A more efficient variant of the Oxford protocol

Nasser Metwally

Sektion Physik, Universität München,
Theresienstraße 37, 80333 München, Germany

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Abstract

An alternative presentation of the Oxford purification protocol is obtained by using dynamical variables. I suggest to introduce the degree of separability as a purification parameter, where the purified state has a smaller degree of separability than the initial one. An improved version of the Oxford protocol is described, in which local unitary transformations optimize each step.

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Some quantum communication proposals require maximally entangled qubit pairs to perform them (see, e.g., [1]). Due to noisy channels, the pairs lose their fidelity partially; dissipative effects of the environment turn pure states into mixed states. The aim is then to purify those states to re-obtain maximally entangled qubit pairs. The entanglement purification that is often required distills a small number of strongly entangled pairs of qubits from a larger number of weakly entangled pairs, by using local quantum operations, classical communications, and measurements.

The first entanglement purification protocol, called IBM protocol has been given by Bennett et al. [2]. It enables one to distill from a large ensemble of entangled states with fidelity greater than 0.5 a smaller ensemble of pairs with fidelity close to unity. Those purified pairs could be used for faithful teleportation. Also Deutsch et al. [3] have formulated another protocol designed for cryptographic purposes; it is called “quantum privacy amplification”, or “Oxford protocol” for short.

Purification under imperfect operations is studied by Giedke et al. [4] who obtain a lower bound for the fidelity, such that purification is possible in the presence of noise. Fidelity is the overlap of the density operator of a pair of qubits with the wanted maximally entangled state.

In this work the dynamical variables of the qubits are used to describe the Oxford protocol, at the relevant example of the so-called Bell-diagonal states and their special kind known as binary states. The degree of separability serves as alternative purification parameter, where a more entangled state has a smaller degree of separability. An improved Oxford protocol is introduced; it converges faster and is more efficient than the original one.

The paper is organized as follows. In Sec. I, Bell-diagonal states are re-considered. In Sec. II, the bilateral controlled NOT (BCNOT) operation is described in terms of the dynamical variables and a description of the Oxford protocol is presented in these variables. In Sec. III, the degree of separability is considered as a purity measure. The two variants of the Oxford protocol are investigated for states of two kinds, binary states and the more general Bell-diagonal states. It should be noted that both variants of the Oxford protocol always produce Bell-diagonal states after the first iteration, irrespective of whether the initial state is of this kind or not. Therefore, we do not need to consider more general states.
I. BELL-DIAGONAL STATES

Analogs of Pauli’s spin operators are, as usual, used for the description of the individual qubits, the hermitian set $\sigma_x, \sigma_y, \sigma_z$ for the first qubit and $\tau_x, \tau_y, \tau_z$ for the second. A “generalized Werner state of the first kind” [5], or “self-transposed state” [6], or “Bell-diagonal state” [3] is given by

$$\rho_{\text{Bell-diag}} = \frac{1}{4}(1 - c_x \sigma_x \tau_x - c_y \sigma_y \tau_y - c_z \sigma_z \tau_z),$$

where

$$1 \geq |c_x| \geq |c_y| \geq |c_z| \geq 0,$$

the order being a matter of convention. This state is separable if it has a positive partial transpose [7], which is the case if either $|c_x| + |c_y| + |c_z| \leq 1$ or $c_x c_y c_z \leq 0$. Otherwise, that is: if $|c_x| + |c_y| + |c_z| > 1$ and $c_x c_y c_z > 0$, the state is non-separable and

$$S = \frac{3}{2} - \frac{1}{2}(|c_x| + |c_y| + |c_z|)$$

is its degree of separability [3, 6]. By a suitable local unitary transformation it can then be arranged that all $c_k$’s are positive. In particular, for $c_x = c_y = c_z = t$, one gets the standard Werner states [8],

$$\rho_{\text{Werner}} = \frac{1}{4}[1 - t(\sigma_x \tau_x + \sigma_x \tau_y + \sigma_x \tau_z)],$$

with $-\frac{1}{3} \leq t \leq 1$. These states are separable for $t \leq \frac{1}{3}$ and non-separable for $t > \frac{1}{3}$ with the degree of separability given by $S=\frac{3}{2}(1 - t)$.

II. OXFORD PROTOCOL

Before performing the Oxford protocol on dynamical variables, one needs to describe the BCNOT operation on those variables. In this operation, both members of one pair are used as source qubits and both qubits from the other pair are used as target qubits. The BCNOT is

$$\text{BCNOT}(\sigma_\mu^{(1)} \sigma_\nu^{(2)}) = \frac{1 + \sigma_\mu^{(1)}}{2} \sigma_\nu^{(2)} + \frac{1 + \sigma_\mu^{(2)}}{2} \sigma_\nu^{(1)}$$

$$+ \frac{1 + \sigma_\mu^{(1)}}{2} \sigma_\nu^{(1)} \frac{1 + \sigma_\mu^{(2)}}{2} \sigma_\nu^{(2)}$$

$$\cdot$$

$$\cdot$$
where the suffixes 1 and 2 refer to the first and the second qubit. Table I shows the effect of the BCNOT operation on the two qubits, that specify $\sigma^{(1)}_\mu$ and $\sigma^{(2)}_\nu$.

In this protocol the users Alice and Bob have a supply of qubit pairs, each pair being in the pure, maximally entangled state,

$$\rho_{\text{ideal}} = \frac{1}{4}(1 + \sigma_x \tau_x - \sigma_y \tau_y + \sigma_z \tau_z).$$

(6)

Because of the noise along the transmission channel, the pairs interact with the environment, so they lose their purity. Assume that Alice and Bob are given an ensemble that consists of two subensembles. Each of those subensembles is made of Bell-diagonal states with different $c_k$'s. Let Alice and Bob pick two different pairs, one from each subensemble,

$$\rho^{(1)} = \frac{1}{4}(1 + c_x \sigma_x^{(1)} \tau_x^{(1)} - c_y \sigma_y^{(1)} \tau_y^{(1)} + c_z \sigma_z^{(1)} \tau_z^{(1)}),$$

$$\rho^{(2)} = \frac{1}{4}(1 + c'_x \sigma_x^{(2)} \tau_x^{(2)} - c'_y \sigma_y^{(2)} \tau_y^{(2)} + c'_z \sigma_z^{(2)} \tau_z^{(2)}),$$

(7)

with fidelities

$$F_1 = \text{tr} \left\{ \rho^{(1)} \rho^{(1)}_{\text{ideal}} \right\} = \frac{1}{4}(1 + c_x + c_y + c_z),$$

$$F_2 = \text{tr} \left\{ \rho^{(2)} \rho^{(2)}_{\text{ideal}} \right\} = \frac{1}{4}(1 + c'_x + c'_y + c'_z).$$

(8)

In the original protocol, Ox, Alice and Bob perform the transformation $U_{12x} = e^{i\pi(\sigma_x - \tau_x)/4}$ on all pairs. This operator changes the positions of $c_y$ and $c_z$ in (6). Then

|       | $1^{(2)}$ | $\sigma_x^{(2)}$ | $\sigma_y^{(2)}$ | $\sigma_z^{(2)}$ |
|-------|-----------|-----------------|-----------------|-----------------|
| $1^{(1)}$ | 1         | $\sigma_x^{(1)}$ | $\sigma_y^{(1)}$ | $\sigma_z^{(1)}$ |
| $\sigma_x^{(1)}$ | $\sigma_x^{(2)}$ | $\sigma_x^{(1)}$ | $\sigma_y^{(2)}$ | $\sigma_z^{(1)}$ |
| $\sigma_y^{(1)}$ | $\sigma_y^{(2)}$ | $\sigma_y^{(1)}$ | $-\sigma_x^{(1)}$ | $\sigma_y^{(2)}$ |
| $\sigma_z^{(1)}$ | $\sigma_z^{(2)}$ | $-\sigma_y^{(1)}$ | $\sigma_z^{(1)}$ | $\sigma_y^{(2)}$ |

Table I: Bilateral CNOT operation between the two qubits which define $\sigma^{(1)}_\mu$ and $\sigma^{(2)}_\nu$. The same table applies for the two qubits $\tau^{(1)}_\mu$ and $\tau^{(2)}_\nu$, where $\mu, \nu=x, y$ and $z$. 

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Alice and Bob perform BCNOT operations on the pairs $\rho^{(1)}$ and $\rho^{(2)}$, followed by measuring the target qubits in the computational basis. For example, they measure the $z$ components of the targets spin, $\sigma_z^{(2)}$ and $\tau_z^{(2)}$. They keep those first pairs for which they get the same measurement results, and discarded the others. The target pairs are always consumed in the process.

In the alternative protocol, $\text{Ox}_2$, one exploits the order specified in (2) and performs BCNOT directly, without first applying $U_{12x}$. The resulting subensemble of good first pairs is characterized by

$$
\rho_{\text{new}} = \frac{1}{4} \left[ 1 + \frac{c_x c'_x + c_y c'_y}{1 + c_z c'_z} \sigma_x \tau_x - \frac{c_z c'_z + c_y c'_y}{1 + c_z c'_z} \sigma_y \tau_y \right. \\
+ \left. \frac{c_z + c'_z}{1 + c_z c'_z} \sigma_z \tau_z \right].
$$

(9)

This is another Bell-diagonal state.

In the standard description of $\text{Ox}_1$ [[1, 3]], certain parameters $A$, $B$, $C$, and $D$ play a central role. Their change under $\text{Ox}_2$ is given by

$$
A = \left\{ \frac{1}{4} \left( 1 + c_x + c_y + c_z \right) \right\} \\
\frac{1}{4} \left( 1 + c'_x + c'_y + c'_z \right) \\
\rightarrow \frac{1}{4N} \left[ (1 + c_z)(1 + c'_z) + (c_x + c_y)(c'_x + c'_y) \right],
$$

(10)

for example, and corresponding expressions apply for $B$, $C$, and $D$. Here $N = \frac{1}{2}(1 + c_z c'_z)$ is the probability that Alice and Bob obtain coinciding outcomes in the measurements of the target pair. If one changes the positions of $c_y$ and $c_z$ and also of $c'_y$ and $c'_z$ in (10), one gets the $A$, $B$, $C$ and $D$ values for $\text{Ox}_1$.

If the two subensembles in (2) are identical, then the protocol works if $F_1 = F_2 > \frac{1}{2}$. In terms of the parameters of (3), this means

$$
|c_x| + |c_y| + |c_z| > 1.
$$

(11)

So, at every step Alice and Bob must check this property. In particular, they need

$$
(|c_x| + |c_y|)^2 - (1 - |c_z|)^2 > 0
$$

(12)

for the first step to be successful.
If the given ensemble does not obey the ordering required by (2), then Alice and Bob use unilateral rotations to bring the state into the wanted form. These are rotations by $\pi$ about the $x, y$ or $z$ axis, namely

$$U_{1x} = \sigma_x, \quad U_{1y} = \sigma_y, \quad U_{1z} = \sigma_z,$$

$$U_{2x} = \tau_x, \quad U_{2y} = \tau_y, \quad U_{2z} = \tau_z,$$

(13)

where $U_1$ and $U_2$ refer to the first and second qubit, respectively. In fact, it is only necessary to ensure that $|c_z|$ is smaller than $|c_x|$ and $|c_y|$; the relative size of $|c_x|$ and $|c_y|$ does not matter.

### III. SEPARABILITY AND PURIFICATION

In this section the degree of separability is used as a purification parameter instead of the fidelity. Also, the behavior of the degree of separability under imperfect operations is investigated. Two cases are considered: Binary states and the more general Bell-diagonal states.

**(1) Binary state with perfect operations:** In this case,

$$\rho_{\text{bin}} = \frac{1}{4}[1 + \sigma_x \tau_x - (2f - 1)\sigma_y \tau_y + (2f - 1)\sigma_z \tau_z],$$

(14)

with the initial degree of separability

$$S_0 = \begin{cases} 
1 & \text{for } 0 < f \leq \frac{1}{2}, \\
2(1 - f) & \text{for } \frac{1}{2} < f < 1.
\end{cases}$$

(15)

Assume that Alice and Bob are given an ensemble of states (14), and they are asked to purify this ensemble. They perform the Ox$_2$ protocol and after one step they get

$$\rho'_{\text{bin}} = \frac{1}{4}\left[1 + \sigma_x \tau_x - \frac{2f - 1}{2f^2 - 2f + 1}\sigma_y \tau_y + \frac{2f - 1}{2f^2 - 2f + 1}\sigma_z \tau_z \right].$$

(16)

The corresponding degree of separability is

$$S_1 = \frac{S_0^2}{1 + (1 - S_0)^2}.$$
After repeating the protocol \( n \) times one gets \( S_n \) as a function of the initial degree of separability \( S_0 \),

\[
S_n = \frac{2}{(2/S_0 - 1)^{2^n} + 1}.
\]  

(18)

From this relation it is clear that \( S_n = 1 \) if \( S_0 = 1 \) and \( S_n \to 0 \) if \( S_0 < 1 \).

(2) Binary state with imperfect operations: In this case the operations are subjected to noise, so that states of two qubit pairs suffer a non-unitary evolution such that

\[
\rho_{12} \to p \rho_{12} + (1 - p) \frac{1}{2} \text{tr}_1 \{\rho_{12}\},
\]

(19)

where \( p \) is called reliability of the imperfect operation. The limit \( p \to 0 \) corresponds to a very noisy channel, while \( p \to 1 \) describes a channel with very little noise. For two pairs in the binary state (14), the map (19) produces

\[
\rho_{\text{bin}}^{\text{noise}} = \frac{1}{4} \left[ 1 + p \sigma_x \tau_x - p(2f - 1) \sigma_y \tau_y + p(2f - 1) \sigma_z \tau_z \right]
\]

(20)

for the “first” pairs. Rather than (15) the initial degree of separability is now

\[
S_0 = \frac{1}{2} \left[ 3 - p(4f - 1) \right].
\]

(21)

Further, the ideal BCNOT operation of (5) is replaced by BCNOT\text{noise},

\[
\text{BCNOT}_{\text{noise}}(.) = p^2 \text{BCNOT}(.) + \frac{1 - p^2}{16},
\]

(22)

where \( . \) is \( \rho^{(1)}/\rho^{(2)} \). Alice and Bob perform the Ox_2 protocol, and after the measurement of the target qubits and discarding of the “bad” first pairs they obtain

\[
\rho = \frac{1}{4} \left[ 1 + p^2 \frac{2f^2 - 2f + 1}{1 - 2p^2 f(1 - f)} \sigma_x \tau_x \\
- p^2 \frac{2f - 1}{1 - 2p^2 f(1 - f)} \sigma_y \tau_y \\
+ p^2 \frac{2f - 1}{1 - 2p^2 f(1 - f)} \sigma_z \tau_z \right],
\]

(23)

for the “good” first pairs. The new degree of separability is

\[
S_{\text{new}} = \frac{1}{2} \left[ 3 - p^2 \frac{2f^2 + 2f - 1}{1 - 2p^2 f(1 - f)} \right].
\]

(24)

(3) Bell-diagonal state: Now consider the ensemble (7) consisting of Bell-diagonal states. In this case the initial degrees of separability are given by

\[
S_0 = \frac{3}{2} - \frac{1}{2} \left( |c_x| + |c_y| + |c_z| \right),
\]

\[
S_0' = \frac{3}{2} - \frac{1}{2} \left( |c'_x| + |c'_y| + |c'_z| \right).
\]

(25)
FIG. 1: The degree of separability $S$ and fidelity $F$ for the two variants of the Oxford protocol. Solid line: original protocol $Ox_1$; dashed line: alternative protocol $Ox_2$.

Alice and Bob perform the $Ox_2$ protocol, and after one step they get

$$S_1 = \frac{3}{2} - \frac{1}{2N}[(|c_x| + |c_y|)(|c'_x| + |c'_y|) + |c_z| + |c'_z|]$$

for the “good” first pairs with $N$ as in (10), or in the presence of noise,

$$N_{\text{noise}} = \frac{1}{4p^2}[1 + p^2(1 + 2|c_z c'_z|)].$$

IV. DISCUSSION AND CONCLUSION

In Fig. 1, the separability $S$ and the fidelity $F$ are plotted as a function of the number of iterations, both one for the original protocol $Ox_1$ and for the alternative protocol, $Ox_2$. 

FIG. 2: Like Fig. 1, but with noise of strength $p = 0.994$. 

Solid line: original protocol $Ox_1$; dashed line: alternative protocol $Ox_2$.
FIG. 3: Number $N$ of pairs needed to create one pair with fidelity $F$, displayed as log$(1 - F)$ vs. log $N$. The initial state of the pairs has fidelity $F_0 = 0.62$.

where one enforces the ordering of (2) in each step. The figure refers to the initial values $(c_x, c_y, c_z) = (0.16, 0.08, 0.84)$ for which $F = 0.52$ is the initial fidelity and $2F + S=2$ holds for all iterations. In this case Alice and Bob use the bilateral rotations to rearrange these three numbers such that $(c_x, c_y, c_z) = (0.84, 0.16, 0.08)$. The figure clearly shows that for Ox$_2$, the fidelity reaches unity much faster than that for Ox$_1$. Moreover, for Ox$_1$ the fidelity decreases and then increases [10], but for Ox$_2$ it increases in each iteration.

The importance of (2) is particularly apparent when one treats the binary state (14), for which two of the $A, B, C, D$ parameters of [1, 3] are positive and the other two vanish. To perform Ox$_1$ successfully, one needs $A, C > 0$ and $B = D = 0$; then Ox$_1$ works and the fidelity increases monotonically. But if one enforces (2), Ox$_2$ works directly, and one doers not have to worry which of the four parameters are non-zero.

In Fig. 3, Ox$_1$ and Ox$_2$ are performed in the presence of noise. The importance of the property (2) is clear: the degree of separability becomes constant faster for Ox$_2$ than for Ox$_1$.

The log-log plot of Fig. 3 shows the number of initial pairs needed to create one pair with fidelity $F$. We see that Ox$_2$ uses up less qubit pairs than Ox$_1$. In addition, Ox$_2$ needs fewer iterations, so that both advantages taken together make Ox$_2$ much more efficient than Ox$_1$.

In summary, in this contribution an alternative form of the Oxford protocol is described for Bell-diagonal states. The final fidelity is obtained as a function of three numbers, $c_x$, $c_y$ and $c_z$. The improvement over the original Oxford protocol is due to the arrangement
of these three numbers in decreasing order. The parameter of the degree of separability is considered as a purification parameter. As the number of iterations increases, the degree of separability decreases.

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