Space Time Codes from Permutation Codes

Oliver Henkel
Fraunhofer German-Sino Lab for Mobile Communications - MCI
Einsteinufer 37, 10587 Berlin, Germany
Email: henkel@hhi.fraunhofer.de

Abstract—A new class of space time codes with high performance is presented. The code design utilizes tailor-made permutation codes, which are known to have large minimal distances as spherical codes. A geometric connection between spherical and space time codes has been used to translate them into the final space time codes. Simulations demonstrate that the performance increases with the block lengths, a result that has been conjectured already in previous work. Further, the connection to permutation codes allows for moderate complex en-/decoding algorithms.

I. INTRODUCTION

In MIMO (Multiple Input Multiple Output) systems space time coding schemes have been proven to be an appropriate tool to exploit the spatial diversity gains. Two distinct scenarios are common, whether the channel coefficients are known (coherent scenario) [1], to the receiver or not (non-coherent scenario) [2]. Prominent coherent codes are the well known Alamouti scheme [3] and general orthogonal designs [4]. A more flexible coding scheme are the so-called linear dispersion codes. They have been introduced in [5] and were further investigated in [6]. A high rate example achieving the diversity multiplexing tradeoff is the recently discovered Golden code [7]. Genuine non-coherent codes have been proposed in [8], but most of the research efforts in the literature focus on differential schemes, introduced in [9], since differential codes usually provide higher data rates than comparable non-differential codes. High performing examples have been constructed in [10], [11],[12],[13]. However, in both cases most research effort has been undertaken for space time block codes with quadratic 2-by-2, resp. \( n_t \)-by-\( n_t \) code matrices (\( n_t \) denotes the number of transmit antennas). Although linear dispersion codes are not restricted to quadratic shape of the design matrices the block length is not a free design parameter when the number of transmit antennas is held fixed (compare the asymptotic guidelines in [6]).

In contrast to that, both coherent and (non-differential) non-coherent case are expected to benefit from coding schemes which use the additional degrees of freedom provided by increasing the block length [14] (whereas \( n_t \) is fixed). This result has originally been developed in the context of packing theory, but in [15] its influence on the performance on space time block codes has been pointed out. Roughly speaking, space time code design can be considered as a constrained sphere packing problem, where the objective (performance gain) can be optimized in a two stage process. Step one aims to construct good packings, step two is concerned with the maximization of the coding gain, given a packing configuration. This method works for the coherent scenario as well as for the non-coherent system.

The present work utilizes the proposed two stage process to construct space time codes for both scenarios. It turns out that the performance in terms of bit error rates of the constructed codes increases with the block length, in accordance to what has been conjectured in [14]. The simulation results show, that it is possible to beat the performance of some optimal conventional 2-by-2 schemes considerably.

The two optimization steps, though different in their nature, are commonly formulated in geometric terms, according to the underlying geometric structures of the coding spaces. While the second step is simply a suitably defined rotation of the data (precoding in some sense), the first step involves geometric and combinatorial aspects. The differential geometric aspects have been already analyzed in previous publications [14], [15], [16], [17], and the contribution of this work has its focus on the combinatorial part, namely the construction of appropriate spherical permutation codes.

Section II introduces the channel model and basic definitions, section III states the code design criteria with emphasis on the aspects which become important for the further development, in particular subsection III-C summarizes the main points. Section IV sketches the results of previous work, namely the differential geometric connection between spherical packings — which occur e.g. in the context optimal sequence design in CDMA systems — and packings on the Stiefel and Grassmann manifolds, the appropriate coding spaces for space time block code design. Then in section V permutation codes enter the stage, since they carry naturally the interpretation as spherical packings. The design of permutation codes yielding large packing distances on spheres with prescribed dimension and rate requirements will be investigated, followed in VI by an analysis of the second optimization step, i.e. the design of an appropriate rotation matrix. Section VII presents simulations of bit error performance and VIII summarizes the work done so far, followed by an outlook to further work.

II. CHANNEL MODEL AND CODING SPACES

Let us assume a MIMO system with \( n_t \) transmit antennas and \( n_r \) receive antennas. The fading statistic is assumed to obey a Rayleigh flat fading model with block length \( T \) of the coherence interval. Then we have the transmission equation

\[
Y = \sqrt{\rho} X H + N
\]
where $X$ denotes the $T$-by-$n_t$ transmit signal with normalized expected power per time step, $H \sim CN(0, 1)$ is the $n_t$-by-$n_r$ circular symmetric complex normal distributed channel matrix, $N \sim CN(0, 1)$ denotes the $T$-by-$n_r$ additive noise, and $Y$ the $T$-by-$n_r$ received signal, where $\rho$ turns out to be the SNR at each receive antenna. The symbol 1 denotes a unit matrix throughout this work, sometimes supplemented by an index indicating the dimension.

Due to the work of Hochwald/Marzetta [18] it is reasonable from a capacity perspective to assume the transmit signals $X$ to have (apart from a scaling factor) unitary columns. More precisely we can write

$$X = \sqrt{\frac{T}{n_t}} \Phi$$

and consider the complex Stiefel manifold

$$V_{n_t,T}^C := \{ \Phi \in \mathbb{C}^{T \times n_t} \mid \Phi^* \Phi = 1_{n_t} \}$$

as the coding space (\( ^* \) denotes the hermitian conjugate). Thus a space time code is considered to be a discrete subset $C \subset V_{n_t,T}^C$ and we define the rate $R$ of the code by

$$R := \frac{1}{T} \log_2 |C|$$

Provided a received signal $\tilde{Y} = \sqrt{\frac{T}{n_t}} \Psi + N$ the maximum likelihood (ML) detection rule reads

$$\Phi_{ML} = \arg \min_{\Phi \in C} \left\| \tilde{Y} - \sqrt{\frac{T}{n_t}} \Phi H \right\|_p$$

where $\left\| A \right\|_p = \sqrt{\text{tr} A^* A}$ denotes the Frobenius norm.

**A. Non-coherent detection**

If the receiver has no information about the fading states the detection is called non-coherent. In this case it is shown in [18], [2], [19] that the coding space is the complex Grassmann manifold

$$G_{n_t,T}^C := \{ \Phi \mid \Phi \in V_{n_t,T}^C \}$$

of $n_t$-dimensional linear complex subspaces of $\mathbb{C}^T$ (\( ^* \) denotes the vector space spanned by the columns of the matrix $\Phi$). One can think of $\Phi$ representing a subspace $\langle \Phi \rangle$, but for a given $\Phi \in V_{n_t,T}^C$, all matrices $\Phi_u$ with arbitrary unitary $n_t$-by-$n_t$ matrix represent the same subspace; therefore the Grassmann manifold is really a coset space of the Stiefel manifold and the choice of a unique representative for each coset is not obvious in general. However, the maximum likelihood detection for non-coherent detection decides on the subspace $\langle \Phi_{ML} \rangle$ represented by

$$\Phi_{ML} = \arg \max_{\langle \Phi \rangle \in C} \left\| \tilde{Y}^* \Phi \right\|_p$$

given a ‘received noisy subspace’ $\langle \tilde{Y} \rangle$ represented by $\tilde{Y} = \sqrt{\frac{T}{n_t}} \Psi + N$. Since the Frobenius norm is unitarily invariant, the ML criterion (7) is independent of the chosen representatives $\Phi$ and $\Psi$, thus (7) provides a well defined measure of subspace correlation. Therefore, the explicit choice of a representative $\Phi$ of $\langle \Phi \rangle \in C$ is irrelevant and we are free to consider non-coherent codes $C$ as subsets of the Stiefel manifold $V_{n_t,T}^C$ rather than subsets of the Grassmann manifold, thinking in terms of representatives. As a notational convention entities from a non-coherent context will be underlined.

**III. Space time code design criteria revisited**

**A. Coherent case:**

The code design aims to maximize an appropriate functional on the set of difference symbols $D := \Phi - \Psi$. Common design criteria arise from the familiar Chernov bound for the pairwise error probability, which has the form $[2]$

$$ch = \frac{1}{2} \prod_{i=1}^{n_t} \left[ 1 + \varrho \sigma_i^2(D) \right]^{-n_r}$$

where $\varrho := \frac{1}{n_t} \sqrt{\rho}$ and $\sigma(A) = (\sigma_1(A), \ldots, \sigma_{n_r}(A))$ generically denotes the vector of singular values of a matrix $A$ in decreasing order. Taking this bound as the target functional it is immediately clear that the code design does not depend on the number of receive antennas, and the objective becomes the maximization of the diversity functional

$$\text{Div} := \prod_{i=1}^{n_t} \left[ 1 + \varrho \sigma_i^2(D) \right] = \sum_{i=0}^{n_t} s_i \varrho^i$$

as well as the diversity product as its leading term

$$p^2 := s_n = \det(\Delta^* \Delta)$$

**B. Non-coherent case:**

Following [2] a similar derivation applies: Defining the codeword difference symbol as $\Delta := \Phi - \Psi$ the Chernov bound now reads

$$ch = \frac{1}{2} \prod_{i=1}^{n_t} \left[ 1 + \varrho(1 - \sigma_i^2(D)) \right]^{-n_r}$$

where $\varrho := \frac{1}{n_t} \sqrt{\rho}$, and the corresponding diversity quantities become

$$\text{Div} := \prod_{i=1}^{n_t} \left[ 1 + \varrho(1 - \sigma_i^2(D)) \right] = \sum_{i=0}^{n_t} s_i \varrho^i$$

with $s_i := \text{sym}_i((1 - \sigma_1^2(D)), \ldots, (1 - \sigma_{n_r}^2(D)))$, and

$$d^2 := s_1 = \left\| \Delta \right\|_F^2$$

$$p^2 := s_n = \det(1 - \Delta^* \Delta)$$
C. Implications for the code design and known results

Coherent and non-coherent diversity functions are homogeneous polynomials, in particular a packing gain $d \rightarrow \alpha d$ (resp. $d \rightarrow \alpha d$), $\alpha > 1$, turns out to be equivalent to coding with effective power $\alpha^2 q$ (resp. $\alpha^2 q$). Thus, the diversity sum, which has been known as a low SNR design criterion in the literature, also scales the SNR itself, and has therefore an impact on the higher order terms in the diversity functional, in particular onto the diversity product. From this insight it is reasonable to consider the code design as a constraint packing problem. This means, that the maximization of diversity can be split up into a two-stage optimization procedure:

1) Find good packings in the coding spaces $V_{n_1,T}^C, G_{n_1,T}^C$.
2) Find a transformation which maps the packings into equivalent packings with maximal diversity product.

Details about the optimality criteria in this context can be found in [15].

Another important point regarding packing gains is the result obtained in [14, Corollary IV.2]: The achievable minimal distances $d^2$, resp. $d^2$, can be lower bounded by a quantity which grows proportionally to $\frac{T}{n_t}$, thus there is a benefit for code designs with large block lengths and the codes constructed in this work benefit considerably in performance as we will see later on.

Since the overall complexity of code design and decoding grows also with large block lengths, in [15, Prop. III.4] the inequality $D_{iv} \leq D_{iv}$ has been established, which is the diversity analogue of the information theoretic inequality $I(X;Y) \leq I(X;Y,H)$. From this one infers immediately that any non-coherent code can be used in a coherent scenario without performance loss. Moreover [15, Thm. III.5] states, that, given a non-coherent code $C$, the set $\{ \Phi u | \Phi \in C, u \in V_T \}$ for any $n_t$-by-$n_t$ coherent code $\tilde{C}$ is actually a coherent space with diversity as least as good as the diversities of $C$ and $\tilde{C}$. This result can be interpreted as a complexity reduction, providing two level code design and decoding algorithms.

IV. Space time packings from spherical codes

Let us start with the proposed first stage optimization procedure for code design, namely the construction of packings in $V_{n_1,T}^C$ resp. $G_{n_1,T}^C$ with large minimal distance. A comprehensive standard source on the general sphere packing problem in Euclidean space is [20]. Unfortunately the methods in [20] rely on the symmetry group of Euclidean space and do not apply to our situation, where the coding spaces are non-flat and the distance metric is nonlinear. Although [21] considers Grassmannian packings, it applies to the real Grassmannian manifold only. Some genuine complex Grassmannian packings have been constructed numerically in [22],[23], and [24] but numerical optimization techniques are computational complex and give only little insight into the construction mechanisms nor do they possess any algebraic structure.

Therefore it would be desirable to find simple model spaces, where structured packings can be constructed and then transformed into packings on the complex Stiefel and Grassmann manifolds. On the one hand this model space must possess a large symmetry group such that some structured packing algorithm may be developed. On the other hand it must be ‘similar’ to the Stiefel and Grassmann manifold in order to construct a mapping which approximately preserves (minimal) distances. In this paper such a model space with corresponding mapping will be presented utilizing the homogeneous structure of the coding spaces (compare [25] for a general introduction to homogeneous spaces or [26] for the homogeneous structure of the (real) Stiefel and Grassmann manifolds). In particular the (complex) Stiefel manifold $V_{n_1,T}^C$ is diffeomorphic to a coset space with respect to the unitary group $U(T)$ of $T$-by-$T$ unitary matrices:

$$V_{n_1,T}^C \cong U(T) \bigg\{( \begin{smallmatrix} 0 & \mathbb{0} \\ \mathbb{0} & U(T-n_1) \end{smallmatrix} \bigg\}$$

whereas $\cong$ means ‘diffeomorphic to’. This fact is due to the symmetry action $\Phi \mapsto ( \begin{smallmatrix} 0 & \mathbb{0} \\ \mathbb{0} & U(T-n_1) \end{smallmatrix} \bigg\} \Phi$ leaving $\{ \begin{smallmatrix} 0 \\ \mathbb{0} \end{smallmatrix} \}$ fixed. Similarly for the (complex) Grassmann manifold $G_{n_1,T}^C$ of $n_t$ dimensional subspaces $\{ \Phi \in \mathbb{C}^T \}$. Since $\Phi \mapsto \{ \Phi \}$ is a projection invariant under all $n_t$-by-$n_t$ unitary basis transformations we obtain the coset representation

$$G_{n_1,T}^C \cong U(T) \bigg\{( U(n_1) \begin{smallmatrix} 0 & \mathbb{0} \\ \mathbb{0} & U(T-n_1) \end{smallmatrix} \bigg\}$$

Homogeneity (or coset structure) means, that any two points can be mapped isometrically into each other, in particular all distance relations are uniquely determined with respect to an arbitrarily chosen reference point (e.g. $\{ \frac{1}{\sqrt{2}} \}$, resp. $\{ \frac{1}{\sqrt{2}} \}$). We will see that homogeneity provides the required ‘similarity’ mentioned above. Let us define $D$ by $D = \dim_{\mathbb{R}} V_{n_1,T}^C = n_t(2T-n_1)$ resp. $D = \dim_{\mathbb{R}} G_{n_1,T}^C = 2n_t(T-n_1)$. The $D$ dimensional sphere $S^D := \{ x \in \mathbb{R}^{D+1} | ||x|| = 1 \} \subset \mathbb{R}^{D+1}$ is also homogeneous, since it has the coset representation

$$S^D \cong V_{1,D+1}^R \cong O(D+1) \bigg\{ \begin{smallmatrix} 0 & \mathbb{0} \\ \mathbb{0} & O(D) \end{smallmatrix} \bigg\}$$

where $O(D)$ denotes the set of $D$-by-$D$ orthogonal matrices. The sphere is highly symmetric and ‘similar’ to our coding spaces, since in [14] a relation between packing densities of the coding spaces and $S^D$ has been established, and in [16], [17] a corresponding mapping of packings $S^D \rightarrow V_{n_1,T}^C$ resp. $S^D \rightarrow G_{n_1,T}^C$ has been defined, utilizing the homogeneous coset structure. Due to the analysis in [14] this mapping is distance preserving up to a positive scaling factor. In summary, spherical codes can be transformed into space time codes with controlled distance loss. Moreover the theory of spherical packings (i.e. packings of spherical caps on $S^D$) is already an item of current research, see e.g. [27], [28]. Nevertheless, here another spherical packing algorithm will be presented to obtain structured and at the same time full rate spherical packings. However, in the space frequency context of MIMO-OFDM systems spherical packings based on lattice constructions have already been investigated [16], [17].

\[ a \] Actually the mapping is appropriately defined on the upper (or lower) hemisphere of $S^D$ only. This is due to the projective nature of $G_{n_1,T}^C$ such that antipodal points on the sphere will be identified under this mapping.
V. SPHERICAL PACKINGS FROM PERMUTATION CODES

A more flexible algebraic tool than lattices to produce spherical packings are groups, i.e. finite subgroups of the orthogonal group. The idea behind it is to take some initial \((D+1)\) dimensional vector of unit norm (s.t. it can be considered as a point on the \(D\) dimensional sphere \(S^D\)). Then let the finite subgroup \(G\) act on the initial vector \(x\) and the outcome is a spherical packing whose constellation size equals the order of \(G\). The optimization procedure to maximize the packing distance involves the choice of the group \(G\) itself and the choice of the initial vector. The packings generated by such a procedure are called geometrically uniform and have been considered recently in a frame theoretic context [29] (see [30] for an introduction to frame theory in communications).

In a broader context the set of vectors (input sequences) obtained as orbits of \((G)\) of some initial vector is called a group code for the Gaussian channel. This class of codes comprises many signal sets that are used in practice, e.g. linear binary codes. In the special case \(G\) consisting of \((D+1)\)-by-\((D+1)\) matrix representations of permutations, the resulting group code is called permutation modulation [31]. Note that in practice only subgroups of the permutation group will be of interest, otherwise the huge number of \(D!\) permutations generate permutation modulations no practical device can handle.

The corresponding spherical packings will be the starting point for the following analysis. In [31] an optimization procedure similar to a Lagrangian method is presented, which solves for the initial vector whose generated permutation modulation has largest minimal distance under the action of a fixed permutation subgroup. The size of the subgroup is specified in terms of the initial vector with appropriate repetitions of its components

\[
x = (\mu_1^{(m_1)}, \ldots, \mu_k^{(m_k)})
\]

where \(\mu_i^{(m_i)}\) denotes \(\mu_i\) repeated \(m_i\) times. Although the analysis in [31] does not provide a complete solution (no solution for the 'Lagrangian' parameters has been given), the method reveals some structure of the optimal initial vector:

The entries \(\mu_i\) are symmetrically arranged around zero and the corresponding weights \(m_i = \left[e^{-(\eta+\mu_i^2)}/\lambda\right]\) are determined according to some discrete Gaussian distribution involving the 'Lagrangian' parameters \((\eta, \lambda)\) [31, Sec. IV]. Plugging this into the constraint equation of the 'Lagrangian' analysis yields, using Maple, complete solutions. Unfortunately due to the integer constraint on the \(m_i\) solutions are possible only for carefully selected parameters. The typical spherical dimensions \(D\) occurring here do not permit solutions with small enough rates. Therefore another strategy has been chosen.

Inspection of the initial solution vectors with lowest possible rate, such that the 'Lagrangian' functional provides a solution, revealed that there are only a few possible alternatives for the choice of \(x\), namely \(x\) is characterized by a large amount of zero components and only a few non-zero ones. The more distinct components in \(x\), the larger the set of distinct permutations (high rate), and the smaller the final minimal distance. Therefore for prescribed dimension and rate the initial vector \(x\) with largest possible number of zero-components has been chosen, such that the rate requirement is satisfied.

Having found an appropriate initial vector the problem of carefully selecting the corresponding permutations remains. Given \(x \in \mathbb{R}^{D+1}\) of the form (19) the corresponding number of distinct permuted versions is (in multi index notation with respect to the vector \(m = (m_1, \ldots, m_k)\))

\[
M :\binom{m_1}{m}! \cdots \binom{m_k}{m}!
\]

Given a prescribed space time code rate \(R\), the corresponding rate of the spherical code is \(r := \frac{R}{T}\) and the required number of permutations is given as \(N = \lfloor 2^{(D+1)r} \rfloor\), where we have chosen the initial vector \(x\) (resp. the vector \(m\)) such that \(N \leq M\) holds. Then the task is, to select \(N\) out of the \(M\) distinct permutations of the multiset \(x\) such that the resulting packing has large minimal distance. Taking the number of transpositions required to transform a permutation \(p\) into another permutation \(q\) as a distance measure between \(p\) and \(q\), the objective is to select \(N\) out of \(M\) multiset permutations with large pairwise distance. In contrast to ordinary permutations the structure of multiset permutations is more complicated, and there seems to be no ranking algorithm available. Nevertheless all multiset permutations can be listed in Gray code order, which is the appropriate ordering with respect to the permutation distance just defined. The algorithm can be obtained as a short C program from the Combinatorial Object Server\(^b\). Then, taking each \(\binom{M}{N}\)'s multiset permutation produced by this algorithm does the job and we end up with the desired spherical packing with large minimal distance, corresponding to the specified rate.

VI. FULL DIVERSITY ROTATION

Let us now come to the second stage of diversity optimization in the sense described in III-C, namely to define a distance preserving mapping which transforms the space time packings into an equivalent packing with maximum diversity product.

To this end we precode the space time code symbols by performing a rotation on the spherical code as follows. As the axis of rotation we choose the 'diagonal' \(c = (1, \ldots, 1) \in \mathbb{R}^{D+1}\). Define a unitary \((D+1)\)-by-\((D+1)\) matrix \(W_c\) by prescribing its first row to be \(e/\sqrt{D+1}\) and for \(j = 2, \ldots, D+1\) its \(j\)th row to be \((1^{j-1}, -(j-1), 0^{(D+1-j)})/\sqrt{j(j-1)}\). Clearly \(c = e_1W_c\) holds with \(e_1 = (1, 0, \ldots, 0)\), thus \(e_1 = eW_c\), where the superscript \(t\) denotes transposition. Suppose we already had defined a rotation matrix \(R_1\) with \(e_1\) as its axis, then we obtain the same rotation about the axis \(e\) as \(R := W_c^tR_1W_c\). The rotation \(R_1\) is constructed easily: Set \(0 = (0^{(D)})\), then \(R_1 = \begin{pmatrix} 0 & 0 \\ 0 & \text{exp}(\alpha X) \end{pmatrix}\) performs a rotation about \(\alpha\) degrees about the axis \(e_1\), where \(X\) being the antisymmetric

\(^b\)the term multiset denotes a set with repeated elements
\(^c\)Programmer: Frank Ruskey / Joe Sawada

http://www.theory.csc.uvic.ca/~cos/inf/mult/Multiset.html
A $D \times D$ matrix with ones on its upper triangular part (which uniformly weights the available degrees of freedom). Figure 1 demonstrates the effect of rotation for some values of $\alpha$ on the performance of a sample non-coherent $8 \times 2$ code of rate $1/2$. Note that without rotation ($\alpha = 0$, thick dashed line) the code does not achieve full diversity order. Trying some values for $\alpha$ reveals some oscillatory behavior of the coding gain (i.e. the value of the diversity). It turns out that for non-coherent codes $\alpha = \frac{7}{4}\pi$ is a good choice, while for coherent codes $\alpha = \pi$ yields good results. If a non-coherent code will be used in the coherent scenario by composing it with some small coherent code (compare III-C), the angle $\alpha = \frac{7}{2}\pi$ remains a good choice.

Fig. 1. Performance of $R = 0.5, 8 \times 2$ space time codes coming from the same spherical code, but precoded with different rotation angles

VII. SIMULATION RESULTS

All simulations have been performed in a scenario with $n_t = 2$ transmit antennas and $n_r = 1$ receive antennas with maximum likelihood decoding. Figure 2 displays the bit error performance of a series of two-stage-optimized non-coherent codes with rate approximately one and block lengths varying from 4 to 12 (continuous lines). The corresponding initial vectors (of dimension $D + 1$) and the number of chosen multiset permutations are $x = (0^{(7)}, 1^{(2)})/\sqrt{2}$, $N = 32$; $x = (-1, 0^{(23)}, 1)/\sqrt{2}$, $N = 512$; $x = (0^{(38)} 1^{(3)})/\sqrt{3}$, $N = 8192$, respectively. The rotation angle is $\alpha = \frac{7}{4}\pi$ and the final space time code is then given as the image of the map $S^D \rightarrow G^C_{nt, T}$ (compare section IV), where now (for $n_t = 2$ fix) $D = 8, 24, 40$ for $T = 4, 8, 12$ respectively. Note that the cardinality of the final space time codes differs from the corresponding spherical code cardinality due to the restriction to one hemisphere of $S^D$, compare footnote a) in section IV (e.g. the spherical code of cardinality $N = 32$ shrunk to a space time code of cardinality 21 only, thus $R \approx 1.1$). The simulation shows that the bit error performance increases with the block length in perfect conformity with the result of earlier work [14], mentioned in III-C. Moreover [12] presented a non-coherent 2-by-2 differential code with optimal diversity sum and diversity product. The performance of this optimal 2-by-2 code is also shown in fig. 2 (thick dashed line). The comparison reveals that the additional degrees of freedom provided by the larger block lengths of the new codes based on permutation codes result in an approximately 2dB performance gain over the 2-by-2 differential code [12]. Note that the non-coherent codes constructed here are not based on a differential transmission scheme. Thus the achieved performance gain over one of the best known differential schemes justifies the research effort for non-differential schemes.

Fig. 2. Non-coherent performance gain with increasing block length, compared to the optimal 2-by-2 differential code

Fig. 3. Coherent performance gain with increasing block length, compared to the well known BPSK Alamouti scheme

Figure 3 displays the bit error performance of a series of two-stage-optimized composed coherent codes with rates ranging from 1.64 to 0.79 and block lengths $T = 4, 8, 16$ (continuous lines). They have been composed from a series of non-coherent codes and a QPSK Alamouti scheme [3]. The non-coherent codes come from corresponding spherical codes of size $N = 8, 32, 512$ (where again some spherical code points have been removed due to the restriction to only one hemisphere) and dimension $D = 8, 24, 56$. Again the bit error performance increases with the block length and comparing
the rate 1.05 8-by-2 code with the 2-by-2 BPSK Alamouti code (thick dashed gray line in fig. 3) shows a performance gain of approximately 2dB. Of course the new codes suffer from a considerably higher decoding complexity compared with the Alamouti scheme, thus there is a tradeoff between performance and signal processing. A more fair comparison incorporating some additional signal processing may be represented by the thick dashed black line in fig. 3. It shows the performance of a 2-by-2 code with optimal diversity sum and diversity product, which is in fact identical to the optimal non-coherent 2-by-2 differential code [12]. This code performs about 1dB better than the Alamouti scheme but compared with the new codes we still obtain a performance gain of approximately 1dB of the new 8-by-2 code over the optimal 2-by-2 code.

**VIII. CONCLUSIONS AND FUTURE WORK**

A new class of space time codes based on spherical permutation codes has been presented. It has been demonstrated that the additional degrees of freedom provided by larger block lengths help to achieve better performance and even beat the bit error performance of 2-by-2 diversity-optimal schemes. The presented construction applies both to coherent and non-coherent code design with a two-stage optimization process which reduces the design complexity by geometrical insights affording algebraic structures.

The inherent design complexity of coherent codes with large block lengths can be further compensated in part by reduction to the design of non-coherent codes, supplemented by small coherent codes. The non-coherent code design in turn is not based on any differential scheme but on the packing theory of the Grassmann manifold. However, the presented construction scheme, in particular the use of permutation codes will be investigated further, in order to obtain low complex decoding algorithms in the future.

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