POSITIVE FORMULÆ FOR K-TYPES OF SL₃(ℝ)-IRREPS
AND A BLATTNER FORMULA FOR SMOOTH K-ORBIT CLOSURES

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To David Vogan for his 60th birthday

ABSTRACT. We prove a version of Blattner’s conjecture, for irreducible subquotients of principal series representations with integral infinitesimal character of a real reductive Lie group whose Beilinson-Bernstein D-module is supported on a K-orbit with smooth closure. (The cases usually considered are closed orbits, or their preimages along G/B → G/P.) We apply this to Gₚ = SL₃(ℝ), where all four K-orbits on G/B have smooth closure, and refine the resulting alternating-sum formulæ to ones with only positive terms.

1. STATEMENT OF RESULTS

Let Gₚ = SL₃(ℝ), with maximal compact Kₚ = SO(3, ℝ) preserving the form with Gram matrix \[
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\]
and flag manifold G/Bₚ ∼= \{(0 < V₁ < V₂ < C³)\}. Let \text{rank}(V) denote the rank on \(V \leq C³\) of the restriction of the K-invariant orthogonal form on \(C³\). Then \(K\) preserves the smooth subvarieties
\[
O₁ := \{(V₁ < V₂) : \text{rank}(V₁) = 0\}, \quad O₂ := \{(V₁ < V₂) : \text{rank}(V₂) = 1\}
\]
and the full set of K-orbit closures is G/Bₚ, O₁, O₂, O₁ ∩ O₂, all smooth.

In (2) we prove a Blattner-type formula for the K-multiplicities in \((g, K)\)-irreps with integral infinitesimal character whose corresponding \(D\)-modules are supported on smooth K-orbit closures. Since the standard modules for the open orbit on SL(3)/B⁻ have composition series whose factors are the irreps for the four orbits, we could hope to define a set \(C\) that computes the SO(3)-multiplicities of the standard module and breaks as \(C = \bigsqcup C_Ο\) giving formulæ for those irreps, and this is our main result (proved in (3)).

**Theorem 1.1.** Fix an infinitesimal character \((a, b) + ρ_ halk; G \in ℤ^2) of Gₚ = SL₃(ℝ). Define the set
\[
C := \{(c, d) \in ℤ^2 : c \equiv a \mod 2, \text{ and if } c = 0 \text{ then } d \equiv b \mod 2\}
\]
and the projection \(π : C \to ℤ, (c, d) \mapsto c + d\). This \(C\) decomposes into four regions:
\[
C_{G/B_ₚ} := \{(c, d) \in C : c \leq a, \ d \leq b\}
\]
\[
C_{O₁} := \{(c, d) \in C : c > a, \ d \leq b\}
\]
\[
C_{O₂} := \{(c, d) \in C : c \leq a, \ d > b\}
\]
\[
C_{O₁∩O₂} := \{(c, d) \in C : c > a, \ d > b\}
\]
The \((sl₃, SO(3))\)-irrep with infinitesimal character \((a, b) + ρ_ G, supported on an orbit closure \(Ο\), contains the SO(3)-irrep of dimension \(2n + 1\) with multiplicity \# (\(π⁻¹(n) \cap C_Ο\)).
Figure 1. The black lines denote the boundaries of the four regions in C. Each alternate line of constant c is crossed out by brown dashed lines, and when c = 0, every other dot is crossed out. When we project the remaining dots vertically to N, the four regions give the multiplicities pictured (counted from the left corners of the regions). The vertical dotted lines go through the corners of $C_{SL_3/B}$ and bound (up to $\rho$-shifts) the regions in the K-multiplicities where the piecewise-quasipolynomial changes.
See figure [1] for a picture of C and its four pieces in the case \((a, b) = (2, 4)\), and the multiplicities induced by their projections to \(\mathbb{N}\). It would be interesting to relate the decomposition of C to the gluing of conormal varieties in \([\text{MaO}]\). Other combinatorial investigations of the Blattner formula appear in \([\text{WZ08, Ha08}]\).

**ACKNOWLEDGMENTS**

This work was inspired by David Vogan, from whom I’ve learned so much – but not the yoga of tempered representations and \(\mathcal{D}\)-modules, which I learned from Peter Trapa, without whom this paper couldn’t have come into existence. (When Peter is unavailable, a nice paper reference is \([\text{Mi93}]\).) Of course, one could use a transitivity argument to again credit David. I also thank Wilfried Schmid for useful discussion about closed orbits in general and the closed \(\text{SO}(3)\)-orbit in particular.

I thank Birgit Speh for explaining to me that in the long history of this subject, pictures such as those in figure [1] to compute \(K\)-multiplicities were essentially unpublishable due to technological restrictions, but were common on David’s blackboard. They may have appeared in her thesis (on \(\text{SL}_3(\mathbb{R})\)-representations), which (unlike \([\text{Ha08}]\)) is unfortunately not available on the MIT website.

Roger Howe pointed me to his paper \([\text{Ho00}]\) which also computes the \(K\)-types. Rather than using the hamfisted character-theoretic approach we take here, he looks much more deeply into operators on the representation itself, finding a basis for each highest weight space.

### 2. A Blattner-type formula for smooth orbit closures

Blattner’s conjecture concerns \(K\)-weight multiplicities, which we will derive from \(T_K\)-weight multiplicities, and those from \(T_K\)-equivariant localization in \(K\)-theory (a slightly nontrivial step, since localization gives us rational functions which carry slightly less information than the relevant power series). There is an extra complication, which is that the \(T_K\)-weight multiplicities are infinite, and we need to split them using an extra action of \(\mathbb{C}^\times\).

#### 2.1. Generalities about equivariant localization and multiplicity functions.

**Proposition 2.1.** Let \(X \subseteq Y\) be a pair of complex varieties, \(Y\) smooth and \(X\) projective, carrying compatible actions of a torus \(T\), with \(Y^T\) finite. Let \(D_X\) be the sheaf of distributions on \(Y\) supported on \(X\) (so, a module over the filtered sheaf \(D_Y\) of differential operators), and \(L\) a \(T\)-equivariant line bundle on \(Y\).

Pick generators for \(D_X\) as a \(D_Y\)-module, so we can define an associated graded \(\text{gr } D_X\), which gives a \((T \times \mathbb{C}^\times)\)-equivariant coherent sheaf on \(\text{Spec}(\text{gr } D_Y) = T^*Y\); here the \(\mathbb{C}^\times\) acts by inverse rescaling action the fibers of \(T^*Y\) (the inverse to make its character \(z\) and not \(z^{-1}\) below). Extend \(L\) to an equivariant line bundle on \(T^*Y\) by pulling it back along the projection \(T^*Y \to Y\). Then each \(H^i(T^*Y; \text{gr } D_X \otimes L)\) has finite-dimensional \(\mathbb{C}^\times\)-weight spaces, hence finite-dimensional \((T \times \mathbb{C}^\times)\)-weight spaces, so its character is a well-defined formal power series. (It does not usually have finite-dimensional \(T\)-weight spaces.)

The \((T \times \mathbb{C}^\times)\)-character on \(\chi(T^*Y; \text{gr } D_X \otimes L) := \sum_i (-1)^i H^i(T^*Y; \text{gr } D_X \otimes L)\), the sheaf Euler characteristic, can be computed by equivariant localization (thereby showing that its power
series sums to a rational function). If \( f \in X^T \) is a smooth point of \( X \), then the contribution to the character from the point \( f \) is

\[
 t^{wt(L_f)} \prod_{\mu \in wts(T|X)} \frac{1}{1-t^{-\mu}} \prod_{\mu \in wts(T|Y/X)} \frac{t^\mu z}{1-t^\mu z}.
\]

Proof. For any coherent \((T \times \mathbb{C}^\times)\)-equivariant sheaf \( \mathcal{F} \) on \( T^*Y \), the localization formula from [192] says we can localize the character around the fixed points \((T^*Y)^{T \times \mathbb{C}^\times} = Y^T_k \),

\[
 \text{Tr}([f, z]|_{|X(T^*Y; \mathcal{F})}) = \sum_{f \in Y^T_k} \frac{L_f}{\prod_{\mu \in wts(T|Y)} (1-t^{-\mu})(1-t^{\mu}z)}, \quad (t, z) \in T_k \times \mathbb{C}^\times
\]

where the two factors in the product in the denominator come from the tangent and cotangent lines in \( T_f(T^*Y) \), and the \( L_f \) are regular functions on \( T \times \mathbb{C}^\times \) derived from \( \mathcal{F} \). (More specifically, we can compute \( L_f \) by picking an equivariant projective resolution of \( \mathcal{F} \) and alternating-sum the characters of the fibers of the resulting vector bundles.) In the case at hand, where \( \mathcal{F} = \text{gr } D_X \otimes \pi^!(\mathcal{L}) \) and \( D_X \) was supported along \( X \), the contribution \( L_f \) vanishes unless \( f \in X \) (and hence \( X^T_k \)).

When \( f \) is a smooth point of \( X \), we can reduce to the local situation \( T_e(V_1 \subset V_2) \) where \( V_1, V_2 \) are vector spaces. There

\[
 \text{Tr} \left( (t, z)|_{|X(T^*V_2; \text{gr } D_{V_1})} \right) = \prod_{\mu \in wts(V_1)} \frac{1}{1-t^{-\mu}} \prod_{\mu \in wts(V_2/V_1)} \frac{t^\mu z}{1-t^\mu z}
\]

which one derives by observing that \( \text{gr } D_{V_1} \) is a free sheaf of rank 1 on the conormal variety \( V_1 \times (V_2/V_1)^* \), with generator of \( T \)-weight \( \sum_{\mu \in wts(V_2/V_1)} \mu \). Letting \( V_1 = T_fX \) and \( V_2 = T_fY \), we get the desired formula.

We actually want to study this representation \( \chi(T^*Y; \text{gr } D_X \otimes \mathcal{L}) \) by its power series, or weight multiplicity function (an \( \mathbb{N} \)-valued function on the weight lattice), not by its character (a rational function on \( T \times \mathbb{C}^\times \)) provided by equivariant localization. The basic problem in going backwards from the character is that the two power series \( \frac{1}{1+x+x^2+\ldots} \) and \( -x^{-1} - x^{-2} - \ldots \) are not equal, but their rational functions \( \frac{1}{1-x} = \frac{x^{-1}}{1+x} \) are. So how to choose a power series given a rational function, especially one presented as a sum of rational functions with even worse denominators?

**Lemma 2.2.** Say \( P \) is a multivariable power series whose exponent vectors live in a cone on which some generic coweight \( \tau \) is proper and bounded above. Assume \( P \) adds up to a rational function \( r \) of the form \( \sum f r_f \) where each \( r_f \)'s denominator is of the form \( \prod (1-t^{-\beta}) \). If we use the identity \( \frac{1}{1-x} = \frac{x^{-1}}{1+x} \) to flip the denominators in each \( r_f \) to make each \( \langle \tau, \beta \rangle > 0 \), and use that choice to thereby choose a power series \( P_f \) for each \( f \), then \( P = \sum f P_f \) as power series (refining the fact that \( r = \sum f r_f \) as rational functions).

Proof. In the one-variable case, this just says that a rational function (with this sort of denominator) has a unique Laurent series expansion. In a quite different situation, two different finitely supported multivariable Laurent “series” can’t have the same associated rational function (the Laurent polynomial).

Now the general case. Since the support is in a proper cone we can perturb \( \tau \) to be rational, then scale it to be integral. Thinking of \( P \) by its coefficients so as a function on
This question is often phrased as one of regularizing the Fourier transform to be able to apply to rational functions. In the cases typically considered (e.g. in [GLS96]) the “series” $P$ is finitely supported and $\tau$ is therefore arbitrary (once generic). In the most familiar example, $P$ is the weight multiplicity function of a finite-dimensional $G$-irrep $V_\lambda$ and localization gives the manifestly Weyl-symmetric formula\[\prod_{\beta \in \Delta_+^G} (1 - t^{-\beta})\] where $\text{Den}(G) = \prod_{\beta \in \Delta_+^G} (1 - t^{-\beta})$ is the Weyl denominator and $\Delta_+^G$ is determined from $\tau$. In this case all denominators flip to the same $\text{Den}(G)$, giving the Weyl character formula and its Fourier transform the Kostant multiplicity formula. But when $P$ is noncompactly supported we will need to choose $\tau$ more carefully.

Nevertheless, start with $\tau$ a generic coweight of $T$, and a finite-dimensional $T$-vector space $V$ with $V^T = 0$. If all of $V$’s weights $wts(V)$ lie in a proper half-space of $T$’s weight lattice, then the $T$-weight multiplicities on the ring of functions on $V$ are finite, and its character is the rational function $\prod_{\mu \in wts(V)} \frac{1}{1 - t^{-\mu}}$. Define
\[
\text{down}(V) := \{\mu \in wts(V) : \langle \tau, \mu \rangle < 0\} \\
\text{ind}(V) := \# \text{down}(V) \\
\rho(V) := \sum_{\mu \in \text{down}(V)} \text{down}(V) \quad \text{so } \langle \tau, \rho(V) \rangle < 0, \text{ unless } \text{down}(V) = \emptyset \\
wts(V)_+ := \{\mu \text{ sign}(\langle \tau, \mu \rangle) : \mu \in wts(V)\}
\]
where $wts(V)$ denotes the set-with-multiplicity of $T$-weights. Call these together the $\tau$-positivity setup, and use them to rewrite the character of the ring of functions on $V$
\[
\prod_{\mu \in wts(V)} \frac{1}{1 - t^{-\mu}} = \prod_{\mu \in wts(V), \langle \mu, \tau \rangle < 0} \frac{1}{1 - t^{-\mu}} \prod_{\mu \in wts(V), \langle \mu, \tau \rangle > 0} \frac{1}{1 - t^{-\mu}} \\
= \prod_{\mu \in wts(V), \langle \mu, \tau \rangle < 0} \frac{1}{1 - t^{-\mu}} \prod_{\mu \in wts(V), \langle \mu, \tau \rangle > 0} \frac{1}{1 - t^{-\mu}} \\
= (-1)^{\text{ind}(V)} \frac{1}{1 - t^0} \prod_{\mu \in wts(V), \langle \mu, \tau \rangle > 0} \frac{1}{1 - t^{-\mu}} \\
= (-1)^{\text{ind}(V)} \frac{1}{1 - t^0} \prod_{\mu \in wts(V)} \frac{1}{1 - t^{-\mu}}.
\]
Let
\[
\kappa_m(\vec{v}; S) := \# \{\text{ways to write } \vec{v} \text{ as a sum of exactly } m \text{ vectors from } S, \text{ with multiplicity}\}
\]

\footnote{We record here the conventions that lead to this formula in this form. The positive roots $\Delta_+^G$ are those in the Borel subgroup $B_G$. Around the basepoint of $G/B_G$, the isotropy weights are $\Delta_+^G$, so the ring of functions on that tangent space or big cell has the character $1/\text{Den}(G)$, which we then twist by $t^\lambda$. Then the other summands are all Weyl translates of this one. When flipping to the standard Weyl character formula, we want each of the $1 - t^{-\beta}$ factors in the denominator to have $\beta \in \Delta_+^G$.}
be a sort of graded vector partition function on a set-with-multiplicity $S$, with
$$\kappa(\vec{v}; S) = \sum_{m \in \mathbb{N}} \kappa_m(\vec{v}; S)$$
the usual one.

**Proposition 2.3.** From the $T$-coweight $\tau$ build a $(T \times \mathbb{C}^\times)$-coweight $(\tau, N \ll 0)$. With this choice, we can Fourier transform the term from proposition 2.1 to the virtual weight multiplicity function
$$(\lambda, d) \mapsto (-1)^{\text{ind}(T_X)} \kappa\left(\lambda - \text{wt}(L_\ell) - \rho(T_X) - \sum_{\mu \in \text{wts}(T_Y/T_X)} \mu, \ d - \text{codim}_Y X; \right.$$
$$\left.\left\{ \left((\mu, 0) : \mu \in \text{wts}_+(T_X) \right) \prod \bigl( (\mu, 1) : \mu \in \text{wts}(T_Y/T_X) \bigr) \right\} \right)$$
and the $(T \times \mathbb{C}^\times)$-weights occurring in $\chi(T^*Y; L \otimes \text{gr} D_X)$ lie in a cone on which the coweight $(\tau, N)$ of $T \times \mathbb{C}^\times$ is proper and bounded above.

**Proof.** The benefit of taking $N \ll 0$ in the coweight $(\tau, N)$ is that the $z$ factors in the second product ensure that none of them need be flipped to satisfy lemma 2.2. The first product we rewrite as explained above:
$$\prod_{\mu \in \text{wts}(T_X)} \frac{1}{1-t^{-\mu}} = (-1)^{\text{ind}(T_X)} t^{\rho(T_X)} \prod_{\mu \in \text{wts}_+(T_X)} \frac{1}{1-t^{-\mu}}$$
so the rational function contributed by proposition 2.1 can be rewritten (with $(\tau, N)$-positive denominators) as
$$(-1)^{\text{ind}(T_X)} t^{\text{wt}(L_\ell) + \rho(T_X)} \prod_{\mu \in \text{wts}_+(T_X)} \frac{1}{1-t^{-\mu}} \prod_{\mu \in \text{wts}(T_Y/T_X)} \frac{t^\mu z}{1-t^\mu z}$$
whose imputed Fourier transform is the formula claimed. \qed

Note that if we try to forget the $\mathbb{C}^\times$-action in the above, we run into a vector partition function on the set $\{-\mu : \mu \in \text{wts}_+(T_X)\} \prod \text{wts}(T_Y/T_X)$, which may very well contain a vector and its negative, making the vector partition function infinite. The retention of the $\mathbb{C}^\times$ therefore looks crucial, but we will sometimes be able to get rid of it, as in the end of theorem 2.2.

### 2.2. A $K$-multiplicity formula for certain $\mathcal{D}$-modules.
Some version of the following argument seems to be well-known to the experts (see e.g. [BS97, p110–111]), at least for proving Blattner’s original conjecture, but we need it in greater generality than we could find in the literature.

**Theorem 2.4.** Let $X \subseteq Y$ be a pair of smooth complex projective varieties, carrying compatible actions of a connected complex reductive group $K$. Assume that $K$’s maximal torus $T_K$ acts on $Y$ (and hence $X$) with isolated fixed points, and that all $K$-stabilizers are solvable. Fix a generic real dominant coweight $\tau$ of $K$, with which to define the positive roots $\Delta^+_\tau$ of $K$ and a $\tau$-positivity setup on each $T_fX$. Then $\text{wts}(T_fX)_+ \supseteq \Delta^+_\tau$ for each $f$. 

Let $D_X$ be the sheaf of distributions on $Y$ supported on $X$, and $L$ a $K$-equivariant line bundle on $Y$. Then for $\lambda$ a dominant $K$-weight, the multiplicity of the finite-dimensional $K$-irrep $V_\lambda$ in the $C^\times$-weight $d$ part of $\chi(T^*Y; \text{gr} D_X \otimes L)$ is

$$\sum_{f \in X^T} (-1)^{\text{ind}(T_fX)} \kappa \left( (\lambda - \text{wt}(L)|_f) - \rho(T_fX) - \sum_{\mu \in \text{wts}(T_fY/T_fX)} \mu, \ t - \text{codim}_Y X \right);$$

$$\{(-\mu, 0) : \mu \in \text{wts}_+(T_fX) \setminus \Delta^K_+ \} \coprod \{((\mu, 1) : \mu \in \text{wts}(T_fY/T_fX))\}.$$

This is $0$ for $d < \text{codim}_Y X$, and nonnegative for $L$ ample enough.

Proof. Now that $X$ is smooth, the localization formula simplifies to proposition 2.1 at every $f \in X^T$, and by lemma 2.2 we obtain the $(T_K \times C^\times)$-multiplicities when we sum up the virtual multiplicity functions from proposition 2.3

$$\sum_{f \in X^T} (-1)^{\text{ind}(T_fX)} \kappa \left( (\lambda - \text{wt}(L)|_f) - \rho(T_fX) - \sum_{\mu \in \text{wts}(T_fY/T_fX)} \mu, \ t - \text{codim}_Y X \right);$$

$$\{(-\mu, 0) : \mu \in \text{wts}_+(T_fX) \} \coprod \{((\mu, 1) : \mu \in \text{wts}(T_fY/T_fX))\}.$$

Note that $t/\text{stab}_f(x) \hookrightarrow T_fX$ as $T_k$-representations, so by the assumption of solvable stabilizers, $\text{wts}(T_fX)$ must contain at least one of $\beta, -\beta$ for each positive root $\beta \in \Delta^K_+$. Hence $\text{wts}(T_fX) \supseteq \Delta^K_+$.

To go from the $(T_K \times C^\times)$-multiplicities to the $(K \times C^\times)$-multiplicities, we have to undo the Kostant multiplicity formula, which amounts to applying difference operators in the directions of $K$'s positive roots. The effect is to remove those vectors from the input to the vector partition function, giving the claimed formula.

On the positive Weyl chamber, that formula computes the $(K \times C^\times)$-multiplicities in $\chi(T^*Y; \text{gr} D_X \otimes L)$. (Outside the positive Weyl chamber it is Weyl-antisymmetric around $-\rho_K$, secretly computing the Euler characteristic of the $n_K$-cohomology.) When $L$ is ample enough, this $\chi = \dim H^0$, hence is nonnegative. \qed

2.3. Closed $K$-orbits. Hereafter $G$ is a complex connected reductive group, with $K$ the identity component of the fixed points of a holomorphic involution $\theta$. The ambient space $Y = G/B^-_G$ is a flag manifold, and $X \subseteq Y$ is a $K$-orbit closure on it. Pick $T_K \leq K$ then $T_G \supseteq T_K$, so we have a restriction map $T_G \rightarrow T_K$ that we won’t require be an isomorphism.

\footnote{As indicated in the previous footnote, this convention leads to the cleanest localization formula.}
Theorem 2.5. If $X$ is a closed $K$-orbit on $G/B_G$, and $L$ is the line bundle on $G/B_G$ corresponding to a dominant $G$-weight $\nu$, then the multiplicity of the $K$-irrep $V_\lambda$ in $\chi(G/B_G; D_X \otimes L)$ is

$$\sum_{w \in W_k} (-1)^{\ell(w)} \chi \left( \lambda + \rho_K - w \cdot (\nu + 2\rho_G - \rho_K) ; w \cdot (\Delta^G_+ \setminus \Delta^K_+) \right)$$

$$= \sum_{w \in W_k} (-1)^{\ell(w)} \chi \left( w \cdot (\lambda + \rho_K) - (\nu + 2\rho_G - \rho_K) ; \Delta^G_+ \setminus \Delta^K_+ \right).$$

For $\nu$ sufficiently dominant, if we refine $\kappa()$ to $\sum_d \kappa_d()$, then each $\kappa_d()$ is individually positive.

In the equal-rank case, this is Blattner’s conjecture for integral infinitesimal character. It is off by a $\rho_G$ from the usual formula because it is stated in terms of the line bundle rather than the infinitesimal character.

Proof. For this we take $\tau$ a generic dominant coweight of $G$, which by restriction (and our choice $T_G \supseteq T_k$) gives a coweight of $K$ as well, defining $\Delta^G_+$ and $\Delta^G_+$. The wt and wts notation hereafter refer to $T_k$-weights.

For this pair $(X, Y)$ every $wts(T_kX)_+ = \Delta^K_+$, so theorem [2.4] becomes

$$\sum_{f \in X^f} (-1)^{\text{ind}(T_kX)} \chi \left( \lambda - \text{wt}(L|_f) - \rho(T_kX) - \sum_{wts(T_kY/T_kX)} \mu ; d - \text{codim}_Y X \right) ; \text{wts}(T_kY/T_kX) \times \{1\}$$

$$= \sum_{f \in X^f} (-1)^{\text{ind}(T_kX)} \chi_{d - \text{codim}_Y X} \left( \lambda - \text{wt}(L|_f) - \rho(T_kX) - \sum_{wts(T_kY/T_kX)} \mu ; \text{wts}(T_kY/T_kX) \right).$$

Now notice that

- $X^f = W_k \cdot B_G/B_G$, so take $f = wB_G/B_G$ hereafter;
- $\text{ind}(T_kX) = \ell(w)$ for $w \in W_k$,
- $\rho(T_kX) = w \cdot \rho_K - \rho_K$, and
- $\sum_{wts(T_kY/T_kX)} \mu = w \cdot 2(\rho_G - \rho_K)$ as $T_k$-weights,

so we can rewrite $\kappa$’s first argument thusly

$$\lambda - \text{wt}(L|_f) - \rho(T_kX) - \sum_{wts(T_kY/T_kX)} \mu = \lambda - w \cdot \nu - (w \cdot \rho_K - \rho_K) - 2w \cdot (\rho_G - \rho_K)$$

$$= (\lambda + \rho_K) - w \cdot (\nu + 2\rho_G - \rho_K)$$
and the $m$th term in the sum becomes
\[
\sum_{w \in W_K} (-1)^{l(w)} k_{d - |\Delta^G \setminus \Delta^K|} \left( (\lambda + \rho_K) - w \cdot (\nu + 2\rho_G - \rho_K) + w \cdot (\Delta^G_+ \setminus \Delta^K_+) \right)
\]

\[
= \sum_{w \in W_K} (-1)^{l(w)} k_{d - |\Delta^G \setminus \Delta^K|} \left( w^{-1} \cdot (\lambda + \rho_K) - (\nu + 2\rho_G - \rho_K) + \Delta^G_+ \setminus \Delta^K_+ \right)
\]

\[
= \sum_{w \in W_K} (-1)^{l(w)} k_{d - |\Delta^K|} \left( w \cdot (\lambda + \rho_K) - (\nu + 2\rho_G - \rho_K) + \Delta^G_+ \setminus \Delta^K_+ \right)
\]

by reindexing.

Since $T_K$ contains regular elements of $G$, the set $\Delta^G_+ \setminus \Delta^K_+$ lives in a proper half-space, so for $d \gg 0$ this vector partition function must vanish. Hence we can sum over $d$ (effectively forgetting the $\mathbb{C}^\times$-action), obtaining the finite $K$-multiplicities claimed.

There is an extension of Blattner’s conjecture in [KV95, p376–377] to the necessarily smooth preimages $O = \pi_F^{-1}(O_{\text{closed}})$ of closed $K$-orbits on $G/Ps$.

2.4. Smooth $K$-orbit closures.

**Theorem 2.6.** Let $X$ be a smooth $K$-orbit closure on $G/B_G^-$, and $\mathcal{L}$ be the line bundle corresponding to the dominant $G$-weight $\nu$. Then the multiplicity of the $K$-irrep $V_\lambda$ in the $\mathbb{C}^\times$-weight $d$ part of $\chi(G/B_G^-; D_X \otimes \mathcal{L})$ is

\[
\sum_{w \in X^d} (-1)^{\text{ind}(T_w X)_K} \left( (\lambda - w \cdot \nu - \rho(T_w X) - 2w \cdot \rho_G + \sum_{\mu \in \text{wts}(T_w X)} \mu, d - \text{codim}_X \right) \bigwedge \{(-\mu, 0) : \mu \in wts_+(T_w X) \setminus \Delta^K_+ \}
\]

where we are identifying $(G/B_G^-)_K \cong W_G$.

**Proof.** We need argue that we are using the correct $D_Y$-module with which to compute the irrep (and not e.g. the standard module, or something in between). Under the correspondence with perverse sheaves, we want to make sure that we’re using the perverse extension to the $K$-orbit closure of the locally constant sheaf from the $K$-orbit. Since the $K$-orbit closure is smooth, the locally constant sheaf on the closure is already perverse, and as the perverse extension is unique we have found it. Converting back to $D_Y$-modules gives us the sheaf of distributions on the orbit closure.

The rest is easy conversion from the general case to $Y = G/B_G^-$, with some special features because $Y$’s tangent bundle is $W_G$-equivariant (or rather, $N(T_G)$-equivariant).

3. $(\mathfrak{sl}_3, \text{SO}(3))$-irreps: proof of theorem [1.1]

We now apply theorem 2.6 to the four $\text{SO}(3)$-orbit closures on $\text{SL}(3)/B_{\text{SL}(3)}^-$. We index $\text{SL}(3)$’s dominant weights by $\mathbb{N}^2$ using the fundamental weights, so $\rho_G = (1, 1)$. But when we index $\text{SO}(3)$’s (not $\text{SL}(2)$’s) dominant weights by $\mathbb{N}$, we miss the $\text{SL}(2)$ weight $\rho_K$, which is at $\frac{1}{2}$ where the physicists (used to) like having it. In these coordinates, the restriction map $r : T^*_G \to T^*_K$ is $(a, b) \mapsto a + b$ (taking the fundamental irreps of $\text{SL}(3)$ to the first, defining, irrep of $\text{SO}(3)$), and $r(\Delta^G_+) \setminus \Delta^K_+ = \{1, 2\}$.
Let $v = (a, b) \in T^*_G$ as in the statement of theorem 1.1. The fixed points $Y^T_k$ are the coordinate flags, where the $SO(3)$-invariant inner product on $\mathbb{C}^3$ is the one where the dual basis to $(e_1, e_2, e_3)$ is $(e_3, e_2, e_1)$.

3.1. **The closed orbit $O_1 \cap O_2$.** In this case we can even apply theorem 2.5, obtaining

$$
\sum_{w \in W_k} (-1)^{\ell(w)} \kappa \left( (\lambda + \rho_k) - w \cdot (v + 2\rho_G - \rho_k) ; w \cdot (\Delta^G_+ \setminus \Delta^k_+) \right)
= \kappa \left( \lambda + \frac{1}{2} - (a + b + 4 - \frac{1}{2}) ; \{1, 2\} \right) - \kappa \left( \lambda + \frac{1}{2} + (a + b + 4 - \frac{1}{2}) ; \{-1, -2\} \right)
= \kappa \left( \lambda - 3 - (a + b) ; \{1, 2\} \right) - \kappa \left( \lambda + 4 + (a + b) ; \{-1, -2\} \right)
= \kappa \left( \lambda - 3 - (a + b) ; \{1, 2\} \right)
$$

since $\lambda, a, b \geq 0$

This vanishes for $\lambda < a + b + 3$, and the values for $\lambda \geq a + b + 3$ are $\lambda \mapsto \lfloor (\lambda - (a + b + 3))/2 \rfloor$, i.e. from $a + b$ they are $0, 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, \ldots$ This easily matches the projection of

$$C_{O_1 \cap O_2} := \{(c, d) \in \mathbb{N}^2 : c > a, \ d > b, \ c \equiv a \mod 2\}$$

from theorem 1.1.

W. Schmid pointed out that the normal bundle to the closed orbit $\equiv \mathbb{C}P^1$ is (by Grothendieck’s theorem) a sum of line bundles, specifically $O(-3) \oplus O(-3)$. However Grothendieck’s theorem fails $SO(3)$-equivariantly, even though it holds $C^\times$-equivariantly [KS98 §4.1]; rather this bundle has one $SO(3)$-invariant subbundle $O(-2)$ with quotient $O(-4)$. If we replace the normal bundle by its associated graded bundle, the symmetric algebra of that is again a sum of line bundles (controlled by the vector partition function on $\{1, 2\}$), to which we can apply Borel-Weil, obtaining this vector partition function formula very directly. Then the vanishing of higher cohomology for the (symmetric algebra of that) associated graded lets us infer that the normal bundle itself gives us the same $H^0$.

For uniformity with the other orbits to come we redo the calculation using the formula from theorem 2.4 directly, which computes the subspace of $C^\times$-weight $d$:

$$
\sum_{f \in X^T} (-1)^{\text{ind}(T_rX)} \kappa \left( (\lambda - \text{wt}(L)|_f) - \rho(T_rX) - \sum_{w \in \text{wts}(T_rY/T_rX)} \mu, \ d = \text{codim}_Y X \right) ;
\{(-\mu, 0) : \mu \in \text{wts}_+ (T_rX) \setminus \Delta^k_+ \} \coprod \{([\mu, 1] : \mu \in \text{wts}(T_rY/T_rX))\}
$$

The $T^*_k$-fixed points on $O_1 \cap O_2$ look like

| $f \in O^T_k$ | $\text{wt}(L)|_f$ | $\text{wts}(T_rX)$ | $\text{wts}(T_rY) \setminus \text{wts}(T_rX)$ | ind$(T_rX)$ | $\rho(T_rX)$ |
|----------------|-----------------|----------------|----------------------------------|-------------|-------------|
| $(e_1 < e_1 + e_2)$ | $(a, b)$ | $\{1\}$ | $\{2, 1\}$ | $0$ | $0$ |
| $(e_3 < e_2 + e_3)$ | $(-b, -a)$ | $\{-1\}$ | $\{-2, -1\}$ | $1$ | $-1$ |

giving two terms

$$
\kappa((\lambda - (a + b) - 0 - 3, d - 2); \{(1, 1), (2, 1)\}) - \kappa((\lambda + (a + b) + 1 + 3, d - 2); \{(-1, 1), (-2, 1)\})
$$

the second of which obviously vanishes, the first simplifying to $\kappa_{d-2}(\lambda - (a + b) - 3; \{1, 2\})$

Now we sum over $d$ to get $\kappa(\lambda - (a + b) - 3; \{1, 2\})$ as expected.
3.2. The orbit $O_1$. Recall $O_1 = \{(V_1 < V_2) : \text{rank}(V_1) = 0\}$. Its $T_k$-fixed points look like

| $f \in O_1^{T_k}$ | $\text{wt}(L|_r)$ | $\text{wt}(T|_X)$ | $\text{wts}(T|_X \setminus \text{wts}(T|_X))$ | $\text{ind}(T|_X)$ | $\rho(T|_X)$ |
|-------------------|-----------------|-----------------|-------------------------------|----------------|------------|
| $(e_1 < e_1 \oplus e_2)$ | $(a, b)$ | $(1, 1)$ | $(2)$ | $0$ | $0$ |
| $(e_1 < e_1 \oplus e_3)$ | $(a + b, -b)$ | $(-1, 1)$ | $(2)$ | $1$ | $-1$ |
| $(e_3 < e_1 \oplus e_3)$ | $(b, -a - b)$ | $(-1, 1)$ | $(-2)$ | $1$ | $-1$ |
| $(e_3 < e_2 \oplus e_3)$ | $(-b, -a)$ | $(-1, -1)$ | $(-2)$ | $2$ | $-2$ |

(where one can figure out the weight on the normal bundle from the fact that the equation defining $O_1$ is quadratic), giving the terms

$$
\kappa(\lambda - (a + b) - 2 - 2(d - 1); \{-2, d - 1\}) - \kappa(\lambda - a - 1 - 2(d - 1); \{-2, d - 1\})
$$

for $d \geq 0$.

Each of the latter two vector partition functions $\kappa(\bar{\nu}; S)$ obviously vanishes, since the first coordinate of $\bar{\nu}$ is strictly positive, but the first coordinate of each $\bar{\nu} \in S$ is nonpositive.

In the first two (and the dropped two) terms, we know that the second vector $(\pm 2, 1)$ in the vector partitions must be used $d - 1$ times. So subtract that off and drop the coordinate:

$$
\kappa(\lambda - (a + b) - 2 - 2(d - 1)); \{-1\}) - \kappa(\lambda - a - 1 - 2(d - 1)); \{-1\})
$$

Now invoke this combinatorial duality to reduce theorem 1.1 for $O_2$ to the just-proven result for $O_1$. 

3.3. Duality and the orbit $O_2$. The duality automorphism $(V_1 < V_2) \mapsto (V_2^\perp < V_1^\perp)$ of $G/B$ acts on K-orbit closures, $O \mapsto O^\ast$, exchanging $O_1, O_2$ and fixing the other two. Pulling back the line bundle switches $(a, b)$ to $(b, a)$, giving $SO(3)$-isomorphic representations. So the K-multiplicity calculation for $O_2$ is easily copied from the one just done for $O_1$. However, the $SO(3)$-multiplicity formulæ are not then visibly the same, which we fix with the following.

**Theorem 3.1.** The map $(c, d) \mapsto \begin{cases} (d, c) & \text{if } d \equiv a \text{ mod } 2 \\ (d + 1, c - 1) & \text{if } d \not\equiv a \text{ mod } 2 \end{cases}$ bijects $C$ with itself, and $C_0(x, y)$ with $C_0^\ast(y, x)$, where $C_0(x, y)$ means $C_0$ with parameters $(a, b) = (x, y)$. Moreover, this duality on $C$ preserves the projection $(c, d) \mapsto c + d$.

We leave the very simple details to the reader. Essentially, the map flips figure 1 upside down; half the dots land on the illegal brown dashed lines, so we jiggle them upward to good positions, leaving holes along the bottom edge where holes belong.

Now invoke this combinatorial duality to reduce theorem 1.1 for $O_2$ to the just-proven result for $O_1$. 

3.4. The open orbit. Now all 3! fixed points are in our orbit closure, looking like

| $f \in \mathcal{O}_X^{T}$ | wt$(\mathcal{L}|_{f})$ | wt$s(\mathcal{T}_{f})$ | ind$(\mathcal{T}_{f})$ | $\rho(\mathcal{T}_{f})$ |
|---|---|---|---|---|
| $(e_1 < e_1 \oplus e_2)$ | $(a, b)$ | $(1, 1, 2)$ | 0 | 0 |
| $(e_1 < e_1 \oplus e_2)$ | $(a + b, -b)$ | $(1, 1, 2)$ | 1 | 1 |
| $(e_2 < e_1 \oplus e_2)$ | $(-a, a + b)$ | $(1, -1, 2)$ | 1 | 1 |
| $(e_2 < e_2 \oplus e_3)$ | $(-a - b, a)$ | $(1, -1, -2)$ | 2 | 3 |
| $(e_3 < e_1 \oplus e_3)$ | $(b, -a - b)$ | $(1, -1, -2)$ | 2 | 3 |
| $(e_3 < e_2 \oplus e_3)$ | $(-b, -a)$ | $(-1, -1, -2)$ | 3 | 4 |

and giving the (essentially Kostant multiplicity) formula

$$\kappa((\lambda - (a + b); -0, d); \{(-1, 0), (-2, 0)\})$$

$\kappa(\lambda - a + 1; (-1, -2))$ $\kappa(\lambda - b + 1; (-1, -2))$ $\kappa((\lambda - (a + b); \{(-1, -2)\}) - \kappa(\lambda - a + 1; \{(-1, -2)\})$ $\kappa((\lambda - (a + b); \{(-1, -2)\})$ $\kappa(\lambda - b + 1; \{(-1, -2)\})$

Each term contributes a multiplicity function like “.., 3, 3, 2, 2, 1, 1, 0, ..” where the 1 is at $a + b$, $a - 1$, or $b - 1$.

Consider the set $D = \{(c, d) \in -\mathbb{N}^2 : c \equiv 0 \mod 2\}$, whose projection $(c, d) \mapsto c + d \in -\mathbb{N}$ also gives this “.., 3, 3, 2, 2, 1, 1, 0, ..” multiplicity function on $-\mathbb{N}$, with the 1 at 0. Then the translate $(a, b) + D$ contains $C_{G/B_G}$; in figure 4, $(a, b) + D$ is the quadrant to the West of $C_{G/B_G}$’s East corner. If we take the difference $(a, b) + D \setminus (a, -1) + D$ of translates, we get a half-infinite strip with the same Northeast side as $C_{G/B_G}$, and this difference projects to the multiplicity function $\kappa((\lambda - (a + b); \{(-1, -2)\}) - \kappa((\lambda - a + 1; \{(-1, -2)\})$.

It remains to subtract off $\kappa((\lambda - b + 1; \{(-1, -2)\})$ using another translate of $D$. There are two cases.

The easy case is $a$ odd, where $(a, b) + D$ contains $(-1, b) + D$, to the Southwest of $C_{G/B_G}$. The subtlety is that $(a, b) + D \setminus (a, -1) + D$ does not contain $(-1, b) + D$, but the missing points $(c, d)$ all have $c + d < 0$, so they don’t affect the point-count in the region we care about. Put another way, if we use $\chi(S)$ to indicate the characteristic function of the set $S$, then

$$+ \chi((a, b) + D) - \chi((a, -1) + D) - \chi((-1, b) + D) = + \chi(C_{O_1 \cap O_2}) \text{ on } (c, d) : c + d \geq 0$$

so the two sides induce the same counts when projected to $\mathbb{N}$.

The tricky case is $a$ even, where we attempt to subtract off $(-2, b + 1) + D$ from $(a, b) + D \setminus (a, -1) + D$. Even on the $c + d \geq 0$ side, we don’t quite get $C_{G/B_G}$, but instead

$$+ \chi((c, d) \in \mathbb{N}^2 : c \equiv a \mod 2, c \leq a, d \leq b) - \chi((-2 - 2n, b + 1) : n \geq 0)$$

or equivalently

$$+ \chi(C_{G/B_G}) + \chi((0, d) \in \mathbb{N}^2 : d \leq b, d \equiv b - 1 \mod 2) - \chi((-2 - 2n, b + 1) : n \geq 0)$$
(again, this computation is only correct over $c + d \geq 0$). If we project the latter two terms to $\mathbb{N}$ using $(c, d) \mapsto c + d$, they become
\[ + \chi(d \in \mathbb{N} : d \leq b, d \equiv b - 1 \mod 2) - \chi(b - 1 - 2n : n \geq 0) \]
and visibly cancel. Hence in either the $a$ even or odd cases, the projection of $C_{G/B}$ correctly computes the $SO(3)$-multiplicities in the finite-dimensional $SL(3)$-irrep $V_{(a,b)}$.

Note that the $SO(3)$-multiplicities in $V_{(a,b)}$ and $V_{(b,a)}$ are the same, which is not manifest from counting points in $C_{G/B}$; this symmetry can be seen combinatorially using the bijection from §3.3.

In the case $a, b$ both even, the $SO(3)$-multiplicities start out $1, 0, 2, 1, \ldots$, i.e. they include every $SO(3)$-irrep in $[0, a + b]$ except for a hole at 1 (the 3-dim representation). Such holes never occur in weight multiplicities or $GL(n)$ tensor product multiplicities [KT99]. Another study of such holes in the Blattner situation appeared in [Ha08].

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