FINITE $p$-GROUPS OF CLASS TWO WITH A LARGE MULTIPLE HOLOMORPH

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Abstract. Let $G$ be any group. The quotient group $T(G)$ of the multiple holomorph by the holomorph of $G$ has been investigated for various families of groups $G$. In this paper, we shall take $G$ to be a finite $p$-group of class two for any odd prime $p$, in which case $T(G)$ may be studied using certain bilinear forms. For any $n \geq 4$, we exhibit examples of $G$ of order $p^n + \binom{n}{2}$ such that $T(G)$ contains a subgroup isomorphic to $GL_n(F_p) \times GL(\binom{n}{2} - n)(F_p)$.

For finite $p$-groups $G$, the prime factors of the order of $T(G)$ which were known so far all came from $p(p-1)$. Our examples show that the order of $T(G)$ can have other prime factors as well. In fact, we can embed any finite group into $T(G)$ for a suitable choice of $G$.

1. Introduction

Let $G$ be any group, and write $S(G)$ for the group of permutations on the set $G$, where we compose maps from left to right. Consider the right regular representation $\rho : G \to S(G)$ defined by

$$x^{\rho(y)} = xy \text{ for all } x, y \in G.$$  

Let $N_{S(G)}(\cdot)$ denote the normalizer operation in $S(G)$, and put

$$\text{Hol}(G) = N_{S(G)}(\rho(G)), \quad \text{NHol}(G) = N_{S(G)}(N_{S(G)}(\rho(G))),$$

which are called the holomorph and multiple holomorph of $G$, respectively. It is well-known that isomorphic regular subgroups of $S(G)$ are conjugates of each other, and since

$$N_{S(G)}(\vartheta^{-1} \rho(G) \vartheta) = \vartheta^{-1} N_{S(G)}(\rho(G)) \vartheta = \vartheta^{-1} \text{Hol}(G) \vartheta$$

holds for any $\vartheta \in S(G)$, we easily deduce that the quotient

$$T(G) = \text{NHol}(G) / \text{Hol}(G)$$


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acts regularly on the set
\[ H(G) = \{ \text{regular subgroups } N \text{ of } S(G) \text{ which are} \]
\[ \text{isomorphic to } G \text{ and satisfy } N_{S(G)}(N) = \text{Hol}(G) \}. \]

We have the alternative description
\[ H(G) = \{ \text{normal regular subgroups of } \text{Hol}(G) \text{ isomorphic to } G \} \]
when \( G \) is finite, which is the case of interest in this paper.

The group \( T(G) \) was first considered in [Mil08] by G. A. Miller, who determined the structure of \( T(G) \) for finite abelian groups \( G \). Later in [Mil51] W. H. Mills computed the structure of \( T(G) \) for finitely generated abelian groups \( G \), which was redone in [CDV17] by an approach using commutative rings. More recently, T. Kohl [Koh15] determined \( T(G) \) for dihedral and generalized quaternion groups \( G \). Since his work, the structure of \( T(G) \) has attracted more attention and has been determined for other new families of groups \( G \) in the past years, such as groups of squarefree order [Tsa20], centerless perfect groups [CDV18], and almost simple groups [Tsa19]. We remark that [CDV18] and [Tsa19] only treated finite groups, but finiteness may be dropped by [Tsa21].

It turns out that \( T(G) \) is elementary 2-abelian for all of the families of groups \( G \) mentioned above. But there also exist groups \( G \) for which \( T(G) \) is not elementary 2-abelian, and most of the examples known so far came from finite \( p \)-groups with \( p \) an odd prime. For example, finite \( p \)-groups of class at most \( p - 1 \) were considered in [Car18, Tsa20], and finite split metacyclic \( p \)-groups in [Tsa22]. It was shown that under suitable hypotheses, the order of \( T(G) \) is divisible by \( p - 1 \) or \( p \). These examples made us wonder whether the order of \( T(G) \) can have prime factors lying outside of \( p(p-1) \), which was the motivation of this paper. We are able to show that the answer is affirmative, and in fact \( T(G) \) can be made arbitrarily large. Our main result is:

**Theorem 1.1.** Let \( p \) be any odd prime and let \( n \geq 4 \) be any integer.

There exists a finite \( p \)-group \( G \) of class two of order \( p^{n+\binom{n}{2}} \) such that \( T(G) \) is a semidirect product of an elementary abelian subgroup of order \( p^{\binom{n}{2} + \binom{n+1}{2}} \)

by a subgroup which is isomorphic to
\[ F_p^{(\binom{n}{2})-n \times n} \ltimes \left( \text{GL}_n(F_p) \times \text{GL}_{(\binom{n}{2})-n}(F_p) \right), \] (1.1)

where the semidirect product in (1.1) is given by
\[ Q^{(A,M)} = M^{-1} Q A \]
for any \( Q \in F_p^{(\binom{n}{2})-n \times n}, A \in \text{GL}_n(F_p), \) and \( M \in \text{GL}_{(\binom{n}{2})-n}(F_p) \).
Since every finite group embeds into $GL_N(\mathbb{F}_p)$ whenever $N$ is large enough, this implies that:

**Corollary 1.2.** Let $p$ be any odd prime and let $H$ be any finite group. For all sufficiently large integers $n$, there exists a finite $p$-group $G$ of class two of order $p^{n+\binom{n}{2}}$ such that $H$ embeds into $T(G)$.

**Remark 1.3.** Let $G$ be one of the groups of Theorem 1.1. Since every pair of regular subgroups in $\mathcal{H}(G)$ normalize each other, our construction yields a large clique in the normalizing graph of $G$ (see [CS22, Section 7]). Equivalently, every pair chosen among the group operations on $G$ associated to the elements of $\mathcal{H}(G)$ yields a bi-skew brace on $G$ [Chi19, Car20], that is, the set of these operations forms a brace block on $G$ [Koc22].

Throughout this paper, we shall use the following notation.

**Notation.** For any group $G$, we write:

- $\exp(G) =$ the exponent of $G$;
- $G' =$ the derived subgroup of $G$;
- $Z(G) =$ the center of $G$;
- $\text{Frat}(G) =$ the Frattini subgroup of $G$;
- $\text{Aut}_c(G) =$ the subgroup of $\text{Aut}(G)$ consisting of the automorphisms which induce the identity on $G/Z(G)$;
- $\text{Aut}_z(G) =$ the subgroup of $\text{Aut}(G)$ consisting of the automorphisms which induce the identity on $Z(G)$.

For any $x, y \in G$, we also write

$$x^y = y^{-1}xy$$ and $$[x, y] = x^{-1}x^y.$$ 

Let us recall that in any group $G$ of class 2, we have

$$(xy)^d = x^dy^d[y, x]^{\binom{d}{2}}$$ for all $x, y \in G$ and $d \in \mathbb{N}$.

We shall use this identity frequently in some of the calculations.

Finally, the symbol $p$ shall always denote an odd prime.

Here is a brief outline of this paper. In Section 2, we shall describe a technique, due to the first-named author [Car18], which allows us to study normal regular subgroups $N$ of $\text{Hol}(G)$ via bilinear forms

$$\Delta : G/Z(G) \times G/G' \rightarrow Z(G)$$

when $G$ is a finite $p$-group of class two. For the purpose of computing $T(G)$, we only want the $N$ that are isomorphic to $G$. In Section 3, we shall discuss the isomorphism class of $N$ in terms of the corresponding bilinear form $\Delta$ when $G' = Z(G)$. This extra assumption allows us to define the notions of symmetric and anti-symmetric on $\Delta$, which make things significantly simpler. In Section 4, we shall study the structure
of $T(G)$ in connection with bilinear forms when $\text{Aut}(G) = \text{Aut}_c(G)$. It turns out that $T(G)$ decomposes as a semidirect product

$$T(G) = \mathcal{S} \rtimes \mathcal{S'},$$

where loosely speaking the part $\mathcal{S}$ comes from symmetric forms and $\mathcal{S'}$ comes from anti-symmetric forms. In Section 5, we shall specialize our findings to a family of groups $G$ constructed in [Car16]. These groups $G$ satisfy $G' = Z(G) = \text{Frat}(G)$ so that the study of the bilinear forms $\Delta$ in question reduces to linear algebra. We shall show that the structure of $T(G)$ is given as in Theorem 1.1 for these groups $G$ under suitable conditions.

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2. MULTIPLE HOLOMORPH VIA BILINEAR FORMS

Let us first recall the following result from [CDV18, Theorem 5.2] and [Car18, Theorem 1.2].

**Theorem 2.1.** Let $G$ be any finite group.

The following data are equivalent.

1. A normal regular subgroup $N$ of $\text{Hol}(G)$.
2. An anti-homomorphism $\gamma : G \to \text{Aut}(G)$ such that
   $$\gamma(x^\beta) = \gamma(x)^\beta \quad \text{for all } x \in G \text{ and } \beta \in \text{Aut}(G).$$
3. A group operation $\circ$ on $G$ such that
   $$\text{Aut}(G) \leq \text{Aut}(G, \circ) \quad \text{and} \quad (xy) \circ z = (x \circ z) \cdot z^{-1} \cdot (y \circ z) \quad \text{for all } x, y, z \in G,$$
   where $z^{-1}$ denotes the inverse of $z$ in $G$. Note that the identity of $(G, \circ)$ coincides with that of $G$.

Moreover, these data are related as follows.

(i) $N = \{\gamma(x)\rho(x) : x \in G\}$.
(ii) $x \circ y = x^{\gamma(y)}y$ for all $x, y \in G$.
(iii) $(G, \circ) \simeq N$ via $x \mapsto \gamma(x)\rho(x)$.

**Definition 2.2.** In the situation of Theorem 2.1, we shall refer to the group $(G, \circ)$ as the circle group.

Theorem 2.1 considers all normal regular subgroups of $\text{Hol}(G)$, but only those which are isomorphic to $G$ correspond to elements of $T(G)$. The connection is given by the following result of [CDV18, Lemma 4.2], which is a rephrasing of a result of [Mil08].

**Proposition 2.3.** Let $G$ be any finite group. Let $N$ be a normal regular subgroup of $\text{Hol}(G)$ with corresponding operation $\circ$ on $G$ such that $N$ is isomorphic to $G$.

For any $\vartheta \in \text{NHol}(G)$ with $1^\vartheta = 1$, the following are equivalent.

(a) $\vartheta : G \to (G, \circ)$ is a group isomorphism.
(b) $\vartheta$ conjugates $\rho(G)$ to $N$, namely $\rho(G)^\vartheta = N$.

For finite $p$-groups of class two, we further have the following linear technique from [Car18, Proposition 2.2] by the first-named author. We shall employ this approach because many of the calculations reduce to linear algebra and hence are much easier.

**Theorem 2.4.** Let $G$ be any finite $p$-group of class two.

There is a one-to-one correspondence between the following:

1. Normal regular subgroups $N$ of $\text{Hol}(G)$ such that
   \[ \gamma(x) \in \text{Aut}_c(G) \cap \text{Aut}_z(G) \text{ for all } x \in G \] 
   \[ \gamma(x) \in \text{Aut}_c(G) \cap \text{Aut}_z(G) \text{ for all } x \in G \] 
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   for the corresponding anti-homomorphism $\gamma : G \to \text{Aut}(G)$.

2. Bilinear forms $\Delta : G/Z(G) \times G/G' \to Z(G)$ such that
   \[ \Delta(x^\beta, y^\beta) = \Delta(x, y)^\beta \text{ for all } x, y \in G \text{ and } \beta \in \text{Aut}(G). \]

The correspondence is given by
   \[ \Delta(x, y) = x^{-1}x^\gamma(y) \text{ for all } x, y \in G, \]
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   \[ \Delta(x, y) = x^{-1}x^\gamma(y) \text{ for all } x, y \in G, \]

   and in this case the corresponding operation $\circ$ on $G$ is given by
   \[ x \circ y = x^{\gamma(y)} y = xy\Delta(x, y) \text{ for all } x, y \in G. \]
   \[ x \circ y = x^{\gamma(y)} y = xy\Delta(x, y) \text{ for all } x, y \in G. \]
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   \[ x \circ y = x^{\gamma(y)} y = xy\Delta(x, y) \text{ for all } x, y \in G. \]

For simplicity, we are writing $\Delta(x, y)$ instead of $\Delta(xZ(G), yG')$.

In what follows, let us assume that $G$ is a finite $p$-group of class two so that Theorem 2.4 applies. For brevity, let us put
   \[ B = \{ \text{bilinear forms } \Delta : G/Z(G) \times G/G' \to Z(G) \text{ satisfying (2.2)} \}, \]
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and note that $B$ is an abelian group under pointwise multiplication in $Z(G)$. Since $G$ has class two, the commutator $[\cdot, \cdot]$ on $G$ is bilinear and we may use it construct elements of $B$ as follows.

**Example 2.5.** For each $c \in \mathbb{Z}$, we have the power bilinear form
   \[ \Delta_{[c]} : G/Z(G) \times G/G' \to Z(G); \Delta_{[c]}(x, y) = [x, y]^c, \]
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   \[ \Delta_{[c]} : G/Z(G) \times G/G' \to Z(G); \Delta_{[c]}(x, y) = [x, y]^c, \]

which clearly satisfies (2.2).

More generally, we can replace the power map $[\cdot, \cdot] \mapsto [\cdot, \cdot]^c$ on $G'$ by any endomorphism on $G'$, as it is suggested by the following.

**Example 2.6.** For each $\sigma \in \text{End}(G')$, we have the bilinear form
   \[ \Delta_{\sigma} : G/Z(G) \times G/G' \to Z(G); \Delta_{\sigma}(x, y) = [x, y]^\sigma, \]
   \[ \Delta_{\sigma} : G/Z(G) \times G/G' \to Z(G); \Delta_{\sigma}(x, y) = [x, y]^\sigma, \]
   \[ \Delta_{\sigma} : G/Z(G) \times G/G' \to Z(G); \Delta_{\sigma}(x, y) = [x, y]^\sigma, \]
   \[ \Delta_{\sigma} : G/Z(G) \times G/G' \to Z(G); \Delta_{\sigma}(x, y) = [x, y]^\sigma, \]

which satisfies (2.2) if and only if
   \[ \sigma \bar{\beta} = \bar{\beta} \sigma \text{ for all } \beta \in \text{Aut}(G), \]
   \[ \sigma \bar{\beta} = \bar{\beta} \sigma \text{ for all } \beta \in \text{Aut}(G), \]
   \[ \sigma \bar{\beta} = \bar{\beta} \sigma \text{ for all } \beta \in \text{Aut}(G), \]
   \[ \sigma \bar{\beta} = \bar{\beta} \sigma \text{ for all } \beta \in \text{Aut}(G), \]

where $\bar{\beta}$ denotes the automorphism on $G'$ induced by $\beta$. 
Consider $\Delta \in B$ and let $\circ$ denote the corresponding operation on $G$. We know that $\Delta$ gives rise to an element of $T(G)$ if and only if $(G, \circ)$ is isomorphic to $G$. In this case, by Proposition 2.3, the corresponding element of $T(G)$ is the coset $\vartheta \text{Hol}(G)$, where

$$\vartheta : G \to (G, \circ)$$

denotes any choice of isomorphism, namely a bijection such that

$$(xy)^\vartheta = x^\vartheta \circ y^\vartheta = x^\vartheta y^\vartheta \Delta(x^\vartheta, y^\vartheta) \text{ for all } x, y \in G. \quad (2.4)$$

We end this section with the following example from [Car18].

**Example 2.7.** For each $c \in \mathbb{Z}$, consider $\Delta_{[c]} \in B$ and let

$$x \circ_{[c]} y = xy\Delta_{[c]}(x, y) = xy[x, y]^c$$

denote the corresponding group operation on $G$. In the case that $c \not\equiv -\frac{1}{2} \pmod{p}$, in other words $2c + 1 \not\equiv 0 \pmod{p}$, we know from [Car18] that $(G, \circ_{[c]})$ is isomorphic to $G$. Explicitly, we may obtain an isomorphism via the power map

$$\vartheta_d : G \to (G, \circ_{[c]}); \quad x^{\vartheta_d} = x^d,$$

where $d \in \mathbb{Z}$ is any integer such that

$$d(2c + 1) \equiv 1 \pmod{\exp(G')},$$

that is $c \equiv -\frac{d-1}{2d} \pmod{\exp(G')}$. The bijectivity of $\vartheta_d$ is clear. For any $x, y \in G$, we have

$$(xy)^{\vartheta_d} = (xy)^d$$

$$= x^d y^d [x, y]^{\frac{d(d-1)}{2d}}$$

$$= x^d y^d [x^d, y^d]^{-\frac{d-1}{2d}}$$

$$= x^{\vartheta_d} y^{\vartheta_d} \Delta_{[c]}(x^{\vartheta_d}, y^{\vartheta_d}),$$

where the second equality holds because $G$ has class two. We then see from (2.4) that $\vartheta_d$ is indeed an isomorphism. The set

$$\{\vartheta_d \text{Hol}(G) : d \in \mathbb{Z} \text{ coprime to } p\},$$

consisting of the cosets defined by these $\vartheta_d$ is precisely the cyclic subgroup of $T(G)$ of order $(p - 1)p^{r-1}$ constructed in [Car18, Proposition 3.1], where $p^r = \exp(G/Z(G))$. We note that $\exp(G/Z(G)) = \exp(G')$ because $G$ is assumed to have class two.

### 3. Isomorphism class of the circle group

Throughout this section, we shall assume that $G$ is a finite $p$-group of class two such that $G' = Z(G)$. We are then looking at the set

$$B = \{\text{bilinear forms } \Delta : G/G' \times G/G' \to G' \text{ satisfying (2.2)}\},$$
which is an abelian group under pointwise multiplication in $G'$. Since both of the arguments come from $G/G''$, the notions of symmetric and anti-symmetric are defined. In particular, we have the subgroup

$$S = \{\Delta \in B : \Delta(y, x) = \Delta(x, y) \text{ for all } x, y \in G\}$$

consisting of the symmetric forms, and similarly the subgroup

$$S' = \{\Delta \in B : \Delta(y, x) = \Delta(x, y)^{-1} \text{ for all } x, y \in G\}$$

consisting of the anti-symmetric forms. It is clear that $S$ and $S'$ intersect trivially. Since every $\Delta \in B$ may be decomposed as

$$\Delta(x, y) = (\Delta(x, y)\Delta(y, x))^{1/2} \cdot (\Delta(x, y)\Delta(y, x)^{-1})^{1/2},$$

we see that $B = S \times S'$ is a direct product of $S$ and $S'$.

Given any $\Delta \in B$ with corresponding operation $\circ$ on $G$, we wish to determine when $(G, \circ)$ is isomorphic to $G$. By the next proposition and corollary, we only need to consider anti-symmetric forms.

**Proposition 3.1.** Let $\Delta_1, \Delta_2 \in B$ be such that $\Delta_1^{-1}\Delta_2$ is symmetric, and let $\circ_1, \circ_2$, respectively, denote their corresponding operations on $G$ as given by (2.3).

The groups $(G, \circ_1)$ and $(G, \circ_2)$ are isomorphic.

**Proof.** Put $\Delta = \Delta_1^{-1}\Delta_2$, which is symmetric by assumption. Define

$$\vartheta : (G, \circ_1) \to (G, \circ_2); \quad x^\vartheta = x\Delta(x, x)^{1/2}.$$

For any $x, y \in G$, we have

$$(x \circ_1 y)^\vartheta = (xy\Delta_1(x, y))^\vartheta$$

$$= xy\Delta_1(x, y)\Delta(xy, xy)^{1/2}$$

$$= xy\Delta_1(x, y)\Delta(x, x)^{1/2}\Delta(y, y)^{1/2}\Delta(x, y)$$

$$= x\Delta(x, x)^{1/2}y\Delta(y, y)^{1/2}\Delta_2(x, y)$$

$$= x^{\vartheta} \circ_2 y^{\vartheta}.$$

It follows that $\vartheta$ is a homomorphism. Since any $x \in \ker(\vartheta)$ must lie in $G'$ and $\vartheta$ is the identity on $G'$, we deduce that $\vartheta$ is injective, and hence bijective because $G$ is finite. This proves that $\vartheta$ is an isomorphism. □

Taking $\Delta_1$ to be the trivial bilinear form, whose corresponding operation $\circ$ coincides with that of $G$, in particular we have:

**Corollary 3.2.** Let $\Delta \in S$ and let $\circ$ denote the corresponding operation on $G$ as given by (2.3).

The groups $G$ and $(G, \circ)$ are isomorphic.
In contrast to symmetric forms, the anti-symmetric forms are much harder to understand, except those considered in Example 2.5. Below, for the sake of completeness, let us just prove several facts that can be established in general.

Let \( \Delta \in B \) and let \( \circ \) denote the corresponding operation on \( G \). For any \( x \in G \), let us write \( x^{-1} \) for its inverse in the group \( (G, \circ) \). A simple calculation shows that

\[
 x^{-1} \Delta(x, x).
\]

For any \( x, y \in G \), their commutator in \( (G, \circ) \) is then given by

\[
 [x, y]_\circ = x^{-1} \circ y^{-1} \Delta(x, y).\]

This implies that we have the inclusions

\[
 (G, \circ)' \leq G' = Z(G) \leq Z(G, \circ). \quad (3.2)
\]

Hence, the group \( (G, \circ) \) is either abelian or has class two, and for it to be isomorphic to \( G \), the inclusions in (3.2) must both be equalities.

**Proposition 3.3.** Let \( \Delta \in S' \) and let \( \circ \) denote the corresponding operation on \( G \) as given in (2.3).

(a) \( (G, \circ) \) is abelian if and only if \( \Delta = \Delta_{[-1/2]} \).

(b) \( (G, \circ)' = G' \) if and only if the image of \( \Delta_{[1/2]} \Delta \) generates \( G' \).

(c) \( Z(G) = Z(G, \circ) \) if and only if \( \Delta_{[1/2]} \Delta \) is non-degenerate, that is, we have \( \Delta_{[1/2]} \Delta(x, y) = 1 \) for all \( y \in G \) only when \( x \in G' \).

**Proof.** Since \( \Delta \) is anti-symmetric, the commutator (3.1) becomes

\[
 [x, y]_\circ = [x, y] \Delta(x, y)^2, \quad \text{or equivalently} \quad [x, y]_{\circ}^{1/2} = \Delta_{[1/2]} \Delta(x, y).
\]

Since \( (G, \circ) \) and \( G \) have the same identity element, it follows that

\[
 (G, \circ) \text{ is abelian } \iff \forall x, y \in G : [x, y]_\circ = 1
 \iff \forall x, y \in G : [x, y] \Delta(x, y)^2 = 1
 \iff \forall x, y \in G : \Delta(x, y) = [x, y]^{-1/2},
\]

which proves (a). Observe that the operation \( \circ \) coincides with that of \( G \) inside \( G' \). From the inclusion \( (G, \circ)' \leq G' \), we then see that

\[
 (G, \circ)' = G' \iff [x, y]_\circ \text{ with } x, y \in G \text{ generate } G'
 \iff [x, y]_{\circ}^{1/2} \text{ with } x, y \in G \text{ generate } G'
 \iff (\Delta_{[1/2]} \Delta)(x, y) \text{ with } x, y \in G \text{ generate } G',
\]
and this yields (b). Finally, notice that $Z(G,\circ)$ is the set consisting of all of the elements $x \in G$ for which

$$[x, y]_\circ = 1,$$

or equivalently $(\Delta_{1/2} \Delta)(x, y) = 1$

holds for all $y \in G$. Since $Z(G) \leq Z(G,\circ)$ and $G' = Z(G)$, we deduce that (c) indeed holds. □

**Example 3.4.** For each $\sigma \in \text{End}(G')$, consider $\Delta_\sigma$ from Example 2.6, and assuming that it does satisfy (2.2), let

$$x \circ_\sigma y = xy \Delta_\sigma(x, y) = xy[x, y]^\sigma$$

denote the corresponding operation on $G$. Note that

$$(\Delta_{1/2} \Delta)(x, y) = [x, y]^{1/2 + \sigma} = [x, y]^{1/2(1+2\sigma)}.$$ 

We then deduce from Proposition 3.3 the following:

(a) If $1 + 2\sigma \not\in \text{Aut}(G')$, then $(G, \circ_\sigma)'$ is a proper subgroup of $G'$, and so $(G, \circ_\sigma)$ cannot be isomorphic to $G$.

(b) If $1 + 2\sigma \in \text{Aut}(G')$, then $(G, \circ_\sigma)' = G'$ holds, and

$$\forall y \in G : (\Delta_{1/2} \Delta)(x, y) = 1 \implies \forall y \in G : [x, y]^{1/(1+2\sigma)} = 1 \implies x \in Z(G) = G',$$

whence $Z(G) = Z(G,\circ)$ holds as well.

However, even though we have equalities

$$(G, \circ_\sigma)' = G' = Z(G) = Z(G,\circ), \quad (3.3)$$

it is unclear at this stage for which $\sigma$ we have that $(G,\circ_\sigma)$ and $G$ are actually isomorphic. The only thing that can be said in general is that they are always isoclinic.

Recall that two groups $\Gamma_1, \Gamma_2$ are isoclinic if there are isomorphisms

$$\varphi : \Gamma_1/Z(\Gamma_1) \rightarrow \Gamma_2/Z(\Gamma_2), \quad \psi : \Gamma_1' \rightarrow \Gamma_2'$$

which are compatible with commutators, namely

$$[\gamma^\varphi, \delta^\varphi] = [\gamma, \delta]^\psi$$

for all $\gamma, \delta \in \Gamma_1$.

We end this section with the next proposition.

**Proposition 3.5.** Let $\sigma \in \text{End}(G')$ be such that $1 + 2\sigma \in \text{Aut}(G')$ and also assume that $\Delta_\sigma$ satisfies (2.2).

The groups $G$ and $(G, \circ_\sigma)$ are isoclinic.

**Proof.** We know that the equalities in (3.3) hold. Take

$$\varphi : G/Z(G) \rightarrow (G, \circ)/Z(G,\circ), \quad \psi : G' \rightarrow (G, \circ)',$$

respectively, to be the identity map and the automorphism $1 + 2\sigma$. It is clear from (2.3) that the operation $\circ$ coincides with that of $G$ when
taken modulo $Z(G)$ as well as when restricted to $G'$. Thus, both $\varphi$ and $\psi$ are isomorphisms. For any $x, y \in G$, it follows from (3.1) that
\[ [x^\varphi, y^\varphi]_0 = [x, y]_0 = [x, y]^{1+2\sigma} = [x, y]^{\psi}, \]
and this proves the claim. \hfill \-box

4. Group structure of the multiple holomorph

Throughout this section, we shall assume that $G$ is a finite $p$-group of class two such that $G' = Z(G)$ and $\text{Aut}(G) = \text{Aut}_c(G)$. These two conditions imply that $\text{Aut}(G) = \text{Aut}_z(G)$ also, and so every normal regular subgroup $N$ of $\text{Hol}(G)$ satisfies (2.1). Moreover, the condition (2.2) is now vacuous, so we are simply looking at the set
\[ B = \{ \text{bilinear forms } \Delta : G/G' \times G/G' \to G' \}. \]
As explained in Section 2, each $\Delta \in B$, with corresponding operation $\circ$ say, gives rise to an element $\vartheta : \text{Hol}(G)$ of $T(G)$ whenever there is an isomorphism $\vartheta : G \to (G, \circ)$. By Theorem 2.4, every element of $T(G)$ arises in this way, and the associated bilinear form is unique.

Now, consider an arbitrary element of $T(G)$ represented as $\vartheta : \text{Hol}(G)$, where $\vartheta : G \to (G, \circ)$ is an isomorphism for the operation $\circ$ corresponding to some $\Delta \in B$. The isomorphism $\vartheta$ induces via restriction isomorphisms
\[ G/G' \to (G, \circ)/(G, \circ)', \quad G' \to (G, \circ)'. \]
But from (3.2), we must have the equalities
\[ (G, \circ)' = G' = Z(G) = Z(G, \circ). \]
Also from (2.3), the operation $\circ$ coincides with that of $G$ when taken modulo $G'$ and when restricted to $G'$. Thus $\vartheta$ induces automorphisms
\[ \text{res}_c(\vartheta) : G/G' \to G/G', \quad \text{res}_z(\vartheta) : G' \to G'. \quad (4.1) \]
Note that they do not depend on the choice of $\vartheta$, for if $\vartheta' : G \to (G, \circ)$ is another isomorphism, then $\vartheta' \vartheta^{-1} \in \text{Aut}(G)$, and by assumption
\[ \text{Aut}(G) = \text{Aut}_c(G) = \text{Aut}_z(G). \]
This gives us a well-defined map
\[ \text{res} : T(G) \to \text{Aut}(G/G') \times \text{Aut}(G'); \quad \text{res} = (\text{res}_c, \text{res}_z), \]
which is clearly a homomorphism.

Observe that the range of $\text{res}$ acts naturally on $B$ via
\[ \Delta^{(\alpha, \beta)}(x, y) = \Delta(x^{\alpha^{-1}}, y^{\alpha^{-1}})^{\beta} \text{ for } \Delta \in B, \alpha \in \text{Aut}(G/G'), \beta \in \text{Aut}(G'). \]
The next proposition shows that the group operations of $T(G)$ and $B$ are related by this action.
**Proposition 4.1.** For $i = 1, 2$, let $\Delta_i \in B$ with corresponding operation $\circ_i$ be such that there exists an isomorphism $\vartheta_i : G \to (G, \circ_i)$.

The bijection $\vartheta_1 \vartheta_2 : G \to (G, \circ)$ is an isomorphism for the operation $\circ$ corresponding to the bilinear form $\Delta_1^{\text{res}(\vartheta_2)} \Delta_2$.

**Proof.** Put $\Delta = \Delta_1^{\text{res}(\vartheta_2)} \Delta_2$, which is explicitly given by

$$\Delta(x, y) = \Delta_1(x^{\vartheta_1 \vartheta_2}, y^{\vartheta_1 \vartheta_2})$$

for all $x, y \in G$.

Clearly $\vartheta_1 \vartheta_2$ is a bijection, and we shall use the criterion (2.4) to check that $\vartheta_1 \vartheta_2$ is a homomorphism. For any $x, y \in G$, we have

$$(xy)^{\vartheta_1 \vartheta_2} = (x^{\vartheta_1} y^{\vartheta_1} \Delta_1(x^{\vartheta_1}, y^{\vartheta_1}))^{\vartheta_2} = x^{\vartheta_1 \vartheta_2} y^{\vartheta_1 \vartheta_2} \Delta_2(x^{\vartheta_1 \vartheta_2}, y^{\vartheta_1 \vartheta_2}) \Delta_1(x^{\vartheta_1}, y^{\vartheta_1})^{\vartheta_2} = x^{\vartheta_1 \vartheta_2} y^{\vartheta_1 \vartheta_2} \Delta(x^{\vartheta_1 \vartheta_2}, y^{\vartheta_1 \vartheta_2}),$$

and the claim now follows. $\square$

Recall that $B = S \times S'$ decomposes as a direct product of the subgroups of symmetric and anti-symmetric forms. Let $S$ and $S'$ denote the subsets of $T(G)$ consisting of all the elements $\vartheta \text{ Hol}(G)$ which arise from symmetric and anti-symmetric forms, respectively. In fact, both $S$ and $S'$ are subgroups of $T(G)$ in view of Proposition 4.1.

**Proposition 4.2.** We have $S = \ker(\text{res})$.

**Proof.** Let $\vartheta \text{ Hol}(G) \in T(G)$, where $\vartheta : G \to (G, \circ)$ is any isomorphism for the operation $\circ$ defined by the corresponding $\Delta \in B$.

If $\Delta$ is symmetric, then we may take $\vartheta$ to be

$$x^{\vartheta} = x^{\Delta(x, x)^{1/2}}$$

for all $x \in G$

by the proof of Proposition 3.1 and clearly $\vartheta \text{ Hol}(G) \in \ker(\text{res})$.

If $\vartheta \text{ Hol}(G) \in \ker(\text{res})$, then

$$[x, y] = [x, y]^{\vartheta} = [x^{\vartheta}, y^{\vartheta}]_\circ = [x, y]_\circ$$

for all $x, y \in G$,

and thus $\Delta$ is symmetric by (3.1). $\square$

Although $B = S \times S'$ decomposes as a direct product, it is not true in general that $T(G) = S \times S'$. But we do have a semidirect product.

**Proposition 4.3.** We have the semidirect product decomposition

$$T(G) = S \rtimes S'.$$

**Proof.** Let $\vartheta \text{ Hol}(G) \in T(G)$, where $\vartheta : G \to (G, \circ)$ is any isomorphism for the operation $\circ$ defined by the corresponding $\Delta \in B$. By Theorem 2.4, the choice of $\Delta \in B$ is unique.

If $\vartheta \text{ Hol}(G) \in S \cap S'$, then $\Delta$ must be the trivial bilinear form, and $\vartheta \in \text{Aut}(G)$ because $\circ$ coincides with the operation of $G$ in this case.
The above shows that $S \cap S' = 1$. Since $S$ is a normal subgroup of $T(G)$ by Proposition 4.2, it remains to show that $T(G) = SS'$. Since $B = S \times S'$, we may decompose

$$\Delta = \Delta_0^{-1} \Delta_2,$$

where $\Delta_0 \in S$ and $\Delta_2 \in S'$. Let $\circ_2$ denote the operation corresponding to $\Delta_2$. Since $\Delta^{-1} \Delta_2 = \Delta_0$ is symmetric, by the proof of Proposition 3.1 we know that

$$(G, \circ) \to (G, \circ_2); \ x \mapsto x\Delta_0(x, x)^{1/2}$$

is an isomorphism. Composing with $\vartheta$, this implies that

$$\vartheta_2 : G \to (G, \circ_2); \ x^{\vartheta_2} = x^{\vartheta} \Delta_0(x^{\vartheta}, x^{\vartheta})^{1/2}$$

is an isomorphism. Put

$$\Delta_1 = \Delta_0^{-\text{res}(\vartheta_2)^{-1}},$$

which clearly lies in $S$, and by Proposition 3.1 we know that

$$\vartheta_1 : G \to (G, \circ_1); \ x^{\vartheta_1} = x\Delta_1(x, x)^{1/2}$$

is an isomorphism for the operation $\circ_1$ corresponding to $\Delta_1$. We have

$$\vartheta \text{ Hol}(G) = \vartheta_1 \vartheta_2 \text{ Hol}(G) = \vartheta_1 \text{ Hol}(G) \cdot \vartheta_2 \text{ Hol}(G) \in SS',$$

by Proposition 4.1 because

$$\vartheta = \vartheta_1 \vartheta_2 \text{ Hol}(G) = \vartheta_1 \text{ Hol}(G) \cdot \vartheta_2 \text{ Hol}(G) \in S S'$$

is in turn given by

$$\Delta = \Delta_0^{-\text{res}(\vartheta_2)^{-1}} \cdot \Delta_1^{\text{res}(\vartheta_2)} \Delta_2,$$

and this completes the proof. \vspace{1em}

Let us compute the conjugation action of $S'$ on $S$. Consider $\vartheta \in S$ and $\vartheta' \in S'$, with corresponding operations $\circ$ and $\circ'$ respectively, such that there exist isomorphisms

$$\vartheta : G \to (G, \circ) \quad \text{and} \quad \vartheta' : G \to (G, \circ').$$

By Proposition 4.1 the bilinear forms corresponding to

$$\vartheta^{\circ^{-1}} \text{ Hol}(G) \quad \text{and} \quad \vartheta \vartheta' \text{ Hol}(G),$$

respectively, are equal to

$$(\Delta')^{-\text{res}(\vartheta)^{-1}} \quad \text{and} \quad \Delta^{\text{res}(\vartheta')} \Delta'.$$

It follows that the bilinear form corresponding to

$$\vartheta^{\circ^{-1}} \vartheta \vartheta' \text{ Hol}(G) = \vartheta^{\circ^{-1}} \text{ Hol}(G) \cdot \vartheta \vartheta' \text{ Hol}(G)$$

is in turn given by

$$(\Delta')^{-\text{res}(\vartheta)^{-1}} \cdot (\Delta^{\text{res}(\vartheta')} \Delta') = \Delta^{\text{res}(\vartheta')} ,$$

where the equality holds because $\text{res}(\vartheta) = 1$ by Proposition 4.2 and $B$ is an abelian group. This means that the action of $S'$ on $S$ agrees with that of $\text{res}(S')$ on $S$. Summarizing, we obtain the following theorem:

**Theorem 4.4.** We have an isomorphism

$$T(G) \simeq S \times \text{res}(S'),$$
Proof. Recall from Corollary 3.2 that every symmetric form gives rise to an element of $T(G)$. Since $S = \ker(\text{res})$ by Proposition 4.2, we see from Proposition 4.1 that $S \simeq S$. Since $S$ intersects trivially with $S'$, we also have $S' \simeq \text{res}(S')$. The claim now follows from Proposition 4.3 and the above calculation.

5. A special family of finite $p$-groups of class two

In this last section, we shall consider the finite $p$-groups of class two constructed in [Car16]. Specifically, let $n \geq 4$ and consider

$$G = \left\langle x_1, x_2, \ldots, x_n : [[x_i, x_j], x_k] = 1 \text{ for all } 1 \leq i, j, k \leq n, \quad x_i^p = \prod_{j<k} [x_j, x_k]^{d_i(j,k)} \text{ for all } 1 \leq i \leq n \right\rangle.$$  

This group has order $p^{n+\binom{n}{2}}$ and has the following properties:

(a) $G' = Z(G) = \text{Frat}(G)$;

(b) $G/G'$ is elementary abelian of order $p^n$ having

$$x_1G', x_2G', \ldots, x_nG'$$

as an $\mathbb{F}_p$-basis;

(c) $G'$ is elementary abelian of order $p\binom{n}{2}$ having

$$[x_j, x_k] \text{ with } 1 \leq j < k \leq n$$

as an $\mathbb{F}_p$-basis.

Since $G$ has class two and $G'$ has exponent $p$, the $p$th power map

$$\pi : G/G' \to G'; \quad (xG')^p = x^p$$

is a well-defined homomorphism, and the $n \times \binom{n}{2}$ matrix

$$(d_i(j,k)), \text{ where } 1 \leq i \leq n \text{ and } 1 \leq j < k \leq n$$

is precisely the matrix of $\pi$ with respect to the bases (5.1) and (5.2).

As shown in [Car16], we may choose $(d_i(j,k))$ in a way such that

$$\text{Aut}(G) = \text{Aut}_c(G),$$

so that the discussion in Section 4 applies. In particular, we have

$$T(G) \simeq S \rtimes \text{res}(S')$$

by Theorem 4.4. For the symmetric part $S$, it is clear that

$$S \simeq \mathbb{F}_p^{\binom{n}{2}}\left(\binom{n+1}{2}\right).$$  

(5.3)
For the anti-symmetric part \( \text{res}(S') \), there is at least a cyclic subgroup of order \( p - 1 \) by Example 2.7. This shows that \( T(G) \) has a subgroup isomorphic to the semidirect product \( \mathbb{F}_p^n \rtimes \mathbb{F}_p^\times \). This is the precisely the content of [Car18, Theorem 5.5], which deals with the same family of groups we are considering here.

However, the anti-symmetric part \( \text{res}(S') \), which is a subgroup of \( \text{Aut}(G/G') \times \text{Aut}(G') \cong \text{GL}_n(\mathbb{F}_p) \times \text{GL}_2(\mathbb{F}_p) \), can potentially be much larger than \( \mathbb{F}_p^\times \). We shall investigate its structure in more detail in the subsequent subsections. For now, let us just show that the anti-symmetric forms are precisely those from Example 2.6 for the groups \( G \) under consideration.

**Proposition 5.1.** We have \( S' = \{ \Delta_\sigma : \sigma \in \text{End}(G') \} \).

*Proof.* Note that we may identify \( G' \) with the exterior square of \( G/G' \) by associating each commutator \([x, y]\) to the wedge product \( x \wedge y \). For any anti-symmetric form \( \Delta \in S' \), by the universal property of the exterior square, we then see that there exists \( \sigma \in \text{End}(G') \) such that \( \Delta = \Delta_\sigma \) commutes, meaning that \( \Delta = \Delta_\sigma \). This proves the claim since the \( \Delta_\sigma \) are clearly all anti-symmetric. \( \Box \)

5.1. **The circle group of anti-symmetric forms.** Let \( \sigma \in \text{End}(G') \) and consider the \( \Delta_\sigma \in S' \) defined in Example 2.6. Let \( x \circ y = x \circ_\sigma y = xy\Delta_\sigma(x, y) = xy[x, y]^\sigma \) denote the corresponding operation on \( G \). We wish to know when the circle group \( (G, \circ) \) is isomorphic to \( G \). We may assume that \( 1 + 2\sigma \in \text{Aut}(G') \), for otherwise \( (G, \circ) \) is not isomorphic to \( G \) by Example 3.4. Then \( (G, \circ)' = G' = Z(G) = Z(G, \circ) \) as noted in 3.3. Although \( (G, \circ) \) need not be isomorphic to \( G \), it still admits a presentation that is analogous to \( G \), as follows.
First of all, the following are immediate from the definition of $\diamondsuit$. In fact, we see from (2.3) that (1) and (2) hold more generally, while (3) holds simply because $\Delta_\sigma$ is anti-symmetric.

1. For all $x, y \in G$, we have $x \diamondsuit y \equiv xy \pmod{G'}$.
2. For all $x, y \in G'$, we have $x \diamondsuit y = xy$.
3. For all $x \in G$ and $k \in \mathbb{Z}$, we have $x \diamondsuit k = x^k$, where $x^k$ denotes the $k$th power of $x$ in the group $(G, \diamondsuit)$.

This implies that there is no need to distinguish the operation $\diamondsuit$ from that of $G$ on the quotient $G/G'$, on the subgroup $G'$, and for arbitrary powers of elements of $G$.

Now, from (3.1) commutators in $(G, \diamondsuit)$ are given by

$$[x, y]_\diamondsuit = [x, y] \Delta_\sigma(x, y) \Delta_\sigma(y, x)^{-1} = [x, y]^{1+2\sigma}, \quad (5.4)$$

and we also know that $(G, \diamondsuit)$ has class two. It then follows that

$$(G, \diamondsuit) = \left\langle x_1, x_2, \ldots, x_n : [[x_i, x_j], x_k]_\diamondsuit = 1 \text{ for all } 1 \leq i, j, k \leq n, \quad x_i^p = \prod_{j<k} [x_j, x_k]_\diamondsuit^{d_{i,j,k}} \text{ for all } 1 \leq i \leq n, \right\rangle,$$

where explicitly, the $n \times \binom{n}{2}$ matrix

$$(d_{i,j,k}^\diamondsuit), \text{ where } 1 \leq i \leq n \text{ and } 1 \leq j < k \leq n$$

is the matrix of $\pi(1 + 2\sigma)^{-1}$ with respect to the bases (5.1) and (5.2). The relations clearly hold in $(G, \diamondsuit)$, and we have equality because the presentation on the right defines a group of the same order as $(G, \diamondsuit)$.

Since $G'$ may be identified with the exterior square of $G/G'$ via the association $[x, y] \mapsto x \wedge y$, each $\alpha \in \text{Aut}(G/G')$ induces

$$\hat{\alpha} \in \text{Aut}(G') \text{ via } [x, y]^{\hat{\alpha}} = [x^\alpha, y^\alpha] \text{ for all } x, y \in G.$$

Using this notation and (4.1), we can prove:

**Proposition 5.2.** Let $\alpha \in \text{Aut}(G/G')$. The following are equivalent.

1. There is an isomorphism $\vartheta : G \to (G, \diamondsuit)$ with $\text{res}_c(\vartheta) = \alpha$.
2. The equality $\alpha^{-1} \pi \hat{\alpha} = \pi(1 + 2\sigma)^{-1}$ holds.

Moreover, in this case, we have

$$\text{res}(\vartheta) = (\text{res}_c(\vartheta), \text{res}_z(\vartheta)) = (\alpha, \hat{\alpha}(1 + 2\sigma)).$$

**Remark 5.3.** In the case that $\sigma = 0$ is the trivial endomorphism, that is, when $(G, \diamondsuit) = G$, this linear criterion has been previously employed in the literature to determine the subgroup of $\text{Aut}(G/G')$ induced by $\text{Aut}(G)$. It was introduced in [DH75] and was later used among others in [Car83, Car16].
Proof. Let us represent $\alpha$ as an $n \times n$ matrix

$$(a_{ij}), \text{ where } 1 \leq i, j \leq n$$

with respect to the basis (5.1). Letting

$$\hat{a}_{(j,k),(s,t)} = a_{js}a_{kt} - a_{jt}a_{ks},$$

we see that $\hat{\alpha}$ is represented by the $\left( \begin{array}{c} n \\ 2 \end{array} \right) \times \left( \begin{array}{c} n \\ 2 \end{array} \right)$ matrix

$$(\hat{a}_{(j,k),(s,t)}), \text{ where } 1 \leq j < k \leq n \text{ and } 1 \leq s < t \leq n$$

with respect to the basis (5.2). We may rewrite the equality in (2) as

$$(a_{ij}) (d_{i,(j,k)}) = (d_{i,(j,k)}) (\hat{a}_{(j,k),(s,t)}) \quad (5.5)$$

in terms of matrices with respect to the bases (5.1) and (5.2).

Let $F$ be the free group on $n$ generators $f_1, f_2, \ldots, f_n$ in the variety of $p$-groups of class two and exponent $p^2$ in which all $p$th powers are central, namely the variety defined by

$$[[F, F], F], \quad [F^p, F].$$

By sending each $f_i$ to $x_i$, we obtain surjective homomorphisms

$$\psi : F \rightarrow G \text{ and } \psi^\circ : F \rightarrow (G, \circ).$$

We may also define a homomorphism $\tilde{\alpha} : F \rightarrow F$ by extending

$$f_i^\tilde{\alpha} = f_i^{a_{1i}^1} f_i^{a_{2i}^2} \cdots f_i^{a_{ni}^n} \text{ for all } 1 \leq i \leq n,$$

where $a_{ij} \in \mathbb{Z}$ denotes a fixed lift of $a_{ij} \in F_p$. Since $\alpha$ is invertible, the image of $\tilde{\alpha}$ contains $f_1, f_2, \ldots, f_n$ modulo $F^p$ and in particular modulo Frat$(F)$. Since Frat$(F)$ is contained in every maximal subgroup of $G$, this implies that $\tilde{\alpha}$ is surjective. Since $F$ is finite, we deduce that $\tilde{\alpha}$ is an isomorphism.

We summarize the above set-up in the following diagram:

$$\xymatrix{ \ker(\psi) & F \ar[l] \ar[r]^\psi & G \ar[d]_{\tilde{\alpha}} \ar@{-->}[r]_{\tilde{\alpha}\psi^\circ} & \ker(\psi^\circ) \ar[l] \ar[r]_{\psi^\circ} & (G, \circ) }$$

The desired map in (1) is simply any homomorphism $\vartheta : G \rightarrow (G, \circ)$ for which the above diagram commutes. Note that $\vartheta$ is necessarily an isomorphism because $\tilde{\alpha}\psi^\circ$ is surjective.

Now, such a homomorphism $\vartheta : G \rightarrow (G, \circ)$ exists if and only if

$$\ker(\psi) \leq \ker(\tilde{\alpha}\psi^\circ),$$
or equivalently
\[
\begin{align*}
[[f_i, f_j], f_k] & = 1 & \text{for all } 1 \leq i, j, k \leq n, \\
(f_j^p)_{\tilde{\alpha}} & = \left( \prod_{j < k} [f_j, f_k]^{d_{i,(j,k)}} \right)_{\tilde{\alpha}} & \text{for all } 1 \leq i \leq n.
\end{align*}
\]

The first relation is trivial because \((G, \circ)\) has class two and
\[
[[f_i, f_j], f_k]_{\tilde{\alpha}} = \left( [[f_i, f_j], f_k]_{\tilde{\alpha}} \right)^{\phi} = [[x_i^a, x_j^a], x_k^a].
\]

For the second relation, since \(F\) has class two, \(p\)th powers are central in \(F\), and \(Z(F)\) has exponent \(p\), the left hand side equals
\[
(f_j^p)_{\tilde{\alpha}} = \left( f_1^{a_{11}} f_2^{a_{12}} \ldots f_n^{a_{1n}} \right)^{\phi} = \left( f_1^{a_{11}} f_2^{a_{12}} \ldots f_n^{a_{1n}} \right)^{\phi} = \left( f_1^{a_{11} \circ a_{21} \circ \ldots \circ a_{n1}} \right)^{\phi} = \prod_{\ell=1}^n \left( \prod_{j<k} [x_j, x_k]^d_{\ell,(j,k)} \right)_{\tilde{\alpha}_{\ell}} = \prod_{j<k} \left( \sum_{i=1}^n a_{i\ell} d_{\ell,(j,k)} \right),
\]

while the right hand side equals
\[
\left( \prod_{j<k} [f_j, f_k]^{d_{i,(j,k)}} \right)_{\tilde{\alpha}} = \left( \prod_{j<k} [f_j, f_k]^{d_{i,(j,k)}} \right)^{\phi} = \left( \prod_{j<k} \prod_{s<t} [f_s, f_t]^{(a_{js} a_{kt} - a_{jts}) d_{i,(j,k)}} \right)^{\phi} = \prod_{s<t, j<k} \left( [x_j, x_k]^{(a_{js} a_{kt} - a_{jts}) d_{i,(s,t)}} \right)^{\phi} = \prod_{j<k} \left( \sum_{s<t} d_{i,(s,t),\tilde{\alpha}_{(s,t),(j,k)}} \right).
\]

By (5.2) and (5.4), we know that
\[
[x_j, x_k]_{\circ}, \text{ where } 1 \leq j < k \leq n
\]
is also an \(F_p\)-basis for \(G'\). We then see that the second relation, when \(1 \leq i \leq n\) is fixed, holds if and only if
\[
\sum_{\ell=1}^n a_{i\ell} d_{\ell,(j,k)} = \sum_{s<t} d_{i,(s,t),\tilde{\alpha}_{(s,t),(j,k)}} \text{ for all } 1 \leq j < k \leq n.
\]

But the sum on the left is the \(i, (j, k)\) entry of the matrix on the left of (5.5), while that on the right is the \(i, (j, k)\) entry of the matrix on the right of (5.5).
We have thus shown that the existence of the \( \vartheta : G \to (G, \circ) \) in (1) is indeed equivalent to the equality in (2). Moreover, in this case, the action of \( \vartheta \) on \( G' \) is given by
\[
[x, y]^{\vartheta} = [x^{\alpha}, y^{\alpha}]^{1 + 2\sigma} = [x, y]^{\hat{\alpha}(1 + 2\sigma)},
\]
and hence \( \text{res}_z(\vartheta) = \hat{\alpha}(1 + 2\sigma) \), as desired. \( \square \)

5.2. **Proof of Theorem 1.1.** By Propositions 5.1 and 5.2, we have\( \text{res}(S') = \{(\alpha, \hat{\alpha}(1 + 2\sigma)) : \alpha \in \text{Aut}(G/G') \text{ and } \sigma \in \text{End}(G') \text{ such that } 1 + 2\sigma \in \text{Aut}(G') \text{ and } \alpha^{-1}\pi\hat{\alpha} = \pi(1 + 2\sigma)^{-1}\} \).

Making a change of variables \( \tau = 1 + 2\sigma \), we then obtain
\[
\text{res}(S') = \{(\alpha, \hat{\alpha}\tau) : \alpha \in \text{Aut}(G/G') \text{ and } \tau \in \text{Aut}(G') \text{ (5.6)}
\]
which satisfy the relation \( \alpha^{-1}\pi\hat{\alpha} = \pi\tau^{-1} \).

Recall that \( \pi : G/G' \to G' \) denotes the \( p \)-th power map, and the structure of \( \text{res}(S') \) depends upon \( \pi \) by the above description. Since \( G \) has class two, the so-called omega subgroup of \( G \) equals the set
\[
\Omega_1(G) = \{x \in G : x^p = 1\}
\]
consisting of the elements of order dividing \( p \). Note that
\[
\Omega_1(G) \text{ is contained in } G' \iff \pi \text{ is injective} \iff (d_{i,(j,k)}) \text{ has full rank.}
\]

Recall that \( (d_{i,(j,k)}) \) is the matrix of \( \pi \) with respect to the bases (5.1) and (5.2). The second equivalence here holds because our matrices act on row vectors from the right. In this case, we can prove:

**Proposition 5.4.** Assume that \( \Omega_1(G) \) is contained in \( G' \).

We have an isomorphism
\[
\text{res}(S') \simeq \mathbb{F}_p^{(n' - n) \times n} \rtimes (\text{GL}_n(\mathbb{F}_p) \times \text{GL}_{n'}(\mathbb{F}_p)),
\]
where \( n' = \binom{n}{2} \) and the semidirect product action is the natural one as given in the statement of Theorem 1.1.

**Proof.** The assumption implies that \( \pi \) is represented by
\[
D = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]
with respect to a suitable choice of bases of \( G/G' \) and \( G' \). In terms of matrices, we then see from (5.6) that
\[
\text{res}(S') \simeq \{(A, \tilde{A}T) : A \in \text{GL}_n(\mathbb{F}_p) \text{ and } T \in \text{GL}_{n'}(\mathbb{F}_p) \text{ (5.1)}
\]
which satisfy the relation \( A^{-1}\tilde{A} = DT^{-1} \).
Here, with respect to the chosen bases, if $A$ is the matrix representing $\alpha \in \text{Aut}(G/G')$ then $\hat{A}$ is the matrix representing $\hat{\alpha} \in \text{Aut}(G')$, while $T$ is the matrix representing $\tau$.

Let us fix an arbitrary $A \in \text{GL}_n(\mathbb{F}_p)$, and we wish to determine all the solutions $T \in \text{GL}_n'(\mathbb{F}_p)$ to the equation

$$A^{-1}D\hat{A} = DT^{-1}, \text{ or equivalently } A^{-1}D = DT^{-1}\hat{A}^{-1}. \quad (5.8)$$

By the shape of the matrix $D$, the left hand side is the block matrix

$$\begin{bmatrix} A^{-1} & 0 \\ -Q & M^{-1} \end{bmatrix}.$$ 

Also, the equation (5.8) only imposes restrictions on and uniquely determines the first $n$ rows of $T^{-1}\hat{A}^{-1}$. In particular, we see that

$$T^{-1}\hat{A}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -Q & M^{-1} \end{bmatrix},$$

where $Q$ is arbitrary but $M$ has to be invertible because we require $T$ to be invertible. We now conclude that the solutions $T \in \text{GL}_n'(\mathbb{F}_p)$ to the equation (5.8) are precisely

$$T = \hat{A}^{-1}\begin{bmatrix} A^{-1} & 0 \\ -Q & M^{-1} \end{bmatrix}^{-1} = \hat{A}^{-1}\begin{bmatrix} A & 0 \\ MQA & M \end{bmatrix},$$

where $Q \in \mathbb{F}_{p(n'-n)}$ and $M \in \text{GL}_{n'-n}(\mathbb{F}_p)$ are arbitrary.

The above shows that elements of $\text{res}(\mathcal{S}')$ may be parametrized by

$$\mathbb{F}_{p(n'-n)} \times \text{GL}_n(\mathbb{F}_p) \times \text{GL}_{n'-n}(\mathbb{F}_p).$$

Let us now compute the structure of $\text{res}(\mathcal{S}')$. For any $Q_1, Q_2 \in \mathbb{F}_{p(n'-n)}$, $A_1, A_2 \in \text{GL}_n(\mathbb{F}_p)$, $M_1, M_2 \in \text{GL}_{n'-n}(\mathbb{F}_p)$, letting $Q = M_2^{-1}Q_1 + Q_2A_1^{-1}$, we have

$$(A_1,\begin{bmatrix} A_1 \\ M_1Q_1A_1 & M_1 \end{bmatrix})\left(A_2,\begin{bmatrix} A_2 \\ M_2Q_2A_2 & M_2 \end{bmatrix}\right) = (A_1A_2,\begin{bmatrix} A_1A_2 \\ M_1M_2QA_1A_2 & M_1M_2 \end{bmatrix}).$$

This means that in terms of the above parametrization, the multiplication of $\text{res}(\mathcal{S}')$ is given by

$$(Q_1, A_1, M_1) \cdot (Q_2, A_2, M_2) = (M_2^{-1}Q_1 + Q_2A_1^{-1}, A_1A_2, M_1M_2).$$

From here, we see that the structure of $\text{res}(\mathcal{S}')$ is as claimed. \qed

One can now appeal to the construction in [Car16] to obtain that the condition $\text{Aut}(G) = \text{Aut}_c(G)$ can always be realized by a choice of a matrix $(d_{i,j,k})$ which has full rank, so that $\Omega_1(G) \leq G'$ as noted in (5.7). Theorem 1.1 is now an immediate consequence of (5.3), Theorem 4.4, and Proposition 5.4.
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