Blowup for $C^2$ Solutions of the $N$-dimensional Euler-Poisson Equations in Newtonian Cosmology

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Abstract

Pressureless Euler-Poisson equations with attractive forces are standard models in Newtonian cosmology. In this article, we further develop the spectral dynamics method and apply a novel spectral-dynamics-integration method to study the blowup conditions for $C^2$ solutions with a bounded domain, $\|X(t)\| \leq X_0$, where $\|\cdot\|$ denotes the volume and $X_0$ is a positive constant. In particular, we show that if the cosmological constant $\Lambda < M/X_0$, with the total mass $M$, then the non-trivial $C^2$ solutions in $\mathbb{R}^N$ with the initial condition $\Omega_{0ij}(x) = \frac{1}{2} \left[ \partial_i u'(0,x) - \partial_j u'(0,x) \right] = 0$ blow up at a finite time.

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Key Words: Euler-Poisson Equations, Newtonian Cosmology, Initial Value Problem, Blowup, Spectral-Dynamics-Integration Method, Attractive Forces, $C^2$ Solutions, Bounded Domain,

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1 Introduction

The evolution of Newtonian cosmology can be modelled by the compressible pressureless Euler-
Poisson equations in dimensionless units:

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0 \\
\rho |u_t + (u \cdot \nabla)u| &= -\rho \nabla \Phi \\
\Delta \Phi(t, x) &= \rho - \Lambda,
\end{align*}
\]  

(1)

where \( \rho = \rho(t, x) \geq 0 \) and \( u = u(t, x) \in \mathbb{R}^N \) are the density and the velocity, respectively, with a background or cosmological constant \( \Lambda \).

The pressureless Euler-Poisson equations are the standard model in cosmology \[1\]. If the Euler-
Poisson equations include the pressure term, then they provide the classical description of galaxies
or gaseous stars in astrophysics \[2\] and \[3\]. For details of the connection between the Euler-Poisson
equations \[1\] and Einstein’s field equations

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu},
\]  

(2)

where \( R_{\mu\nu} \) is the Ricci curvature tensor, \( R \) is the curvature scalar, \( g_{\mu\nu} \) is the metric tensor, \( T_{\mu\nu} \) is
the energy-momentum tensor of the universe, \( G \) is Newton’s gravitational constant and \( c \) is the light
speed, interested readers can refer to Chapter 6 of Longair’s book \[3\]. In addition, for a geometrical
explanation of Newtonian cosmology with a cosmological constant \( \Lambda \), interested readers can refer
to Brauer, Rendall and Reula’s paper \[4\].

For an analysis of stabilities for the related systems, interested readers can refer to \[5\], \[6\], \[7\],
\[8\], \[9\], \[10\], \[11\], \[12\], \[13\], \[14\], \[15\] and \[16\]. In addition, there are explicit blowup or global
(periodical) solutions for the Euler-Poisson systems \[17\], \[18\], \[19\], \[20\] and \[21\].

It should be noted that in 2008, Chae and Tadmor \[13\] determined the finite time blowup for
the pressureless Euler-Poisson equations with attractive forces \[11\] with \( \Lambda = 0 \), under the initial
condition,

\[
S := \{ x_0 \in \mathbb{R}^N \mid \rho_0(x_0) > 0, \ \Omega_0(x_0) = 0, \ \text{div} \ u(0, x_0) < 0 \} \neq \phi,
\]  

(3)

where \( u = (u^1, u^2, ..., u^N) \) and \( \Omega_0(x_0) \) is the re-scaled vorticity matrix defined by \( \Omega_{0,ij}(x_0) = \)
\[ \frac{1}{2} \left[ \partial_i u^j(0, x_0) - \partial_j u^i(0, x_0) \right]. \]

Using spectral dynamics analysis, they identified the Riccati differential inequality

\[ \frac{D \text{div} u(t, x_0(t))}{Dt} \leq -\frac{1}{N} [\text{div} u(t, x_0(t))]^2, \]  

(4)

along the characteristic curve \( \frac{dx_0(t)}{dt} = u(t, x_0(t)) \). The corresponding solution of the inequality (4) blows up at or before \( T = -N/\text{div}u(0, x_0(0)) \) with an initial condition that requires that \( \text{div}u(0, x_0(0)) \) at some non-vacuum state. An improved blowup condition for the Euler-Poisson equations (4) was obtained by Cheng and Tadmor [14] in 2009.

In this article, we modify the spectral dynamics method to introduce a spectral-dynamics-integration method for a bounded domain \( X(t) \), to obtain the new blowup conditions according to the following theorem.

**Theorem 1** For the \( N \)-dimensional Euler-Poisson equations (1), consider \( C^2 \) solutions with a bounded domain: \( \|X(t)\| \leq V_{sup} \), where \( \|\cdot\| \) denotes the volume and \( V_{sup} \) is a positive constant.

We define the weighted functional

\[ H(t) = \int_{X(t)} \text{div} u d\mu_t, \]  

(5)

with the positive measure \( d\mu_t = \rho(t, x(t))dx(t) \). If the initial condition

\[ \Omega_{0ij}(x) = \frac{1}{2} \left[ \partial_i u^j(0, x) - \partial_j u^i(0, x) \right] = 0 \]  

(6)

and any one of the following conditions

1. \( \Lambda < M/V_{sup} \),
2. \( \Lambda \geq M/V_{sup} \) and \( H(0) < -\sqrt{-\frac{M^2N}{V_{sup}} + \Lambda M^2N} \),

with the total mass \( M = \int_{X(0)} \rho(0, x)dx > 0 \) are satisfied, the non-trivial \( C^2 \) solutions blow up at a finite time \( T \).

Here, the functional (5) represents the aggregate density-weighted divergence of the velocity \( u(t, x) \).
2 New Spectral-Dynamics-Integration Method

Before we present the novel spectral-dynamics-integration method, we first quote the following lemma:

Lemma 2 (Proposition 2.2 on page 27 of [22]) Let $S$ be a material system that fills the domain $X(t)$ at time $t$, and let $C$ be a function of class $C^1$ in $t$ and $x$. Then,

$$\frac{d}{dt} \int_{X(t)} C(t, x) \rho(t, x) dx = \int_{X(t)} \frac{DC(t, x)}{Dt} \rho(t, x) dx,$$

(7)

where $(D/Dt) = (\partial/\partial t) + u \cdot \nabla$ is the convective derivative.

In the following proof, we modify the method of spectral dynamics described in [11], [13] and [14] to obtain the different blowup conditions for the $C^2$ solutions.

Proof of Theorem 1. As the mass equation (1):

$$\frac{D\rho}{Dt} + \rho \nabla \cdot u = 0,$$

(8)

with the convective derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (u \cdot \nabla)$$

(9)

could be integrated as:

$$\rho(t, x_0) = \rho_0(x_0(0, x_0)) \exp\left(-\int_0^t \nabla \cdot u(t, x_0(t; x_0)) dt\right) \geq 0$$

(10)

for $\rho_0(x_0(0, x_0)) \geq 0$, the density function $\rho(t, x(t; x))$ generally conserves its non-negative nature.

For the momentum equations (1) and the solutions with non-vacuum, we have

$$u_t + u \nabla \cdot u = -\nabla \Phi.$$  

(11)

We take the divergence to the above equation to obtain:

$$\nabla \cdot (u_t + u \nabla \cdot u) = -\Delta \Phi.$$  

(12)

If the initial condition $\Omega_{0ij}(x) = \frac{1}{2} [\partial_i u^j(0, x) - \partial_j u^i(0, x)] = 0$ is satisfied, we can show by the standard spectral dynamics in [13] and [14] (by directly applying equation (2.6) in [13] or equation
(4.1) in [14]) that
\[
\frac{D}{Dt} \text{div } u(t, x(t)) + \frac{1}{N} [\text{div } u(t, x(t))]^2 \leq -\rho(t, x(t)) + \Lambda. \tag{13}
\]

We notice that the advancement in this article for the new blowup conditions begins here. First, we multiply the density function \(\rho(t, x(t))\) on both sides and take the integration over the domain \(X(t)\) to obtain:
\[
\rho(t, x(t)) \left( \frac{D}{Dt} \text{div } u(t, x(t)) + \frac{1}{N} [\text{div } u(t, x(t))]^2 \right) \leq -[\rho(t, x(t))]^2 + \Lambda \rho(t, x(t)). \tag{14}
\]
\[
\int_{X(t)} \rho \left( \frac{D}{Dt} \text{div } u \right) dx + \frac{1}{N} \int_{X(t)} \rho (\text{div } u)^2 dx \leq - \int_{X(t)} \rho^2 dx + \Lambda \int_{X(t)} \rho dx \tag{15}
\]
\[
\int_{X(t)} \rho \left( \frac{D}{Dt} \text{div } u \right) dx + \frac{1}{N} \int_{X(t)} \rho (\text{div } u)^2 dx \leq - \int_{X(t)} \rho^2 dx + \Lambda M, \tag{16}
\]
where \(M = \int_{X(t)} \rho dx = \int_{X(0)} \rho(0, x(0)) dx(0) > 0\) for non-trivial solutions is the total mass of the fluid.

We apply Lemma 2 with \(C(t, x) := \text{div } u\) to obtain
\[
\frac{d}{dt} \int_{X(t)} \rho \text{div } u dx + \frac{1}{N} \int_{X(t)} \rho (\text{div } u)^2 dx \leq - \int_{X(t)} \rho^2 dx + \Lambda M. \tag{17}
\]
We define the weighted functional
\[
H := H(t) = \int_{X(t)} \text{div } u d\mu_t \tag{18}
\]
with the positive measure \(d\mu_t = \rho(t, x(t)) dx(t)\) for \(\rho(0, x) \geq 0\), with the equation (11) to obtain
\[
\frac{d}{dt} H \leq - \frac{1}{N} \int_{X(t)} (\text{div } u)^2 d\mu_t - \int_{X(t)} \rho^2 dx + \Lambda M. \tag{19}
\]
We can estimate the first term on the right-hand side of the inequality (19) to obtain
\[
\left( \int_{X(t)} \text{div } u d\mu_t \right)^2 = \left( \int_{X(t)} \text{div } u d\mu_t \right)^2 \leq \left( \int_{X(t)} |\text{div } u| d\mu_t \right)^2 \leq M \int_{X(t)} (\text{div } u)^2 d\mu_t. \tag{20}
\]
Using the Cauchy-Schwarz inequality, we get
\[
\int_{X(t)} |\text{div } u| d\mu_t \leq \left( \int_{X(t)} 1^2 d\mu_t \right)^{1/2} \left( \int_{X(t)} (\text{div } u)^2 d\mu_t \right)^{1/2} \tag{21}
\]
\[
\int_{X(t)} |\text{div } u| d\mu_t \leq \left( \int_{X(t)} \rho dx \right)^{1/2} \left( \int_{X(t)} (\text{div } u)^2 d\mu_t \right)^{1/2} = \sqrt{M} \left( \int_{X(t)} (\text{div } u)^2 d\mu_t \right)^{1/2}. \tag{22}
\]
We obtain
\[
\left( \frac{\int_{X(t)} (\nabla u) \, d\mu_t}{M} \right)^2 \leq \int_{X(t)} (\nabla u)^2 \, d\mu_t \tag{23}
\]
\[- \frac{1}{N} \int_{X(t)} (\nabla u)^2 \, d\mu_t \leq - \frac{1}{MN} \left( \int_{X(t)} \nabla u \, d\mu_t \right)^2 = - \frac{H^2}{MN}. \tag{24}
\]

The second term on the right-hand side of the inequality (19) can be determined by
\[
\int_{X(t)} \rho \, dx \leq \left( \int_{X(t)} \rho^2 \, dx \right)^{1/2} \left( \int_{X(t)} \rho \, dx \right)^{1/2} \tag{25}
\]
\[M = \int_{X(t)} \rho \, dx \leq \|X(t)\|^{1/2} \left( \int_{X(t)} \rho^2 \, dx \right)^{1/2} \leq (V_{\text{sup}})^{1/2} \left( \int_{X(t)} \rho^2 \, dx \right)^{1/2} \tag{26}
\]
\[M^2 \leq V_{\text{sup}} \int_{X(t)} \rho^2 \, dx \tag{27}
\]
\[- \int_{X(t)} \rho^2 \, dx \leq - \frac{M^2}{V_{\text{sup}}} \tag{28}
\]
for a bounded domain \( \|X(t)\| \leq V_{\text{sup}} < +\infty. \)

Thus, the inequality (19) becomes
\[
\frac{d}{dt} H \leq - \frac{1}{N} \int_{X(t)} (\nabla u)^2 \, d\mu_t - \int_{X(t)} \rho \, dx + \Lambda M \leq - \frac{H^2}{MN} - \frac{M^2}{V_{\text{sup}}} + \Lambda M. \tag{29}
\]
\[
\frac{d}{dt} H \leq - \frac{H^2}{MN} - \frac{M^2}{V_{\text{sup}}} + \Lambda M. \tag{30}
\]

(1) If \( \Lambda < M/V_{\text{sup}} \), the Riccati inequality (30) can be estimated by
\[
\frac{d}{dt} H \leq - \frac{M^2}{V_{\text{sup}}} + \Lambda M < 0. \tag{31}
\]
Thus, there exists a finite time \( T_0 \), such that
\[
H(T_0) < 0. \tag{32}
\]
By applying the comparison property, we obtain
\[
\left\{ \begin{array}{l}
\frac{d}{dt} H \leq - \frac{H^2}{MN} - \frac{M^2}{V_{\text{sup}}} + \Lambda M \\
H(T_0) < 0.
\end{array} \right. \tag{33}
\]
It is well known that the Riccati inequality (33) blows up at a finite time \( T \).

(2) If \( \Lambda \geq M/V_{\text{sup}} \) and \( H(0) < - \sqrt{- \frac{M^2}{V_{\text{sup}}} + \Lambda M^2 N} \), it is also clear that the solution of the Riccati inequality (30) blows up at a finite time \( T \).

The proof is completed. \( \blacksquare \)
Remark 3  For the one dimensional case, the condition $\Omega_{0ij}(x) = 0$ in Theorem 1 is automatically satisfied.

The corollary below is immediately shown in Theorem 1.

Corollary 4  For $\Lambda = 0$, the non-trivial $C^2$ solutions with the bounded domain $\|X(t)\| \leq V_{\text{sup}}$, of the Euler-Poisson equations (1) in $\mathbb{R}^N$, and the initial condition

$$\Omega_{0ij}(x) = \frac{1}{2} \left[ \partial_i u^j(0, x) - \partial_j u^i(0, x) \right] = 0, \quad (34)$$

blow up at a finite time $T$.

Remark 5  By further requiring the bounded domain $\|X(t)\| \leq V_{\text{sup}}$ and

$$\Omega_{0ij}(x) = \frac{1}{2} \left[ \partial_i u^j(0, x) - \partial_j u^i(0, x) \right] = 0, \quad (35)$$

for the Euler-Poisson equations (1), the main achievement of this spectral-dynamics-integration method is to remove the restriction on $\text{div} u(0, x_0)$ with some point $x_0$, for the positive background constant $\Lambda < M/V_{\text{sup}}$ in Cheng and Tadmor’s paper [14] for obtaining the blowup phenomenon.

For the Euler-Poisson equations (1) with free boundaries, it is possible to establish the existence of the solutions outside the bounded domain $\|X(t)\| \leq V_{\text{sup}}$ after the “blowup” time $T$ in Theorem 1. Therefore, we have the following corollary:

Corollary 6  For the $N$-dimensional Euler-Poisson equations (1), consider the non-trivial global $C^2$ solutions with $\rho(0, x)$ and $u(0, x)$, which lie inside a bounded domain: $\|X(0)\| \leq V_0$, where $\|\cdot\|$ denotes the volume and $V_0$ is a positive constant. We define the weighted functional

$$H(t) = \int_{X(t)} \text{div} u d\mu_t, \quad (36)$$

with the positive measure $d\mu_t = \rho(t, x(t)) dx(t)$. If the initial condition

$$\Omega_{0ij}(x) = \frac{1}{2} \left[ \partial_i u^j(0, x) - \partial_j u^i(0, x) \right] = 0 \quad (37)$$

and any one of the following conditions,

(1) $\Lambda < M/V_0$, 

(2) $\Lambda \geq M/V_0$ and $H(0) < -\sqrt{-\frac{M^2 N}{V_0}} + \Lambda M^2 N$, with the total mass $M = \int_{X(0)} \rho(0, x) dx > 0$ are satisfied, $\|X(t)\|$ cannot be bounded by the constant $V_0$ for all time $t$.

3 Conclusions

In this article, we study the life-span problem of self-gravitational fluids with zero pressure (dust solutions) with a cosmological constant $\Lambda$ and a bounded domain $X(t)$. We apply a new spectral-dynamics-integration method to show that there are blowup phenomena if either the cosmological constant is sufficiently small compared with other parameters of the pressureless Euler-Poisson system (1), or if the weighted functional

$$H(t) = \int_{X(t)} \rho \text{div} u dx$$

(38)

is initially contracting sufficiently fast.

New functional techniques are expected to investigate the possibility of the corresponding blowup phenomena for the Euler-Poisson equations with the pressure term:

$$\left\{ \begin{array}{l}
\rho_t + \nabla \cdot (\rho u) = 0 \\
\rho[u_t + (u \cdot \nabla) u] + K \nabla \rho^\gamma = -\rho \nabla \Phi \\
\Delta \Phi(t, x) = \rho - \Lambda,
\end{array} \right. \quad (39)$$

with constants $K > 0$ and $\gamma \geq 1$.

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