An absolute version of the Gromov–Lawson relative index theorem

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Abstract

A Dirac operator on a complete manifold is Fredholm if it is invertible outside a compact set. Assuming a compact group to act on all relevant structure, and the manifold to have a warped product structure outside such a compact set, we express the equivariant index of such a Dirac operator as an Atiyah–Segal–Singer type contribution from inside this compact set, and a contribution from outside this set. Consequences include equivariant versions of the relative index theorem of Gromov and Lawson, in the case of manifolds with warped product structures at infinity, and the Atiyah–Patodi–Singer index theorem.

Contents

1 Introduction 2

2 The main result 4
  2.1 Dirac operators that are invertible at infinity . . . . 5
  2.2 The index theorem . . . . . . . . . . . . . . . . 6
  2.3 Special cases . . . . . . . . . . . . . . . . . . 9

3 Smooth kernels 10
  3.1 Kernels and projections . . . . . . . . . . . . 10
  3.2 Heat kernels and boundary conditions . . . . . . . . 12
  3.3 Dirichlet heat operators . . . . . . . . . . . . 13

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1 Introduction

Let \( M \) be a complete Riemannian manifold, and \( S \to M \) a Hermitian vector bundle equipped with a Clifford action \( c: TM \to \text{End}(S) \). For a Hermitian connection \( \nabla \) on \( S \) that is compatible with \( c \), we have the Dirac operator \( D = c \circ \nabla \) on \( \Gamma^\infty(S) \). This operator is Fredholm as a self-adjoint, unbounded operator on \( L^2(S) \), under the condition that there are a compact subset \( Z \subset M \) and a \( b > 0 \) such that for all \( s \in \Gamma^\infty_c(S) \) supported outside \( Z \),

\[
\|Ds\|_{L^2} \geq b\|s\|_{L^2}. \tag{1.1}
\]

See \[2, 15\].

A natural index theorem for such operators is Gromov and Lawson’s relative index theorem on page 304 of \[15\]: an expression for the difference of the indices of two such operators \( D \), on possibly different manifolds, that agree outside their respective sets \( Z \), in terms of Atiyah–Singer type integrals on those compact sets.

Another type of index problem that can be put into this context is Atiyah–Patodi–Singer (APS) type index theorems on compact manifolds with boundary. Then one attaches a cylinder to this boundary, to form a noncompact, but complete manifold without boundary, and studies the corresponding index problem there. This was already used in \[3\], but see also for example \[16\].

A widely studied class of index problems for Dirac operators on noncompact manifolds that does not fall into our context is the study of Callias-type operators \[1, 3, 8, 9, 10, 20\]. These have the form \( D + \Phi \), for a vector bundle
endomorphism $\Phi$ that makes the combined operator $D + \Phi$ invertible at infinity in the sense of (1.1), in cases where $D$ itself may not have this property. The operator $D + \Phi$ is not necessarily of the form $c \circ \nabla$, and therefore falls outside our current scope.

Our goal in this paper is to express the index of a Dirac operator that is invertible outside a compact set $Z$, with smooth boundary, as a contribution of an Atiyah–Singer type contribution from $Z$ and a contribution from outside $Z$. We assume that there is a $G$-equivariant isometry $M \setminus Z \cong N \times (0, \infty)$, for a $G$-invariant, compact hypersurface $N \subset M$, with a Riemannian metric of the form

$$e^{2\varphi}(g_N + dx^2),$$

for a function $\varphi$ on $(0, \infty)$. Furthermore, we consider an action by a compact group $G$ on $M$, preserving all relevant structure, and consider the $G$-equivariant index $\text{index}_G(D)$ evaluated at a group element $g \in G$. The index formula, Theorem 2.2, then has the form

$$\text{index}_G(D)(g) = \int_{Z^g} \text{AS}_g(D) + \lim_{N \to \partial Z} \int_0^\infty \int_N \text{tr}(g\lambda^P_s(g^{-1}n,n)) \, dn \, ds. \quad (1.3)$$

Here

- $Z^g$ is the fixed-point set of $g$ in $Z$;
- $\text{AS}_g(D)$ is the Atiyah–Segal–Singer integrand [5, 6] associated to $D$;
- the limit $\lim_{N \to \partial Z}$ means taking compact, $G$-invariant hypersurfaces $N \subset M \setminus Z$ closer and closer to $\partial Z$, in a way made precise in Corollary 2.3;
- $\lambda^P_s$ is the smooth kernel of the operator $e^{-sD_C^T D_C}D_C$, where $e^{-sD_C^T D_C}$ is the heat operator for the restriction $D_C$ of $D$ to $C = M \setminus Z$ with APS-type boundary conditions at $\partial Z$.

The proof of (1.3) includes an argument that the second term on the right hand side converges. Besides the invertibility at infinity of $D$ in the sense of (1.1), we make some functional-analytic assumptions on the restriction of $D$ to $M \setminus Z$ with APS-type boundary conditions. These boundary conditions are important in the follow-up paper [18]. Most of the proof of (1.3) is about splitting up an expression for the left hand side of (1.3) into contributions from inside and outside $Z$, incorporating the APS boundary conditions in a suitable way.
A direct consequence of (1.3) is an equivariant version of the Gromov–Lawson relative index theorem, for manifolds with ends of the type considered: if two operators agree outside compact sets, then the second terms on the right hand side of (1.3) cancel when the difference of their indices is taken. See Corollary 2.5.

Another consequence of (1.3) is Donnelly’s equivariant version of the APS index theorem, see Corollary 2.6. In that setting, we compute the second term on the right hand side of (1.3) using an explicit expression for the heat operator $e^{-sD_CD_C}$ from [1], and obtain (an equivariant version of) the $\eta$-invariant of a Dirac operator on $\partial Z$. Here the APS boundary conditions at $\partial Z$ are important.

The computation in the APS setting raises the question when the contribution from infinity in (1.3) can be computed explicitly, based on knowledge of the behaviour of $M$ and $D$ outside $Z$. In [13], we impose conditions on the function $\varphi$ in (1.2), which are satisfied for the cylindrical ends used in the APS case, and hyperbolic cusps. Then we give more explicit expressions for the contribution from infinity in (1.3).

A result analogous to (1.3) is Theorem 1.1 in [11]. In that result, Dirichlet boundary conditions are used instead of the APS boundary conditions that we need for our purposes. See also Remarks 2.1 and 5.13 for the relevance of these boundary conditions.

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2 The main result

Throughout this paper, $M$ is a complete Riemannian manifold, and $S = S^+ \oplus S^- \to M$ a $\mathbb{Z}/2$-graded Hermitian vector bundle. We denote the Riemannian density on $M$ by $dm$. We also assume that a compact Lie group $G$ acts smoothly and isometrically on $M$, that $S$ is a $G$-equivariant vector bundle and that the action on $G$ preserves the metric and grading on $S$. We fix, once and for all, an element $g \in G$. 
2.1 Dirac operators that are invertible at infinity

We suppose that $S$ is a $G$-equivariant Clifford module, which means that there is a $G$-equivariant vector bundle homomorphism $c: TM \to \text{End}(S)$, with values in the odd-graded endomorphisms, such that for all $v \in TM$,

$$c(v)^2 = -\|v\|^2 \text{Id}_S.$$ 

Let $\nabla$ be a $G$-invariant, Hermitian connection on $S$ that preserves the grading. Suppose that for all vector fields $v$ and $w$ on $M$,

$$[\nabla_v, c(w)] = c(\nabla^{TM}_v w),$$

where $\nabla^{TM}$ is the Levi–Civita connection. Consider the Dirac operator

$$D: \Gamma^\infty(S) \xrightarrow{\nabla} \Gamma^\infty(S \otimes T^* M) \cong \Gamma^\infty(S \otimes TM) \xrightarrow{c} \Gamma^\infty(S). \quad (2.1)$$

It is odd with respect to the grading on $S$; we denote its restrictions to even- and odd-graded sections by $D^\pm$, respectively.

From now on, we assume that $D$ is invertible at infinity, in the sense that there are a $G$-invariant compact subset $Z \subset M$ and a constant $b > 0$ such that for all $s \in \Gamma^\infty_c(S)$ supported in $C := M \setminus Z$,

$$\|Ds\|_{L^2} \geq b\|s\|_{L^2}. \quad (2.2)$$

For $k \geq 1$, let $W^k_D(S)$ be the completion of $\Gamma^\infty_c(S)$ in the inner product

$$(s_1, s_2)^{W^k_D} := \sum_{j=0}^k (D^j s_1, D^j s_2)_{L^2}.$$ 

Because $D$ is invertible at infinity, it is Fredholm as an operator

$$D: W^1_D(S) \to L^2(S).$$

See Theorem 2.1 in [2] or Theorem 3.2 in [15].

There is no loss of generality in assuming that $N := \partial Z$ is a smooth submanifold of $M$. We make this assumption from now on. We assume that there is a $G$-equivariant isometry

$$C \cup N \cong N \times [0, \infty), \quad (2.3)$$

mapping $N$ to $N \times \{0\}$, and where the Riemannian metric on $N \times (0, \infty)$ is of the form

$$e^{2\varphi}(g_N + dx^2), \quad (2.4)$$

where
• \( \varphi \in C^\infty(0, \infty) \);
• \( g_N \) is the restriction of the Riemannian metric to \( N \);
• \( x \) is the coordinate in \((0, \infty)\).

Furthermore, we assume that, with respect to the identification \([2,3]\), we have a \( G \)-equivariant isomorphism of vector bundles
\[
S|_U \cong S|_N \times (0, \infty) \to N \times (0, \infty),
\]
and
\[
D|_U = \sigma \left( f_1 \frac{\partial}{\partial x} + f_2 D_N + f_3 \right), \tag{2.5}
\]
where
• \( \sigma \in \text{End}(S|_N)^G \) is fibre-wise unitary, satisfies \( \sigma^2 = 1 \), and interchanges \( S^+|_N \) and \( S^-|_N \);
• \( f_1, f_2, f_3 \in C^\infty(0, \infty) \);
• \( D_N \) is a \( G \)-equivariant Dirac operator on \( S|_N \) that preserves the grading.

We write \( D^\pm_N \) for the restriction of \( D_N \) to an operator on \( \Gamma^\infty(S^\pm|_N) \).

### 2.2 The index theorem

Because \( N \) is compact, \( D^+_N \) has discrete spectrum. For \( \lambda \in \text{spec}(D^+_N) \), let \( L^2(S^+|_N)_\lambda \) be the corresponding eigenspace. Let \( J \subset \text{spec}(D^+_N) \). Consider the orthogonal projection
\[
P^+: L^2(S^+|_N) \to \bigoplus_{\lambda \in J} L^2(S^+|_N)_\lambda.
\]

We will also use the projection
\[
P^- := \sigma_+ P^+ \sigma^{-1}_+: L^2(S^-|_N) \to \sigma_+ \left( \bigoplus_{\lambda \in J} L^2(S^+|_N)_\lambda \right),
\]
where \( \sigma_+ := \sigma|_{(S^+|_N)} \). Note that \( P^- \) is not necessarily a spectral projection\(^1\) for \( D^-_N \). We combine these to an orthogonal projection
\[
P := P^+ \oplus P^-: L^2(S|_N) \to \bigoplus_{\lambda \in J} L^2(S^+|_N)_\lambda \oplus \sigma_+ \left( \bigoplus_{\lambda \in J} L^2(S^+|_N)_\lambda \right). \tag{2.6}
\]

\(^1\)In the situation of [13], we have \( D^-_N = -\sigma_+ D^+_N \sigma^{-1}_+ \), so \( P^- \) is in fact the spectral projection for \( D^-_N \) onto the eigenspaces corresponding to \(-J\).
We will omit the superscripts \( \pm \) from \( P \) from now on.

Consider the vector bundle

\[ S_C := S|_C \to C = M \setminus Z. \]

For \( k \in \mathbb{N} \), consider the Sobolev space

\[ W^k_D(S_C) := \{ s|_C; s \in W^k_D(S) \}. \]

We denote the subspaces of even- and odd-graded sections by \( W^k_D(S^+_C) \), respectively. We denote the restrictions of \( D \) to operators from \( W^{k+1}_D(S_C) \) to \( W^k_D(S_C) \) by \( D_C \).

Consider the spaces

\[ W^1_D(S^+_C; P) := \{ s \in W^1_D(S^+_C); P(s|_N) = 0 \}; \]
\[ W^1_D(S^-_C; 1 - P) := \{ s \in W^1_D(S^-_C); (1 - P)(s|_N) = 0 \}; \]
\[ W^2_D(S^+_C; P) := \{ s \in W^2_D(S^+_C); P(s|_N) = 0, (1 - P)(D_C s|_N) = 0 \}; \]
\[ W^2_D(S^-_C; 1 - P) := \{ s \in W^2_D(S^-_C); (1 - P)(s|_N) = 0, P(D_C s|_N) = 0 \}. \]

Here we use the fact that there are well-defined, continuous restriction/extension maps \( W^1_D(S_C) \to L^2(S|_N) \). We assume that

- the operators
  \[ D_C^+: W^1_D(S^+_C; P) \to L^2(S^-_C) \quad (2.7) \]
  and
  \[ D_C^-: W^1_D(S^-_C; 1 - P) \to L^2(S^+_C) \quad (2.8) \]
  are invertible; and

- the operators
  \[ D_C^-D_C^+: W^2_D(S^+_C; P) \to L^2(S^+_C) \quad (2.9) \]
  and
  \[ D_C^+D_C^-: W^2_D(S^-_C; 1 - P) \to L^2(S^-_C) \quad (2.10) \]
  are self-adjoint.

**Remark 2.1.** These conditions on the operators (2.7)–(2.10) depend on the set \( J \subset \text{spec}(D^+_N) \) that determines \( P \), and in concrete situations this set should be chosen so that those conditions hold. In the setting of [18], and in Section 5, the conditions are satisfied if \( J \) is the positive part of the spectrum of \( D^+_N \). See also Remark 5.13 for comments on the role of APS-type boundary conditions.
For $s > 0$, let $e^{-sD_{C}D_{C}^{'}}$ be the heat operator for the operator $D_{C}$. We will see in Lemma 4.7 that the operator $e^{-sD_{C}D_{C}^{'}}$ has a smooth kernel $\lambda_{s}^{P}$. For $a' \in (0,1)$, consider the contribution from infinity

\[ A_{g}(D_{C},a') := -f_{1}(a') \int_{0}^{\infty} \int_{N} \text{tr}(g\lambda_{s}^{P}(g^{-1}n,a';n,a')) \, dn \, ds, \quad (2.11) \]

defined whenever the integral in (2.11) converges. If this is the case, then we say that “$A_{g}(D_{C},a')$ converges”.

For a finite-dimensional vector space $V$ with a given representation by $G$, we write $\text{tr}(g|_{V})$ for the trace of the action by $g$ on $V$. The classical equivariant index of the $G$-equivariant Fredholm operator $D$ is

\[ \text{index}_{G}(D) := [\ker(D^{+})] - [\ker(D^{-})] \]

in the representation ring $R(G)$, and $\text{index}_{G}(D)(g)$ is the value of its character at $g$:

\[ \text{index}_{G}(D)(g) := \text{tr}(g|_{\ker(D^{+})}) - \text{tr}(g|_{\ker(D^{-})}). \]

Let $\text{AS}_{g}(D)$ be the Atiyah–Segal–Singer integrand associated to $D$, see for example Theorem 6.16 in [6] or Theorem 3.9 in [5]. It is a differential form of mixed degree on the fixed-point set $M^{g}$ of $g$. The connected components of $M^{g}$ may have different dimensions, and the integral of $\text{AS}_{g}(D)$ over $M^{g}$ is defined as the sum over these connected components of the integral of the component of the relevant degree.

**Theorem 2.2** (Index theorem for Dirac operators invertible at infinity). For all $a' \in (0,\infty)$, the contribution from infinity $A_{g}(D_{C},a')$ converges, and

\[ \text{index}_{G}(D)(g) = \int_{Z^{g}(N^{g} \times (0,a'))} \text{AS}_{g}(D) + A_{g}(D_{C},a'). \quad (2.12) \]

The limit $a' \downarrow 0$ yields a version of Theorem 2.2 that is independent of $a'$.

**Corollary 2.3.** We have

\[ \text{index}_{G}(D)(g) = \int_{Z^{g}} \text{AS}_{g}(D) + \lim_{a' \downarrow 0} A_{g}(D_{C},a'). \quad (2.13) \]

**Proof.** The first term in the right hand side of (2.12) approaches the first term on the right hand side of (2.13) as $a' \downarrow 0$.

8
Remark 2.4. In Corollary 2.3 the limit \( \lim_{a' \downarrow 0} A_g(D_C, a') \) exists, but it generally does not equal the right hand side of (2.11) with \( a' \) replaced by 0. In other words, the limit \( a' \downarrow 0 \) may not be taken inside the integral over \( s \).

Indeed, we will see in Proposition 3.5 that

\[
\int_N \text{tr}(g \Lambda_s^p(g^{-1}n; n, 0)) = 0
\]

for all \( n \in N \). For an explicit example, see Remark 5.12.

2.3 Special cases

An immediate consequence of Theorem 2.2 is the following equivariant version of Gromov and Lawson’s relative index theorem on page 304 of [15]. This shows that, already in the case where \( G \) is trivial, Theorem 2.2 can be viewed as an absolute version of Gromov and Lawson’s result, for manifolds with warped product structures at infinity.

Corollary 2.5. For \( j = 1, 2 \), let \( M_j, S_j, D_j \) and \( Z_j \) be objects like \( M, S, D \) and \( Z \), respectively. Suppose that there is a \( G \)-equivariant isometry \( M_1 \setminus Z_1 \cong M_2 \setminus Z_2 \) that preserves all structure. Then

\[
\text{index}_G(D_1)(g) - \text{index}_G(D_2)(g) = \int_{Z_1}^o \text{AS}_g(D_1) - \int_{Z_2}^o \text{AS}_g(D_2). \tag{2.14}
\]

Proof. In this setting, the number \( A_g((D_1)_{C_1}, a') \) associated to \( D_1 \) as in (2.11) equals the number \( A_g((D_2)_{C_2}, a') \) associated to \( D_2 \) for all \( a' \in (0, 1) \), so the contributions from infinity in Corollary 2.3cancel.
Let $D_N^+$ be the restriction of $D_N$ to even-graded sections. Its $g$-delocalised \( \eta \)-invariant is
\[
\eta_g(D_N^+) := \frac{1}{\sqrt{\pi}} \int_0^\infty \text{Tr}(g \circ e^{-s(D_N^+)^2}) \frac{1}{\sqrt{s}} \, ds,
\]
if this converges.

**Corollary 2.6 (Donnelly–APS index theorem).** The $g$-delocalised $\eta$-invariant of $D_N^+$ converges, and
\[
\text{index}_{\text{APS}}(D_Z)(g) = \int_{Z^g} \text{AS}_g(D) - \frac{1}{2} \eta_g(D_N^+), \tag{2.15}
\]
This is the case of Theorem 1.2 in [13] where $D_N^+$ is invertible. If $D_N^+$ is not invertible, then this result can be obtained in a similar way, through a spectral shift of $D_N^+$. See for example [22], Subsection 5.1 of [18] or Section 6 of [16].

Corollary 2.6 is deduced from Corollary 2.3 in Section 5.

## 3 Smooth kernels

To prepare for the proof of Theorem 2.2 in Section 4, we obtain some results on certain smooth kernel operators. One of these is a relation between heat operators with APS and Dirichlet boundary conditions, Proposition 3.5.

### 3.1 Kernels and projections

A key reason why the index of $D$ splits up into an interior component and a contribution from infinity, as in Theorem 2.2, is a vanishing result for a certain difference of two parametices on $U$, Proposition 4.3. In the proof of that proposition, we use the ‘principle of not feeling the boundary’ to show that heat kernels defined with respect to different boundary conditions are equal in the short time limit. That principle applies to local, e.g. Dirichlet boundary conditions. To apply it in our setting, where we use APS boundary conditions, we analyse how the principle of not feeling the boundary interacts with the projection $P$ in (2.6). This is starts with Lemma 3.1 which we prove in this subsection.

By identifying $S|_N \cong S|_N$ via the Hermitian metric on $S|_N$, we may consider the operator $D_N \otimes D_N$ on $\Gamma^\infty(S|_N \otimes S|_N)$. For any $\kappa \in \Gamma^\infty(S|_N \otimes S|_N)$, we denote the corresponding bounded operator on $L^2(S|_N)$ by $\kappa$ as well. For such a $\kappa$, the composition $\kappa \circ P$, with $P$ as in (2.6), still has a smooth kernel. We denote this kernel by $\kappa \circ P$. 

10
Lemma 3.1. There is a constant $C_1 > 0$, depending only on the dimension and volume of $N$, such that the following holds. Let $\kappa \in \Gamma^\infty(S|_{N} \boxtimes S|_{N}^*)$, and suppose that the corresponding operator is trace-class. Suppose that there are an integer $s > \dim(N)/2$ and a $C_2 > 0$ such that for all $n, n' \in N$,

$$\|(D_N \otimes D_N)^s \kappa(n, n')\| \leq C_2.$$  \hfill (3.1)

Then

$$\left| \int_N \text{tr}(g(\kappa \circ P)(g^{-1} n, n)) \, dn \right| \leq \text{Tr}(||\kappa||) \leq C_1 C_2. \hfill (3.2)$$

Proof. Let $\{e_j\}_{j=1}^\infty$ be a Hilbert basis of $L^2(S|_{N})$ of eigensections of $D_N$, such that the absolute values of the corresponding eigenvalues $\lambda_j$ increase (possibly non-strictly) in $j$. For each $j$, let $e_j^* \in L^2(S|_{N}^*)$ be the section pointwise dual to $e_j$ with respect to the Hermitian metric. Write

$$\kappa = \sum_{j,k=1}^\infty \kappa^{j,k} e_j \otimes e_k^*,$$

for $\kappa^{j,k} \in \mathbb{C}$. Then by (3.1),

$$\sum_{j,k=1}^\infty \lambda_j^{2s} |\lambda_k|^{2s} |\kappa^{j,k}|^2 = \|(D_N \otimes D_N)^s \kappa\|_{L^2(S|_{N} \boxtimes S|_{N})}^2 \leq C_2^2 \text{vol}(N)^2.$$

In particular, we have for all $j, k$, such that $\lambda_j$ and $\lambda_k$ are nonzero,

$$|\kappa^{j,k}| \leq C_2 \text{vol}(N) |\lambda_j|^{-s} |\lambda_k|^{-s}. \hfill (3.3)$$

By Weyl’s law, there is a constant $A > 0$, only depending on the volume and dimension of $N$, such that for all $l$ for which $\lambda_l \neq 0$,

$$|\lambda_l| \geq A l^{1/\dim(N)}.$$

Combining this with (3.3), we find that for all but finitely many $j$ and $k$,

$$|\kappa^{j,k}| \leq C_2 \text{vol}(N) A^{-2s} (jk)^{-s/\dim(N)}. \hfill (3.4)$$

Because $\kappa$ is trace-class, so is $\kappa \circ P$. Hence the left hand side of (3.2) equals

$$|\text{Tr}(g \circ \kappa \circ P)| \leq \text{Tr}(|g \circ \kappa \circ P|) \leq \|g\| \|P\| \text{Tr}(||\kappa||) = \text{Tr}(|\kappa|).$$

Because $2s/ \dim(N) > 1$, the inequality (3.4) implies that

$$\text{Tr}(||\kappa||) = \sum_{j=1}^\infty |\kappa^{j,j}| \leq C_1 C_2,$$

for a constant $C_1$ only depending on the volume and dimension of $N$. \qed
Remark 3.2. The $g$-trace of Definition 4.1 can be generalised to proper actions by noncompact groups, see for example Subsection 2.1 of [16]. The main reason why we assume $G$ to be compact in the current paper is the fact that the $g$-trace of $\kappa \circ P$ on the left hand side of (3.2) then equals $\text{Tr}(g \circ \kappa \circ P)$, as used in the proof of Lemma 3.1.

3.2 Heat kernels and boundary conditions

Let $e^{-tD_C^+D_C^-}$ be the heat operator for (2.10). Recall that $e^{-tD_C^-D_C^+}$ is the heat operator for (2.9).

Lemma 3.3. We have

$$e^{-tD_C^-D_C^+}D_C^- = D_C^-e^{-tD_C^-D_C^+}.$$  

Proof. Let $s \in \Gamma_c^\infty(S_C^\perp)$. For $t > 0$, we write

$$\sigma_t^P := e^{-tD_C^-D_C^+}D_C^-s;$$
$$\sigma_t^{1-P} := D_C^-e^{-tD_C^-D_C^+}s.$$  

The claim is that $\sigma_t^P = \sigma_t^{1-P}$ for all such $s$ and all $t > 0$.

The family of sections $\sigma_t^P$ is the unique solution of the equations

$$\frac{\partial \sigma_t}{\partial t} = D_C^-D_C^+\sigma_t;$$  \hspace{1cm} (3.5)
$$\sigma_0 = D_C^-s;$$  \hspace{1cm} (3.6)
$$P(\sigma_t|_N) = 0;$$  \hspace{1cm} (3.7)
$$(1-P)(D_C^-\sigma_t|_N) = 0.$$  \hspace{1cm} (3.8)

And the family $\sigma_t^{1-P}$ is the unique solution of (3.5), (3.6), (3.7) and

$$(1-P)((D_C^-)^{-1}\sigma_t|_N) = 0.$$  \hspace{1cm} (3.9)

Here we used the assumption that the operator (2.7) is invertible. We claim that (3.5), (3.6) and (3.8) are equivalent to (3.5), (3.6) and (3.9), which implies that $\sigma_t^P = \sigma_t^{1-P}$ for all $t$.

Suppose first that (3.5), (3.6) and (3.9) hold. By (3.5) and (3.9),

$$(1-P)(D_C^+\sigma_t|_N) = (1-P)((D_C^-)^{-1}D_C^-D_C^+\sigma_t|_N)$$
$$= \frac{\partial}{\partial t}(1-P)((D_C^-)^{-1}\sigma_t|_N)$$
$$= 0.$$  

12
so (3.8) holds.

Next, suppose that (3.5), (3.6) and (3.8) hold. Then (3.5) and (3.8) imply that
\[ \frac{\partial}{\partial t} (1 - P)((D_C^-)^{-1}\sigma_t|_N) = (1 - P)(D_C^+\sigma_t|_N) = 0. \]

And by (3.6),
\[ (1 - P)((D_C^-)^{-1}\sigma_0|_N) = P|_N = 0. \]
Here we used that \( s \) is supported in the interior of \( C \). We conclude that \( P((D_C^-)^{-1}\sigma_t|_N) = 0 \) for all \( t \), which is to say that (3.9) holds.

### 3.3 Dirichlet heat operators

The operator \( D_C^2 \) on \( \Gamma^\infty(S_C) \) is symmetric and nonnegative. Hence its Friedrichs extension is self-adjoint. Its domain is defined by Dirichlet boundary conditions:
\[ W^2_D(S_C; F) := \{ s \in W^2_D(S_C); s|_N = 0 \}. \]

We denote the restrictions of the heat operator for this Friedrichs extension to even and odd-graded sections by \( e^{-tD_C^-D_C^+} \) and \( e^{-tD_C^+D_C^-} \), respectively.

Let \( p := \dim(M) \). Consider the bounded operator
\[ P: L^2(S_C) \cong L^2(S|_N) \otimes L^2((0, \infty), e^{p\varphi} \, dx) \xrightarrow{P \otimes 1} \bigoplus_{\lambda \in J} L^2(S|_N)_\lambda \otimes L^2((0, \infty), e^{p\varphi} \, dx) \cong L^2(S_C). \]  

(3.10)

The first and third maps are \( G \)-equivariant, unitary isomorphisms. Here we use the fact that by the form (2.4) of the Riemannian metric on \( C \), the Riemannian density on \( C \) is \( d\text{vol}_N \otimes e^{p\varphi} \, dx \).

**Lemma 3.4.** For all \( t > 0 \),
\[ e^{-tD_C^-D_C^+} P = e^{-tD_C^-D_C^+} P; \]
\[ e_{-tD_C^+D_C^-} (1 - P) = e^{-tD_C^-D_C^+} (1 - P). \]

(3.11) 
(3.12)

**Proof.** We have
\[ \text{im}(P) \cap W^2_D(S_C^+_F; P) = \text{im}(P\zeta) \cap W^2_D(S_C^+_F; F); \]
\[ \text{im}(1 - P) \cap W^2_D(S_C^-; 1 - P) = \text{im}((1 - P)\zeta) \cap W^2_D(S_C^-; F). \]

(3.13)
So the operator (2.9) equals the Friedrichs extension of $D_C^- D_C^+$ on the image of $P$, and the operator (2.10) equals the Friedrichs extension of $D_C^+ D_C^-$ on the image of $1 - P$.

For every eigenvalue $\lambda$ of $D^\pm$, the operator $D^\pm$ restricted to $L^2((S^\pm|_N) \otimes L^2((0, \infty), e^{p\varphi} dx)$ equals $\sigma \otimes (D^\pm)^\lambda$, with

$$(D^\pm)^\lambda := f_1 \frac{\partial}{\partial x} + \lambda f_2 + f_3.$$ 

Let $e^{-t(D_C^\pm)_\lambda(D_C^\pm)^\lambda}_F$ be the corresponding heat operator on $L^2((0, \infty), e^{p\varphi} dx)$, with Dirichlet boundary conditions at $0 \in (0, \infty)$. Then because of (3.13) and the fact that $\sigma^2 = 1$, both sides of (3.11) equal

$$\bigoplus_{\lambda \in J} \text{Id}_{L^2(S^+|_N)_\lambda} \otimes e^{-t(D_C^\pm)_\lambda(D_C^\pm)^\lambda}_F,$$

while both sides of (3.12) equal

$$\bigoplus_{\lambda \notin J} \text{Id}_{L^2(S^-|_N)_\lambda} \otimes e^{-t(D_C^\pm)_\lambda(D_C^\pm)^\lambda}_F.$$ 

Lemmas 3.3 and 3.4 imply the following equality, which is used in the proofs of Propositions 4.5 and 4.6. This equality can also be used to obtain a more computable expression for 2.11, which does not include the APS-type heat operator $e^{-tD_C^- D_C^+}$. This is used in [18].

**Proposition 3.5.** We have

$$e^{-tD_C^+ D_C^-} D_C^- e^{-tD_C^- D_C^+} D_C^+ = e^{-tD_C^- D_C^+} D_C^- P + D_C^- e^{-tD_C^+ D_C^-} D_C^+ (1 - P).$$

### 3.4 Operators on a double

Let $\tilde{M}$ be a compact Riemannian manifold, on which $G$ acts isometrically. Let $\tilde{S} \to \tilde{M}$ be a $G$-equivariant Clifford module, and $\tilde{D}$ a $G$-equivariant Dirac operator acting on sections of $\tilde{S}$. Let $U \subset C$ be the $G$-invariant open subset that is the image of $N \times (0, 2)$ under the isometry (2.3). Let $V \subset U \cong N \times (0, \infty)$ be the image of $N \times (0, 1)$. Suppose that $Z \cup V$ embeds equivariantly and isometrically into $\tilde{M}$, in such a way that $S|_{Z \cup V} = \tilde{S}|_{Z \cup V}$ and $D|_{Z \cup V} = \tilde{D}|_{Z \cup V}$. Such objects can be constructed as follows. Consider the manifold $Z \cup U$ with a Riemannian metric that equals the metric on $M$. 

14
when restricted to \( Z \cup V \), and that equals the product metric \( g_N + dx^2 \) on \( N \times (3/2, 2) \subset U \). Then take the double of this manifold along \( N \times (3/2, 2) \subset U \), and extend \( S \) and \( D \) to it.

The operator \( \tilde{D} \) is self-adjoint and odd-graded. So we have the heat operators \( e^{-t\tilde{D}^\pm \tilde{D}^\pm} \). Let \( \tilde{\kappa}_t \) be the Schwartz kernel of \( e^{-t\tilde{D}^\pm \tilde{D}^\pm} \), and let \( \kappa_t^{F,\pm} \) be the Schwartz kernel of \( e^{-tD_C^\pm D_C^\pm} \). Consider the kernel

\[
K_t^{\pm} := \kappa_t^{F,\pm}|_{U \times U} - \tilde{\kappa}_t^{\pm}|_{U \times U}. 
\tag{3.14}
\]

The principle of not feeling the boundary states that away from a boundary where Dirichlet boundary conditions are imposed, Dirichlet heat kernels approximate heat kernels defined without boundary conditions in the short time limit. We will use the following version of this principle.

**Theorem 3.6 (Principle of not feeling the boundary).** For all \( j,k,l \in \mathbb{Z}_{\geq 0} \),

\[
\| (D_N^k \otimes D_N^l)(K_t^+ \circ D^-) \| = \mathcal{O}(t^j) \quad \text{and} \quad \| (D_N^k \otimes D_N^l)(D^- \circ K_t^+) \| = \mathcal{O}(t^j),
\tag{3.15}
\]
as \( t \downarrow 0 \), uniformly in compact subsets of \( U \times U \). Here \( K_t^+ \circ D^- \) is the kernel of the composition of \( K_t^+ \) with \( D^- \), and \( D^- \circ K_t^+ \) is defined similarly. The operators \( D_N^k \otimes D_N^l \) are applied in the \( N \)-direction in \( U = N \times (0,2) \).

**Proof.** We first compare the heat kernel \( \kappa_t^F \) of \( e^{-tD_C^\pm D_C^\pm} \) with the heat kernel \( \kappa_t \) of \( e^{-tD^2} \) on \( M \). By the principle of not feeling the boundary for the operator \( D \), there are constants \( a, b > 0 \) such that for all and all \( t \in (0,1] \),

\[
\| \kappa_t^F - \kappa_t \| \leq ae^{-b/t},
\]
uniformly in compact subsets of \( U \). See Theorem 1.5 in [19], Theorem 2.1 in [12] or Theorem 2.26 in [21]. Hence, in particular, the two heat kernels have the same asymptotic expansion in \( t \) on \( U \). We can also show this directly, by noting that the terms in these asymptotic expansions are determined by local geometry (see for example Theorem 7.15(i) in [25]), and are therefore equal for \( \kappa_t^F \) and \( \kappa_t \) on \( U \), analogously to the proof of Lemma 5.4 in [16].

Because asymptotic expansions commute with derivatives (see for example Theorem 7.15(ii) in [25]), it follows that for all \( j,k,l \),

\[
\| (D_N^k \otimes D_N^l)((\kappa_t^F - \kappa_t) \circ D^-) \| = \mathcal{O}(t^j) \quad \text{and} \quad \| (D_N^k \otimes D_N^l)(D^- \circ (\kappa_t^F - \kappa_t)) \| = \mathcal{O}(t^j),
\]
uniformly in compact subsets of \( U \).

A similar estimate holds with \( \kappa_t^F \) replaced by \( \tilde{\kappa}_t \), analogously to Lemma 5.4 in [16]. Hence the claim follows by the triangle inequality. \( \square \)
We will only need the weaker version of Theorem 3.6 that the left hand sides of (3.13) are bounded as $t \downarrow 0$.

### 3.5 Parametrices

Recall that $V \cong N \times (0, 1) \subset U$. Let $\varphi_1, \varphi_2, \psi \in C^\infty(M)^G$ be $G$-invariant functions with values in $[0, 1]$, such that

- $\psi|_U, \varphi_1|_U$ and $\varphi_2|_U$ are constant in the $N$-component of $U = N \times (0, 2)$;
- $\varphi_1 \psi = \psi$ and $\varphi_2(1 - \psi) = 1 - \psi$;
- $\psi|_Z = 1$ and $\psi|_{M\setminus V} = 0$;
- $\varphi_1|_{M\setminus V} = 0$ and $\varphi_2|_Z = 0$.

The second and third conditions imply that $\varphi_1|_Z = 1$ and $\varphi_2|_{M\setminus V} = 1$.

Consider the parametrix

$$
\tilde{Q} := 1 - e^{-t\tilde{D}^-\tilde{D}^+} \tilde{D}^- 
$$

(3.16)
of $\tilde{D}^+$. Here the first factor is defined as the functional calculus construction $f(\tilde{D}^-\tilde{D}^+)$ for the bounded continuous function $f(y) = \frac{1 - e^{-is}}{s}$ on $[0, \infty)$. So it is not assumed that $\tilde{D}^-\tilde{D}^+$ is invertible.

Let $(D_C^+)^{-1}$ be the inverse of the operator (2.7). Consider the operators

$$
R := \varphi_1 \tilde{Q} \psi + \varphi_2 (D_C^+)^{-1}(1 - \psi); \\
R' := \psi \tilde{Q} \varphi_1 + (1 - \psi)(D_C^+)^{-1}\varphi_2.
$$

Consider the remainder terms

$$
\tilde{S}_+ := 1 - \tilde{Q} \tilde{D}^+ = e^{-t\tilde{D}^-\tilde{D}^+}; \\
\tilde{S}_- := 1 - \tilde{D}^+ \tilde{Q} = e^{-t\tilde{D}^+\tilde{D}^-},
$$

and

$$
S_+ := 1 - RD^+; \\
S'_+ := 1 - R'D^+; \\
S_- := 1 - D^+R. 
$$

(3.17)

We have the following generalisation of Lemma 5.1 in [16].
Lemma 3.7. With $f_1$ as in (2.5),

\begin{align*}
S_+ &= \varphi_1 \tilde{S}_+ \psi + \varphi_1 \tilde{Q} \sigma' f_1 - \varphi_2 (D_C^+)^{-1} \sigma \psi f_1; \\
S'_+ &= \tilde{S}_+ \varphi_1 + \psi \tilde{Q} \sigma' f_1 + (1 - \psi) (D_C^+)^{-1} \sigma \varphi_2 f_1; \\
S_- &= \varphi_1 \tilde{S}_- \psi - f_1 \varphi'_1 \sigma \tilde{Q} \psi - f_1 \varphi'_2 \sigma (D_C^+)^{-1} (1 - \psi).
\end{align*}

Proof. This is a direct computation, based on the fact that $\tilde{D}$ and $D_C$ both equal (2.5) in the region where the functions $\varphi_j$ and $\psi_j$ are not constant. The difference with Lemma 5.1 in [16] and its proof is that now $[D, \varphi] = \varphi' f_1 \sigma$ for a smooth function $\varphi \in C^\infty(0, \infty) \hookrightarrow C^\infty(U)$, whereas in Lemma 5.1 in [16], the function $f_1$ equals one.

The operators $S_\pm$ and $S'_\pm$ have smooth kernels by the same argument as the proof of Lemma 5.2 in [16].

The inverse $(D_C^+)^{-1}$ is a parametrix of $D_C^+$, but we will also use the parametrix

$$Q_C := \frac{1 - e^{-tD_C}D_C^+}{D_C D_C^+} D_C$$

of $D_C^+$, defined via functional calculus like (3.16).

4 The contribution from infinity

After the preparations in Section 3, we prove Theorem 2.2 in this section, by showing how the contribution from infinity $A_g(D_C, a')$ appears. We first obtain a version of Theorem 2.2 in which the contributions to the index from near $Z$ and from infinity are divided by a smooth cutoff function, this is Theorem 4.14. Then we deduce Theorem 2.2 by letting this smooth function approach a step function in a suitable way.

The proof of Theorem 4.14 is a generalisation of the arguments in Section 5 of [16]. We first investigate convergence of a version of $A_g(D_C, a')$ that involves a smooth cutoff function, and then show how the results from Section 3 can be combined with arguments from [16] to obtain Theorem 4.14.
4.1 The $g$-trace and the $g$-index

As in [16] [17], we will use an extension of the operator trace to operators that are not necessarily trace-class.

**Definition 4.1.** Let $T$ be an operator on $\Gamma^\infty(S)$ with a smooth kernel $\kappa$. Then $T$ is $g$-trace class if the integral
\[
\int_M \text{tr}(g\kappa(g^{-1}m,m)) \, dm
\]
converges absolutely. In that case, the value of this integral is the $g$-trace of $T$, denoted by $\text{Tr}_g(T)$.

If $T$ is trace class and has a smooth kernel, then it is $g$-trace class, and $\text{Tr}_g(T) = \text{Tr}(g \circ T)$.

The $g$-trace has a trace property, see Lemma 3.2 in [16].

**Lemma 4.2.** Let $S, T$ be $G$-equivariant linear operators on $\Gamma^\infty(S)$. Suppose that
- $S$ has a distributional kernel;
- $T$ has a smooth kernel;
- $ST$ and $TS$ are $g$-trace-class.

Then $\text{Tr}_g(ST) = \text{Tr}_g(TS)$.

**Definition 4.3.** Let $F$ be a differential operator on $\Gamma^\infty(S)$. Suppose that $F$ is odd with respect to the grading on $S$, and let $F^+: \Gamma^\infty(S^+) \to \Gamma^\infty(S^-)$ be its restriction to even-graded sections. Then $F$ is $g$-Fredholm if $F^+$ has a parametrix $Q$ such that $1_{S_+} - QF^+$ and $1_{S_-} - F^+Q$ are $g$-trace class. In that case, the $g$-index of $F$ is
\[
\text{index}_g(F) := \text{Tr}_g(1_{S_+} - QF^+) - \text{Tr}_g(1_{S_-} - F^+Q).
\]

The properties of the $g$-index in the following lemma are special cases of Lemmas 2.5 and 2.9 in [16].

**Lemma 4.4.** The $g$-index of a $g$-Fredholm operator is independent of the choice of the parametrix $Q$. If $F$ has finite-dimensional kernel and is $g$-Fredholm, then
\[
\text{index}_g(F) = \text{index}_G(F)(g).
\]
4.2 Convergence of a contribution from infinity

Let the functions \( \varphi_j \) and \( \psi \) be as at the start of Subsection 3.5, and \( f_1 \) as in (2.5). The following result is a variation on Lemma 5.5 in [16]. Because of the boundary conditions used to define the domains of the operators (2.9) and (2.10), we now use the arguments in Subsections 3.1–3.3 to prove this.

Proposition 4.5. The operator \((\varphi_1 \tilde{Q} - \varphi_2 Q_C) \sigma \psi' f_1\) is \(g\)-trace class for all \( t > 0 \), and

\[
\lim_{t \downarrow 0} \text{Tr}_g((\varphi_1 \tilde{Q} - \varphi_2 Q_C) \sigma \psi' f_1) = 0.
\]

Proof. Let \( \varphi \in C^\infty_c(U) \) be such that \( \varphi \psi' = \psi' \), and for \( j = 1, 2 \), \( \varphi \) is constant zero outside the support of \( 1 - \varphi_j \). As in the proof of Lemma 5.5 in [16],

\[
\text{Tr}_g((\varphi_1 \tilde{Q} - \varphi_2 Q_C) \sigma \psi' f_1) = -\int_0^t \text{Tr}_g \left( \varphi \left( e^{-s \tilde{D}^- \tilde{D}^+} \tilde{D}^- - e^{-s D^-_{C} D^+_{C}} D^-_{C} \right) \psi' f_1 \right) ds.
\]

By Proposition 3.5 and the fact that \( \tilde{D}|_U = D_C|_U = D|_U \), this equals

\[
-\int_0^t \text{Tr}_g \left( \varphi \left( e^{-s \tilde{D}^- \tilde{D}^+} - e_F^{-s D^-_{C} D^+_{C}} \right) P D^- \sigma \psi' f_1 \right) ds
-\int_0^t \text{Tr}_g \left( \varphi D^- \left( e^{-s \tilde{D}^+ \tilde{D}^-} - e_F^{-s D^+_{C} D^-_{C}} \right) (1 - P) \sigma \psi' f_1 \right) ds. \tag{4.1}
\]

To be precise, in the first term we use the equality

\[
D^- \sigma \psi' = \sigma \psi' D^- + \sigma c(d\psi')
\]

and apply Proposition 3.5 twice, with \( \zeta = \sigma \psi' \) and with \( \zeta = \sigma c(d\psi') \).

The absolute value of the first term in (4.1) is at most equal to

\[
\int_0^t \int_0^2 |\varphi(x)\psi'(x)f'_1(x)| \int_N \left| \text{tr} \left( gK^+_s D^- \sigma P(g^{-1}n,n,x) \right) \right| dn \, dx \, ds \tag{4.2}
\]

Here \( K^+_s \) is as in (3.14). By Theorem 3.6 the kernel \( K^+_s D^- \sigma \) satisfies the conditions of Lemma 3.1, where the constant \( C_2 \) is independent of \( s \). That lemma therefore implies that the integrand in (4.2) is bounded in \( s \). Hence the integral converges, and goes to zero as \( t \downarrow 0 \).

The argument for the second term in (4.1) is similar, where \( P \) is replaced by \( 1 - P \).
For $t > 0$, we write
\[
A_t^g(D_C, \psi') := \int_t^\infty \text{Tr}_g(\varphi_2 e_P^{-sD_C^--D_C^+} D_C^- \psi' f_1) \, ds,
\]
when this converges.

**Proposition 4.6.** If the operators \((2.7)\) and \((2.8)\) are invertible, then $A_t^g(D_C, \psi')$ converges for all $t > 0$.

**Lemma 4.7.** The operator $e_P^{-sD_C^--D_C^+}$ has a smooth kernel.

**Proof.** For any $k \in \mathbb{N}$, we can form the bounded operator
\[
(D_C^--D_C^+)^k e_P^{-sD_C^--D_C^+}
\]
on $L^2(S^+_C)$, by applying functional calculus to \((2.9)\). So for all such $k$, and all $\xi \in L^2(S^+_C)$, the section $(D_C^-D_C^+)^k (e^{-sD_C^-D_C^+}\xi)$ is in $L^2(S^+_C)$. Hence $e^{-sD_C^-D_C^+}\xi$ is smooth, because $D_C^-D_C^+$ is elliptic. So $e_P^{-sD_C^--D_C^+}$ has a smooth kernel, and therefore so does $e_P^{-sD_C^--D_C^+}$.

We will use an analogue of Theorem 10.24 in [14].

**Lemma 4.8.** Suppose that $D_C^2 \geq b^2$ for $b > 0$. Then for all $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$, there is a $B > 0$ such that on $U$, for all $s \geq 1$,
\[
\| (D_N^{k_1} \otimes D_N^{k_2}) D_C \frac{\partial^{k_3}}{\partial s^{k_3}} \kappa_F^s \| \leq Be^{-bs} \quad \text{and} \quad \| (D_N^{k_1} \otimes D_N^{k_2}) \frac{\partial^{k_3}}{\partial s^{k_3}} \kappa_F^s D_C \| \leq Be^{-bs}.
\]

**Proof.** For a relatively compact open subset $\Omega \subset C$, let $D^{2}_\Omega$ be the Friedrichs extension of $D^2|_{\Gamma^\infty(S|_\Omega)}$. Then $D^2_\Omega$ has discrete spectrum satisfying the Weyl law. Let $\lambda_1(\Omega)$ be the smallest eigenvalue. Let $\kappa_F^s(\Omega)$ be the heat kernel associated to $D^2_\Omega$. Then for all $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$ there is a constant $C_\Omega$ such that for all $n, n' \in N$ and $x, x' \in (0, \infty)$,
\[
\| (D_N^{k_1} \otimes D_N^{k_2}) D \frac{\partial^{k_3}}{\partial s^{k_3}} \kappa_F^s(\Omega, n, x; n', x') \| \leq C_\Omega e^{-s\lambda_1(\Omega)}, \tag{4.3}
\]
and a similar estimate holds with $D \kappa_F^s(\Omega)$ replaced by $\kappa_F^s(\Omega) D$. See e.g. the second-last displayed equation of the proof of Theorem 10.24 in [14] for the scalar case, and without differential operators applied to the heat kernel.
The argument is relatively elementary and also applies in our setting. Indeed, (4.3) follows directly from a decomposition of \( \kappa_s^F, \Omega \) into eigensections of \( D^2 \), as in the third-last displayed equation of the proof of Theorem 10.24 in [14]. Each derivative with respect to \( s \) can be replaced by \( D \) because \( \kappa_s^F \) satisfies the heat equation.

One has \( \lambda_1(\Omega) \leq \lambda_1(\Omega') \) when \( \Omega' \subset \Omega \), so then \( e^{-s \lambda_1(\Omega')} \leq e^{-s \lambda_1(\Omega)} \). This implies that for all such \( \Omega \),

\[ e^{-s \lambda_1(\Omega)} \leq e^{-b^2 s}, \]

and that the lemma is true.

**Proof of Proposition 4.6.** Let \( t \in (0, 1] \). Let \( \kappa_{t, \pm}^P \) be the Schwartz kernel of \( e^{-t D_C^\pm D_C^\pm} \). As in the proof of Proposition 4.5, we estimate

\[
\int_t^\infty \int_{M \setminus Z} \left| \text{tr} \varphi_2 (g \kappa_s^P, D_C^- (g^{-1} m, m)) \psi f_1 \right| ds \, dm \\
\leq \int_t^\infty \int_0^2 \left| \psi'(x) f_1(x) \right| \left| \int_N \text{tr} (g \kappa_s^P, D_C^- (g^{-1} n, x, n, x)) \, dn \right| \, dx \, ds \\
= \int_t^\infty \int_0^2 \left| \psi'(x) f_1(x) \right| \left| \int_N \text{tr} (g \kappa_s^F, D_C^- P + g D_C^- \kappa_s^F, (1 - P)) (g^{-1} n, x, n, x)) \, dn \right| \, dx \, ds.
\]

(4.4)

Here we used Proposition 3.5.

By Lemma 4.8, we may apply Lemma 3.1 to find that there is a constant \( B > 0 \) such that for all \( x \in (0, \infty) \) and all \( s \geq 1 \),

\[
\left| \int_N \text{tr} (g \kappa_s^F, D_C^- P + D_C^- \kappa_s^F, (1 - P)) (g^{-1} n, x, n, x)) \, dn \right| \leq Be^{-b^2 s}.
\]

So the integral in the right hand side of (4.4) converges.

We will deduce convergence of \( \lim_{t \downarrow 0} A_{\psi}'(D_C, \psi') \) from the index formula in the proof of Theorem 4.14 below. But as an aside, we note that this convergence also follows from a generalisation by Zhang of a result by Bismut and Freed, in the case of twisted Spin^r-Dirac operators.

**Proposition 4.9.** If \( D \) is a twisted Spin^r-Dirac operator, then \( \lim_{t \downarrow 0} A_{\psi}'(D_C, \psi') \) converges.
Proof. By Proposition 4.6 and (4.4), it is enough to show that the limit

$$\lim_{t \downarrow 0} \int_{0}^{1} \int_{0}^{2} |\psi'(x)f_1(x)| \left| \int_{N} \text{tr} \left( g\kappa_{s}^{F,+}D_{G}^{-}P + D_{G}^{-}\kappa_{s}^{F,-}(1 - P) \right) (g^{-1}n, x; n, x) \ dn \right| \ dx \ ds$$

converges. By Theorem 3.6 and Lemma 3.1, we may replace $\kappa_{s}$ by $\tilde{\kappa}_{s}$ here. And

$$\tilde{\kappa}_{s}^{+}D_{G}^{-}|U = \tilde{\kappa}_{s}^{+}\tilde{D}^{-}|U = \tilde{D}^{-}\tilde{\kappa}_{s}^{-}|U.$$ 

Arguing similarly for the second term, we find

$$\left( \tilde{\kappa}_{s}^{+}D_{G}^{-}P + D_{G}^{-}\tilde{\kappa}_{s}^{-}(1 - P) \right) |U = \tilde{D}^{-}\tilde{\kappa}_{s}^{-}|U.$$ 

We now use the assumption that $D$, and hence $\tilde{D}$ is a twisted Spin$^c$-Dirac operator. Then (2.2) in [27], generalising the second part of Theorem 2.4 in [7], states that

$$\text{Tr} \left( g\tilde{D}^{-}\tilde{\kappa}_{s}^{-}(g^{-1}n, x; n, x) \right) = O(s^{1/2})$$

as $s \downarrow 0$, uniformly in $(n, x) \in U$. 

4.3 Appearance of a contribution from infinity

Lemma 4.10. Let $\chi \in C_{c}^{\infty}(0, \infty)$. Suppose that $D_{G}^{2} \geq b^{2} > 0$. There is a $B > 0$ such that for all $s > 0$,

$$\text{Tr}(||\chi e_{P}^{-sD_{G}^{-}D_{G}^{+}}D_{G}^{-}\psi'||_{L^{2}}) \leq Be^{-b^{2}s}. \quad (4.5)$$

Proof. There is a bounded linear isomorphism $L^{2}(S|U) \cong L^{2}(S|N) \otimes L^{2}(0, 2)$ with bounded inverse. See Lemma 3.5 in [18] for a stronger statement (a unitary isomorphism), but in the current setting boundedness of $e^{\varphi}$ above and below on $(0, 2)$ is enough. We work with $L^{2}(S|N) \otimes L^{2}(0, 2)$ from now on. Let $\{e_{j}^{N}\}_{j=1}^{\infty}$ be a Hilbert basis of $L^{2}(S|N)$ of eigensections of $D_{N}$, and let $\{e_{j}^{(0,2)}\}_{j=1}^{\infty}$ be a Hilbert basis of $L^{2}(0, 2)$. Then the sections $e_{j,k} := e_{j}^{N} \otimes e_{k}^{(0,2)}$ form a Hilbert basis of $L^{2}(S|N) \otimes L^{2}(0, 2)$.

We use Proposition 3.5 to write

$$e_{P}^{-sD_{G}^{-}D_{G}^{+}}D_{G}^{-}\psi' = (e_{F}^{-sD_{G}^{-}D_{G}^{+}}D_{G}^{-}P + D_{G}^{-}e_{F}^{-sD_{G}^{-}D_{G}^{+}}(1 - P))\psi'.$$
So the left hand side of (4.5) equals
\[
\sum_{j,k=1}^{\infty} \left| \left( e_j^N \otimes e_k^{(0,2)}, \chi e_P^{sD_C^+ D_C^+} e_j^N \otimes e_k^{(0,2)} \right) \right|_{L^2(S|_N) \otimes L^2(0,2)} \\
\leq \max_{x,x' \in (0,2)} \text{Tr} \left( \left| \left( \kappa_+^{F,-} D_C^+ P + D_C^- \kappa_+^{F,-} (1 - P) \right) (-, x; -, x') \right| \right) \\
\sum_{k=1}^{\infty} \left| (e_k^{(0,2)}, \chi)_{L^2(0,2)} (e_k^{(0,2)}, \psi' f_1)_{L^2(0,2)} \right|.
\] (4.6)

We apply Lemma 4.8 and use the resulting bound to apply Lemma 3.1 to the first factor on the right. The second inequality in (3.2) then yields a constant $B_0 > 0$ such that
\[
\max_{x,x' \in (0,2)} \text{Tr} \left( \left| \left( \kappa_+^{F,+} D_C^- P + D_C^+ \kappa_+^{F,+} (1 - P) \right) (-, x; -, x') \right| \right) \leq B_0 e^{-b^2 s}.
\]

Furthermore,
\[
\sum_{k=1}^{\infty} \left| (e_k^{(0,2)}, \chi)_{L^2(0,2)} (e_k^{(0,2)}, \psi' f_1)_{L^2(0,2)} \right| = \text{Tr}(\chi \otimes \psi' f_1),
\]
where $\chi \otimes \psi' f_1$ stands for the operator on $L^2(0,2)$ with smooth kernel $\chi \otimes \psi'$. So the sum over $k$ on the right hand side of (4.6) indeed converges, and the claim follows with $B := B_0 \text{Tr}(\chi \otimes \psi' f_1)$.

**Lemma 4.11.** If $A'_g(D_C, \psi')$ converges, then $\varphi e_{sD_C^- D_C^+} (D_C^+)^{-1} \psi' f_1$ is g-trace class, and its g-trace equals $-A'_g(D_C, \psi')$.

**Proof.** Let $\chi \in C^\infty_c(0, \infty)$ be equal to 1 on supp($\psi'$). Then the all $s > 0$, the operator $\chi e_P^{sD_C^+ D_C^+} (D_C^+)^{-1} \psi' f_1$ has a compactly supported smooth kernel, is trace class, and
\[
A'_g(D_C, \psi') = \int_t^\infty \text{Tr}(\chi e_P^{sD_C^+ D_C^+} D_C^- \psi' f_1) \, ds.
\]

Let $\{e_j\}_{j=1}^{\infty}$ be a Hilbert basis of $L^2(S)$. Then the right hand side equals
\[
\int_t^\infty \sum_{j=1}^{\infty} \left( e_j, \chi e_P^{sD_C^+ D_C^+} D_C^- \psi' f_1 e_j \right)_{L^2(S)} \, ds.
\] (4.7)
This combined integral and sum converges absolutely by Lemma [4.10] and the fact that
\[ |\chi g e^{-sD_C^+D_C^-}D_C^-\psi' f_1| = |\chi e^{-sD_C^+D_C^-}D_C^-\psi' f_1|, \]
because \( \chi \) is \( G \)-invariant, and \( g \) acts unitarily. So (4.7) equals
\[ \sum_{j=1}^{\infty} \int_I^\infty \left( e_j, \chi g e^{-sD_C^+D_C^-}D_C^-\psi' f_1 e_j \right)_{L^2(S)} ds. \]
Using the spectral decomposition of (2.9), we find that this equals
\[ -\sum_{j=1}^{\infty} \left( e_j, \chi g e^{-tD_C^+D_C^-}D_C^-\psi' f_1 e_j \right)_{L^2(S)} = -\text{Tr}(\chi g e^{-tD_C^+D_C^-}D_C^-\psi' f_1) \]
\[ = -\text{Tr}(\varphi_2 e^{-tD_C^+D_C^-}D_C^-\psi' f_1). \]

4.4 An index formula with a smooth cutoff function

**Lemma 4.12.** The operators \( S'_+ \) and \( S_- \) in (3.17) are \( g \)-trace class, and
\[ \text{Tr}_g(S'_+) - \text{Tr}_g(S_-) = \text{Tr}(g \circ e^{-tD^-D^+} \psi) - \text{Tr}(g \circ e^{-tD^+D^-} \psi). \] (4.8)

**Proof.** The proof of Lemma 5.3 in [16] shows that \( S'_+ \) and \( S_- \) are \( g \)-trace class, and that the left hand side of (4.8) equals
\[ \text{Tr}_g(e^{-tD^-D^+} \psi) - \text{Tr}_g(e^{-tD^+D^-} \psi). \]
One then uses the fact that the heat operator \( e^{-tD^2} \) is trace class, and the comment below Definition 4.1 on \( g \)-traces of trace-class operators. \( \square \)

**Proposition 4.13.** If the number \( A^t_g(D_C, \psi') \) in (2.11) converges for all \( t \in (0,1] \), then the operator \( S_+ \) is \( g \)-trace class. And
\[ \text{Tr}_g(S_+) - \text{Tr}_g(S'_+) = A^t_g(D_C, \psi') + F(t), \] (4.9)
for a function \( F \) satisfying \( \lim_{t \downarrow 0} F(t) = 0. \)
Proof. Lemma 3.7 implies that

\[ S_+ - S_+^{'} = (\varphi_1 \tilde{S}_+ \psi - \psi \tilde{S}_+ \varphi_1) - (\psi \tilde{Q} \sigma \varphi_1 f_1 + (1 - \psi)(D_C^+)^{-1} \sigma \varphi_2 f_1) + (\varphi_1 \tilde{Q} \sigma \psi f_1 - \varphi_2 (D_C^+)^{-1} \sigma \psi f_1). \quad (4.10) \]

The first term on the right hand side of (4.10) is g-trace class because it is a smooth kernel operator on the compact manifold \( \tilde{M} \). And its g-trace is zero by the trace property of \( \text{Tr}_g \), Lemma 4.2.

The second term on the right hand side of (4.10) is g-trace class with g-trace zero, because it is a smooth kernel operator that vanishes on the set \( \{(g^{-1} m, m); m \in U\} \). Here we use the facts that \( \tilde{Q} \) and \( (D_C^+)^{-1} \) are pseudodifferential operators, and that the supports of \( \psi \) and \( \varphi_1 \) are \( G \)-invariant and disjoint, and the same is true for the supports of \( (1 - \psi) \) and \( \varphi_2^{'} \). See also the proof of Lemma 5.3 in [16].

The third term on the right hand side of (4.10) equals

\[ (\varphi_1 \tilde{Q} - \varphi_2 Q_C) \sigma \psi f_1 - \varphi_2 ((D_C^+)^{-1} - Q_C) \sigma \psi f_1. \quad (4.11) \]

Proposition 4.5 implies that the first of these terms is g-trace class for all \( t > 0 \), and that its g-trace, which will be \( F(t) \) in (4.9), goes to zero in the limit \( t \downarrow 0 \). The second term in (4.11) equals

\[ -\varphi_2 e^{-tD_C^-D_C^+} (D_C^+)^{-1} \sigma \psi f_1. \]

Hence the claim follows by Lemma 4.11.

We obtain a version of Theorem 2.2 involving the function \( \psi \).

**Theorem 4.14.** The limit \( \lim_{t \downarrow 0} A_t^g(D_C, \psi') \) converges, and

\[ \text{index}_G(D)(g) = \int_M \psi|_M A_S g(D) + \lim_{t \downarrow 0} A_t^g(D_C, \psi'). \quad (4.12) \]

**Proof.** Convergence of \( A_t^g(D_C, \psi') \) for \( t > 0 \) is Proposition 4.6.

The operator \( S_- \) is g-trace class by Lemma 4.12. And the operator \( S_+ \) is g-trace class if \( A_t^g(D_C, \psi') \) converges, by Proposition 4.13. So then \( D^+ \) is g-Fredholm. And because it is also Fredholm in the usual sense, because of (2.2), Lemma 4.4 implies that

\[ \text{index}_G(D)(g) = \text{index}_g(D) = \text{Tr}_g(S_+) - \text{Tr}_g(S_-). \]

The operator \( S_+^{'} \) is g-trace class by Lemma 4.12, so the right hand side equals

\[ (\text{Tr}_g(S_+) - \text{Tr}_g(S_+^{'})) + (\text{Tr}_g(S_+^{'} - \text{Tr}_g(S_-)). \]
By Lemma 4.12 and Proposition 4.13, this equals
\[
\text{Tr}(g \circ e^{-tE^+E'^-} \psi) - \text{Tr}(g \circ e^{-tE^+E'^-} \psi) + A_g^t(D_C, \psi') + F(t),
\]
where \( \lim_{t \downarrow 0} F(t) = 0 \). This total expression is independent of \( t \), and by standard heat kernel asymptotics, the first two terms converge to the first term on the right of (4.12) as \( t \downarrow 0 \). (See e.g. the proof of Theorem 6.16 in [6].) Hence \( \lim_{t \downarrow 0} A_g^t(D_C, \psi') \) converges, and (4.12) holds.

\[4.5\] The limit \( \psi' \to -\delta_{a'} \)

Theorem 4.14 implies Theorem 2.2 via a limit where \( \psi' \) approaches minus the Dirac delta distribution \( \delta_{a'} \) at \( a' \in (0, 1) \). We make this precise in this subsection. For \( a' \in (0, 1) \) and \( t > 0 \), we write
\[
A_g^t(D_C, a') := -f_1(a') \int_t^\infty \int_N \text{tr}(g \lambda_s^P(g^{-1}n, a'; n, a')) dn ds,
\]
when this converges.

**Lemma 4.15.** For all \( a' \in (0, 1) \) and \( t > 0 \), \( A_g^t(D_C, a') \) converges. And for all \( t > 0 \) and \( \varepsilon > 0 \), there is a \( \psi \) as at the start of Subsection 3.5 such that
\[
|A_g^t(D_C, \psi') - A_g^t(D_C, a')| < \varepsilon. \tag{4.14}
\]

**Proof.** For \( s > 0 \) and \( x \in (0, 1) \), we write
\[
F(s, x) := \int_N \text{tr}(g \lambda_s^P(g^{-1}n, x)n, x)f_1(x)) dn.
\]
This defines a smooth function on \((0, \infty) \times (0, 1)\). To avoid complications near the boundaries, we consider functions \( \chi_1 \in C^\infty(0, \infty) \) and \( \chi_2 \in C_c^\infty(0, 1) \) with values in \([0, 1]\), such that
\[
\begin{align*}
\chi_1(s) &= 0 \quad \text{for } s \leq t/2; \\
\chi_1(s) &= 1 \quad \text{for } s \geq t; \\
\chi_2(x) &= 1 \quad \text{for } x \in [a'/2, a''],
\end{align*}
\]
for some \( a'' \in (a'/2, 1) \). Then by Lemma 4.8 and the expression for \( \lambda_s^P \) in terms of Dirichlet heat kernels from Proposition 3.5, the extension of the function \((\chi_1 \otimes \chi_2)f\) by zero outside \((0, \infty) \times (0, 1)\) lies in the Schwartz space \( \mathcal{S}(\mathbb{R}^2) \).
Fix $\varepsilon > 0$. Now the distribution $\delta_{\mathbb{R} \times \{a'\}} \in \mathcal{S}'(\mathbb{R}^2)$ defined by integrating functions over $\mathbb{R} \times \{a'\}$, can be approximated arbitrarily closely by functions $1 \otimes (-\psi')$ in the sense of distributions, with $\psi$ as at the start of Subsection 3.5 and with $\psi'$ supported in $(a'/2, a'')$. So we can choose $\psi$ with these properties such that

$$
\left| \int_{\mathbb{R}^2} \chi_1(s) F(s, x)(-\psi'(x)) \, ds \, dx - \int_{\mathbb{R}} \chi_1(s) F(s, a') \, ds \right| < \varepsilon/2. \tag{4.15}
$$

Here we use the fact that $\chi_2(x) = 1$ for $x \in \text{supp}(\psi')$, so the first integral does not change if we insert $\chi_2(x)$ into the integrand.

By choosing $\chi_1$ supported close enough to $[t, \infty)$, we can ensure that the left hand side of (4.15) is within $\varepsilon/2$ of the left hand side of (4.14). \qed

Proof of Theorem 2.2. To prove that $A_g(D, a')$ converges, our starting point is the fact that $\text{index}_G(D)(g)$ equals (4.13), as shown in the proof of Theorem 4.14. Let $\varepsilon > 0$. By standard heat kernel asymptotics, we can choose $t > 0$ such that

$$
\left| \text{Tr}(g \circ e^{-t\tilde{D}^+\tilde{D}^-} \psi) - \text{Tr}(g \circ e^{-t\tilde{D}^+\tilde{D}^-} \psi) \right| < \varepsilon.
$$

Fix $t > 0$ such that this holds, and also $|F(t)| < \varepsilon$. For $a' \in (0, 1)$, choose $\psi$ as in Lemma 4.15. By choosing $\psi$ so that $\psi'$ is close enough to $\delta_{N \times \{a'\}}$, we can ensure that also

$$
\left| \int_{M^g} \psi|_{M^g} \text{AS}_g(D) - \int_{Z^g \cup (N^g \times (a, a']]} \text{AS}_g(D) \right| < \varepsilon.
$$

Then we find that

$$
\left| \text{index}_G(D)(g) - \int_{Z^g \cup (N^g \times (a, a']]} \text{AS}_g(D) - A_g(D, a') \right| < 4\varepsilon.
$$

So $A_g(D, a') = \lim_{t \to 0} A_g(D, a')$ converges, and (2.12) holds. \qed

5 The Donnelly–APS index theorem

In this section, we use Corollary 2.3 to prove Corollary 2.6. The proof is based on an explicit expression for the APS heat operator $e^{-sD^+D^-}$, given in [4]. In Subsection 5.1 of [15], we give a spectral version of the geometric computation in this section.

27
5.1 Vanishing of an integral

In the proof of Corollary 2.6, we will use the following vanishing result.

**Proposition 5.1.** Let \((\lambda_j)_{j=1}^\infty\) and \((a_j)_{j=1}^\infty\) be sequences in \(\mathbb{R}\) such that \(|\lambda_1| > 0\), and \(|\lambda_j| \leq |\lambda_{j+1}|\) for all \(j\), and such that there are \(c_1, c_2, c_3, c_4 > 0\) such that for all \(j\),

\[
|\lambda_j| \geq c_1 j^{c_2}; \\
|a_j| \leq c_3 j^{c_4}.
\]  

(5.1)

Then for all \(a' > 0\),

\[
\int_0^\infty \sum_{j=1}^\infty \text{sgn}(\lambda_j)a_j \frac{e^{-\lambda_j^2 s} e^{-a'^2/s}}{\sqrt{s}} \left(\frac{a'}{s} - |\lambda_j|\right) ds = 0. \tag{5.2}
\]

The main part of the proof is to show that the order of the integral and sum may be interchanged. The rest of the argument is a substitution.

We use the notation of Proposition 5.1. We first consider a version of (5.2) where the integral over \(s\) starts at a positive number \(t\), which we fix from now on.

**Lemma 5.2.** The integral and sum

\[
\int_t^\infty \sum_{j=1}^\infty \text{sgn}(\lambda_j)a_j \frac{e^{-\lambda_j^2 s} e^{-a'^2/s}}{\sqrt{s}} \left(\frac{a'}{s} - |\lambda_j|\right) ds \tag{5.3}
\]

converge absolutely.

**Proof.** For all \(s \geq t\),

\[
\left|a_j \frac{e^{-\lambda_j^2 s} e^{-a'^2/s}}{\sqrt{s}} \left(\frac{a'}{s} - |\lambda_j|\right)\right| \leq |a_j| e^{-\lambda_j^2 t/2} e^{-\lambda_j^2 s/2} \left(\frac{a'}{t} + |\lambda_j|\right).
\]

The sum

\[
\sum_{j=1}^\infty |a_j| e^{-\lambda_j^2 t/2} \left(\frac{a'}{t} + |\lambda_j|\right)
\]

converges by (5.1). The integral

\[
\int_t^\infty e^{-\lambda_j^2 s/2} ds
\]

converges because \(|\lambda_1| > 0\).

\[\square\]
Lemma 5.3. For all nonzero $\lambda \in \mathbb{R}$,

$$
\int_{t}^{\infty} \frac{e^{-\lambda^2 s} e^{-a'^2/s}}{\sqrt{s}} ds = |\lambda| \int_{0}^{a'^2/\lambda^2 t} \frac{e^{-\lambda^2 s} e^{-a'^2/s}}{\sqrt{s}} ds.
$$

Proof. Substitute $s \mapsto \frac{a'^2}{\lambda^2 s}$.

By Lemmas 5.2 and 5.3 and the Fubini–Tonelli theorem, the expression

$$
\sum_{j=1}^{\infty} \lambda_j a_j \left( \int_{0}^{a'^2/\lambda^2 t} \frac{e^{-\lambda^2 s} e^{-a'^2/s}}{\sqrt{s}} ds - \int_{t}^{\infty} \frac{e^{-\lambda^2 s} e^{-a'^2/s}}{\sqrt{s}} ds \right).
$$

(5.4)

We will bound the term for each $j$ by a summable function independent of $t$, so that the limit $t \downarrow 0$ may be taken inside the sum by dominated convergence. We consider two cases separately: $t \leq a'/|\lambda_j|$ and $t \geq a'/|\lambda_j|$.

Lemma 5.4. For all $t > 0$ and nonzero $\lambda \in \mathbb{R}$ with $t \leq a'/|\lambda|$,

$$
\int_{0}^{t} \frac{e^{-2\lambda^2 s - a'^2/s}}{\sqrt{s}} ds \leq \frac{a'}{|\lambda|} e^{-a'|\lambda|}.
$$

Proof. If $s \leq t \leq a'/|\lambda|$, then $e^{-a'^2/s} \leq e^{-a'|\lambda|}$. So the left hand side is at most equal to $te^{-a'|\lambda|}$, which is at most equal to the right hand side.

Lemma 5.5. For all $t > 0$ and nonzero $\lambda \in \mathbb{R}$ with $t \leq a'/|\lambda|$,

$$
\int_{0}^{a'^2/\lambda^2 t} \frac{e^{-\lambda^2 s} e^{-a'^2/s}}{\sqrt{s}} ds - \int_{t}^{\infty} \frac{e^{-\lambda^2 s} e^{-a'^2/s}}{\sqrt{s}} ds = \int_{0}^{t} \frac{e^{-\lambda^2 s} e^{-a'^2/s}}{\sqrt{s}} \left(1 - \frac{a'}{|\lambda| s} \right) ds.
$$

(5.5)

Proof. If $t \leq a'/|\lambda|$, then $t \leq a'^2/\lambda^2 t$. So the left hand side of (5.5) equals

$$
\int_{0}^{t} \frac{e^{-\lambda^2 s} e^{-a'^2/s}}{\sqrt{s}} ds - \int_{a'^2/\lambda^2 t}^{\infty} \frac{e^{-\lambda^2 s} e^{-a'^2/s}}{\sqrt{s}} ds.
$$

By a substitution $s \mapsto a'^2/\lambda^2 s$ (or by Lemma 5.3),

$$
\int_{a'^2/\lambda^2 t}^{\infty} \frac{e^{-\lambda^2 s} e^{-a'^2/s}}{\sqrt{s}} ds = \frac{a'}{|\lambda|} \int_{0}^{t} \frac{e^{-\lambda^2 s} e^{-a'^2/s}}{\sqrt{s}} ds.
$$

\[\square\]
Lemma 5.6. For all $t > 0$ and nonzero $\lambda \in \mathbb{R}$ with $t \leq a'/|\lambda|$, 

$$
\left| \int_0^{a^2/\lambda^2 t} \frac{e^{-\lambda^2 s}e^{-a^2/s}}{\sqrt{s}} \, ds - \int_t^{\infty} \frac{e^{-\lambda^2 s}e^{-a^2/s}}{\sqrt{s}} \, ds \right| \leq f_1(t, |\lambda|)e^{-a'|\lambda|/2}, \tag{5.6}
$$

for a function $f_1$ that is bounded in $t \in (0, 1]$ and in $|\lambda|$ in a subset of $\mathbb{R}$ with a positive lower bound.

Proof. By Lemma 5.5 and the Cauchy-Schwartz inequality, the left hand side of (5.6) is at most equal to

$$
\left( \int_0^t e^{-2\lambda^2 s}e^{-a^2/s} \, ds \right)^{1/2} \left( \int_0^t \frac{e^{-a^2/s}}{s} \left( 1 - \frac{a'}{|\lambda|s} \right)^2 \, ds \right)^{1/2}.
$$

The second factor is bounded in $t \in (0, 1]$ and in $|\lambda|$ in a set with a positive lower bound. By Lemma 5.4, the first factor is at most equal to

$$
\left( \frac{a'}{|\lambda|e^{-a'|\lambda|}} \right)^{1/2}.
$$

Next, we turn to the case $t \geq a'/|\lambda|$. 

Lemma 5.7. For all nonzero $\lambda \in \mathbb{R}$ and $t \geq a'/|\lambda|$, 

$$
\left| \int_0^{a^2/\lambda^2 t} \frac{e^{-\lambda^2 s}e^{-a^2/s}}{\sqrt{s}} \, ds \right| \leq f_2(|\lambda|)e^{-a'|\lambda|/2}, \tag{5.7}
$$

for a function $f_2$ that is bounded in $|\lambda|$ in a subset of $\mathbb{R}$ with a positive lower bound.

Proof. If $s \leq a^2/\lambda^2 t \leq a'/|\lambda|$, then $e^{-a^2/2s} \leq e^{-a'|\lambda|/2}$. This implies that the left hand side of (5.7) is at most equal to

$$
\frac{a^2}{\lambda^2 t} e^{-a'|\lambda|/2} \max_{0 \leq s \leq a^2/\lambda^2 t} \frac{e^{-a^2/2s}}{\sqrt{s}}. \tag{5.8}
$$

The condition on $t$ implies that $\frac{a^2}{\lambda^2 t} \leq a'/|\lambda|$, so the claim follows. 

\[ \square \]
Lemma 5.8. For all nonzero $\lambda \in \mathbb{R}$ and $t \geq a'/|\lambda|$, 

$$
\left| \int_t^{\infty} e^{-\lambda^2 s} e^{-a^2/s} \frac{ds}{\sqrt{s}} \right| \leq f_3(|\lambda|) e^{-a'|\lambda|/2},
$$

(5.9)

for a function $f_3$ that is bounded in $|\lambda|$ in a subset of $\mathbb{R}$ with a positive lower bound.

Proof. If $s \geq t \geq a'/|\lambda|$, then 

$$
e^{-\lambda^2 s} e^{-a^2/s} \frac{1}{\sqrt{s}} \leq e^{-a'|\lambda|/2} e^{-\lambda^2 s/2} e^{-a^2/s} \frac{1}{\sqrt{s}}.
$$

So the left hand side of (5.9) is at most equal to 

$$e^{-a'|\lambda|/2} \int_0^{\infty} e^{-\lambda^2 s/2} e^{-a^2/s} \frac{ds}{\sqrt{s}}.
$$

And 

$$\int_0^{\infty} e^{-\lambda^2 s/2} e^{-a^2/s} ds \leq \int_0^1 e^{-a^2/s} ds + \int_1^{\infty} e^{-\lambda^2 s/2} ds.
$$

The first term on the right is constant in $\lambda$, the second is bounded in $\lambda$ in sets with positive lower bounds.

We now combine the cases $t \leq a'/|\lambda_j|$ and $t \geq a'/|\lambda_j|$.

Lemma 5.9. For all $t > 0$ and nonzero $\lambda \in \mathbb{R}$, 

$$
\left| \int_0^{a'/\lambda^2} e^{-\lambda^2 s} e^{-a^2/s} \frac{ds}{\sqrt{s}} - \int_t^\infty e^{-\lambda^2 s} e^{-a^2/s} \frac{ds}{\sqrt{s}} \right| \leq f(t, |\lambda|) e^{-a'|\lambda|/2},
$$

for a function $f$ that is bounded in $t \in (0, 1]$ and in $|\lambda|$ in a subset of $\mathbb{R}$ with a positive lower bound.

Proof. For $t > 0$ and nonzero $\lambda \in \mathbb{R}$, let 

$$f(t, |\lambda|) := \max(f_1(t, |\lambda|), f_2(|\lambda|) + f_3(|\lambda|)),
$$

for $f_1$ as in Lemma 5.6, $f_2$ as in Lemma 5.7, and $f_3$ as in Lemma 5.8. Then the claim follows from Lemma 5.6 if $t \leq a'/|\lambda|$ and from Lemmas 5.7 and 5.8 if $t \geq a'/|\lambda|$.
Proof of Proposition 5.1. By Lemmas 5.2 and 5.3 the left hand side of (5.2) equals

\[
\lim_{t \downarrow 0} \sum_{j=1}^{\infty} \lambda_j a_j \left( \int_0^{a^2/\lambda_j^2} e^{-\lambda_j^2 s} e^{-a^2/s} \sqrt{s} \, ds - \int_t^{\infty} e^{-\lambda_j^2 s} e^{-a^2/s} \sqrt{s} \, ds \right).
\]

(5.10)

For every \(j\) separately,

\[
\lim_{t \downarrow 0} \left( \int_0^{a^2/\lambda_j^2} e^{-\lambda_j^2 s} e^{-a^2/s} \sqrt{s} \, ds - \int_t^{\infty} e^{-\lambda_j^2 s} e^{-a^2/s} \sqrt{s} \, ds \right) = 0.
\]

By Lemma 5.9 the term in (5.10) for each \(j\) is bounded by a constant times \(|\lambda_j a_j| e^{-a^2/\lambda_j}|/2\) for all \(t \in (0,1]\). The sum over \(j\) of this bounding expression converges by (5.1). So by dominated convergence, (5.10) equals zero.

5.2 Computation of the contribution from infinity

For every \(\lambda \in \text{spec}(D_N^+)\), let \(\{\varphi^N_\lambda, \ldots, \varphi^m_\lambda\}\) be an orthonormal basis of \(\ker(D_N^+ - \lambda)\). We identify \(S^+_N \cong (S^+_N)^*\) using the metric, so that for all \(n, n' \in N, \lambda \in \text{spec}(D_N^+)\) and \(j \in \{1, \ldots, m_\lambda\}\),

\[
\varphi^j_\lambda(n) \otimes \varphi^j_\lambda(n') \in S^+_n \otimes (S^+_n)^* = \text{Hom}(S^+_n, S^+_n).
\]

In this subsection, we take the subset \(J \subset \text{spec}(D_N^+)\) in the definition of \(P\) in Subsection 2.2 to be the positive part of the spectrum of \(D_N^+\). Then, as in [4], \(P\) is orthogonal projection onto the positive eigenspaces of \(D_N^+\).

Proposition 5.10. The Schwartz kernel \(\lambda^P_s\) of \(e^{-sD_N^+}D_N^+\) is given by

\[
\lambda^P_s(n, y; n', y') = \sum_{\lambda > 0} \sum_{j=1}^{m_\lambda} e^{-\lambda^2 t} \left[ e^{-(y-y')^2/4t} \left( \frac{y-y'}{2t} + \lambda \right) + e^{-(y+y')^2/4t} \left( \frac{y+y'}{2t} - \lambda \right) \right] \varphi^j_\lambda(n) \otimes \varphi^j_\lambda(n')
\]

\[
+ \sum_{\lambda < 0} \sum_{j=1}^{m_\lambda} e^{-\lambda^2 t} \left[ e^{-(y-y')^2/4t} \left( \frac{y-y'}{2t} + \lambda \right) + e^{-(y+y')^2/4t} \left( \frac{y+y'}{2t} + \lambda \right) \right] \varphi^j_\lambda(n) \otimes \varphi^j_\lambda(n')
\]

\[- \sum_{\lambda < 0} \sum_{j=1}^{m_\lambda} \frac{\lambda}{\sqrt{\pi t}} e^{-\lambda(y+y')} e^{-\left(\frac{y+y'}{2t} - \lambda \sqrt{t}\right)^2} \varphi^j_\lambda(n) \otimes \varphi^j_\lambda(n'),
\]

(5.11)

for all \(n, n' \in N\) and \(y, y' \in (0, \infty)\). Here the outer sums run over the positive and negative eigenvalues \(\lambda\) of \(D_N^+\), respectively.
Proof. As before, let \( \kappa^{P+}_s \) be the Schwartz kernel of \( e^{-sD^+_C D^-_C} \). From the explicit heat kernel expression in (2.16) and (2.17) in [4], we have for all \( n, n' \in N \) and \( y, y' \in (0, \infty) \),

\[
\kappa^{P+}_s(n, y; n', y') = \sum_{\lambda>0} \sum_{j=1}^{m_{\lambda}} e^{-\lambda^2 t} \left[ e^{-\frac{(y-y')^2}{4t}} - e^{-\frac{(y+y')^2}{4t}} \right] \varphi^j_\lambda(n) \otimes \varphi^j_\lambda(n')
+ \sum_{\lambda<0} \sum_{j=1}^{m_{\lambda}} \left\{ e^{-\lambda^2 t} \left[ e^{-\frac{(y-y')^2}{4t}} + e^{-\frac{(y+y')^2}{4t}} \right] + \lambda e^{-\lambda(y+y')} \text{erfc}\left( \frac{y+y'}{2\sqrt{t}} - \lambda \sqrt{t} \right) \right\} \varphi^j_\lambda(n) \otimes \varphi^j_\lambda(n'),
\]

where \( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-s^2} ds \) is the complementary error function. In this case,

\[
D^\pm_C = \pm \frac{d}{dy} + D^+_N.
\]

We can find \( \lambda^P_s \) by applying \((D^-_C)^* = D^+_C\) to the second variable in \( \kappa^{P+}_s \). In other words,

\[
\lambda^P_s = \left( 1_{S^+_C} \otimes \left( \frac{\partial}{\partial y'} + D^+_N \right) \right) \kappa^{P+}_s,
\]

where, as above, we view \( \kappa^{P+}_s \) as a section of \( S^+_C \otimes S^+_C \). This implies (5.11) by a direct computation. \qed

**Proposition 5.11.** In the situation of Corollary 2.6, we have for all \( a' > 0 \),

\[
A_g(D_C, a') = -\frac{1}{2} \eta_g(D^+_N).
\]

Proof. The function \( f_1 \) in (2.11) now equals 1. We use Proposition 5.10 and the equality

\[
\sum_{j=1}^{m_{\lambda}} \int_N \text{tr}(g \varphi^j_\lambda(g^{-1} n) \otimes \varphi^j_\lambda(n)) \, dn = \sum_{j=1}^{m_{\lambda}} (g \cdot \varphi^j_\lambda, \varphi^j_\lambda)_{L^2} = \text{tr}(g|_{\text{ker}(D^+_N - \lambda)})
\]
to compute

\[ A_g(D_C, a') = -\int_0^\infty \int_N \text{tr}(g\lambda_s^P (g^{-1} n, a'; n, a')) dn ds\]

\[ = -\int_0^\infty \sum_{\lambda > 0} \frac{e^{-\lambda^2 s}}{\sqrt{4\pi s}} \text{tr}(g|_{\ker(D_N^+ - \lambda)}) \left[ e^{-\frac{s^2 a'}{s}} + \lambda - \lambda e^{-\frac{s^2 a'}{s}} \right] ds\]

\[ - \int_0^\infty \sum_{\lambda < 0} \frac{e^{-\lambda^2 s}}{\sqrt{4\pi s}} \text{tr}(g|_{\ker(D_N^+ + \lambda)}) \left[ -e^{-\frac{s^2 a'}{s}} + \lambda + \lambda e^{-\frac{s^2 a'}{s}} \right] ds\]

\[ + \int_0^\infty \sum_{\lambda < 0} \text{tr}(g|_{\ker(D_N^+ + \lambda)}) \frac{\lambda}{\sqrt{\pi s}} e^{-2\lambda a'} e^{-\left(\frac{s^2 a'}{s} - \lambda\sqrt{s}\right)^2} ds.\]

The last two lines can be simplified to

\[ -\int_0^\infty \sum_{\lambda < 0} \frac{e^{-\lambda^2 s}}{\sqrt{4\pi s}} \text{tr}(g|_{\ker(D_N^+ - \lambda)}) \left[ -e^{-\frac{s^2 a'}{s}} + \lambda - \lambda e^{-\frac{s^2 a'}{s}} \right] ds.\]

We find that

\[ A_g(D_C, a') = -\int_0^\infty \sum_{\lambda \in \text{spec}(D_N^+)} \text{tr}(g|_{\ker(D_N^+ - \lambda)}) \frac{e^{-\lambda^2 s}}{\sqrt{4\pi s}} \lambda ds\]

\[ - \int_0^\infty \sum_{\lambda \in \text{spec}(D_N^+)} \text{sgn}(\lambda) \text{tr}(g|_{\ker(D_N^+ - \lambda)}) \frac{e^{-\lambda^2 s} e^{-\frac{s^2 a'}{s}}}{\sqrt{4\pi s}} \left( \frac{a'}{s} - |\lambda| \right) ds.\] (5.12)

We apply Proposition 5.1 to the last line, with \(\lambda_j\) the \(j\)th eigenvalue of \(D_N^+\) (ordered by absolute values), and \(a_j = \text{tr}(g|_{\ker(D_N^+ - \lambda_j)})\). Then the condition on \(\lambda_j\) in Proposition 5.1 holds by Weyl’s law, and the condition on \(a_j\) holds because

\[ |a_j| \leq \dim(\ker(D_N^+ - \lambda_j)),\]

which grows at most polynomially by Weyl’s law. Hence Proposition 5.1 implies that the last line of (5.12) equals zero. We conclude that

\[ A_g(D_C, a') = -\frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{s}} \text{Tr}(ge^{-s(D_N^+)^2}D_N^+) ds = -\frac{1}{2} \eta_g(D_N^+).\]
Remark 5.12. It follows from Proposition 5.10 that for all \( n, n' \in \mathbb{N} \),
\[
\lambda_s^F(n, 0; n', 0) = 0.
\]
This is consistent with Remark 2.4. This also shows that it is important in Proposition 5.11 that \( a' > 0 \).

Proof of Corollary 2.6. In the situation of Corollary 2.6, \( D \) satisfies the conditions of Theorem 2.2 and Corollary 2.3. In particular, see Propositions 2.5 and 2.12 in [4] for invertibility of (2.7) and (2.8) and self-adjointness of (2.9) and (2.10). See also Theorem X.25 in [24]. (Alternatively, one can use generalisations of these results from [4], Propositions 3.12 and 3.15 in [18].) Because \( \text{index}_G(D)(g) = \text{index}^{\text{APS}}_G(D_Z)(g) \), Corollary 2.3 and Proposition 5.11 imply the claim.

Remark 5.13. The proof of Proposition 5.11 can be simplified somewhat if one uses Proposition 3.5 to replace APS heat operators by Dirichlet heat operators and spectral projections. We have chosen to give a more direct proof, to illustrate the role of APS boundary conditions.

We also see in this example why it is important that the heat operator in (2.11) is defined with APS-type boundary conditions at \( N \). Indeed, consider the expression \( A_g^F(D_C, a') \) analogous to (2.11), with the heat operator \( e^{-sD_C^{-1}D_C^*} \) replaced by the Dirichlet heat operator \( e^{-sD_C^{-1}D_C^*} \). By a similar computation to the one in this subsection, one finds that, in the current setting, \( A_g^F(D_C, a') \) equals a version of (5.12) where the factor \( a' s - |\lambda| \) is replaced by \( \text{sgn}(\lambda) \frac{a'}{s} - |\lambda| \). Because of this difference, Proposition 5.1 does not apply, and \( A_g^F(D_C, a') \) is generally not equal to the delocalised \( \eta \)-invariant in this case. This implies that a version of Theorem 2.2 with \( A_g(D_C, a') \) replaced by \( A_g^F(D_C, a') \) is not true in general.

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