LATTICE $(\Phi^4)_4$ EFFECTIVE POTENTIAL GIVING

SPONTANEOUS SYMMETRY BREAKING AND

THE ROLE OF THE HIGGS MASS

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Abstract

We present a critical reappraisal of the available results on the broken phase of $\lambda(\Phi^4)_4$ theory, as obtained from rigorous formal analyses and from lattice calculations. All the existing evidence is compatible with Spontaneous Symmetry Breaking but dictates a trivially free shifted field that becomes controlled by a quadratic hamiltonian in the continuum limit. As recently pointed out, this implies that the simple one-loop effective potential should become effectively exact. Moreover, the usual naive assumption that the Higgs mass-squared $m_h^2$ is proportional to its “renormalized” self-coupling $\lambda_R$ is not valid outside perturbation theory: the appropriate continuum limit has $m_h$ finite and vanishing $\lambda_R$. A Monte Carlo lattice computation of the $\lambda(\Phi^4)_4$ effective potential, both in the single-component and in the O(2)-symmetric cases, is shown to agree very well with the one-loop prediction. Moreover, its perturbative leading-log improvement (based on the concept of $\lambda_R$) fails to reproduce the Monte Carlo data. These results, while supporting in a new fashion the peculiar “triviality” of the $\lambda(\Phi^4)_4$ theory, also imply that, outside perturbation theory, the magnitude of the Higgs mass does not give a measure
of the observable interactions in the scalar sector of the standard model.

1. Introduction

The so-called effective potential of a quantum field theory is a powerful tool to get informations about the structure of its vacuum, whose analysis, in turn, gives insight into its physical content. A reliable determination of the effective potential requires a critical control of details not directly accessible to a perturbative calculation. A convenient approach to uncovering the non-perturbative features of such details may start by trying to determine the continuum limit of some discrete version of the theory on a lattice. In this approach one exploits the correspondence of an Euclidean (imaginary-time) quantum field theory to a relativistic (real-time) theory, the Schwinger’s functions of the former determining the Green’s functions of the latter. While substantial experience has been already gained in the numerical analysis of lattice field theories, the reliability of statements about their continuum limits is still open to questions, due to the fact that the equivalence of different procedures to get those limits is not yet controlled by theoretically well defined conditions. This amounts to a shift of the problem from the domain of bare computation, including the evaluation of numerical errors and/or approximations, to that of interpretation, exploiting the connections of the numerical results with the formal theory or with suitable models of the continuum limit.

The continuum $\lambda(\Phi^4)_D$ theories of a self-interacting scalar field $\Phi$ in $D$ spacetime dimensions are perhaps the simplest examples to study as limits of their versions on a lattice. Their physical interest stems from their role in the Standard Model, as Spontaneous Symmetry Breaking (SSB) and the Higgs phenomenon determine the masses of the vector bosons and some other essential features of the Electroweak Sector of the model, e.g. with $\Phi$ being a (non hermitian) isospin doublet.

The scalar $\lambda(\Phi^4)_D$ theories are characterized by a rather peculiar $D$-dependence of their dynamical content: they have been proved to be ”trivial” (i.e. equivalent to a generalized free field theory) for $D > 4$, but not so (i.e. truly interacting) for $D < 4$. For $D = 4$ the
situation is a bit less definite, since, instead of a proof, one has several “evidences” (based, of course, on different methods of analysis, whose assumed correctness points out a lack of equivalence) suggesting, in some sense, either “triviality” or “nontriviality”, the latter being, admittedly, an almost ignored option [7].

The commonly accepted evidence about the “triviality” of $\lambda(\Phi^4)_4$ is frequently assumed to support the view that the scalar sector of the standard model, dynamically generating the vector boson masses through the Higgs mechanism [1], can only be an effective theory, valid up to some cutoff scale. Without a cutoff, the argument goes, there would be no scalar self-interactions, and without them there would be no symmetry breaking [8,9]. This point of view also leads to upper bounds on the Higgs mass [8–10].

Recently [11,12], it has been pointed out, on the basis of very general arguments, that SSB is not incompatible with “triviality”, at least in the case of the scalar field theory with quartic self-interactions. One could have a non-zero vacuum expectation value (VEV) of the field, $\langle \Phi \rangle$, yet find only non-interacting, quasi-particle excitations above the vacuum. This picture, while providing a consistent model-theory characterized by the coexistence of “triviality” and of a non-trivial effective potential [13–21], also clarifies the meaning of some explicit studies of “triviality” in the broken phase ($\langle \Phi \rangle \neq 0$) [22].

As discussed in [11,12], since the effective potential determines the zero-momentum 1PI vertices of the symmetric phase [25], its structure might carry some traces of non trivial interactions modifying the zero-momentum-mode sector of the symmetric phase in a fashion which somehow de-stabilizes the symmetric vacuum and gives rise to SSB.

This remark [11,12], immediately brings out the relevance of studying the massless $\lambda(\Phi^4)_4$ theories, as, for them, the zero-momentum ($p^\mu = 0$) is a physical on-shell point (inaccessible to perturbation theory).

If, in the continuum limit, all interaction effects of the symmetric phase can be reduced into a set of Green’s functions expressible in terms of just the first two moments of a Gaussian (functional) distribution, then one gets a description of SSB consistent with “triviality” which does not contradict any analytical or numerical result [13 21 1]. In this case the
shifted field \( h(x) = \Phi(x) - \langle \Phi \rangle \) becomes governed by a quadratic Hamiltonian and the simple “one-loop” potential turns out to be effectively exact. More precisely, the one-loop potential and the gaussian effective potential determine the same relation

\[
m_h^2 = 8\pi^2 v_R^2
\]

between the physical Higgs mass \( m_h \) and the physical field VEV \( v_R \). This turns out to be true as well in the post-gaussian approximations (see, in particular, [27]), where the Higgs propagator \( G(x,y) \) is properly optimized at each value of \( \phi = \langle \Phi \rangle \), by solving the corresponding non-perturbative gap-equation \( G = G_o(\phi) \) [11,12,17–19,21,27]. This result follows from the exact connection of the effective potential \( V_{\text{eff}}(\phi) \) with the effective potential for composite operators \( E[\phi,G] \) [28]

\[
\int d^3x \ V_{\text{eff}}(\phi) = E[\phi,G_o(\phi)]
\]

and from “triviality”, which entails that all interaction effects (apart from those included in the physical vacuum structure) should be reabsorbed into the physical mass of the “generalized free field” \( h(x) \) defined above. In any approximation consistent with the (assumed) exact “triviality” of the theory the mass of the Higgs field cannot represent a measure of the observable interactions in the broken phase which, in the absence of interactions involving other fields, should contain only trivially free excitations.

As we shall explain in sect. 2, this simple picture of an interacting zero-momentum mode that produces a non-trivial shape for the effective potential, giving SSB, co-existing with trivially non-interacting excitations in the broken phase, is not in contradiction with any knowledge derived from both all formally rigorous theorems about \( \lambda(\Phi^4)_D \) in the broken phase and all available lattice calculations.

It is well known, since a long time, that the effective potential of \( \lambda(\Phi^4)_D \) can be analysed in a rather simple way on a lattice by computing the VEV of the field as a function of a (global or uniform) ”external source” strength \( J \) [32,23,24].

We report, in sects.3,4, an analysis of the data obtained from precise Monte Carlo lattice
calculations of the effective potential for both the single component and the O(2)-symmetric $\lambda(\Phi^4)_4$ theory.

We have performed such an analysis for several values of the bare quadratic ($r_o$) and quartic ($\lambda_o$) couplings in the Euclidean discrete action in four dimensions

$$S = \sum_x \left[ \frac{1}{2} \sum_\mu (\Phi_{x+\mu} - \Phi_x)^2 + \frac{r_o}{2} \Phi_x^2 + \frac{\lambda_o}{4} \Phi_x^4 - J\Phi_x \right]$$

(2)

where $x$ stands for a general lattice site and, unless otherwise stated, lattice units are understood.

The numerical results from a version of Metropolis’ Monte Carlo algorithm have been best fitted with analytical forms possibly consistent with ”triviality” and allowing for some suitably chosen “corrections”.

As a matter of fact, the one-loop potential agrees remarkably well with the lattice results, while its perturbative “improvement” fails to reproduce the Monte Carlo data.

Thus, definite support is obtained for the picture proposed in [11,12]. The basic physical interest of our analysis, emphasizing a rather unconventional but theoretically not surprising peculiarity of “renormalization” in this Bose field example, also stems from the new interpretation given to the Higgs mass, no more appearing as proportional to a renormalized coupling constant (whose vanishing is required by “triviality”), with remarkable new implications on the dynamics of the Higgs particle in the Standard Model.

As we shall explain in sect.2, our numerical results agree with those already obtained from lattice calculations, as far as we know, and our theoretical analysis is fully consistent with the general conditions imposed on the theory by the coexistence of SSB and triviality.

Our analysis also shows that “triviality” and perturbation theory, within any attempt to construct a cutoff-independent and non-vanishing renormalized coupling at non zero external momenta $\lambda_R$, are in inner contradiction [29]. The (new) evidence of this incompatibility is given by the lattice computation of the effective potential which turns out to represent just the sum of the classical potential and of the zero-point energy of a free field. Outside perturbation theory, $m_h$ and $\lambda_R$ are not proportional and while the latter vanishes the
former remains an arbitrary parameter. This result has important physical implications as discussed in sect.5 where we shall also propose our conclusions.

2. SSB and “triviality”

In our study of SSB the basic quantity to compute is the VEV of the bare scalar field $\Phi(x)$

$$\langle \Phi \rangle_J = \phi_B(J)$$

in the presence of an “external source” whose strength $J(x) = J$ in Eq.(2) is $x$-independent.

In fact, determining $\phi_B(J)$ at several $J$-values is equivalent [23,24] to inverting the relation

$$J = J(\phi_B) = \frac{dV_{\text{eff}}}{d\phi_B}$$

involving the effective potential $V_{\text{eff}}(\phi_B)$. Actually, starting from the action in Eq.(2), the effective potential of the theory can be rigorously defined, up to an arbitrary integration constant, through the above procedure.

In this framework, the occurrence of SSB is determined by exploring (for $J \neq 0$) the properties of the function

$$\phi_B(J) = -\phi_B(-J)$$

in connection with its behaviour in the limit of vanishing $J$

$$\lim_{J \to 0^\pm} \phi_B(J) = \pm v_B \neq 0$$

over a suitable range of the bare parameters $r_o, \lambda_o$ appearing in the lattice action Eq.(2).

The existence of the $\lambda(\Phi^4)_4$ critical point can be stated, in the framework connecting the lattice theory with its continuum limit, on the basis of rigorous formal arguments [4]. Namely, for any $\lambda_o > 0$, a critical value $r_c = r_c(\lambda_o)$ exists, such that, for $r_o < r_c$, one finds non trivial solutions of Eq.(6). After fixing the vacuum and hence $\langle \Phi \rangle = v_B$, one can define a shifted field
whose VEV vanishes by construction.

As reviewed in the last section of [3], the following statements hold true. For $D > 4$, the continuum limit, precisely defined in Ch.15 of [3], is “trivial” in the sense that the connected n-point Green’s functions ($n \geq 3$) vanish in that limit. Moreover, the shifted field renormalization constant $Z_h$ (as derived from the renormalization of the term containing the field derivatives in the bare action) must be finite. As a consequence, if one assumes a single renormalization constant for both the field VEV $v_B$ and the shifted field $h(x)$, one cannot get both a finite Higgs mass and a finite vacuum expectation value in the continuum limit [3]. Indeed, the bare field VEV $v_B$ becomes infinite in units of the mass $m_h$ of the shifted field which, in this specific context, must be trivially free.

In this picture, where the same “$Z$” controls the scaling properties of both vacuum field and fluctuations, namely

$$Z = Z_h$$

what amounts to agree that

$$v^2_R = \frac{v^2_B}{Z_h}$$

the above quoted statements [3] requiring

$$\frac{m_h^2}{v^2_B} \to 0$$

and the finiteness of $Z_h = Z$, also entail

$$\frac{m^2_h}{v^2_R} = \frac{m^2_h Z_h}{v^2_B} \sim \frac{m^2_R}{v^2_B} \to 0$$

In this approach, where $\frac{v^2_R}{Z_h} \sim v^2_B$ is kept fixed and related to the Fermi constant, $m^2_h$ vanishes in the continuum limit and one concludes that the theory makes sense only in the presence of an ultraviolet cutoff. This point of view, also leads to upper bounds on the Higgs mass [8–10]. In all interpretations of the lattice simulations performed so far, the validity of
Eq.(8) has been assumed. Let us refer to the very complete review by C.B.Lang [1]. There, the value of $Z = Z_h$ is extracted from the large distance decay of the Goldstone boson propagator and used in Eq.(9) to define a renormalized field VEV $v_R$ from the bare field VEV $v_B$ as measured on the lattice. Since the lattice data both support the trend in Eq.(10) and provide $Z_h = 1$ to very good accuracy (see Tab. II of [9]) one deduces (11). The limiting behaviour (11), obtained in this way, is usually interpreted (see [9] Fig.19, caption included) to be a test of the validity of the perturbative relation

$$m_h^2 = \lambda_R v_R^2$$

in which the renormalized coupling $\lambda_R$, at external momenta comparable to the Higgs mass itself, has been introduced.

As discussed in the Introduction, $\lambda_R$ is a spurious concept in a “trivial” theory. On the other hand, without assuming the validity of any perturbative or quasi-perturbative relation, the lattice data confirm the conclusions of [5], namely, in the continuum limit

a) the bare field becomes infinite in units of the Higgs mass, i.e. (10)

b) the shifted field (fields) is (are) trivially free since $Z_h \rightarrow 1$

Point a), supported by theoretical arguments and by the results of lattice simulations, can be taken as a basic condition defining the continuum limit of SSB in $\Phi^4$ theories.

Point b) is completely consistent with the usual treatment, based on the Lehmann spectral representation of the propagator, for a field with vanishing VEV. To this end one identifies [30] the $Z_h$ of the continuum theory, related to the (integral of the) spectral function $\rho(s)$, with the limit-value of its cutoff-dependent form when the ultraviolet regulator is (continuously) removed. Thus $Z_h = 1$ implies $\rho(s)$ to be just a $\delta$-function, since the relevant fields have a vanishing VEV.

Points a) and b) describe two basic outcomes of “triviality” in the broken phase. A consistent description of SSB in the $\lambda(\Phi^4)_4$ theory should explain the non-uniform scaling of the Higgs mass and of the bare vacuum field while preserving, at the same time, the non-interacting nature of the shifted field(s).
Does perturbation theory provide any clue to understand a) and b)? Strictly speaking, the perturbative one-loop $\beta$-function, exhibiting an unphysical Landau pole, does not define any continuum limit. The usually accepted view maintains that the interval between the origin and the Landau pole becomes arbitrarily large in the limit $\lambda_R \to 0$, thus allowing to recover the continuum limit of the theory in agreement with “triviality” while still preserving, at finite ultraviolet cutoff, the validity of the two perturbative relations (9,12). This interpretation of “triviality” can be questioned for the following reasons. In perturbation theory, $\lambda_R$ is the basic Renormalization Group (RG) invariant whose cutoff-independence generates the cutoff-dependence of the bare coupling $\lambda_o = \lambda_o(a)$, with $a$ denoting the lattice spacing equivalent to the ultraviolet cutoff $\Lambda$. Different $\lambda_R$’s define different theories. It would be contradictory, and inconsistent with the original premise that $\lambda_R$ is finite and cutoff-independent, to now conclude that $\lambda_R$ has any implicit or explicit cutoff dependence that causes it to vanish in the limit $a \to 0$. “Triviality” means that $\lambda_R$, which represents a measure of the observable interactions, must vanish identically in order to get a flow of the bare parameters $r_o = r_o(a)$, $\lambda_o = \lambda_o(a)$ extending down to $a = 0$. The one-loop perturbative $\beta$-function, obtained under the assumption $\lambda_R > 0$ (and $\frac{\partial \lambda_o}{\partial a} \ln \frac{\Lambda^2}{m_h^2} < 1$ ) simply does not allow for such an extension. Furthermore, the above perturbative interpretation of “triviality” is inconsistent with explicit two-loop calculations, where the perturbative $\beta$-function exhibits a non trivial ultraviolet fixed point at non zero bare coupling $\lambda_o = \lambda^* > 0$ (let us remark that rigorous arguments [3] can be given to qualify as “spurious” this additional fixed point). In the perturbative approach, if $\lambda_R$ is taken to represent a value of the bare coupling at some finite momentum scale, $\lambda_o = \lambda_o(a)$ flows towards $\lambda^*$ when $a \to 0$ for any $0 < \lambda_R < \lambda^*$.

Moreover, the perturbative explanation of Eq.(10), namely $m_h^2 \to 0$ at $v_B^2 \sim \frac{m_h^2}{Z_h} = fixed$, is certainly not unique. One might as well consider, e.g. the alternative possibility $v_B^2 \to +\infty$, $m_h^2 = fixed$, if the renormalized vacuum field $v_R$ and $v_B$ are non trivially related through a renormalization constant $Z_\phi \neq Z_h$. This alternative, first proposed in [14] and later on thoroughly discussed in [11,12,27], does not violate any basic principle. Let us
emphasize that the vacuum field renormalization $Z_\phi$ has nothing to do, in principle, with the cutoff-dependence of the term containing the field derivatives in the bare action. The usual assumption of a single “field”-renormalization factor derives from giving an operatorial meaning to the field rescaling, namely

\[ \Phi(x) = \sqrt{Z} \Phi_R(x) \]

This relation is a consistent shorthand for expressing the “wave function”-renormalization in a theory allowing an asymptotic Fock representation, but no more so in the presence of SSB, since, then, it overlooks that the shifted field is not defined before fixing the vacuum. The Lehmann spectral-decomposition argument constrains only $Z_h$.

The previous discussion should make clear that the perturbative interpretation (9,12) of the basic results a) and b) might be hardly taken as correct. At the same time, it should make apparent that the problems associated with the peculiar structure of the continuum limit of the $\lambda(\Phi^4)_4$ theory must be faced anew with open minded critical attention.

In the following we shall briefly review the nonperturbative renormalization pattern which arises from the analysis of the effective potential in those approximations which are consistent with “triviality”, as referred to in the Introduction and explained in [11,12]. This will clearly show how the two basic results a) and b) are consistently derived in our approach. The basic difference with respect to perturbation theory, where one attempts to define a continuum limit in the presence of observable interactions, lies in the choice of the quantities that are required to remain finite in the continuum limit: namely, the vacuum energy and the “gap” in the energy spectrum corresponding to the mass of the free-field quanta.

In terms of the bare vacuum field $\phi_B = \langle \Phi \rangle$ and of the bare coupling $\lambda_o$ we obtain the well-known result for the one-loop effective potential

\[ V^{1-\text{loop}}(\phi_B) = \frac{\lambda_o}{4} \phi_B^4 + \frac{\omega^4(\phi_B)}{64\pi^2} \left( \ln \frac{\omega^2(\phi_B)}{\Lambda^2} - \frac{1}{2} \right) \]  

(13) where, by definition, $\omega^2(\phi_B) = 3\lambda_o\phi_B^2$ and $\Lambda$ denotes the Euclidean ultraviolet cutoff equivalent to the lattice spacing ($\Lambda \sim \frac{\pi}{a}$). The minimum condition requiring
\[ m_h^2 = \omega^2(v_B) = 3\lambda_0 v_B^2 = \Lambda^2 \exp(-\frac{16\pi^2}{9\lambda_0}). \tag{14} \]

with \( Z_\phi = \frac{8\pi^2}{3\lambda_0} \), also allows giving the one-loop effective potential in the form
\[
V^{1\text{-loop}}(\phi_B) = \frac{\pi^2 \phi_B^4}{Z_\phi^2} (\ln \frac{\phi_B^2}{v_B} - \frac{1}{2}), \tag{15} \]

Finally, the ground state energy is
\[
W = V^{1\text{-loop}}(v_B) = -\frac{m_h^4}{128\pi^2} \tag{16} \]

By defining the physical vacuum field \( \phi^2_R = \frac{\phi_R^2}{Z_\phi} \) in such a way that all zero-momentum interaction effects are reabsorbed into its normalization, i.e. so that
\[
\frac{d^2V^{1\text{-loop}}}{d\phi^2_R} |_{\phi_R = \pm v_R} = m_h^2 \tag{17} \]

holds true, we get \[11,12,17,19,18\]
\[
V^{1\text{-loop}}(\phi_R) = \pi^2 \phi_R^4 (\ln \frac{\phi_R^2}{v_R^2} - \frac{1}{2}) \tag{18} \]
and
\[
m_h^2 = 8\pi^2 v_R^2 \tag{19} \]

The non-perturbative nature of the vacuum field renormalization \( (Z_\phi \sim 1/\lambda_0) \), first discovered in the gaussian approximation by Stevenson and Tarrach \[14\], should not be confused with the \( h \)-field wave function renormalization. In fact, at one loop, \( h \) is just a free field with mass \( \omega(\phi_B) \), and hence one has trivially \( Z_h = 1 \), in agreement with the basic result b). The structure with \( Z_\phi \neq Z_h \), is allowed by the Lorentz-invariant nature of the field decomposition into \( p_\mu = 0 \) and \( p_\mu \neq 0 \) components \[11,12\]. It is, of course, more general than the one assumed in perturbation theory, and its unconventional appearance is, in a sense, an indication of the peculiar non-perturbative nature of the physical vacuum in the massless \( \lambda(\Phi^4)_4 \) theory, which allows SSB to coexist with “triviality”. Notice that the possibility of a different rescaling for the vacuum field and the fluctuations is somewhat implicit in the
conclusions of the authors of [5] (see their footnote at page 401: ”This is reminiscent of the standard procedure in the central limit theorem for independent random variables with a nonzero mean: we must subtract a mean of order $n$ before applying the rescaling $n^{-1/2}$ to the fluctuation fields”).

To further illustrate the asymmetric treatment of the zero-momentum mode (which is well known to play a special role in a Bose system at zero temperature) one can consider the analogy with the quantum mechanical 3-dimensional $\delta$- potential discussed by Huang [24] and Jackiw [31]. In this example, the exact solution of the Schrödinger equation requires the radial wave-function to vanish at the origin. This condition is automatically satisfied for all partial waves except S-waves. For S-waves it cannot be satisfied if the wave function is required to be continuous at the origin. In that case, there would be no S-waves, and the solutions of the equation would not form a complete set. But a discontinuity at the origin might be acceptable, since the potential is singular there. By defining the $\delta$-potential as the limit of a sequence of square well potentials, it is straightforward to show that, in the limit of zero-range, the S-wave solution vanishes discontinuously at the origin. Not surprisingly, all scattering phase shifts vanish. The analogy with $\lambda\Phi^4$ theory is remarkable. There is no observable scattering, and yet the particle is not completely free. It is free everywhere except at the origin where the S-wave vanishes abruptly. In our field theoretical case, all finite momentum modes are free and as such there is no non-trivial S-matrix. However, the unobservable zero-momentum mode is a truly dynamical entity whose non-trivial self-interaction is not in contradiction with ”triviality” being reflected in the zero-point energy of the free shifted field.

In conclusion, the basic result a) is in complete agreement with the analytical results derived from the one-loop effective potential. Indeed, the finiteness of the RG-invariant ground state energy in Eq.(16) requires that the Higgs mass has also to be finite. In the continuum limit one gets a cutoff-independent $m_h$ and a finite ground state energy in connection with the vanishing of the bare coupling as
\[ \lambda_o = \frac{16\pi^2}{9\ln \frac{\Delta^2}{m_h^2}} \]  

(20)

In this limit, from Eq.(14), one rediscovers, with \( Z_h = 1 \) identically, the trend in (10) in complete agreement with the previous points a) and b). Eqs.(18.19) also hold in the more sophisticated gaussian and postgaussian approximations [11,12,17,18,27]. It is important to note that the logarithmic trend in Eq.(20), controlling the cutoff dependence of the ratio \( \frac{m^2}{v_B} \) (see Eq.(14)), allows to explain the observed logarithmic decrease on the lattice of the ratio \[ \frac{m^2 Z_h}{v_B^2} \sim \frac{m^2}{v_B^2} \sim \frac{1}{\ln \frac{\Delta^2}{m_h^2}} \]  

(see fig.19 of [1]) without any need of introducing the leading-log \( \lambda_R \)

\[ \lambda_R = \frac{\lambda_o}{1 + \frac{9\lambda_o}{16\pi^2} \ln \frac{\Delta^2}{m_h^2}} \]

Indeed, the usual perturbative interpretation of “triviality” (see fig.6 of [1]), based on the assumed reliability of the above equation at \( \lambda_o = +\infty \), i.e.

\[ \lambda_R = \frac{16\pi^2}{9\ln \frac{\Delta^2}{m_h^2}} \]

neglects that the leading-log resummation is not defined in that region and, as previously recalled, contradicts explicit perturbative two-loop calculations.

3. Lattice computation of \( \phi_B(J) \).

The VEV of the bare field \( \langle \Phi \rangle_J = \phi_B(J) \) is the simplest quantity to compute on the lattice, and its \( J \)-dependence can be exploited to get the first derivative of the effective potential, Eq.(4), as already recalled at the beginning of sect.2. In this sense, a computation of the slope of the effective potential, based on the response of the system to an applied external source, has the advantage of being fully model-independent. At the same time, since it is well known [4,32] that the data become unstable for small values of the external source, we have to restrict our analysis to a ”safe” region of \( J \) (in our case \( |J| \geq 0.05 \)) where finite size effects appear to be negligible as checked on several lattices (see below). Indeed, at smaller \( |J| \) the precision of the data becomes uncertain (not only but especially in the
neighborhood of the critical point) and this was confirmed by testing the accuracy of the
data against Eq.(5). The loss of accuracy observed in the neighborhood of $J = 0$ cannot be
accounted for by numerical and statistical errors but is sensitive to finite size effects. In this
way we have been led to avoid any “direct” lattice calculation of $\phi_B(J)$ at $J = 0$, at least
in a “small sized” lattice as $10^4$, and this is the main motivation for our choice of a method
not relying on such “direct” calculations in the analysis of the lattice data.

In order to compare the lattice data with the continuum theory one has to employ some
definite model. As recalled in the Introduction, the equivalence of different procedures to get
this limit is not yet safely controlled by theoretically well defined conditions. For instance,
as discussed by Lang [9] one can use an effective theory (the so called ”chiral perturbation
theory”) to relate the lattice quantities computed in a finite volume at non-zero $J$ to those
at $J = 0$ in an infinite volume. The cutoff-dependence of such “thermodynamic” quantities
can be then compared with models of the continuum limit.

In our case, we shall use two basically different functional forms, based on the one-loop
effective potential and its perturbative leading-log improvement, to fit the measured values of
$\phi_B(J)$ and to extrapolate $\phi_B(J)$ toward $J \to 0^\pm$. Our choice is motivated by the discussion
of SSB and “triviality” presented in sect.2. However, anybody could use our data, collected
in a ”safe” region of $J$ and determined in a model-independent way, to compare with his
own preferred model and draw the appropriate conclusions.

The calculation of $\phi_B(J)$, on the basis of the action given in Eq.(2), has been performed by
using a version of the Metropolis’ Monte Carlo algorithm with periodic boundary conditions,
for several values of the parameters $r_o, \lambda_o, J$ appearing in the discrete action. Other currently
used algorithms, as the cluster method and the hybrid Langevin’s approach, are presently
under our investigation and the results will be compared in a forthcoming paper. However,
possible improvements are only expected in the region of very small $J$ which is not analyzed
here.

The calculations have been carried out on DEC 3000 AXP (Mod.400 and 500) computers.

With the aim of obtaining a reliable control on the precision of the data, we have carefully
analyzed their dependence on the choice of the random number generator, of the “thermalization” process and of the dimension of the sample (number of “sweeps” or “iterations”) entering the determination of the relevant data. The results for \( J = J(\phi_B) \) obtained from runs on our \( 10^4 \) lattice have been compared with those reported in \[23,24\] and the agreement with the plots \( J = J(\phi_B) \) at \( \lambda_0 = 1 \) in Fig.2 of \[23\] (suitably enlarged) has been found better than 1% in the range \(|J| \geq 0.05\). Other comparisons with \[23,24\] have not been attempted in view of their direct determination of \( v_B \) at \( J = 0 \) and of their assumption \( Z_\phi = Z_h \) in the evaluation of the shifted field propagator.

After 60,000 iterations, which correspond to the data reported in Table I, the calculated \( \phi_B(J) \) is stable at the level of the first three significant digits for all \(|J|\)-values. This stability has been controlled with several runs up to 200,000 iterations. As \(|J|\) increases, the stability gets better (down to the level of \( 10^{-4} \) in the explored range of \( J\)-values) and finite size effects appear negligible, as checked by comparing with the runs on a \( 16^4 \) lattice and, subsequently, confirmed by an independent computation performed by P. Cea and L. Cosmai \[33\] reported below.

In order to introduce our analysis of the Monte Carlo data, let us first discuss the predictions from the one-loop potential which, as discussed in sect.2, reflect the “triviality” of the theory. By differentiating Eq.(15), we obtain the “bare source”

\[
J(\phi_B) = \frac{dV^{1-\text{loop}}(\phi_B)}{d\phi_B} = \frac{4\pi^2 \phi_B^3}{Z_\phi^2} \ln \frac{\phi_B^2}{v_B^2},
\]

(21)

which can be directly compared with the lattice results for \( J = J(\phi_B) \). Strictly speaking, just as the effective potential is the convex envelope of Eq.(15) \[33,21\], Eq.(21) is valid only for \(|\phi_B| > v_B\) and \( J(\phi_B) \) should vanish in the presence of SSB in the range \(-v_B \leq \phi_B \leq v_B\).

This means that the average bare field \( \phi_B(J) = \langle \Phi_B \rangle_J \), while satisfying the relation (5), i.e.

\[
\phi_B(J) = -\phi_B(-J)
\]

(5)

for any \( J \neq 0 \), should tend to the limits stated in Eq.(6). However, as discussed above, Eq.(5) is not well reproduced numerically for \(|J| \sim 0.01\) or smaller and we shall have to
restrict our analysis to a “safe” region of \( J \)-values ( \(|J| \geq 0.05\)) allowing to get the values of \( v_B \) and \( Z_\phi \) from a best fit to the lattice data by using Eq.(21) after having identified on the lattice the “massless” regime, i.e. the one corresponding to a renormalized theory with no intrinsic scale in its symmetric phase \( \langle \Phi \rangle = 0 \).

Usually, this would require to find the value of the quadratic bare coupling \( r_o \) for which

\[
\frac{d^2V}{d\phi_B^2}|_{\phi_B=0} = 0
\]

However, the region around \( \phi_B = 0 \) being not directly accessible, we have argued as follows. We start with the general expression \([12]\)

\[
J(\phi_B) = \alpha \phi_B^3 \ln(\phi_B^2/v_B^2) + \beta v_B^2 \phi_B (1 - \phi_B^2/v_B^2), \tag{22}
\]

which is still consistent with “triviality” (corresponding to an effective potential given by the sum of a classical background and the zero point energy of a free field) but allows for an explicit scale-breaking \( \beta \)-term. Setting \( \alpha = 0 \) one obtains a good description of the data in the “extreme double well” limit (\( r_o \) much more negative than \( r_c \), where \( r_c \) corresponds to the onset of SSB) where SSB is a semi-classical phenomenon and the zero-point energy represents a small perturbation. Then we start to increase \( r_o \), at fixed \( \lambda_o \), toward the unknown value \( r_c \) and look at the quality of the fit with \( (\alpha, \beta, v_B) \) as free parameters. The minimum allowed value of \( r_o \) such that the quality of the 2-parameter fit \( (\alpha, \beta = 0, v_B) \) is exactly the same as in the more general 3-parameter case will define the “massless” case so that we can fit the data for \( J = J(\phi_B) \) with our Eq.(21). Finally, Eq.(14) suggests that the vacuum field \( v_B \), in lattice units, vanishes extremely fast when \( \lambda_o \to 0 \). Thus, to avoid that noise and signal become comparable, we cannot run the lattice simulation at very small values of \( \lambda_o \) but have to restrict to values \( \lambda_o \sim 1 \) such that still \( \frac{\lambda_o}{\pi^2} << 1 \) but \( v_B \), in lattice units, is not too small. Smaller values of \( \lambda_o \) \((\sim 0.4-0.6)\), however, should become accessible with the largest lattices available today \((\sim 100^4)\) where one should safely reach values \(|J| \sim 0.001\) being still in agreement with Eq.(5). At \( \lambda_o = 1 \) we have identified the massless regime at a value \( r_o = r_s < r_c \) where \( r_s \sim -0.45 \). By using the accurate weak-coupling relation between
the bare mass and the euclidean cutoff

\[ r_s = -\frac{3\lambda_o}{16\pi^2}\Lambda^2 \]  

(23)

and using the relation \( \Lambda = \pi y_Q \) (where \( y_Q \) is expected to be \( O(1) \)) we obtain \( y_Q \sim 1.55 \). It should be noted that our operative definition of the “massless” regime amounts to determine for weak bare coupling the numerical coefficient relating ultraviolet cutoff and lattice spacing for quadratic divergences (\( \Lambda \sim 4.87 \)). This agrees well with an independent analysis of the Lüscher and Weisz lattice data presented by Brahm [35]. In fact Brahm’s result (in the range \( \lambda_o \leq 10 \)) is \( \Lambda = 4.893 \pm 0.003 \). Also, ref. [35] predicts the massless regime to correspond to \( r_s = -(0.224 \pm 0.001) \) for \( \lambda_o = 0.5 \) in the infinite-volume limit. This implies \( r_s \sim -(0.448 \pm 0.002) \) for \( \lambda_o = 1 \), in excellent agreement with our result \( r_s \sim -0.45 \). Thus, the model-dependence in the definition of the massless regime, introduced by our fitting procedure to Eq.(22), is negligible for \( \frac{\lambda_o}{16\pi^2} << 1 \).

Finally, the massless relation (14) predicts,

\[ (v_B)^{1\text{-loop}} = \frac{\pi y_L}{\sqrt{3\lambda_o}} \exp(-\frac{8\pi^2}{9\lambda_o}) \]  

(24)

and

\[ Z_\phi^{1\text{-loop}} = \frac{8\pi^2}{3\lambda_o}. \]  

(25)

In Eq.(24) we have used \( y_L \) rather than \( y_Q \) since one does not expect precisely the same coefficient to govern the relation between euclidean cutoff and lattice spacing for both quadratic and logarithmic divergences.

The consistency of Eqs.(24,25) with the corresponding values obtained from the best fit to the lattice data checks the validity of the picture given in sect.2 for the coexistence of SSB and “triviality”. We have determined \( y_L \) at \( \lambda_o = 1 \) and, after that, the scaling behaviour of \( v_B \) with the bare coupling has been directly compared with Eq.(24). On the other hand, the prediction in Eq.(25) for \( Z_\phi \), which fixes the absolute normalization of the curve \( J = J(\phi_B) \), only depends on \( \lambda_o \) and can be directly compared with the lattice data in the massless regime.
Our numerical values for $\phi_B(J)$ and the results of the 2-parameter fits to the data by using Eq.(21) for $\lambda_o = 0.8, 1.0$ and 1.2 are shown in Table I together with the one-loop prediction (25). The change of $r_o$ with $\lambda_o$ is computed by using Eq.(23) and our result $r_o a^2 \sim -0.45$ for $\lambda_o = 1$. The reported errors are due to three main sources:

a) the purely statistical errors from the Monte Carlo sampling of the field configurations

b) errors which can be easily recognized as numerical artifacts (in our case they can be estimated by comparing with Eq.(5))

c) finite size effects which we have estimated by computing a few points on a $16^4$ lattice

By fixing $Z_{\phi}$ to its one-loop value (25) and using Eq.(24) for $v_B$ we have determined the value $y_L = 2.07 \pm 0.01 \left( \frac{\Lambda^2}{\alpha(J)} = \frac{6}{27} \right)$ which corresponds to $v_B = (5.814 \pm 0.028) \times 10^{-4}$ at $\lambda_o = 1$. Once we know $y_L$ we can predict the value of $v_B = (v_B)^{Th}$ and compare with the results of the fit to the lattice data at $\lambda_o = 0.8$ and 1.2. The results in Table II show a remarkable agreement of the Monte Carlo data with the one-loop predictions.

As an additional check that our values for $\phi_B(J)$ are not affected by appreciable systematic effects, we have compared with a completely independent Monte Carlo simulation performed by P. Cea and L. Cosmai [33] on a $16^4$ lattice. Their data for $\lambda_o = 1$ and $r_o = -0.45$ are reported in Table III. Note that their errors are statistical only and are smaller by an order of magnitude than those reported in our Table I (where errors of type b) and an estimate of the errors of type c) were included). Still, the quality of the one-loop fit is excellent confirming that finite size effects do not affects the results of the fits.

At this point one may ask: "How does the standard perturbative approach compare with the lattice data?". This is not a trivial question. Indeed, the same lattice data might be consistent with different functional forms thus allowing for different extrapolations at $J = 0$ and, hence, different interpretations. In particular, since our data are collected in the region of small bare coupling ($\frac{\Lambda}{\alpha} << 1$), it is reasonable to check whether the lattice data agree better with the one-loop potential or with its perturbative leading-log improvement, related to the concept of $\lambda_R$. This is even more important in our case because:
i) The leading-log approximation provides a commonly accepted perturbative interpretation of “triviality” (ignoring the objections raised in sect.2)

ii) Just the validity of the leading-log resummation is usually considered as an indication that the one-loop minimum cannot be trusted and there is no SSB in the massless regime \[25\]. If this were true the correct extrapolation at \( J = 0 \) would require \( v_B = 0 \).

However, one should be aware that the higher order corrections to the one-loop potential represent genuine self-interaction effects of the shifted field which we know from “triviality” cannot be physically present in the continuum limit. Were we observing the need for perturbative corrections of the one-loop formula this would contradict all the non perturbative evidence about \( h(x) \) being a generalized free field.

By fitting our lattice data to the leading-logarithmically “improved” version of Eq.(21), namely

\[
J^{LL}(\phi_B) = \frac{\lambda_o \phi_B^3}{1 + \frac{9\lambda_o}{8\pi^2} \ln \frac{\pi x_{LL}}{|\phi_B|}}
\]

(\( x_{LL} \) denoting an adjustable parameter for each \( \lambda_o \)) we find, for 21 degrees of freedom, \((\chi^2)_{LL} = 53, 163, 365\) for \( \lambda_o = 0.8, 1.0, 1.2 \) respectively (to compare with the values \( \chi^2 = 5, 6, 13 \) of the corresponding one-loop 1-parameter fits). The same comparison with the Cea-Cosmai data reported in Table III gives the result

\[
x_{LL} = 1.627 \pm 0.001
\]

\[
\frac{\chi^2}{d.o.f.} = \frac{11100}{9 - 1}
\]

This should be compared with the one-loop results \( \frac{\chi^2}{d.o.f.} = \frac{3.2}{9 - 2} \) reported in Table III for the 2-parameter fit and the value \( \frac{\chi^2}{d.o.f.} = \frac{3.3}{9 - 1} \) for the 1-parameter fit when \( Z_\phi \) is constrained to its one-loop value \( Z_\phi = 26.319 \). These results show that the agreement between lattice data and one-loop formulas are not a trivial test of perturbation theory but, rather, a non perturbative test of “triviality”. They also verify the consistency of our definition of the continuum limit presented in sect.2.
In order to fully appreciate what the above results exactly mean we have reported in Tables IV and V the two theoretical sets $J = J^{Th}(\phi_B)$ (Th=one-loop, leading-log) which best fit the measured values of $\phi_B$ (with their errors $\Delta J$ corresponding to the error in $\phi_B$). Finally, in the last column, we report the quantity $\chi^2 = (J^{\text{EXP}} - J^{Th})^2 / (\Delta J)^2$ where $J = J^{\text{EXP}}$ is the "experimental", input value of $J$ at which the various $\phi_B$ have been computed by Cea and Cosmai. By inspection of Tables IV and V one finds:

1) the leading-log formula is accurate to the level of a few percent. The fit routine tries to reproduce the lattice data with a suitable value of the free parameter $x_{LL}$ within the logarithm, thus optimizing the relation between euclidean cutoff and lattice spacing and effectively reabsorbing also non-leading perturbative corrections. In this case the experimental data and the leading-log curve intersect each other somewhere in between $J=0.5$ and $J = 0.4$. However, their slopes are clearly different as it can be seen from the fact that the theoretical prediction is lower than the experimental data for $J > 0.5$ and higher for $J < 0.4$. Since the Monte Carlo data are so precise the value of the $\chi^2$ is enormous.

2) the one-loop prediction is accurate to a completely different level, namely $O(10^{-4})$. This result cannot be understood from the conventional perturbation theory viewpoint; apparently a cruder approximation reproduces the lattice data remarkably well, while its "pro forma" improvement fails badly! However, if one understands our interpretation of "triviality", then this is precisely what one expects.

In conclusion, the lattice computations nicely confirm the conjecture \cite{11,12} that, as a consequence of "triviality", the one-loop potential becomes effectively exact in the continuum limit. Indeed, there is a well defined region in the space of the bare parameters ($r_o, \lambda_o$), controlled by Eq.(23), where the effective potential is described by its one-loop approximation to very high accuracy. The lattice analysis of the effective potential in this region, i.e. close to the critical line and in the broken phase, where the theory is \textit{known} to become gaussian, should be pursued with larger lattices and, possibly, also with different methods. Indeed, by increasing the lattice size, our analysis can be further extended towards the physically
relevant point $\lambda o_p \to 0^+, r_o \to 0^-$ which corresponds, in the continuum limit of the theory, to “dimensional transmutation” [25] from the classically scale invariant case. Our results confirm the point of view [11,12] that the massless version of the $\lambda(\Phi^4)_4$ theory, although “trivial”, is not “entirely trivial”: it provides, at the same time, SSB and a meaningful continuum limit $\Lambda \to \infty, \lambda_o \to 0^+$ such that the mass of the free shifted field in Eq.(14) is cutoff independent. However, the mass $m_h$ does not represent a measure of any observable interaction since the corresponding field $h(x)$ is described by a quadratic hamiltonian.

4. The O(2)-symmetric case

As discussed in [17,18,11], and explicitly shown in [27], one expects Eqs.(18,19), of the one-component theory, to be also valid for the radial field in the O(N)-symmetric case. This observation originates from ref. [36] which obtained, for the radial field, the same effective potential as in the one-component theory. This is extremely intuitive. The Goldstone-boson fields contribute to the effective potential only through their zero-point energy, that is an additional constant, since, according to “triviality”, they are free massless fields. Thus, in the O(2)-symmetric case, one may take the diagram $(V_{eff}, \phi_B)$ for the one-component theory and “rotate” it around the $V_{eff}$ symmetry axis. This generates a three-dimensional diagram $(V_{eff}, \phi_1, \phi_2)$ where $V_{eff}$, owing to the O(2) symmetry, only depends on the bare radial field,

$$\rho_B = \sqrt{\phi_1^2 + \phi_2^2}$$

(26)

in exactly the same way as $V_{eff}$ depends on $\phi_B$ in the one-component theory; namely $(\omega^2(\rho_B) = 3\lambda_o \rho_B^2)$

$$V^{1-loop}(\rho_B) = \frac{\lambda_o}{4} \rho_B^4 + \frac{\omega^4(\rho_B)}{64\pi^2} \left( \ln \frac{\omega^2(\rho_B)}{\Lambda^2} - \frac{1}{2} \right).$$

(27)

By using Eqs.(14,25) $V^{1-loop}$ can be re-expressed in the form

$$V^{1-loop}(\rho_B) = \frac{\pi^2 \rho_B^4}{Z_{\phi}^{2}}(\ln \frac{\rho_B^2}{v_B^2} - \frac{1}{2}),$$

(28)

By differentiating Eq.(28), we obtain the bare “radial source”
\[
J(\rho_B) = \frac{dV^{1-\text{loop}}(\rho_B)}{d\rho_B} = \frac{4\pi^2 \rho_B^3}{Z_\phi^2} \ln \frac{\rho_B^2}{v_B^2},
\]

which we shall compare with the lattice results for \( J = J(\rho_B) \).

The lattice simulation of the \( O(2) \)-invariant theory has been obtained, with periodic boundary conditions, from the action

\[
\sum_x \left[ \frac{1}{2} \sum_{\mu=1}^{4} \sum_{i=1}^{2} (\Phi_i(x + e_\mu) - \Phi_i(x))^2 + \frac{1}{2} r_o \sum_{i=1}^{2} (\Phi_i^2(x)) + \frac{\lambda_o}{4} \left( \sum_{i=1}^{2} \Phi_i^2(x) \right)^2 - \sum_{i=1}^{2} J_i \Phi_i(x) \right]
\]

where \( \Phi_1 \) and \( \Phi_2 \) are coupled to two constant external sources \( J_1 \) and \( J_2 \). By using \( J_1 = J \cos \theta \) and \( J_2 = J \sin \theta \) it is straightforward to show that, having defined \( \phi_1 = \langle \Phi_1 \rangle_{J_1,J_2} \) and \( \phi_2 = \langle \Phi_2 \rangle_{J_1,J_2} \), the bare radial field

\[
\rho_B = \sqrt{\phi_1^2 + \phi_2^2}
\]

does only depend on \( J \), that is

\[
\rho_B = \rho_B(J)
\]

We started our Monte Carlo simulation on a \( 10^4 \) lattice by investigating first the \((r_o, \lambda_o)\) correlation corresponding to the classically scale-invariant case.

In the \( O(2) \) case, we have used the results of the previous section and of [35] to get the analogous of Eq.(23) for the two-component theory. Since we are considering \( \frac{\lambda_o}{\pi^2} \ll 1 \), this simply introduces a combinatorial factor of \( 4/3 \) so that Eq.(23) becomes

\[
\frac{r_s}{\sqrt{\Lambda^2}} = -\frac{\lambda_o}{4\pi^2} \Lambda^2
\]

Hence, we expect the massless case to correspond to \( r_s \sim -0.6 \) for \( \lambda_o = 1 \). This was confirmed to good accuracy by using the fitting procedure to the analogous of Eq.(22) defined with \( \rho_B \) in the place of \( \phi_B \). Thus, the identification of the massless regime on the lattice appears to obey the simple scaling laws (23,33) and is under theoretical control.

Just like in the single-component case Eq.(5) was poorly reproduced numerically at small values of \( J \), in the \( O(2) \)-symmetric case we find that, at small \( J \), the exact \( \theta \)-independence of
\( \rho_B \) (see Eq.(32)) is poorly reproduced and a “direct” data processing becomes unreliable. We therefore consider a “safe” region of \( J \)-values, \( J \geq 0.05 \), in which the spurious \( \theta \)-dependence gives the data an uncertainty less than \( \pm 3\% \). Fitting the data to Eq.(29) we can infer the values of \( v_B \) and \( Z_\phi \) and compare with Eqs.(24,25).

Our numerical values for \( \rho_B(J) \) in the massless case, obtained with our version of the Metropolis’ Monte Carlo algorithm on the \( 10^4 \) lattice, are reported in Table VI for \( \lambda_o =1.0, 1.5 \) and \( 2.0 \). For each \( \lambda_o \) the corresponding \( r_o \) is computed by using Eq.(33) and our numerical input \( r_s = -0.6 \) for \( \lambda_o =1 \). The errors in Table VI are essentially determined by the observed spurious variation of \( \rho_B \) in the range \( 0 \leq \theta \leq 2\pi \). As discussed above, this is a numerical artifact and should be considered a systematic effect of the Monte Carlo lattice simulation. It is reproduced with three different random number generators, all consistent with the Kolmogorov-Smirnov test at the level \( O(10^{-4}) \). At low \( J \) this systematic effect completely dominates the error; the statistical errors, already after 30,000 iterations, are 4-5 times smaller.

Table VI also reports the \( v_B \) and \( Z_\phi \) values obtained from the two-parameter fits to the data using Eq.(29). The resulting \( Z_\phi \) values agree well with the one-loop prediction of Eq.(25). As in the one-component theory, to perform a more stringent test of the one-loop potential, we have next constrained \( Z_\phi \) to its one-loop value in Eq.(25) and, under such condition, we have done a precise determination of \( v_B \) from a one-parameter fit to Eq.(29). This allows a meaningful comparison with Eq.(24). In this case the value of \( y_L \) derived from the one-parameter fit to the \( O(2) \) lattice data at \( \lambda_o = 1.5 \) and \( r_o = -0.9 \) turns out to be \( y_L = 2.44 \pm 0.03 \) with \( \frac{\chi^2}{d.o.f.} = \frac{4.9}{17} \). In Table VII we show the results of the one-parameter fits to the data at \( \lambda_o =1.0 \) and \( 2.0 \) and the comparison with Eq.(24) for \( y_L = 2.44 \pm 0.03 \). It is apparent from Table VII that the one-loop potential reproduces quite well the scaling law of \( v_B \) with the bare coupling, as previously discovered in the one-component case.

The value of \( y_L \) obtained from the lattice simulation of the \( O(2) \)-symmetric massless \( \lambda \Phi^4 \) theory is \( \sim 17\% \) larger than the value \( y_L = 2.07 \pm 0.01 \) obtained in the one-component case. This is partly due to our choice of fixing the value of \( r_s = -0.6 \) at \( \lambda_o = 1 \) with the simple
combinatorial factor $4/3$ discussed above and ignoring finite size corrections to Eq.(33), see [35].

With a two-component field, the errors (as estimated from the spurious $\theta$-dependence of $\rho_B$) are larger than in the one-component theory. The massless regime, identified on the basis of the $(J, \rho_B)$ correlation, is found in a range of $r_o$-values around $r_o = -0.6$ for $\lambda_o = 1$. Choosing instead $r_s \sim -0.585$ would give $y_L \sim 2.07$.

We have analyzed the O(2) lattice data in the same fashion as already described in the one-component case. Again, if the agreement with the one-loop formulas would be a trivial test of perturbation theory the data should agree at least as well, if not better, with the leading-log formula based on the perturbative $\beta$-function

$$J_{LL}^{LL}(\rho_B) = \frac{\lambda_o \rho_B^3}{1 + \frac{5\lambda_o}{4\pi^2} \ln \frac{x_{LL}}{\rho_B}}$$

($x_{LL}$ denoting an adjustable parameter to be optimized at each $\lambda_o$). However, when we fit the $\lambda_o = 1.0, 1.5$ and 2.0 data to this formula we find, respectively, for 17 degrees of freedom, $(\chi^2)_{LL} = 9, 44$ and 133 (to compare with the values $\chi^2 = 0.8, 3.3$ and 7.3 obtained from the 1-loop one-parameter fits with Eq.(29) when $Z_\phi$ is constrained to its value in Eq.(25)). Note that we are working in a region where the “$\lambda_o \log$” term is not small, being of order unity. Thus, the good agreement between the data and the one-loop formula cannot be due to the higher-order corrections (expected by perturbation theory) being negligibly small; it must be due to their complete absence. Without the “triviality” argument, this would be an incomprehensible miracle.

Finally, to have an idea of the finite size effects, we have repeated the lattice simulation on a $16^4$ lattice for $\lambda_o = 1$ and $r_o = -0.6$. The computation has been performed at the same values $\theta = 30, 60, 90, 120, 150, 180, 210, 240$ and 270 degrees for the angular source and the data are reported in Table VIII. The spurious dependence of the radial field on $\theta$ is slightly smaller than in the $10^4$ case and the agreement with the one loop prediction in Eq.(25) improves.

In conclusion, the good agreement between lattice simulation and eqs. (24,25,29) provides
definite evidence that the dependence of the effective potential on the radial field in the O(2)-symmetric case is remarkably consistent with the expectations \[11,12\] based on the analysis of the one-component theory and with the explicit analytical postgaussian calculation in \[27\] which numerically confirmed, to very high accuracy, the validity of Eq.(19) in the O(2) and O(4) case.

5. Conclusions

We have proposed a critical reappraisal of the available results on the broken phase of \(\lambda(\Phi^4)_4\) theory, suggesting a consistent interpretation of the coexistence of “triviality” and SSB, whose validity is also confirmed by the lattice calculations. All the existing analytical and numerical evidences can be summarized in the two basic points a) and b) illustrated in sect.2. They represent the essential ingredients for a consistent description of SSB in pure \(\lambda(\Phi^4)_4\) theory. They are naturally discovered in those approximations to the effective potential which are consistent with “triviality” (one-loop, gaussian effective potential, postgaussian approximations and so on) where the shifted field(s) appear as generalized free field(s).

As discussed in sect.2, the perturbative interpretation of the lattice data based on the leading-log approximation is by no means unique and suffers of internal theoretical drawbacks. Even more, in sects.3 and 4 we have provided extensive numerical evidence that in the scaling region, clearly identified with our procedure, there is no trace of any Higgs-Higgs or Higgs-Goldstone interactions in the lattice data for \(J = J(\phi_B)\). The measured effective potential, being remarkably consistent with the peculiar exponential decay in Eq.(24) and with Eq.(25), is just the sum of the classical potential and of the zero point energy of a free field. The corrections, expected on the basis of the leading-log approximation, are definitely ruled out by our results.

The simple scaling laws associated with the one-loop potential allow to recover a “not-entirely trivial” continuum limit where, despite of the absence of observable interactions,
one still gets SSB as the basic ingredient to generate the vector boson masses in the standard model of electroweak interactions. The $\lambda(\Phi^4)$ theory is not “entirely trivial” and can be physically distinguished from a free field theory. Indeed it exhibits SSB and a first order phase transition \cite{11,37} at a critical temperature $T_c$ depending on the vacuum energy in eq.(16). Also, the origin of SSB from a classically scale-invariant theory, allows to reconcile the results of ref. \cite{7} with “triviality”.

Let us emphasize, now, that the mass of the Higgs particle does not represent, by itself, a measure of any interaction and the naive proportionality between $m_h$ and $\lambda_R$ is not valid outside perturbation theory. While the latter vanishes the former remains an arbitrary parameter. As discussed in \cite{11,17,12,18} substantial phenomenological implications follow. In our picture, which is consistent with all rigorous and numerical results available today, the Higgs particle can be very heavy and, still, be weakly interacting. In fact, in the absence of the gauge and Yukawa couplings, it would be trivially free. In the most appealing theoretical framework, where SSB is generated from a classically scale-invariant theory, we get $m_h^2 = 8\pi^2 v_R^2$ so that, by relating $v_R$ to the Fermi constant (modulo purely electroweak corrections which are small if the top quark mass is below 200 GeV \cite{11,17,18}) we get $m_h \sim 2$ TeV. In general the Higgs can be lighter or heavier, depending on the size of the scale-breaking parameter in the classical potential \cite{12}. In any case, beyond perturbation theory, its mass is not related to its decay rate through $\lambda_R$.

For instance, consider the Higgs decay width to $W$ and $Z$ bosons. The conventional calculation would give a huge width, of order $G_F m_h^3 \sim m_h$ for $m_h \sim 2$ TeV. However, in a perturbatively renormalizable-gauge calculation of the imaginary part of the Higgs self-energy, this result comes from a diagram in which the Higgs supposedly couples strongly to a loop of Goldstone bosons with a physical strength proportional to its mass squared. On the other hand, if “triviality” is true, all interaction effects of the pure $\lambda\Phi^4$ sector of the standard model have to be reabsorbed into two numbers, namely $m_h$ and $v_R$, and there are no residual interactions. Thus, beyond perturbation theory, that diagram is absent, leaving a width of order $g^2 m_h$. Therefore, if “triviality” is true, the Higgs is a relatively narrow resonance,
decaying predominantly to $t\bar{t}$ quarks. Although the scalar sector must be treated non-perturbatively, one may continue to treat the gauge interactions using perturbation theory. Effectively, then, inclusive electroweak processes can be computed as usual, provided one uses a renormalizable gauge and sets the Higgs self-coupling and its coupling to the Higgs-Kibble ghosts (the would-be Goldstones) to zero. One should avoid the so-called “unitary gauge” where, as well known since a long time \[38\], there is no smooth limit $M_w \to 0$ in perturbation theory, as a consequence of the longitudinal degrees of freedom. The possibility of decoupling the gauge sector from the scalar sector of the standard model in the limit $g \to 0$ is crucially dependent on the fact that gauge invariance is carefully retained at any stage. By computing in a renormalizable $R_\xi$ gauge one never gets effects (such as cross-sections, decay rates,...) growing proportionally to the Higgs mass squared \textit{unless} there is a coupling in the theory proportional to the Higgs mass at zero gauge coupling. Thus, the Higgs phenomenology, on the basis of gauge invariance, depends on the details of the pure scalar sector, namely “triviality” and the description of SSB. We believe that our picture, supported as we have shown by the lattice results for the effective potential, deserves a very serious attention. The conventional wisdom, built up on the perturbative assumption that the Higgs mass is a measure of its physical interactions, runs a serious risk of predicting a whole set of physical phenomena which exist only in perturbation theory.

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| \( J \) | \( \lambda_o = 0.8 \ r_o = -0.36 \) | \( \lambda_o = 1.0 \ r_o = -0.45 \) | \( \lambda_o = 1.2 \ r_o = -0.54 \) |
|---|---|---|---|
| -0.700 | \(-1.0024 \pm 0.0003\) | \(-0.9401 \pm 0.0003\) | \(-0.8935 \pm 0.0003\) |
| -0.600 | \(-0.9540 \pm 0.0003\) | \(-0.8950 \pm 0.0003\) | \(-0.8512 \pm 0.0003\) |
| -0.500 | \(-0.8997 \pm 0.0005\) | \(-0.8444 \pm 0.0003\) | \(-0.8037 \pm 0.0003\) |
| -0.400 | \(-0.8371 \pm 0.0005\) | \(-0.7867 \pm 0.0005\) | \(-0.7493 \pm 0.0005\) |
| -0.300 | \(-0.7635 \pm 0.0010\) | \(-0.7182 \pm 0.0010\) | \(-0.6846 \pm 0.0010\) |
| -0.200 | \(-0.6702 \pm 0.0010\) | \(-0.6313 \pm 0.0010\) | \(-0.6032 \pm 0.0010\) |
| -0.150 | \(-0.6112 \pm 0.0015\) | \(-0.5764 \pm 0.0010\) | \(-0.5513 \pm 0.0010\) |
| -0.125 | \(-0.5764 \pm 0.0015\) | \(-0.5445 \pm 0.0015\) | \(-0.5213 \pm 0.0010\) |
| -0.100 | \(-0.5366 \pm 0.0020\) | \(-0.5077 \pm 0.0020\) | \(-0.4865 \pm 0.0024\) |
| -0.075 | \(-0.4902 \pm 0.0024\) | \(-0.4641 \pm 0.0020\) | \(-0.4459 \pm 0.0024\) |
| -0.050 | \(-0.4299 \pm 0.0026\) | \(-0.4092 \pm 0.0030\) | \(-0.3934 \pm 0.0024\) |
| 0.050 | \(0.4273 \pm 0.0026\) | \(0.4062 \pm 0.0030\) | \(0.3915 \pm 0.0024\) |
| 0.075 | \(0.4877 \pm 0.0024\) | \(0.4621 \pm 0.0020\) | \(0.4435 \pm 0.0024\) |
| 0.100 | \(0.5348 \pm 0.0020\) | \(0.5055 \pm 0.0020\) | \(0.4842 \pm 0.0024\) |
| 0.125 | \(0.5749 \pm 0.0015\) | \(0.5428 \pm 0.0015\) | \(0.5201 \pm 0.0010\) |
| 0.150 | \(0.6093 \pm 0.0015\) | \(0.5754 \pm 0.0010\) | \(0.5505 \pm 0.0010\) |
| 0.200 | \(0.6693 \pm 0.0010\) | \(0.6306 \pm 0.0010\) | \(0.6023 \pm 0.0010\) |
| 0.300 | \(0.7625 \pm 0.0010\) | \(0.7172 \pm 0.0010\) | \(0.6838 \pm 0.0010\) |
| 0.400 | \(0.8367 \pm 0.0005\) | \(0.7863 \pm 0.0005\) | \(0.7488 \pm 0.0005\) |
| 0.500 | \(0.8993 \pm 0.0005\) | \(0.8445 \pm 0.0003\) | \(0.8035 \pm 0.0003\) |
| 0.600 | \(0.9540 \pm 0.0003\) | \(0.8951 \pm 0.0003\) | \(0.8513 \pm 0.0003\) |
| 0.700 | \(1.0024 \pm 0.0003\) | \(0.9402 \pm 0.0003\) | \(0.8937 \pm 0.0003\) |

\( Z_\phi = 33.3 \pm 0.6 \) \( Z_\phi = 26.0 \pm 0.4 \) \( Z_\phi = 21.3 \pm 0.3 \)

\( v_B = (5.9 \pm 2.0) \times 10^{-5} \) \( v_B = (6.9 \pm 1.4) \times 10^{-4} \) \( v_B = (3.1 \pm 0.4) \times 10^{-3} \)

\( Z_\phi^{1-\text{loop}} = 32.9 \) \( Z_\phi^{1-\text{loop}} = 26.3 \) \( Z_\phi^{1-\text{loop}} = 21.9 \)
TABLES

TABLE I. We report the values of $\phi_B(J)$ for the massless case as discussed in text. At the various $(\lambda_o, r_o)$ we also show the results of the 2-parameter fit with Eq.(21) and the one loop prediction (25).
TABLE II. By using Eq.(21), we show the results of the 1-parameter fits for $v_B$ at $\lambda_o = 0.8$ and 1.2 when $Z_\phi$ is constrained to its one-loop value in Eq.(25). We also show the predictions from Eq.(24), $(v_B)^{Th}$, for $y_L = 2.07 \pm 0.01$ as determined from the fit to the data at $\lambda_o = 1.$

| $\lambda_o$  | $r_o$  | $Z_\phi$ | $v_B$          | $\chi^2$ d.o.f |
|-------------|--------|----------|----------------|----------------|
| 0.8         | -0.36  | 32.90 = fixed | $(7.30 \pm 0.03)10^{-5}$ | $\frac{5}{21}$ |
| 1.2         | -0.54  | 21.93 = fixed | $(2.28 \pm 0.01)10^{-3}$ | $\frac{13}{21}$ |

$(v_B)^{Th} = (7.25 \pm 0.03)10^{-5}$ $(v_B)^{Th} = (2.29 \pm 0.01)10^{-3}$
TABLE III. We report the values of $\phi_B(J)$ as obtained by L. Cosmai and P. Cea with their 16$^4$ lattice at $\lambda_o = 1$ and $r_o = -0.45$. Errors are statistical only. We also report the results of the fit with Eq.(29) and the one loop prediction (25).

| $J$  | $\lambda_o = 1.0$ | $r_o = -0.45$ |
|------|-------------------|----------------|
| 0.100 | 0.506086 ± 0.91E - 04 |
| 0.125 | 0.543089 ± 0.82E - 04 |
| 0.150 | 0.575594 ± 0.82E - 04 |
| 0.200 | 0.630715 ± 0.62E - 04 |
| 0.300 | 0.717585 ± 0.52E - 04 |
| 0.400 | 0.786503 ± 0.44E - 04 |
| 0.500 | 0.84473 ± 0.41E - 04  |
| 0.600 | 0.894993 ± 0.39E - 04 |
| 0.700 | 0.940074 ± 0.37E - 04 |

$\chi^2_{d.o.f} = \frac{3.2}{9-2}$

$Z_\phi = 26.323 ± 0.042$

$v_B = (5.783 ± 0.013)10^{-4}$

$Z_{\phi}^{1-loop} = 26.319$
TABLE IV. We report the one-loop theoretical prediction $J_{\text{1-loop}}^\phi (\phi_B)$ based on Eq.(21), with the corresponding error $\Delta J$, which best fits the Monte Carlo data reported in Table III. $Z_\phi = 26.319 = fixed$ and $v_B = 5.7943 \times 10^{-4}$. $\chi^2 = \frac{(J_{\text{1-loop}} - J_{\text{EXP}})^2}{(\Delta J)^2}$.

| $\phi_B$          | $J_{\text{1-loop}}^\phi$ | $J_{\text{EXP}}$ | $\chi^2$ |
|-------------------|---------------------------|-----------------|----------|
| 0.940074 ± 0.37E-04 | 0.699975 ± 0.82E-04       | 0.700           | 0.09     |
| 0.894993 ± 0.39E-04 | 0.600001 ± 0.78E-04       | 0.600           | 0.00     |
| 0.844473 ± 0.41E-04 | 0.500042 ± 0.72E-04       | 0.500           | 0.29     |
| 0.786503 ± 0.44E-04 | 0.400027 ± 0.67E-04       | 0.400           | 0.17     |
| 0.717585 ± 0.52E-04 | 0.299952 ± 0.65E-04       | 0.300           | 0.54     |
| 0.630715 ± 0.62E-04 | 0.199982 ± 0.59E-04       | 0.200           | 0.09     |
| 0.575594 ± 0.82E-04 | 0.150010 ± 0.64E-04       | 0.150           | 0.02     |
| 0.543089 ± 0.82E-04 | 0.124943 ± 0.57E-04       | 0.125           | 0.99     |
| 0.506086 ± 0.91E-04 | 0.100062 ± 0.54E-04       | 0.100           | 1.34     |
TABLE V. The same as in table IV for the leading-log fit.

| $\phi_B$ | $J^{LL}$ | $J^{EXP}$ | $\chi^2$ |
|---------|---------|---------|--------|
| 0.940074 ± 0.37E−04 | 0.696382 ± 0.82E−04 | 0.700 | 1917 |
| 0.894993 ± 0.39E−04 | 0.598116 ± 0.78E−04 | 0.600 | 580 |
| 0.844473 ± 0.41E−04 | 0.499679 ± 0.72E−04 | 0.500 | 19 |
| 0.786503 ± 0.44E−04 | 0.400981 ± 0.67E−04 | 0.400 | 212 |
| 0.717585 ± 0.52E−04 | 0.301938 ± 0.65E−04 | 0.300 | 851 |
| 0.630715 ± 0.62E−04 | 0.202585 ± 0.60E−04 | 0.200 | 1833 |
| 0.575594 ± 0.82E−04 | 0.152692 ± 0.65E−04 | 0.150 | 1675 |
| 0.543089 ± 0.82E−04 | 0.127580 ± 0.57E−04 | 0.125 | 1910 |
| 0.506086 ± 0.91E−04 | 0.102582 ± 0.55E−04 | 0.100 | 2175 |
TABLE VI. The values of $\rho_B(J)$ for the massless case are reported as discussed in the text.
At the various values of $\lambda_o$ and $r_o$ we also show the results of the 2-parameter fits with Eq.(29) and the one loop prediction (25).

| $J$  | $\lambda_o = 1.0 \, r_o = -0.6$ | $\lambda_o = 1.5 \, r_o = -0.9$ | $\lambda_o = 2.0 \, r_o = -1.2$ |
|------|--------------------------------|--------------------------------|--------------------------------|
| 0.050| 0.4110 ± 0.0130                | 0.3850 ± 0.0116                | 0.3753 ± 0.0091                |
| 0.075| 0.4686 ± 0.0086                | 0.4337 ± 0.0086                | 0.4176 ± 0.0069                |
| 0.100| 0.5133 ± 0.0084                | 0.4724 ± 0.0065                | 0.4517 ± 0.0055                |
| 0.125| 0.5495 ± 0.0063                | 0.5048 ± 0.0065                | 0.4814 ± 0.0054                |
| 0.150| 0.5819 ± 0.0060                | 0.5332 ± 0.0053                | 0.5063 ± 0.0042                |
| 0.200| 0.6377 ± 0.0047                | 0.5812 ± 0.0040                | 0.5500 ± 0.0037                |
| 0.250| 0.6844 ± 0.0039                | 0.6217 ± 0.0034                | 0.5875 ± 0.0029                |
| 0.300| 0.7256 ± 0.0030                | 0.6571 ± 0.0026                | 0.6190 ± 0.0029                |
| 0.350| 0.7612 ± 0.0029                | 0.6892 ± 0.0026                | 0.6473 ± 0.0026                |
| 0.400| 0.7938 ± 0.0025                | 0.7178 ± 0.0025                | 0.6731 ± 0.0025                |
| 0.450| 0.8246 ± 0.0024                | 0.7435 ± 0.0025                | 0.6969 ± 0.0023                |
| 0.500| 0.8520 ± 0.0023                | 0.7683 ± 0.0021                | 0.7193 ± 0.0023                |
| 0.550| 0.8785 ± 0.0021                | 0.7911 ± 0.0021                | 0.7399 ± 0.0018                |
| 0.600| 0.9029 ± 0.0019                | 0.8124 ± 0.0020                | 0.7593 ± 0.0017                |
| 0.650| 0.9261 ± 0.0019                | 0.8330 ± 0.0019                | 0.7778 ± 0.0017                |
| 0.700| 0.9481 ± 0.0018                | 0.8520 ± 0.0019                | 0.7953 ± 0.0017                |
| 0.750| 0.9693 ± 0.0018                | 0.8703 ± 0.0017                | 0.8119 ± 0.0015                |
| 0.800| 0.9893 ± 0.0016                | 0.8879 ± 0.0015                | 0.8274 ± 0.0013                |
| $Z_\phi = 25.0 \pm 1.6$ | $Z_\phi = 16.4 \pm 0.7$ | $Z_\phi = 12.3 \pm 0.4$ |
| $v_B = (1.4^{+1.5}_{-0.9}) \times 10^{-3}$ | $v_B = (1.8 \pm 0.6) \times 10^{-2}$ | $v_B = (5.6 \pm 0.8) \times 10^{-2}$ |
| $Z_\phi^{1-\text{loop}} = 26.3$ | $Z_\phi^{1-\text{loop}} = 17.5$ | $Z_\phi^{1-\text{loop}} = 13.1$ |
TABLE VII. By using Eq. (29), we show the results of the 1-parameter fits for $v_B$ at $\lambda_o = 1.0$ and 2.0 when $Z_\phi$ is constrained to its one-loop value in Eq. (25). We also show the predictions from Eq. (24), $(v_B)^{\text{Th}}$, for $y_L = 2.44 \pm 0.03$ as determined from the fit to the data at $\lambda_o = 1.5$.

| $\lambda_o$ | $r_o$ | $\lambda_o$ | $r_o$ |
|-------------|-------|-------------|-------|
| 1.0         | -0.6  | 2.0         | -1.2  |
| $Z_\phi = 26.32 = \text{fixed}$ | $Z_\phi = 13.16 = \text{fixed}$ |
| $v_B = (7.09 \pm 0.11) \times 10^{-4}$ | $v_B = (3.79 \pm 0.03) \times 10^{-2}$ |
| $\chi^2/\text{d.o.f} = 0.8/17$ | $\chi^2/\text{d.o.f} = 7.3/17$ |
| $(v_B)^{\text{Th}} = (6.85 \pm 0.09) \times 10^{-4}$ | $(v_B)^{\text{Th}} = (3.89 \pm 0.05) \times 10^{-2}$ |
TABLE VIII. We report the values of $\rho_B(J)$ obtained on our $16^4$ lattice for $\lambda_o = 1.0$ and $r_o = -0.6$. We also show the results of the 2-parameter fits with Eq.(29) and the one loop prediction (25).

| $J$  | $\lambda_o = 1.0$ $r_o = -0.6$ |
|-----|---------------------------------|
| 0.050 | 0.4101 ± 0.0098                |
| 0.075 | 0.4659 ± 0.0065                |
| 0.100 | 0.5105 ± 0.0052                |
| 0.125 | 0.5470 ± 0.0046                |
| 0.150 | 0.5774 ± 0.0035                |
| 0.250 | 0.6808 ± 0.0023                |
| 0.300 | 0.7235 ± 0.0022                |
| 0.350 | 0.7601 ± 0.0022                |
| 0.400 | 0.7933 ± 0.0022                |
| 0.450 | 0.8237 ± 0.0020                |
| 0.500 | 0.8517 ± 0.0019                |
| 0.550 | 0.8779 ± 0.0018                |
| 0.600 | 0.9025 ± 0.0018                |
| 0.650 | 0.9256 ± 0.0014                |
| 0.700 | 0.9475 ± 0.0013                |
| 0.750 | 0.9681 ± 0.0013                |
| 0.800 | 0.9874 ± 0.0012                |
| $Z_\phi = 26.35 ± 1.33$       |

$v_B = (6.8^{+6.3}_{-3.8}) \times 10^{-4}$

$Z_{\phi}^{1-loop} = 26.32$