Muon decay spin asymmetry

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We compute the spin asymmetry of the muon decay $\mu \to e\bar{\nu}_e\nu_\mu$ through $\mathcal{O}(\alpha^2)$ in perturbative QED. These two-loop corrections are about a factor five (twenty) smaller than the current statistical (systematic) uncertainty of the most precise measurement, performed by the TWIST collaboration. We point out that at $\mathcal{O}(\alpha^2)$ the asymmetry requires a careful definition due to multi-lepton final states and suggest to use familiar QCD techniques to define it in an infra-red safe way. We find that the TWIST measurement of the asymmetry is in excellent agreement with the Standard Model.

I. INTRODUCTION

The muon decay is a paradigm for all charged current flavor transformations. It is a purely leptonic process, $\mu \to e\bar{\nu}_e\nu_\mu$, whose properties can be theoretically predicted with very high precision. Measurements of its lifetime [1] and distributions of the daughter electron [2,4] determine fundamental parameters of the Standard Model and probe its extensions.

Since muons produced in decays of pions are polarized, the angle $\theta$ between the muon spin direction and the daughter electron momentum can be observed. The electron distribution in space is

$$\frac{d\Gamma(\mu^- \to e^-\bar{\nu}_e\nu_\mu)}{d\cos\theta} = \frac{\Gamma + \Gamma_0 A \cos\theta}{2},$$

where $A$ is the asymmetry and $\Gamma_0 = G_F^2 m_\mu^5/(192\pi^3)$ is the muon decay rate in the massless electron limit, and without radiative corrections. $G_F$ is the Fermi constant.

The decay rate of an unpolarized muon decay, given by the $\Gamma$-term in Eq. (1), has been extensively studied both theoretically and experimentally. It was the first decay process of a charged particle to which one-loop [2] and, four decades later, two-loop [6] corrections were computed. Together with the recent measurement [7], these results give the best value of the Fermi constant $G_F$, one of the pillars of precise electroweak studies. Corrections to more differential quantities such as the energy spectrum of electrons, were considered in Refs. [8–11].

The $A$-term in Eq. (1) is less well studied, and is the subject of the present paper. Since $\cos\theta \sim \vec{s} \cdot \vec{p}_e$, it violates parity and, as such, it was central in establishing the structure of the electroweak interaction. Indeed, the two experiments [12, 13] that confirmed Madame Wu’s discovery of the parity non-conservation [14], observed the angular asymmetry of the positron distribution in the antimuon decay.

Before we describe our calculation in detail, we briefly discuss the origin of the simplicity of Eq. (1), neglecting the electron mass and radiative corrections. This simple decay pattern is due to the spin 1/2 of the muon. If we do not observe neutrinos nor the polarization of the daughter electron, two functions of the electron energy fully describe the decay distribution. They are the probability amplitudes $M_\pm$ of the electron emission along the muon spin, and in the opposite direction.

Indeed, the probability amplitude for the emission of the electron in another direction, described by spherical coordinates $\theta$ and $\phi$ with respect to the muon spin, follows from the spin 1/2 rotation,

$$M(\theta, \phi) = \cos\frac{\theta}{2} M_+ + i \sin\frac{\theta}{2} e^{i\phi} M_-.$$  

Since the electron is produced left-handed, the amplitude $M_+$ describes the situation when the electron spin points against the muon spin; thus, the projection of the angular momentum carried by neutrinos on the electron momentum should be minus one, cf. Fig. 1 This is easy to arrange when the neutrinos are flying back-to-back, since the helicities of $\nu_\mu$ and $\bar{\nu}_e$ are opposite, as happens when the neutrinos carry most of the energy and the electron little. For the electrons of the highest energy, $M_+$ vanishes. Conversely, $M_-$ describes the configuration when the electron has the same spin projection as the muon, and the projection of the neutrinos’ angular momentum on the electron direction of motion vanishes. This favors configurations with both neutrinos going in the same direction. Relative to $M_-$, the amplitude $M_+$ contains a factor $\sqrt{2(1-x)}$ (from the Lorentz boost of the polarization vector of the $\nu\bar{\nu}$ pair, treated as a spin-one particle of mass $m_{\nu\bar{\nu}} = \sqrt{1-x} m_\mu$ where $x = 2E_\mu/m_\mu$). $M_+$ therefore vanishes for $x = 1$. As a result, there is a parity violating asymmetry of the electron distribution with respect to the muon spin, favoring the production of high energy electrons in the direction counter to the muon spin. Since the muon decay is suppressed at small electron energies, the asymmetry averaged over electron energies is negative.
Precise studies of angular effects in the muon decay turned out to be challenging for both experiment and theory. Measurements of angular distributions have been performed ever since the pioneering study following the discovery of parity violation. The results are usually presented in terms of the product of the degree of the muon polarization $P$ and $\zeta$, one of the so-called Michel-Kinoshita-Sirlin parameters. It is related to the decay asymmetry by $A = |P\zeta|A_{1\text{b},\text{NLO}}$, where $A_{1\text{b},\text{NLO}}$ is the theoretical prediction for the asymmetry accurate through next-to-leading order in the fine structure constant. A deviation of the measured value of $|P\zeta|$ from unity may be interpreted as the effect of higher-order QED corrections or effects of physics beyond the Standard Model. The current best value is:

$$|P\zeta| = 1.00084^{+0.00029+0.00165}_{-0.00029-0.00063}.$$  

(3)

where the first error is statistical and the second systematic.

Since $\alpha/\pi \sim 2 \cdot 10^{-3}$, where $\alpha$ is the fine structure constant, theoretical prediction for the asymmetry beyond one-loop may be expected to be small. Nevertheless, it is interesting to calculate two-loop effects for a variety of reasons. First, given the definition of $P\zeta$, the two-loop correction to the asymmetry is the first QED effect that may explain a small deviation of $P\zeta$ from 1. Second, there is an intrinsic ambiguity in defining the polarization asymmetry in events with additional electron-positron pairs that appear at NNLO for the first time. This ambiguity leads to the infra-red enhancement of the NNLO QED corrections by $(\alpha/\pi)^2 \ln m^2_{\nu}/m^2_e$, so they might be larger than the naive counting suggests. Careful definition of the asymmetry is needed if we use the massless electron approximation. We discuss this in detail in Section III. Finally, as we explain below, current computational technology makes this formidable calculation possible. This fact is quite impressive since, in general, progress with evaluation QED corrections to the asymmetry was slow. Although the one-loop asymmetry was computed in 1958, its dependence on the electron mass was determined only in 2001, five years after the two-loop effects were obtained for the lifetime.

The remainder of the paper is organized as follows. In the next Section we discuss technical details of the computation. In Section III we discuss multi-electron final states and describe an infra-red safe definition of the asymmetry. In Section IV we provide numerical results for the asymmetry and discuss their significance for the interpretation of measurements.

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1 Note that some QED corrections beyond NLO were included in the description of the electron spectrum in Ref. 16; however, the included corrections do not contribute to the inclusive asymmetry discussed here.
where the complex numbers \( \rho_{\lambda_2;\bar{\lambda}_1,\bar{\lambda}_i} \) describe a rotation between spinor bases. Assuming that \( \vec{n}_i = (\sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i) \), we find

\[
\begin{align*}
\rho_{+,\bar{n}_2,+} &= c_2 s_1 + s_2 s_1 e^{i \varphi_{21}}, \\
\rho_{+,\bar{n}_2,-} &= -c_2 s_1 + s_2 s_1 e^{i \varphi_{21}}, \\
\rho_{-,\bar{n}_2,+} &= -\rho_{+,\bar{n}_2,-}^* e^{i \varphi_{21}}, \\
\rho_{-,\bar{n}_2,-} &= \rho_{+,\bar{n}_2,+}^* e^{i \varphi_{21}},
\end{align*}
\]

(6)

where \( c_2 \equiv \cos \theta_i / 2, s_i \equiv \sin \theta_i / 2 \), and \( \varphi_{21} \equiv \varphi_2 - \varphi_1 \). We use Eq. (5) to translate between the original amplitudes and those where the muon spin quantization axis is suitable for the calculation of the asymmetry.

**III. MULTI-ELECTRON FINAL STATES**

At both the leading and the next-to-leading order in \( \alpha \), muon always decays to a final state with a single electron. But at NNLO, the final state can have an additional electron-positron pair \( \mu^- \rightarrow e^- e^+ \bar{\nu}_e \nu_\mu \). In the approximation when electron mass is neglected, this process is not separately collinear-safe since a collinear \( e^+ e^- \) pair is indistinguishable from a photon.

Moreover, multi-electron final states pose a problem for the computation of the asymmetry: which of the electrons should define the muon quantization axis? The algorithm that selects the quantization axis should be infra-red and collinear safe to ensure the cancellation of singularities in the final result.

Suppose we decide to choose the direction of the hardest electron in the computation of the asymmetry. This choice creates no problem if this electron is produced in the hard \( \mu \rightarrow e\nu\bar{\nu} \) transition. However, if the hardest electron originates from the photon splitting into a collinear \( e^+ e^- \) pair, and if its momentum is picked up as the direction to compute the asymmetry, the counter-term for this amplitude will have the photon momentum as the reference direction for the asymmetry. This counter-term will therefore not cancel with the divergence of the virtual correction where there is just one electron in the final state so that its direction is automatically taken as the quantization axis for the muon spin.

Hence, the issue of the definition of the spin asymmetry is subtle. However, it is similar to infra-red problems encountered in the context of the quark jets forward-backward asymmetry in perturbative QCD [24]. A full solution depends on experimental details, including how electrons and photons are operationally defined. Unfortunately, such details, and especially a discussion of multi-electron final states, are absent in Ref. [16].

To address this issue in a way that is theoretically sound and has a potential to make a contact with experiment, we decided to define the spin asymmetry in terms of infra-red and collinear-safe objects, echoing similar studies of the forward-backward asymmetry in perturbative QCD [25]. To this end, for each muon decay event with an arbitrary final state, we will define a set of electron and photon jets, and then use the hardest among the reconstructed electron jets as the direction to calculate the asymmetry. This is legitimate because Eq. (11) remains valid if we interpret the angle \( \theta \) there as the direction of the electron jet rather than the direction of the electron proper.

The theory of an infra-red safe definition of jet algorithms is well developed in QCD (see e.g. Ref. [26] for a review). However, traditional jet algorithms are flavor-blind which is unacceptable for us since we need well-defined “electron” jets. The required modification was worked out in [27] and we borrow a suitable jet algorithm from that paper. The Durham jet algorithm, that allows tracking the jet “flavor”, is defined by its distance measure that we take to be

\[
y_{ij}^{(F)} = \frac{2(1 - \cos \theta_{ij})}{m^2} \times \left\{ \begin{array}{ll}
\max(E_i^2, E_j^2), & \text{softer of } i, j \text{ is flavored}, \\
\min(E_i^2, E_j^2), & \text{softer of } i, j \text{ is flavorless},
\end{array} \right.
\]

(7)

and by the clustering procedure that we take to be a simple addition of the four-momenta of partons that are re-combined into a jet. When the measure in Eq. (7) is applied to an event in the muon decay, the flavor of a parton is equated to its electric charge and the flavor of a jet is given by the sum of flavors of its constituents.

The procedure is iterative: partons are re-combined into a pre-jet if a distance \( y_{ij}^{(F)} \) between them is smaller than some chosen value \( y \) and the algorithm continues until no further re-combinations are possible.

With this modification, the asymmetry is calculated with respect to the direction of the hardest of the electron jets, if more than one are reconstructed by the jet algorithm, or with respect to the direction of the double-electron jet if both electrons end up in a single jet. In the limit, when the jet resolution parameter vanishes, \( y \to 0 \), the ill-defined no-jet computation of the asymmetry should be recovered. This means that, at order \( \alpha^2 \), the asymmetry contains \( \alpha^2 \ln y \) terms.

To choose the jet resolution parameter \( y \) in a sensible way, we note that a high-energy electron predominantly emits photons in a cone of the size \( \theta \sim m_e / E_e \) around its direction. We imagine that those photons should be treated as part of the electron jet, while photons emitted at larger angles should be distinguishable experimentally. Hence, a physics-motivated choice of the jet resolution parameter is

\[
y \sim \frac{\theta^2 E^2}{m_p^2} \sim \frac{m_e^2}{m_p^2} \sim 2 \cdot 10^{-5}.
\]

(8)
Note that, for this choice of the resolution parameter, the magnitude of $\alpha^2 \ln \frac{1}{y}$ is similar to that of $\alpha^2 \ln \left(m_e^2/m_e^2\right)$ which would have appeared, had the mass of the electron been retained. Although $\ln(10^5) \sim 12 \gg 1$, a resummation of $\alpha^2 \ln \frac{1}{y}$ corrections is not needed because the QED coupling constant is small, so that $\alpha^2 \ln \frac{1}{y} \ll 1$ anyway, and also because the contribution of multi-electron states (the only place where such enhanced corrections appear) to the asymmetry is relatively small.

IV. RESULTS AND DISCUSSION

Choosing the jet resolution parameter $y = 10^{-5}$, we find the asymmetry

$$A = A_0 \left[1 - 2.9451 \bar{a} + 11.2(1) \bar{a}^2\right],$$

where $A_0 = -1/3$ is the leading order asymmetry, $\bar{a} = \bar{\alpha}/\pi$ and $\bar{\alpha}$ is the MS QED coupling renormalized at the scale $\mu = m_\mu$. It is slightly larger than the canonical fine structure constant, $\bar{\alpha} = 1/135.90$. To compare this result with the experimental measurement, we take the ratio of $A$ and $A^{\text{th,NLO}}$, obtained by truncating Eq. (9) at order $\mathcal{O}(\bar{a})$. We find

$$|P_\xi|^\text{th} = 1.00006(1),$$

where the error reflects the sensitivity to $y$, discussed in the next paragraph. This result is approximately one sigma below the measured value of $|P_\xi|$, Eq. (3). The NNLO correction to $P_\xi$ evaluates to $0.6 \cdot 10^{-4}$; it is therefore much smaller than the current experimental precision of $2.9 \cdot 10^{-4}$ (statistical) and $6 \cdot 10^{-4}$ (systematic).

We note that our result for the asymmetry depends on the jet resolution parameter, but this dependence is weak, as we explain below. Indeed, the leading order asymmetry $A_0$ is independent of $y$. The NLO coefficient exhibits a linear dependence on $y$ for small $y$. The dependence of the NNLO correction to the asymmetry on the jet resolution parameter for small $y$ can be approximated by $9.5 - 0.14 \ln y$, as shown in Fig. 2. It follows from that figure that a change in the jet resolution parameter from $10^{-5}$ to $10^{-2}$, changes the second order correction to the asymmetry by 10 percent. As we already noticed, a choice of $y$ is, in some sense, equivalent to understanding a correspondence between the measurement setup and the theoretical calculation reported here. It follows from Fig. 2 that this issue becomes relevant when the precision of the asymmetry measurement becomes comparable to $0.6 \cdot 10^{-5}$, far from the current level.

The acceptance of the TWIST experiment is fairly complicated; electrons are accepted in particular angular and energy regions. It is therefore interesting to understand to what extent the asymmetry depends on the electron energy range selected in the experimental analysis. To find out, we computed four values of the asymmetry that differ by the cut on the minimal energy of the electron jet $E_\text{min} < E_\text{jet}$. We consider four values of the minimal jet energy cut, $E_\text{min} = 10$, 20, 30, 40 MeV. The results are shown in Table II for two values of the jet resolution parameter. It is apparent from that Table that corrections strongly depend on the value of $y$ and on the jet energy interval. This is the consequence of the electron energy not being a collinear-safe observable in $m_e \to 0$ limit, so that NNLO corrections to the electron energy spectrum contain $\ln^2 y$-enhanced terms. As follows from Table II QED corrections are particularly large in case of $y = 10^{-5}$, especially in the limit when the cut on the minimal electron jet energy approaches the kinematic boundary. These results suggest that the asymmetry depends strongly on the electron (or electron jet) energy. This feature will hamper the interpretation of results if improved asymmetry measurements become available, unless the asymmetry can be theoretically computed including the experimental cuts. Since muon decay experiments do not use the concept of lepton jets, fully differential computations with massive electrons may be needed. This remains an interesting challenge for the future.

It is interesting to compare corrections to the asymmetry and corrections to the total rate. Corrections to the rate, computed in Ref. 6, read

$$\Gamma = \Gamma_0 \left[1 - 1.81 \bar{a} + 6.74 \bar{a}^2\right].$$

Figure 2: Dependence of the second order relative correction to the asymmetry on the jet resolution parameter $y$. The solid line is the fit to the function $c_1 + c_2 \ln y$, with $c_1 = 9.5(1)$ and $c_2 = -0.14(1)$.

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2 Note that inclusion of mass $m_e/m_\mu$ corrections at leading and next-to-leading order cannot affect the result for $|P_\xi|$ since its deviation from unity can only start at $\mathcal{O}(\alpha^2)$. Hence, it is consistent to evaluate $|P_\xi|$ in the massless approximation for two first orders in the expansion in the fine structure constant.

3 Analogous logarithmic corrections to the asymmetry at NNLO $\alpha^2 \ln^2 \frac{m_e}{m_\mu}$ were computed in Refs. 6, 10.
Since the decay rate $\Gamma$ is infra-red finite, Eq. (11) is independent of $y$. Comparing Eq. (11) and Eq. (9), we find that corrections to the asymmetry are larger. However, the relative size of subsequent coefficients in the perturbative series is comparable in both cases.

Another source of corrections to the asymmetry and to the total rate are the electron mass effects $O(m_e/m_\mu)$ and it is interesting to compare them with the size of NNLO QED corrections, for both the decay rate and the asymmetry. The correction of order $O(a^2)$ to the total decay width is less important than the effect of the electron mass in the lowest order decay rate. In the asymmetry, the electron mass effect is practically negligible, suppressed by additional two powers of $m_e/m_\mu$ [18]. Thus, for the asymmetry, the radiative corrections are more important than the electron mass effects.

Should future tests of the $V-A$ structure of the muon decay interaction be undertaken, they will not be encumbered with large radiative corrections. However, if the experimental precision reaches the size of the two-loop effects, one will have to carefully match the details of theoretical computations with the experimental setup.

Our approach can also be used to compute the NLO QED corrections to the radiative decay of a polarized muon. Such computation is important for controlling the background to the search of $\mu \to e\gamma$, and the need for such a correction has recently been pointed out [29]. The main modification necessary in that study will be the optimal parametrization of the phase space, convenient for the rather extreme kinematics interesting from the experimental point of view.

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| $E_{\text{min}}$ (MeV) | $a^{(0)}$ | $a^{(1)}$ | $a^{(2)}$ | $\delta_{\text{NLO}}\%$ | $\delta_{\text{NNLO}}\%$ |
|------------------------|---------|---------|---------|----------------|----------------|
| 10                     | 1.01    | 1.01    | 12.6    | 11.7           | -0.9          |
|                        | -4.01   | -3.25   | 12.6    | 11.7           | -0.9          |
| 20                     | 1.05    | 1.05    | 21.1    | 13.8           | -1.3          |
|                        | -5.96   | -3.73   | 21.1    | 13.8           | -1.3          |
| 30                     | 1.05    | 1.05    | 49.3    | 15.9           | -2.1          |
|                        | -9.24   | -4.19   | 49.3    | 15.9           | -2.1          |
| 40                     | 0.87    | 0.87    | 98.4    | 14.1           | -3.2          |
|                        | -11.78  | -3.79   | 98.4    | 14.1           | -3.2          |

Table I: Dependence of the asymmetry $A_{1p,E_{\text{min}}} = -\left(a_0 + a_1 \bar{a} + a_2 \bar{a}^2\right)/3$ on the minimum accepted jet energy $E_{\text{min}}$, for two values of the jet resolution $y = 10^{-5}$ and 10^{-2}. The last two columns show the relative NLO and NNLO effects, in percent. The uncertainty in NNLO coefficients due to numerical integration errors is at a few percent level.

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