Running of Newton’s constant and non integer powers of the d’Alembertian

D. López Nacir * and F. D. Mazzitelli †
Departamento de Física Juan José Giambiagi,
Facultad de Ciencias Exactas y Naturales, UBA,
Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina

The running of Newton’s constant can be taken into account by considering covariant, non local
generalizations of the field equations of general relativity. These generalizations involve nonanalytic
functions of the d’Alembertian, as \((-\square)^{-\alpha}\), with \(\alpha\) a non integer number, and \(\ln(-\square)\). In this paper
we define these non local operators in terms of the usual two point function of a massive field. We
analyze some of their properties, and present specific calculations in flat and Robertson Walker
spacetimes.

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I. INTRODUCTION

In quantum field theory, the coupling constants depend on the energy scale according to the Renormal-
ization Group Equations (RGE). This running behavior is crucial to understand perturbative and nonper-
turbative aspects of elementary particles and their interactions.

Over the years, there have been several attempts to incorporate the running behavior in astrophysics and
cosmology. In the first case, a scale dependent Newton’s constant would explain at least part of the dark
matter. There are proposals in the literature [1] where it is assumed that the gravitational constant around
a compact object depends on the radial coordinate \(r\). The dependence is dictated by the RGE, replacing
the energy scale \(\mu\) by \(1/r\).

In a cosmological framework, the expansion of the universe makes any physical length scale to increase
with time and naively one would expect the coupling constants to become time dependent as the universe
evolves. Hopefully, this time dependence would help to explain the accelerated expansion of the Universe and
to shed light into the dark energy problem. This point of view has been considered by a number of authors,
and several proposals have been analyzed in detail. For example, it was assumed [2] that Newton’s and
cosmological constants depend on \(t\) according to the RGE through the replacement \(\mu \rightarrow 1/t\). The Einstein
equations with these time dependent constants in general do not satisfy Bianchi identities, and therefore
some constraints are needed for mathematical consistency [3]. Related proposals assume that the correct
replacement is obtained with the Hubble “constant” \(\mu \rightarrow H [4]\), the inverse radii of the event horizon [5], or
with the scale factor \(\mu \rightarrow \hat{\mu}/a(t) [6]\). Resumation of the Schwinger DeWitt expansion in the large curvature
limit produce running couplings that depend on the scalar curvature [7].

In field theory, the running of the coupling constants is computed in momentum space, and the relevant
momentum scale is associated to the characteristic energy in a scattering process. However, in astrophysics
and cosmology, one is eventually interested in the running of the couplings in configuration space. As a
trivial property of Fourier transforms, a scale dependent coupling constant in momentum state implies in
general a non local dependence in configuration space. Therefore, if the running of couplings is produced by
quantum effects (they could also be produced by extra dimensions, see below), in principle one should be able
to compute a non local effective action and obtain real and causal effective equations for the gravitational
field.

This program can be explicitly worked out for weak gravitational fields, in the case where quantum fields

* dnacir@df.uba.ar
† fmazzi@df.uba.ar
are free and massless. Indeed, after integrating the quantum degrees of freedom, the effective action \( \hat{S} \) is equivalent to the classical action with the replacement of the coupling constants by the non local kernels \( \hat{\alpha} \)

\[
\hat{\alpha}_i(\hat{\mu}) \rightarrow \alpha_i(-\Box) = \alpha_{i0}(\mu) + c_i \ln \left( \frac{-\Box}{\mu^2} \right)
\]

where \( \mu \) is a reference energy scale and the values of the constants \( c_i \) are fixed by the RGE. Therefore, in this particular example the running of the gravitational constants is correctly and completely taken into account by replacing \( \hat{\mu}^2 = -\Box \) at the level of the effective action (this approach has also been considered in Refs. \[15\]).

The effective field equations for the gravitational field give rise to a quantum corrected Newtonian potential, that in turn can be interpreted as the classical one with a running gravitational constant.

As already stressed, if one assumes that the modifications to General Relativity come from quantum effects, the running behavior of the coupling constants should be contained in the effective action \[12\], along with other quantum corrections. The “Wilsonian” approach, in any of the versions described above, is a way of partially taking into account the effects of quantum fields. The running of the coupling constants is not necessarily logarithmic. For example, renormalization group analysis based on the effective average action of quantum Einstein gravity \[12\] and non perturbative studies in the lattice theory of quantum gravity \[14\], suggests different behaviors for the scale dependence of the gravitational constants, as for instance a power law scaling of Newton’s constant. One could implement this running in either cosmological or astrophysical scenarios following the ideas described above, by an adequate replacement of the momentum by a running in either cosmological or astrophysical scenarios following the ideas described above, by an adequate replacement of the momentum by a running gravitational constant.

For example, in the context of lattice theory of quantum gravity, it has been suggested that the strength of gravitational interactions might slowly increase with distance as a consequence of vacuum polarization effects. Provided that a non trivial ultraviolet fixed point exists, to leading order in the vicinity of such fixed point the scale dependence of the Newton’s constant is characterized by a critical exponent \( \nu = (2\alpha)^{-1} \), and the size of the corrections is set by a non perturbative scale \( L \). In this scenario, the manifestly covariant non local gravitational coupling can be written as \[15\]

\[
G(-\Box) = G_N \left(1 + a_0 L^{-2\alpha} (-\Box + L^{-2})^{-\alpha} \right), \tag{3}
\]

where \( G_N \) is the usual Newton’s constant, \( a_0 \) is a positive number whose typical value was estimated to be of order of \( 10^{-2} \), and estimates for the value of \( \nu^{-1} = 2\alpha \) vary from \( \nu^{-1} \approx 3.0 \) to \( \nu^{-1} \approx 1.7 \). The length scale \( L \) separates the ultraviolet regime where non perturbative corrections can be neglected from long distance regime where such corrections become significant. This scale might be related to the vacuum expectation value of the curvature of the spacetime, and therefore, in cosmological situations, it is generally chosen to have a typical value of order of the Hubble ratio \( H^{-1} \).

Similar proposals have been considered to introduce infrared corrections to General Relativity, which may help to understand the present acceleration of the universe. These are in turn inspired by extra dimensional brane world models (see below) \[16\].

Effective equations like Eq. \(2\) are in general inconsistent, i.e. the Bianchi identity is not satisfied. There is a way out, which consists in considering the effective field equations in the weak field limit. In this case, it is possible to write a non local effective action whose variation gives the effective field equations to linear order in the curvature \[15\]. This procedure is consistent, although restricted to weak fields. A more drastic approach is to assume that Eq. \(2\) is valid to all orders, and that the allowed energy-momentum tensors are only those that satisfy \( (G(-\Box)T_{\mu\nu})_{\,\mu} = 0 \).
Another problem of Eq. (2) is that, unless the non local corrections contained in $G(-\Box)$ are treated perturbatively, the theory will contain ghosts. Indeed, as shown in Ref. [18], at the linearized level the ghost-free theories must be of the form

$$G^{(1)}_{\mu\nu} - m^2(-\Box)(h_{\mu\nu} - \eta_{\mu\nu} h) = 8\pi G_N T_{\mu\nu}$$

where $G^{(1)}_{\mu\nu}$ is the linearized Einstein’s tensor, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $h = \eta^{\mu\nu} h_{\mu\nu}$. Eq. (4) generalizes the Pauli-Fierz theory of massive gravitons [19] to the case of a non local mass $m^2(-\Box)$. Note that the linearization of Eq. (2) is not of the Pauli-Fierz form.

In brane world scenarios, the non local kernels previously considered in the literature involve either arbitrary powers or the logarithm of the d’Alembertian. For example, in order to modify General Relativity at distances larger that a certain scale $L$, it has been assumed that the leading behavior of the kernels is

$$\frac{G_N}{G(-\Box)} \sim \frac{m^2(-\Box)}{\Box} \sim L^{-2\alpha}(-\Box)^{-\alpha}$$

where $\alpha$ is a positive number [20] ($\alpha = 1/2$ for the five dimensional DGP model [16]) and $L$ is a length scale which for cosmological situations is naturally assumed to be of order of $H^{-1}$ . Note that in contrast to the running suggested by non perturbative analysis of quantum effects (see Eq. 3), the Newton’s constant here decrease with distance. As an illustration, a “phenomenological” running coupling of the form

$$\frac{1}{G(-\Box)} = \frac{1}{G_N}(1 + L^{-2\alpha}(-\Box + b^2)^{-\alpha})$$

with $bL \ll 1$, interpolates between the usual Newton’s constant at laboratory distance scales and $G(0) \ll G_N$ at scales much larger than $L$.

In order to analyze the effective field equations it is necessary to define properly the action of an arbitrary power of the d’Alembertian on scalar and tensor functions. This is one of the main goals of the present paper. We will introduce a formal definition based on an integral representation, as it has been previously done with the logarithm of the d’Alembertian [8], and use this representation to obtain the action of the non local operator on arbitrary functions. We will consider both astrophysical and cosmological situations. This problem has been previously considered by Hamber and Williams, who computed $(\Box)^{-\alpha}$ by analytic continuation of the action of $(\Box)^n$, for an integer number $n \geq 1$, to the case $n = -\alpha \in \mathbb{R}$, in Robertson Walker [15] and Schwarzschild metrics [21]. However, this approach has some drawbacks. On the one hand, it is only possible to get an answer for the action of $(\Box)^{-\alpha}$ on some particular functions that depend either on $t$ or $r$. On the other hand, the non local and causal properties of the kernel are not apparent from its definition.

The paper is organized as follows. In the next Section we introduce our formal definition for $(\Box)^{-\alpha}$. In Section we calculate explicitly in flat spacetime, and reobtain previous results based on a different definition. We also show that the distribution $\ln[\Box]$ can be obtained as a particular limiting case. In Section we compute explicitly the action of $(\Box)^{-\alpha}$ on a scalar function independent of the spatial coordinates in Friedman Robertson Walker (FRW) spacetimes. We consider the de Sitter spacetime, and other FRW metrics with initial singularities. We also study the action of the kernel $(\Box + b^2)^{-\alpha}$ on a constant function, and check whether it gives $b^{-2\alpha}$ or not, a result that has been used to argue that Eqs. like (2) can explain the smallness of the cosmological constant. In Section we use the results obtained for the kernel $(\Box)^{-\alpha}$ to derive similar ones for $\ln[\Box]$. As a concrete application, in Section we discuss the effect of a running Newton’s constant on the generation of gravitational waves. We summarize our results in Section.

Throughout the paper we set $\hbar = c = 1$ and adopt the sign convention denoted $(++)$ by Misner, Thorne, and Wheeler [22].
II. FORMAL DEFINITION OF \((-\Box)^{-\alpha}\)

We will define the kernel \((-\Box)^{-\alpha}\) using an integral representation in terms of the two point function for massive fields. The definition is based on the formal identity

\[
(-\Box)^{-\alpha} = \frac{2\sin(\alpha\pi)}{\pi} \int_0^{+\infty} \frac{dm}{m^{1-2\alpha}} \frac{1}{-\Box + m^2}
\]  

valid for \(0 < \alpha < 1\). A similar representation has been previously considered \[8\] to define \(\ln[-\Box\mu^2]\), i.e.

\[
\ln[-\Box\mu^2] = \int_0^{+\infty} dm^2 \left[ -\frac{1}{-\Box + m^2} + \frac{1}{\mu^2 + m^2} \right]
\]  

It is worth noting that both kernels are related: knowing the kernel \((-\Box\mu^2)^{-\alpha}\), it is possible to obtain \(\ln[-\Box\mu^2]\) by considering the identity \(x^{-\alpha} = 1 - \alpha \ln x + O(\alpha^2)\) \[11\].

The tensorial character of the kernels is defined by the one of the two point function of the massive field. When \((-\Box)^{-\alpha}\) acts on a scalar (tensorial) function, \((-\Box + m^2)^{-1}\) represents the propagator of a scalar (tensorial) massive field. In what follows we will mainly consider the action on scalar functions.

The definitions given above are unambiguous in Euclidean space, where \(-\Box + m^2\) is a unique positive definite operator. In Lorentzian spacetimes, it is necessary to impose an additional prescription to select one of the propagators \((-\Box + m^2)^{-1}\) (retarded, advanced, Feynman, etc). As we will be imposing the modified field equations “phenomenologically”, i.e. without considering the full underlying field theory, it is not possible to give such a prescription. An educated guess is to impose the field equation to be causal, and this can be implemented by choosing the retarded propagator in the spectral representations. This is the correct prescription when the effective equations come from quantum corrections, and in particular for weak gravitational fields. In this case, this choice corresponds to the vacuum quantum state of the matter fields \[8\]. For the rest of the paper we will follow this particular prescription, although it is clear that it is not the only possibility. The scale dependence of the Newton constant could be a consequence of the presence of extra dimensions, and in this case it is not even clear that the effective equations on the brane should be causal \[17\]. Moreover, even if the corrections come from matter quantum fields, one could consider more general initial states. These issues deserve further investigation.

Before addressing specific calculations, we introduce the following notation for the kernels: \((-\Box + \mu^2)^{-\alpha} \equiv G^\alpha_\mu(x,x')\) and \(\ln[-\Box\mu^2] \equiv L(x,x').\) Moreover, \(G(M^2, x, x')\) will stand for the retarded propagator of a scalar field of mass \(M\). The action of the kernels on a scalar function \(f\) is explicitly given by

\[
\ln[-\Box\mu^2] f \equiv \int d^4x' \sqrt{|g(x')|} L(x, x') f(x'),
\]

and

\[
(-\Box + \mu^2)^{-\alpha} f \equiv \int d^4x' \sqrt{|g(x')|} G^\alpha_\mu(x, x') f(x'),
\]

where

\[
G^\alpha_\mu(x, x') = \frac{2\sin(\alpha\pi)}{\pi} \int_0^{+\infty} dm m^{1-2\alpha} G(m^2 + \mu^2, x, x').
\]

III. \((-\Box)^{-\alpha}\) IN MINKOWSKI SPACETIME

Let us begin by computing the application of the operator defined in Eq. (10) in Minkowski spacetime to an arbitrary function. For this purpose, we start by considering the retarded propagator for a massive field,
which is twice the real and causal part of the Feynman propagator,
\[ G(m^2 + b^2, x, x') = 2\theta(t - t')\Re[G_F(x, x')] = 2\theta(t - t')\Re\left[ \int \frac{d^3k}{(2\pi)^3} \frac{\exp\{ik \cdot x\}}{k^2 - i\epsilon + m^2 + b^2} \right] \]
\[ = 2\theta(t - t') \int_0^{\infty} \frac{dk}{(2\pi)^3} \frac{\exp\{-\epsilon(t - t')\}}{k^2 + m^2 + b^2} \frac{\sin(|k|\bar{x} - \bar{x}')}{|\bar{x} - \bar{x}'|} \frac{\sin(\sqrt{k^2 + m^2 + b^2}(t - t'))}{\sqrt{k^2 + m^2 + b^2}}, \]
where the positive number \(\epsilon \to 0\) at the end of the calculations.

The generalized Green’s function \(G^\alpha_0(x, x')\) can be then obtained by inserting this result into Eq. (11) and performing the remaining integration. In the case in which \(b = 0\) we get
\[ G^\alpha_0(x, x') = \frac{\Gamma(1/2)\Gamma(\alpha + 1/2)}{\pi^2\Gamma(\alpha - 1)\Gamma(2\alpha)} \theta(t - t' - |\bar{x} - \bar{x}'|)(t - t')^2 - |\bar{x} - \bar{x}'|^2)^{\alpha-1} \exp\{-\epsilon(t - t')\}, \]
which is the same as the corresponding retarded Green’s function obtained by Bollini and Giambiagi in Ref. [24] using a different method. Note that this generalized Green’s function does not satisfy the Huygens’ principle, i.e. its support is not restricted to the surface of the past light cone.

With this result we can apply the \((-\Box)^{-\alpha}\) operator to any test function that depends on the spacetime coordinates. In the case of time independent functions, the time integral of the retarded propagator coincides with the Green’s function of the Laplacian,
\[ \int_{-\infty}^{+\infty} dt' G(m^2 + b^2, x, x') = \int \frac{d^3k}{(2\pi)^3} \frac{\exp\{i\vec{k} \cdot (\bar{x} - \bar{x}')\}}{k^2 + m^2 + b^2}. \]

Therefore, in such cases we obtain
\[ (-\Box + b^2)^{-\alpha} f = \int d^3x' \int \frac{d^3k}{(2\pi)^3} \frac{\exp\{i\vec{k} \cdot (\bar{x} - \bar{x}')\}}{(k^2 + b^2)^\alpha} f(\bar{x}') = \int \frac{d^3k}{(2\pi)^3} \frac{\exp\{i\vec{k} \cdot \bar{x}\}}{(k^2 + b^2)^\alpha} f(\vec{k}), \]
where \(f(\vec{k}) = \int d^3x \exp\{-i\vec{k} \cdot \bar{x}\} f(\bar{x})\).

As one can immediately note from the equation above, the application of \((-\Box + b^2)^{-\alpha}\) to a constant function \(f = 1\) gives \(b^{-2\alpha}\). This result is the one that would be expected by expanding \((-\Box + b^2)^{-\alpha}\) in powers of the derivative operator \(\Box\).

On the other hand, if the test function only depends on the time coordinate we are allowed to perform the integration on the space coordinates, with the result
\[ g_0^\alpha(t, t') \equiv \int d^3x' G_0^\alpha(x, x') = 2^{1/2 - \alpha} b^{1/2 - \alpha} \frac{\Gamma(1/2)}{\Gamma(\alpha)} J_{\alpha-1/2}(b(t - t'))\theta(t - t')(t - t')^{\alpha-1/2}, \]
where \(J\) is the Bessel function of the first kind [25]. For \(b = 0\) it reduces to
\[ \frac{\exp\{-\epsilon(t - t')\}}{\Gamma(2\alpha)} \theta(t - t')(t - t')^{2\alpha - 1}. \]

As particular examples, let us consider the test functions \(f_1(t) = (-t)^\beta\) \((t < 0, \beta < -2\alpha)\) and \(f_2(t) = \exp\{\nu t\} \) \((\nu > 0)\), for which we obtain
\[ (-\Box)^{-\alpha} f_1 = \int_{-\infty}^{+\infty} dt' g_0^\alpha(t, t') f_1(t') = \frac{\Gamma(2\alpha - \beta)}{\Gamma(-\beta)} (-t)^{2\alpha + \beta}, \]
\[ (-\Box)^{-\alpha} f_2 = \int_{-\infty}^{+\infty} dt' g_0^\alpha(t, t') f_2(t') = \nu^{-2\alpha} \exp\{\nu t\}. \]
Note that as the definition we are considering is not applicable to all integer values of \( \nu \), one cannot obtain the second result from the first one by a Taylor expansion of the exponential function. Note also that both results can be obtained from analytic continuation, by considering the application of the derivative operator \((-\Box)^n\) with \( n \in \mathbb{N} \) that yields a real result after replacing \( n \) by \(-\alpha\) [26].

The results of this Section are useful to analyze field equations like Eq. (2) in the weak field limit. We will use them in Section VI to study the generation of gravitational waves in the linearized limit.

IV. \((-\Box)^{-\alpha}\) IN FRW SPACETIMES

In this Section we consider three qualitatively different spatially flat FRW spacetimes. We use coordinates such that

\[
ds^2 = a^2(\eta) \left[ -d\eta^2 + \sum_{i=1}^{3} dx_i^2 \right] = -dt^2 + a^2(t) \sum_{i=1}^{3} dx_i^2,
\]

where \( a \) is the scale factor, \( \eta \) is the conformal time \((d\eta = dt/a(t))\) and \( t \) the cosmic time.

A. De Sitter spacetime

We set the scalar factor to be \( a(\eta) = -(H\eta)^{-1} \), where the constant \( H = \dot{a}/a \) (with a dot denoting differentiation with respect to \( t \)). The retarded propagator for massive scalar field in this spacetime is given by [27]

\[
G(m^2 + b^2, x, x') = \theta(\eta - \eta') R \left\{ \frac{H^2}{8\pi} \left( \frac{1 - \nu}{\cos(\pi\nu)} \right) F \left[ \frac{3}{2} + \nu, \frac{3}{2} - \nu, 1 + \frac{(\eta - \eta' - i\epsilon)^2 - |x - x'|^2}{4\eta\eta'} \right] \right\},
\]

where \( F \) is the Gaus's hypergeometric function [23] and \( \nu^2 = 9/4 - (m^2 + b^2)/H^2 \).

As in this case the integration on the mass parameter \( m \) is very difficult, we restrict ourselves to test functions that depend only on the time coordinate. Then, before performing the integration on the mass we integrate on the space coordinates, which results in

\[
g(m^2 + b^2, \eta, \eta') = \int d^3 x' \sqrt{\lvert g(\eta') \rvert} G(m^2 + b^2, x, x') = \theta(\eta - \eta') \frac{(\eta')^{3/2}}{H^2\nu} \sinh \left( \nu \ln \left( \frac{\eta'}{\eta} \right) \right).
\]

This equation can be integrated on the mass parameter to find

\[
g^\alpha(\eta, \eta') = \frac{2\sin(\alpha\pi)}{\pi} \int_0^{+\infty} dm \ m^{1-2\alpha} g(m^2 + b^2, \eta, \eta').
\]

The result depends on whether \( \nu \) is a real or purely imaginary number. We get

\[
g^\alpha(\eta, \eta') = \theta(\eta - \eta') H^{-2\alpha} \frac{\Gamma(1/2)}{\Gamma(\alpha)} \frac{(\eta')^{3/2}}{(-\eta')^{5/2}} \left( \frac{1}{2} \ln \left( \frac{\eta'}{\eta} \right) \right)^{\alpha-1/2}
\]

\[
\times \left\{ \begin{array}{ll}
\left( \frac{2}{\pi} - \frac{b^2}{4H^4} \right)^{1/4-\alpha/2} I_{\alpha-\frac{1}{2}} \left( \sqrt{\frac{2}{\pi} - \frac{b^2}{4H^4}} \ln \left( \frac{\eta'}{\eta} \right) \right), & b^2 \leq 9/4H^2; \\
\left( \frac{b^2}{4H^4} - \frac{9}{4} \right)^{1/4-\alpha/2} J_{\alpha-\frac{1}{2}} \left( \sqrt{\frac{b^2}{4H^4} - \frac{9}{4}} \ln \left( \frac{\eta'}{\eta} \right) \right), & b^2 \geq 9/4H^2.
\end{array} \right.
\]

where \( I \) is the modified Bessel function of the first kind [25].
As in Minkowski spacetime, applying the operator $(-\Box + b^2)^{-\alpha}$ to a constant $f = 1$ we obtain the “expected” result $b^{-2\alpha}$. As we will see in the next sections, this property is not true in spacetimes with initial singularities. This is important, for example, in the context of models where in order to explain the smallness of the experimentally inferred dark energy density, the conventional laws of Einstein’s gravity are modified at very large scales by introducing non local operators. In such models, gravity is modified so that a large vacuum energy density (identified with the dark energy density) does not give rise to large observable curvature. Phenomenologically, it can be seen as a result of the possibility of having a modified Einstein’s equation like Eq. (2) with the non local effective gravitational coupling such that when it is applied to a smooth and persistent source yields a small contribution due to $G(\Box)1 = G(0) \ll G_N$. For example, in the presence of a vacuum energy density, $T_{\mu\nu} = -\rho c g_{\mu\nu}$, the trace of Eq. (2) becomes

$$R = 32\pi G(-\Box)\rho_c,$$

(25)

so the effective vacuum energy is expected to be $G(0)\rho_c \ll G_N\rho_c$. As a phenomenological approach, the idea of addressing the cosmological problem with non local modification of gravity in the infrared is discussed for example in Ref. [28], where it is argued that acausality is a fundamental feature for this program to work.

The application of $(-\Box)^{-\alpha}$ to $f(\eta) = (-H\eta)^k \ (k < 0)$ yields

$$\int d\eta^\prime \ g_0^\alpha(\eta, \eta^\prime) f(\eta^\prime) = \frac{(-H\eta)^k}{k(k - 3)|\alpha|H^{2\alpha}},$$

(26)

which can also be seen as an analytic continuation of the result of the application of the covariant derivative operator $(-\Box)^n$, with $n$ an integer number, from $n \to -\alpha$ (we remind that we are assuming $0 < \alpha < 1$).

It is useful to note that if we are only interested in applying the kernel $(-\Box + b^2)^{-\alpha}$ to test functions that are independent on the space coordinates we only need to know the zero mode of the scalar field. That is, we can start by working out the solution of the Klein-Gordon equation for a spatially homogeneous scalar field $\phi(\eta) = (2\pi)^{-3/2}\chi(\eta)/a(\eta)$ (with mass $M = (m^2 + b^2)^{1/2}$ and minimally coupled to gravity),

$$\chi''(\eta) + a^2(\eta) \left( m^2 + b^2 - \frac{R}{6} \right) \chi(\eta) = 0,$$

(27)

where a prime means a derivative with respect to $\eta$ and $R = 6a''(\eta)/a^3(\eta)$ is the Ricci scalar. After imposing the Wronskian normalization condition,

$$\chi(\eta)\partial_\eta \chi^*(\eta) - \chi^*(\eta)\partial_\eta \chi(\eta) = i,$$

(28)

the retarded propagator integrated in the space coordinates is determined by

$$g(m^2 + b^2, \eta, \eta^\prime) \equiv \int d^d x \sqrt{|g(\eta^\prime)|} G(m^2 + b^2, x, x') = 2\theta(\eta - \eta^\prime) \sqrt{|g(\eta^\prime)|} \Re \{i(2\pi)^3 \phi(\eta)\phi^*(\eta^\prime)\}.$$

(29)

We can reobtain the result presented in Eq. (22) by starting with the following solution to Eq. (27):

$$\phi(\eta) = \frac{\chi(\eta)}{a(\eta)(2\pi)^{3/2}} = \frac{(-\eta)^{1/2}}{a(\eta)(2\pi)^{3/2}} \begin{cases} \frac{(-\eta)^{\nu} - i(-\eta)^{-\nu}}{\sqrt{\nu}}, & \nu \in \mathbb{R}; \\ \frac{(-\eta)^{\nu}}{\sqrt{\nu^2|\nu|}}, & i\nu \in \mathbb{R}. \end{cases}$$

(30)

As one can easily check, this solution satisfies the Wronskian condition [28]. Then, in order to find the propagator $g(m^2 + b^2, \eta, \eta^\prime)$ given in Eq. (22) we only need to insert this solution into Eq. (29). It is worth noting that the result does not depend on the initial condition for the mode.

For the remaining FRW spacetimes considered below, we restrict ourselves to test functions that depend only on time, and therefore we will follow this last procedure.
B. FRW spacetime with \( a(t) \propto t^{1/2} \)

We now consider a radiation dominated FRW spacetime with scale factor \( a(t) = a_0 t^{1/2} \). This spacetime has an initial singularity and a finite particle horizon.

A solution of Eq. (29) that satisfies the Wronskian normalization condition is given by

\[
\phi(\eta) = \frac{\sqrt{t}}{8\pi a(\eta)} H^{(2)}_1(Mt),
\]

where \( t = a_0^2 \eta^2 / 4, M = \sqrt{m^2 + b^2}, \) and \( H^{(2)} \) is the Hankel function of the second kind \[25\]. Substituting this solution into Eq. (29) we obtain

\[
g(m^2 + b^2, \eta, \eta') = \theta(\eta - \eta') \frac{\pi a_0^3}{2^{9/2} \eta^{7/2}} \left\{ J_{1/4}(Mt) J_{-1/4}(M't) - J_{-1/4}(Mt) J_{1/4}(M't) \right\}.
\]

For \( b = 0 \) the mass integration yields

\[
g^0(\eta, \eta') = \theta(\eta - \eta') \frac{t^{2\alpha - 2\alpha - 2\alpha^2 - 2\alpha^3 \eta^2 - 1}}{(2\alpha - 2\alpha - 2\alpha^2 - 2\alpha^3 \eta^2 - 1)\eta^4} F \left[ \alpha - \frac{1}{4}, \alpha, 2\alpha; 1 - \frac{\eta' \eta}{4} \right].
\]

Contrarily to the previous cases, after applying the \((-\Box)^{-\alpha}\) operator to a constant \( f = 1 \) we obtain a finite result,

\[
\int_0^{+\infty} d\eta' \ g^0(\eta, \eta') = \frac{\Gamma(5/4) t^{2\alpha - 4\alpha} \eta^{5/2} \eta' \eta^{5/2}}{\Gamma(\alpha + 1) \Gamma(5/4 + \alpha)}.
\]

This finite result is a consequence of the fact that this spacetime has an initial singularity since, as it was pointed out in Ref. \[28\], starting at \( t = 0, \eta = 0 \) in this case) it would take causal physics too much time to recognize that the constant was truly a constant. This interpretation may be further supported by computing the application of the \((-\Box + b^2)^{-\alpha}\) to \( f = 1 \),

\[
\int_0^{+\infty} d\eta' \ g^0(\eta, \eta') = \sin(\pi\alpha) t^{2\alpha - 2\alpha + 1/2} \eta^{5/2} \eta',
\]

where \( {}_1 F_2 \) is one of the generalized hypergeometric function \( {}_p F_q \) \[29\]. Then, taking the limit \( t \to +\infty \) we arrive at the expected result

\[
\lim_{t \to +\infty} \int_0^{+\infty} d\eta' \ g^0(\eta, \eta') = b^{-2\alpha}.
\]

On the other hand, despite the initial singularity, applying \((-\Box)^{-\alpha}\) to \( f(\eta) = (a_0^2 \eta^2 / 4)^\beta = t^\beta \) \( (\beta > -2) \) we get

\[
\int_0^{+\infty} d\eta' \ g^0(\eta, \eta') f(\eta') = 4^{-\alpha} t^{2\alpha + \beta} \frac{\Gamma(\beta/2 + 1) \Gamma(\beta/2 + 5/4)}{\Gamma(\beta/2 + \alpha + 1) \Gamma(\beta/2 + 5/4 + \alpha)}.
\]

This result can also be obtained through analytic continuation of the application of covariant derivative operator \((-\Box)^n\) for positive integers \( n \) \[13\].
C. FRW spacetime with $a(t) \propto t^2$

We will now consider a FRW spacetime with an initial singularity but, in contrast to the previous example, without a finite particle horizon. The choice $a(t) = a_0 t^2$ will allow us to find a solution of Eq. (27) in terms of elementary functions. Indeed, a solution that satisfies the Wronskian condition in Eq. (28) is given by

$$
\phi(\eta) = \frac{\exp(-i Mt)}{a(\eta)(2\pi)^{3/2}(Mt)^{3/2}} \sqrt{\frac{M}{2a_0}} [M^2 t^2 - 3(1 + i Mt)],
$$

with $t = -(a_0 \eta)^{-1}$ and $M = \sqrt{m^2 + b^2}$. Inserting this solution into Eq. (29) we find

$$
g(m^2 + b^2, \eta, \eta') = \frac{\theta(\eta - \eta') a^3(\eta')}{a_0 M^5 t^3 t^3} \left\{ \cos(M(t - t')) \left[ 3 M t (M^2 t^2 - 3) - 3 M t' (M^2 t^2 - 3) \right] \right\},
$$

and for $b = 0$, after integrating on the mass parameter, we obtain

$$
g_0(\eta, \eta') = \frac{a(\eta') \theta(\eta - \eta') (t - t')^{2\alpha - 1}}{\Gamma(2\alpha)(1 + 2\alpha)(3 + 2\alpha)} \left[ 3 t^4 + 3 t^5 + 3(2\alpha - 1) 3 t^2 ) + 3) \right],
$$

The application to a constant $f = 1$ yields the result

$$
\int d\eta' g_0(\eta, \eta') = \frac{15 t^{2\alpha}}{\Gamma(2\alpha + 2)(2\alpha + 3)(2\alpha + 5)}.
$$

which is also finite.

As in the previous case, applying $(-\Box + b^2)^{-\alpha}$ to a constant we get

$$
\int d\eta' g_0(\eta, \eta') = \frac{15}{\pi} t^{2\alpha} \sin(2\pi \alpha) \left\{ -\alpha - 2\alpha \right\} F_2 \left[ \frac{5}{2}, \frac{3}{2} + \alpha; \frac{3}{2} + \alpha; \frac{b^2 t^2}{4} \right] + 3 \Gamma(-2 - 2\alpha) \left[ \frac{3}{2}, \frac{3}{2} + \alpha; \frac{3}{2} + \alpha; \frac{b^2 t^2}{4} \right],
$$

and taking the limit $t \to +\infty$ we find

$$
\lim_{t \to +\infty} \int d\eta' g_0(\eta, \eta') = b^{-2\alpha}.
$$

Therefore, the fact that the application of the $(-\Box)^{-\alpha}$ to a constant yields a finite result seems not related to the existence of a particle horizon, but to the existence of a singularity at $t = 0$.

Another interesting property we can point out is that considering the advanced propagator,

$$
g_{adv}(m^2 + b^2, \eta, \eta') = -2 \theta(\eta' - \eta) \sqrt{|g(\eta')|} R \left\{ i(2\pi)^{3} \phi(\eta) \phi^*(\eta') \right\},
$$

instead of the retarded one, we obtain

$$
(-\Box + b^2)^{-\alpha} = b^{-2\alpha},
$$

independently of the value of the initial time $t$.

Finally, for the sake of completeness, we compute the application of $(-\Box)^{-\alpha}$ to $f(\eta) = (a_0 \eta)^{-\beta} = t^\beta$ ($\beta > -2$), with the result

$$
\int d\eta' g_0(\eta, \eta') f(\eta') = 4^{-\alpha} t^{2\alpha + \beta} \frac{\Gamma(3/2 + \beta/2) \Gamma(1 + 3/2 \beta)}{\Gamma(1 + \alpha + \beta/2) \Gamma(1 + \beta/2)}
$$

which can also be obtained by analytic continuation from an integer number $n$ to $-\alpha$. 

V. THE OPERATOR $\ln[-\Box/\mu^2]$

In this Section we compute the application of the operator that can represent the logarithm of the covariant d’Alembertian to scalar test functions that are independent of the space coordinates, for each of the four spacetimes considered above. As described in Section III, we will do that by taking the limit $\alpha \to 0$ in the kernel $(-\Box)^{-\alpha}$.

We begin with Minkowski spacetime. We start by applying the kernel $(-\Box)^{-\alpha}$, given in Eq. (17), to a generic test function $f$ that is independent of the space coordinates,

$$\int_{-\infty}^{t} dt' \mu^{2\alpha} g_0^\alpha(t,t') f(t') = \frac{1}{\Gamma(2\alpha)} \int_{-\infty}^{t} dt' \mu^{2\alpha} \exp\{-\epsilon(t-t')\}(t-t')^{2\alpha-1} f(t'),$$

(47)

where we have added a constant $\mu$ with mass dimensionality.

Before expanding in powers of $\alpha$, and in order to exclude explicitly the local contribution, we perform an integration by parts,

$$\frac{1}{\Gamma(2\alpha)} \int_{-\infty}^{t} dt' \mu^{2\alpha} \frac{\exp\{-\epsilon(t-t')\}}{(t-t')^{1-2\alpha}} f(t') = f(t) \left(\frac{\mu^2}{\epsilon}\right)^{2\alpha} - \frac{1}{\Gamma(2\alpha)} \int_{-\infty}^{t} dt' \mu^{2\alpha} \Gamma(2\alpha,\epsilon(t-t')) \frac{d}{dt'} f(t'),$$

(48)

where $\Gamma(a,x)$ is the incomplete gamma function [23], and we have assumed that $f(t')\Gamma(2\alpha,\epsilon(t-t'))$ goes to zero as $t' \to -\infty$.

After subtracting the local zeroth order contribution $f(t)$ and expanding up to first order in $\alpha$ we find

$$\int_{-\infty}^{t} dt' g_0^\alpha(t,t') f(t') - f(t) \simeq -\alpha \left\{ \ln \left(\frac{\epsilon^2}{\mu^2}\right) + 2 \int_{-\infty}^{t} dt' \Gamma(0,\epsilon(t-t')) \frac{d}{dt'} f(t') \right\}.$$

(49)

Finally, in the limit $\epsilon \to 0$ we obtain

$$\ln \left[-\frac{\Box}{\mu^2}\right] f = -2\gamma f(t) - 2 \int_{-\infty}^{t} dt' \ln (\mu(t-t')) \frac{d}{dt'} f(t'),$$

(50)

where $\gamma$ is the Euler’s constant and we have assumed that $f(t') \to 0$ as $t' \to -\infty$. The same result can also be obtained from the representation proposed for this kernel in Ref. [30].

With the use of this integral representation one can straightforwardly compute the application of $\ln[-\Box/\mu^2]$ to spatially homogeneous functions. For example, for $f(t) = (-t)^{\beta}$ (with $t < 0$ and $\beta < -2\alpha$) it yields

$$\ln \left[-\frac{\Box}{\mu^2}\right] f = (-t)^{\beta} \left(-\ln(t^2\mu^2) + 2\psi(-\beta)\right),$$

(51)

where $\psi(x)$ is the Psi (Digamma) function [23]. Note that it can also be obtained by expanding the result in Eq. (18) in powers of $\alpha$ and extracting the first order contribution.

For the three remaining spacetimes, we can follow the same procedure. Details of the calculations are relegated to the Appendix A.

For the de Sitter spacetime, we arrive at

$$\ln \left[-\frac{\Box}{\mu^2}\right] f = \ln \left(\frac{H^2}{\mu^2}\right) f(\eta) - \int_{-\infty}^{\eta} d\eta' \left\{ \ln \left(\frac{e^\gamma}{3} \ln \left(\frac{\eta'}{\eta}\right)\right) + E_1 \left(-3 \ln \left(\frac{\eta'}{\eta}\right)\right) \right\} \frac{d}{d\eta'} f(\eta'),$$

(52)

where $E_1$ is the exponential-integral function [23] and we have assumed that $f(\eta) \to 0$ as $\eta \to -\infty$. The application of this operator to $f(\eta) = (-H\eta)^{k}$ ($k < 0$) gives the result

$$\ln \left[-\frac{\Box}{\mu^2}\right] f = (-H\eta)^{k} \ln \left(\frac{H^2}{\mu^2} k(k-3)\right).$$

(53)
Likewise, for the radiation dominated FRW spacetime \((a(t) \propto t^{1/2})\) we obtain

\[
\ln \left[ -\frac{\Box}{\mu^2} \right] f = \left[ 4 - 2\gamma - \frac{\pi}{2} - \ln \left( \frac{\mu^2}{2H^2} \right) \right] f(\eta) + \int_0^\eta d\eta' \left\{ 4\eta'^5 F \left[ 1, \frac{5}{4}, \frac{9}{4}, \frac{\eta'}{\eta} \right] - \ln \left( 1 - \frac{\eta'^4}{\eta^4} \right) \right\} \frac{d}{d\eta'} f(\eta'),
\]

with \(f(\eta) \to 0\) as \(\eta \to 0\). After applying it to \(f = t^\beta = (a_0 \eta/2)^{2\beta} (\beta > 0)\) we get

\[
\ln \left[ -\frac{\Box}{\mu^2} \right] f = t^\beta \left\{ \ln \left( \frac{16H^2}{\mu^2} \right) + \psi \left( \frac{\beta}{2} + \frac{5}{4} \right) + \psi \left( \frac{\beta}{2} + 1 \right) \right\}.
\]

(54)

Lastly, in the case of the FRW spacetime with \(a(t) \propto t^2\) the result is

\[
\ln \left[ -\frac{\Box}{\mu^2} \right] f = \left[ \frac{46}{15} - 2\gamma - \ln \left( \frac{4\mu^2}{H^2} \right) \right] f(\eta) - \int_0^t dt' \left\{ \frac{2\mu^5}{5t^5} + \frac{2t^2}{t} + \frac{2\mu^3}{3t^3} + 2 \ln \left( 1 - \frac{t'}{t} \right) \right\} \frac{d}{dt'} f(t'),
\]

(56)

with \(f(t) \to 0\) as \(t \to 0\). For \(f = t^\beta = (-a_0 \eta)^{-\beta} (\beta > 0)\) it yields

\[
\ln \left[ -\frac{\Box}{\mu^2} \right] f = t^\beta \left\{ \ln \left( \frac{H^2}{\mu^2} \right) + \psi \left( \frac{\beta}{2} + \frac{7}{2} \right) + \psi \left( \frac{\beta}{2} + 1 \right) \right\}.
\]

(57)

As we have pointed out in the Introduction, there are different proposals in the literature to take into account the running of coupling constants, which correspond to different replacements of the energy scale \(\bar{\mu}\) by \(1/t, H, \bar{\mu}/a(t)\), etc. In cosmological situations, it is common to assume a power law behavior for both the scale factor and the matter content. In such situations, the relevant results are the ones given in Eqs. (57, 55, 56), which were obtained by applying the corresponding non local kernel to test functions that have a power law dependence on the time coordinate (either \(\eta\) or \(t\)). These results can be written in the form

\[
(-\Box)^{-\alpha} f \propto H^{-2\alpha} f,
\]

(58)

\[
\ln \left[ -\frac{\Box}{\mu^2} \right] f = \ln \left( \frac{H^2}{\mu^2} \right) f + \text{const} \times f.
\]

(59)

Moreover, in de Sitter spacetime, if we consider test functions that have a power law dependence on the conformal time \(\eta\), we can also write the results in the same form (see Eqs. (20a) and (55)). This would suggest that in FRW spacetimes the action of the non local kernel is equivalent to the replacement \(\bar{\mu} \to H\). However, this is not strictly correct. Indeed, the results depend on the test function. On the one hand, the constant of proportionality in Eq. (58) depends on \(\alpha\) and \(\beta\). On the other hand, in de Sitter spacetime, if we apply the non local operator to a function \(f = (-t)^\beta (\beta < 0 \text{ and } t < 0)\) we obtain

\[
\ln \left[ -\frac{\Box}{\mu^2} \right] f = (-t)^\beta \left\{ \ln \left( \frac{H^2}{\mu^2} \right) + F(Ht, \beta) \right\},
\]

(60)

where \(F(Ht, \beta)\) is a complicated function whose functional dependence on time is not logarithmic. For example, in the particular case of \(\beta = -2\) it reduces to

\[
F(Ht, -2) = 2 - \gamma - \ln(-Ht/3) - \exp(-3Ht)(1 + 3Ht)\alpha(3Ht).
\]

(61)

This non logarithmic behavior may have been expected from the fact that, in contrast to the other FRW examples, for de Sitter spacetime one has \(t\) and \(H\) as two different and relevant scales.
VI. GENERATION OF GRAVITATIONAL WAVES

In this Section we analyze the modified Einstein’s equations \[2\] with a running coupling given by Eq. \[\alpha\], for the particular case \(b = 0\). We assume weak gravitational fields. Using an expansion in powers of the curvature, it can be shown that the modified equations follow from the non local action \[17\]

\[
S = - \int d^4x \sqrt{|g|} \left\{ \frac{1}{16\pi G_N} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \frac{1 + F[-\Box]}{\Box} R_{\mu\nu} + \mathcal{O}(R_{\mu\nu}^3) \right\} + S_{\text{matter}},
\]

where \(S_{\text{matter}}\) corresponds to the source of the gravitational waves \(2\delta S_{\text{matter}} = \sqrt{|g|} T_{\mu\nu} \delta g^{\mu\nu}\). Here, the kernel \(F[-\Box] = (\alpha L^2 \Box)^{-\alpha}\) represents a small modification of Einstein’s theory for very large scales.

As anticipated, we work in the weak field limit \(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\), where \(h_{\mu\nu}\) describes a linear perturbation to the Minkowski metric. In the harmonic gauge, up to first order in \(h_{\mu\nu}\), the modified Einstein’s equations become

\[
(1 + F[-\Box]) \Box \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu},
\]

where \(\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu} h_{\gamma\gamma}/2\).

We assume that the inclusion of the kernel \(F[-\Box]\) yields a perturbatively small correction to the standard result. Therefore, we split \(\bar{h}_{\mu\nu} = \bar{h}^{(0)}_{\mu\nu} + \bar{h}^{(1)}_{\mu\nu}\), where \(\bar{h}^{(0)}_{\mu\nu}\) satisfy

\[
\Box \bar{h}^{(0)}_{\mu\nu} = -16\pi G_N T_{\mu\nu},
\]

and \(\bar{h}^{(1)}_{\mu\nu}\) can be perturbatively computed by solving the following equation:

\[
\Box \bar{h}^{(1)}_{\mu\nu} = -(\alpha L^2 \Box)^{-\alpha} \Box \bar{h}^{(0)}_{\mu\nu} = 16\pi G_N (\alpha L^2 \Box)^{-\alpha} T_{\mu\nu} \equiv -16\pi G_N \tilde{T}_{\mu\nu}.
\]

With the use of the retarded Green’s function we can write

\[
\bar{h}^{(1)}_{\mu\nu}(t, \vec{x}) = 4G_N \int \frac{d^4x'}{|\vec{x} - \vec{x}'|} \tilde{T}_{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}'),
\]

where

\[
\tilde{T}_{\mu\nu}(t, \vec{x}) = -\int d^4x' G_0^\alpha(x, x') T_{\mu\nu}(x').
\]

Note that the value of \(\bar{h}^{(1)}_{\mu\nu}\) in the spacetime point \(x\) is determined only by the values that \(\tilde{T}_{\mu\nu}\) takes on the past light cone of \(x\), but in order to compute \(\tilde{T}_{\mu\nu}\) one needs to know all the values of the energy-momentum tensor \(T_{\mu\nu}\) inside that past light cone. Therefore, the solution for the metric perturbation \(\bar{h}_{\mu\nu}\) does not obey the Huygens’ principle. This last issue was analyzed in detail in Ref. \[24\] for different spacetime dimensions.

Finally, working with the Fourier transform and using the generalized Green’s function of Eq. \[15\] one can find

\[
\bar{h}_{\mu\nu}(\omega, \vec{k}) = -16\pi G_N \left\{ 1 - L^{-2\alpha} (\omega + i\epsilon)^2 + |\vec{k}|^2 \right\}^{-\alpha} \frac{T_{\mu\nu}(\omega, \vec{k})}{(\omega + i\epsilon)^2 + |\vec{k}|^2}.
\]

The non locality of the modified Einstein equations has other interesting consequences. It gives rise to corrections to the Newtonian potential which in turn imply that the metric outside a massive object may depend on its inner structure, violating Birkhoff theorem \[31\]. Therefore, we expect the corrections to the metric outside a collapsing star to be time dependent, even in spherically symmetric situations.
VII. CONCLUSIONS

We have presented a detailed derivation of the action of the non local kernels \(-\Box^{-\alpha}\) and \(\ln(-\Box)\) on scalar functions. Our starting point was the representation of both kernels in terms of integrals of the retarded propagator of a massive scalar field.

In Minkowski spacetime, we derived an explicit expression for the kernel \(-\Box^{-\alpha}\). This is relevant for the analysis of the modified Einstein Eq. (2) in the weak field limit. As an application, we obtained formal expressions for the generation of gravitational waves, and showed explicitly that Huygens’ principle is no longer valid when one considers a scale dependence of Newton constant.

In Robertson Walker spacetimes, we considered the action of \(-\Box^{-\alpha}\) on spatially homogeneous time dependent test functions. We presented a method for obtaining general expressions for the action of the non local kernel on such test functions, which only involves the zero mode solution of the Klein-Gordon equation for a massive, minimally coupled scalar field. We specialized the general expressions to particular cases, as test functions defined as powers of the cosmic or conformal times. We have also obtained an expression for the action of \(\ln(-\Box)\) on time dependent test functions by considering the limit \(\alpha \to 0\).

The application of the operator \((-\Box)^{-\alpha}\) to functions that depend on \(t\) in Robertson Walker spacetimes, or functions that depend on \(r\) in static metrics with spherical symmetry, has been considered previously by Hamber and Williams in Refs. [15, 21]. Their approach was based on an analytic continuation of the action of the derivative operator \((-\Box)^n\) for positive integers \(n\) to the case \(n \to -\alpha\). For simple test functions in which the analytic continuation can be carried out, this procedure gives the same answers than the non local kernels defined in this paper. However, it is worth to note that, within our approach based on the integral representation (Eq. (7)), it is in principle possible to evaluate \((-\Box)^{-\alpha}f\) for arbitrary scalar functions in arbitrary spacetimes. Moreover, this representation shows explicitly the non local properties of the kernel and could also be generalized to tensor functions.

In the cases of the FRW spacetimes in which both the scale factor and the test function have a power law dependence on either the cosmic or conformal time, after applying the non local kernels, we have found that the functional dependence on time of the results is the same as the one obtained after the replacement of the energy scale \(\mu\) by the Hubble rate \(H\). However, using as an example the de Sitter spacetime, we have explicitly shown that in general the result depend on the test function and can be very different from the one obtained with the use of any of the replacements mentioned in the Introduction.

We also investigated whether the action of \(1/G(-\Box)\) on a constant test function gives \(1/G(0)\) or not, as naively expected for analytic functions of the d’Alembertian. In particular, we have shown that \((\Box + b^2)^{-\alpha}1 = b^{-2\alpha}\) only for spacetimes without initial singularities, or when considering acausal propagators, and that it seems not to be related to the existence or not of a particle horizon. These results are in tune with the claim [28] that, in the presence of an initial singularity, acausality may be crucial to solve the cosmological constant problem using infrared modifications to General Relativity.

As for future work, we consider that a similar method as the one proposed in this paper could also be used, without much additional difficulty, in static and spherically symmetric spacetimes for test functions that respect such symmetries.

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APPENDIX A: EXPANSION UP TO FIRST ORDER IN $\alpha$

In this appendix we describe some technical details about the computation of the expansion up to first order in $\alpha$ of the application of $(-\Box)^{-\alpha}$ to time dependent functions. The procedure we follow here is the same as the one described in the main text for the Minkowski case.

1. De Sitter spacetime

In order to obtain the result presented in Eq. (52), we start from Eq. (24) (in which the zero mode Green’s function $g^\alpha_0$ is given). We compute a primitive of the Green’s function $g^\alpha_0$ with respect to $\eta'$,

$$ G(\eta, \eta') \equiv \int d\eta' \mu^{2\alpha} g^\alpha_0(\eta, \eta') = \frac{3^{1/2-\alpha} \mu^{2\alpha}}{H^{2\alpha}} \frac{\Gamma(1/2)}{\Gamma(\alpha)} \int^{\eta'} \frac{d\eta''}{-\eta''} \left( \frac{\eta''}{\eta'} \right)^{3/2} I_{\alpha-1/2} \left( \frac{3}{2} \ln \left( \frac{\eta''}{\eta'} \right) \right) \left( \ln \left( \frac{\eta''}{\eta'} \right) \right)^{1/2-\alpha} \text{(A1)} $$

$$ = \frac{2^{\alpha+1/2} \mu^{2\alpha}}{(3H)^{2\alpha}} \left( \ln \eta' / \eta \right)^{3/2} \left( \ln \left( \frac{\eta'}{\eta} \right) \right)^{\alpha+1/2} I_{\alpha-1/2} \left( \frac{3}{2} \ln \left( \frac{\eta'}{\eta} \right) \right) + I_{\alpha+1/2} \left( \frac{3}{2} \ln \left( \frac{\eta'}{\eta} \right) \right), $$

where the second equality follows after the change of variable $s = 3/2 \ln(\eta'/\eta)$ (for the last equality see Ref. [23]).

Assuming that $G(\eta, \eta') f(\eta') \sim (\ln(\eta'/\eta))^{\alpha} f(\eta')$ goes to zero as $\eta' \to -\infty$ ($\alpha \to 0^+$), and expanding $G(\eta, \eta')$ up to first order in $\alpha$,

$$ G(\eta, \eta') \simeq -1 - \alpha \left\{ \ln \left( \frac{e^{\gamma} \mu^{2}}{3H^{2}} \ln \left( \frac{\eta'}{\eta} \right) \right) + E_i \left( -3 \ln \left( \frac{\eta'}{\eta} \right) \right) \right\}, \text{(A2)} $$

we get

$$ \int d\eta' \mu^{2\alpha} g^\alpha_0(\eta, \eta') f(\eta') - f(\eta) \simeq \alpha \int_{-\infty}^{\eta} d\eta' \left\{ \ln \left( \frac{e^{\gamma} \mu^{2}}{3H^{2}} \ln \left( \frac{\eta'}{\eta} \right) \right) + E_i \left( -3 \ln \left( \frac{\eta'}{\eta} \right) \right) \right\} d\eta' f(\eta'), \text{(A3)} $$

where $E_i$ is the exponential-integral function [23]. Then, the required operator [52] is obtaining by comparing this result with the first order of the alpha expansion of $(-\Box)^{-\alpha}$.

2. FRW spacetime with $a(t) \propto t^{1/2}$

We begin by computing a primitive of the Green’s function $g^\alpha_0$ given in Eq. (33) (see Ref. [23]),

$$ G(\eta, \eta') \equiv \int d\eta' \mu^{2\alpha} g^\alpha_0(\eta, \eta') = 4^{1-\alpha} \left( \mu t \right)^{2\alpha} \Gamma(2\alpha) \int_{0}^{\eta'} \eta''^{\alpha-3} \left( \eta'' - \eta''^4 \right)^{2\alpha-1} F \left[ \alpha - \frac{1}{4}, \alpha, 2\alpha; \frac{1}{4} - \frac{\eta''^4}{\eta^4} \right] \text{(A4)} $$

$$ = \frac{\Gamma(5/4) \Gamma(\alpha + 1/2) \left( \mu t \right)^{2\alpha}}{\Gamma(\alpha + 5/4) \Gamma(1/2) \Gamma(2\alpha + 1)} \left( \eta'' - \eta''^4 \right)^{2\alpha} F \left[ \alpha - \frac{1}{4}, \alpha, 2\alpha + 1; 1 - \frac{\eta''^4}{\eta^4} \right]. $$

Then, the integration by part results in

$$ \int d\eta' \mu^{2\alpha} g^\alpha_0(\eta, \eta') f(\eta') = G(\eta, \eta') f(\eta) - \int_{0}^{\eta} d\eta' G(\eta, \eta') \frac{d}{d\eta'} f(\eta'), \text{(A5)} $$
where we have assumed that \( f(\eta')g(\eta, \eta') \sim f(\eta')\eta'^{4} \to 0 \) as \( \eta' \to 0 \). It is straightforward to calculate the expansion up to first order in \( \alpha \) of \( G(\eta, \eta) \),

\[
G(\eta, \eta) \simeq 1 + \alpha(-4 + 2\gamma + \frac{\pi}{2} + \ln(2\mu^{2}t^{2})).
\]  

(A6)

In order to expand \( G(\eta, \eta') \) it is useful to use the following identities (see Ref.27):

\[
F\left[\alpha - \frac{1}{4}, \alpha, 2\alpha + 1; 1 - z\right] = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} \left\{ \frac{\Gamma(5/4)}{\Gamma(\alpha + 5/4)} F\left[\alpha - \frac{1}{4}, \alpha, -\frac{1}{4}; z\right] + \alpha \frac{5}{4} - 4 + 2\gamma + \pi^{2} + \ln(2\mu^{2}t^{2}) \right\},
\]  

(A7)

\[
F\left[\alpha - \frac{1}{4}, \alpha, -\frac{1}{4}; z\right] = 1 + \alpha(1 - 4\alpha) \int_{0}^{z} ds F\left[\alpha + \frac{3}{4}, \alpha + 1, \frac{3}{4}; s\right] \simeq 1 + \alpha \int_{0}^{z} ds F\left[\frac{3}{4}, \frac{3}{4}, \frac{3}{4}; s\right] = 1 - \alpha \ln(1 - z).
\]  

(A8)

Then, we find that up to first order in \( \alpha \)

\[
G(\eta, \eta') \simeq \alpha \left\{ \frac{4\eta'^{5}}{\eta^{5}} F\left[\frac{5}{4}, 1, \frac{9}{4}, \eta^{4}; \eta'\right] - \ln\left(1 - \frac{\eta'^{4}}{\eta^{4}}\right) \right\}.
\]  

(A9)

Inserting (A6) and (A9) into Eq.(A5), and comparing with the expansion of \((-\Box)^{-\alpha}\) we arrive at the result given in Eq.(54).

3. FRW spacetime with \( a(t) \propto t^{2} \)

A primitive of the Green’s function given in Eq.(10) can be obtained as

\[
G(\eta, \eta') \equiv \int d\eta'' \; \mu^{2\alpha}g_{0}(\eta, \eta'') = \int_{0}^{t} dt' \frac{(t - t')^{2\alpha-1} \mu^{2\alpha}}{\Gamma(2\alpha)(1 + 2\alpha)(3 + 2\alpha)}
\times \left[ 3\frac{t'}{t} + 3\frac{t'^{5}}{t^{5}} + 3(2\alpha - 1) \left(\frac{t'^{2}}{t^{2}} + \frac{t'^{4}}{t^{4}}\right) + (3 + 4\alpha(\alpha - 1)) \frac{t'^{3}}{t^{3}} \right],
\]  

(A10)

where we have used that \( a(\eta')d\eta' = dt' \). In this case, it is not difficult to carry out the \( \alpha \) expansion to find that up to first order we have

\[
G(\eta, \eta) \simeq 1 + \alpha \left\{ -\frac{46}{15} + \ln(e^{2\gamma} \mu^{2}t^{2}) \right\},
\]  

(A11)

\[
G(\eta, \eta') \simeq \alpha \left\{ -\frac{2t'}{t} - \frac{2t'^{5}}{5t^{5}} - \frac{2t'^{3}}{3t^{3}} - 2 \ln\left(1 - \frac{t'}{t}\right) \right\},
\]  

(A12)

where we have assumed that \( G(\eta, \eta')f(t') \sim t'^{2}f(t') \to 0 \) as \( t' \to 0 \).

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