Scientific comment on “Tail risk of contagious diseases”

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Abstract

Cirillo and Taleb [1] study the size of major epidemics in human history in terms of the number of fatalities. Using the figures from 72 epidemics, from the plague of Athens (429 BC) to the COVID-19 (2019-2020), they claim that the resulting fatality distribution is “extremely fat-tailed”, i.e., asymptotically a power law. This has important consequences for risk, as the mean value of the fatality distribution becomes infinite. Reanalyzing the same data we find that, although the data may be compatible with a power-law tail, these results are not conclusive, and other distributions, not fat-tailed, could explain the data equally well. Simulation of a log-normally distributed random variable provides synthetic data whose statistics are indistinguishable from the statistics of the empirical data.
Cirillo and Taleb [1] identify “fat-tailed” distributions with regularly varying distributions [2], defined by a complementary cumulative distribution function (or survival function, probability of being above $x$) given by $S_{fat}(x) = \ell(x)/x^\alpha$, with $\alpha$ the exponent (of $S_{fat}(x)$, and $\xi = 1/\alpha$ the tail index) and $\ell(x)$ an unspecified slowly varying function (for example, a function that tends to a constant when $x \to \infty$, but not only). Roughly speaking, a “fat-tailed” distribution becomes a power law asymptotically. Be aware that fat-tailed distributions are long-tailed distributions, which are heavy-tailed distributions in their turn, but not the opposite [2].

As an alternative, we consider the truncated log-normal (ln) distribution, which is not fat-tailed but is subexponential (and therefore long-tailed and heavy-tailed [2]). Its probability density is

$$f_{ln}(x) = \sqrt{\frac{2}{\pi}} \left[ \text{erfc} \left( \frac{\ln u - \mu}{\sqrt{2}\sigma} \right) \right]^{-1} \frac{1}{\sigma x} \exp \left( -\frac{(\ln x - \mu)^2}{2\sigma^2} \right),$$

for $x \geq u$ (and zero otherwise), with $\mu$ and $\sigma$ the mean and standard deviation of the underlying (untruncated) normal distribution, $u$ a lower cut-off, and erfc the complementary error function. The log-normal distribution has been an important competitor of the power law for the size distribution of structures and events in complex systems [3–5].

We fit the truncated log-normal to the epidemic data of Ref. [1] using the method of Ref. [4], obtaining $u = 1000$ (fitting the whole data set), $\mu = 10.47$ and $\sigma = 3.58$ ($p$-value 0.95; scale parameter $e^\mu \simeq 35,200$). The empirical estimation of the probability density of the data together with the obtained fit are shown in Fig. [1]. A comparison with a power-law (pl) fit for the tail ($f_{pl}(x) \propto 1/x^{1+\alpha}$, see Ref. [4]) shows that both fits are very close to each other, but also that the power law, with $u \simeq 33,000$ and $\alpha = 0.344$ ($p$-value 0.21), gives a higher probability than the log-normal for the most extreme events (as expected).

Next we will compare the statistical behavior of the epidemic empirical data of Ref. [1] with that of the simulation of a truncated log-normal distribution (with the values of the parameters given above and $N = 72$ events).

Cirillo and Taleb [1] propose two main ways to check fat-tailness. One of this uses the mean-excess function $\epsilon(u) = \langle x - u|x \geq u \rangle$, where the brackets denote expected value. This is the same as the expected residual size [7] (see also Ref. [8]) used in reliability theory and characterizing the distribution in a way totally equivalent to $f(x)$ or $S(x)$, provided that the first moment of the distribution is finite ($\epsilon(0) = \langle x \rangle$, but note that for $\alpha < 1$ this is not
FIG. 1: Empirical distribution of the number of fatalities for each of the 72 historical epidemics studied in Ref. [1]. A truncated log-normal fit and a power-law tail (starting at $u \simeq 33,000$) are shown as well. (a) Probability density (empirical distribution obtained using logarithmic binning [6]). (b) Complementary cumulative distribution function (survival function).
The other approach in Ref. [1] uses the (partial) maximum-to-sum ratio (the maximum of the \(x\) values divided by the sum of the values). As \(N \to \infty\), this ratio should tend to zero when the mean of the distribution is finite (as it happens with the log-normal but not with the power law when \(\alpha < 1\)). We again compare the empirical data with the simulated data, sorting the simulated data in order that the ranks of the sizes (number of fatalities) follow the same temporal pattern as the empirical data (i.e., the largest simulated event is always put on the 11th position, where the Black Death, the largest event on record, takes
place in the original data, and so on). The results, displayed at Fig. 2(b), show again that
the behavior of synthetic log-normal data is very close to that of the empirical data. Thus,
although the theory teaches us that the maximum-to-sum ratio tends to zero when $N \to \infty$
if the distribution has a finite mean, this convergence can be rather slow, as it happens with
the log-normal distribution for the parameter values that describe the epidemic data.

Now we provide complementary evidence that the log-normal distribution is a good fit
of the epidemic data of Ref. [1]. In fact, the power law can be considered a particular case
of the truncated log-normal (in the same way that the exponential is a particular case of
the truncated normal distribution when $\sigma^2 \to \infty$ and $\mu \to -\infty$ [9]). In this sense, the
log-normal will always provide a better fit. However, on the other hand, it may happen
that this improvement in the fit is not significant, and then the power-law fit suffices for
describing the data. This is something that can be evaluated by a likelihood-ratio (LR) test
[10].

Taking advantage of the fact that the LR between both distributions is a decreasing
function of the logarithmic coefficient of variation (CV) [9], this provides a very simple way
to perform the LR test (without the need of performing maximum-likelihood fitting): critical
values of the LR translate into critical values of the logarithmic CV. When this quantity
is close enough to one, the power-law hypothesis cannot be rejected, and when it departs
significantly from one (from below), the power law is rejected in favor of the log-normal.

The test is performed for different values of the lower cut-off $u$, and the results, for the
complete data set, are displayed on Fig. 2(c). This shows that only for the 21 largest
epidemics the power-law tail is not rejected in favor of the log-normal. The corresponding
cut-off $u$ turns out to be at about 350,000 fatalities. In other words, the 21 epidemics with
more than 350,000 casualties are well described by a power law (the improvement brought
by the log-normal is not significant), but, including events below 350,000, the log-normal fit
is significantly better (for the full range). Applying the same procedure to the log-normally
simulated data reproduces again the pattern obtained for the empirical data, as also shown
in the figure.

Assuming that a power law can describe the largest epidemics (in terms of fatalities),
which would be the value of the corresponding power-law exponent $\alpha$? Above, for
$u \approx 33,000$, we report $\alpha = 0.344$, but for $u \approx 350,000$ the value is larger. In fact, the value of $\alpha$
is not stable at all, growing when the lower cut-off $u$ increases (this is already apparent in the

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results of Cirillo and Taleb [1]), which prevents that one can establish a well-defined exponent [11]. Indeed, Fig. 2(d) shows the resulting exponents $\alpha$ as a function of $u$, comparing the original (empirical) data with the log-normally simulated ones. It is clear that the simulated data provides a pattern very similar to the empirical one, with an increase of the value of the exponent $\alpha$ when $u$ increases. Indeed, this increasing behavior of the fitted exponent is what one expects from a log-normal distribution.

We have shown how the probability distribution of the number of fatalities of historical epidemics can be well explained by a log-normal distribution, which is a distribution that is empirically similar to the power law but quite different from a theoretical point of view (in particular, the mean and all moments of a log-normal distribution are well defined). Our work shows the importance of considering alternative probability models when fitting heavy-tailed distributed data (which is different from “fat-tailed’ data” [2]), as well of the key role of computer simulations to contrast the validity of theoretical results when the number of data is not infinite.

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