The distance spectrum of the complements of graphs of diameter greater than three

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Abstract Suppose $G$ is a connected simple graph with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let $d_G(v_i, v_j)$ be the least distance between $v_i$ and $v_j$ in $G$. Then the distance matrix of $G$ is $D(G) = (d_{ij})_{n \times n}$, where $d_{ij} = d_G(v_i, v_j)$. Since $D(G)$ is a non-negative real symmetric matrix, its eigenvalues can be arranged $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, where eigenvalues $\lambda_1(G)$ and $\lambda_n(G)$ are called the distance spectral radius and the least distance eigenvalue of $G$, respectively. In this paper, we characterize the unique graph whose distance spectral radius attains maximum and minimum among all complements of graphs of diameter greater than three, respectively. Furthermore, we determine the unique graph whose least distance eigenvalue attains minimum among all complements of graphs of diameter greater than three.

Keywords Distance matrix · Diameter · Distance spectral radius · Least distance eigenvalues · Complements of graphs

Mathematics Subject Classification 05C12 · 05C50

1 Introduction

The distance spectral radius of graphs have been studied extensively. S. Bose, M. Nath and S. Paul [2] determined the unique graph with maximal distance spectral radius among graphs without a pendant vertex. A. Ilic [4] obtained the unique graph whose distance spectral radius is maximum among $n$-vertex trees with perfect matching and fixed maximum degree. W. Ning, L. Ouyang and M. Lu [10] characterized the graph with minimum distance spectral radius among trees with given number of pendant vertices. For more about the distance spectra of graphs see the survey [1] as well as the references therein.

The least distance eigenvalues of connected graphs have been also studied. H. Lin [6] gave an upper bound on the least distance eigenvalue and characterized all the connected graphs with the least distance eigenvalue in $[-1 - \sqrt{2}, a]$, where $a$ is the smallest root of $x^3 - x^2 - 11x - 7 = 0$ and $a \in (-1 - \sqrt{2}, -2)$. H. Lin and B. Zhou [8] obtained the trees with the least distance eigenvalues in $[-3 - \sqrt{5}, -2 - \sqrt{2}]$ and the unicyclic and

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bicyclic graphs with least distance eigenvalues in \((-2 - \sqrt{2}, -2.383)\). G. Yu [13] introduced all the graphs with the least distance eigenvalue in \([-2.383, 0]\).

The complement of graph \(G = (V(G), E(G))\) is denoted by \(G^c = (V(G^c), E(G^c))\), where \(V(G^c) = V(G)\) and \(E(G^c) = \{xy \notin E(G) : x, y \in V(G)\}\). Y. Fan, F. Zhang and Y. Wang [3] determined the connected graph with the minimal least eigenvalue among all complements of trees. G. Jiang, G. Yu, W. Sun and Z. Ruan [5] gave the graph with the minimal least eigenvalue among all graphs whose complements are connected and have only two pendent vertices. S. Li and S. Wang [9] introduced the unique connected graph whose least signless Laplacian eigenvalue attains the minimum in the set of the complements of all trees. G. Yu, Y. Fan and M. Ye [12] achieved the unique graph which minimizes the least signless Laplacian eigenvalue among all connected complements of unicyclic graphs.

Currently there is very little research about the distance eigenvalues of complements of graphs. H. Lin and S. Drury [7] characterized the unique graph whose distance spectral radius has maximum and minimum among all complements of trees, and the unique graph whose least distance eigenvalue has maximum and minimum among all complements of trees. R. Qin, D. Li, Y. Chen and J. Meng [11] determined the unique graph which has maximum distance spectral radius among all complements of unicyclic graphs and the unique graph which has maximum least distance eigenvalue among all complements of unicyclic graphs of diameter three.

Let \(G\) be a connected simple graph with the vertex set \(V(G) = \{v_1, v_2, \ldots, v_n\}\). Then the adjacency matrix of \(G\) is \(A(G) = (a_{ij})_{n \times n}\), where \(a_{ij} = 1\) if \(v_i\) is adjacent to \(v_j\), and \(a_{ij} = 0\) otherwise. In this paper, we observe the relations between \(D(G^c)\) and \(A(G)\) and use them to characterize the unique graph whose distance spectral radius attains maximum and minimum among all complements of graphs of diameter greater than three, respectively. Furthermore, we determine the unique graph whose least distance eigenvalue attains minimum among all complements of graphs of diameter greater than three.

2 The distance spectral radius of the complements of graphs of diameter greater than three

Let \(J_n\) be the matrix of order \(n\) whose all entries are 1, and let \(I_n\) be the identity matrix of order \(n\). Suppose \(A = (a_{ij})_{n \times n}\) and \(B = (b_{ij})_{n \times n}\). Then we write \(A = B\) if \(a_{ij} = b_{ij}\), and \(A \geq B\) if \(a_{ij} \geq b_{ij}\). If two vertices \(u\) and \(v\) are adjacent, then we write \(u \sim v\) and otherwise \(u \not\sim v\).

We denote by \(d\) the diameter of \(G\), which is the farthest distance between all pairs of vertices. In this paper we always assume that the diameter \(d(G)\) of \(G\) is greater than three, and so \(G\) and its complement \(G^c\) are both connected. The below Lemma 2.1 reflects the relationship of \(D(G^c)\) and \(A(G)\).

**Lemma 2.1** Suppose \(G\) is a simple graph on \(n\) vertices whose diameter \(d(G)\) is greater than two. Then we have

(I) when \(d(G) > 3\), \(D(G^c) = J_n - I_n + A(G)\).

(II) when \(d(G) = 3\), \(D(G^c) \geq J_n - I_n + A(G)\).

**Proof** Suppose \(d(G) > 3\). Let \(u\) and \(v\) be two vertices of \(G\). If \(u \sim v\), then \(G\) contains a vertex \(w\) which is adjacent to neither \(u\) nor \(v\), and so \(d_{G^c}(u, v) = 2\). So we assume \(u \not\sim v\). Then \(d_{G^c}(u, v) = 1\). This shows \(D(G^c) = J_n - I_n + A(G)\).

Suppose \(d(G) = 3\). Let \(u\) and \(v\) be two vertices of \(G\). If \(u \sim v\), then there are two facts as following. When all vertices are adjacent to either \(u\) or \(v\) in \(G\), \(d_{G^c}(u, v) = 3\). When the vertex \(w\) is adjacent to neither \(u\) nor \(v\) in \(G\), \(d_{G^c}(u, v) = 2\). So we assume \(u \not\sim v\). Then \(d_{G^c}(u, v) = 1\). This shows \(D(G^c) \geq J_n - I_n + A(G)\).

In this section using the relations between \(D(G^c)\) and \(A(G)\) stated in Lemma 2.1 we determine the unique graph whose distance spectral radius attains maximum and minimum among all complements of graphs of diameter greater than three.

Suppose \(G\) is a connected simple graph with the vertex set \(V(G) = \{v_1, v_2, \ldots, v_n\}\). Let \(x = (x_1, x_2, \ldots, x_n)^T\) be an eigenvector of \(D(G)\) with respect to the eigenvalue \(\rho\), where \(x(v_i) = x_i\) \((i = 1, 2, \ldots, n)\). Then we have

\[
\rho x_i = \sum_{v_j \in V(G)} d_{ij} x_j. \tag{1}
\]

Let the vertex \(u\) connect the \(s\) vertices of the complete graph \(K_{n-s}\) and \(v\) connect other \(t = n - 2 - s\) vertices of \(K_{n-s}\). We denote by \(H(s, t)\) the resulting graph. If two graphs \(G\) and \(H\) are isomorphic, then we write \(G \cong H\). Let \(N_G(v)\) be neighbor of the vertex \(v\) in the graph \(G\).
Lemma 2.2 Let $H(s, t)$ be the graph of order $n$. Then we have $\lambda_1(H^c(s, t)) \leq \lambda_1(H^c(\lceil \frac{n}{2} - 1 \rceil, \lfloor \frac{n}{2} - 1 \rfloor))$ with equality if and only if $H(s, t) \equiv H(\lceil \frac{n}{2} - 1 \rceil, \lfloor \frac{n}{2} - 1 \rfloor)$.

Proof Let $k = \lambda_1(H^c(s, t))$. Set $x$ to be the Perron vector of $D(H^c(s, t))$ with respect to $k$. By the symmetry of $H^c(s, t)$ all the vertices in $N_{H(s, t)}(u)$ correspond to the same value $x_1$ and all the vertices in $N_{H(s, t)}(v)$ correspond to the same value $x_2$. Let $x(u) = x_1$ and $x(v) = x_2$. Then from the eigen-equation (1) we have

$$\begin{align*}
kxu &= 2sx_1 + tx_2 + xu, \\
kx1 &= 2ux + 2sx_1 + 3tx_2 + xu, \\
kx2 &= sx_1 + 2sx_1 + 2stx_2, \\
kxv &= sx_1 + sx_2 + 2tx_2.
\end{align*}$$

We can transform the above equations into a matrix equation $(kI_4 - D)x' = 0$, where $x' = (x_u, x_1, x_2, x_v)^T$ and

$$D = \begin{pmatrix}
0 & 2s & t & 1 \\
2 & 2(s - 1) & 3t & 1 \\
1 & 3s & 2(t - 1) & 2 \\
1 & s & 2t & 0
\end{pmatrix}.$$

Let $\phi_{k,t}(\lambda) = \det(I_4 - \lambda D)$. Then

$$\phi_{k,t}(\lambda) = \lambda^4 + (-2s - 2t + 4)\lambda^3 + (-9s - 9t - 5st + 3)\lambda^2 + (-12s - 12t - 4st - 4)\lambda + (-4s - 4t - 4).$$

Therefore, we obtain $\phi_{k,t}(\lambda) - \phi_{k-1,t+1}(\lambda) = \lambda(s - t - 1)(5\lambda + 4)$. Since the path $P_2$ of order 2 is an induced subgraph of $H^c(s, t)$, $D(H^c(s, t))$ contains $D(P_2)$ as a principal submatrix. Whereas $\lambda_1(P_2) = 1$, by Intercalation theorem we have $\lambda_1(H^c(s, t)) > 1$. Without loss of generality we assume $s \leq t$. By computation, $\phi_{k,t}(\lambda) - \phi_{k-1,t+1}(\lambda) < 0$, and so $\lambda_1(H^c(s, t)) > \lambda_1(H^c(s - 1, t + 1))$. Note that $s + t = n - 2$. Then $\lambda_1(H^c(s, t)) \leq \lambda_1(H^c(\lceil \frac{n}{2} - 1 \rceil, \lfloor \frac{n}{2} - 1 \rfloor))$.

\begin{align*}
\lambda_1(G^c) &< \lambda_1(H^c(s, t)) \\
\lambda_1(G^c) &< \lambda_1(H^c(s, t))
\end{align*}

Lemma 2.3 Suppose $G$ is a simple graph of diameter greater than three on $n$ vertices, and let $H(s, t)$ be the graph defined above. Then $\lambda_1(G^c) < \lambda_1(H^c(s, t))$.

Proof Since $d(G) > 3$, there must be two vertices $u$ and $v$ of $G$ such that $d(G)(u, v) > 3$. Clearly, the neighbours $N_G(u)$ and $N_G(v)$ of vertices $u$ and $v$ in the graph $G$ satisfy $N_G(u) \cap N_G(v) \neq \emptyset$. Set $W = V(G) \setminus (N_G(u) \cap N_G(v) \cup \{u, v\})$. Suppose that $s$ and $t$ are two positive integers such that $s \geq |N_G(u)|$, $t \geq |N_G(v)|$ and $s + t = n - 2$. Connecting all pairs of vertices of $G$ but $u$ and $v$, connecting $u$ with $s - |N_G(u)|$ vertices of $W$ and connecting $v$ with other $t - |N_G(v)|$ vertices of $W$, we obtain a graph isomorphic to $H(s, t)$.

Let $x$ be the unit Perron vector of $D(G^c)$ with respect to $\lambda_1(G^c)$. That is, each entry of $x$ is positive and $\|x\| = 1$. By the above construction of $H(s, t)$ we have $x^T A(G)x = \sum_{i,j \in E(G)} x_i x_j < \sum_{i,j \in E(H(s, t))} x_i x_j = x^T A(H(s, t))x$. Note that $d(G) > 3$ and $d(H(s, t)) = 3$. From Lemma 2.1 we have

$$\lambda_1(G^c) = x^T D(G^c)x$$

$$= x^T (J_n - I_n)x + x^T A(G)x$$

$$< x^T (J_n - I_n)x + x^T A(H(s, t))x$$

$$\leq x^T D(H^c(s, t))x.$$
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Theorem 2.7 Suppose $G$ is a simple graph of diameter greater than three on $n$ vertices. Then applying Lemma 2.6 we have the following result.

Let $d_{\lambda}^1(G)$ be the least distance eigenvalue of $G$. Then we have

$$d_{\lambda}^1(G) = x^T D(G)x,$$

where $x$ is the unit eigenvector of $\lambda_1(G)$. Note that $d(G) > 3$ and $d(G) \geq 3$.

From Lemma 2.1 we have

$$\lambda_1(G^c) = x^T D(G^c)x$$

$$= x^T (J_n - I_n)x + x^T A(G)x$$

$$\leq x^T (J_n - I_n)x + x^T A(G)x.$$

By Rayleigh’s theorem, $\lambda_1(G^c) \geq x^T D(G^c)x$, and so $\lambda_1(G^c) < \lambda_1(G^c).$

Let $\mathbb{G}_{n,d}$ be a set of all connected simple graphs of order $n$ with diameter $d$. By repeatedly applying Lemma 2.5 we have the following result.

Lemma 2.6 Let $d \geq 3$. Then we have

$$\max_{G \in \mathbb{G}_{n,d}} \lambda_1(G^c) > \max_{G \in \mathbb{G}_{n,d+1}} \lambda_1(G^c).$$

Since the path $P_n$ of order $n$ has the longest diameter among all simple graphs of order $n$, by repeatedly applying Lemma 2.6 we have the following result.

Theorem 2.7 Suppose $G$ is a simple graph of diameter greater than three on $n$ vertices. Then $\lambda_1(G^c) \geq \lambda_1(P_n^c)$, with equality if and only if $G \cong P_n$.

3 The least distance eigenvalue of the complements of graphs of diameter greater than three

In this section using the relations between $D(G^c)$ and $A(G)$ declared in Lemma 2.1 we determine the unique graph whose least distance eigenvalue attains minimum among all complements of graphs of diameter greater than three.

Let $T(a, b)$ denote the tree obtained from $P_2$ by appending $a$ vertices to one vertex of $P_2$ and $b$ vertices to the other. We denote by $T_1(a, b)$ the tree obtained from $P_3$ by appending $a$ vertices to one end vertex of $P_3$ and $b$ vertices to the other. Let $T_2(a, b)$ be the tree obtained from $T(a, b)$ by appending an additional pendent edge to a pendent vertices of $T(a, b)$. Clearly, $d(T(a + 1, b)) + 1 = d(T_1(a, b)) = d(T_2(a, b)) = 4$.

Lemma 3.1 ([7]) Let $T(a + 1, b), T_1(a, b)$ and $T_2(a, b)$ be three trees of order $n = a + b + 3$. Then

$$\lambda_n(T_1^c(a, b)) > \lambda_n(T_2^c(a, b)) > \lambda_n(T^c(a + 1, b)).$$

The equality holds if and only if $T_1(a, b) \cong T_2(a, b)$.

Suppose $G$ is a simple graph of diameter greater than three with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ ($n \geq 7$). Let $x = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of $D(G^c)$ with respect to $\lambda_n(G^c)$, where $x(i) = x_i$ ($i = 1, 2, \ldots, n$). Write $V_+ = \{v_i \in V(G^c) : x_i > 0\}$, $V_- = \{v_i \in V(G^c) : x_i < 0\}$ and $V_0 = \{v_i \in V(G^c) : x_i = 0\}$. Let $|V_+ \cup V_0| = p$ and $|V_-| = q$. Without loss of generality in what follows we assume $p \geq q$. Note that $p + q = n \geq 7$. We have $p \geq 4$.

Lemma 3.2 Suppose $G$ is a simple graph of diameter greater than three on $n \geq 7$ vertices. If $q = 1$ then $\lambda_n(G^c) > \lambda_n(T^c(n - 3, 1))$.

Proof Note that $q = 1$. We let $V_- = \{v\}$. Since $d(G) > 3$, $G$ contains an induced subgraph $P_4 = v\tilde{u}_1\tilde{u}_2\tilde{u}_3$. Deleting all edges in $G \setminus v$ except $\tilde{u}_1\tilde{u}_2$ and $\tilde{u}_2\tilde{u}_3$ and connecting all pairs of vertices which are not adjacent between the vertex $v$ and all vertices of $G \setminus \{v, \tilde{u}_2, \tilde{u}_3\}$ in $G$, we obtain a graph isomorphic to $T_1(n - 4, 1)$. From the above argument we have

$$x^T A(G)x = \sum_{v_i \in E(G)} x_i x_j \geq \sum_{v_i \in E(T_1(n - 4, 1))} x_i x_j = x^T A(T_1(n - 4, 1))x.$$

Set $x$ to be the unit eigenvector of $D(G^c)$ with respect to $\lambda_n(G^c)$. Note that $d(G) > 3$ and $d(T_1(n - 4, 1)) = 4$.

From Lemma 2.1 we have

$$\lambda_n(G^c) = x^T D(G^c)x$$

$$= x^T (J_n - I_n)x + x^T A(G)x$$

$$= x^T (J_n - I_n)x + x^T A(T_1(n - 4, 1))x$$

$$= x^T D(T_1^c(n - 4, 1))x.$$
By Rayleigh’s theorem we have $\lambda_n(T_1^c(n - 4, 1)) \leq x^TD(T_1^c(n - 4, 1))x$, and so $\lambda_n(G^c) \geq \lambda_n(T_1^c(n - 4, 1))$.

By Lemma 3.1 we have $\lambda_n(T_1^c(n - 4, 1)) \geq \lambda_n(T_2^c(n - 4, 1)) > \lambda_n(T_2^c(n - 3, 1))$. Thus $\lambda_n(G^c) > \lambda_n(T_2^c(n - 3, 1))$. \hfill \[\square\]

Let $p \geq q \geq 2$ and $p + q = n \geq 7$. Then $p \geq 4$. Let $B_1(p, q)$ be the graph obtained from the complete bipartite graph $K_{p,q}$ by deleting the edge $uv$. Suppose $u$ and $w$ are two vertices of the partition $U$ ($|U| = p$) and $v$ belongs to the partition $V$ ($|V| = q$). Deleting all edges of $B_1(p, q)$ which are incident to $w$ except $uv$ we denote by $B_2(p, q)$ the resulting graph. Clearly, $d(B_1(p, q)) + 1 = d(B_2(p, q)) = 4$.

Suppose $S$ is a subset of $V(G)$. Then we denote by $G[S]$ the subgraph of $G$ induced by $S$.

**Lemma 3.3** Suppose $G$ is a simple graph of diameter greater than three on $n \geq 7$ vertices. If $q \geq 2$ then $\lambda_n(G^c) \geq \lambda_n(B_2^c(p, q))$.

**Proof** Set $x$ to be the unit eigenvector of $D(G^c)$ with respect to $\lambda_n(G^c)$. Deleting all edges in $G[V -]$ and $G[V_+ \cup V_0]$ of $G$ we denote by $G'$ the resulting bipartite graph. If $G'$ is connected then $d(G') \geq d(G) > 3$, and so $G'$ contains two vertices $u$ and $w$ such that $d_{G'}(u, w) = 2$. Let $P = uu_1u_2vw$ be the path between $u$ and $w$. Then $u$ and $w$ are in the same partition, say $u$ and $w$ are both contained in $V_+ \cup V_0$. Without loss of generality assume that $x(u) \geq x(w)$. Deleting all edges which are incident to $w$ except $uv$ and connecting all pairs of vertices which are not adjacent between $(V_+ \cup V_0) \setminus \{u\}$ and $V_-$ except $u$ and $v$ in $G'$, we obtain a graph isomorphic to $B_2(p, q)$. From the above construction we have $x^T A(G)x = \sum_{v \in V} x_v x_w \geq \sum_{v \in V} x_v x_w = x^T A(B_2(p, q)) x$.

So we can assume that $G'$ is not connected. Since $G$ is connected, $G'$ must have one nontrivial component, that is, it contains at least one edge. We now distinguish two cases as follows.

**Case 1.** $G'$ has at least two nontrivial components.

Suppose two edges $\hat{u} \hat{v}$ and $\hat{u} \hat{v}'$ belong to two distinct nontrivial components. Without loss of generality we assume that $x(\hat{u}) \geq x(\hat{v}) \geq 0$. Deleting all edges which are incident to $\hat{u}$ except $\hat{u} \hat{v}'$ and connecting all pairs of vertices which are not adjacent between $(V_+ \cup V_0) \setminus \{\hat{u}\}$ and $V_-$ except $\hat{u}$ and $v$ in $G'$, we obtain a graph isomorphic to $B_2(p, q)$. Then $x^T A(G)x \geq x^T A(B_2(p, q)) x$.

**Case 2.** $G'$ has exactly one nontrivial component.

Suppose, without loss of generality, $G'$ has exactly one isolated vertex $\hat{w}$ such that $x(\hat{w}) \geq 0$. Since $d(G) > 3$, there must be an edge $\hat{u} \hat{w}$ and $x(\hat{u}) \geq 0$ such that $\hat{u} \hat{v}$ and $\hat{u} \hat{w}$ are both contained in $V_+ \cup V_0$. Without loss of generality we assume that $x(\hat{u}) \geq x(\hat{w}) \geq 0$. Deleting all pairs of vertices which are not adjacent between $(V_+ \cup V_0) \setminus \{\hat{u}\}$ and $V_-$ except $\hat{u}$ and $v$ and connecting $\hat{w}$ and $\hat{v}$ in $G'$, we obtain a graph isomorphic to $B_2(p, q)$. Then $x^T A(G)x \geq x^T A(B_2(p, q)) x$.

Note that $d(G) > 3$ and $d(B_2(p, q)) = 4$. From Lemma 2.1 and the above arguments we have

$$\lambda_n(G^c) = x^T D(G^c)x$$

$$= x^T(J_n - I_n)x + x^T A(G)x$$

$$\geq x^T(J_n - I_n)x + x^T A(B_2(p, q))x$$

$$= x^T D(B_2^c(p, q))x.$$ 

By Rayleigh’s theorem we have $\lambda_n(B_2^c(p, q)) \leq x^T D(B_2^c(p, q))x$. Therefore, we have $\lambda_n(G^c) \geq \lambda_n(B_2^c(p, q))$. \hfill \[\square\]

**Lemma 3.4** Let $B_2(p, q)$ and $B_1(p, q)$ be two graphs of order $n$. Then we have $\lambda_n(B_1^c(p, q)) < \lambda_n(B_2^c(p, q)) < -3$.

**Proof** Let $\lambda_n$ be the least eigenvalue of $D(B_2^c(p, q))$. Set $x$ to be an eigenvector of $D(B_2^c(p, q))$ with respect to $\lambda_n$. By the symmetry of $B_2^c(p, q)$ all vertices in $U \setminus \{u, w\}$ correspond to the same value $x_1$ and all the vertices in $V \setminus \{v\}$ correspond to the same value $x_2$. Let $x(u) = x_u$, $x(v) = x_v$ and $x(w) = x_w$. Then from the eigen-equation (1) we have

$$\begin{align*}
\lambda_n x_u &= x_v + x_w + (p - 2)x_1 + 2(q - 1)x_2, \\
\lambda_n x_v &= x_u + 2x_w + 2(p - 2)x_1 + (q - 1)x_2, \\
\lambda_n x_w &= x_u + 2x_v + (p - 2)x_1 + (q - 1)x_2, \\
\lambda_n x_1 &= x_u + 2x_v + x_w + (p - 3)x_2 + 2(q - 1)x_2, \\
\lambda_n x_2 &= 2x_u + x_v + x_w + 2(p - 2)x_1 + (q - 2)x_2.
\end{align*}$$

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We can transform the above equation into a matrix equation \((\lambda_n I - D_{B_2}')x' = 0\), where 
\[ x' = (x_u, x_v, x_w, x_1, x_2) \]
and
\[
D_{B_2}' = \begin{pmatrix}
0 & 1 & 1 & 2(q-1) \\
1 & 0 & 2 & (p-2) \\
1 & 2 & 0 & q-1 \\
1 & 2 & 1 & p-3 \\
2 & 1 & 2 & (p-2)
\end{pmatrix} .
\]

Let \(\varphi_{p,q}(\lambda) = \det(I_{B_2}' - D_{B_2}')\). Then we get

\[
\varphi_{p,q}(\lambda) = \lambda^5 - (q - 5 + p)\lambda^4
- (3pq + 4p + q - 10)\lambda^3
- (8pq + 6p - 4q - 8)\lambda^2
- (pq + 10p - 8)\lambda + 3pq - 6p - 2q + 4.
\]

Similarly we have

\[
\phi_{p,q}(\lambda) = \det(I_{B_2}' - D_{B_2}')
= \lambda^4 + (-q + 4 - p)\lambda^3
+ (-8pq + 2p + 2q + 4)\lambda^2
+ (-14pq + 6p + 6q)\lambda - 5pq + 2p + 2q .
\]

By the above two equations we have

\[
\varphi_{p,q}(\lambda) - (\lambda + 1)\varphi_{p,q}(\lambda) = (5pq - 5p - 2q + 2)\lambda^3
+ (14pq - 14p - 4q + 4)\lambda^2
+ (18pq - 18p - 8q + 8)\lambda + 8pq - 8p - 4q + 4.
\]

Since \(P_4\) is an induced subgraph of \(B_2'(p, q)\) and \(B_1'(p, q)\), \(D(P_4)\) is a principal submatrix of \(D(B_2'(p, q))\) and \(D(B_1'(p, q))\). Whereas \(\lambda_{P_4} < -3\), by Interlacing theorem we have \(\lambda_n(B_2'(p, q)) < -3\) and \(\lambda_n(B_1'(p, q)) < -3\). Note that \(p \geq 4\) and \(q \geq 2\). By computation, \(\varphi_{p,q}(\lambda) - (\lambda + 1)\phi_{p,q}(\lambda) > 0\) if \(\lambda < -3\). This implies \(\lambda_n(B_2'(p, q)) > \lambda_n(B_1'(p, q))\). \(\square\)

Let \(T(n - 3, 1)\) be a tree of order \(n\) obtained by appending \(n - 3\) vertices to the vertex \(u\) of the path \(uvw\).

**Lemma 3.5** Let \(B_1(p, q)\) and \(T(n - 3, 1)\) be two graphs of order \(n\) \((= p + q)\). Then \(\lambda_n(B_1'(p, q)) < \lambda_n(T'(n - 3, 1)) < -3\).

**Proof** Let \(\lambda_n\) be the least eigenvalue of \(D(T'(n - 3, 1))\). Set \(x\) to be an eigenvector of \(D(T'(n - 3, 1))\) with respect to \(\lambda_n\). By the symmetry of \(T'(n - 3, 1)\) all vertices in \(N_T(u) \setminus \{v\}\) correspond to the same value \(x_1\). Let \(x(u) = x_a, x(v) = x_v\) and \(x(w) = x_w\). Then from the eigen-equation (1) we have

\[
\begin{align*}
\lambda_n x_a &= 3x_v + 2(n - 3)x_1 + x_w, \\
\lambda_n x_v &= 3x_a + (n - 3)x_1 + 2x_w, \\
\lambda_n x_1 &= 2x_a + x_v + (n - 4)x_1 + x_w, \\
\lambda_n x_w &= x_a + 2x_v + (n - 3)x_1.
\end{align*}
\]

We can transform the above equation into a matrix equation \((\lambda_n I - D_{T'})x' = 0\), where 
\[ x' = (x_u, x_v, x_1, x_w) \]
and
\[
D_{T'} = \begin{pmatrix}
0 & 3 & 2(n - 3) & 1 \\
3 & 0 & n - 3 & 2 \\
2 & 1 & n - 4 & 1 \\
1 & 2 & n - 3 & 0
\end{pmatrix} .
\]

Let \(\psi_{p,q}(\lambda) = \det(I_n\lambda - D_{T'})\). Then

\[
\psi(\lambda) = \lambda^4 + (n + 4)\lambda^3 + (4 - 6n)\lambda^2 + (-6n - 8)\lambda - 12.
\]
Note that $n = p + q$. From the equations (2) and (3) we have

$$\phi_{p,q}(\lambda) - \psi(\lambda) = (-8pq + 8p + 8q)\lambda^2 + (-14pq + 12p + 12q + 8)\lambda - 5pq + 2p + 2q + 12.$$ 

Since $P_4$ is an induced subgraph of $T^c(n - 3, 1)$, $D(T^c(n - 3, 1))$ contains $D(P_4)$ as a principal submatrix. Whereas $\lambda_4(P_4) < -3$, we have $\lambda_n(T^c(n - 3, 1)) < -3$. Recall that $p \geq 4$ and $q \geq 2$. By computation, $\phi_{p,q}(\lambda) - \psi(\lambda) < 0$ if $\lambda < -3$. By Lemma 3.4 we have $\lambda_n(B^c_4(p, q)) < -3$, and so $\lambda_n(T^c(n - 3, 1)) > \lambda_n(B^c_4(p, q))$. 

**Lemma 3.6** Let $B_4(p, q)$ be the graph of order $n$. Then

$$\lambda_n(B^c_4(p, q)) \geq \lambda_n\left(\frac{n}{2} \left\lceil \frac{n}{2} \right\rceil \right).$$

The equality holds if and only if $B_4(p, q) \cong B_1(\left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor)$.

**Proof** By the equation (2) we have

$$\phi_{p,q}(\lambda) - \phi_{p-1,q+1}(\lambda) = (8p - 8q - 8)\lambda^2 + (14p - 14q - 14)\lambda + 5p - 5q - 5.$$ 

Note that $p \geq q$. By computation, $\phi_{p,q}(\lambda) - \phi_{p-1,q+1}(\lambda) \geq 0$ if $\lambda < -3$. By Lemma 3.4 we have $\lambda_n(B^c_4(p, q)) < -3$, and so $\lambda_n(B^c_4(p, q)) > \lambda_n\left(\frac{n}{2} \left\lceil \frac{n}{2} \right\rceil \right)$.

Combining Lemmas 3.2-3.6 we have the following main result.

**Theorem 3.7** Let $G$ be a simple graph of diameter greater than three on $n \geq 7$ vertices. Then $\lambda_n(G^c) > \lambda_n(B^c_4(\left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor))$. 

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