Several classes of two-weight and three-weight linear codes

Dabin Zheng, Qing Zhao, Xiaoqiang Wang

Hubei Province Key Laboratory of Applied Mathematics,
Faculty of Mathematics and Statistics, Hubei University, Wuhan 430062, China

Abstract. Linear codes with few weights have been widely studied due to their applications in secret sharing, authentication codes, association schemes and strongly regular graphs. In this paper, we further construct several classes of new two-weight and three-weight linear codes from defining sets and determine their weight distributions by applications of the theory of quadratic forms and Weil sums over finite fields. Some of the linear codes obtained are optimal or almost optimal with respect to the Griesmer bound. This paper generalizes some results in [18, 21].

Keywords: Linear code, Weight distribution, Exponential sum, Quadratic form

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1 Introduction

Let \( \mathbb{F}_p \) be a finite field of size \( p \), where \( p \) is an odd prime. An \([n,k,d]\) linear code \( C \) over \( \mathbb{F}_p \) is a \( k \)-dimensional subspace of \( \mathbb{F}_p^n \) with minimum distance \( d \). Let \( A_i \) denote the number of codewords with Hamming weight \( i \) in \( C \). The weight enumerator of \( C \) is defined by \( 1 + A_1 x + A_2 x^2 + \cdots + A_n x^n \) and the sequence \((1, A_1, A_2, \cdots, A_n)\) is called the weight distribution of \( C \). If the number of nonzero \( A_i \) in this sequence is equal to \( t \), then we call \( C \) a \( t \)-weight code. The weight distribution of a code not only gives the error correcting ability of the code, but also allows the computation of the error probability of error detection and correction [19]. So, the study of the weight distribution of a linear code is important in both theory and applications. Linear codes with few weights attracts many researchers’ attention due to their wide applications in secret schemes, strongly regular graphs, association schemes and authentication codes. The recent progress on constructions of two-weight and three-weight linear codes can be seen in [9-11], [13-16], [18], [21-23], [24-29], [32-34] and the references therein.

Let \( \mathbb{F}_{p^m} \) be a finite field with \( p^m \) elements, where \( m \) is a positive integer. Let \( \text{Tr} \) denote the trace function from \( \mathbb{F}_q \) to \( \mathbb{F}_p \). From a subset \( D = \{d_1,d_2,\ldots,d_n\} \subset \mathbb{F}_{p^m} \), Ding et al. [6] first defined a generic class of linear codes of length \( n = |D| \) over \( \mathbb{F}_p \) as

\[
C_D = \{ (\text{Tr}(xd_1), \text{Tr}(xd_2), \cdots, \text{Tr}(xd_n)) \mid x \in \mathbb{F}_{p^m} \}.
\]

Here, \( D \) is called the defining set of \( C_D \). This construction is generic in the sense that many classes of known codes could be produced by selecting the defining set \( D \). By application of this technique many good linear codes with few weights have been constructed [7-11], [13-17], [21-23], [29-31]. Motivated by this construction, Jian et al. [18] constructed the linear code

\[
C_D = \left\{ c(a,b) = (\text{Tr}(ax + by))_{(x,y) \in D} : a, b \in \mathbb{F}_{p^m} \right\},
\]

\[\text{1}\]

Corresponding author. E-Mail addresses: dzheng@hubu.edu.cn(D. Zheng), zhaoqing9@126.com(Q. Zhao), waxiqq@163.com(X. Wang)
where
\[ D = \left\{ (x, y) \in \mathbb{F}_{p^m}^2 \setminus \{(0, 0)\} : \operatorname{Tr} \left( x^2 + y^{p^n+1} \right) = 0 \right\}. \]

In this paper, we generalize their results by choosing the defining set:
\[ D = \left\{ (x, y) \in \mathbb{F}_{p^m}^2 \setminus \{(0, 0)\} : \operatorname{Tr} \left( x^{p^k+1} + y^{p^\ell+1} \right) = c, c \in \mathbb{F}_p \right\}, \quad (2) \]

where \( k, \ell \) are nonnegative integers. By using the quadratic form theory and Weil sums, we obtain several classes of two-weight and three-weight linear codes and determine their weight distributions. Moreover, some of linear codes obtained are optimal or almost optimal.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries, which will be used in the following sections. Section 3 investigates the weight distribution of the linear code \( C_D \) and present several classes of two-weight and three-weight linear codes. In Section 4, the punctured version of \( C_D \) is discussed. Section 5 concludes this paper.

## 2 Preliminaries

Throughout this paper, we adopt the following notations unless otherwise stated:

- \( \mathbb{F}_{p^m} \) is a finite field with \( p^m \) elements.
- \( \operatorname{Tr}(\cdot) \) is the trace function from \( \mathbb{F}_{p^m} \) to \( \mathbb{F}_p \).
- \( v_2(\cdot) \) is the 2-adic order function and we denote \( v_2(0) = \infty \).
- \( k \) and \( \ell \) are positive integers, \( \gcd(k, m) = u \) and \( \gcd(\ell, m) = v \).
- \( \zeta_p = e^{\frac{2\pi i}{p}} \) is the primitive \( p \)-th root of unity.

Let \( \psi \) be a multiplicative character of \( \mathbb{F}_{p^m}^* \). The Gaussian sum \( G(\psi) \) is defined by
\[ G(\psi) = \sum_{x \in \mathbb{F}_{p^m}^*} \psi(x) \chi(x), \]

where \( \chi \) be the canonical additive character of \( \mathbb{F}_{p^m} \). The explicit values of Gaussian sums are very difficult to determine and are known for only a few cases. For later use, we state the quadratic Gaussian sums in the following lemma.

**Lemma 1 [20, Theorem 5.15]** Let \( \mathbb{F}_{p^m} \) be a finite field with \( p^m \) elements and \( \eta \) be the quadratic multiplicative character of \( \mathbb{F}_{p^m}^* \). Then
\[ G(\eta) = (-1)^{m-1} \sqrt{(-1)^{(\frac{p-1}{2})m} p^m} = \begin{cases} (-1)^{m-1} p^{\frac{m}{2}}, & \text{if } p \equiv 1 \mod 4, \\ (-1)^{m-1} (\sqrt{-1})^{m} p^{\frac{m}{2}}, & \text{if } p \equiv 3 \mod 4. \end{cases} \]

By identifying the finite field \( \mathbb{F}_{p^m} \) with an \( m \)-dimensional vector space \( \mathbb{F}_p^m \) over \( \mathbb{F}_p \), a function \( f \) from \( \mathbb{F}_{p^m} \) to \( \mathbb{F}_p \) can be viewed as an \( m \)-variable polynomial over \( \mathbb{F}_p \). The function \( f(x) \) is called a quadratic form if it is a homogenous polynomial of degree two as follows:
\[ f(x_1, x_2, \cdots, x_m) = \sum_{1 \leq i \leq j \leq m} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{F}_p, \]

where we fix a basis of \( \mathbb{F}_p^m \) over \( \mathbb{F}_p \) and identify \( x \in \mathbb{F}_{p^m} \) with a vector \( (x_1, x_2, \cdots, x_m) \in \mathbb{F}_p^m \). The rank of the quadratic form \( f(x) \) is defined as the codimension of \( \mathbb{F}_p \)-vector space
\[ V = \left\{ x \in \mathbb{F}_p^m : f(x + z) - f(x) - f(z) = 0, \text{ for all } z \in \mathbb{F}_p \right\}, \]
which is denoted by \( \text{rank}(f) \). Then \( |V| = p^{n-\text{rank}(f)} \).  

For a quadratic form \( Q(x) \) with \( m \) variables over \( \mathbb{F}_p \), there exists a symmetric matrix \( A \) such that \( Q(x) = XAX' \), where \( X = (x_1, x_2, \cdots, x_m) \in \mathbb{F}_p^m \) and \( X' \) denotes the transpose of \( X \). The determinant \( \det(Q) \) of \( Q(x) \) is defined to be the determinant of \( A \), and \( Q(x) \) is nondegenerate if \( \det(Q) \neq 0 \). It is known that there exists a nonsingular matrix \( T \) such that \( TAT' \) is a diagonal matrix \([20]\). Making a nonsingular linear substitution \( X = YT \) with \( Y = (y_1, y_2, \cdots, y_m) \), we have  

\[
Q(x) = YTA'TY' = \sum_{i=1}^r a_i y_i^2, \quad a_i \in \mathbb{F}_p,
\]

where \( r(\leq m) \) is the rank of \( Q(x) \). The following lemma gives a general result on the exponential sums of a quadratic function from \( \mathbb{F}_{p^m} \) to \( \mathbb{F}_p \). These sums are also known as Weil sums.  

**Lemma 2** ([27, Theorems 5.15 and 5.33]) Let \( Q(x) \) be a quadratic function from \( \mathbb{F}_{p^m} \) to \( \mathbb{F}_p \) with rank \( r(\neq 0) \), and \( \eta_0 \) be the quadratic multiplicative character of \( \mathbb{F}_p \). Then \[
\sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{Q(x)} = \begin{cases} 
\eta_0(\Delta)p^{m-\frac{k}{2}}, & \text{if } p \equiv 1 \pmod{4}, \\
(-1)^{\frac{k}{2}} \eta_0(\Delta)p^{m-\frac{k}{2}}, & \text{if } p \equiv 3 \pmod{4},
\end{cases}
\]

where \( \Delta \) is the determinant of \( Q(x) \). Furthermore, for any \( z \in \mathbb{F}_{p^*}, \)

\[
\sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{zQ(x)} = \eta(z) \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{Q(x)},
\]

where \( \eta \) is the quadratic multiplicative character of \( \mathbb{F}_{p^m} \).  

**Lemma 3** ([12]) Let \( \text{Tr}(x^{p^k+1}) \) be the quadratic function from \( \mathbb{F}_{p^m} \) to \( \mathbb{F}_p \) with rank \( r \). Then \[
r = \begin{cases} 
m - 2u, & \text{if } v_2(m) > v_2(k) + 1, \\
m, & \text{otherwise},
\end{cases}
\]

where \( u = \gcd(m, k) \).  

The weight distribution of the linear code \( \mathcal{C}_D \) is related to the rank of the quadratic form \( \text{Tr}(x^{p^k+1} + y^{p^k+1}) \) and the following Weil sum \[
S_k(a, b) = \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}(ax^{p^k+1} + bx)} , a \in \mathbb{F}_{p^m}^*, b \in \mathbb{F}_{p^m}. \quad (3)
\]

When \( b = 0 \), the Weil sum \( S_k(a, 0) \) is as follows.  

**Lemma 4** ([12, Corollary 7.6]) Let \( v_2(\cdot) \) denote the 2-adic order function and \( v_2(0) = \infty \). Let \( \eta \) be the quadratic character of \( \mathbb{F}_{p^m}^* \). For \( a \in \mathbb{F}_{p^m}^* \) we have 

(i) \( \text{If } v_2(m) \leq v_2(k), \text{ then } \)

\[
S_k(a, 0) = \eta(a)(-1)^{m-1}(\sqrt{-1})^{(p-1)2m/p} \frac{p^m}{m+\gcd(2k, m)}. \quad (4)
\]

(ii) \( \text{If } v_2(m) = v_2(k) + 1, \text{ then } \)

\[
S_k(a, 0) = \begin{cases} 
p^{m+\gcd(2k, m)/2}, & \text{if } a^{(p^k-1)(p^m-1)} = -1, \\
p^{-m/2}, & \text{otherwise},
\end{cases}
\]

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(iii) If \( v_2(m) > v_2(k) + 1 \), then

\[
S_k(a, 0) = \begin{cases} 
-p^{m \cdot \text{gcd}(2k, m)} \chi, & \text{if } a^{\frac{(p^k - 1)(p^m - 1)}{p\cdot \text{gcd}(2k, m)-1}} = 1, \\
p^\frac{m}{2}, & \text{otherwise.} 
\end{cases}
\]

When \( b \neq 0 \), the value of \( S_k(a, b) \) is related to the solutions of the polynomial \( a^{p^k} x^{p^{2k}} + ax = 0 \). To this end, we first recall the result on this equation.

Lemma 5 \[4\] Theorem 4.1] Let \( m, k \) be positive integers with \( u = \gcd(m, k) \) and \( a \in \mathbb{F}_{p^m}^* \). The equation

\[
a^{p^k} x^{p^{2k}} + ax = 0
\]

is solvable in \( \mathbb{F}_{p^m}^* \) if and only if \( \frac{u}{m} \) is even and \( \frac{m-1}{u} = (-1)^\frac{p^m-1}{2} \). In such cases, there are \( p^{2u} - 1 \) non-zero solutions.

By application of this lemma, R. S. Coulter determined the possible values of the Weil sum \( S_k(a, b) \) as follows.

Lemma 6 \[4\] Theorem 2] Let \( m, k \) be positive integers with \( u = \gcd(m, k) \). Let \( a \in \mathbb{F}_{p^m}^* \) and \( f(x) = a^{p^k} x^{p^{2k}} + ax \) be a permutation polynomial over \( \mathbb{F}_{p^m} \). Assume that \( x_0 \) is the unique solution of the equation \( f(x) = -b^{p^k} \). The following statements hold.

(i) If \( \frac{u}{m} \) is odd, then

\[
S_k(a, b) = (-1)^{m-1} (\sqrt{-1})^2 p^{m} \chi(a \cdot x_0^{p^{2k} + 1})
\]

\[
= \begin{cases} 
(-1)^{m-1} p^{\frac{m}{u}} \chi(a \cdot x_0^{p^{2k} + 1}), & \text{if } p \equiv 1 \pmod{4}, \\
(-1)^{m-1} (\sqrt{-1})^2 p^{\frac{m}{u}} \chi(a \cdot x_0^{p^{2k} + 1}), & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

(ii) If \( \frac{u}{m} \) is even and \( \frac{m-1}{u} \neq (-1)^\frac{p^m-1}{2} \) and

\[
S_k(a, b) = (-1)^{\frac{m}{u}} p^{\frac{m}{u}} \chi(a \cdot x_0^{p^{2k} + 1}),
\]

where \( \eta \) and \( \chi \) are the quadratic multiplicative and canonical additive character of \( \mathbb{F}_{p^m} \), respectively.

Lemma 7 \[4\] Theorem 2] Let \( m, k \) be positive integers with \( u = \gcd(m, k) \) and \( m \) being even. Assume that \( f(x) = a^{p^k} x^{p^{2k}} + ax \) is not a permutation polynomial over \( \mathbb{F}_{p^m} \), then \( S_k(a, b) = 0 \) unless the equation \( f(x) = -b^{p^k} \) is solvable. If the equation has a solution \( x_0 \), then

\[
S_k(a, b) = -(-1)^{\frac{m}{u}} p^{\frac{m}{u}} \chi(a \cdot x_0^{p^{2k} + 1}).
\]

For later use, we need the following lemma.

Lemma 8 \[15\] Lemma 13] Let \( u = \gcd(m, k) \). If \( \frac{m}{u} \equiv 0 \pmod{4} \), then

\[
\left| \left\{ c \in \mathbb{F}_{p^m} : x^{p^{2k}} + x = c^{p^k} \text{ is solvable in } \mathbb{F}_{p^m} \right\} \right| = p^{m-2u}.
\]

In order to obtain the multiplicity of the weight distribution of the linear codes, we need the Pless power moment identities on linear codes. Let \( C \) be \([n, k]\) code over \( \mathbb{F}_q \), and denote its dual by \( C^\perp \). Let \( A_i \) and \( A_i^\perp \) be the number of codewords of weight \( i \) in \( C \) and \( C^\perp \), respectively. The first two Pless power moment identities are as follows (\[24\], p. 131):

\[
\sum_{i=0}^{n} A_i = p^k;
\]

\[
\sum_{i=0}^{n} i A_i = p^{k-1} (pn - A_1^\perp).
\]
For the linear code $C_D$ defined in (11) with the defining set $D$ in (2), it is easy to verify that $A^k_0 = 0$ if $(0,0) \not\in D$ from the nondegenerate property of the trace function.

The following lemma on the bound of linear codes is well-known.

**Lemma 9 (Griesmer Bound)** If an $[n,k,d]$ p-ary code exists, then

$$n \geq \sum_{i=0}^{k-1} \left\lfloor \frac{d}{p^i} \right\rfloor$$

where the symbol $\lfloor x \rfloor$ denotes the smallest integer greater than or equal to $x$.

3 The weight distribution of $C_D$

In this section, we investigate the weight distribution of the linear code $C_D$ defined in (11), where the defining set $D$ is given in (2). Firstly, we determine the length of $C_D$. Let $m, k$ and $\ell$ be integers with $u = \gcd(m, k)$ and $v = \gcd(m, \ell)$. For convenience, we define the following symbols.

$$\varepsilon_u = \begin{cases} 1, & \text{if } v_2(m) > v_2(u) + 1, \\ 0, & \text{if } v_2(m) \leq v_2(u) + 1, \end{cases} \quad \varepsilon_v = \begin{cases} 1, & \text{if } v_2(m) > v_2(v) + 1, \\ 0, & \text{if } v_2(m) \leq v_2(v) + 1. \end{cases}$$ (4)

**Proposition 10** Let $C_D$ be a linear code defined in (11) with the defining set $D$ given in (2). Let $n = |D|$ be the length of $C_D$. If $c \in F_p^*$, then

$$n = \begin{cases} p^{2m-1} - (-1)^{\frac{(p-1)m}{2}} p^{m-1}, & \text{if } 2v_2(m) = v_2(u) + v_2(v), \\ p^{2m-1} - (-1)^{\frac{(p-1)m}{2}} p^{m-1}, & \text{if } 2v_2(m) > v_2(u) + v_2(v) + 1, \\ p^{2m-1} - p^{m+\xi_u + \xi_v + 1}, & \text{if } 2v_2(m) < v_2(u) + v_2(v) + 1. \end{cases}$$

If $c = 0$, then

$$n = \begin{cases} p^{2m-1} - (-1)^{\frac{(p-1)m}{2}} (p-1) p^{m-1} - 1, & \text{if } 2v_2(m) = v_2(u) + v_2(v), \\ p^{2m-1} - (-1)^{\frac{(p-1)m}{2}} (p-1) p^{m-1} - 1, & \text{if } 2v_2(m) > v_2(u) + v_2(v) + 1, \\ p^{2m-1} + p^{m+\xi_u + \xi_v} - p^{m+\xi_u + \xi_v + 1} - 1, & \text{if } 2v_2(m) > v_2(u) + v_2(v) + 1. \end{cases}$$

**Proof.** By the orthogonal property of additive characters and Lemma 2 we have

$$n = \sum_{(x,y) \in (F_p^m \times F_p^m) \setminus \{(0,0)\}} \frac{1}{p} \sum_{z \in F_p} \zeta_p^{\Tr(x^{k+1} + y^{\ell+1}) - c}$$

$$= \sum_{x,y \in F_p^m} \frac{1}{p} \sum_{z \in F_p} \zeta_p^{\Tr(x^{k+1} + y^{\ell+1}) - c} - \frac{1}{p} \sum_{z \in F_p} \zeta_p^{\bar{z} c}$$

$$= p^{2m-1} + \frac{1}{p} \sum_{z \in F_p} \zeta_p^{-\bar{z} c} \sum_{x \in F_p^m} z^{\Tr(x^{k+1})} \sum_{y \in F_p^m} z^{\Tr(y^{\ell+1})} - \frac{1}{p} \sum_{z \in F_p} \zeta_p^{-\bar{z} c}$$

$$= p^{2m-1} + \frac{1}{p} \sum_{z \in F_p} \zeta_p^{-\bar{z} c} \sum_{x \in F_p^m} \eta^1(z) \zeta_p^{\Tr(x^{k+1})} \sum_{y \in F_p^m} \eta^2(z) \zeta_p^{\Tr(y^{\ell+1})} - \frac{1}{p} \sum_{z \in F_p} \zeta_p^{-\bar{z} c}$$

$$= p^{2m-1} + \frac{1}{p} \sum_{z \in F_p} \zeta_p^{-\bar{z} c} \eta^{(r_1 + r_2)}(z) \sum_{x \in F_p^m} \zeta_p^{\Tr(x^{k+1})} \sum_{y \in F_p^m} \zeta_p^{\Tr(y^{\ell+1})} - \frac{1}{p} \sum_{z \in F_p} \zeta_p^{-\bar{z} c},$$
where \( \eta \) is the quadratic multiplicative character of \( \mathbb{F}_p \), \( r_1 \) and \( r_2 \) are the ranks of the quadratic forms \( \text{Tr} \left( x^{p^k+1} \right) \) and \( \text{Tr} \left( y^{p^f+1} \right) \), respectively. By Lemma 3 it is easy to see that \( \eta^{r_1+r_2}(z) = 1 \) for any \( z \in \mathbb{F}_p^* \) since \( r_1 + r_2 \) is even. Hence,

\[
n = p^{2m-1} + \frac{1}{p} \sum_{x \in \mathbb{F}_p^m} \zeta_p z c \left( \sum_{x \in \mathbb{F}_p^m} \zeta_p \text{Tr}(x^{p^k+1}) \sum_{y \in \mathbb{F}_p^m} \zeta_p \text{Tr}(y^{p^f+1}) \right) - \frac{1}{p} \sum_{x \in \mathbb{F}_p^m} \zeta_p z c \left( \sum_{x \in \mathbb{F}_p^m} \zeta_p \text{Tr}(x^{p^k+1}) \sum_{y \in \mathbb{F}_p^m} \zeta_p \text{Tr}(y^{p^f+1}) \right)
\]

(5)

where

\[
\Omega = \sum_{x \in \mathbb{F}_p^m} \zeta_p \text{Tr}(x^{p^k+1}) \sum_{y \in \mathbb{F}_p^m} \zeta_p \text{Tr}(y^{p^f+1})
\]

Combining with Lemma 1 and 2, the result follows.

It is clear that for any \((a, b) \in \mathbb{F}_p^2 \setminus \{(0, 0)\}\), a codeword in \( C_D \) is

\[
c_{a, b} = (\text{Tr}(ax_0 + by_0), \text{Tr}(ax_1 + by_1), \cdots, \text{Tr}(ax_n + by_n))
\]

where \((x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)\) are all the elements of the defining set \( D \). Clearly, its Hamming weight is as follows:

\[
\text{wt}_H(c(a, b)) = n - N(a, b),
\]

(6)

where \( n \) is the length of linear code \( C_D \) and

\[
N(a, b) = \left| \left\{ (x, y) \in \mathbb{F}_p^2 \setminus \{(0, 0)\} : \text{Tr}(x^{p^k+1} + y^{p^f+1}) = c, \text{Tr}(ax + by) = 0 \right\} \right|
\]

Assume that \( \eta \) is the quadratic multiplicative character of \( \mathbb{F}_p \), \( r_1 \) and \( r_2 \) are the ranks of the quadratic forms of \( \text{Tr} \left( x^{p^k+1} \right) \) and \( \text{Tr} \left( y^{p^f+1} \right) \), respectively. By Lemma 2 we have

\[
N(a, b) = \left( \frac{1}{p} \sum_{x, y \in \mathbb{F}_p^m} \zeta_p \text{Tr}(x^{p^k+1} + y^{p^f+1}) \right) \left( \frac{1}{p} \sum_{z_2 \in \mathbb{F}_p} \text{Tr}(z_2 \text{Tr}(ax + by)) \right)
\]

\[
= p^{2m-2} \left( \sum_{x, y \in \mathbb{F}_p^m} \zeta_p \text{Tr}(x^{p^k+1} + y^{p^f+1}) \right) \left( \sum_{z_2 \in \mathbb{F}_p} \text{Tr}(z_2 \text{Tr}(ax + by)) \right)
\]

(7)
where
\[
\Omega_1 = \sum_{z_1 \in \mathbb{F}_p} \zeta_p^{-z_1c} \sum_{x \in \mathbb{F}_{p^m}} \eta^{z_1}(z_1) \Tr(x^{p^{k+1}}) \sum_{y \in \mathbb{F}_{p^m}} \eta^{z_2}(z_1) \zeta_p(y^{p^{k+1}})
\]
and
\[
\Omega_2 = \sum_{z_1, z_2 \in \mathbb{F}_p} \zeta_p^{-z_1c} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{(z_1 x^{p^{k+1}} + z_2 a x)} \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^{(z_1 y^{p^{k+1}} + z_2 b y)} = \sum_{z_1, z_2 \in \mathbb{F}_p} \zeta_p^{-z_1c} S_k(z_1, z_2 a) S_l(z_1, z_2 b). \tag{9}
\]

Here, \(S_k(z_1, z_2 a)\) and \(S_l(z_1, z_2 b)\) are defined in (4). Hence, in order to determine the possible weights of the linear code \(C_D\), the main task is to consider the possible values of \(\Omega_1\) and \(\Omega_2\).

In the following, we will determine the weight distribution of the linear code \(C_D\) defined in (2) with the defining set \(D\) given in (2).

**Theorem 11.** Let \(C_D\) be a linear code defined in (2) with the defining set \(D\) given in (2). Assume that \(2v_2(m) = v_2(u) + v_2(v)\), then the following statements hold.

1. If \(c = 0\), then \(C_D\) is a \(\left[ p^{2m-1} + (-1)^\frac{(p-1)m}{2} (p-1)p^{m-1} - 1, 2m \right] \) two-weight linear code with weight distribution given in Table 1.

| Weight               | Multiplicity |
|----------------------|--------------|
| 0                    | 1            |
| \((p-1)p^{2m-2}\)    | \(p^{2m-1} + (-1)^\frac{(p-1)m}{2} (p-1)p^{m-1} - 1\) |
| \((p-1)(p^{2m-2} + (-1)^\frac{(p-1)m}{2} p^{m-1})\) | \((p-1)(p^{2m-1} - (-1)^\frac{(p-1)m}{2} p^{m-1})\) |

2. If \(c \in \mathbb{F}_p^*\), then \(C_D\) is a \(\left[ p^{2m-1} - (-1)^\frac{(p-1)m}{2} p^{m-1}, 2m \right] \) two-weight linear code with weight distribution given in Table 2.

| Weight               | Multiplicity |
|----------------------|--------------|
| 0                    | \(1\)        |
| \((p-1)p^{2m-2}\)    | \(\frac{1}{2} \left( (p+1)p^{2m-1} + (-1)^\frac{(p-1)m}{2} (p-1)p^{m-1} \right) - 1\) |
| \((p-1)p^{2m-2} - (-1)^\frac{(p-1)m}{2} 2p^{m-1}\) | \(\frac{1}{2} (p-1) \left( p^{2m-1} - (-1)^\frac{(p-1)m}{2} p^{m-1} \right)\) |

**Proof.** We only prove the case \(c \in \mathbb{F}_p^*\), and the case \(c = 0\) can be shown by the similar way. By Proposition 10, the length of the linear code \(C_D\) is \(n = p^{2m-1} - (-1)^\frac{(p-1)m}{2} p^{m-1}\). From (6) and (7), in order to determine the weight distribution of \(C_D\), we first need to obtain the possible values of \(\Omega_1\) and \(\Omega_2\), where \(\Omega_1\) and \(\Omega_2\) are denoted in (5) and (6), respectively.

Recall that \(u = \gcd(m, k)\) and \(v = \gcd(m, \ell)\). It is clear that \(v_2(m) = v_2(u) = v_2(v)\) since \(2v_2(m) = v_2(u) = v_2(v)\).
v_2(u) + v_2(v)$. By lemmas 34 we have

\[
\Omega_1 = \sum_{z_1 \in \mathbb{F}_p} \zeta_p^{-z_1 c} \sum_{x \in \mathbb{F}_{p^m}} \eta^{r_1}(z_1) \zeta_p^x \sum_{y \in \mathbb{F}_{p^m}} \eta^{r_2}(z_1) \zeta_p^y \\
= \sum_{z_1 \in \mathbb{F}_p} \zeta_p^{-z_1 c} \sum_{x \in \mathbb{F}_{p^m}} \eta^{r_1+c}(z_1) \zeta_p^x \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^y \\
= - \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^x \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^y \\
= - \left( (-1)^{m-1} \left( \frac{(n-1)^2 m}{2} - \frac{p}{4} \right) \right)^2 \\
= - \left( (-1)^{(n-1)m} \right) p^m.
\]

In the following, we determine the possible values of \( \Omega_2 \). As \( \frac{a}{n} \) and \( \frac{b}{m} \) are odd, we verify that \( f_k(x) = z_1^{3k} x^{p+1} + z_1 x \) and \( f_l(x) = z_1^{3l} x^{p+1} + z_1 x \) are permutations over \( \mathbb{F}_{p^m} \) for any \( z_1 \in \mathbb{F}_{p^*} \). Assume that \( \gamma_a \) and \( \gamma_b \) are the solutions of the equations \( x^{p+1} + x = -a^{p+1} \) and \( x^{p+1} + x = -a^{p-1} \), respectively. Set \( z_3 = z_1^{-1}z_2 \). Then \( z_3 \gamma_a \) and \( z_3 \gamma_b \) are the solutions of \( f_k(x) = -(z_2 a)^{p+1} \) and \( f_l(x) = -(z_2 b)^{p-1} \) for any \( z_2 \in \mathbb{F}_{p^*} \), respectively. By Lemma 5 we have

\[
\Omega_2 = \sum_{z_1, z_2 \in \mathbb{F}_p^*} \zeta_p^{-z_1 c} \sum_{x, y \in \mathbb{F}_{p^m}} \eta(x, y) \zeta_p^x \sum_{y \in \mathbb{F}_{p^m}} \eta(x, y) \\
= \sum_{z_1, z_3 \in \mathbb{F}_p^*} \zeta_p^{-z_1 c} \left( (-1)^{m-1} \left( \frac{(n-1)^2 m}{4} \eta(z_1) \left( \frac{(n-1)^2 m}{4} \eta(z_1) \right) \right) \right) \left( (-1)^{m-1} \left( \frac{(n-1)^2 m}{4} \eta(z_1) \left( \frac{(n-1)^2 m}{4} \eta(z_1) \right) \right) \right) \\
= (-1)^{\frac{(n-1)m}{2}} p^m \sum_{z_1, z_3 \in \mathbb{F}_p^*} \zeta_p^{-z_1 c} \left( \sum_{x \in \mathbb{F}_{p^m}} \eta(x) \right) \left( \sum_{y \in \mathbb{F}_{p^m}} \eta(y) \right),
\]

where \( i = \sqrt{-1} \).

If \( \text{Tr} (\gamma_a^{p+1} + \gamma_b^{p+1}) = 0 \), then

\[
\Omega_2 = -(-1)^{\frac{(n-1)m}{2}} (p-1) p^m.
\]

If \( \text{Tr} (\gamma_a^{p+1} + \gamma_b^{p+1}) \neq 0 \) and \( -\frac{c}{\text{Tr} (\gamma_a^{p+1} + \gamma_b^{p+1})} \) is a square in \( \mathbb{F}_p \), then there exist two elements \( z_3, z_3' \in \mathbb{F}_{p^*} \) satisfying the equation \( c + x^{p+1} \sum_{y \in \mathbb{F}_{p^m}} \eta(x) \sum_{y \in \mathbb{F}_{p^m}} \eta(x) = 0 \). So, we have

\[
\Omega_2 = (-1)^{\frac{(n-1)m}{2}} p^m \sum_{z_1, z_3 \in \mathbb{F}_p^*} \zeta_p^{-z_1 c} \left( \sum_{x \in \mathbb{F}_{p^m}} \eta(x) \right) \left( \sum_{y \in \mathbb{F}_{p^m}} \eta(x) \right) \\
= 2(-1)^{\frac{(n-1)m}{2}} (p-1) p^m + (-1)^{\frac{(n-1)m}{2}} p^m \sum_{z_1 \in \mathbb{F}_p^*} \sum_{z_3 \in \mathbb{F}_p^*} \zeta_p^{-z_1 c} \left( \sum_{x \in \mathbb{F}_{p^m}} \eta(x) \right) \left( \sum_{y \in \mathbb{F}_{p^m}} \eta(x) \right) \\
= 2(-1)^{\frac{(n-1)m}{2}} (p-1) p^m - (-1)^{\frac{(n-1)m}{2}} (p-3) p^m \\
= (-1)^{\frac{(n-1)m}{2}} (p+1) p^m.
\]

If \( \text{Tr} (\gamma_a^{p+1} + \gamma_b^{p+1}) \neq 0 \) and \( -\frac{c}{\text{Tr} (\gamma_a^{p+1} + \gamma_b^{p+1})} \) is a nonsquare in \( \mathbb{F}_p \), then \( c + z_3^2 \sum_{x \in \mathbb{F}_{p^m}} \eta(x) \sum_{y \in \mathbb{F}_{p^m}} \eta(x) \neq 0 \) for any \( z_3 \in \mathbb{F}_{p^*} \). So, we get

\[
\Omega_2 = -(-1)^{\frac{(n-1)m}{2}} (p-1) p^m.
\]
Proof. Recall that \( u = \gcd(m, k) \) and \( v = \gcd(m, \ell) \). When \( 2v_2(m) = v_2(u) + v_2(v) + 1 \), there are two cases:

\[
\begin{cases}
  v_2(m) = v_2(u) \text{ and } v_2(m) = v_2(v) + 1, \\
  v_2(m) = v_2(v) \text{ and } v_2(m) = v_2(u) + 1.
\end{cases}
\]

In the following, we only prove the case \( c \in \mathbb{F}_p^* \), \( v_2(m) = v_2(u) \) and \( v_2(m) = v_2(v) + 1 \). The other cases can be shown by the similar way.
Since \( c \in \mathbb{F}_p^* \) and \( 2v_2(m) = v_2(u) + v_2(v) + 1 \), Proposition 10 shows that the length of the code \( C_D \) is

\[
n = p^{2m-1} - (-1)^{\frac{(p-1)m}{2}} p^{m-1}.
\]

In order to determine the weight distribution of \( C_D \), by the similar calculations as in Theorem 11, we first compute the possible values of \( \Omega_1 \) and \( \Omega_2 \), where \( \Omega_1 \) and \( \Omega_2 \) are given in 3 and 4, respectively.

When \( v_2(m) = v_2(u) \) and \( v_2(m) = v_2(v) + 1 \), we have that \( v_2(m) \leq v_2(k) \) and \( v_2(m) = v_2(\ell) + 1 \). By lemmas 3 and 4, we have

\[
\Omega_1 = \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-z_1 c} \eta_1^{1 + z_1} \left( \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{Tr(xz_1^k)} \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^{Tr(yp^\ell)} \right) = -\sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{Tr(xz_1^k)} \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^{Tr(yp^\ell)}
\]

\[
= -\left( \eta(1)(-1)^{m-1}(\sqrt{-1})^{\frac{(p-1)m}{2} \frac{m}{2}} \right) \left( -p^{\frac{m}{2}} \right) = -(1)^{\frac{(p-1)m}{2} \frac{m}{2}} \frac{m}{2}.
\]

Next, we determine the possible values of \( \Omega_2 \). Since \( v_2(m) = v_2(u) \) and \( v_2(m) = v_2(v) + 1 \), \( \frac{m}{2} \) and \( \frac{m}{2} \) are odd. By Lemma 5, we have that \( f_k(x) = z_1^k x^p z_1 x \) and \( f_\ell(x) = z_1^\ell x^p z_1 x \) are permutations over \( \mathbb{F}_{p^m} \) for any \( z_1 \in \mathbb{F}_p^* \). Let \( \gamma_a \) and \( \gamma_b \) be the solutions of the equations \( x^p + x = -a^p \) and \( x^p + x = -a^p \), respectively. Set \( z_3 = z_1^k \). Then \( z_3 \gamma_a \) and \( z_3 \gamma_b \) are the solutions of \( f_k(x) = -(z_3 \gamma_a)^p \) and \( f_\ell(x) = -(z_3 \gamma_b)^p \), respectively. By Lemma 5, we have

\[
\Omega_2 = \sum_{z_1, z_2 \in \mathbb{F}_p^*} \zeta_p^{-z_1 c} S_k(z_1, z_2 a) S_\ell(z_1, z_2 b)
\]

\[
= \sum_{z_1, z_2 \in \mathbb{F}_p^*} \zeta_p^{-z_1 c} \left( (-1)^{m-1}(\sqrt{-1})^{\frac{(p-1)m}{2} \frac{m}{2}} \zeta_p^{Tr(z_1(z_3 \gamma_a)^p)} \right) \left( (-1)^{m-1}(\sqrt{-1})^{\frac{(p-1)m}{2} \frac{m}{2}} \zeta_p^{Tr(z_1(z_3 \gamma_b)^p)} \right)
\]

\[
= -(1)^{\frac{(p-1)m}{2} \frac{m}{2}} \frac{m}{2} \zeta_p^{-z_1 c} \zeta_p^{Tr(z_1(z_3 \gamma_a)^p)} \zeta_p^{Tr(z_1(z_3 \gamma_b)^p)}.
\]

The last equality follows from the facts that \( m \) is even and \( \frac{m}{2} \) is odd. Through the similar analysis as 11-13, we have

\[
\Omega_2 = \begin{cases} 
-\left( \frac{(p-1)m}{4} \right) (p-1)^m, & \text{if } \text{Tr} \left( \gamma_a^{k+1} + \gamma_b^{\ell+1} \right) = 0, \text{ or} \\
\frac{m}{2} \text{ is a nonsquare in } \mathbb{F}_p, \\
\frac{m}{2} \text{ is a square in } \mathbb{F}_p.
\end{cases}
\]

\[
-\left( \frac{(p-1)m}{4} \right) (p+1)^m, & \text{if } \text{Tr} \left( \gamma_a^{k+1} + \gamma_b^{\ell+1} \right) \neq 0 \\
-\frac{m}{2} \text{ is a nonsquare in } \mathbb{F}_p, \\
\frac{m}{2} \text{ is a square in } \mathbb{F}_p.
\end{cases}
\]

(15)

From 9, 11, 13 and 15, we have that the code \( C_D \) has two distinct nonzero weights \( w_1 = (p-1) p^{2m-2} \) and \( w_2 = (p-1)(p+1)^m p^{m-1} \). This shows that \( \text{wt}_H(c(a,b)) > 0 \) for any \( (a,b) \neq (0,0) \). So, the dimension of \( C_D \) is \( 2m \). By applying the first two Pless power moments, we obtain the weight distribution of \( C_D \) in Table 11.

Remark 15 When \( k = 0 \), the weight distribution of \( C_D \) in Table 3 has been given in [15, Table 5]. Theorem 4 generalizes the results of [15] and Table 4 is new.

Example 16 Let \( m = 4, p = 3, k = 0, \ell = 2 \).

(1) If \( c \in \mathbb{F}_p^* \), then \( C_D \) has parameters \( [2160, 8, 1404] \) and weight enumerator \( 1 + 2160x^{1404} + 4400x^{1458} \).
Theorem 17  Let \( C_D \) be a linear code defined in \( \text{(1)} \) with the defining set \( D \) given in \( \text{(2)} \). Assume that \( 2v_2(m) > v_2(u) + v_2(v) + 1 \), then the following statements hold.

1. If \( c = 0 \), then \( C_D \) is a \([p^{2m-1} + p^{m+\varepsilon_u u + \varepsilon_v v} - p^{m+\varepsilon_u u + \varepsilon_v v-1} - 1, 2m]\) three-weight linear code with weight distribution given in Table 5.

Table 5: The weight distribution of \( C_D \) for \( c = 0 \) and \( 2v_2(m) > v_2(u) + v_2(v) + 1 \)

| Weight | Multiplicity |
|--------|--------------|
| \((p-1)(p^{2m-2} + p^{m+\varepsilon_u u + \varepsilon_v v-1} - p^{m+\varepsilon_u u + \varepsilon_v v-2})\) | \(p^2m - p^{2m-2} - p^{m+\varepsilon_u u + \varepsilon_v v} - p^{m+\varepsilon_u u + \varepsilon_v v-1} + 1\) |
| \((p-1)p^{2m-2}\) | \((p^{m-\varepsilon_u u - \varepsilon_v v}) + (p^{m-\varepsilon_u u - \varepsilon_v v-1} + 1)\) |
| \((p-1)(p^{2m-2} + p^{m+\varepsilon_u u + \varepsilon_v v-1})\) | \((p-1)(p^{2m-2} - p^{m-\varepsilon_u u - \varepsilon_v v})\) |

2. If \( c \in \mathbb{F}_{p^2}^* \), then \( C_D \) is a \([p^{2m-1} - p^{m+\varepsilon_u u + \varepsilon_v v-1}, 2m]\) three-weight linear code with weight distribution given in Table 6.

Table 6: The weight distribution of \( C_D \) for \( c \in \mathbb{F}_{p^2}^*\) and \( 2v_2(m) > v_2(u) + v_2(v) + 1 \)

| Weight | Multiplicity |
|--------|--------------|
| \((p-1)(p^{2m-2} - p^{m+\varepsilon_u u + \varepsilon_v v-1})\) | \(p^2m - p^{2m-2} - p^{m+\varepsilon_u u + \varepsilon_v v} - p^{m+\varepsilon_u u + \varepsilon_v v-1} + 1\) |
| \((p-1)p^{2m-2}\) | \(\frac{1}{2}(p+1)p^{2m-2} - p^{m-\varepsilon_u u - \varepsilon_v v} + \frac{1}{2}(p-1)p^{m-\varepsilon_u u - \varepsilon_v v-1} - 1\) |
| \((p-1)(p^{2m-2} - 2p^{m+\varepsilon_u u + \varepsilon_v v-1})\) | \(\frac{1}{2}(p-1)(p^{2m-2} - 2p^{m+\varepsilon_u u + \varepsilon_v v-1})\) |

Proof. Recall that \( u = \gcd(m, k) \) and \( v = \gcd(m, \ell) \). When \( 2v_2(m) > v_2(u) + v_2(v) + 1 \), then \( m \) is even and there are six cases:

\[
\begin{align*}
&v_2(m) = v_2(u) \text{ and } v_2(m) > v_2(v) + 1, \\
v_2(m) = v_2(v) \text{ and } v_2(m) > v_2(u) + 1, \\
v_2(m) = v_2(u) + 1 \text{ and } v_2(m) = v_2(v) + 1, \\
v_2(m) = v_2(u) + 1 \text{ and } v_2(m) > v_2(v) + 1, \\
v_2(m) > v_2(u) + 1 \text{ and } v_2(m) = v_2(v) + 1, \\
v_2(m) > v_2(u) + 1 \text{ and } v_2(m) > v_2(v) + 1.
\end{align*}
\]

In the following, we only prove the case \( c \in \mathbb{F}_{p^2}^* \), \( v_2(m) > v_2(u) + 1 \) and \( v_2(m) > v_2(v) + 1 \). The other cases can be shown by the similar way.

If \( v_2(m) > v_2(u) + 1 \) and \( v_2(m) > v_2(v) + 1 \), then \( \varepsilon_u = 1 \) and \( \varepsilon_v = 1 \) from \( \text{(3)} \). For \( c \in \mathbb{F}_{p^2}^* \), by Proposition \( \text{(1)} \) the length of the code \( C_D \) is \( n = p^{2m-1} - p^{m+u+v-1} \). By the similar calculations as in Theorem \( \text{(1)} \) we first determine the possible values of \( \Omega_1 \) and \( \Omega_2 \), where \( \Omega_1 \) and \( \Omega_2 \) are given in \( \text{(8)} \) and \( \text{(9)} \), respectively.
If \(v_2(m) > v_2(u) + 1\) and \(v_2(m) > v_2(v) + 1\), then \(v_2(m) > v_2(k) + 1\) and \(v_2(m) > v_2(\ell) + 1\). By lemmas 9, 10, we have

\[
\Omega_1 = \sum_{z_1 \in \mathbb{F}_p} \zeta_p^{z_1} \eta u r_1 (z_1) \sum_{x \in \mathbb{F}_{p^m}} \text{Tr}(x^{s_1^2}) \sum_{y \in \mathbb{F}_{p^m}} \text{Tr}(y^{s^2}) \\
= - \sum_{x \in \mathbb{F}_{p^m}} \text{Tr}(x^{s_1^2}) \sum_{y \in \mathbb{F}_{p^m}} \text{Tr}(y^{s^2}) \\
= - \left( -p^{m+\gcd(2k, m)} \right) \left( -p^{m+\gcd(2l, m)} \right) \\
= -p^{m+u+v}.
\]

In the following, we determine the possible values of \(\Omega_2\). As \(v_2(m) > v_2(u) + 1\) and \(v_2(m) > v_2(v) + 1\), by Lemma 5 then \(f_k(x) = z_1^{p^k} x^{p^{2k}} + z_1 x\) and \(f_l(x) = z_1^{p^l} x^{p^{2l}} + z_1 x\) are not permutations over \(\mathbb{F}_{p^m}\) for any \(z_1 \in \mathbb{F}_p^*\). If \(f_k(x) = -(z_2a)^{p^k}\) has no solution in \(\mathbb{F}_{p^m}\) or \(f_l(x) = -(z_2b)^{p^l}\) has no solution in \(\mathbb{F}_{p^m}\), then by Lemma 7 it is easy to see that

\[
\Omega_2 = 0.
\]

Otherwise, assume that \(\gamma_a\) and \(\gamma_b\) are the solutions of the equations \(x^{p^{2k}} + x = -a^{p^k}\) and \(x^{p^{2l}} + x = -a^{p^l}\), respectively. Set \(z_3 = z_1^{-1} z_2\). Then \(z_3 a\) and \(z_3 \gamma_b\) are the solutions of \(f_k(x) = -(z_3 a)^{p^k}\) and \(f_l(x) = -(z_3 b)^{p^l}\) for any \(z_3 \in \mathbb{F}_{p^*}\), respectively. By Lemma 7 we have

\[
\Omega_2 = \sum_{z_1, z_3 \in \mathbb{F}_p^*} \zeta_p^{z_1} \left( -1 \right)^{m+u+v} \chi(z_1(z_3 \gamma_b)^{p^{2k}}) \left( -1 \right)^{m+u+v} \chi(z_1(z_3 \gamma_b)^{p^{2l}}) \\
= p^{m+u+v} \sum_{z_1, z_3 \in \mathbb{F}_p^*} \zeta_p^{z_1} \left( -z_3 \right)^{c+3z_3 \text{Tr}(\gamma_a^{p^{k+1}} + \gamma_b^{p^{l+1}})}.
\]

Through the similar analysis as (11)-(13), we have

\[
\Omega_2 = \begin{cases} 
-(p-1)p^{m+u+v}, & \text{if } \text{Tr}(\gamma_a^{p^{k+1}} + \gamma_b^{p^{l+1}}) = 0, \ \\
\text{or } \text{Tr}(\gamma_a^{p^{k+1}} + \gamma_b^{p^{l+1}}) 
eq 0 \text{ and } -\frac{c}{\text{Tr}^p(\gamma_a^{p^{k+1}} + \gamma_b^{p^{l+1}})} \text{ is a nonsquare in } \mathbb{F}_p^*, \ \\
(p+1)p^{m+u+v}, & \text{if } \text{Tr}(\gamma_a^{p^{k+1}} + \gamma_b^{p^{l+1}}) 
eq 0 \text{ and } -\frac{c}{\text{Tr}^p(\gamma_a^{p^{k+1}} + \gamma_b^{p^{l+1}})} \text{ is a square in } \mathbb{F}_p^*.
\end{cases}
\]

From 9, 10, and 11-13, the linear code \(C_D\) has three distinct nonzero weights: \(w_1 = (p-1)(p^{2m-2} - p^{m+u+v-2})\), \(w_2 = (p-1)p^{2m-2}\) and \(w_3 = (p-1)p^{2m-2} - p^{m+u+v-2}\). This shows that \(w_{1H}(c(a,b)) > 0\) for \((a, b) \neq (0, 0)\). So, the dimension of \(C_D\) is 2m.

To determine the multiplicity of each weight, we first investigate the multiplicity \(A_{w_1}\) of the weight \(w_1\). From above analysis, it is clear that \(\Omega_2 = 0\) if and only if \(f_k(x) = -(z_2 a)^{p^k}\) or \(f_l(x) = -(z_2 b)^{p^l}\) has no solution in \(\mathbb{F}_{p^m}\), which is equivalent to that \(x^{p^{2k}} + x = -a^{p^k}\) or \(x^{p^{2l}} + x = -b^{p^l}\) has no solution in \(\mathbb{F}_{p^m}\) since \(z_1 \in \mathbb{F}_p^*\). So, by Lemma 8 we have

\[
A_{w_1} = \left| \{ (a, b) \in \mathbb{F}_{p^m}^2 \setminus \{ (0, 0) \} : x^{p^{2k}} + x = -a^{p^k} \text{ and } x^{p^{2l}} + x = -b^{p^l} \text{ has no solution in } \mathbb{F}_{p^m} \} \right| \\
+ \left| \{ (a, b) \in \mathbb{F}_{p^m}^2 \setminus \{ (0, 0) \} : x^{p^{2k}} + x = -a^{p^k} \text{ has solutions in } \mathbb{F}_{p^m} \text{ and } x^{p^{2l}} + x = -b^{p^l} \text{ has no solution in } \mathbb{F}_{p^m} \} \right| \\
+ \left| \{ (a, b) \in \mathbb{F}_{p^m}^2 \setminus \{ (0, 0) \} : x^{p^{2k}} + x = -a^{p^k} \text{ has no solution in } \mathbb{F}_{p^m} \text{ and } x^{p^{2l}} + x = -b^{p^l} \text{ has solutions in } \mathbb{F}_{p^m} \} \right| \\
= (p^m - p^{m-2n})(p^m - p^{m-2v}) + (p^m - p^{m-2n})p^{m-2n} + (p^m - p^{m-2n})p^{m-2v} \\
= p^{2m} - p^{2(m-u-v)}.
\]

From \(A_{w_1}\) and the first two Pless power moments, we obtain the weight distribution of \(C_D\) given in Table 8.
Remark 18 The weight distribution of \( C_D \) in Table 7 has been given in [18, Table 5] if \( k = 0 \) and [21, Table 6] if \( k = \ell = \frac{m}{2} \), respectively. Theorem 14 generalizes the results of [18] and [21]. Table 6 is new.

Example 19 Let \( m = 4, p = 3 \).

(1) If \( k = 3, \ell = 1 \) and \( c \in \mathbb{F}_p^* \), then \( \varepsilon_u = \varepsilon_v = 1 \) and \( C_D \) has parameters \([1944, 8, 972] \) and weight enumerator 
\[
1 + 24x^{972} + 6480x^{1296} + 56x^{1458}.
\]

(2) If \( k = 1, \ell = 2 \) and \( c = 0 \), then \( \varepsilon_u = 1, \varepsilon_v = 0 \) and \( C_D \) has parameters \([2348, 8, 1458] \) and weight enumerator 
\[
1 + 260x^{1458} + 5832x^{1566} + 468x^{1620}.
\]

(3) If \( k = 2, \ell = 3 \) and \( c \in \mathbb{F}_p^* \), then \( \varepsilon_u = 0, \varepsilon_v = 1 \) and \( C_D \) has parameters \([2106, 8, 1296] \) and weight enumerator 
\[
1 + 234x^{1296} + 5832x^{1404} + 494x^{1458}.
\]

(4) If \( k = 2, \ell = 2 \) and \( c = 0 \), then \( \varepsilon_u = \varepsilon_v = 0 \) and \( C_D \) has parameters \([2240, 8, 1458] \) and weight enumerator 
\[
1 + 2240x^{1458} + 1512x^{4320}.
\]

4 Punctured codes \( \bar{C}_D \)

In this section, we investigate the punctured linear code \( \bar{C}_D \), which is derived from \( C_D \) by deleting some coordinates of codewords in \( C_D \). Some new two-weight and three-weight linear codes are obtained.

From Theorem 11, Theorem 14, and Theorem 17 it is observed that the Hamming weight of each codeword in \( C_D \) has a common divisor \( p - 1 \) for \( c = 0 \). This indicates that \( C_D \) may be punctured into a shorter one whose weight distribution is derived from that of the original code. To this end, we define an equivalence relation in the set \( D \) as follows. For \((\beta, \gamma), (\delta, \eta) \in D\), we say that \((\beta, \gamma)\) is equivalent to \((\delta, \eta)\) if and only if there exists \( a \in \mathbb{F}_p^* \) such that \((\delta, \eta) = a(\beta, \gamma)\). The elements chosen from each equivalent class in \( D \) consist of a set \( \bar{D} \). It is clear that 
\[
D = \mathbb{F}_p^* D = \{ z(x, y) = (zx, zy) : z \in \mathbb{F}_p^*, (x, y) \in \bar{D} \}. \tag{19}
\]

Then the linear code \( \bar{C}_D \) defined in (11) with the defining set \( \bar{D} \) is a punctured version of \( C_D \), whose parameters are given in the following theorem.

Theorem 20 Let \( \bar{C}_D \) be the linear code defined as above, where \( \bar{D} \) is defined in (17). Then the following statement hold.

(1) If \( 2v_2(m) = v_2(u) + v_2(v) \), then \( \bar{C}_D \) is a \([p^{2m-1}, (p-1)p^{m-1}-1, 2m] \) two-weight linear code with weight distribution given in Table 7.

| Weight | Multiplicity |
|--------|--------------|
| 0      | 1            |
| \( p^{2m-2} \) | \( p^{2m-1} + (-1)^{(p-1)m}(p-1)p^{m-1} - 1 \) |
| \( p^{2m-2} + (-1)^{(p-1)m} \) | \((p-1)(p^{2m-1} - (-1)^{(p-1)m})p^{m-1} - 1 \) |

(2) If \( 2v_2(m) = v_2(u) + v_2(v) + 1 \), then \( \bar{C}_D \) is a \([p^{2m-1}, (p-1)^2p^{m-1} + 1, 2m] \) two-weight linear code with weight distribution given in Table 8.

(3) If \( 2v_2(m) > v_2(u) + v_2(v) + 1 \), then \( \bar{C}_D \) is a \([p^{2m-1}, p^m + \varepsilon_u + \varepsilon_v, 2m] \) three-weight linear code with weight distribution given in Table 7.
Table 8: The weight distribution of $C_\bar{D}$ for $2v_2(m) = v_2(u) + v_2(v) + 1$

| Weight                    | Multiplicity |
|---------------------------|--------------|
| $0$                       | $1$          |
| $p^{2m-2}$                | $p^{2m-1} + (-1)^{(\frac{p-1}{4})m}(p-1)p^{m-1}-1$ |
| $p^{2m-2} + (-1)^{(\frac{p-1}{4})m}p^{m-1}$ | $(p-1)\left(p^{2m-1} - (-1)^{(\frac{p-1}{4})m}p^{m-1}\right)$ |

Table 9: The weight distribution of $C_\bar{D}$ for $2v_2(m) > v_2(u) + v_2(v) + 1$

| Weight                    | Multiplicity |
|---------------------------|--------------|
| $0$                       | $1$          |
| $p^{2m-2} + p^{m+\varepsilon_u u + \varepsilon_v v - 1} - p^{m+\varepsilon_u u + \varepsilon_v v - 2}$ | $p^{2m} - p^{2m-2\varepsilon_u u - 2\varepsilon_v v}$ |
| $p^{2m-2}$                | $(p^{m-\varepsilon_u u - \varepsilon_v v} - 1)(p^{m-\varepsilon_u u - \varepsilon_v v - 1} + 1)$ |
| $p^{2m-2} + p^{m+\varepsilon_u u + \varepsilon_v v - 1}$ | $(p-1)\left(p^{2m-2\varepsilon_u u - 2\varepsilon_v v - 1} - p^{m-\varepsilon_u u - \varepsilon_v v - 1}\right)$ |

**Remark 21** Assume that $m = 2$, $v_2(u) + v_2(v) = 1$ and $p \equiv 3 \pmod{4}$. From Table 8 it is easy to see that $C_\bar{D}$ is a $[p^2 + 1, 4, p^2 - p]$ code. This code is optimal with respect to the Griesmer bound.

**Example 22** Let $m = 2$ and $p = 3$.

1. If $k = 0$ and $\ell = 0$, or $k = 1$ and $\ell = 1$, then $C_\bar{D}$ has parameters $[16, 4, 9]$ and weight enumerator $1 + 32x^9 + 48x^{12}$.

2. If $k = 0$ and $\ell = 1$, or $k = 1$ and $\ell = 0$, then $C_\bar{D}$ has parameters $[10, 4, 6]$ and weight enumerator $1 + 60x^6 + 20x^9$.

All of these codes are optimal with respect to the tables of best codes known maintained at [http://www.codetables.de](http://www.codetables.de).

### 5 Conclusions

This paper continued the work of [20] and [18], and obtained several classes of new two-weight and three-weight linear codes and determined their weight distributions. Moreover, we investigated punctured linear codes $C_\bar{D}$ which are derived from $C_D$ by deleting some coordinates, which included some optimal linear codes. These new two-weight and three-weight linear codes $C_D$ may be applied to construct strongly regular graphs [2] and association schemes [1] with new parameters, respectively. Furthermore, if $m$ is large enough, one easily check that

$$\frac{w_{\min}}{w_{\max}} > \frac{p-1}{p}$$

for the linear codes $C_D$, where $w_{\min}$ and $w_{\max}$ denote the minimum and maximum nonzero weights of $C_D$, respectively. Then the new codes may be used to construct secret sharing schemes with nice access structures [10].

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