Cosmic Chronometers in the $R_h = ct$ Universe

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ABSTRACT

The use of luminous red galaxies as cosmic chronometers provides us with an indispensable method of measuring the universal expansion rate $H(z)$ in a model-independent way. Unlike many probes of the cosmological history, this approach does not rely on integrated quantities, such as the luminosity distance, and therefore does not require the pre-assumption of any particular model, which may bias subsequent interpretations of the data. We employ three statistical tools – the Akaike, Kullback, and Bayes Information Criteria (AIC, KIC and BIC) – to compare the $\Lambda$CDM model and the $R_h = ct$ Universe with the currently available measurements of $H(z)$, and show that the $R_h = ct$ Universe is favored by these model selection criteria. The parameters in each model are individually optimized by maximum likelihood estimation. The $R_h = ct$ Universe fits the data with a reduced $\chi^2_{dof} = 0.745$ for a Hubble constant $H_0 = 63.2 \pm 2.5$ km s$^{-1}$ Mpc$^{-1}$, and $H_0$ is the sole parameter in this model. By comparison, the optimal $\Lambda$CDM model, which has three free parameters (including $H_0 = 68.9 \pm 2.4$ km s$^{-1}$ Mpc$^{-1}$, $\Omega_m = 0.32$, and a dark-energy equation of state $p_{de} = -\rho_{de}$), fits the $H(z)$ data with a reduced $\chi^2_{dof} = 0.777$. With these $\chi^2_{dof}$ values, the AIC yields a likelihood of $\approx 82$ per cent that the distance–redshift relation of the $R_h = ct$ Universe is closer to the correct cosmology, than is the case for $\Lambda$CDM. If the alternative BIC criterion is used, the respective Bayesian posterior probabilities are 91.2 per cent ($R_h = ct$) versus 8.8 per cent ($\Lambda$CDM). Using the concordance $\Lambda$CDM parameter values, rather than those obtained by fitting $\Lambda$CDM to the cosmic chronometer data, would further disfavor $\Lambda$CDM.

Key words: cosmological parameters, cosmology: observations, cosmology: redshift, cosmology: theory, distance scale, galaxies

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1 INTRODUCTION

The expansion of the Universe is now being studied by several methods, including observations of Type Ia SNe (Riess et al. 1998; Perlmutter et al. 1999), weak lensing (Refregier 2003), baryon acoustic oscillations (Seo & Eisenstein 2003; Eisenstein et al. 2005; Pritchard et al. 2007; Percival et al. 2007), and cluster counts (Haiman et al. 2000), among several others. Each of these methods presents its own set of difficulties, among them a dependence on integrated quantities, such as the luminosity distance which, however, is not independent of the assumed cosmology. It is therefore quite difficult to use the data for unbiased, comparative studies to test different expansion histories. This problem is particularly acute in the case of Type Ia SNe, where at least four ‘nuisance’ parameters characterizing the standard candle must be optimized simultaneously with the model’s free parameters, rendering the data compliant to the underlying cosmology (see, e.g., Melia 2012a).

Even so, some progress has been made recently with attempts at comparing predictions of the $R_\text{h} = ct$ Universe (Melia 2007; Melia & Shevchuk 2012) with the data, and with $\Lambda$CDM. The evidence thus far seems to suggest that the $R_\text{h} = ct$ cosmology is a better match to the observations at high redshifts, particularly when it comes to the large-scale fluctuations of the cosmic microwave background (CMB), expressed through its angular correlation function and the apparent alignment of its quadrupole and octopole moments (for a summary of these comparisons, see Melia 2012a).

In the local universe, these two models are virtually indistinguishable, e.g., in predicting a very similar luminosity distance all the way out to a redshift of 6 and beyond. Thus, given the problem of identifying model-independent data through Type Ia supernova observations, it is not easy to evaluate one model against the other on the basis of these measurements alone. However, some clarification begins to emerge beyond a redshift of 6, where the high-$z$ quasars are now known to be accreting at, or near, their Eddington limit (see, e.g., Willott et al. 2010a,b; De Rosa et al. 2011). We showed recently that a Hubble Diagram (HD) constructed from these sources reveals a cosmic expansion fully consistent with the $R_\text{h} = ct$ Universe, assuming a current value of $69 \pm 4$ km s$^{-1}$ Mpc$^{-1}$ for the Hubble constant $H_0$ (Melia 2012b). Interestingly, $\Lambda$CDM can also fit the high-$z$ quasar HD, but only for a very specific set of parameters, including a matter energy density $\Omega_m = 0.27$, scaled to its current critical value. But whereas the $R_\text{h} = ct$ Universe has only one free parameter – the Hubble constant – the standard model has as many as six (depending on how one parametrizes the dark-energy equation of state $w_{\text{de}} \equiv p_{\text{de}}/\rho_{\text{de}}$) – including $H_0$, $\Omega_m$, and $w_{\text{de}}$. The implication of this is that the optimization of $\Lambda$CDM simply forces it to relax to the $R_\text{h} = ct$ expansion profile, which is more robust.
Moreover, though the distance–redshift relationship is essentially the same in these two models (even out to $z \gtrsim 6 - 7$), the age–redshift relationship is not. In fact, these same high-$z$ quasars present a seemingly insurmountable problem for ΛCDM because they suggest that $\sim 10^9 M_\odot$ supermassive black holes appeared only 700–900 Myr after the big bang. Instead, in $R_h = ct$, their emergence at redshift $\sim 6$ corresponds to a cosmic age of $\gtrsim 1.6$ Gyr. This was enough time for them to begin growing from $\sim 5 - 20 M_\odot$ seeds (presumably the remnants of Pop II and III supernovae) at $z \lesssim 15$ (i.e., after the onset of re-ionization) and still reach a billion solar masses by $z \sim 6$ via standard, Eddington-limited accretion (Melia 2013).

This kind of tangible result suggests that the $R_h = ct$ Universe relieves the growing tension between ΛCDM and the observations, but it would be highly beneficial for us to find a way of testing this cosmology – and quantifying its superiority over ΛCDM – by exploiting model-independent data in the nearby Universe. The purpose of this paper is to demonstrate that the use of luminous red galaxies as cosmic chronometers (Jimenez & Loeb 2002) can do just that. We shall show that over the redshift range $0 \lesssim z \lesssim 1.8$, the measured Hubble constant $H(z)$ is fitted better by the $R_h = ct$ model than by ΛCDM; and especially so, if one takes account of the reduction in the number of free parameters. Unlike other indicators that rely on the expansion history of the Universe, the cosmic chronometers may therefore offer us the best evidence yet that $R_h = ct$ is to be preferred over ΛCDM.

We introduce the cosmic chronometers in § 2, and in § 3 discuss the AIC, KIC and BIC tools we use to test ΛCDM and the $R_h = ct$ Universe against these data. The results of our comparison between ΛCDM and $R_h = ct$ are presented in § 4 and discussed in § 5. We conclude in § 6 with a discussion of future prospects.

2 THE COSMIC CHRONOMETERS

Cosmic chronometers offer us the possibility of measuring the differential age of the Universe, circumventing the limitations associated with the use of integrated histories, by directly measuring the derivative $dt/dz$, which represents the change in cosmic time as a function of redshift. And since $H(z) \equiv \dot{a}/a$, in terms of the expansion factor $a(t)$, a measurement of $dt/dz$ directly yields the expansion rate, because

$$H(z) = \frac{\dot{a}}{a} = -\frac{1}{1 + z} \frac{dz}{dt}. \quad (1)$$

For various reasons, the best cosmic chronometers appear to be galaxies that are evolving passively on a time-scale much longer than their age difference. Observations indicate that the
most massive galaxies contain the oldest stellar populations up to redshifts \( z \sim 1 - 2 \) (Dunlop et al. 1996; Spinrad et al. 1997; Cowie et al. 1999; Heavens et al. 2004; Thomas et al. 2005). Less than 1 per cent of the stellar mass in these massive galaxies formed at \( z < 1 \) (Heavens et al. 2004; Panter et al. 2007). In high-density regions (i.e., galaxy clusters), star formation ceased by redshift \( z \sim 3 \) (Thomas et al. 2005), and other massive systems – those with stellar masses \( \gtrsim 5 \times 10^{11} M_\odot \) – finished their star formation activity by \( z \sim 2 \) (Treu et al. 2005).

The empirical evidence therefore suggests that galaxies in the highest density regions of clusters formed their stellar content at \( z \gtrsim 2 \), and have been evolving passively since that time, without any additional episodes of star formation. One can therefore view these galaxies as tracing the ‘red envelope,’ hosting the oldest stars in the Universe at every redshift. Thus, given their viability as cosmic chronometers, a great deal of effort is being expended to calculate \( \frac{dt}{dz} \) – and therefore \( H(z) \) – using their measured properties (see, e.g., Stern et al. 2010; Stern et al. 2012; Moresco et al. 2012a,b).

For example, one of the most direct ways of determining the age of the galaxy is to use the 4000 Å break in its spectrum, which depends linearly on age for old stellar populations (Moresco et al. 2011). This break is a discontinuity of the spectral continuum due to metal absorption lines whose amplitude correlates linearly with the age and metal abundance. If the metallicity is known, then the difference in age between two galaxies is proportional to the difference in their 4000 Å amplitudes.

However, one must also be aware of the fact that many systematic sources of uncertainty can bias this kind of analysis (see, e.g., Moresco et al. 2012b). These include: (1) the degeneracy between the effect of a change in age and an effect due to a change in stellar metallicity or the star formation history; (2) the possible biasing of the estimate of \( H(z) \) by the choice of stellar population synthesis model, used to estimate the age or calibrate the 4000 Å versus age relation; and (3) the possible existence of a progenitor bias (van Dokkum & Franx 1996), in which high-redshift samples of early-type galaxies might not be statistically equivalent to those at low redshifts.

These caveats notwithstanding, one is none the less encouraged by the agreement seen between the results of several different approaches. The data set shown in figure 1, including both \( H(z) \) measurements and error bars, was assembled from the compilations of Simon et al. (2005), Stern et al. (2010), and Moresco et al. (2012a), and spans the redshift range \( 0 \leq z \leq 1.8 \). Together, these compilations paint a fairly consistent picture of the universal expansion, particularly when viewed in terms of the theoretical expectations, which we shall consider shortly, following our discussion of model selection statistics.
3 MODEL SELECTION STATISTICS

To compare the evidence for and against competing models, such as models of the distance–
redshift relationship, the use of the Akaike Information Criterion (AIC) is now common in cos-
mology (see, e.g., Takeuchi 2000; Liddle 2004, 2007; Tan & Biswas 2012). The AIC can be viewed
as an enhanced ‘goodness of fit’ criterion, which extends the usual $\chi^2$ criterion by taking account
of the number of parameters in each model. It prefers models with few parameters to those with
many, unless the latter provide a substantially better fit to the data. This reduces the possibility of
overfitting: the fact that by optimizing over a greater number of parameters, one may simply be
fitting the noise.

As developed (Akaike 1973; see also Burnham & Anderson 2002, 2004), the AIC provides the
relative ranks of two or more competing models, and also a numerical measure of confidence that
each model is the best. These confidences are analogous to likelihoods or posterior probabilities in
traditional statistical inference. But unlike traditional inference methods, the AIC can be applied
to models that are not ‘nested.’ Comparing a pair of models that are nested, in the sense that one is
a specialization of the other, is straightforward: after fitting each model to the data, one computes
the $\chi^2$ per degree of freedom for each, and decides which is a better fit. One can also calculate (say,
by applying an F-test) a likelihood that the simpler model should be rejected, or the likelihood of
the null hypothesis that the simpler model is a better approximation to the ‘true’ one. By exploiting
the AIC one can generalize this procedure: one can compare a pair of models, neither of which is
a specialization of the other; such as $\Lambda$CDM and an alternative model.

The AIC can be applied after regression of the following kind is performed. Suppose that for
values $z_1, \ldots, z_n$ of an independent variable there are measured values $h_1, \ldots, h_n$ of a dependent
one, with (known) error bars $\pm \sigma_1, \ldots, \pm \sigma_n$; and suppose the errors are normally distributed. Sup-
pose that a model $M$ predicts values $\hat{h}_1, \ldots, \hat{h}_n$, computed from a formula $\hat{h}_i = \hat{h}_i(\vec{\beta})$ that involves
a parameter vector $\vec{\beta}$ comprising $k$ unknown parameters, i.e., $\vec{\beta} = (\beta_1, \ldots, \beta_k)$. That is, the data
model $M$ is really a statistical one, of the form

$$ h_i = \hat{h}_i(\vec{\beta}) + \sigma_i Z_i, $$

where $Z_1, \ldots, Z_n$ are independent standard normal random variables. (In the case of linear re-
gression, $\hat{h}_i(\vec{\beta})$ would be $\sum_{j=1}^k X_{ij} \beta_j$ for known coefficients $X_{ij}$; typically, $X_{ij} = \hat{h}^{(j)}(z_i)$ for known
functions $\hat{h}^{(1)}, \ldots, \hat{h}^{(k)}$ of $z$.)
For model $M$, the $\chi^2$ goodness of its fit to the data is given by

$$\chi^2 = \sum_{i=1}^{n} \left( h_i - \hat{h}_i(\beta) \right)^2 / \sigma_i^2,$$

i.e., a (weighted) sum of squared errors, and the reduced $\chi^2$ (i.e., the $\chi^2$ per degree of freedom) by

$$\chi^2_{\text{dof}} = \chi^2 / (n - k).$$

(It is assumed that $n > k$.) The parameters ($\beta_1, \ldots, \beta_k$) are chosen to minimize the $\chi^2$, yielding the best fit to the data. The AIC for the resulting fitted model is then given by

$$\text{AIC} = \chi^2 + 2k.$$

If there are two or more competing models for the data, $M_1, \ldots, M_N$, and they have been separately fitted, the one with the least resulting AIC is assessed as the one most likely to be nearest to the ‘truth,’ i.e., to the unknown model $M$, that generated the data. A more quantitative ranking of models can be computed as follows. If $\text{AIC}_\alpha$ comes from model $M_\alpha$, the unnormalized likelihood that $M_\alpha$ is closest to the truth is the ‘Akaike weight’ $\exp(-\text{AIC}_\alpha/2)$. Informally, $M_\alpha$ has likelihood

$$L(M_\alpha) = \frac{\exp(-\text{AIC}_\alpha/2)}{\exp(-\text{AIC}_1/2) + \cdots + \exp(-\text{AIC}_N/2)}$$

of being the best choice. (The 2’s here could of course be omitted by redefining AIC, but the normalization implicit in Equation (5) is traditional.) In the case of a pair of models $M_1, M_2$, the difference $\text{AIC}_2 - \text{AIC}_1$ determines the extent to which $M_1$ is favored over $M_2$.

It is clear that the $2k$ term in Equation (5), proportional to the parameter count $k$, exponentially disfavors models with too many parameters, though such models can be favored if they do a much better job of fitting the data. The choice of proportionality constant (i.e., 2) is not entirely arbitrary, being based on an argument from information theory that has close ties to statistical mechanics. The following is a brief summary, with many more details to be found in the statistics literature. (The reader should note that most of the literature focuses on the case when the error variances $\sigma_1^2, \ldots, \sigma_n^2$ are both unknown and equal to some common variance $\sigma^2$, a nuisance parameter that must be estimated as part of the fitting process; the setup given in Equations 2–3 is actually simpler.)

Any two statistical models of the data set $(h_1, \ldots, h_n)$, such as a ‘true’ model $M_\ast$ and another model $M$, can be viewed as probability density functions (PDF’s) on $\mathbb{R}^n$, say $f_\ast(h_1, \ldots, h_n)$ and $f(h_1, \ldots, h_n)$, respectively. In information theory one says that the discrepancy of the PDF $f$ from the PDF $f_\ast$, which is a measure of distance, is given by the Kullback–Leibler formula

$$D(M_\ast||M) = \int_{\mathbb{R}^n} dh_1 \ldots dh_n f_\ast(h) \ln \frac{f_\ast(h)}{f(h)} \geq 0$$

(where in an obvious notation, the argument $h$ stands for the entire data set $[h_1, \ldots, h_n]$). In a
thermodynamic interpretation this is a relative entropy in the sense of Boltzmann and Hasenöhrl.

To select the best model $\mathcal{M}$ from a set of candidate models, one would choose the one with the minimum $D(\mathcal{M}_*\|\mathcal{M})$. Of course $\mathcal{M}_*$ is not known, so this cannot be done literally. But the case when $\mathcal{M}$ is a parametrized model, and its parameters are chosen (by minimizing $\chi^2$) to fit a data set generated by $\mathcal{M}_*$, is special. It can be shown that the AIC of the fitted model $\mathcal{M}$ is a good approximation to $2D(\mathcal{M}_*\|\mathcal{M})$, up to an unimportant additive constant. This is especially the case when $\mathcal{M}_*$ is a model of the same type, with unknown parameters ($\beta_1^*, \ldots, \beta_k^*$).

Specifically, $\text{AIC}/2$ is an unbiased estimator of the distance $D(\mathcal{M}_*\|\mathcal{M})$: exactly so for linear regression, and to leading order for non-linear regression. The phrase ‘unbiased estimator’ means that on average the two are the same, where the averaging is over data sets generated by $\mathcal{M}_*$, with PDF $f_*$. Of course the fitted model $\mathcal{M}$ depends on the data set, so in the context of $\mathcal{M}_*$, both $D(\mathcal{M}_*\|\mathcal{M})$ and $\text{AIC}/2$ are random variables. In probabilistic language, the lack of bias means that they have the same expectation.

The extent to which the fitted AIC is an accurate estimate of $2D(\mathcal{M}_*\|\mathcal{M})$, data set by data set, as well as being the same on average, has been investigated theoretically (Yanagihara & Ohmoto 2005). Its variability has also been studied empirically; for example, by repeatedly comparing $\Lambda$CDM to other cosmological models on the basis of data sets generated by a bootstrap method (Tan & Biswas 2012). It is known that the AIC is increasingly accurate when $n$ is large, but it is felt that for all $n$, the magnitude of the difference $\Delta = \text{AIC}_2 - \text{AIC}_1$ should provide a numerical assessment of the evidence that model 1 is to be preferred over model 2. A rule of thumb that has been used in the literature is that if $\Delta \lesssim 2$, the evidence is weak; if $\Delta \approx 3$ or 4, it is mildly strong; and if $\Delta \gtrsim 5$, it is quite strong.

Besides using fixed thresholds, one can weight each candidate model in a Boltzmann-like way by its Akaike weight, i.e., according to Equation (6). For each model $\mathcal{M}_\alpha$, the likelihood $L(\mathcal{M}_\alpha)$, which is determined by the differences between $\text{AIC}_\alpha$ and the AIC’s of the other model(s), is loosely analogous to a posterior probability in statistical inference, despite its not being computed by a Bayesian procedure (no Bayesian prior is involved). But in the absence of a general theory of AIC variability, deciding between models 1 and 2 cannot be viewed as a hypothesis test, at any fixed level of significance such as 0.05.

Several alternatives to the AIC have been considered in the literature. A lesser-known one arises as follows. The discrepancy $D(\mathcal{M}_*\|\mathcal{M})$ is not symmetric in the PDF’s $f_*, f$, and it has been suggested that it should be replaced by a symmetrized version, which is arguably a better tool for distinguishing between data models (Cavanaugh 1999). The unbiased estimator for the
symmetrized version has been given the name KIC (Kullback Information Criterion), and is given by

\[ \text{KIC} = \chi^2 + 3k. \quad (8) \]

The KIC, with \( k \) multiplied by the coefficient 3 rather than 2, disfavors overfitting more than does the AIC, and has been shown to perform favorably against the AIC as a tool for model selection (Cavanaugh 2004). It has long been felt (since, e.g., Bhansali & Downham 1977) that the problem of overfitting may be best dealt with by choosing a coefficient that is larger than 2, and perhaps even than 3. But the AIC and KIC are the only such schemes that follow readily from information theory.

A better-known alternative to the AIC is the BIC (Bayes Information Criterion), which is a misnomer in that it is not based on information theory, but rather on an asymptotic \( (n \to \infty) \) approximation to the outcome of a conventional Bayesian inference procedure for deciding between models (Schwarz 1978). It is defined by

\[ \text{BIC} = \chi^2 + (\ln n)k, \quad (9) \]

and suppresses overfitting very strongly if \( n \) is large. Liddle et al. (2006) and Liddle (2007) make the case for using BIC in cosmological model selection, and it has now been used to compare several popular models against \( \Lambda \)CDM (see, e.g., Shi et al. 2012). However, it should be noted that the monograph of Burnham & Anderson (2002), which popularized Equation (6) for assigning AIC-based likelihoods to models, strongly prefers AIC to BIC as a tool for model selection. They elsewhere note that the AIC can in fact be interpreted in Bayesian terms, as being the consequence of imposing a nonuniform but reasonable choice of prior distribution on the set of candidate models (Burnham & Anderson 2004). Kuha (2004) draws further analogies between AIC and BIC, and argues that they are both valuable tools.

In the comparison below, we employ the AIC, KIC, and BIC. We do not employ the so-called corrected AIC, denoted \( \text{AIC}_c \), which includes a correction term intended to remove bias when \( n \) is small (Burnham & Anderson 2002). The correction term is small (cf. Tan & Biswas 2012). More importantly, the form of this term is appropriate only for data sets without explicit error bars, with the common error variance \( \sigma^2 \) estimated as part of the fitting process (Maier 2013, in preparation).

## 4 A COMPARISON BETWEEN \( \Lambda \)CDM AND \( R_H = CT \)

The \( R_h = ct \) Universe is a flat Friedmann–Robertson–Walker (FRW) cosmology that strictly adheres to the constraints imposed by the simultaneous application of the Cosmological principle.
and Weyl’s postulate (Melia, 2007; Melia & Shevchuk 2012; Melia 2012a). When these ingredients are applied to the cosmological expansion, the gravitational horizon \( R_h = c/H \) must always be equal to \( ct \). This cosmology is therefore very simple, because \( a(t) \propto t \), which also means that \( 1 + z = 1/t \), with the (standard) normalization that \( a(t_0) = 1 \). Therefore in the \( R_h = ct \) Universe, we have the straightforward scaling

\[
H(z) = (1 + z)H_0.
\]

Notice, in particular, that the expansion rate \( H(z) \) in this model has only one free parameter. By comparison, \( \Lambda \)CDM has as many as six parameters (depending on the application), including \( H_0 \), the scaled matter energy density \( \Omega_m \equiv \rho_m/\rho_c \), in terms of the matter energy density \( \rho_m \) and the critical density \( \rho_c \equiv [3c^2/8\pi G]H_0^2 \), and the dark-energy equation of state \( w_{de} = p_{de}/\rho_{de} \).

In this paper, we shall take the minimalist approach and optimize \( \Lambda \)CDM using only these three free parameters. (Using additional parameters would weaken the statistical significance of the fit even further, so by selecting this minimal set, we present \( \Lambda \)CDM in its best possible light.) The Hubble constant in this cosmology is therefore given by

\[
H(z) = H_0 \left[ \Omega_m(1 + z)^3 + \Omega_r(1 + z)^4 + \Omega_{de}(1 + z)^{3(1+w_{de})} \right]^{1/2},
\]

where \( \Omega_r \) and \( \Omega_{de} \) for radiation and dark energy, respectively, are defined analogously to \( \Omega_m \). In addition, we shall assume a flat \( \Lambda \)CDM cosmology, for which \( \Omega_m + \Omega_r + \Omega_{de} = 1 \), thus avoiding the introduction of \( \Omega_{de} \) as an additional free parameter. Of course, \( \Omega_r \sim 6 \times 10^{-5} \) is known from the current temperature (\( \approx 2.7 \) K) of the cosmic microwave background radiation.

For each model \( M_\alpha \) (with \( \alpha = 1, 2 \) specifying the \( R_h = ct \) Universe and \( \Lambda \)CDM, respectively), we optimize the fit by finding the model parameter vector \( \vec{\beta}_\alpha \) that minimizes the \( \chi^2 \). Equivalently, we choose \( \vec{\beta}_\alpha \) to maximize the joint likelihood function

\[
\Phi_\alpha(\vec{\beta}_\alpha) = \prod_{i=1}^n e^{-[H(z_i,H(\vec{\beta}_\alpha))]^2/2\sigma_i^2} \sqrt{2\pi \sigma_i},
\]

where the \( H_i \) are the measured values of the Hubble constant at redshift \( z_i \), and the \( H(z_i,\vec{\beta}_\alpha) \) are the corresponding theoretical values computed from the parameter vector \( \vec{\beta}_\alpha \). For \( \alpha = 1, 2 \), the number of parameters (i.e., the length \( k \) of the vector \( \vec{\beta}_\alpha \)) is respectively 1 and 3, as stated. The fitting is a linear regression in the case of the \( R_h = ct \) Universe and a non-linear one for \( \Lambda \)CDM, as is evident from Equations (10) and (11). The data set \( \{(z_i, H_i, \sigma_i)\}_{i=1}^n \), to which each model is fitted was assembled from the \( H(z) \) compilations of Moresco et al. (2012a), Stern et al. (2010), and Simon et al. (2005), and consists of \( n = 19 \) measured values, each with an error bar.

The results for the \( R_h = ct \) Universe (for which the best fit has \( H_0 = 63.2 \pm 2.5 \) km s\(^{-1}\) Mpc\(^{-1}\))
Figure 1. Nineteen $H(z)$ measurements, with error bars, and comparison with two theoretical models: (solid) the $R_h = ct$ Universe, with its sole parameter $H_0 = 63.2 \pm 2.5$ km s$^{-1}$ Mpc$^{-1}$, and (dashed) the standard $\Lambda$CDM cosmology, assuming a flat Universe, with $\Omega_m = 0.32$, $\Omega_\Lambda = 0.68$, and $H_0 = 68.9 \pm 2.4$ km s$^{-1}$ Mpc$^{-1}$. The reduced $\chi^2_{\text{dof}}$ (with 18 degrees of freedom) for the $R_h = ct$ fit is 0.745. The corresponding value for the optimal $\Lambda$CDM model (with 16 degrees of freedom) is $\chi^2_{\text{dof}} = 0.777$. Even by eye, one can see that $R_h = ct$ is a better fit to the data at $z \gtrsim 0.9$. The reduced $\chi^2$ overall suggests that $R_h = ct$ is at least as good as $\Lambda$CDM; especially, when its having only one parameter is taken into account. We shall see shortly that on statistical grounds, the $R_h = ct$ distance–redshift predictions are in fact more likely than those of $\Lambda$CDM to be closer to the correct cosmology.

It is worth pointing out that the $\Lambda$CDM model optimized for the cosmic chronometer data alone is quite different from the concordance model, characterized by the parameter values $H_0 = 73.8 \pm 2.4$ km s$^{-1}$ Mpc$^{-1}$, $\Omega_m = 0.27$ and $w_{\text{de}} = -1$. Using the concordance $\Lambda$CDM parameter values to fit the cosmic chronometer data yields $\chi^2_{\text{dof}} = 0.9567$, which is acceptable (since $\chi^2_{\text{dof}} \lesssim 1.0$), but
which indicates a noticeably less good fit than both the fits shown in figure 1. By comparing the $R_h = ct$ Universe against the optimized $\Lambda$CDM, rather than against the concordance model, we are once again presenting $\Lambda$CDM in its best possible light.

With $n = 19$ data points and $k = 1$ parameter, the AIC for the optimized $R_h = ct$ Universe is $AIC_1 = 15.41$. For the optimized $\Lambda$CDM, with $k = 3$, the corresponding value is $AIC_2 = 18.432$. The magnitude of the difference $\Delta = AIC_2 - AIC_1$, namely $\Delta = 3.022$, indicates that $M_1$ is to be preferred over $M_2$. According to Equation (6), the likelihood of $R_h = ct$ (i.e., $M_1$) being the correct choice is $L(M_1) = 82$ per cent. For $\Lambda$CDM (i.e., $M_2$), the corresponding value is $L(M_2) = 18$ per cent.

If one uses the KIC and BIC statistics instead of the AIC, but continues to weight the models as in Equation (6), the difference is greater, since $k$ is multiplied by 3 in the former, and by $\ln n \approx 2.9$ in the latter. One finds that $KIC_1 = 16.41$ and $KIC_2 = 21.434$, yielding $L(M_1) = 92.4$ per cent and $L(M_2) = 7.6$ per cent for $R_h = ct$ and $\Lambda$CDM, respectively. And for BIC, the results are $BIC_1 = 16.35$ and $BIC_2 = 21.27$, yielding $L(M_1) = 91.2$ per cent and $L(M_2) = 8.8$ per cent.

According to all three statistics, the predictions of $R_h = ct$ are more likely than those of $\Lambda$CDM to be closer to the correct cosmology. This is notably the case for BIC, for which there is an accepted interpretation of the magnitude of $\Delta = BIC_2 - BIC_1$ in terms of the strength of the evidence against model 2 (Kass & Raftery 1995; Tan & Biswas 2012). If, as here, $\Delta = 4.92$, the evidence against model 2 (i.e., $\Lambda$CDM) would be judged ‘positive’ (the positive range for $\Delta$ extends from 2 to 6, at which point the ‘strong’ range begins).

5 DISCUSSION

Though a measurement of the cosmic expansion rate using early type galaxies is subject to several possible systematic errors, the fact that the inferred values of $H(z)$ are model-independent makes this a highly desirable and meaningful approach for testing different cosmological models. In this paper, we have compared the fits to a data sample drawn from several sources, and have demonstrated that the $R_h = ct$ Universe is more likely than $\Lambda$CDM to account for the observed $H$ versus $z$ profile. In addition, the inferred value of the Hubble constant $H_0$ is consistent with the rate $(69 \pm 4 \text{ km s}^{-1} \text{ Mpc}^{-1})$ emerging from a fit to the high-$z$ quasar Hubble Diagram (Melia 2012b).

This is rather impressive, given that the former corresponds to a probe of the local Universe (at $z \leq 2$), whereas the latter concerns the cosmic expansion at high redshift ($z \geq 6$). In addition, one should not underestimate the fact that in $R_h = ct$, there is only one free parameter. By comparison,
the standard model of cosmology, with as many as six, depending on how one parametrizes the
dark-energy equation of state, fails to account for the appearance of high- \( z \) quasars at redshift \( \gtrsim 6 \) (Melia 2013). This type of comparative analysis therefore supports the suggestion that \( R_h = ct \) is
closer to the correct cosmology than is \( \Lambda CDM \). The growing tension between the predictions of
\( \Lambda CDM \) and the ever improving cosmological data also suggests that the current standard model
may be a useful approximation, but will probably not endure in the long run.

Recently, however, some criticism has been leveled at the \( R_h = ct \) cosmology on the basis of
several claimed inconsistencies, some theoretical, others observational (Bilicki and Seikel 2012).
One of the observational arguments was based on the same cosmic chronometer data we have ad-
dressed in this paper, from which a different conclusion was arrived at from the one obtained above.
However, these earlier results are incorrect: simply, because they were not based on a proper sta-
tistical analysis. Those conclusions appear to have been based on a qualitative inspection by eye.
But clearly, the results presented here show that such an approach does not stand up well to a
quantitative assessment based on comparisons of likelihoods. And since the cosmic chronometer
data favor the \( R_h = ct \) cosmology over \( \Lambda CDM \) when using a simple, direct statistical compari-
son, any higher-order metric employed with the \( H(z) \) data, particularly those designed to test the
parametrization of \( \Lambda CDM \), e.g., the decomposition of density into the three specific components,
matter, radiation, and dark energy, cannot be used to meaningfully constrain the \( R_h = ct \) Uni-
verse. On the contrary, as we have shown here, the cosmic chronometer data – when interpreted
quantitatively – suggest that the \( R_h = ct \) Universe is at least as good as the standard model.

The second observational argument for the criticism was based on the analysis of Type Ia SNe.
However, here too the data were used incorrectly to arrive at an invalid result. The use of Type Ia
supernova data ignored the fact that these were optimized for a pre-assumed \( \Lambda CDM \) cosmology.
Therefore, these cannot be used for a comparative test between different expansion scenarios. A
complete discussion of this problem has already been published in Melia (2012a), so we shall not
reproduce it here.

The danger of using data optimized for \( \Lambda CDM \) to test other cosmologies has also been high-
lighted recently by an examination of the Gamma Ray Burst (GRB) Hubble Diagram (Wei et al.
2013). In this work, the data were recalibrated separately for each model and, though the results
are quite similar, a comparison of the reduced \( \chi^2_{dof} \)’s for \( R_h = ct \) and \( \Lambda CDM \) shows that the data
clearly favor the former over the latter. This result would not have been evident without a recal-
ibration of the data using the \( R_h = ct \) expansion history. Given the preponderance of evidence, it
seems likely that when the Type Ia SN data are calibrated correctly for each cosmology, $R_h = ct$ will emerge as the more likely of the two to be correct.

Finally, it is worth pointing out that the ‘theoretical difficulties’ invoked to argue against $R_h = ct$ are based on a failure to comprehend fully Birkhoff’s theorem and its corollary, and the consequence of Weyl’s postulate on Friedmann–Robertson–Walker (FRW) metrics. It is not the purpose of this paper to correct this misunderstanding, but since it appears to be an issue relevant to the interpretation of cosmic chronometer data, we shall address it here as well.

Birkhoff’s theorem and its corollary (Birkhoff 1923) place no limit on scale, so Bilicki and Seikel’s (2012) assertion that the definition of a Schwarzschild (i.e., a gravitational) radius makes no sense on cosmic dimensions is without foundation. Moreover, one does not ‘define’ a Schwarzschild radius, as was claimed; this scale emerges automatically when one re-writes the metric in terms of observer-dependent coordinates versus the more commonly used co-moving coordinates (see, e.g., Melia & Abdelqader 2009). Many are perhaps not aware of the fact that exactly the same phenomenon occurs when writing the spacetime metric for compact objects. The distinction arises between a free-falling observer and the observer at a fixed radius (and hence accelerated) relative to the central mass. The former is not aware of the gravitational radius that emerges only when the metric is written using rulers and clocks fixed to the latter. In the cosmological context, we are free-falling observers when we write the FRW metric using co-moving coordinates. However, a gravitational radius emerges when we re-write this metric in terms of an observer’s fixed coordinates.

The irony, of course, is that the gravitational radius in cosmology actually first appeared as far back as 1917, though its meaning was not then fully appreciated. de Sitter’s (1917) paper on his now famous metric was originally written in terms of the observer’s fixed coordinates, which included the gravitational radius, since the co-moving coordinates would be introduced by Friedmann only several years later. The argument against the validity of a gravitational horizon in cosmology would therefore imply that de Sitter space is meaningless on large scales. This is simply not true.

And since the meaning and validity of the gravitational radius in cosmology (which, by the way, coincides with the better known Hubble radius) were not appreciated, the consequence of Weyl’s postulate on its permitted rate of expansion was ignored. Since the Hubble radius is a ‘proper’ radius, it has no choice but to expand at a constant rate, as demonstrated by Melia & Shevchuk (2012) and (in the more pedagogical treatment) by Melia (2012c).
Euclid (Laureijs et al. 2011) and BOSS (Eisenstein et al. 2011) should provide thousands of passive galaxies at $z > 0.5$, which will significantly improve the accuracy of $H(z)$ at these higher redshifts. In concert with this improved statistic, it will be essential to understand better if the systematic effects, e.g., the error due to the metallicity and star formation uncertainties, may be controlled and minimized. It is crucial to carry out this arduous work, because these cosmic chronometers are among the few sources that provide us with model-independent data. And as we have seen, only such model-independent data can truly distinguish between competing cosmologies.

We end with a word of caution. It should be evident from the contents of this paper how important it is to use only model-independent data in any comparative analysis between competing cosmologies. In some cases, it is simply not possible to avoid the ‘circularity problem,’ in which a model must be pre-assumed in order to extract the data. This is certainly the case in the Type Ia SN work, but also when dealing with any observations requiring the use of integrated quantities, such as the luminosity distance.

An entirely different approach sometimes used to determine $H(z)$ is based on the identification of Baryon Acoustic Oscillations (BAO) and the Alcock–Paczynski distortion from galaxy clustering. That is, instead of using information on how cosmic time changes with $z$ (as is the case for the data we have used here), this alternative approach measures how ‘standard rulers’ evolve with redshift. The results of these two different methods are sometimes combined to produce an overall $H$ versus $z$ diagram, but there is a good reason to be wary of this procedure: whereas the cosmic chronometers produce model-independent data, the second approach must necessarily assume a particular cosmology and is therefore model-dependent (Blake et al. 2012).

With the BAO method, the cosmic expansion is measured from the growth of structure as a function of redshift. Redshift-space distortions arise because the recession velocities of galaxies, from which distances are inferred, include contributions from both the Hubble flow and from the peculiar velocities driven by the clustering of matter (see, e.g., Hamilton 1998 for a review). The oscillations are modeled via the non-linear evolution of both the matter density and velocity fields, which are quite different between, say, $\Lambda$CDM and $R_0 = ct$ (Melia & Shevchuk 2012). In addition, to compute redshift space separations for each pair of galaxies given their angular coordinates and redshifts, one must adopt a cosmological model for the expansion to relate these quantities to each other.

Unfortunately, this gives rise to a situation not unlike that currently existing with Type Ia SNe.
(Melia 2012a), in which one must simultaneously optimize at least four nuisance parameters incorporated into the description of the measurements, along with the free parameters of the assumed model. One must therefore avoid the use of such model-dependent data in any attempts to directly compare fits using $\Lambda$CDM with those of other models, such as $R_h = c t$. Only the cosmic chronometer data are truly model-independent and therefore appropriate for this purpose.

ACKNOWLEDGMENTS

We are grateful to the many workers who spent an extraordinary amount of effort and time accumulating the data summarized in figure 1. FM is also grateful to Amherst College for its support through a John Woodruff Simpson Lectureship. Part of this work was carried out at Purple Mountain Observatory, Nanjing China.
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