The Chevalley-Shephard-Todd Theorem for Finite Linearly Reductive Group Schemes

Matthew Satriano

1 Introduction

Given a field $k$ and an action of a finite (abstract) group $G$ on a $k$-vector space $V$, we obtain a linear action of $G$ on the polynomial ring $k[V]$. A central theme in Invariant Theory is determining when certain nice properties of a ring with $G$-action are inherited by its invariants. In particular, it is natural to ask when $k[V]^G$ is polynomial. If $G$ acts faithfully on $V$, we say $g \in G$ is a pseudo-reflection (with respect to the action of $G$ on $V$) if $V^g$ is a hyperplane. The classical Chevalley-Shephard-Todd Theorem states

**Theorem 1.1** ([Bo, §5 Thm 4]). If $G \to \text{Aut}_k(V)$ is a faithful representation of a finite group and the order of $G$ is not divisible by the characteristic of $k$, then $k[V]^G$ is polynomial if and only if $G$ is generated by pseudo-reflections.

In this paper we generalize this theorem to the case of finite linearly reductive group schemes. To do so, we first need a notion of pseudo-reflection in this setting.

**Definition 1.2.** Let $k$ be a field and $V$ a finite-dimensional $k$-vector space with a faithful action of a finite linearly reductive group scheme $G$ over $\text{Spec} \, k$. We say that a subgroup scheme $N$ of $G$ is a pseudo-reflection if $V^N$ has codimension 1 in $V$. We define the subgroup scheme generated by pseudo-reflections to be the intersection of the subgroup schemes which contain all of the pseudo-reflections of $G$. We say $G$ is generated by pseudo-reflections if $G$ is the subgroup scheme generated by pseudo-reflections.

Our first main result is then

**Theorem 1.3.** Let $k$ be an algebraically closed field and $V$ a finite-dimensional $k$-vector space with a faithful action of a finite linearly reductive group scheme $G$ over $\text{Spec} \, k$. Then $G$ is generated by pseudo-reflections if and only if $k[V]^G$ is polynomial.

A version of this theorem holds over fields which are not algebraically closed as well; however, the “only if” direction does not hold for finite linearly reductive group schemes in general (see Example 2.3). We instead prove the “only if” direction for the smaller class of stable group schemes, which we now describe (see Proposition 2.1 for examples). Over an algebraically closed field, the class of stable group schemes coincides with that of finite linearly reductive group schemes. Recall from [AOV, Def 2.9] that $G$ is called well-split if it is isomorphic to a semi-direct product $\Delta \rtimes Q$, where $\Delta$ is a finite diagonalizable group scheme and $Q$ is a finite constant tame group scheme; here, tame means that the degree is prime to the characteristic.

**Definition 1.4.** A group scheme $G$ over a field $k$ is called stable if the following two conditions hold:

(a) for all finite field extensions $K/k$, every subgroup scheme of $G_K$ descends to a subgroup scheme of $G$
there exists a finite Galois extension $K/k$ such that $G_K$ is well-split.

**Remark 1.5.** If $G$ is a finite linearly reductive group scheme over a perfect field $k$, then \cite[Lemma 2.11]{AOV} shows that condition (b) above is automatically satisfied.

Theorem 1.3 is then a special case of the following generalization of the Chevalley-Shephard-Todd theorem:

**Theorem 1.6.** Let $k$ be a field and $V$ a finite-dimensional $k$-vector space with a faithful action of a finite linearly reductive group scheme $G$ over Spec$k$. If $G$ is generated by pseudo-reflections, then $k[V]^G$ is polynomial. The converse holds if $G$ is stable.

We also prove a version of this theorem for an action of a finite linearly reductive group scheme on a smooth scheme.

**Definition 1.7.** Given a smooth affine scheme $U$ over Spec$k$ with a faithful action of a finite linearly reductive group scheme $G$ which fixes a field-valued point $x \in U(K)$, we say a subgroup scheme $N$ of $G$ is a pseudo-reflection at $x$ if $N_K$ is a pseudo-reflection with respect to the induced action of $G_K$ on the cotangent space at $x$. We define what it means for $G$ to be generated by pseudo-reflections at $x$ in the same manner as in Definition 1.2.

Theorem 1.6 then has the following corollary.

**Corollary 1.8.** Let $k$ be a field, let $U$ be a smooth affine $k$-scheme with a faithful action by a finite linearly reductive group scheme $G$ over Spec$k$. Let $x \in U(K)$, where $K/k$ is a finite separable field extension, and suppose $x$ is fixed by $G$. If $G$ is generated by pseudo-reflections at $x$, then $U/G$ is smooth at the image of $x$. The converse holds if $G$ is stable.

The second main result of this paper is

**Theorem 1.9.** Let $k$ be a field and let $U$ be a smooth affine $k$-scheme with a faithful action by a stable group scheme $G$ over Spec$k$. Suppose $K/k$ is a finite separable field extension and $G$ fixes a point $x \in U(K)$. Let $M = U/G$, let $M^0$ be the smooth locus of $M$, and let $U^0 = U \times_M M^0$. If $G$ has no pseudo-reflections at $x$, then after possibly shrinking $M$ to a smaller Zariski neighborhood of the image of $x$, we have that $U^0$ is a $G$-torsor over $M^0$.

We remark that in the classical case, Theorem 1.9 follows directly from Corollary 1.8 and the purity of the branch locus theorem \cite[X.3.1]{SGA1}. For us, however, a little more work is needed since $G$ is not necessarily étale.

As an application of Theorem 1.9 we generalize the well-known result (see for example \cite[2.9]{Vi} or \cite[Rmk 4.9]{FMN}) that schemes with quotient singularities prime to the characteristic are coarse spaces of smooth Deligne-Mumford stacks. We say a scheme has linearly reductive singularities if it is étale locally the quotient of a smooth scheme by a finite linearly reductive group scheme. We show that every such scheme $M$ is the coarse space of a smooth tame Artin stack (in the sense of \cite{AOV}) whose stacky structure is supported at the singular locus of $M$. More precisely,

**Theorem 1.10.** Let $k$ be a perfect field and $M$ a $k$-scheme with linearly reductive singularities. Then it is the coarse space of a smooth tame stack $X$ over $k$ such that $f^0$ in the diagram

$$
\begin{array}{c}
\chi^0 \xrightarrow{j_0} \chi \\
\downarrow f^0 \quad \downarrow f \\
M^0 \xrightarrow{j} M
\end{array}
$$

is an isomorphism, where $j$ is the inclusion of the smooth locus of $M$ and $\chi^0 = M^0 \times_M \chi$.  


This paper is organized as follows. In Section 2, we prove the “if” direction of Theorem 1.6 and reduce the proof of the “only if” direction to the special case of Theorem 1.9 in which \( U = V^\vee(V) \) for some \( k \)-vector space \( V \) with \( G \)-action (see the Notation section below). This special case is proved in Section 3. The key input for the proof is a result of Iwanari [Iw, Thm 3.3] which we reinterpret in the language of pseudo-reflections. We finish the section by proving Corollary 1.8. In Section 4, we use Corollary 1.8 to complete the proof of Theorem 1.9. In Section 5, we prove Theorem 1.10.

Acknowledgements. I would like to thank Dustin Cartwright, Ishai Dan-Cohen, Anton Geraschenko, and David Rydh for many helpful conversations. I am of course indebted to my advisor, Martin Olsson, for his suggestions and help in editing this paper. Lastly, I thank Dan Edidin for suggesting that I write up these results in this stand-alone paper rather than include them in [Sa].

Notation. Throughout this paper, \( k \) is a field and \( S = \text{Spec}k \). If \( V \) is a \( k \)-vector space with an action of a group scheme \( G \), then we denote by \( V^\vee(V) \), or simply \( V^\vee \) if \( V \) is understood, the scheme \( \text{Spec}k[V] \) whose \( G \)-action is given by the dual representation on functor points. Said another way, if \( G = \text{Spec}A \) is affine and its action on \( V \) is given by the co-action map \( \sigma : V \longrightarrow V \otimes_k A \), then the co-action map \( k[V] \longrightarrow k[V] \otimes_k A \) defining the \( G \)-action on \( V^\vee \) is given by \( \sum a_i v_i \mapsto \sum a_i \sigma(v_i) \).

All Artin stacks in this paper are assumed to have finite diagonal so that, by Keel-Mori [KM], they have coarse spaces. Given a scheme \( U \) with an action of a finite flat group scheme \( G \), we denote by \( U/G \) the coarse space of the stack \([U/G]\).

If \( R \) is a ring and \( \mathcal{I} \) an ideal of \( R \), then we denote by \( V(\mathcal{I}) \) the closed subscheme of \( \text{Spec}R \) defined by \( \mathcal{I} \).

2 Linear Actions on Polynomial Rings

Our goal in this section is to prove the “if” direction of Theorem 1.6 and show how the “only if” direction follows from the special case of Theorem 1.9 in which \( U = V^\vee \). We begin with examples of stable group schemes and with some basic results about the subgroup scheme generated by pseudo-reflections.

Proposition 2.1. Let \( G \) be a finite group scheme over \( S \). Consider the following conditions:

1. \( G \) is diagonalizable.
2. \( G \) is a constant group scheme.
3. \( k \) is perfect, the identity component \( \Delta \) of \( G \) is diagonalizable, and \( G/\Delta \) is constant.

If any of the above conditions hold, then \( G \) is stable.

Proof. It is clear that finite diagonalizable group schemes and finite constant group schemes are stable, so we consider the last case. Let \( Q = G/\Delta \). Since \( k \) is perfect, the connected-étale sequence

\[ 1 \longrightarrow \Delta \longrightarrow G \longrightarrow Q \longrightarrow 1 \]

is functorially split (see [Ta, 3.7 (IV)]). Let \( K/k \) be a finite extension and let \( P \) be a subgroup scheme of \( G_K \). Letting \( \Delta' = P \cap \Delta_K \) and \( Q' = P/\Delta' \), we have a commutative
diagram

\[
\begin{array}{ccccccc}
1 & \rightarrow & \Delta_K & \rightarrow & G_K & \rightarrow & Q_K & \rightarrow & 1 \\
\varphi & & \downarrow & & \downarrow & & \psi \\
1 & \rightarrow & \Delta' & \rightarrow & P & \rightarrow & Q' & \rightarrow & 1
\end{array}
\]

with exact rows. Since $\Delta$ is connected and has a $k$-point, [EGA4, 4.5.14] shows that $\Delta$ is geometrically connected. In particular, $\Delta_K$ is the connected component of the identity of $G_K$, and so $\Delta'$ is the connected component of the identity of $P$. Therefore, the bottom row of the above diagram is the connected-étale sequence of $P$, and so

\[ P = \Delta' \rtimes Q' \]

as $k$ is perfect. Since $Q'$ is a constant group scheme, it clearly descends to a group scheme over $k$. Similarly, since $\Delta'$ is diagonalizable, it is of the form $\text{Spec} K[A]$ for some finite abelian group $A$, and so it descends to $\text{Spec} k[A]$. To show that $P$ descends, note that its underlying scheme is $\Delta' \times K Q'$ and its group structure is given by an action of $Q'$ on $\Delta'$. This action of $Q'$ is equivalent to an action of $Q'$ on $A$. We can therefore define a group scheme structure on $\text{Spec} k[A] \times_k Q'$ using the same action of $Q'$ on $A$. This group scheme $P_0$ over $k$ then pulls back to $P$ over $K$.

Lastly, we must show that $P_0$ is a subgroup scheme of $G$. Let $*$ denote the action of $Q_K$ (resp. $Q'$) on $\Delta_K$ (resp. $\Delta'$). Since the splitting of the connected-étale sequence of a finite group scheme over a perfect field is functorial, we see that for all $q' \in Q'$ and local sections $\delta'$ of $\Delta'$,

\[ \psi(q') \ast \varphi(\delta') = \varphi(q' \ast \delta') . \]

We therefore obtain a morphism from $P_0$ to $G$ whose pullback to $K$ is the morphism from $P$ to $G_K$.

**Lemma 2.2.** Let $V$ be a finite-dimensional $k$-vector space with a faithful action of a stable group scheme $G$ over $S$, and let $H$ be the subgroup scheme generated by pseudo-reflections. If $K/k$ is a finite field extension, then a subgroup scheme of $G_K$ is a pseudo-reflection if and only if it descends to a pseudo-reflection. Furthermore, $H_K$ is the subgroup scheme of $G_K$ generated by pseudo-reflections.

**Proof.** Note that if $N$ is a pseudo-reflection of $G$, then $N_K$ is a pseudo-reflection of $G_K$, as

\[ (V_K)^N_K = (V^N)_K. \]

Since $G$ is stable, this proves the first claim. The second claim follows from the fact that if $P'$ and $P''$ are subgroup schemes of $G$, then $P'_K$ contains $P''_K$ if and only if $P'$ contains $P''$. 

We remark that even in characteristic zero, Lemma 2.2 is false for general finite linearly reductive group schemes $G$, as the following example shows. Note that this example also shows that the “only if” direction of Theorem 1.6 and of Corollary 1.8 is false for general finite linearly reductive group schemes.

**Example 2.3.** Let $k$ be a field contained in $R$ or let $k = \mathbf{F}_p$ for $p$ congruent to 3 mod 4. Let $K = k(i)$, where $i^2 = -1$, and let $G$ be the locally constant group scheme over $\text{Spec} k$
whose pullback to Spec $K$ is $\mathbb{Z}/2 \times \mathbb{Z}/2$ with the Galois action that switches the two $\mathbb{Z}/2$ factors. Let $g_1$ and $g_2$ be the generators of the two $\mathbb{Z}/2$ factors and consider the action 

$$\rho : G_K \longrightarrow \text{Aut}_K(K^2)$$

on the $K$-vector space $K^2$ given by 

$$\rho(g_1) : (a, b) \mapsto (-bi, ai)$$

$$\rho(g_2) : (a, b) \mapsto (bi, -ai).$$

Then $\rho$ is Galois-equivariant and hence comes from an action of $G$ on $k^2$. Note that $\mathbb{Z}/2 \times 1$ and $1 \times \mathbb{Z}/2$ are both pseudo-reflections of $G_K$, as the subspaces which they fix are $K \cdot (1, i)$ and $K \cdot (1, -i)$, respectively. Since $G_K$ is not a pseudo-reflection, it follows that there are no Galois-invariant pseudo-reflections of $G_K$, and hence, the subgroup scheme generated by pseudo-reflections of $G_K$, however, is $G_K$.

**Corollary 2.4.** If $V$ is a finite-dimensional $k$-vector space with a faithful action of a stable group scheme $G$ over $S$, then the subgroup scheme generated by pseudo-reflections is normal in $G$.

**Proof.** We denote by $H$ the subgroup scheme generated by pseudo-reflections. Let $T$ be an $S$-scheme and let $g \in G(T)$. We must show the subgroup schemes $H_T$ and $gH_Tg^{-1}$ of $G$ are equal. To do so, it suffices to check this on stalks and so we can assume $T = \text{Spec} R$, where $R$ is strictly Henselian. By [AOV, Lemma 2.17], we need only show that these two group schemes are equal over the closed fiber of $T$, so we can further assume that $R = K$ is a field. Since $G$ is finite over $S$, the residue fields of $G$ are finite extensions of $k$. We can therefore assume that $K/k$ is a finite field extension.

By Lemma 2.2, we know that $H_K$ is the subgroup scheme of $G_K$ generated by pseudo-reflections. Note that if $N'$ is a pseudo-reflection of $G_K$, then $gN'g^{-1}$ is as well since

$$V_K^{gN'g^{-1}} = g(V_K^{N'}).$$

As a result, $gH_Kg^{-1} = H_K$, which completes the proof. 

**Lemma 2.5.** Given a finite-dimensional $k$-vector space $V$ with a faithful action of a finite linearly reductive group scheme $G$ over $S$, let $\{N_i\}$ denote the set of pseudo-reflections of $G$ and let $H$ be the subgroup scheme generated by pseudo-reflections. If the intersection of the $k[V]^{N_i}$ in $k[V]$ is polynomial, then 

$$k[V]^H = \bigcap_i k[V]^{N_i}.$$

**Proof.** Let $R = \bigcap_i k[V]^{N_i}$. Consider the functor 

$$G_R : (k\text{-alg}) \longrightarrow (\text{Groups})$$

$$A \longmapsto \{ g \in G(A) \mid g(m) = m \text{ for all } m \in R \otimes_k A \}. $$

By hypothesis, $R = k[r_1, \ldots, r_n]$ for some $r_i$ in $k[V]$. We see then that $G_R$ is the intersection of the stabilizers $G_{r_j}$, and so is represented by a closed subgroup scheme of $G$. Since $G_R$ contains every pseudo-reflection, we see $H \subset G_R$. We therefore have the containments 

$$R \subset k[V]^{G_R} \subset k[V]^H \subset \bigcap_i k[V]^{N_i}$$

from which the lemma follows. 

\[\square\]
If $N$ is any subgroup scheme of $G$, it is linearly reductive by \cite[Prop 2.7]{AOV}. It follows that
\[ V \cong V^N \oplus V/V^N \]
as $N$-representations. If $N$ is a pseudo-reflection, then $\dim_k V/V^N = 1$. Let $v$ be a generator of the 1-dimensional subspace $V/V^N$ and let $\sigma : V \rightarrow V \otimes_k B$ be the coaction map, where $N = \text{Spec } B$. Then via the above isomorphism, $\sigma$ is given by
\[ V^N \oplus V/V^N \longrightarrow (V^N \otimes_k B) \oplus (V/V^N \otimes_k B) \]
for some $b \in B$. It follows that there is a $k$-linear map $h : V \rightarrow B$ such that for all $w \in V$,
\[ \sigma(w) = (w \otimes 1, w \otimes b) \]
for some $b \in B$. If we continue to denote by $\sigma$ the induced coaction map $k[V] \longrightarrow k[V] \otimes_k B$, we see that $h$ extends to a $k[V]^N$-module homomorphism $k[V] \longrightarrow k[V] \otimes_k B$, which we continue to denote by $h$, such that for all $f \in k[V]$,
\[ \sigma(f) - (f \otimes 1) = (v \otimes 1) \cdot h(f). \]

We are now ready to prove the “if” direction of Theorem \ref{thm:1.6}. Our proof is only a slight variant of the proof of the classical Chevalley-Shephard-Todd Theorem presented in \cite{Sm}.

**Proof of “if” direction of Theorem \ref{thm:1.6}** By Lemma \ref{lem:2.5}, it suffices to show that the intersection $R$ of the $k[V]^N$ is polynomial, where $N$ runs through the pseudo-reflections of $G$. Since $k[V]$ is a finite $k[V]^G$-module, it is a finite $R$-module as well. As explained in \cite{Sm}, to show $R$ is polynomial, it suffices to show $\text{Tor}_1^R(k, k[V]) = 0$. Since $R$ is contained in the graded ring $k[V]$, we have an augmentation map $\epsilon : R \rightarrow k$. We therefore have a long exact sequence
\[ 0 \longrightarrow \text{Tor}_1^R(k, k[V]) \longrightarrow \ker \epsilon \otimes_R k[V] \xrightarrow{\phi} R \otimes_R k[V] \xrightarrow{\epsilon \otimes 1} k \otimes_R k[V] \longrightarrow 0. \]

To show $\text{Tor}_1^R(k, k[V]) = 0$, we must show $\phi$ is injective. We in fact show
\[ \phi \otimes 1 : \ker \epsilon \otimes_R k[V] \otimes_k C \longrightarrow k[V] \otimes_k C \]
is injective for all finite-dimensional $k$-algebras $C$. If this is not the case, then the set
\[ \{ \xi \mid C \text{ is a finite-dimensional } k\text{-algebra, } 0 \neq \xi \in \ker \epsilon \otimes_R k[V] \otimes_k C, (\phi \otimes 1)(\xi) = 0 \} \]
is non-empty and we can choose an element $\xi$ of minimal degree, where $\ker \epsilon$ is given its natural grading as a submodule of $k[V]$ and the elements of $C$ are defined to be of degree 0. We begin by showing $\xi \in \ker \epsilon \otimes_R R \otimes_k C$. That is, we show $\xi$ is fixed by all pseudo-reflections.

Let $N = \text{Spec } B$ be a pseudo-reflection. Let $\sigma : k[V] \longrightarrow k[V] \otimes B$ be the coaction map. As explained above, we get a $k[V]^N$-module homomorphism $h : k[V] \longrightarrow k[V] \otimes B$. Note that this morphism has degree -1. Since
\[ (1 \otimes \sigma \otimes 1)(\xi) - \xi \otimes 1 = (1 \otimes h \otimes 1)(\xi) \cdot (1 \otimes v \otimes 1 \otimes 1), \]
the commutativity of
\[
\begin{array}{ccc}
\ker \epsilon \otimes k[V] \otimes B \otimes C & \xrightarrow{\phi \otimes 1 \otimes 1} & k[V] \otimes B \otimes C \\
1 \otimes \sigma \otimes 1 & \downarrow & \\
\ker \epsilon \otimes k[V] \otimes C & \xrightarrow{\phi \otimes 1} & k[V] \otimes C
\end{array}
\]
imply

$$(\phi \otimes 1 \otimes 1)(1 \otimes h \otimes 1)(\xi) \cdot (v \otimes 1 \otimes 1) = 0.$$ 

It follows that $(1 \otimes h \otimes 1)(\xi)$ is killed by $\phi \otimes 1 \otimes 1$. Since $h$ has degree -1, our assumption on $\xi$ shows that $(1 \otimes h \otimes 1)(\xi) = 0$. We therefore have $(1 \otimes \sigma \otimes 1)(\xi) = \xi \otimes 1$, which proves that $\xi$ is $N$-invariant.

Since $G$ is linearly reductive, we have a section of the inclusion $k[V]^G \hookrightarrow k[V]$. We therefore, also obtain a section $s$ of the inclusion $j : R \hookrightarrow k[V]$. Let $\psi : \ker \epsilon \otimes_R R \longrightarrow R$ be the canonical map, and consider the diagram

$$
\begin{array}{ccc}
\ker \epsilon \otimes k[V] \otimes C & \xrightarrow{\phi \otimes 1} & k[V] \otimes C \\
1 \otimes j \otimes 1 & \downarrow & \downarrow \phi \otimes 1 \\
\ker \epsilon \otimes R \otimes C & \xrightarrow{\psi \otimes 1} & R \otimes C
\end{array}
$$

We see that

$$(j \otimes 1)(\psi \otimes 1)(1 \otimes s \otimes 1)(\xi) = (\phi \otimes 1)(1 \otimes j \otimes 1)(1 \otimes s \otimes 1)(\xi) = (\phi \otimes 1)(\xi) = 0.$$ 

But $j \otimes 1$ and $\psi \otimes 1$ are injective, so $(1 \otimes s \otimes 1)(\xi) = 0$. Since $\xi \in \ker \epsilon \otimes_R R \otimes k C$, it follows that $\xi = 0$, which is a contradiction. 

Now that we have proved the “if” direction of Theorem 1.6, we work toward reducing the “only if” direction of this special case of Theorem 1.9. In the classical case, the proof of this statement relies on the fact that $j$ acts faithfully on $V$, and $H$ acts trivially on $V$. In the classical case, the proof of this statement relies on the fact that $G$ has no pseudo-reflection if and only if $V$ is étale in codimension one. In our case, however, this relation between pseudo-reflections and ramification no longer holds. For example, if $k$ has characteristic 2 and $G = \mu_2$ acts on $V = kx \oplus ky$ by sending $x$ to $\zeta x$ and $y$ to $\zeta y$, then $V \to V/G$ is ramified at every height 1 prime, but $G$ has no pseudo-reflections.

Nonetheless, we introduce the following functor which, for our purposes, should be thought of as an analogue of the inertia group. If $v \in V$ and $\mathfrak{P}$ is the ideal of $k[V]$ generated by $v$, then let

$$I_{\mathfrak{P}} : (k\text{-alg}) \longrightarrow (Groups)$$

$$R \longrightarrow \{ g \in G(R) \mid \text{for all } R\text{-algebras } R' \text{ and all } f \in (V \otimes R')^\vee \text{ such that } f(v \otimes 1) = 0, \text{ we have } (g(f))(v \otimes 1) = 0 \}.$$ 

Note that in the above example, $I_{\mathfrak{P}} = 1$ for all homogeneous height one primes $\mathfrak{P}$. So our “inertia groups” do not capture information about ramification, but they are related to pseudo-reflections, as the following lemma shows. 

**Lemma 2.6.** If $G$ acts faithfully on $V$ and stabilizes the closed subscheme $V(\mathfrak{P})$ of $V^\vee$ defined by $\mathfrak{P} = (v)$ for some $v \in V$, then $I_{\mathfrak{P}} = 1$ if and only if $G$ has no pseudo-reflections acting trivially on $\mathfrak{P}$.

**Proof.** Since $G$ stabilizes $V(\mathfrak{P})$, we have a morphism $G \to \text{Aut}(V(\mathfrak{P}))$ of sheaves of groups. We see then that $I_{\mathfrak{P}}$ is a closed subgroup scheme of $G$, as it is the kernel of the above morphism. If $N$ is any pseudo-reflection of $G$ which acts trivially on $V(\mathfrak{P})$, then it is clearly contained in $I_{\mathfrak{P}}$. Conversely, since $I_{\mathfrak{P}}$ is a closed subgroup scheme of $G$ which acts trivially on $V(\mathfrak{P})$, it is a pseudo-reflection. This completes the proof. 

\[7\]
We now prove a general result concerning faithful actions by group schemes.

**Lemma 2.7.** Let $G$ be a finite group scheme which acts faithfully on an affine scheme $U$. If $H$ is a normal subgroup scheme of $G$, then the action of $G/H$ on $U/H$ is faithful.

**Proof.** Let $\mathfrak{X} = U/H$ and let $\pi : U \to U/H$ be the natural map. We must show that if $G'$ is a subgroup scheme of $G$ such that $G'/H$ acts trivially on $U/H$, then $G' = H$. Replacing $G$ by $G'$, we can assume $G' = G$.

Since $G$ acts faithfully on $U$, there is a non-empty open substack of $\mathfrak{X}$ which is isomorphic to its coarse space. That is, we have a non-empty open subscheme $V$ of $U/H$ over which $\pi$ is an $H$-torsor. Let $P = V \times_{U/H} U$. Since $G$ acts on $P$ over $V$, we obtain a morphism

$$s : G \to \text{Aut}(P) = H.$$ 

Note that $s$ is a section of the closed immersion $H \to G$, so $H = G$. 

With the above results in place, we are ready to prove that after quotienting by the subgroup scheme generated by pseudo-reflections, there are no pseudo-reflections in the resulting action. We then use this result to prove the “only if” direction of Theorem 1.6 assuming the special case of Theorem 1.9 in which $U = V^\vee$.

**Proposition 2.8.** Let $G$ be a finite linearly reductive group scheme over $S$ with a faithful action on a finite-dimensional $k$-vector space $V$. Let $U = V^\vee$ and $H$ be the subgroup scheme of $G$ generated by pseudo-reflections. Then the induced action of $G/H$ on $U/H$ is faithful.

**Proof.** By the “if” direction of Theorem 1.6 we have $k[V]^H = k[V]$ for some subvector space $W$ of $k[V]$. The proof of [Ne, Prop 6.19] shows that the degrees of the homogeneous generators of $k[V]^H$ are determined. As a result, the action of $G/H$ on $k[V]$ is linear. Lemma 2.7 further tells us that this action is faithful.

Suppose $N/H$ is a pseudo-reflection at the origin of the $G/H$-action on $U/H$. Then there is some $w \in W$ such that $N/H$ acts trivially on the closed subscheme $V(p)$ of $U/H$ defined by $p = (w)$. Let $H = \text{Spec } A$, $N = \text{Spec } B$, and $N/H = \text{Spec } C$. We have a commutative diagram

$$\begin{array}{ccc}
  k[V] \otimes_k A & \xrightarrow{\sigma_A} & k[V] \\
  \downarrow{i} & & \downarrow{j}
  k[V] & \xrightarrow{\sigma_B} & k[V] \otimes_k B \\
  \downarrow{\sigma_C} & & \downarrow{}
  k[V]^H & \xrightarrow{\sigma} & k[V]^H \otimes_k C
\end{array}$$

where the $\sigma$ denote the corresponding coaction maps. Since $H$ is linearly reductive, there is a section $s$ of $i$. Since $N/H$ is a pseudo-reflection, we see that $w \otimes 1$ divides $\sigma_C(f) - f \otimes 1$ for all $f \in k[V]^H$. In particular, this holds when $f = s(v)$ for any $v \in V$. There is some $v \in V$ such that $\sigma_B(v) - v \otimes 1$ is non-zero, as the action of $G$ on $V$ is faithful. Since $w \otimes 1$ divides $\sigma_C(s(v)) - s(v) \otimes 1$, applying $j$ shows that $w \otimes 1$ divides $\sigma_B(v) - v \otimes 1$. Since $\sigma_B(v) - v \otimes 1$ lies in $V \otimes B$, it follows that $w$ has no higher degree terms; that is, $w \in V$. 

8
Let $P$ be the height one prime of $k[V]$ generated by $w$. Since $j\sigma_C(w) = \sigma_B(i(w))$, we see that $N$ stabilizes the closed subscheme $V(\mathfrak{p})$ of $U$. Note that the diagram

$$
\begin{array}{ccc}
V(\mathfrak{p}) & \longrightarrow & \mathbb{V}^\vee \\
\downarrow & & \downarrow \\
V(p) & \longrightarrow & \mathbb{V}^\vee /H
\end{array}
$$

is cartesian, as $\mathfrak{p} = i(p)k[V]$. Since $H$ is linearly reductive, it follows that $V(\mathfrak{p})/H = V(p)$.

Let $I_\mathfrak{p}$ denote the “inertia group” of the $N$-action on $\mathbb{V}^\vee$. Then, by Lemma 2.6, to complete the proof we must show that $I_\mathfrak{p}$ is non-trivial. Since $I_\mathfrak{p}$ is the kernel of $N \to \text{Aut}(V(\mathfrak{p}))$,

if $I_\mathfrak{p} = 1$, then $N$ acts faithfully on $V(\mathfrak{p})$. Since the action of $N/H$ on $V(p)$ is trivial, Lemma 2.7 shows that $N = H$, and so $N/H$ is not a pseudo-reflection.

Proof of “only if” direction of Theorem 1.6. Let $H$ be the subgroup scheme generated by pseudo-reflections. By the “if” direction, $k[V]^H$ is polynomial and as explained in the proof of Proposition 2.8, the $G/H$-action on $k[V]^H$ is linear. Since $G/H$ acts on $U/H$ without pseudo-reflections at the origin by Proposition 2.8, and since $M = U/G$ is smooth by assumption, Theorem 1.9 implies that $U/H$ is a $G/H$-torsor over $U/G$ after potentially shrinking $U/G$. Since the origin of $U/H$ is a fixed point, we conclude that $G = H$.

3 Theorem 1.9 for Linear Actions on Polynomial Rings

In Section 2, we reduced the proof of the “only if” direction of Theorem 1.6 to

Proposition 3.1. Let $G$ be a stable group scheme over $S$ which acts faithfully on a finite-dimensional $k$-vector space $V$. Then Theorem 1.9 holds when $U = \mathbb{V}^\vee$ and $x$ is the origin.

The proof of this proposition is given in two steps. We handle the case when $G$ is diagonalizable in Subsection 3.1 and then handle the general case in Subsection 3.2 by making use of the diagonalizable case.

3.1 Reinterpreting a Result of Iwanari

The key to proving Proposition 3.1 for diagonalizable $G$ is provided by Theorem 3.3 and Proposition 3.4 of [Iw] after we reinterpret them in the language of pseudo-reflections. We refer the reader to [Iw, p.4-6] for the basic definitions concerning monoids. We recall the following definition given in [Iw, Def 2.5].

Definition 3.2. An injective morphism $i : P \to F$ from a simplicially toric sharp monoid to a free monoid is called a minimal free resolution if $i$ is close and if for all injective close morphisms $i' : P \to F'$ to a free monoid $F'$ of the same rank as $F$, there is a unique morphism $j : F \to F'$ such that $i' = ji$.

Given a faithful action of a finite diagonalizable group scheme $\Delta$ over $S$ on a $k$-vector space $V$ of dimension $n$, we can decompose $V$ as a direct sum of 1-dimensional
\(\Delta\)-representations. Therefore, after choosing an appropriate basis, we have an identification of \(k[V]\) with \(k[N^n]\) and can assume that the \(\Delta\)-action on \(U = \mathbb{V}^\vee\) is induced from a morphism of monoids
\[
\pi : F = N^n \to A,
\]
where \(A\) is the finite abelian group such that \(\Delta\) is the Cartier dual \(D(A)\) of \(A\). We see then that
\[
U/\Delta = \text{Spec} k[P],
\]
where \(P\) is the submonoid \(\{ p \mid \pi(p) = 0 \}\) of \(F\). Note that \(P\) is simplicially toric sharp, that \(i : P \to F\) is close, and that \(A = F^{gp}/i(P^{gp})\).

We now give the relationship between minimal free resolutions and pseudo-reflections.

**Proposition 3.3.** With notation as above, \(i : P \to F\) is a minimal free resolution if and only if the action of \(\Delta\) on \(V = \mathbb{V}^\vee\) has no pseudo-reflections.

**Proof.** If \(i\) is not a minimal free resolution, then without loss of generality, \(i = ji'\), where \(i' : P \to F\) is close and injective, and \(j : F \to F\) is given by
\[
j(a_1, a_2, \ldots, a_n) = (ma_1, a_2, \ldots, a_n)
\]
with \(m \neq 1\). We have then a short exact sequence
\[
0 \to F^{gp}/i'(P^{gp}) \to F^{gp}/i(P^{gp}) \to F^{gp}/(m, 1, \ldots, 1)(F^{gp}) \to 0.
\]
Let \(N\) be the Cartier dual of \(F^{gp}/(m, 1, \ldots, 1)(F^{gp})\), which is a subgroup scheme of \(\Delta\). Letting \(\{x_i\}\) be the standard basis of \(F\), we see that
\[
k[F]^N = k[x_1^m, x_2, \ldots, x_n],
\]
and so \(V^N\), which is the degree 1 part of \(k[F]^N\), has codimension 1 in \(V\). Therefore, \(N\) is a pseudo-reflection.

Conversely, suppose \(N\) is a pseudo-reflection. Since \(N\) is a subgroup scheme of \(\Delta\), it is diagonalizable as well. Let \(N = \text{Spec} k[B]\), where \(B\) is a finite abelian group and let \(\psi : A \to B\) be the induced map. We see that
\[
V^N = \bigoplus_{i \neq j} kx_i
\]
for some \(j\). Without loss of generality, \(j = 1\). It follows then that
\[
\{ f \in F \mid \psi\pi(f) = 0 \} = (m, 1, \ldots, 1)F
\]
for some \(m\) dividing \(|B|\). Since the \(\Delta\) action on \(V\) is assumed to be faithful, we see, in fact, that \(m = |B|\). Therefore, \(i\) factors through \(\cdot(m, 1, \ldots, 1) : F \to F\), which shows that \(i\) is not a minimal free resolution. \(\square\)

Having reinterpreted minimal free resolutions, the proof of Proposition 3.1 for diagonalizable group schemes \(G\) follows easily from Iwanari’s work.

**Proposition 3.4.** Let \(G = \Delta\) be a finite diagonalizable group scheme over \(S\) which acts faithfully on a finite-dimensional \(k\)-vector space \(V\). Then Theorem 1.9 holds when \(U = \mathbb{V}^\vee\) and \(x\) is the origin. In this case it is not necessary to shrink \(M\) to a smaller Zariski neighborhood of the image of \(x\).

**Proof.** Let \(F\) and \(P\) be as above, and let \(X = [U/\Delta]\). By Proposition 3.3, the morphism \(i : P \to F\) is a minimal free resolution. Theorem 3.3 (1) and Proposition 3.4 of [Iw] then show that the natural morphism \(X \times_M M^0 \to M^0\) is an isomorphism. Since \(X \times_M M^0 = [U^0/\Delta]\), we see \(U^0\) is a \(\Delta\)-torsor over \(M^0\). \(\square\)
3.2 Finishing the Proof

The goal of this subsection is to prove Proposition 3.1. The main result used in the proof of this proposition, as well as in the proof of Theorem 1.9, is the following.

**Proposition 3.5.** Let notation and hypotheses be as in Theorem 1.9. Let \( X = U/\Delta \) and \( G = \Delta \times Q \), where \( \Delta \) is diagonalizable and \( Q \) is constant and tame. If in addition to assuming that \( G \) acts without pseudo-reflections at \( x \), we assume that \( \Delta \) is local and that the base change of \( U \) to \( X^{sm} \) is a \( \Delta \)-torsor over \( X^{sm} \), then after possibly shrinking \( M \) to a smaller Zariski neighborhood of the image of \( x \), the quotient map \( f : X \to M \) is unramified in codimension 1.

**Proof.** Let \( g \) be the quotient map \( U \to X \). For every \( q \in Q \), consider the cartesian diagram

\[
\begin{array}{ccc}
  Z_q & \longrightarrow & U \\
  \downarrow & & \downarrow \Delta \\
  U & \longrightarrow & \Delta \\
  \Gamma_q & \longrightarrow & U \times U
\end{array}
\]

where \( \Gamma_q(u) = (u, qu) \). We see that \( Z_q \) is a closed subscheme of \( U \) and that \( Z_q(T) \) is the set of \( u \in U(T) \) which are fixed by \( q \). Let \( Z \) be the closed subset of \( U \) which is the union of the \( Z_q \) for \( q \neq 1 \). Since the action of \( G \) on \( U \) is faithful, \( Z \) is not all of \( U \). Let \( Z' \) be the union of the codimension 1 components of \( Z \). Since \( fg \) is finite, we see that \( fg(Z') \) is a closed subset of \( M \). Moreover, \( fg(Z') \) does not contain the image of \( x \), as \( G \) is assumed to act without pseudo-reflections at \( x \). By shrinking \( M \) to \( M - fg(Z') \), we can assume that no non-trivial \( q \in Q \) acts trivially on a divisor of \( U \).

Let \( U = \text{Spec} R \). The morphism \( f \) is unramified in codimension 1 if and only if the (traditional) inertia groups of all height 1 primes \( p \) of \( R^\Delta \) are trivial. So, we must show that if \( q \in Q \) acts trivially on \( V(p) \), then \( q = 1 \). Since \( g \) is finite, and hence integral, the going up theorem shows that

\[
pR = \mathfrak{P}_1^{e_1} + \cdots + \mathfrak{P}_n^{e_n},
\]

where the \( \mathfrak{P}_i \) are height 1 primes and the \( e_i \) are positive integers. Note that \( X \) is normal and so the complement of \( X^{sm} \) in \( X \) has codimension at least 2. As a result,

\[
h : U \times_X \text{Spec} \mathcal{O}_{X,p} \longrightarrow \text{Spec} \mathcal{O}_{X,p}
\]

is a \( \Delta \)-torsor. Since \( \Delta \) is local, \( h \) is a homeomorphism of topological spaces, so there is exactly one prime \( \mathfrak{P} \) lying over \( p \). We see then that \( U \times_X V(p) = V(\mathfrak{P}^e) \) for some \( e \).

Let \( V(p)^0 \) be the intersection of \( V(p) \) with \( X^{sm} \), and let \( Z^0 = U \times_X V(p)^0 \). Then \( Z^0 \) is a \( \Delta \)-torsor over \( V(p)^0 \). Since \( q \) acts trivially on \( V(p) \), we obtain an action of \( q \) on \( Z^0 \) over \( V(p)^0 \), and hence a group scheme homomorphism

\[
\varphi : Q'_{V(p)^0} \longrightarrow \text{Aut}(Z^0/V(p)^0) = \Delta_{V(p)^0},
\]

where \( Q' \) denotes the subgroup of \( Q \) generated by \( q \). Since \( V(p)^0 \) is reduced, we see that \( \varphi \) factors through the reduction of \( \Delta_{V(p)^0} \), which is the trivial group scheme. Therefore, \( q \) acts trivially on \( Z^0 \).

Since the complement of \( X^{sm} \) in \( X \) has codimension at least 2, and since \( g \) factors as a flat map \( U \to [U/\Delta] \) followed by a coarse space map \( [U/\Delta] \to X \), both of which are
codimesion-preserving (see Definition 4.2 and Remark 4.3 of [FMN]), we see that the complement of $Z^0$ in $V(\mathfrak{g}^e)$ has codimension at least 2. Note that if $Y$ is a normal scheme and $W$ is an open subscheme of $Y$ whose complement has codimension at least 2, then any morphism from $W$ to an affine scheme $Z$ extends uniquely to a morphism from $Y$ to $Z$. Since the action of $q$ on $V(\mathfrak{g}^e)$ restricts to a trivial action on $Z^0$, the action of $q$ on $V(\mathfrak{g}^e)$ is trivial. Therefore, $q$ acts trivially on a divisor of $U$, and so $q = 1$.

**Proof of Proposition 3.4.** Let $k'/k$ be a finite Galois extension such that $G_{K'} \simeq \Delta \times Q$, where $\Delta$ is diagonalizable and $Q$ is constant and tame. Let $S' = \text{Spec } k'$ and consider the diagram

$$
\begin{array}{ccc}
U' & \longrightarrow & U \\
\downarrow & & \downarrow \\
M' & \longrightarrow & M \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S
\end{array}
$$

where the squares are cartesian. We denote by $x'$ the induced $k'$-rational point of $U'$. Since $\Delta$ is the product of a local diagonalizable group scheme and a locally constant diagonalizable group scheme, replacing $k'$ by a further extension if necessary, we can assume that $\Delta$ is local.

Since $G$ is stable, $G_{K'}$ has no pseudo-reflections at $x'$. It follows then from Proposition 3.5 that there exists an open neighborhood $W'$ of $x'$ such that $U' \times_M W' \longrightarrow W'$ is unramified in codimension 1. Since $k'/k$ is a finite Galois extension, replacing $W'$ by the intersection of the $\tau(W')$ as $\tau$ ranges over the elements of $\text{Gal}(k'/k)$, we can assume $W'$ is Galois-invariant. Hence, $W' = W \times_M M'$ for some open subset $W$ of $M$. We shrink $M$ to $W$.

To check that $U^0$ is a $G$-torsor over $M^0$, we can look étale locally. We can therefore assume $S = S'$. Let $X = U/\Delta$, and let $g : U \rightarrow X$ and $f : X \rightarrow M$ be the quotient maps. We denote by $X^0$ the fiber product $X \times_M M^0$ and by $f^0$ the induced morphism $X^0 \rightarrow M^0$.

By Proposition 3.4, we know that the base change of $U$ to $X^{sm}$ is a $\Delta$-torsor over $X^{sm}$. Since $f$ is unramified in codimension 1, we see that $f^0$ is as well. Since $M^0$ is smooth and $X^0$ is normal, the purity of the branch locus theorem [SGA1 X.3.1] implies that $f^0$ is étale, and hence a $Q$-torsor. Since $X^0$ is étale over $M^0$, it is smooth. As a result, $U^0$ is a $\Delta$-torsor over $X^0$ from which it follows that $U^0$ is a $G$-torsor over $M^0$.

This finishes the proof of Proposition 3.4 and hence also of Theorem 1.6. We conclude this section by proving Corollary 1.8.

**Proof of Corollary 1.8.** Let $U = \text{Spec } R$ and $M = U/G$. We denote by $y$ the image of $x$. Since $G$ being generated by pseudo-reflections at $x$ implies that $G_{K}$ is generated by pseudo-reflections at $x$ for arbitrary finite linearly reductive group schemes $G$, and since smoothness of $M$ at $y$ can be checked étale locally, we can assume that $x$ is $k$-rational. Let $V = m_x/m_x^2$ be the cotangent space of $x$. As $G$ is linearly reductive, there is a $G$-equivariant section of $m_x \rightarrow V$. This yields a $G$-equivariant map $\text{Sym}^*(V) \rightarrow R$, which induces an isomorphism $k[[V]] \longrightarrow \mathcal{O}_{U,x}$ of $G$-representations. That is, complete locally, we have linearized the $G$-action. Since $\mathcal{O}_{M,y} = k[[V]]^G$, the corollary follows from Theorem 1.6 as $M$ is smooth at $y$ if and only if $\mathcal{O}_{M,y}$ is a formal power series ring over $k$. □
4 Actions on Smooth Schemes

Having proved Theorem 1.9 for polynomial rings with linear actions, we now turn to the general case. We begin with two preliminary lemmas and a technical proposition.

Lemma 4.1. Let $U$ be a smooth affine scheme over $S$ with an action of a finite diagonalizable group scheme $\Delta$. Then there is a closed subscheme $Z$ of $U$ on which $\Delta$ acts trivially, and with the property that every closed subscheme $Y$ on which $\Delta$ acts trivially factors through $Z$. Furthermore, the construction of $Z$ commutes with flat base change on $U/\Delta$.

Proof. Let $U = \text{Spec } R$ and $\Delta = \text{Spec } k[A]$, where $A$ is a finite abelian group written additively. The $\Delta$-action on $U$ yields an $A$-grading $R = \bigoplus_{a \in A} R_a$.

We see that if $J$ is an ideal of $R$, then $\Delta$ acts trivially on $Y = \text{Spec } R/J$ if and only if $J$ contains the $R_a$ for $a \neq 0$. Letting $I$ be the ideal generated by the $R_a$ for $a \neq 0$, we see that $\text{Spec } R/I$ is our desired $Z$.

We now show that the formation of $Z$ commutes with flat base change. Note that $U/\Delta = \text{Spec } R_0$.

Let $R'_0$ be a flat $R_0$-algebra and let $R' = R'_0 \otimes_{R_0} R$. The induced $\Delta$-action on $\text{Spec } R'$ corresponds to the $A$-grading $R' = \bigoplus_{a \in A} (R'_0 \otimes_{R_0} R_a)$.

Since $R'_0$ is flat over $R_0$, we see that $I \otimes_{R_0} R'_0$ is an ideal of $R'$, and one easily shows that it is the ideal generated by the $R'_a$ for $a \neq 0$.

Recall that if $G$ is a group scheme over a base scheme $B$ which acts on a $B$-scheme $U$, and if $y : T \to U$ is a morphism of $B$-schemes, then the stabilizer group scheme $G_y$ is defined by the cartesian diagram

$$
\begin{array}{ccc}
G_y & \to & G \times_B U \\
\downarrow & & \downarrow \varphi \\
T & \xrightarrow{y \times y} & U \times_B U
\end{array}
$$

where $\varphi(g, u) = (gu, u)$. If $U$ is separated over $B$, then $G_y$ is a closed subgroup scheme of $G_T$.

Lemma 4.2. Let $B$ be a scheme and $G$ a finite flat group scheme over $B$. If $G$ acts on a $B$-scheme $U$, then $U \to U/G$ is a $G$-torsor if and only if the stabilizer group schemes $G_y$ are trivial for all closed points $y$ of $U$.

Proof. The “only if” direction is clear. To prove the “if” direction, it suffices to show that the stabilizer group schemes $G_y$ are trivial for all scheme valued points $y : T \to U$. This is equivalent to showing that the universal stabilizer $G_u$ is trivial, where $u : U \to U$ is the identity map. Since $G_u$ is a finite group scheme over $U$, it is given by a coherent sheaf $\mathcal{F}$ on $U$. The support of $\mathcal{F}$ is a closed subset, and so to prove $G_u$ is trivial, it suffices to check this on stalks of closed points. Nakayama’s Lemma then shows that we need only check the triviality of $G_u$ on closed fibers. That is, we need only check that the $G_y$ are trivial for closed points $y$ of $U$. \qed
**Proposition 4.3.** Let $U$ be a smooth affine scheme over $S$ with a faithful action of a stable group scheme $G$ fixing a $k$-rational point $x$. If $N$ has a pseudo-reflection at $x$, then there is an étale neighborhood $T \to U/G$ of $x$ and a divisor $D$ of $U_T$ defined by a principal ideal on which $N_T$ acts trivially.

**Proof.** Let $M = U/G$ and let $y$ be the image of $x$ in $M$. As in the proof of Corollary 1.8, we have an isomorphism $k[[V]] \to \mathcal{O}_{U,x}$ of $G$-representations, where $V = \mathfrak{m}_x/\mathfrak{m}_x^2$. If $N$ is a pseudo-reflection at $x$, then there is some $v \in V$ such that $N$ acts trivially on the closed subscheme of Spec $k[[V]]$ defined by the prime ideal generated by $v$.

Consider the contravariant functor $F$ which sends an $M$-scheme $T$ to the set of divisors of $U_T$ defined by a principal ideal on which $N_T$ acts trivially. As $F$ is locally of finite presentation and $U \times_M \text{Spec} \mathcal{O}_{M,y} = \text{Spec} \mathcal{O}_{U,x}$, Artin’s Approximation Theorem [Ar] finishes the proof.

We are now ready to prove Theorem 1.9. Our method of proof is similar to that of Proposition 3.1; we first prove the theorem in the case that $G$ is diagonalizable and then make use of this case to prove the theorem in general.

**Proposition 4.4.** Theorem 1.9 holds when $G = \Delta$ is a finite diagonalizable group scheme.

**Proof.** Let $g : U \to M$ be the quotient map. Since any subgroup scheme $N$ of $\Delta$ is again finite diagonalizable, Lemma 3.1 shows that for every $N$, there exists a closed subscheme $Z_N$ of $U$ on which $N$ acts trivially, and with the property that every closed subscheme $Y$ on which $N$ acts trivially factors through $Z_N$. Let $Z$ be the union of the finitely many closed subsets $Z_N$ for $N \neq 1$. Since the action of $\Delta$ on $U$ is faithful, $Z$ has codimension at least 1. Let $Z'$ be the union of all irreducible components of $Z$ which have codimension 1. Since $\Delta$ acts without pseudo-reflections at $x$, we see $x \notin Z'$. Note that $g(Z')$ is closed as $g$ is proper. Since the construction of $Z$ commutes with flat base change on $M$ and since flat morphisms are codimension-preserving, replacing $M$ with $M - g(Z')$, we can assume that there are no non-trivial subgroup schemes of $\Delta$ which fppf locally on $M$ act trivially on a divisor of $U$.

By Lemma 4.2 to show $U^0$ is a $\Delta$-torsor over $M^0$, it suffices to show that for every closed point $y$ of $U$ which maps to $M^0$, the stabilizer group scheme $\Delta_y$ is trivial. Fix such a closed point $y$ and let $T = \text{Spec} k(y)$. Since $T$ is fppf over $S$, we see from Proposition 4.3 that the closed subgroup scheme $\Delta_y$ of $\Delta_T$ acts faithfully on $U_T$ without pseudo-reflections at the $k(y)$-rational point $y'$ of $U_T$ induced by $y$. Since $y$ maps to a smooth point of $M$, it follows that $y'$ maps to a smooth point of $M_T$. Corollary 1.8 then shows that $\Delta_y$ is generated by pseudo-reflections. Since $\Delta_y$ has no pseudo-reflections, it is therefore trivial.

**Proof of Theorem 1.9.** If $G = \Delta \rtimes Q$, where $\Delta$ is diagonalizable and $Q$ is constant and tame, then letting $Z'$ be as in Proposition 4.4 and letting $U$, $X$, $f$, and $g$ be as in the proof of Proposition 3.1, the proof of Proposition 4.4 shows that after replacing $M$ by $M - fg(Z')$, the base change of $U$ to $X^{sm}$ is a $\Delta$-torsor over $X^{sm}$. As in the proof of Proposition 3.1 we can then reduce the general case to the case when $G = \Delta \rtimes Q$, where $\Delta$ is local diagonalizable and $Q$ is constant tame. The last paragraph of the proof of Proposition 3.1 then shows that $U^0$ is a $G$-torsor over $M^0$.

## 5 Schemes with Linearly Reductive Singularities

Let $k$ be a perfect field of characteristic $p$. 
Definition 5.1. We say a scheme $M$ over $S$ has linearly reductive singularities if there is an étale cover \( \{ U_i/G_i \to M \} \), where the $U_i$ are smooth over $S$ and the $G_i$ are linearly reductive group schemes which are finite over $S$.

Note that if $M$ has linearly reductive singularities, then it is automatically normal and in fact Cohen-Macaulay by [HR, p.115].

Our goal in this section is to prove Theorem 1.10, which generalizes the result that every scheme with quotient singularities prime to the characteristic is the coarse space of a smooth Deligne-Mumford stack. We remark that in the case of quotient singularities, the converse of the analogous theorem is true as well; that is, every scheme which is the coarse space of a smooth Deligne-Mumford stack has quotient singularities. It is not clear, however, that the converse of Theorem 1.10 should hold. We know from Theorem 3.2 of [AOV] that $X$ is étale locally $[V/G_0]$, where $G_0$ is a finite flat linearly reductive group scheme over $V/G_0$, but $V$ need not be smooth and $G_0$ need not be the base change of a group scheme over $S$. On the other hand, Proposition 5.2 below shows that $X$ is étale locally $[U/G]$ where $U$ is smooth and $G$ is a group scheme over $S$, but here $G$ is not finite.

Before proving Theorem 1.10, we begin with a technical proposition followed by a series of lemmas.

**Proposition 5.2.** Let $X$ be a tame stack over $S$ with coarse space $M$. Then there exists an étale cover $T \to M$ such that

$$X \times_M T = [U/G_{r,m,T} \rtimes H],$$

where $H$ is a finite constant tame group scheme and $U$ is affine over $T$. Furthermore, $G_{r,m,T} \rtimes H$ is the base change to $T$ of a group scheme $G_{r,m,S} \rtimes H$ over $S$, so $X \times_M T = [U/G_{r,m,S} \rtimes H].$

**Proof.** Theorem 3.2 of [AOV] shows that there exists an étale cover $T \to M$ and a finite flat linearly reductive group scheme $G_0$ over $T$ acting on a finite finitely presented scheme $V$ over $T$ such that

$$X \times_M T = [V/G_0].$$

By [AOV Lemma 2.20], after replacing $T$ by a finer étale cover if necessary, we can assume there is a short exact sequence

$$1 \to \Delta \to G_0 \to H \to 1,$$

where $\Delta = \text{Spec} \mathcal{O}_T[A]$ is a finite diagonalizable group scheme and $H$ is a finite constant tame group scheme. Since $\Delta$ is abelian, the conjugation action of $G_0$ on $\Delta$ passes to an action

$$H \to \text{Aut}(\Delta) = \text{Aut}(A).$$

Choosing a surjection $F \to A$ in the category of $\mathbb{Z}[H]$-modules from a free module $F$, yields an $H$-equivariant morphism $\Delta \hookrightarrow \mathbb{G}_{m,T}^r$. Using the $H$-action on $\mathbb{G}_{m,T}^r$, we define the group scheme $\mathbb{G}_{m,T}^r \rtimes G_0$ over $T$. Note that there is an embedding

$$\Delta \hookrightarrow \mathbb{G}_{m,T}^r \rtimes G_0$$

sending $\delta$ to $(\delta, \delta^{-1})$, which realizes $\Delta$ as a normal subgroup scheme of $\mathbb{G}_{m,T}^r \rtimes G_0$. We can therefore define

$$G := (\mathbb{G}_{m,T}^r \rtimes G_0)/\Delta.$$
One checks that there is a commutative diagram

\[
\begin{array}{c}
1 \\
\downarrow \\
\downarrow \\
G_0 \\
\downarrow \downarrow \\
H \\
\downarrow \downarrow \\
1
\end{array}
\begin{array}{c}
\Delta \\
\downarrow \\
\downarrow \\
\pi \\
\downarrow \downarrow \\
H \\
\downarrow \downarrow \\
1
\end{array}
\begin{array}{c}
\text{where the rows are exact and the vertical arrows are injective.}
\end{array}
\]

We show that étale locally on \( T \), there is a group scheme-theoretic section of \( \pi \), so that \( G = G_{m,T} \rtimes H \). Let \( P \) be the sheaf on \( T \) such that for any \( T \)-scheme \( W \), \( P(W) \) is the set of group scheme-theoretic sections of \( \pi_W : G_W \to H_W \). Note that the sheaf \( \text{Hom}(H,G) \) parameterizing group scheme homomorphisms from \( H \) to \( G \) is representable since it is a closed subscheme of \( G^{\times |H|} \) cut out by suitable equations. We see that \( P \) is the equalizer of the two maps

\[
\begin{array}{c}
\text{Hom}(H,G) \\
\downarrow \\
H^{\times |H|}
\end{array}
\begin{array}{c}
\text{p}_1 \\
\downarrow \\
\text{p}_2 \\
H^{\times |H|} \times H^{\times |H|}
\end{array}
\]

where \( \text{p}_1(\phi) = (\pi(\phi(h)))_h \) and \( \text{p}_2(\phi) = (h)_h \). That is, there is a cartesian diagram

\[
\begin{array}{c}
P \\
\downarrow \\
H^{\times |H|} \\
\downarrow \\
H^{\times |H|} \times H^{\times |H|}
\end{array}
\begin{array}{c}
\Delta \\
\downarrow \\
\text{p}_1, \text{p}_2
\end{array}
\]

Since \( H \) is separated over \( T \), we see that \( P \) is a closed subscheme of \( \text{Hom}(H,G) \). In particular, it is representable and locally of finite presentation over \( T \). Furthermore, \( P \to T \) is surjective as \( [AOV] \) Lemma 2.16 shows that it has a section fpff locally. To show \( P \) has a section étale locally, by \( [EGA4] \) 17.16.3, it suffices to prove \( P \) is smooth over \( T \).

Given a commutative diagram

\[
\begin{array}{c}
X_0 = \text{Spec } A/I \\
\downarrow \\
X = \text{Spec } A \\
\downarrow \\
T
\end{array}
\begin{array}{c}
P \\
\text{P}
\end{array}
\]

with \( I \) a square zero ideal, we want to find a dotted arrow making the diagram commute. That is, given a group scheme-theoretic section \( s_0 : G_{W_0} \to H_{W_0} \) of \( \pi_{W_0} \), we want to find a group scheme homomorphism \( s : G_W \to H_W \) which pulls back to \( s_0 \) and such that \( \pi_W \circ s \) is the identity. Note first that any group scheme homomorphism \( s \) which pulls back to \( s_0 \) is automatically a section of \( \pi_W \) since \( H \) is a finite constant group scheme and \( \pi_W \circ s \) pulls back to the identity over \( W_0 \). By \( [SGA3] \) Exp. III 2.3, the obstruction to lifting \( s_0 \) to a group scheme homomorphism lies in

\[
H^2(H, \text{Lie}(G) \otimes \mathcal{I}),
\]

which vanishes as \( H \) is linearly reductive. This proves the smoothness of \( P \).

To complete the proof of the lemma, let \( U := V \times_{G_0} G \) and note that

\[
\mathcal{X} \times_M T = [V/G_0] = [U/G].
\]
Since $V$ is finite over $T$ and $G$ is affine over $T$, it follows that $U$ is affine over $T$ as well. Replacing $T$ by a finer étale cover if necessary, we have

$$\mathcal{X} \times_M T = [U/G_{m,T} \times H].$$

Lastly, the scheme underlying $G_{m,T} \times H$ is $G_{m,T} \times_T H$ and its group scheme structure is determined by the action $H \to \text{Aut}(G_{m,T})$. Since $\text{Aut}(G_{m,T}) = \text{Aut}(\mathbb{Z}^r)$, we can use this same action to define the semi-direct product $G_{m,S} \times H$ and it is clear that this group scheme base changes to $G_{m,T} \times H$.

**Lemma 5.3.** If $V$ is a smooth $S$-scheme with an action of finite linearly reductive group scheme $G_0$ over $S$, then $[V/G_0]$ is smooth over $S$.

**Proof.** Let $\mathcal{X} = [V/G_0]$. To prove $\mathcal{X}$ is smooth, it suffices to work étale locally on $S$, where, by [AOV Lemma 2.20], we can assume $G_0$ fits into a short exact sequence

$$1 \to \Delta \to G_0 \to H \to 1,$$

where $\Delta$ is a finite diagonalizable group scheme and $H$ is a finite constant tame group scheme. Let $G$ be obtained from $G_0$ as in the proof of Proposition 5.2 and let $U = V \times^{G_0} G$. Since $\mathcal{X} = [U/G]$, it suffices to show $U$ is smooth over $S$. The action of $G_0$ on $V \times G$, given by $g_0 \cdot (v, g) = (v g_0, g_0 g)$, is free as the $G_0$-action on $G$ is free. As a result, $U = [(V \times G)/G_0]$ and $G/G_0 = [G/G_0]$. Since the projection map $p : V \times G \to G$ is $G_0$-equivariant, we have a cartesian diagram

$$\begin{array}{ccc}
V \times G & \xrightarrow{p} & G \\
\downarrow & & \downarrow \\
U & \xrightarrow{q} & G/G_0
\end{array}$$

Since $p$ is smooth, $q$ is as well. Since $G \to [G/G_0] = G/G_0$ is flat and $G$ is smooth, [EGA4 17.7.7] shows that $G/G_0$ is smooth, and so $U$ is as well. \[\square\]

**Lemma 5.4.** Let $X$ be a smooth $S$-scheme and $i : U \hookrightarrow X$ an open subscheme whose complement has codimension at least 2. Let $P$ be a $G$-torsor on $U$, where $G = \mathbb{G}_m^r \rtimes H$ and $H$ is a finite constant étale group scheme. Then $P$ extends uniquely to a $G$-torsor on $X$.

**Proof.** The structure map from $P$ to $U$ factors as $P \to P_0 \to U$, where $P$ is a $\mathbb{G}_m^r$-torsor over $P_0$ and $P_0$ is an $H$-torsor over $U$. Since the complement of $U$ in $X$ has codimension at least 2, we have $\pi_1(U) = \pi_1(X)$ and so $P_0$ extends uniquely to an $H$-torsor $Q_0$ on $X$. Let $i_0 : P_0 \hookrightarrow Q_0$ be the inclusion map. Since $Q_0$ is smooth and the complement of $P_0$ in $Q_0$ has codimension at least 2, the natural map $\text{Pic}(Q_0) \to \text{Pic}(P_0)$ is an isomorphism. It follows that any line bundle over $P_0$ can be extended uniquely to a line bundle over $Q_0$. We can therefore inductively construct a unique lift of $P$ over $X$.

Our proof of the following lemma closely follows that of [FMN Thm 4.6].

**Lemma 5.5.** Let $f : \mathcal{Y} \to M$ be an $S$-morphism from a smooth tame stack $\mathcal{Y}$ to its coarse space which pulls back to an isomorphism over the smooth locus $M^0$ of $M$. If $h : \mathcal{X} \to M$ is a dominant, codimension-preserving morphism (see [FMN Def 4.2]) from a smooth tame stack, then there is a morphism $g : \mathcal{X} \to \mathcal{Y}$, unique up to unique isomorphism, such that $fg = h$. 

17
Proof. We show that if such a morphism \( g \) exists, then it is unique. Suppose \( g_1 \) and \( g_2 \) are two such morphisms. We see then that \( g_1 |_{h^{-1}(M^0)} = g_2 |_{h^{-1}(M^0)} \). Since \( h \) is dominant and codimension-preserving, \( h^{-1}(M^0) \) is open and dense in \( X \). Proposition 1.2 of [FMN] shows that if \( X \) and \( Y \) are Deligne-Mumford with \( X \) normal and \( Y \) separated, then \( g_1 \) and \( g_2 \) are uniquely isomorphic. The proof, however, applies equally well to tame stacks since the only key ingredient used about Deligne-Mumford stacks is that they are locally \([U/G]\) where \( G \) is a separated group scheme.

By uniqueness, to show the existence of \( g \), we can assume by Proposition 5.2 that \( Y = [U/G] \), where \( U \) is smooth and affine, and \( G = G_m \times H \), where \( H \) is a finite constant tame group scheme. Let \( p : V \to X \) be a smooth cover by a smooth scheme. Since smooth morphisms are dominant and codimension-preserving, uniqueness implies that to show the existence of \( g \), we need only show there is a morphism \( g_1 : V \to Y \) such that \( fg_1 = hp \). So, we can assume \( X = V \).

Given a stack \( Z \) over \( M \), let \( Z^0 = M^0 \times_M Z \). Given a morphism \( \pi : Z_1 \to Z_2 \) of \( M \)-stacks, let \( \pi^0 : Z_1^0 \to Z_2^0 \) denote the induced morphism. Since \( f^0 \) is an isomorphism, there is a morphism \( g^0 : V^0 \to Y^0 \) such that \( f^0g^0 = h^0 \). It follows that there is a \( G \)-torsor \( P^0 \) over \( V^0 \) and a \( G \)-equivariant map from \( P^0 \) to \( U^0 \) such that the diagram

\[
\begin{array}{ccc}
P^0 & \longrightarrow & U^0 \\
\downarrow & & \downarrow \\
V^0 & \longrightarrow & Y^0 \\
\downarrow & & \downarrow \\
M^0 & \longrightarrow & \simeq \\
\end{array}
\]

commutes and the square is cartesian. By Lemma 5.4, \( P^0 \) extends to a \( G \)-torsor \( P \) over \( V \).

Note that if \( X \) is a normal algebraic space and \( i : W \hookrightarrow X \) is an open subalgebraic space whose complement has codimension at least 2, then any morphism from \( W \) to an affine scheme \( Y \) extends uniquely to a morphism \( X \to Y \). As a result, the morphism from \( P^0 \) to \( U^0 \) extends to a morphism \( q : P \to U \). Consider the diagram

\[
\begin{array}{ccc}
G \times P & \xrightarrow{id \times q} & G \times U \\
\downarrow & & \downarrow \\
P & \longrightarrow & U \\
\end{array}
\]

where the vertical arrows are the action maps. Precomposing either of the two maps in the diagram from \( G \times P \) to \( U \) by the inclusion \( G \times P^0 \hookrightarrow G \times P \) yields the same morphism. That is, the two maps from \( G \times P \) to \( U \) are both extensions of the same map from \( G \times P^0 \) to the affine scheme \( U \), and hence are equal. This shows that \( q \) is \( G \)-equivariant, and therefore yields a map \( g : V \to Y \) such that \( fg = h \).

Proof of Theorem 1.10. We begin with the following observation. Suppose \( U \) is smooth and affine over \( S \) with a faithful action of a finite linearly reductive group scheme \( G \) over \( S \). Let \( y \) be a closed point of \( U \) mapping to \( x \in U/G \). After making the étale base change \( \text{Spec} k(y) \to S \), we can assume \( y \) is a \( k \)-rational point. Let \( G_y \) be the stabilizer subgroup scheme of \( G \) fixing \( y \). Since

\[
U/G_y \longrightarrow U/G
\]
is étale at $y$, replacing $U/G$ by an étale cover, we can further assume that $G$ fixes $y$. Then by Corollary 1.8 we can assume $G$ has no pseudo-reflections at $y$, and hence, Theorem 1.9 shows that after shrinking $U/G$ about $x$, we can assume that the base change of $U$ to the smooth locus of $U/G$ is a $G$-torsor.

We now turn to the proof. Since $M$ has linearly reductive singularities, there is an étale cover $\{U_i/G_i \to M\}$, where $U_i$ is smooth and affine over $S$ and $G_i$ is a finite linearly reductive group scheme over $S$ which acts faithfully on $U_i$. By the above discussion, replacing this étale cover by a finer étale cover if necessary, we can assume that the base change of $U_i$ to the smooth locus of $U_i/G_i$ is a $G_i$-torsor. Let $M_i = U_i/G_i$ and $X_i = [U_i/G_i]$. We see that the $X_i$ are locally the desired stacks, so we need only glue the $X_i$. Let $M_{ij} = M_i \times_M M_j$ and let $V_i \to X_i$ be a smooth cover. Since $M_{ij}$ is the coarse space of both $X_i \times_M M_{ij}$ and $X_j \times_M M_{ij}$, and since coarse space maps are dominant and codimension-preserving, Lemma 5.5 shows that there is a unique isomorphism of $X_i \times_M M_{ij}$ and $X_j \times_M M_{ij}$. Identifying these two stacks via this isomorphism, let $I_{ij}$ be the fiber product over the stack of $V_i \times_M M_{ij}$ and $V_j \times_M M_{ij}$. We see then that we have a morphism $I_{ij} \to U_i \times_M U_j$. This yields a groupoid
\[
\coprod I_{ij} \to \coprod U_i \times_M U_j,
\]
which defines our desired glued stack $X$. Note that $X$ is smooth and tame by [AOV, Thm 3.2].

References

[Ar] M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 23–58.

[AOV] D. Abramovich, M. Olsson, and A. Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier, to appear, [arXiv:math/0703310](http://arxiv.org/abs/math/0703310)

[Bo] N. Bourbaki, *Groupes et algèbres de Lie*, Ch. V. Hermann, Paris, 1968.

[EGA4] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas IV*. Inst. Hautes Études Sci. Publ. Math. **32** 1967.

[FMN] B. Fantechi, E. Mann, and F. Nironi, *Smooth Toric DM Stacks*, [arXiv:0708.1254v1](http://arxiv.org/abs/0708.1254v1)

[HR] M. Hochster and J. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. **13** (1974), 115–175.

[Iw] I. Iwanari, *Logarithmic geometry, minimal free resolutions and toric algebraic stacks*, to appear, [arXiv:0705.3524](http://arxiv.org/abs/0705.3524), 2007.

[KM] S. Keel and S. Mori, *Quotients by groupoids*. Ann. of Math. (2) **145** (1997), no. 1, 193–213.

[Ne] M. Neusel, *Invariant Theory*, Student Mathematical Library, 36, American Mathematical Society, Providence, RI, 2007.

[Sa] M. Satriano, *de Rham Theory for tame stacks and schemes with linearly reductive singularities*, preprint, 2008.
[SGA1] A. Grothendieck, Revêtements étalés et groupe fondamental, Séminaire de Géométrie Algébrique du Bois Marie 1960–61 (SGA 1), Springer-Verlag, Berlin, 1971.

[SGA3] M. Demazure and A. Grothendieck, Schémas en groupes. I: Propriétés générales des schémas en groupes, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Lecture Notes in Mathematics, Vol. 151, Springer-Verlag, Berlin, 1970.

[Sm] L. Smith, *On the invariant theory of finite pseudoreflection groups*, Arch. Math. (Basel) 44 (1985), no. 3, 225–228.

[Ta] J. Tate, *Finite flat group schemes* in Cornell, Silverman, Stevens: “Modular forms and Fermat’s Last Theorem”, Springer-Verlag, New York, 1997, p. 121-154.

[Vi] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. 97 (1989), no. 3, 613–670.