WIND RIEMANNIAN SPACEFORMS 
AND RANDERS METRICS OF 
CONSTANT FLAG CURVATURE 

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Abstract. Recently, wind Riemannian structures (WRS) have been introduced as a generalization of Randers and Kropina metrics. They are constructed from the natural data for Zermelo navigation problem, namely, a Riemannian metric $g_R$ and a vector field $W$ (the wind), where, now, the restriction of mild wind $g_R(W,W) < 1$ is dropped.

Here, the models of WRS spaceforms of constant flag curvature are determined. Indeed, the celebrated classification of Randers metrics of constant flag curvature by Bao, Robles and Shen [2], extended to the Kropina case in the works by Yoshikawa, Okubo and Sabau [29, 30], can be used to obtain the local classification. For the global one, a suitable result on completeness for WRS yields the complete simply connected models. In particular, any of the local models in the Randers classification does admit an extension to a unique model of wind Riemannian structure, even if it cannot be extended as a complete Finslerian manifold.

Thus, WRS’s emerge as the natural framework for the analysis of Randers spaceforms and, prospectively, wind Finslerian structures would become important for other global problems too. For the sake of completeness, a brief overview about WRS (including a useful link with the conformal geometry of a class of relativistic spacetimes) is also provided.

1. Introduction

Wind Riemannian structures (WRS’s) are generalizations of the indicatrices of both, Randers and Kropina metrics on a manifold $M$, introduced in [7] for several purposes. The simplest one is that they provide a natural framework for modelling Zermelo navigation problem. Namely, consider the motion of a zeppelin: its engine is able to develop a maximum speed with respect to the air, modelled by the indicatrix of a Riemannian metric $g_R$, but the air is moving with respect to earth with a (time-independent) velocity modelled by a vector field $W$. So, the maximum speed of the zeppelin with respect to earth is modelled by $\Sigma = S_R + W$. This is the indicatrix of a Randers metric when $g_R(W,W) < 1$, but when $g_R(W,W) = 1$, it is the indicatrix of a Kropina metric, a singular Finsler metric in the sense that it is not defined in all the directions; the geodesics of such metrics solve Zermelo problem of finding the fastest path between two prescribed points. In the more general case in that no restriction on $g_R(W,W)$ is imposed, $\Sigma$ is a WRS and, as proven in [7], such a WRS admits a notion of geodesic which also solves Zermelo navigation.
However, there is a deeper motivation for studying WRS’s because of the existence of a link between the geometry of such WRS’s and the conformal geometry of a class of relativistic spacetimes, those which are standard with a space-transverse Killing vector field, or SSTK spacetimes.

The existence of a fruitful correspondence between the geometry of Randers manifolds and the conformal geometry of stationary spacetimes (a particular case of SSTK ones) has already been pointed out and systematically exploited in [6, 7, 12, 10, 14] and others. However, the importance of the correspondence in the case of general WRS’s becomes especially useful for the Finslerian framework. Indeed, the existence of apparently singular Finslerian elements for WRS’s (such as regions with a Kropina metric) is reinterpreted from the Lorentzian viewpoint in a completely non-singular way. In particular, geodesics for WRS’s can be seen as projections of lightlike pregeodesics of SSTK spacetimes. Moreover, geodesic completeness of WRS’s becomes equivalent to the global hyperbolicity of the spacetime. Notice that both, lightlike pregeodesics and global hyperbolicity are conformally invariant elements in Lorentzian Geometry.

Our main aim along the present note will be to show an application of WRS’s to Randers metrics of constant flag curvature (CFC). The complete classification of these manifolds was a landmark in Finslerian Geometry, obtained by Bao, Robles and Shen [2]. The solutions admit a neat description, which can be summarized by saying that $g_R$ must have constant curvature and $W$ must be a homothetic vector field. From a global viewpoint, however, there is a striking difference with the classical (simply connected) models of Riemannian manifolds of constant curvature: in the Randers case, some of the models are necessarily incomplete. Namely, some local models cannot be extended to a complete one, and only its inextensibility can be claimed. Nevertheless, we will see that this comes from the fact that $g_R(W,W)$ may not remain bounded by 1, and we will show that all the local models admit a unique extension as a complete simply connected WRS. Moreover, taking into account the classification of Kropina metrics of CFC by Yoshikawa and Okubo [29] and Yoshikawa and Sabau [30] we will provide both, the local and global classification of WRS’s.

This article is organized as follows. In Section 2, the motivations and necessary results on WRS’s are briefly summarized. In Section 3, the classification of WRS’s of CFC is achieved. For this purpose, we summarize first the known results in the Randers [2] and Kropina cases [29, 30] (subsection 3.1), and show how these results can be used for the local classification (subsection 3.2). The global classification is obtained in subsection 3.3. Here, the key issue is completeness. First, we prove a result of independent interest on completeness for WRS’s, Theorem 3.10; the subtleties of this result are stressed in Example 3.11. The global classification is reached in Theorem 3.12, by using the previous result on completeness, the local result and a specific study of the case when $W$ is properly homothetic. Finally, in the appendix (Section 4), we summarize some known results on Killing and homothetic vector fields. The results in subsection 4.1 are useful to understand when Randers models can be naturally extended in WRS’s, and those in subsection 4.2 for a neat description of the Kropina case and the region of transition between mild and strong wind.
\( h(v, v) \geq 0 \) and \( g_R(W, v) > 0 \) region

Figure 1. The diagram on the left represents the domain of \( F_p \) when \( \Lambda(p) < 0 \). In this case, we consider \( g_R \) the Euclidean metric in \( \mathbb{R}^3 \) and \( W = (0, 0, 2) \). The condition \( h(v, v) \geq 0 \) determines that the region must lie in the two cones of \( h \) and the condition \( g_R(W, v) > 0 \) that it must be contained in the half-space \( z > 0 \); so, the cone \( C_p \) is selected. The diagram on the right represents the indicatrix and the domain of \( F_p, z > 0 \), in the Kropina case when \( \Lambda(p) = 0 \) and \( W = (0, 0, 1) \).

2. A Finslerian overview on Wind Riemannian Structures

2.1. Generalizing Randers and Kropina metrics. Let \((g_R, W)\) be Zermelo data in a manifold \( M \), namely, \( g_R \) is a Riemannian metric and \( W \), a vector field on \( M \). Assume that \(|W|_R < 1\), where \(|·|_R\) denotes the pointwise \( g_R\)-norm, and consider the Randers metric \( F \) whose indicatrix \( \Sigma \) at \( p \in M \) is obtained by displacing the \( g_R\)-indicatrix \( S_R \) with the vector \( W_p \), that is, \( \Sigma = S_R + W_p \). This metric is the key of Zermelo navigation problem (the \( F \)-geodesics solve it), and it can be written as:

\[
F(v_p) = \frac{1}{\Lambda(p)} \left( \sqrt{\Lambda(p)}|v_p|_R^2 + g_R(v_p, W_p)^2 - g_R(W_p, v_p) \right), \quad \text{where} \quad \Lambda = 1 - |W_p|_R^2,
\]

for any \( v_p \in T_pM \). Clearly, this expression crashes when \(|W_p|_R = 1\), as \( \Lambda \) vanishes. However, we can rewrite it by removing the root from the numerator and then, \( \Lambda \) from the denominator:

\[
F(v_p) = \frac{|v_p|_R^2}{g_R(W_p, v_p) + \sqrt{\Lambda(p)}|v_p|_R^2 + g_R(v_p, W_p)^2} = \frac{|v_p|_R^2}{g_R(W_p, v_p) + \sqrt{h(v_p, v_p)}}, \quad (1)
\]

where

\[
h(v_p, v_p) = \Lambda(p)|v_p|_R^2 + g_R(v_p, W_p)^2. \quad (2)
\]

This expression for \( F(v_p) \) makes sense for an arbitrary wind \( W \) even if \( \Lambda \) vanishes, suggesting a possibility for the description of the case \(|W|_R \geq 1\). However, a caution should be taken into account: both, \( F(v_p) \) and the expression for \( h \) (which lies inside a root) should be nonnegative. So, restrict the domain of \( F \) by imposing:

\[
\text{if } \Lambda(p) \leq 0 \text{ restrict to } \begin{cases} h(v_p, v_p) \geq 0, & \text{and} \\ g_R(W_p, v_p) > 0. & \end{cases} \quad (3)
\]

These restrictions have the following meaning (see Figure 1). Formula (2) shows
that \( h \) is a signature-changing metric, which becomes Riemannian in the region of mild wind \( |W|_R < 1 \), degenerate when the wind is critical \( |W|_R = 1 \), and Lorentzian of signature \((+,-,\ldots,-)\) when the wind is strong \( |W|_R > 1 \). While in the region \( |W|_R < 1 \) one recovers a Randers metric with indicatrix \( F^{-1}(1) = \Sigma \) \((= S_R + W)\), in the region \( |W|_R = 1 \) \((i.e., \Lambda = 0)\) one has a Kropina metric \( \alpha^2/\beta \) with \( \alpha = |\cdot|_R, \beta = 2g_R(W, \cdot) \). Indeed the restriction \( g_R(W_p, v_p) > 0 \) selects the (pointwise) tangent open half-space \( A_p \) where \( F \) becomes positive, and one can still regard \( \Sigma = S_R + W \) as the indicatrix of \( F \) (up to the “singular” vector \( 0_p \)), Figure 2. In the region \( |W|_R > 1 \), the Lorentzian metric \( h \) determines at every tangent space two (lightlike) cones, and the restrictions (3) have a neat meaning: the allowed \( v_p \) must belong either to one of these two \( h \)-cones \( C_p \) \((the \ cone \ selected \ by \ g_R(W_p, v_p) > 0)\) or to the interior region \( A_p \) determined by it (see Figure 1). So, \( F \) can be regarded as a “conic” Finsler metric defined only on the tangent vectors satisfying (3). Notice that, now, \( \Sigma = S_R + W \) includes the “indicatrix” \( F^{-1}(1) \). Indeed, \( \Sigma_p \cap C_p \) divides \( \Sigma_p \) into two pieces, and \( F^{-1}(1) \) corresponds with one of them; namely, the (strongly) convex one, when looking from infinity into the cone region. Easily, one can check that the other piece is equal to \( F^{-1}_l(1) \) where

\[
F_l(v_p) := \frac{|v_p|_R^2}{g_R(W_p, v_p) - \sqrt{h(v_p, v_p)}}. \tag{4}
\]

Recall that \( F^{-1}_l(1) \) is (strongly) concave and, so, \( F_l \) will be called a Lorentz-Finsler metric. Notice that \( F_l \) is univocally determined by \( F \); consequently, most of our computations will deal only with \( F \). However, \( F_l \) has a nice interpretation for Zermelo navigation under strong wind: the indicatrix of \( F_l \) provides the minimum speed of the moving object at each allowed direction.

Let us summarize the previous approach and introduce suitable conventions:

1. Given a (connected) manifold \( M \), a wind Riemannian structure (WRS) is any hypersurface \( \Sigma \subset TM \) which can be expressed as \( \Sigma = S_R + W \) for some vector field \( W \) and Riemannian metric \( g_R \) (both univocally determined). At each \( p \in M \), \( \Sigma_p \) encloses an open domain \( B_p \) which will be called the unit ball at \( p \).
(2) Such a \( \Sigma \) determines the (possibly signature-changing) metric \( h \) in (2), as well as a domain \( A := \cup_{p \in M} A_p \) included in the slit tangent bundle \( TM \setminus \{0\} \), defined by choosing each \( A_p \) as follows: \( A_p = T_p M \setminus \{0\} \) in the region of mild wind \( (\Lambda(p) > 0) \), \( A_p \) is the open half space determined by (3) in the region of critical wind \( (\Lambda(p) = 0) \), and \( A_p \) is the interior of the solid cone determined by (3) in the region \( M_1 \) of strong wind \( (\Lambda(p) < 0) \).

(3) \( \Sigma \) also determines:

(i) A conic Finsler metric \( F : A \to \mathbb{R} \).
(ii) A Lorentz-Finsler metric \( F_l : A_l \to \mathbb{R} \), where \( A_l \) is the union of the pointwise domains \( A_p \) in the region \( M_1 \) of strong wind.

Moreover, the following three extensions of the domains \( A, A_l \) will be used when necessary:

(a) \( F \) and \( F_l \) are extended continuously on \( M_l \) to the \( h \)-cones \( C_p \subset T_p M \setminus \{0\} \),

(b) \( F_l \) is extended as \( \infty \) outside the region of strong wind, that is, in \( TM \setminus \{0\} \cup TM_l \), and

(c) in the region of critical wind \( (\Lambda(p) = 0) \), we define \( F(0_p) = F_l(0_p) = 1 \) (even though, necessarily, such a choice is discontinuous, see below).

2.2. Wind curves and the appearance of Lorentzian geometry. In the framework of Zermelo navigation, consider two points \( p, q \in M \) and a WRS \( \Sigma \) which provides the maximum velocities at each direction. If a moving object going from \( p \) to \( q \) is represented by the curve \( \gamma : [t_0, t_1] \to M \), where \( t_0, t_1 \) are, respectively, the instant of departure from \( p \) and arrival to \( q \), then its velocity at each instant \( t \) must be an allowed one, that is:

\[
F(\dot{\gamma}(t)) \leq 1 \leq F_l(\dot{\gamma}(t)), \quad \forall t \in [t_0, t_1]. \tag{5}
\]

Any curve in \( M \) satisfying these inequalities will be called a wind curve. Recall that (5) assumes implicitly that \( \dot{\gamma}(t) \) lies in the domains of \( F \) and \( F_l \) explained at the end of the last subsection, so: (a) \( \dot{\gamma}(t) \) is allowed to belong not only to the unit ball \( B_{\gamma(t)} \) but also to its closure (in the slit tangent bundle) \( \Sigma_{\gamma(t)} \), (b) as \( F_l \equiv \infty \) outside \( M_l \), the last inequality in (5) imposes no restriction when the wind is not strong, and (c) the zero velocity \( 0_p \) is excluded in both, the region of strong wind (because it is not an allowed velocity) and the region of mild wind (by convenience, analogous to the restriction to regular curves in Riemannian Geometry); however, the zero-velocity is allowed in the region of critical wind, as it has a special meaning there, namely, it is the minimum allowed velocity in the direction of the wind. The arrival time of the moving object \( t_1 - t_0 \) is bounded by the \( F \) and \( F_l \) lengths of the curves, that is:

\[
\ell_F(\gamma) := \int_{t_0}^{t_1} F(\dot{\gamma}(t)) \, dt \leq t_1 - t_0 \leq \ell_{F_l}(\gamma) := \int_{t_0}^{t_1} F_l(\dot{\gamma}(t)) \, dt.
\]

Obviously, when \( \gamma \) is reparametrized so that \( F(\dot{\gamma}(t)) \equiv 1 \), this corresponds to a moving object which uses the maximum possible velocity along the trajectory of \( \gamma \) so that it spends the minimum possible travel time along that trajectory; analogously, in the case that \( \gamma \) lies entirely in the region of strong wind, a reparametrization with \( F_l(\dot{\gamma}(t)) \equiv 1 \) corresponds to minimum velocity and maximum travel time.

A better insight is obtained by considering the graph \( \{ (t, \gamma(t)), t \in [t_1, t_2] \} \subset \mathbb{R} \times M \) of the wind curve. The allowed velocities for \( \gamma \) at each instant \( t \in \mathbb{R} \) and
each point $p$ are represented by
\begin{equation}
\{(1, v_p) \in T_{(t,p)}(\mathbb{R} \times M) : F(v_p) \leq 1 \leq F_t(v_p)\}.
\end{equation}

Now, notice that the half lines in $T_{(t,p)}(\mathbb{R} \times M)$ which start at 0 and cross any of these allowed velocities provide a solid cone on $T_{(t,p)}(\mathbb{R} \times M)$ (recall Figure 3). Thus, the WRS yields naturally a cone structure on all $\mathbb{R} \times M$. This cone structure is invariant in the $t$-coordinate, as neither the wind $W$ nor the metric $g_R$ are time-dependent. A simple computation shows that the cone structure is equal to the (future) cone structure associated with a Lorentzian metric $g$ on $\mathbb{R} \times M$, namely:
\begin{equation}
g = -(\Lambda \circ \pi)dt^2 + \pi^* \omega \otimes dt + dt \otimes \pi^* \omega + \pi^* g_0
\end{equation}
where $\pi : \mathbb{R} \times M \rightarrow M$ is the natural projection and:
\begin{equation}
\Lambda = 1 - |W|_R^2, \quad \omega = -g_R(W, \cdot), \quad g_0 = g_R.
\end{equation}

Some comments on this Lorentzian metric are in order:

- The metric $g$ is Lorentzian with signature $(-, +, \ldots, +)$. At each $(t, p)$ the non-zero tangent vectors $(\tau, v_p) \in T_{(t,p)}(\mathbb{R} \times M)$ satisfying $g((\tau, v_p), (\tau, v_p)) = 0$ (resp. $< 0$, $> 0$) are called lightlike (resp. timelike, spacelike). The light-like vectors at each $(t, p)$ are distributed in two cones. One of them, the future-directed lightlike cone, contains tangent vectors with $\tau > 0$; the other one is called the past-directed lightlike cone. The future-directed lightlike cone plus the corresponding future-directed timelike vectors (those inside the solid cone) contain the set (6) determined by the wind curves.

- The natural vector field $K = \partial_t$ is Killing for $g$. When $\Lambda > 0$ (resp. $= 0, < 0$), $K$ is future-directed timelike (resp. future-directed lightlike, spacelike), see Figure 3. The natural projection $t : \mathbb{R} \times M \rightarrow \mathbb{R}$ is a time function because it is strictly increasing on any future-directed timelike or lightlike curve; its slices $t =$constant are spacelike hypersurfaces (they inherit a Riemannian metric). According to [8], these spaces are called standard with a spacelike-transverse Killing vector field, or just SSTK spacetimes. They include important families of relativistic spacetimes, as the standard stationary ones (those with $\Lambda > 0$), which correspond to the Randers case under our approach.

- The SSTK spacetime $(\mathbb{R} \times M, g)$ will be canonically associated with a WRS. However, the only relevant properties of the spacetime for our purposes will be the conformally invariant ones$^3$. Indeed, any conformal metric $g^* = \Omega \cdot g$, $\Omega > 0$, will have the same lightlike cones as $g$. Moreover, it is well-known that if two Lorentzian metrics $g, g^*$ share the same timelike cones then they are (pointwise) conformally related through some function $\Omega > 0$ [3, 21].

The correspondence between the WRS and the SSTK spacetime is very fruitful for both, our Finslerian problem and the geometry of relativistic spacetimes. Indeed, this happens even in the particular case $\Lambda > 0$, where the WRS is just a Randers metric and the SSTK spacetime, a standard stationary spacetime (see the detailed study in [7] and further developments such as [12, 14]). However, the correspondence becomes crucial for general WRS. Indeed, the study of WRS

$^3$So, sometimes a different representative of the conformal class of the SSTK may be preferred. For example, the normalization $\Lambda \equiv 1$ was chosen in the case of Randers metrics and standard stationary spacetimes studied in [7]. (This is the reason why we preferred to write the metric $g_0$ in (7) even if it is taken $g_0 = g_R$ later.)
Figure 3. The three possibilities for the cones of the associated SSTK spacetime. At the tangent space of each \( p \), the intersection of the \((n+1)\)-dimensional cone with the hyperplane \( dt_p = 1 \) projects into the indicatrix of a Randers, Kropina or wind-Riemannian metric.

through more classical Finslerian elements such as the metrics \( F, F_i \) presents the important drawback of having "singular" elements (as the 0 vector in the Kropina region) or discontinuous elements (as the sudden jump in the structure of \( A_p \) when \( p \) varies from the region of mild wind to the non-mild one). This reason underlies the difficulties to develop general Zermelo navigation in spite of attempts such as [9]. Nevertheless, the spacetime viewpoint allows both: (i) a description in terms of completely regular and non-singular elements (the metric \( g \)) and (ii) the possibility to derive results for WRS by using results or techniques in the well established conformal theory of spacetimes. Indeed, the general problem of Zermelo navigation can be described and solved satisfactorily by using this correspondence [8]. We will not go deeper here in this correspondence and refer to the exhaustive study in [8]; however, we would like to point out that some of the results below were obtained by using it.

2.3. Balls, geodesics and completeness. Let \( \Sigma \) be a WRS and, for any \( p, q \in M \), let \( C_{p,q}^\Sigma \) denote the set of all the wind curves from \( p \) to \( q \). The forward and backward wind balls of center \( p_0 \in M \) and radius \( r > 0 \) associated with the WRS \( \Sigma \) are, resp:

\[
B^+_\Sigma(p_0, r) = \{ x \in M : \exists \gamma \in C_{p_0,x}^\Sigma, \text{ s.t. } r = b_\gamma - a_\gamma \text{ and } \ell_F(\gamma) < r < \ell_{F_1}(\gamma) \},
\]

\[
B^-\Sigma(p_0, r) = \{ x \in M : \exists \gamma \in C_{x,p_0}^\Sigma, \text{ s.t. } r = b_\gamma - a_\gamma \text{ and } \ell_F(\gamma) < r < \ell_{F_1}(\gamma) \}.
\]
These balls are open [8, Remark 5.2] and their closures are called (forward, backward) closed wind balls, denoted $B^\pm_\Sigma(p_0, r)$. Between these two types of balls, the forward and backward $c$-balls are defined, resp., by:

$$
\hat{B}^+_\Sigma(p_0, r) = \{ x \in M : \exists \gamma \in C^2_{p_0,x}, \text{ s.t. } r = b_\gamma - a_\gamma (so, \ell_F(\gamma) \leq r \leq \ell_F(\gamma)) \},
$$

$$
\hat{B}^-_\Sigma(p_0, r) = \{ x \in M : \exists \gamma \in C^2_{x,p_0}, \text{ s.t. } r = b_\gamma - a_\gamma (so, \ell_F(\gamma) \leq r \leq \ell_F(\gamma)) \}
$$

for $r > 0$; for $r = 0$, by convention $\hat{B}^\pm_\Sigma(p_0, 0) = p_0$ (so that, consistently with our conventions, if $0_{p_0} \in \Sigma_{p_0}$ then $p_0 \in \hat{B}^\pm_\Sigma(p_0, r)$ for all $r \geq 0$). If $\Sigma$ comes from a Randers metric, then $B^\pm_\Sigma(p_0, r)$ coincides with the usual (forward or backward) open balls. However, even in the Riemannian case, one may have $B^\pm_\Sigma(p_0, r) \not\subseteq B^\pm_\Sigma(p_0, r)$ (put $M = \mathbb{R}^2 \setminus \{(1, 0)\}$, $p_0 = (0, 0)$, $r = 2$).

Starting at these notions of balls, geodesics can be defined as follows. A wind curve $\gamma : I = [a, b] \to M$, $a < b$, is called a unit extremizing geodesic if

$$
\gamma(b) \in \hat{B}^+_{\Sigma}(\gamma(a), b - a) \setminus B^+_{\Sigma}(\gamma(a), b - a).
$$

Then, a curve is an extremizing geodesic if it is an affine reparametrization of a unit extremizing geodesic, and it is a geodesic if it is locally an extremizing geodesic.

The geodesics of a WRS coincide, up to a reparametrization, with the projection on $M$ of the future-directed lightlike geodesics of the associated SSTK spacetime $(\mathbb{R} \times M, g)$. This allows one to prove that a curve $\gamma : I \to M$ is a geodesic if an only if it lies in one of the following cases:

1. $\gamma$ is a geodesic of the conic Finsler metric $F$. In this case, $\dot{\gamma}(t)$ lies always in $A$ (it cannot belong to its boundary) and $\gamma$ may lie in the regions of mild, critical or strong wind, eventually crossing them; moreover, $\gamma$ minimizes locally the $F$-length in a natural sense.
2. $\gamma$ is a geodesic of the Lorentz-Finsler metric $F_l$. In this case, $\dot{\gamma}(t)$ lies always in $A^l$ (it cannot belong to its boundary) and $\gamma$ is entirely contained in the region of strong wind $M^l$; moreover, $\gamma$ maximizes locally the $F_l$-length in a natural sense.
3. $\gamma$ is an exceptional geodesic, that is, $\gamma$ is constantly equal to some point $p_0$ with $\Lambda(p_0) = 0$ and $d\Lambda(v_{p_0}) = 0$ for all $v_{p_0}$ satisfying $g_R(W_{p_0}, v_{p_0}) = 0$.
4. $\gamma$ is included in the closure of $M_l$ and it satisfies: (i) whenever $\gamma$ remains in $M_l$, it is a lightlike pregeodesic of the Lorentzian metric $h$ in (2), reparametrized so that $F(\dot{\gamma}) \equiv F_l(\dot{\gamma})$ is a constant $c > 0$ (that is, $\gamma$ is a boundary geodesic in $M_l$, with velocity in the boundary of each solid cone $A_p, p \in M_l$), and (ii) $\gamma$ can reach the boundary $\partial M_l$ (which is included in the critical region $\Lambda = 0$) only at isolated points $s_j \in I, j = 1, 2, ..., where^4 \dot{\gamma}(s_j) = 0$, $d\Lambda$ does not vanish on all the $g_R$-orthogonal to $W_{\gamma(s_j)}$ and the second derivative of $\gamma$ (in one and then in any coordinates) is continuous and does not vanish at $s_j$.

The WRS is (geodesically) complete when its inextendible geodesics are defined on all $\mathbb{R}$; it is called forward (resp. backward) complete when only the upper (resp. lower) unboundedness of their intervals of definition is required. Given $\Sigma$, its reverse WRS is $-\Sigma = SR - W$. Clearly, the reverse parametrization of a geodesic for $\Sigma$ becomes a geodesic for $-\Sigma$ and $\Sigma$ is forward complete iff $-\Sigma$ is backward complete.

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^4Recall that the equality $F(\dot{\gamma}) = F_l(\dot{\gamma}) = c > 0$ close to $s_j$ is compatible with $\dot{\gamma}(s_j) = 0$ because of the Kropina character of $\Sigma$ at $\gamma(s_j)$.
From now on, completeness will mean forward and backward completeness. We have the following characterization extracted from [8, Prop. 6.4].

**Theorem 2.1.** Let \((M, \Sigma)\) be a WRS. The following properties are equivalent:

(i) \(\Sigma\) is geodesically complete,
(ii) \(B^*_{\Sigma}(x,r)\) and \(B_{\Sigma}^{-}(x,r)\) are precompact for every \(x \in M\) and \(r > 0\).
(iii) \(\hat{B}^*_{\Sigma}(x,r)\) and \(\hat{B}_{\Sigma}^{-}(x,r)\) are compact for every \(x \in M\) and \(r > 0\).

In particular, if \(M\) is compact then \(\Sigma\) is geodesically complete.

The following result of completeness will be used throughout the text.

**Theorem 2.2.** Let \((M, \Sigma)\) be a complete WRS and \(W\) a complete homothetic field of the associated conic pseudo-Finsler metrics \(F\) and \(F_l\). Then, \(\Sigma+W\) is a complete WRS.

**Proof.** It is a straightforward consequence of [15, Theorem 1.2].

The equality between \(c\)-balls and closed balls in a (connected) Finslerian manifold becomes equivalent to its convexity (in the sense that any \(p, q \in M\) can be joined by a geodesic of minimum length). In general, a WRS is called \(w\)-convex when this equality between balls hold. In a \(w\)-convex WRS, an \(F\)-extremizing wind geodesic from \(p\) to \(q\) will exist whenever there exists a wind curve starting from \(p\) and ending at \(q\). A complete WRS is always \(w\)-convex, and even in this case some of their points might be non-connectable through wind curves (see Figure 4). These possibilities are very appealing from the SSTK viewpoint, as they are related to the existence of Killing horizons in globally hyperbolic spacetimes and other notions in the relativistic fauna.

3. **Wind Riemannian Structures of constant flag curvature**

3.1. **Randers and Kropina solutions.** Randers manifolds of constant flag curvature have been completely classified by Bao, Robles and Shen [2, Th. 3.1]. This
result has a local nature, even though it can be used to obtain models of inex-
tendible Randers spaceforms (some of them necessarily incomplete as Finslerian
manifolds). Their result can be summarized as follows.

**Theorem 3.1.** A Randers metric $F$ has constant flag curvature $\kappa$ if and only if
its Zermelo data $(g_R, W)$ satisfy the following:

(i) the metric $g_R$ has constant curvature $\kappa + \frac{1}{4} \mu^2$ for some constant $\mu$,
(ii) the wind $W$ satisfies $g_R(W, W) < 1$ and it is $\mu$-homothetic for $g_R$, namely,
$$\mathcal{L}_W g_R = 2\mu g_R,$$
where $\mathcal{L}$ is the Lie derivative.

Moreover, the only complete simply connected Randers spaceforms are:

(0) if $\kappa = 0$, those with Zermelo data given by the Euclidean metric and a
parallel vector field with norm less than 1,
(-) if $\kappa < 0$, the hyperbolic space of constant curvature $\kappa$,
(+) if $\kappa > 0$, those with Zermelo data given by the round sphere of radius $1/\sqrt{\kappa}$
and a Killing vector field with norm less than 1.

Observe that given a vector field in the sphere, there is a multiple of it with
norm less than 1. However, completeness excludes the possibility of being prop-
erly homothetic for $W$, as well as most of the cases of Randers metrics that are
not Riemannian for $\kappa \leq 0$. About the Kropina case with Zermelo data $(g_R, W)$,
Yoshikawa and Okubo [29, Th. 4] solution becomes a natural extension of Bao et
al.’s: $g_R$ must have constant curvature but now $W$ must be a homothetic vector
field satisfying $|W|_R \equiv 1$ (in particular, $W$ is then necessarily Killing, see below).
The corresponding explicit cases were obtained by Yoshikawa, and Sabau [30]. For
the convenience of the reader, we include some results on homothe tic and Killing
vector fields in the Appendix so that one can rewrite the concrete Kropina solutions
[29, 30] as follows.

**Theorem 3.2.** A Kropina manifold $(M, F)$ of constant flag curvature is determined
by Zermelo data $(g_R, W)$ which lie in one of the following two cases:

(i) $g_R$ is flat and $W$ is parallel and unit,
(ii) $g_R$ is locally isometric to an odd round sphere $S^{2m+1}(r)$ and $W$ is a unit
Killing vector field, i.e. a unit Hopf vector field.

Proof. By Proposition 4.3, $W$ must be a Killing field. Moreover, Proposition 4.6
determines the two possibilities of the theorem. $\square$

Again the result has a local nature, but all the local examples of Theorem 3.2
can be extended to complete simply connected models.

**Theorem 3.3.** The complete simply connected Kropina manifolds $(M, F)$ of con-
stant flag curvature lie in one of the following two cases, up to isometries:

(i) $(M, g_R) = \mathbb{R}^n$ is flat and $W$ is a parallel, unit vector field of $\mathbb{R}^n$.
(ii) $(M, g_R) = \mathbb{R}^n = S^{2m+1}(r)$ and $W$ is a unit Hopf vector field.

In particular, $W$ is also a complete geodesic vector field.

**Remark 3.4.** As emphasized in subsection 2.3, completeness always means “for-
ward and backward completeness”. Otherwise, further possibilities appear; for
example, in the case (i) the half space $x_n > 0$ of $\mathbb{R}^n$ with $W = \partial x_n$ is forward
complete.
3.2. Local classification for the WRS case. The following result relates the preservation of WRS’s, Finslerian metrics and Zermelo data in the spirit of [2, Proposition 1] (other background results about the indicatrices of wind Riemannian and wind Finslerian structures can be seen in Prop. 2.12, 2.13 and 2.46 of [8]).

**Lemma 3.5.** Let \((M_i, \Sigma_i), i = 1, 2\) be two manifolds endowed with a WRS, each one with the associated Riemannian metric \(g_i\), wind \(W_i\) and conic pseudo-Finsler metrics \(F^i_1, F^i_2\). For any diffeomorphism \(\phi : M_1 \to M_2\), the following conditions are equivalent:

(a) \(\phi_*(\Sigma_1) = \Sigma_2\).

(b) \(\phi\) is an isometry from \((M, g_1)\) to \((M, g_2)\) and \(\phi_*(W_1) = W_2\).

(c) \(\phi^*(F^2_1) = F^1_1\) and \(\phi^*(l^2_1) = l^1_1\).

In any of the previous cases, we will say that \(\phi\) is a WRS isometry, and \((M_1, \Sigma_1)\) and \((M_2, \Sigma_2)\) are called isometric.

**Proof.** For \((a) \Rightarrow (b)\) observe that \(W_i\) is the centroid of the interior of \(\Sigma_i\), because translating \(\Sigma_i\) with \(-W_i\) produces the sphere \(S_i\) of \(g_i\), which is centered in the origin. Moreover, a linear map preserves the centroid, therefore \(\phi_*(W_1) = W_2\), and then \(\phi_*(S_1) = S_2\), which means that \(\phi\) is an isometry from \((M, g_1)\) to \((M, g_2)\). The implication \((b) \Rightarrow (c)\) follows directly from the expression of \(F^i_1\) and \(F^i_2\) in terms of \(g_i\) and \(W_i\) (see formulas (1), (2), (4)) and \((c) \Rightarrow (a)\) is straightforward. \(\Box\)

The strong convexity (resp. concavity) of \(F\) (resp. \(F_1\)) allows one to define the flag curvature for any flagpole in \(A\) (resp. \(A_1\)). So, one has the natural notion.

**Definition 3.6.** A WRS has constant flag curvature if its associated conic metrics \(F\) and \(F_1\) has constant flag curvatures for all the flagpoles in \(A\) and \(A_1\), resp.

**Remark 3.7.** The relation between the flag curvature of (conic) pseudo-Finsler metrics in the same manifold and with indicatrices which are the same up to translation by an arbitrary homothetic vector of one of the pseudo-Finsler metrics has been studied systematically in [15]. In particular, it is checked that if \((M, g_R)\) has constant curvature and \(W\) is a homothetic vector, then the corresponding WRS, \(\Sigma = S_R + W\), has constant flag curvature, as in the Randers case [15, Corollary 5.1].

**Theorem 3.8** (Local classification of WRS). A WRS \((M, \Sigma)\) has constant flag curvature \(\kappa\) if and only if its Zermelo data \((g_R, W)\) satisfy the following:

(i) the Riemannian metric \(g_R\) has constant curvature \(\kappa + \frac{1}{4} \mu^2\) for some constant \(\mu\),

(ii) the wind \(W\) is \(\mu\)-homothetic for \(g_R\), namely, \(L_W g_R = 2 \mu g_R\), where \(L\) is the Lie derivative.

**Proof.** First, observe that the local computation in [2, Th. 3.1] remains valid for the conic Finsler metric \(F\) in the open subset where \(g_R(W, W) \neq 1\), since in this subset \(F\) is of Randers type. Using this observation and Theorem 3.2 we deduce that locally \(g_R\) has constant curvature and \(W\) is \(\mu\)-homothetic in an open dense subset: the manifold \(M\) except the points in the boundary \(\partial U\) of the region \(U\) where \(g_R(W, W) \neq 1\). As both conditions are closed we deduce that \(g_R\) is of constant curvature and \(W\) is \(\mu\)-homothetic in every connected component of \(M \setminus \partial U\) and, by continuity, on all \(M\). Thus, the result follows taking into account Remark 3.7. \(\Box\)
Remark 3.9. We have used the Kropina classification [29, 30] in our proof. However, one could consider the expression (1) for the conic Finsler metric, which remains valid in all the regions, and try to carry out all the Randers-type computations in order to reprove the Randers-Kropina case.

3.3. The global result. In order to go from the local to the global result, the following theorem on completeness becomes crucial, and the Example 3.11 below is important to know exactly what is going on.

**Theorem 3.10.** Let $\Sigma$ be a WRS determined by Zermelo data $(g_R, W)$.

(i) If $\Sigma$ is complete, then $W$ is complete.

(ii) If $g_R$ is complete and $W$ is a homothetic vector field, then $\Sigma$ is complete. Moreover, if $W$ is properly homothetic (i.e., non-Killing), then $g_R$ is flat.

**Proof.** (i) Notice that each integral curve $\rho$ of $W$ is a wind curve. Thus, if $\rho$ were forward or backward incomplete, then a closed forward or backward ball (centered at $\rho(0)$ and with radius equal to the finite length of $\rho$ towards $+\infty$ or $-\infty$ would be non compact (recall that, by Theorem 2.1, the geodesic completeness of a WRS is equivalent to the precompactness of its balls).

(ii) Assuming that $W$ is complete, the first assertion follows from Theorem 2.2. However, the completeness of $W$ can be deduced from the completeness of $g_R$. Indeed, when $W$ is Killing, such a result follows from the fact that the flow of $W$ must preserve a small closed ball along all its integral curves (for a more general result valid even in the semi-Riemannian case, see [23, Prop. 9.30]). In the case that $W$ is properly homothetic, Tashiro [26, Th. 4.1] (which summarizes a conclusion from previous work by Kobayashi [18], Nomizu [22] and Yano and Tagano [28]) ensures that $(M, g_R)$ is flat (thus, obtaining the second assertion). Therefore, $(M, g_R)$ is globally covered by $\mathbb{R}^n$. But the homothetic vector fields in $\mathbb{R}^n$ are well known and they must be complete.\hfill $\square$

It is worth emphasizing that the completeness of $\Sigma$ does not imply the completeness of $g_R$ even in the Randers case.

**Example 3.11.** Consider on $\mathbb{R}^+$ the Randers metric $R$ determined by Zermelo’s $(g_R = dx^2, W = f \partial_x)$, where $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is any function satisfying the following. Take the intervals $I_k = \left[2^{-(k+1)}, 2^{-k}\right], k = 0, 1, \ldots$ and put

$$f(x) = \begin{cases} 2^{-(4k+1)} - 1 & \text{on } I_{4k} \\ 1 - 2^{-(4k+3)} & \text{on } I_{4k+2} \end{cases}$$

so that $\begin{cases} R(\partial_x) = 2^{4k+1} \\ R(-\partial_x) = 2^{4k+3} \end{cases}$ on $I_{4k}$ with no restriction on $f$ outside the above intervals except being $C^\infty$ and $|f| < 1$ on all $\mathbb{R}^+$. As the length of each interval $I_k$ is $2^{-(k+1)}$, the $R$-length $L$ of the curves $\gamma_{\pm}(s) = \pm s$ satisfies $L(\gamma_{+}|_{t_k}) = 1, L(\gamma_{-}|_{t_{k+2}}) = 1$. This implies the (forward and backward) completeness of $R$, in spite of the incompleteness of $g_R$.\footnote{Notice that they are affine, and all the affine vector fields of $\mathbb{R}^n$ have affine natural coordinates. Thus, their completeness follows by direct integration (or by applying general results such as [25, Theorem 1].)}

\footnote{An alternative way of producing this type of examples appears considering the spacetime viewpoint. Take any SSTK splitting $\mathbb{R} \times M$ with a Cauchy hypersurface $\{0\} \times M$ which inherits an incomplete Riemannian metric $g_R$ (a Cauchy hypersurface $S$ satisfying this property can be easily constructed in Lorentz-Minkowski, and moving $S$ with the flow of the natural timelike parallel vector field $K = \partial_t$ one obtains the required SSTK splitting). The completeness for the induced Randers metric (obtained from the incomplete $g_R$ and the wind $W$ in (7) and (8)) is a...}
Theorem 3.12. The complete simply connected WRS's with constant flag curvature lie in one of the following two exclusive cases, determined by Zermelo data:

(i) \( (M, g_R) \) is a model space of constant curvature and \( W \) is any of its Killing vector fields.

(ii) \( (M, g_R) \) is isometric to \( \mathbb{R}^n \) and \( W \) is a properly homothetic (non-Killing) vector field.

Proof. The fact that the two WRS’s above have constant flag curvature comes from the local result (Theorem 3.8) and their completeness from part (ii) of Theorem 3.10.

Conversely, if \( \Sigma \) is a complete simply connected WRS with constant flag curvature, from the local result we know that locally \( g_R \) has constant curvature (and, so, this happens globally) and \( W \) is a homothetic vector field. Now, we have to check the completeness of \( g_R \). Recall first that \( W \) must be complete (Theorem 3.10-(ii)) and consider two cases.

In the case that \( W \) is Killing for \( g_R \), then it is also Killing for \( \Sigma \), in the sense that its flow is composed by isometries (indeed, it satisfies the condition (b) in Lemma 3.5). Notice that \( g_R \) can be seen as the WRS obtained by the translation \( \Sigma - W \), and the completeness of this WRS follows from Theorem 2.2, since \( \Sigma \) is complete and, as commented above, \( W \) is Killing for \( \Sigma \) and complete because it is Killing for \( g_R \).

In the case that \( W \) is properly homothetic, \( g_R \) must be flat (see part (i) of Proposition 4.3). Let us first see that the completeness of \( W \) and \( \Sigma \) implies that \( W \) must have a zero. Choose any point \( p_0 \) and the integral curve \( \gamma \) of \( W \) through \( p_0 \). We can assume that the \( W \)-flow \( \phi_t \) is homothetic of ratio \( e^{-\mu} < 1 \) for \( t = -1 \) (otherwise, take the flow of \(-W\) and recall the comment about the reverse metric before Theorem 2.1). Then, \( \{p_k := \phi_{-k}(p_0)\}_{k \in \mathbb{N}} \) is a Cauchy sequence for \( g_R \) and the \( g_R \)-length of \( \gamma|_{(-\infty, 0]} \) is finite. Moreover, for some \( t_0 \leq 0 \) one has that \(|W|_{R} < 1/2\) at \( q = \phi_{t_0}(p_0) \) and, then, also on all \( \phi_t(p) \) for \( t \leq t_0 \). Thus, \(|W|_{R} < 1/2\) on a neighborhood \( U \) of \( \gamma|_{(-\infty, t_0]} \) and \( \Sigma \) becomes a Randers metric \( F \) on \( U \). From the definition of \( F \), one has

\[
F(v_x) \leq \frac{|v_x|_{R}}{1 - |Wx|_{R}} \leq 2|v_x|_{R}
\]

on \( U \). Thus, the \( F \)-length of \( \gamma|_{(-\infty, 0]} \) is also finite and \( \{p_k\}_k \) becomes a Cauchy sequence also for \( F \). Observe that there exists \( t_0 < 0 \) such that \( \gamma|_{(-\infty, t_0]} \) is contained in the backward ball \( B_{\Sigma}^-(\gamma(0), t_F(\gamma)) \) (recall that \( F \) is Randers in the neighborhood \( U \) of \( \gamma|_{(-\infty, t_0]} \)), and this ball is, by the hypothesis of completeness of \( \Sigma \) and Theorem 2.1, precompact. Then the Cauchy sequence \( \{p_k\}_k \) will have a limit \( p_{\infty} \) and \( W_{\infty} = 0 \) (on \( p_k \), \(|W|_{R} \) is smaller than \( e^{-k\mu} \rightarrow 0 \), as required. Now, for some small \( \epsilon > 0 \), the closed \( g_R \)-ball \( B(p_{\infty}, \epsilon) \) is compact and, thus, so is \( \phi_k(B(p_{\infty}, \epsilon)) = B(p_{\infty}, e^{k\epsilon}) \) for all \( k > 0 \), which implies that the flat manifold has to be isometric to \( \mathbb{R}^n \).

Finally, the classical models of simply connected spaceforms plus part (ii) of Theorem 3.10 imply that the two stated cases are the unique possibilities of global representation for a WRS spaceform.
Remark 3.13. In the proof of completeness for homothetic $W$ above, it was crucial that the completeness of $\Sigma$ implied the existence of a point invariant by the $W$-flow. Indeed, $\mathbb{R}^n$ minus the origin (or minus any closed set of radial half lines) is an example of flat incomplete manifold with a complete homothetic vector field, namely, $r \partial_r$.

Thus, in the case (ii) one can choose the origin 0 of $\mathbb{R}^n$ as the unique point where $W$ must vanish. Then, for some $\mu \neq 0$, the vector field $Y := W - \mu r \partial_r$ is Killing, vanishing at 0, and tangent to the spheres centred at 0.

4. Appendix: results on homothetic and Killing fields

Next, some results on Killing and homothetic vector fields spread in the literature, which become relevant for our discussions, are collected for the convenience of the reader (see also the summary in [24] for other related results).

4.1. Global and local results. Global results (to be taken into account in the global classification of constant flag WRS) are the following.

Proposition 4.1. Let $(M, g_R)$ be a complete Riemannian manifold.

(i) If $W$ is a homothetic vector field and $g$ is not flat, then $W$ is Killing.

(ii) If $W$ is a homothetic (or, with more generality, an affine) vector field with bounded (pointwise) norm, then $W$ is Killing. Moreover, in the case that $g_R$ is flat then $W$ is parallel.

(iii) If $g_R$ has constant curvature $\kappa \leq 0$ and $W$ is homothetic with bounded norm then $\kappa = 0$ and $W$ is parallel.

Proof. (i) See [19, Lemma VI.2 (p. 242)].

(ii) The first assertion is a well known theorem by Hano [13, Theorem 2] (see also [19, Theorem VI.3.8]), for the second one see [13, Lemma 4].

(iii) We know from (ii) that $W$ is Killing and, in the case $\kappa = 0$, it is parallel. So, passing to the universal covering, it is enough to prove that the norm of any Killing vector field of the hyperbolic space $\mathbb{H}^n$ is unbounded. Looking $\mathbb{H}^n$ as the upper component of the unit timelike vectors of Lorentz-Minkowski $\mathbb{L}^{n+1}$, its Killing vector fields can be seen as the subalgebra of the Killing fields of $\mathbb{L}^{n+1}$ which vanish at the origin (the flow of such Killing fields leaves invariant the unit vectors, so, they preserve and are tangent to $\mathbb{H}^n$). If $S^*$ is one of such vector fields, it can be constructed by taking each skew-adjoint linear matrix $S$ of $\mathbb{L}^{n+1}$ and putting $S_p^* = (Sp)_p$ for all $p \in \mathbb{L}^{n+1}$, where $(Sp)_p$ is obtained by regarding $Sp \in \mathbb{L}^{n+1}$ as a vector in $T_p\mathbb{L}^n$ (see for example [23, Example 9.29]). Notice that these skew-adjoint matrices are:

$S = \begin{pmatrix} 0 & b_1 \ldots b_n \\ b_1 & \vdots & A \\ \vdots & & \vdots \\ b_n \end{pmatrix}$

where $A$ is any skew-symmetric $n \times n$ matrix ($a_{ij} = -a_{ji}$). Then, choosing the points $p^*_p(\lambda) = (\sqrt{1 + \lambda^2}, 0, \ldots, 0, \pm \lambda, 0, \ldots, 0)^t \in \mathbb{H}^n$ for $i = 1, \ldots, n$, and making $\lambda \to +\infty$ one checks that the unique possibility to bound the induced Riemannian norm on all $Sp$ is to put $S = 0$. \qed
Remark 4.2. The previous result shows that, under the completeness of $g_R$, one obtains necessarily a WRS which is not Randers in the following cases:

(a) when the wind vector $W$ is properly homothetic,

(b) in the case of constant curvature $k \leq 0$, when $W$ is Killing but non-parallel.

Indeed, the Randers models of constant flag curvature obtained in [1, 2] fail to be complete in these cases. For the case of positive curvature, of course, one can take any Killing vector field as the wind $W$ and multiply it by a constant so that it becomes mild everywhere (or if it has no zeroes, strong everywhere).

The following local result (with no assumption of completeness) bounds the possible cases of WRS with constant flag curvature.

Proposition 4.3. Let $Z$ be a homothetic vector field in a Riemannian manifold of constant curvature $\kappa$:

(i) When $\kappa \neq 0$, $Z$ is Killing.

(ii) When $\kappa = 0$, $Z$ has constant norm if and only if $Z$ is parallel.

Proof. (i) As observed in [2, §3.2.1], the flow $\psi_t$ of a $\mu$-homothetic field sends the constant curvature $k$ to $e^{2\mu t}k$, and then by the hypothesis of constant curvature, $e^{2\mu t} = 1$ and $\mu = 0$.

(ii) Observe that any homothetic vector field is affine. Then, by a direct local computation in natural orthonormal coordinates of $\mathbb{R}^n$, any affine vector field $Z$ and its norm are written as

$$Z = \sum_{j=1}^{n} \left( a_j + \sum_{i=1}^{n} a_j^i x_i \right) \partial_j, \quad |Z|^2 = \sum_{j=1}^{n} \left( a_j + \sum_{i=1}^{n} a_j^i x_i \right)^2,$$

and the matrix $(a_j^i)$ vanishes by equating $\frac{\partial^2}{\partial x_k^2} |Z|^2 = 0$ for $k = 1, \ldots, n$. □

Remark 4.4. Observe that the part (i) of Proposition 4.3 can be generalized in several directions. For example, Knebelman [16, Theorem 4 and below] showed that any affine vector field in a non-Ricci flat Einstein manifold is Killing, and Knebelman and Yano [17, Theorem 2] that, for non-zero constant scalar curvature, any homothetic vector field is Killing.

4.2. Further results for Killing vectors with constant norm. Once the homothetic case was ruled out by Proposition 4.3, the hypothesis of constant norm for Killing vector fields becomes essential for the possible Kropina models. So, the following results are in order.

Lemma 4.5. A Killing vector field $Z$ on a Riemannian manifold has constant norm if and only if it is geodesic. In this case $g(\nabla_X Z, \nabla_Y Z) = R(X, Z, Z, Y)$ for all vector fields $X, Y$ on $M$.

Proof. The first assertion follows from $g(\nabla_Z Z, X) = -g(\nabla_X Z, Z) = -\frac{1}{2} X(g(Z, Z))$. The second one is a straightforward computation (see [4, Lemma 3, Prop. 1]). □

Proposition 4.6. Let $Z$ be a Killing vector field of constant norm 1 on a Riemannian manifold:

(i) $Z$ cannot exist for negative curvature, and must be parallel for 0 curvature.

(ii) $Z$ cannot exist in manifolds of positive curvature of even dimension.
In round spheres of dimension odd $S^{2n+1}(r)$, the vector field $Z$ is a Hopf vector field, i.e., one of the natural unit Killing vector fields tangent to the fiber in Hopf fibration.

Proof. (i) Using the formula in the previous lemma
\[ g(\nabla_X Z, \nabla_X Z) = R(X, Z, Z, X). \]
In the case of negative curvature, a contradiction is obtained by choosing $X \neq 0$ orthogonal to $Z$. The flat case follows applying this formula to all $X$.

(ii) As $\nabla Z$ is a skew adjoint operator ($X \mapsto \nabla_X Z$) that can be restricted to the (odd dimensional) orthogonal to $Z$, it has a 0 eigenvalue, and one of such eigenvectors $X$ yields a contradiction in the formula above.\footnote{This proof follows the spirit of Berger's one \cite{Ber} to prove that any Killing vector field on an even dimensional compact manifold of positive curvature must have a zero.}

(iii) The proof (for radius $r = 1$) is in \cite[Theorem 1]{GM} (see also \cite[§3]{G}).

Recall that the exception in the case (iii) corresponds to a natural unit Killing vector field tangent to the fiber in Hopf fibration which is extended to a fibration in any odd dimension $S^{2n+1} \to CP^n$ (see for instance \cite[Vol. II, p. 135]{B}) thus providing all the possible examples in the odd dimensional case.

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