Local trace formulae and scaling asymptotics in Toeplitz quantization

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Abstract

A trace formula for Toeplitz operators was proved by Boutet de Monvel and Guillemin in the setting of general Toeplitz structures. Here we give a local version of this result for a class of Toeplitz operators related to continuous groups of symmetries on quantizable compact symplectic manifolds. The local trace formula involves certain scaling asymptotics along the clean fixed locus of the Hamiltonian flow of the symbol, reminiscent of the scaling asymptotics of the equivariant components of the Szegő kernel along the diagonal.

1 Introduction

A trace formula for Toeplitz operators was proved by Boutet de Monvel and Guillemin in the setting of general Toeplitz structures [BG], following antecedents for Laplacians [C], and more generally for positive pseudodifferential operators [DG]. The aim of the present paper is to give a local version of the trace formula in terms of suitable scaling limits for a special, but geometrically interesting, class of Toeplitz operators in the context of positive line bundles.

As in [P2], we shall adapt the conceptual framework of [Z], [BSZ] and [SZ], where scaling limits are studied building on the microlocal theory of the Szegő kernel in [BS], and pair this approach with classical arguments from [H1], [DG], [GS]. We remark that in this setting scaling limits are generally taken with respect to the discrete parameter indexing the isotype for the circle action on X; in the present situation, we shall instead consider scaling limits with respect to the continuous auxiliary parameter in the trace formula asymptotics.

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Although quite restrictive, the class of Toeplitz operators in point is nonetheless rather natural in geometric quantization (and algebraic geometry), since it is related to continuous 1-parameter groups of symmetries preserving the quantization setup. Thus the following approach applies in particular whenever a compact Lie group acts on a quantizable compact symplectic manifold in an Hamiltonian fashion, for in this case one can find a $G$-invariant and compatible almost complex structure, and adopt as quantizations the spaces of sections defined by the deformation of the $\overline{\partial}$ complex introduced in [BG]. The theory in [SZ] provides a generalization to this context of the microlocal description of the Szegő kernel in [BS]. For ease of exposition, we shall confine ourselves to the more familiar complex projective case.

Thus let $M$ be a $d$-dimensional complex projective manifold, and let $A$ be an ample line bundle on it. Let $h$ be an Hermitian metric on $A$, and $\nabla$ be the unique covariant derivative compatible with the holomorphic and Hermitian structures. Assume, as we may, that $\nabla$ has curvature $\Theta = -2i\omega$, where $\omega$ is a Kähler form. Endowed with the volume form $dV_M =: (1/d!) \omega^d$, $M$ has total volume $\text{vol}(M) = (\pi d/d!) \int_M c_1(A)^d$.

Our focus here is on Hamiltonians generating 1-parameter flows of holomorphic symplectomorphisms of $M$. More precisely, given a real function $f \in \mathcal{C}^\infty(M)$ let $\upsilon_f$ be its Hamiltonian vector field in the symplectic structure $2\omega$; since $M$ is compact, $\upsilon_f$ generates a 1-parameter group of Hamiltonian symplectomorphisms $\phi^M_\tau : M \rightarrow \text{Symp}(M)$.

**Definition 1.1.** We shall call $f$ a compatible Hamiltonian if every $\phi^M_\tau : M \rightarrow M$ is holomorphic.

For instance, if $M = \mathbb{CP}^1$ and $A$ is the hyperplane bundle endowed with the Fubini-Study metric, then $f(\begin{bmatrix} z_0 & z_1 \end{bmatrix}) =: (k|z_0|^2 + l|z_1|^2) / (|z_0|^2 + |z_1|^2)$ is compatible for any pair of integers $k, l$. More generally, let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$, and let $G \times M \rightarrow M$ be an holomorphic Hamiltonian action. Any $\xi \in \mathfrak{g}$ induces a vector field $\xi_M$ on $M$. If $\Phi : M \rightarrow \mathfrak{g}^*$ is the moment map and $f =: \langle \Phi, \xi \rangle$, then $\upsilon_f = \xi_M$ and $f$ is compatible.

If $f$ is compatible, $\phi^M_\tau$ induces a 1-parameter group of unitary automorphisms $\psi^{(k)}_\tau : H^0(M, A^{\otimes k}) \rightarrow H^0(M, A^{\otimes k})$ for $k = 0, 1, 2, \ldots$; here $H^0(M, A^{\otimes k})$ is the space of global holomorphic sections of $A^{\otimes k}$, endowed with the natural Hermitian product induced by $h$ and $dV_M$. In fact, $\phi^M_\tau$ lifts to a 1-parameter group of holomorphic bundle automorphism $\phi^A_\tau : A \rightarrow A$, and $\psi^{(1)}_\tau(s) = \phi^A_\tau \circ s \circ \phi^M_{-\tau}$ for $s \in H^0(M, A)$; similarly for $k \geq 2$.

Let $\mathcal{H}(A) =: \bigoplus_k H^0(M, A^{\otimes k})$ be the Hilbert space direct sum; then $\psi_\tau =: \bigoplus_k \psi^{(k)}_\tau : \mathcal{H}(A) \rightarrow \mathcal{H}(A)$ is a unitary isomorphism. If $f > 0$, the trace
of $\psi_\tau$ is a tempered distribution on $\mathbb{R}$, and its singular support is a set of periods for an appropriate Hamiltonian flow; the trace formula describes its singularities at each isolated period (the flow in point is not the flow of $\xi_f$ on $M$, but is closely related to it). These singularities are encapsulated in certain asymptotic expansions, whose coefficients relate to the dynamics of the closed trajectories of the given period; in particular, the leading term is given by Poincaré map data.

To make this more precise, it is convenient to lift the problem to the unit circle bundle $A^* \supseteq X \overset{\pi}{\rightarrow} M$. The connection form $\alpha$ on $X$ is then a contact form, and $d\mu_X = (1/2\pi) \alpha \wedge \pi^*(dv_X)$ is a volume form. We shall henceforth identify functions, densities and half-densities on $X$.

The Hamiltonian vector field $v_f$ lifts to a contact vector field $\tilde{v}_f$ on $X$; let $\phi^X_\tau : X \rightarrow X$ be the associated 1-parameter group of contactomorphisms. Pull-back by $\phi^X_\tau$, $(\phi^X_\tau)^* : L^2(X) \rightarrow L^2(X)$, is a unitary isomorphism.

In addition, the hypothesis that $\phi^M_\tau$ be holomorphic implies that $(\phi^X_\tau)^*$ leaves the Hardy space $H(X) \subseteq L^2(X)$ invariant. Under the natural unitary isomorphism $H(X) \cong \mathcal{S}(A)$, $\psi_\tau$ corresponds to $(\phi^X_\tau)^*$. Let $U_H(\tau) : H(X) \rightarrow H(X)$ be the unitary operator induced by $(\phi^X_\tau)^*$, and let us extend $U_H(\tau)$ to $L^2(X)$ by declaring it to vanish on the orthocomplement $H(X)^\perp$. In other words, $U_H(\tau) = (\phi^X_\tau)^* \circ \Pi$, where $\Pi : L^2(X) \rightarrow H(X)$ is the orthogonal projector. Basic results on wave fronts imply that $U_H(\tau)$ extends to a continuous operator $\mathcal{D}'(X) \rightarrow \mathcal{D}'(X)$ [14]; we shall also denote by $U_H(\tau) \in \mathcal{D}'(X \times X)$ its distributional kernel. As a differential operator on $X$, $iv_f$ leaves $H(X) \cap C^\infty(X)$ invariant; therefore, its restriction is the first order self-adjoint Toeplitz operator $T_f = iv_f \circ \Pi$, which has a positive symbol. Let $\lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of $T_f$ acting on $H(X)$ [BG], so that $e^{i\lambda_\tau}$ are the eigenvalues of $U_H(\tau)$. The trace in point is the tempered distribution $\sum_j e^{i\lambda_\tau} \tau$ on $\mathbb{R}$.

The asymptotic estimates describing the singularities of $\sum_j e^{i\lambda_\tau}$ involve certain functions $S_\chi \in C^\infty(X \times X)$. More precisely, for every $\chi \in C^\infty_0(\mathbb{R})$ the averaged operator

$$S_\chi =: \int_{-\infty}^{+\infty} \chi(\tau)U_H(\tau)\,d\tau$$

is smoothing, and thus (with abuse of language) has kernel $S_\chi \in C^\infty(X \times X)$. In particular,

$$\left< \sum_j e^{i\lambda_\tau}, \chi \right> = \text{trace}(S_\chi) = \int_X S_\chi(x,x)\,dV_X(x).$$

**Definition 1.2.** $\text{Per}_X(f) \subseteq \mathbb{R}$ is the set of periods of $\phi^X$. Thus $\tau \in \text{Per}_X(f)$ if and only if $\phi^X_\tau(x) = x$ for some $x \in X$. 

3
If \( \tau \in \text{Per}_X(f) \), then \( \tau \) is also a period of \( \phi^M \), but the converse needn’t be true. However, let \( \text{Fix}(\phi^M_\tau) \subseteq M \) and \( \text{Fix}(\phi^X_\tau) \subseteq X \) be the fixed loci of \( \phi^M_\tau \) and \( \phi^X_\tau \), respectively; if \( \tau \in \text{Per}_X(f) \), then by \( S^1 \)-invariance \( \phi^X_\tau \) is an \( S^1 \)-bundle over a union of connected components of \( \text{Fix}(\phi^M_\tau) \).

Now \( \text{Per}_X(f) \) contains the singular support of \( \sum_j e^{i\lambda_j} \tau \). Suppose \( \tau_0 \) is an isolated point of \( \sum_j e^{i\lambda_j} \tau \), and \( \chi \) is a bump function with \( \chi(\tau_0) = 1 \) and supported in a small open neighborhood of \( \tau_0 \). Then \( \chi(\tau) \sum_j e^{i\lambda_j} \tau \) equals \( \sum_j e^{i\lambda_j} \tau \) near \( \tau_0 \), and is compactly supported and non-singular away from \( \tau_0 \); therefore, the singularity of \( \sum_j e^{i\lambda_j} \tau \) at \( \tau_0 \) is characterized by the asymptotics of the Fourier transform of \( \chi(\tau) \sum_j e^{i\lambda_j} \tau \), viz. of the trace of \( S^X_{\chi e^{-i\lambda(\cdot)}} \) as \( \lambda \to \infty \).

Here we shall localize the problem, and study the pointwise asymptotics of the kernel of \( S^X_{\chi e^{-i\lambda(\cdot)}} \); this amounts to considering certain scaling limits in the neighborhood of the fixed locus of \( \phi^X_{\tau_0} \). A global trace formula follows by integration.

The following estimates are phrased in terms of local Heisenberg coordinates on \( X \) \([SZ]\). Having fixed a system of Heisenberg local coordinates \((\theta, v)\) centered at \( x \in X \), we shall follow \([SZ]\) and write \( x + (\theta, v) \) for the point with local coordinates \((\theta, v)\), and \( x + v \) for \( x + (0, v) \); we have \( x + (\theta, v) = r_x \theta(x + v) \), where \( r : S^1 \times X \to X \) is the circle action. A choice of Heisenberg local coordinates on \( X \) centered at \( x_0 \) includes a choice of preferred (not necessarily holomorphic) local coordinates on \( M \) centered at \( m_0 =: \pi(x_0) \), whence of a unitary isomorphism \( T_{m_0}M \cong \mathbb{C}^d \). With this implicit, the expression \( x + v \) may be used for either \( v \in T_{m_0}M \) or \( v \in \mathbb{C}^d \).

Before stating the Theorem, let us introduce two further pieces of notation.

First, following \([SZ]\), we shall denote by \( \psi_2 : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C} \) the smooth function:

\[
\psi_2(u, w) =: i \Im (u \cdot \overline{w}) - \frac{1}{2} \|u - v\|^2,
\]

where \( u, v \in \mathbb{C} \) is the standard scalar product of vectors \( a, b \in \mathbb{C}^d \), so that \( a \cdot \overline{b} \) is their standard Hermitian product \((a \cdot b =: \Re(a \cdot \overline{b}) \in \mathbb{R} \) will denote their standard Euclidean product as vectors in \( \mathbb{R}^{2d} \)).

If \( f \) is compatible, and \( \tau_0 \in \text{Per}_X(f) \) has a clean fixed locus, then \( \text{Fix}(\phi^M_{\tau_0}) \subseteq M \) is a complex submanifold, hence for any \( x_0 \in \text{Fix}(\phi^X_{\tau_0}) \) its tangent space at \( \pi(x_0) \), \( T_{\pi(x_0)} \left( \text{Fix}(\phi^M_{\tau_0}) \right) \), is a complex vector subspace of the tangent space \( T_{\pi(x_0)}M \) to \( M \). Let us denote by \( T_{\pi(x_0)} \left( \text{Fix}(\phi^M_{\tau_0}) \right)^\perp \subseteq T_{\pi(x_0)}M \) the orthocomplement.

**Theorem 1.1.** Assume that \( f \in C^\infty(M) \) is positive and compatible. Let \( \nu_f \) be the contact vector field on \( X \) induced by \( f \), and let \( \lambda_1 \leq \lambda_2 \leq \cdots \) be the
eigenvalues of the Hermitian operator $i\tilde{\nu}_f$ acting on the Hardy space $H(X)$. Then the singular support of the tempered distribution $\sum e^{i\lambda_j \tau}$ is contained in $\text{Per}_X(f)$. Furthermore, assume that $\tau_0 \in \text{Per}_X(f)$ is an isolated period with clean fixed locus $\text{Fix}(\phi_{\tau_0}^X) \subseteq X$. Then there exists $\epsilon > 0$ such that for all $\chi \in C^\infty((\tau_0 - \epsilon, \tau_0 + \epsilon))$ and $C > 0$ the following holds.

1. $S_{\chi e^{-i\lambda \tau}}(x, x) = O(\lambda^{-\infty})$ uniformly on $X \times X$ as $\lambda \to -\infty$.

2. $S_{\chi e^{-i\lambda \tau}}(x, x) = O(\lambda^{-\infty})$ uniformly for $\text{dist}_X(x, \text{Fix}(\phi_{\tau_0}^X)) \geq C\lambda^{-7/18}$ as $\lambda \to +\infty$.

3. Uniformly in $x_0 \in \text{Fix}(\phi_{\tau_0}^X)$ and $u \in T_{\pi(x_0)}M$ with

$$\|u\| \leq C\lambda^{1/9}$$

and $u \in T_{\pi(x_0)}(\text{Fix}(\phi_{\tau_0}^M))^\perp$, as $\lambda \to +\infty$ we have an asymptotic expansion

$$S_{\chi e^{-i\lambda \tau}} \left( x_0 + \frac{u}{\sqrt{\lambda}}, x_0 + \frac{u}{\sqrt{\lambda}} \right) \sim \frac{2\pi e^{-i\lambda \tau}}{f(m_0)^{d+1}} \left( \frac{\lambda}{\pi} \right)^d e^{f(m_0)^{-1} \psi_2(d_{m_0} \phi_{\tau_0}^M(u), u)} \chi(\tau_0) \cdot \left[ 1 + \sum_{j=1}^{+\infty} \lambda^{-j/2} G_j(x_0, u) \right].$$

where $G_j(x_0, u)$’s are polynomials in $u$ depending smoothly on $x_0$. More precisely, in the given range the $N$-th remainder is uniformly $O(\lambda^{-aN})$ for some $a > 0$.

4. Let us write

$$S_{\chi e^{-i\lambda \tau}} \left( x_0 + \frac{u}{\sqrt{\lambda}}, x_0 + \frac{u}{\sqrt{\lambda}} \right) = \mathcal{E}_\lambda(u) + \mathcal{O}_\lambda(u),$$

where $\mathcal{E}_\lambda$ and $\mathcal{O}_\lambda$ are even and odd functions of $u$, respectively. Then the asymptotic expansion of $\mathcal{E}_\lambda(u)$ as $\lambda \to +\infty$ is obtained from (1) by collecting all integer powers of $\lambda$; similarly, the asymptotic expansion of $\mathcal{O}_\lambda(u)$ is obtained by collecting all fractional (non-integer) powers of $\lambda$.

We obtain a global trace formula by integration. To state this, we need a further piece of notation. In the hypothesis of the Theorem, $X_{\tau_0} := \text{Fix}(\phi_{\tau_0}^X)$ is a union of connected components $X_{\tau_0,j}$, $1 \leq j \leq N_{\tau_0}$, of real dimension $\dim(X_{\tau_0,j}) = 2f_j + 1$; thus, $f_j$ is the complex dimension of the connected component $M_{\tau_0,j} := \pi(X_{0j})$ of $M_{\tau_0} := \text{Fix}(\phi_{\tau_0}^M)$. 

5
If \( m_0 \in M_{\tau_0,j} \), let \( N_{m_0} =: (T_{m_0}M_{\tau_0,j})^\perp \subseteq T_{m_0}M \) be the normal space to \( M_{\tau_0,j} \) at \( m_0 \). As the fixed locus is clean, the differential \( d_{m_0}\phi^M_{\tau_0} \) restricts to a unitary isomorphism \( d_{m_0}\phi^M_{\tau_0}|_{N_{m_0}} : N_{m_0} \to N_{m_0} \) such that \( id_{N_{m_0}} - d_{m_0}\phi^M_{\tau_0}|_{N_{m_0}} \) is an isomorphism. The determinant of the latter isomorphism only depends on \( j \). We may set \( c(\tau_0, j) =: \det \left( id_{N_{m_0}} - d_{m_0}\phi^M_{\tau_0}|_{N_{m_0}} \right) \) for \( m_0 \in M_{\tau_0,j} \).

Let \( dV_{M_{\tau_0,j}} \) be the volume form on \( M_{\tau_0,j} \) given by restriction of \( \omega \wedge f_0 \big/ f_0 ! \).

**Corollary 1.1.** In the hypothesis of Theorem 1.1, and with the above notation, the following holds. As \( \lambda \to -\infty \) we have

\[
\int_X S_X e^{-i\lambda (\cdot, \cdot)}(x, x) \, dV_X(x) = O \left( \lambda^{-\infty} \right);
\]

as \( \lambda \to +\infty \), on the other hand, we have

\[
\int_X S_X e^{-i\lambda (\cdot, \cdot)}(x, x) \, dV_X(x) = \sum_{j=1}^{N_{\tau_0}} I_j(\chi, \lambda),
\]

where each \( I_j \) admits an asymptotic expansion

\[
I_j(\chi, \lambda) \sim 2\pi e^{-i\lambda \tau_0} \left( \frac{\lambda}{\pi} \right)^{f_j} \frac{\chi(\tau_0)}{c(\tau_0, j)} \cdot \left( \int_{M_{\tau_0,j}} \frac{1}{f(m)^{j+1}} \, dV_{M_{\tau_0,j}}(m) \right) \cdot \left( 1 + \sum_{j=1}^{+\infty} \lambda^{-j} p_j \right).
\]

The following remarks are in order.

The Corollary expresses \( \Gamma(\lambda) =: \text{trace} \left( S_X e^{-i\lambda (\cdot, \cdot)} \right) \) as an asymptotic expansion with general term a multiple of \( e^{-i\lambda \tau_0} \lambda^a \), where \( a \) is a descending sequence of integers. Since \( \Gamma(\lambda) \) is the Fourier transform of \( \chi(\tau) \text{ trace} (U(\tau)) \), the latter is given by an asymptotic expansion in decreasing powers of \( (\tau - \tau_0 + i0)^{-1} \). More precisely, there is one such expansion for each \( X_{0j} \), and the leading term of the \( j \)-th expansion is a scalar multiple of \( (\tau - \tau_0 + i0)^{-f_j+1} \) (cfr [BG]).

The local asymptotics related to the singularity at \( \tau = 0 \) have already been studied in a more general setting in [P2].

With obvious changes, the proof of Theorem 1.1 applies to more general scaled pairs of the form \( (x_0 + w/\sqrt{\lambda}, x_0 + u/\sqrt{\lambda}) \).
For a slight generalization of Theorem 1.1, one may consider any $S^1$-invariant first order self-adjoint Toeplitz operator $T = \Pi \circ i\tilde{\upsilon}_f + T'$, with $T'$ of degree 0; the proof carries over to this situation, but the leading order term in the asymptotic expansion will now depend on the subprincipal symbol of $T$ through an oscillatory factor.

2 Preliminaries.

The cotangent bundle $T^*X$ is endowed with the standard symplectic structure $\omega_{\text{stan}}$. Explicitly, suppose given local coordinates $q$ on an open subset $U \subseteq X$, and let $(q, p)$ the corresponding local coordinates on $TU \subseteq TX$, so that $(q_0, p_0)$ corresponds to the tangent vector $p_0 \cdot (\partial/\partial q)|_{q_0}$; then $\omega_{\text{stan}} = dp \wedge dq$ on $TU$.

Let $\Sigma = \{(x, r\alpha_x) : x \in X, r > 0\}$ be the closed symplectic cone in $T^*X \setminus \{0\}$ generated by the connection 1-form. Thus $\Sigma \cong X \times \mathbb{R}_+$ canonically. Let $\theta$ be the ‘circle’ coordinate on $X$ (locally defined) and let $r$ be the ‘cone’ coordinate on $\Sigma$; then the restriction of $\omega_{\text{stan}}$ to $\Sigma$ is $\omega_{\Sigma} = 2r\omega + dr \wedge d\theta$ (symbols of pull-back are omitted).

We adopt the convention that the Hamiltonian vector field $\upsilon_g$ of a smooth function $g$ with respect to a symplectic structure $\Omega$ is defined by $\Omega(\iota(\upsilon_g), u) = dg(u)$ for any tangent vector $u$. The Hamiltonian vector field on $\Sigma$ of $\tilde{\upsilon}_f = rf_\Sigma$ is then $\tilde{v}_f = v^\Sigma_f - f(\partial/\partial \theta)$, where $v^\Sigma_f$ is the horizontal lift of $v_f$ and $\partial/\partial \theta$ is the infinitesimal generator of the $S^1$-action (with some ambiguity, $\tilde{v}_f$ denotes both the contact lift of $v_f$ to $X$ and its further Hamiltonian lift to $\Sigma$; strictly speaking, the latter is $(\tilde{v}_f, 0)$).

The 1-parameter group of symplectomorphisms $\phi^\Sigma_{\tau} : \Sigma \to \Sigma$ generated by $\tilde{v}_f$ is $\phi^\Sigma_{\tau}(x, r\alpha_x) = (\phi^X_{\tau}(x), r\alpha_{\phi^X_{\tau}(x)})$.

Viewed as a differential operator on $X$, by our hypothesis $\tilde{v}_f$ commutes with $\Pi$ and is elliptic on $\Sigma$, since its symbol there is $-i\tilde{f}$; therefore, it remains elliptic in a conic neighborhood of $\Sigma$ in $T^*X \setminus \{0\}$. Hence $i\tilde{v}_f$, restricted to $H(X)$ is an elliptic self-adjoint Toeplitz operator $T_f$ of the first order, with symbol $\sigma_{T_f} = \tilde{f} : \Sigma \to \mathbb{R}$. By the proof of Lemma 12.2 of [BG], there exists a self-adjoint first order pseudodifferential operator $Q$ on $X$ with everywhere positive principal symbol $q > 0$, which commutes with $\Pi$, equals $i\tilde{v}_f$ on $H(X)$ (thus $T_f = Q \circ \Pi$), and is such that $Q - i\tilde{v}_f$ is microlocally smoothing on a conic neighborhood of $\Sigma$.

The operator $U_H(\tau)$, on the other hand, may be written $U_H(\tau) = U(\tau) \circ \Pi$, where...
where $U(\tau) = e^{i\tau Q}$; by the previous discussion, its wave front is

$$\text{WF}(U_H(\tau)) = \left\{ (x, r\alpha_x, y, -r\alpha_y) : x \in X, y = \phi^{-1}_X(x), r > 0 \right\},$$

and so the singular support is $\text{sing.(supp}(U_H(\tau)) = \text{graph} (\phi^{-1}_X)$. Proposition 12.4 of [BG] shows by a functorial description of $\sum e^{i\lambda_j \tau}$ that its singular support is contained in the set of periods of $\phi^{-1}_X$, which is the same as the set of periods of $\phi^{-1}_X$.

Now suppose $x \not\in \text{Fix}(\phi^{-1}_X)$, that is, $(x, x) \not\in \text{graph} (\phi^{-1}_X)$. By continuity, for some $\epsilon > 0$ we have $(x, x) \not\in \text{graph} (\phi^{-1}_X)$ for any $\tau \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$. As a distribution on $\mathbb{R} \times X \times X$, $U_H$ is $C^\infty$ on $X' \times X'$, where $X' \subseteq X$ is an appropriate open neighborhood of $X$. If $\chi \in C^\infty_0((\tau_0 - \epsilon, \tau_0 + \epsilon))$, then $\chi(\tau) U_H (\tau, x', x'')$ is $C^\infty$ on $X' \times X'$; hence its Fourier transform in $\tau$ is of rapid decrease in $\lambda$ uniformly in $(x', x'') \in X' \times X''$, where $X'' \subseteq X'$ is an open neighborhood of $x$. In particular,

$$S_{\chi e^{-i\lambda \cdot}} (x, x) = O \left( \lambda^{-\infty} \right)$$

as $\lambda \to \infty$, uniformly in $x$ with $\text{dist}_X (x, \text{Fix}(\phi^{-1}_X)) \geq \delta$ for any fixed $\delta > 0$ (here $\text{dist}_X$ is the Riemanian distance function on $X$), for any $\chi$ with sufficiently small compact support near $\tau_0$.

### 3 Proof of Theorem 1.1.

We shall adapt the approach used in [P2] for the singularity at $\tau = 0$ to the case of a general period (and thus occasionally refer to arguments in [P2]).

When working locally on $X$, looking for asymptotic expansions we may replace $U(\tau)$ and $\Pi$ by their respective representations as Fourier integral operators. Thus we shall write [DG]

$$U(\tau)(x, y) = \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{2d+1}} e^{i[\phi(\tau, x, \eta) - y \cdot \eta]} a(\tau, x, y, \eta) \, d\eta,$$

where $a(\tau, \cdot, \cdot) \in S^0_{cl}$, and the phase $\phi(\tau, \cdot, \cdot)$ is a local generating function for the 1-parameter group $\phi^{-1}_{\tau} \cdot X$ of homogeneous Hamiltonian symplectomorphism of $T^*X$ generated by $-q$. Working with $\tau$ close to $\tau_0$, we change variable $\tau \sim \tau + \tau_0$, so that

$$\phi(\tau + \tau_0, x, \eta) = \phi(\tau_0, x, \eta) + \tau q(x, d_x \phi(\tau_0, x, \eta)) + O \left( \tau^2 \right) \| \eta \|.$$  \hspace{1cm} (2)

On the other hand, $\Pi$ is an FIO with complex phase of the form

$$\Pi(x, y) = \int_{0}^{+\infty} e^{i\psi(t, y)} s(t, x, y) \, dt,$$
where \( s \) is a classical symbol \( s(t, x, y) \sim \sum_{j \geq 0} s_j(x, y) t^{d-j} \), and the Taylor expansion of \( \psi \) along the diagonal is determined by the Kähler metric [BS].

Let \( \sim \) stand for ‘equal asymptotics as \( \lambda \to \infty \). Given any \( x \in X \), let \( g \) be a positive smooth cut-off function on \( X \), identically equal to 1 near \( x \), and with support contained in a sufficiently open neighborhood \( X_1 \subseteq X \); arguing as in [P2], Lemma 2.1 we get

\[
S_{X, e^{-i\lambda(\cdot)}}(x, x)
\]

\[
\sim \frac{1}{(2\pi)^{2d+1}} \int_{X_1} \int_{-\epsilon}^{\epsilon} e^{-i\lambda(\tau + \tau_0)} \chi(\tau + \tau_0) g(z) U(\tau + \tau_0)(x, z) \Pi(z, x) \, d\mu_X(z) \, d\tau
\]

\[
\sim \frac{1}{(2\pi)^{2d+1}} \int_{X_1} \left[ \int_{\mathbb{R}^{2d+1}} \int_{-\epsilon}^{\epsilon} e^{i\Psi(\tau, x, z, \eta, \lambda)} B(\tau, x, z, \eta) \, d\tau \, d\eta \right] \cdot \Pi(z, x) \, d\mu_X(z),
\]

where

\[
B(\tau, x, z, \eta) = \chi(\tau + \tau_0) g(z) a(\tau + \tau_0, x, z, \eta),
\]

and in view of [18]

\[
\Psi(\tau, x, z, \eta, \lambda) =: \varphi(\tau + \tau_0, x, \eta) - z \cdot \eta - \lambda(\tau + \tau_0)
\]

\[
= \varphi(\tau_0, x, \eta) + \tau q(x, d_x \varphi(\tau_0, x, \eta)) + O(\tau^2) \|\eta\| - y \cdot \eta - \lambda(\tau + \tau_0).
\]

Now as \( \lambda \to -\infty \) since \( q \) is elliptic and positive for sufficiently small \( \epsilon \) we have

\[
\partial_{\tau} \Psi = q(x, d_x \varphi(\tau_0, x, \eta)) + O(\tau) \|\eta\| - \lambda \geq C_1 (\|\eta\| + |\lambda|)
\]

for some fixed \( C_1 > 0 \); by repeated integration by parts in \( d\tau \) we conclude

\[
S_{X, e^{-i\lambda(\cdot)}}(x, x) = O(\lambda^{-\infty}) \text{ uniformly in } x \in X \text{ as } \lambda \to -\infty.
\]

Next we consider the asymptotics for \( \lambda \to +\infty \). By the discussion in §1, we are reduced to considering the problem in the neighborhood of \( \text{Fix}(\phi^X_{\tau_0}) \), so we fix \( x_0 \in \text{Fix}(\phi^X_{\tau_0}) \) and a suitably small open neighborhood \( X_1 \subseteq X \) of \( x_0 \), and consider the asymptotics for \( x \in X_1 \). Let there be given on \( X_1 \) a system of Heisenberg local coordinates \((\theta, \nu) : X_1 \to (-\delta, \delta) \times B_{2d}(0, \delta) \) centered at \( x_0 \) for some \( \delta > 0 \), where \( B_{2d}(0, \delta) \subseteq \mathbb{R}^{2d} \cong \mathbb{C}^d \) is the open ball of radius \( \delta > 0 \) centered at the origin. In particular, in the associated cotangent coordinates \((x_0, \alpha_{x_0}) \in T^*X \) corresponds to \(((0, 0), (1, 0))\).

To begin with, we only lose a smoothing term, hence a negligible contribution to the asymptotics, if we multiply the amplitude \( a \) of \( U \) by a radial function \( b(\eta) \) identically equal to 0 near the origin and to 1 for \( \|\eta\| \gg 0 \). We may then assume without loss that \( a \) vanishes identically near the origin as a function of \( \eta \).

Let \( S_1, S_2 \subseteq S^{2d} \) be open subsets covering \( S^{2d} \) with \((1, 0) \notin \overline{S_2} \), and let \( \gamma_1 + \gamma_2 = 1 \) be a partition of unity on \( S^{2d} \) subordinate to the open
cover \( \{S_1, S_2\} \). Also, for \( j = 1, 2 \) let \( U^{(j)}(\tau) \) be defined as \( U(\tau) \), but with the amplitude \( a \) multiplied by \( \rho(z) \gamma_j(\eta/\|\eta\|) \), and let \( S^{(j)}_{\chi e^{-i\lambda\tau}} \) be defined as \( S_{\chi e^{-i\lambda\tau}} \) with \( U^{(j)}(\tau) \) in place of \( U(\tau) \). Then clearly

\[
S_{\chi e^{-i\lambda\tau}}(x, x) \sim \sum_{j=1}^{2} S^{(j)}_{\chi e^{-i\lambda\tau}}(x, x). \tag{4}
\]

Now we claim that, perhaps after replacing \( X_1 \) with a smaller open neighborhood and \( \epsilon \) with a smaller positive real number, the second summand on the right hand side of (4) gives a negligible contribution to the asymptotics, uniformly in \( x \in X_1 \).

To see this, consider the operator giving rise to the second summand:

\[
S^{(j)}_{\chi e^{-i\lambda\tau}} = \int_{-\epsilon}^{+\epsilon} e^{-i\lambda\tau} \left( U^{(2)}(\tau + \tau_0) \circ \Pi \right) \, d\tau. \tag{5}
\]

With abuse of language, let us mix intrinsic notation and expressions in local coordinates. Thus we shall provisionally write \((x, \eta)\) for the cotangent vector in \( T^*X \) with base \( x \in X \) and fiber coordinates \( \eta \) in the chosen system of Heisenberg local coordinates. The wave front of \( U^{(2)}(\tau + \tau_0) \circ \Pi \) is contained in

\[
\{(\phi_{T^*X}^\tau (x, \eta), (x, -\eta)) : x \in X_1, \eta \in S_{-}\}\.
\]

By construction, the intersection of the locus \( \{(x, \eta) : x \in X_1, \eta \in S_{-}\} \) with the unit sphere bundle has positive distance from \((x_0, \alpha x_0)\); therefore, perhaps after restricting \( X_1 \), it has positive distance from \( \Sigma \). Therefore \( U^{(2)} \circ \Pi \) is \( C^\infty \) on \((\tau_0 - \epsilon, \tau_0 + \epsilon) \times X \times X \), so \( \chi \cdot U^{(2)} \circ \Pi \) is \( C^\infty \) on \( \mathbb{R} \times X \times X \). Thus its Fourier transform is rapidly decreasing, as claimed.

Hence \( S_{\chi e^{-i\lambda\tau}}(x, x) \sim S^{(j)}_{\chi e^{-i\lambda\tau}}(x, x) \) for \( x \in X_1 \) as \( \lambda \to +\infty \).

Let \( F \in C^\infty_0((0, +\infty)) \) be identically 1 in \((1/C_2, C_2)\) for some \( C_2 \gg 0 \). Returning to (3), let us incorporate \( \gamma_1(\eta/\|\eta\|) \) in the definition of \( S^{(1)}_{\chi e^{-i\lambda\tau}}(x, x) \) in the amplitude \( B \); in distributional short-hand,

\[
S_{\chi e^{-i\lambda\tau}}(x, x) \sim \frac{1}{(2\pi)^{2d+1}} \int_{X_1} \left[ \int_{\mathbb{R}^{2d+1}} \int_{-\epsilon}^{\epsilon} \right] e^{i\Psi(\tau, x, z, \eta, \lambda)} B(\tau, x, z, \eta) F\left( \frac{\|\eta\|}{\lambda} \right) \, d\tau \, d\eta \right] \cdot \Pi(z, x) \, d\mu_X(z)
\]

\[
+ \frac{1}{(2\pi)^{2d+1}} \int_{X_1} \left[ \int_{\mathbb{R}^{2d+1}} \int_{-\epsilon}^{\epsilon} \right] e^{i\Psi(\tau, x, z, \eta, \lambda)} B(\tau, x, z, \eta) \left[ 1 - F\left( \frac{\|\eta\|}{\lambda} \right) \right] \, d\tau \, d\eta \right] \cdot \Pi(z, x) \, d\mu_X(z).
\]

Now \( |\partial_i \Psi| \geq C_3(\|\eta\| + \lambda) \) for some \( C_3 > 0 \) where \( 1 - F(\eta/\lambda) \neq 0 \); therefore the second summand is \( O(\lambda^{-\infty}) \) as \( \lambda \to +\infty \) uniformly in \( x \in X \).
Inserting in (3) the description of $\Pi$ as an FIO we get for $x \in X_1$

$$S_{\chi e^{-i\lambda t}} (x, x) \sim \frac{1}{(2\pi)^{2d+1}} \int_{X_1} \int_0^{+\infty} \int_{-\epsilon}^{\epsilon} \int_{\mathbb{R}^{2d+1}} e^{i\Phi_1(x,z,t,\tau,\eta,\lambda)} \cdot A(x,z,t,\tau,\eta,\lambda) \, d\mu_X(z) \, dt \, d\tau \, d\eta,$$

where

$$\Phi_1 (x, u, z, t, \tau, \eta, \lambda) = \varphi (\tau + \tau_0, x, \eta) - z \cdot \eta + t \psi (z, x) - \lambda (\tau + \tau_0), \quad (8)$$

and

$$A(x,z,t,\tau,\eta,\lambda) = : \chi (\tau + \tau_0) g(z) a (\tau + \tau_0, x, z, \eta) s (t, z, x) \gamma_1 \left( \frac{\eta}{\|\eta\|} \right) F \left( \frac{\|\eta\|}{\lambda} \right).$$

Apply the change of variables $t \sim \lambda t$, $\eta \sim \lambda r \omega$, where $r > 0$ and $\omega \in S^{2d} \subseteq \mathbb{R}^{2d+1} \cong \mathbb{R} \times \mathbb{R}^{2d}$; thus $\omega = (\omega_0, \omega_1)$, where $\omega_0 \in \mathbb{R}$, $\omega_1 \in \mathbb{R}^{2d}$ and $\omega_0^2 + \|\omega_1\|^2 = 1$. We can rewrite (7) as

$$S_{\chi e^{-i\lambda t}} (x, x) \sim 2\pi e^{-i\lambda \tau_0} \left( \frac{\lambda}{2\pi} \right)^{2d+2} \int_{X_1} \int_{0}^{+\infty} \int_{-\epsilon}^{\epsilon} \int_{\mathbb{R}^{2d}} e^{i\lambda \Phi_2 (x,z,t,\tau,r,\omega)} \cdot A(x,z,\lambda t,\tau,\lambda r \omega,\lambda) \cdot r^{2d} \, d\mu_X(z) \, dt \, d\tau \, dr \, d\omega,$$

where

$$\Phi_2 (x, z, t, \tau, r, \omega, \lambda) = : r \left[ \varphi (\tau + \tau_0, x, \omega) - z \cdot \omega \right] + t \psi (z, x) - \tau. \quad (11)$$

**Lemma 3.1.** Perhaps after further replacing $S_1$ and $X_1$ with smaller open neighborhoods of $(1, 0) \in S^{2d}$ and $x_0 \in X$, respectively, we have

$$\varphi (\tau + \tau_0, x, \omega) = \phi^X_{-\tau} (x) \cdot \omega$$

for $x \in X_1$, $\omega \in S_1$, $|\tau| < \epsilon$.

**Proof.** For any $\tau$, $\varphi (\tau, x, \omega)$ is the local generating function for the Hamiltonian flow at time $-\tau$ of the principal symbol $q$ of $Q$ on $T^* X$. In Heisenberg local coordinates, pairs $(x, \omega) \in X_1 \times S_1$ belong to a conic open neighborhood of $(x_0, \alpha x_0) \in \Sigma$. On a certain conic neighborhood of $\Sigma$, on the other hand, $Q$ is microlocally equivalent to $i\tilde{\nu}_f$, hence $q$ equals the symbol of $i\tilde{\nu}_f$ there. On the same conic neighborhood of $\Sigma$, therefore, the Hamiltonian flow of $q$ is the Hamiltonian flow of the symbol of $i\tilde{\nu}_f$. However, the latter is the cotangent lift of the flow of $\tilde{\nu}_f$ on $X$.

Q.E.D.
Corollary 3.1. In the same range,
\[ \Phi_2 (x, z, t, \tau, r, \omega, \lambda) = r \left( \phi^X_{(r + \tau_0)}(x) - z \right) \cdot \omega + t \psi (z, x) - \tau. \]

Remark 3.1. Before proceeding we note the following:

- In view of the radial factor \( F(r) \), integration in \( dr \) is now restricted to the interval \( 1/C_4 \leq r \leq C_4 \) for some \( C_4 \gg 0 \); integration in \( d\omega \) is restricted to the open neighborhood \( S_1 \subseteq S_{2d} \) of \( (1, 0) \).
- Arguing as in Lemmata 2.3 and 2.4 of [P2], upon introducing a cut-off in \( t \) we can further restrict integration in \( dt \) to an interval of the form \( 1/C_5 \leq t \leq C_5 \) for some \( C_5 \gg 0 \).

We can now prove statement 2 in the Theorem.

For fixed \( C > 0 \), as \( \lambda \to +\infty \) we consider the family of loci \( V_\lambda \subseteq X \) given by the set of all \( x \in X \) such that \( \text{dist}_X (x, \text{Fix} (\phi^X_{\tau_0})) > C \lambda^{-7/18} \). Let \( X_0 \subseteq X \) be the connected component of \( \text{Fix} (\phi^X_{\tau_0}) \) through \( x_0 \), and set \( F_0 =: \pi (X_0) \); thus \( F_0 \) is the connected component of \( \text{Fix} (\phi^M_{\tau_0}) \) through \( m_0 = \pi (x_0) \), and \( X_0 = \pi^{-1}(F_0) \) is an \( S^1 \)-bundle over \( F_0 \). Since \( \pi \) is a Riemannian submersion, we also have \( \text{dist}_M (m, \text{Fix} (\phi^M_{\tau_0})) > C \lambda^{-7/18} \) if \( m = \pi (x) \). As \( F_0 \) is a fixed clean locus for \( \phi^M_{\tau_0} \), perhaps after replacing \( C \) with a smaller positive constant we may assume that in the same range we have

\[ \text{dist}_X (x, \phi^X_{\tau_0}(x)) \geq \text{dist}_M (m, \phi^M_{\tau_0}(m)) > C \lambda^{-7/18}. \]

Lemma 3.2. Perhaps after further decreasing \( C \) and \( \epsilon \), we also have (with \( m = \pi (x) \))

\[ \text{dist}_X (x, \phi^X_{\tau}(x)) \geq \text{dist}_M (m, \phi^M_{\tau}(m)) > C \lambda^{-7/18}, \]

for all \( x \in V_\lambda \) and \( \tau \in (\tau_0 - \epsilon, \tau_0 + \epsilon) \).

Proof. The statement of the Lemma is intrinsic, but for the proof it is convenient to make a specific choice of Heisenberg local coordinates.

The chosen Heisenberg local coordinates centered at \( x_0 \) imply the prior choice of a system of preferred local coordinates on \( M \) centered at \( m_0 =: \pi (x_0) \), where \( \pi : X \to M \) is the projection.

Let \( r \) be the complex dimension of the complex submanifold \( F_0 \subseteq M \), and \( s =: d - r \) its codimension. For \( m \in F_0 \) let \( T_m \subseteq T_m M \) and \( N_m =: T^\perp_m \subseteq T_m M \) be the tangent and normal subspaces of \( F_0 \) at \( m \), respectively. Furthermore, let \( \exp^{tg}_{m_0} : T_{m_0} \to F_0 \) be the exponential map at \( m_0 \) of the Riemannian
manifold \( F_0 \), and for any \( m \in F_0 \) let \( \exp_{m}^{\text{nor}} : N_m \to M \) be the restriction to \( N_m \) of the exponential map of \( M \).

Let us choose an orthonormal basis of \( T_{m_0} \), so as to unitarily identify \( T_{m_0} \cong \mathbb{C}^r \); also, let us choose a local unitary trivialization of the normal bundle of \( F_0 \) in the neighborhood of \( m_0 \), so as to have smoothly varying unitarily isomorphisms \( N_{m_0} \cong \mathbb{C}^s \), for \( m \in F_0 \) near \( m_0 \). In particular, by taking the normal component of the differential we have an induced unitary isomorphism

\[
d_{m_0}^{M} : \mathbb{C}^s \to \mathbb{C}^s,
\]

with no eigenvalue equal to 1.

With these isomorphisms implicit, a preferred local coordinate chart on \( M \) centered at \( m_0 \), and defined on some open ball centered at the origin in \( \mathbb{C}^r \times \mathbb{C}^s \cong \mathbb{C}^d \) may then be taken

\[
\zeta(a, b) =: \exp_{\exp_{m_0}^{\text{nor}}(a)}^{\text{nor}}(b).
\]

In this chart \( T_{m_0} \cong \mathbb{C}^r \oplus \{0\} \) and \( N_{m_0} \cong \{0\} \oplus \mathbb{C}^s \) unitarily. Let us set \( m_0 + (a, b) =: \zeta(a, b) \). Since the flow leaves \( F_0 \) invariant,

\[
\phi_{\tau + m_0}^{M}(m_0) = \phi_{\tau}^{M}(m_0) = m_0 + (v(\tau), 0),
\]

where \( v(\tau) = \tau v_f(m_0) + o(\tau) \in \mathbb{C}^r \); similarly, if \( b \in \mathbb{C}^s \) then clearly \( \exp_{m_0}^{\text{nor}}(b) = m_0 + (0, b) \). On the other hand, since \( \phi_{\tau}^{M} \) preserves the normal geodesics at \( m_0 \), we have

\[
\phi_{m_0}^{M}(m_0 + (0, b)) = m_0 + (0, d_{m_0}^{M}(b)).
\]

Therefore, for \( \tau \) close to 0 we have

\[
\phi_{\tau + m_0}^{M}(m_0 + b) = m_0 + \left( v(\tau), d_{m_0}^{M}(b) \right) + O(\tau b).
\]

Now the previous construction may be locally smoothly deformed with \( m_0 \in F_0 \); furthermore, \( \text{Fix} \left( \phi_{\tau}^{X} \right) = \pi^{-1}(F_0) \). If \( \text{dist}_{X} \left( x, \phi_{\tau}^{X}(x) \right) > C\lambda^{-7/18} \)
and \( m = \pi(x) \) is in a neighborhood of \( F_0 \), then \( m = m_0 + (0, b) \) for some \( m_0 \in F_0 \) and \( b \in \mathbb{C}^s \) with \( \|b\| > (C/2)\lambda^{-7/18} \). Since a preferred local coordinate system is isometric at the origin, we get for sufficiently small \( \epsilon \) and \( |\tau| < \epsilon \)

\[
\text{dist}_{M} \left( m, \phi_{\tau + m_0}^{M}(m) \right) \geq \frac{1}{2} \left[ \|v(\tau)\| + \|\phi_{m_0}^{M}(b) - b\| \right] + O(\tau b) \\
\geq D\|b\| \geq D'\lambda^{-7/18}.
\]
Given that $\pi$ is a Riemannian submersion, and $\phi^X_\tau$ covers $\phi^M_\tau$, we have
\[
\text{dist}_X \left( x, \phi^X_\tau(x') \right) \geq \text{dist}_M \left( m, \phi^M_{\tau+n_0}(m) \right) \geq D' \lambda^{-7/18},
\]
for some $D' > 0$.

On the way to prove statement 2, in the situation of Lemma 3.2 let us rewrite (10) as
\[
S_x e^{-i\lambda t} \left( x, x \right) \sim 2\pi e^{-i\lambda t_0} \left( \frac{\lambda}{2\pi} \right)^{2d+2} \int_{X_1} \left[ \int_{1/D}^D \int_{1/D}^D \int_{S^{2d}} \int_{-\epsilon}^\epsilon e^{i\lambda t_2(x,z,t,\tau,\omega,\lambda)} \cdot A \left( x, z, \lambda t, \lambda \tau, \lambda r, \lambda \omega, \lambda \right) \cdot r^{2d} dt dr d\omega d\tau \right] \text{d}\mu_X(z),
\]
for some $D \gg 0$. Next we split the outer integral as the sum of two terms: one over those $z$ with
\[
\text{dist}_M(z, x) > C \frac{\lambda^{-7/18}},
\]
and one over those $z$ with
\[
\text{dist}_M(z, x) \leq C \frac{\lambda^{-7/18}}{2},
\]
here, for simplicity, we have denoted by $\text{dist}_M$ the pull-back to $X \times X$ of the distance function on $M \times M$. On the domain (13), integration by parts in $t$ implies that the corresponding contribution to the asymptotics is $O \left( \lambda^{-\infty} \right)$; in fact, a slight modification of the proof of Lemma 2.5 of [P2] shows that each integration introduces a factor $\frac{1}{\lambda} \lambda^7/9 = \lambda^{-2/9}$. On the other hand, in view of Lemma 3.2 on the domain (14) for $|\tau| < \epsilon$ we get
\[
\text{dist}_X \left( \phi^X_{-(\tau+n_0)}(x), z \right) \geq \text{dist}_M \left( \phi^M_{-(\tau+n_0)}(x), z \right) \geq \text{dist}_M \left( \phi^X_{-(\tau+n_0)}(x), z \right) \geq C \frac{\lambda^{-7/18}}{2}.
\]
Now in view of Corollary 3.1 where this holds we have $\|\partial_\omega \Phi_2\| \geq C' \lambda^{-7/18}$; multidimensional integration by parts in $\omega$, therefore, introduces at each step a factor $\frac{1}{\lambda} \lambda^{7/18} = \lambda^{-11/18}$. We conclude that also in this case the contribution to the asymptotics is $O \left( \lambda^{-\infty} \right)$.

Having established the second statement in the Theorem, let us focus on the third. To this end, it suffices to show that there exist $a, b \in \mathbb{R}$ with $a > 0$
such that for every integer $N > 0$ as $\lambda \to +\infty$ we have

$$S_{\chi e^{-i\lambda}} \left( x_0 + \frac{u}{\sqrt{\lambda}}, x_0 + \frac{u}{\sqrt{\lambda}} \right)$$

$\sim  2\pi e^{-i\lambda \tau_0} \left( \frac{\lambda}{2\pi} \right)^{2d+2}$

$$\times \int X_1 \int_{1/D} \int X_2 \int_{1/D} \int \int_{S^{2d}} e^{i\lambda \Phi_2(x_0 + \frac{u}{\sqrt{\lambda}}, z, \lambda t, \lambda r, \omega)} \cdot A \left( x_0 + \frac{u}{\sqrt{\lambda}}, z, \lambda t, \lambda r, \omega, \lambda \right)$$

$$\cdot d^2 \mu_X (z) dt dr d\tau d\omega.$$

Given Lemma 3.1 for $\omega \in S_1$ and $\tau \sim 0$ we have

$$\varphi (\tau + \tau_0, x, \omega) = \phi_{X_{(\tau + \tau_0)}} (x) \cdot \omega$$

$$= \phi_{X_{\tau_0}} (x) \cdot \omega - \tau \tilde{v}_f (x) \cdot \omega + O (\tau^2).$$

The exponential map exp$_{m_0} : T_{m_0} M \to M$, composed with some unitary isomorphism $\mathbb{C}^d \cong T_{m_0} M$ and restricted to some open ball $B_{2d}(0, \delta) \subseteq \mathbb{R}^{2d} \cong \mathbb{C}^d$, provides a set of preferred local coordinates for $M$ at $m_0 = \pi(x_0)$. From now on, we shall assume for convenience that these are the preferred local coordinates underlying the chosen Heisenberg local coordinates.

Since $\phi_{\tau_0}^M (m_0) = m_0$ and $\phi_{\tau_0}^M$ is a Riemannian isometry,

$$\phi_{\tau_0}^M (m_0 + \frac{u}{\sqrt{\lambda}}) = m_0 + \frac{1}{\sqrt{\lambda}} d_{m_0} \phi_{\tau_0}^M (u).$$

By Lemma 2.5 of [P1] we then have

$$\phi_{\tau_0}^X (x_0 + \frac{u}{\sqrt{\lambda}}) = \left( \vartheta \left( \frac{u}{\sqrt{\lambda}} \right), m_0 + \frac{1}{\sqrt{\lambda}} d_{m_0} \phi_{\tau_0}^M (u) \right),$$

where $\vartheta$ vanishes to third order at the origin.

In Heisenberg local coordinates, $z = m_0 + (\theta, \nu)$ and $d\mu_X (z) = \mathcal{V}(\theta, \nu) d\theta d\nu$, where $\mathcal{V}(\theta, 0) = 1/(2\pi).$
Recalling Corollary 3.1,
\[ \Phi_2 \left( x_0 + \frac{u}{\sqrt{\lambda}}, m_0 + (\theta, v), t, \tau, r, \omega \right) \]
\[ = r \left[ \theta \left( \frac{u}{\sqrt{\lambda}} \right) - \theta \right] \omega_0 + r \left( \frac{1}{\sqrt{\lambda}} d_{m_0} \phi_{m_0}^M (u) - v \right) \cdot \omega_1 \]
\[ + t \psi_1 \left( x_0 + (\theta, v), x_0 + \frac{u}{\sqrt{\lambda}} \right) - \tau. \]

Remark 3.2. Let us pause to note the following:

- If \( \|u\| \leq C \lambda^{1/9} \) and \( \|v\| \geq 3C \lambda^{7/18} \), say, then \( \text{dist}_M (x_0 + (\theta, v), x_0 + u/\sqrt{\lambda}) \geq C \lambda^{-7/18} \) for \( \lambda \gg 0 \); here \( \text{dist}_M \) is the pull-back of the distance function on \( M \).

Given this, as in the argument in the proof of Lemma 2.5 of [P2] integration by parts in \( dt \) shows that the contribution to the asymptotics coming from the locus \( \|v\| \geq 3C \lambda^{7/18} \) is \( O(\lambda^{-\infty}) \).

Accordingly, only a rapidly decreasing contribution is lost if an appropriate cut-off of the form \( \gamma (\lambda^{7/18} v) \) is absorbed into the amplitude, with \( \gamma \in C_0^\infty (C^1) \) identically equal to one near the origin.

- Let us perform the coordinate change \( v \mapsto v/(r \sqrt{\lambda}) \), so that integration in \( dv \) is now over a ball of radius \( O(\lambda^{1/9}) \). The volume element becomes
\[ d\mu_X (z) = \frac{1}{r^{2d+1}} \psi_1 \left( \frac{\theta}{r \sqrt{\lambda}}, \frac{v}{r \sqrt{\lambda}} \right) d\theta d\nu. \]

- By the computations in §3 of [SZ] we have
\[ t \psi_1 \left( x_0 + \left( \theta, \frac{v}{r \sqrt{\lambda}} \right), x_0 + \frac{u}{\sqrt{\lambda}} \right) \]
\[ = it \left[ 1 - e^{i\theta} \right] - \frac{it}{\lambda} \psi_2 \left( \frac{v}{r}, u \right) e^{i\theta} + t R_2^\psi \left( \frac{v}{r \sqrt{\lambda}}, \frac{u}{\sqrt{\lambda}} \right), \]
where \( \psi_2 (a, b) = i \Im (a \cdot \overline{b}) - (1/2) \| a - b \|^2 \) and \( R_2^\psi \) vanishes to third order at the origin.

The upshot is that with some manipulations (17) may be rewritten
\[ S_{\lambda} e^{-i\lambda (\cdot)} \left( x_0 + \frac{u}{\sqrt{\lambda}}, x_0 + \frac{u}{\sqrt{\lambda}} \right) \sim e^{-i\lambda \gamma_0} \frac{1}{(2\pi)^{2d+1}} \lambda^{d+2} \]
\[ \cdot \int_{S^{2d+1}} \int_{S^{2d+1}} e^{-i \nabla \omega} e^i \psi_{u,v} \cdot B_{u,v,w,v} \cdot d\theta dt d\tau dr \]
\[ \cdot \gamma \left( \lambda^{-1/6} v \right) d\nu d\omega, \]
\[ \cdot \psi \left( \frac{u}{\sqrt{\lambda}}, \frac{v}{r \sqrt{\lambda}} \right) d\nu d\omega, \]
\[ \cdot \gamma \left( \lambda^{-1/6} v \right) d\nu d\omega, \]
where for $s \in \mathbb{C}^d$, $\|s\| < \delta$ we have set
\[
\Psi_{s,\omega}(\theta, t, \tau, r) = -r\theta\omega_0 - \tau r \cdot v_f(x_0 + s) \cdot \omega + O(\tau^2) \cdot r + it \left[1 - e^{i\theta}\right] - \tau + r d\Phi_X^{-n_0}(s) \cdot \omega_1,
\]
and
\[
B_{u,v,\omega,\lambda}(\theta, t, \tau, r) =: e^{it\psi_2(v/r, u/\sqrt{\lambda})} \cdot A \left( x_0 + \frac{u}{\sqrt{\lambda}}, x_0 + \left( \frac{\theta}{r \sqrt{\lambda}}, \frac{v}{r \sqrt{\lambda}} \right), \lambda t, \tau, \lambda r \omega, \lambda \right) \psi_3 \left( \frac{v}{r \sqrt{\lambda}}, u/\sqrt{\lambda} \right)
\]
with $Q$ vanishing to third order at the origin as a function of $v, u$ (and depending on the other variables as well). In particular, $\lambda Q(v/\sqrt{\lambda}, u/\sqrt{\lambda})$ is bounded in our range.

We shall estimate asymptotically the inner integral by viewing it as depending parametrically on $v, \omega_1$ and $s$, which will then be set equal to $u/\sqrt{\lambda}$. Let us define
\[
I(s, u, v, \omega, \lambda) := \int_{-a}^a \int_{1/D}^D \int_{-\epsilon}^\epsilon \int_{1/D}^D e^{i\lambda\Psi_{s,\omega}} \cdot B_{u,v,\omega,\lambda} \, d\theta \, dt \, d\tau \, dr,
\]
and first consider the case $s = 0$. We have
\[
\Psi_{0,\omega}(\theta, t, \tau, r) = -r\theta\omega_0 - \tau r v_f(x_0) \cdot \omega + O(\tau^2) \cdot r + it \left[1 - e^{i\theta}\right] - \tau.
\]
We have $v_f(x_0) \cdot (1, 0) = -f(m_0) < 0$, hence perhaps after restricting $S_1$ we may assume that $-v_f(x_0) \cdot \omega > \delta' > 0$ for any $\omega \in S_1$. A straightforward computation shows that

**Lemma 3.3.** The phase $\Psi_{0,\omega}$ has a unique stationary point
\[
(\theta_{0,\omega}, t_{0,\omega}, \tau_{0,\omega}, r_{0,\omega}) = \left(0, -\omega_0/(v_f(x_0) \cdot \omega), 0, -1/(v_f(x_0) \cdot \omega)\right).
\]
Furthermore, if $\Psi''_{0,\omega}$ is the Hessian at the critical point then
\[
\det \left( \frac{\lambda\Psi''_{0,\omega}}{2\pi i} \right) = \left( \frac{\lambda}{2\pi} \right)^4 (v_f(x_0) \cdot \omega)^2.
\]
In particular, the critical point is non-degenerate.
Therefore, for \( s \sim 0 \) the phase \( \Psi_{s,\omega} \) has a unique stationary point \( c(s,\omega) = (\theta_{s,\omega}, t_{s,\omega}, \tau_{s,\omega}, \rho_{s,\omega}) \), again non degenerate; furthermore, one can see by direct inspection that \( c(s,\omega) \) is real.

By the stationary phase Lemma, for every integer \( N > 0 \) we have \( I(s, \nu, \omega) = S_N(s, \nu, \omega) + R_N(s, \nu, \omega) \), where the partial sum \( S_N \) and the remainder \( R_N \) are as follows. First,

\[
S_N(s, \nu, \omega) = \left( \frac{2\pi}{\lambda} \right)^2 \gamma(s, \omega) e^{i\lambda \Psi_{s,\omega}} (c(s,\omega)) \cdot \sum_{j=0}^{N} \lambda^{-j} L_j(B_{u,\nu,\omega,\lambda}) \tag{26}
\]

where \( \gamma(0, \omega) = -1/(v_f(x_0) \cdot \omega) \), \( L_0 \) is the identity and any \( L_j \) is a differential operator of degree \( 2j \) in \( \theta, t, \tau, \rho \), with coefficients depending on \( s \) and \( \omega \).

On the other hand (see Theorem 7.7.5 of [H3], §5 of [SZ]),

\[
|R_N(s, \nu, \omega)| \leq \lambda^{-(N+1)} C_N \sup_{|\alpha| < 2N+2} \left\{ \|D^\alpha_{\theta,t,\tau,\rho} B_{u,\nu,\omega}\| \right\} \tag{27}
\]

In the exponent of \( e^{i\nu \omega(s,\omega)} + d\lambda R^\omega \left( (\nu/\sqrt{\lambda}) \cdot \omega \right) \), the second summand is bounded for \( \|u\|, \|v\| \leq \lambda^{1/6} \). Therefore, (27) implies

\[
|R_N(s, \nu, \omega)| \leq C_N \lambda^{d-3-N} (\|u\| + \|v\|)(4N+4) e^{-a \|v-u\|^2}
\]

\[
\leq C'_N \lambda^{(d-7/9)\cdot(5/9)N}.
\]

Since integration in \( d\nu \) in (22) is over a ball of radius \( O(\lambda^{1/9}) \), the overall contribution of \( R_N \) is \( O\left( \lambda^{2d/9} \cdot \lambda^{d+2} \cdot \lambda^{(d-7/9)\cdot(5/9)N} \right) = O\left( \lambda^{(20/9)d+11/9-(5/9)N} \right) \).

We omit the proof of the following:

**Lemma 3.4.** We have \( \tau_{s,\omega} \in \mathbb{R} \) and

\[
\tau_{s,\omega} = (d_{m_0} \phi_{-\tau_0}^M(s) \cdot \omega_1) / (v_f(x_0 + s) \cdot \omega) + \omega_1 A(s, \omega) \omega_1,
\]

where \( B(s, \omega) \) vanishes to second order at the origin as a function of \( s \).

Taylor expanding \( 1/(v_f(x_0 + s) \cdot \omega) \) in \( s \) at \( s = 0 \), we get

\[
\tau_{s,\omega} = (d_{m_0} \phi_{-\tau_0}^M(s) \cdot \omega_1) / (v_f(x_0) \cdot \omega) + \omega_1 A(\omega, s) \omega_1,
\]

where again \( A(\omega, s) \) vanishes to second order at the origin as a function of \( s \). Consequently,

\[
i\lambda \Psi_{u/\sqrt{\lambda}, \omega} (c \left( \frac{u}{\sqrt{\lambda}}, \omega \right)) = -i\lambda \tau_{u/\sqrt{\lambda}, \omega} \tag{28}
\]

\[
= -i\sqrt{\lambda} (d_{m_0} \phi_{-\tau_0}^M(u) \cdot \omega_1) / (v_f(x_0) \cdot \omega) + i\omega_1 A_2(u, \omega) \omega_1 + i\lambda P \left( \frac{u}{\sqrt{\lambda}}, \omega \right),
\]

18
where $A_2$ collects the second order terms in $A$ (as functions of $s$), while $P(s, \omega)$ vanishes to third order at the origin $s = 0$, and vanishes identically for $\omega_1 = 0$. In particular, $i\lambda P(u/\sqrt{\lambda}, \omega)$ remains bounded in the given range.

Given this, (22) may be rewritten

$$
S_{\lambda} e^{-i\lambda\epsilon} \left( x_0 + \frac{u}{\sqrt{\lambda}}, x_0 + \frac{u}{\sqrt{\lambda}} \right) = O\left( \lambda^{(20/9)d+4/9-(5/9)N} \right) \quad (29)
$$

$$
+ \frac{e^{-i\lambda\tau_0}}{(2\pi)^{2d-1}} \lambda^d \int_{C^d} \int_{S^{2d}} e^{i\sqrt{\lambda} \Phi_u(v, \omega)} \cdot {K_N}(\lambda, u, v, \omega) \cdot \gamma(\lambda^{-1/9} v) \, dv \, d\omega,
$$

where

$$
\Phi_u(v, \omega) = - \left[ v + \frac{1}{v f(x_0)} \cdot dm_0 \phi_{m_0}(u) \right] \cdot \omega_1, \quad (30)
$$

$$
{K_N}(\lambda, u, v, \omega) = e^{t_\omega \psi_2(\nu, r_\omega, u)} f^{A_2(u, \omega)} \gamma(\nu \sqrt{\lambda}, \omega_1) \quad (31)
$$

$$
\cdot e^{i\lambda R(v/\sqrt{\lambda} u/\sqrt{\lambda} \omega)} \sum_{j=0}^N \lambda^{-j} L_j \left( \tilde{B}_{u,v,\omega,\lambda} \right) \bigg|_{\varepsilon(u/\sqrt{\lambda} \omega)}
$$

where we have set $t_\omega = t(0, \omega)$, $r_\omega = r(0, \omega)$, and $R(\cdot, \cdot, \cdot)$ vanishes to third order at the origin.

Setting $\mu = \sqrt{\lambda}$, we may interpret (29) as an oscillatory integral in $\mu$ with phase $\Phi_u$.

Now $\Phi_u$ has a unique critical point $(v_{cr}, \omega_{1cr})$, given by

$$
v_{cr} = \frac{1}{f(m_0)} dm_0 \phi_{m_0}(u), \quad \omega_{1cr} = 0,
$$

which is also nondegenerate: if $\Phi_u''$ denotes the Hessian of $\Phi_u$ at this critical point, then

$$
\det\left( \frac{\mu \Phi_u''}{2\pi i} \right) = \left( \frac{\mu}{2\pi i} \right)^{4d} = \frac{\lambda^{2d}}{(2\pi)^{4d}}.
$$

Furthermore, since $\omega$ varies in a small neighborhood of $(1, 0)$, $-1/(v f(x_0) \cdot \omega)$ is close to $1/f(m_0)$. Therefore, upon introducing a further cut-off, we may restrict integration in $dv$ to a small open neighborhood of $f(m_0)^{-1} dm_0 \phi_{m_0}(u)$, for elsewhere each integration by parts in $d\omega$ introduces a factor $\lambda^{-1/2} \lambda^{2/9} = \lambda^{-5/18}$, given that $A_2$ is quadratic in $u$, and $\|u\| \leq C \lambda^{1/9}$.

Applying the stationary phase Lemma, one sees that the second summand on the right hand side of (29) may be rewritten

$$
R' + 2\pi e^{-i\lambda\tau_0} \sum_{j=0}^N \lambda^{-j/2} \tilde{L}_j(K_N) \bigg|_{v=v_{cr}, \omega=0} \quad (32)
$$

19
for certain operators $\tilde{L}_j$ of degree $2j$ in $v, \omega_1$, and a remainder $R_N$ that may be estimated by a modification of (27). More precisely, since each $\omega$ derivative brings down a quadratic factor in $\omega$, we have

$$|R_N| \leq C_N \lambda_d^d \mu^{-(N+1)} \left(\lambda^{2/9}\right)^{2N+1} \lambda^d = C_N \lambda^{2d-5/18-N/18}.$$  

On the other hand, in view of (24) and (31), the second summand in (32) may be rearranged as a sum of terms of the form

$$\lambda^{-k/2} e^{f(m_0)^{-1} \psi_2 \left( d_{m_0} \phi_{M_{\tau_0}} (u), u \right)} G_k(u) \cdot P_k \left( x_0, \frac{u}{\sqrt{\lambda}} \right) \cdot e^{i f(m_0)^{-1} \lambda \left( R_0 \left( d_{m_0} \phi_{M_{\tau_0}} (u)/\sqrt{\lambda}, u/\sqrt{\lambda} \right) + \theta \left( u/\sqrt{\lambda} \right) \right)},$$  

(33)

where $k$ is an integer and $G_k$ a polynomial.

We now use the assumption that $u \in T_{m_0} M_{\rho_0}^\perp$, and that $d_{m_0} \phi_{M_{\tau_0}} - I$ is invertible on $T_{m_0} M_{\rho_0}^\perp$. This implies that $e^{f(m_0)^{-1} \psi_2 \left( d_{m_0} \phi_{M_{\tau_0}} (u), u \right)}$ is bounded by $C e^{-a \|u\|^2}$ for some $C, a > 0$. Therefore, if we insert the Taylor expansion for the last two factors, we obtain an expansion for (33) with an $N$-th step remainder bounded by (for some polynomial $T_{kN}$)

$$\lambda^{-k+N/2} T_{kN}(u) e^{-a \|u\|^2} \leq C' \lambda^{-k+N/2} e^{-(a/2) \|u\|^2}$$  

(34)

for some $C' > 0$.

This implies for (32) an asymptotic expansion in descending powers of $\lambda^{-1/2}$, as stated in the Theorem. We now focus on the leading term. To this end, let us recall the asymptotic expansions for the classical symbols in the FIO’s describing $\Pi$ and $U(\tau)$:

$$s(x, y, t) \sim \sum_{j \geq 0} s_j(x, y) t^{d-j} \quad \text{and} \quad a(\tau, x, z, \eta) \sim \sum_{j \geq 0} a_j(\tau, x, z, \eta),$$  

(35)

where $s_0(x_0, x_0) = \pi^{-d}$, and $a_j$ is homogeneous of degree $-j$ in $\eta$. Collecting homogeneous terms of the same degree we get an asymptotic expansion

$$a \left( \tau_0, x_0 + \frac{u}{\sqrt{\lambda}}, x_0 + \frac{v}{\sqrt{\lambda}}, \lambda r_{u/\sqrt{\lambda}, (1, 0)} \omega \right) \sim \left( \frac{\lambda}{\pi} \right)^d \frac{1}{f(m_0)^{d+1}} a_0(\tau_0, x_0, x_0, (1 : 0)) + \sum_{j \geq 1} \lambda^{d-j/2} R_j.$$  

(36)

We have $\gamma(0, (1, 0)) = 1/f(m_0)$ in (26), and $V(\theta, 0) = 1/(2\pi)$ in (24).

Therefore, we are left with a leading term

$$\chi(\tau_0) e^{-i \lambda \tau_0} \left( \frac{\lambda}{\pi} \right)^d \frac{1}{f(m_0)^{d+1}} e^{f(m_0)^{-1} \psi_2 \left( d_{m_0} \phi_{-\tau_0} (u), u \right)} a_0(\tau_0, x_0, x_0, (1 : 0)).$$

The description of the leading coefficient is completed by the following:
Lemma 3.5. \( a_0(\tau_0, x_0, (1, 0)) = 2\pi. \)

Proof. The operators in our construction act on half-densities through the trivialization of the half-density bundle offered by the volume form \( d\mu_X. \) By assumption, the latter is invariant under \( \tilde{v}_f, \) that is, \( \mathcal{L}_{\tilde{v}_f} (d\mu_X) = 0, \) where \( \mathcal{L} \) is the Lie derivative. A straightforward computation then shows that the subprincipal symbol of \( \tilde{v}_f, \) viewed as an operator on half-densities, vanishes identically.

Now \( Q \) is microlocally equivalent to \( i\tilde{v}_f \) in a conic neighborhood of \( \Sigma, \) and therefore in the same neighborhood its subprincipal symbol also vanishes. The discussion in \( \S 6 \) of [DG] then implies that in a conic neighborhood of \( (x_0, \alpha x_0, x_0, -\alpha x_0) \in \Sigma \) in the wave front of \( U(\tau_0) \) the principal symbol of \( U(\tau) \) equals the natural section of \( \Omega_{1/2} \otimes L \) (the tensor product of the half-density and Maslov line bundles).

By the theory in \( \S 4.1 \) of [H2], \( U(\tau_0) \) in local coordinates near \( x_0 \) has the form \( I(x, y) \sqrt{dx} \sqrt{dy}, \) where

\[
I(x, y) = \int_{\mathbb{R}^{2d+1}} e^{i\phi(x, y, \eta)} b(x, y, \eta) |D(\phi)|^{1/2} \, d\eta,
\]

where \( \phi(x, y, \eta) =: \varphi(x, \eta) - y \cdot \eta, \) \( D(\phi) \) is the determinant in Proposition 4.1.3 of [H2], and \( b(\tau_0, \cdot, \cdot) = 1 + b'(\cdot, \cdot), \) with \( b' \in S^{-1}_{cl}. \) Since however we are using the expression of \( U(\tau) \) in terms of the trivialization of \( \Omega_{1/2} \) given by \( |d\mu_X|, \) we should write this as

\[
I(x, y) = U(\tau_0)(x, y) \sqrt{V_X(x)} \sqrt{V_X(y)} \sqrt{dx} \sqrt{dy},
\]

so that

\[
U(\tau_0)(x, y) = \int_{\mathbb{R}^{2d+1}} e^{i\phi(x, y, \eta)} b(x, y, \eta) |D(\phi)|^{1/2} V_X(x)^{-1/2} V_X(y)^{-1/2} \, d\eta.
\]

Therefore, \( a_0(x, x, (1, 0)) = 2\pi |D(\phi)(\tau_0, x_0, \eta)|^{1/2}. \)

To determine \( D(\phi), \) given (13), (19) and (20) we write with \( x = x_0 + (\theta, \psi): \)

\[
\phi(\tau_0, x_0 + (\theta, \psi), y, \eta) = \varphi(\tau_0, x_0 + (\theta, \psi), \eta) - y \cdot \eta = (\theta + \psi(\psi)) \eta_0 + d_{m\phi_{\tau_0}}(\psi) \cdot \eta_1 - y \cdot \eta.
\]

Thus we obtain

\[
D(\phi)(\tau_0, x_0, x_0, \eta) = \det \begin{pmatrix} \phi_n^m & \phi_n^x \\ \phi_g^m & \phi_g^y \end{pmatrix} = \det \begin{pmatrix} 0 & A \\ -I_{2d+1} & 0 \end{pmatrix},
\]

21
where $A$ is the $(1 + 2d) \times (1 + 2d)$ matrix given by

$$A = \begin{pmatrix} 1 & 0 \\ \text{Jac}_{m_0} (\phi_{-t_0}) & 0' \end{pmatrix}. $$

The latter matrix has determinant 1, since $\phi_{-t_0}$ is a Riemannian isometry and and fixes $m_0$. 

Q.E.D.

We shall now prove the last statement of the Theorem. Let us write

$$S_{\chi, \lambda, \mu} \left( x_0 + \frac{u}{\sqrt{\lambda}}, x_0 + \frac{u}{\sqrt{\lambda}} \right) = \mathcal{E}(x_0, u) + \mathcal{D}(x_0, u), \tag{38} $$

where $\mathcal{E}$ and $\mathcal{D}$ are even and odd functions of $u$, respectively. Thus $\mathcal{E}$ (respectively, $\mathcal{D}$) admits an asymptotic expansion in descending powers of $\lambda^{-1/2}$, whose coefficients are even (respectively, odd) polynomials in $u$, and the claim is that in this expansion only integral (respectively, fractional) powers of $\lambda$ occur.

To see this, recall that the presence of fractional powers of $\lambda$ in the asymptotic expansion of the Theorem originates from applying the stationary phase Lemma in $\mu = \sqrt{\lambda}$ in (32) and further Taylor expanding in $u/\sqrt{\lambda}$ the coefficients of the result, with remainders as in (34). Now the same expansion may be obtained as follows.

First we apply Taylor expansion in (31) in $u/\sqrt{\lambda}$ and $v/\sqrt{\lambda}$; if $v$ is sufficiently close to $v_{cr} = f(m_0)^{-1} d_{m_0} \phi_{-t_0} (u)$, we have a remainder estimate similar to (34). The general term in this expansion will be sum of contributions of the form $\lambda^{-(a+b)/2} F_{a,b}(u, v, \omega_1)$, where $l$ is an integer and $F_{a,b}(u, v, \omega_1)$ is bihomogeneous of bidegree $(a, b)$ in $(u, v)$.

Next we use the stationary phase Lemma in $\mu$. This can be done as above applying the operators $L_i$ in $v$, $\omega_1$, and setting $v = f(m_0)^{-1} d_{m_0} \phi_{-t_0} (u)$, $\omega_1 = 0$. However, it will simplify the present discussion to proceed in the following equivalent manner. First we make the change of variable $v \sim v + (v_f(x_0) \cdot \omega)^{-1} d_{m_0} \phi_{-t_0} (u)$, $\omega_1 \sim \omega_1$, which turns the phase $\Phi_u$ in (30) into the quadratic phase $-v \cdot \omega_1$ and each $F_{a,b}$ in

$$F_{a,b}(u, v, \omega_1) = F_{a,b} \left( u, v - (v_f(x_0) \cdot \omega)^{-1} d_{m_0} \phi_{-t_0} (u), \omega_1 \right).$$

The new phase has the nondegenerate critical point $v = \omega_1 = 0$. Then we apply the stationary phase Lemma in $\mu = \sqrt{\lambda}$ in the new variables. We obtain an asymptotic expansion given by a linear combination of terms of the following form:

$$\mu^{-t} \lambda^{-(a+b)/2} \cdot \frac{\partial^2}{\partial v_{c_1} \partial \omega_d} \circ \cdots \circ \frac{\partial^2}{\partial v_{c_t} \partial \omega_d} \left( e^{i\omega \psi_2 (v'/r_\omega, u) + iw_1 A_2 (\omega, u) \omega_1 \tilde{F}_{a,b}} \right)_{v=0, \omega_1=0},$$

where $A$ is the $(1 + 2d) \times (1 + 2d)$ matrix given by

$$A = \begin{pmatrix} 1 & 0 \\ \text{Jac}_{m_0} (\phi_{-t_0}) & 0' \end{pmatrix}. $$

The latter matrix has determinant 1, since $\phi_{-t_0}$ is a Riemannian isometry and and fixes $m_0$. 

Q.E.D.
where $\mathbf{v}' := \mathbf{v} - \left( v_f(x_0) \cdot \omega \right)^{-1} d_{m_0} \phi_M^{M}(u)$. 

Now one can see that the latter expression splits as a sum of terms of the form $\exp(-t/2)G(u) e^{(m_0)^{-1} t^2 (d_{m_0} \phi_M^{M}(u), u)}$, where $G$ is homogeneous of degree $2k + a + b - t$ in $u$, for some integer $k$. Thus $G$ is even if and only if $a + b + t$ is even, so that only integral powers of $\lambda$ contribute to the asymptotic expansion of $\mathcal{E}$.

By the same token, only fractional (non-integral) powers of $\lambda$ contribute to the asymptotic expansion of $\mathcal{O}$.

This completes the proof of the Theorem.

Q.E.D.

4 Proof of Corollary 1.1.

We want to obtain a global trace formula, that is, an asymptotic expansion for $\int X S_{x} e^{-i\lambda(x)}(x, x) \, d\mu_X(x)$. We start off by noticing that the integral is rapidly decreasing for $\lambda \to -\infty$ by the first statement of the Theorem, and that for $\lambda \to +\infty$ integration may be localized near $X_0$ by the second statement. Our next step will be to insert the local expansion in the third statement of the Theorem within the integral, and this requires making sense of the expression $x + \mathbf{v}$ for a variable $x \in X_{\tau_0}$. This can be done by smoothly deforming with $x$ the construction of Heisenberg local coordinates centered at $x$ $[\mathcal{P}1]$. Recall that $M_{\tau_0} := \text{Fix} \left( \phi_{\tau_0}^{K} \right) \subseteq M$ and $X_{\tau_0} := \text{Fix} \left( \phi_{\tau_0}^{X} \right) \subseteq X$, so that $X_{\tau_0} = \pi^{-1}(M_{\tau_0})$.

Let us fix attention on a connected component $Y = X_{\tau_0 j}$ of $X_{\tau_0}$ at a time, and let us set $N = \pi(Y)$. Thus $N$ is a connected component of $M_{\tau_0}$ and we shall denote its complex dimension by $f$. Clearly $Y =: \pi^{-1}(N)$.

Consider a finite cover $N = \bigcup_{i} N_i$ by coordinate charts $\beta_i : B_{2\delta_0}(\delta) \to N_i$, and suppose given on each $N_i$ a local section $s_i$ of $A$ of unit norm. This induces a local chart $\tilde{\beta}_i : (-\pi, \pi) \times B_{2\delta_0}(\delta) \to Y_i =: \pi^{-1}(N_i)$, given by $\tilde{\beta}_i(\theta, r) =: e^{i\theta} \cdot s_i(\beta_i(r))$.

Upon choosing the $N_i$’s sufficiently small, we may assume given for each $i$ a smooth map $\Upsilon_i : X_i \times (-\pi, \pi) \times B_{2\delta}(0) \to X$, such that for every $x \in X_i$ the partial map $\Upsilon_i(x, \cdot, \cdot)$ is a Heisenberg local chart for $X$. We may as well assume, setting $x + (\theta, \mathbf{v}) =: \Upsilon(x, \theta, \mathbf{v})$, that $x + (\theta, \mathbf{v}) = (x + (\theta, 0)) + (0, \mathbf{v})$. Let $c := d-f$ be the complex codimension of $N$. In terms of the isomorphism $\mathbb{C}^d \cong \mathbb{C}^f \oplus \mathbb{C}^c$, we may assume in addition that $\Upsilon_i(x, \theta, (r, 0)) \in Y$ for every $r \in B_{2\delta_0}(\delta)$.
We get a coordinate chart
\[ \widetilde{\Upsilon}_i : (-\pi, \pi) \times B_{2\delta} (\delta) \times B_{2\delta} (\delta) \to X, \]
\[ (\theta, \mathbf{r}, \mathbf{u}) \mapsto \Upsilon_i \left( \beta_i (\theta, \mathbf{r}), 0, (0, \mathbf{u}) \right) = \beta_i (\theta, \mathbf{r}) + (0, \mathbf{u}) \]
with range an open neighborhood \( X_i \subseteq X \) of \( Y_i \). In a natural sense, \( \widetilde{\Upsilon}_i \) is ‘Heisenberg in the normal direction’. Composing with \( \beta_i^{-1} \), we may repackage this as a diffeomorphism
\[ \widetilde{\Upsilon}_i : Y_i \times B_{2\delta} (\delta) \to X_i, \quad y \mapsto y + (0, \mathbf{u}). \]
Notice that the path \( y + t \, (0, \mathbf{u}) \) meets \( Y_i \) at \( y \) for \( t = 0 \) with normal velocity. Since in the following we shall only use these normal displacements, we shall simplify notation and simply write \( y + \mathbf{u} \) for \( y + (0, \mathbf{u}) \). Let us write
\[ \widetilde{\Upsilon}_j (d\mu_X) = \mathcal{U}(y, \mathbf{u}) \, d\mathbf{u} \, d\mu_Y (y), \]
d where \( d\mu_Y \) is the natural volume form on the open subset \( Y_i \subseteq Y \) (induced by the form \( \omega^f / f! \) on \( N \) and the connection form); by construction, \( \mathcal{U}(y, 0) = 1 \) for every \( y \in Y_i \). Perhaps after composing \( \widetilde{\Upsilon}_j \) with a suitable change of variables of the form \( \mathbf{u}' = \mathbf{u}'(y, \mathbf{u}), y' = y \), we may further assume that \( \mathcal{U}(x, \mathbf{u}) = 1 \) identically.
Finally, let \( \{ \gamma_k \} \) be a partition of unity on \( N \) subordinate to the open cover \( \{ N_k \} \); this may also be regarded as an \( S^1 \)-invariant partition of unity on \( Y \) subordinate to the open cover \( \{ Y_k \} \). Given a tubular contraction \( X' =: \bigcup_k X_k \to Y \), the \( \gamma_k \)'s may be naturally extended to a partition of unity of \( X' \). We may also arrange that \( \gamma_k (y + \mathbf{u}) = \gamma_k (y) \) for all \( y \in Y \) and sufficiently small \( \mathbf{u} \in \mathbb{C}^c \).
We may assume that the open neighborhood \( X' \) of \( Y \) has positive distance from the other connected components of \( X_n \). By the second statement of the Theorem, therefore, \( S_{x e^{-i\lambda \cdot \gamma}} (x, x) = O (\lambda^{-\infty}) \) on \( X' \) if \( x = y + \mathbf{u} \in X' \) and \( \| \mathbf{u} \| \geq C \lambda^{-1/4} \). Thus, for an appropriate radial bump function \( \eta \) on \( \mathbb{C}^c \) identically equal to 1 near the origin, we have
\[ \int_{X'} S_{x e^{-i\lambda \cdot \gamma}} (x, x) \, d\mu_X (x) = \sum_k \int_{X_k} \gamma_k (x) \cdot S_{x e^{-i\lambda \cdot \gamma}} (x, x) \, d\mu_X (x) \quad (39) \]
\[ \sim \sum_k \int_{Y_k \times B_{2\delta} (\delta)} \eta (\lambda^{1/18} \mathbf{u}) \, \gamma_k (y + \mathbf{u}) \cdot S_{x e^{-i\lambda \cdot \gamma}} (y + \mathbf{u}, y + \mathbf{u}) \, \tilde{\Upsilon}_k (d\mu_X) \]
\[ = \sum_k \int_{Y_k} \gamma_k (y) \left[ \int_{B_{2\delta} (\delta)} \eta (\lambda^{1/18} \mathbf{u}) \cdot S_{x e^{-i\lambda \cdot \gamma}} (y + \mathbf{u}, y + \mathbf{u}) \, d\mu_X (y) \right] \, d\mu_Y (y), \]
Let us estimate asymptotically each of the summands in the last line of (39). By the change of variables \( \mathbf{u} \sim \mathbf{u} / \sqrt{\lambda} \), the \( k \)-th summand transforms
to

\[ \lambda^{-c} \int_{Y_k} \gamma_k (y) \cdot \left[ \int_{C^c} \eta (\lambda^{-1/9} u) \, S_{\chi e^{-i\lambda_0}} \left( y + \frac{u}{\sqrt{\lambda}}, y + \frac{u}{\sqrt{\lambda}} \right) d\mu (y) \right] d\nu (y). \]

Integration in the inner integral is now over a ball of radius \( O \left( \lambda^{1/9} \right) \) centered at the origin in \( C^c \). Since the remainder at the \( N \)-th step in the asymptotic expansion for \( S_{\chi e^{-i\lambda_0}} \left( y + u/\sqrt{\lambda}, y + u/\sqrt{\lambda} \right) \) given by the Theorem is \( O \left( \lambda^{-aN} \right) \) for some \( a > 0 \), the expansion may integrated term by term. Furthermore, only the even part in the expansion gives a non-vanishing contribution, so by item 4 in the Theorem integration yields an asymptotic expansion in descending powers of \( \lambda \).

The leading order term in the resulting asymptotic expansion is determined by computing

\[ \frac{2\pi e^{-i\lambda\tau_0}}{f(n)^{d+1}} \left( \frac{\lambda}{\pi} \right)^d \chi (\tau_0) \lambda^{-c} \int_{C^c} e^{f(n)^{-1} \psi_2 \left( d_n \phi_{-\tau_0} (u), u \right)} \eta (\lambda^{-1/9} u) \, du, \quad (40) \]

where \( n = \pi (y) \).

Since \( u \) is a normal vector, and by assumption \( id - d_n \phi_{-\tau_0} \) is invertible on the normal space to the fixed locus, \( \Re \left( \psi_2 \left( d_n \phi_{-\tau_0} (u), u \right) \right) < -c \| u \|^2 \) for some \( c > 0 \) and every \( u \in C^c \). Therefore, only a rapidly decreasing contribution is lost if \( \eta (\lambda^{-1/9} u) \) is replaced by 1 in (40) (and similarly in the lower terms).

Performing the change of variable \( u = \sqrt{f(n)} v \), as in the derivation of (64) in [P1] we obtain

\[ \int_{C^c} e^{f(n)^{-1} \psi_2 \left( d_n \phi_{-\tau_0} (u), u \right)} \, du = f(n)^c \int_{C^c} e^{\psi_2 \left( d_n \phi_{-\tau_0} (v), v \right)} \, dv \]

\[ = f(n)^c \int_{C^c} e^{\psi_2 \left( d_n \phi_{-\tau_0} (v), v \right)} \, dv \]

\[ = f(n)^c \cdot \frac{\pi^c}{\det \left( \left[ \left. id - d_n \phi_{-\tau_0} \right|_{N_n} \right] \right)}. \]

The leading order terms of the expansion is then the integral over \( N \) of

\[ \frac{2\pi e^{-i\lambda \tau_0}}{f(m)^{d+1}} \left( \frac{\lambda}{\pi} \right)^f \chi (\tau_0) \left( \frac{\pi}{\det \left( \left[ \left. id - d_n \phi_{-\tau_0} \right|_{N_n} \right] \right)} \right), \]

as claimed.

The proof is completed by repeating this argument over the set of all connected components.

Q.E.D.

25
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