Weight functions and Drinfeld currents

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Abstract

A universal weight function for a quantum affine algebra is a family of functions with values in a quotient of its Borel subalgebra, satisfying certain coalgebraic properties. In representations of the quantum affine algebra it gives off-shell Bethe vectors and is used in the construction of solutions of the qKZ equations. We construct a universal weight function for each untwisted quantum affine algebra, using projections onto the intersection of Borel subalgebras of different types, and study its functional properties.

1 Introduction

The first step of the nested Bethe ansatz method ([KR]) consists in the construction of certain rational functions with values in a representation of a quantum affine algebra or its rational or elliptic analogue. In the case of the quantum affine algebra \( U_q(\widehat{\mathfrak{gl}}_2) \) these rational functions, known as the off-shell Bethe vectors, have the form \( B(z_1) \cdots B(z_n) v \), where \( B(u) = T_{12}(u) \) is an element of the monodromy matrix (this is a generating series for elements in the algebra) and \( v \) is a highest weight vector of a finite dimensional representation of \( U_q(\widehat{\mathfrak{gl}}_2) \). For the quantum affine algebra \( U_q(\widehat{\mathfrak{g}}_N) \), the off-shell Bethe vectors are constructed in [KR] by an inductive procedure (the induction is over \( N \)). These Bethe vectors were then used (under the name ‘weight functions’) in the construction of solutions of the \( q \)-difference Knizhnik-Zamolodchikov equation ([TV1, S]). Inductive procedures for the construction of Bethe vectors were also used in rational models (where the underlying symmetry algebra is a Yangian) in [BF] and [ABFR], for \( \mathfrak{g} = \mathfrak{sl}_N \); in these cases the Bethe vectors were expressed explicitly in the quasi-classical limit or using the Drinfeld twist (see Section 3.5). Bethe vectors for rational models with \( \mathfrak{g} = \mathfrak{o}(n) \) and \( \mathfrak{sp}(2k) \) were studied in [R]; the twisted affine case \( A_2^{(2)} \) was treated in [T].
Despite their complicated inductive definition, the weight functions enjoy nice properties, which do not depend on induction steps. These are coalgebraic properties, which relate the weight function in a tensor product of representations with weight functions in the tensor components ([TV1]).

The goal of this paper is to give a direct construction of weight functions, independent of inductive procedures. For this purpose we introduce the notion of a universal weight function. This is a family of formal Laurent series with values in a quotient of the Borel subalgebra of the quantum affine algebra, satisfying certain coalgebraic properties. The action of the universal weight function on a highest weight vector defines a weight function with values in a representation of the quantum affine algebra, which enjoys coalgebraic properties as in [TV1].

It is well-known that quantum affine algebras, as well as affine Kac-Moody Lie algebras, admit two different realizations ([D2]). In the first realization, the quantum affine algebra is generated by Chevalley generators, satisfying $q$-analogues of the defining relations for Kac-Moody Lie algebras ([D1]). In the second realization ([D2]), generators are the components of the Drinfeld currents, and the relations are deformations of the loop algebra presentation of the affine Lie algebra. The quantum affine algebra is equipped with two coproducts (‘standard’ and ‘Drinfeld’), each of which expresses simply in the corresponding realization. Both realizations are related to weight functions: on the one hand, the weight function satisfies coalgebraic properties with respect to the standard coproduct structure; on the other hand, the notion of a highest weight vector is understood in the sense of the ‘Drinfeld currents’ presentation.

Our construction of a universal weight function is based on the use of deep relations between the two realizations. This connection was done in several steps. The isomorphism between the algebra structures of both sides was proved in [D2] and [DF]. The coalgebra structures were then related in [KT2, KT3, Be, Da, DKP1]; there it was proved that the ‘standard’ and ‘Drinfeld’ coproducts are related by a twist, which occurs as a factor in the decomposition of the $R$-matrix for the ‘standard’ coproduct. A further description of this twist (in the spirit of the Riemann problem in complex analysis) was suggested in [ER] and developed in [EF, E2, DKP]. Each realization determines a decomposition of the algebra as the product of two opposite Borel subalgebras (so there are 4 Borel subalgebras). In its turn, each Borel subalgebra decomposes as the product of its intersections with the two Borel subalgebras of the other type, and determines two projection operators which map it to these intersections. The twist is then equal to the image of the tensor of the bialgebra pairing between opposite Borel subalgebras by the tensor product of opposite projections.

In this paper, we give a new proof of these results. For this, we prove a general result on twists of the double of a finite dimensional Hopf algebra $A$, arising from the decomposition of $A$ as a product of coideals (Subsection 2.1); this result has a graded analogue (Subsection 2.2), which can be applied e.g. to quantum Kac-Moody algebras with their standard coproduct. More importantly, this result has a topological version (Subsection 2.3). In Section 3, we show how this topological version implies that the Drinfeld and standard coproducts are related by the announced twist.

The main result of this paper is Theorem 3. It says that the collection of images of products of Drinfeld currents by the projection defines a universal weight function. This allows to compute the weight function explicitly when $g = sl_2$ or $sl_3$ (see [KP1]), or
when \( q = 1 \) (see identity (1.30)), using techniques of complex analysis and conformal algebras ([DK]). We give a conjecture on the general form of the universal weight function. We describe functional properties of the weight functions at the formal level, using techniques of ([1]). We also prove more precise rationality results in two cases: (a) in the case of finite dimensional modules, as a consequence of the conjecture on the form of the universal weight function; (b) in the case of lowest weight modules, the rationality follows (unconditionally) from a grading argument.

The paper is organized as follows. In Section 2, we prove results on twists of double Hopf algebras. In Section 3, we recall the definition of the untwisted quantum affine algebra \( U_q(\mathfrak{g}) \), of its coproducts, the construction of the Cartan-Weyl basis and its relation with the currents realization of \( U_q(\mathfrak{g}) \), following ([KT1], [KT2]); we also reprove the twist relation between the two (Drinfeld and standard) coproducts, using Section 2. In particular, we introduce Borel subalgebras of different types and the related projection operators. Their definition, relies on a generalization of the convexity property of the Cartan-Weyl generators to ‘circular’ Cartan-Weyl generators, see ([KT1]); their properties are proved in the Appendix.

2 Twists of doubles of Hopf algebras

2.1 The finite dimensional case

Let \( A \) be a finite dimensional Hopf algebra. Assume that \( A_1, A_2 \) are subalgebras of \( A \) such that:\(^1\) (a) the map \( m_A : A_1 \otimes A_2 \rightarrow A \) is a vector space isomorphism, (b) \( A_1 \) (resp., \( A_2 \)) is a left (resp., right) coideal of \( A \), i.e., \( \Delta_A(A_1) \subset A \otimes A_1, \quad \Delta_A(A_2) \subset A_2 \otimes A \).

Let \( P_i : A \rightarrow A_i \) be the linear maps such that \( P_i(a_1a_2) = a_1 \varepsilon_A(a_2), \quad P_i(a_1a_2) = \varepsilon_A(a_1)a_2 \) for \( a_i \in A_i \). Then we have \( m_A \circ (P_1 \otimes P_2) \circ \Delta_A = \text{id}_A \).

Let \( D \) be the double of \( A \) and let \( R \in D^{2 \otimes 2} \) be its \( R \)-matrix. Set \( R_i := (P_i \otimes \text{id})(R) \).

The above identity, together with \( (\Delta_A \otimes \text{id})(R) = R_1^{1,3}R_2^{2,3} \), implies that \( R = R_1R_2 \).

Let us set\(^2\) \( B := A^{\text{cop}}, \quad B_1 := (A_1^{A_2})^\perp \subset B, \quad B_2 := (A_1^{A_2})^\perp \subset B. \)

Theorem 1. (see [ER], [ER], [DKP])

1) \( B_1, B_2 \) are subalgebras of \( B \); the subalgebra \( B_1 \) (resp., \( B_2 \)) is a left (resp., right) coideal of \( B \), i.e., \( \Delta_B(B_1) \subset B \otimes B_1, \quad \Delta_B(B_2) \subset B_2 \otimes B, \) and \( m_B : B_2 \otimes B_1 \rightarrow B \) is a vector space isomorphism.

2) Define \( P'_i : B \rightarrow B_i \) by \( P'_2(b_2b_1) = b_2 \varepsilon_B(b_1), \quad P'_1(b_2b_1) = \varepsilon_B(b_2)b_1 \). Then \( R_i = (\text{id}_A \otimes P_{3-i})(R) \), for \( i = 1, 2 \). In fact, \( R_i = (P_i \otimes P_{3-i})(R) \in A_i \otimes B_{3-i} \).

3) \( R_2 \) is a cocycle for \( D \), i.e., \( R_2^{1,2}(\Delta_D \otimes \text{id}_D)(R_2) = R_2^{2,3}(\text{id}_D \otimes \Delta_D)(R_2) \). It follows that \( D \), equipped with the coproduct \( R_2^1 \Delta_D(x) := R_2 \Delta_D(x)R_2^{-1} \), is a quasitriangular Hopf algebra (which we denote by \( R_2^D \)) with \( R \)-matrix \( R_2^{2,1}R_1 \).

\(^1\)If \( X \) is a Hopf algebra, we denote by \( m_X, \Delta_X, S_X, \varepsilon_X, 1_X \) its operations.

\(^2\)For \( X \) a Hopf algebra, \( X^{\text{cop}} \) means \( X \) with opposite coproduct; if \( Y \subset X \), then \( Y^* := Y \cap \text{Ker}(\varepsilon_X) \).
4) \( m_D(A_i \otimes B_i) = m_D(B_i \otimes A_i) \) for \( i = 1, 2 \), so \( D_i := m_D(A_i \otimes B_i) \subset D \) are subalgebras of \( D \).

5) \( A_i, B_i \) have the following coideal properties: \( R^2 \Delta_D(A_1) \subset A_1 \otimes D_1, R^2 \Delta_D(B_1) \subset D_1 \otimes B_1, R^2 \Delta_D(A_2) \subset A_2 \otimes D_2, R^2 \Delta_D(B_2) \subset D_2 \otimes B_2. \)

6) \( D_i \) are Hopf subalgebras of \( R^2 D \). The quasitriangular Hopf algebra \( R^2 D \) is isomorphic to the double of \( (D_1, (R^2 \Delta_D)|_{D_1}) \), whose dual algebra with opposite coproduct is \( (D_2, (R^2 \Delta_D)|_{D_2}) \).

**Proof.** 1) For \( X, Y \subset A \), set \( XY := m_A(X \otimes Y) \).

We have \((\text{id} \otimes \varepsilon_A) \circ \Delta_A = \text{id}_A\), which implies that \( \Delta_A(A_i^e) \subset A \otimes A_i^e + A_i^e \otimes 1_A. \)

Then \( \Delta_A(A_i^e A_i^e) \subset \Delta_A(A_i^e) \Delta_A(A_i^e) \subset (A \otimes A_i^e + A_i^e \otimes 1_A)(A_2 \otimes A) \subset A \otimes A_i^e A_i^e + A_i^e A_i^e \otimes A. \)

Now \( A = A_1 A_2 \), and \( A_i^e A_1 = A_i^e \), so \( A_i^e A = A_i^e A_2 \), which implies that \( A_i^e A_2 \) is a two-sided coideal of \( A \). This implies that \( B_1 \) is a subalgebra of \( B \). In the same way, \( B_2 \) is a subalgebra of \( B \).

\( A_i^e A_2 \) is a right ideal of \( A \), which implies that \( B_1 \) is a left coideal of \( B \). In the same way, \( B_2 \) is a right coideal of \( B \).

We now show that \( m_B : B_2 \otimes B_1 \to B \) is a vector space isomorphism. We have vector space isomorphisms \( B_i \cong A_i^{3-i} \), for \( i = 1, 2 \), induced by \( A = A_1 \oplus A_1 A_2^e \) and \( A = A_2 \oplus A_1^e A_2 \). So we will prove that the transposed map \( A \xrightarrow{m_B^t} A_1 \otimes A_2 \) is an isomorphism.

Let \( b_i \in A_i^{3-i} \approx B_{3-i} \). When viewed as elements of \( B = A^* \), \( b_i \) satisfy \( \langle b_i, a_1 a_2 \rangle = \langle b_i, a_1 \rangle \varepsilon(a_2), \langle b_i, a_1 a_2 \rangle = \varepsilon(a_1) \langle b_i, a_2 \rangle \). Then \( \langle b_1 b_2, a_1 a_2 \rangle = \langle b_1 \otimes b_2, a_1^{(1)} a_2^{(1)} \otimes a_1^{(2)} a_2^{(2)} \rangle = \langle b_1, a_1^{(1)} \rangle \varepsilon(a_2^{(1)}) \varepsilon(a_2^{(2)}) = \langle b_1, a_1 \rangle \langle b_1, a_2 \rangle \). So the composed map \( A_1 \otimes A_2 \xrightarrow{m_B} A \xrightarrow{m_B^t} A_1 \otimes A_2 \) is the identity, which implies that \( A \xrightarrow{m_B} A_1 \otimes A_2 \), and therefore \( m_B : B_2 \otimes B_1 \to B \), is an isomorphism.

2) Let \( R_i^t \in A_i \otimes B_{3-i} \) be the canonical element arising from the isomorphism \( B_i \cong A_i^{3-i} \). Let us show that \( R = R_1^t R_2^t, R_i = R_i^t \).

We have \( R_i^t R_i^t \in A \otimes B \). For \( a \in A \), let us compute \( \langle R_i^t R_i^t, \text{id} \otimes a \rangle \). We assume that \( a = a_1 a_2 \), with \( a_i \in A_i \). Then \( \langle R_i^t R_i^t, \text{id} \otimes a \rangle = \langle R_i^t R_i^t, \text{id} \otimes a_1 a_2 \rangle = \langle (R_i^t)^{1,2}(R_i^t)^{1,3}, \text{id} \otimes a_1^{(1)} a_2^{(1)} \otimes a_1^{(2)} a_2^{(2)} \rangle = \langle (R_i^t)^{1,2}(R_i^t)^{1,3}, \text{id} \otimes a_1^{(1)} \varepsilon_A(a_2^{(1)}) \varepsilon_A(a_2^{(2)}) \rangle = a_1 a_2 = a \). So \( R_i^t R_i^t \in A \otimes B \) is the canonical element, so it is equal to \( R \).

Now \( R_i = (P_i \otimes \text{id}_B)(R) = R_i^t \), since we have \( \langle \varepsilon_A \otimes \text{id}_B \rangle(R_i) = 1_A \). We also compute \( (\text{id} \otimes P_{3-i}^t)(R) = R_i \) and \( (P_i \otimes P_{3-i}^t)(R) = R_i \).

3) We first prove that for any \( a \in A, b \in B \), we have

\[
\langle a^{(1)}, b^{(1)} \rangle b^{(2)} a^{(2)} = a^{(1)} b^{(1)} \langle a^{(2)}, b^{(2)} \rangle.
\] (2.1)

Recall the multiplication formula in the quantum double \([D1]:\)

\[
b a = \langle S_D(a^{(1)}), b^{(1)} \rangle \langle a^{(3)}, b^{(3)} \rangle a^{(2)} b^{(2)}.
\] (2.2)

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3We denote by \( a \otimes b \mapsto \langle a, b \rangle = \langle b, a \rangle \) the pairing \( A \otimes B \to \mathbb{C} \).
Using this equality, we may write the right hand side of (2.1) as follows:
\[
\langle a^{(1)}, b^{(1)} \rangle \langle S_D(a^{(2)}), b^{(2)} \rangle \langle a^{(4)}, b^{(4)} \rangle a^{(3)} b^{(3)} = \langle a^{(1)} \otimes S_D(a^{(2)}), \Delta_B(b^{(1)}) \rangle \langle a^{(4)}, b^{(3)} \rangle a^{(3)} b^{(2)}
\]
\[
\epsilon(a^{(1)}) \epsilon(b^{(1)}) \langle a^{(3)}, b^{(3)} \rangle a^{(2)} b^{(2)} = \langle a^{(2)}, b^{(2)} \rangle a^{(1)} b^{(1)}
\]
which coincides with the left hand side of equality (2.1).

We now prove the cocycle relation \( R_2^{1,2} R_2^{12,3} = R_2^{2,3} R_2^{1,23} \) (the coproduct is \( \Delta_D \)). Both sides of this identity belong to \( A_2 \otimes D \otimes B_1 \). Using the pairing, we will identify them with linear maps \( B_1 \otimes A_2 \rightarrow D \). Let us compute the pairing of both sides of this equality with \( b_1 \otimes \text{id} \otimes a_2 \) for arbitrary \( b_1 \in B_1 \) and \( a_2 \in A_2 \). For the left hand side we have
\[
\langle R_2^{1,2} R_2^{12,3}, b_1 \otimes \text{id} \otimes a_2 \rangle = \langle R_2(a_2^{(1)} \otimes a_2^{(2)}), b_1 \otimes \text{id} \rangle = \sum_i \langle r_i' a_2^{(1)} \otimes r_i'' a_2^{(2)}, b_1 \otimes \text{id} \rangle
\]
\[
= \sum_i \langle r_i' \otimes a_1^{(1)} b_1^{(2)} \otimes b_1^{(1)}, r_i'' a_2^{(2)} \rangle = \langle a_2^{(1)} b_1^{(1)} b_2^{(2)} \rangle a_2^{(2)} .
\]
Here \( R_2 = \sum_i r_i' \otimes r_i'' \). On the other hand, for the right hand side we obtain
\[
\langle R_2^{2,3} R_2^{1,23}, b_1 \otimes \text{id} \otimes a_2 \rangle = \langle R_2(b_1^{(1)} \otimes b_2^{(2)}), \text{id} \otimes a_2 \rangle = \sum_i \langle r_i' b_1^{(1)} \otimes r_i'' b_2^{(2)}, \text{id} \otimes a_2 \rangle
\]
\[
= \sum_i r_i' \otimes b_1^{(1)} \langle r_i'' \otimes b_2^{(2)}, a_1^{(1)} a_2^{(2)} \rangle = a_2^{(1)} b_1^{(1)} a_2^{(2)} b_2^{(2)} ,
\]
where we used the fact that \( R_2 \) is the pairing tensor between the subalgebras \( A_2 \) and \( B_1 \). The cocycle identity now follows from (2.1).

Here is another proof of 3). Recall that \( R \) is invertible, and \( R^{-1} = (S_D \otimes \text{id}_D)(R) \). It follows that \( R_1 \) and \( R_2 \) are invertible. Let us show that \( R_1^{-1} \in A_1 \otimes B_2 \). We first show that \( R_1^{-1} \in D \otimes B_2 \). For this, we let \( a_1 \in A_1, a_2^\varepsilon \in A_2^\varepsilon \) and we compute:
\[
\langle R_1^{-1}, \text{id} \otimes a_1 a_2^\varepsilon \rangle = \langle R_2 R_1^{-1}, \text{id} \otimes a_1 a_2^\varepsilon \rangle = \langle R_2^{1,2}(R_1^{-1})^{1,3}, \text{id} \otimes (a_1 a_2^\varepsilon)^{(1)} \otimes (a_1 a_2^\varepsilon)^{(2)} \rangle
\]
\[
= \langle R_2, \text{id} \otimes (a_1 a_2^\varepsilon)^{(1)} \rangle \langle R_1^{-1}, \text{id} \otimes (a_1 a_2^\varepsilon)^{(2)} \rangle
\]
\[
= P_2((a_1 a_2^\varepsilon)^{(1)}) S_A((a_1 a_2^\varepsilon)^{(2)}) = P_2(a_1^{(1)})(a_2^\varepsilon)^{(1)} S_A((a_2^\varepsilon)^{(2)}) S_A(a_1^{(2)}) = 0,
\]
which proves that \( R_1^{-1} \in D \otimes B_2 \). In the same way, one proves that \( R_1^{-1} \in A_1 \otimes D \), so \( R_1^{-1} \in A_1 \otimes B_2 \), and then \( R_1^{-1} \in A_2 \otimes B_1 \).

Let us set \( \Phi := R_2^{2,3} R_2^{1,23}(R_2^{1,2} R_2^{12,3})^{-1} \) (the coproduct is \( \Delta_D \)). Then \( \Phi \in A_2 \otimes D \otimes B_1 \). Using the quasitriangular identities satisfied by \( R \), we get \( \Phi = (R_1^{1,23} R_2^{2,3})^{-1} R_1^{12,3} R_2^{1,2} \), where the coproduct is now \( \Delta_D^D(x) = R \Delta_D(x) R^{-1} \). The last identity implies that \( \Phi \in A_1 \otimes D \otimes B_2 \). Since \( A_1 \cap A_2 = C_1 A_1 \) and \( B_1 \cap B_2 = C_1 B \), we get \( \Phi \in 1_D \otimes D \otimes 1_D \). The pentagon identity satisfied by \( \Phi \) then implies that \( \Phi = 1_D^D \).

4) Recall that for \( a \in A, b \in B \), we have
\[
ab = \langle b^{(1)}, a^{(1)} \rangle \langle b^{(3)}, S_A(a^{(3)}) \rangle b^{(2)} a^{(2)}, \quad ba = \langle b^{(1)}, S_A(a^{(1)}) \rangle \langle b^{(3)}, a^{(3)} \rangle a^{(2)} b^{(2)}.
\]

Let us set \( XY := m_D(X \otimes Y) \), for \( X, Y \subset D \). We will show that \( B_1 A_1 \subset A_1 B_1 \). Let \( a_1 \in A_1, b_1 \in B_1 \). Then \( b_1 a_1 = \langle b_1^{(1)}, S_A(a_1^{(1)}) \rangle \langle b_1^{(3)}, a_1^{(3)} \rangle a_1^{(2)} b_1^{(2)} \); we have \( \otimes_{i=1}^3 a_i^{(i)} \in 5 \).
\[ A^{\otimes 2} \otimes A_1 \text{ and } \otimes^3_{i=1} b^{(i)}_1 \in B^{\otimes 2} \otimes B_1, \text{ and since } \langle b, a \rangle = \varepsilon_B(b)\varepsilon_A(a) \text{ for } a \in A_1, b \in B_1, \text{ we have } b_1 a_1 = \langle b^{(1)}_1, S_A(a^{(1)}_1) \rangle a^{(2)}_1 b^{(2)}_1; \text{ now } b^{(1)}_1 \otimes b^{(2)}_1 \in B \otimes B_1 \text{ and } a^{(1)}_1 \otimes a^{(2)}_1 \in A \otimes A_1, \text{ which implies that } b_1 a_1 \in A_1 B_1, \text{ as wanted. (One proves in the same way that } B_1 A_1 \subset A_1 B_1. )\]

It follows that \( D_1 := A_1 B_1 \) is a subalgebra of \( D \). In the same way, one shows that \( D_2 := A_2 B_2 \) is a subalgebra of \( D \).

Note that as above, for \( a_1 \in A_1, b_1 \in B_1 \), we have

\[ a_1 b_1 = \langle b^{(1)}_1, a^{(1)}_1 \rangle \langle b^{(3)}_1, S_A(a^{(3)}_1) \rangle b^{(2)}_1 a^{(2)}_1 = \langle b^{(1)}_1, a^{(1)}_1 \rangle b^{(2)}_1 a^{(2)}_1, \]

so that \( B_1 A_1 = A_1 B_1 \simeq A_1 \otimes B_1 \). In the same way, \( B_2 A_2 = A_2 B_2 \simeq A_2 \otimes B_2 \).

5) We have \( R^2_2 \Delta_D(A_1) = R^2_2 \Delta_D(A_1) R^{-1}_2 \subset R^2_2 (A \otimes A_1) R^{-1}_2 \subset A \otimes D_1 \), since \( R^2_2 \in A_2 \otimes B_1 \). On the other hand, \( R^2_2 \Delta_D(A_1) = R^{-1}_2 \Delta^2_D(1) R^2_1 \subset R^{-1}_2 (A_1 \otimes A) R^1_1 \subset A_1 \otimes D_1 \), since \( R^2_1 \in A_1 \otimes B_2 \). Finally, \( R^2_2 \Delta_D(A_1) \subset A_1 \otimes D_1 \). The other inclusions are proved similarly.

6). 4) and 5) imply that \( D_1 \) are Hopf subalgebras of \( D \).

We have now \( R^2_2 R^1_1 \in D_1 \otimes D_2 \). It is a nondegenerate tensor, as it is inverse to the pairing \( D_1 \otimes D_2 \simeq B_1 A_1 \otimes A_2 B_2 \simeq (A_1 \otimes B_1) \otimes (A_2 \otimes B_2) \to \mathbb{C} \), given by the tensor product of the natural pairings \( A_1 \otimes B_2 \to \mathbb{C}, A_2 \otimes B_1 \to \mathbb{C} \).

Let us prove that \( m_D : D_1 \otimes D_2 \to D \) is a vector space isomorphism. The map \( A_1 \otimes B_1 \otimes A_2 \otimes B_2 \simeq A_1 B_1 \otimes A_2 B_2 \simeq D_1 \otimes D_2 \to D \simeq A \otimes B \) is given by \( a_1 \otimes b_1 \otimes a_2 \otimes b_2 \mapsto \langle b^{(1)}_1, a^{(1)}_1 \rangle \langle b^{(3)}_1, a^{(3)}_1 \rangle a_1 a_2 \otimes b_1 b_2 \). One checks that the inverse map is given by \( A_1 \otimes A_2 \otimes B_1 \otimes B_2 \simeq A \otimes B \to D_1 \otimes D_2 \) using the same formula, replacing \( \Delta_D \) by \( R^2_2 \Delta_D \).

The statement is now a consequence of the following fact: let \( (H, R_H) \) be a quasitriangular Hopf algebra, and \( H_i, i = 1, 2 \) be Hopf subalgebras, such that \( R_H \in H_1 \otimes H_2 \) is nondegenerate and \( m_H : H_1 \otimes H_2 \to H \) is a vector space isomorphism, then \( H_2 = H_1^{s\text{cop}} \) and \( H \) is the double of \( H_1 \) (indeed, since \( R_H \) is nondegenerate, it sets up a vector space isomorphism \( H_1 \simeq H_2^{s\text{cop}} \), and since it satisfies the quasitriangularity equations, this is an isomorphism \( H_1 \simeq H_2^{s\text{cop}} \) of Hopf algebras; we are then in the situation of the theorem of [DI] on doubles). \( \square \)

### 2.2 The graded case

In the case when \( A \) is a Hopf algebra in the category of \( \mathbb{N} \)-graded vector spaces with finite dimensional components, the results of the previous section can be generalized as follows.

Let \( (\alpha_{ij})_{1 \leq i,j \leq r} \) be a nondegenerate matrix, let \( A' \) be a \( \mathbb{N} \)-graded braided Hopf algebra, with finite dimensional components and \( A'[0] \simeq \mathbb{C} \), where the braiding is defined by \( (q^{\alpha_{ij}})_{1 \leq i,j \leq r} \). Let \( A := A' \otimes \mathbb{C}[\mathbb{Z}^r] \) be the corresponding Hopf algebra. Let \( B' \) be the graded dual to \( A' \) and \( B \) be the corresponding Hopf algebra. We then have a nondegenerate Hopf pairing \( A \otimes B \to \mathbb{C} \). Let \( D \) be the quotient of the bicrossproduct of \( A \) and \( B \) by the diagonal inclusion of \( \mathbb{C}[\mathbb{Z}^r] \).

To explain in what space \( R \) lies, we introduce the following notion. If \( V = \bigoplus_{n \in \mathbb{Z}} V[n] \) is a \( \mathbb{Z} \)-graded vector space, set \( V^{\otimes k} := \prod_{n_1,\ldots,n_k \in \mathbb{Z}} V[n_1] \otimes \cdots \otimes V[n_k] \), let \( V^{\otimes > k} \subset V^{\otimes k} \) be the set of all combinations \( \sum v_1 \otimes \cdots \otimes v_k \), such that there exists a constant \( c_1 \) and
functions \(c_2(n_1), \ldots, c_k(n_1, \ldots, n_{k-1})\), such that \(\deg(v_1) \geq c_1, \deg(v_2) \geq c_2(\deg(v_1)), \ldots, \deg(v_k) \geq c_k(\deg(v_1), \ldots, \deg(v_{k-1}))\).

Define \(R_1 \in A_1 \otimes \triangleright B_2\) as the tensor of the pairing \(\langle -,-\rangle_1\), \(R_2' \in A_2' \otimes \triangleright B_1\) as the tensor of \(\langle -,-\rangle_2\) and \(R_2\) as the tensor of \(\langle -,-\rangle_0\). Then \(R_2 = q^{r_0}\), where \(r_0\) is inverse to the matrix \((\alpha_{ij})\), and the tensor of the \(\langle -,-\rangle_2\) is \(R_2 := R_2' R_0\) (it belongs to a suitable extension of \(A_2 \otimes \triangleright B_1\)). The \(R\)-matrix of \(D\) is then \(R = R_1 R_2\) (also in a suitable extension of \(A \otimes \triangleright B\)).

In Theorem 1, 1) is unchanged; 2) is unchanged, with the addition that \(B_i\) are now graded subalgebras of \(B\); in 3), the cocycle identity holds in \(D^{\otimes > 3}\) and the next statement is that \(R_2 \Delta_D\) defines a topological bialgebra structure, i.e., we have an algebra morphism \(R_2 \Delta_D : D \to D^{\otimes > 2}\) and coassociativity is an identity of maps \(D \to D^{\otimes > 3}\); 4) is unchanged; 5) has to be understood in the topological sense; and 6) has to be replaced by the statement that \(D_i\) are topological subbialgebras of \(R_2 D\).

**Example.** One may take \(A = U_q(b_+)\), where \(b_+\) is the Borel subalgebra of a Kac-Moody Lie algebra, equipped with the principal grading. In some cases, the twisted bialgebra \(R_2 D\) is an ordinary Hopf algebra, i.e., \(R_2 \Delta : D \to D^{\otimes 2}\).

### 2.3 The topological case

If \(V = \bigoplus_{n \in \mathbb{Z}} V[n]\) is a \(\mathbb{Z}\)-graded vector space, let \(V^{\otimes > k}\) be the image of \(V^{\otimes k}\) by \(v_1 \otimes \ldots \otimes v_k \mapsto v_k \otimes \ldots \otimes v_1\). We define \((V^{\otimes > k})[n]\) as the part of \(V^{\otimes k}\) of total degree \(n\) and \((V^{\otimes > k})_{fs} := \bigoplus_{n \in \mathbb{Z}} (V^{\otimes > k})[n]\) the ‘finite support’ part of \(V^{\otimes k}\). We define \((V^{\otimes > k})_{fs}\) similarly. Then if \(V\) is a \(\mathbb{Z}\)-graded algebra, we have algebra inclusions \(V^{\otimes k} \subset (V^{\otimes > k})_{fs} \subset V^{\otimes < k}\) and \(V^{\otimes k} \subset (V^{\otimes > k})_{fs} \subset V^{\otimes > k}\).

We will make the following assumptions.

\begin{enumerate}[(H1)]
  \item \(D\) is a \(\mathbb{Z}\)-graded topological bialgebra. Here topological means that the coproduct is an algebra morphism \(\Delta_D : D \to (D^{\otimes > 2})_{fs}\) of degree 0 (then the coassociativity is an equality of maps \(D \to (D^{\otimes < 3})_{fs}\)).
  \item \(D \supset A, B \supset \mathbb{C}[\mathbb{Z}^*]\), where \(A, B\) are \(\mathbb{Z}\)-graded topological subbialgebras of \(D\) and \(\mathbb{C}[\mathbb{Z}^*] \subset D_0\) is equipped with its standard bialgebra structure. We assume that the product map yields an isomorphism of vector spaces \(A \otimes \mathbb{C}[\mathbb{Z}^*] \to B \to D\), and that we have a nondegenerate bialgebra pairing \(\langle -,-\rangle : A \otimes B^{opp} \to \mathbb{C}\) of degree 0; this means that
    \begin{align*}
      \langle a_1 a_2, b \rangle = \langle a_1 \otimes a_2, b^{(2)} \otimes b^{(1)} \rangle, \quad \langle a, b_1 b_2 \rangle = \langle a^{(1)} \otimes a^{(2)}, b_1 \otimes b_2 \rangle, \quad (2.3)
    \end{align*}
    \begin{enumerate}
      \item \(a_i \in A_i, \ b_i \in B_i\), and \(\langle a, 1_D \rangle = \varepsilon_D(a), \ \langle 1_D, b \rangle = \varepsilon_D(b)\); we further require that the identity (2.1)
        \begin{align*}
          \langle a^{(1)}, b^{(1)} \rangle b^{(2)} a^{(2)} = a^{(1)} b^{(1)} (a^{(2)}, b^{(2)})
        \end{align*}
    \end{enumerate}
        holds. (One checks that all the sums involved in these identities are finite.)
  \item \(A \supset A_1, A_2\), where \(A_i\) are \(\mathbb{Z}\)-graded subalgebras of \(A\), such that the nontrivial components of \(A_1\) (resp., \(A_2\)) are in degrees \(\geq 0\) (resp., \(\leq 0\)), and the product map \(A_1 \otimes A_2 \to A\) is a linear isomorphism.
\end{enumerate}
Moreover, let us denote by case of $R$ holds in Lemma 2.1.

If we define $\langle -,-\rangle_1$ as the restriction of $\langle -,-\rangle$ to $A_i \otimes B_{3-i} \to \mathbb{C}$ ($i = 1, 2$), then this assumption is equivalent to the identity

$$\langle a_1a_2, b_2b_1 \rangle = \langle a_1, b_2 \rangle_1 \langle a_2, b_1 \rangle_2,$$

where $a_i \in A_i$, $b_i \in B_i$ ($i = 1, 2$).

The degree zero components are $A_2[0] = B_1[0] = \mathbb{C}[Z^r]$, and $A_1[0] = B_2[0] = \mathbb{C}$. We also assume that $A_2, B_1$ contain graded subalgebras $A_2', B_1'$, such that the product induces linear isomorphisms $A_2' \otimes \mathbb{C}[Z^r] \cong A_2, B_1' \otimes \mathbb{C}[Z^r] \cong B_1$ (so $A_2'[0] = B_1'[0] = \mathbb{C}$). We assume that the homogeneous components of $A_1, A_2', B_1', B_2$ are finite dimensional.

Let us denote by $\langle -,-\rangle_0$ the restriction of $\langle -,-\rangle$ to $A_2[0] \otimes B_1[0] \cong \mathbb{C}[Z^r] \otimes ^2 \to \mathbb{C}$; we assume that it has the form $\langle \delta_i, \delta_j \rangle_0 = q^{\alpha_{ij}}, q \in \mathbb{C}^\times$ and the matrix $(\alpha_{ij})$ is nondegenerate (here $\delta_i$ is the $i$th basis vector of $Z^r$). Let us denote by $\langle -,-\rangle_2'$ the restriction of $\langle -,-\rangle_2$ to $A_2' \otimes B_1' \to \mathbb{C}$. We assume that

$$\langle a_2' a_0, b'_1 b_0 \rangle_2 = \langle a_2', b'_1 \rangle_2 \langle a_0, b_0 \rangle_0$$

where $a_2' \in A_2', b_1' \in B_1'$, $a_0 \in A_2[0]$ and $b_0 \in B_1[0]$. We assume that the pairings $\langle -,-\rangle_1'$ and $\langle -,-\rangle_2'$ are non-degenerate, in the sense that each pairing between each pair of finite-dimensional homogeneous components of opposite degrees is non-degenerate.

Define $R_1 \in A_1 \otimes B_2$ as the tensor of the pairing $\langle -,-\rangle_1$, $R_2' \in A_2' \otimes B_1'$ as the tensor of $\langle -,-\rangle_2'$ and $R_0$ as the tensor of $\langle -,-\rangle_0$. Then $R_0 = q^{r_0}$, where $r_0$ is inverse to the matrix $(\alpha_{ij})$, and the tensor of the $\langle -,-\rangle_2$ is $R_2 := R_2' R_0 \in A_2 \otimes B_1$.

Actually, $R_1, R_2$ have degree 0, so we have $R_1 \in (A_1 \otimes B_2)_{f_s}, R_2 \in (A_2 \otimes B_1)_{f_s}$. Since $R_1$ has the form $1 + \sum_{i>0} a_i \otimes b_i$, where $\deg(a_i) = - \deg(b_i) = i$, $R_1$ is invertible in $A_1 \otimes B_2$. In the same way, $R_2$ is invertible.

**Lemma 2.1.** $R_2$ is a cocycle for $\Delta_D$, i.e., the identity

$$R_2^{1,2}(\Delta_D \otimes \text{id})(R_2) = R_2^{2,3}(\text{id} \otimes \Delta_D)(R_2)$$

holds in $D^{\otimes< 3}$. In the same way, $R_1^{-1}$ is a cocycle for $\Delta_D^{2,1}$.

**Proof.** The proof for $R_2$ is the same as the first proof of 3) in Theorem In the case of $R_1^{-1}$, we similarly prove that $(\Delta_D^{2,1} \otimes \text{id})(R_1) R_1^{-1} = (\text{id} \otimes \Delta_D^{2,1})(R_1) R_1^{2,3}$ and take inverses. □
We therefore obtain topological bialgebra structures \( \Delta : D \to D^{\otimes 2} \) and \( \hat{\Delta} : D \to D^{\otimes 2} \), defined by

\[
\Delta(x) := R_2 \Delta_D(x) R_2^{-1}, \quad \hat{\Delta}(x) = R_1^{-1} \Delta_D(x)^{21} R_1.
\]

We now prove:

**Theorem 2.** \( \Delta \) and \( \hat{\Delta} \) actually take their values in \( D^{\otimes 2} \), and are equal as maps \( D \to D^{\otimes 2} \).

Let us first briefly summarize the proof. For \( x \in A \), \( \Delta(x) \in A \otimes_\mathbb{Z} D \) while \( \hat{\Delta}(x) \in A \otimes_\mathbb{R} D \). We pair both elements with \( b \otimes \text{id} \), where \( b \in B \). Then \( \Delta(x) \), \( \hat{\Delta}(x) \) define elements of two completions \( \text{Hom}_+(B, \hat{D}) \) of the same convolution algebra \( \bigoplus_{(i,j) \in \mathbb{Z}^2} \text{Hom}(B_i, D_j) \). Using identities in these convolution algebras, and computing degrees carefully to prove that certain maps, a priori valued in \( D^{\otimes 3} \) or \( D^{\otimes 3} \), take in fact their values in \( D^{\otimes 2} \), we prove identity \( \hat{\Delta}(x) = \Delta(x) \), which implies that the pairings of \( \Delta(x) \) and \( \hat{\Delta}(x) \) with \( b \otimes \text{id} \) are the same. This implies \( \Delta(x) = \hat{\Delta}(x) \in D^{\otimes 2} \); the proof with \( x \) replaced by \( y \in B \) is similar.

**Proof.** We will consider the convolution algebra \( \bigoplus_{(i,j) \in \mathbb{Z}^2} \text{Hom}(B_i, D_j) \), where the product is \((f_1 \ast f_2)(b) := f_1(b^{(2)}) f_2(b^{(1)})\). This is an associative algebra with identity element \( 1_\ast : b \mapsto \varepsilon(b) 1_D \). This algebra is bigraded by \( \mathbb{Z}^2 \). The convolution product can be extended as follows. Let \( \hat{D} := \prod_{i \in \mathbb{Z}} D_i \), then \( \text{Hom}(B, \hat{D}) = \prod_{(i,j) \in \mathbb{Z}^2} \text{Hom}(B_i, D_j) \).

If \( f = \prod_{i,j} f_{i,j} \in \text{Hom}(B, \hat{D}) \), we define the support of \( f \) as \( \text{supp}(f) := \{(i,j) | f_{i,j} \neq 0\} \).

Then if \( f_1, f_2 \in \text{Hom}(B, \hat{D}) \) are such that the sum map \( \text{supp}(f_1) \times \text{supp}(f_2) \to \text{supp}(f_1) + \text{supp}(f_2) \) has finite fibers, then the convolution \( f_1 \ast f_2 \in \text{Hom}(B, \hat{D}) \) is defined, and has support contained in \( \text{supp}(f_1) + \text{supp}(f_2) \). One checks that the convolution in \( \text{Hom}(B, \hat{D}) \) is associative in the restricted sense that if \( S_1 \times S_2 \times S_3 \to S_1 + S_2 + S_3 \) has finite fibers, where \( S_i := \text{supp}(f_i) \), then \((f_1 \ast f_2) \ast f_3 = f_1 \ast (f_2 \ast f_3)\).

In particular, we define \( \text{Hom}_+(B, \hat{D}) \) (respectively, \( \text{Hom}_-(B, \hat{D}) \)) as the subset of \( \text{Hom}(B, \hat{D}) \) of all the elements \( f \) such that \( \text{supp}(f) \) is contained is some part of \( \mathbb{Z}^2 \) of the form \( S + N(1,1) \) (resp., \( S + N(-1,-1) \)),

\[
\forall n \in \mathbb{N}, \forall a \in B, \quad f(a \otimes d) = f(a) \otimes d.
\]

The intersection of \( \text{Hom}_+(B, \hat{D}) \) and \( \text{Hom}_-(B, \hat{D}) \) is \( \text{Hom}(B, D) \).

One checks that one has algebra injections \( (A \otimes_{\mathbb{Z}} D)_{fs} \to \text{Hom}_+(B, \hat{D}) \) and \( (A \otimes_{\mathbb{R}} D)_{fs} \to \text{Hom}_-(B, \hat{D}) \) (denoted \( x \mapsto [x] \)), extending the map \( A \otimes D \to \text{Hom}(B, \hat{D}) \), \( a \otimes d \mapsto (b \mapsto (a,b)d) \). The intersection of \( (A \otimes_{\mathbb{Z}} D)_{fs} \) and \( (A \otimes_{\mathbb{R}} D)_{fs} \) in \( \text{Hom}(B, \hat{D}) \) respectively coincide with \( P_2 \) and \( P_1 \), where \( P_2(b_2 b_1) = b_2 \varepsilon(b_1) \) and \( P_1(b_2 b_1) = \varepsilon(b_2) b_1 \).

Let \( f := [R_1^{-1}] \in \text{Hom}_-(B, \hat{D}) \) and \( g := [R_2^{-1}] \in \text{Hom}_+(B, \hat{D}) \). Then \( f \ast P_2 \) and \( P_2 \ast f \) are defined, and \( f \ast P_2 = P_2 \ast f = 1_\ast \). This means that for any \( b \in B \), \( f(b^{(2)}) P_2(b^{(1)}) = P_2(b^{(2)}) f(b^{(1)}) = \varepsilon(b) \). For \( b = b_2 b_1 \), this means that \( f(b^{(2)})(b^{(1)}) P_2(b^{(1)}) = \varepsilon(b) \) and \( P_2(b^{(2)})(b^{(1)}) P_2(b^{(1)}) = \varepsilon(b) \). The first equality says in particular (setting \( b_1 = 1 \))

\[\mathbb{N}(a,b) = \{(0,0), (a,b), (2a, 2b), \ldots\}\]
that $f(b_2^{(2)}) P_2(b_2^{(2)}) = \varepsilon(b_2)$; plugging this in the second equality, we get $f(b_2) \varepsilon(b_1) = f(b_2^{(2)}) \varepsilon(b_2^{(1)} b_1) = f(b_2^{(2)}) P_2(b_2^{(2)}) f(b_2^{(1)}) b_1 = \varepsilon(b_2^{(2)}) f(b_2^{(1)}) b_1 = f(b_2) b_1$, so
\[
f(b_2 b_1) = f(b_2) \varepsilon(b_1).
\]
In addition, we have
\[
f(b_2^{(2)}) b_1^{(1)} = P_2(b_2^{(2)}) f(b_2^{(1)}) = \varepsilon(b_2).
\]
These identities imply
\[
f(b_2^{(2)}) b_1^{(1)} = P_1(b).
\] (2.6)
In the same way, one shows that
\[
g(b_2 b_1) = \varepsilon(b_2) g(b_1), \quad g(b_2^{(2)}) P_1(b_1^{(1)}) = b_1^{(2)} g(b_1^{(1)}) = \varepsilon(b_1),
\]
where all the a priori infinite sums reduce to finite sums.

Then we compute
\[
\langle \Delta(x), b \otimes \text{id} \rangle = (P_1 \ast [\Delta(x)] \ast g)(b) = P_1(b^{(3)})[\Delta_D(x)](b^{(2)}) g(b^{(1)})
\]
\[
= P_1(b^{(2)} b_1^{(3)}) \langle x^{(1)}, b^{(2)} b_1^{(2)} \rangle x^{(2)} g(b_1^{(1)}) = P_1(b^{(2)}) b_1^{(3)} \langle x^{(1)}, b_1^{(2)} b_1^{(2)} \rangle x^{(2)} g(b_1^{(1)})
\]
\[
= P_1(b^{(2)}) b_1^{(3)} \langle x^{(1)}, b_1^{(2)} \rangle \langle x^{(2)}, b_1^{(2)} \rangle x^{(3)} g(b_1^{(1)})
\]
where the fourth equality uses $b_1^{(3)} \in B_1$, $b_1^{(1)} \in B_2$, and the last equality uses the bialgebra pairing rules (2.3), which do not introduce infinite sums.

We finally get
\[
\langle \Delta(x), b_2 b_1 \otimes \text{id} \rangle = P_1(b^{(2)}) \langle x^{(1)}, b_1^{(1)} \rangle b_1^{(3)} \langle x^{(2)}, b_1^{(2)} \rangle x^{(3)} g(b_1^{(1)}),
\]
and similarly
\[
\langle \tilde{\Delta}(x), b_2 b_1 \otimes \text{id} \rangle = f(b^{(3)}) \langle x^{(2)}, b_2^{(2)} \rangle x^{(1)} b_1^{(1)} \cdot \langle x^{(3)}, b_1^{(2)} \rangle P_2(b_1^{(1)}).
\]
We now prove that
\[
P_1(b^{(2)}) \langle x^{(1)}, b_1^{(1)} \rangle x^{(2)} = f(b^{(3)}) \langle x^{(2)}, b_2^{(2)} \rangle x^{(1)} b_1^{(1)}.
\] (2.7)
We first show that both sides are finite sums. Let $x$ and $b_2$ be of fixed degree (we denote by $|x|$ the degree of $x \in D$). Then for some constant $c$, we have $|b_2^{(1)}|, |x^{(1)}| \leq c$, $|b_2^{(2)}| = |b_2| - |b_2^{(1)}|$, $|x^{(2)}| = |x| - |x^{(1)}|$, so the l.h.s. reduces to the sum of contributions with $|b_2^{(1)}|, |x^{(1)}| \in \{-c, \ldots, c-1, c\}$ and is a finite sum. Let us show that the r.h.s. is a finite sum. For some constant $c'$ and function $c''(n)$, we have $|b_2^{(1)}|, |x^{(1)}| \leq c'$, $|b_2^{(2)}| \leq c''(|b_2^{(1)}|), |x^{(2)}| = |x| - |x^{(1)}|$ and $|b_2^{(3)}| = |b_2| - |b_2^{(1)}| - |b_2^{(2)}|$. The nontrivial contributions are for $|x^{(2)}| + |b_2^{(2)}| = 0$ and $|b_2^{(3)}| \leq 0$ (as supp($f$) $\subset \mathbb{N}(-1, -1)$). These conditions impose $|b_2^{(1)}| + |b_2^{(2)}| \geq 0$, so $|b_2^{(2)}| \geq -c'$, which leaves only finitely many possibilities for $|b_2^{(2)}|$, then $|b_2^{(1)}| \geq -|b_2^{(2)}|$, which leaves only finitely many possibilities for $(|b_2^{(1)}|, |b_2^{(2)}|, |b_2^{(3)}|)$. This also implies that only finitely many $(|x^{(1)}|, |x^{(2)}|)$ contribute.

We have therefore proven that there are linear maps $F, G : B_2 \otimes A \to D$, $F(x \otimes b_2) := P_1(b_2^{(2)}) \langle x^{(1)}, b_2^{(1)} \rangle x^{(2)}$ and $G(x \otimes b_2) := f(b_2^{(3)}) \langle x^{(2)}, b_2^{(2)} \rangle x^{(1)} b_2^{(1)}$. 

\
Similarly, one proves that there exists a unique linear map \( f : B_2 \otimes A \rightarrow D \), such that 
\[
\begin{align*}
   f(b_2 \otimes x) &= P_1(b_2^{(2)}) \langle x, b_2^{(1)} \rangle.
\end{align*}
\]
Then the image of the composed map \( B_2 \otimes A \xrightarrow{id \otimes \Delta_D} B_2 \otimes A^{\otimes \leq 2} \xrightarrow{f \circ \text{inc}} D^{\otimes \leq 2} \) (inc is the canonical inclusion) is contained in \( D^{\otimes 2} \), so its composition with \( m_D : D^{\otimes 2} \rightarrow D \) is well-defined. Then \( F = m_D \circ (f \otimes \text{inc}) \circ (\text{id} \otimes \Delta_D) \), and (2.7) expresses as \( F = G \).

One checks that there are maps \( u, v : A \otimes B \rightarrow D \), such that \( u(a \otimes b) := \langle a^{(1)}, b^{(1)} \rangle b^{(2)} a^{(2)} \) and \( v(a \otimes b) := \langle a^{(2)}, b^{(2)} \rangle a^{(1)} b^{(1)} \); (2.1) can then be expressed by the equality \( u = v \).

As above, one checks that the composed map \( A \otimes B \xrightarrow{id \otimes \Delta_D} A \otimes B^{\otimes \leq 2} \xrightarrow{\text{inc} \otimes f} D^{\otimes \leq 2} \) actually takes its values in \( D^{\otimes 2} \) for \( w = u \) or \( v \). This means that the equality
\[
\langle x^{(1)}, b^{(1)}_1 \rangle b^{(2)}_2 x^{(2)} \otimes f(b^{(3)}_2) = \langle x^{(2)}, b^{(2)}_1 \rangle x^{(1)} b^{(1)}_2 \otimes f(b^{(3)}_2)
\]
takes place in \( D^{\otimes 2} \). Applying \( m_D \) after transposing the factors, we get the identity in \( D \)
\[
f(b^{(3)}_2) \langle x^{(1)}, b^{(1)}_1 \rangle b^{(2)}_2 x^{(2)} = f(b^{(3)}_2) \langle x^{(2)}, b^{(2)}_1 \rangle x^{(1)} b^{(1)}_2,
\]
which according to (2.7) yields (2.7).

One proves similarly that the following is an equality between finite sums
\[
b_1^{(3)} \langle x^{(1)}, b^{(1)}_1 \rangle x^{(2)} g(b^{(1)}_1) = x^{(1)} \langle x^{(2)}, b^{(2)}_1 \rangle P_2(b^{(1)}_1),
\]
which one expresses as \( F' = G' \), where \( F', G' : A \otimes B \rightarrow D \) are given by
\[
F'(x \otimes b_2) := b_1^{(3)} \langle x^{(1)}, b^{(1)}_1 \rangle x^{(2)} g(b^{(1)}_1) \quad \text{and} \quad G'(x \otimes b_2) := x^{(1)} \langle x^{(2)}, b^{(2)}_1 \rangle P_2(b^{(1)}_1).
\]

As before, there exists a unique linear map \( g : A \otimes B_1 \rightarrow D \), such that \( g(x \otimes b_1) = \langle x, b^{(2)}_1 \rangle P_2(b^{(1)}_1) \). Then the image of the composed map \( A \otimes B_1 \xrightarrow{\Delta_D \otimes \text{id}} A^{\otimes \leq 2} \otimes B_1 \xrightarrow{\text{inc} \otimes g} D^{\otimes \leq 2} \) is contained in \( D^{\otimes 2} \), so its composition with \( m_D : D^{\otimes 2} \rightarrow D \) is well-defined, and \( G' = m_D \circ (\text{inc} \otimes g) \circ (\Delta_D \otimes \text{id}) \).

We then consider the composed map \( B \otimes A \otimes B \xrightarrow{id \otimes \Delta_D} B \otimes A^{\otimes \leq 3} \otimes B \xrightarrow{\text{inc} \otimes g} D^{\otimes \leq 3} \) (where \( \Delta_D^{(2)} = (\Delta_D \otimes \text{id}) \circ \Delta_D \)); this map is
\[
b_2 \otimes x \otimes b_1 \mapsto P_1(b^{(2)}_2) \langle x^{(1)}, b^{(1)}_2 \rangle x^{(2)} \otimes \langle x^{(3)}, b^{(2)}_1 \rangle P_2(b^{(1)}_1) \tag{2.8}
\]
and actually takes its values in \( D^{\otimes 3} \), since for \( |x| \) and \( |b_2| \) fixed, we have \( |b^{(1)}_1|, |b^{(1)}_2|, |x^{(1)}| \leq c \) for some \( c \), and the nontrivial contributions are for \( |b^{(1)}_2| = -|x^{(1)}| \geq -c \), which leaves only finitely many possibilities for \( (|b^{(1)}_2|, |b^{(2)}_1|) \), and therefore also for \( |x^{(1)}| \). Now for each such \( |x^{(1)}| \), we have \( |x^{(2)}| \leq c'(|x^{(1)}|) \) for some function \( c'(n) \), and \( |x^{(3)}| = |x| - |x^{(1)}| - |x^{(2)}| \geq |x| - c - c'(|x^{(1)}|) \). On the other hand, \( |b^{(2)}_1| = |b_1| - |b^{(1)}_1| \geq |b_1| - c \), since we must have \( |x^{(3)}| + |b^{(2)}_1| = 0 \), this leaves only finitely many possibilities for \( (|b^{(2)}_1|, |x^{(3)}|) \). Finally, we have finitely many possibilities for \( (|b^{(1)}_1|, |b^{(2)}_1|) \) and for \( (|x^{(1)}|, |x^{(2)}|) \), hence for \( (|x^{(1)}|, |x^{(2)}|, |x^{(3)}|) \). So the r.h.s. of (2.8) is a finite sum and belongs to \( D^{\otimes 3} \).

We then consider the map
\[
m_D^{(2)} \circ (f \otimes \text{id} \otimes g) \circ (\text{id} \otimes \Delta_D^{(2)} \otimes \text{id}) : B \otimes A \otimes B \rightarrow D
\]
(where \(m_D^{(2)} = (m_D \otimes \text{id}) \circ m_D\)). On one hand, we have

\[
m_D^{(2)} \circ (f \otimes \text{id} \otimes g) \circ (\text{id} \otimes \Delta_D \otimes \text{id}) = m_D \circ (F \otimes g) \circ (\text{id} \otimes \Delta_D \otimes \text{id}) = m_D \circ (G \otimes g) \circ (\text{id} \otimes \Delta_D \otimes \text{id});
\]
on the other hand, we have

\[
m_D^{(2)} \circ (f \otimes \text{id} \otimes g) \circ (\text{id} \otimes \Delta_D \otimes \text{id}) = m_D \circ (f \otimes G') \circ (\text{id} \otimes \Delta_D \otimes \text{id}) = m_D \circ (f \otimes F') \circ (\text{id} \otimes \Delta_D \otimes \text{id});
\]

so

\[
m_D \circ (G \otimes g) \circ (\text{id} \otimes \Delta_D \otimes \text{id}) = m_D \circ (f \otimes F') \circ (\text{id} \otimes \Delta_D \otimes \text{id}). \tag{2.9}
\]

Explicitly,

\[
P_1(b_2^{(2)}) \langle x^{(1)}, b_2^{(1)} \rangle \cdot (x^{(2)} \cdot x^{(3)} \cdot b_1^{(2)}) P_2(b_1^{(1)}) = P_1(b_2^{(2)}) \langle x^{(1)}, b_2^{(1)} \rangle \cdot b_1^{(3)} \langle x^{(2)}, b_2^{(2)} \rangle x^{(3)} g(b_1^{(1)})
\]
and

\[
(P_1(b_2^{(2)}) \langle x^{(1)}, b_2^{(1)} \rangle \cdot x^{(2)} \cdot x^{(3)} \cdot b_1^{(2)}) P_2(b_1^{(1)}) = f(b_2^{(3)}) \langle x^{(2)}, b_2^{(2)} \rangle x^{(1)} b_1^{(1)} \cdot x^{(3)} \cdot b_1^{(2)}) P_2(b_1^{(1)}),
\]
so \(2.9\) is rewritten as

\[
P_1(b_2^{(2)}) \langle x^{(1)}, b_2^{(1)} \rangle \cdot b_1^{(3)} \langle x^{(2)}, b_2^{(2)} \rangle x^{(3)} g(b_1^{(1)}) = f(b_2^{(3)}) \langle x^{(2)}, b_2^{(2)} \rangle x^{(1)} b_1^{(1)} \cdot x^{(3)} \cdot b_1^{(2)}) P_2(b_1^{(1)}),
\]
i.e., \(\langle \Delta(x), b_2 b_1 \otimes \text{id} \rangle = \langle \tilde{\Delta}(x), b_2 b_1 \otimes \text{id} \rangle\). This means that \([\Delta(x)] = [\tilde{\Delta}(x)]\). Since the intersection of \((A \otimes < D)_f\) and \((A \otimes > D)_f\) in \(\text{Hom}(B, \hat{D})\) is \(A \otimes D\), we get \(\Delta(x) = \tilde{\Delta}(x) \in A \otimes D\).

One proves similarly that for \(y \in B, \Delta(y) = \tilde{\Delta}(y) \in D \otimes B\) by using the convolution algebra \(\prod_{(i,j) \in \mathbb{Z}^2} \text{Hom}(A_i, D_j)\), where the product is given by \((f_1 * f_2)(a) = f_1(a^{(1)}) f_2(a^{(2)})\).

\[\square\]

**Proposition 2.2.** \(\Delta\) defines a (nontopological) bialgebra structure on \(D\), quasitriangular with \(R\)-matrix \(R_2^{2,1} R_1 \in D^{\otimes>2}\) (the quasitriangular identities are satisfied in \(D^{\otimes>3}\)).

**Proof.** Let us prove that for \(x \in D\),

\[
R_2^{2,1} R_1 \Delta(x) = \Delta^{2,1}(x) R_2^{2,1} R_1 \tag{2.10}
\]

(equality in \(D^{\otimes>2}\)). We have \(\Delta(x) = \tilde{\Delta}(x) = R_1^{-1} \Delta_D^{2,1}(x) R_1 \in D^{\otimes>2}\) so the l.h.s. is equal to \(R_2^{2,1} \Delta_D^{2,1}(x) R_1 \in D^{\otimes>2}\).

On the other hand, \(\Delta^{2,1}(x) = R_2^{2,1} \Delta_D^{2,1}(x) R_2^{2,1} \in D^{\otimes>2}\) so the r.h.s. is equal to \(R_2^{2,1} \Delta_D^{2,1}(x) R_1 \in D^{\otimes>2}\). This proves \(2.10\).

We now prove the quasitriangular identity

\[
(\Delta \otimes \text{id})(R) = R^{1,3} R^{2,3} \tag{2.11}
\]
in \(D^{\otimes>3}\). Recall that \(R_2^{1,2}(\Delta_D \otimes \text{id})(R_2) = R_2^{2,3}(\text{id} \otimes \Delta_D)(R_2)\) (equality in \(D^{\otimes<3}\)), which gives by applying the transposition \(x \mapsto x^{3,2,1}\)

\[
R_2^{3,2}(\text{id} \otimes \Delta_D^{2,1})(R_2^{2,1}) = R_2^{2,1}(\Delta_D^{2,1} \otimes \text{id})(R_2^{2,1}) \tag{2.12}
\]
(equality in \( D^{\otimes 3} \)). On the other hand, recall that
\[
(\Delta_{D}^{2,1} \otimes \text{id})(R_{1})R_{1}^{1,2} = (\text{id} \otimes \Delta_{D}^{2,1})(R_{1})R_{1}^{2,3}
\]
(equality in \( D^{\otimes 3} \)). Taking the product of (2.12) written in opposite order with (2.13), we get
\[
R_{2}^{2,1}(\Delta_{D}^{2,1} \otimes \text{id})(R)R_{1}^{1,2} = R_{2}^{3,2}(\text{id} \otimes \Delta_{D}^{2,1})(R)R_{1}^{2,3}.
\]
Equality (2.11) then follows from \( R_{1}^{1,2}\Delta(x) = \Delta_{D}^{2,1}(x)R_{1}^{1,2} \) (equality in \( D^{\otimes 2} \)). The proof of the identity \( (\text{id} \otimes \Delta)(R) = R_{1}^{1,3}R_{1}^{1,2} \) is similar. \( \square \)

**Proposition 2.3.** \( D_{1} := A_{1}B_{1}, \; D_{2} := A_{2}B_{2} \) are subbialgebras of \( D \). We also have
\[
\Delta(A_{1}) \subset A_{1} \otimes D_{1}, \quad \Delta(B_{1}) \subset D_{1} \otimes B_{1}, \quad \Delta(A_{2}) \subset A_{2} \otimes D_{2}, \quad \Delta(B_{2}) \subset D_{2} \otimes B_{2}.
\]

**Proof.** We first prove that \( B_{1}A_{1} \subset A_{1}B_{1} \). Set \( R(a, b) := \langle a^{(1)}, b^{(1)} \rangle b^{(2)}a^{(2)} - \langle a^{(2)}, b^{(2)} \rangle a^{(1)}b^{(1)} \) and \( S(a, b) := ba - \langle S_{D}(a^{(1)}), b^{(2)} \rangle \langle a^{(3)}, b^{(3)} \rangle a^{(2)}b^{(2)} \). We have \( R(a, b) = \langle a^{(1)}, b^{(1)} \rangle S(a^{(2)}, b^{(2)}) \) and \( S(a, b) = \langle a^{(1)}, S_{D}(b^{(1)}) \rangle R(a^{(2)}, b^{(2)}) \) so since \( R(a, b) = 0 \), we get \( S(a, b) = 0 \). Therefore
\[
ba = \langle S_{D}(a^{(1)}), b^{(2)} \rangle \langle a^{(3)}, b^{(3)} \rangle a^{(2)}b^{(2)}.
\]
If now \( a \in A_{1}, \; b \in B_{1} \), we get \( a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \in A^{\otimes 2} \otimes \ll A_{1} \) and \( b^{(1)} \otimes b^{(2)} \otimes b^{(3)} \in B^{\otimes 2} \otimes \ll B_{1} \). For \( x \in A_{1}, \; y \in B_{1} \), we have \( \langle x, y \rangle = \varepsilon(x)\varepsilon(y) \), so \( ba = \langle S_{D}(a^{(1)}), b^{(1)} \rangle \varepsilon(a^{(3)})\varepsilon(b^{(3)})a^{(2)}b^{(2)} = \langle S_{D}(a^{(1)}), b^{(1)} \rangle a^{(2)}b^{(2)} \). Now since \( \Delta_{D}(a_{1}) \in A \otimes \ll A_{1} \) and \( \Delta_{D}(b_{1}) \in B \otimes \ll B_{1} \), we get \( ba \) \( \subset A_{1}B_{1} \), as wanted. This implies that \( D_{1} := A_{1}B_{1} \) is a subalgebra of \( D \).

In the same way, we prove that \( D_{2} \) is a subalgebra of \( D \).

Let us prove that \( \Delta(A_{1}) \subset A_{1} \otimes D_{1} \). For \( x \in A_{1}, \; \Delta(x) = R_{1}R_{1}^{-1}R_{2} \) and since \( D_{1} \) is an algebra, \( \Delta(x) \in A \otimes D_{1} \). On the other hand, \( \Delta(x) = R_{1}^{-1}R_{2}^{-1} \Delta_{D}(x)R_{1} \) \( \subset D_{1} \otimes D \). Since \( \Delta(x) = \Delta_{D}(x) \in A \otimes D \) and \( (A \otimes D) \cap (A \otimes D_{1}) = A \otimes D_{1} \), we get \( \Delta(x) \in A \otimes D_{1} \).

In the same way, \( \Delta(x) \in A_{1} \otimes D \). Then \( (A \otimes D_{1}) \cap (A_{1} \otimes D) = A_{1} \otimes D_{1} \), which implies \( \Delta(x) \in A_{1} \otimes D_{1} \), as wanted. The other inclusions (2.14) are proved in the same way.

Since \( D_{2} \) are generated by \( A_{i}, \; B_{i} \), these inclusions imply that \( D_{i} \) are subbialgebras of \( (D, \Delta) \).

We will show that the quantum affine algebras, equipped with their currents co-products, are examples of the situation of Subsection 2.3.

**3 Quantum affine algebra** \( U_{q}(\hat{\mathfrak{g}}) \)

In this paper, \( q \) is a complex number, which is neither 0 nor a root of unity.

**3.1 Chevalley-type presentation of** \( U_{q}(\hat{\mathfrak{g}}) \)

Let \( \mathfrak{g} \) be a simple Lie algebra; let \( r \) be its rank and let \( (h_{i,j})_{i,j=1,...,r} \) be its Cartan matrix. Let \( (a_{i,j})_{i,j=0,...,r} \) be the Cartan matrix of the affine Lie algebra \( \hat{\mathfrak{g}} \). We denote by
\[ \Pi = \{ \alpha_1, ..., \alpha_r \} \] is the set of positive simple roots of \( \mathfrak{g} \) and by \( \hat{\Pi} = \{ \alpha_0, \alpha_1, ..., \alpha_r \} \) the set of positive simple roots of \( \hat{\mathfrak{g}} \). The symmetrized Cartan matrix of \( \hat{\mathfrak{g}} \) is \( ((\alpha_i, \alpha_j))_{i,j=0,...,r} \); we have \((\alpha_i, \alpha_j) = d_i a_{i,j} = d_j a_{j,i} \) (where \( d_i = 1, 2 \) or 3 are coprime). Let \( \delta \) be the minimal positive imaginary root of \( \hat{\mathfrak{g}} \), so \( \delta = \sum_{i=0}^{r} n_i \alpha_i \), \( n_i \in \mathbb{Z}_{\geq 0} \), \( n_0 = 1 \). Let \[ \begin{bmatrix} n \end{bmatrix}_q = \frac{[n]!}{[q]^! [n-q]^!}, [n]_q! = [1][2]_q \cdots [n]_q, [n]_q = q^n - q^{-n}, q_\alpha = q^{\frac{\alpha(\alpha)}{2}}, q_i = q^\alpha_i = q^{d_i}. \]

The quantum (untwisted) affine Lie algebra \( U_q(\hat{\mathfrak{g}}) \) is generated by the Chevalley generators \( e_{\pm \alpha_i}, k_{\pm 1}^{\pm 1} \) \((i = 0, ..., r)\), the grading elements \( q^{\pm d} \), and the central elements \( k_{\delta}^{\pm 1/2} \), subject to the relations

\[ [q^d, k_{\alpha_i}] = [k_{\alpha_i}, k_{\alpha_j}] = 0, \quad q^d e_{\pm \alpha_i} q^{-d} = q^{\pm \delta_i,0} e_{\pm \alpha_i}, \quad k_{\alpha_i} e_{\pm \alpha_j} k_{\alpha_i}^{-1} = q_{\pm \alpha_j} e_{\pm \alpha_j}, \]

\[ (k_{\delta}^{\pm 1/2})^2 = \prod_{i=0}^{r} k_{\alpha_i}^{\pm n_i}, \quad q^d q^{-d} = k_{\alpha_i} k_{\alpha_i}^{-1} = k_{\alpha_i}^{-1} k_{\alpha_i} = k_{\delta}^{1/2} k_{\delta}^{-1/2} = k_{\delta}^{-1/2} k_{\delta}^{1/2} = 1, \]

\[ [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q_i - q_i^{-1}}, \quad (3.15) \]

\[ \sum_{r+s=-1, \alpha_i} (-1)^r e_{\pm \alpha_i} e_{\pm \alpha_i} e_{\pm \alpha_i} = 0, \quad i \neq j, \quad \text{where} \quad e_{(k)}^{(r)} = \frac{e_{\pm \alpha_i}}{[k]_q!}. \]

The \textit{standard Hopf structure} of \( U_q(\hat{\mathfrak{g}}) \) is given by the formulas:

\[ \Delta^{\text{std}}(q^{\pm d}) = q^{\pm d} \otimes q^{\pm d}, \quad \Delta^{\text{std}}(k_{\pm 1}^{\pm 1}) = k_{\pm 1}^{\pm 1} \otimes k_{\pm 1}^{\pm 1}, \quad \Delta^{\text{std}}(k_{\delta}^{\pm 1/2}) = k_{\delta}^{\pm 1/2} \otimes k_{\delta}^{\pm 1/2}, \]

\[ \Delta^{\text{std}}(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + k_{\alpha_i} \otimes e_{\alpha_i}, \quad \Delta^{\text{std}}(e_{-\alpha_i}) = 1 \otimes e_{-\alpha_i} + e_{-\alpha_i} \otimes k_{\alpha_i}^{-1}, \]

\[ \varepsilon(q^{\pm d}) = 1, \quad \varepsilon(e_{\pm \alpha_i}) = 0, \quad \varepsilon(k_{\pm 1}^{\pm 1}) = 1, \quad \varepsilon(k_{\delta}^{\pm 1/2}) = 1, \]

\[ S(e_{\alpha_i}) = -k_{\alpha_i}^{-1} e_{\alpha_i}, \quad S(e_{-\alpha_i}) = -e_{-\alpha_i} k_{\alpha_i}, \quad S(k_{\pm 1}^{\pm 1}) = k_{\mp 1}^{\mp 1}, \quad S(q^{\pm d}) = q^{\mp d}, \quad S(k_{\delta}^{\pm 1/2}) = k_{\delta}^{\mp 1/2}, \quad (3.16) \]

where \( \Delta^{\text{std}}, \varepsilon \) and \( S \) are the coproduct, counit and antipode maps respectively.

Let \( U_q(\mathfrak{h}) \) be the Cartan subalgebra of \( U_q(\hat{\mathfrak{g}}) \). It is generated by the elements \( k_{\alpha_i}^{\pm 1} \) \((i = 0, ..., r)\) and \( q^{\pm d} \). Denote by \( U_q(\mathfrak{b}_+ \mathfrak{b}_-) \) the subalgebra of \( U_q(\hat{\mathfrak{g}}) \) generated by the elements \( e_{\alpha_i}, k_{\alpha_i}^{\pm 1} \) \((i = 0, ..., r)\), \( k_{\delta}^{\pm 1/2} \) and \( q^{\pm d} \), and by \( U_q(\mathfrak{b}_+) \) the subalgebra of \( U_q(\hat{\mathfrak{g}}) \) generated by the elements \( e_{-\alpha_i}, k_{\alpha_i}^{\pm 1} \) \((i = 0, ..., r)\), \( k_{\delta}^{\pm 1/2} \) and \( q^{\pm d} \).

The algebras \( U_q(\mathfrak{b}_+) \) are Hopf subalgebras of \( U_q(\hat{\mathfrak{g}}) \) with respect to the standard coproduct \( \Delta^{\text{std}} \). They are \( q \)-deformations of the enveloping algebras of opposite Borel subalgebras of Lie algebra \( \hat{\mathfrak{g}} \). We call them the \textit{standard Borel subalgebras}. Moreover, \( U_q(\mathfrak{b}_-) \) is the dual, with opposite coproduct, of \( U_q(\mathfrak{b}_+) \), and \( U_q(\hat{\mathfrak{g}}) \otimes U_q(\mathfrak{h}) \) is the double of \( U_q(\mathfrak{b}_+) \) (where \( U_q(\mathfrak{h}) \) is equipped with the standard structure, for which \( k_{\alpha_i}^{\pm 1} \) is primitive).

The algebras \( U_q(\mathfrak{b}_+) \) contain subalgebras \( U_q(\mathfrak{n}_+) \), which are generated by the elements \( e_{\pm \alpha_i}, i = 0, ..., r \). The subalgebra \( U_q(\mathfrak{n}_+) \) is a left coideal of \( U_q(\mathfrak{b}_+) \) with respect to standard coproduct and the subalgebra \( U_q(\mathfrak{n}_+) \) is a right coideal of \( U_q(\mathfrak{b}_-) \) with respect to the same coproduct, that is

\[ \Delta^{\text{std}}(U_q(\mathfrak{n}_+)) \subset U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{n}_+), \quad \Delta^{\text{std}}(U_q(\mathfrak{n}_-)) \subset U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{b}_-). \]
The algebras $U_q(n)_{\pm}$ are $q$-deformations of the enveloping algebras of the standard pro-nilpotent subalgebras of the affine Lie algebra $\hat{g}$.

3.2 Cartan-Weyl basis of $U_q(\hat{g})$

We now construct a Cartan-Weyl (CW) basis of $U_q(\hat{g})$. Let us denote by $\Sigma_+ \subset \Sigma$ the set of positive (resp., all) roots of $g$ and by $\hat{\Sigma}_+ \subset \hat{\Sigma}$ the set of positive (resp., all) roots of $\hat{g}$, so $\Sigma_+ \subset \Sigma, \hat{\Sigma}_+ \subset \hat{\Sigma}$. Recall that $\alpha_0$ is the affine positive simple root and let $\delta$ be the minimal positive imaginary root, so $\delta = \alpha_0 + \theta$, where $\theta$ is the longest root of $\Sigma_+$.

We consider the following class of normal orderings on $\hat{\Sigma}_+$. Let $\hat{W}$ be the Weyl group of $\hat{g}$. It contains the Weyl group $W$ of $g$ and the normal subgroup $Q$, which is the set of all elements having only finitely many conjugates. There is a unique group morphism $Q \to \mathfrak{h}^*$, $p \mapsto \bar{p}$, such that the action of $p \in Q$ on $\mathfrak{h}^*$ is the translation by $\bar{p}$. This map is injective and identifies $Q$ with a subgroup $\hat{Q}$ of $\mathfrak{h}^*$.

Choose $p \in Q$ such that $(\bar{p}, \alpha_i) > 0$ for any $i = 1, ..., r$. Choose a reduced decomposition $p = s_{\alpha_0} s_{\alpha_1} \cdots s_{\alpha_{m-1}}$ such that $\alpha_{i_0} = \alpha_0$ is the affine positive root. Extend the sequence $i_0, i_1, ..., i_m$ to a periodic sequence

$$\ldots, i_{-1}, i_0, i_1, \ldots, i_n, \ldots$$

(3.17)

satisfying the conditions $i_n = i_{n+m}$ for any $n \in \mathbb{Z}$. We then set

$$\gamma_1 := \alpha_{i_1}, \gamma_2 := s_{\alpha_{i_1}}(\alpha_{i_2}), \ldots, \gamma_k := s_{\alpha_{i_1}} \cdots s_{\alpha_{i_{k-1}}}(\alpha_{i_k}) \text{ for } k \geq 1;$$

and

$$\gamma_0 := \alpha_{i_0}, \gamma_{-1} := s_{\alpha_{i_0}}(\alpha_{i_{-1}}), \ldots, \gamma_{-\ell} := s_{\alpha_{i_0}} \cdots s_{\alpha_{i_{-\ell}}}(\alpha_{i_{-\ell}}) \text{ for } \ell \geq 0. \quad (3.18)$$

Then $[\mathfrak{B}, \mathfrak{D}]$ the order $\gamma_1 \prec \gamma_2 \prec \ldots \prec \gamma_n \prec \ldots \prec \delta \prec 2\delta \prec \ldots \prec \gamma_{-n} \prec \ldots \prec \gamma_{-1} \prec \gamma_0$ is normal and satisfies the condition

$$l\delta + \alpha \prec (m+1)\delta \prec (n+1)\delta - \beta, \quad (3.19)$$

for any positive roots $\alpha, \beta \in \Sigma_+$, and any $l, m, n \geq 0$. From now on, we fix a normal ordering $\prec$ on $\hat{\Sigma}_+$, given by the procedure above.

Recall that the principal degree deg is the linear additive map $\mathbb{Z}[\Pi] \to \mathbb{Z}$, such that $\text{deg}(\alpha_i) = 1$ for $i = 0, \ldots, r$. We first construct the CW generators $e_\gamma$, for $\gamma \in \hat{\Sigma}_+$ and $\text{deg}(\gamma) \leq \text{deg}(\delta) - 1$. For $\gamma \in \hat{\Pi}$, $e_\gamma$ is equal to the corresponding Chevalley generator of $U_q(\hat{g})$. The CW generator $e_\gamma$ is then constructed by induction on the degree of $\gamma$ as follows: if $\gamma \in \hat{\Sigma}_+$ and $\text{deg}(\gamma) \leq \text{deg}(\delta) - 1$, we let $[\alpha, \beta]$ be minimal for the inclusion, among the set of all segments $[\alpha', \beta']$, where $\alpha', \beta' \in \hat{\Sigma}_+$ are such that $\gamma = \alpha' + \beta'$ (we define the segment $[\alpha', \beta']$). We then set

$$e_\gamma := [e_\alpha, e_\beta]_{q^{-1}}, \quad e_{-\gamma} := [e_{-\beta}, e_{-\alpha}]_q, \quad (3.20)$$

where$^5$ the $q$-commutator $[e_\alpha, e_\beta]_q$ means $[e_\alpha, e_\beta]_q = e_\alpha e_\beta - q^{\langle \alpha, \beta \rangle} e_\beta e_\alpha$.

---

$^5$One can introduce the algebra $\hat{U}_q(\hat{g})$ over the ring $\mathbb{C}[\hat{g}, \hat{g}^{-1}, 1/(\hat{g}^n - 1); n \geq 1]$ with the same generators and relations as $U_q(\hat{g})$; one can show using the CW basis that this is a free module over this ring, and $U_q(\hat{g})$ is its specialization. $\hat{U}_q(\hat{g})$ is equipped with the Cartan antiinvolution $x \mapsto x^*$, defined by $e_{\pm \alpha_i}^* = e_{\mp \alpha_i}, k_{\alpha_i}^* = k_{-\alpha_i}^{-1}, \hat{q}^* = \hat{q}^{-1}$. Then the analogues of $e_\pm \gamma$ satisfy $e_{-\gamma} = e_\gamma^*$. 

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The basis elements defined in this way coincide (up to normalization) with those defined by the braid group action, using the same ordering on roots. Since the latter basis is convex, \( e_\gamma \) defined above is independent (up to normalization) of the choice of the segment \([\alpha, \beta]\), and depends only on the ordering of roots. Moreover, it has been shown in [Da], Proposition 11 that \( e_\delta - e_\alpha \) is also independent on the choice of a normal ordering (up to normalization).

We then put

\[
e^{(i)}_\delta = e^{(i)}_\delta = [e_\alpha, e_{\delta - \alpha}]_{q^{-1}}, \quad e^{(i)}_{-\delta} = [e_{\alpha - \delta}, e_{-\alpha}]_q
\]

and by induction for all \( k > 0 \)

\[
e^{(i)}_\pm (\alpha \pm k\delta) = \pm \frac{1}{2} \left[ e^{(i)}_\pm (\alpha \pm (k-1)\delta), e^{(i)}_\pm \right], \quad e^{(i)}_{\pm (\delta - \alpha) \pm k\delta} = \pm \frac{1}{2} \left[ e^{(i)}_{\pm \delta}, e^{(i)}_{\pm (\delta - \alpha) \pm (k-1)\delta} \right],
\]

\[
e^{(i)}_{k\delta} = [e_{\alpha \pm (k-1)\delta}, e_{\delta - \alpha}]_{q^{-1}}, \quad e^{(i)}_{-k\delta} = [e_{-\delta + \alpha + k}, e_{-\delta - \alpha (k-1)\delta}]_q
\]

Then we apply procedure (3.20) again to construct the remaining (real root) CW generators. As before, if \( \gamma \) is real, then \( e_\gamma \) depends only on the choice of a normal ordering (up to normalization); the \( e_{n\delta \pm \alpha} \) and the \( e^{(i)}_{n\delta} \) are independent on the choice of this ordering (up to normalization).

The CW generators of the Borel subalgebras \( U_q(\mathfrak{b}_\pm) \) satisfy the following properties (see [KT2] Bel [Da])

\[
k_\alpha e_\beta k^{-1}_\alpha = q^{(\alpha, \beta)} e_\beta, \quad [e_\alpha, e_{-\alpha}] = a(\alpha) \frac{k_\alpha - k^{-1}_\alpha}{q - q^{-1}},
\]

\[
[e_{\pm \alpha}, e_{\pm \beta}]_q^{-1} = \sum_{\gamma_1, \ldots, \gamma_m \in \Sigma_+} C^{(\gamma)}_{\{n_j \}} (q) e^{n_{\gamma_1}}_\pm e^{n_{\gamma_2}}_\pm \cdots e^{n_{\gamma_m}}_\pm, \quad \alpha, \beta \in \Sigma_+ \quad (3.22)
\]

where in \((3.22)\) the sum is over all \( \gamma_1, \ldots, \gamma_m \in \Sigma_+ \), \( n_1, \ldots, n_m > 0 \) such that \( \alpha < \gamma_1 < \gamma_2 < \cdots < \gamma_m < \beta \) and \( \sum_j n_j \gamma_j = \alpha + \beta \) and the coefficients \( C^{(\gamma)}_{\{n_j \}} (q) \) and \( a(\alpha) \) are rational functions in \( q \) in \( \mathbb{Q}[q, q^{-1}, 1/(q^n - 1); n \geq 1] \). The elements \( k_\alpha, \alpha \in \widehat{\Sigma} \) are defined according to the prescriptions \( k_{\alpha + \beta} = k_\alpha k_\beta, \ k_\alpha = k_{\alpha}^{-1} \), if \( \alpha = \pm \alpha_i, i = 0, \ldots, r \). The analogues of \( e_{\pm \alpha} \) in \( \hat{U}_q(\mathfrak{g}) \) also satisfy \( e_{-\alpha} = e^*_\alpha \).

The commutators \( [e_\alpha, e_{-\beta}] \) and \( [e_{-\alpha}, e_\beta] \), where \( \alpha, \beta \in \Sigma_+ \), satisfy properties analogous to \((3.22)\), but the structure coefficients \( C^{(\gamma)}_{\{n_j \}} \) belong to \( U_q(\mathfrak{g}) \). One can construct slightly different generators in the CW basis such that property \((3.22)\) is still valid with scalar structure constants. For this, the normal ordering \( \alpha \) in the system \( \Sigma_+ \) of positive roots is extended to a ‘circular’ normal ordering (which is not a total order) of the system \( \Sigma \) of all roots of \( \mathfrak{g} \). This order \( \alpha \prec \beta \) is defined by: for \( \alpha, \beta \in \Sigma_+ \), \( (\alpha \prec_c \beta) \Leftrightarrow (\alpha \prec \beta) \Leftrightarrow (-\alpha \prec_c -\beta) \) and \( (\alpha \prec_c -\beta) \Leftrightarrow (\beta \prec \alpha) \Leftrightarrow (-\alpha \prec_c \beta) \). The

As before, the analogues of \( e^{(i)}_{\pm \delta} \) in \( \hat{U}_q(\mathfrak{g}) \) satisfy \( e^{(i)}_{-\delta} = (e^{(i)}_\delta)^* \).
Proposition 3.1. For any $\alpha, \beta \in \hat{\Sigma}$, such that $\alpha \prec_c \beta$,

\[
[\hat{e}_\alpha, \hat{e}_\beta]_{q^{-1}} = \sum_{\{\gamma\}, \{n_j\}} C^{\{\gamma\}}_{\{n_j\}}(q) \hat{e}_{n_1}^{\gamma_1} \hat{e}_{n_2}^{\gamma_2} \cdots \hat{e}_{n_m}^{\gamma_m}, \quad \text{if } \beta \in \hat{\Sigma}_+,
\]

\[
[\hat{e}_\alpha, \hat{e}_\beta]_{q^{-1}} = \sum_{\{\gamma\}, \{n_j\}} C^n_{\{\gamma\}}(q) \hat{e}_{n_1}^{\gamma_1} \hat{e}_{n_2}^{\gamma_2} \cdots \hat{e}_{n_m}^{\gamma_m}, \quad \text{if } \beta \in -\hat{\Sigma}_+,
\]

where the sums in (3.25) and (3.26) are over all $\gamma_1, \gamma_2, \ldots, \gamma_m$, $n_1, n_2, \ldots, n_m$ such that $\alpha \prec_c \gamma_1 \prec_c \gamma_2 \cdots \prec_c \gamma_m \prec_c \beta$ (meaning $\alpha \prec_c \gamma_1$, $\gamma_1 \prec_c \gamma_2$, etc.) and $\sum_j n_j \gamma_j = \alpha + \beta$; $C^{\{\gamma\}}_{\{n_j\}}(q)$ and $C^{\gamma}_{\{n_j\}}(q)$ are Laurent polynomials of $q$.

Proof. See the Appendix. \[\square\]

For $\alpha, \beta \in \hat{\Sigma}$ such that $\alpha \prec_c \beta$, define $[\alpha, \beta] := \{\gamma \in \hat{\Sigma} | \alpha \preceq_c \gamma \preceq_c \beta\}$. 

Figure 1: The left figure shows the circle ordering of the system $\hat{\Sigma}$ of $U_q(\hat{g})$. It is such that $\gamma_1 + m_1 \delta \prec_c m_2 \delta \prec_c (m_3 + 1) \delta - \gamma_2 \prec_c -\gamma_3 - m_4 \delta \prec_c -m_5 \delta \prec_c -(m_6 + 1) \delta + \gamma_4 \prec_c \gamma_5 + m_1 \delta$, where $\gamma_i \in \Sigma_+$ and $m_i > 0$. In the right figure, line 1 shows the decomposition $\hat{\Sigma} = \hat{\Sigma}_+ \cup (-\hat{\Sigma}_+)$, related to the subalgebras $U_q(n_\pm)$. Line 2 shows the decomposition $\Sigma = \Sigma_E \cup \Sigma_F$, related to the currents Borel subalgebras $U_E$ and $U_F$ discussed in Section 3.4.

order $\prec_c$ can be described as follows. Suppose that $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_N \prec \cdots$ is the normal ordering of $\Sigma_+$. Then we put all the roots of $\Sigma$ on the circle clockwise in the following order (see Fig. 1):

\[
\gamma_1, \gamma_2, \ldots, \gamma_N, \ldots, -\gamma_1, -\gamma_2, \ldots, -\gamma_N, \ldots,
\]

and say that the root $\gamma' \in \Sigma$ precedes the root $\gamma'' \in \Sigma$, $\gamma' \prec_c \gamma''$, if the segment $[\gamma', \gamma'']$ in the circle (3.23) does not contain any of the opposite roots $-\gamma'$ or $-\gamma''$.
We then define $U_q^{[\alpha,\beta]}(\mathfrak{g}) \subset U_q(\mathfrak{g})$ as the subalgebra generated by the $\hat{e}_\gamma, \gamma \in [\alpha, \beta]$, if $\beta \in \widehat{\Sigma}_+$, and as the subalgebra generated by the $\hat{e}_\gamma, \gamma \in [\alpha, \beta]$, if $\beta \in -\widehat{\Sigma}_+$. If $\alpha, \beta \in \widehat{\Sigma}$ are such that $\alpha \prec_c \beta$, we define the intervals $[\alpha, \beta] := \{\gamma \in \widehat{\Sigma}| \alpha \preceq_c \gamma \preceq_c \beta\}$ and $\langle \alpha, \beta \rangle := \{\gamma \in \widehat{\Sigma}| \alpha \prec_c \gamma \preceq_c \beta\}$. We then define the subalgebras $U_q^{[\alpha,\beta]}(\mathfrak{g})$, $U_q^{[\alpha,\beta]}(\mathfrak{g})$ and $U_q^{[\alpha,\beta]}(\mathfrak{g}) \subset U_q(\mathfrak{g})$ as above (using $\hat{e}_\gamma$ or $\hat{e}_\gamma$, depending on whether $\beta \in \widehat{\Sigma}_+$ or $\beta \in -\widehat{\Sigma}_+$). Relations (3.25), (3.26) imply that the algebra $U_q^{[\alpha,\beta]}(\mathfrak{g})$ admits a Poincaré-Birkhoff-Witt (PBW) basis, formed by the ordered monomials $\hat{e}_{\gamma_1} \hat{e}_{\gamma_2} \cdots \hat{e}_{\gamma_m}$, where $\alpha = \gamma_1 \prec_c \gamma_2 \prec_c \cdots \prec_c \gamma_m = \beta$, if $\beta \in \widehat{\Sigma}_+$, and by the monomials $\hat{e}_{\gamma_1} \hat{e}_{\gamma_2} \cdots \hat{e}_{\gamma_m}$, where $\alpha = \gamma_1 \prec_c \gamma_2 \prec_c \cdots \prec_c \gamma_m = \beta$, if $\beta \in -\widehat{\Sigma}_+$. Moreover, if $\alpha, \beta, \gamma \in \widehat{\Sigma}$ are such that $\alpha \prec_c \beta$, $\beta \prec_c \gamma$ and $\alpha \prec_c \gamma$, and either $\beta, \gamma \in \widehat{\Sigma}_+$ or $\beta, \gamma \in -\widehat{\Sigma}_+$, then the product of $U_q^{[\alpha,\beta]}(\mathfrak{g})$ induces vector space isomorphisms

$$U_q^{[\alpha,\beta]}(\mathfrak{g}) \otimes U_q^{[\alpha,\beta]}(\mathfrak{g}) \simeq U_q^{[\alpha,\beta]}(\mathfrak{g}), \quad U_q^{[\alpha,\beta]}(\mathfrak{g}) \otimes U_q^{[\alpha,\beta]}(\mathfrak{g}) \simeq U_q^{[\alpha,\beta]}(\mathfrak{g}),$$

$$U_q^{[\alpha,\beta]}(\mathfrak{g}) \otimes U_q^{[\alpha,\beta]}(\mathfrak{g}) \simeq U_q^{[\alpha,\beta]}(\mathfrak{g}), \quad U_q^{[\alpha,\beta]}(\mathfrak{g}) \otimes U_q^{[\alpha,\beta]}(\mathfrak{g}) \simeq U_q^{[\alpha,\beta]}(\mathfrak{g}).$$

(3.27)

Then $U_q(n_+)$ coincides with $U_q^{\widehat{\Sigma}_+}(\mathfrak{g}) = U_q^{[\alpha_1,\delta-\theta]}(\mathfrak{g})$, where $\alpha_1$ is the first root (necessarily simple) of the normal ordering $\prec$ of $\widehat{\Sigma}_+$. The algebra $U_q(n_-)$ coincides with $U_q^{-\widehat{\Sigma}_+}(\mathfrak{g}) = U_q^{[-\alpha_1,-\delta+\theta]}(\mathfrak{g})$.

The segment $\widehat{\Sigma}_+ = [\alpha_{i_1}, \delta - \theta]$ is a disjoint union

$$\widehat{\Sigma}_+ = \Sigma_{+,e} \sqcup \Sigma_{+,F}.$$

Here

$$\Sigma_{+,e} = \{\gamma + m\delta| \gamma \in \Sigma_+, m \geq 0\}, \quad \Sigma_{+,F} = \{m\delta|m > 0\} \sqcup \{-\gamma + m\delta| \gamma \in \Sigma_+, m > 0\}.$$  

(3.28)

Analogously, $-\widehat{\Sigma}_+ = [-\alpha_{i_1}, -\delta + \theta]$ is a disjoint union

$$-\widehat{\Sigma}_+ = \Sigma_{-,E} \sqcup \Sigma_{-,F},$$

where

$$\Sigma_{-,E} = -\Sigma_{+,F}, \quad \Sigma_{-,F} = -\Sigma_{+,e}.$$  

(3.29)

The segments (3.28) and (3.29) can be united in different ways, composing segments $\Sigma_E$ and $\Sigma_F$:

$$\Sigma_E = \Sigma_{+,e} \sqcup \Sigma_{-,E} = \{n\delta, \gamma + m\delta| \gamma \in \Sigma_+, n < 0, m \in \mathbb{Z}\},$$

$$\Sigma_F = \Sigma_{+,F} \sqcup \Sigma_{-,F} = \{n\delta, -\gamma + m\delta| \gamma \in \Sigma_+, n > 0, m \in \mathbb{Z}\}.$$  

(3.30)

The subalgebras related to the segments (3.28), (3.29) and (3.30) play a crucial role in our further constructions.
3.3 The ‘currents’ presentation of $U_q(\hat{g})$

In this presentation (D2), $U_q(\hat{g})$ is generated by the central elements $C^\pm$, the grading elements $q^\pm d$, and by the elements $e_\alpha[n], f_\alpha[n]$ (where $\alpha \in \Pi$, $n \in \mathbb{Z}$) and $k_\alpha^{\pm 1}, h_\alpha[n]$ (where $n \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \Pi$). These elements are gathered into generating functions

$$e_\alpha(z) = \sum_{n \in \mathbb{Z}} e_\alpha[n] z^{-n}, \quad f_\alpha(z) = \sum_{n \in \mathbb{Z}} f_\alpha[n] z^{-n},$$

$$\psi^\pm_\alpha(z) = \sum_{n > 0} \psi^\pm_\alpha[n] z^n = k_\alpha^{\pm 1} \exp \left( \pm (q_\alpha - q_\alpha^{-1}) \sum_{n > 0} h_\alpha \{n\} z^n \right),$$

such that $q^d q^{-d} = q^{-d} q^d = CC^{-1} = C^{-1} C = 1$ and $q^d a(z) q^{-d} = a(q^{-1} z)$ for any of these generating functions, which we will call currents. Currents are labeled by the simple roots $\alpha \in \Pi$ of $g$.

The defining relations of $U_q(\hat{g})$ in the ‘currents’ presentation are ($\alpha, \beta \in \Pi$):

$$(z - q^{(\alpha,\beta)} w) e_\beta(z) e_\beta(w) = e_\beta(w) e_\beta(z) (q^{(\alpha,\beta)} z - w),$$

$$(z - q^{-(\alpha,\beta)} w) f_\beta(z) f_\beta(w) = f_\beta(w) f_\beta(z) (q^{-(\alpha,\beta)} z - w),$$

$$(z - q^{(\alpha,\beta)} C z^{-1} - w) e_\beta(z) = \frac{q^{(\alpha,\beta)} C z^{-1} - w}{C z^{-1} - q^{(\alpha,\beta)} w} e_\beta(w),$$

$$(z - q^{-(\alpha,\beta)} C z^{-1} - w) f_\beta(z) = \frac{q^{-(\alpha,\beta)} C z^{-1} - w}{C z^{-1} - q^{-(\alpha,\beta)} w} f_\beta(w),$$

$$\left[ e_\alpha(z), f_\beta(w) \right] = \frac{\delta_{\alpha,\beta}}{q_\alpha - q_\alpha^{-1}} \left( \delta(z/C^2 w) \psi^+_\alpha(C^{-1} z) - \delta(C^2 z/w) \psi^-_\alpha(C^{-1} w) \right).$$

$$\sum_{r=0}^{n_{ij}} (-1)^r \left[ \begin{array}{c} n_{ij} \\ r \end{array} \right] \text{Sym}_{z_1, \ldots, z_{n_{ij}}} e_{\pm \alpha_1}(z_1) \cdots e_{\pm \alpha_r}(z_r) e_{\pm \alpha}(w) \exp(\pm \alpha_{r+1} \cdots \pm \alpha_{n_{ij}}) = 0.$$

Here $\alpha_i \neq \alpha_j$, $n_{ij} = 1 - \alpha_i \cdot \alpha_j$, $\delta(z) = \sum_{k \in \mathbb{Z}} z^k$.

We now describe the isomorphism of the two realizations (D2, CP). Suppose that the root vector $e_\theta \in g$, corresponding to the longest root $\theta \in \Sigma_+$, is presented as a multiple commutator $e_\theta = \lambda [e_{\alpha_1}, \ldots, e_{\alpha_r}, e_{\alpha_j}, \ldots]$ for some $\lambda \in \mathbb{C}$. The isomorphism is given by the assignment

$$k^{\pm 1}_\alpha \mapsto k^{\pm 1}_\alpha, \quad e_\alpha \mapsto e_\alpha[0], \quad e_{-\alpha} \mapsto f_\alpha[0], \quad i = 1, \ldots, r,$$

$$k^{\pm 1}_\alpha \mapsto C^{\pm 2} k^{\pm 1}_\alpha, \quad k^{\pm 1/2}_\delta \mapsto C^{\pm 1}, \quad q^{\pm d} \mapsto q^{\pm d},$$

$$e_{a_0} \mapsto \mu S^+_i S^-_r \cdots S^-_n (f_{\alpha_j}[1]), \quad e_{-a_0} \mapsto \lambda S^+_i S^-_r \cdots S^-_n (e_{\alpha_j}[-1]).$$

Here $S^\pm_i \in \text{End}(U_q(\hat{g}))$ are the following operators of adjoint action (with respect to the coproducts $\Delta^\text{std}$ and $(\Delta^\text{std})^{2,1}$, see (3.16)):

$$S^+_i(x) = e_{\alpha_i}[0] x - k_{\alpha_i} x k_{\alpha_i}^{-1} e_{\alpha_i}[0], \quad S^-_i(x) = x f_{\alpha_i}[0] - f_{\alpha_i}[0] k_{\alpha_i} x k_{\alpha_i}^{-1},$$

$$\sum_{r=0}^{n_{ij}} (-1)^r \left[ \begin{array}{c} n_{ij} \\ r \end{array} \right] \text{Sym}_{z_1, \ldots, z_{n_{ij}}} e_{\pm \alpha_1}(z_1) \cdots e_{\pm \alpha_r}(z_r) e_{\pm \alpha}(w) \exp(\pm \alpha_{r+1} \cdots \pm \alpha_{n_{ij}}) = 0.$$
and the constant \( \mu \) is determined by the condition that relation (3.16) for \( i = j = 0 \) remains valid in the image.

We now describe the inverse isomorphism. Let \( \pi : \{\alpha_1, \ldots, \alpha_r\} \mapsto \{0, 1\} \) be the map such that \( \pi(\alpha_i) \neq \pi(\alpha_j) \) for \( (\alpha_i, \alpha_j) \neq 0 \) and \( \pi(\alpha_i) = 0 \).

**Proposition 3.2.** (see [KT1, Be, Da]) The inverse of isomorphism (3.32) is such that

\[
e_{\alpha_i}[n] \mapsto (-1)^{n \pi(\alpha_i)} \hat{e}_{\alpha_i + n \delta} = \left\{ \begin{array}{ll}
(-1)^{n \pi(\alpha_i)} e_{\alpha_i + n \delta}, & n \geq 0, \\
-(1)^{n \pi(\alpha_i)} k_{\alpha_i - n \delta} e_{\alpha_i + n \delta}, & n < 0,
\end{array} \right.
\]
\[
f_{\alpha_i}[n] \mapsto (-1)^{n \pi(\alpha_i)} \hat{e}_{-\alpha_i + n \delta} = \left\{ \begin{array}{ll}
(-1)^{n \pi(\alpha_i)} e_{-\alpha_i + n \delta}, & n \leq 0, \\
-(1)^{n \pi(\alpha_i)} e_{-\alpha_i + n \delta} k_{\alpha_i - n \delta}, & n > 0,
\end{array} \right.
\]

\[
\psi^+_{\alpha_i}[0] \mapsto k_{\alpha_i}, \hspace{1cm} \psi^-_{\alpha_i}[0] \mapsto k_{\alpha_i}^{-1}, \hspace{1cm} C^{\pm 1} \mapsto k^{\pm 1/2}_{\delta},
\]
\[
\psi^+_{\alpha_i}[n] \mapsto (q - q^{-1})(-1)^{n \pi(\alpha_i)} k_{\alpha_i} k_{\delta}^{n \beta} e_{\alpha_i + n \delta}, \hspace{1cm} n > 0,
\]
\[
\psi^-_{\alpha_i}[n] \mapsto -(q - q^{-1})(-1)^{n \pi(\alpha_i)} k_{\alpha_i}^{-1} k_{\delta}^{-n \beta} e_{\alpha_i + n \delta}, \hspace{1cm} n < 0.
\]

The relation between the imaginary root generators \( e_{\pm \alpha_i + n \delta} \) and \( e_{\pm \delta} \) is given by formulas (3.21).

Note that the root vectors \( e_{\pm \alpha_i + n \delta}, n \in \mathbb{Z} \), as well as the imaginary root generators do not depend on the choice of a normal ordering of \( \hat{G} \), satisfying the condition (3.19) ([Ja], Proposition 11). So the identification of Proposition 3.2 does not depend on such a normal ordering.

The ‘currents’ bialgebra structure \( \Delta^{(D)} \) on \( U_q(\hat{g}) \) is given by:

\[
\Delta^{(D)}(q^{\pm d}) = q^{\pm d} \otimes q^{\pm d}, \hspace{1cm} \Delta^{(D)}(C^{\pm 1}) = C^{\pm 1} \otimes C^{\pm 1},
\]
\[
\Delta^{(D)}(\psi^\pm_{\alpha})(z) = \psi^\pm_{\alpha}(C^{\pm 1}_{\delta} z) \otimes \psi^\pm_{\alpha}(C^{\mp 1}_{\delta} z),
\]
\[
\Delta^{(D)}(e_{\alpha})(z) = e_{\alpha}(z) \otimes 1 + \psi^-(C_{\delta} z) \otimes e_{\alpha}(C^{2}_{\delta} z),
\]
\[
\Delta^{(D)}(f_{\alpha})(z) = 1 \otimes f_{\alpha}(z) + f_{\alpha}(C^{2}_{\delta} z) \otimes \psi^+(C^{2}_{\delta} z),
\]

where \( C_{1} = C \otimes 1 \) and \( C_{2} = 1 \otimes C \). The counit map is given by:

\[
\varepsilon(e_{\alpha}(z)) = \varepsilon(f_{\alpha}(z)) = 0, \hspace{1cm} \varepsilon(\psi^\pm_{\alpha}(z)) = \varepsilon(q^{\pm d}) = \varepsilon(C^{\pm 1}) = 1.
\]

The principal degree on \( U_q(\hat{g}) \) is defined by \( |e_i[n]| = n\nu + 1, |k_{i}^{\pm 1}| = |C| = 0, |f_i[n]| = n\nu - 1, |h_{i}[n]| = n\nu \), where \( \nu = \sum_{i=0}^{r} n_i \) (recall that \( \delta = \sum_{i=0}^{r} n_i \alpha_i \)); then \( \Delta^{(D)} : U_q(\hat{g}) \to U_q(\hat{g}) \otimes \mathbb{C}^{2} \) is a topological bialgebra, in the sense of Section 2.3 (H1). The identification with the situation of this Section is \( D = U_q(\hat{g}), \Delta_D = \Delta^{(D)} \).

The coproducts \( \Delta^{std} \) of Subsection 3.1 and \( \Delta^{(D)} \) are related by a twist which can be described explicitly (see Proposition 3.6 and [KT1]).

### 3.4 Intersections of Borel subalgebras

We first describe the Borel subalgebras related to the ‘currents’ realization of \( U_q(\hat{g}) \).
Let $U_F$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $U_q(\mathfrak{h})$ and the elements $f_\alpha[n]$ (where $\alpha \in \Pi$, $n \in \mathbb{Z}$) and $h_\alpha[m]$ (where $\alpha \in \Pi$, $m > 0$). In the circular description, this is the subalgebra of $U_q(\mathfrak{g})$ associated with the segment $\Sigma_F$ (see (3.31)). Analogously, we define $U_E$ as the subalgebra of $U_q(\mathfrak{g})$ generated by $U_q(\mathfrak{h})$ and the elements $e_\alpha[n]$ (where $\alpha \in \Pi$, $n \in \mathbb{Z}$) and $h_\alpha[m]$ (where $\alpha \in \Pi$, $m < 0$). It corresponds to the subalgebra of $U_q(\mathfrak{g})$ associated to the segment $\Sigma_E$. Both $U_F$ and $U_E$ are Hopf subalgebras of $U_q(\mathfrak{g})$ with respect to the coproduct $\Delta^{(D)}$. We call them the currents Borel subalgebras. We also define $U_f \subset U_F$ as the subalgebra generated by the elements $f_\alpha[n]$ (where $\alpha \in \Pi$, $n \in \mathbb{Z}$) and $U_c \subset U_E$ as the subalgebra generated by the elements $e_\alpha[n]$ (where $\alpha \in \Pi$, $n \in \mathbb{Z}$).

We have

$$U_q(\mathfrak{g}) \simeq U_E \otimes_{U_q(\mathfrak{h})} U_F,$$

where the isomorphism is induced by the product map. So $D = U_q(\mathfrak{g})$, $A := U_E$, $B := U_F$, $\mathbb{C}[\mathbb{Z}'] := U_q(\mathfrak{h})$ satisfy the first part of (H2) in Section 2.3.

We define $U^0_F \subset U_F$, $U^0_E \subset U_E$ as the algebras with the same generators, except $U_q(\mathfrak{h})$.

The relations (3.37) and (3.38) imply that the subalgebra $U_f$ is a right coideal of $U_F$ with respect to $\Delta^{(D)}$, and the subalgebra $U_c$ is a left coideal of $U_E$ with respect to $\Delta^{(D)}$:

$$\Delta^{(D)}(U_f) \subset U_f \otimes U_F, \quad \Delta^{(D)}(U_c) \subset U_E \otimes U_c.$$  \hfill (3.39)

There is a unique bialgebra pairing $\langle -, - \rangle : U_E \otimes U_F \rightarrow \mathbb{C}$, expressed by

$$\langle e_\alpha(z), f_\beta(w) \rangle = \frac{\delta_{\alpha,\beta} \delta(z/w)}{q^{1/2} - q^{-1/2}}, \quad \langle \psi^-_\alpha(z), \psi^+_\beta(w) \rangle = \frac{q^{(\alpha,\beta)} - z/w}{1 - q^{(\alpha,\beta)} z/w}, \quad \langle c, d \rangle = 1$$

in terms of generating series; all other pairings between generators are zero.

**Proposition 3.3.** This pairing satisfies identity (2.14) (see (H2)).

**Proof.** Let us set $R(a, b) := a^{(1)} b^{(1)} (a^{(2)}, b^{(2)}) - b^{(2)} a^{(2)} (a^{(1)}, b^{(1)})$ for $a \in U_E$, $b \in U_F$. One checks that $R(a, b) = 0$ if $a, b$ are generators of $U_E$, $U_F$. Moreover, we have the identities

$$R(aa', b) = a^{(1)} R(a', b^{(1)}) (a^{(2)}, b^{(2)}) + R(a, b^{(2)}) a^{(2)} (a^{(1)}, b^{(1)}),$$

$$R(a, bb') = R(a^{(1)}, b) b^{(1)} (a^{(2)}, b^{(2)}) + b^{(2)} R(a^{(2)}, b') (a^{(1)}, b^{(1)}).$$

Reasoning by induction on the length of $a$ and $b$ (expressed as products of generators), these identities imply that $R(a, b) = 0$ for any $a, b$. \hfill $\square$

Therefore $\langle -, - \rangle$ satisfies the hypothesis (H2). One also checks that it satisfies hypothesis (H6).

We will be interested in intersections of Borel subalgebras of different types. Denote by $U^+_F$, $U^-_F$, $U^+_E$, $U^-_E$ and $U^+_c$, $U^-_c$ the following intersections of the standard and currents Borel algebras:

$$U^-_F = U_F \cap U_q(n_-), \quad U^+_F = U_F \cap U_q(b_+), \quad U^0_+ = U^+_F \cap U_q(n_+),$$

$$U^+_E = U_E \cap U_q(n_+), \quad U^-_E = U_E \cap U_q(b_-), \quad U^0_+ = U^+_E \cap U_q(n_-). \quad (3.40)$$
The notation is such that the upper sign says in which standard Borel subalgebra 
$U_q(b_{\pm})$ the given algebra is contained and the lower letter says in which currents Borel
subalgebra ($U_F$ or $U_E$) it is contained. This letter is capital if the subalgebra contains
imaginary root generators $h_i[n]$ and the Cartan subalgebra $U_q(b)$ and is small otherwise.
In the notation of Section 3.2 the subalgebras $U_f^-$ and $U_f^+$ correspond to the segments
$\Sigma_{-,f}$ and $\Sigma_{+,f}$; the subalgebras $U_E^-$ and $U_E^+$ correspond to the segments $\Sigma_{-,E}$ and $\Sigma_{+,e}$.

**Proposition 3.4.** The product in $U_E$ sets up an isomorphism of vector spaces $U_E \simeq U_E^+ \otimes U_E^-$. The product in $U_F$ sets up an isomorphism of vector spaces $U_F \simeq U_F^+ \otimes U_F^-$. 

**Proof.** This follows from (3.27). 

We will set $A_1 := U_e^+, A_2 := U_e^-, B_1 := U_f^+, B_2 := U_f^-$. Then the hypothesis (H3) is satisfied. We will set $B'_1 := U_F^0+, A'_2 := U_E^0-$, so that (H5) is satisfied.

The next proposition describes a family of generators of the intersections of Borel subalgebras.

**Proposition 3.5.**

(i) The algebra $U_e^+$ (resp., $U_e^-$) is generated by the elements $e_i[n]$ (resp., $f_i[-n]$), where $i \in \{1, \ldots, r\}$, $n \geq 0$.

(ii) The algebra $U_F^+$ (resp., $U_F^-$) is generated by $U_q(h)$, the elements $e_i[n], h_i[m]$, where $n, m < 0$, $i \in \{1, \ldots, r\}$, and by the root vectors $e_{\gamma - \delta}$, $\gamma \in \Sigma_+$ (resp., $U_q(h)$, the elements $f_i[n], h_i[m]$, where $n, m > 0$, $i \in \{1, \ldots, r\}$, and the root vectors $e_{\delta - \gamma}$, $\gamma \in \Sigma_+$).

**Proof.** The PBW result shows that a basis for $U_e^+$ is given by the ordered monomials in the $\hat{e}_\gamma$, $\gamma \in \Sigma_+$. These generators are expressed via those listed in the Proposition, by means of successive applications of relations (3.25) and (3.26). The proof is the same in the other cases.

Note that the generators listed in the Proposition do not depend on a choice of the normal ordering, satisfying (5.19). This was already stated in Section 3.2 for all of them, except for the root vectors $e_{\pm(\delta - \gamma)}$, $\gamma \in \Sigma_+$. They are constructed as successive $q$-commutators of the type $[e_{\pm \gamma'}, e_{\pm(\delta - \gamma')}]_{q^{2n+1}}$. By induction of the height of $\gamma$, one proves that these $q$-commutators define the same root vectors.

### 3.5 Relation between $\Delta^{std}$ and $\Delta^{(D)}$

**Proposition 3.6.** We have $\Delta^{(D)}(U_q(b_-)) \subset U_q(b_-) \otimes U_q(g)$ and $\Delta^{(D)}(U_q(b_+)) \subset U_q(g) \otimes U_q(b_+)$.

**Proof.** We prove the first statement. It suffices to show that $\Delta^{(D)}(e_{-\alpha}) \in U_q(b_-) \otimes U_q(g)$, for $i = 0, \ldots, r$. When $i \in \{1, \ldots, r\}$, $\Delta^{(D)}(e_{-\alpha_i}) = \Delta^{(D)}(f_{\alpha_i}[0]) = 1 \otimes f_{\alpha_i}[0] + \sum_{k \geq 0} f_{\alpha_i}[-k] \otimes \psi_{\alpha_i}^+ [k] C^{-2n+k} \in U_q(b_-) \otimes U_q(g)$.

When $i = 0$, $\Delta^{(D)}(e_{-\alpha_0}) = \Delta^{(D)}(S_i^+ \cdots S_n^+(e_{\alpha_1}[-1]))$. Let us prove by descending induction on $k$ that $\Delta^{(D)}(S_i^+ \cdots S_n^+(e_{\alpha_1}[-1])) \in U_E^- \otimes U_q(g)$. This is true for $k = n + 1$ since $\Delta^{(D)}(e_{\alpha_j}[-1]) = e_{\alpha_j}[-1] \otimes 1 + \sum_{s \geq 0} \psi_{\alpha_j}[-s] C^{s+2} \otimes e_{\alpha_j}[s - 1]$. If now
\[ x = S_{i_{k+1}}^{+} \cdots S_{i_{k}}^{+}(e_{\alpha_i}[-1]) \] is such that \( \Delta^{(D)}(x) \in U_E^- \otimes U_q(\mathfrak{g}) \), we have \( \Delta^{(D)}(S_{i_{k}}^{+}(x)) = \Delta^{(D)}(e_{i_{k}}[0]x - q^{\alpha_i(x)}xe_{i_{k}}[0]) = [\Delta^{(D)}(e_{i_{k}}[0]), x]_q = [e_{i_{k}}[0] \otimes 1 + \sum_{s \geq 0} \psi_{\alpha_i}[-s]C^{-s} \otimes e_{i_{k}}[0], \Delta^{(D)}(x)]_q \equiv [e_{i_{k}}[0] \otimes 1, \Delta^{(D)}(x)]_q \text{ modulo } U_E^- \otimes U_q(\mathfrak{g}) \) (here \( [x] \) is the degree of \( x \)). The result now follows from the induction assumption \( \Delta^{(D)}(x) \in U_E^- \otimes U_q(\mathfrak{g}) \) and (3.25).

The proof of the second statement is similar. \( \Box \)

**Corollary 3.7.** The intersections of Borel subalgebras have the following coideal properties:

\[
\Delta^{(D)}(U_F^+) \subset U_F^+ \otimes U_F^+ \quad \Delta^{(D)}(U_F^-) \subset U_F^- \otimes U_F^- \quad (3.41)
\]

\[
\Delta^{(D)}(U_E^+) \subset U_E^+ \otimes U_E^+ \quad \Delta^{(D)}(U_E^-) \subset U_E^- \otimes U_E^- \quad (3.42)
\]

*Proof.* This follows directly from (3.39) and Proposition 3.6. \( \Box \)

This means that \( (D, \Delta^{(D)}), A_i, B_i \) satisfy the hypothesis (H4).

We can therefore apply Theorem 2 to to \( (D, \Delta_D) = (U(q(\hat{\mathfrak{g}}), \Delta^{(D)}), A = U_E, A_1 = U_E^+, A_2 = U_E^+), B = U_F, B_1 = U_F^+, B_2 = U_F^-, \mathbb{C}[\hat{\mathbb{Z}}^+] = U_q(\mathfrak{h}) \).

We denote by \( R_1 \in U_E^+ \otimes U_F^+ \) and \( R_2 \in U_E^- \otimes U_F^- \) the analogues of \( R_1, R_2 \). Then \( \mathcal{R}_2 \) is a cocycle for \( (U_q(\hat{\mathfrak{g}}), \Delta^{(D)}) \), so we get a Hopf algebra structure on \( U_q(\hat{\mathfrak{g}}) \), given by \( \Delta^{tw}(x) = \mathcal{R}_2 \Delta^{(D)}(x) \mathcal{R}_2^{-1} \) (\( \Delta^{tw} \) is the analogue of \( \Delta \) of section 2.3), quasitriangular with R-matrix \( \mathcal{R}^{tw} = \mathcal{R}_2^{2,1} \mathcal{R}_1 \).

A proof of the following result (based on the braid group action) was first given in [KT1].

**Proposition 3.8.** \( \Delta^{tw} = (\Delta^{std})^{2,1} \), therefore we have \( \Delta^{(D)}(x) = \mathcal{R}_2^{-1}(\Delta^{std})^{2,1}(x) \mathcal{R}_2 \). We also have \( \Delta^{(D)}(x) = \mathcal{R}_1^{2,1} \Delta^{std}(x)(\mathcal{R}_1^{2,1})^{-1} \) (3.43)

*Proof.* \( \Delta^{tw} \) defines a Hopf structure on \( U_q(\hat{\mathfrak{g}}) \), for which \( U_q(b_+) = A_1 B_1 \) and \( U_q(b_-) = A_2 B_2 \) are Hopf subalgebras. Since \( \mathcal{R}_2 \) is invariant under \( U_q(\mathfrak{h}) \), we have \( \Delta^{tw}(k_{a_1}^{1/2}) = (k_{a_1}^{1/2})^{\otimes 2} \) and \( \Delta^{tw}(k_{a_1}^{-1/2}) = (k_{a_1}^{-1/2})^{\otimes 2} \).

Since \( \Delta^{(D)} \) and \( \mathcal{R} \) are homogeneous for the principal degree, the same holds for \( \Delta^{tw} \). Moreover, \( \mathcal{R}_2 \) are invariant under \( \Delta^{(D)}(x) = [\Delta^{(D)}(x)]_q \mathcal{R}_2^{-1}(\Delta^{std}(x)) \mathcal{R}_2 \) and \( \Delta^{tw}(e_{\alpha_i}) = e_{\alpha_i} \otimes k_{\alpha_i} + \text{degree 0} \otimes \text{degree 1} \).

On the other hand, \( \Delta^{tw}(x) = \mathcal{R}_2 \Delta^{(D)}(x) \mathcal{R}_2^{-1} = \mathcal{R}_2^{-1}(\Delta^{(D)})^{2,1} \mathcal{R}_2 \), and since \( \mathcal{R}_1 = 1 + (\text{positive principal degree}) \otimes (\text{negative principal degree}) \), we find \( \Delta^{tw}(e_{\alpha_i}) = e_{\alpha_i} \otimes k_{\alpha_i} + \text{degree 0} \otimes \text{degree 1} \).

Combining these results, we get \( \Delta^{tw} = (\Delta^{std})^{2,1} \).

Then \( \Delta^{std}(x) = \mathcal{R}_2^{2,1}(\Delta^{(D)})^{2,1}(x)(\mathcal{R}_2^{2,1})^{-1} = (\mathcal{R}_1^{2,1})^{-1}(\Delta^{D})^{2,1} \mathcal{R}_1 \), which implies (3.43).

*Proposition 3.9.** The intersections of Borel subalgebras have the following coideal properties:

\[
\Delta^{std}(U_F^+) \subset U_F^+ \otimes U_q(b_+), \quad \Delta^{std}(U_F^-) \subset U_F^- \otimes U_q(b_-), \quad (3.44)
\]

\[
\Delta^{std}(U_E^+) \subset U_q(b_+) \otimes U_E^+, \quad \Delta^{std}(U_E^-) \subset U_q(b_-) \otimes U_E^-. \quad (3.45)
\]
This is a translation of Proposition 2.3.
We will denote by $P^\pm$ the projection operators of the Borel subalgebra $U_F$, corresponding to the decomposition $U_F = U_F^+ \oplus U_F^-$. So for any $f_+ \in U_F^+$ and any $f_- \in U_F^-$,

$$P^+(f_- f_+) = \varepsilon(f_-) f_+, \quad P^-(f_- f_+) = f_- \varepsilon(f_+) .$$

(3.46)

The operator $P^+$ will also be denoted by $P$ (without index).

4 Universal weight functions

4.1 The definition

Denote by $(U^+_e)^\varepsilon$ the augmentation ideal of $U^+_e$, i.e., $(U^+_e)^\varepsilon = U_e \cap \ker(\varepsilon)$. Let $J$ be the left ideal of $U_q(b_\pm)$ generated by $(U^+_e)^\varepsilon$:

$$J = U_q(b_\pm) (U^+_e)^\varepsilon = \sum_{\alpha \in \Pi, n \geq 0} U_q(b_\pm) e_\alpha[n].$$

(4.1)

Recall that the $\psi^+_\alpha[0]^{\pm 1}$ and $\psi^+_\alpha[n]$ and $C^{\pm 1}$ (where $i \in \{1, ..., r\}$, $n > 0$) commute in $U_q(\widehat{\mathfrak{g}})$.

Proposition 4.1. $J$ is a coideal of $U_q(b_\pm)$, i.e., $\Delta^\text{std}(J) \subset J \otimes U_q(b_\pm) + U_q(b_\pm) \otimes J$. The space $J$ is also stable under right multiplication by the $\psi^+_\alpha[0]^{\pm 1}$, $\psi^+_\alpha[n]$ (where $i \in \{1, ..., r\}$, $n > 0$), i.e., $J \psi^+_\alpha[0]^{\pm 1} \subset J$, $J \psi^+_\alpha[n] \subset J$.

It follows that $U_q(b_\pm)/J$ is both a coalgebra and a right module over $\mathbb{C}[\psi^+_\alpha[0]^{\pm 1}, \psi^+_\alpha[n], C^{\pm 1}; i \in \{1, ..., r\}, n > 0]$.

Proof. The second part of (3.35) implies that for $n \geq 0$, $\Delta^\text{std}(e_\alpha[n]) \in U_q(b_\pm) \otimes U^+_e$. Moreover, $(\text{id} \otimes \varepsilon) \circ \Delta^\text{std} = \text{id}$ implies that $\Delta^\text{std}(e_\alpha[n]) - e_\alpha[n] \otimes 1 \in U_q(b_\pm) \otimes \ker \varepsilon$. Therefore $\Delta^\text{std}(e_\alpha[n]) - e_\alpha[n] \otimes 1 \in U_q(b_\pm) \otimes (U^+_e)^\varepsilon$, so $\Delta(e_\alpha[n]) \in J \otimes U_q(b_\pm) + U_q(b_\pm) \otimes J$. This implies the coideal property of $J$. The second property follows from the commutation relation of the currents $\psi^+_\alpha(w)$ and $e_\alpha(z)$.

Define an ordered $\Pi$-multiset as a triple $\bar{I} = (I, \prec, \iota)$, where $(I, \prec)$ is a finite, totally ordered set, and $\iota : I \to \Pi$ is a map (the ‘coloring map’). Ordered $\Pi$-multisets form a category, where a morphism $\bar{I} \to \bar{I}' = (I', \prec', \iota')$ is a map $m : I \to I'$, compatible with the order relations and such that $\iota' \circ m = m \circ \iota$.

If $\bar{I} = (I, \prec, \iota)$ is an ordered $\Pi$-multiset, then a partition $I = I_1 \sqcup I_2$ gives rise to ordered $\Pi$-multisets $\bar{I}_i = (I_i, \prec_i, \iota_i)$, $i = 1, 2$, where $\prec_i$ and $\iota_i$ are the restrictions of $\prec$ and $\iota$ to $I_i$.

If $\bar{I} = (I, \prec, \iota)$ is an ordered $\Pi$-multiset, and $I = \{i_1, ..., i_n\}$, with $i_1 \prec ... \prec i_n$, then we attach to $\bar{I}$ an ordered set of variables $(t_i)_{i \in I} = (t_{i_1}, ..., t_{i_n})$. The ‘color’ of $t_k$ is $\iota(i_k) \in \Pi$.

For any vector space $V$ denote by $V[[t_1^{\pm 1}, ..., t_n^{\pm 1}]]$ the vector space of all formal series

$$\sum_{(k_1, ..., k_n) \in \mathbb{Z}^n} A_{k_1, ..., k_n} t_1^{k_1} \cdots t_n^{k_n}, \quad A_{k_1, ..., k_n} \in V ,$$

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and by \( V((t_1)) \cdots ((t_n)) \) its subspace corresponding to the maps \( (k_1, \ldots, k_n) \mapsto A_{k_1, \ldots, k_n} \), such that there exists an integer \( a_1 \) and integer valued functions \( a_2(k_1), a_3(k_1, k_2), \ldots, a_n(k_1, \ldots, k_{n-1}) \) (depending on the map \( (k_1, \ldots, k_n) \mapsto A_{k_1, \ldots, k_n} \)), such that \( A_{k_1, \ldots, k_n} = 0 \) whenever \( k_1 > a_1 \), or \( k_2 > a_2(k_1) \), or \( k_3 > a_3(k_1, k_2), \ldots, \) or \( k_n > a_n(k_1, \ldots, k_{n-1}) \). A bilinear map \( V \otimes W \to Z \) gives rise to a bilinear map \( V((t_1)) \cdots ((t_n)) \times W((t_1)) \cdots ((t_n)) \to Z((t_1)) \cdots ((t_n)) \) using the multiplication of formal series.

A universal weight function is an assignment \( I \mapsto W_I \), where \( I = (I, <, \iota) \) is an ordered \( \Pi \)-multiset and \( W_I((t_i)_{i \in I}) \in (U_q(\mathfrak{b}_+)/J)((t_i^{-1})) \cdots ((t_i^{-1})) \) \( (I = \{i_1, \ldots, i_n\} \), with \( i_1 < \ldots < i_n \) \) is such that

(a) (functoriality) If \( f : \tilde{I} \to \tilde{J} \) is an isomorphism of ordered \( \Pi \)-multisets, then \( W_f((t_i)_{i \in \tilde{I}}) = W_f((t_{j^{-1}(i)})_{j \in \tilde{J}}) \);

(b) \( W_\emptyset = 1 \), where \( W_\emptyset \) is the series corresponding to \( I \) equal to the empty set;

(c) The series \( W_f(t_{i_1}, \ldots, t_{i_n}) \) satisfies the relation

\[
\Delta^{std}(W_f((t_i)_{i \in I})) = \sum_{I = I_1 \sqcup I_2} (W_{I_1}((C_2^2t_i)_{i \in I_1}) \otimes W_{I_2}((t_i)_{i \in I_2})) \times \\
\times (1 \otimes \prod_{i \in I_1} \psi_{(i,i)}^+(Ct_i)) \times \prod_{k,l : k < l, i_k \in I_1, i_l \in I_2} \frac{q^{-(\sigma_i(i_k), \sigma_i(i_l))} - t_{i_l}/t_{i_k}}{1 - q^{-(\sigma_i(i_k), \sigma_i(i_l))}t_{i_l}/t_{i_k}} (4.2)
\]

where \( \Delta^{std} \) is the coproduct of \( U_q(\mathfrak{b}_+)/J \). Here \( C_2 = 1 \otimes C \), where \( C \) is the central element of \( U_q(\hat{\mathfrak{g}}) \), see Section 3.3. The summation in (4.2) runs over all possible decompositions of the set \( I \) into two disjoint subsets \( I_1 \) and \( I_2 \).

**Remark.** Let \( U'_q(\hat{\mathfrak{g}}) \) be the subquotient of \( U_q(\hat{\mathfrak{g}}) \) with trivial central element \( C = 1 \) and dropped gradation element. Let \( U'_q(\mathfrak{b}_+) \) be the corresponding Borel subalgebra of \( U'_q(\hat{\mathfrak{g}}) \). The analogue of Proposition 4.1 holds with \( J \) replaced by its analogue \( J' \). The notion of universal weight function makes sense for the algebra \( U_q(\hat{\mathfrak{g}}) \) as well. The conditions (a), (b) remain unchanged, the relation (4.2) of condition (c) is now

\[
\Delta^{std}(W_f((t_i)_{i \in I})) = \sum_{I = I_1 \sqcup I_2} (W_{I_1}((t_i)_{i \in I_1}) \otimes W_{I_2}((t_i)_{i \in I_2})) \times \\
\times (1 \otimes \prod_{i \in I_1} \psi_{(i,i)}^+(t_i)) \times \prod_{k,l : k < l, i_k \in I_1, i_l \in I_2} \frac{q^{-(\sigma_i(i_k), \sigma_i(i_l))} - t_{i_l}/t_{i_k}}{1 - q^{-(\sigma_i(i_k), \sigma_i(i_l))}t_{i_l}/t_{i_k}} (4.3)
\]

where \( \Delta^{std} \) is the coproduct of \( U'_q(\mathfrak{b}_+)/J' \).

### 4.2 Vector-valued weight functions

Let \( V \) be a representation of \( U_q(\hat{\mathfrak{g}}) \) and \( v \) be a vector in \( V \). We call \( v \) a singular weight vector of weight \( \{\lambda_i(z), i = 1, \ldots, r\} \) if

\[ e_\alpha[n]v = 0, \quad \psi^+_{\alpha_i}(z)v = \lambda_i(z)v, \quad i \in \{1, \ldots, r\}, \quad n \geq 0, \]
where \( \lambda_i(z) \in \mathbb{C}[[z^{-1}]]^\times \). Then \( V \) is called a representation with singular weight vector \( v \in V \) if it is generated by \( v \) over \( U'_q(\widehat{g}) \). It is clear that the ideal \( J \) defined by (4.1) annihilates any singular weight vector, i.e., \( Jv = 0 \).

For \( V \) a representation generated by the singular weight vector \( v \) and \( I \) an ordered \( \Pi \)-multiset, we define a \( V \)-valued function

\[
w^I_V((t_i)_{i \in I}) = W_I((t_i)_{i \in I})v
\]

which is a \( V \)-valued Laurent formal series. For \( I = \emptyset \) we set \( w^0_V = v \). Let \( V_i (i = 1, 2) \) be representations generated by the singular weight vectors \( v_i \), with series \( \{\lambda^{(i)}(z)\alpha \in \Pi\} \). Then the coproduct property (1.3) of the universal weight function yields the following property of the \( V_1 \otimes V_2 \)-valued function

\[
w^I_{V_1 \otimes V_2}((t_i)_{i \in I}) = \sum_{I = I_1 + I_2} w^{I_1}_{V_1}((t_i)_{i \in I_1}) \otimes w^{I_2}_{V_2}((t_i)_{i \in I_2}) \times
\]

\[
\prod_{i \in I_1} \lambda^{(2)}_{i(i)}(t_i) \prod_{k, l \leq I, i_k, i_l \in I_2} \frac{q^{-\alpha_{i(k)} - \alpha_{i(l)}} t_{i_k} - t_{i_l}}{t_{i_k} - q^{-\alpha_{i(k)} - \alpha_{i(l)}} t_{i_l}}.
\]

A collection of \( V \)-valued functions \( w^I_V((t_i)_{i \in I}) \) for all possible \( \Pi \)- multisets \( I \), and representations \( V \) with a singular weight vector \( v \), is called a vector-valued weight function or simply a weight function, if satisfies the relations (1.3), the initial condition \( w^\emptyset_V = v \), and depends only on the isomorphism class of the ordered \( \Pi \)-multiset \( I \).

It is clear how to modify relation (4.5) to define a vector-valued weight function of the algebra \( U_q(\widehat{g}) \). Any universal weight function determines a vector-valued weight function by relation (4.4).

### 4.3 Main Theorems

Let \( I = (I, \prec, \iota) \) be an ordered \( \Pi \)-multiset. If \( I = \{i_1, \ldots, i_n\} \), with \( i_1 \prec \ldots \prec i_n \), we set

\[
W_I((t_i)_{i \in I}) := P \left( f_{i(i_1)}(t_{i_1}) \cdots f_{i(i_n)}(t_{i_n}) \right).
\]

**Theorem 3.** The map \( I \mapsto W_I \) defined by (4.0) is a universal weight function.

Note that the statement of Theorem 3 is valid for both algebras \( U_q(\widehat{g}) \) and \( U'_q(\widehat{g}) \).

**Proof.** First one should check that \( W_I((t_i)_{i \in I}) \in U_q(b_\pm)((t_{i_1}^{-1}) \cdots (t_{i_n}^{-1})) \). This follows from the fact that for any \( x \in U_F \) and \( \alpha_i \in \Pi \), there exists an integer \( M \), such that for any \( n > M \) we have \( P(x f_{\alpha_i}[-n]) = 0 \). We now prove this fact. Define a degree on \( U_F \) by \( \deg(h_{\alpha}[n]) = n \) \( (n > 0) \), \( \deg(U_q(b)) = 0 \), \( \deg(f_{\alpha}[n]) = n \) \( (n \in \mathbb{Z}) \). Then \( U_F^+, U_F^- \subset U_F \) are graded subalgebras. Moreover, the nontrivial homogeneous components of \( U_F^\pm \) all have degree \( \geq 0 \). We have \( U_F = U_F^+ \oplus (U_F^-)^* U_F^+ \). Then we take \( M \) to be the largest degree of a nonzero homogeneous component of \( x \).

Using the definition of \( P \), one also checks that

\[
W_I((t_i)_{i \in I}) \in t_{i_1}^{-1} U_q(b_\pm)((t_{i_1}^{-1}) \cdots (t_{i_n}^{-1}))[[t_{i_1}^{-1}]].
\]
Let us now show that $\tilde{I} \to W_I$ satisfies conditions (a)–(c) of Section 4.1. Conditions (a) and (b) of Section 4.1 are trivially satisfied. Let us show that (c) is satisfied. Theorem 4 and the following relations

\[ \Delta^{(D)} f_\alpha(z) = 1 \otimes f_\alpha(z) + f_\alpha(C_2^2 z) \otimes \psi^+(C_2 z), \]
\[ \psi^+(z) f_\beta(w) (\psi^+(z))^{-1} = \frac{q^{-(\alpha, \beta)} C^{-1} - w/z}{1 - q^{-(\alpha, \beta)} C^{-1} w/z} f_\beta(w), \]

imply that relations (4.2) are satisfied modulo $U_q(\mathfrak{g}) \otimes J$. But both sides of (4.2) belong to $U_q(b_+) \otimes U_q(b_+)$. Thus they coincide modulo $U_q(b_+) \otimes J$. \hfill \Box

So, we reduced the proof of Theorem 1 to the following statement. Recall first that $\Delta^{std} \circ P$ defines a map $U_F \to U_F^+ \otimes U_q(b_+)$, and $P^{\otimes 2} \circ \Delta^{(D)}$ defines a map $U_F^+ \to (U_F^+)^{\otimes 2}$.

**Theorem 4.** For any element $f \in U_F$

\[ \Delta^{std}(P(f)) \equiv (P \otimes P)(\Delta^{(D)}(f)) \mod U_F^+ \otimes J, \quad (4.7) \]

where $J$ is the left ideal of $U_q(b_+)$ defined by (4.1).

**Proof of Theorem 4.** We have $U_F = U_F^+ U_F^-$, therefore $U_F = U_F^+ \oplus (U_F^-)^2 U_F^+$. So we will prove (4.7) in the two following cases: (a) $f \in U_F^+$, and (b) $f = xy$, where $x \in (U_F^-)^e$ and $y \in U_F^+$.

Assume first that $f \in U_F^+$. According to (3.41), $\Delta^{(D)}(f) \in U_F \otimes U_F^+$, therefore

\[ (P \otimes P)(\Delta^{(D)}(f)) = (P \otimes \text{id})(\Delta^{(D)}(f)). \]

According to (3.43), we have $\Delta^{(D)}(x) = \mathcal{R}_1^{2,1}_1 \Delta^{std}(x) (\mathcal{R}_1^{2,1}_1)^{-1}$ for any $x \in U_q(\mathfrak{g})$, where $(\mathcal{R}_1^{2,1})^\pm \subset U_F^+ \otimes U_F^-$. It follows that

\[ (P \otimes \text{id})(\Delta^{(D)}(f)) = (P \otimes \text{id})(\mathcal{R}_1^{2,1}_1 \Delta^{std}(f)(\mathcal{R}_1^{2,1}_1)^{-1}). \]

Now $\Delta^{std}(f) \in U_F^+ \otimes U_q(b_+)$, therefore $\Delta^{std}(f)(\mathcal{R}_1^{2,1}_1)^{-1} \in U_F \otimes U_q(b_+)$; it follows that $(P \otimes \text{id})(\Delta^{std}(f)(\mathcal{R}_1^{2,1}_1)^{-1})$ is well defined. Now $\mathcal{R}_1^{2,1} = 1 \otimes 1 + (U_F^-)^e \otimes U_F^+$, therefore

\[ (P \otimes \text{id})(\mathcal{R}_1^{2,1}_1 \Delta^{std}(f)(\mathcal{R}_1^{2,1}_1)^{-1}) = (P \otimes \text{id})(\Delta^{std}(f)(\mathcal{R}_1^{2,1}_1)^{-1}). \]

Since $(\mathcal{R}_1^{2,1}_1)^{-1} \subset 1 \otimes 1 + U_F^- \otimes (U_F^+)^e \subset 1 \otimes 1 + U_F^- \otimes J$, we have

\[ (P \otimes \text{id})(\Delta^{std}(f)(\mathcal{R}_1^{2,1}_1)^{-1}) \equiv (P \otimes \text{id})(\Delta^{std}(f)) \mod U_F \otimes J. \]

Recall now that $\Delta^{std}(f) \in U_F^+ \otimes U_q(b_+)$. This implies that

\[ (P \otimes \text{id})(\Delta^{std}(f)) = \Delta^{std}(f). \]

Finally, since $f \in U_F^+$, we have $f = P(f)$, therefore

\[ \Delta^{std}(f) = \Delta^{std}(P(f)). \]

\footnote{Recall that if $A \subset U_q(\mathfrak{g})$, we set $A^e := A \cap \text{Ker}(\epsilon)$}
Combining the above equalities and congruence, we get the desired congruence, when \( f \in U^+_F \).

Assume now that \( f = xy \), where \( x \in (U^-_F)^e \) and \( y \in U^+_F \). According to (3.11), \( \Delta^{(D)}(x) \in U^+_F \); on the other hand, the identities \( (\epsilon^{(D)} \otimes \text{id}) \circ \Delta^{(D)} = \text{id} \) and \( \epsilon = \epsilon^{(D)} \) imply that \( \Delta^{(D)}(x) \in 1 \otimes x + \operatorname{Ker}(\epsilon) \otimes U_q(\hat{G}) \); all this implies that \( \Delta^{(D)}(x) = 1 \otimes x + \sum_i a_i \otimes b_i \), where \( a_i \in (U^-_F)^e \) and \( b_i \in U_F \).

Therefore
\[
(P \otimes \text{id})(\Delta^{(D)}(xy)) = (P \otimes \text{id})((1 \otimes x + \sum_i a_i \otimes b_i)\Delta^{(D)}(y)).
\]

Recall that \( \Delta^{(D)}(y) \in U_F \otimes U^+_F \). Now since \( a_i \in (U^-_F)^e \), we have \( P(a_i\eta) = 0 \) for any \( \eta \in U_F \), which implies that \( (P \otimes \text{id})(a_i \otimes b_i)\Delta^{(D)}(y) = 0 \). Therefore
\[
(P \otimes \text{id})(\Delta^{(D)}(xy)) = (P \otimes \text{id})((1 \otimes x)\Delta^{(D)}(y)).
\]

Applying \( \text{id} \otimes P \) to this identity, we get
\[
(P \otimes P)(\Delta^{(D)}(xy)) = (P \otimes \text{id}) \circ (\text{id} \otimes P)((1 \otimes x)\Delta^{(D)}(y)).
\]

Now since \( x \in (U^-_F)^e \), we have \( P(xy) = 0 \) for any \( \eta \in U_F \), which implies that \( (\text{id} \otimes P)((1 \otimes x)\Delta^{(D)}(y)) = 0 \). Therefore
\[
(P \otimes P)(\Delta^{(D)}(xy)) = 0.
\]

Since \( P(xy) = 0 \), we get
\[
\Delta^{std}(P(xy)) = (P \otimes P)(\Delta^{(D)}(xy)),
\]
which proves the desired congruence for the elements of the form \( xy, x \in (U^-_F)^e, y \in U^+_F \).

4.4 Functional properties of the universal weight function

Let \( \bar{I} = (I, \prec, t) \) be an ordered \( \Pi \)-multiset, let \( n := |I| \) and let \( \sigma \in S_n \) be a permutation. Let \( \bar{I}^\sigma = (I, \prec, t) \) be the \( \Pi \)-multiset such that if \( I = \{i_1, \ldots, i_n\} \), where \( i_1 \prec \ldots \prec i_n \), then \( i_{\sigma(1)} \prec \ldots \prec i_{\sigma(n)} \). We call \( I^\sigma \) a permutation of \( \bar{I} \).

**Proposition 4.2.** There exists a collection of formal functions
\[
\underline{W}_{n_1, \ldots, n_r} \in (U_q(\mathfrak{b}_+)/J)[[u_j^{(s)}] s \in \{1, \ldots, r\}, j \in \{1, \ldots, n_i\}],
\]
symmetric in each group of variables \( (u_j^{(s)})_{j=1,\ldots, n_i} \), such that
\[
W_{\bar{I}}((t_i)_{i \in I}) = \frac{\epsilon(\sigma) \prod_{s=1}^r \prod_{i,j \in I, i \prec j} (t_i^{-1} - t_j^{-1})}{\prod_{i \in I} t_i \prod_{i,j \in I, i \prec j} (t_i^{-1} - q^{(\sigma(i), \sigma(j))}t_j^{-1})} \underline{W}_{n_1, \ldots, n_r}((t_j^{-1})_{j_i \in I_1}, \ldots, (t_j^{-1})_{j_r \in I_r}),
\]

(4.8)
where $I_s := e^{-1}(\alpha_s)$ (it is an ordered set), and $\sigma \in S_n$ is the shuffle permutation of $\{1, \ldots, n\}$ given by $\{1, \ldots, n\} \simeq I_1 \sqcup I_2 \sqcup \cdots \sqcup I_r \rightarrow I \simeq \{1, \ldots, n\}$, where the first and last bijections are ordered, the order relation on $I_1 \sqcup \cdots \sqcup I_r$ is such that $I_1 \prec \cdots \prec I_r$, and the map $I_1 \sqcup \cdots \sqcup I_r \rightarrow I$ is such that its restriction to each $I_s$ is the natural injection.

Proof. For any ordered $\Pi$-multiset $\vec{I}$, denote by $F_{\vec{I}}(t_{i_1}, \ldots, t_{i_n}) \in U_q(b_+)[[t_{i_1}^\pm, \ldots, t_{i_n}^\pm]]$ the series

$$F_{\vec{I}}(t_{i_1}, \ldots, t_{i_n}) = f_{l(i_1)}(t_{i_1}) \cdots f_{l(i_n)}(t_{i_n}),$$

and by $A_{\vec{I}}(t_{i_1}, \ldots, t_{i_n}) \in \mathbb{C}[t_{i_1}^{-1}, \ldots, t_{i_n}^{-1}]$ the product

$$A_{\vec{I}}(t_{i_1}, \ldots, t_{i_n}) = \prod_{1 \leq k < l \leq n} (t_{i_k}^{-1} - q^{l(i_k)l(i_l)}t_{i_l}^{-1}). \quad (4.9)$$

Let $\vec{I} \mapsto G_{\vec{I}}$ be an assignment taking an ordered $\Pi$-multiset $\vec{I} = (I, \prec, \iota)$ (with $I = \{i_1, \ldots, i_n\}$ and $i_1 \prec \cdots \prec i_n$) to $G_{\vec{I}}(t_{i_1}, \ldots, t_{i_n}) \in V[[t_{i_1}^\pm, \ldots, t_{i_n}^\pm]]$. We say that $\vec{I} \mapsto G_{\vec{I}}$ is antisymmetric, iff for any $\vec{I}$ and any $\sigma \in S_n$ (where $n = |I|$), we have $G_{\vec{I}}((t_{i_{\sigma(i)}})_{i \in I}) = \epsilon(\sigma)G_{\vec{I}}((t_{i_{\iota(i)}})_{i \in I})$.

The defining relations (3.61) imply that the assignment $\vec{I} \mapsto \overline{F}_{\vec{I}}$ is antisymmetric, where

$$\overline{F}_{\vec{I}}(t_{i_1}, \ldots, t_{i_n}) = A_{\vec{I}}(t_{i_1}, \ldots, t_{i_n})F_{\vec{I}}(t_{i_1}, \ldots, t_{i_n})$$

takes its values in $U_q(b_+)[[t_{i_1}^\pm, \ldots, t_{i_n}^\pm]]$.

Applying $P$, we get that the assignment $\vec{I} \mapsto \overline{W}_{\vec{I}}$ is antisymmetric, where

$$\overline{W}_{\vec{I}}(t_{i_1}, \ldots, t_{i_n}) := A_{\vec{I}}(t_{i_1}, \ldots, t_{i_n})W_{\vec{I}}(t_{i_1}, \ldots, t_{i_n});$$

its takes its values in $(U_q(b_+)/J)[[t_{i_1}^\pm, \ldots, t_{i_n}^\pm]]$.

According to the proof of Theorem 3, we have

$$W_{\vec{I}}(t_{i_1}, \ldots, t_{i_n}) \in t_{i_1}^{-1}(U_q(b_+)/J)((t_{i_1}^{-1})) \cdots ((t_{i_n}^{-1}))[t_{i_1}^{-1}],$$

therefore $\overline{W}_{\vec{I}}(t_{i_1}, \ldots, t_{i_n}) := A_{\vec{I}}W_{\vec{I}}(t_{i_1}, \ldots, t_{i_n})$ takes its values in the same space. The antisymmetry of $\vec{I} \mapsto \overline{W}_{\vec{I}}$ then implies that it takes its values in the intersection of all the $t_{i_1}^{-1}(U_q(b_+)/J)((t_{i_1}^{-1})) \cdots ((t_{i_n}^{-1}))[t_{i_1}^{-1}],$, where $\sigma \in S_n$, i.e., in

$$(t_{i_1} \cdots t_{i_n})^{-1}(U_q(b_+)/J)[[t_{i_1}^{-1}, \ldots, t_{i_n}^{-1}]].$$

If now $V$ is a vector space and $\vec{I} \mapsto v_{\vec{I}}(t_{i_1}, \ldots, t_{i_n}) \in V[[t_{i_1}^\pm, \ldots, t_{i_n}^\pm]]$ is antisymmetric, one shows that there exists a family of formal series $v_{i_1, \ldots, i_r}(u_{1}^{(1)}, \ldots, u_{n_t}^{(r)}) \in V[[u_{1}^{(1)}, \ldots, u_{n_t}^{(r)}]]$, symmetric in each group of variables $u_{a_i^{(i)}}$ for fixed $i$, such that

$$v_{\vec{I}}(t_{i_1}, \ldots, t_{i_n}) = \epsilon(\sigma)\prod_{s=1}^{r} \prod_{k,l \in I_s, k<l} (t_k^{-1} - q^{-l(i_k)l(i_l)})v_{I_1, \ldots, I_r}(t_{i_1\in I_1}, \ldots, (t_{i_r\in I_r})).$$

The result follows. \(\square\)

Let $s, t \in \{1, \ldots, r\}$, with $s \neq t$. Let $m := 1 - a_{\alpha_s\alpha_t}$, where $(a_{\alpha\beta})_{\alpha,\beta \in \Pi}$ is the Cartan matrix of $\mathfrak{g}$. Let $k_1, \ldots, k_m \in \{1, \ldots, n_s\}$ be distinct, and let $l \in \{1, \ldots, n_t\}$. Let $H_{(k_0, \ldots, k_m), l} \subset \bigoplus_{s=1}^{r} \mathbb{C}u_{a_s^{(s)}}$ be the subspace of all $(u_{j}^{(s)})_{s \in \{1, \ldots, r\}, j \in \{1, \ldots, n_s\}}$, such that

$$u_l^{(t)} = q^{-\frac{m-1}{2}(a_{\alpha_s^{(s)}}, a_{\alpha_t^{(s)}})}u_{k_1}^{(s)} = q^{-\frac{m-3}{2}(a_{\alpha_s^{(s)}}, a_{\alpha_t^{(s)}})}u_{k_2}^{(s)} = \cdots = q^{-\frac{m-1}{2}(a_{\alpha_s^{(s)}}, a_{\alpha_t^{(s)}})}u_{k_m}^{(s)}.$$
Proposition 4.3. The restriction of $\prod_{n_1, \ldots, n_r}$ to $H^*_{(k_1, \ldots, k_m), l}$ is identically zero.

Proof. The proof is the same as the proof of the similar statement in [E1], and is based on the quantum Serre relations.

When $q = 1$, $W_I((t_i)_{i \in I})$ can be computed as follows. Set

$$W^\text{Lie}_I((t_i)_{i \in I}) := \frac{[f_{i(n_1)}, f_{i(n_2)}, \ldots, f_{i(n_n)}](t_i)}{(-1 + t_{i_1}/t_{i_2}) \cdot \ldots \cdot (-1 + t_{i_l}/t_{i_1})},$$

where for $x, y \in g$, we set $[x, y](t) = [x[0], y(t)] = [x(t), y[0]]$; then $x(t) = \sum_{n \in \mathbb{Z}} x[n] t^{-n},$ and $x^+(t) := \sum_{n > 0} x[n] t^{-n}$. Then

$$W_I((t_i)_{i \in I}) = \sum_{s \geq 0, (I_1, \ldots, I_s) \vdash I_1 \sqcup \ldots \sqcup I_s, \text{min}(I_1) < \ldots < \text{min}(I_s)} W^\text{Lie}_I((t_i)_{i \in I_1}) \cdots W^\text{Lie}_I((t_i)_{i \in I_s}),$$

(4.10)

where the sum is over all the partitions $I = I_1 \sqcup \ldots \sqcup I_s$, such that $\text{min}(I_1) < \ldots < \text{min}(I_s)$, and $I_i = (I_i, \prec_i, t_i)$ is the ordered $\Pi$-multiset induced by $I_i$.

For $\alpha \in \Pi$, set where $f^+_\alpha(z) := \sum_{n > 0} f^\alpha[z^n] z^{-n}$.

Conjecture 4.4. $W_I((t_i)_{i \in I})$ is a linear combination of noncommutative polynomials in the $f^\alpha(q^k t_i)$, $f^\alpha[0]$ ($\alpha \in \Pi$, $k \in \mathbb{Z}$), where the coefficients have the form $P((t_i)_{i \in I}) / \prod_{i,j \in I, i < j} (t_i - q^{-(\alpha(i), \alpha(j))} t_j)$, and $P((t_i)_{i \in I})$ is a polynomial of degree $|I|(|I| - 1)/2$.

We have $[[f_{i(n_1)}, f_{i(n_2)}], \ldots, f_{i(n_n)}]^+(t) = [[f^+_{i(n_1)}(t), f_{i(n_2)}[0]], \ldots, f_{i(n_n)}[0]]$, hence the Conjecture is true for any $g$ when $q = 1$. According to [KP1], it is also true when $g = sl_2$ or $sl_3$ for any $q \neq 0$. For example, we have in $U_q(sl_2)$ (see [KP1])

$$P(f^\alpha(t_1) f^\alpha(t_2)) = f^\alpha(t_1) f^\alpha(t_2) - \frac{(q - q^{-1}) t_1}{q t_1 - q^{-1} t_2} (f^\alpha(t_1))^2.$$

Remark. (4.10) is also valid if $U(\mathfrak{n}_-[z, z^{-1}])$ is replaced by $U(\mathfrak{n}_-[z, z^{-1}])$, where $\mathfrak{n}_-$ is the free Lie algebra with generators $f_\alpha, \alpha \in \Pi$; this algebra is presented by the relations $(z - w)[f_\alpha(z), f_\beta(w)] = 0$ for any $\alpha, \beta \in \Pi$, so the Serre relations do not play a role in the derivation of (4.10) (where $q = 1$). However, the results of [KP1] use the quantum Serre relations.

Let now $V$ be a finite dimensional representation of $U_q(\mathfrak{g})$ with singular weight vector $v$. Let $I$ be an ordered $\Pi$-multiset and $W^I_V((t_i)_{i \in I})$ be the vector-valued weight function

$$w^I_V((t_i)_{i \in I}) := P(f_{i(n_1)}(t_i) \cdots f_{i(n_n)}(t_i)) v.$$

(4.11)

Proposition 4.5. Assume that Conjecture 4.4 is true. Then $w^I_V((t_i)_{i \in I})$ is the Laurent expansion of a rational function on $\mathbb{C}^n$. There exist rational functions

$$\overline{w}_{n_1, \ldots, n_r}((u_j^{(1)})_{j=1, \ldots, n_1}, \ldots, (u_j^{(r)})_{j=1, \ldots, n_r}),$$

such that the analogue of identity (4.8) holds.

Each function $\overline{w}_{n_1, \ldots, n_r}$ is symmetric in each group of variables $(u_j^{(s)})_{j=1, \ldots, n_s}$. Its only singularities are poles at $u_j^{(s)} \in S_s$, where $S_s \subset \mathbb{C}^s$ is a finite subset of $\mathbb{C}^s$. It vanishes on the spaces $H^*_{(k_1, \ldots, k_m), l}$. 

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Proof. According to the theory of Drinfeld polynomials, the image of \( f_\alpha^+(t) \) in \( \text{End}(V) \) is a rational function in \( t \) with poles in \( \mathbb{C}^\times \). It follows from Conjecture 4.4 that \( w^T ((t_i)_{i \in I}) \) is the Laurent expansion of a rational function, which is regular except for (a) simple poles at \( t_i = q^{-(\iota(i),\iota(j))} t_j \), where \( i < j \), and (b) poles at \( t_i \in S_i \), where \( S_i \subset \mathbb{C} \) is a finite subset. The form of \( W_f \) proved in Proposition 4.2 also implies that \( w^T \) vanishes on the hyperplanes \( t_i = t_j \), where \( \iota(i) = \iota(j) \) (as a formal function, hence as a rational function), and Proposition 4.3 implies that \( w^T \) vanishes on the spaces \( H^{\alpha\beta}_{(k_1,...,k_m),l} \) (as a formal function, hence as a rational function).

Define then \( \overline{w}_{n_1,...,n_r} := \overline{W}_{n_1,...,n_r} v \). Then the analogue of (4.8) holds. It follows from the properties of \( w^T \) that \( \overline{w}_{n_1,...,n_r} \) is rational, with the announced poles structure. Since the Laurent expansion of \( \overline{w}_{n_1,...,n_r} \) is symmetric in each group of variables, so is \( \overline{w}_{n_1,...,n_r} \) itself. \( \square \)

5 Relation to the off-shell Bethe vectors

In this section, we relate the universal weight functions to the off-shell Bethe vectors, in the case of the quantum affine algebra \( U'_q(\widehat{sl}_2) \). The algebra \( U'_q(\widehat{sl}_2) \) is generated by the modes of the currents \( e(z), f(z) \) and \( \psi^\pm(z) \). We will need only the commutation relations between the currents \( f(z), f(w) \) and \( f(z), \psi^+(w) \):

\[
(qz - q^{-1}w)f(z)f(w) = (q^{-1}z - qw)f(w)f(z)
\]

\[
\psi^+(z)f(w) = \frac{q^{-2} - w/z}{1 - q^{-2}w/z} f(w)\psi^+(z)
\]

Using formula (5.1) we may calculate the projection \( P(f(z_1) \cdots f(z_n)) \).

The algebra \( U'_q(\widehat{sl}_2) \) also has a realization in terms of \( L \)-operators ([RS]):

\[
L^\pm(z) = \begin{pmatrix} 1 & f^\pm(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^\pm(zq^{-2})^{-1} & 0 \\ 0 & k^\pm(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^\pm(z) & 1 \end{pmatrix} = \begin{pmatrix} A^\pm(z) & B^\pm(z) \\ C^\pm(z) & D^\pm(z) \end{pmatrix}
\]

which satisfy

\[
R(u/v) \cdot (L'(u) \otimes 1) \cdot (1 \otimes L'(v)) = (1 \otimes L'(v)) \cdot (L'(u) \otimes 1) \cdot R(u/v)
\]

with \( \epsilon, \epsilon' \in \{+, -\} \), and

\[
R(z) = (qz - q^{-1}) (E_{11} \otimes E_{11} + E_{22} \otimes E_{22}) + (z - 1) (E_{11} \otimes E_{12} + E_{22} \otimes E_{11}) + (q - q^{-1}) (zE_{12} \otimes E_{21} + E_{21} \otimes E_{12})
\]

\((E_{ij} \text{ denotes the matrix unit})\).

According to [DE], the Gauss coordinates of the \( L \)-operators are related to the currents as follows

\[
e(z) = e^+(z) - e^-(z), \quad f(z) = f^+(z) - f^-(z), \quad \psi^\pm(z) = (k^\pm(zq^{-2})k^\pm(z))^{-1}
\]

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Let $v$ be a vector such that $C^+(z)v = 0$. The vector-valued function
\[ w(z_1, \ldots, z_n) = B^+(z_1) \cdots B^+(z_n)v \] is called an off-shell Bethe vector. Using the equality $B^+(z) = f^+(z)k^+(z)$ and the relation (3.2) we may present the product (5.3) in terms of the product of the half-currents $f^+(z)$. This gives the relation
\[ B^+(z_1) \cdots B^+(z_n) = \prod_{i<j} \frac{qz_i - q^{-1}z_j}{z_i - z_j} P(f(z_1) \cdots f(z_n)) \prod_{i=1}^n k^+(z_i) \] which shows the relation between the off-shell Bethe vectors and the weight function (1.2). For a general quantum affine algebra, the calculation of the weight functions given by the universal weight function (1.6) is a complicated and interesting problem. Such calculations for quantum affine algebras $U_q(\mathfrak{g})$ will be studied in [KPT].

Appendix

Here we give a proof of the properties (3.25) and (3.26) of circular Cartan-Weyl generators, see Section 3.2. The proof uses the braid group approach to the CW generators, which we describe first.

Let $T_i : U_q(\hat{\mathfrak{g}}) \to U_q(\hat{\mathfrak{g}})$, $i = 0, 1, \ldots, r$, be the Lusztig automorphisms [L2], defined by the formulas
\[
T_i(e_{\alpha_i}) = -e_{-\alpha_i}k^{-1}_{\alpha_i}, \quad T_i(e_{\alpha_j}) = \sum_{p+s=-\alpha_{i,j}} (-1)^p q^s e^{(p)}_{\alpha_i} e^{(s)}_{\alpha_j} \quad i \neq j, \quad (6.5)
\]
\[
T_i(e_{-\alpha_i}) = -k_{\alpha_i}e_{\alpha_i}, \quad T_i(e_{-\alpha_j}) = \sum_{p+s=-\alpha_{i,j}} (-1)^p q^{-s} e^{(s)}_{-\alpha_i} e^{(p)}_{-\alpha_j} \quad i \neq j \quad (6.6)
\]
where $e^{(p)}_{\pm \alpha_i} = e^{p}_{\pm \alpha_i}/[p]_{q!}$.

We attach to the periodic sequence $\ldots, i_{-1}, i_0, i_1, \ldots, i_n, \ldots$ given by (3.13) the sequence $(w_n)_{n \in \mathbb{Z}}$ of elements of the Weyl group, given by $w_0 = w_1 = 1, w_{k+1} = w_k s_i k$ for $k > 0$, and $w_{l-1} = s_i w_l$ for $l \leq 0$. Let $\gamma_k$ be the corresponding positive real roots (3.18). We have a normal ordering $\gamma_1 < \gamma_2 < \ldots < \delta < 2\delta < \ldots < \gamma_{-1} < \gamma_0$ of the system $\sum_k$.

We define real root vectors $e_{\pm \gamma_k}$, where $k > 0$ and $e_{\pm \gamma_l}$, where $l \leq 0$ by the relations
\[ e_{\pm \gamma_k} = T_{w_k}(e_{\pm \alpha_k}), \quad e_{\pm \gamma_l} = T_{w_l}^{-1}(e_{\pm \alpha_l}) \] that is, $e_{\pm \gamma_n} = e_{\pm \alpha_n}$ for $n = 0, 1$; $e_{\pm \gamma_k} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(e_{\pm \alpha_k})$ for $k > 1$, and $e_{\pm \gamma_l} = T_{t_0}^{-1} T_{t_{-1}}^{-1} \cdots T_{t_{-l+1}}^{-1}(e_{\pm \alpha_k})$ for $l < 0$. The imaginary root vectors are defined by the relations (3.22) and (3.21). The imaginary root vectors, related to positive roots, generate an abelian subalgebra $U^+_n \subset U_q(n_+)$. It is characterized by the properties
\[ p \in U^+_n \iff T_{w_k}^{-1}(p) \in U_q(n_+) \quad \text{and} \quad T_{w_l}(p) \in U_q(n_+) \quad \text{for all} \quad k > 0, l \leq 0. \]
The root vectors $6.7$, $3.21$ satisfy the property $3.22$ (see $13e$) and thus coincide, up to normalization, with the CW generators of Section $3.2$

Let $c$ be an integer $> 0$. Let $\ldots, j_{-1}, j_0, j_1, \ldots$ be the periodic sequence defined by the rule $j_n = i_{n-c}$ for all $n \in \mathbb{Z}$, $\{\tilde{w}_n\}$ the related sequence of elements of the Weyl group, given by the rule $\tilde{w}_0 = \tilde{w}_1 = 1$, $\tilde{w}_{k+1} = \tilde{w}_k s_{j_k}$ for $k > 0$, and $\tilde{w}_{-1} = s_{j_1} \tilde{w}_1$ for $l < 0$. Let $\{\tilde{\gamma}_n, n \in \mathbb{Z}\}$ be the corresponding sequence of real positive roots, $\tilde{\gamma}_k = \tilde{w}_k (\alpha_{j_k})$, if $k \geq 0$ and $\tilde{\gamma}_l = \tilde{w}_{-l}^{-1} (\alpha_{j_l})$, if $l \leq 0$. Let $\{\tilde{e}_j\}$ be the CW generators, built by the braid group procedure, related to the sequence $\{j_k\}$; $\tilde{e}_{\pm \tilde{\gamma}_k} = T_{\tilde{w}_k} (e_{\pm \alpha_{j_k}})$ if $k \geq 1$, and $\tilde{e}_{\pm \tilde{\gamma}_l} = T_{\tilde{w}_{-l}}^{-1} (e_{\pm \alpha_{j_l}})$ if $l \leq 0$. Let $\tilde{U}^+_{\text{Im}}$ be the subalgebra of $U_q(n_+)$, generated by the imaginary root vectors $\tilde{e}_{n\hat{i}}$, $i = 1, \ldots, r, n > 0$.

We have the correspondence:

$$\tilde{\gamma}_n = \left\{ \begin{array}{ll} s_{\alpha_{1-c}} \cdots s_{\alpha_{1}} (\gamma_{n-c}), & n \neq 1, 2, \ldots, c, \\ s_{\alpha_{1-c}} \cdots s_{\alpha_{1}} (-\gamma_{n-c}), & n = 1, 2, \ldots, c. \end{array} \right. \quad (6.9)$$

$$\tilde{e}_{\gamma_n} = T_{i_{1-c}} \cdots T_{i_0} (\tilde{e}_{\gamma_{-c}}), \quad n \in \mathbb{Z}. \quad (6.10)$$

Indeed, for $n \neq 1, \ldots, c$ we have $\tilde{e}_{\gamma_n} = T_{i_{1-c}} \cdots T_{i_0} (e_{\gamma_{-c}})$ by the construction. For $n = 1, \ldots, c$ we have $\tilde{e}_{\gamma_n} = T_{i_{1-c}} \cdots T_{i_0} (T_{i_{n-c}}^{-1} T_{i_{n-c}}^{-1} T_{w_0^{-c}}) (e_{\alpha_{n-c}})$, which is equal to $T_{i_{1-c}} \cdots T_{i_0} (\tilde{e}_{\gamma_{-c}})$ by (6.5) and (6.6). The relation (6.10) follows from the description (6.3) of the algebra $U^+_{\text{Im}}$ and its analogue for the algebra $U^+_{\text{Im}}$.

Let $\prec$ be the normal ordering of the system $\tilde{\Sigma}_+$, related to the sequence $\{i_n\}$, $\prec_c$ the corresponding circular order in $\tilde{\Sigma}$, and $\prec$ the normal ordering of $\tilde{\Sigma}_+$, related to the sequence $\{j_n\}$. For any $\tilde{\alpha}, \tilde{\beta} \in \tilde{\Sigma}_+$ we have the correspondence:

$$\tilde{\alpha} \prec \tilde{\beta} \Leftrightarrow \alpha \prec_c \beta, \quad (6.11)$$

where $\alpha = s_{\alpha_{1}} \cdots s_{\alpha_{i-1}} (\tilde{\alpha})$, and $\beta = s_{\alpha_{1}} \cdots s_{\alpha_{i-1}} (\tilde{\beta})$.

Consider the relation (3.22) for CW generators, related to the sequence $\{j_n\}$:

$$[\tilde{e}_{\tilde{\alpha}}, \tilde{e}_{\tilde{\beta}}]_{q-1} = \sum C^{(\tilde{\nu}_j)}_{\{n_j\}} (q) \tilde{e}^{n_{1}}_{\nu_1} \tilde{e}^{n_{2}}_{\nu_2} \cdots \tilde{e}^{n_{m}}_{\nu_m},$$

with $\tilde{\alpha} \prec \tilde{\nu}_1 \prec \cdots \prec \tilde{\nu}_m \prec \tilde{\beta}$, where $C^{(\tilde{\nu}_j)}_{\{n_j\}} (q) \in \mathbb{C}[q, q^{-1}, 1/(q^n - 1); n \geq 1]$. Due to (6.9), (6.10), (6.8), automorphism properties of the maps $T_i$, and commutativity of imaginary root vectors, this is equivalent to the relation on circular generators:

$$[\tilde{e}_{\alpha}, \tilde{e}_{\beta}]_{q-1} = \sum \tilde{C}^{(\nu_j)}_{\{n_j\}} (q) \tilde{e}^{n_{1}}_{\nu_1} \tilde{e}^{n_{2}}_{\nu_2} \cdots \tilde{e}^{n_{m}}_{\nu_m}, \quad (6.12)$$

with $\alpha \prec_c \nu_1 \prec_c \cdots \prec_c \nu_m \prec_c \beta$, where $\alpha = s_{\alpha_{1}} \cdots s_{\alpha_{i-1}} (\tilde{\alpha})$, and $\beta = s_{\alpha_{1}} \cdots s_{\alpha_{i-1}} (\tilde{\beta})$; $C^{(\nu_j)}_{\{n_j\}} (q) \in \mathbb{C}[q, q^{-1}, 1/(q^n - 1); n \geq 1]$. This is a particular case of the relation (3.22), when the root $\alpha$ satisfies the condition $-\delta \prec_c \alpha$ and $\beta$ is positive.

Let $d$ be an integer $> 0$. Let now $\{j_n\}$ be a periodic sequence, related to the sequence (3.17) by the rule $j_n = i_{n+d}$ for all $n$, $\{\tilde{w}_n\}$ the related sequence of elements of the Weyl group, $\{\tilde{\gamma}_n, n \in \mathbb{Z}\}$ the corresponding sequence of real positive roots. Let $\{\tilde{e}_{\pm \gamma}\}$ be CW generators, built by braid group procedure, related to the sequence
\{j_k\}, and \(\hat{U}_\pm\) the subalgebras of \(U_q(n_\pm)\), generated by imaginary root vectors \(\hat{e}_{\pm n}\), \(i = 1, \ldots, r, n > 0\).

We have now the following correspondence:

\[
\tilde{\gamma}_{n-d} = \left\{ \begin{array}{ll} 
 s_{\alpha_1} \cdots s_{\alpha_d}(\gamma_n), & n \neq 1, 2, \ldots, d, \\
 s_{\alpha_1} \cdots s_{\alpha_d}(-\gamma_n), & n = 1, 2, \ldots, d.
\end{array} \right.
\]  

(6.13)

\[
\tilde{e}_{\pm \gamma_{n-d}} = T_{i_d}^{-1} \cdots T_{i_1}^{-1}(e_{\pm \gamma_n}), \
 n \in \mathbb{Z}.
\]  

(6.14)

\[
\tilde{U}_\pm = T_{i_d}^{-1} \cdots T_{i_1}^{-1}(U_\pm),
\]  

(6.15)

where the temporary real root generators \(\tilde{e}_{-\gamma}\) are given by the prescription \(\tilde{e}_{\pm \gamma_n} = e_{\pm \gamma_n}\) for \(n \neq 1, 2, \ldots, d\) and \(\tilde{e}_{\pm \gamma_n} = T_{w_n} T_{i_n}(e_{\pm \alpha_n})\) for \(n = 1, 2, \ldots, d\). Again, the normal ordering \(\prec\), attached to the sequence \(\{j_n\}\), is in accordance with the circular ordering \(\prec\): \(\tilde{\alpha} \prec \tilde{\beta} \iff \alpha \prec c \beta\), where \(\alpha = s_{\alpha_1} \cdots s_{\alpha_d}(\tilde{\alpha})\), and \(\beta = s_{\alpha_1} \cdots s_{\alpha_d}(\tilde{\beta})\). Consider the relation (3.22) for CW generators \(\tilde{e}_\gamma\), related to negative roots \(\tilde{\alpha}\) and \(\tilde{\beta}\):

\[
[\tilde{e}_\alpha, \tilde{e}_\beta]_q = \sum C^{(\nu)}_{(m)}(q) \tilde{e}^{\nu_1} \tilde{e}^{\nu_2} \cdots \tilde{e}^{\nu_m},
\]  

(6.16)

with \(\tilde{\alpha}, \tilde{\beta}, \nu_i, \in -\hat{\Sigma}_\pm\), so that \(-\tilde{\alpha} \prec -\nu_1 \prec \cdots \prec -\nu_m \prec -\tilde{\beta}\). Due to (6.14), (6.15), it is equivalent to

\[
[\tilde{e}_\alpha, \tilde{e}_\beta]_q = \sum \tilde{C}^{(\nu)}_{(m)}(q) \tilde{e}^{\nu_1} \tilde{e}^{\nu_2} \cdots \tilde{e}^{\nu_m},
\]  

(6.17)

with \(\alpha \prec c \nu_1 \prec c \cdots \prec c \nu_m \prec c \beta\), where \(\alpha = s_{\alpha_1} \cdots s_{\alpha_d}(\tilde{\alpha})\), and \(\beta = s_{\alpha_1} \cdots s_{\alpha_d}(\tilde{\beta})\). Since all the roots in (6.16) are negative, the collection of the roots \(\{\alpha, \beta, \nu_1, \ldots, \nu_m\}\) in (6.17) contains negative roots and some positive roots belonging to the set \(\{\gamma_1, \ldots, \gamma_d\}\).

Note, that \(\tilde{e}_\nu = e_\nu\), if \(\nu\) is negative, and \(\tilde{e}_\nu = -k_\nu e_\nu\), if \(\nu \in \{\gamma_1, \ldots, \gamma_d\}\). Now we apply to (6.16) the following automorphism of the algebra \(U_q(\mathfrak{g})\):

\[
e_{-\gamma} \mapsto -k_\gamma e_{-\gamma}, \quad e_\gamma \mapsto -e_\gamma k^{-1}_\gamma, \quad k_\gamma \mapsto k_\gamma, \quad \text{for all } \gamma \in \hat{\Sigma}_c.
\]

One can see, that this automorphism transforms the relation (6.17) to the particular case of (3.22):

\[
[\tilde{e}_\alpha, \tilde{e}_\beta]_q = \sum \tilde{C}^{(\nu)}_{(m)}(q) \tilde{e}^{\nu_1} \tilde{e}^{\nu_2} \cdots \tilde{e}^{\nu_m},
\]  

(6.18)

with \(\alpha \prec c \nu_1 \prec c \cdots \prec c \nu_m \prec c \beta\) and \(\tilde{C}^{(\nu)}_{(m)}(q) \in \mathbb{C}[q, q^{-1}, 1/(q^n - 1); n \geq 1]\), when the root \(\beta\) satisfies the condition \(\beta \prec c \delta\) and \(\alpha\) is negative. The relations (6.12) and (6.17) imply (3.25) in full generality. The proof of (3.26) is analogous.

Remark. There are analogues of all the relations (3.22), (3.25) and (3.26), in which the order of the products (equivalently, the order of the root vectors) in the monomials in the right hand sides are reversed. To derive them, it is sufficient to first apply the Cartan antiinvolution * to (3.22), and then to apply the arguments of the Appendix to the result.

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