Regularity for radial solutions of degenerate fully nonlinear equations.

I. Birindelli, F. Demengel

Abstract

In this paper we prove the $C^{1,\beta}$ regularity of the solutions of radial solutions for fully nonlinear degenerate equations.

1 Introduction

In this paper we prove the regularity of radial solutions of

$$ F(x, \nabla u, D^2 u) = f, $$

where $F$ is a fully nonlinear degenerate elliptic operator, homogenous of degree 1 in the Hessian and homogenous of some degree $\alpha > -1$ in the gradient, which is elliptic when the gradient is not null. Precise conditions on $F$ will be stated in the next section.

The class of operators we consider includes:

$$ F(\nabla u, D^2 u) = |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2 u) $$

where $\mathcal{M}_{a,A}$ is one of the Pucci operators (i.e. either $\mathcal{M}_{a,A} = \mathcal{M}^+_a,A$ or $\mathcal{M}_{a,A} = \mathcal{M}^-_{a,A}$);

$$ F(\nabla u, D^2 u) = \Delta_{\alpha+2} u = \text{div}(|\nabla u|^\alpha \nabla u) $$

or, more in general,

$$ F(\nabla u, D^2 u) = |\nabla u|^\alpha (p_1 \text{tr}(D^2 u) + p_2 D^2 u \frac{\nabla u}{|\nabla u|} \cdot \frac{\nabla u}{|\nabla u|}) $$

with $p_1 > 0$ and $p_1 + p_2 > 0$. 
In a previous paper [3] we proved that for $\alpha \in (-1, 0]$ all solutions are $C^{1,\beta}$ if $F$ satisfies

$$F(\nabla u, D^2 u) = |\nabla u|^\alpha \tilde{F}(D^2 u), \quad (1.1)$$

(if $\tilde{F}$ is concave we obtained that the solutions are $C^{2,\beta}$).

Here we prove that even for $\alpha > 0$, the radial solutions are $C^{1,\frac{1+\alpha}{1+\alpha}}$ everywhere, and if the dependence on the Hessian is convex, in points where the derivative is not zero, the solutions are $C^{2,\beta}$. Observe that where the radial derivative is zero the Hölder continuity of the first derivative is optimal. Indeed it is easy to see that $u(r) = r^{\frac{1+2\alpha}{1+\alpha}}$ is a viscosity solution of

$$|\nabla u|^\alpha \mathcal{M}_{a,A}^+(D^2 u) = c$$

for $c = \left(\frac{\alpha+2}{\alpha+1}\right)^{\alpha+1} A(\frac{1}{1+\alpha} + N - 1)$.

Beside its intrinsic interest, the regularity question was raised naturally while proving the simplicity of the principal eigenfunctions. In recent years, the concept of principal eigenvalue has been extended to fully nonlinear operators, by means of the maximum principle (see [1]). The values

$$\lambda^+(\Omega) = \sup\{ \lambda, \exists \phi > 0 \text{ in } \Omega, F(x, \nabla \phi, D^2 \phi) + \lambda \phi^{1+\alpha} \leq 0 \text{ in } \Omega\}$$

$$\lambda^-(\Omega) = \sup\{ \lambda, \exists \psi < 0 \text{ in } \Omega, F(x, \nabla \psi, D^2 \psi) + \lambda |\psi|^\alpha \psi \geq 0 \text{ in } \Omega\}$$

are generalized eigenvalues in the sense that there exists a non trivial solution to the Dirichlet problem

$$F(\nabla \phi, D^2 \phi) + \lambda^\pm(\Omega)|\phi|^\alpha \phi = 0 \text{ in } \Omega, \ \phi = 0 \text{ on } \partial\Omega.$$  

In [3], we proved that, for $F$ satisfying (1.1), these eigenfunctions are simple as long as $\partial \Omega$ has only one connected component. This result extends to the situation where $\partial \Omega$ has at most two connected components when the dimension is 2, the proof uses the fact that the eigenfunctions are $C^{1,\beta}$.

Let us emphasize that regularity results for degenerate elliptic operators that are not in divergence form are in general very difficult. The difficulty comes from the fact that difference of solutions are not sub or super solutions of some elliptic equation. As an example, let us recall that for the infinity Laplacian $\Delta_\infty$ the solutions are known to be in $C^{1,\beta}$ for small $\beta > 0$ only in dimension 2, and only for $f \equiv 0$, see [12].
On the other hand, N. Nadirashvili, S. Vladut in [13] prove $C^2$ regularity of radial solutions for a large class of fully nonlinear operators uniformly elliptic.

Of course we use viscosity solutions, and since, as can easily be imagined, the difficulties arise where the derivative is zero, our first concern is to check that if $u' \neq 0$ in the viscosity sense at some point, then this holds in a neighborhood, and furthermore, in that neighborhood the solution is $C^1$. Then we treat the points where $u' = 0$ in the viscosity sense. The proof relies only on the comparison principle, Hopf principle, the regularity results of [9] and [6], together with some classical analysis.

2 Hypothesis and known results.

In all the paper we suppose that $\Omega$ is a ball or an annulus. We shall consider solutions of the following equation

$$F(x, \nabla u, D^2 u) = f(|x|).$$  \hspace{1cm} (2.2)

The operator $F$ is supposed to be continuous on $\Omega \times (\mathbb{R}^N)^* \times S$, where $S$ is the space of $N \times N$ symmetric matrices and to satisfy:

(H1) For some $\alpha > -1$, for all $x \in \Omega$, for all $p \neq 0$ and $N \in S$ and for all $t \in \mathbb{R}$ and $\mu > 0$, $F(x, tp, \mu N) = |t|^\alpha \mu F(x, p, N)$.

(H2) $F$ is fully nonlinear elliptic, i.e there exist some positive constants $a$ and $A$, such that for any $M \in S$ and $N \geq 0$ in $S$, one has

$$a|p|^\alpha tr(N) \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha tr(N).$$

(H3) Furthermore $F$ is an Hessian operator, i.e. for any $M \in S$ and $O \in \mathcal{O}(n, \mathbb{R})$, $F(Ox^t, Op^t, OMO) = F(x, p, M)$.

(H4) There exists $\nu > 0$ and $\kappa \in [1/2, 1]$ such that for all $|p| = 1$, $|q| \leq \frac{1}{2}$, $M \in S$

$$|F(x, p + q, M) - F(x, p, M)| \leq \nu|q|^\kappa |M|$$

which implies by homogeneity that for all $p \neq 0$, $|q| \leq \frac{|p|}{2}$, $M \in S$

$$|F(x, p + q, M) - F(x, p, M)| \leq \nu |q|^\kappa |p|^\alpha - \kappa |M|$$
We need to precise what we mean by viscosity solutions:

**Definition 2.1** Let \( \Omega \) be a domain in \( \mathbb{R}^N \), let \( g \) be a continuous function on \( \Omega \times \mathbb{R} \), then \( v \), continuous in \( \Omega \) is called a viscosity super-solution (respectively sub-solution) of \( F(x, \nabla u, D^2u) = g(x, u) \) if for all \( x_0 \in \Omega \),

- Either there exists an open ball \( B(x_0, \delta) \), \( \delta > 0 \) in \( \Omega \) on which \( v \) equals to a constant \( c \) and \( 0 \leq g(x, c) \) for all \( x \in B(x_0, \delta) \) (respectively \( 0 \geq g(x, c) \) for all \( x \in B(x_0, \delta) \)).

- Or \( \forall \varphi \in C^2(\Omega) \), such that \( v - \varphi \) has a local minimum (respectively local maximum) at \( x_0 \) and \( \nabla \varphi(x_0) \neq 0 \), one has \( F(x_0, \nabla \varphi(x_0), D^2\varphi(x_0)) \leq g(x_0, v(x_0)) \). (respectively \( F(x_0, \nabla \varphi(x_0), D^2\varphi(x_0)) \geq g(x_0, v(x_0)) \)).

A viscosity solution is a function which is both a super-solution and a sub-solution.

Let us observe that in the case where \( \alpha > 0 \), the operator is well defined everywhere, and then it is a natural question to ask if the viscosity solutions in the sense above are the same as the viscosity solutions in the usual sense. The answer is yes as is proved in the appendix of this paper.

From now on we suppose that \( \alpha \geq 0 \) and that the solutions are “radial”. Let us observe that the hypothesis that \( F \) be a fully nonlinear elliptic hessian operator implies that there exists some operator \( H \) defined on \( \mathbb{R}^+ \times \mathbb{R}^2 \), such that if \( u(x) = g(r) \) is radial and \( C^2, F(x, \nabla u, D^2u) = H(r, g'', g') \) with

\[
|g'|^\alpha \left( A \left( (g'')^- + \frac{N-1}{r} (g')^- \right) + a \left( (g'')^+ + \frac{N-1}{r} (g')^+ \right) \right) 
\leq H(r, g'', g') 
\leq |g'|^\alpha \left( A \left( (g'')^+ + \frac{N-1}{r} (g')^+ \right) + a \left( (g'')^- + \frac{N-1}{r} (g')^- \right) \right)
\]

We now recall some known results concerning the operators considered:

**Proposition 2.2** ([2]) Suppose that \( \Omega \) is a bounded open set. Suppose that \( f \) and \( g \) are continuous on \( \Omega \) and \( f \geq g \). Assume that \( \beta \) is some continuous and non decreasing function on \( \mathbb{R} \), and that \( u \) and \( v \) are continuous respectively sub- and super-solutions of the equation

\[
F(x, \nabla u, D^2u) - \beta(u) \geq f
\]
\[ F(x, \nabla v, D^2 v) - \beta(v) \leq g \]

with \( u \leq v \) on \( \partial \Omega \). Then if \( f > g \) in \( \Omega \), or if \( f \geq g \) but \( \beta \) is increasing, \( u \leq v \) inside \( \Omega \).

We shall also need the Lipschitz regularity of the solutions:

**Proposition 2.3 ([2])** Suppose that \( \Omega \) is an open bounded regular domain of \( \mathbb{R}^N \). Suppose that \( u \) is a function in \( C(\Omega) \) which is a solution of

\[
\begin{cases}
F(x, \nabla u, D^2 u) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

then \( u \) is Lipschitz continuous.

**Remark 2.4** We shall use this Proposition for radial solutions, so we shall fix \( r_o \), and use the previous Proposition for \( u - u(r_o) \) which is a solution of the above homogeneous Dirichlet problem on \( B(0, r_o) \).

In all the paper, \( f \) denotes some continuous and radial function.

### 3 \( C^1 \) regularity

In all the sequel we denote for simplicity

\[ F[u] := H(r, u'', \frac{u'}{r}) \]

and \( u \) is supposed to be a continuous radial solution of \( F[u] = f(r) \) on either the interval \( [0, R) \) or the interval \( (R_1, R_2) \).

**Definition 3.1** For any \( (p, q) \in \mathbb{R}^2 \), we define the paraboloid

\[ w(p, q, r)(s) = u(r) + p(s - r) + \frac{q}{2}(s - r)^2. \]

We also give the following

**Definition 3.2** For a Lipschitz continuous function \( u \), we define the following so called derivative numbers of \( u \) :

\[ \lambda_g(r_1) = \liminf_{r \to r_1, r < r_1} \frac{u(r) - u(r_1)}{r - r_1}, \]
\[ \Lambda_g(r_1) = \limsup_{r \to r_1, r < r_1} \frac{u(r) - u(r_1)}{r - r_1}, \]
\[ \lambda_d(r_1) = \liminf_{r \to r_1, r > r_1} \frac{u(r) - u(r_1)}{r - r_1}, \]
\[ \Lambda_d(r_1) = \limsup_{r \to r_1, r > r_1} \frac{u(r) - u(r_1)}{r - r_1}. \]

Clearly for \( r_1 = 0 \), only the "right" derivatives are defined.

We shall say that \( u'(\bar{r}) > 0 \) in the viscosity sense (respectively \( u'(\bar{r}) < 0 \)) if
\[ \inf(\lambda_g(\bar{r}), \lambda_d(\bar{r})) > 0 \] (respectively if \( \sup(\Lambda_d(\bar{r}), \Lambda_g(\bar{r})) < 0 \)).

On the opposite we shall say that \( u'(\bar{r}) = 0 \) in the viscosity sense if one has
\[ \lambda_g(\bar{r})\lambda_d(\bar{r}) \leq 0 \] and \( \Lambda_d(\bar{r})\Lambda_g(\bar{r}) \geq 0 \).

**Remark 3.3** Let us note first that all the numbers defined above exist and are finite for \( u \) a solution of (2.2) since the solutions are known to be Lipschitz. Furthermore we proved in [4] that since \( f \) is bounded, \( \Lambda_g \leq \lambda_d \), \( \Lambda_d \geq \lambda_g \). Finally, if all these numbers coincide on \( \bar{r} \), \( u' \) exists on \( \bar{r} \) in the classical sense.

We begin with a simple lemma

**Lemma 3.4** Suppose that \( u \) is a radial continuous viscosity solution of \( F[u] = f \) on \([0, R)\), then \( u'(0) \) exists and it is zero.

**Remark 3.5** We want to point out that for radial function, i.e. for the continuous functions \( u \) defined on some ball of \( \mathbb{R}^N \), such that there exists \( v \) continuous on \([0, r]\), with \( u(x) = v(|x|) \), in order to test on points \( x \neq 0 \), it is sufficient to use test functions which are radial. A consequence of Lemma 3.4 and Definition 2.7 is that we do not need to test at the point zero. As a consequence \( u \) is a viscosity supersolution of \( F[u] = f \) in \( B(0, R) \) if and only if \( u'(0) \) exists and is zero, and for all \( r \neq 0 \) and for all \( \varphi \) which is \( C^2 \) around \( \bar{r} \neq 0 \) which touches \( u \) by below on \( \bar{r} \)
\[ F[\varphi](\bar{r}) \leq f(\bar{r}). \]

**Proof of Lemma 3.4.** We want to prove that
\[ \Lambda_d(0) = \lambda_d(0) = 0. \]

Suppose that \( \Lambda_d(0) = m > 0 \). Let \( m' < m \) arbitrary close to it. Choose \( \delta \) small enough in order that
\[ a(m')^{1+\alpha} \frac{N - 1}{\delta} > |f|_{\infty}. \]
By hypothesis, for such $\delta$, there exists $r \in [0, \delta]$ such that
\[
\frac{u(r) - u(0)}{r} \geq m' \text{ i.e. } u(r) \geq u(0) + m'r.
\]
Let $w := w(m', 0, 0)$ so that $w(0) = u(0)$, $w(r) \leq u(r)$, and
\[
F[w](s) \geq a(m')^{1+\alpha}\frac{N-1}{s} \geq f(s) \text{ on } [0, \delta].
\]
Then, by Proposition 2.2, $w(s) \leq u(s)$, in $[0, r]$. Hence
\[
\frac{u(s) - u(0)}{s} \geq m', \text{ and } \lambda_d(0) \geq m'.
\]
This implies that $\Lambda_d(0) = \lambda_d(0)$.

We now suppose that $\Lambda_d(0) \leq 0$, so either $\lambda_d(0) = 0$ and then they are equal and we are done, or $\lambda_d(0) = -m < 0$.

Let $0 < m' < m$ arbitrary close to it. Choose $\delta$ as above, by hypothesis, there exists $r \in [0, \delta]$, such that
\[
\frac{u(r) - u(0)}{r} \leq -m'.
\]
Let $w \equiv w(-m', 0, 0)$, then $w(0) = u(0)$, $w(r) \geq u(r)$ and
\[
F[w](s) \leq -a(m')^{1+\alpha}\frac{N-1}{s} \leq f(s) \text{ on } [0, \delta].
\]
Then by Proposition 2.2, $w(s) \geq u(s)$ and then for all $s \in [0, \delta]$,
\[
\frac{u(s) - u(0)}{s} \leq -m'.
\]
This implies that $\Lambda_d(0) = \lambda_d(0)$. And the existence of the derivative at zero is proved.

We still need to prove that it is zero. Suppose by contradiction that it is not, one can suppose that it is positive, the other case can be done with obvious changes.

Let $\delta$ be small enough that
\[
\frac{a(u'(0))^{1+\alpha}}{\delta^{2+\alpha}} > f(r) \text{ for } r < \delta.
\]
The function
\[
\varphi(x) = u(0) + \frac{u'(0)}{2}x_1 + \frac{u'(0)}{4\delta}x_2^2.
\]
touches \( u \) from below at zero and can be used as a test function and by definition of a supersolution
\[
\frac{a(u'(0))^{1+\alpha}}{\delta^{2(1+\alpha)}} \leq F[\varphi](0) \leq f(0),
\]
a contradiction with the choice of \( \delta \). This ends the proof.

We now state the main result of this section.

**Theorem 3.6** Suppose that \( u \) is a radial solution of \( F[u] = f \). Then \( u \) is \( C^1 \).

The proof of Theorem 3.6 relies on Proposition 3.7, Corollary 3.8 and Proposition 3.9.

**Proposition 3.7** Suppose that \( u \) is a radial solution of \( F[u] = f \) such that in \( \bar{r} \) one of the derivative numbers is different from zero.

Then in a neighborhood of \( \bar{r} \), \( u' \) exists in the classical sense and the function \( u' \) is continuous in \( \bar{r} \).

**Corollary 3.8** If, at \( r_1 \), one of the derivative numbers is zero, then \( u'(r_1) \) exists and it is zero.

**Proof of Proposition 3.7** There are, in theory, 8 cases to treat, because each of the derivative number may be either positive or negative. But in fact considering the function \( v = -u \), that satisfies
\[
G[v] := -F[-v] = -f,
\]
it is enough to consider only half of the cases.

What we want to prove is that, for any \( \bar{r} \), as in Proposition 3.7
\[
\exists \delta > 0, \lambda_d(r) = \lambda_g(r) = \Lambda_d(r) = \Lambda_g(r), \forall r \in (\bar{r} - \delta, \bar{r} + \delta). \tag{3.3}
\]

**Claim 1: \( \Lambda_d(\bar{r}) = k < 0 \) implies the thesis (3.3).**

We first prove that
\[
\Lambda_d(\bar{r}) < \mu < 0 \Rightarrow \exists \delta > 0, \sup(\lambda_d, \lambda_g, \Lambda_d, \Lambda_g)(r) < \mu < 0, \forall r \in (\bar{r} - \delta, \bar{r} + \delta). \tag{3.4}
\]
Let \( \mu' \in ]\Lambda_d, \mu[ \), and let \( \delta_1 \) be small enough in order that \( \frac{\mu'}{1 - \sqrt{\delta_1}} > \Lambda_d \). Let \( \delta_2 < \delta_1 \) be small enough in order that
\[
\frac{a|\mu|^{1+\alpha}}{(1 - \sqrt{\delta_2})\sqrt{\delta_2}} \geq |f|_\infty. \tag{3.5}
\]
Let $\delta_3 < \bar{r}$ such that for $\delta < \inf(\delta_1, \delta_2, \delta_3)$
\[
\frac{u(\bar{r} + \delta) - u(\bar{r})}{\delta} \leq \frac{\mu'}{1 - \sqrt{\delta_1}}.
\]
Fixe such a $\delta$, by continuity of $u$, there exists $\delta_4$ such that for $r \in [\bar{r} - \delta_4, \bar{r} + \delta_4[$
\[
\frac{u(r + \delta) - u(r)}{\delta} \leq \frac{\mu}{1 - \sqrt{\delta_1}}.
\]
For any such $r$ let
\[
w := w(\mu, \frac{\mu}{(1 - \sqrt{\delta})\sqrt{\delta}}, r).
\]
Then
\[
w(r) = u(r), \quad w(r + \delta) \geq u(r + \delta),
\]
and using (3.5) it is easy to check that $w$ is a supersolution in $[r, r + \delta]$. From this, using Proposition 2.2 one gets that $w(s) \geq u(s)$ on $[r, r + \delta]$ and then
\[
\frac{u(s) - u(r)}{s - r} \leq \mu
\]
which implies that $\Lambda_d(r) \leq \mu$. Exchanging the roles of $r$ and $s$ one obtains also $\Lambda_g(r) \leq \mu$. This proves (3.4).

To complete the proof of Claim 1 we prove that
\[
0 > \Lambda_d(\bar{r}) > \mu \Rightarrow \exists \delta > 0, \inf(\lambda_d, \lambda_g, \Lambda_d, \Lambda_g)(r) > \mu, \forall r \in (\bar{r} - \delta, \bar{r} + \delta). \quad (3.7)
\]
Indeed (3.4) and (3.7) imply the thesis (3.3).

The proof of (3.7) proceeds in a similar fashion then that of (3.4), we give the detail of the computation for completeness sake. Let $\delta_1 < \inf(\frac{\bar{r}}{2}, 1)$ be small enough in order that $\frac{\mu}{1 + \sqrt{\delta_1}} < \Lambda_d$, that
\[
\frac{8A(N - 1)}{\bar{r}} < \frac{a}{\sqrt{\delta_1}} \quad \text{and that} \quad \frac{|\mu|^{1+\alpha}a}{2(1 + \sqrt{\delta_1})^{1+\alpha}\sqrt{\delta_1}} > |f|_\infty.
\]
As above there exists $\delta_4$ such that for $\delta < \delta_1$ small enough and for $r \in [\bar{r} - \delta_4, \bar{r} + \delta_4[$,
\[
\frac{u(r + \delta) - u(r)}{\delta} > \frac{\mu}{1 + \sqrt{\delta_1}}.
\]
We define, on $[r, r + \delta]$, 

$$w := w(\mu, \frac{|\mu|}{(1 + \sqrt{\delta})\sqrt{\delta}}, r)$$

then $w(r) = u(r)$, $w(r + \delta) \leq u(r + \delta)$ and, using (3.8), $w$ is a sub-solution. Then 

$$w(s) \leq u(s) \text{ on } [r, r + \delta].$$

Finally one gets that there exists $\delta_4$, and $\delta$ such that for $r \in \bar{r} - \delta_4, \bar{r} + \delta_4$, and $s \in [r, r + \delta]$ 

$$u(s) \geq u(r) + \mu(s - r),$$

hence 

$$\lambda_d(r) \geq \mu.$$ 

Exchanging the roles of $r$ and $s$, one gets $\lambda_g(r) \geq \mu$. This proves (3.7) and it ends the proof of Claim 1.

Observe that in fact, (3.4) and (3.7) prove more than Claim 1 because they imply that the derivative of $u$ is continuous in points where the derivative is not zero.

Claim 1 implies

Claim 2: $\lambda_d(\bar{r}) = k > 0$ implies the thesis (3.3).

Indeed for $v = -u$, 

$$\lambda_{d,u}(\bar{r}) = -\Lambda_{d,v}(\bar{r})$$

(here we have added in the index the function for which the derivative number is computed).

The proofs of the following claims are similar to the proof of (3.4) or (3.7) but, of course, each case needs a different choice of function $w$, so we give the details for completeness sake.

Claim 3: $\Lambda_d(\bar{r}) = k > 0$ implies the thesis (3.3).

Suppose that $0 < \mu < \Lambda_d$, and let $\delta_1$ be so that $\frac{\mu^{1+\alpha}}{(1-\sqrt{\delta_1})\sqrt{\delta_1}} \geq |f|_{\infty}$. Let $\delta_2 < \delta_1$ be so that $\frac{\mu}{(1-\sqrt{\delta_2})\sqrt{\delta_2}} < \Lambda_d$. As it is done before let $\delta_3 < \inf(\delta_1, \delta_2)$ be so that for $\delta < \delta_3$, 

$$\frac{u(\bar{r} + \delta) - u(\bar{r})}{\delta} > \frac{\mu}{1 - \sqrt{\delta_2}}.$$ 

Finally let $\delta_4$ so that for $r \in [\bar{r} - \delta_4, \bar{r} + \delta_4]$ one has $\frac{u(r + \delta) - u(r)}{\delta} > \frac{\mu}{1 - \sqrt{\delta_2}}$. Then define $w := w(\mu, \frac{\mu}{(1-\sqrt{\delta_2})\sqrt{\delta}}, r)$. Then $w(r) = u(r)$, $w(r + \delta) \leq u(r + \delta)$ and $w$ is a sub-solution.
Hence, by Proposition 2.2, \( w(s) \leq u(s) \) on \( [r, r + \delta] \), which implies that for \( r \in ]\bar{r} - \delta_4, \bar{r} + \delta_4[ \),

\[
\inf(\Lambda_d, \Lambda_g, \lambda_g, \lambda_d)(r) \geq \mu.
\]

We are back to the hypothesis of Claim 2, which implies the thesis.

Claim 4: \( \Lambda_g(\bar{r}) = k < 0 \) implies the thesis (3.3).

Let \( \mu \) be such that \( 0 > \mu > \Lambda_g(\bar{r}) \). Let \( \delta_1 < \frac{\bar{r}}{2}, \sqrt{\delta_1} < \inf(1, \frac{ar}{8A(N-1)}), \Lambda_g(\bar{r}) < \frac{\mu}{1 - \sqrt{\delta_1}} \), and such that

\[
\frac{|\mu|^{1+\alpha}a}{2(1 - \sqrt{\delta_1})\sqrt{\delta_1}} > |f|_\infty.
\]

As before there exists \( \delta_4 < \delta_1 \) such that for \( r \in ]\bar{r} - \delta_4, \bar{r} + \delta_4[ \) and for \( \delta < \delta_4 \) one has

\[
\frac{u(r - \delta) - u(r)}{-\delta} \leq \frac{\mu}{1 - \sqrt{\delta_1}}.
\]

Then

\[
w(s) := w(\mu, \frac{|\mu|}{(1 - \sqrt{\delta})\sqrt{\delta}}, r)(s)
\]

is a subsolution which satisfies \( w(r) = u(r), w(r - \delta) \leq u(r - \delta) \). Then, by Proposition 2.2 \( w(s) \leq u(s) \) in \( (r - \delta, r) \). This in turn implies that

\[
\sup(\lambda_d, \lambda_g, \Lambda_d, \Lambda_g)(r) < \mu < 0.
\]

We are once again in the hypothesis of Claim 1 and we are done.

Claim 5: \( \Lambda_g(\bar{r}) = k > 0 \) implies the thesis (3.3).

Let \( \mu \) be so that \( \Lambda_g > \mu > 0 \). Let \( \delta_1 \) be such that \( \delta_1 < \frac{\bar{r}}{2}, \frac{\mu}{1 - \sqrt{\delta_1}} < \Lambda_g \),

\[
\sqrt{\delta_1} < \inf(1, \frac{ar}{8A(N-1)}) \quad \text{and} \quad \frac{\mu^{1+\alpha}a}{2(1 - \sqrt{\delta_1})\sqrt{\delta_1}} \geq |f|_\infty.
\]

As before for \( \delta \) fixed, \( \delta < \delta_1 \), there exists some \( \delta_4 < \delta_1 \) such that for \( r \in ]\bar{r} - \delta_4, \bar{r} + \delta_4[ \),

\[
\frac{u(r - \delta) - u(r)}{-\delta} \geq \frac{\mu}{1 - \sqrt{\delta_1}}.
\]

We define \( w := w(\mu, \frac{\mu}{(1 - \sqrt{\delta})\sqrt{\delta}}, r) \), then \( w(r) = u(r), w(r - \delta) \geq u(r - \delta) \) and by the assumptions \( w \) is a supersolution on \( (r - \delta, r) \). Once more this implies that

\[
\inf(\lambda_d, \lambda_g, \Lambda_d, \Lambda_g)(r) > \mu > 0, \forall r \in (\bar{r} - \delta, \bar{r} + \delta).
\]
And we conclude with Claim 2.

Again, using \( v = -u \), the Claims 3, 4 and 5 give that respectively \( \lambda_d(\bar{r}) = k < 0 \) or \( \lambda_g(\bar{r}) = k < 0 \) or \( \lambda_g(\bar{r}) = k > 0 \) imply the thesis (3.3). And this ends the proof.

Proof of Corollary 3.8. By Proposition 3.7 if one of the derivative numbers has a sign then \( u'(r_1) \) exists and it is different from zero, which contradicts our hypothesis, so all four derivative numbers are zero and \( u'(r_1) = 0 \) in the classical sense.

We finally give the last step of the proof of Theorem 3.6:

Proposition 3.9 \( u' \) is continuous on points where \( u' = 0 \).

Proof. We treat separately the case \( \bar{r} = 0 \) and \( \bar{r} \neq 0 \). In the latter case, let \( \epsilon > 0 \) and \( \delta_1 < \inf(\frac{\bar{r}}{2}, \frac{a\epsilon^{1+\alpha}}{2^{1+\alpha}|f|_\infty}) \), such that for \( \delta < \delta_1 \),

\[
\frac{u(r-\delta) - u(r)}{(-\delta)} \leq \frac{\epsilon}{2}.
\]

Fixing such \( \delta \), let \( \delta_2 < \delta_1 \) such that for \( r \in [\bar{r} - \delta_2, \bar{r} + \delta_2] \), by continuity,

\[
\frac{u(r-\delta) - u(r)}{(-\delta)} \leq \frac{\epsilon}{2}.
\]

Let \( w := w(\epsilon, \frac{\epsilon}{2}, \bar{r}) \). Then \( w(r) = u(r), w(r - \delta) \leq u(r - \delta) \) and, by the hypothesis on \( \delta \), \( w \) is a sub-solution on \( [r - \delta, r] \). Then \( w(s) \leq u(s) \) for \( r \in [\bar{r} - \delta_2, \bar{r} + \delta_2] \), and \( s \in [r - \delta, r] \). By passing to the limit when \( s \) goes to \( r \) it gives

\[
u'(r) \leq \epsilon.
\]

We now prove that for all \( \epsilon > 0 \), there exists a neighborhood of \( \bar{r} \) where \( u' \geq -\epsilon \).

Let, as above,

\[
\delta_1 < \inf(\frac{\bar{r}}{2}, \frac{a\epsilon^{1+\alpha}}{2^{1+\alpha}|f|_\infty}),
\]

such that for \( \delta < \delta_1 \),

\[
\frac{u(r-\delta) - u(r)}{(-\delta)} \leq \frac{\epsilon}{4}.
\]

Fixing such \( \delta \), let \( \delta_2 < \delta_1 \) such that for \( r \in [\bar{r} - \delta_2, \bar{r} + \delta_2] \), by continuity,

\[
\frac{u(r-\delta) - u(r)}{(-\delta)} \geq \frac{-\epsilon}{2}.
\]

Let \( w := w(-\epsilon, \frac{\epsilon}{2}, \bar{r}) \). Then \( w(r - \delta) \geq u(r - \delta) \), \( w(r) = u(r) \) and \( w \) is a supersolution on \( [r - \delta, r] \). Using again Proposition 2.2

\[
w(s) \geq u(s),
\]
and passing to the limit \( u'(r) \geq -\epsilon \).

We now consider the case where \( \bar{r} = 0 \). We want to prove the inequality \( |u'| \leq \epsilon \) in a neighborhood of zero.

Take any \( \epsilon > 0 \), by Lemma 3.4 there exists \( \delta_\epsilon > 0 \) such that
\[
\frac{|u(\delta) - u(0)|}{\delta} \leq \frac{\epsilon}{2}, \quad \forall \delta \in (0, \delta_\epsilon).
\]

Let \( \delta \leq \min\left(\frac{a\epsilon^{1+n}(N-1)}{2|f|_\infty}, \delta_\epsilon\right) \), by the continuity of \( u \), there exists \( \delta_1 < \delta \) such that for \( r \in [\delta, \delta + \delta_1] \),
\[
\frac{u(r) - u(r - \delta)}{\delta} \geq -\epsilon.
\]

Let us consider \( w := w(-\epsilon, 0, r) \). Then \( u(r) = w(r), \, w(r - \delta) \geq u(r - \delta) \) and, with our choice of \( \delta \), \( w \) is a supersolution in \([r - \delta, r]\). By Proposition 2.2 we have obtained that
\[
u(s) \leq w(s) \text{ on } [r - \delta, r] \]
and then
\[
u'(r) \geq -\epsilon, \quad \forall r \in [\delta, \delta + \delta_1].
\]

In particular, \( u'(\delta) \geq -\epsilon \) for any \( \delta \leq \min\left(\frac{a\epsilon^{1+n}(N-1)}{2|f|_\infty}, \delta_\epsilon\right) \).

In a similar fashion, for any \( \delta \leq \min\left(\frac{a\epsilon^{1+n}(N-1)}{2|f|_\infty}, \delta_\epsilon\right) \) and for all \( r \in [\delta, \delta + \delta_1] \)
\[
u'(r) \leq \epsilon.
\]

This is the desired result and it ends both the proof of Proposition 3.9 and the proof of Theorem 3.6.

4 \( C^{1, \beta} \) regularity.

Observe that we now know that \( u' \) is continuous, so we can consider \( \tilde{F}(x, D^2v) := F(x, \nabla u, D^2v) \) and clearly \( u \) is a solution of
\[
\tilde{F}(x, D^2v) := f(x).
\]

**Theorem 4.1** Suppose that \( N \leq 3 \) or for any \( N > 3 \) that \( M \mapsto F(x, p, .) \) is convex or concave and that \( u \) is a radial solution of \( F[u] = f \). Then \( u \) is \( C^{1+\frac{\beta}{1+n}} \) everywhere and is \( C^2 \) on points where the derivative is different from zero.
The case where the derivative is different from zero is easy to treat:

**Proposition 4.2** Suppose that for \( r_0 > 0 \), \( u'(r_0) \neq 0 \) in the viscosity sense. Then, on a neighborhood around \( r_0 \), \( u \) is \( C^{1,\beta} \) for some \( \beta \), and if \( N \leq 3 \) or for any dimension when \( \tilde{F} \) is convex or concave, \( u \) is \( C^{2,\beta} \).

**Proof of Proposition 4.2.** Observe that condition [(H4)] implies that we are in the hypothesis of \([7]\) in \( B_{r_0+\delta} \setminus B_{r_0-\delta} \) for some \( \delta > 0 \). Hence \( u \) is \( C^{1,\beta} \) on that annulus.

Similarly if \( F \) is concave or convex we are in the hypothesis of \([11]\) and \([7]\) and \( u \) is \( C^{2,\beta} \).

Note that when \( N \leq 3 \), the \( C^{2,\beta} \) regularity holds without the convexity or concavity assumption thanks to \([13]\).

To prove the \( C^{1,1_{\alpha}} \) regularity result on any point, including those with the derivative equal to zero, we begin to establish a technical lemma.

**Lemma 4.3** Suppose that \( u \) is a radial \( C^2 \) viscosity solution of (2.2), and that on \([r,s], 0 < r < s, \ u' > 0 \). Then

\[
|u'|^\alpha u'(s) \leq |u'|^\alpha u'(r) + (1 + \alpha) \int_r^s \epsilon_{a,A}(f(t)) dt
\]

(4.9)

where \( \epsilon_{a,A}(x) = \frac{x^+}{a} - \frac{x^-}{A} ; \) furthermore, for \( \gamma = \frac{A}{a}(N - 1)(1 + \alpha) \)

\[
|u'|^\alpha u'(s) \geq \left( \frac{r}{s} \right)^\gamma |u'|^\alpha u'(r) - \frac{|f|_\infty (1 + \alpha) s}{A(N - 1)(1 + \alpha) + A} \left( 1 - \left( \frac{r}{s} \right)^{\gamma+1} \right).
\]

(4.10)

If \( u' < 0 \) on \([r,s], 0 < r < s \), then

\[
|u'|^\alpha u'(s) \geq |u'|^\alpha u'(r) + (1 + \alpha) \int_r^s \epsilon_{A,a}(f(t)) dt
\]

(4.11)

and

\[
|u'|^\alpha u'(s) \leq \left( \frac{r}{s} \right)^\gamma |u'|^\alpha u'(r) + \frac{|f|_\infty (1 + \alpha) s}{A(N - 1)(1 + \alpha) + a} \left( 1 - \left( \frac{r}{s} \right)^{\gamma+1} \right).
\]

(4.12)

**Proof:** We start by supposing that \( u' > 0 \) in \((r,s)\). Since \( u'' \) is continuous then \((u'')^{-1}(\mathbb{R}^+)\) is an open set of \( \mathbb{R}^+ \) and there exists a union of numerable open sets \( \bigcup_{n \in \mathbb{N}} [r_n, r_{n+1}] \), with \([r,s] = \bigcup_{n \in \mathbb{N}} [r_n, r_{n+1}] \) where \( u'' \) is of constant sign on each interval \([r_n, r_{n+1}] \). By redefining in an obvious fashion the end points of the intervals...
one can suppose that \( r, s = \cup_{n \in \mathbb{Z}} [r_n, r_{n+1}] \) with \( u'' \geq 0 \) on \([r_{2p}, r_{2p+1}]\) and \( u'' \leq 0 \) on \([r_{2p+1}, r_{2p+2}]\).

Then, in \([r_{2p}, r_{2p+1}]\),

\[
a(u'' + \frac{(N-1)}{r} u')|u'|^\alpha \leq f(r) \leq A(u'' + \frac{(N-1)}{r} u')|u'|^\alpha
\] (4.13)

and, in \([r_{2p+1}, r_{2p+2}]\),

\[
(Au'' + \frac{a(N-1)}{r} u')|u'|^\alpha \leq f(r) \leq (au'' + \frac{A(N-1)}{r} u')|u'|^\alpha.
\] (4.14)

We begin to prove (4.9) using the inequality on the left of \( f \). In each case one can drop the term \( |u'|^\alpha u' \), hence integrating and using \( \frac{\int A}{\int a} \leq \epsilon_{a,A}(f) \), this imply (4.9) on \([r_{2p}, r_{2p+1}]\) and on \([r_{2p+1}, r_{2p+2}]\).

Let now \( P \) arbitrary large negative and \( N \) arbitrary large positive, \( r_P \) close to \( r \) and \( r_N \) close to \( s \),

\[
|u'|^\alpha u'(r_N) \leq |u'|^\alpha u'(r_{N-1}) + (1 + \alpha) \int_{r_N}^{r_N} \epsilon_{a,A}(f(t)) dt
\]

\[
\leq |u'|^\alpha u'(r_P) + (1 + \alpha) \sum_{r_n}^{r_{n+1}} \int_{r_n}^{r_{n+1}} \epsilon_{a,A}(f(t)) dt
\]

and one obtains (4.9) by passing to the limit when \( P \) and \( N \) go respectively to \(-\infty\) and \(+\infty\).

We now prove (4.10). In \([r_{2p+1}, r_{2p+2}]\), since \( u'' \leq 0 \) we multiply the second inequality of (4.14) by \( \frac{(1+\alpha)}{a} \frac{\int A(t)}{\int a(t)} := \frac{(1+\alpha)}{a} r^\gamma \) and integrating one gets

\[
r_{2p+2}^\gamma |u'|^\alpha u'(r_{2p+2}) \geq r_{2p+1}^\gamma |u'|^\alpha u'(r_{2p+1}) + \int_{r_{2p+1}}^{r_{2p+2}} (1 + \alpha) f(t)t^\gamma dt.
\]

Hence dividing by \( r_{2p+2}^\gamma \), using \( f \geq -|f|_\infty \) one gets

\[
|u'|^\alpha u'(r_{2p+2}) \geq \left( \frac{r_{2p+1}}{r_{2p+2}} \right)^\gamma |u'|^\alpha u'(r_{2p+1}) - \frac{(1 + \alpha)}{a} |f|_\infty \int_{r_{2p+1}}^{r_{2p+2}} \left( \frac{t}{r_{2p+2}} \right)^\gamma dt.
\]

Similarly, if \( u'' > 0 \), multiplying (4.13) by \( (1 + \alpha) r^{(N-1)(1+\alpha)} := (1 + \alpha) r^{-\nu_1} \) one gets

\[
|u'|^\alpha u'(r_{2p+1}) \geq \left( \frac{r_{2p}}{r_{2p+1}} \right)^\gamma |u'|^\alpha u'(r_{2p}) - \frac{(1 + \alpha)}{A} |f|_\infty \int_{r_{2p}}^{r_{2p+1}} \left( \frac{t}{r_{2p+1}} \right)^\gamma dt
\]

\[
\geq \left( \frac{r_{2p}}{r_{2p+1}} \right)^\gamma |u'|^\alpha u'(r_{2p}) - \frac{(1 + \alpha)}{a} |f|_\infty \int_{r_{2p}}^{r_{2p+1}} \left( \frac{t}{r_{2p+1}} \right)^\gamma dt.
\]
We have used the fact that $\gamma = (N - 1)(1 + \alpha) < (N - 1)(1 + \alpha)\frac{A}{a} := \gamma$ and $A(N - 1)(1 + \alpha) + A \geq A(N - 1)(1 + \alpha) + a$.

Using the same decomposition of $|r|, s = \cup_{n \in \mathbb{Z}} [r_n, r_{n+1}]$, with $u''$ of constant sign in each interval, for $P$ large negative and $N$ large positive, $r_P$ close to $r$ and $r_N$ close to $s$, one has

$$|u'|^{\alpha}u'(r_N) \geq \left(\frac{r_N - 1}{r_N}\right)^{\gamma} |u'|^{\alpha}u'(r_{N-1}) - \frac{|f|_{\infty}(1 + \alpha)}{a} \int_{r_{N-1}}^{r_N} \left(\frac{t}{r_N}\right)^{\gamma} dt \geq \left(\frac{r_N - 1}{r_N}\right)^{\gamma} \left(\frac{r_{N-1}}{r_N}\right)^{\gamma} |u'|^{\alpha}u'(r_{N-2})$$

$$- \left(\frac{r_N - 1}{r_N}\right)^{\gamma} \frac{|f|_{\infty}(1 + \alpha)}{a} \int_{r_{N-2}}^{r_{N-1}} \left(\frac{t}{r_N}\right)^{\gamma} dt - \frac{|f|_{\infty}(1 + \alpha)}{a} \int_{r_{N-1}}^{r_N} \left(\frac{t}{r_N}\right)^{\gamma} dt$$

$$= \left(\frac{r_N - 1}{r_N}\right)^{\gamma} |u'|^{\alpha}u'(r_{N-2}) - \frac{|f|_{\infty}(1 + \alpha)}{a} \int_{r_{N-2}}^{r_N} \left(\frac{t}{r_N}\right)^{\gamma} dt - \frac{|f|_{\infty}(1 + \alpha)}{a} \int_{r_{N-1}}^{r_N} \left(\frac{t}{r_N}\right)^{\gamma} dt$$

$$\geq \left(\frac{r_{P}}{r_N}\right)^{\gamma} |u'|^{\alpha}u'(r_P) - \frac{|f|_{\infty}(1 + \alpha)}{a} \int_{r_P}^{r_N} \left(\frac{t}{r_N}\right)^{\gamma} dt.$$ 

By passing to the limit when $P$ and $N$ go to $-\infty$ and $+\infty$ one obtains (4.10).

The inequalities (4.11) and (4.12) can of course be proved either in the same manner or considering $v = -u$ as the solution of

$$G[v] = -f$$

and $G[v] = -F[-v]$ which possesses the same properties as $F$.

**Proposition 4.4** The solutions of $F[u] = f$ are $C^{1, \frac{1}{1+\alpha}}$.

**Proof.** Let $r_1 > 0$ such that $u'(r_1) = 0$, and let $\frac{r_1}{2} < r < r_1$. We shall prove that

$$|u'|^{\alpha+1}(r) \leq \frac{2^{\gamma-1}(\gamma + 1)|f|_{\infty}(1 + \alpha)}{A}(r_1 - r).$$

(4.15)

For that aim, suppose first that $u'(r) > 0$ and let $s$ be the first point between $r_1$ and $r$, so that $u'(s) = 0$. Then $u' > 0$ between $s$ and $r$ and inequality (4.10), with $\gamma = \frac{A(N - 1)(1 + \alpha)}{a}$, becomes

$$|u'|^{\alpha}u'(s) = 0 \geq \left(\frac{r}{s}\right)^{\gamma} |u'|^{\alpha}u'(r) - \frac{|f|_{\infty}(1 + \alpha)s}{A(N - 1)(1 + \alpha) + a} \left(1 - \left(\frac{r}{s}\right)^{\gamma+1}\right)$$

16
and then

\[ |u'|^\alpha u'(r) \leq \left( \frac{s}{r} \right)^\gamma \frac{|f|_\infty (1 + \alpha)}{A(N - 1)(1 + \alpha) + a} \left( 1 - \left( \frac{r}{s} \right)^{\gamma + 1} \right) \]

\[ \leq (\gamma + 1) \frac{s^{\gamma - 1}}{r^{\gamma - 1}} \frac{|f|_\infty (1 + \alpha)}{A(N - 1)(1 + \alpha) + a} (s - r) \]

\[ \leq 2^{\gamma - 1} (\gamma + 1) \frac{|f|_\infty (1 + \alpha)}{A(N - 1)(1 + \alpha) + a} (r_1 - r). \]

From this one gets that

\[ |u'(r)| \leq C(r_1 - r)^{\frac{1}{\gamma + \alpha}}. \]

The case where \( u'(r) < 0 \) can be done similarly.

We now consider the right of \( r_1 \). This proof still holds when \( r_1 = 0 \). Suppose \( s > r_1 \), we want to prove that

\[ |u'|^{\alpha + 1}(s) \leq \frac{(1 + \alpha)|f|_\infty}{a} (s - r). \]

For that aim suppose that \( u'(s) > 0 \). Let \( r \) be the last point in \( ]r_1, s[ \) such that \( u'(r) = 0 \). Then \( u' > 0 \) on \( ]r, s[ \) and by (4.13)

\[ |u'|^\alpha u'(s) \leq 0 + (1 + \alpha) \int_r^s \epsilon_{a,A}(f). \]

This implies

\[ |u'|^{\alpha + 1}(s) \leq \frac{(1 + \alpha)|f|_\infty}{a} (s - r) \leq \frac{(1 + \alpha)|f|_\infty}{a} (s - r_1). \]

5 Appendix : The equivalence of definitions of viscosity solution in the case \( \alpha > 0 \).

We have the following equivalence result

**Proposition 5.1** If \( F \) satisfies (H1) and (H2) with \( \alpha \geq 0 \), the viscosity solutions in the classical meaning are the same as the viscosity solutions in the sense of Definition 2.1.

**Remark 5.2** Let us note that the Definition 2.1 presents an advantage with regards to the classical definition since it allows to not test points where the gradient of a test function is zero.
**Proof of Proposition 5.1.**

We assume that $u$ is a supersolution in the sense of Definition 2.1 and we want to prove that it is a supersolution in the classical sense. We suppose that for $x_o \in \Omega$ there exists $M \in S$ such that

$$u(x) \geq u(x_o) + \frac{1}{2} (M(x - x_o), (x - x_o)) + o(|x - x_o|^2) := \phi(x). \quad (5.16)$$

Let us observe first that one can suppose that $M$ is invertible, since if it is not, it can be replaced by $M_{n} = M - \frac{1}{n} I$ which satisfies (5.16) and tends to $M$. Without loss of generality we will suppose that $x_o = 0$.

Let $k > 2$ and $R > 0$ such that

$$\inf_{x \in B(0, R)} \left( u(x) - \frac{1}{2} |M x| + |x|^k \right) = u(0)$$

where the infimum is strict. We choose $\delta < R$ such that $(2\delta)^{k-2} < \frac{\inf_\phi \phi_n (M)}{2\epsilon}$. Let $\epsilon$ be such that

$$\inf_{|x| > \delta} \left( u(x) - \frac{1}{2} (Mx, x) + |x|^k \right) = u(0) + \epsilon$$

and let $\delta_2 < \delta$ and such that $k(2\delta)^{k-1}\delta_2 + |M|_{\infty}(\delta_2^2 + 2\delta_2) < \frac{\epsilon}{4}$. Then, for $x$ such that $|x| < \delta_2$,

$$\inf_{|y| \leq \delta} \left\{ u(y) - \frac{1}{2} (My, y) + |y|^k \right\} \leq \inf_{|y| \leq \delta} \left\{ u(y) - \frac{1}{2} (My, y) + |y|^k \right\} + \frac{\epsilon}{4}$$

$$= u(0) + \epsilon / 4$$

and on the opposite

$$\inf_{|y| > \delta} \left\{ u(y) - \frac{1}{2} (My, y) + |y|^k \right\} \geq \inf_{|y| > \delta} \left\{ u(y) - \frac{1}{2} (My, y) + |y|^k \right\} - \frac{\epsilon}{4}$$

$$> u(0) + \frac{3\epsilon}{4}.$$ 

Since the function $u$ is supposed to be non locally constant, there exist $x_\delta$ and $y_\delta$ in $B(0, \delta_2)$ such that

$$u(x_\delta) > u(y_\delta) - \frac{1}{2} (M(x_\delta - y_\delta), x_\delta - y_\delta) + |x_\delta - y_\delta|^k$$

and then the infimum $\inf_{y, |y| \leq \delta} \left\{ u(y) - \frac{1}{2} (M(x_\delta - y), x_\delta - y) + |x_\delta - y|^k \right\}$ is achieved on some point $z_\delta$ different from $x_\delta$. This implies that the function

$$\varphi(z) := u(z_\delta) + \frac{1}{2} (M(x_\delta - z), x_\delta - z) - |x_\delta - z|^k - \frac{1}{2} (M(x_\delta - z), x_\delta - z) + |x_\delta - z|^k$$

is invertible, since if it is not, it
touches \( u \) by below at the point \( z_\delta \). But

\[
\nabla \varphi(z_\delta) = M(z_\delta - x_\delta) - k|x_\delta - z_\delta|^{k-2}(z_\delta - x_\delta),
\]

cannot be zero since, if it was, \( z_\delta - x_\delta \) would be an eigenvector for the eigenvalue \( k|x_\delta - z_\delta|^{k-2} \) which is supposed to be strictly less than any eigenvalue of \( M \).

We have obtained that \( \nabla \varphi(z_\delta) \neq 0 \) and then, since \( u \) is a supersolution in the sense of Definition 2.1,

\[
F(z_\delta, M(z_\delta - x_\delta) - k|x_\delta - z_\delta|^{k-2}(z_\delta - x_\delta), M - \frac{d^2}{dz^2}(|x_\delta - z|^k)(z_\delta)) \leq g(z_\delta, u(z_\delta)).
\]

By passing to the limit we obtain

\[
0 \leq g(0, u(0)),
\]

which is the desired conclusion.

Of course we can do the same for sub-solutions.

\section*{References}

[1] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, \textit{The principal eigenvalue and maximum principle for second-order elliptic operators in general domains}, Comm. Pure Appl. Math. \textbf{47} (1994), no. 1, 47–92.

[2] I. Birindelli, F. Demengel, \textit{Eigenvalue, maximum principle and regularity for fully non-linear homogeneous operators}, Comm. Pure and Appl. Analysis, \textbf{6} (2007), 335-366.

[3] I. Birindelli, F. Demengel, \textit{Regularity and uniqueness of the first eigenfunction for singular fully nonlinear operators}, J. Differential Equations \textbf{249} (2010) 1089-1110.

[4] I. Birindelli, F. Demengel, \textit{Uniqueness of the first eigenfunction for fully non-linear equations: the radial case}, Journal for Analysis and its Applications, \textbf{29} (2010), 75-88.

[5] X. Cabré, L. Caffarelli, \textit{Regularity for viscosity solutions of fully nonlinear equations} \( F(D^2u) = 0 \), Topological Meth. Nonlinear Anal. \textbf{6} (1995), 31-48.
[6] X. Cabré, L. Caffarelli, *Interior $C^2$ regularity theory for a class of nonconvex fully nonlinear elliptic equations* J. Maths Pures Appl. (9) **82** (2003) 573-612.

[7] L. Caffarelli, *Interior a Priori Estimates for Solutions of Fully Nonlinear Equations*, The Annals of Mathematics, Second Series, **130** (1989), 189-213.

[8] L. Caffarelli, X. Cabré, Fully-nonlinear equations Colloquium Publications 43, American Mathematical Society, Providence, RI, 1995.

[9] I. Capuzzo Dolcetta, A. Vitolo, *Glaesers type gradient estimates for non-negative solutions of fully nonlinear elliptic equations* Discrete Contin. Dyn. Syst. **28** (2010), 539-557.

[10] G. Davila, P. Felmer, A. Quaas, *Harnack Inequality for singular fully nonlinear operators and some existence’s results*. Calculus of Variations and PDE, Vol. **39**, (2010), 557-578.

[11] L.C. Evans, *Classical Solutions of Fully Nonlinear, Convex, Second-Order Elliptic Equations*, Comm. on Pure and Applied Mathematics, **25**, (1982)333-363.

[12] L.C. Evans, O. Savin, *$C^{1,\alpha}$ Regularity for infinity harmonic functions in two dimensions*. Calculus of Variations and PDE, **32** (2008), 325-347.

[13] N. Nadirashvili, S. Vladut, *On Axially Symmetric Solutions of Fully Nonlinear Elliptic Equations*, Math. Z. to appear.