New theory of diffusive and coherent nature of optical wave via a quantum walk

Yusuke Ide  
Department of Information Systems Creation, Faculty of Engineering,  
Kanagawa University Kanagawa, Yokohama 221-8686, Japan

Norio Konno  
Department of Applied Mathematics, Faculty of Engineering,  
Yokohama National University Hodogaya, Yokohama 240-8501, Japan

Shigeki Matsutani  
Industrial Mathematics, National Institute of Technology, Sasebo College,  
1-1, Okishin-machi, Sasebo, Nagasaki 857-1193, Japan and  
Institute of Mathematics for Industry, Kyushu University,  
Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan

Hideo Mitsuhashi  
Faculty of Education, Utsunomiya University  
Utsunomiya, Tochigi 321-8505, Japan

(Dated: August 8, 2018)

We propose a new theory on a relation between diffusive and coherent nature in one dimensional wave mechanics based on a quantum walk. It is known that the quantum walk in homogeneous matrices provides the coherent property of wave mechanics. Using the recent result of a localization phenomenon in a one-dimensional quantum walk (Konno Quantum Inf. Proc. (2010) 9 405-418), we numerically show that the randomized localized matrices suppress the coherence and give diffusive nature.

*ide@kanagawa-u.ac.jp  
konno@ynu.ac.jp  
smatsu@sasebo.ac.jp  
mitsu@cc.utsunomiya-u.ac.jp
I. INTRODUCTION

Feynman presented the nature of wave mechanics based on his picture of path integral [FH]. At a point in the space the value of a wave function which is a complex number is determined by the summation of contributions over all paths from all other point in the space [FH]. In his lecture for an ordinary audience [FH], he explained the behavior of the light ray stemmed from a point to another point along the line of his picture and showed that the picture reproduces the Fermat principle of the light. Though Feynman mentioned nature of the quantum mechanics, the natural correspondence between quantum mechanics and optics as wave mechanics [BW] means that nature of light is coherence and thus the interference naturally appears in optics. A path from a point to another point, in one-dimensional case, consists of transmission and reflection at each point, and for each path, a complex value is determined. By summing the complex values over all possible paths, the value of the wave function at the point is determined.

In optics, the transmission and reflection in one-dimensional system is described by $S$-matrix in the transfer matrix theory [C, L]. Crook proved that the transfer matrix theory and path integral (summation) are equivalent statically [C].

Recently such a picture is obtained in quantum walks mathematically. The reviews for developments of the theory of quantum walks are, for example, Kempe [Kem], Kendon [Kem], Venegas-Andraca [VA1, VA2], Konno [K3], Manouchehri and Wang [MW], Portugal [P]. Konno analytically showed that the homogeneous scattering by Hadamard type matrix at each point forms the “wavefront” in quantum walks [K1, K3]. It could be interpreted as a rigorous mathematical proof of the picture in Feynman’s lecture [FH]. Quantum walks provide the nature of the wave mechanics, in which the complex number plays a crucial role. As the probability density corresponds to the intensity of light, the results of quantum walks exhibit the coherent nature, the existence of the wavefront, and the Fermat principle of light in a one-dimensional system as in Sec III. Further as the optical ray has the finite speed, quantum walks naturally include the time-development and have the finite speed.

On the other hand, when we consider coloring of the colored materials, e.g., paint, ink in a media, colored fabric and so on, the color is determined by the diffusive reflection of the light with the intensity distribution of the visible wavelengths. The incident white rays with flat spectrum distribution penetrates into a colored material and then some of its spectrum are absorbed and scattered. We observe the out-going diffusive reflection colored rays whose distribution of the intensity on the visible wavelengths differs from white one. The Kubelka-Munk model which is a one-dimensional model describes well the nature of coloring of diffusive reflection of the optical rays except structural color. We decompose the white ray to each monochromatic ray of each wavelength and the behavior of each monochromatic ray is governed by a differential equation [KM] (See Appendix B). In the equation, we handle the light intensity of a fixed spectrum as a real number rather than a complex number at each point. In other words, in the model, coherence and interference of the light are neglected though coherence is the most important nature of the light as mentioned above. It implies that in the coloring, the real valued intensity plays much more important roles than complex valued properties of the coherence. The real valued ray is based on the diffusive light theory [D, KM]. Light is scattered by small particles and voids in the media many times, e.g., following the Mie scattering theory and absorbed by colored materials. It is said that these scattering and absorption make the coherent light forget its coherence and become the diffusive light.

It is a natural problem when the optical wave forgets its coherence. We numerically consider this problem using quantum walks and the correspondence between quantum mechanics and optics as in Sec III.

One of the answers was obtained by Ribeiro, Milman, and Mosseri [RMM], Mackay, Bartlett, Stephenson and Sanders [MBSS] and Kempe [Kem]. They studied the quantum random walk which is a model of quantum walks with randomized quantum-coins matrices. The randomized quantum-coins matrices cause quantum walks to lose its coherence of wave nature and we have the diffusive and localized distributions. However they assumed that the matrices change their value randomly in space and time. The assumption differs from the above situation. Small particles, voids and colored materials in a solid are static.

Another answer was obtained by Haney and van Wijk [HvW]. They studied the wave transport in the random media and revised the Kubelka-Munk equation (they also called it O’Doherty-Anstey formula [HvWS]) based on the theoretical investigation of Goedecke [G] and the numerical model proposed by Weaver and his coauthors [LW, W1, W2, WB]. The numerical model is a coupled spring model with randomized spring constants and reproduces many interesting phenomenon, such as Anderson’s localization phenomenologically [LW, W1, W2, WB]. Since Goedecke showed that under the assumption that randomness makes the wave lose its coherency, the transfer matrix theory is connected with the Kubelka-Munk theory, Haney and van Wijk used the transfer matrix theory phenomenologically and derived a time-development equation as a revision of the formula in the Kubelka-Munk theory. They analyzed their numerical computations of the random coupled spring model and showed that the equation reproduces the numerical results.

Though the numerical results are similar to ours in this article, the problem is not solved in the framework of the
Let us consider the infinite one-
and further we consider matrices which consist of the random B-
later we consider optical wave using quantum walks.
that such impurities suppress the coherent nature and the distribution of quantum walks behaves like diffusive one.
the solid. It implies one of disordered models of quantum walk [K2, RMM, MBSS, Kem]. There we numerically show matrices and consider quantum walks over them. These impurities are static like the voids, and the small particles in the homogeneous material. We set sufficiently many B-impurities uniformly randomly in homogeneous matrices as a background, whose density $p = 0.05$ in 6000 points. Mathematically it means the Poisson point process of the localized matrix of the B-impurity in the homogeneous material. We fix these quantum coin matrices and consider quantum walks over them. These impurities are static like the voids, and the small particles in the solid. It implies one of disordered models of quantum walk [K2, RMM, MBSS, Kem]. There we numerically show that such impurities suppress the coherent nature and the distribution of quantum walks behaves like diffusive one.
In other words, we provide a new phenomena of the connection between the coherency and diffusive nature of optical wave using quantum walks.
The contents in this article is as follows: Sec II gives a brief review of the quantum walks including the recent results of localization, which we call A- and B-impurities, [K3]. In Sec III we interpret the models and results in quantum walks as optical systems. Sec IV gives the numerical results to answer the problem mentioned above. We also give the short reviews of the correspondences among transfer matrix theory, path integral (path summation), and quantum walks in [VA1, VA2, K3, MW, P]. In fact, using quantum walks, many physical phenomenon were re-investigated and reevaluated rigorously.
Using the correspondence between optical transfer matrix theory and quantum walks, we naturally introduced the time development (see Appendix A). Thus we will investigate the problem in the framework of quantum walks rigorously.
Recently Konno gave an interesting fact on the localization in quantum walks [K5] that a single quantum coin matrix causes the localization of the density. We call the matrix B-impurity as in Sec II.
In this article, we employ this model and consider the quantum coin matrices which consist of the random B-impurities in the homogeneous material. We set sufficiently many B-impurities uniformly randomly in homogeneous Hadamard matrices as a background, whose density $p = 0.05$ in 6000 points. Mathematically it means the Poisson point process of the localized matrix of the B-impurity in the homogeneous material. We fix these quantum coin matrices and consider quantum walks over them. These impurities are static like the voids, and the small particles in the solid. It implies one of disordered models of quantum walk [K2, RMM, MBSS, Kem]. There we numerically show that such impurities suppress the coherent nature and the distribution of quantum walks behaves like diffusive one.
In other words, we provide a new phenomena of the connection between the coherency and diffusive nature of optical wave using quantum walks.

### II. SHORT REVIEW OF QUANTUM WALKS

Following [K3, VA2], we show a short review of quantum walks in this section. Let us consider the infinite one-dimensional lattice

$$
\mathcal{N} = \{ na \mid n \in \mathbb{Z} \}, \quad \mathcal{E} = \{ [na, (n+1)a] \mid n \in \mathbb{Z} \},
$$

and the set of the complex two dimensional vector valued functions of $\mathcal{N}$, denoted by $\mathcal{S}(\mathcal{N}) := \{ \sigma \}$,

$$
\sigma = \left\{ \sigma_n := \begin{pmatrix} \sigma_n^{(+)} \\ \sigma_n^{(-)} \end{pmatrix} \mid \sigma_{\pm,n} \in \mathbb{C}, n \in \mathbb{Z} \right\}.
$$

Here $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of integers, real numbers, and complex numbers respectively. Further we consider a subset $\mathcal{S}_0(\mathcal{N})$ of $\mathcal{S}(\mathcal{N})$,

$$
\mathcal{S}_0(\mathcal{N}) = \left\{ \sigma \in \mathcal{S}(\mathcal{N}) \mid |\sigma| := \sum_i (|\sigma_i^{(+)}|^2 + |\sigma_i^{(-)}|^2) = 1 \right\},
$$

and $p : \mathcal{S}_0(\mathcal{N}) \to \mathcal{F}_0(\mathcal{N}) (p(\sigma) = \{|\sigma_i^{(+)}|^2 + |\sigma_i^{(-)}|^2\}_i)$, where

$$
\mathcal{F}_0(\mathcal{N}) = \left\{ d_n \in \mathbb{R} \mid n \in \mathbb{Z}, d_n \geq 0, \sum_{n \in \mathbb{Z}} d_n = 1 \right\}.
$$

An element $\sigma$ of $\mathcal{S}_0(\mathcal{N})$ means a quantum state in quantum walks and $p(\sigma)$ means the probability density of the state $\sigma$. Later we consider $p(\sigma)$ as the intensity of the light using the correspondence between quantum mechanics
and optics. We consider a linear map from $S_0(N)$ to $S_0(N)$ which is given by

$$\tau = \{ \tau_n := \begin{pmatrix} \tau_n^{++} & \tau_n^{+-} \\ \tau_n^{-+} & \tau_n^{--} \end{pmatrix} \mid \tau_n \in U(2), \ n \in \mathbb{Z} \}$$

with its action $\tau \sigma$, 

$$(\tau \sigma)_n = \begin{pmatrix} \tau_{n-1}^{++} \sigma_{n-1}^{++} + \tau_{n-1}^{+-} \sigma_{n-1}^{+-} \\ \tau_{n+1}^{-+} \sigma_{n+1}^{-+} + \tau_{n+1}^{--} \sigma_{n+1}^{--} \end{pmatrix}.$$  \hspace{1cm} (II.1)

The set of the linear maps is denoted by $\mathcal{U}(N)$. $U(2)$ means the set of unitary matrix, i.e., for every $A \in U(2)$, its hermite conjugate $A^*$ is identified with its inverse matrix. The unitary matrix is referred the quantum-coin matrix and the element $\mathcal{U}(N)$ is the quantum-coin operator.

When an element $\tau \in \mathcal{U}(N)$ is homogeneous case or the constant for every $n$, the behavior is completely determined by Konno [K1, K3]. When every $\tau_n$ of $\tau \in \mathcal{U}(N)$ is given by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

which is called Hadamard type. We let it be denoted by $\tau^H \in \mathcal{U}(N)$. The probability density $p(\sigma)$ for the initial state,

$$\sigma_n = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & n = 0, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{otherwise}, \end{cases}$$

is illustrated in Figure 1 at $t = 3000$. For this case, the limit behavior of the probability density is given by [K1, K3],

$$\lim_{t \to \infty} P(u \frac{X_t}{t} \leq v) = \int_u^v \frac{1}{\pi \sqrt{1 - x^2 \sqrt{1 - 2x^2}}} dx,$$  \hspace{1cm} (II.3)

if the initial state $t = 0$ is emitted from the point $n = 0$.

As mentioned in Introduction, the probability density has the wavefront and recovers the dispersion relation of the light ray, which is consistent with the picture of Feynman [F, FH]. In other words, the peaks in the envelop of (II.3) and Figure 1 represent the wavefront of the ballistic motion for the initial state (II.2). Using the correspondence between quantum mechanics and optics, it is interpreted as the wavefront of the ballistic motion of light whose speed is $1/\sqrt{2}$. The probability density is regarded as the intensity of light ray as in Sec. III and A.

![Figure 1](image-url)  \hspace{1cm} FIG. 1. The Probability density of the Hadamard process at $t = 3000$.

The probability density is symmetric with respect to the origin as in Figure 1 were we set the origin as the center of the system. We introduce the center of gravity of the half-side,

$$\text{COG}(t) := \frac{\sum_{n \geq 0} n \sigma_n^* \sigma_n}{\sum_{n \geq 0} \sigma_n^* \sigma_n}. \hspace{1cm} (II.4)$$
We numerically compute the time dependence of the center of gravity of the half-side, which is displayed in Figure 2. Then the numerical computation shows that its value is given by

$$\text{COG}(t) = \beta_0 t^\alpha$$

where $\alpha_0 = 1$. It is proved strictly using the equation (II.3).

FIG. 2. The time dependence of the center of gravity of the half side of the density distribution of Hadamard process.

A. Localization in Quantum Walk

Recently Konno also investigated the behaviors of quantum walks for the configurations of $\tau^A$ and $\tau^B \in U(N)$:

$$\tau^A_n = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\gamma} & 1 \\ 1 & -e^{-i\gamma} \end{pmatrix}, & \text{for } n = 0, \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, & \text{otherwise}, \end{cases}$$

$$\tau^B_n = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\gamma} \\ e^{-i\gamma} & -1 \end{pmatrix}, & \text{for } n = 0, \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, & \text{otherwise}. \end{cases}$$

In this article, we call the former case A-type $[\text{K4}]$ and the latter one B-type $[\text{K5}]$. Konno proved that A-type does not cause the localization whereas B-type does $[\text{K4, K5}]$. Further we also call the matrix $\tau^A_0$ A-impurity and the matrix $\tau^B_0$ B-impurity here.

We compute these quantum walks for 6000 points numerically. The numerical computations reproduce the results of Konno on the probability densities of A-type and B-type matrices. We computed $\gamma = 0.3$ cases of A-type in Figure 3 (a) and B-type in Figure 3 (b) for the initial state (II.2). The probability densities are plotted there. In the A-type case, we cannot observe the localization of the probability densities whereas in the B-type case, we find the localization.

The asymmetry in Figure 3 (b) comes from the phase $\gamma$; if we put $\gamma = -0.3$, we obtained its reflection image as in Figure 3 (c). We show explicit states around $n = 0$ of a few time steps in Tables 1 and 2. In Figure 3, we also observe the coherent wave, which is ballistic like Figure 1, besides the localized one which is localized in the center. It is natural because the wave must obey the superposition principle.

III. QUANTUM WALK FROM OPTICAL POINT OF VIEW

In this section, let us consider the phenomena in quantum walks from the viewpoint of optics. They correspond to ideal optical layers whose interfaces agree with $\mathcal{N}$ as in Figure 4 (a) $[\text{BW}]$. In the layers, the light is reflected at the interface with a complex reflection rate $r$ or transmits to next layer with transmission rate $t$. 

![Image of Figure 2](image-url)
The Probability densities of the A-type and B-type processes: (a) is the A-type process of $\gamma = 0.3$ at $t = 3000$, and (b) and (c) are the B-type processes of $\gamma = 0.3$ and $\gamma = -0.3$ at $t = 3000$ respectively.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$t$ & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline
0 & \(0\) & \(0\) & \(1\) & \(0\) & \(0\) & \(0\) \\
1 & \(0\) & \(0\) & \(0\) & \(\delta + 1\) & \(0\) & \(0\) \\
2 & \((-1 + \frac{1}{2})\) & \(0\) & \((1 - \frac{1}{2})\) & \(0\) & \(\delta + 1\) & \(0\) \\
3 & \(0\) & \((-1 + \frac{1}{2})\) & \(0\) & \(2\delta\) & \(\delta + 1\) & \(0\) \\
\hline
\end{tabular}
\caption{Table: $2^{(t+1)/2} \sigma_n$ A-impurity $\delta = e^{i\gamma}$}
\end{table}

It implies that we regard a quantum-coin matrix in quantum walks as a $S$-matrix in the scalar optics (see A). We consider an optical ray lit at the position $n = 0$ and time $t = 0$. By letting its wavevector be $k_\alpha$ for $n$-th segment in $\mathcal{E}$, the phase is shifted to $e^{ik_\alpha a}$ when it passes through a layer. The finite speed of the light restricts the possible paths among the interfaces. The amplitude of the light at the $m$-step of time direction at $n$-th interface is denoted by $\sigma_{n,m}$. Since the complex amplitude of the light is given by the two states, i.e., the left-direction and the right-direction (see A),

$$\psi(x, t = m) = \sigma_{n,m}^{(+)} e^{ik_{n-1}(x-na)} + \sigma_{n,m}^{(-)} e^{-ik_{n}(x-na)},$$

the time evolution $\tau \in \mathcal{U}(\mathcal{N})$ is regarded as

\begin{align*}
\sigma_{n,m}^{(+)} &= \tilde{r}_{n-1} \sigma_{n-1,m-1}^{(+)} + \tilde{r}_{n-1} \sigma_{n-1,m-1}^{(-)} \\
\sigma_{n,m}^{(-)} &= \tilde{r}_{n+1} \sigma_{n+1,m-1}^{(+)} + \tilde{r}_{n+1} \sigma_{n+1,m-1}^{(-)}.
\end{align*}

(III.1)

From the optical viewpoint, the coefficients $\tilde{r}_n$ and $\tilde{t}_n$ of $\tau$'s are regarded as reflection and transmission ratios respectively, as illustrated in Figure 4 (b). The probability density of quantum walks corresponds to the intensity of
TABLE II. Table: \(2^{(t+1)/2}\sigma_n\) B-impurity \(\delta = e^{it}\)

| \(t\) | \(-1\)  | 0    | 1    |
|-------|--------|------|------|
| 0     | \(
\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\) | \(
\begin{pmatrix} 1 \\ 1 \end{pmatrix}
\) | \(
\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\) |
| 1     | \(
\begin{pmatrix} 0 \\ i/\pi - 1 \end{pmatrix}
\) | \(
\begin{pmatrix} 1/\pi - 1 \\ 0 \end{pmatrix}
\) | \(
\begin{pmatrix} 1 + \delta \\ 0 \end{pmatrix}
\) |
| 2     | \(
\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\) | \(
\begin{pmatrix} 1/\pi - 1 \\ 1 + \delta \end{pmatrix}
\) | \(
\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\) |
| 3     | \(
\begin{pmatrix} -1 + 1 \\ i/\pi^2 - 1 - \delta \end{pmatrix}
\) | \(
\begin{pmatrix} 1 \\ 0 \end{pmatrix}
\) | \(
\begin{pmatrix} 1 + \delta \end{pmatrix}
\) |

FIG. 4. Optical Layer

The optical wave. The condition \(|\tilde{t}_n\tilde{r}_n - \tilde{r}_n\tilde{r}_n'| = 1\) means the energy conservation. If \(\tilde{r}_n\) and \(\tilde{t}_n\) are complex numbers, they mean the phase shift at the interface.

The homogeneous case in which the transmission rate is equal to the reflection rate, it corresponds to the Hadamard process. The peak in the distribution in Figure 4 could be considered as the wavefront, which speed \(c\) differs from the original speed \(c_0\) of the particle but is \(c = c_0/\sqrt{2}\). It reminds us of the fact that the group velocity in general differs from the speed of the phase velocity.

The \(A\)-impurity means that the transmission ratio in a point differs from the others whereas \(B\)-impurity corresponds to the fact that the reflection ratio in the point differs from the others. Thus due to the \(B\)-impurity, the optical lengths in the both sides loops in Figure 4(c) differ from the Hadamard case and it behaves like the Fabry-Perot interferometer [BW]. Thus \(B\)-impurity causes the localization. The light is confined there from the optical viewpoint, though it is studied rigorously in [K3].
FIG. 5. The intensity distribution of the optical wave of the point process with $\gamma = 0.3$ of the seed of the pseudo-random = 2: (a) $t = 500$, (b) $t = 1000$ and (c) $t = 2000$.

FIG. 6. The average of the intensity distribution of the optical wave of the point process with $\gamma = 0.3$: (a) $t = 0$, (b) $t = 500$, (c) $t = 1000$ and (d) $t = 2000$.

IV. LOCALIZATION IN POISSON POINT PROCESS OF THE B-IMPURITY

In this section, we consider a disordered model of quantum walks for a random configuration of the $B$-impurities,

$$\tau_n = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\gamma} \\ e^{-i\gamma} & -1 \end{pmatrix}, & \text{for } n \in \mathcal{N}_0, \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, & \text{otherwise.} \end{cases}$$

where the positions $\mathcal{N}_0$ are given in uniformly random. More precisely the configuration obeys the Poisson point process. In the numerical computations, we used the pseudo-randomness of a certain seed $i_S$. However if two impurities are assigned a single point, we set one of them to the nearest empty points. We avoid the multiple occupation of the impurities.

Then we show that such impurities localize the wave and make it forget its wave nature.

The number of points or matrices is 6000. The number of $B$-impurities is 300 whose phase $\gamma$ is 0.3. The density $p$ of $B$-impurities is fixed as 0.05. The initial state of every case is the same as (II.2). We computed the quantum walks numerically as in Figure 5. We set 100 different configurations using different seeds $i_S$’s, and computed 100 cases of the time-evolutions of the probability densities of the quantum walks. We averaged the densities over 100 seeds of the pseudo-randomness and obtained the time-evolutions of the averaged densities as in Figure 6. With the passage
FIG. 7. The average of the intensity distributions of the optical wave of the point process at $t = 1000$, $t = 2000$ and $t = 3000$. (a),(b) and (c) correspond to $\gamma = 0.2$. (d),(e) and (f) to $\gamma = 0.3$, and (g),(h) and (i) to $\gamma = 0.5$.

FIG. 8. The time dependence of the standard deviation of the density distribution for $\gamma = 0.0$, 0.2, 0.3 and 0.5.

of time, the shape of the density differs from that of Hadamard type as in Figure 1. There exists a localized state around the center but it is extended with the passage of time.

We further handled other two cases; the number of the points and the density of B-impurities are the same as above but the phase $\gamma$ are 0.2 and 0.5. The computation results of the averaged densities are displayed in Figure 7. They are symmetric and show that the larger $\gamma$ is, the more the density is localized. Localized densities bereave the coherency of the density.

Following the arguments in \cite{MBSS, RMM}, we investigated the standard deviations of these density distributions as in Figure 8. However the standard deviations exhibit that the density distributions behave like stationary, though the localized states change their shapes as in Figure 7. Since each density distribution is symmetric with respect to the origin, we compute the time dependence of the center of gravity of the half-side $|H|_1$ as which is displayed in Figure 9.

The behaviors of the cases $\gamma > 0$ in Figure 9 are contrast to the time dependence of the center of gravity of the
FIG. 9. The time dependence of the center of gravity of the half side of the density distribution for $\gamma = 0.0, 0.2, 0.3$ and 0.5.

Hadamard process $\gamma = 0$. It is emphasized that Figure 9 represents well the properties of the behaviors in Figure 7. The time-evolution of the center of gravity is expressed well by

$$\text{COG}(t) = \beta t^{\alpha(t)},$$

(IV.1)

where $\alpha(t) \in [0, 1]$. Since $\alpha(t)$ is expressed by

$$\alpha(t) = \frac{t}{\text{COG}(t)} \frac{d\text{COG}(t)}{dt},$$

we display $\alpha(t)$ by replacing the differential by the finite difference procedure as in Figure 10.

FIG. 10. The order $\alpha$ of the time dependence of the center of gravity of the half side of the density distribution. The gray curves are given by (IV.3).

For the formal heat kernel, $\frac{1}{\sqrt{4\pi t}} e^{x^2/4t}$, its center of gravity is given as

$$\text{COG}_{HK}(t) = \frac{\int_0^\infty \frac{x}{\sqrt{4\pi t}} e^{-x^2/4t} dx}{\int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} dx}$$

$$= \frac{\int_0^\infty \frac{x}{\sqrt{2\pi}} e^{-x^2/2t} dx}{\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2t} dx}$$

$$= \sqrt{\frac{t}{2\pi}}.$$  

(IV.2)

and thus the diffusion process has the parameter $\alpha = \alpha_{HK} = \frac{1}{2}$ of COG.

In the plots in Figure 10 we can approximate them by the formula

$$\alpha(t) = \frac{1}{\kappa t + 1}.$$  

(IV.3)
In Figure 10, gray curves are given by the equation (IV.3): \( \kappa = 0.05 \) for \( \gamma = 0.2 \) case, \( \kappa = 0.0012 \) for \( \gamma = 0.3 \) case, and \( \kappa = 0.003 \) for \( \gamma = 0.5 \) case. The numerical computations show that they vanish asymptotically. Since in Figure 9, \( \alpha(t) \) of \( \gamma = 0.2 \) has the region whose the exponent \( \alpha = 1/2 = \alpha_{HK} \), the light behaves like diffusion process in the region. However it is a transient. In other words, due to the impurities, the transport states changes from the coherent one to the localized one and in its transient, the state like the diffusion appears.

It means that our model shows the coexistence of the diffusion light and the coherence light. With progress of time, the ratio of the coexistence changes.

Further Figure 11 shows that the probability density or the intensity is expressed well by the Laplace distribution,

\[
P(x)dx = \frac{A}{2\delta_t} \exp\left(-\frac{|x-x_0|}{\delta_t}\right) dx. \quad (IV.4)
\]

It might be contrast to the fact that the quantum random walk is expressed by the binomial distribution. Behind it, we have

\[
\left| \frac{d}{dx} P \right| = -\frac{1}{\delta_t} P.
\]

It is decay by \(-1/\delta_t\), which is obtained in [HvW, (A.8)].

**FIG. 11.** The average of the intensity distributions of the optical wave of the point process with \( \gamma = 0.5 \) at (a) \( t = 1000 \), (b) \( t = 2000 \) and (c) \( t = 3000 \).

However even for the \( \gamma = 0.5 \) case, the penetration to outside does not finish because the total density inside \([-50, 50]\) decreases as in Figure 12. In fact, (IV.3) shows that the behavior of the intensity becomes stationary following the power law. It means that the localized state changes slowly.

**FIG. 12.** Time dependence of the total density inside \([-50, 50]\) for the point process \( \gamma = 0.5 \)

In [G], Goedecke assumed that due to the randomness, the correlation \( \langle \sigma_n^{(+)} \sigma_n^{(-)} \rangle \) vanishes and derived the Kubelka-Munk equation. Here \( \langle \sigma_n^{(+)} \sigma_n^{(-)} \rangle \) is the correlation averaged over the space direction. Further in [HvW], under the same assumption, Haney and van Wijk derived the time development equation which they called the modified Kubelka-Munk equation.
Thus we also compute the correlation but due to symmetry of this system, the correlation always vanishes. Hence we computed the time-dependence of the real part of the correlation quantity of the half-side,

$$\eta := \text{Re} \sum_{n \geq 0} \sigma_n^{(+)} \sigma_n^{(-)}$$

(IV.5)

which is illustrated in Figure 13. Whereas the real part $\eta$ of $\langle \sigma_n^{(+)} \sigma_n^{(-)} \rangle$ is constant for $\gamma = 0$ case, it decreases with the passage of time for $\gamma \neq 0$ case. Further the larger $\gamma$ is, the correlation $\eta$ decreases more rapidly. The insight of Goedecke in [G] means that these systems in this article change from the coherent one to diffusive one.

V. DISCUSSION

The time dependence of the distribution in the Poisson point process of B-impurities is quite different from the pure Hadamard process. It shows that the B-impurities suppress the coherent properties whereas the pure Hadamard process shows the coherent nature.

Further in the intermediate process, the ray is decomposed to the localized one and the coherent one. It means the coexistence of the coherent state and diffusive state. Thus it implies that the behavior of the light ray in the real material could be approximated well by the coherent light governed by Fermat principle and diffusive light.

For example, in the real material, there are so many impurities and thus, these numerical computations support the coexistence of two facts that 1) the coloring is predicted well by Kubelka-Munk formula [D], in which the coherence of the wave is neglected, and 2) the light is expressed well by the coherent wave as in [F].

The obtained results are very similar to the numerical computational results in [HvW]. They studied the same problem based on the wave equation. However since it is not clear the relation between the continuum space-time version of quantum walks and wave equation, in this stage, we cannot give the further investigation on the relation to the results in [HvW] rigorously.

However in the framework of the path integral method and transfer matrix theory, we could give comments on our results. Our model in quantum walks is naturally obtained as the extension of the path integral and transfer matrix theory as in [A]. We computed the correlation (IV.5) of the wave function as in Figure 13. Goedecke derived the Kubelka-Munk equation from the transfer matrix theory by assuming the fact that the randomness makes the correlation $\langle \sigma_n^{(+)} \sigma_n^{(-)} \rangle$ vanish [G]. Though we do not give the direct relation to these investigations, our computations in our model of quantum walks also provide the justification of his assumption.

The advantage of our model in quantum walks is to give rigorous arguments without problems of discretization as mentioned in Introduction. It is simple and rigorous though we did not give their explicit relations to the wave equation as the continuum limit. Thus it should be regarded as a toy model but its connectivity with the transfer matrix theory and path integral is natural. Thus we consider that this model gives a new theory in optics to analyze several fundamental problems including this decoherent problem as quantum walks provide rigorous reinvestigations on crucial physical phenomena.

Since it is known that the structural color is caused by the interference of the light, if we unify the coloring in the diffusive reflection and the structural color, our theory has an effect on such investigation.

Further we should note that our investigation might be related to the random lasers [CLFAR].
ACKNOWLEDGMENTS

This work is dedicated to Professor Kenichi Tamano. The authors would like to thank him for encouragements, suggestions and profound discussions. This work has been done in his seminar which started 1988.

Y. I. was supported by the Grant-in-Aid for Young Scientists (B) of Japan Society for the Promotion of Science (Grant No. 16K17652). N. K. was supported by the Grant-in-Aid for Challenging Exploratory Research of Japan Society for the Promotion of Science (Grant No. 15K13443). S. M. was supported by the Grant-in-Aid for Scientific Research (C) of Japan Society for the Promotion of Science (Grant No. 16K05187). H. M. was supported by the Grant-in-Aid for Scientific Research (C) of Japan Society for the Promotion of Science (Grant No. 16K05249).

The authors are grateful to the referees for critical and helpful suggestions, especially for pointing out the reference [HvW].

Appendix A: Transfer matrix, S-matrix and quantum walk

In this appendix, we review the transfer matrix theory with S-matrix and the path integral (path summation) method, and show their connection with quantum walks.

![Diagram of an optical system](image)

**FIG. 14.** Optical system

We consider the scalar Helmholtz equation, which is also the fundamental equation of the Kirchhoff wave optics [BW]. In this article, we deal with its one-dimensional case,

\[
\frac{d^2}{dz^2} \psi + \frac{\omega^2 \eta(x)}{c^2} \psi = 0, \quad (A.1)
\]

where \( \eta \) is the optical index and \( \omega \) is the frequency. As we displayed it in Figure 14(a), in order to solve this equation, we divide the one-dimensional region to \( N \) pieces with the interval \( a, [na, (n+1)a), (n = 0, 1, 2, \cdots, N) \) and assume that \( \eta \) is constant at each \([na, (n+1)a)\). In order to solve (A.1), we set

\[
\psi(x) = u_n e^{ik_n(x-na)} + d_n e^{-ik_n(x-an)}, \quad x \in [na, (n+1)a),
\]

where \( k_n^2 = \omega^2 \eta(x)/c^2 \). At each \( x \in 2an, (n = 0, 1, 2, \cdots, N) \), the solution of the Helmholtz equation must satisfy

\[
\psi(na) = \lim_{x \to na-0} \psi(x), \quad \left( \frac{d}{dx} \right) (na) = \lim_{x \to na-0} \frac{d}{dx} \psi(x).
\]

These mean

\[
\begin{pmatrix} 1 & 1 \\ k_n & -k_n \end{pmatrix} \begin{pmatrix} u_n \\ d_n \end{pmatrix} = \begin{pmatrix} \alpha_{n-1} & 1/\alpha_{n-1} \\ k_{n-1}\alpha_{n-1} & -k_{n-1}/\alpha_{n-1} \end{pmatrix} \begin{pmatrix} u_{n-1} \\ d_{n-1} \end{pmatrix}, \quad (A.2)
\]

where \( \alpha_n = e^{ik_na} \). By letting

\[
\hat{r}_n := \frac{2k_n}{(k_n + k_{n-1})}, \quad \hat{t}_n := \frac{(k_n - k_{n-1})}{k_n + k_{n-1}},
\]
difference between the reflection ratios, i.e., we sum the amplitudes over all possible paths and by using the relation , here we use the facts that \( r_n \) and \( t_n \) are the reflection and the transmission ratios respectively. In this case, \( r_n = r_n' = \tilde{r}_n \), \( t_n' = \tilde{t}_n \) and \( t_n = \frac{k_n-1}{k_n+k_n-1} \). It is turned out that \( S_n \) is an unitary matrix and the total intensity of the incoming rays is equal to that of the outgoing rays, i.e., \(|u_n|^2 + |d_n-1|^2 = |u_{n-1}|^2 + |d_n|^2\). Using the transfer matrices \( T_{0,N} = T_{0,1} T_{1,2} \cdots T_{N-1,N} T_{N,1} \), we also have
\[
\begin{pmatrix} u_N \\ d_N \end{pmatrix} = S_{0,N} \begin{pmatrix} u_0 \\ d_0 \end{pmatrix},
\]
which is obtained like \( S_n \) for \( T_n \). For example, \( N = 2 \) case illustrated in Figure 14 (b),
\[
S_{0,1} = \begin{pmatrix} t_2(1 + r_1 r_2 a_2^2)^{-1} t_1 & r_2 + t_2 t_2' (1 + r_1 r_2 a_2^2)^{-1} \\ r_1 + t_1 t_1' (1 + r_1 r_2 a_2^2)^{-1} & t_2(1 + r_1 r_2 a_2^2)^{-1} t_1 \end{pmatrix}. \tag{A.4}
\]
Here we use the facts that \( r_n^2 t_n t_n' = 1 \) and \( r_n' = -r_n \).

In the path integral method, we sum the amplitude over all possible paths as in [C] noting that there is the phase difference between the reflection ratios, i.e., \( r_n' = -r_n \). The amplitude is determined by the multiplication of \( r_n \)’s, \( t_n \’s \) and the phase factor \( e^{ik_L} \), where \( L \) is the length of the path. For example, for the simple case as in Figure 14 we sum the amplitudes over all possible paths and by using the relation,
\[
(1 - r_1 r_2 a_2^2 + (r_1 r_2 a_2^2)^2) \cdots = \frac{1}{1 + r_1 r_2 a_2^2},
\]
we recover \( A.4 \) even for the path integral (path summation) method. Crook proved that the results of the path integral method are completely identified with those of the S-matrix method for more general cases [C].

However these are static. In quantum walks, we introduce the time parameter \( m \). Using the fact that \(|u_n|^2 + |d_{n-1}|^2 = |u_{n-1}|^2 + |d_n|^2\), let \( \sigma_{n,m}^{(\pm)} = \begin{pmatrix} u_{n-1} & 0 \\ 0 & d_{n-1} \end{pmatrix} \). Noting the relations,
\[
\begin{pmatrix} u_{n-1} & 0 \\ 0 & d_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha_{n-1} t_{n-1} & \alpha_{n-1} r_{n-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_{n-2} & 0 \\ 0 & d_{n-2} \end{pmatrix},
\]
\[
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_n t_{n+1} & \alpha_n r_{n+1} \\ \alpha_n' r_{n+1} & \alpha_n' t_{n+1} \end{pmatrix} \begin{pmatrix} u_n & 0 \\ 0 & d_n \end{pmatrix}
\]
we obtain the time-evolution [33] by letting \( \tilde{t}_n = \alpha_n t_n \), \( \tilde{r}_n = \alpha_{n-1} t_n \tilde{r}_n = \alpha_n r_n \) and \( \tilde{r}_n' = \alpha_{n-1} r_n' \). It is obvious that the associated matrix is unitary.
Appendix B: Short Review of Kubelka-Munk Theory

In [KM], Kubelka and Munk investigated the diffusive light whose value is given by real number in paint layer. They considered the one-dimensional half space whose negative is for downward region which consists of paint layer. They modeled the diffusive light by the downward-going light \( i \) and the upward-going light \( j \) which interchanges by scattering and absorption for the scattering ration \( s \) and an absorption ratio \( k \):

\[
\frac{d}{dx} \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} s + k & -s \\ s & -(s + k) \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix}.
\]

Since this matrix denoted by \( S \) has the property,

\[
S^2 = \begin{pmatrix} (s + k)^2 - s^2 & 0 \\ 0 & (s + k)^2 - s^2 \end{pmatrix},
\]

we have

\[
e^{Sx} = \cosh(\sqrt{(s + k)^2 - s^2}x)I + \frac{\sinh(\sqrt{(s + k)^2 - s^2}x)}{\sqrt{(s + k)^2 - s^2}} S,
\]

and

\[
\begin{pmatrix} i(x) \\ j(x) \end{pmatrix} = \exp(Sx) \begin{pmatrix} i_0 \\ j_0 \end{pmatrix},
\]

where \( I \) is the unit matrix and \( i_0 \) and \( j_0 \) are the intensity of each light at \( x = 0 \).

Noting asymptotically

\[
\cosh(\sqrt{(s + k)^2 - s^2}x)/\sinh(\sqrt{(s + k)^2 - s^2}x) \to -1
\]

for \( x \to -\infty \), if we assume, for example, that \( i(x) \) vanishes at \( x \to -\infty \), we have

\[
j_0 = \left(1 + \frac{k}{s} - \sqrt{\frac{k^2}{s^2} + \frac{2k}{s}}\right) i_0.
\]

For the case, the ratio of the intensities of the diffusive light and the incoming ray,

\[
R_\infty = \frac{j_0}{i_0},
\]

is expressed by

\[
R_\infty = \left(1 + \frac{k}{s} - \sqrt{\frac{k^2}{s^2} + \frac{2k}{s}}\right), \quad \frac{k}{s} = \frac{(1 - R_\infty)^2}{2R_\infty}.
\]

By observing the reflection ratio \( R_\infty \) for an ideal case for monochromatic light with the wave length \( \lambda \), we determine the ratio \( k/s \) each \( \lambda \). Using \( k/s \), we can predict the coloring for the light. For example, if we use \( n \) paints \((k_i, s_i)\) in layers with depth \( d_i \), we have the formula

\[
\begin{pmatrix} i(\sum_i d_i) \\ j(\sum_i d_i) \end{pmatrix} = e^{S_1d_1} \cdots e^{S_{n}d_{n}} \begin{pmatrix} i_0 \\ j_0 \end{pmatrix},
\]

where \( S_i \) is the matrix \( S \) consisting of \((k_i, s_i)\). If \( \begin{pmatrix} i(\sum_i d_i) \\ j(\sum_i d_i) \end{pmatrix} \) is determined by a boundary condition, we have

\[
\begin{pmatrix} i_0 \\ j_0 \end{pmatrix} = e^{-S_1d_1} \cdots e^{-S_{n}d_{n}} \begin{pmatrix} i(\sum_i d_i) \\ j(\sum_i d_i) \end{pmatrix}.
\]

This shows the reflection ratio of the light for layers. This theory is a nice theory to predict the coloring in nature and has been used in painting and printing industry including cosmetics for protection of the ultraviolet rays.

However these rays are given by real numbers whereas the light is basically given by the complex number as in [F]. The difference are concerned.
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