Ideals generated by traces in the symplectic reflection algebra $H_{1,\nu_1,\nu_2}(I_2(2m))$. II.

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Abstract

The associative algebra of symplectic reflections $\mathcal{H} := H_{1,\nu_1,\nu_2}(I_2(2m))$ based on the group generated by the root system $I_2(2m)$ has two parameters, $\nu_1$ and $\nu_2$. For every value of these parameters, the algebra $\mathcal{H}$ has an $m$-dimensional space of traces. A given trace $tr$ is called degenerate if the associated bilinear form $B_{tr}(x,y) = tr(xy)$ is degenerate. Previously, there were found all values of $\nu_1$ and $\nu_2$ for which there are degenerate traces in the space of traces, and consequently the algebra $\mathcal{H}$ has a two-sided ideal. We proved earlier that any linear combination of degenerate traces is a degenerate trace. It turns out that for certain values of parameters $\nu_1$ and $\nu_2$, degenerate traces span a 2-dimensional space. We prove that non-zero traces in this 2d space generate three proper ideals of $\mathcal{H}$.

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1 Introduction

This paper is a continuation of [3]; we advise the reader to recall [3].

2 The associative algebra $H_{1,\nu_1,\nu_2}(I_2(n))$ with $n$ even

2.1 The group $I_2(2m)$

Definition 2.1. We denote the finite subgroup of $O(2, \mathbb{R})$, generated by the root system $I_2(2m)$, also by $I_2(2m)$. It consists of $2m$ reflections $R_k$ and $2m$ rotations $S_k := R_kR_0$, where $S_0 = S_{2m}$ is the unit in $I_2(2m)$. The indices of $S_k$ and $R_k$ belong to $\mathbb{Z}/n\mathbb{Z}$. These elements satisfy the relations

$$R_kR_l = S_{k-l}, \quad S_kS_l = S_{k+l}, \quad R_kS_l = R_{k+l}, \quad S_kR_l = R_{k+l}. \quad (2.1.1)$$

Evidently, the $R_{2k}$ belong to one conjugacy class and the $R_{2k+1}$ belong to another class. The rotations $S_k$ and $S_l$ constitute a conjugacy class if $k + l = 2m$.

It is convenient to use the following basis $L_p$, $Q_p$ instead of $R_k$ and $S_k$ in the group algebra $\mathbb{C}[I_2(2m)]$:

Definition 2.2.

$$L_p := \frac{1}{2m} \sum_{k=0}^{2m-1} \lambda^{kp} R_k, \quad Q_p := \frac{1}{2m} \sum_{k=0}^{2m-1} \lambda^{-kp} S_k, \quad \lambda = \exp \left( \frac{\pi i}{m} \right), \quad p = 0, \ldots, 2m - 1. \quad (2.1.2)$$

2.2 Symplectic reflection algebra $H_{1,\nu_1,\nu_2}(I_2(2m))$

For a general definition of Symplectic reflection algebra, see, e.g., [1]. Here we restrict ourselves to $H_{1,\nu_1,\nu_2}(I_2(2m))$ only, introducing, for convenience, the new parameters

$$\mu_0 := m(\nu_1 + \nu_2), \quad \mu_1 := m(\nu_1 - \nu_2). \quad (2.2.1)$$

Here and below we use the following notation. Let $I$ be a logical expression; set

$$\delta_I := \begin{cases} 1, & \text{if } I \text{ is true;} \\ 0, & \text{if } I \text{ is false; } \end{cases} \quad (2.2.2)$$

e.g. $\delta_{i=j} = \delta_{i,j}$ (the Kronecker delta), $\delta_{p \geq q} = 1$ if $p \geq q$ and $\delta_{p < q} = 0$ if $p < q$.

Definition 2.3. The symplectic reflection algebra $\mathcal{H} := H_{1,\nu_1,\nu_2}(I_2(2m))$ is an associative algebra of polynomials in $a^\alpha, b^\alpha$, where $\alpha = 0, 1$, with coefficients in $\mathbb{C}[I_2(2m)]$, satisfying the relations

$$R_k a^\alpha = -\lambda^k b^\alpha R_k, \quad R_k b^\alpha = -\lambda^{-k} a^\alpha R_k, \quad (2.2.3)$$

$$S_k a^\alpha = \lambda^{-k} a^\alpha S_k, \quad S_k b^\alpha = \lambda^k b^\alpha S_k,$$
and

\[ [a^\alpha, b^\beta] = \varepsilon^{\alpha\beta} (1 + \mu_0 L_0 + \mu_1 L_m), \]
\[ [a^\alpha, a^\beta] = \varepsilon^{\alpha\beta} (\mu_0 L_1 + \mu_1 L_{m+1}), \]
\[ [b^\alpha, b^\beta] = \varepsilon^{\alpha\beta} (\mu_0 L_{-1} + \mu_1 L_{m-1}), \]

where \( \varepsilon^{\alpha\beta} \) is the skew-symmetric tensor with \( \varepsilon^{01} = 1 \).

The relations (2.2.3) imply

\[ L_p a^\alpha = -b^\alpha L_{p+1}, \quad L_p b^\alpha = -a^\alpha L_{p-1}, \]
\[ Q_p a^\alpha = a^\alpha Q_{p+1}, \quad Q_p b^\alpha = b^\alpha Q_{p-1}, \]
\[ L_k L_l = \delta_{k=-l} Q_l, \quad L_k Q_l = \delta_{k=l} L_l, \]
\[ Q_k L_l = \delta_{k=-l} L_l, \quad Q_k Q_l = \delta_{k=l} Q_l, \]

### 2.3 Subalgebra of singlets

The algebra \( \mathcal{H} \) contains the Lie subalgebra \( sl_2 \) of inner derivations with the generating elements

\[ T^{\alpha\beta} := \frac{1}{2}(\{a^\alpha, b^\beta\} + \{b^\alpha, a^\beta\}) \]

which act on \( \mathcal{H} \) as follows

\[ f \mapsto [f, T^{\alpha\beta}] \quad \text{for each} \quad f \in \mathcal{H}. \]

We say that the element \( f \in \mathcal{H} \) is a singlet if \( [f, T^{\alpha\beta}] = 0 \) for each \( \alpha, \beta \) and denote the subalgebra consisting of all the singlets in \( \mathcal{H} \) by \( \mathcal{H}_0 \).

Let the skew-symmetric tensor \( \varepsilon_{\alpha\beta} \) be such that \( \varepsilon_{01} = 1 \) and \( \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} = \delta_{\gamma}^{\alpha} \). Set

\[ s := \sum_{\alpha, \beta = 0,1} \frac{1}{4i} (\{a^\alpha, b^\beta\} - \{b^\alpha, a^\beta\}) \varepsilon_{\alpha\beta}. \]

Then

\[ [s, Q_p] = [s, S_k] = [T^{\alpha\beta}, s] = 0, \]
\[ s L_p = -L_p s, \quad s R_k = -R_k s, \]
\[ (s - i(\mu_0 L_0 + \mu_1 L_m)) a^\alpha = a^\alpha (s + i(\mu_0 L_0 + \mu_1 L_m)). \]

**Proposition 2.4.** If \( f \in \mathcal{H}_0 \), then \( f \) has the form

\[ f = \sum_{p=0}^{2m-1} (\phi_p Q_p + \psi_p L_p), \quad \text{where} \quad \phi_p, \psi_p \in \mathbb{C}[s]. \]
3 Ideals generated by degenerate traces

We call the trace $tr$ degenerate if symmetric invariant bilinear form $B_{tr} : B_{tr}(x, y) = tr(xy)$ is degenerate.

It is clear that the kernel of $B_{tr}$ is an ideal in $H$, we will denote it $I_{tr}$.

It is shown in [2] that for any degenerate trace $tr$, the ideal $I_{tr}$ is completely determined by $I_{tr} \cap H_0$.

3.1 The values of the trace on $\mathbb{C}[I_2(2m)]$

According to the general result of [4], the restriction of the trace to $\mathbb{C}[I_2(2m)]$ is completely defined by the value of this trace on the conjugacy classes without the eigenvalue $+1$ in their spectra.

The group $I_2(2m)$ has the following $m$ conjugacy classes without the eigenvalue $+1$ in their spectra: $m - 1$ classes consisting of two elements,

$$\{S_p, S_{n-p}\}, \text{ where } p = 1, ..., m - 1,$$

and one class consisting of 1 element,

$$\{S_m\}.$$  

The values of the trace on these conjugacy classes

$$s_k := tr(S_k) = tr(S_{n-k}), \quad s_{2m-k} = s_k, \quad k = 1, ..., m,$$

completely define the trace on $H$, and therefore the dimension of the space of traces is equal to $m$.

Besides, the group $I_2(2m)$ has two conjugacy classes each having one eigenvalue $+1$ in its spectrum:

$$\{R_{2l} | l = 0, ..., m - 1\}, \quad \{R_{2l+1} | l = 0, ..., m - 1\},$$

and one conjugacy class with two eigenvalues $+1$ in its spectrum: $\{S_0\}$.

The traces on these conjugacy classes (see [3]) are equal to

$$tr(R_{2l}) = -2\nu_2 X_1 - 2\nu_1 X_2, \quad (l = 0, 1, ..., m - 1),$$

$$tr(R_{2l+1}) = -2\nu_1 X_1 - 2\nu_2 X_2, \quad (l = 0, 1, ..., m - 1),$$

$$tr(S_0) = 2(\nu_1^2 + \nu_2^2) m X_1 + 4\nu_1\nu_2 m X_2,$$

where

$$X_1 := \sum_{l=1}^{m-1} s_{2l} \sin^2 \left(\frac{\pi l}{m}\right), \quad X_2 := \sum_{l=0}^{m-1} s_{2l+1} \sin^2 \left(\frac{\pi (2l + 1)}{2m}\right).$$

We note also that

$$tr(L_0) = -\frac{\mu_0}{m} (X_1 + X_2), \quad tr(L_m) = -\frac{\mu_1}{m} (X_1 - X_2), \quad tr(L_p) = 0 \text{ for } p \neq 0, \text{ (3.1.9)}$$

$$tr(S_0) = -\mu_0 tr(L_0) - \mu_1 tr(L_m).$$  

(3.1.10)
3.2 Generating functions of the trace

Set \( L := \mu_0 L_0 + \mu_1 L_m \).

For each trace \( \text{tr} \), we define the following set of generating functions on \( \mathcal{H} \):
\[
F_p(t) := \text{tr}(\exp(t(s - iL))Q_p), \tag{3.2.1}
\]
\[
\Psi_p(t) := \text{tr}(\exp(tL_p)), \tag{3.2.2}
\]
where \( p = 0, \ldots, 2m - 1 \). From \( sL_p = -L_p s \) and definition of the trace it follows that
\[
\Psi_p(t) = \Psi_p(0). \tag{3.2.3}
\]

We also consider the functions \( \Phi_p(t) := \text{tr}(\exp(t(s + iL))Q_p) \) related with the functions \( F_p \) by the formula
\[
\Phi_p(t) = F_p(t) + 2i\Delta_p(t), \text{where } \Delta_p(t) = \delta_p \sin(\mu_0 t)\text{tr}(L_0) + \delta_{m-p} \sin(\mu_1 t)\text{tr}(L_m). \tag{3.2.4}
\]

In [3] we derived the following system of equations
\[
\frac{d}{dt}F_p - \lambda^k \frac{d}{dt}F_{p+1} = iF_p + i\lambda \frac{d}{dt}F_{p+1} + 2i\lambda \frac{d}{dt}(e^{it}\Delta_{p+1}). \tag{3.2.5}
\]

Next, consider the Fourier transform of (3.2.5),
\[
G_k := \sum_{p=0}^{2m-1} \lambda^{kp} F_p, \text{ where } k = 0, \ldots, 2m - 1, \tag{3.2.6}
\]
\[
\tilde{\Delta}_k := \sum_{p=0}^{2m-1} \lambda^{kp} \Delta_{p+1} = \lambda^{-k} \left( \sin(\mu_0 t)\text{tr}(L_0) + \lambda^{km} \sin(\mu_1 t)\text{tr}(L_m) \right),
\]
where \( k = 0, \ldots, 2m - 1 \) and \( \lambda := e^{i\pi/m} \),
\[
\tag{3.2.7}
\]
which satisfies the equation
\[
\frac{d}{dt}G_k = i\lambda^k + e^{it}G_k + \frac{2i\lambda^k}{\lambda^k - e^{it}} \frac{d}{dt}(e^{it}\tilde{\Delta}_k) \tag{3.2.8}
\]
with initial conditions
\[
G_k(0) = s_k, \text{ where } k = 0, \ldots, 2m - 1, \text{ and } s_k = s_{2m-k} \tag{3.2.9}
\]
and where the \( s_k \) are defined by Eq. (3.1.3) for \( k = 1, \ldots, 2m - 1 \), and \( s_0 := \text{tr}(S_0) \) is defined by Eq. (3.1.7). The value \( s_0 \) depends linearly on \( s_k \), where \( k = 1, \ldots, m \) (see Eq. (3.1.7) and take in account the relations \( s_k = s_{2m-k} \)).

The solution of the equations for \( G_k \) has the form:
\[
G_k(t) = \frac{e^{it}f_k(t)}{(e^{it} - \lambda^k)^2}, \tag{3.2.10}
\]
where
\begin{align*}
f_k(t) &= \frac{2\lambda^k}{m}X_+[1 - \cos(t\mu_0)] + (-1)^k\frac{2\lambda^k}{m}X_-[1 - \cos(t\mu_1)] + \\
&\quad + \frac{2i}{m}(e^{it} - \lambda^k)[\mu_0X_+\sin(t\mu_0) + (-1)^k\mu_1X_-\sin(t\mu_1)] - 4\lambda^k s'_k. \quad (3.2.11)
\end{align*}

Here
\begin{align*}
X_\pm &:= X_1 \pm X_2, \quad (3.2.12) \\
s'_k &:= s_k \sin^2 \left( \frac{\pi k}{2m} \right), \quad k = 1, \ldots, 2m - 1, \quad s'_0 = 0. \quad (3.2.13)
\end{align*}

Note that the functions \( G_k \) in Eq (3.2.10) do not depend on the signs of \( \mu_0 \) and \( \mu_1 \).
So we can consider \( \mu_0 \) and \( \mu_1 \) positive if they are non-zero.

### 3.3 The degeneracy conditions for the trace

It is proved in [3] that the trace \( tr \) is degenerate if and only if the functions \( F_p \) (and, as a consequence, \( G_k \)) have the form \( \sum_{i=0}^{n_p} \exp(t\omega_i)q_i(t) \), where the \( q_i \) are polynomials. Further, these conditions yield that the trace \( tr \) is degenerate if and only if the parameters \( s'_k \) satisfy the following system of linear equations
\begin{align*}
\left( 1 - \cos \left( \frac{\pi}{m} k \mu_0 \right) \right) X_+ + (-1)^k \left( 1 - \cos \left( \frac{\pi}{m} k \mu_1 \right) \right) X_- &= 2ms'_k, \quad k = 1, \ldots, 2m - 1, \quad (3.3.1) \\
s'_{2m-r} &= s'_r, \quad r = 1, \ldots, m, \quad (3.3.2) \\
X_\pm &= X_1 \pm X_2, \quad (3.3.3) \\
X_1 &= \sum_{1 \leq l \leq m-1} s'_{2l}, \quad (3.3.4) \\
X_2 &= \sum_{0 \leq l \leq m-1} s'_{2l+1}, \quad (3.3.5)
\end{align*}

and the parameters \( \mu_0 \) and \( \mu_1 \) are defined from the condition that this system has a nonzero solution.

**Theorem 3.1.** Let \( m \geq 2 \). Then the system of equations (3.3.1)-(3.3.5) has nonzero solutions at the following values of the parameters \( \mu_0 \) and \( \mu_1 \) only:
\begin{align*}
\mu_0 \in \mathbb{Z} \setminus m\mathbb{Z}, & \quad \mu_1 \in \mathbb{Z} \setminus m\mathbb{Z}, \quad (3.3.6) \\
\mu_0 \in \mathbb{Z} \setminus m\mathbb{Z}, & \quad \text{any } \mu_1, \quad (3.3.7) \\
\mu_1 \in \mathbb{Z} \setminus m\mathbb{Z}, & \quad \text{any } \mu_0, \quad (3.3.8) \\
\mu_0 = \pm \mu_1 + m(2l + 1), & \quad l = 0, \pm 1, \pm 2, \ldots \quad (3.3.9)
\end{align*}

Here,

1. In case (3.3.6), the system of equations (3.3.1)-(3.3.5) has a two-parameter family of solutions;
2. In case (3.3.7), if $\mu_1 \notin \mathbb{Z} \setminus m\mathbb{Z}$, then the system of equations (3.3.1)-(3.3.5) has a one-parameter family of solutions with $X_- = 0$.

3. In case (3.3.8), if $\mu_0 \notin \mathbb{Z} \setminus m\mathbb{Z}$, then the system of equations (3.3.1)-(3.3.5) has a one-parameter family of solutions with $X_+ = 0$.

4. In case (3.3.9), if $\mu_0, \mu_1 \notin \mathbb{Z} \setminus m\mathbb{Z}$, then the system of equations (3.3.1)-(3.3.5) has a one-parameter family of solutions with $X_1 = 0$.

4 Explicit expressions for generating functions $F_p$.

Consider the associative algebra $\mathcal{H}$ with parameters $\nu_1, \nu_2$ such that

$$\mu_0 \in \mathbb{Z} \setminus m\mathbb{Z}, \quad \mu_1 \in \mathbb{Z} \setminus m\mathbb{Z}$$

and find explicit expressions for generating functions $F_p$.

4.1 Explicit expressions for generating functions $G_k$ of degenerate trace.

Recall, that

$$G_k = \sum_{p=0}^{n-1} \lambda^{kp} F_p, \quad F_p = \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-kp} G_k. \quad (4.1.1)$$

Introduce the notation

$$z := e^{it}, \quad z_k := z^{\mu_k} \quad (k = 0, \ldots, n-1), \quad (4.1.2)$$

and substitute the expression (3.3.1) into the formula (3.2.10). Since $\mu_0 \in \mathbb{Z} \setminus m\mathbb{Z}$ and $\mu_1 \in \mathbb{Z} \setminus m\mathbb{Z}$, we can introduce the polynomials $P_{\mu_0}$ and $P_{\mu_1}$ of variable $x$ by the formula

$$P_{\mu}(x) := \frac{x}{(1-x)^2} (\mu(x-1) - (x^{\mu} - 1)).$$

The polynomial $P_{\mu}$ is a polynomial of $x$ and can be represented in the form

$$P_{\mu}(x) = \begin{cases} 0 & \text{if } \mu = 1, \\ -\sum_{q=\mu-1}^{1} (\mu - q) x^q & \text{if } \mu > 1. \end{cases}$$

Now we can notice that the functions $G_k$ can be expressed in the form

$$G_k(t) = G_k^{(+)}(t) X_+ + G_k^{(-)}(t) X_-,$$  \quad (4.1.3)

where

$$G_k^{(+)}(t) = -\mu_0 z^{\mu_0} + z^{\mu_0} P_{\mu_0}(z^{-1} \lambda^k) + z^{-\mu_0} P_{\mu_0}(z \lambda^{-k}),$$  \quad (4.1.4)

$$G_k^{(-)}(t) = -(1)^k \mu_1 z^{\mu_1} + (1)^k z^{\mu_1} P_{\mu_1}(z^{-1} \lambda^k) + (1)^k z^{-\mu_1} P_{\mu_1}(z \lambda^{-k}),$$  \quad (4.1.5)
or

\[
G_k^{(+)}(t) = \frac{1}{i} \frac{d}{dt} \left( -1 \sum_{q=-\mu_0}^{-1} z^{-q-\mu_0} \lambda^{kq} - \frac{1}{i} \frac{d}{dt} \left( z^{\mu_0} + \sum_{q=1}^{q=\mu_0-1} z^{\mu_0-q} \lambda^{kq} \right) \right), \quad (4.1.6)
\]

\[
G_k^{(-)}(t) = (-1)^k \frac{1}{i} \frac{d}{dt} \left( \sum_{q=-\mu_1}^{-1} z^{-q-\mu_1} \lambda^{kq} - (-1)^k \frac{1}{i} \frac{d}{dt} \left( z^{\mu_1} + \sum_{q=1}^{q=\mu_1-1} z^{\mu_1-q} \lambda^{kq} \right) \right). \quad (4.1.7)
\]

Now we can write down the functions \( F_p(t) \) in the form

\[
F_p(t) = F_p^{(+)}(t) X_+ + F_p^{(-)}(t) X_-, \quad (4.1.8)
\]

where

\[
F_p^{(\pm)}(t) = \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-kp} G_k^{(\pm)}. \quad (4.1.9)
\]

First, introduce the values

\[
d_{\alpha} = \left[ \frac{\mu_\alpha}{n} \right], \quad \text{where } \alpha = 0, 1, \text{ and } [x] \text{ is an integer part of } x, \quad (4.1.10)
\]

\[
\tilde{\mu}_\alpha : = \mu_\alpha - n d_{\alpha}, \quad 1 \leq \tilde{\mu}_\alpha \leq n - 1. \quad (4.1.11)
\]

Each value \( q \in \{-\mu_\alpha, ..., \mu_\alpha\} \) can be represented in the form

\[
q = sn + q',
\]

where

\[
s \in \{-d_{\alpha} - 1, ..., d_{\alpha}\}, \quad (4.1.12)
\]

\[
q' \in \{0, ..., n - 1\}. \quad (4.1.13)
\]

Now we can write

\[
F_p^{(+)}(z) = \frac{1}{i} \frac{d}{dt} \left( \sum_{s=-d_0}^{s=-1} z^{-\mu_0-sn-p} \delta_{\mu_0>n} + z^{-\mu_0+(d_0+1)n-p} \delta_{n-1\geq p\geq n-\tilde{\mu}_0} \right) - \frac{1}{i} \frac{d}{dt} \left( \sum_{s=0}^{s=d_0-1} z^{-\mu_0-sn-p} \delta_{\mu_0>0} + \sum_{s=0}^{s=d_0-1} z^{-\mu_0-sn-p} \delta_{\mu_0>0} \delta_{\mu_0>n} + z^{-\mu_0-d_0n-p} \delta_{1\leq p\leq \tilde{\mu}_0} \right). \quad (4.1.14)
\]

Further,

\[
F_p^{(-)}(z) = \frac{1}{i} \frac{d}{dt} \left( \sum_{s=-d_1}^{s=-1} z^{-\mu_1-sn-p} \delta_{\mu_1>n} + z^{-\mu_1+(d_1+1)n-p} \delta_{n-1\geq p\geq n-\tilde{\mu}_1} \right)
\]

\[
- \frac{1}{i} \frac{d}{dt} \left( \sum_{s=0}^{s=d_1-1} z^{-\mu_1-sn-p} \delta_{\mu_1>0} + \sum_{s=0}^{s=d_1-1} z^{-\mu_1-sn-p} \delta_{\mu_1>0} \delta_{\mu_1>n} + z^{-\mu_1-d_1n-p} \delta_{1\leq p\leq \tilde{\mu}_1} \right), \quad (4.1.15)
\]
where
\[ \rho(p) = \begin{cases} p + m & \text{if } 0 \leq p < m, \\ p - m & \text{if } m \leq p < 2m, \end{cases} \]  
(4.1.16)

it is clear that \( \rho(\rho(p)) = p \).

**Proposition 4.1.** The functions \( F_p^{(\pm)} \) have the following form
\[ F_p^{(\pm)} = i \frac{d}{dt} \sum_{\ell = -\ell_{\pm}}^{\ell_{\pm}} a_{p,\ell}^{\pm} e^{\ell_{\pm}}, \]  
(4.1.17)

where \( \ell_+ = \mu_0, \ell_- = \mu_1 \), and the \( a_{p,\ell}^{\pm} \) are equal to either \( \text{sgn}(\ell) \) or 0.

**Proposition 4.2.** If there exist \( p \neq 0, m \) and \( \ell \) such that \( a_{p,\ell}^{+} \neq 0 \) and \( a_{p,\ell}^{-} \neq 0 \), then there exists an odd integer \( o \) such that \( \mu_0 - \mu_1 = om \).

### 4.2 Generating functions of the trace II

Now define one more set of generating functions of the trace, connected with Eq(4.2.1):
\[ \tilde{F}_p(t) := \text{tr}(\exp(ts)Q_p), \]  
(4.2.1)

where \( p = 0, \ldots, 2m - 1 \).

Evidently, \( \tilde{F}_p(t) = F_p(t) \) if \( p \neq 0, m \) since \( LQ_p = 0 \) if \( p \neq 0, m \).

Let \( \text{tr} \) be a degenerate trace. Then, it is possible to express \( \tilde{F}_p \) via \( F_p \) for \( p = 0, m \).

### 4.3 The generating functions \( \tilde{F}_0 = \text{tr}(\exp(ts)Q_0) \) and \( \tilde{F}_m = \text{tr}(\exp(ts)Q_m) \) for the degenerate trace

Let \( \mu_0, \mu_1 \in \mathbb{Z} \setminus m\mathbb{Z} \). Express \( \tilde{F}_p \) via \( F_p \) for \( p = 0, m \).

**Proposition 4.3.** Let \( p = 0, m \), then \( \tilde{F}_p \) is an even function of \( t \):
\[ \tilde{F}_p = \text{tr}(\cosh(ts)Q_p). \]  
(4.3.1)

Indeed, \( \tilde{F}_p = \text{tr}(\cosh(ts)Q_p + \sinh(ts)Q_p) \) and \( \text{tr}(\sinh(ts)Q_p) = 0 \) since
\[
\text{tr}(\sinh(ts)Q_p) = \text{tr}((\sinh(ts)L_p)L_p) = \text{tr}(L_p(\sinh(ts)L_p)) \\
= \text{tr}((\sinh(-ts)L_p)L_p) = \text{tr}((-(\sinh(ts)L_p))L_p) = \text{tr}(-\sinh(ts)Q_p).
\]

Now, decompose \( F_0 \):
\[ F_0 = \text{tr} \left( e^{(s-i\mu)L_p}Q_0 \right) = F_0^{\text{even}} + F_0^{\text{odd}}, \]  
(4.3.2)
where

\[ F_0^{even} = \text{tr} \left( \sum_{s=0}^{\infty} \frac{1}{(2s)!} (t(s - i \mu L_0))^{2s} Q_0 \right) = \text{tr} \left( \sum_{s=0}^{\infty} \frac{1}{(2s)!} t^{2s} (s^2 - \mu^2)^s Q_0 \right), \quad (4.3.3) \]

\[ F_0^{odd} = \text{tr} \left( \sum_{s=0}^{\infty} \frac{1}{(2s+1)!} (t(s - i \mu L_0))^{2s+1} Q_0 \right) = \text{tr} \left( \sum_{s=0}^{\infty} \frac{1}{(2s+1)!} t^{2s+1} (s^2 - \mu_0^2)^s (s - i \mu_0 L_0) Q_0 \right) = \text{tr} \left( \sum_{s=0}^{\infty} \frac{1}{(2s+1)!} t^{2s+1} (s^2 - \mu_0^2)^s (-i \mu_0) \text{tr} L_0 \right) \]

\[ = \sinh(-it \mu_0) \text{tr} L_0 = -\frac{1}{2}(z^{\mu_0} - z^{-\mu_0}) \text{tr} L_0. \quad (4.3.4) \]

Eq. \((4.1.17)\) implies that

\[ F_{odd} = \frac{1}{2} \left( \sum_{\ell=\mu}^{-\mu} a_0^{\ell} z^\ell - \sum_{\ell=-\mu}^{\mu} a_0^{\ell} \tilde{z}^\ell \right). \quad (4.3.5) \]

Comparing Eq. \((4.3.5)\) with Eq. \((4.3.4)\) we see that

\[ a_0^{\ell} = a_0^{-\ell}, \quad \text{if} \ \ell \neq \mu, \ \ell \neq -\mu, \]

\[ a_0^{\mu} = a_0^{-\mu} = -\text{tr} L_0, \quad (4.3.6) \]

and

\[ F_{even} = \frac{1}{2} a_0^{\mu} (y^\mu + y^{-\mu}) + \frac{1}{2} \sum_{\ell=0}^{\mu-1} a_0^{\ell} (y^\ell + y^{-\ell}) = a_0^{\mu} \cosh(it \mu) + \sum_{\ell=0}^{\mu-1} a_0^{\ell} \cosh(it \ell). \quad (4.3.7) \]

**Proposition 4.4.**

\[ \tilde{F}^{\text{tr}}_0(t) = a_0^{\mu} + \sum_{\ell=0}^{\mu-1} a_0^{\ell} \cosh \left( t \sqrt{\mu^2 - \ell^2} \right). \]

**Proof.** Taking Proposition \(4.3\) into account let us expand Eq \((4.3.1)\) into the Taylor series:

\[ \tilde{F}^{\text{tr}}_0(t) = \sum_{s=0}^{\infty} a_{2s} \frac{t^{2s}}{(2s)!}, \]

where \(a_{2s} := \text{tr}(s^{2s} Q_0)\) for \(s = 0, 1, 2, \ldots\).
Eq (4.3.3) implies
\[ a_{2s} = \left( \frac{d^2}{dt^2} + \mu^2 \right)^s F_{\text{even}}|_{t=0}, \]
and Eq (4.3.7) implies
\[ a_{2s} = \left\{ \begin{array}{ll} a_0^0 + \sum_{\ell=0}^{\mu-1} \alpha_\ell^0 & \text{if } s = 0 \\ \sum_{\ell=0}^{\mu-1} \alpha_\ell^0 (-\ell^2 + \mu^2)^s & \text{if } s \neq 0. \end{array} \right. \]
So
\[ \tilde{F}_{0}^{tr}(t) = \sum_{s=0}^{\infty} a_{2s} \frac{t^{2s}}{(2s)!} = a_0^0 + \sum_{\ell=0}^{\mu-1} \alpha_\ell^0 \cosh \left( t \sqrt{\mu^2 - \ell^2} \right). \]

4.4 Expressions for \( \tilde{F}_{0}^{tr} = \text{tr} (\exp(ts)Q_0) \) and \( \tilde{F}^{tr} = \text{tr} (\exp(ts)Q_m) \) for the degenerate trace
For \( F_{0}^{(+)} \), \( F_{0}^{(-)} \), \( F_{m}^{(+)} \) and \( F_{m}^{(-)} \), we find the following expressions:
\[
F_{0}^{(+)}(z) = \frac{1}{i} \frac{d}{dt} \left( \sum_{s=-d_0}^{s=-1} z^{-\mu_0-sn} \delta_{p=n} \right) - \frac{1}{i} \frac{d}{dt} \left( \sum_{s=0}^{d_0} z^{\mu_0-sn} \delta_{p=0} \right) = -\frac{1}{i} \frac{d}{dt} z^{\mu_0} + \left( -\sum_{s=1}^{d_0} z^{\mu_0-sn} \delta_{p=n} + \sum_{s=-d_0}^{s=-1} z^{-\mu_0-sn} \delta_{p=n} \right) = -\frac{1}{i} \frac{d}{dt} z^{\mu_0} + \left( \sum_{s=1}^{d_0} z^{\mu_0-sn} \delta_{p=n} + \sum_{s=1}^{s=d_0} z^{-\mu_0+sn} \delta_{p=n} \right) = -\frac{1}{i} \frac{d}{dt} z^{\mu_0} + \left( \sum_{s=1}^{d_0} \sinh(it(-\mu_0 + sn)) \delta_{p=n} \right), \tag{4.4.1} \]
\[
F_0^{(-)}(z) = \frac{1}{i} \frac{d}{dt} \left( \sum_{s=-d_1}^{s=-1} z^{-\mu_1 - sn - m} \delta_{\mu_1 > n} + z^{-\mu_1 + (d_1 + 1)n - m} \delta_{n-1 \geq m \geq n - \tilde{\mu}_1} \right) \\
- \frac{1}{i} \frac{d}{dt} \left( \sum_{s=d_0}^{d_1-1} z^{-\mu_1 - sn - m} \delta_{\mu_1 > n} + z^{\mu_1 - d_1 n - m} \delta_{1 \leq m \leq \tilde{\mu}_1} \right) \\
= \frac{1}{i} \frac{d}{dt} \left( \sum_{s=-d_1}^{s=d_1} z^{-\mu_1 - sn - m} - \sum_{s=0}^{d_1-1} z^{\mu_1 - sn - m} \right) \delta_{\mu_1 > n} \\
+ \frac{1}{i} \frac{d}{dt} \left( z^{-\mu_1 + (d_1 + 1)n - m} - z^{\mu_1 - d_1 n - m} \right) \delta_{1 \leq m \leq \tilde{\mu}_1} \\
= \frac{2}{i} \frac{d}{dt} \left( \sum_{s=1}^{s=d_1} \sinh(it(-\mu_1 + sn - m)) \right) \delta_{\mu_1 > n} \\
+ \frac{2}{i} \frac{d}{dt} \sinh(it(-\mu_1 + d_1 n + m)) \delta_{m \leq \tilde{\mu}_1}, \quad (4.4.2)
\]

\[
F_m^{(+)}(z) = \frac{1}{i} \frac{d}{dt} \left( \sum_{s=-d_0}^{s=-1} z^{-\mu_0 - sn - m} \delta_{\mu_0 > n} + z^{-\mu_0 + (d_0 + 1)n - m} \delta_{n-1 \geq m \geq n - \tilde{\mu}_0} \right) \\
- \frac{1}{i} \frac{d}{dt} \left( \sum_{s=0}^{s=d_0-1} z^{-\mu_0 - sn - m} \delta_{\mu_0 > n} + z^{\mu_0 - d_0 n - m} \delta_{1 \leq m \leq \tilde{\mu}_0} \right) \\
= \frac{1}{i} \frac{d}{dt} \sum_{s=1}^{s=d_0} \left( z^{-\mu_0 + sn - m} - z^{\mu_0 - sn + m} \right) \delta_{\mu_0 > n} \\
+ \frac{1}{i} \frac{d}{dt} \left( z^{-\mu_0 + d_0 n + m} - z^{\mu_0 - d_0 n - m} \right) \delta_{1 \leq m \leq \tilde{\mu}_0} \\
= \frac{2}{i} \frac{d}{dt} \sum_{s=1}^{s=d_0} \sinh(it(\mu_0 - sn + m)) \delta_{\mu_0 > n} \\
+ \frac{2}{i} \frac{d}{dt} \sinh(it(-\mu_0 + d_0 n + m)) \delta_{m \leq \tilde{\mu}_0}, \quad (4.4.3)
\]
\[
F_m^{(-)}(z) = \frac{1}{i} \frac{d}{dt} \left( \sum_{s=-d_1}^{s=1} z^{-\mu_1 - sn} \delta_{\mu_1 > n} \right) - \frac{1}{i} \frac{d}{dt} \left( \sum_{s=0}^{d_1} z^{\mu_1 - sn} \right)
= - \frac{1}{i} \frac{d}{dt} \mu_1 + \frac{1}{i} \frac{d}{dt} \left( \sum_{s=1}^{d_1} z^{-\mu_1 + sn} \delta_{\mu_1 > n} \right)
- \frac{1}{i} \frac{d}{dt} \left( \sum_{s=1}^{d_1} z^{\mu_1 - sn} \delta_{\mu_1 > n} \right)
= - \frac{1}{i} \frac{d}{dt} \mu_1 + \frac{2}{i} \frac{d}{dt} \sum_{s=1}^{d_1} \sinh \left( it \left( -\mu_1 + sn \right) \right) \delta_{\mu_1 > n}.
\]

(4.4.4)

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