution, $\hat{\Pi}^{ON}_{\text{pot}}(t)$, satisfying

$$\int_{-\infty}^{\infty} dt \hat{\Pi}^{ON}_{\text{pot}}(t) = \int dk |\tilde{\psi}(k)T(k)|^2. \quad (50)$$

In terms of operators this would require an operator, $\hat{\Pi}^{ON}$, satisfying

$$\int_{-\infty}^{\infty} dt \hat{\Pi}^{ON} = \int dk |T(k)|^2 |k\rangle \langle k|. \quad (51)$$
With Eqs. (18), (20) and (46) it is easily seen that in this case the kernel of $\hat{B}^{-1/2}$ takes the form

$$b(k, k')^{-1/2} = \sqrt{\frac{\hbar}{2\pi m}},$$  

(52)

which yields

$$F_\epsilon(k, k') = f_\epsilon(k, k') \rightarrow T(k)T(k'), \quad \epsilon \rightarrow 0. \quad (53)$$

The modified distribution $\Pi_{\text{pol}}^\infty(t)$ is then given by

$$\Pi_{\text{pol}}^\infty(t) = \frac{\hbar}{2\pi m} \int dk |\tilde{\psi}(k)e^{-i(k^2/2m}\sqrt{k} T(k)}|^2,$$  

(54)

which satisfies Eq. (50) and gives the joint probability density for both arrival and transmission. Normalizing this to 1 by hand just yields the distribution $\Pi_{\text{pol}}^\infty(t)$ of Eq. (49). Therefore $\Pi_{\text{pol}}^\infty(t)$ can be understood as a conditional probability density for the arrival of the particle under the condition that it has been transmitted through the potential barrier.

**IV. MEAN ARRIVAL TIMES**

To compare our result of tunneling times with previous works, it is useful to consider not only the arrival time distribution but also mean arrival times. We restrict our analysis to a wave packet that comes from the far left and collides with a rectangular potential barrier at $a \leq x \leq b$. The arrival time behind the barrier is measured at $x = 0$ (Fig. 1b).

With $T(k) = |T(k)|\exp(i\Phi_T(k))$ the mean arrival time, $(t) = \int dt \Pi_{\text{pol}}^\infty(t)$, of the distribution in Eq. (47) is given by

$$(t) = m \int \frac{dk}{\hbar} |\tilde{\psi}(k)|^2 \frac{|x_0| + \Phi_T(k)}{k},$$  

(55)

where $x_0 < 0$ denotes the initial value for the mean position of the wave packet.

This result for $(t)$ can be understood as the average of the “phase times” (the time required for a freely moving particle plus Wigner’s time delay [50]) over the initial state. In contrast, previous proposals for the mean arrival time are written in terms of an average over the transmitted state [27, 46–48], which is just the first moment of the arrival time distribution of Eq. (49). These results are not contradictory but correspond to different state preparations.

The dependence with the potential height is shown in Fig. 4, where we have plotted $(t)$ versus $U$ for fixed barrier width $l$. In the free limit $U \rightarrow 0$, $(t)$ approaches an “averaged free arrival time”, $|x_0|(v^{-1})$, and for $U \rightarrow \infty$ it approaches the “Hartman time” $(|x_0| - l)(v^{-1})$, where $v^{-1} = \int dk |\tilde{\psi}(k)|mk^{-1}/\hbar$.

For analyzing the dependence with the barrier width $l$ let us heuristically define a “mean tunneling time” as an easily calculable quantity obtained by subtracting from $(t)$ the classical time for crossing the non-potential region with average momentum $k_0$,

$$\tau = (t) - \frac{m(|x_0| - l)}{\hbar k_0}.$$  

(56)

The Hartman effect occurs when $\tau$ in a tunneling collision of a quantum particle with an opaque square barrier becomes essentially independent of the barrier width $l$ [46].

This is shown in Fig. 5, where we have plotted the tunneling time $\tau$ versus the potential width $l$. For thin barriers $\tau$ is above the free traversal time, as shown in the inset, but for increasing $l$ there is a sudden transition from a positive delay to a negative one for increasing $U$. It is clearly visible, that in the negative delay regime the tunneling time is nearly constant for increasing $l$ (Hartman regime). In contrast we have plotted the tunneling time with respect to the arrival time distribution $\Pi_{\text{pol}}^\infty(t)$ of Eq. (49), $\tau_T$, which is related to the proposals of Ref. [13, 27, 31]. Here the behavior of $\tau$ for thin barriers is similar, but for increasing $l$ the tunneling time first slowly decreases and gets actually negative. This has caused many discussions and warnings that the interpretation of the “extrapolated phase times” as an actual tunneling time for the transmitted particle is unjustified [51, 52]. This question is not of our concern here though. The point we want to stress is that for very thick barriers the Hartman effect vanishes for $\tau_T$, which for widths larger than a critical barrier length grows linearly [47]. This is related to the influence of the exponentially decaying $|T(k)|$ in Eq. (49), which causes a domination of the above-threshold components of the wave packet [27, 47]. By contrast, $\tau$, obtained with a different

![FIG. 4: The mean arrival time $(t)$ of Eq. (55) behind a barrier of fixed width $l = 10$ (solid), $l = 5$ (dashed) as a function of the barrier height $U$. The initial wave packet is a minimal uncertainty Gaussian with $x_0 = -50$, $v_0 = 1$. $\Delta x = 10$ in atomic units $m = \hbar = 1$. For $U \rightarrow 0$ one has the free arrival time $|x_0|(v^{-1})$, and for $U \rightarrow \infty$ the arrival time approaches the Hartman time $t_H = (|x_0| - l)(v^{-1})$.](image)
preparation procedure does not show any transition to the classical-like, ultra-opaque regime. The intuitive explanation is that operator normalization compensates for all detection losses due to the \( l \)-dependence of \(|T(k)|\), so that the result is never dominated by above-the-barrier components.

A crucial ingredient for obtaining the present results is the way in which detection losses are compensated by operator normalization. Reasonable as this modification may be, there remains to be seen how this can be interpreted from an operational point of view, i.e. how such a major intervention could be performed in practice. Operator normalization can be viewed as a state modification, \(|\psi\rangle \rightarrow B^{-1/2}|\psi\rangle\), and we note that the modified state is not normalized. The physical process leading from the original state to the modified one may be a “filtering” preparation by scattering. However, the unbounded nature of the operator \( B^{-1/2} \) restricts such a method to states with a low-energy cut-off. The feasibility of the transformation in a more general case is an open question and further investigation is required.

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**APPENDIX A: TRANSFER MATRICES**

To calculate the transmission and reflection coefficients of the one dimensional potential problems of Fig. 1 or similar piecewise constant potentials we use the transfer matrix method of Ref. [53] and define “matching” matrices

\[
\mathbf{M}_i(x) = \begin{pmatrix} e^{ik_i x} & e^{-ik_i x} \\ k_i e^{ik_i x} & -k_i e^{-ik_i x} \end{pmatrix},
\]

for each section \( i \) of constant potential \( V_i \), where

\[
k_i = [k^2 - 2mV_i/\hbar^2]^{1/2},
\]

with \( \text{Im} k_i \geq 0 \), and \( k_i > 0 \) for positive \( V_i \). We shall always use the index 0 for the leftmost section, and number the rest rightwards and consecutively. We shall also denote by \( x_{i+1} \) the boundary point between sections \( i \) and \( i + 1 \), so that the matching conditions take the general form

\[
\mathbf{M}_i(x_{i+1}) \begin{pmatrix} A_i^+ \\ A_i^- \end{pmatrix} = \mathbf{M}_{i+1}(x_{i+1}) \begin{pmatrix} A_{i+1}^+ \\ A_{i+1}^- \end{pmatrix},
\]

Multiplying both sides by \( \mathbf{M}_i^{-1}(x_{i+1}) \),

\[
\begin{pmatrix} A_i^+ \\ A_i^- \end{pmatrix} = \mathbf{T}(i, i + 1) \begin{pmatrix} A_{i+1}^+ \\ A_{i+1}^- \end{pmatrix}
\]

where the transfer matrix connecting the amplitudes of regions \( i \) and \( i + 1 \) is given by

\[
\mathbf{T}(i, i + 1) = \mathbf{M}_i^{-1}(x_{i+1}) \mathbf{M}_{i+1}(x_{i+1})
\]
One may similarly obtain transfer matrices for non contiguous regions $i$ and $j \geq i+2$ by multiplying all the intermediate one-step transfer matrices,

$$T(i, j) = T(i, i+1) T(i+1, i+2) \cdots T(j-1, j). \quad (A6)$$

In the following subsections we discuss the cases needed for the present paper. Note however that the transfer matrix method is of general applicability and may be adapted to continuous potentials leading in that case to differential equations for the amplitudes.

### 1. Free particles

For the free arrival time problem (Fig. 1a) one has three regions with $k_0 = k_2 = k$, $k_1 = q_e$, and matching points at $x_1 = -\epsilon$ and $x_2 = \epsilon$. From the matching matrices one obtains

$$T(0, 2) = \left( \begin{array}{cc} \cos(2q_e \epsilon) - \frac{i}{2} \left( \frac{k}{mV_2} + \frac{q_e}{m} \right) \sin(2q_e \epsilon) \\ \frac{i}{2} \left( \frac{k}{mV_2} - \frac{q_e}{m} \right) \sin(2q_e \epsilon) \end{array} \right) e^{2ik\epsilon}$$

and

$$T(1, 2) = \frac{1}{2} \left( \begin{array}{cc} (1 + \frac{k}{mV_2})e^{i(k-q_e)\epsilon} & (1 - \frac{k}{mV_2})e^{-i(k+q_e)\epsilon} \\ (1 - \frac{k}{mV_2})e^{i(k+q_e)\epsilon} & (1 + \frac{k}{mV_2})e^{-i(k-q_e)\epsilon} \end{array} \right). \quad (A8)$$

This second matrix is useful to obtain the wave function in the absorbing region. In the limits $\epsilon \to 0$ and $\epsilon \to 0$ one has

$$\lim_{\epsilon \to 0} T(0, 2) = \left( \begin{array}{cc} 1 + \frac{mV_2L_2}{\hbar^2} & \frac{mV_2L_2}{\hbar^2} \\ -\frac{mV_2L_2}{\hbar^2} & 1 \end{array} \right) \quad (A9)$$

$$\lim_{\epsilon \to 0} T(1, 2) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \quad (A10)$$

$$\lim_{\epsilon \to 0} T(1, 2) = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \quad (A11)$$

For waves incident from the left ($x < 0$) the boundary conditions are $A_0^+ = 1$ and $A_2^- = 0$ and there results $A_2^+ = [T(0, 2)_{11}]^{-1}$. Match $A_0^- = T(0, 2)_{21}A_2^+$, $A_1^+ = T(1, 2)_{11}A_2^+$ and $A_1^- = T(1, 2)_{21}A_2^+$.

### 2. Tunneling particles

In the case of a tunneling particle (Fig. 1b) with $a < b < 0$ one has to consider boundary conditions at $x_1 = a$, $x_2 = b$, $x_3 = -\epsilon$ and $x_4 = \epsilon$. $T(2, 4)$ is the same as the transfer matrix obtained in the previous section, Eq. (A7), for the regions on both sides of the absorbing potential so we only need to calculate $T(0, 2)$,

$$T(0, 2) = \left( \begin{array}{cc} e^{ikl}(\cos(xl) - \frac{i}{2}\left(\frac{k}{m} + \frac{q_e}{m}\right)\sin(xl)) \\ \frac{i}{2}e^{iks}(\frac{k}{m} - \frac{q_e}{m})\sin(xl) \end{array} \right) \quad (A13)$$

where $l = b - a$ and $s = a + b$. Since we are interested in the wave function inside the measurement region ($-\epsilon, \epsilon$), the corresponding amplitudes are given from the amplitudes of the rightmost region using the matrix in Eq. (A8) as before.

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