RESONANCE PROJECTORS AND ASYMPTOTICS FOR $r$-NORMALLY HYPERBOLIC TRAPPED SETS

SEMYON DYATLOV

Abstract. We prove an asymptotic formula for the number of scattering resonances in a strip near the real axis when the trapped set is $r$-normally hyperbolic with $r$ large and a pinching condition on the normal expansion rates holds. Our dynamical assumptions are stable under smooth perturbations and motivated by the setting of black holes. The key tool is a Fourier integral operator which microlocally projects onto the resonant states in the strip. In addition to Weyl law, this operator provides new information about microlocal concentration of resonant states.

For a semiclassical Schrödinger operator $h^2 \Delta_g + V(x), V \in C^\infty(X; \mathbb{R})$, on a compact Riemannian manifold $(X, g)$ the Weyl law (see for example [DiSj, Theorem 10.1]) provides an asymptotic for the number of eigenvalues $\lambda_j(h)$ as $h \to 0$:

$$#(\lambda_j(h) \in [\alpha_0, \alpha_1]) = (2\pi h)^{-n} (\text{Vol}_\sigma(p_V^{-1}([\alpha_0, \alpha_1])) + O(h)).$$  

(1.1)

Here $n$ is the dimension of $X$, $p_V(x, \xi) = |\xi|^2_g + V(x)$ is the (semiclassical) principal symbol of our Schrödinger operator, defined on the cotangent bundle $T^*X$, and $\text{Vol}_\sigma$ is the symplectic volume on $T^*X$.

A natural generalization of eigenvalues to noncompact manifolds are resonances, the poles of the meromorphic continuation of the resolvent to the lower half-plane $\{\text{Im} \omega \leq 0\} \subset \mathbb{C}$, see (1.3) and §§4.3, 4.4. However, there are very few results giving Weyl asymptotics of resonances in the style of (1.1). The first one is probably due to Regge [Re], with some of the following results including [Zw87, SjVo, SjZw99, Sj11, FaTs1] – see the discussion of related work below.

This paper provides a new Weyl asymptotic formula for resonances, under the assumption that the trapped set is $r$-normally hyperbolic and expansion rates satisfy a pinching condition – see Theorems 1 and 2. These dynamical assumptions are motivated by the study of black holes, see [KoSc]; this continues the previous work of the author [Dy11a, Dy11b, Dy12], and the application to stationary perturbations of Kerr–de Sitter black holes will be given in [Dy13]. See also [GSWW] for applications of normally hyperbolic trapping to molecular dynamics. Because the imaginary part of a resonance can be interpreted as the exponential decay rate of the corresponding linear wave, we study long-living resonances, i.e. those in strips of size $Ch$ around the real axis. More precisely, we establish an asymptotic formula for the number of resonances in a band located between two resonance free strips.
Setup. To illustrate the results, we consider here the setting of semiclassical Schrödinger operators on \( X = \mathbb{R}^n \), studied in detail in §4.3:

\[
P_V := \hbar^2 \Delta + V(x), \quad V \in C^\infty_0(\mathbb{R}^n; \mathbb{R}).
\] (1.2)

Here \( \Delta = -\sum_j \partial_{x_j}^2 \) is the Euclidean Laplacian. The results apply under the more general assumptions of §§4.1 and 5.1, in particular in the setting of even asymptotically hyperbolic manifolds – see §4.4 and Appendix A. Resonances are the poles of the meromorphic continuation of the resolvent

\[
R_V(\omega) = (P_V - \omega^2)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n), \quad \text{Im} \omega > 0,
\] (1.3)

across the ray \((0, \infty) \subset \mathbb{C}\), as a family of operators \( L^2_{\text{comp}}(\mathbb{R}^n) \to H^2_{\text{loc}}(\mathbb{R}^n) \). For the proofs, it is convenient to consider a different operator with the same set of poles

\[
\mathcal{R}(\omega) = \mathcal{P}(\omega)^{-1} : \mathcal{H}_2 \to \mathcal{H}_1,
\] (1.4)

where \( \mathcal{H}_1 = H^2_h(\mathbb{R}^n) \) is a semiclassical Sobolev space, \( \mathcal{H}_2 = L^2(\mathbb{R}^n) \), and \( \mathcal{P}(\omega) : \mathcal{H}_1 \to \mathcal{H}_2 \) is constructed from \( P_V \) using the method of complex scaling (see §4.3).

To formulate dynamical assumptions, let \( p_V(x, \xi) = |\xi|^2 + V(x) \), fix energy intervals \([\alpha_0, \alpha_1] \subset [\beta_0, \beta_1] \subset (0, \infty)\), put \( p = \sqrt{p_V} \) on \( p_V^{-1}([\beta_0^2, \beta_1^2]) \) (see (4.4) for the general case) and define the incoming/outgoing tails \( \Gamma_\pm \) and the trapped set \( K \) as

\[
\Gamma_\pm := \{ \rho \in p_V^{-1}([\beta_0^2, \beta_1^2]) \mid \exp(tH_\rho)(\rho) \not\to \infty \text{ as } t \to \mp \infty \}, \quad K := \Gamma_+ \cap \Gamma_-.
\]

Here \( \exp(tH_\rho) \) denotes the Hamiltonian flow of \( p \). We assume that (see §5.1 for details) \( \Gamma_\pm \) are sufficiently smooth codimension one submanifolds intersecting transversely at \( K \), which is symplectic, and the flow is \( r\text{-normally hyperbolic} \) for large \( r \) in the sense that the minimal expansion rate \( \nu_{\text{min}} \) of the flow \( \exp(tH_\rho) \) in the directions transverse to \( K \) is much greater than the maximal expansion rate \( \mu_{\text{max}} \) along \( K \) – see (5.1), (5.3), (5.4). These assumptions are stable under small smooth perturbations of the symbol \( p \), using the results of [HiPuSh] – see §5.2.

Distribution of resonances. Let \( \nu_{\text{max}} \) be the maximal expansion rate of the flow \( \exp(tH_\rho) \) in the directions transverse to the trapped set, see (5.2). The following theorem provides a resonance free region with a polynomial resolvent bound:

**Theorem 1.** Let the assumptions of §§4.1 and 5.1 hold and fix \( \varepsilon > 0 \). Then for

\[
\text{Re} \omega \in [\alpha_0, \alpha_1], \quad \text{Im} \omega \in [-(\nu_{\text{min}} - \varepsilon)h, 0] \setminus \frac{1}{2}(-\nu_{\text{max}} + \varepsilon)h, -(\nu_{\text{min}} - \varepsilon)h), \quad (1.5)
\]

\( \omega \) is not a resonance and we have the bound\(^1\)

\[
\| \mathcal{R}(\omega) \|_{\mathcal{H}_2 \to \mathcal{H}_1} \leq Ch^{-2}. \quad (1.6)
\]

\(^1\)The estimate (1.6) implies, in the case (1.2), cutoff resolvent bounds \( \| \chi R_V(\omega) \chi \|_{L^2 \to H^2_h} = \mathcal{O}(h^{-2}) \) for any fixed \( \chi \in C^\infty_0(\mathbb{R}^n) \).
In particular, we get a resonance free strip \( \{ \text{Im} \omega > -\frac{\nu_{\text{min}} - \varepsilon}{2} h \} \), recovering in our situation the results of [GeSj87, WuZw11, NoZw13] – see below for a detailed discussion.

Under the pinching condition

\[ \nu_{\text{max}} < 2\nu_{\text{min}}, \quad (1.7) \]

we get a *second* resonance free strip \( \{ \text{Im} \omega \in \left[ -(\nu_{\text{min}} - \varepsilon)h, -(\nu_{\text{max}} + \varepsilon)h/2 \right] \} \). We can then count the resonances in the band between the two strips, see Figure 1(a):

**Theorem 2.** Let the assumptions of §§4.1 and 5.1 and the condition (1.7) hold. Fix \( \varepsilon > 0 \) such that \( \nu_{\text{max}} + \varepsilon < 2(\nu_{\text{min}} - \varepsilon) \). Then, with \( \text{Res} \) denoting the set of resonances counted with multiplicities (see (4.3)),

\[
\#(\text{Res} \cap \{ \text{Re} \omega \in [\alpha_0', \alpha_1'], \ \text{Im} \omega \in \frac{1}{2}[-(\nu_{\text{max}} + \varepsilon)h, -(\nu_{\text{min}} - \varepsilon)h] \}) = (2\pi h)^{1-n}(\text{Vol}_\sigma(K \cap p^{-1}([\alpha_0', \alpha_1'])) + o(1)),
\]

as \( h \to 0 \), for every \([\alpha_0', \alpha_1'] \subset (\alpha_0, \alpha_1)\) such that \( p^{-1}(\alpha_0') \cap K \) has zero measure in \( K \). Here \( \text{Vol}_\sigma \) denotes the symplectic volume on \( K \), defined by \( d\text{Vol}_\sigma = \sigma_{S}^{n-1}/(n-1)! \).

A band structure similar to the one exhibited in Theorems 1 and 2, with Weyl laws in each band, has been obtained in [FaTs1] for a related setting of Anosov diffeomorphisms, see the discussion below.

**The resonance projector.** The key tool in proving Theorems 1 and 2 is a microlocal projector \( \Pi \) corresponding to resonances in the band (1.8). We construct it as a *Fourier integral operator* (see §3.2), associated to the canonical relation \( \Lambda^\circ \subset T^*X \times T^*X \) defined as follows. Let \( \mathcal{V}_\pm \subset T\Gamma_\pm \) be the symplectic complements of \( T\Gamma_\pm \) in \( T_{\Gamma_\pm}(T^*X) \).
For some neighborhoods $\Gamma^0_\pm, K^0$ of $K \cap p^{-1}([\alpha_0, \alpha_1])$ in $\Gamma_\pm, K$, respectively, we can define the projections $\pi_\pm : \Gamma^0_\pm \to K^0$ along the flow lines of $V_\pm$—see \S 5.4. We define (see also [BFRZ])

$$\Lambda^0 := \{(\rho_-, \rho_+) \in \Gamma^0_- \times \Gamma^0_+ \mid \pi_-(\rho_-) = \pi_+(\rho_+)\}.$$  \hfill (1.9)

Then $\Lambda^0$ is a canonical relation, see \S 5.4; it is pictured on Figure 1(b).

We now construct an operator $\Pi$ with the following properties (see Theorem 3 in \S 7.1 for details, including a uniqueness statement):

1. $\Pi$ is a compactly supported Fourier integral operator associated to $\Lambda^0$;
2. $\Pi^2 = \Pi + \mathcal{O}(h^\infty)$ microlocally near $K \cap p^{-1}([\alpha_0, \alpha_1])$;
3. $[P, \Pi] = \mathcal{O}(h^\infty)$ microlocally near $K \cap p^{-1}([\alpha_0, \alpha_1])$.

Here $P$ is a pseudodifferential operator equal to $\sqrt{F_V}$ microlocally in $p^{-1}([\beta_0, \beta_1])$ (see Lemma 4.3 for the general case). Conditions (2) and (3) mimic idempotency and commutation properties of spectral projectors of self-adjoint operators.

The operator $\Pi$ is constructed iteratively, solving a degenerate transport equation on each step, with regularity of resulting functions guaranteed by $r$-normal hyperbolicity. The obtained operator provides a rich microlocal structure, which makes it possible to locally relate our situation to the Taylor expansion, ultimately proving Theorems 1 and 2. See \S 2.1 for a more detailed explanation of the ideas behind the proofs.

**Related work.** A particular consequence of Theorem 1 is a resonance free strip

$$\{\text{Im } \omega > -\frac{\kappa_{\text{min}} - \varepsilon}{2} h\}.$$  \hfill (1.10)

For normally hyperbolic trapped sets, such strips (also called spectral gaps) have been obtained by Gérard–Sjöstrand [GéSj88] for operators with analytic coefficients and possibly non-smooth $\Gamma_\pm$; Wunsch–Zworski [WuZw11] for sufficiently smooth $\Gamma_\pm$, without specifying the size of the gap; and Dolgopyat [Do], Liverani [Li], and Tsujii [Ts] for contact Anosov flows. The recent preprint of Nonnenmacher and Zworski [NoZw13] gives a gap of optimal size for a variety of normally hyperbolic trapped sets with very weak assumptions on the regularity of $\Gamma_\pm$; in our special case, the gap of [NoZw13] coincides with the one given by Theorem 1. For a related, yet quite different, case of hyperbolic trapped sets (where the flow is hyperbolic in all directions, but no assumptions are made on the regularity of $\Gamma_\pm$ and $K$), such gaps are known under a pressure condition, see [NoZw09] and the references given there.

Upper bounds for the number of resonances in strips near the real axis have been proved in different situations, both for normally hyperbolic and for hyperbolic trapping, by Sjöstrand [Sj90], Guillopé–Lin–Zworski [GuLiZw], Sjöstrand–Zworski [SjZw07], Nonnenmacher–Sjöstrand–Zworski [NoSjZw1, NoSjZw2], Faure–Sjöstrand [FaSj], Datchev–Dyatlov [DaDy], and Datchev–Dyatlov–Zworski [DaDyZw]; see [NoSjZw2] or [DaDy] for a more detailed overview. The optimal known bounds follow the fractal Weyl law,

$$\#(\text{Res} \cap \{\text{Re } \omega \in [\alpha_0, \alpha_1], \ |\text{Im } \omega| \leq C_0 h\}) \leq C h^{-1-\delta}.$$  \hfill (1.10)
Here $C_0$ is any fixed number and $2\delta + 2$ is bigger than the upper Minkowski dimension of the trapped set $K$ (inside $T^* X$), or equal to it if $K$ is of pure dimension. In our case, $\dim K = 2n - 2$, therefore the Weyl law (1.8) saturates the bound (1.10).

Much less is known about lower bounds for hyperbolic or normally hyperbolic trapped sets – some special completely integrable cases were studied in particular by Gérard–Sjöstrand [GéSj87], Sá Barreto–Zworski [SáZw], and the author [Dy12], a lower bound with a smaller power of $h^{-1}$ than (1.10) for certain hyperbolic surfaces was proved by Jakobson–Naud [JaNa], and Weyl laws have been established in some situations in [SjZw99, SjVo, FaTs1, FaTs2, FaTs3] – see below. It has been conjectured [No, Definition 6.1] that for $C_0$ large enough, a lower bound matching (1.10) holds, but no such bound for non-integer $\delta$ has been proved so far.

There also exists a Weyl asymptotic for surfaces with cusps, see Müller [Mii]; in this case, the infinite ends of the manifold are so narrow that almost all trajectories are trapped, and the Weyl law in strips coincides with the Weyl law in disks, with a power $h^{-n}$. Other Weyl asymptotics in large regions in the complex plane have been obtained by Zworski [Zw87] for one-dimensional potential scattering and by Sjöstrand [Sj11] for Schrödinger operators with randomly perturbed potentials.

Finally, some situations where resonances form several bands of different depth were studied in [SjZw99, StVo, SjVo, FaTs1, FaTs2, FaTs3]. Sjöstrand–Zworski [SjZw99] showed existence of cubic bands of resonances for strictly convex obstacles, under a pinching condition on the curvature, with a Weyl law in each band. Stefanov–Vodev [StVo] studied the elasticity problem outside of a convex obstacle with Neumann boundary condition and showed existence of resonances $O((\Re \omega)^{-\infty})$ close to the real line and a gap below this set of resonances; a Weyl law for resonances close to the real line was proved by Sjöstrand–Vodev [SjVo]. A case bearing some similarities to the one considered here, namely contact Anosov diffeomorphisms, has been studied by Faure–Tsujii [FaTs1]; their upcoming work [FaTs2, FaTs3] will handle contact Anosov flows – the latter can be put in the framework of §4.1 using the work of Faure–Sjöstrand [FaSj].

The results of [FaTs1, FaTs2, FaTs3] for the dynamical setting include, under a pinching condition, the band structure of resonances (with the first band analogous to the one in Theorem 2) and Weyl asymptotics in each band; the trapped set has to be normally hyperbolic, symplectic, and smooth, however the manifolds $\Gamma_{\pm}$ need only have Hölder regularity, and no assumption of $r$-normal hyperbolicity is made. These considerably weaker assumptions on regularity are crucial for Anosov flows and maps, as one cannot even expect $\Gamma_{\pm}$ to be $C^2$ in most cases. The lower regularity is in part handled by conjugating $P(\omega)$ by the exponential of an escape function, similar to the one in [DaDyZw, Lemma 4.2] – this reduces the analysis to an $O(h^{1/2})$ sized neighborhood of the trapped set. It then suffices to construct only the principal part of the projector $\Pi$ to first order on the trapped set; such projector is uniquely defined
locally on $K$ (by putting the principal symbol to be equal to 1 on $K$), without the need for the global construction of §7.1 or the transport equation (2.2). The present paper however was motivated by resonance expansions on perturbations of slowly rotating black holes, where the more restrictive $r$-normal hyperbolicity assumption is satisfied and it is important to have an operator $\Pi$ defined to all orders in $\hbar$ and away, as well as on, the trapped set. Another advantage of such a global operator is the study of resonant states, see §8.5.

2. Outline of the paper

In this section, we explain informally the ideas behind the construction of the projector $\Pi$ and the proofs of Theorems 1 and 2, list some directions in which the results could possibly be improved, and describe the structure of the paper.

2.1. Ideas of the proofs and concentration of resonant states.

Construction of $\Pi$. An important tool is the model case (see §6.1)

$$X = \mathbb{R}^n, \quad \Gamma_0^- = \{x_n = 0\}, \quad \Gamma_0^+ = \{\xi_n = 0\}, \quad \Pi^0 f(x', x_n) = f(x', 0).$$

Any operator satisfying properties (1) and (2) of $\Pi$ listed in the introduction can be microlocally conjugated to $\Pi^0$ (see Proposition 6.3 and part 2 of Proposition 6.9). However, there is no canonical way of doing this, and to construct $\Pi$ globally, we need to use property (3), which eventually reduces to solving the transport equation on $\Gamma_\pm$

$$H_p a = f, \quad a|_K = 0,$$

where $f$ is a given smooth function on $\Gamma_\pm$ with $f|_K = 0$. The solution to (2.2) exists and is unique for any normally hyperbolic trapped set, by representing $a(\rho)$ as an exponentially converging integral of $f$ over the forward ($\Gamma_-$) or backward ($\Gamma_+$) flow line of $H_p$ starting at $\rho$. However, to know that $a$ lies in $C^r$ we need $r$-normal hyperbolicity (see Lemma 5.2). This explains why $r$-normal hyperbolicity, and not just normal hyperbolicity, is needed to construct the operator $\Pi$.

Proof of Theorem 1. The proof in §8 is based on positive commutator arguments, with additional microlocal structure coming from the projector $\Pi$ and the annihilating operators $\Theta_\pm$ discussed below. However, here we present a more intuitive (but harder to make rigorous) argument based on propagation by

$$U(t) = e^{-itP/\hbar},$$

which is a Fourier integral operator quantizing the Hamiltonian flow $e^{iH_p}$ (see Proposition 3.1). Note that we use not the original operator $P(\omega)$, but the operator $P$ constructed in Lemma 4.3, equal to $\sqrt{P_V}$ for the case (1.2); this means that $U(t)$ is the wave, rather than the Schrödinger, propagator. We will only care about the behavior of $U(t)$ near the trapped set; for this purpose, we introduce a pseudodifferential cutoff
analog of (2.3) is Proposition 8.1, and of (2.4), Proposition 8.2): for $t > 0$, $K$ semiclassic wavefront set (as discussed in §3.1) is contained in a small neighborhood of $K \cap p^{-1}([\alpha_0, \alpha_1])$, Theorem 1 follows from the following two estimates (a rigorous analog of (2.3) is Proposition 8.1, and of (2.4), Proposition 8.2): for $t > 0$,

$$\| \mathcal{X} U(t)(1 - \Pi) f \|_{L^2} \leq (C h^{-1} e^{-\nu_{\min} \epsilon/2}t + \mathcal{O}(h^\infty)) \| f \|_{L^2}, \tag{2.3}$$

$$C^{-1} e^{-\nu_{\max} \epsilon/2} \| \mathcal{X} \Pi f \|_{L^2} - \mathcal{O}(h^\infty) \| f \|_{L^2} \leq \| \mathcal{X} U(t) \Pi f \|_{L^2} \leq C e^{-\nu_{\min} \epsilon/2} \| \mathcal{X} \Pi f \|_{L^2} + \mathcal{O}(h^\infty) \| f \|_{L^2}. \tag{2.4}$$

The estimates (2.3) and (2.4) are of independent value, as they give information about the long time behavior of solutions to the wave equation, resembling resonance expansions of linear waves; an application to black holes will be given in [Dy13]. Note however that these estimates are nontrivial only when $t = \mathcal{O}(\log(1/h))$, because of the $\mathcal{O}(h^\infty)$ error term.

The resonance free region (1.5) of Theorem 1 is derived from here as follows. Assume that $\omega$ is a resonance in (1.5). Then there exists a resonant state, namely a function $u \in \mathcal{H}_1$ such that $\mathcal{P}(\omega) u = 0$ and $\| u \|_{\mathcal{H}_1} \sim 1$. We formally have $U(t) u = e^{-i\omega t/h} u$. Also, $u$ is microlocalized on the outgoing tail $\Gamma_+$, which is propagated by the flow $e^{t H_p}$ towards infinity; this means that if $f := \mathcal{X}_1 u$ for a suitably chosen pseudodifferential cutoff $\mathcal{X}_1$, then $\Pi u = \Pi f + \mathcal{O}(h^\infty)$ and for $t > 0$,

$$U(t) f = e^{-i\omega t/h} f + \mathcal{O}(h^\infty) \text{ microlocally near } \text{WF}_h(\mathcal{X}).$$

Since $\Pi$ commutes with $P$ modulo $\mathcal{O}(h^\infty)$, it also commutes with $U(t)$, which gives

$$\mathcal{X} U(t)(1 - \Pi) f = e^{-i\omega t/h} \mathcal{X}(1 - \Pi) f + \mathcal{O}(h^\infty),$$

$$\mathcal{X} U(t) \Pi f = e^{-i\omega t/h} \mathcal{X} \Pi f + \mathcal{O}(h^\infty).$$

Since $\text{Im} \omega \geq -(\nu_{\min} - \epsilon) h$, we take $t = N \log(1/h)$ for arbitrarily large constant $N$ in (2.3) to get $\| \mathcal{X}(1 - \Pi)f \|_{L^2} = \mathcal{O}(h^\infty)$. Since $\text{Im} \omega \notin (-\nu_{\max} + \epsilon) h/2, -(\nu_{\min} - \epsilon) h/2)$, by (2.4) we get $\| \mathcal{X} \Pi f \|_{L^2} = \mathcal{O}(h^\infty)$. Together, they give $\| \mathcal{X} f \|_{L^2} = \mathcal{O}(h^\infty)$, implying by standard outgoing estimates (see Lemma 4.6) that $\| u \|_{\mathcal{H}_1} = \mathcal{O}(h^\infty)$, a contradiction.

We now give an intuitive explanation for (2.3) and (2.4). We start by considering the model case (2.1), with the pseudodifferential cutoff $\mathcal{X}$ replaced by the multiplication operator by some $\chi \in C_0^\infty(\mathbb{R}^n)$. For the operator $P$, we consider the model (somewhat inappropriate since the actual Hamiltonian vector field $H_p$ is typically nonvanishing on $K$, contrary to the model case, but reflecting the nature of the flow in the transverse directions) $P = x_n \cdot h D x_n - i h/2$; here the term $-i h/2$ makes $P$ symmetric. We then have in the model case, $p = x_n \xi_n, e^{t H_p}(x, \xi) = (x', e^t x_n, \xi', e^{-t} \xi_n)$, $\nu_{\min} = \nu_{\max} = 1$, and

$$U(t) f(x', x_n) = e^{-t/2} f(x', e^{-t} x_n).$$
Then (2.3) (in fact, a better estimate with $e^{-3t/2}$ in place of $e^{-t}$ – see the possible improvements subsection below) follows by Taylor expansion at $x_n = 0$. More precisely, we use the following form of this expansion: for $f \in C_0^\infty(\mathbb{R}^n)$,

$$(1 - \Pi^0)f = x_n \cdot g, \quad g(x', x_n) := \frac{f(x', x_n) - f(x', 0)}{x_n},$$

and one can show that $\|g\|_{L^2} \leq C h^{-1}\|f\|_{H^1_t}$; the factor $h^{-1}$ coming from taking one nonsemiclassical derivative to obtain $g$ from $f$ (see Lemma 6.12). Then $\chi U(t)(1 - \Pi^0)f = \chi U(t)x_n U(-t)U(t)g$, where (by a special case of Egorov’s theorem following by direct computation) $\chi U(t)x_n U(-t)$ is a multiplication operator by

$$\chi U(t)x_n U(-t) = \chi(x)e^{-t}x_n = \mathcal{O}(e^{-t});$$

this shows that $\|\chi U(t)(1 - \Pi^0)f\|_{L^2} \leq Ce^{-t}\|g\|_{L^2} \leq Ch^{-1}e^{-t}\|f\|_{H^1_t}$ and (2.3) follows.

To show (2.4) in the model case, we start with the identity

$$\|\chi U(t)\Pi^0f\|_{L^2} = \|\chi_t\Pi^0f\|_{L^2}, \quad \chi_t := U(-t)\chi U(t).$$

If $\chi \in C_0^\infty(\mathbb{R}^n)$, then $\chi_t(x) = \chi(x', e^tx_n)$ has shrinking support as $t \to \infty$. To compare $\|\chi_t\Pi^0f\|_{L^2}$ to $\|\Pi^0f\|_{L^2}$, we use the following fact:

$$hD_{x_n}\Pi^0f = 0.$$  \hspace{1cm} (2.7)

This implies that for each $a(x) \in C_0^\infty(\mathbb{R}^n)$, the inner product $\langle a\Pi^0f, \Pi^0f \rangle$ depends only on the function $b(x') = \int_{\mathbb{R}} a(x', x_n)dx_n$; writing $\|\chi\Pi^0f\|_{L^2}$ and $\|\chi_t\Pi^0f\|_{L^2}$ as inner products, we get $\|\chi_t\Pi^0f\|_{L^2}^2 = e^{-t}\|\Pi^0f\|_{L^2}^2$ and (2.4) follows.

The proofs of (2.3) and (2.4) in the general case work as in the model case, once we find appropriate replacements for differential operators $x_n$ and $hD_{x_n}$ in (2.5) and (2.7). It turns out that one needs to take pseudodifferential operators $\Theta_{\pm}$ solving, microlocally near $K \cap p^{-1}([\alpha_0, \alpha_1])$,

$$\Pi\Theta_{\pm} = \mathcal{O}(h^\infty), \quad \Theta_{\pm}\Pi = \mathcal{O}(h^\infty),$$

then $\Theta_{\pm}$ is a replacement for $x_n$ and $\Theta_{\pm}$, for $hD_{x_n}$. Note that $\Theta_{\pm}$ are not unique, in fact solutions to (2.8) form one-sided ideals in the algebra of pseudodifferential operators – see §§6.4 and 7.2. The principal symbols of $\Theta_{\pm}$ are defining functions of $\Gamma_{\pm}$.

**Concentration of resonant states.** As a byproduct of the discussion above, we obtain new information about microlocal concentration of resonant states, that is, functions $u \in \mathcal{H}_1$ such that $\mathcal{P}(\omega)u = 0$ and $\|u\|_{x_1} \sim 1$. It is well-known (see for example [NoZw09, Theorem 4]) that the wavefront set of $u$ is contained in $\Gamma_+ \cap p^{-1}(\text{Re}\, \omega)$. The new information we obtain is that if $\omega$ is a resonance in the band given by Theorem 2 (that is, $\text{Im}\, \omega > -\nu_{\min} - \varepsilon \hbar$), then by (2.3), $u = \Pi u + \mathcal{O}(h^\infty)$ microlocally near $K$. Then by (2.8), $\Theta_+u = \mathcal{O}(h^\infty)$ near $K$, that is, $u$ solves a pseudodifferential equation; note that the Hamiltonian flow lines of the principal symbol of $\Theta_+$ are transverse to the trapped set. This implies in particular that any corresponding semiclassical
defect measure is determined uniquely by a measure on the trapped set which is conditionally invariant under $H_p$, similarly to the damped wave equation. See Theorem 4 in §8.5 for details.

**Proof of Theorem 2.** We start with constructing a well-posed Grushin problem, representing resonances as zeroes of a certain Fredholm determinant $F(\omega)$. Using complex analysis (essentially the argument principle), we reduce counting resonances to computing a contour integral of the logarithmic derivative $F'(\omega)/F(\omega)$, which, taking $\nu_- = -(\nu_{\max} + \varepsilon)/2$, $\nu_+ = -(\nu_{\min} - \varepsilon)/2$, is similar to (see §10 for the actual expression)

$$
\frac{1}{2\pi i} (I_- - I_+) = \int_{\text{Im} \omega = h \nu_{\pm}} \chi(\omega) \text{Tr}(\Pi R(\omega)) d\omega
$$

for some cutoff function $\chi(\omega)$. The integration is over the region where Theorem 1 gives polynomial bounds on the resolvent $R(\omega)$, and we can use the methods developed for the proof of this theorem to evaluate both integrals, yielding Theorem 2. An important additional tool, explaining in particular why the two integrals do not cancel each other, is microlocal analysis in the spectral parameter $\omega$, or equivalently a study of the essential support of the Fourier transform of $\Pi R(\omega)$ in $\omega$ – see §§8.4 and 10.

2.2. **Possible improvements.** First of all, it would be interesting to see if one could construct further bands of resonances, lying below the one in Theorem 2. One expects these bands to have the form

$$
\{\text{Im} \omega \in [-(k + 1/2)(\nu_{\max} + \varepsilon)h, -(k + 1/2)(\nu_{\min} - \varepsilon)h], \ k \in \mathbb{Z}, \ k \geq 0,
$$

and to have a Weyl law in the $k$-th band under the pinching condition $(k + 1/2)\nu_{\max} < (k + 3/2)\nu_{\min}$. Note that the presence of the second band of resonances improves the size of the second resonance free strip in Theorem 1 and gives a weaker pinching condition $\nu_{\max} < 3\nu_{\min}$ for the Weyl law in the first band. The proofs are expected to work similarly to the present paper, if one constructs a family of operators $\Pi_0 = \Pi, \Pi_1, \ldots, \Pi_k$ such that $\Pi_j$ is $h^{-j} \times$ Fourier integral operator associated to $\Lambda^0, \Pi_j \Pi_k = O(h^\infty)$, and $[P, \Pi_j] = O(h^\infty)$ (microlocally near $K \cap p^{-1}(\alpha_0, \alpha_1]$). However, the method of §7.1 does not apply directly to construct $\Pi_k$ for $k > 0$, since one cannot conjugate all $\Pi_j$ to the model case, which is the base of the crucial Proposition 6.9.

Another direction would be to consider the case when the operator $P$ is quantum completely integrable on the trapped set (a notion that needs to be made precise), and derive a quantization condition for resonances like the one for the special case of black holes [SáZw, Dy12]. The author also believes that the results of the present paper should be adaptable to the situation when $\Gamma_\pm$ have codimension higher than 1, which makes it possible to revisit the distribution of resonances generated by one closed hyperbolic trajectory, studied in [GéSj87].
An interesting special case lying on the intersection of the current work and \([\text{FaTs}1, \text{FaTs}2, \text{FaTs}3]\) is given by geodesic flows on compact manifolds of constant negative curvature; the corresponding manifolds \(\Gamma_{\pm}\) and \(K\) are smooth in this situation. While \(r\)-normal hyperbolicity does not hold (in fact, \(\mu_{\max} = \nu_{\min} = \nu_{\max}\)), the rigid algebraic structure of hyperbolic quotients suggests that one could still look for the projector \(\Pi\) as a (smooth) Fourier integral operator – in terms of the construction of \(\S\) 7.1, the transport equation (2.2), while not yielding a smooth solution for an arbitrary choice of the right-hand side \(f\), will have a smooth solution for the specific functions \(f\) arising in the construction.

Finally, a natural question is improving the \(o(1)\) remainder in the Weyl law (1.8). Obtaining an \(O(h^\delta)\) remainder for \(\delta < 1\) does not seem to require conceptual changes to the microlocal structure of the argument; however, for the \(O(h)\) remainder of Hörmander [Hö] or the \(o(h)\) remainder of Duistermaat–Guillemin [DuGu], one would need a finer analysis of the interaction of the operator \(\Pi\) with the Schrödinger propagator, and more assumptions on the flow on the trapped set might be needed. Moreover, the complex analysis argument of \(\S\) 11 does not work in the case of an \(O(h)\) remainder; a reasonable replacement would be to adapt to the considered case the work of Sjöstrand [Sj00] on the damped wave equation.

2.3. Structure of the paper.

- In \(\S\) 3, we review the tools we need from semiclassical analysis.
- In \(\S\) 4, we present a framework which makes it possible to handle resonances and the spatial infinity in an abstract fashion. The assumptions we make are listed in \(\S\) 4.1, followed by some useful lemmas (\(\S\) 4.2) and applications to Schrödinger operators (\(\S\) 4.3) and even asymptotically hyperbolic manifolds (\(\S\) 4.4).
- In \(\S\) 5, we study \(r\)-normally hyperbolic trapped sets, stating the dynamical assumptions (\(\S\) 5.1), discussing their stability under perturbations (\(\S\) 5.2), and deriving some corollaries (\(\S\) 5.3–5.5).
- In \(\S\) 6, we study in detail Fourier integral operators associated to \(\Lambda^0\), and in particular properties of operators solving \(\Pi^2 = \Pi + O(h^\infty)\).
- In \(\S\) 7, we construct the projector \(\Pi\) and the annihilating operators \(\Theta_{\pm}\).
- In \(\S\) 8, we prove Theorem 1, establish microlocal estimates on the resolvent, and study the microlocal concentration of resonant states (\(\S\) 8.5).
- In \(\S\) 9, we formulate a well-posed Grushin problem for \(\mathcal{P}(\omega)\), representing resonances as zeroes of a certain Fredholm determinant.
- In \(\S\) 10, we prove a trace formula for \(\mathcal{R}(\omega)\) microlocally on the image of \(\Pi\).
- In \(\S\) 11, we prove the Weyl asymptotic for resonances (Theorem 2).
- In Appendix A, we provide an example of an asymptotically hyperbolic manifold satisfying the dynamical assumptions of \(\S\) 5.1.
3. Semiclassical preliminaries

In this section, we review semiclassical pseudodifferential operators, wavefront sets, and Fourier integral operators; the reader is directed to [Zw, DiSj] for a detailed treatment and [HöIII, HöIV, GrSj] for the closely related microlocal case.

3.1. Pseudodifferential operators and microlocalization. Let $X$ be a manifold without boundary. Following [Zw, §9.3 and 14.2], we consider the symbol classes $S^k(T^*X)$, $k \in \mathbb{R}$, consisting of smooth functions $a$ on the cotangent bundle $T^*X$ satisfying in local coordinates

$$\sup_h \sup_{x \in K} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{\alpha\beta K} \langle \xi \rangle^{-|\beta|},$$

for each multiindices $\alpha, \beta$ and each compact set $K \subset X$. The corresponding class of semiclassical pseudodifferential operators is denoted $\Psi^k(X)$. The residual symbol class $h^\infty S^{-\infty}$ consists of symbols decaying rapidly in $h$ and $\xi$ over compact subsets of $X$; the operators in the corresponding class $h^\infty \Psi^{-\infty}$ have Schwartz kernels in $h^\infty C^\infty(X \times X)$. Operators in $\Psi^k$ are bounded, uniformly in $h$, between the semiclassical Sobolev spaces $H^s_h, \text{comp}(X) \rightarrow H^{s-k}_h, \text{loc}(X)$, see [Zw, (14.2.3)] for the definition of the latter.

Note that for noncompact $X$, we impose no restrictions on the behavior of symbols as $x \to \infty$. Accordingly, we cannot control the behavior of operators in $\Psi^k(X)$ near spatial infinity; in fact, a priori we only require them to act $C^\infty_0(X) \rightarrow C^\infty_0(X)$ and on the spaces of distributions $\mathcal{E}'(X) \rightarrow \mathcal{D}'(X)$. However, each $A \in \Psi^k(X)$ can be written as the sum of an $h^\infty \Psi^{-\infty}$ remainder and an operator properly supported uniformly in $h$ – see for example [HöIII, Proposition 18.1.22]. Properly supported pseudodifferential operators act $C^\infty_0 \rightarrow C^\infty_0$ and $C^\infty \rightarrow C^\infty$ and therefore can be multiplied with each other, giving an algebra structure on the whole $\Psi^k$, modulo $h^\infty \Psi^{-\infty}$.

To study the behavior of symbols near fiber infinity, we use the fiber-radial compactified cotangent bundle $\overline{T}^*X$, a manifold with boundary whose interior is diffeomorphic to $T^*X$ and whose boundary $\partial \overline{T}^*X$ is diffeomorphic to the cosphere bundle over $X$ – see for example [Va10, §2.2]. We will restrict ourselves to the space of classical symbols, i.e. those having an asymptotic expansion

$$a(x, \xi; h) \sim \sum_{j \geq 0} h^j a_j(x, \xi),$$

with $a_j \in S^{k-j}$ classical in the sense that $\langle \xi \rangle^{j-k} a_j$ extends to a smooth function on $\overline{T}^*X$. The principal symbol $\sigma(A) := a_0 \in S^k$ of an operator is defined independently of quantization. We say that $A \in \Psi^k$ is elliptic at some $(x, \xi) \in \overline{T}^*X$ if $\langle \xi \rangle^{-k} \sigma(A)$ does not vanish at $(x, \xi)$.

Another invariant object associated to $A \in \Psi^k(X)$ is its wavefront set $WF_h(A)$, which is a closed subset of $\overline{T}^*X$; a point $(x, \xi) \in \overline{T}^*X$ does not lie in $WF_h(A)$ if and
only if there exists a neighborhood $U$ of $(x, \xi)$ in $\mathcal{T}^* X$ such that the full symbol of $A$ (in any quantization) is in $h^\infty S^{-\infty}$ in this neighborhood. Note that $\text{WF}_h(A) = \emptyset$ if and only if $A = \mathcal{O}(h^\infty)_{\mathcal{P}^{-\infty}}$. We say that $A_1 = A_2 + \mathcal{O}(h^\infty)$ microlocally in some $U \subset \mathcal{T}^* X$, if $\text{WF}_h(A - B) \cap U = \emptyset$.

We denote by $\Psi^\comp(X)$ the space of all operators $A \in \Psi^0(X)$ such that $\text{WF}_h(A)$ is a compact subset of $\mathcal{T}^* X$, in particular not intersecting the fiber infinity $\partial \mathcal{T}^* X$. Note that $\Psi^\comp(X) \subset \Psi^k(X)$ for all $k \in \mathbb{R}$.

**Tempered distributions and operators.** Let $u = u(h)$ be an $h$-dependent family of distributions in $\mathcal{D}'(X)$. We say that $u$ is $h$-tempered (or polynomially bounded), if for each $\chi \in C^\infty_0(X)$, there exists $N$ such that $\|\chi u\|_{H^N_h} = \mathcal{O}(h^{-N})$. The class of $h$-tempered distributions is closed under properly supported pseudodifferential operators. For an $h$-tempered $u$, define the wavefront set $\text{WF}_h(u)$, a closed subset of $\mathcal{T}^* X$, as follows: $(x, \xi) \in \mathcal{T}^* X$ does not lie in $\text{WF}_h(u)$ if and only if there exists a neighborhood $U$ of $(x, \xi)$ in $\mathcal{T}^* X$ such that for each properly supported $A \in \Psi^0(X)$ with $\text{WF}_h(A) \subset U$, we have $Au = \mathcal{O}(h^\infty)_{C^\infty}$. We have $\text{WF}_h(u) = \emptyset$ if and only if $u = \mathcal{O}(h^\infty)_{C^\infty}$. We say that $u = v + \mathcal{O}(h^\infty)$ microlocally on some $U \subset \mathcal{T}^* X$ if $\text{WF}_h(u - v) \cap U = \emptyset$.

Let $X_1$ and $X_2$ be two manifolds. An operator $B : C^\infty_0(X_1) \to \mathcal{D}'(X_2)$ is identified with its Schwartz kernel $\mathcal{K}_B(y, x) \in \mathcal{D}'(X_2 \times X_1)$:

$$Bf(y) = \int_{X_1} \mathcal{K}_B(y, x)u(x) \, dx, \quad u \in C^\infty_0(X_1).$$  \hspace{1cm} (3.1)

Here we assume that $X_1$ is equipped with some smooth density $dx$; later, we will also assume that densities on our manifolds are specified when talking about adjoints.

We say that $B$ is $h$-tempered if $\mathcal{K}_B$ is, and define the wavefront set of $B$ as

$$\text{WF}_h(B) := \{(x, \xi, y, \eta) \in \mathcal{T}^* (X_1 \times X_2) \mid (y, \eta, x, -\xi) \in \text{WF}_h(\mathcal{K}_B)\}. \hspace{1cm} (3.2)$$

If $B \in \Psi^k(X)$, then the wavefront set of $B$ as an $h$-tempered operator is equal to its wavefront set as a pseudodifferential operator, under the diagonal embedding $\mathcal{T}^* X \to \mathcal{T}^* (X \times X)$.

3.2. Lagrangian distributions and Fourier integral operators. We now review the theory of Lagrangian distributions; for details, the reader is directed to [Zw, Chapters 10–11], [GuSt90, Chapter 6], or [VuNg, §2.3], and to [HöIV, Chapter 25] or [GrSj, Chapters 10–11] for the closely related microlocal setting. Here, we only present the relatively simple local part of the theory; geometric constructions of invariant symbols will be done by hand when needed, without studying the structure of the bundles obtained (see §6.2). For a more complete discussion, see for example [DyGu, §3].
A semiclassical Lagrangian distribution locally takes the form

\[ u(x; h) = (2\pi h)^{-m/2} \int_{X \times \mathbb{R}^m} e^{\frac{i}{h} \Phi(x, \theta)} a(x, \theta; h) \, d\theta. \]  

(3.3)

Here \( \Phi \) is a nondegenerate phase function, i.e. a real-valued function defined on an open subset of \( X \times \mathbb{R}^m \), for some \( m \), such that the differentials \( d(\partial_{\theta_1} \Phi), \ldots, d(\partial_{\theta_m} \Phi) \) are linearly independent on the critical set

\[ C_\Phi := \{(x, \theta) \mid \partial_{\Phi} \Phi(x, \theta) = 0\}. \]

The amplitude \( a(x, \theta; h) \) is a classical symbol (that is, having an asymptotic expansion in nonnegative integer powers of \( h \) as \( h \to 0 \)) compactly supported inside the domain of \( \Phi \). The resulting function \( u(x; h) \) is smooth, compactly supported, \( h \)-tempered, and

\[ \text{WF}_h(u) \subset \{(x, \partial_x \Phi(x, \theta)) \mid (x, \theta) \in C_\Phi \cap \text{supp} \, a\}. \]  

(3.4)

We say that \( \Phi \) generates the (immersed, and we shrink the domain of \( \Phi \) to make it embedded) Lagrangian submanifold

\[ \Lambda_\Phi := \{(x, \partial_x \Phi(x, \theta)) \mid (x, \theta) \in C_\Phi\}; \]

note that \( \text{WF}_h(u) \subset \Lambda_\Phi \). Moreover, if we restrict \( \Phi \) to \( C_\Phi \) and pull it back to \( \Lambda_\Phi \), then \( d\Phi \) equals the canonical 1-form \( \xi \, dx \) on \( \Lambda_\Phi \).

In general, assume that \( \Lambda \) is an embedded Lagrangian submanifold of \( T^*X \) which is moreover exact in the sense that the canonical form \( \xi \, dx \) is exact on \( \Lambda \); we fix an antiderivative on \( \Lambda \), namely a function \( F \) such that \( \xi \, dx = dF \) on \( \Lambda \). (This is somewhat similar to the notion of Legendre distributions, see [MeZw, §11].) Then we say that a compactly supported \( h \)-tempered family of distributions \( u \) is a (compactly microlocalized) Lagrangian distribution associated to \( \Lambda \), if \( u \) can be written as a finite sum of expressions (3.3), with phase functions \( \Phi_j \) generating open subsets of \( \Lambda \), plus an \( \mathcal{O}(h^\infty)c_\infty \) remainder, where \( \Phi_j \) are normalized (by adding a constant) so that the pullback to \( \Lambda \) of the restriction of \( \Phi_j \) to \( C_{\Phi_j} \) equals \( F \). (Without such normalization, passing from one phase function to the other produces a factor \( e^{is} \) for some constant \( s \), which does not preserve the class of classical symbols – this is an additional complication of the theory compared to the nonsemiclassical case.) Denote by \( I_{\text{comp}}(\Lambda) \) the class of all Lagrangian distributions associated to \( \Lambda \). For \( u \in I_{\text{comp}}(\Lambda) \), we have \( \text{WF}_h(u) \subset \Lambda \); in particular, \( \text{WF}_h(u) \) does not intersect the fiber infinity \( \partial T^*X \).

If now \( X_1, X_2 \) are two manifolds of dimensions \( n_1, n_2 \) respectively, and \( \Lambda \subset T^*X_1 \times T^*X_2 \) is an exact canonical relation (with some fixed antiderivative), then an operator \( B : C^\infty(X_1) \to C_0^\infty(X_2) \) is called a (compactly microlocalized) Fourier integral operator associated to \( \Lambda \), if its Schwartz kernel \( K_B(y, x) \) is \( h^{-(n_1+n_2)/4} \) times a Lagrangian distribution associated to

\[ \{(y, \xi, x, -\xi) \in T^*(X_1 \times X_2) \mid (x, \xi, y, \eta) \in \Lambda\}. \]
We write $B \in I_{\text{comp}}(\Lambda)$; note that $\text{WF}_h(B) \subset \Lambda$. A particular case is when $\Lambda$ is the graph of a canonical transformation $\varpi : U_1 \to U_2$, with $U_j$ open subsets in $T^*X_j$. Operators associated to canonical transformations (but not general relations!) are bounded $H^s_h \to H^{s'}_h$ uniformly in $h$, for each $s, s'$.

Compactly microlocalized Fourier integral operators associated to the identity transformation are exactly compactly supported pseudodifferential operators in $\Psi^{\text{comp}}(X)$. Another example of Fourier integral operators is given by Schrödinger propagators, see for instance [Zw, Theorem 10.4] or [DyGu, Proposition 3.8]:

**Proposition 3.1.** Assume that $P \in \Psi^{\text{comp}}(X)$ is compactly supported, $\text{WF}_h(P)$ is contained in some compact subset $V \subset T^*X$, and $p = \sigma(P)$ is real-valued. Then for $t \in \mathbb{R}$ bounded by any fixed constant, the operator $e^{-itP/h} : L^2(X) \to L^2(X)$ is the sum of the identity and a compactly supported operator microlocalized in $V \times V$. Moreover, for each compactly supported $A \in \Psi^{\text{comp}}(X)$, $Ae^{-itP/h}$ and $e^{-itP/h}A$ are smooth families of Fourier integral operators associated to the Hamiltonian flow $e^{iH_p} : T^*X \to T^*X$.

Here we put the antiderivative $F$ for the identity transformation to equal zero, and extend it to the antiderivative $F_t$ on the graph of $e^{iH_p}$ by putting

$$F_t(\gamma(0), \gamma(t)) := tp(\gamma(0)) - \int_{\gamma([0,t])} \xi \, dx$$

for each flow line $\gamma$ of $H_p$. The corresponding phase function is produced by a solution to the Hamilton–Jacobi equation [Zw, Lemma 10.5].

We finally discuss products of Fourier integral operators. Assume that $B_j \in I_{\text{comp}}(\Lambda_j)$, $j = 1, 2$, where $\Lambda_1 \subset T^*X_1 \times T^*X_2$ and $\Lambda_2 \subset T^*X_2 \times T^*X_3$ are exact canonical relations. Assume moreover that $\Lambda_1, \Lambda_2$ satisfy the following transversality assumption: the manifolds $\Lambda_1 \times \Lambda_2$ and $T^*X_1 \times \Delta(T^*X_2) \times T^*X_3$, where $\Delta(T^*X_2) \subset T^*X_2 \times T^*X_2$ is the diagonal, intersect transversely inside $T^*X_1 \times T^*X_2 \times T^*X_2 \times T^*X_3$, and their intersection projects diffeomorphically onto $T^*X_1 \times T^*X_3$. Then $B_2B_1 \in I_{\text{comp}}(\Lambda_2 \circ \Lambda_1)$, where

$$\Lambda_2 \circ \Lambda_1 := \{ (\rho_1, \rho_3) \mid \exists \rho_2 \in T^*X_2 : (\rho_1, \rho_2) \in \Lambda_1, (\rho_2, \rho_3) \in \Lambda_2 \},$$

and, if $F_j$ is the antiderivative on $\Lambda_j$, then $F_1(\rho_1, \rho_2) + F_2(\rho_2, \rho_3)$ is the antiderivative on $\Lambda_2 \circ \Lambda_1$. See for example [HöIIV, Theorem 25.2.3] or [GrSj, Theorem 11.12] for the closely related microlocal case, which is adapted directly to the semiclassical situation.

The transversality condition is always satisfied when at least one of the $\Lambda_j$ is the graph of a canonical transformation. In particular, one can always multiply a pseudodifferential operator by a Fourier integral operator, and obtain a Fourier integral operator associated to the same canonical relation.

---

\[2\] [Zw, Theorem 10.4] is stated for self-adjoint $P$, rather than operators with real-valued principal symbols; however, the proof works similarly in the latter case, with the transport equation acquiring an additional zeroth order term due to the subprincipal part of $P$. 
3.3. Basic estimates. In this section, we review some standard semiclassical estimates, parametrices, and microlocalization statements.

Throughout the section, we assume that \(k, s \in \mathbb{R}, P, Q \in \Psi^k(X)\) are properly supported and \(u, f\) are \(h\)-tempered distributions on \(X\), in the sense of §3.1.

We start with the elliptic estimate, see for instance [Dy12, Proposition 2.2]:

**Proposition 3.2.** (Elliptic estimate) Assume that \(Pu = f\). Then:

1. If \(A, B \in \Psi^0(X)\) are compactly supported and \(P, B\) are elliptic on \(WF_h(A)\), then
   \[
   \|Au\|_{H^k_h} \leq C\|Bf\|_{H^{s-k}h} + \mathcal{O}(h^\infty). \tag{3.6}
   \]

2. We have
   \[
   WF_h(u) \subset WF_h(f) \cup \{\langle \xi \rangle^{-k}\sigma(P) = 0\}. \tag{3.7}
   \]

Proposition 3.2 is typically proved using the following fact, which is of independent interest:

**Proposition 3.3.** (Elliptic parametrix) If \(V \subset T^*X\) is compact and \(P\) is elliptic on \(V\), then there exists a compactly supported operator \(P' \in \Psi^{-k}(X)\) such that \(PP' = 1 + \mathcal{O}(h^\infty), P'P = 1 + \mathcal{O}(h^\infty)\) microlocally near \(V\). Moreover, \(\sigma(P') = \sigma(P)^{-1}\) near \(V\).

We next give a version of propagation of singularities which allows for a complex absorbing operator \(Q\), see for instance [Va10, §2.3]:

**Proposition 3.4.** (Propagation of singularities) Assume that \(\sigma(P)\) is real-valued, \(\sigma(Q) \geq 0\), and \((P \pm iQ)u = f\). Then:

1. If \(A_1, A_2, B \in \Psi^0(X)\) are compactly supported and for each flow line \(\gamma(t)\) of the Hamiltonian field \(\pm \langle \xi \rangle^{1-k}H_{\sigma(P)}\) such that \(\gamma(0) \in WF_h(A_1)\), there exists \(t \geq 0\) such that \(A_2\) is elliptic at \(\gamma(t)\) and \(B\) is elliptic on the segment \(\gamma([0, t])\), then
   \[
   \|A_1u\|_{H^k_h} \leq C\|A_2u\|_{H^k_h} + Ch^{-1}\|Bf\|_{H^{k-1}h} + \mathcal{O}(h^\infty). \tag{3.8}
   \]

2. If \(\gamma(t), 0 \leq t \leq T,\) is a flow line of \(\pm \langle \xi \rangle^{1-k}H_{\sigma(P)}\), then
   \[
   \gamma([0, T]) \cap WF_h(f) = \emptyset, \quad \gamma(T) \not\in WF_h(u) \implies \gamma(0) \not\in WF_h(u).
   \]

For \(Q = 0\), Proposition 3.4 can be viewed as a microlocal version of uniqueness of solutions to the Cauchy problem for hyperbolic equations; a corresponding microlocal existence fact is given by

**Proposition 3.5.** (Hyperbolic parametrix) Assume that \(\sigma(P)\) is real-valued, \(WF_h(f) \subset T^*X\) is compact, \(U, V \subset T^*X\) are compactly contained open sets, and for each flow line \(\gamma(t)\) of the Hamiltonian field \(H_{\sigma(P)}\) such that \(\gamma(0) \in WF_h(f)\), there exists \(t \in \mathbb{R}\) such that \(\gamma(t) \in U\) and \(\gamma(s) \in V\) for all \(s\) between 0 and \(t\).
Then there exists an $h$-tempered family $v(h) \in C_0^\infty(X)$ such that \( \text{WF}_h(v) \subset V \) and
\[
\|v\|_{L^2} \leq C h^{-1} \|f\|_{L^2}, \quad \|Pv\|_{L^2} \leq C \|f\|_{L^2}, \quad \text{WF}_h(Pv - f) \subset U.
\]

**Proof.** By applying a microlocal partition of unity to $f$, we may assume that there exists $T > 0$ (the case $T < 0$ is considered similarly and the case $T = 0$ is trivial by putting $v = 0$) such that for each flow line $\gamma(t)$ of $H_{\sigma(P)}$ such that $\gamma(0) \in \text{WF}_h(f)$, we have $\gamma(T) \in U$ and $\gamma([0,T]) \subset V$. Take $\varepsilon \in (0,T)$ such that $\gamma([T-\varepsilon,T]) \subset U$ for each such $\gamma$. Since $V$ is compactly contained in $T^*X$, we may assume that $P$ is compactly supported and $P \in \Psi^{\text{comp}}(X)$. We then take $\chi \in C_0^\infty(-\infty,T)$ such that $\chi = 1$ near $[0,T-\varepsilon]$ and put
\[
v := \frac{i}{h} \int_0^T \chi(t) e^{-itP/h} f \, dt.
\]
Then $\|v\|_{L^2} \leq C h^{-1} \|f\|_{L^2}$ and $\text{WF}_h(v) \subset V$ by Proposition 3.1. Integrating by parts, we compute
\[
Pv = -\int_0^T \chi(t) \partial_t e^{-itP/h} f \, dt = f + \int_0^T (\partial_t \chi(t)) e^{-itP/h} f \, dt;
\]
therefore, $\|Pv\|_{L^2} \leq C \|f\|_{L^2}$ and by Proposition 3.1, $\text{WF}_h(Pv - f) \subset U$. \hfill \Box

We also need the following version of the sharp Gårding inequality, see [Zw, Theorem 4.32] or [Dy11a, Proposition 5.2]:

**Proposition 3.6.** (Sharp Gårding inequality) Assume that $A \in \Psi^{\text{comp}}(X)$ is compactly supported and $\text{Re} \, \sigma(A) \geq 0$ near $\text{WF}_h(u)$. Assume also that $B \in \Psi^{\text{comp}}(X)$ is compactly supported and elliptic on $\text{WF}_h(A) \cap \text{WF}_h(u)$. Then
\[
\text{Re} \langle Au, u \rangle \geq -Ch \|B u\|_{L^2}^2 - O(h^\infty).
\]

4. Abstract framework near infinity

In this section, we provide an abstract microlocal framework for studying resonances; the general assumptions are listed in §4.1. Rather than considering resonances as poles of the meromorphic continuation of the cutoff resolvent, we define them as solutions of a nonselfadjoint eigenvalue problem featuring a holomorphic family of Fredholm operators, $\mathcal{P}(\omega)$. We assume that the dependence of the principal symbol of $\mathcal{P}(\omega)$ on $\omega$ can be resolved in a convex neighborhood $\mathcal{U}$ of the trapped set, yielding the $\omega$-independent symbol $p$ (and the operator $P$ later in Lemma 4.3). Finally, we require the existence of a semiclassically outgoing parametrix for $\mathcal{P}(\omega)$, resolving it modulo an operator microlocalized near the trapped set.

In §4.2, we derive several useful corollaries of our assumptions, making it possible to treat spatial infinity as a black box in the following sections. Finally, in §§4.3 and 4.4, we provide two examples of situations when the assumptions of §4.1 (but not
necessarily the dynamical assumptions of §5.1) are satisfied: Schrödinger operators on \( \mathbb{R}^n \), studied using complex scaling, and Laplacians on even asymptotically hyperbolic manifolds, handled using [Va10, Va11].

### 4.1. General assumptions

Assume that:

1. \( X \) is a smooth \( n \)-dimensional manifold without boundary, possibly noncompact, with a prescribed volume form;
2. \( \mathcal{P}(\omega) \in \Psi^k(X) \) is a family of properly supported semiclassical pseudodifferential operators depending holomorphically on \( \omega \) lying in an open simply connected set \( \Omega \subset \mathbb{C} \) such that \( \mathbb{R} \cap \Omega \) is connected, with principal symbol \( p(x, \xi, \omega) \);
3. \( \mathcal{H}_1, \mathcal{H}_2 \) are \( h \)-dependent Hilbert spaces such that \( H_{h,\text{comp}}^N(X) \subset \mathcal{H}_j \subset H_{h,\text{loc}}^{-N}(X) \) for some \( N \), with norms of embeddings \( O(h^{-N}) \), and \( \mathcal{P}(\omega) \) is bounded \( \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) with norm \( O(1) \);
4. for some fixed \( [\alpha_0, \alpha_1] \subset \mathbb{R} \cap \Omega \) and \( C_0 > 0 \), the operator \( \mathcal{P}(\omega) : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) is Fredholm of index zero in the region

\[
\text{Re} \, \omega \in [\alpha_0, \alpha_1], \quad |\text{Im} \, \omega| \leq C_0 h. \tag{4.1}
\]

Together with invertibility of \( \mathcal{P}(\omega) \) in a subregion of (4.1) proved in Theorem 1, by Analytic Fredholm Theory [Zw, Theorem D.4] our assumptions imply that

\[
\mathcal{R}(\omega) := \mathcal{P}(\omega)^{-1} : \mathcal{H}_2 \rightarrow \mathcal{H}_1 \tag{4.2}
\]

is a meromorphic family of operators with poles of finite rank for \( \omega \) satisfying (4.1). Resonances are defined as poles of \( \mathcal{R}(\omega) \). Following [GoSi, Theorem 2.1], we define the multiplicity of a resonance \( \omega_0 \) as

\[
\frac{1}{2\pi i} \text{Tr} \oint_{\omega_0} \mathcal{P}(\omega)^{-1} \partial_\omega \mathcal{P}(\omega) \, d\omega. \tag{4.3}
\]

Here \( \oint_{\omega_0} \) stands for the integral over a contour enclosing \( \omega_0 \), but no other poles of \( \mathcal{R}(\omega) \). Since \( \mathcal{R}(\omega) \) has poles of finite rank, we see that the integral in (4.3) yields a finite dimensional operator on \( \mathcal{H}_1 \) and thus one can take the trace. The fact that the resulting multiplicity is a positive integer will follow for example from the representation of resonances as zeroes of a Fredholm determinant, in part 1 of Proposition 9.5. See also [Sj00, Appendix A].

We next fix a ‘physical region’ \( \mathcal{U} \) in phase space, where most of our analysis will take place, in particular the intersection of the trapped set with the relevant energy shell will be contained in \( \mathcal{U} \). The region \( \mathcal{U} \) will be contained in a larger region \( \mathcal{U}' \), which is used to determine when trajectories have escaped from \( \mathcal{U} \). (See (4.16) and (4.21) for the definitions of \( \mathcal{U}, \mathcal{U}' \) for the examples we consider.) We assume that:

5. \( \mathcal{U}' \subset T^* X \) is open and bounded, and each compactly supported \( A \in \Psi^{\text{comp}}(X) \) with \( \text{WF}_h(A) \subset \mathcal{U}' \) is bounded \( L^2 \to \mathcal{H}_j, \mathcal{H}_j \to L^2, \, j = 1, 2, \) with norm \( O(1) \).
(6) $\mathcal{P}(\omega)^* = \mathcal{P}(\omega) + \mathcal{O}(h^{\infty})$ microlocally in $\mathcal{U}'$ when $\omega \in \mathbb{R} \cap \Omega$;
(7) for each $(x, \xi) \in \mathcal{U}'$, the equation $p(x, \xi, \omega) = 0$, $\omega \in \Omega$ has unique solution
$$\omega = p(x, \xi).$$

Moreover, $p(x, \xi) \in \mathbb{R}$ and $\partial_\omega p(x, \xi, p(x, \xi)) < 0$ for $(x, \xi) \in \mathcal{U}'$;
(8) $\mathcal{U} \subset \mathcal{U}'$ is a compactly contained open subset, whose closure $\overline{\mathcal{U}}$ is relatively convex with respect to the Hamiltonian flow of $p$, i.e. if $\gamma(t), 0 \leq t \leq T$, is a flow line of $H_p$ in $\mathcal{U}'$ and $\gamma(0), \gamma(T) \in \overline{\mathcal{U}}$, then $\gamma([0, T]) \subset \overline{\mathcal{U}}$;

Note that for $\omega \in \mathbb{R} \cap \Omega$, Hamiltonian flow lines of $p$ in $\mathcal{U}' \cap p^{-1}(\omega)$ are rescaled Hamiltonian flow lines of $p(\cdot, \omega)$ in $\{\rho \in \mathcal{U}' | p(\rho, \omega) = 0\}$. The symbol $p$ is typically the square root of the principal symbol of the original Laplacian or Schrödinger operator, see (4.17) and (4.22).

We can now define the incoming/outgoing tails $\Gamma_\pm \subset \overline{\mathcal{U}}$ as follows: $\rho \in \overline{\mathcal{U}}$ lies in $\Gamma_\pm$ if and only if $e^{\mp \imath t H_p}(\rho)$ stays in $\mathcal{U}$ for all $t \geq 0$. Define the trapped set as
$$K := \Gamma_+ \cap \Gamma_-.$$ 

Note that $\Gamma_\pm$ and $K$ are closed subsets of $\overline{\mathcal{U}}$ (and thus the sets $\Gamma_\pm$ defined here are smaller than the original $\Gamma_\pm$ defined in the introduction), and $e^{\imath t H_p}(\Gamma_\pm) \subset \Gamma_\pm$ for $\mp t \geq 0$, thus $e^{\imath t H_p}(K) = K$ for all $t$. We assume that, with $\alpha_0, \alpha_1$ defined in (4.1),

(9) $K \cap p^{-1}([\alpha_0, \alpha_1])$ is a nonempty compact subset of $\mathcal{U}$.

Finally, we assume the existence of a semiclassically outgoing parametrix, which will make it possible to reduce our analysis to a neighborhood of the trapped set in §4.2:

(10) $\mathcal{Q} \in \Psi^{\text{comp}}(X)$ is compactly supported, WF$_h(\mathcal{Q}) \subset \mathcal{U}$, and the operator
$$\mathcal{R}'(\omega) := (\mathcal{P}(\omega) - i \mathcal{Q})^{-1} : \mathcal{H}_2 \to \mathcal{H}_1$$
satisfies, for $\omega$ in (4.1),
$$\|\mathcal{R}'(\omega)\|_{\mathcal{H}_2 \to \mathcal{H}_1} \leq C h^{-1};$$

(11) for $\omega$ in (4.1), $\mathcal{R}'(\omega)$ is semiclassically outgoing in the following sense: if $(\rho, \rho') \in \text{WF}_h(\mathcal{R}'(\omega))$ and $\rho, \rho' \in \mathcal{U}'$, there exists $t \geq 0$ such that $e^{t H_p}(\rho) = \rho'$ and $e^{s H_p}(\rho) \in \mathcal{U}'$ for $0 \leq s \leq t$. (See Figure 2(a) below.)

4.2. Some consequences of general assumptions. In this section, we derive several corollaries of the assumptions of §4.1, used throughout the rest of the paper.

Global properties of the flow. We start with two technical lemmas:

Lemma 4.1. Assume that $\rho \in \Gamma_\pm$. Then as $t \to \mp \infty$, the distance $d(e^{\mp \imath t H_p}(\rho), K)$ converges to zero.
Proof. We consider the case $\rho \in \Gamma_-$. Put $\gamma(t) := e^{tH}(\rho)$, then $\gamma(t) \in \Gamma_-$ for all $t \geq 0$. Assume that $d(\gamma(t), K)$ does not converge to zero as $t \to +\infty$, then there exists a sequence of times $t_j \to +\infty$ such that $\gamma(t_j)$ does not lie in a fixed neighborhood of $K$. By passing to a subsequence, we may assume that $d(\gamma(t_j), K) \to 0$ as $t \to +\infty$. Then $\rho_\infty \notin \Gamma_+; \therefore$ there exists $T \geq 0$ such that $e^{-TH}(\rho_\infty) \notin \overline{U}$. For $j$ large enough, we have $\gamma(t_j - T) = e^{-TH}(\gamma(t_j)) \notin \overline{U}$ and $t_j \geq T$; this contradicts convexity of $\overline{U}$ (assumption (8)). \hfill \Box

Lemma 4.2. Assume that $U_1$ is a neighborhood of $K$ in $\overline{U}$. Then there exists a neighborhood $U_2$ of $K$ in $\overline{U}$ such that for each flow line $\gamma(t)$, $0 \leq t \leq T$ of $H_\rho$ in $\overline{U}$, if $\gamma(0), \gamma(T) \in U_2, \text{ then } \gamma([0, t]) \subset U_1$.

Proof. Assume the contrary, then there exist flow lines $\gamma_j(t)$, $0 \leq t \leq T$, in $\overline{U}$, such that $d(\gamma_j(0), K) \to 0$, $d(\gamma_j(T_j), K) \to 0$, yet $\gamma_j(t_j) \notin U_1$ for some $t_j \in [0, T_j]$. Passing to a subsequence, we may assume that $\gamma_j(t_j) \to \rho_\infty \in \overline{U} \setminus K$. Without loss of generality, we assume that $\rho_\infty \notin \Gamma_+$. Then there exists $T > 0$ such that $e^{-TH}(\rho_\infty) \in U' \setminus \overline{U}$, and thus $e^{-TH}(\gamma_j(t_j)) \notin \overline{U}$ for $j$ large enough. Since $\gamma_j([0, T_j]) \subset \overline{U}$, we have $t_j \leq T$. By passing to a subsequence, we may assume that $t_j \to t_\infty \in [0, T]$. However, then $\gamma_j(0) \to e^{-t_\infty H}(\rho_\infty)$, which implies that $e^{-t_\infty H}(\rho_\infty) \in \Gamma_+$, contradicting the fact that $\rho_\infty \notin \Gamma_+$. \hfill \Box

Resolution of dependence on $\omega$. We reduce the operator $\mathcal{P}(\omega)$ microlocally near $U$ to an operator of the form $P - \omega$, see also [IaSjZw, §4]:

Lemma 4.3. There exist:

- a compactly supported $P \in \Psi^{\text{comp}}(X)$ such that $P^* = P$ and $\sigma(P) = p$ near $U$, where $p$ is defined in (4.4), and

- a family of compactly supported operators $S(\omega) \in \Psi^{\text{comp}}(X)$, holomorphic in $\omega \in \Omega$, with $S(\omega)^* = S(\omega)$ for $\omega \in \mathbb{R} \cap \Omega$ and $S(\omega)$ elliptic near $U$, such that

$$\mathcal{P}(\omega) = S(\omega)(P - \omega)S(\omega) + O(h^\infty) \text{ microlocally near } U. \quad (4.8)$$

Proof. We argue by induction, constructing compactly supported operators $P_j, S_j(\omega) \in \Psi^{\text{comp}}(X)$, such that $P_j^* = P_j$, $S_j^*(\omega) = S_j(\omega)$ for $\omega \in \mathbb{R} \cap \Omega$, and $\mathcal{P}(\omega) = S_j(\omega)(P_j - \omega)S_j(\omega) + O(h^{j+1})$ microlocally near $U$. It will remain to take the asymptotic limit.

For $j = 0$, it suffices to take any $P_0, S_0(\omega)$ such that $\sigma(P_0) = p$ and $\sigma(S_0(\omega))(\rho) = s_0(\rho, \omega)$ near $U$, where (with $p(\cdot, \omega)$ denoting the principal symbol of $\mathcal{P}(\omega)$)

$$p(\rho, \omega) = s_0(\rho, \omega)^2(p(\rho) - \omega), \quad \rho \in U';$$

the existence of such $s_0$ and the fact that it is real-valued for real $\omega$ follows from assumption (7).
By propagation of singularities (Proposition 3.4) applied to (4.9), we see that
Proof. Assume that $P_j, S_j(\omega)$ for some $j \geq 0$, we construct $P_{j+1}, S_{j+1}(\omega)$. We have $\mathcal{P}(\omega) = S_j(\omega)(P_j-\omega)S_j(\omega) + h^{2j+1}R_j(\omega)$ microlocally near $U$, where $R_j(\omega) \in \Psi^{\text{comp}}$ is a holomorphic family of operators and, by assumption (6), $R_j(\omega)^* = R_j(\omega) + \mathcal{O}(h^\infty)$ microlocally near $U$ when $\omega \in \mathbb{R} \cap \Omega$. We then put $P_{j+1} = P_j + h^{2j+1}A_j, S_{j+1}(\omega) = S_j(\omega) + h^{2j+1}B_j(\omega)$, where $\sigma(A_j) = p_j, \sigma(B_j(\omega))(\rho) = s_j(\rho, \omega)$ near $U$ and
$$\sigma(R_j)(\rho, \omega) = 2s_0(\rho, \omega)s_j(\rho, \omega)(p(\rho) - \omega) + s_0(\rho, \omega)^2 p_j(\rho), \quad \rho \in U'.$$
The existence of $s_j(\rho, \omega), p_j(\rho)$ and the fact that $p_j(\rho) \in \mathbb{R}$ and $s_j(\rho, \omega) \in \mathbb{R}$ for $\rho$ near $U$ and $\omega \in \mathbb{R} \cap \Omega$ follow from assumption (7). In particular, we put $p_j(\rho) = \sigma(R_j)(\rho, p(\rho))/s_0(\rho, p(\rho))^2$. \hfill \Box

Note that, if $u(h) \in \mathcal{H}_1, f(h) \in \mathcal{H}_2$ have norms polynomially bounded in $h$ (and in light of assumption (3) are $h$-tempered in the sense of §3.1), and $\mathcal{P}(\omega)u = f$, then
$$(P - \omega)S(\omega)u = S'(\omega)f + \mathcal{O}(h^\infty) \quad \text{microlocally near } U, \quad (4.9)$$
where $S'(\omega) \in \Psi^{\text{comp}}(X)$ is an elliptic parametrix of $S(\omega)$ microlocally near $U$, constructed in Proposition 3.3.

**Microlocalization of $\mathcal{R}(\omega)$.** Next, we use the semiclassically outgoing parametrix $\mathcal{R}'(\omega)$ from (4.6) to derive a key restriction on the wavefront set of functions in the image of $\mathcal{R}(\omega)$, see Figure 2(b):

**Lemma 4.4.** Assume that $u(h) \in \mathcal{H}_1, f(h) \in \mathcal{H}_2$ have norms polynomially bounded in $h, \mathcal{P}(\omega)u = f$ for some $\omega = \omega(h)$ satisfying (4.1), and $\text{WF}_h(f) \subset U$. Then for each $\rho \in \text{WF}_h(u) \cap \overline{U}$, if $\gamma(t) = e^{th\nu}(\rho)$ is the corresponding maximally extended flow line in $\overline{U}'$, then either $\gamma(t) \in \overline{U}$ for all $t \leq 0$ or $\gamma(t) \in \text{WF}_h(f)$ for some $t \leq 0$.

**Proof.** By propagation of singularities (Proposition 3.4) applied to (4.9), we see that either $\gamma(t) \in \overline{U}$ for all $t \leq 0$, or $\gamma(t) \in \text{WF}_h(f)$ for some $t \leq 0$, or there exists $t \leq 0$ such that $\gamma(t) \in \text{WF}_h(u) \cap (\overline{U}' \setminus \overline{U})$; we need to exclude the third case. However, in this...
case by convexity of $\overline{U}$ (assumption (8)), $\gamma(t-s) \notin \overline{U}$ for all $s \geq 0$; by assumption (11), and since $u = R'(\omega)(f-iQu)$ with $WF_h(f-iQu) \subset U$, we see that $\gamma(t) \notin WF_h(u)$, a contradiction.

It follows from Lemma 4.4 that any resonant state, i.e. a function $u$ such that $\|u\|_{H^1} \sim 1$ and $\mathcal{P}(\omega)u = 0$, has to satisfy $WF_h(u) \cap U \subset \Gamma_+$. The next statement improves on the parametrix $\mathcal{R}'(\omega)$, inverting the operator $\mathcal{P}(\omega)$ outside of any given neighborhood of the trapped set. One can see this as a geometric control statement (see for instance [BuZw, Theorem 3]).

**Lemma 4.5.** Let $W \subset U$ be a neighborhood of $K \cap p^{-1}([\alpha_0, \alpha_1])$ (which is a compact subset of $U$ by assumption (9)), and assume that $f(h) \in \mathcal{H}_2$ has norm bounded polynomially in $h$ and each $\omega = \omega(h)$ is in (4.1). Then there exists $v(h) \in \mathcal{H}_1$, with $f - \mathcal{P}(\omega)v$ compactly supported in $X$ and

$$\|v\|_{H^1} \leq Ch^{-1}\|f\|_{H^2}, \quad \|\mathcal{P}(\omega)v\|_{H^2} \leq C\|f\|_{H^2}, \quad WF_h(f - \mathcal{P}(\omega)v) \subset W.$$  

**Proof.** First of all, take compactly supported $Q' \in \Psi^\text{comp}(X)$ such that $WF_h(Q') \subset U$ and $Q' = 1$ microlocally near $WF_h(Q)$ (with $Q$ defined in assumption (10)), and put

$$v_1 := (1 - Q')\mathcal{R}'(\omega)f.$$  

Then by (4.7), $\|v_1\|_{H^1} \leq Ch^{-1}\|f\|_{H^2}$ and $\mathcal{P}(\omega)v_1 = f_1$, where

$$f_1 = (1 - Q' - [\mathcal{P}(\omega), Q']\mathcal{R}'(\omega) + (1 - Q')iQ\mathcal{R}'(\omega))f.$$  

Since $(1 - Q')iQ = O(h^\infty)_{c_0}$, by (4.7) we find $\|f_1\|_{H^2} \leq C\|f\|_{H^2}$, $f - f_1$ is compactly supported, and $WF_h(f - f_1) \subset WF_h(Q')$. It is now enough to prove our statement for $f - f_1$ in place of $f$; therefore, we may assume that $f$ is compactly supported and

$$WF_h(f) \subset WF_h(Q').$$

Since $WF_h(Q')$ is compact, by a microlocal partition of unity we may assume that $WF_h(f)$ is contained in a small neighborhood of some fixed $\rho \in WF_h(Q') \subset U$. We now consider three cases:

**Case 1:** $\rho \notin p^{-1}([\alpha_0, \alpha_1])$. Then the operator $\mathcal{P}(\omega)$ is elliptic at $\rho$, therefore we may assume it is elliptic on $WF_h(f)$. The function $v$ is then obtained by applying to $f$ an elliptic parametrix of $\mathcal{P}(\omega)$ given in Proposition 3.3; we have $f - \mathcal{P}(\omega)v = O(h^\infty)_{c_0}$.  

**Case 2:** $\rho \in \Gamma_+ \cap p^{-1}([\alpha_0, \alpha_1])$. By Lemma 4.1, there exists $t \geq 0$ such that $e^{\mathcal{H}_p} \in W$. We may then assume that $e^{\mathcal{H}_p}(WF_h(f)) \subset W$, and $v$ is then constructed by Proposition 3.5, using (4.8); we have $WF_h(v) \subset U$ and $WF_h(f - \mathcal{P}(\omega)v) \subset W$.  

**Case 3:** $\rho \notin \Gamma_+$. Then there exists $t \geq 0$ such that $e^{\mathcal{H}_p}(\rho) \in U \setminus \overline{U}$. As in case 2, subtracting from $v$ the parametrix of Proposition 3.5, we may assume that $f$ is instead
microlocalized in a neighborhood of $e^{tH_\rho}(\rho)$. Now, put $v = R'(\omega)f$, with $R'(\omega)$ defined in (4.6); then $\|v\|_{H_2} \leq Ch^{-1}\|f\|_{H_2}$ by (4.7) and

$$f - P(\omega)v = -iQv.$$  

However, by assumption (11), and by convexity of $U$ (assumption (8)), we have $WF_h(Q) \cap WF_h(v) = \emptyset$ and thus $f - P(\omega)v = O(h^\infty)_{C_0^\infty}$.

Finally, we can estimate the norm of $u \in H_1$ by the norm of $P(\omega)u$ and the norm of $u$ microlocally near the trapped set. This can be viewed as an observability statement (see for instance [BuZw, Theorem 2]).

**Lemma 4.6.** Let $A \in \Psi^{\text{comp}}(X)$ be compactly supported and elliptic on $K \cap p^{-1}(\alpha_0, \alpha_1)$. Then we have for any $u \in H_1$ and any $\omega$ in (4.1),

$$\|u\|_{H_1} \leq C\|Au\|_{L^2} + Ch^{-1}\|P(\omega)u\|_{H_2}.  \quad (4.10)$$

**Proof.** By rescaling, we may assume that $u = u(h)$ has $\|u\|_{H_1} = 1$ and put $f = P(\omega)u$. Take a neighborhood $W$ of $K \cap p^{-1}(\alpha_0, \alpha_1)$ such that $A$ is elliptic on $W$. Replacing $u$ by $u - v$, where $v$ is constructed from $f$ in Lemma 4.5, we may assume that $WF_h(f) \subset W$.

Take $Q', Q'' \in \Psi^{\text{comp}}(X)$ compactly supported, with $WF_h(Q') \subset U$, $Q'' = 1 + O(h^\infty)$ microlocally near $WF_h(Q')$, and $Q' = 1 + O(h^\infty)$ microlocally near $WF_h(Q)$ (with $Q$ defined in assumption (10)). Then by the elliptic estimate (Proposition 3.2),

$$\|Q'u\|_{H_1} \leq C\|Q''u\|_{L^2} + O(h^\infty), \quad (4.11)$$

$$\|P(\omega), Q'u\|_{H_2} \leq Ch\|Q''u\|_{L^2} + O(h^\infty). \quad (4.12)$$

Now,

$$(1 - Q')u = R'(\omega)((1 - Q')f - [P(\omega), Q']u - iQ(1 - Q')u);$$

since $iQ(1 - Q') = O(h^\infty)_{\Psi^{-\infty}}$, we get by (4.7) and (4.12),

$$\|(1 - Q')u\|_{H_1} \leq C\|Q''u\|_{L^2} + Ch^{-1}\|f\|_{H_2} + O(h^\infty);$$

by (4.11), it then remains to prove that

$$\|Q''u\|_{L^2} \leq C\|Au\|_{L^2} + Ch^{-1}\|f\|_{H_2} + O(h^\infty).$$

By a microlocal partition of unity, it suffices to estimate $\|Bu\|_{L^2}$ for $B \in \Psi^{\text{comp}}(X)$ compactly supported with $WF_h(B)$ in a small neighborhood of some $\rho \in WF_h(Q'') \subset U$. We now consider three cases:

**Case 1:** $\rho \notin p^{-1}(\alpha_0, \alpha_1)$. Then $P(\omega)$ is elliptic at $\rho$, therefore we may assume it is elliptic on $WF_h(B)$. By Proposition 3.2, we get $\|Bu\|_{L^2} \leq C\|f\|_{H_2} + O(h^\infty)$.

**Case 2:** there exists $t \leq 0$ such that $e^{tH_\rho}(\rho) \in W$, therefore we may assume that $e^{tH_\rho}(WF_h(B)) \subset W$. Since $A$ is elliptic on $W$, by Proposition 3.4 together with (4.8), we get $\|Bu\|_{L^2} \leq C\|Au\|_{L^2} + Ch^{-1}\|f\|_{H_2} + O(h^\infty)$. 

**Case 3:** if $\gamma(t) = e^{tH_\rho}(\rho)$ is the maximally extended trajectory of $H_\rho$ in $U'$, then $\rho \in p^{-1}([\alpha_0, \alpha_1])$ and $\gamma(t) \not\in W$ for all $t \leq 0$. By Lemma 4.1, we have $\rho \not\in \Gamma_+$. Since $WF_h(f) \subset W$, Lemma 4.4 implies that $\rho \not\in WF_h(u)$. We may then assume that $WF_h(B) \cap WF_h(u) = \emptyset$ and thus $\|Bu\|_{L^2} = \mathcal{O}(h^\infty)$. \hfill \square

4.3. **Example: Schrödinger operators on** $\mathbb{R}^n$. In this section, we consider the case described in the introduction, namely a Schrödinger operator on $X = \mathbb{R}^n$ with

$$P_V = h^2 \Delta + V(x),$$

where $\Delta$ is the Euclidean Laplacian and $V \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$. We will explain how this case fits into the framework of §4.1.

To define resonances for $P_0$, we use the method of complex scaling of Aguilar–Combes [AgCo], which also applies to more general operators and potentials – see [SjZw91], [Sj97], and the references given there. Take $R > 0$ large enough so that

$$\text{supp } V \subset \{|x| < R/2\}.$$ Fix the deformation angle $\theta \in (0, \pi/2)$ and consider a deformation $\Gamma_{\theta,R} \subset \mathbb{C}^n$ of $\mathbb{R}^n$ defined by

$$\Gamma_{\theta,R} := \{x + iF_{\theta,R}(x) \mid x \in \mathbb{R}^n\},$$

where $F_{\theta,R} : \mathbb{R}^n \to \mathbb{R}^n$ is defined in polar coordinates $(r, \varphi) \in [0, \infty) \times S^{n-1}$ by

$$F_{\theta,R}(r, \varphi) = (f_{\theta,R}(r), \varphi),$$

and the function $f_{\theta,R} \in C^\infty([0, \infty))$ is chosen so that (see Figure 3(a))

$$f_{\theta,R}(r) = 0, \quad r \leq R; \quad f_{\theta,R}(r) = r \tan \theta, \quad r \geq 2R;$$

$$f'_{\theta,R}(r) \geq 0, \quad r \geq 0; \quad \{f'_R = 0\} = \{f_R = 0\}. $$

Note that

$$\Gamma_{\theta,R} \cap \{|\text{Re } z| \leq R\} = \mathbb{R}^n \cap \{|\text{Re } z| \leq R\};$$

$$\Gamma_{\theta,R} \cap \{|\text{Re } z| \geq 2R\} = e^{i\theta}\mathbb{R}^n \cap \{|\text{Re } z| \geq 2R\}. $$

Define the deformed differential operator $\tilde{P}_V$ on $\Gamma_{\theta,R}$ it as follows: $\tilde{P}_V = P_V$ on $\mathbb{R}^n \cap \Gamma_{\theta,R}$, and on the complementing region $\{|\text{Re } z| > R\}$, it is defined by the formula

$$\tilde{P}_V(v) = \sum_{j=1}^n (hD_{z_j})^2 \tilde{v}|_{\Gamma_{\theta,R}},$$

for each $v \in C_0^\infty(\Gamma_{\theta,R} \cap \{|\text{Re } z| > R\})$ and each almost analytic continuation $\tilde{v}$ of $v$ (that is, $\tilde{v}|_{\Gamma_{\theta,R}} = v$ and $\partial_z \tilde{v}$ vanishes to infinite order on $\Gamma_{\theta,R}$ – the existence of such continuation follows from the fact that $\Gamma_{\theta,R}$ is totally real, that is for each $z \in \Gamma_{\theta,R}$, $T_z \Gamma_{\theta,R} \cap iT_z \Gamma_{\theta,R} = 0$). We identify $\Gamma_{\theta,R}$ with $\mathbb{R}^n$ by the map

$$\iota : \mathbb{R}^n \to \Gamma_{\theta,R} \subset \mathbb{C}^n, \quad \iota(x) = x + iF_{\theta,R}(x),$$
so that \( \tilde{P}_V \) can be viewed as a second order differential operator on \( \mathbb{R}^n \). Then in polar coordinates \((r, \varphi)\), we can write for \( r > R \),

\[
\tilde{P}_V = \left( \frac{1}{1 + if'_{\theta,R}(r)} h D_r \right)^2 - \frac{(n-1)i}{(r + if_{\theta,R}(r))(1 + if'_{\theta,R}(r))} h^2 D_r + \frac{\Delta_\varphi}{(r + if_{\theta,R}(r))^2},
\]

with \( \Delta_\varphi \) denoting the Laplacian on the round sphere \( S^{n-1} \). We have

\[
\sigma(\tilde{P}_V) = \frac{|\xi_r|^2}{(1 + if'_{\theta,R}(r))^2} + \frac{|\xi_\varphi|^2}{(r + if_{\theta,R}(r))^2} + V(r, \varphi).
\] (4.13)

Fix a range of energies \([\alpha_0, \alpha_1] \subset (0, \infty)\) and a bounded open set \( \Omega \subset \mathbb{C} \) such that (see Figure 3(b))

\[
[\alpha_0, \alpha_1] \subset \Omega, \quad \overline{\Omega} \subset \{-\theta < \arg \omega < \pi - \theta\}.
\]

For \( \omega \in \Omega \), define the operator

\[
\mathcal{P}(\omega) = \tilde{P}_V - \omega^2 : \mathcal{H}_1 \to \mathcal{H}_2, \quad \mathcal{H}_1 := H^2_0(\mathbb{R}^n), \quad \mathcal{H}_2 := L^2(\mathbb{R}^n).
\]

Then \( \mathcal{P}(\omega) \) is Fredholm \( \mathcal{H}_1 \to \mathcal{H}_2 \) for \( \omega \in \Omega \). Indeed,

\[
\mathcal{P}(\omega) = \cos^2 \theta e^{-2i\theta} h^2 \Delta - \omega^2 \quad \text{on \(|x| \geq 2R\)},
\]

thus \( \mathcal{P}(\omega) \) is elliptic on \(|x| \geq 2R\), as well as for \(|\xi|\) large enough, in the class \( S(\langle \xi \rangle^2) \) of \( [Zw, \S 4.4.1] \) (this class incorporates the behavior of symbols as \( x \to \infty \), in contrast with those used in \( \S 3.1 \)). Using a construction similar to Lemma 3.3, but with symbols in the class \( S(\langle \xi \rangle^{-2}) \), we can define a parametrix near (both spatial and fiber) infinity, \( \mathcal{R}_\infty(\omega) \), with \( \| \mathcal{R}_\infty \|_{L^2(\mathbb{R}^n) \to H^2_0(\mathbb{R}^n)} = O(1) \) and

\[
\mathcal{R}_\infty(\omega) \mathcal{P}(\omega) = 1 + Z(\omega) + O(h^\infty)_{H^2_0(\mathbb{R}^n) \to H^2_0(\mathbb{R}^n)}, \quad \mathcal{P}(\omega) \mathcal{R}_\infty(\omega) = 1 + Z'(\omega) + O(h^\infty)_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)},
\] (4.14)

where \( Z(\omega), Z'(\omega) \in \Psi^{\text{comp}}(\mathbb{R}^n) \) are compactly supported inside \(|x| < 2R + 1\). Since \( 1 + O(h^\infty) \) is invertible and \( Z(\omega), Z'(\omega) \) are compact, we see that \( \mathcal{P}(\omega) \) is indeed Fredholm \( \mathcal{H}_1 \to \mathcal{H}_2 \). We have thus verified assumptions (1)–(4) of \( \S 4.1 \).
The identification of the poles of $\mathcal{R}(\omega)$ with the poles of the meromorphic continuation of the resolvent $R_V(\omega) = (P_V - \omega^2)^{-1}$ defined in (1.3) from $\{\text{Im} \, \omega > 0\}$ to $\Omega$, and in fact, the existence of such a continuation, follows from the following formula (implicit in [Sj97], and discussed in [TaZw]): if $\chi \in C_0^\infty(\mathbb{R}^n)$, $\text{supp} \, \chi \subset B(0, R)$, then

$$\chi \mathcal{R}(\omega) \chi = \chi R_V(\omega) \chi. \quad (4.15)$$

This is initially valid in $\Omega \cap \{\text{Im} \, \omega > 0\}$ so that the right-hand side is well-defined, and then by analytic continuation in the region where the left hand side is meromorphic.

Now, we take intervals $[\alpha_0, \alpha_1] \subset [\beta_0, \beta_1] \subset \Omega \cap (0, \infty)$ and put

$$U' := \{ |x| < R, |\xi|^2 + V(x) \in ((\beta_0')^2, (\beta_1')^2) \},$$

$$U := \{ |x| < 3R/4, |\xi|^2 + V(x) \in ((\beta_0^2, \beta_1^2) \}. \quad (4.16)$$

Note that $\mathcal{P}(\omega) = P_V - \omega^2$ in $U'$; this verifies assumptions (5) and (6). Assumption (7) is also satisfied, with

$$p(x, \xi) = \sqrt{|\xi|^2 + V(x)}, \quad (x, \xi) \in U'. \quad (4.17)$$

The operators $P$ and $S(\omega)$ from Lemma 4.3 take the form, microlocally near $U$,

$$P = \sqrt{P_V}, \quad S(\omega) = \sqrt{\sqrt{P_V} + \omega}. \quad (4.18)$$

Here the square root is understood in the microlocal sense: for an operator $A \in \Psi^k(X)$ with $\sigma(A) > 0$ on $U'$, we define the microlocal square root $\sqrt{A} \in \Psi^{\text{comp}}(X)$ of $A$ in $U'$ as the (unique modulo $\mathcal{O}(h^\infty)$ microlocally in $U'$) operator such that $(\sqrt{A})^2 = A + \mathcal{O}(h^\infty)$ microlocally in $U'$ and $\sigma(\sqrt{A}) = \sqrt{\sigma(A)}$. See for example [GrSj, Lemma 4.6] for details of the construction of the symbol.

Assumption (8), namely convexity of $\overline{U}$, is satisfied since for each $(x, \xi) \in U'$, if $|x| \geq R/2$ and $H_p|x|^2 = 0$ at $(x, \xi)$, then $H_p^2|x|^2 > 0$ at $(x, \xi)$; therefore, the function $|x|^2$ cannot attain a local maximum on a trajectory of $e^{tH_p}$ in $U' \setminus \overline{U}$. Same observation shows assumption (9); in fact, $K \subset \{|x| \leq R/2\}$.

Finally, for assumptions (10) and (11), we take any compactly supported $Q \in \Psi^{\text{comp}}(X)$ such that $\text{WF}_h(Q) \subset U$ and

$$\sigma(Q) \geq 0 \quad \text{everywhere}; \quad \sigma(Q) > 0 \quad \text{on } p^{-1}([\alpha_0, \alpha_1]) \cap \{|x| \leq R/2\}. \quad (4.19)$$

To verify assumption (10), consider an arbitrary family $u = u(h) \in H^2_h(\mathbb{R}^n)$, with norm bounded polynomially in $h$, and put

$$f = (\mathcal{P}(\omega) - iQ)u,$$
where $\omega$ satisfies (4.1). By (4.13), and since $\text{Im} \omega = \mathcal{O}(h)$, we find
\[
\text{Im} \sigma(\mathcal{P}(\omega)) \leq 0 \quad \text{everywhere;}
\]
\[
\{\langle \xi \rangle^{-2} \sigma(\mathcal{P}(\omega)) = 0 \} \subset \{F_{\theta,R}(x) = 0\}.
\]
Note also that $\sigma(\mathcal{P}(\omega)) = |\xi|^2 + V(x) - \omega^2$ on $\{F_{\theta,R}(x) = 0\}$. Together with the convexity property of $|\xi|^2$ mentioned above, we see that for each $\rho \in T^*_X$, there exists $t \leq 0$ such that $\mathcal{P}(\omega) - iQ$ is elliptic at $\exp(tH_{\text{Re}\sigma(\mathcal{P}(\omega))}(\rho))$. Since $\text{Im} \sigma(\mathcal{P}(\omega) - iQ) \leq 0$ everywhere, by propagation of singularities with a complex absorbing term (Proposition 3.4) and the elliptic estimate (Proposition 3.2) we get
\[
\|Z(\omega)u\|_{H^2_h} \leq Ch^{-1}\|f\|_{L^2} + \mathcal{O}(h^\infty),
\]
where $Z(\omega)$ is defined in (4.14). Then by (4.14),
\[
\|u\|_{H^2_h(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)} + \|Z(\omega)u\|_{H^2_h} + \mathcal{O}(h^\infty) \leq Ch^{-1}\|f\|_{L^2(\mathbb{R}^n)} + \mathcal{O}(h^\infty),
\]
proving the estimate (4.7) of assumption (10).

Assumption (11) is proved in a similar fashion: assume that $\text{WF}_h(f) \subset U'$ and $\rho' \in \text{WF}_h(u) \cap U'$. Denote $\gamma(t) = \exp(tH_{\text{Re}\sigma(\mathcal{P}(\omega))})(\rho')$. Then there exists $t_0 \geq 0$ such that $\mathcal{P}(\omega) - iQ$ is elliptic at $\gamma(-t_0)$. By Proposition 3.4, we see that either $\exp(-tH_{\text{Re}\sigma(\mathcal{P}(\omega))})(\rho') \in \text{WF}_h(f)$ for some $t \in [0, t_0]$ or $\exp(-t_0H_{\text{Re}\sigma(\mathcal{P}(\omega))})(\rho') \in \text{WF}_h(u)$, in which case this point also lies in $\text{WF}_h(f)$ by Proposition 3.2; therefore, $\gamma(-t) \in \text{WF}_h(f)$ for some $t \geq 0$. Let $t_1$ be the minimal nonnegative number such that $\gamma(-t_1) \in \text{WF}_h(f)$; we may assume that $t_1 > 0$. Since $\gamma([-t_1, 0])$ does not intersect $\text{WF}_h(f)$, it also does not intersect the elliptic set of $\mathcal{P}(\omega)$; therefore, $\gamma([-t_1, 0]) \subset \{F_{\theta,R}(x) = 0\}$ and thus $\sigma(\mathcal{P}(\omega)) = p^2 - \omega^2$ on $\gamma([-t_1, 0])$. It follows that $e^{-tH}(\rho') \in \text{WF}_h(f)$ for some $t \geq 0$, as required.

### 4.4. Example: even asymptotically hyperbolic manifolds.

In this section, we define resonances, in the framework of §4.1, for an $n$-dimensional complete noncompact Riemannian manifold $(M, g)$ which is asymptotically hyperbolic in the following sense: $M$ is diffeomorphic to the interior of a smooth manifold with boundary $\overline{M}$, and for some choice of the boundary defining function $\tilde{x} \in C^\infty(\overline{M})$ and the product decomposition $\{\tilde{x} < \varepsilon\} \sim [0, \varepsilon) \times \partial \overline{M}$, the metric $g$ takes the following form in $\{0 < \tilde{x} < \varepsilon\}$:
\[
g = \frac{d\tilde{x}^2 + g_1(\tilde{x}, \tilde{y}, dy)}{\tilde{x}^2}. \quad (4.19)
\]
Here $g_1$ is a family of Riemannian metrics on $\partial \overline{M}$ depending smoothly on $\tilde{x} \in [0, \varepsilon)$. We moreover require that the metric is even in the sense that $g_1$ is a smooth function of $\tilde{x}^2$.

To put the Laplacian $\Delta_g$ on $M$ into the framework of §4.1, we use the recent construction of Vasy [Va11]. We follow in part [DaDy, §4.1], see also [DaDy, Appendix B] for a detailed description of the phase space properties of the resulting operator in
a model case. Take the space \( \overline{M} \) obtained from \( M \) by taking the new boundary
defining function \( \mu = \tilde{x}^2 \) and put (see [Va11, §3.1])
\[
P_1(\omega) = \mu^{-1/2} - \frac{n+1}{4} e^{i \omega \theta} (h^2(\Delta_g) - (n-1)^2/4 - \omega^2)e^{-i \omega \phi} \mu^{-1/2 + \frac{n+1}{4}}.
\]
Here \( \phi \) is a smooth real-valued function on \( M \) such that
\[
e^{\phi} = \mu^{1/2}(1 + \mu)^{-1/4} \quad \text{on } \{0 < \mu < \delta_0\},
\]
where \( \delta_0 > 0 \) is a small constant; the values of \( \phi \) on \( \{\mu \geq \delta_0\} \) are chosen as in the
paragraph preceding [Va11, (3.14)]. We can furthermore choose \( \mu \) and \( \phi \) to be equal
to 1 near the set \( \{\tilde{x} > \varepsilon_0/2\} \), for any fixed \( \varepsilon_0 > 0 \) (and \( \delta_0 \) chosen small depending
on \( \varepsilon_0 \)) so that
\[
P_1(\omega) = h^2(\Delta_g) - (n-1)^2/4 - \omega^2 \quad \text{on } \{\tilde{x} > \varepsilon_0/2\}.
\]
(4.20)
The differential operator \( P_1(\omega) \) has coefficients smooth up to the boundary of \( \overline{M} \); then it is possible to find a compact \( n \)-dimensional manifold \( X \) without boundary such
that \( \overline{M} \) embeds into \( X \) as \( \{\mu \geq 0\} \) and extend \( P_1(\omega) \) to an operator
\( P_2(\omega) \in \Psi^2(X) \), see [Va11, §3.5] or [DaDy, Lemma 4.1]. Finally, we fix a complex absorbing operator
\( Q \in \Psi^2(X) \), with Schwartz kernel supported in the nonphysical region \( \{\mu < 0\} \),
satisfying the assumptions of [Va11, §3.5]. We now fix an interval \( [\alpha_0, \alpha_1] \subset (0, \infty) \),
take \( \Omega \subset \mathbb{C} \) a small neighborhood of \( [\alpha_0, \alpha_1] \), and put
\[
\mathcal{P}(\omega) := P_2(\omega) - iQ, \quad \omega \in \Omega.
\]
Fix \( C_0 > 0 \), take \( s > C_0 + 1/2 \), and put \( \mathcal{H}_2 = H^{s-1}_h(X) \) and
\[
\mathcal{H}_1 = \{u \in H^s_h(X) \mid P_2(1)u \in H^{s-1}_h(X)\}, \quad \|u\|^2_{\mathcal{H}_1} = \|u\|^2_{H^s_h(X)} + \|P_2(1)u\|^2_{H^{s-1}_h(X)}.
\]
It is proved in [Va11, Theorem 4.3] that for \( \omega \) satisfying (4.1), the operator \( \mathcal{P}(\omega) : \mathcal{H}_1 \to \mathcal{H}_2 \) is Fredholm of index zero; therefore, we have verified assumptions (1)–(4)
of §4.1. The poles of \( \mathcal{R}(\omega) = \mathcal{P}(\omega)^{-1} \) coincide with the poles of the meromorphic
continuation of the Schwartz kernel of the resolvent
\[
R_g(\omega) := (h^2(\Delta_g) - (n-1)^2/4 - \omega^2)^{-1} : L^2(M) \to L^2(M), \quad \text{Im} \omega > 0,
\]
to the entire \( \mathbb{C} \), first constructed in [MaMe] with improvements by [Gu] – see [Va11, Theorem 5.1].

We can now proceed similarly to §4.3, using that the regions \( \{\tilde{x} > \varepsilon_0\} \) are geodesically
convex for \( \varepsilon_0 > 0 \) small enough (see for instance [DyGu, Lemma 7.1]). Fix small
\( \varepsilon_0 > 0 \), take any intervals
\[
[\alpha_0, \alpha_1] \subset [\beta_0, \beta_1] \subset [\beta'_0, \beta'_1] \subset \Omega \cap (0, \infty),
\]
and define
\[
\mathcal{U}' := \{\tilde{x} > \varepsilon_0/2, \ |\xi|_g \in (\beta'_0, \beta'_1)\}, \quad \mathcal{U} := \{\tilde{x} > \varepsilon_0, \ |\xi|_g \in (\beta_0, \beta_1)\}.
\]
(4.21)
As in §4.3, assumptions (5)–(9) hold, with
\[ p(x, \xi) = |\xi|_g. \] (4.22)
The operators \( P \) and \( S(\omega) \) constructed in Lemma 4.3 are given microlocally near \( U \) by
\[ P = \sqrt{h^2\Delta_g - (n-1)^2/4}, \quad S(\omega) = \sqrt{h^2\Delta_g - (n-1)^2/4 + \omega}, \]
with the square roots defined as in (4.18).

Finally, for assumptions (10) and (11), take \( Q \in \Psi^{\text{comp}}(X) \) with \( \text{WF}_h(Q) \subset U \) and
\[ \sigma(Q) \geq 0 \text{ everywhere}; \quad \sigma(Q) > 0 \text{ on } p^{-1}([\alpha_0, \alpha_1]) \cap \{ \tilde{x} \geq 2\varepsilon_0 \}. \]
Then assumption (10) follows from [Va11, Theorem 4.8]. To verify assumption (11), we modify the proof of [Va11, Theorem 4.9] as follows: assume that \( \gamma(t) \) is not elliptic at \( \gamma' \), since otherwise \( \gamma' \in \text{WF}_h(f) \). If \( \gamma(t) \) is the bicharacteristic of \( \sigma(P_2(\omega)) \) starting at \( \gamma' \), then (see [Va11, (3.32) and the end of §3.5]) either \( \gamma(t) \) converges to the set \( L_+ \subset \partial T^* X \cap \{ \mu = 0 \} \) of radial points as \( t \to -\infty \), or \( Q \) is elliptic at \( \gamma(-t_0) \) for some \( t_0 > 0 \). In the first case, \( \gamma(-t_0) \notin \text{WF}_h(u) \) for \( t_0 > 0 \) large enough by the radial points argument [Va11, Proposition 4.5]; in the second case, by Proposition 3.2 we see that if \( \gamma(-t_0) \in \text{WF}_h(u) \), then \( \gamma(-t_0) \in \text{WF}_h(f) \). Combining this with Proposition 3.4, we see that there exists \( t_1 \geq 0 \) such that \( \gamma(-t_1) \in \text{WF}_h(f) \). Since \( \gamma(0), \gamma(-t_1) \in U' \), and \( U' \) is convex with respect to the bicharacteristic flow of \( \sigma(P_2(\omega)) \) (the latter being just a rescaling of the geodesic flow pulled back by a certain diffeomorphism), we see that \( \gamma([-t_1, 0]) \subset U' \). Now, by (4.20), \( \gamma([-t_1, 0]) \) is a flow line of \( H_{\rho^*} \); therefore, for some \( t \geq 0 \), \( e^{-tH_{\rho^*}(\rho')} \in \text{WF}_h(f) \), as required.

5. \( r \)-NORMALLY HYPERBOLIC TRAPPED SETS

In this section, we state the dynamical assumptions on the flow near the trapped set \( K \), namely \( r \)-normal hyperbolicity, and define the expansion rates \( \nu_{\min}, \nu_{\max} \) (§5.1). We next establish some properties of \( r \)-normally hyperbolic trapped sets: existence of special defining functions \( \varphi_{\pm} \) of the incoming/outgoing tails \( \Gamma_{\pm} \) near \( K \) (§5.3), existence of the canonical projections \( \pi_{\pm} \) from open subsets \( \Gamma_{\pm}^d \subset \Gamma_{\pm} \) to \( K \) and the canonical relation \( \Lambda^o \) (§5.4), and regularity of solutions to the transport equations (§5.5).

5.1. Dynamical assumptions. Let \( U \subset U' \) be the open sets from §4.1, and \( p \in C^\infty(U'; \mathbb{R}) \) be the function defined in (4.4). Consider also the incoming/outgoing tails \( \Gamma_{\pm} \subset \overline{U} \) and the trapped set \( K = \Gamma_+ \cap \Gamma_- \) defined in (4.5). We assume that, for a large fixed integer \( r \) depending only on the dimension \( n \) (see Figure 4(a)),
Consider one-dimensional subbundles $V_{\pm} \subset T\Gamma_{\pm}$ defined as the symplectic complements of $T\Gamma_{\pm}$ in $T\Gamma_{\pm}(T^*X)$ (see Figure 4(b)); they are invariant under the flow $e^{tH_p}$. By assumption (2), we have $T_K\Gamma_{\pm} = V_{\pm}|_K \oplus TK$. Define the minimal expansion rate in the normal direction, $\nu_{\min}$, as the supremum of all $\nu$ for which there exists a constant $C$ such that

$$\sup_{\rho \in K} \|de^{\mp tH_p}(\rho)|_{V_{\pm}}\| \leq Ce^{-\nu t}, \quad t > 0. \tag{5.1}$$

Here $\| \cdot \|$ denotes the operator norm with respect to any smooth inner product on the fibers of $T(T^*X)$. Similarly we define the maximal expansion rate in the normal direction, $\nu_{\max}$, as the infimum of all $\nu$ for which there exists a constant $c > 0$ such that

$$\inf_{\rho \in K} \|de^{\mp tH_p}(\rho)|_{V_{\pm}}\| \geq ce^{-\nu t}, \quad t > 0. \tag{5.2}$$

Since $e^{tH_p}$ preserves the symplectic form $\sigma_S$, which is nondegenerate on $V_{\pm}|_K \oplus V_{\pm}|_K$, it is enough to require (5.1) and (5.2) for a specific choice of sign.

We assume $r$-normal hyperbolicity:
(3) Let \( \mu_{\text{max}} \) be the maximal expansion rate of the flow along \( K \), defined as the infimum of all \( \mu \) for which there exists a constant \( C \) such that

\[
\sup_{\rho \in K} \|d e^{tH_p}(\rho)|_{TK}\| \leq C e^{\mu |t|}, \quad t \in \mathbb{R}.
\]

Then

\[
\nu_{\text{min}} > r \mu_{\text{max}}.
\]

Assumption (3), rather than a weaker assumption of normal hyperbolicity \( \nu_{\text{min}} > 0 \), is needed for regularity of solutions to the transport equations, see Lemma 5.2 below. The number \( r \) depends on how many derivatives of the symbols constructed below are needed for the semiclassical arguments to work. In the proofs, we will often take \( r = \infty \), keeping in mind that a large fixed \( r \) is always enough.

5.2. Stability. We now briefly discuss stability of our dynamical assumptions under perturbations; a more detailed presentation, with applications to general relativity, will be given in [Dy13]. Assume that \( p_s \), where \( s \in \mathbb{R} \) varies in a neighborhood of zero, is a family of real-valued functions on \( U' \) such that \( p_0 = p \) and \( p_s \) is continuous at \( s = 0 \) with values in \( C^\infty(U') \). Assume moreover that conditions (8) and (9) of §4.1 are satisfied with \( p \) replaced by any \( p_s \). Here \( \Gamma_\pm \) and \( K \) are replaced by the sets \( \Gamma_\pm(s) \) and \( K(s) \) defined using \( p_s \) instead of \( p \). We claim that assumptions (1)–(3) of §5.1 are satisfied for \( p_s, \Gamma_\pm(s), K(s) \) when \( s \) is small enough.

We use the work of Hirsch–Pugh–Shub [HiPuSh] on stability of \( r \)-normally hyperbolic invariant manifolds. Assumptions (1)–(3) imply that the flow \( e^{tH_p} \) is eventually absolutely \( r \)-normally hyperbolic on \( K \) in the sense of [HiPuSh, Definition 4]. Then by [HiPuSh, Theorem 4.1], for \( s \) small enough, \( \Gamma_\pm(s) \) and \( K(s) \) are \( C^r \) submanifolds of \( T^*X \), which converge to \( \Gamma_\pm \) and \( K \) in \( C^r \) as \( s \to 0 \). It follows immediately that conditions (1) and (2) are satisfied for small \( s \).

To see that condition (3) is satisfied for small \( s \), as well as stability of the pinching condition (1.7) under perturbations, it suffices to show that, with \( \nu_{\text{min}}(s), \nu_{\text{max}}(s), \mu_{\text{max}}(s) \) defined using \( e^{tH_p}, \Gamma_\pm(s), K(s) \),

\[
\lim_{s \to 0} \nu_{\text{min}}(s) \geq \nu_{\text{min}}, \quad \lim_{s \to 0} \nu_{\text{max}}(s) \leq \nu_{\text{max}}, \quad \lim_{s \to 0} \mu_{\text{max}}(s) \leq \mu_{\text{max}}.
\]

(5.5) \( \quad \) (5.6) \( \quad \) (5.7)

We show (5.5); the other two inequalities are proved similarly. Fix a smooth metric on the fibers of \( T(T^*X) \). Take arbitrary \( \varepsilon > 0 \), then for \( T > 0 \) large enough, we have

\[
\sup_{\rho \in K} \|d e^{\pm TH_p}(\rho)|_{\nu_\pm}\| \leq e^{-(\nu_{\text{min}} - \varepsilon)T}.
\]
Fix $T$; since $p_s$, $\Gamma_\pm(s)$, $K(s)$, and the corresponding subbundles $V_\pm(s)$ depend continuously on $s$ at $s = 0$, we have for $s$ small enough,

$$\sup_{\rho \in K(s)} \|de^{t\pi H_{p_s}(\rho)}|_{V_\pm(s)}\| \leq e^{-(\nu_{\text{min}} - \epsilon/2)T}.$$ 

Since $e^{tH_{p_s}}$ is a one-parameter group of diffeomorphisms, we get

$$\sup_{\rho \in K(s)} \|de^{t\pi H_{p_s}(\rho)}|_{V_\pm(s)}\| \leq Ce^{-(\nu_{\text{min}} - \epsilon/2)t}, \quad t \geq 0;$$

therefore, $\nu_{\text{min}}(s) \geq \nu_{\text{min}} - \epsilon/2$ for $s$ small enough and (5.5) follows.

5.3. Adapted defining functions. In this section, we construct special defining functions $\varphi_\pm$ of $\Gamma_\pm$ near $K$. We will assume below that $\Gamma_\pm$ are smooth; however, if $\Gamma_\pm$ are $C^r$ with $r \geq 1$, we can still obtain $\varphi_\pm \in C^r$. A similar construction can be found in [WuZw11, Lemma 4.1].

**Lemma 5.1.** Fix $\epsilon > 0$.\footnote{The parameter $\epsilon$ is fixed in Theorem 1; it is also taken small enough for the results of §5.5 to hold.} Then there exist smooth functions $\varphi_\pm$, defined in a neighborhood of $K$ in $U'$, such that for $\delta > 0$ small enough, the set

$$U_\delta := \overline{U} \cap \{|\varphi_+| \leq \delta, |\varphi_-| \leq \delta\}, \quad (5.8)$$

is a compact subset of $U$ when intersected with $p^{-1}([\alpha_0, \alpha_1])$, and:

1. $\Gamma_\pm \cap U_\delta = \{\varphi_\pm = 0\} \cap U_\delta$, and $d\varphi_\pm \neq 0$ on $U_\delta$;
2. $H_p\varphi_\pm = \mp c_\pm \varphi_\pm$ on $U_\delta$, where $c_\pm$ are smooth functions on $U_\delta$ and, with $\nu_{\text{min}}, \nu_{\text{max}}$ defined in (5.1), (5.2),

$$\nu_{\text{min}} - \epsilon < c_\pm < \nu_{\text{max}} + \epsilon \quad \text{on } U_\delta; \quad (5.9)$$
3. the Hamiltonian field $H_{\varphi_\pm}$ spans the subbundle $V_\pm$ on $\Gamma_\pm \cap U_\delta$ defined before (5.1);
4. $\{\varphi_+, \varphi_-\} > 0$ on $U_\delta$;
5. $U_\delta$ is convex, namely if $\gamma(t), 0 \leq t \leq T$, is a Hamiltonian flow line of $p$ in $\overline{U}$ and $\gamma(0), \gamma(T) \in U_\delta$, then $\gamma([0, T]) \subset U_\delta$.

**Proof.** Since $\Gamma_\pm$ are orientable, there exist defining functions $\tilde{\varphi}_\pm$ of $\Gamma_\pm$ near $K$; that is, $\tilde{\varphi}_\pm$ are smooth, defined in some neighborhood $U$ of $K$, and $d\tilde{\varphi}_\pm \neq 0$ on $U$ and $\Gamma_\pm \cap U = \overline{U} \cap \{\tilde{\varphi}_\pm = 0\}$. Since $K$ is symplectic, by changing the sign of $\tilde{\varphi}_-$ if necessary, we can moreover assume that $\{\tilde{\varphi}_+, \tilde{\varphi}_-\} > 0$ on $K$.

Since $e^{tH_p}(\Gamma_\pm) \subset \Gamma_\pm$ for $\mp t \geq 0$, we have $H_p\tilde{\varphi}_\pm = 0$ on $\Gamma_\pm$; therefore,

$$H_p\tilde{\varphi}_\pm = \mp \tilde{c}_\pm \tilde{\varphi}_\pm,$$

where $\tilde{c}_\pm$ are smooth functions on $U$. The functions $\tilde{c}_\pm$ control how fast $\varphi_\pm$ decays along the flow as $t \to \pm \infty$. The constants $\nu_{\text{min}}$ and $\nu_{\text{max}}$ control the average decay rate; to construct $\varphi_\pm$, we will modify $\tilde{\varphi}_\pm$ by averaging along the flow for a large time.
For any $\rho \in \Gamma_\pm \cap U$, the kernel of $d\tilde{\varphi}_\pm(\rho)$ is equal to $T_p\Gamma_\pm$; therefore, the Hamiltonian fields $H_{\tilde{\varphi}_\pm}$ span $\mathcal{V}_\pm$ on $\Gamma_\pm \cap U$. We then see from the definitions (5.1), (5.2) of $\nu_{\min}, \nu_{\max}$ that there exists a constant $C$ such that, with $(e^{tH_{\rho}})_*H_{\tilde{\varphi}_\pm} \in \mathcal{V}_\pm$ denoting the push-forward of the vector field $H_{\tilde{\varphi}_\pm}$ by the diffeomorphism $e^{tH_{\rho}}$,

$$C^{-1}e^{-(\nu_{\max}+\varepsilon/2)t} \leq \frac{(e^{tH_{\rho}})_*H_{\tilde{\varphi}_\pm}}{H_{\tilde{\varphi}_\pm}} \leq Ce^{-(\nu_{\min}-\varepsilon/2)t} \quad \text{on } K, \quad t \geq 0.$$ 

Now, we calculate on $K$,

$$\partial_t((e^{tH_{\rho}})_*H_{\tilde{\varphi}_\pm}) = \pm(e^{tH_{\rho}})_*[H_{\rho}, H_{\tilde{\varphi}_\pm}]$$

$$= -(e^{tH_{\rho}})_*H_{\bar{c}_\pm \tilde{\varphi}_\pm} = -(\bar{c}_\pm \circ e^{tH_{\rho}})(e^{tH_{\rho}})_*H_{\tilde{\varphi}_\pm}.$$ 

Combining these two facts, we get for $T > 0$ large enough,

$$\nu_{\min} - \varepsilon < \langle \bar{c}_\pm \rangle_T < \nu_{\max} + \varepsilon \quad \text{on } K,$$

where $\langle \cdot \rangle_T$ stands for the ergodic average on $K$:

$$\langle f \rangle_T := \frac{1}{T} \int_0^T f \circ e^{tH_{\rho}} \, dt.$$ 

Fix $T$. We now put $\varphi_\pm := e^{tf_\pm} \cdot \tilde{\varphi}_\pm$, where $f_\pm$ are smooth functions on $U$ with

$$f_\pm = \frac{1}{T} \int_0^T (T-t)\bar{c}_\pm \circ e^{tH_{\rho}} \, dt \quad \text{on } K,$$

so that $H_{\rho}f_\pm = \langle \bar{c}_\pm \rangle_T - \bar{c}_\pm$ on $K$. Then $\varphi_\pm$ satisfy conditions (1)–(4), with

$$c_\pm = \mp \frac{H_{\rho}\varphi_\pm}{\varphi_\pm} = \langle \bar{c}_\pm \rangle_T \in (\nu_{\min} - \varepsilon, \nu_{\max} + \varepsilon)$$

on $K$, and thus on $U_\delta$ for $\delta$ small enough.

To verify condition (5), fix $\delta_0 > 0$ small enough so that $\pm H_{\rho}\varphi_\pm^2 \leq 0$ on $U_{\delta_0}$. By Lemma 4.2, for $\delta$ small enough depending on $\delta_0$, for each Hamiltonian flow line $\gamma(t)$, $0 \leq t \leq T$, of $p$ in $\overline{U}$, if $\gamma(0), \gamma(T) \in U_\delta$, then $\gamma([0, T]) \subset U_\delta$. Since $\pm \partial_t \varphi_\pm(\gamma(t))^2 \leq 0$ for $0 \leq t \leq T$ and $|\varphi_\pm(\gamma(t))| \leq \delta$ for $t = 0, T$, we see that $\gamma([0, T]) \subset U_\delta$. \hfill \square

5.4. The canonical relation $\Lambda^0$. We next construct the projections $\pi_\pm$ from subsets $\Gamma_\pm \subset \Gamma_\pm$ to $K$. Fix $\delta_0, \delta_1 > 0$ small enough so that Lemma 5.1 holds with $\delta_0$ in place of $\delta$ and $K \cap p^{-1}(\{\alpha_0 - \delta_1, \alpha_1 + \delta_1\})$ is a compact subset of $\mathcal{U}$ (the latter is possible by assumption (9) in §4.1), consider the functions $\varphi_\pm$ from Lemma 5.1 and put

$$\Gamma_\pm^0 := \Gamma_\pm \cap p^{-1}(\{\alpha_0 - \delta_1, \alpha_1 + \delta_1\} \cap \{\varphi_\pm < \delta_0\}), \quad K_\pm^0 := K \cap p^{-1}(\{\alpha_0 - \delta_1, \alpha_1 + \delta_1\}),$$

(5.10)

so that $K_\pm^0 = \Gamma_\pm^0 \cap \Gamma_\pm^0$ and, for $\delta_0$ small enough, $\Gamma_\pm^0 \subset \mathcal{U}$. Note that, by part (2) of Lemma 5.1, the level sets of $p$ on $\Gamma_\pm$ are invariant under $H_{\varphi_\pm}$ and $e^{tH_{\rho}}(\Gamma_\pm^0) \subset \Gamma_\pm^0$ for $\mp t \geq 0$. 

By part (4) of Lemma 5.1, \( \Gamma^\circ_{\pm} \) is foliated by trajectories of \( H_{\varphi_{\pm}} \) (or equivalently, by trajectories of \( \mathcal{V}_{\pm} \)), moreover each trajectory intersects \( K \) at a single point. This defines projection maps

\[
\pi_{\pm} : \Gamma^\circ_{\pm} \to K^\circ,
\]
mapping each trajectory to its intersection with \( K \). The flow \( e^{tH_p} \) preserves the subbundle \( \mathcal{V}_{\pm} \) generated by \( H_{\varphi_{\pm}} \), therefore

\[
\pi_{\pm} \circ e^{tH_p} = e^{tH_p} \circ \pi_{\pm}, \quad t \geq 0.
\] (5.11)

Now, define the \( 2n \)-dimensional submanifold \( \Lambda^\circ \subset T^*X \times T^*X \) by

\[
\Lambda^\circ := \{ (\rho_-, \rho_+) \in \Gamma^\circ_+ \times \Gamma^\circ_- \mid \pi_-(\rho_-) = \pi_+(\rho_+) \}.
\] (5.12)

We claim that \( \Lambda^\circ \) is a canonical relation. Indeed, it is enough to prove that \( \sigma_S|_{TT^*_\pm} = \pi^*_\pm(\sigma_S|_{TK^\circ}) \), where \( \sigma_S \) is the symplectic form on \( T^*M \). This is true since the Hamiltonian flow \( e^{tH_p} \) preserves \( \sigma_S \) and \( \mathcal{V}_{\pm}|_K \) is symplectically orthogonal to \( TK \).

5.5. The transport equations. Finally, we use \( r \)-normal hyperbolicity to establish existence of solutions to the transport equations, needed in the construction of the projector \( \Pi \) in §7.1. We start by estimating higher derivatives of the flow. Take \( \delta_0, \Gamma^\circ_\pm, K^\circ \) from §5.4 and identify \( \Gamma^\circ_\pm \sim K^\circ \times (-\delta_0, \delta_0) \) by the map

\[
\rho_\pm \in \Gamma^\circ_\pm \mapsto (\pi_\pm(\rho_\pm), \varphi_{\pm}(\rho_\pm)).
\] (5.13)

Denote elements of \( K^\circ \times (-\delta_0, \delta_0) \) by \( (\theta, s) \) and the flow \( e^{tH_p} \) on \( \Gamma^\circ_\pm, \exists t \geq 0, \) by (recall (5.11))

\[
e^{tH_p} : (\theta, s) \mapsto (e^{tH_p}(\theta), \psi^t_\pm(\theta, s)).
\]

Note that \( \psi^t_\pm(\theta, 0) = 0 \). We have the following estimate on higher derivatives of the flow on \( K^\circ \) (in any fixed coordinate system), see for example [DyGu, Lemma C.1] (which is stated for geodesic flows, but the proof applies to any smooth flow):

\[
\sup_{\theta \in K^\circ} |\partial^\alpha_\theta e^{tH_p}(\theta)| \leq C_\alpha e^{(|\alpha|\mu_{\text{max}} + \tilde{\varepsilon})|t|}, \quad t \in \mathbb{R}.
\] (5.14)

Here \( \mu_{\text{max}} \) is defined by (5.3), \( \tilde{\varepsilon} > 0 \) is any fixed constant, and \( C_\alpha \) depends on \( \tilde{\varepsilon} \). We choose \( \tilde{\varepsilon} \) small enough in (5.17) below and the constant \( \varepsilon > 0 \) in Lemma 5.1 is small depending on \( \tilde{\varepsilon} \).

Next, we estimate the derivatives of \( \psi^t_\pm \). We have, with \( c_\pm \) defined in part (2) of Lemma 5.1,

\[
\partial_t \psi^t_\pm(\theta, s) = \pm c_\pm(e^{tH_p}(\theta), \psi^t_\pm(\theta, s)) \psi^t_\pm(\theta, s).
\]

Then

\[
\partial_t(\partial^k_\theta \partial^\alpha_\theta \psi^t_\pm(\theta, s)) = \pm c_\pm(e^{tH_p}(\theta), 0) \partial^k_\theta \partial^\alpha_\theta \psi^t_\pm(\theta, s) + \ldots,
\]

where \( \ldots \) is a linear combination, with uniformly bounded variable coefficients depending on the derivatives of \( c_\pm \), of expressions of the form

\[
\partial^\beta_\theta e^{tH_p}(\theta) \cdots \partial^\gamma_\theta e^{tH_p}(\theta) \partial^k_\theta \partial^\alpha_\theta \psi^t_\pm(\theta, s) \cdots \partial^\gamma_\theta \partial^k_\theta \psi^t_\pm(\theta, s),
\]
where $\beta_1 + \cdots + \beta_m + \gamma_1 + \cdots + \gamma_l = \alpha$, $k_1 + \cdots + k_l = k$, and $|\beta_j|, |\gamma_j| + k_j > 0$. Moreover, if $l = 0$ or $l + m = 1$, then the corresponding coefficient is a bounded multiple of $\psi_\pm^k(\theta, s)$. It now follows by induction from (5.9) that

$$\sup_{\theta \in K^o, |s| < \delta_0} |\partial_s^k \partial_\theta^\alpha \psi_\pm^t(\theta, s)| \leq C_{\alpha k} e^{(|\alpha|\mu_{\max} - \nu_{\min} + \bar{\epsilon}) t}, \quad t \geq 0.$$ 

(5.15)

We can now prove the following

**Lemma 5.2.** Assume that (5.4) is satisfied, with some integer $r > 0$. Let $f \in C^{r+1}(\Gamma^o_\pm)$ be such that $f|_K = 0$. Then there exists unique solution $a \in C^r(\Gamma^o_\pm)$ to the equation

$$H_p a = f, \quad a|_{K^o} = 0.$$ 

(5.16)

**Proof.** Using (5.4), choose $\bar{\epsilon} > 0$ so that

$$r\mu_{\max} - \nu_{\min} + \bar{\epsilon} < 0.$$ 

(5.17)

Any solution to (5.16) satisfies for each $T > 0$,

$$a = a \circ e^{\mp TH_p} \pm \int_0^T f \circ e^{\mp tH_p} \, dt.$$ 

Since $a|_{K^o} = 0$, by letting $T \to +\infty$ we see that the unique solution to (5.16) is

$$a = \pm \int_0^\infty f \circ e^{\mp tH_p} \, dt.$$ 

(5.18)

The integral (5.18) converges exponentially, as

$$|f \circ e^{\mp tH_p}(\theta, s)| \leq C |\psi_\pm^t(\theta, s)| \leq Ce^{-(\nu_{\min} - \bar{\epsilon}) t}.$$ 

To show that $a \in C^r$, it suffices to prove that when $|\alpha| + k \leq r$, the integral

$$\int_0^\infty \partial_s^k \partial_\theta^\alpha (f \circ e^{\mp tH_p}) \, dt$$

converges uniformly in $s, \theta$. Given (5.17), it is enough to show that

$$\sup_{\theta, s} |\partial_s^k \partial_\theta^\alpha (f \circ e^{\mp tH_p})(\theta, s)| \leq C_{\alpha k} e^{(|\alpha|\mu_{\max} - \nu_{\min} + \bar{\epsilon}) t}, \quad t \geq 0.$$ 

(5.19)

To see (5.19), we use the chain rule to estimate the left-hand side by a sum of terms of the form

$$\partial_\theta^m \partial_s^l f(e^{\mp tH_p}(\theta, s)) \partial_\theta^{\beta_1} e^{\mp tH_p}(\theta) \cdots \partial_\theta^{\beta_m} e^{\mp tH_p}(\theta) \partial_\theta^{\gamma_1} \psi_\pm^t(\theta, s) \cdots \partial_\theta^{\gamma_l} \partial_s^{k_1} \psi_\pm^t(\theta, s)$$

where $\beta_1 + \cdots + \beta_m + \gamma_1 + \cdots + \gamma_l = \alpha$, $k_1 + \cdots + k_l = k$, and $|\beta_j|, |\gamma_j| + k_j > 0$. For $l = 0$, we have $|\partial_\theta^\alpha f \circ e^{\mp tH_p}| = O(e^{-(\nu_{\min} - \bar{\epsilon}) t})$ and (5.19) follows from (5.14). For $l > 0$, (5.19) follows from (5.14) and (5.15).
6. Calculus of microlocal projectors

In this section, we develop tools for handling Fourier integral operators associated to the canonical relation $\Lambda^\circ$ introduced in §5.4. We will not use the operator $P$ or the global dynamics of the flow $e^{tH_p}$; we will only assume that $X$ is an $n$-dimensional manifold and

- $\Gamma_\pm \subset T^*X$ are smooth orientable hypersurfaces;
- $\Gamma_\pm$ intersect transversely and $K^\circ := \Gamma_+ \cap \Gamma_-$ is symplectic;
- if $\mathcal{V}_\pm \subset TT_\pm \subset T(T^*X)$, then each maximally extended flow line of $\mathcal{V}_\pm$ on $\Gamma_\pm$ intersects $K^\circ$ at precisely one point, giving rise to the projection maps $\pi_\pm : \Gamma_\pm \to K^\circ$;
- the canonical relation $\Lambda^\circ \subset T^*(X \times X)$ is defined by
  \[ \Lambda^\circ = \{ (\rho_-, \rho_+) \in \Gamma_\pm \times \Gamma_\pm \mid \pi_-(\rho_-) = \pi_+(\rho_+) \}; \]
- the projections $\tilde{\pi}_\pm : \Lambda^\circ \to \Gamma_\pm$ are defined by
  \[ \tilde{\pi}_\pm(\rho_-, \rho_+) = \rho_\pm. \] (6.1)

If we only consider a bounded number of terms in the asymptotic expansions of the studied symbols, and require existence of a fixed number of derivatives of these symbols, then the smoothness requirement above can be replaced by $C^r$ for $r$ large enough depending only on $n$.

We will study the operators in the class $I_{\text{comp}}(\Lambda^\circ)$ considered in §3.2. The antiderivative on $\Lambda^\circ$ (see §3.2) is fixed so that it vanishes on the image of the embedding

\[ j_K : K^\circ \to \Lambda^\circ, \quad j_K(\rho) = (\rho, \rho); \] (6.2)

this is possible since $j_K^*(\eta dy - \xi dx) = 0$ and the image of $j_K$ is a deformation retract of $\Lambda^\circ$.

We are particularly interested in defining invariantly the principal symbol $\sigma_\Lambda(A)$ of an operator $A \in I_{\text{comp}}(\Lambda^\circ)$. This could be done using the global theory of Fourier integral operators; we take instead a more direct approach based on the model case studied in §6.1. The principal symbols on a neighborhood $\tilde{\Lambda}$ of a compact subset $\tilde{K} \subset K^\circ$ are defined as sections of certain vector bundles in §6.2.

We are also interested in the symbol of a product of two operators in $I_{\text{comp}}(\Lambda^\circ)$. Note that such a product lies again in $I_{\text{comp}}(\Lambda^\circ)$, since $\Lambda^\circ$ satisfies the transversality condition with itself and, with the composition defined as in (3.5), $\Lambda^\circ \circ \Lambda^\circ = \Lambda^\circ$.

To study the principal symbol of the product, we again use the model case – see Proposition 6.5.

Next, in §6.3, we study idempotents in $I_{\text{comp}}(\Lambda^\circ)$, microlocally near $\tilde{K}$, proving technical lemmas need in the construction of the microlocal projector $\Pi$ in §7. Finally,
Then we get (where the symbols quantized by $Op$ From here, using stationary phase expansions similarly to [Zw, Theorems 4.11 and 4.12], testing formulas, see [Zw, Theorem 4.19]:

6.1. Model case. We start with the model case

$$X := \mathbb{R}^n, \quad \Gamma_0^+ := \{x_n = 0\}, \quad \Gamma_0^- := \{x_n = 0\}. \quad (6.3)$$

Then $K^0 = \{x_n = \xi_n = 0\}$ is canonically diffeomorphic to $T^*\mathbb{R}^{n-1}$. If we denote elements of $\mathbb{R}^{2n} \simeq T^*\mathbb{R}^n$ by $(x', x_n, \xi', \xi_n)$, with $x', \xi' \in \mathbb{R}^{n-1}$, then the projection maps $\pi_{\pm} : \Gamma_0^\pm \to K_0$ take the form

$$\pi_+(x, \xi', 0) = (x', 0, \xi', 0), \quad \pi_-(x', 0, \xi) = (x', 0, \xi', 0),$$

and the map

$$\phi : (x, \xi) \mapsto (x', 0, \xi; x, \xi', 0) \in T^*(\mathbb{R}^n \times \mathbb{R}^n) \quad (6.4)$$

gives a diffeomorphism of $\mathbb{R}^{2n}$ onto the corresponding canonical relation $\Lambda^0$.

Basic calculus. For a Schwartz function $a(x, \xi) \in \mathcal{S}(\mathbb{R}^{2n})$, define its $\Lambda^0$-quantization $Op^\Lambda_h(a) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by the formula

$$Op^\Lambda_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x'=\xi'-y\xi)}a(x, \xi)u(y) \, dyd\xi. \quad (6.5)$$

The operator $Op^\Lambda_h(a)$ will be a Fourier integral operator associated to $\Lambda^0$, see below for details. We also use the standard quantization for pseudodifferential operators [Zw, §4.1.1], where $a(x, \xi; h) \in C^\infty(\mathbb{R}^{2n})$ and all derivatives of $a$ are bounded uniformly in $h$ by a fixed power of $1 + |x|^2 + |\xi|^2$:

$$Op_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\xi}a(x, \xi)u(y) \, dyd\xi. \quad (6.6)$$

The symbol $a$ can be extracted from $Op^\Lambda_h(a)$ or $Op_h(a)$ by the following oscillatory testing formulas, see [Zw, Theorem 4.19]:

$$Op^\Lambda_h(a)(e^{\frac{i}{h}x'\xi}) = e^{\frac{i}{h}x'\xi'}a(x, \xi), \quad \xi \in \mathbb{R}^n, \quad (6.7)$$

$$Op_h(a)(e^{\frac{i}{h}x\xi}) = e^{\frac{i}{h}x\xi}a(x, \xi), \quad \xi \in \mathbb{R}^n. \quad (6.8)$$

From here, using stationary phase expansions similarly to [Zw, Theorems 4.11 and 4.12], we get (where the symbols quantized by $Op_h^\Lambda$ are Schwartz)

$$Op_h^\Lambda(a)\, Op_h^\Lambda(b) = Op_h^\Lambda(a\#^\Lambda b), \quad (6.9)$$

$$Op_h^\Lambda(a)\, Op_h(b) = Op_h^\Lambda(a\#_b), \quad (6.10)$$

$$Op_h(b)\, Op_h^\Lambda(a) = Op_h^\Lambda(a\#_b), \quad (6.11)$$
where the symbols $a^\Lambda b, a_{\#b}, a_{b\#} \in \mathcal{S}(\mathbb{R}^{2n})$ have asymptotic expansions

$$a^\Lambda b(x, \xi) \sim \sum_{\alpha} \frac{(-ih)^{|\alpha|}}{\alpha!} \partial_{\xi}^\alpha a(x, \xi') \partial_x^\alpha b(x', 0, \xi),$$  \hspace{1cm} (6.12)

$$a_{\#b}(x, \xi) \sim \sum_{\alpha} \frac{(-ih)^{|\alpha|}}{\alpha!} \partial_{\xi}^\alpha a(x, \xi) \partial_x^\alpha b(x', 0, \xi),$$  \hspace{1cm} (6.13)

$$a_{b\#}(x, \xi) \sim \sum_{\alpha} \frac{(-ih)^{|\alpha|}}{\alpha!} \partial_{\xi}^\alpha b(x, \xi') \partial_x^\alpha a(x, \xi).$$  \hspace{1cm} (6.14)

Finally, the operators $\text{Op}_h^\Lambda(a)$ are bounded $L^2 \to L^2$ with norm $O(h^{-1/2})$:

**Proposition 6.1.** If $a \in \mathcal{S}(\mathbb{R}^{2n})$, then there exists a constant $C$ such that

$$\| \text{Op}_h^\Lambda(a) \|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq Ch^{-1/2}.$$  

**Proof.** Define the semiclassical Fourier transform

$$\hat{u}(\xi) := (2\pi h)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{i}{h} y \cdot \xi} u(y) \, dy,$$

then $\|\hat{u}\|_{L^2} = \|u\|_{L^2}$ and

$$\text{Op}_h^\Lambda(a) u(x) = (2\pi h)^{-1/2} \int_{\mathbb{R}} v(x, \xi_n) \, d\xi_n,$$

where

$$v(x, \xi_n) := (2\pi h)^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} e^{\frac{i}{h} x' \cdot \xi'} a(x, \xi', \xi_n) \hat{u}(\xi', \xi_n) \, d\xi'.$$

Using the $L^2$-boundedness of pseudodifferential operators on $\mathbb{R}^{n-1}$, we see that for each $(x_n, \xi_n) \in \mathbb{R}^{2}$,

$$\|v(\cdot, x_n, \xi_n)\|_{L^2_{x'}} \leq F(x_n, \xi_n) \|\hat{u}(\cdot, \xi_n)\|_{L^2_{\xi'}},$$

where $F(x_n, \xi_n)$ is bounded by a certain $\mathcal{S}(\mathbb{R}^{2n-2})$ seminorm of $a(\cdot, x_n, \cdot, \xi_n)$. Then $F$ is rapidly decaying on $\mathbb{R}^2$ and for any $N$,

$$\|v(\cdot, \xi_n)\|_{L^2_{x}} \leq C(\xi_n)^{-N} \|\hat{u}(\cdot, \xi_n)\|_{L^2_{\xi'}}.$$

Therefore,

$$\| \text{Op}_h^\Lambda(a) u(x) \|_{L^2} \leq Ch^{-1/2} \int_{\mathbb{R}} \|v(\cdot, \xi_n)\|_{L^2_{x}} \, d\xi_n \leq Ch^{-1/2} \|u\|_{L^2}$$

as required. \qed

**Microlocal properties.** For $a \in \mathcal{S}(\mathbb{R}^{2n})$, the operator $\text{Op}_h^\Lambda(a)$ is $h$-tempered as defined in Section 3.1. Moreover, the following analog of (3.4) follows from (6.10) and (6.11):

$$\text{WF}_h(\text{Op}_h^\Lambda(a)) \subset \phi(\text{supp} \, a) \subset \Lambda^0,$$  \hspace{1cm} (6.15)

with $\phi$ defined by (6.4).
For \( a \in C^\infty_0(\mathbb{R}^{2n}) \), we use (3.3) to check that \( \text{Op}_h^\Lambda(a) \) is, modulo an \( \mathcal{O}(h^\infty)_{\mathcal{S}' \rightarrow \mathcal{S}} \) remainder, a Fourier integral operator in the class \( I^\text{comp}(\Lambda^0) \) defined in §3.1.

We will also use the operator \( \text{Op}_h^\Lambda(1) : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n) \) defined by

\[
\text{Op}_h^\Lambda(1)f(x) = f(x',0), \quad f \in C^\infty(\mathbb{R}^n). \tag{6.16}
\]

Since (6.5) was defined only for Schwartz symbols, we understand (6.16) as follows: if \( a \in C^\infty_0(\mathbb{R}^{2n}) \) is equal to 1 near some open set \( U \subset \mathbb{R}^{2n} \), then the operator \( \text{Op}_h^\Lambda(1) \) defined in (6.16) is equal to the operator \( \text{Op}_h^\Lambda(a) \) defined in (6.5), microlocally near \( \phi(U) \subset T^*(\mathbb{R}^n \times \mathbb{R}^n) \). Moreover, \( \text{WF}_h(\text{Op}_h^\Lambda(1)) \cap T^*(\mathbb{R}^n \times \mathbb{R}^n) \subset \Lambda^0 \). To see this, it is enough to note that for \( a \in C^\infty_0(\mathbb{R}^{2n}) \) and \( \chi \in C^\infty_0(\mathbb{R}^n) \), we have \( \chi \text{Op}_h^\Lambda(1) \text{Op}_h(a) = \text{Op}_h^\Lambda(\tilde{a}) \), where \( \tilde{a}(x,\xi) = \chi(x)a(x',0,\xi) \in C^\infty_0(\mathbb{R}^{2n}) \) and \( \text{Op}_h(\tilde{a}) \) is defined using (6.5).

**Canonical transformations.** We now study how \( \text{Op}_h^\Lambda(a) \) changes under quantized canonical transformations preserving its canonical relation (see §3.2). Let \( U, V \subset \mathbb{R}^{2n} \) be two bounded open sets and \( \varkappa : U \rightarrow V \) a symplectomorphism such that

\[
\varkappa(\Gamma^0_\pm \cap U) = \Gamma^0_\pm \cap V,
\]

with \( \Gamma^0_\pm \) given by (6.3). We further assume that for each \((x',\xi') \in T^*\mathbb{R}^{n-1}\), the sets \( \{x_n \mid (x',x_n,\xi,0) \in U\} \) and \( \{\xi_n \mid (x',0,\xi',\xi_n) \in U\} \), and the corresponding sets for \( V \), are either empty or intervals containing zero, so that the maps \( \pi_\pm : U \cap \Gamma^0_\pm \rightarrow U \cap K^0 \) are well-defined. Since \( \varkappa \) preserves the subbundles \( \mathbb{V}_\pm \), it commutes with the maps \( \pi_\pm \) and thus preserves \( \Lambda^0 \); using the map \( \phi \) from (6.4), we define the open sets \( \hat{U}, \hat{V} \subset \mathbb{R}^{2n} \) and the diffeomorphism \( \hat{\varkappa} : \hat{U} \rightarrow \hat{V} \) by

\[
\hat{U} := \phi^{-1}(U \times U), \quad \hat{V} := \phi^{-1}(V \times V), \quad \phi \circ \hat{\varkappa} = \varkappa \circ \phi.
\]

**Proposition 6.2.** Let \( B, B' : C^\infty(\mathbb{R}^n) \rightarrow C^\infty_0(\mathbb{R}^n) \) be two compactly microlocalized Fourier integral operators associated to \( \varkappa \) and \( \varkappa^{-1} \), respectively,\(^4\) such that

\[
\begin{align*}
BB' &= 1 + \mathcal{O}(h^\infty), & \text{microlocally near } V', \\
B'B &= 1 + \mathcal{O}(h^\infty), & \text{microlocally near } U',
\end{align*}
\tag{6.17}
\]

for some open \( U' \subset U \), \( V' \subset V \) such that \( \varkappa(U') = V' \). Then for each \( a \in C^\infty_0(\hat{V}) \),

\[
B' \text{Op}_h^\Lambda(a)B = \text{Op}_h^\Lambda(a_\varkappa) + \mathcal{O}(h^\infty)_{\mathcal{S}' \rightarrow \mathcal{S}},
\]

for some classical symbol \( a_\varkappa \) compactly supported in \( \hat{U} \), and

\[
a_\varkappa(x,\xi) = \gamma^+_\varkappa(x,\xi')\gamma^-_\varkappa(x',\xi)a(\hat{\varkappa}(x,\xi)) + \mathcal{O}(h) \quad \text{on } \phi^{-1}(U' \times U'),
\tag{6.18}
\]

where \( \gamma^\pm_\varkappa \) are smooth functions on \( U \cap \Gamma_\pm \) depending on \( \varkappa, B, B' \) with \( \gamma^\pm_\varkappa|_{K^0 \cap U'} = 1 \).

\(^4\)The choice of antiderivative (see §3.2) is irrelevant here, since the phase factor in \( B \) resulting from choosing another antiderivative will be cancelled by the phase factor in \( B' \).
Proof. Assume first that \( \varphi \) has a generating function \( S(x, \eta) \):
\[
\varphi(x, \xi) = (y, \eta) \iff \xi = \partial_x S(x, \eta), \quad y = \partial_\eta S(x, \eta).
\]
If \( \mathcal{D}_S \subset \mathbb{R}^{2n} \) is the domain of \( S \), then for each \( (x', \eta') \in T^* \mathbb{R}^{n-1} \), the sets \( \{x_n \mid (x', x_n, \eta', 0) \in \mathcal{D}_S \} \) and \( \{\eta_n \mid (x', 0, \eta'_n, \eta_n) \in \mathcal{D}_S \} \) are either empty or intervals containing zero. Since \( \varphi \) preserves \( \Gamma_{\pm} \), we find \( \partial_{\eta_n} S(x', 0, \eta) = \partial_{x_n} S(x, \eta', 0) = 0 \) and thus
\[
S(x, \eta', 0) = S(x', 0, \eta) = S(x', 0, \eta', 0). \tag{6.19}
\]
We can write, modulo \( \mathcal{O}(h^\infty)_{\varphi \to \varphi} \) errors,
\[
Bu(y) = (2\pi h)^{-n} \int e^{\frac{i}{h}(y \eta - S(x, \eta))} b(x, \eta; h) u(x) dx d\eta,
\]
\[
B' u(x) = (2\pi h)^{-n} \int e^{\frac{i}{h}(S(x, \eta) - y \eta)} b'(x, \eta; h) u(y) dy d\eta,
\]
where \( b, b' \) are compactly supported classical symbols and by (6.17) the principal symbols \( b_0 \) and \( b'_0 \) have to satisfy for \( (x, \xi) \in U' \),
\[
b_0(x, \eta)b'_0(x, \eta) = |\det \partial^2_{x\eta} S(x, \eta)|. \tag{6.20}
\]
We can now use oscillatory testing (6.7) to get
\[
a_{\varphi}(x, \xi) := e^{-\frac{i}{h}x' \xi} B' \text{Op}_h^A(a) B(e^{\frac{i}{h}x \xi})
\]
\[
= (2\pi h)^{-2n} \int e^{\frac{i}{h}(-x' \xi + S(x, \eta) - y \eta + y' \eta' - S(x', \eta) + \tilde{x} \xi)} b'(x, \tilde{\eta}; h) a(y, \eta) b(\tilde{x}, \eta; h) dy d\tilde{\eta} d\eta d\tilde{x}.
\]
We analyse this integral by the method of stationary phase; this will yield that \( a_{\varphi} \) is a classical symbol in \( h \), compactly supported in \( \tilde{U} \) modulo an \( \mathcal{O}(h^\infty)_{\varphi \to \varphi} \) error, and thus \( B' \text{Op}_h^A(a) B = \text{Op}_h^A(a_{\varphi}) \).

The stationary points are given by
\[
\tilde{\eta} = (\eta', 0), \quad \tilde{x} = (x', 0), \quad (y, \eta) = \tilde{\varphi}(x, \xi).
\]
The value of the phase at stationary points is zero due to (6.19). To compute the Hessian, we make the change of variables \( \tilde{\eta} = \tilde{\eta} + (\eta', 0) \). We can then remove the variables \( y, \tilde{\eta} \) and pass from the original Hessian to \( \partial^2_{\eta' \eta'} S(x, \eta', 0) - \partial^2 S(x', 0, \eta) \), where the first matrix is padded with zeros. Since \( \partial_{\eta_n} S(x', 0, \eta) = 0 \), we have \( \partial^2_{\eta_n \eta_n} S = \partial^2_{x_n x_n} S = \partial^2_{x_n \eta'_n} S = 0 \) at \( (x', 0, \eta) \), therefore we can remove the \( x_n, \eta_n \) variables, with a multiplicand of \( (\partial^2_{\eta' \eta'} S(x', 0, \eta))^2 \) in the determinant. Next, by (6.19) \( \partial^2_{\eta' \eta'} S(x', 0, \eta) = \partial^2_{\eta' \eta'} S(x', \eta, 0) \); therefore, the Hessian has signature zero and determinant
\[
(\partial^2_{x_n \eta_n} S(x', 0, \eta) \det \partial^2_{x' \eta'} S(x', 0, \eta))^2.
\]
Since \( \partial^2_{x_n \eta_n} S(x', 0, \eta) = 0 \), this is equal to \( (\det \partial^2_{x\eta} S(x', 0, \eta))^2 \). Therefore, we get (6.18) with
\[
\gamma^+_{\varphi}(x, \xi') \gamma^-_{\varphi}(x', \xi) = \frac{b'_0(x, \eta', 0)b_0(x', 0, \eta)}{|\det \partial^2_{x\eta} S(x', 0, \eta)|} = \frac{b'_0(x, \eta', 0)}{b_0(x', 0, \eta)}.
\]
here \((y, \eta) = \tilde{\pi}(x, \xi)\) and the last equality follows from (6.20). We then find
\[
\gamma_+^+(x, \xi') = b_0'(x, \eta', 0)/b_0'(x', 0, \eta, 0), \quad \gamma_-^-(x', \xi) = b_0'(x', 0, \eta', 0)/b_0(x', 0, \eta).
\] (6.21)

We now consider the case of general \(\kappa\). Using a partition of unity for \(a\), we may assume that the intersection \(U \cap K^0\) is arbitrary small. We now represent \(\kappa\) as a product of several canonical relations, each of which satisfies the conditions of this Proposition and has a generating function; this will finish the proof.

First of all, consider a canonical transformation of the form
\[
(x, \xi) \mapsto (y, \eta), \quad (y', \eta') = \kappa(x', \xi'), \quad (y_0, \eta_0) = (x_0, \xi_0),
\] (6.22)
with \(\kappa\) a canonical transformation on \(T^*\mathbb{R}^{n-1} \simeq K^0\). We can write \(\kappa\) locally as a product of canonical transformations close to the identity, each of which has a generating function – see [Zw, Theorems 10.4 and 11.4]. If \(\tilde{S}(x', \eta')\) is a generating function for \(\kappa\), then \(\tilde{S}(x', \eta') + x_n\eta_n\) is a generating function for (6.22).

Multiplying our \(\kappa\) by a transformation of the form (6.22) with \(\kappa = (\kappa|_{K^0})^{-1}\), we reduce to the case
\[
\kappa(x', 0, \xi', 0) = (x', 0, \xi', 0) \quad \text{for} \quad (x', 0, \xi', 0) \in U \cap K^0.
\]
If \(\kappa(x, \xi) = (y(x, \xi), \eta(x, \xi))\), since \(\kappa\) commutes with \(\pi_\pm\) we have
\[
y'(x, \xi', 0) = y'(x', 0, \xi) = x',
\]
\[
\eta'(x, \xi', 0) = \eta'(x', 0, \xi) = \xi'.
\] (6.23)

We now claim that \(\kappa\) has a generating function, if we shrink \(U\) to be a small neighborhood of \(U \cap (\Gamma^0_+ \cup \Gamma^0_-)\) (which does not change anything since \(\text{Op}_h^A(a)\) is microlocalized in \(\Gamma^0_- \times \Gamma^0_+\)). For that, it is enough to show that the map
\[
\psi: (x, \xi) \mapsto (x, \eta(x, \xi))
\]
is a diffeomorphism from \(U\) onto some open subset \(\mathcal{O}_S \subset \mathbb{R}^{2n}\).

We first show that \(\psi\) is a local diffeomorphism near \(\Gamma^0_\pm\); that is, the differential \(\partial \eta\) is nondegenerate on \(\Gamma^0_\pm\). By (6.23), \(\partial_{x', \xi'}(y', \eta')\) equals the identity on \(\Gamma^0_+ \cup \Gamma^0_-\); moreover, on \(\Gamma^0_+\), we have \(\partial_{x, \xi} \eta_n = 0\) and \(\partial_{x_n} (y', \eta') = 0\) and on \(\Gamma^0_-\), we have \(\partial_{x', \xi} \eta_n = 0\) and \(\partial_{\xi_n} (y', \eta') = 0\). It follows that on \(\Gamma^0_+ \cup \Gamma^0_-\), \(\det \partial \eta = \det \partial \eta_n\), and since \(\kappa\) is a diffeomorphism, \(0 \neq \det \partial \psi(x, \xi)(y, \eta) = \partial_{x, \xi} y_n \cdot \partial_{\xi_n} \eta_n\), yielding \(\det \partial \eta \neq 0\).

It remains to note that \(\psi\) is one-to-one on \(\Gamma^0_+ \cup \Gamma^0_-\), which follows immediately from the identities \(\psi(x, \xi', 0) = (x, \xi', 0)\) and \(\psi(x', 0, \xi) = \kappa(x', 0, \xi)\). \qed

6.2. General case. We now consider the case of general \(\Gamma^0_\pm, K^0, \Lambda^0\), satisfying the assumptions from the beginning of \(\S 6\). We start by shrinking \(\Gamma^0_\pm\) so that our setup can locally be conjugated to the model case of \(\S 6.1\). (The set \(\tilde{K}\) will be chosen in \(\S 7.1\).)

**Proposition 6.3.** Let \(\tilde{K} \subset K^0\) be compact. Then there exist \(\tilde{\delta} > 0\) and
• a finite collection of open sets $U_i \subset T^* X$, such that

$$\hat{K} \subset \hat{K} := \bigcup_i K_i, \quad K_i := K^\circ \cap U_i.$$ 

• symplectomorphisms $\kappa_i$ defined in a neighborhood of $U_i$ and mapping $U_i$ onto

$$V_\delta := \{|(x', \xi')| < \delta, \ |x_n| < \tilde{\delta}, \ |\xi_n| < \tilde{\delta}\} \subset T^* \mathbb{R}^n,$$

such that, with $\Gamma^0_\pm$ defined in (6.3),

$$\kappa_i(U_i \cap \Gamma^\circ_\pm) = V_\delta \cap \Gamma^0_\pm;$$

• compactly microlocalized Fourier integral operators

$$B_i : C^\infty(X) \rightarrow C^\infty_0(\mathbb{R}^n), \quad B'_i : C^\infty(\mathbb{R}^n) \rightarrow C^\infty_0(X),$$

associated to $\kappa_i$ and $\kappa_i^{-1}$, respectively, such that

$$B_iB'_i = 1 \quad \text{near } V_\delta, \quad B'_iB_i = 1 \quad \text{near } U_i.$$ (6.25)

Proof. It is enough to show that each point $\rho \in K^\circ$ has a neighborhood $U_\rho$ and a symplectomorphism $\kappa_\rho : U_\rho \rightarrow V_\rho \subset T^* \mathbb{R}^n$ such that $\kappa_\rho(U_\rho \cap \Gamma^\circ_\pm) = V_\rho \cap \Gamma^0_\pm$; see for example [Zw, Theorem 11.5] for how to construct the operators $B_i, B'_i$ locally quantizing the canonical transformations $\kappa_\rho, \kappa_\rho^{-1}$.

By the Darboux theorem [Zw, Theorem 12.1] (giving a symplectomorphism mapping an arbitrarily chosen defining function of $\Gamma^\circ_\pm$ to $x_n$), we can reduce to the case $\rho = 0 \in T^* \mathbb{R}^n$ and $\Gamma^\circ_\pm = \{x_n = 0\}$ near 0. Since $\Gamma^0_+ \cap \Gamma^\circ_+ = K^\circ$ is symplectic, the Poisson bracket of the defining function $x_n$ of $\Gamma^\circ_+$ and any defining function $\varphi_+$ of $\Gamma^0_+$ is nonzero at 0; thus, $\partial_{\xi_n} \varphi_+(0) \neq 0$ and we can write $\Gamma^0_+$ locally as the graph of some function:

$$\Gamma^0_+ = \{\xi_n = F(x, \xi')\}.$$

Put $\varphi'_+(x, \xi) = \xi_n - F(x, \xi')$, then $\{\varphi'_+, x_n\} = 1$. It remains to apply the Darboux theorem once again, obtaining a symplectomorphism preserving $x_n$ and mapping $\varphi'_+$ to $\xi_n$. □

We now consider the sets

$$\hat{\Gamma}_\pm := \bigcup_i \Gamma^i_\pm, \quad \Gamma^i_\pm := \Gamma^\circ_\pm \cap U_i,$$

$$\hat{\Lambda} := \bigcup_i \Lambda_i, \quad \Lambda_i := \{(\rho_-, \rho_+) \in \Lambda^\circ \mid \rho_\pm \in \Gamma^i_\pm\}. \quad (6.26)$$

Let $\hat{\Gamma}_\pm \subset \hat{\Gamma}_\pm$ be compact, with $\pi_\pm(\hat{\Gamma}_\pm) = \hat{K}$ and for each $\rho \in \hat{K}$, the set $\pi_\pm^{-1}(\rho) \cap \hat{\Gamma}_\pm$ is a flow line of $\mathcal{V}_\pm$ containing $\rho$. Define the compact set

$$\hat{\Lambda} := \{(\rho_-, \rho_+) \in \Lambda^\circ \mid \rho_\pm \in \hat{\Gamma}_\pm\} \quad (6.27)$$
and assume that $\tilde{\Gamma}_\pm$ are chosen so that $\tilde{\Lambda} \subset \tilde{\Lambda}$. The goal of this subsection is to obtain an invariant notion of the principal symbol of Fourier integral operators in $I_{\text{comp}}(\Lambda^0)$, microlocally near $\tilde{\Lambda}$.

Define the diffeomorphisms $\tilde{\varphi}_i : \Lambda_i \to V_j$ by the formula

$$(\varphi_i(\rho_-), \varphi_i(\rho_+)) = \phi(\tilde{\varphi}_i(\rho_-), \rho_+)$$

for all $\rho_- \in \Lambda_i$. Here $\phi$ is defined in (6.4).

Consider some $A \in I_{\text{comp}}(\Lambda^0)$, then $B_i A B'_i$ is a Fourier integral operator associated to the model canonical relation $\Lambda^0$ from §6.1 (with the antiderivatives on $\Lambda^0$ and $\Lambda^0$ chosen in the beginning of §6). Therefore, there exists a compactly supported classical symbol $a^i(x, \xi; h)$ on $\mathbb{R}^{2n}$ such that, with $\text{Op}_h^A$ defined in (6.5),

$$B_i A B'_i = \text{Op}_h^A(a^i) + O(h^\infty),$$

By (6.25), we find

$$A = B'_i \text{Op}_h^A(a^i) B_i + O(h^\infty)$$

microlocally near $\Lambda_i$. Define the function $a^i \in C^\infty(\Lambda_i)$ using the principal symbol $\tilde{a}^i_0$ by

$$a^i = \tilde{a}^i_0 \circ \tilde{\varphi}_i.$$

By Proposition 6.2, applied to the Fourier integral operators $B_j B'_i$ and $B_i B'_j$ quantizing $\varphi = \varphi_j \circ \varphi_i^{-1}$ and $\varphi_i^{-1}$, respectively, with $U' = \varphi_i(U_i \cap U_j)$, $V' = \varphi_j(U_i \cap U_j)$ we see that whenever $\Lambda_i \cap \Lambda_j \neq \emptyset$, we have

$$a^i|_{\Lambda_i \cap \Lambda_j} = (\gamma^i_{ij} \otimes \gamma^j_{ij}) a^j|_{\Lambda_i \cap \Lambda_j},$$

(6.29)

where $\gamma^i_{ij}$ and $\gamma^j_{ij}$ are smooth functions on $\Gamma^i_\pm \cap \Gamma^j_\pm$ and $\gamma^i_{ij}|_K = 1$. Moreover, $\gamma^i_{ji} = (\gamma^i_{ij})^{-1}$ and $\gamma^i_{ij} \gamma^j_{jk} = \gamma^j_{ik}$ on $\Gamma^i_\pm \cap \Gamma^j_\pm \cap \Gamma^k_\pm$ (this can be seen either from the fact that the formulas (6.29) for different $i, j$ have to be compatible with each other, or directly from (6.21)). Therefore, we can consider smooth line bundles $\mathcal{E}_\pm$ over $\tilde{\Gamma}_\pm$ with smooth sections $e^i_\pm$ of $\mathcal{E}_{\pm}|_{\Gamma^i_\pm}$ such that $e^i_\pm = \gamma^i_{ij} e^j_\pm$ on $\Gamma^i_\pm \cap \Gamma^j_\pm$ – see for example [HöI, §6.4].

Define the line bundle $\mathcal{E}$ over $\tilde{\Lambda}$ using the projection maps from (6.1):

$$\mathcal{E} = (\tilde{\pi}^+ \mathcal{E}^-) \otimes (\tilde{\pi}^+ \mathcal{E}^+)$$

and for $A \in I_{\text{comp}}(\Lambda^0)$, the symbol $\sigma_A(\Lambda) \in C^\infty(\tilde{\Lambda}; \mathcal{E})$ by the formula

$$\sigma_A(\Lambda)|_{\Lambda_i} = a^i(\tilde{\pi}^+ e^i_\pm \otimes \tilde{\pi}^+ e^i_\pm).$$

Note that the bundle $\mathcal{E}$ can be studied in detail using the global theory of Fourier integral operators (see for instance [HöIV, §25.1]). However, the situation in our special case is considerably simplified, since the Maslov bundle does not appear.
We have $\sigma_\Lambda(A) = 0$ near $\Lambda$ if and only if $A \in hI_{\text{comp}}(\Lambda^\circ)$ microlocally near $\Lambda$. Moreover, for all $a \in C^\infty(\Lambda; \mathcal{E})$, there exists $A \in I_{\text{comp}}(\Lambda^\circ)$ such that $\sigma_\Lambda(A) = a$ near $\Lambda$.

The restrictions $\mathcal{E}_\pm|_K$ are canonically trivial; that is, for $a_\pm \in C^\infty(\Gamma_\pm; \mathcal{E}_\pm)$, we can view $a_\pm|_K$ as a function on $K$, by taking $e^i_i|_K_i = 1$. The bundles $\mathcal{E}_\pm$ are trivial:

**Proposition 6.4.** There exist sections $a_\pm \in C^\infty(\Gamma_\pm; \mathcal{E}_\pm)$, nonvanishing near $\Gamma_\pm$ and such that $a_\pm|_K = 1$ near $K$.

**Proof.** Since $\gamma_{ij}^\pm$ is a nonvanishing smooth function on $\Gamma_\pm \cap \Gamma_\mp$ such that $\gamma_{ij}^\pm|_{\Gamma_i \cap \Gamma_j} = 1$, we can write

$$\gamma_{ij}^\pm = \exp(f_{ij}^\pm),$$

where $f_{ij}^\pm$ is a uniquely defined function on $\Gamma_\pm \cap \Gamma_\mp$, such that $f_{ij}^\pm|_{\Gamma_i \cap \Gamma_j} = 0$. We now put near $\Gamma_\pm$,

$$a_\pm|_{\Gamma_i} = \exp(b_i^\pm e_i^\pm),$$

where $b_\pm \in C^\infty(\Gamma_\pm^i)$ are such that near $\Gamma_\pm$ and $K$ respectively,

$$(b_i^\pm - b_j^\pm)|_{\Gamma_i \cap \Gamma_j} = f_{ij}^\pm, \quad b_\pm|_{\Gamma_i} = 0.$$

Such functions exist since $f_{ij}^\pm$ is a cocycle:

$$f_{ii}^\pm = f_{ij}^\pm + f_{ji}^\pm = 0; \quad f_{ij}^\pm + f_{jk}^\pm = f_{ik}^\pm \quad \text{on } \Gamma_i^\pm \cap \Gamma_j^\pm \cap \Gamma_k^\pm$$

and since the sheaf of smooth functions is fine; more precisely, if $1 = \sum_i \chi_i$ is a partition of unity on $\Gamma_\pm$, with $\text{supp} \chi_i \subset \Gamma_i^\pm$, we put

$$b_i^\pm = \sum_k \chi_k f_{ik}^\pm. \quad \Box$$

We now state the properties of the calculus, following directly from (6.9)–(6.11), the general theory of Fourier integral operators, and Egorov’s Theorem [Zw, Theorem 11.1] (see the beginning of §6 for multiplying two elements of $I_{\text{comp}}(\Lambda^\circ)$):

**Proposition 6.5.** Assume that $A_1, A_2 \in I_{\text{comp}}(\Lambda^\circ), P \in \Psi^k(X)$. Then $A_1 A_2, A_1 P, PA_1$ lie in $I_{\text{comp}}(\Lambda^\circ)$, and

$$\sigma_\Lambda(A_1 A_2)(\rho_-, \rho_+) = \sigma_\Lambda(A_2)(\rho_, \pi_-(\rho_-)) \otimes \sigma_\Lambda(A_1)(\pi_+(\rho_+), \rho_+), \quad (6.31)$$

$$\sigma_\Lambda(A_1 P)(\rho_-, \rho_+) = \sigma(P)(\rho_-) \cdot \sigma_\Lambda(A_1)(\rho_-, \rho_+), \quad (6.32)$$

$$\sigma_\Lambda(P A_1)(\rho_-, \rho_+) = \sigma(P)(\rho_+) \cdot \sigma_\Lambda(A_1)(\rho_-, \rho_+). \quad (6.33)$$

Here in (6.31), $\sigma_\Lambda(A_2)(\rho_-, \pi_-(\rho_-))$ and $\sigma_\Lambda(A_1)(\pi_+(\rho_+), \rho_+)$ are considered as sections of $\mathcal{E}_-$ and $\mathcal{E}_+$, respectively.

We next give a parametrix construction for operators of the form $1 - A$, with $A \in I_{\text{comp}}(\Lambda^\circ)$, needed in §9:
Proposition 6.6. Let $A \in I_{\text{comp}}(\Lambda^o)$ and assume that
$$\text{WF}_h(A) \subset \hat{\Lambda}; \quad \sigma_\Lambda(A)|_{\hat{\Lambda}} \neq 1 \text{ everywhere.}$$
Then there exists $B \in I_{\text{comp}}(\Lambda^o)$ with $\text{WF}_h(B) \subset \hat{\Lambda}$, and such that
$$(1 - A)(1 - B) = 1 + O(h^\infty), \quad (1 - B)(1 - A) = 1 + O(h^\infty).$$
Moreover, $B$ is uniquely defined modulo $O(h^\infty)$ and
$$\sigma_\Lambda(B)(\rho_-, \rho_+) = \frac{\sigma_\Lambda(A)(\rho_-, \pi_-(\rho_-)) \otimes \sigma_\Lambda(A)(\pi_+(\rho_+), \rho_+)}{\sigma_\Lambda(A)(\pi_-(\rho_-), \pi_+(\rho_+))} - \sigma_\Lambda(A)(\rho_-, \rho_+). \quad (6.34)$$

Proof. Take any $B_1 \in I_{\text{comp}}(\Lambda^o)$ with $\text{WF}_h(B_1) \subset \hat{\Lambda}$ and symbol given by (6.34). By (6.31), $(1 - A)(1 - B_1) = 1 - hR$, for some $R \in I_{\text{comp}}(\Lambda^o)$ with $\text{WF}_h(R) \subset \hat{\Lambda}$. Define $B_2 \in I_{\text{comp}}(\Lambda^o)$ by the asymptotic Neumann series
$$-B_2 \sim \sum_{j \geq 1} h^j R^j.$$ 
Define $B \in I_{\text{comp}}(\Lambda^o)$ by the identity $1 - B = (1 - B_1)(1 - B_2)$, then $(1 - A)(1 - B) = 1 + O(h^\infty)$. Similarly, we construct $B' \in I_{\text{comp}}(\Lambda^o)$ such that $(1 - B')(1 - A) = 1 + O(h^\infty)$. A standard algebraic argument, see for example the proof of [HöIII, Theorem 18.1.9], shows that $B' = B + O(h^\infty)$ and both are determined uniquely modulo $O(h^\infty)$. \hfill \Box

We finish this subsection with a trace formula for operators in $I_{\text{comp}}(\Lambda^o)$, used in §10:

Proposition 6.7. Assume that $A \in I_{\text{comp}}(\Lambda^o)$ and $\text{WF}_h(A) \subset \hat{\Lambda}$. Then, with $d\text{Vol}_\sigma = \sigma_S^{n-1}/(n - 1)!$ denoting the symplectic volume form and $j_K : K^\circ \to \Lambda^o$ defined in (6.2),
$$(2\pi h)^{-1} \text{Tr} A = \int_{\hat{\Lambda}} \sigma_\Lambda(A) \circ j_K d\text{Vol}_\sigma + O(h).$$

Proof. By a microlocal partition of unity, we reduce to the case when $\text{WF}_h(A)$ lies entirely in one of the sets $\Lambda_i$ defined in (6.26). If $\tilde{a}_i$ is defined by (6.28), then by the cyclicity of the trace, $\text{Tr} A = \text{Tr} \text{Op}_h^\Lambda(\tilde{a}_i) + O(h^\infty)$. It remains to note that for any $a(x, \xi) \in C_0^\infty(\mathbb{R}^{2n})$,
$$(2\pi h)^{-1} \text{Tr} \text{Op}_h^\Lambda(a) = \int_{\mathbb{R}^{2n-2}} a(x', 0, \xi', 0) dx'd\xi' + O(h),$$
seen directly from (6.5) by the method of stationary phase in the $x_n, \xi_n$ variables. \hfill \Box

6.3. Microlocal idempotents. In this subsection, we establish properties of microlocal idempotents associated to the Lagrangian $\Lambda^o$ considered in §6.2, microlocally on the compact set $\hat{\Lambda}$ defined in (6.27). We use the principal symbol $\sigma_\Lambda$ constructed in (6.30).

Definition 6.8. We call $A \in I_{\text{comp}}(\Lambda^o)$ a microlocal idempotent of order $k > 0$ near $\hat{\Lambda}$, if $A^2 = A + O(h^k)_{I_{\text{comp}}(\Lambda^o)}$ microlocally near $\hat{\Lambda}$ and $\sigma_\Lambda(A)$ does not vanish on $\hat{\Lambda}$. 
In the following Proposition, part 1 is concerned with the principal part of the idempotent equation; part 2 establishes a normal form for microlocal idempotents, making it possible to conjugate them microlocally to the operator $\text{Op}_0^\Lambda(1)$ from (6.16). Part 3 is used to construct a global idempotent of all orders in Proposition 6.10 below, while part 4 establishes properties of commutators used in the construction of §7.

**Proposition 6.9.** 1. $A \in I_{\text{comp}}(\Lambda^0)$ is a microlocal idempotent of order 1 near $\hat{\Lambda}$ if and only if near $\hat{\Lambda}$,

$$\sigma_\Lambda(A)(\rho_-, \rho_+) = a_0^-(\rho_-) \otimes a_0^+(\rho_+)$$

for some sections $a_0^+ \in C^\infty(\hat{\Gamma}_\pm; \mathcal{E}_\pm)$ nonvanishing near $\hat{\Gamma}_\pm$ and such that $a_0^+|_{\hat{\Gamma}} = 1$ near $\hat{\Gamma}$. Moreover, $a_0^\pm$ are uniquely determined by $A$ on $\hat{\Gamma}_\pm$.

2. If $A, B \in I_{\text{comp}}(\Lambda^0)$ are two microlocal idempotents of order $k > 0$ near $\hat{\Lambda}$, then there exists an operator $Q \in \Psi^\text{comp}(X)$, elliptic on $\hat{\Gamma}_+ \cup \hat{\Gamma}_-$ and such that $B = QAQ^{-1} + \mathcal{O}(h^k)I_{\text{comp}}(\Lambda^0)$ microlocally near $\hat{\Lambda}$. Here $Q^{-1}$ denotes an elliptic parametrix of $Q$ constructed in Proposition 3.3.

3. If $A \in I_{\text{comp}}(\Lambda^0)$ is a microlocal idempotent of order $k > 0$ near $\hat{\Lambda}$, and $A^2 - A = h^kR_k + \mathcal{O}(h^\infty)$ microlocally near $\hat{\Lambda}$ for some $R_k \in I_{\text{comp}}(\Lambda^0)$, then for $\rho_+ \in \hat{\Gamma}_+$ near $\hat{\Gamma}$,

$$\sigma_\Lambda(R_k)(\pi_+(\rho_+), \rho_+) = \sigma_\Lambda(R_k)(\pi_+(\rho_+), \pi_+(\rho_+)) \cdot a_0^+(\rho_+),$$

$$\sigma_\Lambda(R_k)(\rho_-, \pi_-(\rho_-)) = \sigma_\Lambda(R_k)(\pi_-(\rho_-), \pi_-(\rho_-)) \cdot a_0^-(\rho_-),$$

with $a_0^\pm$ defined in (6.35).

4. If $A \in I_{\text{comp}}(\Lambda^0)$ is a microlocal idempotent of all orders near $\hat{\Lambda}$, $P \in \Psi^\text{comp}(X)$ is compactly supported, and $[P, A] = h^kS_k + \mathcal{O}(h^\infty)$ microlocally near $\hat{\Lambda}$ for some $S_k \in I_{\text{comp}}(\Lambda^0)$, then near $\hat{\Lambda}$,

$$\sigma_\Lambda(S_k)(\rho_-, \rho_+) = a_0^-(\rho_-) \otimes \sigma_\Lambda(S_k)(\pi_+(\rho_+), \rho_+) + \sigma_\Lambda(S_k)(\rho_-, \pi_-(\rho_-)) \otimes a_0^+(\rho_+).$$

In particular, $\sigma_\Lambda(S_k) \circ j_K = 0$ near $\hat{K}$, with $j_K : K^c \to \Lambda^c$ defined in (6.2).

**Proof.** In this proof, all the equalities of operators in $I_{\text{comp}}(\Lambda^0)$ and the corresponding symbols are presumed to hold microlocally near $\hat{\Lambda}$.

1. By (6.31), we have $A^2 = A + \mathcal{O}(h)$ if and only if

$$\sigma_\Lambda(A)(\rho_-, \rho_+) = \sigma_\Lambda(A)(\rho_-, \pi_-(\rho_-)) \otimes \sigma_\Lambda(A)(\pi_+(\rho_+), \rho_+).$$

In particular, restricting to $\hat{\Gamma}$, we obtain $\sigma_\Lambda(A) = \sigma_\Lambda(A)^2$ near $\hat{K}$. Since $\sigma_\Lambda(A)$ is nonvanishing, we get $\sigma_\Lambda(A)|_{\hat{\Gamma}} = 1$ near $\hat{K}$. It then remains to put $a_0^-(\rho_-) = \sigma_\Lambda(A)(\rho_-, \pi_-(\rho_-))$ and $a_0^+(\rho_+) = \sigma_\Lambda(A)(\pi_+(\rho_+), \rho_+)$.

2. We use induction on $k$. For $k = 1$, we have by (6.32) and (6.33),

$$\sigma_\Lambda(QAQ^{-1})(\rho_-, \rho_+) = \frac{\sigma(Q)(\rho_+)}{\sigma(Q)(\rho_-)}\sigma_\Lambda(A)(\rho_-, \rho_+).$$
If $a_0^\pm$ and $b_0^\pm$ are given by (6.35), then it is enough to take any $Q$ with
\[ \sigma(Q)|_{\Gamma_-} = a_0^- / b_0^-, \quad \sigma(Q)|_{\Gamma_+} = b_0^+ / a_0^+, \]
(6.37) this is possible since the restrictions of $a_0^\pm$ and $b_0^\pm$ to $\tilde{K}$ are equal to 1.

Now, assuming the statement is true for $k \geq 1$, we prove it for $k + 1$. We have $B = \hat{Q}AQ^{-1} + \mathcal{O}(h^k)$ for some $\hat{Q} \in \Psi^{\text{comp}}$ elliptic on $\Gamma_+ \cup \Gamma_- $; replacing $A$ by $\hat{Q}AQ^{-1}$, we may assume that $B = A + \mathcal{O}(h^k)$. Then $B - A = h^k \Gamma_k$ for some $\Gamma_k \in I^{\text{comp}}(\Lambda^0)$; since both $A$ and $B$ are microlocal idempotents of order $k + 1$, we find $\Gamma_k = AR_k + R_kA + \mathcal{O}(h)$ and thus by (6.31),

\[ \sigma_\Lambda(\Gamma_k)(\rho_-, \rho_+) = a_0^-(\rho_-) \sigma_\Lambda(\Gamma_k)(\pi_+(\rho_+), \rho_+) + \sigma_\Lambda(\Gamma_k)(\rho_-, \pi_-(\rho_-)) \otimes a_0^+(\rho_+). \]
(6.38)

Take $Q = 1 + h^k Q_k$ for some $Q_k \in \Psi^{\text{comp}}$, then $Q^{-1} = 1 - h^k Q_k + \mathcal{O}(h^{k+1})$ and

\[ QAQ^{-1} = A + h^k [Q_k, A] + \mathcal{O}(h^{k+1}). \]

Now, $B = QAQ^{-1} + \mathcal{O}(h^{k+1})$ if and only if

\[ (\sigma(Q_k)(\rho_+) - \sigma(Q_k)(\rho_-))a_0^- (\rho_-) \otimes a_0^+(\rho_+) = \sigma_\Lambda(\Gamma_k)(\rho_-, \rho_+). \]

By (6.38), it is enough to choose $Q_k$ such that for $\rho_{\pm} \in \tilde{\Gamma}_{\pm}$,

\[ \sigma(Q_k)(\rho_-) = - \frac{\sigma_\Lambda(\Gamma_k)(\rho_-, \pi_-(\rho_-))}{a_0^-(\rho_-)}, \quad \sigma(Q_k)(\rho_+) = \frac{\sigma_\Lambda(\Gamma_k)(\pi_+(\rho_+), \rho_+)}{a_0^+(\rho_+)}, \]

this is possible since $\sigma_\Lambda(\Gamma_k) \circ j_K = 0$ (with $j_K$ defined in (6.2)) as follows from (6.38).

3. Since this is a local statement, we can use (6.28) to reduce to the model case of §6.1. Using part 2 and the fact that the operator $\text{Op}_{h^0}^\Lambda(1)$ considered in (6.16) is a microlocal idempotent of all orders, we can write

\[ A = Q \text{Op}_{h^0}^\Lambda(1)Q^{-1} + h^k A_k, \]

for some elliptic $Q \in \Psi^{\text{comp}}$ and $A_k \in I^{\text{comp}}(\Lambda^0)$. Then

\[ \Gamma_k = Q \text{Op}_{h^0}^\Lambda(1)Q^{-1} A_k + A_k Q \text{Op}_{h^0}^\Lambda(1)Q^{-1} - A_k + \mathcal{O}(h); \]
(6.36) follows by (6.31) since $\sigma_\Lambda(Q \text{Op}_{h^0}^\Lambda(1)Q^{-1}) = \sigma_\Lambda(A)$ is given by (6.35).

4. As in part 3, we reduce to the model case of §6.1 and use part 2 to write

\[ A = Q \text{Op}_{h^0}^\Lambda(1)Q^{-1} + \mathcal{O}(h^\infty); \]

then

\[ [P, A] = Q[Q^{-1} P Q, \text{Op}_{h^0}^\Lambda(1)]Q^{-1} + \mathcal{O}(h^\infty). \]

Put $\tilde{P} = Q^{-1} PQ$; by (6.13) and (6.14) we have $[\tilde{P}, \text{Op}_{h^0}^\Lambda(1)] = \text{Op}_{h^0}^\Lambda(s \circ \phi)$, where $\phi$ is given by (6.4) and

\[ s(\rho_-, \rho_+; h) = \tilde{p}(\rho_+; h) - \tilde{p}(\rho_-; h), \]

where $\tilde{P} = \text{Op}_{h^0}(\tilde{p})$; thus

\[ s(\rho_-, \rho_+; h) = s(\pi_+(\rho_+), \rho_+; h) + s(\rho_-, \pi_-(\rho_-); h). \]

It remains to conjugate by $Q$, keeping in mind (6.37).
We can use part 3 of Proposition 6.9, together with the triviality of the bundles $\mathcal{E}_\pm$, to show existence of a global idempotent, which is the starting point of the construction in §7.

**Proposition 6.10.** There exists a microlocal idempotent $\tilde{\Pi} \in I_{\text{comp}}(\Lambda^\circ)$ of all orders near $\hat{\Lambda}$.

**Proof.** We argue inductively, constructing microlocal idempotents $\tilde{\Pi}_k$ of order $k$ for each $k$ and taking the asymptotic limit. To construct $\tilde{\Pi}_1$, we use part 1 of Proposition 6.9; the existence of symbols $a_0^\pm$ was shown in Proposition 6.4.

Now, assume that $\tilde{\Pi}_k$ is a microlocal idempotent of order $k > 0$. By part 3 of Proposition 6.9, we have $\tilde{\Pi}_k^2 - \tilde{\Pi}_k = h^k R_k + \mathcal{O}(h^\infty)$ microlocally near $\hat{\Lambda}$, where $R_k \in I_{\text{comp}}(\Lambda^\circ)$ and $r_k = \sigma_{\Lambda}(R_k)$ satisfies (6.36). Put $\Pi_{k+1} = \tilde{\Pi}_k + h^k B_k$, for some $B_k \in I_{\text{comp}}(\Lambda^\circ)$. We need to choose $B_k$ so that microlocally near $\hat{\Lambda}$,

$$R_k + \tilde{\Pi}_k B_k + B_k \tilde{\Pi}_k - B_k = \mathcal{O}(h).$$

Taking $b_k = \sigma_{\Lambda}(B_k)$, by (6.31) this translates to

$$b_k(\rho_-,\rho_+) = a_0^-(\rho_-) \otimes b_k(\pi_+(\rho_+),\rho_+) + b_k(\rho_-,\pi_-(\rho_-)) \otimes a_0^+(\rho_+) + r_k(\rho_-,\rho_+).$$

By (6.36), it is enough to take any $b_k^\pm \in C^\infty(\hat{\Gamma}_\pm; \mathcal{E}_\pm)$ such that near $\hat{K}$, $b_k^\pm|_{\hat{K}} = -r_k \circ j_K$, with $j_K$ defined in (6.2) (for example, $b_k^+ = -(r_k \circ j_K \circ \pi_+ a_0^+) \otimes$, and put

$$b_k(\rho_-,\rho_+) := a_0^-(\rho_-) \otimes b_k^+(\rho_+) + b_k^-(\rho_-) \otimes a_0^+(\rho_+) + r_k(\rho_-,\rho_+). \quad \square$$

### 6.4. Annihilating ideals.

Assume that $\Pi \in I_{\text{comp}}(\Lambda^\circ)$ is a microlocal idempotent of all orders near the set $\hat{\Lambda}$ introduced in (6.27), see Definition 6.8. We are interested in the following equations:

\begin{align*}
\Pi \Theta_+ &= \mathcal{O}(h^\infty) \quad \text{microlocally near } \hat{\Lambda}, \\
\Theta_+ \Pi &= \mathcal{O}(h^\infty) \quad \text{microlocally near } \hat{\Lambda},
\end{align*}

where $\Theta_\pm$ are pseudodifferential operators. The solutions to (6.39) form a right ideal and the solutions to (6.40) form a left ideal in the algebra of pseudodifferential operators. Moreover, by (6.32), (6.33), each solution $\Theta_\pm$ to the equations (6.39), (6.40) satisfies $\sigma(\Theta_\pm)|_{\hat{\Gamma}_\pm} = 0$ near $\hat{\Gamma}_\pm$ and each $\Theta_\pm$ such that $WF_h(\Theta_\pm) \cap \hat{\Gamma}_\pm = \emptyset$ solves these equations.

Note that in the model case of §6.1, with $\Pi$ equaling the operator $\text{Op}_h(1)$ from (6.16), and with the quantization procedure $\text{Op}_h$ defined in (6.6), the set of solutions to (6.39) is the set of operators $\text{Op}_h(\theta_-)$ with $\theta_-|_{x_n=0} = 0$; that is, the right ideal generated by the operator $x_n$. The set of solutions to (6.40) is the set of operators $\text{Op}_h(\theta_+)$ with $\theta_+|_{\xi_n=0} = 0$; that is, the left ideal generated by the operator $hD_{x_n}$. This follows...
from the multiplication formulas (6.13) and (6.14), together with the multiplication formulas for the standard quantization \([Z_w, (4.3.16)]\).

We start by showing that our ideals are principal in the general setting:

**Proposition 6.11.** 1. For each defining functions \(\varphi_\pm\) of \(\Gamma_\pm\) near \(\hat{\Gamma}_\pm\), there exist operators \(\Theta_\pm\) solving (6.39), (6.40), such that \(\sigma(\Theta_\pm) = \varphi_\pm\) near \(\hat{\Gamma}_\pm\). Such operators are called basic solutions of the corresponding equations.

2. If \(\Theta_\pm, \Theta'_\pm\) are solutions to (6.39), (6.40), and moreover \(\Theta_\pm\) are basic solutions, then there exist \(Z_\pm \in \Psi^{\text{comp}}\) such that \(\Theta'_\pm = \Theta_\pm Z_\pm + \mathcal{O}(h^\infty)\) microlocally near \(\hat{\Gamma}_-\) and \(\Theta'_+ = Z_+ \Theta_+ + \mathcal{O}(h^\infty)\) microlocally near \(\hat{\Gamma}_+\).

**Proof.** We concentrate on the equation (6.39); (6.40) is handled similarly. Since the equations (6.39) and \(\Theta' = \Theta_- Z_-\) are linear in \(\Theta_-, \Theta', Z_-, \) respectively, we can use (6.28) and a pseudodifferential partition of unity to reduce to the model case of §6.1. Using part 2 of Proposition 6.9, we can furthermore assume that \(\Pi = \sigma\Pi\).

To show part 1, in the model case, we can take \(\Theta_- = \operatorname{Op}_h(\varphi_-)\), where \(\varphi_-(x, \xi)\) is the given defining function of \(\{x_n = 0\}\). For part 2, if \(\Theta_- = \operatorname{Op}_h(\varphi_-)\) and \(\Theta'_- = \operatorname{Op}_h(\varphi'_-)\), then we can write microlocally near \(\hat{\Gamma}_-\), \(\Theta_- = x_n Y_- + \mathcal{O}(h^\infty)\), where \(Y_- \in \Psi^{\text{comp}}\) is elliptic on \(\hat{\Gamma}_-\); in fact, \(Y_- = \operatorname{Op}_h(\varphi_-/x_n)\). Similarly, we can write \(\Theta'_- = x_n Y'_- + \mathcal{O}(h^\infty)\) microlocally near \(\hat{\Gamma}_-\), for some \(Y'_- \in \Psi^{\text{comp}}\); it remains to put \(Z_- = Y_-^{-1} Y'_-\) microlocally near \(\hat{\Gamma}_-\). \(\square\)

For the microlocal estimate on the kernel of \(\Pi\) in §8.2, we need an analog of the following fact:

\[
f \in C^\infty(\mathbb{R}^n) \implies f(x) - f(x', 0) = x_n g(x), \quad g \in C^\infty(\mathbb{R}^n),
\]

where \(f(x', 0)\) is replaced by \(\Pi f\) and multiplication by \(x_n\) is replaced by a basic solution to (6.39). We start with a technical lemma for the model case:

**Lemma 6.12.** Consider the operator \(\Xi_0 : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)\) defined by

\[
\Xi_0 f(x', x_n) = \frac{f(x', x_n) - f(x', 0)}{x_n} = \int_0^1 (\partial_{x_n} f)(x', tx_n) dt.
\]

Then:

1. \(\Xi_0\) is bounded \(H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\) and thus \(\|\Xi_0\|_{H^1 \to L^2} = \mathcal{O}(h^{-1})\).

2. The wavefront set WF\(_h\)(\(\Xi_0\)) defined in §3.1 satisfies\(^5\)

\[
\text{WF}_h(\Xi_0) \cap T^*(\mathbb{R}^n \times \mathbb{R}^n) \subset \Delta(T^*\mathbb{R}^n) \cup \Lambda^0 \cup \{(x', 0, \xi, x', 0, \xi', t\xi_n) \mid (x', \xi) \in \mathbb{R}^{2n-1}, t \in [0, 1]\},
\]

\(^5\)It would be interesting to understand the microlocal structure of \(\Xi_0\), starting from the fact that its wavefront set lies in the union of three Lagrangian submanifolds.
where $\Delta(T^*\mathbb{R}^n) \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ is the diagonal and $\Lambda^0$ is defined using (6.3).

**Proof.** 1. Put $\lambda_t f(x', x_n) = (\partial_{x_n} f)(x', tx_n)$; then
\[
\|\xi_0 f\|_{L^2} \leq \int_0^1 \|\lambda_t f\|_{L^2} dt \leq \int_0^1 t^{-1/2} \|f\|_{H^1} dt \leq 2\|f\|_{H^1}.
\]

2. Denote elements of $T^*\mathbb{R}^n \times \mathbb{R}^n$ by $(x, \xi, \eta, \eta)$. If $\chi \in C_0^\infty(\mathbb{R})$ is supported away from zero, then, with $\text{Op}_h^A(1)$ defined in (6.16),
\[
\chi(x_n)\xi_0 = \frac{\chi(x_n)}{x_n}(1 - \text{Op}_h^A(1)).
\]
Since $\chi(x_n)/x_n$ is a smooth function, the identity operator has wavefront set on the diagonal, and $\text{WF}_h(\text{Op}_h^A(1)) \cap T^*(\mathbb{R}^n \times \mathbb{R}^n) \subset \Lambda^0$, we find
\[
\text{WF}_h(\xi_0) \cap T^*(\mathbb{R}^n \times \mathbb{R}^n) \cap \{y_n \neq 0\} \subset \Delta(T^*\mathbb{R}^n) \cup \Lambda^0.
\]
Similarly, one has $\xi_0\chi(x_n) = \chi(x_n)/x_n$; therefore,
\[
\text{WF}_h(\xi_0) \cap \{x_n \neq 0\} \subset \Delta(T^*\mathbb{R}^n).
\]
To handle the remaining part of the wavefront set, take $a, b \in C_0^\infty(T^*\mathbb{R}^n)$ such that
\[
(x', tx_n, \xi) \in \text{supp } a, \ t \in [0, 1] \implies (x, \xi', t\xi_n) \notin \text{supp } b.
\]
We claim that for any $\psi \in C_0^\infty(\mathbb{R}^n)$,
\[
\text{Op}_h(b)\psi \xi_0 \text{Op}_h(a)\psi = \mathcal{O}(h^\infty);
\]
indeed, the Schwartz kernel of this operator is
\[
K(y, x) = (2\pi h)^{-2n} \int_{\mathbb{R}^{3n} \times [0, 1]} e^{i\pi((y-z)\eta + (z'-x')\xi' + (tz_n-x_n)\xi_n)}
\]
\[
b(y, \eta)\psi(z)(ih^{-1}\eta_n a(z', t\xi_n, \xi) + (\partial_{z_n} a)(z', t\xi_n, \xi))\psi(x) d\xi d\eta dz dt.
\]
The stationary points of the phase in the $(\xi, \eta, z)$ variables are given by
\[
z = y, \ x' = y', \ x_n = ty_n, \ \eta' = \xi', \ \eta_n = t\xi_n
\]
and lie outside of the support of the amplitude; by the method of nonstationary phase in the $(\xi, \eta, z)$ variables, the integral is $\mathcal{O}(h^\infty)_{C^\infty}$. Now, (6.42) implies that
\[
\text{WF}_h(\xi_0) \cap T^*(\mathbb{R}^n \times \mathbb{R}^n) \cap \{x_n = y_n = 0\}
\]
\[
\subset \{(x', 0, \xi, x', 0, \xi', t\xi_n) | (x', \xi) \in \mathbb{R}^{2n-1}, \ t \in [0, 1]\},
\]
which finishes the proof. 

The microlocal analog of (6.41) in the general case is now given by
Proposition 6.13. Let $\Pi \in I_{\text{comp}}(\Lambda^0)$ be a microlocal idempotent of all orders near $\hat{\Lambda}$ and $\Theta_-$ be a basic solution to (6.39), see Proposition 6.11. Then there exists an operator $\Xi : C^\infty(X) \to C^\infty_0(X)$ such that:

1. $\text{WF}_h(\Xi)$ is a compact subset of $T^*(M \times M)$ and $\|\Xi\|_{L^2 \to L^2} = O(h^{-1})$;

2. $\text{WF}_h(\Xi) \subset \Delta(T^*M) \cup \Lambda^0 \cup \Upsilon$, where $\Delta(T^*M) \subset T^*M \times T^*M$ is the diagonal and $\Upsilon$ consists of all $(\rho_-, \rho'_-)$ such that $\rho_-, \rho'_- \in \Gamma_-$ and $\rho'_-$ lies on the segment of the flow line of $\mathcal{V}_-$ between $\rho_-$ and $\pi_-(\rho_-)$;

3. $1 - \Pi = \Theta_- \Xi + O(h^\infty)$ microlocally near $\hat{K} \times \hat{K}$.

Proof. By (6.28) and a microlocal partition of unity, we can reduce to the model case of §6.1. Moreover, by part 2 of Proposition 6.9, we may conjugate by a pseudodifferential operator to make $\Pi = \text{Op}_h(1)$. Finally, by part 2 of Proposition 6.11 we can multiply $\Theta_-$ on the right by an elliptic pseudodifferential operator to make $\Theta_- = \text{Op}_h(x_n)$. Then we can take $\Xi = A\Xi_0 A$, with $\Xi_0$ defined in Lemma 6.12 and $A \in \Psi^{\text{comp}}(\mathbb{R}^n)$ compactly supported, with $A = 1 + O(h^\infty)$ microlocally near $\hat{K}$. □

7. The projector $\Pi$

In this section, we construct the microlocal projector $\Pi$ near a neighborhood $\hat{W}$ of $K \cap p^{-1}([\alpha_0, \alpha_1])$ discussed in the introduction (Theorem 3 in §7.1). In §7.2, we study the annihilating ideals for $\Pi$ in $\hat{W}$ using §6.4.

7.1. Construction of $\Pi$. Assume that the conditions of §§4.1 and 5.1 hold. Consider the sets $\Gamma_\pm$ and $K^o = \Gamma_+^o \cap \Gamma_-^o$ defined in (5.10) and let $\Lambda^o$ be given by (5.12). Put

$$\hat{K} := K \cap p^{-1}([\alpha_0 - \delta_1/2, \alpha_1 + \delta_1/2]) \subset K^o,$$

here $\delta_1$ is defined in §5.4. The sets $\Gamma_\pm^o$ satisfy the assumptions listed in the beginning of §6, as follows from §§5.1 and 5.4.

We choose $\delta > 0$ small enough so that Lemma 5.1 holds (we will impose more conditions on $\delta$ in §7.2) and consider the sets

$$\hat{W} := U_\delta \cap p^{-1}([\alpha_0 - \delta_1/2, \alpha_1 + \delta_1/2]),$$

$$\hat{\Gamma}_\pm^o := \Gamma_\pm^o \cap \hat{W}, \quad \hat{\Lambda} := \Lambda^o \cap (\hat{W} \times \hat{W}).$$

(7.1)

Here $U_\delta$ is defined in (5.8). We now apply Proposition 6.3; for $\delta$ small enough, $\hat{W}, \hat{\Gamma}_\pm^o$ are compact and $\hat{\Gamma}_\pm^o, \hat{\Lambda}$ satisfy the conditions listed after (6.26). Then (6.30) defines the principal symbol $\sigma_\Lambda(A)$ on a neighborhood of $\hat{\Lambda}$ in $\Lambda^o$ for each $A \in I_{\text{comp}}(\Lambda^o)$.

Theorem 3. Let the assumptions of §§4.1 and 5.1 hold for all $r$, let $\Lambda^o$ be defined in (5.12) and $\hat{\Lambda} \subset \Lambda^o$ be given by (7.1). Then there exists $\Pi \in I_{\text{comp}}(\Lambda^o)$, uniquely
defined modulo $\mathcal{O}(h^\infty)$ microlocally near $\hat{\Lambda}$, such that the principal symbol of $\Pi$ is nonvanishing on $\hat{\Lambda}$ and, with $P \in \Psi^{\text{comp}}(X)$ defined in Lemma 4.3,

$$\Pi^2 - \Pi = \mathcal{O}(h^\infty) \quad \text{microlocally near } \hat{\Lambda},$$

(7.2) $$[P, \Pi] = \mathcal{O}(h^\infty) \quad \text{microlocally near } \hat{\Lambda}.$$  

(7.3)

Same can be said if we replace $\mathcal{O}(h^\infty)$ above by $\mathcal{O}(h^N)$, require that the full symbol of $\Pi$ lies in $C^{3N}$ for some large $N$ (rather than being smooth), and the assumptions of §5.1 hold for $r$ large enough depending on $N$.

**Proof.** We argue by induction, finding a family $\Pi_k$, $k \geq 1$, of microlocal idempotents of all orders near $\hat{\Lambda}$ (see Definition 6.8) such that $[P, \Pi_k] = \mathcal{O}(h^{k+1})$ microlocally near $\hat{\Lambda}$, and taking their asymptotic limit to obtain $\Pi$.

We first construct $\Pi_1$. Take the microlocal idempotent of all orders $\tilde{\Pi} \in \Psi^{\text{comp}}(\Lambda^o)$ near $\hat{\Lambda}$ constructed in Proposition 6.10. Since the Hamilton field of $p = \sigma(P)$ is tangent to $\Gamma_\pm$, $dp$ is annihilated by the subbundles $\mathcal{V}_\pm$ from §5.4; therefore, $p(\rho_\pm) = p(\pi_\pm(\rho_\pm))$, $\rho_\pm \in \Gamma^o_\pm$;

by (6.32) and (6.33), $[P, \tilde{\Pi}] = \mathcal{O}(h)$ microlocally near $\hat{\Lambda}$. We write $[P, \tilde{\Pi}] = hS_0$ microlocally near $\hat{\Lambda}$, where $S_0 \in \Psi^{\text{comp}}(\Lambda^o)$ and by part 4 of Proposition 6.9,

$$\sigma_\Lambda(S_0)(\rho_-, \rho_+) = \tilde{a}_0^-(\rho_-) \otimes s_0^+(\rho_+) + s_0^-(\rho_-) \otimes \tilde{a}_0^+(\rho_+),$$

(7.4) with $s_0^\pm \in \mathcal{C}^\infty(\tilde{\Gamma}_\pm; \mathcal{E}_\pm)$ vanishing on $K$ near $\hat{K}$ and $\tilde{a}_0^\pm \in \mathcal{C}^\infty(\tilde{\Gamma}_\pm; \mathcal{E}_\pm)$ giving the principal symbol of $\tilde{\Pi}$ by (6.35). Here $\Gamma_\pm$ are the neighborhoods of $\Omega_\pm$ in $\Gamma^o_\pm$ defined in (6.26).

We look for $\Pi_1$ in the form

$$\Pi_1 = e^{Q_0} \tilde{\Pi} e^{-Q_0},$$

(7.5) where $Q_0 \in \Psi^{\text{comp}}(X)$ is compactly supported and thus $e^{\pm Q_0}$ are pseudodifferential (see for example [Dy12, Proposition 2.7]). We calculate microlocally near $\hat{\Lambda}$,

$$e^{-Q_0}[P, \Pi_1]e^{Q_0} = [e^{-Q_0}Pe^{Q_0}, \tilde{\Pi}] = hS_0 + [[P, Q_0], \tilde{\Pi}] + \mathcal{O}(h^2).$$

Here we use that $e^{-Q_0}Pe^{Q_0} = P + [P, Q_0] + \mathcal{O}(h^2)$. By (7.4), (6.32), (6.33),

$$\sigma_\Lambda(S_0 + h^{-1}[[P, Q_0], \tilde{\Pi}]) (\rho_-, \rho_+)$$

$$= \tilde{a}_0^-(\rho_-) \otimes (s_0^+(\rho_+) + iH_p\sigma(Q_0)(\rho_+)\tilde{a}_0^+(\rho_+))$$

$$+(s_0^-(\rho_-) + iH_p\sigma(Q_0)(\rho_-)\tilde{a}_0^-(\rho_-)) \otimes \tilde{a}_0^+(\rho_+).$$

It is thus enough to take any $Q_0$ such that for the restrictions $q_0^\pm = \sigma(Q_0)|_{\Gamma_\pm}$, the following transport equations hold near $\hat{\Gamma}_\pm$:

$$H_p q_0^\pm = \mp is_0^\pm/\tilde{a}_0^\pm, \quad q_0^\pm|_{\tilde{K}} = 0.$$  

(7.6)
Such \( q_0^\pm \) exist and are unique and smooth enough by Lemma 5.2, giving \( \Pi_1 \). Note that Lemma 5.2 can be applied near \( \hat{\Gamma}_\pm \), instead of the whole \( \Gamma_\pm^o \), since \( e^{itH_p}(\hat{\Gamma}_\pm) \subset \hat{\Gamma}_\pm \) for \( \mp t \geq 0 \) by part (2) of Lemma 5.1.

Now, assume that we have constructed \( \Pi_k \) for some \( k > 0 \). Let \( a_0^\pm \) be the components of the principal symbol of \( \Pi_k \) given by (6.35). Then microlocally near \( \hat{\Lambda} \), \( [P, \Pi_k] = \hbar^{k+1}S_k \), where \( S_k \in I_{\text{comp}}(\Lambda^o) \) and by part 4 of Proposition 6.9,

$$\sigma(\Lambda)(s_-(\rho_-, \rho_+)) = a_0^-(\rho_-) \otimes s_k^+(\rho_+) + s_k^-(\rho_-) \otimes a_0^+(\rho_+),$$

where \( s_k^\pm \in C^\infty(\hat{\Gamma}_\pm; \mathcal{E}_\pm) \) vanish on \( K \) near \( \hat{K} \). We then take

$$\Pi_{k+1} = (1 + \hbar^kQ_k)\Pi_k(1 + \hbar^kQ_k)^{-1}$$

(7.7)

where \( Q_k \) is a compactly supported pseudodifferential operator. Microlocally near \( \hat{\Lambda} \),

$$[P, \Pi_{k+1}] = \hbar^{k+1}S_k + \hbar^k[[P, Q_k], \Pi_k] + \mathcal{O}(\hbar^{k+2}).$$

Therefore, \( q_k^\pm = \sigma(Q_k)|_{\hat{\Gamma}_\pm} \) need to satisfy the transport equations near \( \hat{\Gamma}_\pm \)

$$H_\rho q_k^\pm = \mp is_k^\pm/a_k^\pm, \quad q_k^\pm|_{\hat{K}} = 0.$$  

(7.8)

Such \( q_k^\pm \) exist and are unique and smooth enough again by Lemma 5.2, giving \( \Pi_{k+1} \).

To show that the operator \( \Pi \) satisfying (7.2) and (7.3) is unique microlocally near \( \hat{\Lambda} \), we show by induction that each such \( \Pi \) satisfies \( \Pi = \Pi_k + \mathcal{O}(\hbar^k) \) microlocally near \( \hat{\Lambda} \). First of all, \( \Pi \) has the form (7.5) for some operator \( Q_0 \) microlocally near \( \hat{\Lambda} \), by part 2 of Proposition 6.9; moreover, by the proof of this fact, we can take \( \sigma(Q_0)|_{\hat{K}} = 0 \) near \( \hat{K} \). Now, \( \sigma(Q_0)|_{\hat{\Gamma}_\pm} \) are determined uniquely by the transport equations (7.6), and this gives \( \Pi = \Pi_1 + \mathcal{O}(\hbar) \) microlocally near \( \hat{\Lambda} \). Next, if \( \Pi = \Pi_k + \mathcal{O}(\hbar^k) \) for some \( k > 0 \), then, as follows from the proof of Part 2 of Proposition 6.9, \( \Pi \) has the form (7.7) for some operator \( Q_k \) microlocally near \( \hat{\Lambda} \), such that \( \sigma(Q_k)|_{\hat{K}} = 0 \) near \( \hat{K} \). Then \( \sigma(Q_k)|_{\hat{\Gamma}_\pm} \) are determined uniquely by the transport equations (7.8), and this gives \( \Pi = \Pi_{k+1} + \mathcal{O}(\hbar^{k+1}) \) microlocally near \( \hat{\Lambda} \).  

\[ \Box \]

### 7.2. Annihilating ideals

Let \( \Pi \in I_{\text{comp}}(\Lambda^o) \) be the operator constructed in Theorem 3. In this section, we construct pseudodifferential operators \( \Theta_\pm \) annihilating \( \Pi \) microlocally near \( \hat{\Lambda} \); they are key for the microlocal estimates in §8. More precisely, we obtain

**Proposition 7.1.** If \( \delta > 0 \) in the definition (7.1) of \( \hat{W} \) is small enough, then there exist compactly supported \( \Theta_\pm \in \Psi_{\text{comp}}(X) \) such that:

1. \( \Pi \Theta_- = \mathcal{O}(\hbar^\infty) \) and \( \Theta_+ \Pi = \mathcal{O}(\hbar^\infty) \) microlocally near \( \hat{\Lambda} \);
2. \( \sigma(\Theta_\pm) = \varphi_\pm \) near \( \hat{W} \), with \( \varphi_\pm \) defined in Lemma 5.1.
(3) if $P$ is the operator constructed in Lemma 4.3, then

$$[P, \Theta_-] = -i\hbar \Theta_- Z_- + O(h^{\infty}), \quad [P, \Theta_+] = i\hbar Z_+ \Theta_+ + O(h^{\infty})$$

(7.9)

microlocally near $\hat{W}$, where $Z_\pm \in \Psi^{\text{comp}}(X)$ are compactly supported and $\sigma(Z_\pm) = c_\pm$ near $\hat{W}$, with $c_\pm$ defined in Lemma 5.1;

(4) if $\text{Im} \Theta_+ = \frac{1}{2i}(\Theta_+ - \Theta_+^*)$ and $\zeta = \sigma(h^{-1} \text{Im} \Theta_+)$, then

$$H_p \zeta = -c_+ \zeta - \frac{1}{2}\{\varphi_+, c_+\} \quad \text{on } \Gamma_+ \text{ near } \hat{W};$$

(7.10)

(5) there exists an operator $\Xi : C^\infty(X) \to C^\infty_0(X)$, satisfying parts 1 and 2 of Proposition 6.13 and such that

$$1 - \Pi = \Theta_- \Xi + O(h^{\infty}) \quad \text{microlocally near } \hat{W} \times \hat{W}. \quad (7.11)$$

Proof. The operators $\Theta_\pm$ satisfying conditions (1) and (2) exist by part 1 of Proposition 6.11. Next, since $[P, \Pi] = O(h^{\infty})$ microlocally near $\hat{\Lambda}$, we find

$$\Pi[P, \Theta_-] = O(h^{\infty}), \quad [P, \Theta_+] \Pi = O(h^{\infty})$$

microlocally near $\hat{\Lambda}$; condition (3) now follows from part 2 of Proposition 6.11. The symbols $\sigma(Z_\pm)$ can be computed using the identity $H_p \varphi_\pm = \mp c_\pm \varphi_\pm$ from part (2) of Lemma 5.1. Condition (5) follows immediately from Proposition 6.13, keeping in mind that by making $\delta$ small we can make $\hat{W}$ contained in an arbitrary neighborhood of $\hat{K}$.

Finally, we verify condition (4). Taking the adjoint of the identity $[P, \Theta_+] = i\hbar Z_+ \Theta_+ + O(h^{\infty})$ and using that $P$ is self-adjoint, we get microlocally near $\hat{W}$,

$$[P, \Theta_+] = i\hbar \Theta_+^* Z_+^*.$$ 

Therefore, microlocally near $\hat{W}$

$$2[P, h^{-1} \text{Im} \Theta_+] = Z_+ \Theta_+ - \Theta_+^* Z_+^* = [Z_+, \Theta_+] + 2i((\text{Im} \Theta_+) Z_+ + \Theta_+ \text{Im} Z_+).$$

By comparing the principal symbols, we get (7.10). \qed

8. Resolvent estimates

In this section we give various estimates on the resolvent $\mathcal{R}(\omega)$, in particular proving Theorem 1. In §8.1, we reduce Theorem 1 to a microlocal estimate in a neighborhood of the trapped set, which is further split into two estimates: on the kernel of the projector $\Pi$ given by Theorem 3, proved in §8.2, and on the image of $\Pi$, proved in §8.3. In §8.4 we obtain a restriction on the wavefront set $\mathcal{R}(\omega)$ in $\omega$ on the image of $\Pi$, needed in §10. Finally, in §8.5, we discuss the consequences of our methods for microlocal concentration of resonant states and the corresponding semiclassical measures.
8.1. **Reduction to the trapped set.** We take $\delta > 0$ small enough so that the results of §7.1, 7.2 hold, and define following (7.1) (with $\delta_1$ chosen in §5.4),

$$\hat{W} := U_\delta \cap p^{-1}([\alpha_0 - \delta_1 / 2, \alpha_1 + \delta_1 / 2]), \quad W' := U_{\delta/2} \cap p^{-1}([\alpha_0 - \delta_1 / 4, \alpha_1 + \delta_1 / 4]), \quad (8.1)$$

so that $W'$ is a neighborhood of $K \cap p^{-1}([\alpha_0, \alpha_1])$ compactly contained in $\hat{W}$. Here $U_\delta$ is defined in (5.8).

For the reductions of this subsection, it is enough to assume that $\omega$ satisfies (4.1). The region (1.5) will arise as the intersection of the regions (8.9) and (8.11) where the two components of the estimate will hold.

To prove Theorem 1, it is enough to show the estimate

$$\|\tilde{u}\|_{H^1} \leq C h^{-2} \|\tilde{f}\|_{H^2} + O(h^\infty) \quad (8.2)$$

for each $\tilde{u} = \tilde{u}(h) \in H_1$ with $\|\tilde{u}\|_{H_1}$ bounded polynomially in $h$ and for $\tilde{f} = \mathcal{P}(\omega)\tilde{u}$, where $\omega = \omega(h)$ satisfies (1.5).

Subtracting from $\tilde{u}$ the function $v$ constructed in Lemma 4.5, we may assume that

$$WF_h(\tilde{f}) \subset W'.$$

Let $S(\omega)$ be the operator constructed in Lemma 4.3, $S'(\omega)$ be its elliptic parametrix near $\mathcal{U} \supset \hat{W}$ constructed in Lemma 3.3, and put

$$u := S(\omega)\tilde{u}, \quad f := S'(\omega)\tilde{f},$$

so by (4.9), for the operator $P$ constructed in Lemma 4.3,

$$\quad \quad \quad (P - \omega)u = f \quad \text{microlocally near } \hat{W}, \quad WF_h(f) \subset \hat{W}. \quad \quad \quad (8.3)$$

By ellipticity (Proposition 3.2) and since $WF_h(f) \subset W'$,

$$WF_h(u) \cap \hat{W} \subset p^{-1}([\alpha_0 - \delta_1 / 4, \alpha_1 + \delta_1 / 4]). \quad \quad \quad (8.4)$$

Let $\varphi_\pm$ be the functions constructed in Lemma 5.1. By Lemma 4.4, $u$ satisfies the conditions (see Figure 5)

$$WF_h(u) \cap \hat{W} \subset \{|\varphi_\pm| \leq \delta/2\}, \quad \quad \quad (8.5)$$

$$WF_h(u) \cap \Gamma^\infty \subset W'. \quad \quad \quad (8.6)$$

Indeed, if $\rho \in WF_h(u) \cap \mathcal{U}$, then either $\rho \in \Gamma_+$ (in which case (8.5) and (8.6) follow immediately) or there exists $T \geq 0$ such that for $\gamma(t) = e^{itH_\rho}(\rho)$, $\gamma([-T, 0]) \subset \widehat{\mathcal{U}}$ and $\gamma(-T) \in WF_h(f) \subset W'$. In the second case, if $\rho \in \widehat{\mathcal{W}}$, then by convexity of $U_\delta$ (part (5) of Lemma 5.1) we have $\gamma([-T, 0]) \subset \widehat{\mathcal{W}}$. To show (8.5), it remains to use that $H_\rho \varphi_+^2 \leq 0$ on $\widehat{\mathcal{W}}$, following from part (2) of Lemma 5.1. For (8.6), note that if $\rho \in \Gamma_-$, then $\gamma(-T) \in \Gamma_- \cap W'$; however, $e^{itH_\rho}(\Gamma_- \cap W') \subset \Gamma_- \cap W'$ for all $t \geq 0$ and thus $\rho \in W'$. 


By Lemma 4.6, we reduce (8.2) to
\[ \| A_1 u \|_{L^2} \leq C h^{-2} \| f \|_{L^2} + O(h^\infty), \] (8.7)
where \( A_1 \in \Psi^{\text{comp}}(X) \) is any compactly supported operator elliptic on \( W' \).

Now, let \( \Pi \in I^{\text{comp}}(\Lambda^o) \) be the operator constructed in Theorem 3 in §7.1. Note that
\[ (P - \omega) \Pi u = \Pi f + O(h^\infty) \] microlocally near \( \hat{W} \), (8.8)
since \([P, \Pi] = O(h^\infty)\) microlocally near \( \hat{W} \times \hat{W} \), \( \text{WF}_h(\Pi) \subset \Lambda^o \subset \Gamma_- \times \Gamma_+ \), and by (8.6).

We finally reduce (8.7) to the following two estimates, which are proved in the following subsections:

**Proposition 8.1.** Assume that \( u, f \) are \( h \)-tempered families satisfying (8.3)–(8.6) and
\[ \Re \omega \in [\alpha_0, \alpha_1], \quad \Im \omega \in [-(\nu_{\min} - \varepsilon)h, C_0 h]. \] (8.9)
Then there exists compactly supported \( A_1 \in \Psi^{\text{comp}}(X) \) elliptic on \( W' \) such that
\[ \| A_1 (1 - \Pi) u \|_{L^2} \leq C h^{-1} \| \Xi f \|_{L^2} + O(h^\infty), \] (8.10)
where \( \Xi \) is the operator from part (5) of Proposition 7.1; note that by part 1 of Proposition 6.13, \( \| \Xi \|_{L^2 \to L^2} = O(h^{-1}). \)

**Proposition 8.2.** Assume that \( u, f \) are \( h \)-tempered families satisfying (8.3)–(8.6) and
\[ \Re \omega \in [\alpha_0, \alpha_1], \quad \Im \omega \in [-C_0 h, C_0 h] \setminus \left( -\frac{\nu_{\max} + \varepsilon}{2} h, \frac{\nu_{\min} - \varepsilon}{2} h \right), \] (8.11)
Then there exists compactly supported \(A_1 \in \Psi^{\text{comp}}(X)\) elliptic on \(W'\) such that
\[
\|A_1 \Pi u\|_{L^2} \leq Ch^{-1}\|\Pi f\|_{L^2} + \mathcal{O}(h^{\infty}). \tag{8.12}
\]
Note that by Proposition 6.1 and the reduction to the model case of §6.2, we have
\[
\|\Pi\|_{L^2 \rightarrow L^2} = \mathcal{O}(h^{-1/2}).
\]

8.2. **Estimate on the kernel of \(\Pi\).** In this section, we prove Proposition 8.1, which is a microlocal estimate on the kernel of \(\Pi\) (or equivalently, on the image of \(1 - \Pi\)). We will use the identity (7.11) together with the commutator formula (7.9) to effectively shift the spectral parameter to the upper half-plane, where a standard positive commutator argument gives us the estimate.

By (8.8), we have microlocally near \(\hat{W}\),
\[
(P - \omega)(1 - \Pi)u = (1 - \Pi)f + \mathcal{O}(h^{\infty}) \tag{8.13}
\]
Let \(\Theta_- \in \Psi^{\text{comp}}(X)\) and \(\Xi\) be the operators constructed in Proposition 7.1, and denote
\[
v := \Xi u, \quad g := \Xi f.
\]
Then microlocally near \(\hat{W}\),
\[
(1 - \Pi)u = \Theta_- v, \quad (1 - \Pi)f = \Theta_- g. \tag{8.14}
\]
Indeed, by part 2 of Proposition 6.13, (8.6), and the fact that \(\text{WF}_h(\Pi) \subset \Lambda^0 \subset \Gamma_- \times \Gamma_+^c\), we see that \(1 - \Pi = \Theta_- \Xi + \mathcal{O}(h^{\infty})\) microlocally near \((\text{WF}_h(u) \setminus \hat{W}) \times \hat{W}\), since each of the featured operators is microlocalized away from this region. Combining this with (7.11), we see that \(1 - \Pi = \Theta_- \Xi + \mathcal{O}(h^{\infty})\) microlocally near \(\text{WF}_h(u) \times \hat{W}\), and thus also near \(\text{WF}_h(f) \times \hat{W}\), yielding (8.14).

By part 2 of Proposition 6.13 together with (8.4)–(8.6) and part (4) of Lemma 5.1,
\[
\text{WF}_h(v) \cup \text{WF}_h(g) \subset p^{-1}([a_0 - \delta_1/4, a_1 + \delta_1/4]), \tag{8.15}
\]
\[
(\text{WF}_h(v) \cup \text{WF}_h(g)) \cap \hat{W} \subset \{\varphi_+ \leq \delta/2\}. \tag{8.16}
\]
We now obtain a differential equation on \(v\); the favorable imaginary part of the operator in this equation, coming from commuting \(\Theta_-\) with \(P\), is the key component of the proof.

**Proposition 8.3.** Let \(Z_-\) be the operator from (7.9). Then microlocally near \(\hat{W}\),
\[
(P - i h Z_- - \omega)v = g + \mathcal{O}(h^{\infty}). \tag{8.17}
\]

**Proof.** Given (8.14), the equation (8.13) becomes \((P - \omega)\Theta_- v = \Theta_- g + \mathcal{O}(h^{\infty})\) microlocally near \(\hat{W}\). Using (7.9), we get microlocally near \(\hat{W}\),
\[
\Theta_-(P - i h Z_- - \omega)v = \Theta_- g + \mathcal{O}(h^{\infty}).
\]
To show (8.17), it remains to apply propagation of singularities (part 2 of Proposition 3.4), for the operator $\Theta_-$. Indeed, by part (4) of Lemma 5.1, for each $\rho \in \hat{W}$, there exists $t \geq 0$ such that the Hamiltonian trajectory $\{ e^{\i h\varphi_-} (\rho) | 0 \leq s \leq t \}$ lies entirely inside $\hat{W}$ and $e^{\i h\varphi_-} (\rho)$ lies in $\{ \varphi_+ = -\delta \}$ and by (8.16) does not lie in $WF_h((- P - i h Z_- - \omega) v - \Theta_- g)$.

We now use a positive commutator argument. Take a self-adjoint compactly supported $X_- \in \Psi_{\text{comp}}(X)$ such that $WF_h(X_-)$ is compactly contained in $\hat{W}$ and $\sigma(X_-) = \chi(\varphi_-)$ near $\hat{W} \cap WF_h(v)$, where $\varphi_-$ is defined in Lemma 5.1, $\chi \in C_0^\infty(-\delta, \delta)$, $s\chi'(s) \leq 0$ everywhere, and $\chi = 1$ near $[-\delta/2, \delta/2]$. This is possible by (8.15) and (8.16). Put $\text{Im} \omega = h\nu$; by (8.17) and since $P$ is self-adjoint,

$$\text{Im}(X_- v, g) = \frac{h}{2} \langle (Z^* X_- + X_- Z_- + 2\nu X_-) v, v \rangle + \frac{1}{2h} \langle [P, X_-] v, v \rangle + O(h^\infty) = h\langle Y_- v, v \rangle + O(h^\infty),$$

where $Y_- \in \Psi_{\text{comp}}(X)$ is compactly supported, $WF_h(Y_-) \subset WF_h(X_-) \subset \hat{W}$ and, using the function $c_-$ from part (2) of Lemma 5.1 together with part (3) of Proposition 7.1, we write near $\hat{W} \cap WF_h(v)$,

$$\sigma(Y_-) = (c_- + \nu)\chi(\varphi_-) - \frac{1}{2} H_p \chi(\varphi_-) = (c_- + \nu)\chi(\varphi_-) - \frac{1}{2} c_- \varphi_- \chi'(\varphi_-).$$

However, $\nu \geq - (\nu_{\text{min}} - \varepsilon)$ by (8.9) and by (5.9), $c_- > (\nu_{\text{min}} - \varepsilon)$ on $\hat{W}$; therefore

$$\sigma(Y_-) \geq 0 \quad \text{near } WF_h(v), \quad \sigma(Y_-) > 0 \quad \text{near } WF_h(v) \cap W'. \tag{8.19}$$

To take advantage of (8.19), we use the following combination of sharp Gårding inequality with propagation of singularities:

**Lemma 8.4.** Assume that $Z, Q \in \Psi_{\text{comp}}(X)$ are compactly supported, $WF_h(Z), WF_h(Q)$ are compactly contained in $\hat{W}$, $Z^* = Z$, and

$$\sigma(Z) \geq 0 \quad \text{near } WF_h(v), \quad \sigma(Z) > 0 \quad \text{near } WF_h(v) \cap W'. \tag{8.20}$$

Then

$$\|Qv\|^2_{L^2} \leq C \langle Zv, v \rangle + Ch^{-2} \|g\|^2_{L^2} + O(h^\infty). \tag{8.21}$$

**Proof.** Without loss of generality, we may assume that $Q$ is elliptic on $WF_h(Z) \cup W'$. There exists compactly supported $Q_1 \in \Psi_{\text{comp}}(X)$, elliptic on $W'$, such that $\sigma(Z - Q_1 Q_1) \geq 0$ near $WF_h(v)$ and $Q$ is elliptic on $WF_h(Q_1)$. Applying sharp Gårding inequality (Proposition 3.6) to $Z - Q_1^* Q_1$, we get

$$\|Q_1 v\|^2_{L^2} \leq C \langle Zv, v \rangle + Ch \|Qv\|^2_{L^2} + O(h^\infty).$$
Now, by (8.15), (8.16), and since $H_p\varphi^2 > 0$ on $\hat{\mathcal{W}} \setminus \Gamma_-$ by part (2) of Lemma 5.1, each backwards flow line of $H_p$ starting on $\text{WF}_h(Q)$ reaches either $\text{WF}_h(Q_1)$ or the complement of $\text{WF}_h(v)$, while staying in $\hat{\mathcal{W}}$; by propagation of singularities (Proposition 3.4) applied to (8.17),

$$\|Qv\|_{L^2} \leq C\|Q_1v\|_{L^2} + Ch^{-1}\|g\|_{L^2} + \mathcal{O}(h^\infty).$$

(8.22)

Combining (8.21) and (8.22), we get (8.20).

Now, there exists $A_1 \in \Psi^\text{comp}(X)$ compactly supported, elliptic on $\hat{\mathcal{W}}$ and with $\text{WF}_h(A_1)$ compactly contained in $\hat{\mathcal{W}}$, such that the estimate

$$|\langle \mathcal{X}_-v, g \rangle| \leq \bar{\varepsilon}h\|A_1v\|^2_{L^2} + C\bar{\varepsilon}h^{-1}\|g\|^2_{L^2} + \mathcal{O}(h^\infty)$$

(8.23)

holds for each $\bar{\varepsilon} > 0$ and constant $C_{\bar{\varepsilon}}$ dependent on $\bar{\varepsilon}$. Taking $\bar{\varepsilon}$ small enough and combining (8.18), (8.20) (for $Z = \mathcal{V}_-$ and $Q = A_1$), and (8.23), we arrive to

$$\|A_1v\|_{L^2} \leq Ch^{-1}\|g\|_{L^2} + \mathcal{O}(h^\infty).$$

Since $(1 - \Pi)u = \Theta_-v$ microlocally near $\hat{\mathcal{W}}$, we get (8.10).

8.3. Estimate on the image of $\Pi$. In this section, we prove Proposition 8.2, which is a microlocal estimate on the image of $\Pi$. We will use the pseudodifferential operator $\Theta_+$ microlocally solving $\Theta_+\Pi = \mathcal{O}(h^\infty)$ to obtain an additional pseudodifferential equation satisfied by elements of the image of $\Pi$. This will imply that for a pseudodifferential operator $A$ microlocalized near $K$, the principal part of the expression $\langle A\Pi u(h), \Pi u(h) \rangle$ depends only on the integral of $\sigma(A)$ over the flow lines of $\mathcal{V}_+$, with respect to an appropriately chosen measure. A positive commutator estimate finishes the proof.

By (8.8), we have microlocally near $\hat{\mathcal{W}}$,

$$(P - \omega)\Pi u = \Pi f + \mathcal{O}(h^\infty).$$

(8.24)

Let $\Theta_+ \in \Psi^\text{comp}(X)$ be the operator constructed in Proposition 7.1, then by (8.6),

$$\Theta_+\Pi u = \mathcal{O}(h^\infty) \quad \text{microlocally near} \quad \hat{\mathcal{W}}.$$

(8.25)

We start with

**Lemma 8.5.** Let $\zeta := \sigma(h^{-1} \text{Im} \Theta_+)$. Take the function $\psi$ on $\Gamma_+ \cap \hat{\mathcal{W}}$ such that

$$\{\varphi_+, \psi\} = 2\zeta, \quad \psi|_K = 0.$$

(8.26)

Assume that $A \in \Psi^\text{comp}(X)$ satisfies $\text{WF}_h(A) \subseteq \hat{\mathcal{W}}$ and

$$\int (e^{\psi}\sigma(A)) \circ e^{sH_+} \, ds = 0 \quad \text{on} \quad K.$$

(8.27)

The integral in (8.27), and all similar integrals in this subsection, is taken over the interval corresponding to a maximally extended flow line of $H_{\varphi_+}$ in $\Gamma_+ \cap \hat{\mathcal{W}}$. 
Then there exists compactly supported $A_0 \in \Psi^{\text{comp}}(X)$ with $\text{WF}_h(A_0) \subseteq \widehat{W}$ such that
\[
|\langle A\Pi u, \Pi u \rangle| \leq Ch\|A_0\Pi u\|_{L^2}^2 + \mathcal{O}(h^\infty).
\]

Proof. By (8.27), there exists $q \in C_0^\infty(\widehat{W})$ such that $\{\varphi_+, e^\psi q\} = e^\psi \sigma(A)$ on $\Gamma_+$. We can rewrite this as
\[
\{\varphi_+, q\} + 2\zeta q = \sigma(A) \quad \text{on } \Gamma_+.
\] (8.28)

Take $Q, Y \in \Psi^{\text{comp}}(X)$ microlocalized inside $\widehat{W}$ and such that $\sigma(Q) = q$ and
\[
\sigma(A) = \{\varphi_+, q\} + 2\zeta q + \sigma(Y)\varphi_+.
\]

Then $A = (ih)^{-1}(Q\Theta_+ - \Theta_+^* Q) + Y\Theta_+ + \mathcal{O}(h)\Psi^{\text{comp}}$ and thus for some $A_0$,
\[
\langle A\Pi u, \Pi u \rangle = \langle Q\Theta_+\Pi u, \Pi u \rangle - \langle Q\Pi u, \Theta_+\Pi u \rangle + \langle Y\Theta_+\Pi u, \Pi u \rangle + \mathcal{O}(h)\|A_0\Pi u\|_{L^2}^2 + \mathcal{O}(h^\infty).
\]

The first three terms on the right-hand side are $\mathcal{O}(h^\infty)$ by (8.25).

Now, take compactly supported self-adjoint $X_+ \in \Psi^{\text{comp}}(X)$ such that $\text{WF}_h(X_+)$ is compactly contained in $\widehat{W}$ and the symbol $\chi_+ := \sigma(X_+)$ satisfies $\chi_+ \geq 0$ everywhere, $\chi_+ > 0$ on $W'$, and
\[
\int (e^{\psi}\chi_+) \circ e^{sH_{\varphi_+}} \, ds = 1 \quad \text{on } K \cap p^{-1}([\alpha_0 - \delta_1/4, \alpha_1 + \delta_1/4]).
\] (8.29)

Putting $\text{Im } \omega = h\nu$, we have by (8.24)
\[
\text{Im} \langle X_+\Pi u, \Pi f \rangle = h\nu\langle X_+\Pi u, \Pi u \rangle + \frac{1}{2i}\langle [P, X_+]\Pi u, \Pi u \rangle + \mathcal{O}(h^\infty)
\]
\[
= h\langle \mathcal{Y}_+\Pi u, \Pi u \rangle + \mathcal{O}(h^\infty),
\] (8.30)

where $\mathcal{Y}_+ \in \Psi^{\text{comp}}(X)$ is compactly supported, $\text{WF}_h(\mathcal{Y}_+) \subset \text{WF}_h(X_+) \subset \widehat{W}$, and
\[
\sigma(\mathcal{Y}_+) = \nu\chi_+ - H_p\chi_+/2.
\]

We now want to use Lemma 8.5 together with Gårding inequality to show that $\langle \mathcal{Y}_+\Pi u, \Pi u \rangle$ has fixed sign, positive for $\nu \geq -(\nu_{\min} - \varepsilon)/2$ and negative for $\nu \leq -(\nu_{\max} + \varepsilon)/2$. For that, we need to integrate $\sigma(\mathcal{Y}_+)$ over the Hamiltonian flow lines of $\varphi_+$ on $\Gamma_+$, with respect to the measure from (8.27). This relies on

Lemma 8.6. If $c_+$ is defined in Lemma 5.1, then
\[
\int (e^{\psi}H_p\chi_+) \circ e^{sH_{\varphi_+}} \, ds = -c_+ \quad \text{on } K \cap p^{-1}([\alpha_0 - \delta_1/4, \alpha_1 + \delta_1/4]).
\] (8.31)
Proof. By part (2) of Lemma 5.1, we have on $\Gamma_+ \cap \hat{W}$

$$(e^{s H_{\varphi^+}})_* \partial_s (e^{-s H_{\varphi^+}})_* H_p = -[H_p, H_{\varphi^+}] = c_+ H_{\varphi^+}.$$  

Therefore, we can write (at $\rho \in K$ and $s$ such that $e^{s H_{\varphi^+}}(\rho) \in \hat{W}$)

$$(e^{-s H_{\varphi^+}})_* H_p = H_p + w(s) H_{\varphi^+} \quad \text{on } K$$

where $w(s)$ is the smooth function on $K \times \mathbb{R}$ given by

$$\partial_s w(s) = c_+ \circ e^{s H_{\varphi^+}}, \; w(0) = 0.$$

Now, differentiating (8.29) along $H_p$ and integrating by parts, we have on $K \cap p^{-1}([\alpha_0 - \delta_1/4, \alpha_1 + \delta_1/4])$

$$\int (H_p(e^s \chi_+)) \circ e^{s H_{\varphi^+}} \, ds = \int (H_p + w(s) \partial_s) ((e^s \chi_+) \circ e^{s H_{\varphi^+}}) \, ds$$

$$= - \int (e^s c_+ \chi_+) \circ e^{s H_{\varphi^+}} \, ds;$$

therefore,

$$\int (e^s H_p \chi_+) \circ e^{s H_{\varphi^+}} \, ds = - \int (e^s (c_+ + H_p \psi) \chi_+) \circ e^{s H_{\varphi^+}} \, ds. \quad (8.32)$$

Now, we find on $\Gamma_+ \cap \hat{W}$ by (8.26) and (7.10),

$$H_{\varphi^+} H_p \psi = (H_p + c_+) H_{\varphi^+} \psi = 2(H_p + c_+) \zeta = -H_{\varphi^+} c_+.$$

We have on $K \cap \hat{W}$, $H_p \psi = 0$; thus

$$c_+ + H_p \psi = c_+ \circ \pi_+ \quad \text{on } \Gamma_+ \cap \hat{W}$$

and by (8.32) and (8.29), on $K \cap p^{-1}([\alpha_0 - \delta_1/4, \alpha_1 + \delta_1/4])$,

$$\int (e^s H_p \chi_+) \circ e^{s H_{\varphi^+}} \, ds = - c_+ \int (e^s \chi_+) \circ e^{s H_{\varphi^+}} \, ds = - c_+.$$

This finishes the proof of (8.31). \qed

Using (8.29), (8.31), and Lemma 8.5 (taking into account (8.4)), we find for some compactly supported $A_1 \in \Psi^{\text{comp}}$ with $\text{WF}_h(A_1) \subset \hat{W}$ and $A_1$ elliptic on $W' \cup \text{WF}_h(\chi_+)$,

$$\langle \mathcal{Y}_+ \Pi u, \Pi u \rangle = \langle \mathcal{Z}_+ \Pi u, \Pi u \rangle + \mathcal{O}(h) \|A_1 \Pi u\|_{L^2}^2 + \mathcal{O}(h^\infty)$$

where $\mathcal{Z}_+ \in \Psi^{\text{comp}}(X)$ is any self-adjoint compactly supported operator with $\text{WF}_h(\mathcal{Z}_+) \subset \hat{W}$ and

$$\sigma(\mathcal{Z}_+) = (\nu + (c_+ \circ \pi_+)/2) \chi_+ \quad \text{on } \Gamma_+.$$

Then by (8.30),

$$\text{Im} \langle \mathcal{X}_+ \Pi u, \Pi f \rangle = h \langle \mathcal{Z}_+ \Pi u, \Pi u \rangle + \mathcal{O}(h^2) \|A_1 \Pi u\|_{L^2}^2 + \mathcal{O}(h^\infty). \quad (8.33)$$
Now, by (5.9), $\nu_{\text{min}} - \varepsilon < c_+ < \nu_{\text{max}} + \varepsilon$ on $K$, therefore, keeping in mind that $\text{WF}_h(\Pi u) \subset \Gamma_+$, we find

$$\sigma(Z_+) \geq 0 \text{ near } \text{WF}_h(\Pi u) \text{ for } \nu \geq - (\nu_{\text{min}} - \varepsilon)/2, \quad (8.34)$$

$$\sigma(Z_+) \leq 0 \text{ near } \text{WF}_h(\Pi u) \text{ for } \nu \leq - (\nu_{\text{max}} + \varepsilon)/2. \quad (8.35)$$

Moreover, in both cases $\sigma(Z_+) \neq 0$ on $\text{WF}_h(\Pi u) \cap W'$. We now combine sharp Gårding inequality and propagation of singularities for the operator $\Theta_+$:

**Lemma 8.7.** Assume that $Z, Q \in \Psi^\text{comp}(X)$ are compactly supported, $\text{WF}_h(Z), \text{WF}_h(Q)$ are compactly contained in $\hat{W}$, $Z^* = Z$, and

$$\sigma(Z) \geq 0 \text{ near } \text{WF}_h(\Pi u), \quad \sigma(Z) > 0 \text{ near } \text{WF}_h(\Pi u) \cap W'.$$

Then

$$\|Q \Pi u\|_{L^2}^2 \leq C\langle Z \Pi u, \Pi u \rangle + O(h^\infty). \quad (8.36)$$

**Proof.** We argue similarly to the proof of Lemma 8.4, with (8.22) replaced by

$$\|Q \Pi u\|_{L^2} \leq C\|Q_1 \Pi u\|_{L^2} + O(h^\infty). \quad (8.37)$$

The estimate (8.37) follows from propagation of singularities (Proposition 3.4) applied to (8.25). Indeed, by part (4) of Lemma 5.1 together with (8.4), for each $\rho \in \hat{W} \cap \text{WF}_h(\Pi u) \subset \Gamma_+$, there exists $t \in \mathbb{R}$ such that $e^{tH_{\rho}}(\rho) \in W'$ and $e^{sH_{\rho}}(\rho) \in \hat{W}$ for each $s$ between 0 and $t$. □

Using (8.36) (for $Z = \pm Z_+, Q = A_1$), (8.33), and an analog of (8.23), we complete the proof of (8.12).

### 8.4. Microlocalization in the spectral parameter.

In this section, we provide a restriction on the wavefront set of solutions to the equation $(P - \omega)u = f$ in the spectral parameter $\omega$, needed in §10. We use the method of §8.3, however since $\text{Re}\omega$ is now a variable, we will get an extra term coming from commutation with the multiplication operator by $\omega$. Because of the technical difficulties of studying operators on product spaces (namely, a pseudodifferential operator on $X$ does not give rise to a pseudodifferential operator on $X \times (\alpha_0, \alpha_1)$ since the corresponding symbol does not decay under differentiation in $\xi$ and thus does not lie in the class $S^k$ of §3.1), we use the Fourier transform in the $\omega$ variable.

**Proposition 8.8.** Fix $\nu \in [-C_0, C_0]$ and put $\omega = \alpha + ih\nu$, where $\alpha \in (\alpha_0, \alpha_1)$ is regarded as a variable. Assume that $u(x, \alpha; h) \in C([\alpha_0, \alpha_1]; H_1)$, $f(x, \alpha; h) \in C([\alpha_0, \alpha_1]; H_2)$ have norms bounded polynomially in $h$, satisfying (8.3)–(8.6) uniformly in $\alpha$. Define the semiclassical Fourier transform

$$\hat{u}(x, s; h) = \int_{\alpha_0}^{\alpha_1} e^{-i\alpha s} u(x, \alpha; h) \, d\alpha, \quad (8.38)$$
and \(\hat{f}(x,s;h)\) accordingly. Then there exists \(A_1 \in \Psi^{\text{comp}}(X)\) elliptic on \(W'\) such that:

1. If \(\nu \geq -(\nu_{\min} - \varepsilon)/2\), then for any fixed \(s_0 \in \mathbb{R}\),
   \[
   \|\Pi\hat{f}\|_{L^2_s((-\infty,s_0))L^2_x(X)} = \mathcal{O}(h^\infty) \implies \|A_1\Pi\hat{u}(s_0)\|_{L^2_x(X)} = \mathcal{O}(h^\infty). \tag{8.39}
   \]
2. If \(\nu \leq -(\nu_{\max} + \varepsilon)/2\), then for any fixed \(s_0 \in \mathbb{R}\),
   \[
   \|\Pi\hat{f}\|_{L^2_s(s_0,\infty)L^2_x(X)} = \mathcal{O}(h^\infty) \implies \|A_1\Pi\hat{u}(s_0)\|_{L^2_x(X)} = \mathcal{O}(h^\infty). \tag{8.40}
   \]

Proof. We consider case 1; case 2 is handled similarly using (8.35) instead of (8.34). Since \(u(\alpha), f(\alpha)\) are \(h\)-tempered uniformly in \(\alpha\), their Fourier transforms \(\hat{u}(s), \hat{f}(s)\) are \(h\)-tempered and satisfy (8.3)–(8.6) in the \(L^2\) sense in \(s\); therefore, the corresponding \(\mathcal{O}(h^\infty)\) errors will be bounded in \(L^2\) for expressions linear in \(\hat{u}, \hat{f}\) and in \(L^1_s\) for expressions quadratic in \(\hat{u}, \hat{f}\). We also note that for each \(j\), the derivatives \(\partial^j\hat{u}(s), \partial^j\hat{f}(s)\) are \(h\)-tempered uniformly in \(s \in \mathbb{R}\) and also in the \(L^2\) sense in \(s\).

Taking the Fourier transform of (8.8), we get
   \[
   (hD_s + P - ih\nu)\Pi\hat{u}(s) = \Pi\hat{f}(s) + \mathcal{O}(h^\infty)_{L^2_s(\mathbb{R})} \text{ microlocally near } \hat{W}. \tag{8.41}
   \]

We use the operators \(X_+, Z_+, A_1\) from \S 8.3. Similarly to (8.33), we find
\[
\text{Im} \langle X_+\Pi\hat{u}(s), \Pi\hat{f}(s) \rangle = \frac{h}{2} \partial_s \langle X_+\Pi\hat{u}(s), \Pi\hat{u}(s) \rangle \\
+ h \langle Z_+\Pi\hat{u}(s), \Pi\hat{u}(s) \rangle + \mathcal{O}(h^2)\|A_1\Pi\hat{u}(s)\|^2_{L^2_s} + \mathcal{O}(h^\infty)_{L^1_s(\mathbb{R})}.
\]

Integrating this over \(s \in (-\infty, s_0]\), by the assumption of (8.39), we find
\[
\langle X_+\Pi\hat{u}(s_0), \Pi\hat{u}(s_0) \rangle + 2 \int_{-\infty}^{s_0} \langle Z_+\Pi\hat{u}(s), \Pi\hat{u}(s) \rangle ds \\
\leq C h\|A_1\Pi\hat{u}(s)\|^2_{L^2_s((-\infty,s_0))L^2_x} + \mathcal{O}(h^\infty). \tag{8.42}
\]

Applying Lemma 8.7 to \(Q = A_1\) and \(Z = Z_+, X_+\), and using (8.34), we get
\[
\|A_1\Pi\hat{u}(s)\|^2_{L^2_x} \leq C \langle Z_+\Pi\hat{u}(s), \Pi\hat{u}(s) \rangle + \mathcal{O}(h^\infty)_{L^1_s(\mathbb{R})}, \tag{8.43}
\]
\[
\|A_1\Pi\hat{u}(s_0)\|^2_{L^2_x} \leq C \langle X_+\Pi\hat{u}(s_0), \Pi\hat{u}(s_0) \rangle + \mathcal{O}(h^\infty). \tag{8.44}
\]

Combining (8.42) with (8.43), integrated over \(s \in (-\infty, s_0]\), and (8.44), we get the conclusion of (8.39). \(\square\)

8.5. Localization of resonant states. In this section, we study an application of the estimates of the preceding subsections to microlocal behavior of resonant states, namely elements of the kernel of \(P(\omega)\) for a resonance \(\omega\). Assume that we are given a sequence \(h_j \to 0\), and \(\omega(h) \in \mathbb{C}, \hat{u}(h) \in \mathcal{H}_1\), defined for \(h\) in this sequence, such that
\[
P(\omega)\hat{u} = 0, \quad \|\hat{u}\|_{\mathcal{H}_1} = 1;
\]
\[
\text{Re} \omega \in [\alpha_0, \alpha_1], \quad \text{Im} \omega \in \{-(\nu_{\min} - \varepsilon)h, C_0h\}. \tag{8.45}
\]
the condition on $\omega$ is just (8.9). We also use the operators $S(\omega)$ and $P$ from Lemma 4.3 and put

$$u := S(\omega)\tilde{u},$$

so that

$$(P - \omega)u = O(h^\infty) \text{ microlocally near } U.$$  

(8.47)

We say that the sequence $u(h_j)$ converges to some Radon measure $\mu$ on $T^*X$, and we call $\mu$ the semiclassical defect measure of $u$ (see [Zw, Chapter 5]) if for each compactly supported $A \in \Psi^0(M)$, we have

$$\langle Au, u \rangle \to \int_{T^*M} \sigma(A) d\mu \text{ as } h_j \to 0.$$  

(8.48)

Such $\mu$ is necessarily a nonnegative measure, see [Zw, Theorem 5.2].

**Theorem 4.** Let $\tilde{u}(h)$ be a sequence of resonant states corresponding to some resonances $\omega(h)$, as in (8.45), and $u$ defined in (8.46). Take the neighborhood $\hat{W}$ of $K \cap p^{-1}([\alpha_0, \alpha_1])$ defined in (7.1). Then:

1. $\text{WF}_h(\tilde{u}) \cap U \subseteq \Gamma_+ \cap p^{-1}([\alpha_0, \alpha_1])$;
2. for each $A_1 \in \Psi^{\text{comp}}(X)$ elliptic on $K \cap p^{-1}([\alpha_0, \alpha_1])$, there exists a constant $c > 0$ independent of $h$ such that $\|A_1u\|_{L^2} \geq c$;
3. $u = \Pi u + O(h^\infty)$ and $\Theta_+ u = O(h^\infty)$ microlocally near $\hat{W}$, where $\Pi$ is constructed in Theorem 3 in §7.1 and $\Theta_+$ is the pseudodifferential operator from Proposition 7.1;
4. there exists a smooth family of smooth measures $\mu_\rho$, $\rho \in K \cap p^{-1}([\alpha_0, \alpha_1])$, on the flow line segments $\pi_+^{-1}(\rho) \cap \hat{W} \subset \Gamma_+$ of $V_+$, independent of the choice of $u$, such that if $u$ converges to some measure $\mu$ on $T^*M$ in the sense of (8.48), and $\text{Re } \omega(h_j) \to \omega_\infty$, $h^{-1} \text{Im } \omega(h_j) \to \nu$ as $h_j \to 0$, then $\mu|_{\hat{W}}$ has the form

$$\mu|_{\hat{W}} = \int_{K \cap p^{-1}(\omega_\infty)} \mu_\rho d\hat{\mu}(\rho),$$

(8.49)

for some nontrivial measure $\hat{\mu}$ on $K \cap p^{-1}(\omega_\infty)$, such that for each $b \in C^\infty(K),$

$$\int_{K \cap p^{-1}(\omega_\infty)} H_p b - (2\nu + c_+) b d\hat{\mu} = 0,$$

(8.50)

with the function $c_+$ defined in Lemma 5.1.

**Remark.** The equation (8.50) is similar to the equation satisfied by semiclassical defect measures for eigenstates for the damped wave equation, see [Zw, (5.3.21)].

**Proof.** Part (1) follows immediately from Lemma 4.4, part (2) follows from Lemma 4.6 and implies that $\mu|_{\hat{W}}$ is a nontrivial measure in part (4). By the discussion in §8.1, $u$ satisfies (8.3)–(8.6), with $f = 0$. The first statement of part (3) then follows from Proposition 8.1. Indeed, we have $(1 - \Pi)u = O(h^\infty)$ microlocally near the set $W'$
introduced in (8.1); it remains to apply propagation of singularities (Proposition 3.4) to (8.13), using Lemma 4.1. The second statement of part (3) now follows from (8.25).

Finally, we prove part (4). First of all, \( \mu|_{U} \) is supported on \( \Gamma_{+} \) by part (1), and on \( p^{-1}(\omega_{\infty}) \) by (8.47) and the elliptic estimate (Proposition 3.2; see also [Zw, Theorem 5.3]). Next, note that by Lemma 8.5 and since \( u = \Pi u + \mathcal{O}(h^{\infty}) \) microlocally near \( \hat{W} \), we have for each \( a \in C_{0}^{\infty}(\hat{W}) \) and the function \( \psi \) given by (8.26),

\[
\int (e^{\psi}a)(e^{sH_{+}}(\rho)) \, ds = 0 \quad \text{for all } \rho \in K \cap p^{-1}(\omega_{\infty}) \implies \int a \, d\mu = 0.
\]

This implies (8.49), with

\[
\int a \, d\mu_{\rho} := \int (e^{\psi}a)(e^{sH_{+}}(\rho)) \, ds, \quad a \in C_{0}^{\infty}(\hat{W}), \quad \rho \in K \cap p^{-1}(\omega_{\infty}).
\]

To see (8.50), we note that by (8.47), for each \( a \in C_{0}^{\infty}(\hat{W}) \) we have (see the derivation of [Zw, (5.3.21)])

\[
\int H_{p}a - 2na \, d\mu = 0. \tag{8.51}
\]

Put \( b(\rho) = \int a \, d\mu_{\rho} \) for \( \rho \in K \cap p^{-1}(\omega_{\infty}) \). Similarly to Lemma 8.6 (replacing 1 by \( b(\rho) \) on the right-hand side of (8.29)), we compute

\[
\int H_{p}a \, d\mu_{\rho} = H_{p}b(\rho) - c_{+}(\rho)b(\rho), \quad \rho \in K \cap p^{-1}(\omega_{\infty})
\]

and (8.50) follows by (8.51).

\[\square\]

9. **Grushin problem**

In this section, we construct a well-posed Grushin problem for the scattering resolvent, representing resonances in the region (8.9) as zeroes of a certain determinant \( F(\omega) \) defined in (9.25) below. Together with the trace formulas of §10, this makes possible the proof of the Weyl law in §11.

We assume that the conditions of §§4.1 and 5.1 hold, fix \( \varepsilon > 0 \) (to be chosen in Theorem 2), and use the neighborhoods \( W' \subset \hat{W} \) of \( K \cap p^{-1}([\alpha_{0}, \alpha_{1}]) \) defined in (8.1); let \( \delta, \delta_{1} > 0 \) be the constants used to define these neighborhoods. Take compactly supported \( Q_{1}, Q_{2} \in \Psi^{\comp}(X) \) such that (with \( U_{\delta} \) defined in Lemma 5.1)

\[
Q_{1} = 1 + \mathcal{O}(h^{\infty}) \quad \text{microlocally near } U_{\delta/4} \cap p^{-1}([\alpha_{0} - \delta_{1}/6, \alpha_{1} + \delta_{1}/6]),
\]

\[
Q_{2} = 1 + \mathcal{O}(h^{\infty}) \quad \text{microlocally near } U_{\delta/3} \cap p^{-1}([\alpha_{0} - \delta_{1}/5, \alpha_{1} + \delta_{1}/5]),
\]

\[
WF_{h}(Q_{1}) \subset U_{\delta/3} \cap p^{-1}([\alpha_{0} - \delta_{1}/5, \alpha_{1} + \delta_{1}/5]), \quad WF_{h}(Q_{2}) \subset W'.
\]

We will impose more restrictions on \( Q_{1} \) later in Lemma 9.2.
Using the operator $\mathcal{P}(\omega) : \mathcal{H}_1 \to \mathcal{H}_2$ from §4.1 and the operator $\mathcal{S}(\omega)$ constructed in Lemma 4.3, define the holomorphic family of operators

$$
\mathcal{G}(\omega) := \begin{pmatrix} \mathcal{P}(\omega) & \mathcal{S}(\omega)Q_1\Pi Q_2 \\ Q_1\Pi Q_2\mathcal{S}(\omega) & 1 - Q_1\Pi Q_2 \end{pmatrix} : \mathcal{H}_1 \oplus L^2(X) \to \mathcal{H}_2 \oplus L^2(X).
$$

Here $\Pi \in I_{\text{comp}}(\Lambda^\circ)$ is the operator constructed in Theorem 3 in §7.1; it is a microlocal idempotent commuting with the operator $P$ from Lemma 4.3 microlocally near the set $\hat{\Lambda} = \Lambda^\circ \cap (\hat{W} \cap \hat{W})$. Note that, since $Q_1, Q_2$ are microlocalized away from fiber infinity, $\mathcal{G}(\omega)$ is a compact perturbation of $\mathcal{P}(\omega) \oplus 1$, and therefore Fredholm of index zero.

In this section, we will prove

**Proposition 9.1.** There exists a global constant $^6 N$ such that for $\omega$ satisfying (8.9),

$$
\|\mathcal{G}(\omega)^{-1}\|_{\mathcal{H}_2 \oplus L^2 \to \mathcal{H}_1 \oplus L^2} = \mathcal{O}(h^{-N}).
$$

Moreover, if

$$
\mathcal{G}(\omega)^{-1} = \begin{pmatrix} \mathcal{R}_{11}(\omega) & \mathcal{R}_{12}(\omega) \\ \mathcal{R}_{21}(\omega) & \mathcal{R}_{22}(\omega) \end{pmatrix},
$$

then $\mathcal{R}_{22}(\omega) = 1 - L_{22}(\omega) + \mathcal{O}(h^\infty)_{\mathcal{D}' \to \mathcal{E}^*_0}$, where $L_{22}(\omega) \in I_{\text{comp}}(\Lambda^\circ)$ is microlocalized inside $\hat{\Lambda}$ and the symbol $\sigma_\Lambda(L_{22})$ defined in (6.30) satisfies

$$
\sigma_\Lambda(L_{22}(\omega))(\rho, \rho) = \frac{\sigma(Q_1)(\rho)^2 + (p(\rho) - \omega)\sigma(Q_1)(\rho)}{\sigma(Q_1)(\rho)^2 + (p(\rho) - \omega)(\sigma(Q_1)(\rho) - 1)}, \quad \rho \in \hat{K}.
$$

To prove (9.2), we consider families of distributions $u(h) \in \mathcal{H}_1$, $f(h) \in \mathcal{H}_2$, $v(h), g(h) \in L^2(X)$, bounded polynomially in $h$ in the indicated spaces and satisfying $\mathcal{G}(u, v) = (f, g)$, namely

$$
P(\omega)u + \mathcal{S}(\omega)Q_1\Pi Q_2v = f,
$$

$$
Q_1\Pi Q_2\mathcal{S}(\omega)u + (1 - Q_1\Pi Q_2)v = g.
$$

Note that by (4.9), (9.5) implies

$$
(P - \omega)\mathcal{S}(\omega)u + Q_1\Pi Q_2v = \mathcal{S}'(\omega)f + \mathcal{O}(h^\infty) \quad \text{microlocally near } \mathcal{U}.
$$

Here $\mathcal{S}'(\omega)$ is an elliptic parametrix of $\mathcal{S}(\omega)$ near $\mathcal{U}$ constructed in Proposition 3.3.

To show (9.2), it is enough to establish the bound

$$
\|u\|_{\mathcal{H}_1} + \|v\|_{L^2} \leq C h^{-N}(\|f\|_{\mathcal{H}_2} + \|g\|_{L^2}) + \mathcal{O}(h^\infty).
$$

We start with a technical lemma:

**Lemma 9.2.** There exists $Q_1 \in \Psi_{\text{comp}}(X)$ satisfying (9.1) and such that

$$
\sigma(Q_1)^2 + (p - \omega)(\sigma(Q_1) - 1) \neq 0 \quad \text{on } K \text{ for all } \omega \in [\alpha_0, \alpha_1].
$$

---

6A more careful analysis, as in §8, could give the optimal value of $N$; we do not pursue this here since the value of $N$ is irrelevant for our application in §11.
Proof. It suffices to take $Q_1$ such that $\sigma(Q_1)|_K = \psi(p)$, where $\psi \in C^\infty_0(\alpha_0 - \delta_1/5, \alpha_1 + \delta_1/5)$ is equal to 1 near $[\alpha_0 - \delta_1/6, \alpha_1 + \delta_1/6]$ and
\[
\psi(\lambda)^2 + (\lambda - \omega)(\psi(\lambda) - 1) \neq 0, \quad \lambda \in \mathbb{R}, \; \omega \in [\alpha_0, \alpha_1]. \tag{9.10}
\]
We now show that such $\psi$ exists. The equation (9.10) holds automatically for $\lambda \notin (\alpha_0 - \delta_1/5, \alpha_1 + \delta_1/5)$, as $\psi = 0$ there and the left-hand side of (9.10) equals $\omega - \lambda \neq 0$. This however also shows that a real-valued $\psi$ with the desired properties does not exist. We take $\Re \psi \in C^\infty_0(\alpha_0 - \delta_1/5, \alpha_1 + \delta_1/5)$ equal to 1 near $[\alpha_0 - \delta_1/6, \alpha_1 + \delta_1/6]$ and take values in $[0, 1]$ and $\Im \psi \in C^\infty_0(\alpha_1 + \delta_1/6, \alpha_1 + \delta_1/5)$ a nonnegative function to be chosen later. Then the left-hand side of (9.10) is equal to 1 for $\lambda \in [\alpha_0 - \delta_1/6, \alpha_1 + \delta_1/6]$ and is positive for $\lambda \in [\alpha_0 - \delta_1/5, \alpha_0 - \delta_1/6]$. Next, the imaginary part of (9.10) is
\[
\Im \psi(\lambda)(2 \Re \psi(\lambda) + \lambda - \omega).
\]
Since $2 \Re \psi(\lambda) + \lambda - \omega > 0$ for $\lambda \in [\alpha_1 + \delta_1/6, \alpha_1 + \delta_1/5]$, it remains to take $\Im \psi(\lambda) > 0$ on a large compact subinterval of $(\alpha_1 + \delta_1/6, \alpha_1 + \delta_1/5)$; then $\psi$ satisfies (9.10). \qed

Using Lemma 9.2, we determine $v$ microlocally outside of the elliptic region:

**Proposition 9.3.** Let $Q_1$ be chosen in Lemma 9.2. Then there exist $L^e_{21}(\omega), L^e_{22}(\omega) \in I^\comp_{\Lambda^0}(\Lambda^0)$ holomorphic in $\omega$, microlocalized inside $\hat{\Lambda}$, and such that for all $u, v, f, g$ satisfying (9.5), (9.6),
\[
v = L^e_{21}f + (1 - L^e_{22})g \tag{9.11}
\]
microlocally outside of $\Gamma_+ \cap \widehat{W} \cap p^{-1}([\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8])$. Moreover, $\sigma_\Lambda(L^e_{22})$ satisfies (9.4) for $\rho \notin p^{-1}([\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8])$.

Proof. Using Proposition 3.3, construct compactly supported $R^e(\omega) \in \Psi^\comp(X)$ such that $R^e(\omega)(P - \omega) = 1 + O(h^\infty)$ microlocally near $\widehat{W} \setminus p^{-1}(\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8)$. By (9.7), we get
\[
S(\omega)u = R^e(\omega)(S'(\omega)f - Q_1\Pi Q_2v) + O(h^\infty)
\]
microlocally near $\widehat{W} \setminus p^{-1}(\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8)$. Substituting this into (9.6), we get
\[
(1 - L')v = g - Q_1\Pi Q_2R^e(\omega)S'(\omega)f + O(h^\infty) \tag{9.12}
\]
microlocally outside of $\Gamma_+ \cap \widehat{W} \cap p^{-1}([\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8])$, where $L' = Q_1\Pi Q_2(1 + R^e(\omega)Q_1\Pi Q_2)$ \in $I^\comp(\Lambda^0)$ and $\WF_h(L') \subset \hat{\Lambda}$.

Let $\sigma_\Lambda(L')$ be the symbol of $L'$, defined in (6.30). By (6.31)–(6.33), and since $\sigma_\Lambda(\Pi)|_K = 1$ near $\widehat{W}$ (see part 1 of Proposition 6.9 or §7.1), we find for $\rho \in K \setminus p^{-1}(\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8)$,
\[
\sigma_\Lambda(L'(\rho, \rho) = \sigma(Q_1)(\rho)(1 + \sigma(Q_1)(\rho)/(p(\rho) - \omega));
\]

it follows from (9.9) that
\[
\sigma_\Lambda(L'|_K \neq 1 \quad \text{outside of } p^{-1}(\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8). \tag{9.13}
\]
By Proposition 6.6, there exists $L_{\S2}^2(\omega) \in \mathcal{I}_{\text{comp}}(\Lambda^c)$, with $\text{WF}_h(L_{\S2}^2) \subset \hat{\Lambda}$, such that $(1 - L_{\S2}^2)(1 - L') = 1 + \mathcal{O}(h^\infty)$ microlocally outside of $p^{-1}([\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8])$, and note that the symbol $\sigma_{\Lambda}(L_{\S2}^2)$ satisfies (9.4) for $\rho \notin p^{-1}([\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8])$ by (6.34). By (9.12), we get (9.11) with $L_{\S12}^2(\omega) = -(1 - L_{\S2}^2(\omega))Q_1\Pi Q_2 R^e(\omega)S'(\omega)$. \hfill \Box

By Proposition 9.3, replacing $v$ by $A_v$, where $A_v \in \Psi^{\text{comp}}(X)$ is compactly supported, $\text{WF}_h(A_v) \subset \mathcal{U} \cap p^{-1}([\alpha_0 - \delta_1/7, \alpha_1 + \delta_1/7])$, and $A_v = 1 + \mathcal{O}(h^\infty)$ near $\hat{\mathcal{W}} \cap p^{-1}([\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8])$, we see that it is enough to prove (9.8) in the case

$$\text{WF}_h(v) \subset \mathcal{U} \cap p^{-1}([\alpha_0 - \delta_1/7, \alpha_1 + \delta_1/7]).$$

(9.14)

Using Lemma 4.5, consider $u' \in \mathcal{H}_1$ such that $\|u'\|_{\mathcal{H}_1} \leq C h^{-1} \|f\|_{\mathcal{H}_2}$ and $\text{WF}_h(\mathcal{P}(\omega)u' - f) \subset \text{WF}_h(Q_1) \cap p^{-1}([\alpha_0 - \delta_1/7, \alpha_1 + \delta_1/7])$. Subtracting $u'$ from $u$, we see that is suffices to prove (9.8) for the case

$$\text{WF}_h(f) \subset \text{WF}_h(Q_1) \cap p^{-1}([\alpha_0 - \delta_1/7, \alpha_1 + \delta_1/7]).$$

(9.15)

By (9.14), the wavefront set of $\mathcal{P}(\omega)u = f - \mathcal{S}(\omega)Q_1 \Pi Q_2 u$ satisfies (9.15). Arguing as in §8.1, and keeping in mind (9.7), we see that $u$ satisfies (8.4)–(8.6); in fact, (8.4) can be strengthened to

$$\text{WF}_h(u) \cap \hat{\mathcal{W}} \subset p^{-1}([\alpha_0 - \delta_1/7, \alpha_1 + \delta_1/7]).$$

(9.16)

and (8.6) can be strengthened to

$$\text{WF}_h(u) \cap \Gamma^c_{\delta/3} \subset U_{\delta/3} \cap p^{-1}([\alpha_0 - \delta_1/7, \alpha_1 + \delta_1/7]).$$

(9.17)

We can now solve for $v$:

**Proposition 9.4.** Assume that $u, v, f, g$ satisfy (9.5), (9.6), (9.14), (9.15). Then

$$v = Q_1\Pi S'(\omega)f + (1 - Q_1(P - \omega + 1)\Pi Q_2)g + \mathcal{O}(h^\infty)c_0^\infty.$$  

(9.18)

**Proof.** Since $\Pi^2 = \Pi + \mathcal{O}(h^\infty)$ microlocally near $\hat{\mathcal{W}} \times \hat{\mathcal{W}}$ and $Q_1 = 1 + \mathcal{O}(h^\infty)$ microlocally near $K \cap p^{-1}([\alpha_0 - \delta_1/6, \alpha_1 + \delta_1/6])$, we have

$$\Pi Q_1 \Pi = \Pi + \mathcal{O}(h^\infty) \quad \text{microlocally near } (\mathcal{W} \cap p^{-1}([\alpha_0 - \delta_1/6, \alpha_1 + \delta_1/6])) \times \hat{\mathcal{W}}. \quad (9.19)$$

We rewrite (9.6) as

$$Q_1\Pi Q_2(\mathcal{S}(\omega)u - g) + (1 - Q_1\Pi Q_2)(v - g) = 0. \quad (9.20)$$

It follows immediately that $\text{WF}_h(v - g) \subset \text{WF}_h(Q_1)$ and thus $Q_2(v - g) = v - g + \mathcal{O}(h^\infty)c_0^\infty$. Also, by (9.6), (9.14), and (9.16), $\text{WF}_h(g) \subset \mathcal{U} \cap p^{-1}([\alpha_0 - \delta_1/7, \alpha_1 + \delta_1/7])$. Applying $\Pi$ to (9.20) and using (9.14), (9.16), and (9.19), we get $\Pi Q_2 S(\omega)u - \Pi Q_2 g = \mathcal{O}(h^\infty)$ microlocally near $\hat{\mathcal{W}}$. By (9.17), we have $\Pi Q_2 S(\omega)u = \Pi S(\omega)u + \mathcal{O}(h^\infty)c_0^\infty$; therefore,

$$\Pi S(\omega)u = \Pi Q_2 g + \mathcal{O}(h^\infty) \quad \text{microlocally near } \hat{\mathcal{W}}. \quad (9.21)$$
Then (9.20) becomes
\[ v = Q_1 \Pi Q_2 v + (1 - Q_1 \Pi Q_2)g + \mathcal{O}(h^\infty)_{C_0^\infty}. \] (9.22)
Applying \( \Pi \) to (9.7), using that \([P, \Pi] = \mathcal{O}(h^\infty)\) microlocally near \( \hat{W} \times \hat{W} \), and keeping in mind (9.17), we get
\[ (P - \omega)\Pi S(\omega)u + \Pi Q_2 v = \Pi S'(\omega)f + \mathcal{O}(h^\infty) \] microlocally near \( \hat{W} \). (9.23)
Together, (9.21) and (9.23) give
\[ \Pi Q_2 v = \Pi S'(\omega)f - (P - \omega)\Pi Q_2 g + \mathcal{O}(h^\infty) \] microlocally near \( \hat{W} \).
By (9.22), we now get (9.18).

By Proposition 9.4, we see that
\[ \|v\|_{L^2} \leq Ch^{-N}(\|f\|_{H^2} + \|g\|_{L^2}) + \mathcal{O}(h^\infty). \] (9.24)
By Proposition 8.1 (using (9.7) instead of (8.3)), we get for some \( A_1 \in \Psi^{\text{comp}}(X) \) elliptic near \( W' \),
\[ \|A_1(1 - \Pi)S(\omega)u\|_{L^2} \leq Ch^{-N}(\|f\|_{H^2} + \|g\|_{L^2}) + \mathcal{O}(h^\infty). \]
Combining this with (9.21), we estimate \( \|A_1u\|_{L^2} \) by the right-hand side of (9.24).
Applying Lemma 4.6 to (9.5), we can estimate \( \|u\|_{H^1} \) by the same quantity, completing the proof of (9.8).

It remains to describe the operator \( R_{22} \) from (9.3). We assume that \( u, v, f, g \) satisfy (9.5), (9.6) and \( f = 0 \); then \( R_{22}g = v \). By Proposition 9.3, \( v = (1 - L_{22}')g + \mathcal{O}(h^\infty) \) microlocally outside of \( \hat{W} \cap p^{-1}(\alpha_0 - \delta_1/8, 1 + \delta_1/8) \); it then suffices to describe \( v \) microlocally near \( \hat{W} \cap p^{-1}(\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8) \). Let \( A_\epsilon \) be the operator introduced before (9.14) and \( R'(\omega) \) be an elliptic parametrix for \( P - \omega \) constructed in the proof of Proposition 9.3. Replacing \( (u, v) \) by \( (u + S'(\omega)R'(\omega)Q_1 \Pi Q_2(1 - A_\epsilon)v, A_\epsilon v) \), we may assume that (9.14) and (9.15) hold, and in fact the resulting \( f \) is \( \mathcal{O}(h^\infty)_{C_0^\infty} \) and the resulting \( g \) coincides with the original \( g \) microlocally near \( \hat{W} \cap p^{-1}(\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8) \).
By Proposition 9.4, we now get for the original \( v \) and \( g \),
\[ v = (1 - Q_1(P - \omega + 1)\Pi Q_2)g + \mathcal{O}(h^\infty) \] microlocally near \( \hat{W} \cap p^{-1}(\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8) \).

Note that \( Q_1(P - \omega + 1)\Pi Q_2 \in I_{\text{comp}}(\Lambda^0) \) and its principal symbol satisfies (9.4) in \( p^{-1}(\alpha_0 - \delta_1/8, \alpha_1 + \delta_1/8) \), since \( \sigma(Q_1)|_\lambda = 1 \) in that region. This finishes the proof of Proposition 9.1.

By Proposition 9.1, \( R_{22}(\omega) - 1 \) is a compactly supported operator mapping \( H^{-N}_h \to H^N_h \) for all \( N \), therefore it is trace class. We can then define the determinant (see for instance [Ta, (A.6.38)])
\[ F(\omega) := \det R_{22}(\omega), \] (9.25)
which is holomorphic in the region (8.9) and $F(\omega) = 0$ if and only if $R_{22}(\omega)$ is not invertible (see [Ta, Proposition A.6.16]). The key properties of $F$ needed in §11 are established in

**Proposition 9.5.** 1. Resonances in the region (8.9) coincide (with the multiplicities defined in (4.3)) with zeroes of $F(\omega)$.

2. For some constants $C$ and $N$, we have $|F(\omega)| \leq e^{Ch^{-N}}$ for $\omega$ in (8.9), and $|F(\omega)| \geq e^{ch^{-N}}$ for $\omega$ in the resonance free region (1.5).

3. For $\omega$ in the resonance free region (1.5), we have

$$\frac{\partial_\omega F(\omega)}{F(\omega)} = - \text{Tr}((1 - Q_1\Pi Q_2 - Q_1\Pi S(\omega)\mathcal{R}(\omega)S(\omega)Q_1\Pi Q_2)\partial_\omega L_{22}(\omega)) + O(h^\infty).$$

Here $L_{22}(\omega)$ is defined in Proposition 9.1.

**Proof.** 1. By Schur’s complement formula [Zw, (D.1.1)], and since $G(\omega)$ is invertible by Proposition 9.1, we know that $P(\omega)$ is invertible if and only if $R_{22}(\omega)$ is, and in fact

$$P(\omega)^{-1} = R_{11}(\omega) - R_{12}(\omega)R_{22}(\omega)^{-1}R_{21}(\omega). \quad (9.26)$$

To see that the multiplicity of a resonance $\omega_0$ defined by (4.3) coincides with the multiplicity of $\omega_0$ as a zero of the function $F(\omega)$ (and in particular, to demonstrate that the multiplicity defined by (4.3) is a positive integer), it is enough to show that

$$\frac{1}{2\pi i} \int_{\omega_0} P(\omega)^{-1}\partial_\omega P(\omega) \, d\omega = \frac{1}{2\pi i} \int_{\omega_0} R_{22}(\omega)^{-1}\partial_\omega R_{22}(\omega) \, d\omega; \quad (9.27)$$

indeed, since $\partial_\omega R_{22}(\omega)$ is trace class, we can put the trace inside the integral on the right-hand side of (9.27), yielding $\partial_\omega F(\omega)/F(\omega)$; therefore, the right-hand side gives the multiplicity of $\omega_0$ as a zero of $F(\omega)$ by the argument principle.

Since $\partial_\omega (G(\omega)^{-1}) = -G(\omega)^{-1}(\partial_\omega G(\omega))G(\omega)^{-1}$, we have

$$\partial_\omega R_{22}(\omega) = -R_{21}(\omega)(\partial_\omega P(\omega))R_{12}(\omega) + A(\omega)R_{22}(\omega) + R_{22}(\omega)B(\omega),$$

where $A(\omega), B(\omega) : L^2(X) \to L^2(X)$ are bounded operators holomorphic at $\omega_0$. By (9.26), (9.27) follows from the two identities

$$\text{Tr} \int_{\omega_0} R_{12}(\omega)R_{22}(\omega)^{-1}R_{21}(\omega)\partial_\omega P(\omega) \, d\omega = \text{Tr} \int_{\omega_0} R_{22}(\omega)^{-1}R_{21}(\omega)(\partial_\omega P(\omega))R_{12}(\omega) \, d\omega,$$

$$\text{Tr} \int_{\omega_0} R_{22}(\omega)^{-1}(A(\omega)R_{22}(\omega) + R_{22}(\omega)B(\omega)) \, d\omega = 0.$$

Both of them follow from the cyclicity of the trace, replacing $R_{22}(\omega)^{-1}$ by its finite-dimensional principal part at $\omega_0$ and putting the trace inside the integral.

2. By Proposition 9.1, the trace class norm $\|R_{22}(\omega) - 1\|_\text{Tr}$ is bounded polynomially in $h$. Using the bound $|\det(1 + T)| \leq e^{\|T\|_\text{Tr}}$ (see for example [Ta, (A.6.44)]), we get
\[ |F(\omega)| \leq e^{C h^{-N}}. \] By Theorem 1, we have \( \|R(\omega)\|_{H_2\to H_1} \leq C h^{-2} \) when \( \omega \) satisfies (1.5).

Using Schur’s complement formula again, we get
\[ R_{22}(\omega)^{-1} = 1 - Q_1 \Pi Q_2 - Q_1 \Pi Q_2 S(\omega) R(\omega) S(\omega) Q_1 \Pi Q_2. \] (9.28)

Then \( \|R_{22}(\omega)^{-1} - 1\|_\text{Tr} \leq C h^{-N} \) and thus \( |F(\omega)|^{-1} = |\det(R_{22}(\omega)^{-1})| \leq e^{C h^{-N}}. \)

3. By Proposition 9.1, we have \( \partial_\omega R_{22}(\omega) = -\partial_\omega L_{22}(\omega) + O(h^\infty)_{D^*\to C^\infty}, \) thus
\[ \frac{\partial_\omega F(\omega)}{F(\omega)} = -\text{Tr}(R_{22}(\omega)^{-1} \partial_\omega L_{22}(\omega)) + O(h^\infty). \]

By (9.28), it then suffices to prove that
\[ \text{Tr}(Q_1 \Pi (1 - Q_2) S(\omega) R(\omega) S(\omega) Q_1 \Pi Q_2 \partial_\omega L_{22}(\omega)) = O(h^\infty). \]

For that, it suffices to show that the intersection of the wavefront set of the operator on the left-hand side with the diagonal in \( T^*X \) is empty. We assume the contrary, then there exists \( \rho \in T^*X \) such that
\[ (\rho, \rho) \in \text{WF}_h(Q_1 \Pi (1 - Q_2) S(\omega) R(\omega) S(\omega) Q_1 \Pi Q_2 \partial_\omega L_{22}(\omega)). \]

Since both \( \Pi \) and \( \partial_\omega L_{22} \) are microlocalized inside \( \Lambda^0 \subset \Gamma^0 \cap \Gamma^0_+ \), we see that \( \rho \in K^0 = \Gamma^0_+ \cap \Gamma^0_\infty \). There exists \( \rho' \in T^*X \) such that
\[ (\rho, \rho') \in \text{WF}_h(S(\omega) R(\omega) S(\omega) Q_1 \Pi Q_2 \partial_\omega L_{22}(\omega)), \quad (\rho', \rho) \in \text{WF}_h(Q_1 \Pi (1 - Q_2)). \]

For any \( h \)-tempered \( f \in L^2(X) \), we have \( \text{WF}_h(S(\omega) Q_1 \Pi Q_2 \partial_\omega L_{22}(\omega) f) \subset \Gamma^0_+ \cap \hat{W} \), therefore by Lemma 4.4 we have \( \text{WF}_h(R(\omega) S(\omega) Q_1 \Pi Q_2 \partial_\omega L_{22}(\omega) f) \cap U \subset \Gamma_+ \). It follows that \( \rho' \in \Gamma_+ \). Since \( (\rho', \rho) \in \text{WF}_h(Q_1 \Pi (1 - Q_2)) \), we see that \( \rho' = \rho \in K^0 \). However, then \( \rho \in \text{WF}_h(Q_1) \cap \text{WF}_h(1 - Q_2) \), which is impossible since \( Q_2 = 1 + O(h^\infty) \) microlocally near \( \text{WF}_h(Q_1) \).

\[ \square \]

10. Trace formula

In this section, we establish an asymptotic expansion for contour integrals of the logarithmic derivative of the determinant \( F(\omega) \) of the effective Hamiltonian of the Grushin problem of §9, defined in (9.25). By Proposition 9.5, this reduces to computing contour integrals of operators of the form \( \Pi R(\omega) \), where \( \Pi \) is the projector constructed in Theorem 3 in §7.1. This in turn is done by approximating \( R(\omega) \) microlocally on the image of \( \Pi \) by pseudodifferential operators, using Schrödinger propagators and microlocalization in the spectral parameter established in §8.4.

We operate under the pinching condition (1.7) of Theorem 2, namely \( \nu_{\text{max}} < 2\nu_{\text{min}} \), and choose \( \varepsilon > 0 \) such that \( \nu_{\text{max}} + \varepsilon < 2(\nu_{\text{min}} - \varepsilon) \). Take \( \chi \in C^\infty_0(\alpha_0, \alpha_1) \) with \( \alpha_0, \alpha_1 \) from (4.1). Consider an almost analytic extension \( \tilde{\chi}(\omega) \) of \( \chi \), that is \( \tilde{\chi} \in C^\infty(\mathbb{C}) \) such that \( \tilde{\chi}|_\mathbb{R} = \chi \) and \( \partial_\omega \tilde{\chi}(\omega) = O(|\text{Im} \omega|^{\infty}) \). We may take \( \tilde{\chi} \) such that \( \text{supp}(\tilde{\chi}) \subset \{ \text{Re} \omega \in (\alpha_0, \alpha_1) \} \).
The main result of this section is

**Proposition 10.1.** Take

\[ \nu_\pm \in \left[ - (\nu_{\min} - \varepsilon), \frac{\nu_{\max} + \varepsilon}{2} \right], \quad \nu_+ \in \left[ - \frac{\nu_{\min} - \varepsilon}{2}, C_0 \right]. \quad (10.1) \]

Let \( F(\omega) \) be defined in (9.25) and put

\[ I^\pm := (2\pi h)^{n-1} \int_{\text{Im} \omega = h\nu_\pm} \bar{\chi}(\omega) \frac{\partial_\omega F(\omega)}{F(\omega)} \, d\omega. \quad (10.2) \]

Then, with \( d \text{Vol}_\sigma = \sigma^{n-1} / (n-1)! \) the symplectic volume form,

\[ I^- + I^+ = 2\pi i \int_K \chi(p) \, d \text{Vol}_\sigma + O(h). \quad (10.3) \]

**Remark.** More precise trace formulas are possible; in particular, one can get a full asymptotic expansion in \( h \) of each of \( I^\pm \). For simplicity, we prove here a less general version which suffices for the analysis of §11.

The key feature of the expansions for the integrals (10.2), which produces a nontrivial asymptotics for resonances in Theorem 2, is that the principal part of \( I^\pm \) depends on the sign of \( \pm \). The reason for this dependence is the difference of directions for propagation in the resolvent approximation \( R^\pm \psi(\omega) \) of Proposition 10.2 for the two cases; this in turn is explained by the difference between (8.39) and (8.40), which is due to the difference of the signs of the ‘commutator’ \( Z_+ \) between (8.34) and (8.35).

We start the proof by using Proposition 8.8 to replace \( R(\omega) \) in the formula for \( \partial_\omega F(\omega) / F(\omega) \) from Proposition 9.5 by an operator \( R^\pm(\omega) \) obtained by integrating the Schrödinger propagator \( e^{-it(P-\omega)/h} \) over a bounded range of times \( t \).

**Proposition 10.2.** Fix \( \psi \in C^\infty_0(\mathbb{R}) \) such that \( \psi = 1 \) near zero. For \( \omega \in \mathbb{C} \), define the operators \( R^\pm(\omega) : L^2(X) \to L^2(X) \) by

\[ R^+(\omega) := \frac{i}{h} \int_{-\infty}^{0} e^{is(P-\omega)/h} \psi(s) \, ds; \quad (10.4) \]

\[ R^-(\omega) := -\frac{i}{h} \int_{0}^{\infty} e^{is(P-\omega)/h} \psi(s) \, ds. \quad (10.5) \]

Then, if \( \text{supp} \, \psi \) is contained in a small enough neighborhood of zero,

\[ I^\pm = -(2\pi h)^{n-1} \text{Tr} \int_{\text{Im} \omega = h\nu_\pm} \bar{\chi}(\omega)(1 - Q_1 \Pi Q_2) \]

\[- Q_1 R^\pm(\omega) \Pi Q_1 \Pi Q_2 \partial_\omega L_{22}(\omega) \, d\omega + O(h^{\infty}). \quad (10.6) \]
Proof. We concentrate on the case of $\mathcal{I}^+_\chi$, the case of $\mathcal{I}^-_\chi$ is handled similarly, using (8.40) in place of (8.39). We denote $\omega = \alpha + i\nu_\pm$, where $\alpha \in (\alpha_0, \alpha_1)$. By part 3 of Proposition 9.5, it suffices to prove the trace norm bound

$$\left\| \int_{\text{Im}\omega = h\nu_+} \check{\chi}(\omega)Q_1(\Pi S(\omega)R(\omega)S(\omega) - R^+_\psi(\omega)\Pi)Q_1\Pi \partial_{\omega} L_{22}(\omega) \, d\omega \right\|_{\text{Tr}} = O(h^\infty).$$

Since the operator on the left-hand side is compactly supported and microlocalized away from the fiber infinity, it is enough to prove an estimate of the $L^2$ operator norm instead of the trace class norm. Take arbitrary $h$-independent family $\tilde{f} = \tilde{f}(h) \in L^2(X)$ with $\|\tilde{f}\|_{L^2} \leq 1$ and put

$$f(\alpha) := \check{\chi}(\omega)Q_1\Pi Q_2 \partial_{\omega} L_{22}(\omega)\tilde{f}, \quad u(\alpha) := S(\omega)R(\omega)S(\omega)f(\alpha).$$

Then $f(x, \alpha)$ is compactly supported in both $x \in X$ and $\alpha \in (\alpha_0, \alpha_1)$, $\|f\|_{L^2_\alpha L^2_x}$ is polynomially bounded in $h$, and $\text{WF}_h(f(\alpha)) \subset \mathbb{I} \cap W'$. Since $R(\omega)|_{\mathcal{H}_2 \to \mathcal{H}_1} = O(h^{-2})$ by Theorem 1, we see that $u(\alpha) \in \mathcal{H}_2$ is compactly supported in $\alpha \in (\alpha_0, \alpha_1)$ and the norm $\|u\|_{L^2_\alpha L^2_x}$ is bounded polynomially in $h$. Using Lemma 4.4 similarly to §8.1, we see that $u, f$ satisfy (8.3)–(8.6), uniformly in $\alpha$.

It now suffices to prove that for each choice of $\tilde{f}$, independent of $\alpha$, we have

$$\int_{\alpha_0}^{\alpha_1} Q_1(\Pi u(\alpha) - R^+_\psi(\omega)\Pi f(\alpha)) \, d\alpha = O(h^\infty)_{L^2}.$$  \hspace{1cm} (10.7)

Define the semiclassical Fourier transforms $\hat{u}(s), \hat{f}(s)$ by (8.38). Then (10.7) becomes

$$Q_1\left(\Pi \hat{u}(0) - \frac{i}{h} \int_{-\infty}^{0} e^{is(P-i\nu_+)/h}\psi(s)\Pi \hat{f}(s) \, ds \right) = O(h^\infty)_{L^2}.$$  \hspace{1cm} (10.8)

By (8.41) and Proposition 3.1, we find microlocale near $W'$,

$$\Pi \hat{u}(0) = \frac{i}{h} \int_{-\infty}^{0} e^{is(P-i\nu_+)/h}\psi(s)\Pi \hat{f}(s) - i\hbar \psi'(s)\Pi \hat{u}(s) \, ds + O(h^\infty).$$  \hspace{1cm} (10.9)

Take $\tilde{\epsilon} > 0$ such that $\psi = 1$ near $[-\tilde{\epsilon}, \tilde{\epsilon}]$, so that $\psi'(s)$ is compactly supported in $\{|s| > \tilde{\epsilon}\}$. Since $\chi(\omega)$ and $\partial_{\omega} L_{22}(\omega)$ depend smoothly on $\alpha$, we see that $\|\partial_{\alpha}^j f(\alpha)\|_{L^2_\alpha L^2_x} = O(h^{-1/2})$ for all $j$. By repeated integration by parts, we get

$$\|\hat{f}(s)\|_{L^2_x((-\infty, -\tilde{\epsilon}) \cup (\tilde{\epsilon}, \infty))} = O(h^\infty).$$

Then by (8.39), $\Pi \hat{u}(s) = O(h^\infty)$ microlocale near $W'$ locally uniformly in $s \in (-\infty, -\tilde{\epsilon}]$, and thus $Q_1 e^{is(P-i\nu_+)/h}\Pi \hat{u}(s) = O(h^\infty)_{L^2}$ uniformly in $s \in (-\infty, 0] \cap \text{supp} \psi'$. By (10.9), we now get (10.8). \hspace{1cm} \Box

Now, note that, since the expression under the integral in (10.6) is almost analytic in $\omega$, we can replace the integral over $\text{Im} \omega = h\nu_\pm$ by the integral over the real line,
with an $O(h^\infty)$ error. Then

$$\mathcal{I}_-^- - \mathcal{I}_+^+ = (2\pi h)^{n-1} \text{Tr} \mathcal{A}_\chi + O(h^\infty),$$

$$\mathcal{A}_\chi := \int_\mathbb{R} \chi(\alpha) \partial_\alpha L_{22}(\alpha) Q_1(\mathcal{R}_\psi^- + \mathcal{R}_\psi^+ - \mathcal{R}_\psi)(\alpha) \Pi Q_1 \Pi Q_2 \, d\alpha.$$ 

Proposition 10.1 now follows from Proposition 6.7, the fact that $\text{WF}_h(\mathcal{A}_\chi) \subset \hat{W} \times \hat{W}$, and the following

**Proposition 10.3.** The operator $\mathcal{A}_\chi$ lies in $I_{\text{comp}}(\Lambda^\circ)$ and its principal symbol, as defined by (6.30), satisfies $\sigma_\Lambda(\mathcal{A}_\chi) \circ j_K = 2\pi i \chi(p)$, with $j_K : K^\circ \to \Lambda^\circ$ defined in (6.2).

**Proof.** Given the multiplication formula (6.31), the fact that $\sigma(Q_1) = \sigma(Q_2) = 1$ and $\sigma_\Lambda(\Pi) \circ j_K = 1$ on $K \cap p^{-1}(\{0, 1\})$ and $\text{supp} \chi \subset (0, 1)$, it is enough to prove the proposition with $\mathcal{A}_\chi$ replaced by

$$\mathcal{A}'_\chi := -\frac{i}{h} \int_{\mathbb{R}^2} e^{-isq/h} \chi(\alpha) \partial_\alpha L_{22}(\alpha) Q_1 e^{isP/h} \psi(s) \, ds \, d\alpha.$$ 

Denote $\mathcal{L}(\alpha) = \chi(\alpha) \partial_\alpha L_{22}(\alpha) Q_1$; it is an operator in $I_{\text{comp}}(\Lambda^\circ)$. By applying a microlocal partition of unity to $\mathcal{L}(\alpha)$, we may reduce to the case when both $\mathcal{L}(\alpha)$ and $e^{isP/h}$ have local parametrizations (see (3.3) for the first one and for example [Zw, Theorem 10.4] for the second one)

$$\mathcal{L}(\alpha) u(x) = (2\pi h)^{-(N+n)/2} \int_{\mathbb{R}^{N+n}} e^{\frac{i}{h} \Phi(x,y,\theta)} a(x, y, \theta, \alpha; h) u(y) \, dy \, d\theta,$$

$$e^{isP/h} u(y) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(S(y,\zeta,s) - z,\zeta)} b(y, \zeta, s; h) u(z) \, dz \, d\zeta.$$ 

Here $S(y,\zeta,s) = y \cdot \zeta + sp(y,\zeta) + O(s^2)$ and $b(y,\zeta,0;0) = 1$. Then $\mathcal{A}'_\chi$ takes the form

$$\mathcal{A}'_\chi u(x) = -ih^{-1}(2\pi h)^{-(N+3n)/2} \int_{\mathbb{R}^{N+3n}} e^{\frac{i}{h}(\Phi(x,y,\theta) + S(y,\zeta,s) - z,\zeta - \sigma_\alpha)} a(x, y, \theta, \alpha; h) b(y, \zeta, s; h) \psi(s) u(z) \, dy \, d\theta \, dz \, d\zeta \, ds \, d\alpha.$$ 

We now apply the method of stationary phase in the $y, \zeta, s, \alpha$ variables. The stationary points are given by $s = 0, \alpha = p(z, \zeta), y = z, \zeta = -\partial_\zeta \Phi(x, z, \theta)$. We get

$$\mathcal{A}'_\chi u(x) = -2\pi i(2\pi h)^{-(N+n)/2} \int_{\mathbb{R}^{N+n}} \Phi(x, z; \theta, h) u(z) \, d\theta \, dz,$$

where $c$ is a classical symbol and $c(x, z, \theta; 0) = a(x, z, \theta, p(z, -\partial_\zeta \Phi(x, z, \zeta); 0)$. It follows that $\mathcal{A}'_\chi \in I_{\text{comp}}(\Lambda^\circ)$ and $\sigma_\Lambda(\mathcal{A}'_\chi)(\rho_-, \rho_+) = -2\pi i \sigma_\Lambda(\mathcal{L}(\rho))(\rho_-, \rho_+)$. By (9.4), $\sigma_\Lambda(L_{22}(\alpha))(\rho, \rho) = p(\rho) - \alpha + 1$ when $\rho \in K \cap p^{-1}(\{0, 1\})$, and thus $\sigma_\Lambda(\partial_\alpha L_{22}(\alpha))(\rho, \rho) = -1$. Therefore, we find $\sigma_\Lambda(\mathcal{A}'_\chi)(\rho, \rho) = 2\pi i \chi(p(\rho))$ for $\rho \in K$. \qed
In this section, we prove Theorem 2, using the Grushin problem from §9, the trace formula of §10, and several tools from complex analysis. The argument below is quite standard, see for instance [Ma, Sj97, Sj01], and is simplified by the fact that we do not aim for the optimal $O(h)$ remainder in the Weyl law, instead carrying out the argument in a rectangle of width $\sim 1$ and height $\sim h$. For more sophisticated techniques needed to obtain the optimal remainder, see [Sj00].

First of all, by Proposition 9.5, resonances in the region of interest are (with multiplicities) the zeroes of the holomorphic function $F(\omega)$ defined in (9.25). Take $\alpha''_0 \in (\alpha_0, \alpha'_0)$, $\alpha''_1 \in (\alpha'_1, \alpha_1)$. Fix $\nu_\pm$ satisfying (10.1) and let $\{\omega_j\}_{j=1}^{M(h)}$ denote the set of zeroes (counted with multiplicities) of $F(\omega)$ in the region (see Figure 6)

$$\Omega(h) := \{\text{Re} \omega \in [\alpha''_0, \alpha''_1], \text{Im} \omega \in [\nu_- h, \nu_+ h]\}$$

By part 2 of Proposition 9.5 and Jensen’s inequality, see for example [DaDy, §2], we have the polynomial bound, for some $N, C$,

$$M(h) \leq C h^{-N}. \quad (11.1)$$

By a standard argument approximating the indicator function of $[\alpha'_0, \alpha'_1]$ by smooth functions from above and below, it is enough to prove that for each $\chi \in C_0^\infty(\alpha_0, \alpha_1)$,

$$(2\pi h)^{n-1} \sum_{j=1}^{M(h)} \chi(\text{Re} \omega_j) = \int_K \chi(p) d\text{Vol}_\sigma + O(h). \quad (11.2)$$

Let $\tilde{\chi}(\omega)$ be an almost analytic continuation of $\chi$, as discussed in the beginning of §10. We may assume that $\text{supp} \tilde{\chi} \subset \{\text{Re} \omega \in (\alpha''_0, \alpha''_1)\}$. 

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**Figure 6.** The contour $\partial \Omega(h)$ (in blue). The horizontal shaded region is $\{\text{Im} \omega \in (-\nu_{\text{max}} + \varepsilon)h/2, -(\nu_{\text{min}} - \varepsilon)h/2\}$, where Theorem 1 does not provide polynomial resolvent bounds; the vertical shaded region is the support of $\tilde{\chi}$. 

11. Weyl law for resonances
By Proposition 10.1, we have (with the integral over the vertical parts of \( \partial \Omega(h) \) vanishing since \( \tilde{\chi} = 0 \) there)
\[
\left( \frac{2\pi h}{2\pi i} \right)^{n-1} \int_{\partial \Omega(h)} \tilde{\chi}(\omega) \frac{\partial_\omega F(\omega)}{F(\omega)} \, d\omega = \int_K \chi(p) \, d\text{Vol}_\sigma + \mathcal{O}(h). \tag{11.3}
\]
By Lemma \( \alpha \) in [Ti, §3.9] and the exponential estimates of part 2 of Proposition 9.5 (splitting the region \( \Omega(h) \) into boxes of size \( h \) and applying Lemma \( \alpha \) to each of these boxes, transformed into the unit disk by the Riemann mapping theorem), we have for some fixed \( N \),
\[
\frac{\partial_\omega F(\omega)}{F(\omega)} = \sum_{j=1}^{M(h)} \frac{1}{\omega - \omega_j} + G(\omega); \quad G(\omega) = \mathcal{O}(h^{-N}), \quad \omega \in \Omega(h) \cap \text{supp} \tilde{\chi}.
\]
Applying Stokes theorem to (11.3) (over the contour comprised of \( \partial \Omega(h) \) minus the sum of circles of small radius \( r \) centered at each \( \omega_j \), and letting \( r \to 0 \)) we get
\[
(2\pi h)^{n-1} \sum_{j=1}^{M(h)} \tilde{\chi}(\omega_j) = \int_K \chi(p) \, d\text{Vol}_\sigma - \frac{(2\pi h)^{n-1}}{2\pi i} \int_{\Omega(h)} \frac{\partial_\omega F(\omega)}{F(\omega)} \partial_\omega \tilde{\chi}(\omega) \, d\bar{\omega} \wedge d\omega + \mathcal{O}(h).
\]
Since \( \tilde{\chi} \) is almost analytic and \( \Omega(h) \) lies \( \mathcal{O}(h) \) close to the real line, we have \( \partial_\omega \tilde{\chi}(\omega) = \mathcal{O}(h^\infty) \) for \( \omega \in \Omega(h) \). Therefore, the second integral on the right-hand side is \( \mathcal{O}(h^\infty) \) and we get
\[
(2\pi h)^{n-1} \sum_{j=1}^{M(h)} \tilde{\chi}(\omega_j) = \int_K \chi(p) \, d\text{Vol}_\sigma + \mathcal{O}(h).
\]
Since \( \tilde{\chi}(\omega) = \chi(\text{Re}\,\omega) + \mathcal{O}(h) \) for \( \omega \in \Omega(h) \), we get
\[
(2\pi h)^{n-1} \sum_{j=1}^{M(h)} \chi(\text{Re}\,\omega_j) = \int_K \chi(p) \, d\text{Vol}_\sigma + \mathcal{O}(h(1 + h^{n-1}M(h))). \tag{11.4}
\]
Since one can take \( \chi \) to be any compactly supported function on \( (\alpha_0, \alpha_1) \), and \( M(h) = \mathcal{O}(h^{-N}) \) for some fixed \( N \) and any choice of \( (\alpha''_0, \alpha''_1) \), by induction we see from (11.4) that \( M(h) = \mathcal{O}(h^{1-n}) \). Given this bound, (11.4) implies (11.2), which finishes the proof.

**Appendix A. Example of a manifold with \( r \)-normally hyperbolic trapping**

In this Appendix, we provide a simple example of an even asymptotically hyperbolic manifold (as defined in §4.4) whose geodesic flow satisfies the dynamical assumptions of §5.1 and the pinching condition (1.7), therefore our Theorems 1–4 apply. This example is a higher dimensional generalization of the hyperbolic cylinder, considered for instance in [DaDy, Appendix B].
The resonances for the provided example can be described explicitly via the eigenvalues of the Laplacian on the underlying compact manifold $N$, using separation of variables. However, our results apply to small perturbations of the metric (see §5.2), as well as to subprincipal perturbations in the considered operator, when separation of variables no longer takes place.

Let $(N, \tilde{g})$ be a compact $n-1$ dimensional Riemannian manifold (at the end of this appendix, we will impose further conditions on $\tilde{g}$). We consider the manifold $M = \mathbb{R}_r \times N_\theta$ with the metric

$$g = dr^2 + \cosh^2 r \tilde{g}(\theta, d\theta).$$

Then $M$ has two infinite ends $\{r = \pm \infty\}$; near each of these ends, one can represent it as an even asymptotically hyperbolic manifold by taking the boundary defining function $\tilde{x} = e^{\pm r}$:

$$g = \frac{d\tilde{x}^2}{\tilde{x}^2} + \frac{(1 + \tilde{x}^2)^2}{4\tilde{x}^2} \tilde{g}(\theta, d\theta).$$

The resonances for the Laplace–Beltrami operator on $M$ therefore fit into the framework of §4.1, as demonstrated in §4.4. The associated flow $e^{tH_p}$ is the geodesic flow on the unit cotangent $S^*M$, extended to a homogeneous flow of degree zero on the complement of the zero section in $T^*M$.

We now verify the assumptions of §5.1. If $\xi_r, \xi_\theta$ are the momenta dual to $r, \theta$, then

$$p^2 = \xi_r^2 + \cosh^{-2} r \tilde{g}^{-1}(\theta, \xi_\theta),$$

where $\tilde{g}^{-1}$ is the dual metric to $g$, defined on the fibers of $T^*N$. We then have

$$H_p r = \frac{\xi_r}{p}, \quad H_p \xi_r = \frac{p^2 - \xi_r^2}{p} \tanh r.$$

The trapped set $K$ and the incoming/outgoing tails $\Gamma_\pm$ are given by

$$\Gamma_\pm = \{\xi_r = \pm p \tanh r\}, \quad K = \{r = 0, \xi_r = 0\},$$

or strictly speaking, by the intersections of the sets above with the set $\mathcal{U}$ from (4.21).

Consider the following defining functions of $\Gamma_\pm$:

$$\varphi_\pm = \xi_r \mp p \tanh r,$$

then $\{\varphi_+, \varphi_-\}|_K = 2p$ and thus assumptions (1) and (2) of §5.1 are satisfied. Next,

$$H_p \varphi_\pm = \mp c_\pm \varphi_\pm, \quad c_\pm = 1 \pm \frac{\xi_r}{p} \tanh r.$$

In particular, $c_\pm|_K = 1$ and, arguing as in the proof of Lemma 5.1, we get

$$\nu_{\min} = \nu_{\max} = 1.$$

In particular, the pinching condition (1.7) is satisfied.
Finally, in order for the $r$-normal hyperbolicity assumption (3) of §5.1 to be satisfied, we need to make $\mu_{\text{max}} \ll 1$, with $\mu_{\text{max}}$ defined in (5.3). This is a condition on the underlying compact Riemannian manifold $(N, \tilde{g})$, since $\mu_{\text{max}}$ is the maximal expansion rate of the geodesic flow of $\tilde{g}$ on the unit cotangent bundle $S^*N$. To satisfy this condition, we can start with an arbitrary compact Riemannian manifold and multiply its metric by a large constant $C^2$; indeed, if $\varphi_t$ is the geodesic flow on the original manifold, then $\varphi_{C^{-1}t}$ is the geodesic flow on the rescaled manifold and the resulting $\mu_{\text{max}}$ is divided by $C$.

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References

[AgCo] Jacques Aguilar and Jean-Michel Combes, A class of analytic perturbations for one-body Schrödinger Hamiltonians, Comm. Math. Phys. 22(1971), no. 4, 269–279.
[BFRZ] Jean–François Bony, Setsuro Fujiié, Thierry Ramond, and Maher Zerzeri, Spectral projection, residue of the scattering amplitude, and Schrödinger group expansion for barrier-top resonances, Ann. Inst. Fourier 61(2011), no. 4, 1351–1406.
[BuZw] Nicolas Burq and Maciej Zworski, Control for Schrödinger operators on tori, Math. Res. Lett. 19(2012), no. 2, 309–324.
[DaDy] Kiril Datchev and Semyon Dyatlov, Fractal Weyl laws for asymptotically hyperbolic manifolds, to appear in Geom. Funct. Anal., arXiv:1206.2255.
[DaDyZw] Kiril Datchev, Semyon Dyatlov, and Maciej Zworski, Sharp polynomial bounds on the number of Pollicott–Ruelle resonances, to appear in Erg. Theory Dyn. Syst., arXiv:1208.4330.
[DiSj] Mounz Dimassi and Johannes Sjöstrand, Spectral asymptotics in the semi-classical limit, London Mathematical Society Lecture Note Series 268, Cambridge University Press, 1999.
[Do] Dmitry Dolgopyat, On decay of correlations in Anosov flows, Ann. of Math. (2) 147(1998), no. 2, 357–390.
[DuGu] Johannes Duistermaat and Victor Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, Invent. Math. 29(1975), no. 1, 39–79.
[Dy11a] Semyon Dyatlov, Quasi-normal modes and exponential energy decay for the Kerr–de Sitter black hole, Comm. Math. Phys. 306(2011), 119–163.
[Dy11b] Semyon Dyatlov, Exponential energy decay for Kerr–de Sitter black holes beyond event horizons, Math. Res. Lett. 18(2011), 1023–1035.
[Dy12] Semyon Dyatlov, Asymptotic distribution of quasi-normal modes for Kerr–de Sitter black holes, Annales Henri Poincaré 13(2012), 1101–1166.
[Dy13] Semyon Dyatlov, Resonance expansions in general relativity, Ph.D. thesis, in preparation.
[DuGu] Semyon Dyatlov and Colin Guillarmou, Microlocal limits of plane waves and Eisenstein functions, preprint, arXiv:1204.1305.
[FaSj] Frédéric Faure and Johannes Sjöstrand, *Upper bound on the density of Ruelle resonances for Anosov flows*, Comm. Math. Phys. **308** (2011), no. 2, 325–364.

[FaTs1] Frédéric Faure and Masato Tsujii, *Prequantum transfer operator for Anosov diffeomorphism (preliminary version)*, preprint, arXiv:1206.0282.

[FaTs2] Frédéric Faure and Masato Tsujii, *Band structure of the Ruelle spectrum of contact Anosov flows*, preprint, arXiv:1301.5525.

[FaTs3] Frédéric Faure and Masato Tsujii, *Spectrum and zeta function of contact Anosov flows*, in preparation.

[GéSj87] Christian Gérard and Johannes Sjöstrand, *Semiclassical resonances generated by a closed trajectory of hyperbolic type*, Comm. Math. Phys. **108** (1987), no. 3, 391–421.

[GéSj88] Christian Gérard and Johannes Sjöstrand, *Resonances en limite semi-classique et exposants de Lyapunov*, Comm. Math. Phys. **116** (1988), no. 2, 193–213.

[GoSi] Israel C. Gohberg and Efim I. Sigal, *An operator generalization of the logarithmic residue theorem and Rouché’s Theorem*, Mat. Sb. **84(126)** (1971), no. 4, 607–629.

[GSWW] Arseni Goussev, Roman Schubert, Holger Waalkens, and Stephen Wiggins, *Quantum theory of reactive scattering in phase space*, Adv. Quant. Chem. **60** (2010), 269–332.

[GrSj] Alain Grigis and Johannes Sjöstrand, *Microlocal analysis for differential operators: an introduction*, London Mathematical Society Lecture Note Series **196**, Cambridge University Press, 1994.

[Gu] Colin Guillarmou, *Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds*, Duke Math. J. **129** (2005), no. 1, 1–37.

[GuSt90] Victor Guillemin and Shlomo Sternberg, *Geometric asymptotics*, AMS, 1990.

[GuLiZw] Laurent Guillopé, Kevin K. Lin, and Maciej Zworski, *The Selberg zeta function for convex co-compact Schottky groups*, Comm. Math. Phys. **245** (2004), no. 1, 149–176.

[HiPuSh] Morris W. Hirsch, Charles C. Pugh, and Michael Shub, *Invariant manifolds*, Lecture Notes in Mathematics **583**, Springer Verlag, 1977.

[Hö] Lars Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218.

[HöI] Lars Hörmander, *The Analysis of Linear Partial Differential Operators. I. Distribution Theory and Fourier Analysis*, Springer Verlag, 1990.

[HöIII] Lars Hörmander, *The Analysis of Linear Partial Differential Operators. III. Pseudo-Differential Operators*, Springer Verlag, 1994.

[HöIV] Lars Hörmander, *The Analysis of Linear Partial Differential Operators. IV. Fourier Integral Operators*, Springer Verlag, 1994.

[IaSjZw] Alexei Iantchenko, Johannes Sjöstrand, and Maciej Zworski, *Birkhoff normal forms in semiclassical inverse problems*, Math. Res. Lett. **9**(2002), no. 2–3, 337–362.

[JaNa] Dmitry Jakobson and Frédéric Naud, *Lower bounds for resonances of infinite-area Riemann surfaces*, Anal. PDE **3**(2010), no. 2, 207–225.

[KoSc] Kostas D. Kokkotas and Bernd Schmidt, *Quasi-normal modes of stars and black holes*, Living Rev. Relativity **2**(1999), 2; http://relativity.livingreviews.org/Articles/lrr-1999-2/

[Li] Carlangelo Liverani, *On contact Anosov flows*, Ann. of Math. **159**(2004), no. 3, 1275–1312.

[Ma] Alexander S. Markus, *Introduction to the spectral theory of polynomial operator pencils*, Translations of Mathematical Monographs **71**, AMS, 1998.

[MaMc] Rafe R. Mazzeo and Richard B. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Func. Anal. **75**(1987), no. 2, 260–310.

[MeZw] Richard B. Melrose and Maciej Zworski, *Scattering metrics and geodesic flow at infinity*, Invent. Math. **124**(1996), no. 1–3, 389–436.
[Mü] Werner Müller, *Spectral geometry and scattering theory for certain complete surfaces of finite volume*, Invent. Math. **109**(1992), no. 2, 265–305.

[No] Stéphane Nonnenmacher, *Spectral problems in open quantum chaos*, Nonlinearity **24**(2011), R123–R167.

[NoSjZw1] Stéphane Nonnenmacher, Johannes Sjöstrand, and Maciej Zworski, *From open quantum systems to open quantum maps*, Comm. Math. Phys. **304**(2011), no. 1, 1–48.

[NoSjZw2] Stéphane Nonnenmacher, Johannes Sjöstrand, and Maciej Zworski, *Fractal Weyl law for open quantum chaotic maps*, to appear in Ann. of Math. (2), arXiv:1105.3128.

[NoZw09] Stéphane Nonnenmacher and Maciej Zworski, *Quantum decay rates in chaotic scattering*, Acta Math. **203**(2009), no. 2, 149–233.

[NoZw13] Stéphane Nonnenmacher and Maciej Zworski, *Decay of correlations for normally hyperbolic trapping*, preprint.

[Re] Tullio Regge, *Analytic properties of the scattering matrix*, Nuovo Cimento **8**(1958), 671–679.

[SáZw] Antônio Sá Barreto and Maciej Zworski, *Distribution of resonances for spherical black holes*, Math. Res. Lett. **4**(1997), no. 1, 103–121.

[Sj90] Johannes Sjöstrand, *Geometric bounds on the density of resonances for semiclassical problems*, Duke Math. J. **60**(1990), no. 1, 1–57.

[Sj97] Johannes Sjöstrand, *A trace formula and review of some estimates for resonances*, in *Microlocal analysis and spectral theory* (Lucca, 1996), 377–437, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 490, Kluwer Acad. Publ., Dordrecht, 1997.

[Sj00] Johannes Sjöstrand, *Asymptotic distribution of eigenfrequencies for damped wave equations*, Publ. Res. Inst. Math. Sci. **36**(2000), no. 5, 573–611.

[Sj01] Johannes Sjöstrand, *Resonances for bottles and trace formulae*, Math. Nachr. **221**(2001), 95–149.

[Sj11] Johannes Sjöstrand, *Weyl law for semi-classical resonances with randomly perturbed potentials*, preprint, arXiv:1111.3549.

[SjVo] Johannes Sjöstrand and Georgi Vodev, *Asymptotics of the number of Rayleigh resonances*, with an appendix by Jean Lannes, Math. Ann. **309**(1997), no. 2, 287–306.

[SjZw91] Johannes Sjöstrand and Maciej Zworski, *Complex scaling and the distribution of scattering poles*, J. Amer. Math. Soc. **4**(1991), no. 4, 729–769.

[SjZw99] Johannes Sjöstrand and Maciej Zworski, *Asymptotic distribution of resonances for convex obstacles*, Acta Math. **183**(1999), no. 2, 191–253.

[SjZw07] Johannes Sjöstrand and Maciej Zworski, *Fractal upper bounds on the density of semiclassical resonances*, Duke Math. J. **137**(2007), no. 3, 381–459.

[StVo] Plamena Stefanov and Georgi Vodev, *Distribution of resonances for the Neumann problem in linear elasticity outside a strictly convex body*, Duke Math. J. **78**(1995), no. 3, 677–714.

[TaZw] Siu-Hung Tang and Maciej Zworski, *From quasimodes to resonances*, Math. Res. Lett. **5**(1998), no. 3, 261–272.

[Ta] Michael E. Taylor, *Partial Differential Equations I. Basic theory*, Springer, 1996.

[Ti] Edward Charles Titchmarsh, *The Theory of the Riemann Zeta-function*, second edition, revised by David Rodney Heath-Brown, Oxford University Press, 1986.

[Ts] Masato Tsujii, *Contact Anosov flows and the FBI transform*, preprint, arXiv:1010.0396.

[Va10] András Vasy, *Microlocal analysis of asymptotically hyperbolic and Kerr–de Sitter spaces*, to appear in Invent. Math., arXiv:1012.4391v2.
[Va11] András Vasy, *Microlocal analysis of asymptotically hyperbolic spaces and high energy resolvent estimates*, Inverse Problems and Applications. Inside Out II, edited by Gunther Uhlmann, Cambridge University Press, MSRI publications 60(2012), arXiv:1104.1376v2.

[VuNg] San Vũ Ngọc, *Systèmes intégrables semi-classiques: du local au global*, Panoramas et Synthèses 22, 2006.

[WuZw11] Jared Wunsch and Maciej Zworski, *Resolvent estimates for normally hyperbolic trapped sets*, Ann. Henri Poincaré, 12(2011), no. 7, 1349–1385.

[Zw87] Maciej Zworski, *Distribution of poles for scattering on the real line*, J. Funct. Anal. 73(1987), no. 2, 277–296.

[Zw] Maciej Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics 138, AMS, 2012.

E-mail address: dyatlov@math.berkeley.edu

Department of Mathematics, University of California, Berkeley, CA 94720, USA