ON A LIST OF ORDINARY DIFFERENTIAL EQUATIONS PROBLEMS *

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Abstract

This evocative essay focuses on the mathematical activities witnessed by the author along 1962-64 at IMPA. The list of research problems proposed in September 1962 by Mauricio Peixoto at the Seminar on the Qualitative Theory of Differential Equations is pointed out as a landmark for the genesis of the research interest in the Qualitative Theory of Differential Equations and Dynamical Systems in Brazil.

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Just as every human undertaking pursues certain objectives, so also mathematical research requires its problems. It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon.

D. Hilbert, ICM, Paris. 1900.

1 The diary of a journey.

To find an old mathematics notebook, protected from total deterioration by a plastic cover, was like an encounter with the diary of a distant journey. The contact with its yellowish pages triggered an avalanche of recollections of the years 1962 – 64, when I was a doctoral student in Brazil. Its sequential structure prompted the reconstruction of the chronology of my initiation into mathematical research.

The essential landmarks – the stations – of a mathematical peregrination, starting from a bookish approach, heading toward an attempt to tackle research problems, pulsed latent in the rough writing and scribbled drawings. Some pages were missing, a few of them had faded away.
2 An afternoon in September 1962.

In a seminar room of the Institute of Pure and Applied Mathematics (IMPA), in the Botafogo quarter of the city of Rio de Janeiro, gathered a group of around ten people: mathematicians, research fellows of disparate backgrounds, candidates to become mathematicians, and one or two voyeurs.

Figure 1: Present view of the facade of the two store house, located at the corner of the streets São Clemente and Sorocaba, hosting IMPA in 1962 – 64.

The occasion was the Seminar on the Qualitative Theory of Differential Equations (QTDE), directed by Prof. Mauricio M. Peixoto (hereafter Prof. Peixoto, or simply Peixoto) who had announced the following title for his lecture: “Open Problems on The Qualitative Theory of Differential Equations”. The seminar activity had been interrupted for a few sessions. It was restarting after the return of Prof. Peixoto who had travelled to attend the International Congress of Mathematicians at Stockholm, August 15 - 22.

At that time Prof. Peixoto was the only Research Director resident at IMPA. Prof. Leopoldo Nachbin (1922 - 1993), also a research Director, was on leave of absence.

Peixoto had very explicit views concerning Mathematics learning in graduate level. They where well known to most participants: “This science is assimilated through solving problems and thinking”, he had clearly stated, contesting insinuations of those who favored the preponderance of more intensive lecture courses and bookish learning. This
contrasted with the naive and easy going vision I had partially acquired in my undergraduate predominantly reading contact with mathematics.

Excitement and a tense expectation could be noticed in the audience. For the most experienced participants of the Seminar, the time to face true research problems had arrived. As a newcomer into advanced studies, and the youngest of all, I could be implicitly regarded outside such group.

3 A flash of mathematics at IMPA in 1962 - 63.

Among the experienced researchers, besides Prof. Peixoto who stayed at IMPA the whole period from 1962 to 1964, were Elon Lima (1930 - 2017), Djairo de Figueiredo (1934 - ) and Otto Endler (1929 - 1988) who sojourned for shorter periods. Elon stayed the whole first semester and Djairo only a couple of months. Otto stayed along 1962 and part of 1963.

Few lecturers visited the Institute. Among them I mention below those which captured my interest.

Charles Pugh (1940 - ). Subject: Closing Lemma, crucial in the work of Peixoto.

Gilberto Loibel (1932 - 2013). Subject: Stratified Sets, introduced by René Thom (1923 - 2004), founder, together with Hassler Whitney, of the Theory of Singularities of Mappings.

Wilhelm Klingenberg (1924 - 2010). Subject: Closed Geodesics, located in the intersection of Differential Geometry, Differential Equations and Calculus of Variations.

At that time, IMPA did not have a minimal program of courses to be offered along the year. This activity depended on the research fellows present, in a quickly changing regime.

However, there was a permanent basis providing scientific stability to the Institute along those years, around Prof. Peixoto research project. Furthermore, due to an agreement with the University of Brazil, later denominated UFRJ (Federal University of Rio de Janeiro), a doctoral program started at IMPA in 1963.

This pioneer research project, under the leadership of Peixoto, is the landmark of the systematic interest in Dynamical Systems (DS) in Brazil.

This was the first explicit effort to stimulate the initiation into research in this area of mathematics in Brazil.
I had the unique chance and privilege to be part of the first group of doctoral candidates, under the supervision of Prof. Peixoto.

After the influential works of the American mathematician Stephen Smale (sixties and seventies) the denomination DS nominally assimilated a significant part of the QTDE from which the separating border is not well defined. See Smale’s 1967 landmark article [40]. DS is also the name of a famous book of George Birkhoff, printed in 1927.

Ordered by the preponderance they had in my initiation into research, below I list some of the courses and seminars held at IMPA in 1962:

1. Seminar on the Qualitative Theory of Differential Equations;
2. Seminar on Differentiable Manifolds following the inspiring Porto Alegre Elon Lima book [22];
3. Seminar based on the reading of the, now classic, book of J. Dieudonné: Foundations of Modern Analysis [7];
4. Course on Algebraic Topology, taught by Peixoto, based on the books of Cairns and Hocking - Young;
5. Course on Multilinear Algebra and Exterior Differential Calculus, based on Bourbaki and Flanders, taught by Elon Lima;
6. Seminar on the reading of the book “Lectures on Ordinary Differential Equations” by W. Hurewicz [17].

I list bellow, with no ordering, some of the research fellows and habituées of IMPA, related to Prof. Peixoto Project, in 1962 - 63:

Ivan Kupka (1938 - ),
Maria Lúcia Alvarenga (1937 - ),
Jacob Palis (1940 - ),
Lindolpho C. Dias (1930 - ),
Alcilea Augusto, (1937 - ),
Aristides C. Barreto, (1935 - 2000).

Much of the contents of the activities 1 to 5 listed above was new to me. I devoted to them special attention, with library search and extensive complementary readings. In this endeavor the friendly interaction with Ivan Kupka, by far the most knowledgeable in the group, was auspicious.

1This abridged version concerns activities on the QTDE. A wider description can be found in [54].
Figure 2: **At the entrance of IMPA, August 1962.**

(a) Top row, left to right: Lindolpho Dias, M. Peixoto, Ivan Kupka, E. Isla, Aristides Barreto, Jacob Palis, Eliana Rocha, Alciéa Augusto; descending the right margin and continuing clockwise on the bottom margin: C. Marquez, M. L. Alvarenga, M. H. Cerqueira, J. Sotomayor, Lia Velloso and Adarcy P. Costa. Three gentlemen sitting at the center, descending counterclockwise: L. Nachbin, H. Machado and J. A. Barroso.

(b) Top row from left to right: M. Peixoto, E. Isla, Celina Marquez, Lindolpho Dias and Jacob Palis. L. Nachbin at center, with necktie. Bottom right corner: Alciéa Augusto, M. L. Alvarenga, Adarcy P. Costa.

4 **Peixoto’s Seminar on QTDE, 1962.**

Along my sojourn at IMPA, 1962-64, the most remarkable of all the activities in which I engaged was the Seminar on the Qualitative Theory of Ordinary Differential Equations. Following the epistolary reading directions sent to me by Peixoto in 1961, recounted in the evocative essay “Mathematical Encounters” [55], with the inspiring master presentations of Coddington and Levinson [6] and Hurewicz [17], I had already been initiated into the first steps of the Qualitative Theory. The phase portrait, limit cycles and singular points local structures: saddles, nodes, foci and centers, were found there. A bright synthesis of these elements in the Poincaré - Bendixson Theorem complemented the introductory contact.

While the subjects presented and discussed in the Seminar did not require a heavy
background knowledge, their in depth appreciation depended on a level of maturity and on an inquisitive disposition, beyond the initiation outlined above, to which I had not yet been exposed.

4.1 Non singular ODEs on the Klein Bottle and the Torus.

The first lecture at the Seminar TQEDO, was delivered by Peixoto at the beginning of April. The subject was *The Theorem of Kneser*, whose conclusion is that “Every vector field with no singular points in the Klein Bottle has a periodic orbit”. See [19].

At the two next meetings of the seminar, Peixoto presented the basic theory of the rotation number following the last chapter of Coddington and Levinson.

The second presentation included also a geometric construction of the example of Denjoy of a \( C^1 \) non singular vector field for which all its orbits cluster in a closed invariant set, which transversally is a Cantor set, with no proper subset sharing these properties.

Such a set is denominated “minimal non-trivial”. The “trivial minimal” sets are the singular points –equilibria– and periodic orbits.

4.2 Invariant Manifolds.

Elon Lima continued the Seminar. He presented part of Chap. 13 of Coddington and Levinson [6] which contains the Theory of Invariant Manifolds. It deals with the \( n \)-dimensional generalizations of “saddles” for singular points –equilibria– and periodic orbits. Actually, these dynamical objects are called “hyperbolic”, a name coined by Smale. This is one of the most technical matters of the book, which follows the approach of the German mathematician Oskar Perron (1880 –1975).

In 1970, C. Pugh, whose name will appear later in this essay, and M. Hirsh made a substantial extension of the Invariant Manifold Theory elaborating ideas in the works of the French mathematician Jacques Hadamard (1865 - 1963). See [16]. Years later, I had a better assimilation of [16] and, in 1979, included it in [47]. In 1973, I had used it to give a conceptual proof of the smoothness of the flow of a vector field [44].

Meanwhile, the approach of Perron, also in [6], was elaborated by C. Irwen using the Implicit Function Theorem in Banach Spaces. In [6] is used the method of successive approximations. A version of this idea can be found also in Melo and Palis [27].
4.3 A detailed presentation of three papers of Peixoto on Structural Stability.

Along part of May and June, Alciléa Augusto delivered a series of presentations, very detailed and carefully prepared, covering Peixoto’s papers: [29], [30] and [31]. The last paper, however, involved difficulties, particularly on non-orientable surfaces. This matter evolved into the so called The Closing Lemma Problem, of present research interest, extrapolating the domain of Classical Analysis.

Rather than outlining the individual contents of these papers, I include below a personal appreciation on the subject, with non-exhaustive references.

5 A Glimpse into Structural Stability.

The concept of Structural Stability was established during the collaboration of the Russian mathematicians A. Andronov (1901 - 1952) and L. Pontryagin (1908 -1988) that started in 1932 [36]. It first appeared in their research note published in 1937. Andronov (who was also a physicist) founded the very important Gorkii School of Dynamical Systems. He left a remarkable mathematical heritage, highly respected both in Russia and in the West, [14]. By 1932 Pontryagin was an already famous Topologist who had started to teach differential equations and had voiced his interest in studying applied problems.

Structural Stability is a consequence of the encounter of two mathematical cultures, See discussion in Sec.13.

For a dynamic model –that is, a differential equation or system \( x' = f(x) \)– to faithfully represent a phenomenon of the physical world, it must have a certain degree of stability. Small perturbations, unavoidable in the recording of data and experimentation, should not affect its essential features. Mathematically this is expressed by the requirement that the phase portrait of the model, which is the geometric synthesis of the system, must be topologically unchanged by small perturbations. In other words, the phase portraits of \( f \) and \( f + \Delta f \) must agree up to a homeomorphism of the form \( I + \Delta I \), where \( I \) is the identity transformation of the phase space of the system and \( ||\Delta I|| \) is small. A homeomorphism of the form \( I + \Delta I \) is called an \( \epsilon \)-homeomorphism if \( ||\Delta I|| < \epsilon \); that is, it moves points at most \( \epsilon \) units from their original positions.

Andronov and Pontryagin stated a characterization of structurally stable systems on a disk in the plane. This work was supported by the analysis of numerous concrete models of mechanical systems and electrical circuits, performed by Andronov and his associates [1].
The concept of structural stability, initially called robustness, represents a remarkable evolution of the continuation method of Poincaré.

When the American mathematician S. Lefschetz translated the writings of Andronov and his collaborators from Russian to English [1], he changed the name of the concept to the more descriptive one it has today [1]. He also stimulated H. B. de Baggis to work on a proof of the main result as stated by Andronov and Pontryagin.

Peixoto improved the results of the Russian pioneers in several directions.

For example, he introduced the space $X_r$ of all vector fields of class $C^r$, and established the openness and genericity of structurally stable vector fields on the plane and on orientable surfaces. He also removed the $\epsilon$-homeomorphism requirement from the original definition, proving that it is equivalent to the existence of any homeomorphism. This was a substantial improvement of the Andronov-Pontryagin planar theory.

The transition from the plane to surfaces, as in Peixoto’s work, takes us from classical ODEs to the modern theory of Dynamical Systems, from Andronov and Pontryagin to D. V. Anosov and S. Smale.

It has also raised delicate problems—for instance, the closing lemma— that have challenged mathematicians for decades [15].

In [41] S. Smale regards Peixoto’s structural stability theorem as the prototypical example and fundamental model to follow for global analysis.

6 Open Problems in ODEs, September 1962.

After a concise, though very emphatic, introduction about the importance of attacking research problems, Prof. Peixoto began to enumerate and discuss five of them.

6.1 First order structurally stable systems.

Consider the complement $X^r_1$ of the set $\Sigma^r$ of $C^r$-structurally stable vector fields, relative to the set $X^r$ of all vector fields on a compact two-dimensional manifold. Let $X^r_1$ be endowed with the induced $C^r$ topology. Characterize the set $\Sigma^r_1$ of those vector fields that are structurally stable with respect to arbitrarily small perturbations inside $X^r_1$.

This problem goes back to a 1938 research announcement of A. A. Andronov and E. A. Leontovich [3], [4]. They formulated a characterization of $\Sigma^r_1$ for a compact region in the plane. This step points toward a systematic study of the bifurcations (qualitative changes) that occur in families of vector fields as they cross $X^r_1$. In the research announcement
–contained in a dense four pages note– they stated that the most stable bifurcations occur in $\Sigma^r_1$, [45], [48].

6.2 The problem of the arc.

Prove or disprove that a continuous curve (an arc) in the space $X^r$ of vector fields of class $C^r$ on the sphere can be arbitrarily well-approximated by a continuous curve that meets only finitely many bifurcation points; that is, points outside the set of structurally stable vector fields, at which qualitative changes occur.

Later research established that $X^r_1$ enjoys great transversal complexity, which grows quickly with the dimension of the phase domain.

This knowledge became apparent after the work of S. Smale [40] and also Newhouse [26] and Palis -Takens [28], among others.

The understanding of the phenomenon of persistent accumulation of bifurcations implies that the problem of the arc as stated above has a negative answer, [42]. However, after removing the requirement of the approximation, Peixoto and S. Newhouse proved...
that every pair of structurally stable vector fields is connected by an arc that meets only finitely many bifurcation points. See [35].

6.3 The classification problem.

Use combinatorial invariants to classify the connected components of the open set of structurally stable vector fields.

The essential difficulty of this problem is to determine when two structurally stable vector fields agree up to a homeomorphism that preserves their orbits and is isotopic to the identity.

Some years later, Peixoto himself worked on this problem, [33].

6.4 The existence of nontrivial minimal sets.

Do invariant perfect sets (that is, sets that are nonempty, compact, and transversally totally discontinuous) exist for differential equations of class $C^2$ on orientable two-dimensional manifolds?

This problem goes back to H. Poincaré and A. Denjoy and was known to experts.

It was solved in the negative direction by A. J. Schwartz [39]. Peixoto presented this result from a preprint that he received in November 1962.

6.5 Structurally stable second order differential equations.

For equations of the form $x'' = f(x, x')$ (more precisely, for systems of the form $x' = y$, $y' = f(x, y))$, characterize structural stability, and prove the genericity of structural stability, in the spirit of Peixoto’s results for vector fields on two-dimensional manifolds.

Problems 6.2 to 6.4 were assigned, in one-to-one correspondence, to the senior participants of the seminar. The first and last problems were held in reserve for a few months.

In Sec. [11] I will recount how I was conduced to obtain Peixoto’s support to attack problem 6.1 on his list.

When, years later, I read the proposal of the famous 1900 Hilbert Problems, the words in the epigraph made me evoke the above mentioned introit of Peixoto’s list of problems in September 1962, which then, keeping in mind the enormous difference in proportions, struck me as a distant echo of Hilbert’s
words in Paris 1900, that reverberated along decades before reaching the tropics.

7 Peixoto’s seminar, last 1962 sessions.

The last lecture in the Seminar was delivered by Peixoto. It was based on a preprint of A. J. Schwartz [39], which solved in the negative problem 6.4.

Before this, toward the end of October, Ivan Kupka delivered a series of very technical lectures about Invariant Manifolds along arbitrary orbits, not necessarily periodic.

He mentioned the work of Oskar Perron as the main reference for the hyperbolicity hypotheses adopted.

To mitigate the concern of most participants in face with abundant analytical technicalities involved in the rough presentations, Peixoto started a parallel series of very informal tutorial discussions on Kupka’s lectures.

Everybody freely expressed their disparate attempts to explain geometrically the ideas as well as the long chains of inequalities involved.

Maybe this convivial contact and my participation in the discussions, freely formulating hunches, established a more direct channel of communication between me and Peixoto, which so far had been a very formal one.

8 A good research problem.

After the presentations of Peixoto’s work, timely commented by himself, complemented with considerable struggle with the bibliography, it was possible to have a panoramic view of a fascinating piece of knowledge. It was a sample of the evolution of mathematical ideas with an intriguing historical background, mathematically deep but essentially accessible, whose consolidation had seen the light in the last four years, concomitantly with my learning at university level, of the principles of Mathematical Analysis, Geometry and Differential Equations.

Complementing the seminar lectures I had made substantial readings related to the QTDE and the Calculus and Geometry on Banach Spaces. I devoted considerable attention to Sard Theorem, which I first read in the presentation, [18], of Prof. Edson Júdice of the University of Minas Gerais (U.F.M.G), donated to me by the author by a recommendation of Aristides Barreto, with whom I became close friend. I also studied
the book by S. Lang on Differentiable Manifolds, modeled on Banach spaces [21].

Along this endeavor I had rewarding discussions on mathematical subjects with Ivan Kupka.

With great profit I read the enlightening lecture notes Introdução à Topologia Dif-
ferencial, [23] by Elon Lima, which anticipated in several years Milnor’s “Topology from a
differentiable viewpoint” [21]. After this, I studied substantial parts of L. S. Pontrjagin,
[37], “Smooth Manifolds and its Applications to Homotopy Theory”, which I consider
the original source for the application of Differential Analysis to the study problems in
Toplogy.

Some undefined intuitive and esthetic considerations, and a certain overestimation of
my readings about Sard’s Theorem, led me to the hunch that, if not all, part of Peixoto’s
genericity Theorem of Structurally Stable Systems could be obtained from an appropriate
infinite dimensional version of this theorem. Being undefined the involved domain and
range spaces.

In discussions with Kupka, I had learned of an extension in this direction due to Smale.

I approached Peixoto and shared with him my naive expectations. He made no com-
ments. However, toward the end of november he handed me a copy of the Andronov -
Leontovich four page note [3].
He said: “Do not loose this. It is important. It is a good problem”.

Despite the technical difficulties, enhanced by, at that time, lack of bibliography on
bifurcations, the sensation of possessing a research problem, produced in me the mixed
feelings of a naive fulfillment and of overwhelming responsibility.

I left Rio de Janeiro for the extended summer break of 1963. In my mind I carried a
new sense of mathematical awareness. In my bag, packed in a plastic cover, travelled the
note of Andronov - Leontovich.

Remark 8.1. Concerning the desideratum of providing a proof of Structural Stability
Theorem genericity theorem with Sard’s Theorem, I mention that in the decade of 1970,
when I began to lecture on the subject for wide audiences, I felt the need, and found,
a direct, self-contained, transparent proof that worked for the plane and for polynomial
vector fields. The course in the 1981, 13th Brazilian Mathematics Colloquium, [48], was
an opportunity to communicate the new proof that used only an elementary form of the
one-dimensional Sard’s Theorem. An abridged version was also published in [49].
9 Back to IMPA in 1963.

In December 1962 I took the final examinations of a few courses I had pending to fulfill the number of academic credits required to get the Mathematics Bachelor degree from the National University of San Marcos, Lima, Peru.

Along January and February I worked as teaching assistant in a Summer Mathematics School for the training of high school teachers. In my spare time, and full time on March, I scribbled piles of pages of calculations and drawings attempting to decipher the statements in Andronov-Leontovich note.

Arriving to IMPA at the end of March, I had some time to discuss with Ivan Kupka the outcome of my summer struggle with the note of Andronov-Leontovich. Prof. Peixoto had scheduled me to present a report on the subject in May.

The possibility of adapting the methods of his works to get some form of density of $\Sigma^r_1$ in $\mathcal{X}^r_1$, as proposed in problem 6.1, was raised along the preparation and in discussions after the seminar sessions. This point is not mentioned in [3] and [4].

At the expense of considerable work, it seemed possible to adapt the methods in the works of Peixoto to the formulation extracted from the note of Andronov-Leontovich, extending it from the plane to surfaces.

However, while attempting to complement my personal studies by reading whatever work containing some material on bifurcations that fell in my hands, among which were [25] and [38], I was being led to the suspicion that problems 6.1 and 6.2 were intimately interconnected.

This intuition received a mathematical formulation during July, catalized by events that took place at the IV Brazilian Mathematical Colloquium.

10 The IV Brazilian Mathematical Colloquium, 1963.

At the Fourth Brazilian Mathematics Colloquium, in July 1963, [5], [54], everyone working under Peixoto’s supervision made reports in a session of short communications. Only Ivan Kupka delivered a plenary lecture.

Below, in free translation, the titles of the communications:

Maria Lúcia Alvarenga: “Planar Structural Stability”.

Alciléa Augusto: “Parametric Structural Stability”,
Aristides Barreto: “Structural Stability of Equations of the form $x'' = f(x, x')$.” “Higher order Structural Stability”, my communication, closed the list.

The part of the work I did at IMPA that I consider innovative, concerning the structure of smooth Banach sub-manifold of the class of Andronov - Leontovich vector fields, had not yet been conceived. However its main ideas emerged during the Colloquium. I explain this point now.

In the first semester of 1963, Kupka had spent a couple of months at the University of Columbia, N.Y., where Smale worked. He brought the, now classic, paper of Palais and Smale “A Morse Theory for Infinite dimensional manifolds”. Encouraged by Peixoto, he presented a Plenary Lecture about it. The title in [5] is “Counter example of Morse - Sard Theorem for the case of Infinite Dimensional Manifolds”, though he spent most of his time explaining the Palais-Smale Theory.

After an enlightening presentation, he concluded:

“The introduction of a new Theory must be accompanied by a solid justification. The one I presented today has applications to Calculus of Variations, Control Theory and Differential Geometry, among other subjects.”

In her short presentation Alcilea Augusto exposed the generic finitude for the encounter of an arc of vector fields with the class of those whose equilibria have vanishing Jacobian. She used an elementary version of Thom Transversality Theorem analogous to the one used to prove the invariance of the Euler - Poincaré - Hopf characteristic of a smooth manifold, expressed as the sum of the indexes of the singularities of a vector field with non-vanishing Jacobians. At that time I had seen this procedure clearly explained in Lima’s lecture notes [23].

Added to the examples of one parameter bifurcations I had scribbled from books on non-linear mechanics and oscillations, such as [25] [2], and the re-reading of [3], the ensemble of the lectures of A. Augusto and I. Kupka, impacted my vision of problem 6.1. The following intuition, or desideratum, struck me: The First and Second Problems were intimate parts of the same problem. The suitable synthesis of this association should be presented in terms of infinite dimensional sub-manifolds and transversality to sub-manifolds in the Banach space of all vector fields tangent to a surface.

In fact, the part of the class $X_1'$ to be crossed by an arc, or curve of vector fields, needed to have a smooth structure, as that of a hypersurface, to express the crossing, or bifurcation, as a transversal intersection, thus unifying in one concept –the codimension– all the diverse dynamical phenomena, including the global ones, such as the non-hyperbolic
periodic orbits, homoclinic and heteroclinic orbits, and not only the punctual ones, such as the singularities.

The suitable approach to express this structure had to include the mathematical objects such as those appearing in Kupka’s lecture: infinite dimensional manifolds with tangent spaces that could be used to express infinitesimally the transversal crossing with $X^r_1$.

The expression in paper form of this intuition had to wait some years to see the light.

I will mention only two other lectures:
- M. Peixoto, An elementary proof of the Euler-Poincaré formula on Surfaces (“Uma prova elementar da formula de Euler-Poincaré em Superfícies”).
- Charles Pugh, “The Closing Lemma”.

This Lemma is in fact an open problem whose statement for class $C^s$ and dimension $n$ is as follows:

“Every $C^r$ vector field on a compact $n$-dimensional manifold $M$ having a non trivial recurrent through $p \in M$ orbit can be arbitrarily approximated in the $C^s$, $s \leq r$, topology by one which has through $p$ a periodic orbit.”

In his lecture for the 1963 Colloquium Pugh presented a particular case for $n = 2$ and $s = 1$. Later he extended his analysis for arbitrary $n$. Carlos Gutierrez made remarkable contributions to the case of $n = 2$, $s > 1$, where, it is still open, in most non-orientable surfaces [15].

The Closing Lemma Problem is a question that stems from Peixoto’s works.

Proportionally, regarding by subjects, the presence of Structural Stability and related topics –the school founded by Peixoto– in the ensemble of presentations in the IV Brazilian Mathematical Colloquium was overwhelming.

11 A thesis project based on the good problem in [8].

By the end of November 1963, the fellow that, on March 1962 had arrived to IMPA with a bookish mathematical knowledge [55], seemed somewhat distant. Prompted by the unfolding of Peixoto’s Seminar, specially by the ODE Open Problems Session, outlined in Sec. 6, he had experienced an upgrade on the amplitude of his mathematical imagery and on the profundity and extension of his knowledge. The readings performed and the mathematical events at the IV Bras. Math. Colloq., outlined in Sect. 10 had a radical
influence on his view of the problems of Peixoto presented in Sec. 6.

It was clear then that Problems 1 and 2 of Peixoto’s List were linked by the differentiable structure of the extended class of Andronov - Leontovich that I denoted $\Sigma_r^1$. The detailed analysis of this structure, however, depended on making it explicit in several instances.

Peixoto agreed with my doctoral dissertation project consisting on the extension to surfaces of the class $\Sigma_r^1$, its smooth structure and its density inside $X_r^1$.

An explicit counter example for the generic finitude of planar bifurcations was easy to find, reconsidering in terms of transversality and further elaborating the results concisely expressed by the Russian pioneers. See Sotomayor [42].

12 The 1964 mathematical works.

In the written composition of my doctoral thesis (DT) was deposited most of what I had learned along 1962 - 63:
The Calculus in Banach spaces and manifolds,
The invariant manifolds a la Coddington-Levinson,
The Structural Stability papers of Peixoto,
The personal digest I had scribbled on the understanding of Andronov - Leontovich (AL) announcement note [4].

Keeping in mind the analogy with the previous works of Andronov and Pontrjagin, as improved by Peixoto, the note of AL, [3], can be outlined as consisting of:

1.- An axiomatic definition for the class $\Sigma_r^1$ as the part of $X_r^1 = X^r \setminus \Sigma^r$ that violate minimally the conditions of Andronov- Pontrjagin and Peixoto that define $\Sigma^r$.

2.- A definition of the class $S_r^1$ of the systems in $X_r^1$ that are structurally stable under small perturbations inside $X_r^1$.

3.- The statement identifying $\Sigma_r^1$ with $S_r^1$.

I transliterated the terminology for the systems in [3] as being “first order structurally stable”. However Russian translators use the name : “first order structurally unstable”. In fact, they are the “most stable among the unstable ones”. See Andronov-Leontovich et al [4], where the proofs of the planar theorems of (AP) and (AL) were published in 1971.

The programatic analogy between Andronov - Leontovich and Andronov - Pontryagin, in planar domains, is clear. It strikes as natural to extend it to Peixoto’s surface domains. However, DT takes this analogy further and prepares the way for a geometric synthesis.
of the Generic Bifurcations.

In fact, it establishes the openness and density of $\mathcal{S}_r^1$ relative to the sub-space $\mathcal{X}_r^1$. It also endows it with the structure of smooth co-dimension one sub-manifold of the Banach space $\mathcal{X}^r$.

This last, analytic and geometric, aspect of DT has no parallel in the Russian works on the subject. It makes possible to regard geometrically the simple bifurcations as the points of transversal intersection of a curve of systems with the sub-manifold $\Sigma_1^r$.

In DT are calculated the tangent spaces to each piece of $\Sigma_1^r$. For the cases of the homoclinic and heteroclinic connections of saddle points, the functional whose kernel defines the pertinent tangent space, is expressed in terms of an improper convergent integral, which corresponds to the Melnikov Integral when restricted to vector fields, i.e. autonomous systems. In 1964 no reference was known to me.

Sotomayor \[51\] contains a study of the characterization of First Order Structural Stability in terms of Regularity of $\mathcal{X}_r^1$.

The work of Ivan Kupka achieved celebrity after Peixoto published \[32\], which unified the versions of Smale, for diffeomorphisms, and that of Kupka, for flows, and coined the name Kupka-Smale Systems for those systems whose singularities and periodic orbits are all hyperbolic and all pairs of associated stable and unstable manifolds meet transversally.

This work of Peixoto provided me with the language and methods that I had missed in 1964, for the extension of the class of Andronov Leontovich to a strictly larger immersed manifold, containing properly the imbedded one, whose structure was established in DT. Concerning this immersed manifold, the transversality to it gives the generic position of an arc with a dense, in $\mathcal{X}_r^1$, smooth part. See Sotomayor \[45\].

The work of Aristides Barreto studied the systems of the form $x' = y, y' = f(x, y)$, with $f$ periodic in $x$. There, he characterized those which are structurally stable. As far as I know this is the first work on Structural Stability on non compact manifolds, the cylinder in this case. In \[50\] I presented a compact version of the solution problem 6.5 in Peixoto’s list.
13 Concluding Comments.

13.1 Timeline focused in this essay, with some extrapolation.

Looking in retrospect one may be tempted to think that some of the subjects presented in the seminar, Sec. 11, had, already in 1962-63, some scent of a distant past. However, it cannot be denied that they also glimpsed into the future. In fact, for the next three or more decades, they had current interest for an active line for research training to work in Dynamical Systems, touching its kernel. Fundamental work on these subjects was done along the forthcoming years, reaching relatively recent ones. On this matter allow me to evoke the following universal words:

*If we wish to foresee the future of mathematics,*
*our proper course is to study the history and*
*present condition of the science. For us mathematicians,*
*is not this procedure to some extent professional?*
*We are accustomed to extrapolation, which is a method*
*of deducing the future from the past and the present;*
*and since we are well aware of its limitations, we run no risk*
*of deluding ourselves as to the scope of the results it gives us.*

H. Poincaré, in The Future of Mathematics,
read by G. Darboux in Rome, ICM, 1908.

Figure 4: Timeline with colored landmarks, weighted by the size of stars, with organizing center on the years 1962 – 64.

Brazil (red): Peixoto’s works, Seminar and Symposium;
France (black): Poincaré QTDE;
Russia (blue): The Gorkii School Landmark;
USA (green): Lefschetz, 1949-52, and Smale, Visit to IMPA, 1961, Seminar in Berkeley: 1966-67, and his landmark papers Differentiable Dynamical System, 1967, and What is Global Analysis?, 1969.
13.2 Some Inquisitive comments.

The mathematical concept of Structural Stability could hardly have stemmed, in isolation, in the offices of Mathematicians, pure or applied, or in the laboratories of Physicists or at the workshops of Engineers. Something deeper and innovative happened in the collaboration of Andronov and Pontrjagin.

An effective collaboration involves the intellectual affinity of spirits. In this case, involving Andronov, Physicist, with exceptional mathematical knowledge, engaged in the research of the modeling of mechanisms, [14], and Pontrjagin, distinguished Mathematician, with remarkable contributions in Topology, interested in engaging himself in applied problems, [36].

How do the transition from concrete examples and technological needs are processed into seminal mathematical concepts and, afterwards, to pertinent theorems?

This is a central question of the Psychology of the Creative Process, whose basis and analysis have been addressed by [13] and [20], among others.

"... The creative act, by connecting previously unrelated dimensions of experience, enables the authors to attain a higher level of mental evolution. It is an act of liberation – the defeat of habit by originality."

A. Koestler, (1964). The Act of Creation, (p. 96). London: Hutchinson and Co.

However, once formulated in the domain of Mathematics, the concepts and theorems are amenable to generalizations, extensions and refinements, in style and essence. Thus, they allow their elaboration by Mathematicians, with their phantasies and the creative flight of their imagination.

There are several stages in this transition in the realm of the evolution of mathematical ideas around Structural Stability, its extensions and generalizations. Maybe the first one, after the Russian pioneers, is that of Solomon Lefschetz, responsible for its diffusion in the West and for coining its expressive name, re-baptizing, the original Robust Systems given by the pioneers [1]. On this line of presentation, besides Peixoto, already cited, the names of Smale, Anosov, Arnold, Thom and Mather, among others, should be mentioned, thus extrapolating the realm of Differential Equations and Dynamical Systems.

What, in an attempt of expository simplification, I referred to above as the outcome of the encounter of two distinct mathematical cultures: knowledgable expertise and mathematical talent, [14], [36], may, perhaps, be better explained in the delicate threshold between Mathematics and Art.
Mathematics, rightly viewed, possesses not only truth, but supreme beauty, a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.

B. Russell, (1919) “The Study of Mathematics”, Mysticism and Logic and Other Essays. Longman. p. 60.

The mathematical and philosophical implications of Structural Stability, can be appreciated in its extensions to higher dimensional Dynamical Systems and to other domains of the Analysis on Manifolds, such as the Singularities of Differentiable Mappings and the Theory of Catastrophes, [56] [46], and Multiparametric Bifurcations, [8] [9] [10], as well as to Classical Differential Geometry, such as the configurations of principal curvature lines and umbilic points on surfaces [52], [53], [11], [12].

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