THE MÉNAGE PROBLEM WITH A KNOWN MATHEMATICIAN

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Abstract. We give a solution of the following combinatorial problem: "A known mathematician N found himself with his wife among the guests, which were n(≥ 3) married couples. After seating the ladies on every other chair at a circular table, N was the first offered to choose an arbitrary chair but not side by side with his wife. To find the number of ways of seating of other men after N chose a chair (under the condition that no husband is beside his wife)." We discuss also the problem: "For which values of n the number of ways of seating of other men does not depend on a choice by N his chair?"

1. Introduction

In 1891, Lucas [2] formulated the following ”ménage problem”:

Problem 1. To find the number $M_n$ of ways of seating $n$ married couples at a circular table, men and women in alternate positions, so that no husband is next to his wife.

After seating the ladies by $2n!$ ways we have

$$M_n = 2n!U_n,$$

where $U_n$ is the number of ways of seating men.

Earlier Muir [3] solved a problem posed by Tait (cf. [4]): to find the number $H_n$ of permutations $\pi$ of \{1, ..., n\} for which $\pi(i) \neq i$ and $\pi(i) \neq i+1$ (mod $n$), $i = 1, ..., n$. By a modern language, $H_n = \text{per}(J_n - I - P)$, where $J_n$ is $n \times n$ matrix composed by 1's only, $I = I_n$ is the identity matrix and $P = P_n$ is the incidence matrix corresponding to the cycle (1, 2, ..., n) (cf. [3]). Simplifying Muir’s solution, Cayley [1] found a very simple recursion for $H_n$: $H_2 = 0, H_3 = 1,$ and for $n \geq 4$,

$$1.2\quad (n-2)H_n = n(n-2)H_{n-1} + nH_{n-2} + 4(-1)^{n+1}.$$

Only in 1934 due to celebrated research by Touchard [9], it became clear that

$$1.3\quad H_n = U_n$$
and thus formulas (1.1)-(1.2) give a recursion solution of the ménage problem. Moreover, Touchard gave a remarkable explicit formula

\[
U_n = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} (2n-k) (n-k)!
\]

A beautiful proof of (1.4) with help of the rook technique one can find in [5].

The first terms of the sequence \(\{U_n\}\), for \(n \geq 2\), are (cf. [8])

\[
0, 1, 2, 13, 80, 579, 4378, 34387, 439792, 439792, 4890741, 59216642, ...
\]

Note that formulas for \(U_n\) in other forms are given by Wayman and Moser [10] and Shevelev [6].

In the present paper we study the following problem.

**Problem 2.** A known mathematician \(N\) found himself with his wife among the guests, which were \(n(\geq 3)\) married couples. After seating the ladies on every other chair at a circular table, \(N\) was the first offered to choose an arbitrary chair but not side by side with his wife. To find the number of ways of seating of other men, after \(N\) chose a chair, under the condition that no husband is beside his wife.

We also discuss a close problem:

**Problem 3.** For which values of \(n\) the number of ways of seating of other men in Problem 2 does not depend on a choice by mathematician \(N\) his chair?

2. A COMMENT TO REPRESENTATION OF SOLUTION OF PROBLEM 1 BY \(\text{per}(J_n - I - P)\) IN CONNECTION WITH PROBLEM 2

Denote \(2n\) chairs at a circular table by the symbols

\[
1, \bar{1}, 2, \bar{2}, ..., n, \bar{n}
\]

over clockwise. Ladies occupy either chairs \(\{1, ..., n\}\) or chairs \(\{\bar{1}, ..., \bar{n}\}\). Let they occupy, say, chairs \(\{\bar{1}, ..., \bar{n}\}\). Then to every man we give a number \(i\), if his wife occupies the chair \(\bar{i}\). Now the \(i\)-th man, for \(i = 1, ..., n - 1\), can occupy every chair except of chairs \(i, i+1\), while the \(n\)-th man cannot occupy chairs \(n\) and 1. Denoting in the corresponding \(n \times n\) incidence matrix the prohibited positions by 0’s and other positions by 1’s, we obtain the matrix \(J_n - I - P\). Now, evidently, to every seating the men corresponds a diagonal of 1’s in this matrix. This means that

\[
U_n = \text{per}(J_n - I - P).
\]

Let in Problem 2 the wife of mathematician \(N\) occupy, say, chair \(\bar{1}\).
Let us measure the distance between N and his wife via the number of spaces between the separating them chairs over clockwise. Now, if N occupies the \( r \)-th chair, then the distance equals to \( r - 1 \). In the incidence matrix, the \( r \)-th chair of the first man corresponds to position (1, \( r \)). Denote the matrix obtained by the removing the first row and the \( r \)-th column of the matrix \( J_n - I - P \) by \((J_n - I - P)[1\mid r]\). Then, we obtain the following lemma.

**Lemma 1.** If N chose a chair at the distance \( r - 1 \) from his wife, then the number of seating of other men equals to \( \text{per}((J_n - I - P)[1\mid r]) \).

Note that, if to consider numeration 2.1 over counterclockwise, then we obtain a quite symmetric result in which \( r \) corresponds to \( n - r + 3 \), \( r = 3, ..., n \), such that as a corollary of Lemma 1 we have

\[
\text{per}((J_n - I - P)[1\mid r]) = \text{per}((J_n - I - P)[1\mid n-r+3]), \quad r = 3, ..., n.
\]

### 3. Rook lemmas

Here we place several results of the classic Kaplansky-Riordan rook theory (cf. [5], Ch. 7-8).

Let \( M \) be a rectangle (quadratic) (0,1)-matrix \( M \).

**Definition 1.** The polynomial

\[
R_{M}(x) = \sum_{j=0}^{n} \nu_{j}(M)x^{j}
\]

where \( \nu_{0} = 1 \) and \( \nu_{j} \) is the number of ways of putting \( j \) non-taking rooks on positions 1’s of \( M \), is called rook polynomial.

Note that \( n \) is the maximal number for which there exists at least one possibility to put \( n \) non-taking rooks on positions 1’s of \( M \).

**Lemma 2.** If \( M \) is a quadratic matrix with the rook polynomial (3.1), then

\[
\text{per}(J_n - M) = \sum_{j=0}^{n} (-1)^{j}\nu_{j}(M)(n - j)!
\]

**Definition 2.** Two submatrices \( M_{1} \) and \( M_{2} \) of (0,1)-matrix \( M \) are called disjunct if no 1’s of \( M_{1} \) in the same row or column as those of \( M_{2} \).

From Definition 2 the following lemma evidently follows.

**Lemma 3.** If (0,1)-matrix \( M \) consists of two disjunct submatrices \( M_{1} \) and \( M_{2} \), then

\[
R_{M}(x) = R_{M_{1}}(x)R_{M_{2}}(x).
\]
Consider a position \((i, j)\) of 1 in matrix \(M\). Denote \(M^{(0(i,j))}\) the matrix obtained from \(M\) after replacing 1 in position \((i, j)\) by 0. Denote \(M^{(i,j)}\) the matrix obtained from \(M\) by removing the \(i\)-th row and \(j\)-column.

**Lemma 4.** We have
\[
R_M(x) = xR_{M(i,j)} + R_{M^{(0(i,j)))}}.
\]

Consider so-called simplest connected staircase \((0, 1)\)-matrices. Such matrix is called \(k\)-staircase, if the number of its 1’s equals to \(k\). For example, the following several matrices are 5-staircase:
\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

and the following matrices are 6-staircase:
\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

**Lemma 5.** For every \(k \geq 1\), all \(k\)-staircase matrices \(M\) have the same rook polynomial
\[
(3.5) \quad R_M(x) = \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k - i + 1}{i} x^i.
\]

4. **Solution of Problem 2**

According to Lemma 2, in order to calculate permanent of matrix \((J_n - I - P)[1\mid r]\), we can find rook polynomial of matrix \(J_{n-1} - (J_n - I - P)[1\mid r]\). We use an evident equation
\[
(4.1) \quad J_{n-1} - (J_n - I - P)[1\mid r] = (I_n + P)[1\mid r].
\]

Pass from matrix \((I_n + P)\) to matrix \((I_n + P)[1\mid r]\). We have (here \(n = 10\), \(r = 5\))
\[
(4.2) \quad \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \rightarrow
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Now we use Lemma 4 to the latter matrix in case $i = n, \ j = 1$. Denote

$$A = ((I_n + P)[1\ |\ r])^{(n-1,1)}, \ B = ((I_n + P)[1\ |\ r])^{(0(n-1,1))}. \tag{4.3}$$

According to (3.4), we have

$$R_{(I_n + P)[1\ |\ r]}(x) = xR_A(x) + R_B(x). \tag{4.4}$$

Note that matrix $A$ has the form (here $n = 10, \ r = 5$)

$$A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \tag{4.5}$$

which is $(n - 2) \times (n - 2)$ matrix with $2n - 6$ 1’s. This matrix consists of two disjunct matrices: $(r - 2) \times (r - 2)$ matrix $A_1$ of the form (here $r = 5$)

$$A_1 = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}, \tag{4.6}$$

which is $2r - 5$-staircase matrix, and $(n - r) \times (n - r)$ matrix (here $n = 10, \ r = 5$)

$$A_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}, \tag{4.7}$$

which is $2(n - r) - 1$-staircase matrix.

Thus, by Lemmas $\text{[3]}$ and $\text{[5]}$ we have

$$R_A(x) = \sum_{i=0}^{r-2} \binom{2r - i - 4}{i} x^i \sum_{i=0}^{n-r} \binom{2(n - r) - i}{i} x^i \tag{4.8}$$

Note that, since $\binom{n}{n} = 0$, then we write formally the lower limit in interior sum $j = 0$. Furthermore, matrix $B$ has the form (here $n = 10, \ r = 5$)
which is \((n - 1) \times (n - 1)\) matrix with \(2n - 5\) 1’s. This matrix consists of two disjunct matrices: \((r - 2) \times (r - 1)\) matrix \(B_1\) of the form (here \(r = 5\))

\[
B_1 = \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

which is \(2r - 5\)-staircase matrix, and \((n - r + 1) \times (n - r)\) matrix (here \(n = 10, \ r = 5\))

\[
B_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

which is \(2(n - r)\)-staircase matrix.

Thus, by Lemmas 3 and 5, we have

\[
R_B(x) = \sum_{i=0}^{r-2} \binom{2r - i - 4}{i} x^i \sum_{j=0}^{n-r+1} \binom{2(n-r) - j + 1}{j} x^j.
\]

Note that, since \(\binom{n-r}{n-r+1} = 0\), then we write formally the upper limit in interior sum \(j = n - r + 1\). Now, using Lemma 4 for \(M = (I_n + P)[1\mid r]\), from (4.7) and (4.11) we find

\[
R_{(I_n+P)[1\mid r]}(x) = \sum_{i=0}^{r-2} \binom{2r - i - 4}{i} x^i \sum_{j=0}^{n-r+1} \binom{2(n-r) - j + 2}{j} x^j
\]

\[
= \sum_{k=0}^{n-1} x^k \sum_{i=0}^{\min(k, \ r-2)} \binom{2r - i - 4}{i} \binom{2(n-r) - k + i + 2}{k - i}.
\]

Note that in the interior sum in (4.13) it is sufficient to take summation over interval \([\max(r + k - n - 1, 0), \min(k, r - 2)]\). Thus, by Lemma 2 and
(4.1), we have

\[ \text{per}((J_n - I - P)[1\mid r]) = \]

\[ \sum_{k=0}^{n-1} (-1)^k (n-k-1)! \sum_{i=\max(r+k-n-1, 0)}^{\min(k, r-2)} \binom{2r - i - 4}{i} \binom{2(n-r) - k + i + 2}{k - i}. \]

By Lemma 1, formula (4.14) solves Problem 2.

\[ \square \]

**Remark 1.** It is well known (\[5\], Ch. 8), that if in Problem 1 to replace a circular table by a straight one, than the incidence matrix of the problem is obtained from \(J_n - I - P\) by removing 1 in position \((n, 1)\). Therefore, a solution of the corresponding problem to Problem 2 for a fixed \(r \geq 3\), is given by \(\text{per}(J_{n-1} - B)\), where \(B\) is the matrix (4.9). Thus, by Lemma 2 and (4.12), we analogously have

\[ \text{per}(J_{n-1} - B) = \]

\[ \sum_{k=0}^{n-2} (-1)^k (n-k-1)! \sum_{i=\max(r+k-n, 0)}^{\min(k, r-2)} \binom{2r - i - 4}{i} \binom{2(n-r) - k + i + 1}{k - i}. \]

5. **Discussion of Problem 3**

Expanding \(U_n = \text{per}(J_n - I - P)\) over the first row, we have

\[ U_n = \sum_{r=3}^{n} \text{per}((J_n - I - P)[1\mid r]). \]

According to Lemma 1 in conditions of Problem 3, a value of \(\text{per}((J_n - I - P)[1\mid r])\) does not depend on \(r\). This means that, by (5.1), we have

\[ \text{per}((J_n - I - P)[1\mid r]) = \frac{U_n}{n-2}, \quad r = 3, \ldots, n. \]

Note that, in view of (2.3), it is sufficient to consider in (5.2) \(r = 3, \ldots, \left\lfloor \frac{n+3}{2} \right\rfloor\).

For the first time, Problem 3 was announced by the author in \[7\] with conjecture that the solution supplies the set of those \(n\) for which \(n-2|U_n\). Such solutions were verified for \(n = 3, 4, 6\). Let us show that this conjecture is not true. Reducing (4.14) for \(r = 3\), let us find a necessary condition for the suitable \(n\).
Lemma 6. If, for a given \( n \), Problem 3 is solved in affirmative, then we have

\[
\sum_{k=0}^{n-3} (-1)^k \binom{2n - k - 4}{k} (n - k - 2)! (n - k - 2) \frac{U_n}{n - 2}.
\]

Proof. By (5.2) and (4.14) for \( r = 3 \), we have

\[
\frac{U_n}{n - 2} = \text{per}((J_n - I - P)[1\ 3]) =
\]

\[
\sum_{k=0}^{n-1} (-1)^k (n - k - 1)! B_{n, k},
\]

where

\[
B_{n, k} = \min(k, 1) \sum_{i = \max(k-n+2, 0)} (2 - i) \binom{2n - 4 - k + i}{k - i}, \quad k = 0, \ldots, n - 1.
\]

It is easy to see that

\[
B_{n, 0} = B_{n, n - 1} = 1;
\]

\[
B_{n, k} = \binom{2n - 4 - k}{k} + \binom{2n - 3 - k}{k - 1}, \quad k = 1, \ldots, n - 2.
\]

Therefore, by (5.4), we have

\[
\frac{U_n}{n - 2} = (n - 1)! + (-1)^{n-1} +
\]

\[
\sum_{k=1}^{n-2} (-1)^k \left( \binom{2n - 4 - k}{k} + \binom{2n - 3 - k}{k - 1} \right) (n - k - 1)! =
\]

\[
(n - 1)! + (-1)^{n-1} +
\]

\[
\sum_{k=1}^{n-2} (-1)^k \binom{2n - 4 - k}{k} (n - k - 1)! - \sum_{k=0}^{n-3} (-1)^k \binom{2n - 4 - k}{k} (n - k - 2)! =
\]

\[
(n - 1)! - (n - 2)! +
\]
\[
\sum_{k=1}^{n-3} (-1)^k \binom{2n - 4 - k}{k} ((n - k - 1)! - (n - k - 2)!) 
\]
and (5.3) follows. ■

However, for \( n = 10 \), \( \frac{U_n}{n!} = 54974 \), but the left hand side of (5.3) equals to 54888.

**Conjecture 1.** Set \{3, 4, 6\} contains only solutions of Problem 3.

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