The study of collective modes is a fundamental component of many-particle physics, because for every spontaneously broken continuous symmetry there are low-energy modes that emerge as expected from Goldstone’s theorem [1], and additional higher-energy excitations such as the Higgs mode [2, 3]. Collective modes are essential in understanding a variety of systems ranging from condensed matter (quantum magnets, superconductors) [4, 5], high energy physics (standard nuclear matter, quantum chromodynamics) [6, 7], and astrophysics (neutron stars, black holes) [8, 9] to atomic (Bose and Fermi superfluids) physics [10, 11]. Unfortunately, in condensed matter it is not easy to tune parameters such as interactions, density, and dimensionality over a wide range, in high energy physics it is very difficult, and in astrophysics it is impossible. However, in atomic physics this is relatively easy via well established techniques [12, 13]. This makes it possible to investigate collective modes in superfluids, particularly important because they reveal the effects of quantum fluctuations above the superfluid ground state.

Superfluids in 2D are inherently different from their 3D counterparts, due to the importance of fluctuations [14, 15] leading to a Berezinskii-Kosterlitz-Thouless (BKT) transition [16, 17]. In the context of ultracold atoms, the desire to study 2D Fermi superfluids is driven not only by connections to high-temperature superconductors [18–21], but also by the high degree of experimental control that allows the measurement of the equation of state [22–24], the observation of the BKT transition [25, 26], and the examination of collective modes [27–30].

Ultracold fermions with tunable interactions in nearly 2D configurations were studied using harmonic traps and optical lattices [31–34]. With the very recent advent of box potentials, it is now possible to study experimentally homogeneous 2D fermions [28, 30, 35]. Inspired by recent measurements of collective excitations via Bragg-spectroscopy [30, 36, 37], we investigate the low-energy collective modes of 2D s-wave Fermi superfluids in box potentials, and find very good agreement with experiments. We establish that mean field (saddle point) results in 2D [38] produce incorrect values of the chemical potential and lead to the erroneous conclusion that the sound velocity is a constant throughout the BCS to Bose evolution [39–41]. In sharp contrast, we show that the inclusion of quantum fluctuations [42] is crucial to produce physically acceptable results for the dispersion of collective modes and leads to a varying speed of sound in the crossover from BCS to Bose regimes at low temperatures [43]. Furthermore, we demonstrate that phase and modulus fluctuations of the pairing order parameter become increasingly more coupled with growing interaction strength. Importantly, we clarify the longstanding confusion about the difference between the resulting sound mode arising from the broken U(1) symmetry and Landau’s phenomenological first sound.

Based on weakly coupled s-wave Fermi superfluids and a linear dispersion of the collective mode, it has been long thought [44] that the low-energy collective modes in neutral Fermi superfluids are strongly damped (due to Landau damping) when the energy of the collective mode is sufficiently large to reach the pairbreaking energy threshold. In 3D, this view is still valid, even when taking into account the changing concavity of the dispersion [45, 46]. However, we show that the situation is fundamentally different in 2D, where the inclusion of the ubiquitous bound states and of higher-order momentum corrections to the collective mode dispersion show that the collective mode energy never reaches the two-particle continuum, and thus there is no damping of the collective mode at the Gaussian level for s-wave superfluids.

**Hamiltonian:** To analyze the low-energy collective modes of 2D s-wave Fermi superfluids in box potentials, we start from the Hamiltonian density

\[
\mathcal{H} = \frac{\hbar^2}{2m} \left( \nabla^2 - i \nabla \right) \psi_s(r) - g \psi_s(r) \psi_s^\dagger(r) \psi_s^\dagger(r) \psi_s^\dagger(r),
\]

where \( \psi_s(r) \) is a fermion field operator with spin \( s \) at posi-
tion \( r \). The first term is the kinetic energy and the second represents local attractive interactions. The associated action is \( S(\psi^\dagger, \psi) = \int d^3r \{ \psi^\dagger(r) [\hbar c - \mu] \psi(r) + \mathcal{H}(r) \} \), where \( r = (r, \tau) \), \( \int d^3r = \int^\beta d\tau \int d^2r \), \( \beta = \hbar/k_B T \), and \( \mu \) is the chemical potential. The grand canonical partition function of the system is \( Z = \int D\psi^\dagger D\psi e^{-S/k_B} \).

We introduce the Hubbard-Stratonovich complex pair field \( \Phi(r) \) to decouple the contact interactions and integrate out the fermionic fields to obtain an effective action \( S_{\text{eff}}(\psi^\dagger, \Phi) \). We write \( \Phi(r) = [\Phi(r)] e^{i\theta(r)} \) in terms of its modulus \( |\Phi(r)| = |\Delta| [1 + \lambda(r)] \) and phase \( \theta(r) \), and expand \( S_{\text{eff}}(\psi^\dagger, \Phi) \) up to quadratic order in both the phase \( \theta(r) \) and modulus fluctuations \( \lambda(r) \) around the saddlepoint \( |\Phi_{\text{sp}}(r)| = |\Delta| \). The resulting Gaussian action is

\[
S_{\text{eff}} = S_{\text{sp}} + \beta |\Delta|^2 \sum_q (\bar{\theta}_{-q} - \bar{\lambda}_{-q}) M(q) \left( -\bar{\theta}_q / \bar{\lambda}_q \right),
\]

where \( q = (q_L, q_V) \) and \( \nu = 2\pi n / \beta \) are bosonic Matsubara frequencies, \( S_{\text{sp}} = \beta \sum_q (\bar{\xi}_q - \bar{E}_q + \bar{\lambda}_q |\Delta|^2) / 2q \) is the saddlepoint action, and \( M(q) \) is the symmetric Gaussian fluctuation matrix \([47]\). Using the analytic continuation \( i\nu \rightarrow \omega + i\delta \), the matrix elements of \( M(q) \), at zero temperature, are

\[
M_{\pm} = \int \frac{d^2k}{4\pi^2} \left[ \frac{E_+ + E_- E_+ E_- + \xi_+ \xi_- \pm |\Delta|^2}{2E_k} \right] + \frac{1}{2E_k},
\]

\[
M_{-} = \int \frac{d^2k}{4\pi^2} \left[ \frac{\hbar \omega - E_+ \xi_+ - \xi_-}{2E_k E_+} \right] ,
\]

where \( E_k = \sqrt{\xi_k^2 + |\Delta|^2} \) is the quasiparticle energy, \( \xi_k = \epsilon_k - \mu \) is the energy of the free fermions of mass \( m \) \( \epsilon_k = \hbar^2 |k|^2 / 2m \) with respect to \( \mu \), and \( |\Delta| \) is the modulus of the order parameter. We have used the shorthand notations \( E_{\pm} = E_k e^{\pm iq/2} \) and \( \xi_{\pm} = \xi_k e^{\pm iq/2} \). \( M_{++} = M_{\theta\theta}, M_{--} = M_{\lambda\lambda}, \) and \( M_{+-} = M_{\theta\lambda} \).

**Equation of state**: In Eq. (2), the action \( S_{\text{eff}} \) is fully characterized by \( |\Delta| \) and \( \mu \) or by the dimensionless parameters \( x = \mu/|\Delta| \) and \( |\Delta|/\epsilon_F \), where \( \epsilon_F \) is the Fermi energy for a specified density \( n = k_F^3 / (2\pi)^3 \), with \( k_F \) being the Fermi momentum. However, to study the evolution from the BCS to the Bose limit, it is experimentally more relevant to relate \( |\Delta| \) and \( \mu \) to the 2D scattering length \( a \) and the scattering density \( n \). The order parameter is found from \( \partial |\Delta| / \partial |\Delta| \mid_{T, V} = 0 \), where \( \Omega_{\text{sp}} = S_{\text{sp}} / \beta \) is the saddlepoint thermodynamic potential. Replacing the interaction strength \( g \) in favor of the (positive) two-body binding energy \( \epsilon_b \), using the Lippmann-Schwinger relation \( L^2 \gamma / g = \sum_k 1 / (2\epsilon_k + \epsilon_b) \) \([48]\), one finds \( |\Delta| = \sqrt{\epsilon_b (2\mu + \epsilon_b) \Theta (2\mu + \epsilon_b)} \), which is explicitly only a function of \( \mu \) and \( \epsilon_b \). The chemical potential can be found by solving the saddlepoint number equation \( n_{\text{sp}} = -\partial |\Delta| / \partial |\Delta| \mid_{T, V} / L^2 \), while fixing the density \( n_{\text{sp}} = n = k_F^3 / (2\pi)^3 \), resulting in \( \mu_{\text{sp}} = \epsilon_F - \epsilon_b / 2 \), which substituted in the order parameter relation leads to \( |\Delta|_{\text{sp}} = \sqrt{2\epsilon_F \epsilon_b} \). These expressions are connected to the 2D scattering length \( a \) via the relation \( \epsilon_b = n_{\text{sp}} / \exp(2\gamma_E + 2\ln k_F a) \), with \( \gamma_E \approx 0.577 \) the Euler-Mascheroni constant \([49]\). However, as discussed below, this analysis leads to the unphysical result of a constant sound velocity \( c = v_F \) \([50, 51]\) and 2D \([42, 48]\). Here, \( n_{\text{sp}} = -\partial |\Delta| / \partial |\Delta| \mid_{T, V} / L^2 \) must be always positive and \( n(\mu) = k_F^3 / (2\pi)^3 \) must be solved numerically, leading to a significant reduction of \( \mu \) in the Bose regime \([52]\), see Fig. 1.

**Collective Modes**: As seen from Eq. (2), the fluctuation matrix \( M(q) \) acts as the inverse propagator of modulus and phase fluctuations. The collective mode frequency \( \omega_\mathbf{q} \) is found from the poles of \( [M(q)]^{-1} \) or equivalently from \( \text{det} M(q, \omega_\mathbf{q}) = 0 \) \([47]\). As shown in the Supplemental Material, at \( T = 0 \), these poles are identical to the poles of the density-density response function, as probed by experiments measuring the dynamical structure factor \([30]\). A real solution is found below the two-particle continuum \( \epsilon_c(q) = \min_{E_+} (E_+ + E_-) \), above which it is energetically more favorable to break pairs. In a 2D Fermi gas, \( \omega_\mathbf{q} \) can hit the two-particle continuum at some finite value of \( \mathbf{q} \) causing damping \([45]\). However, for a 2D Fermi gas, a real \( \omega_\mathbf{q} < \epsilon_c(\mathbf{q}) \) is found for all values of \( \mathbf{q} \), given that a two-body bound state always exists for a 2D contact potential \([53]\). This physics arises from \( M_{++} = M_{\theta\theta} \), which always diverges when \( \hbar \omega \rightarrow \epsilon_c \), making \( \text{det} M(q, \epsilon_c) = 0 \) impossible, and thus there is no damping of the mode at the Gaussian level (See Supplemental Material for an in-depth discussion on damping).

For arbitrary \( \mathbf{q} \) there is no general analytical solution for \( \omega_\mathbf{q} \), but we obtain numerical results shown as solid red lines in Fig. 2. We compare our results to the measured
spectrum from Ref. [30], and find that \( \omega_q \) follows closely the maximum of the dynamical structure factor, without any fitting parameters. Moreover, it can be seen that \( \omega_q \) avoids the two-particle continuum \( \epsilon_{c}(q) \) and that a solution exists for all \( q \).

Although numerical solutions are useful for comparison to recent experiments [28, 30], analytical insight is essential to understand the underlying physics. To reveal the interplay between modulus and phase fluctuations, we show next that their coupling increases dramatically as the system evolves from the BCS to the Bose regime. We invert \( M(q) \) and obtain the propagators in Fourier space, which in the long wavelength limit become

\[
\left( \begin{array}{c} \langle \theta_q \theta_q \rangle \\ \langle \lambda_q \lambda_q \rangle \\ \langle \lambda_q \theta_q \rangle \\ \langle \lambda_q \lambda_q \rangle \end{array} \right) = \frac{|\Delta|^2}{h^2 c^2 q^2 - h^2 \omega^2} \left( \begin{array}{c} \chi_{\theta\theta} \\ -i \hbar \omega \chi_{\theta\lambda} \\ \chi_{\lambda\lambda} - \hbar^2 \omega^2 \chi_{\lambda\lambda} \end{array} \right).
\]

The different \( \chi \) coefficients defined in this equation are shown in Fig. 3(a) as a function of interaction parametrized by the ratio \( x = \mu/|\Delta| \). They are explicitly given by \( \chi_{\theta\theta} = 4\pi \), \( \chi_{\theta\lambda} = 2\pi(-x + \sqrt{1 + x^2})/|\Delta| \), \( \chi_{\lambda\lambda} = h^2 \pi \sqrt{1 + x^2}/2m|\Delta| \), and \( \chi_{\lambda\lambda} = \pi/|\Delta|^2 \). Most notably, the solid red line in Fig. 3(a) shows \( \chi_{\theta\lambda} \), which controls the coupling between phase and modulus. Notice that \( \chi_{\theta\lambda} \) is large in the Bose regime (\( \mu/|\Delta| \ll -1 \)), indicating that phase and modulus are strongly mixed, while it is negligible in the BCS regime (\( \mu/|\Delta| \gg 1 \)), showing that phase and modulus are essentially decoupled. For \( \omega > 0 \), the pole of \( [M(|q|, \omega)]^{-1} \) occurs at \( \omega_q = c|q| \), with

\[
2mc^2 = \mu + \sqrt{\mu^2 + |\Delta|^2},
\]

where \( c \) is the sound velocity associated with the broken U(1) symmetry. This expression includes modulus and phase fluctuations of the order parameter, and explicitly illustrates the dependence of \( c \) on \( \mu \). In Fig. 3(b), we show the behavior of \( c/v_F \) at three levels of approximation: the dotted green line includes only phase fluctuations, the dashed blue line includes phase and modulus fluctuations with the saddlepoint value of \( \mu \), while the solid red line includes phase and modulus fluctuations with a selfconsistent Gaussian fluctuation value of \( \mu \). It is clear that the first two levels of approximation [39, 40] give completely unphysical results, in particular predicting the constant value \( c = v_F/\sqrt{2} \) for any coupling in the saddlepoint approximation, as quoted in the literature [39, 40]. However, we show here that the inclusion of phase and modulus fluctuations with the correct \( \mu \) leads to the appropriate behavior of \( c \) both in the Bose and BCS limits, giving results that are surprisingly close to the experimentally measured isentropic sound velocity [28], that is typically a good estimate for Landau’s first sound.

The sound mode arising from the broken U(1) symmetry should not be confused with Landau’s first or second sound [55], as they are fundamentally different. Our \( T = 0 \) microscopic collective mode can be directly observed in measurements of the dynamic structure factor, and exists in the collisionless regime. Conversely, first and second sound result from a phenomenological decomposition of the superfluid into two components, and exist only in the hydrodynamic regime [56, 57]. In clarifying the difference between the broken-U(1) and Landau’s first sound, we show that Landau’s first sound velocity is always larger than \( c \), see Supplemental Material. We also note that the isentropic sound velocity [28] is not the same as the sound velocity that can be extracted from the dynamical structure factor [30].

Further insight is gained by studying the long wavelength limit of \( S_{\text{eff}} \) in Eq. (2) by expanding the fluctuation matrix \( M(q, \omega) \) for small \( q \) and \( \omega \). Performing an inverse Fourier transform back to real space, the second term in Eq. (2) reduces to

\[
S_g = \frac{1}{2} \int d^3r \left\{ \rho_s (\nabla \theta)^2 + A(\hbar \partial_r \theta)^2 + iD \hbar \partial_r \theta + C \lambda^2 \right\}.
\]

The first two coefficients are the \( T = 0 \) superfluid density \( 2m\rho_s/\hbar^2 = n_{\text{eff}}/2 \) and the compressibility \( A = (1 + x/\sqrt{1 + x^2})m/8\pi\hbar^2 \), while \( D = m|\Delta|/\sqrt{1 + x^2}/2\pi\hbar^2 \) controls the phase-modulus coupling, and \( C = |\Delta|^2(1 + x/\sqrt{1 + x^2})m/2\pi\hbar^2 \) describes the mass term for the modulus fluctuations. Neglecting modulus fluctuations (\( \lambda =
FIG. 3. (a) Different \( \chi \) coefficients vs. \( \mu/|\Delta| \) or ln \( k_F a \) as defined in Eq. (3). (b) sound velocity \( c/V_F \) vs. ln \( k_F a \): the dotted green line includes phase-only fluctuations with saddlepoint \( \mu \), diverging in the Bose limit; blue dashed line includes phase and modulus fluctuations with saddlepoint \( \mu \), giving always a constant value; solid red line combines phase and modulus fluctuations with Gaussian \( \mu \) self-consistently. The dot-dashed magenta and black lines show the results in the BCS and Bose limits, respectively. We compare our broken-U(1) sound velocity \( c \) with the experimental results of the isentropic sound velocity \( u_s \) from Ref. [28] (blue circles), and with \( c \) from Eq. (4) and the Monte Carlo equation of state from Ref. [54] (dashed yellow line). See an in-depth discussion of the conceptual difference between \( c \) and \( u_s \) in the Supplemental Material.

\[0\) leads to a sound velocity \( c = \sqrt{\rho_s/\bar{A}/\hbar} = (\mu^2 + |\Delta|^2)^{1/4}/\sqrt{m} \), shown as the dotted green line in Fig. 3(b), which diverges in the Bose limit. However, when \( \lambda \neq 0 \), the coupling between modulus and phase renormalizes \( \bar{A} \). Integrating out \( \lambda \) leads to a renormalized phase-only action with unchanged superfluid density \( \rho_s R = \rho_s \), but renormalized compressibility \( \bar{A}_R = \bar{A} + D^2/4C = m/4\pi\hbar^2 \). This renormalization leads to the corrected speed \( c = \sqrt{\rho_s R/\bar{A}_R/\hbar} \) given by Eq. (4), as expected. This moreover leads to the conclusion that the collective mode studied here is neither a Goldstone (pure phase) nor Higgs (pure modulus) mode, because the mixing of phase and modulus cannot be neglected.

Change in concavity: To investigate the low-energy collective modes beyond linear dispersion, it is necessary to expand \( M_{\pm \pm} \) and \( M_{\pm \mp} \) up to sixth order in \( \omega \) and \( |q| \). In this case, the condition \( \det M(q, \omega_q) = 0 \) leads to

\[ \omega_q = c|q| \left[ 1 + \frac{\gamma}{8} \left( \frac{\hbar|q|}{mc} \right)^2 + \frac{\eta}{16} \left( \frac{\hbar|q|}{mc} \right)^4 + \mathcal{O}\left( \frac{\hbar|q|}{mc} \right)^6 \right], \tag{6} \]

where the coefficients of the cubic and quintic order corrections have analytic expressions

\[\gamma = \frac{1}{24} \left( 1 - 4x^2 - x \frac{7 + 4x^2}{\sqrt{1 + x^2}} \right), \tag{7}\]

\[\eta = -\frac{365 + 2802x^2 + 2048x^4 + 160x^6}{23040(1 + x^2)^{5/2}} - \frac{685 + 1813x^2 + 1064x^4 + 80x^6}{11520(1 + x^2)^{3/2}}. \tag{8}\]

In Fig. 2, we show \( \omega_q \) for various interaction regimes and different levels of approximation. The solid red curves represent the full numerical solutions, while the other curves represent the linear (dashed yellow) and cubic (dot-dashed green) approximations in \(|q|\) with the Gaussian-corrected \( \mu \). The dotted line represents the numerical phase-only \((\lambda = 0)\) dispersion \( \omega_{qPO} \) using the saddlepoint value of \( \mu \). Notice that \( \omega_{qPO} \) severely overestimates the correct \( \omega_q \) in the Bose regime, that is, \( \omega_{qPO} \gg \omega_q \), while in the BCS limit \( \omega_{qBCS} \approx \omega_q \), because the modulus and phase fluctuations are nearly decoupled. The panels in Fig. 2 show that there is a change in curvature in the solid red lines, also found in 3D Fermi gases [37, 45–47], where the dispersion \( \omega_q \) is supersonic \((\gamma > 0)\) in the Bose regime shown in panel (a), and subsonic \((\gamma < 0)\) in the BCS regime shown in panel (d), where it bends downwards due to the pairbreaking continuum. The coefficients of the nonlinear terms play a significant role at larger momenta. While the cubic correction gives a good approximation for the large momentum behavior in the Bose limit, as one moves towards the BCS regime, progressively higher-order terms are needed to produce the appropriate behavior.

The coefficients \( \gamma \) and \( \eta \) are presented in Fig. 4 as function of ln \( k_F a \). In all panels, \( \gamma \) and \( \eta \) are evaluated at different levels of approximation: the dotted green lines describe phase-only fluctuations using the saddlepoint \( \mu \), the dashed blue lines include modulus and phase fluctuations using the saddlepoint \( \mu \), and the solid red lines include modulus and phase fluctuations using the Gaussian \( \mu \). Panels (a)-(b) ((c)-(d)) show \( \gamma \) and \( \eta \) in units that elucidate their limiting behavior in the Bose (BCS) regime given by the dot-dashed black (magenta) lines.

The concavity of \( \omega_q \) is controlled by \( \gamma \), which changes from \( \gamma > 0 \) (convex) to \( \gamma < 0 \) (concave) in the Bose and BCS regimes, respectively. The parameter \( \gamma \) changes sign at \( x = \sqrt{2(\sqrt{13} - 7)/12} \approx 0.133 \), corresponding to ln \( k_F a \approx 0.65 \) using the Gaussian \( \mu \). In this case \((\gamma = 0)\), the first correction to the linear spectrum is a quintic \((|q|^5)\) term controlled by \( \eta \). Although the behavior of \( \gamma \) and \( \eta \) is similar to the 3D results in the Bose limit [46], in the rest of the crossover the 2D case is qualitatively different, where \( \eta \) always stays negative because of the strong level repulsion with the two-particle continuum, due to the existence of two-body bound states for all interactions.
In the Bose limit \( x \ll -1 \), expanding the matrix elements of \( M(q, \omega) \) to order \( (\Delta/|\mu|)^2 \) and to lowest order in \( \hbar |q|/\sqrt{2m|\mu|} \) leads to

\[
\hbar^2 (\omega_q^2)^2 = \frac{\hbar^2 q^2}{2m_B} \left( \frac{\hbar^2 q^2}{2m_B} + 2m_B c_B^2 \right),
\]

where \( c_B = |\Delta|/\sqrt{2m_B|\mu|} \) is the Bogoliubov speed of sound, and \( m_B = 2m \) is the boson mass. Using the saddlepoint \( \mu \) leads to the incorrect value \( c_B = v_F/\sqrt{2} \), while using the Gaussian corrected \( \mu \) leads to \( c_B = v_F/\sqrt{8|\ln k_Fa|} \) in the Bose limit. The Bogoliubov-Popov interaction energy \( 2n_B V_B(0) = 2m_B c_B^2 = E_F/|\ln k_Fa| \), with boson density \( n_B = n/2 \) leads to the boson-boson interaction parameter \( V_B(0) = (E_F/n)/|\ln k_Fa| \). The values of \( \gamma \) and \( \eta \) from Eq. (9) are equal to limiting results obtained from Eqs. (7-8), that is, \( \gamma \to 1/4 \), and \( \eta \to -1/128 \), as seen in Figs. 4(a)-(b).

In the BCS limit \( x \gg 1 \), the saddlepoint and Gaussian correction tend to the same results, as fluctuations are less important. Rescaling energies by \( |\Delta| \), such that \( \hbar \omega_q/|\Delta| \) tends to a universal function of \( \hbar c|q|/|\Delta| \), leads to \( (\Delta/mc^2)^2 \gamma \to -1/3 \) and \( (\Delta/mc^2)^4 \eta \to -1/72 \), as revealed in Figs. 4(c)-(d). This is a consequence of the two-particle continuum pushing down the collective mode branch [45, 46]. In this case, the expansion in \( |q| \) is limited to \( \hbar c|q| \leq 2|\Delta| \) and \( c \to v_F/\sqrt{2} \).

**Conclusions:** We analyzed low-energy collective modes of 2D \( s \)-wave Fermi superfluids from the BCS to the Bose regime giving excellent results when compared to Bragg spectroscopy experiments in 2D box potentials. We showed that quantum fluctuations in the phase and modulus of the pairing order parameter are absolutely necessary to give physically acceptable chemical potential and sound velocity. We presented analytical results for the change in concavity of the collective mode dispersion from convex to concave as contact interactions are changed from the BCS to the Bose regime. The dispersion never hits the two-particle continuum threshold, due to the existence of two-body bound states for arbitrarily small attractive \( s \)-wave interactions in 2D.

**Acknowledgments:** We thank the Belgian American Educational Foundation for support. We also thank Henning Moritz and Lennart Sobirey for sharing their experimental data.

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Supplemental Material for
“Effects of quantum fluctuations on the low-energy collective modes
of two-dimensional superfluid Fermi gases from the BCS to the Bose Limit”

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(Dated: September 19, 2023)

In this supplemental material, we provide details of a few fundamental points that were made in the main text. We correct some myths that have been perpetuated in the literature. The first one is the issue of damping of the low-energy collective mode at zero temperature (T = 0) in 2D. The second point is that the poles of the structure factor at T = 0 correspond to the broken U(1) collective mode and not to Landau’s first sound. The third issue is that the sound velocity of the broken U(1) collective mode at T = 0 is always quantitatively smaller than the isothermal or isentropic sound velocities, which are often used as approximations for Landau’s first sound.

EFFECTS OF DAMPING: 2D VERSUS 3D

In our discussion about damping, we limit ourselves to the case of neutral s-wave superfluids with short-ranged interactions at T = 0, since we only computed the broken U(1) collective mode dispersion for this case.

There is a long-standing result in the literature stating that the broken U(1) collective mode at T = 0 always exhibits damping at sufficiently weak interactions due to pair-breaking [1]. This expectation is based on linear extrapolations of the broken U(1) collective modes ω_{col}(q) = c|q| into higher energies, suggesting that the collective mode energy crosses the two-quasiparticle continuum threshold 2|Δ| in 2D and 3D [1–3] for sufficiently weak interactions, where Cooper pairs may be easily broken. Such approximation suggests that for |q| > 2|Δ|/c, where c is the speed of sound, damping occurs. This result is strongly based on two assumptions. First, that one has a BCS superfluid, and second that the linear approximation ω_{col}(q) = c|q| for the collective mode dispersion still holds for |q| > 2|Δ|/c.

Recent investigations [3] in 3D included higher order corrections in |q| leading to ω_{col}(q) = c|q| + γ|q|^3 + δ|q|^5 and showed that indeed there is a pair breaking regime as naively expected from the linear extrapolation. However, the the value of the wavenumber |q| where the pair-breaking threshold is reached, gets pushed to higher values. This reduction of the damping region due to pair-breaking was attributed to the change in concavity of the dispersion relation in the BCS regime [3].

In 2D, our results show that the inclusion of higher order corrections has an even stronger effect, that is, the change in concavity completely suppresses the decay of the collective mode into the two-quasiparticle continuum for all q and all interaction strengths. Therefore, the naive expectation [1], based on the linear extrapolation, that the broken U(1) collective mode always decays into two quasiparticles at sufficiently large wavevectors and energies for weak interactions is wrong. The key insight on damping provided by our work is that the strong change in the concavity of ω_{col}(q), that avoids the two-quasiparticle threshold altogether, is due to the existence of stable two-body bound states for arbitrarily weak s-wave interactions in 2D. This means that the decay mechanism of the broken U(1) collective modes into two quasiparticles is completely suppressed in 2D by the ubiquitous presence of two-body bound states, in sharp contrast with the 3D situation where decay into the two-quasiparticle continuum still occurs.

Having shown that decay of the broken U(1) collective mode into two-quasiparticles (pair breaking) is forbidden in 2D at T = 0, a natural question to ask is: Are there additional decay (damping) mechanisms that may arise? The answer to this question is simple. To Gaussian order in the order parameter fluctuations, the only other decay mechanism is Landau damping, which occurs at T ≠ 0 and is due to quasiparticle-quasihole processes, both in 2D and 3D. Since we are at T = 0, the Landau damping mechanism is thermally suppressed and does not emerge in our calculation.

Beyond Gaussian order there are possible damping mechanisms such as those proposed by Beliaev [4], where a collective mode (phonon) can decay into two collective modes (phonons) with lower energies, or by Landau-Khalatnikov [5], where two collective modes (phonons) can decay into two other collective modes (phonons). In the absence of disorder, the damping of low-energy collective modes (phonons) is determined by collective-mode-collective-mode (phonon-phonon) interactions that conserve momentum and energy. Therefore, the damping rate depends strongly on the curvature of the collective mode (phonon) dispersion relation [6, 7]. The concavity change in the collective mode dispersion, from concave (BCS) to convex (Bose), can lead to different outcomes. Ultracold atom Fermi superfluids are believed to be in the collisionless regime where ω_{col}(q)τ ≫ 1, with τ being a characteristic collision time [8, 9]. Within the concave
region of $\omega_{\text{col}}(q)$, Beliaev processes are kinematically suppressed, but Landau-Khalatnikov damping can exist \cite{10}. In the convex regime, the two processes may contribute to damping of the collective modes. However, we do not address effects beyond Gaussian order in our work.

Having clarified the issue of damping of collective modes in 2D at $T = 0$, we shown next that the poles of the structure factor are not given by Landau’s first sound at $T = 0$, but rather by the broken U(1) collective mode that we calculate.

**DENSITY-DENSITY RESPONSE**

In this section, we show that the poles of the density-density correlation function are identical to the poles of the modulus-phase fluctuation matrix at zero temperature. Therefore, the collective modes detected by the dynamical structure factor in the superfluid state at $T = 0$ are the ones that arise from the spontaneously broken U(1) symmetry.

We obtain the density-density correlation function by adding a source term

$$S_J(\bar{\psi}_s, \psi_s) = \int_0^\beta d\tau \int d^D x J(x) \sum_s \bar{\psi}_s(x) \psi_s(x)$$

(S.1)

to the action $S$ corresponding to the Hamiltonian density $\mathcal{H}$ given in Eq. (1) of the main text. The source field $J(x)$, with dimensions of energy, couples to the local density $\rho(x) = \sum_s \bar{\psi}_s(x) \psi_s(x)$, where $x = (x, \tau)$. Integrating the fermions leads to the second-order contribution

$$S_2 = \frac{\beta}{2} \sum_q \Phi^T(q) M(q) \Phi(q),$$

(S.2)

where the vector $\Phi^T = (\eta_R, \eta_I, J)$ is three-dimensional, $T$ means transposition, $\eta_R$ and $\eta_I$ are the real and imaginary parts of the fluctuations in the order parameter, and $q = (q, i\omega)$. The matrix correlating $\Phi^T$ and $\Phi$ is

$$M = \begin{pmatrix} M_{RR} & M_{RI} & M_{RJ} \\ M_{IR} & M_{II} & M_{IJ} \\ M_{JR} & M_{JI} & M_{JJ} \end{pmatrix},$$

(S.3)

has dimensions of inverse energy, and the first $2 \times 2$ block represents the order parameter fluctuation matrix

$$\tilde{M} = \begin{pmatrix} M_{RR} & M_{RI} \\ M_{IR} & M_{II} \end{pmatrix}. $$

(S.4)

Integration over the order parameter fluctuations produces the $JJ$ action

$$S_{JJ} = \frac{\beta}{2} \sum_q J(-q) \chi_{\rho\rho}(q) J(q),$$

(S.5)

that is second order in the source $J$, leading to the density-density correlation function

$$\chi_{\rho\rho}(q) = M_{JJ}(q) - \frac{D_{JJ}(q)}{\text{det} M(q)},$$

(S.6)

that has dimensions of inverse energy. The term $M_{JJ}(q)$ has no poles for low frequencies at $T = 0$, thus the poles of $\chi_{\rho\rho}(q)$ are determined solely by the integer-order zeros of $\text{det} M(q) = 0$, with $D_{JJ}(q)$ playing a role in determining the residue of the poles obtained via the analytic continuation $q = (q, i\omega) \rightarrow (q, \omega + i\delta)$.

This expression shows that the low-frequency poles of $\chi_{\rho\rho}(q, \omega + i\delta)$ are the same as those from the inverse fluctuation matrix $\tilde{M}^{-1}$, which describes the order parameter fluctuations. The result in Eq. (S.6) is mathematically the same as that found in the literature \cite{2, 11, 12} using standard linear response theory. The major advantage of our approach is that it shows explicitly that the low-frequency poles of $\chi_{\rho\rho}(q, \omega + i\delta)$ and of the order parameter fluctuation matrix $M(q, \omega + i\delta)$ are exactly the same. The explicit expressions of $M_{JJ}(q)$ and $D_{JJ}(q)$ are given in the Appendix.

At $T = 0$, the density-density structure factor is

$$S(q, \omega) = -\frac{1}{\pi} \text{Im} \chi_{\rho\rho}(q, \omega + i\delta),$$

(S.7)

implying that the peaks of $S(q, \omega)$ correspond to the poles $\omega_{\text{col}}(q)$ of $\chi_{\rho\rho}(q, \omega + i\delta)$, and are therefore a manifestation of the broken U(1) symmetry of the superfluid state. This proves that what $S(q, \omega)$ measures at $T = 0$ is microscopically equivalent to the collective modes revealed in the order parameter fluctuation matrix discussed in the main text.

There have been claims in the literature that Landau’s two-fluid model can be used to describe collective modes of Fermi superfluids detected in $S(q, \omega)$ \cite{13}, provided that the normal and superfluid components are in local hydrodynamic equilibrium. Such phenomenological construction requires short collision times between excitations in the normal fluid and an artificial separation between normal and superfluid components.

The artificiality of Landau’s phenomenological theory was thoroughly discussed by Feynman \cite{14}, who criticized the approach, because physically there is only one fluid and not two. What happens microscopically is that the fluid changes its properties below the critical temperature. Nevertheless, practitioners of Landau’s approach like to justify the separability of a single fluid into a normal and superfluid component by assuming that the frequency of collective modes $\omega_{\text{col}}(q)$ must be much smaller than a typical relaxation rate $1/\tau$, that is, $\omega_{\text{col}}(q) \tau \ll 1$.

Furthermore, it has been argued that local hydrodynamic equilibrium in ultracold atoms is very difficult to achieve because the scattering length and densities are not sufficiently large \cite{13}. In the case of ultracold fermions, it
was assumed that this may be possible near unitarity, but apparently nowhere else [13]. However, experimentally, these systems seem to be in the colisionless regime \( \omega_{\text{col}}(q) \tau \gg 1 \) as \( T \to 0 \) [8–10].

The question then arises: Is the broken U(1) collective mode at \( T = 0 \), revealed by the dynamical structure factor, in any way related to Landau’s first sound? This issue is addressed next.

**Broken U(1) vs. Landau First Sound**

We established unequivocally that the low-frequency collective modes extracted experimentally from \( S(q, \omega) \) at \( T = 0 \) are indeed the broken U(1) symmetry modes discussed in the main text. At unitarity, Landau’s first and second sounds are assumed to be collective modes that arise from the poles of \( S(q, \omega) \) [13] provided that \( \omega_{\text{col}}(q) \tau \ll 1 \), but often, in conversations, Landau’s first sound is identified as the collective mode that is measured from \( S(q, \omega) \) throughout the BCS-Bose evolution without regard to the required conditions for reaching the hydrodynamic limit. For us, this is a serious concern.

Keeping the previous statements in mind, it is of fundamental importance to ask the question: Are the broken U(1) and Landau’s first sound the same at \( T = 0 \)? Our answer is that these sound velocities are not the same either conceptually or quantitatively. Landau’s first sound does not invoke broken U(1) symmetry in a fluid that changes its properties below the critical temperature \( T_c \), rather it relies on the phenomenological construction of two fluids with different densities, where the superfluid density vanishes at and above \( T_c \). In Landau’s approach there is no broken symmetry explicitly invoked.

According to Landau’s original work [15], the first sound velocity \( u_1 \) is

\[
  u_1^2 = \frac{b}{2} - \sqrt{\left( \frac{b}{2} \right)^2 - g}, \tag{S.8}
\]

while the second sound velocity \( u_2 \) is

\[
  u_2^2 = \frac{b}{2} - \sqrt{\left( \frac{b}{2} \right)^2 - g}. \tag{S.9}
\]

The first coefficient appearing above is

\[
  b = u_S^2 = \left( \frac{\partial P}{\partial \rho} \right)_S, \tag{S.10}
\]

representing the square of the isentropic sound velocity \( u_S \), where \( P \) is the pressure, \( \rho \) is the mass density and \( S \) is the entropy. The second coefficient is

\[
  g = \frac{T S^2}{C_V} \frac{\rho_s}{\rho_n} \left( \frac{\partial P}{\partial \rho} \right)_T, \tag{S.11}
\]

having dimensions of velocity to the fourth power, where \( T \) is temperature, \( C_V \) is the heat capacity at constant volume \( V \), \( \rho_s \) is the superfluid density, and \( \rho_n \) is the normal density. The first few factors of \( g \) can be grouped into a squared velocity

\[
  u_c^2 = \frac{T S^2}{C_V} \frac{\rho_s}{\rho_n}, \tag{S.12}
\]

while the last factor in \( g \) is

\[
  u_T^2 = \left( \frac{\partial P}{\partial \rho} \right)_T, \tag{S.13}
\]

where \( u_T \) is the isothermal sound velocity.

In the regions of phase space where the superfluid has a small coefficient of thermal expansion \( \alpha \), the first sound velocity is approximately the isentropic sound velocity, that is, \( u_1 \approx u_S \) [15, 16] since \( g \ll (b/2)^2 \) in Eq. (S.9). The thermodynamic relation \( u_S^2 = \gamma u_T^2 \), where \( \gamma = C_P/C_V \) is the ratio between the heat capacities at constant pressure \( (C_P) \) and at constant volume \( (C_V) \), guarantees that \( u_S \geq u_T \) since \( \gamma \geq 1 \).

The microscopic dynamical structure factor \( S(q, \omega) \) can only be related to the phenomenological first and second sounds in the hydrodynamic regime [17, 18] where \( \omega \tau \ll 1 \). Here, \( \tau \) is a temperature dependent relaxation time, which is usually a high power of inverse temperature [17, 18], when the artificial separation between normal and superfluid components may have a microscopic justification, at least for bosonic systems [17, 18]. The inverse relaxation time can be parametrized as \( \tau^{-1} = \omega_0 (T/T_c)^\ell \), where \( \omega_0 \) is a characteristic relaxation frequency of the system, \( T_c \) is the superfluid transition temperature and \( \ell \) is typically in the interval \( 4 \leq \ell \leq 6 \) in 3D or \( 3 \leq \ell \leq 5 \) in 2D. The hydrodynamic condition requires that \( \omega \ll \omega_0 (T/T_c)^\ell \). Strictly at \( T = 0 \), the hydrodynamic limit defined above does not exist. Therefore, at \( T = 0 \), our microscopic collective mode describing the broken U(1) symmetry cannot be Landau’s first sound. Using the Hohenberg-Martin terminology, our microscopic broken U(1) collective mode arises as a pole of \( S(q, \omega) \) in the collisionless regime \( \omega \tau \gg 1 \), which seems to be the experimental situation in ultracold fermions [8–10].

The question then becomes: Is there a relationship between Landau’s first sound and the broken U(1) collective mode at \( T = 0 \)? In the next section we will address this question via the compressibility sum rule.

**Compressibility Sum Rule**

The dynamical structure factor in Eq. (S.7) satisfies the compressibility sum rule

\[
  \lim_{q \to 0} \int_{-\infty}^{+\infty} d\omega \frac{S(q, \omega)}{\omega} = \frac{1}{2mu_T^2}. \tag{S.14}
\]
Using the explicit expression for $\chi_{\rho\rho}(q)$ in Eq. (S.6) we can show that
\[
\lim_{q \to 0} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} S(q, \omega) = \frac{1}{2mc^2} \left| \frac{\Delta}{\mu} \right| F(|\Delta|/\mu),
\]
(S.15)
where $F(|\Delta|/\mu)$ is a strictly positive dimensionless function that depends only on the ratio $|\Delta|/\mu$. The first contribution in Eq. (S.15) comes from the pole of $S(q, \omega)$, while the one containing $F(|\Delta|/\mu)$ arises from the background. Combining Eqs. (S.14) and (S.15) we obtain the relation
\[
\left( \frac{u_T}{c} \right)^2 = 1 + 2ma^2 T \frac{|\Delta|}{\mu^2 + |\Delta|^2} F(|\Delta|/\mu),
\]
(S.16)
indicating that the isothermal sound velocity $u_T$ is always larger than the broken U(1) sound velocity $c$. In general, $u_T \geq c$, where the equality sign only occurs for $|\Delta|/\mu \to 0$, when the effects of the broken U(1) symmetry are negligible (extreme BCS regime or the deep Bose limit) or for $|\Delta| = 0$ strictly, when there is no broken U(1) symmetry.

Since the isentropic sound velocity is $u_S = \gamma u_T$, then the inequality $u_S \geq u_T \geq c$ holds. If we assume that the hydrodynamic limit can be reached in the limit of $T \to 0$, and take $u_T \approx u_S$ as argued for 4He [16] and for unitary Fermi superfluids [13], then it is clear that the first sound $u_1$ is not quantitatively the same as the broken U(1) sound velocity $c$, but in the limit of $T \to 0$ there is a potential relationship between the two conceptually distinct quantities. However, we would like to emphasize that at $T = 0$ the poles revealed by $S(q, \omega)$ arise from the broken U(1) collective modes and not from Landau's first sound.

In Fig. 1, we plot $c$ and $u_T$ to illustrate the quantitative difference between the two sound velocities. Notice that $u_T$ is always larger than $c$ throughout the BCS-Bose evolution, and thus, $u_T \approx u_S$ is also greater than $c$. The two quantities, $u_T$ and $c$, become only asymptotically the same when $|\Delta|/\mu \to 0$, that is, either in the extreme BCS regime or in the deep Bose limit. At $T = 0$ and with $\gamma > 1$, this implies generally that $u_1 \approx u_S > c$.

The discussion above also illuminates the difference between the experimental measurement of the isentropic sound velocity $u_S$ [19], where phase imprinting produces a pressure wave that propagates through the sample at constant entropy, and the experimental measurement of the broken-U(1) sound velocity $c$, which we calculate at $T = 0$, revealed by the dynamical structure factor [20]. Measurements of $S(q, \omega)$ have not been performed at momentum modulus lower than $|q|/k_F \leq 0.14$, but the broken-U(1) sound velocity [20] appears to compare well with its isentropic counterpart [19]. We would like to emphasize that a more accurate measurement of $c$ should reveal that it is not only conceptually distinct from $u_S$, but also quantitatively different, obeying the relation $c \leq u_S$ discussed above.

**APPENDIX**

In this appendix, we give explicit expressions for the terms appearing in $\chi_{\rho\rho}(q)$ given in Eq. (S.6).

The first term is
\[
M_{11}(q, i\omega) = \sum_{k} \frac{U_{11}(k_+, k_-)}{(i\omega)^2 - (E_{k_+} + E_{k_-})^2},
\]
(S.17)
where the coherence factor is
\[
U_{11}(k_+, k_-) = \left[ E_{k_+} E_{k_-} - \xi_{k_+} \xi_{k_-} + |\Delta_0|^2 \right] L(k_+, k_-),
\]
(S.18)
with coefficient
\[
L(k_+, k_-) = \frac{E_{k_+} + E_{k_-}}{E_{k_+} E_{k_-}}.
\]
(S.19)
The second term, introduced in Eq. (S.6), is
\[
D_{11}(q, i\omega) = M_{11}^2 + M_{11}^2 M_{RR} - 2M_{11} M_{R1} M_{11},
\]
(S.20)
Each term appearing in $D_{11}$ is listed below. The RJ matrix element is
\[
M_{R1}(q, i\omega) = \sum_{k} \left[ \frac{U_{R1}(k_+, k_-)}{(i\omega)^2 - (E_{k_+} + E_{k_-})^2} \right],
\]
(S.21)
with the coherence factor
\[
U_{R1}(k_+, k_-) = i\omega|\Delta_0| L(k_+, k_-).
\]
(S.22)
The II matrix element is
\[
M_{II}(q, i\omega) = \sum_{k} \left[ \frac{1}{E_k} + \frac{U_{II}(k_+, k_-)}{(i\omega)^2 - (E_{k_+} + E_{k_-})^2} \right],
\]
(S.23)
with coherence factor
\[
U_{II}(k_+, k_-) = \left[ E_{k_+} E_{k_-} + \xi_{k_+} \xi_{k_-} - |\Delta_0|^2 \right] L(k_+, k_-).
\]
(S.24)
The IJ matrix element is
\[
M_{IJ}(q, i\omega) = \sum_{k} \frac{U_{IJ}(k_+, k_-)}{(i\omega)^2 - (E_{k_+} + E_{k_-})^2},
\]
(S.25)
where the coherence factor is
\[ U_{IJ}(k_+, k_-) = |\Delta_0|(\xi_{k_+} + \xi_{k_-})L(k_+, k_-). \]  
(S.26)

The RR matrix element is
\[ M_{RR}(q, i\omega) = \sum_k \left[ \frac{1}{E_k} + \frac{U_{RR}(k_+, k_-)}{(i\omega)^2 - (E_{k_+} + E_{k_-})^2} \right], \]
where the coherence factor is
\[ U_{RR}(k_+, k_-) = \left[ E_{k_+}E_{k_-} + \xi_{k_+}\xi_{k_-} + |\Delta_0|^2 \right] L(k_+, k_-). \]  
(S.28)

The last matrix element needed is
\[ M_{RI}(q, i\omega) = \sum_k \frac{U_{RI}(k_+, k_-)}{(i\omega)^2 - (E_{k_+} + E_{k_-})^2}, \]
where the coherence factor is
\[ U_{RI}(k_+, k_-) = \frac{i\omega(\xi_{k_+}E_{k_-} + \xi_{k_-}E_{k_+})}{E_{k_+}E_{k_-}}. \]  
(S.30)

The matrix elements given in this appendix are necessary to obtain the results discussed above.

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