Information-theoretic principle entails orthomodularity of a lattice

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Quantum logical axiomatic systems for quantum theory usually include a postulate that a lattice under consideration is orthomodular. We propose a derivation of orthomodularity from an information-theoretic axiom. This provides conceptual clarity and removes a long-standing puzzle about the meaning of orthomodularity.

1 INTRODUCTION

There exist many axiomatic systems from which one derives the formalism of quantum theory. Some of these systems employ the concepts of orthodox quantum logic, i.e. of the theory of orthomodular lattices. For reconstruction of quantum theory, orthomodularity of the lattice is a necessary, although not a sufficient condition. In the axiomatizations it is commonly postulated or derived from formal assumptions about relations between elements of the lattice. The work of Beltrametti and Cassinelli is a notable exception: they give a conceptual justification of orthomodularity. Namely, they argue that “orthomodularity corresponds to the survival... of a notion of the logical conditional, which takes the place of the classical implication associated with Boolean algebra” A similar motivation was chosen by Jauch and Piron, who reformulated orthomodularity as a condition that if a
proposition is greater than another one in the lattice, then they must be compatible, i.e. their span with the operations of join and meet and the orthogonal complement must give a Boolean algebra. Although his own motivation of orthomodularity is different, Drieschner gives a useful shorthand to this idea: “if $x$ implies $y$, they have to be compatible”. However, to follow Jauch and Piron, one has to make a substantial intuitive assumption: in order to use the derivation of orthomodularity via the existence of Boolean subalgebras, one must separately introduce a notion of compatibility of propositions. It is clear that compatibility refers to the concept of complementarity in quantum physics, and thus one uses an intuitive feature motivated by quantum theory, yet before deriving this theory and for deriving it. Such an early recourse to the quantum theoretic intuition is certainly an unwelcome factor.

In the present article we give a different conceptual justification of orthomodularity of a lattice in the framework of the information-theoretic approach to quantum theory. This approach consists in viewing quantum theory as a theory about, or of, information. Constraints imposed on the kind of information available in the theory lead then to a particular, rather than general, theory of information, and one seeks for a set of constraints which will produce quantum theory. To express the constraints mathematically, a certain formalism must be employed, e.g. quantum logical or algebraic.

In the quantum logical formalism, we show that orthomodularity arises as a consequence of the following information-theoretic axiom:

**Axiom I.** There is a maximum amount of relevant information that can be extracted from a system.

This axiom was first formulated by Rovelli who took inspiration from the work of Wheeler. Rovelli linked it to the idea of maximal “informational capacity” of a system and argued informally that Axiom already adds the Planck constant to physics. He, however, did not discuss the meaning, nor treat rigorously, what we take to be the crucial term in the formulation of the axiom: relevant information. After introducing basic concepts of the theory of orthomodular lattices in Section we propose in Section a formal definition of the notion of relevance of information. This leads to a key result in Section in which we prove that the lattice of yes-no questions is orthomodular.
2 ELEMENTS OF LATTICE THEORY

Lattice $\mathcal{L}$ is a partially ordered set in which any two elements $x, y$ have a supremum $x \lor y$ and an infimum $x \land y$. Equivalently, one can require that a set $\mathcal{L}$ be equipped with two idempotent, commutative, and associative operations $\lor, \land : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, which satisfy $x \lor (y \land x) = x$ and $x \land (y \lor x) = x$. The partial ordering is then defined by $x \leq y$ if $x \land y = x$. The largest element in the lattice, if it exists, is denoted by 1, and the smallest one, if it exists, by 0.

Lattice is called complete when every subset of $\mathcal{L}$ has a supremum as well as an infimum. Complete lattice always contains elements 0 and 1. An atom of lattice $\mathcal{L}$ is an element $a$ for which $0 \leq x \leq a$ implies that $x = 0$ or $x = a$. A lattice with 0 is called atomic if for every $x \neq 0$ in $\mathcal{L}$ there is an atom $a \neq 0$ such that $a \leq x$.

Lattice is called distributive if

$$x \lor (y \land z) = (x \lor y) \land (x \lor z).$$

(1)

One can weaken the distributivity condition by requiring (1) only if $x \leq z$. Thus, a lattice is called modular if for all $y$

$$x \leq z \Rightarrow x \lor (y \land z) = (x \lor y) \land z.\quad (2)$$

Orthocomplemented lattice is a lattice $\mathcal{L}$ which carries orthocomplementation, i.e. a map $x \mapsto x^\perp$, satisfying for all $x, y \in \mathcal{L}$ the following conditions:

(a) $x^{\perp\perp} = x$,  (b) $x \leq y \iff y^\perp \leq x^\perp$,  (c) $x \land x^\perp = 0$,  (d) $x \lor x^\perp = 1$. In an orthocomplemented lattice hold de Morgan laws

$$1^\perp = 0; \quad 0^\perp = 1; \quad (x \lor y)^\perp = x^\perp \land y^\perp; \quad (x \land y)^\perp = x^\perp \lor y^\perp.\quad (3)$$

By further weakening modularity condition (2), one arrives at the following definition:

**Definition 1.** Orthocomplemented lattice $\mathcal{L}$ is called orthomodular if condition (2) holds for $y = x^\perp$, that is,

$$x \leq z \Rightarrow x \lor (x^\perp \land z) = z.\quad (4)$$
It is instructive to give the following reformulation of the condition of orthomodularity, for which we offer a new proof.

**Lemma 2.** An orthocomplemented lattice $L$ is orthomodular if and only if $x \leq z$ and $x^\perp \wedge z = 0$ imply $x = z$.

**Proof.** If the lattice is orthomodular, i.e. (4) holds, and if $x^\perp \wedge z = 0$, then $z = x \vee 0 = x$. To prove the converse, it suffices to show that if the lattice is not orthomodular then there exist elements $x$ and $z$ such that

$$x \leq z, \quad x^\perp \wedge z = 0, \quad x \neq z. \quad (5)$$

Let us use the notation $x < z$ if $x \leq z$ and $x \neq z$. We can then rewrite (5) as

$$x < z, \quad x^\perp \wedge z = 0. \quad (6)$$

Assume that the lattice is not orthomodular. In virtue of Definition 1 there exist elements $y$ and $z$ such that

$$y \leq z, \quad y \lor (y^\perp \wedge z) \neq z. \quad (7)$$

Now recall that in any lattice holds

$$a \leq b \Rightarrow (c \wedge b) \lor a \leq (c \lor a) \wedge b \quad \forall c. \quad (8)$$

Put in (8) $a = y$, $b = z$, $c = y^\perp$. Follows that

$$(y^\perp \wedge z) \lor y \leq (y^\perp \lor y) \wedge z. \quad (9)$$

In the right-hand side replace $y^\perp \lor y$ by 1, and $1 \wedge z = z$. Equation (9) then takes the form

$$(y^\perp \wedge z) \lor y \leq z. \quad (10)$$

From equations (10) and (7) one obtains that

$$(y^\perp \wedge z) \lor y < z. \quad (11)$$

On the other hand, from de Morgan laws one has

$$z \wedge (y \lor (y^\perp \wedge z))^\perp = z \wedge (y^\perp \wedge (y^\perp \wedge z)) = z \wedge (y^\perp \wedge (y \lor z^\perp)) = (z \wedge y^\perp) \wedge (y \wedge z^\perp) = (z \wedge y^\perp) \wedge (z \wedge y^\perp)^\perp = 0. \quad (12)$$
Now put \( x = y \lor (y^\perp \land z) \). Equations (11) and (12) can be rewritten as

\[
x < z, \quad x^\perp \land z = 0.
\] (13)

This is exactly what was required in (6).

3 INFORMATION-THEORETIC APPROACH

The fundamental notion of measurement in the information-theoretic approach is represented as a yes-no question. Information is then brought about in the answer to a yes-no question. Input of the information-theoretic approach to the derivation of orthomodularity is therefore limited to the choice of the departure point: It is a set of yes-no questions that can be asked to the system. We postulate that this set is an orthocomplemented complete atomic lattice \( L \), with orthogonal complementation denoting negation of a question.

At the level of ordinary linguistic usage of words, assume that the information obtained from a question \( a \) is relevant for the observer. We are looking for ways to make it irrelevant. This can be achieved by asking some new question \( b \) that will make \( a \) irrelevant. Consider, for instance, \( b \) such that it entails the negation of \( a \): \( b \rightarrow \neg a \). If the observer asks the question \( a \) and obtains an answer to \( a \), but then asks a genuine new question \( b \), it means, by virtue of what “genuine” commonly signifies, that the observer expects either a positive or a negative answer to \( b \). This, in turn, is only possible if information \( a \) is no more relevant; indeed, otherwise the observer would be bound to always obtain the negative answer to \( b \). Consequently, we say that that, by asking \( b \), the observer makes the question \( a \) irrelevant. Such considerations of common linguistic usage of words motivate the following formal definition of relevance.

**Definition 3.** Question \( b \) is called irrelevant with respect to question \( a \) if \( b \land a^\perp \neq 0 \). Otherwise question \( b \) is called relevant with respect to question \( a \).

What does Definition 3 mean with regard to the best known lattice, i.e. the Hilbert lattice or the lattice of all closed subspaces of the Hilbert space?
Figure 1: Notion of relevance. Order in the lattice is denoted by solid lines and grows from bottom to top, i.e. $0 \leq a \leq b$, etc. If there exists $c \neq 0$ such that $c \leq b$ and $c \leq a^\perp$, then question $b$ is irrelevant with respect to question $a$, i.e. in $b$ “is contained a component” of non-$a$, and consequently, by asking $b$, one renders information of the question $a$ irrelevant.
If \( x \in \mathcal{L} \) is a closed subspace of the Hilbert space \( H \), then its orthogonal complement \( x^\perp \in \mathcal{L} \) satisfies \( x \oplus x^\perp = H \) and \( \dim x = \text{codim } x^\perp \). Assume, as on Figure 1 that two Hilbert lattice elements \( a \) and \( b \) are such that \( a < b \). Then \( \dim b > \dim a \) and, consequently, there exist closed subspaces of \( b \) not contained in \( a \). Call one such subspace \( c \). In virtue of the definition of orthogonality, one has \( a \oplus a^\perp = H \), and this implies that \( c \cap a^\perp \neq \emptyset \). Therefore, \( b \land a^\perp \neq 0 \). We obtain that if \( b > a \) in a Hilbert lattice, then \( b \) is always irrelevant with respect to \( a \). Generally, as follows from Lemma 2 and Definition 3, in an orthomodular lattice there exist no questions relevant with respect to a given one, which at the same time are strictly greater than this question.

Relevance then becomes trivially assimilated to orthogonality in the case of Hilbert lattices. Any closed subspace \( c \) such that \( c \subseteq a \) provides a lattice element \( c \) relevant with respect to \( a \), and all other lattice elements are irrelevant with respect to \( a \). If, for example, one is concerned with the Hilbert space of spin projections, with respect to \( \sigma \), only operators which commute with it are relevant, i.e. \( \sigma \) itself and the null projector. The interest in the notion of relevance is therefore motivated, not by the Hilbert lattices, but by the generic case where relevance is not a mere reformulation of set-theoretic inclusion and of the compatibility of propositions. While some theorists deliberately choose not to cross this limit, the property of orthomodularity can be properly derived only if one goes beyond the standard Hilbert lattice vision.

An example when the notion of relevance is non-trivial is given on Figure 2. This is a non-orthomodular complete atomic lattice, in which \( b \) is relevant with respect to \( a \), although \( b \geq a \) holds. In general, there exist infinitely many non-orthomodular complete atomic orthocomplemented lattices, which can be for example constructed by slightly violating Greechie’s procedure. All such lattices are “wild” from the point of view of quantum theory, and it is necessary to give an argument excluding them from consideration.

Apart from relevance, another concept mentioned in Axiom 1 which necessitates a formal definition, is the amount of information. In general, when relevance of information is not preserved throughout several acts of asking questions, i.e. when, as new information comes in, some older information ceases to be relevant, it is impossible to dissociate the amount of information generated by a new question from the considerations of relevance of this new
question with respect to the previously asked ones. Influence of every new question on the total amount of relevant information is therefore contextual: it depends on which questions have been asked before. It is easy, though, to define one particular property of the amount of information in an exceptional case when the contextual dependence disappears. Thus, we assume that the amount of information is a monotonously growing function of the number of answers to yes-no questions that the observer obtains, in the exceptional case when none of the new questions render irrelevant any of the previous questions. In other words, if the observer keeps all the old information, then the amount of information grows as new information comes in.

Going back to the phrase in the first paragraph in which we introduced the set of yes-no questions, we interpret the words “can be asked” so that one is concerned with possible acts of bringing about information, i.e. the lattice contains all possible questions, most of which, indeed, will never be asked. Applied to the amount of information, such a lattice construction leads to a further assumption that there always are sufficiently many questions as to bring in, potentially, any amount of information permitted \textit{a priori} by the

![Figure 2: An orthocomplemented non-orthomodular lattice.](image)
theory. In particular, there always exists a question such that it brings the maximum amount of relevant information to be had about the system.

4 ORTHOMODULARITY OF A LATTICE

Axiom I ensures that cases as the one shown on Figure 2 do not appear in quantum theory, i.e. that all important lattices are orthomodular. This is achieved in virtue of the following theorem.

Theorem 4. \( L \) is an orthomodular lattice.

Proof. By Axiom I there exists a finite upper bound of the amount of relevant information. Let this be an integer \( N \). Select an arbitrary question \( a \) and consider a question \( \tilde{a} \) such that

\[
\{a, \tilde{a}\}
\]

bring the maximum amount of relevant information, i.e. \( N \) bits. Notation \( \{\ldots\} \) means a sequence of questions that are asked one after another. Because all information here is relevant, we have by the definition of relevance that

\[
\tilde{a} \land a^\perp = 0.
\]

(15)

We now use Lemma 2. It is sufficient to show that \( a \leq b \) and \( a^\perp \land b = 0 \) imply \( a = b \). Note first that the second condition means, by Definition 3, that \( b \) is relevant with respect to \( a \). Since \( a \leq b \), we obtain that

\[
b^\perp \leq a^\perp.
\]

(16)

Using this result and the result of Equation 15 we derive

\[
\tilde{a} \land b^\perp = 0.
\]

(17)

This, in turn, means that question \( \tilde{a} \) is relevant with respect to \( b \).

Now suppose, contrary to what is needed, that \( b > a \) and consider the following sequence of questions:

\[
\{a, b, \tilde{a}\}
\]

(18)
From Equations 15 and 17 follows that relevance is not lost in this sequence of question, i.e. all later information is relevant with respect to all earlier information. However, while relevance is preserved, this sequence, in virtue of the fact that \( a \neq b \), brings about more information than sequence (14). It means that we have constructed a setting in which the amount of relevant information is strictly greater than \( N \) bits, causing a contradiction with the initial assumption. Consequently, \( a = b \) and the lattice \( \mathcal{L} \) is orthomodular. \( \square \)

5 CONCLUSION

Theorem 1 provides a solution to the long-standing puzzle about the meaning of orthomodularity. Historically, orthomodularity appeared at a late stage of quantum logical research, the first invention being the theory of von Neumann algebras,\(^{31}\) second the theory of von Neumann modular lattices,\(^{32}\) and only third the theory of orthomodular lattices. Its origin can be traced back to Husimi’s work in which he was predominantly concerned with deriving non-Boolean logic from empirical facts,\(^{33}\) and in full orthomodularity was for the first time discussed much later.\(^{34}\) Thus, appearing as the weakest lattice constraint and as a result of the historic evolution of quantum logic, orthomodularity was easily taken for granted in axiomatization attempts. Reviving recently the interest in quantum logic, information-theoretic approach permits to bridge this conceptual gap. From the information-theoretic viewpoint orthomodularity must be seen as a consequence of the finite amount of relevant information.

ACKNOWLEDGEMENTS

The initial idea of proof of Lemma 2 is due to Prof. V. A. Franke. Many thanks to Carlo Rovelli and Jeffrey Bub for their insightful comments.
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