Twisted $K$-Theory and Loop Groups

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Abstract. Twisted $K$-theory has received much attention recently in both mathematics and physics. We describe some models of twisted $K$-theory, both topological and geometric. Then we state a theorem which relates representations of loop groups to twisted equivariant $K$-theory. This is joint work with Michael Hopkins and Constantin Teleman.

The loop group of a compact Lie group $G$ is the space of smooth maps $S^1 \to G$ with multiplication defined pointwise. Loop groups have been around in topology for quite some time [Bo], and in the 1980s were extensively studied from the point of view of representation theory [Ka], [PS]. In part this was driven by the relationship to conformal field theory. The interesting representations of loop groups are projective, and with fixed projective cocycle $\tau$ there is a finite number of irreducible representations up to isomorphism. Considerations from conformal field theory [V] led to a ring structure on the abelian group $R^\tau(G)$ they generate, at least for transgressed twistings. This is the Verlinde ring. For $G$ simply connected $R^\tau(G)$ is a quotient of the representation ring of $G$, but that is not true in general. At about this time Witten [W] introduced a three-dimensional topological quantum field theory in which the Verlinde ring plays an important role. Eventually it was understood that the fundamental object in that theory is a “modular tensor category” whose Grothendieck group is the Verlinde ring. Typically it is a category of representations of a loop group or quantum group.

For the special case of a finite group $G$ the topological field theory is specified by a certain cocycle on $G$ and the category can be calculated explicitly [F1]. We identified it as a category of representations of a Hopf algebra constructed from $G$, thus directly linking the Chern-Simons lagrangian and quantum groups. Only recently did we realize that this category has a description in terms of twisted equivariant $K$-theory, and it was natural to guess that the Verlinde ring for arbitrary $G$ has a similar description. Ongoing joint work with Michael Hopkins and Constantin Teleman.

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Teleman has confirmed this description. We can speculate further and hope that twisted $K$-theory provides a construction of the modular tensor category, and perhaps even more of the three-dimensional topological field theory. In another direction the use of $K$-theory may shed light on Verlinde’s formula for certain Riemann-Roch numbers. In any case our result fits well with other uses of $K$-theory in representation theory [CG], for example in the geometric Langlands program.

The physics motivation for the main theorem is discussed in [F2], [F3]. Here, in §3, we explain the statement of our result in mathematical terms; the proof will appear elsewhere. As background we describe some concrete topological models of twisted $K$-theory in §1, and give a twisted version of the Chern-Weil construction in §2.

As mentioned above the work I am discussing is being carried out with Michael Hopkins and Constantin Teleman. I thank them for a most enjoyable collaboration.

§1 Twistings of $K$-theory

Let $X$ be a reasonable compact space, say a finite CW complex. Then isomorphism classes of complex vector bundles over $X$ form a semigroup whose group completion is the $K$-theory group $K^0(X)$. This basic idea was introduced by Grothendieck in the context of algebraic geometry [BS], and was subsequently transported to topology by Atiyah and Hirzebruch [AH]. Vector bundles are local—they can be cut and glued—and in the topological realm this leads to a cohomology theory. In particular, there are groups $K^n(X)$ defined for $n \in \mathbb{Z}$. Historically, $K$-theory was the first example of a generalized cohomology theory, and it retains the features of ordinary cohomology with one notable exception: the cohomology of a point is nontrivial in all even degrees, as determined by Bott periodicity. There are many nice spaces $B$ which represent complex $K$-theory in the sense that $K^0(X)$ is the set of homotopy classes of maps from $X$ to $B$. A particularly nice choice [A1, Appendix], [J] is the space $B = \text{Fred}(H)$ of Fredholm operators on a separable complex Hilbert space $H$. Thus a map $T: X \to \text{Fred}(H)$ determines a $K$-theory class on $X$. It is convenient to generalize and allow the Hilbert space $H$ to vary as follows. A Fredholm complex is a graded Hilbert space bundle $E^\bullet = E^0 \oplus E^1 \to X$ together with a fiberwise Fredholm map $E^0 T E^1$, and it also represents an element of $K^0(X)$. Another innovation was the introduction by Atiyah and Segal [S2] of the equivariant $K$-theory groups $K_G^\bullet(X)$ for a compact space $X$ which carries the action of a compact Lie group $G$. The basic objects are $G$-equivariant vector bundles $E \to X$, and $K^0_G(X)$ is the group completion of the set of equivalence classes. For example, if $X$ is a point then the equivariant $K$-theory is the representation ring $K_G$ of the compact Lie group $G$; in general, $K_G(X)$ is a $K_G$-module.

As a first example of twisted $K$-theory we consider twisted versions of $K_G$. A twisting $\tau$ is a

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$^1$One [S1] can allow more general topological vector spaces and complexes which are nonzero in degrees other than 0 and 1.
central extension

$$1 \rightarrow \mathbb{T} \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

where $\mathbb{T} \subset \mathbb{C}$ is the circle group of unit norm complex numbers. Then the twisted representation "ring" $K_{\tilde{G}}^\tau$ is the group completion of equivalence classes of complex representations of $\tilde{G}$ on which the central $\mathbb{T}$ acts by scalar multiplication. Twisted $K$-theory is not a ring, but rather $K_{\tilde{G}}^\tau$ is a $K_G$-module. A map of twistings, i.e., an isomorphism of central extensions, determines an isomorphism of twisted $K$-groups. So twisted $K$-theory is determined up to noncanonical isomorphism by the equivalence class of the twisting. For example, $K_{SO(3)} \cong \mathbb{Z}[s]$ is the polynomial ring on a single generator, the 3-dimensional defining representation. Up to equivalence there is a single nontrivial central extension

$$\tau = \{ 1 \rightarrow \mathbb{T} \rightarrow U(2) \rightarrow SO(3) \rightarrow 1 \}$$

which is induced by the inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{T}$ from the extension

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow SU(2) \rightarrow SO(3) \rightarrow 1.$$

Virtual representations of $U(2)$ on which the center acts naturally correspond 1:1 with virtual representations of $SU(2)$ on which the central element acts as $-1$. Now $K_{SU(2)} \cong \mathbb{Z}[t]$, where $t$ is the defining 2-dimensional representation, and we identify $K_{SO(3)} \cong K_{SU(2)}$ as the subgroup of odd polynomials in $t$; the $K_{SO(3)}$-module structure is $s \cdot t = t^3 - t$, by the Clebsch-Gordon rule, and $K_{SO(3)}^\tau$ is a free module of rank one.

More generally, on a $G$-space $X$ a “cocycle” $\tau$ for the equivariant cohomology group $H^3_G(X; \mathbb{Z})$ defines a twisted $K$-theory group $K_{\tau}^G(X)$ which is a module over $K_0^G(X)$. A better point of view is that $\tau$ is a cocycle, or geometric representative, of a class in $H^1_G(X; A_G)$, where $A_G$ is a group of automorphisms of equivariant $K$-theory. We will not try to make “automorphism of equivariant $K$-theory” precise here, but content ourselves with a concrete model, first in the nonequivariant case. Take $\text{Fred}(H)$ to be the classifying space of $K$-theory. Then the group $A = PGL(H)$ acts as automorphisms by conjugation. Since $\tilde{A} = GL(H)$ is contractible [K], the quotient $A = \tilde{A}/\mathbb{C}^\times$ is homotopy equivalent to $\mathbb{CP}^\infty$, and so $H^1(X; A) \cong H^3(X; \mathbb{Z})$. A twisting $\tau$ can be taken to be a principal $A$-bundle $\pi: P \rightarrow X$. The action of $A$ on $\text{Fred}(H)$ defines an associated bundle $\text{Fred}(H)_P \rightarrow X$, and the twisted $K$-group $K^\tau_\tau(X)$ is the group of homotopy classes of sections of this bundle [A2]. A section is an $A$-equivariant map $T: P \rightarrow \text{Fred}(H)$, and as before it is convenient to let the Hilbert space vary. Thus define a $P$-twisted Fredholm complex to be an $\tilde{A}$-equivariant graded Hilbert space bundle $E^\bullet \rightarrow P$, where the center $\mathbb{C}^\times \subset \tilde{A}$ acts by scalar multiplication, together with a fiberwise $\tilde{A}$-equivariant Fredholm operator $E^0 \xrightarrow{T} E^1$. Then $(E^\bullet, T)$ represents an element of $K^\tau(X)$.

It is perhaps unsettling that the model is infinite dimensional, but that is unavoidable unless the class of the twisting in $H^3(X; \mathbb{Z})$ is torsion.
There are many other models of twistings and twisted $K$-theory. For example, gerbes are geometric representatives of elements in degree three integral cohomology; see [H], [B], [M] for example. In a Čech description we have a covering $X = \bigcup_i U_i$ of $X$ by open sets and a complex line bundle $L_{ij} \to U_i \cap U_j$ on double intersections. There is further cocycle data on triple intersections. In a model of $K$-theory on which line bundles act as automorphisms this can be used to define twisted $K$-theory; see [BCMMS] for example. In fact, the “group” $\mathbb{Z}/2\mathbb{Z} \times \mathbb{C}\mathbb{P}^\infty$ of graded lines act as automorphisms of $K$-theory, so there is a larger group of (equivalence classes of) twistings

\[ H^1(X; \mathbb{Z}/2\mathbb{Z} \times \mathbb{C}\mathbb{P}^\infty) \cong H^1(X; \mathbb{Z}/2\mathbb{Z}) \times H^3(X; \mathbb{Z}). \]

We remark that there is a natural group structure on (1), but it is not the product. In topology these twisted versions of $K$-theory, at least for torsion twistings, were introduced by Donovan and Karoubi [DK], who also considered real versions. There is another viewpoint and generalization of $K$-theory using $C^*$-algebras, and in that context twisted $K$-theory was discussed by Rosenberg [R] for both torsion and nontorsion twistings. Twisted versions of $K$-theory have appeared recently in various parts of geometry and index theory, for example in [LU], [AR], [To], [NT], [MMS].

A generalization of the previous model is useful. Here a twisting is a quartet $\tau = (\mathcal{G}, \epsilon, \tilde{\mathcal{G}}, P)$:

\[
\begin{align*}
\mathcal{G} & \quad \text{topological group} \\
\epsilon: \mathcal{G} & \to \mathbb{Z}/2\mathbb{Z} \quad \text{homomorphism termed the \textit{grading}} \\
\tilde{\mathcal{G}} & \to \mathcal{G} \quad \text{central extension by} \ T \\
P & \to X \quad \text{principal} \ \mathcal{G}\text{-bundle}
\end{align*}
\]

We require the existence of a homomorphism $\tilde{\mathcal{G}} \to GL(H)$ which is the identity on the central $T$. Let $\tilde{\mathcal{G}}_0 \to \mathcal{G}_0$ be the restriction of the central extension over the subgroup $\mathcal{G}_0 = \epsilon^{-1}(0)$. Then the equivalence class of $\tau$ is the obstruction in (1) to restricting/lifting $P$ to a principal $\mathcal{G}_0$ bundle. Our previous construction is the special case $G = PGL(H)$ and $\epsilon$ trivial. An element of $K^\tau(X)$ is represented by a $\tilde{\mathcal{G}}$-equivariant $\mathbb{Z}/2\mathbb{Z}$-graded Fredholm complex over $P$, where the action of $\tilde{\mathcal{G}}$ is compatible with the grading $\epsilon$. If a compact Lie group $G$ acts on $X$ then it is easy to extend this to a model of equivariant twistings and equivariant $K$-theory.

As an illustration of a nontrivial $H^1$-twisting, consider $X = pt$ and $\tau = (O(2), \epsilon, O(2) \times T, O(2))$ with nontrivial grading $\epsilon$. The representation ring of $O(2)$ may be written

\[ K_{O(2)} \cong \mathbb{Z}[\sigma, \delta] / (\sigma(\delta - 1), \delta^2 - 1), \]

where $\delta$ is the one-dimensional sign representation and $\sigma$ the standard two dimensional representation. Then the twisted $K$-group $K^\tau_{O(2)}$ is a module over $K_{O(2)}$ with a single generator $t$ and the
relation $\delta \cdot t = t$. There is a nontrivial odd twisted $K$-group $K_{O(2)}^{\tau+1}$ which is also a module with a single generator $u$; the relations are $\delta \cdot u = -u$ and $\sigma \cdot u = 0$.

Many topological properties of $K$-theory, including exact and spectral sequences, have straightforward analogs in the twisted case. This is easiest to see from the homotopy-theoretic view of cohomology theories, and so applies to twisted cohomology theories in general. Computations are usually based on these sequences. A more specialized result is the completion theorem in equivariant $K$-theory [AS]; its generalization to the twisted case has some new features [FHT]. The Thom isomorphism theorem fits naturally into the twisted theory [DK]. Let $V \to X$ be a real vector bundle of finite rank, which for convenience we suppose endowed with a metric. There is an associated twisting
\[ \tau(V) = (O(n), \epsilon, \text{Pin}^c(n), O(V)), \]
where $O(V) \to X$ is the orthonormal frame bundle of $V$ and $\epsilon: O(n) \to \mathbb{Z}/2\mathbb{Z}$ is the nontrivial grading. The isomorphism class of $\tau(V)$ is the pair of Stiefel-Whitney classes $(w_1(V), W_3(V))$. Denote
\[ K^V(X) = K^{\text{rank}(V)+\tau(V)}(X) \]
as the degree-shifted twisted $K$-theory. Then the Thom isomorphism theorem asserts that the natural map
\[ K^{q+V}(X) \to K^{q+2 \text{rank}(V)}(V) \]
is an isomorphism. We remark that if $X$ is a smooth compact manifold, then Poincaré duality identifies $K^{TX}(X)$ with the $K$-homology group $K_0(X)$. Also, we can define $K^V(X)$ for virtual bundles $V$.

The Chern character maps twisted $K$-theory to a twisted version of real cohomology, as we explain next in the context of Chern-Weil theory.

\[ \text{§2 A differential-geometric model} \]

For simplicity we work in the nonequivariant context. Our thoughts here were stimulated by reading [BCMMS].

Let $\mathfrak{a} = \mathfrak{pgl}(H)$ denote the Lie algebra of $\hat{A} = PGL(H)$ and $\mathfrak{a} = \mathfrak{gl}(H)$ the Lie algebra of $A = GL(H)$. They fit into the exact sequence of Lie algebras
\[ 0 \to \mathbb{C} \to \hat{\mathfrak{a}} \to \mathfrak{a} \to 0. \]
A linear map \( L: \mathfrak{a} \to \tilde{\mathfrak{a}} \) is a splitting if the composition with \( \tilde{\mathfrak{a}} \to \mathfrak{a} \) is the identity. A splitting \( L \) determines a closed right-invariant 2-form on \( A \) which represents the generator of \( H^2(A; \mathbb{Z}) \); its value on right-invariant vector fields \( \xi, \eta \) is

\[
[L(\xi), L(\eta)] - L([\xi, \eta]).
\]

Let \( \pi: P \to X \) be a principal \( A \)-bundle. We now add two pieces of geometric data:

\[
\begin{align*}
\Theta & \in \Omega^1(P; \mathfrak{a}) \quad \text{connection on } P \to X \\
L: P & \to \text{Hom}(\mathfrak{a}, \tilde{\mathfrak{a}}) \quad A\text{-invariant map into splittings}
\end{align*}
\]

Both are sections of affine space bundles over \( X \), so can be constructed using partitions of unity. As usual, define the curvature

\[
F_\Theta = d\Theta + \frac{1}{2}[\Theta \wedge \Theta].
\]

It is an \( \mathfrak{a} \)-valued 2-form on \( P \). Introduce the scalar 2-form

\[
\beta = (dL(\Theta) + \frac{1}{2}[L(\Theta) \wedge L(\Theta)]) - L(F_\Lambda).
\]

Then one can check that \( \beta \) is transgressive. In other words, \( d\beta = \pi^*\eta \) for a closed scalar 3-form \( \eta \in \Omega^3(X) \). The de Rham cohomology class of \( \eta \) in \( H^3(X; \mathbb{R}) \) represents the image in real cohomology of the isomorphism class of the twisting \( P \to X \).

Now let \( E^0 \stackrel{T}{\to} E^1 \) be a twisted Fredholm complex over \( P \). Thus \( E^i \to P \) are \( \tilde{A} \)-equivariant Hilbert space bundles, with \( \mathbb{C}^\times \subset \tilde{A} \) acting by scalar multiplication, and \( T \) is an \( \tilde{A} \)-equivariant Fredholm map. The \( \tilde{A} \) action determines an \( \tilde{A} \)-invariant partial covariant derivative on \( E^\bullet \to P \) along the fibers of \( \pi: P \to X \). Again we introduce differential-geometric data:

\[
\nabla^\bullet \quad \tilde{A}\text{-invariant extension of the partial covariant derivative}
\]

Such an extension is a section of an affine space bundle over \( X \), so can be constructed via a partition of unity. Introduce a formal parameter \( u \) of degree 2 and its inverse \( u^{-1} \) of degree \(-2\), and so the graded ring of \( \mathbb{R}[[u, u^{-1}]]\)-valued differential forms. (One can identify \( u \) as a generator of the \( K \)-theory of a point.) Following Quillen [Q] define the \( \tilde{A} \)-invariant superconnection

\[
D = \begin{pmatrix} \nabla^0 & uT^* \\ T & \nabla^1 \end{pmatrix}
\]
on $E^\bullet \to X$. Its usual curvature $D^2 \in \Omega(P; \text{End } E \otimes \mathbb{R}[u, u^{-1}])^2$ is an $A$-invariant form of total degree 2. However, it does not descend to the base $X$. Instead, one can check that the *twisted* curvature

$$F(E^\bullet, T, \nabla^\bullet) := D^2 - \beta \cdot \text{id}$$

is $A$-invariant and basic, so descends to an element of $\Omega(X; \text{End } E \otimes \mathbb{R}[u, u^{-1}])^2$. (Note that $\text{End } E^\bullet \to P$ descends to a graded vector bundle on $X$, since the center of $\tilde{A}$ acts trivially.) It does not, however, satisfy a Bianchi identity, since $\beta$ is not closed.

The Chern character form

$$(6) \quad \text{ch}(E^\bullet, T, \nabla^\bullet) := \text{Tr} \exp(u^{-1}F)$$

is an element in $\Omega(X; \mathbb{R}[[u, u^{-1}]])^0$ of total degree 0. Here we assume favorable circumstances in which the graded trace $\text{Tr}$ is finite. For example, if the twisting class is torsion then we can take $E^\bullet \to P$ finite dimensional. Or, if the superconnection comes from a family of elliptic operators as in [Bi] then the graded trace exists. The Chern character form (6) is not closed in the usual sense, but rather

$$(d + u^{-1}\eta) \text{ch}(E^\bullet, T, \nabla^\bullet) = 0.$$  

The differential $d + u^{-1}\eta$ on $\Omega(X; \mathbb{R}[[u, u^{-1}]])^\bullet$ computes a twisted version of real cohomology which is the codomain of the Chern character.

This construction works with little change for the more general twistings (2).

The differential geometric model we have outlined not only gives geometric representatives of twisted topological $K$-theory classes, but also geometric representatives of twisted *differential* $K$-theory classes. Similarly, the geometric twistings (5) give geometric models for differential cohomology classes. See [HS] for foundations of (untwisted) differential cohomology theories in general, and [L] for the basics differential $K$-theory.

§3 Loop groups

Let $G$ be a compact Lie group. The loop group $LG$ of $G$ is the space of smooth maps $S^1 \to G$. There is a twisted description we use instead. Namely, let $R \to S^1$ be a principal $G$-bundle and $LG_R$ the group of gauge transformations, i.e., the space of smooth sections of the bundle of groups $G_R \to S^1$ associated to $R$. A trivialization of $R$ gives an isomorphism $LG_R \cong LG$. Note that $R$ is necessarily trivializable if $G$ is connected, and in general the topological class of $R$ is labeled by a conjugacy class in $\pi_0 G$. Let $G(R) \subset G$ be the union of components which comprise that conjugacy class. The theory of loop groups $LG_R$ is described in [PS], [Ka], and some specific further developments appear in [FF], [We].

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One salient feature of loop groups is the existence of nontrivial central extensions

\[ 1 \to T \to LG_R^+ \to LG_R \to 1. \]

As in (2) we may also consider gradings \( \epsilon: LG_R \to \mathbb{Z}/2\mathbb{Z} \). We call the pair \( \tau = (LG_R^+, \epsilon) \) a graded central extension and denote it simply as \( LG_R^+ \). It has an invariant in

\[ H^1_G(G(R); \mathbb{Z}/2\mathbb{Z}) \times H^3_G(G(R); \mathbb{Z}) \]

as follows. Fix a basepoint \( s \in S \) and consider the product

\[ P = A_R \times R_s \]

of the space of connections \( A_R \) on \( R \) and the fiber \( R_s \) at the basepoint. Then \( LG_R \) acts freely on \( P \) with quotient the holonomy map

\[ \text{hol}: P \to G(R). \]

Furthermore, the \( G \) action on \( G(R) \) by conjugation lifts to the \( G \) action on the \( R_s \) factor of \( P \), where it commutes with the \( LG_R \) action. Thus we have an equivariant twisting

\[ \tau = (LG_R, \epsilon, LG_R^+, P) \]

whose isomorphism class, called the level, lies in (7). (The basepoint is not necessary. It is more natural to replace (8) by \( A_R \times R \) and \( LG_R \) by the group of bundle automorphisms which cover rotations in the base \( S^1 \).

As a warm-up to representations of loop groups, recall the Borel-Weil construction of representations of a compact connected Lie group \( G \). Let \( T \subset G \) be a maximal torus, and \( F = G/T \) the flag manifold. Then \( F \) admits \( G \)-invariant complex structures. Fix one. A character \( \lambda \) of \( T \) determines a holomorphic line bundle \( L_\lambda \to F \), and the standard construction takes the induced virtual representation of \( G \) to be \( \oplus_q (-1)^q H^q(F; L_\lambda) \), which is the \( G \)-equivariant index of the \( \bar{\partial} \) operator. We can use the Dirac operator instead of the \( \bar{\partial} \) operator. This has the advantage that no complex structure need be chosen, though it can be described in holomorphic terms as the \( \rho \)-shift \( L_\lambda \to L_\lambda \otimes K_F^{1/2} \), where \( K_F^{1/2} \to F \) is a square root of the canonical bundle. More generally, if \( Z \subset G \) is the centralizer of any subtorus of \( T \), and \( F' = G/Z \) the corresponding generalized flag manifold, then there is a Dirac induction map

\[ K_Z \cong K_G(F') \to K_G. \]
A representation of \( Z \) defines a \( G \)-equivariant vector bundle on \( F' \), and (9) is the equivariant index of the Dirac operator with coefficients in this bundle.

A similar construction works for loop groups. Fix a conjugacy class \( C \subset G(R) \). The group \( LG_R \times G \) acts transitively on \( \text{hol}^{-1}(C) \). Let \( Z_C \) denote the stabilizer at some point; it embeds isomorphically into either factor of \( LG_R \times G \). Introduce the flag manifold \( F_C = \text{hol}^{-1}(C)/G \); then the loop group \( LG_R \) acts transitively \( F_C \) with stabilizer the image of \( Z_C \) in \( LG_R \). On the other hand, the image of \( Z_C \) in \( G \) is the centralizer of an element in \( C \). The geometry of \( F_C \) is similar to that of finite dimensional flag manifolds [F4]. An important special case is \( C = \{1\} \) and \( R \to S^1 \) trivial. Then the flag manifold is the “loop Grassmannian” \( F = LG/G \).

There is a Dirac induction for loop groups which generalizes (9). For this we introduce spinors on \( LG_R \) and a canonical graded central extension. Consider the principal \( SO(g \oplus \mathbb{R}) \)-bundle associated to \( R \to S^1 \) via the twisted adjoint homomorphism

\[
G \to SO(g \oplus \mathbb{R}) \\
g \mapsto \text{Ad}_g \oplus \det \text{Ad}_g
\]

It is trivializable, and any trivialization induces an isomorphism \( LG_R \to LSO(N) \) for \( N = \dim(G) + 1 \). There is a canonical graded central extension of \( LSO(N) \) from the spin representation, and it pulls back to the desired canonical graded central extension \( LG^\sigma_R(G) \to LG_R \). The spin representation itself defines (projective) spinors on \( LG_R \). For a conjugacy class \( C \subset G(C) \) the embedding \( i_C : Z_C \hookrightarrow LG_R \) induces a graded central extension \( Z_C^{i_C^*\sigma_R(G)} \) which may be described instead by the real adjoint representation \( z_C \), viewed as a \( Z_C \)-equivariant vector bundle over a point (see (3)). In this form it carries a degree shift, and so it is natural to also include a degree shift in \( \sigma_R(G) \). We do not specify it precisely, but remark that its parity agrees with that of \( \dim Z_C \).

If \( G \) is simple, connected, and simply connected and \( R \to S^1 \) is trivial, then \( \sigma_R(G) \) has degree shift \( \dim G \) and the \( H^3 \) component of the level in (7) is the dual Coxeter number of \( G \) times a generator; the \( H^1 \) component vanishes.

For a fixed graded central extension \( LG^*_R \) of \( LG_R \) there is a finite set of isomorphism classes of irreducible positive energy representations of \( LG^*_R \) on which the center acts by scalar multiplication. Let \( R^*(G) \) denote the abelian group generated by these equivalence classes. It is natural to extend this to a \( \mathbb{Z} \)-graded group with \( mod 2 \) periodicity and possibly nontrivial groups in odd degrees. Now we describe Dirac induction for loop groups. As in the first map of (9) a representation of \( Z_C \) defines an \( LG^*_R \)-equivariant vector bundle over the flag manifold \( F_C \). However, we are interested in \( LG^*_R \)-equivariant vector bundles, so need to start with a representation of the central extension of \( Z_C \) defined by \( i_C^* \tau \). Finally, spinors on the flag manifold \( F_C \) may be constructed from spinors on \( LG_R \) by “subtracting” spinors on the adjoint representation of \( Z_C \), and this imposes an additional twisting. Altogether, then, Dirac induction is a map

\[
K_{Z_C}^{i_C^*\tau - 3\sigma_R} \to R^* - \sigma_R(G)(G(R)).
\]
The adjoint shift $\sigma_R(G)$ on the right hand side means that we obtain representations of the fiber product of $LG_R^* \to LG_R$ with the inverse of $LG_R^{\sigma^R(G)} \to LG_R$, including a degree shift. For connected, simply connected $G$ it suffices to consider $C = \{1\}$, since then (10) is surjective, but this is not true in general.

The inclusion $i_C : C \hookrightarrow G(R)$ induces a pushforward in twisted $K$-theory (cf. (4)):

$$K_{ZC}^{i_C^*\tau G\tau C} \cong K_G^{i_C^*\tau G\tau G + T_{G\tau G}^C} (C) \to K\tilde{\tau}_G(G(R)).$$

The maps (10) and (11) give, for each conjugacy class $C$, a correspondence between certain representations of the loop group and a twisted $K$-theory group. Our main result is

**Theorem 1.** These correspondences induce an isomorphism of abelian groups

$$R^\tau_{\tau - \sigma^R(G)}(G(R)) \to K\tilde{\tau}_G(G(R)).$$

There is a transgression from $H^4(BG;\mathbb{Z})$ to levels (7) with trivial first component. More generally, there is an extension

$$0 \to H^4(BG;\mathbb{Z}) \to E^4(BG) \to H^2(BG;\mathbb{Z}/2\mathbb{Z}) \to 0,$$

and elements of $E^4(BG)$ transgress to general levels for all loop groups $LG_R$ simultaneously. If $G$ is connected then any $R \to S^1$ is trivializable, and both sides of (12) have a ring structure for any fixed $R$. For arbitrary $G$ the sum of each side of (12) over representatives of each topological type of $R \to S^1$ has a ring structure. The multiplication on representations is the *fusion product* [V], [F], [T]; on twisted $K$-theory it is the pushforward by multiplication $G \times G \to G$ or equivalently the Pontrjagin product in $K$-homology.

**Theorem 2.** If the level of the twisting $\tilde{\tau} + \tau(TG)$ is transgressed, then the isomorphisms (12) are compatible with the ring structure.

Our proof reduces both sides of (12) to the statement for tori, where there is a direct argument.

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