Green-Schwarz Superstring Action for \((p, q)\)-Strings from a Wrapped Supermembrane on a 2-Torus

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We consider a wrapped supermembrane around non-trivial two cycles of a 2-torus. We exam-ine the double dimensional reduction and the T-dual transformation to deduce Green-Schwarz type IIB superstring action for \((p, q)\)-strings directly from the wrapped supermembrane on the 2-torus. The resulting action has the couplings with both the NSNS- and the RR-background fields and has the tension of the \((p, q)\)-string.

Subject Index: 120, 125, 129

§1. Introduction

The supermembrane in eleven dimensions1) is expected to play an important role to understand the fundamental degrees of freedom in M-theory. In fact, it was shown that the wrapped supermembrane in a \(S^1\)-compactified eleven dimensions is related to the type IIA superstring in ten dimensions by means of the double dimensional reduction.2) Meanwhile type IIB superstring is related to type IIA superstring via T-duality, or the type IIA superstring on \(\mathbb{R}^9 \times S^1\) leads to the type IIB superstring on \(\mathbb{R}^{10}\) in the shrinking limit of the \(S^1\). Accordingly, the supermembrane wrapped on a vanishing 2-torus is reduced to the type IIB superstring in ten dimensions.

The transformation rule for the NSNS fields under T-duality in type II super-string theory is given in the sigma-model with (at least) one isometry direction, which is the well-known Buscher’s rule.3) The generalized Buscher’s rule, which is the transformation rule under T-duality not only for the NSNS fields but also for the RR fields in type II superstring, was derived at the level of the low energy effective action of type II string theory.4,5) In addition, the generalized Buscher’s rule was also derived for the type II Green-Schwarz superstring action6) which is obtained by means of the double dimensional reduction of the wrapped supermembrane up to quadratic order in the anti-commuting superspace coordinates.7)

It is well-known that at low-energy level the type IIB superstring theory has a duality group of \(SL(2, \mathbb{R})\) which is broken down to \(SL(2, \mathbb{Z})\) by the quantum effect. Schwarz showed an \(SL(2, \mathbb{Z})\) family of string solutions in type IIB supergravity,8) which couples to both the NSNS and the RR background fields. The \((p, q)\)-string8,9)
is considered to be the bound state of fundamental strings (F-strings) and D1-branes (D-strings) in type IIB superstring theory. Then, $SL(2, \mathbb{Z})$-covariant string actions were proposed.\textsuperscript{10, 11)} Meanwhile, the supermembrane which is wrapping $p$-times around one of the two compact directions and $q$-times around the other direction gives a $(p, q)$-string, however the direct derivation of the action was not given. Recently the bosonic sector of the type IIB Green-Schwarz superstring action for $(p, q)$-string was derived directly from the wrapped supermembrane action on a 2-torus.\textsuperscript{12)}

In this paper we shall proceed to the analysis with the anti-commuting superspace coordinates being recovered. We use the normal coordinates in the superfield formulation\textsuperscript{13, 14)} of the supermembrane in a supergravity background. We consider a supermembrane wrapped around non-trivial two cycles on a 2-torus. In fact, the action is expanded with respect to the anti-commuting coordinates $\theta$ up to quadratic order.\textsuperscript{7)} We shall take the shrinking volume limit of the 2-torus to approach type IIB superstring theory along the line of Ref. 6). Then, we deduce $(p, q)$-strings in type IIB superstring theory from the wrapped supermembrane in the limit. We shall see that the string carries $p$-times the unit NSNS 2-form charge and $q$-times the unit RR 2-form charge as well, which indicates that the deduced string is, in fact, a $(p, q)$-string in type IIB superstring theory.

The plan of this paper is as follows. In the next section, we set up the supermembrane compactified on $T^2$ up to quadratic order in $\theta$. In §3 we shall carefully rewrite the eleven-dimensional supergravity background fields compactified on a 2-torus and consider the double dimensional reduction along an oblique direction of the 2-torus. In §4, we consider the T-dual of the derived superstring action along the other compact direction of the 2-torus to deduce the action of $(p, q)$-strings. The final section contains some discussion.

\section*{2. Supermembrane in 11-dimensional superspace}

The action of a supermembrane coupled to an eleven-dimensional supergravity background is given by\textsuperscript{1)}

$$S = T \int d\sigma^0 \int_0^{2\pi} d\sigma^1 d\sigma^2 \left[ -\frac{1}{2} \sqrt{-\hat{\gamma}^{ij}} \hat{N}_i^A \hat{N}_j^B \eta_{AB} - \frac{1}{2} \sqrt{-\hat{\gamma}} - \frac{1}{3!} \hat{c}^{ijk} \partial_i Z^M \partial_j Z^N \partial_k Z^P \hat{C}_{PNM} \right],$$

(2.1)

where $T$ is the tension of supermembrane,\textsuperscript{*}) $\hat{C}_{MNP}(Z)$ is the super three-form,

$$\hat{N}^A_i = (\partial_i Z^M) \hat{E}^A_M,$$

(2.2)

$\hat{\gamma}_{ij}$ ($i, j = 0, 1, 2$) is the worldvolume metric, $\hat{\gamma} = \det \hat{\gamma}_{ij}$, the target space is a supermanifold with the superspace coordinates $Z^M = (X^M, \theta^\alpha)$ ($M = 0, \cdots, 10$, $\alpha = 1, 2, 3$).

\textsuperscript{*}) The eleven-dimensional Planck length $l_{11}$ is defined by $T = \left( \frac{2\pi}{l_{11}} \right)^{-2} l_{11}^{-1}$. 

\[ \text{Downloaded from https://academic.oup.com/ptp/article-abstract/121/3/445/1867138 by guest on 27 July 2018} \]
1, \cdots, 32). Furthermore, with the tangent superspace index $\hat{A} = (A, a)$, $\hat{E}_M \hat{A}$ is the supervielbein and $\eta_{AB}$ is the tangent space metric in eleven dimensions. The mass dimensions of the worldvolume parameters $\sigma^i$ and the eleven-dimensional background fields $(G_{MN}, \hat{C}_{MNP})$ are 0, while that of the worldvolume metric $\hat{\gamma}_{ij}$ is $-2$. Note that the variation w.r.t. $\hat{\gamma}_{ij}$ yields the induced metric

$$\hat{\gamma}_{ij} = \hat{\Pi}_i^A \hat{\Pi}_j^B \eta_{AB}, \quad (2.3)$$

and plugging it back into the original action leads to the Nambu-Goto form

$$S = T \int d\sigma^0 \int_0^{2\pi} da^1 d\sigma^2 \left[ -\sqrt{-\det(\hat{\Pi}_i^A \hat{\Pi}_j^B \eta_{AB})} - \frac{1}{3!} \epsilon^{ijk} \partial_i Z^M \partial_j Z^N \partial_k Z^P \hat{C}_{P^NMQ} \right]. \quad (2.4)$$

In fact, it is convenient to work in the Nambu-Goto action when we carry out the double dimensional reduction in the next section. Note that the action (2.4) has a fermionic gauge symmetry, or the $\kappa$-symmetry

$$\delta_\kappa Z^M \hat{E}_M^A = 0, \quad \delta_\kappa Z^M \hat{E}_M^a = (1 + \hat{\Gamma}_K)^a_b \kappa^b, \quad (2.5)$$

where

$$\hat{\Gamma}_K = \frac{1}{6} \frac{1}{\sqrt{-\det \hat{\gamma}_{ij}}} \epsilon^{ijk} \hat{\Pi}_i^A \hat{\Pi}_j^B \hat{\Pi}_k^C \hat{\gamma}_{ABC}, \quad (2.6)$$

$\hat{\gamma}_{ij}$ is the induced metric (2.3) and the parameter $\kappa$ is a 32-component spacetime Majorana spinor and a worldvolume scalar. In fact, the $\kappa$-symmetry also provides that the background supergeometry is constrained,\(^1\) which is equivalent to the on-shell $D=11$ supergravity background.\(^{15}\)

The supervielbein and the super three-form are explicitly given to $O(\theta^2)$ in the fermionic coordinates.\(^7\) Setting the fermionic background fields to zero, we have\(^a\)

$$\hat{E}_M^A = \hat{e}_M^A + i \bar{\theta} \Gamma^A \hat{\Omega}_M \theta, \quad \hat{E}_M^a = (\hat{\Omega}_M \theta)^a, \quad \hat{E}_A^a = -i (\bar{\theta} \Gamma^A)_{\alpha}, \quad \hat{E}_a^a = \delta_\alpha^a + M_\alpha^a, \quad (2.7)$$

and\(^b\)

$$\hat{C}_{MNP} = \hat{A}_{MNP} + 3i \bar{\theta} \hat{\Gamma}_{[MP} \hat{\Omega}_{N]} \theta, \quad \hat{C}_{M\alpha} = -i (\bar{\theta} \hat{\Gamma}_M)_{\alpha}, \quad (2.8)$$

where

$$\hat{\Omega}_M = \frac{1}{4} \hat{\omega}_M^{BC} \Gamma_{BC} - \hat{T}_M, \quad (2.9)$$

$$\hat{T}_M = \frac{1}{288} (\hat{\Gamma}_M^{NPQR} - 8 \delta_{M}^{[N} \hat{\Gamma}^{PQR]} \hat{C}_{NPQR}) \hat{F}_{NPQR}, \quad (2.10)$$

\(^a\) $M_\alpha^\alpha$ is of $O(\theta^2)$, however, the explicit form is not used in our analysis.

\(^b\) Symmetrization $[\cdots \cdots]$ and anti-symmetrization $(\cdots \cdots)$ of the indices are made with unit weight, $A_{[M B_N]} = (1/2)(A_M B_N - B_N A_M)$, etc. (see Appendix A).
\( \hat{e}^A_M \) is the eleven-dimensional bosonic vielbein (also called the \( e \)-bein) and hence \( G_{MN} = \hat{e}^A_M \hat{e}^B_N \eta_{AB} \). \( \hat{A}_{MNP} \) is the bosonic three-form and its field strength \( \hat{F}_{MNP} = 4 \hat{\theta}_{[M} \hat{A}_{NPQ]} \), \( \Gamma_A \) is the gamma matrix in eleven dimensions,\(^*) \( \hat{\theta} = \imath \theta C = \imath \theta \Gamma^0 \), \( \Gamma_A \Gamma_2 \cdots \Gamma_n \equiv \Gamma_{[A_1} \Gamma_{A_2} \cdots \Gamma_{A_n]} \), \( \hat{\Gamma}_M \equiv \hat{e}^A_M \hat{\Gamma}_A \), and \( \hat{\omega}^{AB}_M \) is the torsion free spin connection

\[
\hat{\omega}^{AB}_M = \hat{e}^{[A|N|} (\partial_M \hat{e}^B_N - \partial_N \hat{e}^B_M) - \frac{1}{2} \hat{e}^{[A|N|} e^B_P (\partial_P \hat{e}_C^N - \partial_N \hat{e}_C^P) \hat{e}_M^C . \tag{2.11}
\]

It is helpful to define the objects with the tangent space indices as follows:

\[
\hat{\Omega}_A = \hat{\theta}^A_M \hat{\Omega}_M , \quad \hat{\omega}^{BC}_A = \hat{\omega}^A_M \hat{\mu}^{BC}_M , \quad \hat{F}^{ABCD} = \hat{\theta}^A_M \hat{e}^N_B \hat{e}^P_C \hat{e}^Q_D \hat{F}_{MNPQ} , \tag{2.12}
\]

where \( \hat{\theta}^A_M \) is the inverse of \( \hat{e}^A_M \). In fact, it is easy to work with the tangent space indices rather than with the target space indices when we calculate the component field expansion to quadratic order in \( \theta \). Consequently, the supermembrane action in \( \theta^2 \)-order is given by**

\[
S = T \int d^3 \sigma \left[ -\sqrt{-\det G_{ij}} - \frac{1}{6} \epsilon^{ijk} \hat{C}_{kji} \right] = T \int d^3 \sigma \left[ -\sqrt{-\det G_{ij}} \hat{G}_{ij} - i \sqrt{-\det G_{ij}} \hat{\theta} (1 - \hat{\Gamma}_M) \hat{\Gamma}^i \hat{D}_i \hat{\theta} \right] , \tag{2.13}
\]

where

\[
G_{ij} = G_{ij} + 2i \hat{\theta} \hat{\Gamma}_{(i} \hat{D}_{j)} \hat{\theta} , \quad \hat{C}_{kji} = \hat{A}_{kij} + 3i \hat{\theta} \hat{\Gamma}_{ij} \hat{D}_k \hat{\theta} , \tag{2.14}
\]

\[
G_{ij} = \partial_i X^M \partial_j X^N G_{MN} , \quad \hat{A}_{kij} = \partial_i X^M \partial_j X^N \partial_k X^P \hat{A}_{MNP} , \tag{2.15}
\]

\[
\hat{\Gamma}_i = \partial_i X^M \hat{\Gamma}_M , \quad \hat{\Gamma}^i = \hat{G}^{ij} \hat{\Gamma}_j , \tag{2.16}
\]

\[
\hat{\Omega}_i = \partial_i X^M \hat{\Omega}_M , \quad \hat{D}_i = \partial_i + \hat{\Omega}_i , \tag{2.17}
\]

\[
\hat{\Gamma}_M = \frac{1}{6} \sqrt{-\det G_{ij}} \epsilon^{ijk} \hat{\Gamma}_{ijk} . \tag{2.18}
\]

§3. Double dimensional reduction

We consider a wrapped supermembrane action compactified on a 2-torus. We shall take the shrinking limit of the 2-torus, or make the double dimensional reduction,\(^2\) and perform the T-dual transformation to deduce the \( (p,q) \)-string action directly from the supermembrane action (2.1) or (2.4). We take the 10th and 9th directions to compactify on \( T^2 \), whose radii are \( L_1 \) and \( L_2 \), respectively. In taking the shrinking volume limit of the 2-torus, we keep the ratio of the radii finite, or fix the moduli of \( T^2 \),

\[
g_b \equiv \frac{L_1}{L_2} : \text{finite.} \quad (L_1, L_2 \to 0) \tag{3.1}
\]

\(^*)\) The gamma matrices satisfy \( \{ \Gamma_A, \Gamma_B \} = 2 \eta_{AB} \) and the Dirac conjugate for a general spinor \( \psi \) is \( \hat{\psi} = i \psi \Gamma^0 \).

\(^**)\) Note that the fermionic coordinate \( y \) in Ref. 16) corresponds to \(-i\sqrt{2} \theta\).
Considering the line element on the 2-torus
\[ ds_{T^2}^2 = G_{uv} dX^u dX^v = \left(G_{99} - \frac{(G_{910})^2}{G_{1010}}\right)(dX^9)^2 + G_{1010} \left(dX^{10} + \frac{G_{910}}{G_{1010}} dX^9\right)^2, \] (3.2)
where \( u, v = 9, 10 \), we shall impose that the target space coordinates satisfy the following boundary conditions\(^{12)\)}

\[ \sqrt{\hat{G}_{1010}} X^{10}(\sigma^1, \sigma^2 + 2\pi) = 2\pi w_1 L_1 p + \sqrt{\hat{G}_{1010}} X^{10}(\sigma^1, \sigma^2), \]
\[ \sqrt{\hat{G}_{99}} - \frac{(\hat{G}_{910})^2}{G_{1010}} X^9(\sigma^1, \sigma^2 + 2\pi) = 2\pi w_1 L_2 q + \sqrt{\hat{G}_{99}} - \frac{(\hat{G}_{910})^2}{G_{1010}} X^9(\sigma^1, \sigma^2), \]
\[ \sqrt{\hat{G}_{1010}} X^{10}(\sigma^1 + 2\pi, \sigma^2) = 2\pi w_2 L_1 r + \sqrt{\hat{G}_{1010}} X^{10}(\sigma^1, \sigma^2), \]
\[ \sqrt{\hat{G}_{99}} - \frac{(\hat{G}_{910})^2}{G_{1010}} X^9(\sigma^1 + 2\pi, \sigma^2) = 2\pi w_2 L_2 s + \sqrt{\hat{G}_{99}} - \frac{(\hat{G}_{910})^2}{G_{1010}} X^9(\sigma^1, \sigma^2), \] (3.3)

where\(^*)\)
\[ pr + qs = 0, \quad ps - qr = n_c > 0, \quad (p, q, r, s \in \mathbb{Z}, \ w_1 \in \mathbb{N} \setminus \{0\}, \ w_2 \in \mathbb{Z} \setminus \{0\}) \] (3.4)
and \( \hat{G}_{1010} \), \( \hat{G}_{99} \) and \( \hat{G}_{910} \) stand for the asymptotic constant values of the metric.\(^{**)}\)

Equation (3.3) can be written by
\[ X^{10}(\sigma^1, \sigma^2) = R_1 (w_1 p \sigma^2 + w_2 r \sigma^1) + Y^1(\sigma^1, \sigma^2), \]
\[ X^9(\sigma^1, \sigma^2) = R_2 (w_1 q \sigma^2 + w_2 s \sigma^1) + Y^2(\sigma^1, \sigma^2), \] (3.5)
with
\[ Y^\xi(\sigma^1 + 2\pi, \sigma^2) = Y^\xi(\sigma^1, \sigma^2 + 2\pi) = Y^\xi(\sigma^1, \sigma^2), \quad (\xi = 1, 2) \] (3.6)
and\(^{***)}\)
\[ R_1 \equiv \frac{L_1}{\sqrt{\hat{G}_{1010}}}, \quad R_2 \equiv \frac{L_2}{\sqrt{\hat{G}_{99}} - \frac{(\hat{G}_{910})^2}{G_{1010}}} \] (3.7)

The other fields satisfy the periodic boundary conditions, \( X^0(\sigma^1 + 2\pi, \sigma^2) = X^0(\sigma^1, \sigma^2 + 2\pi) = X^0(\sigma^1, \sigma^2), \) etc. The above expressions represent that the supermembrane is wrapping \( w_1p \)-times around one of the two compact directions (the \( X^{10} \)-direction) and \( w_1q \)-times around the other direction (the \( X^9 \)-direction), or \( w_1 \)-times around \((p, q)\)-cycle along the \( \sigma^2 \)-direction on the worldsheet. And it is also wrapping \( w_2 \)-times around \((r, s)\)-cycle along the \( \sigma^1 \)-direction. These two cycles are

\(^*\) We may assume \( n_c > 0 \) and \( w_1 > 0 \) without loss of generality since we can flip the signs of \((p, q) \to (-p, -q) \) for \( w_1 \) and \((r, s) \to (-r, -s) \) for \( n_c \) if necessary. Furthermore, we can see that Eq. (3.4) leads to \((r, s) = n(-q, p) \) \((n \in \mathbb{N})\).

\(^{**}\) Precisely speaking, \( \hat{G}_{uv} \) \((u, v = 9, 10) \) should satisfy \( \partial_i \hat{G}_{uv} = 0 \) and (3.10) as well.

\(^{***}\) We shall see \( R_1 = L_1 e^{-2\phi_0/3} \) from Eq. (C.1) where \( \phi_0 \) is the asymptotic value of the type IIA dilaton background and hence M/IIA-relation, or 11d/IIA-SUGRA-relation, leads to \( R_1 = \ell_{11}^{-1} \) (the eleven-dimensional Planck length).
orthogonal to each other and intersect at least once. Thus, this wrapped supermembrane is expected to give the \((p, q)\)-string\(^8,^{12}\). In fact, we shall see below that the \((p, q)\)-string comes out through the double dimensional reduction.

Now that we shall adopt the double dimensional reduction technique\(^2\) to deduce \((p, q)\)-strings. First we determine the spacetime direction to be aligned with one of the worldvolume coordinate, or we fix the gauge. We define \(X^y\) and \(X^z\) by an \(SO(2)\) rotation of the target space,

\[
\begin{pmatrix}
X^z \\
X^y
\end{pmatrix} = O_{(p,q)} \begin{pmatrix}
X^{10} \\
X^9
\end{pmatrix},
\]

where

\[
O_{(p,q)} = \frac{1}{c_{pq}} \begin{pmatrix}
p & q \\
-q & p
\end{pmatrix} \equiv \begin{pmatrix}
\hat{p} & \hat{q} \\
-\hat{q} & \hat{p}
\end{pmatrix} \in SO(2), \quad c_{pq} \equiv \sqrt{p^2 + q^2}.
\]

By using the relations between the eleven-dimensional supergravity fields and the \(S^1\)-compactified type IIB ones,\(^4,^5\) we have

\[
\sqrt{G_{99} - \frac{(G_{1010})^2}{G_{1010}}} = e^{-\varphi} \rightarrow \sqrt{\hat{G}_{99} - \frac{(\hat{G}_{1010})^2}{\hat{G}_{1010}}} = e^{-\varphi_0} = g_{\hat{b}}^{-1} = \frac{L_2}{L_1},
\]

where \(\varphi\) is the type IIB dilaton background and \(\varphi_0\) is its asymptotic constant value. Thus, Eq. (3.10) leads to

\[
R_1 = R_2 \equiv R_B.
\]

Then we have

\[
\begin{align*}
X^z &= w_1 c_{pq} R_B \sigma^2 + \hat{p} Y^1(\sigma^i) + \hat{q} Y^2(\sigma^i), \\
X^y &= \frac{w_2 p_c R_B}{c_{pq}} \sigma^1 - \hat{q} Y^1(\sigma^i) + \hat{p} Y^2(\sigma^i).
\end{align*}
\]

The target space metric and the background 3-form field are transformed under the \(SO(2)\) rotation in Eq. (3.8) as \((\hat{M}, \hat{N}, \hat{P}, \hat{Q}) = 0, 1, 2, \ldots, 8, y, z\)

\[
\hat{G}_{\hat{M}\hat{N}} = G_{MN} \frac{\partial X^M}{\partial X^\hat{M}} \frac{\partial X^N}{\partial X^\hat{N}}, \quad \hat{A}_{\hat{M}\hat{N}\hat{P}} = A_{MN\hat{P}} \frac{\partial X^M}{\partial X^\hat{M}} \frac{\partial X^N}{\partial X^\hat{N}} \frac{\partial X^\hat{P}}{\partial X^P}.
\]

In addition, we shall define the objects in the rotated coordinate system as follows:

\[
\hat{\Gamma}^\hat{M} = \tilde{e}_A^\hat{M} \Gamma_A, \quad \hat{\Omega}^\hat{M} = \tilde{e}_A^\hat{M} \Omega_A, \\
\tilde{\omega}_{\hat{M}\hat{N}\hat{P}} = \tilde{e}_A^\hat{M} \omega_A^\hat{N}\hat{P}, \quad \hat{F}_{\hat{M}\hat{N}\hat{P}\hat{Q}} = \tilde{e}_A^\hat{M} \tilde{e}_B^\hat{N} \tilde{e}_C^\hat{P} \tilde{e}_D^\hat{Q} \hat{F}_{ABC\hat{D}}.
\]

Let us choose the Kaluza-Klein condition for the target space metric and the bosonic vielbein in the rotated coordinate system. A suitable choice is \((\hat{\mu}, \hat{\nu}) = 0, 1, \cdots, 8, y\) and \(\mu, \nu = 0, 1, \cdots, 8\)

\[
\hat{G}_{\hat{M}\hat{N}} \equiv e^{-\Phi} \begin{pmatrix}
g_{\hat{\mu}\hat{\nu}} + e^{2\Phi} \hat{A}_{\hat{\mu}} A_{\hat{\nu}} & e^{2\Phi} A_{\hat{\mu}} \\
e^{2\Phi} \hat{A}_{\hat{\nu}} & e^{2\Phi}
\end{pmatrix}
\]

\[
\tilde{e}_A^\hat{M} \rightarrow \tilde{e}_A^\hat{M} \quad \hat{e}_A^\hat{M} \chi_A \,,
\]
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\[
\begin{pmatrix}
\frac{1}{\sqrt{G_{zz}}} \tilde{g}_{\mu\nu} + \frac{1}{G_{zz}} \tilde{G}_{\mu z} \tilde{G}_{\nu z} & \frac{1}{\sqrt{G_{zz}}} \tilde{g}_{\mu y} + \frac{1}{G_{zz}} \tilde{G}_{\mu z} \tilde{G}_{y z} \\
\frac{1}{\sqrt{G_{zz}}} \tilde{g}_{y y} + \frac{1}{G_{zz}} \tilde{G}_{y y} \tilde{G}_{y z} & \tilde{G}_{y z}
\end{pmatrix},
\]

(3.15)

and

\[
\tilde{e}_M^A = e^{-\frac{\phi}{6}} \begin{pmatrix} \tilde{e}_M^\mu & e^{\phi} \tilde{A}_\mu \\
0 & e^{\frac{\phi}{6}} \end{pmatrix}, \quad \tilde{e}_A^M = e^{\frac{\phi}{3}} \begin{pmatrix} \tilde{e}_A^\mu & -e^{\phi} \tilde{A}_\mu \\
0 & e^{-\frac{\phi}{6}} \end{pmatrix},
\]

(3.16)

where \(\tilde{e}_M^\mu\) and \(\tilde{g}_{\mu\nu}\) are the vielbein (zehnbein) and the target space metric in ten dimensions, respectively, \(\tilde{A}_\mu\) is a Kaluza-Klein vector field and \(\tilde{\phi}\) is a scalar field which is reduced to the type IIA dilaton in the case of \((p, q) = (1, 0)\) in Eq. (3.9).

Equation (3.16) implies that the spinor \(\theta\) should be rescaled by \(e^{-\tilde{\phi}/6}\) in ten dimensions, which is understood as follows. The gamma matrices in eleven dimensions are split into the ten-dimensional gamma matrices and the rest, \(\{\Gamma_A\} = \{\Gamma, \Gamma_{10}\}\). Let us consider the supersymmetry transformation, \(\delta X^M = i \epsilon \Gamma^M \theta = i e_A^M \epsilon \Gamma^A \theta, \quad \delta \theta = \epsilon\). Then, we have \(\delta X^{10} = i e_A^{10} \epsilon \Gamma^{10} \theta = i e^{\tilde{\phi}/3} e_A^{10} \epsilon \Gamma^{10} \theta, \quad \delta \theta = \epsilon\). Once we impose that the form of the transformation in eleven dimensions is preserved in ten dimensions, we should rescale the Majorana spinors \(\theta \rightarrow e^{-\tilde{\phi}/6} \theta\) and \(\epsilon \rightarrow e^{-\tilde{\phi}/6} \epsilon\). In addition, we also define the following objects in ten dimensions

\[
\tilde{\Omega}_\mu \equiv \tilde{e}_M^\mu \Omega_M, \quad \Omega_\mu \equiv e^{-\tilde{\phi}/3} \tilde{\Omega}_\mu, \quad \Omega_{10} \equiv e^{-\tilde{\phi}/3} \tilde{\Omega}_{10},
\]

(3.17)

and these are related to \(\{\tilde{\Omega}_M\} = \{\tilde{\Omega}_\mu, \tilde{\Omega}_{10}\}\) in Eq. (3.14) as

\[
\tilde{\Omega}_\mu = \tilde{e}_M^A \tilde{\Omega}_A = \tilde{\Omega}_\mu + e^{\tilde{\phi}} \tilde{A}_\mu \Omega_{10}, \quad \tilde{\Omega}_{10} = \tilde{e}_A^A \tilde{\Omega}_A = e^{\tilde{\phi}} \Omega_{10}.
\]

(3.18)

The supervielbein analog of the Kaluza-Klein condition (3.16) is given by \((\tilde{m} = (\tilde{\mu}, \alpha))\)

\[
\tilde{E}_M^\tilde{A} = \begin{pmatrix} \tilde{E}_m^\tilde{\mu} & \tilde{E}_m^a \psi^a \\
\tilde{E}_z^\tilde{\mu} & \tilde{E}_z^a \psi^a \end{pmatrix} = \tilde{\phi}^{-\frac{1}{3}} \begin{pmatrix} \tilde{E}_m^\tilde{\mu} & \tilde{E}_m^a + \tilde{C}_m^a \psi^a \\
0 & \tilde{\psi}^a \end{pmatrix},
\]

(3.19)

which implies

\[
\tilde{e}_z^\tilde{\mu} = 0, \quad \tilde{\theta} \Gamma^{\tilde{\mu}} \tilde{\Omega}_{\tilde{\mu}} \theta = 0. \quad (\tilde{\theta} \Gamma^{\tilde{\mu}} \Omega_{10} \theta = 0)
\]

(3.20)

Now we shall make a (partial) gauge choice of (cf. Ref. 2)

\[
X^z = \frac{L_1 w_1 c_{pq}}{\sqrt{C_{1010}}} \sigma^2 \equiv C_{pq} \sigma^2,
\]

(3.21)

or the \(X^z\)-direction is aligned with one of the space direction \(\sigma^2\) of the worldvolume. Then the dimensional reduction is achieved by imposing the following conditions on the target superspace coordinates and the background fields,

\[
\frac{\partial}{\partial \sigma^2} Z^\tilde{m} = 0,
\]

(3.22)

\[
\frac{\partial}{\partial X^z} \tilde{G}_{\tilde{M} \tilde{N}} = \frac{\partial}{\partial X^z} \tilde{A}_{\tilde{M} \tilde{N} \tilde{P}} = 0.
\]

(3.23)
Thus the induced metric on the worldvolume is given by\(^2\) \((i, j = 0, 1)\)

\[
\hat{\gamma}_{ij} = \hat{\Pi}_i^A \hat{\Pi}_j^B \eta_{AB} = \Phi^2 (\gamma_{ij} + \Phi^2 C_i C_j - \Phi^2 C_i - \Phi^2 C_j), \tag{3.24}
\]

where

\[
\Phi^4 = C_{pq} \phi^4, \quad \Phi^4 C_i = C_{pq} \phi^4 \partial_i X^m \bar{C}_m, \quad \gamma_{ij} = C_{pq} \hat{\Pi}_i^q \hat{\Pi}_j^p \eta_{pq}, \quad (\hat{\Pi}_i^p \equiv \partial_i Z^m \bar{E}_m^p)
\tag{3.25}
\]

and (up to quadratic order in \(\theta\) with the fermionic background fields being zero)\(^*\)

\[
\hat{\phi}_\alpha^\dagger = (\hat{E}_z^1)^{\dagger} = e^{\hat{\phi}} \left( 1 + \frac{i}{2} \bar{\theta} \Gamma_\alpha \Omega_{10} \theta \right),
\]

\[
\hat{e}_\mu^\hat{r} = \Phi^2 \hat{e}_\mu^\hat{r} = \left( 1 + \frac{i}{2} \bar{\theta} \Gamma_\mu \Omega_{10} \theta \right) \hat{e}_\mu^\hat{r} + i \bar{\theta} \Gamma^\hat{r} \Omega_\mu \theta,
\]

\[
\hat{E}_\alpha^\hat{r} = \Phi^2 \hat{E}_\alpha^\hat{r} = -e^{\hat{\phi}} i (\bar{\theta} \Gamma^\hat{r})_\alpha,
\]

\[
\hat{\Pi}_i^\hat{r} = \partial_i Z^m \hat{E}_m^\hat{r} = \partial_i X^\mu \left\{ \hat{e}_\mu^\hat{r} \left( 1 + \frac{i}{2} \bar{\theta} \Gamma_{10} \Omega_\theta \right) + i \bar{\theta} \Gamma^\hat{r} \Omega_\mu \theta \right\} + i \bar{\theta} \Gamma^\hat{r} \partial_i \theta. \tag{3.26}
\]

We have

\[
\sqrt{-\det \hat{\gamma}_{ij}} = \sqrt{-\det \gamma_{ij}}. \tag{3.27}
\]

Thus, by the double dimensional reduction of Eqs. (3.21)–(3.23), the supermembrane action (2.4) is reduced to

\[
S_{ddr} = 2\pi T \int d\sigma^0 \int_0^{2\pi} d\sigma^1 C_{pq} \left[ -\sqrt{-\det \hat{\zeta}_{ij} - \frac{1}{2} \epsilon_{ij} \hat{B}_{ji} \right], \tag{3.28}
\]

where

\[
\hat{\zeta}_{ij} = \hat{g}_{ij} + \hat{Q}_{ij} + 2i \bar{\theta} \Gamma_\dot{i} (\partial_\dot{j}) \theta, \tag{3.29}
\]

\[
\hat{B}_{ij} = \hat{A}_{ij} - \hat{P}_{ij} - 2i \bar{\theta} \Gamma_\dot{i} \Gamma_{10} \partial_\dot{j} \theta, \tag{3.30}
\]

the indices \(i, j\) of \(\hat{g}_{ij}, \hat{Q}_{ij}\), etc. mean

\[
\hat{g}_{ij} \equiv \partial_i X^\mu \partial_j X^\nu \hat{g}_{\mu\nu}, \quad \hat{Q}_{ij} \equiv \partial_i X^\mu \partial_j X^\nu \hat{Q}_{\mu\nu}, \quad \hat{\Gamma}_i \equiv \partial_i X^\mu \hat{e}_\mu^\hat{r} \Gamma^\hat{r}, \quad \text{etc.}, \tag{3.31}
\]

and

\[
\hat{A}_{\rho\sigma} = \hat{\Lambda}_{\dot{\rho} \dot{\sigma} \nu z}, \quad \hat{Q}_{\rho\sigma} = \hat{e}_{\rho}^\hat{r} \hat{e}_{\sigma}^\hat{s} \hat{Q}_{\hat{r}\hat{s}}, \quad \hat{P}_{\rho\sigma} = \hat{e}_{\rho}^\hat{r} \hat{e}_{\sigma}^\hat{s} \hat{P}_{\hat{r}\hat{s}}, \quad \hat{Q}_{\hat{r}\hat{s}} = i \bar{\eta}_{\hat{r}\hat{s}} \bar{\theta} \Gamma_{10} \Omega_{10} \theta + 2i \bar{\theta} \Gamma_\dot{r} (\partial_\dot{s}) \theta, \quad \hat{P}_{\hat{r}\hat{s}} = -i \bar{\theta} \Gamma_{\hat{r}\hat{s}} \Omega_{10} \theta - 2i \bar{\theta} \Gamma_{10} \Gamma_{\hat{r}} \Omega_{\dot{3}} \theta. \tag{3.32}
\]

\(^*\) Note that we have taken the rescaling of \(\theta, \theta \rightarrow e^{-\hat{\phi}^2/6} \theta.\)
This reduced action (3.28) naturally inherits \( \kappa \)-symmetry. In fact, Eq. (2.5) leads to the transformation law which leaves the action (3.28) of order up to quadratic in \( \theta \) invariant

\[
\delta_{\kappa} \theta = (1 + \Gamma_{F}) \kappa, \quad \delta_{\kappa} X^{\hat{\mu}} = -i \bar{\theta} \Gamma^{\hat{\mu}} (1 + \Gamma_{F}) \kappa, \quad \delta_{\kappa} \Phi_{bg} = \delta_{\kappa} X^{\hat{\mu}} \partial_{\hat{\mu}} \Phi_{bg},
\]

(3.33)

where \( \Phi_{bg} \) stands for a general field of supergravity background and

\[
\Gamma_{F} = \frac{1}{2} \frac{1}{\sqrt{-\det g_{ij}}} \epsilon^{ij} \Gamma_{ij} \Gamma^{10}.
\]

(3.34)

Similarly, the SUSY transformation, which leaves the action (3.28) invariant, is given by

\[
\delta_{\epsilon} \theta = \epsilon, \quad \delta_{\epsilon} X^{\hat{\mu}} = i \bar{\epsilon} \Gamma^{\hat{\mu}} \theta, \quad \delta_{\epsilon} \Phi_{bg} = \delta_{\epsilon} X^{\hat{\mu}} \partial_{\hat{\mu}} \Phi_{bg}.
\]

(3.35)

By introducing the worldsheet metric \( \tilde{\gamma}_{ij} \), Eq. (3.28) can be rewritten in the Polyakov form as usual

\[
S_{ddr} = \frac{2\pi T}{2} \int d\sigma^{0} \int^{2\pi} d\sigma^{1} C_{pq} \left[ -\sqrt{-\tilde{\gamma}} \tilde{\gamma}^{ij} \left( \partial_{i} X^{\hat{\mu}} \partial_{j} X^{\hat{\nu}} \tilde{E}_{\hat{\mu} \hat{\nu}} + 2 \partial_{i} X^{\hat{\mu}} \tilde{G}_{\hat{j} \hat{\mu}} \right) + \epsilon^{ij} \left( \partial_{i} X^{\hat{\mu}} \partial_{j} X^{\hat{\nu}} \tilde{B}_{\hat{\mu} \hat{\nu}} - 2 \partial_{i} X^{\hat{\mu}} \tilde{B}_{\hat{j} \hat{\mu}} \right) \right],
\]

(3.36)

where

\[
\tilde{E}_{\hat{\mu} \hat{\nu}} = \tilde{g}_{\hat{\mu} \hat{\nu}} + \tilde{Q}_{\hat{\mu} \hat{\nu}}, \quad \tilde{B}_{\hat{\mu} \hat{\nu}} = \tilde{A}_{\hat{\mu} \hat{\nu}} - \tilde{P}_{\hat{\mu} \hat{\nu}},
\]

(3.37)

are the super metric and the super 2-form, respectively, and

\[
\tilde{G}_{\hat{j} \hat{\mu}} = \tilde{e}_{\hat{\mu}} \tilde{G}_{\hat{j} \hat{\nu}}, \quad \tilde{B}_{\hat{j} \hat{\mu}} = \tilde{e}_{\hat{\mu}} \tilde{B}_{\hat{j} \hat{\nu}},
\]

\[
\tilde{G}_{\hat{j} \hat{\nu}} = i \bar{\theta} \Gamma_{\hat{\nu}} \partial_{\hat{j}} \theta, \quad \tilde{B}_{\hat{j} \hat{\nu}} = i \bar{\theta} \Gamma_{\hat{\nu}} \Gamma_{10} \partial_{\hat{j}} \theta.
\]

(3.38)

As was pointed out in 2), this action (3.36) has conformal invariance. When we consider the T-dual transformation between type IIA/IIB superstring theories, it is, in fact, convenient to work with the action of the Polyakov type. Equation (3.36) can be written by

\[
S_{ddr} = \frac{2\pi T}{2} \int d\sigma^{0} \int^{2\pi} d\sigma^{1} C_{pq} \left[ -\sqrt{-\tilde{\gamma}} \tilde{\gamma}^{ij} \left( \partial_{i} X^{\hat{\mu}} \partial_{j} X^{\hat{\nu}} \tilde{G}_{\hat{\mu} \hat{\nu}} + \epsilon^{ij} \partial_{i} X^{\hat{\mu}} \partial_{j} X^{\hat{\nu}} \tilde{B}_{\hat{\mu} \hat{\nu}} \right) \right],
\]

(3.39)

where

\[
\tilde{G}_{\hat{\mu} \hat{\nu}} = \tilde{g}_{\hat{\mu} \hat{\nu}} + \tilde{Q}_{\hat{\mu} \hat{\nu}} + 2i \bar{\theta} \Gamma_{\hat{\nu}} \partial_{\hat{\nu}} \theta, \quad \tilde{B}_{\hat{\mu} \hat{\nu}} = \tilde{A}_{\hat{\mu} \hat{\nu}} - \tilde{P}_{\hat{\mu} \hat{\nu}} - 2i \bar{\theta} \Gamma_{\hat{\mu}} \Gamma_{10} \partial_{\hat{\nu}} \theta.
\]

(3.40)

We shall give \( Q_{\hat{x} \hat{s}} \) and \( P_{\hat{s} \hat{s}} \) in Eq. (3.32) more explicitly with the background fields. As we noted, it is easy to work out with the tangent space indices rather
than with the target space indices. Hence we shall calculate the decomposition of the 4-form field strength $\tilde{F}_{ABCD}$ and the spin connection $\tilde{\omega}_A^{BC}$ in Eq. (3-14) under the Kaluza-Klein condition (3-16). The decomposition of the 4-form field strength

$$\{\tilde{F}_{ABCD}\} = \{\tilde{F}_{\tilde{r}\tilde{s}\tilde{t}\tilde{u}}, \tilde{F}_{\tilde{r}\tilde{s}\tilde{t}\tilde{10}}\},$$

(3-41)
is

$$\tilde{F}_{\tilde{r}\tilde{s}\tilde{t}\tilde{u}} = e^{\frac{1}{4} \phi} \tilde{e}^{\tilde{r}} \tilde{e}^{\tilde{s}} \tilde{e}^{\tilde{t}} \tilde{e}^{\tilde{u}} \tilde{F}_{\tilde{r}\tilde{s}\tilde{t}\tilde{u}} \equiv e^{\frac{1}{4} \phi} \tilde{F}_{\tilde{r}\tilde{s}\tilde{t}\tilde{u}}, \quad \tilde{F}_{\tilde{r}\tilde{s}\tilde{t}\tilde{10}} = e^{\frac{1}{4} \phi} \tilde{e}^{\tilde{r}} \tilde{e}^{\tilde{s}} \tilde{e}^{\tilde{t}} \tilde{e}^{\tilde{u}} \tilde{H}_{\tilde{r}\tilde{s}\tilde{t}\tilde{10}} \equiv e^{\frac{1}{4} \phi} \tilde{H}_{\tilde{r}\tilde{s}\tilde{t}\tilde{10}},$$

(3-42)

where

$$\tilde{F}_{\tilde{r}\tilde{s}\tilde{t}\tilde{u}} \equiv \tilde{F}_{\tilde{r}\tilde{s}\tilde{t}\tilde{u}} + 4 \tilde{A}_{\tilde{r}\tilde{s}\tilde{t}\tilde{u}} \tilde{H}_{\tilde{r}\tilde{s}\tilde{t}\tilde{10}}, \quad \tilde{H}_{\tilde{r}\tilde{s}\tilde{t}\tilde{10}} \equiv \tilde{F}_{\tilde{r}\tilde{s}\tilde{t}\tilde{10}}.$$

(3-43)

Similarly, the spin connection

$$\{\tilde{\omega}_A^{BC}\} = \{\tilde{\omega}_{\tilde{r}}^{\tilde{s}\tilde{t}}, \tilde{\omega}_{\tilde{r}}^{\tilde{s}\tilde{10}}, \tilde{\omega}_{\tilde{10}}^{\tilde{s}\tilde{t}}, \tilde{\omega}_{\tilde{10}}^{\tilde{s}\tilde{10}}\},$$

(3-44)
is given by

$$\tilde{\omega}_{\tilde{r}}^{\tilde{s}\tilde{t}} = e^{\frac{1}{4} \phi} \left(\tilde{\omega}_{\tilde{r}}^{\tilde{s}\tilde{t}} - \frac{2}{3} \delta_{\tilde{r}}^{\tilde{s}} e^{\tilde{a}} \delta_{\tilde{b}} \tilde{\phi}\right), \quad \tilde{\omega}_{\tilde{r}}^{\tilde{s}\tilde{10}} = \frac{1}{2} e^{\frac{1}{4} \phi} \tilde{e}^{\tilde{p}} \tilde{e}^{\tilde{q}} \tilde{F}_{\tilde{p}\tilde{q}},$$

$$\tilde{\omega}_{\tilde{10}}^{\tilde{s}\tilde{t}} = -\frac{1}{2} e^{\frac{1}{4} \phi} \tilde{e}^{\tilde{p}} \tilde{e}^{\tilde{q}} \tilde{e}^{\tilde{r}} \tilde{e}^{\tilde{t}} \tilde{F}_{\tilde{p}\tilde{q}}, \quad \tilde{\omega}_{\tilde{10}}^{\tilde{s}\tilde{10}} = -\frac{2}{3} e^{\frac{1}{4} \phi} \tilde{e}^{\tilde{p}} \tilde{e}^{\tilde{q}} \delta_{\tilde{r}} \tilde{\phi},$$

(3-45)

where $\tilde{\omega}_{\tilde{r}}^{\tilde{s}\tilde{t}} = \tilde{e}^{\tilde{p}} \tilde{\omega}^{\tilde{r}}_{\tilde{p}} \tilde{\omega}^{\tilde{s}}_{\tilde{q}} \tilde{\omega}^{\tilde{t}}_{\tilde{p}}$, $\tilde{\omega}^{\tilde{s}\tilde{t}}$ is the torsion free spin connection, which is made of $\tilde{e}^{\tilde{p}}_{\tilde{r}}$, and

$$\tilde{F}_{\tilde{p}\tilde{q}} = 2 \partial_{[\tilde{p}} \tilde{A}_{\tilde{q}]}$$

(3-46)
is the field strength of the Kaluza-Klein vector field in Eq. (3-16).

Then, $\{\hat{\Omega}_A\} = \{\hat{\Omega}_{\tilde{r}}, \hat{\Omega}_{\tilde{10}}\}$ in Eq. (2-12) is calculated explicitly by using (3-42), (3-45) and (3-46), and hence we have (cf. Appendix B)

$$Q_{\tilde{r}\tilde{s}} = Q^{(2)}_{\tilde{r}\tilde{s}} - Q^{(4)}_{\tilde{r}\tilde{s}} + \frac{i}{2} s^{IJ} \tilde{\theta}_I \tilde{\Gamma}_{\tilde{r}} \tilde{H}_{\tilde{s}} \tilde{\theta}_J \tilde{H}_{\tilde{s}} \tilde{\theta}_J \tilde{H}_{\tilde{s}},$$

$$P_{\tilde{r}\tilde{s}} = P^{(2)}_{\tilde{r}\tilde{s}} - P^{(4)}_{\tilde{r}\tilde{s}} + \frac{i}{2} \tilde{\theta}_I \tilde{\Gamma}_{\tilde{r}} \tilde{\theta}_J \tilde{\phi} \tilde{H}_{\tilde{s}} \tilde{\theta}_J \tilde{\phi} \tilde{H}_{\tilde{s}} \tilde{\theta}_J \tilde{\phi} \tilde{H}_{\tilde{s}},$$

$$Q_{\tilde{10}\tilde{s}} = i \tilde{\theta}_I \tilde{\Gamma}_{\tilde{10}} \tilde{\theta}_J \tilde{\phi},$$

$$P_{\tilde{10}\tilde{s}} = i s^{IJ} \tilde{\theta}_I \tilde{\Gamma}_{\tilde{10}} \tilde{\theta}_J \tilde{\phi},$$

(3-47)

where $I, J = +, -, s^{++} - 1 = s^{--} + 1 = s^{+-} = s^{-+} = 0,$

$$Q^{(n)}_{\tilde{r}\tilde{s}} = \frac{i}{2n!} e^{\phi} \tilde{\theta}_I \Gamma_{\tilde{r}} \tilde{F}^{\tilde{r}_1 \cdots \tilde{r}_n} \tilde{\theta}_J \tilde{F}^{\tilde{r}_1 \cdots \tilde{r}_n}, \quad (n = 2, 4)$$

$$P^{(n)}_{\tilde{r}\tilde{s}} = \frac{i}{2n!} e^{\phi} \tilde{\theta}_I \Gamma_{\tilde{r}} \tilde{F}^{\tilde{r}_1 \cdots \tilde{r}_n} \tilde{\theta}_J \tilde{F}^{\tilde{r}_1 \cdots \tilde{r}_n}, \quad (n = 2, 4)$$

(3-48)

and $\theta_\pm$ are the 16-component positive and negative chiral spinors, respectively

$$\theta_\pm = \Gamma_\pm \theta, \quad \Gamma_\pm = \frac{1}{2} (1 \pm \Gamma_0).$$

(3-49)
Then, Eq. (3.28) can also be written by (see Appendix D)

\[
S_{ddr} = 2\pi T \int d^2\sigma C_{pq} \left[ -\sqrt{-g} - \frac{1}{2} \epsilon^{ij} \tilde{A}_{ji} - 2i \sqrt{-g} \tilde{\theta} P_{(-)} \Gamma^i \nabla_i \tilde{\theta} \right],
\]

where \( \tilde{g} = \text{det} \tilde{g}_{ij} \) and

\[
P_{(-)} = \frac{1}{2} \left( 1 - \frac{1}{2\sqrt{-g}} \epsilon^{ij} \Gamma_{ij} \Gamma^{10} \right).
\]

\[
\nabla_i = D_i + \frac{1}{8} \phi \left( \frac{1}{2!} \Gamma^{\tilde{r}\tilde{s}} \Gamma_{10} \tilde{F}_{\tilde{r}\tilde{s}} - \frac{1}{4!} \Gamma^{\tilde{r}\tilde{s}\tilde{t}\tilde{u}} \tilde{F}_{\tilde{r}\tilde{s}\tilde{t}\tilde{u}} \right) \Gamma_i + \frac{1}{8} \Gamma^{\tilde{r}\tilde{s}} \Gamma_{10} \tilde{H}_{\tilde{r}\tilde{s}},
\]

\[
D_i = \partial_i + \frac{1}{4} \partial_i X^\mu \Gamma_{\tilde{r}\tilde{s}} \tilde{\omega}_{\mu \tilde{r}\tilde{s}},
\]

\[
\tilde{H}_{\tilde{r}\tilde{s}} = \partial_i X^{\tilde{r}} \tilde{\epsilon}_i \tilde{H}_{\tilde{r}\tilde{s}}, \quad \Gamma_i = \partial_i X^\mu \Gamma_\mu, \quad \Gamma^a = \tilde{g}^{ij} \Gamma_j.
\]

\[\text{§4.} \quad (p, q)\text{-string from wrapped supermembrane}\]

In this section we derive the \((p, q)\)-string action from the reduced supermembrane action in Eq. (3.36). The action has an abelian isometry associated with the other compactified \(X^y\)-direction, and we can make a dual transformation as is the case with sigma models. Introducing a variable \(X^9\), which will be seen to be dual to \(X^y\), Eq. (3.36) (or (3.39)) can be rewritten in a classically equivalent form

\[
S_{ddr} = -\frac{2\pi T}{2} \int d\sigma^0 \int_0^{2\pi} d\sigma^1 C_{pq} \left[ \sqrt{-\tilde{g}} \tilde{\gamma}^{ij} \left( \partial_i X^\mu \partial_j X^\nu \tilde{G}_{\mu\nu} + 2\partial_i X^\mu \Xi_j \tilde{G}_{\mu y} \right) \right.
\]

\[
+ \Xi_i \Xi_j \tilde{G}_{y y} - \tilde{\epsilon}^{ij} \left( \partial_i X^\mu \partial_j X^\nu \tilde{B}_{\mu y} + 2\partial_i X^\mu \Xi_j \tilde{B}_{\mu y} \right) - 2 \epsilon^{ij} X^9 \partial_i \Xi_j \right] \quad (4.1),
\]

since the variation w.r.t. \(X^9\) leads to \(\epsilon^{ij} \partial_i \Xi_j = 0\) or \(\Xi_j = \partial_j X^y\) and hence Eq. (3.36) can be reproduced. On the other hand, assuming that all the fields are independent of \(\Xi_j\) (or \(X^y\)), the variation w.r.t. \(\Xi_i\) leads to

\[
\Xi_i = \frac{\tilde{g}_{ij} \epsilon^{kj}}{\tilde{G}_{y y}} \left( \partial_k X^\mu \tilde{B}_{\mu y} - \partial_k X^9 \right) - \frac{\tilde{G}_{\mu y}}{\tilde{G}_{y y}} \partial_i X^\mu.
\]

Plugging Eq. (4.2) into Eq. (4.1) we have

\[
S_{ddr} = -\frac{2\pi T}{2} \int d\sigma^0 \int_0^{2\pi} d\sigma^1 C_{pq} \left[ \sqrt{-\tilde{g}} \tilde{\gamma}^{ij} \left( \partial_i X^\mu \partial_j X^\nu G_\mu^\nu \right. \right.
\]

\[
- \epsilon^{ij} \left( \partial_i X^\mu \partial_j X^\nu B_\mu^\nu + 2\partial_i X^\mu \tilde{B}_\mu^\nu \right)
\]

\[
\quad - \epsilon^{ij} \left( \partial_i X^\mu \partial_j X^\nu E_\mu^\nu + 2\partial_i X^\mu \tilde{G}_\mu^\nu \right) \right], 
\]

\[\text{\footnote{\textsuperscript{\*}} We assume that the background fields are independent of } X^9 \text{ in Eq. (4.1).}\]
4.1. Note that in the case of \((X^\mu, X^9) = (X^\mu, \tilde{X}^9)\) and the ten-dimensional dual fields

\[
\begin{align*}
G'_{\mu\nu} &= \tilde{G}_{\mu\nu} + \tilde{G}_{yy}^{-1}(\tilde{B}_{\mu y}\tilde{B}_{\nu y} - \tilde{G}_{\mu y}\tilde{G}_{\nu y}), & G'_{\mu 9} &= -\tilde{G}_{yy}^{-1}\tilde{B}_{\mu y}, & G'_{9 9} &= \tilde{G}_{yy}^{-1}, \\
B'_{\mu\nu} &= \tilde{B}_{\mu\nu} - 2\tilde{G}_{yy}^{-1}\tilde{B}_{[\mu y}\tilde{G}_{\nu]y}, & B'_{\mu 9} &= -\tilde{G}_{yy}^{-1}\tilde{G}_{\mu y},
\end{align*}
\]

or

\[
\begin{align*}
E'_{\mu\nu} &= \tilde{E}_{\mu\nu} + \tilde{E}_{yy}^{-1}(\tilde{B}_{\mu y}\tilde{B}_{\nu y} - \tilde{E}_{\mu y}\tilde{E}_{\nu y}), & E'_{\mu 9} &= -\tilde{E}_{yy}^{-1}\tilde{B}_{\mu y}, & E'_{9 9} &= \tilde{E}_{yy}^{-1}, \\
B'_{\mu\nu} &= \tilde{B}_{\mu\nu} - \tilde{E}_{yy}^{-1}(\tilde{B}_{\mu y}\tilde{E}_{\nu y} - \tilde{B}_{\nu y}\tilde{E}_{\mu y}), & B'_{\mu 9} &= -\tilde{E}_{yy}^{-1}\tilde{E}_{\mu y}, \\
G'_{j\mu} &= \tilde{G}_{j\mu} + \tilde{E}_{yy}^{-1}(\tilde{B}_{\mu y}\tilde{B}_{j y} - \tilde{E}_{\mu y}\tilde{G}_{j y}), & G'_{j 9} &= -\tilde{E}_{yy}^{-1}\tilde{B}_{j y}, \\
B'_{j\mu} &= \tilde{B}_{j\mu} + \tilde{E}_{yy}^{-1}(\tilde{B}_{\mu y}\tilde{G}_{j y} - \tilde{B}_{j y}\tilde{E}_{\mu y}), & B'_{j 9} &= -\tilde{E}_{yy}^{-1}\tilde{G}_{j y}.
\end{align*}
\]

4.1. \(\theta^0\)-order action

The \(\theta^0\)-order part of Eq. (4.4) is given by

\[
\begin{align*}
g'_{\mu\nu} &= \tilde{g}_{\mu\nu} + \tilde{g}_{yy}^{-1}(\tilde{A}_{\mu y}\tilde{A}_{\nu y} - \tilde{g}_{\mu y}\tilde{g}_{\nu y}), & g'_{\mu 9} &= -\tilde{g}_{yy}^{-1}\tilde{A}_{\mu y}, & g'_{9 9} &= \tilde{g}_{yy}^{-1}, \\
A'_{\mu\nu} &= \tilde{A}_{\mu\nu} - 2\tilde{g}_{yy}^{-1}\tilde{A}_{[\mu y}\tilde{g}_{\nu]y}, & A'_{\mu 9} &= -\tilde{g}_{yy}^{-1}\tilde{g}_{\mu y}.
\end{align*}
\]

Note that in the case of \((p, q) = (1, 0)\) in Eq. (3.9), Eq. (4.6) is reduced to the ordinary Buscher’s T-dual rule for the NSNS sector.\(^3\) However, since we have rotated the target space coordinates in Eq. (3.8) by the \(SO(2)\) matrix in Eq. (3.9), the T-dual transformation rule (4.6) of order zero in \(\theta\) includes not only the NSNS fields but also the RR fields, or the RR 2-form, which will be seen below.

Now that we consider T-dual for the background fields in Eq. (3.36) (or Eq. (4.3)). Since we regard \(X^9\) (not \(X^z\)) as the 11th direction, we should take T-dual along the \(X^9\)-direction (not \(X^y\)-direction) to transform type IIA superstring theory to type IIB superstring theory. Then we can rewrite the background fields in terms of those of the type IIB supergravity as follows (cf. Appendix C),\(^*)

\[
\begin{align*}
\tilde{g}_{\mu\nu} &= \Delta(\tilde{p} q)(j_{\mu\nu} - \frac{j_{9\mu}j_{9\nu}}{j_{99}} - B^{(pq)}_{9\mu}B^{(pq)}_{9\nu}), \\
\tilde{g}_{\mu y} &= \frac{B^{(pq)}_{9\mu}}{j_{99}}, \\
\tilde{g}_{yy} &= \frac{1}{\Delta(\tilde{p} q)}j_{99}, \\
\tilde{A}_{\mu\nu} &= \Delta(\tilde{p} q)(\tilde{A}^{(pq)}_{\mu\nu} + \tilde{p} A_{\mu\nu}) = \Delta(\tilde{p} q)\left(B^{(pq)}_{\mu\nu} + \frac{2B^{(pq)}_{9\mu}j_{9\nu}}{j_{99}}\right), \\
\tilde{A}_{\mu yz} &= \Delta(\tilde{p} q)A_{\mu yz} = -\frac{j_{9\mu}}{j_{99}}.
\end{align*}
\]

\(^*)\text{Equation (4.8) leads to:}

\[
\begin{align*}
g'_{\mu\nu} &= \Delta(\tilde{p} q)j_{\mu\nu}, & g'_{\mu 9} &= \Delta(\tilde{p} q)j_{\nu 9}, & g'_{9 9} &= \Delta(\tilde{p} q)j_{9 9}, \\
A'_{\mu\nu} &= \Delta(\tilde{p} q)B^{(pq)}_{\mu\nu}, & A'_{\mu 9} &= -\Delta(\tilde{p} q)B^{(pq)}_{\mu 9}.
\end{align*}
\]
where
\[ \Delta_{(p\bar{q})} = \sqrt{(\tilde{p} + \tilde{q} d)^2 + e^{-2\tilde{q} q^2}}, \] (4.9)

\( B_{(p\bar{q})}^{(1)} \) and \( B_{(p\bar{q})}^{(2)} \) are the NSNS and the RR second-rank antisymmetric tensors, respectively, \( J_{\mu\nu} \) is the metric in type IIB supergravity, \( l = G_{910}/G_{1010} = A_9 \) and
\[ B_{(p\bar{q})} = \frac{\tilde{p} B_{(1)}^{(1)}(\mu\nu) + \tilde{q} B_{(2)}^{(2)}}{\Delta_{(p\bar{q})}} . \] (4.10)

Then, plugging these equations into Eq. (4.6), the \( \theta^0 \)-order part of the action in Eq. (4.3) is reduced to
\[ S_{ddr} |_{\theta^0} = -\frac{2\pi T}{2} \int d\sigma^0 \int_0^{2\pi} d\sigma^1 C_{pq} \Delta_{(p\bar{q})} \left[ \sqrt{-\tilde{g} \tilde{\gamma}^{ij} \partial_i X^\mu \partial_j X^\nu J_{\mu\nu} - e^{\tilde{q} \tilde{\gamma}} \partial_i X^\mu \partial_j X^\nu B_{(p\bar{q})} \right] . \] (4.11)

We regard \( X^{10} \) as the 11th direction, therefore the type IIA string tension \( T_s \) is given by
\[ 2\pi L_1 T / \sqrt{G_{1010}} \] since the eleven-dimensional metric \( G_{MN} \) is converted to the type IIA metric \( g_{\mu\nu} \) by the relation \( G_{\mu\nu} = g_{\mu\nu} / \sqrt{G_{1010}} \). Furthermore, putting the background fields \( \{ l, \varphi \} \) to be the asymptotic constant values \( \{ l_0, \varphi_0 \} \), respectively and hence \( e^{\varphi_0} = g_s^{\text{IIB}} \), we have
\[ 2\pi T C_{pq} \Delta_{(p\bar{q})} \rightarrow w_1 T_s \sqrt{(p + q l_0)^2 + e^{-2\varphi_0} q^2} \equiv w_1 T_{pq} , \] (4.12)
where \( T_{pq} \) is the tension of a \( (p, q) \)-string in type IIB superstring theory.\(^8\) We can see that both the NSNS and the RR antisymmetric tensors have coupled to \( X^{10} \) in Eq. (4.11), which implies that the reduced action (4.11) is, in fact, that of \( (p, q) \)-strings. Note that \( w_1 \) is just the number of copies of the resulting \( (p, q) \)-string. If we allow \( q \) to be zero and take \( (p, q, r, s) = (1, 0, 0, 1) \), we have the fundamental strings in type IIB superstring theory. On the other hand, \( (p, q, r, s) = (0, 1, 1, 0) \) leads to the strings which couple minimally with the RR B-field, i.e., the D-strings.

### 4.2. \( \theta^2 \)-order action

We next proceed to consider the \( \theta^2 \)-order terms in Eq. (4.5),
\[ Q'_{\mu\nu} = \tilde{Q}_{\mu\nu} - 2\tilde{g}_{yy}^{-1} (\tilde{A}_{(\mu|y} | \tilde{P}_{\nu)y} + \tilde{g}_{(\mu|y} | \tilde{Q}_{\nu)y}) - \tilde{g}_{yy}^{-2} \tilde{Q}_{yy} (\tilde{A}_{\mu} \tilde{A}_{\nu} - \tilde{g}_{\mu} \tilde{g}_{\nu}) , \]
\[ Q'_{\mu 9} = -\tilde{g}_{yy}^{-1} (\tilde{P}_{\mu y} - \tilde{g}_{yy}^{-1} \tilde{Q}_{yy} \tilde{A}_{\mu y}) , \quad Q'_{99} = -\tilde{g}_{yy}^{-2} \tilde{Q}_{yy} , \]
\[ P'_{\mu\nu} = \tilde{P}_{\mu\nu} - 2\tilde{g}_{yy}^{-1} (\tilde{A}_{(\mu|y} | \tilde{Q}_{\nu)y} + \tilde{P}_{(\mu|y} | \tilde{g}_{\nu)y}) + \tilde{g}_{yy}^{-1} \tilde{Q}_{yy} (\tilde{A}_{|y} | \tilde{\gamma}_{\nu}) , \]
\[ P'_{\mu 9} = \tilde{g}_{yy}^{-1} (\tilde{Q}_{\mu y} - \tilde{g}_{yy}^{-1} \tilde{Q}_{yy} \tilde{g}_{\mu y}) , \]
\[ G'_{\mu\nu} = \tilde{G}_{\mu\nu} + \tilde{g}_{yy}^{-1} (\tilde{A}_{\mu} \tilde{B}_{\nu} - \tilde{G}_{\mu\nu} \tilde{g}_{\nu}) , \quad G'_{\mu 9} = -\tilde{g}_{yy}^{-1} \tilde{B}_{\mu y} , \]
\[ B'_{\mu\nu} = \tilde{B}_{\mu\nu} + \tilde{g}_{yy}^{-1} (\tilde{A}_{\mu} \tilde{B}_{\nu} - \tilde{B}_{\mu\nu} \tilde{g}_{\nu}) , \quad B'_{\mu 9} = -\tilde{g}_{yy}^{-1} \tilde{G}_{\mu y} . \] (4.13)

We shall plug Eq. (3.47) into Eq. (4.13) and rewrite them by using the type IIB superstring variables. Note that \( \tilde{Q}_{\mu\nu}, \tilde{P}_{\mu\nu}, \text{etc.} \) include the vielbein \( \mathcal{e}^\mu_{\tilde{\mu}} \), which is
transformed under T-dual according to Eq. (4.8), and hence we need the T-dual transformation rule of $\tilde{e}_\mu^\hat{\nu}$. In fact, Eq. (4.8) can be written by

$$J_{\hat{\mu}\hat{\nu}} = (Q^{-1})_\mu^\hat{\nu} \tilde{g}_{\hat{\mu}\hat{\nu}} (tQ)^{\hat{\nu}}_\rho, \quad \tilde{f}_{\hat{\mu}\hat{\nu}} = (tQ)_\mu^\hat{\nu} \tilde{g}^{\hat{\mu}\hat{\nu}} (Q)^\hat{\nu}_\rho,$$

where

$$(Q^{-1})_\mu^\hat{\nu} = \Delta^{-1/2}_{(\hat{\rho}\hat{\sigma})} \left( \begin{array}{cc} \delta_\mu^\nu & -\tilde{g}_{\mu y} + \tilde{A}_{\mu y} \\ 0 & \tilde{g}^{-1}_{yy} \end{array} \right) = \Delta^{-1/2}_{(\hat{\rho}\hat{q})} \left( \begin{array}{cc} \delta_\mu^\nu & -\Delta_{(\hat{\rho}\hat{q})}(J_{\hat{\rho}y} - B_{y\hat{q}}^{(pq)}) \\ 0 & \Delta_{(\hat{\rho}\hat{q})}J_{\hat{q}y} \end{array} \right),$$

$$Q_\mu^\hat{\nu} = \Delta^{1/2}_{(\hat{\rho}\hat{q})} \left( \begin{array}{cc} \delta_\mu^\nu & \tilde{g}_{\mu y} + \tilde{A}_{\mu y} \\ 0 & \tilde{g}^{-1}_{yy} \end{array} \right) = \Delta^{1/2}_{(\hat{\rho}\hat{q})} \left( \begin{array}{cc} \delta_\mu^\nu & -\Delta_{(\hat{\rho}\hat{q})}(J_{\hat{\rho}y} - B_{y\hat{q}}^{(pq)}) \\ 0 & \Delta^{-1}_{(\hat{\rho}\hat{q})}J_{\hat{q}y} \end{array} \right).$$

Thus we can take the transformation rules for the vielbeins as follows:

$$e_\mu^\hat{\rho} = (Q^{-1})_\mu^\hat{\rho} \tilde{e}_\mu^\hat{\sigma}, \quad \tilde{e}_\mu^\hat{\rho} = (Q)_\mu^\hat{\rho} e_\mu^\hat{\sigma},$$

$$e_\hat{\rho}^\mu = e_\hat{\rho}^\sigma (Q)_\sigma^\mu, \quad \tilde{e}_\hat{\rho}^\mu = e_\hat{\rho}^\sigma (Q^{-1})_\sigma^\mu.$$

We shall rewrite the 4-form and 3-form field strengths in Eq. (3.42) and the 2-form field strength in (3.46). Since we have

$$\{ \tilde{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + 4\tilde{A}_{\hat{\mu}}(\tilde{H}_{\hat{\nu}\hat{\rho}\hat{\sigma}}) \} = \{ \tilde{F}_{\mu\nu\rho\sigma} + 4\tilde{A}_{\mu}(\tilde{H}_{\nu\rho\sigma}) \} + 3\tilde{A}_{\mu}(\tilde{H}_{\nu\rho\sigma} - \tilde{A}_{\hat{y}} \tilde{H}_{\mu\nu\rho\sigma}) \} = \{ \tilde{F}_{\mu\nu\rho\sigma} \},$$

$$\{ \tilde{F}_{\hat{\mu}\hat{\nu}\hat{\rho}} \} = \{ \tilde{F}_{\mu\nu\rho} \},$$

the 4-form and 3-form field strengths with the tangent indices become

$$\tilde{F}_{\hat{r}\hat{s}\hat{t}\hat{u}} = \tilde{e}_{\hat{r}}^\mu \tilde{e}_{\hat{s}}^\nu \tilde{e}_{\hat{t}}^\rho \tilde{e}_{\hat{u}}^\sigma (\tilde{F}_{\mu\nu\rho\sigma} + 4\tilde{A}_{\mu}(\tilde{H}_{\nu\rho\sigma}))$$

$$= -\Delta^{-2}_{(\hat{\rho}\hat{q})} e_\mu^\hat{\rho} e_s^\hat{\nu} e_t^\hat{\sigma} e_u^\hat{\tau} (\tilde{F}_{\mu\nu\rho\sigma} + 4\tilde{A}_{\mu}(\tilde{H}_{\nu\rho\sigma})) + 4\Delta^{-1}_{(\hat{\rho}\hat{q})} e_r^\hat{\rho} e_s^\hat{\nu} e_t^\hat{\sigma} e_u^\hat{\tau} J_{\hat{\rho}y} \tilde{H}_{\mu\nu\rho\sigma} \} = \Delta^{3/2}_{(\hat{\rho}\hat{q})} J_{\hat{r}\hat{s}\hat{t}\hat{u}},$$

$$\tilde{H}_{\hat{r}\hat{s}\hat{t}} = \tilde{e}_{\hat{r}}^\mu \tilde{e}_{\hat{s}}^\nu \tilde{e}_{\hat{t}}^\rho \tilde{H}_{\mu\nu\rho},$$

$$= \Delta^{3/2}_{(\hat{\rho}\hat{q})} J_{\hat{r}\hat{s}\hat{t}} - 3\Delta^{-1/2}_{(\hat{\rho}\hat{q})} e_r^\hat{\rho} e_s^\hat{\nu} e_t^\hat{\sigma} J_{\hat{\rho}y} (\Delta^{-1}_{(\hat{\rho}\hat{q})} h_{\mu\nu\rho\sigma} + 2\partial_{\mu\nu\rho\sigma}),$$

and similarly, the 2-form field strength of $\tilde{A}_\mu$ in Eq. (3.46) is given by

$$\tilde{F}_{\hat{r}\hat{s}} = \tilde{e}_{\hat{r}}^\mu \tilde{e}_{\hat{s}}^\nu \tilde{F}_{\mu\nu} = e_r^\mu e_s^\nu (\Delta^{-1}_{(\hat{\rho}\hat{q})} f_{\mu\nu\rho\sigma} + 2J_{\hat{r}\hat{s}\\hat{\rho}\\hat{\sigma}}),$$

where

$$b_{\hat{\mu}hat{\nu}}^{(\hat{\rho})} = \hat{\rho} B_{\mu\nu}^{(1)} + \hat{q} B_{\mu\nu}^{(2)},$$

$$h_{\hat{\mu}hat{\nu}}^{(\hat{\rho})} = 3\partial_{\mu\nu} b_{\hat{\mu}hat{\nu}}^{(\hat{\rho})},$$

$$f_{\mu\nu\rho\sigma}^{(5)} = 5\partial_{\mu\nu} b_{\mu\nu\rho\sigma}^{(\hat{\rho})} - 15\partial_{\mu\nu} b_{\mu\nu\rho\sigma}^{(-\hat{\rho})} + 15 b_{\mu\nu\rho\sigma}^{(-\hat{\rho})}\partial_{\mu\nu}\partial_{\rho\sigma} b_{\mu\nu\rho\sigma}^{(\hat{\rho})},$$

$$f_{\mu\nu\rho\sigma}^{(3)} = 3\partial_{\mu\nu} b_{\mu\nu\rho\sigma}^{(-\hat{\rho})} - 3\partial_{\mu\nu} b_{\mu\nu\rho\sigma}^{(\hat{\rho})},$$

$$f_{\hat{\mu}}^{(1)} = \partial_{\hat{\mu}} l.$$
Note that $f^{(5)}$ is selfdual ($f^{(5)} = * f^{(5)}$)

$$f^{(5)}_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} = \frac{1}{5!} \varepsilon_{\mu_1 \mu_2 \cdots \mu_{10} / \nu_{10}} f^{(5)}_{\nu_{10} \nu_{11} \nu_{12} \nu_{13} \nu_{14} \nu_{15}}.$$  \hfill (4.22)

Consequently, we have

\[
\Gamma^{\hat{f}_{r_1 r_2 r_3 r_4}} = 10 f^{(5)}_{r_1 r_2 r_3 r_4 g} \Gamma^{r_1 r_2 r_3 r_4 g}_{\nu_{99} + 8 e_9^r s_{98} r^s e_9 s \Gamma^{r_1 r_2 r_3 r_4 g}_{\nu_{99} - 2 e_9^r s_{98} r^s e_9 s \Gamma^{r_1 r_2 r_3 r_4 g}_{\nu_{99}}}} f^{(5)}_{r_1 r_2 r_3 r_4 g}, \hfill (4.23)
\]

where the index $\hat{9}$ stands for the ninth-direction of the tangent space. In the above we have used the relations below,\(^*\)

\[
\tilde{F}_{\mu \nu \rho \sigma} = 4 \partial_{[\mu} \tilde{A}_{\nu \rho \sigma]} \quad \tilde{A}_{\mu \nu} = D_{9 \mu \nu} + \frac{3}{2} \left( \left( b_{9 \mu \nu}^\dagger b_{9 \nu \mu}^\dagger + \frac{2 b_{9 \mu \nu}^\dagger}{J_{99}} b_{9 \nu \mu}^\dagger - (b_{9 \mu \nu}^\dagger) \leftrightarrow (b_{9 \nu \mu}^\dagger) \right) \right),
\]

\[
\tilde{F}_{\mu \nu \rho \sigma} = 4 \partial_{[\mu} \tilde{A}_{\nu \rho \sigma]} = 3 \partial_{[\mu} \tilde{A}_{\nu \rho \sigma]} = \tilde{A}_{\mu \nu} = b_{\mu \nu},
\]

\[
\tilde{H}_{\mu \nu} = -b_{\nu \mu},
\]

\[
\tilde{A}_{\mu} = \tilde{A}_{\nu} = \tilde{A}_{\rho} = \tilde{A}_{\sigma} = \tilde{A}_{\nu},
\]

\[
\tilde{A}_{y} = \tilde{A}_{z} = \tilde{A}_{y} = \tilde{A}_{z} = \delta(\tilde{\nu} - \tilde{\mu}) (\tilde{\mu} - \tilde{\nu}) + \tilde{\mu} \tilde{\nu} e^{-2 \varphi} \delta \frac{2}{J_{99}} \left( (\tilde{\nu} - \tilde{\mu}) (\tilde{\mu} - \tilde{\nu}) + \tilde{\mu} \tilde{\nu} e^{-2 \varphi} \right) \hfill (4.24)
\]

Note that in the case of $(p, q) = (1, 0)$, $b_{\mu \nu}^\dagger$ is reduced to the ordinary type IIB NSNS 2-form $B_{\mu \nu}^{(1)}$ and $A_{\hat{p} \hat{q}} = \Delta_{\hat{p} \hat{q}} = 1$.

The spin connection $\tilde{\omega}_{\hat{s}}^{\hat{t} \hat{i}}$ is rewritten by

\[
\tilde{\omega}_{\hat{s}}^{\hat{u} \hat{v}} = \Delta^{-1/2}_{\hat{p} \hat{q}} \omega_{\hat{s}}^{\hat{u} \hat{v}} + \frac{1}{2} \Delta^{-1/2}_{\hat{p} \hat{q}} \left( \varepsilon_{\hat{s} \hat{t} \hat{u} \hat{v}} \varepsilon_{\hat{t} \hat{v} \hat{u} \hat{t}} + \varepsilon_{\hat{s} \hat{t} \hat{u} \hat{v}} \varepsilon_{\hat{t} \hat{v} \hat{u} \hat{t}} \right) (2 \partial_{\hat{s}} \hat{v} \hat{t} + \hat{s} \hat{t} e^{-2 \varphi} \partial_{\hat{v}} \Delta_{\hat{p} \hat{q}}).
\]

The gamma matrices with the target space indices of the dual space are defined by

\[
\Gamma^{\hat{\mu}} = \Gamma^{\hat{\nu}} e_{\hat{\nu}}^{\hat{\mu}}, \quad \Gamma_{\hat{\mu}} = e_{\hat{\mu}}^{\hat{\nu}} \Gamma_{\hat{\nu}} = J_{\hat{\mu} \hat{\nu}} \Gamma_{\hat{\nu}}, \hfill (4.26)
\]

and Eq. (4.17) leads to

\[
\tilde{\Gamma}^{\mu} = \Gamma^{\hat{\nu}} e_{\hat{\nu}}^{\hat{\mu}} = \Delta^{-1/2}_{\hat{p} \hat{q}} \Gamma^{\hat{\nu}} e_{\hat{\nu}}^{\hat{\mu}} = \Delta^{-1/2}_{\hat{p} \hat{q}} \Gamma_{\hat{\nu}} \hfill (4.27)
\]

\(^*\) We have assumed that the backgrounds are independent of both $X^y$ and $X^9$ in (4.1).
Similarly to the rescaling in the course of the double dimensional reduction in the previous section, the spinors \( \theta_\pm \) are rescaled by \( \Delta_{(\hat{p}\hat{q})}^{1/4} \) to maintain the canonical form of the supersymmetry transformation under T-dual. In fact, we shall define the spinors \( \theta_{1,2} \) in type IIB superstring theory as

\[
\theta_1 = \Delta_{(\hat{p}\hat{q})}^{-1/4} \Omega_\chi \theta_+, \quad \theta_2 = \Delta_{(\hat{p}\hat{q})}^{-1/4} \theta_-, \quad (4.28)
\]

where

\[
\Omega_\chi = \frac{1}{\sqrt{g_{99}}} \Gamma^{10} \Gamma_9, \quad (\Omega_\chi^2 = -1) \quad (4.29)
\]

and hence

\[
\bar{\theta}_1 = -\bar{\theta}_+ \Omega_\chi^{-1/4}, \quad \bar{\theta}_2 = \bar{\theta}_- \Delta_{(\hat{p}\hat{q})}^{-1/4}. \quad (4.30)
\]

Note that \( \theta_{1,2} \) satisfy

\[
\Gamma_- \theta_\xi = \theta_\xi, \quad \bar{\theta}_\xi \Gamma_+ = \bar{\theta}_\xi, \quad \Gamma_+ \theta_\xi = \bar{\theta}_\xi \Gamma_- = 0. \quad (\xi = 1, 2) \quad (4.31)
\]

One comment is in order: The \( \Omega_\chi \) (4.29) can be written by

\[
\Omega_\chi = \frac{1}{\sqrt{g_{99}}} \Gamma^{10} \Gamma_9 = \frac{1}{\sqrt{g_{yy}}} \Gamma^{10} \Gamma_y, \quad (4.32)
\]

which is because Eq. (4.17) leads to \( j_{99}^{-1/2} e_{\hat{q}}^\vec{\gamma} = g_{yy}^{-1/2} e_{\hat{q}}^\vec{\gamma} \).

Now that we shall give \( Q', P', G', B' \) in (4.13) by the type IIB variables. With the tangent space variables,

\[
Q'_{\mu\nu} = e_{\hat{p}}^\vec{\gamma} e_{\hat{p}}^\vec{\nu} Q^\prime_{\hat{r}\hat{s}}, \quad P'_{\mu\nu} = e_{\hat{p}}^\vec{\gamma} e_{\hat{p}}^\vec{\nu} P^\prime_{\hat{r}\hat{s}}, \quad G'_{\hat{r}\hat{s}} = e_{\hat{p}}^\vec{\gamma} G^\prime_{\hat{r}\hat{s}}, \quad B'_{\hat{r}\hat{s}} = e_{\hat{p}}^\vec{\gamma} B^\prime_{\hat{r}\hat{s}}, \quad (4.33)
\]

Eq. (4.13) becomes

\[
Q^\prime_{\hat{r}\hat{s}} = \Delta_{(\hat{p}\hat{q})}\left\{ Q_{\hat{r}\hat{s}} + 2j_{99} e_{\hat{y}} e_{\hat{y}} (P_{\hat{r}\hat{s}} - Q_{\hat{r}\hat{s}}) \right\}, \quad (4.34)
\]

Then we have

\[
Q^\prime_{\hat{r}\hat{s}} = Q^{(1)}_{\hat{r}\hat{s}} - Q^{(3)}_{\hat{r}\hat{s}} + Q^{(5)}_{\hat{r}\hat{s}} + \frac{i}{4} s^{\xi \zeta} \bar{\theta}_\xi \Gamma_\chi \hat{\theta}_\eta \Gamma^{(pq)}_{\hat{r}\hat{s}} \hat{h}^{\hat{y}} (99) \hat{u} \hat{v} \Gamma^{10} \Gamma_9, \quad (4.35)
\]

\[
P^\prime_{\hat{r}\hat{s}} = P^{(1)}_{\hat{r}\hat{s}} - P^{(3)}_{\hat{r}\hat{s}} + P^{(5)}_{\hat{r}\hat{s}} - \frac{i}{4} s^{\xi \zeta} \bar{\theta}_\xi \Gamma_\chi \hat{\theta}_\eta \Gamma^{(pq)}_{\hat{r}\hat{s}} \hat{h}^{\hat{y}} (99) \hat{u} \hat{v} \Gamma^{10} \Gamma_9, \quad (4.36)
\]

\[
G^\prime_{\hat{r}\hat{s}} = i \Delta_{(\hat{p}\hat{q})} \delta^{\xi \zeta} \bar{\theta}_\xi \Gamma_\chi \hat{\theta}_\eta - i \Delta_{(\hat{p}\hat{q})} \delta^{\xi \zeta} \bar{\theta}_\xi \Gamma_\chi \hat{\theta}_\eta X_{\hat{r}\hat{s}}^{\hat{y}} \hat{y} (99) \hat{u} \hat{v} \Gamma^{10} \Gamma_9, \quad (4.37)
\]

\[
B^\prime_{\hat{r}\hat{s}} = i \Delta_{(\hat{p}\hat{q})} \delta^{\xi \zeta} \bar{\theta}_\xi \Gamma_\chi \hat{\theta}_\eta - i \Delta_{(\hat{p}\hat{q})} \delta^{\xi \zeta} \bar{\theta}_\xi \Gamma_\chi \hat{\theta}_\eta X_{\hat{r}\hat{s}}^{\hat{y}} \hat{y} (99) \hat{u} \hat{v} \Gamma^{10} \Gamma_9, \quad (4.38)
\]
where \( \xi, \zeta = 1, 2 \) and \( s^{11} = -s^{22} = 1, s^{12} = s^{21} = 0 \) (cf. (4.21)),

\[
Q^{(n)}_{\hat{r}\hat{s}} = \left\{ \begin{array}{ll}
-\frac{i}{2 \cdot n!} e^{\varphi(n)} \bar{\theta}_1 \Gamma(\hat{r}_1 \cdots \hat{r}_n \Gamma_{\hat{s}}) \theta_2 f_{\hat{r}_1 \cdots \hat{r}_n}^{(n)} , & (n = 1, 3) \\
-\frac{i}{4 \cdot n!} e^{\varphi(n)} \bar{\theta}_1 \Gamma(\hat{r}_1 \cdots \hat{r}_n \Gamma_{\hat{s}}) \theta_2 f_{\hat{r}_1 \cdots \hat{r}_n}^{(n)} , & (n = 5)
\end{array} \right.
\]

\[
P^{(n)}_{\hat{r}\hat{s}} = \left\{ \begin{array}{ll}
-\frac{i}{2 \cdot n!} e^{\varphi(n)} \bar{\theta}_1 \Gamma(\hat{r}_1 \cdots \hat{r}_n \Gamma_{\hat{s}}) \theta_2 f_{\hat{r}_1 \cdots \hat{r}_n}^{(n)} , & (n = 1, 3) \\
-\frac{i}{4 \cdot n!} e^{\varphi(n)} \bar{\theta}_1 \Gamma(\hat{r}_1 \cdots \hat{r}_n \Gamma_{\hat{s}}) \theta_2 f_{\hat{r}_1 \cdots \hat{r}_n}^{(n)} , & (n = 5)
\end{array} \right.
\]

(4.39)

\( \varphi(n) \) is a “coupling parameter” given by

\[
\varphi(n) = \varphi + \frac{7 - n}{2} \log \Delta(\hat{p}\hat{q}) ,
\]

(4.40)

and \( \omega_{\hat{r}\hat{s}} \) is the spin connection in type IIB theory in ten dimensions. Note that \( \varphi(n) \) and \( h_{\hat{r}\hat{s}} \) are reduced to the type IIB dilaton and the field strength of the NSNS 2-form field, respectively, in the case of \( (p, q) = (1, 0) \), or the fundamental string.

Then, the \( \theta^2 \) order part of (4.3) is given by

\[
S_{dhr}|_{\vartheta^2} = -2\pi i T \int d\sigma^0 \int_0^{2\pi} d\sigma^1 C_{pq} \left[ \Delta(\hat{p}\hat{q}) (\sqrt{-\gamma} \bar{\gamma}^{ij} \delta^{\xi\zeta} + e^{ij} s^{\xi\zeta}) \partial_{\hat{r}} X^{\bar{\mu}_{\hat{s}} \hat{\xi}_{\hat{s}}} \Gamma_{\hat{s}} \partial_{\hat{s}} \theta_{\hat{r}} \theta_{\hat{s}} \hat{p}_{\hat{q}} \right.

+ \frac{1}{2} \left\{ \frac{1}{4} (\sqrt{-\gamma} \bar{\gamma}^{ij} s^{\xi\zeta} + e^{ij} \delta^{\xi\zeta}) \partial_{\hat{r}} X^{\bar{\mu}_{\hat{s}} \hat{\xi}_{\hat{s}}} \partial_{\hat{s}} X^{\bar{\mu}_{\hat{r}} \hat{\xi}_{\hat{r}}} \Gamma_{\hat{r}} \partial_{\hat{r}} \theta_{\hat{s}} h_{\hat{r}\hat{s}}^{(pq)}

+ (\sqrt{-\gamma} \bar{\gamma}^{ij} + e^{ij}) \partial_{\hat{r}} X^{\bar{\mu}_{\hat{s}} \hat{\xi}_{\hat{s}}} \partial_{\hat{s}} X^{\bar{\mu}_{\hat{r}} \hat{\xi}_{\hat{r}}} \left( \frac{e^{\varphi(1)}}{2} \bar{\theta}_1 \Gamma_{\hat{r}} \Gamma_{\hat{p}_{\hat{1}}} \Gamma_{\hat{s}} \theta_2 f_{\hat{p}_{\hat{1}}}^{(1)}

- \frac{e^{\varphi(3)}}{2 \cdot 3!} \bar{\theta}_1 \Gamma_{\hat{r}} \Gamma_{\hat{p}_{\hat{1}} \hat{p}_{\hat{2}}} \Gamma_{\hat{s}} \theta_2 f_{\hat{p}_{\hat{1}} \hat{p}_{\hat{2}}}^{(3)}

- \frac{e^{\varphi(5)}}{4 \cdot 5!} \bar{\theta}_1 \Gamma_{\hat{r}} \Gamma_{\hat{p}_{\hat{1}} \cdots \hat{p}_{\hat{5}}} \Gamma_{\hat{s}} \theta_2 f_{\hat{p}_{\hat{1}} \cdots \hat{p}_{\hat{5}}}^{(5)}

+ \frac{1}{2} e^{ij} s^{\xi\zeta} \partial_{\hat{r}} X^{\bar{\mu}_{\hat{s}} \hat{\xi}_{\hat{s}}} \partial_{\hat{s}} X^{\bar{\mu}_{\hat{r}} \hat{\xi}_{\hat{r}}} \Gamma_{\hat{r}} \partial_{\hat{r}} \theta_{\hat{s}} \Delta(\hat{p}\hat{q}) \right) \right\} ,
\]

(4.41)

where

\[
D_{\hat{r}} = \partial_{\hat{r}} + \frac{1}{4} \partial_{\hat{r}} X^{\bar{\mu}_{\hat{s}}} \omega_{\hat{s}} \Gamma_{\hat{s}} .
\]

(4.42)

As is noted in Ref. 6), the derivative for chiral spinor appears only in the covariant derivative. Note that in the case of \( (p, q) = (1, 0) \), or fundamental string, the resulting action (4.41) is reduced to the one in Ref. 6). In that case we can see that the fundamental string couples with the RR 1-, 3- and 5-form fields with strength \( e^{\varphi} \). In the case of \( (p, q) = (0, 1) \), or D-string, the strengths of the coupling with the RR 1-, 3- and 5-form field strengths are given by \( e^{-2\varphi}, e^{-\varphi}, 1 \), respectively. Putting (4.11)

\* The extra factor 1/2 for \( n = 5 \) is due to the self-duality.
and (4.41) together, we have explicitly the Green-Schwarz type \((p,q)\)-string action of order up to quadratic in \(\theta\) in type IIB superstring

\[
S_{ddr} = -\frac{2\pi T}{2} \int d\sigma^0 \int_0^{2\pi} d\sigma^1 C_{pq} \Delta_{(p\bar{q})} \left[ \sqrt{-\gamma} \gamma^{ij} \partial_i X^\mu \partial_j X^\nu \gamma_{\mu\nu} - e^{ij} \partial_i X^\mu \partial_j X^\nu B_{\mu\nu}^{(pq)} + 2i(\sqrt{-\gamma} \gamma^{ij} \delta^{\xi\zeta} + e^{ij} s^{\xi\zeta}) \partial_i X^\mu \partial_j X^\nu \theta_\xi \Gamma_{\mu} \Gamma_{\nu} \theta_\psi \Delta_{(p\bar{q})} \right]
\]

(4.3)

Finally in this subsection, we also give the action (4.3) in Nambu-Goto form up to quadratic in \(\theta\) by integrating out the worldsheet metric \(\hat{\gamma}_{ij}\) in (4.3), or in (4.43) (cf. Appendix E)

\[
S_{ddr} = 2\pi T \int d^2 \sigma C_{pq} \Delta_{(p\bar{q})} \left[ -\sqrt{-j} + \frac{1}{2} e^{ij} B_{ij}^{(pq)} - 2i\sqrt{-j} e^{ij} \hat{\theta} P_+ \hat{\theta} \right] \Delta_{(p\bar{q})},
\]

(4.44)

where

\[
P_+ = \frac{1}{2} \left( 1 + \frac{1}{2\sqrt{-j}} \sigma_3 \otimes e^{ij} \Gamma_{ij} \Gamma^{10} \right),
\]

\[
D_i = \partial_i + \frac{1}{4} \partial_i X^\mu \omega_{\mu} \Gamma_{r\bar{s}} ,
\]

\[
\nabla_i = D_i - \frac{1}{4} \partial_i (\ln \Delta_{(p\bar{q})}) \Gamma_{i\mu} \Gamma_\mu + \frac{1}{8 \Delta_{(p\bar{q})}} \sigma_3 \otimes \Gamma_{\mu} \gamma_{pq} h_{\mu\bar{\nu}}
\]

\[
+ 4 e^{\phi} \left[ i \sigma_2 \otimes \Delta_{(p\bar{q})} \Gamma_{i\mu} f_{\mu\bar{\nu}}^{(1)} - (1 - \Delta_{(p\bar{q})}) \frac{\Delta_{(p\bar{q})}}{3!} \Gamma_{i\mu\bar{\nu}} f_{\mu\bar{\nu}}^{(3)} \right]
\]

\[
\chi = \text{det} j_{ij}, \quad \Gamma = \partial_i X^\mu \Gamma_{i\mu},
\]

\[
\gamma_{ij} = \partial_i X^\mu \partial_j X^\nu \gamma_{\mu\nu}, \quad B_{ij}^{(pq)} = \partial_i X^\mu \partial_j X^\nu B_{\mu\nu}^{(pq)}, \quad h_{\mu\bar{\nu}}^{(pq)} = \partial_i X^\mu h_{\mu\bar{\nu}}^{(pq)}
\]

\(\sigma_{\{1,2,3\}}\) are Pauli matrices and

\[
\vartheta = \left( \vartheta_1 \vartheta_2 \right), \quad \theta = \left( \theta_1 \theta_2 \right).
\]

4.3. Fermionic symmetry

First, we show that the \((p,q)\)-string action (4.44) has the \(\kappa\)-symmetry which is really inherited from the \(\kappa\)-symmetry in the supermembrane (2.5). In fact, we shall
analyse the $\kappa$-symmetry of (4·1) along the argument in Ref. 18). Since (4·1) is equivalent to (3·28) when $\epsilon^{ij}\partial_i\Xi_j = 0$ and (3·28) is invariant under the $\kappa$-transformation (3·33), the variation of (4·1) under the $\kappa$-transformation should be proportional to $\epsilon^{ij}\partial_i\Xi_j$,

$$\delta_\kappa S_{ddr} = -2\pi T \int d^2 \sigma C_{pq} \epsilon^{ij}\partial_i\Xi_j (\delta_\kappa X^\mu \Delta_\mu + i\bar{\theta} \Gamma_y \Gamma^{10}\delta_\kappa \theta - \delta_\kappa X^9)$$

$$= 2\pi T \int d^2 \sigma C_{pq} \epsilon^{ij}\partial_i\Xi_j \{i\bar{\theta}(\Gamma^\mu \Delta_\mu - \Gamma_y \Gamma^{10})(1 + \Gamma_F)\kappa + \delta_\kappa X^9\},$$

(4·51)

where

$$\theta = \theta_+ + \theta_-, \quad \kappa = \kappa_+ + \kappa_-.$$  

(4·52)

Thus, $X^9$ should be transformed as

$$\delta_\kappa X^9 = -i\bar{\theta}(\Gamma^\mu \Delta_\mu - \Gamma_y \Gamma^{10})(1 + \Gamma_F)\kappa \equiv -i\bar{\theta} M(1 + \Gamma_F)\kappa.$$  

(4·53)

We shall rewrite Eq. (4·53) with the dual variables, or the $(p, q)$-string fields and the supergravity background fields. Since the spinors are converted as in Eq. (4·28), we shall write in the matrix form (4·50). Then, we have$^*$ (cf. Appendix F)

$$\delta_\kappa X^9 = -i\Delta^{1/2}_{(j\bar{q})} \bar{\theta} \left( \begin{array}{cc} -\Omega_\chi M(1 + \Gamma_F)\Omega_\chi & -\Omega_\chi M(1 + \Gamma_F) \\ M(1 + \Gamma_F)\Omega_\chi & M(1 + \Gamma_F) \end{array} \right) \kappa$$

$$= -i\bar{\theta} \Gamma^{9}_{j\bar{q}} \left( \begin{array}{cc} 1 - \Omega_\chi \Gamma_F \Omega_\chi & 0 \\ 0 & 1 + \Gamma_F \end{array} \right) \kappa' \equiv -i\bar{\theta} \Gamma^{9}_{j\bar{q}}(1 - \Gamma_{IB})\kappa',$$

(4·55)

where

$$\theta = \left( \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right), \quad \kappa = \left( \begin{array}{c} \kappa_1 \\ \kappa_2 \end{array} \right), \quad \kappa' = \frac{\Gamma_-}{(1 - \delta_{j\bar{q}}^{-1} \Omega_\chi \Gamma_F \Omega_\chi)} \left( \begin{array}{c} (1 - \epsilon_{j\bar{q}} \Omega_\chi \Gamma'_F) \kappa_1 \\ (1 + \epsilon_{j\bar{q}} \Omega_\chi \Gamma'_F) \kappa_2 \end{array} \right),$$

(4·56)

$$\Gamma_B = \frac{1}{2} \frac{1}{\sqrt{-j}} \epsilon^{ij}\Gamma^{10}_{ij}, \quad \Gamma_{IB} = \sigma_3 \otimes \Gamma_B,$$

(4·57)

and we have used the following relations,

$$M(\Gamma^\mu \Delta_\mu - \Gamma_y \Gamma^{10}) = \Delta_{(j\bar{q})}^{-1/2}(\Gamma^{9}_{j\bar{q}} + 2\delta_{j\bar{q}}^{-1/2} \Omega_\chi \Gamma_+),$$

(4·58)

$$\Omega_\chi \Gamma^{9}_{j\bar{q}} = \Gamma^{9}_{j\bar{q}} \Omega_\chi + 2\delta_{j\bar{q}}^{-1/2} \Gamma^{10},$$

(4·59)

$^*$ In rewriting the relations with the dual variables we have used the relation (cf. (4·2))

$$\partial_i X^y \sim \frac{\hat{\mathcal{G}}^{ij}}{\mathcal{G}^{ij}_{yy}} \partial_i X^y - \partial_i \mathcal{X}^9 - \frac{\hat{\mathcal{G}}^{ij}_{yy}}{\mathcal{G}^{ij}_{yy}} \partial_i X^9.$$  

(4·54)
\[(1 + \Gamma_F) \Gamma_- = (1 + \Gamma_B) \frac{1 + \epsilon^{ij} j_{ij} \gamma \Omega_\chi}{\sqrt{-j_{ij} j_{ij}}} \Gamma_- , \quad (4.60)\]

\[(1 - \Omega_\chi \Gamma_F \Omega_\chi) \Gamma_- = (1 - \Gamma_B) \frac{1 - \epsilon^{mn} j_{mn} \gamma \Omega_\chi}{\sqrt{-j_{mn} j_{mn}}} \Gamma_- . \quad (4.61)\]

We also calculate $\delta_{\kappa} X^\mu (= \delta_{\kappa} X^\mu)$ to get

\[\delta_{\kappa} X^\mu = -i(\bar{\theta}_+ + \bar{\theta}_-) \Gamma^\mu (1 + \Gamma_F)(\kappa_+ + \kappa_-) = -i \Delta^{1/2} (\bar{\theta}_1 \bar{\theta}_2) \left( -\Omega_\chi \Gamma^\mu (1 + \Gamma_F) \Omega_\chi \Omega_\chi \Gamma^\mu (1 + \Gamma_F) \Gamma^\mu (1 + \Gamma_F) \right) \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = -i \bar{\theta} \Gamma^\mu (1 - \Gamma_{\text{IB}}) \kappa', \quad (4.62)\]

where the following relations have been used,

\[\Omega_\chi \Gamma^\mu = \Gamma^\mu \Omega_\chi , \quad \Gamma^\mu = \Delta^{-1/2} \Gamma^\mu . \quad (4.63)\]

The $\kappa$-transformations of the fermionic coordinates are given by

\[\delta_{\kappa} \theta_1 = \delta_{\kappa} \left( \Delta^{-1/4} \Omega_\chi \theta_+ \right) = (1 - \Omega_\chi \Gamma_F \Omega_\chi) \kappa_1 + O(\theta^2) , \]

\[\delta_{\kappa} \theta_2 = \delta_{\kappa} \left( \Delta^{-1/4} \theta_- \right) = (1 + \Gamma_F) \kappa_2 - \frac{1}{4} \delta_{\kappa} \left( \ln \Delta_{(\bar{p} \bar{q})} \right) \theta_2 = (1 + \Gamma_F) \kappa_2 + O(\theta^2) , \quad (4.64)\]

and hence we arrive at the conclusion that the $(p,q)$-string action (4.44) is invariant under the $\kappa$-transformation, which is really inherited from that of the supermembrane

\[\delta_{\kappa} \theta = (1 - \Gamma_{\text{IB}}) \kappa' , \quad \delta_{\kappa} X^\mu = -i \bar{\theta} \Gamma^\mu (1 - \Gamma_{\text{IB}}) \kappa' , \quad \delta_{\kappa} \Phi_{bg} = \delta_{\kappa} X^\mu \partial_\mu \Phi_{bg} . \quad (4.65)\]

Next, we consider the SUSY transformation. Similarly, the variation of (4.1) under the SUSY transformation should be proportional to $\epsilon^{ij} \partial_i \Xi_j$ and it is calculated as (cf. Eq. (3.35))

\[\delta_{\epsilon} S_{ddr} = -2\pi T \int d^2 \sigma C_{pq} \epsilon^{ij} \partial_i \Xi_j (\delta_{\epsilon} X^\mu \bar{\Delta}_{\mu y} + i \bar{\theta} \Gamma^\mu \Gamma_{10} \delta_{\epsilon} \theta - \delta_{\epsilon} X^9) \]

\[= 2\pi T \int d^2 \sigma C_{pq} \epsilon^{ij} \partial_i \Xi_j \left\{ i \bar{\theta} (\Gamma^\mu \bar{\Delta}_{\mu y} - \Gamma_y \Gamma_{10}) \epsilon + \delta_{\epsilon} X^9 \right\} , \quad (4.66)\]

and hence

\[\delta_{\epsilon} X^9 = -i(\bar{\theta}_+ + \bar{\theta}_-) M(\epsilon_+ + \epsilon_-) \]

\[= -i \left( \bar{\theta}_1 \bar{\theta}_2 \right) \left( -\Omega_\chi \left\{ \Gamma^9 + 2 j_{99}^{-1/2} \Omega_\chi \Gamma_+ \right\} \Omega_\chi \begin{pmatrix} 0 \\ \Gamma^9 + 2 j_{99}^{-1/2} \Omega_\chi \Gamma_+ \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \right) \]

\[= -i \bar{\theta} \Gamma^9 \epsilon = i \bar{\epsilon} \Gamma^9 \theta . \quad (4.67)\]

We also have

\[\delta_{\epsilon} X^\mu = i(\bar{\epsilon}_+ + \bar{\epsilon}_-) \Gamma^\mu (\theta_+ + \theta_-) = i \begin{pmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \end{pmatrix} \bar{\epsilon} \Gamma^\mu \left( \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \right) . \quad (4.68)\]
Thus, the \((p, q)\)-string action (4.44) is invariant under the following SUSY transformation, which is inherited from the superdiffeomorphism in the supermembrane action,

\[
\delta_\epsilon \theta = \epsilon, \quad \delta_\epsilon X^{\hat{\mu}} = i\epsilon \Gamma^{\hat{\mu}} \theta, \quad \delta_\epsilon \Phi_{bg} = \delta_\epsilon X^{\hat{\mu}} \partial_{\hat{\mu}} \Phi_{bg}.
\] (4.69)

§5. Summary and discussion

In this paper we have explicitly derived the \((p, q)\)-string action of the Green-Schwarz type from the supermembrane action up to quadratic order in the anti-commuting supercoordinate in the bosonic curved background. We have also shown that both the \(\kappa\)-symmetry and the supersymmetry in the \((p, q)\)-string action are really inherited from the \(\kappa\)-symmetry and the (super) diffeomorphism in the supermembrane action, respectively. In fact, we have first studied the double dimensional reduction of the wrapped supermembrane compactified on a 2-torus up to quadratic order of the anti-commuting coordinate.\(^1\) Next, we applied the T-dual transformation and explicitly derived the type IIB Green-Schwarz superstring action for the \((p, q)\)-string in Eq. (4.44).\(^{**}\) This indicates that the supermembrane actually includes a \((p, q)\)-string as an excitation mode or object. The \((1,0)\)-string (F-string) is, of course, a fundamental mode in the weak coupling region \(g_{s\text{IIB}} \ll 1\), while the \((0,1)\)-string (D-string) in the strong coupling region \(g_{s\text{IIB}} \gg 1\) for \(l = 0\). However, the valid region to treat the \((p, q)\)-string perturbatively is still obscure and it is deserved to be investigated further.\(^{***}\)

In this paper we have considered classically to approach the boundary of vanishing cycles of the 2-torus with the wrapped supermembrane. On the other hand, Refs. 23) and 24) studied quantum mechanical justification of the double dimensional reduction in Ref. 2). In those references, the Kaluza-Klein modes associated with the \(\sigma^2\)-coordinate were not removed classically, but they were integrated in the path integral formulation of the wrapped supermembrane theory. Similar quantum mechanical investigation of the double dimensional reduction adopted in this paper deserves to be investigated.

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\(^{1}\) The procedure of the double dimensional reduction here was realized on the bosonic sector of the matrix-regularized wrapped supermembrane on \(\mathbb{R}^9 \times T^{210}\) relying on the technique given in Refs. 20)–22).

\(^{**}\) In the case of \((p, q) = (1, 0)\), or the fundamental string, the resulting action is reduced to the one in Ref. 6).

\(^{***}\) Of course, a BPS saturated classical solution of the \((p, q)\)-string action (4.44) is valid irrespective of the value of the string coupling \(g_{s\text{IIB}}\).
Appendix A

Notation

11d super spacetime indices:

\[ \hat{M} = (M, \alpha), \]
\[ M, N, P, Q = 0, 1, \cdots, 8, 9, 10, \]
\[ \alpha, \beta, \gamma = 1, 2, \cdots, 32. \] (A.1)

11d tangent superspace indices:

\[ \hat{A} = (A, a), \]
\[ A, B, C = 0, 1, \cdots, 8, 9, 10, \]
\[ a, b, c = 1, 2, \cdots, 32. \] (A.2)

11d rotated spacetime indices:

\[ \tilde{M}, \tilde{N}, \tilde{P}, \tilde{Q} = 0, 1, \cdots, 8, y, z. \] (A.3)

10d spacetime and tangent space indices:

\[ \hat{\mu}, \hat{\nu} = 0, 1, \cdots, 8, 9, \]
\[ \hat{r}, \hat{s} = 0, 1, \cdots, 8, 9. \] (A.4)

10d rotated super spacetime indices:

\[ \tilde{m} = (\tilde{\mu}, \alpha), \]
\[ \tilde{\mu}, \tilde{\nu} = 0, 1, \cdots, 8, y. \] (A.5)

9d spacetime and tangent space indices:

\[ \mu, \nu = 0, 1, \cdots 8, \]
\[ r, s = 0, 1, \cdots 8. \] (A.6)

The worldvolume and worldsheet indices:

\[ \hat{i}, \hat{j}, \hat{k} = 0, 1, 2, \]
\[ i, j, k = 0, 1. \] (A.7)

Target space metrics:

\[ G = \text{11d target space metric}, \]
\[ \tilde{G} = \text{11d rotated target space metric}, \]
\[ g = \text{10d IIA target space metric}, \]
\[ \tilde{g} = \text{10d IIA rotated target space metric}, \]
\[ j = \text{10d IIB target space metric}. \] (A.8)
Worldvolume and worldsheet metrics:

\[ \hat{\gamma} = \text{membrane worldvolume metric}, \]
\[ \gamma = \text{string worldsheet metric}. \]

(A.9)

(Anti-)symmetrization w.r.t. indices:

\[ A_{[\mu B \nu]} = \frac{1}{2} (A_\mu B_\nu - A_\nu B_\mu), \]
\[ A_{[\mu B \nu C \rho]} = \frac{1}{3!} (A_\mu B_\nu C_\rho + A_\nu B_\rho C_\mu + A_\rho B_\mu C_\nu - A_\mu B_\rho C_\nu - A_\rho B_\nu C_\mu - A_\nu B_\mu C_\rho), \]
\[ A_{[\mu B \nu | C \rho]} = \frac{1}{2} (A_\mu B_\nu C_\rho - A_\rho B_\nu C_\mu), \]
\[ A_{(\mu B \nu)} = \frac{1}{2} (A_\mu B_\nu + A_\nu B_\mu), \quad \text{etc.} \]

(A.10)

Appendix B

Bispinor Formula

The charge conjugate matrix \( C \) satisfies

\[ C \Gamma_A C^{-1} = -t \Gamma_A, \quad t C = -C, \]

or

\[ C_{\alpha\gamma}(\Gamma_A)_{\gamma\tau}C_{\tau\beta} = -(\Gamma_A)_{\beta\alpha}, \quad C_{\alpha\beta} = -C_{\beta\alpha}. \]

(B.1)

Equation (B.1) leads to

\[ \bar{\theta} \Gamma_{A_1 A_2 \cdots A_n} \psi = (-1)^{n(n+1)/2} \bar{\psi} \Gamma_{A_1 A_2 \cdots A_n} \theta, \]

where \( \theta \) and \( \psi \) are 32-component Majorana spinor in eleven dimensions and

\[ \bar{\theta} = i t \theta C = i t \theta \Gamma^0. \quad (\bar{\theta}_\alpha = i \theta^\beta C_{\beta\alpha}) \]

(B.3)

Note that for \( n = 0, 3, 4, 7, 8 \) and \( n = 1, 2, 5, 6, 9, 10 \), the bispinor products Eq. (B.3) are symmetric and antisymmetric, respectively and

\[ (\Gamma_{A_1 A_2 \cdots A_n})_{\alpha\beta} = \begin{cases} -(\Gamma_{A_1 A_2 \cdots A_n})_{\beta\alpha}, & (n = 0, 3, 4, 7, 8) \\ (\Gamma_{A_1 A_2 \cdots A_n})_{\beta\alpha}, & (n = 1, 2, 5, 6, 9, 10) \end{cases} \]

(B.5)

where the spinor indices are lowered and raised by \( C \)

\[ M_{\alpha\beta} = C_{\alpha\gamma} M_{\gamma\beta}, \]
\[ M^{\alpha\beta} = M^{\alpha\gamma} C_{\gamma\beta}, \]
\[ M_{\alpha}^{\beta} = C_{\alpha\gamma} M_{\gamma}^{\delta} C^{\delta\beta}. \]

(B.6)

(B.7)

(B.8)

In particular we have

\[ (\Gamma^A)_{\alpha\beta} = (\Gamma^A)_{\beta\alpha}. \]

(B.9)
Putting $\theta = \psi$ in Eq. (B.3) we have the identity
\[ \bar{\theta}\Gamma^{A_1A_2\cdots A_n}\theta = 0, \quad (n = 1, 2, 5, 6, 9, 10) \] (B.10)
For the chiral-projected spinors $\psi_\pm, \theta_\pm$ (cf. Eq. (3.49)), Eq. (B.1) leads to
\[ \bar{\psi}_\pm\Gamma^{A_1A_2\cdots A_n}\theta_\pm = 0, \quad (n = 2N) \]
\[ \bar{\psi}_\pm\Gamma^{A_1A_2\cdots A_n}\theta_\mp = 0, \quad (n = 2N + 1) \] (B.11)
and from Eqs. (B.3) and (B.11) we have
\[
\bar{\theta}+\Gamma^{A_1A_2\cdots A_n}\psi_+ = \begin{cases} \bar{\psi}_-\Gamma^{A_1A_2\cdots A_n}\theta_+, & (n = 0, 4, 8) \\ -\bar{\psi}_-\Gamma^{A_1A_2\cdots A_n}\theta_+, & (n = 2, 6, 10) \end{cases} \\
\bar{\theta}+\Gamma^{A_1A_2\cdots A_n}\psi_- = \begin{cases} -\bar{\psi}_+\Gamma^{A_1A_2\cdots A_n}\theta_+, & (n = 1, 5, 9) \\ \bar{\psi}_+\Gamma^{A_1A_2\cdots A_n}\theta_+. & (n = 3, 7) \end{cases} 
\] (B.12)

We also write down some useful formulas for the $\Gamma$ matrices
\[
\Gamma_\hat{\varphi}\Gamma^{\hat{\varphi}_1\hat{\varphi}_2\cdots\hat{\varphi}_n} = \Gamma_\hat{\varphi}^{\hat{\varphi}_1\cdots\hat{\varphi}_n} + \sum_i (-1)^{i-1}\delta_{\hat{\varphi}}^{\hat{\varphi}_i}\Gamma_\hat{\varphi}^{\hat{\varphi}_1\cdots\hat{\varphi}_{i-1}\hat{\varphi}_i\cdots\hat{\varphi}_n}, \\
\Gamma_\hat{\varphi}_1\Gamma_\hat{\varphi}_2\cdots\hat{\varphi}_n = \Gamma_\hat{\varphi}_1\hat{\varphi}_2\cdots\hat{\varphi}_n + 2\sum_i (-1)^i\delta_{\hat{\varphi}}^{\hat{\varphi}_i}\Gamma_\hat{\varphi}_1\cdots\hat{\varphi}_{i-1}\hat{\varphi}_i\cdots\hat{\varphi}_n \\
+ 2\sum_{i<j} (-1)^{i+j}\delta_{\hat{\varphi}}^{\hat{\varphi}_i}\delta_{\hat{\varphi}}^{\hat{\varphi}_j}\Gamma_\hat{\varphi}_1\cdots\hat{\varphi}_{i-1}\hat{\varphi}_i\hat{\varphi}_j\cdots\hat{\varphi}_n. 
\] (B.13)

### Appendix C

**11d vs 10d Background Fields**

The 11-dimensional metric can be written by
\[
G_{MN} = e^{-\frac{2}{3}\phi} \begin{pmatrix} g_{\hat{\mu}\hat{\nu}} + e^{2\phi} A_{\hat{\mu}}A_{\hat{\nu}} & e^{2\phi} A_{\hat{\mu}} \\ e^{2\phi} A_{\hat{\nu}} & e^{2\phi} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{G_{1010}}} g_{\hat{\mu}\hat{\nu}} + \frac{1}{G_{1010}} G_{\hat{\mu}_{10}} G_{\hat{\nu}_{10}} G_{1010} & G_{\hat{\mu}_{10}} \\ G_{\hat{\nu}_{10}} & G_{1010} \end{pmatrix}, \tag{C.1}
\]
and the third-rank and forth-rank antisymmetric tensors $\hat{A}_{MNP}, \hat{F}_{MNPQ}$ are decomposed as
\[
\{\hat{A}_{MNP}\} = \{\hat{A}_{\mu\nu\rho}, \hat{A}_{\mu\nu10}, \hat{A}_{\mu\nu9}, \hat{A}_{\mu910}\} = \{C_{\mu\nu\rho}, B_{\mu\nu}, C_{\mu9}, B_{\mu9}\}, \tag{C.2}
\]
\[
\{\hat{F}_{MNPQ}\} = \{\hat{F}_{\mu\nu\rho\sigma}, \hat{F}_{\mu\nu\rho10}\} = \{F_{\mu\nu\rho\sigma}, H_{\mu\nu\rho}\} = \{4\partial_{[\mu}C_{\nu\rho\sigma]}, 3\partial_{\mu}B_{\nu\rho}\}. \tag{C.3}
\]
Those fields are related to those in type IIB theory as
\[
g_{\mu\nu} = J_{\mu\nu} - \frac{g^{(1)}_{\mu\nu} - B_{\mu9}^{(1)} B_{\nu9}^{(1)}}{J_{99}}, \tag{C.4}
\]
\[ g_{9\mu} = \frac{B_{9\mu}^{(1)}}{g_{99}}, \quad (C.5) \]
\[ g_{99} = \frac{1}{g_{99}}, \quad (C.6) \]
\[ C_{\mu \nu 9} = B_{\mu \nu}^{(2)} + \frac{2B_{9[\mu\nu]9}}{g_{99}}, \quad (C.7) \]
\[ C_{\mu \nu \rho} = D_{9\mu \nu \rho} + \frac{3}{2} \left\{ \left( B_{9[\mu\nu]}^{(1)} B_{\rho9}^{(2)} + \frac{2B_{9[\mu\nu]}^{(1)} B_{[\rho9]}^{(2)}}{g_{99}} \right) - (1 \leftrightarrow 2) \right\}, \quad (C.8) \]
\[ B_{\mu \nu} = B_{\mu \nu}^{(1)} + \frac{2B_{9[\mu\nu]9}}{g_{99}}, \quad (C.9) \]
\[ B_{9\mu} = \frac{g_{9\mu}}{g_{99}}, \quad (C.10) \]
\[ A_\mu = -B_{9\mu}^{(2)} + lB_{9\mu}^{(1)}, \quad (C.11) \]
\[ A_9 = l, \quad (C.12) \]
\[ \phi = \varphi - \frac{1}{2} \ln g_{99}. \quad (C.13) \]

C.1. Metric in the rotated coordinate

On the other hand, the 9–10 rotated metric is given by \((\tilde{M}, \tilde{N} = 0, 1, \cdots, 8, y, z)\)

\[
\tilde{G}_{\tilde{M}\tilde{N}} = G_{MN} \frac{\partial X^M}{\partial \tilde{X}^\tilde{M}} \frac{\partial X^N}{\partial \tilde{X}^\tilde{N}}
\]
\[
= \begin{pmatrix}
\frac{1}{\sqrt{G_{zz}}} \tilde{g}_{\mu\nu} + \frac{1}{G_{zz}} \tilde{G}_{\mu z} \tilde{G}_{\nu z} & \frac{1}{\sqrt{G_{zz}}} \tilde{g}_{\mu y} + \frac{1}{G_{zz}} \tilde{G}_{\mu z} \tilde{G}_{y z} & \tilde{G}_{\mu z} \\
\frac{1}{\sqrt{G_{zz}}} \tilde{g}_{y \nu} + \frac{1}{G_{zz}} \tilde{G}_{y z} \tilde{G}_{\nu z} & \frac{1}{\sqrt{G_{zz}}} \tilde{g}_{y y} + \frac{1}{G_{zz}} \tilde{G}_{y z} \tilde{G}_{y z} & \tilde{G}_{y z} \\
\tilde{G}_{\nu z} & \tilde{G}_{y z} & \tilde{G}_{zz}
\end{pmatrix} \quad (C.14)
\]

Thus, we have

\[
\tilde{G}_{zz} = \tilde{q}^2 G_{99} + 2\tilde{p}\tilde{q} G_{910} + \tilde{p}^2 G_{1010} = e^{4\varphi/3} j_{99}^{-2/3} \{ (\hat{p} + \hat{q})^2 + e^{-2\varphi} \tilde{q}^2 \}, \quad (C.15)
\]
\[
\tilde{G}_{yy} = e^{4\varphi/3} j_{99}^{-2/3} \{ (\hat{q} - \hat{p})^2 + \tilde{p}^2 e^{-2\varphi} \}, \quad (C.16)
\]
\[
\tilde{G}_{yz} = e^{4\varphi/3} j_{99}^{-2/3} \{ (\hat{p} - \hat{q})(\hat{q} + \hat{p}) + \hat{q}\hat{p} e^{-2\varphi} \}. \quad (C.17)
\]

Furthermore,

\[
\tilde{G}_{\mu y} = \hat{p} G_{\mu 9} - \hat{q} G_{\mu 10} = \frac{1}{\sqrt{G_{zz}}} \tilde{g}_{\mu y} + \frac{1}{G_{zz}} \tilde{G}_{\mu z} \tilde{G}_{y z}, \quad (C.18)
\]

and hence

\[
\tilde{g}_{\mu y} = \frac{1}{\sqrt{G_{zz}}} \left( \tilde{G}_{\mu y} \tilde{G}_{zz} - \tilde{G}_{\mu z} \tilde{G}_{y z} \right) = \frac{B_{9\mu}^{(pq)}}{g_{99}}, \quad (C.19)
\]
where, as they are given in (4.9) and (4.10),
\[
\Delta_{(\hat{p}\hat{q})} = \sqrt{(\hat{p} + \hat{q})^2 + e^{-2e^2\hat{q}^2}}, \quad B_{\mu\nu}^{(pq)} = \Delta_{(\hat{p}\hat{q})}^{-1}(\hat{p}B_{\mu\nu}^{(1)} + \hat{q}B_{\mu\nu}^{(2)}).
\]  
(C.20)
Similarly, we have
\[
\bar{g}_{yy} = \frac{1}{\Delta_{(\hat{p}\hat{q})}j_{99}}, \quad (C.21)
\]
\[
\bar{g}_{\mu\nu} = \Delta_{(\hat{p}\hat{q})}\left(j_{\mu\nu} - \frac{j_{9\mu}j_{9\nu}}{j_{99}} + \frac{B_{\mu\mu}^{(pq)}B_{\nu\nu}^{(pq)}}{j_{99}}\right). \quad (C.22)
\]
Note that
\[
\sqrt{\frac{\hat{G}_{zz}}{\bar{G}_{1010}}} = \Delta_{(\hat{p}\hat{q})}. \quad (C.23)
\]

Appendix D
IIA Action (3.50)
First, the \(\theta^0\)-order part of (3.28) is given by
\[
S_{ddr}|_{\theta^0} = 2\pi T \int d^2\sigma C_{pq} \left[ -\sqrt{-\bar{g}} - \frac{1}{2} \epsilon^{ij} \tilde{A}_{ji} \right]. \quad (D.1)
\]
Next, we shall calculate the \(\theta^2\)-order part of (3.28). Due to
\[
\tilde{\theta}_+ \Gamma_{\hat{r}} \Gamma_{\hat{u}\hat{v}} \Gamma_{\hat{s}} \theta_- = -\bar{\theta}_- \Gamma_{\hat{s}} \Gamma_{\hat{u}\hat{v}} \Gamma_{\hat{r}} \theta_+, \quad \tilde{\theta}_+ \Gamma_{\hat{r}} \Gamma_{\hat{u}_1\hat{u}_2\hat{u}_3\hat{u}_4} \Gamma_{\hat{s}} \theta_- = \bar{\theta}_- \Gamma_{\hat{s}} \Gamma_{\hat{u}_1\hat{u}_2\hat{u}_3\hat{u}_4} \Gamma_{\hat{r}} \theta_+, \quad (D.2)
\]
we have
\[
Q_{\tilde{r}s}^{(2)} = -\frac{i}{4 \cdot n!} \epsilon^{\tilde{r} \tilde{s}} \bar{\theta} \Gamma_{\tilde{r}} \Gamma_{\tilde{u}\tilde{v}} \Gamma_{\tilde{s}} \Gamma_{\tilde{r}}^{10} \theta \bar{F}_{\tilde{u}\tilde{v}},
\]
\[
Q_{\tilde{r}s}^{(4)} = \frac{i}{4 \cdot n!} \epsilon^{\tilde{r} \tilde{s}} \bar{\theta} \Gamma_{\tilde{r}} \Gamma_{\tilde{u}_1\tilde{u}_2\tilde{u}_3\tilde{u}_4} \Gamma_{\tilde{s}} \Gamma_{\tilde{r}}^{10} \theta \bar{F}_{\tilde{u}_1\tilde{u}_2\tilde{u}_3\tilde{u}_4},
\]
\[
P_{\tilde{r}s}^{(2)} = \frac{i}{4 \cdot n!} \epsilon^{\tilde{r} \tilde{s}} \bar{\theta} \Gamma_{\tilde{r}} \Gamma_{\tilde{u}\tilde{v}} \Gamma_{\tilde{s}} \Gamma_{\tilde{r}} \Gamma_{\tilde{r}_1\tilde{r}_4} \theta \bar{F}_{\tilde{u}\tilde{v}},
\]
\[
P_{\tilde{r}s}^{(4)} = -\frac{i}{4 \cdot n!} \epsilon^{\tilde{r} \tilde{s}} \bar{\theta} \Gamma_{\tilde{r}} \Gamma_{\tilde{u}_1\tilde{u}_2\tilde{u}_3\tilde{u}_4} \Gamma_{\tilde{s}} \Gamma_{\tilde{r}_1\tilde{r}_4} \Gamma_{\tilde{r}}^{10} \theta \bar{F}_{\tilde{r}_1\tilde{r}_4}. \quad (D.3)
\]
Then, the \(\theta^2\)-order part of (3.28) is calculated as follows:
\[
S_{ddr}|_{\theta^2} = \frac{2\pi T}{2} \int d^2\sigma C_{pq} \left[ -\sqrt{-\bar{g}} \bar{g}^{ij}(\bar{Q}_{ij} + 2i\bar{\theta}\Gamma_{ij}\partial^j\theta) - \epsilon^{ij}(-\bar{P}_{ij} - 2i\bar{\theta}\Gamma_{ij}\Gamma_{10}\partial^j\theta) \right]
\]
\[
= \frac{2\pi T}{2} \int d^2\sigma C_{pq} \left[ -\sqrt{-\bar{g}} \bar{g}^{ij}\left\{2i\bar{\theta}\Gamma_{ij}\partial^j\theta + \frac{i}{2} \bar{\theta} \Gamma_{ij} \Gamma_{\tilde{r}s} \theta \bar{\omega}_j \tilde{s} + \frac{i}{8} \epsilon^{\tilde{r} \tilde{s}} \bar{\theta} \Gamma_{\tilde{r}} \Gamma_{\tilde{s}} \Gamma_{\tilde{r}_1\tilde{r}_2} \Gamma_{\tilde{s}_1\tilde{s}_2} \Gamma_{\tilde{r}_3\tilde{r}_4} \Gamma_{\tilde{s}_3\tilde{s}_4} \Gamma_{\tilde{r}}^{10} \theta \bar{F}_{\tilde{r}_1\tilde{r}_2\tilde{s}_1\tilde{s}_2} + \frac{i}{4} \bar{\theta} \Gamma_{\tilde{r}} \Gamma_{\tilde{s}} \Gamma_{\tilde{r}_1\tilde{r}_2\tilde{s}_1\tilde{s}_2} \Gamma_{\tilde{r}}^{10} \theta \bar{H}_{\tilde{r}_1\tilde{r}_2\tilde{s}_1\tilde{s}_2} \right\} \right]
\]
Then, the \( \theta \)-order part of the action (4.41) is calculated in the matrix form (cf. (4.44)) as follows:

\[
S_{ddr}|_{\theta^2} = -2\pi i T \int d^2 \sigma \ C_{pq} \left[\tilde{\theta} \sigma_3 \otimes \sqrt{-g} J^{ij} + \sigma_3 \otimes \epsilon^{ij} \right] \tilde{F}_{ij} \theta
+ \frac{1}{2} \left\{ \frac{1}{4} \tilde{\theta} (\sigma_3 \otimes \sqrt{-g} J^{ij} + \epsilon^{ij}) \tilde{\Gamma}^{(pq)}_{\mu \nu} \theta \bar{F}_{\mu \nu}^{(pq)}
+ \frac{e^{\phi(1)}}{4} \tilde{\theta} (i \sigma_2 \otimes \sqrt{-g} J^{ij} + \sigma_1 \otimes \epsilon^{ij}) \tilde{\Gamma}^{\mu_1 \mu_2 \mu_3}_{\mu \nu} \theta f_{\mu_1 \mu_2 \mu_3}^{(1)}
+ \frac{e^{\phi(3)}}{4 \cdot 3!} \tilde{\theta} (i \sigma_2 \otimes \sqrt{-g} J^{ij} + \sigma_1 \otimes \epsilon^{ij}) \tilde{\Gamma}^{\mu_1 \mu_2 \mu_3}_{\mu \nu} \theta f_{\mu_1 \mu_2 \mu_3}^{(3)}
+ \frac{e^{\phi(5)}}{8 \cdot 5!} \tilde{\theta} (i \sigma_2 \otimes \sqrt{-g} J^{ij} + \sigma_1 \otimes \epsilon^{ij}) \tilde{\Gamma}^{\mu_1 \ldots \mu_5}_{\mu \nu} \theta f_{\mu_1 \ldots \mu_5}^{(5)}
- \frac{1}{2} \epsilon^{ij} \tilde{\theta} \sigma_3 \tilde{\Gamma}^{\mu \nu \rho} \theta \bar{F}_{\mu \nu \rho} \Delta (pq) \right\}
\]

\[
= -2\pi i T \int d^2 \sigma \ C_{pq} \left[\tilde{\theta} \sigma_3 \otimes \sqrt{-g} J^{ij} + \sigma_3 \otimes \epsilon^{ij} \right] \tilde{D}_{ij} \theta
+ \frac{1}{2} \left\{ \frac{1}{4} \tilde{\theta} (\sqrt{-g} J^{ij} + \sigma_3 \otimes \epsilon^{ij}) (\sigma_3 \otimes \tilde{\Gamma}^{\rho\sigma}) \theta h_{ij}^{(pq)} \right\}
\]
Note that

\[ \lambda \equiv \frac{\theta}{\sqrt{-g}} \Gamma_j \Gamma^{10} \kappa, \quad \gamma^i_{\pm j} \equiv \frac{1}{2} \left( \delta^i_j \pm \frac{\epsilon^{ik} \bar{g}_{kj}}{\sqrt{-g}} \right). \] (F.1)

Note that \( \gamma^i_{\pm j} \) satisfies

\[ \gamma^i_{\pm j} \gamma^j_{\mp k} = \gamma^i_{\pm k}, \quad \gamma^i_{\pm j} \gamma^j_{\mp k} = 0, \quad \gamma^i_{\pm j} + \gamma^i_{-j} = \delta^i_j, \] (F.2)

and also

\[ \epsilon^{ij} \gamma^l_{\pm i} \gamma^l_{\pm j} = 0, \quad \epsilon^{ij} \gamma^k_{\pm i} \gamma^l_{\mp j} = \epsilon^{jk} \Gamma_{kl}, \quad \epsilon^{ij} \gamma^k_{\pm i} \gamma^l_{\pm j} = \epsilon^{k \mp j} \gamma^l_{\pm i}. \] (F.3)

Then, we have

\[ \Gamma_i (\gamma^i_{+j} \Gamma_- + \gamma^i_{-j} \Gamma_+) \lambda^j = (1 + \Gamma_F) \kappa, \] (F.4)
or

\[ \Gamma_i \gamma_{i+}^i \lambda_{i+}^i = (1 + \Gamma_F) \kappa_+ \quad (\lambda_{i+}^i \equiv \Gamma_+ \lambda_i) \]  

(F.5)

We should note that the projection \( \gamma_{i+}^i \) can be written, up to the equations of motion of the auxiliary variables, by

\[ \gamma_{i+}^i = \frac{1}{2} \left( \delta_{ij} + \frac{\epsilon_{ik} \gamma_{kj}}{\sqrt{-g}} \right) = \frac{1}{2} \left( \delta_{ij} \pm \frac{\epsilon_{ik} \gamma_{kj}}{\sqrt{-g}} \right). \]  

(F.6)

Since \( \Xi_i \) in Eq. (4.2) can be calculated as

\[ \Xi_i = -\gamma_{i+}^i \Delta_{(pq)} \left( \partial_\mu \hat{g}^{pq} + B_{i+}^{(pq)} \partial_\mu \hat{X}^{\dot{p} \dot{q}} \right) 
+ \gamma_{i-}^i \Delta_{(pq)} \left( \partial_\mu \hat{g}^{pq} - B_{i-}^{(pq)} \partial_\mu \hat{X}^{\dot{p} \dot{q}} \right) + O(\theta^2), \]  

we have

\[ \Gamma_i = \partial_\mu \hat{X}^{\dot{p} \dot{q}} \left( \gamma_{i+}^i \Gamma_{i+} - \gamma_{i-}^i \Omega_{i-} \Gamma_{i+} \right) \]  

(F.8)

where use has been made of \( \Xi_i \simeq \partial_i X^y \). And then,

\[ \Gamma_i (\gamma_{i+}^i \Gamma_{i+} + \gamma_{i-}^i \Gamma_{i-}) \lambda^i = \Delta_{(pq)}^{1/2} (\gamma_{i+}^i \Gamma_{i+} - \gamma_{i-}^i \Omega_{i-} \Gamma_{i+} \lambda^i). \]  

(F.9)

Due to (F.8) \( \lambda^i \) is rewritten by

\[ \lambda^i = \frac{\epsilon_{ij}}{\sqrt{-g}} \Delta_{(pq)}^{1/2} (\gamma_{i-}^k \Gamma_{i-} - \gamma_{i+}^k \Omega_{i+} \Gamma_{i-} \lambda^i) \Gamma_{10} \kappa, \]  

(F.10)

and hence

\[ \gamma_{i+}^i \lambda^i = \frac{\Delta_{(pq)}^{1/2} \epsilon_{ij}}{\sqrt{-g}} \gamma_{i+}^k \Gamma_{i+} \Gamma_{10} \kappa, \quad \gamma_{i-}^i \lambda^i = -\frac{\Delta_{(pq)}^{1/2} \epsilon_{ij}}{\sqrt{-g}} \gamma_{i-}^k \Omega_{i-} \Gamma_{i+} \Gamma_{10} \kappa. \]  

(F.11)

Now that we shall evaluate \( \epsilon_{ij} \Gamma_{ij} \). We have

\[ \epsilon_{ij} \Gamma_{ij} = \Delta_{(pq)} \epsilon_{ij} \Gamma_{ij} \left( 1 + \frac{2 \epsilon_{ij} \gamma_{i+}^k \gamma_{j+}^l \gamma_{ij}^9}{\sqrt{-g_{99}}} \left( \Gamma_{99} + j_{99}(1 + \Gamma_B \Gamma_{10}) \right) \right), \]  

(F.12)

where

\[ j_{99} \equiv \partial_\mu \hat{X}^{\dot{p} \dot{q}} j_{99}, \quad \Gamma_{99} \equiv \partial_\mu \hat{X}^{\dot{p} \dot{q}} \Gamma_{99}. \]  

(F.13)

Up to the equations of motion of the auxiliary variables, we also have

\[ \tilde{g}_{ij} = \Delta_{(pq)} \left( 1 - j^{kl} \partial_\mu \hat{X}^{\dot{p} \dot{q}} \partial_\nu \hat{X}^{\dot{p} \dot{q}} j_{99} \right) + O(\theta^2). \]  

(F.14)

Thus, \( \Gamma_F \) (3.34) is written by (cf. (F.12))

\[ \Gamma_F = \Gamma_B \mathcal{J}, \]  

(F.15)

where

\[ \mathcal{J} \equiv 1 + \frac{2 \epsilon_{ij} \gamma_{i+}^k \gamma_{j+}^l \gamma_{ij}^9}{\sqrt{-g_{99}}} \left\{ \Gamma_{99} + j_{99}(1 + \Gamma_B \Gamma_{10}) \right\} \]  

\[ 1 - j^{kl} j_{99}, \]  

(F.16)
Now that we shall calculate the projection,

\[ 1 + \Gamma_F = \frac{1 + \Gamma_B}{2} (1 + \mathcal{J}) + \frac{1 - \Gamma_B}{2} (1 - \mathcal{J}). \]  

(F.17)

First of all, we have

\[ 1 - \mathcal{J} = \frac{1 - \Gamma_B \Gamma^{10}}{2} \frac{2 \epsilon^{ij} \gamma^k_{-i} \gamma^l_{+j} j_{l9} \Gamma_{k} j_{9} \Gamma_{k}}{\sqrt{-j_{j99}(1 - j_{k9} j_{99})}} + O(\theta^2), \]  

(F.18)

where use has been made of (F.3) and

\[ \gamma^i_{-k}(1 + \Gamma_B \Gamma^{10}) L_i^k = \gamma^i_{+k}(1 - \Gamma_B \Gamma^{10}) L_i^k = 0. \]  

(F.19)

Thus we have

\[ (1 + \Gamma_B \Gamma^{10})(1 - \mathcal{J}) = 0, \]  

(F.20)

and (F.17) leads to

\[ (1 + \Gamma_F) \Gamma_- = \frac{1 + \Gamma_B}{2} (1 + \mathcal{J}) \Gamma_- . \]  

(F.21)

In addition, we have

\[ (1 - \Omega \chi \Gamma_F \Omega \chi) \Gamma_- = \{1 - \Omega \chi \Gamma_B \Gamma^{10} \Omega \chi - \Omega \chi (1 - \mathcal{J}) \Omega \chi \} \Gamma_- . \]  

(F.22)

Since we have

\[ \Omega \chi \Gamma_B \Gamma^{10} \Omega \chi = -\Gamma_B \Gamma^{10} + \frac{2 \epsilon^{ij} j_{j9}}{\sqrt{-j_{j99}}} \Gamma_{k} \Gamma^{10} \Omega \chi , \]  

(F.23)

\[ \Omega \chi (1 - \mathcal{J}) \Omega \chi = \left(1 - \Gamma_B \Gamma^{10} + \frac{2 \epsilon^{mn} j_{m9} j_{n9}}{\sqrt{-j_{j99}}} \Gamma_{m} \Gamma^{10} \Omega \chi \right) \frac{\epsilon^{ij} \gamma^k_{-i} \gamma^l_{+j} j_{l9} \Gamma_{k} \Gamma^{10} \Omega \chi}{\sqrt{-j_{j99}(1 - j_{k9} j_{99})}}, \]  

(F.24)

we obtain

\[ (1 - \Omega \chi \Gamma_F \Omega \chi) \Gamma_- = (1 - \Gamma_B) \frac{1 - \epsilon^{mn} j_{m9} \Gamma_{n9} \Omega \chi}{1 - j_{k9} j_{99}} \Gamma_- . \]  

(F.25)

where use has been made of (F.19) and

\[ \epsilon^{mn} j_{m9} \Gamma_{n9} \Gamma^{10} \Omega \chi \left(1 + \frac{\epsilon^{ij} \gamma^k_{-i} \gamma^l_{+j} j_{l9} \Gamma_{k} \Gamma^{10} \Omega \chi}{\sqrt{-j_{j99}(1 - j_{k9} j_{99})}} \right) \]

\[ = P_+ \frac{\epsilon^{mn} j_{m9} \Gamma_{n9} \Gamma^{10} \Omega \chi}{1 - j_{k9} j_{99}} + P_- \frac{\epsilon^{mn} j_{m9} \Gamma_{n9} \Gamma^{10} \Omega \chi}{1 - j_{k9} j_{99}}, \]  

(F.26)

where

\[ P_{\pm} \equiv \frac{1 \pm \Gamma_B \Gamma^{10}}{2}. \]  

(F.27)

That is, Eqs. (F.21) and (F.25) lead to Eq. (4.55).
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