Fermi-Pasta-Ulam recurrence and modulation instability

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We give a qualitative explanation of the analog of the Fermi-Pasta-Ulam (FPU) recurrence in a one-dimensional focusing nonlinear Schrödinger equation (NLSE). That recurrence can be considered as a result of the nonlinear development of modulation instability. All known exact localized solitons-type solutions describing propagation on the background of the modulationally unstable condensate show the recurrence to the condensate state after its interaction with solitons. The condensate state locally recovers its original form with the same amplitude but a different phase after soliton leave its initial region. This is the analog of the FPU recurrence for the NLSE. Based on the integrability of the NLSE, we demonstrate that the FPU recurrence takes place not only for condensate but for more general solution in the form of the cnoidal wave. This solution is periodic in space and can be represented as a solitonic lattice. That lattice reduces to isolated soliton solution in the limit of large distance between solitons. The lattice transforms into the condensate in the opposite limit of dense soliton packing. The cnoidal wave is also modulationally unstable due to soliton overlapping. This instability at the linear stage does not provide the cnoidal wave recurrence. The recurrence happens at the nonlinear stage of the modulation instability. From the practical point of view the latter property is very important, especially for the fiber communication systems which use soliton as an information carrier.

1. The phenomenon of recurrence in nonlinear systems with many degrees of freedom was first observed in numerical experiment by Fermi, Pasta and Ulam [1] in 1954. The idea of Fermi was to address how randomization due to the nonlinear interaction leads to the energy equipartition between large number of degrees of freedom in the mechanical chain. The chain in [1] had a quadratic nonlinearity and included 64 oscillators supplemented with long-wave initial conditions. Instead of the energy equipartition, numerical experiments showed that after a finite time a recurrence to the initial data was achieved accompanied by a quasi-periodic energy exchange between several initially exited modes. That recurrence phenomenon became known as the Fermi-Pasta-Ulam (FPU) problem and has been one of the most attractive subjects for numerous investigations. Later, mainly by efforts of N. Zabusky, these results were reproduced by means of more powerful computers. Besides, there were observed many other peculiarities in this problem (for details, see the original papers by Zabusky (1962) [2], Zabusky & Kruskal (1965) [3], and Zabusky & Deem (1967) [4]). It was a time of forerunner of the era of integrability for nonlinear systems.

Since the discovery of the Inverse Scattering Transform (IST), which was first applied to the KDV equation by Gardner, Greene, Kruskal and Miura [5], and later to the nonlinear Schrodinger equation (NLSE) by Zakharov and Shabat [6], many aspects of the FPU recurrence became more clear. In 1971 Zakharov and Faddeev [7] proved that the KDV equation, which, in particular, can be obtained from the FPU system in the continuous limit for waves propagated in one direction, represents completely integrable Hamiltonian system. In 1974 Zakharov [8] demonstrated that the so-called nonlinear string equation (sometimes called as the Boussinesq equation) also belongs to the systems integrable by the IST. That equation is close to the continuous limit of the FPU system. According to [8] the long-time randomization for the FPU system can be explained by the ”distance” of that system to the nearest fully integrable one. In such a case its dynamics will follow in accordance with the nearest integrable system until deviations from the integrable trajectory reach values of order 1. This time can be taken as an estimate randomization time.

Now one can find many review papers devoted to the FPU recurrence phenomenon (see, e.g. [9-11]). The main conjecture of all of them is that the recurrence is a pure nonlinear phenomenon mainly intrinsic to the integrable models. It is worth noting that the FPU recurrence was also intensively studied experimentally. The first experimental demonstration in optical fibers of the FPU recurrence was presented by Van Simaeys et al. [12] showing that the dynamics of optical fields in the region of small third dispersion is described with a good accuracy by the NLSE and connected with modulation instability (MI). For current experimental situation, see [13] and references therein.

The main aim of this paper is to explain how the recurrence phenomenon appears within the one-dimensional NLSE

\[ i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0. \]  

(1)

It is well-known that in many physical applications this equation can be derived from the averaging of the equations of motion for solutions in the form of quasimonochromatic waves in a weakly nonlinear regime (see, e.g. [24]). Therefore very often the NLSE is considered to be the envelope equation. In the optical context, \( \psi \)
in Eq. (11) is a dimensionless amplitude of an electromagnetic wave packet in a reference frame moving with the group velocity. It is well-known that this equation with a reasonable accuracy describes propagation of optical solitons in fibers where they can be used as an information carrier [14]. Note that the current practical needs in fiber communications requires an increase in the information rates and consequently a denser packing of information. In such a case, soliton overlapping in soliton-based fiber communications becomes very important factor. Fortunately, the NLSE has an exact solution in the form of the soliton train, i.e. the so-called cnoidal wave. This is a whole family depending on several parameters. In particular, the soliton solution itself belongs to this family because it can be obtained from the cnoidal wave in the limit of an infinite spatial period. Another limit of the cnoidal wave is a solution in the form of a monochromatic wave or condensate. Such solution is unstable with respect to the modulation instability, also known as the Benjamin-Feir instability [13] (see, also [16]). Less is known about such instability for the cnoidal wave [17]. The growth rate in this case can be found exactly by means of the linearized dressing procedure [18] and expressed in terms of σ- and ζ- Weierstrass functions. When the distance between solitons becomes large enough, the maximal instability growth rate turns out to be exponentially small, but increases with the distance decrease [17]. For denser information packing the modulation instability can destroy information and therefore one needs to develop a nonlinear theory. From another side, a linear theory can not explain the FSU recurrence in the NLS system. Here we explain the FSU recurrence from the fully nonlinear theory.

At the present time, there are known many exact solutions of the NLSE which describe propagation of solitons/breathers on the condensate background. Such solution was constructed for a first time in [19] and later in many other papers (see [20–22] and references therein). All these solutions show that after a while the condensate recovers its amplitude but has a different (but constant) phase. This is the analog of the FPU recurrence for the NLSE. In this paper we give a qualitative explanation of the FPU analog for cnoidal waves. For fiber communications the latter means that such FPU recurrence can ensure the preservation of information, in spite of the MI existence.

2. We start from a stationary solution of Eq. (11) in the form \( \psi = e^{i\lambda^2 t} \psi_0(x) \) assuming for simplicity \( \psi_0(x) \) to be real. Then \( \psi_0(x) \) is defined from the Newton equation, \( \psi_0'' = -\partial U/\partial \psi_0 \), where \( U = 1/2 (-\lambda^2 \psi_0^2 + \psi_0^4) \) is the potential and \( \psi_0 \) has a meaning of time. In this case the stationary NLSE has a first integral, i.e. the energy
\[
\varepsilon = \frac{(\psi_0')^2}{2} + U(\psi_0)
\] (2) that allows one to find a solution depending on two parameters \( \varepsilon \) and \( \lambda \). At \( \varepsilon = 0 \) integration of this equation yields the well-known soliton solution
\[
\psi_0 = \lambda \text{sech}[\lambda(x - x(0))],
\] (3)
where \( x(0) \) is the coordinate of the soliton center of mass. (We remind that we consider here pure real \( \psi_0 \). In the general situation, soliton [45] has a constant phase multiplier.) When the energy reaches minimal value in the potential \( U(\psi_0) \), \( \varepsilon_{\text{min}} = -\lambda^2/8 \), we arrive at another well-known solution in the form of condensate with the constant amplitude,
\[
\psi_0 = \lambda/\sqrt{2},
\] (4)
Indeed, solutions (3) and (4) represent two limits of the two-parameters family. This is the so-called cnoidal wave which can be expressed in terms of elliptic functions. By introducing intensity \( I = \psi_0^2 \) and then shifting, \( I = -[\psi(x - i\omega') - \lambda^2/3] \), Eq. (2) transforms into the equation for the elliptic Weierstrass function:
\[
(\psi')^2 + U_\psi = 0,
\]
where \( U_\psi = -4(\varepsilon - e_1)(\varepsilon - e_2)(\varepsilon - e_3) \) has a meaning of a new ”potential energy” for trajectories related to a (new) ”energy” equal to zero. Here \( e_{1,2,3} \) are values of \( \phi \) in points \( z = \omega, \omega + i\omega', i\omega' \), besides, \( e_1 > e_2 > e_3 \) and \( e_1 + e_2 + e_3 = 0 \). In the given case they are equal to \( e_1 = \lambda^2/3, e_2 = -\lambda^2/6 + \sqrt{\lambda^2/4 + 2\varepsilon}, e_3 = -\lambda^2/6 - \sqrt{\lambda^2/4 + 2\varepsilon} \). The Weierstrass elliptic function is known as a double-periodic analytical function with periods \( 2\omega \) (along real axis) and \( 2i\omega' \) (along imaginary axis). Oscillations between zero points \( e_2 \) and \( e_3 \) in the ”potential” \( U_\psi \) defines the real period \( 2\omega \). The oscillations between \( e_1 \) and \( e_2 \) in imaginary ”time” \( x \rightarrow iy \) yields another period \( 2i\omega' \). As known [24] (see also [25]), the Weierstrass elliptic function can be represented in the form of the solitonic lattice:
\[
\psi(x - i\omega') = -\mu^2 \left\{ \sum_{n=-\infty}^{\infty} \text{sech}^2 [\mu (x - 2n\omega)] \right\}
+2 \sum_{n=1}^{\infty} \text{cosech}^2 [\mu (2n\omega)] - \frac{1}{3},
\]
where \( \mu = \pi / (2\omega') \). Respectively, for the intensity \( I \) we have
\[
I = \mu^2 \sum_{n=-\infty}^{\infty} \left\{ \text{sech}^2 [\mu (x - 2n\omega)] + \text{cosech}^2 [\mu (2n - 1) \omega] \right\}.
\]
In this lattice, the amplitude and inverse width for each soliton coincide and both equal to \( \mu \), in correspondence with (3). Hence, one can easily see that the soliton solution can be obtained as the limit of large spatial period, \( \omega/\omega' \rightarrow \infty \) when \( e_1 \rightarrow e_2 \).
Intensity $I$ reaches its minimum at the points $x_n = (2n + 1)\omega$ corresponding to half-distance between neighboring solitons:

$$I_{\text{min}} = \mu^2 \sum_{n = -\infty}^{\infty} \left\{ \text{sech}^2 [\mu (x - 2n\omega)] + \text{cosech}^2 [\mu (2n - 1)\omega] \right\}.$$  

This constant pedestal is a result of overlapping between solitons. At large distance between solitons, their overlapping is weak and, by this reason, $I_{\text{min}}$ becomes exponentially small:

$$I_{\text{min}} = \frac{4\pi^2}{(\omega')^2} \exp \left( -\frac{\pi \omega}{\omega'} \right).$$

In the opposite limit, $\omega'/\omega \to \infty$, when the size of soliton in the lattice tends to infinity, the overlapping between solitons becomes the main factor defining the cnoidal wave form: the function $\varphi(x - i\omega')$ in this case tends to the constant value equal to $-\lambda^2/6$, corresponding to the condensate solution $\Psi$, plus small harmonic oscillations,

$$\varphi(x - i\omega') \simeq -\lambda^2/6 + \sqrt{\epsilon - \epsilon_{\text{min}}} \cos k_0 x \quad (5)$$

with $k_0 = \pi/\omega = \sqrt{2}\lambda$. In this limit $\epsilon_2 \to \epsilon_3$.

3. It is well-known (see, e.g. [16]) that the condensate solution is unstable relative to the modulation instability with growth rate

$$\gamma = k\sqrt{2\lambda^2 - k^2}, \quad (6)$$

which vanishes at $k_0 = \sqrt{2}\lambda$ corresponding to stationary oscillations $\varphi$. It is less known that the cnoidal wave is also unstable relative to the condensate solution $\Psi$. To find the growth rate in this case one needs to solve the NLSE linearized on the background of the cnoidal wave by setting

$$\psi(x, t) = \psi_0(x) e^{\lambda x t} + \phi,$$

where $\phi$ is a small perturbation. Linearization of the NLSE gives the system of coupled PDEs of the second order. As it was shown in [17], using the IST simplifies significantly solution of this linear problem, in particular, for the linearized NLSE this problem reduces to solution of the first order PDEs. Following [17] we here show how this system arises from the auxiliary linear differential equations.

The NLSE can be represented as a compatibility condition for two linear equations [4],

$$\frac{\partial \Psi}{\partial x} = L\Psi = i(\lambda \sigma_3 + \hat{\psi})\Psi, \quad (7)$$

$$\frac{\partial \Psi}{\partial t} = A\Psi = i(2\lambda^2 \sigma_3 + 2\lambda \psi + \hat{\mathcal{Q}})\Psi, \quad (8)$$

where $\Psi$ is the two-component vector function, $\lambda$ is the spectral parameter and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\psi} = \begin{pmatrix} 0 & \psi^* \\ \psi & 0 \end{pmatrix}, \quad \hat{\mathcal{Q}} = \begin{pmatrix} -|\psi|^2 & -i\psi_x \\ i\psi_x & |\psi|^2 \end{pmatrix}.$$ 

As soon as $\psi$ satisfies the NLSE, a compatible solution of this overdetermined system exists for all $\lambda$. Consider now two linear equations for the $2 \times 2$ matrix function $\Phi$:

$$\frac{\partial \Phi}{\partial x} = L\Phi - \Phi L, \quad \frac{\partial \Phi}{\partial t} = A\Phi - \Phi A,$$

which are also compatible due to (4) and (5). Then perturbation $\phi$ is defined by means of the relation

$$\begin{pmatrix} 0 & \phi^* \\ \phi & 0 \end{pmatrix} = [\sigma_3, \Phi].$$

That can also be verified by direct calculation as follows. First, the diagonal part of the compatibility equations and the spectral parameter $\lambda$ must be excluded from this system. Second, a simple algebra then yields the linearized NLS equation.

This scheme is nothing more than the linearized version of the Zakharov-Shabat dressing procedure [26]. This version, in fact, was introduced for a first time in 1974 [25] and developed later in [18].

To find $\Phi$ for the cnoidal wave, one needs first to exclude a time dependence from the matrices $L$ and $A$ by means of simple transformation (rotation) making these matrices to be periodic functions in $x$: $L \to L_0(x)$ and $A \to A_0(x)$. Thus, a solution of this system is necessary to be sought in the form $\Phi_0(t, x) = \Theta(x)e^{i\gamma t}$, where $\Theta(x)$ is of the Bloch form with real quasi momentum $p$, $\Theta(x) = \theta_p e^{ip x}$, $\theta_p(x+2\omega) = \theta_p(x)$. Solvability condition to this transformed system gives the dispersion relation

$$\gamma = \gamma(p)$$

which is defined from a solution of pure algebraic equation

$$\gamma \Theta = [A_0, \Theta].$$

Omitting then all quite simple calculations (for more details, see [11]) we present the final answer for the maximal growth rate when the distance between solitons becomes large enough:

$$\gamma_{\text{max}} = 8 \left( \frac{\pi}{\omega} \right)^2 \exp \left( -\frac{\pi \omega}{\omega'} \right).$$

This expression shows that $\Gamma$ is exponentially small in this case but grows with the spatial period decrease.

In another limit of the condensate, we arrive at the classical expression for the MI growth rate given by [3].

4. What happens at the nonlinear stage of the MI for the cnoidal wave? We know that, according to Zakharov and Shabat [4] (see, also [27]), the phase space of the NLSE represents discrete number of degrees of freedom which are solitons (these are the most nonlinear objects) and solutions corresponding to the continuous spectrum. Moreover, we know that collisions between solitons are elastic and pairwise. A scattering of two solitons results only in changing of two their parameters, namely the center of soliton mass and its phase as follows

$$\Delta x_1^{(0)} = \frac{1}{2\eta_1} \log \left| \frac{\lambda_1 - \lambda_2^*}{\lambda_1 - \lambda_2} \right|^2, \quad \Delta \phi_1^{(0)} = 2\arg \left( \frac{\lambda_1 - \lambda_2^*}{\lambda_1 - \lambda_2} \right).$$
where $\Delta x_1^{(0)}$ and $\Delta \phi_1^{(0)}$ are respectively shifts in mass center and phase for the first soliton, $\lambda_{1,2}$ are eigenvalues corresponding to the first and second solitons, $\eta = \text{Im } \lambda > 0$ and $\lambda^*$ means complex conjugation of $\lambda$. Analogous formula can be written for the second soliton.

The cnoidal wave is of the form of the solitonic lattice. Therefore any soliton from the lattice after interaction with a soliton propagating along the cnoidal wave will undergo the same shift for its center of mass and phase. This means that after scattering of the propagating soliton with the lattice, the cnoidal wave will restore its previous form (up to the definite spatial and phase shifts). Evidently, the same statement will be valid for condensate as the partial solution of the cnoidal wave. The interaction of condensate with any soliton after its propagation will restore amplitude of the condensate but its (constant) phase will be different from the initial value.

Scattering of a soliton with the non-soliton part also retains the soliton form unchanged except shifts of both center of mass of the soliton and its phase. Thus, the cnoidal wave subject to the modulation instability, at the nonlinear stage of the modulation instability development, should recover its form together with some phase and spatial shifts.

This is the qualitative explanation of the FSU recurrence for the cnoidal wave and for the condensate, in particular. It is necessary to underline that the same phenomenon takes place for the KDV cnoidal wave that was found by Mikhailov and the author in 1974 [25]. Similar to the KDV case, an exact solution describing propagation of soliton on the cnoidal wave background can be found analytically for the NLSE also. For instance, it can be done by means of the dressing procedure which is however beyond the scope of this paper. We remark only that this (still unknown) solution will represent a defect (or point dislocation) of the cnoidal wave. These objects are non-stationary characterizing by two different frequencies. Like for the KDV cnoidal wave, these defects propagate along the wave with some mean velocity. In the partial case of the condensate these solitons were well studied analytically (see, e.g., [22] and references therein).

In conclusion, we have presented qualitative explanation for the analog of the FPU recurrence for the cnoidal waves in the presence of perturbations, which are not assumed to be small. In this meaning the recurrence for the condensate during nonlinear development of modulation instability represents the partial case of the recurrence for the more general solution in the form of the cnoidal wave. The conjecture made in this paper is based on the integrability of the NLSE [3], in particular, due to the elastic and pairwise collisions between solitons and between solitons and non-soliton part. It is necessary to underline that the same phenomenon takes place for the KDV cnoidal wave that was found in 1974 [25]. The arguments presented here for the NLSE can be directly applied to explain recurrence for the KDV cnoidal wave.

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