ON COMPACTNESS CONDITIONS FOR THE p-LAPLACIAN

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Abstract. We investigate the geometry and validity of various compactness conditions (e.g. Palais-Smale condition) for the energy functional

$$J_{\lambda_1}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \lambda_1 \int_{\Omega} |u|^p \, dx - \int_{\Omega} fu \, dx$$

for $u \in W^{1,p}_0(\Omega), 1 < p < \infty$, where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $f \in L^\infty(\Omega)$ is a given function and $-\lambda_1 < 0$ is the first eigenvalue of the Dirichlet $p$-Laplacian $\Delta_p$ on $W^{1,p}_0(\Omega)$.

1. Introduction. In this article we follow and generalize the work of Drábek and Takáč [4]. We investigate the geometry and behavior of Palais-Smale sequences of the energy functional

$$J_{\lambda_1}(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \lambda_1 \int_{\Omega} |u|^p \, dx - \int_{\Omega} fu \, dx$$

on $W^{1,p}_0(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $-\lambda_1$ is the first eigenvalue of the Dirichlet $p$-Laplacian $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ on $W^{1,p}_0(\Omega)$ and $f \in L^\infty(\Omega)$ is a given function. This energy functional is related to the Dirichlet boundary value problem

$$\begin{cases}
-\Delta_p u - \lambda_1 |u|^{p-2} u = f & \text{in } \Omega; \\
u = 0 & \text{on } \partial\Omega.
\end{cases}$$

The first eigenvalue $\lambda_1$ of $-\Delta_p$ is given by

$$\inf \left\{ \int_{\Omega} |\nabla u|^p \, dx : u \in W^{1,p}_0(\Omega) \wedge \int_{\Omega} |u|^p \, dx = 1 \right\}.$$ 

The first eigenvalue $\lambda_1$ is simple and we will denote the associated eigenfunction by $\varphi_1$. This eigenfunction can be normalized by $\varphi_1 > 0$ in $\Omega$ and $\|\varphi_1\|_{L^p(\Omega)} = 1$, by Anane [1]. Moreover $\varphi_1 \in L^\infty(\Omega)$.

Motivated by the Fredholm alternative we investigate the situation when the right hand side function $f$ and $\varphi_1$ are orthogonal in the $L^2(\Omega)$ sense. To this end

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we define
\[ L^2(\Omega)^\perp = \{ u \in L^2(\Omega) : \int_\Omega u \varphi_1 \, dx = 0 \} , \]
\[ M^1,L^2 := \{ u \in L^2(\Omega) : \int_\Omega uv \, dx = 0 \forall v \in M \} . \]

It follows that \( L^2(\Omega) = \text{Lin}\{ \varphi_1 \} \oplus L^2(\Omega)^\perp \). Similarly we can adapt this decomposition to \( W^{1,p}_0(\Omega) \) for \( p > 2 \) thanks to \( \varphi_1 \in W^{1,p}_0(\Omega) \) and \( W^{1,p}_0(\Omega) \subset L^2(\Omega) \). This allows us to uniquely split any function \( u \) from \( W^{1,p}_0(\Omega) \) in the following way
\[ u = \tau \varphi_1 + u^\perp , \]
where \( \tau \in \mathbb{R} \) and \( \int_\Omega u^\perp \varphi_1 \, dx = 0 \).

In our study of \( J_{\lambda_1} \) we will focus on understanding its nonlinear part \( E \).

\[ E(u) := \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\lambda_1}{p} \int_\Omega |u|^p \, dx. \]

The functional \( E \) is positively \( p \)-homogeneous and nonnegative by the Poincaré inequality. Moreover \( E(u) = 0 \) if and only if \( u \) is a real multiple of \( \varphi_1 \).

In this article we will work with several different compactness conditions. Apart from the standard \textit{Palais-Smale} condition we will use its generalizations - \textit{Cerami} and \( C^{\psi(\cdot)} \) conditions. In the following definitions we will always assume that \( F \in C^1(X, \mathbb{R}) \), where \( X \) is a Banach space.

**Definition 1.1 (Palais-Smale condition).** We say that the functional \( F \) satisfies the \textit{Palais-Smale} condition at the level \( c \) \((PS_c)\) if every sequence \( \{x_n\} \subset X \) such that
\[ \lim_{n \to \infty} F(x_n) = c, \]
\[ \lim_{n \to \infty} \|F'(x_n)\| = 0 \]
possesses a convergent subsequence.

We say that \( F \) satisfies the \textit{Palais-Smale} condition \((PS)\) if it satisfies the \( PS_c \) condition for all \( c \in \mathbb{R} \).

**Definition 1.2 (Cerami condition).** We say that the functional \( F \) satisfies the \textit{Cerami} condition at the level \( c \) \((C_c)\) if every sequence \( \{x_n\} \subset X \) such that
\[ \lim_{n \to \infty} F(x_n) = c, \]
\[ \lim_{n \to \infty} \|F'(x_n)\|(1 + \|x_n\|) = 0 \]
possesses a convergent subsequence.

We say that \( F \) satisfies the \textit{Cerami} condition \((C)\) if it satisfies the \( C_c \) condition for all \( c \in \mathbb{R} \).

**Definition 1.3 \((C^{\psi(\cdot)}_c)\) condition.** Let \( \psi(x) : \mathbb{R}^+ \to \mathbb{R}^+ \) be a positive nonincreasing lipschitz continuous function satisfying
\[ \int_1^\infty \psi(x) \, dx = \infty. \]
We say that the functional $F$ satisfies $C_c^{\psi(\cdot)}$ condition at the level $c$ if every sequence $\{x_n\} \subset X$ such that
\[
\lim_{n \to \infty} F(x_n) = c, \quad (5)
\]
\[
\lim_{n \to \infty} \|F'(x_n)\| \frac{1}{\psi(||x_n||)} = 0 \quad (6)
\]
possesses a convergent subsequence.

We say that $F$ satisfies the $C^{\psi(\cdot)}$ condition if it satisfies the $C_c^{\psi(\cdot)}$ condition for all $c \in \mathbb{R}$.

**Remark 1.** Of course we have $PS_c \Rightarrow C_c \Rightarrow C_c^{\psi(\cdot)}$, $PS \Rightarrow C \Rightarrow C^{\psi(\cdot)}$. Moreover all of these conditions are sufficient to prove the deformation lemma at the level $c$ [5].

The article is organized as follows. In the next section we introduce our hypotheses and tools. Then we compare our new results with the previous ones. Finally we present proofs of our results.

2. **Hypotheses, notation.** All Banach and Hilbert spaces used in this article are real. We will use standard inner product in $L^2(\Omega)$ defined by $\langle u, v \rangle := \int_{\Omega} uv \, dx$.

This inner product inducys duality between spaces $L^p(\Omega)$ and $L^{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $1 < p, p' < \infty$, and between spaces $W_0^{1,p}(\Omega)$ and $W_0^{1,p'}(\Omega)$ as well.

We will always assume the following

**Hypothesis 1.** If $N \geq 2$ then $\Omega$ is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial \Omega$ is a compact manifold of class $C^{1, \alpha}$ for some $\alpha \in (0, 1)$, and $\Omega$ satisfies also the interior sphere condition at every point of $\partial \Omega$. If $N = 1$ then $\Omega$ is a bounded open interval in $\mathbb{R}$.

Next we define set
\[
U := \{ x \in \Omega : \nabla \varphi_1(x) \neq 0 \}.
\]

We also often work with the function $a \mapsto \frac{1}{p} |a|^p : \mathbb{R}^N \to \mathbb{R}$ together with its first and second Fréchet derivatives $a \mapsto |a|^{p-2}a : \mathbb{R}^N \to \mathbb{R}^N$ and $a \mapsto A(a) : \mathbb{R}^N \to \mathbb{R}^{N \times N}$, respectively, where $A(0) = 0$ and for $0 \neq a \in \mathbb{R}^N$
\[
A(a) := |a|^{p-2} \left( I + (p-2) \frac{a \otimes a}{|a|^2} \right).
\]

In the case of $N = 1$ the preceding expression reduces to
\[
A(a) = (p-1)|a|^{p-2}.
\]

Notice that $I + (p-2) \frac{a \otimes a}{|a|^2}$ is a positive definite, symmetric $N \times N$ matrix with the eigenvalues 1 and $p - 1$. The "elliptic" degeneracy of the matrix $A(a)$ is expressed by the inequalities
\[
\min\{1, p-1\} |a|^{p-2} |v|^2 \leq \langle A(a)v, v \rangle_{\mathbb{R}^N} \leq \max\{1, p-1\} |a|^{p-2} |v|^2 \quad (7)
\]
for all $a, v \in \mathbb{R}^N$, $a \neq 0$.

Next we define the function space $D_{\varphi_1}$ which naturally arises during our investigation of the problem. There are major differences in its definition and properties.
in the cases $1 < p < 2$ and $p > 2$. The norm on this space is given (in both cases) by the following expression
\[
\|u\|_{D_{\varphi_1}} := \left( \int_\Omega |\nabla \varphi_1|^{p-2} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.
\] (8)

In the case $p > 2$ the expression (8) is also a norm on $W_0^{1,p}(\Omega)$ by Takáč [7] and we denote by $D_{\varphi_1}$ the completion of $W_0^{1,p}(\Omega)$ under this norm. The embedding $D_{\varphi_1} \hookrightarrow L^2(\Omega)$ is compact.

In the case $1 < p < 2$ we define $u \in D_{\varphi_1}$ if and only if $u \in W_0^{1,2}(\Omega)$, $\nabla u(x) = 0$ for almost all $x \in \Omega \setminus U$ and its $D_{\varphi_1}$ norm given by (8) is finite. This space endowed with its norm $\|\cdot\|_{D_{\varphi_1}}$ is continuously embedded into $W_0^{1,2}(\Omega)$.

Next we consider the case $p > 2$. To get any information on the behavior of the functional $E$ we pick an arbitrary function $\phi \in W_0^{1,p}(\Omega)$ and use the second order Taylor formula in its energy form [3, Proposition 3.2.27]
\[
E(\varphi_1 + \phi) = E(\varphi_1 + \phi) - E(\varphi_1) = \frac{1}{p} \int_\Omega |\nabla (\varphi_1 + \phi)|^p \, dx - \frac{\lambda_1}{p} \int_\Omega |\varphi_1 + \phi|^p \, dx
\]
where $Q_\phi$ is a symmetric bilinear form on $[W_0^{1,p}(\Omega)]^2$ defined as follows
\[
Q_\phi(w_1, w_2) := \int_\Omega \left( \int_0^1 |A(\nabla (\varphi_1 + s\phi))(1-s)ds\right) \nabla w_1, \nabla w_2 \right) \, dx
\]
\[\quad - \lambda_1 (p-1) \int_\Omega \left( \int_0^1 |\varphi_1 + s\phi|^{p-2}(1-s)ds\right) w_1 w_2 \, dx
\]
for $w_1, w_2 \in W_0^{1,p}(\Omega)$. In particular, for $\phi \equiv 0$ one has
\[
Q_0(w_1, w_2) = \frac{1}{2} \int_\Omega \left( A(\nabla \varphi_1) \nabla w_1, \nabla w_2 \right) \, dx - \lambda_1 \frac{p}{2} \int_\Omega \varphi_1^{p-2} w_1 w_2 \, dx.
\]

The quadratic form $Q_0$ is positive semidefinite, i.e., $Q_0(v, v) \geq 0 \forall v \in W_0^{1,p}(\Omega)$. Furthermore $Q_0$ is closable in $L^2(\Omega)$ and the domain of its closure is equal to $D_{\varphi_1}$.

To be able to conclude for which $v \in D_{\varphi_1}$ we have $Q_0(v, v) = 0$ we must also make the following hypothesis:

**Hypothesis 2.** If $N \geq 2$ and $\partial \Omega$ is not connected, then there is no function $v \in D_{\varphi_1}$ : $Q_0(v, v) = 0$ with the following four properties:

- $v = \varphi_1 \chi_S$ a.e. in $\Omega$, where $S \subset \Omega$ is Lebesgue measurable and $0 < |S| < |\Omega|$;
- $\bar{S}$ is connected and $\bar{S} \cap \partial \Omega \neq \emptyset$;
- if $V$ is a connected subset of $U$, the either $V \subset S$ or $V \subset \Omega \setminus S$;
- $(\partial S) \cap \Omega \subset \Omega \setminus \overline{S}$.

It has been conjectured in [7, Section 2.1] that Hypothesis 2 always holds true provided Hypothesis 1 is satisfied. Also this hypothesis is always satisfied in the case $1 < p < 2$.

**Proposition 1.** Let $p > 2$ and assume both Hypotheses 1 and 2. Then a function $v \in D_{\varphi_1}$ satisfies $Q_0(v, v) = 0$ if and only if $v = k\varphi_1$ for some constant $k \in \mathbb{R}$.

**Proof.** [7, Proposition 4.4]
Remark 2. Preceding quadratization procedure can be done even in the case 1 < p < 2. Although, since we can’t differentiate the functional E twice in general, we must restrict ourselves to the space $D_{\varphi_1}$. We will not go into detail since we do not use it in this article.

In the similar way we approach the Fréchet derivate $E'$. Let $\phi$ and $v$ be arbitrary functions from $W_0^{1,p}(\Omega)$. To compute $E'(\varphi_1 + \phi)v$ we use the first order Taylor formula [3, Theorem 3.2.6]

$$E'(\varphi_1 + \phi)v = E'(\varphi_1 + \phi)v - E'(\varphi_1)v$$

$$= \int_\Omega |\nabla (\varphi_1 + \phi)|^{p-2} (\nabla (\varphi_1 + \phi), \nabla v) \, dx - \lambda_1 \int_\Omega |\varphi_1 + \phi|^{p-2}(\varphi_1 + \phi)v \, dx$$

$$= X_\phi (\phi, v),$$

where $X_\phi$ is the symmetric bilinear form on $|W_0^{1,p}(\Omega)|^2$ defined as follows

$$X_\phi(w_1, w_2) := \int_\Omega \left\langle \left( \int_0^1 |A(\nabla (\varphi_1 + s\phi))| \, ds \right) \nabla w_1, \nabla w_2 \right\rangle \, dx$$

$$- \lambda_1 (p-1) \int_\Omega \left( \int_0^1 |\varphi_1 + s\phi|^{p-2} \, ds \right) w_1 w_2 \, dx$$

for $w_1, w_2 \in W_0^{1,p}(\Omega)$. In particular, for $\phi \equiv 0$ one has

$$X_0(w_1, w_2) = \int_\Omega \langle A(\nabla \varphi_1) \nabla w_1, \nabla w_2 \rangle \, dx - \lambda_1 (p-1) \int_\Omega \varphi_1^{p-2}w_1 w_2 \, dx,$$

$$= 2Q_0(w_1, w_2).$$

In the case $p > 2$ we will often use the following improved Poincaré inequality by Fleckinger and Takáč [6]:

There exists a positive constant $c = c(p, \Omega)$ such that for any function $u \in W_0^{1,p}(\Omega)$, $u = \tau \varphi_1 + u^\perp$, we have

$$E(u) \geq c \left( \tau |\nabla \varphi_1|^{p-2} \int_\Omega |\nabla \varphi_1|^{p-2} |\nabla u^\perp|^2 \, dx + \int_\Omega |\nabla u^\perp|^p \, dx \right).$$

3. Results. We start with a short survey of previous results. The following theorem was proved by Drábek and Takáč in [4].

Theorem 3.1 (Drábek, Takáč). Let $1 < p < \infty$, $p \neq 2$, moreover assume hypothesis 1 holds and $0 \neq f = f^+ \in L^\infty(\Omega)^\perp$. If $p > 2$ then assume also hypothesis 2 holds. If $1 < p < 2$ then assume $f^+ \not\in D_{\varphi_1}^{1,2}$. Then

a) For $p > 2$ and $-\infty < c < 0$ the functional $J_{\lambda_1}$ satisfies the PS condition at the level $c$.

b) For $p > 2$ the functional $J_{\lambda_1}$ does not satisfy the PS condition at the level $c = 0$.

c) For $p > 2$ and $0 < c < \infty$ every PS sequence for $J_{\lambda_1}$ at the level $c$ such that \{J_{\lambda_1}(u_n)\} is bounded in $L^\infty(\Omega)$ possesses a convergent subsequence.

d) For $1 < p < 2$ and arbitrary $c \in \mathbb{R}$ every PS sequence at the level $c$ such that \{J_{\lambda_1}^c(u_n)\} is bounded in $L^\infty(\Omega)$ possesses a convergent subsequence.
Notice the assumptions in parts c) and d) that require \( \{J_{\lambda_i}(u_n)\} \) bounded in \( L^\infty(\Omega) \). These assumptions are artificial in essence - with their help the authors were able to use boundary regularity results to prove the theorem although these conditions do not arise naturally. Our main aim was to remove these extra conditions. We were successful in the case \( p > 2 \) (part a) of the following theorem) and only partially successful in the case \( 1 < p < 2 \) (part d). Moreover for \( p > 2 \) we verified that the unbounded \( PS \) sequence at the level \( c = 0 \) constructed in [4] violates not only the \( PS \) condition but also the more general \( C^{\psi(\cdot)} \) condition as well (part b).

**Theorem 3.2** (Main results summary). Let \( 1 < p < \infty, \ p \neq 2, \) and assume hypothesis 1 and 0 \( \neq f = f^+ \in L^\infty(\Omega)^\perp \). If \( p > 2 \) then assume also hypothesis 2. If \( 1 < p < 2 \) then assume \( f^+ \in D_{\psi_1}^{1,2} \). Then

a) For \( p > 2 \) and \( c \neq 0 \) the functional \( J_{\lambda_1} \) satisfies the \( PS \) condition at the level \( c \).

b) For \( p > 2 \) the functional \( J_{\lambda_1} \) does not satisfy the \( C^{\psi(\cdot)} \) condition at the level \( c = 0 \).

c) For \( 1 < p < 2 \) and arbitrary \( c \in \mathbb{R} \) every \( PS \) sequence at the level \( c \) such that \( \{J_{\lambda_i}(u_n)\} \) is bounded in \( L^\infty(\Omega) \) possesses a convergent subsequence.

d) For \( 1 < p < 2 \) and \( c > 0 \) the functional \( J_{\lambda_1} \) satisfies \( C \) condition at the level \( c \).

Of some note are also estimates on behavior of \( E \) and \( E' \) given by lemma (4.4).

4. **Proofs.** As written in the Introduction section we used the second order Taylor expansion to estimate values of \( E \) and \( E' \). To this end we start with several lemmas that help us to understand the geometry of the functional \( E \). The next one is a minor but important improvement of [4, Proposition 3.6]. It gives us information on how growths in \( \tau \) and \( \alpha \) and their relation affect values of \( E(\tau \varphi_1 + \alpha u^\perp) \).

**Lemma 4.1.** Let \( p > 2 \) and \( u^\perp \) be an arbitrary nonzero function from \( W^{1,p}_0(\Omega)^\perp \). Assume both hypotheses 1 and 2 are satisfied. Then

\[
E(\tau \varphi_1 + \alpha(\tau) u^\perp) = |\tau|^{-2} \alpha^2(\tau) (Q_0(u^\perp, u^\perp) + o(1)) \quad \text{for} \quad |\tau| \to \infty \land \frac{\alpha(\tau)}{\tau} \to 0.
\]

Notice that \( Q_0(u^\perp, u^\perp) \neq 0 \) by \( u^\perp \neq 0 \).

**Proof.** Let us write \( u = \tau \varphi_1 + \alpha(\tau) u^\perp = \tau (\varphi_1 + v^\perp) \), where \( v^\perp := \frac{\alpha(\tau)}{\tau} u^\perp \) for \( \tau \neq 0 \). Using \( p \)-homogeneity of \( E \) and Taylor formulas as described in the Notation section we have

\[
E(\tau \varphi_1 + \alpha(\tau) u^\perp) = |\tau|^p E(\varphi_1 + v^\perp) = |\tau|^p Q_{v^\perp}(u^\perp, u^\perp)
\]

\[
= |\tau|^{p-2} \alpha^2(\tau) Q_{v^\perp}(u^\perp, u^\perp)
\]

\[
= |\tau|^{p-2} \alpha^2(\tau) (Q_0(u^\perp, u^\perp) + o(1))
\]

as \( |\tau| \to \infty \land \frac{\alpha(\tau)}{\tau} \to 0 \), where the last equality follows from

\[
\lim_{|\tau| \to \infty} Q_{v^\perp}(u^\perp, u^\perp) = Q_0(u^\perp, u^\perp) + o(1). \quad (10)
\]

We prove (10) using the Lebesgue dominated convergence theorem. Without loss of generality we may assume \( \frac{\alpha(\tau)}{\tau} < 1 \). Hence \( v^\perp = \frac{\alpha(\tau)}{\tau} u^\perp \) satisfies \( |v^\perp| < |u^\perp| \) in \( \Omega \). Next we can construct an integrable majorant exactly in the same way as in the proof of [4, Proposition 3.6]. See also the following proof of lemma (4.3) where we are proving a more general result than (10) for \( p > 2 \). \( \square \)
Remark 3. For $p > 2$ it follows from the improved Poincaré inequality that the seemingly very limiting assumptions $\frac{\alpha(\tau)}{\tau} \to 0$ will hold for $PS$ sequences.

Analogously we can prove the following lemma dealing with the Fréchet derivative $E'$.

**Lemma 4.2.** Let $p > 2$ and $u^\perp, v^\perp$ be arbitrary nonzero functions from $W_0^{1,p}(\Omega)^\perp$. Assume both hypotheses 1 and 2 are satisfied. Then

$$E'(\nu_\tau + \alpha(\tau)u^\perp)v^\perp = |\tau|^{p-2}\alpha(\tau)(X_0(u^\perp, v^\perp) + o(1)) \quad \text{for } |\tau| \to \infty \land \frac{\alpha(\tau)}{\tau} \to 0.$$

Motivated by the two previous lemmas we are interested in the situation when the function $u^\perp$ is not fixed. To this end we proved the following.

**Lemma 4.3.** Let $p > 2$ and $\{u^\perp_n\}$ and $\{v^\perp_n\}$ be bounded sequences of functions from $W_0^{1,p}(\Omega)^\perp$, moreover assume both hypotheses 1 and 2 are satisfied and $\|v_n\|_{W_0^{1,p}(\Omega)} \to 0$. Then there exist a subsequence $\{v^\perp_{n_k}\}$ of $\{v^\perp_n\}$ such that

$$Q_{v^\perp_{n_k}}(u^\perp_n, u^\perp_n) = Q_0(u^\perp_n, u^\perp_n) + o(1) \quad \text{for } n \to \infty,$$

$$X_{v^\perp_{n_k}}(u^\perp_n, u^\perp_n) = X_0(u^\perp_n, u^\perp_n) + o(1) \quad \text{for } n \to \infty.$$

**Proof.** We will prove only the first part of the assertion. The other would be proved analogously.

We start by subtracting $Q_0(u^\perp_n, u^\perp_n)$ from $Q_{v^\perp_n}(u^\perp_n, u^\perp_n)$ and using the Hólder’s inequality.

$$|Q_{v^\perp_n}(u^\perp_n, u^\perp_n) - Q_0(u^\perp_n, u^\perp_n)|$$

$$\leq \left| \int_\Omega \left( \int_0^1 A(\nabla(\varphi_1 + sv^\perp_n))(1-s) \, ds - \frac{1}{2} A(\nabla \varphi_1) \right) \langle \nabla u^\perp_n, \nabla v^\perp_n \rangle \, dx \right|$$

$$+ \lambda_1(p-1) \left| \int_\Omega \left( \int_0^1 |\varphi_1 + sv^\perp_n|^p - 2(1-s) \, ds - \frac{1}{2} \varphi_1^p \right) (u^\perp_n)^2 \, dx \right|$$

$$\leq \left[ \int_\Omega \left( \int_0^1 A(\nabla(\varphi_1 + sv^\perp_n))(1-s) \, ds - \frac{1}{2} A(\nabla \varphi_1) \right) \left| \nabla u^\perp_n \right|^p \, dx \right]^{\frac{p-2}{p}} \left| u^\perp_n \right|^2_{W_0^{1,p}(\Omega)}$$

$$+ \lambda_1(p-1) \left[ \int_\Omega \left( \int_0^1 |\varphi_1 + sv^\perp_n|^p - 2(1-s) \, ds - \frac{1}{2} \varphi_1^p \right) |v^\perp_n|^2 \, dx \right]^{\frac{p-2}{p}} \left| u^\perp_n \right|^2_{L^p(\Omega)}$$

Next we show that square brackets in the previous expression converge to 0 as $n \to \infty$. This together with boundedness of the sequence $\{u^\perp_n\}$ in $W_0^{1,p}(\Omega)$ will complete the proof. Again we use Lebesgue dominated convergence theorem. From $\|v^\perp_n\|_{W_0^{1,p}(\Omega)} \to 0$ and the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ we conclude that there exist a subsequence $\{v^\perp_{n_k}\}$ such that both $|\nabla v^\perp_{n_k}(x)|$ and $v^\perp_{n_k}(x)$ converge pointwise to 0 a.e. in $\Omega$. Moreover there exist $L^p$ integrable majorants $h_1(x)$ resp. $h_2(x)$ of both sequences [3, Remark 1.2.18]. Moreover thanks to (7) and the
inequalities from [7] we have

$$\left| \int_0^1 A(\nabla(\varphi_1 + sv_{n_k}^\perp)) (1 - s) \, ds - \frac{1}{2} A(\nabla\varphi_1) \right|$$

\[ \leq (p - 1) \int_0^1 |\nabla(\varphi_1 + sv_{n_k}^\perp)|^{p-2}(1 - s) \, ds + \frac{p - 1}{2} |\nabla\varphi_1|^{p-2} \]
\[ \leq (p - 1)(|\nabla\varphi_1| + \frac{1}{2}|\nabla v_{n_k}^\perp|)^{p-2} \leq (p - 1)(|\nabla\varphi_1| + \frac{1}{2}h_1)^{p-2} := g_1, \]
\[ \left| \int_0^1 |\varphi_1 + sv_{n_k}^\perp|^{p-2}(1 - s) \, ds - \frac{1}{2} \varphi_1^{p-2} \right| \]
\[ \leq (\varphi_1 + \frac{1}{2}|v_{n_k}^\perp|)^{p-2} \leq (\varphi_1 + \frac{1}{2}h_2)^{p-2} := g_2. \]

Obviously, we have $g_i \in L^{\frac{p}{p-2}}(\Omega), \ i = 1, 2$ and so the assumptions of the Lebesque dominated convergence theorem are met. 

Putting together lemma (4.3) with the ideas from the proof of the lemma (4.1) we can prove the following generalized estimates of $E$ and $E'$.

**Lemma 4.4.** Let $p > 2$ and $\{u_n^\perp\}$ and $\{v_n^\perp\}$ be bounded sequences of functions from $W_0^{1,p}(\Omega)^\perp$. Assume both hypotheses 1 and 2 are satisfied. Then there exists a subsequence $\{u_{n_k}^\perp\}$ of $\{u_n^\perp\}$ such that

$$E(\tau\varphi_1 + \alpha(\tau)u_{n_k}^\perp) = |\tau|^{p-2} \alpha^2(\tau) \left(Q_0(u_{n_k}^\perp, u_{n_k}^\perp) + o(1)\right) \quad \text{for } |\tau| \to \infty \wedge \frac{\alpha(\tau)}{\tau} \to 0,$$

$$E'(\tau\varphi_1 + \alpha(\tau)u_{n_k}^\perp)v_{n_k}^\perp = |\tau|^{p-2} \alpha(\tau) \left(X_0(u_{n_k}^\perp, v_{n_k}^\perp) + o(1)\right) \quad \text{for } |\tau| \to \infty \wedge \frac{\alpha(\tau)}{\tau} \to 0.$$

Now we can proceed to the proof of the theorem (3.2).

**Proof.** Part (a). In the first step we show that every PS sequence must be bounded. We argue by contradiction. Let us assume that there exists an unbounded PS sequence $\{u_n\}$. We split functions from this sequence into a real multiple of $\varphi_1$ and a real multiple of a function $u_n^\perp$ that are $L^2(\Omega)$ orthogonal. To avoid any confusion we always assume that $\|u_n^\perp\|_{W_0^{1,p}(\Omega)} = 1 \ \forall n \in \mathbb{N}$ to make the split unique. So for some numbers $\tau_n$ and $\alpha_n$ we have

$$u_n = \tau_n\varphi_1 + \alpha_n u_n^\perp \ \forall n \in \mathbb{N}.$$

Next we define

$$v_n^\perp := \frac{\alpha_n u_n^\perp}{\tau_n} \ \forall n \in \mathbb{N}.$$

Since $\{u_n\}$ is a PS sequence we also have

$$J_{\varphi_1}(u_n) \to c, \quad J'_{\varphi_1}(u_n) \to o \ \text{in } W_0^{-1,p'}(\Omega). \quad (11)$$

From the improved Poincaré inequality and (11) it follows that the sequence $\{\alpha_n\}$ must be bounded. So, obviously, we have $\tau_n \to \infty$ and we can use the lemma
It follows from the lemma 1 that the function \( \alpha u \) want to show that

\[
\{ \frac{u_0}{n} \to \perp \}
\]

Putting

\[
\{ \frac{u_0}{n} \to \perp \}
\]

\[
\omega \to \{ \frac{u_0}{n} \to \perp \}
\]

So for \( n \to \infty \) we have

\[
J_{\varphi_1}(u_n) = \tau_n|\alpha_n^2(Q_0(u_n^+, u_n^-) + \omega(1)) - \frac{\alpha_n}{\Omega} \int u_n^+ dx \to c,
\]

(13)

\[
J'(u_n)\alpha_n u_n^+ = (\tau_n|\alpha_n^2(2Q_0(u_n^+, u_n^-) + \omega(1)) - \frac{\alpha_n}{\Omega} \int u_n^+ dx \to 0.
\]

Next we want to show that \( \alpha_n \int_\Omega f u_n^+ dx \to 0 \). We must consider two cases. Either \( \exists c_1 > 0 : Q_0(u_n^+, u_n^-) > c_1 \forall n \in \mathbb{N} \) or we can assume that \( Q_0(u_n^+, u_n^-) \to 0 \). In the first case the required assertion follows from \( \alpha_n \to 0 \) due to the equation (13). In the latter case notice that the sequence \( \{ u_n \} \) is bounded in \( W_0^{1, p}(\Omega) \). So for \( n \to \infty \) we have that weakly converges to some function \( u_0^+ \in W_0^{1, p}(\Omega) \). Due to the compact embedding \( W_0^{1, p}(\Omega) \to L^2(\Omega) \) we have \( u_n^+ \to u_0^+ \) in \( L^2(\Omega) \). Moreover since \( Q_0 \) is closable in \( L^2(\Omega) \) we have \( Q_0(u_n^+, u_n^-) = 0 \). It follows from the lemma 1 that the function \( u_n^+ \) is a real multiple of \( \varphi_1 \). But \( u_0^+ \) and \( \varphi_1 \) are orthogonal so \( u_n^+ \equiv 0 \). That means \( \int_\Omega f u_n^+ dx \to \int_\Omega f u_0^+ dx = 0 \) and boundedness of the sequence \( \alpha_n \) gives us the required assertion.

Putting \( \alpha_n \int_\Omega f u_n^+ dx \to 0 \) together with (13) and (14) we arrive at

\[
E(u_n) = \tau_n|\alpha_n^2 Q_0(u_n^+, u_n^-) \to c \neq 0,
\]

(15)

\[
E'(u_n)\alpha_n u_n^+ = (\tau_n|\alpha_n^2(2Q_0(u_n^+, u_n^-) + \omega(1)) - \frac{\alpha_n}{\Omega} \int u_n^+ dx \to 0.
\]

(16)

But this is a contradiction since it is obvious that \( 0 \leq Q_0(u_n^+, u_n^-) \leq X_{v_n^+}(u_n^+, u_n^-) \).

The sequence \( \{ u_n \} \) must be bounded. Next we show that from every bounded PS sequence we can pick a convergent subsequence.

From boundedness of \( \{ u_n \} \) it follows that there exists its subsequence, denoted again by \( \{ u_n \} \), that weakly converges in \( W_0^{1, p}(\Omega) \) to some function \( u_0 \). Now we want to show that \( u_n \to u_0 \) strongly. There are several ways to show this. We use an argument similar to those used in investigating monotone operators.

Let us define \( v_n := u_n - u_0 \). We have \( J'_{\varphi_1}(u_n) v_n \to 0 \) and so

\[
\int_\Omega |\nabla u_n|^p - \lambda_1 \int_\Omega |u_n|^{p-2} u_n (u_n - u_0) dx
\]

\[
- \int_\Omega f (u_n - u_0) dx \to 0.
\]

(17)

From the definition of \( \{ v_n \} \) it follows that this sequence is weakly convergent to zero function in \( W_0^{1, p}(\Omega) \) and the compact embedding \( W_0^{1, p}(\Omega) \to L^p(\Omega) \) yields
\( v_n \to 0 \) in \( L^p(\Omega) \). So we have
\[
\int_{\Omega} |u_n|^{p-2} u_n (u_n - u_0) \, dx \to 0,
\]
\[
\int_{\Omega} f(u_n - u_0) \, dx \to 0.
\]
That means (17) reduces to
\[
\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n (\nabla u_n - \nabla u_0) \, dx \to 0.
\]
Since \( v_n \to 0 \) we also have
\[
\int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 (\nabla u_n - \nabla u_0) \, dx \to 0.
\]
Now we can subtract the previous two equations and use the Hölder inequality
\[
\int_{\Omega} (|\nabla u_n|^{p-2} - |\nabla u_0|^{p-2}) \nabla u_n \nabla u_0 \, dx \to 0
\]
\[
\geq \|u_n\|_{W_0^{1,p}(\Omega)}^{1} - \|u_0\|_{W_0^{1,p}(\Omega)}^{1} - \|u_n\|_{W_0^{1,p}(\Omega)}^{1} + \|u_0\|_{W_0^{1,p}(\Omega)}^{1}
\]
\[
= \left( \|u_n\|_{W_0^{1,p}(\Omega)}^{1} - \|u_0\|_{W_0^{1,p}(\Omega)}^{1} \right) \left( \|u_n\|_{W_0^{1,p}(\Omega)}^{1} - \|u_0\|_{W_0^{1,p}(\Omega)}^{1} \right) \geq 0.
\]
So \( \|u_n\|_{W_0^{1,p}(\Omega)} \to \|u_0\|_{W_0^{1,p}(\Omega)} \) and thanks to the uniform convexity of \( W_0^{1,p}(\Omega) \) we have \( u_n \to u_0 \).

Part (b). We will verify that the unbouded PS sequence constructed in [4] violates \( C^{(c)} \) condition as well. The idea behind constructing such a PS sequence is the following. Let us recall that for given \( \tau \in \mathbb{R} \) the functional \( u^\tau \to J_{\varphi_1}(\tau \varphi_1 + u^\tau) \) defined on \( W_0^{1,p}(\Omega) \) is weakly coercive and its global minimizer \( u^\tau \) satisfies the following equations
\[
\begin{align*}
-\Delta_p (\tau \varphi_1 + u^\tau) &= f^\tau + \varsigma_\tau \varphi_1 & \text{in } \Omega, \\
\varphi_1 &= 0 & \text{on } \partial \Omega, \\
\langle u^\tau, \varphi_1 \rangle &= 0
\end{align*}
\]
for some Lagrange multiplier \( \varsigma_\tau \).

Next we take an arbitrary real sequence \( \{\tau_n\} \) such that \( \tau_n \to \infty \) and for simplicity’s sake we denote \( u^\tau_{\tau_n} = u_{\tau_n} \) and \( \varsigma_{\tau_n} = \varsigma_n \). It can be shown [8, Proposition 6.1] that \( \varsigma_{\tau_n} \to 0 \), \( J_{\varphi_1}(\tau_n \varphi_1 + u_{\tau_n}) \to 0 \) and
\[
J'_{\varphi_1}(\tau_n \varphi_1 + u_{\tau_n}) \phi = \varsigma_n \int_\Omega \varphi_1 \phi
\]
for arbitrary test function \( \phi \in W_0^{1,p}(\Omega) \) [4, Proposition 4.4].

Hence \( \{u_{\tau_n}\} \) is an unbounded PS sequence since the right hand side of (18) converges to 0. To prove that the \( C^{(c)} \) condition is violated as well we need to
investigate how quick this convergence is. The answer is given by the following estimate from [8, Proposition 6.1].

\[
\lim_{n \to \infty} c_n |\tau_n|^{p-2}\tau_n = (p-2)\|\varphi_1\|^2_{L^p(\Omega)} Q_0(w, w),
\]

where \( w \in D_{\varphi_1} \) is the unique solution to

\[
\begin{align*}
-\text{div}(A(\nabla \varphi_1) \nabla w) &= \lambda_1 \varphi_1^{p-2} w + f \quad \text{in } \Omega; \\
w &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Moreover we have \( Q_0(w, w) > 0 \).

From (18), (19) and \( \|u_n^+\|_{W^{1,p}_0(\Omega)} \to 0 \) (check the proof of [4, Proposition 4.4]) it follows that there exist positive constants \( c_1 \) and \( c_2 \) such that

\[
\|J'_{\varphi_1}(\tau_n \varphi_1 + u_n^+)\|_{W^{-1,p'}(\Omega)} \leq c_1 |\tau_n|^{1-p} \leq c_2 |\tau_n \varphi_1 + u_n^+|_{W^{1,p}_0(\Omega)}^{1-p}.
\]

So \( \|J'_{\varphi_1}(\tau_n \varphi_1 + u_n^+)\|_{W^{-1,p'}(\Omega)} \) converges to 0 too fast and the \( C^\psi(.) \) condition cannot hold.

Part (c). There was no improvement in this part.

Part (d). Again we argue by contradiction. Let \( \{u_n\} \) be a \( C \) sequence for the functional \( J_{\lambda_1} \). We have

\[
J_{\lambda_1}(u_n) - E(u_n) = \int u_n f \, dx \to c > 0,
\]

\[
J'_{\lambda_1}(u_n) - E'(u_n) u_n = \int u_n f \, dx \to 0.
\]

Since \( E(u_n) = \frac{1}{p} E'(u_n) u_n \) we get

\[
\int u_n f \, dx \to \frac{pc}{1-p},
\]

\[
E'(u_n) u_n \to \frac{pc}{1-p},
\]

\[
E(u_n) \to \frac{c}{1-p}.
\]

For positive \( c \) the expression \( \frac{c}{1-p} \) is negative but the Poincaré inequality gives \( E(u) \geq 0 \quad \forall u \in W^{1,p}_0(\Omega) \) and that is a contradiction. That means there exist no \( C \) sequences for \( J_{\lambda_1} \) at positive levels and the Cerami condition holds trivially. \( \square \)

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