A Thin Fundamental Set for $SL(2, \mathbb{Z})$

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Abstract

Let $\Gamma = SL(2, \mathbb{Z})$ and $G = SL(2, \mathbb{R})$. Let $g = kan$ be the Iwasawa decomposition. Let $\epsilon$ be a small positive number. In this paper, we construct a fundamental set $F_{\epsilon}$ such that the $k$-component of $g \in F_{\epsilon}$ is within the $\epsilon$-distance from the identity. We further prove an inequality for the $L^2$-norm of functions on $G/\Gamma$.

1 Introduction

We start with the projective group $PSL(2, \mathbb{R})$. Any element $g \in PSL(2, \mathbb{R})$ has an Iwasawa decomposition $kan$ with $k \in PSO(2), a \in A, n \in N$, where $A$ consists of diagonal matrices with positive entries $(a, a^{-1})$ and $N$ consists of upper triangular unipotent matrices parametrized by $t \in \mathbb{R}$. Let $PSL(2, \mathbb{Z})$ be the modular group consisting of all matrices in $PSL(2, \mathbb{R})$ with integer entries. Automorphic forms on $PSL(2, \mathbb{R})/PSL(2, \mathbb{Z})$ play a central role in many branches of mathematics [2]. Their analytic properties were often obtained by analysis on the fundamental set $F$, explicitly

\[
\{kan : k \in PSO(2), |t| \leq \frac{1}{2}, a^{-4} + t^2 \geq 1\}.
\]

Here the fundamental set has a cusp at 0. This is consistent with [6] and [3], but differs from the one more commonly used by an inversion $a \rightarrow a^{-1}$ ([5]).

One advantage of using $F$ as the fundamental set, is that analysis based on $F$ will often involve computations on $K$-finite functions which can be expressed as hypergeometric functions $_{2}F_{1}$ or $_{1}F_{1}$ ([10]). However, the behavior of hypergeometric functions can be very complicated and precise computations are often impossible ([11]). In this paper, we shall construct a fundamental set that is not $K$-invariant. This fundamental set $F_{\epsilon}$ will only involve a small neighborhood of the compact group $K$. This small neighborhood of $K$ can be made infinitesimally small.

To state our main result, we let $G = SL(2, \mathbb{R})$ and $\Gamma = SL(2, \mathbb{Z})$. Fix the standard Iwasawa decomposition $KAN$ with $N$ the unipotent upper triangular matrices parametrized by $t \in \mathbb{R}$, $K = SO(2)$ parametrized by $\theta \in \mathbb{R}/\mathbb{Z}$. Our main result can be stated as follows.

*Key word: Iwasawa decomposition, Siegel set, $SL(2, \mathbb{Z})$, fundamental domain
Theorem 1.1 Let $\mathcal{F}_c = \bigcup_{i=1}^{3} \mathcal{F}_c^i \subseteq G$ with
\[ \mathcal{F}_c^1 = \{ g = \text{kan} : |\theta| < \epsilon, a^2 \leq \csc(\epsilon + |\theta|), |t| \leq \frac{1}{2} \}; \]
\[ \mathcal{F}_c^2 = \{ g = \text{kan} : |\theta| < \epsilon, \csc(\epsilon+|\theta|) \leq a^2 \leq \csc(\epsilon-|\theta|), \sgn(\theta) t \in [-a^{-2} \cot(\epsilon-|\theta|), (1-\sqrt{1-a^{-4}})] \}; \]
\[ \mathcal{F}_c^3 = \{ g = \text{kan} : |\theta| < \epsilon, \csc(\epsilon-|\theta|) \leq a^2 \leq \cot(\epsilon-\theta)+\cot(\epsilon+\theta), t \in [1-a^{-2} \cot(\epsilon+\theta), a^{-2} \cot(\epsilon-\theta)] \}; \]
\[ \mathcal{F}_c^4 = \{ g = \text{kan} : |\theta| < \epsilon, \csc(\epsilon-|\theta|) \leq a^2, t \in [\sqrt{1-a^{-4}} - 1, 1 - \sqrt{1-a^{-4}}] \}. \]

The natural map $\pi|_{\mathcal{F}_c} : \mathcal{F}_c \subseteq G \rightarrow G/\Gamma$ is surjective. $\pi|_{\mathcal{F}_c}$ is injective on the interior of $\mathcal{F}_c$ and finite on the boundary of $\mathcal{F}_c$.

The fundamental set $\mathcal{F}_c$ is contained entirely in $K_AN$ with $K_c$ the segment of $SO(2)$ within $\epsilon$-distance from the identity. See Theorem 6.1 for a stronger statement.

The main idea of the proof is to interpret $G/\Gamma$ as the space of unital lattices in $\mathbb{R}^2$, namely those lattices with two generators that span an area of 1. Every fundamental set for $G/\Gamma$ essentially corresponds to a parametrization of the unital lattices. In the classical reduction theory, the parametrization of a unital lattice is based on the minimal element in the unital lattice (3). In this paper, we modify the classical reduction theory by imposing a restriction on the $K$-component. Then the relations between $a$-component and $t$-component become a lot more complicated, but still tractable. We prove a sufficient and necessary condition that dictates the relation between $a$ and $t$. This leads us to the construction of $\mathcal{F}_c$.

At the end of this paper, we also prove some integral inequality similar to (8). These bounds relate the $L^2$-norm on certain “conic” region to the $L^2$-norm on the fundamental set.

Finally, we shall remark that most results in this paper should generalize to $SL(n, \mathbb{Z})$ and perhaps to all congruence subgroups. It will require deeper studies on unital lattices in $\mathbb{R}^n$. Even though the precise statement for $\mathcal{F}_c$ will be difficult to write down, the Siegel sets of the same type may be obtained which will provide an equivalent norm for automorphic representations. Hence $\mathcal{F}_c$ or related Siegel set is potentially useful in the study of automorphic functions. The fundamental set $\mathcal{F}_c$ may also be studied from a topological viewpoint. It is not clear whether it can offer anything new.

Let $\lceil x \rceil$ be the ceiling function, namely the smallest integer bigger than or equal to $x$.

2 Parametrization of $SL(2, \mathbb{R})$: Setup

Let $g \in SL(2, \mathbb{R})$. Let
\[ g = (u, v), \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \]

Let
\[ u^+ = \begin{pmatrix} \frac{u_1}{\sqrt{u_1^2 + u_2^2}} \\ \frac{u_2}{\sqrt{u_1^2 + u_2^2}} \end{pmatrix}. \]

Then $\|u^+\| = \|u\|^{-1}$ and $v = u^+ + tu$. The Iwasawa decomposition of $SL(2, \mathbb{R})$ is given by
\[ g = \left( \frac{u}{\|u\|}, \frac{u^+}{\|u^+\|} \right) \left( \begin{array}{cc} \|u\| & 0 \\ 0 & \|u^+\| \end{array} \right) \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right). \]
We may write this decomposition traditionally as \( g = kan \) with
\[
k = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad a = \begin{pmatrix} ||u|| & 0 \\ 0 & ||u^\perp|| \end{pmatrix}, \quad n = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.
\]

Here \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \) and \( t \in \mathbb{R} \). We may abuse the notation by writing \( a = ||u|| \). Notice that \( ||u^\perp|| = ||u||^{-1} = a^{-1} \) and
\[
\begin{pmatrix} u & u^\perp \\ ||u|| & ||u^\perp|| \end{pmatrix} = \begin{pmatrix} \frac{u}{||u||} & \frac{u^\perp}{||u||} \\ \frac{u}{||u||} & -\frac{u^\perp}{||u||} \end{pmatrix} \in SO(2).
\]

Since \( v = u^\perp + tu \), \( \langle v, u \rangle = t \langle u, u \rangle \). Hence \( t = \frac{\langle v, u \rangle}{\langle u, u \rangle} \).

Fix \( K = SO(2) \) and \( G = SL(2, \mathbb{R}) \). We also have a variant of the Iwasawa decomposition \( G = KNA \). The advantage of \( KNA \) decomposition is that the product of the invariant measures on \( K \), \( N \) and \( A \) is an invariant measure of \( G \). To distinguish \( KNA \) decomposition from \( KAN \) decomposition, we write \( g = kn(T)a \) in contrast with \( g = kan(t) \). In both decompositions, \( k \) and \( a \) remain the same. The parameter \( T \) is related to \( t \) by \( T = a^2 t \). In fact, \( T = \langle v, u \rangle \).

In any case, all parameters pertaining to \( g \in G, u, v, u^\perp, t, a, T, k, n \) should be understood as
\[
\begin{align*}
u(g), v(g), u^\perp(g), t(g), a(g), T(g), \theta(g), k(g), n(g)
\end{align*}
\]
for a fixed \( g \).

### 3 Parametrization of unital lattice in \( \mathbb{R}^2 \): \( K \)-invariant view

Let \( \Gamma = SL(2, \mathbb{Z}) \). By a rank 2 lattice in \( \mathbb{R}^2 \), we meant any additive subgroup of \( \mathbb{R}^2 \) with 2 linearly independent generators. We call it a unital lattice if any two generators of the lattice span a parallelogram of area 1. Denote the space of unital lattice in \( \mathbb{R}^2 \) by \( \mathcal{U} \). Let \( \mathcal{L} \in \mathcal{U} \). Then any pair of positively oriented generators \( (u, v) \) of \( \mathcal{L} \) corresponds an element \( (u, v) \) of \( G \) in a one-to-one fashion. Hence, the group \( G \) parametrizes all positively oriented generators of all \( \mathcal{L} \in \mathcal{U} \).

Let the group \( SL(2, \mathbb{Z}) \) acts on each lattice \( \mathcal{L} \in \mathcal{U} \) by changing the generators:
\[
(u, v) \rightarrow (pu + qv, ru + sv) = (u, v) \begin{pmatrix} p & r \\ q & s \end{pmatrix}, \quad \forall \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \Gamma.
\]

Thus permuting the lattice points in \( \mathcal{L} \). Clearly any two positively oriented generators of \( \mathcal{L} \) differs by an action of \( \gamma \in \Gamma \) and vice versa. Hence \( \mathcal{U} \) can be identified with \( G/\Gamma \) with \( \Gamma \) acting from the right. We have the natural projection
\[
\pi : G \rightarrow \mathcal{U}.
\]

The fiber will be the matrices \( \{(u, v)\} \) given by any two positively oriented generators \( (u, v) \) of \( \mathcal{L} \). Furthermore, \( \mathcal{U} \) has a natural fibration \( \mathcal{U} \):
\[
\mathbb{Z}^2 \rightarrow \mathcal{U} \rightarrow \mathcal{U}
\]

with the group \( \Gamma \) acting on the fiber as additive group automorphisms.

Now we seek to parametrize \( \mathcal{U} \). This is more or less equivalent to finding a fundamental set of \( G/\Gamma \) in the classical sense. For a comprehensive account of fundamental set for reductiove groups,
See [2] and the references therein. The fundamental sets in [2] are in fact the fundamental sets of $K \setminus G/\Gamma$. They can be viewed as a $K$-invariant set in $G/\Gamma$ by the pullback map of the projection $G \to K \setminus G$. Due to the action of the nontrivial center of $G$, the pullback will be a double cover of $G/\Gamma$ in the interior. In order to provide insight for the construction of non $K$-invariant fundamental set, we shall now review the basic ideas of the reduction theory in a $K$-invariant fashion.

For each $L \subseteq \mathbb{U}$, we shall now review the basic ideas of the reduction theory in $G/\Gamma$. We have the following is well-known.

**Theorem 3.1** The natural map $\pi|_F : F \subset G \to G/\Gamma$ is surjective. It is two-to-one in the interior of $F$ and finite on the boundary of $F$.

*Proof:* We already showed that if $\|u\|$ is minimal in $L$, then

$$a^2 \leq \frac{2}{\sqrt{3}}, \quad t^2 \geq 1 - a^{-4}, \quad t^2 \geq 1$$

The converse is true, but not quite trivial. Suppose that $g = \text{kan}$ with $a, t$ satisfy the above properties. For each $m \in \mathbb{Z}$, define the line

$$u^{(m)} = mu^1 + R u.$$ 

Then $L \subseteq \cup_{m \in \mathbb{Z}} u^{(m)}$. If $|m| \geq 2$, then any $w \in u^{(m)}$ satisfies

$$\|w\|^2 \geq 4\|u^1\|^2 = 4\|u\|^2 - 2 = 2\frac{\sqrt{3}}{2} > \frac{2}{\sqrt{3}} \geq \|u\|^2.$$ 

For $m = \pm 1$, $(u, v)$ or $(u, -v)$ is in $G$. Then $v = \pm u^1 + tu$. Our earlier discussion showed that $\|v\| \geq \|u\|$ for any $v \in u^{(\pm 1)} \cap L$. For $m = 0$, $w \in u^{(0)} \cap L$ means $w = ku$ for some $k \in \mathbb{Z}$. Hence $\|w\| \geq \|u\|$ unless $w = 0$.
Combining with the cases \(m = 0\) or \(m = \pm 1\), we see that the minimal vector \(u\) is unique up to a \(\pm\) sign if \(\frac{1}{t} > t^2 > 1 - \|u\|^{-4}\). Therefore \(\pi_F : F \to G/\Gamma\) is two-to-one in the interior of \(F\). Over the boundary of \(F\), it is not hard to see that \(\deg(\pi_F) \leq 6\). When \(t^2 = 1 - a^{-4}\), the vectors \(u\) and \(v\) form an isosceles triangle and the degree of \(\pi_F\) over these points is 4 with one exception: \(t^2 = 1 - a^{-4} = \frac{1}{4}\). This happens when \(L\) is generated by an equilateral triangle and \(deg = 6\) in this case. Of course, over \(t = \pm \frac{1}{2}\), the degree of \(\pi_F\) is also 4 with the equilateral triangle case as the exception. □

**Corollary 3.1** Let \(L \in U\). If \(\Phi(L) < 1\), then there are only two vectors \(\pm u \in L\) such that \(\|u\| = \Phi(L)\).

The boundary of \(F\) can be seen more easily on the upper half plane model of the symmetric space \(K \setminus G\). The curve \(t^2 = 1 - a^{-4}\) corresponds to a segment of the unit circle and \(t = \pm \frac{1}{2}\) corresponds to two straight lines \(x = \pm \frac{1}{2}\). Also if we use the projective group \(PSL(2, \mathbb{R})\), then map \(\pi_F\) will be one-to-one in the interior of \(F\) and have degree at most 3 over the boundary.

Now we shall divide \(F\) into two regions:

\[
F^1 = \{g = kan \mid a \leq 1, |t| \leq \frac{1}{2}\};
\]

\[
F^2 = \{g = kan \mid 1 \leq a \leq (\frac{4}{3})^\frac{1}{4}, 1 - a^{-4} \leq t^2 \leq \frac{1}{4}\}.
\]

Then \(F = F^1 \cup F^2\).

4 *non* \(K\)-invariant Parametrization of unital lattices in \(\mathbb{R}^2\): main result

When we parametrize \(U\) in the last section, we allow the parameter \(\theta \in \mathbb{R}/2\pi\mathbb{Z}\) to be arbitrary. Fix \(0 < \epsilon \leq \frac{\pi}{6}\). We now consider only \(k\) with \(|\theta| \leq \epsilon\). Let \(C_\epsilon\) be defined as the open cone

\[
\left\{\begin{pmatrix} x \\ y \end{pmatrix} : x > 0, \frac{|y|}{x} < \tan \epsilon \right\} \subseteq \mathbb{R}^2.
\]

Let \(B(r) = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\right\}\).

Fix \(L \in U\). Let

\[
\Phi_\epsilon(L) = \min\{|\|u\| : u \in L \cap C_\epsilon\}.
\]

**Lemma 4.1** \(\Phi_\epsilon(L)\) exists.

Proof: Pick a generator \(v \in L\). Consider the lines \(\{v^{(m)} : m \in \mathbb{Z}\}\). There are infinitely many \(m_i\) with \(v^{(m_i)} \cap C_\epsilon \neq \emptyset\). If one of \(v^{(m_i)} \cap C_\epsilon\) is of infinite length, then there must be infinitely many lattice points on \(v^{(m_i)} \cap C_\epsilon\). If not, the lengths of \(v^{(m_i)} \cap C_\epsilon\), in ascending order, are of arithmetic progression, thus go to infinity. Recall that \(L \subseteq \cup v^{(m)}\) and \(L \cap v^{(m_i)}\) are equally spaced for all \(m_i\). There must be infinitely many lattice points on these line segments. This show that the set \(\{|\|u\| : u \in L \cap C_\epsilon\}\) is infinite. Therefore \(I = \inf\{|\|u\| : u \in L \cap C_\epsilon\}\) exists.
To show that $\Phi_c(L)$ exists, we must show that this infimum is a minimum. If not, there are infinitely many lattice point $w_i$ such that $\|w_i\| \in [0, I + 1]$. Then there must be infinitely many lattice points in the ball $B(I + 1)$. This is not possible because $B(I + 1)$ is compact. □

Now fix $u \in L$ such that $\|u\| = \Phi_c(L)$. Strictly speaking, we should write $u_c(L)$ for $u$. To simply our notation, we may write $u$ or $u_c$, with the understanding $\epsilon$ and $L$ are fixed. With $u_c$ selected for $L \in U$, we have the $k$ parameter $(\cos \theta, \sin \theta) = u/\|u\|$ and $a = \|u\|$. For the $t$ parameter, we choose $t \in \mathbb{R}/\mathbb{Z}$.

Define

$$F^1_\epsilon = \{ g = \text{kan} : |\theta| < \epsilon, a^2 \leq \csc(\epsilon + |\theta|), |t| \leq \frac{1}{2} \};$$

$$F^2_\epsilon = \{ g = \text{kan} : |\theta| < \epsilon, \csc(\epsilon+|\theta|) \leq a^2 \leq \csc(\epsilon-|\theta|), t \in [-\text{sgn}(\theta)a^{-2} \cot(\epsilon+|\theta|), \text{sgn}(\theta)(1-\sqrt{1-a^{-4}})] \};$$

$$F^3_\epsilon = \{ g = \text{kan} : |\theta| < \epsilon, \csc(\epsilon-|\theta|) \leq a^2 \leq \cot(\epsilon-\theta)+\cot(\epsilon+\theta), t \in [1-a^{-2} \cot(\epsilon+\theta), a^{-2} \cot(\epsilon-\theta)] \};$$

Define $F_\epsilon = \bigcup_{i=1}^{4} F^i_\epsilon \subseteq G$.

**Theorem 4.1** The natural map $\pi|_{\mathcal{F}_\epsilon} : \mathcal{F}_\epsilon \subseteq G \rightarrow G/\Gamma$ is surjective. $\pi|_{\mathcal{F}_\epsilon}$ is injective on the interior of $\mathcal{F}_\epsilon$ and finite on the boundary of $\mathcal{F}_\epsilon$.

The condition that $\epsilon \leq \frac{\pi}{6}$ allows us to claim that

$$\csc(\epsilon - |\theta|) \leq \cot(\epsilon - |\theta|) + \cot(\epsilon + |\theta|).$$

Hence $F^3_\epsilon$ is not empty.

## 5 Parametrization of $\mathcal{U}$: necessary and sufficient condition for $t$

Let $L$ be a unital lattice in $\mathbb{R}^2$. Fix a $u$ in $L \cap C_\epsilon$ such that $\|u\| = \Phi_c(L)$. $u$ must be primitive, namely there is no $u' \in L$ such that $u = q u'$ with $|q| > 1$. Then all lattice points of $L$ must lay on one of the lines $u^{(m)}$ with $m \in \mathbb{Z}$. The lattice points on the line $u^{(m)}$ must be of the form $nu^t + (t + q)u$ for $t \in \mathbb{R}$ and $q \in \mathbb{Z}$. Hence $t$ is in $\mathbb{R}/\mathbb{Z}$. In this section, we choose $t \in [0, 1]$. Let $v = u^t + tu$. Then $L = Zu + Zv$. Clearly, $u$ and $t$ parametrizes $L$. For each $(a, \theta)$ determined by $u$, we need to find the range of $t$, i.e., a necessary and sufficient condition for $t$ such that

for any $l \in L \cap C_\epsilon$, $\|l\| \geq \|u\|$.

Let us call this property $\epsilon$. In contrast to the $K$-invariant parametrization where $\|u\|$ is bounded from above, $\|u_c(L)\|$ is unbounded.

Define the (vertical) stripe

$$S = \{ su^t + tu \mid s \in \mathbb{R}, t \in (0, 1) \}.$$

**Lemma 5.1** Let $v = u^t + tu \in L$ with $t \in [0, 1]$. Then property $\epsilon$ is equivalent to the condition that for any $l \in C_\epsilon \cap L \cap S$, $\|l\| \geq \|u\|$.
Proof: For \( w = su^+ + tu \) with \( t \leq 0 \), \( w \notin C_\epsilon \). For \( w = su^+ + tu \) with \( t \geq 1 \), \( \|w\| \geq \|u\| \). Hence we are only concerned with \( w \in C_\epsilon \cap L \) with \( w = su^+ + tu \ (t \in (0,1)) \). The equivalence with property \( \epsilon \) follows immediately. \( \square \)

Even though our lemma provide a necessary and sufficient condition, it is hard to manage the lattice points in \( S \cap C_0^\epsilon \). If we fix a lattice point \( v = u^+ + tu \), then the number of \( pu +qv \) in the triangle \( S \cap C_0^\epsilon \) may depend on how \( t \) is located near the rational points \( \mathbb{Q} \). This turns out to be a difficult problem.

Instead, we shall consider a necessary condition, namely, for all lattice points \( l \in u^{(\pm)} \cap L \cap C_\epsilon \), \( \|l\| \geq \|u\| \).

**Lemma 5.2** Let \( u_\epsilon(L) = a \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \) with \( \theta \in [0, \epsilon) \). Fix \( t_\epsilon(L) \in [0,1] \). Write \( u,t \) for \( u_\epsilon(L), t_\epsilon(L) \).

1. For any \( l \in L \cap C_\epsilon \cap u^{(1)} \), \( \|l\| \geq \|u\| \) if and only if one of the following is true
   (a) If \( a^2 \leq \csc(\epsilon - \theta) \), then \( t \in [0,1] \);
   (b) If \( a^2 \geq \csc(\epsilon - \theta) \), then \( t \in [0, a^{-2} \cot(\epsilon - \theta)] \cup [\sqrt{1-a^{-4}}, 1] \).

2. For any \( l \in L \cap C_\epsilon \cap u^{(-1)} \), \( \|l\| \geq \|u\| \) if and only if one of the following is true
   (a) If \( a^2 \leq \csc(\epsilon + \theta) \), then \( t \in [0,1] \);
   (b) If \( a^2 \geq \csc(\epsilon + \theta) \), then \( t \in [1 - a^{-2} \cot(\epsilon + \theta), 1] \cup [0,1 - \sqrt{1-a^{-4}}] \).

Proof: Let \( v = u^+ + tu \). By the proof of Lemma 5.1, we only need to find those \( t \in [0,1] \) such that either \( v \notin C_\epsilon \), or \( v \in C_\epsilon \) and \( \|v\| \geq a \). By basic trigonometry,

\[
v \notin C_\epsilon \iff \frac{\|tu\|}{\|u^+\|} \leq \cot(\epsilon - \theta) \iff t \leq a^{-2} \cot(\epsilon - \theta).
\]

On the other hand, if \( v \in C_\epsilon \),

\[
\|v\|^2 = t^2 \|u\|^2 + \|u^+\|^2 = a^2 t^2 + a^{-2} \geq a^2 = \|u\|^2 \iff t^2 \geq 1 - a^{-4}.
\]

Hence either \( t \in [0, a^{-2} \cot(\epsilon - \theta)] \cap [0,1] \) or \( t \in [\sqrt{1-a^{-4}}, 1] \). If \( a^{-2} \csc(\epsilon - \theta) \geq 1 \), the union of these two sets is \([0,1]\). If \( a^{-2} \csc(\epsilon - \theta) \leq 1 \), then the union of these two sets is \([0, a^{-2} \cot(\epsilon - \theta)] \cup [\sqrt{1-a^{-4}}, 1] \). The first statement is proved.

For the second statement, let \( v = -u^+ + (1-t)u \). Then \( 1 - t \) should satisfy similar inequalities for the angle \( \epsilon + \theta \). The second statement follows immediately. \( \square \)

Certainly, in order that \((u_\epsilon, t_\epsilon)\) parametrizes \( L \), both conditions (1) and (2) must be met. We have

**Corollary 5.1** Fix a \( u \in L \) such that \( \|u\| = \Phi_\epsilon(L) \). Suppose that \( \theta \in [0,\epsilon) \) and \( t_\epsilon(L) \in [0,1] \). Then

\[
t_\epsilon(L) \in ([0, a^{-2} \cot(\epsilon - \theta)] \cup [\sqrt{1-a^{-4}}, 1]) \cap ([1 - a^{-2} \cot(\epsilon + \theta), 1] \cup [0,1 - \sqrt{1-a^{-4}}]) \cap [0,1].
\]

What is difficult and perhaps surprising is that these conditions turn out to be sufficient.

We consider the horizontal direction (along \( u \)). Define \( u^+ \) to be the half plane \( \{ w \in \mathbb{R}^2 : \langle w, u^+ \rangle > 0 \} \) and \( u^- \) to be \( \{ w \in \mathbb{R}^2 : \langle w, u^- \rangle < 0 \} \). Then \( \mathbb{R}^2 = u^+ \cup u^0 \cup u^- \).
Lemma 5.3 Suppose that \( u = a \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \) with \( \theta \in [0, \epsilon) \) and \( a > 1 \). If
\[
t \in ([0, a^{-2} \cot(\epsilon - \theta)] \cup \left[\sqrt{1 - a^{-4}}, 1\right]) \cap ([1 - a^{-2} \cot(\epsilon + \theta), 1] \cup [0, 1 - \sqrt{1 - a^{-4}}]) \cap [0, 1],
\]
then the lattice \( \mathcal{L} \) generated by \( u \) and \( u^+ + tu \) is in \( \mathcal{U} \) and satisfies the property that \( \|u\| = \Phi_c(\mathcal{L}) \), the minimal norm of all lattice points in \( \mathcal{L} \cap C_c \).

**Remark:** When \( a \leq 1 \), Lemma 5.2 says that \( t \in \mathbb{R}/\mathbb{Z} \) can be arbitrary. This also turns out to be sufficient. More precisely, the set
\[
\{\theta \in (-\epsilon, \epsilon), \quad a \leq 1, \quad t \in \mathbb{R}/\mathbb{Z}\}
\]
is already contained in the fundamental set \( \mathcal{F} \). The corresponding \( u \) satisfies \( \|u\| = \phi(\mathcal{L}) = \phi_c(\mathcal{L}) \).

**Proof:** The key here is to treat \( \mathcal{L} \cap u^+ \) separately. Consider the half plane \( u^+ \). Set
\[
t \in [0, a^{-2} \cot(\epsilon - \theta)] \cup \left[\sqrt{1 - a^{-4}}, 1\right]
\]
and \( v = u^+ + tu \). Define the line
\[
v^{(m)} = mu + Rv, \quad (m \in \mathbb{Z}).
\]
The lattice \( \mathcal{L} \) is contained in the union of the lines \( v^{(m)} \) with \( m \in \mathbb{Z} \). In fact, we have
\[
\mathcal{L} \cap u^+ = \mathbb{Z}^+ v + Zu.
\]

1. Suppose that \( t \in [0, a^{-2} \cot(\epsilon - \theta)] \). Then \( v = u^+ + tu \) lies outside the cone \( C_c \). For \( m \leq 0 \), \( v^{(m)} \cap u^+ \) lies entirely outside the cone \( C_c \). It suffice to show that for \( m \in \mathbb{Z}^+ \),
\[
w \in v^{(m)} \cap u^+ \cap \mathcal{L}, \quad \|w\| \geq \|u\|.
\]
Observe that \( w = mu + jv \) for \( j \in \mathbb{Z}^+ \) and
\[
\|mu + jv\| = \|(m + jt)u + ju^+\| \geq \|u\|.
\]
Hence for any \( w \in C_c \cap u^+ \cap \mathcal{L}, \|w\| \geq \|u\| \).

2. Suppose that \( t' \in [\sqrt{1 - a^{-4}}, 1] \). Let \( t = t' - 1 \). Then \( t \in [\sqrt{1 - a^{-4}} - 1, 0] \). Let \( v = u^+ + tu \).
Consider the semilattice \( \mathbb{Z}^+ v + Zu \). This is the same semilattice as if we use \( v = u^+ + t'u \).
(a) for \( m \leq 0 \) the lattice points in \( v^{(m)} \cap u^+ \) are of the form \( mu + Z^+ v \). They lay entirely outside \( C_c \).
(b) For \( m > 1 \) and \( m, j \in \mathbb{Z}^+ \), we have \( mu + jv = (m + jt)u + ju^+ \). If \( |m + jt| \geq 1 \), then \( \|mu + jv\| \geq \|u\| \) automatically. If \( m + jt \in (-1, 1) \), then \( jt < 1 - m \). Hence
\[
j > \frac{1 - m}{t} = \frac{-m - 1}{-t} \geq \frac{1}{1 - \sqrt{1 - a^{-4}}} > a^4.
\]
It follows that \( \|mu + jv\| \geq \|ju^+\| > a^3 > a = \|u\| \).
(c) For \( m = 1 \), we would like to show that for \( j \geq 1 \),
\[
\|u + jv\|^2 = \|u + jv\|^2 = (1 +jt)^2u + jv^2 = (1 + jt)^2a^2 + j^2a^{-2} \geq a^2 = \|u\|^2.
\]
This is equivalent to \( (1 + j)^2 \geq 1 - j^2a^{-4} \). This follows from the fact that
\[
2t + j(t^2 + a^{-4}) \geq 2t + t^2 + a^{-4} = (1 + t)^2 - 1 + a^{-4} \geq 0.
\]
Hence for any \(w \in C_\epsilon \cap u^+ \cap L\), \(\|w\| \geq \|u\|\).

Next consider the half plane \(u^−\). Set

\[
t' \in [0, a^{-2} \cot(\epsilon + \theta)] \cup [\sqrt{1-a^{-4}}, 1]
\]

and \(v' = -u^\perp + t'u\). By essentially the same argument,

\[
\|u\| \leq \min\{\|w\| : w \in u^− \cap L \cap C_\epsilon\}.
\]

The only difference is that the angle \(\epsilon - \theta\) becomes \(\epsilon + \theta\) because in \(u^−\), the angle between the boundary of \(C_\epsilon\) and \(u\) is \(\epsilon + \theta\). Notice this angle is less than \(\pi/3\). Since \(\det(u, v') = -1\) in this setting, we switch back to the positive orientation and obtain \(v = u^\perp + (1-t')u\). Since \(t \in \mathbb{R}/\mathbb{Z}\) is chosen to be in \([0, 1]\), we have

\[
t \in ([1 - a^{-2} \cot(\epsilon + \theta), 1] \cup [0, 1 - \sqrt{1-a^{-4}}])
\]

if \(a^{-2} \cot(\epsilon + \theta) \leq \sqrt{1-a^{-4}}\); \(t \in [0, 1]\) if \(a^{-2} \cot(\epsilon + \theta) \geq \sqrt{1-a^{-4}}\).

Observe that the range of \(t\) we obtain is precisely the range of \(t\) specified in Lemma 5.3 \(\Box\)

### 6 Proof of the main result

We shall prove our main theorem for \(\theta \in [0, \epsilon)\). For \(\theta \in (-\epsilon, 0]\), the proof is similar.

Let \(\theta \in [0, \epsilon)\). If \(a = \Phi_\epsilon(L) \leq 1\), by Lemma 5.2 \(t \in [0, 1]\). Conversely, for any such pair of \((a, t) \in (0, 1) \times [0, 1]\), there is one unique lattice with generators \((u, v)\) satisfying \(\|u\| = a\) and \(v = u^\perp + tu\). In fact, this part of \(\mathcal{F}_\epsilon\) overlaps with a small section of \(K\)-invariant \(\mathcal{F}^1\).

We shall now figure out precisely the range of \(t\) for \(a > 1\). Write

\[
I_{a, \theta} = ([0, a^{-2} \cot(\epsilon - \theta)] \cup [\sqrt{1-a^{-4}}, 1]) \cap ([1 - a^{-2} \cot(\epsilon + \theta), 1] \cup [0, 1 - \sqrt{1-a^{-4}}]) \cap [0, 1].
\]

1. If \(a^2 \leq \csc(\epsilon + \theta)\), then \(a^2 \leq \csc(\epsilon - \theta)\). We have

\[
a^{-2} \cot(\epsilon - \theta) \geq a^{-2} \cot(\epsilon + \theta) \geq \sqrt{1-a^{-4}}.
\]

Hence \(t \in [0, 1]\). Combined with the \(a \leq 1\) case, \(L \in \mathcal{F}_\epsilon^1\).

2. If \(\csc(\epsilon + \theta) \leq a^2 \leq \csc(\epsilon - \theta)\), then

\[
a^{-2} \cot(\epsilon - \theta) \geq \sqrt{1-a^{-4}} \geq a^{-2} \cot(\epsilon + \theta).
\]

We have \(t \in [1 - a^{-2} \cot(\epsilon + \theta), 1] \cup [0, 1 - \sqrt{1-a^{-4}}]\). Since \(t \in \mathbb{R}/\mathbb{Z}\), we can make this set into a single interval: \(t \in [-a^{-2} \cot(\epsilon + \theta), 1 - \sqrt{1-a^{-4}}]\). We obtain \(L \in \mathcal{F}_\epsilon^2\).

3. Suppose that \(a^2 > \csc(\epsilon - \theta)\). Then

\[
I_{a, \theta} = ([0, a^{-2} \cot(\epsilon - \theta)] \cup [\sqrt{1-a^{-4}}, 1]) \cap ([1 - a^{-2} \cot(\epsilon + \theta), 1] \cup [0, 1 - \sqrt{1-a^{-4}}])
\]
We check that
\[ a^{-2} \cot(\epsilon - \theta) \geq 1 - \sqrt{1 - a^{-4}}, \quad \sqrt{1 - a^{-4}} \geq 1 - a^{-2} \cot(\epsilon + \theta) \]
by setting \( a^{-2} = \sin \alpha \) and \( 0 < \alpha < \epsilon - \theta \). Then these two inequalities can be derived easily from the fact that
\[ \cot(\epsilon - \theta) \geq \cot(\epsilon + \theta) \geq \cot(\frac{\pi}{3}) = \frac{1}{\sqrt{3}} \geq \tan(\frac{\epsilon}{2}) \geq \tan(\frac{\alpha}{2}) \].

Hence
\[ I_{a,\theta} = [\sqrt{1 - a^{-4}}, 1] \cup [0, 1 - \sqrt{1 - a^{-4}}] \cup ([0, a^{-2} \cot(\epsilon - \theta)] \cap [1 - a^{-2} \cot(\epsilon + \theta), 1]). \]
If \( a^2 \geq \cot(\epsilon + \theta) + \cot(\epsilon - \theta) \), then \( I_{a,\theta} = [\sqrt{1 - a^{-4}}, 1] \cup [0, 1 - \sqrt{1 - a^{-4}}] \). If \( a^2 \leq \cot(\epsilon + \theta) + \cot(\epsilon - \theta) \), then
\[ I_{a,\theta} = [\sqrt{1 - a^{-4}}, 1] \cup [0, 1 - \sqrt{1 - a^{-4}}] \cup [1 - a^{-2} \cot(\epsilon + \theta), a^{-2} \cot(\epsilon - \theta)]. \]

Using a shift, we combine \([0, 1 - \sqrt{1 - a^{-4}}]\) with \([\sqrt{1 - a^{-4}} - 1, 0]\) and obtain
\[ t \in [\sqrt{1 - a^{-4}} - 1, 1 - \sqrt{1 - a^{-4}}]. \]
Hence we obtain \( \mathcal{F}^3_\epsilon \) and \( \mathcal{F}^4_\epsilon \). We shall remark that \( \cot(\epsilon + \theta) + \cot(\epsilon - \theta) \geq \csc(\epsilon - \theta) \) since \( \epsilon \leq \frac{\pi}{3} \). Therefore \( \mathcal{F}^3_\epsilon \) is nonempty.

\[ \square \]

It is interesting to try to compare \( \mathcal{F}_\epsilon \) with \( \mathcal{F} \). Perhaps the most distinctive feature is that there is another ”cusp” of the shape \(|t| \leq 1 - \sqrt{1 - a^{-4}} \geq \frac{1}{2}a^{-4} \) as \( a \to \infty \) in \( \mathcal{F}_\epsilon \).

We can now rewrite \( \mathcal{F}_\epsilon \) in the \( KNA \) decomposition.
\[ \mathcal{F}^1_\epsilon = \{ g = kan : |\theta| < \epsilon, a^2 \leq \csc(\epsilon + |\theta|), |T| \leq \frac{1}{2}a^2 \}; \]
\[ \mathcal{F}^2_\epsilon = \{ g = kan : |\theta| < \epsilon, \csc(\epsilon + |\theta|) \leq a^2 \leq \csc(\epsilon - |\theta|), \text{sgn}(\theta)T \in [-\cot(\epsilon + |\theta|), (a^2 - \sqrt{a^4 - 1})] \}; \]
\[ \mathcal{F}^3_\epsilon = \{ g = kan : |\theta| < \epsilon, \csc(\epsilon - |\theta|) \leq a^2 \leq \cot(\epsilon - \theta) + \cot(\epsilon + \theta), T \in [a^2 - \cot(\epsilon + \theta), \cot(\epsilon - \theta)] \}; \]
\[ \mathcal{F}^4_\epsilon = \{ g = kan : |\theta| < \epsilon, \csc(\epsilon - |\theta|) \leq a^2, T \in [\sqrt{a^4 - 1} - a^2, a^2 - \sqrt{a^4 - 1}] \}. \]
For \( \mathcal{F}^3_\epsilon \) and \( \theta \in [0, \epsilon) \), we can shift the parameter \( T \) to \((- \cot(\epsilon + \theta), \cot(\epsilon - \theta) - a^2)\).

**Theorem 6.1** Let \( \epsilon \in (0, \frac{\pi}{3}) \). The fundamental set can be chosen inside
\[ \{ \theta \in (-\epsilon, \epsilon), a \in (0, \infty), |T| \leq \cot(\epsilon + |\theta|) \} \subseteq G. \]

Proof: Since \( \epsilon + |\theta| < \frac{\pi}{3} \), in \( \mathcal{F}^1_\epsilon \),
\[ |T| \leq \frac{1}{2}a^2 \leq \frac{1}{2} \csc(\epsilon + |\theta|) < \cot(\epsilon + |\theta|). \]
In \( \mathcal{F}^2_\epsilon \), since \( a^2 \geq \csc(\epsilon + |\theta|) \),
\[ a^2 - \sqrt{a^4 - 1} \leq \cot(\epsilon + |\theta|). \]
Hence $|T| \leq \cot(\epsilon + |\theta|)$. In $\mathcal{F}_\epsilon^3$, if $\theta \in (-\epsilon, 0)$, then $T \in [a^2 - \cot(\epsilon + \theta), \cot(\epsilon - \theta)]$. Clearly $|T| \leq \cot(\epsilon + |\theta|)$. If $\theta \in [0, \epsilon)$, then $T \in (-\cot(\epsilon + \theta), \cot(\epsilon - \theta) - a^2)$. We also have $|T| \leq \cot(\epsilon + |\theta|)$.

In $\mathcal{F}_\epsilon^4$, we clearly have $a^2 - \sqrt{a^4 - 1} \leq \cot(\epsilon + |\theta|)$.

Hence $\mathcal{F}_\epsilon$ can be put entirely inside

\[ \{ \theta \in (-\epsilon, \epsilon), a \in (0, \infty), |T| \leq \cot(\epsilon + |\theta|) \} . \]

When $a^2 \in [\csc(\epsilon + |\theta|), \cot(\epsilon + \theta) + \cot(\epsilon - \theta)]$, this is the best approximation. For a small or big, we have two cusps: one at 0 and one at $\infty$. □

Notice that $\tan(\epsilon + |\theta|) \geq \tan(\epsilon) \geq \epsilon$. Hence $\cot(\epsilon + \theta) \leq \frac{1}{\epsilon}$. In $\mathcal{F}_\epsilon$, there is a "duality" between the $k$ parameter and $T$ parameter, namely as the size of $|\theta|$ shrinks, the range of $T$ can increase to size $\frac{1}{\epsilon}$.

Now we shall give an integral inequality on functions on $G/\Gamma$. Since $\cot(\epsilon + |\theta|) \leq \cot(\epsilon) < \frac{1}{\epsilon}$, the following theorem is an immediate consequence of Theorem 6.1.

**Theorem 6.2** Let $f \in L^2(G/\Gamma)$. Then

\[
\int_{-\epsilon}^{\epsilon} \int_{-\frac{\epsilon}{3}}^{\frac{\epsilon}{3}} \int_0^\infty |f(k_\theta n_T a)|^2 \frac{da}{a} dT d\theta \geq \| f \|^2_{L^2} .
\]

Here $k_\theta n_T a$ is the KNA decomposition and $\frac{da}{a} dT d\theta$ is the $G$-invariant measure.

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