Classical Bounded and Almost Periodic Solutions to Quasilinear First-Order Hyperbolic Systems in a Strip

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Abstract

We consider boundary value problems for quasilinear first-order one-dimensional hyperbolic systems in a strip. The boundary conditions are supposed to be of a smoothing type, in the sense that the $L^2$-generalized solutions to the initial-boundary value problems become eventually $C^2$-smooth for any initial $L^2$-data. We investigate small global classical solutions and obtain the existence and uniqueness result under the condition that the evolution family generated by the linearized problem has exponential dichotomy on $\mathbb{R}$. We prove that the dichotomy survives under small perturbations in the leading coefficients of the hyperbolic system. Assuming that the coefficients of the hyperbolic system are almost periodic, we prove that the bounded solution is almost periodic also.

Key words: quasilinear first-order hyperbolic systems, smoothing boundary conditions, exponential dichotomy, robustness, bounded classical solutions, almost periodic solutions

1 Introduction

1.1 Problem setting and main result

We consider first-order quasilinear hyperbolic systems of the following type

$$\partial_t u + A(x, t, u)\partial_x u + B(x, t, u)u = f(x, t), \quad x \in (0, 1),$$

subjected to the (nonlocal) reflection boundary conditions

$$u_j(0, t) = \sum_{k=m+1}^{n} p_{jk}u_k(0, t) + \sum_{k=1}^{m} p_{jk}u_k(1, t), \quad 1 \leq j \leq m,$$

$$u_j(1, t) = \sum_{k=m+1}^{n} p_{jk}u_k(0, t) + \sum_{k=1}^{m} p_{jk}u_k(1, t), \quad m < j \leq n,$$

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where \( u = (u_1, \ldots, u_n) \) and \( f = (f_1, \ldots, f_n) \) are vectors of real-valued functions, \( A = \text{diag}(A_1, \ldots, A_n) \) and \( B = \{B_{jk}\}_{j,k=1}^n \) are matrices of real-valued functions, \( 0 \leq m \leq n \) are fixed integers, and \( p_{jk} \) are real constants.

The purpose of the paper is to establish conditions ensuring existence and uniqueness of small global classical (continuously differentiable) solutions to the problem (1.1)–(1.2) in the strip

\[ \Pi = \{(x, t) \in \mathbb{R}^2 : 0 < x < 1\}. \]

If the coefficients of the hyperbolic system are almost periodic (or periodic) in \( t \), we prove that the bounded solution is almost periodic (respectively, periodic) also.

Denote by \( \| \cdot \| \) the Euclidian norm in \( \mathbb{R}^n \). Given a (closed) domain \( \Omega \subset \mathbb{R}^l \), let \( BC(\Omega; \mathbb{R}^n) \) be the Banach space of all bounded and continuous maps \( u : \Omega \to \mathbb{R}^n \) with the usual sup-norm

\[ \| u \|_{BC(\Omega; \mathbb{R}^n)} = \sup \{ \| u(z) \| : z \in \Omega \}. \]

Similarly one can introduce the space \( BC^k(\Omega; \mathbb{R}^n), k = 1, 2 \), of bounded and \( k \)-times continuously differentiable functions.

Suppose that the coefficients of the system (1.1) satisfy the following conditions.

**\((H1)\)** There exists \( \delta_0 > 0 \) such that

- for all \( j \leq n \) and \( k \leq n \) the coefficients \( A_j(x, t, v) \) and \( B_{jk}(x, t, v) \) have bounded and continuous partial derivatives up to the second order in \( (x, t) \in \Pi \) and in \( v \in \mathbb{R}^n \) with \( \| v \| \leq \delta_0 \),
- there exists \( \Lambda_0 > 0 \) such that
  \[
  \inf \{ A_j(x, t, v) : (x, t) \in [0, 1] \times \mathbb{R}, \| v \| \leq \delta_0, 1 \leq j \leq m \} \geq \Lambda_0,
  \sup \{ A_j(x, t, v) : (x, t) \in [0, 1] \times \mathbb{R}, \| v \| \leq \delta_0, m + 1 \leq j \leq n \} \leq -\Lambda_0,
  \inf \{|A_j(x, t, v) - A_k(x, t, v)| : (x, t) \in [0, 1] \times \mathbb{R}, \| v \| \leq \delta_0, 1 \leq j \neq k \leq n \} \geq \Lambda_0. \quad (1.3)
  \]

**\((H2)\)** For all \( j \leq n \) the functions \( f_j(x, t) \) have bounded and continuous partial derivatives up to the second order in \( (x, t) \in \Pi \).

Along with the nonlinear system (1.1), consider its linearized version at \( u = 0 \), namely

\[
\partial_t u + a(x, t) \partial_x u + b(x, t) u = 0, \quad x \in (0, 1), \quad (1.4)
\]

where \( a(x, t) = A(x, t, 0) \) and \( b(x, t) = B(x, t, 0) \). Supplement the system (1.4) with the boundary conditions (1.2) and the initial conditions

\[ u(x, s) = \varphi(x), \quad x \in [0, 1], \quad (1.5) \]

where \( s \in \mathbb{R} \) is an arbitrary fixed initial time.
We will work with the evolution family generated by the problem (1.4), (1.2), (1.5) and defined on $L^2((0,1);\mathbb{R}^n)$. To introduce the evolution family, let us define the notion of an $L^2$-generalized solution.

Let $C^1_0([0,1];\mathbb{R}^n)$ be the space of continuously differentiable functions on $[0,1]$ with compact support in $(0,1)$. It is evident that the functions from $C^1_0([0,1];\mathbb{R}^n)$ fulfill the zero-order and the first-order compatibility conditions between (1.5) and (1.2). Hence, due to Theorem 3.1 in Section 3.1, if $\varphi \in C^1_0([0,1];\mathbb{R}^n)$, then the problem (1.4), (1.2), (1.5) has a unique classical solution.

**Definition 1.1** Let $\varphi \in L^2((0,1);\mathbb{R}^n)$. A function $u \in C([s,\infty),L^2((0,1);\mathbb{R}^n))$ is called an $L^2$-generalized solution to the problem (1.4), (1.2), (1.5) if for any sequence $\varphi_l \in C^1_0([0,1];\mathbb{R}^n)$ with $\varphi_l \to \varphi$ in $L^2((0,1);\mathbb{R}^n)$ the sequence $u_l \in C^1([0,1] \times [s,\infty);\mathbb{R}^n)$ of classical solutions to (1.4), (1.2), (1.5) with $\varphi$ replaced by $\varphi_l$ fulfills the convergence

$$\|u(\cdot, \theta) - u_l(\cdot, \theta)\|_{L^2((0,1);\mathbb{R}^n)} \to 0 \text{ as } l \to \infty,$$

uniformly in $\theta$ varying in the range $s \leq \theta \leq s + T$, for every $T > 0$.

As usual, by $\mathcal{L}(X,Y)$ we denote the space of linear bounded operators from $X$ into $Y$, and write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$. Note that the assumption (H1) (especially, (1.3)) entails that

$$\inf \{a_j(x,t) : (x,t) \in [0,1] \times \mathbb{R}, 1 \leq j \leq m\} \geq \Lambda_0,$$

$$\sup \{a_j(x,t) : (x,t) \in [0,1] \times \mathbb{R}, m + 1 \leq j \leq n\} \leq -\Lambda_0,$$

$$\inf \{|a_j(x,t) - a_k(x,t)| : (x,t) \in [0,1] \times \mathbb{R}, 1 \leq j \neq k \leq n\} \geq \Lambda_0. \tag{1.6}$$

**Theorem 1.2** [20] Suppose that the coefficients $a$ and $b$ of the system (1.4) have bounded and continuous partial derivatives up to the first order in $(x,t) \in \overline{\Pi}$. If the inequalities (1.6) are fulfilled, then, given $s \in \mathbb{R}$ and $\varphi \in L^2((0,1);\mathbb{R}^n)$, there exists a unique $L^2$-generalized solution $u : \mathbb{R}^2 \to \mathbb{R}^n$ to the problem (1.4), (1.2), (1.5). Moreover, the map

$$\varphi \mapsto U(t,s)\varphi := u(\cdot,t)$$

from $L^2((0,1);\mathbb{R}^n)$ to itself defines a strongly continuous, exponentially bounded evolution family $U(t,s) \in \mathcal{L}(L^2((0,1);\mathbb{R}^n))$, which means that

- $U(t,t) = I$ and $U(t,s) = U(t,r)U(r,s)$ for all $t \geq r \geq s$,
- the map $(t,s) \in \mathbb{R}^2 \mapsto U(t,s)\varphi \in L^2((0,1);\mathbb{R}^n)$ is continuous for all $t \geq s$ and each $\varphi \in L^2((0,1);\mathbb{R}^n)$,
- there exist $K \geq 1$ and $\nu \in \mathbb{R}$ such that

$$\|U(t,s)\|_{\mathcal{L}(L^2((0,1);\mathbb{R}^n))} \leq Ke^{\nu(t-s)} \text{ for all } t \geq s. \tag{1.7}$$
We will consider boundary conditions ensuring that the regularity of solutions to the initial boundary value problem for the linearized system increases in a finite time. In other words, we assume that the system (1.4), (1.2), (1.5) has a smoothing property of the following kind, see [17, 18, 20].

**Definition 1.3** Let \( Y \hookrightarrow Z \) be continuously embedded Banach spaces and, for each \( s \) and \( t \geq s \), \( V(t, s) \in \mathcal{L}(Z) \). The two-parameter family \( \{ V(t, s) \}_{t \geq s} \) is called smoothing from \( Z \) to \( Y \) if there is \( T > 0 \) (smoothing time) such that \( V(t, s) \in \mathcal{L}(Z, Y) \) for all \( t \geq s + T \).

Now we introduce the following condition ensuring a smoothing property of the evolution family generated by the problem (1.4), (1.2), (1.5) (see Theorem 3.4 below).

\[(H3)\] \( p_{i_{1}i_{2}i_{3}} \cdots p_{i_{n}i_{n+1}} = 0 \) for all tuples \( (i_{1}, i_{2}, \ldots, i_{n+1}) \in \{1, \ldots, n\}^{n+1} \).

**Definition 1.4** [2, 14] An evolution family \( \{ U(t, s) \}_{t \geq s} \) on a Banach space \( X \) is said to have an exponential dichotomy on \( \mathbb{R} \) (with an exponent \( \alpha > 0 \) and a bound \( M \geq 1 \)) if there exists a projection-valued function \( P : \mathbb{R} \to \mathcal{L}(X) \) such that the function \( t \mapsto P(t)x \) is continuous and bounded for each \( x \in X \), and for all \( t \geq s \) the following hold:

(i) \( U(t, s)P(s) = P(t)U(t, s) \);

(ii) \( U(t, s)(I - P(s)) \) is invertible as an operator from \( \text{Im}(I - P(s)) \) to \( \text{Im}(I - P(t)) \) with the inverse denoted by \( U(s, t) \);

(iii) \( \| U(t, s)P(s) \|_{\mathcal{L}(X)} \leq Me^{-\alpha(t-s)} \);

(iv) \( \| U(s, t)(I - P(t)) \|_{\mathcal{L}(X)} \leq Me^{-\alpha(t-s)} \).

We are prepared to state the main result of the paper.

**Theorem 1.5** Suppose that the assumptions \((H1)-(H3)\) are fulfilled. Moreover, suppose that the evolution family \( \{ U(t, s) \}_{t \geq s} \) in \( L^{2}((0, 1); \mathbb{R}^{n}) \) generated by the linearized problem (1.4), (1.2) has an exponential dichotomy on \( \mathbb{R} \). Then the following is true:

(i) There exist \( \varepsilon > 0 \) and \( \delta > 0 \) such that for all \( f \in BC^{2}_{\overline{1};\mathbb{R}^{n}} \) with \( \| f \|_{BC^{2}_{\overline{1};\mathbb{R}^{n}}} \leq \varepsilon \) there exists a unique classical solution \( u \) to the problem (1.1), (1.2) such that \( \| u \|_{BC^{1}(\overline{1};\mathbb{R}^{n})} \leq \delta \).

(ii) If the coefficients \( A(x, t, v), B(x, t, v), \) and \( f(x, t) \) are Bohr almost periodic in \( t \) uniformly in \( x \in [0, 1] \) and \( v \) with \( \| v \| \leq \delta_{0} \) (respectively, \( T \)-periodic in \( t \)), then the bounded classical solution to the problem (1.1), (1.2) is Bohr almost periodic in \( t \) (respectively, \( T \)-periodic in \( t \)) as well.

The paper is organized as follows. In Section 2 we describe our approach and discuss the assumptions of our main Theorem 1.5. In Section 3 we obtain a general result about robustness of the exponential dichotomy for the linearized problem under small perturbations of all coefficients in the hyperbolic system. In Section 4 we prove the equivalence of the PDE setting (1.1), (1.2), (1.5) and the corresponding abstract setting. The main result of the paper, stated in Theorem 1.5, is proved in Section 5.
2 Motivation and comments

An overview of known existence-uniqueness results on global regular solutions for first-order one-dimensional hyperbolic systems of quasilinear equations can be found e.g. in [25, 26].

2.1 Our setting

2.1.1 Quasilinear system (1.1)

Quasilinear hyperbolic system (1.1) is written in the canonical form of Riemann invariants. Specifically, the matrix $A$ is diagonal and each equation of the system consists of partial derivatives of a single unknown function only. R. Courant and P. Lax [7] showed that many quasilinear one-dimensional hyperbolic systems can be written in the form of (1.1). Moreover, B. Rozhdestvenskii and N. Yanenko [32] showed that even more general nonlinear systems of the kind $\partial_t u = g(x, t, u, \partial_x u)$ are reducible to (1.1).

2.1.2 Smoothing boundary conditions

Assumption (H3) on the boundary conditions (1.2) is essentially used in the proofs of the robustness Theorem 3.8 and the main Theorem 1.5. It turns out that (H3) has an algebraic characterization being equivalent to

$$\text{tr} \left( \sum_{k=1}^{n} W^k \right) = 0,$$

where $W$ is the $n \times n$-matrix with entries $w_{jk} = |p_{jk}|$. This characterization implies that the assumption (H3) is efficiently checkable. The proof of the equivalence of (H3) and (2.1) and a comprehensive discussion of boundary conditions of this type can be found in [19]. Problems (1.1), (1.2) satisfying Assumption (H3) occur in chemical kinetics [34, 35], population dynamics [28, 29], boundary feedback control theory [1, 6, 13, 29], and inverse problems [33]. A collection of examples from these areas can be found in [20].

2.2 Robustness of exponential dichotomy

Since the nonlinear coefficients $A$ and $B$ are a source of different singularities, global classical solvability requires assumptions preventing shocks and blow-ups. To construct small global regular solutions to the quasilinear system (1.1), (1.2), we assume the smallness of the right-hand sides and a regular behavior of the linearized system. The latter is ensured by the existence of the exponential dichotomy on $\mathbb{R}$ for the evolution family on $L^2((0, 1); \mathbb{R}^n)$. A crucial technical tool in our analysis is the robustness of exponential dichotomy for perturbations of $a$ and $b$. Though robustness issue has extensively been studied in the literature [3, 4, 14, 27, 31], none of these results is applicable to hyperbolic PDEs with unbounded perturbations. D. Henry established a general sufficient condition of the robustness for abstract
evolution equations (see Theorem 3.6 below). Attempts to apply this approach to hyperbolic PDEs meet complications caused by loss of regularity. It turns out that the loss of regularity is unavoidable for perturbations of the coefficients $a_j$ (unbounded perturbations). In [23] these complications are overcome for the boundary conditions of the smoothing type in the space of continuous functions. Here we extend our approach to the $L^2$-setting. In [15] this issue is addressed for the periodic boundary conditions. For more general boundary conditions the robustness issue for hyperbolic PDEs remains unexplored.

2.3 Our approach and the choice of spaces

In the proof of our main Theorem 1.5 we use an iteration procedure to construct classical (continuously differentiable) solutions. Each iteration is a $C^2$-solution to the corresponding linear problem with coefficients depending on the preceding iteration. To solve this linear problem, we put it into an abstract $L^2$-setting, which is provable to be equivalent to the $L^2$-setting in the sense of Definition 1.1. Using an $L^2$-setting instead of the smooth setting enables us to use appropriate results from the abstract theory of evolution semigroups. Due to the robustness Theorem 3.8, the homogeneous version of the linear problem under consideration has an exponential dichotomy on $\mathbb{R}$. Consequently, the nonhomogeneous problem admits a unique solution given by Green’s formula whenever the right-hand side belongs to the domain of the corresponding evolution family. This means that, working in the spaces of continuous functions, the right-hand sides have to satisfy compatibility conditions for all $t \in \mathbb{R}$ (what cannot be fulfilled on each step of our iteration procedure), while working in $L^2$ the compatibility conditions are not needed at all. Finally, we show that the $L^2$-solution is actually in $C^2$, for which we use the smoothing property provided by Theorem 3.4.

2.4 Time-periodic problems

A natural way of proving the existence of time-periodic solutions is provided by local smooth continuation theory and bifurcation theory. In [22] a generalized implicit function theorem is established to prove the existence of time-periodic solutions and in [21] the Lyapunov-Schmidt reduction is adapted to prove the existence of Hopf bifurcations for semilinear hyperbolic problems. We suggest another approach and provide a constructive method of getting periodic solutions, for quasilinear hyperbolic PDEs. Existence of time-periodic solutions to nonlinear hyperbolic PDEs is a challenging problem going back to the classical work by E. Fermi, et al. [11]. Verificating the hypothesis of P. Dedye [8] numerically, they observed the existence of time-periodic solutions in nonlinear hyperbolic problems.

Analysis of time-periodic solutions to hyperbolic PDEs usually meets a complication known as a problem of small divisors. However, if boundary conditions are of smoothing type, then this problem does not appear at all. For the discussion of this point see [19].
2.5 Verification of the assumption that the linearized problem has an exponential dichotomy on $\mathbb{R}$

While it is relatively easy to verify the assumptions (H1), (H2), and (H3) of our main Theorem 1.5, it is not so trivial to verify the remaining assumption that the evolution family $U(t,s)$ has an exponential dichotomy on $\mathbb{R}$, see e.g. [15]. In particular, if the coefficients $a$ and $b$ do not depend on $t$, then for the problem (1.4), (1.2) (where the boundary conditions are considered to be of a smoothing type) falls into the scope of the spectral mapping theorem for eventually differentiable $C_0$-semigroups. This means that the exponential dichotomy is described by spectral properties of the corresponding operator, what is described in detail in the next example.

Example 2.1 Our aim is to show that the assumption about the existence of an exponential dichotomy does not contradict to (H1), (H2), and (H3). Furthermore, we will show that the dichotomy is not necessarily trivial or, in other words, the dichotomous system (1.4), (1.2) is not necessarily exponentially stable.

Consider the following $2 \times 2$-first-order quasilinear hyperbolic system depending on a real parameter $\lambda$:

\begin{align*}
\partial_t u_1 + A_1(x, t, u)\partial_x u_1 &= (\lambda + B_{11}(x, t, u))u_1 + (B_{12}(x, t, u) - 1)u_2 + f_1(x, t) \\
\partial_t u_2 + A_2(x, t, u)\partial_x u_2 &= B_{21}(x, t, u)u_1 + B_{22}(x, t, u)u_2 + f_2(x, t)
\end{align*}

(2.2)

with the boundary conditions

\begin{align*}
u_1(0, t) &= 0, \quad u_1(1, t) = u_2(1, t), \quad (2.3)
\end{align*}

where $A_1(x, t, 0) \equiv 1$, $A_2(x, t, 0) \equiv -1$, and $B_{ij}(x, t, 0) \equiv 0$ with $i, j = 1, 2$. Assume that the functions $A_j$, $B_{jk}$, and $f_j$ fulfill the conditions (H1) and (H2), what causes that the coefficients of (2.2) fulfill (H1) and (H2) as well. It is evident that the boundary conditions (2.3) fulfill the condition (H3).

In order to check the remaining assumption, let us consider the linearized problem

\begin{align*}
\partial_t u_1 + \partial_x u_1 &= \lambda u_1 - u_2, \quad \partial_t u_2 - \partial_x u_2 = 0, \\
u_1(0, t) &= 0, \quad u_1(1, t) = u_2(1, t).
\end{align*}

(2.4) (2.5)

Our aim is to state conditions on $\lambda$ under which the system (2.4)–(2.5) is dichotomous on $\mathbb{R}$. The corresponding eigenvalue problem reads

\begin{align*}
v_1' - \lambda v_1 + v_2 &= \mu v_1, \quad v_2' = \mu v_2, \quad x \in (0, 1), \\
v_1(0) &= 0, \quad v_1(1) - v_2(1) = 0.
\end{align*}

(2.6)

$\mu$ being the spectral parameter. It is easy to verify that there do not exist real eigenvalues to (2.6) and that (2.6) is equivalent to

\begin{align*}
v_1(x) &= \frac{c}{\lambda - 2\mu} \left( e^{\mu x} - e^{(\lambda - \mu)x} \right), \quad v_2(x) = ce^{\mu x}, \\
e^{\lambda - 2\mu} &= 2\mu - \lambda + 1.
\end{align*}
Here \( c = v_2(0) \) is a nonzero complex constant. Setting \( \lambda - 2\mu = \xi + i\eta \) with \( \xi \in \mathbb{R} \) and (without loss of generality) \( \eta > 0 \), we get

\[
\eta = \sqrt{e^{2\xi} - (1 - \xi)^2}
\]

and

\[
\sin \sqrt{e^{2\xi} - (1 - \xi)^2} = -\sqrt{1 - e^{-2\xi}(1 - \xi)^2}.
\] (2.7)

It is easy to see that equation (2.7) has (besides of the solution \( \xi = 0 \)) a countable number of solutions \( 0 < \xi_0 < \xi_1 \ldots \) tending to \( \infty \). Hence, the spectrum of (2.6) consists of countably many geometrically simple eigenvalues

\[
\mu_j^\pm(\lambda) = \frac{1}{2} \left( \lambda - \xi_j \pm i\sqrt{e^{2\xi_j} - (1 - \xi_j)^2} \right).
\]

If \( \lambda \neq \xi_j \) for all \( j \in \mathbb{N} \), then the real parts of all eigenvalues are not equal to zero. By Theorem 3.4, the evolution semigroup on \( L^2((0, 1); \mathbb{R}^n) \) generated by the linearized problem (2.4), (2.5) is eventually differentiable and, hence by [9, p. 281, Corollary 3.12], satisfies the spectral mapping theorem. This entails that the system (2.4), (2.5) is exponentially dichotomous on \( \mathbb{R} \) with an exponent \( \alpha = \alpha(\lambda) \) fulfilling the inequality \( \alpha(\lambda) \leq \min_j |\lambda - \xi_j| \).

Furthermore, if \( \xi_k < \lambda < \xi_{k+1} \), then the system (2.2)–(2.3) has a \( k \)-dimensional unstable submanifold.

### 3 Robustness of exponential dichotomy

#### 3.1 Auxiliary statements

We start with providing existence-uniqueness results for the homogeneous system (1.4) and its non-homogeneous version

\[
\partial_t u + a(x, t)\partial_x u + b(x, t)u = f(x, t), \quad x \in (0, 1),
\] (3.1)

both subjected to the boundary conditions (1.2) and the initial conditions (1.5).

Given \( s \in \mathbb{R} \), denote

\[
\Pi_s = \{(x, t) : 0 < x < 1, s < t < \infty\}.
\]

The existence and uniqueness of classical and piecewise smooth solutions to initial-boundary value hyperbolic problems is proved in [16]. We summarize the needed results in the following theorem.

**Theorem 3.1** Suppose that the coefficients \( a \) and \( b \) of the system (1.4) are continuous and have bounded and continuous first-order partial derivatives in \( x \). Moreover, suppose that the condition (1.6) is fulfilled. Let \( s \in \mathbb{R} \) be arbitrarily fixed and \( \varphi \in C^1([0, 1]; \mathbb{R}^n) \).
(i) If \( f \) is continuous and has bounded and continuous first-order partial derivatives in \( x \), and \( \varphi \) fulfills the zero order compatibility conditions

\[
\varphi_j(0) = \sum_{k=1}^{m} p_{jk} \varphi_k(1) + \sum_{k=m+1}^{n} p_{jk} \varphi_k(0), \quad 1 \leq j \leq m,
\]

\[
\varphi_j(1) = \sum_{k=1}^{m} p_{jk} \varphi_k(1) + \sum_{k=m+1}^{n} p_{jk} \varphi_k(0), \quad m < j \leq n,
\]

then in \( \Pi_s \) there exists a unique continuous solution \( u(x,t) \) to the problem (3.1), (1.2), (1.5) that is a piecewise continuously differentiable function (further referred to as piecewise continuously differentiable solution).

(ii) If \( \varphi \) fulfills the zero order compatibility conditions (3.2) and the first order compatibility conditions

\[
\psi_j(0) = \sum_{k=1}^{m} p_{jk} \psi_k(1) + \sum_{k=m+1}^{n} p_{jk} \psi_k(0), \quad 1 \leq j \leq m,
\]

\[
\psi_j(1) = \sum_{k=1}^{m} p_{jk} \psi_k(1) + \sum_{k=m+1}^{n} p_{jk} \psi_k(0), \quad m < j \leq n,
\]

where

\[
\psi(x) = -(a(x,s) \partial_x + b(x,s)) \varphi(x),
\]

then in \( \Pi_s \) there exists a unique classical solution \( u(x,t) \) to the problem (1.4), (1.2), (1.5). Moreover, there are constants \( K_1 \geq 1 \) and \( \nu_1 > 0 \) not depending on \( s, t, \) and \( \varphi \) such that

\[
\| u(\cdot,t) \|_{C^1([0,1];\mathbb{R}^n)} \leq K_1 e^{\nu_1(t-s)} \| \varphi \|_{C^1([0,1];\mathbb{R}^n)} \quad \text{for} \quad t \geq s.
\]

Similarly to the homogeneous problem (1.4), (1.2), (1.5), we introduce the notion of an \( L^2 \)-generalized solution for the non-homogeneous problem (3.1), (1.2), (1.5).

**Definition 3.2** Let \( \varphi \in L^2((0,1);\mathbb{R}^n) \). A function \( u \in C ([s,\infty),L^2 ((0,1);\mathbb{R}^n)) \) is called an \( L^2 \)-generalized solution to the problem (3.1), (1.2), (1.5) if for any sequence \( \varphi^l \in C^1_0([0,1];\mathbb{R}^n) \) with \( \varphi^l \to \varphi \) in \( L^2((0,1);\mathbb{R}^n) \) the sequence \( u^l \) of piecewise continuously differentiable solutions to (3.1), (1.2), (1.5) with \( \varphi(x) \) replaced by \( \varphi^l(x) \) fulfills the convergence

\[
\| u(\cdot,\theta) - u^l(\cdot,\theta) \|_{L^2((0,1);\mathbb{R}^n)} \to 0 \quad \text{as} \quad l \to \infty,
\]

uniformly in \( \theta \) varying in the range \( s \leq \theta \leq s + T \), for every \( T > 0 \).

We will use the following variant of the existence-uniqueness result stated in [20, Theorem 2.3], for the case of the non-homogeneous system (3.1).
Theorem 3.3 Suppose that the coefficients $a$, $b$, and $f$ of the system (3.1) have bounded and continuous partial derivatives up to the first order in $(x,t) \in \Pi$. Moreover, suppose that the condition (1.6) is fulfilled. Then, given $s \in \mathbb{R}$ and $\varphi \in L^2((0,1);\mathbb{R}^n)$, there exists a unique $L^2$-generalized solution $u : \mathbb{R}^2 \to \mathbb{R}^n$ to the problem (3.1), (1.2), (1.5).

The proof of this theorem repeats the proof of [20, Theorem 2.3].

As it follows from the results of [17, 18, 20], the problems (1.4), (1.2), (1.5) and (3.1), (1.5), (1.2) have a smoothing property, described in the next two theorems.

Theorem 3.4 Let the assumption $\text{(H3)}$ and the conditions of Theorem 1.2 be fulfilled. Then there exists $d > 0$ not depending on $s \in \mathbb{R}$ such that

(i) the evolution family $\{U(t,\tau)\}_{\tau \geq 0}$ on $L^2((0,1);\mathbb{R}^n)$ generated by (1.4), (1.2) is smoothing from $L^2((0,1);\mathbb{R}^n)$ to $C^1([0,1];\mathbb{R}^n)$, with smoothing time equal to $2d$.

(ii) if $a$ and $b$ have bounded and continuous partial derivatives up to the second order in $(x,t) \in \Pi$, then the evolution family $\{U(t,\tau)\}_{\tau \geq 0}$ on $L^2((0,1);\mathbb{R}^n)$ generated by (1.4), (1.2) is smoothing from $L^2((0,1);\mathbb{R}^n)$ to $C^2([0,1];\mathbb{R}^n)$, with smoothing time equal to $3d$.

Theorem 3.5 Let the assumption $\text{(H3)}$ and the conditions of Theorem 1.2 be fulfilled. Let $s \in \mathbb{R}$ be arbitrary fixed. If $f \in BC^1(\Pi;\mathbb{R}^n)$, then the $L^2$-generalized solution $u(x,t)$ to the problem (3.1), (1.2), (1.5) is $C^1$-smooth after time $s + 2d$ and satisfies the estimate

$$\|u(.t)\|_{C^1([0,1];\mathbb{R}^n)} \leq L \left( \|\varphi\|_{L^2((0,1);\mathbb{R}^n)} + \|f\|_{BC^1(\Pi;\mathbb{R}^n)} \right), \quad s + 2d \leq t \leq s + 3d. \quad (3.3)$$

If $f \in BC^2(\Pi;\mathbb{R}^n)$, then $u(x,t)$ is $C^2$-smooth after time $s + 3d$ and satisfies the estimate

$$\|u(.,s+3d)\|_{C^2([0,1];\mathbb{R}^n)} \leq L \left( \|\varphi\|_{L^2((0,1);\mathbb{R}^n)} + \|f\|_{BC^2(\Pi;\mathbb{R}^n)} \right). \quad (3.4)$$

Here the constant $L > 0$ depends on $d$ but does not depend on the initial time $s \in \mathbb{R}$, the initial function $\varphi \in L^2((0,1);\mathbb{R}^n)$, and the coefficient $f$.

One of our main technical tools is the robustness of an exponential dichotomy on $\mathbb{R}$ (Theorem 3.8 below). To prove this result, we will check the following modification of the sufficient condition established by D. Henry in [14, Theorem 7.6.10], see [23, Theorem 2.3].

Theorem 3.6 Let $X$ be a Banach space. Assume that the evolution operator $U(t,s) \in \mathcal{L}(X)$ has an exponential dichotomy on $\mathbb{R}$ with an exponent $\alpha$ and a bound $M$. Assume also that $\|U(t,s)\|_{\mathcal{L}(X)}$ is bounded by a constant over all $s,t$ such that $0 \leq t - s \leq 1$. Then there exist positive $\eta$, $T$, $\alpha_1 \leq \alpha$, and $M_1 \geq M$ such that every perturbed evolution operator $\tilde{U}(t,s) \in \mathcal{L}(X)$ with

$$\|U(t,s) - \tilde{U}(t,s)\|_{\mathcal{L}(X)} < \eta, \quad \text{whenever} \quad t - s = T$$

has an exponential dichotomy on $\mathbb{R}$ with an exponent $\alpha_1$ and a bound $M_1$. 

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In the proof of the robustness Theorem 3.8, by technical reasons instead of the constructive condition (H3) we will use a non-constructive condition stated below as (H3)′. Our nearest goal is to introduce (H3)′ and to show that (H3) entails (H3)′.

To this end, let us introduce a weak formulation of the problem (1.4), (1.2), (1.5) using integration along characteristic curves. For given $j \leq n$, $x \in [0,1]$, and $t \in \mathbb{R}$, the $j$-th characteristic of (1.4) passing through the point $(x,t) \in \Pi_s$ is defined as the solution $\xi \in [0,1] \mapsto \omega_j(\xi) = \omega_j(\xi, x, t) \in \mathbb{R}$ of the initial value problem
\[
\partial_\xi \omega_j(\xi, x, t) = \frac{1}{a_j(\xi, \omega_j(\xi, x, t))}, \quad \omega_j(x, x, t) = t. \tag{3.5}
\]

Due to the assumption (1.6), the characteristic curve $\tau = \omega_j(\xi, x, t)$ reaches the boundary of $\Pi_s$ in two points with distinct ordinates. Let $x_j(x,t)$ denote the abscissa of that point whose ordinate is smaller. Remark that the value of $x_j(x,t)$ does not depend on $x, t$ if $t > s + \frac{1}{\Lambda_0}$.

More precisely, it holds
\[
x_j(x,t) = x = \begin{cases} 0 & \text{if } 1 \leq j \leq m \\ 1 & \text{if } m < j \leq n \end{cases} \quad \text{for } t > \frac{1}{\Lambda_0}. \tag{3.6}
\]

Write
\[
c_j(\xi, x, t) = \exp \int_\xi^x \left[ \frac{b_{jj}}{a_j} \right] (\eta, \omega_j(\eta)) d\eta, \quad d_j(\xi, x, t) = \frac{c_j(\xi, x, t)}{a_j(\xi, \omega_j(\xi))}.
\]

Introduce a linear bounded operator $R : BC(\Pi; \mathbb{R}^n) \mapsto BC(\mathbb{R}; \mathbb{R}^n)$ by
\[
(Ru)_j(t) = \sum_{k=m+1}^n p_{jk}u_k(0,t) + \sum_{k=1}^m p_{jk}u_k(1,t), \quad j \leq n, \tag{3.7}
\]
and an affine bounded operator $Q : BC(\Pi_s; \mathbb{R}^n) \mapsto BC(\Pi_s; \mathbb{R}^n)$ by
\[
(Qu)_j(x,t) = \begin{cases} c_j(x_j(x,t), t)(Ru)_j(\omega_j(x_j(x,t))) & \text{if } x_j(x,t) \notin (0,1), \\ c_j(x_j(x,t), t, \varphi_j(x_j(x,t))) & \text{if } x_j(x,t) \in (0,1), \end{cases} \tag{3.8}
\]
being defined on the affine subspace of $BC(\Pi_s; \mathbb{R}^n)$ of functions satisfying the initial condition (1.5).

A $C^1$-map $u : \Pi_s \to \mathbb{R}^n$ is a classical solution to (3.1), (1.2), (1.5) if and only if it satisfies the following system of integral equations
\[
u_j(x,t) = (Qu)_j(x,t) - \int_{x_j(x,t)}^x d_j(\xi, x, t) \left[ \sum_{k \neq j} b_{jk}u_k - f_j \right](\xi, \omega_j(\xi)) d\xi, \quad j \leq n. \tag{3.9}
\]
A $C$-map $u : \Pi_s \to \mathbb{R}^n$ is called a continuous solution to (3.1), (1.2), (1.5) in $\Pi_s$ if it satisfies (3.9) in $\Pi_s$.

Introduce a linear bounded operator $S$ from $BC(\mathbb{R}; \mathbb{R}^n)$ to $BC(\Pi; \mathbb{R}^n)$ by

$$(Sv)_j(x,t) = c_j(x_j, x, t)v_j(\omega_j(x_j, x, t)), \quad j \leq n,$$

where $x_j$ is given by (3.6). A sufficient condition ensuring a smoothing property of the evolution family generated by (1.4), (1.2), (1.5) (see Theorem 3.4) can now be formulated as follows:

$$(H3)' \ (SR)^nu \equiv 0 \text{ for all } u \in C(\Pi; \mathbb{R}^n).$$

This condition also means that every (continuous) solution to the decoupled system (1.4) ($b_{jk} = 0$ for all $k \neq j$) with the boundary and the initial conditions (1.2) and (1.5) stabilizes to zero in a finite time.

**Lemma 3.7** Condition $(H3)'$ follows from Condition $(H3)$.

**Proof.** First show that the lemma is true for $n = 2$. The condition $(H3)'$ for $n = 2$ can be written as follows:

$$c_j(x_j, x, t)(RSRu)_j(\omega_j(x_j, x, t)) \equiv 0 \quad \text{for all } (x, t) \in \Pi, \ u \in BC(\Pi; \mathbb{R}^2), \ j \leq 2,$$

that is equivalent to

$$(RSRu)(t) \equiv 0 \quad \text{for all } t \in \mathbb{R}, \ u \in BC(\Pi; \mathbb{R}^2). \quad (3.11)$$

We have

$$(RSRu)_j(t) = \sum_{k=1}^{2} p_{jk}(SRu)_k(1 - x_k, t)$$

$$= \sum_{k=1}^{2} p_{jk}c_k(x_k, 1 - x_k, t)(Ru)_k(\omega_k(x_k, 1 - x_k, t))$$

$$= \sum_{k=1}^{2} p_{jk}c_k(x_k, 1 - x_k, t) \sum_{i=1}^{2} p_{ki}u_i(1 - x_i, \omega_k(x_k, 1 - x_k, t)). \quad (3.12)$$

At the same time, the condition $(H3)$ for $n = 2$ reads

$$p_{jk}p_{ki} = 0 \quad \text{for all } j, k, i \leq 2. \quad (3.13)$$

As a consequence, the equations (3.12), (3.13), and (3.11) entail the desired statement for $n = 2$. 

The proof for $n = 3$ uses a similar argument. The analogs of (3.12) and (3.13) read

$$(RSRSRu)_j(t) = \sum_{k=1}^{3} p_{jk} c_k(x_k, 1 - x_k, t) \sum_{i=1}^{3} p_{ki} c_i(x_i, 1 - x_i, \omega_k(x_k, 1 - x_k, t))$$

$$\times \sum_{s=1}^{3} p_{is} u_s(1 - x_s, \omega_i(x_i, 1 - x_i, \omega_k(x_k, 1 - x_k, t)))$$

(3.14)

and

$$p_{jkp_kp_is} = 0 \quad \text{for all } j, k, i, s \leq 3,$$

(3.15)

respectively. On the account of (3.14), one can easily see that (3.15) implies $(\textbf{H3})'$ for $n = 3$.

Proceeding similarly, one can easily obtain the desired statement for an arbitrary fixed $n \in \mathbb{N}$. \hfill $\Box$

### 3.2 Robustness Theorem

We here address the issue of robustness of the exponential dichotomy for the linearized problem (1.4), (1.2), with respect to perturbations of the coefficients $a$ and $b$. To this end, along with the system (1.4) we will consider its perturbed version

$$\partial_t v + (a(x, t) + \tilde{a}(x, t)) \partial_x v + (b(x, t) + \tilde{b}(x, t))v = 0, \quad x \in (0, 1),$$

(3.16)

where $\tilde{a} = \text{diag}(\tilde{a}_1, \ldots, \tilde{a}_n)$ and $\tilde{b} = \{\tilde{b}_{jk}\}_{j,k=1}^{n}$ are matrices of real-valued functions. Suppose that the entries of $\tilde{a}$ and $\tilde{b}$ have bounded and continuous partial derivatives in $x$ and $t$ up to the second order.

Fix $\varepsilon_0$ to be so small that for all $\tilde{a}$ and $\tilde{b}$ with $\|\tilde{a}\|_{BC^1([0,1]; \mathbb{R}^n)} \leq \varepsilon_0$ and $\|\tilde{b}\|_{BC([0,1]; \mathbb{R}^n)} \leq \varepsilon_0$ the coefficients of the system (3.16) fulfill the assumptions of Theorem 1.2 with $a$ and $b$ replaced by $a + \tilde{a}$ and $b + \tilde{b}$, respectively. This means that the perturbed problem (3.16), (1.2) generates the evolution family on $L^2((0, 1); \mathbb{R}^n)$ (see Theorem 1.2), which will be referred to as $\{(\tilde{U}(t, s))_{t \geq s}\}$. We also suppose that the assumption $(\textbf{H3})$ is fulfilled. Then Theorem 3.4(5) guarantees that the families $\{U(t, s)\}_{t \geq s}$ and $\{\tilde{U}(t, s)\}_{t \geq s}$ have a smoothing property in the sense of Definition 1.3.

**Theorem 3.8** Assume that the evolution family $U(t, s)$ has an exponential dichotomy on $\mathbb{R}$ with an exponent $\alpha > 0$ and a bound $M \geq 1$. Then the value of $\varepsilon_0 > 0$ can be chosen so small that for all $\tilde{a}$ and $\tilde{b}$ with $\|\tilde{a}\|_{BC^1([0,1]; \mathbb{R}^n)} \leq \varepsilon_0$ and $\|\tilde{b}\|_{BC([0,1]; \mathbb{R}^n)} \leq \varepsilon_0$ the evolution family $\tilde{U}(t, s)$ has an exponential dichotomy on $\mathbb{R}$ with an exponent $\alpha_1 \leq \alpha$ and a bound $M_1 \geq M$ depending on $\varepsilon_0$ but not on $\tilde{a}$ and $\tilde{b}$.

**Proof.** We check the sufficient conditions for the robustness of exponential dichotomies given in Theorem 3.6. Since the evolution family $U(t, s)$ is exponentially bounded, the uniform boundedness of $\|U(t, s)\|_{L(X)}$ over all $s, t$ such that $0 \leq t - s \leq 1$ follows directly
from the estimate (1.7) and the assumption (H1). It remains to prove that there exists a function \( \beta : [0, 1] \rightarrow \mathbb{R} \) with \( \beta(\varepsilon) \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \), such that for all \( \tilde{a} \) and \( \tilde{b} \) with \( \|\tilde{a}\|_{BC^1(\Pi; \mathbb{M}_n)} \leq \varepsilon \) and \( \|\tilde{b}\|_{BC(\Pi; \mathbb{M}_n)} \leq \varepsilon \) we have

\[
\|(U - \tilde{U})(s + T, s)\|_{L^2((0,1); \mathbb{R}^n))} \leq \beta(\varepsilon)
\]  

(3.17)

for some \( T > 0 \).

Recall that the operator \( Q \) given by (3.8) is defined on the affine subspace of \( BC \left( \Pi_s; \mathbb{R}^n \right) \) of functions satisfying the initial condition (1.5). It is important to note that \( Q \) maps this subspace into itself. Due to (H1), (H3), and Lemma 3.7, one can fix some \( d > 0 \) such that \([Q^n u](x, s + d) = [(SR)^n u](x, s + d) \equiv 0\) for all \( x \in [0, 1] \) and \( s \in \mathbb{R} \). Moreover, the value of \( d \) remains the same, whenever the operators \( S, Q, \) and \( R \) are perturbed by means of replacing \( a \) and \( b \) by \( a + \tilde{a} \) and \( b + \tilde{b} \) such that \( \|\tilde{a}\|_{BC^1(\Pi; \mathbb{M}_n)} \leq \varepsilon \) and \( \|\tilde{b}\|_{BC(\Pi; \mathbb{M}_n)} \leq \varepsilon \). On the account of Theorem 3.4 and Lemma 3.7, we conclude that \( U(s + d, s) \in \mathcal{L}\left(L^2((0, 1); \mathbb{R}^n), C\left([0, 1]; \mathbb{R}^n\right)\right)\) and \( U(s + 2d, s) \in \mathcal{L}\left(L^2\left((0, 1); \mathbb{R}^n\right), C^1\left([0, 1]; \mathbb{R}^n\right)\right)\).

Fix an arbitrary \( s \in \mathbb{R} \). Note that any initial function \( \varphi \in C^1_0\left([0, 1]; \mathbb{R}^n\right) \), for which we have \( \varphi(0) = \varphi(1) = 0 \), satisfies both the zero-order and the first-order compatibility conditions between (1.2) and (1.5). Therefore, for given \( \varphi \in C^1_0\left([0, 1]; \mathbb{R}^n\right) \), by Theorem 3.1 there exist unique classical solutions \( u \) and \( v \) to the problems (1.4), (1.2), (1.5) and (3.16), (1.2), (1.5), respectively.

Due to Theorem 1.1 and the fact that the space \( C^1_0\left([0, 1]; \mathbb{R}^n\right) \) is dense in \( L^2\left((0, 1); \mathbb{R}^n\right) \), the desired estimate (3.17) will be proved if we derive the bound

\[
\|(u - v)(\cdot, s + 3d)\|_{L^2((0,1); \mathbb{R}^n)} \leq \beta(\varepsilon)\|\varphi\|_{L^2((0,1); \mathbb{R}^n)}
\]  

(3.18)

uniformly in \( s \in \mathbb{R}, \varphi \in C^1_0\left([0, 1]; \mathbb{R}^n\right) \), and \( \tilde{a}, \tilde{b} \in BC^1\left(\Pi; \mathbb{M}_n\right) \) with \( \|\tilde{a}\|_{BC^1(\Pi; \mathbb{M}_n)} \leq \varepsilon \) and \( \|\tilde{b}\|_{BC(\Pi; \mathbb{M}_n)} \leq \varepsilon \). In (3.18) the number \( T \) is taken to be \( 3d \) by technical reasons.

We split the derivation of the estimate (3.18) into a sequence of steps.

**Step1.** Derivation of an equation for \( (u - v)|_{\Pi_{s+2d}} \). By the smoothing property, after the time \( t = s + 2d \) the solutions \( u \) and \( v \) are continuously differentiable and, therefore, satisfy pointwise the systems (1.4) and (3.16), respectively. Our starting point is that the difference \( u - v \) fulfills the equation

\[
(\partial_t + a(x, t)\partial_x + b(x, t))(u - v) = \tilde{a}(x, t)\partial_x v + \tilde{b}(x, t)v, \quad (x, t) \in \Pi_{s+2d}
\]

and the boundary conditions

\[
(u_j - v_j)(0, t) = (R(u - v))_j(t), \quad 1 \leq j \leq m, \quad t \geq s,
\]

\[
(u_j - v_j)(1, t) = (R(u - v))_j(t), \quad m < j \leq n, \quad t \geq s,
\]

This implies the operator equality

\[
(u - v)|_{\Pi_{s+2d}} = (SR)(u - v) + D(u - v) + F(\tilde{a}\partial_x v) + F(\tilde{b}v),
\]

(3.19)
where the operators $S$ and $R$ are given by (3.10), (3.7), respectively, and $D, F : BC(\bar{\Pi}_s; \mathbb{R}^n) \to BC(\bar{\Pi}_s; \mathbb{R}^n)$ are linear bounded operators defined by

$$
(Dw)_j(x,t) = -\int_{x_j(x,t)}^x d_j(\xi, x, t) \sum_{k=1}^n b_{jk}(\xi, \omega_j(\xi)) w_k(\xi, \omega_j(\xi)) d\xi, \quad j \leq n,
$$

$$
(Ff)_j(x,t) = \int_{x_j(x,t)}^x d_j(\xi, x, t) f_j(\xi, \omega_j(\xi)) d\xi, \quad j \leq n.
$$

Since $u - v$ occurs in both sides of (3.19), this equation can be iterated. Note that $D$ operates with $u - v$ on a different (shifted) domain. Hence, such iteration is possible only on a subdomain of $\bar{\Pi}_{s+2d}$. Specifically, $n$ iterations are possible on $\bar{\Pi}_{s+3d}$ and, doing so, on the first step we obtain

$$(u - v)_{|\bar{\Pi}_{s+3d}} = (SR)^2(u - v) + (I + SR)D(u - v) + (I + SR)F(\tilde{a}\partial_x v) + (I + SR)F(\tilde{b}v).$$

Iterating this, that is, substituting (3.19) into the last equation once and once again, in the $n$-th step we meet the property $(H3)'$, resulting in the identity

$$(SR)^n(u - v) \equiv 0.$$

Consequently, we get

$$(u - v)_{|\bar{\Pi}_{s+3d}} = \sum_{i=0}^{n-1} (SR)^i D(u - v) + \sum_{i=0}^{n-1} (SR)^i F(\tilde{a}\partial_x v) + \sum_{i=0}^{n-1} (SR)^i F(\tilde{b}v).$$

This gives us the desired formula

$$(u - v)(x, s + 3d) = \left[ \sum_{i=0}^{n-1} (SR)^i D(u - v) + \sum_{i=0}^{n-1} (SR)^i F(\tilde{a}\partial_x v) + \sum_{i=0}^{n-1} (SR)^i F(\tilde{b}v) \right](x, s + 3d). \quad (3.20)$$

To prove the estimate (3.18), we derive appropriate smallness bounds for each of the three summands in the right hand side of (3.20) separately.

Step 2. Obtaining an upper bound of the type $\beta(\varepsilon)\|\varphi\|_{L^2((0,1);\mathbb{R}^n)}$ for the second and the third summands in (3.20). Given $s < \tau < \infty$, denote

$$\Pi_s^* = \{(x, t) : 0 < x < 1, s < t < \tau\}.$$
the boundedness of the operators $S$ and $R$, and the smallness of $\tilde{a}$ and $\tilde{b}$. The desired bound for the third summand is a simple consequence of the smallness of $\tilde{a}$ and $\tilde{b}$.

To estimate the first summand in the right-hand side of (3.20), it suffices to derive a smallness bound for $D(u - v)$.

**Step 3. Derivation of an operator equation for $D(u - v)$.** The continuous solutions $u$ and $v$ on $\mathbb{R}^{s+3d}$ satisfy the operator equations

$$u = Qu + Du, \quad v = Qv + Dv,$$

where the operators $Q, \tilde{Q}, D,$ and $\tilde{D}$ are restricted to the subspace of $C(\mathbb{R}^{s+3d}; \mathbb{R}^n)$ of functions satisfying the initial condition $u = v = 0$. Note that the operators $Q, \tilde{Q}, D,$ and $\tilde{D}$ map $C(\mathbb{R}^{s+3d}; \mathbb{R}^n)$ into itself. Thus, for the difference $u - v$ we have

$$u - v = Q(u - v) + (Q - \tilde{Q})v + D(u - v) + (D - \tilde{D})v,$$

hence

$$D(u - v) = DQ(u - v) + (Q - \tilde{Q})v + D(u - v) + (D - \tilde{D})v.$$  

(3.21)

Substitute (3.21) into the first summand in the right-hand side of (3.22) and rewrite the last equation with respect to the new variable $w = D(u - v)$. We get

$$w = DQ^2(u - v) + D(I + Q)(Q - \tilde{Q})v + D(I + Q)w + D(I + Q)(D - \tilde{D})v.$$  

(3.22)

Continuing in this fashion (again substituting (3.21) into the first summand in the right-hand side of (3.23)), in the $n$-th step we arrive at the formula

$$w = DQ^n(u - v) + D\sum_{i=0}^{n-1} Q^i(Q - \tilde{Q})v + D\sum_{i=0}^{n-1} Q^iw + D\sum_{i=0}^{n-1} Q^i(D - \tilde{D})v.$$  

(3.23)

Furthermore, combining the condition $(H3)'$ and the fact that $(u - v)(\cdot, s) \equiv 0$ on $[0, 1]$, we conclude that $[Q^n(u - v)](x, t) \equiv 0$ on $\mathbb{R}^{s+3d}$. The resulting equation for $w$ restricted to $\mathbb{R}^{s+3d}$ reads

$$w = D\sum_{i=0}^{n-1} Q^i(Q - \tilde{Q})v + D\sum_{i=0}^{n-1} Q^i(D - \tilde{D})v + D\sum_{i=0}^{n-1} Q^iw.$$  

(3.24)

**Step 4. Obtaining an upper bound of the type $\beta(\varepsilon)\|\varphi\|_{L^2((0,1); \mathbb{R}^n)}$ for $w = D(u - v)$**. Next we prove that there exists a function $\beta_0 : [0, 1] \to \mathbb{R}$ with $\beta_0(\varepsilon) \to 0$ as $\varepsilon \to 0$, for which we have the estimate

$$\max_{s \leq t \leq s+3d} \|w(\cdot, t)\|_{L^2((0,1); \mathbb{R}^n)} \leq \beta_0(\varepsilon)\|\varphi\|_{L^2((0,1); \mathbb{R}^n)},$$  

(3.25)
being uniform in $s \in \mathbb{R}$, $\varphi \in C^1_0([0,1];\mathbb{R}^n)$, and $\tilde{a}, \tilde{b} \in BC^1(\bar{\Pi};\mathbb{M}_n)$ with $\|\tilde{a}\|_{BC^1(\Pi;\mathbb{M}_n)} \leq \varepsilon$ and $\|\tilde{b}\|_{BC(\Pi;\mathbb{M}_n)} \leq \varepsilon$.

By technical reasons, we rewrite the integral operator $D$ in the following equivalent form, obtained using integration along characteristic curves in $t$ (rather than in $x$)

$$(Dw)_j(x,t) = -\int_{t_j(x,t)}^t \exp\int_t^\tau b_{jj}(\sigma_j(\eta),\eta)\,d\eta \sum_{k \neq j} b_{jk}(\sigma_j(\tau),\tau)w_k(\sigma_j(\tau),\tau)\,d\tau,$$

where

$$\tau \in \mathbb{R} \mapsto \sigma_j(\tau) = \sigma_j(\tau,x,t) \in [0,1]$$

is the inverse form of the $j$-th characteristic of (1.1) passing through the point $(x,t) \in \bar{\Pi}$, $t_j(x,t)$ is the minimum value of $\tau$ at which the characteristic $\tau = \sigma_j(\tau,x,t)$ reaches $\partial \Pi_s$. The function $\sigma_j(\tau)$ is the solution to the initial value problem

$$\partial_\tau \sigma_j(\tau,x,t) = a_j(\sigma_j(\tau,x,t),\tau), \quad \sigma_j(t,x,t) = x.$$

Therefore, the estimate (3.25) follows from the Gronwall’s inequality applied to (3.24), provided the first two summands satisfy an upper bound of the type $\beta(\varepsilon)\|\varphi\|_{L^2([0,1];\mathbb{R}^n)}$.

The rest of the proof consists in deriving the desired upper bound for the first two summands in the right-hand side of (3.24). In Steps 5–8 we get the desired bound for the second summand, while in Step 9 we get it for the first summand.

**Step 5. Derivation of a representation formula for the second summand in (3.24).** Remark that the main technicalities appear already in the case $i = 0$ and the proof for $i \geq 1$ uses a similar argument. Hence, let $i = 0$ and estimate the summand $D(D - \tilde{D})v$.

In what follows, we will use the following notation. The $j$-th characteristic of (3.16) passing through the point $(x,t) \in \bar{\Pi}_s$ is defined as the solution $\xi \in [0,1] \mapsto \bar{\omega}_j(\xi) = \bar{w}_j(\xi,x,t) \in \mathbb{R}$ of the initial value problem

$$\partial_\xi \bar{w}_j(\xi,x,t) = \frac{1}{[a_j + \tilde{a}_j](\xi,\bar{w}_j(\xi,x,t))}, \quad \bar{w}_j(x,x,t) = t.$$  \hfill (3.26)

Write

$$\bar{c}_j(\xi,x,t) = \exp\int_x^\xi \left[ \frac{b_{jj} + \tilde{b}_{jj}}{a_j + \tilde{a}_j} \right](\eta,\bar{w}_j(\eta))\,d\eta, \quad \bar{d}_j(\xi,x,t) = \frac{\bar{c}_j(\xi,x,t)}{[a_j + \tilde{a}_j](\xi,\bar{w}_j(\xi))}.$$

Introduce the linear bounded operator $\tilde{D} : BC(\bar{\Pi}_s;\mathbb{R}^n) \to BC(\bar{\Pi}_s;\mathbb{R}^n)$ and the affine bounded operator $\tilde{Q} : BC(\bar{\Pi}_s;\mathbb{R}^n) \to BC(\bar{\Pi}_s;\mathbb{R}^n)$ by

$$\left( \tilde{D}w \right)_j(x,t) = -\int_{\tilde{x}_j(x,t)}^x \bar{d}_j(\xi,x,t) \sum_{k \neq j} \left[ b_{jk} + \tilde{b}_{jk} \right](\xi,\bar{w}_j(\xi))w_k(\xi,\bar{w}_j(\xi))\,d\xi,$$

$$\left( \tilde{Q}u \right)_j(x,t) = \begin{cases} \bar{c}_j(\tilde{x}_j(x,t),x,t)(Ru)_j(\bar{w}_j(\tilde{x}_j(x,t),x,t)) & \text{if } \tilde{x}_j(x,t) \notin (0,1), \\ \bar{c}_j(\tilde{x}_j(x,t),x,t)(\varphi_j(\tilde{x}_j(x,t)) & \text{if } \tilde{x}_j(x,t) \in (0,1), \\ \end{cases}$$
where \( \bar{x}_j(x,t) \) denotes the abscissa of the point with the smallest ordinate, at which the characteristic curve \( \tau = \bar{\omega}_j(\xi, x, t) \) reaches the boundary of \( \Pi_s \). Set

\[
d_{jki}(\xi, \eta, x, t) = d_j(\xi, x, t)d_k(\eta, \xi, \omega_j(\xi))b_{ki}(\eta, \omega_k(\eta, \xi, \omega_j(\xi))),
\]

\[
\tilde{d}_{jki}(\xi, \eta, x, t) = d_j(\xi, x, t)d_k(\eta, \xi, \omega_j(\xi))b_{ki}(\xi, \omega_j(\xi)) \left[ b_{ki} + \tilde{b}_{ki} \right] (\eta, \bar{\omega}_k(\eta, \xi, \omega_j(\xi))).
\]

Then we have

\[
\left[ (D^2 - D\bar{D})v \right](x, t) = \sum_{k \neq j} \sum_{i \neq k} \int_{x_j(x,t)}^{x_k(\xi,\omega_j(\xi))} d_{jki}(\xi, \eta, x, t)v_i(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \, d\eta d\xi
\]

\[
- \sum_{k \neq j} \sum_{i \neq k} \int_{x_j(x,t)}^{x_k(\xi,\omega_j(\xi))} \tilde{d}_{jki}(\xi, \eta, x, t)v_i(\eta, \bar{\omega}_k(\eta, \xi, \omega_j(\xi))) \, d\eta d\xi
\]

\[
+ \sum_{k \neq j} \sum_{i \neq k} \int_{x_j(x,t)}^{x_k(\xi,\omega_j(\xi))} \left( d_{jki}(\xi, \eta, x, t) - \tilde{d}_{jki}(\xi, \eta, x, t) \right) v_i(\eta, \bar{\omega}_k(\eta, \xi, \omega_j(\xi))) \, d\eta d\xi
\]

\[
+ \sum_{k \neq j} \sum_{i \neq k} \int_{x_j(x,t)}^{x_k(\xi,\omega_j(\xi))} d_{jki}(\xi, \eta, x, t) \left( v_i(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) - v_i(\eta, \bar{\omega}_k(\eta, \xi, \omega_j(\xi))) \right) \, d\eta d\xi.
\]

Let us estimate each of the three summands in the right hand side separately.

**Step 6. Obtaining some technical inequalities.** Due to the regularity and the boundedness assumptions on the coefficients \( a, \tilde{a}, b, \) and \( \bar{b} \), we have

\[
\max_{\xi,\eta \in [0,1]} \| \bar{\omega}_k(\eta, \xi, \omega_j(\xi, \cdot, \cdot)) - \omega_k(\eta, \xi, \omega_j(\xi, \cdot, \cdot)) \|_{C(\Pi^{+3d};\mathbb{R}^n)} \leq \tilde{\beta}(\varepsilon),
\]

\[
\max_{\xi \in [0,1]} \| \bar{x}_k(\xi, \omega_j(\xi, \cdot, \cdot)) - x_k(\xi, \omega_j(\xi, \cdot, \cdot)) \|_{C(\Pi^{+3d};\mathbb{R}^n)} = \| \bar{x}_k - x_k \|_{C(\Pi^{+3d};\mathbb{R}^n)} \leq \tilde{\beta}(\varepsilon),
\]

\[
\max_{\xi,\eta \in [0,1]} \| \frac{d}{d\xi} \bar{\omega}_k(\eta, \xi, \omega_j(\xi, \cdot, \cdot)) - \frac{d}{d\xi} \omega_k(\eta, \xi, \omega_j(\xi, \cdot, \cdot)) \|_{C(\Pi^{+3d};\mathbb{R}^n)} \leq \tilde{\beta}(\varepsilon),
\]

\[
\max_{\xi \in [0,1]} \| \frac{d}{d\xi} \bar{x}_k(\xi, \omega_j(\xi, \cdot, \cdot)) - \frac{d}{d\xi} x_k(\xi, \omega_j(\xi, \cdot, \cdot)) \|_{C(\Pi^{+3d};\mathbb{R}^n)} \leq \tilde{\beta}(\varepsilon)
\]

for all \( s \in \mathbb{R} \), for all \( \tilde{a} \) and \( \bar{b} \) with \( \| \tilde{a} \|_{BC^1(\Pi;\mathbb{R}^n)} \leq \varepsilon \) and \( \| \bar{b} \|_{BC(\Pi;\mathbb{R}^n)} \leq \varepsilon \), for all \( j, k \leq n \), and for a function \( \tilde{\beta} : [0,1] \to \mathbb{R} \) approaching zero as \( \varepsilon \to 0 \). In order to prove (3.28), we use the equations (3.5) and (3.26) and obtain

\[
\frac{d}{d\eta} (\omega_k(\eta) - \bar{\omega}_k(\eta)) = \frac{ak(\eta, \bar{\omega}_k(\eta)) - ak(\eta, \omega_k(\eta)) + \tilde{a}k(\eta, \omega_k(\eta))}{ak(\eta, \omega_k(\eta))(ak(\eta, \bar{\omega}_k(\eta)) + \tilde{a}k(\eta, \omega_k(\eta)))}, \quad \omega_k(x) - \bar{\omega}_k(x) = 0.
\]
Application of (1.6) gives
\[
|\omega_k(\eta) - \tilde{\omega}_k(\eta)| \leq \frac{1}{\Lambda_0} \left| \int_x^\eta \left( \|a_k\|_{BC^1(M)} |\omega_k(\eta_1) - \tilde{\omega}_k(\eta_1)| + \|\tilde{a}_k\|_{BC(M)} \right) d\eta_1 \right|.
\]
The Gronwall’s inequality yields
\[
|\omega_k(\eta) - \tilde{\omega}_k(\eta)| \leq \frac{1}{\Lambda_0} \|\tilde{a}_k\|_{BC(M)} \exp \left\{ \frac{\|\eta - x\|}{\Lambda_0} \|a_k\|_{BC^1(M)} \right\},
\]
implying (3.28)\(_1\).

To derive (3.28)\(_2\), we proceed similarly, but now we consider the initial value problem for the difference \(\tilde{\sigma}_k(\tau, x, t) - \sigma_k(\tau, x, t)\), namely
\[
\frac{d}{d\tau}(\tilde{\sigma}_k(\tau) - \sigma_k(\tau)) = a_k(\tilde{\sigma}_k(\tau), \tau) + \tilde{a}_k(\tilde{\sigma}_k(\tau), \tau) - a_k(\sigma_k(\tau), \tau), \quad \tilde{\sigma}_k(t) - \sigma_k(t) = 0.
\]

It remains to recall that, if \(0 < x_k(x, t) < 1\), then \(x_k(x, t) = \sigma_k(s, x, t)\).

The estimate (3.28)\(_3\) follows directly from (3.28)\(_1\) and the smallness of \(\tilde{a}\) and \(\tilde{b}\).

To prove (3.28)\(_4\), we use the identities
\[
\partial_x \tilde{\omega}_k(\xi, x, t) = -\frac{1}{a_k(x, t) + \tilde{a}_k(x, t)} \exp \int_x^\xi \frac{[\partial_x a_k + \partial_x \tilde{a}_k](\eta, \tilde{\omega}_k(\eta))}{[a_k + \tilde{a}_k]^2(\eta, \tilde{\omega}_k(\eta))} d\eta, \quad (3.29)
\]
\[
\partial_t \tilde{\omega}_k(\xi, x, t) = \exp \int_x^\xi \frac{[\partial_t a_k + \partial_t \tilde{a}_k](\eta, \tilde{\omega}_k(\eta))}{[a_k + \tilde{a}_k]^2(\eta, \tilde{\omega}_k(\eta))} d\eta \quad (3.30)
\]
and do the following calculations:
\[
\frac{d}{d\xi}(\omega_k(\eta, \xi, \omega_j(\xi)) - \tilde{\omega}_k(\eta, \xi, \omega_j(\xi))) = \partial_2[\omega_k - \tilde{\omega}_k](\eta, \xi, \omega_j(\xi))
\]
\[
+ \frac{1}{a_j(\xi, \omega_j(\xi))} \partial_3[\omega_k - \tilde{\omega}_k](\eta, \xi, \omega_j(\xi))
\]
\[
= \frac{1}{[a_k + \tilde{a}_k](\xi, \omega_j(\xi))} \exp \int_x^\xi \frac{\partial_2[a_k + \tilde{a}_k](\eta_1, \omega_k(\eta_1, \xi, \omega_j(\xi)))}{[a_k + \tilde{a}_k]^2(\eta_1, \omega_k(\eta_1, \xi, \omega_j(\xi)))} d\eta_1
\]
\[
- \frac{1}{a_k(\xi, \omega_j(\xi))} \exp \int_x^\xi \frac{\partial_2[a_k + \tilde{a}_k](\eta_1, \omega_k(\eta_1, \xi, \omega_j(\xi)))}{a_k^2(\eta_1, \omega_k(\eta_1, \xi, \omega_j(\xi)))} d\eta_1
\]
\[
- \frac{1}{a_j(\xi, \omega_j(\xi))} \left( \exp \int_x^\xi \frac{\partial_2[a_k + \tilde{a}_k](\eta_1, \omega_k(\eta_1, \xi, \omega_j(\xi)))}{[a_k + \tilde{a}_k]^2(\eta_1, \omega_k(\eta_1, \xi, \omega_j(\xi)))} d\eta_1 \right)
\]
\[
- \exp \int_x^\xi \frac{\partial_2[a_k + \tilde{a}_k](\eta_1, \omega_k(\eta_1, \xi, \omega_j(\xi)))}{a_k^2(\eta_1, \omega_k(\eta_1, \xi, \omega_j(\xi)))} d\eta_1 \right).\]

Here and in what follows \(\partial_i\) denotes the partial derivative with respect to the \(i\)-th argument. The estimate (3.28)\(_4\) is now an easy consequence of the inequality \(\|\tilde{a}\|_{BC(H;M_a)} + \|\partial_t \tilde{a}\|_{BC(H;M_a)} \leq \varepsilon\).
Finally, to prove (3.28)_5, we take into account the equalities
\[ \partial_x \sigma_k(\tau, x, t) = \exp \int_t^\tau \partial_1 a_k(\sigma_k(\eta, x, t), \eta) d\eta, \]
\[ \partial_0 \sigma_k(\tau, x, t) = -a_k(x, t) \exp \int_t^\tau \partial_1 a_k(\sigma_k(\eta, x, t), \eta) d\eta, \]
which yields
\[ \frac{d}{d\xi} (\sigma_k(\tau, \xi, \omega_j(\xi)) - \bar{\sigma}_k(\tau, \xi, \omega_j(\xi))) = \partial_2 [\sigma_k - \bar{\sigma}_k](\tau, \xi, \omega_j(\xi)) \]
\[ + \frac{1}{a_j(\xi, \omega_j(\xi))} \partial_3 [\sigma_k - \bar{\sigma}_k](\tau, \xi, \omega_j(\xi)) \]
\[ = \exp \int_t^\tau \partial_1 a_k(\sigma_k(\eta_1, \xi, \omega_j(\xi)), \eta_1) d\eta_1 - \exp \int_t^\tau \partial_1 [a_k + \bar{a}_k](\sigma_k(\eta_1, \xi, \omega_j(\xi)), \eta_1) d\eta_1 \]
\[ - \frac{1}{a_j(\xi, \omega_j(\xi))} \left[ a_k(\xi, \omega_j(\xi)) \exp \int_t^\tau \partial_1 a_k(\sigma_k(\eta_1, \xi, \omega_j(\xi)), \eta_1) d\eta_1 \right. \]
\[ \left. - [a_k + \bar{a}_k](\xi, \omega_j(\xi)) \exp \int_t^\tau \partial_1 [a_k + \bar{a}_k](\sigma_k(\eta_1, \xi, \omega_j(\xi)), \eta_1) d\eta_1 \right]. \]

Similarly to the above, the estimate (3.28)_5 now follows directly from the bound \( \| \tilde{a} \|_{BC(\Pi; \mathbb{M}_a)} \) + \( \| \partial_x \tilde{a} \|_{BC(\Pi; \mathbb{M}_a)} \) \leq \varepsilon.

Since \( \tilde{\beta}(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), below we suppose that \( \tilde{\beta}(\varepsilon) < 1 \).

**Step 7.** Obtaining an upper bound of the type \( \beta(\varepsilon) \| \varphi \|_{L^2((0,1); \mathbb{R}^n)} \) for the first and the second summands in the right-hand side of (3.27). For the integrals in the first summand we use (3.28)_1 and the Cauchy-Schwarz inequality, obtaining
\[
\left| \int_{x_j(x,t)}^{x_k(\xi,\omega_j(\xi))} \tilde{d}_{jki}(\xi, \eta, x, t) v_i(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\eta d\xi \right|
\leq \max_{\xi, \eta, x \in [0,1]} \left| \tilde{d}_{jki}(\xi, \eta, x, t) \right| \max_{\eta \in [0,1]} \left| \int_{x_j(x,t)}^{x} \int_{y}^{y+\tilde{\beta}(\varepsilon)} \left| v_i(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \right| d\eta d\xi \right|
= \max_{\xi, \eta, x \in [0,1]} \left| \tilde{d}_{jki}(\xi, \eta, x, t) \right|
\times \max_{\eta \in [0,1]} \left\{ \int_{y}^{y+\tilde{\beta}(\varepsilon)} \left| v_i(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \right| d\eta \right\}. \tag{3.31}
\]

For a fixed \( \eta \), let us change the variables
\[ \xi \to \theta = \tilde{\omega}_k(\eta, \xi, \omega_j(\xi)). \tag{3.32} \]
Taking into account the equalities (3.29) and (3.30), from (3.32) we get

\[
d\theta = \left( \partial_2 \tilde{\omega}_k(\eta, \xi, \omega_j(\xi)) + \partial_3 \tilde{\omega}_k(\eta, \xi, \omega_j(\xi)) \partial_\xi \omega_j(\xi) \right) d\xi
\]

\[
= \left[ \frac{a_k + \tilde{a}_k - a_j}{a_j(a_k + \tilde{a}_k)} \right] (\xi, \omega_j(\xi)) \partial_2 \tilde{\omega}_k(\eta, \xi, \omega_j(\xi)) d\xi. \tag{3.33}
\]

As it follows from (3.33), the change of variables (3.32) is non-degenerate for all \( \xi, x \in [0, 1] \) and \( t \in [s, s + 3d] \) whenever \([a_k + \tilde{a}_k] (\xi, \omega_j(\xi)) - a_j(\xi, \omega_j(\xi)) \neq 0\). Remark that the last condition is true due to the assumption (H1) and the choice of \( \varepsilon_0 \). Denote the inverse of (3.32) by \( \tilde{x}(\theta) = \tilde{x}(\theta, \eta, x, t) \). One can see that \( \tilde{x}(\theta, \eta, x, t) \) is continuous in all its arguments. Therefore, changing the variables according to (3.32), the double integral in the right-hand side of (3.31) reads

\[
\int_y^{y+\tilde{\beta}(\varepsilon)} \int_{\tilde{w}_k(\eta, x_j(x,t), \omega_j(x_j(x,t)))} \left| \frac{a_j(a_k + \tilde{a}_k)}{a_k + \tilde{a}_k - a_j} \right| (\tilde{x}(\theta), \omega_j(\tilde{x}(\theta)))
\]

\[
\times [\partial_3 \tilde{w}_k(\eta, \tilde{x}(\theta), \omega_j(\tilde{x}(\theta)))]^{-1} v_i(\eta, \theta) \ d\theta \ d\eta
\]

\[
\leq C_1 \max_{\theta \in [s, s + 3d]} \int_y^{y+\tilde{\beta}(\varepsilon)} |v_i(\eta, \theta)| \ d\eta \leq C_1 \tilde{\beta}(\varepsilon) \max_{\theta \in [s, s + 3d]} \left( \int_y^{y+\tilde{\beta}(\varepsilon)} |v_i(\eta, \theta)|^2 \ d\eta \right)^{1/2}
\]

\[
\leq KC_1 \tilde{\beta}(\varepsilon) e^{3d\nu} \| \varphi \|_{L^2((0,1); \mathbb{R}^n)} \leq \beta_1(\varepsilon) \| \varphi \|_{L^2((0,1); \mathbb{R}^n)},
\]

where

\[
C_1 = \max_{\eta, \xi \in [0, 1]} \sup_{t \in \mathbb{R}} \left| \tilde{w}_k(\eta) - \tilde{w}_k(\eta, x_j(x,t), \omega_j(x_j(x,t))) \right|
\]

\[
\times \max_{\eta, \xi \in [0, 1]} \sup_{t \in \mathbb{R}} \left| \frac{a_j(a_k + \tilde{a}_k)}{a_k + \tilde{a}_k - a_j} \right| (\tilde{x}(\theta), \omega_j(\tilde{x}(\theta))) \left[ \partial_3 \tilde{w}_k(\eta, \tilde{x}(\theta), \omega_j(\tilde{x}(\theta))) \right]^{-1}
\]

and the function \( \beta_1 : [0, 1] \rightarrow \mathbb{R} \) approaches zero as \( \varepsilon \rightarrow 0 \). Here we used the assumption (1.6) and the estimate (1.7) about the exponential boundedness of the evolution operator. The desired estimate for the first summand in (3.27) is derived.

Similar estimate for the second summand in (3.27) immediately follows from the assumption (1.6) and the estimates (3.28).

**Step 8. Obtaining an upper bound of the type \( \beta(\varepsilon) \| \varphi \|_{L^2((0,1); \mathbb{R}^n)} \) for the third summand in the right-hand side of (3.27).** Fix arbitrary \( 1 \leq j, k \leq m \) (for the other \( j, k \) we proceed similarly) and use the mean value theorem and the estimates (3.28). This results in the following representation of the third summand, which will be denoted by \( I_1(x,t) \):

\[
I_1(x,t) = \int_{x_j(x,t)}^x \int_{x_k(\xi, \omega_j(\xi))}^{\xi} d_j k_i(\xi, \eta, x, t) \left( \omega_k(\eta, \xi, \omega_j(\xi)) - \tilde{\omega}_k(\eta, \xi, \omega_j(\xi)) \right)
\]

\[
\times \int_0^1 \partial_2 v_i \left( \eta, \gamma \omega_k(\eta, \xi, \omega_j(\xi)) + (1 - \gamma) \tilde{\omega}_k(\eta, \xi, \omega_j(\xi)) \right) d\gamma \ d\eta \ d\xi. \tag{3.34}
\]
Using the notation

\[
\rho(\xi, \eta, x, t, \gamma) = \gamma \left[ \frac{a_k - a_j}{a_j a_k} \right] (\xi, \omega_j(\xi)) \partial_j \omega_k(\eta, \xi, \omega_j(\xi))
\]

\[
+ (1 - \gamma) \left[ \frac{a_k + \tilde{a}_k - a_j}{a_j (a_k + \tilde{a}_k)} \right] (\xi, \omega_j(\xi)) \partial_j \tilde{\omega}_k(\eta, \xi, \omega_j(\xi)),
\]

we have

\[
\frac{d}{d\xi} v_i(\eta, \gamma \omega_k(\eta, \xi, \omega_j(\xi)) + (1 - \gamma) \tilde{\omega}_k(\eta, \xi, \omega_j(\xi))) = \partial_2 v_i(\eta, \gamma \omega_k(\eta, \xi, \omega_j(\xi)) + (1 - \gamma) \tilde{\omega}_k(\eta, \xi, \omega_j(\xi))) \rho(\xi, \eta, x, t, \gamma).
\] (3.35)

Remark that \(\rho(\xi, \eta, x, t, \gamma) \neq 0\) for all \(\xi, \eta, x \in [0, 1]\), \(t \in [s, s + 3d]\), and \(\gamma \in [0, 1]\), since our assumptions imply that \((a_k + \tilde{a}_k - a_j)(\xi, \omega_j(\xi)) \neq 0\) for all \(\|a_k\|_{BC(\Omega)} \leq \varepsilon_0\). Note also that \(\partial_2 \omega_k\) and \(\partial_2 \tilde{\omega}_k\) are strictly positive, see (3.30). On the account of (3.35), the expression (3.34) can be rewritten as follows:

\[
I_1(x, t) = \int_{x_j(x,t)}^x \frac{d}{d\xi} \left( \int_{x_k(\xi,\omega_j(\xi))}^\xi d_{jk_i}(\xi, \eta, x, t) (\omega_k(\eta, \xi, \omega_j(\xi)) - \tilde{\omega}_k(\eta, \xi, \omega_j(\xi))) \right) d\xi d\eta \left( \int_{x_\xi(\xi,\omega_j(\xi))}^\xi d\eta \left( \int_{x_{\eta,\omega_j(\xi)}}^\eta \left( \int_{x_{\xi,\omega_j(\xi)}}^\xi d\xi \right) \right) \right)
\]

\[
\times \rho^{-1}(\xi, \eta, x, t, \gamma) v_i(\eta, \gamma \omega_k(\eta, \xi, \omega_j(\xi)) + (1 - \gamma) \tilde{\omega}_k(\eta, \xi, \omega_j(\xi))) \left( \int_{x_j(x,t)}^x \frac{d}{d\xi} d_{jk_i}(\xi, \eta, x, t) \right) d\gamma d\xi d\eta d\xi.
\] (3.36)

Denote by \(x_{jk}(\theta, x, t)\) the \(x\)-coordinate of the point where the characteristics \(\omega_j(\xi, x, t)\) and \(\omega_k(\xi, \theta, s)\) intersect (if they do), that is

\[
\omega_j(x_{jk}(\theta, x, t), x, t) = \omega_k(x_{jk}(\theta, x, t), \theta, s).
\] (3.37)

Suppose for definiteness that \(a_j(x, t) > a_k(x, t)\) (the case of \(a_j(x, t) < a_k(x, t)\) is similar). Since \(x_k(\xi, \omega_j(\xi)) = 0\) for all \(\xi \in [x_j(x,t), x_{jk}(0, x, t)]\), the integral over the interval \([x_j(x,t), x_{jk}(0, x, t)]\) in the second summand of (3.36) disappears. Furthermore, if \(x_{jk}(0, x, t) \notin [x_j(x,t), x]\), then evidently the integral over \([x_j(x,t), x]\) in this summand disappears. We therefore need to estimate the second summand in (3.36) whenever \(x_{jk}(0, x, t) \in \)
where the derivative \( \partial \) can be computed using the identity \( \omega_k(x_k, \omega_j(\xi)), \xi, \omega_j(\xi) \equiv s \). Indeed, this and (3.5) yield
\[
\frac{dx_k(\xi, \omega_j(\xi))}{d\xi} = -a_k(x_k(\xi, \omega_j(\xi)), \omega_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)))
\]
\[
\times (\partial_2 \omega_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) + a_j(\xi, \omega_j(\xi)) \partial_3 \omega_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi))) \tag{3.39}
\]
where \( \partial_2 \omega_k \) and \( \partial_3 \omega_k \) are given by the formulas (3.29) and (3.30), respectively.

Using the change of variables \( \xi \rightarrow \theta = x_k(\xi, \omega_j(\xi)) \) with the inverse \( \xi = x_{jk}(\theta, x, t) \), one rewrites (3.38) in the form
\[
\int_0^{x_k(\theta, x, t)} \int_0^1 \frac{dx_k(\xi, \omega_j(\xi))}{d\xi} d_{jki}(\xi, \theta, x, t) [\omega_k - \tilde{\omega}_k](\theta, \xi, \omega_j(\xi))
\]
\[
\times \rho^{-1}(\xi, \theta, x, t) v_i(\xi, \theta, \xi, \omega_j(\xi), \omega_j(\xi)) \bigg|_{\xi=x_{jk}(\theta, x, t)}
\]
\[
\times \frac{\partial}{\partial \theta} x_{jk}(\theta, x, t) d\gamma d\theta, \tag{3.40}
\]
where the derivative \( \frac{\partial}{\partial \theta} x_{jk}(\theta, x, t) \) can be easily computed from the identity (3.37) as
\[
\frac{\partial}{\partial \theta} x_{jk}(\theta, x, t) = \frac{\partial_2 \omega_k(x_{jk}(\theta, x, t), \theta, s)[a_k a_j](x_{jk}(\theta, x, t), \omega_j(x_{jk}(\theta, x, t), x, t))}{[a_k - a_j](x_{jk}(\theta, x, t), \omega_j(x_{jk}(\theta, x, t), x, t))}.
\]

Taking into account (3.40), the expression \( I_1 \) given by (3.36) now reads
\[
I_1(x, t) = \int_0^x \int_{x_{jk}(\theta, x, t)} d_{jki}(x, \eta, x, t)(\omega_k(\eta) - \tilde{\omega}_k(\eta))
\]
\[
\times \int_0^1 \rho^{-1}(x, \eta, x, t, \gamma) v_i(\eta, \gamma \omega_k(\eta) + (1 - \gamma)\tilde{\omega}_k(\eta)) d\gamma d\eta
\]
\[
+ \int_0^{x_k(x, t)} \int_0^1 \frac{dx_k(\xi, \omega_j(\xi))}{d\xi} x_k(\xi, \omega_j(\xi)) d_{jki}(\xi, \eta, x, t)[\omega_k - \tilde{\omega}_k](\theta, \xi, \omega_j(\xi))
\]
\[
\times \rho^{-1}(\xi, \theta, x, t, \gamma) \frac{\partial x_{jk}(\theta, x, t)}{\partial \theta} v_i(\xi, \gamma \omega_k + (1 - \gamma)\tilde{\omega}_k)(\theta, \xi, \omega_j(\xi)) \bigg|_{\xi=x_{jk}(\theta, x, t)} d\gamma d\theta
\]
\[
- \int_{x_j(t)}^{x} \int_{x_k(\xi,\omega_j(\xi))}^{\xi} \int_0^1 \frac{d}{d\xi} \left[ d_{jki}(\xi, \eta, t)(\omega_k(\eta, \xi, \omega_j(\xi)) - \bar{\omega}_k(\eta, \xi, \omega_j(\xi))) \right. \\
\times \rho^{-1}(\xi, \eta, x, t, \gamma) \left. v_i(\eta, \gamma \omega_k(\eta, \xi, \omega_j(\xi)) + (1 - \gamma)\omega_k(\eta, \xi, \omega_j(\xi))) \right] d\gamma d\eta d\xi. \quad (3.41)
\]

We are prepared to derive the desired upper bound for \(|I_1|\). To this end, we use the estimates (1.6), (1.7), (3.28) and apply the Cauchy-Schwarz inequality to (3.41). As a result, we derive the estimate

\[
|I_1(x, t)| \leq C_2 \tilde{\beta}(\varepsilon) \max_{\theta \in [s, s + 3d]} \left( \int_0^1 |v_i(\eta, \theta)|^2 d\eta \right)^{1/2} \leq KC_2 \tilde{\beta}(\varepsilon) e^{3d\nu} \|\varphi\|_{L^2((0,1);\mathbb{R}^n)},
\]
the constant \(C_2\) being independent of \(s\), \(\varphi\), and \(\varepsilon\).

**Step 9. Obtaining an upper bound of the type \(\beta(\varepsilon)\|\varphi\|_{L^2((0,1);\mathbb{R}^n)}\) for the first summand in the right-hand side of (3.24).** Again, we consider the case \(i = 0\) and estimate \(D(\bar{Q} - Q)v\) (the proof of \(i \geq 1\) uses a similar arguments). Our starting point is the formula

\[
[D(\bar{Q} - Q)v]_j(x, t) = \\
= \sum_{k \neq j} \int_{x_j(x, t)}^{x} d_j(\xi, x, t)b_{jk}(\xi, \omega_j(\xi)) \left[ (\bar{Q} - Q)v \right]_k(\xi, \omega_j(\xi)) d\xi \\
= \sum_{k \neq j} \int_{x_j(x, t)}^{x} d_j(\xi, x, t) \left( \bar{c}_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) - c_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) \right) (\bar{P}v_k)(\xi, \omega_j(\xi)) d\xi \\
+ \sum_{k \neq j} \int_{x_j(x, t)}^{x} d_j(\xi, x, t)b_{jk}(\xi, \omega_j(\xi))c_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) \left[ (\bar{P} - P)v_k \right] (\xi, \omega_j(\xi)) d\xi, \quad (3.42)
\]

where

\[
(Pv)_k(x, t) = \begin{cases} 
(Rv)_k(w_k(x_k(x, t))), & x_k(x, t) \notin (0, 1), \\
\varphi_k(x_k(x, t)), & x_k(x, t) \in (0, 1),
\end{cases} \quad (3.43)
\]

while \(\bar{P}\) is given by the formula (3.43) with \(\tilde{\omega}_k\) and \(\tilde{x}_k\) in place of \(\omega_k\) and \(x_k\), respectively. Next, we use (3.28) to conclude that for all \(j, k \leq n\)

\[
\max_{\xi, x \in [0,1]} \max_{s \leq t \leq s + 3d} \left| \bar{c}_k(\bar{x}_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) - c_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) \right| \leq C_3 \tilde{\beta}(\varepsilon), \quad (3.44)
\]

where \(C_3\) does not depend on \(s\) and \(\varepsilon\). Applying the inequality (3.44) to the first sum in the right-hand side of (3.42) and using the bound (1.7), we estimate the absolute value of this summand from above by \(C_4 \tilde{\beta}(\varepsilon)\), where the positive constant \(C_4\) does not depend on \(s\) and \(\varepsilon\).

Now we aim at estimating the second sum in (3.42), denoted further by

\[
I_2(x, t) = \sum_{k \neq j} I_{2k}(x, t).
\]
To this end, fix $j, k \leq m$ (for the other $j, k$ we proceed similarly), and let $\tilde{x}_{jk}(\theta, x, t)$ denote the value of $\xi$ at which the characteristics $\omega_j(x, t)$ and $\tilde{\omega}_k(\xi, \theta, s)$ intersect (if they do). Note that $\tilde{x}_{jk}(\theta, x, t)$ fulfills the equation

$$\omega_j(\tilde{x}_{jk}(\theta, x, t), x, t) = \tilde{\omega}_k(\tilde{x}_{jk}(\theta, x, t), \theta, s).$$

Suppose that $a_j(x, t) > a_k(x, t)$ (the case $a_j(x, t) < a_k(x, t)$ is treated similarly). Then

$$I_{2k}(x, t) = \int_{x_{jk}(x, t)}^{\min\{x_{jk}(0, x, t), \tilde{x}_{jk}(0, x, t)\}} \left( (Rv)_{jk}(\tilde{\omega}_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) - (Rv)_{jk}(\omega_k(x, \omega_j(\xi)), \xi, \omega_j(\xi)) \right) d\xi$$

where $j, k$ and for the other $j, k$ we proceed similarly.

To estimate the second summand $I_{2k_2}$, first derive the bound

$$|x_{jk}(0, x, t) - \tilde{x}_{jk}(0, x, t)| \leq \beta_2(\varepsilon), \quad (3.46)$$

where $\beta_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. Recall that we are in the case $j, k \leq m$ and $a_j(x, t) > a_k(x, t)$, and for the other $j, k$ we proceed similarly.

Characteristic functions $\sigma_k(\tau, 0, s)$ and $\tilde{\sigma}_k(\tau, 0, s)$ are solutions to the initial value problems

$$\frac{d\xi}{d\tau} = a_k(\xi, \tau), \quad x(s) = 0 \quad (3.47)$$

and

$$\frac{d\xi}{d\tau} = a_k(\xi, \tau) + \tilde{a}_k(\xi, \tau), \quad x(s) = 0, \quad (3.48)$$

respectively. Changing the variables $(x, \tau) \to (y, \theta)$ by $x = y, \tau = \omega_j(y, 1, \theta)$, the equations (3.47) and (3.48) can be transformed as follows:

$$\frac{dy}{d\theta} = \left. \frac{a_j(y, \tau) a_k(y, \tau) \partial_\tau \omega_j(y, 1, \theta)}{a_j(y, \tau) - a_k(y, \tau)} \right|_{\tau = \omega_j(y, 1, \theta)} \quad (3.49)$$

and

$$\frac{dy}{d\theta} = \left. \frac{a_j(y, \tau) (a_k(y, \tau) + \tilde{a}_k(y, \tau)) \partial_\tau \omega_j(y, 1, \theta)}{a_j(y, \tau) - a_k(y, \tau) - \tilde{a}_k(y, \tau)} \right|_{\tau = \omega_j(y, 1, \theta)}, \quad (3.50)$$
respectively. Write \( \theta_0 = \omega_j(1,0,s) \) and estimate the difference of solutions \( y_1(\theta) \) and \( y_2(\theta) \) with the same initial values \( y_1(\theta_0) = y_2(\theta_0) = 0 \) to the equations (3.49) and (3.50), respectively. We have

\[
\frac{dy_1}{d\theta} - \frac{dy_2}{d\theta} = -\frac{a_j^2(y_2, \tau_2)\tilde{a}_k(y_2, \tau_2)\partial_3\omega_j(y_2, 1, \tau_2)}{a_j(y_1, \tau_1) - a_k(y_1, \tau_1)} - \frac{a_j(y_2, \tau_2)\partial_3\omega_j(y_2, 1, \theta)}{a_j(y_2, \tau_2) - a_k(y_2, \tau_2)},
\]

where \( \tau_1 = \omega_j(y_1, 1, \theta) \), \( \tau_2 = \omega_j(y_2, 1, \theta) \). By (1.6), \( |a_j - \tilde{a}_k| \geq \Lambda_0 \). Using the Gronwall’s argument, we derive

\[
|y_1(\theta) - y_2(\theta)| \leq C_5\|\tilde{a}\|_{BC(\Pi;M_\alpha)}, \ \theta \in [s, s + 3d],
\]

where positive constant \( C_5 \) does not depend on \( s \). Geometrically, \( y_1(\theta) \) is the abscissa of the point where characteristics \( \omega_j(y_1, 1, \theta) \) and \( \sigma_k(\tau, 0, s) \) intersect. Given \( (x, t) \), write \( \theta_1 = \omega_j(1, x, t) \). Then \( y_1(\theta_1) = x_jk(0, x, t) \) and \( y_2(\theta_1) = \tilde{x}_jk(0, x, t) \). This yields the desired estimate (3.46) for all \( \|\tilde{a}\|_{BC(\Pi;M_\alpha)} \leq \varepsilon \).

Now, using the mean value theorem and the exponential estimate (1.7), we easily get

\[
|I_{2k2}(x, t)| \leq C_6(\varepsilon)e^{3d\nu}\|\varphi\|_{L^2((0,1);\mathbb{R}^n)} \leq \beta_3(\varepsilon)\|\varphi\|_{L^2((0,1);\mathbb{R}^n)},
\]

where \( C_6 \) does not depend on \( \varepsilon, \varphi, k \) and \( s \), while the function \( \beta_3 : [0,1] \rightarrow \mathbb{R} \) approaches zero as \( \varepsilon \rightarrow 0 \).

Returning to (3.45), we proceed with the summand

\[
|I_{2k3}(x, t)| = \left| \int_0^x d_j(\xi, x, t)b_jk(\xi, \omega_j(\xi))c_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi))
\right|
\]

Using the notation (see (3.39))

\[
\rho_k(\xi, x, t, \gamma) = \frac{d}{d\xi} \left( \gamma \tilde{x}_k(\xi, \omega_j(\xi)) + (1 - \gamma)x_k(\xi, \omega_j(\xi)) \right)
\]

\[
= -\gamma \left[ \frac{a_k + \tilde{a}_k - a_j}{a_j(a_k + \tilde{a}_k)} \right] (\xi, \omega_j(\xi))
\]

\[
\times \partial_3\tilde{\omega}_k(\tilde{x}_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi))[a_k + \tilde{a}_k][\tilde{x}_k(\xi, \omega_j(\xi)), \tilde{\omega}_k(\tilde{x}_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi))]
\]

\[
-(1 - \gamma) \left[ \frac{a_k - a_j}{a_j a_k} \right] (\xi, \omega_j(\xi))
\]

\[
\times \partial_3\omega_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi))a_k(x_k(\xi, \omega_j(\xi)), \omega_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi))),
\]

(3.51)
we get
\[
\varphi'_k(\gamma \bar{x}_k(\xi, \omega_j(\xi))) + (1 - \gamma)x_k(\xi, \omega_j(\xi)) = \rho_k^{-1}(\xi, x, t, \gamma) \frac{d}{d\xi} \varphi_k(\gamma \bar{x}_k(\xi, \omega_j(\xi))) + (1 - \gamma)x_k(\xi, \omega_j(\xi)).
\]
Notice that \(\rho_k(\xi, x, t, \gamma) \neq 0\) for all \(\xi, x \in [0, 1], t \in [s, s + 3d], \gamma \in [0, 1],\) and \(k \leq n.\) Hence,
\[
|I_{2k3}(x, t)| \leq \left| \int_0^x \frac{d}{d\xi} \left[ d_j(\xi, x, t)b_{jk}(\xi, \omega_j(\xi))c_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) \times(\bar{x}_k(\xi, \omega_j(\xi)) - x_k(\xi, \omega_j(\xi))) \right] d\xi \right|
\]
Further,
\[
|I_{2k31}(x, t)| = \left| \frac{1}{a_j(x, t)}b_{jk}(x, x, t)c_k(x_k(x, x, t), x, t)(\bar{x}_k(x, t) - x_k(x, t)) \times(\bar{x}_k(x, t) - x_k(x, t)) | \rho_k^{-1}(x, x, t, \gamma) \varphi_k(\gamma \bar{x}_k(x, t) + (1 - \gamma)x_k(x, t))d\gamma \right|
\]
where \(y = y(x, t) = \max\{x_{jk}(0, x, t), \bar{x}_{jk}(0, x, t)\}.\) Next,
\[
\max_{s \leq t \leq s + 3d} \left\| I_{2k31}(\cdot, t) \right\|_{L^2((0, 1); \mathbb{R}^n)}^2 \leq 2 \max_{s \leq t \leq s + 3d} \int_0^1 \left[ \int_0^1 \frac{b_{jk}(x, t)}{a_j(x, t)}c_k(x_k(x, t), x, t) \times(\bar{x}_k(x, t) - x_k(x, t)) | \rho_k^{-1}(x, x, t, \gamma) \varphi_k(\gamma \bar{x}_k(x, t) + (1 - \gamma)x_k(x, t))d\gamma \right]^2 dx
\]
\[
+ 2 \max_{s \leq t \leq s + 3d} \int_0^1 \left[ d_j(y, x, t)b_{jk}(y, \omega_j(y))c_k(x_k(y, \omega_j(y)), y, \omega_j(y)) \times(\bar{x}_k(y, \omega_j(y)) - x_k(y, \omega_j(y))) \times(\bar{x}_k(y, \omega_j(y)) - x_k(y, \omega_j(y))) \times(\bar{x}_k(y, \omega_j(y)) - x_k(y, \omega_j(y))) | \rho_k^{-1}(y, x, t, \gamma) \varphi_k(\gamma \bar{x}_k(y, \omega_j(y)) + (1 - \gamma)x_k(y, \omega_j(y)))d\gamma \right]^2 dx,
\]
Changing the variables

\[ x \to z = z(x, t, \gamma) = \gamma x_k(x, t) + (1 - \gamma) x_k(x, t) \]  

(3.52)

and

\[ x \to \eta = \eta(x, t, \gamma) = \gamma x_k(y, \omega_j(y)) + (1 - \gamma) x_k(y, \omega_j(y)) \]  

(3.53)

in the first and in the second summands, respectively, we get

\[
\max_{s \leq t \leq s+3d} \| I_{\nu_{k1}}(\cdot, t) \|_{L^2((0,1); \mathbb{R}^n)}^2 \\
\leq 2\tilde{\beta}^2(\varepsilon) \max_{s \leq t \leq s+3d} \int \gamma x_k(y(1, t), \omega_j(y(1, t)), 1, t) + (1 - \gamma) x_k(y(1, t), \omega_j(y(1, t)), 1, t) \\
\times a_k(Z, \eta) b_jk(Z, \eta) c_k(x_k(y(y(Y, t)), \omega_j(y(Y, t), Y, t)), y(Y, t), \omega_j(y(Y, t), Y, t)) \\
\times \int_0^1 \rho_k^{-1}(y(Y, t), Y, t, \gamma) \varphi_k(\eta) d\gamma \right]^2 \partial_y Z(\eta, t, \gamma) d\eta,
\]  

(3.54)

where \( Z = Z(z, t, \gamma) \) and \( Y = Y(\eta, t, \gamma) \) are inverses to (3.52) and (3.53), respectively. Moreover, similarly to (3.51),

\[
\partial_z Z(x, t, \gamma) = \frac{d}{dx} \left[ \gamma x_k(x, t) + (1 - \gamma) x_k(x, t) \right] \\
= -\gamma \partial_2 \tilde{\omega}_k(x_k(x, t), a_k + \tilde{a}_k) (\tilde{x}_k(x, t), \tilde{\omega}_k(x_k(x, t))) \\
-(\gamma - 1) \partial_2 \omega_k(x_k(x, t)) a_k(x_k(x, t), \omega_k(x_k(x, t)))
\]

and

\[
\partial_1 \eta(x, t, \gamma) = \frac{d}{dx} \left( \gamma x_k(y, \omega_j(y)) + (1 - \gamma) x_k(y, \omega_j(y)) \right) \\
= \left( -\gamma \left[ \frac{a_k + \tilde{a}_k - a_j}{a_j(a_k + \tilde{a}_k)} \right] (y, \omega_j(y)) \right) \\
\times \partial_1 \tilde{\omega}_k(x_k(y, \omega_j(y)), y, \omega_j(y)) [a_k + \tilde{a}_k] (\tilde{x}_k(y, \omega_j(y)), \tilde{\omega}_k(x_k(y, \omega_j(y)), y, \omega_j(y))) \\
-(\gamma - 1) \left[ \frac{a_k - a_j}{a_j a_k} \right] (y, \omega_j(y)) \\
\times \partial_1 \omega_k(x_k(y, \omega_j(y)), y, \omega_j(y)) a_k(x_k(y, \omega_j(y)), \omega_k(x_k(y, \omega_j(y)), y, \omega_j(y))) \right) \frac{\partial y(x, t)}{\partial x},
\]
where, on the account of (3.37),

\[
\frac{\partial y(x, t)}{\partial x} = \begin{cases} 
\frac{\partial x_{jk}(0, x, t)}{\partial x} & \text{if } y = x_{jk}(0, x, t), \\
\frac{\partial \tilde{x}_{jk}(0, x, t)}{\partial x} & \text{if } y = \tilde{x}_{jk}(0, x, t), 
\end{cases}
\]

Due to (3.29) and (1.6), the right hand side is bounded uniformly in \( (x, t) \in \Pi \) and \( s \in \mathbb{R} \).

A similar argument is applied also to \( \frac{\partial x_{jk}(0, x, t)}{\partial x} \).

As it now easily follows from (3.54),

\[
\max_{s \leq t \leq s + 3d} \| I_{2k31}(\cdot, t) \|_{L^2((0,1);\mathbb{R}^n)} \leq \beta_4(\varepsilon) \| \varphi \|_{L^2((0,1);\mathbb{R}^n)},
\]

where the function \( \beta_4(\varepsilon) \) approaches zero as \( \varepsilon \to 0 \).

The summand \( I_{2k32} \) can be treated similarly, this time using the change of variables

\[
\xi \to \eta(\xi, x, t, \gamma) = \gamma \tilde{x}_k(\xi, \omega_j(\xi)) + (1 - \gamma) x_k(\xi, \omega_j(\xi)).
\]

Therewith we complete estimation of the summand \( I_{2k3} \).

Returning to the formula (3.45) again, we are left with the summand \( I_{2k1} \), for which we will use the same argument as for \( I_{2k3} \). Indeed, the mean value theorem yields the representation

\[
I_{2k1}(x, t) = \int_{x_{jk}(0, x, t)}^{\min\{x_k(0, x, t), \tilde{x}_{jk}(0, x, t)\}} d_i(\xi, x, t) b_{jk}(\xi, \omega_j(\xi)) c_k(\xi, \omega_j(\xi), \xi, \omega_j(\xi))
\]

\[
\times \left[ \tilde{\omega}_k(\tilde{x}_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) - \omega_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) \right]
\]

\[
\times \int_0^1 \frac{d}{d\tau} \left[ \sum_{i=m+1}^{n} p_{ki} v_i(0, \tau(\gamma, \xi)) + \sum_{i=1}^{m} p_{ki} v_i(1, \tau(\gamma, \xi)) \right] d\gamma d\xi = \sum_{i=1}^{n} I_{2k1i}(x, t),
\]

where \( \tau(\gamma, \xi) = \gamma \tilde{\omega}_k(\tilde{x}_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) + (1 - \gamma) \omega_k(x_k(\xi, \omega_j(\xi)), \xi, \omega_j(\xi)) \). Fix \( i \leq m \) (for \( m + 1 \leq i \leq n \) we use the same argument) and proceed with the summand \( I_{2k1i} \). Similarly to the above, first note the identity

\[
\frac{d}{d\xi} v_i(1, \tau(\gamma, \xi)) = \frac{d}{d\tau} v_i(1, \tau(\gamma, \xi)) \partial_\tau \tau(\gamma, \xi),
\]

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and, hence,
\[ \frac{d}{d\tau} v_i(1, \tau(\gamma, \xi)) = [\partial_\xi \tau(\gamma, \xi)]^{-1} \frac{d}{d\xi} v_i(1, \tau(\gamma, \xi)). \]

Substituting the latter into the summand \( I_{2k_{11}} \) and integrating by parts, we easily arrive at the desired estimate for this summand.

Summarizing, the final estimate for \( I_2 \) is as follows:
\[
\max_{s \leq t \leq s + 3d} \| I_2(\cdot, t) \|_{L^2((0,1);\mathbb{R}^n)} \leq \beta_5(\varepsilon) \| \varphi \|_{L^2((0,1);\mathbb{R}^n)},
\]
where the function \( \beta_5(\varepsilon) \) approaches zero as \( \varepsilon \to 0 \). This means that we finish with the upper bound for the first summand in (3.24).

The proof is therewith complete. \( \square \)

4 Abstract setting

4.1 Formulation of the abstract problem

Let us write down the linear nonhomogeneous problem (3.1), (1.2), (1.5) in the form of an abstract evolution equation in \( L^2((0,1);\mathbb{R}^n) \). As usually, by \( H^1((0,1);\mathbb{R}^n) \) we denote the Sobolev space of all functions \( u \in L^2((0,1);\mathbb{R}^n) \) whose distributional derivative \( u' \) is in \( L^2((0,1);\mathbb{R}^n) \). Denote \( v(t) = (u_1(0,t), \ldots u_m(0,t), u_{m+1}(1,t), \ldots u_n(1,t)) \)

and define a one-parameter family of operators \( A(t) \) from \( L^2((0,1);\mathbb{R}^n) \) to \( L^2((0,1);\mathbb{R}^n) \) for each \( t \in \mathbb{R} \) by
\[
(A(t)u)(x) = \left( -a(x,t) \frac{\partial}{\partial x} - b(x,t) \right) u,
\]

with the domain
\[
D(A(t)) = \{ u \in H^1((0,1);\mathbb{R}^n) : v(t) = (Ru)(t) \} \subset L^2((0,1);\mathbb{R}^n),
\]
where the operator \( R \) is given by (3.7). Note that \( D(A(t)) = D \) is independent of \( t \).

Writing \( u(t) \) and \( f(t) \), we mean bounded and continuous maps \( u : \mathbb{R} \to L^2((0,1);\mathbb{R}^n) \) and \( f : \mathbb{R} \to L^2((0,1);\mathbb{R}^n) \) defined by \( [u(t)](x) = u(x,t) \) and \( [f(t)](x) = f(x,t) \), respectively. In this notation, the problem (3.1), (1.2), (1.5) can be written in the abstract form
\[
\frac{d}{dt} u = A(t)u + f(t), \quad u(s) = \varphi \in L^2((0,1);\mathbb{R}^n).
\]

Given \( \varphi \in D \), a function \( u \in C([s,\infty);L^2((0,1);\mathbb{R}^n)) \) is called a \textit{classical solution to the abstract problem} (4.1) if \( u \) is continuously differentiable in \( L^2((0,1);\mathbb{R}^n) \) for \( t > s \), \( u(t) \in D \) for \( t > s \) and (4.1) is satisfied in \( L^2((0,1);\mathbb{R}^n) \).
4.2 Equivalence between the original and the abstract problem settings

Here we show that, if \( \varphi \in D \), then the \( L^2 \)-generalized solution to the problem (3.1), (1.2), (1.5) is a classical solution to the abstract problem (4.1) and vice versa.

**Theorem 4.1** Suppose that \( a, b \in BC^1(\overline{\Pi}; \mathbb{M}_n) \), \( f \in BC^1(\overline{\Pi}; \mathbb{R}_n) \), and the condition (1.6) is fulfilled. If \( \varphi \in D \) and \( u(x, t) \) is the \( L^2 \)-generalized solution to the problem (3.1), (1.2), (1.5), then the function \( u(t) \) such that \( [u(t)](x) := u(x, t) \), is a classical solution to the abstract problem (4.1). Vice versa, if \( u(t) \) is a classical solution to the abstract problem (4.1), then \( u(x, t) := [u(t)](x) \) is an \( L^2 \)-generalized solution to the problem (3.1), (1.2), (1.5).

The proof of the theorem is based on Lemmas 4.2–4.5 below.

**Lemma 4.2** Let the initial function \( \varphi \) belongs to \( C^1([0,1]; \mathbb{R}^n) \) and fulfills the zero order compatibility conditions (3.2). Then there exist constants \( K_2, \nu_2 \) such that the piecewise continuously differentiable solution \( u \) to the problem (3.1), (1.2), (1.5) (ensured by Theorem 3.1 (i)) fulfills the estimate

\[
\|u(\cdot, t)\|_{H^1((0,1); \mathbb{R}^n)} + \|\partial_t u(\cdot, t)\|_{L^2((0,1); \mathbb{R}^n)} \leq K_2 e^{\nu_2(t-s)} \left( \|\varphi\|_{H^1((0,1); \mathbb{R}^n)} + \sup_{t \in \mathbb{R}} \|f(\cdot, t)\|_{L^2((0,1); \mathbb{R}^n)} + \sup_{t \in \mathbb{R}} \|\partial_t f(\cdot, t)\|_{L^2((0,1); \mathbb{R}^n)} \right) \tag{4.2}
\]

for all \( t \geq s \).

**Proof.** We proceed similarly to [20, Lemma 4.2]. Take a scalar product of (3.1) and \( u \) in \( \mathbb{R}^n \) and integrate the resulting system over the domain \( \Pi_s^t \). We get

\[
\int \int_{\Pi_s^t} \left( \frac{\partial}{\partial \theta} (u, u) + \frac{\partial}{\partial x} (au, u) \right) \, dx \, d\theta = \int \int_{\Pi_s^t} (-2(bu, u) + (\partial_x au, u) + 2(f, u)) \, dx \, d\theta.
\]

Here and in what follows, \((\cdot, \cdot)\) denotes the scalar product in \( \mathbb{R}^n \). Applying Green’s formula to the left hand side, we obtain

\[
\|u(\cdot, t)\|_{L^2((0,1); \mathbb{R}^n)}^2 + \int_s^t \left( \sum_{j=1}^n a_j(1, \theta) u_j^2(1, \theta) - \sum_{j=1}^n a_j(0, \theta) u_j^2(0, \theta) \right) \, d\theta = \|\varphi\|_{L^2((0,1); \mathbb{R}^n)}^2 + \int \int_{\Pi_s^t} (-2(bu, u) + (\partial_x au, u) + 2(f, u)) \, dx \, d\theta. \tag{4.3}
\]

Suppose first that the boundary conditions (1.2) are dissipative, i.e.

\[
\sum_{j=1}^m a_j(1, t) u_j^2(1, t) - \sum_{j=m+1}^n a_j(0, t) u_j^2(0, t) + \sum_{j=m+1}^n a_j(1, t) (Ru_j)^2(t) - \sum_{j=1}^m a_j(0, t) (Ru_j)^2(t) \geq 0. \tag{4.4}
\]
Then from (4.3) we have

\[ \|u(\cdot, t)\|_{L^2((0,1);\mathbb{R}^n)}^2 \leq \|\varphi\|_{L^2((0,1);\mathbb{R}^n)}^2 + \int \int_{\Pi_t} \left|((\partial_x a - 2b)u, u) + 2(f, u)\right| \, dx \, d\theta \]

\[ \leq \|\varphi\|_{L^2((0,1);\mathbb{R}^n)}^2 + (t - s) \sup_{t \in \mathbb{R}} \|f(\cdot, t)\|_{L^2((0,1);\mathbb{R}^n)}^2 + \kappa_1 \int_s^t \|u(\cdot, \theta)\|_{L^2((0,1);\mathbb{R}^n)}^2 \, d\theta, \tag{4.5} \]

where \( \kappa_1 = n \|\partial_x a - 2b\|_{BC(\Pi;\mathbb{R}^n)} + 1. \)

Let us show that the inequality (4.4), supposed above, causes no loss of generality. Let \( \mu_j(x, t) \) be arbitrary smooth functions satisfying the conditions

\[ \inf_{\Pi_t} |\mu_j| > 0, \quad \sup_{\Pi_t} |\mu_j| < \infty \quad \text{for all } j \leq n. \]

The change of each variable \( u_j \) to \( v_j = \mu_j u_j \) brings the system (1.4) to

\[ \partial_t v_j + a_j(x, t) \partial_x v_j - \frac{\partial_t \mu_j + a_j(x, t) \partial_x \mu_j}{\mu_j} v_j + \sum_{k=1}^n b_{jk} \frac{\mu_j}{\mu_k} v_k = 0 \tag{4.6} \]

and the boundary conditions (1.2) to

\[ v_j(0, t) = \sum_{k=1}^m p_{jk} \frac{\mu_j(0, t)}{\mu_k(1, t)} v_k(1, t) + \sum_{k=m+1}^n p_{jk} \frac{\mu_j(0, t)}{\mu_k(0, t)} v_k(0, t), \quad 1 \leq j \leq m, \]

\[ v_j(1, t) = \sum_{k=1}^m p_{jk} \frac{\mu_j(1, t)}{\mu_k(1, t)} v_k(1, t) + \sum_{k=m+1}^n p_{jk} \frac{\mu_j(1, t)}{\mu_k(0, t)} v_k(0, t), \quad m < j \leq n. \tag{4.7} \]

Note that the resulting system (4.6), (4.7) is of the type (1.4), (1.2), and the inequality (4.4) for it reads

\[ \sum_{j=1}^m a_j(1, t) v_j^2(1, t) - \sum_{j=m+1}^n a_j(0, t) v_j^2(0, t) \]

\[ + \sum_{j=m+1}^n a_j(1, t) \left[ \sum_{k=m+1}^n p_{jk} \frac{\mu_j(1, t)}{\mu_k(0, t)} v_k(0, t) + \sum_{k=1}^m p_{jk} \frac{\mu_j(1, t)}{\mu_k(1, t)} v_k(1, t) \right]^2 \]

\[ - \sum_{j=1}^m a_j(0, t) \left[ \sum_{k=m+1}^n p_{jk} \frac{\mu_j(0, t)}{\mu_k(0, t)} v_k(0, t) + \sum_{k=1}^m p_{jk} \frac{\mu_j(0, t)}{\mu_k(1, t)} v_k(1, t) \right]^2 \geq 0. \tag{4.8} \]

One can easily see that the functions \( \mu_j \) can be chosen so that the left hand side of (4.8) is a non-negative definite quadratic form with respect to \( v_j(1, t), j \leq m \) and \( v_j(0, t), m + 1 \leq j \leq n \). This finishes the proof of the desired statement.

Further we will estimate \( \|\partial_t u(\cdot, t)\|_{L^2((0,1);\mathbb{R}^n)} \). With this aim, set

\[ v = \partial_t u, \]

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where \( \partial_t \) denotes the distributional derivative. Formal differentiation of (3.1) and (1.2) in \( t \) (in a distributional sense) combined with (3.1) gives

\[
\partial_t v + a \partial_x v + (b - a^{-1} \partial_t a) v + (\partial_t b - a^{-1} \partial_t a b) u = \partial_t f - a^{-1} \partial_t a f
\]  

(4.9)

and

\[
v_j(0, t) = \sum_{k=m+1}^{n} p_{jk}v_k(0, t) + \sum_{k=1}^{m} p_{jk}v_k(1, t) \quad 1 \leq j \leq m, \\
v_j(1, t) = \sum_{k=1}^{m} p_{jk}v_k(0, t) + \sum_{k=m+1}^{n} p_{jk}v_k(1, t) \quad m < j \leq n,
\]  

(4.10)

all the equalities being understood in the distributional sense. We endow the system (4.9)–(4.10) with initial conditions

\[
v(x, s) = -a(x, s)\varphi'(x) - b(x, s)\varphi(x) + f(x, s).
\]  

(4.11)

Note that (4.9)–(4.11) is the initial-boundary value problem with respect to \( v \).

Fix an arbitrary \( t \geq s \). As it follows from Theorem 3.1, the vector-function \( v \) is piecewise continuous in \( \Pi_{t}^{s} \), with a finite number of first order discontinuities (if any) along certain characteristic curves. The union of those characteristic curves will be denoted by \( J \). From the equation (4.9) we conclude that the generalized directional derivatives

\[
z_j = \partial_t v_j + a_j \partial_x v_j
\]

are continuous functions on \( \Pi_{t}^{s} \setminus J \), with possible first order discontinuities on \( J \). This means that the system (4.9) is satisfied pointwise everywhere on \( \Pi_{t}^{s} \setminus J \), while the system (4.10) is satisfied everywhere on \([s, t]\) excepting a finite number of points.

Consequently, we have the following pointwise identity on \( \Pi_{t}^{s} \setminus J \):

\[
z + (b - a^{-1} \partial_t a) v + (\partial_t b - a^{-1} \partial_t a b) u = \partial_t f - a^{-1} \partial_t a f.
\]  

(4.12)

Multiplying (4.12) by \( v \) and integrating the resulting system over the domain \( \Pi_{t}^{s} \), we get

\[
\int \int_{\Pi_{t}^{s}} (z, v) \, dx d\theta = -\int \int_{\Pi_{t}^{s}} ((b - a^{-1} \partial_t a) v + (\partial_t b - a^{-1} \partial_t a b) u, v) \, dx d\theta \\
+ \int \int_{\Pi_{t}^{s}} (\partial_t f - a^{-1} \partial_t a f, v) \, dx d\theta.
\]  

(4.13)

Since \( C^1 \left( \Pi_{s}^{t}; \mathbb{R}^n \right) \) is densely embedded into \( L^2 \left( \Pi_{s}^{t}; \mathbb{R}^n \right) \), there is a sequence \( v^l \in C^1 \left( \Pi_{s}^{t}; \mathbb{R}^n \right) \), \( l \in \mathbb{N} \), such that

\[
v^l \to v \text{ in } L^2 \left( \Pi_{s}^{t}; \mathbb{R}^n \right) \text{ as } l \to \infty
\]  

(4.14)

Let us show that

\[
\langle \partial_t v^l + a \partial_x v^l, \varphi \rangle_{L^2} \to \langle z, \varphi \rangle_{L^2} \text{ for all } \varphi \in L^2 \left( \Pi_{s}^{t}; \mathbb{R}^n \right),
\]  

(4.15)
where $\langle \cdot, \cdot \rangle_{L^2} : L^2(\Pi_s^t; \mathbb{R}^n) \times L^2(\Pi_s^t; \mathbb{R}^n) \to \mathbb{R}$ denotes a scalar product in $L^2(\Pi_s^t; \mathbb{R}^n)$. Indeed, due to (4.14), for any $\varphi \in C_0^\infty(\Pi_s^t; \mathbb{R}^n)$ we have

$$\langle \partial_t v^l + a \partial_x v^l, \varphi \rangle_{L^2} = -\langle v^l, \partial_t \varphi + \partial_x (a \varphi) \rangle_D$$

$$\to -\langle v, \partial_t \varphi + \partial_x (a \varphi) \rangle_D = \langle \partial_t v + a \partial_x v, \varphi \rangle_D = \langle z, \varphi \rangle_{L^2},$$

where $\langle \cdot, \cdot \rangle_D : \mathcal{D}'(\Pi_s^t; \mathbb{R}^n) \times \mathcal{D}(\Pi_s^t; \mathbb{R}^n) \to \mathbb{R}$ denotes a dual pairing in $\mathcal{D}'$ and $\partial_t$ and $\partial_x$ are understood in a distributional sense. As the space $C_0^\infty(\Pi_s^t; \mathbb{R}^n)$ is dense in $L^2(\Pi_s^t; \mathbb{R}^n)$, the desired assertion (4.15) follows.

On the account of (4.15), it holds

$$\int \int_{\Pi_s^t} (z, v) \, dx \, d\theta$$

$$= \lim_{r \to \infty} \int \int_{\Pi_s^t} (\partial_t v^r + a \partial_x v^r, v) \, dx \, d\theta + \lim_{l \to \infty} \int \int_{\Pi_s^t} (\partial_t v^l + a \partial_x v^l, v^l) \, dx \, d\theta$$

$$= -\lim_{r \to \infty} \lim_{l \to \infty} \int \int_{\Pi_s^t} (v^r, \partial_t v^l + a \partial_x v^l) \, dx \, d\theta + \lim_{l \to \infty} \lim_{r \to \infty} \int_0^1 (v^r, v^l) \, dx$$

$$\to \int \int_{\Pi_s^t} (v, \partial_x v^l + a \partial_x v) \, dx \, d\theta - \int \int_{\Pi_s^t} (v, a v) \, dx \, d\theta$$

$$+ \int_0^1 \sum_{j=1}^n [v_j^2(x, t) - v_j^2(x, s)] \, dx + \int_s^t \left( \sum_{j=1}^n a_j(1, \theta)v_j^2(1, \theta) - \sum_{j=1}^n a_j(0, \theta)v_j^2(0, \theta) \right) \, d\theta.$$

Consequently,

$$2 \int \int_{\Pi_s^t} (z, v) \, dx \, d\theta = -\int \int_{\Pi_s^t} (v, \partial_x a v) \, dx \, d\theta + \int_0^1 \sum_{j=1}^n [v_j^2(x, t) - v_j^2(x, s)] \, dx$$

$$+ \int_s^t \left( \sum_{j=1}^n a_j(1, \theta)v_j^2(1, \theta) - \sum_{j=1}^n a_j(0, \theta)v_j^2(0, \theta) \right) \, d\theta. \quad (4.16)$$

Combining (4.16) with (4.13), we have

$$\|v(\cdot, t)\|^2_{L^2((0,1); \mathbb{R}^n)} + \int_s^t \left( \sum_{j=1}^n a_j(1, \theta)v_j^2(1, \theta) - \sum_{j=1}^n a_j(0, \theta)v_j^2(0, \theta) \right) \, d\theta$$

$$= \|a(\cdot, s)\varphi + b(\cdot, s)\varphi - f(\cdot, s)\|^2_{L^2((0,1); \mathbb{R}^n)} + \int \int_{\Pi_s^t} \left( [\partial_x a - 2b + 2a^{-1}\partial_x a] v, v \right) \, dx \, d\theta$$

$$-2 \int \int_{\Pi_s^t} \left( [\partial_t b - a^{-1}\partial_x a b] u, v \right) \, dx \, d\theta + 2 \int \int_{\Pi_s^t} \left( \partial_t f - a^{-1}\partial_x a f, v \right) \, dx \, d\theta. \quad (4.17)$$

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We now use the dissipativity condition (4.4) (similarly to the above, this causes no loss of generality). The equation (4.17) yields

\[
\begin{align*}
\|v(\cdot, t)\|_{L^2((0,1);\mathbb{R}^n)}^2 d\theta &\leq \|a(\cdot, s)\varphi + b(\cdot, s)\varphi - f(\cdot, s)\|_{L^2((0,1);\mathbb{R}^n)}^2 \\
+ \kappa_2 \int_s^t \|f(\cdot, \theta)\|_{L^2((0,1);\mathbb{R}^n)}^2 d\theta + \int_s^t \|\partial_t f(\cdot, \theta)\|_{L^2((0,1);\mathbb{R}^n)}^2 d\theta \\
+ \kappa_3 \left( \int_s^t \|u(\cdot, \theta)\|_{L^2((0,1);\mathbb{R}^n)}^2 d\theta + \int_s^t \|v(\cdot, \theta)\|_{L^2((0,1);\mathbb{R}^n)}^2 d\theta \right),
\end{align*}
\]

(4.18)

where the constants \(\kappa_2\) and \(\kappa_3\) depend on \(a\) and \(b\) but not on \(f\) and \(\varphi\).

Furthermore, we sum up (4.5) and (4.18). After applying the Gronwall’s argument to the resulting inequality, we get the bound

\[
\|u(\cdot, t)\|_{L^2((0,1);\mathbb{R}^n)} + \|\partial_t u(\cdot, t)\|_{L^2((0,1);\mathbb{R}^n)} \leq K_{22}e^{\nu_2(t-s)}\left(\|\varphi\|_{H^1((0,1);\mathbb{R}^n)} \right)
+ \sup_{t \in \mathbb{R}} \|f(\cdot, t)\|_{L^2((0,1);\mathbb{R}^n)} + \sup_{t \in \mathbb{R}} \|\partial_t f(\cdot, t)\|_{L^2((0,1);\mathbb{R}^n)}
\]

for all \(t \geq s\) and some positive constants \(K_{22}\) and \(\nu_2\).

A similar estimate for \(\|\partial_x u(\cdot, t)\|_{L^2((0,1);\mathbb{R}^n)}\) easily follows from (3.1). This completes the proof of (4.2). \(\square\)

**Lemma 4.3** Let \(\varphi \in D\). Then
(i) the continuous solution \(u\) to the problem (3.1), (1.2), (1.5) belongs to \(C([s, t], H^1((0,1);\mathbb{R}^n)) \cap \) and to \(C^1([s, t], L^2((0,1);\mathbb{R}^n))\);
(ii) the function \(U(t,s)\varphi\) for \(t \geq s\) is continuously differentiable in \(t\) and satisfies the homogeneous abstract equation (4.1) in \(L^2((0,1);\mathbb{R}^n))\).

**Proof.** Define \(\tilde{\varphi}(x) = \varphi(0) + x(\varphi(1) - \varphi(0))\) for \(x \in [0, 1]\). Note that \(\varphi - \tilde{\varphi} \in H^1_0((0,1);\mathbb{R}^n)\), see [10, p. 259]. Therefore, there exists a sequence \(\varphi_0 \in C_0^\infty([0,1];\mathbb{R}^n)\) approaching \(\varphi - \tilde{\varphi}\) in \(H^1((0,1);\mathbb{R}^n)\). It follows that the sequence \(\varphi^l = \varphi_0 + \tilde{\varphi}\) approaches \(\varphi\) in \(H^1((0,1);\mathbb{R}^n)\).

By Theorem 3.1 (ii) and Lemma 4.2, the piecewise continuously differentiable solution \(u^l\) to the problem (3.1), (1.2), (1.5) with \(\varphi^l\) in place of \(\varphi\) satisfies the estimate (4.2) with \(u = u^l\) and \(\varphi = \varphi^l\). This entails the convergence

\[
\max_{s \leq \tau \leq t} \|u^m(\cdot, \tau) - u^l(\cdot, \tau)\|_{H^1((0,1);\mathbb{R}^n)} + \max_{s \leq \tau \leq t} \|\partial_t u^m(\cdot, \tau) - \partial_t u^l(\cdot, \tau)\|_{L^2((0,1);\mathbb{R}^n)} \to 0 \quad (4.19)
\]

as \(m, l \to \infty\) for each \(t > s\). Consequently, the sequence \(u^l\) converges in \(C([s, t], H^1((0,1);\mathbb{R}^n)) \cap C^1([s, t], L^2((0,1);\mathbb{R}^n))\). This proves Claim (i).

Claim (ii) now easily follows from Claim (i). \(\square\)

\[35\]
Lemma 4.4 Let $\varphi \in D$. A function $u$ is the $L^2$-generalized solution to the problem (3.1), (1.2), (1.5) (see Theorem 3.3) if and only if it is the continuous solution to the problem (3.1), (1.2), (1.5) (see Theorem 3.2 (ii)).

Proof. Necessity. Notice first that similarly to the proof of Lemma 4.3 there is a sequence $\varphi^l \in C^1([0,1];\mathbb{R}^n)$ approaching $\varphi$ in $H^1((0,1);\mathbb{R}^n))$. We use the convergence (4.19) for the piecewise continuously differentiable solution $u^l$ to the problem (3.1), (1.2), (1.5) with $\varphi^l$ in place of $\varphi$. It follows that, given $s < t$, $u^l$ converges in $C([0,1] \times [s,t];\mathbb{R}^n)$. This means that the $L^2$-generalized solution has, in fact, better regularity.

Now, since $u^l$ for each $l \in \mathbb{N}$ is a continuous solution, it satisfies the system

$$
u^l_j(x,t) = (Qu^l)_j(x,t) - \int_{x_j(x,t)}^{x} d_j(\xi, x, t) \left( \sum_{k \neq j} b_{jk}(\xi, \omega_j(\xi)) u^l_k(\xi, \omega_j(\xi)) - f(\xi, \omega_j(\xi)) \right) d\xi$$

for all $j \leq n$, where the operator $Q$ is given by (3.8). Letting $l \to \infty$ finishes the proof of the necessity.

Sufficiency. We use the fact that the problem (3.1), (1.2), (1.5), according to Theorem 3.3, has a unique $L^2$-generalized solution $\bar{u}$. By uniqueness, $\bar{u}$ is the limit of $u^l$ in the sense of Definition 3.2. Due to (4.19), $\bar{u}$ is a continuous function that coincides with $u$. \qed

Lemma 4.5 Let $\varphi \in D$. A continuous function $u$ is the continuous solution to the problem (3.1), (1.2), (1.5) if and only if $u$ satisfies (3.1) in a distributional sense and (1.2) and (1.5) pointwise.

Proof. Necessity. Let $u$ be the continuous solution to the problem (3.1), (1.2), (1.5). It is straightforward to check that $u$ fulfills (1.2) and (1.5). It remains to show that $u$ satisfies (3.1) in a distributional sense. Fix arbitrary $j \leq n$ and $s < t$ and take an arbitrary sequence $u^l \in C^1([0,1] \times [s,t];\mathbb{R}^n)$ approaching $u$ in $C([0,1] \times [s,t];\mathbb{R}^n)$. Then for any smooth function $\phi : (0,1) \times (s,t) \to \mathbb{R}$ with compact support we have

$$\left\langle (\partial_t + a_j \partial_x)u_j, \phi \right\rangle = \left\langle u_j, -\partial_t \phi - \partial_x (a_j \phi) \right\rangle = \lim_{l \to \infty} \left\langle u^l_j, -\partial_t \phi - \partial_x (a_j \phi) \right\rangle = \lim_{l \to \infty} \left\langle (Qu^l)_j(x,t) \right\rangle$$

$$- \int_{x_j(x,t)}^{x} d_j(\xi, x, t) \sum_{k \neq j} b_{jk}(\xi, \omega_j(\xi)) u^l_k(\xi, \omega_j(\xi)) - f^l_j(\xi, \omega_j(\xi)) d\xi, -\partial_t \phi - \partial_x (a_j \phi) \right\rangle$$

$$= \lim_{l \to \infty} \left\langle - \sum_{k=1}^{n} b_{jk}(x,t) u^l_k(x,t) + f^l_j(x,t), \phi \right\rangle = \left\langle - \sum_{k=1}^{n} b_{jk}(x,t) u_k + f_j(x,t), \phi \right\rangle$$

as desired. Here we used the formula

$$(\partial_t + a_j(x,t) \partial_x)\psi(\omega_j(\xi, x, t)) = 0$$

being true for all $j \leq n$, $\xi, x \in [0,1]$, $t \geq s$, and for any $\psi \in C^1(\mathbb{R})$. 

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Sufficiency. Assume that a continuous vector-function \( u \) satisfies (3.1) in a distributional sense and (1.2) and (1.5) pointwise. Note the identity

\[
(\partial_t + a_j(x, t)\partial_x) x_j(x, t) = 0.
\] (4.21)

In the domain \( \{(x, t) \in \Pi_s : t > \omega_j(x_j(x, t), x, t)\} \) it is obvious. In the domain \( \{(x, t) \in \Pi_s : t < \omega_j(x_j(x, t), x, t)\} \) this identity easily follows from the identity \( \omega_j(x_j(x, t), x, t) = s \), after applying the operator \( \partial_t + a_j(x, t)\partial_x \) to both sides and using the equation (4.20).

On the account of (4.20) and (4.21), we rewrite the system (3.1) in the form

\[
(\partial_t + a_j(x, t)\partial_x) \left( c^{-1}_j(x_j(x, t), x, t) u_j \right) = c^{-1}_j(x_j(x, t), x, t) \left( -\sum_{k \neq j} b_{jk}(x, t) u_k + f_j(x, t) \right),
\] (4.22)

without destroying the equalities in the sense of distributions. To prove that \( u \) satisfies (3.9) pointwise, we use the constancy theorem of distribution theory claiming that any distribution on an open set with zero generalized derivatives is a constant on any connected component of the set. Hence, due to (4.22), for each \( j \leq n \) the expression

\[
c^{-1}_j(x_j(x, t), x, t) \left( u_j(x, t) + \int_{x_j(x, t)}^{x} d_j(\xi, x, t) \left[ \sum_{k \neq j} b_{jk}(x, t) u_k - f_j(\xi, \omega_j(\xi)) d\xi \right] \right)
\] (4.23)

is a constant along the characteristic curve \( \omega_j(\xi, x, t) \). In other words, the distributional directional derivative \( (\partial_t + a_j(x, t)\partial_x) \) of the function (4.23) is equal to zero. Since (4.23) is a continuous function, \( c_j(x_j(x, t), x, t) = 1 \), and the trace \( u_j(x_j(x, t), t) \) is given by means of (1.2) and (1.5), it follows that \( u \) satisfies the system (3.9) pointwise, as desired. □

Proof of Theorem 4.1. Given \( \varphi \in D \), let \( u \) be the \( L^2 \)-generalized solution to the problem (3.1), (1.2), (1.5). Due to Lemmas 4.4 and 4.5, this solution satisfies (3.1) in a distributional sense and (1.2) and (1.5) pointwise. By Lemma 4.3, the distributional derivatives \( \partial_t u \) and \( \partial_t u \) belong in fact to \( C([s,t], L^2((0,1); \mathbb{R}^n)) \). Consequently, \( u(t) \) is a classical solution to the abstract problem (4.1).

The converse follows from the uniqueness of the classical solution to the abstract problem (4.1). □

5 Proof of the main Theorem 1.5

5.1 Bounded Solutions

Here we prove Theorem 1.5 (i). Suppose that the unperturbed linear system (1.4), (1.2) is exponentially dichotomous with an exponent \( \alpha > 0 \), a bound \( M \geq 1 \), and with the dichotomy projectors \( P(t), t \in \mathbb{R} \). By Theorem 3.8, there exist \( \varepsilon_0 > 0, \alpha_1 \leq \alpha, \) and \( M_1 \geq M \) such that for all \( \tilde{a} \) and \( \tilde{b} \) with \( \|\tilde{a}\|_{BC^1(\Pi; M_n)} \leq \varepsilon_0 \) and \( \|\tilde{b}\|_{BC^1(\Pi; M_n)} \leq \varepsilon_0 \) the perturbed system
(3.16), (1.2) is exponentially dichotomous with the exponent $\alpha_1$ and the bound $M_1$. Since the functions $A$ and $B$ are $C^2$-smooth, there exists positive $\delta \leq \delta_0$ such that

$$\sup\{|A_j(x, t, v) - A_j(x, t, 0)| : (x, t) \in \overline{\Pi}, \|v\| \leq \delta, 1 \leq j \leq n\} \leq \varepsilon_0,$$

$$\sup\{|B_{jk}(x, t, v) - B_{jk}(x, t, 0)| : (x, t) \in \overline{\Pi}, \|v\| \leq \delta, 1 \leq j, k \leq n\} \leq \varepsilon_0.$$

Then, given $\varphi \in BC^1(\overline{\Pi}; \mathbb{R}^n)$, the system

$$\partial_t u + A(x, t, \varphi)\partial_x u + B(x, t, \varphi)u = 0$$

with boundary conditions (1.2) has the exponential dichotomy with the constants $\alpha_1$ and $M_1$ whenever $\|\varphi\|_{BC^1(\overline{\Pi}; \mathbb{R}^n)} \leq \delta$. The proof will be based on the following iteration procedure. Put $u_0(x, t) \equiv 0$. We will obtain the iteration $u_{k+1}(x, t)$ as the unique $BC^2(\overline{\Pi}; \mathbb{R}^n)$-smooth bounded solution to the linear system

$$\partial_t u + a^k(x, t)\partial_x u + b^k(x, t)u = f(x, t), \quad k = 0, 1, 2, \ldots,$$  \hspace{1cm} (5.24)

with the boundary conditions (1.2). Here $a^k(x, t) = A(x, t, u^k(x, t))$ and $b^k(x, t) = B(x, t, u^k(x, t))$.

We divide the proof into three claims.

Claim 1. Suppose that

$$\|f\|_{BC^2(\overline{\Pi}; \mathbb{R}^n)} \leq \frac{1}{L} \left(\frac{2M_1}{\alpha_1} + 1\right)^{-1} \delta,$$  \hspace{1cm} (5.25)

where the constant $L$ is defined in Theorem 3.5. Then there exists a sequence $u^k$ of $C^2$-solutions to (5.24), (1.2) such that $\|u^k\|_{BC^2(\overline{\Pi}; \mathbb{R}^n)} \leq \delta$ for all $k$.

The proof will be done using induction in $k$. To treat the base case $k = 0$, let us construct $u^1(x, t)$. Consider (5.24), (1.2) for $k = 0$ and switch to the abstract problem setting. Recall the equivalence of both settings is proved in Section 4.2. Since $A(x, t, 0) = a(x, t)$ and $B(x, t, 0) = b(x, t)$, the homogeneous system (5.24), (1.2) (or, the same, its abstract version (4.1) with $f = 0$) is dichotomous by the assumption. This implies (see [2]) that the nonhomogeneous system (4.1) has a unique bounded $L^2$-generalized solution $u^1(t)$ given by

$$u^1(t) = \int_{-\infty}^{\infty} G_0(t, s)f(s)ds,$$  \hspace{1cm} (5.26)

where $U_0(t, s) = U(x, t)$ is the evolution operator generated by the linear system (1.4), (1.2) and

$$G_0(t, s) = \begin{cases} U_0(t, s)P(s), & t \geq s, \\ U_0(t, s)(I - P(s)), & t < s \end{cases}$$

is the corresponding Green function satisfying the inequality

$$\|G_0(t, s)\|_{L^2((0, 1); \mathbb{R}^n)} \leq Me^{-\alpha_1|t-s|}, \quad t, s \in \mathbb{R}.$$
Moreover, we have
\[
\|u^1(t)\|_{L^2((0,1);\mathbb{R}^n)} \leq \int_{-\infty}^{\infty} \|G_0(t,s)\|_{L((0,1);\mathbb{R}^n)} \|f(s)\|_{L^2((0,1);\mathbb{R}^n)} ds \\
\leq \frac{2M_1}{\alpha_1} \|f\|_{BC^2(\Pi;\mathbb{R}^n)}. \tag{5.27}
\]

Let us show that \( u^1 \) actually has \( C^2 \)-regularity. With this aim, let us rewrite \( u^1(t) \) in the form (see [14, p. 228])
\[
u^1(t) = U_0(t,t_0)u^1(t_0) + \int_{t_0}^{t} U_0(t,s)f(s)ds, \quad t \geq t_0. \tag{5.28}
\]

Given an arbitrary \( t_0 \in \mathbb{R} \), the function \( U_0(t,t_0)u^1(t_0) \) is an \( L^2 \)-generalized solution to the equation (4.1) with \( f = 0 \) (or, the same, to the system (1.4), (1.2)) with the initial value \( u^1(t_0) \). By Theorem 3.5, the function \([U_0(t,t_0)u^1(t_0)](x)\) has a \( C^2 \)-regularity for \( t \geq t_0+3d, x \in [0,1] \). Since the map \( f : \mathbb{R} \to L^2((0,1);\mathbb{R}^n) \) is differentiable, the second summand in (5.28), denoted by \( w(t) \), is a classical solution to the abstract equation (4.1) subjected to the initial condition \( w(t_0) = 0 \) (see, e.g. [30, p. 147], [24, p. 197]). Due to Theorem 4.1, the function \( w(t) \) is a classical solution of (4.1) if and only if it is an \( L^2 \)-generalized solution to the problem (3.1), (1.2). By Theorem 3.5, the function \([w(t)](x)\) has a \( C^2 \)-regularity for \( t \geq t_0+3d, x \in [0,1] \).

As \( t_0 \in \mathbb{R} \) is arbitrary, \( u^1(x,t) \) has \( C^2 \)-regularity in the whole domain \( \Pi \). Due to the inequalities (3.4) and (5.27), it satisfies the following smoothing estimate:
\[
\|u^1(\cdot,t)\|_{C^2([0,1];\mathbb{R}^n)} \leq L \left( \|u^1(t-3d)\|_{L^2((0,1);\mathbb{R}^n)} + \|f\|_{BC^2(\Pi;\mathbb{R}^n)} \right) \\
\leq L \left( \frac{2M_1}{\alpha_1} + 1 \right) \|f\|_{BC^2(\Pi;\mathbb{R}^n)}.
\]

If \( f \) fulfills (5.25), then \( \|u^1\|_{BC^2(\Pi;\mathbb{R}^n)} \leq \delta \). As a consequence, the linear system (5.24), (1.2) for \( k = 1 \) is exponentially dichotomous, with the same constants \( \alpha_1 \) and \( M_1 \).

Assuming that Claim 1 is true for some \( k \geq 1 \), let us prove it for \( k+1 \). Suppose that \( u^k \) is found such that \( \|u^k\|_{BC^2(\Pi;\mathbb{R}^n)} \leq \delta \). Then the homogeneous system (5.24) (1.2) has an exponential dichotomy with the same constants \( \alpha_1 \) and \( M_1 \). Consider
\[
u^{k+1}(t) = \int_{-\infty}^{\infty} G_k(t,s)f(s)ds,
\]
where
\[
G_k(t,s) = \begin{cases} U_k(t,s)P_k(s), & t \geq s, \\
U_k(t,s)(I-P_k(s)), & t < s,
\end{cases}
\]
\( U_k(t,s) \) is the evolution operator generated by the linear homogeneous system (5.24), (1.2), and \( P_k \) and \( I-P_k \) are the corresponding dichotomy projectors. The Green function \( G_k(t,s) \) satisfies the inequality
\[
\|G_k(t,s)\|_{L((0,1);\mathbb{R}^n)} \leq M_1 e^{-\alpha_1|t-s|}, \quad t, s \in \mathbb{R}.
\]
Similarly to the above, we see that $u^{k+1}$ is $C^2$ smooth. Moreover, due to (5.25), the function $u^{k+1}$ fulfills the estimate
\[
\|u^{k+1}\|_{BC^2(\Pi;\mathbb{R}^n)} \leq L \left( \frac{2M_1}{\alpha_1} + 1 \right) \|f\|_{BC^2(\Pi;\mathbb{R}^n)} \leq \delta.
\]

Claim 1 is therewith proved.

Claim 2. Let
\[
\varepsilon = \min \left\{ \delta L^{-1} \left( \frac{2M_1}{\alpha_1} + 1 \right)^{-1}, (L^2N)^{-1} \left( \frac{2M_1}{\alpha_1} + 1 \right)^{-2} \right\}.
\]
If $\|f\|_{BC^2(\Pi;\mathbb{R}^n)} \leq \varepsilon$, then the sequence $\{u^k\}$ converges in $BC^1(\overline{\Pi};\mathbb{R}^n)$. The difference $w^{k+1} = u^{k+1} - u^k$ belongs to $BC^2(\overline{\Pi};\mathbb{R}^n)$ and satisfies the system
\[
\partial_t w^{k+1} + a^k(x,t)\partial_x w^{k+1} + b^k(x,t) w^{k+1} = f^k(x,t),
\]
with the boundary conditions (1.2), where
\[
f^k(x,t) = - (b^k(x,t) - b^{k-1}(x,t)) u^k(x,t) - (a^k(x,t) - a^{k-1}(x,t)) \partial_x u^k(x,t).
\]
The right-hand side of (5.30) is $C^1$-smooth in $x$ and $t$ and satisfies the estimate
\[
\|f^k\|_{BC^1(\Pi;\mathbb{R}^n)} \leq N_1 \|u^k\|_{BC^2(\Pi;\mathbb{R}^n)} \|w^k\|_{BC^1(\Pi;\mathbb{R}^n)},
\]
where the constant $N_1$ depends on $A(x,t,u^k)$ and $B(x,t,u^k)$ but not on $w^k$. Additionally, since the estimate (5.29) is uniform in $k$, $N_1$ can be chosen common for all $k \in \mathbb{Z}$.

Analogously to (5.26) and (5.27), $w^{k+1} : \mathbb{R} \to L^2((0,1);\mathbb{R}^n)$ reads
\[
w^{k+1}(t) = \int_{-\infty}^{\infty} G_k(t,s)f^k(s)ds,
\]
and satisfies the estimate
\[
\|w^{k+1}(t)\|_{L^2((0,1);\mathbb{R}^n)} \leq \int_{-\infty}^{\infty} \|G_k(t,s)\|_{L(L^2((0,1);\mathbb{R}^n))} \|f^k\|_{BC^1(\Pi;\mathbb{R}^n)} ds
\leq \frac{2M_1}{\alpha_1} \|u^k\|_{BC^2(\Pi;\mathbb{R}^n)} \|w^k\|_{BC^1(\Pi;\mathbb{R}^n)}.
\]

Now, consider $w^{k+1}(x,t)$ as a solution to the initial-boundary value problem (5.30), (1.2) with the initial value $w^{k+1}(t-2d)$. Using Theorem 3.5 and the inequalities (5.29) and (5.31), we get
\[
\|w^{k+1}(t)\|_{BC^1([0,1];\mathbb{R}^n)} \leq L \left( \|w^{k+1}(t-2d)\|_{L^2((0,1);\mathbb{R}^n)} + \|f^k\|_{BC^1(\Pi;\mathbb{R}^n)} \right)
\leq LN_1 \left( \frac{2M_1}{\alpha_1} + 1 \right) \|u^k\|_{BC^2(\Pi;\mathbb{R}^n)} \|w^k\|_{BC^1(\Pi;\mathbb{R}^n)}
\leq L^2 N_1 \left( \frac{2M_1}{\alpha_1} + 1 \right)^2 \|f\|_{BC^2(\Pi;\mathbb{R}^n)} \|w^k\|_{BC^1(\Pi;\mathbb{R}^n)}.
\]
If
\[ L^2 N_1 \left( \frac{2M_1}{\alpha_1} + 1 \right)^2 \| f \|_{BC^2(\Pi; \mathbb{R}^n)} < 1 \]  
then the sequence \( \{ w^k \} \) tends to zero in \( BC^1(\Pi; \mathbb{R}^n) \). Consequently, if \( \| f \|_{BC^2(\Pi; \mathbb{R}^n)} \leq \varepsilon \), then \( f \) fulfills the inequalities (5.25) and (5.32), which implies that the sequence \( u^k \) converges in \( BC^1(\Pi; \mathbb{R}^n) \) to some function \( u^* \in BC^1(\Pi; \mathbb{R}^n) \). It is a simple matter to show that the function \( u^* \) is a classical solution to the problem (1.1), (1.2) and satisfies the following estimate:
\[ \| u^* \|_{BC^1(\Pi; \mathbb{R}^n)} \leq L \left( \frac{2M_1}{\alpha_1} + 1 \right) \| f \|_{BC^2(\Pi; \mathbb{R}^n)} \leq \delta. \]  

Claim 3. If \( \| f \|_{BC^2(\Pi; \mathbb{R}^n)} \leq \varepsilon \), then the classical solution \( u^* \) to the problem (1.1), (1.2) satisfying the bound (5.33) is unique. On the contrary, suppose that \( \tilde{u} \) is a solution to the problem (1.1), (1.2) different from \( u^* \), such that \( \| \tilde{u} \|_{BC^1(\Pi; \mathbb{R}^n)} \leq \delta \). Then the linear system
\[ \partial_t u + \tilde{a}(x,t) \partial_x u + \tilde{b}(x,t) u = 0 \]
with the boundary conditions (1.2), where \( \tilde{a}(x,t) = A(x,t, \tilde{u}(x,t)) \), \( \tilde{b}(x,t) = B(x,t, \tilde{u}(x,t)) \), is exponentially dichotomous with the same constants \( \alpha_1 \) and \( M_1 \). Clearly, the difference
\[ \tilde{w}^{k+1} = \tilde{u} - u^{k+1} \]
satisfies the system
\[ \partial_t u + \tilde{a}(x,t) \partial_x u + \tilde{b}(x,t) u = \tilde{f}^{k+1}(x,t) \]
with the boundary conditions (1.2), where
\[ \tilde{f}^{k+1}(x,t) = \left( b^k(x,t) - \tilde{b}(x,t) \right) u^{k+1}(x,t) + (a^k(x,t) - \tilde{a}(x,t)) \partial_x u^{k+1}(x,t). \]
Similarly to the above, the function \( \tilde{f}^{k+1}(x,t) \) is \( C^1 \)-smooth in \( x \) and \( t \) and satisfies estimate
\[ \| \tilde{f}^{k+1} \|_{BC^1(\Pi; \mathbb{R}^n)} \leq N_1 \| u^{k+1} \|_{BC^2(\Pi; \mathbb{R}^n)} \| \tilde{w}^k \|_{BC^1(\Pi; \mathbb{R}^n)}. \]
Applying the same estimates as for \( w^k \), we derive the bound
\[ \| \tilde{w}^{k+1}(t) \|_{BC^1([0,1]; \mathbb{R}^n)} \leq L^2 N_1 \left( \frac{2M_1}{\alpha_1} + 1 \right)^2 \| f \|_{BC^2(\Pi; \mathbb{R}^n)} \| \tilde{w}^k \|_{BC^1(\Pi; \mathbb{R}^n)}. \]
Combining it with (5.32), we get the convergence \( \| \tilde{w}^k(t) \|_{BC^1(\Pi; \mathbb{R}^n)} \to 0 \) as \( k \to \infty \). Consequently, \( \tilde{u}(x,t) = u^*(x,t) \), a contradiction.
5.2 Almost Periodic Solutions

Here we prove Theorem 1.5 (5.2 Almost Periodic Solutions), the almost periodic case.

Recall that a continuous function $f : \mathbb{R} \to X$ with values in a Banach space $X$ is called a Bohr almost periodic if for every $\varepsilon > 0$ there exists a relatively dense set of $\varepsilon$-almost periods of $f$, i.e., for every $\varepsilon > 0$ there exists a positive number $l$ such that every interval of length $l$ contains a number $\tau$ such that

$$
\|f(t + \tau) - f(t)\|_X < \varepsilon \quad \text{for all} \quad t \in \mathbb{R}.
$$

As shown in Section 5.1, we are done if we show that, under the assumption that the coefficients $A(x, t, v), B(x, t, v)$, and $f(x, t)$ are almost periodic in $t$, the constructed solution $u^*(x, t)$ is almost periodic in $t$ as well. We use the fact that the limit of a uniformly convergent sequence of almost periodic functions is almost periodic [5]. This means that it suffices to show that the approximating sequence $\{u^k\}$ is a sequence of almost periodic functions.

We will also use the fact that if a function $w(x, t)$ has bounded and continuous partial derivatives up to the second order in $x \in [0, 1]$ and in $t \in \mathbb{R}$ and is Bohr almost periodic in $t$ uniformly with respect to $x$, then the partial derivatives $\partial_x w(x, t)$ and $\partial_t w(x, t)$ are also Bohr almost periodic in $t$ uniformly in $x$.

The almost periodicity of $\partial_t w(x, t)$ follows from [12, Theorem 1.16] and its proof. To prove the almost periodicity of $\partial_x w(x, t)$, we use a similar argument. Let $x \in [0, 1/2]$. Consider the sequence of almost periodic functions in $t$

$$
w^k(x, t) = k \left[ w(x + 1/k, t) - w(x, t) \right], \quad k \geq 2
$$

and prove that it tends to $\partial_x w(x, t)$ as $k \to \infty$, uniformly in $t \in \mathbb{R}$ and $x \in [0, 1/2]$. Indeed,

$$
k \left[ w(x + 1/k, t) - w(x, t) \right] - \partial_x w(x, t) = k \int_0^{1/k} (\partial_x w(x + \xi, t) - \partial_x w(x, t)) d\xi.
$$

Since $w \in BC^2(\overline{\Pi}; \mathbb{R}^n)$, then for any $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$
|\partial_x w(x + \xi, t) - \partial_x w(x, t)| < \varepsilon
$$

for all $\xi \in [0, 1/k_0]$, uniformly in $x \in [0, 1/2]$ and $t \in \mathbb{R}$. This means that the sequence of almost periodic functions $\{w^k(x, t)\}$ tends to $\partial_x w(x, t)$ uniformly in $x \in [0, 1/2]$ and $t \in \mathbb{R}$. Hence, the function $\partial_x w(x, t)$ is almost periodic in $t$ uniformly in $x \in [0, 1/2]$.

For $x \in [1/2, 1]$ we proceed similarly considering the sequence

$$
w^k(x, t) = k \left[ w(x - 1/k, t) - w(x, t) \right], \quad k \geq 2.
$$

Summarizing, this shows the almost periodicity of $\partial_x w(x, t)$.

Turning back to the almost periodicity of $u^*$, first recall that $u^0 \equiv 0$. Assuming that the solution $u^k(x, t)$ to (5.24) is almost periodic in $t$ uniformly in $x$, let us prove that $u^{k+1}(x, t)$ is almost periodic also. Fix an arbitrary $\varepsilon > 0$. Let $h$ be an $\varepsilon$-almost period of almost
periodic in $t$ functions $f(x,t)$, $a^k(x,t) = A(x,t, u^k(x,t))$, $b^k(x,t) = B(x,t, u^k(x,t))$ as well as their derivatives in $x$ and $t$. Then the differences $\tilde{a}^k(x,t) = a^k(x,t + h) - a^k(x,t)$ and $\tilde{b}^k(x,t) = b^k(x,t + h) - b^k(x,t)$ satisfy the inequalities

$$\|\tilde{a}^k\|_{BC^1(\Pi; \mathbb{M}_n)} \leq \varepsilon, \quad \|\tilde{b}^k\|_{BC^1(\Pi; \mathbb{M}_n)} \leq \varepsilon. \quad (5.34)$$

We are done if we prove that $h$ is an almost period of the function $u^{k+1}(x,t)$.

It is known that the functions $u^{k+1}(x,t)$ and $u^{k+1}(x,t + h)$ are unique bounded solutions, respectively, to the system (5.24) with the boundary conditions (1.2) and to the system

$$\partial_t u + a^k(x,t)\partial_x u + b^k(x,t + h)u = f(x,t + h)$$

with the boundary conditions (1.2). The difference $z^{k+1}(x,t) = u^{k+1}(x,t) - u^{k+1}(x,t + h)$ satisfies the system

$$\partial_t z^{k+1} + a^k(x,t)\partial_x z^{k+1} + b^k(x,t)z^{k+1} = g^k(x,t)$$

subjected to (1.2), where

$$g^k(x,t) = -(b^k(x,t) - b^k(x,t + h))u^{k+1}(x,t + h) - (a^k(x,t) - a^k(x,t + h))\partial_x u^{k+1}(x,t + h) + f(x,t) - f(x,t + h).$$

The function $g^k(x,t)$ is $C^1$-smooth in $x$ and $t$ and, due to (5.29) and (5.34), satisfies the bound

$$\|g^k\|_{BC^1(\Pi; \mathbb{R}^n)} \leq \varepsilon(d + 1). \quad (5.35)$$

Analogously to (5.26) and (5.27), $z^{k+1} : \mathbb{R} \rightarrow L^2((0,1); \mathbb{R}^n)$ is defined by the formula

$$z^{k+1}(t) = \int_{-\infty}^{\infty} G_k(t,s)g^k(s)ds.$$

Moreover, the following estimate is true:

$$\|z^{k+1}(t)\|_{L^2((0,1); \mathbb{R}^n)} \leq \int_{-\infty}^{\infty} \|G_k(t,s)\|_{L(L^2((0,1); \mathbb{R}^n))}\|g^k\|_{BC^1(\Pi; \mathbb{R}^n)}ds \leq \frac{2M_1}{\alpha_1} (\delta + 1)\varepsilon, \quad (5.36)$$

the bound being uniform in $t \in \mathbb{R}$. We combine the representation

$$z^{k+1}(t) = U_k(t,t - 2d)z^{k+1}(t - 2d) + \int_{t - 2d}^{t} U_k(t,s)g^k(s)ds$$

with Lemma 3.5 and the inequalities (5.35) and (5.36). This yields the desired estimate

$$\|z^{k+1}(t)\|_{C^1([0,1]; \mathbb{R}^n)} \leq L \left( \|z^{k+1}(t - 2d)\|_{L^2((0,1); \mathbb{R}^n)} + \|g^k\|_{BC^1(\Pi; \mathbb{R}^n)} \right) \leq \varepsilon L(\delta + 1) \left( \frac{2M_1}{\alpha_1} + 1 \right) = L_1 \varepsilon$$

or, the same,

$$\|u^{k+1}(t) - u^{k+1}(t + h)\|_{C^1([0,1]; \mathbb{R}^n)} \leq L_1 \varepsilon,$$

the constant $L_1$ being independent of $\varepsilon$ and $t \in \mathbb{R}$. This finishes the proof of the almost periodicity of $u^{k+1}(x,t)$.
5.3 Periodic Solutions

If the coefficients $A(x, t, v)$, $B(x, t, v)$, and $f(x, t)$ are $T$-periodic in $t$, then each constructed iteration $u^k$ is in fact a unique solution to a linear dichotomous problem with $T$-periodic in $t$ coefficients. This yields the $T$-periodicity in $t$ of $u^k$ and, hence the $T$-periodicity in $t$ of the limit function $u^*$. The proof of Theorem 1.5 is complete.

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