Non-smooth analysis method in optimal investment- a BSDE approach

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Abstract

In this paper, our aim is to investigate necessary conditions for optimal investment. We model the wealth process by Backward differential stochastic equations (shortly for BSDE) with or without constraints on wealth and portfolio process. The constraints can be very general thanks the non-smooth analysis method we adopted.

Keywords: Backward stochastic differential equation, Constraint, Non-smooth analysis, Optimal investment.

1 Introduction

In the sequel, let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a standard Brownian motion $W$. For a fixed real number $T > 0$, we consider the filtration $\mathcal{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ which is generated by $W$ and augmented by all $P$-null sets. The filtered probability space $(\Omega, \mathcal{F}, P)$ satisfies the usual conditions.

Given an initial capital $x$, the investor’s investment process is said to vary against some kind of Backward Differential Stochastic Equations (short for BSDE),

$$y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T z_s^* dW_s, 0 \leq t \leq T$$

(1)

with $y_0 \leq x$, where $g(\omega, t, y, z) : \Omega \times [0, T] \times R \times R^d \rightarrow R$ is a function satisfying uniformly Lipschitz condition, i.e., there exists a positive constant $M$ such that for all $(y_1, z_1), (y_2, z_2) \in R^d$,

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq M(|y_1 - y_2| + |z_1 - z_2|)$$

(A1)

and

$$g(\cdot, 0, 0) \in H^2_T(R),$$

(A2)

where $H^2_T(R^d)$ denotes the space of predictable process $\varphi : \Omega \times [0, T] \rightarrow R^d$ satisfying $\|\varphi\|^2 = E \int_0^T |\varphi(s)|^2 ds < +\infty$. We call $y_t$ wealth process and $z_t$ portfolio process.

The BSDE approach is a backward view for investment. For comparison, if we take a forward view for the equation (1), then the complicated process $z_t$ acts as a control. However, by the theory of BSDE, $z_t$ is determined by the terminal value $\xi$ via a one-one correspondence. Thus the BSDE approach has the virtue to handle similar control problem taking $\xi$ as a control instead. Moreover, if we consider the function $y_0 := \xi^2_{T,T}(\xi)$ induced by BSDE with terminal value $\xi$, a terminal perturbation method, which was first used in Bielecki et al. [1], can be used to analyze the optimal investment problem. Along with this line, later in many years, Ji and Peng [8] used it to obtain a necessary condition via Ekeland variation principle. In this paper, as a generalization, we study optimal investment problems by non-smooth analysis method, which makes more general optimal problems inside our consideration.

Supposing that the investor has initial wealth $x$, he invest it in the financial market according to the equation (1). By the above analysis, his investment strategy is determined by all available terminal value for him. Let

$$\mathcal{A}(x) := \{\xi \in L^2_T(R) | y_0 \leq x\},$$

then our problem is

$$\min_{\xi \in \mathcal{A}(x)} \rho(\xi),$$

(2)

where $\rho(\cdot)$ is a function defined on $L^2_T(R)$ which usually represents a risk measure. However in our paper, it can be a general Lipschitz function.
Sometimes, one ask \((y_t, z_t)\) satisfy some constraint condition
\[
y(t), z(t) \in \Gamma_t, \quad \text{a.e., a.s. on } [0, T] \times \Omega,
\]
where \(\Gamma_t := \{(y, z) \mid \phi(t, y, z) = 0 \} \subset R \times R^d\) and \(\phi\) satisfies conditions (A1) and (A2). In such constrained case, the investment model should be changed to a Constrained Backward Differential Equation (shortly for CBSDE) as follows,
\[
y_t = \xi + \int_t^T g(s, y_s, z_s)ds + C_T - C_t - \int_t^T z_s^* dW_s, 0 \leq t \leq T,
\]
where \(C_t\) is an increasing RCLL (right continuous and left limit exists) process with limited initial capital \(C_0 = 0\), \(y_t\) is often called a super-solution of BSDE in the literature.

The idea of constrained investment comes from incompleteness or other constraints on investment in financial market. In such case, super-hedging strategies are often adopted. Corresponding to such strategies, the minimal super-solution defined as follow is meaningful.

**Definition 1.1.** \((g_t\text{-solution})\) A \(g\)-super-solution \((y_t, z_t, C_t)\) is said to be the minimal solution, given \(y_T = \xi\), subjected to the constraint \((C)\) if for any other \(g\)-super-solution \((y_t', z_t', C_t')\) satisfying \((C)\) with \(y_T' = \xi\), we have \(y_t \leq y_t'\) a.e., a.s.. We call the minimal solution \(g_t\text{-solution}\) and denote it as \(y_t := E_t^{g,\phi}(\xi)\). In no constrained case, i.e. when \(\phi(t, y, z) = 0\) \(P\text{-a.s.}\) for any \(t \in [0, T]\), we denote it as \(E_t^{g}(\xi)\) for convenience.

Our problem in the constrained case is similar to \([2]\) but change \(A := \{\xi \in L^2_T(R) \mid y_0 \leq x\}\) to \(A^\phi(x) := \{\xi \in L^2_T(R) \mid E_{0,T}^{g,\phi}(\xi) \leq x\}\), that is to minimize
\[
\min_{\xi \in A^\phi(x)} \rho(\xi).
\]

Our paper is organized as follows. In section 2, we first study optimal investment problem without constraints on wealth and portfolio process. With the help of non-smooth analysis, we obtain a necessary condition for an optimal solution, which generalize those obtained in Ji and Peng \([8]\). Secondly, we continue to consider constrained case. We point out serious difficulties we met in this case and discussed such problems briefly, more details and fully discussion about such constrained problem will be included in our future papers. In sections 3, we give some examples to verify our analysis. At last section, some necessary backgrounds about non-smooth analysis are gathered.

## 2 Maximum principle for the optimal investment problem

In this section, we aim to derive some necessary conditions for the optimality of our problem. Suppose the wealth process of an investor evolving according to \([1]\) with limited initial capital \(x\). The optimal problem \([2]\) is a constrained problem. Just as usual,

Suppose no constraints on wealth and portfolio process, by an exact penalization method used in non-smooth analysis, we need to assume that
i) The risk measure \(\rho(\cdot)\) is Lipschitz
ii) If we write \(y_0 \equiv E_{0,T}^{g,\phi}(\cdot)\) as a function of terminal value, it is Lipschitz.

If no constraints on wealth and portfolio process, by the theory of BSDE, \(E_{0,T}^{g,\phi}(\cdot)\) is obviously Lipschitz, see E. Pardoux, S.G. Peng \([7]\) for example.

**Proposition 2.1.** Suppose \(g\) satisfies conditions (A1) and (A2), \(\xi_i \in L^2_T(R), (y_i^0, z_i^1), i = 1, 2\) are solutions of \([1]\) with terminal values \(\xi_i\), then there exists a constant \(C > 0\) such that
\[
|y_0^2 - y_0^1|^2 \leq CE|\xi_2 - \xi_1|^2
\]

In order to use results \([1, 2]\) in appendix, we need to show

**Lemma 2.1.** Suppose that \(g\) satisfies conditions (A1) and (A2), then for any \(\xi^* \in L^2_T(R)\), we have \(0 \notin \partial^\phi E_{0,T}^{g}(\xi^*)\).

The proof is very similar to the proof of the strict comparison theorem of BSDE.

**Proof.** Suppose on the contrary \(0 \in \partial^\phi E_{0,T}^{g}(\xi^*)\). Let \(f(\cdot) = E_{0,T}^{g}(\cdot)\), then
Theorem 2.1. Suppose that $\mathcal{E}_{0,T}^\eta$ is not empty for any $\xi$ and $\eta$ holds for any $\xi$ and $\eta$. Let $\hat{g}$ be a solution of BSDE via \((1)\) is Lipschitz, then according to in appendix, $\partial^o \mathcal{E}_{0,T}^\eta(\xi)$ is not empty for any $\xi$, and in fact, we have the following results.

**Theorem 2.1.** Suppose that $g$ and $\xi$ are the standard parameters for BSDE, \((y_t, z_t)\) is a solution of BSDE with terminal value $\xi$. Let $f(\xi) = y_0$, then $f(\cdot)$ is Lipschitz on Hilbert space $L^2_{\mathcal{P}}(R)$, and

$$
\partial^o f(\xi) \subset \int_0^T (\partial^o g(t, y_t, z_t), (\hat{y}_t, \hat{z}_t))dt - \int_0^T \hat{z}_t^* dW_t. \tag{5}
$$

The meaning of the equation above is that, for any $\eta \in \partial^o f(\xi)$, there exists $(\varphi_t, \psi_t) \in \partial^o g(t, y_t, z_t)$, such that for any $\zeta \in L^2_{\mathcal{P}}(R)$,

$$
\langle \eta, \zeta \rangle = \hat{y}_0 = \zeta + \int_0^T (\varphi_t \hat{y}_t + \psi_t \hat{z}_t)dt - \int_0^T \hat{z}_t^* dW_t,
$$

where $(\hat{y}_t, \hat{z}_t)$ is the solution of BSDE generated by $h(t, y, z) = \varphi y + \psi z$ with terminal value $\zeta$.

**Proof.** Let $\hat{g}(t, \hat{y}, \hat{z}) = g^o(t, y_t, z_t; \hat{y}_t, \hat{z}_t)$, where $(\hat{y}_t, \hat{z}_t) \in R \times R^d$, then by the Definition of generalized directional derivative, $\hat{g}(t, \hat{y}, \hat{z})$ is Lipschitz homogeneous and convex in $(\hat{y}, \hat{z})$ and

$$
\hat{g}(t, \hat{y}, \hat{z}) = \max_{(\varphi_t, \psi_t) \in \partial^o g(t, y_t, z_t)} \langle (\varphi_t, \psi_t), (\hat{y}, \hat{z}) \rangle. \tag{6}
$$

For $\zeta \in L^2_{\mathcal{P}}(R)$, the BSDE generated by $\hat{g}$ with terminal value $\zeta$ evolves as follows

$$
\hat{y}_t = \zeta + \int_t^T \hat{g}(t, \hat{y}_t, \hat{z}_t)dt - \int_t^T \hat{z}_t^* dW_t, 0 \leq t \leq T. \tag{7}
$$

Let $\hat{f}(\zeta) = \hat{y}_0$, then $\hat{f}$ is homogeneous and convex. Let

$$
M := \{\eta \in L^2_{\mathcal{P}}(R) | \langle \eta, \zeta \rangle = \hat{y}_0, (\varphi_t, \psi_t) \in \partial^o g(t, y_t, z_t)\}, \tag{8}
$$

where $(\hat{y}_t, \hat{z}_t)$ is the solution of following BSDE

$$
\hat{y}_t = \zeta + \int_t^T (\varphi_t \hat{y}_t + \psi_t \hat{z}_t)dt - \int_t^T \hat{z}_t^* dW_t, 0 \leq t \leq T. \tag{9}
$$

By the comparison theorem of BSDE, for any $\eta \in M$, one has $\hat{f}(\zeta) \geq \langle \eta, \zeta \rangle$, and by \[(9), \hat{f}(\zeta) = \max_{\eta \in M} \langle \eta, \zeta \rangle,

\] thus $\partial \hat{f}(0) = M$. By now, if we can proof $\partial^o \mathcal{E}_{0,T}^\eta(\xi) = \partial \hat{f}(0)$, then the theorem is proved. But in fact, by the continuous dependence theorem and comparison proposition, $f^o(\xi, \zeta) = \hat{f}(\zeta)$ can be obtained easily. \[ \square \]

Based on the above results in non-smooth analysis, we get a necessary condition for the optimality of $\xi^*$.

**Theorem 2.2.** Suppose that $\rho(\cdot)$ is a Lipschitz function. If $\xi^*$ is an optimal solution of \[(2)\] then for some $\lambda$, there exist $\zeta \in \partial^o \rho(\xi^*)$ and $\eta \in \partial^o \mathcal{E}_{0,T}^\eta(\xi^*)$ such that

$$
\zeta + \lambda \eta = 0
$$

holds.
Proof. By Lemma 2.1, we have $0 \notin \partial^o E_{0,T}^g(\xi^*)$ and thus Proposition 1.2 can be used to deduce that
\[ N_C(\xi^*) \subset \bigcup_{\lambda \geq 0} \lambda \partial^o E_{0,T}^g(\xi^*). \]

If $\xi^*$ is an optimal solution of (2) satisfying $E_{0,T}^g(\xi^*) = x$, then by the Fermat condition, there exists a nonnegative number $\lambda \geq 0$ and some $\zeta \in \partial^o \rho(\xi^*)$, $\eta \in \partial^o E_{0,T}^g(\xi^*)$ such that
\[ \zeta + \lambda \eta = 0. \]

Now supposing $\tilde{x} = E_{0,T}^g(\xi^*) < x$, then we can set
\[ \tilde{C} := \{ \xi \in L_T^2(R)| E_{0,T}^g(\xi) \leq \tilde{x} \} \]
and solve optimal problem on $\tilde{C}$. It is easy to see that $\xi^*$ is optimal on $\tilde{C}$ if it is optimal on $C$ for $\rho(\cdot)$. \qed

In Ji and Peng [8], they assume that the generator is continuously differentiable with variables, in this special case, we can get an explicit form of $E_{0,T}^g(\cdot)$. To do so, we need a notation named strict differentiable for a function in Banach Space and a related theorem.

Definition 2.1 (Strict differentiable, Clark[2]). A function $f(\cdot)$ defined on Banach space $X$ is called strict differentiable at $x \in X$ if there exists $x^* \in X^*$ such that
\[ \lim_{y \to x, t \to 0^+} \frac{f(y + td) - f(y)}{t} = (x^*, d) \]
exists in any direction $d \in X$.

Theorem 2.3. (Clark[2]) A function $f(\cdot)$ defined on Banach space $X$ is strict differentiable at $x \in X$ as in the above definition, then $\partial f(x) = \{x^*\}$.

Based on the above notations and results, we have

Lemma 2.2. If $g$ is continuously differentiable in $(y, z) \in R \times R^d$ with bounded derivatives, $(y_t = E_{0,t}^g(\xi), z_t)$ is the solution of BSDE with terminal value $\xi$, then
\[ \partial^o E_{0,T}^g(\xi) = \{q_T\}, \]
where $q_T \in L_T^2(R), \forall \eta \in L_T^2(R), \langle q_T, \eta \rangle = \tilde{y}_0 = E_{0,T}^g(\xi), \tilde{g}(t, y, z) = g_y(y_t, z_t) + g_z(y_t, z_t)z,$
\[ \tilde{y}_t = \eta + \int_t^T \tilde{g}(s, \tilde{y}_s, \tilde{z}_s)ds - \int_t^T \tilde{z}_s dW_s, 0 \leq t \leq T, \] (10)
i.e., $\tilde{y}_0$ is the solution of BSDE with generator $\tilde{g}(t, y, z)$.

Proof. By Ji and Peng [8], if $g$ is continuously differentiable in $(y, z) \in R \times R^d$ with bounded derivatives, then $E_{0,T}^g(\cdot)$ is strict differentiable and for any $\eta \in L_T^2(R)$,
\[ \lim_{\zeta \to \xi, t \to 0^+} \frac{E_{0,T}(\zeta + t\eta) - E_{0,T}(\zeta)}{t} = \tilde{y}_0. \]
It is obviously that the function $\tilde{y}_0 = E_{0,T}^g(\eta)$ deduced by (10) is linear continuous on $L_T^2(R)$, hence by the Riesz representation theorem, there exists $q_T \in L_T^2(R)$ such that $\langle q_T, \eta \rangle = E_{0,T}^g(\eta)$ holds for any $\eta \in L_T^2(R)$ and the corresponding sub-differential set contains only one element $q_T$.

Corollary 2.1. Suppose $g(t, y, z)$ has bounded continuous derivatives in $(y, z)$ and $(y_t = E_{0,t}^g(\xi), z_t)$ is a solution of BSDE with terminal value $\xi$. If $\xi^*$ is an optimal solution of (2), then there exists $\zeta \in \partial^o \rho(\xi^*)$ and some positive number $\lambda$ such that
\[ \zeta + \lambda q_T = 0, \]
where $q_T$ is obtained by the Riesz representation theorem through (10).

The key points for our successes to use non-smooth results are Proposition 2.1 and Lemma 2.1 and we can take $h(\cdot)$ as $E_{0,T}^g(\cdot)$ in Proposition 1.2. But in constrained case, the function $E_{0,T}^g(\cdot)$ fails in both Proposition 2.1 and Lemma 2.1. In such case, we try to describe $\partial^o C(\xi)$ or $N_C(\xi)$ for $\xi \in C$ directly, where $C := A(\xi) := \{ \xi \in L_T^2(R)| E_{0,T}^g(\xi) \leq x \}$. Thanks to the lower-semi continuity of $E_{0,T}^g(\cdot)$, the constrained set $C$ in our optimal problem (1) is closed and many results about distance function $d_C(\cdot)$ of closed set $C$ thus can be
3 Examples

In this section, some examples are proposed to illustrate the obtained result. In Ji and Peng [8], they considered the following optimal problem to find $\xi^*$ such that

$$J(\xi) := E[u(\xi)]$$

is minimized under the following constraints

$$\begin{cases}
E[\varphi(\xi)] = c, \\
\mathcal{E}^g_{0,T}(\xi) = x, \\
\xi \in U,
\end{cases}$$

where

$$U = \{\xi | \xi \in L^2_T(\Omega), \xi \geq 0, \quad \text{a.s.}\}$$

and functions $u, \varphi$ are both continuous differentiable with bounded derivatives. In the framework of Ji and Peng [8], if we take the notations of non-smooth analysis, Ji and Peng [8] obtained the following result.

**Theorem 3.1.** If $\xi^*$ is an optimal solution of (11), then there exist number $h^1$ and non-positive number $h$ such that

$$hu_\xi(\xi^*(w)) + h^1 \varphi_\xi(\xi^*(w)) + q_T \geq 0, \forall w \in M, \text{a.s.,}$$

$$hu_\xi(\xi^*(w)) + h^1 \varphi_\xi(\xi^*(w)) + q_T(w) = 0, \forall w \in M^c, \text{a.s.,}$$

where $q_T$ is the terminal value of BSDE (9), $M := \{w | \xi^*(w) = 0\}$.

Since $E[\varphi(\xi)] = c$ is a constant, then we can let $\rho(\xi) := E[u(\xi) + \varphi(\xi)]$ and transfer (11) to our optimal problem (12). For $\rho(\xi) := E[u(\xi) + \varphi(\xi)]$, we have the following proposition.

**Proposition 3.1.** If $u(x) : R \to R$ is continuous differentiable with bounded derivatives, then the function $f(\xi) = E[u(\xi)]$ defined on $L^2_T(R)$ is absolutely differentiable, and its sub-differential is $u_\xi(\xi)$.

**Proof.** It can be proved by Fubini theorem and bounded convergence theorem.

Noting that when $u$ and $\varphi$ are both continuously differentiable, by the above Proposition, $\rho(\xi)$ is absolutely differentiable, then there exists only one element in $\partial \rho(\xi)$, i.e., $u_\xi(\xi) + \varphi_\xi(\xi)$, where $u_\xi(\cdot), \varphi_\xi(\cdot)$ is the corresponding derivative. At the same time, by Lemma 2.2 when the generator $g$ of BSDE is continuously differentiable, the sub-differential of the function $\mathcal{E}^g_{0,T}(\xi)$ deduced by BSDE contains only $q_T$, then by Corollary 2.1, there exists a number $\lambda$, such that

$$q_T + \lambda (u_\xi(\xi) + \varphi_\xi(\xi)) = 0. \quad (14)$$

We consider a similar optimal investment problem by non-smooth analysis via BSDE approach.

**Example 3.1.** Minimize

$$\min E[\xi^2] - c^2 \quad (15)$$

in a set of variables satisfying the following constraints

$$\begin{cases}
E[\xi] = c, \\
\mathcal{E}^g_{0,T}(\xi) \leq x, \\
\xi \in L^2_T(R), \quad \xi \geq 0, \quad \text{a.s.,}
\end{cases}$$

where $g(t, y, z) = r(t)y + \theta(t)z, r(t)$ and $\theta(t)$ are coefficients derived from financial market satisfying suitable measurable and integrable conditions.

In this example, since $E[\xi] = c$ is a constant, for any $b \in R$, we take $\rho(\xi) := E[\xi^2 + b\xi]$ and it is obviously absolutely differentiable, then the sub-differential at $\xi^*$ only contains $2\xi^* + b$. It is easy to get $\partial \mathcal{E}^g_{0,T}(\xi^*) = \{q_T\}$, where ($\hat{y}_t, \hat{z}_t$) is the solution of following BSDE

$$\hat{y}_t = \xi^* + \int_t^T (r(s)\hat{y}_s + \theta(s)\hat{z}_s)ds - \int_t^T \hat{z}_s^*dW_s, 0 \leq t \leq T.$$

Then, if $\xi^*$ is an optimal solution of this example, then by Theorem 2.2 or Corollary 2.1, there exists a number $\lambda_b$ such that

$$2\xi^* + b + \lambda_b q_T = 0$$

holds.

The virtue of our method can help us consider the optimal problem when the expectation is not lower than any time in future.
Example 3.2. Finding an optimal $\xi^*$ in the following set
\[
\begin{align*}
\{ & E[\xi] \geq c, \\
& \mathcal{E}_{0,T}^g(\xi) \leq x, \\
& \xi \in L_2^2(R), \quad \xi \geq 0, \text{ a.s.},
\end{align*}
\]
to minimize
\[
\min E[\xi^2] - E^2[\xi].
\]
(16)

Because the constraint on the expectation is not a constant, we take $\rho(\xi) := E[\xi^2] - E^2[\xi]$. We combine the constraints on the expectation and initial value of investment together to get the following new constraint
\[
A := \{ \xi \in L_2^2(R) | h(\xi) \leq 0 \},
\]
where $h(\xi) := \max\{ \mathcal{E}_{0,T}^g(\xi) - x, -E[\xi] + c \}$.

Lemma 3.1. (Clark [2]) Supposing that $\{ f_i, i = 1, 2, \cdots, n \}$ is a set of Lipschitz functions, we define
\[
f(x) := \max\{ f_i(x) | i = 1, 2, \cdots, n \}.
\]

Let $I(x)$ be the subset of index satisfying $f_i(x) = f(x)$, then
\[
\partial f(x) \subset \text{co}\{ \partial f_i(x) : i \in I(x) \}.
\]
(17)

Furthermore, if $f_i$ is normal, then the equality holds, where coA is the convex hull of A.

By the Lemma stated above and Theorem 2.2, we have the following theorem.

Theorem 3.2. If $\xi^*$ is an optimal solution of Example 3.2, then there exist a nonnegative number $\lambda$ and $a \in [0, 1]$ such that
\[
\begin{align*}
2(\xi^* + E[\xi^*]) + \lambda q_T &= 0, \\
2(\xi^* + E[\xi^*]) - \lambda &= 0, \\
2(\xi^* + E[\xi^*]) + \lambda((1-a)q_T - a) &= 0,
\end{align*}
\]
where $\mathcal{E}_{0,T}^g(\xi)$ is a solution of Example 3.3, then there exist a nonnegative number $\lambda$ and $a \in [0, 1]$ such that
\[
\begin{align*}
2y_T + \lambda q_T &= 0, \\
2y_T - \lambda &= 0, \\
2y_T + \lambda((1-a)q_T - a) &= 0,
\end{align*}
\]
and $(y_t, z_t)$ is the solution of BSDE generated by $g(t, y, z)$ with terminal value $\xi^*$. Thus we have the following result.

Theorem 3.3. If $\xi^*$ is an optimal solution of Example 3.3, then there exist a nonnegative number $\lambda$ and $a \in [0, 1]$ such that
\[
\begin{align*}
2y_T + \lambda q_T &= 0, \\
2y_T - \lambda &= 0, \\
2y_T + \lambda((1-a)q_T - a) &= 0,
\end{align*}
\]
4 Appendix: some results about non-smooth analysis

Suppose that $X$ is a Banach space, $X^*$ is its dual space. A function $f : X \to R$ is called Lipschitzian if
\[ |f(x_1) - f(x_2)| \leq M||x_1 - x_2|| \tag{19} \]
holds for some $M > 0$, where $|| \cdot ||$ is the norm in $X$.

The generalized directional derivative of $f$ at $x$, denoted as $f^\circ(x; v)$, is defined as
\[ f^\circ(x; v) := \limsup_{y \to x, t \to 0} \frac{f(y + tv) - f(y)}{t} \tag{20} \]
where $y$ is a vector in $X$, $t$ is a positive number.

Obviously, $f^\circ(x; v)$ is homogeneous and sub-linear on $X$, then by Banach Theorem, the generalized derivative set of $f$ at $x$
\[ \partial^\circ f(x) := \{ \zeta \in X^* | \zeta(v) \leq f^\circ(x; v), \forall v \in X \} \tag{21} \]
is nonempty and weak star compact in $X^*$.

By definition, the Fermat optimal principle $0 \in \partial^\circ f(x_0)$ holds when $f(x)$ attains extreme at some point $x_0 \in X$,
\[ f(x) \geq f(x_0), \quad \forall x \in X. \]

Now, we recall more results in non-smooth analysis. For more details, one can see Clark et al. [3].

Given a set $C \subset X$, the distance function $d_C(x) : X \to R$ is defined as
\[ d_C(x) := \inf\{||y - x||, y \in C\}. \]

The following lemma transfers the constrained problem to the unconstrained case.

**Lemma 4.1.** (Exact penalization) Suppose that $f$ is a Lipschitz function with coefficient $K$ defined on $S$, $x \in C \subset S$ and $f$ takes its minimum value at $x$ on $C$. Then, for any $\tilde{K} \geq K$, $g(y) = f(y) + \tilde{K}d_C(y)$ attains minimum value at $x$ on $S$. On the contrary, if $\tilde{K} > K$ and $C$ is closed, then the minimum point of $g$ on $S$ must belong to $C$.

Contingent and normal derivative for a set $C$ are defined by the distance function, see Clark et al. [3] for details.

**Definition 4.1.** Assume $x \in C$, if $d_C^\circ(x; v) = 0$, then $v$ is said to be a contingent derivative at $x \in X$. We denote the set of contingent derivatives as $T_C(x)$. By polarity, we define the normal derivative set as
\[ N_C(x) := \{ \zeta \in X^* | \zeta(v) \leq 0, \quad \forall v \in T_C(x) \}. \]

By the above definition, we have the following proposition.

**Proposition 4.1.** Supposing that $x \in C$, then it holds that
\[ N_C(x) = cl \left\{ \bigcup_{\lambda \geq 0} \lambda \partial^\circ d_C(x) \right\}, \tag{22} \]
where $cl$ means the weak star closure.

For $C = A(x)$, if $\xi^*$ is an optimal solution of [2], then the Fermat condition
\[ 0 \in \partial^\circ \rho(\xi^*) + N_C(\xi^*). \tag{23} \]
holds.

For a special kind of set $C$, we have the following result.

**Proposition 4.2.** Suppose that $h$ is Lipschitz in a neighborhood of $x$ and $0 \notin \partial^\circ h(x)$, if $C = \{ y \in X : h(y) \leq h(x) \}$, then it holds that
\[ N_C(x) \subset \bigcup_{\lambda \geq 0} \lambda \partial^\circ h(x). \tag{24} \]
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