A Model for Daily Global Stock Market Returns

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Abstract

Most stock markets are open for 6-8 hours per trading day. The Asian, European and American stock markets are separated in time by time-zone differences. We propose a statistical dynamic factor model for a large number of daily returns across multiple time zones. Our model has a common global factor as well as continent factors. Under a mild fixed-signs assumption, our model is identified and has a structural interpretation. We derive the asymptotic theories of the quasi-maximum likelihood estimator (QMLE) of our model. As QMLE is inefficient by definition in this article, we outline three related estimators for practical use; Monte Carlo simulations reveal that two of them work well. We then apply our model to two real data sets - the equity portfolio returns of Japan, Europe and US and MSCI equity indices of 41 developed and emerging markets. Some new insights about linkages between different markets are drawn. Last, a Bayesian estimator (i.e., the Gibbs sampling) is also explained and suitable for estimation when the number of stocks is not too big.

1 Introduction

Although the Asian, European and American stock markets are separated in time by substantial time-zone differences (Table 1 lists the trading hours of the world’s top ten stock exchanges in terms of market capitalisation), no doubt that they are becoming more and more globalised. Linkages between different markets were particularly evident during stressful times like the financial crisis in 2008 and COVID-19 outbreak in 2020. Recent three decades have witnessed a heightening interest in measuring and modelling such linkages, whether dubbed as the stock market integration, international return spillovers, cross-market correlations etc. Gagnon and Karolyi (2006) and Sharma and Seth (2012) have carefully reviewed the literature and categorized these studies according to methodologies, data sets and findings.

All the existing studies either examined a small number of entities such as a few market indices or exchange rates, or ignored the time-zone differences whenever using daily data.

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It is far better studying a large number of entities to aggregate information. Moreover, the use of daily closing prices while ignoring the time-zone differences causes the so-called stale-price problem (Martens and Poon (2001), Connor, Goldberg, and Korajczyk (2010, p.42-44)). The standard approach is to include the lead and lagged covariances to correct for the stale pricing effect. This approach does not allow one to identify the source of variation or its relative impacts.

We build a framework to model the correlations of the daily stock returns in different markets across multiple time zones. The machinery will be a statistical dynamic factor model, which enables us to work with a large number of entities. To make the framework tractable, we make the following modeling assumption: All the markets belong to one of three continents: Asia (A), Europe (E) and America (U). Within a calendar day, the Asian markets close first, followed by the European and then American markets. The closing times are ordered as follows:

\[
\begin{align*}
A & \quad E & \quad U & \quad A & \quad E & \quad U & \quad A & \quad \cdots \\
t & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots
\end{align*}
\]

Note that the unit of \(t\) is not a day, but a fraction of a day. This framework could be applied to three markets only (i.e., one market in each continent), or to the case where some continent contains several markets. For the latter, we shall assume that differences in the closing times of different markets in the same continent can be ignored.

Let \(p_{c,t}\) denote the log price of stock \(i\) in continent \(c\) at time \(t\) for \(c = A, E, U\). In this framework, we shall assume that there are no weekends. Nevertheless, we do not observe \(p_{c,t}\) for two scenarios:

(i) Missing because of non-synchronized trading. That is, \(t\) might not correspond to the closing times of continent \(c\). For example, we do not observe \(p_{E,3}\) for any stock \(i\) in the European continent because of non-synchronized trading.

(ii) Missing because of some specific reasons. The reasons could be continent-specific (e.g., Chinese New Year, Christmas), market-specific (e.g., national holidays) or stock-specific (e.g., general meetings of shareholders).

We shall rule out scenario (ii) for the time being and address it to some extent in Appendix A.5. Define the log 24-hr return \(y_{c,t} := p_{c,t} - p_{c,t-3}\) for \(c = A, E, U\). Assume that the observed log 24-hr returns follow:

\[
y_{A,t} = \sum_{j=0}^{2} z_{i,j}^{A} f_{g,t-j} + z_{i,3}^{A} f_{A,t} + e_{i,t}^{A}, \quad i = 1, \ldots, N_{A}, \quad t = 1, 4, 7, \ldots, T - 2
\]

\[
y_{E,t} = \sum_{j=0}^{2} z_{i,j}^{E} f_{g,t-j} + z_{i,3}^{E} f_{E,t} + e_{i,t}^{E}, \quad i = 1, \ldots, N_{E}, \quad t = 2, 5, 8, \ldots, T - 1
\]

\[
y_{U,t} = \sum_{j=0}^{2} z_{i,j}^{U} f_{g,t-j} + z_{i,3}^{U} f_{U,t} + e_{i,t}^{U}, \quad i = 1, \ldots, N_{U}, \quad t = 3, 6, 9, \ldots, T
\]

where \(T\) is a multiple of 3, \(f_{g,t}\) is the scalar unobserved global factor, and \(f_{A,t}, f_{E,t}\) and \(f_{U,t}\) are the scalar unobserved continental factors. Assume \(f_{g,t} = 0\) for \(t \leq 0\). The dynamics of these factors are to be specified in Section 2. The approach of having global and continental factors, in some respects, resembles the GVAR modeling approach (Pesaran,
Schuermann, and Weiner (2004)), which was developed to model world low frequency macroeconomic series. Here we do not have so many relevant variables beyond the prices themselves and hence we focus on the unobserved factors. Note that at every $t$ we only observe the log 24-hr returns for one continent. This model reflects a situation in which new global information represented by the global factor affects all three continents simultaneously, but is only revealed in stock returns in three continents sequentially as three continents open in turn and trade on the new information (Koch and Koch (1991, p.235)).

Our framework is also closely related to the nowcasting framework (Giannone, Reichlin, and Small (2008), Banbura, Giannone, Modugno, and Reichlin (2013), Aruoba, Diebold, and Scotti (2009) etc). In the nowcasting literature, researchers used factor models to extract the information contained in the data published early and possibly at higher frequencies than the target variable of interest in order to forecast the target variable. Here if we make additional assumptions on the data generating processes of the unobserved log 24-hr returns, we could also obtain their corresponding forecasts in the same spirit as nowcasting; this is the similarity. The difference is that, as we shall point out in Section 2, model (1.1) is identified under a mild fixed-signs assumption (Assumption 2.2) and hence has a structural interpretation, whereas in the nowcasting literature, identification of factor models is usually not addressed, and factor models are mere dimension-reducing tools with no structural interpretations. In some sense, our model belongs to the class of structural dynamic factor models (Stock and Watson (2016)).

On the theoretical side, research about estimation of large factor models via the likelihood approach has matured over the last decade. The likelihood approach enjoys several advantages such as efficiency compared to the principal components method (Banbura et al. (2013, p.204)). Doz, Giannone, and Reichlin (2012) established an average rate of convergence of the estimated factors using a quasi-maximum likelihood estimator (QMLE) via the Kalman smoother. However, there is a rotation matrix attached to the estimated factors as the authors did not address identification of factor models. Also they
did not derive consistency for the estimated factor loadings, or the limiting distributions of any estimate.

Bai and Li (2012) took a different approach to study large exact factor models. They treated factors as fixed parameters instead of random vectors. One nice thing about this approach is that the theoretical results obtained hold for any dynamic pattern of factors. Bai and Li (2012) obtained consistency, the rates of convergence and limiting distributions of the maximum likelihood estimators (MLE) of the factor loadings, idiosyncratic variances, and sample covariance matrix of factors. In fact, Bai and Li (2012) called their estimators QMLE instead of MLE. We decided to re-label them as MLE since we intend to use the phrase QMLE for another purpose to be made specific shortly. Factors are then estimated via a generalised least squares (GLS) method. Bai and Li (2016) generalised the results of Bai and Li (2012) to large approximate factor models.

In practice, instead of maximising a likelihood and finding MLE, people usually use the EM algorithm together with the Kalman smoother to estimate the model, say (1.1). Note that the EM algorithm runs only for a finite number of iterations, and it might converge to a local maximum of the likelihood instead of the global maximum. Thus, the EM estimator is only an approximation to the MLE estimator. The theoretical properties of the EM estimator for large factor models remained largely unknown until recently. In a breakthrough study, Barigozzi and Luciani (2022) showed that the EM and MLE estimators are asymptotically equivalent. Moreover, Barigozzi and Luciani (2022) established consistency, the rates of convergence and limiting distributions of the EM estimators of the loadings and factors.

These asymptotic results of large factor models are not directly applicable to our model because the proofs of these results are identification-scheme dependent. In particular, Bai and Li (2012), Bai and Li (2016) established their results under five popular identification schemes, none of which is consistent with our model (1.1). It took us a considerable amount of work to derive the corresponding large-sample theories in our framework. In order to have an identification scheme consistent with our model (1.1) and at the same time utilise the theories of Bai and Li (2012), we could only impose some, not all, of the restrictions implied by our model to derive the first-order conditions (FOC) of the log-likelihood. It is because of this reason we call the estimator satisfying these first-order conditions QMLE instead of MLE. We establish consistency, the rates of convergence and asymptotic distributions for the QMLE estimators of the idiosyncratic variances, and consistency, the rates of convergence and asymptotic representations for the QMLE estimators of the factor loadings and parameter characterising the dynamics of the factors. The asymptotic representation of the estimated factors is also provided.

The QMLE estimators, except those for the idiosyncratic variances, are in nature inefficient, so we next propose three practically usable estimators, either which build on QMLE or whose standard errors could be approximated by a procedure involving the asymptotic representations of QMLE. The first estimator (QMLE-delta) builds on QMLE and relies on delta method to achieve a smaller standard error. The second estimator is the usual EM estimator. We use the asymptotic representations of QMLE to approximate the standard error of the EM estimator. The third estimator (QMLE-res) estimates a ”two-day” form of our model (see (4.1)) via the EM algorithm. The only non-standard feature is that we treat \( \{f_t\}_{t=1}^{T_f} \) defined in (4.1) as i.i.d. when setting up the likelihood.

The full-fledged theoretical results of these three estimators are difficult to establish, so we evaluate these estimators by Monte Carlo simulations. Indeed Monte Carlo simulations reveal that the EM and QMLE-res estimators perform well in terms of the root mean
square errors, average of the standard errors across the Monte Carlo samples, and coverage probability of the constructed confidence interval.

We then apply our model to two real data sets. The first data set is the equity portfolio returns of Japan, Europe and US; that is, one market per continent. Our methodology easily quantifies how much the global factor loaded on the returns during a particular fraction of a calendar day, as well as the relative importance of the global and continental factors. We also uncover some interesting time-series patterns. The second data set is MSCI equity indices of the 41 developed and emerging markets. Taking the Asian-Pacific continent as an example, we find that Mainland China and Hong Kong have particularly high loadings on the global factors during the US trading time. Japan have high loadings on the continental factor but small idiosyncratic variances, while other markets have small loadings on the continental factor.

In fact we also propose a Bayesian approach (i.e., the Gibbs sampling) to estimate our model. However, the Gibbs sampling is computationally intensive and feasible only for a small number of entities. In some unreported Monte Carlo simulations, we find that the proposed Bayesian estimator works well for $N_c = 20$ but not so well for $N_c = 200$ for $c = A, E, U$. We use this Bayesian estimator to re-estimate the model for the first real data set and obtain results similar to those obtained by the EM estimator.

To sum up, we contribute to methodologies by providing a new modelling framework for daily global stock market returns. Our framework could easily handle a large number of stocks and at the same time take into account of the time-zone differences. Under a mild fixed-signs assumption, our model is identified and has a structural interpretation. We contribute to theories by deriving the asymptotic properties of QMLE. The machinery will be the theoretical results of Bai and Li (2012), but we demonstrate how their results could be adapted for new identification schemes. We contribute to the applied work by proposing several practically usable estimators and validate their performances via Monte Carlo simulations. When applying our model to two real data sets, we draw some new insights about linkages between different stock markets.

The rest of the article is structured as follows. In Section 2 we discuss the model and identification while in Section 3 we introduce the EM estimator. Section 4 gives the large sample theories of the QMLE estimators of our model, as well as proposes three estimators for practical use. Section 5 conducts Monte Carlo simulations assessing those proposed estimators, and Section 6 presents two empirical applications of our model. Section 7 concludes. All the proofs and other results are put in Appendix.

2 The Model

2.1 Notation

For $x \in \mathbb{R}^n$, let $\|x\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}$ denote the Euclidean ($\ell_2$) norm. Let $A$ be an $m \times n$ matrix. Let $\text{vec} A$ denote the vector obtained by stacking the columns of $A$ one underneath the other. The commutation matrix $K_{m,n}$ is an $mn \times mn$ orthogonal matrix which translates $\text{vec} A$ to $\text{vec}(A^\top)$, i.e., $\text{vec}(A^\top) = K_{m,n} \text{vec}(A)$. If $A$ is a symmetric $n \times n$ matrix, its $n(n-1)/2$ supradiagonal elements are redundant in the sense that they can be deduced from symmetry. If we eliminate these redundant elements from $\text{vec} A$, we obtain a new $n(n+1)/2 \times 1$ vector, denoted $\text{vech} A$. They are related by the full-column-rank, $n^2 \times n(n+1)/2$ duplication matrix $D_n$: $\text{vec} A = D_n \text{vech} A$. Conversely,
vech $A = D_n^+ \text{vec } A$, where $D_n^+$ is $n(n + 1)/2 \times n^2$ and the Moore-Penrose generalized inverse of $D_n$.

Given a vector $v$, diag$(v)$ creates a diagonal matrix whose diagonal elements are elements of $v$. We use $p(\cdot)$ to denote the (asymptotic) probability density function. $[x]$ denotes the greatest integer strictly less than $x \in \mathbb{R}$ and $\lfloor x \rfloor$ denotes the smallest integer greater than or equal to $x \in \mathbb{R}$. Landau (order) notation in this article, unless otherwise stated, should be interpreted in the sense that $N, T \to \infty$ jointly. We use $C$ or $C$ with number subscripts to denote \textit{absolute} positive constants (i.e., constants independent of anything which is a function of $N$ and/or $T$); identities of such $C$s might change from one place to another.

2.2 The Model Setup

Define $T_A := \{1, 4, 7, \ldots, T - 2\}$, $T_E := \{2, 5, 8, \ldots, T - 1\}$ and $T_U := \{3, 6, 9, \ldots, T\}$. Since we shall assume that all three continent factors are uncorrelated over time, we could just define one factor $f_C,t$ such that $f_C,t$ is equal to $f_A,t, f_E,t, f_U,t$ for $t \in T_A, T_E, T_U$, respectively, for the purpose of reducing the number of state variables later on. Stacking all the stocks in the Asian continent, we have

$$y_t^A = Z^A \alpha_t + e_t^A \quad t \in T_A$$

where

$$y_t^A := \begin{bmatrix} y_{1,t}^A \\ \vdots \\ y_{N_A,t}^A \end{bmatrix}, \quad e_t^A := \begin{bmatrix} e_{1,t}^A \\ \vdots \\ e_{N_A,t}^A \end{bmatrix}, \quad \alpha_t := \begin{bmatrix} f_{g,t} \\ f_{g,t-1} \\ f_{g,t-2} \\ f_{C,t} \end{bmatrix}$$

$$Z^A := \begin{bmatrix} z_{1,0}^A & z_{1,1}^A & z_{1,2}^A & z_{1,3}^A \\ \vdots & \vdots & \vdots & \vdots \\ z_{N_A,0}^A & z_{N_A,1}^A & z_{N_A,2}^A & z_{N_A,3}^A \end{bmatrix} =: \begin{bmatrix} z_0^A & z_1^A & z_2^A & z_3^A \end{bmatrix},$$

and $f_{g,t}, f_{C,t} := 0$ for $t \leq 0$. That is, $y_t^A, e_t^A$ are $N_A \times 1$ vectors, $\alpha_t$ is $4 \times 1$, and $Z^A$ is $N_A \times 4$. Similarly,

$$y_t^E = Z^E \alpha_t + e_t^E \quad t \in T_E$$

$$y_t^U = Z^U \alpha_t + e_t^U \quad t \in T_U,$$

where

$$Z^E := \begin{bmatrix} z_{1,0}^E & z_{1,1}^E & z_{1,2}^E & z_{1,3}^E \\ \vdots & \vdots & \vdots & \vdots \\ z_{N_E,0}^E & z_{N_E,1}^E & z_{N_E,2}^E & z_{N_E,3}^E \end{bmatrix} =: \begin{bmatrix} z_0^E & z_1^E & z_2^E & z_3^E \end{bmatrix}$$

$$Z^U := \begin{bmatrix} z_{1,0}^U & z_{1,1}^U & z_{1,2}^U & z_{1,3}^U \\ \vdots & \vdots & \vdots & \vdots \\ z_{N_U,0}^U & z_{N_U,1}^U & z_{N_U,2}^U & z_{N_U,3}^U \end{bmatrix} =: \begin{bmatrix} z_0^U & z_1^U & z_2^U & z_3^U \end{bmatrix}$$

The model is dynamic in the sense that $\alpha_t$ can exhibit certain dynamics. We shall specify an AR(1) process for $f_{g,t}$:

$$f_{g,t+1} = \phi f_{g,t} + \eta_{g,t} \quad |\phi| < 1 \quad (2.1)$$

$$f_{C,t+1} = \eta_{C,t}$$
for \( t = 0, 1, \ldots, T - 1 \). The efficient markets hypothesis (along with a time invariant risk premium) predicts that \( \phi = 0 \).

Although some of the aforementioned studies allow dynamics of factors, say, factors following an AR(1) process, strictly speaking those factor models in those studies are not dynamic factor models in the sense that the lagged factors are not allowed to enter the equation relating factors to the observed series (Bai and Wang (2015)). One exception is Barigozzi and Luciani (2022). Thus in this article we will only refer factor models allowing the lagged factors to enter the observation equation as dynamic factor models.

**Assumption 2.1.** (i) The idiosyncratic components are i.i.d. across time: \( \{ e_t^c \}_{t \in T} \overset{i.i.d.}{\sim} N(0, \Sigma) \), where \( \Sigma_c := \text{diag}(\sigma_{c,1}^2, \ldots, \sigma_{c,N_c}^2) \) for \( c = A, E, U \). Moreover, \( e_t^A, e_t^E, e_t^U \) are mutually independent for all possible \( i \) and \( t \). Moreover, \( \mathbb{E}[e_t^c] = 0 \) for all \( i \) and \( t \), \( c = A, E, U \), and some \( C < \infty \).

(ii) Assume that \( \eta_t := (\eta_{t,1}, \eta_{t,2}) \overset{i.i.d.}{\sim} N(0, I_2) \), for \( t = 0, 1, \ldots, T - 1 \). Moreover, \( \{ \eta_t \}_{t=1}^T \) are independent of \( \{ e_t^c \}_{t \in T} \) for \( c = A, E, U \).

Assumption 2.1(i) is the same as Assumption B of Bai and Li (2012). We make the assumption of diagonality of \( \Sigma \) for simplicity as our model is already quite involved so we refrain from complicating the model unnecessarily. Assumption 2.1(ii) is a white-noise assumption on the innovations of the factors.

We now cast the model in the state space form

\[
y_t = Z_t \alpha_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, \Sigma) \quad (2.2)
\]

for \( t = 1, \ldots, T \), where \( y_t = y_t^c, Z_t = Z^c, \varepsilon_t = e_t^c, \Sigma_t = \Sigma_c \) if \( t \in T_c \), for \( c = A, E, U \). This is a non-standard dynamic factor model. The non-standard features are: (1) The factor loading matrix \( Z_t \) is switching among three states \( \{ Z^A, Z^E, Z^U \} \). (2) The column dimensions of \( y_t, Z_t, \varepsilon_t \) are switching among \( \{ N_A, N_E, N_U \} \). (3) The covariance matrix of \( \varepsilon_t \) is switching among \( \{ \Sigma_A, \Sigma_E, \Sigma_U \} \).

### 2.3 The Structural Interpretation

In general, for a static factor model, say, \( y_t = Z \alpha_t + \varepsilon_t \), further identification restrictions are needed in order to separately identify \( Z \) and \( \alpha_t \) from the term \( Z \alpha_t \). In particular, \( Z \alpha_t = \tilde{Z} \hat{\alpha}_t \) for any \( 4 \times 4 \) invertible matrix \( C \) such that \( \tilde{Z} := ZC^{-1} \) and \( \hat{\alpha}_t := C \alpha_t \); we need \( 4^2 \) identification restrictions so that the only admissible \( C \) is an identity matrix. A classical reference on this issue would be Anderson and Rubin (1956). These restrictions have been ubiquitous in the literature (e.g., Bai and Li (2012), Bai and Li (2016)). One exception is Bai and Wang (2015); Bai and Wang (2015) pointed out that by relying on the dynamic equation of \( f_{t,1} \), such as (2.1), one could use far less identification restrictions to identify the model. In our case, we shall only make the following mild assumption to identify the model.

**Assumption 2.2.** Estimators of \( z^A_0, z^A_1, z^A_2, z^A_3, z^E_0, z^E_1 \) have the same column signs as those of \( z^A_0, z^A_1, z^A_2, z^A_3, z^E_0, z^E_1 \).

Bai and Li (2012) have made similar assumption for their identification schemes (IC2, IC3 and IC5) (see Bai and Li (2012, p.445, p.463)).
Lemma 2.1. The parameters of the dynamic factor model ((2.2), (2.1)) are identified under Assumption 2.2.

Our model ((2.2), (2.1)) has a structural interpretation under Assumption 2.2, because one could not freely insert a rotation matrix between \( Z_t \) and \( \alpha_t \). In other words, under Assumption 2.2 we are not estimating the rotations of \( Z_t \) or \( \alpha_t \); we are estimating the true \( Z_t \) and \( \alpha_t \) of the data generating process. This is a novel feature of our model.

3 Estimation

In this section, we shall outline one frequentist approach to estimate the model - the Expectation-Maximisation (EM) algorithm. In Appendix A.2, we also provide a Bayesian approach (i.e., the Gibbs sampling) to estimate the model; the Bayesian approach is computationally intensive and feasible only for small \( N_c \).

The EM algorithm consists of an E-step and an M-step. In the E-step, we evaluate a conditional expectation of a complete log-likelihood function while in the M-step we maximize it. To give the starting values of parameters in the EM algorithm, we first use MLE to estimate a restricted version of model ((2.2), (2.1)). Take the Asian continent as an example. All the elements of \( z_{0A}^t, z_{1A}^t, z_{2A}^t \) are set to one scalar, all the elements of \( z_3^A \) are set to one scalar, and all the diagonal elements of \( \Sigma^A \) are set to one scalar. This will give reasonably good starting values. We do not use the Principal Component (PC) estimator as the starting values because in finite samples the PC estimator will not ensure that \( \hat{\alpha}_{t+1}^{PC} = \hat{\alpha}_{t+2}^{PC} \) for all \( t \), where \( \hat{\alpha}_t^{PC} \) is the PC estimator of \( \alpha_t \). We shall assume \( N := N_A = N_E = N_U \) in this subsection.

3.1 E-Step

Define \( \theta := \{\phi, Z^c, \Sigma_c, c = A, E, U\}, \Xi := (\alpha_1, \ldots, \alpha_T)\) and \( Y_{1:T} := \{y_1^t, \ldots, y_T^t\} \). One could show that the log-likelihoods of \( \Xi \) and \( Y_{1:T} | \Xi \) are

\[
\ell(\Xi; \theta) = -T \log(2\pi) - \frac{1}{2} \sum_{t=0}^{T-1} [(f_{g(t)} - \phi f_{g(t)})^2 + f_{C(t)}^2]
\]

\[
\ell(Y_{1:T}|\Xi; \theta) = -\frac{TN}{2} \log(2\pi) - \frac{1}{2} \sum_{t=1}^{T} \log |\Sigma_t| - \frac{1}{2} \sum_{t=1}^{T} (y_t - Z_t \alpha_t)^\top \Sigma_t^{-1} (y_t - Z_t \alpha_t).
\]

The complete log-likelihood function of model ((2.2), (2.1)) is hence (omitting constant)

\[
\ell(\Xi, Y_{1:T}; \theta) = \ell(Y_{1:T}|\Xi; \theta) + \ell(\Xi; \theta)
\]

\[
= -\frac{1}{2} \sum_{t=1}^{T} \left( \log |\Sigma_t| + \epsilon_t^\top \Sigma_t^{-1} \epsilon_t \right) - \frac{1}{2} \sum_{t=0}^{T} \eta_{g,t}^2 =: -\frac{1}{2} \sum_{t=1}^{T} \left( \ell_{1,t} + \ell_{2,t} \right)
\]

where \( \ell_{1,t} := \log |\Sigma_t| + \text{tr} \left( \epsilon_t \epsilon_t^\top \Sigma_t^{-1} \right) \), and \( \ell_{2,t} = \eta_{g,t}^2 \). Let \( \hat{E} \) denote the expectation with respect to the conditional density \( p(\Xi|Y_{1:T}; \theta^{(i)}) \) at \( \hat{\theta}^{(i)} \), where \( \hat{\theta}^{(i)} \) is the estimate of \( \theta \).
This is the so-called "E" step of the EM algorithm. \( \hat{E} \left[ \ell(\Xi, Y_{1:T}; \theta) \right] \) could be computed using the Kalman smoother (KS; see Section A.4 for the formulas of the KS).

### 3.2 M-Step

The M step involves maximising \( \hat{E} \left[ \ell(\Xi, Y_{1:T}; \theta) \right] \) with respect to \( \theta \). This is usually done by computing

\[
\frac{\partial \hat{E} \left[ \ell(\Xi, Y_{1:T}; \theta) \right]}{\partial \theta}
\]

and setting the preceding display to zero to obtain the estimate \( \hat{\theta}^{(i+1)} \) of \( \theta \) for the \( (i+1) \)th iteration of the EM algorithm.

#### 3.2.1 M-Step of \( Z_t \) and \( \Sigma_t \)

We now consider \( \Sigma_t \) and \( \Sigma_t \) to minimize \( \hat{E} \sum_{t=1}^{T} \ell_{1,t} \). Recall that \( Z_t = Z_c, \Sigma_t = \Sigma_c \) if \( t \in T_c \) for \( c = A, E, U \). Without loss of generality, we shall use the Asian continent to illustrate the procedure. We now find values of \( Z_t \) and \( \Sigma_t \) to minimise \( \hat{E} \sum_{t \in T_A} \ell_{1,t} \).

Since \( \varepsilon_t = y_t - Z_t \alpha_t \) and \( \eta_t = R^t (\alpha_{t+1} - \mathcal{T} \alpha_t) \) (see (A.17)), we have

\[
\sum_{t \in T_A} \ell_{1,t} = \frac{T}{3} \log |\Sigma_A| + \sum_{t \in T_A} \text{tr} \left( \left[ y_t y_t^\top - 2Z_A \alpha_t y_t^\top + Z_A \alpha_t \alpha_t^\top Z_A^\top \right] \Sigma_A^{-1} \right)
\]

and hence

\[
\hat{E} \sum_{t \in T_A} \ell_{1,t} = \frac{T}{3} \log |\Sigma_A| + \sum_{t \in T_A} \text{tr} \left( \left[ \hat{E} y_t y_t^\top - 2Z_A \hat{E}[\alpha_t] y_t^\top + Z_A \hat{E}[\alpha_t \alpha_t^\top] Z_A^\top \right] \Sigma_A^{-1} \right)
\]

We now consider \( Z_A \), and take differential of (3.2) with respect to \( Z_A \):

\[
d \hat{E} \sum_{t \in T_A} \ell_{1,t} = \sum_{t \in T_A} \text{tr} \left( \left[ -2dZ_A \hat{E}[\alpha_t] y_t^\top + 2dZ_A \hat{E}[\alpha_t \alpha_t^\top] Z_A^\top \right] \Sigma_A^{-1} \right)
\]

\[
= -2 \sum_{t \in T_A} \text{tr} \left( dZ_A \left[ \hat{E}[\alpha_t] y_t^\top - \hat{E}[\alpha_t \alpha_t^\top] Z_A^\top \right] \Sigma_A^{-1} \right),
\]

whence we have

\[
\hat{Z}_A = \sum_{t \in T_A} \hat{E}[\alpha_t \alpha_t^\top] \left( \sum_{t \in T_A} \hat{E}[\alpha_t \alpha_t^\top] \right)^{-1}.
\]

We next consider \( \Sigma_A \). Define

\[
C_A := \sum_{t \in T_A} \left[ y_t y_t^\top - 2Z_A \hat{E}[\alpha_t] y_t^\top + Z_A \hat{E}[\alpha_t \alpha_t^\top] Z_A^\top \right]
\]

\[
\hat{C}_A := \sum_{t \in T_A} \left[ y_t y_t^\top - 2\hat{Z}_A \hat{E}[\alpha_t] y_t^\top + \hat{Z}_A \hat{E}[\alpha_t \alpha_t^\top] \hat{Z}_A^\top \right]
\]
Then (3.2) can be written as
\[
\hat{E} \sum_{t \in T_{A}} \ell_{1,t} = \frac{T}{3} \log |\Sigma_{A}| + \text{tr}(C_{A} \Sigma_{A}^{-1})
\]

Take the differential of \( \hat{E} \sum_{t \in T_{A}} \ell_{1,t} \) with respect to \( \Sigma_{A} \)
\[
d\hat{E} \sum_{t \in T_{A}} \ell_{1,t} = \frac{T}{3} \text{tr}(\Sigma_{A}^{-1} d\Sigma_{A}) - \text{tr}(\Sigma_{A}^{-1} C_{A} \Sigma_{A}^{-1} d\Sigma_{A}) = \text{tr} \left( \Sigma_{A}^{-1} \left[ \frac{T}{3} \Sigma_{A} - C_{A} \right] \Sigma_{A}^{-1} d\Sigma_{A} \right)
\]

whence we have, recognising diagonality of \( \Sigma_{A} \),
\[
\frac{\partial \hat{E} \sum_{t \in T_{A}} \ell_{1,t}}{\partial \Sigma_{A}} = \Sigma_{A}^{-1} \left[ \frac{T}{3} \Sigma_{A} - C_{A} \right] \Sigma_{A}^{-1} \circ I_{N_{A}}
\]

where \( \circ \) denotes the Hadamard product. The first-order condition of \( \Sigma_{A} \) is
\[
\tilde{\Sigma}_{A} = \frac{3}{T} (\tilde{C}_{A} \circ I_{N}).
\]

### 3.2.2 M-Step of \( \phi \)

We now find value of \( \phi \) to minimize \( \hat{E} \sum_{t=1}^{T} \ell_{2,t} \). We have
\[
\hat{E} \sum_{t=1}^{T} \ell_{2,t} = \sum_{t=1}^{T} \hat{E} \eta_{2, t-1}^{2} = \sum_{t=1}^{T} \left( \hat{E} [f_{g,t}^{2}] - 2 \phi \hat{E} [f_{g,t} f_{g,t-1}] + \phi^{2} \hat{E} [f_{g,t-1}^{2}] \right),
\]

whence the first order condition of \( \phi \) gives
\[
\tilde{\phi} = \left( \sum_{t=1}^{T} \hat{E} [f_{g,t-1}^{2}] \right)^{-1} \sum_{t=1}^{T} \hat{E} [f_{g,t} f_{g,t-1}].
\]

**Remark 3.1.** As mentioned before, our model is perfectly geared for the scenario of missing observations due to non-synchronized trading (scenario (i)). In Appendix A.5, we discuss how to alter the EM algorithm if we include the scenario of missing observations due to continent-specific reasons such as continent-wide public holidays (e.g., Chinese New Year). That is, both scenario (i) and a specific form of scenario (ii) are present in the data. We do not consider other forms of scenario (ii) - missing observations due to market-specific, stock-specific reasons - in this article.

### 4 Large Sample Theories

In this section, we present the large sample theories of the QMLE estimators of our model ((2.2), (2.1)). In this article, we shall make a distinction between QMLE and MLE in the sense that QMLE does not use all the information implied by the model when setting up the likelihood. Hence in this article QMLE is inefficient, but it does allow us to utilize the theoretical results of Bai and Li (2012). In particular, we establish the asymptotic distribution for the QMLE estimator of \( \Sigma_{c} \), and asymptotic representations for the QMLE estimators of \( Z_{c} \) and \( \phi \). In some unreported Monte Carlo simulations we found that the QMLE estimators of \( Z_{c} \) and \( \phi \) are too inefficient for practical use, so we outline in Section 4.2 three practically usable estimators either which build on QMLE or whose standard errors could be approximated by a procedure involving the asymptotic representations of QMLE. Without loss of generality, again we shall assume \( N := N_{A} = N_{E} = N_{U} \) in this section.
4.1 The QMLE

In order to utilize the theoretical results of Bai and Li (2012), we shall re-write our model (2.2, 2.1) in the following “two-day” form:

\[
\begin{align*}
\hat{y}_t & = \Lambda f_t + e_t \\
\end{align*}
\]  \hspace{1cm} (4.1)

for \( t = 1, 2, \ldots, T/6 \), where we define \( \ell := 6(t - 1) + 1, \)

\[
\begin{bmatrix}
\hat{y}_t \\
\hat{y}_{t+1} \\
\vdots \\
\hat{y}_{t+3} \\
\hat{y}_{t+4} \\
\hat{y}_{t+5}
\end{bmatrix} = \begin{bmatrix} y_A^A \\ y_{E+1}^E \\ y_{U+1}^U \\ y_A^A \\ y_{E+2}^E \\ y_{U+2}^U \\ y_A^A \\ y_{E+3}^E \\ y_{U+3}^U \\ y_A^A \\ y_{E+4}^E \\ y_{U+4}^U \\ y_A^A \\ y_{E+5}^E \\ y_{U+5}^U \\ e_{\ell+5} \\ e_{\ell+4} \\ e_{\ell+3} \\ e_{\ell+2} \\ e_{\ell+1} \\ e_{\ell}
\end{bmatrix} + \begin{bmatrix} f_{g,\ell+5} \\ f_{g,\ell+4} \\ f_{g,\ell+3} \\ f_{g,\ell+2} \\ f_{g,\ell+1} \\ f_{g,\ell} \\ f_{g,\ell-1} \\ f_{g,\ell-2} \\ f_{C,\ell+5} \\ f_{C,\ell+4} \\ f_{C,\ell+3} \\ f_{C,\ell+2} \\ f_{C,\ell+1} \\ f_{C,\ell}
\end{bmatrix}
\]  \hspace{1cm} (4.2)

\[ \Lambda = \text{some function}(Z^A, Z^E, Z^U) \]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & z_0^A \\ 0 & 0 & 0 & 0 & 0 & z_1^A \\ 0 & 0 & 0 & 0 & 0 & z_2^A \\ 0 & 0 & 0 & 0 & 0 & z_3^A \\ 0 & 0 & 0 & 0 & 0 & z_0^E \\ 0 & 0 & 0 & 0 & 0 & z_1^E \\ 0 & 0 & 0 & 0 & 0 & z_2^E \\ 0 & 0 & 0 & 0 & 0 & z_3^E \\ 0 & 0 & 0 & 0 & 0 & z_0^U \\ 0 & 0 & 0 & 0 & 0 & z_1^U \\ 0 & 0 & 0 & 0 & 0 & z_2^U \\ 0 & 0 & 0 & 0 & 0 & z_3^U
\end{bmatrix}
\]

where \( z_k^c \) is \( N \times 1 \) for \( k = 0, 1, 2, 3 \) and \( c = A, E, U \).

Note that with this “two-day” form, we could resort to other identification scheme instead of Assumption 2.2 to identify the model parameters in this section. We define the following quantities

\[
S_{yy} := \frac{1}{T_f} \sum_{t=1}^{T_f} y_t y_T \\ S_{yy} := \Lambda M \Lambda^T + \Sigma_{ee} \hspace{1cm} T_f := \frac{T}{6} \\
\Sigma_{ee} := \text{diag}(\sigma_1^2, \ldots, \sigma_N^2, \ldots, \sigma_1^2, \ldots, \sigma_2^2, \ldots, \sigma_3^2, \ldots, \sigma_4^2, \ldots, \sigma_5^2, \ldots, \sigma_6^2, \ldots)
\]

\[
M := \text{diag}(\sigma_1^2, \ldots, \sigma_N^2, \ldots, \sigma_1^2, \ldots, \sigma_2^2, \ldots, \sigma_3^2, \ldots, \sigma_4^2, \ldots, \sigma_5^2, \ldots, \sigma_6^2, \ldots)
\]
Given the assumption of $|\phi| < 1$ in (2.1), we have $M = O(1)$ and $M^{-1} = O(1)$.

The log-likelihood of $\{\mathbf{y}_t\}_{t=1}^T$, scaled by $1/(NT_f)$ and omitting the constant, is

$$
\frac{1}{NT_f} \ell (\{\mathbf{y}_t\}_{t=1}^T; \theta) = -3 \log(2\pi) - \frac{1}{2N} \log |\Sigma_{yy}| - \frac{1}{2N} \text{tr}(S_{yy}\Sigma_{yy}^{-1}).
$$

(4.4)

We shall only utilise the information that $M$ is symmetric, positive definite and that $\Sigma_{ee}$ is diagonal to derive generic first-order conditions (FOCs). The word generic means that specific forms of $\Lambda$ given by (4.2) and of $M$ given by (4.3) are not utilised to derive the FOCs. In Appendix A.6, we derive such FOCs and identify the QMLE $\hat{\Lambda}$ and $\hat{\Sigma}$ after imposing $14^2$ identification restrictions. The QMLE $\hat{\Lambda}$, $\hat{M}$, $\hat{\Sigma}_{ee}$ satisfy

$$
\hat{\Lambda}^T \hat{\Sigma}_{yy}^{-1} (S_{yy} - \hat{\Sigma}_{yy}) = 0
$$

$$
\text{diag}(\hat{\Sigma}_{yy}^{-1}) = \text{diag}(\hat{\Sigma}_{yy}^{-1} S_{yy} \hat{\Sigma}_{yy}^{-1})
$$

(4.5)

where $\hat{\Sigma}_{yy} := \hat{\Lambda} M \hat{\Lambda}^T + \hat{\Sigma}_{ee}$.

Display (4.5) is the same as (2.7) and (2.8) of Bai and Li (2012). Bai and Li (2012) considered five identification schemes, none of which is consistent with $\Lambda$ and $M$ defined in (4.2) and (4.3), respectively. Actually $\Lambda$ and $M$ give more than $14^2$ restrictions, but in order to have a solution for the generic FOCs, we could only impose $14^2$ restrictions on $\Lambda$ and $M$. We call the resulting estimators QMLE rather than MLE because we did not impose all the restrictions implied by $\Lambda$ and $M$ (i.e., implied by our model). How to select these $14^2$ restrictions from those implied by $\Lambda$ and $M$ is crucial because we cannot afford imposing a restriction which is not instrumental for the proofs later. We shall painstakingly explain our procedure in the proof of Proposition 4.1. Our procedure is quite ingenious and does not exist in the proofs of Bai and Li (2012).

Note that $\Lambda$ consists of six row blocks of dimension $N \times 14$. Let $\lambda_{k,j}$ denote the $j$th row of the $k$th row block of $\Lambda$. In other words, $\lambda_{k,j}^T$ refers to the factor loadings for the $j$th Asian asset in day one, while $\lambda_{s,j}$ refers to the factor loadings for the $j$th European asset in day two. Hence we can also use $\{\lambda_{k,j}, \sigma_{k,j}, \hat{M} : k = 1, \ldots, 6, j = 1, \ldots, N\}$ to denote the QMLE.

**Assumption 4.1.**

(i) The factor loadings $\lambda_{k,j}$ satisfy $||\lambda_{k,j}||_2 \leq C$ for all $k$ and $j$.

(ii) Assume $C^{-1} \leq \sigma_{k,j}^2 \leq C$ for all $k$ and $j$. Also $\hat{\sigma}_{k,j}^2$ is restricted to a compact set $[C^{-1}, C]$ for all $k$ and $j$.

(iii) $\hat{M}$ is restricted to be in a set consisting of all positive definite matrices with all the elements bounded in the interval $[C^{-1}, C]$.
Suppose that $Q := \lim_{N \to \infty} \frac{1}{N} \Lambda^{\top} \Sigma^{-1}_{ee} \Lambda$ is a positive definite matrix.

Assumption 4.1 is standard in the literature of factor models and has been taken from the assumptions of Bai and Li (2012).

**Proposition 4.1.** Suppose that Assumptions 2.1, 4.1 hold. When $N, T_f \to \infty$, with the identification condition outlined in the proof of this proposition (i.e., $1^2$ particular restrictions imposed on $\hat{\Lambda}$ and $\hat{M}$), and the requirement that $\hat{\Lambda}$ and $\Lambda$ have the same column signs, we have

$$\hat{\lambda}_{k,j} - \lambda_{k,j} = o_p(1)$$

$$\frac{1}{6N} \sum_{k=1}^{6} \sum_{j=1}^{N} (\hat{\sigma}_{k,j}^2 - \sigma_{k,j}^2)^2 = o_p(1)$$

$$\hat{M} - M = o_p(1).$$

for $k = 1, \ldots, 6, j = 1, \ldots, N$.

Display (4.6) and (4.8) establish consistency for the individual loading estimator $\hat{\lambda}_{k,j}$ and $\hat{M}$, respectively, while display (4.7) establishes some average consistency for $\{\hat{\sigma}_{k,j}^2\}$.

The proof of Proposition 4.1 is based on Proposition 5.1 of Bai and Li (2012), but is considerably more involved because our identification scheme is non-standard (see the proof of Proposition 4.1 for details.)

**Theorem 4.1.** Under the assumptions of Proposition 4.1, we have

$$\|\hat{\lambda}_{k,j} - \lambda_{k,j}\|^2 = O_p(T_f^{-1})$$

$$\frac{1}{6N} \sum_{k=1}^{6} \sum_{j=1}^{N} (\hat{\sigma}_{k,j}^2 - \sigma_{k,j}^2)^2 = O_p(T_f^{-1})$$

$$\hat{M} - M = O_p(T_f^{-1/2}).$$

for $k = 1, \ldots, 6, j = 1, \ldots, N$.

Theorem 4.1 resembles Theorem 5.1 of Bai and Li (2012) and establishes the rate of convergence for the QMLE. The only difference is while Bai and Li (2012) only established an average rate of convergence for $\{\hat{\lambda}_{k,j}\}$, we managed to establish a rate of convergence for the individual loading estimator $\hat{\lambda}_{k,j}$.

**Theorem 4.2.** Under the assumptions of Proposition 4.1, we have

$$\sqrt{T_f} (\hat{\sigma}_{k,j}^2 - \sigma_{k,j}^2) \overset{d}{\to} N(0, 2\sigma_{k,j}^4)$$

for $k = 1, \ldots, 6, j = 1, \ldots, N$.

Theorem 4.2 is the same as Theorem 5.4 of Bai and Li (2012) and establishes the asymptotic distribution of $\hat{\sigma}_{k,j}^2$.

**Theorem 4.3.** Suppose that the assumptions of Proposition 4.1 hold. For $k = 1, \ldots, 6$, $j = 1, \ldots, N$, we have
Theorem 4.4. Suppose that the assumptions of Proposition 4.1 hold and restrictions, implied by our model ((2.2), (2.1)). The same reasoning applies to (4.13).

As in Bai and Li (2012), \( f_t \) could be estimated by the generalized least squares (GLS):

\[
\hat{f}_t = (\hat{\Lambda}^{1}\hat{\Sigma}^{-1}_{ee}\hat{\Lambda})^{-1}\hat{\Lambda}^{1}\hat{\Sigma}^{-1}_{ee}\hat{y}_t.
\]

**Theorem 4.4.** Suppose that the assumptions of Proposition 4.1 hold and \( \sqrt{N}/T_f \to 0 \), \( N/T_f \to \Delta \in [0, \infty) \). Then we have

\[
\sqrt{N}(\hat{f}_t - f_t) = -\sqrt{\Delta}/T_f A^{1}\hat{f}_t + Q^{-1}\frac{1}{\sqrt{N}}\hat{\Lambda}^{1}\hat{\Sigma}^{-1}_{ee}e_t + o_p(1),
\]

where \( Q \) is defined in Assumption 4.1.
Theorem 4.4 gives the asymptotic representation of the GLS \( \hat{f}_t \). We did not give its asymptotic distribution for the same reason mentioned in the discussion following Theorem 4.3.

### 4.2 Three Estimators for Practical Use

Although QMLE of the factor loadings and \( \phi \) are inefficient and of limited practical use, their asymptotic representations could serve as an stepping stone for us to find better estimators. In this subsection, we will present three estimators for practical use. These three estimators either build on QMLE or have standard errors that could be approximated by a procedure involving the asymptotic representations of QMLE. The full-fledged theoretical results of these estimators are difficult to establish, so these estimators will be evaluated by the Monte Carlo simulations in Section 5.

The first estimator builds on QMLE and enjoys a smaller standard error than that of QMLE; we call this QMLE-delta because the technique which we employ to achieve the smaller standard error is delta method. The second estimator is the EM estimator outlined in Section 3. We will try to approximate its standard error so that one could conduct inference in practice. The third estimator estimates the "two-day" form of our model (4.1) via the EM algorithm. All the restrictions implied by \( \Lambda \) and \( M \) defined in (4.2) and (4.3), respectively, will be taken into account by this EM algorithm. The only nonstandard feature is that we shall treat \( \{f_t\}_{t=1}^{T_f} \) as i.i.d. when setting up the likelihood. We call the third estimator QMLE-res.

In Appendix A.16, we provide a concise comparison between all the estimators introduced in this article for readers’ reference.

#### 4.2.1 QMLE-delta

Recall that QMLE only used \( 14^2 \) restrictions and there are additional restrictions implied by our model ((2.2), (2.1)). In this subsubsection, we propose an improved estimator (QMLE-delta) by including some of these additional restrictions via delta method. Suppose that we have QMLE ready; the computational issues of QMLE are addressed in Appendix A.13. There is no need finding an improved estimator for \( \Sigma_c \) as one could rely on Theorem 4.2 to conduct inference, so we shall set QMLE-delta of \( \Sigma_c \) as QMLE of \( \Sigma_c \).

We consider factor loadings first. For concreteness, suppose that we are interested in inference of the factor loadings of the fifth stock in the Asian, European and American continents, respectively (i.e., \( z_{c,5}, z_{E,5}, z_{U,5} \) for \( c = A, E, U \)). Define \( \hat{b} := (\hat{\lambda}_{1,5}, \hat{\lambda}_{2,5}, \hat{\lambda}_{3,5}, \hat{\lambda}_{4,5}, \hat{\lambda}_{5,5}, \hat{\lambda}_{6,5})^\top \), \( b := (\lambda_{1,5}, \lambda_{2,5}, \lambda_{3,5}, \lambda_{4,5}, \lambda_{5,5}, \lambda_{6,5})^\top \) and

\[
c := \left( T_f^{-1/2} \sum_{t=1}^{T_f} f_t^\top e_{5,t}, T_f^{-1/2} \sum_{t=1}^{T_f} f_t^\top e_{N+5,t}, \ldots, T_f^{-1/2} \sum_{t=1}^{T_f} f_t^\top e_{N+5,t} \right)^\top
\]

Note that (4.12) implies

\[
\sqrt{T_f}(\hat{b} - b) = \sqrt{T_f}(I_6 \otimes A)b - (I_6 \otimes M^{-1})c + o_p(1).
\]

The central limit theorem implies

\[
\sqrt{T_f}(\hat{b} - b) \xrightarrow{d} N(0, B)
\]
where \( B := \text{var} \left[ \lim_{T_1 \to \infty} \sqrt{T_f} (I_6 \otimes A) \mathbf{b} \right] + \text{var} \left[ \lim_{T_1 \to \infty} (I_6 \otimes M^{-1}) \mathbf{c} \right] \) is an \( 84 \times 84 \) positive definite, symmetric matrix. As mentioned in the discussion following Theorem 4.3, such asymptotic distribution has limited use because of its large standard deviations. We now seek an improved estimator for \( \mathbf{z} \).

According to (4.2), we know that only the 7 \( - k \), 8 \( - k \), 9 \( - k \), 15 \( - k \) th elements of \( \mathbf{z} \) are non-zero for \( k \) = 1, \ldots, 6. Define the following selection matrices:

\[
S_1 := \begin{pmatrix} S_{1,nz} & 0 & 0 & 0 & 0 \\ 0 & S_{2,nz} & 0 & 0 & 0 \\ 0 & 0 & S_{3,nz} & 0 & 0 \\ 0 & 0 & 0 & S_{4,nz} & 0 \\ 0 & 0 & 0 & 0 & S_{5,nz} \\ 0 & 0 & 0 & 0 & 0 & S_{6,nz} \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & S_{4,nz} \\ 0 & 0 & 0 & 0 & 0 & S_{5,nz} \\ 0 & 0 & 0 & 0 & 0 & S_{6,nz} \end{pmatrix}
\]

\[
S_2 := \begin{pmatrix} S_{1,z} & 0 & 0 & 0 & 0 \\ 0 & S_{2,z} & 0 & 0 & 0 \\ 0 & 0 & S_{3,z} & 0 & 0 \\ 0 & 0 & 0 & S_{4,z} & 0 \\ 0 & 0 & 0 & 0 & S_{5,z} \\ 0 & 0 & 0 & 0 & S_{6,z} \end{pmatrix}
\]

(4.15)

where submatrices \( \{S_{k,nz}\}_{k=1}^6 \) and \( \mathbf{0} \) in \( S_1 \) and \( S_3 \) are \( 4 \times 14 \) with \( S_{k,nz} \) being the \( 7 - k \), 8 \( - k \), 9 \( - k \), 15 \( - k \) th rows of \( I_{14} \) for \( k \) = 1, \ldots, 6, and submatrices \( \{S_{k,z}\}_{k=1}^6 \) and \( \mathbf{0} \) in \( S_2 \) are \( 10 \times 14 \) with \( S_{k,z} \) being the submatrix of \( I_{14} \) after deleting its \( 7 - k \), 8 \( - k \), 9 \( - k \), 15 \( - k \) th rows for \( k \) = 1, \ldots, 6. In other words, \( S_{k,z} \) denotes the \( 10 \times 14 \) selection matrix which extracts out the 10 zero elements of \( \mathbf{z} \), while \( S_{k,nz} \) denotes the \( 4 \times 14 \) selection matrix which extracts out the 4 non-zero elements of \( \mathbf{z} \).

We hence have

\[
\begin{pmatrix} S_1 \\ S_2 \\ S_3 - S_1 \end{pmatrix} \sqrt{T_f} (\hat{\mathbf{b}} - \mathbf{b}) \overset{d}{\to} N \left( \begin{pmatrix} 0 \\ S_1 S_2 \end{pmatrix}, \begin{pmatrix} S_1 B (S_3 - S_1) ^\top & S_1 B (S_3 - S_1) ^\top \\ S_2 B (S_3 - S_1) ^\top & S_2 B (S_3 - S_1) ^\top \end{pmatrix} \right) \]

Re-partition the preceding display into

\[
\sqrt{T_f} \begin{pmatrix} \hat{\mathbf{b}}_{n_z} \\ \hat{\mathbf{b}}_z \end{pmatrix} - \begin{pmatrix} \mathbf{b}_{n_z} \\ \mathbf{0} \end{pmatrix} \overset{d}{\to} N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} B_{11} \\ B_{12} \end{pmatrix} \right)
\]

(4.16)

where \( \hat{\mathbf{b}}_{n_z} := S_1 \hat{\mathbf{b}}, \hat{\mathbf{b}}_z := [(S_2 \hat{\mathbf{b}}) ^\top, (S_3 - S_1) \mathbf{b}] ^\top, \mathbf{b}_{n_z} := S_1 \mathbf{b}, B_{11} := S_1 S_1 ^\top, B_{12} := (S_1 S_1 ^\top, S_1 B (S_3 - S_1) ^\top), \) and

\[
B_{22} := \begin{pmatrix} S_2 B (S_3 - S_1) ^\top & S_2 B (S_3 - S_1) ^\top \\ (S_3 - S_1) B_s ^\top & (S_3 - S_1) B (S_3 - S_1) ^\top \end{pmatrix}.
\]

Note that \( \mathbf{b}_{n_z} \) contains all the non-zero elements in \( \mathbf{b} \), and \( B_{11} \) is the asymptotic covariance matrix of \( \hat{\mathbf{b}}_{n_z} \).
Now we are ready to state an improved estimator of $b_{nz}$. Note that (4.16) implies

$$\sqrt{T_f} \left[ \left( \begin{array}{c} \hat{b}_{nz} \\ -B_{12}B_{22}^{-1}\hat{b}_z \end{array} \right) - \left( \begin{array}{c} b_{nz} \\ 0 \end{array} \right) \right] \xrightarrow{d} N \left( \left( \begin{array}{c} \frac{B_{11}}{\sqrt{N}} \\ -\frac{B_{12}B_{22}^{-1}B_{12}}{\sqrt{N}} \end{array} \right) \right)$$

whence we have

$$\sqrt{T_f} \left[ (\hat{b}_{nz} - B_{12}B_{22}^{-1}\hat{b}_z) - b_{nz} \right] \xrightarrow{d} N \left( 0, B_{11} - B_{12}B_{22}^{-1}B_{12} \right).$$

Thus we propose $\hat{b}_{nz} := b_{nz} - \hat{B}_{12}B_{22}^{-1}\hat{b}_z$, where $\hat{B}_{12} := (S_1BS_1^T, S_1\hat{B}(S_3 - S_1)^T)$,

$$\hat{B}_{22} := \left( \begin{array}{cc} S_2\hat{B}S_2^T & S_2\hat{B}(S_3 - S_1)^T \\ (S_3 - S_1)\hat{B}S_2^T & (S_3 - S_1)\hat{B}(S_3 - S_1)^T \end{array} \right),$$

and $\hat{B}$ is a parametric bootstrap estimator of $B$. Then

$$\sqrt{T_f} \left[ \hat{b}_{nz} - b_{nz} \right] = \sqrt{T_f} \left[ (\hat{b}_{nz} - B_{12}B_{22}^{-1}\hat{b}_z + B_{12}B_{22}^{-1}\hat{b}_z - \hat{B}_{12}B_{22}^{-1}\hat{b}_z) - b_{nz} \right]$$

$$= \sqrt{T_f} \left[ (\hat{b}_{nz} - B_{12}B_{22}^{-1}\hat{b}_z) - b_{nz} \right] + o_p(1) \xrightarrow{d} N \left( 0, B_{11} - B_{12}B_{22}^{-1}B_{12} \right),$$

where the second equality is due to $(B_{12}B_{22}^{-1} - \hat{B}_{12}\hat{B}_{22}^{-1})\sqrt{T_f}\hat{b}_z = (B_{12}B_{22}^{-1} - \hat{B}_{12}\hat{B}_{22}^{-1})O_p(1) = o_p(1)$ if we assume that the parametric bootstrap estimator $\hat{B}$ is consistent (i.e., $\hat{B} - B = o_p(1)$). It is obvious that $\hat{b}_{nz}$ has a smaller asymptotic variance than that of $\hat{b}_{nz}$. These theoretical results are not complete in the sense that we did not prove consistency of the parametric bootstrap estimator $\hat{B}$.

QMLE-delta for $\phi$ is provided in Appendix A.14.

### 4.2.2 The Standard Error for the EM Estimator

In this subsubsection, we give a procedure to approximate the standard errors of the EM estimators defined in Section 3. Barigozzi and Luciani (2022) showed that the EM and MLE are asymptotically equivalent under their assumptions (see their Lemmas 22 and 23). Thus in principle we should use the standard errors of MLE as those of the EM estimators. However, the large sample theories of MLE of our model still remain as a formidable, if not impossible, task to be completed in the future. We need to have a procedure to approximate the standard errors of MLE.

**For $\hat{\phi}$.** By relying on the first-order condition of MLE with respect to $\phi$, we propose a parametric bootstrap to approximate the sampling distribution of $\hat{\phi}$ defined in (3.4). It can be shown that the first-order condition of MLE with respect to $\phi$ is

$$\frac{1}{N^{1/T_f}}\frac{\partial}{\partial \theta} \left\{ \sum_{t=1}^{T_f} \frac{1}{2N} \right\} \left( \frac{\partial}{\partial \theta} \left( \frac{1}{2N} \right) \right) = \frac{1}{2N} \text{tr} \left[ \Lambda^{-1}(\Sigma_{yy}^{-1}S_{yy}^{-1} - \Sigma_{yy}^{-1})\Lambda K(\phi) \right]$$

where $K(\phi)$ is the $14 \times 14$ derivative matrix $\partial M/\partial \phi$. Given the EM estimators $\{\tilde{Z}^c, \tilde{\Sigma}_c, \tilde{\phi}_h\}$ for $c = A, E, U$, in each bootstrap replication, simulate $\{y_t\}$ and hence calculate $S_{yy}$. Then use the numeric means to find $\tilde{\phi}$ which solves

$$\frac{1}{2N} \text{tr} \left[ \tilde{\Lambda}_b^{-1}(\tilde{\Sigma}_{yy,b}^{-1}S_{yy,b}^{-1} - \tilde{\Sigma}_{yy,b}^{-1})\tilde{\Lambda}_b K(\tilde{\phi}_b) \right] = 0,$$

where $\tilde{\Lambda}_b$ and $\tilde{\Sigma}_{yy,b}$ are estimates of $\Lambda$ and $\Sigma_{yy}$ computed using $\{\tilde{Z}^c, \tilde{\Sigma}_c\}$ and $\tilde{\phi}_b$. If we have $B$ parametric bootstrap replications, then the standard deviation of $\{\tilde{\phi}_b\}_{b=1}^B$ could be used as the approximation for the standard error of $\tilde{\phi}$. 

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For $\tilde{Z}^c$. We propose a procedure to approximate the standard errors of $\tilde{Z}^c$ for $c = A, E, U$ defined in (3.3). The idea is to have an improved estimator which builds on the QMLE but incorporates all the restrictions implied by the factor loading matrix $\Lambda$ defined in (4.2). Since this improved estimator has incorporated all the information contained in $\Lambda$, its standard error should approximate that of the MLE. The machinery which we shall employ to get the improved estimator is the Bayes theorem.

Let $a := \text{vec} A = Ja^*$ with $a^*$ being the elements of $A$ with an order bigger than $o_p(T_f^{-1/2})$.\footnote{In unreported calculations, we found that there are 4 elements in $A$ which are of an order $o_p(T_f^{-1/2})$.} (Also note that $a^* = J^T a$.) For large $T_f$, we have

$$M^{-1} \frac{1}{\sqrt{T_f}} \sum_{t=1}^{T_f} f_t e_{(k-1)N+j,t} \overset{d}{\to} N(0, \sigma_{k,j}^2 M^{-1}).$$

For $k = 1, 2, 3$, (4.12) hence implies

$$\sqrt{T_f} \begin{pmatrix} \hat{\lambda}_{k,j} - \lambda_{k,j} \\ \hat{\lambda}_{k+3,j} - \lambda_{k+3,j} \end{pmatrix} \overset{d}{\to} N \left( \begin{pmatrix} \sqrt{T_f} A \lambda_{k,j} \\ \sqrt{T_f} A \lambda_{k+3,j} \end{pmatrix}, \begin{bmatrix} \sigma_{k,j}^2 M^{-1} & 0 \\ 0 & \sigma_{k,j}^2 M^{-1} \end{bmatrix} \right). \quad (4.17)$$

Define

$$G_k := \begin{pmatrix} S_{k,z} & 0 \\ 0 & S_{k+3,z} \\ S_{k,nz} & -S_{k+3,nz} \end{pmatrix},$$

where $S_{k,z}, S_{k,nz}$ are defined in (4.15). Note that $G_k (\lambda_{k,j}^T, \lambda_{k+3,j}^T)^T = 0$. As $T_f \to \infty$

$$\sqrt{T_f} a^* \overset{d}{\to} N(0, W_a) \quad (4.18)$$

$$p(\sqrt{T_f} a^*) \propto \exp \left( -\frac{1}{2} a^T W_a^{-1} a^* \right)$$

where the convergence in distribution is because $A$ could be expressed into some known (but complicated) additive function involving elements of (4.14), and $p(\cdot)$ denotes the asymptotic density.

Let

$$\hat{\omega}_{k,j} := \sqrt{T_f} \begin{pmatrix} \hat{\lambda}_{k,j} - \lambda_{k,j} \\ \hat{\lambda}_{k+3,j} - \lambda_{k+3,j} \end{pmatrix}.$$

Pre-multiplying (4.17) by $G_k$, we have

$$G_k \hat{\omega}_{k,j} | a^* \overset{d}{\to} N \left( G_k \begin{pmatrix} \sqrt{T_f} A \lambda_{k,j} \\ \sqrt{T_f} A \lambda_{k+3,j} \end{pmatrix}, G_k \begin{bmatrix} \sigma_{k,j}^2 M^{-1} & 0 \\ 0 & \sigma_{k,j}^2 M^{-1} \end{bmatrix} G_k^T \right) = N \left( G_k \sqrt{T_f} \begin{pmatrix} \lambda_{k,j}^T \otimes I_{14} & \lambda_{k+3,j} \otimes I_{14} \\ \lambda_{k,j}^T \otimes I_{14} \end{pmatrix} J a^*, \sigma_{k,j}^2 g_k (I_2 \otimes M^{-1}) G_k^T \right) \cdot$$
Then the asymptotic density of $G_k \hat{\omega}_{k,j}$ conditional on $\alpha^*$ is
\[
p(G_k \hat{\omega}_{k,j} | \alpha^*) = (2\pi)^{-\frac{d}{2}} |\sigma_{k,j}^2 G_k(I_2 \otimes M^{-1})G_k^\top|^{-1/2} \times \\
\exp\left\{-\frac{1}{2} \left( G_k \hat{\omega}_{k,j} - G_k \mathbf{v}^T \begin{bmatrix} \lambda_{k,j}^T \otimes I_{14} \\ \lambda_{k+3,j}^T \otimes I_{14} \end{bmatrix} J \alpha^*\right)^T \sigma_{k,j}^{-2} [G_k(I_2 \otimes M^{-1})G_k^\top]^{-1} \left( G_k \hat{\omega}_{k,j} - G_k \mathbf{v}^T \begin{bmatrix} \lambda_{k,j}^T \otimes I_{14} \\ \lambda_{k+3,j}^T \otimes I_{14} \end{bmatrix} J \alpha^*\right)\right\}
\times \exp\left\{-\frac{1}{2} \left( G_k \hat{\omega}_{k,j} - G_k \mathbf{v}^T \begin{bmatrix} \lambda_{k,j}^T \otimes I_{14} \\ \lambda_{k+3,j}^T \otimes I_{14} \end{bmatrix} J \alpha^*\right)^T \sigma_{k,j}^{-2} [G_k(I_2 \otimes M^{-1})G_k^\top]^{-1} \left( G_k \hat{\omega}_{k,j} - G_k \mathbf{v}^T \begin{bmatrix} \lambda_{k,j}^T \otimes I_{14} \\ \lambda_{k+3,j}^T \otimes I_{14} \end{bmatrix} J \alpha^*\right)\right\}
\]

Note that $\{G_k \hat{\omega}_{k,j}\}_{k=1,j=1}^{k=3,j=N}$ are conditionally (given $\alpha^*$) asymptotically independent across $k$ and $j$ because of (4.12) and Assumption 2.1. Then we have
\[
p\left(G_k \hat{\omega}_{k,j} \mid k=3,j=N \mid \alpha^*\right) = \prod_{k=1}^3 \prod_{j=1}^N p(G_k \hat{\omega}_{k,j} | \alpha^*) \times \\
\exp\left\{-\frac{1}{2} \sum_{k,j} \left( G_k \hat{\omega}_{k,j} - G_k \mathbf{v}^T \begin{bmatrix} \lambda_{k,j}^T \otimes I_{14} \\ \lambda_{k+3,j}^T \otimes I_{14} \end{bmatrix} J \alpha^*\right)^T \sigma_{k,j}^{-2} [G_k(I_2 \otimes M^{-1})G_k^\top]^{-1} \left( G_k \hat{\omega}_{k,j} - G_k \mathbf{v}^T \begin{bmatrix} \lambda_{k,j}^T \otimes I_{14} \\ \lambda_{k+3,j}^T \otimes I_{14} \end{bmatrix} J \alpha^*\right)\right\}
\times \exp\left\{-\frac{1}{2} \left( G_k \hat{\omega}_{k,j} - G_k \mathbf{v}^T \begin{bmatrix} \lambda_{k,j}^T \otimes I_{14} \\ \lambda_{k+3,j}^T \otimes I_{14} \end{bmatrix} J \alpha^*\right)^T \sigma_{k,j}^{-2} [G_k(I_2 \otimes M^{-1})G_k^\top]^{-1} \left( G_k \hat{\omega}_{k,j} - G_k \mathbf{v}^T \begin{bmatrix} \lambda_{k,j}^T \otimes I_{14} \\ \lambda_{k+3,j}^T \otimes I_{14} \end{bmatrix} J \alpha^*\right)\right\}
\times \exp\left(\Psi_1 \alpha^* - \frac{1}{2} \alpha^T \Psi_2 \alpha^*\right)
\] (4.19)

where
\[
\Psi_1 := \sum_{k=1}^3 \sum_{j=1}^n \left( G_k \hat{\omega}_{k,j} \right)^T \sigma_{k,j}^{-2} [G_k(I_2 \otimes M^{-1})G_k^\top]^{-1} S_k \mathbf{v}^T \begin{bmatrix} \lambda_{k,j}^T \otimes I_{14} \\ \lambda_{k+3,j}^T \otimes I_{14} \end{bmatrix} J
\]
\[
\Psi_2 := \left\{ W_3^{-1} + \sum_{k=1}^3 \sum_{j=1}^n J^T \begin{bmatrix} \lambda_{k,j}^T \otimes I_{14} \\ \lambda_{k+3,j}^T \otimes I_{14} \end{bmatrix} G_k^\top \sigma_{k,j}^{-2} [G_k(I_2 \otimes M^{-1})G_k^\top]^{-1} S_k \begin{bmatrix} \lambda_{k,j}^T \otimes I_{14} \\ \lambda_{k+3,j}^T \otimes I_{14} \end{bmatrix} J \right\}
\]

Thus (4.19) implies
\[
\sqrt{T_f} \alpha^* \mid \{G_k \hat{\omega}_{k,j}\}_{k=1,j=1}^{k=3,j=N} \text{ is asymptotically distributed as } N\left(\Psi_1 \Psi_2^{-1}, \Psi_2^{-1}\right).
\]

Note that (4.12) could be written into:
\[
\sqrt{T_f}(\hat{\lambda}_{k,\ell} - \lambda_{k,\ell}) = \sqrt{T_f}(\lambda_{k,\ell}^T \otimes I_{14})J \alpha^* + M^{-1} \frac{1}{\sqrt{T_f}} \sum_{t=1}^{T_f} f_t \epsilon(k-1)N + \alpha_p(1).
\]

For $\ell > 4$, we have that
\[
\sqrt{T_f}(\hat{\lambda}_{k,\ell} - \lambda_{k,\ell}) \mid \{G_k \hat{\omega}_{k,j}\}_{k=1,j=1}^{k=3,j=N} \text{ is asymptotically distributed as }^3
\]
\[
N\left( (\lambda_{k,\ell}^T \otimes I_{14}), J \Psi_1 \Psi_2^{-1}(\lambda_{k,\ell}^T \otimes I_{14}), J \Psi_2^{-1}(\lambda_{k,\ell}^T \otimes I_{14}) + \sigma_{k,\ell}^2 M^{-1}\right).
\]

Since $G_k(\lambda_{k,j}^T, \lambda_{k+3,j}^T)^T = 0$, one could impose restrictions on $\lambda_{k,j}$, $\lambda_{k+3,j}$ so that $G_k \hat{\omega}_{k,j} = 0$ for $k = 1, 2, 3, j = 1, \ldots, N$. Thus, for $\ell > 4, \sqrt{T_f}(\hat{\lambda}_{k,\ell} - \lambda_{k,\ell}) \mid \{G_k \hat{\omega}_{k,j} = 0\}_{k=1,j=1}^{k=3,j=N} \text{ is asymptotically distributed as }$
\[
N\left( 0, (\lambda_{k,\ell}^T \otimes I_{14}), J \Psi_2^{-1} J^T(\lambda_{k,\ell}^T \otimes I_{14}) + \sigma_{k,\ell}^2 M^{-1}\right).
\]

Thus $(\lambda_{k,\ell}^T \otimes I_{14}), J \Psi_2^{-1} J^T(\lambda_{k,\ell}^T \otimes I_{14}) + \sigma_{k,\ell}^2 M^{-1}$ could be used as an approximation of the asymptotic variance matrix of the corresponding EM estimators.

\footnote{For $\ell \leq 4$, there is one more term in the asymptotic variance matrix (i.e., the asymptotic covariance between $T_f^{1/2}(\lambda_{k,\ell}^T \otimes I_{14})J \alpha^*$ and $M^{-1} T_f^{-1/2} \sum_{t=1}^{T_f} f_t \epsilon(k-1)N + \alpha_p(1)$), but the reasoning is the same.}
For $\hat{\Sigma}_c$. We shall approximate the standard error of $\hat{\sigma}_{c,j}^2$ by $\hat{\sigma}_{c,j}^2 \sqrt{2/T_f}$ (see Theorem 4.2). This is because the QMLE of $\Sigma_c$ has already incorporated the fact of diagonality of $\Sigma_c$ for $c = A, E, U$, so the standard error of the QMLE of $\sigma_{c,j}^2$ should approximate that of the MLE reasonably well.

4.2.3 QMLE-res

In this subsubsection, we propose to estimate the "two-day" form of our model (4.1) via the EM algorithm. We shall incorporate all the restrictions, implied by $\Lambda$ and $M$ defined in (4.2) and (4.3), respectively, into the EM algorithm. This EM estimator differs from the EM estimator defined in Section 3 in the sense that we shall treat $\{f_t\}_{t=1}^{T_f}$ as i.i.d. when setting up the likelihood. The idea is that although we ignore the autocorrelation of $\{f_t\}_{t=1}^{T_f}$, the $14 \times 1$ vector $f_t$ should contain enough information for estimating $\phi$. We call this estimator QMLE-res.

One could show that the log-likelihoods of $\{y_t\}_{t=1}^{T_f}$ and $\{f_t\}_{t=1}^{T_f}$ are, respectively,

$$
\ell(\{y_t\}_{t=1}^{T_f} | \{f_t\}_{t=1}^{T_f}; \theta) = -\frac{T_f}{2} \log(2\pi) - \frac{T_f}{2} \log |\Sigma_{cc}| - \frac{1}{2} \sum_{t=1}^{T_f} (y_t - \Lambda f_t)^\top \Sigma_{cc}^{-1} (y_t - \Lambda f_t)
$$

$$
\ell(\{f_t\}_{t=1}^{T_f}; \theta) = -\frac{T_f}{2} \log(2\pi) - \frac{T_f}{2} \log |M| - \frac{1}{2} \sum_{t=1}^{T_f} f_t^\top M^{-1} f_t,
$$

The complete log-likelihood function 1) of the "two-day" form of our model (4.1) while ignoring the autocorrelation of $\{f_t\}_{t=1}^{T_f}$ is hence (omitting constant)

$$
\ell(\{y_t\}_{t=1}^{T_f}, \{f_t\}_{t=1}^{T_f}; \theta) = \ell(\{y_t\}_{t=1}^{T_f} | \{f_t\}_{t=1}^{T_f}; \theta) + \ell(\{f_t\}_{t=1}^{T_f}; \theta)
$$

$$
= -\frac{T_f}{2} \log |\Sigma_{cc}| - \frac{1}{2} \sum_{t=1}^{T_f} (y_t - \Lambda f_t)^\top \Sigma_{cc}^{-1} (y_t - \Lambda f_t) - \frac{T_f}{2} \log |M| - \frac{1}{2} \sum_{t=1}^{T_f} f_t^\top M^{-1} f_t
$$

$$
= -\frac{1}{2} \left[ \sum_{t=1}^{T_f} \tilde{e}_{1,t} + \sum_{t=1}^{T_f} \tilde{e}_{2,t} \right] \quad (4.20)
$$

where

$$
\sum_{t=1}^{T_f} \tilde{e}_{1,t} := T_f \log |\Sigma_{cc}| + \sum_{t=1}^{T_f} \text{tr} \left[ (y_t - \Lambda f_t)(y_t - \Lambda f_t)^\top \Sigma_{cc}^{-1} \right]
$$

$$
\sum_{t=1}^{T_f} \tilde{e}_{2,t} := T_f \log |M| + \sum_{t=1}^{T_f} \text{tr} \left[ f_t f_t^\top M^{-1} \right].
$$

Let $\mathbb{E}$ denote the expectation with respect to the conditional density $p(\{y_t\}_{t=1}^{T_f}, \{f_t\}_{t=1}^{T_f}; \theta^{(i)})$ at $\bar{\theta}^{(i)}$, where $\bar{\theta}^{(i)}$ is the estimate of $\theta$ from the $i$th iteration of the EM algorithm. Taking such expectation on both sides of (4.20), we hence have

$$
\mathbb{E} \left[ \ell(\{y_t\}_{t=1}^{T_f}, \{f_t\}_{t=1}^{T_f}; \theta) \right] = -\frac{1}{2} \left( \bar{\mathbb{E}} \sum_{t=1}^{T_f} \tilde{e}_{1,t} + \mathbb{E} \sum_{t=1}^{T_f} \tilde{e}_{2,t} \right).
$$
This is the E-step. We now find values of $Z^c$ to minimise $\bar{E} \sum_{t=1}^{T_f} \vec{e}_{1,t}$. We could show that

$$\bar{E} \sum_{t=1}^{T_f} \vec{e}_{1,t} \propto \sum_{t=1}^{T_f} \left[ -2(Z^A)^1 L_1 \bar{E}[f_i \hat{y}_{1,1}] / \sigma_{1,1}^2 + (Z^A)^1 L_1 \bar{E}[f_i f_i^\top] L_1^\top (Z^A)^1 / \sigma_{1,1}^2 \right]$$

$$+ \sum_{t=1}^{T_f} \left[ -2(Z^A)^1 L_4 \bar{E}[f_i \hat{y}_{1;3N+1}] / \sigma_{1,1}^2 + (Z^A)^1 L_4 \bar{E}[f_i f_i^\top] L_4^\top (Z^A)^1 / \sigma_{1,1}^2 \right].$$

Taking the differential with respect to $(Z^A)^1$, recognising the derivative and setting that to zero, we have

$$(\hat{Z}^A)^1 = \left[ \sum_{t=1}^{T_f} \left( L_1 \bar{E}[f_i f_i^\top] L_1^\top + L_4 \bar{E}[f_i f_i^\top] L_4^\top \right) \right]^{-1} \left[ \sum_{t=1}^{T_f} \left( L_1 \bar{E}[f_i \hat{y}_{1,1}] + L_4 \bar{E}[f_i \hat{y}_{1;3N+1}] \right) \right].$$

In a similar way, we can obtain the QMLE-res for other factor loadings.

We now find values of $\Sigma_{ee}$ to minimise $\bar{E} \sum_{t=1}^{T_f} \vec{e}_{1,t}$. We could show that

$$\bar{E} \sum_{t=1}^{T_f} \vec{e}_{1,t} \propto T_f \log |\Sigma_{ee}| + \text{tr} \left[ C_e \Sigma_{ee}^{-1} \right] = T_f \sum_{k=1}^{3N} 2 \log \sigma_k^2 + \sum_{k=1}^{3N} C_{e,k,k} + C_{e,3N+k,3N+k}$$

where $C_e := \sum_{t=1}^{T_f} (\hat{y}_t \hat{y}_t^\top - 2 \bar{E}[f_i \hat{y}_t] + \bar{E}[f_i f_i^\top]) \Lambda^\top$, and the single-index $\sigma_k^2$ is defined as $\sigma_k^2 := \frac{C_{e,k,k} + C_{e,3N+k,3N+k}}{2}$. Taking the derivative with respect to $\sigma_k^2$ and setting that to zero, we have

$$\bar{\sigma}_k^2 = \frac{1}{T_f} C_{e,k,k} + C_{e,3N+k,3N+k}.$$

We will provide the formulas for $\bar{E}[f_i f_i^\top]$ and $\bar{E}[f_i \hat{y}_t]$ in Appendix A.12.

Next, we find values of $\phi$ to minimise $\bar{E} \sum_{t=1}^{T_f} \vec{e}_{2,t}$. It is difficult to derive the analytical solution for $\phi$ so we will obtain $\bar{\phi}$ in a numerical way.

The standard errors of QMLE-res are calculated in the same way as those of the EM estimators (see Section 4.2.2 for details). The idea is that since QMLE-res has incorporated all the restrictions implied by our model, its standard errors should be close to those of MLE.

5 Monte Carlo Simulations

In this section, we shall conduct Monte Carlo simulations to evaluate the performances of our proposed estimators. We specify the following values for the parameters: $N = 50, 200$, $T = 750$ (around one year’s trading data), 1500, 2250, $\phi = 0.3$. For $c = A, E, U$, $\Sigma_{c,ii}$ are drawn from uniform[0.5, 1] for $i = 1, \ldots, N$, and

$$z_{c}^i, j = 0.25 + 0.25a_{c, i, j} + 0.25b_{c, i} + 0.25d_{c, j}$$

4Note that $k \mapsto (\lfloor \frac{k}{N} \rfloor, k - \lfloor \frac{k}{N} \rfloor N)$ is a bijection from $\{1, \ldots, 6N\}$ to $\{1, \ldots, 6\} \times \{1, \ldots, N\}.$
where \( \{a_{c,i,j}\}_{i=1,j=0}^{N,j=3}, \{b_{c,i}\}_{i=1}^{N} \) and \( \{d_{c,j}\}_{j=0}^{3} \) are all drawn from uniform\([0, 1]\). After the log 24-hr returns are generated, the econometrician only observes those log 24-hr returns whose \( ts \) correspond to the closing times of their belonging continents. The econometrician is aware of the structure of the true model ((2.2), (2.1)), but does not know the values of those parameters. In particular, he is aware of diagonality of \( \Sigma_A, \Sigma_E, \Sigma_U \).

For the EM algorithm, the starting values of the parameters are estimated according to Section 3. The number of the Monte Carlo samples is chosen to be 100. From these 100 Monte Carlo samples, we calculate the following three quantities for evaluation:

(i) the root mean square errors (RMSE),

(ii) the average of the standard errors (Ave.se) across the Monte Carlo samples.

(iii) the coverage probability (Cove) of the confidence interval formed by the point estimate \( \pm 1.96 \times \) the standard error. The standard errors differ across the Monte Carlo samples.

For a particular \( j \) and \( c \), it is impossible to present a evaluation criterion of \( z_{c,j}^i \) for all \( i \), so we only report the average value for the vector \( z_{c,j}^i \). Likewise, we report the average value for the diagonals of \( \{ \Sigma_c : c = A, E, U \} \). Tables 2 and 3 report these results. To save space, we only present the results for \( T = 750, 2250 \) (the results for \( T = 1500 \) are available upon request). We see that the EM and QMLE-res estimators are very similar in terms of those three evaluation criteria, especially for the case \( N = 200 \) and \( T = 2250 \). In terms of the RMSE, the EM and QMLE-res are much better than the QMLE-delta estimator. In terms of Ave.se, the QMLE-delta has larger standard errors than those of the EM and QMLE-res; this is particularly so for the parameter \( \phi \). This is probably because the QMLE with 14^2 restrictions (before the delta-method improvement) has very large standard errors.

When the dimension \( N \) increases from 50 to 200, we obtain smaller RMSE and Ave.se and better coverage. Similarly, when \( T \) increases from 750 to 2250, we also obtain smaller RMSE and Ave.se and better coverage. In particular, the coverage of \( \phi \) in the EM increases from 0.76 for \( T = 750, N = 50 \) to 0.93 for \( T = 2250, N = 200 \). The coverage of \( \phi \) in the QMLE-res also increases from 0.67 for \( T = 750, N = 50 \) to 0.93 for \( T = 2250, N = 200 \). All three estimators estimate the idiosyncratic variance \( \Sigma_c \) quite well and have rather good coverage.
|       | EM   | QMLE-res | QMLE-delta |
|-------|------|----------|------------|
|       | RMSE | Ave.se   | Cove       | RMSE | Ave.se | Cove    | RMSE | Ave.se | Cove      |
| $z_{A0}$ | 0.1579 | 0.0935 | 0.7538     | 0.1804 | 0.0925 | 0.6524 | 0.2694 | 0.1270 | 0.6174     |
| $z_{A1}$ | 0.1846 | 0.1180 | 0.7952     | 0.1972 | 0.1164 | 0.7676 | 0.4211 | 0.3288 | 0.8390     |
| $z_{A2}$ | 0.1622 | 0.1147 | 0.8384     | 0.1882 | 0.1125 | 0.7476 | 0.3715 | 0.3076 | 0.8616     |
| $z_{A3}$ | 0.1061 | 0.0983 | 0.9350     | 0.1351 | 0.0977 | 0.8408 | 0.3510 | 0.2950 | 0.8544     |
| $z_{E0}$ | 0.1470 | 0.1029 | 0.8326     | 0.1775 | 0.1004 | 0.7238 | 0.3109 | 0.2181 | 0.7926     |
| $z_{E1}$ | 0.1936 | 0.1156 | 0.7784     | 0.2157 | 0.1106 | 0.6492 | 0.3332 | 0.2569 | 0.8364     |
| $z_{E2}$ | 0.1622 | 0.1147 | 0.8384     | 0.1882 | 0.1125 | 0.7476 | 0.3715 | 0.3076 | 0.8616     |
| $z_{E3}$ | 0.1061 | 0.0983 | 0.9350     | 0.1351 | 0.0977 | 0.8408 | 0.3510 | 0.2950 | 0.8544     |
| $z_{U0}$ | 0.1558 | 0.1026 | 0.8130     | 0.1585 | 0.1014 | 0.8052 | 0.3574 | 0.2318 | 0.7604     |
| $z_{U1}$ | 0.1370 | 0.0884 | 0.8434     | 0.1570 | 0.0882 | 0.7980 | 0.2327 | 0.1700 | 0.8104     |
| $z_{U2}$ | 0.1528 | 0.1103 | 0.8456     | 0.1842 | 0.1075 | 0.7570 | 0.3730 | 0.2901 | 0.8318     |
| $z_{U3}$ | 0.1676 | 0.1126 | 0.8142     | 0.1859 | 0.1098 | 0.7592 | 0.3268 | 0.2697 | 0.8694     |
| $\Sigma c$ | 0.0718 | 0.0547 | 0.8598     | 0.0909 | 0.0545 | 0.7468 | 0.1501 | 0.0752 | 0.6598     |
| $\phi$  | 0.0799 | 0.0766 | 0.9416     | 0.0866 | 0.0762 | 0.9214 | 0.2778 | 0.2495 | 0.8936     |
|       |       |       |            | 0.0923 | 0.0740 | 0.8812 | 0.3221 | 0.2774 | 0.8822     |
|       |       |       |            | 0.0641 | 0.0611 | 0.9392 | 0.2552 | 0.2385 | 0.9070     |
|       |       |       |            | 0.0770 | 0.0635 | 0.8898 | 0.2554 | 0.1693 | 0.7764     |
|       |       |       |            | 0.0940 | 0.0729 | 0.8618 | 0.2233 | 0.1942 | 0.8978     |
|       |       |       |            | 0.0768 | 0.0649 | 0.9086 | 0.2580 | 0.1737 | 0.7886     |
|       |       |       |            | 0.0556 | 0.0539 | 0.9434 | 0.1971 | 0.1307 | 0.7294     |
|       |       |       |            | 0.0843 | 0.0705 | 0.8972 | 0.2834 | 0.2286 | 0.8466     |
|       |       |       |            | 0.0900 | 0.0709 | 0.8686 | 0.2692 | 0.2250 | 0.8822     |
|       |       |       |            | 0.0764 | 0.0689 | 0.9176 | 0.2529 | 0.2196 | 0.9028     |
|       |       |       |            | 0.0702 | 0.0593 | 0.9026 | 0.2084 | 0.1908 | 0.8978     |
|       |       |       |            | 0.0412 | 0.0386 | 0.9059 | 0.0771 | 0.0515 | 0.7713     |
| $\Sigma c$ | 0.0407 | 0.0387 | 0.9112     | 0.0446 | 0.0378 | 0.7200 | 0.1320 | 0.1527 | 0.9200     |

Table 2: RMSE, Ave.se and Cove stand for the root mean square errors, average of the standard errors across the Monte Carlo samples, and the coverage probability of the confidence interval formed by the point estimate ± 1.96× the standard error, respectively. The EM, QMLE-res and QMLE-delta stand for the EM estimator, QMLE estimator with all the restrictions, QMLE estimator with the delta-method improvement (see Section 4.2).
Table 3: RMSE, Ave.se and Cove stand for the root mean square errors, average of the standard errors across the Monte Carlo samples, and the coverage probability of the confidence interval formed by the point estimate $\pm 1.96 \times$ the standard error, respectively. The EM, QMLE-res and QMLE-delta stand for the EM estimator, QMLE estimator with all the restrictions, QMLE estimator with the delta-method improvement (see Section 4.2).

|       | T=750, N=200 |       |       |       |
|-------|---------------|-------|-------|-------|
|       | EM            | QMLE-res | QMLE-delta |
|       | RMSE | Ave.se | Cove  | RMSE | Ave.se | Cove  | RMSE | Ave.se | Cove  |
| $z_0^A$ | 0.1034 | 0.0863 | 0.8893 | 0.1045 | 0.0860 | 0.8885 | 0.1967 | 0.1381 | 0.8113 |
| $z_1^A$ | 0.1015 | 0.0917 | 0.9239 | 0.1049 | 0.0915 | 0.9100 | 0.3095 | 0.3508 | 0.9687 |
| $z_2^A$ | 0.0961 | 0.0882 | 0.9249 | 0.1057 | 0.0879 | 0.8992 | 0.3175 | 0.3315 | 0.9549 |
| $z_3^A$ | 0.0814 | 0.0835 | 0.9547 | 0.0923 | 0.0836 | 0.9231 | 0.2528 | 0.2811 | 0.9567 |
| $z_0^E$ | 0.0873 | 0.0896 | 0.9550 | 0.0901 | 0.0891 | 0.9469 | 0.3233 | 0.2727 | 0.9484 |
| $z_1^E$ | 0.1011 | 0.0962 | 0.9350 | 0.1112 | 0.0949 | 0.9020 | 0.3092 | 0.3425 | 0.9656 |
| $z_2^E$ | 0.0994 | 0.0887 | 0.9456 | 0.0973 | 0.0883 | 0.9238 | 0.3214 | 0.2911 | 0.9309 |
| $z_3^E$ | 0.0837 | 0.0818 | 0.9431 | 0.0895 | 0.0816 | 0.9254 | 0.2050 | 0.2048 | 0.9313 |
| $z_0^U$ | 0.1002 | 0.0930 | 0.9290 | 0.1053 | 0.0925 | 0.9139 | 0.2913 | 0.3325 | 0.9699 |
| $z_1^U$ | 0.0999 | 0.0900 | 0.9222 | 0.0983 | 0.0895 | 0.9254 | 0.3014 | 0.3213 | 0.9673 |
| $z_2^U$ | 0.0790 | 0.0817 | 0.9572 | 0.0843 | 0.0816 | 0.9396 | 0.2348 | 0.2671 | 0.9647 |
| $\Sigma^c$ | 0.0678 | 0.0657 | 0.9141 | 0.0691 | 0.0656 | 0.9116 | 0.1360 | 0.0819 | 0.7008 |
| $\phi$ | 0.0503 | 0.0298 | 0.7600 | 0.0556 | 0.0298 | 0.6900 | 0.1313 | 3.9825 | 0.9900 |

|       | T=2250, N=200 |       |       |       |
|-------|---------------|-------|-------|-------|
|       | EM            | QMLE-res | QMLE-delta |
|       | RMSE | Ave.se | Cove  | RMSE | Ave.se | Cove  | RMSE | Ave.se | Cove  |
| $z_0^A$ | 0.0535 | 0.0504 | 0.9344 | 0.0631 | 0.0504 | 0.8780 | 0.1245 | 0.0764 | 0.7512 |
| $z_1^A$ | 0.0587 | 0.0553 | 0.9354 | 0.0625 | 0.0554 | 0.9148 | 0.2435 | 0.2785 | 0.9656 |
| $z_2^A$ | 0.0505 | 0.0534 | 0.9611 | 0.0527 | 0.0533 | 0.9522 | 0.2602 | 0.2671 | 0.9470 |
| $z_3^A$ | 0.0427 | 0.0498 | 0.9771 | 0.0487 | 0.0498 | 0.9541 | 0.2076 | 0.2366 | 0.9568 |
| $z_0^E$ | 0.0483 | 0.0539 | 0.9731 | 0.0479 | 0.0537 | 0.9717 | 0.2454 | 0.2023 | 0.8946 |
| $z_1^E$ | 0.0570 | 0.0589 | 0.9589 | 0.0619 | 0.0584 | 0.9370 | 0.2451 | 0.2683 | 0.9602 |
| $z_2^E$ | 0.0491 | 0.0532 | 0.9636 | 0.0497 | 0.0531 | 0.9623 | 0.2399 | 0.2201 | 0.9326 |
| $z_3^E$ | 0.0446 | 0.0487 | 0.9673 | 0.0472 | 0.0486 | 0.9546 | 0.1622 | 0.1616 | 0.9292 |
| $z_0^U$ | 0.0486 | 0.0538 | 0.9727 | 0.0514 | 0.0537 | 0.9608 | 0.2400 | 0.2403 | 0.9374 |
| $z_1^U$ | 0.0601 | 0.0566 | 0.9343 | 0.0569 | 0.0563 | 0.9482 | 0.2275 | 0.2514 | 0.9637 |
| $z_2^U$ | 0.0546 | 0.0543 | 0.9503 | 0.0510 | 0.0541 | 0.9625 | 0.2358 | 0.2381 | 0.9487 |
| $z_3^U$ | 0.0419 | 0.0486 | 0.9757 | 0.0457 | 0.0486 | 0.9621 | 0.1814 | 0.2032 | 0.9562 |
| $\Sigma^c$ | 0.0391 | 0.0384 | 0.9259 | 0.0392 | 0.0384 | 0.9251 | 0.0653 | 0.0521 | 0.8492 |
| $\phi$ | 0.0240 | 0.0241 | 0.9300 | 0.0249 | 0.0241 | 0.9300 | 0.0782 | 30.2019 | 0.8900 |
6 Empirical Applications

In this section, we present two empirical applications of our model. Section 6.1 is about modelling equity portfolio returns of Japan, Europe and US. That is, one market per continent. Section 6.2 studies MSCI equity indices of the developed and emerging markets (41 markets in total). Given the similar performances of the EM and QMLE estimators in the Monte Carlo simulations, we shall only use the EM estimator here.

6.1 An Empirical Study of Three Markets

We now apply our model to equity portfolios of three continents/markets: Japan, Europe and US. Take Japan as an example. First, we consider six equity portfolios constructed by intersections of 2 size groups (small (S) and big (B)) and 3 book-to-market equity ratio (B/M) groups (growth (G), neutral (N) and value (V)), in the spirit of Fama and French (1993). We denote the six portfolios SG, SN, SV, BG, BN and BV. Second, in a similar manner we consider six equity portfolios constructed by intersections of 2 size groups (small (S) and big (B)) and 3 momentum groups (loser (L), neutral (N) and winner (W)). We denote these six portfolios SL, SN, SW, BL, BN and BW. We downloaded the daily value-weighted portfolio returns (in percentage points) from Kenneth R. French’s website.

Note that these returns are not log returns, so strictly speaking our model does not apply. Moreover we demeaned and standardised the daily value-weighted portfolio returns so that the returns have sample variances of one. In Appendix A.15, we show that our model could still be applied using some innocuous approximations.

We next discuss how to interpret the factor loadings of our model. Let \( \dot{y}_{i,t} \) denote the standardised return of portfolio \( i \) of region \( c \) on day \( t \). Recall that

\[
\dot{y}_{i,t} = \sum_{j=0}^{2} z_{i,j}^{c} f_{g,t-j}^{c} + z_{i,3}^{c} f_{C,t}^{c} + e_{i,t}^{c},
\]

var\( (f_{g,t}) = (1 - \phi^2)^{-1} \) and var\( f_{C,t} = 1 \). Thus an additional standard-deviation increase in \( f_{g,t-1} \) predicts \( z_{i,1}^{c} \sqrt{1 - \phi^2} \) standard-deviation increase in the standardised return \( \dot{y}_{i,t} \), while an additional standard-deviation increase in \( f_{C,t} \) predicts \( z_{i,3}^{c} \) standard-deviation increase in the standardised return \( \dot{y}_{i,t} \).

We then give a formula for variance decomposition. Recall (A.17): \( \alpha_{t+1} = T\alpha_{t} + R\eta_{t} \), \( \eta_{t} \sim N(0, I_{2}) \). We first calculate the unconditional variance of \( \alpha_{t} \); it can be shown that

\[
\text{var}(\alpha_{t}) = \text{unvec}\left\{ [I_{16} - T \otimes T]^{-1}(R \otimes R) \text{vec}(I_{2}) \right\} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \phi & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Consider

\[
\dot{y}_{i,t} = \begin{bmatrix}
z_{i,0}^{c} & z_{i,1}^{c} & z_{i,2}^{c} & z_{i,3}^{c}
\end{bmatrix} \alpha_{t} + e_{i,t}^{c}.
\]

\(^{5}\)https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/
Then we have
\[
\text{var}(\hat{y}_{i,t}^c) = \begin{bmatrix}
z_{i,0}^c & z_{i,1}^c & z_{i,2}^c & z_{i,3}^c \\
\end{bmatrix}
\begin{bmatrix}
\frac{1}{1-\phi^2} & \frac{\phi}{1-\phi^2} & \frac{\phi^2}{1-\phi^2} & 0 \\
\frac{\phi}{1-\phi^2} & \frac{1}{1-\phi^2} & \frac{\phi^2}{1-\phi^2} & 0 \\
\frac{\phi^2}{1-\phi^2} & \frac{1}{1-\phi^2} & \frac{1}{1-\phi^2} & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
z_{i,0}^c \\
z_{i,1}^c \\
z_{i,2}^c \\
z_{i,3}^c \\
\end{bmatrix}
+ \text{var}(e_{i,t}^c)
\]

\[
= \frac{1}{1-\phi^2} \left[ z_{i,0}^{c,2} + z_{i,1}^{c,2} + z_{i,2}^{c,2} + 2\phi z_{i,1}^{c} z_{i,0}^{c} + 2\phi^2 z_{i,1}^{c} z_{i,2}^{c} + 2\phi^2 z_{i,2}^{c} z_{i,0}^{c} \right]
\]

\[
+ \var\left( z_{i,3}^{c,2} \right) + \sigma_{c,i}^{2}.
\]

(6.1)

For fixed \( \text{var}(\hat{y}_{i,t}^c) \) and \( \sigma_{c,i}^{2} \), a small \( z_{i,3}^{c,2} \) means that the variance of the return is largely explained by the global factor.

### 6.1.1 Two Periods of Five-Year Data

We estimate the model using two periods of data (20110103-20151231; 20160104-20201231) by the EM algorithm. We take care of the missing returns due to continent-specific reasons using the technique outlined in Section A.5. The starting values of the parameters in the EM algorithm are estimated according to Section 3.

The EM estimates of the factor loading matrices are reported in Table 4. We first examine the six portfolios constructed by intersections of size and book-to-market equity ratio (B/M) groups. In 2016-2020 the Japanese standardised returns were more likely to be affected by the global factor during the US trading time (\( z_1^A \)) and less likely to be affected by the global factor during the European trading time (\( z_2^A \)) than they were in 2011-2015. Take the Japanese SG portfolio as an example. In 2011-2015, an additional standard-deviation increase in \( f_{g,t-1} \) predicts \( 0.24/\sqrt{1-0.1654^2} = 0.2424 \) standard-deviation increase in the standardised return \( \hat{y}_{SG,t}^A \), while an additional standard-deviation increase in \( f_{A,t-2} \) predicts \( 0.51/\sqrt{1-0.1654^2} = 0.5171 \) standard-deviation increase in the standardised return \( \hat{y}_{SG,t}^A \). In 2016-2020, an additional standard-deviation increase in \( f_{g,t-1} \) predicts \( 0.71/\sqrt{1-0.2323^2} = 0.73 \) standard-deviation increase in the standardised return \( \hat{y}_{SG,t}^A \), while an additional standard-deviation increase in \( f_{A,t-2} \) predicts \( 0.26/\sqrt{1-0.2323^2} = 0.2673 \) standard-deviation increase in the standardised return \( \hat{y}_{SG,t}^A \).

In 2011-2015, an additional standard-deviation increase in \( f_{C,t} \) predicts 0.75 standard-deviation increase in the standardised return, while in 2016-2020, an additional standard-deviation increase in \( f_{C,t} \) only predicts 0.63 standard-deviation increase in the standardised return.

For the European portfolios, the standardised returns were less likely to be affected by the global factor during the European and US trading times (\( z_{1}^E, z_{2}^E \)), and more likely to be affected by the continental factor (\( z_{3}^E \)) than they were in 2011-2015. Because of the much larger continental loadings in 2016-2020, one could argue that the European portfolios became less integrated into the global market than they were in 2011-2015. Take the European BG portfolio as an example. In 2011-2015, an additional standard-deviation increase in \( f_{g,t} \) predicts \( 0.47/\sqrt{1-0.1654^2} = 0.4766 \) standard-deviation increase in the standardised return. In 2016-2020, an additional standard-deviation increase in \( f_{g,t} \) predicts \( 0.18/\sqrt{1-0.2323^2} = 0.1851 \) standard-deviation increase in the standardised return.
### Size and B/M portfolios in 2011-2015

| A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|
| z^0_0 & z^1_1 & z^2_2 & z^3_3 & z^E_0 & z^E_1 & z^E_2 & z^U_0 & z^U_1 & z^U_2 & z^U_3 |
| SG & 0.16 & 0.24 & 0.51 & 0.75 & 0.43 & 0.54 & 0.57 & 0.48 & 0.45 & 0.46 & 0.49 & 0.48 |
| SN & 0.18 & 0.30 & 0.63 & 0.73 & 0.44 & 0.54 & 0.57 & 0.52 & 0.47 & 0.48 & 0.52 & 0.48 |
| SV & 0.17 & 0.31 & 0.66 & 0.68 & 0.43 & 0.53 & 0.57 & 0.48 & 0.47 & 0.48 & 0.52 & 0.46 |
| BG & 0.20 & 0.39 & 0.85 & 0.34 & 0.47 & 0.58 & 0.70 & 0.16 & 0.41 & 0.56 & 0.66 & 0.15 |
| BN & 0.18 & 0.38 & 0.89 & 0.31 & 0.50 & 0.60 & 0.71 & 0.17 & 0.41 & 0.58 & 0.72 & 0.09 |
| BV & 0.18 & 0.37 & 0.85 & 0.34 & 0.56 & 0.65 & 0.20 & 0.42 & 0.55 & 0.63 & 0.18 & 0.83 |

### Size and B/M portfolios in 2016-2020

| A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|
| z^0_0 & z^1_1 & z^2_2 & z^3_3 & z^E_0 & z^E_1 & z^E_2 & z^U_0 & z^U_1 & z^U_2 & z^U_3 |
| SG & 0.30 & 0.71 & 0.26 & 0.63 & 0.23 & 0.53 & 0.26 & 0.77 & 0.50 & 0.29 & 0.60 & 0.42 |
| SN & 0.31 & 0.82 & 0.55 & 0.34 & 0.54 & 0.24 & 0.78 & 0.40 & 0.50 & 0.64 & 0.47 & 0.84 |
| SV & 0.30 & 0.83 & 0.56 & 0.46 & 0.42 & 0.55 & 0.23 & 0.73 & 0.30 & 0.62 & 0.64 & 0.42 |
| BG & 0.33 & 0.88 & 0.34 & 0.18 & 0.54 & 0.21 & 0.68 & 0.55 & 0.18 & 0.72 & 0.01 & 0.85 |
| BN & 0.35 & 0.92 & 0.57 & 0.20 & 0.36 & 0.58 & 0.18 & 0.68 & 0.41 & 0.54 & 0.74 & 0.12 |
| BV & 0.33 & 0.85 & 0.70 & 0.08 & 0.52 & 0.57 & 0.16 & 0.60 & 0.30 & 0.71 & 0.72 & 0.12 |

### Size and momentum portfolios in 2011-2015

| A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|
| z^0_0 & z^1_1 & z^2_2 & z^3_3 & z^E_0 & z^E_1 & z^E_2 & z^U_0 & z^U_1 & z^U_2 & z^U_3 |
| SL & 0.08 & 0.27 & 0.26 & 0.63 & 0.89 & 0.37 & 0.53 & 0.72 & 0.26 & 0.37 & 0.41 & 0.69 & 0.42 |
| SN & 0.06 & 0.25 & 0.27 & 0.91 & 0.42 & 0.44 & 0.79 & 0.16 & 0.39 & 0.53 & 0.53 & 0.46 |
| SW & 0.05 & 0.22 & 0.26 & 0.87 & 0.47 & 0.34 & 0.80 & 0.06 & 0.43 & 0.59 & 0.35 & 0.55 |
| BL & 0.12 & 0.28 & 0.33 & 0.79 & 0.39 & 0.57 & 0.64 & 0.14 & 0.30 & 0.48 & 0.74 & 0.15 |
| BN & 0.08 & 0.27 & 0.36 & 0.81 & 0.49 & 0.47 & 0.71 & 0.05 & 0.34 & 0.74 & 0.49 & 0.12 |
| BW & 0.05 & 0.26 & 0.36 & 0.78 & 0.55 & 0.31 & 0.75 & 0.24 & 0.38 & 0.77 & 0.25 & 0.28 |

### Size and momentum portfolios in 2016-2020

| A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|
| z^0_0 & z^1_1 & z^2_2 & z^3_3 & z^E_0 & z^E_1 & z^E_2 & z^U_0 & z^U_1 & z^U_2 & z^U_3 |
| SL & 0.23 & 0.47 & 0.34 & 0.84 & 0.63 & 0.85 & 0.35 & -0.19 & 0.52 & 0.75 & 0.37 & 0.43 |
| SN & 0.21 & 0.47 & 0.24 & 0.88 & 0.41 & 0.95 & 0.44 & -0.25 & 0.59 & 0.60 & 0.42 & 0.47 |
| SW & 0.23 & 0.45 & 0.07 & 0.84 & 0.24 & 0.94 & 0.47 & -0.20 & 0.65 & 0.34 & 0.43 & 0.44 |
| BL & 0.19 & 0.46 & 0.44 & 0.73 & 0.68 & 0.84 & 0.28 & 0.05 & 0.56 & 0.83 & 0.35 & 0.22 |
| BN & 0.22 & 0.50 & 0.29 & 0.80 & 0.44 & 0.97 & 0.41 & 0.17 & 0.71 & 0.66 & 0.40 & 0.16 |
| BW & 0.21 & 0.49 & 0.04 & 0.77 & 0.14 & 0.94 & 0.45 & 0.11 & 0.87 & 0.25 & 0.34 & 0.01 |

Table 4: The factor loadings estimated by the EM algorithm. To save space, we do not report the standard errors, which are around 0.02, with minimal 0.0084 and maximal 0.0404. Almost all factor loadings are significant at 1% significance level. Entries with ♦ are significant at 5% significance level; entries with † are insignificant at 10%.
In 2011-2015, an additional standard-deviation increase in $f_{C,t}$ predicts 0.16 standard-deviation increase in the standardised return, while in 2016-2020 an additional standard-deviation increase in $f_{C,t}$ predicts 0.68 standard-deviation increase in the standardised return.

For the US portfolios, the standardised returns in 2015-2020 to a large extent became slightly less affected by the global factor during the US trading time ($z_{0}^U$), and became slightly more affected by the global factor during the Asian trading time ($z_{1}^A$) than they were in 2011-2015. Take the US SN portfolio as an example. In 2011-2015, an additional standard-deviation increase in $f_{g,t}$ predicts $0.47/\sqrt{1-0.1654^2} = 0.4766$ standard-deviation increase in the standardised return, while an additional standard-deviation increase in $f_{g,t-2}$ predicts $0.52/\sqrt{1-0.1654^2} = 0.5273$ standard-deviation increase in the standardised return. In 2016-2020, an additional standard-deviation increase in $f_{g,t}$ predicts $0.40/\sqrt{1-0.2323^2} = 0.4113$ standard-deviation increase in the standardised return, while an additional standard-deviation increase in $f_{g,t-2}$ predicts $0.64/\sqrt{1-0.2323^2} = 0.6580$ standard-deviation increase in the standardised return. For the US portfolios, the loadings for the continental factor have decreased slightly; in particular the loading of the BG portfolio has decreased from 0.15 to something statistically insignificant.

Over the two periods of five years, a few general patterns emerge. First, within the same B/M ratio category, the B portfolio has much smaller loadings for the continental factor but larger loadings for the global factor than the S portfolio. In particular, the variances of the standardised returns of the US B portfolios could largely be explained by the global factor in light of (6.1). Second, within the same size category, the value portfolio is more affected by the global factor during the European trading time than the growth portfolio across the three continents.

We next examine the six portfolios constructed by intersections of size and momentum groups. The Japanese portfolios in general were less affected by the global factor during the Asian trading time ($z_{1}^A$), particularly in 2011-2015. The effect of the global factor during the US trading time on the Japanese portfolios has almost doubled ($z_{1}^U$). For example, in 2011-2015 an additional standard-deviation increase in $f_{g,t-1}$ predicts 0.27 standard-deviation increase in the standardised return of the Japanese BN portfolio, while in 2016-2020, the same increase predicts $0.5/\sqrt{1-0.3079^2} = 0.5255$ standard-deviation increase in the standardised return.

For the European portfolios, the standardised returns in 2015-2020 became more affected by the global factor during the Asian trading time ($z_{1}^A$), and became less affected by the global factor during the US trading time ($z_{1}^U$) than they were in 2011-2015. Take the European BW portfolio as an example. In 2011-2015, an additional standard-deviation increase in $f_{g,t-1}$ predicts 0.31 standard-deviation increase in the standardised return, while an additional standard-deviation increase in $f_{g,t-2}$ predicts 0.75 standard-deviation increase in the standardised return. In 2016-2020, an additional standard-deviation increase in $f_{g,t-1}$ predicts $0.94/\sqrt{1-0.3079^2} = 0.9880$ standard-deviation increase in the standardised return, while an additional standard-deviation increase in $f_{g,t-2}$ predicts $0.45/\sqrt{1-0.3079^2} = 0.4730$ standard-deviation increase in the standardised return.

For the US portfolios, the standardised returns in 2015-2020 became more affected by the global factor during the US trading time ($z_{0}^U$). Take the US BL portfolio as an example. In 2011-2015, an additional standard-deviation increase in $f_{g,t}$ predicts 0.30 standard-deviation increase in the standardised return, while in 2016-2020 the same increase predicts $0.56/\sqrt{1-0.3079^2} = 0.5886$ standard-deviation increase in the stan-
For the size-momentum portfolios, one consistent pattern across the three continents is that in 2011-2015 within the same size category, the winner (W) portfolio was less affected by the global factor during the Asian trading time than the loser (L) portfolio. In 2016-2020, again within the same size category, the W portfolio was less affected by the global factor during the European trading time than the L portfolio.

The estimated $\Sigma_A, \Sigma_E, \Sigma_U$ together with their standard errors are reported in Table 5. In general, the idiosyncratic variances for both the size-B/M and size-momentum portfolios are quite small. For the size-B/M portfolios, within the B/M ratio category, the B portfolio tends to have higher idiosyncratic variances than the S portfolio. Two exceptions are the Japanese growth group in 2011-2015 and the US growth group in 2016-2020. In addition, within the same size group, to a large extent the neutral (N) portfolio has much smaller idiosyncratic variances than the growth (G) portfolio or value (V) portfolio. For the size-momentum portfolios, the Japanese portfolios have bigger idiosyncratic variances than the European and US portfolios; the only exception is the small-losers (SL) portfolio in 2016-2020.

The $\hat{\phi}$ in the application of size-B/M portfolios is significantly negative in both five-year periods. The value is -0.1654 with a standard error of 0.0266 in 2011-2015, and -0.2323 with a standard error of 0.0219 in 2016-2020. The $\hat{\phi}$ in the application of size-momentum portfolios is statistically insignificant with a value of 0.0051 in 2011-2015, and significantly negative in 2016-2020, with a value of -0.3079 and a standard error of 0.0229.

### 6.1.2 Time Series Patterns

In this subsection, we use the EM algorithm to estimate the model using twenty periods of one-year data (1991-2020). For simplicity, we only consider the size-B/M portfolios. The detailed point estimates and their standard errors are available upon request; here...
### Idiosyncratic variances of size and B/M portfolios in 2011-2015

|       | SG  | SN  | SV  | BG  | BN  | BV  |
|-------|-----|-----|-----|-----|-----|-----|
| Japan | 0.1294 | 0.0074 | 0.0300 | 0.0645 | 0.0156 | 0.0752 |
|       | (0.0031) | (0.0017) | (0.0014) | (0.0027) | (0.0023) | (0.0027) |
| Europe| 0.0454 | 0.0000 | 0.0530 | 0.0599 | 0.0000 | 0.0937 |
|       | (0.0020) | (0.0014) | (0.0025) | (0.0023) | (0.0021) | (0.0033) |
| US    | 0.0680 | 0.0000 | 0.0309 | 0.0739 | 0.0000 | 0.1067 |
|       | (0.0020) | (0.0011) | (0.0012) | (0.0028) | (0.0029) | (0.0039) |

### Idiosyncratic variances of size and B/M portfolios in 2016-2020

|       | SG  | SN  | SV  | BG  | BN  | BV  |
|-------|-----|-----|-----|-----|-----|-----|
| Japan | 0.1366 | 0.0021 | 0.0301 | 0.1572 | 0.0562 | 0.0606 |
|       | (0.0046) | (0.0021) | (0.0014) | (0.0061) | (0.0040) | (0.0055) |
| Europe| 0.0666 | 0.0117 | 0.0368 | 0.2047 | 0.1098 | 0.1295 |
|       | (0.0011) | (0.0012) | (0.0017) | (0.0088) | (0.0050) | (0.0048) |
| US    | 0.0766 | 0.0000 | 0.0238 | 0.0000 | 0.0404 | 0.0257 |
|       | (0.0029) | (0.0013) | (0.0014) | (0.0042) | (0.0019) | (0.0021) |

### Idiosyncratic variances of size and momentum portfolios in 2011-2015

|       | SL  | SN  | SW  | BL  | BN  | BW  |
|-------|-----|-----|-----|-----|-----|-----|
| Japan | 0.0735 | 0.0337 | 0.1300 | 0.1878 | 0.1519 | 0.1988 |
|       | (0.0035) | (0.0024) | (0.0038) | (0.0091) | (0.0056) | (0.0083) |
| Europe| 0.0302 | 0.0304 | 0.0664 | 0.1128 | 0.0546 | 0.0014 |
|       | (0.0042) | (0.0027) | (0.0033) | (0.0056) | (0.0031) | (0.0097) |
| US    | 0.0315 | 0.0267 | 0.0000 | 0.0724 | 0.0000 | 0.0573 |
|       | (0.0021) | (0.0012) | (0.0018) | (0.0027) | (0.0028) | (0.0031) |

### Idiosyncratic variances of size and momentum portfolios in 2016-2020

|       | SL  | SN  | SW  | BL  | BN  | BW  |
|-------|-----|-----|-----|-----|-----|-----|
| Japan | 0.0533 | 0.0239 | 0.1119 | 0.1908 | 0.1347 | 0.2023 |
|       | (0.0024) | (0.0018) | (0.0051) | (0.0073) | (0.0052) | (0.0078) |
| Europe| 0.0590 | 0.0071 | 0.0598 | 0.0721 | 0.0018 | 0.0867 |
|       | (0.0028) | (0.0022) | (0.0022) | (0.0029) | (0.0035) | (0.0045) |
| US    | 0.0443 | 0.0177 | 0.0457 | 0.0723 | 0.0539 | 0.0000 |
|       | (0.0023) | (0.0016) | (0.0024) | (0.0030) | (0.0030) | (0.0068) |

Table 5: The idiosyncratic variances estimated by the EM algorithm. The standard errors are in parentheses.

### Table 6: Parameter $\phi$ estimated by the EM algorithm. The standard errors are in parentheses.

|       | Size and B/M | Size and momentum |
|-------|--------------|------------------|
| 2011-2015 | 2016-2020 | 2011-2015 | 2016-2020 |
| $\phi$    | -0.1654 | -0.2323 | 0.0051 | -0.3079 |
|          | (0.0266) | (0.0219) | (0.0273) | (0.0229) |
we only discuss the main findings.

First, across the three continents, the B portfolios tended to have smaller loadings for the continental factor but larger loadings for the global factor than the S portfolios. The Japanese idiosyncratic variances are in general larger than those of the US and Europe. This is especially so in 1999-2001 and 2019-2020. These observations are consistent with the observations based on five-year data reported in the previous subsection.

Second, we discuss some year-specific patterns:

(i) In 1998, the Japanese portfolios have particularly large loadings for the global factor during the Asian trading time, but small loadings for the global factor during the US trading time. This could be interpreted as the effect of the Asian financial crisis.

(ii) During the 2007-2008 financial crisis, the Japanese portfolios have large loadings for the global factors during the European trading time. The European portfolios have small loadings for the continental factor but large loadings for the global factor during the US trading time. This could be interpreted as the spread of the US subprime mortgage crisis.

(iii) In 2017-2018, the Japanese portfolios have large loadings for the global factor during the US trading time but small loadings for the continental factor. The European portfolios have large loadings for the continental factor but small loadings for the global factor during the Asian and US trading times. The US portfolios have large loadings for the global factor during the Asian trading time. This could be interpreted as Japan and US markets being more integrated during this period but not so for the Europe. This could be due to the Sino-US trade war.

(iv) In 2020, the European portfolios have quite small loadings for the continental factor. Compared with the Japanese and European portfolios, the US portfolios have relatively constant loadings for the global factor during the three trading periods of a day.

Last, we re-estimate the model using ten periods of two-year data (1991-2020) and compute the variance decomposition using (6.1). The decompositions are reported in Figure 1. The blue solid and red dashed lines depict the variance proportions of the global and continental factors, respectively. The magenta dotted lines represent the realized volatility (divided by two for a better layout) computed using the daily returns. We find that the continental factor accounts for a decreasing share of the variance of the portfolio returns of the US big stocks in the past 30 years. In the 1990s, the global factor only accounted for a small share of the total variance for the US. During the turbulent years such as the 2008 financial crisis, the global factor tended to account for a larger share of the total variance for the European and US portfolios.

6.2 An Empirical Study of Many Markets

We also apply our model to MSCI equity indices of the developed and emerging markets (41 markets in total). The daily indices are obtained from https://www.msci.com/end-of-day-data-search. There are 6 indices for each market: Large-Growth, Mid-Growth, Small-Growth, Large-Value, Mid-Value, and Small-Value, all in USD currency. According to the closing time of each market, we categorize these markets into 3 continents: Asia-Pacific, Europe and America. Since the closing time of Israel market is both far away
from Asia-Pacific and Europe, we exclude it from our sample. We use the four-year data of January 1st 2018 to February 21st 2022. Indices which started after January 1st 2018 are excluded. We estimate the model with the EM algorithm, the estimated $\phi$ is 0.3588 with a standard error 0.0267. In the following, we examine the factor loading estimates for the three continents.

Table 7 reports the estimates of the factor loadings and idiosyncratic variances for the Asian-Pacific continent. We present all the indices for Mainland China, Hong Kong and Japan, but only Middle-Value and Middle-Growth indices for other Asian-Pacific markets in the interest of space. There are several findings. First, Mainland China and Hong Kong have particularly high loadings on the global factors during the US trading time (i.e., $z_{1A}^A$). Second, these loadings are higher for the Growth indices than for the Value indices in Hong Kong. Third, Japan have high loadings on the continental factor (i.e, $z_{2A}^A$) but small idiosyncratic variances. Fourth, considering other Asian-Pacific markets as well, we find that the Growth indices in general have larger idiosyncratic variances than the Value indices do. Fifth, most markets have larger loadings on the global factor during the US trading time, but small loadings on the continental factor except Japan.

Table 8 reports the estimates of the factor loadings and idiosyncratic variances for the European continent. We present all the indices for the UK, Germany and France, but only Mid-Value and Mid-Growth indices for other European markets in the interest of space. Most European markets have the largest loadings on the global factor during the Asian trading time ($z_{1E}^E$). The developed European markets have large loadings on the continental factor, but the emerging European markets have rather small loadings on the continental factor.

Table 9 reports the estimates of the factor loadings and idiosyncratic variance for the American continent. The US and Canada have very small magnitudes of the factor loadings on the continental factor, while Brazil have large factor loadings on the continental factor (with magnitudes greater than 1). Moreover, the emerging American markets also have higher loadings on the global factor during the Asian trading time (i.e., $z_{2U}^U$) and small loadings on the global factor during the American and European trading times (i.e., $z_{0U}^U, z_{1U}^U$), while this pattern does not hold for the developed markets.
Table 7: The EM estimates of the factor loadings and idiosyncratic variances are reported for selected indices. LV, LG, MV, MG, SV, SG stand for Large-Value, Large-Growth, Middle-Value, Middle-Growth, Small-Value and Small-Growth, respectively. The standard errors are in parentheses. The $z_0^A$, $z_1^A$ and $z_2^A$ stand for the factor loadings on the global factor during the Asian Pacific, American, and European trading times, respectively. The $z_3^A$ stands for the factor loadings on the Asian Pacific continental factor.
| Country     | MV     | MG     | SV     | LG     | LV     |
|-------------|--------|--------|--------|--------|--------|
| AUSTRIA     | 0.096  | 0.751  | 0.116  | 0.462  | 0.967  |
| BELGIUM     | 0.028  | 0.451  | 0.112  | 0.408  | 0.342  |
| DENMARK     | -0.061 | 0.392  | 0.023  | 0.450  | 3.618  |
| FINLAND     | 0.150  | 0.556  | 0.191  | 0.478  | 0.879  |
| IRELAND     | 0.170  | 0.516  | 0.052  | 0.709  | 1.775  |
| ITALY       | 0.087  | 0.551  | 0.013  | 0.608  | 0.615  |
| NETHER.     | 0.220  | 0.514  | 0.149  | 0.475  | 0.389  |
| NORWAY      | 0.071  | 0.607  | 0.068  | 0.391  | 0.784  |
| SPAIN       | 0.019  | 0.533  | 0.045  | 0.563  | 0.351  |
| SWEDEN      | 0.134  | 0.491  | 0.156  | 0.512  | 0.551  |
| SWITZER.    | 0.015  | 0.405  | 0.137  | 0.428  | 0.205  |
| CZECH       | -0.094 | 0.402  | 0.071  | 0.108  | 0.754  |
| EGYPT       | 0.214  | -0.148 | 0.242  | 0.014  | 2.877  |
| GREECE      | -0.053 | 0.579  | 0.199  | 0.398  | 2.680  |
| POLAND      | -0.120 | 0.692  | 0.084  | 0.089  | 1.437  |
| RUSSIA      | -0.147 | 0.672  | -0.003 | -0.006 | 1.190  |
| SOUTH       | -0.223 | 1.237  | 0.048  | -0.078 | 0.631  |
| TURKEY      | -0.285 | 1.332  | -0.310 | -0.350 | 5.793  |

| Country     | SV     | MG     | SV     | LG     | LV     |
|-------------|--------|--------|--------|--------|--------|
| AUSTRIA     | 0.170  | 0.559  | 0.093  | 0.401  | 1.352  |
| BELGIUM     | 0.140  | 0.711  | 0.075  | 0.571  | 1.045  |
| DENMARK     | 0.128  | 0.396  | 0.135  | 0.479  | 0.579  |
| FINLAND     | 0.227  | 0.566  | 0.160  | 0.495  | 0.835  |
| IRELAND     | 0.122  | 0.398  | 0.140  | 0.358  | 0.915  |
| ITALY       | 0.300  | 0.598  | 0.198  | 0.667  | 0.503  |
| NETHER.     | 0.052  | 0.392  | 0.066  | 0.397  | 0.419  |
| NORWAY      | 0.083  | 0.616  | 0.105  | 0.377  | 0.673  |
| PORTUGAL    | 0.048  | 0.616  | 0.066  | 0.328  | 0.774  |
| SPAIN       | 0.132  | 0.395  | 0.145  | 0.554  | 0.993  |
| SWEDEN      | 0.184  | 0.616  | 0.245  | 0.549  | 0.326  |
| SWITZER.    | 0.140  | 0.421  | 0.183  | 0.442  | 0.220  |
| GREECE      | 0.028  | 0.609  | 0.168  | 0.245  | 2.476  |
| POLAND      | -0.035 | 0.784  | 0.229  | 0.198  | 1.196  |
| RUSSIA      | -0.113 | 0.679  | -0.001 | -0.037 | 1.077  |
| SOUTH       | -0.295 | 1.376  | -0.012 | -0.137 | 1.115  |
| TURKEY      | -0.168 | 1.054  | -0.176 | -0.243 | 3.063  |

Table 8: The EM estimates of the factor loadings and idiosyncratic variances are reported for selected indices. LV, LG, MV, MG, SV, SG stand for Large-Value, Large-Growth, Middle-Value, Middle-Growth, Small-Value and Small-Growth, respectively. The standard errors are in parentheses. The $z_{E,i}^F$, $z_i^F$ and $z_i^F$ stand for the factor loadings on the global factor during the European, Asian Pacific and American trading times, respectively. The $z_i^F$ stands for the factor loadings on the European continental factor.
| Country | LV | LG | MV | MG | SV | SG |
|---------|----|----|----|----|----|----|
| USA     |    |    |    |    |    |    |
| LV      | 0.124 (0.035) | 0.522 (0.029) | 0.255 (0.037) | 0.021 (0.030) | 0.128 (0.008) |
| LG      | 0.183 (0.045) | 0.778 (0.033) | 0.231 (0.047) | 0.015 (0.037) | 0.135 (0.008) |
| MV      | 0.132 (0.034) | 0.536 (0.027) | 0.226 (0.036) | 0.004 (0.029) | 0.107 (0.007) |
| MG      | 0.183 (0.039) | 0.762 (0.026) | 0.230 (0.042) | -0.006 (0.031) | 0.044 (0.003) |
| SV      | 0.114 (0.040) | 0.630 (0.031) | 0.248 (0.041) | 0.011 (0.033) | 0.143 (0.009) |
| SG      | 0.144 (0.043) | 0.814 (0.028) | 0.230 (0.045) | 0.003 (0.033) | 0.067 (0.004) |
| CANADA  |    |    |    |    |    |    |
| LV      | 0.065 (0.034) | 0.246 (0.034) | 0.400 (0.035) | 0.021 (0.031) | 0.221 (0.014) |
| LG      | 0.072 (0.035) | 0.285 (0.034) | 0.334 (0.036) | 0.025 (0.032) | 0.230 (0.014) |
| MV      | 0.090 (0.045) | 0.263 (0.045) | 0.502 (0.046) | 0.031 (0.042) | 0.408 (0.025) |
| MG      | 0.100 (0.043) | 0.303 (0.042) | 0.348 (0.044) | 0.063 (0.040) | 0.369 (0.023) |
| SV      | 0.023 (0.038) | 0.173 (0.039) | 0.437 (0.039) | 0.081 (0.036) | 0.307 (0.019) |
| SG      | 0.066 (0.046) | 0.215 (0.045) | 0.340 (0.046) | 0.049 (0.042) | 0.439 (0.027) |
| BRAZIL  |    |    |    |    |    |    |
| LV      | 0.160 (0.059) | 0.140 (0.075) | 0.720 (0.061) | 1.504 (0.053) | 0.694 (0.043) |
| LG      | 0.160 (0.043) | 0.096 (0.060) | 0.639 (0.045) | 1.343 (0.037) | 0.333 (0.021) |
| MV      | 0.160 (0.045) | 0.131 (0.062) | 0.584 (0.047) | 1.412 (0.040) | 0.375 (0.023) |
| MG      | 0.154 (0.037) | 0.103 (0.054) | 0.551 (0.040) | 1.299 (0.032) | 0.241 (0.015) |
| SV      | 0.113 (0.032) | 0.112 (0.053) | 0.582 (0.035) | 1.410 (0.026) | 0.151 (0.009) |
| SG      | 0.135 (0.046) | 0.114 (0.065) | 0.572 (0.048) | 1.545 (0.040) | 0.380 (0.024) |
| MEXICO  |    |    |    |    |    |    |
| LV      | 0.158 (0.078) | -0.014 (0.081) | 0.766 (0.075) | 0.250 (0.073) | 0.137 (0.078) |
| LG      | 0.160 (0.070) | 0.027 (0.072) | 0.680 (0.070) | 0.258 (0.065) | 1.088 (0.067) |
| MV      | 0.120 (0.071) | -0.011 (0.073) | 0.640 (0.072) | 0.260 (0.067) | 1.155 (0.072) |
| MG      | 0.016 (0.133) | -0.043 (0.133) | 0.084 (0.133) | 0.037 (0.125) | 4.094 (0.253) |
| SV      | 0.173 (0.070) | 0.061 (0.071) | 0.590 (0.070) | 0.262 (0.065) | 1.087 (0.067) |
| SG      | 0.130 (0.068) | 0.018 (0.070) | 0.595 (0.068) | 0.219 (0.064) | 1.045 (0.065) |
| CHILE   |    |    |    |    |    |    |
| LV      | 0.133 (0.067) | -0.050 (0.069) | 0.768 (0.067) | 0.147 (0.063) | 1.003 (0.062) |
| LG      | 0.131 (0.070) | -0.032 (0.073) | 0.829 (0.070) | 0.246 (0.065) | 1.084 (0.067) |
| MV      | 0.062 (0.071) | -0.095 (0.073) | 0.773 (0.071) | 0.131 (0.067) | 1.143 (0.071) |
| MG      | 0.091 (0.083) | -0.085 (0.085) | 0.752 (0.083) | 0.148 (0.078) | 1.563 (0.097) |
| SV      | 0.064 (0.083) | -0.026 (0.085) | 0.796 (0.083) | 0.116 (0.078) | 1.546 (0.096) |
| SG      | 0.042 (0.077) | -0.086 (0.079) | 0.785 (0.077) | 0.088 (0.073) | 1.351 (0.084) |
| COLOMBIA|    |    |    |    |    |    |
| LV      | 0.123 (0.069) | 0.129 (0.071) | 0.686 (0.070) | 0.217 (0.065) | 1.052 (0.065) |
| LG      | 0.047 (0.092) | 0.323 (0.092) | 0.759 (0.092) | 0.197 (0.085) | 1.823 (0.113) |
| SV      | 0.150 (0.087) | 0.080 (0.087) | 0.531 (0.087) | 0.113 (0.081) | 1.700 (0.105) |
| SG      | 0.177 (0.085) | 0.061 (0.086) | 0.509 (0.085) | 0.200 (0.080) | 1.653 (0.102) |
| PERU    |    |    |    |    |    |    |
| LG      | 0.201 (0.060) | 0.299 (0.061) | 0.547 (0.061) | 0.183 (0.056) | 0.751 (0.047) |

Table 9: The EM estimates of the factor loadings and idiosyncratic variances are reported for selected indices. LV, LG, MV, MG, SV, SG stand for Large-Value, Large-Growth, Middle-Value, Middle-Growth, Small-Value and Small-Growth, respectively. The standard errors are in parentheses. The $z_U^1$, $z_U^2$ and $z_U^3$ stand for the factor loadings on the global factor during the American, European and Asian Pacific trading times, respectively. The $z_U^3$ stands for the factor loadings on the American continental factor.
7 Conclusion

In this article we propose a new framework to model a large number of daily stock returns across different time zones. The asymptotic theories of QMLE of our model are carefully derived. We propose three usable estimators and find that the EM and QMLE-res estimators work well in Monte Carlo simulations. Last, we use the EM estimator to estimate our model in two real data sets - the equity portfolio returns of Japan, Europe and US and MSCI equity indices of 41 developed and emerging markets. Some new insights about linkages between different stock markets are drawn. One future research direction is to work out the full-fledged theoretical results of, say, the EM estimator. Perhaps one could try to adapt the results of Barigozzi and Luciani (2022).

A Appendix

A.1 Proof of Lemma 2.1

Proof of Lemma 2.1. This proof is inspired by that of Bai and Wang (2015). Suppose that Assumption 2.2 hold. Fix a particular $t$. Recall (2.2):

$$y_t = Z_t \begin{pmatrix} f_{g,t} \\ f_{g,t-1} \\ f_{g,t-2} \\ f_{C,t} \end{pmatrix} + \varepsilon_t \quad f_{g,t+1} = \phi f_{g,t} + \eta_{g,t} \quad f_{C,t+1} = \eta_{C,t}.$$ 

Note that

$$f_{g,t} = \phi^3 f_{g,t-3} + \phi^2 \eta_{g,t-3} + \phi \eta_{g,t-2} + \eta_{g,t-1}$$

$$f_{g,t-1} = \phi^2 f_{g,t-3} + \phi \eta_{g,t-3} + \eta_{g,t-2}$$

$$f_{g,t-2} = \phi f_{g,t-3} + \eta_{g,t-3}.$$ 

Since $Z_t$ assumes one of $\{Z^A, Z^E, Z^U\}$, we need to consider three $4 \times 4$ rotation matrices represented by:

$$\Delta_1 := \begin{bmatrix} A_1 & B_1 & C_1 & O_1 \\ D_1 & E_1 & F_1 & P_1 \\ G_1 & H_1 & I_1 & Q_1 \\ R_1 & S_1 & T_1 & W_1 \end{bmatrix}, \Delta_2 := \begin{bmatrix} A_2 & B_2 & C_2 & O_2 \\ D_2 & E_2 & F_2 & P_2 \\ G_2 & H_2 & I_2 & Q_2 \\ R_2 & S_2 & T_2 & W_2 \end{bmatrix}, \Delta_3 := \begin{bmatrix} A_3 & B_3 & C_3 & O_3 \\ D_3 & E_3 & F_3 & P_3 \\ G_3 & H_3 & I_3 & Q_3 \\ R_3 & S_3 & T_3 & W_3 \end{bmatrix}. $$
Consider

\[
\begin{bmatrix}
A_1 & B_1 & C_1 & O_1 \\
D_1 & E_1 & F_1 & P_1 \\
G_1 & H_1 & I_1 & Q_1 \\
R_1 & S_1 & T_1 & W_1 \\
\end{bmatrix}
\begin{bmatrix}
f_{g,t} \\
f_{g,t-1} \\
f_{g,t-2} \\
f_{C,t} \\
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{f}_{g,t} \\
\tilde{f}_{g,t-1} \\
\tilde{f}_{g,t-2} \\
\tilde{f}_{C,t} \\
\end{bmatrix}
\quad (A.1)
\]

\[
\begin{bmatrix}
A_3 & B_3 & C_3 & O_3 \\
D_3 & E_3 & F_3 & P_3 \\
G_3 & H_3 & I_3 & Q_3 \\
R_3 & S_3 & T_3 & W_3 \\
\end{bmatrix}
\begin{bmatrix}
f_{g,t-1} \\
f_{g,t-2} \\
f_{g,t-3} \\
f_{C,t-1} \\
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{f}_{g,t-1} \\
\tilde{f}_{g,t-2} \\
\tilde{f}_{g,t-3} \\
\tilde{f}_{C,t-1} \\
\end{bmatrix}
\quad (A.2)
\]

\[
\begin{bmatrix}
A_2 & B_2 & C_2 & O_2 \\
D_2 & E_2 & F_2 & P_2 \\
G_2 & H_2 & I_2 & Q_2 \\
R_2 & S_2 & T_2 & W_2 \\
\end{bmatrix}
\begin{bmatrix}
f_{g,t-2} \\
f_{g,t-3} \\
f_{g,t-4} \\
f_{C,t-2} \\
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{f}_{g,t-2} \\
\tilde{f}_{g,t-3} \\
\tilde{f}_{g,t-4} \\
\tilde{f}_{C,t-2} \\
\end{bmatrix}
\quad .
\]

Considering (A.1) and (A.2), we have

\[
\tilde{f}_{g,t-1} = D_1 f_{g,t} + E_1 f_{g,t-1} + F_1 f_{g,t-2} + P_1 f_{C,t} = A_3 f_{g,t-1} + B_3 f_{g,t-2} + C_3 f_{g,t-3} + O_3 f_{C,t-1}
\]

whence we have

\[
0 = [D_1 \phi^3 + (E_1 - A_3) \phi^2 + (F_1 - B_3) \phi - C_3] f_{g,t-3} + [D_1 \phi^2 + (E_1 - A_3) \phi + (F_1 - B_3)] \eta_{g,t-3} + [D_1 \phi + (E_1 - A_3)] \eta_{g,t-2} + D_1 \eta_{g,t-1} + P_1 \eta_{C,t-1} - O_3 \eta_{C,t-2}.
\]

Note that each of \( \eta_{g,t-1}, \eta_{C,t-1}, \eta_{C,t-2} \) is uncorrelated with any other term on the right hand side of the preceding display. We necessarily have \( D_1 \eta_{g,t-1} = 0, P_1 \eta_{C,t-1} = 0 \) and \( O_3 \eta_{C,t-2} = 0 \) because of the non-zero variance. Equivalently, we have \( D_1 = P_1 = O_3 = 0 \). Likewise, we deduce that \( E_1 = A_3, F_1 = B_3 \) and \( C_3 = 0 \). Next, note that

\[
\tilde{f}_{g,t-2} = G_1 f_{g,t} + H_1 f_{g,t-1} + I_1 f_{g,t-2} + Q_1 f_{C,t} = D_3 f_{g,t-1} + E_3 f_{g,t-2} + F_3 f_{g,t-3} + P_3 f_{C,t-1}
\]

whence we have

\[
0 = [G_1 \phi^3 + (H_1 - D_3) \phi^2 + (I_1 - E_3) \phi - F_3] f_{g,t-3} + [G_1 \phi^2 + (H_1 - D_3) \phi + (I_1 - E_3)] \eta_{g,t-3} + [G_1 \phi + (H_1 - D_3)] \eta_{g,t-2} + G_1 \eta_{g,t-1} + Q_1 \eta_{C,t-1} - P_3 \eta_{C,t-2}.
\]

Using a similar trick, we deduce that

\[
G_1 = Q_1 = P_3 = 0, \quad H_1 = D_3 = 0, \quad I_1 = E_3, \quad F_3 = 0.
\]

The rotation matrices \( \Delta_1, \Delta_3 \) are deduced to

\[
\Delta_1 :=
\begin{bmatrix}
A_1 & B_1 & C_1 & O_1 \\
0 & E_1 & F_1 & 0 \\
0 & H_1 & I_1 & 0 \\
R_1 & S_1 & T_1 & W_1 \\
\end{bmatrix}
\quad (A.4)
\]

\[
\Delta_3 :=
\begin{bmatrix}
E_1 & F_1 & 0 & 0 \\
H_1 & I_1 & 0 & 0 \\
G_3 & H_3 & I_3 & Q_3 \\
R_3 & S_3 & T_3 & W_3 \\
\end{bmatrix}
\quad .
\]

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Applying the trick to (A.2) and (A.3), we deduce
\[ \Delta_3 := \begin{bmatrix} A_3 & B_3 & C_3 & O_3 \\ 0 & E_3 & F_3 & 0 \\ 0 & H_3 & I_3 & 0 \\ R_3 & S_3 & T_3 & W_3 \end{bmatrix} \]  \hspace{1cm} (A.6)
\[ \Delta_2 := \begin{bmatrix} E_3 & F_3 & 0 & 0 \\ H_3 & I_3 & 0 & 0 \\ G_2 & H_2 & I_2 & Q_2 \\ R_2 & S_2 & T_2 & W_2 \end{bmatrix} \] . \hspace{1cm} (A.7)

Applying the trick to (A.3) and (A.1), we deduce
\[ \Delta_2 := \begin{bmatrix} A_2 & B_2 & C_2 & O_2 \\ 0 & E_2 & F_2 & 0 \\ 0 & H_2 & I_2 & 0 \\ R_2 & S_2 & T_2 & W_2 \end{bmatrix} \]  \hspace{1cm} (A.8)
\[ \Delta_1 := \begin{bmatrix} E_2 & F_2 & 0 & 0 \\ H_2 & I_2 & 0 & 0 \\ G_1 & H_1 & I_1 & Q_1 \\ R_1 & S_1 & T_1 & W_1 \end{bmatrix} \] . \hspace{1cm} (A.9)

Comparing (A.4) and (A.9), we have
\[ A_1 = E_2, \quad B_1 = F_2, \quad H_2 = 0, \quad I_2 = E_1, \quad G_1 = 0, \quad F_1 = 0. \]
Comparing (A.5) and (A.6), we have
\[ H_1 = 0, \quad F_3 = 0, \quad I_3 = E_2 = A_1. \]
Comparing (A.7) and (A.8), we have
\[ A_2 = E_3 = I_1, \quad B_2 = F_3 = 0, \quad E_2 = A_1, \quad I_2 = E_1, \quad H_3 = 0, \quad F_2 = 0. \]

Thus the rotation matrices \( \Delta_1, \Delta_2, \Delta_3 \) are reduced to
\[ \Delta_1 := \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & 0 & I_1 & 0 \\ R_1 & S_1 & T_1 & W_1 \end{bmatrix}, \quad \Delta_2 := \begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & E_1 & 0 \\ R_2 & S_2 & T_2 & W_2 \end{bmatrix}, \quad \Delta_3 := \begin{bmatrix} E_1 & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ R_3 & S_3 & T_3 & W_3 \end{bmatrix}. \]

Note that
\[ R_1 \phi^3 + S_1 \phi^2 + T_1 \phi \ f_{g,t} + S_1 f_{g,t-1} + T_1 f_{g,t-2} + W_1 f_{C,t} = \tilde{f}_{c,t}, \]
whence we have
\[ (R_1 \phi^3 + S_1 \phi^2 + T_1 \phi) f_{g,t-3} + (R_1 \phi^2 + S_1 \phi + T_1) \eta_{g,t-3} + (R_1 \phi + S_1) \eta_{g,t-2} + R_1 \eta_{g,t-1} + W_1 \eta_{C,t-1} = \tilde{\eta}_{c,t-1}. \]
Since \( \eta_{g,t} \) and \( \eta_{C,t} \) are uncorrelated, we have \( R_1 = S_1 = T_1 = 0 \). Then \( \Delta_1 \) is reduced to
\[
\Delta_1 = \begin{bmatrix}
A_1 & 0 & 0 & 0 \\
0 & E_1 & 0 & 0 \\
0 & 0 & I_1 & 0 \\
0 & 0 & 0 & W_1
\end{bmatrix}.
\]

Applying the similar trick, we have
\[
\Delta_2 := \begin{bmatrix}
I_1 & 0 & 0 & 0 \\
0 & A_1 & 0 & 0 \\
0 & 0 & E_1 & 0 \\
0 & 0 & 0 & W_2
\end{bmatrix}, \quad \Delta_3 := \begin{bmatrix}
E_1 & 0 & 0 & 0 \\
0 & I_1 & 0 & 0 \\
0 & 0 & A_1 & 0 \\
0 & 0 & 0 & W_3
\end{bmatrix}.
\]

We have
\[
\tilde{f}_{g,t} = A_1 f_{g,t} = A_1 (\phi f_{g,t-1} + \eta_{g,t-1}) = A_1 \phi f_{g,t-1} + A_1 \eta_{g,t-1} \quad \text{var}(A_1 \eta_{g,t-1}) = 1 \\
\tilde{f}_{g,t-1} = E_1 f_{g,t-1} = E_1 (\phi f_{g,t-2} + \eta_{g,t-2}) = E_1 \phi f_{g,t-2} + E_1 \eta_{g,t-2} \quad \text{var}(E_1 \eta_{g,t-2}) = 1 \\
\tilde{f}_{g,t-2} = I_1 f_{g,t-2} = I_1 (\phi f_{g,t-3} + \eta_{g,t-3}) = I_1 \phi f_{g,t-3} + I_1 \eta_{g,t-3} \quad \text{var}(I_1 \eta_{g,t-3}) = 1 \\
\tilde{f}_{c,t-1} = W_1 f_{C,t} = W_1 \eta_{C,t-1} \quad \text{var}(W_1 \eta_{C,t-1}) = 1 \\
\tilde{f}_{c,t-2} = W_2 f_{C,t-2} = W_2 \eta_{C,t-3} \quad \text{var}(W_2 \eta_{C,t-3}) = 1 \\
\tilde{f}_{c,t-3} = W_3 f_{C,t-3} = W_3 \eta_{C,t-3} \quad \text{var}(W_3 \eta_{C,t-3}) = 1
\]

We hence deduce that \( A_1 = \pm 1, E_1 = \pm 1, I_1 = \pm 1 \) and \( W_i = \pm 1 \) for \( i = 1, 2, 3 \). Requiring that estimators of \( z_A^0, z_A^1, z_A^2, z_A^3, z_E^0, z_E^1 \) have the same column signs as those of \( z_0^0, z_1^0, z_2^0, z_3^0, z_0^E, z_1^E \) ensures that \( A_1 = 1, E_1 = 1, I_1 = 1 \) and \( W_i = 1 \) for \( i = 1, 2, 3 \). Thus \( \Delta_1, \Delta_2, \Delta_3 \) are reduced to identity matrices. Note that the proof works for both \( \phi = 0 \) and \( \phi \neq 0 \).

\section*{A.2 Bayesian Estimation Using the Gibbs Sampling}

In this subsection, we outline a Bayesian procedure (i.e., the Gibbs sampling) to estimate the factor model ((2.2), (2.1)). The Gibbs sampling consists of the following steps:

1. Get the starting values of the model parameters for the Gibbs sampling.
2. Conditional on the model parameters and the observed data, draw the factors.
3. Conditional on the factors and the observed data, draw the model parameters.
4. Return to step 2.

After a large number of steps, the collection of drawn factors will give the posterior distribution of the factors given the data, while the collection of drawn parameters will give the posterior distribution of the parameters given the data. We shall now explain steps 2-3 in detail.
A.2.1 Step 2

Let $Y_{1:t} := \{y^1_t, \ldots, y^T_t\}$ and $Y_0 = \emptyset$. For

$$\boldsymbol{\alpha}_t = \begin{bmatrix} f_{g,t} \\ f_{g,t-1} \\ f_{g,t-2} \\ f_{C,t} \end{bmatrix}, \quad \boldsymbol{\alpha}_{t+1} := \begin{bmatrix} f_{g,t+1} \\ f_{g,t} \\ f_{g,t-1} \\ f_{C,t+1} \end{bmatrix},$$

only the last 2 elements of $\boldsymbol{\alpha}_t$ are unknown, i.e., $f_{g,t-2}, f_{C,t}$, once $\boldsymbol{\alpha}_{t+1}$ is given. We want to sample $\boldsymbol{\alpha}_t$ from the joint distribution

$$p(\boldsymbol{\alpha}_T, \boldsymbol{\alpha}_{T-1}, \ldots, \boldsymbol{\alpha}_1 | Y_{1:T})$$

$$= p(\boldsymbol{\alpha}_T | Y_{1:T}) p(\boldsymbol{\alpha}_{T-1} | \boldsymbol{\alpha}_T, Y_{1:T}) p(\boldsymbol{\alpha}_{T-2} | \boldsymbol{\alpha}_{T-1}, \boldsymbol{\alpha}_T, Y_{1:T}) \cdots p(\boldsymbol{\alpha}_1 | \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_T, Y_{1:T})$$

$$= p(\boldsymbol{\alpha}_T | Y_{1:T}) \prod_{t=1}^{T-1} p(\boldsymbol{\alpha}_t | \boldsymbol{\alpha}_{t+1}, Y_{1:t}) = p(\boldsymbol{\alpha}_T | Y_{1:T}) \prod_{t=1}^{T-1} p(\boldsymbol{\alpha}_t | \boldsymbol{\alpha}_{t+1}, Y_{1:t})$$

where the last equality is due to the Markov structure of the state space system (Carter and Kohn (1994) Lemma 2.1).

We first sample

$$\boldsymbol{\alpha}_T | Y_{1:T} \sim N(a_{T|T}, P_{T|T})$$

where $a_{T|T} := \mathbb{E}[\boldsymbol{\alpha}_T | Y_{1:T}]$ and $P_{T|T} := \text{var}(\boldsymbol{\alpha}_T | Y_{1:T})$. These are obtained from Kalman filter (please refer to Section A.4 for details). Then we may sample $\boldsymbol{\alpha}_t$ from $p(\boldsymbol{\alpha}_t | \boldsymbol{\alpha}_{t+1}, Y_{1:t})$, $t = T - 1, \ldots, 3$. Given $\boldsymbol{\alpha}_{t+1}$, only the last 2 elements of $\boldsymbol{\alpha}_t$, i.e., $f_{g,t-2}, f_{C,t}$, are random, which can be drawn from $p(f_{g,t-2}, f_{C,t} | \boldsymbol{\alpha}_{t+1}, Y_{1:t})$ for $t = T - 1, \ldots, 3$. The conditional density $p(f_{g,t-2}, f_{C,t} | \boldsymbol{\alpha}_{t+1}, Y_{1:t})$ can be written as

$$p(f_{g,t-2}, f_{C,t} | \boldsymbol{\alpha}_{t+1}, Y_{1:t}) = p(f_{g,t-2}, f_{C,t} | f_{g,t+1}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t})$$

$$= p(f_{g,t-2} | f_{C,t}, f_{g,t+1}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t}) p(1:C,t | f_{g,t+1}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t}). \quad (A.10)$$

We consider the first term of (A.10).

$$p(f_{g,t-2} | f_{C,t}, f_{g,t+1}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t}) = \frac{p(f_{g,t-2}, f_{C,t+1} | f_{g,t+1}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t})}{p(f_{g,t+1} | f_{C,t}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t})}$$

$$\propto p(f_{g,t-2}, f_{g,t+1} | f_{C,t}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t})$$

$$= p(f_{g,t+1} | f_{g,t-2}, f_{C,t}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t}) p(f_{g,t-2} | f_{C,t}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t}) \quad (A.11)$$

Consider the second term of (A.11) first.

$$p(f_{g,t-2} | f_{C,t}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t}) = p(f_{g,t-2} | f_{C,t}, f_{g,t}, f_{g,t-1}, Y_{1:t})$$

Note that we can obtain the distribution $\boldsymbol{\alpha}_t | Y_{1:t}$ via Kalman filter:

$$\begin{bmatrix} f_{g,t} \\ f_{g,t-1} \\ f_{g,t-2} \\ f_{C,t} \end{bmatrix} | Y_{1:t} = \boldsymbol{\alpha}_t | Y_{1:t} \sim N(\mu_{\alpha_t}, P_{\alpha_t}) =: N \left( \begin{bmatrix} a_{\alpha_t,1} \\ a_{\alpha_t,2} \\ a_{\alpha_t,3} \\ a_{\alpha_t,4} \end{bmatrix}; P_{\alpha_t} \right) = N \left( \begin{bmatrix} P_{\alpha_t,11} & P_{\alpha_t,12} & P_{\alpha_t,13} & P_{\alpha_t,14} \\ P_{\alpha_t,21} & P_{\alpha_t,22} & P_{\alpha_t,23} & P_{\alpha_t,24} \\ P_{\alpha_t,31} & P_{\alpha_t,32} & P_{\alpha_t,33} & P_{\alpha_t,34} \\ P_{\alpha_t,41} & P_{\alpha_t,42} & P_{\alpha_t,43} & P_{\alpha_t,44} \end{bmatrix} \right),$$

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for $t = T - 1, \ldots, 3$. Reshuffle the elements of $\alpha_t$ a little bit:

$$\begin{bmatrix} f_{C,t} \\ f_{g,t} \\ f_{g,t-1} \\ f_{g,t-2} \end{bmatrix} \sim N(P_e \alpha_{t|t}, P_e P_e^T) = N \left( \begin{bmatrix} a_{t|t,4} \\ a_{t|t,1} \\ a_{t|t,2} \\ a_{t|t,3} \end{bmatrix} \begin{bmatrix} P_{t|t,44} & P_{t|t,41} & P_{t|t,42} & P_{t|t,43} \\ P_{t|t,14} & P_{t|t,11} & P_{t|t,12} & P_{t|t,13} \\ P_{t|t,24} & P_{t|t,21} & P_{t|t,22} & P_{t|t,23} \\ P_{t|t,34} & P_{t|t,31} & P_{t|t,32} & P_{t|t,33} \end{bmatrix} \right),$$

(A.12)

where $P_e$ is a permutation matrix

$$P_e = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$  

Partition (A.12) accordingly:

$$\begin{bmatrix} f_{C,t} \\ f_{g,t} \\ f_{g,t-1} \\ f_{g,t-2} \end{bmatrix} \sim N \left( \begin{bmatrix} a_{t|t,a} \\ a_{t|t,b} \end{bmatrix} \begin{bmatrix} P_{t|t,aa} & P_{t|t,ab} \\ P_{t|t,ba} & P_{t|t,bb} \end{bmatrix} \right)$$

where $\alpha_{t|t,a} := (a_{t|t,1}, a_{t|t,2})^\top$, $a_{t|t,b} := a_{t|t,3}$, $P_{t|t,bb} := (P_{t|t,33}, P_{t|t,23})^\top$, $P_{t|t,ba} = P_{t|t,ab}^T$ and

$$P_{t|t,aa} := \begin{pmatrix} P_{t|t,44} & P_{t|t,41} & P_{t|t,42} \\ P_{t|t,14} & P_{t|t,11} & P_{t|t,12} \\ P_{t|t,24} & P_{t|t,21} & P_{t|t,22} \end{pmatrix}. $$

Then

$$f_{g,t-2|f_{C,t}, f_{g,t}, f_{g,t-1}, Y_{1:t}} \sim N(c_t, C_t) \quad t = T - 1, \ldots, 3$$

where

$$c_t = a_{t|t,b} + P_{t|t,ba} P_{t|t,aa}^{-1} \left[ \begin{bmatrix} f_{C,t} \\ f_{g,t} \\ f_{g,t-1} \end{bmatrix} - a_{t|t,a} \right]$$

$$C_t = P_{t|t,bb} - P_{t|t,ba} P_{t|t,aa}^{-1} P_{t|t,ab}. $$

We now consider the first term of (A.11). Note that $f_{g,t+1} = \phi f_{g,t} + \eta_{g,t}$, $\eta_{g,t} \sim N(0, 1)$. We hence have $f_{g,t+1|f_{g,t-2}, f_{C,t}, f_{g,t}, f_{g,t-1}, f_{C,t-1}, Y_{1:t}} \sim N(\phi f_{g,t}, 1)$, which is independent of $f_{g,t-2}$. We hence have

$$p(f_{g,t-2|f_{C,t}, f_{g,t+1}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t}}) \propto N(c_t, C_t).$$

(A.13)

Consider the second term of (A.10).

$$p(f_{C,t}|f_{g,t+1}, f_{g,t}, f_{g,t-1}, f_{C,t+1}, Y_{1:t}) = p(f_{C,t}|f_{g,t+1}, f_{g,t}, f_{g,t-1}, Y_{1:t}) = p(f_{C,t}|f_{g,t}, f_{g,t-1}, Y_{1:t}).$$

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Using (A.12), we have
\[
\begin{bmatrix}
  f_{C,t} \\
  f_{g,t} \\
  f_{g,t-1}
\end{bmatrix}_{\text{Y}_{1:t}} \sim N\left(\begin{bmatrix}
a_{t|1,4} \\
a_{t|1,1} \\
a_{t|2}
\end{bmatrix}, \begin{bmatrix}
P_{t|t,44} & P_{t|t,41} & P_{t|t,42} \\
P_{t|t,14} & P_{t|t,11} & P_{t|t,12} \\
P_{t|t,24} & P_{t|t,21} & P_{t|t,22}
\end{bmatrix}\right)
\]
\]
where \( a_{t|t,c} = a_{t|t,4}, \) \( a_{t|t,d} := (a_{t|t,1}, a_{t|t,2})^T, \) \( P_{t|t,cc} := P_{t|t,44}, \) \( P_{t|t,cd} := (P_{t|t,41}, P_{t|t,42}), \) \( P_{t|t,dc} = P_{t|t,cd}^T, \) and
\[
P_{t|t,dd} := \begin{bmatrix} P_{t|t,11} & P_{t|t,12} \\
P_{t|t,21} & P_{t|t,22}
\end{bmatrix}.
\]
Then
\[
f_{C,t\mid f_{g,t}, f_{g,t-1}, Y_{1:t}} \sim N(d_t, D_t) \quad t = T - 1, \ldots, 3 \tag{A.14}
\]
where
\[
d_t = a_{t|t,c} + P_{t|t,cd}P_{t|t,dd}^{-1}\begin{bmatrix} f_{g,t} \\
f_{g,t-1} \end{bmatrix} - a_{t|t,d}
\]
\[
D_t = P_{t|t,cc} - P_{t|t,cd}P_{t|t,dd}^{-1}P_{t|t,dc}.
\]

In sum, conditional on the model’s parameters, the following recursion describes how to draw from \( p(\alpha_T, \alpha_{T-1}, \ldots, \alpha_1 \mid Y_{1:T}) \):

(a) We first sample
\[
\alpha_T \mid Y_{1:T} \sim N(\alpha_{T \mid T}, P_{T \mid T})
\]
where \( \alpha_{T \mid T} := \mathbb{E}[\alpha_T \mid Y_{1:T}] \) and \( P_{T \mid T} := \text{var}(\alpha_T \mid Y_{1:T}) \).

(b) Consider \( t = T - 1, \ldots, 3 \). For each fixed \( t \), first draw \( f_{C,t} \) from \( N(d_t, D_t) \) as in (A.14), and then draw \( f_{g,t-2} \) from \( N(c_t, C_t) \) as in (A.13).

(c) For \( t = 2 \), draw \( f_{C,2} \) from \( N(d_2, D_2) \) as in (A.14).

(d) For \( t = 1 \), note that
\[
\begin{bmatrix}
f_{C,1} \\
f_{g,1}
\end{bmatrix}_{\text{Y}_1} \sim N\left(\begin{bmatrix}
a_{1|1,4} \\
a_{1|1,1}
\end{bmatrix}, \begin{bmatrix}
P_{1|1,44} & P_{1|1,41} \\
P_{1|1,14} & P_{1|1,11}
\end{bmatrix}\right),
\]
whence we have
\[
f_{C,1 \mid f_{g,1}, f_{g,0}, Y_1} = f_{C,1 \mid f_{g,1}, Y_1} \sim N(d_1, D_1),
\]
where
\[
d_1 = a_{1|1,4} + P_{1|1,41}P_{1|1,11}^{-1}(f_{g,1} - a_{1|1,1})
\]
\[
D_1 = P_{1|1,44} - P_{1|1,41}P_{1|1,11}^{-1}P_{1|1,14}.
\]
Thus draw \( f_{C,1} \) from \( N(d_1, D_1) \).

Steps (a)-(d) finish one round of sampling from \( p(\alpha_T, \alpha_{T-1}, \ldots, \alpha_1 \mid Y_{1:T}) \).
A.2.2 Step 3

In this subsubsection, we consider how to draw model parameters conditional on the factors and observed data. We first provide two auxiliary results.

Lemma A.1.

$$\text{tr}(Z^T B Z C) = (\text{vec}Z)^T (C^T \otimes B) \text{vec}Z.$$  

Proposition A.1. If a positive random variable $X$ has a density that is proportional to $x^{-a-1}e^{-\frac{b}{x}}$, then $X \sim \text{inverse-gamma}(a,b)$ and $Y := 1/X \sim \text{gamma}(a,b)$. The matlab syntax is

```
y = random('Gamma',a,b)
x = 1/y
```

Draw $Z^A, Z^E, Z^U, \Sigma_A, \Sigma_E, \Sigma_U$ We first consider how to draw $Z^A, Z^E, Z^U, \Sigma_A, \Sigma_E, \Sigma_U$. Consider

$$y_{i}^A = Z_i \alpha_t + e_t^A = Z^A \alpha_t + e_t^A \quad t \in T_A \quad \text{(A.15)}$$

$$y_{i}^E = Z_i \alpha_t + e_t^E = Z^E \alpha_t + e_t^E \quad t \in T_E$$

$$y_{i}^U = Z_i \alpha_t + e_t^U = Z^U \alpha_t + e_t^U \quad t \in T_U.$$  

We shall use (A.15) to illustrate the procedure. We assume that little is known a priori about $Z^A$ and $\Sigma_A$. We hence use Jeffrey’s priors:

$$p(Z^A) = \text{constant} \quad p(\Sigma_A) \propto |\Sigma_A|^{-1}.$$

Moreover, we assume that $Z^A$ and $\Sigma_A$ are independent, so

$$p(Z^A, \Sigma_A) = p(Z^A)p(\Sigma_A) \propto |\Sigma_A|^{-1}$$

$$p(Z^A|\Sigma_A) \propto 1.$$  

Stacking (A.15), we have

$$Y^A = \left[\begin{array}{c}
y_{i}^{A^T} \\
y_{i}^{A^T}
\end{array}\right] = \left[\begin{array}{c}
\alpha_{1}^T \\
\alpha_{4}^T \\
\alpha_{7}^T \\
\vdots \\
\alpha_{T-2}^T
\end{array}\right] Z^{A^T} + \left[\begin{array}{c}
e_{1}^{A^T} \\
e_{4}^{A^T} \\
e_{7}^{A^T} \\
\vdots \\
e_{T-2}^{A^T}
\end{array}\right] =: Y^A_A, Z^{A^T} + E_A \quad \text{(A.16)}$$

Suppose that we observe $\{\alpha_t\}_{t=1}^T$. How do we estimate $Z^{A^T}$? The OLS estimate would be

$$\hat{Z}^{A^T} = \left(\sum_{t \in T_A} \alpha_t \alpha_t^T\right)^{-1} \sum_{t \in T_A} \alpha_t y_{t}^{A^T} = (\Xi_A^T \Xi_A)^{-1}\Xi_A^T Y^A.$$

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We can hence calculate
\[
p(Y^A|\Xi_A, Z^A, \Sigma_A) = \prod_{t \in T_A} \frac{1}{(2\pi)^{\frac{N_A}{2}} |\Sigma_A|^{1/2}} \exp \left[ -\frac{1}{2} (y_t^A - Z^A \alpha_t)^T \Sigma_A^{-1} (y_t^A - Z^A \alpha_t) \right]
\]
\[
= (2\pi)^{-\frac{N_A}{2}} |\Sigma_A|^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} \text{tr} \left( \sum_{t \in T_A} (y_t^A - Z^A \alpha_t)(y_t^A - Z^A \alpha_t)^T \Sigma_A^{-1} \right) \right]
\]
\[
= (2\pi)^{-\frac{N_A}{2}} |\Sigma_A|^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} \text{tr} \left( \sum_{t \in T_A} (y_t^A - \bar{Z}^A \alpha_t)(y_t^A - \bar{Z}^A \alpha_t)^T \Sigma_A^{-1} \right) \right]
\]
\[
\cdot \exp \left[ -\frac{1}{2} \text{tr} \left( (\bar{Z}^A - Z^A) \Xi_A^T \Xi_A (\bar{Z}^A - Z^A)^T \Sigma_A^{-1} \right) \right]
\]
\[
= (2\pi)^{-\frac{N_A}{2}} |\Sigma_A|^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} \text{tr} \left( S_A \Sigma_A^{-1} \right) \right] \cdot \exp \left[ -\frac{1}{2} \text{tr} \left( (\bar{Z}^A - Z^A) \Xi_A^T \Xi_A (\bar{Z}^A - Z^A)^T \Sigma_A^{-1} \right) \right],
\]
where
\[
S_A := \sum_{t \in T_A} (y_t^A - \bar{Z}^A \alpha_t)(y_t^A - \bar{Z}^A \alpha_t)^T = \bar{E}_A^T \bar{E}_A = (Y^A - \Xi_A \bar{Z}^A)^T (Y^A - \Xi_A \bar{Z}^A).
\]

We can hence calculate the posterior
\[
p(Z^A, \Sigma_A|Y^A, \Xi_A) \propto p(Z^A, \Sigma_A, Y^A|\Xi_A) = p(Z^A, \Sigma_A|\Xi_A)p(Y^A|Z^A, \Sigma_A, \Xi_A)
\]
\[
\propto p(Z^A, \Sigma_A)p(Y^A|Z^A, \Sigma_A, \Xi_A)
\]
\[
\propto |\Sigma_A|^{-T/2} \exp \left[ -\frac{1}{2} \text{tr} \left( S_A \Sigma_A^{-1} \right) \right] \cdot \exp \left[ -\frac{1}{2} \text{tr} \left( (\bar{Z}^A - Z^A) \Xi_A^T \Xi_A (\bar{Z}^A - Z^A)^T \Sigma_A^{-1} \right) \right],
\]
where the second \(\propto\) is due to that knowing \(\Xi_A\) does not shed any information on \(Z^A, \Sigma_A\).

Viewing the preceding display as a prior probability density function (pdf), we see that it factors into a normal part for \(Z^A\) given \(\Sigma_A\) and a marginal pdf for \(\Sigma_A\).\(^6\)

\[
p(Z^A|\Sigma_A, Y^A, \Xi_A) \propto |\Sigma_A|^{-\frac{T}{2}} \exp \left[ -\frac{1}{2} \text{tr} \left( (\bar{Z}^A - Z^A) \Xi_A^T \Xi_A (\bar{Z}^A - Z^A)^T \Sigma_A^{-1} \right) \right]
\]
\[
= |\Sigma_A|^{-\frac{T}{2}} \exp \left[ -\frac{1}{2} \left[ \text{vec}(\bar{Z}^A - Z^A)^T (\Sigma_A^{-1} \otimes \Xi_A^T \Xi_A) \text{vec}(\bar{Z}^A - Z^A) \right] \right]
\]
via Lemma A.1 and
\[
p(\Sigma_A|Y^A, \Xi_A) \propto |\Sigma_A|^{-\frac{T/2+2}{2}} \exp \left[ -\frac{1}{2} \text{tr} \left( S_A \Sigma_A^{-1} \right) \right] = \prod_{i=1}^{N_A} (\sigma_{A,i}^2)^{-\frac{T/2+2}{2}} \exp \left[ -\frac{1}{2} \frac{1}{\sigma_{A,i}^2} \right]
\]
\[
= \prod_{i=1}^{N_A} (\sigma_{A,i}^2)^{-\frac{T/2+4}{2}} \cdot \frac{1}{\sigma_{A,i}^2} \cdot \frac{1}{(2/S_{A,ii})}
\]
\(^6\)To see why the exponent of \(|\Sigma_A|\) in \(p(Z^A|\Sigma_A, Y^A, \Xi_A)\) is \(-4/2\): The covariance matrix of \(\text{vec}(\bar{Z}^A)\) is \(\Sigma_A \otimes (\Xi_A^T \Xi_A)^{-1}\). When calculating \(p(Z^A|\Sigma_A, Y^A, \Xi_A)\), we have a term:
\[
\frac{1}{|\Sigma_A \otimes (\Xi_A^T \Xi_A)^{-1}|^{-\frac{1}{2}}} = |\Sigma_A \otimes (\Xi_A^T \Xi_A)^{-1}|^{-\frac{1}{2}} \propto |\Sigma_A|^{-\frac{1}{2}}.
\]
whence we have
\[
\begin{align*}
\text{vec}(Z^A)|\Sigma_A, Y^A, \Xi_A &\sim N(\text{vec}(\tilde{Z}^A), \Sigma_A \otimes (\Xi_A \Xi_A)^{-1}) \\
p(\sigma_{A,i}^2|Y^A, \Xi_A) &\propto (\sigma_{A,i}^2)^{-T/3-4} \exp \left[ -\frac{1}{\sigma_{A,i}^2 (2/S_{A,ii})} \right]
\end{align*}
\]
for \(i = 1, \ldots, N_A\). Thus,
\[
\begin{align*}
\sigma_{A,i}^2|Y^A, \Xi_A &\sim \text{inverse-gamma} \left( \frac{T}{3} - 4 \cdot \frac{2}{S_{A,ii}} \right) \\
\sigma_{A,i}^{-2}|Y^A, \Xi_A &\sim \text{gamma} \left( \frac{T}{3} - 4 \cdot \frac{2}{S_{A,ii}} \right).
\end{align*}
\]
via Proposition A.1.

**Draw \(\phi\)** We now consider how to draw \(\phi\) conditional on the factors and observed data. Recall (2.1): \(f_{g,t+1} = \phi f_{g,t} + \eta_{g,t}\). Define
\[
\ddot{\phi} := \left( \sum_{t=1}^{T-1} f_{g,t}^2 \right)^{-1} \sum_{t=1}^{T-1} f_{g,t} f_{g,t+1}.
\]
Recalling (3.1), we have
\[
p(\Xi|\phi) \propto \prod_{t=0}^{T-1} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{(f_{g,t+1} - \phi f_{g,t})^2}{2} \right] = (2\pi)^{-\frac{T}{2}} \exp \left[ -\frac{1}{2} S_{f,g} \right] \exp \left[ -\frac{1}{2} (\ddot{\phi} - \phi)^2 \sum_{t=0}^{T-1} f_{g,t}^2 \right],
\]
where \(S_{f,g} := \sum_{t=0}^{T-1} (f_{g,t+1} - \ddot{\phi} f_{g,t})^2\), and the last equality is due to the identity:
\[
\sum_{t=0}^{T-1} (f_{g,t+1} - \phi f_{g,t})^2 = \sum_{t=0}^{T-1} (f_{g,t+1} - \ddot{\phi} f_{g,t})^2 + \sum_{t=0}^{T-1} (\ddot{\phi} - \phi)^2 f_{g,t}^2.
\]
Assume that \(p(\phi) = \text{constant}\). We can calculate
\[
p(\phi, \Xi) \propto p(\Xi|\phi) \propto (2\pi)^{-\frac{T}{2}} \exp \left[ -\frac{1}{2} S_{f,g} \right] \exp \left[ -\frac{1}{2} (\ddot{\phi} - \phi)^2 \sum_{t=0}^{T-1} f_{g,t}^2 \right]
\]
\[
\times \exp \left[ -\frac{1}{2} \cdot \frac{(\ddot{\phi} - \phi)^2}{1/\sum_{t=0}^{T-1} f_{g,t}^2} \right],
\]
whence we have
\[
\phi|\Xi \sim N \left( \ddot{\phi}, \frac{1}{\sum_{t=0}^{T-1} f_{g,t}^2} \right).
\]
We will discard the draw if the stationarity condition \(|\phi| < 1\) is not satisfied and re-draw from the preceding display until we obtain a draw satisfying the stationarity condition.
A.2.3 An Application

The Bayesian approach is computationally intensive and feasible only for small $N_c$. In some unreported Monte Carlo simulations, we found that the proposed Bayesian estimator works well for $N_c = 20$ but not so well for $N_c = 200$ for $c = A, E, U$.

Here we use the proposed Bayesian estimator to re-estimate the model in the empirical study in Section 6.1. The starting values of the model parameters for the Gibbs sampling are set to those of the EM algorithm. The length of the Markov chain is chosen to be 50,000 with a burn-in period of 40,000. After the burn-in period, we store the every 10th draw; the posterior distributions are formed based on these stored 1000 draws. The results are reported in Tables 10, 11 and 12. One can see that these results are similar to those obtained by the EM estimator in Section 6.1. The only major difference is that the Bayesian estimator gives a significant -0.1495 for $\phi$ for size-momentum portfolios in 2011-2015 (see Table 12). However, it is important to stress that two valid estimation procedures cannot produce identical results in the finite samples.
Table 10: The factor loadings estimated by the Bayesian posterior means. To save space, we do not report the standard deviations of the posterior distributions, with a mean 0.0151, a minimum $2.0699 \times 10^{-6}$ and a maximum 0.0359. An entry with superscript ♣ means that zero is within the interval of this entry (i.e., the posterior mean) ±1.96×the standard deviation of the posterior distribution.
### Idiosyncratic variances of size and B/M portfolios in 2011-2015

|        | SG  | SN  | SV  | BG  | BN  | BV  |
|--------|-----|-----|-----|-----|-----|-----|
| Japan  | 0.1297 | 0.0001 | 0.0338 | 0.0642 | 0.0102 | 0.0760 |
|        | (0.0050) | (0.0002) | (0.0014) | (0.0029) | (0.0019) | (0.0036) |
| Europe | 0.0455 | 0.0000 | 0.0536 | 0.0599 | 0.0000 | 0.0946 |
|        | (0.0018) | (0.0000) | (0.0022) | (0.0023) | (0.0000) | (0.0038) |
| US     | 0.0680 | 0.0000 | 0.0314 | 0.0756 | 0.0000 | 0.1094 |
|        | (0.0027) | (0.0001) | (0.0013) | (0.0029) | (0.0000) | (0.0043) |

### Idiosyncratic variances of size and B/M portfolios in 2016-2020

|        | SG  | SN  | SV  | BG  | BN  | BV  |
|--------|-----|-----|-----|-----|-----|-----|
| Japan  | 0.1398 | 0.0000 | 0.0328 | 0.1090 | 0.0000 | 0.1075 |
|        | (0.0054) | (0.0001) | (0.0014) | (0.0045) | (0.0000) | (0.0045) |
| Europe | 0.0647 | 0.0116 | 0.0349 | 0.1932 | 0.1047 | 0.1289 |
|        | (0.0028) | (0.0013) | (0.0017) | (0.0083) | (0.0044) | (0.0056) |
| US     | 0.0776 | 0.0016 | 0.0227 | 0.0547 | 0.0292 | 0.0361 |
|        | (0.0041) | (0.0017) | (0.0019) | (0.0060) | (0.0027) | (0.0033) |

### Idiosyncratic variances of size and momentum portfolios in 2011-2015

|        | SL  | SN  | SW  | BL  | BN  | BW  |
|--------|-----|-----|-----|-----|-----|-----|
| Japan  | 0.0714 | 0.0297 | 0.1214 | 0.1835 | 0.1507 | 0.1971 |
|        | (0.0035) | (0.0029) | (0.0057) | (0.0084) | (0.0069) | (0.0088) |
| Europe | 0.0674 | 0.0000 | 0.0586 | 0.0876 | 0.0000 | 0.0790 |
|        | (0.0028) | (0.0001) | (0.0026) | (0.0037) | (0.0000) | (0.0036) |
| US     | 0.0437 | 0.0227 | 0.0131 | 0.0694 | 0.0199 | 0.0417 |
|        | (0.0029) | (0.0016) | (0.0023) | (0.0051) | (0.0036) | (0.0037) |

### Idiosyncratic variances of size and momentum portfolios in 2016-2020

|        | SL  | SN  | SW  | BL  | BN  | BW  |
|--------|-----|-----|-----|-----|-----|-----|
| Japan  | 0.0503 | 0.0222 | 0.1104 | 0.1876 | 0.1316 | 0.1957 |
|        | (0.0026) | (0.0022) | (0.0051) | (0.0082) | (0.0057) | (0.0083) |
| Europe | 0.0505 | 0.0109 | 0.0552 | 0.0674 | 0.0000 | 0.0843 |
|        | (0.0030) | (0.0022) | (0.0031) | (0.0035) | (0.0000) | (0.0037) |
| US     | 0.0421 | 0.0250 | 0.0449 | 0.0767 | 0.0000 | 0.1084 |
|        | (0.0031) | (0.0019) | (0.0038) | (0.0036) | (0.0000) | (0.0053) |

Table 11: The idiosyncratic variances estimated by the Bayesian posterior means. The standard deviations of the posterior distributions are in parentheses.

|       | Size and B/M | Size and momentum |
|-------|--------------|-------------------|
| 2011-2015 | 2016-2020 | 2011-2015 | 2016-2020 |
| φ     | -0.1599  | -0.1492   | -0.1495  | -0.2399  |
|       | (0.0178) | (0.0238) | (0.0367) | (0.0209) |

Table 12: Parameter φ estimated by the Bayesian posterior means. The standard deviations of the posterior distributions are in parentheses.
A.3 Motivation of the EM Algorithm

In this subsection, we shall review the motivation of the EM algorithm. The log-likelihood of \( Y_{1:T} \) is

\[
\ell(Y_{1:T}; \theta) = \log p(Y_{1:T}; \theta) = \log \int p(Y_{1:T}|\Xi; \theta)p(\Xi; \theta)d\Xi.
\]

Given \( \tilde{\theta}^{(i)} \), we could compute

\[
\ell(Y_{1:T}; \theta) - \ell(Y_{1:T}; \tilde{\theta}^{(i)}) = \log \int \left( \frac{p(\Xi|Y_{1:T}, \tilde{\theta}^{(i)})}{p(\Xi|Y_{1:T}, \tilde{\theta}^{(i)})} \right) p(Y_{1:T}|\Xi; \theta)p(\Xi; \theta)d\Xi - \log p(Y_{1:T}; \tilde{\theta}^{(i)})
\]

\[
= \log \mathbb{E} \left[ \frac{p(Y_{1:T}|\Xi; \theta)p(\Xi; \theta)}{p(\Xi|Y_{1:T}, \tilde{\theta}^{(i)})} \right] - \log p(Y_{1:T}; \tilde{\theta}^{(i)}) = \log \mathbb{E} \left[ \frac{p(Y_{1:T}|\Xi; \theta)p(\Xi; \theta)}{p(\Xi|Y_{1:T}, \tilde{\theta}^{(i)})} \right] - \mathbb{E} \left[ \log p(Y_{1:T}; \tilde{\theta}^{(i)}) \right]
\]

\[
\geq \mathbb{E} \left[ \log \frac{p(Y_{1:T}|\Xi; \theta)p(\Xi; \theta)}{p(\Xi|Y_{1:T}, \tilde{\theta}^{(i)})} \right] - \mathbb{E} \left[ \log p(Y_{1:T}; \tilde{\theta}^{(i)}) \right] = \mathbb{E} \left[ \log \frac{p(Y_{1:T}|\Xi; \theta)p(\Xi; \theta)}{p(\Xi|Y_{1:T}, \tilde{\theta}^{(i)})} \right] - \mathbb{E} \left[ \log p(Y_{1:T}; \tilde{\theta}^{(i)}) \right]
\]

whence we have

\[
\ell(Y_{1:T}; \theta) \geq \ell(Y_{1:T}; \tilde{\theta}^{(i)}) + \mathbb{E} \left[ \log \frac{p(Y_{1:T}|\Xi; \theta)p(\Xi; \theta)}{p(\Xi|Y_{1:T}, \tilde{\theta}^{(i)})} \right] =: B(\theta, \tilde{\theta}^{(i)}).
\]

We see that \( B(\theta, \tilde{\theta}^{(i)}) \) is a lower bound for \( \ell(Y_{1:T}; \theta) \), and \( \ell(Y_{1:T}; \tilde{\theta}^{(i)}) = B(\tilde{\theta}^{(i)}, \tilde{\theta}^{(i)}) \). Thus we would like to choose \( \theta \) to maximise \( B(\theta, \tilde{\theta}^{(i)}) \):

\[
\tilde{\theta}^{(i+1)} = \arg \max_{\theta} B(\theta, \tilde{\theta}^{(i)}) = \arg \max_{\theta} \mathbb{E} \left[ \ell(Y_{1:T}, \Xi; \theta) \right].
\]

A.4 Formulas for the Kalman Filter and Smoother

In this subsection, we shall give the recursive formulas for the Kalman filter and smoother, which will be used in the Bayesian estimation and EM algorithm, respectively.

A.4.1 The Kalman Filter

Recall the model \((2.2), (2.1)\):

\[
\begin{align*}
y_t &= Z_t \alpha_t + \epsilon_t, \quad \epsilon_t \sim N(0, \Sigma_t) \quad t = 1, \ldots, T, \\
f_{g,t+1} &= \phi f_{g,t} + \eta_{g,t}, \quad |\phi| < 1 \quad t = 0, 1, \ldots, T - 1, \\
f_{C,t+1} &= \eta_{C,t} \quad t = 0, 1, \ldots, T - 1.
\end{align*}
\]

As a result, we could write out a dynamic equation for \( \alpha_t \):

\[
\alpha_{t+1} = \begin{bmatrix} f_{g,t+1} \\ f_{g,t} \\ f_{g,t-1} \\ f_{C,t+1} \end{bmatrix} = \begin{bmatrix} \phi & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{g,t} \\ f_{g,t-1} \\ f_{g,t-2} \\ f_{C,t} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \eta_t =: \mathcal{T} \alpha_t + R \eta_t \quad (A.17)
\]
for $t = 0, 1, \ldots, T - 1$. Let $Y_{1:t} := \{y_t^1, \ldots, y_t^T\}$ and $Y_0 = \emptyset$. Define

\[
\alpha_{t|t-1} := \mathbb{E}[\alpha_t|Y_{1:t-1}] \quad P_{t|t-1} := \text{var}(\alpha_t|Y_{1:t-1}) \\
\alpha_{t|t} := \mathbb{E}[\alpha_t|Y_{1:t}] \quad P_{t|t} := \text{var}(\alpha_t|Y_{1:t}).
\]

Given that $\alpha_0 = 0$, it can be calculated that

\[
\alpha_{1|0} = 0 \quad P_{1|0} = RR^T.
\]

Define

\[
\nu_t := y_t - \mathbb{E}[y_t|Y_{1:t-1}] = y_t - Z_t \alpha_{t|t-1} = Z_t(\alpha_t - \alpha_{t|t-1}) + \varepsilon_t.
\]

That is, $\nu_t$ is the one-step ahead forecast error of $y_t$ given $Y_{1:t-1}$. When $Y_{1:t-1}$ and $\nu_t$ are fixed, then $Y_{1:2}$ is fixed. When $Y_{1:t}$ is fixed, then $Y_{1:t-1}$ and $\nu_t$ are fixed. Thus

\[
\alpha_{t|t} = \mathbb{E}[\alpha_t|Y_{1:t}] = \mathbb{E}[\alpha_t|Y_{1:t-1}, \nu_t] \\
\alpha_{t+1|t} = \mathbb{E}[\alpha_{t+1}|Y_{1:t}, \nu_t] \\
\mathbb{E}[\nu_t|Y_{1:t-1}] = \mathbb{E}[Z_t(\alpha_t - \alpha_{t|t-1}) + \varepsilon_t|Y_{1:t-1}] = 0.
\]

We have

\[
\alpha_t \mid Y_{1:t-1} \sim N\left(\begin{pmatrix} \alpha_{t|t-1} \\ 0 \end{pmatrix}, \begin{pmatrix} P_t & \text{cov}(\alpha_t, \nu_t|Y_{1:t-1}) \\ \text{cov}(\alpha_t, \nu_t|Y_{1:t-1})^T & F_t \end{pmatrix}\right)
\]

where

\[
F_t := \text{var}(\nu_t|Y_{1:t-1}) = \text{var}(Z_t(\alpha_t - \alpha_{t|t-1}) + \varepsilon_t|Y_{1:t-1}) = Z_t P_t Z_t^\top + \Sigma_t \\
\text{cov}(\alpha_t, \nu_t|Y_{1:t-1}) = \mathbb{E}[\alpha_t \nu_t^T|Y_{1:t-1}] = \mathbb{E}\left[Z_t(\alpha_t - \alpha_{t|t-1}) + \varepsilon_t|Y_{1:t-1}\right] = P_t Z_t.
\]

Thus invoking a lemma on multivariate normal, we have

\[
\alpha_{t|t} = \alpha_{t|t-1} + P_t Z_t^\top F_t^{-1} \nu_t \\
P_{t|t} = P_t - P_t Z_t^\top F_t^{-1} Z_t P_t.
\]

(A.18)

We now develop the recursions for $\alpha_{t+1|t}$ and $P_{t+1}$.

\[
\alpha_{t+1|t} = \mathbb{E}[\alpha_{t+1}|Y_{1:t}] = \mathbb{E}[\mathcal{T} \alpha_t + R \eta_t|Y_{1:t}] = \mathcal{T} \alpha_{t|t} \\
P_{t+1} = \text{var}(\alpha_{t+1}|Y_{1:t}) = \text{var}(\mathcal{T} \alpha_t + R \eta_t|Y_{1:t}) = \mathcal{T} P_{t|t} \mathcal{T}^\top + RR^T
\]

for $t = 1, \ldots, T - 1$.

**A.4.2 The Kalman Smoother**

We present the formulas for the Kalman smoother here. For the derivations, see Durbin and Koopman (2012).

\[
L_t := \mathcal{T} - (\mathcal{T} P_t \mathcal{T}^\top F_t^{-1}) Z_t \\
\varsigma_{t-1} := Z_t^\top F_t^{-1} \nu_t + L_t \varsigma_t \\
N_{t-1} := Z_t^\top F_t^{-1} Z_t + L_t^\top N_t L_t \\
\alpha_{t|T} := \mathbb{E}[\alpha_t|Y_{1:T}] = \alpha_{t|t-1} + P_t \varsigma_{t-1} \\
P_{t|T} := \text{var}(\alpha_t|Y_{1:T}) = P_t - P_t N_{t-1} P_t \\
\text{cov}(\alpha_t, \alpha_{t+1}|Y_{1:T}) = P_t L_t^\top (I_4 - N_t P_{t+1});
\]

for $t = T, \ldots, 1$, initialised with $\varsigma_T = 0$ and $N_T = 0$. 

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Table 13: Letters $A, E, U$ denote the times when we should theoretically observe the closing prices of stocks in the Asian continent (A), European continent (E), and American continent (U), respectively. Status Trade means that the stock market opens normally, while status NA means that it is a time when the stock market closes because of non-synchronised trading; status Closure means that the stock market does not open for trading because of public holidays, while status Re-open means the stock market re-opens after public holidays.

### A.5 Missing Because of Continent-Specific Reasons

In this subsection, we discuss how to alter the EM algorithm if we include the scenario of missing observations due to continent-specific reasons such as continent-wide public holidays (e.g., Chinese New Year).

Suppose that continent $c$ last traded at $t = t_1 - 3$, did not open for trading at $t = t_1, t_1 + 3, \ldots, t_1 + 3(\tau - 1)$, and re-opened for trading at $t_1 + 3\tau$, where $\tau$ is some integer $\geq 1$. There are actually four statuses for this continent at a particular $t$: Trade, NA, Closure and Re-open. Status Trade means that the stock market opens normally (e.g., $t = t_1 - 3$), while status NA means that it is a time when the stock market closes because of non-synchronised trading (e.g., $t = t_1 - 2$); status Closure means that the stock market does not open for trading because of public holidays (e.g., $t = t_1$), while status Re-open means the stock market re-opens after public holidays (e.g., $t = t_1 + 3\tau$).

Define $y_t^*$ such that

$$y_t^* = \begin{cases} y_t & \text{if Status}_t = \text{Trade or NA} \\ \varepsilon_t^* & \text{if Status}_t = \text{Closure} \\ \sum_{i=0}^{\tau-1} y_{t_1+3i} & \text{if Status}_t = \text{Re-open} \end{cases}$$

(A.19)

where $\varepsilon_t^*$ is to be defined shortly. The idea is that $y_{t_1+3\tau}^*$ is the actual observed returns on $t = t_1 + 3\tau$. We now give a concrete example to illustrate this. Suppose that the European continent did not open for trading on $t = 32, 35$ because of some public holiday. Then the values of $y_t^*$ are given in Table 13.

Recall (2.2) and (A.17):

$$y_t = Z_t \alpha_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, \Sigma_t)$$

$$\alpha_{t+1} = T_t \alpha_t + R_t \eta_t, \quad \eta_t \sim N(0, I_2).$$

We now write down the state space model for $y_t^*$.

$$y_t^* = Z_t^* \alpha_t^* + \varepsilon_t^*, \quad \varepsilon_t \sim N(0, \Sigma_t^*)$$

$$\alpha_{t+1}^* = T_t^* \alpha_t^* + R_t^* \eta_t, \quad \eta_t \sim N(0, I_2).$$

(A.20)

Parameters $Z_t^*, \alpha_t^*, \Sigma_t^*, T_t^*, R_t^*$ should be chosen in such a way that (A.20) is consistent
with (A.19). In our example, $Z^*_t, \alpha^*_t, \Sigma^*_t, T^*_t, R^*_t$ take the following values:

$$\alpha^*_t = \alpha_t \quad t = 29, 30, 31, 32, 39$$

$$\alpha^*_{33} = \begin{bmatrix} \alpha_{33} \\ 0_{4 \times 1} \\ \alpha_{32} \\ 0_{4 \times 1} \end{bmatrix}, \alpha^*_{34} = \begin{bmatrix} \alpha_{34} \\ 0_{4 \times 1} \\ \alpha_{32} \\ 0_{4 \times 1} \end{bmatrix}, \alpha^*_{35} = \begin{bmatrix} \alpha_{35} \\ 0_{4 \times 1} \\ \alpha_{32} \\ 0_{4 \times 1} \end{bmatrix}$$

$$\alpha^*_{36} = \begin{bmatrix} \alpha_{36} \\ 0_{4 \times 1} \\ \alpha_{32} + \alpha_{35} \\ 0_{4 \times 1} \end{bmatrix}, \alpha^*_{37} = \begin{bmatrix} \alpha_{37} \\ 0_{4 \times 1} \\ \alpha_{32} + \alpha_{35} \\ 0_{4 \times 1} \end{bmatrix}, \alpha^*_{38} = \begin{bmatrix} \alpha_{38} \\ 0_{4 \times 1} \\ \alpha_{32} + \alpha_{35} \\ 0_{4 \times 1} \end{bmatrix}.$$

$$Z^*_t = \begin{cases} Z_t & t = 29, 30, 31, 39 \\ 0_{N \times 4} & t = 32 \\ 0_{N \times 16} & t = 35 \\ \begin{bmatrix} Z_t & 0_{4 \times 12} \\ Z_t & 0_{4 \times 4} \end{bmatrix} & t = 33, 34, 36, 37 \\ \end{cases}$$

$$T^*_t = \begin{cases} T & t = 29, 30, 31, 39 \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_4 & 0 \\ 0 & 0 & I_4 \end{bmatrix} & t = 33, 34, 36, 37 \\ \begin{bmatrix} T & 0 & 0 \\ 0 & I_4 & 0 \\ 0 & 0 & I_4 \end{bmatrix} & t = 33, 34, 36, 37 \\ \end{cases}$$

$$\Sigma^*_t = \begin{cases} \Sigma_t & t \notin \{32, 35, 38\} \\ 0.01 & t \in \{32, 35\} \\ 3\Sigma_t & t = 38 \end{cases}$$

$$R^*_t = \begin{cases} R & t \notin \{32, 33, \ldots, 37\} \\ \begin{bmatrix} R \\ 0_{12 \times 4} \end{bmatrix} & t \in \{32, 33, \ldots, 37\} \end{cases}.$$

The main idea of the state space representation for $y^*_t$ is to extend the state variable from dimension 4 to dimension 16 when needed. The first 4 elements of $\alpha^*_t$ are $\alpha_t$, while the other elements of $\alpha^*_t$ are used to store the states that accumulated during the public holiday.\(^7\)

\(^7\)The three continents in our model may have different but overlapping periods of public holidays. One alternative way is to set the dimension of $\alpha^*_t$ to 16 for all $t$. We could then use the 5th - 8th, 9th - 12th, 13th - 16th elements of $\alpha^*_t$ to store the Asian, European, and American accumulated states, respectively for the whole sample. The disadvantage of this treatment is that the KF and KS will be inefficient since the last 12 elements of $\alpha^*_t$ would often be zero. Hence, instead of keeping the dimension of $\alpha^*_t$ to be 16 for all $t$, we only extend the dimension to 16 when needed.
We shall write $Z^*_t = Z_t A_t$, where

$$
A_t = \begin{cases} 
0_{4 \times 1} & \text{if Status}_t = \text{Closure} \\
0_{4 \times 12} \sum_{j=1}^3 \omega_{t-j} & \text{if Status}_t = \text{Re-open} \\
I_4 & \text{if Status}_t = \text{NA} \text{ and } (\omega_{t-1} = 1 \text{ or } \omega_{t-2} = 1) \\
Z_t & \text{otherwise} \\
\end{cases}
$$

Similarly, we have

$$
\Sigma^*_t = \begin{cases} 
\Sigma_t & \text{Status}_t \in \{\text{NA, Trade}\} \\
0.01 & \text{Status}_t = \text{Closure} \\
3\Sigma_t & \text{Status}_t = \text{Re-open} \\
\end{cases}
$$

$$
R^*_t = \begin{cases} 
R & \text{if Status}_t = \text{Closure} \\
0_{12 \times 1} & \text{if Status}_t = \text{NA} \text{ and } (\omega_{t-1} = 1 \text{ or } \omega_{t-2} = 1) \\
R & \text{otherwise} \\
\end{cases}
$$

$$
T^*_t = [\omega_t \cdot v_t^T \otimes I_4 \sum_{j=0}^2 \omega_{t-j} I_4 \sum_{j=0}^2 \omega_{t-j} \sum_{j=1}^3 \omega_{t-j}].
$$

The EM algorithm outlined before could then be applied to the state space model of $y^*_t$. The principle remains unchanged. Again we use the European continent to illustrate and suppose that its stock market does not open for $\tau + 1$ days because of some public holiday. Equation (3.2) takes the following form:

$$
\hat{E} \sum_{t \in T_E} \ell_{1,t} = \sum_{t \in T_E} \log |\Sigma^*_t| + \sum_{t \in T_E} \text{tr} \left( \left[ y^*_t (y^*_t)^\top - 2Z^*_t \hat{E}[\alpha^*_t] (y^*_t)^\top + Z^*_t \hat{E}[\alpha^*_t \alpha^*_t\top] Z_t^\top \right] \Sigma_t^{-1} \right)
$$

$$
= \sum_{t \in T_E} \left\{ \log |\Sigma_t(\tau_t + 1)| + \text{tr} \left( \left[ y^*_t (y^*_t)^\top - 2Z_t \hat{E}[\alpha^*_t] (y^*_t)^\top + Z_t \hat{E}[\alpha^*_t \alpha^*_t\top] A_t^\top Z_t^\top \right] \Sigma_t^{-1} \right) \right\}
$$

$$
= \sum_{t \in T_E} \text{tr} \left( \left[ y^*_t (y^*_t)^\top \frac{(1 - \omega_t)}{(\tau_t + 1)} - 2Z_t A_t \hat{E}[\alpha^*_t] A_t^\top \frac{(1 - \omega_t)}{(\tau_t + 1)} + Z_t \hat{E}[\alpha^*_t \alpha^*_t\top] A_t^\top Z_t^\top \frac{(1 - \omega_t)}{(\tau_t + 1)} \right] \Sigma_t^{-1} \right)
$$

$$
+ \sum_{t \in T_E} (1 - \omega_t) \log |\Sigma_t| + \text{constant},
$$

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where \( \tau_t = \tau \) if Status\(_t = \text{Re-open} \) and \( \tau_t = 0 \) if otherwise, and \( \tau \) is the number of days of the closure due to the public holiday. That is, the terms \( y_{1t}^* y_{1t}^T (1 - \omega_t) \) in \( (3.2) \) are replaced by \( y_{1t}^* y_{1t}^T (1 - \omega_t) \), where \( \omega_t = \omega \). Hence, we have

\[
\hat{Z}^E = \sum_{t \in T_E} \left( \hat{\mathbb{E}}[y_{1t}^* \alpha_t^\top] A_t^T \left( 1 - \omega_t \right) \left( \sum_{t \in T_E} A_t \hat{\mathbb{E}}[\alpha_t^* \alpha_t^\top] A_t^T \left( 1 - \omega_t \right) \right)^{-1} \right)
\]

\[
\hat{C}_E := \sum_{t \in T_E} \frac{1}{\tau_t + 1} \left[ y_{1t}^* y_{1t}^T - 2 \hat{Z}^E A_t \hat{\mathbb{E}}[\alpha_t^*] y_{1t}^T + \hat{Z}^E A_t \hat{\mathbb{E}}[\alpha_t^* \alpha_t^\top] A_t^T \hat{Z}^E \right]
\]

\[
\hat{\Sigma}_E = \frac{1}{\sum_{t \in T_E} (1 - \omega_t)} (\hat{C}_E \circ I_{N_E}).
\]

The formula of \( \hat{\phi} \) in \( (3.4) \) remains unchanged, since \( \hat{\mathbb{E}} \sum_{t=1}^{T} \ell_{2,t} \) remains unchanged.

### A.6 First-Order Conditions of (4.4)

In this subsection, we derive (4.5). Note that we only utilise information that \( M \) is symmetric, positive definite and that \( \Sigma_{ee} \) is diagonal to derive the first-order conditions; no specific knowledge of \( \Lambda \) or \( M \) is utilised to derive the first-order conditions. Cholesky decompose \( M: M = LL^\top \), where \( L \) is the unique lower triangular matrix with positive diagonal entries. Thus

\[
\Sigma_{yy} = \Lambda M \Lambda^\top + \Sigma_{ee} = \Lambda L L^\top \Lambda^\top + \Sigma_{ee} = BB^\top + \Sigma_{ee},
\]

where \( B := \Lambda L \). Recall the log-likelihood function (4.4) omitting the constant:

\[
-\frac{1}{2N} \log |\Sigma_{yy}| - \frac{1}{2N} \text{tr}(S_{yy}^{-1}) = -\frac{1}{2N} \log |BB^\top + \Sigma_{ee}| - \frac{1}{2N} \text{tr} \left( S_{yy} \left[ BB^\top + \Sigma_{ee} \right]^{-1} \right).
\]

Take the derivatives of the preceding display with respect to \( B \) and \( \Sigma_{ee} \). The FOC of \( \Sigma_{ee} \) is:

\[
\text{diag}(\hat{\Sigma}_{yy}^{-1}) = \text{diag}(\Sigma_{yy}^{-1} S_{yy} - \hat{\Sigma}_{yy}),
\]

where \( \hat{\Sigma}_{yy} := \hat{B} B^\top + \hat{\Sigma}_{ee} \). The FOC of \( B \) is

\[
\hat{B}^\top \Sigma_{yy}^{-1} (S_{yy} - \hat{\Sigma}_{yy}) = 0.
\]

Note that (A.22, A.23) has 6\( N(14 + 1) \) equations, while \( B, \Sigma_{ee} \) has 6\( N(14 + 1) \) parameters. Thus \( \hat{B}, \hat{\Sigma}_{ee} \) can be uniquely solved. Then we need identification conditions to kick in. Even though we could uniquely determine \( \hat{B} \), we cannot uniquely determine \( \hat{\Lambda}, \hat{M} \). This is because

\[
\hat{B} B^\top = \hat{\Lambda} \hat{M} \hat{\Lambda}^\top = \hat{\Lambda} \hat{M} \hat{\Lambda}^\top = \hat{\Lambda} C^{-1} \hat{M} (C^{-1})^\top C^\top \hat{\Lambda}^\top
\]

where \( \hat{\Lambda} := \hat{\Lambda} C \) and \( \hat{M} := C^{-1} \hat{M} (C^{-1})^\top \) for any \( 14 \times 14 \) invertible \( C \).\(^8\) We hence need to impose \( 14^2 \) identification restrictions on the estimates of \( \Lambda \) and \( M \) to rule out the rotational indeterminacy. After imposing the \( 14^2 \) restrictions, we obtain the unique estimates, say, \( \hat{\Lambda}, \hat{M} \) (and hence \( \hat{L} \)). Substituting \( \hat{B} = \hat{\Lambda} \hat{L} \) into (A.23), we have

\[
\hat{\Lambda}^\top \Sigma_{yy}^{-1} (S_{yy} - \hat{\Sigma}_{yy}) = 0.
\]

\(^8\)Note that we could find at least one pair \((\hat{\Lambda}, \hat{M})\) satisfying \( \hat{B} B^\top = \hat{\Lambda} \hat{M} \hat{\Lambda}^\top \): \( \hat{\Lambda} = \hat{B} \) and \( \hat{M} = I_{14} \).
A.7 Proof of Proposition 4.1

As Bai and Li (2012) did, we use a superscript "*" to denote the true parameters, $\Lambda^*, \Sigma_{ee}, M^*$ etc. The parameters without the superscript "*" denote the generic parameters in the likelihood function. Note that the proof of (4.7) is exactly the same as that of Bai and Li (2012), so we omit the details here.

Define

$$
\dot{H} := (\hat{\Lambda}^\top \hat{\Sigma}_{ee}^{-1} \hat{\Lambda})^{-1} \\
A := (\hat{\Lambda} - \Lambda^*)^\top \hat{\Sigma}_{ee}^{-1} \hat{\Lambda} \dot{H} \\
K := \hat{M}^{-1}(MA - A^\top MA).
$$

(A.24)

Our assumptions satisfy those of Bai and Li (2012), so (A17) of Bai and Li (2012) still holds (in our notation):

$$
\dot{\lambda}_{k,j} - \lambda^*_{k,j} = K \lambda^*_{k,j} + o_p(1),
$$

(A.25)

for $k = 1, \ldots, 6$ and $j = 1, \ldots, N$. As mentioned before, $\Lambda^*$ (i.e., $\{\lambda^*_{k,j} : k = 1, \ldots, 6, j = 1, \ldots, N\}$) defined (4.2) gives more than $14^2$ restrictions, but in order to utilise the theories of Bai and Li (2012) we shall only impose $14^2$ restrictions on $\{\lambda^*_{k,j} : k = 1, \ldots, 6, j = 1, \ldots, N\}$. How to select these $14^2$ restrictions from those implied by $\{\lambda^*_{k,j}\}$ are crucial because we cannot afford imposing a restriction which is not instrumental for the proofs later. The idea is that one restriction should pin down one free parameter in $K$. We shall now explain our procedure. Write (A.25) in matrix form:

$$
\dot{\Lambda}_k - \Lambda^*_k = K \lambda^*_k + o_p(1),
$$

(A.26)

where $\dot{\Lambda}_k := (\dot{\lambda}_{k,1}, \ldots, \dot{\lambda}_{k,N})$ and $\Lambda^*_k := (\lambda^*_{k,1}, \ldots, \lambda^*_{k,N})$ are $14 \times N$ matrices. For a generic matrix $C$, let $C_{x,y}$ denote the matrix obtained by intersecting the rows and columns whose indices are in $x$ and $y$, respectively; let $C_{x,*}$ denote the matrix obtained by extracting the rows whose indices are in $x$ while $C_{*,y}$ denote the matrix obtained by extracting the columns whose indices are in $y$.

A.7.1 Step I Impose Some Zero Restrictions in $\{\Lambda^*_k\}_{k=1}^6$

Let $a \subset \{1, 2, \ldots, 14\}$ and $c \subset \{1, \ldots, N\}$ be two vectors of indices, whose identities vary from place to place. From (A.26), we have

$$
(\dot{\Lambda}_k - \Lambda^*_k)_{a,c} = K_{a,*} \lambda^*_{k,*c} + o_p(1) = K_{a,b} \lambda^*_{k,b,c} + K_{a,-b} \lambda^*_{k,-b,c} + o_p(1) = K_{a,b} \lambda^*_{k,b,c} + o_p(1)
$$

(A.27)

$$
= K_{a,b_1} \lambda^*_{k,b_1,c} + K_{a,b_2} \lambda^*_{k,b_2,c} + o_p(1)
$$

where $b \subset \{1, 2, \ldots, 14\}$ is chosen in such a way such that $\lambda^*_{k,-b,c} = 0$ for each of the steps below. $-b$ denotes the complement of $b$, and $b_1 \cup b_2 = b$ with the cardinality of $b_2$ equal to the cardinality of $c$.

In each of the sub-step of step I, we shall impose $\dot{\lambda}_{k,a,c} = 0$. Step I is detailed in Table 14, and we shall use step I.1 to illustrate. For step I.1, $\dot{\lambda}_{k,a,c} = 0$ means

$$
\begin{bmatrix}
\dot{\lambda}^T_{1,1} \\
\dot{\lambda}^T_{1,2} \\
\dot{\lambda}^T_{1,3} \\
\dot{\lambda}^T_{1,4}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & - & - & - & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & - & - & - & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & - & - & - & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & - & - & - & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

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| Step | k | c       | a               | b₁   | b₂       |
|------|---|---------|-----------------|------|----------|
| I.1  | 1 | {1,2,3,4} | {1,2,3,4,5,9,10,11,12,13} | ∅    | {6,7,8,14} |
| I.2  | 6 | {1,2,3,4} | {4,5,6,7,8,10,11,12,13,14} | ∅    | {1,2,3,9}  |
| I.3  | 4 | {1,2,3}   | {6,7,8,10,12,13,14}       | 3    | {4,5,11}  |
| I.4  | 4 | {1,2,3,4} | {1,2,9}           | ∅    | {3,4,5,11} |
| I.5  | 3 | {1,2,3}   | 3                | 6    | {4,5,12}  |
| I.6  | 3 | {1,2}     | {7,8,14}         | {4,5} | {6,12}    |
| I.7  | 2 | {1,2}     | {4,11}           | {6,7} | {5,13}    |
| I.8  | 2 | {1,2}     | {8,14}           | {5,6} | {7,13}    |
| I.9  | 5 | {1,2}     | {1,9}            | {3,4} | {2,10}    |
| I.10 | 5 | {1,2}     | {5}              | {2,3} | {4,10}    |
| I.11 | 5 | {1,2}     | {11}             | {2,3} | {4,10}    |
| I.12 | 5 | 1         | {6,7,8,12,13,14}  | {2,3,4} | 10       |
| I.13 | 3 | 1         | {1,2,9,10,11,13}  | {4,5,6} | 12       |
| I.14 | 2 | 1         | {1,2,3,9,10,12}   | {5,6,7} | 13       |

Table 14: Step I.

This means (A.27) holds with LHS being \( \hat{\Lambda}_k - \Lambda_k \)\(a,c\) = 0, where

\[
k = 1, \quad c = \{1, 2, 3, 4\}, \quad a = \{1, 2, 3, 4, 5, 9, 10, 11, 12, 13\}, \quad b_1 = \emptyset, \quad b_2 = \{6, 7, 8, 14\}.
\]

Note that \(c = \{1, 2, 3, 4\}\) is arbitrary and could be replaced with any other \(c \subseteq \{1, \ldots, N\}\) with cardinality being 4. The crucial point is that \(c\) needs to be chosen such that \(\Lambda_{k,b_2,c}\) is invertible. This is an innocuous requirement given large \(N\), so we shall make this assumption implicitly for the rest of the article. Solving (A.27) gives \(K_{a,b_2} = o_p(1)\).

### A.7.2 Step II Impose Some Equality Restrictions in \(\{\Lambda_k^e\}_{k=1}^6\)

#### (II.1) Note that (A.26) implies

\[
\hat{\Lambda}_{6,9,c} - \Lambda_{6,9,c}^e = K_{9,x} \Lambda_{6,x,c}^e + o_p(1) = K_{9,1} \Lambda_{6,1,c}^e + K_{9,9} \Lambda_{9,9,c}^e + o_p(1) \quad x \in \{1, 2, 3, 9\}
\]

\[
\hat{\Lambda}_{3,12,c} - \Lambda_{3,12,c}^e = K_{12,y} \Lambda_{3,y,c}^e + o_p(1) = K_{12,12} \Lambda_{3,12,c}^e + o_p(1) \quad y \in \{4, 5, 6, 12\}.
\]

We then impose \(\hat{\Lambda}_{6,9,c} = \hat{\Lambda}_{3,12,c}\) for \(c = \{1, 2\}\). The preceding display implies

\[
K_{9,1} \Lambda_{6,1,c}^e + (K_{9,9} - K_{12,12}) \Lambda_{6,9,c}^e = o_p(1)
\]

whence we have \(K_{9,1} = o_p(1)\) and \(K_{9,9} - K_{12,12} = o_p(1)\).

#### (II.2) We impose \(\hat{\Lambda}_{4,11,c} = \hat{\Lambda}_{1,14,c}\) for \(c = \{1, 2\}\). Repeating the procedure in step II.1, we have \(K_{14,8} = o_p(1)\) and \(K_{11,11} - K_{14,14} = o_p(1)\) in the same way.

#### (II.3) We impose \(\hat{\Lambda}_{6,1,c} = \hat{\Lambda}_{3,4,c}\) for \(c = \{1, 2\}\), and have \(K_{1,1} - K_{4,4} = o_p(1)\), \(K_{1,9} - K_{4,12} = o_p(1)\).

#### (II.4) We impose \(\hat{\Lambda}_{6,2,c} = \hat{\Lambda}_{3,5,c}\) for \(c = \{1, 2, 3\}\), and have \(K_{2,1} = o_p(1)\), \(K_{2,2} - K_{5,5} = o_p(1)\), \(K_{2,9} - K_{5,12} = o_p(1)\).
(II.5) We impose \( \hat{\Lambda}_{a,3,c} = \hat{\Lambda}_{3,6,c} \) for \( c = \{1, 2, 3, 4\} \), and have \( K_{3,1} = o_p(1), K_{3,2} = o_p(1), K_{3,3} - K_{6,6} = o_p(1), K_{3,9} - K_{6,12} = o_p(1) \).

(II.6) We impose \( \hat{\Lambda}_{1,8,c} = \hat{\Lambda}_{4,5,c} \) for \( c = \{1, 2\} \), and have \( K_{8,8} - K_{5,5} = o_p(1), K_{8,14} - K_{5,11} = o_p(1) \).

(II.7) We impose \( \hat{\Lambda}_{1,7,c} = \hat{\Lambda}_{4,4,c} \) for \( c = \{1, 2, 3\} \), and have \( K_{7,8} = o_p(1), K_{7,7} - K_{4,4} = o_p(1), K_{7,14} - K_{4,11} = o_p(1) \).

(II.8) We impose \( \hat{\Lambda}_{1,6,c} = \hat{\Lambda}_{4,3,c} \) for \( c = \{1, 2, 3\} \), and have \( K_{6,7} = o_p(1), K_{6,8} = o_p(1), K_{6,14} - K_{3,11} = o_p(1) \).

### A.7.3 Step III: Impose Some Restrictions in \( M^* \)

After steps I and II, \( K \) is reduced to

\[
K = \begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} \\ 0 & \bar{K}_{22} \end{bmatrix} + o_p(1),
\]

where

\[
\bar{K}_{11} = \begin{bmatrix} K_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{1,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_{2,2} & 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
\bar{K}_{12} = \begin{bmatrix} K_{4,12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{5,12} & K_{2,10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{6,12} & K_{3,10} & K_{3,11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{4,10} & K_{4,11} & K_{4,12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{5,11} & K_{5,12} & K_{5,13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{6,12} & K_{6,13} & K_{3,11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{7,13} & K_{4,11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{5,11} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \bar{K}_{22} = \begin{bmatrix} K_{12,12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{10,10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{11,11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{12,12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{13,13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{11,11} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]

In the paragraph above (A16) of Bai and Li (2012), they showed \( A = O_p(1) \). Given Assumption 4.1(iii), we have \( K = O_p(1) \). Next, (A16) of Bai and Li (2012) still holds and could be written as

\[
\hat{M} - (I_{14} - A^T)M^*(I_{14} - A) = o_p(1).
\]  

(A.29)

Since \( M^* \) and \( \hat{M} \) are of full rank (Assumption 4.1(iii)), (A.29) implies that \( I_{14} - A \) is of full rank. Write (A.29) as

\[
\hat{M}(K + I_{14}) - (I_{14} - A^T)M^* = o_p(1).
\]

As Bai and Li (2012) did in their (A20), we could premultiply the preceding display by \([(I_{14} - A^T)M^*]^{-1}\) to arrive at (after some algebra and relying on (A.29))

\[
(I_{14} - A)(K + I_{14}) - I_{14} = o_p(1).
\]

(A.30)
Likewise, partition $A$ into $8 \times 8, 8 \times 6, 6 \times 8$ and $6 \times 6$ submatrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. $$

Then (A.30) could be written into

$$
(I_8 - A_{11})(\overline{K}_{11} + I_8) - I_8 = o_p(1) \tag{A.31}
$$

$$
(I_8 - A_{11})\overline{K}_{12} - A_{12}(\overline{K}_{22} + I_6) = o_p(1) \tag{A.32}
$$

$$
-A_{21}(\overline{K}_{11} + I_8) = o_p(1) \tag{A.33}
$$

$$
-A_{21}\overline{K}_{12} + (I_6 - A_{22})(\overline{K}_{22} + I_6) - I_6 = o_p(1). \tag{A.34}
$$

Consider (A.31) first. Since $I_8 + \overline{K}_{11}$ is diagonal, we deduce that the diagonal elements of $I_8 + \overline{K}_{11}$ could not converge to 0, and $A_{11}$ converges to a diagonal matrix. Using the fact that the diagonal elements of $I_8 + \overline{K}_{11}$ could not converge to 0, (A.33) implies

$$A_{21} = o_p(1),$$

and $A_{21}\overline{K}_{12} = o_p(1)O_p(1) = o_p(1)$. Then (A.34) is reduced to

$$
(I_6 - A_{22})(\overline{K}_{22} + I_6) - I_6 = o_p(1).
$$

Since $\overline{K}_{22} + I_6$ is diagonal, we deduce that the diagonal elements of $\overline{K}_{22} + I_6$ could not converge to 0, and $A_{22}$ should converge to a diagonal matrix as well. To sum up

$$I_8 + \overline{K}_{11} \quad I_8 - A_{11} \quad I_6 + \overline{K}_{22} \quad I_6 - A_{22}
$$

are diagonal or diagonal in the limit, and invertible in the limit (i.e., none of the diagonal elements is zero in the limit). Moreover, (A.31) implies

$$
(I_8 + \overline{K}_{11})^{-1} = (I_8 - A_{11}) + o_p(1). \tag{A.35}
$$

Via (A.30), (A.29) implies

$$
(I_{14} + K^T)\hat{M}(I_{14} + K) - M^* = o_p(1). \tag{A.36}
$$

Partition $\hat{M}$ into $8 \times 8, 8 \times 6, 6 \times 8$ and $6 \times 6$ submatrices:

$$\hat{M} = \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix}. $$

Then (A.36) could be written as

$$
(I_8 + \overline{K}_{11})\hat{M}_{11}(I_8 + \overline{K}_{11}) - \Phi^* = o_p(1) \tag{A.37}
$$

$$
(I_8 + \overline{K}_{11})\hat{M}_{12} + (I_8 + \overline{K}_{11})\hat{M}_{12}(I_6 + \overline{K}_{22}) = o_p(1) \tag{A.38}
$$

$$
[\overline{K}_{12}\hat{M}_{11} + (I_6 + \overline{K}_{22})(\hat{M}_{21})] (\overline{K}_{11} + I_8) = o_p(1) \tag{A.39}
$$

$$
\overline{K}_{12} (\hat{M}_{12}(I_6 + \overline{K}_{22}) + \hat{M}_{11}\overline{K}_{12}) + (I_6 + \overline{K}_{22}) (\hat{M}_{22}(I_6 + \overline{K}_{22}) + \hat{M}_{21}\overline{K}_{12}) - I_6 = o_p(1). \tag{A.40}
$$
Step III.1 Considering (A.37), we have
\[
(1 + K_{3,3})^2 \hat{M}_{6,6} = \frac{1}{1 - \phi^*} + o_p(1)
\]
\[
(1 + K_{1,1})^2 \hat{M}_{4,4} = \frac{1}{1 - \phi^*} + o_p(1)
\]
\[
(1 + K_{2,2})^2 \hat{M}_{5,5} = \frac{1}{1 - \phi^*} + o_p(1).
\]
Imposing $\hat{M}_{4,4} = \hat{M}_{6,6}$, we have
\[
\hat{M}_{4,4} [(1 + K_{1,1})^2 - (1 + K_{3,3})^2] = o_p(1).
\]
Since (A.37) implies that $\hat{M}_{4,4} \neq o_p(1)$. The preceding display implies
\[
K_{3,3} = K_{1,1} + o_p(1), \quad \text{or} \quad 1 + K_{3,3} = -(1 + K_{1,1}) + o_p(1).
\]
Likewise, imposing $\hat{M}_{4,4} = \hat{M}_{5,5}$, we have
\[
K_{2,2} = K_{1,1} + o_p(1), \quad \text{or} \quad 1 + K_{2,2} = -(1 + K_{1,1}) + o_p(1).
\]
Thus, there are four cases:
(a) $K_{2,2} = K_{1,1} + o_p(1)$ and $K_{3,3} = K_{1,1} + o_p(1)$
(b) $K_{2,2} = K_{1,1} + o_p(1)$ and $1 + K_{3,3} = -(1 + K_{1,1}) + o_p(1)$
(c) $1 + K_{2,2} = -(1 + K_{1,1}) + o_p(1)$ and $K_{3,3} = K_{1,1} + o_p(1)$
(d) $1 + K_{2,2} = -(1 + K_{1,1}) + o_p(1)$ and $1 + K_{3,3} = -(1 + K_{1,1}) + o_p(1)$

Irrespective of case, (A.37) is reduced to $(1 + K_{1,1})^2 \hat{M}_{11} = \Phi^* + o_p(1)$, whence we have
\[
(1 + K_{1,1})^2 \hat{M}_{4,4} = \frac{1}{1 - \phi^*} + o_p(1)
\]
\[
(1 + K_{1,1})^2 \hat{M}_{6,6} = \frac{\phi^*}{1 - \phi^*} + o_p(1).
\]
Imposing $\hat{M}_{4,4} - \hat{M}_{6,6} = 1$, we have
\[
(1 + K_{1,1})^2 = 1 + o_p(1)
\]
whence we have $K_{1,1} = o_p(1)$ or $1 + K_{1,1} = -1 + o_p(1)$. Suppose that $1 + K_{1,1} = -1 + o_p(1)$. Then (A.35) implies $(A_{11})_{1,1} = 2 + o_p(1)$. Note that the identification scheme which we employ in Proposition 4.1 only identifies $\Lambda^*$ up to a column sign change. Thus by assuming that $\hat{\Lambda}$ and $\Lambda^*$ have the same column signs, we can rule out the case $(A_{11})_{1,1} = 2 + o_p(1)$ (Bai and Li (2012, p.445, p.463)). Thus we have $K_{1,1} = o_p(1)$ and hence rule out cases (b)-(d). To sum up, we have
\[
K_{1,1} = o_p(1), \quad \hat{K}_{11} = o_p(1), \quad \hat{M}_{11} = \Phi^* + o_p(1), \quad A_{11} = o_p(1).
\]
Step III.2 Now (A.38) is reduced to
\[ \hat{M}_{12}(I_6 + K_{22}) = -\Phi^* K_{12} + o_p(1) \]

Impose three more restrictions: Assume the 4th-6th elements of the fourth column of \( \hat{M}_{12} \) are zero; that is \( \hat{M}_{4,12} = \hat{M}_{5,12} = \hat{M}_{6,12} = 0 \). This implies that the corresponding three elements of \( \Phi^* K \) are \( o_p(1) \):

\[
\frac{1}{1 - \phi^2} \begin{bmatrix}
K_{4,12} + \phi^* K_{5,12} + \phi^2 K_{6,12} \\
\phi^* K_{4,12} + K_{5,12} + \phi^* K_{6,12} \\
\phi^2 K_{4,12} + \phi^* K_{5,12} + K_{6,12}
\end{bmatrix} = o_p(1)
\]

whence we have \( K_{4,12} = K_{5,12} = K_{6,12} = o_p(1) \). Similarly, assuming the 2nd-4th elements of the second column of \( \hat{M}_{12} \) are zero, we could deduce that \( K_{2,10} = K_{3,10} = K_{4,10} = o_p(1) \); assuming the 3rd-5th elements of the third column of \( \hat{M}_{12} \) are zero, we could deduce that \( K_{3,11} = K_{4,11} = K_{5,11} = o_p(1) \); assuming \( \hat{M}_{5,13} = \hat{M}_{6,13} = \hat{M}_{7,13} = 0 \), we could deduce that \( K_{5,13} = K_{6,13} = K_{7,13} = o_p(1) \). We hence obtain

\( K_{12} = o_p(1) \).

Step III.3 With \( K_{12} = o_p(1) \), (A.40) is reduced to
\[ (I_6 + K_{22}) \hat{M}_{22}(I_6 + K_{22}) - I_6 = o_p(1). \]

Since \( I_6 + K_{22} \) is diagonal, \( \hat{M}_{22} \) is asymptotically diagonal. Imposing that the 2nd-5th diagonal elements of \( \hat{M}_{22} \) are 1 (i.e., \( \hat{M}_{j,j} = 1 \) for \( j = 10, 11, 12, 13 \)), we have

\[
(1 + K_{10,10})^2 - 1 = o_p(1) \\
(1 + K_{11,11})^2 - 1 = o_p(1) \\
(1 + K_{12,12})^2 - 1 = o_p(1) \\
(1 + K_{13,13})^2 - 1 = o_p(1)
\]

whence we have \( K_{j,j} = o_p(1) \) or \( 1 + K_{j,j} = -1 + o_p(1) \) for \( j = 10, 11, 12, 13 \). Likewise, the case \( 1 + K_{j,j} = -1 + o_p(1) \) is ruled out. Thus

\[ K_{22} = o_p(1), \quad A_{22} = o_p(1), \quad \hat{M}_{22} = I_6 + o_p(1). \]

Then (A.38) and (A.39) imply \( \hat{M}_{12} = o_p(1) \) and \( \hat{M}_{21} = o_p(1) \), respectively. Also (A.32) implies \( A_{12} = o_p(1) \). To sum up, we have

\[ K = o_p(1), \quad A = o_p(1), \quad \hat{M} = M^* + o_p(1). \]

Substituting \( K = o_p(1) \) into (A.25), we have

\[ \hat{\lambda}_{k,j} - \lambda^*_{k,j} = o_p(1), \]

for \( k = 1, \ldots, 6 \) and \( j = 1, \ldots, N \).
A.8 Proof of Theorem 4.1

Given consistency, we can drop the superscript from the true parameters for simplicity. The proof of Theorem 4.1 resembles that of Theorem 5.1 of Bai and Li (2012). Most of the proof of Theorem 5.1 of Bai and Li (2012) is insensitive to the identification condition; the only exception is their Lemma B5. Thus we only need to prove the result of their Lemma B5 under our identification condition. That is, we want to prove

$$MA = O_p(T_f^{-1/2}) + O_p\left(\left[ \frac{1}{6N} \sum_{k=1}^{6} \sum_{j=1}^{N} (\hat{\sigma}_{k,j}^2 - \sigma_{k,j}^2)^2 \right]^{1/2} \right) =: O_p(\hat{\circ}). \quad (A.41)$$

Following the approach of Bai and Li (2012) and using their Lemmas B1, B2, B3, one could show that

$$\hat{\lambda}_{k,j} - \lambda_{k,j} = K\lambda_{k,j} + O_p(\hat{\circ}).$$

Using the same approach we adopted in the proof of Proposition 4.1, we arrive at

$$K = \left[ \begin{array}{cc} \bar{K}_{11} & \bar{K}_{12} \\ 0 & \bar{K}_{22} \end{array} \right] + O_p(\hat{\circ}),$$

where $\bar{K}_{11}, \bar{K}_{12}, \bar{K}_{22}$ are defined in (A.28), and

$$\hat{M} - (I_{14} - A^T)M(I_{14} - A) = O_p(\hat{\circ}) \quad (A.42)$$

$$(I_{14} - A)(K + I_{14}) - I_{14} = O_p(\hat{\circ}) \quad (A.43)$$

Note that $\sqrt{1 + O_p(\hat{\circ})} = 1 + O_p(\hat{\circ})$ and $(1 + O_p(\hat{\circ}))^{-1} = 1 + O_p(\hat{\circ})$ because of the generalised Binomial theorem and that $O_p(\hat{\circ}) = o_p(1)$. Then one could repeat the argument in the proof of Proposition 4.1 to arrive at

$$K = O_p(\hat{\circ}), \quad A = O_p(\hat{\circ}), \quad \hat{M} = M + O_p(\hat{\circ}) \quad (A.44)$$

$$\hat{\lambda}_{k,j} - \lambda_{k,j} = O_p(\hat{\circ}), \quad MA = O_p(\hat{\circ}) \quad (A.45)$$

for $k = 1, \ldots, 6$ and $j = 1, \ldots, N$.

A.9 Proof of Theorem 4.2

Equation (C4) of the supplement of Bai and Li (2012) still holds in our case since its derivation does not involve identification conditions (page 17 of the supplement of Bai and Li (2012)); in our notation it is

$$\hat{\sigma}_{m}^2 - \sigma_{m}^2 = \frac{1}{T_f} \sum_{t=1}^{T_f} (\hat{e}_{m,t}^2 - \sigma_{m}^2) + o_p(T_f^{-1/2})$$

for $m = 1, \ldots, 6N$, where the single-index $\sigma_m^2$ is defined as $\sigma_m^2 := \sigma_{\lfloor \frac{m}{N} \rfloor, m-\lfloor \frac{m}{N} \rfloor}^2$; interpret $\hat{\sigma}_{m}^2$ similarly. Thus theorem 4.2 follows.
A.10 Proof of Theorem 4.3

Pre-multiply \( \hat{M} \) to (A14) of Bai and Li (2012) and write in our notation:

\[
\hat{M} \left( \hat{\lambda}_{k,j} - \lambda_{k,j} \right) = M(\hat{\Lambda} - \Lambda)\Sigma_{ee}^{-1} \hat{\Lambda} \hat{H} \lambda_{k,j}
\]

\[
- \hat{H} \hat{\Lambda}^{\top} \Sigma_{ee}^{-1} (\hat{\Lambda} - \Lambda) M(\hat{\Lambda} - \Lambda) \Sigma_{ee}^{-1} \hat{\Lambda} \hat{H} \lambda_{k,j}
\]

\[
- \hat{H} \hat{\Lambda}^{\top} \Sigma_{ee}^{-1} \Lambda \left( \frac{1}{T_f} \sum_{t=1}^{T_f} f_t e_t^1 \right) \Sigma_{ee}^{-1} \hat{\Lambda} \hat{H} \lambda_{k,j}
\]

\[
- \hat{H} \hat{\Lambda}^{\top} \Sigma_{ee}^{-1} \left( \frac{1}{T_f} \sum_{t=1}^{T_f} e_t f_t^1 \right) \Lambda^{\top} \Sigma_{ee}^{-1} \hat{\Lambda} \hat{H} \lambda_{k,j}
\]

\[
- \hat{H} \left( \sum_{m=1}^{6N} \sum_{t=1}^{6N} \frac{1}{\sigma_m^2 \sigma_t^2} \hat{\lambda}_m \hat{\lambda}_t^\top \frac{1}{T_f} \sum_{t=1}^{T_f} \left[ e_{m,t} e_{t,t} - E(e_{m,t} e_{t,t}) \right] \right) \hat{H} \lambda_{k,j}
\]

\[
+ \hat{H} \left( \sum_{m=1}^{6N} \frac{1}{\sigma_m^4} \hat{\lambda}_m \hat{\lambda}_m^{\top} \left( \hat{\Sigma}_{ee} - \sigma_m^2 \right) \right) \hat{H} \lambda_{k,j}
\]

\[
+ \hat{H} \hat{\Lambda}^{\top} \Sigma_{ee}^{-1} \left( \frac{1}{T_f} \sum_{t=1}^{T_f} e_t f_t^1 \right) \lambda_{k,j} + \hat{H} \hat{\Lambda}^{\top} \Sigma_{ee}^{-1} \Lambda \left( \frac{1}{T_f} \sum_{t=1}^{T_f} f_t e(k-1)_{N+j,t} \right)
\]

\[
+ \hat{H} \left( \sum_{t=1}^{6N} \frac{1}{\sigma_t^2} \hat{\lambda}_t \frac{1}{T_f} \sum_{t=1}^{T_f} \left[ e_{m,t} e(k-1)_{N+j,t} - E(e_{m,t} e(k-1)_{N+j,t}) \right] \right) - \hat{H} \lambda_{k,j} \frac{1}{\sigma_{k,j}^2} (\sigma_{k,j}^2 - \sigma_{k,j}^2),
\]

where \( e_{i,t} \) denotes the \( i \)th element of \( e_t \), the single-index \( \lambda_m \) is defined as \( \lambda_m := \lambda_m(\mathbb{R}_m, \mathbb{R}_m) \); interpret \( \hat{\lambda}_m, \sigma_m^2, \sigma_t^2 \) similarly.

Consider the right hand side of (A.46). The third and fourth terms are \( o_p(T_f^{-1/2}) \) by Lemma C1(e) of Bai and Li (2012). The fifth term is \( o_p(T_f^{-1/2}) \) by Lemma C1(d) of Bai and Li (2012). The sixth term is \( o_p(T_f^{-1/2}) \) by Lemma C1(f) of Bai and Li (2012). The seventh term is \( o_p(T_f^{-1/2}) \) by Lemma C1(e) of Bai and Li (2012). The ninth term is \( o_p(T_f^{-1/2}) \) by Lemma C1(c) of Bai and Li (2012). The tenth term is \( o_p(T_f^{-1/2}) \) by Theorem 4.2. Thus (A.46) becomes

\[
\hat{M} \left( \hat{\lambda}_{k,j} - \lambda_{k,j} \right) = MA \lambda_{k,j} - A^\top MA \lambda_{k,j} + \hat{H} \hat{\Lambda}^{\top} \Sigma_{ee}^{-1} \Lambda \left( \frac{1}{T_f} \sum_{t=1}^{T_f} f_t e(k-1)_{N+j,t} \right) + o_p(T_f^{-1/2}).
\]

Substituting (4.10) into (A.41), we have \( O_p(\hat{\phi}) = O_p(T_f^{-1/2}) \). Thus \( A = O_p(T_f^{-1/2}) \) via (A.45). The preceding display hence becomes

\[
\hat{\lambda}_{k,j} - \lambda_{k,j} = \hat{M}^{-1} MA \lambda_{k,j} + \hat{M}^{-1} \hat{H} \hat{\Lambda}^{\top} \Sigma_{ee}^{-1} \Lambda \left( \frac{1}{T_f} \sum_{t=1}^{T_f} f_t e(k-1)_{N+j,t} \right) + o_p(T_f^{-1/2}).
\]

Note that

\[
\hat{H} \hat{\Lambda}^{\top} \Sigma_{ee}^{-1} \Lambda = (\hat{\Lambda}^{\top} \Sigma_{ee}^{-1} \hat{\Lambda})^{-1} \hat{\Lambda}^{\top} \Sigma_{ee}^{-1} \hat{\Lambda} = (\hat{\Lambda}^{\top} \Sigma_{ee}^{-1} \hat{\Lambda} + \hat{\Lambda}^{\top} \Sigma_{ee}^{-1} (\Lambda - \hat{\Lambda}))
\]

\[
= I_{14} + (\hat{\Lambda}^{\top} \Sigma_{ee}^{-1} \hat{\Lambda})^{-1} \hat{\Lambda}^{\top} \Sigma_{ee}^{-1} (\Lambda - \hat{\Lambda}) = I_{14} + \hat{H} \Lambda^{\top} \Sigma_{ee}^{-1} (\Lambda - \hat{\Lambda}) = I_{14} + O_p(T_f^{-1/2})
\]

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where the last equality is due to Lemma C1(a) of Bai and Li (2012). Substituting the preceding display into (A.47), we have

\[
\hat{\lambda}_{k,j} - \lambda_{k,j} = \hat{M}^{-1}MA\lambda_{k,j} + \hat{M}^{-1}M\lambda_{k,j} + \hat{M}^{-1}O_p(T_f^{-1/2}) \left( \frac{1}{T_f} \sum_{t=1}^{T_f} f_t e_{(k-1)N+j,t} \right) + o_p(T_f^{-1/2})
\]

where \(\hat{M} = 0\) for each of the sub-steps of Step I, \(-b\) denotes the complement of \(b\), and \(b_1 \cup b_2 = b\) with the cardinality of \(b_2\) equal to the cardinality of \(c\) and \(A_{a,b_1}\) containing solved elements for each of the sub-steps of Step I. Step I is again detailed in Table 14. Imposing \(\hat{\lambda}_k = 0\) and solving (A.48) gives

\[
A_{a,b_2} = - \left( A_{a,b_1} \Lambda_{k,b_1,c} + F_{k,a,c} \right) \Lambda_{k,b_2,c}^{-1} + o_p(T_f^{-1/2}).
\]
A.10.2 Step II Impose $\hat{M}_{i,j} = 1$ for $j = 10, 11, 12, 13$

Consider (A13) of Bai and Li (2012) and write in our notation:

$$\hat{M} - M = -\hat{H}\hat{\Lambda}^\top\hat{\Sigma}_{ee}^{-1}(\hat{\Lambda} - \Lambda)M - M(\hat{\Lambda} - \Lambda)^\top\hat{\Sigma}_{ee}^{-1}\hat{\Lambda}\hat{H}$$

$$+ \hat{H}\hat{\Lambda}^\top\hat{\Sigma}_{ee}^{-1}(\hat{\Lambda} - \Lambda)M(\hat{\Lambda} - \Lambda)^\top\hat{\Sigma}_{ee}^{-1}\hat{\Lambda}\hat{H}$$

$$+ \hat{H}\hat{\Lambda}^\top\hat{\Sigma}_{ee}^{-1}\Lambda \left( \frac{1}{T_f} \sum_{t=1}^{T_f} f_t e_t^\top \right) \hat{\Sigma}_{ee}^{-1}\hat{\Lambda}\hat{H}$$

$$+ \hat{H} \left( \sum_{m=1}^{6N} \sum_{t=1}^{6N} \frac{1}{\sigma^2_m \sigma^2_t} \hat{\lambda}_m \hat{\lambda}_t \frac{1}{T_f} \sum_{t=1}^{T_f} \left[ e_{m,t} e_{t,t} - E(e_{m,t} e_{t,t}) \right] \right) \hat{H}$$

$$- \hat{H} \left( \sum_{m=1}^{6N} \frac{1}{\sigma^4_m} \hat{\lambda}_m^2 (\sigma^2_m - \sigma^2_m) \right) \hat{H}. \quad \text{(A.50)}$$

In the paragraph following (A.46), we already showed that the last four terms of the right hand side of the preceding display are $o_p(T_f^{-1/2})$. Since $A = O_p(T_f^{-1/2})$, $A^\top MA = o_p(T_f^{-1/2})$. Thus, (A.50) becomes

$$\hat{M} - M + A^\top M + MA = o_p(T_f^{-1/2}). \quad \text{(A.51)}$$

Imposing $\hat{M}_{10,10} = 1$, the (10, 10)th element of the preceding display satisfies

$$o_p(T_f^{-1/2}) = \hat{M}_{10,10} - 1 + M_{10,10} + (A_{10,10})^\top M_{10,10} = 2A_{10,10}$$

whence $A_{10,10} = o_p(T_f^{-1/2})$. In the similar way, imposing $\hat{M}_{11,11}, \hat{M}_{12,12}, \hat{M}_{13,13} = 1$, we could deduce that $A_{11,11}, A_{12,12}, A_{13,13} = o_p(T_f^{-1/2})$.

A.10.3 Step III Impose Some Equality Restrictions in $\{\Lambda_k\}_{k=1}^{6}$

(III.1) Note that (A.48) implies

$$\hat{\Lambda}_{6,9,c} - \Lambda_{6,9,c} = A_{9,x} \Lambda_{6,x,c} + F_{6,9,c} + o_p(T_f^{-1/2}) \quad x = \{1, 2, 3, 9\}$$

$$= A_{9,[1,9]} \Lambda_{6,[1,9],c} + A_{9,[2,9]} \Lambda_{6,[2,9],c} + F_{6,9,c} + o_p(T_f^{-1/2})$$

$$\hat{\Lambda}_{3,12,c} - \Lambda_{3,12,c} = A_{12,y} \Lambda_{3,y,c} + F_{3,12,c} + o_p(T_f^{-1/2}) \quad y = \{4, 5, 6, 12\}.$$ We then impose $\hat{\Lambda}_{6,9,c} = \hat{\Lambda}_{3,12,c}$ for $c = \{1, 2\}$; that is, the loadings of the continent factor on day one and day two are the same for the first two American assets. The preceding display implies

$$A_{9,[1,9]} \Lambda_{6,[1,9],c} = A_{12,y} \Lambda_{3,y,c} + F_{3,12,c} - A_{9,[2,9]} \Lambda_{6,[2,9],c} - F_{6,9,c} + o_p(T_f^{-1/2})$$

whence we have

$$A_{9,[1,9]} = (A_{12,y} \Lambda_{3,y,c} + F_{3,12,c} - A_{9,[2,9]} \Lambda_{6,[2,9],c} - F_{6,9,c}) \Lambda_{6,[1,9],c}^{-1} + o_p(T_f^{-1/2}).$$
(III.2) Note that (A.48) implies
\[
(\hat{A}_1 - \Lambda_1)_{14,c} = A_{14,x}\Lambda_{1,x,c} + F_{1,14,c} + o_p(T_f^{-1/2}) \quad x = \{6, 7, 8, 14\}
\]
\[
= A_{14,14}\Lambda_{1,14,c} + A_{14,6,7}\Lambda_{1,6,7},c + F_{1,14,c} + o_p(T_f^{-1/2})
\]
\[
(\hat{A}_4 - \Lambda_4)_{11,c} = A_{11,y}\Lambda_{4,y,c} + F_{4,11,c} + o_p(T_f^{-1/2}) \quad y = \{3, 4, 5, 11\}.
\]
We then impose \(\hat{A}_{1,14,c} = \hat{A}_{4,11,c}\) for \(c = \{1, 2\}\); that is, the loadings of the continent factor on day one and day two are the same for the first two Asian assets. The preceding display implies
\[
A_{14,14}\Lambda_{1,14,c} = A_{11,y}\Lambda_{4,y,c} + F_{4,11,c} - A_{14,6,7}\Lambda_{1,6,7},c - F_{1,14,c} + o_p(T_f^{-1/2})
\]
whence we have
\[
A_{14,14} = (A_{11,y}\Lambda_{4,y,c} + F_{4,11,c} - A_{14,6,7}\Lambda_{1,6,7},c - F_{1,14,c})^\dagger + o_p(T_f^{-1/2})
\]

A.10.4 Step IV Impose Some Restrictions in \(M\)
Imposing \(\hat{M}_{4,12} = \hat{M}_{5,12} = \hat{M}_{6,12} = 0\). Considering the (4, 12)th, (5, 12)th and (6, 12)th elements of the left hand side of (A.51), we have
\[
M_{4,12}A_{12,4} + A_{12,4} = o_p(T_f^{-1/2})
\]
\[
M_{5,12}A_{12,5} + A_{12,5} = o_p(T_f^{-1/2})
\]
\[
M_{6,12}A_{12,6} + A_{12,6} = o_p(T_f^{-1/2})
\]

with the only unknown elements \(A_{4,12}, A_{5,12}\) and \(A_{6,12}\). The preceding display could be written as
\[
M_{a,a}A_{a,12} + M_{a,b}A_{b,12} + (A_{12,a})^T = o_p(T_f^{-1/2}),
\]
where \(a := \{4, 5, 6\}, b := \{1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 14\}\). Thus, we obtain
\[
A_{a,12} = -(M_{a,a})^{-1} (M_{a,b}A_{b,12} + (A_{12,a})^T)
\]
Similarly, imposing \(\hat{M}_{2,10} = \hat{M}_{3,10} = \hat{M}_{4,10} = 0\), we could solve \(A_{[2,3,4],10}\). Imposing \(\hat{M}_{3,11} = \hat{M}_{4,11} = \hat{M}_{5,11} = 0\), we could solve \(A_{[3,4,5],11}\). Imposing \(\hat{M}_{5,13} = \hat{M}_{6,13} = \hat{M}_{7,13} = 0\), we could solve \(A_{[5,6,7],13}\).

A.10.5 Step V Impose Some Restrictions in \(M\)
(V.1) Consider the (4, 4)th and (6, 6)th elements of (A.51):
\[
\hat{M}_{4,4} - 1/(1 - \phi^2) + 2M_{4,4}A_{4,4} + 2M_{4,-4}A_{-4,4} = o_p(T_f^{-1/2})
\]
\[
\hat{M}_{6,6} - 1/(1 - \phi^2) + 2M_{6,6}A_{6,6} + 2M_{6,-6}A_{-6,6} = o_p(T_f^{-1/2})
\]
Imposing \(\hat{M}_{4,4} = \hat{M}_{6,6}\), we could arrange the preceding display into
\[
M_{4,4}A_{4,4} + M_{4,-4}A_{-4,4} = M_{6,6}A_{6,6} + M_{6,-6}A_{-6,6} + o_p(T_f^{-1/2}).
\]
\(\text{(A.52)}\)
Next, consider the (4,4)th and (6,4)th elements of (A.51):

\[
\begin{align*}
\hat{M}_{4,4} - 1/(1 - \phi^2) + 2M_{4,4}A_{4,4} + 2M_{4,-4}A_{-4,4} &= o_p(T_f^{-1/2}) \\
\hat{M}_{6,4} - \phi^2/(1 - \phi^2) + M_{6,\bullet}A_{\bullet,4} + (A_{\bullet,6})^TM_{\bullet,4} &= o_p(T_f^{-1/2}).
\end{align*}
\]

Imposing \( \hat{M}_{4,4} - \hat{M}_{6,4} = 1 \), we could arrange the preceding display into

\[
2M_{4,4}A_{4,4} + 2M_{4,-4}A_{-4,4} = M_{6,\bullet}A_{\bullet,4} + (A_{\bullet,6})^TM_{\bullet,4} + o_p(T_f^{-1/2}).
\]

(A.53)

Since there are only two unknown elements in (A.52) and (A.53) (i.e., \( A_{4,4} \) and \( A_{6,6} \)), we could thus solve them. Write (A.52) and (A.53) in matrix

\[
\begin{pmatrix}
M_{4,4} \\
2M_{4,4} - M_{6,4} - M_{6,6}
\end{pmatrix}
\begin{pmatrix}
A_{4,4} \\
A_{6,6}
\end{pmatrix}
= 
\begin{pmatrix}
M_{6,-6}A_{-6,6} - M_{4,-4}\Lambda A_{-4,4} \\
M_{4,-6}A_{-6,6} - (2M_{4,-4} - M_{6,-4}) A_{-4,4}
\end{pmatrix}.
\]

That is,

\[
\begin{pmatrix}
A_{4,4} \\
A_{6,6}
\end{pmatrix}
= 
\begin{pmatrix}
M_{4,4} \\
2M_{4,4} - M_{6,4} - M_{6,6}
\end{pmatrix}^{-1}
\begin{pmatrix}
M_{6,-6}A_{-6,6} - M_{4,-4}\Lambda A_{-4,4} \\
M_{4,-6}A_{-6,6} - (2M_{4,-4} - M_{6,-4}) A_{-4,4}
\end{pmatrix}.
\]

(V.2) Consider the (4,4)th and (5,5)th elements of (A.51):

\[
\hat{M}_{4,4} - 1/(1 - \phi^2) + 2M_{4,\bullet}A_{\bullet,4} = o_p(T_f^{-1/2}) \\
\hat{M}_{5,5} - 1/(1 - \phi^2) + 2M_{5,5}A_{5,5} + 2M_{5,-5}A_{-5,5} = o_p(T_f^{-1/2}).
\]

Imposing \( \hat{M}_{4,4} = \hat{M}_{5,5} \), we could solve the preceding display for \( A_{5,5} \):

\[
A_{5,5} = (M_{4,\bullet}A_{\bullet,4} - M_{5,-5}A_{-5,5}) / M_{5,5} + o_p(T_f^{-1/2}).
\]

A.10.6 Step VI Impose Some Equality Restrictions in \( \{\Lambda_k\}_k^{6} \)

(VI.1) Note that (A.48) implies

\[
(\hat{\Lambda}_6 - \Lambda_6)_{1,c} = A_{1,x}\Lambda_{6,x,c} + F_{6,1,c} + o_p(T_f^{-1/2}) \quad x = \{1, 2, 3, 9\} \\
(\hat{\Lambda}_3 - \Lambda_3)_{4,c} = A_{4,y}\Lambda_{3,y,c} + F_{3,4,c} + o_p(T_f^{-1/2}) \quad y = \{4, 5, 6, 12\}.
\]

We then impose \( \hat{\Lambda}_{6,1,c} = \hat{\Lambda}_{3,4,c} \) for \( c = \{1, 2\} \); that is, the loadings of the continent factor on day one and day two are the same for the first two American assets. The preceding display implies

\[
A_{1,[1,9]}\Lambda_{6,[1,9],c} = A_{4,y}\Lambda_{3,y,c} + F_{3,4,c} - A_{1,[2,3]}\Lambda_{6,[2,3],c} - F_{6,1,c} + o_p(T_f^{-1/2})
\]

whence we have

\[
A_{1,[1,9]} = (A_{4,y}\Lambda_{3,y,c} + F_{3,4,c} - A_{1,[2,3]}\Lambda_{6,[2,3],c} - F_{6,1,c}) \Lambda_{6,[1,9],c}^{-1} + o_p(T_f^{-1/2}).
\]

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(VI.2) Note that (A.48) implies

\[
(\hat{\Lambda} - \Lambda_6)_{2,c} = A_{2,c} \Lambda_{6,c} + F_{6,c} + o_p(T_f^{-1/2}) \quad x = \{1, 2, 3, 9\} \\
= A_{2,[1,2,9]} \Lambda_{6,[1,2,9]} + A_{2,3} \Lambda_{6,3} + F_{6,2} + o_p(T_f^{-1/2}) \\
(\hat{\Lambda} - \Lambda_3)_{5,c} = A_{5,c} \Lambda_{3,c} + F_{3,5,c} + o_p(T_f^{-1/2}) \quad y = \{4, 5, 6, 12\}.
\]

We then impose \(\hat{\Lambda}_{6,2,c} = \hat{\Lambda}_{3,5,c}\) for \(c = \{1, 2, 3\}\); that is, the first lagged loadings of the global factor on day one and day two are the same for the first two American assets. The preceding display implies

\[
A_{2,[1,2,9]} \Lambda_{6,[1,2,9],c} = A_{5,y} A_{3,y,c} + F_{3,5,c} - A_{2,3} \Lambda_{6,3,c} - F_{6,2,c} + o_p(T_f^{-1/2})
\]

whence we have

\[
A_{2,[1,2,9]} = (A_{5,y} A_{3,y,c} + F_{3,5,c} - A_{2,3} \Lambda_{6,3,c} - F_{6,2,c}) \Lambda_{6,[1,2,9],c}^{-1} + o_p(T_f^{-1/2}).
\]

(VI.3) Note that (A.48) implies

\[
(\hat{\Lambda} - \Lambda_6)_{2,c} = A_{2,x} \Lambda_{6,x} + F_{6,2} + o_p(T_f^{-1/2}) \quad x = \{1, 2, 3, 9\} \\
= A_{2,[1,2,9]} \Lambda_{6,[1,2,9],c} + A_{2,3} \Lambda_{6,3,c} + F_{6,2,c} + o_p(T_f^{-1/2}) \\
(\hat{\Lambda} - \Lambda_3)_{5,c} = A_{5,y} \Lambda_{3,y,c} + F_{3,5,c} + o_p(T_f^{-1/2}) \quad y = \{4, 5, 6, 12\}.
\]

We then impose \(\hat{\Lambda}_{6,2,c} = \hat{\Lambda}_{3,5,c}\) for \(c = \{1, 2, 3\}\); that is, the first lagged loadings of the global factor on day one and day two are the same for the first three American assets. The preceding display implies

\[
A_{2,[1,2,9]} \Lambda_{6,[1,2,9],c} = A_{5,y} A_{3,y,c} + F_{3,5,c} - A_{2,3} \Lambda_{6,3,c} - F_{6,2,c} + o_p(T_f^{-1/2})
\]

whence we have

\[
A_{2,[1,2,9]} = (A_{5,y} A_{3,y,c} + F_{3,5,c} - A_{2,3} \Lambda_{6,3,c} - F_{6,2,c}) \Lambda_{6,[1,2,9],c}^{-1} + o_p(T_f^{-1/2}).
\]

(VI.4) Note that (A.48) implies

\[
(\hat{\Lambda} - \Lambda_6)_{3,c} = A_{3,x} \Lambda_{6,x} + F_{6,3,c} + o_p(T_f^{-1/2}) \quad x = \{1, 2, 3, 9\} \\
(\hat{\Lambda} - \Lambda_3)_{6,c} = A_{6,y} \Lambda_{3,y,c} + F_{3,6,c} + o_p(T_f^{-1/2}) \quad y = \{4, 5, 6, 12\}.
\]

We then impose \(\hat{\Lambda}_{6,3,c} = \hat{\Lambda}_{3,6,c}\) for \(c = \{1, 2, 3, 4\}\); that is, the second lagged loadings of the global factor on day one and day two are the same for the first four American assets. The preceding display implies

\[
A_{3,x} \Lambda_{6,x,c} = A_{6,y} A_{3,y,c} + F_{3,6,c} - F_{6,3,c} + o_p(T_f^{-1/2})
\]

whence we have

\[
A_{3,x} = (A_{6,y} A_{3,y,c} + F_{3,6,c} - F_{6,3,c}) \Lambda_{6,x,c}^{-1} + o_p(T_f^{-1/2})
\]

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Note that \((A.48)\) implies

\[
\hat{\Lambda}_1 - \Lambda_1 = A_{6,x} \Lambda_{1,x,c} + F_{1,6,c} + o_p(T_f^{-1/2}) \quad x = \{6, 7, 8, 14\}
\]

\[
A_{8,[8,14]} \Lambda_{1,[8,14],c} + A_{8,[6,7]} \Lambda_{1,[6,7],c} + F_{1,8,c} + o_p(T_f^{-1/2})
\]

\[
\hat{\Lambda}_4 - \Lambda_4 = A_{5,y} \Lambda_{4,y,c} + F_{4,5,c} + o_p(T_f^{-1/2}) \quad y = \{3, 4, 5, 11\}.
\]

We then impose \(\hat{\Lambda}_{1,8,c} = \hat{\Lambda}_{4,5,c} \) for \(c = \{1, 2\}\); that is, the second lagged loadings of the global factor on day one and day two are the same for the first two Asian assets. The preceding display implies

\[
A_{8,[8,14]} \Lambda_{1,[8,14],c} = A_{5,y} \Lambda_{4,y,c} + F_{4,5,c} - A_{8,[6,7]} \Lambda_{1,[6,7],c} - F_{1,8,c} + o_p(T_f^{-1/2})
\]

whence we have

\[
A_{8,[8,14]} = (A_{5,y} \Lambda_{4,y,c} + F_{4,5,c} - A_{8,[6,7]} \Lambda_{1,[6,7],c} - F_{1,8,c}) \Lambda_{1,[8,14],c}^{-1} + o_p(T_f^{-1/2}).
\]

Note that \((A.48)\) implies

\[
\hat{\Lambda}_1 - \Lambda_1 = A_{7,x} \Lambda_{1,x,c} + F_{1,7,c} + o_p(T_f^{-1/2}) \quad x = \{6, 7, 8, 14\}
\]

\[
A_{7,[7,8,14]} \Lambda_{1,[7,8,14],c} + A_{7,6} \Lambda_{1,6,c} + F_{1,7,c} + o_p(T_f^{-1/2})
\]

\[
\hat{\Lambda}_4 - \Lambda_4 = A_{4,y} \Lambda_{4,y,c} + F_{4,4,c} + o_p(T_f^{-1/2}) \quad y = \{3, 4, 5, 11\}.
\]

We then impose \(\hat{\Lambda}_{1,7,c} = \hat{\Lambda}_{4,4,c} \) for \(c = \{1, 2, 3\}\); that is, the first lagged loadings of the global factor on day one and day two are the same for the first three Asian assets. The preceding display implies

\[
A_{7,[7,8,14]} \Lambda_{1,[7,8,14],c} = A_{4,y} \Lambda_{4,y,c} + F_{4,4,c} - A_{7,6} \Lambda_{1,6,c} - F_{1,7,c} + o_p(T_f^{-1/2})
\]

whence we have

\[
A_{7,[7,8,14]} = (A_{4,y} \Lambda_{4,y,c} + F_{4,4,c} - A_{7,6} \Lambda_{1,6,c} - F_{1,7,c}) \Lambda_{1,[7,8,14],c}^{-1} + o_p(T_f^{-1/2}).
\]

Note that \((A.48)\) implies

\[
\hat{\Lambda}_1 - \Lambda_1 = A_{6,x} \Lambda_{1,x,c} + F_{1,6,c} + o_p(T_f^{-1/2}) \quad x = \{6, 7, 8, 14\}
\]

\[
A_{6,[7,8,14]} \Lambda_{1,[7,8,14],c} + A_{6,6} \Lambda_{1,6,c} + F_{1,6,c} + o_p(T_f^{-1/2})
\]

\[
\hat{\Lambda}_4 - \Lambda_4 = A_{3,y} \Lambda_{4,y,c} + F_{4,3,c} + o_p(T_f^{-1/2}) \quad y = \{3, 4, 5, 11\}.
\]

We then impose \(\hat{\Lambda}_{1,6,c} = \hat{\Lambda}_{4,3,c} \) for \(c = \{1, 2, 3\}\); that is, the contemporaneous loadings of the global factor on day one and day two are the same for the first three Asian assets. The preceding display implies

\[
A_{6,[7,8,14]} \Lambda_{1,[7,8,14],c} = A_{3,y} \Lambda_{4,y,c} + F_{4,3,c} - A_{6,6} \Lambda_{1,6,c} - F_{1,6,c} + o_p(T_f^{-1/2})
\]

whence we have

\[
A_{6,[7,8,14]} = (A_{3,y} \Lambda_{4,y,c} + F_{4,3,c} - A_{6,6} \Lambda_{1,6,c} - F_{1,6,c}) \Lambda_{1,[7,8,14],c}^{-1} + o_p(T_f^{-1/2})
\]
A.10.7 To Sum Up

Recall that (A.48) implies that for \( k = 1, \ldots, 6, j = 1, \ldots, N, \)

\[
\sqrt{T_f}(\hat{\lambda}_{k,j} - \lambda_{k,j}) = \sqrt{T_f}A\lambda_{k,j} + M^{-1} \frac{1}{\sqrt{T_f}} \sum_{t=1}^{T_f} f_t e_{(k-1)N+j,t} + o_p(1). \tag{A.54}
\]

From (A.51), we have

\[
\sqrt{T_f}(\hat{M} - M) = -\sqrt{T_f} (A^T M + MA) + o_p(1).
\]

A.11 Proof of Theorem 4.4

\[
\sqrt{N}(\hat{f}_t - f_t) = -\sqrt{N} (\hat{\Lambda}^{\top} \hat{\Sigma}_{ee}^{-1} \hat{\Lambda})^{-1} \hat{\Lambda}^{\top} \hat{\Sigma}_{ee}^{-1} (\hat{\Lambda} - \Lambda) f_t + \sqrt{N} (\hat{\Lambda}^{\top} \hat{\Sigma}_{ee}^{-1} \hat{\Lambda})^{-1} \hat{\Lambda}^{\top} \hat{\Sigma}_{ee}^{-1} e_t
\]

\[
= -\sqrt{N} \frac{T_f}{T_f} A^{\top} f_t + \sqrt{N} \left( 1 - \frac{1}{N} \hat{\Lambda}^{\top} \hat{\Sigma}_{ee}^{-1} \hat{\Lambda} \right) \left( \frac{1}{N} \hat{\Lambda}^{\top} \hat{\Sigma}_{ee}^{-1} f_t + O_p \left( N^{-1/2} T_f^{-1/2} \right) + O_p(T_f^{-1}) \right).
\tag{A.55}
\]

Lemma D1 of Bai and Li (2012) still holds in our setting and it reads, in our notation:

\[
\frac{1}{N} \hat{\Lambda}^{\top} \hat{\Sigma}_{ee}^{-1} e_t = \frac{1}{N} \Lambda^{\top} \Sigma_{ee}^{-1} e_t + O_p \left( N^{-1/2} T_f^{-1/2} \right) + O_p(T_f^{-1}). \tag{A.56}
\]

\[
\left( \frac{1}{N} \hat{\Lambda}^{\top} \hat{\Sigma}_{ee}^{-1} \hat{\Lambda} \right)^{-1} = Q^{-1} + o_p(1). \tag{A.57}
\]

Substituting (A.56) and (A.57) into (A.55), we have

\[
\sqrt{N}(\hat{f}_t - f_t)
\]

\[
= -\sqrt{N} \frac{T_f}{T_f} A^{\top} f_t + \sqrt{N} \left( Q^{-1} + o_p(1) \right) \left[ \frac{1}{N} \Lambda^{\top} \Sigma_{ee}^{-1} e_t + O_p \left( N^{-1/2} T_f^{-1/2} \right) + O_p(T_f^{-1}) \right]
\]

\[
= -\sqrt{N} \frac{T_f}{T_f} A^{\top} f_t + (Q^{-1} + o_p(1)) \left[ \frac{1}{\sqrt{N}} \Lambda^{\top} \Sigma_{ee}^{-1} e_t + O_p \left( T_f^{-1/2} \right) + O_p(\sqrt{NT_f^{-1}}) \right]
\]

\[
= -\sqrt{\Delta} \sqrt{T_f} A^{\top} f_t + (Q^{-1} + o_p(1)) \left[ \frac{1}{\sqrt{N}} \Lambda^{\top} \Sigma_{ee}^{-1} e_t + O_p \left( T_f^{-1/2} \right) + o_p(1) \right]
\]

\[
= -\sqrt{\Delta} \sqrt{T_f} A^{\top} f_t + Q^{-1} \frac{1}{\sqrt{N}} \Lambda^{\top} \Sigma_{ee}^{-1} e_t + o_p(1), \tag{A.58}
\]

where the third equality is due to \( \sqrt{N}/T_f \to 0, N/T_f \to \Delta \) and \( \sqrt{T_f} A^{\top} f_t = O_p(1) \), and the last equality uses the fact that \( Q^{-1} = O_p(1) \) and \( N^{-1/2} \Lambda^{\top} \Sigma_{ee}^{-1} e_t = O_p(1) \) by the central limit theorem.

A.12 Formulas for \( \mathbb{E}[f_t f_t^{\top}] \) and \( \mathbb{E}[f_t g_t^{\top}] \)

In this subsection, we give the formulas for \( \mathbb{E}[f_t f_t^{\top}] \) and \( \mathbb{E}[f_t g_t^{\top}] \) defined in Section 4.2.3.

We know that

\[
\left( \begin{array}{c}
\mathbf{f}_t \\
\mathbf{y}_t
\end{array} \right) \sim N \left( \left( \begin{array}{c}
0 \\
0
\end{array} \right), \left( \begin{array}{cc}
M & MA^{\top} \\
AM & \Sigma_{yy}
\end{array} \right) \right).
\]

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Recall that in Section 4.2.3 we treat \( \{f_t\}_{t=1}^{T_f} \) as i.i.d. Thus, according to the conditional distribution of the multivariate normal, we have

\[
E[f_t|\{\hat{y}_t\}_{t=1}^{T_f}; \theta^{(i)}] = M \Lambda^T \Sigma_{yy}^{-1} y_t
\]

\[
\text{var}[f_t|\{\hat{y}_t\}_{t=1}^{T_f}; \theta^{(i)}] = M - M \Lambda^T \Sigma_{yy}^{-1} \Lambda M + M \Lambda^T \Sigma_{yy}^{-1} S_{yy} \Sigma_{yy}^{-1} \Lambda M.
\]

We then can show that

\[
\frac{1}{T_f} \sum_{t=1}^{T_f} \hat{E}[\hat{y}_t f_t^T] = S_{yy} \Sigma_{yy}^{-1} \Lambda M
\]

\[
\frac{1}{T_f} \sum_{t=1}^{T_f} \hat{E}[f_t f_t^T] = M - M \Lambda^T \Sigma_{yy}^{-1} \Lambda M + M \Lambda^T \Sigma_{yy}^{-1} S_{yy} \Sigma_{yy}^{-1} \Lambda M.
\]

(A.59)

**A.13 Computation of QMLE**

In this subsection, we provide a way to compute the QMLE defined in Section 4.1. Again we will rely on the EM algorithm.

(i) In the E step, calculate \( T_f^{-1} \sum_{t=1}^{T_f} \hat{E}[\hat{y}_t f_t^T] \) and \( T_f^{-1} \sum_{t=1}^{T_f} \hat{E}[f_t f_t^T] \) as in (A.59).

(ii) In the M step, obtain the factor loading estimates similar to those of QMLE-res by imposing the factor loading restrictions within the \( 14^2 \) restrictions. We only have to change the selection matrices, say, \( L_1 \) and \( L_4 \) defined in Section 4.2.3, accordingly.

(iii) Iterate steps (i) and (ii) until the estimates \( \hat{\Lambda}, \hat{\Sigma}_{ee}, \hat{M} \) satisfy (4.5) reasonably well.

(iv) Rotate the converged \( \hat{\Lambda} \) and \( \hat{M} \) so that the rotated \( \hat{\Lambda} \) and \( \hat{M} \) satisfy all the \( 14^2 \) restrictions. This could only be done numerically. In particular, we define a distance function which measures the distance between the restricted elements in \( \hat{\Lambda} \) and \( \hat{M} \) and their corresponding rotated elements.

**A.14 QMLE-delta for \( \phi \)**

In this subsection, we will provide the QMLE-delta for \( \phi \). We shall state it as a theorem:

**Theorem A.1.** Suppose that the assumptions of Proposition 4.1 hold. Then we have

\[
\sqrt{T_f} \text{vech} (\hat{M} - M) \xrightarrow{d} N (0, \mathcal{M})
\]

where \( \mathcal{M} \) is \( 105 \times 105 \) and defined

\[
\mathcal{M} := 4D_{14}^T (I_{14} \otimes M) \left[ \lim_{T_f \to \infty} \text{var} \left( \sqrt{T_f} \text{vec} A \right) \right] (I_{14} \otimes M) D_{14}^T.
\]

Note that \( \text{vech}(M) = h(\phi) \), where \( h(\cdot) : \mathbb{R} \to \mathbb{R}^{105} \). Define the following QMLE-delta estimator

\[
\tilde{\phi} = \arg \min_{\phi} \left[ \text{vech}(\hat{M}) - h(\phi) \right]^T \left[ \text{vech}(\hat{M}) - h(\phi) \right].
\]

(A.60)
Then we have

\[ \sqrt{T_f}(\hat{\phi} - \phi) \overset{d}{\to} N(0, \mathcal{O}), \]

where

\[ \mathcal{O} := \left[ \frac{\partial h(\phi)}{\partial \phi^\intercal} \frac{\partial h(\phi)}{\partial \phi} \right]^{-1} \frac{\partial h(\phi)}{\partial \phi^\intercal} \mathcal{M} \frac{\partial h(\phi)}{\partial \phi} \left[ \frac{\partial h(\phi)}{\partial \phi^\intercal} \frac{\partial h(\phi)}{\partial \phi} \right]^{-1}. \]

In the preceding theorem, although choosing \( \mathcal{M}^{-1} \) as the weighting matrix in (A.60) gives a more efficient estimator, we recommend not to do this as this requires inverting an estimated \( \mathcal{M} \).

**Proof of Theorem A.1.** Recall (4.13):

\[ \sqrt{T_f}(\hat{M} - M) = -\sqrt{T_f}(A^T M + MA) + o_p(1), \]

whence we have

\[
\begin{align*}
\sqrt{T_f} \text{vech}(\hat{M} - M) & = -\sqrt{T_f} \text{vech} \left( A^T M + MA \right) + o_p(1) \\
& = -\sqrt{T_f} D^+_{14} \left[ (M \otimes I_{14}) \text{vec}(A^T) + (I_{14} \otimes M) \text{vec} A \right] + o_p(1) \\
& = -\sqrt{T_f} D^+_{14} \left[ (M \otimes I_{14})K_{14,14} \text{vec} A + (I_{14} \otimes M) \text{vec} A \right] + o_p(1) \\
& = -\sqrt{T_f} D^+_{14} \left[ K_{14,14}(I_{14} \otimes M) \text{vec} A + (I_{14} \otimes M) \text{vec} A \right] + o_p(1) \\
& = -2\sqrt{T_f} D^+_{14}D_{14}^+ (I_{14} \otimes M) \text{vec} A + o_p(1) \\
& = -2D^+_{14}(I_{14} \otimes M) \sqrt{T_f} \text{vec} A + o_p(1)
\end{align*}
\]

where the second equality is due to symmetry of \( A^T M + MA \), and the fifth and seventh equalities are due to properties of \( K_{14,14} \). Similar to (4.18), we have

\[ \sqrt{T_f} \text{vec} A \overset{d}{\to} N \left( 0, \lim_{T_f \to \infty} \text{var} \left( \sqrt{T_f} \text{vec} A \right) \right), \]

whence we have

\[ \sqrt{T_f} \text{vech}(\hat{M} - M) \overset{d}{\to} N \left( 0, \mathcal{M} \right), \]

where \( \mathcal{M} \) is \( 105 \times 105 \) and defined as

\[ \mathcal{M} := 4D^+_{14}(I_{14} \otimes M) \left[ \lim_{T_f \to \infty} \text{var} \left( \sqrt{T_f} \text{vec} A \right) \right] (I_{14} \otimes M)D^+_{14}. \]

Note that \( \text{vech}(M^*) = h(\phi^*) \), where \( h(\cdot) : \mathbb{R} \to \mathbb{R}^{105} \). Let \( W \) denote a \( 105 \times 105 \) weighting matrix and define the following minimum distance estimator

\[ \tilde{\phi} := \arg \min_{\phi} \left[ \text{vech}(\hat{M}) - h(\phi) \right]^\intercal W \left[ \text{vech}(\hat{M}) - h(\phi) \right]. \]

The minimum distance estimator \( \tilde{\phi} \) satisfies the first-order condition:

\[ \frac{\partial h(\tilde{\phi})}{\partial \phi^\intercal} W \left[ \text{vech}(\hat{M}) - h(\tilde{\phi}) \right] = 0. \quad \text{(A.61)} \]

Do a Taylor expansion

\[ h(\tilde{\phi}) = h(\phi^*) + \frac{\partial h(\tilde{\phi})}{\partial \phi} (\tilde{\phi} - \phi^*), \]

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where \( \hat{\phi} \) is a mid-point between \( \tilde{\phi} \) and \( \phi^* \). Substituting the preceding display into (A.61), we have

\[
\frac{\partial h(\hat{\phi})}{\partial \hat{\phi}^\top} W \left[ \text{vech}(\hat{M}) - h(\phi^*) - \frac{\partial h(\hat{\phi})}{\partial \hat{\phi}} (\hat{\phi} - \phi^*) \right] = 0
\]

whence we have

\[
\sqrt{T_f}(\hat{\phi} - \phi^*) = \left[ \frac{\partial h(\hat{\phi})}{\partial \hat{\phi}^\top} W \frac{\partial h(\hat{\phi})}{\partial \hat{\phi}} \right]^{-1} \frac{\partial h(\hat{\phi})}{\partial \hat{\phi}^\top} W \left[ \text{vech}(\hat{M}) - h(\phi^*) \right]
\]

\[
\overset{d}{\rightarrow} \left[ \frac{\partial h(\phi^*)}{\partial \phi^\top} W \frac{\partial h(\phi^*)}{\partial \phi} \right]^{-1} \frac{\partial h(\phi^*)}{\partial \phi^\top} W N(0, M),
\]

where the convergence in distribution follows from consistency of \( \hat{\phi} \) (i.e., \( \hat{\phi} \overset{p}{\rightarrow} \phi^* \)). The proof of consistency of a low-dimensional minimum distance estimator is a classic result, so we omit the details. Although setting \( W = M^{-1} \) gives the most efficient minimum distance estimator, we recommend to set \( W = I_{105} \) to avoid inverting an estimated \( M \). In this case, we have

\[
\sqrt{T_f}(\hat{\phi} - \phi^*) \overset{d}{\rightarrow} N(0, \mathcal{O}),
\]

where

\[
\mathcal{O} := \left[ \frac{\partial h(\phi^*)}{\partial \phi^\top} \frac{\partial h(\phi^*)}{\partial \phi} \right]^{-1} \frac{\partial h(\phi^*)}{\partial \phi^\top} M \frac{\partial h(\phi^*)}{\partial \phi} \left[ \frac{\partial h(\phi^*)}{\partial \phi^\top} \frac{\partial h(\phi^*)}{\partial \phi} \right]^{-1}.
\]

We call this \( \hat{\phi} \) the QMLE-delta estimator of \( \phi \). \qed

### A.15 Validity of Our Model to Standardised Portfolio Returns

In this subsection, we discuss the validity of applying our model to standardised portfolio returns. The daily value-weighted portfolio return \( R_t \) on day \( t \) is calculated using the following formula:

\[
R_t = \sum_{i=1}^{M} w_{i,t-1} \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}} \approx \sum_{i=1}^{M} w_{i,t-1} \log \frac{P_{i,t}}{P_{i,t-1}} = \log \left( \prod_{i=1}^{M} \frac{P_{i,t}}{P_{i,t-1}} \right)
\]

\[
= \log \left( \prod_{i=1}^{M} P_{i,t-1}^{w_{i,t-1}} \right) - \log \left( \prod_{i=1}^{M} P_{i,t-1}^{w_{i,t-1}} \right) 
\approx \log \left( \prod_{i=1}^{M} P_{i,t-1}^{w_{i,t-1}} \right) - \log \left( \prod_{i=1}^{M} P_{i,t-1}^{w_{i,t-1}} \right)
\]

where \( P_{i,t} \) denotes the closing price of stock \( i \) on day \( t \), \( w_{i,t-1} \) denotes the market capitalisation of stock \( i \) divided by the market capitalisation of the portfolio on day \( t - 1 \), and \( M \) denotes the number of the stocks in the portfolio. The first approximation sign will hold on the assumption that daily returns are often small, and the second approximation sign will hold on the assumption that weights do not change much over a day. The preceding display shows that \( \log \left( \prod_{i=1}^{M} P_{i,t-1}^{w_{i,t-1}} \right) \) could be interpreted as the log closing price of the portfolio on day \( t \), and we could hence fit \( R_t \) with our model:

\[
R_t = \mu + \sum_{j=0}^{2} z_j f_{g,t-j} + z_3 f_{C,t} + \epsilon_t
\]
where $z_j$ are scalars, and $e_t$ is a random variable. Let $\bar{R} := T^{-1} \sum_{t=1}^{T} R_t$ and $se(R) := T^{-1} \sum_{t=1}^{T} (R_t - \bar{R})^2$. Thus

$$\frac{R_t - \bar{R}}{se(R_t)} \approx \sum_{j=0}^{2} z_j \frac{f_{g,t-j}}{\text{var}(R_t)} + \frac{z_3}{\text{var}(R_t)} f_{C,t} + \frac{e_t}{\text{var}(R_t)} =: \sum_{j=0}^{2} z_j^* f_{g,t-j} + z_3^* f_{C,t} + e_t^*.$$

We hence see that our model could be applied to standardised portfolio returns in practice by relying on a few innocuous approximations.

### A.16 Relationship between Various Estimators

**The EM estimator.** Use the EM algorithm to optimize the likelihood function of $y_c^t = Z^c \alpha_t + e_c^t$, where $\alpha_t := [f_{g,t}, f_{g,t-1}, f_{g,t-2}, f_{C,t}]^\top$, $f_{g,t+1} = \phi f_{g,t} + \eta_{g,t}$ and $f_{C,t+1} = \eta_{C,t}$. This estimator should be asymptotically equivalent to MLE.

**MLE-two day.** Use the EM algorithm to optimize the likelihood function of $\hat{y}_t = \Lambda f_t + e_t$, where $\text{var}(f_t) = M$. This is equivalent to the preceding EM estimator, if we impose all the restrictions implied by $\Lambda$ and $M$, and implied by autocorrelation between $f_t$ and $f_{t-1}$. This estimator is not used in our article, but it will help us better understand the connection between MLE and QMLE.

**QMLE-res.** This is the QMLE estimator of $\hat{y}_t = \Lambda f_t + e_t$, incorporating all the restrictions implied by $\Lambda$ and $M$. However, autocorrelation between $f_t$ and $f_{t-1}$ is ignored, and we treat $\{f_t\}_{t=1}^{T_f}$ as i.i.d over $t$.

**QMLE.** This is the QMLE estimator of $\hat{y}_t = \Lambda f_t + e_t$ with the specific set of $14^2$ restrictions mentioned in the article. Autocorrelation between $f_t$ and $f_{t-1}$ is ignored, and we treat $\{f_t\}_{t=1}^{T_f}$ as i.i.d over $t$.

**Bai and Li (2012)’s QMLE.** This is the QMLE estimator of $\hat{y}_t = \Lambda f_t + e_t$ with specific sets of $14^2$ restrictions consistent with the five identification schemes of Bai and Li (2012). Autocorrelation between $f_t$ and $f_{t-1}$ is ignored, and $\{f_t\}_{t=1}^{T_f}$ is assumed i.i.d over $t$. Unfortunately, our model is not consistent with any one of the five identification schemes.

**QMLE-delta.** Something in-between QMLE and QMLE-res. Besides those $14^2$ restrictions of QMLE, we include some of the additional restrictions implied by $\Lambda$ and $M$ via delta method.

**Bayesian.** Use the Gibbs sampling to estimate $\alpha_t := [f_{g,t}, f_{g,t-1}, f_{g,t-2}, f_{C,t}]^\top$, $f_{g,t+1} = \phi f_{g,t} + \eta_{g,t}$ and $f_{C,t+1} = \eta_{C,t}$.

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