Slicing theorems and rigidity phenomena for self-affine carpets

Amir Algom

Abstract
Let \( F \) be a Bedford–McMullen carpet defined by independent exponents. We prove that \( \dim B(\ell \cap F) \leq \max\{\dim^* F - 1, 0\} \) for all lines \( \ell \) not parallel to the principal axes, where \( \dim^* \) is Furstenberg’s star dimension (maximal dimension of a microset). We also prove several rigidity results for incommensurable Bedford–McMullen carpets, that is, carpets \( F \) and \( E \) such that all defining exponents are independent: Assuming various conditions, we find bounds on the dimension of the intersection of such carpets, show that self-affine measures on them are mutually singular, and prove that they do not embed affinely into each other.

We obtain these results as an application of a slicing theorem for products of certain Cantor sets. This theorem is a generalization of the results of Shmerkin [Ann. of Math. (2) 189 (2019) 319–391] and Wu [Ann. of Math. (2) 189 (2019) 707–751], which proved Furstenberg’s slicing conjecture [Problems in analysis (ed. R. C. Gunning; Princeton University Press, Princeton, NJ, 1970) 41–59].

1. Introduction
Let \( F \subset \mathbb{R}^2 \) be a set, and let \( \ell \subset \mathbb{R}^2 \) be an affine line. One of the classic questions in geometric measure theory involves studying the dimension of \( F \cap \ell \), as we go over all the lines in the plane. It is natural to parametrize a line in the plane by its slope (an element in \( \mathbb{R} \cup \{\infty\} \), where \( \infty \) corresponds to lines parallel to the y-axis) and its intercept (an element in \( \mathbb{R} \)). The most general result in this direction, known as Marstrand’s slicing theorem, asserts that for any fixed slope \( u \),

\[
\dim_H F \cap \ell_{u,t} \leq \max\{\dim_H F - 1, 0\} \quad \text{for Lebesgue almost every } t,
\]

where \( \dim_H \) denotes the Hausdorff dimension and \( \ell_{u,t} \) is the line with slope \( u \) and intercept \( t \).

While (1) predicts the dimension of the intersection of \( F \) with a typical line \( \ell \), it is a challenging problem to understand the intersection of \( F \) with a fixed line \( \ell \). However, when the set \( F \) has some arithmetic or dynamical origin, it is sometimes possible to say something beyond (1).

Indeed, for an integer \( 2 \leq m \in \mathbb{N} \), define the \( m \)-fold map of the unit interval

\[
T_m : [0,1] \rightarrow [0,1], \quad T_m(x) = m \cdot x \mod 1.
\]

(2)

When we say that a line is not principal, we mean that its slope is in \( \mathbb{R} \setminus \{0\} \), that is, it is not parallel to the principal axes of \( \mathbb{R}^2 \). The following conjecture, known also as Furstenberg’s slicing conjecture, is an example of the approach described in the previous paragraph.

Conjecture 1.1 (Furstenberg, [15]). Let \( \emptyset \neq X, Y \subseteq [0,1] \) be the closed sets that are invariant under \( T_m \) and \( T_n \), respectively. If \( \frac{\log n}{\log m} \notin \mathbb{Q} \), then for every non-principal line \( \ell \),

\[
\dim_H \ell \cap (X \times Y) \leq \max\{\dim_H X + \dim_H Y - 1, 0\}.
\]
Recently, two landmark papers have proven, simultaneously and independently, this conjecture to be correct: One of them, by Shmerkin [26], proved it by computing the $L^q$ dimensions of all the projections of products of invariant Cantor–Lebesgue measures. The second approach, by Wu [27], followed initially along the original idea of Furstenberg by constructing a stationary distribution on the space of measures on slices of $X \times Y$. Wu then applied Sinai’s factor theorem, ‘forcing’ many slices of large dimension to pass through a small region in the unit square, which yielded the conjecture. In this paper, we shall take after Wu’s approach.

The objectives of this paper are threefold. The first is to generalize the phenomenon predicted by Conjecture 1.1 to more general product sets, and in particular, to products of sets that are not necessarily $T_m$ invariant for some $m$ (the results of Shmerkin and Wu do not apply for these sets). The second objective is to apply these results in order to prove slicing theorems for Bedford–McMullen carpets with independent exponents. The third objective is to apply the results on slicing theorems for product sets in order to prove some rigidity results in the class of Bedford–McMullen carpets. Namely, for two carpets that are incommensurable in a sense that will be defined below (and satisfy some other varying conditions), we bound non-trivially the dimension of their intersection, show that a large class of self-affine measures on them are mutually singular, and show that they do not embed affinely into one another.

In the subsequent section, we outline our results in the context of the latter two objectives, which form the main results of this paper. The section following it outlines our results in the context of the first objective, which forms our main technical tool.

1.1. Main results

Our main results are about geometric properties of Bedford–McMullen carpets. These are defined as follows: let $m \neq n$ be integers greater than one, and denote for every integer, $[n] := \{0, \ldots, n-1\}$. We shall always assume $m > n$. Let

$$\Gamma \subseteq \{0, \ldots, m-1\} \times \{0, \ldots, n-1\} = [m] \times [n],$$

and define

$$F = \left\{ \left( \sum_{k=1}^{\infty} x_k m^k, \sum_{k=1}^{\infty} y_k n^k \right) : (x_k, y_k) \in \Gamma \right\}.$$ 

$F$ is then called a Bedford–McMullen carpet with defining exponents $m, n$, and allowed digit set $\Gamma$. For every $j \in [n]$, let

$$\Gamma_j := \{ i \in [m] : (i, j) \in \Gamma \} \subseteq [m].$$

We shall always assume that our carpets do not lie on a single vertical or horizontal line. When we have two carpets $F$ and $E$, we shall denote the set of allowed digits of $E$ by $\Lambda$.

1.1.1. Dimension of slices through Bedford–McMullen carpets. We denote by $P_2 : \mathbb{R}^2 \to \mathbb{R}$ the principal projection $P_2(x, y) = y$. We shall use the same notation for the coordinate projection in $([m] \times [n])^N$.

**Theorem 1.2.** Let $F$ be a Bedford–McMullen carpet with exponents $(m, n)$ such that $\frac{\log m}{\log n} \notin \mathbb{Q}$. Let $\ell$ be any non-principal line in the plane. Then

$$\text{dim}_B(\ell \cap F) \leq \max \left\{ \text{dim}_H P_2(F) + \max_{i \in [n]} \frac{\log |\Gamma_i|}{\log m} - 1, 0 \right\}.$$
The bound obtained in Theorem 1.2 comes from the star dimension of the carpet \( F \), a notion introduced by Furstenberg in [16]: For any set \( A \) we define
\[
\dim^* A := \sup \{ \dim_H M : M \text{ is a microset of } A \},
\]
where microsets of \( A \) are limits in the Hausdorff metric of ‘blow-up’ of increasingly small balls about points in \( A \) (for a formal definition of a microset, and some discussion of them, see Section 2.2). Now, in [23], Mackay proved that for a Bedford–McMullen carpet \( F \),
\[
\dim^* F = \dim_H P_2(F) + \max_{i \in [n]} \frac{\log |\Gamma_i|}{\log m}.
\]
Thus, Theorem 1.2 implies that \( \overline{\dim}_{H}(\ell \cap F) \leq \max\{\dim^* F - 1, 0\} \) for any non-principal line \( \ell \).

Also, notice that if for every \( i \neq j \in P_2(\Gamma) \), we have \( |\Gamma_i| = |\Gamma_j| \) and then it is known that \( \dim_H F = \dim^* F \) (this follows from the original works of McMullen [25] and Bedford [5], see also a proof in [6]). Therefore, in this situation, we recover the ‘optimal’ bound, in the sense of (1) and Conjecture 1.1. However, in general \( \dim_H F < \dim^* F \), and we do not know whether Theorem 1.2 can be optimized to give that \( \dim_H F - 1 \) bounds the dimension of any non-principal slice.

1.1.2. Rigidity phenomena in the class of Bedford–McMullen carpets. Let \( F \) and \( E \) be two Bedford McMullen carpets with defining exponents \((m_1, n_1)\) and \((m_2,n_2)\), respectively, and allowed digits sets \( \Gamma \) and \( \Lambda \).

**Definition 1.3.** We shall say that \( F \) and \( E \) are incommensurable if
\[
\frac{\log m_1}{\log m_2}, \frac{\log m_1}{\log n_2}, \frac{\log m_1}{\log m_2}, \frac{\log n_1}{\log n_2}, \frac{\log n_1}{\log m_2}
\]
are all not in \( \mathbb{Q} \).

In this section, we shall describe several results about geometric rigidity of incommensurable Bedford–McMullen carpets. The following result gives a bound on the dimension of intersections of such carpets. When we write \( \dim \) we always mean Hausdorff dimension.

**Theorem 1.4.** Let \( F \) and \( E \) be two incommensurable Bedford–McMullen carpets. Let \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) be an affine map.

1. If the linear part of \( g \) is given by a diagonal matrix, then
\[
\dim^*(g(F) \cap E) \leq \max_{(i,j) \in [n_1] \times [n_2]} \left\{ \frac{\log |\Gamma_i|}{\log m_1} + \frac{\log |\Lambda_j|}{\log m_2} - 1, 0 \right\} + \max \{ \dim P_2(F) + \dim P_2(E) - 1, 0 \}.
\]

2. If the linear part of \( g \) is given by an anti-diagonal matrix, then
\[
\dim^*(g(F) \cap E) \leq \max_{i \in [n_1]} \left\{ \frac{\log |\Gamma_i|}{\log m_1} + \dim P_2(E) - 1, 0 \right\} + \max_{j \in [n_2]} \left\{ \dim P_2(F) + \frac{\log |\Lambda_j|}{\log m_2} - 1, 0 \right\}.
\]

Theorem 1.4 is related to a long line of research about intersections of Cantor sets. Notable related works include, for example, those of Shmerkin [26] and Wu [27] that proved Conjecture 1.1, the work of Feng, Huang and Rao [12] and the work of Elekes, Keleti and Máté.
By this theorem, it is quite easy to see that the assumption that the carpets are incommensurable cannot be lifted from Theorem 1.4.

Next, we discuss self-affine measures on Bedford–McMullen carpets. First, for any integer $n \geq 2$, we define a map $\pi_n : [n]^N \to [0,1]$ by

$$
\pi_n(\xi) = \sum_{i=1}^{\infty} \frac{\xi_i}{n^i}, \quad \xi = (\xi_i) \in [n]^N.
$$

(6)

A self-affine measure $\mu$ on a Bedford–McMullen carpet $F$ is the push-forward $\pi_{m_1} \times \pi_{m_2}(\nu)$ of a Bernoulli measure $\nu \in P(\Gamma^N)$ (that is, a stationary product measure), where $P(X)$ denotes the family of probability measures on a Borel space $X$.

**Theorem 1.5.** Let $F$ and $E$ be two incommensurable Bedford–McMullen carpets, and let $\mu \in P(F)$ and $\nu \in P(E)$ be two self-affine measures. Let $\kappa := \max\{\dim_H \mu, \dim_H \nu\}$.

(1) If

$$
\kappa > \max_{(i,j) \in [m_1] \times [m_2]} \left\{ \frac{\log |\Gamma_i|}{\log m_1} + \frac{\log |A_j|}{\log m_2} - 1, 0 \right\} + \max \{\dim P_2(F) + \dim P_2(E) - 1, 0\}.
$$

Then for any affine map $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that the linear part of $g$ is a diagonal matrix, the measures $g\mu$ and $\nu$ are mutually singular.

(2) If

$$
\kappa > \max_{i \in [m_1]} \left\{ \frac{\log |\Gamma_i|}{\log m_1} + \dim P_2(E) - 1, 0 \right\} + \max_{j \in [m_2]} \left\{ \frac{\log |A_j|}{\log m_2} + \dim P_2(F) - 1, 0 \right\}.
$$

Then for any affine map $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that the linear part of $g$ is an anti-diagonal matrix, the measures $g\mu$ and $\nu$ are mutually singular.

For the definition of the dimension of a measure, we refer the reader to Section 2.1. Theorem 1.5 is an analogue in higher dimension of a theorem of Hochman ([19, Theorem 1.4]). By this theorem, if $\frac{\log m}{\log n} \notin \mathbb{Q}$, then any diffeomorphic image of an ergodic $T_m$ invariant measure on $\mathbb{R}/\mathbb{Z}$ and any ergodic $T_n$ invariant measure on $\mathbb{R}/\mathbb{Z}$ are mutually singular, assuming that both have intermediate dimension (recall the definition of the $m$-fold map of the interval $T_m$ from (2)).

Finally, we discuss affine embeddings of incommensurable Bedford–McMullen carpets. Let $F$ and $E$ be two Bedford–McMullen carpets. We say that $F$ may be affinely embedded into $E$ if there exists an invertible affine map $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(F) \subseteq E$.

**Theorem 1.6.** Let $F$ and $E$ be two incommensurable Bedford–McMullen carpets. Assume that $\min_{i \in [n_1]}|\Gamma_i| > 1$, and that $\dim E < 2$. Then $F$ does not admit an affine embedding into $E$.

Theorem 1.6 is related to the recently developed theory of affine embeddings of Cantor sets. The first to study such problems (for self-similar sets) were Feng, Huang and Rao in [12]. In the same paper, they formulated a conjecture, stating that if one self-similar set embeds into the other, then every one of its contraction ratios should be algebraically dependent on the contractions of the other set. This conjecture was resolved for homogeneous self-similar sets in dimension 1 by Shmerkin and Wu, in the papers proving Conjecture 1.1, but remains open in general (for some partial results, see also [1, 2, 13]). There is a clear relation between this conjecture and Theorem 1.6: our theorem says that if $F$ embeds into $E$, then the eigenvalues of the matrices in a generating iterated function system (IFS) for $F$ are dependent on those of $E$, which is an analogue (in an appropriate sense) of the latter conjecture.
Finally, we do not know whether the assumptions on the dimensions of $F$ and $E$ are a by-product of our proof, or form genuine obstructions. The assumption that the carpets are incommensurable cannot be even slightly weakened in the general case, as the following example shows. Consider the carpet $F$ defined by the exponents $(3,2)$ and the digit set

$$\Gamma = \{(0,0), (0,1), (2,0)\},$$

and let $E$ be the carpet defined by exponents $(5,3)$ and the digit set

$$\Lambda = \{(i,0), (j,2), (1,1) : 0 \leq i, j \leq 4\}.$$ 

Notice that $\dim^* E = 2$. Then, although $\frac{\log 5}{\log 3} \notin \mathbb{Q}$, it is not hard to see that we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot F \subset E.$$

1.2. A slicing theorem for products of Cantor sets

We obtain the results of Section 1.1 as applications of the following slicing theorem. Let us first describe its setup. Let $m_1 > m_2 \geq 2$ and $n_1, n_2 \geq 2$ be integers. Unless stated otherwise, we always assume $\theta := \frac{\log m_2}{\log m_1} \notin \mathbb{Q}$. For every $i \in [n_1]$, we associate a subset $\emptyset \neq \Gamma_i \subseteq [m_1]$, and for every $j \in [n_2]$, we associate a subset $\emptyset \neq \Lambda_j \subseteq [m_2]$. We always assume that there exists some $i \in [n_1]$ such that $\Gamma_i \neq [m_1]$, and similarly a $j \in [n_2]$ such that $\Lambda_j \neq [m_2]$. Our setup (and notation) is motivated by Bedford–McMullen carpets, and the notation we have used for them in Section 1.1, in particular (3).

Thus, given $\omega \in [n_1]^N$ and $\eta \in [n_2]^N$, we define product sets

$$\tilde{F}_\omega = \prod_{i=1}^\infty \Gamma_{\omega_i} \subseteq [m_1]^N, \quad \tilde{E}_\eta = \prod_{i=1}^\infty \Lambda_{\eta_i} \subseteq [m_2]^N. \quad (7)$$

In particular, for $\omega \in [n_1]^N$ and $\eta \in [n_2]^N$, we have

$$\pi_{m_1}(\tilde{F}_\omega) = \left\{ \sum_{i=1}^\infty \frac{x_i}{m_1^i} : x_i \in \Gamma_{\omega_i} \right\}, \quad \pi_{m_2}(\tilde{E}_\eta) = \left\{ \sum_{i=1}^\infty \frac{y_i}{m_2^i} : y_i \in \Lambda_{\eta_i} \right\},$$

where the maps $\pi_{m_i}$ were defined in (6).

**Theorem 1.7.** (1) Let $\ell \subset \mathbb{R}^2$ be a non-principal line, and let $(\omega, \eta) \in [n_1]^N \times [n_2]^N$. Then

$$\bar{\dim}_B \left( \pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta) \right) \cap \ell \leq \max_{i \in [n_1], j \in [n_2]} \left( \frac{\log |\Gamma_i|}{\log m_1} + \frac{\log |\Lambda_j|}{\log m_2} - 1, 0 \right).$$

(2) Let $u \in \mathbb{R} \setminus \{0\}$, and let $\alpha_1$ and $\alpha_2$ be Bernoulli measures on $[n_1]^N$ and $[n_2]^N$, respectively. Then there exists a measurable set $A(u, \alpha_1, \alpha_2) \subseteq [n_1]^N \times [n_2]^N$ of full $\alpha_1 \times \alpha_2$ measure such that for all $(\omega, \eta) \in A(u, \alpha_1, \alpha_2)$, and for any line $\ell$ with slope $u$,

$$\bar{\dim}_B \left( \pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta) \right) \cap \ell \leq \max \left\{ \dim_H \pi_{m_1}(\tilde{F}_\omega) + \dim_H \pi_{m_2}(\tilde{E}_\eta) - 1, 0 \right\}.$$

If for all $i \neq j \in [n_1]$ we have $|\Gamma_i| = |\Gamma_j|$, then $\dim_H \pi_{m_1}(\tilde{F}_\omega) = \frac{\log |\Gamma_i|}{\log m_1}$, and similarly for $\pi_{m_2}(\tilde{E}_\eta)$ if $|\Lambda_j|$ is constant for all $j \in [n_2]$. Thus, by Theorem 1.7 part (1), we recover many new explicit examples of product sets satisfying the Furstenberg slicing bound, in the sense of Conjecture 1.1. Moreover, by this observation and an approximation argument, it is possible to show that Theorem 1.7 implies Conjecture 1.1. However, as our method is based on Wu’s method from [27], this does not yield a new proof.
1.3. On the proof of Theorem 1.7

Let $m_1 > m_2 \geq 2$ be integers such that $\theta := \frac{\log m_2}{\log m_1} \notin \mathbb{Q}$. First, let $\emptyset \neq X, Y \subseteq [0, 1]$ be two closed sets that are $T_{m_1}$ and $T_{m_2}$ invariant sets, respectively. Let $\ell \cap (X \times Y)$ be any non-principal slice through the corresponding product set. In [27], Wu proved $\dim \ell \cap (X \times Y) \leq \max\{\dim X + \dim Y - 1, 0\}$ (and thus Conjecture 1.1) by first constructing a well-structured measure (a CP distribution) on the space of measures on slices of $X \times Y$. Two key features of this measure are that its marginal on the slopes of these slices is the Lebesgue measure, and that almost all of these slices have at least the same dimension as the original slice $\ell \cap (X \times Y)$. The construction of such a measure, originally due to Furstenberg in [15], relies on the following observation: For every $t \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, define a map $\Phi_t : [0, 1]^2 \to [0, 1]^2$, by

$$\Phi_t(z) = \begin{cases} (T_{m_1}(z_1), T_{m_2}(z_2)) & \text{if } t \in [1 - \theta, 1) \\ (z_1, T_{m_2}(z_2)) & \text{if } t \in [0, 1 - \theta). \end{cases}$$

Notice that, if $m_1^\ell$ is the slope of $\ell$, then the map $\Phi_t$ transforms our slice into a finite family of slices through $X \times Y$, such that their slope corresponds to the translation by $\theta$ in $\mathbb{T}$ of $t$, and at least one has the same dimension as the original slice.

Wu then proceeded to apply Sinai’s factor theorem, allowing him to show that many slices that are both of dimension at least $\dim \ell \cap (X \times Y)$, and such that their slopes correspond to sets of arbitrarily large density in an equidistributed sequence in $\mathbb{T}$, pass through a small region in the unit square. This yielded the desired bound on $\dim \ell \cap (X \times Y)$ by a Fubini-type argument.

We take a similar approach, but we construct our CP distribution on a larger parameter space: The space of non-principal slices of all product sets in the family

$$\left\{ \pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta) : (\omega, \eta) \in [n_1]^N \times [n_2]^N \right\}.$$

We also define, for $t \in \mathbb{T}$, a map $\sigma_t : [n_1]^N \to [n_1]^N$ by

$$\sigma_t(\omega) = \begin{cases} \sigma(\omega) & \text{if } t \in [1 - \theta, 1) \\ \omega & \text{if } t \in [0, 1 - \theta). \end{cases}$$

The basic observation behind our approach is that now, for any non-principal slice $\ell \cap (\pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta))$ through any product set in our family, the map $\Phi_t$ transforms this slice into a finite family of slices through $\pi_{m_1}(\tilde{F}_{\sigma_t(\omega)}) \times \pi_{m_2}(\tilde{E}_{\sigma_t(\eta)})$, where $\sigma : [n_1]^N \to [n_1]^N$ is the left shift. It is still true that their slopes correspond to the original slope translated by $\theta$ in $\mathbb{T}$, and at least one has the same dimension as the original slice. Notice that this is a slice through (possibly a different) product set in our family.

An application of Sinai’s factor theorem yields a similar conclusion to that of Wu’s, that many slices in this family that are both of dimension at least $\dim \ell \cap (\pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta))$, and such that their slopes correspond to sets of arbitrarily large density in an equidistributed sequence in $\mathbb{T}$, pass through a small region in the unit square. Moreover, using this idea we can also show that the amount of product sets in our family being sliced in this procedure is not too large (in some sense), allowing for a Fubini argument (similar, but more complicated, than that of Wu’s), to be performed.

However, unless we have some additional information about the $(\omega, \eta)$ from Theorem 1.7 part (1) (as we do in Theorem 1.7 part (2)), we cannot control which product sets will play a part in the end game of this procedure. This explains the bound appearing in part (1) of the theorem (which is the ‘worst case scenario’, the largest possible box dimension of a product set in our family).
**Organization**
In Section 2, we survey some relevant definitions and results about dimension theory of sets and measures, and about CP distributions. We then proceed to prove, in Section 3, Theorems 1.2, 1.4, 1.5 and 1.6, assuming that Theorem 1.7 is correct. The subsequent sections are then devoted to the proof of Theorem 1.7 and related constructions.

**Notation**
For the reader’s convenience, we summarize some of our main notation conventions in the following table. We remark that these notations will be used for the proofs in Section 3. For some minor variations related to the proof of Theorem 1.7, see the remark after the table.

| Notation | Interpretation |
|----------|----------------|
| \([m], m \in \mathbb{N}\) | The set \(\{0, \ldots, m - 1\}\) |
| \(T^m : [0, 1] \to [0, 1]\) | The map \(T^m(x) = m \cdot x \mod 1\) |
| \(\dim P(X)\) | Hausdorff dimension |
| \(P_1, P_2\) | The projections \(\mathbb{R}^2 \to \mathbb{R}, P_2(x, y) = y, P_1(x, y) = x\) |
| \(F, E\) | Bedford–McMullen carpets |
| \(\Gamma, \Lambda\) | Sets of digits defining \(F\) and \(E\), \(\Gamma \subseteq [m_1] \times [n_1], \Lambda \subseteq [m_2] \times [n_2]\) |
| \(\pi_m\) | The ‘projection’ \(\pi_m : [m]^N \to [0, 1], \pi_m(\omega) = \sum_{k=1}^\infty \omega_k\) |
| \(\pi_m \times \pi_n\) | \(\pi_m \times \pi_n : ([m] \times [n])^N \to [0, 1], \pi_m \times \pi_n(\omega, \eta) = (\pi_m(\omega), \pi_n(\eta))\) |
| \(\tilde{F}\) | Symbolic version of \(F, \tilde{F} = \Gamma^N \subseteq ([m_1] \times [n_1])^N\) |
| \(\tilde{E}_y\) | Horizontal slice at height \(y\), \(\{x \in \mathbb{R} : (x, y) \in F\}\) |
| \(\Gamma_j, j \in [n_1]\) | \(\{i \in [m_1] : (i, j) \in \Gamma\}\) |
| \(\Lambda_j, j \in [n_2]\) | \(\{i \in [m_2] : (i, j) \in \Lambda\}\) |
| \(\tilde{F}_\omega, \omega \in [n_1]^N\) | \(\tilde{F}_\omega = \prod_{i=1}^N \Gamma_{\omega_i}\) |
| \(\tilde{E}_\eta, \eta \in [n_2]^N\) | \(\tilde{E}_\eta = \prod_{i=1}^N \Gamma_{\eta_i}\) |
| \(\alpha\) | Shift operator on a symbolic space of the form \([m]^N\) |
| \(D_p, p \geq 2\) | The \(p\)-adic partition of \(\mathbb{R}^d, D_p = \{\prod_{i=1}^d \left[ \frac{z_i}{p} \cdot \frac{p+1}{p} \right] : (z_1, \ldots, z_d) \in \mathbb{Z}^d\}\) |
| \(T^k\) | The partition of \([n]^N\) into cylinders of length \(k\) |
| \(R_\theta : T \to T, \theta \in T\) | The rotation \(R_\theta(t) = t + \theta \mod 1\) |

Recall that in the setting of Theorem 1.7, we always assume that \(m_1 > m_2 \geq 2\) and \(n_1, n_2 \geq 2\) are integers, and that \(\theta := \frac{\log m_2}{\log m_1} \notin \mathbb{Q}\). For every \(i \in [n_1]\), we associate a subset \(\emptyset \neq \Gamma_i \subseteq [m_1]\), and for every \(j \in [n_2]\), we associate a subset \(\emptyset \neq \Lambda_j \subseteq [m_2]\). Apart from this, we use the same notation as the table above. In particular, given \(\omega \in [n_1]^N\) and \(\eta \in [n_2]^N\), we define the product sets \(\tilde{F}_\omega\) and \(\tilde{E}_\eta\) exactly as in the table above.

2. **Preliminaries**

Let \(X\) be a metric space. The set of Borel probability measures on \(X\) will be denoted by \(P(X)\). In this paper, all measures are Borel probability measures.

2.1. **Some notions of dimension of sets and measures**

For a set \(A\) in some metric space, we use the standard notation \(\dim_H A\) for the Hausdorff dimension of \(A\), and \(\dim_{UB}(A)\) for the upper box dimension of \(A\). See, for example, Falconer’s book [8] for some exposition on these concepts.

Next, let \(\mu\) be a Borel probability measure on some metric space. For every \(x \in \text{supp}(\mu)\), we define the pointwise (exact) dimension of \(\mu\) at \(x\) as

\[
\dim(\mu, x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},
\]
where \( B(x, r) \) denotes the closed ball of radius \( r \) about \( x \). If the limit does not exist, we define the upper and lower pointwise dimensions of \( \mu \) at \( x \) as the corresponding \( \limsup \) and \( \liminf \).

We also define the (lower) Hausdorff dimension of the measure \( \mu \) as

\[
\dim(\mu) = \inf \{ \dim_H A : \mu(A) > 0 \}.
\]

If the pointwise dimension of \( \mu \) exists at almost every \( x \in \text{supp}(\mu) \) and is constant almost surely, then this constant value is known to equal \( \dim(\mu) \). For proofs and some more discussion, see, for example, \([9]\) or \([10]\).

Next, we discuss entropy of measures and entropy dimension. First, let \( \mu \) be a Borel probability measure on some metric space. Let \( \mathcal{A} \) denote a countable (or finite) partition of the underlying space. Then the entropy of \( \mu \) with respect to \( \mathcal{A} \) is defined as

\[
H(\mu, \mathcal{A}) = -\sum_{A \in \mathcal{A}} \mu(A) \cdot \log \mu(A)
\]

with the convention \( 0 \log 0 = 0 \).

Let us now define the entropy dimension of a measure \( \mu \in \mathcal{P}(\mathbb{R}^d) \). For every integer \( p \geq 0 \), let \( \mathcal{D}_p \) denote the \( p \)-adic partition of \( \mathbb{R}^d \), that is,

\[
\mathcal{D}_p = \left\{ \prod_{i=1}^d \left[ \frac{z_i}{p}, \frac{z_i + 1}{p} \right) : (z_1, \ldots, z_d) \in \mathbb{Z}^d \right\}.
\]

The entropy dimension of \( \mu \) is defined as

\[
\dim_e(\mu) = \lim_{k \to \infty} \frac{1}{k \log 2} H(\mu, \mathcal{D}_{2^k}),
\]

provided that the limit exists. If the limits do not exist, the upper and lower entropy dimensions of \( \mu \) are defined as the corresponding \( \limsup \) and \( \liminf \).

Next, let \( n \geq 2 \) and consider the symbolic space \([n]^\mathbb{N}\), with the usual product topology. For every finite word \( u \in [n]^k \) for some \( k \in \mathbb{N} \), we associate its length, defined by \( |u| = k \), and a cylinder set defined by

\[
[u] := \{ \omega \in [n]^\mathbb{N} : (\omega_1, \ldots, \omega_k) = u \}.
\]

Though this coincides with the notation \([n]\), which notion is meant will be clear from context. Let \( \mathcal{I}_n^k \) denote the partition of \([n]^\mathbb{N}\) into cylinders of length \( k \). For a measure \( \mu \in P([n]^\mathbb{N}) \) we define the entropy dimension of \( \mu \) as

\[
\dim_e(\mu) = \lim_{k \to \infty} \frac{1}{k \log 2} H(\mu, \mathcal{I}_n^k),
\]

provided that the limit exists. If the limits does not exist, the upper and lower entropy dimensions of \( \mu \) are defined as the corresponding \( \limsup \) and \( \liminf \).

Finally, let \( \mu \in P(\mathbb{R}^d) \) or \( \mu \in P([n]^\mathbb{N}) \) for some \( n \geq 2 \). Then, if \( \mu \) is supported on a set \( A \),

\[
\dim(\mu) \leq \dim_e(\mu) \leq \overline{\dim}_e(\mu) \leq \dim_H A.
\] (8)

If \( \mu \) is exact dimensional, then

\[
\dim(\mu) = \dim_e(\mu).
\]

For proofs and more discussion of these concepts, see \([10, 24]\).
2.2. Star dimension, microsets and covariance of microsets

Let $X$ be a compact metric space, which in practice will be either $[-1,1]^2$ or a symbolic space of the form $[n]^N$. If $X = [-1,1]^2$, we shall use the Euclidean norm $\| \cdot \|$, and in the space $[n]^N$ we consider, for some $\rho \in (0,1)$, the metric $d_\rho$ on $[n]^N$, defined by

$$d_\rho(x,y) = \rho^{\min\{k: \ x_k \neq y_k\}}.$$  \hspace{1cm} (9)

Let $\text{cpt}(X)$ denote the set of non-empty closed subsets of $X$. For $A, B \in \text{cpt}(X)$ and $\epsilon > 0$, define

$$A_\epsilon = \{ x \in X : \exists a \in A, d(x,a) < \epsilon \}.$$  

The Hausdorff distance between $A$ and $B$ is defined by

$$d_H(A,B) = \inf \{ \epsilon > 0 : \ A \subseteq B_\epsilon, \ B \subseteq A_\epsilon \}.$$  

Then $(\text{cpt}(X), d_H(\cdot,\cdot))$ is a compact metric space (see, for example, [6, the appendix]).

Now, let us restrict to $X = [-1,1]^2$. Let $F \subseteq [0,1]^2$ be a compact set. A set $A \subseteq [-1,1]^2$ is called a miniset of $F$ if $A \subseteq (a \cdot F + t) \cap [-1,1]^2$ for some $a \geq 1, t \in \mathbb{R}$. A microset of $F$ is a limit of minisets of $F$ in the Hausdorff metric on subsets of $[-1,1]^2$. Let $\mathcal{G}_F$ denote the family of all microsets of $F$. Recall, from (4), that the star dimension of $F$ is defined as

$$\dim^* F = \sup \{ \dim_H A : \ A \in \mathcal{G}_F \}.$$  

It is known that this supremum is in fact a maximum, obtained by the dimension of a limit of non-degenerate minisets, that is, minisets of the form $\{ (a_k \cdot F + t_k) \cap [-1,1]^2 : a_k \to \infty \}$ such that $a_k \to \infty$.

For a proof, see [6, Lemma 2.4.4].

We shall also consider a special type of minisets and microsets. Let $m > 1$ and fix $x \in F$. An $m$-adic mini-set of $F$ about $x$ is a set of the form

$$[m^k(F-x)] \cap [-1,1]^2 \in \text{cpt}([-1,1]^2), \text{ where } k \in \mathbb{N}. \hspace{1cm} (10)$$

An $m$-adic microset of $F$ about $x$ is a limit of such sets as $k \to \infty$ (there is always a converging subsequence by the compactness of $\text{cpt}([-1,1]^2)$).

One of the many reasons it is interesting to study microsets is their nice behaviour with respect to affine (and more generally, smooth) embeddings. Namely, an affine embedding of one set into another set induces a corresponding affine embedding of their microsets:

**Proposition 2.1.** Let $m_1, m_2 > 1$ be integers, and let $g(x) = Ax + t$ be an invertible affine map of $\mathbb{R}^2$ such that $c \cdot A([-1,1]^2) \subseteq [-1,1]^2$ for all $c \in [1, m_1]$. Let $F, E \subseteq [-1,1]^2$ be compact, and suppose that $g(F) \subseteq E$. Let $x \in F$ and set $y = g(x) \in E$. Suppose that for some sequence $\{n_k\} \subseteq \mathbb{N}$

$$\lim_{k \to \infty} \left[ m_1^{n_k \log m_2 / \log m_1} (F-x) \right] \cap [-1,1]^2 = T, \quad \text{and} \quad \lim_{k \to \infty} [m_2^{n_k} (E-y)] \cap [-1,1]^2 = T'.$$

Then there exists some constant $c \in [1, m_1]$ such that $c \cdot A(T) \subseteq T'$.

We refer to this phenomenon as ‘covariance of microsets’. We omit the proof, since it is rather similar to the proof of [3, Proposition 4.3]. The assumption $c \cdot A([-1,1]^2) \subseteq [-1,1]^2$ for all $c \in [1, m_1]$ is needed for certain algebraic manipulations to work out. Without it, we can obtain a similar result of the form $A(T) \subseteq T' \cdot c'$, and $c'$ can be bounded in terms of the operator norm of the matrix $A$. 

2.3. CP distributions

2.3.1. Dynamical systems. In this paper, a measure-preserving system is a quadruple \((X, B, T, \mu)\), where \(X\) is a compact metric space, \(B\) is the Borel sigma algebra, \(T : X \to X\) is a measure preserving map, that is, \(T\) is Borel measurable and \(T \mu = \mu\). Since we always work with the Borel sigma-algebra, we shall usually just write \((X, T, \mu)\).

A class of examples of a dynamical systems are symbolic dynamical systems: We take \(X = [n]^N\) for some \(n\), and we take \(T = \sigma\) to be the shift map \(\sigma : [n]^N \to [n]^N\) defined by \(\sigma(\omega) = \xi\) where \(\xi(k) = \omega(k+1)\) for every \(k\). A special case is when \(\mu\) is a Bernoulli measure: that is, \(\mu = p^N\) where \(p\) is the probability vector \(p \in P([n])\). These systems are also called Bernoulli shifts.

A dynamical system is ergodic if and only if the only invariant sets are trivial. That is, if \(B \in B\) satisfies \(T^{-1}(B) = B\), then \(\mu(B) = 0\) or \(\mu(B) = 1\). A dynamical system is called weakly mixing if for any ergodic dynamical system \((Y, S, \nu)\), the product system \((X \times Y, T \times S, \mu \times \nu)\) is also ergodic. In particular, weakly mixing systems are ergodic. Moreover, if both \((X, T, \mu)\) and \((Y, S, \nu)\) are weakly mixing, then their product system is also weakly mixing. A class of examples of weakly mixing systems is given by Bernoulli shifts.

A useful tool that will appear frequently in this paper is the ergodic decomposition theorem:

**Theorem 2.2.** Let \((X, T, \mu)\) be a dynamical system. Then there is a map \(X \to P(X)\), denoted by \(\mu \mapsto \mu^x\), such that:

1. the map \(x \mapsto \mu^x\) is measurable with respect to the sub-sigma algebra \(\mathcal{E}\) of \(T\) invariant sets;
2. \(\mu = \int \mu^x d\mu(x)\);
3. for \(\mu\) almost every \(x\), \(\mu^x\) is \(T\) invariant, ergodic and supported on the atom of \(\mathcal{E}\) that contains \(x\). The measure \(\mu^x\) is called the ergodic component of \(x\).

Another useful notion is that of generic points in a dynamical system \((X, T, \mu)\). We say that a point \(x \in X\) is generic with respect to \(\mu\) if

\[
\frac{1}{N} \sum_{i=0}^{N-1} \delta_{T^i x} \to \mu, \quad \text{where } \delta_y \text{ is the dirac measure on } y \in X,
\]

in the weak-* topology. By the ergodic theorem, if \(\mu\) is ergodic, then \(\mu\) almost every \(x\) is generic for \(\mu\).

Finally, we discuss generators. Let \(\mathcal{A}\) be a finite partition of \(X\). Let \(\mathcal{A}_k = \bigvee_{i=0}^{k-1} T^{-i} \mathcal{A}\) denote the coarsest common refinement of \(\mathcal{A}, T^{-1} \mathcal{A}, \ldots, T^{-k+1} \mathcal{A}\). The sequence \(\mathcal{A}_k\) is called the filtration generated by \(\mathcal{A}\) with respect to \(T\). For every \(k \geq 1\) and \(x \in X\), let \(\mathcal{A}_k(x)\) denote the unique element of \(\mathcal{A}_k\) that contains \(x\).

Now, if the smallest sigma algebra that contains \(\mathcal{A}_k\) for all \(k\) is the Borel sigma algebra, we say that \(\mathcal{A}\) is a generator for \((X, T, \mu)\). By the Kolmogorov–Sinai theorem, if \(\mathcal{A}\) is a generator, then

\[
\lim_{k \to \infty} \frac{1}{k} H(\mu, \mathcal{A}_k) = \sup \lim_{k \to \infty} \frac{1}{k} H(\mu, D_k).
\]

where the supremum is taken over all countable partitions \(\mathcal{D}\) of \(X\). The common value described above is called the entropy of the dynamical system \((X, T, \mu)\) and is denoted by \(h(\mu, T)\).

2.3.2. CP distributions on symbolic spaces. The theory of CP distributions, that we discuss in this section, originated implicitly with Furstenberg in [15]. It was then reintroduced by Furstenberg in [16], and has since been used by many authors, notably by Hochman and
Shmerkin in [20]. In particular, CP distributions will play a crucial role in the proof of Theorem 1.7, as they do in Wu’s work [27]. In this section, we follow closely [27, Section 3].

As is standard in this context, if $X$ is a metric space, then elements of $P(X)$ are called measures, and elements of $P(P(X))$, measures on the space of measures, are called distributions.

Let $m_1, m_2 \geq 2$ and let $X = ([m_1] \times [m_2])^N$ (the theory extends to any finite alphabet, but this model will suffice for us). Fix $\rho \in (0, 1)$ and consider the metric $d_\rho$ on $X$ (recall (9)). Let

$$\Omega = \{(\mu, x) \in P(X) \times X : x \in \text{supp}(\mu)\}.$$  

We define the magnification operator $M : \Omega \to \Omega$ by

$$M(\mu, x) = (\mu^{[x_1]}, \sigma(x))$$

where $[x_1] = \{y \in X : y_1 = x_1\}$, and $\mu^{[x_1]} = \frac{\sigma(\mu|_{[x_1]})}{\mu([x_1])}$.

It is clear that $M$ is continuous, and that $M(\Omega) \subseteq \Omega$. For any distribution $Q \in P(\Omega)$, let $Q_1$ denote its marginal on the measure coordinate. We shall say that $Q$ is adapted if for every $f \in C(X)$,

$$\int f(\mu, x) dQ(\mu, x) = \int \left(\int f(\mu, x) d\mu(x)\right) dQ_1(\mu).$$

In particular, if $Q$ is adapted, then a property holds $Q$ almost surely if and only if it holds for $Q_1$ almost every $\mu$, and for $\mu$ almost every $x$.

**Definition 2.3.** A distribution $Q \in P(\Omega)$ is called a CP-distribution if it is $M$ invariant and adapted.

A CP-distribution $Q$ is called ergodic if the underlying dynamical system $(\Omega, M, Q)$ is ergodic. If it is not ergodic, its ergodic decomposition provides us with ergodic CP distributions:

**Proposition 2.4.** The ergodic components of a CP-distribution are, almost surely, themselves ergodic CP-distributions.

A proof is indicated by Furstenberg in [16] (after Proposition 5.1), and can be deduced from [18, Theorem 1.3].

We proceed to collect some useful properties of CP distributions.

**Proposition 2.5** [16]. Let $Q$ be an ergodic CP-distribution. Then $Q_1$ almost every measure $\mu$ is exact dimensional with dimension

$$\dim \mu = \frac{1}{\log \rho^{-1}} \int -\log \nu([x_1]) dQ(\nu, x).$$

For an ergodic CP distribution $Q$, $\dim Q$ denotes this (almost surely) constant value.

Next, let $x \in X$, and denote $[x^1] = \{y \in X : (y_1, \ldots, y_k) = (x_1, \ldots, x_k)\}$. We also denote

$$\mu^{[x^1]} = \frac{\sigma(\mu|_{[x^1]})}{\mu([x^1])}.$$  

It follows from the ergodic theorem that if $Q$ is an ergodic CP distribution, then $Q_1$ almost every $\mu$ generates $Q_1$ in the sense that for $\mu$ almost every $x$

$$\frac{1}{N} \sum_{i=1}^N \delta_{\mu[x^1]} \to Q_1$$

in the weak-* topology. Measures that satisfy this will be called generic for $Q_1$. 

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The following proposition was proved by Wu in [27]. Recall that $T_{m_i}^k$ denotes the $k$th-generation cylinder partition of $[m_i]^2$ for $i = 1, 2$. Thus, $T_{m_1}^k \times T_{m_2}^k$ is the $k$th-generation cylinder partition of the space $X$.

**Proposition 2.6** [27, Proposition 3.7]. Let $Q$ be an ergodic CP distribution with $\dim Q = q > 0$. For every $\epsilon > 0$, there exists $k_0(\epsilon) \in \mathbb{N}$ such that for each $\mu$ that is generic for $Q_1$ and for $\mu$ almost every $x$,

$$\liminf_{N \to \infty} \frac{1}{N} \left| \left\{ 1 \leq p \leq N : \max_{u \in ([m_1] \times [m_2])^{k_0(\cdot)}} \mu^{([x_p])}([u]) \leq \epsilon \right\} \right| > 1 - \epsilon,$$

and

$$\liminf_{N \to \infty} \frac{1}{N} \left| \left\{ 1 \leq p \leq N : H(\mu^{[x_p]}, T_{m_1}^p \times T_{m_2}^p) \geq p \cdot (q \cdot \log \rho^{-1} - \epsilon) \right\} \right| > 1 - \epsilon, \quad \text{for all } p \geq k_0(\epsilon).$$

Moreover, this is true for all pairs $(\mu, x)$ satisfying (11) (with $k_0$ depending on $(\mu, x)$).

Finally, in practice, we shall construct a CP distribution on a space of the form \(([m_1] \times [m_2])^{N_0}\), where $N_0 = \mathbb{N} \cup \{0\}$. It is not hard to see how the discussion in this section generalizes to that situation.

**2.3.3. CP distributions on Euclidean spaces.** The CP distributions discussed in the previous section have many applications for problems in geometric measure theory. To make the connection, we introduce the Euclidean version of CP distributions, which are closely related to symbolic CP distributions. In this section, we partially follow [14, Section 2.1]. We introduce the theory only in $\mathbb{R}^2$, where we shall use it.

Let $B \subset \mathbb{R}^2$ be a box, that is, a product of intervals (open, closed or half open). Let $T_B : \mathbb{R}^2 \to \mathbb{R}^2$ denote the orientation preserving affine map

$$T_B(x) = \frac{1}{\sqrt{|B|}} (x - \min B),$$

where $|B|$ is volume of $B$ and $\min B$ is the minimal element of $B$ with respect to the lexicographic order (so it is the lower left corner of the box). We define the normalized box $B^* = T_B(B)$, so that $|B^*| = 1$. If $\mu \in P(\mathbb{R}^2)$ and $B$ is a box with $\mu(B) > 0$, we write

$$\mu^B = \frac{1}{\mu(B)} T_B(\mu|_B) \in P(B^*).$$

Next, we define partition operators and filtrations. Let $\mathcal{E}$ be a collection of boxes in $\mathbb{R}^2$. A partition operator $\Delta$ on $\mathcal{E}$ associates to every $B \in \mathcal{E}$ a partition $\Delta B \subset \mathcal{E}$ of $B$ such that, for every homothety $S : \mathbb{R}^2 \to \mathbb{R}^2$, we have $S(\Delta B) = \Delta(SB)$. For every $B \in \mathcal{E}$, the partition operator $\Delta$ defines a filtration of $B$ by

$$\Delta^0(B) = \{B\}, \quad \Delta^{k+1}(B) = \{\Delta(A) : A \in \Delta^k(B)\}.$$

A partition operator $\Delta$ is called $\delta$-regular if for any $B \in \mathcal{E}$, there is a constant $c > 1$ such that for all $k \in \mathbb{N}$, any element $A \in \Delta^k(B)$ contains a ball of radius $\delta^k/c$ and is contained in a ball of radius $c\delta^k$.

For example, for every $m \geq 2$ we define the base $m$ partition operator on $\mathcal{E} = \{[u, v]^2 : u < v\}$ by defining

$$\Delta_m([0, 1]^2) = \left\{ \left[ \frac{k_1}{m} \frac{k_1 + 1}{m} \right] \times \left[ \frac{k_2}{m} \frac{k_2 + 1}{m} \right] : 0 \leq k_1, k_2 < m - 1, \quad k_1, k_2 \in \mathbb{Z} \right\},$$

and extending (by invariance) to all cubes. Notice that this operator is $\frac{1}{m}$ regular.
DEFINITION 2.7. Fix a collection of boxes $\mathcal{E}$ and define a state space
$$\Theta = \{(B, \mu) : \mu \in P(B^*), B \in \mathcal{E}\}.$$  
A $\delta$-regular CP-chain $Q$ with respect to a $\delta$-regular partition operator $\Delta$ is a stationary Markov process on the state space $\Theta$ with the Markov kernel
$$F(B, \mu) = \sum_{A \in \Delta(B^*)} \mu(A) \delta_{(A, \mu_A)}, \quad (B, \mu) \in \Theta.$$  
Thus, by definition, if $Q$ is the unique stationary distribution with respect to the chain, then $(\Theta^N, \sigma, \hat{Q})$ is a dynamical system, where $\hat{Q}$ is the extension of $Q \in P(\Theta)$ to a measure on $\Theta^N$, generated by running the Markov chain starting from $Q$. We abuse notation and refer to $\hat{Q}$ as $Q$. Thus, $Q$ is ergodic if this system is ergodic.

DEFINITION 2.8. Let $Q$ be a CP chain as above. We abuse notation and write $Q$ for the distribution of its measure component. Given $B \in \mathcal{E}$, the CP chain $Q$ is generated by $\mu \in P(B^*)$ if at $\mu$ almost every $x \in B^*$
$$\frac{1}{N} \sum_{k=0}^{N-1} \delta_{\mu^{\Delta_k(B)(x)}} \to Q$$  
in the weak star topology, and for any $q \in \mathbb{N}$,
$$\frac{1}{N} \sum_{k=0}^{N-1} \delta_{\mu^{\Delta^q_k(B)(x)}} \to R_q$$  
converges to some (possibly different) distribution $R_q$.

2.3.4. Continuous time scaling scenery. To prove Theorem 1.5, we shall require the notion of the continuous scaling scenery of a measure $\mu \in P([0, 1]^2)$ at a point $x \in \text{supp}(\mu)$. First, we define the scaling and translation maps $S_t, T_x : \mathbb{R}^2 \to \mathbb{R}^2$ by
$$S_t(y) = e^t \cdot y, \quad T_x(y) = y - x.$$  
We also define the restriction and normalization operator
$$\nu \in P(\mathbb{R}^2) \mapsto \nu^\square := \left(\frac{\nu}{\nu([-1, 1]^2)}\right)|_{-1, 1]^2}, \text{ assuming } 0 \in \text{supp}(\nu).$$

DEFINITION 2.9 [17, 18]. Let $\mu \in P([0, 1]^2)$ and let $x \in \text{supp}(\mu)$.

(1) We define the parameterized family of measures $\mu_{x,t} = (S_t \circ T_x(\mu))^{\square}$. This family is called the scenery of $\mu$ at $x$.

(2) For every $T > 0$, we define the scenery distribution
$$\langle \mu \rangle_{x,T} = \frac{1}{T} \int_0^T \delta_{\mu_{x,t}} dt \in P(P([-1, 1]^2)).$$

(3) If $\langle \mu \rangle_{x,T} \to P$ as $T \to \infty$, we say that $\mu$ generates $P$ at $x$.

One of the main advantages of zooming into a measure in this way is that it is done in a coordinate-free way. An example of how this is useful is the following lemma.

LEMMA 2.10 [18]. Let $\mu \in P([0, 1]^2)$ be a Borel probability measure such that for $\mu$ almost every $x \in [0, 1]^2$, $\langle \mu \rangle_{x,T} \to P$, for some $P \in P(P([0, 1]^2))$.  

(1) If  \( \nu \ll \mu \), then  \( \nu \) generates  \( P \) at almost every  \( x \).

(2) Let  \( g \in \text{diff}(\mathbb{R}^2) \). Then for  \( g \nu \) almost every  \( g(x) \), 
\[ (g \mu)_{g(x), T} \to (Dg(x))^{\Box} P, \]
where \((Dg(x))^{\Box}\) transforms measures by first pushing them forward via \( Dg(x) \) and then applying \( \Box \).

The following theorem, due to Gavish [17], and in greater generality to Hochman [18], shows that a measure that generates an ergodic CP distribution also generates a distribution in the sense of Definition 2.9. Moreover, using the centring operation (see [18]), we are able to relate the two distributions:

THEOREM 2.11 [17, 18]. Let  \( \mu \in P([0,1]^2) \) be a measure that generates an ergodic CP distribution  \( Q \) in the sense of Definition 2.8. Then  \( \mu \) generates a distribution  \( P \in P(P([0,1]^2)) \) at  \( \mu \) almost every  \( x \), in the sense of Definition 2.9. Moreover, there exists a distribution  \( R \) on triplets of the form  \( (\rho, \nu, x) \) as follows.

1. The first coordinate  \( \rho \) is distributed according to  \( P \).
2. The second coordinate  \( \nu \) is distributed according to  \( Q \), and  \( x \) is distributed according to  \( \nu \).
3. For  \( R \)-almost every such triplet, there exist  \( r(\rho, \nu, x) > 0 \) and  \( t > 0 \) such that
\[ g_{B(0, r)} \ll S_t \circ T_r(\nu). \]

2.4. Bedford–McMullen carpets

2.4.1. Iterated function systems. Let  \( \Phi = \{ \phi_i \}_{k=1}^l, l \in \mathbb{N}, l \geq 2 \) be a family of contractions  \( \phi_i : \mathbb{R}^d \to \mathbb{R}^d, d \geq 1 \). The family  \( \Phi \) is called an IFS, the term being coined by Hutchinson [21], who defined them and studied some of their fundamental properties. In particular, he proved that there exists a unique compact  \( \emptyset \neq F \subset \mathbb{R}^d \) such that  \( F = \bigcup_{i=1}^l \phi_i(F) \).  \( F \) is called the attractor of  \( \Phi \), and  \( \Phi \) is called a generating IFS for  \( F \). A set  \( F \subset \mathbb{R}^d \) will be called self-similar if there exists a generating IFS  \( \Phi \) for  \( F \) such that  \( \Phi \) consists only of similarity mappings. Similarly, if  \( \Phi \) consists only of affine maps, then we say that  \( F \) is a self-affine set.

The self-similar sets we shall encounter in this paper are deleted digit sets: for an integer  \( n \geq 2 \), Let  \( D \subseteq [n] \). Define an IFS  \( \Phi = \{ f_i \}_{i \in D} \), where
\[ \forall i \in D, \forall x \in \mathbb{R}, \quad f_i(x) = \frac{x + i}{n}. \]

The attractor of  \( \Phi \) is called a deleted digit set. These sets are quite nice. For example, if  \( K \) is a deleted digit set, then
\[ \dim_H K = \dim_B K = \dim^* K = \frac{\log |D|}{\log n}. \]

Finally, we discuss self-similar measures on deleted digit sets. Let  \( K \) be a deleted digit set as above. A measure  \( \mu \in P(K) \) is called a self-similar measure if there exists a fully supported Bernoulli measure \( \alpha \in P(D^n) \) such that  \( \pi_n \alpha = \mu \) (recall the map  \( \pi_n \) from (6)). These measures are known to be exact dimensional (in much greater generality, see [11]) of dimension  \( \dim K \).

2.4.2. Bedford–McMullen carpets. We now recall some basic concepts regarding Bedford–McMullen carpets. We follow the terminology of [3], which motivates our notation with regard to Theorem 1.7. Recall the definition of a Bedford–McMullen carpet \( F \) with defining exponents \( m, n \) and allowed digit set \( \Gamma \) from Section 1.1. Notice that if \( F \) is a Bedford–McMullen carpet, then both  \( P_1(F), P_2(F) \) are deleted digit sets. Also, note that \( F \) is a self-affine set generated...
by an IFS consisting of maps whose linear parts are diagonal matrices. Specifically, $F$ is the attractor of $\Phi = \{ \phi_{(i,j)} \}_{(i,j) \in \Gamma}$ where

$$
\phi_{(i,j)}(x, y) = \left( \frac{x + i}{m}, \frac{y + j}{n} \right) = \left( \frac{i}{m}, \frac{j}{n} \right) \cdot (x, y) + \left( \frac{i}{m}, \frac{j}{n} \right).
$$

(12)

Recall that when we have two carpets $F$ and $E$, we shall denote the set of allowed digits of $E$ by $\Lambda$. We also define, by a slight abuse of notation, the projection $\pi : \big[ [m] \times [n] \big]^\mathbb{N} \to \{ \rho \times \{ \eta \} \}^\mathbb{N}$ by identifying $\big( [m] \times [n] \big)^\mathbb{N}$ with $\big[ [m] \times [n] \big]^\mathbb{N}$. Then $\tilde{F} = \pi \subseteq \big( [m] \times [n] \big)^\mathbb{N}$ is a shift invariant subset satisfying $\pi_m \times \pi_n(\tilde{F}) = F$. As before, this may not be an injection, even though it is surjective, and $\tilde{F} \subseteq (\pi_m \times \pi_n)^{-1}(F)$, but the two sets might not be equal. Recall that for $y \in P_2(F)$, we defined $F_y$ as the horizontal slice $F_y = \{ x \in \mathbb{R} : (x, y) \in F \}$. Note that $F_y \times \{ y \} = F \cap (\mathbb{R} \times \{ y \})$. In the symbolic context, for an infinite sequence $\omega \in [n]^\mathbb{N}$, we define the symbolic slice corresponding to $\omega$ by

$$
\bar{F}_\omega = \{ \eta \in \big[ [m] \big]^\mathbb{N} : (\eta, \omega) \in \tilde{F} \} = \prod_{i=1}^\infty \Gamma_{\omega_i},
$$

where for $i \in [n]$, $\Gamma_i$ was defined in Section 1.1. Notice that this coincides with the definition of the infinite product sets from (7).

Note that

$$
\pi_m(\bar{F}_\omega) \subseteq F_{\pi_n(\omega)},
$$

but the two sets might not be equal if $\pi_n(\omega) \in [0, 1]$ admits another base-$n$ expansion in $\tilde{F}$. But we always have that

$$
F_y = \bigcup_{\omega \in \pi_m^{-1}(y)} \pi_m(\bar{F}_\omega).
$$

This is a union of at most two sets (again, if one pre-image of $y$ is not in $\tilde{F}$, the corresponding term in the union is empty). Given $\omega$, we have

$$
\pi_m(\bar{F}_\omega) = \left\{ \sum_{k=1}^\infty \frac{x_k}{m^k} : x_k \in \Gamma_{\omega_k} \right\}.
$$

We also have an elementary expression for the Hausdorff dimension of projections of symbolic slices: given $\omega \in [n]^\mathbb{N}$, by Billingsley’s lemma,

$$
\dim_H \pi_m(\bar{F}_\omega) = \liminf_{k \to \infty} \frac{\sum_{i=1}^k \log |\Gamma_{\omega_i}|}{k \log m} \leq \max_{i \in [n]} \frac{\log |\Gamma_i|}{\log m}.
$$

(13)

If, in addition, $\omega$ is generic with respect to some ergodic measure $\alpha \in P([n]^\mathbb{N})$, then by the ergodic theorem,

$$
\dim_H \pi_m(\bar{F}_\omega) = \sum_{i=0}^{n-1} \alpha([i]) \frac{\log |\Gamma_{\omega_i}|}{\log m} = \dim_B \pi_m(\bar{F}_\omega).
$$

2.4.3. Microsets of Bedford–McMullen carpets. In [4], Bandt and Käenmäki had studied the structure of microsets of a general class of self-affine carpets, where the point of magnification is drawn according to a self-affine measure. Now, Let $F$ be a Bedford–McMullen
carpet, and suppose that $F$ is not a self-similar set. In our recent work with Hochman [3], we were able to characterize $m$-adic microsets of $F$ about any point in $F$ (a similar characterization was given roughly at the same time in [22] by Käenmäki, Ojala and Rossi). As this characterization is key for our present work, we briefly recall it.

For $\omega \in [n]^\mathbb{N}$ and $s \in [0,1)$, we define an $(\omega,s)$-set to be a set of the form

$$\left(\begin{array}{c} 1 \\ 0 \\ n^s \end{array}\right) \cdot \left(\pi_m(F_\omega) \times P_2(F) + z\right),$$

which is contained in $[-2,2]^2$. For a fixed $(\omega,s)$, a set $Y \subseteq [-1,1]^2$ is an $(\omega,s)$-multiset if there are finitely many $(\omega,s)$-sets $Y_1, \ldots, Y_N$ and $z \in \pi_m(F_\omega) \times P_2(F)$, such that

$$\left(\begin{array}{c} 1 \\ 0 \\ n^s \end{array}\right) \cdot \left(\pi_m(F_\omega) \times P_2(F) - z\right) \cap [-1,1]^2 \subseteq Y \subseteq \bigcup_{i=1}^N Y_i \cap [-1,1]^2.$$

Finally, for $\omega \in [n]^\mathbb{N}$, let $S(\omega) \subseteq [n]^\mathbb{N} \times \mathbb{T}$ denote the set

$$S(\omega) = \{ (\xi,s) \in [n]^\mathbb{N} \times \mathbb{T} : (\sigma^{l_k} \omega, l_k \log_n m) \to (\xi,s) \text{ for some } l_k \to \infty \},$$

that is, $S(\omega)$ is the set of accumulation points of the orbit of $(\omega,0)$ under the transformation $(\xi,s) \mapsto (\sigma \xi, s + \log_n m)$.

For $\omega \in [n]^\mathbb{N}$, let $\bar{\omega} = \omega$ if $\pi_n(\omega)$ has a unique base-$n$ expansion, and otherwise let $\bar{\omega}$ be the other expansion. Recall the definition of $m$-adic microsets from (10).

**Theorem 2.12** [3, Theorem 4.2]. Fix $f = (x,y) \in F$ with $y \neq 0,1$ and let $\omega \in \pi_n^{-1}(y)$. Then for every $m$-adic microset $T$ about $f$, there exists $(\xi,s) \in S(\omega)$ such that $T$ is a non-empty union of a $(\xi,s)$-multiset and a $(\bar{\xi},s)$-multiset. Conversely, if $(\xi,s) \in S(\omega)$, then there is an $m$-adic microset set $T$ about $f$ which is a union of this type.

In the special case when $y = 0$ or $y = 1$, the same is true but omitting the $(\bar{\xi},s)$-multiset from the union.

In applications, we shall either not care about the identity of the limit point $(\xi,s)$, provided in the theorem, or else we will control it by starting with $y$ whose expansions are suitably engineered.

For general microsets, we have the following result.

**Theorem 2.13.** Let $M_k \subseteq [-1,1]^2$ be a sequence of mini-sets of $F$ of the form

$$\alpha_k(F - z_k) \cap [-1,1]^2, \quad z_k \in \mathbb{R}^2, \alpha_k \to \infty.$$

Then for every limit $M$ of $M_k$ in the Hausdorff metric, there is some $p \in \mathbb{N}$ such that

$$M \subseteq \bigcup_{i=1}^p \left(\begin{array}{c} a_i \\ 0 \\ b_i \end{array}\right) \cdot Y_i + t_i,$$

where for every $i$, $Y_i$ is an $(\omega_i,s_i)$ set for some $(\omega_i,s_i) \in [n]^\mathbb{N} \times \mathbb{T}$, $a_i, b_i > 0$ and $t_i \in \mathbb{R}^2$.

The theorem follows by inspecting the proof of [3, Theorem 4.2], which deals with the case when there is some $z \in \mathbb{R}^2$ such that all the translations $z_k$ from Theorem 2.13 are equal to $z$. Indeed, one notes that the results of Section 7.2, most notably a rescaled version of Corollary 7.5, generalize to this situation, with some minor modifications.

2.4.4. **CP distributions generated by self-affine measures on Bedford–McMullen carpets.** Let $F$ be a Bedford–McMullen carpet with exponents $(m,n)$, and allowed digits set
$\Gamma \subseteq [m] \times [n]$. Recall that $\mu \in P([0,1]^2)$ is a self-affine measure on $F$ if there exists some Bernoulli measure $\nu \in P(\Gamma^{\mathbb{N}})$ such that

$$\mu = \pi_m \times \pi_n(\nu).$$

Notice that $P_2\mu$ is a self-similar measure on the deleted digit set $P_2(F)$.

Given any measure $\mu \in P([0,1]^2)$, we denote by $\{\mu_\eta\}$ the family of conditional measures obtained by disintegrating $\mu$ according to the coordinate projection $P_2 : \mathbb{R}^2 \to \mathbb{R}$, $P_2(x,y) = y$. We also have a corresponding family of conditional measures $\{\nu_\omega\}$ associated with any measure $\nu \in P(\Gamma^{\mathbb{N}})$, obtained by disintegrating $\nu$ according to the coordinate projection $(\eta, \omega) \mapsto \omega$.

The following theorem, due to Fraser, Ferguson and Sahlsten, shows that self-affine measures on Bedford–McMullen carpets generate ergodic CP distributions, in the sense of Definition 2.7.

**Theorem 2.14** [14]. Let $\mu$ be a self-affine measure on a Bedford–McMullen carpet $F$ with exponents $(m,n)$. Then there is a family of boxes $E$ and a $\delta$-regular partition operator $\Delta$ such that $\mu$ generates an ergodic CP-distribution in the sense of Definition 2.8.

The measure component of the CP-distribution is the distribution of the measures of the form

$$\left(\begin{array}{cc} m^t & 0 \\ 0 & m^{-t}\end{array}\right)(\mu_y \times P_2\mu),$$

where $t \in [0,1)$ is distributed according to Lebesgue if $\frac{\log n}{\log m} \notin \mathbb{Q}$, and otherwise according to some periodic measure with respect to the translation of $\mathbb{T}$ by $\frac{\log n}{\log m}$, and $\mu_\eta$ is a conditional measure of $\mu$ with respect to the projection $P_2$, where $y$ is drawn according to $P_2\mu$.

Finally, let $\nu \in P(\Gamma^{\mathbb{N}})$ be a Bernoulli measure and let $\mu \in P(F)$ be the corresponding self-affine measure on $F$. We have

$$\mu_y = \pi_m \nu_{\pi_n^{-1}(y)} \text{ for } P_2\mu \text{ almost every } y. \quad (17)$$

Thus, letting $y = \pi_n(\omega)$ that satisfy (17), as long as $y$ does not belong to the countably many points of the form $y = \frac{j}{n^k}$ for some $j \in \mathbb{N}$ (which is of measure zero), we have

$$\text{supp}(\mu_y) = F_y = \pi_m(F_\omega) = \left\{ \sum_{k=1}^\infty \frac{x_k}{m^k} : x_k \in \Gamma_{\omega_k} \right\},$$

3. **Proof of the main results**

3.1. **Proof of Theorem 1.2**

Let $F$ be a Bedford–McMullen carpet with exponents $(m,n)$ such that $\frac{\log n}{\log m} \notin \mathbb{Q}$. Let $\ell$ be a non-principal line such that $F \cap \ell \neq \emptyset$. We aim to prove that

$$\dim^* F \cap \ell := \sup \{ \dim_H M : \text{M is a microset of } F \cap \ell \} \leq \max_{i \in [n]} \left\{ \frac{\log |\Gamma_i|}{\log m} + \dim P_2(F) - 1, 0 \right\}.$$

This will suffice for the proof of Theorem 1.2, since $\overline{\dim}_B F \cap \ell \leq \dim^* F \cap \ell$ (this inequality is true for any bounded set, see [6, Lemma 2.4.4]).

So, let $M$ be a microset of $F \cap \ell$. Then it is not hard to see that, by definition, there exists a microset $M'$ of $\ell$ such that $M \subseteq M'$. Similarly, there exists a microset $M''$ of $F$ such that $M \subseteq M''$. It follows that $M \subseteq M' \cap M''$. 
Now, on the one hand, every microset of \( \ell \) is contained within a line \( \ell' \) that has the same slope as \( \ell \) (so it is still a non-principal line). On the other hand, by Theorem 2.13, \( M'' \) is contained within a finite union of sets of the form
\[
\bigcup_{i=1}^{p} \left( \left( a_i 0 \right) \cdot \pi_m(\tilde{F}_{\omega_i}) \times P_2(F) + t_i \right),
\]
where \( p < \infty \), \( t_i \in \mathbb{R}^2 \), \( (a_i, b_i) \in \mathbb{R}^2 \setminus \{(0,0)\} \), \( \omega_i \in [n]^\mathbb{N} \).

Combining these observations, we see that
\[
M \subseteq M' \cap M'' \subseteq \bigcup_{i=1}^{p} \left( \left( a_i 0 \right) \cdot \pi_m(\tilde{F}_{\omega_i}) \times P_2(F) + t_i \right) \cap \ell'
\]
and therefore
\[
\dim_H M \leq \max \dim_H \left( \left( a_i 0 \right) \cdot \pi_m(\tilde{F}_{\omega_i}) \times P_2(F) + t_i \right) \cap \ell'
\]
\[
= \max \dim_H \left( \pi_m(\tilde{F}_{\omega_i}) \times P_2(F) \right) \cap \ell''_i,
\]
where \( \ell''_i \) is the corresponding non-principal affine line. An application of Theorem 1.7 shows that
\[
\dim_H M \leq \max_{i \in [n]} \left\{ \frac{\log|\Gamma_i|}{\log m_1} + \dim P_2(F) - 1, 0 \right\},
\]
as required.

3.2. Proof of Theorem 1.4

Let \( F \) and \( E \) be two incommensurable Bedford–McMullen carpets, with exponents \((m_1, n_1), (m_2, n_2)\) respectively. Recall that we denote by \( \Gamma \subseteq [m_1] \times [n_1] \) and \( \Lambda \subseteq [m_2] \times [n_2] \) the allowed digits sets that define \( F \) and \( E \), respectively. Let \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) be an invertible affine map, such that its linear part is a diagonal matrix. We prove that
\[
\dim^* g(F) \cap E \leq \max_{(i,j) \in [m_1] \times [n_2]} \left\{ \frac{\log|\Gamma_i|}{\log m_1} + \frac{\log|\Lambda_j|}{\log m_2} - 1, 0 \right\} + \max(\dim P_2(F) + \dim P_2(E) - 1, 0).
\]

To this end, let \( M \) be a microset of \( g(F) \cap E \). Then, on the one hand, \( M \) is contained within a microset of \( g(F) \). Since \( F \) and \( g(F) \) are affine images of each other, by an analogue of Proposition 2.1, every microset of \( g(F) \) is contained within an image of a microset of \( F \) under an affine map with the same linear part as \( g^{-1} \). Since the linear part of \( g^{-1} \) is diagonal, and by Theorem 2.13, we know that
\[
M \subseteq \bigcup_{i=1}^{p} \left( \left( a_i 0 \right) \cdot \pi_{m_1}(\tilde{F}_{\omega_i}) \times P_2(F) + t_i \right),
\]
where \( p < \infty \), \( t_i \in \mathbb{R}^2 \), \( (a_i, b_i) \in \mathbb{R}^2 \setminus \{(0,0)\} \), \( \omega_i \in [n_1]^\mathbb{N} \). On the other hand, \( M \) is also contained within a microset of \( E \), so
\[
M \subseteq \bigcup_{j=1}^{q} \left( \left( c_j 0 \right) \cdot \pi_{m_2}(\tilde{E}_{\eta_j}) \times P_2(E) + t'_j \right),
\]
where \( q < \infty \), \( t'_j \in \mathbb{R}^2 \), \( (c_j, d_j) \in \mathbb{R}^2 \setminus \{(0,0)\} \), \( \eta_j \in [n_2]^\mathbb{N} \).
It follows that $M$ is contained within a finite union of sets of the form
\[ M_{i,j} = \left( \left( a_i \begin{pmatrix} 0 \\ b_i \end{pmatrix} \cdot \pi_{m_1}(\tilde{F}_{\omega_i}) \times P_2(F) + t_i \right) \cap \left( \left( c_j \begin{pmatrix} 0 \\ d_j \end{pmatrix} \cdot \pi_{m_2}(\tilde{E}_{\eta_j}) \times P_2(E) + t'_j \right) \right) \]
for some $1 \leq i \leq p, 1 \leq j \leq q$. Rewriting the equation above, we have $M_{i,j} = P_1(M_{i,j}) \times P_2(M_{i,j})$, where
\[ P_1(M_{i,j}) = (a_i \cdot \pi_{m_1}(\tilde{F}_{\omega_i}) + P_1(t_i)) \cap (c_j \cdot \pi_{m_2}(\tilde{E}_{\eta_j}) + P_1(t'_j)) \]
and
\[ P_2(M_{i,j}) = (b_i \cdot P_2(F) + P_2(t_i)) \cap (d_j \cdot P_2(E) + P_2(t'_j)). \]

Finally, $P_1(M_{i,j})$ corresponds to a non-principal slice in the product set $\pi_{m_1}(\tilde{F}_{\omega_i}) \times \pi_{m_2}(\tilde{E}_{\eta_j})$. By Theorem 1.7, we see that
\[ \dim_B(P_1(M_{i,j})) \leq \max_{(i,j) \in [n_1] \times [n_2]} \left\{ \frac{\log |\Gamma_i|}{\log m_1} + \frac{\log |\Lambda_j|}{\log m_2} - 1, 0 \right\}. \]

In addition, $P_2(M_{i,j})$ corresponds to a non-principal slice in the product set $P_2(F) \times P_2(E)$, so by Theorem 1.7 (or by the main results of [27] and [26])
\[ \dim_B(P_2(M_{i,j})) \leq \max \{ \dim P_2(F) + \dim P_2(E) - 1, 0 \}. \]

Since $M \subseteq \bigcup M_{i,j}$ (a finite union), and $M_{i,j} = P_1(M_{i,j}) \times P_2(M_{i,j})$, combining the last two displayed equations completes the proof by the finite stability of $\dim_B$.

The second part of Theorem 1.4, where the linear part of $g$ is an anti-diagonal matrix, follows by a similar argument.

3.3. Proof of Theorem 1.5

Let $\mu \in P(F)$ and $\nu \in P(E)$ be self-affine measures that satisfy the conditions of Theorem 1.5 part (1). Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine map such that its linear part is given by a diagonal matrix. Suppose towards a contradiction that the conclusion of Theorem 1.5 part (1) is false. Then there is a mutually non-null set $A$ such that $(g\mu)|_A \sim \nu|_A$.

Now, by Theorem 2.14, $\mu$ generates an ergodic CP distribution $Q_1$ in the sense of Definition 2.8. Therefore, by Theorem 2.11, $\mu$ generates a distribution $W_1 \in P(P([0,1]^2))$ at $\mu$ almost every point $x$, in the sense of Definition 2.9. By Lemma 2.10 part (2), $g\mu$ generates the push-forward of this distribution $D(g)(x)W_1$, at $g\mu$ almost every point. By Lemma 2.10 part (1), $(g\mu)|_A \ll \mu$, so $(g\mu)|_A$ generates the same distribution at almost every point in $A$.

By a completely analogous argument, $\nu$ generates an ergodic CP distribution $Q_2$. Therefore, $\nu$ generates a distribution $W_2 \in P(P([0,1]^2))$ at $\nu$ almost every point. Since $\nu|_A \ll \nu$, by Lemma 2.10, $\nu|_A$ generates the same distribution.

Thus, the assumption that $g\mu|_A \sim \nu|_A$ implies, via Lemma 2.10 part (2), that there exists some $x \in A$ such that $LW_1 = W_2$, where $L := D(g)(x) \in GL(\mathbb{R}^2)$ is a diagonal matrix by assumption. Let us denote this common distribution by $P$.

Therefore, by Theorem 2.11, for $P$ almost every $\rho$, there is a $r_1 > 0$ such that $\rho|_{B(0,r_1)} \ll S_t \circ T_{\alpha}(\Lambda\alpha)$, where $\alpha$ is a $Q_1$ typical measure, $t > 0$ and $x \in \text{supp}(\alpha)$. Similarly, for $P$ almost every $\rho$, there is a $r_2 > 0$ such that $\rho|_{B(0,r_2)} \ll S_u \circ T_{\beta}(\beta)$, where $\beta$ is a $Q_2$ typical measure, $u > 0$ and $y \in \text{supp}(\beta)$. Thus, for $P$ almost every $\rho$, there is a small ball such that $\rho|_{B(0,r)}$ is absolutely continuous with respect to both $S_t \circ T_{\alpha}(\Lambda\alpha)$ and $S_u \circ T_{\beta}(\beta)$. Let us select such a measure $\rho$ and corresponding measures $\alpha$ and $\beta$.

Moreover, we may assume that the selected $\alpha$ and $\beta$ satisfy $\dim \alpha = \dim \mu$ and $\dim \beta = \dim \nu$. This is because by Theorem 2.14 $\dim \alpha = \dim \mu$ for $Q_1$ almost every $\alpha$, and $\dim \beta = \dim \nu$ for
Q₂ almost every β. Combining this with Theorem 2.11 shows that we can work with measures satisfying this property in the previous paragraph.

Let B denote the support of ρ|_{B(0,r)}. Then both \( S_t \circ T_x(\alpha)(B) > 0 \) and \( S_u \circ T_y(\beta)(B) > 0 \). It follows that

\[
\dim_H B \geq \max \{ \dim S_t \circ T_x(\alpha), S_u \circ T_y(\beta) \} \geq \max \{ \dim \alpha, \dim \beta \} = \max \{ \dim \mu, \dim \nu \}.
\]

(18)

On the other hand,

\[
B \subseteq e^{-t} \cdot (L(\supp(\alpha)) - x), \quad \text{and} \quad B \subseteq e^{-u}(\supp(\beta) - y).
\]

Therefore,

\[
\dim_H B \leq \dim_H (e^{-t} \cdot (L(\supp(\alpha)) - x)) \cap (e^{-u}(\supp(\beta) - y))
\]

\[
= \dim_H (e^{-t+u} \cdot (L(\supp(\alpha)) - x)) \cap (\supp(\beta) - y)
\]

\[
= \dim_H (e^{-t+u} \cdot (L(\supp(\alpha)) - t)) \cap (\supp(\beta)),
\]

where \( t = x + e^{-u}y \). Recalling Theorem 2.14, we can deduce that \( \dim_H B \) is bounded above by

\[
\overline{\dim}_B \left( e^{-t+u} \left( \begin{array}{cc} L_1 & 0 \\ 0 & L_2 \end{array} \right) \left( \begin{array}{cc} m_1^{s/2} & 0 \\ 0 & m_1^{-s/2} \end{array} \right) \cdot F_y \times P_2(F) + t \right) \cap \left( \begin{array}{cc} m_2^{-r/2} & 0 \\ 0 & m_2^{r/2} \end{array} \right) \cdot E_z \times P_2(E)
\]

\[
= \overline{\dim}_B (g_1(F_y) \times g_2(P_2(F))) \cap (E_z \times P_2(E)) = \overline{\dim}_B (g_1(F_y) \cap E_z) \times (g_2 \circ P_2(F) \cap P_2(E))
\]

for suitable non-degenerate affine maps \( g_1, g_2 : \mathbb{R} \to \mathbb{R} \), where \( y \in P_2(F) \) and \( z \in P_2(E) \). By well-known properties of the upper box dimension, we find that

\[
\dim_H B \leq \overline{\dim}_B (g_1(F_y) \cap E_z) + \overline{\dim}_B ((g_2 \circ P_2(F)) \cap P_2(E)).
\]

(19)

We obtain our desired contradiction by applying Theorem 1.7 to bound the right-hand side (RHS) of equation (19) from above, and using (18) to bound the left-hand side (LHS) of (19) from below.

The proof of Theorem 1.5 part (2) is analogous.

3.4. Proof of Theorem 1.6

Let \( F \) and \( E \) be two incommensurable Bedford–McMullen carpets. Recall that we are assuming that there exists some \( 0 \leq i \leq n_1 - 1 \) such that \( |\Gamma_i| \geq 2 \) and that \( \dim^* E < 2 \). Suppose, towards a contradiction, that there exists an invertible affine map \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( g(F) \subseteq E \). We denote by \( A \in GL(\mathbb{R}^2) \) the linear part of \( g \).

Let \( \omega = (i, i, \ldots) \in [n_1]^\mathbb{N} \). Then \( \pi_m(\tilde{F}_\omega) \) is equal to a self-similar set \( K \subseteq [0,1] \), where \( K \) is generated by the self-similar IFS \( \{ f_j \}_{j \in \Gamma_i} \), defined by

\[
\forall j \in \Gamma_i, \forall x \in \mathbb{R}, \quad f_j(x) = \frac{x + j}{m_1}.
\]

Thus, \( K \) is a deleted digit set, so we have

\[
\dim K = \frac{\log |\Gamma_i|}{\log m_1} > 0.
\]
Now, let \( y = \pi_{n_1}(\omega) \), fix \((x, y) \in (\pi_{m_1}(F_\omega), y) \subset F \) and let \( g(x, y) = (w, z) \in E \). Consider the following two sequences of \( m_1 \)-adic minisets of \( E \) and \( F \), respectively,

\[
M_k = (m_1^k(E - (w, z))) \cap (-1, 1)^2, \quad R_k = (m_1^k(F - (x, y))) \cap (-1, 1)^2.
\]

We can find a subsequence \( \{n_k\} \subseteq \mathbb{N} \) such that both \( M_{n_k} \) and \( R_{n_k} \) converge. By applying Proposition 2.1 and Theorem 2.12 along this subsequence, we find that, since \( \omega \) is a fixed point for the shift on \([n_1]^N\)

\[
A((K \times P_2(F) - t) \cap (-1, 1)^2) \subseteq \bigcup_{i=1}^p \left( a_i \cdot \pi_{m_2}(\tilde{E}_{\eta_i}) \times P_2(E) + t_i \right), \quad \text{(20)}
\]

where \( p < \infty, \eta_i \in [n_2]^N, t_i \in \mathbb{R}^2, (a_i, b_i) \in \mathbb{R}^2 \setminus \{(0, 0)\} \) and \( t \in \pi_{m_1}(\tilde{F}_\omega) \times P_2(F) \) (we absorb the \( c \) from Proposition 2.1 that should appear on the LHS into the matrices on the RHS).

By (5), the assumption \( \dim^* E < 2 \) implies that either \( \dim P_2(E) < 1 \) or \( \max_{j \in [n_2]} |\Lambda_j| < m_2 \). If \( \max_{j \in [n_2]} |\Lambda_j| < m_2 \), then by projecting both sides of (20) by \( P_1 \) and using the fact that \( A \) is invertible, there exist \( a_1, a_2 \in \mathbb{R} \) not both zero such that

\[
\emptyset \neq a_1 \cdot (K - t_1) \cap (-1, 1) + a_2 \cdot (P_2(F) - t_2) \cap (-1, 1) \subseteq \bigcup_{i=1}^p \left( a_i \cdot \pi_{m_2}(\tilde{E}_{\eta_i}) + t_i \right).
\]

Since both \( K \) and \( P_2(F) \) are self-similar sets, we see that either

\[
\phi(K) \subseteq \bigcup_{i=1}^p \left( a_i \cdot \pi_{m_2}(\tilde{E}_{\eta_i}) + t_i \right), \quad \text{or} \quad \psi(P_2(F)) \subseteq \bigcup_{i=1}^p \left( a_i \cdot \pi_{m_2}(\tilde{E}_{\eta_i}) + t_i \right),
\]

where either \( \phi \) or \( \psi \) are invertible similarity maps \( \mathbb{R} \to \mathbb{R} \). However, since \( \frac{\log m_1}{\log m_2}, \frac{\log n_1}{\log m_2} \notin \mathbb{Q} \), both options lead to a contradiction, since, for example, if the first option holds, then by Theorem 1.7 part (1),

\[
0 < \overline{\dim_B} K = \overline{\dim_B}(\phi(K))
\]

\[
= \overline{\dim_B}\left( \phi(K) \cap \left( \bigcup_{i=1}^p \left( a_i \cdot \pi_{m_2}(\tilde{E}_{\eta_i}) + t_i \right) \right) \right)
\]

\[
\leq \dim_H K + \max_{j \in [n_2]} \frac{\log |\Lambda_j|}{\log m_2} - 1
\]

\[
< \dim_H K
\]

since \( \max_{j \in [n_2]} \frac{\log |\Lambda_j|}{\log m_2} < 1 \) for all \( j \in [n_2] \).

If \( \dim P_2(E) < 1 \), then we follow a similar argument, projecting both sides of (20) by \( P_2 \) this time, and using the fact that both \( \frac{\log m_1}{\log m_2}, \frac{\log n_1}{\log m_2} \notin \mathbb{Q} \) and that \( A \) is invertible.

4. A CP chain on the space of slices of a family of product sets

4.1. Some notations and preliminaries

We now begin the proof of Theorem 1.7. Recall the notation introduced before Theorem 1.7. In particular, we always assume \( m_1 > m_2 \) and \( \frac{\log m_2}{\log m_1} = \theta \notin \mathbb{Q} \). Let \( R_\theta : \mathbb{T} \to \mathbb{T} \) denote the irrational rotation \( R_\theta(t) = t + \theta \mod 1 \). Let \( X = ([m_1] \times [m_2])^{100}, \) where \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \).

For \( t \in \mathbb{T} \), define a map \( \sigma_t : [n_1]^N \to [n_1]^N \) by \( \sigma_t(\omega) = \begin{cases} \sigma(\omega) & \text{if } t \in [1 - \theta, 1) \\ \omega & \text{if } t \in [0, 1 - \theta) \end{cases} \) and let \( X_t = (\sigma_t) \cdot X \).

We now begin the proof of Theorem 1.7. Recall the notation introduced before Theorem 1.7. In particular, we always assume \( m_1 > m_2 \) and \( \log m_2 \). Let \( R_\theta : \mathbb{T} \to \mathbb{T} \) denote the irrational rotation \( R_\theta(t) = t + \theta \mod 1 \). Let \( X = ([m_1] \times [m_2])^{100}, \) where \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \).

For \( t \in \mathbb{T} \), define a map \( \sigma_t : [n_1]^N \to [n_1]^N \) by \( \sigma_t(\omega) = \begin{cases} \sigma(\omega) & \text{if } t \in [1 - \theta, 1) \\ \omega & \text{if } t \in [0, 1 - \theta) \end{cases} \) and let \( X_t = (\sigma_t) \cdot X \).
Next, let \( t \in \mathbb{T} \). We define a map \( T \to [2]^{\mathbb{N}_0} \) by \( t \mapsto v_t \), where
\[
v_t(n) = 1 \text{ if and only if } R^\theta_n(t) \in [1 - \theta, 1).
\]
Notice that this is an injection. On the space \([2]^{\mathbb{N}_0}\), we use the metric \( d(x, y) = m_2^{-\min\{k : x_k \neq y_k\}} \).

Thus, the map \( t \mapsto v_t \) induces a metric \( d_\theta \) on the image of \( \mathbb{T} \) in \([2]^{\mathbb{N}_0}\) by taking
\[
d_\theta(v_s, v_t) = m_2^{-\min\{k \geq 0 : v_t(k) \neq v_s(k)\}}.
\]

**Notation 4.1.** We denote by \( S \) the closure with respect to the metric \( d_\theta \) on \([2]^{\mathbb{N}_0}\), of the image of \( \mathbb{T} \) under the map \( t \mapsto v_t \).

Notice that not every \( \tau \in S \) has some \( t \in \mathbb{T} \) such that \( \tau = v_t \). Indeed, this follows by noting that, for a sequence \( \{t_k\} \subset \mathbb{T} \), if \( v_{t_k} \) converges to \( v_t \) in \( d_\theta \) for some \( t \in \mathbb{T} \), then \( t_k \) converges to \( t \) in the usual metric on \( \mathbb{T} \). Thus, for the sequence \( t_k = 1 - \theta - \frac{1}{k} \), \( v_{t_k} \) has no \( d_\theta \) limits coming from elements of \( \mathbb{T} \). For if it had one, then it would have to be \( v_{1 - \theta} \). But \( v_{1 - \theta}(1) = 1 \neq 0 = v_{1 - \theta}(1) \) for all \( k \) large enough, a contradiction.

We define a partition \( \mathcal{C} \) of \( \mathbb{T} \) in the following manner: \( \mathcal{C} = \{(1 - \theta, 1), [0, 1 - \theta)\} \). We also denote, for every \( k \in \mathbb{N} \), the partition \( \mathcal{C}_k = \bigvee_{i=0}^{\infty} R^{-i}_\theta \mathcal{C} \). Notice that the elements of \( \mathcal{C}_k \) are half closed half open intervals.

Next, for \( \tau \in S \), we define
\[
r_k(\tau) = |0 \leq i \leq k - 1 : \tau_i = 1|, \tag{21}
\]
and for \( t \in \mathbb{T} \), we abuse notation and write \( r_k(t) := r_k(v_t) \).

**Claim 4.2.** (1) There exists some integer \( C > 1 \) such that for every \( \tau \in S \) and every \( k \in \mathbb{N} \),
\[
|r_k(\tau) - k \cdot \theta| \leq C
\]
(2) Let \( t \in \mathbb{T} \). Assume that for every \( k \), \( t \) is not an endpoint of an interval in \( \mathcal{C}_k \). Suppose that a sequence \( t_k \) converges to \( t \) in the usual metric on \( \mathbb{T} \). Then \( v_{t_k} \) converges to \( v_t \) in \( S \).

For a proof, see Section 7.

4.2. **Symbolic setting**

Let \( \tau \in S \), and set \( Z(\tau) = \{n \geq 0 : \tau_n = 1\} \). Write the elements of \( Z(\tau) \) in increasing order \( x_1(\tau) < x_2(\tau) < \ldots \).

**Definition 4.3.** For every \( \tau \in S \) and \((\omega, \eta) \in [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}} \), we define the set \( X_{\tau, \omega, \eta} \subseteq X \) as
\[
\prod_{i=0}^{\infty} (T_{\omega, \tau, i} \times \Lambda_{\eta, i+1}),
\]
where \( T_{\omega, \tau, i} = \Gamma_{i+1} \) if \( i \in Z(\tau) \) and \( T_{\omega, \tau, i} = \{1\} \) otherwise. We also define

- a metric on \( X_{\tau, \omega, \eta} \) by taking \( d(x, y) = m_2^{-\min\{k \geq 0 : x_k \neq y_k\}} \);
- a map \( \pi_{\tau, \omega, \eta} : X_{\tau, \omega, \eta} \to \pi_{m_1}(\hat{F}_\omega) \times \pi_{m_2}(\hat{E}_\eta) \) by taking
\[
\pi_{\tau, \omega, \eta}(a_n, b_n) = \left( \sum_{k=0}^{\infty} \frac{a_{x_k}(\tau)}{m_1^{k+1}} \sum_{k=0}^{\infty} \frac{b_k}{m_2^{k+1}} \right).
\]

Note that this is a surjective map.
Lemma 4.4. (1) Suppose that \( \tau_k, \tau \in S \) and \( \omega_k, \omega \in [n_1]^N \) and \( \eta_k, \eta \in [n_2]^N \) are such that \( \tau_k \to \tau \) in \( d_\theta \), and \( (\omega_k, \eta_k) \to (\omega, \eta) \) in \( [n_1]^N \times [n_2]^N \). Then \( X_{\tau_k, \omega_k, \eta_k} \to X_{\tau, \omega, \eta} \) in the Hausdorff metric, and \( \pi_{\tau_k, \omega_k, \eta_k} \to \pi_{\tau, \omega, \eta} \) uniformly.

(2) \( \exists C_1 > 0 \) such that the maps \( \pi_{\tau, \omega, \eta} \) are uniformly \( C_1 \)-Lipschitz.

(3) \( \exists C_2 > 0 \) such that \( \forall \tau \in S \) and \( (\omega, \eta) \in [n_1]^N \times [n_2]^N \) and all \( A \subseteq X_{\tau, \omega, \eta} \), we have
\[
N(A, m_{2^{-k}}^+) \leq C_2 \cdot N(\pi_{\tau, \omega, \eta}(A), m_{2^{-k}}^+)
\]
for all \( k \in \mathbb{N} \), where \( N(\cdot, m_{2^{-k}}^+) \) on the LHS denotes the number of \( k \)-level cylinders \( A \) intersects, and on the RHS the number of \( m_{2^{-k}}^+ \)-adic squares needed to cover a set in \([0, 1]^2\).

(4) For all \( \tau \in S \) and \( (\omega, \eta) \in [n_1]^N \times [n_2]^N \), and all \( A \subseteq X_{\tau, \omega, \eta} \), \( \dim_H(A) = \dim_H(\pi_{\tau, \omega, \eta}(A)) \).

Proof. The proof follows along the same lines of [27, Lemma 4.1]. The key idea here is that for any cylinder \([u]\) of length \( k \) in \( X_{\tau, \omega, \eta} \), the set \( \pi_{\tau, \omega, \eta}(\cdot) \) contained in at most four boxes of side length \( m_{2^{-k}}^+ \times m_{2^{-k}}^+ \), since by Claim 4.2 the length on the \( x \)-axis is \( m_{1^{-k}} - C \leq m_{1^{-k}}^{1-\theta} \leq m_{1^{-k}}^{1-\theta+C} \).

Next, define for every \( t \in \mathbb{T} \), a map \( \Phi_t : [0, 1]^2 \to [0, 1]^2 \), by
\[
\Phi_t(z) = \begin{cases} (T_{m_2}(z_1), T_{m_2}(z_2)) & \text{if } t \in [1-\theta, 1) \\ (z_1, T_{m_1}(z_2)) & \text{if } t \in [0, 1-\theta) \end{cases}
\]

Lemma 4.5. If \( \ell \) is a line with slope \( m_1^{0.5} \), \( t \in \mathbb{T} \), through \([0, 1]^2\), then \( \Phi_t(\ell) \) consists of a finite number of lines of slope \( m_1^{0.5} \) (here we think of \( \mathbb{T} \) as \([0, 1] \) with addition (mod 1)).

Next, let \( \ell_{u,z} \) denote the line through \( z \) with slope \( u \). Define
\[
\mathcal{F} \subseteq \text{cpc}(X) \times X \times \mathbb{T} \times [n_1]^N \times [n_2]^N
\]
by
\[
\mathcal{F} = \left\{ (A, x, t, \tau, \omega, \eta) : x \in A \subseteq X_{\tau, \omega, \eta}, \quad \pi_{\tau, \omega, \eta}(A) \subseteq [\pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta)] \cap \ell_{m_1^{0.5}, \pi_{\tau, \omega, \eta}(x)} \right\}
\]
(22)

Note that for every line \( \ell_{m_1^{0.5}, z} \) for some \( t \in \mathbb{T} \) and \( z \in [\pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta)] \cap \ell_{m_1^{0.5}, z} \), for all \( x \in \pi_{\ell_{u,z}}(z) \), we have
\[
(\pi_{\ell_{u,z}}^{-1}(\pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta)) \cap \ell_{m_1^{0.5}, \pi_{\ell_{u,z}}(x)}), x, t, v_t, \omega, \eta) \in \mathcal{F}
\]

Lemma 4.6. (1) If \( (A, x, t, v_t, \omega, \eta) \in \mathcal{F} \), then
\[
(\sigma(A \cap [x_1]), \sigma(x), R_0(t), \sigma(v_t), \sigma(\omega), \sigma(\eta)) \in \mathcal{F}.
\]

(2) If \( (A_k, x_k, t_k, \tau_k, \omega_k, \eta_k) \to (A, x, t, \tau, \omega, \eta) \) and \( (A_k, x_k, t_k, \tau_k, \omega_k, \eta_k) \in \mathcal{F} \) for all \( k \), then \( (A, x, t, \tau, \omega, \eta) \in \mathcal{F} \).

Proof. Let \( x \in X_{\ell_{u,z}} \). Then \( \sigma(x) \in X_{\ell_{R_0(t)}, \sigma(\omega), \sigma(\eta)} = X_{\ell_{\sigma(v_t)}, \sigma(\omega), \sigma(\eta)} \). In particular,
\[
\Phi_t(\pi_{\ell_{u,z}}(x)) = \pi_{\sigma(v_t), \sigma(\omega), \sigma(\eta)}(\sigma(x)).
\]
It follows that
\[
\Phi_t(\pi_{\ell_{u,z}}(A \cap [x_1])) = \pi_{\sigma(v_t), \sigma(\omega), \sigma(\eta)}(\sigma(A \cap [x_1])).
\]
Also, since $F_\omega$ and $E_\eta$ are product sets,
\[
\Phi_1(\pi_{m_1}(F_\omega) \times \pi_{m_2}(E_\eta)) = \pi_{m_1}(F_{\sigma_1(\omega)}) \times \pi_{m_2}(E_{\sigma_2(\eta)}).
\]
This implies part (1). Part (2) follows from part (1) of Lemma 4.4. \qed

4.3. Construction of a CP-distribution

Consider the space $Y = P(X) \times X \times T \times S \times [n_1]^N \times [n_2]^N$. Define $\hat{M} : Y \to Y$ by
\[
\hat{M}(\mu, x, t, \tau, \omega, \eta) = (\mu^{[x_0]}, \sigma(x), R_0(t), \sigma(\tau), \sigma(\omega), \sigma(\eta))
\]
(recall the definition of $\mu^{[x_0]}$ from Section 2.3.2). This rather cumbersome space comes from mostly natural geometric considerations: the first two coordinates of $Y$ are the usual (symbolic) setting for a CP distribution, as in Definition 2.3. These will describe measures on slices of $\pi_{m_1}(F_\omega) \times \pi_{m_2}(E_\eta)$. The following two coordinates, $T \times S$, capture the slope of the slice, where the $S$ coordinate (the only ‘unnatural’ coordinate) is needed for compactness reasons. The final two coordinates capture the $(\omega, \eta)$ that corresponds to the set $\pi_{m_1}(F_\omega) \times \pi_{m_2}(E_\eta)$ in the family that is being sliced.

Note that $M$ is not continuous; the set of its discontinuity points is contained in $P(X) \times X \times \{0, 1 - \theta\} \times S \times [n_1]^N \times [n_2]^N$. This is because the skew-product $\sigma : T \times [n_1]^N \to [n_1]^N$ is continuous at all points except at $\{0, 1 - \theta\} \times [n_1]^N$.

We shall say that $P \in P(Y)$ is globally adapted if for every $f \in C(Y)$,
\[
\int f(\mu, x, t, \tau, \omega, \eta)dP(\mu, x, t, \tau, \omega, \eta) = \int \left( \int f(\mu, x, t, \tau, \omega, \eta)d\mu(x) \right)dP_{1,3,4,5,6}(\mu, t, \tau, \omega, \eta),
\]
where $P_{1,3,4,5,6}$ denote the corresponding marginal of $P$ on the coordinates $1, 3, 4, 5, 6$. This means that if a property holds $P$ almost everywhere, then it holds for $P_{1,3,4,5,6}$ almost every $(\mu, t, \tau, \omega, \eta)$ and for $\mu$ almost every $x$.

For $P \in P(Y)$ define $H(P) = \int \frac{1}{\log m_2} \log \mu([x_1])dP_{1,2}(\mu, x)$.

**Proposition 4.7.** Suppose that $\exists (t_0, \omega_0, \eta_0) \in T \times [n_1]^N \times [n_2]^N$ such that there exists a line $\ell_1$ of slope $m_1\| \ell_1$ that satisfies
\[
\dim_B(\pi_{m_1}(F_{\omega_0}) \times \pi_{m_2}(E_{\eta_0})) \cap \ell_1 = \gamma > 0.
\]

Then there exists $Q \in P(Y)$ that is $\hat{M}$ invariant, $H(Q) = \gamma$, and $Q$ satisfies (23) for all $f \in C(Y)$.

In particular, $Q_{1,2}$ is a CP-distribution. Moreover:

1. Recall the definition of $F$ from (22), and let
\[
D_F = \bigcup_{(A, x, t, \tau, \omega, \eta) \in F} P(A) \times \{x\} \times \{t\} \times \{\tau\} \times \{\omega\} \times \{\eta\}.
\]
Then $Q$ is supported on $D_F$. Thus, $Q$ almost every ergodic component is supported there. Moreover,
\[
Q(\{(\mu, x, t, \tau, \omega, \eta) \in D_F : \tau = v_1\}) = 1.
\]

2. There is a measurable set $E_\gamma \subseteq Y$ such that $Q(E_\gamma) > 0$, and for $Q$ almost every $(\mu, x, t, \tau, \omega, \eta) \in E_\gamma$, $Q_{1,2}^{(\mu, x, t, \tau, \omega, \eta)}$ (the marginal of the corresponding ergodic component of $Q$ on the first two coordinates) is an ergodic CP chain of dimension $\geq \gamma$. 


Proof. Let \( E = \pi_{v_{0}}^{-1} \omega_{0}, \eta_{0}((\pi_{m_{1}}(F_{\omega_{0}}) \times \pi_{m_{2}}(\tilde{E}_{\eta_{0}})) \cap \ell) \). By Lemma 4.4 parts (2) and (3), we have \( \overline{\dim}_{p}(E) = \gamma \) in the space \( X_{v_{0}, \omega_{0}, \eta_{0}} \). Thus, there exists \( n_{k} \to \infty \) such that

\[
\lim_{k} \frac{\log N(E, m_{2}^{-nk})}{-n_{k} \log m_{2}} = \gamma. \tag{25}
\]

Define a sequence of measures \( \{\mu_{k}\}_{k} \) on \( E \) by setting

\[
\mu_{k} = \frac{1}{N(E, m_{2}^{-nk})} \sum_{u:|u|=n_{k},|u| \cap E \neq \emptyset} \delta_{x_{u}}
\]

for some \( x_{u} \in [u] \cap E \). We also define

\[
P_{k} = \frac{1}{N(E, m_{2}^{-nk})} \sum_{u:|u|=n_{k},|u| \cap E \neq \emptyset} \delta(\mu_{k}, x_{u}, t_{0}, n_{t_{0}}, \omega_{0}, \eta_{0})
\]

\[
Q_{k} = \frac{1}{n_{k}} \sum_{i=0}^{n_{k}-1} \hat{M}^{i}P_{k}.
\]

Note that by the construction of \( Q_{k} \), for all \( f \in C(Y) \), (23) holds.

Note that \( H(Q_{k}) \to \gamma \) as \( k \) grows to \( \infty \). The proof of this fact can be found in [27, Section 4.2]. Now, by the compactness of \( P(Y) \), we may find a subsequence such that \( Q_{k} \to Q \in P(Y) \). It follows that \( H(Q) = \gamma \).

We claim that \( Q \) is \( \hat{M} \) invariant. Indeed, we note that \( Q_{\beta} \) is a measure that is invariant under the irrational rotation \( R_{\theta} \). Therefore, this must be the Lebesgue measure on \( T \). Thus,

\[
Q(\text{ discontinuities of } \hat{M}) \leq Q(P(X) \times X \times [0,1-\theta) \times S \times [n_{1}]^{N} \times [n_{2}]^{N}) = Q_{\beta}([0,1-\theta]) = 0.
\]

It follows that \( Q \) is a measure such that an orbit of \( \hat{M} \) equidistributes for, and by the above calculation \( \hat{M} \) is continuous up to a \( Q \) null set. Therefore, \( Q \) is \( \hat{M} \) invariant. Finally, \( Q \) satisfies (23) since each \( Q_{k} \) does.

Let \( Q = \int Q^{(\mu, x, t, \tau, \omega, \eta)}dQ(\mu, x, t, \tau, \omega, \eta) \) denote the ergodic decomposition of \( Q \). Define

\[
E_{\gamma} = \{ (\mu, x, t, \tau, \omega, \eta) \in Y : H(Q^{(\mu, x, t, \tau, \omega, \eta)}) \geq \gamma \}.
\]

Since \( H(Q) = \gamma \), we have \( Q(E_{\gamma}) > 0 \), and for \( Q \) almost every \( (\mu, x, t, \tau, \omega, \eta) \in E_{\gamma}, Q^{(\mu, x, t, \tau, \omega, \eta)} \) is an ergodic CP distribution of dimension \( \geq \gamma \) (by Proposition 2.4).

Next, by Lemma 4.6, \( D_{\mathcal{F}} \) is closed and \( Q_{k} \) is supported on \( D_{\mathcal{F}} \) for all \( k \). It follows that \( Q \) is supported on \( D_{\mathcal{F}} \). Thus, \( Q \) almost every ergodic component is supported there.

Finally, we prove equation (24). Let \( A \) denote the countable set of all endpoints of the intervals in the partitions \( \mathcal{C}_{k} \) of \( T \) for all \( k \geq 0 \), defined before Claim 4.2. Note that this a countable set. Now, consider the set

\[
B = \{ (\mu, x, t, \tau, \omega, \eta) \in D_{\mathcal{F}} \cap \text{supp}(Q) : t \notin A \}.
\]

Since the projection from the space \( Y \) to its third coordinate is continuous, and since \( A \subset T \) is countable (and hence measurable), it follows that \( B \) is a measurable set. We now prove that if \( (\mu, x, t, \tau, \omega, \eta) \in B \), then \( \tau = v_{t} \).

\[\text{1In fact, this requires some work. Specifically, use the fact that if } \mu_{k} \to \mu \text{ in a compact metric space } X, \text{ then } \text{supp}(\mu) \subseteq \{ x : \text{lim sup } d_{X}(z, \text{supp}(\mu_{k})) = 0 \}, \text{ and Lemma 4.4 part (1).} \]
Fix \((\mu, x, t, \tau, \omega, \eta) \in B\). It is well known that since \(Q\) is the weak limit of the distributions \(Q_k\) (defined earlier), then

\[
\text{supp}(Q) \subseteq \left\{ z \in Y : \limsup_{k} d_Y(z, \text{supp}(Q_k)) = 0 \right\}.
\]

Thus, there exists a sequence \(n_k\) and elements \((\mu_{n_k}, x_{n_k}, t_{n_k}, \tau_{n_k}, \omega_{n_k}, \eta_{n_k}) \in \text{supp}(Q_{n_k})\) that converge to \((\mu, x, t, \tau, \omega, \eta)\). In particular, \(t_{n_k}\) converges to \(t\) in \(T\) and \(\tau_{n_k}\) converges to \(\tau\) in \(S\).

Since these are elements in the support of \(Q_k\), it follows that

\[
t_{n_k} = R_{\theta_{n_k}}(t_0) = n_k \cdot \theta + t_0 \mod 1,
\]

\[
\tau_{n_k} = \sigma_{n_k}(v_{t_0}) = v_{n_k \cdot \theta + t_0} \mod 1 = v_{t_{n_k}}.
\]

Now, since \(t_{n_k}\) converges to \(t\) in \(T\), and since \(t \notin A\), we may apply part (2) of Claim 4.2 and see that \(v_{t_{n_k}} = \tau_{n_k}\) converges to \(v_t\). Therefore, \(\tau = v_t\).

Finally,

\[
Q(B^C) = Q(\{((\mu, x, t, \tau, \omega, \eta) \in D_F : \ t \in A)\} = Q_3(A) = 0,
\]

since the marginal of \(Q\) on the third coordinate is the Lebesgue measure and \(A\) is countable. Also, notice that

\[
\{((\mu, x, t, \tau, \omega, \eta) \in D_F : \ \tau = v_t\} = B \cup \{((\mu, x, t, \tau, \omega, \eta) \in D_F : \ \tau = v_t, t \in A\},
\]

and since both sets on the RHS are measurable, so is the set on the LHS.

Notice that Proposition 4.7 assumes nothing about \((\omega_0, \eta_0)\), and thus forms the first step towards the proof of Theorem 1.7 part (1). We next discuss some improvements of the above proposition when we can control \(\omega_0\) and \(\eta_0\) in the statement. Consider the compact metric space \(T \times [n_1]^N \times [n_2]^N\). Define a map

\[
Z : T \times [n_1]^N \times [n_2]^N \to T \times [n_1]^N \times [n_2]^N
\]

by taking

\[
Z(t, \omega, \eta) = (R_{\theta}(t), \sigma(t), \sigma(\eta)).
\]

The following proposition was proved during the proof of [14, Proposition 4.3].

**Theorem 4.8 [14].** Let \(\alpha_1, \alpha_2\) be Bernoulli measures on \([n_1]^N\), \([n_2]^N\), respectively. Then for every \(t \in T\) there is a set of full \(\alpha_1 \times \alpha_2\) measure \(A\), such that for all \((\omega, \eta) \in A\), we have

\[
\frac{1}{N} \sum_{i=0}^{N-1} \delta_{Z^i(t, \omega, \eta)} \to \lambda \times \rho \times \alpha.
\]

In particular, the measure preserving system \((T \times [n_1]^N \times [n_2]^N, Z, \lambda \times \alpha_1 \times \alpha_2)\) is ergodic, where \(\Lambda\) is the Lebesgue measure on \(T\).

The following corollary is thus a result of the previous corollary, and the construction carried out in Proposition 4.7.

**Corollary 4.9.** Let \(Q\) be the CP-distribution built in Proposition 4.7. Suppose that the \((\omega_0, \eta_0) \in [n_1]^N \times [n_2]^N\) appearing in the statement of Proposition 4.7 are typical with respect to \(t_0\) and some Bernoulli measures \(\alpha_1 \in P([n_1]^N)\) and \(\alpha_2 \in P([n_2]^N)\), in the sense of Theorem 4.8.
Let \( Q_{3,5,6} \) denote the joint distribution of \( Q \) on the third coordinate, the fifth coordinate and the sixth coordinate. Then

\[
Q_{3,5,6} = \lambda \times \alpha_1 \times \alpha_2 \in P(T \times \{n_1\}^{N} \times \{n_2\}^{N}).
\]

5. A skew product dynamical system

5.1. The transformation \( U \)

Define \( U : [0,1]^2 \times T \times \{n_1\}^{N} \times \{n_2\}^{N} \to [0,1]^2 \times T \times \{n_1\}^{N} \times \{n_2\}^{N} \) by setting

\[
U(z,t,\omega,\eta) = (\Phi_t(z), R_\theta(t), \sigma_t(\omega), \sigma(t)).
\]

Recall that

\[
r_k(t) = |\{0 \leq i \leq k-1 : R_\theta^i(t) \in [1-\theta,1)\}|.
\]

We denote by \( U^{t,k}(z) \) the first coordinate of the map \( U^k(z,t,\omega,\eta) \), and its third coordinate by \( \sigma^{t,k}(\omega) \). Notice that

\[
\sigma^{t,k}(\omega) := \sigma_{R_\theta^{k-1}(t)} \circ \cdots \circ \sigma_{R_\theta(t)} \circ \sigma_t(\omega) = \sigma_{r_k(t)}(\omega). \quad (26)
\]

Also, recalling the definition of the maps \( T_{n_1} \) from (2)

\[
U^{t,k}(z) = \Phi_{R_\theta^{k-1}(t)} \circ \cdots \circ \Phi_{R_\theta(t)} \circ \Phi_t(z) = (T_{r_k(t)}^{t,k}(z_1), T_{m_2}^{t,k}(z_2)), \text{ for } z = (z_1, z_2). \quad (27)
\]

We now define a sequence of partitions of \([0,1]^2 \times T \times \{n_1\}^{N} \times \{n_2\}^{N}\) as follows: Let \( D_{m_1} \) and \( D_{m_2} \) be the \( m_1 \)-adic and \( m_2 \)-adic partitions, respectively, of \([0,1]\). Recall that \( C = \{0,1-\theta,1-\theta,1\} \) is the partition of \( T \) we previously defined, and that \( T_{n_1}^k \) is the \( k \)-generation cylinder partition of \([n_1]\). Let

\[
B_1 = [D_m \times D_n] \times C \times T_{n_1}^1 \times T_{n_2}^1.
\]

and for \( k \geq 2 \), let

\[
B_k = \bigvee_{m=0}^{k-1} U^{-m}B_1.
\]

Let us make the following observations. For \( k \geq 2 \), let \( C_k = \bigvee_{m=0}^{k-1} R_\theta^{-m}C \), and notice that \( \bigvee_{m=0}^{k-1} \sigma^{-m}T_{n_2}^1 = T_{n_2}^k \).

- For \( k \geq 1 \) and \( t \in T \), define \( T_{n_1}^{t,k} := \bigvee_{m=0}^{k-1} (\sigma^t)^{-1}T_{n_1}^1 \). By (26) we have \( T_{n_1}^{t,k} = T_{r_k(t)}^1 \).
- For \( k \geq 1 \) and \( t \in T \), define \( A^{t,k} = \bigvee_{m=0}^{k-1} (U^t)^{-1} (D_{m_1} \times D_{m_2}) \). By (27) we have

\[
A^{t,k} = D_{m_1}^{r_k(t)} \times D_{m_2}^{r_k(t)}.
\]

- Note that if \( t, t' \in T \) belong to the same atom of \( C_k \), then \( A^{t,k} = A^{t',k} \) and \( T_{n_1}^{t,k} = T_{n_1}^{t',k} \), since this means that \( r_i(t) = r_i(t') \) for all \( 0 \leq i \leq k-1 \).
- Every atom of \( B_k \) has the form \( A \times C \times I \times J \) for \( J \in T_{n_2}^k \), \( C \subset C_k \) and \( A \subset A^{t,k} \) and \( I \in T_{n_1}^k \), for some \( t \in C \).

The following lemma is modelled after [27, Lemma 5.1]. We defer its proof to Section 7.

**Lemma 5.1.** (1) Let \((t,v_1,\omega,\eta) \in T \times S \times \{n_1\}^{N} \times \{n_2\}^{N}\) and \( x \in X_{v_1,\omega,\eta} \). If \( \pi_{v_1,\omega,\eta}(x) \) is in the interior of \( A^{t,k}(\pi_{v_1,\omega,\eta}(x)) \), then the set \( \pi_{v_1,\omega,\eta}([x_{0}^{k-1}]) \) is contained within \( A^{t,k}(\pi_{v_1,\omega,\eta}(x)) \) except possibly at the boundary points of \( A^{t,k}(\pi_{v_1,\omega,\eta}(x)) \).
In this section, we construct a family of invariant measures on \([0, 1]^{1/2}\). If \(\mu\) is not atomic, then for \(\mu\) almost every \(x\) and all \(k \geq 1\),
\[
\pi_{\sigma^k(\nu), \sigma^k(\nu)}\left(\frac{\sigma^k(\mu[x_0^{k-1}])}{\mu([x_0^{k-1}])}\right) = U^{t,k}\left(\frac{\pi_{\nu, \nu, \nu}(A^{t,k}(\nu, \nu, \nu(x)))}{\pi_{\nu, \nu, \nu}(A^{t,k}(\nu, \nu, \nu(x)))}\right).
\]

Let \(\nu \in P([0, 1]^2)\) and \(z \in \text{supp}(\nu)\). Denote
\[
\nu^{A^{t,k}(z)} = U^{t,k}\left(\frac{\nu|^{A^{t,k}(z)}}{\nu(A^{t,k}(z))}\right).
\]
Note that if \(\nu \in P(\ell \cap [\pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta)])\) with \(\ell\) being a line with slope \(m_1\), then
\[
\nu^{A^{t,k}(z)} \in P(\ell' \cap [\pi_{m_1}(\tilde{F}_{\sigma^t, k}(\omega)) \times \tilde{E}_{\sigma^k(\eta)}]),
\]
where \(\ell'\) has slope \(m_1^{R_k(t)}\).

5.2. Construction of \(U\) invariant measures

In this section, we construct a family of \(U\) invariant measures on \([0, 1]^2 \times \mathbb{T} \times [n_1]^N \times [n_2]^N\) by transferring information from an ergodic component of the CP-distribution \(Q\) constructed in Proposition 4.7, in a similar spirit to [27], Proposition 5.3. Unlike the proof in [27], we do this by considering the intensity measure of some of the ergodic components of the CP-chain \(Q\).

**Theorem 5.2.** For \(Q_{1,3,4,5,6}\) almost every \((\mu, t, \tau, \omega, \eta)\) and \(\mu\) almost every \(x\) such that \((\mu, x, t, \tau, \omega, \eta) \in E_y\), let
\[
\nu^{(\mu, x, t, \tau, \omega, \eta)} := \int (\pi_{\tau', \xi, \zeta}(\nu) \times \delta_{\eta} \times \delta_{\tau} \times \delta_{\xi})dQ^{(\mu, x, t, \tau, \omega, \eta)}_{1,3,4,5,6}(\nu, s, \tau', \xi, \zeta).
\]
Then this is a measure on \([0, 1]^2 \times \mathbb{T} \times [n_1]^N \times [n_2]^N\) that is \(U\) invariant.

**Proof.** First, notice that by equation (24), we have that for \(Q\) almost every \((\mu, x, t, \tau, \omega, \eta)\),
\[
\nu^{(\mu, x, t, \tau, \omega, \eta)} = \int (\pi_{\nu, \nu, \nu}(\nu) \times \delta_{\eta} \times \delta_{\tau} \times \delta_{\xi})dQ^{(\mu, x, t, \tau, \omega, \eta)}_{1,3,4,5,6}(\nu, s, v_s, \xi, \zeta),
\]
(that is, \(\tau' = v_s\) in (32)) since the set where \(\tau' \neq v_s\) has \(Q\) measure 0. So, fix such an element in \(E_y\). In this proof, we denote the corresponding ergodic component \(Q^{(\mu, x, t, \tau, \omega, \eta)}\) by \(R\), and its marginal \(R_{1,3,4,5,6}\) by \(R'\).

Now, let \(f \in C([0, 1]^2 \times \mathbb{T} \times [n_1]^N \times [n_2]^N)\). Then:
\[
\int f d\nu^{(\mu, x, t, \tau, \omega, \eta)} = \int \left(\int f(z, s, \xi, \zeta)d\pi_{s, \xi, \zeta}(\nu)(z)\right)dR'(\nu, s, v_s, \xi, \zeta)
\]
\[
= \int \left(\int f(z, s, \xi, \zeta)d\pi_{s, \xi, \zeta}(\nu)(z)\right)dR(\nu, y, s, v_s, \xi, \zeta)
\]
\[
= \int \left(\int f(z, s, \xi, \zeta)d\pi_{s, \xi, \zeta}(\nu)(z)\right)d\tilde{M}R(\nu, y, s, v_s, \xi, \zeta)
\]
\[
= \int \left(\int f(z, R_0(s), \sigma_s(\xi), \sigma(\zeta))d\pi_{\sigma(s), \xi, \zeta}(\nu)(z)\right)dR(\nu, y, s, v_s, \xi, \zeta),
\]
where the second equality follows since the function we are integrating against does not depend on the second coordinate \(y\) (in the space \(Y\)), the third equality follows from \(\tilde{M}\).
invariance and the fourth equality follows from changing variables via \( M \). We continue our calculation:

\[
\begin{align*}
&= \int \left( \int \left( \int f(z, R\theta(s), \sigma_s(\xi), \sigma(\zeta))d\nu_{vR\theta(s), \sigma_s(\xi), \sigma(\zeta)}(\nu_{[y_0]})(z) \right) d\nu(y) \right) dR'(\nu, s, v_s, \xi, \zeta) \\
&= \int \left( \sum_{i,j} \int f(z, R\theta(s), \sigma_s(\xi), \sigma(\zeta))d\nu_{vR\theta(s), \sigma_s(\xi), \sigma(\zeta)}(\nu_{[i,j]})(z) \nu_{[i,j]}(\nu_{[i,j]}) \right) dR'(\nu, s, v_s, \xi, \zeta) \\
&= \int \left( \sum_{i,j} \int f(z, R\theta(s), \sigma_s(\xi), \sigma(\zeta))d\nu_{vR\theta(s), \sigma_s(\xi), \sigma(\zeta)}(\nu_{[i,j]})(z) \right) dR'(\nu, s, v_s, \xi, \zeta),
\end{align*}
\]

where the first equality above follows since \( \sigma(v_s) = v_{R\theta(s)} \) and from the adaptedness\(^1\) of \( R \), the second equality since there are finitely many options for \( y_0 \) (so that the integrand above is a simple function with respect to \( y \)) and the third from the definition of \( \nu_{[i,j]} \). Next,

\[
\begin{align*}
&= \int \left( \sum_{i,j} \int f(z, R\theta(s), \sigma_s(\xi), \sigma(\zeta))d\Phi_s \frac{\left( \pi_{v_s, \xi, \zeta} \nu \right)_{A_{[i,j]}^1(\pi_{v_s, \xi, \zeta}([i,j]))}}{\pi_{v_s, \xi, \zeta} \nu (A_{[i,j]}^1(\pi_{v_s, \xi, \zeta}([i,j])))}(z) \right) dR'(\nu, s, v_s, \xi, \zeta),
\end{align*}
\]

where by the notation \( A_{[i,j]}^1(\pi_{v_s, \xi, \zeta}([i,j])) \) we mean the unique partition element of \( A_{[i,j]}^1 = D_{m_2} \times D_{m_2} \) that contains the elements \( \pi_{v_s, \xi, \zeta}(y) \) for all \( y \in [i, j] \) (except for maybe on the measure zero boundary of the cell). Notice that we have applied (30) for \( k = 1 \), using that generically, we have \( \dim \nu \geq \gamma > 0 \) in the above integral (since we are working with an ergodic component with positive entropy). Next,

\[
\begin{align*}
&= \int \left( \sum_{i,j} \int f(\Phi_s(z), R\theta(s), \sigma_s(\xi), \sigma(\zeta))d\frac{\left( \pi_{v_s, \xi, \zeta} \nu \right)_{A_{[i,j]}^1(\pi_{v_s, \xi, \zeta}([i,j]))}}{\pi_{v_s, \xi, \zeta} \nu (A_{[i,j]}^1(\pi_{v_s, \xi, \zeta}([i,j])))}(z) \right) dR'(\nu, s, v_s, \xi, \zeta),
\end{align*}
\]

where we have changed variables in the \( z \) coordinate using \( \Phi_s \). Therefore, we arrive at

\[
\begin{align*}
&= \int \left( \sum_{i,j} \int f \circ U(z, s, \xi, \zeta)d\frac{\left( \pi_{v_s, \xi, \zeta} \nu \right)_{A_{[i,j]}^1(\pi_{v_s, \xi, \zeta}([i,j]))}}{\pi_{v_s, \xi, \zeta} \nu (A_{[i,j]}^1(\pi_{v_s, \xi, \zeta}([i,j])))}(z) \right) dR'(\nu, s, v_s, \xi, \zeta).
\end{align*}
\]

Finally, by retracing our steps, and using (30), we see that

\[
\begin{align*}
&= \int (f \circ U)dR' = \int (f \circ U)d\nu_{[\mu, t, \zeta, \omega, \eta]}.
\end{align*}
\]

which implies the \( U \) invariance of our measure. \( \square \)

5.3. Some properties of our \( U \) invariant measures

Fix an element \((\mu_0, x_0, t_0, \tau_0, \omega_0, \eta_0) \in E_\gamma \) such that \( Q_{1,2}^{(\mu_0, x_0, t_0, \tau_0, \omega_0, \eta_0)} \) is an ergodic CP distribution of dimension \( \geq \gamma \), and

\[
\nu_{\infty} = \nu_{(\mu_0, x_0, t_0, \tau_0, \omega_0, \eta_0)} = \int (\pi_{v_1, \omega, \eta})_{\mu \times \delta_t \times \delta_\omega \times \delta_\eta}dQ_{1,3,4,5,6}^{(\mu_0, x_0, t_0, \tau_0, \omega_0, \eta_0)}(\mu, t, v_1, \omega, \eta)
\]

is \( U \) invariant.

\(^1\)Notice that the function \((\nu, s, v_s, \xi, \zeta) \mapsto \pi_{v_s, \xi, \zeta} \times \delta_s \times \delta_\zeta \times \delta_s \) is \( Q \) almost everywhere continuous.
We record some other useful properties of the measure \(\nu_\infty\) and the partitions \(B_k\), defined in (28). We defer the proof to Section 7.

**Proposition 5.3.** (1) The partitions \(B_k\) generate the Borel sigma algebra of \([0,1]^2 \times T \times [n_1]^N \times [n_2]^N\).

(2) For every \(k \in \mathbb{N}\) and every element \(B \in B_k\), we have \(\nu_\infty(\partial B) = 0\).

(3) Suppose that the \((\omega_0,\eta_0)\) in the assumption of Proposition 4.7 are \(\alpha_1 \times \alpha_2\) typical for \(t_0\) in the sense of Theorem 4.8, for some Bernoulli measures \(\alpha_1 \in P([n_1]^N)\) and \(\alpha_2 \in P([n_2]^N)\). Then we may assume the marginal of \(\nu_\infty\) on the third component and fourth component gives full measure to the set of \(\alpha_1 \times \alpha_2\) generic points, with respect to the product system \(([n_1]^N \times [n_2]^N, \sigma \times \sigma, \alpha_1 \times \alpha_2)\).

We next outline another important property of measure \(\nu_\infty\). By applying Proposition 2.6 to the ergodic CP distribution \(Q_{1,2}^{(\mu_0, x_0, t_0, \tau_0, \omega_0, \eta_0)}\), we see that: for any \(\epsilon > 0\) there exists \(k_0(\epsilon)\) such that for \(Q^{(\mu_0, x_0, t_0, \tau_0, \omega_0, \eta_0)}\) almost every \(\mu\) and \(\mu\) almost every \(x\), we have

\[
\liminf_N \frac{1}{N} \left\{ 1 \leq k \leq N : \max_{|u|=k_0(\epsilon)} |\mu|^{x_0^{-1}}([u]) \leq \epsilon \text{ and } H(\mu, I_{m_1}^p \times I_{m_2}^p) \geq p \cdot (\gamma \log m_2 - \epsilon) \right\} > 1 - 2\epsilon, \text{ for any } p \geq k_0(\epsilon).
\]

By applying part (3) of Lemma 4.4, we see that for any \(\epsilon > 0\) there is some \(\delta(\epsilon) > 0\) and some \(k_1(\epsilon) \in \mathbb{N}\) such that for \(Q_{1,2}^{(\mu_0, x_0, t_0, \tau_0, \omega_0, \eta_0)}\) almost every \(\mu\) and \(\pi_{v_1, \omega, \eta} \mu\) almost every \(z\), we have

\[
\liminf_N \frac{1}{N} \left\{ 1 \leq k \leq N : \sup_{y \in [0,1]^2} (\pi_{v_1, \omega, \eta} \mu)^{A_{v_1}^k}(z) \leq \epsilon \text{ and } H(\pi_{v_1, \omega, \eta} \mu, D_{2p} \times D_{2p}) \geq p \cdot (\gamma \log 2 - 2\epsilon) \right\} > 1 - 2\epsilon, \text{ for any } p \geq k_1(\epsilon).
\]

In particular, the above is true for \(Q_{1,2,3,4,5,6}^{(\mu_0, x_0, t_0, \tau_0, \omega_0, \eta_0)}\) almost every \((\mu, t, v_1, \omega, \eta)\) and \(\pi_{v_1, \omega, \eta} \mu\) almost every \(z\). On the other hand, by the definition of the measure \(\nu_\infty\), selecting \((z, t, \omega, \eta)\) according to \(\nu_\infty\), we can be done by first drawing \((\mu, t, v_1, \omega, \eta)\) according to \(Q_{1,2,3,4,5,6}^{(\mu_0, x_0, t_0, \tau_0, \omega_0, \eta_0)}\) and then selecting \(z\) according to \(\pi_{v_1, \omega, \eta} \mu\). Thus, we have the following proposition.

**Proposition 5.4.** The measure \(\nu_\infty\) satisfies the following property: for every \(\epsilon > 0\) there exists \(\delta(\epsilon) > 0\) and \(k_1(\epsilon)\) such that \(\nu_\infty\) almost every \((z, t, \omega, \eta)\), there exists a measure \(\mu \in P(X)\) such that

(1) \(\pi_{v_1, \omega, \eta} \mu \in P(\ell \cap \pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta))\) for some line \(\ell\) with slope \(m_1^1\);

(2) the property (34) holds for \(\pi_{v_1, \omega, \eta}\) and for \(z\).

Finally, if \(\nu_\infty\) is not ergodic, we may move to an ergodic component of \(\nu_\infty\). So, we have proved the following theorem.

**Theorem 5.5.** There exists an ergodic \(U\)-invariant measure \(\nu_\infty\) that satisfies the properties stated in Propositions 5.4 and 5.3.
6. Proof of Theorem 1.7

6.1. An application of the Sinai factor theorem

Recall that a sequence \( \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{T} \) is uniformly distributed (UD) if for every sub-interval \( J \subseteq \mathbb{T} \) we have
\[
\frac{1}{N} |\{0 \leq k \leq N - 1 : x_k \in J\}| \to \lambda(J),
\]
where \( \lambda \) is the Lebesgue measure on \( \mathbb{T} \).

In [27], Wu was able to prove the following theorem by using the Sinai factor theorem.

**Theorem 6.1** [27, Theorem 6.1]. Let \((X, T, \mu)\) be an ergodic measure preserving system. Let \( A \) be a generator with finite cardinality, and let \( \{A_k\}_k \) denote the filtration generated by \( A \) and \( T \). Suppose that \( \mu(\partial A) = 0 \) for every \( k \in \mathbb{N} \) and every \( A \in A_k \). Let \( \beta \notin \mathbb{Q} \).

Then for any \( \epsilon > 0 \) there exists another disjoint family of measurable sets \( \{\tilde{C}_i\}_{i=1}^{N(k,\epsilon)}, \tilde{C}_i \subset X \), such that:

1. \( \mu(\bigcup \tilde{C}_i) > 1 - \epsilon; \)
2. for every \( 1 \leq i \leq N(k,\epsilon), \{|A \in A_k : C_i \cap A\}| \leq e^{k+\epsilon}; \)
3. there exists another disjoint family of measurable sets \( \{\hat{C}_i\}_{i=1}^{N(k,\epsilon)}, \hat{C}_i \subset X \), s.t. for every \( 1 \leq i \leq N(k,\epsilon) \) we have:
   - \( C_i \subseteq \hat{C}_i \),
   - \( \mu(C_i) \geq (1 - \epsilon)\mu(\hat{C}_i) \),
   - for \( \mu \) almost every \( x \) we have that the sequence
     \[
     \left\{ R^k_{\beta}(0) \in \mathbb{T} : k \in \mathbb{N} \text{ and } T^k(x) \in \hat{C}_i \right\}
     \]
   is UD.

6.2. Extracting geometric information from Theorem 6.1

The following proposition is modelled after [27, Proposition 7.1]. As in [27], we denote the coordinate projections\(^\dagger\) of \([0, 1]^2 \times \mathbb{T} \times [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}}\) by \( \Pi_i \), for \( i = 1, 2, 3, 4 \), and similarly \( \Pi_{1,2} \) and \( \Pi_{3,4} \).

**Proposition 6.2.** There exists a constant \( C > 0 \) such that for every \( \epsilon > 0 \) there is some \( r_0 = r_0(\epsilon) \) and \( k_5 = k_5(\epsilon) \in \mathbb{N} \) such that for every \( k \geq k_5(\epsilon) \) the following is true.

For \( n_\infty \) almost every \((z,t,\omega,\eta)\), we can find a measure \( \nu \in P([0,1]^2) \), a measurable set \( D \subseteq [0,1]^2 \times \mathbb{T} \times [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}} \) and a set \( N \subseteq \mathbb{N} \) such that:

1. \( \nu \in P(\ell \cap [\pi_{m_1}(\tilde{F}_{\omega}) \times \pi_{m_2}(\tilde{E}_\eta)]) \) for some line \( \ell \) with slope \( m_1^\dagger \);  
2. we have
\[
\frac{1}{k} \log N_{2-k}(D_1) \leq C \cdot \left( \epsilon + \frac{1}{k} \right),
\]
where \( \Pi_1(D) = D_1 \), and \( N_{2-k}(A) \) is the number of \( k \)-level dyadic boxes \( A \) intersects. In addition,
\[
\frac{1}{k \log 2} \log N_{2-k}(\pi_{m_1} \times \pi_{m_2}(\tilde{D}_{3,4})) \leq C \cdot \left( \epsilon + \frac{1}{k} \right) + \frac{\max_{(i,j) \in [n_1] \times [n_2]} |\Gamma_{i,j}|}{\log m_1} + \frac{|\Lambda_j|}{\log m_2},
\]
where \( \Pi_{3,4}(D) = D_{3,4} \) and \( \tilde{D}_{3,4} = \bigcup_{(\omega,\eta) \in D_{3,4}} \tilde{F}_{\omega} \times \tilde{E}_\eta \subseteq [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}} \);

\(^\dagger\)For example, \( \Pi_1(z,t,\omega,\eta) = z \).
(3) for every $p \in \mathbb{N}$ we have $U_p(z, t, \omega, \eta) \in D$;

(4) $\lambda(\{ R_0^p(t) : p \in N\}) \geq 1 - C \cdot \epsilon$;

(5) for every $p \in \mathbb{N}$,

$$\inf_{y \in \mathbb{R}^d} \frac{1}{k \log 2} \log H_{\nu A^i \rho(z)}(B(y, r_0)^c, D_{2^k}) \geq \gamma - C \cdot \sqrt{\epsilon};$$

(6) suppose that $(\omega_0, \eta_0)$ from the condition in the statement of Proposition 4.7 is typical with respect to $t_0$ and a product of Bernoulli measures $\alpha_1 \times \alpha_2 \in P([n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}})$, in the sense of Theorem 4.8. Then for $\nu_\infty$ almost every $(z, t, \omega, \eta)$ we can construct sets and measures with all the above properties, with the additional property that

$$\frac{1}{k \log 2} \log N_{2^{-k}}(\pi m_1 \times \pi m_2(\tilde{D}_{i,3,4}))$$

$$\leq C \cdot \left( \epsilon + \frac{1}{k} \right) + \sum_{i=0}^{n_1-1} \alpha_1([i]) \cdot \frac{\log |\Gamma_{i}|}{\log m_1} + \sum_{j=0}^{n_2-1} \alpha_2([j]) \cdot \frac{\log |A_j|}{\log m_2},$$

where $D_{i,3,4} = \bigcup_{k \in \mathbb{N}} \Pi_{3,4} U^k(z, t, \omega, \eta) \subseteq [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}}$ and $\tilde{D}_{i,3,4} = \bigcup_{(\omega, \eta) \in D_{i,3,4}} \tilde{F}_\omega \times \tilde{E}_\eta$.

We shall require two lemmas for the proof. Both can be found in [27]. For a set $O \subset \mathbb{N}$ we denote the density of $O$ in $\mathbb{N}$ by

$$d(O, \mathbb{N}) := \lim_{N \to \infty} \frac{1}{N} |O \cap [0, N - 1]|$$

(35)

If the limit does not exist, we call the lim sup the upper density of $O$ in $\mathbb{N}$ which we denote by $\overline{d}(O, \mathbb{N})$, and the lim inf the lower density of $O$ in $\mathbb{N}$, denoted by $\underline{d}(O, \mathbb{N})$.

**Lemma 6.3** [27, Lemma 7.2]. Let $\{x_k\} \subset \mathbb{T}$ be UD. Let $O \subseteq \mathbb{N}$. Then

$$\lambda(\{x_k : x_k \in O\}) \geq \overline{d}(O, \mathbb{N}).$$

**Lemma 6.4** [27, Lemma 7.3]. Let $\mu \in P(\mathbb{R}^d)$ and fix $0 < \delta < 1$. If $\sup_{y \in \mathbb{R}^d} \mu(B(y, \delta)) \leq \epsilon$, then for all $k \in \mathbb{N}$ such that $2^{-k} \leq \delta$ we have

$$\inf_{y \in \mathbb{R}^d} H(\mu|_{B(y, r_0)^c}, D_{2^k}) \geq H(\mu, D_{2^k}) - C_1 \cdot k \sqrt{\epsilon}.$$ 

For some constant $C_1$ that depends only on $d$.

We now prove Proposition 7.1, under the additional assumption that $(\omega_0, \eta_0)$ from Lemma 4.7 are typical with respect to a product of Bernoulli measures $\alpha_1 \times \alpha_2$ and $t_0 \in \mathbb{T}$, in the sense of part (6). If this is not the case, then proof follows along the same lines, and is actually easier. Let $\epsilon > 0$.

**Choice of the integer $k_5$ and $r_0$**

By Theorem 5.5, $\nu_\infty$ is ergodic and satisfies Proposition 5.4. Put $r_0(\epsilon) := \delta(\epsilon)$, where $\delta(\epsilon)$ is the number from Proposition 5.4. Recall the partition $B_1$ of $[0, 1]^2 \times \mathbb{T} \times [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}}$, defined in (28). Recall that $B_1$ is a partition of finite cardinality, and that by Proposition 5.3, $\nu_\infty(\partial B) = 0$, for all $B \in B_k$ and all $k \geq 1$. We may thus apply Theorem 6.1 to the dynamical system $([0, 1]^2 \times \mathbb{T} \times [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}}, U, \nu_\infty)$.

In addition, for every $i \in [n_1]$ define continuous functions $f_i : [0, 1]^2 \times \mathbb{T} \times [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}} \to \mathbb{R}$ by $f_i(z, t, \omega, \eta) = 1_{(\omega_1 = i)}(\omega)$. Let $f_i^k$ be the ergodic average (with respect to $([n_1]^{\mathbb{N}}, \sigma)$,

$$f_i^k(z, t, \omega, \eta) = \frac{1}{[\theta \cdot k]} \sum_{p=0}^{[\theta \cdot k]-1} f_i(\sigma^p(\omega)).$$
Similarly, for every $j \in [n_2]$, define continuous functions $g_j : [0, 1]^2 \times T \times [n_1]^N \times [n_2]^N \to \mathbb{R}$ by $g_j(z, t, \omega, \eta) = 1_{\{n_1 = j\}}(\omega)$. Let $g_j^k$ be the ergodic average (with respect to $([n_2]^N, \sigma)$),

$$g_j^k(z, t, \omega, \eta) = \frac{1}{k} \sum_{p=0}^{k-1} g_j(\sigma^p(\eta)).$$

By Proposition 5.3, $\Pi_{3,4} \nu_\infty$ almost every $(\omega, \eta)$ is generic with respect to the product system $([n_1]^N \times [n_2]^N, \sigma \times \sigma, \alpha_1 \times \alpha_2)$. So for $\nu_\infty$ almost every $(\omega, \eta)$, $\omega$ is generic for $([n_1]^N, \sigma, \alpha_1)$ and $\eta$ is generic for $([n_2]^N, \sigma, \alpha_2)$, and therefore for every $(i, j) \in [n_1] \times [n_2]$,

$$\lim_k f_j^k(z, t, \omega, \eta) = \int 1_{\{\omega_1 = 1\}}(\omega) d\alpha_1(\omega) = \alpha_1([i]), \quad \text{and similarly } \lim_k g_j^k(z, t, \omega, \eta) = \alpha_2([j]).$$

Thus, by $n_1 \cdot n_2$ applications of Egorov’s theorem, we may find an integer $k_3(\epsilon)$ such that

$$V = \left\{ (z, t, \omega, \eta) : \forall k > k_3(\epsilon), \forall (i, j) \in [n_1] \times [n_2], \quad \left| f_j^k(z, t, \omega, \eta) - \alpha_1([i]) \right| < \frac{\epsilon}{n_1 \cdot n_2}, \quad \text{and} \quad \left| g_j^k(z, t, \omega, \eta) - \alpha_2([j]) \right| < \frac{\epsilon}{n_1 \cdot n_2} \right\}$$

has measure $\nu_\infty(V) \geq 1 - \epsilon$.

Let $k_2(\epsilon)$ be the integer provided by Theorem 6.1. Let $k_1(\epsilon)$ be the integer from Proposition 5.4. Let $k_4(\epsilon)$ be such that $2^{-k} \leq \delta$ for all $k \geq k_4$. Let

$$k_5 = \max \left\{ k_1(\epsilon), k_2(\epsilon) \cdot \frac{\log m_2}{\log 2}, k_3(\epsilon) \cdot \frac{\log m_2}{\log 2}, k_4(\epsilon) \right\}.$$ 

We will show that $k_5$ can be taken to be the integer promised in the statement of Proposition 6.2.

Construction of the sets $N$ and $D$

Let $k \geq k_5$. Define $\tilde{k} = [k \cdot \frac{\log 2}{\log m_2}] + 1$. Then $\tilde{k} \geq k_2(\epsilon), k_3(\epsilon)$. By Theorem 6.1 we can find disjoint families $\{C_i\}_{i=1}^{\tilde{k}, \epsilon}, \{\tilde{C}_i\}_{i=1}^{\tilde{k}, \epsilon}$ of measurable subsets of $[0, 1]^2 \times T \times [n_1]^N \times [n_2]^N$ satisfying the conditions of Theorem 6.1 with respect to the partition $B_{\tilde{k}}$.

Let $A' \subset [0, 1]^2 \times T \times [n_1]^N \times [n_2]^N$ denote the set of $(z, t, \omega, \eta)$ such that:

- the sequence

$$\left\{ R_j^k(t) \in T : k \in \mathbb{N} \text{ and } U^k(z, t, \omega, \eta) \in \tilde{C}_i \right\}$$

is UD for every $1 \leq i \leq N(\tilde{k}, \epsilon)$;

- there exists a measure $\mu = \mu_{z, t, \omega, \eta}$ such that $\pi_{v_\ell, \omega, \eta} \mu \in P(\ell \cap [\pi_{m_1}(\bar{F}_\omega) \times \pi_{m_2}(\bar{E}_\eta)])$ for some line $\ell$ with slope $m_1^*$, and (34) holds for $\pi_{v_\ell, \omega, \eta} \mu$ and $z$.

By Theorem 6.1 part (3) and by Proposition 5.4, since $k \geq k_1(\epsilon)$ and by the choice of $r_0(\epsilon)$, $\nu_\infty(A') = 1$.

Next, for $(z, t, \omega, \eta)$ and $1 \leq i \leq N(\tilde{k}, \epsilon)$, define the sequences of visiting times

$$B(C_i, z, t, \omega, \eta) = \left\{ k \in \mathbb{N} : U^k(z, t, \omega, \eta) \in C_i \right\},$$

$$B(\tilde{C}_i, z, t, \omega, \eta) = \left\{ k \in \mathbb{N} : U^k(z, t, \omega, \eta) \in \tilde{C}_i \right\},$$

$$B(V, z, t, \omega, \eta) = \left\{ k \in \mathbb{N} : U^k(z, t, \omega, \eta) \in V \right\}.$$ 

Recall the definition of the density of a set of integers from (35). Let $A''$ be the set of all $(z, t, \omega, \eta)$ such that for all $i$

$$d(B(C_i, z, t, \omega, \eta), N) = \nu_\infty(C_i), \quad d(B(\tilde{C}_i, z, t, \omega, \eta), N) = \nu_\infty(\tilde{C}_i),$$
and
\[ d(B(V, z, t, \omega, \eta), N) = \nu_\infty(V) \geq 1 - \epsilon. \]

Then the ergodicity of \( \nu_\infty \) implies that \( \nu_\infty(A'') = 1 \). Let \( A = A' \cap A'' \), then \( \nu_\infty(A) = 1 \).

Let \((z, t, \omega, \eta) \in A'.\) Then \((z, t, \omega, \eta) \in A', \) so there exists \( \mu = \mu_{z, t, \omega, \eta} \) such that
\[ \pi_{\nu, z, t, \omega, \eta, \mu} \in P(\ell \cap [\pi_{m_1} (\hat{F}_\omega) \times \pi_{m_2} (\hat{E}_\eta)]) \]
for some line \( \ell \) with slope \( m'_1 \), and (34) holds for \( \pi_{\nu, z, t, \omega, \eta, \mu} \) and \( z \). Denote \( \nu = \pi_{\nu, z, t, \omega, \eta}. \) By the choice of \( r_0(\epsilon) = \delta(\epsilon), \) as \( k \geq k_1(\epsilon), \) and by (34), the set
\[ A(\nu, z, t, \omega, \eta) \]
has lower density \( \geq 1 - 2\epsilon \) in \( N \).

Since \( d(B(V, z, t, \omega, \eta), N) \geq 1 - \epsilon, \) it follows, by the inclusion-exclusion principle, that the set
\[ B(V, z, t, \omega, \eta) \cap A(\nu, z, t, \omega, \eta) \]
has lower density at least \( 1 - 3 \cdot \epsilon \) in \( N \).

On the other hand, by Theorem 6.1 part (1), the density of \( \bigcup_{i=1}^{N(\bar{k}, \epsilon)} B(C_i, z, t, \omega, \eta) \) in \( N \) is at least \( 1 - \epsilon. \) Notice that the sets \( B(C_i, z, t, \omega, \eta) \) are disjoint. It follows that there exists at least one \( 1 \leq i_0 \leq N(\bar{k}, \epsilon) \) such that\(^1\)
\[ d((B(V, z, t, \omega, \eta) \cap A(\nu, z, t, \omega, \eta)) \cap B(C_{i_0}, z, t, \omega, \eta), B(C_{i_0}, z, t, \omega, \eta)) \geq 1 - 4\epsilon. \]
We thus set \( D = C_{i_0} \) and \( N = A(\nu, z, t, \omega, \eta) \cap B(V, z, t, \omega, \eta) \cap B(C_{i_0}, z, t, \omega, \eta). \)

**Proof of the Proposition 6.2 part (4)**

**Lemma 6.5.** \( \lambda(\{R_k^b(t) : k \in N\}) > 1 - 5\epsilon. \)

**Proof.** Since \( C_{i_0} \subset \hat{C}_{i_0}, \) by the choice of \( A'' \) and Theorem 6.1 part (3), we have
\[ d(B(C_{i_0}, z, t, \omega, \eta), B(\hat{C}_{i_0}, z, t, \omega, \eta)) = d(B(C_{i_0}, z, t, \omega, \eta), N) \cdot d(B(\hat{C}_{i_0}, z, t, \omega, \eta), N)^{-1} \geq 1 - \epsilon. \]
Therefore,
\[ d(N, B(\hat{C}_{i_0}, z, t, \omega, \eta)) \geq d(N, B(C_{i_0}, z, t, \omega, \eta)) \cdot d(B(C_{i_0}, z, t, \omega, \eta), B(\hat{C}_{i_0}, z, t, \omega, \eta)) \]
\[ \geq (1 - 4\epsilon) \cdot (1 - \epsilon). \]
Since \((z, t, \omega, \eta) \in A', \) then \( \{R_k^b(t) : t \in T : k \in B(\hat{C}_{i_0}, z, t, \eta)\} \) is UD. It follows from Lemma 6.3 that
\[ \lambda(\{R_k^b(t) : k \in N\}) \geq (1 - 4\epsilon)(1 - \epsilon) > 1 - 5\epsilon. \]

**Proof of Proposition 6.2 parts (2) and (6)**

\(^1\)If \( S_1, S_2 \subseteq N, \) we define \( d(S_1, S_2) = \liminf_N \frac{S_1 \cap (S_2)\cap [N])}{S_2 \cap [N]).}
Claim 6.6. For \( i = 3, 4 \) let \( \Pi_i(D) = D_i \), and let \( D'_i = \bigcup_{k \in \mathbb{N}} \Pi_i(U^k(z, t, \omega, \eta)) \subset D_i \). Define
\[
\pi_{m_1}(\tilde{F}_{D'_i}) := \bigcup_{\xi \in D'_i} \pi_{m_1}(\tilde{F}_\xi), \quad \text{and} \quad \pi_{m_2}(\tilde{E}_{D'_i}) := \bigcup_{\xi \in D'_i} \pi_{m_2}(\tilde{E}_\xi).
\]
Then, for some constant \( C_2 \) that does not depend on \( k \) or \( \epsilon \),
\[
\frac{\log N_{2^{-k}}(\pi_{m_1}(\tilde{F}_{D'_i}))}{k \log 2} \leq \sum_{i=0}^{n_1-1} \alpha_1([i]) \cdot \frac{\log |\Gamma_1|}{\log m_1} + C_2 \cdot \left( \epsilon + \frac{1}{k} \right) \tag{38}
\]
and
\[
\frac{\log N_{2^{-k}}(\pi_{m_2}(\tilde{E}_{D'_i}))}{k \log 2} \leq \sum_{j=0}^{n_2-1} \alpha_2([j]) \cdot \frac{\log |\Lambda_j|}{\log m_2} + C_2 \cdot \left( \epsilon + \frac{1}{k} \right). \tag{39}
\]

Proof. We first study \( D_3 \). By Theorem 6.1 part (2) and the choice of \( D \),
\[
|\{B \in B_k : D \cap B \neq \emptyset\}| \leq e^{e \cdot k}.
\]
Let \( t_0 \in \mathbb{T} \). Then, by the last displayed equation, the definition of the partitions \( B_k \), and recalling that \( \mathcal{I}_{n_1}^p \) is the \( p \)-level cylinder partition of \([n_1]^{\mathbb{N}}\),
\[
\left| \left\{ I \in \mathcal{I}_{n_1}^{[\tilde{k} \cdot \theta] - C} : D_3 \cap I \neq \emptyset \right\} \right| \leq \left| \bigcup_{t \in \mathcal{C}_k} \left\{ I \in \mathcal{I}_{n_1}^{r_k(t)} : D_3 \cap I \neq \emptyset \right\} \right| \leq e^{e \cdot k}.
\]
By Claim 4.2, we know that there exists some constant \( C \in \mathbb{N} \) such that for every \( t \)
\[
r_k(t) \geq \tilde{k} \cdot \theta - C \geq [\tilde{k} \cdot \theta] - C.
\]
It follows that
\[
N(D_3, \mathcal{I}_{n_1}^{[\tilde{k} \cdot \theta] - C}) := \left| \left\{ I \in \mathcal{I}_{n_1}^{[\tilde{k} \cdot \theta] - C} : D_3 \cap I \neq \emptyset \right\} \right| \leq \left| \left\{ I \in \mathcal{I}_{n_1}^{r_k(t_0)} : D_3 \cap I \neq \emptyset \right\} \right| \leq e^{e \cdot k}.
\]
Since \( \tilde{k} = [k \frac{\log 2}{\log m_2}] + 1 \), we see that, as \( \theta = \frac{\log m_2}{\log m_1} \),
\[
[\tilde{k} \cdot \theta] \geq k \frac{\log 2}{\log m_1} - 1 \geq k \frac{\log 2}{\log m_1} - C \geq k \frac{\log 2}{\log m_1} - C,
\]
since \( C > 1 \).

Now, recall that \( D'_3 = \bigcup_{k \in \mathbb{N}} \Pi_3(U^k(z, t, \omega, \eta)) \subset D_3 \). Then, as \( \tilde{k} \geq k_3(\epsilon) \), by the definition of \( B(V, z, t, \omega, \eta) \) and of the set \( V \) (recall (36)), we have, for every \( \xi \in D'_3 \) and every \( 0 \leq i \leq n - 1 \),
\[
\left| \left\{ 1 \leq j \leq \left[ k \frac{\log 2}{\log m_1} - 2C : \xi_j = i \right] \right\} \right| \leq \left| \left\{ 1 \leq j \leq [\tilde{k} \cdot \theta] : \xi_j = i \right\} \right| \leq [\theta \cdot \tilde{k}] \cdot \alpha_1([i]) + [\theta \cdot \tilde{k}] \cdot \epsilon. \tag{40}
\]
We can now calculate. Define for every \( \xi \in D'_3 \)
\[
\pi_{m_1}(\tilde{F}_\xi) := \left\{ \sum_{i=1}^{[\tilde{k} \cdot \theta] - 2C} \frac{x_i}{m_1} : x_i \in \Gamma_\xi \right\}, \quad \text{and} \quad \pi_{m_2}(\tilde{F}_{D'_3}) := \bigcup_{\xi \in D'_3} \pi_{m_1}(\tilde{F}_\xi).
\]
Notice that $\pi_m(\hat{F}_{D'})$ is actually a finite union of sets of the form $\pi_m(\hat{F}_\xi)$, and that, by considering $m_1$-adic rationals,

$$N\left(\pi_m(\hat{F}_{D'}), m_1^{\left[k \log_2 \frac{2}{m_1}\right]-2C}\right) \leq 3 \cdot N\left(\pi_m(\hat{F}_{D'}), m_1^{\left[k \log_2 \frac{2}{m_1}\right]-2C}\right).$$

(41)

Recall that for a set $A \subseteq \mathbb{R}$, $N(A,p)$ denotes the number of $p$-adic intervals $A$ intersects, and that $N_{2-h}(A) := N(A, 2^h)$. Let $C_1 \in \mathbb{N}$ be such that $2^{C_1} > m_1$. Then,

$$N_{2-k}(\pi_m(\hat{F}_{D'})) \leq 2^{C_1+1} \cdot N\left(\pi_m(\hat{F}_{D'}), m_1^{\left[k \log_2 \frac{2}{m_1}\right]-2C}\right)$$

$$\leq 2^{C_1+1} \cdot 3 \cdot N\left(\pi_m(\hat{F}_{D'}), m_1^{\left[k \log_2 \frac{2}{m_1}\right]-2C}\right) \cdot m_1^{2C}$$

$$\leq 2^{C_1+1} \cdot 3 \cdot N\left(D_3, I_{m_1}^{\left[k \log_2 \frac{2}{m_1}\right]-2C}\right) \cdot \max_{\xi \in D_3} N\left(\pi_m(\hat{F}_\xi), m_1^{\left[k \log_2 \frac{2}{m_1}\right]-2C}\right) \cdot m_1^{2C}$$

$$\leq 2^{C_1+1} \cdot 3 \cdot e^{\frac{m_2}{m_1}} \cdot \prod_{i=0}^{n_1-1} |\Gamma_i| [\theta, k \cdot \alpha_1(|i|) + [\theta, k] \cdot \epsilon] \cdot m_1^{2C},$$

where the first two inequalities follow by adjusting the diameters of the sets we are covering with, the third one follows from (41), the fourth follows since there are at most $N(D_3, I_{m_1}^{\left[k \log_2 \frac{2}{m_1}\right]-C})$ sets in the union $\pi_m(\hat{F}_{D'})$, the fifth follows from the definition of $\pi_m(\hat{F}_\xi)$ and the last inequality follows from (40). Finally, taking log and dividing by $k \log 2$, recalling that $\theta = \frac{\log m_2}{\log m_1}$, yields (38).

For $D_4$, we follow a similar argument. The main difference is that for $D_4$, by the definition of $B_k$,

$$\left|\left\{J \in I_{n_2}^k : D_4 \cap J \neq \emptyset\right\}\right| \leq e^{\frac{m_2}{m_1}},$$

since $\prod_i B_k = I_{n_2}^k$, the cylinder partition of generation $k$. In addition, as $k \geq k_3(\epsilon)$, by the definition of $B(V, z, t, \omega, \eta)$ and of the set $V$ (recall (36)), we have, for every $\zeta \in D_4'$ and every $0 \leq i \leq n_2 - 1$,

$$\left|\left\{1 \leq j \leq k : \zeta_j = i\right\}\right| \leq k \cdot \alpha_2(|i|) + k \cdot \epsilon.$$

Thus, by a similar argument to the one proving (38), we see that

$$N_{2-k}(\pi_m(\hat{F}_{D'})) \leq 2^{C_1+1} \cdot 3 \cdot e^{\frac{m_2}{m_1}} \cdot \prod_{i=0}^{n_2-1} |\Lambda_i| [k \cdot \alpha(|i|) + k \cdot \epsilon] \cdot m_2$$

Taking log and dividing by $k \log 2$, this yields (39).

Recall that we want to bound

$$\frac{1}{k \log 2} \log N_{2-k}(\pi_m(\hat{F}_{D'})),$$

\[\square\]
where \( D'_{3,4} = \bigcup_{k \in \mathcal{N}} \Pi_{3,4} U^k(z, t, \omega, \eta) \) and \( \tilde{D}'_{3,4} = \bigcup_{(\omega, \eta) \in \tilde{D}'_{3,4}} \tilde{F}_\omega \times \tilde{E}_\eta \) and
\[
\pi_{m_1} \times \pi_{m_2}(\tilde{D}'_{3,4}) = \bigcup_{(\omega, \eta) \in \tilde{D}'_{3,4}} \pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta).
\]
It follows by definition that \( \pi_{m_1} \times \pi_{m_2}(\tilde{D}'_{3,4}) \subseteq \pi_{m_1}(\tilde{F}_{D'_4}) \times \pi_{m_2}(\tilde{E}_{D'_4}) \). Thus,
\[
\frac{1}{k} \log N_{2-k} \left( \pi_{m_1} \times \pi_{m_2}(\tilde{D}'_{3,4}) \right) \leq \frac{1}{k} \log N_{2-k} \left( \pi_{m_1}(\tilde{F}_{D'_4}) \right) + \frac{1}{k} \log N_{2-k} \left( \pi_{m_2}(\tilde{E}_{D'_4}) \right),
\]
and the result follows by Claim 6.6.

**Remaining proofs**

The rest of the proofs are similar to those appearing in [27, Proposition 7.1]. In particular, Lemma 6.3 is needed to prove part (5), and the remaining case of part (2) follows by an argument similar to Claim 6.6. In each case, we get a constant \( C \) multiplying \( \epsilon \) and \( \frac{1}{k} \) that does not depend on \( k \) or \( \epsilon \). Taking the maximal such constant, we obtain Proposition 6.2. We omit the rest of the details.

### 6.3. Proof of Theorem 1.7

We begin by relating our assumptions from Theorem 1.7 to those of Proposition 4.7, and hence to the subsequent results.

**Lemma 6.7.** Let \( \ell \subseteq \mathbb{R}^2 \) be a non-principal line of positive slope. Suppose that for some \( (\omega, \eta) \in [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}} \) we have
\[
\gamma := \dim_B \left( \pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta) \right) \bigcap \ell > 0.
\]
Then \( \exists(t_0, \omega_0, \eta_0) \in T \times [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}} \) and a line \( \ell' \) of slope \( m_1^{\mathbb{N}} \) such that
\[
\dim_B \left( \pi_{m_1}(\tilde{F}_{\omega_0}) \times \pi_{m_2}(\tilde{E}_{\eta_0}) \right) \cap \ell' \geq \gamma > 0.
\]
Moreover, if \( \alpha_1 \times \alpha_2 \in P([n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}}) \) is a product of Bernoulli measures, then there is a set \( A \) of full \( \alpha_1 \times \alpha_2 \) measure such that: If \( (\omega, \eta) \in A \), then we may take \((t_0, \omega_0, \eta_0)\) to be generic with respect to the measure preserving system \((T \times [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}}, Z, \lambda \times \alpha_1 \times \alpha_2)\), discussed in Theorem 4.8.

The case of a negative slope can be treated in a completely analogous way. We defer the proof to Section 7.

Now, we want to show that if (42) holds, then \( \gamma + 1 \leq \max_{(i, j) \in [n_1] \times [n_2]} \frac{\log |\Gamma_{i,j}|}{\log m_1} + \frac{\log |\Lambda_{i,j}|}{\log m_2} \).

We also want to show that under the additional assumption that \( (\omega_0, \eta_0) \) are typical with respect to \( t_0 \) and a product of Bernoulli measures \( \alpha_1 \times \alpha_2 \in P([n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}}) \) (in the sense of Theorem 4.8), then
\[
1 + \gamma \leq \sum_{i=0}^{n_1-1} \frac{\log |\Gamma_i|}{\log m_1} \cdot \alpha_1([i]) + \sum_{i=0}^{n_2-1} \frac{\log |\Lambda_i|}{\log m_2} \cdot \alpha_2([i]).
\]
We shall prove the latter assertion. The other assertion follows from a similar argument.

To this end, let \( \epsilon > 0 \), and let \( r_5 = r_9(\epsilon), k_5 = k_9(\epsilon) \) be as in Proposition 6.2. Let \( k \geq k_5 \). Choose a point \( (z, t, \omega, \eta) \in [0,1]^2 \times T \times [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}} \), a measure \( \nu \in P([0,1]^2) \), a set \( D \subseteq [0,1]^2 \times T \times [n_1]^{\mathbb{N}} \times [n_2]^{\mathbb{N}} \) and \( N \subseteq \mathbb{N} \) with the properties stated in Proposition 7.1.
Lemma 6.8. For all \( p \in \mathcal{N} \),
\[
\inf_{y \in \mathbb{R}^2} \frac{1}{k \log 2} \log N_{2^{-k}}(\text{supp}(\nu^{A_{t,p}(z)}) \setminus B(y, r_0)) \geq \gamma - o(1), \quad \text{as } \epsilon \to 0 \text{ and } k \to \infty. \tag{43}
\]

Proof. This is a consequence of property (5) of Proposition 6.2 as proven in [27, equation (7.3)].

Let \( K := \pi_{m_1} \times \pi_{m_2}(\tilde{D}'_{3,4}) \) be a union of product sets, where \( D'_{3,4} = \bigcup_{p \in \mathcal{N}} \Pi_{3,4} U^p(z, t, \omega, \eta) \) and \( \tilde{D}'_{3,4} \subseteq [m_1]^N \times [m_2]^N \) is as in Proposition 6.2 part (6) (and part (2)).

Lemma 6.9. For all \( p \in \mathcal{N} \), \( \nu^{A_{t,p}(z)} \) is supported on a slice \( \ell' \cap K \) of \( K \) of slope \( m_{R^p(t)} \), and \( U_{t,p}(z) \in \text{supp}(\nu^{A_{t,p}(z)}) \cap D_1 \) is a point on this line. In particular,
\[
\inf_{y \in \mathbb{R}^2} \frac{1}{k \log 2} \log N_{2^{-k}}(\ell' \cap K \setminus B(y, r_0)) \geq \gamma - o(1), \quad \text{as } \epsilon \to 0 \text{ and } k \to \infty. \tag{44}
\]

Proof. Since \( \nu \in P((\pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta)) \cap \ell) \) for a line \( \ell \) of slope \( m'_{\ell} \), the measure \( \nu^{A_{t,p}(z)} \) is supported on a slice \( (\pi_{m_1}(\tilde{F}_{\sigma^{t,p}(\omega)}) \times \pi_{m_2}(\tilde{E}_{\sigma^{t,p}(\eta)})) \cap \ell', \) where \( \ell' \) has slope \( m_{R^p(t)} \). In addition, for every \( p \in \mathcal{N} \), \( \Pi_1(U^p(z, t, \omega, \eta)) \in \Pi_1(D) \) and \( \Pi_{3,4}(U^p(z, t, \omega, \eta)) = (\sigma^p_t(\omega), \sigma^p_t(\eta)) \in D'_{3,4} \).

So,
\[
\text{supp}(\nu^{A_{t,p}(z)}) \subseteq \ell' \cap (\pi_{m_1}(\tilde{F}_{\sigma^{t,p}(\omega)}) \times \pi_{m_2}(\tilde{E}_{\sigma^{t,p}(\eta)})) \subseteq \ell' \cap K,
\]
and \( U_{t,p}(z) \in \text{supp}(\nu^{A_{t,p}(z)}) \cap \Pi_1(D) \neq \emptyset \), since \( z \in \text{supp}(\nu) \). The last assertion is thus a consequence Lemma 6.8.

To sum up, for every \( \epsilon > 0 \) and large enough \( k \), we have produced sets \( K \subseteq [0,1]^2, D_1 \subseteq [0,1]^2 \) and \( V = \{ R^p(t) : p \in \mathcal{N} \} \) such that:

1. \( \lambda(V) \geq 1 - o(1) \), as \( \epsilon \to 0 \) and \( k \to \infty \);
2. we have
\[
\frac{1}{k} \log N_{2^{-k}}(D_1) = o(1), \quad \text{as } \epsilon \to 0 \text{ and } k \to \infty
\]
and
\[
\frac{1}{k \log 2} \log N_{2^{-k}}(K) \leq \sum_{i=0}^{n_1-1} \frac{\log |\Gamma_i|}{\log m_1} \cdot \rho([i]) + \sum_{i=0}^{n_2-1} \frac{\log |\Lambda_i|}{\log m_2} \cdot \alpha([i]) + o(1),
\]
as \( \epsilon \to 0 \) and \( k \to \infty \);
3. by Lemma 6.9, for every \( v \in V \), there is a line \( \ell' \) of slope \( m^v \) that intersects \( D_1 \cap K \), such that (44) holds.

Let \( K' = K - D_1 = \{ k - d : k \in K, d \in D_1 \} \). By items (1) and (3) above, for all \( v \in V \) there is a line \( \ell \) of slope \( m^v \) satisfying (44), passing through sufficiently many \( k \)-level dyadic cubes containing the origin. Thus,
\[
\frac{\log N_{2^{-k}}(K')} {k \log 2} \geq 1 + \gamma - o(1), \quad \text{as } \epsilon \to 0 \text{ and } k \to \infty.
\]
Since for any two sets \( A, B \subseteq \mathbb{R}^2 \), there is a constant \( C(2) \) such that
\[
N_{2^{-k}}(A + B) \leq C(2) N_{2^{-k}}(A) \cdot N_{2^{-k}}(B).
\]
We deduce from item (2) above that
\[
\sum_{i=0}^{n_1-1} \frac{\log |\Gamma_i|}{\log m_1} \cdot \alpha_1([i]) + \sum_{i=0}^{n_2-1} \frac{\log |\Lambda_i|}{\log m_2} \cdot \alpha_2([i]) \geq \frac{\log N_{2-k}(K')}{k \log 2} \geq 1 + \gamma - o(1),
\]
as \(\epsilon \to 0\) and \(k \to \infty\).

Taking \(\epsilon \to 0\) and \(k \to \infty\) yields the theorem.

7. Remaining proofs

**Proof of Claim 4.2**

**Proof.** For the first item, if \(\tau = v_t\) for some \(t \in \mathbb{T}\), then the existence of such a constant \(C\) is well known. Otherwise, \(\tau = \lim_p v_{t_p}\) for some \(t_p \in \mathbb{T}\), and let \(k \in \mathbb{N}\). Find \(p_0\) such that for all \(p > p_0\), \(d_0(t_p, \tau) < m_2^{-k}\). This means that the first \(k\) digits of \(\tau\) agree with the first \(k\) digits of \(v_{t_p}\) for all \(p > k\). For any such \(p\)
\[
|r_k(\tau) - k \cdot \theta| \leq |r_k(\tau) - r_k(t_p)| + |r_k(t_p) - k \cdot \theta| \leq 0 + C.
\]
As required.

For the second item, let \(p \in \mathbb{N}\) and let \(C_p(t)\) be the unique element of the partition \(C_p\) that contains \(t\). Since \(t\) is not an endpoint of \(C_p(t)\), \(t\) belongs to the interior of that interval. Since \(t_k\) converges to \(t\), there is some \(k_0\) such that for all \(k > k_0\), \(t_k\) also belongs to the interior of \(C_p(t)\). By noting that this means that \(v_{t_k}\) and \(v_t\) share the same first \(p\) digits, we see that \(d_0(v_{t_k}, v_t) \leq \frac{1}{n^p}\), which is sufficient for the claim.

**Proof of Lemma 5.1**

**Proof.** Part (1) is an immediate corollary of (29). For part (2), notice that \(\pi_{v_t, \omega, \eta, \mu}\) is a measure supported on some slice of \(\pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta) \subseteq [0,1]^2\), of the form \(\ell_{\mu, z}\), for \(z \in \pi_{m_1}(\tilde{F}_\omega) \times \pi_{m_2}(\tilde{E}_\eta)\). Notice that for every \(k \geq 1\) for every atom \(A \in \mathcal{A}^{t,k}\), \(A\) being a rectangle and \(\text{supp}(\pi_{v_t, \omega, \eta, \mu})\) being contained on a line, we have \(|\text{supp}(\pi_{v_t, \omega, \eta, \mu}) \cap \partial A| \leq 2\). As \(\mu\) is not atomic, and the map \(\pi_{v_t, \omega, \eta}\) is finite to 1, \(\pi_{v_t, \omega, \eta, \mu}(\partial A) = 0\). It follows that for \(\mu\) almost every \(x\) and all \(k \geq 1\) we have
\[
\pi_{v_t, \omega, \eta, \mu}(\mu_{[x_0^{k-1}]}(x)) = \pi_{v_t, \omega, \eta, \mu}|_{\mathcal{A}^{t,k}(\pi_{v_t, \omega, \eta}(x))}.
\]
Finally, for all \(t \in \mathbb{T}, \tau \in \mathbb{S}, (\omega, \eta) \in ([n_1] \times [n_2])^\mathbb{N}\) and \(x \in X_{t, \omega}\), we have (by the proof of Lemma 4.6)
\[
U^k(\pi_{\tau, \omega, \eta}(x), t, \omega, \eta) = (U^{t,k}(\pi_{\tau, \omega, \eta}(x)), R^k(t)(\sigma^{t,k}(\omega), \sigma^k(\eta)))
\]
combining the last two calculation yields part (2) of the lemma.

**Proof of Proposition 5.3**

**Proof.** Part (1) is an easy consequence of the fact that, as \(k\) grows to infinity, the maximal diameter of an element in the partition \(B_k\) converges to 0. For the second part, let \(k \in \mathbb{N}\) and
fix an element $A \times C \times I \times J \in \mathcal{B}_k$ where $J \in \mathcal{T}_{n_2}$, $C \in \mathcal{C}_k$, $A \in \mathcal{A}^{t,k}$ and $I \in \mathcal{T}_{m_1}^{t,k}$, for some $t \in C$. Since $\partial(I) = \partial(J) = \emptyset$, we have $\partial(I \times J) = \emptyset$, since

$$\partial(I \times J) = (\partial(I) \times J) \cup (I \times \partial(J)) = \emptyset.$$  

Therefore, by two application of the 'product rule' for boundary of product sets

$$\partial(A \times C \times I \times J) \subseteq \partial(A \times C) \times [n_1]^N \times [n_2]^N = (\partial A \times C \times [n_1]^N \times [n_2]^N) \bigcup (A \times \partial C \times [n_1]^N \times [n_2]^N).$$

Thus, by Boole's inequality,

$$\nu_\infty(\partial(A \times C \times I \times J)) \subseteq \nu_\infty(\partial A \times C \times [n_1]^N \times [n_2]^N) + \nu_\infty(A \times \partial C \times [n_1]^N \times [n_2]^N).$$

Now, the first summoned on the RHS above is 0. This is because $Q_{(\mu_0, \tau_0)}^{(\nu_0, \tau_0)}$ almost every $\mu$ has positive and exact dimension. By Lemma 4.4, it follows that $\pi_{\tau, \omega, \mu}$ is also exact dimensional with positive dimension, and is therefore not atomic. It is also supported on a line, and $\partial A$ is a union of four lines. Thus, $\pi_{\tau, \omega, \mu}(\partial A)$ is 0 almost surely (this is not too different from the proof of Lemma 5.1). The second summoned is trivially 0 since the marginal on the second coordinate of $\nu_\infty$ is $\lambda$, and $\partial C$ consists of two points.

For part (3), we make use of Corollary 4.9. By this corollary, and our assumptions, we have that the joint distribution of $Q$ on $\mathbb{T} \times [n_1]^N \times [n_2]^N$ is $\lambda \times \alpha_1 \times \alpha_2$. So, we may assume that we chose $Q^{(\mu_0, \tau_0), (\nu_0, \eta_0)}$ so that it gives full mass to the set

$$\{ (\mu, t, \tau, \omega, \eta) : \tau = v_1, \ (\omega, \eta) \text{ are } \alpha_1 \times \alpha_2 \text{ generic } \}.$$  

This can be done since $Q$ gives this set full measure. \qed

**Proof of Lemma 6.7**

Define functions $\phi_1, \phi_2 : [0, 1]^2 \to [0, 1]^2$ by

$$\phi_1(x, y) = (T_{m_1}(x), T_{m_2}(y)), \text{ } \phi_2(x, y) = (x, T_{m_2}(y)).$$

Let $u > 0$ denote the slope of $\ell$. Then $\phi_1(\ell)$ is a finite family of lines through $[0, 1]^2$, all with slope $u \cdot \frac{m_2}{m_1}$, and at least one of these lines intersects $\pi_{m_1}(\tilde{F}_\omega(\omega)) \times \pi_{m_2}(\tilde{E}_\eta(\eta))$ in a set of upper box dimension $\geq \gamma$. Similarly, $\phi_2(\ell)$ is a finite family of lines through $[0, 1]^2$, all with slope $u \cdot \frac{m_1}{m_2}$, and at least one of these lines intersects $\pi_{m_1}(\tilde{F}_\omega(\omega)) \times \pi_{m_2}(\tilde{E}_\eta(\eta))$ in a set of upper box dimension $\geq \gamma$.

Since $\log \frac{m_2}{m_1} \notin \mathbb{Q}$, the set $\{ u \cdot \frac{m_2}{m_1} : k \geq n \}$ is dense in $(0, \infty)$. Therefore, there exists $k \geq n$ such that $u \cdot \frac{m_2}{m_1} = m_1^{t_0} \in (1, m_1)$. By $n$ applications of $\phi_1$ to $\ell$ followed by $k - n$ applications $\phi_2$ to the resulting line, we see that there exists a line $\ell'$ of slope $m_1^{t_0}$ that intersects $\pi_{m_1}(\tilde{F}_{\sigma^n(\omega)}) \times \pi_{m_2}(\tilde{E}_{\sigma^k(\eta)})$ in a set of dimension $\geq \gamma$. Denote $\omega_0 = \sigma^n(\omega)$, $\eta_0 = \sigma^k(\eta)$.

Finally, by Theorem 4.8, there is a set $A' \subseteq [n_1]^N \times [n_2]^N$ satisfying $\alpha_1 \times \alpha_2(A') = 1$ such that for every $(\xi, \zeta) \in A'$, $(t_0, \xi, \zeta)$ is generic with respect to the system $(\mathbb{T} \times [n_1]^N \times [n_2]^N, Z, \lambda \times \alpha_1 \times \alpha_2)$. Define $\mathcal{A} = \sigma^{-n} \times \sigma^{-k}(A')$. Then since $\sigma^n \alpha_1 = \alpha_1$ and $\sigma^k \alpha_2 = \alpha_2$, the product $\sigma^k \alpha_1 \times \sigma^n$ preserves the measure $\alpha_1 \times \alpha_2$. Therefore, $\alpha_1 \times \alpha_2(\sigma^{-n} \times \sigma^{-k}(A')) = 1$. Finally, if $(\omega, \eta) \in A$, then $(\omega_0, \eta_0) = (\sigma^n(\omega), \sigma^k(\eta)) \in A'$, so $(\omega_0, \eta_0)$ satisfies that $(t_0, \omega_0, \eta_0)$ is generic. \qed

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Amir Algom
Einstein Institute of Mathematics
Edmond J. Safra Campus The Hebrew University of Jerusalem
Givat Ram
Jerusalem 9190401
Israel

and

Department of Mathematics
The Pennsylvania State University
McAllister Building, University Park
State College, PA 16802
USA

amir.algom@mail.huji.ac.il