UNBOUNDED ABSOLUTE WEAK DUNFORD-PETTIS AND UNBOUNDED ABSOLUTE WEAK COMPACT OPERATORS

NAZIFE ERKURTUN-ÖZCAN(1), NIYAZI ANIL GEZER(2), OMID ZABETI(3,∗)

Abstract. In this paper, using the concept of unbounded absolute weak convergence (uaw, for short) in a Banach lattice, we define two classes of continuous operators, named uaw-Dunford-Pettis and uaw-compact operators. We investigate some properties and relations between them. In particular, we consider some situations under which, the adjoint or the modulus of an uaw-Dunford-Pettis or uaw-compact operator inherits a similar property. In addition, we look into some connections between compact operators, weakly compact operators, and Dunford-Pettis ones with uaw-versions of these operators. Many examples are given to illustrate the essential conditions, as well.

1. Introduction and Preliminaries

The notion of uo-convergence under the name individual convergence was initially introduced in [11] and "uo-convergence" is proposed firstly in [4]. Recently, various types of interesting papers about uo-convergence in Banach lattices have been announced by several authors (see [6, 7, 8] for more expositions on these results). Un-convergence was introduced by Troitsky in [14] and further investigated in [5, 9]. Unbounded convergent net in term of weak convergence, uaw-convergence, was introduced by Zabeti and considered in [15].

Let $E$ be a Banach lattice. For a net $(x_\alpha)$ in $E$, if there is a net $(u_\gamma)$, possibly over a different index set, with $u_\gamma \downarrow 0$ and for every $\gamma$ there exists $\alpha_0$ such that $|x_\alpha - x| \leq u_\gamma$ whenever $\alpha \geq \alpha_0$, we say that $(x_\alpha)$ converges to $x$ in order, in notation, $x_\alpha \overset{o}{\rightharpoonup} x$. A net $(x_\alpha)$ in $E$ is said to be unbounded order convergent (uo-convergent, in brief) to $x \in E$ if for each $u \in E^+$, the net $(|x_\alpha - x| \wedge u)$ converges to zero in order. It is called unbounded norm convergent (un-convergent, for short) if $\||x_\alpha - x| \wedge u\| \to 0$. For a version of an unbounded convergent net in term of weak convergence, a net $(x_\alpha)$ in a Banach lattice $E$ is said to be unbounded absolutely weakly convergent to $x \in E$ if for each positive $u \in E$, one has $|x_\alpha - x| \wedge u \overset{w}{\rightharpoonup} 0$. In a recent paper [15], several properties of uaw-convergence have been investigated. In particular, order continuous Banach lattices and reflexive ones are characterized in terms of uaw-convergent nets. In addition, it is shown that the uaw-convergence is topological.

In this note, by an operator, we mean a bounded operator between Banach lattices, unless otherwise explicitly stated.

Compact operators are known to have important directions in both theory and applications. In this paper, the concept of an uaw-compact operator is defined. An operator $T: X \to E$, where $X$ is a Banach space and $E$ is a Banach lattice, is said to be (sequentially) uaw-compact if $T(B_X)$ is relatively (sequentially) uaw-compact where $B_X$ denotes the closed unit ball of the Banach space $X$. Equivalently, for every bounded net $(x_\alpha)$ (respectively, every bounded sequence $(x_n)$) its image has a subnet (respectively, subsequence), which is uaw-compact. We further say that the operator $T$ is un-compact if $T(B_X)$ is relatively un-compact in $E$. In [9], some properties of un-compact operators are studied.

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∗ Corresponding author.

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Moreover, we consider an $uaw$-version of Dunford-Pettis operators. For the general theory of Dunford-Pettis operators, reader is referred to \[10\] \[13\]. Suppose $E$ is again a Banach lattice and $X$ is a Banach space. We say that $T$: $E \to X$ is an $uaw$-Dunford-Pettis operator if for every norm bounded sequence $(x_n)$ in $E$, $x_n \overset{uaw}{\to} 0$ implies $\|T(x_n)\| \to 0$.

In the present paper, we investigate relationships between compact and Dunford-Pettis operators in the $uaw$-version. Some properties of $uaw$-compact and $uaw$-Dunford-Pettis operators are studied. In particular, we consider some conditions, for them, the adjoint or the modulus of an $uaw$-compact or $uaw$-Dunford-Pettis operator has again a similar property. In addition, various examples are given to make the concepts and hypotheses more understandable.

Denote by $B_{UDP}(E), B_{DP}(E), K_{uaw}(E), K_{un}(E)$ the spaces of all $uaw$-Dunford-Pettis, Dunford-Pettis, $uaw$-compact and $un$-compact operators on a Banach lattice $E$, respectively. For other necessary terminology on vector and Banach lattice, we refer the reader to \[1\] \[2\]. In this paper, all vector lattices are assumed to be Archimedean.

2. Main Results

Proposition 1. Suppose that $E$ is a Banach lattice whose dual space is order continuous and $X$ is a Banach space. Then, every Dunford-Pettis operator $T$: $E \to X$ is $uaw$-Dunford-Pettis.

Proof. Suppose $T \in B_{DP}(E, X)$ and $(x_n)$ is a norm bounded sequence in $E$ which is $uaw$-convergent to zero. By \[15\] Theorem 7, it is weakly convergent. By the assumption, $\|T(x_n)\| \to 0$, as desired.

Note that order continuity of $E'$ is essential in Proposition 1 and can not be dropped. To see this, consider the identity operator $I$ on $\ell_1$. It follows from the Schur property of $\ell_1$ that $I$ is Dunford-Pettis. However it can not be $uaw$-Dunford-Pettis as the $uaw$-null sequence $(e_1)$ formed by the standard basis of $\ell_1$ is not norm convergent to zero. In addition, it can be easily seen that every $uaw$-Dunford-Pettis operator is automatically continuous but the converse is not true, in general; again, consider the identity operator on $\ell_1$.

Remark 1. Suppose that $E$ is an $AM$-space and $X$ is a Banach space. Using Proposition 1 it can be seen that an operator $T$: $E \to X$ is $uaw$-Dunford-Pettis if and only if it is Dunford-Pettis. Suppose further that $E$ is an atomic order continuous Banach lattice. It follows from \[12\] Proposition 2.5.23 that an operator $T$: $E \to X$ is $uaw$-Dunford-Pettis, then it is a Dunford-Pettis operator.

It is known that every compact operator is Dunford-Pettis. In the following example, we show that in the case of an $uaw$-Dunford-Pettis operator, the situation is different.

Example 1. Let $T$: $\ell_1 \to \mathbb{R}$ be defined by $T((x_n)) = \sum_{n=1}^{\infty} x_n$ for every $(x_n) \in \ell_1$. Since $T$ is of finite rank, it is compact. It follows by considering the standard basis of $\ell_1$ that $T$ can not be an $uaw$-Dunford-Pettis operator.

A typical example of a Dunford-Pettis operator which is not compact is the identity operator on $\ell_1$ because of the Schur property. But this operator does not do the job for the $uaw$-case since it is not also $uaw$-Dunford-Pettis. Nevertheless, there is a good news if one considers the Lozanovsky-like example as it is described in \[2\] Page 289, Exercise 10.

Example 2. Consider the operator $T$: $C[0, 1] \to c_0$ given by

$$T(f) = \left( \int_0^1 f(t) \sin t \, dt, \int_0^1 f(t) \sin 2t \, dt, \ldots \right)$$

for every $f \in C[0, 1]$. Denote by $f_n \in C[0, 1]$ a norm bounded sequence for which $f_n \overset{uaw}{\to} 0$ holds. It follows from \[15\] Theorem 7 that $f_n \overset{uaw}{\to} 0$ and that $\|T(f_n)\| = \sup_{n \geq 1} |\int_0^1 f_n(t) \sin mt \, dt| \to 0$. Hence, the noncompact operator $T$ is an $uaw$-Dunford-Pettis operator.

As in \[9\] Proposition 9.1, we have the same conditions for $uaw$-compactness and sequentially $uaw$-compactness of an operator.
Proposition 2. Let $T: E \to F$ be two operators between Banach lattices.

(i) If $E$ has a quasi-interior point then $T$ is uaw-compact iff it is sequentially uaw-compact;
(ii) If $E$ is order continuous and $T$ is uaw-compact then $T$ is sequentially uaw-compact;
(iii) If $E$ is an atomic KB-space then $T$ is uaw-compact and sequentially uaw-compact.

Let us continue with several ideal properties.

Proposition 3. Let $S: E \to F$ and $T: F \to G$ be two operators between Banach lattices.

i. If $T$ is (sequentially) uaw-compact and $S$ is continuous then $TS$ is (sequentially) uaw-compact.
ii. If $T$ is an uaw-Dunford-Pettis operator and $S$ is either (sequentially) un-compact or uaw-compact then $TS$ is compact.
iii. If $T$ is uaw-Dunford-Pettis and $S$ is Dunford-Pettis then $TS$ is Dunford-Pettis.
iv. If $T$ and $S$ are uaw-Dunford-Pettis, So is $TS$.

Proof. (i). We prove the results for the sequence case. For nets, the proof is similar. Suppose $(x_n) \subseteq E$ is a bounded sequence. By the assumption, the sequence $(S(x_n))$ is also norm bounded. Therefore, there is a subsequence $TS(x_{n_k})$ which is uaw-convergent.

(ii). Suppose $(x_n)$ is a bounded sequence in $E$. There is a subsequence $(x_{n_k})$ such that $S(x_{n_k}) \xrightarrow{\text{uaw}} x$ for some $x \in F$. Thus, by the hypothesis, $\|T(S(x_{n_k})) - T(S(x))\| \to 0$, as desired.

(iii). Suppose $(x_n)$ is a sequence in $E$ which is weakly null. By the assumption, $\|S(x_n)\| \to 0$. Again, this implies that $\|TS(x_n)\| \to 0$.

(iv). Suppose $(x_n)$ is a norm bounded sequence in $E$ which is uaw-null. By the hypothesis, $\|S(x_n)\| \to 0$. So, $S(x_n) \xrightarrow{\text{uaw}} 0$. Therefore, $\|T(S(x_n))\| \to 0$, as desired. \qed

Corollary 1. Suppose $E$ is a Banach lattice. Then $B_{\text{uaw}}(E)$ is a subalgebra of $B(E)$.

In general, we have $K(E) \subseteq K_{un}(E) \subseteq K_{\text{uaw}}(E)$. In the next discussion, we show that not every uaw-compact operator is un-compact.

Example 3. The inclusion $\ell_2 \hookrightarrow \ell_\infty$ is weakly compact by [2 Theorem 5.24]. Hence, it is uaw-compact because of the range of the operator. However it is not un-compact. Since by [9 Theorem 2.3], it should be compact which is not possible.

Remark 2. $K_{un}(E)$ and $K_{\text{uaw}}(E)$ are not order closed in the usual order of the space of all continuous operators on $E$, as shown by [9 Example 9.3]; see also [15 Theorem 4].

Following results are motivated by the Krengel’s Theorem, see [2 Theorem 5.9].

Theorem 1. If $E$ is an AL-space and $F$ is a Banach lattice whose dual space is order continuous. Then every (sequentially) uaw-compact operator $T$ from $E$ into $F$ has a (sequentially) uaw-compact adjoint.

Proof. We prove the result for sequentially compact operators. The case for nets is similar. Let $T: E \to F$ be an uaw-compact operator. For every norm bounded sequence $(x_n)$ in $E$, the sequence $T(x_n)$ has a subsequence $T(x_{n_k})$ which is convergent in the uaw-topology. By [15 Theorem 7], the subsequence is weakly convergent. This implies that the operator $T$ is weakly compact. By the Gantmacher’s theorem [2 Theorem 5.23], it follows that $T'$ is weakly compact. Since range of $T'$ is an AM-space, it is uaw-compact. \qed

Corollary 2. Suppose $E$ is an AL-space, $F$ is an AM-space, and $T \in B(E,F)$. Then, $T$ is uaw-compact if and only if so is $T'$.

Remark 3. Note that order continuity of $F'$ is essential and can not be removed. Consider the identity operator on $\ell_1$. One may verify that it is uaw-compact; for $\ell_1$ is an atomic KB-space, therefore using [9 Theorem 7.5] and [15 Theorem 4], yield the desired result. But its adjoint is the identity operator on $\ell_\infty$ which is not uaw-compact.
Theorem 2. If \( E \) is an AL-space and \( F \) is a reflexive Banach lattice. Then every order bounded \( uaw \)-compact operator \( T \) from \( E \) into \( F \), has a weakly compact modulus.

Proof. By Theorem 1 if \( T \) is \( uaw \)-compact then \( T' \) is an \( uaw \)-compact operator. Note that \( E' \) is an AM-space. So, the operator \( T' \) is weakly compact and the result follows from [2, Theorem 5.35].

Proposition 4. Let \( E \) be a Banach lattice whose dual space is atomic and order continuous. Also let \( F \) be a Banach lattice whose dual is order continuous. Then, every (sequentially) un-compact operator (\( uaw \)-compact operator) \( T: E \to F \) has a (sequentially) un-compact (\( uaw \)-compact, respectively) adjoint operator \( T': F' \to E' \).

Proof. For any norm bounded sequence \( (x_n) \) in \( E \), the sequence \( (T(x_n)) \) has a subsequence which is un-convergent to zero by \( uaw \)-compactness. By [5, Theorem 6.4], it is weakly convergent. Hence, the operator \( T \) is weakly compact. It follows from Gantmacher’s theorem that \( T' \) is weakly compact. By [9, Proposition 4.16], the operator \( T' \) is \( uaw \)-compact. Same result holds for the \( uaw \)-compact operator \( T \) using this fact that by [15, Theorem 4], in order continuous Banach lattices, the \( uaw \)-topology and the \( uaw \)-topology agree.

Recall that, see [2] for details, an operator \( T: E \to F \) is \( M \)-weakly compact if for every norm bounded disjoint sequence \( (x_n) \) one has \( \|Tx_n\| \to 0 \); see [3] for a recent progress on this topic.

Proposition 5. If \( T: E \to F \) is an \( uaw \)-Dunford-Pettis operator then \( T \) is \( M \)-weakly compact; in particular, it is weakly compact. Also, if \( F' \) is order continuous and \( T: E \to F \) is an \( uaw \)-compact operator, then \( T \) is weakly compact.

Proof. If \( (x_n) \) is norm bounded disjoint sequence in \( E \), by [15, Lemma 2], \( x_n \xrightarrow{uaw} 0 \). Hence, \( \|Tx_n\| \to 0 \). The second part follows from [15, Theorem 7].

Similar to the case of usual compact operators and Dunford-Pettis ones, it might seem at the first glance that every \( uaw \)-compact operator is \( uaw \)-Dunford-Pettis; the following example is surprising.

Example 4. The inclusion \( \ell_2 \hookrightarrow \ell_\infty \) is weakly compact by [2, Theorem 5.24]. Previously we showed that this operator is \( uaw \)-compact. However it is not \( uaw \)-Dunford-Pettis. For, the standard basis \( (e_n) \) is \( uaw \)-null but it is not norm convergent to zero.

Also, the other implication may fail, as well.

Example 5. Consider the inclusion map \( J: L^\infty[0,1] \to L^1[0,1] \). It follows from [2, Subsection 5.2, Exercise 7] that \( J \) is weakly compact. In fact, \( J \) is \( uaw \)-Dunford-Pettis. To see this suppose \( (f_n) \) is a norm bounded sequence which converges to zero in the \( uaw \)-topology, by [15, Theorem 7], it follows that it is weakly convergent. Since \( L^1[0,1] \subseteq (L^\infty[0,1])' \) and the constant function one lies in \( L^1[0,1] \), we conclude that \( \|f_n\| \to 0 \), as claimed. However \( J \) is not \( uaw \)-compact, since the norm bounded sequence \( (r_n) \) of the Rademacher’s functions does not have any \( uaw \)-convergent subsequence.

For the \( uaw \)-convergence, we have \( x_\alpha \xrightarrow{uaw} x \) in \( E \) if and only if \( |x_\alpha - x| \xrightarrow{uaw} 0 \); see [15, Lemma 1]. It allows one to reduce \( uaw \)-convergence to the \( uaw \)-convergence of positive nets to zero.

Proposition 6. Let \( T: E \to F \) be a positive \( uaw \)-Dunford-Pettis operator between Banach lattices with \( F \) Dedekind complete. Then the Kantorovich-like extension \( S: E \to F \) defined via

\[
S(y) = \sup \left\{ T(y \land y_\alpha); (y_\alpha) \subseteq E_+, y_n \xrightarrow{uaw} 0 \right\}
\]

for \( y \in E_+ \) is again \( uaw \)-Dunford-Pettis.

Proof. Suppose \( y, z \in E_+ \). Then

\[
S(y + z) = \sup \{ T((y + z) \land \gamma_\alpha) \} \leq \sup \{ T(y \land \gamma_\alpha) \} + \sup \{ T(z \land \gamma_\alpha) \} \leq S(y) + S(z),
\]
Proposition 8. For every norm bounded $uaw$ sequence $(x_n)$ in which, $(\gamma_n)$ is a positive sequence that is $uaw$-null. On the other hand,

$$T(y \land \alpha_n) + T(z \land \beta_n) = T(y \land \alpha_n + z \land \beta_n) \leq T((y + z) \land (\alpha_n + \beta_n)) \leq S(y + z),$$

provided that two positive sequences $(\alpha_n), (\beta_n)$ are $uaw$-null so that $S(y) + S(z) \leq S(y + z)$. Therefore, by the Kantorovich extension Theorem [2, Theorem 1.10], $S$ extends to a positive operator. Denote by $S$ the extended operator $S: E \rightarrow F$.

We show that $S$ is also $uaw$-Dunford-Pettis. Suppose the norm bounded sequence $(y_n) \subseteq E_+$ is $uaw$-null. Therefore, we have

$$\|S(y_n)\| = \sup_m \|T(y_n \land \alpha_m)\| \leq \|T(y_n)\| \rightarrow 0,$$

in which $(\alpha_m)$ is a positive sequence in $E$ which is convergent to zero in the $uaw$-topology. □

In the following example, we show that adjoint of an $uaw$-Dunford-Pettis operator need not be a Dunford-Pettis operator.

Example 6. Consider the operator $T$ given in Example 2. We claim that its adjoint is not $uaw$-Dunford-Pettis. The adjoint $T^*: \ell_1 \rightarrow M[0, 1]$ is defined via $T^*((x_n))(f) = \sum_{n=1}^{\infty} x_n (\int_0^1 f(t) \sin nt \, dt)$, in which $M[0, 1]$ is the space of all regular Borel measures on $[0, 1]$. Note that the standard basis $(e_n)$ is $uaw$-null. For each $n \in \mathbb{N}$, put $f_n(t) = \sin nt$. Then we have

$$\|T^*(e_n)\| = \|T^*(e_n)(f_n)\| = \int_0^1 (\sin nt)^2 \, dt \geq \frac{1}{2}.$$

In the next example, we show that adjoint of a non $uaw$-Dunford-Pettis operator can be $uaw$-Dunford-Pettis.

Example 7. Consider the operator $T: \ell_1 \rightarrow L^2[0, 1]$ defined by $T((x_n)) = \left( \sum_{n=1}^{\infty} x_n \chi_{[0,1]} \right)$ for all $(x_n) \in \ell_1$ where $\chi_{[0,1]}$ denotes the characteristic function of $[0,1]$. The operator $T$ is compact but it is not $uaw$-Dunford-Pettis. Its adjoint $T^*: L^2[0,1] \rightarrow \ell_\infty$ is compact, and hence, it is Dunford-Pettis. Since the range space is an $AM$-space, by Proposition 4 we conclude that it is $uaw$-Dunford-Pettis.

Remark 4. One may verify that every positive operator which is dominated by a positive $uaw$-Dunford-Pettis operator is again $uaw$-Dunford-Pettis. Therefore, if $T$ is an operator whose modulus is $uaw$-Dunford-Pettis, it can be easily seen that $T$ is also $uaw$-Dunford-Pettis. For the converse, we have the following. Recall that the norm of a Banach lattice $E$ is weakly sequentially Fatou if there exists a positive constant $K$ such that whenever $(a_n)$ is an upward sequence with supremum $m$, we have $\|m\| \leq K \sup_n \|a_n\|, n \in \mathbb{N}$.

Proposition 7. Suppose $E$ and $F$ are Banach lattices with $F$ Dedekind complete and weakly sequentially Fatou. If $T$ and $S$ are two order bounded $uaw$-Dunford-Pettis operators, so is $T \lor S$.

Proof. By the Riesz-Kantorovich formula, for each positive $x$, we have

$$(T \lor S)(x) = \sup \{T(u) + S(v) : u, v \geq 0, u + v = x\}.$$ 

For every norm bounded $uaw$-null sequence $(x_n) \subseteq E_+$, consider norm bounded positive sequences $(u_n), (v_n)$ such that $x_n = u_n + v_n$, both of them are norm bounded and $uaw$-null. Therefore, $\|T(u_n)\| \rightarrow 0$ and $\|S(v_n)\| \rightarrow 0$. So,

$$\|(T \lor S)(x_n)\| \leq K \sup \{\|T(u_n) + S(v_n)\| : u_n, v_n \geq 0, u_n + v_n = x_n\} \leq \|T(u_n)\| + \|S(v_n)\| \rightarrow 0.$$ 

Finally, we investigate closeness properties of $B_{UDP}(E)$.

Proposition 8. $B_{UDP}(E)$ is closed subalgebra of $B(E)$. 
Proof. Suppose \((T_m)\) is sequence of \(uaw\)-Dunford-Pettis operators which is convergent to the operator \(T\). We show that \(T\) is also \(uaw\)-Dunford-Pettis. Assume that \((x_n)\) is a bounded \(uaw\)-null sequence in \(E\). Given any \(\varepsilon > 0\). There is an \(m_0\) such that \(\|T_m - T\| < \frac{\varepsilon}{2}\) for each \(m > m_0\). Fix an \(m > m_0\). For sufficiently large \(n\), we have \(\|T_m(x_n)\| < \frac{\varepsilon}{2}\). Therefore,
\[
\|T(x_n)\| < \|T_m - T\| + \|T_m(x_n)\| < \varepsilon.
\]
\(\square\)

The class of all \(uaw\)-Dunford-Pettis operators is not order closed. Consider the following.

Example 8. Put \(E = c_0\). Suppose \(P_n\) is the projection on the \(n\)-th first components. Each \(P_n\) is finite rank operator so that Dunford-Pettis. By Proposition 3 it is \(uaw\)-Dunford-Pettis. Also, \(P_n \rightharpoonup I\), where \(I\) denotes the identity operator on \(E\). But \(I\) is not \(uaw\)-Dunford-Pettis as the standard basis \((e_i)\) is \(uaw\)-null but not norm convergent to zero.

References

[1] Y. Abramovich and C.D. Aliprantis, An invitation to Operator theory, Vol. 50. Providence, RI: American Mathematical Society, 2002.
[2] C.D. Aliprantis and O. Burkinshaw, Positive operators, Springer, 2006.
[3] E. Bayram and A. W. Wickstead, \(L\)-weakly and \(M\)-weakly compact operators and the center, Archiv der Mathematik, to appear.
[4] R. DeMarr, Partially ordered linear spaces and locally convex linear topological spaces, Illinois J. Math., 8, (1964), pp. 601-606.
[5] Y. Deng, M O’Brien, and V. G. Troitsky, Unbounded norm convergence in Banach lattices, Positivity, \textbf{21}(3) (2017), pp. 963–974.
[6] N. Gao, Unbounded order convergence in dual spaces, J. Math. Anal. Appl., \textbf{419}(2014), pp 347-354.
[7] N. Gao, V. G. Troitsky, and F. Xanthos, \(Uo\)-convergence and its applications to Cesaro means in Banach lattices, Israel J. Math., \textbf{220} (2017), pp. 649–689.
[8] N. Gao and F. Xanthos, Unbounded order convergence and application to martingales without probability, J. Math. Anal. Appl., \textbf{415} (2014), pp 931–947.
[9] M. Kandić, M.A.A. Marabeh, and V. G. Troitsky, Unbounded norm topology in Banach lattices, J. Math. Anal. Appl., \textbf{451}(1)(2017), pp. 259–279.
[10] P. Meyer-Nieberg, \textit{Banach lattices}, Springer-Verlag, Berlin, 1991.
[11] H. Nakano, Ergodic theorems in semi-ordered linear spaces, Ann. of Math. \textbf{49 (2)}, (1948), pp. 538-556.
[12] P. Nieberg, \textit{Banach lattices}, Springer-Verlag, Berlin, 1991.
[13] H.H. Schaefer, \textit{Banach lattices and positive operators}, Springer-Verlag, Berlin, 1974.
[14] V.G. Troitsky, Measures of non-compactness of operators on Banach lattices, Positivity, \textbf{8}(2), (2004), pp. 165178.
[15] O. Zabeti, Unbounded Absolute Weak Convergence in Banach Lattices, Positivity, to appear. arXiv: 1608.02151v6.

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\begin{itemize}
\item[1] Department of Mathematics, Faculty of Science, Hacettepe University, Ankara, 06800, Turkey. \\
\textit{E-mail address: erkurun.ozcan@hacettepe.edu.tr}
\item[2] Department of Aeronautical Engineering, University of Turkish Aeronautical Association, Ankara, Turkey. \\
\textit{E-mail address: anilgezer@gmail.com}
\item[3] Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, P.O. Box: 98135-674, Zahedan, Iran. \\
\textit{E-mail address: o.zabeti@gmail.com}
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