Direct numerical reconstruction of conductivities in three dimensions using scattering transforms

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Abstract
A direct three-dimensional EIT reconstruction algorithm based on complex geometrical optics solutions and a nonlinear scattering transform is presented and implemented for spherically symmetric conductivity distributions. The scattering transform is computed both with a Born approximation and from the forward problem for purposes of comparison. Reconstructions are computed for several test problems. A connection to Calderón’s linear reconstruction algorithm is established, and reconstructions using both methods are compared.

1. Introduction
The reconstruction of conductivity distributions in two or three dimensions from measurements of the current density-to-voltage map is known as electrical impedance tomography, or EIT, and has applications in medical imaging, nondestructive testing and geophysics. For the 3D bounded domain considered here, medical applications include head imaging and the detection of breast tumors. See, for example, [Hol05] for a survey of clinical applications of EIT. In this work, we consider a bounded domain in $\mathbb{R}^3$ and present a direct reconstruction algorithm and its numerical implementation on the unit sphere. The theoretical foundation of the method dates back more than 20 years to a series of papers by Sylvester–Uhlmann [SU87], Novikov [Nov88], Nachman–Sylvester–Uhlmann [NSU88] and Nachman [Nac88]. The algorithm makes use of complex geometrical optics (CGO) solutions to the Schrödinger equation and uses the inverse scattering method. This is described in detail in section 2 of this paper.

The inverse conductivity problem was first formulated mathematically by Calderón [Cal80] as follows. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a simply connected, bounded domain with a smooth boundary $\partial \Omega$, and let $\gamma \in L^\infty(\Omega)$ denote the conductivity distribution. Assume that
there exists $C > 0$ such that for $x \in \Omega$, $C^{-1} \leq \gamma(x) \leq C$. The electric potential $u$ arising from the application of a known voltage to the boundary of $\Omega$ is modeled by the generalized Laplace equation with the Dirichlet boundary condition

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \quad u = f \text{ on } \partial \Omega. \tag{1}$$

The Dirichlet-to-Neumann map $\Lambda_\gamma$ is defined by

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega}. \tag{2}$$

Thus, $\Lambda_\gamma$ represents static electrical boundary measurements: it maps an applied voltage distribution on the boundary to the resulting current flux through the boundary. Calderón [Cal80] posed the question of whether the conductivity $\gamma$ is uniquely determined by the Dirichlet-to-Neumann map, and if so, how to reconstruct the conductivity. He gave an affirmative answer to the uniqueness question for the linearized problem and gave a reconstruction algorithm for that case. His algorithm is described in section 2.3 of this paper.

The uniqueness question for $\gamma \in L^\infty(\Omega)$ is still open in $\mathbb{R}^3$, but has been solved recently by Astala and Päivärinta [AP06a, AP06b] for a bounded domain in $\mathbb{R}^2$, sharpening the previous results due to Nachman [Nac96] in which $\gamma \in W^{2, p}(\Omega)$, $p > 1$, and Brown and Uhlmann [BU97] in which $\gamma \in W^{1, p}(\Omega)$, $p > 2$. In three dimensions, the uniqueness problem was solved for smooth conductivities in [SU87]. At the time of this publication, in $\mathbb{R}^3$ the uniqueness results with lowest regularity are [BT03] with $\gamma \in W^{3/2, p}(\Omega)$, $p > 2n$, and [PPU03] with $\gamma \in W^{3/2, \infty}(\Omega)$.

Most existing 3D EIT reconstruction algorithms are linear or iterative, minimizing a functional that describes the nearness of the predicted voltages to the measured data in a given norm with one or more regularization terms. In contrast, the algorithm presented here is direct and fully nonlinear. It is similar to the 2D D-bar algorithms based on the works [Nac96] and [BU97, KT04], which were first implemented in [MS03, SMI00, SMI01, Knu03]. In these initial works the Born approximation to the CGO solutions is used in the computation of the scattering transform. It was used successfully on experimental tank data in, for example, [IMNS04, EM09] and human chest data in [IMNS06, DM10]. This inspired the approach in section 2.2 of this paper in which the Born approximation is used in the 3D direct algorithm. For further reading on 2D D-bar algorithms, the reader is referred to [KLMS07] in which the application to discontinuous conductivity distributions is specifically addressed, and [KLMS09] in which a rigorous regularization framework is established using the full scattering transform. Calderón’s method has also recently been used for the reconstruction from experimental data in both 2D [BM08] and 3D [BTJS08].

In this work we assume the conductivity $\gamma \in C^2(\Omega)$, we take $\Omega$ to be the unit sphere in $\mathbb{R}^3$ and assume $\gamma = 1$ near $\partial \Omega$. The smoothness assumption on $\gamma$ is necessary. The choice $\gamma = 1$ near $\partial \Omega$ avoids the extra step of reconstructing $\gamma$ on the boundary. A method for doing this is given in [Nac88], and if the values of $\gamma$ and $\frac{\partial \gamma}{\partial \nu}$ have been computed on the boundary, then the Dirichlet-to-Neumann map for the Schrödinger equation, as defined in section 2.1, can be computed by equation (4). However, reconstructions that instead use the best constant-conductivity fit to the measured data have been used successfully on experimental data in 2D (see, for example [IMNS04, IMNS06]), which inspire the simplification here. See [ST] for an implementation in 2D of the boundary reconstruction method described in [Nac96] or [NRST10] for the reconstruction of $\gamma$ and $\frac{\partial \gamma}{\partial \nu}$ on $\partial \Omega$ in $\mathbb{R}^3$. The choice of $\Omega$ as the unit sphere is convenient for our numerical simulations, since there we restrict ourselves to spherically symmetric conductivities. We stress in particular that the theory is valid in more complex geometries.
The outline of the paper is as follows. In section 2.2 we describe a direct reconstruction algorithm with a linearizing assumption tantamount to a Born approximation. That approach is referred to as the $t^{\exp}$ approach, consistent with the notation used in the 2D D-bar algorithms. An explicit connection to the linearized method of Calderón is established in section 2.3. The reconstruction of the conductivity in the 2D D-bar method described in the works above is achieved by taking a small frequency limit in a D-bar equation for the CGO solutions to directly obtain $\gamma(x)$. In contrast, here we have to take a high complex frequency limit. A D-bar equation for the 3D problem is utilized in [CKS06], resulting in a promising but more complicated approach than the one studied here. The numerical implementation of that approach is left for future work. In section 3 we consider the case of spherically symmetric conductivities and show symmetry properties in the scattering transform. We also show how the Dirichlet-to-Neumann map can be represented and approximated in that case. Details on the numerical implementation are found in section 4. Numerical examples are found in section 5.

2. The reconstruction methods

In this section we describe the nonlinear reconstruction method, the $t^{\exp}$ approach, and the relationship to Calderón’s linearized method.

2.1. The nonlinear reconstruction method

The method was developed in [NSU88, Nac88, Nov88, SU87]; here we provide a brief outline. The reader is referred to [Nac88] for rigorous proofs. The equations closely parallel those of the 2D problem (note that [Nac88] precedes that work), and so readers familiar with that case will recognize the notation and functions involved.

The initial step is to transform the conductivity equation into a Schrödinger equation. Indeed, if $u$ satisfies (1) then $v = \gamma^{1/2}u$ satisfies

$$(-\Delta + q)v = 0 \text{ in } \Omega \quad \text{with} \quad q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}.$$  (3)

Note that $q = 0$ near $\partial\Omega$. The Dirichlet-to-Neumann map for equation (3) is defined by

$$\Lambda_q f = \frac{\partial v}{\partial \nu}|_{\partial\Omega},$$

where now $v$ satisfies (3) with $v|_{\partial\Omega} = f$. In general the maps $\Lambda_\gamma$ and $\Lambda_q$ are related by

$$\Lambda_q = \gamma^{-1/2} \left( \Lambda_\gamma + \frac{1}{2} \frac{\partial \gamma}{\partial v} \right) \gamma^{-1/2}.$$  (4)

The assumption that $\gamma = 1$ in a neighborhood of $\partial\Omega$ simplifies (4) to $\Lambda_q = \Lambda_\gamma$. To define the CGO solutions, introduce a complex frequency parameter $\xi \in \mathbb{C}^3$ and define the set

$$\mathcal{V} = [\mathbb{C}^3 \setminus \{0\} : \xi \cdot \xi = 0].$$  (5)

Then, $e^{i\xi} \xi$ is harmonic in $\mathbb{R}^3$ if and only if $\xi \in \mathcal{V}$. For $\xi \in \mathbb{R}^3$, introduce the subset of $\mathcal{V}$ given by

$$\mathcal{V}_\xi = \{ \xi \in \mathcal{V} : (\xi + \xi)^2 = 0 \}.$$  (6)

Note that $\xi \cdot \xi = (\xi + \xi)^2 = 0$ gives an explicit characterization of $\mathcal{V}_\xi$ in terms of an auxiliary vector $\xi^\perp \in \mathbb{R}^3$ with $\xi^\perp \cdot \xi = 0$. Indeed suppose $\xi, \xi \in \mathbb{R}^3$. Then, $\xi = \xi + i\xi \in \mathcal{V}_\xi$ if and
only if
\begin{equation}
\begin{aligned}
\zeta_R &= -\xi/2 + \xi^\perp, \\
\zeta_I \cdot \xi &= \zeta_I \cdot \xi^\perp = 0, \\
|\zeta_I| &= |\zeta_R|.
\end{aligned}
\end{equation}

Since \( q = 0 \) in a neighborhood of \( \partial\Omega \), one can extend \( q = 0 \) into \( \mathbb{R}^3 \setminus \overline{\Omega} \). The CGO solutions \( \psi(x, \zeta) \) to the Schrödinger equation solve
\begin{equation}
( -\Delta + q(x) ) \psi(x, \zeta) = 0, \quad x \in \mathbb{R}^3, \quad \zeta \in \mathcal{V},
\end{equation}
and behave like \( e^{i \xi \cdot \cdot} \) for \( |\zeta| \) large. More precisely, define
\begin{equation}
\mu(x, \zeta) = \psi(x, \zeta) e^{-i x \cdot \zeta}.
\end{equation}
Then \( \mu - 1 \) approaches zero in a certain sense as either \( |x| \) or \( |\zeta| \) tends to infinity, see [Nac88, SU87]. Note that \( \psi(x, \zeta) \) grows exponentially for \( x \cdot \text{Im} \zeta < 0 \). The function \( \mu \) satisfies
\begin{equation}
( -\Delta - 2i \zeta \cdot \nabla + q ) \mu(x, \zeta) = 0 \text{ in } \mathbb{R}^3.
\end{equation}

Denote by \( G_\zeta \) the Faddeev Green’s function defined by
\begin{equation}
G_\zeta(x) = e^{i \xi \cdot \cdot} g_\zeta(x), \quad \text{where } g_\zeta(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{i \xi \cdot \cdot}}{|\xi|^2 + 2\xi \cdot \zeta} \, d\xi,
\end{equation}
where the integral is understood in the sense of the Fourier transform defined on the space of tempered distributions. The functions \( G_\zeta, g_\zeta \) are Green’s function for the Laplace equation and the conjugate Laplace equation respectively, i.e.
\begin{equation}
(\Delta + 2i \zeta \cdot \nabla + q ) g_\zeta = -\delta_0 \quad \text{and} \quad \Delta G_\zeta = -\delta_0.
\end{equation}
Then, (9) and the asymptotic condition for \( \mu \) are equivalent to the Faddeev–Lippmann–Schwinger equation
\begin{equation}
(I + g_\zeta \ast (q \cdot \cdot)) \mu = 1 \text{ in } \mathbb{R}^3.
\end{equation}

Estimates for the operator \( g_\zeta \ast (q \cdot \cdot) \) for large \( \zeta \) ([SU87]) and small \( \zeta \) ([CKS06]) give the existence and uniqueness of \( \mu \) (and therefore \( \psi \)) for any sufficiently large or small \( \zeta \in \mathcal{V} \).

The key intermediate object in the reconstruction method is the so-called non-physical scattering transform of the potential \( q \) defined for \( \xi \in \mathbb{R}^3 \) and sufficiently large or small \( \zeta \in \mathcal{V} \) by
\begin{equation}
t(\xi, \zeta) = \int_{\Omega} e^{-i \xi \cdot (\cdot + \zeta)} \psi(x, \zeta)q(x) \, dx.
\end{equation}
Integrating by parts and assuming that \( \zeta \in \mathcal{V} \) we find that
\begin{equation}
t(\xi, \zeta) = \int_{\partial\Omega} e^{-i \xi \cdot (\cdot + \zeta)} (\Lambda_q - \Lambda_0) \psi(x, \zeta) \, d\sigma(x).
\end{equation}

Thus, we require \( \psi|_{\partial\Omega} \) in order to compute the scattering transform from the Dirichlet-to-Neumann map. It turns out that \( \psi|_{\partial\Omega} \) satisfies a uniquely solvable Fredholm integral equation of the second kind on \( \partial\Omega \) [Nac88, Nov88], namely
\begin{equation}
\psi(x, \zeta) + \int_{\partial\Omega} G_\zeta(x - \tilde{x})(\Lambda_q - \Lambda_0) \psi(\tilde{x}, \zeta) \, d\sigma(\tilde{x}) = e^{i \xi \cdot \cdot}, \quad x \in \partial\Omega.
\end{equation}
Equation (13) together with \( \psi \sim e^{i \xi \cdot \cdot} \) gives a way to compute the Fourier transform \( \hat{q} \) of the potential from the scattering transform by taking the large frequency limit
\begin{equation}
\lim_{|\zeta| \to \infty} t(\xi, \zeta) = \hat{q}(\xi).
\end{equation}

Summary of the reconstruction method.
(i) Solve the boundary integral equation (15) for $\psi|_{\partial\Omega}$.
(ii) Compute $t(\xi, \zeta)$ for $\xi \in \mathbb{R}^3$, $\zeta \in V_{\xi}$ by (14).
(iii) Compute $\hat{q}(\xi)$ from (16).
(iv) Compute $q$ by inverting the Fourier transform.
(v) Compute $\gamma$ by solving $-\Delta \sqrt{\gamma} + q \sqrt{\gamma} = 0$ in $\Omega$, $\sqrt{\gamma}|_{\partial\Omega} = 1$.

We stress that the ill-posedness of the inverse problem is in this algorithm isolated in the first step.

2.2. The reconstruction method using $t^{\exp}$

Inspired by the $t^{\exp}$ approximation in the 2D D-bar method, an analogous approach can be taken in 3D. Approximating $\psi(x, \zeta)$ on the boundary by its asymptotic behavior $e^{ix \cdot \zeta}$ eliminates the need for the ill-posed first step. We define for $\xi \in \mathbb{R}^3$, $\zeta \in V_{\xi}$

$$t^{\exp}(\xi, \zeta) = \int_{\partial \Omega} e^{-i x \cdot (\xi + \zeta)} (\Lambda q - \Lambda_0) e^{ix \cdot \zeta} d\sigma(x).$$

(17)

This approximation is tantamount to a linearization of the first step in the reconstruction algorithm above around $\gamma = 1$. Using $t^{\exp}$ for $t$ in (16) gives the following simple reconstruction algorithm.

(i) Compute $t^{\exp}(\xi, \zeta)$ for $\xi \in \mathbb{R}^3$, $\zeta \in V_{\xi}$ by (17).
(ii) Compute $\hat{q}^{\exp}(\xi) = \lim_{|\zeta| \to \infty} t^{\exp}(\xi, \zeta)$ by (18).
(iii) Compute $\gamma^{\exp}$ by solving $-\Delta \sqrt{\gamma^{\exp}} + q^{\exp} \sqrt{\gamma^{\exp}} = 0$ in $\Omega$, $\sqrt{\gamma^{\exp}}|_{\partial\Omega} = 1$.

It is not guaranteed from the theory that the limit in (18) is well defined. In our numerical simulations we will compute $t^{\exp}(\xi, \zeta)$ for a fixed but large value of $\zeta$. This will numerically define $\hat{q}^{\exp}(\xi)$.

2.3. Calderón’s linearized reconstruction method

Several properties of $t^{\exp}$ can be established from an analysis comparing this approach to that of Calderón. In [KLMS07] a connection was established between the 2D D-bar method based on the global uniqueness proof by Nachman [Nac96] and Calderón’s linearized reconstruction method.

Define a function $u^{\exp}(x, \zeta)$ as the unique solution to the boundary value problem,

$$\nabla \cdot \sqrt{\gamma} \nabla u^{\exp}(x, \zeta) = 0, \quad x \in \Omega, \quad \zeta \in \mathbb{C}^3$$

$$u^{\exp}|_{\partial\Omega} = e^{ix \cdot \zeta}.$$ Integration by parts in equation (17) results in a formula for $t^{\exp}$ defined in terms of $\gamma$ in the interior

$$t^{\exp}(\xi, \zeta) = \int_{\Omega} (\gamma - 1) \nabla u^{\exp}(x, \zeta) \cdot \nabla e^{-i x \cdot (\xi + \zeta)} \, dx.$$ (19)

Write $u^{\exp} = e^{ix \cdot \zeta} + \delta u$ for $\delta u \in H_0^1(\Omega)$. Then, $\delta u$ satisfies

$$\nabla \cdot (\gamma \nabla \delta u) = -\nabla \cdot ((\gamma - 1) \nabla e^{ix \cdot \zeta})$$

and one can estimate

$$\|\delta u\|_{H^1(\Omega)} \lesssim C \|((\gamma - 1) \nabla e^{ix \cdot \zeta})\|_{L^2(\Omega)} \lesssim |\xi| \|\gamma - 1\|_{L^\infty(\Omega)} e^{\xi |B|},$$

(21)
where $R$ is the radius of the smallest ball containing $\Omega$. From (19) we then get

$$\mathbf{t}^{\exp}(\xi, \zeta) = \int_{\Omega} (\gamma - 1) \nabla (e^{ix\cdot\zeta} + \delta u) \cdot \nabla e^{-ix\cdot(\xi+\zeta)} \, dx$$

$$= \int_{\Omega} (\gamma - 1) \nabla e^{ix\cdot\zeta} \cdot \nabla e^{-ix\cdot(\xi+\zeta)} \, dx + R(\xi, \zeta)$$

$$= (\xi \cdot \zeta) \int_{\Omega} (\gamma - 1) e^{-ix\cdot(\xi+\zeta)} \, dx + R(\xi, \zeta),$$

where the remainder term

$$R(\xi, \zeta) = \int_{\Omega} (\gamma - 1) \nabla \delta u \cdot \nabla e^{-ix\cdot(\xi+\zeta)} \, dx.$$

Since $(\xi + \zeta)^2 = \zeta^2 = 0$ we have $-\xi^2 = 2\xi \cdot \zeta$ and hence

$$\mathbf{t}^{\exp}(\xi, \zeta) = -\frac{|\xi|^2}{4} (\gamma - 1)(\xi) + R(\xi, \zeta),$$

where $\hat{\cdot}$ indicates the Fourier transform. The remainder is $O(|\zeta|)$ for $\zeta$ small, which can be seen from (21). This fact suggests that we use the minimal $\zeta \in V_{\xi}$, that is,

$$\zeta_\xi = -\frac{\xi}{2} + i\zeta I \quad \text{with} \quad \zeta I \cdot \xi = 0, \quad |\zeta I| = \frac{|\xi|}{2}.$$

Moreover, with this particular choice we can divide in (22) by $|\xi|^2$ as the following proposition shows.

**Proposition 2.1.** Suppose $\gamma \in L^\infty(\Omega)$. Then,

$$|\mathbf{t}^{\exp}(\xi, \zeta_\xi)| = O(|\xi|^2)$$

for small $|\xi|$.

**Proof.** Note that $|\zeta_\xi|^2 = |\xi|^2/2$. Since $\Lambda_{\gamma}$ maps constant functions to zero and has its range inside the space of mean free distributions in $H^{-1/2}(\partial\Omega)$, we have that for small $|\xi|$,

$$|\mathbf{t}^{\exp}(\xi, \zeta_\xi)| = \int_{\partial\Omega} (e^{-ix\cdot(\xi+\zeta_\xi)} - 1)(\Lambda_{\gamma} - \Lambda_1)(e^{ix\cdot\zeta_\xi} - 1) \, d\sigma(x) \leq C|\xi + \zeta_\xi||\zeta_\xi|,$$

and hence the particular form of $\zeta_\xi$ gives the conclusion. \(\square\)

With the particular choice $\zeta = \zeta_\xi$ in (22) we now neglect the term $R(\xi, \zeta_\xi)$ and divide by $-|\xi|^2$, which gives

$$-2\frac{\mathbf{t}^{\exp}(\xi, \zeta_\xi)}{|\xi|^2} \approx (\gamma - 1)(\xi).$$

Introduce $\chi_B(\xi)$, the characteristic function on the ball $|\xi| < B$. With this function we remove high frequencies and invert the Fourier transform. This results in a linear reconstruction algorithm

$$\gamma^{\text{app}}(x) = 1 - \frac{2}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\mathbf{t}^{\exp}(\xi, \zeta_\xi)}{|\xi|^2} e^{ix\cdot\xi} \chi_B(\xi) \, d\xi.$$  

(23)

This formula is equivalent to the second inversion formula obtained by Calderón [Cal80, p 72].

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In summary the linear reconstruction algorithm consists of two steps.

(i) Compute $t^{exp}(\xi, \zeta \xi)$ by (17).

(ii) Compute the reconstruction by (23).

This method is truly a linearization of the nonlinear reconstruction method outlined in section 2.1. As explained above the mapping $\Lambda \rightarrow t^{exp}$ is for fixed $\zeta$ a linearization of the mapping $\Lambda \rightarrow t$. Moreover, linearization of the final step on page 7 is tantamount to solving for $v = \sqrt{\gamma} - 1$ the equation

$$(-\Delta + q)v = -q \text{ in } \mathbb{R}^3.$$  

Using the Fourier transform the solution is (up to the terms linear in $q$) approximated by

$$\hat{v}(\xi) \approx -\frac{\hat{q}(\xi)}{|\xi|^2}.$$  

Thus, when we use $t^{exp}(\xi, \zeta \xi)\chi_B(\xi)$ as an approximation of $\hat{q}(\xi)$ we find

$$\sqrt{\gamma} - 1 \approx -\frac{1}{(2\pi)^3} \int \frac{t^{exp}(\xi, \zeta \xi)}{|\xi|^2} e^{i\xi \cdot \xi} \chi_B(\xi) d\xi.$$  

Now adding 1 to both sides and squaring up to linear order, i.e. approximating $(1 - p)^2 = 1 - 2p + p^2 \approx 1 - 2p$, give exactly (23).

3. The case of a spherically symmetric conductivity

As a test problem we chose spherically symmetric conductivities in the unit sphere. The benefits of this choice are the symmetric properties of the scattering transform as well as the possibility to formulate the Dirichlet-to-Neumann map explicitly in terms of the eigenvalues and eigenfunctions. Both the symmetric properties of the scattering transform and the eigenvalues and eigenfunctions are described in detail in this section. Moreover, the Dirichlet-to-Neumann map is described explicitly in terms of eigenvalues and eigenfunctions, which in this case are the spherical harmonics. These properties will be derived in this section.

3.1. Symmetry in the scattering transform

The Fourier transform of a spherically symmetric function is spherically symmetric itself. For the scattering transforms $t$ and $t^{exp}$ we have similar properties. In the following we will tacitly assume that $\zeta$ is either small or large. This is needed to circumvent possible exceptional points where $t(\xi, \zeta)$ is not well defined, see [Nac88] and [CKS06].

**Proposition 3.1.** Let $S$ be an arbitrary orthogonal real matrix, and suppose $q(x) = q(Sx)$ for $x \in \Omega$. Then

$$t(\xi, \zeta) = t(S\xi, S\zeta), \quad t^{exp}(\xi, \zeta) = t^{exp}(S\xi, S\zeta).$$

In particular,

$$t(\xi, \zeta_1) = t(\xi, \zeta_2), \quad t^{exp}(\xi, \zeta_1) = t^{exp}(\xi, \zeta_2)$$

for all $\zeta_1, \zeta_2 \in V_\xi$.

**Proof.** Which proves (24). We will prove the result for $t$ only; for $t^{exp}$ the reasoning is similar. By the uniqueness of the CGO solutions, the rotational invariance of the Laplace operator and
the symmetry in $q$ we have $\psi(x, \xi) = \psi(Sx, S\xi)$. Consider the integral (13) and make the change of variables $ST y = x$:

\[
\begin{align*}
t(\xi, \zeta) &= \int_{\Omega} e^{-ix \cdot (\xi + \zeta)} \psi(x, \zeta) q(x) \, dx \\
&= \int_{S\Omega} e^{-iS^T y \cdot (\xi + \zeta)} \psi(S^T y, \zeta) q(S^T y) \, d(S^T y) \\
&= \int_{\Omega} e^{-iy \cdot (\xi + \zeta)} \psi(y, S\xi) q(y) \, dy \\
&= t(S\xi, S\zeta).
\end{align*}
\]

To prove (25) fix $\xi \in \mathbb{R}^3$ and take $\zeta_1, \zeta_2 \in V_\xi$ with $|\zeta_1| = |\zeta_2|$. Then for $j = 1, 2$

\[
\begin{align*}
\zeta_j &= \text{Re}(\zeta_j) + i \text{Im}(\zeta_j), \quad \text{with} \quad \text{Re}(\zeta_j) = -\frac{\xi}{2} + \xi_j^\perp \cdot \xi = 0, \\
\text{Im}(\zeta_j) \cdot \xi &= \text{Im}(\zeta_j) \cdot \xi_j^\perp = 0.
\end{align*}
\]

Define a linear transformation $S$ by $S\xi = \xi, S\xi_j^\perp = \xi_j^\perp, S(\xi \times \xi_j^\perp) = \xi \times \xi_j^\perp$. Then $S$ is an orthogonal real matrix, and (25) follows from (24).

The Fourier transform of a real and even function is real itself. For the scattering transform we have the following equivalent property.

**Proposition 3.2.** Suppose $q(x)$ is real and even. Then for $\xi \in \mathbb{R}^3$, $\zeta \in V_\xi$

\[
t(\xi, \zeta) = t(\xi, \zeta),
\]

with $\overline{\cdot}$ indicating the complex conjugate.

**Proof.** We will again only show the properties for $t$. From equation (9) satisfied by $\mu$ and the uniqueness of the CGO solutions, it follows that if $q$ is even then $\mu(-x, \zeta) = \mu(x, -\zeta)$. Moreover, if $q$ is real, then $\mu(x, \zeta) = \mu(x, -\zeta)$. Hence, if $q$ is both even and real then

\[
\begin{align*}
\overline{t}(\xi, \zeta) &= \int_{\Omega} e^{ix \cdot \xi} q(x) \mu(x, \zeta) \, dx = \int_{\Omega} e^{-i(-x) \cdot \xi} q(x) \mu(x, -\zeta) \, dx \\
&= \int_{\Omega} e^{-iy \cdot \xi} q(y) \mu(y, \zeta) \, dy \\
&= t(\xi, \zeta).
\end{align*}
\]

We now have a corollary for spherically symmetric potentials.

**Corollary 3.3.** Suppose $q$ is spherically symmetric. Then,

\[
\overline{t}(\xi, \zeta) = t(\xi, \zeta), \quad \overline{t^{\text{exp}}}(\xi, \zeta) = t^{\text{exp}}(\xi, \zeta), \quad \xi \in \mathbb{R}^3, \ z \in V_\xi.
\]

**Proof.** There exists an orthogonal real matrix $S$ such that $S(\xi) = \xi, S(\zeta) = \overline{\zeta}$, and hence from (24) we have $t(\xi, \zeta) = t(\xi, \overline{\zeta})$. Equation (26) now implies the result for $t$. For $t^{\text{exp}}$ the result follows similarly.
3.2. Eigenfunctions and eigenvalues for the Dirichlet-to-Neumann map

We use the same ideas for the computation of eigenvalues for the 3D problem that were used for the 2D problem in [SMI00].

**Proposition 3.4.** Let $D$ be the unit disk and suppose $\gamma(x)$ is spherically symmetric. Then the eigenfunctions of $\Lambda_1$ are the spherical harmonics $Y^m_l$.

**Proof.** When $\gamma$ is spherically symmetric it follows from separation of variables that the solution to $\nabla \cdot \gamma \nabla u_{lm} = 0$ with $u_{lm}|_{\partial D} = Y^m_l$ is

$$u_{lm} = R_l(r)Y^m_l(\theta, \phi),$$

where $R_l(r)$ solves an Euler type equation. Thus,

$$\Lambda_1 Y^m_l(\theta, \phi) = \frac{\partial R_l}{\partial r}|_{r=1} Y^m_l(\theta, \phi) = \lambda_l Y^m_l(\theta, \phi).$$

□

Note that $\lambda$ is independent of $m$ since $R_l$ is independent of $m$.

3.3. Approximation of eigenvalues and eigenfunctions of the Dirichlet-to-Neumann map

Next we will consider how to approximate the eigenvalues for the special case of a constant conductivity $\gamma = 1$. The particular form of $R_l$ gives the following result.

**Proposition 3.5.** The eigenvalues of $\Lambda_1$ are given by $\lambda_l = l$. In the case of a piecewise constant radially symmetric conductivity, the eigenvalues can be computed recursively. Suppose $0 = r_0 < r_1 < r_2 < \cdots < r_{N-1} < r_N = 1$ and for $j = 1, 2, \ldots, N$

$$\gamma(x) = \gamma_j > 0, \quad |x| \in [r_{j-1}, r_j].$$

**Proposition 3.6.** Suppose $\gamma$ is given by (29). Then the eigenvalues of $\Lambda_1$ are given by

$$\lambda_0 = 0, \quad \lambda_l = \frac{l(l + 1)}{1 + C_{N-1}}, \quad l > 0,$$

where $C_j = w_j \rho_j + \beta_j \gamma_j, \quad \rho_1 = 1, \quad \rho_j = \frac{C_{j-1} + w_j}{C_{j-1} + \beta_j}, \quad \beta_1 = \frac{l(l + 1)}{1} \quad \text{and} \quad w_j = r_j^{-(2l+1)}$.

**Proof.** Since $Y^m_0$ is a constant, $\lambda_0 = 0$. The solution to $\nabla \cdot \gamma \nabla u_{lm} = 0, \quad u_{lm}|_{\partial D} = Y^m_l$ is given by (27) with $R_l(r) = A_l r^l + B_l r^{-(l+1)}$ for $r_{j-1} \leq r < r_j, \quad j = 1, \ldots, N$. The coefficients $A_l$ and $B_l$ are determined by matching the Dirichlet and Neumann conditions at the $r_j, \quad j = 1, \ldots, N - 1$. The outermost Dirichlet condition (at $r = 1$) gives $1 = A_N + B_N$ which leads to the following eigenvalue expression:

$$\lambda_l = \gamma \frac{\partial u_{lm}}{\partial r}|_{\partial D}$$

$$= lA_N - (l + 1)B_N$$

$$= l - (2l + 1)B_N.$$  

(30)

Moreover, by induction it follows that $A_j = B_j C_{j-1}$ for $j = 2, \ldots, N$. Again using the Dirichlet condition from the boundary $1 = A_N + B_N$, we get $B_N = (C_{N-1} + 1)^{-1}$ which leads to the expression of the eigenvalue as stated in the theorem. □
By [SCII91] if conductivities \( \gamma_L \) and \( \gamma_U \) are such that \( \gamma_L(r) \leq \gamma_U(r) \) for all \( r \), then the eigenvalues \( \lambda_L^l \) and \( \lambda_U^l \) of their corresponding Dirichlet-to-Neumann maps satisfy \( \lambda_L^l \leq \lambda_U^l \). This gives a means for finding lower and upper bounds on the eigenvalues of a smooth function by finding the eigenvalues of the piecewise constant function, \( \gamma_L \) and \( \gamma_U \) that satisfy \( \gamma_L(r) \leq \gamma(r) \leq \gamma_U(r) \).

4. Implementation details

4.1. Numerical method for computing the scattering transform \( t(\xi, \zeta) \)

We compute the scattering transform \( t(\xi, \zeta) \) from the definition (13) as a comparison to the \( t^{exp} \) approximation and to study the reconstructions from an accurate scattering transform. The computation requires that we solve the Lippmann–Schwinger equation (12) for \( \mu(x, \zeta) \). Hence we require

- a method of computation for the Faddeev Green’s function in three dimensions,
- a numerical method for the solution of (12) and
- numerical quadrature for computing \( t(\xi, \zeta) \) from (13).

We describe each of these in turn.

4.1.1. Computation of the Faddeev Green’s function. The Faddeev Green’s function was defined in equations (10). The effect of scaling and rotation of \( \zeta \) on \( G_\zeta \) was analyzed in [CKS06], and it was shown that when \( \zeta \) satisfying \( \zeta \cdot \zeta = 0 \) is decomposed in the form

\[
\zeta = \kappa(k_\perp + ik),
\]

where \( k_\perp, k \in \mathbb{R}^3, |k_\perp| = |k| = 1, k \cdot k_\perp = 0 \) and \( |\zeta| = \sqrt{2}\kappa \),

\[
g_\zeta(x) = \kappa^{-2} g_{k_\perp + ik}(\kappa x).
\]

Furthermore, if \( S \) is an orthogonal real matrix, then

\[
g_\zeta(x) = g_{S\zeta}(Sx).
\]

Combining (32) and (33) yields the formula

\[
g_\zeta(x) = \kappa g_{e_1 + ie_2}(\kappa Sx),
\]

where the first and second columns of \( S \) are \( k \) and \( k_\perp \) respectively. This formula shows that it is sufficient to compute \( g_{e_1 + ie_2} \).

To compute \( g_{e_1 + ie_2} \) we use formula (6.4) of [New89]:

\[
g_{e_1 + ie_2}(x) = \frac{e^{-r(x_1^2 + x_2^2)}}{4\pi r} - \frac{1}{4\pi} \int_s^1 \frac{e^{-ru(x_1^2 + x_2^2)}}{\sqrt{1-u}} J_1(r\sqrt{1-u^2}) \, du,
\]

where \( J_1 \) denotes the Bessel function of the first kind of order 1. Here \( r = |x| \) and \( s = \hat{x} \cdot e_2 = x/|x| \cdot e_2 \). Since the function \( J_1(t)/t \) is continuous on the interval \([0, \infty)\) (in particular at \( t = 0 \)), we will approximate the integral in (35) by a simple midpoint Riemann sum

\[
\int_s^1 \frac{e^{-ru(x_1^2 + x_2^2)}}{\sqrt{1-u}} J_1(r\sqrt{1-u^2}) \, du \approx \sum_{j=1}^N \frac{e^{-ru(j) + x_2^2 - i\xi_1}}{\sqrt{1-u(j)^2}} J_1(r\sqrt{1-u(j)^2})h,
\]

where \( N \) is the number of discretization points, \( h = (1-s)/N \) and \( u(j) = s + (j-1/2)h \), \( j = 1, 2, \ldots, N \).
4.1.2. The computation of complex geometrical optics. Having computed the Faddeev Green’s function we now turn to the numerical solution of the integral equation (12) for $\mu(\cdot, \zeta)$. We will use a method due to Vainikko [Vai00] for solving Lippmann–Schwinger equations; see also [Hoh01, KMS04] for implementations in different contexts. The main idea is to transform (12) to a multiperiodic integral equation in $\mathbb{R}^3$, which can be solved efficiently using FFT.

Let $\Omega_\rho = \{ x \in \mathbb{R}^3 \mid |x| \leq \rho \}$. Then by assumption $\text{supp}(q) \subset \Omega \subset \Omega_1$. Extend the potential $q$ and the Green’s function $g_\zeta$ to $\Omega_2$ such that

$$ q^0(x) = \begin{cases} q(x), & x \in \Omega, \\ 0, & x \in \Omega_2 \setminus \Omega, \end{cases} \quad g^0_\zeta(x) = \begin{cases} g_\zeta(x), & x \in \Omega, \\ 0, & x \in \Omega_2 \setminus \Omega, \end{cases} $$

and then extend $q^0$ and $g^0_\zeta$ to $\mathbb{R}^3$ as periodic functions in all variables with the period equal to 4. Instead of (12) we consider the periodic integral equation

$$ \mu^p(x, \zeta) + \int_{\mathbb{R}^3} g^0_\zeta(x - y)q^0(y)\mu^p(y, \zeta)\,dy = 1. \quad (36) $$

This equation is uniquely solvable since (12) is, and moreover one can show that on $\Omega$ we have

$$ \mu^p(x, \zeta) = \mu(x, \zeta), \quad x \in \Omega. $$

In order to solve (36) numerically define

$$ Z^3_N = \{ j \in \mathbb{Z}^3 \mid -N/2 < j_k < N/2, \ k = 1, \ldots, 3 \} $$

and the computational grid

$$ C_N = hZ^3_N, $$

where $h = 4/N$ specifies the discretization fineness. Define the grid approximation $\phi_N$ of a continuous function $\phi \in C(\Omega_2)$ by

$$ \phi_N(jh) = \phi(jh) $$

and the grid approximation $g_N$ of $g^0_\zeta$ (which is smooth except for a singularity at the origin) for fixed $\zeta$ by

$$ g_N(jh) = \begin{cases} 0, & j = 0 \\ g^0_\zeta(jh), & \text{otherwise}. \end{cases} $$

The convolution operator appearing in (36)

$$ K\phi(x) = \int_{\mathbb{R}^3} g^0_\zeta(x - y)\phi(y)\,dy $$

is now discretized by trigonometric collocation, which, using the discrete Fourier transform $\mathcal{F}_N$, gives

$$ K_N\phi_N(jh) = \mathcal{F}_N^{-1}(\hat{g}^0_N \cdot \hat{\phi}_N). $$

Here $\cdot$ denotes pointwise multiplication. In practice the discrete Fourier transform can be implemented efficiently using FFT (with proper zero-padding) in $\mathcal{O}(N^3 \log(N))$ arithmetic operations. The total discretization of (36) now reads

$$ \mu_N + K_N(q_N\mu_N) = 1, $$

where the right-hand side denotes the vector of size $N^3$ with all entries equal to 1. This discrete linear system is solved numerically in Matlab using the iterative algorithm GMRES [SS86], without setting up a matrix for the linear map $K_N(q_N\cdot)$.
4.1.3. The scattering transform. For purposes of reconstruction we are interested in the limit
of \( t(\xi, \zeta) \), \( \xi \in \mathbb{R}^3 \), \( \zeta \in \mathcal{V}_\zeta \), when \( |\zeta| \) goes to infinity. Thus, a large value of \( \zeta \) is chosen (this is
discussed further in section 5.2), and the approximation \( \mu_N \) to \( \mu(x, \zeta) \) on the computational
grid \( \mathcal{C}_N \) is computed as described in the previous section. For the case when \( t(\xi, \zeta) \) is being
computed by definition (for purposes of comparison with \( t^{\exp} \) and \( \hat{q} \)), it is then straightforward
to evaluate \( \rho(\xi, \zeta) \) by using numerical integration in the following form of (13):

\[
t(\xi, \zeta) = \int_{\Omega} e^{-ix \cdot \xi} \mu(x, \zeta)q(x) \, dx.
\]

(37)

In this implementation we have used a simple midpoint quadrature rule.

4.2. Numerical method for computing \( t^{\exp} \) for spherically symmetric conductivities

For the calculation of \( t^{\exp} \) we expand \( e^{ix \cdot \zeta} \) in terms of spherical harmonics\(^5\) and \( e^{-ix \cdot (\xi + \zeta)} \) in
terms of the spherical harmonics conjugates:

\[
e^{-ix \cdot (\xi + \zeta)} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}(\xi, \zeta) Y_{lm}^* (\theta, \phi),
\]

\[
e^{ix \cdot \xi} = \sum_{k=0}^{\infty} \sum_{n=-k}^{k} b_{kn}(\zeta) Y_{kn} (\theta, \phi).
\]

Using these expansions leads to

\[
t^{\exp}(\xi, \zeta) = \sum_{l,m,k,n} a_{lm}(\xi, \zeta) b_{kn}(\zeta) \int_{\partial D} [Y_{lm}^*(\theta, \phi)]^* (\lambda_k - \Lambda_0) Y_{kn} (\theta, \phi) \, d\sigma.
\]

(38)

In the special case of spherically symmetric conductivities we can use the knowledge of the
eigenvalues of the Dirichlet-to-Neumann maps, \( \Lambda_i Y_{lm}^*(\theta, \phi) = \lambda_i Y_{lm}^* \), to simplify the
calculation of \( t^{\exp} \). In particular we get

\[
t^{\exp}(\xi, \zeta) = \sum_{l,m,k,n} a_{lm}(\xi, \zeta) b_{kn}(\zeta) \int_{\partial D} [Y_{lm}^*(\theta, \phi)]^* (\lambda_i - \lambda_k - k) Y_{kn} (\theta, \phi) \, d\sigma
\]

\[
= \sum_{l,m,k,n} a_{lm}(\xi, \zeta) b_{kn}(\zeta) (\lambda_i - \lambda_k - k) \int_{\partial D} [Y_{lm}^*(\theta, \phi)]^* Y_{kn} (\theta, \phi) \, d\sigma
\]

\[
= \sum_{l,m} a_{lm}(\xi, \zeta) b_{lm}(\zeta) (\lambda_i - l).
\]

(39)

The last equality comes from the orthonormality of the spherical harmonics. Equation (39)
can be easily calculated if the coefficients \( a_{lm} \) and \( b_{lm} \) are available. In this work, these
coefficients were calculated with a software package called ‘S2kit’ which are C routines that
can be accessed from Matlab. Detailed information can be found in [HRKM03].

4.3. Computation of the conductivity

After taking the high frequency limit in (16) and (18) we calculate the inverse Fourier transform
to get \( q(x) \) and \( q^{\exp}(x) \). The integral in the inverse Fourier transform is here computed
numerically using a simple Riemann sum. To get the conductivity \( \gamma \) we need to solve the
boundary value problem \( \Delta \gamma^{1/2} = q^{1/2} \) with \( \gamma^{1/2} \mid_{\partial \Omega} = 1 \). This was realized with the
standard Green’s function for the Laplace equation in three dimensions. Using symmetries
reduces the problem to a single integral.

\(^5\) We use here the normalized spherical harmonics given by \( Y_{lm}^*(\theta, \phi) = N_{lm} P_{lm}^*(\cos \theta) e^{im\phi} \), where \( N_{lm} \) are
normalization factors and \( P_{lm}^* \) are associated Legendre functions.
4.4. Numerical implementation of Calderón’s method

Calderón’s method based on (23) is simply implemented by evaluating the integral using numerical quadrature.

5. Results

5.1. Examples

The conductivity distributions we will use in the examples are smooth, spherically symmetric and constant one near $\partial \Omega$. They are given by

$$\gamma(x) = (\alpha \Psi(|x|) + 1)^2$$

$$\Psi(r) = \begin{cases} 
  e^{-\frac{r^2}{\sigma^2}} & \text{for } -d < r < d \\
  0 & \text{otherwise},
\end{cases}$$

(40)

where $0 < d \leq 1$ is a parameter determining the support of $\Psi$. The parameter $\alpha$ regulates the amplitude of $\gamma$, which is largest at $r = 0$ with amplitude $(\alpha + 1)^2$. A similar function was used [SMI00] as an example for the two-dimensional problem.

5.2. The scattering transform

Let us fix $d = 0.9$, $\alpha = 0.3$ in (40). To approximate the limit of $t(\xi, \zeta)$, $\xi \in \mathbb{R}^3$, $\zeta \in \mathcal{V}_\xi$, when $|\zeta|$ goes to infinity, we compute $t$ for a large value of $|\zeta|$. For purposes of illustration, here we compute $t(\xi, \zeta)$ for fixed $\xi = (10, 0, 0)$ and varying $\zeta \in \mathcal{V}_\xi$ with $8 < |\zeta| < 50$. We use a discretization level in the algorithm corresponding to $N = 2^6$. In addition we compute $t^{\exp}(\xi, \zeta)$ by (39) using the first 30 eigenvalues of the Dirichlet-to-Neumann map. We truncate the sum of the spherical harmonics at $l = 30$, which means we use approximately the first 900 spherical harmonics. As a benchmark we compute $\hat{q}(\xi)$. The results are shown in figure 1.

We know from corollary 3.3 that $t$ and $t^{\exp}$ are real and this is consistent with our numerical results. The data verify that for our example $t(\xi, \zeta)$ converges to $\hat{q}(\xi)$ as $|\zeta| \to \infty$. We observe that $t^{\exp}$ is independent of the magnitude of $\zeta \in \mathcal{V}_\xi$, until it diverges due to numerical instability. The same phenomena appears in other examples and with different values of $\xi$. We believe that this phenomena has to do with the special class of spherically symmetric conductivities considered here.

Next we compare $t(\xi, \zeta)$ and $t^{\exp}(\xi, \zeta)$ for different values of $\xi$. For each $\xi = s[1, 0, 0]$, $s \in [0, 50]$, we fix $\zeta \in \mathcal{V}_\xi$ with $|\zeta| = 50$. We compute $t(\xi, \zeta)$ using a discretization level with $N = 2^6$. $t^{\exp}$ is computed with the parameters as above. As a benchmark we compute $\hat{q}(\xi)$. The results are displayed in figure 2. The difference in $\hat{q}(\xi)$ and $t(\xi, \zeta)$ is very small. $t^{\exp}(\xi)$ is displayed only for $0 \leq |\xi| \leq 32$ since the calculation becomes numerically unstable and blows up for $|\xi| > 32$. One observes a good agreement of all three curves for $|\xi| \geq 20$. Close to $|\xi| = 0$ the approximation $t^{\exp}$ is close to zero and differs from the correct values.

5.3. The reconstructions

Evaluating the inverse Fourier transform of the numerically computed $t^{\exp}(\xi)$ and $t(\xi, \zeta)$ gives two approximations of $\hat{q}(x)$ which are displayed in figure 3. The approximation calculated from $t(\xi, \zeta)$ differs as expected only slightly from the actual value. The approximation $q^{\exp}$ of
Figure 1. $t(\xi, \zeta)$, $t_{\text{exp}}(\xi, \zeta)$ calculated for fixed $\xi = (10, 0, 0)$ and varying $|\zeta|$. Here $d = 0.9$ and $\alpha = 0.3$.

Figure 2. Scattering data $t$, $t_{\text{exp}}$ and $\hat{q}$ ($\hat{q}_{\text{hat}}$) with $d = 0.9$ and $\alpha = 0.3$. For each $\xi, \zeta \in V_\xi$ is chosen such that $|\zeta| = 50$. The Fourier transform $\hat{q}$ virtually coincides with $t(\xi, \zeta)$.

$q$ calculated from $t_{\text{exp}}$ (and hence from the boundary data) is quite different from $q$. For $x$ near the boundary, the $q_{\text{exp}}(x)$ is quite accurate, but for $x$ near zero there are large discrepancies, especially in the magnitude. Looking at the scattering data in figure 2, one sees two features most likely responsible for that difference. The first one is the differences in the values of $t_{\text{exp}}(\xi)$ for $\xi$ close to zero compared to $\hat{q}(\xi)$. The second is the truncation of $t_{\text{exp}}(\xi)$ due to numerical instability for large $\xi$ values. More details on the influence of the truncation of the scattering data are provided in section 5.4.
Also in figure 3 we display three reconstructions of the conductivity distribution. The first reconstruction of $\gamma$ is from $t(\xi, \zeta)$. Since $t(\xi, \zeta)$ is computed from the forward problem, it may be expected that this reconstruction would be very close to the actual value, as it is. The second reconstruction is $\gamma^{\exp}(x)$ from $t^{\exp}$, and the third reconstruction $\gamma^{\text{app}}$ is from the linear method (23). Considering the relatively large difference in the magnitude of $q^{\exp}(x)$, the reconstruction $\gamma^{\exp}$ is surprisingly good. Also $\gamma^{\text{app}}$ is a fairly good reconstruction. A positive aspect in both reconstructions is that we get $\gamma \equiv 1$ close to the boundary. Moreover, the overall shape is also fairly well reconstructed.

5.4. The influence of the truncation of the scattering data

When we reconstructed $q^{\exp}$ and $\gamma^{\exp}$ we truncated the scattering data $t^{\exp}$ due to numerical instabilities. In this section we investigate the influence on the reconstructions of the truncation of the true scattering data $t(\xi, \zeta)$. Figure 4 shows $t(\xi, \zeta)$ and the reconstructions $q(x)$ and $\gamma(x)$ for different truncations of $t(\xi, \zeta)$, namely at $\xi = R$ for $R = 15, 25, 50$. We have chosen $\zeta \in \mathcal{V}_{\xi}$ with $|\zeta| = 50$. The actual potential and conductivity are almost identical to the curves corresponding to $R = 50$. It is evident that the amount of truncation of the scattering transform influences the reconstruction, and that a very poorly reconstructed $q$ can still result in a good approximation of $\gamma$.

This suggests that for the reconstruction of $\gamma$ the values of the scattering data for small $\xi$ are very important. This is analogous to observations made in the 2D case [MS03].

5.5. Influence of the support and magnitude of $\gamma(x)^{1/2} - 1$

So far we have used fixed values for $d$ and $\alpha$, which determine the support and the magnitude of $\gamma(x)^{1/2} - 1$. Figure 5 displays the reconstructions $\gamma^{\exp}$ and $\gamma^{\text{app}}$ of $\gamma(x)$ from $t^{\exp}(\xi)$ for different choices of support $d$ and magnitude $\alpha$. Each row corresponds to a certain $d$-value and each column to a specific $\alpha$-value. For small support and small magnitude we get good reconstructions, but the quality changes dramatically with larger amplitude and larger support. Especially $\gamma^{\exp}$ does not recover the actual conductivity very well for the large amplitude $\alpha = 0.9$. 

Figure 3. Left: reconstructions of $q(x)$ by taking the inverse Fourier transform of $t(\xi, \zeta)$ and $t^{\exp}(\xi, \zeta)$ for $\alpha = 0.3$ and $d = 0.9$. Right: reconstructions of $\gamma$ from $t$, $\gamma^{\exp}$ and $\gamma^{\text{app}}$ compared to actual conductivity for $\alpha = 0.3$ and $d = 0.9$. $\gamma$ from $t$ nearly coincides with the actual conductivity.
Figure 4. Left: reconstructed Schrödinger potential with truncation of $t$ at $R = 15, 25$ and 50. Right: reconstructions of $\gamma$.

Figure 5. Reconstructions of conductivities of varying support and magnitude: each row corresponds to a specific support $d$ and each column corresponds to a specific magnitude of $\gamma$. The dash-dotted curves are the $\gamma^{exp}$ reconstructions, the dashed curves are the $\gamma^{app}$ reconstructions and solid curves are the actual conductivities $\gamma$.

5.6. Reconstructions from noisy data

We consider reconstructions with Calderón’s method from noisy data as follows. Define a perturbation of the Dirichlet-to-Neumann map by

$$\Lambda^\epsilon_q = \Lambda_q + N,$$
where $N$ is a random operator with the relative norm $\varepsilon$. Since we think of our measurements in terms of the difference map $\Lambda_q - \Lambda_0$ we take the operator norm of $N$ such that

$$\varepsilon = \frac{\|N\|}{\|\Lambda_q - \Lambda_0\|}.$$ 

This is tantamount to perturbing $t^\exp$ as follows:

$$t^\exp_\varepsilon(\xi, \zeta) = \int_{\partial\Omega} e^{-i\xi \cdot \sigma} \left( \Lambda_q^\varepsilon - \Lambda_0 \right) e^{i\zeta \cdot \sigma} d\sigma(x)$$

$$= t^\exp(\xi, \zeta) + \int_{\partial\Omega} e^{-i\xi \cdot \sigma} N e^{i\zeta \cdot \sigma} d\sigma(x),$$

where $t^\exp$ is defined as in (17). Representing $N$ as a matrix of Gaussian random noise with mean 0 and standard deviation 1, we scale by choosing $\varepsilon$ to achieve the desired noise level and compute $t^\exp_\varepsilon$ by numerical integration.

The perturbed scattering transforms with noise level $\varepsilon = 1\%, 2\%$ and $5\%$ are found in figure 6. As in the 2D case, the computed scattering transforms blow up at a smaller and smaller radius as the noise level increases, while retaining nearly the same values for small $|\xi|$ values. The corresponding reconstructions were computed with truncation radii $|\xi| = 12, |\xi| = 9$ and $|\xi| = 8$ for noise levels of $1\%, 2\%$ and $5\%$, respectively. The reconstructions are surprisingly robust in the presence of noise with little change in the spatial resolution, and some loss of accuracy in the reconstructed amplitude of $\gamma$.

6. Conclusions

In this work a direct method based on [Nac88] for reconstructing a 3D conductivity distribution from the Dirichlet-to-Neumann map was implemented and tested on noise-free and noisy data. A linearizing approximation to the scattering transform, denoted $t^\exp$, was studied and compared to Calderón’s reconstruction algorithm. Reconstructions of spherically symmetric conductivities in the unit sphere were computed using the $t^\exp$ approximation, Calderón’s method and a scattering transform computed from the definition requiring knowledge of the actual Schrödinger potential. The latter case served as a benchmark to study the quality of reconstructions for which the actual scattering transform is known. It was shown that very accurate reconstructions can be obtained from accurate knowledge of the scattering transform.
It was found that in contrast to the 2D case, the $t^{\infty}$ approximation is inaccurate near the origin, and this results in poor approximations to the magnitude of the conductivity. However, the support of $\gamma - 1$ and the boundary value $\gamma = 1$ was well approximated by all three methods. Truncating the computed scattering transform in the computations was found to have a profound effect on the reconstructed Schrödinger potential $q$, but the effect on the reconstructed conductivity $\gamma$ was less dramatic. The method proved to be quite robust in the presence of noise. In summary, it appears that the use of the full scattering transform in this method is a promising approach for 3D reconstructions, while linearizations may lead to significant inaccuracies in the reconstructed amplitudes.

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