Blow-up for a non-linear stable non-Gaussian process in fractional time

Soveny Solís¹ · Vicente Vergara²

Received: 9 March 2022 / Revised: 28 March 2023 / Accepted: 29 March 2023 / Published online: 13 April 2023
© Diogenes Co.Ltd 2023

Abstract
The behaviour of solutions for a non-linear diffusion problem is studied. A subordination principle is applied to obtain the variation of parameters formula in the sense of Volterra equations, which leads to the integral representation of a solution in terms of the fundamental solutions. This representation, the so-called mild solution, is used to investigate some properties about continuity and non-negativeness of solutions as well as to prove a Fujita type blow-up result. Fujita’s critical exponent is established in terms of the parameters of the stable non-Gaussian process and a result for global solutions is given.

Keywords Non-Gaussian process (primary) · Blow-up of solutions · Mild solutions · Volterra equations · Subordination principle

Mathematics Subject Classification 35B44 (primary) · 35C15 · 35E05 · 60J35

1 Introduction

Let \( \gamma > 1 \). We consider the following Cauchy problem

\[
\partial_t^{\alpha}(u - u_0)(t, x) + \Psi_{\beta}(-i \nabla)u(t, x) = |u(t, x)|^{\gamma-1}u(t, x), \quad t > 0, \quad x \in \mathbb{R}^d,
\]

\[
u(t, x)|_{t=0} = u_0(x) \geq 0, \quad x \in \mathbb{R}^d, \tag{1.1}
\]

---

¹ Escuela Superior Politécnica del Litoral, Facultad de Ciencias Naturales y Matemáticas, Km 30.5 Vía Perimetral, Guayaquil, Ecuador
² Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Víctor Lamas 1290, Concepción, Chile

Springer
where $\partial_t^\alpha$ is the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ given by

$$\partial_t^\alpha v = \frac{d}{dt} \int_0^t g_{1-\alpha}(t-s)v(s)\,ds =: \frac{d}{dt}(g_{1-\alpha} \ast v)(t),$$

with $g_\rho(t) := \frac{1}{\Gamma(\rho)} t^{\rho-1}$ and the Euler Gamma function $\Gamma(\cdot)$. The notation $\partial_t^\alpha (u - u_0)(t, x)$ is understood as $\partial_t^\alpha u(t, x) - \partial_t^\alpha (1) \times u_0(x)$ and the function $u_0$ stands for the initial data in a certain Lebesgue space. The term $\Psi_\beta(-i\nabla)$ is a singular integral operator of constant order $\beta \in (0, 2)$ with symbol $\psi$, that is

$$\Psi_\beta(-i\nabla)v(x) = F_{\xi \to x}^{-1} \{ \psi(\xi)(Fv)(\xi) \}, \ v \in C^\infty_0(\mathbb{R}^d),$$

where $F$ is the Fourier transform in $\mathbb{R}^d$ and $F^{-1}$ is its inverse. As usual, $C^\infty_0(\mathbb{R}^d)$ denotes the space of test functions on $\mathbb{R}^d$. The symbol $\psi$ is a measurable function on $\mathbb{R}^d$ given by

$$\psi(\xi) = ||\xi||^\beta \omega_\mu(\xi) \frac{\xi}{||\xi||}, \ \xi \in \mathbb{R}^d,$$

where

$$\omega_\mu(\theta) := \int_{S^{d-1}} |\theta \cdot \eta|^\beta \mu(d\eta), \ \theta \in S^{d-1},$$

with $||\xi|| = \sqrt{\xi_1^2 + \cdots + \xi_d^2}$ being the standard Euclidean norm. Here, $\mu(d\eta)$ is a centrally symmetric finite (non-negative) Borel measure defined on the unitary sphere $S^{d-1}$, called spectral measure, and $\omega_\mu(\cdot)$ is a continuous function on $S^{d-1}$, see e.g. [22, Section 1.8]. Whenever $\mu(d\eta) = \varrho(\eta)d\eta$, where $\varrho$ is a continuous function on $S^{d-1}$, we will refer to $\varrho$ as the density of $\mu$. Some restrictions on the function $\varrho$ may be required for the lower bound and behaviour of the fundamental solutions; see, e.g. [20, Section 5.2]. More precisely, our basic hypothesis throughout the paper is the following:

\begin{itemize}
  \item[(H_1)] The spectral measure $\mu$ has a strictly positive density, such that the function $\omega_\mu$ is strictly positive and $(d + 1 + [\beta])$-times continuously differentiable on $S^{d-1}$.
\end{itemize}

We denote by (H_2) to refer to (H_1) whenever we need to assume that $\omega_\mu$ is $(d + 2 + [\beta])$-times continuously differentiable on $S^{d-1}$, $[\beta]$ being the maximal integer not exceeding the real number $\beta$. The considerations just made above have been taken from [22, Proposition 4.5.1] and [22, Theorem 4.5.1], for $d = 1$ and $d > 1$ respectively. We want to point out that the condition of strict positivity on $\omega_\mu$ in (H_1), guarantees that the support of the measure $\mu$ on $S^{d-1}$ is not contained in any hyperplane of $\mathbb{R}^d$ ([22, Section 4.5]).

In this work we are concerned with studying the blow-up phenomena for reaction-diffusion equations like (1.1) considering a non-regular class of solutions instead of the classical calls, with a temporal fractional derivative and a pseudo-differential operator.
related to a stochastic process. Blow-up results for differential equations with the so-called *Caputo fractional derivative* ([17, Section 2.4] with $0 < \alpha < 1$) have been studied e.g., in [38].

Since $\psi$ is a Lévy-Khintchine symbol with index of stability $\beta$, it is well known that the corresponding stochastic process is called a localised Feller-Courrège process and therefore it makes sense to use the notation $\Psi_\beta(-i\nabla)$ for the associated generator (see, e.g. [20, Chapter 6, Appendices C and D]).

As an important concept associated with $\alpha$, let us mention the *mean squared displacement* (MSD) or the centred second moment, which describes how fast is the dispersion of the particles in a random process. In [16, Lemma 2.1], the authors proved that the MSD governed by the equation

$$\partial_t^\alpha (u - u_0) - \Delta u = 0$$

specifically turns out to be $\frac{2d}{\Gamma(1+\alpha)} t^\alpha$, $t > 0$, $0 < \alpha < 1$. In the literature one traditionally finds that anomalous diffusion refers to this power-law. See, e.g. [3, 25, 26, 34] and references therein. However, in our case, the Cauchy problem (1.1) does not possess a finite MSD. This can be directly checked by using the definition of MSD ([16, expression (6)]) and similar arguments as in the proof of [16, Lemma 2.1] or [35, Theorem 2.8].

In this setting, the Green function of the Cauchy problem $\frac{\partial u}{\partial t} + \Psi_\beta(-i\nabla)u = 0$ is non-Gaussian and it is interpreted as the transition probability density of the corresponding stable non-Gaussian process [21, Chapter 7]. The study of these processes and their generalizations is motivated by the increasing use in the mathematical modeling of processes in engineering, natural sciences and economics. See, e.g. [3, 4, 27] and [39, Chapter 1]. Similar to the case of Gaussian processes, which have been widely studied (see, e.g. [2, 9, 11–13]), it arises an interest in qualitative properties, blow-up and asymptotic behaviour for the solutions of non-Gaussian ones. For instance, in the case $\beta = 2$ and $\omega_\mu \equiv 1$ we see that the operator, namely $\Psi_2(-i\nabla)$, becomes the negative Laplacian $(-\Delta)$ with symbol $\psi(\xi) = \|\xi\|^2$. The blow-up of the solution to the corresponding ordinary differential equation

$$\partial_t u(t, x) + (-\Delta)u(t, x) = u(t, x)^\gamma, \quad t > 0, \; x \in \mathbb{R}^d,$$

$$u(t, x)|_{t=0} = u_0(x) \geq 0, \; x \in \mathbb{R}^d,$$

was investigated by Fujita in 1966 ([10]). Since then, many other researchers have explored blow-up phenomena (see, e.g. [18, 23, 28, 32, 36, 38]). An interesting generalization that includes a Riemann-Liouville fractional integral in the non-linear term, on the right hand side, can be found in [24]. Following the analysis of this phenomenon, in this work we show that the non-linearity of (1.1) leads to the blow-up of positive solutions in a finite time.

For this purpose, we say that a function $u : [0, T) \times \mathbb{R}^d \to \mathbb{R}$ *blows-up* at the finite time $T$ if

$$\lim_{t \to T^-} \|u(t)\|_\infty = +\infty,$$
and thus our main result is stated as follows.

**Theorem 1** Let $\alpha \in (0, 1)$ and $\beta \in (0, 2)$. Assume the hypothesis $(H_1)$ holds. Suppose that $\alpha = \frac{\beta}{2}$, that $1 < p < \infty$ and that $u_0 \in L_p(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ is a non-negative function. If $1 < \gamma < 1 + \frac{\beta}{d}$, then all non-trivial non-negative solutions of (1.1) admit the representation (3.1) can only be local. If $\gamma = 1 + \frac{\beta}{d}$, then the non-trivial non-negative solutions can only be local whenever the initial condition is sufficiently large. Moreover, if additionally $u_0 \in L_\infty(\mathbb{R}^d)$, then any positive mild solution of (1.1) blows-up in finite time.

Since the literature on blow-up theorems of Fujita type is quite extensive, we do not attempt to review it in this paper. Nevertheless, let us emphasize that the relation $\alpha = \frac{\beta}{2}$ pays a crucial role in the proof of Theorem 1, which makes a similarity with what Fujita (1928-) found in 1966 for the case $\alpha = 1$ working in the Gaussian framework when $\beta = 2$ and $\omega_\mu \equiv 1$. For our proof, we exploit the representation of the Volterra equations in the sense of Prüss as well as the theory of completely positive kernels of type $(PC)$, like $g_\rho$ with $\rho \in (0, 1)$ (see, e.g. [30]). We also deal with the theory of pseudo-differential operators that generate a sub-Markovian semigroup on $L_p(\mathbb{R}^d)$, under the condition that the symbol $\psi: \mathbb{R}^d \to \mathbb{C}$ is a continuous and negative definite function; see [14, Examples 4.1.12 and 4.1.13]. In our case, $\psi$ satisfies such condition ([19, Formula 1.9], [14, Theorem 3.6.11 and Lemma 3.6.8]).

Other results in [15, Section 2] and [22, Section 8.2], are also particularly important for our work. Here, the authors show that the linear Cauchy problem

$$
\partial_t^\alpha (u - u_0)(t, x) + \Psi_\beta(-i\nabla)u(t, x) = f(t, x), \quad t > 0, \ x \in \mathbb{R}^d,
$$

$$
u(t, x)|_{t=0} = u_0(x), \quad x \in \mathbb{R}^d,
$$

admits a pair of fundamental solutions $(Z, Y)$, given by

$$Z(t, x) := \frac{1}{\alpha} \int_0^\infty G(t^\alpha s, x)s^{-1 - \frac{1}{\alpha}} G_\alpha\left(1, s^{-\frac{1}{\alpha}}\right) ds \quad (1.2)$$

and

$$Y(t, x) := \int_0^\infty t^{\alpha - 1} G(t^\alpha s, x)s^{-\frac{1}{\alpha}} G_\alpha\left(1, s^{-\frac{1}{\alpha}}\right) ds, \quad (1.3)$$

where $G$ stands for the Green function that solves the problem

$$\partial_t G(t, x) + \Psi_\beta(-i\nabla)G(t, x) = 0, \quad t > 0, \ x \in \mathbb{R}^d,$n

$$G(t, x)|_{t=0} = \delta_0(x), \quad x \in \mathbb{R}^d, \quad (1.4)$$

$\delta_0$ being the Dirac delta distribution, and $G_\alpha(\cdot, \cdot)$ is the Green function that solves the problem

$$\partial_t v(t, s) + \frac{d^\alpha}{ds^\alpha} v(t, s) = 0, \quad t > 0, \ s \in \mathbb{R}, \ G_\alpha(0, s) = \delta(s),$$

[Springer]
where \( \alpha \in (0, 1) \) and
\[
\frac{d^\alpha}{ds^\alpha} f(s) := \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(s - \tau) - f(s)}{\tau^{1+\alpha}} d\tau,
\]
see [22, Formulas (1.111) and (2.74)]. From [35, Lemma 2.15], we know that the fundamental solutions \((Z, Y)\) given by (1.2)-(1.3) satisfy the relation
\[
Y(\cdot, x) = \frac{d}{dt} (g_\alpha * Z(\cdot, x)), \quad t > 0, \quad x \in \mathbb{R}^d \setminus \{0\},
\]
which is crucial for the integral representation (3.1) below.

This paper is organized as follows. In Section 2 we have compiled some properties of the fundamental solutions \(Z, Y\). In Section 3 we derive the integral representation of a solution to (1.1) in the sense of Definition 1. The main result of this section is given by Theorem 2. In Section 4 we prove two results on continuity and non-negativeness of the local solutions, given by Theorems 3 and 4 respectively. The last section is devoted to prove our main result related to the blow-up of positive solutions, stated in Theorem 1. We also give a Fujita type result for global solutions in Theorem 5.

2 Preliminaries

In what follows we use the notations \( f \asymp g \) and \( f \lesssim g \) in \( D \), which means that there exists constants \( C, C_1, C_2 > 0 \) such that \( C_1 g \leq f \leq C_2 g \) and \( f \leq C g \) in \( D \), respectively. Such constants may change line by line. We also use the notation \( \Omega = \|x\|^\beta t^{-\alpha} \) for \( x \in \mathbb{R}^d \) and \( t > 0 \).

The following result summarizes the two-sided estimates for \( Z \). For its proof see [15, Theorem 2].

**Proposition 1** Let \( \alpha \in (0, 1) \) and \( \beta \in (0, 2) \). Assume the hypothesis \((H_1)\) holds. Then there exists a positive constant \( C \) such that for \((t, x) \in (0, \infty) \times \mathbb{R}^d\) the following two-sided estimates for \( Z \) hold. For \( \Omega \leq 1 \),
\[
Z(t, x) \asymp C t^{-\frac{ad}{p}} \quad \text{if } d < \beta,
\]
\[
Z(t, x) \asymp C t^{-\alpha} (|\log(\Omega)| + 1) \quad \text{if } d = \beta,
\]
\[
Z(t, x) \asymp C t^{-\frac{ad}{p}} \Omega^{1-\frac{d}{p}} \quad \text{if } d > \beta.
\]

For \( \Omega \geq 1 \),
\[
Z(t, x) \asymp C t^{-\frac{ad}{p}} \Omega^{1-\frac{d}{p}}.
\]

In the same way we have derived the two-sided estimates for \( Y \).

**Proposition 2** Under the same assumptions as Proposition 1, the following two-sided estimates for \( Y \) hold. For \( \Omega \leq 1 \),
\[
Y(t, x) \asymp C t^{-\frac{ad}{p}} \Omega^{1-\frac{d}{p}}.
\]
\[ Y(t, x) \asymp Ct^{-\frac{ad}{p} + \alpha - 1} \quad \text{if} \quad d < 2\beta, \]
\[ Y(t, x) \asymp Ct^{-\alpha - 1}(|\log(\Omega)| + 1) \quad \text{if} \quad d = 2\beta, \]
\[ Y(t, x) \asymp Ct^{-\frac{ad}{p} + \alpha - 1} \Omega^{\frac{d}{p}} \quad \text{if} \quad d > 2\beta. \]

For \( \Omega \geq 1 \),
\[ Y(t, x) \asymp Ct^{-\frac{ad}{p} + \alpha - 1} \Omega^{1 - \frac{d}{p}}. \]

**Proof** The assertions follow from straightforward computations made in the proof of the previous estimates for \( Z \), in [15, Theorem 2]. There, the authors used the fact that the asymptotic behaviour of \( G_\alpha \) is the same as for the density \( w_\alpha \) given in [15, Proposition 1] (with the skewness of the distribution that equals to 0) by

\[ w_\alpha(\tau) \sim C \begin{cases} \tau^{-1-\alpha} & \text{as} \quad \tau \to \infty, \\ f_\alpha(\tau) := \tau^{-\frac{2}{\alpha}(1-\alpha)} e^{-c_\alpha \tau^{-\alpha}} & \text{as} \quad \tau \to 0, \end{cases} \]

where \( c_\alpha = (1 - \alpha)\alpha^{\frac{\alpha}{\alpha}} \). See e.g., [22, Proposition 2.4.1] and [39, Theorem 2.5.2] for more details.

Keeping this in mind, we see that a difference between the functions \( Z \) and \( Y \), given by (1.2) and (1.3) respectively, is the factor \( s^{-1} \) inside the improper Riemann integral of \( Z \). Thus, we only need to check the corresponding two-sided estimates for

\[ \int_0^\infty G(t^\alpha s, x)s^{-\frac{1}{\alpha}} G_\alpha \left(1, s^{-\frac{1}{\alpha}}\right) ds, \]

which can be written equivalently as

\[ \int_0^\infty G(t^\alpha s, x)s^{-\frac{1}{\alpha}} G_\alpha \left(1, s^{-\frac{1}{\alpha}}\right) s ds \asymp I_1 + I_2. \]

Here, similar to the integrals that are used in [15, expression (33)],

\[ I_1 := \int_0^1 \min \left(t^{-\frac{ad}{p}} \Omega^{-\frac{1-d}{p}} s, t^{-\frac{ad}{p}} s^{-\frac{d}{p}}\right) s ds \quad (2.1) \]

and

\[ I_2 := \int_1^\infty \min \left(t^{-\frac{ad}{p}} \Omega^{-\frac{1-d}{p}} s, t^{-\frac{ad}{p}} s^{-\frac{d}{p}}\right) s^{-1-\frac{1}{\alpha}} f_\alpha(s^{-\frac{1}{\alpha}}) s ds, \quad (2.2) \]

where

\[ \min \left(t^{-\frac{ad}{p}} \Omega^{-\frac{1-d}{p}} s, t^{-\frac{ad}{p}} s^{-\frac{d}{p}}\right) = \begin{cases} t^{-\frac{ad}{p}} \Omega^{-\frac{1-d}{p}} s, & \text{for} \quad s < \Omega, \\ t^{-\frac{ad}{p}} s^{-\frac{d}{p}}, & \text{for} \quad s \geq \Omega, \end{cases} \]
as in [15, expression (32)]. Next, we need to analyse the two-sided estimates for $I_j$, $j = 1, 2$. The case $\Omega \leq 1$ yields

$$I_1 = t^{-\frac{ad}{\beta}} \int_0^\Omega s^{2-d} \, ds + t^{-\frac{ad}{\beta}} \int_\Omega^1 s^{1-d} \, ds \leq \frac{1}{3} t^{-\frac{ad}{\beta}} \int_0^\Omega s^{2-d} \, ds + t^{-\frac{ad}{\beta}} \int_\Omega^1 s^{1-d} \, ds.$$  

The last integral requires the sub-cases $d < 2\beta$, $d = 2\beta$ and $d > 2\beta$:

$$t^{-\frac{ad}{\beta}} \int_\Omega^1 s^{1-d} \, ds = \begin{cases} t^{-\frac{ad}{\beta}} \frac{1}{2} \left( 1 - \Omega^{2-d} \frac{1}{\beta} \right), & \text{for } d < 2\beta, \\ t^{-2\alpha} |\log(\Omega)|, & \text{for } d = 2\beta, \\ t^{-\frac{ad}{\beta}} \frac{1}{\beta-2} \left( \Omega^{2-d} \frac{1}{\beta} - 1 \right), & \text{for } d > 2\beta. \end{cases}$$

For $I_2$, since $\Omega \leq 1$, we have that

$$I_2 = t^{-\frac{ad}{\beta}} \int_1^\infty s^{-\frac{d}{\beta}} s^{-1-\frac{1}{\alpha}} f_\alpha \left( s^{-\frac{1}{\alpha}} \right) \, ds = t^{-\frac{ad}{\beta}} \int_1^\infty s^{-\frac{d}{\beta}} s^{-\frac{1}{\alpha}} \frac{2-\alpha}{2\alpha(1-\alpha)} e^{-(1-\alpha)\alpha s^{\frac{1}{\alpha}}} \, ds.$$  

We point out that the improper integral is convergent due to the Laplace method for integrals (see e.g., [15, (A1)]). Therefore, if $d < 2\beta$ we find that

$$C t^{-\frac{ad}{\beta}} = I_2 \leq I_1 + I_2 = \frac{1}{3} t^{-\frac{ad}{\beta}} \Omega^{2-d} \frac{1}{\beta} + t^{-\frac{ad}{\beta}} \frac{1}{2-\frac{d}{\beta}} \left( 1 - \Omega^{2-d} \frac{1}{\beta} \right) + C t^{-\frac{ad}{\beta}} \lesssim t^{-\frac{ad}{\beta}},$$

if $d = 2\beta$ we obtain

$$I_1 + I_2 = \frac{1}{3} t^{-2\alpha} + t^{-2\alpha} |\log(\Omega)| + C t^{-2\alpha}$$

and $d > 2\beta$ implies that

$$\frac{1}{3} t^{-\frac{ad}{\beta}} \Omega^{2-d} \frac{1}{\beta} \leq I_1 \leq I_1 + I_2 = \frac{1}{3} t^{-\frac{ad}{\beta}} \Omega^{2-d} \frac{1}{\beta} + t^{-\frac{ad}{\beta}} \frac{1}{\beta-2} \left( \Omega^{2-d} \frac{1}{\beta} - 1 \right) + C t^{-\frac{ad}{\beta}} \lesssim t^{-\frac{ad}{\beta}} \Omega^{2-d} \frac{1}{\beta}.$$  

\( \varepsilon \) Springer
Blow-up for a non-linear...

Since the additional factor $t^{\alpha - 1}$ is a constant for the integral of $Y$, the estimates hold for $\Omega \leq 1$. Now, the case $\Omega \geq 1$ implies that

$$I_1 = t^{-\frac{ad}{p}} \Omega^{-\frac{1}{p}} \int_0^1 s^2 ds = \frac{1}{2} t^{-\frac{ad}{p}} \Omega^{-\frac{1}{p}}$$

and

$$I_2 = t^{-\frac{ad}{p}} \Omega^{-\frac{1}{p}} \int_1^\Omega s^{-\frac{1}{\alpha}} f_\alpha \left(s^{-\frac{1}{\alpha}}\right) ds + t^{-\frac{ad}{p}} \int_\Omega^\infty s^{-\frac{d}{p} - \frac{1}{\alpha}} f_\alpha \left(s^{-\frac{1}{\alpha}}\right) ds.$$

We see that

$$I_2 \leq t^{-\frac{ad}{p}} \Omega^{-\frac{1}{p}} \int_1^\Omega s^{-\frac{1}{\alpha}} f_\alpha \left(s^{-\frac{1}{\alpha}}\right) ds + t^{-\frac{ad}{p}} \Omega^{-\frac{1}{p}} \int_\Omega^\infty s^{-\frac{d}{p} - \frac{1}{\alpha}} f_\alpha \left(s^{-\frac{1}{\alpha}}\right) ds = C_1 t^{-\frac{ad}{p}} \Omega^{-\frac{1}{p}}.$$

On the other hand,

$$I_2 \geq t^{-\frac{ad}{p}} \Omega^{-\frac{1}{p}} \int_1^\Omega s^{-\frac{d}{p} - \frac{1}{\alpha}} f_\alpha \left(s^{-\frac{1}{\alpha}}\right) ds + t^{-\frac{ad}{p}} \Omega^{-\frac{1}{p}} \int_\Omega^\infty s^{-\frac{d}{p} - \frac{1}{\alpha}} f_\alpha \left(s^{-\frac{1}{\alpha}}\right) ds = C_2 t^{-\frac{ad}{p}} \Omega^{-\frac{1}{p}}.$$

These bounds show that $I_1 + I_2 \asymp t^{-\frac{ad}{p}} \Omega^{-\frac{1}{p}}$ for $\Omega \geq 1$. The factor $t^{\alpha - 1}$ completes the proof.

\[\square\]

\textbf{Remark 1} We note a singularity at the origin with respect to the spatial variable for $Z$, whenever $d \geq \beta$, and for $Y$ whenever $d \geq 2\beta$. It is well known that this type of singularities occurs in the equations of fractional evolution in time, even if $\beta = 2$ and $\omega_{\mu \nu} \equiv 1$.

In order to formulate our results, we also recall the following properties of the fundamental solutions $Z$ and $Y$, Lemmata 1-4 (see proofs in [35, Section 2]).

\textbf{Lemma 1} Under the same assumptions as Proposition 1, there exists a positive constant $C$ for all $t_1, t_2 > 0$ and $x \in \mathbb{R}^d$, such that there exists $t_c > 0$, between $t_1$ and $t_2$,
and the following estimates for $Z$ hold with $\Omega_c = \|x\|^\beta t_c^{-\alpha}$. For $\Omega_c \leq 1$,

$$|Z(t_1, x) - Z(t_2, x)| \leq C|t_1 - t_2| \begin{cases} \frac{-ad}{p} - 1 & \text{if } d < \beta, \\ t_c^{-\alpha - 1}(\log(\Omega_c) + 1) & \text{if } d = \beta, \\ \Omega_c - \frac{d}{p} & \text{if } d > \beta, \end{cases}$$

and for $\Omega_c \geq 1$,

$$|Z(t_1, x) - Z(t_2, x)| \leq C|t_1 - t_2|t_c^{\frac{-ad}{p} - 1} \Omega_c^{1 - \frac{d}{p}}.$$

**Lemma 2** Under the same assumptions as Proposition 1, there exists a positive constant $C$ for all $t_1, t_2 > 0$ and $x \in \mathbb{R}^d$, such that there exists $t_c > 0$, between $t_1$ and $t_2$, and the following estimates for $Y$ hold with $\Omega_c = \|x\|^\beta t_c^{-\alpha}$. For $\Omega_c \leq 1$,

$$|Y(t_1, x) - Y(t_2, x)| \leq C|t_1 - t_2| \begin{cases} \frac{-ad}{p} + \alpha - 2 & \text{if } d < 2\beta, \\ t_c^{\frac{-ad}{p} + \alpha - 2}(\log(\Omega_c) + 1) & \text{if } d = 2\beta, \\ \Omega_c^{2 - \frac{d}{p}} & \text{if } d > 2\beta, \end{cases}$$

and for $\Omega_c \geq 1$,

$$|Y(t_1, x) - Y(t_2, x)| \leq C|t_1 - t_2|t_c^{\frac{-ad}{p} + \alpha - 2} \Omega_c^{1 - \frac{d}{p}}.$$

**Lemma 3** Let $\alpha \in (0, 1)$ and $\beta \in (0, 2)$. Assume the hypothesis $(H_2)$ holds. Then there exists a positive constant $C$ for all $t > 0$ and $x_1, x_2 \in \mathbb{R}^d$, such that there exists $\zeta$ in the open segment connecting $x_1$ and $x_2$, and the following estimates for $Z$ hold with $\Omega_\zeta = \|\zeta\|^\beta t^{-\alpha}$. For $\Omega_\zeta \leq 1$,

$$|Z(t, x_1) - Z(t, x_2)| \leq C\|x_1 - x_2\|t^{\frac{-\alpha(d+1)}{p}} \Omega_\zeta^{1 - \frac{d+1}{p}}$$

and for $\Omega_\zeta \geq 1$,

$$|Z(t, x_1) - Z(t, x_2)| \leq C\|x_1 - x_2\|t^{\frac{-\alpha(d+1)}{p}} \Omega_\zeta^{1 - \frac{d+1}{p}}.$$

**Lemma 4** Under the same assumptions as Lemma 3, then there exists a positive constant $C$ for all $t > 0$ and $x_1, x_2 \in \mathbb{R}^d$, such that there exists $\zeta$ in the open segment connecting $x_1$ and $x_2$, and the following estimates for $Y$ hold with $\Omega_\zeta = \|\zeta\|^\beta t^{-\alpha}$. For $\Omega_\zeta \leq 1$,

$$|Y(t, x_1) - Y(t, x_2)| \leq C\|x_1 - x_2\| \begin{cases} t^{\frac{-\alpha(d+1)}{p} + \alpha - 1} & \text{if } d + 1 < 2\beta, \\ t^{\frac{-\alpha(d+1)}{p} + \alpha - 1}(\log(\Omega_\zeta) + 1) & \text{if } d + 1 = 2\beta, \\ \Omega_\zeta^{2 - \frac{d+1}{p}} & \text{if } d + 1 > 2\beta, \end{cases}$$

\(\copyright\) Springer
and for $\Omega_\zeta \geq 1$,

$$|Y(t, x_1) - Y(t, x_2)| \leq C\|x_1 - x_2\| t^{-\frac{\alpha(d+1)}{p} + \alpha - 1} \Omega_\zeta^{-\frac{d+1}{p}}.$$ 

**Lemma 5** Under the same assumptions as Lemma 3, then there exists a positive constant $C$ for all $t > 0$ and $x_1, x_2 \in \mathbb{R}^d$, such that the estimate

$$\|Y(t, \cdot - x_1) - Y(t, \cdot - x_2)\|_q \leq C \|x_1 - x_2\| \|\nabla Y(t, \cdot)\|_q$$

$$\lesssim \|x_1 - x_2\| t^{-\frac{\alpha d}{p} (1 - \frac{1}{q}) - \frac{\beta}{q} + \alpha - 1} \tag{2.3}$$

is true for $1 \leq q < \kappa'$, where $\kappa' := \left\{ \begin{array}{ll} \frac{d}{d+1-2\beta}, & d+1 > 2\beta \text{ and } \beta > \frac{1}{2}, \\ \infty, & d+1 \leq 2\beta. \end{array} \right.$

In the case of $d+1 < 2\beta$, (2.3) remains true for $q = \infty$.

**Proof** It follows from the same arguments as in [35, Lemma 6.1] but using the bounds given in Lemma 4.

It is worth mentioning that all these estimates have been thoroughly investigated using the Zolotarev-Pollard formula for Mittag-Leffler functions $E_\alpha$, which is valid for the case $0 < \alpha < 1$ (see [15, Section 2] and [22, Proposition 8.1.1]). To our knowledge this type of representation has not been explored explicitly in the literature for the case $\alpha > 1$, however, we refer the reader to [7] and [5] for the study of evolution equations with a Caputo fractional derivative of order $1 < \alpha < 2$.

**Definition 1** Let $\alpha \in (0, 1)$, $\beta \in (0, 2)$ and $\gamma > 1$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Suppose that $1 < p < \infty$ and that $u_0 \in L_p(\mathbb{R}^d)$ is a non-negative function. A function $u$ is called a local solution of (1.1), if there exists $T > 0$ such that

(i) $u \in C([0, T]; L_p(\mathbb{R}^d)) \cap L_\infty((0, T) \times \mathbb{R}^d)$,

(ii) $u$ satisfies (1.1) in $[0, T]$.

A function $u$ is called a global solution of (1.1) if (i)-(ii) are satisfied for any $T > 0$. We say that $u$ is a mild solution of (1.1) if $u \in C([0, T]; L_p(\mathbb{R}^d)) \cap L_\infty((0, T) \times \mathbb{R}^d)$ and it satisfies the integral equation

$$u(t, x) = \int_{\mathbb{R}^d} Z(t, x - y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y)u(s, y)|^\gamma - 1 u(s, y)dyds$$

for all $x \in \mathbb{R}^d$ and $0 \leq t < T$.

At this point, we mention that problems like (1.1) have been studied in [35, Section 5]. Under suitable conditions on $\alpha$, $\beta$, $\gamma$ and $p$, together with other parameters, the authors find positive, local and global solutions.
3 Representation of solution in its integral form

In this section we analyse the conditions under which a local solution $u$ of (1.1), in the sense of Definition 1, can be represented as

$$u(t, x) = \int_{\mathbb{R}^d} Z(t, x - y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y)|u(s, y)|^{\gamma - 1}u(s, y)dyds$$

(3.1)

for all $x \in \mathbb{R}^d$ and $0 \leq t < T$.

Although the subordination principle employed here follows directly from [6, Chapter 3], for instance, the point we want to emphasize is the relation (1.5), between the fundamental solutions $Z$ and $Y$ in the context of non-Gaussian process, which leads to the main result of this section.

First, we recall that the symbol $\psi(\xi)$ is a continuous and negative definite function. Thereby, from [14, Example 4.6.29] we know that $(-\Psi_\beta(-i\nabla), C_0^\infty(\mathbb{R}^d))$ satisfies, for any $1 < p < \infty$, the Dirichlet condition

$$\int_{\mathbb{R}^d} (-\Psi_\beta(-i\nabla)f)(x)((f - 1)^+)^{p-1}(x)dx \leq 0, \quad f \in C_0^\infty(\mathbb{R}^d),$$

and consequently it is $L_p(\mathbb{R}^d)$-dissipative ([14, Propositions 4.6.11 and 4.6.12]). In fact, the density of $C_0^\infty(\mathbb{R}^d)$ in $L_p(\mathbb{R}^d)$ implies that $(-\Psi_\beta(-i\nabla), C_0^\infty(\mathbb{R}^d))$ is closable and its closure $(A, D(A))$ generates a sub-Markovian semigroup $\{T_t\}_{t \geq 0}$ on $L_p(\mathbb{R}^d)$ which is a strongly continuous contraction semigroup ([14, Lemma 4.1.36, Theorems 4.1.33 and 4.6.17, Definition 4.1.6]). Besides, $A$ is densely defined on $L_p(\mathbb{R}^d)$ ([14, Corollary 4.1.15]). On the other hand, it is well known that $g_\alpha$ is a completely positive function and belongs to $L_1(\mathbb{R}^+)$. We denote $u(t) = u(t, \cdot)$ and $|u|^{\gamma - 1}(t) = |u(t, \cdot)|^{\gamma - 1}$. Since $u$ and $u_0$ satisfy Definition 1, if $u_0 \in L_\infty(\mathbb{R}^d)$ we observe that $g_\alpha * |u|^{\gamma - 1}u(t) \in L_p(\mathbb{R}^d)$ for $0 \leq t < T$. Equation (1.1) can be written as the Volterra equation

$$u(t) = u_0 + g_\alpha * |u|^{\gamma - 1}u(t) + g_\alpha * Au(t), \quad 0 < t < T,$$

(3.2)

which admits a resolvent $\{S(t)\}_{t \geq 0}$ in $L_p(\mathbb{R}^d)$ ([31, Theorems 4.1 and 4.2]). From [31, Corollary 4.5] we have that

$$S(t) = -\int_0^\infty T_t w(t; d\tau), \quad t > 0,$$

where $w$ is the propagation function associated with $g_\alpha$. In order to describe this resolvent, we use the representation

$$T_t f(\cdot) = \int_{\mathbb{R}^d} G(t, \cdot - y)f(y)dy, \quad f \in D(A),$$
the function $G$ being the fundamental solution of the problem (1.4) (see [29, Section 1.2 Theorem 2.4 (c) and Section 4.1 Theorem 1.3]). For $v \in D(A)$ we see that

$$S(t)v = -\int_0^\infty T_\tau v \ w(t; d \tau)$$

$$= -\int_0^\infty G(\tau, \cdot) \ast v \ w(t; d \tau)$$

and using the Fourier transform we obtain

$$\mathcal{F}(S(t)v) = -\int_0^\infty e^{-\tau \psi(\xi)} \hat{w}(t; d \tau)$$

$$= s(t, \psi(\xi)) \hat{v}$$

$$= \hat{Z}(t, \xi) \hat{v}$$

with a kernel $s$ that comes via scalar Volterra equations (see [31, Proposition 4.9], [30, Sections 2 and 3]). This implies that

$$S(t)v = Z(t, \cdot) \ast v$$

and the boundedness of $S(t)$ leads to an extension to all of $L^p(\mathbb{R}^d)$.

Let $0 < t < T$. If $u(s) \in D(A), 0 \leq s \leq t$, identity (3.2) and [31, Proposition 1.1, Definition 1.3] yield

$$1 \ast u(t) = \int_0^t u(s) ds$$

$$= \int_0^t (S(t - s)u(s) - A(g_\alpha \ast S)(t - s)u(s)) ds$$

$$= \int_0^t S(t - s)u(s) ds - \int_0^t (g_\alpha \ast S)(t - s)Au(s) ds$$

$$= \int_0^t S(s)u(t - s) ds - \int_0^t S(s)(g_\alpha \ast Au)(t - s) ds$$

$$= \int_0^t S(s)(u(t - s) - (g_\alpha \ast Au)(t - s)) ds$$

$$= \int_0^t S(s) \left( u_0 + g_\alpha \ast |u|^{\gamma - 1} u(t - s) \right) ds$$

and thus we get the variation of parameters formula for (3.2) given by

$$u(t) = \frac{d}{dt} \int_0^t S(s) \left( u_0 + g_\alpha \ast |u|^{\gamma - 1} u \right)(t - s) ds.$$
We note that
\[
\frac{d}{dt} \int_0^t S(s)u_0 ds = S(t)u_0 = Z(t, \cdot)\ast u_0.
\]

By proceeding as in the proof of [35, Lemma 5.1], but working with the $L^p(\mathbb{R}^d)$ space, using the relation (1.5) and the fact that $\sup_{0 \leq t < T} \|u|^{\gamma-1}u(t)\|_\infty < \infty$, we show that
\[
\frac{d}{dt} \int_0^t S(s) \left(g_\alpha \ast |u|^{\gamma-1}u\right)(t-s) ds = \int_0^t Y(t-s, \cdot)\ast |u|^{\gamma-1}u(s, \cdot) ds.
\]

**Theorem 2** Let $\alpha \in (0, 1)$ and $\beta \in (0, 2)$. Assume the hypothesis $({\mathcal H}_1)$ holds. Let $\gamma > 1$ and suppose that $1 < p < \infty$. Let $u_0 \in D(A) \cap L_\infty(\mathbb{R}^d)$ be a non-negative function. If $u$ is a local solution in the sense of Definition 1 for some $T > 0$ and $u(t) \in D(A)$ for all $0 \leq t < T$, then $u$ admits the representation (3.1).

## 4 Continuity and non-negativeness of solution in $[0, T) \times \mathbb{R}^d$

Let $u$ be a local solution of (1.1). In this section we show that $u$ is a continuous and non-negative function on $[0, T) \times \mathbb{R}^d$, for some $T > 0$. For this purpose, the representation (3.1) obtained in the previous section is particularly important. Besides, we need the following technical result.

**Lemma 6** Let $d \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\beta \in (0, 2)$. Assume the hypothesis $({\mathcal H}_1)$ holds. If $f$ is a continuous and bounded function on $\mathbb{R}^d$, then $Z(t, \cdot)\ast f \to f$ uniformly on compact sets whenever $t \to 0$.

**Proof** From [35, Lemma 2.12 and Formula (2.20)] we know that $g(x) := Z(1, x)$, $x \in \mathbb{R}^d$, satisfies all assumptions of [8, Theorem 1.6] with $\epsilon = t^{\beta}$. \(\Box\)

In what follows we use the parameter $\kappa := \begin{cases} d \beta, & d > \beta, \\ 1, & \text{otherwise} \end{cases}$ which sets a condition on $p$ for the existence of some $q \geq 1$ such that
\[
\frac{1}{p} + \frac{1}{q} = 1
\]
and the $L_q$-norm for $Y(t, \cdot)$, $t > 0$, is reached. Indeed, by choosing $\kappa < p < \infty$ we obtain that $1 < q < \infty$ whenever $\kappa = 1$ and $1 < q < \frac{d}{d-\beta}$ whenever $\kappa = \frac{d}{\beta}$. This implies that $q < \kappa_2$, with $\kappa_2$ as in [35, Theorem 2.10].

**Theorem 3** Let $\alpha \in (0, 1)$ and $\beta \in (1, 2)$. Assume the hypothesis $({\mathcal H}_2)$ holds. Let $\gamma > 1$ and suppose that $\max(1, \kappa) < p < \infty$. Let $u_0 \in D(A) \cap L_\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ be a non-negative function. If $u$ is a local solution in the sense of Definition 1 for some $T > 0$ and $u(t) \in D(A)$ for all $0 \leq t < T$, then $u \in C([0, T) \times \mathbb{R}^d)$. \(\Box\) Springer
Proof From Theorem 2 it follows that the local solution $u$ has the form

$$u(t, x) = \int_{\mathbb{R}^d} Z(t, x - y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y)|u|^{\gamma - 1}u(s, y)dyds,$$

$x \in \mathbb{R}^d$, $0 \leq t < T$. We define

$$u_1(t, x) := \int_{\mathbb{R}^d} Z(t, x - y)u_0(y)dy$$

and

$$u_2(t, x) := \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y)|u|^{\gamma - 1}u(s, y)dyds.$$

We shall show that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|u_j(t, x) - u_j(t_0, x_0)| < \epsilon, \forall (t, x) \in B((t_0, x_0), \delta) \subset [0, T) \times \mathbb{R}^d,$$

for $j \in \{1, 2\}$. Let $x_0 \in \mathbb{R}^d$ and $0 < t_0 < T$. We suppose $t_0 < t < T$ without loss of generality. For $u_1$ we see that

$$|u_1(t, x) - u_1(t_0, x_0)| \leq \int_{\mathbb{R}^d} |Z(t, x - y) - Z(t_0, x_0 - y)|u_0(y)dy$$

$$\leq \int_{\mathbb{R}^d} |Z(t, x - y) - Z(t_0, x - y)|u_0(y)dy$$

$$+ \int_{\mathbb{R}^d} |Z(t_0, x - y) - Z(t_0, x_0 - y)|u_0(y)dy$$

$$\lesssim \|u_0\|_{\infty} \int_{\mathbb{R}^d} |Z(t, x - y) - Z(t_0, x - y)|dy$$

$$+ \|u_0\|_{\infty} \int_{\mathbb{R}^d} |Z(t_0, x - y) - Z(t_0, x_0 - y)|dy$$

$$\lesssim \|u_0\|_{\infty} |t - t_0|t_0^{-1} + \|u_0\|_{\infty} \|x - x_0\|t_0^{-\frac{\alpha}{\beta}},$$

where the last estimates follow from [35, Theorem 2.13 and Lemma 6.1], respectively. Thus,

$$|u_1(t, x) - u_1(t_0, x_0)| \lesssim |t - t_0|t_0^{-1} + \|x - x_0\|t_0^{-\frac{\alpha}{\beta}}$$

and we can take a ball in $\mathbb{R}^d$ of radius $C^{-1}\epsilon t_0^{\frac{\alpha}{\beta}}$ centered at $x_0$, and an interval in $[0, T)$ of radius $C^{-1}\epsilon t_0$ centered at $t_0$, where $C$ is the constant of the estimate.

For the continuity of $u_1$ in $(0, x_0)$ we have that

Springer
We note that, by Lemma 6, the continuity and boundedness of $u_0$ imply the uniform limit on compact subsets of $\mathbb{R}^d$ for the first term as $t \to 0$. By choosing a sufficiently small $\delta$ we get the desired result.

Next, we analyse the continuity of $u_2$. We see that

$$|u_2(t, x)| \leq \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y)|u(s, y)|^{\gamma} \, dy \, ds$$

$$\leq \sup_{0 \leq s \leq t} \|u(s)\|_{\infty}^{\gamma} \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y) \, dy \, ds$$

$$\leq \sup_{0 \leq s \leq t} \|u(s)\|_{\infty}^{\gamma} \int_0^t (t - s)^{\alpha - 1} \frac{1}{\Gamma(\alpha)} \, ds$$

$$\leq \sup_{0 \leq s \leq t} \|u(s)\|_{\infty}^{\gamma} t^\alpha \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)}.$$

This proves that

$$\lim_{t \to 0} u_2(t, x) = 0$$

uniformly on $\mathbb{R}^d$.

Now, let $x_0 \in \mathbb{R}^d$ and $0 < t_0 < T$. Again, we suppose $t_0 < t < T$ without loss of generality. We find that

$$|u_2(t, x) - u_2(t_0, x)|$$

$$\leq |u_2(t, x) - u_2(t_0, x)| + |u_2(t_0, x) - u_2(t_0, x)|$$

$$\leq \int_0^{t_0} \int_{\mathbb{R}^d} Y(s, x - y)|u|^{\gamma - 1}u(t - s, y) - |u|^{\gamma - 1}u(t_0 - s, y) \, dy \, ds$$

$$+ \int_{t_0}^t \int_{\mathbb{R}^d} Y(s, x - y)|u(t - s, y)|^{\gamma} \, dy \, ds$$

$$+ \int_0^{t_0} \int_{\mathbb{R}^d} |Y(t_0 - s, x - y) - Y(t_0 - s, x_0 - y)||u(s, y)|^{\gamma} \, dy \, ds$$

$$\lesssim \gamma \sup_{0 \leq s \leq t} \|u(s)\|_{\infty}^{\gamma - 1} \int_0^{t_0} \int_{\mathbb{R}^d} Y(s, x - y)|u(t - s, y) - u(t_0 - s, y)| \, dy \, ds$$

$$+ \sup_{0 \leq s \leq t} \|u(s)\|_{\infty}^{\gamma} \int_{t_0}^t \int_{\mathbb{R}^d} Y(s, x - y) \, dy \, ds$$
\[
\begin{align*}
&+ \sup_{0 \leq s \leq t} \|u(s)\|_{L^\infty}^{\gamma} \int_0^{t_0} \int_{\mathbb{R}^d} |Y(t_0 - s, x - y) - Y(t_0 - s, x_0 - y)| dy ds \\
&\lesssim \gamma \sup_{0 \leq s \leq t} \|u(s)\|_{L^\infty}^{\gamma - 1} \int_0^{t_0} \|Y(s, \cdot)\|_p \|u(t - s) - u(t_0 - s)\|_{L^\infty} ds \\
&+ \sup_{0 \leq s \leq t} \|u(s)\|_{L^\infty}^\gamma \int_0^t s^{\alpha - 1} ds \\
&+ \sup_{0 \leq s \leq t} \|u(s)\|_{L^\infty}^\gamma \int_0^{t_0} \|x - x_0\| (t_0 - s)^{-\frac{\alpha}{p} + \alpha - 1} ds,
\end{align*}
\]

where the last integral is estimated by Lemma 5. For estimating the first term, we use the continuity of \(u\) with respect to the norm topology on \(L_p(\mathbb{R}^d)\) and Young’s convolution inequality, i.e.,

\[
\begin{align*}
&\int_0^{t_0} \|Y(s, \cdot)\|_p \|u(t - s) - u(t_0 - s)\|_{L^\infty} ds \\
&\lesssim \int_0^{t_0} \|Y(s, \cdot)\|_q \|u(t - s) - u(t_0 - s)\|_p ds \\
&\lesssim \epsilon \int_0^{t_0} s^{-\frac{\alpha d}{pp} + \alpha - 1} ds.
\end{align*}
\]

Thus,

\[
|u_2(t, x) - u_2(t_0, x_0)| \lesssim \epsilon t_0^{\alpha - \frac{\alpha d}{pp}} + (t_0^\alpha - t_0^\alpha) + \|x - x_0\| t_0^{-\frac{\alpha}{p}}.
\]

\[\square\]

The second result of this section is the following.

**Theorem 4** Let \(\alpha \in (0, 1)\) and \(\beta \in (0, 2)\). Assume the hypothesis \((\mathcal{H}_1)\) holds. Let \(\gamma > 1\) and suppose that \(1 < p < \infty\). Let \(u_0 \in D(A) \cap L_\infty(\mathbb{R}^d)\) be a non-negative function. If \(u\) is a local solution in the sense of Definition I for some \(T > 0\) and \(u(t) \in D(A)\) for all \(0 \leq t < T\), then there exists \(0 < T* \leq T\) such that \(u\) is non-negative in \([0, T*) \times \mathbb{R}^d\).

**Proof** We define the operator

\[
\mathcal{M}v(t, x) := \int_{\mathbb{R}^d} Z(t, x - y)v_0(y)dy + \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y)g(v(s, y))dy ds
\]

on the Banach space \(L_\infty((0, T) \times \mathbb{R}^d)\), where \(g\) is a non-decreasing Lipschitz function with \(g(0) = 0\) and \(v_0 \in L_\infty(\mathbb{R}^d)\). As in the proof of [1, Lemma 1.3], we derive that the operator \(\mathcal{M}\) has a unique fixed point \(v\). Furthermore, \(v \geq w\) whenever \(v_0 \geq w_0\), where \(w\) is the fixed point associated with \(w_0 \in L_\infty(\mathbb{R}^d)\). Our aim now is to apply this result to a sequence of functions \(g_n\), such that for each \(n \in \mathbb{N}\) they have the same properties as \(g\) but with the additional constraint that their structure approximates the non-linear term \((\cdot)^\gamma\) on \([0, \infty)\). In accordance with our particular situation with \(\gamma > 1\),
we need a sequence that allows us to control the derivative of the function $(\cdot)^\gamma$. For that purpose, we define

$$g_n(r) := \begin{cases} 
0 & \text{if } r < 0, \\
r^\gamma & \text{if } 0 \leq r \leq n, \\
an - bn e^{-r} & \text{if } r > n,
\end{cases}$$

where $a_n, b_n$ are positive constants that guarantee the existence of $g_n' \geq 0$ on $\mathbb{R}$ a.e. By construction we have that for all $n \in \mathbb{N}$ the constant Lipschitz of $g_n$ is $\gamma n^{\gamma - 1}$, $g_n(0) = 0$ and $g_n(r) = r^\gamma$ for $0 \leq r \leq n$. Therefore, there exists a unique function $u_n \in L_\infty((0, T) \times \mathbb{R}^d)$ such that $0 \leq u_n$ and

$$u_n(t, x) = \int_{\mathbb{R}^d} Z(t, x - y) \left( u_0 + \frac{1}{n} \right) (y) dy + \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y) g_n(u_n(s, y)) dy ds,$$

for $x \in \mathbb{R}^d$ and $0 < t < T$. Since $\frac{1}{n} \geq \frac{1}{n+1}$, we have that $u_{n+1} \leq u_n$. Thus, for almost every $(t, x) \in (0, T) \times \mathbb{R}^d$, the sequence of real numbers $(u_n(t, x))_{n \in \mathbb{N}}$ is decreasing and bounded from below by zero. Consequently, we can define the function

$$\tilde{u}(t, x) = \lim_{n \to \infty} u_n(t, x)$$

a.e. in $(0, T) \times \mathbb{R}^d$. On the other hand, we have that

$$\|u_n(t)\|_\infty \leq \left\| u_0 + \frac{1}{n} \right\|_\infty + \frac{\gamma n^{\gamma - 1}}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1}\|u_n(s)\|_\infty ds$$

and Gronwall’s inequality (see [37, Corollary 2]) yields

$$\|u_n(t)\|_\infty \leq \left\| u_0 + \frac{1}{n} \right\|_\infty E_{\alpha, 1} \left( \gamma n^{\gamma - 1} t^\alpha \right) \leq \left\| u_0 + \frac{1}{n} \right\|_\infty E_{\alpha, 1} \left( \gamma n^{\gamma - 1} T^\alpha \right), \quad 0 < t < T.$$

Now, for small enough $0 < T^* \leq T$ we can find $N \in \mathbb{N}$ such that

$$\left\| u_0 + \frac{1}{N} \right\|_\infty E_{\alpha, 1} \left( \gamma N^{\gamma - 1} (T^*)^\alpha \right) \leq N.$$

Therefore, for all $n \geq N$ it follows that $u_n(t, x) \leq N$, for $x \in \mathbb{R}^d$ and $0 < t < T^*$. This shows that

$$u_n(t, x) = \int_{\mathbb{R}^d} Z(t, x - y) \left( u_0 + \frac{1}{n} \right) (y) dy + \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y) u_n(s, y)^\gamma dy ds$$
whenever \( n \geq N \). We note that the non-linear integral term is dominated by \( N^\gamma \) and the dominated convergence theorem implies that

\[
\tilde{u}(t, x) = \int_{\mathbb{R}^d} Z(t, x - y) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y) \tilde{u}(s, y)^\gamma dy ds.
\]

Next, we show that \( u = \tilde{u} \) a.e. in \((0, T^*)\). Indeed,

\[
|u(t, x) - \tilde{u}(t, x)| \leq \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y) \left| |u|^{\gamma - 1} u(s, y) - |\tilde{u}|^{\gamma - 1} \tilde{u}(s, y) \right| dy ds
\]

\[
= \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y) \left| |u|^\gamma - 1 u(s, y) - |\tilde{u}|^{\gamma - 1} \tilde{u}(s, y) \right| dy ds
\]

\[
\lesssim \sup_{0 \leq s < T^*} \left( \|u(s)\|_\infty^{\gamma - 1} + \|\tilde{u}(s)\|_\infty^{\gamma - 1} \right)
\]

\[
\times \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y)|u(s, y) - \tilde{u}(s, y)|dy ds
\]

\[
\leq C(T^*) \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y)\|u(s) - \tilde{u}(s)\|_\infty dy ds
\]

\[
\leq C(T^*) \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} \|u(s) - \tilde{u}(s)\|_\infty ds
\]

and thus

\[
\|u(t) - \tilde{u}(t)\|_\infty \leq C(T^*) \int_0^t \frac{(t - s)^{\alpha - 1} \|u(s) - \tilde{u}(s)\|_\infty}{\Gamma(\alpha)} ds.
\]

By Gronwall’s inequality we conclude the desired result. \( \square \)

5 Proof of the main result Theorem 1

**Proof** Firstly, we get the following estimates. Let \( t > 0 \). Using the bounds given in Proposition 1, it is clear that

\[
Z(t, x - y) \geq C t^{-\frac{ad}{\beta}} e^{-\frac{\|x - y\|^2}{4t}}, \quad \Omega \leq 1.
\]

If \( \Omega \geq 1 \), we have that

\[
Z(t, x - y) \geq C t^{-\frac{ad}{\beta}} \Omega^{-1 - \frac{d}{\beta}}
\]

\[
= C t^{-\frac{ad}{\beta}} t^{\alpha + \frac{ad}{\beta}} \|x - y\|^{-\beta - d}
\]

\[
= C t^{-\frac{ad}{\beta}} t^{\alpha + \frac{ad}{\beta}} (2\sqrt{t})^{-\beta - d} \left( \frac{\|x - y\|}{2\sqrt{t}} \right)^{-\beta - d}
\]
\[ \geq C t^{-\frac{ad}{\beta}} t^{\alpha - \frac{ad}{\beta}} (2\sqrt{t})^{-\beta - d} e^{-\frac{\|x-y\|^2}{4t}}, \]

whenever \( d \leq 3 \). For larger dimensions, it is always possible to find a suitable constant \( K > 1 \), depending on \( \beta \) and \( d \), such that

\[ (\|x-y\| \sqrt{t})^{-\beta - d} e^{-\frac{\|x-y\|^2}{4t}} \]

whenever \( d \leq 3 \). For larger dimensions, it is always possible to find a suitable constant \( K > 1 \), depending on \( \beta \) and \( d \), such that

\[ (\|x-y\| \sqrt{t})^{-\beta - d} e^{-\frac{\|x-y\|^2}{4t}} \]

From the hypothesis \( \alpha = \frac{\beta}{2} \), it follows that

\[ Z(t, x - y) \geq C t^{-\frac{ad}{\beta}} e^{-\frac{\|x-y\|^2}{4t}}, \Omega \geq 1, \]

which means that

\[ Z(t, x - y) \geq C_1 t^{-\frac{ad}{\beta}} e^{-\frac{\|x-y\|^2}{4t}}, \]

for all \( t > 0 \) and \( x, y \in \mathbb{R}^d \), with \( C_1 = \frac{C}{2^{\beta+d}} \).

We may assume without loss of generality that the constant \( C \) of the Proposition 2 is the same as that of the Proposition 1. In this way, we have also derived

\[ Y(t-s, x - y) \geq C_1 (t-s)^{-\frac{ad}{\beta}} e^{-\frac{\|x-y\|^2}{4(t-s)}}, \]

for all \( 0 \leq s < t \) and \( x, y \in \mathbb{R}^d \).

Now, we proceed by contradiction. We suppose that there exists a global non-trivial solution \( u \) of (1.1), according to Definition 1. In this case, \( u_0(y_0) > 0 \) for some \( y_0 \in \mathbb{R}^d \). The continuity of \( u_0 \) implies that

\[ u_0(y) > C_0, \quad \forall y \in B(y_0, \delta), \]

with some \( \delta > 0 \) and \( C_0 = \frac{u_0(y_0)}{2} \).

The representation (3.1) for \( u \) is

\[ u(t, x) = \int_{\mathbb{R}^d} Z(t, x - y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} Y(t-s, x - y)u(s, y)\gamma dyds \]

for all \( x \in \mathbb{R}^d \) and \( 0 < t < T \). We note that, given the assumption made, \( T \) can be arbitrarily large. As in Section 3, we define

\[ u_1(t, x) := \int_{\mathbb{R}^d} Z(t, x - y)u_0(y)dy \]

and

\[ u_2(t, x) := \int_0^t \int_{\mathbb{R}^d} Y(t-s, x - y)u(s, y)\gamma dyds. \]
Using (5.1), it follows that
\[
\begin{align*}
    u_1(t, x) &\geq C_1 t^{-\frac{ad}{p}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy \\
    &\geq C_1 C_0 t^{-\frac{ad}{p}} \int_{B(y_0, \delta)} e^{-\frac{|x-y_0|^2}{4t}} dy \\
    &\geq C_1 C_0 t^{-\frac{ad}{p}} e^{-\frac{|x-y_0|^2}{2t}} \int_{B(y_0, \delta)} e^{-\frac{|y-y_0|^2}{2t}} dy 
\end{align*}
\]
and we obtain
\[
    u_1(t, x) \geq C_2 t^{-\frac{ad}{p}} e^{-\frac{|x|^2}{t}}, \quad t > 1, \quad x \in \mathbb{R}^d.
\] (5.3)

Let \( H \) be the heat kernel
\[
    H(t, x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^d.
\]

Using the fact that
\[
    \int_{\mathbb{R}^d} H(t, x) dx = 1,
\]
we define the function
\[
    F(t) = \int_{\mathbb{R}^d} H(t, x) u(t, x) dx, \quad t > 0,
\] (5.4)
and splitting the integral into two parts we see that
\[
    F(t) = \int_{\mathbb{R}^d} H(t, x) u_1(t, x) dx + \int_{\mathbb{R}^d} H(t, x) u_2(t, x) dx.
\]

In the first integral we use the estimate (5.3), for obtaining
\[
    F(t) \geq C_3 t^{-\frac{ad}{p}} + \int_{\mathbb{R}^d} H(t, x) u_2(t, x) dx
\]
whenever \( t > 1. \)

In the second integral, we use the fact that (see [35, Theorem 2.14])
\[
    \frac{1}{g_\alpha(t)} \int_{\mathbb{R}^d} Y(t, x) dx = 1, \quad t > 0.
\]

Jensen’s inequality and Fubini’s theorem yield
\[
    \int_{\mathbb{R}^d} H(t, x) u_2(t, x) dx
\]
= \int_{\mathbb{R}^d} H(t, x) \left[ \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y)u(s, y)'^{y} \, dy \, ds \right] \, dx

= \int_0^t g_\alpha(t - s) \int_{\mathbb{R}^d} H(t, x) \left[ \int_{\mathbb{R}^d} \frac{1}{g_\alpha(t - s)} Y(t - s, x - y)u(s, y)'^{y} \, dy \right] \, dx \, ds

\geq \int_0^t g_\alpha(t - s) \int_{\mathbb{R}^d} H(t, x) \left[ \int_{\mathbb{R}^d} \frac{1}{g_\alpha(t - s)} Y(t - s, x - y)u(s, y)dy \right]^{y} \, dx \, ds

= \int_0^t (g_\alpha(t - s))^{1 - y} \int_{\mathbb{R}^d} H(t, x) \left[ \int_{\mathbb{R}^d} Y(t - s, x - y)u(s, y)dy \right]^{y} \, dx \, ds

\geq \int_0^t (g_\alpha(t - s))^{1 - y} \left\{ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} H(t, x)Y(t - s, x - y)dx \right\} u(s, y)dy \right\}^{y} \, ds.

The expression in the square brackets can be estimated with (5.2), i.e.,

\int_{\mathbb{R}^d} H(t, x)Y(t - s, x - y)dx

\geq C_1(t - s)^{-\frac{\alpha d}{2} + \alpha - 1} \int_{\mathbb{R}^d} H(t, x)e^{-\frac{|x - y|^2}{4(t - s)}} \, dx

= C_1(4\pi s)^{-\frac{d}{2}} e^{-\frac{|s - y|^2}{4s}} \left( \frac{S}{t} \right)^{\frac{d}{2}} (t - s)^{\alpha - 1}

Proceeding in the same way as in [12, page 42], with \( \alpha = \frac{d}{2} \), we get

\int_{\mathbb{R}^d} H(t, x)Y(t - s, x - y)dx \geq C_4(4\pi s)^{-\frac{d}{2}} e^{-\frac{|s - y|^2}{4s}} \left( \frac{S}{t} \right)^{\frac{d}{2}} (t - s)^{\alpha - 1}

and thus

\left\{ \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} H(t, x)Y(t - s, x - y)dx \right] u(s, y)dy \right\}^{y}

\geq C_4^y (t - s)^{(\alpha - 1)y} \left( \frac{S}{t} \right)^{\frac{d}{2} y} \left\{ \int_{\mathbb{R}^d} (4\pi s)^{-\frac{d}{2}} e^{-\frac{|s - y|^2}{4s}} u(s, y)dy \right\}^{y}

= C_4^y (t - s)^{(\alpha - 1)y} \left( \frac{S}{t} \right)^{\frac{d}{2} y} F^y (s)

for 0 < s < t.

It follows that

\int_{\mathbb{R}^d} H(t, x)u_2(t, x)dx \geq C_4^y \int_0^t (g_\alpha(t - s))^{1 - y} (t - s)^{(\alpha - 1)y} \left( \frac{S}{t} \right)^{\frac{d}{2} y} F^y (s)ds

Springer
and hence
\[ F(t) \geq C_3 \frac{d}{t^2} + C_5 \frac{f^{\alpha-1}}{t^{2\gamma}} \int_0^t s^{\frac{d}{2\gamma}} F^{\gamma}(s) ds \]
for all \( t > 1 \). Consequently,
\[ t^{\frac{d}{2\gamma}} t^{1-\alpha} F(t) \geq C_3 t^{\frac{d}{2\gamma} (\gamma-1)} t^{1-\alpha} + C_5 \int_0^t s^{\frac{d}{2\gamma}} F^{\gamma}(s) ds. \tag{5.5} \]

Defining the r.h.s. of this expression as \( f(t) \), \( t > 1 \), we have that
\[ f(t) \geq C_3 t^{\frac{d}{2\gamma} (\gamma-1)} t^{1-\alpha} \tag{5.6} \]
and that
\[ f'(t) \geq C_5 t^{\frac{d}{2\gamma}} F^{\gamma}(t). \tag{5.7} \]

From (5.5) it follows that
\[ f'(t) f^{-\gamma}(t) \geq C_5 t^{\frac{d}{2\gamma} (1-\gamma)-(1-\alpha)\gamma} \]
and
\[ \int_t^T f'(s) f^{-\gamma}(s) ds \geq C_5 \int_t^T s^{\frac{d}{2\gamma} (1-\gamma)-(1-\alpha)\gamma} ds \]
with \( T > t \). From here, we get that
\[ \frac{f^{1-\gamma}(t)}{\gamma-1} \geq C_5 \int_t^T s^{\frac{d}{2\gamma} (1-\gamma)-(1-\alpha)\gamma} ds \]
and using (5.6) we also obtain the estimate
\[ \frac{f^{1-\gamma}(t)}{\gamma-1} \leq C_3^{1-\gamma} t^{\frac{d}{2} (1-\gamma)^2-(1-\alpha)(\gamma-1)}. \]

This implies that
\[ \frac{C_3^{1-\gamma}}{\gamma-1} t^{\frac{d}{2} (1-\gamma)^2-(1-\alpha)(\gamma-1)} \geq C_5 \int_t^T s^{\frac{d}{2\gamma} (\gamma-1)-(1-\alpha)\gamma} ds. \tag{5.8} \]
Next we analyse the r.h.s. of (5.8), according to the following cases with \( a := d - 2(1 - \alpha) \).

For the case \( 1 < \gamma \leq \frac{a}{d} + \frac{2}{d\gamma} \), we have

\[
\gamma \leq \frac{a}{d} + \frac{2}{d\gamma} \Rightarrow d\gamma^2 \leq a\gamma + 2
\]

\[
\Leftrightarrow d\gamma^2 + 2(1 - \alpha)\gamma - d\gamma - 2 \leq 0
\]

\[
\Leftrightarrow -\frac{d\gamma}{2}(\gamma - 1) - (1 - \alpha)\gamma + 1 \geq 0,
\]

which yields a contradiction for large enough \( T \).

For the case \( \frac{a}{d} + \frac{2}{d\gamma} < \gamma < \frac{a}{d} + \frac{2}{d} \), we write the expression (5.8) as

\[
C_3^{1-\gamma} (t - \frac{d}{2})^{(1-\gamma)^2 - (1-\alpha)(\gamma - 1)} \geq C_5 \frac{t^{-\frac{d}{2}(\gamma - 1) - (1 - \alpha)\gamma + 1} - t^{-\frac{d}{2}(\gamma - 1) - (1 - \alpha)\gamma + 1}}{\frac{d}{2}(\gamma - 1) + (1 - \alpha)\gamma - 1}.
\]

Besides

\[
\gamma < \frac{a}{d} + \frac{2}{d} \Rightarrow d\gamma < d - 2(1 - \alpha) + 2
\]

\[
\Leftrightarrow -1 < -\frac{d}{2}(\gamma - 1) - (1 - \alpha)
\]

\[
\Leftrightarrow \frac{d\gamma}{2}(\gamma - 1) + (1 - \alpha)\gamma - 1 < \frac{d}{2}(\gamma - 1)^2 + (1 - \alpha)(\gamma - 1),
\]

which is a contradiction for large enough \( t \) and \( T \to \infty \).

For the critical case \( \gamma = 1 + \frac{\beta}{d} \), we use the facts that

\[
u(t, x)^\gamma \geq u_1(t, x)^\gamma
\]

and

\[
u(t, x) \geq u_2(t, x),
\]

together with the estimates (5.2) and (5.3). Therefore, for \( t > 2 \), we get

\[
\nu(t, x) \geq \int_1^2 \int_{\mathbb{R}^d} \frac{t}{1} \frac{1}{s} Y(t - s, x - y)\nu(s, y)^\gamma dy ds
\]

\[
\geq C_1 C_2^\gamma \int_1^2 \frac{t}{1} \frac{1}{s} \frac{e^{-\frac{1}{4(t-s)^2} s}}{s} e^{-\frac{1}{4(t-s)^2} \frac{\|y\|^2}{s}} dy ds
\]

\[
= C_1 C_2^\gamma \int_1^2 \frac{t}{1} \frac{1}{s} \frac{e^{-\frac{1}{4(t-s)^2} s}}{s} \frac{1}{(t-s)\frac{d}{2}s} \frac{1}{(t-s)\frac{d}{2}s} \frac{e^{-\frac{1}{4(t-s)^2} \frac{\|y\|^2}{s}}}{t} dy ds
\]

\[
\Box \text{ Springer}
\]
\[
\geq C_1 C_2^{\gamma} \frac{t^d}{t^{2\gamma}} e^{-\frac{\|x\|^2}{t}} \int_1^t \frac{1}{s} (t-s)^{\alpha-1} t^d \, ds \geq C_6 \frac{t^d}{t^{2\gamma}} e^{-\frac{\|x\|^2}{t}} \int_1^t \frac{1}{s} (t-s)^{\alpha-1} \, ds
\]

and hence
\[
\geq C_6 \frac{t^d}{t^{2\gamma}} e^{-\frac{\|x\|^2}{t}} \int_1^t \frac{1}{t-s} \, ds
\]

Using this and (5.4), we obtain that
\[
F(t) \geq \frac{C_7}{t^{2\gamma}} \ln \left(2 - \frac{2}{t}\right). \tag{5.9}
\]

Now,
\[
t^d \frac{d}{dt} t^{1-\gamma} F(t) = \frac{1}{2} t^d \gamma t^{1-\alpha} F(t) + \frac{1}{2} t^d \gamma t^{1-\alpha} F(t)
\]
\[
\geq \frac{C_7}{2} t^d \gamma t^{1-\alpha} \ln \left(2 - \frac{2}{t}\right) + \frac{C_5}{2} \int_0^t s t^d \gamma F'(s) \, ds,
\]

where (5.9) yields the bound for the first term and the second term comes from (5.5). The critical value of \(\gamma\) yields
\[
\geq \frac{C_7}{2} t^d \gamma t^{1-\alpha} \ln \left(2 - \frac{2}{t}\right) + \frac{C_5}{2} \int_0^t s t^d \gamma F'(s) \, ds.
\]

Defining the r.h.s. of this expression as the new \(f(t)\), \(t > 1\), we proceed as before but using
\[
f(t) \geq C_8 t \ln \left(2 - \frac{2}{t}\right)
\]

and
\[
f'(t) \geq C_9 t^{\frac{d}{2\gamma}} F(t)
\]

instead of (5.6) and (5.7), respectively, with \(C_8 = \frac{C_7}{2}\) and \(C_9 = \frac{C_5}{2}\). The resulting expression, instead of (5.8), is
\[
\geq \frac{C_8}{\gamma - 1} t^1 \gamma \ln^{1-\gamma} \left(2 - \frac{2}{t}\right) \geq C_9 \int_t^T s^{-\frac{d}{2\gamma} t^1 \gamma} \, ds
\]
or, in this case,
\[
\frac{C_8^{1-\gamma}}{\gamma - 1} t^{1-\gamma} \ln^{1-\gamma} \left(2 - \frac{2}{t}\right) \geq C_9 \int_t^T s^{-\gamma} ds.
\]

This implies, as \( T \to \infty \), that
\[
C_8^{1-\gamma} \ln^{1-\gamma} \left(2 - \frac{2}{t}\right) \geq C_9,
\]
which is a contradiction whenever the initial condition is sufficiently large at the point \( y_0 \).

So far we note that in this proof we do not require that \( u \) satisfies (1.1). Hence, any positive mild solution \( u \) can only be local under the assumptions of Theorem 1. In this context, let
\[
\tilde{T} = \sup \left\{ T > 0 : u \in C([0, T]; L_p(\mathbb{R}^d)) \cap L_\infty((0, T) \times \mathbb{R}^d) \right\}
\]

is a positive mild solution of (1.1).

Previous work implies that \( \tilde{T} < +\infty \). Suppose that \( \lim_{t \to \tilde{T}^-} \|u(t)\|_\infty < +\infty \). Since \( u_0 \in L_\infty(\mathbb{R}^d) \), it follows that there exists \( M > 0 \) such that \( \|u(t)\|_\infty \leq M \) for all \( t \in (0, \tilde{T}) \). We choose a sequence \( t_n \to \tilde{T} \) as \( n \to \infty \), with \( t_n < \tilde{T} \) for all \( n \in \mathbb{N} \). We suppose \( \frac{1}{2} \tilde{T} < t_m < t_n \) without loss of generality, with \( n, m \geq N \) for some \( N \in \mathbb{N} \). As in the proof of [35, Theorem 3.1], we find that
\[
\|u(t_n) - u(t_m)\|_p \lesssim (t_n - t_m)^{-1} \|u_0\|_p + M^{\gamma - 1} \int_0^{t_m} \|Y(t_n - s) - Y(t_m - s)\|_1 \|u(s)\|_p ds
\]
\[
+ M^{\gamma - 1} \int_{t_m}^{t_n} \|Y(t_n - s)\|_1 \|u(s)\|_p ds.
\]

On the other hand, for any \( t \in [0, \tilde{T}) \) we see that
\[
\|u(t)\|_p \leq \|u_0\|_p + \frac{M^{\gamma - 1}}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \|u(s)\|_p ds
\]
and Gronwall’s inequality ([37, Corollary 2]) yields
\[
\|u(t)\|_p \leq \|u_0\|_p E_{\alpha,1}(M^{\gamma - 1} t^\alpha), \quad 0 \leq t < \tilde{T}.
\]

This shows that \( \|u(t)\|_p \leq \|u_0\|_p E_{\alpha,1}(M^{\gamma - 1} \tilde{T}^\alpha) =: K \) for all \( t \in [0, \tilde{T}) \). Thus,
\[
\|u(t_n) - u(t_m)\|_p \lesssim (t_n - t_m)^{-1} \|u_0\|_p
\]
\[ + M^{\gamma - 1} K \int_0^{t_m} \| Y(t_n - s) - Y(t_m - s) \|_1 ds \]
\[ + M^{\gamma - 1} K \int_{t_m}^{t_n} \| Y(t_n - s) \|_1 ds. \]

The integral over \([0, t_m]\) can be estimated, using \([35, \text{Theorems 2.10 and 2.14}]\), as follows:
\[
\int_0^{t_m} \| Y(t_n - s) - Y(t_m - s) \|_1 ds = \int_0^{t_m} \| Y(t_n - t_m + s) - Y(s) \|_1 ds
\]
\[ \leq \int_0^{t_n - t_m} \| Y(t_n - t_m + s) - Y(s) \|_1 ds + \int_{t_n - t_m}^{\infty} \| Y(t_n - t_m + s) - Y(s) \|_1 ds
\]
\[ \leq \int_0^{t_n - t_m} \| Y(t_n - t_m + s) \|_1 ds + \int_0^{t_n - t_m} \| Y(s) \|_1 ds
\]
\[ + \int_{t_n - t_m}^{\infty} \| Y(t_n - t_m + s) - Y(s) \|_1 ds
\]
\[ \lesssim (t_n - t_m)^{\alpha - 1} ds + \int_0^{t_n - t_m} s^{\alpha - 1} ds + \int_{t_n - t_m}^{\infty} (t_n - t_m)s^{\alpha - 2} ds
\]
\[ \lesssim (t_n - t_m)^{\alpha}. \]

Consequently,
\[ \| u(t_n) - u(t_m) \|_p \lesssim (t_n - t_m)^{\tilde{T} - 1} \| u_0 \|_p + M^{\gamma - 1} K (t_n - t_m)^{\alpha} \]
and thus \((u(t_n))_{n \in \mathbb{N}}\) represents a Cauchy sequence in \(L_p(\mathbb{R}^d)\). We define \(u(\tilde{T}) := \lim_{t \to \tilde{T}^-} u(t)\). From \([33, \text{Theorem 3.12}]\) it follows that \(\| u(\tilde{T}) \|_\infty \leq M\) and that \(u(\tilde{T}) \geq 0\). Next, as in the proof of \([38, \text{Theorem 3.2}]\), we define the operator
\[
\mathcal{M}v(t) := Z(t)\ast u_0 + \int_0^{\tilde{T}} Y(t - s)\ast u'(s) ds + \int_0^{t'} Y(t - s)\ast |u(s)|^{\gamma - 1} v(s) ds
\]
on the Banach space
\[ E_\tau = C([\tilde{T}, \tilde{T} + \tau]; L_p(\mathbb{R}^d)) \cap L_\infty([\tilde{T}, \tilde{T} + \tau] \times \mathbb{R}^d), \]
with some \(\tau > 0\) and the norm
\[ \| v \|_{E_\tau} = \sup_{t \in [\tilde{T}, \tilde{T} + \tau]} \| v(t) \|_p + \sup_{(t, x) \in [\tilde{T}, \tilde{T} + \tau] \times \mathbb{R}^d} |v(t, x)|. \]
It is straightforward to see that $\mathcal{M} : E_\tau \to E_\tau$ is well defined and that $\mathcal{M} v(\tilde{T}) = u(\tilde{T})$. Besides, for $v, w \in E_\tau$ we have that

$$|\mathcal{M} v(t, x) - \mathcal{M} w(t, x)| \leq \|\mathcal{M} v(t) - \mathcal{M} w(t)\|\infty \leq \int_{\tilde{T}}^{t} \|Y(t-s)\|_1 \|v(s)\|^{\gamma-1} v(s) - \|w(s)\|^{\gamma-1} w(s)\|\infty ds \leq (\|v\|_{E_\tau} + \|w\|_{E_\tau})^{\gamma-1} \int_{\tilde{T}}^{t} (t-s)^{\alpha-1} \|v(s) - w(s)\|\infty ds \leq (\|v\|_{E_\tau} + \|w\|_{E_\tau})^{\gamma-1} \|v - w\|_{E_\tau}^{\alpha},$$

and hence

$$\|\mathcal{M} v(t) - \mathcal{M} w(t)\|\infty \leq (\|v\|_{E_\tau} + \|w\|_{E_\tau})^{\gamma-1} \|v - w\|_{E_\tau}^{\alpha}, \quad t \in [\tilde{T}, \tilde{T} + \tau).$$

Similarly,

$$\|\mathcal{M} v(t) - \mathcal{M} w(t)\|_p \leq (\|v\|_{E_\tau} + \|w\|_{E_\tau})^{\gamma-1} \|v - w\|_{E_\tau}^{\alpha}, \quad t \in [\tilde{T}, \tilde{T} + \tau].$$

Therefore, there exists $C_{10} > 0$ such that

$$\|\mathcal{M} v - \mathcal{M} w\|_{E_\tau} \leq C_{10} \tau^{\alpha} (\|v\|_{E_\tau} + \|w\|_{E_\tau})^{\gamma-1} \|v - w\|_{E_\tau}, \quad v, w \in E_\tau. \quad (5.10)$$

We also find that

$$\left\| Z(t) * u_0 + \int_0^\tilde{T} Y(t-s) * u^\gamma(s) ds \right\|\infty \leq \|Z(t) * u_0\|\infty + \int_0^\tilde{T} \|Y(t-s)\|_1 \|u^\gamma(s)\|\infty ds \leq \|u_0\|\infty + M^\gamma \int_0^\tilde{T} (t-s)^{\alpha-1} ds \leq \|u_0\|\infty + M^\gamma \tau^{\alpha} \leq \|u_0\|\infty + M^\gamma \tilde{T}^\alpha$$

and that

$$\left\| Z(t) * u_0 + \int_0^\tilde{T} Y(t-s) * u^\gamma(s) ds \right\|_p \leq \|u_0\|_p + M^{\gamma-1} K \tilde{T}^\alpha,$$
that is, there exists $C_{11} > 0$ such that
\[
\left\| Z(t) \star u_0 + \int_0^\tilde{T} Y(t-s) \star u^\gamma(s) ds \right\|_{E_\tau} 
\leq C_{11} \left( \|u_0\|_\infty + \|u_0\|_p + M^\gamma - 1 (M + K) \tilde{T}^\alpha \right).
\]

Let $R = 2C_{11} \left( \|u_0\|_\infty + \|u_0\|_p + M^\gamma - 1 (M + K) \tilde{T}^\alpha \right)$. If we consider the closed ball
\[
B_{E_\tau} := \{ w \in E_\tau : \|w\|_{E_\tau} \leq R \},
\]
then estimates (5.10), with $v = 0$, and (5.11) show that $M : B_{E_\tau} \to B_{E_\tau}$ is a contraction whenever $\tau$ is small enough (see [35, Theorem 3.1]), thus showing that $M$ has a unique fixed point $w' \in B_{E_\tau}$. Moreover, since $u \geq 0$ we obtain that $w' \geq 0$ in $[\tilde{T}, \tilde{T} + \tau) \times \mathbb{R}^d$ following the same arguments as in the proof of Theorem 4, but one must now use the fact that
\[
v_n(t) = Z(t) \star \left( u_0 + \frac{1}{n} \right) + \int_0^\tilde{T} Y(t-s) \star \left( u + \frac{1}{n} \right)^\gamma(s) ds
+ \int_t^\tilde{T} Y(t-s) \star g_n(v_n(s)) ds,
\]
for all $n \in \mathbb{N}$ and $v_n \in L_\infty(\tilde{T}, \tilde{T} + \tau) \times \mathbb{R}^d$. However, this leads to a contradiction with the definition of $\tilde{T}$, and therefore $\lim_{t \to \tilde{T}^-} \|u(t)\|_\infty = +\infty$. \qed

The final result of this section deals with the case $\gamma > 1 + \frac{\beta}{d}$. For this purpose, as in Section 4, we set
\[
\kappa = \begin{cases} 
\frac{d}{\beta}, & d > \beta, \\
1, & \text{otherwise}. 
\end{cases}
\]
We also define $H^\beta_2(\mathbb{R}^d) := C_{0,\infty}^\infty(\mathbb{R}^d) \|\|_{\psi_\beta, L_2}$, with the closure being respect to the graph norm $\|\cdot\|_2^2 = \|\cdot\|^2 + \|\psi_\beta(-i \nabla)(\cdot)\|^2_2$.

**Theorem 5** Let $\alpha \in (0, 1)$ and $\beta \in (0, 2)$. Assume the hypothesis $(H_1)$ holds. Suppose that $\gamma > 1 + \frac{\beta}{d}$, that max $\left( 1, \frac{d(\gamma - 1)}{\beta} \right) < p < \infty$ and that $1 = p' < \frac{d}{\beta} (\gamma - 1)$ whenever $d < \beta$, or $\frac{d}{\beta} < p' < \frac{d}{\beta} (\gamma - 1)$ whenever $d \geq \beta$. If $u_0 \in L_1(\mathbb{R}^d) \cap H^\beta_2(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$ is sufficiently small and non-negative, then there exists a global solution $u$ to (1.1) in the sense of Definition 1 and the optimal time decay estimate
\[
\|u(t)\|_1 + t^{\frac{ad}{p} \left( \frac{1}{p} - \frac{1}{p'} \right)} \|u(t)\|_p + t^{\frac{ad}{p'}} \|u(t)\|_\infty \lesssim \left( \|u_0\|_1 + \|u_0\|_p + \|u_0\|_\infty \right)
\]
is true for all $t \geq 1$. 

\[\text{Springer}\]
Remark 2. Whenever \( d \leq \beta \), the existence of parameter \( p' \) follows from the fact that \( \gamma > 1 + \frac{\beta}{d} \). However, in the case \( d > \beta \) one can not generally guarantee the existence of \( p' \).

**Proof** We consider the Banach space

\[
E := C([0, \infty); L_p(\mathbb{R}_d) \cap L_1(\mathbb{R}_d)) \cap L_\infty((0, \infty); L_\infty(\mathbb{R}_d)),
\]

with the norm

\[
\|v\|_E := \sup_{t \geq 0} \left( \frac{ad}{p'} \left( \frac{1}{p'} - 1 \right) \|v(t, \cdot)\|_p + \|v(t, \cdot)\|_1 \right) + \sup_{t > 0} \{t\}^{\frac{ad}{p'}} \|v(t, \cdot)\|_\infty,
\]

where \( \{t\} := \sqrt{1 + t^2} \).

We define on \( E \) the operator

\[
\mathcal{M}(v)(t, x) := \int_{\mathbb{R}_d} Z(t, x - y)u_0(y)dy
\]

\[
+ \int_0^t \int_{\mathbb{R}_d} Y(t - s, x - y)|v(s, y)|^{\gamma-1}v(s, y)dyds
\]

and similar arguments as in [35, Sections 3 and 4] show that

\[
\mathcal{M}(v) \in C([0, \infty); L_p(\mathbb{R}_d) \cap L_1(\mathbb{R}_d))
\]

and that

\[
\|Z(t, \cdot)\|_{L_\infty} \leq \|Z(t, \cdot)\|_1 \|u_0\|_\infty = \|u_0\|_\infty, \quad t > 0.
\]

For \( 0 < t \leq 1 \) we have that

\[
\left\| \int_0^t Y(t - s, \cdot)\|v(s, \cdot)\|^{\gamma-1}v(s, \cdot)ds \right\|_\infty
\]

\[
\leq \int_0^t \|Y(t - s, \cdot)\|_1 \|v(s, \cdot)\|^{\gamma-1}v(s, \cdot)\|_\infty ds
\]

\[
\lesssim \sup_{(t, x) \in [0, 1] \times \mathbb{R}_d} |v(t, x)|^{\gamma} \int_0^t (t - s)^{\alpha-1} ds
\]

\[
\lesssim \sup_{(t, x) \in [0, 1] \times \mathbb{R}_d} |v(t, x)|^{\gamma}
\]

and for \( t > 1 \) we obtain (see [35, Section 4])

\[
\left\| \int_0^t Y(t - s, \cdot)\|v(s, \cdot)\|^{\gamma-1}v(s, \cdot)ds \right\|_\infty
\]
\[
\leq \int_0^t \| Y(t - s, \cdot) \|_{p'} \| v(s, \cdot) \|_{p'} ds
\]
\[
\lesssim \| v \|_p^\gamma \int_0^t (t - s)^{\frac{-ad}{p} + \alpha - 1} s^{\frac{-ad}{p}} (s)^{\frac{-ad}{p} (\gamma - 1)} \left( \frac{1}{p'} - \frac{1}{p} \right) ds
\]
\[
\lesssim \| v \|_E \| v \|_{p'} \gamma - 1 \lesssim \| v \|_E.
\]

This proves that
\[
\mathcal{M}(v) \in L_\infty((0, \infty); L_\infty(\mathbb{R}^d)).
\]

Besides, as in [35, Section 4] one finds that
\[
\| Z \ast u_0 \|_E \leq C_1 \left( \| u_0 \|_1 + \| u_0 \|_p + \| u_0 \|_\infty \right)
\]
and that the operator \( \mathcal{M} \) is a contraction in the closed ball \( B_R = \{ v \in E : \| v \|_E \leq R \} \) of radius \( R = 2C_1 \left( \| u_0 \|_1 + \| u_0 \|_p + \| u_0 \|_\infty \right) \). Consequently there exists a fixed point \( \tilde{u} \) which is unique in \( E \) because of Gronwall’s inequality ([37, Corollary 2]).

Let \( T > 0 \). We define the Volterra equation
\[
u(t) = u_0 + g_\alpha \ast |\tilde{u}|^{\gamma - 1} \tilde{u}(t) + g_\alpha \ast Au(t), \quad 0 \leq t \leq T,
\]
and by proceeding as in [35, Section 5], since \( u_0 \in H^\beta_2(\mathbb{R}^d) \), we find that there exists a unique strong solution \( u \in L_2([0, T]; H^\beta_2(\mathbb{R}^d)) \), and it satisfies the variation of parameters formula
\[
u(t) = \frac{d}{dt} \int_0^t S(s) \left( u_0 + g_\alpha \ast |\tilde{u}|^{\gamma - 1} \tilde{u} \right) (t - s) ds.
\]

On the other hand, similar arguments as in [35, Lemma 5.1] show that the fixed point \( \tilde{u} \) satisfies
\[
\tilde{u}(t) = \frac{d}{dt} \int_0^t S(s) \left( u_0 + g_\alpha \ast |\tilde{u}|^{\gamma - 1} \tilde{u} \right) (t - s) ds
\]
and therefore \( \tilde{u} = u \). This holds for any \( T > 0 \) which implies that \( u \) is global. \( \square \)

**Remark 3** Since \( u_0 \in L_\infty(\mathbb{R}^d) \), Theorem 4 guarantees the positivity of the global solution \( u \) on \([0, T)\) for some \( T > 0 \).

**Acknowledgements** The authors thank the Chilean Research Grant “Fondo Nacional de Desarrollo Científico y Tecnológico”, FONDECYT 1190255, for partially supporting this research. The authors are very grateful to the referees for their valuable comments and suggestions, which helped to improve the quality of the paper.
Declarations

Conflict of interest  The authors declare that they have no conflict of interest.

References

1. Aguirre, J., Escobedo, M.: A Cauchy problem for $u_t - \Delta u = u^p$ with $0 < p < 1$ asymptotic behaviour of solutions. Annales de la faculté des sciences de Toulouse 5e série 8(2), 175–203 (1986-1987)
2. Aronson, D.: Bounds for the fundamental solution of a parabolic equation. Bull. Amer. Math. Soc. 73(6), 890–896 (1967)
3. Awad, E., Metzler, R.: Closed-form multi-dimensional solutions and asymptotic behaviours for sub-diffusive processes with crossovers: II. Accelerating case. J. Phys. A: Math. Theor. 55(20), 205003 (2022)
4. Awad, E., Sandev, T., Metzler, R., Chechkin, A.: Closed-form multi-dimensional solutions and asymptotic behaviors for subdiffusive processes with crossovers: I. Retarding case. Chaos, Solitons & Fractals 152, 111357 (2021)
5. Bai, Z., Sun, S., Du, Z., et al.: The Green function for a class of Caputo fractional differential equations with a convection term. Fract. Calc. Appl. Anal. 23(3), 787–798 (2020). https://doi.org/10.1515/fca-2020-0039
6. Bazhlekova, E.: Fractional Evolution Equations in Banach Spaces, Dissertation. Technische Universiteit Eindhoven (2001). https://pure.tue.nl/ws/portalfiles/portal/2442305/200113270.pdf
7. Bazhlekova, E.: Subordination in a class of generalized time-fractional diffusion-wave equations. Fract. Calc. Appl. Anal. 21(4), 869–900 (2018). https://doi.org/10.1515/fca-2018-0048
8. Folland, G.B.: Lectures on Partial Differential Equations. Tata Institute of Fundamental Research, India (1983)
9. Friedman, A.: Partial Differential Equations of Parabolic Type. Robert E. Krieger Publishing Company, Florida (1983)
10. Fujita, H.: On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sect. I(13), 109–124 (1966)
11. Hayashi, N., Kaikina, E., Naumkin, P., Shishmarev, I.: Asymptotics for Dissipative Nonlinear Equations. Springer-Verlag, Berlin Heidelberg (2006)
12. Hu, B.: Blow-up Theories for Semilinear Parabolic Equations. Springer-Verlag, Berlin/Heidelberg (2011)
13. Ilyin, A., Kalashnikov, A., Oleynik, O.: Second order linear equations of parabolic type. Journal of Mathematical Sciences 108(4), 435–542 (2002)
14. Jacob, N.: Pseudo-differential Operators and Markov Processes, vol. I. Imperial College Press, World Scientific Publishing CO (2001)
15. Johnston, I., Kolokoltsov, V.: Green’s function estimates for time-fractional evolution equations. Fractal Fract. 3(2), 36 (2019)
16. Kemppainen, J., Siljander, J., Vergara, V., Zacher, R.: Decay estimates for time-fractional and other non-local in time subdiffusion equations in $\mathbb{R}^d$. Math. Ann. 366(3), 941–979 (2016)
17. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier Science Limited (2006)
18. Kirane, M., Laskri, Y., Tatar, N.: Critical exponents of Fujita type for certain evolution equations and systems with spatio-temporal fractional derivatives. J. Math. Anal. Appl. 312(2), 488–501 (2005)
19. Kolokoltsov, V.: Symmetric stable laws and stable-like jump-diffusions. Proceedings of the London Mathematical Society 80(3), 725–768 (2000)
20. Kolokoltsov, V.: Semiclassical Analysis for Diffusions and Stochastic Processes. Springer, Berlin/New York (2000)
21. Kolokoltsov, V.: Markov Processes. Semigroups and Generators. Walter de Gruyter GmbH and Co. KG, Berlin/New York (2011)
22. Kolokoltsov, V.: Differential Equations on Measures and Functional Spaces. Birkhäuser Advanced Texts Basler Lehrbücher, e-book (2019)
23. Li, L., Liu, J., Wang, L.: Cauchy problems for Keller-Segel type time-space fractional diffusion equation. Journal of Differential Equations 265(3), 1044–1096 (2018)
24. Li, Y., Zhang, Q.: Blow-up and global existence of solutions for a time fractional diffusion equation. Fract. Calc. Appl. Anal. 21(6), 1619–1640 (2018). https://doi.org/10.1515/fca-2018-0085
25. Luchko, Y.: Multi-dimensional fractional wave equation and some properties of its fundamental solution. Communications in Applied and Industrial Mathematics 6(1), e-485 (2014)
26. Mainardi, F., Mura, A., Pagnini, G., Gorenflo, R.: Sub-diffusion equations of fractional order and their fundamental solutions. In: Taş, K., Tenreiro Machado, J.A., Baleanu, D. (eds) Mathematical Methods in Engineering, pp. 23–55, Springer, Dordrecht (2007). https://doi.org/10.1007/978-1-4020-5678-9_3
27. Metzler, R., Klafter, J.: The random walk’s guide to anomalous diffusion: a fractional dynamics approach. Physics Reports 339(1), 1–77 (2000)
28. Na, Y., Zhou, M., Zhou, X., Gai, G.: Blow-up theorems of Fujita type for a semilinear parabolic equation with a gradient term. Advances in Difference Equations 2018(1), 128 (2018)
29. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag, New York (1983)
30. Pozo, J., Vergara, V.: Fundamental solutions and decay of fully non-local problems. Discrete and Continuous Dynamical Systems - A 39(1) (2018)
31. Prüss, J.: Evolutionary Integral Equations and Applications. Birkhäuser Verlag, Switzerland (1993)
32. Quittner, P., Souplet, P.: Superlinear Parabolic Problems Blow-up. Global Existence and Steady States. Birkhäuser Verlag AG, Germany (2007)
33. Rudin, W.: Análisis Real y Complejo. McGraw-Hill, España (1987)
34. Schneider, W., Wyss, W.: Fractional diffusion and wave equations. J. Math. Phys. 30(1), 134 (1989)
35. Solís, S., Vergara, V.: A non-linear stable non-Gaussian process in fractional time. Topol. Methods Nonlinear Anal. 59(2B), 987–1028 (2022)
36. Vergara, V., Zacher, R.: Stability, instability, and blowup for time fractional and other nonlocal in time semilinear subdiffusion equations. Journal of Evolution Equations 17(1), 599–626 (2017)
37. Ye, H., Gao, J., Ding, Y.: A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. 328(2), 1075–1081 (2007)
38. Zhang, Q., Sun, H.: The blow-up and global existence of solutions of Cauchy problems for a time fractional diffusion equation. Topol. Methods Nonlinear Anal. 46(1), 69–92 (2015)
39. Zolotarev, V.: One-dimensional Stable Distributions. American Mathematical Society, USA (1986)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.