ADAPTED θ-SHEME AND ITS ERROR ESTIMATES FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper we propose a new kind of high order numerical scheme for backward stochastic differential equations (BSDEs). Unlike the traditional θ-scheme, we reduce truncation errors by taking θ carefully for every subinterval according to the characteristics of integrands. We give error estimates of this nonlinear scheme and verify the order of scheme through a typical numerical experiment.

Key words. backward stochastic differential equations, Crank-Nicolson scheme

AMS subject classifications. 60H35, 65C20, 60H10

1. Introduction. Let (Ω, ℱ, P) be a probability space, T > 0 a finite time and ℱt0≤t≤T a filtration satisfying the usual conditions. Let (Ω, ℱ, P, ℱt0≤t≤T) be a complete, filtered probability space on which a standard d-dimensional Brownian motion Wt = (W1t, W2t, ···, Wdt)T is defined and ℱ0 contains all the P-null sets of ℱ. Let L2 = L2(0, T) be the set of all ℱt-adapted mean-square-integrable processes.

The general form of backward stochastic differential equation (BSDE) is

(1.1) yt = ξ + \int_t^T f(s, ys, zs)ds - \int_t^T zs dWs, \quad t ∈ [0, T]

where the generator f = f(t, y, z) is a vector function valued in Rm and is ℱt-adapted for each (y, z) and the terminal variable ξ ∈ L2 is ℱT-measurable. A process (yt, zt) : [0, T] × Ω → Rm × Rm×d is called an L2-solution of the BSDE (1.1) if it is ℱt-adapted, square integrable and satisfies the equation.

In 1990, Pardoux and Peng first proved in [7] the existence and uniqueness of the solution of general nonlinear BSDEs and afterwards there has been very active research in this field with many applications. (4)

In this paper we assume that the terminal condition is a function of WT, i.e. ξ = ϕ(WT) and the BSDE (1.1) has a unique solution (yt, zt). It was shown in [8] that the solution (yt, zt) of (1.1) can be represented as

(1.2) yt = u(t, Wt), zt = ∇x u(t, Wt), \quad ∀t ∈ [0, T)

where u(t, x) is the solution of the parabolic partial differential equation

(1.3) \frac{∂u}{∂t} + \frac{1}{2} \sum_{i=1}^d \frac{∂^2 u}{∂x_i^2} + f(t, u, ∇x u) = 0

with the terminal condition u(T, x) = ϕ(x), and ∇x u is the gradient of u with respect to the spatial variable x. The smoothness of u depends on f and ϕ.

Although BSDEs have very important applications in many fields such as mathematical finance and stochastic control, it is well known that it is difficult to obtain analytical solutions except some special cases and there have been many works on numerical methods to get approximate solution. A four step algorithm was proposed

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in [6] to solve a class of more general equations called forward-backward stochastic differential equations (FBSDEs) and in [2] a numerical method based on binomial approach was proposed. Besides, there are very interesting numerical methods for BSDEs ([1, 3, 5, 9]).

In 2006, Zhao proposed a new kind of numerical method for BSDE in [10], it is called \( \theta \)-scheme and was proved to be very effective through many experiments. This \( \theta \)-scheme is simple in form, stable and fairly accurate. In [13] Zhao et al. proved that \( \theta \)-scheme is of first-order when \( \theta \neq \frac{1}{2} \) and of second-order when \( \theta = \frac{1}{2} \) (Crank-Nicolson scheme or C-N scheme) in the case where the generator is independent of \( z \), and in [11] they proved the same result in the case of general generators. So we can not expect the high order convergence with this “traditional” \( \theta \)-scheme which uses a fixed \( \theta \) all the time. Afterwards, there have been works on high order scheme and in [14] a family of multi-step schemes were proposed. But to one’s regret this multi-step scheme is not stable for \( z \) and the convergence order was not satisfactory. In [12] a generalized \( \theta \)-scheme that is more flexible with some additional coefficients was proposed but the convergence order was still about 2 or 3.

In this paper we propose a new kind of high order numerical scheme for BSDEs. This new scheme is similar to the traditional \( \theta \)-scheme and it could achieve high order convergence rate. The main idea is to take \( \theta \) carefully for every subinterval according to the characteristics of the integrand. We call this scheme “adapted \( \theta \)-scheme” because we pick various \( \theta \) at every subinterval according to the integrand. To the best to our knowledge, this kind of scheme has not been proposed before.

We consider the case where the generator \( f \) is independent of \( z \). We assume that \( f \) and \( \varphi \) are all bounded, smooth enough and their derivatives are also bounded as in [13].

The rest of this paper is organized as follows. In Section 2 we explain the idea of the adapted \( \theta \)-scheme through the approximation of the integral of real functions. In Section 3, we propose a new kind of discrete scheme for BSDEs based on the adapted \( \theta \)-scheme. In Section 4, we give error estimates of the new scheme theoretically. In Section 5, we give a numerical experiment for a typical BSDE to demonstrate the high order convergence of our scheme. In Section 6, some conclusions are given.

2. Approximation of integral based on the adapted \( \theta \)-scheme.

2.1. The case where the derivatives of integrand are known. Let \( f(t) : [a, b] \to \mathbb{R} \) be \( q \) times continuously differentiable on \( [a, b] \), \( q + 1 \) times differentiable on \( (a, b) \) and assume \( f, f', \cdots, f^{(q+1)} \) are all bounded. Now we consider the approximation of the integral of \( f(t) \) on \( [a, b] \).

\[
I = \int_a^b f(s)ds
\]

Let \( a = t_0 < t_1 < \cdots < t_N = b \) be an equidistant partition of \( [a, b] \) and \( h = \frac{T}{N} \). We approximate the integral on the subinterval \( [t_n, t_{n+1}] \), \( I_n = \int_{t_n}^{t_{n+1}} f(s)ds \) by

\[
\hat{I}_n(\theta) = [\theta f(t_n) + (1 - \theta)f(t_{n+1})]h
\]
From Taylor expansion we have

$$\Delta I_n(\theta) = \hat{I}_n(\theta) - I_n = \int_{t_n}^{t_{n+1}} [\theta(f(s) - f(t_n)) + (1 - \theta)(f(s) - f(t_{n+1}))] ds$$

$$= \theta \int_{t_n}^{t_{n+1}} [f'(t_n)(s - t_n) + \cdots + f^{(q)}(t_n)\frac{(s - t_n)^q}{q!}] ds$$

$$+ (1 - \theta) \int_{t_n}^{t_{n+1}} [f'(t_{n+1})(s - t_{n+1}) + \cdots + f^{(q)}(t_{n+1})\frac{(s - t_{n+1})^q}{q!}] ds + R_n(\theta)$$

where

$$R_n(\theta) = \int_{t_n}^{t_{n+1}} \left[ \theta f^{(q+1)}(\alpha_1)\frac{(s - t_n)^{q+1}}{(q+1)!} + (1 - \theta) f^{(q+1)}(\alpha_2)\frac{(s - t_{n+1})^{q+1}}{(q+1)!} \right] ds$$

$$= \theta f^{(q+1)}(\alpha_1) \frac{h^{q+2}}{(q+2)!} + (1 - \theta) f^{(q+2)}(\alpha_2) \frac{(-1)^q h^{q+2}}{(q+2)!}$$

for some $\alpha_1, \alpha_2 \in [t_n, t_{n+1}]$ and we deduce

$$\Delta I_n(\theta) - R_n(\theta) =$$

$$= \theta \int_{t_n}^{t_{n+1}} \sum_{k=1}^{q} \frac{f^{(k)}(t_n)(s - t_n)^k}{k!} ds + (1 - \theta) \int_{t_n}^{t_{n+1}} \sum_{k=1}^{q} \frac{f^{(k)}(t_{n+1})(s - t_{n+1})^k}{k!} ds$$

$$= \theta \sum_{k=1}^{q} \frac{[f^{(k)}(t_n) + (-1)^{k+1} f^{(k)}(t_{n+1})] h^{k+1}}{(k+1)!} - \sum_{k=1}^{q} \frac{(-1)^{k+1} f^{(k)}(t_{n+1}) h^{k+1}}{(k+1)!}.$$

So if we take $\theta$ as

$$\theta_n^q = \frac{\sigma_n^q}{\rho_n^q} = \frac{\sum_{k=1}^{q} \frac{(-1)^{k+1} f^{(k)}(t_{n+1}) h^{k+1}}{(k+1)!}}{\sum_{k=1}^{q} \frac{[f^{(k)}(t_n) + (-1)^{k+1} f^{(k)}(t_{n+1})] h^{k+1}}{(k+1)!}}$$

then $\Delta I_n(\theta_n^q) = R_n(\theta_n^q)$. (Note that we assumed that the denominator is not zero.)

The truncation error becomes

$$|\Delta I_n(\theta_n^q)| = |R_n(\theta_n^q)| \leq \frac{h^{q+2}}{(q+2)!} \max\{f^{(q+1)}(\alpha_1), f^{(q+1)}(\alpha_2)\} (2|\theta_n^q| + 1)$$

$$\leq (2|\theta_n^q| + 1) C_{q+1} h^{q+2}$$

where $C_{q+1}$ is a constant which depends only on the bound of the $(q+1)$th derivative of $f$.

**Definition 2.1** (Validity of subinterval in integral approximation).

*For a constant $L_\theta > 0$, the subinterval $[t_n, t_{n+1}](0 \leq n \leq N - 1)$ is said to be valid if

$$\rho_n^q \neq 0, \quad |\theta_n^q| \leq L_\theta$$

where $\rho_n^q, \theta_n^q$ are defined in (2.2)
Now we take \( \theta_n = \theta_n^q \) for valid subintervals and \( \theta_n = \frac{1}{2} \) for invalid ones in the adapted \( \theta \)-scheme. Then the truncation error in valid subintervals satisfies

\[
|\Delta I_n| = |\Delta I_n(\theta_n^q)| \leq (2L_\theta + 1)C_{q+1}h^{q+2}
\]

and in the invalid subintervals it is equal to the C-N scheme and satisfies

\[
|\Delta I_n| \leq C_2h^3.
\]

So if there are \( M \) invalid subintervals, the overall truncation error satisfies

\[
|\Delta I| \leq \sum_{n=0}^{N-1} |\Delta I_n| \leq (N - M)(2L_\theta + 1)C_{q+1}h^{q+2} + MC_2h^3
\]

We call the above method “the \( q \)th order adapted \( \theta \)-scheme” for the approximation of the integral.

\( \theta_n^q \) for \( q = 1, 2, 3 \) are as follows.

- For \( q = 1 \)

\[
\theta_n^1 = \frac{f'(t_{n+1})}{f'(t_n) + f'(t_{n+1})}
\]

- For \( q = 2 \)

\[
\theta_n^2 = \frac{3f'(t_{n+1}) - f''(t_{n+1})h}{3[f'(t_n) + f'(t_{n+1})] + [f''(t_n) - f''(t_{n+1})]h}
\]

- For \( q = 3 \)

\[
\theta_n^3 = \frac{12f'(t_{n+1}) - 4f''(t_{n+1})h + f'''(t_{n+1})h^2}{12[f'(t_n) + f'(t_{n+1})] + 4[f''(t_n) - f''(t_{n+1})]h + [f'''(t_n) + f'''(t_{n+1})]h^2}
\]

Now let us discuss under what conditions the subinterval \([t_n, t_{n+1}]\) is valid. In the case of \( q = 1 \), if \( f'(t_n)f'(t_{n+1}) \geq 0 \) we clearly have \( \theta_n^1 \in [0, 1] \). And if \( f'(t_n)f'(t_{n+1}) < 0 \), there exists \( s \in [t_n, t_{n+1}] \) such that \( f'(s) = 0 \) by intermediate-value theorem, and we deduce that the number of invalid subintervals does not exceed the number of subintervals that have points at which \( f'(t) \) becomes zero.

Likewise in the case of \( q > 1 \), if

\[
\left( \sum_{k=1}^{q} \frac{f^{(k)}(t_n)}{(k+1)!}h^{k+1} \right) \left( \sum_{k=1}^{q} \frac{(-1)^{k+1}f^{(k)}(t_{n+1})}{(k+1)!}h^{k+1} \right) \geq 0
\]

then \( \theta_n^q \in [0, 1] \) and \([t_n, t_{n+1}]\) is valid. From the fact that \( \theta_n^q \to \theta_n^1 \) as \( h \to 0 \), it would be similar to the case of \( q = 1 \) when \( h \) is small enough.

After all, we could say that the subintervals around zero points of \( f'(t) \) are likely to be invalid. As the validity of subintervals depend on the partition, it is difficult to obtain the general relationship between the numbers of valid ones and invalid ones. But if there are a finite number of zero points of \( f'(t) \) in \([a, b]\), the ratio of invalid ones to valid ones would be smaller as we increase the size of the partition.
2.2. The case where the derivatives of integrand are not known. As we shall see later, in the case of BSDEs we do not know the precise derivatives of the integrands. So we discuss about the approximation of \( \theta_n^q \) in (2.2). We approximate the derivatives of \( f(t) \) by the ones of the Lagrange interpolation polynomial. Assume that \( f(t) \) is \( q+1 \) times differentiable and \( f(t_i), i = 0 \cdots q \) are given. If we let

\[
I_i(t) = \frac{\Pi(t)}{(t-t_i)\Pi'(t_i)}, \quad \Pi(t) = \Pi_{i=0}^q(t-t_i)
\]

the Lagrange interpolation polynomial \( L(t) \) can be expressed in the form

\[
L(t) = \sum_{i=0}^q I_i(t)f(t_i)
\]

and the deviation is given by

\[
L(t) - f(t) = \frac{f^{(q+1)}(\xi)\Pi(t)}{(q+1)!}, \quad t_0 \leq \xi \leq t_n.
\]

For \( n \leq N - q - 1 \), we define \( L_n(t) \) as the Lagrange interpolation polynomial based on \( q+1 \) pairs \( (t_{n+k}, f(t_{n+k})): 1 \leq k \leq q + 1 \).

Now we approximate \( f^{(k)}(t_n) \) and \( f^{(k)}(t_{n+1}) \) \( (n \leq N - q - 1) \) in (2.2) as follows.

\[
f^{(k)}(t_n) \approx \bar{f}^{(k)}(t_n) = L_n^{(k)}(t_n)
\]

\[
f^{(k)}(t_{n+1}) \approx \bar{f}^{(k)}(t_{n+1}) = L_n^{(k)}(t_{n+1})
\]

Then \( \bar{f}^{(k)}(t_n), \bar{f}^{(k)}(t_{n+1}) \) can be written as

\[
\bar{f}^{(k)}(t_n) = h^{-k} \sum_{j=1}^{q+1} t_{kj} f(t_{n+j}), \quad \bar{f}^{(k)}(t_{n+1}) = h^{-k} \sum_{j=1}^{q+1} t_{kj}^2 f(t_{n+j})
\]

where \( t_{kj}, t_{kj}^2 (1 \leq k \leq q, 1 \leq j \leq q + 1) \) are coefficients of \( f(t_{n+j}) \) in (2.12) and (2.13), respectively.

From (2.11), we get

\[
\bar{f}^{(k)}(t_n) - f^{(k)}(t_n) = C_1 h^{q-k+1}, \quad \bar{f}^{(k)}(t_{n+1}) - f^{(k)}(t_{n+1}) = C_2 h^{q-k+1}
\]

where \( C_1, C_2 \) are constants that depend only on the bound of \( f^{(q+1)} \).

Now we approximate \( \theta_n^q \) as follows.

\[
\theta_n^q \approx \tilde{\theta}_n^q = \frac{\tilde{\sigma}^q_n}{\rho^q_n} = \frac{\sum_{k=1}^q (-1)^{k+1} \bar{f}^{(k)}(t_{n+1}) h^{k+1}}{\sum_{k=1}^q [\bar{f}^{(k)}(t_n) + (-1)^{k+1} \bar{f}^{(k)}(t_{n+1})] h^{k+1}}
\]

If we let \( r_j = \sum_{k=1}^q (-1)^{k+1} t_{kj}^2, s_j = \sum_{k=1}^q \frac{t_{kj}^2}{(k+1)!} \) we have

\[
\tilde{\theta}_n^q = \frac{\tilde{\sigma}^q_n}{\rho^q_n} = \frac{\sum_{j=1}^{q+1} r_j f(t_{n+j})}{\sum_{j=1}^{q+1} (r_j + s_j) f(t_{n+j})} \quad (0 \leq n \leq N - q - 1)
\]
and from (2.15) one can see that

$$|\tilde{\theta}_n^q - \theta_n^q| \leq C_1 h^{q+2}, \quad |\tilde{\sigma}_n^q - \sigma_n^q| \leq C_2 h^{q+2}$$

where $C_1, C_2$ are constants that depend only on the bound of $f^{(q+1)}$.

So the deviation of $\theta_n^q$ from $\theta_n^q$ is

$$|\tilde{\theta}_n^q - \theta_n^q| = \left| \frac{\tilde{\sigma}_n^q}{\rho_n^q} - \frac{\sigma_n^q}{\rho_n^q} \right| = \left| \frac{\rho_n^q (\tilde{\sigma}_n^q - \sigma_n^q) + \sigma_n^q (\rho_n^q - \tilde{\rho}_n^q)}{\rho_n^q \tilde{\rho}_n^q} \right|$$

$$\leq \left| \frac{\tilde{\sigma}_n^q - \sigma_n^q}{\rho_n^q} \right| + \left| \theta_n^q \frac{\tilde{\rho}_n^q - \rho_n^q}{\rho_n^q} \right| \leq \frac{1}{\rho_n^q} (C_1 h^{q+2} + |\theta_n^q| C_2 h^{q+2})$$

$$\leq \frac{1}{\rho_n^q} (C_1 h^{q+2} + |\theta_n^q - \tilde{\theta}_n^q| C_2 h^{q+2} + |\tilde{\theta}_n^q| C_2 h^{q+2})$$

If we assume that there exist $L_\theta > 0$ and $L_\rho > 0$ such that $|\tilde{\theta}_n^q| \leq L_\theta$ and $|\tilde{\rho}_n^q|^{-1} \leq L_\rho$ for all $n$, we have

$$|\tilde{\theta}_n^q - \theta_n^q| \leq L_\rho (C_1 h^{q+2} + |\theta_n^q - \tilde{\theta}_n^q| C_2 h^{q+2} + L_\theta C_2 h^{q+2})$$

and

$$(1 - L_\rho C_2 h^{q+2}) |\tilde{\theta}_n^q - \theta_n^q| \leq L_\rho (C_1 h^{q+2} + L_\theta C_2 h^{q+2}).$$

Now we can pick $C_0 \in (0, 1)$ such that

$$1 - L_\rho C_2 h^{q+2} \geq C_0$$

by choosing $h$ small enough and for this $C_0$ we deduce

$$|\tilde{\theta}_n^q - \theta_n^q| \leq \frac{L_\rho}{C_0} (C_1 + L_\theta C_2) h^{q+2} \leq C h^{q+2}$$

where $C$ is a constant that depends only on $L_\rho, L_\theta$ and the bound of $f^{(q+1)}$.

**Definition 2.2** (Validity of subinterval in integral approximation using approximate derivatives).

For constants $L_\theta > 0$ and $L_\rho > 0$, the subinterval $[t_n, t_{n+1}]$ with $0 \leq n \leq N - q - 1$ is said to be **valid** if

$$|\tilde{\theta}_n^q| \leq L_\theta, \quad |\tilde{\rho}_n^q|^{-1} \leq L_\rho$$

where $\tilde{\theta}_n^q, \tilde{\rho}_n^q$ are defined in (2.17)

Now as in Subsection 2.1, we take $\theta_n = \tilde{\theta}_n^q$ for valid subintervals and $\theta_n = \frac{1}{2}$ for invalid ones.

Then the truncation error in the valid subinterval $[t_n, t_{n+1}]$ becomes

$$|I_n - \hat{I}_n(\tilde{\theta})| \leq |I_n - \tilde{I}_n(\tilde{\theta})| + |\tilde{I}_n(\tilde{\theta}) - \hat{I}_n(\tilde{\theta})|$$

$$\leq (2L_\theta + 1) C_q h^{q+2} + |f(t) - f(t + h)||\theta_q - \tilde{\theta}_0|h$$

$$\leq (2L_\theta + 1) C_q h^{q+2} + C_1 L_\rho (1 + L_\theta) h^{q+4}$$

$$\leq C h^{q+2}$$
that is
\begin{equation}
|I_n - \hat{I}_n(\tilde{\theta})| \leq C h^{q+2}.
\end{equation}

If we assume that the truncation errors for \( n > N - q - 1 \) do not exceed \( C h^{q+2} \) and the number of invalid subintervals is \( M \), the overall error does not exceed
\begin{equation}
(N - M)C_1 h^{q+2} + MC_2 h^3.
\end{equation}

We give \( \tilde{\theta}_n^q \) for \( q = 1, 2, 3, 4 \) below. (For simplicity we write \( f(t_{n+j}) \) as \( f_j \).)

- For \( q = 1 \)
\begin{equation}
\tilde{\theta}_n^1 = \frac{f_1 - f_2}{2(f_1 - f_2)} = \frac{1}{2}
\end{equation}

- For \( q = 2 \)
\begin{equation}
\tilde{\theta}_n^2 = \frac{11f_1 - 16f_2 + 5f_3}{12(2f_1 - 3f_2 + f_3)}
\end{equation}

- For \( q = 3 \)
\begin{equation}
\tilde{\theta}_n^3 = \frac{31f_1 - 59f_2 + 37f_3 - 9f_4}{24(3f_1 - 6f_2 + 4f_3 - f_4)}
\end{equation}

- For \( q = 4 \)
\begin{equation}
\tilde{\theta}_n^4 = \frac{1181f_1 - 2774f_2 + 2616f_3 - 1274f_4 + 251f_5}{720(4f_1 - 10f_2 + 10f_3 - 5f_4 + f_5)}
\end{equation}

Note that the case of \( q = 1 \) is equivalent to the C-N scheme.

2.3. An Example. We test the efficiency of the adapted \( \theta \)-scheme through the approximation of the integral of \( f(t) = t^3 e^{-(t-\frac{1}{2})^2} \) on \([-3, 3]\). We try C-N scheme and the adapted \( \theta \)-scheme of order 2, 3(using (2.23) and (2.24)) for comparison. We set \( L_\rho = 1e + 8, L_\theta = 1 \) and compare the errors increasing the size of partition from \( 2^7 \) to \( 2^{12} \). As the size of the partition grows \( \rho \) becomes very small, so we take \( L_\rho \) large to reduce the number of invalid subintervals. The size of partitions and the number of invalid subintervals in the experiment are shown in Table 2.1. In the table, we denote by TOT and INV the size of partition and the number of invalid subintervals, respectively. In this example \( f'(t) \) is zero at a point in \( \{-1, 0, \frac{3}{2}\} \) and the invalid subintervals appear around \(-1\) and 0. Note that compared to the size of partitions.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\text{No} & \text{TOT} & \text{INV (}\!q = 2\!) & \text{INV (}\!q = 3\!) \\
\hline
1 & 128 & 1 & 0 \\
2 & 256 & 2 & 1 \\
3 & 512 & 1 & 0 \\
4 & 1024 & 2 & 1 \\
5 & 2048 & 1 & 0 \\
6 & 4096 & 2 & 0 \\
\hline
\end{tabular}
\caption{The size of partitions and the number of invalid subintervals}
\end{table}
being increased, the number of invalid subintervals does not exceed 2. (On the other hand we repeat the same experiment increasing the size of partition from $3^4$ to $3^{10}$ and no invalid subintervals appear.) Figure 2.1 shows the convergence result in log scale. The convergence rate (denoted by CR) is obtained by linear least square fitting. These CR values are consistent with the theoretical result. Especially one can see that the error for 3rd order adapted $\theta$-scheme rises a little at the fourth point and this is because there is an invalid subinterval at Experiment 4. (see Table 2.1) We perform some more tests for various types of integrands and get similar results.

Fig. 2.1. Errors for approximation of integral using the adapted $\theta$-scheme

3. Discrete scheme for BSDEs based on the adapted $\theta$-scheme. In this section, we propose a discrete scheme for BSDE:

\begin{equation}
y_t = \varphi(W_T) + \int_t^T f(s, y_s) ds - \int_t^T z_s dW_s, \quad t \in [0, T]
\end{equation}

For the sake of simplicity, we assume that everything is one-dimensional, i.e. $m = d = 1$, but all discussions can be generalized to the multi-dimensional case easily.

Let $0 = t_0 < \cdots < t_N = T$ be an equidistant partition of the time interval $[0, T]$ and $t_{n+1} - t_n = h = \frac{T}{N}$.

In $[t_n, t_{n+1}]$ the BSDE (3.1) can be written as follows.

\begin{equation}
y_{t_n} = y_{t_{n+1}} + \int_{t_n}^{t_{n+1}} f(s, y_s) ds - \int_{t_n}^{t_{n+1}} z_s dW_s
\end{equation}

Let $\mathcal{F}_r^t \ (t \leq s \leq T)$ be a $\sigma$-field generated by the Brownian motion $\{x + W_r - W_t, t \leq r \leq s\}$ starting from the time-space point $(t, x)$ and $E_{t_n}^{t,x}[X] := E[X | \mathcal{F}_{t_n}^{t,x}], E_{t}^{x}[X] := E[X | \mathcal{F}_{t}^{t,x}]$ as in [11, 13].

Taking $E_{t_n}^{t,x} [\cdot]$ to the both sides of (3.2) leads to

\begin{equation}
y_{t_n} = E_{t_n}^{t,x}[y_{t_{n+1}}] + \int_{t_n}^{t_{n+1}} E_{t_n}^{t,x} [f(s, y_s)] ds
\end{equation}

where the integrand $E_{t_n}^{t,x} [f(s, y_s)]$ is a deterministic function of $s$. 
Now we introduce the variational equation of (3.1) as follows.

\[
\nabla y_t = \varphi_x(W_T) + \int_t^T f_y(s, y_s) \nabla y_s ds - \int_t^T \nabla z_s dW_s
\]

where \(\varphi_x, f_y\) are the partial derivatives of \(\varphi, f\) with respect to \(x, y\) and \(\nabla y_s, \nabla z_s\) are the variations of \(y_s, z_s\) with respect to spatial variable \(x\). (See [13])

From (1.2) we have

\[
z_t = \nabla y_t
\]

and (3.4) can be written as

\[
z_t = \varphi_x(W_T) + \int_t^T f_y(s, y_s) z_s ds - \int_t^T \nabla z_s dW_s
\]

So we have

\[
z_{tn} = z_{tn+1} + \int_{tn}^{t_{n+1}} f_y(s, y_s) z_s ds - \int_{tn}^{t_{n+1}} \nabla z_s dW_s
\]

and taking \(E_{tn}^x[\cdot]\) to the both sides leads to

\[
z_{tn} = E_{tn}^x[z_{tn+1}] + \int_{tn}^{t_{n+1}} E_{tn}^x[f_y(s, y_s) z_s] ds
\]

In [11, 13] the integrals in (3.3) and (3.7) were replaced by the approximation based on \(\theta\)-scheme to get the discrete scheme, and in [14] the integrands were replaced by their interpolation polynomials resulting in multi-step scheme.

The traditional \(\theta\)-scheme is based on

\[
\int_{tn}^{t_{n+1}} E_{tn}^x[f_y(s, y_s) z_s] ds = h(\theta f(t_n, y_{tn}) + (1 - \theta) E_{tn}^x[f(t_{n+1}, y_{tn+1})]) + R_y^n
\]

(3.8)

\[
\int_{tn}^{t_{n+1}} E_{tn}^x[f_y(s, y_s) z_s] ds = h(\theta f_y(t_n, y_{tn}) z_{tn} + (1 - \theta) E_{tn}^x[f_y(t_{n+1}, y_{tn+1}) z_{tn+1}]) + R_z^n
\]

(3.9)

where \(\theta\) is a global constant and we have the reference equation as follows.

\[
y_{tn} = E_{tn}^x[y_{tn+1}] + h(\theta f(t_n, y_{tn}) + (1 - \theta) E_{tn}^x[f(t_{n+1}, y_{tn+1})]) + R_y^n
\]

\[
z_{tn} = E_{tn}^x[z_{tn+1}] + h(\theta f_y(t_n, y_{tn}) z_{tn} + (1 - \theta) E_{tn}^x[f_y(t_{n+1}, y_{tn+1}) z_{tn+1}]) + R_z^n
\]

(3.10)

\[(0 \leq n \leq N - 1).\]

The discrete scheme based on (3.10) is

\[
\begin{align*}
y^n & = E_{tn}^x[y^{n+1}] + h(\theta f(t_n, y^n) + (1 - \theta) E_{tn}^x[f(t_{n+1}, y^{n+1})]) + R_y^n \\
z^n & = E_{tn}^x[z^{n+1}] + h(\theta f_y(t_n, y^n) z^n + (1 - \theta) E_{tn}^x[f_y(t_{n+1}, y^{n+1}) z^{n+1}]) + R_z^n \\
y^N & = \varphi_x(W_T) \\
z^N & = \frac{d}{dx}(W_T)
\end{align*}
\]

(3.11)
and it is proved that this scheme achieves the best convergence rate, 2, when \( \theta = \frac{1}{2} \).

Here we approximate the integrals in (3.8) and (3.9) using the \( q \)th order adapted \( \theta \)-scheme which needs \( q + 1 \) times differentiability of the integrands and the following lemma guarantees this. ([13])

**Lemma 3.1.** Let \( \Delta^2_t W_s = x + W_s - W_t \) and \( g(s, x), v(s, x), w(s, x) \) be certain functions and \( G(s), \hat{G}(s) \) be as follows.

\[
G(s) = E^x_t [g(s, v(s, \Delta^2_t W_s))], \quad \hat{G}(s) = E^x_t [g(s, v(s, \Delta^2_t W_s))w(s, \Delta^2_t W_s)]
\]

If there exists a positive integer \( m \) such that for all \( \beta_1, \beta_2 \) satisfying \( 0 \leq \beta_1 \leq m+1, 0 \leq \beta_2 \leq 2m + 1, \beta_1 + \beta_2 \leq 2m + 1 \) the derivatives

\[
\frac{\partial^{\beta_1+\beta_2} g(s, x)}{\partial^2_t s \partial^2 x}, \quad \frac{\partial^{\beta_1+\beta_2} v(s, x)}{\partial^2_t s \partial^2 x}, \quad \frac{\partial^{\beta_1+\beta_2} w(s, x)}{\partial^2_t s \partial^2 x}
\]

are continuous and bounded, then \( G(s), \hat{G}(s) \) are \( m \) times continuously differentiable and the derivatives are also bounded.

**Proof.** Applying Ito’s formula to \( g(s, v(s, \Delta^2_t W_s)) \) and \( g(s, v(s, \Delta^2_t W_s)) \cdot w(s, \Delta^2_t W_s) \) repeatedly, the proof is straightforward. \( \square \)

So under the assumption that the parameters of (3.1) are smooth enough, the integrands \( E^x_t [f(s, y_s)] \) and \( E^x_t [f_y(s, y_s)z_s] \) are also smooth enough and we can apply the adapted \( \theta \)-scheme of a proper order.

Based on the \( q \)th order adapted \( \theta \)-scheme we have the reference equations as follows.

\[
y_{tn} = E^{x}_{tn}[y_{tn+1}] + h \left( \tilde{\theta}^y_n(x) f(t_n, y_{tn}) + (1 - \tilde{\theta}^y_n(x)) E^{x}_{tn}[f(t_{n+1}, y_{tn+1})] \right) + R^y_n
\]

\[
z_{tn} = E^{x}_{tn}[z_{tn+1}] + h \left( \tilde{\theta}^z_n(x) f_y(t_n, y_{tn}) z_{tn} + (1 - \tilde{\theta}^z_n(x)) E^{x}_{tn}[f_y(t_{n+1}, y_{tn+1}) z_{tn+1}] \right) + R^z_n
\]

(0 \leq n \leq N - 1)

where

\[
R^y_n = \int_{t_n}^{t_{n+1}} E^{x}_{tn}[f(s, y_s)]ds - h \left( \tilde{\theta}^y_n(x) f(t_n, y_{tn}) + (1 - \tilde{\theta}^y_n(x)) E^{x}_{tn}[f(t_{n+1}, y_{tn+1})] \right)
\]

\[
R^z_n = \int_{t_n}^{t_{n+1}} E^{x}_{tn}[f_y(s, y_s)z_s]ds - h \left( \tilde{\theta}^z_n(x) f_y(t_n, y_{tn}) z_{tn} + (1 - \tilde{\theta}^z_n(x)) E^{x}_{tn}[f_y(t_{n+1}, y_{tn+1}) z_{tn+1}] \right)
\]

and \( \tilde{\theta}^y_n(x), \tilde{\theta}^z_n(x) \) are defined as follows.
From this reference equation we can get a discrete scheme of (3.1) as follows

\[ \tilde{\theta}^y_n(x) = \frac{\tilde{\sigma}^y_n(x)}{\tilde{\rho}^y_n(x)} = \frac{E^{x}_{t_n} \left[ \sum_{j=1}^{q+1} r_j f(t_{n+j}, y_{n+j}) \right]}{E^{x}_{t_n} \left[ \sum_{j=1}^{q+1} (r_j + s_j) f(t_{n+j}, y_{n+j}) \right]} \]

(3.14)

\[ \tilde{\theta}^z_n(x) = \frac{\tilde{\sigma}^z_n(x)}{\tilde{\rho}^z_n(x)} = \frac{E^{x}_{t_n} \left[ \sum_{j=1}^{q+1} r_j f(t_{n+j}, y_{n+j}) z_{t_{n+j}} \right]}{E^{x}_{t_n} \left[ \sum_{j=1}^{q+1} (r_j + s_j) f(t_{n+j}, y_{n+j}) z_{t_{n+j}} \right]} \]

(0 \leq n \leq N - q - 1)

Note that \( \tilde{\theta}^y_n \) depends on the space point \( x \) because the integrand depends on \( x \), that is, even in the same time interval \( \tilde{\theta}^y_n \) and \( \tilde{\theta}^z_n \) differ according to the space point. If any of them is not well defined, we use \( \frac{1}{2} \) as in Subsection 2.1 and Subsection 2.2.

**Definition 3.2 (Validity of subinterval in reference equations).**

For constants \( L_\rho > 0 \) and \( L_\theta > 0 \), the subinterval \([t_n, t_{n+1}) (0 \leq n \leq N - q - 1)\) is said to be **valid** if

\[
|\tilde{\rho}^y_n(x)|^{-1} \leq L_\rho, |\tilde{\theta}^y_n(x)| \leq L_\theta
\]

|\tilde{\rho}^z_n(x)|^{-1} \leq L_\rho, |\tilde{\theta}^z_n(x)| \leq L_\theta
\]

for all \( x \), where \( \tilde{\rho}^y_n, \tilde{\theta}^y_n, \tilde{\rho}^z_n, \tilde{\theta}^z_n \) are defined in (3.14).

From this reference equation we can get a discrete scheme of (3.1) as follows

\[
y^n = E^{x}_{t_n} [y^{n+1}] + h(\tilde{\theta}^y_n f(t_n, y^n) + (1 - \tilde{\theta}^y_n) E^{x}_{t_n} [f(t_{n+1}, y^{n+1})])
\]

(3.15)

\[
z^n = E^{x}_{t_n} [z^{n+1}] + h(\tilde{\theta}^z_n f_y(t_n, y^n) z^n + (1 - \tilde{\theta}^z_n) E^{x}_{t_n} [f_y(t_{n+1}, y^{n+1}) z^{n+1}])
\]

(0 \leq n \leq N - q - 1)

where

\[
\tilde{\theta}^y_n(x) = \frac{\tilde{\sigma}^y_n(x)}{\tilde{\rho}^y_n(x)} = \frac{E^{x}_{t_n} \left[ \sum_{j=1}^{q+1} r_j f(t_{n+j}, y_{n+j}) \right]}{E^{x}_{t_n} \left[ \sum_{j=1}^{q+1} (r_j + s_j) f(t_{n+j}, y_{n+j}) \right]} \]

(3.16)

\[
\tilde{\theta}^z_n(x) = \frac{\tilde{\sigma}^z_n(x)}{\tilde{\rho}^z_n(x)} = \frac{E^{x}_{t_n} \left[ \sum_{j=1}^{q+1} r_j f(t_{n+j}, y_{n+j}) z_{t_{n+j}} \right]}{E^{x}_{t_n} \left[ \sum_{j=1}^{q+1} (r_j + s_j) f(t_{n+j}, y_{n+j}) z_{t_{n+j}} \right]} \]

(0 \leq n \leq N - q - 1)

Here we assume that we have approximations \((y^l, z^l)_{N-q \leq j \leq N}\) using any other numerical methods, for example C-N scheme.

We call the discrete scheme (3.15) the \( \text{“}(q\text{th order}) \text{ adapted } \theta\text{-scheme}” \) for BSDE (3.1).
DEFINITION 3.3 (Validity of subinterval in the discrete scheme).
For constants \( L_\rho > 0 \) and \( L_\theta > 0 \), the subinterval \([t_n, t_{n+1}][0 \leq n \leq N - q - 1]\) is said to be valid if
\[
|\hat{\rho}^\rho_n(x)|^{-1} \leq L_\rho, \quad |\hat{\rho}^\theta_n(x)| \leq L_\theta \\
|\hat{\rho}^\rho_n(x)|^{-1} \leq L_\rho, \quad |\hat{\rho}^\theta_n(x)| \leq L_\theta
\]
for all \( x \), where \( \hat{\rho}^\rho_n, \hat{\rho}^\theta_n, \hat{\rho}^\rho_n, \hat{\rho}^\theta_n \) are defined in (3.16).

We note that in invalid subintervals, we use \( \frac{1}{2} \) for \( \theta \).
In the case of \( q = 2 \), from (2.23), \( \hat{\rho}^\rho_n(x), \hat{\rho}^\theta_n(x) \) can be written as follows.
\[
\hat{\rho}^\rho_n(x) = E^F_n [11f(t_{n+1}, y^{n+1}) - 16f(t_{n+2}, y^{n+2}) + 5f(t_{n+3}, y^{n+3})] \\
12E^F_n [2f(t_{n+2}, y^{n+2}) + f(t_{n+3}, y^{n+3})]
\]
\[
\hat{\rho}^\theta_n(x) = E^F_n [11f_1(t_{n+1}, y^{n+1}) \rho_{n+1} - 16f_2(t_{n+2}, y^{n+2}) z^{n+2} + 5f_3(t_{n+3}, y^{n+3}) z^{n+3}] \\
12E^F_n [2f_1(t_{n+2}, y^{n+2}) z^{n+2} + f_2(t_{n+3}, y^{n+3}) z^{n+3}]
\]

Remark 1. The adapted \( \theta \)-scheme is similar to the multistep scheme proposed in [14] that uses approximations at several points, but our new scheme is nonlinear and is stable for both \( y \) and \( z \) assuming that every subinterval is valid.
In fact, the scheme (3.15) can be written as
\[
\begin{align*}
y^n &= E^\theta_n [y^{n+1}] + hF(t_0, y^n, y^{n+1}, \ldots, y^{n+q+1}) \\
z^n &= E^\theta_n [z^{n+1}] + hG(t_0, y^n, y^{n+1}, \ldots, y^{n+q+1}, z^n, z^{n+1}, \ldots, z^{n+q+1})
\end{align*}
\]
and one can see that under the assumption that every subinterval is valid \( F \) and \( G \) are Lipschitz continuous. Furthermore, if \( f \equiv 0 \), \( F = 0 \) and \( G = 0 \). So this scheme is stable from the theory of numerical ODEs.

4. Error estimates of adapted \( \theta \)-scheme. In this section we give error estimates of the adapted \( \theta \)-scheme proposed in Section 3. First we make some assumptions as follows.

**Assumption 1.** The functions \( \varphi \) and \( f \) in (3.1) are bounded, smooth enough with bounded derivatives.

**Assumption 2.** For certain constants \( L_\rho > 0 \) and \( L_\theta > 0 \), every subinterval \([t_n, t_{n+1}][0 \leq n \leq N - q - 1]\) is valid in the sense of Definition 3.2 and Definition 3.3.

We need the Assumption 2 only for simplicity and in the case where there are invalid intervals we could get results similar to (2.21).

**Lemma 4.1.** Let \( R^y_n, R^z_n \) be truncation errors defined in (3.13). Under the Assumption 1 and 2 we have
\[
|R^y_n| \leq C h^{q+2}, |R^z_n| \leq C h^{q+2}
\]
where \( C \) is a constant that depends only on \( L_\rho, L_\theta, T \) and bounds of \( f, \varphi \) and their derivatives.

**Proof.** It can be easily proved using Taylor expansion as in Section 2. ☐

Let \( y_t, z_t(t \in [0, T]) \) and \( y^n, z^n(0 \leq n \leq N - q - 1) \) be solutions of BSDE (3.1) and the discrete scheme (3.15) respectively. Let \( e^y_n = y_t - y^n, e^z_n = z_t - z^n \) and
and using the assumptions we can deduce that
\[ e^n_{\theta_y} = \hat{\theta}^n_y - \tilde{\theta}^n_y, e^n_{\theta_z} = \hat{\theta}^n_z - \tilde{\theta}^n_z \] for \( n = 0 \cdots N - q - 1 \).

**Lemma 4.2.** Under the Assumption 1 and 2 the following estimate holds true:

\[
|e^n_{\theta_y}| \leq C \sum_{j=1}^{q+1} E_{t_n} [ |e^{n+j}_y| ]
\]

where \( C \) is a constant that depends only on \( L_\rho, L_\theta, T \) and bounds of \( f \) and their derivatives.

**Proof.** From (3.14) and (3.16) we have

\[
\tilde{\rho}_y^n = \sum_{j=1}^{q+1} (r_j + s_j) f(t_{n+j}, y_{n+j}), \quad \tilde{\sigma}_y^n = \sum_{j=1}^{q+1} r_j f(t_{n+j}, y_{n+j})
\]

and

\[
|\tilde{\rho}_y^n - \tilde{\rho}_y^n| \leq L_f E_{t_n} [ \sum_{j=1}^{q+1} |r_j + s_j||e^{n+j}_y| ]
\]

\[
|\tilde{\sigma}_y^n - \tilde{\sigma}_y^n| \leq L_f E_{t_n} [ \sum_{j=1}^{q+1} |r_j||e^{n+j}_y| ]
\]

where \( L_f \) is a Lipschitz constant of \( f \).

So if we set \( \gamma = \max_{j=1\cdots q+1} \{ |r_j| + |s_j| \} \) we have

\[
|\tilde{\rho}_y^n - \tilde{\rho}_y^n| \leq \gamma L_f E_{t_n} [ \sum_{j=1}^{q+1} |e^{n+j}_y| ]
\]

\[
|\tilde{\sigma}_y^n - \tilde{\sigma}_y^n| \leq \gamma L_f E_{t_n} [ \sum_{j=1}^{q+1} |e^{n+j}_y| ]
\]

and using the assumptions we can deduce that

\[
|e^n_{\theta_y}| = |\hat{\theta}^n_y - \tilde{\theta}^n_y| \leq \left| \hat{\theta}^n_y - \tilde{\theta}^n_y \right| + \left| \tilde{\theta}^n_y - \tilde{\theta}^n_y \right|
\]

\[
\leq \left| \tilde{\rho}_y^n \right|^{-1} \left( \left| \tilde{\rho}_y^n - \tilde{\theta}^n_y \right| + \left| \tilde{\theta}^n_y - \tilde{\theta}^n_y \right| \right)
\]

\[
\leq L_\rho \left( \left| \tilde{\rho}_y^n - \tilde{\theta}^n_y \right| + L_\theta \left| \tilde{\theta}^n_y - \tilde{\theta}^n_y \right| \right)
\]

\[
\leq \gamma L_\rho (1 + L_\theta) L_f E_{t_n} [ \sum_{j=1}^{q+1} |e^{n+j}_y| ] \leq CE_{t_n} [ \sum_{j=1}^{q+1} |e^{n+j}_y| ]
\]

which completes the proof. \( \square \)
Theorem 4.3. Suppose that Assumption 1 and 2 hold and that the initial approximation satisfies
\[
\max_{N-q \leq n \leq N} E[|y_n - y^n|] = O(h^{q+1})
\]
Then for sufficiently small time steps \(h\), it holds that
\[
\sup_{0 \leq n \leq N} E[|y_n - y^n|] \leq C h^{q+1}
\]
where \(C\) is a constant that depends only on \(L, L_0, T\) and bounds of \(f, \varphi\) and their derivatives.

Proof. For \(0 \leq n \leq N - q - 1\), from (3.12) and (3.15) we have
\[
e^y_n = E_{t_n}^x [e^{y,n+1}_y] + h \bar{\theta}_n^y (f(t_n, y_{t_n}) - f(t_n, y^n)) + hf(t_n, y_{t_n})(\bar{\theta}_n^y - \bar{\theta}_n^n) + h(1 - \bar{\theta}_n^n)E_{t_n}^x [f(t_{n+1}, y_{t_{n+1}}) - f(t_{n+1}, y^{n+1})] + hE_{t_n}^x [f(t_{n+1}, y_{t_{n+1}})](\bar{\theta}_n^y - \bar{\theta}_n^n) + R_n^y
\]
Let \(L_f\) be the Lipschitz constant of \(f\), then from the assumptions and Lemma 4.2 we deduce
\[
|e^n_y| \leq E_{t_n}^x [|e^{y,n+1}_y|] + hL_0L_f |e_n^y| + h(1 + L_0)L_f E_{t_n}^x [|e^{y,n+1}_y|] + 2hC_0 |e_n^y| + |R_n^y|
\]
\[
\leq E_{t_n}^x [|e^{y,n+1}_y|] + hL_0L_f |e_n^y| + h(1 + L_0)L_f E_{t_n}^x [|e^{y,n+1}_y|] + 2hC_0C_1 \sum_{j=1}^{q+1} E_{t_n}^x [|e^{y,n+j}_y|] + |R_n^y|
\]
\[
\leq E_{t_n}^x [|e^{y,n+1}_y|] + hC_2 \sum_{j=n}^{n+q+1} E_{t_n}^x [|e^j_y|] + C_3 h^{q+2}
\]
where \(C_0\) is a constant that depends only on the bound of \(f\), \(C_1\) is a constant determined from Lemma 4.2, \(C_2 = (1 + L_0)L_f + 2C_0C_1\) and \(C_3\) is a constant determined from Lemma 4.1.
Likewise, we have
\[
|e^{n+1}_y| \leq E_{t_n}^x [|e^{y,n+2}_y|] + hC_2 \sum_{j=n+1}^{n+q+2} E_{t_n}^x [|e^j_y|] + C_3 h^{q+2}
\]
\[
|e^{n+2}_y| \leq E_{t_n}^x [|e^{y,n+3}_y|] + hC_2 \sum_{j=n+2}^{n+q+3} E_{t_n}^x [|e^j_y|] + C_3 h^{q+2}
\]
\:::
\[
|e^{N-q-1}_y| \leq E_{t_n}^x [|e^{y,N-q}_y|] + hC_2 \sum_{j=n}^{N} E_{t_n}^x [|e^j_y|] + C_3 h^{q+2}
\]
Adding up the above inequalities gives
\[
|e^n_y| \leq E_{t_n}^x [|e^{y,N-q}_y|] + hC_2q \sum_{j=n}^{N} E_{t_n}^x [|e^j_y|] + NC_3 h^{q+2}
\]
\[
\leq hC_4 \sum_{j=n}^{N} E_{t_n}^x [|e^j_y|] + C_3 h^{q+2}
\]
where we use the assumption on the initial values and $\frac{T}{N}$. So we have

$$|e^n_y| \leq \frac{hC_4}{1 - hC_4} \sum_{j=n+1}^{N} E_{T_n}^x[|e^j_y|] + \frac{C_5}{1 - hC_4} h^{q+1}$$

Now for sufficiently small time steps $h$ that satisfy $1 - hC_4 > C_6$ for a fixed constant $C_6 \in (0, 1)$, we have

$$|e^n_y| \leq hC \sum_{j=n+1}^{N} E_{T_n}^x[|e^j_y|] + Ch^{q+1}$$

where $C$ is a generic constant that does not depend on the partition.

Let $\zeta_n = hC \sum_{j=n}^{N} E_{T_n}^x[|e^j_y|] + Ch^{q+1}$ then $|e^n_y| \leq \zeta_{n+1}$ and we deduce

$$\zeta_n = hC \sum_{j=n}^{N} E_{T_n}^x[|e^j_y|] + Ch^{q+1} = hC|e^n_y| + \zeta_{n+1} \leq (1 + hC)\zeta_{n+1}$$

$$\leq \cdots \leq (1 + hC)^{N-n} \zeta_{N-q} \leq \left(1 + \frac{Tc}{N}\right)^{N-n} \zeta_{N-q} \leq e^{Tc} \zeta_{N-q}$$

From the assumptions on the initial values we have

$$E\zeta_{N-q} = hC \sum_{j=N-q}^{N} E[|e^j_y|] + Ch^{q+1} \leq Ch^{q+1}$$

which says

$$E|e^n_y| \leq Ch^{q+1}$$

and the proof is completed.

To get error estimates for $z$ we introduce the following lemma (see [5] for details) to show the boundedness of $z_t$.

**Lemma 4.4.** Let $z_t$ be the solution of (3.1). Then under Assumption 1, the following estimate holds true:

$$\sup_{t \in [0, T]} E[z_t] \leq C$$

where $C$ is a constant depending only on $T$, upper bounds of the functions $f$ and $\varphi$, and their derivatives.

Using Theorem 4.3 and Lemma 4.4, we repeat the procedure of the proof of Lemma 4.2 to get the following estimates.

**Lemma 4.5.** Under Assumption 1 and 2, the following estimate holds true:

$$|e^n_{\theta_j}| \leq C \sum_{j=1}^{q+1} E_{T_n}^x[|e^{n+j}_z|] + Ch^{q+1}$$

where $C$ is a constant depending only on $L_\rho, L_\theta$, upper bounds of the function $f$ and its derivatives.
Using Lemma 4.5, we repeat the procedure of proof for Theorem 4.3 to get the following estimates.

**Theorem 4.6.** Suppose that Assumption 1 and 2 hold and that the initial values \( y^n, z^n (N-q \leq n \leq N) \) satisfy

\[
\max_{N-q \leq n \leq N} E[|y_{tn} - y^n|] = O(h^{q+1}), \quad \max_{N-q \leq n \leq N} E[|z_{tn} - z^n|] = O(h^{q+1})
\]

Then for sufficiently small time steps \( h \), it holds that

\[
\sup_{0 \leq n \leq N} E[|z_{tn} - z^n|] \leq Ch^{q+1}.
\]

where \( C \) is a constant that depends only on \( L_\rho, L_\theta, T \) and bounds of \( f, \varphi \) and their derivatives.

5. A Numerical Experiment. In this section, we present a typical numerical experiment to demonstrate the effect of the adapted \( \theta \)-scheme. We approximate the conditional expectation by Gauss-Hermite quadrature which has been proved to be efficient in many works, and get estimates at nongrid space points by Lagrange interpolation. We take uniform partitions in both time and space in the experiment and choose the space step size \( \Delta x \) so that the local error in space to be balanced with the local error in time. When the \( q \)th order adapted \( \theta \)-scheme is used, the local error in time is \( O(h^{q+2}) \) and the local error in space from \( r \)th order polynomial interpolation is \( O((\Delta x)^{r+1}) \). So we set \( \Delta x = h^{\frac{q+2}{2+r}} \). We set the number of the Gauss-Hermite quadrature points to be big enough, 8, so that the error contributed by the use of the Gauss-Hermite quadrature rule does not affect measurements of the convergence rate CR.

The BSDE to approximate is as follows. (See [13].)

\[
\begin{align*}
-dy_t &= (-y_t^3 + 2.5y_t^2 - 1.5y_t)dt - z_t dW_t \\
y_T &= \frac{\exp(W_T+T)}{\exp(W_T+T)+1}
\end{align*}
\]

The analytic solution of this BSDE is given as

\[
\begin{align*}
y_t &= \frac{\exp(W_t+t)}{\exp(W_t+t)+1} \\
z_t &= \frac{\exp(W_t+t)+1}{\exp(W_t+t)}
\end{align*}
\]

and the exact solution at \( t = 0 \) is \( (\frac{1}{2}, \frac{1}{4}) \).

We set \( T = 1 \) and measure the error at \( t = 0 \) increasing the size of partitions from \( 2^3 \) to \( 2^7 \). For comparison we try C-N scheme and the adapted \( \theta \)-scheme of orders 2, 3, 4 and the errors for \( y \) and \( z \) are shown in Table 5.1 and Table 5.2, respectively. (In the table we denote the \( q \)th order adapted \( \theta \)-scheme by ‘Ada \( q \).’) The convergence rate CR is obtained by using linear least squares fitting. The convergence rate of the \( q \)th order adapted \( \theta \)-scheme is about \( q + 1 \) theoretically and the experiment result is consistent with the theoretical ones. Note that both the errors for \( y \) and \( z \) converge at almost the same rate. We set \( L_\theta = 10, L_\rho = 1e + 30 \) in the experiment, we set \( L_\rho \) large enough to reduce the number of invalid subintervals.

6. Conclusions. In this paper we proposed a new kind of high-order numerical scheme for BSDEs, called the adapted \( \theta \)-scheme. Unlike the well-known traditional
θ-scheme, it reduces truncation errors by taking θ adaptively for every subinterval according to the characteristics of the integrand. We gave error estimates of this scheme in the case where the generator $f$ is independent of $z$ and verified the efficiency of our scheme through a typical numerical experiment. This new scheme is similar to the multistep scheme proposed in [14] but this scheme is nonlinear and the stability is guaranteed under some reasonable assumptions. The main con of the adapted θ scheme is that it is not clear for which types of functions the time intervals would be valid. But we think the idea of the adapted θ scheme would still be useful for other numerical fields.

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