WHEN THE ZARISKI SPACE IS A NOETHERIAN SPACE

DARIO SPIRITO

Abstract. We characterize when the Zariski space $\text{Zar}(K|D)$ (where $D$ is an integral domain, $K$ is a field containing $D$ and $D$ is integrally closed in $K$) and the set $\text{Zar}_{\text{min}}(L|D)$ of its minimal elements are Noetherian spaces.

1. Introduction

The Zariski space $\text{Zar}(K|D)$ of the valuation ring of a field $K$ containing a subring $D$ was introduced by O. Zariski (under the name abstract Riemann surface) during its study of resolution of singularities $[21, 23]$. In particular, he introduced a topology on $\text{Zar}(K|D)$ (which was later called Zariski topology) and proved that it makes $\text{Zar}(K|D)$ into a compact space $[26$, Chapter VI, Theorem 40]. Later, the Zariski topology on $\text{Zar}(K|D)$ was studied more carefully, showing that it is a spectral space in the sense of Hochster $[14]$, i.e., that there is a ring $R$ such that the spectrum of $R$ (endowed with the Zariski topology) is homeomorphic to $\text{Zar}(K|D)$ $[4, 5, 6]$. This topology has also been used to study representations of an integral domain by intersection of valuation rings $[16, 17, 18]$ and, for example, in real and rigid algebraic geometry $[15, 21]$.

In $[22]$, it was shown that in many cases $\text{Zar}(D)$ is not a Noetherian space, i.e., there are subspaces of $\text{Zar}(D)$ that are not compact. In particular, it was shown that $\text{Zar}(D) \setminus \{V\}$ (where $V$ is a minimal valuation overring of $D$) is often non-compact: for example, this happens when $\dim(V) > 2 \dim(D)$ $[22$, Proposition 4.3] or when $D$ is Noetherian and $\dim(V) \geq 2$ $[22$, Corollary 5.2].

In this paper, we study integral domains such that $\text{Zar}(D)$ is a Noetherian space, and, more generally, we study when the Zariski space $\text{Zar}(K|D)$ is Noetherian. We show that, if $D = F$ is a field, then $\text{Zar}(K|F)$ can be Noetherian only if the transcendence degree of $K$ over $F$ is at most 1 and, when $\text{trdeg}_F K = 1$, we characterize when this happens in terms of the extensions of the valuation domains of $F[X]$, where $X$ is an element of $K$ transcendental over $F$ (Proposition 4.2).

Date: July 25, 2018.

2010 Mathematics Subject Classification. 13F30, 13A15, 13A18.

Key words and phrases. Zariski space; Noetherian space; pseudo-valuation domains.
In Section 5, we study the case where $K$ is the quotient field of $D$: we first consider the local case, showing that if Zar($D$) is Noetherian then $D$ must be a pseudo-valuation domain (Theorem 5.8) and, subsequently, we globalize this result to the non-local case, showing that Zar($D$) is Noetherian if and only if so are Spec($D$) and Zar($D_M$), for every maximal ideal $M$ of $D$ (Theorem 5.11 and Corollary 5.12). We also prove the analogous results for the set Zar$_{\text{min}}(K|D)$ of the minimal elements of Zar($K|D$).

2. Background

2.1. Overrings and the Zariski space. Let $D$ be an integral domain and let $K$ be a ring containing $D$. We define Over($K|D$) as the set of rings contained between $D$ and $K$. The Zariski topology on Over($K|D$) is the topology having, as a subbasis of closed sets, the sets in the form

$$B(x_1, \ldots, x_n) := \{ V \in \text{Over}(K|D) \mid x_1, \ldots, x_n \in V \},$$

as $x_1, \ldots, x_n$ range in $K$. If $K$ is the quotient field of $D$, an element of Over($K|D$) is called an overring of $D$.

If $K$ is the quotient field of $D$, a subset $X \subseteq \text{Over}(K|D)$ is a locally finite family if every $x \in D$ (or, equivalently, every $x \in K$) is a non-unit in only finitely many $T \in \text{Over}(K|D)$.

If $K$ is a field containing $D$, the Zariski space of $D$ in $K$ is the set of all valuation domains containing $D$ and whose quotient field is $K$; we denote it by Zar($K|D$). The Zariski topology on Zar($K|D$) is simply the Zariski topology inherited from Over($K|D$). If $K$ is the quotient field of $D$, then Zar($K|D$) will simply be denoted by Zar($D$), and its elements are called the valuation overrings of $D$.

Under the Zariski topology, Zar($K|D$) is compact [26, Chapter VI, Theorem 40].

We denote by Zar$_{\text{min}}(K|D)$ the set of minimal elements of Zar($K|D$), with respect to containment. If $V$ is a valuation domain, we denote by $m_V$ its maximal ideal. Given $X \subseteq \text{Zar}(D)$, we define

$$X^\dagger := \{ V \in \text{Zar}(D) \mid V \supseteq W \text{ for some } W \in X \}.$$

Since a family of open sets is a cover of $X$ if and only if it is a cover of $X^\dagger$, we have that $X$ is compact if and only if $X^\dagger$ is compact.

If $X$ is a subset of Zar($D$), we denote by $A(X)$ the intersection $\bigcap\{ V \mid V \in X \}$, called the holomorphy ring of $X$ [20]. Clearly, $A(X) = A(X^\dagger)$.

The center map is the application

$$\gamma: \text{Zar}(K|D) \longrightarrow \text{Spec}(D)$$

$$V \mapsto m_V \cap D.$$

If Zar($K|D$) and Spec($D$) are endowed with the respective Zariski topologies, the map $\gamma$ is continuous ([26, Chapter VI, §17, Lemma]...
1) or [4, Lemma 2.1]), surjective (this follows, for example, from [2, Theorem 5.21] or [11, Theorem 19.6]) and closed [4, Theorem 2.5].

In studying \( \text{Zar}(K|D) \), it is usually enough to consider the case where \( D \) is integrally closed in \( K \); indeed, if \( \overline{D} \) is the integral closure of \( D \) in \( K \), then \( \text{Zar}(K|D) = \text{Zar}(K|\overline{D}) \).

2.2. **Noetherian spaces.** A topological space \( X \) is **Noetherian** if its open sets satisfy the ascending chain condition, or equivalently if all its subsets are compact. If \( X = \text{Spec}(R) \) is the spectrum of a ring, then \( X \) is a Noetherian space if and only if \( R \) satisfies the ascending chain condition on radical ideals; in particular, the spectrum of a Noetherian ring is always a Noetherian space. If \( \text{Spec}(R) \) is Noetherian, then every ideal of \( R \) has only finitely many minimal primes (see e.g. the proof of [3, Chapter 4, Corollary 3, p.102] or [2, Chapter 6, Exercises 5 and 7]).

Every subspace and every continuous image of a Noetherian space is again Noetherian; in particular, if \( \text{Zar}(D) \) is Noetherian then so are \( \text{Zar}_{\text{min}}(D) \) and \( \text{Spec}(D) \) [22, Proposition 4.1].

2.3. **Kronecker function rings.** Let \( K \) be the quotient field of \( D \). For every \( V \in \text{Zar}(D) \), let \( V^b := V[X]_{mV[X]} \subseteq K(X) \). If \( \Delta \subseteq \text{Zar}(D) \), the **Kronecker function ring** of \( D \) with respect to \( \Delta \) is \( \text{Kr}(D, \Delta) := \bigcap \{ V^b \mid V \in \Delta \} \);

we denote \( \text{Kr}(D, \text{Zar}(D)) \) simply by \( \text{Kr}(D) \).

The ring \( \text{Kr}(D, \Delta) \) is always a Bézout domain whose quotient field is \( K(X) \), and, if \( \Delta \) is compact, the intersection map \( W \mapsto W \cap K \) establishes a homeomorphism between \( \text{Zar}(\text{Kr}(D, \Delta)) \) and the set \( \Delta^\uparrow \) [4, 5, 6]. Since \( \text{Kr}(D, \Delta) \) is a Prüfer domain, furthermore, \( \text{Zar}(\text{Kr}(D, \Delta)) \) is homeomorphic to \( \text{Spec}(\text{Kr}(D, \Delta)) \); hence, \( \text{Spec}(\text{Kr}(D, \Delta)) \) is homeomorphic to \( \Delta^\uparrow \), and asking if \( \text{Zar}(D) \) is Noetherian is equivalent to asking if \( \text{Spec}(\text{Kr}(D)) \) is Noetherian or, equivalently, if \( \text{Kr}(D) \) satisfies the ascending chain condition on radical ideals.

See [11, Chapter 32] or [10] for general properties of Kronecker function rings.

2.4. **Pseudo-valuation domains.** Let \( D \) be an integral domain with quotient field \( K \). Then, \( D \) is called a **pseudo-valuation domain** (for short, PVD) if, for every prime ideal \( P \) of \( D \), whenever \( xy \in P \) for some \( x, y \in K \), then at least one between \( x \) and \( y \) is in \( P \). Equivalently, \( D \) is a pseudo-valuation domain if and only if it is local and its maximal ideal \( M \) is also the maximal ideal of some valuation overring \( V \) of \( D \) (called the valuation domain **associated to** \( D \)) [12, Corollary 1.3 and Theorem 2.7]. If \( D \) is a valuation domain, then it is also a PVD, and the associated valuation ring is \( D \) itself.
The prototypical examples of a pseudo-valuation domain that is not a valuation domain is the ring $F + XL[[X]]$, where $F \subseteq L$ is a field extension; its associated valuation domain is $L[[X]]$.

3. Examples and reduction

The easiest case for the study of the topology of Zar($D$) is when $D$ is a Prüfer domain, i.e., when $D_M$ is a valuation domain for every maximal ideal $M$ of $D$.

**Proposition 3.1.** Let $D$ be a Prüfer domain. Then:

(a) Zar($D$) is a Noetherian space if and only if Spec($D$) is Noetherian;

(b) Zar$_{min}$($D$) is Noetherian if and only if Max($D$) is Noetherian.

**Proof.** Since $D$ is Prüfer, the center map $\gamma :$ Zar($D$) $\rightarrow$ Spec($D$) is a homeomorphism [4, Proposition 2.2]. This proves the first claim; the second one follows from the fact that the minimal valuation overrings of $D$ correspond to the maximal ideals. □

Another example of a domain that has a Noetherian Zariski space is the pseudo-valuation domain $D := \mathbb{Q} + Y\mathbb{Q}(X)[[Y]]$, where $X, Y$ are indeterminates on $\mathbb{Q}$, since in this case Zar($D$) can be written as the union of the quotient field of $D$ and two sets homeomorphic to Zar($\mathbb{Q}[X]$) $\simeq$ Spec($\mathbb{Q}[X]$), which are Noetherian; from this, it is possible to build examples of non-Prüfer domain whose Zariski spectrum is Noetherian, and having arbitrary finite dimension [22, Example 4.7].

More generally, we have the following result, which is probably well-known.

**Lemma 3.2.** Let $D$ be an integral domain, and suppose that a prime ideal $P$ of $D$ is also the maximal ideal of a valuation overring $V$ of $D$. Then, the quotient map $\pi : V \rightarrow V/P$ establishes a homeomorphism between $\{W \in$ Zar($D$) $| W \subseteq V\}$ and Zar($V/P|D/P$), and between Zar$_{min}$($D$) and Zar$_{min}$($V/P|D/P$).

**Proof.** Consider the set Over($V|D$) and Over($V/P|D/P$). Then, the map

$$\overline{\pi} : \text{Over}(V|D) \rightarrow \text{Over}(V/P|D/P)$$

$$A \mapsto \pi(A) = A/P$$

is a bijection, whose inverse is the map sending $B$ to $\pi^{-1}(B)$. Furthermore, it is a homeomorphism: indeed, if $x \in V/P$ then $\overline{\pi}^{-1}(B(x)) = B(y)$, for any $y \in \pi^{-1}(x)$, while if $x \in V$ then $\overline{\pi}(B(x)) = B(\pi(x))$.

The condition on $P$ implies that $D$ is the pullback of the diagram

$$\begin{array}{ccc}
D & \xrightarrow{\pi} & D/P \\
\downarrow & & \downarrow \\
V & \xrightarrow{\pi} & V/P;
\end{array}$$
hence, every $A \in \text{Over}(V|D)$ arises as a pullback. By [8, Theorem 2.4(1)], $A$ is a valuation domain if and only if $\pi(A)$ is a valuation domain and $V/P$ is the quotient field of $\pi(A)$; hence, $\bar{\pi}$ restricts to a bijection between $\text{Zar}(D) \cap \text{Over}(V|D) = \{W \in \text{Zar}(D) \mid W \subseteq V\}$ and $\text{Zar}(V/P|D/P)$. Furthermore, since $\bar{\pi}$ is a homeomorphism, so is its restriction. The claim about $\text{Zar}(D)$ and $\text{Zar}(V/P|D/P)$ is proved; the claim for the space of minimal elements follows immediately.  

**Proposition 3.3.** Let $D$ be an integral domain, and let $L$ be a field containing $D$. Then, there is a ring $R$ such that:
\begin{itemize}
  \item $\text{Zar}(L|D) \simeq \text{Zar}(R) \setminus \{F\}$, where $F$ is the quotient field of $R$;
  \item $\text{Zar}_{\min}(L|D) \simeq \text{Zar}_{\min}(R)$.
\end{itemize}

**Proof.** Let $X$ be an indeterminate over $L$, and define $R := D + XL[[X]]$. Then, the prime ideal $P := XL[[X]]$ of $R$ is also a prime ideal of the valuation domain $L[[X]]$; by Lemma 3.2, it follows that $\text{Zar}(L|D) \simeq \Delta := \{W \in \text{Zar}(R) \mid W \subseteq L[[X]]\}$. Furthermore, every valuation overring $V$ of $R$ contains $XL[[X]]$, and thus it is either in $\Delta$ or properly contains $L[[X]]$; however, since $L[[X]]$ has dimension 1, the latter case is possible only if $V = L((X))$ is the quotient field of $R$. The first claim is proved, and the second follows easily.  

While Proposition 3.3 shows that (theoretically) we only need to consider spaces of valuation overrings, it is usually easier to not be restricted to this case; the following Proposition 3.4 is an example, as will be the analysis of field extensions done in Section 4.

**Proposition 3.4.** Let $D$ be an integral domain that is not a field, let $K$ be its quotient field and $L$ a field extension of $K$. If $\text{trdeg}_KL \geq 1$, then $\text{Zar}(L|D)$ and $\text{Zar}_{\min}(L|D)$ are not Noetherian.

**Proof.** If $\text{trdeg}_KL \geq 1$, there is an element $X \in L \setminus K$ that is not algebraic over $L$. If $\text{Zar}(L|D)$ is Noetherian, so is its subset $\text{Zar}(L|D[X])$, and thus also $\text{Zar}(K(X)|D[X]) = \text{Zar}(D[X])$, which is the (continuous) image of $\text{Zar}(L|D[X])$ under the intersection map $W \mapsto W \cap K(X)$. However, since $D$ is not a field, $\text{Zar}(D[X])$ is not Noetherian by [22, Proposition 5.4]; hence, neither $\text{Zar}(L|D)$ can be Noetherian.

Consider now $\text{Zar}_{\min}(L|D)$: it projects onto $\text{Zar}_{\min}(K(X)|D)$, and thus we can suppose that $L = K(X)$. Let $V$ be a minimal valuation overring of $D$: then, there is an extension $W$ of $V$ to $L$ such that $X$ is the generator of the maximal ideal of $W$; furthermore, $W$ belongs to $\text{Zar}_{\min}(K(X)|D)$. In particular, $\text{Spec}(W) \setminus \text{Max}(W)$ has a maximum, say $P$. Let $\Delta := \text{Zar}(L|D) \setminus \{W\}$: then, $\Delta$ can be written as the union of $\Lambda := (\text{Zar}_{\min}(L|D) \setminus \{W\})^\dagger$ and $\{W_P\}^\dagger$. The latter is compact since $\{W_P\}$ is compact; if $\text{Zar}_{\min}(L|D) \setminus \{W\}$ were compact, so would be $\Lambda$. In this case, also $\Delta$ would be compact, against the proof of [22, Proposition 5.4]. Hence, $\Delta$ is not compact, and so $\text{Zar}_{\min}(L|D)$ is not Noetherian.  

\[\square\]
4. Field extensions

In this section, we consider a field extension $F \subseteq L$ and analyze when the Zariski space $\text{Zar}(L|F)$ and its subset $\text{Zar}_{\text{min}}(L|F)$ are Noetherian. By Proposition 3.3, this is equivalent to studying the Zariski space of the pseudo-valuation domain $F + XL[[X]]$.

This problem naturally splits into three cases, according to whether the transcendence degree of $L$ over $F$ is 0, 1 or at least 2. The first and the last cases have definite answers, and we collect them in the following proposition. Part (b) is a slight generalization of [22, Corollary 5.5(b)].

**Proposition 4.1.** Let $F \subseteq L$ be a field extension.

(a) If $\text{trdeg}_F L = 0$, then $\text{Zar}(L|F) = \{L\} = \text{Zar}_{\text{min}}(L|D)$, and in particular they are Noetherian.

(b) If $\text{trdeg}_F L \geq 2$, then $\text{Zar}(L|F)$ and $\text{Zar}_{\text{min}}(L|F)$ are not Noetherian.

**Proof.** (a) is obvious. For (b), let $X, Y$ be elements of $L$ that are algebraically independent. Then, the intersection map $\text{Zar}_{\text{min}}(L|F) \rightarrow \text{Zar}_{\text{min}}(F(X,Y)|F)$ is surjective, and thus it is enough to prove that $\text{Zar}_{\text{min}}(F(X,Y)|F)$ is not Noetherian.

Let $V \in \text{Zar}_{\text{min}}(F(X,Y)|F)$ and, without loss of generality, suppose $X, Y \in V$. Let $\Delta := \text{Zar}_{\text{min}}(F(X,Y)|F) \setminus \{V\}$. Then, $\Lambda := \text{Zar}(F(X,Y)|F) \setminus \{V\}$ is the union of $\Delta$ and a finite set (the valuation domains properly containing $V$). If $\Delta$ were compact, so would be $\Lambda$; hence, so would be $\Lambda \cap \text{Zar}(F[X,Y])$ (since both $\Lambda$ and $\text{Zar}(F[X,Y])$ would be closed in the inverse topology; see e.g. [6, Remark 2.2 and Proposition 2.6]). However, $\Lambda \cap \text{Zar}(F[X,Y]) = \text{Zar}(F[X,Y]) \setminus \{V\}$, which is not compact by the proof of [22, Proposition 5.4]. Hence, $\Lambda$ is not compact, and thus $\Delta$ cannot be compact. Hence, $\text{Zar}_{\text{min}}(F(X,Y)|F)$ is not Noetherian. \qed

On the other hand, the case of transcendence degree 1 is more subtle. In [22 Corollary 5.5(a)], it was showed that $\text{Zar}(L|F)$ is Noetherian if $L$ is finitely generated over $F$; we now state a characterization.

**Proposition 4.2.** Let $F \subseteq L$ be a field extension such that $\text{trdeg}_F L = 1$. Then, the following are equivalent:

(i) $\text{Zar}(L|F)$ is Noetherian;

(ii) $\text{Zar}_{\text{min}}(L|F)$ is Noetherian;

(iii) for every $X \in L$ transcendental over $F$, every valuation on $F[X]$ has only finitely many extensions to $L$;

(iv) there is an $X \in L$, transcendental over $F$, such that every valuation on $F[X]$ has only finitely many extensions to $L$;

(v) for every $X \in L$ transcendental over $F$, the integral closure of $F[X]$ in $L$ has Noetherian spectrum;

(vi) there is an $X \in L$, transcendental over $F$, such that the integral closure of $F[X]$ in $L$ has Noetherian spectrum.
Proof. Every valuation domain of $L$ containing $F$ must contain the algebraic closure of $F$ is $L$; hence, without loss of generality we can suppose that $F$ is algebraically closed in $L$.

(i) $\Rightarrow$ (ii) is obvious; (ii) $\Rightarrow$ (i) follows since (being trdeg$_F L = 1$) $\text{Zar}(L|F) = \text{Zar}_{\text{min}}(L|F) \cup \{L\}$.

(i) $\Rightarrow$ (iii) Take $X \in L \setminus F$, and suppose there is a valuation $w$ on $F[X]$ with infinitely many extensions to $L$; let $W$ be the valuation domain corresponding to $w$. Then, the integral closure $\overline{W}$ of $W$ in $L$ would have infinitely many maximal ideals. Since every maximal ideal of $\overline{W}$ contains the maximal ideal of $W$, the Jacobson radical $J$ of $\overline{W}$ contains the maximal ideal of $W$, and in particular it is nonzero. It follows that $J$ has infinitely many minimal primes; hence, $\text{Max}(\overline{W})$ is not a Noetherian space. However, $\text{Max}(\overline{W})$ is homeomorphic to a subspace of $\text{Zar}(L|F)$, which is Noetherian by hypothesis; this is a contradiction, and so every valuation has only finitely many extensions.

(iii) $\Rightarrow$ (v). Let $T$ be the integral closure of $F[X]$. If $\text{Spec}(T)$ is not Noetherian, then $T$ is not locally finite; i.e., there is an $\alpha \in T$ such that there are infinitely many maximal ideals of $T$ containing $\alpha$. Consider the norm $N(\alpha)$ of $\alpha$ over $F[X]$, i.e., the product of the algebraic conjugates of $\alpha$ over $F[X]$. Then, $N(\alpha) \neq 0$, and it is both an element of $F[X]$ (being equal to the constant term of the minimal polynomial of $F[X]$ over $\alpha$) and an element of every maximal ideal containing $\alpha$ (since all the conjugates are in $T$). Since every maximal ideal of $F[X]$ is contained in only finitely many maximal ideals of $T$ (since a maximal ideal of $F[X]$ correspond to a valuation $v$ and the maximal ideals of $T$ containing it to the extensions of $v$), it follows that $N(\alpha)$ is contained in infinitely many maximal ideals of $F[X]$. However, this contradicts the Noetherianity of $\text{Spec}(F[X])$; hence, $\text{Spec}(T)$ is Noetherian.

Now (iii) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (vi) are obvious, while the proof of (iv) $\Rightarrow$ (vi) is exactly the same as the previous paragraph; hence, we need only to show (vi) $\Rightarrow$ (i); the proof is similar to the one of [22, Corollary 5.5(a)].

Let $X \in L$, $X$ transcendental over $F$, be such that the spectrum of the integral closure $T$ of $F[X]$ is Noetherian. Since $X$ is transcendental over $F$, there is an $F$-isomorphism $\phi$ of $F(X)$ sending $X$ to $X^{-1}$; moreover, we can extend $\phi$ to an $F$-isomorphism $\overline{\phi}$ of $L$. Since $\phi(F[X]) = F[X^{-1}]$, the integral closure $T$ of $F[X]$ is sent by $\overline{\phi}$ to the integral closure $T'$ of $F[X^{-1}]$; in particular, $T \simeq T'$, and $\text{Spec}(T) \simeq \text{Spec}(T')$. Thus, also $\text{Spec}(T')$ is Noetherian, and so is $\text{Spec}(T) \cup \text{Spec}(T')$. Furthermore, $\text{Zar}(T) \simeq \text{Spec}(T) \simeq \text{Spec}(L|F[X])$, and analogously for $T'$; hence, $\text{Zar}(T) \cup \text{Zar}(T')$ is Noetherian. But every $W \in \text{Zar}(L|F)$ contains at least one between $X$ and $X^{-1}$, and thus $W$ contains $F[X]$ or $F[X^{-1}]$; i.e., $W \in \text{Zar}(T)$ or $W \in \text{Zar}(T')$. Hence, $\text{Zar}(L|F) = \text{Zar}(T) \cup \text{Zar}(T')$ is Noetherian. □
We remark that there are field extensions that satisfy the conditions of Proposition 4.2 without being finitely generated. For example, if \( L \) is purely inseparable over some \( F(X) \), then every valuation on \( F(X) \) extends uniquely to \( L \), and thus condition [iii] of the previous proposition is fulfilled; more generally, each valuation on \( F(X) \) extends in only finitely many ways when the separable degree \( [L : F(X)] \) is finite [11, Corollary 20.3]. There are also examples in characteristic 0: for example, [19, Section 12.2] gives examples of non-finitely generated algebraic extension \( F \) of the rational numbers such that every valuation on \( \mathbb{Q} \) has only finitely many extensions to \( F \). The same construction works also on \( \mathbb{Q}(X) \), and if \( L \) is such an example then \( \mathbb{Q} \subseteq L \) will satisfy the conditions of Proposition 4.2.

5. The domain case

We now want to study when the space \( \text{Zar}(D) \) is Noetherian, where \( D \) is an integral domain; without loss of generality, we can suppose that \( D \) is integrally closed, since \( \text{Zar}(D) = \text{Zar}(\overline{D}) \). We start by studying intersections of Noetherian families of valuation rings.

Recall that a treed domain is an integral domain whose spectrum is a tree (i.e., such that, if \( P \) and \( Q \) are non-comparable prime ideals, then they are coprime). In particular, every Prüfer domain is treed.

**Lemma 5.1.** Let \( R \) be a treed domain. If \( \text{Max}(R) \) is Noetherian, then every ideal of \( R \) has only finitely many minimal primes.

*Proof.* Let \( I \) be an ideal of \( R \), and let \( \{ P_\alpha \mid \alpha \in A \} \) be the set of its minimal prime ideals. For every \( \alpha \), choose a maximal ideal \( M_\alpha \) containing \( P_\alpha \); note that \( M_\alpha \neq M_\beta \) if \( \alpha \neq \beta \), since \( R \) is treed. Let \( \Lambda \) be the set of the \( M_\alpha \).

Let \( X \subseteq \Lambda \), and define \( J(X) := \bigcap \{ IR_M \mid M \in X \} \cap R \): we claim that, if \( M \in \Lambda \), then \( J(X) \subseteq M \) if and only if \( M \in X \). Indeed, clearly \( J(X) \) is contained in every element of \( X \). On the other hand, suppose \( N \in \Lambda \setminus X \). Since \( \text{Max}(R) \) is Noetherian, \( X \) is compact, and thus also \( \{ R_M \mid M \in X \} \) is compact; by [7, Corollary 5],

\[
J(X)R_N = \left( \bigcap_{M \in X} IR_M \right) R_N \cap R_N = \bigcap_{M \in X} IR_M R_N \cap R_N
\]

Since \( M, N \in \Lambda \), no prime contained in both \( M \) and \( N \) contains \( I \); hence, \( IR_M R_N \) contains 1 for each \( M \in X \). Therefore, 1 \( \in J(X)R_N \), i.e., \( J(X) \nsubseteq N \).
Hence, every subset \( X \) of \( \Lambda \) is closed in \( \Lambda \), since it is equal to the intersection between \( \Lambda \) and the closed set of \( \text{Spec}(R) \) determined by \( J(X) \). Since \( \Lambda \) is Noetherian, it follows that \( \Lambda \) must be finite; hence, also the set of minimal primes of \( I \) is finite. The claim is proved. \( \square \)

**Lemma 5.2.** Let \( D \) be an integral domain with quotient field \( K \), and let \( V, W \in \text{Zar}(D) \). If \( VW = K \), then \( V^bW^b = K(X) \).

**Proof.** Let \( Z := V^bW^b \). Then, since \( \text{Zar}(D) \) and \( \text{Zar}(\text{Kr}(D)) \) are homeomorphic, \( Z = (Z \cap K)^b \); however, \( K \subseteq VW \subseteq V^bW^b \), and thus \( Z \cap K = K \). It follows that \( Z = K^b = K(X) \), as claimed. \( \square \)

A consequence of Lemma 5.1 is the following generalization of [16, Theorem 3.4(2)].

**Theorem 5.3.** Let \( \Delta \subseteq \text{Zar}(D) \) be a Noetherian space, and suppose that \( VW = K \) for every \( V \not= W \) in \( \Delta \). Then, \( \Delta \) is a locally finite space.

**Proof.** Let \( \Delta^b := \{ V^b \mid V \in \Delta \} \), and let \( R := \text{Kr}(D, \Delta) \); then (since, in particular, \( \Delta \) is compact), \( \text{Zar}(R) \) is equal to \( (\Delta^b)^\uparrow \).

Since \( R \) is a Bézout domain, it follows that \( \text{Spec}(R) \simeq (\Delta^b)^\uparrow \), while \( \text{Max}(R) \simeq \Delta^b \); in particular, \( \text{Max}(R) \) is Noetherian, and thus by Lemma 5.1 every ideal of \( R \) has only finitely many minimal primes. However, since \( V^bW^b = K(X) \) for every \( V \not= W \) in \( \Delta \) (by Lemma 5.2), it follows that every nonzero prime of \( \bar{R} \) is contained in only one maximal ideal; therefore, every ideal of \( R \) is contained in only finitely many maximal ideals, and thus the family \( \{ R_M \mid M \in \text{Max}(R) \} \) is locally finite. This family coincides with \( \Delta^b \); since \( \Delta^b \) is locally finite, also \( \Delta \) must be locally finite, as claimed. \( \square \)

We say that two valuation domains \( V, W \in \text{Zar}(D) \setminus \{ K \} \) are dependent if \( VW \not= K \). Since \( \text{Zar}(D) \) is a tree, being dependent is an equivalence relation on \( \text{Zar}(D) \setminus \{ K \} \); we call an equivalence class a dependency class. If \( \text{Zar}(D) \) is finite-dimensional (i.e., if every valuation overring of \( D \) has finite dimension) then the dependency classes of \( \text{Zar}(D) \) are exactly the sets in the form \( \{ W \in \text{Zar}(D) \mid W \subseteq V \} \), as \( V \) ranges among the one-dimensional valuation overrings of \( D \).

Under this terminology, the previous theorem implies that, if \( D \) is local and \( \text{Zar}(D) \) is Noetherian, then \( \text{Zar}(D) \) can only have finitely many dependency classes: indeed, otherwise, we could form a Noetherian but not locally finite subset of \( \text{Zar}(D) \) by taking one minimal overring in each dependency class, against the theorem. We actually can say (and will need) something more.

Given a set \( X \subseteq \text{Zar}(D) \), we define \( \text{comp}(X) \) as the set of all valuation overrings of \( D \) that are comparable with some elements of \( X \); i.e.,

\[
\text{comp}(X) := \{ W \in \text{Zar}(D) \mid \exists V \in X \text{ such that } W \subseteq V \text{ or } V \subseteq W \}.
\]
If $X = \{V\}$ is a singleton, we write $\text{comp}(V)$ for $\text{comp}(X)$. Note that, for every subset $X$, $\text{comp}(\text{comp}(X)) = \text{Zar}(D)$, since $\text{comp}(X)$ contains the quotient field of $D$.

The purpose of the following propositions is to show that, if $D$ is local and Zar($D$) is Noetherian, then Zar($D$) can be written as comp($W$) for some valuation overring $W \neq K$. The first step is showing that Zar($D$) is equal to comp($X$) for some finite $X$.

**Proposition 5.4.** Let $D$ be a local integral domain. If Zar$_{\text{min}}(D)$ is Noetherian, then there are valuation overrings $W_1, \ldots, W_n$ of $D$, $W_i \neq K$, such that Zar($D$) = comp($W_1$) $\cup \cdots \cup$ comp($W_n$).

**Proof.** Let $R := \text{Kr}(D)$ be the Kronecker function ring of $D$. Then, the extension $N := MR$ of the maximal ideal $M$ of $D$ is a proper ideal of $R$, and the prime ideals containing $N$ correspond to the valuation overrings of $R$ where $N$ survives, i.e., to the valuation overrings of $D$ centered on $M$.

Since Zar$_{\text{min}}(D)$ is Noetherian, so is Max($R$); since $R$ is treed (being a Bézout domain), by Lemma 5.1, $N$ has only finitely many minimal primes. Thus, there are finitely many valuation overrings of $D$, say $W_1, \ldots, W_n$, such that every $V \in$ Zar$_{\text{min}}(D)$ is contained in one $W_i$. We claim that Zar($D$) = comp($W_1$) $\cup \cdots \cup$ comp($W_n$). Indeed, let $V$ be a valuation overring of $D$. Since Zar($D$) is compact, $V$ contains some minimal valuation overring $V'$, and by construction $V' \in$ comp($W_i$) for some $i$; in particular, $W_i \supseteq V'$. The valuation overrings containing $V'$ (i.e., the valuation overrings of $V''$) are linearly ordered; thus, $V$ must be comparable with $W_i$, i.e., $V \in$ comp($W_i$). The claim is proved. $\square$

The following result can be seen as a generalization of the classical fact that, if $X = \{V_1, \ldots, V_n\}$ is finite, then Zar($A(X)$) is the union of the various Zar($V_i$) (since $A(X)$ will be a Prüfer domain and its localization at the maximal ideals will be a subset of $X$).

**Proposition 5.5.** Let $D$ be an integral domain and let $X \subseteq$ Zar($D$) be a finite set. Then, Zar($A(\text{comp}(X))$) = comp($X$).

**Proof.** Since comp($V$) $\subseteq$ comp($W$) if $V \subseteq W$, we can suppose without loss of generality that the elements of $X$ are pairwise incomparable. Let $X = \{V_1, \ldots, V_n\}$, $A_i := A(\text{comp}(V_i))$ and let $A := A(\text{comp}(X)) = A_1 \cap \cdots \cap A_n$. Note that $D \subseteq A$, and thus the quotient field of $A$ coincides with the quotient field of $D$ and of the $V_i$.

If $V \in$ comp($X$), then clearly $A \subseteq V$; thus, comp($X$) $\subseteq$ Zar($A$).

Conversely, let $V \in$ Zar($A$), and let $m_i$ be the maximal ideal of $V_i$. Then, $m_i \subseteq W$ for every $W \in$ comp($V_i$); in particular, $m_i \subseteq A_i$. Therefore, $P := m_1 \cap \cdots \cap m_n \subseteq A$; since $A \subseteq V$, this implies that $PV \subseteq V$.

Suppose $V \notin$ comp($X$), and let $T := V \cap V_1 \cap \cdots \cap V_n$. Since the rings $V, V_1, \ldots, V_n$ are pairwise incomparable, $T$ is a Bézout domain whose
localizations at the maximal ideals are \( V, V_1, \ldots, V_n \). In particular, \( V \) is flat over \( T \), and each \( m_i \) is a \( T \)-module; hence,

\[
P V = \left( \bigcap_{i=1}^{n} m_i \right) V = \bigcap_{i=1}^{n} m_i V.
\]

Since \( V \) is not comparable with \( V_i \), for each \( i \), the set \( m_i \) is not contained in \( V_i \); in particular, the family \( \{ m_i V \mid i = 1, \ldots, n \} \) is a family of \( V \)-modules not contained in \( V_i \). Since the \( V \)-submodules of the quotient field \( K \) are linearly ordered, the family has a minimum, and thus \( \bigcap_{i=1}^{n} m_i V \) is not contained in \( V_i \). However, this contradicts \( PV \subseteq V_i \); hence, \( V \) must be in \( \text{comp}(X) \), and \( \text{Zar}(A) = \text{comp}(X) \). \( \square \)

The proof of part \([\text{a}]\) of the following proposition closely follows the proof of \([13, \text{Proposition 1.19}]\).

**Proposition 5.6.** Let \( X := \{ V_1, \ldots, V_n \} \) be a finite family of valuation overrings of the domain \( D \), and suppose that \( V_i V_j = K \) for every \( i \neq j \), where \( K \) is the quotient field of \( D \). Let \( A_i := A(\text{comp}(V_i)) \), and let \( A := A(\text{comp}(X)) \). Then:

(a) each \( A_i \) is a localization of \( A \);
(b) for each ideal \( I \) of \( A \), there is an \( i \) such that \( IA_i \neq A_i \);
(c) if \( i \neq j \), then \( A_i A_j = K \).

**Proof.** \([\text{a}]\) By induction and symmetry, it is enough to prove that \( B := A_2 \cap \cdots \cap A_n \) is a localization of \( A \). Let \( J \) be the Jacobson radical of \( B \); then, \( J \neq (0) \), since it contains the intersection \( m_2 \cap \cdots \cap m_n \). Furthermore, if \( W \neq K \) is a valuation overring of \( V_1 \), then \( J \not\subseteq W \), since otherwise (as in the proof of Proposition 5.5) \( m_2 \cap \cdots \cap m_n \) would be contained in \( m_W \cap (W \cap V_2 \cap \cdots \cap V_n) \), against the fact that \( \{ W, V_2, \ldots, V_n \} \) are independent valuation overrings.

Hence, for every such \( W \) we can apply \([13, \text{Proposition 1.13}]\) to \( D := B \cap W \), obtaining that \( B \) is a localization of \( D \), say \( B = S^{-1} D \), where \( S \) is a multiplicatively closed subset of \( D \); in particular, there is a \( s_W \in S \cap m_W \). Each \( s_W \) is in \( B \cap A_1 = A \) (since \( m_W \) is contained in every member of \( \text{comp}(V_1) \)); let \( T \) be the set of all \( s_W \). Then,

\[
T^{-1} A = T^{-1}(B \cap A_2) = T^{-1} B \cap T^{-1} A_1.
\]

Each \( s_W \) is a unit of \( B \), and thus \( T^{-1} B = B \). On the other hand, no valuation overring \( W \neq K \) of \( V_i \) can be an overring of \( T^{-1} A_1 \), since \( T \) contains \( s_W \), which is inside the maximal ideal of \( W \). Since \( \text{Zar}(A_1) = \text{comp}(V_1) \), it follows that \( T^{-1} A_1 = K \), and thus \( T^{-1} A = B \); in particular, \( B \) is a localization of \( A \).

\([\text{b}]\) Without loss of generality, we can suppose \( I = P \) to be prime. There is a valuation overring \( W \) of \( A \) whose center on \( A \) is \( P \); since \( \text{Zar}(A) = \text{comp}(X) \) by Proposition 5.5, there is a \( V_i \) such that \( W \in \text{comp}(V_i) \). Hence, \( PA_i \neq A_i \).
By Proposition 5.5, \( \text{Zar}(A_i) \cap \text{Zar}(A_j) = \{K\} \). It follows that \( K \) is the only common valuation overring of \( A_i A_j \); in particular, \( A_i A_j \) must be \( K \). \( \square \)

By [23, Proposition 4.3], Proposition 5.6 can also be rephrased by saying that the set \( \{A_1, \ldots, A_n\} \) is a Jaffard family of \( A \), in the sense of [9, Section 6.3].

**Proposition 5.7.** Let \( D \) be an integrally closed domain; suppose that \( \text{Zar}(D) = \text{comp}(V_1) \cup \cdots \cup \text{comp}(V_n) \), where \( X := \{V_1, \ldots, V_n\} \) is a family of incomparable valuation overrings of \( D \) such that \( V_i V_j = K \) if \( i \neq j \). Then:

(a) the restriction of the center map \( \gamma \) to \( X \) is injective;
(b) \( |\text{Max}(D)| \geq |X| \).

**Proof.**

(a) If \( P \) is the image of both \( V_i \) and \( V_j \), then \( P \) survives in both \( A_i \) and \( A_j \); however, since \( A_i \) and \( A_j \) are localizations of \( A \) (Proposition 5.4(a)), \( A_P \) would be a common overring of \( A_i \) and \( A_j \), against the fact that \( A_i A_j = K \) (Proposition 5.4(c)). Therefore, the center map is injective on \( X \).

(b) Let \( M \) be a maximal ideal: then, there is a unique \( i \) such that \( MA_i \neq A_i \). In particular, \( M \) can contain only one element of \( \gamma(X) \), namely \( \gamma(V_i) \); thus, \( |\text{Max}(D)| \geq |\gamma(X)| = |X| \), as claimed. \( \square \)

We are ready to prove the pivotal result of the paper.

**Theorem 5.8.** Let \( D \) be an integrally closed local domain. If \( \text{Zar}_{\text{min}}(D) \) is a Noetherian space, then \( D \) is a pseudo-valuation domain.

**Proof.** Since \( D \) is local, by Proposition 5.4 there are \( W_1, \ldots, W_n \), not equal to \( K \), such that \( \text{Zar}(D) = \text{comp}(W_1) \cup \cdots \cup \text{comp}(W_n) \). By eventually passing to bigger valuation domains, we can suppose without loss of generality that \( W_i W_j = K \) if \( i \neq j \); since \( D \) is local, by Proposition 5.7(b) we have \( 1 \geq n \), and so \( \text{Zar}(D) = \text{comp}(V) \) for some \( V \neq K \).

Let \( \Delta \) be the set of \( W \in \text{Zar}(D) \) such that \( \text{comp}(W) = \text{Zar}(D) \); then, \( \Delta \) is a chain, and thus it has a minimum in \( \text{Zar}(D) \), say \( V_0 \) (explicitly, \( V_0 \) is the intersection of the elements of \( \Delta \)); furthermore, clearly \( V_0 \in \Delta \). Since \( V \in \Delta \), we have \( V_0 \subseteq V \), and in particular \( V_0 \neq K \). Let \( M \) be the maximal ideal of \( V_0 \): then, \( M \) is contained in every \( W \in \text{comp}(V_0) = \text{Zar}(D) \), and thus \( M \subseteq D \).

Consider now the diagram

\[
\begin{array}{ccc}
D & \xrightarrow{\pi} & D/M \\
\downarrow & & \downarrow \\
V_0 & \xrightarrow{\pi} & V_0/M.
\end{array}
\]
Clearly, \( D = \pi^{-1}(D/M) \); let \( F_1 \) be the quotient field of \( D/M \). By Lemma 3.2, the set of minimal valuation overrings of \( D \) is homeomorphic to \( \text{Zar}_{\text{min}}(V_0/M|D/M) \), which thus is Noetherian; by Proposition 3.4, it follows that either \( D/M \) is a field and \( \text{trdeg}_{D/M}(V_0/M) = 1 \) (in which case \( D \) is a pseudo-valuation domain with associated valuation domain \( V_0 \)) or \( \text{trdeg}_{F_1}(V_0/M) = 0 \).

In the latter case, we note that \( D/M \) is integrally closed in \( V_0/M \), since \( D/M \) is the intersection of all the elements of \( \text{Zar}(V_0/M|D/M) \); hence, \( V_0/M \) is the quotient field of \( D/M \). If \( D/M \) is not a field, by the same argument of the first part of the proof it follows that \( \text{Zar}(D/M) = \text{comp}(W_0) \) for some valuation overring \( W_0 \neq F_1 \); however, this contradicts the choice of \( V_0 \), because \( \pi^{-1}(W_0) \) would be comparable with every element of \( \text{Zar}(D) \). Hence, it must be \( V_0/M = D/M \); that is, \( D \) is a valuation domain and, in particular, a pseudo-valuation domain. □

With this result, we can find the possible structures of \( \text{Zar}(D) \) and \( \text{Zar}_{\text{min}}(D) \), when \( D \) is local and \( \text{Zar}_{\text{min}}(D) \) is Noetherian. Indeed, \( D \) is a pseudo-valuation domain; let \( V \) be its associated valuation overring. Then, we have two cases: either \( D = V \) (i.e., \( D \) itself is a valuation domain) or \( D \neq V \).

In the first case, \( \text{Zar}_{\text{min}}(D) \) is a singleton, while \( \text{Zar}(D) \) is homeomorphic to \( \text{Spec}(D) \); in particular, \( \text{Zar}(D) \) is linearly ordered, and it is a Noetherian space if and only if \( \text{Spec}(D) \) is Noetherian.

In the second case, we can separate \( \text{Zar}(D) \) into two parts: \( \text{Zar}_{\text{min}}(D) \) and \( \Delta := \text{Zar}(D) \setminus \text{Zar}_{\text{min}}(D) \). The former must be isomorphic to \( \text{Zar}_{\text{min}}(L|F) = \text{Zar}(L|F) \setminus \{L\} \) (where \( F \) and \( L \) are the residue fields of \( D \) and \( V \), respectively); on the other hand, the latter is linearly ordered, and is composed by the valuation overrings of \( V \), so in particular it is homeomorphic to \( \text{Spec}(V) \), which is (set-theoretically) equal to \( \text{Spec}(D) \). In other words, \( \text{Zar}(D) \) is composed by a long “stalk” (\( \Delta \)), under which there is an infinite family of minimal valuation overrings. In particular, we get the following.

**Proposition 5.9.** Let \( D, V, F, L \) as above. Then:

(a) \( \text{Zar}_{\text{min}}(D) \) is Noetherian if and only if \( \text{Zar}(L|F) \) is Noetherian.

(b) \( \text{Zar}(D) \) is Noetherian if and only if \( \text{Zar}(L|F) \) and \( \text{Spec}(V) \) are Noetherian.

**Proof.** If \( \text{Zar}_{\text{min}}(D) \) is Noetherian, then \( \text{Zar}_{\text{min}}(L|F) \) is Noetherian as well. By Propositions 4.1 and 4.2 \( \text{Zar}(L|F) \) is Noetherian.

If \( \text{Zar}(D) \) is Noetherian, so are \( \text{Spec}(D) = \text{Spec}(V) \) and \( \Delta \simeq \text{Zar}(L|F) \) (in the notation above). Conversely, if \( \text{Zar}(L|F) \) and \( \text{Spec}(V) \) are Noetherian then so are \( \text{Zar}_{\text{min}}(D) \) and \( \Delta \), and thus also \( \text{Zar}_{\text{min}}(D) \cup \Delta = \text{Zar}(D) \) is Noetherian. □
Furthermore, we can now apply Propositions 4.11 and 4.12 to characterize when Zar\((L|F)\) is Noetherian (see the following Corollary 5.12).

We now study the non-local case.

**Lemma 5.10.** Let \(D\) be an integral domain such that \(D_M\) is a PVD for every \(M \in \text{Max}(D)\) and, for every \(M\), let \(V(M)\) be the valuation overring associated to \(D_M\). Then, the space \(\{V(M) \mid M \in \text{Max}(D)\}\) is homeomorphic to \(\text{Max}(D)\).

**Proof.** Let \(\Delta := \{V(M) \mid M \in \text{Max}(D)\}\). If \(\gamma\) is the center map, then \(\gamma(V(M)) = M\) for every \(M\); thus, \(\gamma\) restricts to a bijection between \(\Delta\) and \(\text{Max}(D)\). Since \(\gamma\) is continuous and closed, it follows that it is a homeomorphism.

**Theorem 5.11.** Let \(D\) be an integrally closed domain. Then:

(a) \(\text{Zar}_{\text{min}}(D)\) is Noetherian if and only if \(\text{Max}(D)\) is Noetherian and \(\text{Zar}_{\text{min}}(D_M)\) is Noetherian for every \(M \in \text{Max}(D)\);

(b) \(\text{Zar}(D)\) is Noetherian if and only if \(\text{Spec}(D)\) is Noetherian and \(\text{Zar}(D_M)\) is Noetherian for every \(M \in \text{Max}(D)\).

**Proof.**

(a) If \(\text{Zar}_{\text{min}}(D)\) is Noetherian, then \(\text{Max}(D)\) is Noetherian since it is the image of \(\text{Zar}_{\text{min}}(D)\) under the center map, while each \(\text{Zar}_{\text{min}}(D_M)\) is Noetherian since they are subspaces of \(\text{Zar}_{\text{min}}(D)\).

Conversely, suppose that \(\text{Max}(D)\) is Noetherian and that \(\text{Zar}(D_M)\) is Noetherian for every \(M \in \text{Max}(D)\). By the latter property and Theorem 5.8, every \(D_M\) is a PVD; by Lemma 5.10, the space \(\Delta := \{V(M) \mid M \in \text{Max}(D)\}\) (in the notation of the lemma) is homeomorphic to \(\text{Max}(D)\), and thus Noetherian. Let \(\beta\) be the map sending a \(W \in \text{Zar}_{\text{min}}(D)\) to \(V(\mathfrak{m}_W \cap D)\).

Let \(X\) be any subset of \(\text{Zar}_{\text{min}}(D)\), and let \(\Omega\) be an open cover of \(X\); without loss of generality, we can suppose \(\Omega = \{B(f_\alpha) \mid \alpha \in A\}\), where the \(f_\alpha\) are elements of \(K\). Then, \(\Omega\) is also a cover of \(X' := \{\beta(V) \mid V \in X\}\); since \(X'\) is compact (being a subset of the Noetherian space \(\Delta\)), there is a finite subfamily of \(\Omega\), say \(\Omega' := \{B(f_1), \ldots, B(f_n)\}\), that covers \(X'\). Fix thus an \(i\), let \(f := f_i\), and let \(I := (D :_D f)\) be the conductor ideal. For every \(M \in \text{Max}(D)\), let \(Z(M) := \gamma^{-1}(M) \cap X_i = \{V \in X_i \mid m_W \cap D = M\}\), where \(\gamma\) is the center map. The union of the \(Z(M)\) is \(X_i\); we separate the cases \(I \not\subseteq M \) and \(I \subseteq M\).

If \(I \not\subseteq M\), then \(1 \in ID_M = (D_M :_{D_M} f)\), and thus \(f \in D_M\); hence, in this case \(B(f)\) contains \(Z(M)\).

Suppose \(I \subseteq M\); clearly, we can suppose \(Z(M) \neq \emptyset\). We claim that in this case \(M\) is minimal over \(I\). Indeed, if there is a \(V \in Z(M)\) then \(f \in V\), and thus \(f \in \beta(V)\); therefore, \(f \in D_P\) for every prime ideal \(P \subseteq M\) (since \(D_P \supseteq \beta(V)\) for every such \(P\), and thus \(I \not\subseteq P\).
Therefore, $M$ is minimal over $I$. By Lemma 5.1, $I$ has only finitely many minimal primes; hence, there are only finitely many $M$ such that $I \subseteq M$ and $Z(M) \neq \emptyset$. For each of these $M$, the set of valuation domains in $X$ centered on $M$ is a subset of $\text{Zar}_{\text{min}}(D_M)$, and thus it is compact; hence, for each of them, $\Omega$ admits a finite subcover $\Omega(M)$. It follows that $\Omega_i := \{B(f)\} \cup \bigcup \Omega(M)$ is a finite subset of $\Omega$.

Since $\Omega$ was arbitrary, $\text{Zar}_{\text{min}}(D)$ is Noetherian.

(b) If $\text{Zar}(D)$ is Noetherian, then $\text{Spec}(D)$ and every $\text{Zar}(D_M)$ are Noetherian.

Conversely, suppose that $\text{Spec}(D)$ is Noetherian and that $\text{Zar}(D_M)$ is Noetherian for every $M \in \text{Max}(D)$. By the previous point, $\text{Zar}_{\text{min}}(D)$ is Noetherian. Furthermore, if $P \in \text{Spec}(D) \setminus \text{Max}(D)$ then $D_P$ is a valuation domain; hence, $\text{Zar}(D) \setminus \text{Zar}_{\text{min}}(D)$ is homeomorphic to $\text{Spec}(D) \setminus \text{Max}(D)$, which is Noetherian by hypothesis. Being the union of two Noetherian subspaces, $\text{Zar}(D)$ itself is Noetherian.

Corollary 5.12. Let $D$ be an integral domain that is not a field, and let $L$ be a field containing $D$; suppose that $D$ is integrally closed in $L$. Then, $\text{Zar}(L|D)$ (respectively $\text{Zar}_{\text{min}}(L|D)$) is Noetherian if and only if the following hold:

- $L$ is the quotient field of $D$;
- $\text{Spec}(D)$ is Noetherian (resp., $\text{Max}(D)$ is Noetherian);
- for every $M \in \text{Max}(D)$, the ring $D_M$ is a pseudo-valuation domain such that $\text{Zar}(L|F)$ is Noetherian, where $F$ is the residue field of $D_M$ and $L$ is the residue field of the associated valuation overring of $D_M$.

Proof. Join Proposition 3.4, Theorem 5.11 and Proposition 5.9.

For our last result, we recall that the valuative dimension $\text{dim}_v(D)$ of an integral domain $D$ is the supremum of the dimensions of the valuation overrings of $D$; a domain $D$ is called a Jaffard domain if $\text{dim}(D) = \text{dim}_v(D) < \infty$, while it is a locally Jaffard domain if $D_P$ is a Jaffard domain for every $P \in \text{Spec}(D)$ [1]. Any locally Jaffard domain is Jaffard, while the converse does not hold [1, Example 3.2]. The class of Jaffard domains includes, for example, (finite-dimensional) Noetherian domain, Prüfer domains and universally catenarian domains.

Proposition 5.13. Let $D$ be an integrally closed integral domain of finite dimension, and suppose that $\text{Zar}_{\text{min}}(D)$ is a Noetherian space. Then:

(a) $\text{dim}_v(D) \in \{\text{dim}(D), \text{dim}(D) + 1\}$;
(b) $D$ is locally Jaffard if and only if $D$ is a Prüfer domain.
Proof. (a) Let $M$ be a maximal ideal of $D$. Then, $\text{Zar}_{\text{min}}(D_M)$ is Noetherian, and thus $D_M$ is a pseudo-valuation domain; by Proposition 2.9, $\dim_v(D_M) = \dim(D_M) + \text{trdeg}_FL$, where $F$ is the residue field of $D_M$ and $L$ is the residue field of the associated valuation ring of $D_M$. By Propositions 5.9 and 4.1, $\text{trdeg}_FL \leq 1$, and thus $\dim_v(D_M) \leq \dim(D_M) + 1$. Hence, $\dim_v(D) \leq \dim(D) + 1$; since $\dim_v(D) \geq \dim(D)$ always, we have the claim.

(b) If $D$ is a Pr"ufer domain then it is locally Jaffard. Conversely, if $D$ is locally Jaffard, then $\dim_v(D_P) = \dim(D_P)$ for every prime ideal $P$ of $D$. Take any maximal ideal $M$, and let $F, L$ as above; using $\dim_v(D_M) = \dim(D_M) + \text{trdeg}_FL$, it follows that $\text{trdeg}_FL = 0$. Since $D$ (and so $D_M$) is integrally closed, it must be $F = L$, i.e., $D_M$ itself is a valuation domain. Therefore, $D$ is a Pr"ufer domain. □

Note that there are domains $D$ that are Jaffard domains and have $\text{Zar}(D)$ Noetherian, but are not Pr"ufer domains. Indeed, the construction presented in [1, Example 3.2] gives a ring $R$ with two maximal ideals, $M$ and $N$, such that $R_M$ is a two-dimensional valuation ring while $R_N$ is a one-dimensional pseudo-valuation domain with $\dim_v(R_N) = 2$; in particular, it is a Jaffard domain that is not Pr"ufer. Choosing $k = K(Z_1)$ in the construction (or, more generally, choosing $k$ such that $K(Z_1, Z_2)$ is finite over $k$), the Zariski space of $R_N$ is Noetherian (being homeomorphic to $\text{Zar}(K(Z_1, Z_2)[k])$, which is Noetherian by Proposition 1.2), and thus $\text{Zar}(R)$ is Noetherian.

References

[1] David F. Anderson, Alain Bouvier, David E. Dobbs, Marco Fontana, and Salah Kabbaj. On Jaffard domains. Exposition. Math., 6(2):145–175, 1988.
[2] M. F. Atiyah and I. G. Macdonald. Introduction to Commutative Algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
[3] Nicolas Bourbaki. Commutative Algebra, Chapters 1–7. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1989. Translated from the French, Reprint of the 1972 edition.
[4] David E. Dobbs, Richard Fedder, and Marco Fontana. Abstract Riemann surfaces of integral domains and spectral spaces. Ann. Mat. Pura Appl. (4), 148:101–115, 1987.
[5] David E. Dobbs and Marco Fontana. Kronecker function rings and abstract Riemann surfaces. J. Algebra, 99(1):263–274, 1986.
[6] Carmelo A. Finocchiaro, Marco Fontana, and K. Alan Loper. The constructible topology on spaces of valuation domains. Trans. Amer. Math. Soc., 365(12):6199–6216, 2013.
[7] Carmelo A. Finocchiaro and Dario Spirito. Topology, intersections and flat modules. Proc. Amer. Math. Soc., 144(10):4125–4133, 2016.
[8] Marco Fontana. Topologically defined classes of commutative rings. Ann. Mat. Pura Appl. (4), 123:331–355, 1980.
[9] Marco Fontana, Evan Houston, and Thomas Lucas. Factoring Ideals in Integral Domains, volume 14 of Lecture Notes of the Unione Matematica Italiana. Springer, Heidelberg; UMI, Bologna, 2013.
Marco Fontana and K. Alan Loper. An historical overview of Kronecker function rings, Nagata rings, and related star and semistar operations. In Multiplicative ideal theory in commutative algebra, pages 169–187. Springer, New York, 2006.

Robert Gilmer. Multiplicative Ideal Theory. Marcel Dekker Inc., New York, 1972. Pure and Applied Mathematics, No. 12.

John R. Hedstrom and Evan G. Houston. Pseudo-valuation domains. Pacific J. Math., 75(1):137–147, 1978.

William Heinzer. Noetherian intersections of integral domains. II. 311:107–119, 1973.

Melvin Hochster. Prime ideal structure in commutative rings. Trans. Amer. Math. Soc., 142:43–60, 1969.

Roland Huber and Manfred Knebusch. On valuation spectra. In Recent advances in real algebraic geometry and quadratic forms (Berkeley, CA, 1990/1991; San Francisco, CA, 1991), volume 155 of Contemp. Math., pages 167–206. Amer. Math. Soc., Providence, RI, 1994.

Bruce Olberding. Noetherian spaces of integrally closed rings with an application to intersections of valuation rings. Comm. Algebra, 38(9):3318–3332, 2010.

Bruce Olberding. Affine schemes and topological closures in the Zariski-Riemann space of valuation rings. J. Pure Appl. Algebra, 219(5):1720–1741, 2015.

Bruce Olberding. Topological aspects of irredundant intersections of ideals and valuation rings. In Multiplicative Ideal Theory and Factorization Theory: Commutative and Non-Commutative Perspectives. Springer Verlag, 2016.

Paulo Ribenboim. The Theory of Classical Valuations. Springer Monographs in Mathematics. Springer-Verlag, New York, 1999.

Peter Roquette. Principal ideal theorems for holomorphy rings in fields. J. Reine Angew. Math., 262/263:361–374, 1973. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday.

Niels Schwartz. Compactification of varieties. Ark. Mat., 28(2):333–370, 1990.

Dario Spirito. Non-compact subsets of the Zariski space of an integral domain. Illinois J. Math., 60(3-4):791–809, 2016.

Dario Spirito. Jaffard families and localizations of star operations. J. Commut. Algebra, to appear.

Oscar Zariski. The reduction of the singularities of an algebraic surface. Ann. of Math. (2), 40:639–689, 1939.

Oscar Zariski. The compactness of the Riemann manifold of an abstract field of algebraic functions. Bull. Amer. Math. Soc., 50:683–691, 1944.

Oscar Zariski and Pierre Samuel. Commutative Algebra. Vol. II. Springer-Verlag, New York, 1975. Reprint of the 1960 edition, Graduate Texts in Mathematics, Vol. 29.

Dipartimento di Matematica e Fisica, Università degli Studi “Roma Tre”, Roma, Italy

E-mail address: spirito@mat.uniroma3.it