On the problem of neural network decomposition into some subnets

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Abstract

An artificial neural network is usually treated as a whole system, being characterized by its ground state (the global minimum of the energy functional), the set of fixed points, their basins of attraction, etc. However, it is quite natural to suppose that a large network may consist of a set of almost autonomous subnets. Each subnet works independently (or almost independently) and analyzes the same pattern from other points of view. It seems that it is a proper model for the natural neural networks. We discuss the problem of decomposition of a neural network into a set of weakly coupled subnets. The used technique is similar to the method for the extremal grouping of parameters, proposed by E.M.Braverman (1970).

I. HOPFIELD’S MODEL OF A NEURAL NETWORK

A neural network of size $n$ is a set of $n$ connected spin variables (spins) $\sigma_i$; each $\sigma_i$ can be either $1$ or $-1$:

$$\sigma_i = \{\pm 1\}, \quad i = 1, 2, ... n. \quad (1)$$
The interaction between spins is described by a connection matrix. Let $J_{ii'}$ be the connection strength between the spins $\sigma_i$ and $\sigma_{i'}$, and let $\sigma_i(t)$ be the value of $i$th spin at time $t$, then

$$h_i(t) = \sum_{i'=1}^{n} J_{ii'} \cdot \sigma_{i'}(t)$$

represents the local field that the spin $\sigma_i$ experiences at time $t$. Under the action of this field the new value of the spin $\sigma_i$ at the next moment $t + 1$ is:

$$\sigma_i(t + 1) = \begin{cases} 
\sigma_i(t), & \text{if } h_i(t) \cdot \sigma_i(t) \geq 0 \\
-\sigma_i(t), & \text{if } h_i(t) \cdot \sigma_i(t) < 0
\end{cases}$$

The vectors which coordinates are $\{\pm 1\}$ only is called the configuration vectors. We denote the configuration vectors by small Greek letters.

It is convenient to describe the state of the network at time $t$ by $n$-dimensional configuration vector

$$\vec{\sigma}(t) = (\sigma_1(t), \sigma_2(t), \ldots, \sigma_n(t)).$$

If we introduce the connection matrix $\mathbf{J} = (J_{ii'})_1^n$ and define the quadratic form

$$E(t) = -\sum_{i,i'=1}^{n} J_{ii'} \cdot \sigma_i(t) \cdot \sigma_{i'}(t) = -(\mathbf{J}\vec{\sigma}(t), \vec{\sigma}(t)),$$

then it is easy to show that for any symmetrical connection matrix $\mathbf{J}$ the overturn of a spin $\sigma_i(t)$, which value does not coincide with the sign of $h_i(t)$, leads to the decrease of $E(t)$:

$$E(t + 1) = E(t) + 4 \cdot \sigma_i(t) \cdot h_i(t).$$

$E(t)$ can be interpreted as the energy of the state $\vec{\sigma}(t)$. As the number of network states is finite and the $i$th spin does not turn over if $h_i(t) = 0$, it is obvious that the final state of the network would be a state which corresponds to a minimum (may be local) of the energy $E(t)$. In such a state every spin $\sigma_i$ will be aligned with its local field $h_i$ and there will be no

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1For the sake of simplicity we suppose that there is no self-interaction in the system: $J_{ii} = 0 \forall i$. 

further evolution of the network. These states are called the \textit{fixed points} of the network. Consequently, if the configuration vector \( \vec{\sigma}^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*) \) is a fixed points, then

\[
\sigma_i^* = \text{sgn} \left( \sum_{i'=1}^{n} J_{ii'} \cdot \sigma_{i'}^* \right), \quad i = 1, 2, \ldots, n.
\]  

In what follows the configuration vectors which are fixed points will be marked by superscripts “*”.

Let’s define a neural network which is called Hopfield’s network. Let \( p \) be a number of preassigned configuration vectors \( \vec{\xi}^{(l)} \), which are called the \textit{memorized patterns}:

\[
\vec{\xi}^{(l)} = (\xi_1^{(l)}, \xi_2^{(l)}, \ldots, \xi_n^{(l)}), \quad l = 1, 2, \ldots, p.
\]  

(The superscripts numerate the vectors from \( \mathbb{R}^n \) and the subscripts numerate their coordinates. Usually it is assumed that \( p < n \) or even \( p << n \)). J.Hopfield [1] proposed to use the connection matrix of the form:

\[
J_{ii'} = \begin{cases} 
\sum_{l=1}^{p} \xi_i^{(l)} \xi_{i'}^{(l)}, & i \neq i' \\
0, & i = i', \quad i, i' = 1, 2, \ldots, n.
\end{cases}
\]

The matrix \( J \) (8) is a symmetric matrix with zero diagonal elements. Then, the fixed points are the minima of the energy \( E \) given by Eq.(4). If we define \((p \times n)\)-matrix \( \Xi \) with \( p \) memorized patterns (7) as the rows,

\[
\Xi = \begin{pmatrix} \vec{\xi}^{(1)} \\
\vec{\xi}^{(2)} \\
\vdots \\
\vec{\xi}^{(p)} \end{pmatrix} = \begin{pmatrix} \xi_1^{(1)} & \xi_2^{(1)} & \cdots & \xi_n^{(1)} \\
\xi_1^{(2)} & \xi_2^{(2)} & \cdots & \xi_n^{(2)} \\
\vdots & \vdots & \cdots & \vdots \\
\xi_1^{(p)} & \xi_2^{(p)} & \cdots & \xi_n^{(p)} \end{pmatrix}
\]

then the expression for the connection matrix takes the form

\[
J = \Xi^T \cdot \Xi - p \cdot I,
\]

where \((n \times p)\)-matrix \( \Xi^T \) is the transpose of matrix \( \Xi \) and \( I \) is the unit matrix in the space \( \mathbb{R}^n \). Therefore the searching of the fixed points of Hopfield’s network reduces to the maximization of the functional

\[
(\Xi^T \cdot \Xi \vec{\sigma}, \vec{\sigma}) = \| \Xi \vec{\sigma} \|^2.
\]
But this problem can be reformulated, if \( n \) \( p \)-dimensional vectors \( \vec{\xi}_i \), which are the columns of matrix \( \Xi \) are introduced:

\[
\vec{\xi}_i = \begin{pmatrix}
\xi_i^{(1)} \\
\xi_i^{(2)} \\
\vdots \\
\xi_i^{(p)} 
\end{pmatrix} \in \mathbb{R}^p, \quad i = 1, 2, \ldots, n.
\]  (11)

In contrast to \( n \)-dimensional vectors \( \vec{\xi}^{(l)} \) defined by Eq.(7), here the subscripts numerate the vectors \( \vec{\xi}_i \) from \( \mathbb{R}^p \) and the superscripts numerate their coordinates.

It is easy to see, that the problem of maximization of the functional \( \| \Xi \vec{\sigma} \|^2 \) takes the form:

\[
\| \sum_{i=1}^{n} \sigma_i \cdot \vec{\xi}_i \| \to \text{max}, \quad \text{where } \sigma_i = \{ \pm 1 \} \quad \forall i.
\]  (12)

In other words, we have to find out such a weighted sum of the \( p \)-dimensional vectors \( \vec{\xi}_i \) with the weights are equal \( \{ \pm 1 \} \), which length would be maximal. In what follows the expression (12) would be the start point of our consideration.

II. FACTOR ANALYSIS AND EXTREMAL GROUPING OF PARAMETERS

The problem (12) is a special case of the problem which is well-known for the centroid method of the factor analysis [4]. The basic idea of the factor analysis is to replace the great number of the parameters, which describe the objects under investigation, by a considerably lesser set of specially constructed characteristics provided that such replacement would not lead to the loss of the essential information about these objects.

The formalization of this idea can be done in the following way. Let us have \( p \) objects which are represented by the vectors \( \vec{x}^{(l)} = (x_{1}^{(l)}, x_{2}^{(l)}, \ldots, x_{n}^{(l)}), \ l = 1, 2, \ldots, p \) in the space \( \mathbb{R}^n \). Let’s consider the \( (p \times n) \)-matrix \( X \), which rows are the object-vectors \( \vec{x}^{(l)} \). (This matrix is an analog of the matrix \( \Xi \) (9), but now the matrix elements can be an arbitrary real numbers, and not \( \pm 1 \) only.) On the other hand the matrix \( X \) can be described as the
matrix which columns are the parameter-vectors $\vec{x}_i$:

$$
\vec{x}_i = \begin{pmatrix}
  x_i^{(1)} \\
  x_i^{(2)} \\
  \vdots \\
  x_i^{(p)}
\end{pmatrix}, \quad i = 1, 2, \ldots, n.
$$

(We recall that the vectors from the space $\mathbb{R}^n$ are numerated by superscripts: $l = 1, \ldots, p$, and the vectors from the space $\mathbb{R}^p$ by subscripts: $i = 1, \ldots, n$.)

If a relatively small number $t$ ($t << n$) of such $p$-dimensional vectors $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_t$ can be found, that the parameter-vectors $\vec{x}_i$ can be represented in the form

$$
\vec{x}_i = \sum_{s=1}^{t} a_{is} \cdot \vec{f}_s + \vec{a}_i, \quad i = 1, 2, \ldots, n,
$$

where the remainders $\vec{a}_i$ are small in some sense and can be omitted, then the objects can be described by the characteristics $\vec{f}_s$ instead of the initially used parameters $\vec{x}_i$. Indeed, due to the smallness of the remainders $\vec{a}_i$, characteristics $\vec{f}_s$ adequately describe the investigated phenomenon. But it is much more convenient to work if the number of the parameters is considerably reduced. The characteristics $\vec{f}_s$ are called the essential factors.

The various models of the factor analysis differ in the forms in which the factors $\vec{f}_s$ are sought and the sense in which the smallness of $\vec{a}_i$ is understood. In the centroid method the first factor $\vec{f}_1$ is sought as a linear combination $\sum_{i=1}^{n} \sigma_i \cdot \vec{x}_i$ of the parameters $\vec{x}_i$ with the weights $\sigma_i = \{\pm 1\}$, that have a maximal length

$$
\vec{f}_1 \propto \sum_{i=1}^{n} \sigma_i^* \cdot \vec{x}_i, \quad \text{where} \quad \| \sum_{i=1}^{n} \sigma_i^* \cdot \vec{x}_i \| = \max_{\sigma_i = \{\pm 1\}} \| \sum_{i=1}^{n} \sigma_i \cdot \vec{x}_i \|.
$$

(13)

The comparison of Eq.(12) and Eq.(13) shows that the problem of the network fixed points searching is equivalent to the construction of the first centroid factor for the set of the $p$-dimensional vectors $\vec{\xi}_i$ (11).

In the centroid method after the construction of the first factor $\vec{f}_1$, the vectors $b_1 \cdot \vec{f}_1$, where $b_1, i = 1, 2, \ldots, n$ are some coefficients, are subtracted from each parameter-vector $\vec{x}_i$. In such a way we obtain a new set of vectors $\vec{x}_i' = \vec{x}_i - b_1 \cdot \vec{f}_1$ for which their own factor is
constructed by analogy. This factor would be the second factor for the initial parameters $\vec{x}_i$. This process will be repeated till the vectors which are obtained after the next step would be small enough. For details see \cite{2-3}.

An important generalization of the factor analysis was the idea of the extremal grouping of the parameters suggested by E.M. Braverman in 1970 \cite{3}. Braverman introduced a model of the factor analysis where an essentially nonuniform distribution of the vectors $\vec{x}_i$ in the space $\mathbb{R}^p$ was taken into account.

Indeed, if the number $n$ of the parameter-vectors is very large, it is possible that they can be divided into some compact groups such that the vectors joined into one group are "strongly correlated" with each other and are "weakly correlated" with the parameters included into other groups. Then it is reasonable to construct the factors not for the full set of the parameter-vectors, but for every compact group separately. If these groups are compact enough, we can restrict ourselves with the first factor of each group only. To divide the parameter-vectors into these compact groups, Braverman suggested an approach connected with the maximization of a certain functional depending both on the grouping of the parameters and on the choice of the factors.

Let’s write down Braverman’s functional. Let $p$-dimensional vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ be divided into current disjoint groups $A_1, A_2, \ldots, A_t$:

$$A_1 \cup A_2 \cup \ldots \cup A_t = \{1, 2, \ldots, n\}.$$  

For every group $A_s$ the first centroid factor can be constructed as the solution of the problem:

$$\| \sum_{i \in A_s} \sigma^*_i \cdot \vec{x}_i \| = \max_{\sigma_i = \{\pm 1\}} \| \sum_{i \in A_s} \sigma_i \cdot \vec{x}_i \|. \quad (14)$$

Then, the partition into $t$ the most compact groups is obtained as a result of the maximization of the functional:

$$M(A_1, A_2, \ldots, A_t) = \| \sum_{i \in A_1} \sigma^*_i \cdot \vec{x}_i \| + \| \sum_{i \in A_2} \sigma^*_i \cdot \vec{x}_i \| + \ldots + \| \sum_{i \in A_t} \sigma^*_i \cdot \vec{x}_i \| \to \max \quad (15)$$

where $\sigma^*_i$ are the solutions of the problem (14) for every group $A_s$, $s = 1, 2, \ldots, t$. We want to notice, that, though the problem of maximization of the functional (15) is very hard, the
method for the extremal grouping of parameters was successfully used for various problems in engineering, economics, sociology, psychology and other fields [3,4].

III. NEURAL NETWORKS DECOMPOSITION INTO SOME SUBNETS

Let in Eqs.(14),(15) the vectors $\vec{x}_i$ be replaced by the vectors $\vec{\xi}_i$ from Eq.(11), i.e. only the vectors with the coordinates $\{\pm 1\}$ are under consideration. Then, in the framework of the neural network paradigm, the problem (14),(15) can be interpreted as the problem of the grouping of the network neurons into some connected groups.

Indeed, natural networks have evident differential structure: different neuron groups have different functions, they respond for the regulation/analysis of different aspects of a complicate pattern which is worked over by the network. To some extent every such neuron group can be treated as an autonomous neural network of the smaller size which is dealing with some specific features of the pattern.

Let a network be consisted of some groups of neurons (subnets) $A_1, A_2, \ldots, A_t$. There is one universal mechanism for the functioning of all network neurons: a spin $\sigma_i$ turns over if its sign does not coincide with the sign of the field $h_i$ acting on this spin. However, it is reasonable to assume that the incoming excitations from the neurons belonging to the same group as the neuron $\sigma_i$ affect this neuron stronger then the excitations from the neurons of other groups (those, which analyze the same pattern from other points of view). This hierarchy of excitations can be modelled in different ways. As an initial model it can be assumed that:

$$h_i(t) = \sum_{i' \in A_s} J_{ii'} \cdot \sigma_{i'}(t)$$

(16)

where the summation is taken over all neurons belonging to the same group $A_s$ as the $i$th neuron.

The subnet consisting of the neurons from the group $A_s$ is evolving to one of its fixed points. This leads us to the problem (14). And the network as a whole is acting so, that
the composite functional

\[ M(\{\sigma_i\}_{i=1}^n) = \| \sum_{i \in A_1} \sigma_i \cdot \vec{\xi}_i \| + \| \sum_{i \in A_2} \sigma_i \cdot \vec{\xi}_i \| + \ldots + \| \sum_{i \in A_t} \sigma_i \cdot \vec{\xi}_i \| \rightarrow \max \]  

would be maximized.

We have discussed the situation when the neurons are already decomposed into groups \( A_1, A_2, \ldots, A_t \). If the structure of the groups is unknown, but their number \( t \) is fixed, it is necessary to maximize the functional (17) with respect to the structure of the groups \( A_s \) as well as with respect to all the weights \( \sigma_i \) inside every group. In this case Eqs.(14),(15), where the vectors \( \vec{\xi}_i \) have to be substituted instead of the vectors \( \vec{x}_i \), describe the optimal decomposition of the network into \( t \) autonomous subnets.

Here some remarks must be done. Firstly, it is easy to see, that when the number of the groups \( t \) increases, the functional \( M(A_1, \ldots, A_t) \) (15) is nondecreasing (it follows from the triangle inequality). This functional attains it’s global maximum when the number of the groups \( t \) is equal \( n \). However, it is a trivial decomposition. Simple geometric arguments show that when a group of strongly correlated vectors \( \vec{x}_i \) is divided into two subgroups, the functional (15) increases negligibly. So, the problem is not to get the global maximum of the functional (15), but to obtain such a number \( t^* \) of the groups beginning with which the further increase of the number of the groups would not lead to the substantial increase of this functional. About these \( t^* \) groups we can speak as about the proper number of the subnets which constitute the initial \( n \)-network.

Furthermore, it is reasonable to try to interpret the specific characteristics of each obtained subnet in meaning terms. In other words, we can try to understand what kind of pattern’s characteristics are analyzed by each particular subnet, i.e. we must determine what kind of neurons are joined in the group. On this step the monograph [4], which reflects the accumulated experience in this field, can be useful.

Secondly, the above mentioned program can be fulfilled only if we are able to solve two problems: A) to find out the compact groups of the vectors \( \vec{\xi}_i \); B) to determine the optimal configuration \( \{\sigma^*_i\}, i \in A_s \), for each group. What concerns the problem B, actually all the
attempts to create an effective algorithm for the maximization of the functional \((\vec{J} \vec{\sigma}, \vec{\sigma})\) are devoted to this problem.

The problem \(A\) is much less studied and seems to be more complicated. Usually, it is solved by step by step transferring of \(p\)-dimensional vectors from one group to another, and the comparison of the values of the functional \((15)\) for the consequently obtained grouping. When \(n\) is rather large, in such a way only the determination of the local maximum of the functional \(M\) is guaranteed. We know not so much papers \([3,4]\), devoted exactly to the problem of finding of the global maximum of a functional of type \((15)\). In these papers the general case of vectors \(\vec{x}_i\) with \textit{real} coordinates is studied. As for neural networks, the vectors \(\vec{\xi}_i\) are specific: their coordinates are \(\{\pm 1\}\). It can be hope that the specific character of the vectors \(\vec{\xi}_i\) would make it possible to present effective method for the searching of the compact groups.

And the last remark. Although the proposed approach was formulated for Hopfield’s model, it can be generalized for the case of an arbitrary symmetric connection matrix: it is sufficient to replace in Eqs. (14), (15) and (17) the term

\[ \| \sum_{i \in A_s} \sigma_i^* \cdot \vec{x}_i \| \]

by

\[ \left( \sum_{i,i' \in A_s} J_{ii'} \cdot \sigma_i \cdot \sigma_{i'} \right)^{1/2}, \]

and all reasoning are valid.

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REFERENCES

[1] J.J.Hopfield. Proc. Natl. Acad. Sci. USA 79, 2554 (1982); 81, 3088 (1984).

[2] H.Harman. Modern Factor Analysis. Univ. Chicago Press, 1960.

[3] E.M.Braverman. Automation and Remote Control, No.1, 1970, 108-116.

[4] E.M.Braverman, I.B.Muchnik. The structural methods of the empirical dates processing (in russian), 1983.

[5] V.M.Bukhshtaber, V.K.Maslov. Factor analysis and extremal problems on the Grassmann manifolds. In: Mathematical methods of the economical problems solving. Moscow, Nauka, 1977, #7, pp.87-102 (in russian).

[6] L.B.Litinskii. Automation and Remote Control, No.3, 1985, p.p. 158-161.