Connections between Mean-Field Game and Social Welfare Optimization

Sen Li, Wei Zhang and Lin Zhao

Abstract—This paper studies the connection between a class of mean-field games and a social welfare optimization problem. We consider a mean-field game in function spaces with a large population of agents, and each agent seeks to minimize an individual cost function. The cost functions of different agents are coupled through a mean-field term that depends on the mean of the population states. We show that under some mild conditions any ϵ-Nash equilibrium of the mean-field game coincides with the optimal solution to a social welfare optimization problem, and this holds true even when the individual cost functions and action spaces are non-convex. This connection not only enables us to evaluate the efficiency of the mean-field equilibrium, but also leads to several important implications on the existence, uniqueness, and computation of the mean-field equilibrium. In particular, it recovers and extends the existence and uniqueness conditions of the mean-field equilibrium in the literature. The connection also indicates that the computation of the mean-field equilibrium can be cast as a social welfare optimization problem, which can be efficiently solved by various optimization techniques when it is convex.

I. INTRODUCTION

The mean-field games study the interactions among a large population of strategic agents, whose decision making is coupled through a mean-field term that depends on the statistical information of the overall population [1], [2], [3], [4], [5]. When the population size is large, each individual agent has a negligible impact on the mean-field term. This enables characterizing the game equilibrium via the interactions between the agent and the mean-field, instead of focusing on detailed interactions among all the agents. This idea was originally formalized in a series of seminal papers by Lasry and Lions [1], [2] and by Huang et al. [4], [6], where the mean-field equilibrium was characterized as the solution to an equation system that couples a backward Hamilton-Jacobi-Bellman equation and a forward Fokker-Planck-Kolmogorov equation. These seminal results attracted considerable research effort on various aspects of mean-field games. In particular, many works focused on the coupled equation systems to study the existence [6], [7], uniqueness [1], [8], and computation [9], [10], [11] of the mean-field equilibrium. For a more comprehensive review, please refer to [12] and [13]. Another strand of works extended the mean-field game model to more general settings such as heterogeneous agents [4], major-minor player model [14], [15], [16], extended mean-field games [17], etc. Furthermore, mean-field games also find abundant applications in economics [18], [19], crowd and population dynamics [20], demand response [21], [22], [23], networking [24], coupled oscillators [25], to name a few.

For many applications, it is important to analyze and quantify the efficiency of the mean-field equilibrium as compared to some socially optimal solutions. Along this line, the authors in [26] and [27] showed that the coordinator can design a mean-field game with an equilibrium that asymptotically achieves social optima as the population size goes to infinity. This result is true only when each agent in the game is cooperative. In the non-cooperative game setting, a recent work [28] showed that the Nash equilibrium of an electric vehicle charging game is socially optimal as the number of agents tends to infinity, under the assumption that the underlying game is a potential game. However, when the mean-field game is non-cooperative and does not admit a potential function, the mean-field equilibrium is shown to be inefficient in general. For instance, [29] employed a variational approach to study the efficiency loss of mean-field equilibria for a synchronization game among oscillators. In [30], a mean-field congestion game was formulated, and numerical results were presented to show that the mean-field equilibrium is inefficient in general. In addition, [31] derived conditions under which the mean-filed equilibrium is efficient. However, since these conditions are quite restrictive, instead of indicating the efficiency of the mean-field equilibrium, they are more useful in evaluating what happens in an inefficient equilibrium.

This paper studies the connection between mean-field games and social welfare optimization problems. We consider a class of mean-field games in vector spaces (potentially infinite-dimensional) with a large population of non-cooperative agents. Each agent seeks to minimize a cost function coupled with other agents through a mean-field term that depends on the average of the population states. The key contribution of the paper lies in establishing the connection between the mean-field game and a modified social welfare optimization problem. This connection not only enables the formal evaluation of the efficiency of the mean-field equilibrium, but also has several important implications on the existence, uniqueness, and computation of the mean-field equilibrium. Specifically, these contributions are summarized as follows:

- First, different from existing literature, we show that under some mild conditions, the mean-field equilibrium is actually efficient with respect to a modified social welfare optimization problem. This connection not only enables the formal evaluation of the efficiency of the mean-field equilibrium, but also has several important implications on the existence, uniqueness, and computation of the mean-field equilibrium. Specifically, these contributions are summarized as follows:
The connection between the mean-field equations and the is characterized by a set of coupled equations in Section III. The solution of the game is formulated in Section II. The solution of the game so that the resulting mean-field equilibrium is socially optimal.

Second, we show that the mean-field equilibrium exists if the associated social welfare optimization problem has strong duality. In addition, the other direction also holds under an additional monotonicity condition on the mean-field coupling term. Different from many existing works [1], [8], [32], [33], we do not assume the convexity of the cost functions, and our condition for the existence of the mean-field equilibrium is both necessary and sufficient. Thus, our results on the existence of mean-field equilibria are much stronger than the existing results for the class of mean-field games studied in the paper.

Third, we show that the mean-field equilibrium is unique if the corresponding social welfare optimization problem is strictly convex. This recovers the results of some works on the uniqueness of the mean-field equilibrium [1], [22], providing a novel interpretation of these results.

Fourth, our result implies that computing the mean-field equilibrium is equivalent to solving a social welfare optimization problem. When this optimization problem is convex, various efficient algorithms can be employed to compute the mean-field equilibrium of the game. According to our results, some existing methods on computing the mean-field equilibrium [9], [23] can be interpreted as certain primal-dual algorithms in solving the associated social welfare optimization problem. To improve these algorithms, we provide an example of using the alternating direction method of multipliers [34] to compute the mean-field equilibrium. Simulation result shows that the proposed algorithm converges faster than the existing operator-based method.

We emphasize that in general, the mean-field game considered in this paper is not a potential game. Therefore, our result provides a principled approach to study the properties of the game equilibrium, where no powerful tool is available. On the other hand, under some strong conditions (i.e., the mean field coupling term is linear), the mean-field game may reduce to a potential game. However, even in this case, our approach still has advantage over solving the game using potential functions, since it enables decentralized implementation and enjoys better scalability. More discussions can be found in Section IV.

The rest of the paper proceeds as follows. The mean-field game is formulated in Section II. The solution of the game is characterized by a set of coupled equations in Section III. The connection between the mean-field equations and the social welfare optimization problem is studied in Section IV. The implications of the connections are discussed in Section V, followed by case studies in Section VI.

II. MEAN-FIELD GAMES IN FUNCTION SPACES

This section formulates the class of mean-field games to be studied in this paper. Different from many existing works, we will describe the mean-field games in vector spaces (possibly infinite-dimensional). We will show that such formulation includes many important classes of mean-field games as special cases [4], [9], [23], [35]. Our vector space formulation allows us to more directly focus on aspects and challenges related to strategic interactions among different decision makers without worrying about specific details and unnecessary technical conditions regarding individual dynamics and decisions.

A. The Mean-Field Game

We consider a general mean-field game in a vector space among \( N \) agents. Each agent \( i \) is associated with a state variable \( x_i \), a control input \( u_i \in U_i \) and a noise input \( \pi_i \), where \( U_i \) is an arbitrary vector space, and \( \pi_i \) is a random element in a measurable space \( (\Pi, B_i) \) with an underlying probability space \( (\Omega, F, P) \). The state of each agent is determined by the control and noise according to the following mapping \( f_i : U_i \times \Pi_i \rightarrow \mathcal{X} \):

\[
x_i = f_i(u_i, \pi_i), \quad u_i \in U_i,
\]

where \( x_i \) is a random element that takes value in the space \( \mathcal{X} \). To ensure that \( x_i \) is well-defined, we impose the following assumption on \( f_i(u_i, \pi_i) \):

**Assumption 1.** For each \( u_i \in U_i, f_i(u_i, \pi_i(\cdot)): \Omega \rightarrow \mathcal{X} \) is a measurable mapping with respect to \( F/\mathcal{Z} \), where \( \mathcal{Z} \) is a \( \sigma \)-algebra on \( \mathcal{X} \).

Under Assumption 1, \( x_i : \Omega \rightarrow \mathcal{X} \) is a measurable mapping with respect to \( F/\mathcal{Z} \). Therefore, \( x_i \) is a well-defined random element that takes value in \( \mathcal{X} \). On the space \( \mathcal{X} \), we define an inner product and a norm. In particular, denote the inner product as \( x \cdot y \) for \( x, y \in \mathcal{X} \), and define the norm as \( ||x|| = \sqrt{x \cdot x} \). We assume that \( \mathcal{X} \) is complete.

**Assumption 2.** \( \mathcal{X} \) is a Hilbert space.

The completeness of \( \mathcal{X} \) is mainly used to induce a special form of duality theory in vector spaces. More elaborations can be found in Remark 6.

Throughout the paper, we assume that \( x_i \) and \( x_j \) are uncorrelated. In other words, we have \( E(x_i \cdot x_j) = E x_i \cdot E x_j \) for any \( i \neq j \). In addition, we assume the state \( x_i \) has bounded second moment, i.e., there exists \( C \geq 0 \) such that \( E||x_i||^2 \leq C \) for all \( i = 1, \ldots, N \). In this case, the admissible control set can be defined as \( \mathcal{U}_i = \{ u_i \in U_i | x_i = f_i(u_i, \pi_i), E||x_i||^2 \leq C \} \).

For each agent \( i \), there is a cost function on the system state and control input. The costs of different agents are coupled...
through a mean-field term that depends on the average of the population state, which can be written as follows:

\[
J_i(x_i, u_i, \bar{x}) = \mathbb{E}(V_i(x_i, u_i) + F(\bar{x}) \cdot x_i + G(\bar{x})), \tag{2}
\]

where \(\bar{x} \in \mathcal{X}\) is the average of the population state, i.e.,

\[
\bar{x} \triangleq \frac{1}{N} \sum_{i=1}^{N} x_i, \quad F : \mathcal{X} \rightarrow \mathcal{X}
\]

is the mean-field coupling term, and \(G : \mathcal{X} \rightarrow \mathbb{R}\) is the cost associated with the mean-field term. We impose the following regularity conditions on \(J_i\).

**Assumption 3.** (i) \(F(\cdot)\) is globally Lipschitz continuous on \(\mathcal{X}\) with constant \(L\), (ii) \(G(\bar{x})\) is Fréchet differentiable on \(\mathcal{X}\), and the gradient of \(G(\bar{x})\) at 0 is bounded, i.e., \(||\nabla G(0)|| < \infty\). (iii) the gradient of \(G(\cdot)\) is globally Lipschitz continuous on \(\mathcal{X}\) with constant \(\beta\), i.e., \(||\nabla G(x) - \nabla G(y)|| \leq \beta||x - y||, \forall x, y \in \mathcal{X}\).

The mean-field game can be then formulated as:

\[
\min_{u_i} \mathbb{E}(V_i(x_i, u_i) + F(\bar{x}) \cdot x_i + G(\bar{x})) \tag{3a}
\]

s.t. \(x_i = f_i(u_i, \pi_i), \quad u_i \in \mathcal{U}_i, \tag{3b}\)

where \(\bar{x} \triangleq \frac{1}{N} \sum_{i=1}^{N} x_i, \text{ and } F(\bar{x}) \cdot x_i\) can be either interpreted as the price multiplied by quantity [10], [36], [37] or part of the quadratic penalty of the deviation of the system state from the population mean [4], [26]. This structure of the cost function (3) is quite general. It captures a large body of problems that frequently arise in various applications [4], [5], [21], [23], [26], [35], [38], [39].

**Remark 1.** When the formulated problem (3) is used to describe a dynamic game, different types of information structures may be involved [40], such as open-loop information, closed-loop information, feedback information, among others. We remark that the mean-field game model (3) can accommodate these information structures by implicitly incorporating them in the control spaces. For instance, under open-loop information structure, the control space \(\mathcal{U}_i\) consists of all open-loop control trajectories. Under feedback information structure, the control space \(\mathcal{U}_i\) consists of all feedback control policies. Interested readers can refer to the example in Section 7B for more details.

**Remark 2.** A notable difference between our proposed mean-field game and many related works is that we consider a finite number of agents. In many classic mean-field game models [7], [5], a typical assumption is that there is a continuum of decision-makers, which is inspired by the continuum particle model in fluid dynamics. However, in practice, the decision making problem with a continuum of decision-makers is uncommon in engineering applications, and the continuum model is often used as an approximation of some finite system [41]. Therefore, in these cases, our finite model is directly applicable and better incorporates the heterogeneity of the agents.

There are several solution concepts for the game problem, such as Nash equilibrium, Bayesian Nash equilibrium, dominant strategy equilibrium, among others. In the context of mean-field games, we usually relax the Nash equilibrium solution concept by assuming that each agent is indifferent to an arbitrarily small change \(\epsilon\). This solution concept is referred to as the \(\epsilon\)-Nash equilibrium, formally defined as follows:

**Definition 1.** \((u_1^\epsilon, \ldots, u_N^\epsilon)\) is an \(\epsilon\)-Nash equilibrium of the game (3) if the following inequality holds

\[
J_i(u_i^\epsilon, u_{-i}^\epsilon) \leq J_i(u_i, u_{-i}) + \epsilon \tag{4}
\]

for all \(i = 1, \ldots, N\), and all \(u_i \in \mathcal{U}_i\), where \(u_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)\) and \(J_i(u_i, u_{-i})\) is the compact notation for (2) after plugging (1) in (2).

At an \(\epsilon\)-Nash equilibrium, each agent can lower his cost by at most \(\epsilon\) via deviating from the equilibrium strategy, given that all other players follow the equilibrium strategy. Therefore, the agents are motivated to play the equilibrium strategy if they are indifferent to a change of \(\epsilon\) in their cost.

**B. Examples**

The proposed mean-field game problem (3) provides a unifying framework that incorporates the formulations of a large body of literature. As the problem is defined in general vector spaces, it includes both discrete-time [9], [23] and continuous-time system [4] as special cases, and addresses both deterministic and stochastic cases. The rest of this subsection uses two examples to illustrate the generality of the proposed formulation.

1) **Discrete-time deterministic game:** The first example is a deterministic mean-field game in discrete-time [9]. The particular form of the game is given as follows:

\[
\min_{x_i} \Vert x_i \Vert_2^2 + \Vert x - \bar{x} \Vert_2^2 \tag{5}
\]

where \(x_i \in \mathbb{R}^s\) is the system state, \(\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i\) is the average state, \(\Vert x_i \Vert_2^2\) stands for \(x_i^T Q x_i\), \(C \in \mathbb{R}^{s \times s}\), and \(Q\) and \(\Delta\) are symmetric positive definite matrices of appropriate dimensions. Although the problem (5) is formulated in the static form, it captures a class of finite-dimensional linear quadratic games, which can be transformed to (5) by plugging the linear dynamics in the cost function (9).

Next, we show that the above game problem can be formulated as (3). Note that (5) is the degenerate case of (5), where the disturbance term \(\pi_i\) degenerates to 0, the state space and the control space are both \(\mathbb{R}^s\), and \(x_i = u_i\), i.e., \(f_i\) is the identity function. If we expand the norm and combining similar terms in (5), then we can transform (5) to the following form:

\[
\min_{x_i} x_i^T (Q + \Delta) x_i + 2c^T x_i + 2\bar{x}^T (C - \Delta) x_i + \bar{x}^T \Delta \bar{x} \tag{6}
\]

Comparing (6) to (3), we have \(V_i(x_i, u_i) = x_i^T (Q + \Delta) x_i + 2c^T x_i, \quad F(\bar{x}) = 2(C - \Delta)\bar{x}, \text{ and } G(\bar{x}) = \bar{x}^T \Delta \bar{x}\). In this case, it is easy to verify that the game problem (6) satisfies Assumption 1-3. Therefore, (6) is a special case of our proposed mean-field game (3).
2) Continuous-time stochastic game: The second example is a linear quadratic Gaussian (LQG) game considered in Section II-A of \cite{4}: \[
\min_{\{u_i(t), t \geq 0\}} \mathbb{E} \int_0^\infty e^{-\rho t} \left[ (x_i(t) - v(t))^2 + ru_i(t)^2 \right] dt
\] \[
\text{s.t.} \quad dx_i(t) = u_i(t) dt + \sigma_i d\pi_i(t) \quad \forall i \in \mathbb{N}, \quad u_i(t) \in \mathbb{R}, \quad x_i(t) \in \mathbb{R}, \quad i = 1, \ldots, N,
\] where \(x_i(t)\) and \(u_i(t)\) denote the state and control for the \(i\)th agent at time \(t\), \(\pi_i(t)\) is a standard scalar Brownian motion, \(v(t) \triangleq \eta - \gamma \sum_{i=1}^N x_i(t)\) is the mean-field term, and \(\rho, \gamma, \eta > 0\) are real constants. It is assumed that the Brownian motions \(\pi_i = \{\pi_i(t), t \geq 0\}\) and \(\pi_j = \{\pi_j(t), t \geq 0\}\) are independent for any \(i \neq j\). In addition, letting \(u_i = \{u_i(t), t \geq 0\}\), the authors in \cite{4} defined the admissible control set as \(\mathcal{U}_i = \{u_i(t)\mid u_i(t)\text{ is adapted to} \sigma - \text{algebra} \sigma(x_i(0), \pi_i(s), s \leq t), \mathbb{E} \int_0^\infty e^{-\rho t} |x_i(t) + u_i(t)|^2 dt < \infty\}\). In this case, let \(x = \{x(t) \in \mathbb{R}, t \geq 0\}\), then the state space \(\mathcal{X}\) can be taken as \(\mathcal{X} = \{x \mid \mathbb{E} \int_0^\infty e^{-\rho t} |x(t)|^2 dt < \infty\}\).

Next, we show that problem (7) is a special case of the formulated problem (3). For this purpose, we need to transform (7) in the form of (3) and verify Assumption 1 holds.

First, we note that the stochastic differential equation (9) can be explicitly solved as:
\[
\dot{x}_i(t) = \int_0^t u_i(s) ds + \sigma_i \int_0^t d\pi_i(s).
\]
This equation indicates that \(x_i\) is determined by \(\{u_i(s), 0 \leq s \leq t\}\), thus \(f_i\) can be defined according to (9). In addition, \(x_i(t)\) is clearly well-defined, and Assumption 1 holds.

Second, for any \(x \in \mathcal{X}\) and \(y \in \mathcal{X}\), we define their inner product as
\[
x \cdot y = \int_{[0, \infty)} e^{-\rho t} x(t) y(t) dt,
\]
and define the norm as \(|x| = \sqrt{x \cdot x}\). Under this norm, we can show that \(\mathcal{X}\) is complete, thus Assumption 2 is satisfied.

**Lemma 1.** The space \(\mathcal{X}\) is a Hilbert space.

**Proof.** See Appendix A.

Third, under the inner product (10), the objective function of the problem (7) can be transformed to the form of (3):
\[
\mathbb{E} \left[ V_i(x_i, u_i) + F(\bar{x}) \cdot x_i + G(\bar{x}) \right],
\]
where \(V_i(x_i, u_i) = |x_i|^2 + \int_0^\infty e^{-\rho t} u_i(t)^2 dt, F(\bar{x}) = 2\gamma \bar{x} - 2\eta I, G(\bar{x}) = ||\gamma - \eta I||^2, \) and \(I\) denotes the unit vector in \(\mathcal{X}\). We note that \(F(\bar{x}) = 2\gamma \bar{x} - 2\eta\) is globally Lipschitz continuous, and \(\nabla G(\bar{x}) = 2(\gamma \bar{x} - \eta I)\) is also globally Lipschitz continuous. This indicates that Assumption 3 is satisfied. Therefore, the game problem in (7) is a special case of the mean-field game (3).

**Remark 3.** In this example, the admissible control space is defined as the control strategies adapted to the filtration \(\sigma(x_i(0), \pi_i(s), s \leq t)\). This represents the closed-loop perfect state information structure \(\mathcal{F}_t\), i.e., \(u_i(t)\) depends on \(\{x_i(s), 0 \leq s \leq t\}\). We comment that the information structure of the mean-field game (3) can be much more general than closed-loop perfect state information structure. For instance, if \(U_i\) only contains the control strategies that depend on the initial state \(x_i(0)\), then it represents the open-loop information structure. If each \(u_i\) in \(U_i\) satisfies that \(u_i(\cdot)\) depends on \(x_i(\cdot)\), then it corresponds to the close-loop feedback information structure. Such generalization is possible as long as the admissible control space \(U_i\) remains a vector space.

**C. Goal of this Paper**

The objective of this paper is to study the connection between the mean-field game (3) and the social welfare optimization problem. In our context, we say a mean-field equilibrium is efficient if it maximizes the social welfare, or equivalently, minimizes the social cost (negative of social welfare). In most of the existing literature, a natural candidate for the social welfare is simply the sum of individual utilities. Unfortunately, it is shown that under this social welfare, the mean-field equilibrium is in general not efficient \(\cite{30}\), and research effort has largely focused on characterizing and bounding the gap between the equilibrium solution and the efficient solution \(\cite{4, 29}\).

In this paper, we approach this problem from a different angle. Instead of characterizing the gap, we ask the question of whether the mean-field equilibrium can be efficient for a modified social welfare. In other words, we would like to construct a social welfare optimization problem with some modified social welfare such that its optimal solution coincides with the mean-field equilibrium.

In the rest of this paper, we will answer this question in two steps. First, we focus on the mean-field game (3) and characterize its equilibrium as the solution to a set of mean-field equations. Second, we construct a social welfare optimization problem, and show that the solution to the mean-field equations coincides with the solution to the social welfare optimization problem.

**III. CHARACTERIZING THE ϵ-NASH EQUILIBRIUM**

In this section, we derive a set of mean-field equations that characterize the \(\epsilon\)-Nash equilibrium of the mean-field game (3). The ideas of the derivation are similar to that in \cite{4} and \cite{9}. However, our result is more general than \cite{4} and \cite{9} as we consider more general mean-field games in function spaces. See Remark 4 for detailed discussions on the differences between our results and several others in the literature.

To study the mean-field equilibrium, we note that the cost function in (2) of the individual agent is only coupled through the mean-field terms \(F(\bar{x})\) and \(G(\bar{x})\). In the large population case, the impact of the control input for a single agent on
the coupling term vanishes as the population size goes to infinity. Therefore, we can approximately treat the mean-field terms $F(\bar{x})$ and $G(\bar{x})$ as given, and each agent then faces an optimal response problem defined as follows:

$$\mu_i(y) \in \arg \min_{u_i} \mathbb{E} \left( V_i(x_i, u_i) + y \cdot x_i \right) \quad \text{s.t. } x_i = f_i(u_i, \pi_i), \ u_i \in \mathcal{U}_i, \tag{12}$$

where we use a deterministic value $y \in \mathcal{X}$ to replace $F(\bar{x})$. In (12), $\mu_i(y)$ denotes the optimal solution to the optimal response problem parameterized by $y$, and $G(\bar{x})$ is regarded as a constant in (12) that can be ignored. To avoid triviality, we impose the following assumption on (12):

**Assumption 4.** For any $y \in \mathcal{X}$, the optimal response problem (12) admits at least one solution.

Assumption 4 imposes some mild regularity conditions on the functional $V_i(x_i, u_i)$ and the admissible control set $\mathcal{U}_i$. For instance, the solution to (12) exists if $V_i$ is continuous and $\mathcal{U}_i$ is compact. However, the solution to (12) may also exist beyond these cases. Therefore, to maintain generality, we do not present the detailed technical condition for Assumption 4 to hold. On the other hand, the optimal response problem (12) may have multiple optimal solutions. In this case, $\mu_i(y)$ can be any one of these solutions.

Based on this approximation, $y$ generates a collection of agent responses. Ideally, the value $y$ should guide the individual agents to choose a collection of optimal responses $\mu_i(y)$ which, in return, collectively generate the mean-field term $F(\cdot)$ that is close to the approximation $y$. This suggests that we use the following equation systems to characterize the equilibrium of the mean-field game:

$$\begin{cases} 
\mu_i(y) \in \arg \min_{u_i \in \mathcal{U}_i} \mathbb{E} \left( V_i(f_i(u_i, \pi_i), u_i) + y \cdot f_i(u_i, \pi_i) \right) \tag{14} \\
x_i = f_i(\mu_i(y), \pi_i) \tag{15} \\
y = F \left( \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} x_i \right) \tag{16}
\end{cases}$$

where the mean-field term $y$ induces a collection of responses that generate $y$. In the rest of this subsection we show that the solution to this equation system is an $\epsilon$-Nash equilibrium of the mean-field game (3), and $\epsilon$ goes to 0 as $N$ tends to infinity. For this purpose, we first prove the following two lemmas. These two lemmas are mainly used to set up the stage for the main result that will be introduced later.

**Lemma 2.** If there exists $C > 0$ such that $\mathbb{E} \left| x_i \right|^2 \leq C$ for all $i = 1, \ldots, N$, and $F(\cdot)$ is globally Lipschitz continuous, then the following relation holds for each agent $i$:

$$\left| \mathbb{E} \left( F(\bar{x}) \cdot x_i \right) - \mathbb{E} (\bar{F} x_i) \cdot \mathbb{E} x_i \right| \leq \epsilon, \tag{17}$$

where $\bar{x} \triangleq \frac{1}{N} \sum_{i=1}^{N} x_i$ and $0 < \epsilon = O \left( \frac{1}{\sqrt{N}} \right)$.

**Proof.** See Appendix B. \hfill \square

The other lemma shows that removing the decision of a single agent does not significantly affect the value of $\mathbb{E} G(\cdot)$:

**Lemma 3.** Define $\bar{x} \triangleq \frac{1}{N} \sum_{i=1}^{N} x_i$, and let $\bar{x}_{-i} \triangleq \frac{1}{N} \sum_{j \neq i} x_j$, where $x_i$ denotes the state trajectory corresponding to $u_i$ in $\mathcal{U}_i$, then we have the following relation:

$$\left| \mathbb{E} G(\bar{x}) - \mathbb{E} G(\bar{x}_{-i}) \right| \leq \epsilon, \tag{18}$$

and $0 < \epsilon = O \left( \frac{1}{\sqrt{N}} \right)$.

**Proof.** See Appendix C. \hfill \square

Using the results of Lemma 2 and Lemma 3, we can show that the solution to the equation system (14)-(16) is an $\epsilon_N$-Nash equilibrium of the mean-field game (3), and $0 < \epsilon = O \left( \frac{1}{\sqrt{N}} \right)$.

**Theorem 1.** The solution to the equation system (14)-(16), if exists, is an $\epsilon_N$-Nash equilibrium of the mean-field game (3), and $0 < \epsilon = O \left( \frac{1}{\sqrt{N}} \right)$.

**Proof.** For notation convenience, we denote the solution to the equation system (14)-(16) as $u_i^*, x_i^*$ and $y^*$, where $x_i^*$ is the state trajectory corresponding to $u_i^*$. According to Definition 1 to prove this theorem, we need to show that:

$$\mathbb{E} \left( V_i(x_i^*, u_i^*) + F(\bar{x}^*) \cdot x_i^* + G(\bar{x}^*) \right) \leq \epsilon +$$

$$\mathbb{E} \left( V_i(x_i, u_i) + F \left( \frac{1}{N} x_i + \bar{x}_{-i}^* \right) \cdot x_i + G \left( \frac{1}{N} x_i + \bar{x}_{-i}^* \right) \right) \tag{19}$$

for all $u_i \in \mathcal{U}_i$, where $\epsilon = O \left( \frac{1}{\sqrt{N}} \right)$, $\bar{x}^* \triangleq \frac{1}{N} \sum_{i=1}^{N} x_i^*$, $\bar{x}_{-i}^* \triangleq \frac{1}{N} \sum_{j \neq i} x_j^*$, and $x_i$ is the state trajectory corresponding to $u_i$. Based on Lemma 2 we have

$$F(\mathbb{E} \bar{x}) \cdot \mathbb{E} x_i - O \left( \frac{1}{\sqrt{N}} \right) \leq F(\mathbb{E} \bar{x}) \cdot \mathbb{E} x_i + \mathbb{E} (\bar{F} x_i) \cdot \mathbb{E} x_i + O \left( \frac{1}{\sqrt{N}} \right). \tag{20}$$

Based on Lemma 3 we have

$$\mathbb{E} G(\bar{x}_{-i}) - O \left( \frac{1}{\sqrt{N}} \right) \leq \mathbb{E} G(\bar{x}) \leq \mathbb{E} G(\bar{x}_{-i}) + O \left( \frac{1}{\sqrt{N}} \right). \tag{21}$$

Apply (20) and (21) to the left-hard side of (19), then the left-hand side of (19) is upper bounded as follows:

$$\mathbb{E} \left( V_i(x_i^*, u_i^*) + F(\bar{x}^*) \cdot x_i^* + G(\bar{x}^*) \right) \leq O \left( \frac{1}{\sqrt{N}} \right) +$$

$$\mathbb{E} V_i(x_i, u_i) + F \left( \frac{1}{N} x_i + \bar{x}_{-i}^* \right) \cdot x_i + G \left( \frac{1}{N} x_i + \bar{x}_{-i}^* \right) \tag{22}$$

Applying (20) and (21) to the right-hard side of (19), the right-hand side of (19) is then lower bounded as follows:

$$\mathbb{E} \left( V_i(x_i, u_i) + F \left( \frac{1}{N} x_i + \bar{x}_{-i}^* \right) \cdot x_i + G \left( \frac{1}{N} x_i + \bar{x}_{-i}^* \right) \right) \geq \mathbb{E} V_i(x_i, u_i) + F \left( \frac{1}{N} \mathbb{E} x_i + \sum_{j \neq i} x_j^* \right) \cdot \mathbb{E} x_i +$$

$$\mathbb{E} G(\bar{x}_{-i}) - O \left( \frac{1}{\sqrt{N}} \right). \tag{23}$$
Based on (22) and (23), to prove that (19) holds, it suffices to show that

$$\mathbb{E} V_i(x_i^*, u_i^*) + F\left(\frac{1}{N}\mathbb{E} \sum_{i=1}^{N} x_i^*\right) \cdot \mathbb{E} x_i^* \leq O\left(\frac{1}{\sqrt{N}}\right) +$$

$$\mathbb{E} V_i(x_i, u_i) + F\left(\frac{1}{N}\mathbb{E} (x_i + \sum_{j \neq i} x_j^*)\right) \cdot \mathbb{E} x_i. \quad (24)$$

Since $||\mathbb{E} x_i||$ is bounded (see proof for Lemma 2) and $F(\cdot)$ is Lipschitz continuous with constant $L \geq 0$, we have:

$$F\left(\frac{1}{N}\mathbb{E} (x_i + \sum_{j \neq i} x_j^*)\right) \cdot \mathbb{E} x_i \leq$$

$$\frac{L}{N} \left|\mathbb{E} x_i - \mathbb{E} x_i^*\right|||\mathbb{E} x_i|| = O\left(\frac{1}{\sqrt{N}}\right). \quad (25)$$

Therefore, combining (24) and (25), it suffices to show that:

$$\mathbb{E} V_i(x_i^*, u_i^*) + F\left(\frac{1}{N}\mathbb{E} \sum_{i=1}^{N} x_i^*\right) \cdot \mathbb{E} x_i^* \leq$$

$$\mathbb{E} V_i(x_i, u_i) + F\left(\frac{1}{N}\mathbb{E} \sum_{i=1}^{N} x_i^*\right) \cdot \mathbb{E} x_i + O\left(\frac{1}{\sqrt{N}}\right).$$

Note that based on (16), $F\left(\frac{1}{N}\mathbb{E} \sum_{i=1}^{N} x_i^*\right) = y^*$, which is equivalent to:

$$\mathbb{E} V_i(x_i^*, u_i^*) + y^* \cdot \mathbb{E} x_i^* \leq \mathbb{E} V_i(x_i, u_i) + y^* \cdot \mathbb{E} x_i + O\left(\frac{1}{\sqrt{N}}\right).$$

This obviously holds based on (14), which completes the proof.

Theorem 1 indicates that each agent is motivated to follow the equilibrium strategy $u_i^*$ as deviating from this strategy can only decrease the individual cost by a negligible amount $\epsilon$. Furthermore, this $\epsilon$ can be arbitrarily small, if the population size is sufficiently large. Note that the mean-field equation system (14)-(16) is not the unique way to characterize the $\epsilon$-Nash equilibrium of the mean-field game (3). The game may have other $\epsilon$-Nash equilibria with different values of $\epsilon$. However, in this paper, we only focus on these mean-field equations (14)-(16). In the rest of this paper, the mean-field equilibrium of the game (3) always refers to the solution to the mean-field equations (14)-(16).

**Remark 4.** Compared to similar works in [4] and [9], Theorem 1 generalizes the result from several perspectives. First, these works mainly focus on linear quadratic problems, while we consider a more general mean-field game that includes them as our special cases. Therefore, our result applies to more general cases (including non-convex cost functions) than quadratic individual costs. Second, the mean-field coupling term $F(\cdot)$ is assumed to be affine in [4] and [9], while we relax it to be Lipschitz continuous. Third, we consider the problem in function spaces, which provides a unifying framework that includes both discrete-time system and continuous-time system as special cases, and accommodates both deterministic and stochastic games.

**Remark 5.** In the linear quadratic case, the mean-field equations (17)-(19) recover the result in [4], which is an important seminal result in mean-field games. At the first look, it may appear that these two works are different: we consider a finite number of agents, while in [4] the authors considered a continuum model in the derivation of the mean-field equations. This leads to slightly different forms of the mean-field equations: we use empirical mean $\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} x_i$ to represent the mean-field, while [4] assumes a distribution over the agent parameter and uses integral over this parameter to represent the mean. However, note that when the number of agent goes to infinity, these two results are the same: the empirical mean converges to the integral (see Assumption H3 in [4]). Therefore, the mean-field equations in [4] can be regarded as the limiting case of our mean-field equations (17)-(19) when $N$ approaches infinity.

**IV. Connection to Social Welfare Optimization**

This section focuses on the connection between the mean-field game (3) and the social welfare optimization problem. Such connection is typically referred to as “efficiency”: we say that the mean-field equilibrium is efficient if it maximizes the social welfare. In the literature, some attempts have been made to draw connections between the mean-field game and the social welfare optimization problem. Most of these works consider the social welfare optimization problem to be maximizing the total utility (or equivalently, minimizing the total cost) of all agents, which in our context can be formulated as follows:

$$\min_{(u_1, \ldots, u_N)} \sum_{i=1}^{N} \mathbb{E} (V_i(x_i, u_i) + F(\bar{x}) \cdot x_i + G(\bar{x})) \quad (26)$$

s.t. $x_i = f_i(u_i, \pi_i), \quad u_i \in \bar{U}_i, \quad \forall i = 1, \ldots, N. \quad (27)$

Since the cost function (26) represents the total cost of the entire population, from the efficiency point of view, it is desirable to have the mean-field equilibrium to be the optimal solution to (26). However, it has been shown in multiple works that this statement is not true in general [4, 29, 30]. Therefore, many existing works along this line have focused on characterizing the gap between the mean-field equilibrium and the optimal solution to (26).

Different from all these works, we will construct a social welfare optimization problem so that the mean-field equilibrium achieves exact social optima. This can be done by introducing a virtual agent in the system with a cost function $\phi : \mathcal{X} \to \mathbb{R}$, and consider the following social welfare optimization problem that includes this virtual cost:

$$\min_{u_1, \ldots, u_N, z} \mathbb{E} \left(\sum_{i=1}^{N} V_i(x_i, u_i) + \phi(z)\right) \quad (28)$$

s.t. $\left\{\begin{array}{l}
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} x_i = z, \\
x_i = f_i(u_i, \pi_i), \quad u_i \in \bar{U}_i, \quad \forall i = 1, \ldots, N
\end{array}\right. \quad (29)$
where \( z \) is the decision of the virtual agent. Compared to the classical social welfare optimization problem, the constructed problem has an augmented decision variable, and introduces an additional constraint \( z = \frac{1}{N} \sum_{i=1}^{N} E x_i \) regarding this decision variable. This constraint is inspired by the supply-demand model in microeconomics, where the virtual agent acts as a single supplier, and all other agents can be regarded as the demands. This constraint requires that the supply and the demands are balanced.

In the remainder of the section, we will establish conditions under which we can draw connections between the mean-field equilibrium and the solution to the social welfare optimization problem. Finding such conditions is useful from at least two perspectives. First, if we are given a mean-field game, we can construct the social welfare optimization problem to evaluate the efficiency of the mean-field equilibrium. Second, if we are given a social welfare optimization problem, then we can design the mean-field game so as to control the population to operate at the socially optimal point.

### A. Connections under Strong Duality

This subsection shows that the solution to the social welfare optimization problem is a mean-field equilibrium to the game under the condition that the social welfare optimization problem has strong duality.

For this purpose, we first introduce the concept of strong duality for the social welfare optimization problem. With slight abuse of notation, we drop the dependence of the objective function of on \( x \), and compactly denote as follows:

\[
P^* = \min_{u,z} J_s(u,z) \tag{30}
\]

\[
s.t. \quad \begin{cases} 
g(u,z) = 0 \\
    z \in \mathcal{X}, u_i \in \mathcal{U}_i, \forall i = 1, \ldots, N, \tag{31}
\end{cases}
\]

where \( P^* \) is the optimal value of the social welfare optimization problem, \( u = (u_1, \ldots, u_N) \) is the vector of control inputs, \( J_s(u,z) = \mathbb{E} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{N} f_i(u_i, \pi_j) + \phi(z) \right) \), and \( g(u,z) = \mathbb{E} \sum_{i=1}^{N} f_i(u_i, \pi_i) - N z \). Using this notation, the Lagrangian of problem can be defined as follows:

\[
L(u,z,\lambda) = J_s(u,z) + \lambda \cdot g(u,z). \tag{32}
\]

where \( \lambda \in \mathcal{X} \) is the Lagrange multiplier for the constraint \( g(u,z) = 0 \).

**Remark 6.** In a more general setting, the Lagrange multiplier is in the dual space of \( \mathcal{X} \), and the inner product term in should be replaced with a bounded linear operator evaluated at point \( g(u,z) \) [Chap. 8]. However, in our problem, since the state space \( \mathcal{X} \) is a Hilbert space, we can select the dual space of \( \mathcal{X} \) to be itself, and the bounded linear operator reduces to the inner product on \( \mathcal{X} \). Therefore, the expression relies on the fact that \( \mathcal{X} \) is a Hilbert space. When \( \mathcal{X} \) is not a complete space, the mean-field equations can not be connected to the social welfare optimization problem.

Given the Lagrangian, we first treat the multiplier as given and define the mapping \( D : \mathcal{X} \to \mathbb{R} \):

\[
D(\lambda) = \inf_{u,z} L(u,z,\lambda), \tag{33}
\]

then the dual problem of the social welfare optimization problem is defined as follows:

\[
D^* = \max_{\lambda \in \mathcal{X}} D(\lambda) \tag{34}
\]

where \( D^* \) is the optimal value of the dual problem. When the dual problem admits a solution, and the optimal value of the dual problem coincides with that of the primal problem, then we say the optimization problem has strong duality. Formally, we define it as follows:

**Definition 2.** The optimization problem has strong duality if \( P^* = D^* \) and there exists \( \lambda^* \in \mathcal{X} \) such that \( D^* = D(\lambda^*) \).

Note that the definition of strong duality not only requires the duality gap to be zero, but also requires the dual problem to have a finite solution \( \lambda^* \). This is slightly stronger than only requiring zero duality gap between the primal problem and the dual problem. In general, the existence of a finite multiplier to can be easily guaranteed under mild constraint qualifications (e.g., Slater’s condition) [Chap. 8]. In our paper, when we say that the social welfare optimization problem has strong duality, it indicates the problem already satisfies certain constraint qualifications so that the solution to exists.

Under strong duality, we can establish connections between the mean-field equilibrium and the social welfare optimization problem. These connections are summarized in the next a few theorems and corollaries, which are the main results of this paper.

**Theorem 2.** Let \( \phi : \mathcal{X} \to \mathbb{R} \) be a Fréchet differentiable functional such that \( \nabla \phi(z) = NF(z), \forall z \in \mathcal{X} \). Assume that the social welfare optimization problem has strong duality, then any socially optimal solution to is a mean-field equilibrium to the game.

**Proof.** Since the social welfare optimization has strong duality, then there exists \( \lambda^* \) such that \( P^* = D^* = D(\lambda^*) \). Note that due to weak duality, this indicates that \( \lambda^* \) is the optimal solution to the dual problem, i.e., \( D^* = \inf_{u_i \in \mathcal{U}_i, \ldots, u_N \in \mathcal{U}_N} \mathbb{E} L(u,z,\lambda^*) \). Let \( (u^*, z^*) \) be the optimal solution to , then \( (u^*, z^*) \) satisfies the constraint \( z^* = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} f_i(u_i^*, \pi_i) \), and we have the following inequalities:

\[
D^* = \inf_{u_i \in \mathcal{U}_i, \ldots, u_N \in \mathcal{U}_N, z \in \mathcal{X}} L(u,z,\lambda^*) \leq L(u^*, z^*, \lambda^*) = J_s(u^*, z^*) + \lambda^* \cdot g(u^*, z^*) = J_s(u^*, z^*) = P^*. \tag{35}
\]
Due to strong duality, \( P^* = D^* \). Therefore, equality holds in (35), indicating that \((u^*, z^*)\) satisfies the following:

\[
(u^*, z^*) \in \arg \min_{u_i \in \bar{U}_i, \ldots, u_N \in \bar{U}_N, z \in \mathcal{X}} L(u, z, \lambda^*). \tag{36}
\]

Since \( L \) can be decomposed in terms of \( u^i \) and \( z \), the relation (36) is equivalent to the following:

\[
\begin{align*}
&u_i^* \in \arg \min_{u_i \in \bar{U}_i} \mathbb{E} \left( V_i(f_i(u_i, \pi_i), u_i) + \lambda^* \cdot f_i(u_i, \pi_i) \right) \tag{37} \\
z^* \in \arg \min_{z \in \mathcal{X}} \phi(z) - N\lambda^* \cdot z
\end{align*}
\]

The first-order optimality condition of (38) yields \( \nabla \phi(z^*) = N\lambda^* \). Since \( \nabla \phi(z) = NF(z) \), we have \( F(z^*) = \lambda^* \).

Therefore, the above equation sets can be reduced to the following:

\[
\begin{align*}
u_i^* &\in \arg \min_{u_i \in \bar{U}_i} \mathbb{E} \left( V_i(f_i(u_i, \pi_i), u_i) + \lambda^* \cdot f_i(u_i, \pi_i) \right) \tag{39} \\
\lambda^* &= F \left( \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} f_i(u_i^*, \pi_i) \right)
\end{align*}
\]

It can be verified that (39), (40) is equivalent to the mean-field equations (14)-(16). Therefore, \((u_1^*, \ldots, u_N^*)\) is a mean-field equilibrium. This completes the proof.

Theorem 2 shows that any socially optimal solution is a mean-field equilibrium. Note that this relation only holds from one direction: it does not necessarily mean that any mean-field equilibrium is also socially optimal. However, the other direction of the relation also holds when the mean-field equations have a unique solution. This can be summarized in the following corollary:

**Corollary 1.** Let \( \phi : \mathcal{X} \to \mathbb{R} \) be a Fréchet differentiable functional such that \( \nabla \phi(z) = NF(z) \), \( \forall z \in \mathcal{X} \). Assume that the social welfare optimization problem (28) has strong duality, and the mean-field game (28) has a unique mean-field equilibrium, then the mean-field equilibrium to (3) is the globally optimal solution to the social welfare optimization problem (28).

The proof of the corollary follows easily from Theorem 2 and is therefore omitted. Corollary 1 can be used to check the efficiency of the mean-field equilibrium when the mean-field equations admit at most one solution.

**B. Special Case with Monotone Mean-Field Coupling**

In general, the mean-field equations may admit multiple solutions. According to Theorem 2, the best mean-field equilibrium among these solutions is the optimal solution to the social welfare optimization problem, but there may exist other mean-field equilibria that are not socially optimal. In this subsection, we show that this complication can be resolved if the following monotonicity condition is imposed on the mean-field coupling term \( F(\cdot) \):

**Definition 3** (monotone mean-field coupling). The mean-field coupling term \( F(x) \) is non-decreasing with respect to \( x \in \mathcal{X} \) if \( (F(x) - F(x')) \cdot (x - x') \geq 0 \) for any \( x, x' \in \mathcal{X} \).

Throughout the rest of the paper, if \( F(\cdot) \) is non-decreasing, then we say that the mean-field coupling term \( F(\cdot) \) is monotone. Under this condition, we can derive a stronger result than Theorem 2 where the relation between the mean-field equilibrium and the socially optimal solutions can go either way:

**Theorem 3.** Let \( \phi : \mathcal{X} \to \mathbb{R} \) be a Fréchet differentiable functional such that \( \nabla \phi(z) = NF(z) \), \( \forall z \in \mathcal{X} \). Assume that the social welfare optimization problem (28) has strong duality, and assume that \( F(\cdot) \) is non-decreasing, then \( (u_1^*, \ldots, u_N^*) \) is the mean-field equilibrium to (28) if and only if it is the globally optimal solution to the social welfare optimization problem (28).

Proof. Based on Theorem 2, the socially optimal solution is a mean-field equilibrium. Therefore, it suffices to show the other direction also holds. For notational convenience, let \( \bar{u} = (\bar{u}_1, \ldots, \bar{u}_N) \) be the solution to the mean-field equations (14)-(16). Define \( \bar{z} = \frac{1}{N} \sum_{i=1}^{N} f_i(\bar{u}_i, \pi_i) \), and let \( \bar{y} = F(\bar{z}) \).

Since \( F(z) \) is non-decreasing, \( \phi(z) \) is convex. Therefore, \( \nabla \phi(\bar{z}) = N\bar{y} \) indicates that:

\[
\bar{z} \in \arg \min_{z \in \mathcal{X}} \phi(z) - N\bar{y} \cdot z. \tag{41}
\]

Due to (14), we also have:

\[
u_i \in \arg \min_{u_i \in \bar{U}_i} \mathbb{E} \left( V_i(f_i(u_i, \pi_i), u_i) + \bar{y} \cdot f_i(u_i, \pi_i) \right). \tag{42}
\]

The above two equations together indicate that \((\bar{u}, \bar{z})\) is the optimal solution to the following optimization problem:

\[
\begin{align*}
\min_{u_i, z} &\sum_{i=1}^{N} \mathbb{E} V_i(x_i, u_i) + \phi(z) + \bar{y} \cdot \sum_{i=1}^{N} \mathbb{E} x_i - Nz \\
\text{s.t.} &\quad x_i = f_i(u_i, \pi_i) \\
&\quad u_i \in \bar{U}_i, z \in \mathcal{X}.
\end{align*}
\]

In other words, \((\bar{u}, \bar{z})\) satisfies:

\[
(\bar{u}, \bar{z}) \in \arg \min_{u_1 \in \bar{U}_1, \ldots, u_N \in \bar{U}_N, z \in \mathcal{X}} L(u, z, \bar{y}). \tag{45}
\]

Note that due to weak duality, we have:

\[
L(\bar{u}, \bar{z}, \bar{y}) \leq D^* \leq P^*. \tag{46}
\]

On the other hand, we also have:

\[
L(\bar{u}, \bar{z}, \bar{y}) = J_s(\bar{u}, \bar{z}) + \bar{y} \cdot g(\bar{u}, \bar{z}) = J_s(\bar{u}, \bar{z}) \geq P^*, \tag{47}
\]

where the last inequality is due to the fact that \( P^* \) is the minimum value of \( J_s(u, z) \) among all \((u, z)\) such that \( z = \frac{1}{N} \sum_{i=1}^{N} f_i(u_i, \pi_i) \), and \((\bar{u}, \bar{z})\) is one of them. Combining (46) and (47), we have \( L(\bar{u}, \bar{z}, \bar{y}) = P^* \), thus \((\bar{u}, \bar{z})\) is the globally optimal solution to the social welfare optimization problem (28). This completes the proof.\qed
This theorem establishes equivalence between the mean-field equilibrium and the socially optimal solutions. When the mean-field term is non-decreasing, the solution set of the mean-field equations is the same as that of the social welfare optimization problem as long as (28) has strong duality. Regarding this result, an interesting special case is where optimization problem as long as (28) has strong duality. Furthermore, according to Theorem 3, the solution set of the mean-field equations is the same as that of the social welfare optimization problem. When the mean-field term is non-decreasing, the solution set of the mean-field equilibrium and the socially optimal solutions. When Assumption 5 hold, then (u*1,..., u*N) is the mean-field equilibrium to (3) if and only if it is the globally optimal solution to the social welfare optimization problem (28).

This corollary can be easily proved based on Theorem 3; the non-empty interior condition ensures that Slater’s condition is satisfied, and Assumption 5 guarantees that the social welfare optimization problem (28) is convex with respect to (x1,...,xN,z). Therefore, based on duality theory [42, p. 224], the social welfare optimization problem (28) has strong duality. Furthermore, according to Theorem 3, the solution set of the mean-field equations is the same as that of the social welfare optimization problem.

In the more general case, Assumption 5 may not be satisfied, and the social welfare optimization problem may be non-convex with respect to (x1,...,xN,z). Therefore, to apply the result of the theorems, we need to check whether the social welfare optimization problem has strong duality, especially when Assumption 5 is not satisfied. Along this direction, many sufficient conditions have been developed to check the strong duality for non-convex optimization [43, 44, 45]. Due to space limit, we will not present the technical details of these works. Instead, we will provide an example in Section VI, where the social welfare optimization problem does not satisfy Assumption 5 but Theorem 2 and Theorem 3 can still be applied.

C. Relation to Potential Game

It is well-known that for a potential game, the equilibrium is a minimum point of the corresponding potential function [46, p.100]. However, the converse is not necessarily true: if there is an optimization problem with the same solution as the Nash equilibrium of a game, it does not mean the game has a potential [47]. In this subsection, we show that although the mean-field equilibrium (3) is an optimasolution to (28), the mean-field game (3) is not a potential game. To this end, we construct an example, where each agent has the following objective function

$$\min_{x_i \geq 1} (x_i - 1)^2 + x_i \log x_i,$$  \hfill (48)

Note that on the one hand, it is easy to verify that (48) satisfies Assumption 5. Therefore, based on Theorem 2, the mean-field equilibrium is equivalent to the modified social welfare optimization problem (28). On the other hand, based on Theorem 4.5 in [48], it is clear that (48) is not a potential game.

We comment that under fairly strong assumptions (i.e., F(\bar{x}) = c \sum_{i=1}^{N} x_i), the mean-field game may reduce to a potential game. In this case, the minimum point of the potential function is the Nash equilibrium of the mean-field game, while the optimal solution to (28) is an \epsilon-Nash equilibrium of the game. In other words, if we solve the potential minimization problem, the exact Nash equilibrium can be obtained, and if we solve the social welfare optimization, an approximate Nash equilibrium is obtained. From the computational perspective, it is more attractive to use the social welfare optimization instead of the potential function. This is because the computational complexity of solving the potential minimization problem increases as the number of agents increases, while the social welfare optimization enables a decentralized scheme where the computation time is irrelevant with respect to the number of agents in the game [23]. This property is important in large-scale game problems, and more details on the computation of mean-field equilibria can be found in the next section.

Remark 7. In this paper, we formulate the mean-field term F(\cdot) to depend on the average of the population state. However, the proposed method still works when the mean-field is the average of the control decisions. In this case, the individual cost functional (2) is defined as

$$J_i(x_i, u_i, F(m), G(m)) = V_i(x_i, u_i) + F\left(\frac{1}{N} \sum_{i=1}^{N} u_i\right),$$

Similar result can be obtained using the same approach.

V. ILLUSION ON EXISTENCE, UNIQUENESS AND COMPUTATION

The results introduced in the previous section have some interesting implications on the existence, uniqueness and computation of the mean-field equilibrium. These implications advance the corresponding results in the literature to more general cases. We will discuss these implications in this section.

A. Existence of the Mean-Field Equilibrium

The existence and uniqueness of the mean-field equilibrium is a problem of fundamental importance in mean-field games. This problem has been extensively studied in the literature [11, 8, 32, 43, 49], and many of these works are based on fixed point analysis. In this section, we provide a
novel approach inspired by the connection between the mean-field game and the social welfare optimization problem. We will start with the following result:

**Theorem 4.** There exists a mean-field equilibrium to (3) if the social welfare optimization problem (28) has strong duality, where \( \phi(\cdot) \) is such that \( \nabla \phi(z) = NF(z) \), \( \forall z \in \mathcal{X} \).

The proof of this theorem directly follows from Theorem 2, since the optimal solution to (28) is a mean-field equilibrium, if \( (28) \) admits a solution, then the mean-field equations also admit a solution. An interesting fact about Theorem 4 is that it draws connections between the existence of the mean-field equilibrium and the strong duality of (28), which enables us to check the existence of the mean-field equilibrium by verifying the strong duality of (28). Compared to other works on the existence of mean-field equilibrium [1], [8], [32], [33], [49], our result does not need to assume the agent cost functions to be convex with respect to the control, which is a typical assumption in the literature. Therefore, we can generalize existing results to the case where the individual cost functions are non-convex and the corresponding social welfare optimization problem has strong duality [43], [44].

Theorem 4 provides a sufficient condition for the existence of the mean-field equilibrium. In fact, we can show that this condition is also necessary under additional assumptions on the mean-field coupling term \( F(\cdot) \). This is summarized in the following theorem:

**Theorem 5.** Assume that \( F(\cdot) \) is non-decreasing, then there exists a mean-field equilibrium to the game (3) if and only if the social welfare optimization problem (28) has strong duality, where \( \phi(\cdot) \) is such that \( \nabla \phi(z) = NF(z) \), \( \forall z \in \mathcal{X} \).

The proof of Theorem 5 is a byproduct of the proof of Theorem 4, which directly follows from [46] and [47]. We comment that most related results in existing works only have sufficient conditions for the existence of the mean-field equilibrium [11], [8], [32], [33], [49]. In view of this fact, the significance of Theorem 5 is that it provides an existence condition that is both necessary and sufficient. Using this result, we can not only show that the mean-field equilibrium exists under strong duality, but also show that the mean-field equilibrium does not exist when strong duality does not hold for (28).

### B. Uniqueness of the Mean-Field Equilibrium

In addition to existence, we can also derive a sufficient condition for the uniqueness of the mean-field equilibrium. We summarize it as follows:

**Theorem 6.** Assume that the interior of set \( \bar{U}_i \) is non-empty for all \( i = 1, \ldots, N \), then there is a unique mean-field equilibrium to the game (3) if the social welfare optimization problem (28) is strictly convex with respect to \( (u_1, \ldots, u_N, z) \), where \( \phi(\cdot) \) is such that \( \nabla \phi(z) = NF(z) \), \( \forall z \in \mathcal{X} \).

The proof of this result follows easily from Corollary 2 under the assumptions, the solution set to the mean-field equations is equivalent to that of the social welfare optimization problem. As (28) is strictly convex, it has a unique solution. Therefore, the mean-field equations also have a unique solution.

The result of this theorem has interesting connections to many existing works on the uniqueness of the mean-field equilibrium. Although most of these works focus on continuum mass model, if we consider the finite counterpart of these works and adapt their models to our context, then we can roughly divide the conditions in these works in three categories. First, the cost function \( V_i(f(u_i, \pi_i), u_i) \) is assumed to be strictly convex with respect to \( u_i \), and the coupling term \( F(\cdot) \) is non-decreasing [8], [32], [33]. Second, the cost function \( V_i(f(u_i, \pi_i), u_i) \) is convex with respect to \( u_i \), and the coupling term \( F(\cdot) \) is strictly increasing [17], [49]. Third, the cost function is at least convex, and the coupling term is at least non-decreasing, but either one of them holds strictly [11]. We note that these conditions can be all recovered by Theorem 6; it is not hard to verify that all these conditions essentially ensure the social welfare optimization problem (28) to be strictly convex with respect to \( (u_1, \ldots, u_N, z) \). According to Theorem 6, there is a unique mean field equilibrium for the mean-field game. Therefore, Theorem 6 transforms the uniqueness of the mean-field equilibrium to the strict convexity of an optimization problem, providing a novel interpretation of these existing results. On the other hand, we also point out that there are other works that prove uniqueness using different methods [4], [50], which can not be interpreted by Theorem 6.

To summarize, to this point, we have established the connections between the mean-field equilibrium and the social welfare optimization problem, and discussed the implications on the existence and uniqueness of the mean-field equilibrium. To better understand these results, we graphically summarize these connections using a diagram in Figure 1. In this figure, S.O. stands for the socially optima solution to (28), MFE stands for the mean-field equilibrium, the arrow indicates the conclusion we can draw, and the text on the arrow denotes the conditions we need to draw the conclusion.

### C. Computation of the Mean-Field Equilibrium

Aside from existence and uniqueness, another important implication of our result is on the computation of the mean-field equilibrium. Since the mean-field game can be connected to the social welfare optimization problem (28), we can compute the mean-field equilibrium by solving the corresponding social welfare optimization problem. When problem (28) is convex, there are many efficient algorithms to compute its solutions. In this subsection, we will present a primal-dual algorithm [42], Chap. 10] to compute the mean-field equilibrium, and we will show that many algorithms in the literature for computing the mean-field equilibrium are equivalent to the primal-dual algorithm for solving the corresponding social welfare optimization problem.
We consider a mean-field game (3) that satisfies Assumption 5. According to Corollary 2 to compute the mean-field equilibrium, we can construct the corresponding social welfare optimization problem and solve it using a primal-dual algorithm. The details of the algorithm are summarized in Algorithm 1. To implement the algorithm, we first construct the social welfare optimization problem (28) by finding the virtual cost $\phi$. After this, an initial guess for the Lagrangian multiplier $\lambda$ is broadcast to all the agents. Each agent can then independently solve the convex optimization problem (49), while the virtual supplier solves the optimization problem (50) for a given $\lambda$. The solutions of the cost minimization problems are collected and used to update the dual $\lambda$ according to (51). The updated dual variable is then broadcast to the agents again and this procedure is iterated until it converges.

Based on [42, Chap. 10] and Corollary 2, it is easy to prove that the algorithm converges to the mean-field equilibrium of (3).

**Proposition 1.** Algorithm 1 converges to the mean-field equilibrium of (3) if Assumption 5 is satisfied and the step-size $\nu_k$ satisfies $\lim_{k \to \infty} \nu_k = 0$ and $\lim_{k \to \infty} \sum_{m=1}^{k} t_m = \infty$.

It can be verified that the proposed algorithm includes many existing ways to compute the mean-field equilibrium as special cases. For instance, the algorithms proposed in [9] and [23] are equivalent to the primal-dual algorithm with a scaled stepsize in (51). Specifically, a finite-horizon deterministic linear quadratic mean-field game was considered in [9], and an iterative algorithm was proposed to compute its mean-field equilibrium. The first step of the algorithm solves the same optimal response problem as (49), and the second step updates $z$ according to:

$$z^k = z^{k-1} + \nu^k \left( \frac{1}{N} \sum_{i=1}^{N} f_i(u_i^k) - z^{k-1} \right),$$  

where $\nu^k$ is the step size. We comment that this is a scaled version of the Algorithm 1. This is because in the linear quadratic game, $\phi(\cdot)$ is a quadratic function, and (50) indicates that there is a positive definite matrix $A$ such that $\nabla \phi(\cdot) = NF(\cdot)$.

Algorithm 1 The Primal-Dual Algorithm to Compute the Mean-Field Equilibrium

**Initialization:** the mean-field game (3).
1. Construct (28) by finding $\phi(\cdot)$ that $\nabla \phi(\cdot) = NF(\cdot)$.
2. Generate initial guess for the Lagrange Multiple, $\lambda^0$.
3. for $k = 1, 2, \ldots$, do
   4. Update the individual decisions by solving:
      $$u_i^k = \arg \min_{u_i} \mathbb{E} \left( V_i(f_i(u_i, \pi_i), u_i) + \lambda^{k-1} \cdot f_i(u_i, \pi_i) \right)$$  

5. Update the virtual supplier decision:
   $$z^{k-1} = \arg \min_{z \in \mathcal{X}} \phi(z) - N \lambda^{k-1} \cdot z,$$

6. Update the dual variable according to:
   $$\lambda^k = \lambda^{k-1} + \nu^k \left( \frac{1}{N} \mathbb{E} f_i(u_i^k, \pi_i) - z^{k-1} \right),$$

7. end for

**Output:** the collective decisions $(u_1, \ldots, u_N)$.

$Az^k = \lambda^k$ for all $k$. Therefore, the update equation (51) in Algorithm 1 can be written as:

$$z^k = z^{k-1} + A^{-1} \nu^k \left( \frac{1}{N} \mathbb{E} f_i(u_i^k, \pi_i) - z^{k-1} \right),$$

which is equivalent to (52) with $A^{-1} \nu^k = \nu^k$.

VI. CASE STUDIES

This section presents two examples to show how our results can be used to study the properties of the mean-field equilibrium. The first example is a special case where the mean-field coupling term $F(\cdot)$ is non-decreasing. In this case, we can show that the mean-field equilibrium is equivalent to socially optimal solution if the social welfare optimization problem has strong duality. The second example presents a more general case where the monotonicity condition is not satisfied. In this case, we can still draw connections between the mean-field equilibrium and the social welfare optimization problem using the result of Theorem 2 and Corollary 1.

A. Example: Special Case with Monotone Mean-Field Coupling

The first example considers the problem of coordinating the charging of a population of electric vehicles (EV) [9]. Each EV is modeled as a linear dynamic system, and the objective is to acquire a charge amount within a finite horizon while minimizing the charging cost. The charging cost of each EV is coupled through the electricity price, which is an
affine function of the average of charging energy. This leads to the following game problem [9]:

$$\min_{u_i} \eta|u_i - z|^2 + 2\gamma(z + c)^T u_i$$  \hspace{1cm} (53)

subject to:

$$\begin{align*}
z &= \frac{1}{N} \sum_{i=1}^{N} u_i, \\
x_i(t+1) &= x_i(t) + u_i(t), \\
0 \leq x_i(t) \leq \bar{x}_i, 0 \leq u_i(t) \leq \bar{u}_i, \sum_{i=1}^{T} u_i(t) = \gamma_i,
\end{align*}$$

where $0 < \eta \ll \gamma$, $x_i(t) \in \mathbb{R}$ is the state of charge (scaled by the capacity) of the EV battery, $u_i(t)$ denotes the charging energy during the $t$th control period, $\bar{x}_i$ is the battery capacity, and the electricity price is $2\gamma(z + c)$.

In this mean-field game, the coupling term $F(z) = 2(\gamma - \eta)z$ is increasing with respect to $z$, thus the monotonicity condition is satisfied. We also note that (53) is slightly different from (5) in the sense that the coupling term depends on the average of control instead of the state. However, based on Remark 7 there is no essential difference between these two formulations, and our result applies universally.

The focus of this case study is to investigate the efficiency, existence, uniqueness, and computation of the mean-field equilibrium to (53). To this end, we first construct the social welfare optimization problem for (53) by finding the function $\phi(\cdot)$ such that $F(z) = \frac{1}{N} \nabla \phi(z)$. Since $F(z) = 2(\gamma - \eta)z$, we have $\phi(z) = N(\gamma - \eta)z^2$. This indicates that the social welfare optimization problem for (53) is as follows:

$$\min_{(u_1, \ldots, u_N, z)} \sum_{i=1}^{N} (\eta|u_i|^2 + 2\gamma c^T u_i) + N(\gamma - \eta)z^2$$  \hspace{1cm} (54)

subject to:

$$\begin{align*}
z &= \frac{1}{N} \sum_{i=1}^{N} u_i, \\
x_i(t+1) &= x_i(t) + u_i(t), \quad \forall i \in I, \quad \forall k \in K \\
0 \leq x_i(t) \leq \bar{x}_i, 0 \leq u_i(t) \leq \bar{u}_i, \sum_{i=1}^{T} u_i(t) = \gamma_i,
\end{align*}$$

where $I = \{1, \ldots, N\}$ and $K = \{1, \ldots, K\}$. It can be verified that the game problem (53) satisfies Assumption 2. According to Corollary 2 the control decision $(u_1, \ldots, u_N)$ is a mean-field equilibrium for (53) if and only if it is an optimal solution to (54). In addition, since (54) is strictly convex with respect to $(u_1, \ldots, u_N, z)$, based on Theorem 2 the mean-field equilibrium exists and is unique.

Next, we study the computation of the mean-field equilibrium to (53). In [9], the mean-field equilibrium of (53) is computed by an iterative algorithm. In the algorithm, each agent takes $z$ as given and solve the problem (53) to obtain an optimal control $u_i$. Based on $u_i$, $z$ is updated according to (52). Each agent then takes $z'$ as given to repeat the first step. This algorithm converges to the mean-field equilibrium of (53) if $\lim_{k \to \infty} \nu_k = 0$ and $\lim_{k \to \infty} \sum_{m=1}^{k} \nu_m = \infty$. Under this choice of step-size, the algorithm is referred to as the Mann iteration, and we will use it as the benchmark.

In this paper, we propose to compute the mean-field equilibrium using the connections between the mean-field game and the social welfare optimization problem. Based on Corollary 2 the mean-field equilibrium can be computed by solving the convex problem (54). This problem can be efficiently solved by alternating direction method of multipliers (ADMM) [34]. In the rest of this subsection, we will use numerical simulation to compare the performance of the proposed ADMM algorithm with the benchmark algorithm (Mann iteration).

In the numerical simulation, we generate 100 sets of EV parameters over 36 control periods, and each period spans 5 minutes. The heterogeneous parameters, including the capacity of the EV batteries, the maximum charging rate of the batteries, and the other parameters in the objective function are all generated based on uniform distributions. We run both Mann iterations and ADMM for 200 iterations, and the simulation results are shown in Fig. 2 - Fig. 4. In Fig. 2 we randomly select an EV and show its trajectory of state of charge under the ADMM solution. To compare the performance between ADMM and the Mann iteration, we show $||u_i||$ for a randomly selected EV (Fig. 3) and the average control decision $||z||$ (Fig. 4) over each iteration. Based on the simulation results, the ADMM algorithm convergences to the optimal solution after about 50 iterations, while the Mann iteration converges after 100 iterations. Therefore, the proposed ADMM converges faster than Mann iteration. We emphasize that the algorithm converges only if both $z$ and $u_i$ converge. In Fig. 4 it seems that the Mann iteration converges faster than ADMM. As a matter of fact, although $||z||$ converges fairly fast, $z$ does not converge until after 100 iterations. This has been verified in the codes.

B. Example: Non-Monotone Mean-Field Coupling

The second example presents a general case where the mean-field coupling term is not monotone. In the rest of this subsection, we show how to use the result of Theorem 2 and Corollary 1 to draw connections between the mean-field equilibrium and the social welfare optimization problem.
Consider a game with \( N \) agents, where each agent \( i \) wants to minimize the following objective function:

\[
\min_{x_i \in \mathbb{R}} (x_i - \alpha_i)^2 + \beta x_i \sin \left( \frac{\bar{x} - 1}{N} \sum_{i=1}^{N} \alpha_i \right),
\]

where \( \alpha_i \) is a scalar and \( \beta > 0 \) is a positive constant. In this example, the mean-field coupling term is \( F(\bar{x}) = \beta \sin \left( \frac{\bar{x} - 1}{N} \sum_{i=1}^{N} \alpha_i \right) \). It is clear that the monotonicity condition does not hold.

Next, we will construct the corresponding social welfare optimization problem for (55). To this end, consider a virtual supply cost that satisfies \( \nabla \phi(z) = NF(z) \). This gives \( \phi(z) = -N\beta \cos \left( z - \sum_{i=1}^{N} \alpha_i \right) \), and the social welfare optimization problem is as follows:

\[
\min_{x_1, \ldots, x_N, \alpha_1, \ldots, \alpha_N} \sum_{i=1}^{N} (x_i - \alpha_i)^2 - N\beta \cos \left( z - \sum_{i=1}^{N} \alpha_i \right)
\]

s.t. \( z = \frac{1}{N} \sum_{i=1}^{N} x_i \); \( x_i = \mathbb{R}, z \in \mathbb{R}, \forall i = 1, \ldots, N \).

The above problem is non-convex with respect to \( (x_1, \ldots, x_N, z) \), but we can still show that the primal problem social (56) has the same optimal value as its dual problem.

**Lemma 4.** The social welfare optimization problem (56) has strong duality.

**Proof.** To prove strong duality, according to Definition 2, it suffices to show that there exists \( \lambda^* \) such that \( P^* = D^* = D(\lambda^*) \). To show this, we first note that since \( \beta > 0 \), the cost function of primal problem (56) is lower bounded by \( -N\beta \):

\[
\sum_{i=1}^{N} (x_i - \alpha_i)^2 - N\beta \cos \left( z - \sum_{i=1}^{N} \alpha_i \right) \geq -N\beta
\]

It can be verified that when \( x_i = \alpha_i \) and \( z = \frac{1}{N} \sum_{i=1}^{N} \alpha_i \), the cost function of (56) equals \( -N\beta \) and the constraints are satisfied. Therefore, \( -N\beta \) is the optimal value for the primal problem. According to Definition 2, it suffices to find \( \lambda^* \) such that the minimum value of \( L(u, z, \lambda^*) \) is also \( -N\beta \). Let \( \lambda^* = 0 \), then the Lagrangian dual \( L(u, z, 0) \) corresponds to the following problem:

\[
\min_{x_1, \ldots, x_N, \alpha_1, \ldots, \alpha_N} \sum_{i=1}^{N} (x_i - \alpha_i)^2 - N\beta \cos \left( z - \sum_{i=1}^{N} \alpha_i \right)
\]

s.t.: \( x_i = \mathbb{R}, z \in \mathbb{R}, \forall i = 1, \ldots, N \).

The optimal value of (58) is clearly \( -N\beta \). Therefore, strong duality holds.

Based on Theorem 2 if the social welfare optimization problem (56) has strong duality, then any solution to (56) is a mean-field equilibrium.

We note that this relation only holds from one direction: there may exist a mean-field equilibrium which is not socially optimal. In section IV, we showed that this can be resolved when the mean-field equilibrium is unique. In our example, we can show that this is true under some conditions.

**Lemma 5.** If \( 0 < \beta < 2 \), then the mean-field equations for (55) admit a unique solution.

**Proof.** To prove the uniqueness, the idea is to construct a contraction mapping, whose fixed point is the solution to the mean-field equations. In particular, we first regard \( m \) as given and solve the problem (55) to derive the optimal solution as \( x_i^* = \alpha_i - \frac{1}{2} \beta \sin \left( \frac{\bar{x} - 1}{N} \sum_{i=1}^{N} \alpha_i \right) \). Then we define a function \( T : \mathbb{R} \to \mathbb{R} \) that maps \( \bar{x} \) to the average of \( x_i^* \):

\[
T(\bar{x}) = \frac{1}{N} \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \beta \sin \left( \frac{\bar{x} - 1}{N} \sum_{i=1}^{N} \alpha_i \right).
\]

It can be verified that the mean-field equilibrium is the fixed point of this mapping. Since \( |\sin(x) - \sin(y)| \leq |x - y| \), we have \( T(m_1) - T(m_2) \leq \frac{1}{2} \beta |m_1 - m_2| \) for any \( m_1, m_2 \in \mathbb{R} \). Therefore, as \( \beta < 2 \), \( T(\bar{x}) \) is a contraction mapping, and it has a unique fixed point. This completes the proof.

Based on Corollary 1 and Lemma 5, it is clear that the mean-field equilibrium to (55) is socially optimal when \( 0 < \beta < 2 \).

To summarize, the connection between the mean-field equilibrium to (55) and the social welfare optimization problem is as follows:

**Theorem 7.** If \( \beta > 0 \), the any socially optimal solution of (56) is a mean-field equilibrium to (55). In addition, if \( 0 < \beta < 2 \), then the mean-field equilibrium to (55) is the globally optimal solution to the social welfare optimization problem (56).

The proof of this theorem follows from Lemma 4, Lemma 5, Theorem 2 and Corollary 1.

**VII. CONCLUSION**

This paper studies the connections between a class of mean-field games and the social welfare optimization problem. We showed that the mean-field equilibrium is the optimal solution to a social welfare optimization problem, and this holds for both convex and non-convex individual cost functions and action spaces. The result enables us to evaluate the efficiency of the mean-field equilibria, and it also provides interesting implication on the existence, uniqueness and computation of the mean-field equilibrium. Numerical simulations are presented to validate the proposed approach. Future work includes extending the proposed approach to the case of infinitely many agents and more general formulations where the mean-field term depends on the probability distribution of the population state.

**APPENDIX**

**A. Proof of Lemma 7**

It is well-known that based on the Riesz-Fischer theorem [51, p.148], the \( L^2 \) space is complete. We show that \( \mathcal{X} \) is
isomorphic to $L^2$, and therefore it is also complete [52 p.20]. For this purpose, we define the following mapping: 
$g : L^2 \rightarrow \mathcal{X}$ that satisfies $g(\cdot) = \{g_l(\cdot), t \in T\}$ and 
g($l(t)) = e^{rt/2}|t|$ for any $l \in L^2$, where $l = \{l(t), t \in T\}$. This is a linear surjective mapping and it can be verified that 
g($l(t)) \cdot g(l(2t)) = l_1(t) \cdot l_2(t)$, where the left-hand side inner product is defined as $\langle \cdot \rangle$ on $\mathcal{X}$ and the right-hand side inner product is defined on the $L^2$ space in the canonical form. This indicates that $L^2$ and $\mathcal{X}$ are isomorphic, which completes the proof.

B. Proof of Lemma 2

Proof. To prove this result, it is clear that we have:

$$\|E(F(\bar{x}) \cdot x_i) - E(F(\bar{x})) \cdot Ex_i\| = |I_1 + I_2| \leq |I_1| + |I_2|,$$

where we define $I_1 = E(F(\bar{x}) \cdot x_i) - E(F(\bar{x})) \cdot Ex_i$ and $I_2 = E(F(\bar{x}) \cdot Ex_i) - F(\bar{x}) \cdot Ex_i$. Then it suffices to show that both $|I_1|$ and $|I_2|$ converge to 0 at the rate of $\frac{1}{\sqrt{N}}$. To show this, we first note that:

$$|I_1| = E(F(\bar{x}) \cdot x_i) - E(F(\bar{x})) \cdot Ex_i = |I_3 + I_4| \leq |I_3| + |I_4|,$$

where $I_3 = E(F(\bar{x}) \cdot x_i) - E(F(\bar{x} - x_i) \cdot x_i)$ and $I_4 = E(F(\bar{x} - x_i) \cdot x_i) - E(F(\bar{x}) \cdot x_i)$. Since $F(\cdot)$ is Lipschitz continuous with the constant $L \geq 0$, and the second moment of $x_i$ is bounded, we have:

$$|I_3| = E(F(\bar{x} - x_i) \cdot x_i) \leq \frac{L}{N} \|x_i\|^2 \leq \frac{LC}{N}.$$

In addition, as $x_i$ is uncorrelated with $\bar{x} - x_i$, we have:

$$|I_4| = |E(F(\bar{x} - x_i) \cdot x_i) - E(F(\bar{x}) \cdot x_i)| = E(F(\bar{x} - x_i) \cdot x_i - E(F(\bar{x}) \cdot x_i) \cdot Ex_i) \leq \frac{L}{N} \|Ex_i\|^2.$$

Note that $E|\|x_i\|^2$ is bounded, and thus $\|Ex_i\|$ is also bounded:

$$\|Ex_i\| \leq \|x_i\| = \sqrt{\|x_i\|^2 - E(\|x_i\|^2 - Ex_i^2)} \leq \sqrt{\|x_i\|^2} \leq \sqrt{C}. \quad (60)$$

This indicates that $|I_4| \leq \frac{2LC}{N}$. Therefore, $|I_1| \leq |I_3| + |I_4| \leq \frac{2LC}{N}$. To show that $|I_2|$ converges to 0 at the rate of $\frac{1}{\sqrt{N}}$, we define a random variable $r_N = F(\bar{x}) - F(\bar{x})$. Note that:

$$I_2 \leq \|E(F(\bar{x}) - F(\bar{x}))\| \|Ex_i\| = \|Er_N\| \|Ex_i\|. \quad (61)$$

Since $\|Er_N\| \leq E|r_N|$ and $\|Ex_i\| \leq \sqrt{C}$, the inequality (61) reduces to $I_2 \leq \sqrt{C}E|r_N|$. Therefore, for our purpose, it suffices to show that $E|r_N| = O(\frac{1}{\sqrt{N}})$. Since $F(\cdot)$ is Lipschitz continuous, we have:

$$\|r_N\| = \|F(\bar{x}) - F(\bar{x})\| \leq L\|\bar{x} - \bar{x}\|.$$

Therefore, it suffices to show that $E|\bar{x} - \bar{x}| = O(\frac{1}{\sqrt{N}})$.

To prove this, we note that since $x_i$ and $x_j$ are uncorrelated, thus we have the following relation:

$$E|\bar{x} - \bar{x}|^2 = \frac{1}{N^2} \sum_{i=1}^{N} \|x_i - x_j\|^2 < \frac{1}{N^2} \sum_{i=1}^{N} \|x_i\|^2 \leq \frac{NC}{N^2} = \frac{C}{N}.$$

Therefore, similar to (60), we have

$$E|\bar{x} - \bar{x}| \leq \sqrt{E|\bar{x} - \bar{x}|^2} \leq \sqrt{\frac{C}{N}}.$$

This completes the proof.

C. Proof of Lemma 3

Proof. Since $\nabla G(\cdot)$ is Lipschitz continuous with constant $\beta$, for any $x, y \in \mathcal{X}$, we first show that:

$$G(x) - G(y) \leq \nabla G(y) \cdot (x - y) + \frac{\beta}{2} \|x - y\|^2. \quad (62)$$

To prove (62), we note that the Lipschitz continuity of $\nabla G(\cdot)$ together with Cauchy-Schwarz inequality imply that

$$(\nabla G(x) - \nabla G(y)) \cdot (x - y) \leq \beta \|x - y\|^2 \quad (63)$$

Construct a new function $T(x) = \frac{\beta}{2} \|x\|^2 - G(x)$. It can be verified that (63) is equivalent to $\nabla T(x) \cdot \nabla T(y) \cdot (x - y) \leq 0$, thus $T(x)$ is convex. The first-order condition for convexity of $T$ implies that $T(y) \geq T(x) + \nabla T(x) \cdot (y - x)$. This is exactly equivalent to (62), thus inequality (62) holds.

Based on (62), we have:

$$E|G(\bar{x}) - G(\bar{x} - x)| \leq E \left( \nabla G(\bar{x}) \cdot \bar{x} - \frac{x}{N} \right) + \frac{\beta}{2} \frac{E|x|^2}{N} \leq E \left( \|\nabla G(\bar{x})\|^2 \frac{x}{N} \right) + \frac{\beta}{2} \frac{E|x|^2}{N}. \quad (64)$$

Since derivative of $G(\cdot)$ is Lipschitz continuous with constant $\beta$, we have $\|\nabla G(\cdot)\| \leq \|\nabla G(0)\| + \beta \|\bar{x}\|$. Plugging this in (64), we have the following relations:

$$E|G(\bar{x}) - G(\bar{x} - x)| \leq \frac{\|\nabla G(0)\|}{N} E|x| + \frac{\beta}{N} \frac{E\|\bar{x}\|^2}{N^2} + \frac{\beta}{2} \frac{E|x|^2}{N}. \quad (65)$$

The right-hand side of the above inequality consists of three terms. We will show that all the three terms converge to 0 at the rate of $1/N$. For notation convenience, let the three terms be $T_1^N = \frac{\|\nabla G(0)\|}{N} E|x|$, $T_2^N = \frac{\beta}{N} \frac{E\|\bar{x}\|^2}{N^2}$.

Since $\frac{\|\nabla G(0)\|}{N}$ converges to 0 at the rate of $1/N$. For notation convenience, let the three terms be $T_1^N = \frac{\|\nabla G(0)\|}{N} E|x|$, $T_2^N = \frac{\beta}{N} \frac{E\|\bar{x}\|^2}{N^2}$.

Since $\frac{\|\nabla G(0)\|}{N}$ converges to 0 at the rate of $1/N$. For notation convenience, let the three terms be $T_1^N = \frac{\|\nabla G(0)\|}{N} E|x|$, $T_2^N = \frac{\beta}{N} \frac{E\|\bar{x}\|^2}{N^2}$.

Since $\frac{\|\nabla G(0)\|}{N}$ converges to 0 at the rate of $1/N$. For notation convenience, let the three terms be $T_1^N = \frac{\|\nabla G(0)\|}{N} E|x|$, $T_2^N = \frac{\beta}{N} \frac{E\|\bar{x}\|^2}{N^2}$.

Since $\frac{\|\nabla G(0)\|}{N}$ converges to 0 at the rate of $1/N$. For notation convenience, let the three terms be $T_1^N = \frac{\|\nabla G(0)\|}{N} E|x|$, $T_2^N = \frac{\beta}{N} \frac{E\|\bar{x}\|^2}{N^2}$.
and $T^N_3 = \frac{\beta}{2N^2} \mathbb{E}[|x_i|^2]$. In the proof of Lemma 3, we showed that $\mathbb{E}[|x_i|^2] \leq C$. Therefore, we have $T^N_1 < \frac{\sqrt{C}||\nabla G(0)||}{N}$. Due to Assumption 3, $||\nabla G(0)||$ is bounded, thus $T^N_1 = O\left(\frac{1}{N}\right)$. As for the second term, we have:

$$T^N_2 = \frac{\beta}{N} \mathbb{E}\left[\left|\mathbf{x}(t)|x_i|\right|\middle|\Omega_t\right] = \frac{\beta}{N^2} \mathbb{E}\left[\sum_{j=1}^{N} |x_j||x_i|\right]$$

$$= \frac{\beta}{N^2} \left(\mathbb{E}[|x_i|^2] + \sum_{j \neq i} \mathbb{E}[|x_i||x_j|]\right)$$

$$= \frac{\beta}{N^2} \left(\mathbb{E}[|x_i|^2] + \sum_{j \neq i} \mathbb{E}[|x_i||x_j|]\right) \leq \frac{\beta C}{N^2}$$

Furthermore, since $\mathbb{E}[|x_i|^2] \leq C$, we have $T^N_2 = \frac{\beta C}{2N^2}$, thus $T^N_1 + T^N_2 + T^N_3 = O\left(\frac{1}{N}\right)$. This completes the proof. □

**References**

[1] J. M. Lasry and P. L. Lions. Mean field games. *Japanese Journal of Mathematics, 2*(1):229–260, 2007.

[2] J. M. Lasry and P. L. Lions. Jeux à champ moyen. i–le cas stationnaire. *Comptes Rendus Mathématique, 343*(9):619–625, 2006.

[3] J. M. Lasry and P. L. Lions. Jeux à champ moyen. ii–horizon fini et contrôle optimal. *Comptes Rendus Mathématique, 343*(10):679–684, 2006.

[4] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized c-Nash equilibria. *IEEE Transactions on Automatic Control, 52*(9):1560–1571, 2007.

[5] O. Guéant, J. M. Lasry, and P. L. Lions. Mean field games and applications. In *Paris-Princeton lectures on mathematical finance*, pages 205–266. Springer, 2011.

[6] M. Huang, R. P. Malhamé, P. E. Caines, et al. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information and Systems, 6*(3):221–252, 2006.

[7] R. Ferreira and D. Gomes. Existence of weak solutions to stationary mean-field games through variational inequalities. *arXiv preprint arXiv:1512.05828*, 2015.

[8] S. A. Ahuja. Wellposedness of mean field games with common noise under a weak monotonicity condition. *SIAM Journal on Control and Optimization, 54*(1):30–48, 2016.

[9] S. Grammatico, F. Parise, M. Colombino, and J. Lygeros. Decentralized convergence to Nash equilibria in constrained deterministic mean field control. *IEEE Transactions on Automatic Control, 61*(11):3315–3329, 2016.

[10] A. Lachapelle, J. Salomon, and G. Turinici. Computation of mean field equilibria in economics. *Mathematical Models and Methods in Applied Sciences, 20*(04):567–588, 2010.

[11] Y. Achdou and I. Capuzzo-Dolcetta. Mean field games: numerical methods. *SIAM Journal on Numerical Analysis, 48*(3):1136–1162, 2010.

[12] D. Lackier. Stochastic differential mean field game theory. PhD thesis, PRINCETON UNIVERSITY, 2015.

[13] A. Bensoussan, J. Frehse, P. Yam, et al. Mean field games and mean field type control theory, volume 101. Springer, 2013.

[14] M. Huang. Large-population LQG games involving a major player: the Nash certainty equivalence principle. *SIAM Journal on Control and Optimization, 58*(3):1560–1571, 2010.

[15] A. Bensoussan, M. Chau, and S. Yam. Mean field games with a dominating player. *Applied Mathematics and Optimization, pages 1–38, 2015.*
[42] D. G. Luenberger. *Optimization by vector space methods.* John Wiley & Sons, 1997.

[43] F. Flores-Bazan and G. Mastroeni. Strong duality in cone constrained nonconvex optimization. *SIAM Journal on Optimization, 23*(1):153–169, 2013.

[44] G. Carcamo and F. Flores-Bazan. Strong duality and KKT conditions in nonconvex optimization with a single equality constraint and geometric constraint. *Mathematical Programming,* pages 1–32, 2016.

[45] F. Flores-Bazan, F. Flores-Bazan, and C. Vera. A complete characterization of strong duality in nonconvex optimization with a single constraint. *Journal of Global Optimization, 53*(2):185–201, 2012.

[46] J. P. Hespanha. An introductory course in noncooperative game theory. 2013.

[47] R. Johari and J. N. Tsitsiklis. Parameterized supply function bidding: Equilibrium and efficiency. *Operations research,* 59(5):1079–1089, 2011.

[48] D. Monderer and L. S. Shapley. Potential games. *Games and economic behavior,* 14(1):124–143, 1996.

[49] P. Cardaliaguet. Notes on mean field games. Technical report, 2010.

[50] M. Huang, R. P. Malhamé, and P. E. Caines. Nash equilibria for large-population linear stochastic systems of weakly coupled agents. In *Analysis, Control and Optimization of Complex Dynamic Systems,* pages 215–252. Springer, 2005.

[51] H. L. Royden and P. Fitzpatrick. *Real analysis,* volume 198. Macmillan New York, 1988.

[52] J. B. Conway. *A course in functional analysis,* volume 96. Springer Science & Business Media, 2013.