Finite Symmetry of Leptonic Mass Matrices

C.S. Lam

Department of Physics, McGill University
Montreal, Q.C., Canada H3A 2T8

and

Department of Physics and Astronomy,
University of British Columbia,
Vancouver, BC, Canada V6T 1Z1

Email: Lam@physics.mcgill.ca

Abstract

We search for possible symmetries present in the leptonic mixing data from $SU(3)$ subgroups of order up to 511. Theoretical results based on symmetry are compared with global fits of experimental data in a chi-squared analysis, yielding the following results. There is no longer a group that can produce all the mixing data without a free parameter, but a number of them can accommodate the first or the second column of the mixing matrix. The only group that fits the third column is $\Delta(150)$. It predicts $\sin^2\theta_{13} = 0.11$ and $\sin^2\theta_{23} = 0.94$, in good agreement with experimental results.
I. INTRODUCTION

In the days when the reactor angle of neutrino mixing was thought to be zero and the atmospheric angle maximal, mixings could be taken to be tri-bimaximal, and explained by a $S_4$ symmetry without any free parameter, or groups containing it [1]. Now that both global fits and direct measurements show the reactor angle to be non-negligible and the atmospheric angle possibly non-maximal [2–4], many suggestions have been advanced to explain the new data [5, 6]. In this paper we investigate whether a finite symmetry still exists to accommodate them.

The group theory of mixing reviewed in Sec. 2 will be used to extract columns of possible mixing matrices allowed by a finite group $\mathcal{G}$. With the help of the powerful group software GAP [7], we examine all finite subgroups of $SU(3)$ up to order 511 and compare their predictions with global fits of experimental measurements. Since phases are unknown, only absolute values of the mixing matrix elements are used. The experimental data and the global analysis used in a chi-squared comparison will be discussed in Sec. 3. The result presented in Sec. 4 can be summarized as follows. No group can simultaneously accommodate all three columns of the mixing matrix, like $S_4$ was able to do for the tri-bimaximal mixing. Many groups can accommodate the first or the second column, but the success is not necessarily trustworthy because those globally fitted matrix elements depend on the unknown CP phase. For the third column, only the group $\Delta(150)$ works and it predicts $\sin^2 2\theta_{13} = 0.11$ and $\sin^2 2\theta_{23} = 0.94$, in good agreement with experimental data. In Sec. 5, after a review of the mass matrices in the presence of $A_4$ and $S_4$, we will present the form of the mass matrices for $\Delta(150)$.

We close this section with a brief remark comparing the group theoretical method used here and other approaches. Like texture zeros, both are theories of fermion mass matrices within the Standard Model. There is no need to introduce additional Higgs or valons, together with their alignments. Clebsch-Gordan coefficients are not needed. Unlike the texture-zero approach, mixing parameters are determined by symmetry here and not by ratios of fermionic masses. The group-theory approach is consistent with dynamical models based on horizontal symmetry if the valon alignments are given by the invariant eigenvectors of the residual symmetries discussed in the next section. Such alignments can be obtained from group-invariant potentials in the weak-coupling approximation [8].
II. GROUP THEORY OF MIXING

Every mixing produces a $Z_2 \times Z_2$ symmetry in the neutrino Majorana mass matrix and a $Z_n$ symmetry in the left-handed charged-lepton mass matrix. The group theory of mixing to be reviewed below \cite{1,8,9} is based on the simple assumption that these natural symmetries are the residual symmetries of a horizontal symmetry group.

Let $M_\nu$ be the symmetric mass matrix of active neutrinos, and $M_e := M_e^\dagger M_e$ be the hermitian mass matrix of left-handed charged leptons. In the basis where $M_e$ is diagonal, the neutrino mixing matrix $U$ renders $U^\dagger M_\nu U$ diagonal. Let $u_1, u_2, u_3$ be the three columns of $U$. Then the unitary matrices defined by

$$G_1 = +u_1^\dagger u_1 - u_2^\dagger u_2 - u_3^\dagger u_3,$$

$$G_2 = -u_1^\dagger u_1 + u_2^\dagger u_2 - u_3^\dagger u_3,$$

$$G_3 = -u_1^\dagger u_1 - u_2^\dagger u_2 + u_3^\dagger u_3$$

mutually commute and commute with $U$. They satisfy $G_i^2 = 1$, $G_i G_j = G_k$ ($i, j, k$ different), and $G_i^T M_\nu G_i = M_\nu$. Thus they generate a $Z_2 \times Z_2$ symmetry of $M_\nu$. In the mean time, since $M_e$ is diagonal, every unitary diagonal matrix $F$ commutes with it, giving $F^\dagger M_e F = M_e$. If $F^n = 1$, then $F$ generates a symmetry group $Z_n$ of $M_e$. We will assume the eigenvalues of $F$ to be non-degenerate so that $F$ diagonal forces $M_e$ to be diagonal.

Since these symmetries are always present, additional input is required to nail down the mixing. To this end we shall assume $F$ and $G$ to be residual symmetries of some finite group $G$, in the sense that both are members of the group. In that case the group structure imposes a correlation between $F$ and $G$, allowing $G$ and hence $U$ to be determined when $F$ is given in its diagonal form. We call $G$ a partial-symmetry group if it contains $F$ and one $G_i$, and a full-symmetry group if it contains $F$ and two mutually commuting $G_i$’s. It does not matter which two to choose because the third one, being the product of the first two, must also be in $G$.

Conversely, given a finite group $G$, any of its order-2 elements is a candidate of $G_i$, and any element with order larger than 2 is a candidate of $F$. In a 3-dimensional irreducible unitary representation of $G$, the invariant eigenvector (the one with eigenvalue +1) of $G_i$ in the $F$-diagonal basis constitutes a column $u_i$ of a possible mixing matrix $U$. In order to ensure that $G_i$ has one +1 eigenvalue and two −1 eigenvalues as in (1), we shall confine to
those representations of $G_i$ with character $-1$. Going through all combinations of $F$, $G_i$, and 3-dimensional irreducible representations, we get all possible mixing-matrix columns allowed by this group. We shall refer to these theoretically predicted mixing columns as mixing vectors, to distinguish them from the globally fitted experimental mixing columns. As long as $\mathcal{G}$ is a finite group, the number of allowed mixing vectors is finite, though this number could be large for a large group. Two such vectors $u_i$ and $u_j$ may fit into the same mixing matrix $U$ if and only if $G_i$ and $G_j$ commute.

Within this scheme, the order of $\mathcal{G}$ must be even because it must contain at least one order-2 member $G_i$. If it is to be a full-symmetry group, then its order must be divisible by 4 because there must be two distinct order-2 elements $G_i$ present. This last condition is necessary but not sufficient for a full-symmetry group also requires those three order-2 members to mutually commute.

Since phases of neutrino mixing have not been measured, only the absolute values of these columns are needed for experimental comparison. Moreover, group theory can never know how to label the neutrino flavor states nor the mass eigenstates, hence any mixing vector from group theory can be used to compare with any of the three columns of the experimental mixing matrix, and the rows may be permuted any way we want before the comparison.

There is a large number of allowed mixing vectors for a large group $\mathcal{G}$ to make the comparison a daunting task to do by hand. Fortunately, a powerful free software GAP \cite{7} is available to help us. Note however that the irreducible representations given in GAP may not be unitary so to use it we must first obtain from it the corresponding unitary representations.

In Sec 4, we will use the recipe outlined here to obtain the allowed mixing vectors of all finite subgroups of $SU(3)$ with an order less than 512.

\section{III. Experimental Data and Global Fits}

Recent experiments give the following values for the reactor angle $\theta_{13}$ \cite{4}:

\begin{align*}
\sin^2 2\theta_{13} & = 0.089 \pm 0.010 \pm 0.005 \quad \text{(Daya Bay)} \\
\sin^2 2\theta_{13} & = 0.109 \pm 0.030 \pm 0.025 \quad \text{(Double Chooz)} \\
\sin^2 2\theta_{13} & = 0.113 \pm 0.013 \pm 0.019 \quad \text{(RENO)}
\end{align*}
\[ \sin^2 2\theta_{13} = 0.104 \pm 0.060 \pm 0.045 \] (T2K, normal hierarchy)
\[ \sin^2 2\theta_{13} = 0.128 \pm 0.070 \pm 0.055 \] (T2K, inverted hierarchy).

A preliminary result from MINOS also shows that the atmospheric mixing may not be maximal:
\[ \sin^2 2\theta_{23} = 0.96 \pm 0.04 \] (MINOS, \(\nu\))
\[ \sin^2 2\theta_{23} = 0.97 \pm 0.03/0.08 \] (MINOS, \(\overline{\nu}\)).

These results are to some extent anticipated by global fits of the data. The absolute values of their mixing-matrix elements are shown below:

\[
|U^{aN}| = \begin{pmatrix}
.814 \pm .010 & .558 \pm .014 & .161 \pm .011 \\
.327 \pm .036 & .645 \pm .113 & .691 \pm .046 \\
.480 \pm .026 & .522 \pm .118 & .705 \pm .045 \\
\end{pmatrix}
\] (4)

\[
|U^{aI}| = \begin{pmatrix}
.813 \pm .010 & .558 \pm .014 & .164 \pm .011 \\
.485 \pm .022 & .500 \pm .041 & .718 \pm .041 \\
.322 \pm .032 & .663 \pm .031 & .676 \pm .043 \\
\end{pmatrix}
\] (5)

\[
|U^{bN}| = \begin{pmatrix}
.822 \pm .010 & .547 \pm .015 & .157 \pm .010 \\
.354 \pm .098 & .698 \pm .060 & .623 \pm .022 \\
.446 \pm .099 & .462 \pm .080 & .766 \pm .018 \\
\end{pmatrix}
\] (6)

\[
|U^{bI}| = \begin{pmatrix}
.822 \pm .010 & .547 \pm .015 & .157 \pm .010 \\
.348 \pm .096 & .694 \pm .058 & .631 \pm .025 \\
.451 \pm .093 & .469 \pm .078 & .760 \pm .021 \\
\end{pmatrix}, \quad \text{[global]} 
\] (7)

where \(N\) and \(I\) stand for normal and inverted hierarchies, respectively, and \(a, b\) are respectively the results taken from [2] and [3]. Since phases are not yet measured, only absolute values of the matrix elements are listed and compared. Note that the 22, 23, 32, 33 matrix elements depend on the unknown CP phase \(\delta\), resulting in relatively large errors and may therefore be somewhat unreliable.

We use the chi-square measure
\[
\chi^2 = \sum_{i=1}^{3} (|c_i| - |U_{ij}|)^2 / 2\sigma_{ij}^2 \quad \text{[chi2]} 
\] (8)
to gauge the goodness of a theoretically predicted mixing vector \( c = (c_1, c_2, c_3)^T \), where 
\[
|U_{ij}| \pm \sigma_{ij}^T
\]
is taken from one of the four globally fitted mixing matrices in (7). A factor of 2 is included in the definition to simulate the two degrees of freedom in a normalized column, but since it is the relative size of \( \chi^2 \) that will be invoked, it does not matter whether we drop that factor or not. The result of these fits for finite subgroups of \( SU(3) \) will be discussed in the next section. GAP is used to produce these results, but as remarked in the last section, the irreducible representations given by GAP are not necessarily unitary, so they have to be rendered unitary first before the mixing vectors \( c \) can be computed.

To have a standard for comparison, we list in Table I the chi-square of each column of the tri-bimaximal matrix. The absolute values of its third (bimaximal) column is 
\[
(0, 0.707, 0.707)^T \sim (0, 1, 1)^T,
\]
that of its second (trimaximal) columns is 
\[
(0.577, 0.577, 0.577)^T \sim (1, 1, 1)^T,
\]
and that of its first column is 
\[
(0.816, 0.408, 0.408)^T \sim (2, 1, 1)^T.
\]

| column | mixing vector | a\(N\) | a\(I\) | b\(N\) | b\(I\) |
|--------|---------------|--------|--------|--------|--------|
| 1      | \((2, 1, 1)^T\) | 4.12   | 9.82   | 3.65   | 4.00   |
| 2      | \((1, 1, 1)^T\) | 2.85   | 6.62   | 34.7   | 26.1   |
| 3      | \((0, 1, 1)^T\) | 110    | 119    | 126    | 122    |

Table I. The \( \chi^2 \)-values of the columns of a tri-bimaximal matrix

With the newly measured reactor angle, the third column having a \( \chi^2 \)-value over 100 is clearly unacceptable. The fit to the first and second columns are much more tolerable, but to some extent that may be due to the large errors associated with the unmeasured CP phase appearing in these two columns. In what follows we will reject all fits with \( \chi^2 > 7 \); those that survive in Table I are underlined for easy comparison.

IV. FINITE SUBGROUPS OF \( SU(3) \)

Finite subgroups of \( SU(3) \) with a three-dimensional irreducible representation and an order less than 512 are tabulated in [10], and reproduced here in Columns A and B of Table II. Column A gives the designation of a group in the Small Group Library of GAP; the first of the pair is the order of the group, and the second is the GAP-assigned number among groups of that order. Column B gives the popular name of the group, if there is one. If
the group is known under different names, then several of these may be given. Column C indicates whether the group contains $A_4$ or $S_4$ as a subgroup. A symbol $\circ$ indicates that it contains $A_4$, and a symbol $\bullet$ indicates that it contains $S_4$, which then must also contain $A_4$.

If $A_4$ is a subgroup, the group must contain the (unnormalized) trimaximal mixing vector $(1,1,1)^T$. If $S_4$ is a subgroup, then it must contain both $(1,1,1)^T$ and $(2,1,1)^T$, with the corresponding $\chi^2$ given in Table I. Since these $\chi^2$ may be considered as reasonable, we can use the group as a partial-symmetry group to build a neutrino mass matrix, with $(2,1,1)^T$ in the first column of its mixing matrix, or $(1,1,1)^T$ in the second column. This strategy has been used, for example, in refs. [6, 8]. The corresponding mass matrices are also discussed in Sec. V below. However, we must not use both of them simultaneously, for if we do so then the group becomes a full-symmetry group and the mixing matrix is automatically TBM, giving rise to a zero reactor angle.

Two other symbols also appear in Column C. The symbol $\times$ is used to indicate groups of odd order, which contains no element of order 2, and therefore will be ignored from now on. The symbol $p$ is used to identify groups that can only be partial-symmetry groups. These groups do not contain two mutually commuting order-2 elements so they can never serve as a full-symmetry group.

For each group of even order, we compute all its mixing vectors $c = (c_1, c_2, c_3)^T$ using the recipe discussed in Sec. 2, then its $\chi^2$. We reject cases where $\chi^2 > 7$ for all four global fits. Otherwise the values $|c_1|, |c_2|, |c_3|$ are listed in Column D, with the minimal $\chi^2$ among the four global fits appearing in Column E, and the corresponding global fit in Column F. The column that it fits, namely, $j$ of $|U_{ij}|$ in [8], appears in column G. For groups containing $A_4$ as a proper subgroup, the $(1,1,1)^T$ mixing vector is understood and will not be listed. For groups containing $S_4$ as a proper subgroup, neither the $(2,1,1)^T$ nor the $(1,1,1)^T$ appears explicitly. The symbol $-$ is used to indicate that there is no fit whatsoever with $\chi^2 < 7$.

The results appearing in Tables IIa, IIb, IIc can be summarized as follows. With so many groups and so many possible mixing vectors for each group, it is somewhat surprising that so few passes the experimental test. Besides the $(2,1,1)^T$ mixing of the first column for groups containing $S_4$, and the $(1,1,1)^T$ mixing of the second column for groups containing $A_4$, there are only a few that fit the first or the second column, albeit with a better $\chi^2$. The only group that really fits the third column is $\Delta(150)$, with a mixing vector $(.170, .607, .777)^T$, which gives rise to $\sin^2 2\theta_{13} = 0.11$ and $\sin^2 2\theta_{23} = 0.94$, in good agreement with direct
measurements [4]. The only other group that fits the third column with a $\chi^2 < 7$ is the group $\Delta(294)$, but it yields too small a reactor angle with $\sin^2 2\theta_{13} = 0.06$ and $\sin^2 2\theta_{23} = 0.97$.
| $A$    | $B$    | $C$         | $D$         | $E$  | $F$  | $G$ |
|--------|--------|-------------|-------------|------|------|-----|
| [12,3] | $A_4, T$ | $\circ$    | [.577, .577, .577] | 2.85 | $aN$ | 2   |
| [21,1] | $T_7$   | $\times$   |            |      |      |     |
| [24,12]| $S_4, O, \Delta(24)$ | $\bullet$ | [.816, .408, .408] | 3.65 | $bN$ | 1   |
| [27,3] | $\Delta(27)$ | $\times$   |            |      |      |     |
| [39,1] | $T_{13}$ | $\times$   |            |      |      |     |
| [48,3] | $\Delta(48)$ | $\circ$    |            |      |      |     |
| [54,8] | $\Delta(54)$ | $p$        |            |      |      |     |
| [57,1] | $T_{19}$ | $\times$   |            |      |      |     |
| [60,5] | $A_5, I, \Sigma(60)$ | $\circ$    |            |      |      |     |
| [75,2] | $\Delta(75)$ | $\times$   |            |      |      |     |
| [81,9] |        |            |            |      |      |     |
| [84,11]|        |            |            |      |      |     |
| [93,1] | $T_{31}$ | $\times$   |            |      |      |     |
| [96,64]| $\Delta(96)$ | $\bullet$ |            |      |      |     |
| [108,15]| $\Sigma(36\varphi)$ | $p$        |            |      |      |     |
| [108,22]| $\Delta(108)$ | $\circ$    |            |      |      |     |
| [111,1] | $T_{37}$ | $\times$   |            |      |      |     |
| [129,1] | $T_{43}$ | $\times$   |            |      |      |     |
| [147,1] | $T_{49}$ | $\times$   |            |      |      |     |
| [147,5] | $\Delta(147)$ | $\times$   |            |      |      |     |

Table IIa. Comparison of predictions of $SU(3)$ subgroups with experimental data.
| A        | B                      | C          | D          | E     | F   | G |
|----------|------------------------|------------|------------|-------|-----|---|
| [150, 5] | $\Delta(150)$         | $p$        | [.812, .332, .480] | .018  | $aN$| 1 |
|          |                        |            | [.812, .480, .332] | .086  | $aI$| 1 |
|          |                        |            | [.170, .607, .777] | 1.25  | $bN$| 3 |
| [156, 14]| $\circ$               |            |            |       |     |   |
| [162, 14]| $p$                   |            |            |       |     |   |
| [168, 42]| $\Sigma(168), PSL(3, 2)$ | $\bullet$ | [.815, .363, .452] | .267  | $bN$| 1 |
|          |                        |            | [.815, .452, .363] | .269  | $bI$| 1 |
| [183, 1 ]| $T_{61}$              | $\times$  |            |       |     |   |
| [189, 8 ]|                        | $\times$  |            |       |     |   |
| [192, 3 ]| $\Delta(192)$        | $\circ$   |            |       |     |   |
| [201, 1 ]| $T_{67}$              | $\times$  |            |       |     |   |
| [216, 88]| $\Sigma(72\varphi)$ | $p$        |            |       |     |   |
| [216, 95]| $\Delta(216)$        | $\bullet$ |            |       |     |   |
| [219, 1 ]| $T_{73}$              | $\times$  |            |       |     |   |
| [228, 11]|                        | $\circ$   |            |       |     |   |
| [237, 1 ]| $T_{79}$              | $\times$  |            |       |     |   |
| [243, 26]| $\Delta(243)$        | $\times$  |            |       |     |   |
| [273, 3 ]| $T_{91}$              | $\times$  |            |       |     |   |
| [273, 4 ]| $T'_{91}$             | $\times$  |            |       |     |   |
| [291, 1 ]| $T_{97}$              | $\times$  |            |       |     |   |
| [294, 7 ]| $\Delta(294)$        | $p$        | [.814, .460, .351] | 1.16  | $aI$| 1 |
|          |                        |            | [.814, .351, .460] | .312  | $bI$| 1 |
|          |                        |            | [.707, .500, .500] | 4.95  | $bI$| 2 |
|          |                        |            | [.122, .638, .760] | 5.80  | $bI$| 3 |

Table IIb. Comparison of predictions of $SU(3)$ subgroups with experimental data
### Table IIc. Comparison of predictions of SU(3) subgroups with experimental data

| $A$      | $B$      | $C$  | $D$      | $E$  | $F$ | $G$ |
|----------|----------|------|----------|------|-----|-----|
| [300, 43]| $\Delta(300)$ | $\circ$ |          |      |     |     |
| [309, 1]| $T_{103}$ | $\times$ |          |      |     |     |
| [324, 50]|          | $\circ$ |          |      |     |     |
| [327, 1]| $T_{109}$ | $\times$ |          |      |     |     |
| [336, 67]|          | $\circ$ |          |      |     |     |
| [351, 8]|          | $\times$ |          |      |     |     |
| [363, 2]| $\Delta(363)$ | $\times$ |          |      |     |     |
| [372, 11]|          | $\circ$ |          |      |     |     |
| [381, 1]| $T_{127}$ | $\times$ |          |      |     |     |
| [384, 567]| $\Delta(384)$ | $\bullet$ [.810, .312, .497] | [.810, .497, .312] | .188 | $aN$ | 1   |
| [399, 3]| $T_{133}$ | $\times$ |          |      |     |     |
| [399, 4]| $T'_{193}$ | $\times$ |          |      |     |     |
| [417, 1]| $T_{139}$ | $\times$ |          |      |     |     |
| [432, 103]| $\Delta(432)$ | $\circ$ |          |      |     |     |
| [444, 14]|          | $\circ$ |          |      |     |     |
| [453, 1]| $T_{151}$ | $\times$ |          |      |     |     |
| [471, 1]|          | $\times$ |          |      |     |     |
| [486, 61]| $\Delta(486)$ | $p$ [.804, .279, .525] | 1.41 | $aN$ | 1   |
| [489, 1]| $T_{163}$ | $\times$ |          |      |     |     |
| [507, 1]| $T_{169}$ | $\times$ |          |      |     |     |
| [507, 5]| $\Delta(507)$ | $\times$ |          |      |     |     |

V. MASS MATRICES

If $F$ and $G$ are the residual symmetries, then the invariant conditions $F^\dagger \overline{M}_e F = \overline{M}_e$ and $G^\dagger \overline{M}_\nu G = \overline{M}_\nu$ can be used to determine the form of these two left-handed mass matrices. In this section we will illustrate this procedure by determining the mass matrices of (groups containing) $S_4$, and the group $\Delta(150)$. 

11
For $S_4$ in the $F$-diagonal representation, $\overline{M}_e = \text{diag}(\alpha, \delta, \phi)$ is parametrized by three real numbers $\alpha, \delta, \phi$ which can be used to fit the squared masses of charged leptons. It is well known that $\overline{M}_\nu$ has a 2-3 and a magic symmetry parametrized by three complex parameters that can be used to fit the neutrino masses and Majorana phases.

For $\Delta(150)$, it turns out that $G$ is very complicated in the $F$-diagonal representation, but fairly simple in the representation given by GAP, so the mass matrices will be given in that representation. As a warm up, let us see what the mass matrices of $S_4$ look like in the GAP-representation when $S_4$ is taken both as a partial-symmetry group and a full-symmetry group.

In the GAP representation of $S_4$, $F$ and $G_i$ can be taken to be

$$F = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$G_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Since $\overline{M}_e$ is hermitian and $\overline{M}_\nu$ is symmetric, they can be parametrized in the following way:

$$\overline{M}_e = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta^* & \delta & \epsilon \\ \gamma^* & \epsilon^* & \phi \end{pmatrix}, \quad \overline{M}_\nu = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

where $\alpha, \delta, \phi$ are real and the rest of the parameters are generally complex. Putting in the invariant conditions, it is easy to see that $\gamma = \beta^*$, and $\phi = \delta = \alpha$. For $\overline{M}_e$, we are therefore left with one real parameter $\alpha$ and one complex parameter $\beta$, which can be used to fit the three real mass squares $m_e^2, m_\mu^2, m_\tau^2$. For $\overline{M}_\nu$, the invariant conditions for $G = G_1$ give $d = a$ and $e = -c$, for $G_2$ give $c = e = 0$, and for $G_3$ give $d = a$ and $e = c$. In each case when $S_4$ is treated as a partial-symmetry group, there are four parameters left which can be used to determine the three neutrino masses and Majorana phases, as well as the remaining mixing parameters. If $S_4$ is taken to be the full-symmetry group, when the equalities for $G_1, G_2, G_3$ are simultaneously satisfied, then the mixing is automatically tri-bimaximal, and
the neutrino mass matrix

\[
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

(12)

is parametrized by three parameters which can be used to determine the neutrino masses and Majorana phases.

For the partial-symmetry group \( \Delta(150) \) which gives rise to the desired reactor and atmospheric angles, the residual symmetries in the GAP representation can be taken to be

\[
M^{S_4}_\nu = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & f
\end{pmatrix}
\]

(13)

where \( \phi_5 = \exp(2\pi i/5) \) is the fifth root of unity. With the invariant conditions, we can derive the mass matrices in the GAP representation to be

\[
M_e = \begin{pmatrix}
\alpha & \beta & \beta^* \\
\beta^* & \alpha & \beta \\
\beta & \beta^* & \alpha
\end{pmatrix}
\]

\[
M^{\Delta(150)}_\nu = \begin{pmatrix}
a & b & e\phi_5^2 \\
b & a\phi_5 & e \\
e\phi_5^2 & e & f
\end{pmatrix}
\]

(14)

The charged-lepton mass matrix is identical that of \( S_4 \), and the neutrino mass matrix would become the neutrino mass matrix of the full-mixing \( S_4 \) if we put \( e \to 0 \) and \( \phi_5 \to 1 \). It has one more complex parameter \( e \) compared to the \( S_4 \) case because in this case the solar angle and the CP phase have to be fitted.

VI. SUMMARY

We have used the group theory of mixing to determine whether any of the finite subgroups of \( SU(3) \) up to order 511 can be a symmetry group of neutrino mixing. We conclude that none could be a full-symmetry group, but several may serve as a partial-symmetry group for column 1 or column 2 of the mixing matrix. The only group where the third column can be accommodated is \( \Delta(150) \), which yields \( \sin^2 2\theta_{13} = 0.11 \) and \( \sin^2 2\theta_{23} = 0.94 \), in good agreement with direct experimental measurements and global fits. The three-dimensional representation of \( \Delta(150) \) where this occurs is complex, so it may be difficult to produce a dynamical model in this case.
I am grateful to Profs. A. Hulpke and D. Pasechnik for their help in using GAP, and to
Prof. John McKay for discussions of finite group theory.

[1] C.S. Lam, Phys. Rev. Lett. 101 (2008), 121602; Phys. Rev. D78 (2008) 073015.
[2] D.V. Forero, M. Trtola, J.W.F. Valle, arXiv 1205.4018.
[3] G.L. Fogli, E. Lisi, A. Marrone, D. Montanino, A. Palazzo, A.M. Rotunno, arXiv1205.5254.
[4] The following results on $\theta_{13}$ and $\theta_{23}$ were reported in the Neutrino 2012 Conference in Kyoto,
June 3-9, 2012. M. Ishitsuka, ‘Double Chooz results’; S-B. Kim, ‘RENO results’; D. Dwyer,
‘Daya Bay results’; T. Nakaya, ‘Results from T2K’; R. Nichol, ‘Results from MINOS’.
[5] N. Qin, B-Q. Ma, Phys. Lett. B702 (2011) 143; Y-j. Zheng, B-Q. Ma, Eur. Phys. J. Plus
127(1) (2012) 7; Z-z. Xing, arXiv:1106.3244 X-G. He, A. Zee, arXiv:1106.435; D. Meloni,
arXiv:1107.0221 S. Morisi, K.M. Patel, E. Peinado, Phys. Rev. D84 (2011) 053002;
W. Chao, Y-j. Zheng, arXiv:1107.0738 S. Dev, S. Gupta, R.R. Gautam, Phys. Lett. B704
(2011) 527; P.S. Bhupal Dev, R.N. Mohapatra, M. Severson, arXiv:1107.2378 R.A. Toorop,
F. Feruglio, C. Hagedorn, arXiv:1107.3486 M-C Chen, K.T. Mahanthappa, arXiv:1107.3856
Y.H. Ahn, H-Y. Cheng, S. Oh, arXiv:1107.4549 S.F. King, C. Luhn, JHEP 1109(2011) 042;
Q-H. Cao, S. Khalil, E. Ma, H. Okada, arXiv:1108.0570 T. Schwetz, M. Tortola, J.W.F.
Valle, arXiv:1108.1376 Riazuddin, arXiv:1108.1469 T. Araki, C-Q. Geng, arXiv:1108.3175
S. Antusch, S.F. King, C. Luhn, M. Spinrath, arXiv:1108.4278 M-C. Chen, K.T. Mahanthappa,
A. Meroni, S.T. Petcov, arXiv:1109.0731 P.O. Ludl, S. Morisi, E. Peinado, Nucl. Phys.
B857 (2012) 411; G. Blankenburg, S. Morisi, JHEP 1201 (2012) 016; S. Verma,
Nucl. Phys. B854 (2012) 340; N. Okada, Q. Shafi, arXiv:1109.4963 A. Aranda, C. Bonilla,
Alma D. Rojas, Phys. Rev. D85 (2012) 036004; G-J. Ding, L.L. Everett, A.J. Stuart,
arXiv:1110.1688 M.J Baker, J. Bordes, H.M. Chan, S.T. Tsou, arXiv:1110.3951 K.N.
Deepthi, S. Gollu, R. Mohanta, arXiv:1111.2781 S.F. King, C. Luhn, arXiv:1112.1959
D.A. Eby, P.H. Frampton, arXiv:1112.2675 J. Heeck, W. Rodejohann, JHEP 1202 (2012)
094; S. Gupta, A.S. Joshipura, K.M. Patel, Phys. Rev. D85 (2012) 031903; P.S. Bhupal
Dev, B. Dutta, R.N. Mohapatra, M. Severson, arXiv:1202.4012 P.M. Ferreira, L. Lavoura,
arXiv:1202.4024 X. Zhang, B-Q. Ma, arXiv:1202.4258 I.K. Cooper, S.F. King, C. Luhn,
JHEP 1206 (2012) 130; X. Zhang, Y-j. Zheng, B-Q. Ma, Phys. Rev. D85 (2012) 097301; K.
Siyeon, arXiv:1203.1593; Z-z. Xing, arXiv:1203.1672; Y-L. Wu, arXiv:1203.2382; Y. BenTov, A. Zee, arXiv:1203.2671; H-J. He, X-J. Xu, arXiv:1203.2908; D. Meloni, arXiv:1203.3126; Y.H. Ahn, S.K. Kang, arXiv:1203.4185; B. Grinstein, M. Trott, arXiv:1203.4410; H. Fritzsch, arXiv:1203.4460; I.d.M. Varzielas, G.G. Ross, arXiv:1203.6636; C. Hagedorn, D. Meloni, arXiv:1204.0715; M. Fukugita, Y. Shimizu, M. Tanimoto, T.T. Yanagida, arXiv:1204.2389; Y. Farzan, E. Ma, arXiv:1204.4890; B. Brahmachari, A. Raychaudhuri, arXiv:1204.5619; S.F. King, arXiv:1205.0506; S. Zhou, arXiv:1205.0761; J. Barranco, D. Delepine, L. Lopez-Lozano, arXiv:1205.0859; S. Antusch, C. Gross, V. Maurer, C. Sluka, arXiv:1205.1051; C. Hagedorn, S.F. King, C. Luhn, arXiv:1205.3114; G. Altarelli, F. Feruglio, L. Merlo, E. Stamou, arXiv:1205.4670; F. Gonzalez Canales, A. Mondragon, M. Mondragon, arXiv:1205.4755; G. Altarelli, F. Feruglio, L. Merlo, arXiv:1205.5133; F. Bazzocchi, L. Merlo, arXiv:1205.5135; A. Meroni, S.T. Petcov, M. Spinrath, arXiv:1205.5241; M. J. Baker, J. Bordes, H.M. Chan, S.T. Tsou, arXiv:1206.0199; X. Zhang, B-Q. Ma, arXiv:1206.0519; A. Damanik, arXiv:1206.0987; E. Ma, A. Natale, A. Rashed, arXiv:1206.1570; S.M. Boucenna, S. Morisi, M. Tortola, J.W.F. Valle, arXiv:1206.2555; N. Haba, R/ Takahashi, arXiv:1206.2793; M. Gupta, G. Ahuja, arXiv:1206.3844; D.V. Ahluwalia, arXiv:1206.4779; A.G. Dias, A.C.B. Machado, C.C. Nishi, arXiv:1206.6362; S. Verma, arXiv:1206.6583; P.M. Ferreira, W. Grimus, L. Lavoura, P.O. Ludl, arXiv:1206.7072; G. Altarelli, F. Feruglio, I. Masina, L. Merlo, arXiv:1207.0587; Y.H. Ahn, S. Baek, P. Gondolo, arXiv:1207.1229; W. Grimus, L. Lavoura, arXiv:1207.1678; H.B. Benaoum, arXiv:1207.1967; Z-h. Zhao, arXiv:1207.2545; S.F. King, C. Luhn, A.J. Stuart, arXiv:1207.5741; Y. BenTov, X-G. He, A. Zee, arXiv:1208.1062; K. Siyeon, arXiv:1208.2645; R.N. Mohapatra, C.C. Nishi, arXiv:1208.2875.

[6] R.d.A. Toorop, F. Feruglio, C. Hagedorn, arXiv:1112.1340; D. Hernandez, A.Yu. Smirnov, arXiv:1204.0445; W. Rodejohann, H. Zhang, arXiv:1207.1225.

[7] www.gap-system.org.

[8] C.S. Lam, Phys. Rev. D83 (2011) 113002; and to be published.

[9] C.S. Lam, Phys. Rev. D74 (2006) 113004; Phys. Lett. B640 (2006) 260; Phys. Lett. B656 (2007) 193; Int. J. Mod. Phys. A23 (2008) 3371.

[10] P.O. Ludl, J. Phys. A43 (2010) 395204; Erratum-ibid. A44 (2011) 139501.