Mesonic Correlation Functions in the Random Instanton Vacuum

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Abstract
A general model-independent discussion of mesonic correlation functions is given. We derive new inequalities, including one stronger than Weingarten’s inequality. Mesonic correlation functions are calculated in the random instanton vacuum and are compared with phenomenological expectations and lattice results. Both diagonal and non-diagonal correlators of all strange and light flavored currents, as well as the most important unflavored ones are considered. Our results are used to extract the masses and the coupling constants of the corresponding mesons. Not only the qualitative behaviour is reproduced in all channels, but in several channels the model works with amazing accuracy up to distances of 1.5 \( fm \).
1. Introduction

1.1. Motivations and outline of the paper

Correlation functions are a major tool for understanding the structure of any type of matter, solid or liquid. The QCD vacuum, being a complicated ensemble of interacting quark and gluon fields, has also been studied this way in the framework of various theoretical approaches based directly on quantum field theory, especially, by numerical simulations on a lattice. However, in the past, those studies were mainly focused on the long-distance behaviour of these functions, which is determined by the masses of the lowest hadronic states. Only recently was it realized, that extremely important dynamical information is contained in the behaviour of these functions at smaller distances of \( x \sim 0.1 - 1.0 \text{ fm} \).

Another explanation of why these point-to-point correlation functions are so fundamental for QCD goes as follows. Due to confinement, one cannot directly study quark-quark scattering, but the same information is contained in the correlation functions. At the same time, these functions can also be obtained by inserting a set of physical (asymptotic) states and using dispersion integrals. In this way, one can learn a lot about the space-time structure of the effective interquark interaction. The available phenomenological information on correlation functions, as well as the current theoretical situation, was recently reviewed by one of us [1].

Although some questions related to the short-distance behaviour of the correlators can be traced back to 'current algebra', the Wilson operator product expansion (OPE) and other developments in the sixties, a systematic study of the relevant non-perturbative QCD dynamics was initiated by the so called QCD sum rules [7]. Using the OPE as a practical tool to extract information on correlators at small distances, these authors have found that it works nicely in some cases, but fails in others.

In short, the general lesson one can draw from these studies is that the short range behaviour of the correlation functions is qualitatively different for different channels. Non-perturbative effects cause deviations from the perturbative behaviour, that differ drastically in sign and magnitude for different channels. The resulting picture obviously
contradicts to any simplistic model of strong interactions. In particular, universal color confinement is far from being the only non-perturbative force between quarks and antiquarks.

The main objective of the present work is to investigate the role of instanton-generated effects. By now, it is well known, that at least some of their qualitative features (briefly reviewed in section 1.6) are in good correspondence with empirical observations for the scalar and pseudo-scalar channels. Moreover, numerical studies in the framework of the 'interacting instanton approximation' (IIA) [20] show very promising results, outperforming the QCD sum rules even for the vector and axial correlators. In this work, we report on much more quantitative results for many of these functions and establish more definitely the limitations of this approach.

Another motivation for these studies is that one can get a similar set of results from lattice simulations. The first work in this direction has been reported recently [4]. In the last section of this work, the comparison of the results obtained within these two approaches shows that they are surprisingly consistent with each other.

Many details of our approach have already been explained in the first paper of this series [5], so we do not need to repeat them here. Let us only remind, that we use the simplest possible model, the 'random instanton vacuum' (RIV), as a kind of 'benchmark' for future investigations of instanton dynamics. The main result of this first paper is the derivation of a practical approximate formula for the quark propagator. More specific results for the averaged propagator can, roughly speaking, be interpreted as the appearance of some 'quark effective mass', which happens to be about 300 $MeV$ in agreement with the old 'constituent quark model'. However, as we will show below, it does not mean that this model provides a reasonable approximation to correlation functions. The quark propagators contain 'hidden components'! They are not seen in the average propagators, but they show up if the square (for mesonic correlators) or the cube (for baryonic correlators) of the propagators is averaged over the gauge field configurations. In other words, we have found a strong interaction between quarks, which is not small as compared with the 'constituent masses' and is strongly channel dependent.

The paper is organized as follows. In section 1.2 we remind some general 'kinematical' properties of the correlators and introduce the currents and the terminology used. Then,
in section 1.3, we specify how the various correlators are connected to the propagator decomposition into spin-color components. These formulae can be used, \textit{e.g.}, to make the simplest 'vacuum dominance approximation', which allows us to get at least the correct sign of the non-perturbative corrections to various correlators induced by chiral symmetry breaking. In this subsection we also derive some general inequalities between hadronic correlators, including one which is stronger than Weingarten’s inequality \cite{weingarten}. In section 1.4, we give schematic spectral functions for the various flavored channels. A general discussion of unflavored correlators is given in section 1.5. We finish the introductory part of the paper by reminding the reader of the qualitative features of the instanton-induced effects on correlators (see subsection 1.6).

The main body of the paper is essentially a one-by-one discussion of our results for the various channels, starting with a rather complete set of ‘flavored’ currents (section 2) supplemented by the most interesting ‘unflavored’ channels (section 3). In all cases our results are compared with phenomenological expectations (based on a detailed review \cite{review}) or, if they are absent, with the available predictions from QCD sum rules. The comparison with lattice data and a general discussion of the results (including specific fits of masses and coupling constants for various mesons) can be found in section 4.

1.2. General properties of the correlators

First of all, correlators of two currents can be divided into \textit{diagonal} and \textit{non-diagonal} correlators, depending on whether they involve the same currents or two different currents. It is also convenient to recognize two different types of currents, which we call ‘flavored’ and ‘unflavored’ currents. The correlation function of the flavored currents (\textit{e.g.} $\bar{u}d$ or $\bar{u}s$, but also $\bar{u}u - \bar{d}d$) involves only diagrams in which both quark and anti-quark propagate between $x$ and $y$. The correlation function of the unflavored currents (like $\bar{u}u$) also receive contributions from \textit{two-loop} diagrams, in which one quark line goes from $x$ to $x$ and another quark line goes from $y$ to $y$. The one-loop case, which is much simpler (\textit{e.g.} two-loop correlators are considered rarely in lattice QCD calculations), will be studied first. In line with this, we differ from the usual $SU(3)$ notation and organization in which
the correlators are classified according to their $SU(3)$--representation. Therefore, the $\eta$ and $\eta'$ mesons are discussed separately (see sections 1.5 and 3.1) rather than together with the $\pi$ and $K$ correlators. Another important channel in which 'two-loop diagrams' are crucial is the iso-scalar scalar $\sigma$ channel (see sections 1.5 and 3.2).

In this paper we will consider the flavored currents

\begin{align}
 j^S &= \bar{u}d, \\
 j^P &= \bar{u}i\gamma_5d, \\
 j^V_\mu &= \bar{u}\gamma_\mu d, \\
 j^A_\mu &= \bar{u}\gamma_\mu\gamma_5d, \\
 j^T_{\mu\nu} &= \bar{u}\sigma_{\mu\nu}d,
\end{align}

and their counterpart with the $u$–quark interchanged by an $s$–quark. They obey the hermiticity relation

\[(\bar{d}\Gamma u)^\dagger = \bar{u}\Gamma d.\]  

The study of unflavored currents will be restricted to the following (Hermitian) currents:

\begin{align}
 j^\sigma &= \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d), \\
 j^1 &= \frac{1}{\sqrt{3}}(\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d + \bar{s}i\gamma_5 s), \\
 j^8 &= \frac{1}{\sqrt{6}}(\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d - 2\bar{s}i\gamma_5 s), \\
 j^\omega_\mu &= \frac{1}{\sqrt{2}}(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d), \\
 j^\phi_\mu &= \bar{s}\gamma_\mu s, \\
 j^{f_1}_\mu &= \bar{s}\gamma_\mu\gamma_5 s.
\end{align}

The two-loop diagrams contributing to the last three currents will be ignored (Zweig’s rule), and therefore they will be discussed together with the flavored currents.

The correlators are defined by the time-ordered product

\[\Pi(x) \equiv <0|T j_A(x) j_B^\dagger(0)|0>,\]
where the Dirac indices are suppressed. If we use completeness, and PCT and translation invariance of the vacuum state, the correlator can be written in terms of a spectral representation as

$$\Pi(x) = \int \frac{d^4q}{(2\pi)^3} (\theta(x_0)e^{-iqx} + \theta(-x_0)e^{iqx})\delta(q^2 - \sigma^2)\rho(\sigma^2)d\sigma^2, \quad (1.5)$$

where $\rho(\sigma^2)$ is the spectral function defined by

$$\rho(\sigma^2) = (2\pi)^3 \sum_n \delta^4(\sigma - q_n) <0|j_A(0)|n><n|j_B^\dagger(0)|0>, \quad (1.6)$$

with Dirac indices as given by the two currents. The integral over $d^4q$ yields a Feynman propagator of mass $m^2 = \sigma^2$, which for spacelike distances is given by

$$D(\sigma, x) = \frac{\sigma}{4\pi^2x}K_1(\sigma x), \quad (1.7)$$

where $K_1(\sigma x)$ is a modified Bessel function. The integral in (1.5) can be rewritten as

$$\Pi(x) = \int_0^\infty d\sigma^2 D(\sigma, x)\rho(\sigma^2). \quad (1.8)$$

For spacelike $x$, the r.h.s of (1.8) can be analytically continued to Euclidean spacetime without hitting any singularities. This equation can thus be used to relate the spectral function defined in Minkowsky space-time to a correlator in Euclidean space-time. For later use we also give its Euclidean Fourier transform:

$$\int d^4x e^{iqx}\Pi(x) = \int_0^\infty ds \frac{\rho(s)}{q^2 + s}. \quad (1.9)$$

On the other hand, again using PCT invariance, the spectral function can be written in terms of the discontinuity of the correlator in Minkowsky space-time

$$2\pi\rho(q^2) = 2\text{Im}i\int d^4xe^{iqx} <0|Tj_A(x)j_B^\dagger(0)|0>. \quad (1.10)$$

The Euclidean Fourier transform of (1.8) with the spectral density replaced by the integral in (1.10) yields the standard dispersion relation, in which the correlator evaluated at Euclidean momenta is given by an integral transform of the imaginary part of the correlator in Minkowsky space time.

Finally, let us consider the matrix element of the current in (1.6) in more detail. For the currents defined in (1.1) obeying the Hermiticity relation (1.2) and the hermitian
currents (1.3) the relations

\[
\langle 0|d\Gamma u|n \rangle^* = \langle 0|\bar{u}\gamma_5 \Gamma \gamma_5 d|n \rangle, \quad (1.11a)
\]
\[
\langle 0|\bar{u}\Gamma u|n \rangle^* = \langle 0|\bar{u}\gamma_5 \Gamma \gamma_5 u|n \rangle. \quad (1.11b)
\]

follow from PCT invariance. Therefore, the matrix elements of a Hermitian current are real if the gamma matrix structure is spin flipping, i.e. \( \Gamma = S,\ P \) or \( T \), and they are imaginary if the gamma matrix is spin nonflipping, i.e., \( \Gamma = V \) or \( A \).

### 1.3. Propagator decomposition and inequalities

In the previous paper [5] we already mentioned, that the quark propagator contains both chirality-flipping and non-flipping components. Some of them survive the averaging over configurations, whereas others, called the 'hidden' components, are nonzero only if a square (or cube) of the propagator is averaged. The 'hidden' components generate the effective interaction between quarks.

Thus, the study of mesonic or baryonic correlators actually reveals these 'hidden components’. Generally speaking, they are by no means small as compared with the ‘visible’ components. In other words, the interquark interaction is generally of the same order of magnitude as the 'constituent quark masses' studied in [5].

In this section we specify the 'kinematics' of this statement for various channels, and briefly consider the relevant dynamics. The correlators, evaluated in Euclidean space-time, can be expressed in terms of the Euclidean propagator \( S_{\mu\nu}^{kl}(x,y) \). The general structure of the correlator is \( \Pi = \langle \text{Tr}(S(x,y)\Gamma S(y,x)\Gamma) \rangle \), where the average is over all gauge field configurations. Specifically, we discuss the following diagonal flavored correlators:

\[
\Pi_{S} = \langle \text{Tr}(S(x,y)S(y,x)) \rangle, \quad (1.12a)
\]
\[
\Pi_{P} = \langle \text{Tr}(S(x,y)i\gamma_5 S(y,x)i\gamma_5) \rangle, \quad (1.12b)
\]
\[
\Pi_{V}^{\mu} = \langle \text{Tr}(S(x,y)\gamma_\mu S(y,x)\gamma_\mu) \rangle, \quad (1.12c)
\]
\[
\Pi_{A}^{\mu\mu} = \langle \text{Tr}(S(x,y)\gamma_\mu \gamma_5 S(y,x)\gamma_\mu \gamma_5) \rangle. \quad (1.12d)
\]
\[ \Pi_{\mu\nu}^T = \langle \text{Tr}(S(x, y)\sigma_{\mu\nu}S(y, x)\sigma_{\mu\nu}) \rangle. \] (1.12e)

As we will see below, only two non-diagonal flavored correlators are non-zero. They are given by:

\[ \Pi_{\mu}^{FA} = \langle \text{Tr}(S(x, y)i\gamma_5S(y, x)\gamma_\mu\gamma_5) \rangle, \] (1.13a)
\[ \Pi_{\nu}^{VT} = \langle \text{Tr}(S(x, y)\gamma_\mu S(y, x)\sigma_{\mu\nu}) \rangle. \] (1.13b)

The propagator can be decomposed as

\[ S_{\mu\nu}^{kl} = \sum_i a_{i\mu\nu}^{kl} \Gamma_i, \] (1.14)

where the \( \Gamma_i \) are a complete set of 16 Hermitian gamma-matrices given by,

\[ \Gamma_i = 1, \gamma_5, \gamma_\mu, i\gamma_\mu\gamma_5, \text{ and } i\gamma_\mu\gamma_\nu (\mu \neq \nu). \] (1.15)

The coefficients \( a_i \) are \( SU(3) \)-color matrices. Before rewriting the correlators in terms of the \( a_i \) we point out a relation given by Weingarten [3] for the Euclidean propagator in backward direction

\[ S(x, y) = -\gamma_5 S^\dagger(y, x)\gamma_5, \] (1.16)

which allows us to express the diagonal correlators defined above in terms of the positive definite quantities

\[ Q_i = 4 \sum_{kl} |a_i^{kl}|^2. \] (1.17)

In the \( V, A \) and \( T \) case the Dirac sum over all \( Q_i \) belonging to a specific structure, is denoted by \( Q_V, Q_A \) and \( Q_T \), respectively.

The pseudoscalar (\( \pi \)) correlator, \( \Pi^P \), is simply given by the sum of all coefficients squared

\[ \Pi^P = Q_S + Q_P + Q_V + Q_A + Q_T. \] (1.18)

In the scalar (\( \delta \)) case, the correlator \( \Pi^S \) is given by the difference of the spin non-flipping and the spin flipping components:

\[ \Pi^S = -Q_S - Q_P + Q_V + Q_A - Q_T. \] (1.19)
As a result, we find Weingarten’s inequality that the pseudoscalar correlator should exceed the scalar one. From the experimental values of the pion and the scalar meson masses it is clear that this inequality should be satisfied at large distances. What is nontrivial is that it holds at all distances. One expects that for \( x > 0.5 \text{ fm} \) the scalar correlator is practically zero, which implies there should be a very precise compensation between different components of the propagator.

The vector \((\rho)\) correlator, \(\Pi^V_{\mu\nu}\), and the axial \((a_1)\) correlator, \(\Pi^A_{\mu\nu}\), are given by

\[
-\Pi^V_{\mu\nu} = +4Q_S - 4Q_P + 2Q_V - 2Q_A, \tag{1.20a}
\]

\[
-\Pi^A_{\mu\nu} = -4Q_S + 4Q_P + 2Q_V - 2Q_A. \tag{1.20b}
\]

At small distances both correlators are dominated by the contribution of the free propagator and are very similar. This is no longer the case at distances of the order of 1 \( \text{fm} \) which implies a fine tuning of the components of the propagator. The tensor correlator, given by

\[
\Pi^T_{\mu\nu\rho\sigma} = -6Q_S - 6Q_P + 2Q_T, \tag{1.21}
\]

does not receive a contribution from the free propagator and is less singular at short distances than the four other correlators. The above relations can be inverted, which allows us to express the components \(Q_i\) in terms of the mesonic correlation functions. Only the pion correlator always enters with a positive coefficient which implies that, if there is only one Goldstone boson, it necessary should be the pion. This is in agreement with the Vafa-Witten theorem that vector symmetries cannot be spontaneously broken.

Note that these relations imply the following inequalities,

\[
\Pi^P > -\frac{3}{8}\Pi^V_{\mu\nu} - \frac{1}{8}\Pi^A_{\mu\nu}, \tag{1.22a}
\]

\[
\Pi^P > -\frac{1}{8}\Pi^V_{\mu\nu} - \frac{3}{8}\Pi^A_{\mu\nu}, \tag{1.22b}
\]

which are as strong as Weingarten’s inequality. From the expressions for the \(S\), \(P\) and \(T\) channels we can immediately derive the inequality

\[
\Pi^P > \Pi^S + \Pi^T_{\mu\nu\rho\sigma}, \tag{1.23}
\]

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which is stronger than Weingarten’s inequality. Because both the vector and axial mesons couple to the tensor current (see section 1.4) it follows that the pion mass is not only lighter than the scalar mesons but also lighter than the vector and axial mesons, i.e., \( m_\pi < m_\rho \) and \( m_\pi < m_{a_1} \), which again confirms the Vafa-Witten theorem.

In the chiral limit, the pion is a Goldstone mode whereas the mesons in the other 4 channels remain massive. This requires a precise cancellation at large separations. More precisely, up to exponentially small corrections we have

\[
Q_S(x) = Q_P(x) = \frac{1}{4}Q_V(x) = \frac{1}{4}Q_A(x) = \frac{1}{6}Q_T(x), \quad \text{for} \quad x \to \infty. \tag{1.24}
\]

In other words, all components \( Q_i \), in the tensor decomposition of the propagator, converge to the same limit at large separations.

As a useful practical application of these general relations, we derive the corrections of the quark-condensate to the free propagator. In the so called ‘vacuum dominance’ approximation, the propagator is color diagonal and simplifies to

\[
S = S_0 + \frac{i}{12} < \bar{\psi}\psi > |. \tag{1.25}
\]

If the end points are taken along the \( \tau \)-direction, the free propagator is given by

\[
S_0(\tau) = \frac{i \gamma_0}{2\pi^2 \tau^3}. \tag{1.26}
\]

For the correlators in the different channels one finds

\[
\Pi^S(\tau) = \frac{3}{\pi^4 \tau^6} - \frac{1}{12} < \bar{\psi}\psi > |^2, \tag{1.27a}
\]

\[
\Pi^P(\tau) = \frac{3}{\pi^4 \tau^6} + \frac{1}{12} < \bar{\psi}\psi > |^2, \tag{1.27b}
\]

\[-\Pi^{V}_{\mu\nu}(\tau) = \frac{6}{\pi^4 \tau^6} + \frac{1}{3} < \bar{\psi}\psi > |^2, \tag{1.27c}
\]

\[-\Pi^{A}_{\mu\nu}(\tau) = \frac{6}{\pi^4 \tau^6} - \frac{1}{3} < \bar{\psi}\psi > |^2, \tag{1.27d}
\]

\[
\Pi^{T}_{\mu\nu\rho\sigma}(\tau) = -\frac{1}{2} < \bar{\psi}\psi > |^2. \tag{1.27e}
\]

\footnote{The reader should be warned, that ‘vacuum dominance’ as introduced in \cite{7} does not lead to the corrections we derive now, but rather to a kind of radiative correction to it. The reason is that we consider the OPE in space-time representation and include contributions that are ‘regular’ at \( x = y \), while in \cite{1} the OPE was studied at large momentum transfer, which includes only singular terms. See \cite{1} for further discussion.}
For completeness, let us also mention the results for the nonzero non-diagonal correlators:

\[
\Pi^\mu_{PA}(\tau) = \frac{2\delta_{\mu0}}{2\pi^2\tau^3} | < \bar{\psi}\psi > |, \tag{1.28a}
\]

\[
\Pi^\mu_{VT}(\tau) = \frac{6i\delta_{\nu0}}{2\pi^2\tau^3} | < \bar{\psi}\psi > |. \tag{1.28b}
\]

As we will see below, in all channels the sign of the corrections to the free quark contribution coincides with the experimental trends. Thus, although the vacuum dominance approximation does not provide a quantitative explanation of the behavior of the correlators it certainly gives some qualitative insight into it.

Finally, we derive an inequality that can only be proved under a very plausible but not general condition. Let us consider the difference \( Q_1 - Q_5 \). It can be written as

\[
Q_1 - Q_5 = \frac{1}{4} < \text{Tr}(1 - \gamma_5)S\text{Tr}^*(1 + \gamma_5)S > ,
\]

where the average is over all gauge field configurations. If the fermionic modes of opposite chirality are uncorrelated, this average can be factorized as

\[
Q_1 - Q_5 = \frac{1}{4} < \text{Tr}(1 - \gamma_5)S > < \text{Tr}^*(1 + \gamma_5)S > ,
\]

\[
= \frac{1}{4} < \text{Tr}S > < \text{Tr}^*S > ,
\]

where the second equality can be established because \( < \text{Tr}\gamma_5S > = 0 \) for a parity invariant vacuum state. In terms of the mesonic correlation function the positivity of \( Q_1 - Q_5 \) leads to the inequality

\[
\Pi_V(x) \geq \Pi_A(x),
\]

which is obeyed by our numerical results presented below. One also has the inequality \( \Pi_V(q) \geq \Pi_A(q) \), which can be proved rigorously for Euclidean momenta [23].

The conditions for (1.30) are realized for a random ensemble of instantons. In the opposite case, of an ensemble of instanton—anti-instanton molecules, we have

\[
\text{Tr}S = \text{Tr}\gamma_5S
\]

before averaging over the collective coordinates. Therefore, \( Q_1 = Q_5 \), and the axial and vector correlators are degenerate.
1.4. Spectral functions for flavored correlators

Next we discuss the spectral functions for the different flavored correlators. They will be approximated by the sum of one or two meson resonances in the form of a $\delta$ function and the contribution from the continuum. First, we consider the $S, P$-case in which the coupling of the current to the resonance state is defined by the matrix element

$$<0|j^{S,P}(x)|p> = \lambda^{S,P} \frac{1}{\sqrt{(2\pi)^3}} e^{-ipx}. \quad (1.33)$$

The total spectral function is then given by (see ref. [21])

$$\rho^{S,P}(s) = \lambda^{2}_{S,P} \delta(s - m^{2}_{S,P}) + \frac{3s}{8\pi^2} \theta(s - E_{0, S,P}). \quad (1.34)$$

The continuum contribution can be obtained perturbatively. However, it also follows from asymptotic freedom. At short distances the correlator approaches the free correlator given by $3/\pi^4 x^6$. Alternatively, the continuum contribution is given by the integral

$$\int_{E_0}^{\infty} ds D(\sqrt{s}, x) \rho(s). \quad (1.35)$$

For small $x$ the main contribution comes from large $s$. In this region $\rho(s) \sim s$ on dimensional grounds, and by an explicit calculation of the integral one finds $\rho(s) = (3/8\pi^2)s$.

In the vector channel the spectral function is transverse and can be written as

$$\rho^{V}_{\mu\nu} = (-g_{\mu\nu} + \frac{q_{\mu} q_{\nu}}{q^2}) \rho^{V}_{T}(q^2). \quad (1.36)$$

Its scalar part $\rho^{V}_{T}(q^2)$ is positive definite [21]. The coupling constant of the vector current to the $\rho$ meson is defined by

$$<0|j^{V}(x)|\rho p> = i\lambda_{\rho} \frac{\epsilon_{\mu}}{\sqrt{(2\pi)^3}} e^{-ipx} \quad (1.37)$$

where $\epsilon_{\mu}$ is the polarization vector of the $\rho$–meson. Using the same approximation as in the $S, P$-case the total spectral function is given by

$$-\rho^{V}_{\mu\mu}(s) = 3\lambda^{2}_{\rho} \delta(s - m^{2}_{\rho}) + \frac{3s}{4\pi^2} \theta(s - E_{0, \rho}). \quad (1.38)$$

For the axial vector channel we have the additional complication that also the longitudinal combination contributes to the spectral function,

$$\rho^{A}_{\mu\nu} = (-g_{\mu\nu} + \frac{q_{\mu} q_{\nu}}{q^2}) \rho^{A}_{T}(q^2) + q_{\mu} q_{\nu} \rho^{A}_{L}(q^2), \quad (1.39)$$
where both $\rho^A(q^2)$ and $\rho_L^A(q^2)$ are positive definite. The longitudinal component of the current couples to the pion. The relevant matrix element is conventionally written as

$$<0|j^A_\mu(x)|\pi p> = if_\pi p_\mu \frac{1}{\sqrt{(2\pi)^3}} e^{-ipx},$$

with the experimental value of $f_\pi = 133.7 \pm 0.15 MeV$. The spectral function in this case is given by

$$-\rho^A_{\mu\nu}(s) = -f_\pi^2 m_\pi^2 \delta(s - m_\pi^2) + 3\lambda_1^2 \delta(s - m_{a_1}^2) + \frac{3s}{4\pi^2} \theta(s - E_{0a_1}),$$

where the coupling constant of the $a_1$ meson to the axial current is defined as

$$<0|j^{a_1}_\mu(x)|a_1 p> = i\lambda_{a_1} \frac{\epsilon_\mu}{\sqrt{(2\pi)^3}} e^{-ipx}.$$  

Our last diagonal correlator is the tensor correlator. In this case the contribution of the free quark propagator vanishes which allows us to extract the parameters of the resonances in an much 'cleaner' way. The tensor structure of this current has two 3-vectors (like the electric and magnetic field) with opposite $P-$parity, and thus couples to both vector and axial mesons. One can introduce the corresponding coupling constants as

$$<0|j_{\mu\nu}|\rho> = \tilde{f}_\rho (\epsilon_\mu p_\nu - \epsilon_\nu p_\mu) \frac{1}{\sqrt{(2\pi)^3}} e^{-ipx},$$

$$<0|j_{\mu\nu}|a_1> = \tilde{f}_{a_1} \epsilon_{\mu\nu\alpha\beta} p_\alpha \epsilon_\beta \frac{1}{\sqrt{(2\pi)^3}} e^{-ipx},$$

resulting in the spectral function

$$\rho^T_{\mu\nu\rho\sigma}(s) = 6f_\rho^2 m_\rho^2 \delta(s - m_\rho^2) + 6f_{a_1}^2 m_{a_1}^2 \delta(s - m_{a_1}^2).$$

Let us now briefly mention the non-diagonal correlators. We have 5 different flavored currents and therefore 10 different pairs. The combinations $SP$, $SA$, $PV$, $VA$ and $AT$ are zero as a consequence of the parity invariance of the QCD action. The simplest way to show this is to take the separation between the two currents in the 4-direction. The desired result then follows immediately from a parity transformation in the path integral for the average correlator. Two combinations, $ST$ and $PT$, are zero because an anti-symmetric tensor cannot be constructed out of only one vector. The $SV$ correlator is zero.
as a result of the conservation of the vector current. Formally, this can be proved from
the Ward identities \( <T \bar{u} u(x) \bar{u} \gamma_\mu u(0)> = 0 \) and \( <T \bar{u} u(x) \bar{d} \gamma_\mu d(0)> = 0 \).

Only the \( PA \) and \( VT \) non-diagonal correlators are non-zero. In both cases the con-
tribution of the free propagator to the corresponding spectral function vanishes, and
consequently, the continuum contribution is absent. If 'pion dominance' is used we find

\[
\rho^{PA}_\mu(q^2) = ip_\mu \lambda f_\pi \delta(q^2 - m_\pi^2).
\]

(1.45)

In the \( VT \)-case, the natural approximation is to saturate the intermediate state ex-
pansion by the \( \rho \)-meson. The spectral function at low momenta is then given by

\[
\rho^{VT}_{\mu\nu}(q^2) = 3ip_\nu \lambda_\rho \tilde{f}_\rho \delta(q^2 - m_\rho^2).
\]

(1.46)

1.5. Unflavored correlators

We consider correlators of the unflavored currents defined in (1.3). The general struc-
ture of the correlator of \( \bar{u} \Gamma_A u \) and \( \bar{u} \Gamma_B u \) is given by

\[
\Pi = <\text{Tr} S(x,y) \Gamma_A S(y,x) \Gamma_B > + <\text{Tr} \Gamma_A S(x,x) \text{Tr} \Gamma_B S(y,y) > .
\]

(1.47)

The characteristic feature is the appearance of a two-loop or annihilation diagram. The
one-loop diagram is absent if the flavor content of the two currents is different.

The vector and axial traces, \( \text{Tr} \gamma_\mu S(x,y) \) and \( \text{Tr} \gamma_\mu \gamma_5 S(x,y) \) are not well defined for
\( y \to x \). Both traces have to be regularized by the inclusion of the path ordered exponential
\( P \exp -i \int_x^y A_\mu dx_\mu \) and the subtraction of the free propagator (which only contributes in
the vector channel). In the axial case there will be an additional contribution due to the
chiral anomaly.

However, according to Zweig’s rule, all flavor changing interactions are strongly sup-
pressed in vector and axial channels. Therefore, we will ignore the corresponding con-
tributions in our calculations below and approximate the unflavored axial and vector
correlators by its one-loop contributions only.

The situation is different in the scalar, pseudoscalar and tensor channels. In this case
the annihilation diagrams are finite. Let us consider the correlator of the current

\[
\vec{j}_{u}^{\text{even}}(x) = \bar{u} \Gamma^{\text{even}} u(x),
\]

(1.48)
with $\Gamma^{\text{even}}$ equal to 1, $i\gamma_5$ or $\sigma_{\mu\nu}$, and a similarly defined current $j_d^{\text{even}}$. Only the annihilation diagrams contribute:

$$\Pi^{\text{even}}_{ud} = - <\text{Tr}S_u(x, x)\Gamma^{\text{even}} \text{Tr}S_d(y, y)\Gamma^{\text{even}} > .$$

(1.49)

According to the identity (1.16) we have

$$\text{Tr} \Gamma^{\text{even}} S(x, x) = - \text{Tr} \Gamma^{\text{even}} S(x, x),$$

(1.50)

which implies that the trace is real in the pseudoscalar case and purely imaginary in the scalar and the tensor channel. We thus find that the two-loop pseudo-scalar correlator is negative for $x$ close to $y$ whereas the contribution from this region to the scalar and tensor 2-loop correlator is positive.

Next, we derive the non-diagonal spectral functions for the scalar and pseudoscalar flavored currents. In the scalar case $\bar{u}u$ and $\bar{d}d$ can be written in terms of the sum and difference of $j^\sigma$ and $j^\delta$ (see eq. (1.3)). Because the contribution of the free quark propagator vanishes we find the spectral function

$$\rho^{S, 2-\text{loop}}(s = q^2) = (2\pi)^3\delta^4(q) <0|j^\sigma|0><0|j^\sigma|0> + \lambda_2^2\delta(s - m_\sigma^2) - \lambda_5^2\delta(s - m_\sigma^2),$$

(1.51)

which indeed gives rise to a positive correlator at short distances (the first term in the r.h.s. has to be manipulated carefully).

The $\bar{u}i\gamma_5u$ and $\bar{d}i\gamma_5d$ currents can be written as a linear combination of $j^1$, $j^8$ and $j^\pi$ which allows to express the two-loop pseudoscalar correlator as

$$\Pi^{P}_{ud} = - \frac{1}{2} <0|Tj^{\pi^0}(x)j^{\pi^0}(y)|0> + \frac{1}{6} <0|Tj^8(x)j^8(y)|0> + \frac{1}{3} <0|Tj^1(x)j^1(y)|0>.$$  

(1.52)

Again, the free quark contribution to the correlator vanishes which allows to write to corresponding spectral function as

$$\rho^{P 2-\text{loop}}(s) = - \frac{1}{2} \lambda_8^2\delta(s - m_8^2) + \frac{1}{6} \lambda_2^2\delta(s - m_\sigma^2) + \frac{1}{3} \lambda_5^2\delta(s - m_1^2).$$  

(1.53)

Let us, for simplicity, consider the limit in which all quark masses are equal. Then $m_{\pi^0} = m_8$ and $\lambda_{\pi^0} = \lambda_8$. The $SU(3)-$singlet spectral function, $\rho^{P}$, is then given by the
\[
\rho'(s) = \rho^{P\,1-loop}(s) + 3\rho^{P\,2-loop}(s) \\
= \lambda_1^2 \delta(s - m_1^2) + \frac{3s}{8\pi^2} \theta(s - E_0 P),
\]

where the 1-loop pseudoscalar spectral function coincides with the flavored pseudoscalar spectral function obtained in eq. (1.34). Several important conclusions can be drawn from this result. First, the continuum threshold in the \( \eta' \)–channel is the same as for other pseudoscalar mesons. Second, the \( \eta' \)–mass is determined by the two-loop spectral function, whereas the mass of the other pseudoscalar mesons follows form the 1-loop spectral function. In the \( \eta' \)–channel the contributions of all other pseudoscalar mesons cancel because of a detailed compensation between 1-loop and 2-loop contribution. At relevant distances of the order of 1 \( fm \) the pion correlator is several orders of magnitudes larger than \( \eta' \)–correlator, and therefore both 1-loop and 2-loop contributions have to be calculated extremely accurately in order to determine the mass of the \( \eta' \) particle.

Finally, we show that in the quenched approximation the positivity of the \( \eta' \)–correlator is violated. The simplest argument goes as follows. Suppose, that all quark masses are equal. Then the singlet \( \eta' \)–correlator is given by the sum of the one-loop pseudoscalar correlator and \( N_f \) times the two-loop correlator \( \Pi^{P}_{ud} \). In the quenched approximation both contributions do not depend on \( N_f \), and it is clear that if the two contributions balance each other at one value of \( N_f \), the correlator becomes negative for \( N_f + 1 \).

### 1.6. Qualitative role of the instanton-induced interactions

Before going into the numerical calculations of multi-instanton effects, let us briefly remind the reader of the qualitative effects of the one-instanton gauge field fluctuations. The main physical phenomenon, tunneling through a barrier that separates gauge fields of different topology, is related to a rearrangement of the light quark states: some of them ‘dive into the Dirac sea’ during this process, whereas others ‘emerge’ from it. This process is described by the so called ‘t Hooft effective Lagrangian \[8\]

\[
L_{\text{eff}} \sim \prod_f (\bar{q}_f \psi_0)(\bar{\psi}_0 q_f)
\]

(1.55)
where the quark fields $q_f$ of various flavors ($f = u, d, s$) are projected onto the so called fermionic zero mode $\psi_0(x)$, which is the solution of the Dirac equation $\hat{D}\psi_0(x) = 0$ in the field of an instanton. The zero modes play the role of wave functions of the states, in which the quarks are produced or absorbed during tunneling. They depend in a known way on collective coordinates of the instanton, its position, size and color orientation.

An important observation is that the chirality of the zero modes is directly related to the topological charge of the gauge field: there is only a left-handed fermion zero mode for an instanton and only a right-handed fermion zero mode for an anti-instanton. For anti-quarks, the opposite holds true. As a result, if one ignores quark masses, it is impossible to close the loop and return to the emitted quark because its chirality flipped on the way. Therefore, in the chiral limit, 'solitary' instantons are absent. They only appear in groups of zero total topological charge, exchanging produced quarks among themselves. Obviously, in order to break chiral symmetry, such 'group' has to be infinitely large, percolating the entire space-time volume of the Universe. The low-lying fermion modes become 'collectivized', and occupy the entire space-time as well, instead of being localized near a particular instanton.

At this point, one can ask the question: in what sense the 'one instanton approximation' can be formulated, if instantons cannot be isolated and the low-lying fermion modes are delocalized? Well, provided that the instantons still form a reasonably dilute ensemble, the delocalized modes can be written as a superposition of localized zero modes, $\psi_\lambda(x) = \sum_n C^\lambda_n \psi_0(x - z_n)$, where $z_n$ is the center of the instanton. The coefficients behave as $C^\lambda_n \sim 1/\sqrt{V}$, but, because $\psi(x) \sim 1/x^3$ for large $x$, only nearby instantons contribute to $|\psi_\lambda(x)|^2$, and for a reasonably dilute ensemble $|\psi_\lambda(x)|^2$ is peaked at the center of the instantons. For the same reason, provided the end points $x$ and $y$ (for which correlator is evaluated) are located within the radius of a single instanton $j$, the quark propagator is dominated by the term $[\psi_0(x - z_j)\psi_0^*(y - z_j)][\sum_\lambda |C^\lambda_j|^2/(\lambda + im)]$. The first factor, projected on quark fields, reproduces the original 't Hooft interaction. At the same time, the second factor, (which for a single instanton in the perturbative vacuum is $1/m$ (because there is only one zero mode) is now different: it has a non-singular chiral limit resulting from the many different modes that contribute to the sum over $\lambda$. Since this factor does not depend on $x - y$, it and can be ignored for qualitative studies of instanton effects, as
if only one instanton was present.

A phenomenological discussion of the instanton-induced four-fermion interaction was first made for the pseudoscalar channels in [9], and similar studies have eventually led to the 'instanton liquid model' [10], [11]. Below we make it even more convincing, by considering instanton-induced effects in the scalar channels as well [1].

The specific flavor and chirality-flipping structure of the 't Hooft interaction leads to the following important conclusions:

1. The effect of the interaction is of first order for the scalar and pseudoscalar correlators, but is absent in the vector and axial channels.
2. The sign of the corrections is opposite for scalar and pseudoscalar channels.
3. Due to its flavor structure, \( \bar{u}u \bar{d}d \), the one-instanton corrections have opposite signs for isospin 1 and 0 correlators, which are essentially given by \( (\bar{u}u \pm \bar{d}d)^2 \).

The first point agrees with phenomenological observations [12], according to which much stronger deviations from the asymptotic freedom are seen in spin-zero channels than in spin-one channels. The last two statements provide a qualitative understanding of the short distance behavior of all spin-zero correlators. Ignoring for simplicity the strange quarks, we have four channels with different parity and isospin, which we denote as \( \pi (I^P = 1^-) \), \( \eta' (I^P = 0^-) \), \( \sigma (I^P = 0^+) \) and \( \delta (I^P = 1^+) \). One observes that in the one-instanton approximation the corrections in the \( \pi \) and \( \sigma \) channels have the same sign as free propagation, which implies a larger correlator and an attractive interaction. The opposite, with a repulsive interaction, is found in the two other cases, the \( \eta' \) and the \( \delta \). This behavior of these four channels is indeed seen in the real world, which shows that, by studying instanton-induced interactions, we are on the right track.

In principle, one could proceed along these lines, in which the 't Hooft interaction is treated as an effective multi-fermion interaction that can be taken into account systematically in perturbation theory. However, we do not follow this approach, because we will include all such diagrams by an exact diagonalization of the Dirac operator in the subspace of zero-modes.

2. The 'flavored' mesonic channels
In this and the following section we present our numerical results obtained for an ensemble of 128 instantons and 128 anti-instantons in a periodic box of $(3.36)^3 \times 6.72 \, fm^4$. The end points of the correlator are taken along the longest axis, in which way most of the finite size effects due to periodicity are suppressed even for the maximum considered distance of $1.615 \, fm$. The average correlators have been obtained by averaging over 50 different gauge field configurations, distributed according to the invariant measure of the integration over the collective coordinates, and, for each configuration, by averaging over 100 different points $x$ for a fixed value of $x - y$. The size of the instantons is taken fixed at $\rho = 0.35 \, fm$.

The definition of the quark propagator in the 'random instanton vacuum' was explained in detail in [5]. Let us only comment here that for the present calculations the quark masses have the value of 10 $MeV$ for $u$ and $d$, and 140 $MeV$ for $s$. The former is still larger than the physical mass\footnote{Therefore, for a physical quark mass of $m_u + m_d = 11 \, MeV$, our pion mass should be $m_\pi^{\text{calculated}} = m_\pi^{\text{experimental}}(20/11)^{0.5}$.} but it is about as small as we can tolerate in order to be insensitive the the artifacts \footnote{Therefore, for a physical quark mass of $m_u + m_d = 11 \, MeV$, our pion mass should be $m_\pi^{\text{calculated}} = m_\pi^{\text{experimental}}(20/11)^{0.5}$.} in the spectrum of the Dirac operator at small virtualities. The latter is not fitted to any particular quantity but is taken as a 'common sense value'.

In order to check the volume dependence of the correlators, we also have performed calculations with fewer instantons. A significant volume dependence is only found for the pion and, to a much lesser extent, for the kaon. For example, decreasing the number of instantons from 256 to 64 at constant density, raises the pseudoscalar correlator at 1.5 $fm$ by about a factor of 2.5, whereas 2/3 of this factor occurs by decreasing the number of instantons from 128 to 64. Under the same conditions, the kaon correlator changes by only 25 percent. The volume variation from decreasing the number of instantons by a factor of two is inside the error bars for all other mesonic correlators that have been studied.

2.1. Two pseudoscalar channels: $\pi$ and $K$
In this section we consider correlators of the flavored currents
\[ j_\pi = \frac{1}{\sqrt{2}}(\bar{u}i\gamma^5 u - \bar{d}i\gamma^5 d), \]  
\[ j_K = \bar{u}i\gamma^5 s. \]  
(2.1a)  
(2.1b)

It is obvious, that as these pseudoscalars are the lowest excitations of the QCD vacuum, the Goldstone modes, they can tell us a great about its long-range structure, like for example the chiral condensate. However, unbeknownst is that these channels provide very significant information about the short-range vacuum structure as well.

Another, but closely related, exceptional feature of the pseudoscalar correlators is that all components of the propagator enter with the same positive sign (see eq. (1.18)). Therefore, they are the easiest to measure: no cancellations take place. For the same reason, though, the systematic error due to the finite volume is largest.

In Fig. 1, our results for the \( \pi \) and \( K \) correlators are compared with phenomenological expectations. As usual, they are normalized with respect to the free quark correlator,
\[ \Pi_0(x) = \frac{3}{\pi^4 x^6}, \]  
(2.2)

which corresponds to the simple 1-loop diagram. Therefore, the fact that our points approach 1 at small distances is a general consequence of 'asymptotic freedom', and should be independent of the non-perturbative vacuum fields. What is studied is the deviations from this simple behaviour.

The solid (phenomenological) curves have been obtained from the expression
\[ \frac{\Pi^P(x)}{\Pi_0^P(x)} = \lambda_P^2 D(m_P, x) \frac{\pi^4 x^6}{3} + \frac{\pi^2 x^6}{4} \int_{E_0} \infty dE E^3 D(E, x)(1 + \frac{\alpha_S(E)}{\pi}), \]  
(2.3)

that can be obtained from the spectral function (1.34) and the free propagator (see \[1\] for an explanation of the perturbative correction). The parameters involved, namely the coupling of the pseudoscalar current to these particles, and the ‘continuum threshold’ \( E_0 \) above which the quarks are assumed to be produced as free ones are given by:
\[ \lambda_\pi = (480 \, MeV)^2, \quad \lambda_K = 1.24 \lambda_\pi, \quad E_0 = 1.4 \, GeV. \]  
(2.4)

Unfortunately, the uncertainty in the absolute value of quark condensate translates into an about 50% uncertainty of the couplings. The uncertainty in \( E_0 \) is important only in a
very limited range of distances of about $0.3 - 0.5 \, fm$, where resonance and non-resonance contributions are comparable. For a more detailed analysis of these parameters we refer to ref. [1].

The dashed line, shown for comparison, represents the 'vacuum dominance' approximation (see above). Although this correction has 'good intentions', the correct sign, it describes neither our points nor the experimental curve more accurately. More elaborate OPE formulae fail in these cases as well (see [12], and [1]).

The parameters of (2.3) without the perturbative correction can also be obtained from a fit to our data points. The results are shown in Table 1. For the pion both the mass and the coupling constant agree perfectly with experiment. For the K meson the mass is right where it should be, but the coupling constant is too small. For a comparison to other channels and a general discussion of the parameters we refer to section 4.2.

2.2. Vector channels: $\rho$, $K^*$ and $\phi$

We use vector currents with the same normalization as in [1]:

$$j_{\mu}^{\rho} = \frac{1}{\sqrt{2}} \bar{u} \gamma_{\mu} u - \bar{d} \gamma_{\mu} d \quad \text{or} \quad \bar{u} \gamma_{\mu} d,$$  

$$j_{\mu}^{\omega} = \frac{1}{\sqrt{2}} \bar{u} \gamma_{\mu} u + \bar{d} \gamma_{\mu} d,$$  

$$j_{\mu}^{\phi} = \bar{s} \gamma_{\mu} s,$$  

$$j_{\mu}^{K^*} = \bar{u} \gamma_{\mu} s.$$  

The phenomenological expectations of the correlators can be obtained from the spectral function (see section 1.2). However, in order to make contact with experiment, one uses that the spectral density is proportional to the ratio

$$R_i(s) = \frac{\sigma_{e^+e^- \rightarrow i}(s)}{\sigma_{e^+e^- \rightarrow e^+e^-}(s)}$$  

\[2.6\]

Note, that in this case the disagreement with experiment is not affected by the uncertainty in the value of the quark condensate, because both the contribution of the pion and this correction are proportional to its square.

\[3\]Warning: other authors, e.g. [3], use a different normalization. Any definition that includes the charge of the quarks is confusing if one compares different channels.
of the $e^+e^-$ annihilation cross section into hadrons with quantum numbers $i$ and the cross section for muon pair production. Neglecting the muon mass, the latter is just

$$\sigma_{e^+e^- \rightarrow \mu^+\mu^-} = (4\pi\alpha^2/3s).$$

The trace if the spectral function in the vector channel is related to this ratio by

$$2\pi\rho^V_{\mu\mu}(s) = -s\frac{R^V(s)}{\pi}, \quad (2.7)$$

which after substitution in the dispersion relation leads to

$$<0|Tj^V_\mu(x)j^{V\dagger}_\mu(0)|0> = -\frac{1}{2\pi^2} \int ds D(s^{1/2}, x) R^V(s). \quad (2.8)$$

This relation can be used to extract the ‘experimental definition’ of the correlation function.

Information on strange vector channel, marked by $K^*(892)$, is derived from the weak decay process $\tau \rightarrow \nu_\tau + \text{hadrons}$ due to Cabbibo mixing. We also include the double-strange $\phi$ channel. Strictly speaking, it is related to both one-loop and two-loop diagrams, but according to experimental observations (the famous ‘Zweig rule’), all flavor-changing transitions are strongly suppressed in vector channels, and the two-loop contribution cannot be a significant correction. All final states with a kaon pair are included (see details in [1]) in the phenomenological curves. Also the $\omega$ and $\rho$ correlators only differ by strongly suppressed flavor changing two-loop diagrams. Within our accuracy this difference is not visible, and therefore we only show results for the $\rho$—correlator.

Numerical results for the vector mesons are shown by the solid lines in Fig. 2. As usual, correlators are plotted in a normalized way, namely as $\Pi_{\mu\mu}(x)/\Pi^0_{\mu\mu}(x)$, where $\Pi^0_{\mu\mu}(x) = -6/\pi^4x^6$ corresponds to the simple loop diagram, describing free propagation of a pair of massless quarks.

The general striking observation in all vector channels [3] is that the contributions to the spectral function of the lowest meson and other states complement each other in such a way, that the ratio $\Pi(x)/\Pi^0(x)$ remains close to one up to the distances as large as $1.5 \text{ fm}$. This phenomenon, called superduality, generalizes a well-known ‘quark-meson duality’ argument to much larger distances. From an experimental point of view it means some ‘fine tuning’ of the parameters of all vector states. On the other hand, in the language of field theory it implies that in the vector channels, for some (so far unknown!)
all non-perturbative corrections to the free quark propagators miraculously cancel each other, until the correlator has dropped by a few orders of magnitude.

Results of our calculations are compared with the phenomenological expectations (solid curves in Fig. 2) obtained from the experimental $R$–ratio as discussed above. First of all, 'superduality' is qualitatively reproduced by the calculated points. Keeping in mind that the correlators drop by several orders of magnitude over the range of distances under study, even the quantitatively agreement is actually very good. The fact that all our curves are somewhat below the experimental ones came as no surprise: in the 'instanton vacuum' under consideration there are no radiative corrections to correlators. To first order, they are known to be $(1 + \alpha_s(x)/\pi + ...)$, which means an increase in the value of the correlator value by about 10 percent. If the next terms produce a smaller or similar correction, the agreement with data is actually nearly perfect.

Let us also comment that the splitting between the three vector channels of flavor content $\bar{u}d$, $\bar{u}s$ and $\bar{s}s$ is surprisingly small up to $x = 1.5$ fm. As the strange quark mass is generally far from being negligible (e.g. in the way it affects the propagators), this observation means that the $O(m_s)$ terms have the tendency to cancel among themselves. The theoretical reasons for this are unclear. At large distances, $x \sim 1.2$ fm, a splitting between the different vector mesons appears, but the order is wrong: in the instanton vacuum the $\rho$ meson is heavier than the $\phi$ meson. One may also look at our calculated results in a different way: instead of comparing them to the 'phenomenological correlator', they can be fitted directly to some parametrization, which allows us to extract some 'hadronic parameters'. As in the previous section, we use the standard three-parameter expression, which now looks as follows:

$$\frac{\Pi_{\mu\mu}^V(x)}{\Pi_{\mu\mu}^0(x)} = 3\lambda^2 D(m,x) \frac{\pi^4 x^6}{6} + \frac{\pi^2 x^6}{4} \int_{E_0}^\infty dEE^3 D(E,x).$$  \hspace{1cm} (2.9)$$

The coupling constant is defined as in eq. (1.37) which differs from the definition that is used for extracting coupling constants from the data. We think that in this way the comparison between the various channels can be made most unbiasedly.

---

5 As was discussed briefly in section 1.6, a partial explanation of this phenomenon is provided by the instanton-based theory: at the one-instanton level there is no correction to vector (and axial) correlators, as opposed to scalar and pseudoscalar ones.

6 The 1-loop radiative corrections are included for the $\rho$ and the $a_1$ channel, but are absent in case of the $\phi$ meson, see [1].

7 In [3] dimensionless coupling constants are used, which are related to ours e.g. as $\lambda_\rho = \sqrt{2m_\rho^2/g^\rho}$
In particular, in [1] we have used the matrix elements of electromagnetic current

\[
< 0 | j^{\text{e.m.}}_\mu | \text{meson} > = f^{\text{e.m.}}_{\text{meson}} \epsilon_\mu \tag{2.10}
\]

which then lead to a universal relation containing a well measured quantity, the electromagnetic branching width:

\[
f^{\text{e.m.}}_{\text{meson}}^2 = \frac{3m_{\text{meson}} \Gamma(\text{meson} \to e^+ e^-)}{4\pi \alpha^2}. \tag{2.11}
\]

As the electromagnetic current contains factors related to the electric charges of quarks

\[
j^{\text{e.m.}}_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s + \cdots = \frac{1}{\sqrt{2}} j^\rho_\mu + \frac{1}{\sqrt{2}} j^\omega_\mu - \frac{1}{3} j^\phi_\mu + \cdots \tag{2.12}
\]

we have to remove them which, in absence of mixing between the different channels, leads to the coupling constants

\[
\lambda_\rho = \sqrt{2} f\rho m_\rho = (409 \pm 5 \text{ MeV})^2, \tag{2.2a}
\]
\[
\lambda_\omega = 3\sqrt{2} f\omega m_\omega = (390 \pm 5 \text{ MeV})^2, \tag{2.2b}
\]
\[
\lambda_\phi = 3 f_\phi m_\phi = (492 \pm 15 \text{ MeV})^2. \tag{2.2c}
\]

The \(K^*\) coupling constant, \(f_{K^*}\), is defined in the same way as \(f_\rho\) and can be obtained from the weak \(\tau-\text{decay}\) (see [1]). For \(\lambda_{K^*}\) one finds

\[
\lambda_{K^*} = \sqrt{2} f_{K^*} m_{K^*} = (448 \pm 25 \text{ MeV})^2. \tag{2.14}
\]

In table 1 these accurately known coupling constants are compared with our calculated results.

### 2.3. Axial correlators: \(a_1, K_1\) and \(f_1\)

Currents and correlators in these channels are defined in exactly the same way as for the three vector channels in the previous section with the obvious substitution \(\gamma_\mu \rightarrow \gamma_\mu^A\).
Again, we assume that Zweig’s rule is very accurate so that annihilation diagrams can be ignored in the $\bar{ss}$-type channel.

However, as we have seen in section 1.2, there is a significant difference: the axial correlator $\Pi^A_{\mu\nu}(q)$ has a non-zero longitudinal part, proportional to $q_{\mu}q_{\nu}$. For zero quark masses it only contains the contribution of massless Goldstone modes, pions etc.. For non-zero quark masses, axial currents are not conserved, and a massive pion contributes to the spectral function. Thus, generally speaking, in the axial cases one has two different correlation functions which can be split in, for example, transverse and longitudinal parts.

In this work we concentrate on one particular combination: the trace $\Pi^A_{\mu\mu}$. The reason is that at this moment we are mainly interested in the $a_1$ meson, and not in the pion which, in this combination, is suppressed by the small parameter $m_\pi^2$ and is unimportant at not too large distances. The details of the extraction of the relevant $a_1$ parameters from the $\tau$ lepton decay data can be found in [1]. The normalized correlator can be represented as follows:

\[
\frac{\Pi^A_{\mu\mu}(x)}{\Pi^A_{\mu\mu}(0)} = 3\lambda^2_{a_1} D(m_{a_1}, x) \frac{\pi^4 x^6}{6} - f_\pi^2 m_\pi^2 D(m_\pi, x) \frac{\pi^4 x^6}{6} + \frac{\pi^2 x^6}{4} \int_{E_0}^{\infty} dE E^3 D(E, x),
\]

where the last term represents the non-resonance contribution. It is given approximately by its asymptotic form with some threshold. Unfortunately, the $\tau$ lepton is not heavy enough to allow its determination, and we use two 'reasonable' values, $E_0=1.5$ GeV and $E_0 =1.7$ GeV, to show the sensitivity of the correlator to this parameter (see two solid curves in Fig. 3). Also in this case the 1-loop radiative corrections (not displayed in (2.16)) are included in the phenomenological curves.

---

8The reader should be warned, that the sign of the pion term in this formula in [1] is wrong. As a result, this correlation function changes sign and becomes negative at large distances, where the pion term becomes dominant.
Our calculated results for the three flavored axial vector channels are presented by the data points in Fig. 3. As for the channels considered above, the agreement between calculations and phenomenology is amazingly good. All general features, including the fact that axial mesons are significantly heavier than the corresponding vector mesons, are obviously reproduced. Moreover, the numerical agreement with expectations at small $x$ would be nearly perfect, if radiative corrections would have been taken into account.

Let us also comment, that we find surprisingly close results for $\bar{u}d$, $\bar{u}s$ and $\bar{s}s$ type correlators in a wide range of distances, until approximately $x = 1 \text{ fm}$. It means that, as in the case of the vector channels, in the whole region the $O(m_s)$ terms have a tendency to cancel among themselves.

At large distances, above 1 $\text{fm}$, the contribution of the much lighter pseudoscalars (which we tried to suppress, by taking the trace) still becomes dominant. Somewhat surprisingly, the data look good enough for the extraction of their coupling constants and masses. Using the expression given above we have made a five parameter fit with results shown by the dashed-dotted lines in fig. 3.

The fitted parameters of axial mesons and non-resonance continuum are listed in Table 1 and are discussed, together with others, in section 4.2. Here we only present the results and some comments concerning the pseudoscalars.

For the pion the fitted values, corresponding to minimal $\chi^2$ (and to the curve in Fig. 3), are as follows:

$$m_{\pi} = 252 \pm 15 \text{MeV}, \quad f_{\pi} = 110 \text{MeV}. \quad (2.17)$$

The latter value can be directly compared with the well known experimental result of 131 $\text{MeV}$. However, in order to compare the calculated pion mass to its experimental value, one should make an adjustment for the fact that we cannot take as small quark masses as required by the real world. We remind that our value for $(m_u + m_d)_{\text{calc}} = 20 \text{MeV}$, while experimentally $(m_u + m_d)_{\text{exp}} \approx 11 \text{MeV}$, and therefore we should extrapolate the calculated pion mass to $222 \sqrt{11/20 \text{MeV}} = 178 \pm 10 \text{MeV}$ which is not far from the observed value in the pseudoscalar channel of $142 \pm 14 \text{MeV}$. In other words, the model does reproduce the right magnitude of the mass-independent coefficient $m_{\pi}^2/(m_u + m_d)$ for small enough quark masses.
The corresponding fitted values of the parameters in (2.16) for the kaon are:

\[ m_K = 464 \pm 15\text{MeV}, \quad f_K = 100\text{MeV}. \]  

(2.18)

The mass is right on the experimental value, but the decay constant is about a factor 2 smaller than needed.

For the 'purely strange' pseudoscalar \( \eta_s \) we find from the fit:

\[ m_{\eta_s} = 553 \pm 15\text{MeV}, \quad f_{\eta_s} = 90\text{MeV}. \]  

(2.19)

As this combination is actually some mixture of \( \eta \) and \( \eta' \), with the two-loop effects so far ignored (see below), we do not think these results have a direct physical meaning, but we present them for completeness.

2.4. Two 'flavored' scalars

At this point we would like to make some remarks on the accuracy of our calculations. All calculations have been performed in the quenched approximation where the small eigenvalues are not suppressed by the fermion determinant. In the one-instanton approximation the vector and axial correlators have nevertheless a finite chiral limit, and we expect that the enhancement in the spectrum for small virtualities \[ \beta \] does not significantly alter the correlator. However, in the case of the scalar, pseudoscalar and tensor correlators, the chiral limit cannot be taken in the one-instanton approximation, and therefore we should not be surprised if we are unable to reproduce the experimental results with the physical values of the light quark masses. Because the pseudoscalar correlator is the sum of all 'hidden components' the relative error is still small, and, in section 2.1, we have found that the phenomenological curves can be obtained with somewhat larger quark masses. On the other hand, in the case of the flavored scalar correlators, the first-order instanton-induced interaction is 'repulsive at small distances (see section 1.6). At larger distances, other corrections (\textit{e.g.} multi-instanton ones) become important, but these correlation functions are small as compared to those considered previously. At a distance of the order of 1 \text{fm} a cancellation of a couple of orders of magnitude has to
take place between the spin flipping and the spin non-flipping components of the propagator. We necessary will find a large systematic error not only because of the anomaly in the spectrum at small virtualities, but also because we use approximate propagators. In the tensor channel the instanton induced interactions are attractive. Consequently, we have much less cancellations and much more accurate results can be obtained. The non-diagonal correlators are less singular in the chiral limit (see, for example, the vacuum dominance results), and also in this case we expect smaller errors than in the scalar case.

After these warnings, we are ready to confront our results for the scalar correlator. Our results for the $\bar{u}d$ and $\bar{u}s$ channels are shown by the data points in Fig. 4. One observes that our data decrease more rapidly than in any other channel considered above. In fact, our data even ‘overshoot’ this trend. The ‘random instanton vacuum’ produces a too strong repulsion in these channels: the correlators become negative (since we do not trust the results for the correlator in this region we did not even plot those points). This violates the positivity of the scalar correlator and it points to a defect of the model. Indeed, as we have stressed above, all calculations have been performed in the quenched approximation.

Apart from the $a_0(980)$ particle (which is believed to be mainly an $\eta\pi$ or a $\bar{K}K$ system with flavor composition essentially given by $\bar{u}d\bar{s}s$), no flavored scalar mesons are listed by Particle Data Group, so, probably, they do not exist at all! However, continuum (multi-meson) states should of course exist, and, for comparison, we have plotted some ‘expectations’ in Fig. 4. We have assumed the absence of resonances and have pushed continuum threshold up to $E_0 = 2$ GeV, which we consider as a kind of its upper limit. Therefore, the solid curve represents the most rapidly decreasing correlator, on the edge of what seems reasonable.

2.5. Tensor and non-diagonal correlators

This section deals with a set of correlators which are only rarely discussed in literature, e.g. they are not mentioned in reviews on the subject. However, as we are going to show shortly, they are actually quite useful, because all of them can be described very well by
the dominance of one intermediate state. Therefore, they are much more suitable for the
determination of hadronic masses than all standard correlators.

The first example is the correlator of two ‘tensor currents’ $j_{\mu\nu} = \bar{u}\sigma_{\mu\nu}d$ (and also
its strange analog, $d \rightarrow s$). As the two indices are anti-symmetric in this case, the
intermediate physical states are not the usual tensor mesons, of course, but rather some
specific polarization states of vector and axial mesons.

Contrary to all other diagonal correlators considered above, these tensor correlators
do not have a free-quark contribution at small distances: the corresponding loop diagram
vanishes kinematically because of the Dirac trace alone. (Therefore, one hopes to obtain
the contribution of resonances in a much ‘cleaner’ way.) The coupling constants were
defined in (1.43), and, in this case the normalized correlator simply follows from eq.
(1.44) with result

$$\frac{\Pi^{T}_{\mu\nu\mu\nu}(x)}{\Pi^{T}_{0}(x)} = 2\pi^{4}x^{6}\tilde{f}_{\rho}^{2}m_{\rho}^{2}D(m_{\rho}, x) + 2\pi^{4}x^{6}\tilde{f}_{a_{1}}^{2}m_{a_{1}}^{2}D(m_{a_{1}}, x). \quad (2.20)$$

Our data are shown in Fig. 5 together with an approximate fit, corresponding to
$\tilde{f}_{\rho} = 100\ MeV, \tilde{f}_{a_{1}} = 0.0\ MeV$ and experimental $\rho$ and $a_{1}$ masses. Thus, this correlator
is completely dominated by a single rho meson pole.

Our next topic is the non-diagonal vector-tensor correlator. This is the only non-
diagonal correlator that was considered in the literature in connection with QCD sum
rules [13, 14, 15]. It is related to the baryon magnetic moments and other electromagnetic
properties of hadrons. In particular, the susceptibility of the quark condensate $\chi(q^{2})$ is
defined by the small-$q$ limit of the integral

$$\int d^{4}xe^{iqx}\Pi^{VT}_{\mu} = 3i\chi(q^{2}) < \bar{q}q > q_{\mu}. \quad (2.21)$$

The correlator can also be expressed in terms of the spectral function. When substitute
the $\rho$–meson dominance formula (see eq. (1.46)) in eq. (1.9) we find the sum-rule

$$\chi(q^{2}) = \frac{\lambda_{\rho}\tilde{f}_{\rho}}{< \bar{q}q > (q^{2} + m_{\rho}^{2})}. \quad (2.22)$$

For phenomenological values of $m_{\rho}, \lambda_{\rho}$ and $\tilde{f}_{\rho}$ and our fitted value of $\tilde{f}_{\rho}$ we find $\chi(0) =
-2.0\ GeV^{-2}$ which differs from the corresponding value $\chi(0) = -3.3\ GeV^{-2}$ obtained in
[14]. This deviation originates in the coupling constant $\tilde{f}_{\rho}$ which was estimated with the
help of vacuum dominance in \([14]\), i.e., \(\lambda_\rho \bar{f}_\rho = 2 < \bar{q} q >\), resulting in \(\bar{f}_\rho = 165 \text{ MeV}\) as opposed to our value of 100 \(\text{MeV}\). A more elaborate QCD sum rule calculation taking into account two additional \(\rho\)–meson resonances \([14]\) gives a slightly larger value of \(\chi_{\text{e.m.}} = -4.4 \text{GeV}^{-2}\) (at the normalization point \(\mu_0 = 1 \text{GeV}\)).

More explicitly, they used the spectral function

\[
\rho_\mu(q^2) = 3i q_\mu [c_\rho \delta(q^2 - m_\rho^2) + c_\rho' \delta(q^2 - m_\rho'^2) + c_\rho'' \delta(q^2 - m_\rho''^2)],
\]

(2.23)

where the three terms contain the contributions from the three lowest \(\rho\) resonances, with masses of 780, 1250 and 1600 \(\text{MeV}\), respectively. The constants are related to the coupling constants and are equal to \(c_\rho = 2.9|< \bar{q} q >|, c_\rho' = -1.1|< \bar{q} q >|\) and \(c_\rho'' = 0.6|< \bar{q} q >|\).

The ratio of the corresponding \(VT\) correlator to the free pseudoscalar correlator \(\Pi_0^P\) given by (for \(x_0 > 0\))

\[
\frac{\Pi_{\mu\nu\mu\nu}^V(x)}{\Pi_0^P(x)} = -\pi^4 x^6 [c_\rho \frac{dD(m_\rho, x)}{dx_\mu} + c_\rho' \frac{dD(m_\rho', x)}{dx_\mu} + c_\rho'' \frac{dD(m_\rho'', x)}{dx_\mu}]
\]

(2.24)

is represented by the solid curve in Fig. 6a. Let us remind, there are no free parameters and that we have an absolute normalization. In the same figure, these predictions are compared with our results. One should note that although the two calculations are completely different in nature the agreement is quite reasonable.

At small distances the agreement is exceptionally good. Of course, this is not accidental: QCD sum rules are based on the OPE, and should do better in this region. In fact, the OPE dictates that the short distance behaviour of \(\Pi^{VT}\) is given by the vacuum dominance formula (1.28b). Therefore, the excellent agreement at very small distances is essentially due to the fact that we reproduce the right magnitude of the quark condensate.

A natural generalization of electromagnetic susceptibility of the quark condensate to the case of a \textit{weak external axial current} \([16]\) leads to another non-diagonal correlator: the pseudoscalar-axial correlator. It can be used to evaluate the axial coupling constants of various baryons (see also \([17]\)), but presently we are interested in the correlator by itself.

One can certainly argue that in this case one knows at least its long-range part, which follows from the pion intermediate state. Moreover, there are no unknown constants because the coupling of the pion to axial current is nothing else but the famous pion decay constant (see eq. (1.38)). Using 'pion dominance' for this non-diagonal correlator
we find the expression
\[
\frac{\Pi^{PA}_\mu(x)}{\Pi^{P}_0(x)} = \frac{\pi^4 \alpha^6}{3} \left[ -\lambda_\pi f_\pi \frac{dD(m_\pi, x)}{dx_\mu} \theta(x_0) + \lambda_\pi f_\pi \frac{dD(m_\pi, x)}{dx_\mu} \theta(-x_0) \right].
\]
(2.25)

In Fig. 6b, we show the numerical data together with the above formula with a fitted value of \(f_\pi\) which shows agreement at all distances. If one takes \(m_\pi = 191\ MeV\) (because \(m_u + m_d = 20\ MeV\) and \(\lambda_\pi = (510\ MeV)^2\), as fitted from pseudoscalar correlator, the result for pion decay constant reads
\[
f_\pi = 110 \pm 17\ MeV
\]
(2.26)
which is the same value as obtained from the axial channel.

The final remarks of this chapter deal with scalar-vector correlator. Our calculations show results that are consistent with zero at all distances. The difference between SV and PA correlators is seen from the following argument. Suppose that some intermediate scalar (or pseudoscalar) particle contributes to it, then we have a contribution of the type (2.25). Let us now take another derivative \(\partial_\mu\) of the correlator: when it acts on the propagator it produces the mass squared of the intermediate state. But, in the chiral limit, both the vector and axial currents are conserved: the result should be zero. The dilemma is then as follows: (i) the coupling constant is zero, or (ii) the particle mass is zero. In the PA channel the second alternative is realized, whereas in the SV case only the first one is possible.

Therefore, a zero SV correlator implies that the vector current is conserved in our calculations (see section (1.4)). Of course, this holds true rigorously if a complete set of states is used in the propagator. In practice, we work with an incomplete set of states, so the vanishing results are actually a non-trivial test of our calculations.

3. Correlators of some 'unflavored' currents

3.1. Pseudoscalar channels \(\eta\) and \(\eta'\)

As explained above, both one-loop diagrams and 'two-loop' (or annihilation) diagrams should be included in the 'unflavored' channels (see section 1.5). It implies that the propagators should be defined at zero separations between its end points, or that the
regular part of the propagator should be separated from the singular part. Generally speaking, this is non-trivial: for example, the regular part of the axial current should reproduce correctly the axial anomaly. Fortunately, significant simplifications take place for pseudoscalar currents because the Dirac trace kills all chirality non-flipping terms in the propagator. In practice, the 'double-loop' diagram was calculated from the zero-mode part of the propagator only, which is non-singular by construction.

Let us first explain the definition of the correlators. For two massless flavors, we can distinguish two different components

\[ A(x) = < T \bar{u} i \gamma_5 d(x) \bar{d} i \gamma_5 u(0) >, \]  
\[ B(x) = < T \bar{u} i \gamma_5 u(x) \bar{d} i \gamma_5 d(0) >, \]  

related to one and two-loop diagrams respectively. Inclusion of a massive s quark brings in three more functions,

\[ C(x) = < T \bar{u} i \gamma_5 u(x) \bar{s} i \gamma_5 s(0) >, \]  
\[ D(x) = < T \bar{s} i \gamma_5 s(x) \bar{s} i \gamma_5 s(0) >, \]  
\[ E(x) = < T \bar{u} i \gamma_5 s(x) \bar{s} i \gamma_5 u(0) >, \]  

with \( D(x) \) possessing both contributions, and \( C(x), E(x) \) only one of them. All 5 functions can be obtained from correlators in the four light mesonic channels \( \pi, K, \eta \) and \( \eta' \), plus one non-diagonal correlator \( \eta - \eta' \).

Although we do not really use the strange quark mass as a small parameter, we still use the \( SU(3) \) singlet and octet currents defined in eq. (1.3). In this section we focus on the following 3 correlators

\[ \Pi^{11}(x) = < T j_1(x) j_1(0) >= \frac{1}{3} (2A + 4B + 4C + D), \]  
\[ \Pi^{88}(x) = < T j_8(x) j_8(0) >= \frac{1}{3} (A + 2B - 4C + 2D), \]  
\[ \Pi^{18}(x) = -< T j_1(x) j_8(0) >= \frac{\sqrt{2}}{3} (-A - 2B + C + D). \]
Note, that in the $SU(3)$ symmetric case ($m_s = m_u = m_d$), $B = C$ and $D = A + B$. Therefore $\Pi^{88} = A$ and $\Pi^{18} = 0$, as they should: the $\eta$ and $\pi^+$ channels are the same, and there is no mixing. However, due to the chiral anomaly, $\Pi^{11} = A + 3B \neq \Pi^{88}$.

In Fig. 7, our results are compared with the phenomenological expectations [1]. The diagonal $\eta$ and $\eta'$ correlators are shown in Fig. 7a. The obvious observation is that while the model is doing an excellent job in reproducing the $\eta$ curve, it certainly 'overshoots' in the $\eta'$ case: the positivity condition of the correlator is violated. In the $\eta$ case the two-loop contributions nearly cancel among themselves (exact in the chiral limit), whereas all one-loop contributions add up with the same sign. The $\eta'$-meson remains massive in the chiral limit which requires a delicate balance between the positive one-loop and negative two-loop contributions (see section 1.5). The latter are closely related to the correlations of the topological charge density, which are not implemented properly in the 'random model'.

In Fig. 7b we show our results for $\eta - \eta'$ mixed correlator. Let us remind that idea of mixing was introduced in the days of non-relativistic quark model. One can imagine that the singlet and octet states $\eta_1$ and $\eta_8$, produced by the singlet and octet currents at low energies, are a mixture of the physical $\eta$ and $\eta'$ states. The non-diagonal correlator should look as

$$\Pi^{18} = \lambda_\eta \lambda_{\pi'} \cos \theta \sin \theta [D(m_\eta, x) - D(m_{\pi'}, x)] + \text{(mixing in the continuum)}. \quad (3.4)$$

The curve shown for comparison uses $\lambda_\eta = 1.3\lambda_\pi$, $\lambda_{\pi'} = 0.7\lambda_\pi$ and a angle of $\theta = 20^\circ$. The sign and qualitative behaviour is right, but the angle (or couplings?) seems to be larger than expected. However, much more work is needed in order to understand this mixing phenomenon properly.

### 3.2. The isoscalar scalar (or $\sigma$) channel

This channel is different from all others considered above, because the corresponding current

$$j^\sigma = \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d) \quad (3.5)$$
has a non-zero vacuum expectation value. As a result, the correlator in question has a non-zero disconnected part

$$\Pi^\sigma(x) \rightarrow 2|<\bar{q}q>|^2$$

that dominates at large distances.

Therefore, our data are plotted in two different ways. In Fig. 8a we show the measured signal, normalized, as usual, to the perturbative correlator. The solid line shows the phenomenologically expected contribution of $2\pi^4|<\bar{\psi}\psi>|^2x^6/3$ (for $<\bar{\psi}\psi>=-(240\,MeV)^3$). One observes that the correlator exceeds the perturbative correlator by a big factor, and that our points are somewhat above the phenomenologically expected (solid) curve. The reason is that our model produces a slightly larger value of the condensate, namely, $|<\bar{\psi}\psi>| = 2.22 fm^{-3} = (257 MeV)^3$, which perfectly agrees with the large distance behavior of our points.

However, the disconnected signal is not very interesting by itself. The physical excited states are related to the ‘connected’ part of the correlator. Unfortunately, as shown in Fig. 8b, the accuracy at large distances is greatly reduced after the subtraction of the disconnected part. It is important to note, that the connected part of the correlator is still quite large. This means that the effective interquark interactions are indeed strongly attractive. Of course, this is hardly surprising because in the one-instanton approximation this should be the case (see section 1.6).

As can be seen from the dashed lines in Fig. 8b, the connected part can be very well fitted by our standard three-parameter fit, resonance plus continuum as in (1.34). The following values of the parameters were obtained:

$$m_\sigma = 543\,MeV, \quad \lambda_\sigma = (500\,MeV)^2, \quad E_0 = 1160\,MeV$$

(3.7)

Those parameters show that the ‘instanton vacuum’ contains the ‘sigma meson’ the famous enhancement in the $\pi\pi$ cross section which plays a prominent role in nuclear physics and is the basis of the ‘sigma model’ of Gell-Mann and Levy.

Moreover, the particular parameter determined best of all is the sigma coupling constant. It is interesting to note, that it happens to be the largest of all mesons. It implies

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9 Of course, on the basis of these data one cannot answer the old question, whether the ‘sigma-meson’ is a resonance or just a well-correlated $\pi\pi$ pair, but one definitely sees that in order to reproduce our results for the correlator one should have a peak in spectral density around 600 MeV.
a very compact sigma particle, obviously due to the very strong attractive interaction in this channel.

4. Discussion

4.1. Comparison with other works

In the discussion above we have compared our results for correlation functions with phenomenology. Now, we are going to compare them with other theoretical results. There generally are three different categories: (i) the operator product expansion (OPE) in the context of QCD sum rules with many parameters (or ‘condensates’); (ii) other instanton-based calculations; and (iii) lattice QCD calculations.

We will not discuss the OPE-based expressions, partly because it was done in detail in [1], and partly because we already discussed its simplest version (based on the quark condensate contribution) in section 1.4. For the tensor and non-diagonal correlators we have given a more elaborate discussion because no other sources of information are available.

The instanton-induced effects were originally discussed in [9, 10] to the first-order in instanton density. In section 1.6 we have given a qualitative discussion which is even somewhat wider than those works, because it also includes the scalar channels.

Further developments include the analytic attempts to account for the multi-instanton effects by means of summing a subset of diagrams [24, 28, 18, 25]. The first numerical study of correlators in the interacting instanton vacuum [26] showed surprisingly good results. We do not discuss them here for two reasons. (i) Within their much larger statistical and systematical uncertainty the old calculations agree with the present ones. (ii) A much more accurate and detailed study of the ‘interacting instantons’ vacuum in under way and will be discussed elsewhere [19].

In the last part of this section we compare our results with the first lattice calculations of point-to-point correlation functions, which were recently reported in [4]. Results for four non-strange channels, shown in Fig. 9, show a qualitative agreement in all cases. In the $\pi$ and $\rho$ channels one even finds, somewhat curiously, that the phenomenological curve is in between the two calculations. The axial channel agrees very well, while in the isovector
scalar ($\delta$) case our results definitely 'overshoot' the repulsion (see discussion above).

As one compares these two sets of data, resulting from completely different theoretical approaches, one should keep in mind that the plotted curves do not really represent the measured signal: for convenience of representation the correlators are normalized with respect to the perturbative correlators. Actually the measured correlators change by several orders of magnitude, so their deviation at the largest distances by, say, a factor of two is in fact a surprisingly good agreement. As we will show in our next paper of this series [27], this agreement is observed in the baryonic channels as well.

The reason for the intriguing agreement between the two set of correlators can and should be investigated further. The most obvious way to proceed is, of course, to 'cool' the lattice configurations until one is left with mostly instantons, and then to redo the measurements of the correlators. If the resulting correlators happen to be about the same, our main point the dominant role of the instanton-based quark interactions in the QCD vacuum, would finally be demonstrated explicitly.

4.2. The fitted masses and coupling constants

In our discussion above we have concentrated on correlation functions as such and compared results directly with phenomenological curves [1] based on the integrated physical spectral density or lattice data [4]. We do believe it is the best way to confront the model under consideration with reality.

The extraction of hadronic parameters from correlation functions by fitting is easily beset with systematic errors, for example, because they are very sensitive to the parametrization of the fitting function. We have performed two types of fits: (i) using the data points with statistical errors only (for results given in Table 1), and (ii) with statistical errors provided they are larger than 5 percent, otherwise they are put to be 5 percent. The difference is related to a different weight assigned to the small $x$ region, for which the statistical errors are very small. Generally, the difference in the fitted values is of the order of 10 percent, except in special cases like the $\rho$ mesons, where the $\chi^2$ has a complicated valley in the parameter space, so that small changes in the calculated points
may drastically change the fitted parameters. Lattice data typically resolve such problem by fitting only the long-distance part of the correlators, ignoring the small $x$ behaviour.

Having said this, we still would like to report the results of such fits. All points, from $x = 0$ until the largest $x$, are fitted with 'pole plus continuum' expressions (two poles for axials), with only statistical errors included. A subset of results for diagonal 'flavored' correlators is listed in Table 1. Let us make the following comments.

- The masses of the lowest resonances in these channels, ranging from light pions to heavy axials, are very well reproduced by the model. The largest deviation, for the $\rho$ meson, is about 15 percent.

- The coupling constants $\lambda$ are somewhat smaller than the experimental values. It is probably due to the fact that the model does not have confinement, which effectively cuts off the tails of the wave functions and makes the particle more compact. The largest deviation is observed for the $\phi$ meson which is However, with our definition\(^{10}\) of $\lambda$, the model correctly reproduces the following natural trend: the coupling constants decrease from the 'most attractive' channels ($\sigma, \pi$) to the 'least attractive' ones (vectors and axials). In the most 'repulsive' channels (flavored scalars) the resonances are absent altogether.

- In contrast to the parameters of the lowest excitations, the position of 'continuum threshold' is surprisingly channel independent. Unfortunately, we do not have much phenomenological information about these important parameters (except for vector channels). However, if we assume that the threshold coincides with the position of the next to lowest state, the primed resonance, these numbers can be compared with the experimentally known 'primed' resonances. Indeed, in all known cases they are in the region around 1200 MeV, as the fitted thresholds indicate. Thus, one may speculate that the RIV model does predict the correct position of the 'continuum threshold' as well.

\(^{10}\)Let us remind the reader, that it differs from standard definition in literature. This allows us to compare different channels in a more straightforward way.
4.3. Summary

We have reported a detailed calculation of a wide set of mesonic correlation functions in the ‘random instanton vacuum’. The original idea was to use a crude model as a benchmark for further studies of interacting instantons, with the parameters (the instanton density and a typical radius) as suggested a decade ago [10], without any further adjustments.

However, the results obtained show that this model works amazingly well, in particular for the pseudoscalar correlators. Asymptotic freedom is recovered at small $x$, and at intermediated distances the correlation function can be parametrized by a spectral function with one pole (two for axials) and a continuum contribution. At the moment it is not clear why the phenomenological correlators are reproduced so well in spite of the apparent absence of such usual ingredients as confining forces and perturbative effects.

In fact, not only the qualitative behaviour of all correlators at distances as large as 2 $fm$ is reproduced, but in a few cases, when a quantitative analysis is possible, we found agreement with the physical parameters at the 10-20 percent level. Thus, it is certainly a useful tool, so wider use of the model seems to be justified.

At the same time, in two channels with a strong repulsive interaction, the $\eta'$ and the $\delta$, the results are not satisfactory: the model correctly predicts such repulsion but overestimates its magnitude. We are looking forward to more realistic calculations with 'interacting instantons' and hope to improve on this situation.

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| channel $j^P$ | $m_{\text{res}}$ [MeV] | $\sqrt{\lambda}$ [MeV] | $E_0$ [MeV] | comment       |
|-------------|-------------------|-----------------|-------------|--------------|
| $\pi, 0^-$  | 142 ± 14          | 510±20          | 1360 ± 100  | this work    |
|             | 138               | 480             |             | phenomenology|
| $K, 0^-$    | 482 ± 12          | 467±20          | 1350±50     | this work    |
|             | 495               | 595             |             | phenomenology|
| $\eta, 0^-$ | 500               | 462             | 1250        | this work    |
|             | 548 .8            |                 |             | phenomenology|
| $\sigma, 0^+$| 543               | 500             | 1160        | this work    |
| $\rho, 1^-$ | 950±100           | 390±20          | 1500±100    | this work    |
|             | 780               | 409± 5          |             | phenomenology|
| $K^*, 1^-$  | 860±15            | 341±20          | 1300±50     | this work    |
|             | 892               | 448±25          |             | phenomenology|
| $\phi, 1^-$ | 850±50            | 280±20          | 1000±40     | this work    |
|             | 1020              | 492±15          |             | phenomenology|
| $a_1, 1^+$  | 1132±50           | 305±20          | 1100±50     | this work    |
|             | 1260              | 400             |             | phenomenology|
| $K_1, 1^+$  | 1170±50           | 302±20          | 1120±50     | this work    |
|             | 1270              |                 |             | phenomenology|
| $f_1, 1^+$  | 1210±50           | 293±20          | 1200±50     | this work    |
|             | 1285              |                 |             | phenomenology|

**Table 1**

The mass $m_{\text{res}}$, the coupling constant $\lambda$ and the threshold energy $E_0$ of some mesons in absolute units fitted according to the three-parameter formula for the spectral function. The pion mass is adjusted to the physical quark masses under the assumption that it is proportional to $(m_u + m_d)^{1/2}$. Whenever available the parameters are compared with experimental numbers.
Figure Captions

Fig. 1. Pseudoscalar correlation functions with quantum numbers of the pion and the kaon normalized to free massless quark correlator in the same channel plotted versus the distance $x$ in $fm$. Our results are represented by the points. The phenomenological curves (as derived in [1]) are given by the two solid lines. The dashed line corresponds to the 'vacuum dominance' estimate (see text). Two dash-dotted lines show the fit described in the text.

Fig. 2. Correlators for vector channels with quantum numbers of $\rho$, $K^*$ and $\phi$ mesons normalized to the free massless quark correlator in the same channel plotted versus the distance $x$, in $fm$. Our results shown by squares, triangles and hexagons, respectively, should be compared with phenomenological expectations [1] (solid curves). The dashed line corresponds to the 'vacuum dominance' estimate.

Fig. 3. Correlators for axial channels normalized to the free massless quark correlator in the same channel plotted versus the distance $x$ in $fm$. Squares, triangles and hexagons show our results for the $\bar{u}d$, $\bar{u}s$ and $\bar{s}s$ channels, in this order. The first one should be compared with the phenomenological expectation given by the region in between the two solid curves [1]. The 'vacuum dominance' estimate is represented by the dashed curve. Three dash-dotted lines show the fits described in the text.

Fig. 4. Correlators for isovector scalar channels normalized to the free massless quark correlator in the same channel plotted versus the distance $x$ in $fm$. Squares and triangles show our results for the $\bar{u}d$ and $\bar{u}s$ currents, respectively. The 'lower limit' to the phenomenological expectation (see text) is shown by the solid curve, while the dashed one corresponds to the 'vacuum dominance' estimate. The dash-dotted lines just guide the eye.

Fig. 5. Correlators of the tensor current $j_{\mu\nu}$ normalized to the free pseudoscalar massless quark correlator plotted versus the distance $x$ in $fm$. Squares, triangles and hexagons show our results for the $\bar{u}d$, $\bar{u}s$ and $\bar{s}s$ channels, respectively. The 'vacuum dominance' estimate is represented by the dashed curve, and the dash-dotted curve shows the fit.
described in the text.

Fig. 6. The vector-tensor (a) and the pseudoscalar-axial (b) correlators in units of the pseudoscalar free quark correlator plotted versus the distance $x$ in $fm$. The parameters of the solid curve in (a) are those obtained from QCD sum rules [14]. The dash-dotted line in (b) shows the fit described in the text.

Fig. 7. Correlators for the isoscalar pseudoscalar channels $\eta$ and $\eta'$ normalized to free pseudoscalar massless quark propagator plotted versus distance $x$ in $fm$. In figure (a) we show the correlators of the $SU(3)$ singlet and octet currents (open and closed squares), while (b) represents the singlet-octet mixed correlator. Phenomenological expectations (see [1]) are shown by the solid curves. The dash-dotted curve in (a) is our fit for $\eta$, and the dashed one shows the contribution of the $\eta'$ to the singlet correlator according to [1].

Fig. 8. Correlators for the isoscalar scalar $\sigma$ channel normalized to the free massless quark correlator in the same channel plotted versus the distance $x$ in $fm$. The full correlator is shown in Figure (a), whereas (b) only has the connected part. The solid curve in (a) is the phenomenological expectation for the disconnected part. The long and short-dashed curves in (b) show the total correlator and the $\sigma$—meson contribution to the fit, respectively.

Fig. 9. Comparison between our results for the correlation functions (triangles) and the first lattice results [4] (squares). The four channels considered correspond to the isovector (e.g. $\bar{u}d$) pseudoscalar, vector, scalar and axial the cases. The phenomenological result (solid lines) were derived in [1]. All correlators are normalized with respect to the free quark correlator in the same channel and the distance on the $x$—axis is in $fm$.  

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