The most general fourth order theory of Gravity at low energy

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The Newtonian limit of the most general fourth order gravity is performed with metric approach in the Jordan frame with no gauge condition. The most general theory with fourth order differential equations is obtained by generalizing the $f(R)$ term in the action with a generic function containing other two curvature invariants: Ricci square $(R_{\alpha\beta}R^{\alpha\beta})$ and Riemann square $(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})$. The spherically symmetric solutions of metric tensor yet present Yukawa-like spatial behavior, but now one has two characteristic lengths. At Newtonian order any function of curvature invariants gives us the same outcome like the so-called Quadratic Lagrangian of Gravity. From Gauss-Bonnet invariant one have the complete interpretation of solutions and the absence of a possible third characteristic length linked to Riemann square contribution. From analysis of metric potentials, generated by point-like source, one has a constraint condition on the derivatives of $f$ with respect to scalar invariants.

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I. INTRODUCTION

In recent years, the effort to give a physical explanation to the today observed cosmic acceleration [1] has attracted a good amount of interest in $f(R)$-gravity, where $f$ is a generic function of Ricci scalar $R$, considered as a viable mechanism to explain the cosmic acceleration by extending the geometric sector of field equations without the introduction of dark matter and dark energy. Other issues, od astrophysical nature, as the observed Pioneer anomaly problem [2] can be framed into the same approach [3] and then, apart the cosmological dynamics, a systematic analysis of such theories urges at short scale and in the low energy limit.

While it is very natural to extend the theory of General Relativity (GR) to theories with additional geometric degrees of freedom, recent attempts focused on the old idea [4] of modifying the gravitational Lagrangian in a purely metric framework, leading to higher-order field equations. Due to the increased complexity of the field equations in this framework, the main body of works dealt with some formally equivalent theories, in which a reduction of the order of the field equations was achieved by considering the metric and the connection as independent objects [5].

In addition, other authors exploited the formal relationship to scalar-tensor theories to make some statements about the weak field regime [6], which was already worked out for scalar-tensor theories [7]. Also a Post-Newtonian parameterization with metric approach in the Jordan Frame has been considered [8].

In this paper, we show the theory of Gravity induced by a most general fourth order theory obtained by using all curvature invariants. Precisely we show for a generic $f(X,Y,Z)$-theory, where $X = R$, $Y = R_{\alpha\beta}R^{\alpha\beta}$ and $Z = R_{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ with $R_{\alpha\beta}$ Ricci tensor and $R_{\alpha\beta\gamma\delta}$ Riemann tensor, the modifications to standard gravitational mechanics at Newtonian order. From the usual small velocity and weak field limit approach [9] we find the field equations and since the differential equations are liner we obtain a general solution of metric tensor by Green functions method and demonstrate that any $f(X,Y,Z)$-theory corresponds to so-called Quadratic Lagrangian ($f(X,Y) = a_1 R + a_2 R^2 + a_3 R_{\alpha\beta}R^{\alpha\beta}$). Initially the metric tensor is spherically symmetric and time depending, but in this limit the dependence is missing (we need the post-Newtonian order to fix a possible time dependence). So we recover also a partial outcome about the Birkhoff theorem.

The metric potentials have two characteristic lengths depending on the value of derivatives of $f$ with respect to curvature invariants and only in GR are equal. The general solutions are calculated for a point-like source and since the theory is linear, the gravitational potential can be obtained for any matter distribution.

With this general approach and by adding other curvature invariants to action, this paper summarizes and generalizes the topics of previously papers [9-11].

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II. THE NEWTONIAN LIMIT OF $f(X,Y,Z)$-GRAVITY

Let us start with a general class of fourth order theories given by the action

$$A = \int d^4x \sqrt{-g} \left[ f(X,Y,Z) + \mathcal{L}_m \right]$$

(1)

where $f$ is an unspecified function of curvature invariants $X$, $Y$ and $Z$. The term $\mathcal{L}_m$ is the minimally coupled ordinary matter contribution. In the metric approach, the field equations are obtained by varying (1) with respect to $g_{\mu\nu}$. We get

$$H_{\mu\nu} = f_X R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - f_{X;\mu\nu} + g_{\mu\nu} \Box f_X + 2f_Y R_{\mu \rho}^\alpha R^\alpha_{\rho \nu} - 2[f_Y R^\alpha_{(\mu ;\nu)}]_\alpha + \Box [f_Y R_{\mu \nu}] + [f_Y R_{\alpha \beta}]^{\alpha \beta} g_{\mu \nu} + 2f_Z R_{\mu \alpha \beta \gamma} R^{\alpha \beta \gamma} - 4[f_Z R_{\mu \alpha \beta \gamma}]_{\alpha \beta} = \mathcal{X} T_{\mu \nu}$$

(2)

where $T_{\mu \nu} = -\frac{1}{2} \mathcal{X} (\mathcal{X} + \frac{4}{3} f_{TT})^{-1}$ is the energy-momentum tensor of matter, $f_X = \frac{df}{dX}$, $f_Y = \frac{df}{dY}$, $f_Z = \frac{df}{dZ}$, $\Box = \Box^\alpha_{\alpha \beta}$, and $\mathcal{X} = 8\pi G^1$. The conventions for Ricci’s tensor is $R_{\mu \nu} = R^\sigma_{\mu \sigma \nu}$, while for the Riemann tensor is $R^{\alpha \beta \mu \nu} = \Gamma^\alpha_{\beta \nu,\mu} + \ldots$. The affinities are the usual Christoffel’s symbols of the metric: $\Gamma^\alpha_{\beta \gamma} = \frac{1}{2} g^{\alpha \sigma} (g_{\alpha \gamma,\beta} + g_{\beta \gamma,\alpha} - g_{\alpha \beta,\gamma})$. The adopted signature is $(+ - - -)$ (see for the details [12]). The trace of field equations (2) is the following

$$H = g^{\alpha \beta} H_{\alpha \beta} = f_X R + 2f_Y R_{\alpha \beta} R^{\alpha \beta} + 2f_Z R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} - 2f + \Box [3f_X + f_Y R] + 2((f_Y + 2f_Z) R^{\alpha \beta})_{\alpha \beta} = \mathcal{X} T$$

(3)

where $T = T^\sigma_{\sigma \nu}$ is the trace of energy-momentum tensor.

The paradigm of Newtonian limit is starting from a develop of metric tensor (and of all additional quantities in the theory) with respect to dimensionless quantity $v$ and considering only first term of $tt$- and $ij$-component of metric tensor $g_{\mu \nu}$ (for details see [10]). The develop of metric tensor is as follows

$$g_{\mu \nu} \sim \left(1 + g^{(2)}_{tt}(t,x) + g^{(4)}_{tt}(t,x) + \ldots + g^{(3)}_{ti}(t,x) + \ldots - \delta_{ij} + g^{(2)}_{ij}(t,x) + \ldots \right)$$

(4)

The set of coordinates adopted is $x^\mu = (t,x^1,x^2,x^3)$. The curvature invariants $X$, $Y$, $Z$ become

$$\begin{align*}
X & \sim X^{(2)} + X^{(4)} + \ldots \\
Y & \sim Y^{(4)} + Y^{(6)} + \ldots \\
Z & \sim Z^{(4)} + Z^{(6)} + \ldots
\end{align*}$$

(5)

The function $f$ can be developed as

$$f(X,Y,Z) \sim f(0) + f_X(0)X^{(2)} + \frac{1}{2} f_{XX}(0)X^{(2)} + f_X(0)X^{(4)} + f_Y(0)Y^{(4)} + f_Z(0)Z^{(4)} + \ldots$$

(6)

and analogous relations for partial derivatives of $f$ are obtained. From lowest order of field equations (2) we have

$$f(0) = 0$$

(7)

1 Here we use the convention $c = 1$.
2 The Greek index runs between 0 and 3; the Latin index between 1 and 3.
Not only in \( f(R) \)-gravity \[10, 13\] but also in \( f(X,Y,Z) \)-theory a missing cosmological component in the action \[11\] implies that the space-time is asymptotically Minkowskian. The equations \[2\] and \[3\] at \( O(2) \) - order become\(^3\)

\[
\begin{aligned}
H^{(2)}_{tt} &= f_X(0)R^{(2)}_{tt} - [f_Y(0) + 4f_Z(0)]X^{(2)} - [f_{XX}(0) + \frac{f_Y(0)}{2}]X^{(2)} = \mathcal{X} T^{(0)}_{tt} \\
H^{(2)}_{ij} &= f_X(0)R^{(2)}_{ij} - [f_Y(0) + 4f_Z(0)]X^{(2)} + \frac{f_Y(0)}{2}X^{(2)} \delta_{ij} + [f_{XX}(0) + \frac{f_Y(0)}{2}]X^{(2)} \delta_{ij} - f_{XX}(0)X^{(2)} \cdot X^{(2)} + \\
&+ [f_Y(0) + 4f_Z(0)]R^{(2)}_{m,j,m} + f_Y(0)R^{(2)}_{m,j,m} = 0 \\
H^{(2)} &= -f_X(0)X^{(2)} - 3f_{XX}(0) + 2f_Y(0) + 2f_Z(0)X^{(2)} = \mathcal{X} T^{(0)}
\end{aligned}
\]

where \( \Delta \) is the Laplacian in the flat space. By introducing the quantities

\[
\begin{aligned}
m_1^2 &= -\frac{f_X(0)}{3f_{XX}(0) + 2f_Y(0) + 2f_Z(0)} \\
m_2^2 &= \frac{f_X(0)}{f_Y(0) + 4f_Z(0)}
\end{aligned}
\]

we get three differential equations for curvature invariant \( X^{(2)} \), \( tt \)- and \( ij \)-component of Ricci tensor \( R^{(2)}_{\mu \nu} \)

\[
\begin{aligned}
(\Delta - m_2^2)R^{(2)}_{tt} + \left[ \frac{m_2^2}{2} - \frac{m_2^2 + 2m_2}{6m_1^2} \Delta \right]X^{(2)} &= -\frac{m_2^2X^{(2)}}{f_X(0)} T^{(0)}_{tt} \\
(\Delta - m_2^2)R^{(2)}_{ij} + \left[ \frac{m_2^2}{6m_1^2} - \frac{m_2^2 + 2m_2}{6m_1^2} \Delta \right] \delta_{ij}X^{(2)} &= 0 \\
(\Delta - m_1^2)X^{(2)} &= \frac{m_2^2X^{(2)}}{f_X(0)} T^{(0)}
\end{aligned}
\]

We note that in the case of \( f(R) \)-theory we obtained a characteristic length \( (m_1^{-1}) \) on which the Ricci scalar evolves, but in \( f(X,Y,Z) \)-theory we have an additional characteristic length \( (m_2^{-1}) \) on which the Ricci tensor evolves. The solution for curvature invariant \( X^{(2)} \) in third line of \[13\] is

\[
X^{(2)}(t, x) = \frac{m_1^2X^{(2)}}{f_X(0)} \int d^3x' G_1(x, x') T^{(0)}(t, x')
\]

where \( G_1(x, x') \) is the Green function of field operator \( \Delta - m_1^2 \). The solution for \( g^{(2)}_{tt} \), by remembering \( R^{(2)}_{tt} = \frac{1}{2} \Delta g^{(2)}_{tt} \), is the following

\[
g^{(2)}_{tt}(t, x) = \frac{1}{2\pi} \int d^3x' \left[ \frac{2}{|x - x'|} \frac{X^{(2)}}{f_X(0)} \left( \frac{m_2^2X^{(2)}}{f_X(0)} t, x'' \right) - \frac{f_X(0)}{6f_X(0)} X^{(2)}(t, x'') + \frac{m_2^2 - m_1^2}{6} X^{(2)}(t, x'') \right]
\]

where \( G_2(x, x') \) is the Green function of field operator \( \Delta - m_2^2 \). The expression \[12\] is the "modified" gravitational potential (here we have a factor 2) for \( f(X,Y,Z) \)-gravity. The solution for the gravitational potential \( g^{(2)}_{tt} / 2 \) has a Yukawa-like behaviors \[10\] depending by a characteristic lengths on whose it evolves.

The \( ij \)-component of Ricci tensor in terms of metric tensor \[4\] is

\[
R^{(2)}_{ij} = \frac{1}{2} \delta^{(2)}_{ij, mm} - \frac{1}{2} \delta^{(2)}_{ij, mj} - \frac{1}{2} \delta^{(2)}_{ij, im} - \frac{1}{2} \delta^{(2)}_{ij, ij} + \frac{1}{2} \delta^{(2)}_{mm, ij}
\]

\(^3\) We used the properties: \( R_{\alpha \beta}^{\alpha \beta} = \frac{1}{2} \Box R \) and \( R_{\mu \alpha}^{\alpha \beta} = R_{\mu \alpha}^{\alpha \beta} - \Box R_{\mu \nu} \).
and if we use the harmonic gauge condition \( g^{\alpha \beta} \Gamma^\mu_{\alpha \beta} = 0 \) the \( g^{(2)}_{ij} \) becomes \( R_{ij} (2) \). \( R_{ij} (2) |_{HG} = \frac{1}{2} g^{(2)}_{ij,mm} = \frac{1}{2} \Delta g^{(2)}_{ij} \).

The general solution for \( g^{(2)}_{ij} \) from \( (10) \), in the harmonic gauge, is

\[
g^{(2)}_{ij} |_{HG} = \frac{1}{2\pi} \int d^3x' d^3x' \frac{g^{(2)} G_2(x', x'')}{|x - x'|} \left[ \frac{m_1^2 - m_2^2}{3m_1^2} \delta_{ij} - \left( \frac{m_2^2}{2} - \frac{m_1^2 + 2m_2^2}{6m_1^2} \triangle x'' \right) \delta_{ij} \right] X^{(2)}(x'')
\]

(14)

While if we hypothesize \( g^{(2)}_{ij} = 2\psi \delta_{ij} \), we have \( R^{(2)}_{ij} = \Delta \psi \delta_{ij} + (\psi - \phi)_{,ij} \) and the second field equation of \( (10) \) becomes

\[
\begin{align*}
\frac{\Delta \psi}{d^2x'^2} &= \int d^3x' G_2(x, x') \left( \frac{m_2^2}{2} - \frac{m_1^2 + 2m_2^2}{6m_1^2} \triangle x' \right) X^{(2)}(x') \\
(\psi - \phi)_{,ij} &= \frac{m_1^2 - m_2^2}{3m_1^2} \int d^3x' G_2(x, x') X^{(2)}_{,i'j'}(x')
\end{align*}
\]

(15)

Then the general solution for \( g^{(2)}_{ij} \) from \( (10) \), without gauge condition and by using the first line of \( (15) \), is

\[
g^{(2)}_{ij} = 2\psi \delta_{ij} = -\frac{\delta_{ij}}{2\pi} \int d^3x' d^3x'' \frac{G_2(x', x'')}{|x - x'|} \left( \frac{m_2^2}{2} - \frac{m_1^2 + 2m_2^2}{6m_1^2} \triangle x'' \right) X^{(2)}(x'')
\]

(16)

and the second line of \( (15) \) is only a constraint condition for metric potentials. In fact from its trace we have

\[
\Delta (\phi - \psi) = \frac{m_1^2 - m_2^2}{3m_1^2} \int d^3x' G_2(x, x') \triangle x' X^{(2)}(x')
\]

(17)

and we can affirm that only in GR the metric potentials \( \phi \) and \( \psi \) are equals.

### III. CONSIDERATIONS ABOUT POINT-LIKE SOLUTION

Let us consider a point-like source with mass \( M \). Then we have \( T^{(0)}_{tt}(t, x) = T^{(0)}_{,tt}(t, x) = M \delta(x) \), and if we choose \( m_1^2 > 0 \) and \( m_2^2 > 0 \), the Green functions \( G_i \) become \( G_i(x, x') = -\frac{1}{4\pi} e^{-m_i|x - x'|} \). The curvature invariant \( X^{(2)} \) \( (14) \) and the metric potentials \( \phi \) \( (12) \) and \( \psi \) \( (10) \) are

\[
X^{(2)} = -\frac{r_g m_1^2 e^{-m_1|x|}}{f_X(0)} |x|
\]

(18)

\[
\phi = -\frac{GM}{f_X(0)} \left[ \frac{1}{|x|} + \frac{e^{-m_1|x|}}{3|x|} - \frac{4e^{-m_2|x|}}{3|x|} \right]
\]

(19)

\[
\psi = -\frac{GM}{f_X(0)} \left[ \frac{1}{|x|} - \frac{e^{-m_1|x|}}{3|x|} - \frac{2e^{-m_2|x|}}{3|x|} \right]
\]

(20)

where \( r_g = 2GM \) is the Schwarzschild radius. The modified gravitational potential by \( f(R) \)-theory is further modified by the presence of functions of \( R_{\alpha \beta} R^{\alpha \beta} \) and \( R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} \). The curvature invariant \( X^{(2)} \) (the Ricci scalar) presents a massive propagation and when \( f(X, Y, Z) \rightarrow f(R) \) we find the mass definition \( m_2^2 = -f'(R = 0)/3f'''(R = 0) \) \( (10), (11) \) and propagation mode with \( m_2 \) disappear. Obviously the expressions \( (19) \) and \( (20) \) satisfy the constraint

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4 We choose a system of isotropic coordinates.
FIG. 1: Plot of metric potential $\phi$ (19). $m_2 = \xi m_1$ and $m_1 = .1$ (dotted line), $m_1 = \xi m_2$ and $m_2 = .1$ (dashed line). The behavior of GR is shown by the solid line. The dimensionless quantity $\xi$ runs between $0 \div 10$ with step 2. The dimension of $m_1$ and $m_2$ is the inverse of length. We set $f_X(0) = 1$.

FIG. 2: Plot of metric potential $\psi$ (20). $m_2 = \xi m_1$ and $m_1 = .1$ (dotted line), $m_1 = \xi m_2$ and $m_2 = .1$ (dashed line). The behavior of GR is shown by the solid line. The dimensionless quantity $\xi$ runs between $0 \div 10$ with step 2. The dimension of $m_1$ and $m_2$ is the inverse of length. We set $f_X(0) = 1$.

condition (17). In FIGs. 1 and 2 we report the spatial behavior of metric potentials for some values interval of parameters $m_1$ and $m_2$.

The same outcome can be obtained by considering the so-called Quadratic Lagrangian $\mathcal{L} = \sqrt{-g}(a_1 R + a_2 R^2 + a_3 R_{\alpha\beta}R^{\alpha\beta})$ where $a_1$, $a_2$ and $a_3$ are constants. In this case [11] we found two characteristic lengths $\left| \frac{a_2}{a_1} \right|^{1/2}$ and the Newtonian limit of theory implied as solution the equations (19) and (20). We can affirm, then, the Newtonian limit of any $f(X,Y,Z)$-theory can be reinterpreted by introducing the Quadratic Lagrangian and the coefficients have to satisfy the following relations

$$a_1 = f_X(0), \quad a_2 = \frac{1}{2} f_{XX}(0) - f_Z(0), \quad a_3 = f_Y(0) + 4f_Z(0)$$ (21)

A first considerations about (21) is regarding the characteristic lengths induced by $f(X,Y,Z)$-theory. The second length $m_2^{-1}$ is originated from the presence, in the Lagrangian, of Ricci and Riemann tensor square, but also a theory containing only Ricci tensor square could show the same outcome (it is successful replacing the coefficients $a_i$ of Quadratic Lagrangian or renaming the function $f$). Obviously the same is valid also with the Riemann tensor square alone. Then a such modification of theory enables a massive propagation of Ricci Tensor and, as it is well known in the literature, a substitution of Ricci Scalar with any function of Ricci scalar enables a massive propagation of Ricci scalar. We can, then, affirm that an hypothesis of Lagrangian containing any function of only Ricci scalar
and Ricci tensor square is not restrictive and only the experimental constraints can fix the arbitrary parameters.

A second consideration is starting from the Gauss–Bonnet invariant defined by the relation $G_{GB} = X^2 - 4Y + Z$ [14]. In fact the induced field equations satisfy in four dimensions the following condition

$$H_{\mu\nu}^{GB} = H_{\mu\nu}^X - 4H_{\mu\nu}^Y + H_{\mu\nu}^Z = 0$$

(22)

and by substituting them at Newtonian level ($H_{tt}^Z \sim -4\Delta R^{(2)}_{tt}$) in the equations (2) we find the field equations (ever at Newtonian Level) of Quadratic Lagrangian.

A third and last consideration is about the solutions (19) and (20). When we perform the limit in the origin $|x| = 0$ we don’t have the divergency. In fact we find

$$\lim_{|x|\to0} \phi = \frac{m_1 - 4m_2}{3}, \quad \lim_{|x|\to0} \psi = -\frac{m_1 + 2m_2}{3}$$

(23)

and only if we remove in the action (1) the dependence on the Ricci square or Riemann square we get the known divergence of GR. For a physical interpretation of solution (19) and (20), When we perform the limit in the origin $|x| = 0$ we don’t have the divergency. In fact we find

$$f_{XX}(0) + f_Y(0) + 2f_Z(0) < 0$$

(24)

In the case of $f(R)$-gravity ($f_Y(0) = f_Z(0) = 0$) we reobtain the same condition among the first and second derivatives of $f$ with respect to curvature invariants

IV. CONCLUSIONS

In this paper the theory of Gravity induced by a most general fourth order theory obtained by using all curvature invariants has been considered. By adding these curvature invariants, this paper summarized and generalized the topics of previously papers [9–11]. In fact for a generic $f(X,Y,Z)$-theory, at Newtonian level, it is successful considering only the so-called Quadratic Lagrangian. All contributions to field equation due by curvature invariant Riemann square can be expressed by other two curvature invariants (Ricci tensor square and Ricci scalar square) via Gauss-Bonnet invariant.

The spherically symmetric solutions of metric tensor at $O(2)$-order show a Yukawa-like dependence only by two characteristic lengths and not by three (because we have three curvature invariants in the action). No gauge condition has been considered and the solution of $ij$-component of metric tensor is general. Is general also the solution of $tt$-component of metric tensor, since it is gauge free (but only at Newtonian order).

Furthermore, generally it has been shown that for a $f(X,Y,Z)$-gravity, but the same is valid also for $f(R)$-gravity, the metric potentials are not equal. Only in the limit $f \to R$ we obtain the outcome of GR. This aspect with the consequences of Birkhoff and Gauss theorem are the principal differences between a fourth order gravity and GR.

The general solutions are calculated for a point-like source and since the theory is linear, the gravitational potential can be obtained for any matter distribution. The metric potentials don’t have the divergency in $|x| = 0$ for a point-like source, and by requiring a right physical interpretation of solution we get a constraint on the derivatives of $f$ with respect to curvature invariants. The constraint condition is compatible with respect to one obtained for $f(R)$-gravity. Besides a such class of theory have free parameters and only the experimental evidence can fix them.

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