Remarks on the notion of quantum integrability

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Abstract. We discuss the notion of integrability in quantum mechanics. Starting from a review of some definitions commonly used in the literature, we propose a different set of criteria, leading to a classification of models in terms of different integrability classes. We end by highlighting some of the expected physical properties associated with models fulfilling the proposed criteria.

Keywords: quantum integrability (Bethe ansatz)
1. Introduction

Classical mechanics is a subject with a unique level of maturity: it is one of the most enjoyable to learn, and through the beauty and power of its formalism is often considered as a prototypical example of ‘how things should ideally be done’ in physics. One of the most powerful concepts in the study of the dynamics of classical systems is the notion of integrability (in the sense of Liouville, see e.g. [1]), namely that if a system with \( n \) degrees of freedom (i.e. with \( 2n \)-dimensional phase space) possesses \( n \) independent first integrals of motion in involution (i.e. Poisson-commuting), then the system is integrable by quadratures. The meaning of ‘integrable’ here is thus transparent, namely that the differential equations describing the time evolution can be explicitly integrated using action-angle variables. The solutions of the equations of motion thus display periodic motion on tori in phase space, and ergodicity is absent, in contrast to non-integrable models which explore phase space densely in the course of time. Besides providing explicit solutions for the time evolution, the classical notion of integrability thus partitions classical models into separate classes of integrable and non-integrable models with manifestly different physical behaviours.

It is thus a surprising (and insufficiently known) fact that translating the notion of integrability to the quantum context has faced numerous pitfalls, and remains to this day a subject of debate. This leads to some unfortunate widespread confusion, since integrability is mentioned very often in current discussions and publications concerning, among other themes, in- and out-of-equilibrium dynamics, relaxation and thermalization of many-body quantum systems under current theoretical and experimental investigation. If quantum integrability is ill-defined, how can we thus invoke it at all?

Questioning the precise meaning of ‘quantum integrability’ has been done on many occasions. Nearly two decades ago, in an eminently readable article, Weigert [2] summarized some fundamental issues and discussed the shortcomings of commonly used
definitions. Delving further into the details is, however, not usually done in research articles, but rather in private discussions or proceedings of lectures given by eminent researchers in the field, e.g. [3]. It is also a subject of ongoing work (see for a recent example [4]). Since quantum integrability was very often mentioned and discussed by many participants during the StatPhys 24 conference, we found it appropriate to use the occasion offered by these proceedings to share a few hopefully worthwhile thoughts, observations, suggestions and conclusions on this important theme.

The paper is organized as follows. We first put the problem in context, highlighting precisely what the problem is, and what kind of solution would ideally be required. In section 3, we review many definitions commonly found in the literature, and collect our thoughts and comments on each of them. After summarizing the conclusions reached, we propose a new categorization in section 4, and provide examples of where known models fit within our scheme in section 5. The physical consequences of our definition are discussed in section 6, which is followed by our conclusion.

2. Motivations

We can begin with the simplest and most important questions. Do we not have a proper definition already? Why is this question interesting and important? The first question will be answered in the negative in section 3. The second question is best answered by reflecting on the classical case: since the presence or absence of integrability in a classical system is associated with such drastic differences in physical behaviour, the lack of a proper understanding of the quantum equivalent inevitably means that we must be ‘missing out’ on some important properties and features. The lack of a quantum equivalent to the KAM theorem [5]–[7] (on the stability of quasi-periodic motion in the presence of small perturbations) is possibly the most striking illustration of this point, and makes it difficult to extract hard statements on the equilibration and thermalization of many-body quantum systems.

The lack of correspondence between classical and quantum integrability, which we will discuss further below, leads us to ask more basic questions about the differences between classical and quantum systems [8]. A first point worth remembering is that quantum mechanics differs markedly from classical mechanics in the way it counts degrees of freedom. In quantum mechanics, discretization of levels means that we can comfortably work with finite-dimensional Hilbert spaces: spins, bound atomic levels, etc have eigenstates which we can label with a discrete quantum number taking a finite set of values, and we typically say that the number of degrees of freedom of a quantum system is the dimensionality of its Hilbert space. By contrast, in classical mechanics, we count degrees of freedom by specifying how many pairs of conjugate phase space variables are necessary to specify the configuration of a system. Each variable can take on a continuum of values. In any quantum-classical correspondence, we would thus associate the number of classical degrees of freedom with the multiplicity of infinities of the dimension of the Hilbert space. There thus cannot be a classical equivalent to a quantum system with a finite-dimensional Hilbert space, and this already means that classical integrability is insufficient as a basis for defining quantum integrability in general.

When thinking about conserved charges, the notion of Liouville integrability includes a specification of how many independent charges we need, namely a number identical to
the number of degrees of freedom $n$ (in which case it is said that the system possesses a complete set of charges, or is completely integrable). If we can provide more than $n$ charges, the system is said to be superintegrable; if we can produce $2n$ charges, the system is maximally superintegrable (see e.g. [9, 10] and references therein). One semantic pitfall is associated with the word ‘complete’. Namely, one fundamental notion in quantum mechanics is that of a complete set of commuting observables (CSCO), namely a set of commuting quantum operators whose eigenvalues are sufficient to uniquely specify a state in Hilbert space. In the context of integrability, the word ‘complete’ takes on a different meaning: for a fully non-degenerate system, a single operator (the Hamiltonian) already forms a CSCO. The cardinality of a CSCO is thus patently not the number of conserved charges we should be looking for in the quantum case. We should of course be looking for a maximal Abelian subalgebra of quantum operators in Hilbert space, meaning that we should be able to display a number of commuting operators coinciding with the dimensionality of the Hilbert space in order to call our set of charges ‘complete’. To avoid this pitfall, we will thus rather talk about ‘maximal’ sets than ‘complete’ ones.

Before going further with our discussion of quantum integrability, it is worthwhile following the example of [2] and formulating a number of requirements for a meaningful and useful definition of this concept. Most importantly:

1. it should be unambiguous;
2. it should partition the set of all possible quantum models into distinct classes;
3. different classes of models should display distinguishable physical behaviour.

In addition to these, we could formulate a number of extra requirements, namely:
(a) the contact with the classical limit should be natural; (b) the contact with integrable field theory should be natural; (c) the different classes should be (algorithmically) distinguishable, i.e. it should be easily feasible to determine which class a model belongs to, etc. These are, however, less crucial criteria than the ones we have selected.

3. Common definitions used in the literature

In this section, we initiate our discussion by summarizing a number of definitions of quantum integrability encountered in the literature. We briefly comment on each one.

**QI:N.** A system is quantum integrable QI:N if it possesses a maximal set of independent commuting quantum operators $Q_\alpha$, $\alpha = 1, \ldots, \dim(\mathcal{H})$.

Allowing for a bit of flexibility in the precise terms used, this is (at least in spirit) overwhelmingly the most common definition of quantum integrability encountered in the literature. It has the appeal of being directly related to the classical notion of integrability, in the sense of being essentially a word-for-word translation after replacing Poisson brackets with commutators.

Definition QI:N is given the label N for a simple reason: it is too naive. Its fatal flaw is absolutely trivial: all quantum models associated with (limits of) finite-dimensional Hilbert spaces fall under the label QI:N. By the spectral theorem, all Hermitian Hamiltonians are readily diagonalizable; one thus obtains $\dim(\mathcal{H})$ orthogonal state vectors $|\Psi_\alpha\rangle$ from which one can build projectors $Q_\alpha = |\Psi_\alpha\rangle\langle\Psi_\alpha|$, the set of which constitutes a maximal independent commuting set. So is every quantum system we can
think of to be called integrable? Well, a court jester would amuse himself doing precisely this, but we have to reject this pathway by invoking one of the requirements we had about a proper definition, namely that it should separate models into distinct classes. Definition QI:N blatantly fails to do this, and is thus to be rejected as being formally useless.

Loopholes of definition QI:N were also extensively discussed in [2]. Loophole (A) there corresponds to the flaw mentioned above. A second loophole mentioned is associated with the notions of ‘maximal’ and ‘independent’ quantum operators: a theorem of von Neumann [11] states that it is possible to encode any number of commuting Hermitian operators into a single Hermitian operator $Q$ (in other words, any operator $Q_\alpha$ can then be viewed as a function $Q_\alpha = f_\alpha(Q)$), so the very basic notion of the number of independent operators actually seems ill-defined. Going further, since we allegedly cannot even properly count the number of charges we have, we are then prevented from honestly declaring that a set be maximal.

While we cannot repair the first fatal flaw of definition QI:N, let us address the second point in more detail. We do not dispute the validity of von Neumann’s theorem; however, we do not agree that it is of relevance here. First of all, thinking about the counting of conserved charges which can be defined as how one would represent them in terms of matrices in the eigenbasis, there is no doubt that the number required to call a set maximal coincides with the dimensionality of Hilbert space $\dim(\mathcal{H})$, since this is the number of independent diagonal entries. As far as counting is concerned, the notion of algebraic independence (that is, the set of charges does not obey any nontrivial polynomial equation) is sufficient to make it well-defined, and is already in use in the literature (see for example [9] and references therein).

We thus have to look for something beyond the naive definition. Of course, one of the main assumptions was that in practice we could actually diagonalize the Hamiltonian to obtain the charges as projectors. This suggests a more pragmatic definition:

**QI:ES.** A system is quantum integrable QI:ES if it is exactly solvable, in other words if we can construct its full set of eigenstates explicitly.

While this reminds us of the action-angle variables in the classical case, the reader will probably agree that this washed-down definition does not take us very far. This definition could be further categorized according to which method is employed to obtain the eigenstates: Fourier transform for free theories, Bethe ansatz for specific models (although the completeness of the set of Bethe eigenstates is not formally proven for all models), etc. We also reject this definition on the grounds that it does not fulfil all our criteria: the third, in particular, is hard to relate to.

**QI:HO.** A system is quantum integrable QI:HO if it can be mapped to harmonic oscillators.

This is not really practical: such a mapping is hard to construct explicitly, even for models we know how to solve exactly. Anyway, the existence of such a mapping is guaranteed by the fact that all $C^*$-algebras are unitarily equivalent.

A related definition is the one used in [12], where spin systems are classified as integrable if there exists an operator mapping leading to a Hamiltonian expressible in terms of action operators (i.e. $J_j$).

In the mathematical literature, there exists a so-called algebro-geometric definition of quantum complete integrability [13], implying that a system is algebraically integrable.
(i.e. its eigenvalue problem is solvable by quadratures, mirroring the classical case). This allows nice things such as the application of the logic of Darboux transformations in the quantum context (see e.g. [14]) to generate nontrivial models. This definition applies naturally to quantum models with continuous degrees of freedom (the notion of ‘complete’ is then like in the classical case), but we find it to be too close to the classical definition to be sufficiently general.

Instead of looking at exact solvability, the way out might be to look more precisely at the nature of the wavefunctions. This is in particular what the Bethe ansatz teaches us: scattering in one-dimensional integrable theories has the remarkable feature of being factorizable in two-body scatterings. This forms the basis for Sutherland’s definition:

\textbf{QI:ND}. A system is quantum integrable if the scattering it supports is non-diffractive [15].

This definition is very appealing. Although it does not directly mention the existence or the number of conserved charges, one can easily construct those from the set of conserved momenta, for example. It relates more or less directly to the classical definition, but is also well-defined, testable (at least in principle), and physically meaningful in the sense that non-diffractive scattering can be translated directly into non-ergodicity. It encompasses relativistic integrable field theory, where this non-diffraction condition is simply the factorization condition.

One shortcoming is that it relies on observing the effects of scattering in the ‘asymptotic region’ in the space of states (where all particles are clearly separated), and is thus only suited for models defined directly in the continuum. While we find this definition the best one available, we would rather find non-diffractive scattering as a consequence of integrability, rather than as its defining feature.

We can go further and try to formulate other ‘physical’ definitions, for example

\textbf{QI:ELS} A system is quantum integrable if its energy level statistics is Poissonian.

Looking at the energy level statistics of quantum models is another very appealing way to address the question of integrability, since it directly connects with classical mechanics via Berry and Tabor’s semiclassical reasoning [16]. This showed that the energy level statistics of generic quantum integrable systems is Poissonian. Since this is based on semiclassical reasoning, it only concerns quantum systems with continuous degrees of freedom (and thus infinite-dimensional Hilbert spaces). This was, however, further investigated for quantum lattice models [17]–[19] such as Heisenberg, $t-J$ and Hubbard-like models, explicitly verifying that for non-integrable cases the statistics becomes Wigner’s GOE. For Richardson–Gaudin-like models, the degree of freedom afforded by the large number of tunable internal parameters means that essentially any distribution can be mimicked; however, these quickly become Poissonian upon turning the interaction on [20]. We also refer the reader to the recent paper [21] for recent examples and more extensive literature citations. A related definition is:

\textbf{QI:LC} A system is quantum integrable if it shows level crossings (i.e. does not show level repulsion).

Here, the absence of level crossings in, for example, numerical solutions of finite models is then interpreted as an observation of non-integrability (see for example [22, 23] and references therein). This definition really pertains to families of models with a tunable parameter, and again cannot be sufficiently general.
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Overall, we would rather view these features of energy level distributions used in \( \text{QI:ELS} \) and \( \text{QI:LC} \) as consequences of integrability, rather than as definitions. In any case, there exist counterexamples, for example the Richardson-like models mentioned above, and Haldane–Shastry-type models [24]–[30], the main conclusion there being that the energy level statistics was neither Poissonian nor of the Wigner type. There is thus no universally acceptable definition of the form \( \text{QI:ELS} \); on the other hand, \( \text{QI:LC} \) formally only makes sense when discussing models with tunable parameters, which cannot cover all cases of interest.

3.1. Where does this leave us?

Let us attempt a summary of the important lessons of the preceding discussion. Most importantly, the mere existence of conserved charges (which for any model with a finite Hilbert space is a trivial consequence of the spectral theorem) cannot be a sufficiently meaningful criterion to classify quantum models in different classes. It is therefore imperative to look into the actual structure of the conserved charges when trying to distinguish classes of quantum models. But the structure of quantum operators really depends on the representation we use to write them down. We can always choose to work in the Hamiltonian eigenbasis, in which the charges are diagonal, and we can even define them such that they have a single matrix entry (this is the case for the projectors onto eigenstates). Therefore, the existence of a basis in which the charges look simple again does not provide us with a classification criterion, and a practical definition of quantum integrability cannot exist which does not identify a ‘preferred’ basis in which the structure of the charges must be looked at. In view of our (admittedly, optional) desire to keep a semblance of quantum-classical correspondence, we could restrict ourselves, for example, to ‘real space’ and ‘momentum space’ bases, or more generally accept that a definition of quantum integrability is only valid in a specified basis.

As we have mentioned before, the counting of the number of conserved charges is also a subtle point leading to various pitfalls, as our discussion about von Neumann’s theorem illustrated. A further pitfall occurs when counting is mentioned together with a physically meaningful concept, that of locality: there is a simple contradiction in terms when talking about a ‘maximal’ (or the unfortunately more commonly used ‘complete’) set of ‘local’ conserved charges. Enforcing ‘locality’ restricts the number of operators which can be defined to a number which is negligible in comparison to the required number for ‘maximalness’ (the dimensionality of the Hilbert space). It is thus patently impossible to have a maximal set of local charges, and one of the two concepts has to give way: either our required set is not maximal, or it contains non-local charges. This confusion probably originates from careless interpretations of the integrable quantum field theory literature, by confusing an infinite set of charges with a maximal one. This is also mentioned in the context of (the infinite size limit of) integrable lattice models: for example, in the fundamental paper [31], the set of conserved charges for the infinite Heisenberg chain was shown to form a maximal Abelian family, and this was equated (in the first paragraph of main text) to an ‘explicit, complete set of mutually commuting operator invariants which have local densities . . .’. We take issue with this use of words, and prefer to rephrase this in terms of the weaker statement that there then exist an infinite number of conserved charges which can be written as integrals of local densities, without specifying that this set
is ‘maximal’ (it is not; it only is ‘maximal’ with respect to local operators). This leaves us with the question: how many charges do we need? The lesson from the earlier discussion is that we can probably drop the ‘maximality’ requirement and still make meaningful statements.

With this in mind, let us now attempt to formulate a definition which takes care of all these issues.

4. Alternate definition of quantum integrability

We define a size sequence as an infinite sequence of strictly increasing integers \( (N_1, N_2, N_3, \ldots) \), \( N_1 < N_2 < N_3 < \cdots \). To each given size \( N_a \) in the size sequence, we associate a Hilbert space \( \mathcal{H}^{(N_a)} \) obtained from tensoring \( N_a \) elemental Hilbert spaces \( \mathcal{H}_j, j = 1, \ldots, N_a \). We assume that each elemental Hilbert space is finite-dimensional, \( \text{dim}(\mathcal{H}_j) \equiv d_j < \infty \), so \( \text{dim}(\mathcal{H}^{(N_a)}) = \prod_{j=1}^{N_a} d_j = d^{(N_a)} \) is also finite. As we move up the size sequence, we assume that we simply tensor in additional elemental spaces, \( \mathcal{H}^{(N_{a+1})} = \mathcal{H}^{(N_a)} \otimes_{j=N_a+1}^{N_{a+1}} \mathcal{H}_j \).

Operators in \( \mathcal{H}_j \) can be represented by \( d_j \times d_j \) Hermitian matrices, which can be decomposed in a basis \( e_j^i, i = 1, \ldots, d_j^2 \). Operators \( Q^{(N_a)} \) in \( \mathcal{H}^{(N_a)} \) can thus be decomposed in the \( (d^{(N_a)})^2 \) basis matrices \( e_1^{i_1} \cdots e_1^{i_d} = \otimes_{j=1}^{N_a} e_j^{i_j} \) (which we call the preferred basis), \( Q^{(N_a)} = \sum_{i_1 \cdots i_N} Q^{(N_a)}_{i_1 \cdots i_N} e_1^{i_1} \cdots e_{N_a}^{i_{N_a}} \). For a given operator \( Q \) in a given preferred basis \( e_1^{i_1} \cdots e_{N_a}^{i_{N_a}} \), we denote the number of nonzero entries \( Q^{(N_a)}_{i_1 \cdots i_{N_a}} \) as \( N_e(Q^{(N_a)}) \). For a size sequence of operators \( (Q^{(N_1)}, Q^{(N_2)}, Q^{(N_3)}, \ldots) \), we define the density character as the nature\(^2\) of the minimal function \( f(N_a) \) such that \( N_e(Q^{(N_a)}) < f(N_a) \forall a \).

We can now proceed to define a corresponding size sequence of Hermitian operators \( (H^{(N_1)}, H^{(N_2)}, H^{(N_3)}, \ldots) \), which we interpret as Hamiltonians acting in their respective Hilbert space \( \mathcal{H}^{(N_a)} \). We assume that the Hamiltonians in the size sequence are built according to some simple and meaningful algorithm \( \mathcal{A} : \mathcal{H}^{(N_a)} \rightarrow H^{(N_a)} \). We restrict ourselves to Hamiltonians having polynomial density character, in other words to Hamiltonians which can be expressed in terms of a finite number of sums over elemental labels.

Being Hermitian and finite, each Hamiltonian \( H^{(N_a)} \) obeys the spectral theorem and therefore possesses a complete set of eigenvectors. This also means that it automatically possesses a maximal set of conserved charges \( \{ Q^{(N_a)}_\alpha \}, \alpha = 1, \ldots, d^{(N_a)} \) in involution. We can arrange the labelling of the charges in such a way that meaningful finite size sequences \( (Q^{(N_a)}_\alpha, Q^{(N_{a+1})}_\alpha, Q^{(N_{a+2})}_\alpha, \ldots) \), for all individual \( Q^{(N_a)}_\alpha \) with \( \alpha = 1, \ldots, \alpha_{\text{max}} \leq d^{(N_a)} \) can be defined.

Definition. We will call a Hamiltonian \( H \) \( O(f(N)) \) quantum integrable if it is a member of a sequence \( (H^{(N_1)}, H^{(N_2)}, H^{(N_3)}, \ldots) \) of operators having \( O(f(N)) \) density character in the preferred basis, for which it is possible to define a sequence of sets of operators \( \{Q^{(N_1)}\}, \{Q^{(N_2)}\}, \{Q^{(N_3)}\}, \ldots \) such that

1 Contrary to what might be assumed, there is no implicit restriction to one-dimensional systems.

2 In our definition the terminology we use for \( O(f(N)) \) is identical to that used for classifying algorithmic time complexity. We can thus talk about linear, sublinear, quasi-polynomial, subexponential, etc. If needed, a term proportional to the unit operator should be subtracted when establishing the density character of an operator.
(1) all operators $Q^{(N_a)}_\alpha$ in $\{Q^{(N_a)}\}$ commute with each other and with their Hamiltonian $H^{(N_a)}$;
(2) the operators in $\{Q^{(N_a)}\}$ are algebraically independent;
(3) the cardinality $C^{(N_a)}$ of the set $\{Q^{(N_a)}\}$ becomes unbounded in the infinite size limit$^3$;
(4) each member $Q^{(N_a)}_\alpha$, $\alpha = 1, \ldots, C^{(N_a)}$ of the set $\{Q^{(N_a)}\}$ can be embedded within a sequence of operators $(Q^{(N_1)}_\alpha, Q^{(N_2)}_\alpha, Q^{(N_3)}_\alpha, \ldots)$ with $O(f(N))$ density character in the preferred basis.

We can thus talk about models in the linear, polynomial, quasi-polynomial, etc., integrability classes. For the preferred basis, while our formulation is purposefully left generic, we are of course mainly thinking about a real space based basis (in this case, the Fourier basis is also automatically compatible).

The reader might feel that this definition falls a bit ‘out of the blue’. We will attempt to make its meaning clearer by first specifying which classes known models fall into, and thereafter discussing in more physical terms how this definition should be understood.

5. Which integrability class do known models belong to?

Having suggested that quantum models can be classified into different categories, our task now turns to giving explicit examples.

Free theories

There is little doubt that free theories should be considered integrable in some way, and that they should be the simplest types of integrable models available. Imagine thus that we have a lattice model for a set of particles, with specified hopping amplitudes (with no restriction on dimensionality or on these being local and/or long-range) and without any form of interaction. Diagonalization can then be trivially achieved via Fourier transformation. Conserved charges can be directly built from the operators representing occupation of the Fourier modes. If we imagine, for example, a size sequence in which we double the system size at each step and always use periodic boundary conditions, we can embed each such conserved charge into a sequence of constant density character. We thus say that free hopping Hamiltonians are constant quantum integrable in the Fourier basis. Looked at in the real space basis, we can simply Fourier transform the conserved charges: these then become $O(N)$, and we can thus say that free hopping Hamiltonians are linear quantum integrable in the sites basis.

Fundamental models from quantum inverse scattering

Going beyond free theories, the most straightforward class of models to discuss are those which fit within the scheme of the quantum inverse scattering method (see [32] and references therein), in other words which are treatable with the algebraic Bethe ansatz. This relies on the existence of $R$-matrices solving the Yang–Baxter equation

$$R_{12}(\lambda, \mu)R_{13}(\lambda, \nu)R_{23}(\mu, \nu) = R_{23}(\mu, \nu)R_{13}(\lambda, \nu)R_{12}(\lambda, \mu).$$  \hspace{1cm} (1)

$^3$ Notice the fact that we do not require the set of conserved charges to be maximal.
The existence of monodromy matrices $T(\lambda)$ obeying the intertwining relation
\begin{equation}
R_{12}(\lambda, \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R_{12}(\lambda, \mu)
\end{equation}
then allows us to construct integrable models and their conserved charges by taking logarithmic derivatives of the transfer matrix $\tau(\lambda) \equiv \text{Tr} T(\lambda)$: the set of operators
\begin{equation}
Q_\alpha = c_\alpha \frac{d}{d\lambda} \ln \tau(\lambda) \bigg|_{\lambda=\xi}
\end{equation}
(where $\xi$ is some fixed evaluation parameter and $c_\alpha$ are arbitrary numerical constants) automatically constitutes an Abelian set by virtue of (2). It is reasonable to assume that the set of charges so obtained is maximal, though this remains a conjecture. This logic allows us to construct integrable quantum lattice models with local Hamiltonians by starting from a monodromy matrix which is a product of local operators, for example Heisenberg chains [33] and more general models [34,35]. Interestingly, the search for a classification of all possible solutions to the Yang–Baxter equation was initiated in works such as [36,37] but remains open to this day.

For the explicit case of the Heisenberg spin chain
\begin{equation}
H_{XXX} = J \sum_{j=1}^{N} S_j \cdot S_{j+1}
\end{equation}
(which has linear density character) the locality of the simplest conserved charges in the infinite system (here, locality means that the conserved charges can be expressed as integrals of local densities in the continuum limit) was proven in [38], the structure being explicitly of the form
\begin{equation}
Q_n = \sum_{\{j_1, \ldots, j_n-1\}} G^T_{n-1}(j_1, \ldots, j_{n-1}), \quad G^T_{n}(j_1, \ldots, j_n) = 0, \quad |j_1 - j_n| \geq n
\end{equation}
where the summation is over ordered subsets $\{j_1, \ldots, j_{n-1}\}$ of the chain and $G^T$ is a translationally invariant function. The set of local charges (which, we remind the reader, cannot be the maximal set of conserved charges if $n/N \to 0$ where $N$ is the total number of sites) was shown in [31] to be maximally Abelian on the subspace of states where all but a finite number of spins point in the same direction. For finite systems, the detailed structure of the conserved charges was systematically studied in a series of papers by Grabowski and Mathieu [39]–[41], and the explicit structure was found both for periodic and open boundary conditions. For example, for the $XXX$ chain, the simplest charge above the Hamiltonian is
\begin{equation}
Q_3 = \sum_j S_j \cdot (S_{j+1} \times S_{j+2}).
\end{equation}
A very interesting further paper by the same authors [40] suggests using the existence of a single higher local conserved charge as a litmus test for integrability. The conjectures offered in this paper are that: (1) a necessary condition for integrability is the existence of a conserved charge made of terms coupling at most three sites, and (2) a sufficient condition is the existence of a boost operator $B$ (in Baxter’s logic [42]) generating higher charges according to $[B, Q_n] = Q_{n+1}$. The authors recover many known models using this
logic, but while their ideas are very appealing and suggestive, they unfortunately did not produce new previously unknown cases.

By looking at the structure of the conserved charges above, we can immediately see that each model obtained from a transfer matrix composed of products of quasi-local operators has conserved charges having linear density character in the sites basis. All of these are thus linear quantum integrable in the sites basis. Note that a similar reasoning holds for inhomogeneous models, which can even be used to simulate disordered systems [43]. The list of what we call linear quantum integrable models thus includes many famous lattice models such as Heisenberg chains (also of higher spin), the \(t-J\) and Hubbard models. It also includes variants such as restricted Bose–Hubbard with up to two bosons per site, or low-density limits of such lattice models yielding continuum counterparts like the Lieb–Liniger model [44].

**Haldane–Shastry-type models**

An interesting generalization of the Heisenberg chain is the Haldane–Shastry long-range interacting model [45, 46],

\[
H_{\text{HS}} = \sum_{j_1, j_2=1}^{N} \frac{z_{j_1} z_{j_2}}{z_{j_1 j_2} z_{j_2 j_1}} S_{j_1} \cdot S_{j_2}
\]

(7)
in which \(z_j \equiv e^{2\pi i j / N}\) and \(z_{j_1 j_2} = z_{j_1} - z_{j_2}\). Since it involves a double sum over sites of simple operator combinations, the Hamiltonian of the Haldane–Shastry model is now an operator with quadratic density character. The conserved charges of the Haldane–Shastry chain can be constructed explicitly [47]. The first one above the Hamiltonian can be written

\[
Q_3 = i \sum_{j_1, j_2, j_3} \frac{z_{j_1} z_{j_2} z_{j_3}}{z_{j_1 j_2} z_{j_1 j_3} z_{j_2 j_3}} S_{j_1} \cdot (S_{j_2} \times S_{j_3}).
\]

(8)

Since this charge involves three space summations, it belongs to a size sequence with cubic density character. Each successive conserved charge includes one additional summation over sites. A specific conserved charge \(Q_n\) thus has density character \(O(N^n)\). Some care must thus be taken to characterize the class of integrability this model belongs to according to our definition. The cardinality of the set of conserved charges becomes unbounded as \(N \to \infty\), but a restricted set of charges up to some \(Q_{n_{\max}}\) can only be embedded in operator size sequences of density character \(O(N^{n_{\max}})\). To fulfill condition (3) of our definition, the Haldane–Shastry model must thus display a form of integrability which is distinct from the previous fundamental lattice models (like e.g. the Heisenberg chain): it cannot be a polynomial quantum integrable model. We can fit our definition of quantum integrability by choosing, for example, the density character \(O(N^{10 N}) \sim O(e^{10 N})\), which would make the Haldane–Shastry chain a quasi-polynomial quantum integrable model.

The Haldane–Shastry chain, which according to our definition thus displays a weaker form of integrability than the Heisenberg model, for example, is part of a more general family of long-range models of Calogero–Sutherland type [48]–[50], containing many generalizations [51]–[53]. Note that this class of models contains Heisenberg spin chains as a simple limit, and also more generally fits within the Yang–Baxter logic [54], but the transfer matrix now gives rise to the generators of a Yangian, a set of non-local charges...
which commute with all local conserved charges but not among themselves. Quasi-local conserved charges, including the one given above, can be obtained by expanding the so-called quantum determinant of the transfer matrix [47].

As mentioned before, the energy level statistics of Haldane–Shastry-type models is neither Poissonian nor of Wigner type. We propose to understand this fact via the observation that this family of models fall in the class of quasi-polynomial quantum integrable, rather than linear or polynomial.

**Richardson–Gaudin type models**

Yet another family of models which are interesting to discuss are the Richardson-like models [55]–[59]. Similarly to the Haldane–Shastry-type models, the Hamiltonian has quadratic density character. However, the conserved charges are simpler [58], and are of linear density character. We can thus put this family of models in the class of linear quantum integrable in the sites basis.

Finally, in a generic model (which still fulfils QI:N), it can be expected that all charges will be of exponential density character. We would thus by convention say that if a model is not at least subexponential integrable, it simply should be called non-integrable.

We could carry on mentioning other specific models. However, for the sake of compactness, and given the fact that the examples we have given clearly illustrate our idea, we now turn to the more crucial question of the physical interpretation of the new definition.

### 6. Physical meaning of the new definition

One of our requirements was that the classification scheme should be physically meaningful. We have already mentioned a few features traditionally associated with integrable models, such as factorized scattering, non-diffraction, particular energy level statistics, etc. The main theme which we would, however, like to understand better is how the classical notion of (non-)ergodicity translates to the quantum case.

In this context, the appropriate starting point for discussion is probably Mazur’s inequality [60], which has already proven useful in a number of cases, for example in [61]. Let us recall its statement [60,62,63]. Consider an operator $A$, and consider the canonical average $\langle \Delta A(0) \Delta A(t) \rangle$ where $\Delta A = A - \langle A \rangle$. If $\{ \bar{Q}_\alpha \}$, $\alpha = 1, \ldots, N_Q$ is a set of constants of motion, the long-time average correlation obeys

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle \Delta A(0) \Delta A(t) \rangle \geq \sum_{\alpha=1}^{N_Q} \left| \frac{\langle (\Delta A) \bar{Q}_\alpha \rangle^2}{\langle \bar{Q}_\alpha^2 \rangle} \right|^2$$

in which the charges are defined in such a way as to be ‘orthogonal’ for the chosen form of averaging,

$$\langle \bar{Q}_\alpha \bar{Q}_\beta \rangle = \delta_{\alpha\beta} \langle \bar{Q}_\alpha^2 \rangle$$

(note that we write the charges here as $\bar{Q}_\alpha$ to distinguish them from the charges we have used in section 4: our charges do not depend on the averaging procedure chosen, while the $\bar{Q}_\alpha$ charges do by implementation of (10)). The inequality becomes an equality if the set of charges used is maximal. The operator $A$ is called ergodic if

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle \Delta A(0) \Delta A(t) \rangle = 0,$$
in other words if it is orthogonal to all the conserved charges. The physical picture is that all correlations have dissipated in Hilbert space, and no quasi-periodic behaviour can be identified. Note the fact that in order to prove non-ergodicity, it is sufficient to find a single conserved charge having a finite overlap with the operator of interest (more elaborately, an infinite set of vanishing contributions summing up to a finite value), thereby making the right-hand side of (9) non-vanishing.

How does our definition of integrability relate to Mazur’s inequality? Let us make the following reasoning. We can first of all generalize Mazur’s inequality to a generic energy-diagonal expectation value,

\[ \langle \cdots \rangle \rightarrow \langle \cdots \rangle_f \equiv \frac{1}{Z_f} \sum_\alpha f_\alpha \langle \cdots | \alpha \rangle, \quad Z_f \equiv \sum_\alpha f_\alpha \]  

(12)

in which \( \alpha \) labels eigenstates and \( f_\alpha \) are specified real parameters. The form of the distribution is important: Mazur’s inequality trivializes to an empty statement if the average is taken over a single state, since the conserved charge eigenvalues then cancel in the numerator and denominator in each term of the right-hand side. We thus focus our attention on distributions with at least some measure of extensivity.

Let us consider physically meaningful operators \( A \) which are represented by simple matrices in the preferred basis. Such operators would thus be represented as

\[ A = \sum_{i_1, \ldots, i_N} A_{i_1 \cdots i_N} e^{i_1 \cdots i_N} \]  

(13)

with \( N_e(A) \) (using the terminology of section 4) being for example such that \( A \) is of polynomial density character (this would thus include all operators made of finite products of on-site operators or the Fourier transform of such objects).

Let us now consider computing the building blocks of Mazur’s inequality, namely the expectation values

\[ \langle (\Delta A) \bar{Q}_\alpha \rangle_f. \]  

(14)

This is really a measure of the ‘overlap’ of the operator \( \Delta A \) with the conserved charges we choose to work with. Following [62], this can be translated at the operator level for an ‘orthogonal’ (obeying (10)) set of charges \( \bar{Q}_\alpha \). One can write

\[ A = \sum_\alpha a_\alpha \bar{Q}_\alpha + A' \]  

(15)

where \( a_\alpha \) are c-numbers (implicitly depending on the averaging scheme chosen) and \( A' \) is an operator which is off-diagonal in energy, and therefore doesn’t contribute to Mazur’s inequality. The ratios in (9) are then simply related to the coefficients

\[ a_\alpha = \frac{\langle (\Delta A) \bar{Q}_\alpha \rangle_f}{\langle \bar{Q}_\alpha^2 \rangle_f}. \]  

(16)

The next steps in the discussion depend on the integrability class our system falls into. Let us discuss the generic non-integrable case first. In this case, all the conserved charges \( Q_\alpha \) are expressed in the preferred basis in terms of dense matrices with \( O(\dim(H)^2) \) entries, and thus have exponential density character. Pick any one of these charges, say \( Q_{\alpha_1} \), and use it as the basis for constructing the set \( \bar{Q}_\alpha \) of orthogonal charges to be used. The
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exponential density character of \( \bar{Q}_\alpha \) charge means that the coefficient \( a_{\alpha_1} \) in (15) will be exponentially small. Picking a second charge \( Q_{\alpha_2} \), using an orthogonalization procedure to satisfy (10), one remains (except for utterly contrived cases) with an exponential density character charge \( \bar{Q}_{\alpha_2} \) having vanishing overlap \( a_{\alpha_2} \) with our operator \( A \). This simply carries on, and no charge can be found which overlaps with \( A \), so Mazur’s inequality indicates ergodicity.

Let us now turn to the case of an integrable model according to our definition (for example, a linear quantum integrable model, for which we have many charges \( Q_\alpha \) at our disposal which can be written as single summations of local operator products). By their nature, these charges can be expected to have a finite overlap with operator \( A \). Enforcing the orthogonality condition (10) does not change the density character of the charges, so we obtain a set of orthogonal charges \( \bar{Q}_\alpha \) whose (summed) overlaps have the potential to give a finite value to the right-hand side of Mazur’s inequality. This must of course be checked in individual cases, but the door is clearly left open by the conserved charges’ simple structure, related (in the sense of their simple density character) to that of operator \( A \). The same conclusions would hold for other integrability classes, for appropriate operators having a sufficient degree of similarity to the (simple) conserved charges.

Let us make some simple and obvious remarks. First of all, of course, ergodicity only makes sense in the infinite system size limit. For a finite system, in parallel to the naive definition \( \text{QI:N} \) of quantum integrability, any quantum system will possess ‘reasonable’ observables failing to show ergodicity in Mazur’s sense; these observables will, however, not be physically meaningful for what we have referred to as non-integrable systems, since they must have exponential density character. One thus cannot say that a system is ergodic or not, but rather only that a given observable \( A \) is, for a given averaging in a given system. Our definition offers a way of understanding the origin of this ergodicity, or of its absence, by looking at the detailed structure of the conserved charges, and their relationship with the observables considered within the adopted averaging scheme. Our scheme of separating models into different integrability classes in fact leads us to expect that ergodicity manifests itself differently in each of these classes. We will further characterize this in subsequent publications.

Another context in which the role of quantum integrability is frequently discussed is the dynamics resulting from a non-equilibrium initial state \([64]–[67]\). A common way to bring a system out of equilibrium is by means of a so-called quantum quench \([68]\). Consider a Hamiltonian \( H_\lambda \) which depends on some parameter \( \lambda \). The quantum quench now consists of preparing the system in an eigenstate of \( H_\lambda \), followed by instantaneously changing the parameter \( \lambda \to \lambda' \). If \([H_\lambda, H_{\lambda'}] \neq 0\), one expects complicated non-equilibrium dynamics to ensue. To clarify this, let \( |\phi\rangle \) be an eigenstate of \( H_\lambda \) before the quench. We can now express \( |\phi(t)\rangle \) for \( t > 0 \) as \( e^{-iH_{\lambda'}t} |\phi\rangle = \sum_n c_n |n\rangle e^{-iE_n t} \), where the summation is over all eigenstates of \( H_{\lambda'} \) and the coefficients \( c_n = \langle n|\phi\rangle \) represent the overlaps between the initial state \( |\phi\rangle \) and the eigenstates \( |n\rangle \) of \( H_{\lambda'} \). The complicated form of \( |\phi(t)\rangle \) makes it hard to make predictions at short time scales. It is common to simplify the problem by considering the long-time average of an observable \( A \),

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \phi(t)|A|\phi(t)\rangle = \sum_n |c_n|^2 \langle n|A|n\rangle
\]

(17)
which has the same energy-diagonal form as (9)\textsuperscript{4}. For most quenches, we can also make the assumption that the distribution \{\vert c_n \vert^2\} is narrow in energy. For a sufficiently large system one can try to approximate the distribution \{\vert c_n \vert^2\} by a canonical ensemble
\begin{equation}
\sum_n \vert c_n \vert^2 \langle n \vert A \vert n \rangle^2 = \frac{1}{Z} \text{Tr} \{ A e^{-\beta H} \}
\end{equation}
where the inverse temperature \( \beta \) is determined from the initial condition
\begin{equation}
\langle \phi(0) \vert H \vert \phi(0) \rangle = \frac{1}{Z} \text{Tr} \{ H e^{-\beta H} \}
\end{equation}
and \( Z \) is the usual partition function \( Z = \text{Tr} \{ e^{-\beta H} \} \). This approximation, while successful for some cases, fails in general for integrable systems, this failure being typically attributed to the presence of additional conserved charges. It is thus natural and appealing to generalize the canonical ensemble by the so-called generalized Gibbs ensemble [69],
\begin{equation}
\langle A \rangle_{\text{GGE}} = \frac{1}{Z} \text{Tr} \{ A e^{-\sum_j \beta_j Q_j} \},
\end{equation}
the generalized partition function \( Z \) being now computed as \( Z = \text{Tr} e^{-\sum_i \beta_i Q_i} \). Initial (quench-time) conditions fix the Lagrange multipliers \( \beta_n \) via the self-consistency
\begin{equation}
\langle \phi(0) \vert Q_i \vert \phi(0) \rangle = \frac{1}{Z} \text{Tr} \{ Q_i e^{-\sum_j \beta_j Q_j} \}.
\end{equation}
Using all available conserved charges as per the original formulation of this approach (in other words by adopting the QI:N view of integrability), one might consider the eigenstate projectors \( Q_n = \vert n \rangle \langle n \vert \) as a maximal set of conserved charges to be exploited. While this would lead to an exact description, this is of limited use since in this case \( e^{-\beta_n Q_n} = \vert c_n \vert^2 \) are simply the overlap coefficients mentioned above, involving \( O(\dim(H)) \) parameters. What is lacking in the description of the generalized Gibbs ensemble is thus a prescription for which conserved charges should actually be included in the scheme. Since we assumed that the distribution \{\vert c_n \vert^2\} is narrow in energy, one should include (besides the inevitable Hamiltonian) conserved charges that have a small variance in the distribution \{\vert c_n \vert^2\}. In contrast, if a conserved charge \( Q_n \) has large variance, its Lagrange multiplier \( \beta_n \) would be negligible, and including this charge in the scheme would not provide meaningful state selection. For an integrable system as defined in section 4, many conserved charges having a similar character as the Hamiltonian are available (these having non-negligible Lagrange multipliers), and one cannot generally neglect their effect on the distribution. This is not a behaviour exclusive to integrable systems; recent examples [70] are known of far from integrable models retaining memory of initial conditions, with their long-time states still described by a maximum entropy or generalized Gibbs ensemble (the latter was also recently shown to be a useful concept in the context of quenches in integrable field theory [71]). The intuitive picture which we can, however, propose is that in the generic non-integrable case most of the conserved charges will have an exponentially large variance for this particular distribution, and can thus be disregarded. Our proposed definition of quantum integrability thus leads us to suggest using restricted generalized Gibbs ensembles, which we will return to in future work.

\textsuperscript{4} As in Mazur’s inequality, in general the summation also contains off-diagonal expectation values \( \langle n \vert A \vert m \rangle \) as a result of degenerate energy levels. We neglect these here for simplicity.
7. Conclusions

One of the difficulties in discussing quantum integrability is of course that it can very quickly feel at best confusing, and at worst rather pedantic. We feel that these proceedings offered the right kind of forum for this contribution, in view of the importance of an increased clarity of concepts for use in discussions about quantum many-body equilibration dynamics. Instead of attempting to define ‘quantum integrability’, it might have been more reasonable and less confusing to simply introduce new terminology, avoiding the keyword ‘integrability’ altogether. We have tried to mitigate this risk by suggesting the association of the terminology of algorithmic time complexity with any specific mention of quantum integrability. If we say then that the Heisenberg chain is linear quantum integrable, it will be clear that the notions and definition of our paper are the ones that are referred to, and what consequences we associate with such a categorization.

One point which we should clarify is how we reconcile the current concepts with integrable field theories (ITF) and conformal field theories. The point is that we have considered in detail how the morphology of the conserved charges of a given model behaves as the system size is increased, i.e. as one moves towards a continuum/field theory limit. If a model is classified as linear integrable in the sites basis within our scheme, it will automatically lead to a field theory having an infinite number of conserved charges expressible as integrals of local densities by choosing an $o(N)$ subset of charges in the finite lattice regularization.

In the beginning, we listed our three main requirements: that a proper definition of quantum integrability should be unambiguous, and should lead to different classes of models, these having different discernible types of physical behaviour. On this last point, while we have provided heuristic arguments, the specific and detailed consequences of our definition of quantum integrability have not yet been fully worked out to the level of general theorems. This first involves a rather immense amount of work of looking at specific examples in detail, which we are pursuing at this time. This will teach us whether our definition actually has interesting substance, which is perhaps not completely convincing at this point. While we think that our definition is not ambiguous, the level of mathematical rigour we have adopted is rather low in comparison to what is achievable in the field of integrable models, but should be looked at in correspondence to the level of discussion on algorithmic time complexity. We can thus be satisfied overall that our initial objectives feel as if they have been met. Much work remains to be done, and we will extract more specific conclusions on particular classes of models and observables in future publications.

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