Consensus Algorithms and the Decomposition-Separation Theorem of Markov Chains

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Abstract

Convergence properties of time inhomogeneous Markov chain based discrete time linear consensus algorithms are analyzed. Provided that a so-called infinite jet flow property is satisfied by the underlying chains, necessary conditions for both consensus and multiple consensus are established. A recent extension to non homogeneous Markov chains of the classical Kolmogorov-Doeblin decomposition-separation homogeneous Markov chain results, is then employed to show that the obtained necessary conditions are also sufficient when the chain is of class $\mathcal{P}^*$, as defined by Touri and Nedić. It is also shown that Sonin’s theorem leads to a rediscovery and generalization of most of the existing results in the literature.

I. INTRODUCTION

In this paper, we deal with the limiting behavior of a general linear consensus algorithm in discrete time. Let $M = \{1, \ldots, N\}$ be the set of agents. We consider $N$-agent systems with linear update equation:

$$X(n + 1) = A(n)X(n), \forall n \geq 0$$

where $X(n), n \geq 0$, is the vector of states and $\{A_{N \times N}(n)\}$ is a chain of stochastic matrices, i.e, each row of $A(n), \forall n \geq 0$, sums up to 1. Since $A(n)$ is a stochastic matrix for every $n$, sequence $\{X(n)\}$ forms a backward Markov chain.

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For the purposes of this paper, chain \( \{A(n)\} \) in (1) will be said to form the update basis of the consensus algorithm. If all components of \( X(n) \) asymptotically converge to the same limit, irrespective of the time index \( n \) or the values at which they are initialized, unconditional consensus is said to occur. Furthermore, if there exists a fixed partition of the of the \( N \) agents such that unconditional consensus occurs for the corresponding subvectors of \( X(n) \), then unconditional multiple consensus is said to occur. The subsets in the partition are then said to form consensus clusters. It is well known [5] that under dynamics (1), unconditional consensus is equivalent to the backward ergodicity of chain \( \{A(n)\} \), i.e. the property that backward products converge to matrices with identical rows. Furthermore, [4], [18], establish that a consensus algorithm with update chain \( A(n) \) will induce multiple consensus if \( A(n) \) is so-called class ergodic i.e. (i) for every \( n_0 \geq 0 \), the product \( A(n)A(n-1) \cdots A(n_0) \) converges, as \( n \to \infty \), (ii) set \( M \) can be partitioned into ergodic classes whereby \( i, j \in M \) belong to the same ergodic class if the difference between the \( i \)th and \( j \)th rows of the mentioned product vanishes, as \( n \to \infty \). Under multiple consensus, the agents’ indices within the ergodic classes are the same as those within consensus clusters.

Sonin [15] in his so-called Decomposition-Separation (D-S) Theorem, suggests an elegant and illuminating physical interpretation of the dynamics in (1), which we now report for completeness; Starts with a forward propagating Markov chain \( \{m(n)\} \) with \( \{P_{N \times N}(n)\} \) as its transition chain:

\[
m^t(n+1) = m^t(n)P(n), \forall n \geq 0
\]

where superindex \( t \) indicates transposition. Interpret \( m_i(n), i \in M, n \geq 0 \), as the volume of some liquid, say water for example, in a cup \( i \) (out of \( N \) cups) at time \( n \geq 0 \), while \( P_{ij}(n)m_i(n) \) is the volume of liquid transferred from cup \( i \) to cup \( j \) at time \( n \). The volume of liquid in cup \( i, \forall i \in M \), is assumed to be initialized as \( m_i(0) \) at \( n = 0 \). Now, let \( X_i(n), i \in M, n \geq 0 \), be the concentration of a certain substance, such as sugar, vodka, etc., within the liquid of cup \( i \) at time \( n \). We first assume that the volume of each cup is non zero at all times in order to make the concentration well-defined. Moreover, assume that for every \( i \in M, X_i(0) \) is initialized as \( X_i(0) \) at time \( n = 0 \). It is not difficult to show that:

\[
X_i(n+1) = \sum_{j \in M} P_{ij}(n)m_j(n)X_j(n)/m_i(n+1), \forall i \in M, \forall n \geq 0.
\]
Let
\[ X \triangleq \left[ X_1 \cdots X_N \right]^t \] (4)
and matrix \( A_{N \times N}(n) \) with entries \( A_{ij}(n) \), \( i, j \in M \), be defined by:
\[ A_{ij}(n) = P_{ji}(n)m_j(n)/m_i(n+1), \forall n \geq 0. \] (5)

From Eqs. (3), (4), and (5), we conclude:
\[ X(n+1) = A(n)X(n), \forall n \geq 0. \] (6)

Since \( A(n) \) is stochastic for every \( n \geq 0 \), \( \{X(n)\} \) forms a backward Markov chain with transition chain \( \{A(n)\} \) (notice the evolution is described by a right hand multiplication by a column vector instead of the usual left hand multiplication by a row vector), as in (1). Without the non zero volume assumption, \( \{A(n)\} \) is a chain of stochastic matrices with entries satisfying the following relation:
\[ m_i(n+1)A_{ij}(n) = m_j(n)P_{ji}(n), \forall i, j \in M, \forall n \geq 0 \] (7)

The D-S Theorem describes the limiting behavior of both \( m(n) \) and \( X(n) \), as \( n \) grows large. However, to take advantage of the D-S Theorem in a general consensus algorithm (1), one has to first answer the following questions: Starting with \( \{A(n)\} \), is it always possible to find an associated forward Markov chain \( \{m(n)\} \) with transition chain \( \{P(n)\} \) satisfying Eq. (7)? And how, if so? As discussed in this paper, due to existence of a so-called absolute probability sequence for \( \{A(n)\} \) as proved by Kolmogorov [11], one could show the existence of the desired chains satisfying Eq. (7).

Linear consensus algorithms of type (1) and their convergence properties have gained increasing attention in the past decade. They were first introduced in [6], where the author considered the case when \( \{A(n)\} \) is time-invariant. Later, more general cases were considered in [1], [3], [5], [7], [8], [10], [12], [14], [20], [21]. The authors aimed at identifying sufficient conditions for consensus to occur, i.e., states asymptotically converging to the same value. Besides consensus, multiple consensus has been the subject of many articles, e.g., [4], [9], [13], [17], [19]. Considering the work on linear consensus algorithms, [4], [9], [19] appear to provide the largest class of chains \( \{A(n)\} \), for which consensus or multiple consensus occurs in a system with dynamics (1). In this paper, it is established that based on the D-S Theorem, all previous results can be subsumed. Furthermore, inspired by [18], and recalling the notion of jets.
in Markov chains from [2], we introduce a property of chains resulting in necessary conditions for unconditional occurrence of consensus or multiple consensus in (1). We also establish that under an additional assumption, the so-called \( P^* \) property [18], these necessary conditions also become sufficient.

The rest of the paper is organized as follows. In Section 2, we state necessary conditions for class-ergodicity and ergodicity of a chain. The D-S Theorem and its application in a general linear consensus algorithm are discussed in Section 3. In Section 4, we show how the D-S Theorem relates to and generalizes most of the existing results in the literature on convergence properties of linear consensus algorithms. Concluding remarks end the paper in Section 5.

II. THE INFINITE JET-FLOW PROPERTY

Inspired by work by Blackwell [2] as reported in [15], and Touri [18], in this section we introduce a property of chains of stochastic matrices, here called the infinite jet-flow property, leading to necessary conditions for ergodicity and class-ergodicity of the chain.

**Definition 1:** For a given finite set \( M = \{1, \ldots, N\} \), a jet \( J \) in \( M \) is a sequence \( \{J(n)\} \) of subsets of \( M \). A jet \( J \) in \( M \) is called a proper jet if \( \emptyset \neq J(n) \subseteq M, \forall n \geq 0 \). Moreover, for a jet \( J \), jet-limit \( J^* \) denotes the limit of the sequence \( \{J(n)\} \) as \( n \) grows large, if it exists, in the sense that the sequence becomes constant after a finite time. When the elements of the subsets sequence are all identical to a subset \( S \) of \( M \), the jet will be called, jet \( S \).

**Definition 2:** A tuple of jets \( (J^1, \ldots, J^c) \) is a jet-partition of \( M \), if \( (J^1(n), \ldots, J^c(n)) \) is a partition of \( M \) for every \( n \geq 0 \).

**Definition 3:** Let chain \( \{A(n)\} \) of stochastic matrices be given. For any two disjoint jets \( J^s \) and \( J^k \) in \( M \), \( U_{A}(J^s, J^k) \) or simply \( U(J^s, J^k) \) when no ambiguity results, denotes the total interactions between the two jets over the infinite time interval, as defined by:

\[
U(J^s, J^k) = \sum_{n=0}^{\infty} \left[ \sum_{i \in J^s(n+1)} \sum_{j \in J^k(n)} A_{ij}(n) + \sum_{i \in J^k(n+1)} \sum_{j \in J^s(n)} A_{ij}(n) \right]
\]

Moreover, \( U_{A_n}(J^s, J^k) \) or simply \( U_n(J^s, J^k) \), denotes the interactions between the two jets at time \( n \). More specifically,

\[
U_n(J^s, J^k) = \sum_{i \in J^s(n+1)} \sum_{j \in J^k(n)} A_{ij}(n) + \sum_{i \in J^k(n+1)} \sum_{j \in J^s(n)} A_{ij}(n)
\]
Definition 4: The complement of jet $J$ in $M$, denoted by $M \setminus J$ or simply $\bar{J}$, is sequence $\{M \setminus J_n\}$.

Definition 5: A chain $\{A(n)\}$ of stochastic matrices is said to have the infinite jet-flow property if for every proper jet $J$ in $M$, $U(J, \bar{J})$ is unbounded.

Definition 6: \cite{19} For a chain $\{A(n)\}$ of stochastic matrices, we define its infinite flow graph, $G_A(M, E)$, by a simple graph of size $N$, such that

$$E = \{(i, j)|i, j \in M, i \neq j, \sum_{n=0}^{\infty} (A_{ij}(n) + A_{ji}(n)) = \infty\} \quad (10)$$

The set of nodes of each connected component of $G_A$ is called an island of $\{A(n)\}$.

Theorem 1: A chain $\{A(n)\}$ of stochastic matrices is class-ergodic only if the infinite jet-flow property holds over each island of $\{A(n)\}$.

Proof: Assume that on the contrary, $\{A(n)\}$ is class-ergodic, yet some jet $J$ in an island $I$ of $\{A(n)\}$ is such that $U_A(J, I \setminus J)$ is bounded. Thus, $U_A(J, M \setminus J)$ is bounded as well, since $I$ is an island of $\{A(n)\}$. Form a chain $\{B(n)\}$, an $l_1$-approximation of $\{A(n)\}$ (see \cite{17} for a definition of $l_1$-approximation), by eliminating interactions between $J$ and $M \setminus J$ at all times. From \cite{17, Lemma 1}, it is known that $l_1$-approximations do not influence the ergodic classes of a chain. Therefore, $\{B(n)\}$ remain class-ergodic. On the other hand, $U_B(J, M \setminus J) = 0$. Given two distinct arbitrary constants, $\alpha_1$ and $\alpha_2$, let states of a multi-agent system, $Y_i(n), 1 \leq i \leq N$, evolve via dynamics $Y(n+1) = B(n)Y(n), \forall n \geq 0$, and be initialized at: $Y_i(0) = \alpha_1$ if $i \in J(0)$, and $Y_i(0) = \alpha_2$ otherwise. We simply conclude that for every $n \geq 0$, we have: $Y_i(n) = \alpha_1$ if $i \in J(n)$, and $Y_i(n) = \alpha_2$ otherwise. Since $\{B(n)\}$ is class-ergodic, $\lim_{n \to \infty} Y_i(n)$ exists for every $i \in M$ and the consensual agents can be grouped into clusters sharing the same limit and forming an ergodic class. Since the elements in $\{J(n)\}$ are always associated with the same value of $Y$ for any $n$, they will asymptotically belong to a fixed limiting cluster $S^*$, namely agents for which $Y_i(n)$ converges to $\alpha_1$. Since $J$ is a proper jet in $I$, we have: $\emptyset \neq S^* \subset I$. Consider now jet $S^*$ on island $I$. From the definition of an island, we know that $U_B(J^*, I \setminus J^*)$ is unbounded. This is in contradiction with $U_B(J, I \setminus J) = U_B(J, M \setminus J) = 0$, and completes the proof.

Corollary 1: A chain $\{A(n)\}$ of stochastic matrices is ergodic only if it has the infinite jet-flow property.
Proof: This is an immediate result of Theorem 1 if we note that the infinite flow property of \( \{ A(n) \} \), is equivalent to existence of a single island, and is therefore necessary for ergodicity of \( \{ A(n) \} \) (see [16]).

Later on, we shall establish that the infinite jet-flow property is also sufficient for ergodicity provided \( \{ A(n) \} \) is in class \( \mathcal{P}^* \), as defined in [19].

**Definition 7:** A jet \( J \) in \( M \) is called a leader if the total influence of \( \bar{J} \) on \( J \) is finite over the infinite time interval, i.e.,

\[
\sum_{n=0}^{\infty} \sum_{i \in J(n+1)} \sum_{j \in \bar{J}(n)} A_{ij}(n) < \infty.
\]

**Theorem 2:** A chain \( \{ A(n) \} \) of stochastic matrices is ergodic only if no two disjoint leaders in \( M \) exist.

**Proof:** Assume that on the contrary, there exist two disjoint leaders \( J^1 \) and \( J^2 \) in \( M \). Similar to the proof of Theorem 1, form chain \( \{ B(n) \} \), an \( l_1 \)-approximation of \( \{ A(n) \} \), by eliminating the influence of \( \bar{J}^s \) on \( J^s \), \( s = 1, 2 \), at all times. Recall that \( \{ A(n) \} \) and \( \{ B(n) \} \) will share the same ergodicity properties. Let states of a multi-agent system, \( Y_i(n) \), \( 1 \leq i \leq N \), evolve via dynamics \( Y(n+1) = B(n)Y(n) \), \( \forall n \geq 0 \), and be initialized such that for every \( i \in J^s(0) \) \( (s = 1, 2) \), \( Y_i(0) = \alpha_s \), where \( \alpha_1 \neq \alpha_2 \). Then, for every \( n \geq 0 \), we have: \( Y_i(n) = \alpha_s, \forall i \in J^s(n) \) \( (s = 1, 2) \). Since \( \alpha_1 \neq \alpha_2 \), consensus does not occur. Consequently, chain \( \{ B(n) \} \) and thus \( \{ A(n) \} \) could not possibly be ergodic.

**III. RELATIONSHIP TO THE D-S THEOREM**

Consider a multi-agent system with states evolving according to linear algorithm (1), where \( \{ A(n) \} \) is a chain of stochastic matrices. Based on the work of Kolmogorov [11], we know that chain \( \{ A(n) \} \) admits an absolute probability sequence \( \pi(n) \) (although we do not know a priori how to construct it). Thus, there exists a sequence \( \{ \pi(n) \} \) of probability distribution vectors such that

\[
\pi^t(n+1)A(n) = \pi^t(n), \forall n \geq 0.
\]

The transition matrix sequence \( \{ P(n) \} \) of the forward propagating chain associated with \( \{ A(n) \} \) and \( \{ \pi(n) \} \) as in (7), must be such that

\[
\pi_i(n)P_{ij}(n) = \pi_j(n+1)A_{ji}(n), \forall i, j \in M, \forall n \geq 0.
\]
More specifically, if $\pi_i(n) \neq 0$, then:

$$P_{ij}(n) = \pi_j(n+1)A_{ji}(n)/\pi_i(n) \quad (14)$$

while if $\pi_i(n) = 0$ for some $i$ and $n \geq 0$, we choose $P_{ij}(n)$’s arbitrarily such that

$$\sum_{j=1}^{N} P_{ij}(n) = 1. \quad (15)$$

Note that in the former case ($\pi_i(n) = 0$), Eq. (15) is automatically satisfied, implying that $P(n)$ is a stochastic matrix for every $n \geq 0$. It is easy to see that:

$$\pi^t(n)P(n) = \pi^t(n+1), \forall n \geq 0. \quad (16)$$

Thus, $\{\pi(n)\}$ forms a non-homogeneous forward propagating Markov chain. Let $V(J^s, J^k)$ denote the total flow between two arbitrary jets $J^s$ and $J^k$ in $M$ over the infinite time interval as defined by:

$$V(J^s, J^k) = \sum_{n=0}^{\infty} \left[ \sum_{i \in J^s(n)} \sum_{j \in J^s(n+1)} r_{ij}(n) + \sum_{i \in J^s(n)} \sum_{j \in J^k(n+1)} r_{ij}(n) \right] \quad (17)$$

where

$$r_{ij}(n) = \pi_i(n)P_{ij}(n) = \pi_j(n+1)A_{ji}(n). \quad (18)$$

Value $r_{ij}(n)$ can be interpreted as the absolute joint probability of being in $i$ at time $n$ and $j$ at time $n+1$. Recalling $U$ from Eq. (8), we note that for every $J^s, J^k$ in $M$, $V(J^s, J^k) \leq U(J^s, J^k)$. Sonin in his elegant work [15] characterizes the limiting behavior of the two sequences $\{\pi(n)\}$ and $\{X(n)\}$ (evolving via Eq. (1)) in the so-called D-S Theorem as the following:

**Theorem 3: (Sonin’s D-S Theorem)** There exists an integer $c$, $1 \leq c \leq N$, and a decomposition of $M$ into jet-partition $J^0, J^1, \ldots, J^c$, $J^k = \{J^k(n)\}$, such that irrespective of the particular time or state at which $X_i$’s are initialized,

(i) For every $k$, $1 \leq k \leq c$, there exist constants $\pi_k^*$ and $X_k^*$, such that

$$\lim_{n \to \infty} \sum_{i \in J^k(n)} \pi_i(n) = \pi_k^* \quad (19)$$

and

$$\lim_{n \to \infty} X_{i_n}(n) = X_k^* \text{ for every sequence } \{i_n\}, i_n \in J^k(n). \quad (20)$$

Furthermore, $\lim_{n \to \infty} \sum_{i \in J^0(n)} \pi_i(n) = 0$. 

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(ii) For every distinct \( k, s, 0 \leq k, s \leq c \): \( V(J^k, J^s) < \infty \).

(iii) This decomposition is unique up to jets \( \{J(n)\} \) such that for any \( \{\pi(n)\} \) we have:

\[
\lim_{n \to \infty} \sum_{i \in J(n)} \pi_i(n) = 0 \text{ and } V(J, M \setminus J) < \infty. \tag{21}
\]

IV. RELATIONSHIP TO PREVIOUS WORK

A. The Class \( P^* \) \[19\]

In the following, we apply Sonin’s D-S Theorem to chains in class \( P^* \) as first defined by Touri and Nedić \[19\]. According to \[19\] Definition 3, chain \( \{A(n)\} \) is said to be in class \( P^* \) if it admits an absolute probability sequence uniformly bounded away from zero, i.e., there exists \( p^* > 0 \) such that

\[
\pi_i(n) \geq p^*, \forall i \in M, \forall n \geq 0. \tag{22}
\]

It is immediately implied that in the jet decomposition of the D-S Theorem, there is no \( J^0 \). Otherwise, \( \lim_{n \to \infty} \sum_{i \in P(n)} \pi_i(n) \) would be bounded away from zero by at least \( p^* \), which is in contradiction with the D-S Theorem. Therefore, there is a jet-partition of \( M \) into jets \( J^1, \ldots, J^c \) such that for every \( k = 1, \ldots, c \), \( \lim_{n \to \infty} X_{i_n}(n) = X_k^* \) for every sequence \( \{i_n\} \), where \( i_n \in J^k(n) \). Thus, we have the following theorem for chains in class \( P^* \).

**Theorem 4:** Consider a multi-agent system with dynamics \[1\], where chain \( \{A(n)\} \) is in class \( P^* \). Then, the set of accumulation points of states is finite.

**Proof:** Obvious if we note that \( \{X_k^*|1 \leq k \leq c\} \) form the set of accumulation points of states.

**Lemma 1:** If \( \{A(n)\} \in P^* \), then for every two jets \( J^1 \) and \( J^2 \) in \( M \), \( V(J^1, J^2) = \infty \) if and only if \( U(J^1, J^2) = \infty \).

**Proof:** The result is obvious if one notes that \( p^* U(J^1, J^2) \leq V(J^1, J^2) \leq U(J^1, J^2) \).

**Theorem 5:** A chain \( \{A(n)\} \in P^* \) is class-ergodic if and only if the infinite jet-flow property holds over each island of \( \{A(n)\} \). Moreover, in case of class-ergodicity of \( \{A(n)\} \), islands are the ergodic classes of \( \{A(n)\} \).

**Proof:** The necessity follows from Theorem \[1\]. We now show that that if \( \{A(n)\} \in P^* \) is class-ergodic, islands are ergodicity classes of \( \{A(n)\} \). If \( \{A(n)\} \) is class ergodic, jets in the D-S Theorem become time-invariant after a finite time. Thus, the jet-limits exist and are ergodicity classes of \( \{A(n)\} \). If \( i \) and \( j \) belong to the same jet-limit, they are in the same island since
they are in the same ergodic class of \( \{ A(n) \} \) (\cite{17}, Lemma 2). Conversely, assume that \( i \) and \( j \) are neighbors in the infinite flow graph, i.e., \( \sum_{n=0}^{\infty} (a_{ij}(n) + a_{ji}(n)) = \infty \). If \( i \) and \( j \) belong to different jet-limits \( J^i, J^j \), then \( U(J^i, J^j) \) is unbounded. Thus, from Lemma \[ \ref{thm:jet-limit} \] \( V(J^i, J^j) \) is unbounded as well, which contradicts property (ii) in the D-S theorem. Therefore, every two neighbors in the infinite flow graph belong to the same jet-limit. Consequently, every \( i \) and \( j \) in the same island must be in the same jet-limit.

To prove the sufficiency, we assume that the infinite jet-flow property holds over each island. Let \((J^1, \ldots, J^c)\) be the jet decomposition in the D-S Theorem and for every \( k = 1, \ldots, c \), \( \lim_{n \to \infty} X_{i_n}(n) = X^*_i \) for every sequence \( \{i_n\} \), where \( i_n \in J^k(n) \). Let \( I \) be an arbitrary island. We wish to show that for every \( i \in I \), \( \lim_{n \to \infty} X_i(n) \) exists. Pick an arbitrary jet \( J^k \) among \( J^1, \ldots, J^c \). We show that for infinite times \( n \): \( I \cap J^k(n) = \emptyset \) or \( I \). Indeed, assume instead that this behavior occurs only a finite number of times \( t \), denoted \( n_1, \ldots, n_t \). We form a jet \( J \) in \( I \) such that:

\[
J(n) = I \cap J^k(n), \quad \text{if } n \neq n_i, 1 \leq i \leq t
\]  

(23)

Since the infinite jet-flow property holds over \( I \), \( V(J, I \setminus J) \) is unbounded, and so is \( U(J, I \setminus J) \). On the other hand, except for a finite number of time indices \( n = n_i, 1 \leq i \leq t \), \( U_n(J, I \setminus J) \leq U_n(J^k, M \setminus J^k) \). This implies that \( U(J^k, M \setminus J^k) \) is unbounded, and so is \( V(J^k, M \setminus J^k) \) under the assumption that the chain is in class \( \mathcal{P}^* \), which is in contradiction with the D-S Theorem. Therefore, \( I \cap J^k(n) = \emptyset \) or \( I \) happens infinite times. This means that either one or both of the events \( I \cap J^k(n) = \emptyset \) or \( I \cap J^k(n) = I \) occurs infinitely often. We now show that there are at most a finite number of times \( n \) such that \( I \subseteq J^k(n) \) and \( I \not\subseteq J^k(n + 1) \). Indeed, denote:

\[
\epsilon \triangleq \frac{1}{3} \min\{|X^*_s - X^*_l| | 1 \leq s \neq l \leq c\},
\]  

(24)

there exists \( T_\epsilon \geq 0 \) such that:

\[
|X_i(n) - X^*_i| < \epsilon, \quad \forall l = 1, \ldots, c, \forall i \in J^l(n), \forall n \geq T_\epsilon
\]  

(25)

For some given \( n \geq T_\epsilon \) assume that: \( I \subseteq J^k(n) \) and \( I \not\subseteq J^k(n + 1) \). Then, there exists \( i \in I \) such that \( i \in J^k(n) \setminus J^k(n + 1) \). In view of \[ \ref{eq:20} \] and \[ \ref{eq:21} \], we must have:

\[
\left| \sum_{j \neq J^k(n)} A_{ij}(n)(X_j(n) - X_i(n)) \right| \geq \epsilon
\]  

(26)
On the other hand,

\[
| \sum_{j \notin J^k(n)} A_{ij}(n)(X_j(n) - X_i(n)) | \leq \sum_{j \notin J^k(n)} A_{ij}(n) |X_j(n) - X_i(n)| \leq L \sum_{j \notin J^k(n)} A_{ij}(n) \tag{27}
\]

where

\[
L \triangleq \max \{X_j(n) - X_i(n), |i, j \in M, n \geq 0\} \tag{28}
\]

Note that the maximum exists, since states are updated via a convex combination of previous states. Eqs. (26) and (27) imply:

\[
\sum_{j \notin J^k(n)} A_{ij}(n) \geq \frac{\epsilon}{L} \tag{29}
\]

Therefore,

\[
\sum_{i \in I} \sum_{j \notin I} A_{ij}(n) \geq \sum_{j \notin I} A_{ij}(n) \geq \sum_{j \notin J^k(n)} A_{ij}(n) \geq \frac{\epsilon}{L} \tag{30}
\]

Since \(U(I, M \setminus I) < \infty\), inequality (30) can only occur for finitely many times \(n\). This shows that if \(I \subseteq J^k(n)\) happens infinite times, then there exists \(T\) such that \(I \subseteq J^k(n)\) for every \(n \geq T\). Consequently, \(\lim_{n \to \infty} X_i(n)\) exists, \(\forall i \in I\), and is equal to \(X^*_k\). Therefore, assume that for a fixed island \(I\), \(I \subseteq J^k(n)\) happens only finite times for every \(k, 1 \leq k \leq c\). Thus, \(I \cap J^k(n) = \emptyset\) must happen infinite times, for every \(k, 1 \leq k \leq c\). We show that such an event leads to the following contradiction: For every \(k, 1 \leq k \leq c\), there exists \(T_k \geq 0\) such that \(I \cap J^k(n) = \emptyset\), \(\forall n \geq T_k\). The proof is established by induction on \(k\). With no loss of generality, assume that \(X^*_1 < \cdots < X^*_k\).

\(k = 1\): Recalling \(\epsilon\) and \(T_\epsilon\) from Eqs. (24) and (25), assume that for a fixed \(n \geq T_\epsilon\) we have \(I \cap J^1(n) = \emptyset\) and \(I \cap J^1(n + 1) \neq \emptyset\). Thus, there exists \(i \in I\) such that \(i \in J^1(n + 1) \setminus J^1(n)\). Therefore,

\[
\sum_{j \in J^1(n)} |A_{ij}(n)(X_j(n) - X_i(n))| \geq \epsilon \tag{31}
\]

Noting that \(J^1(n) \subseteq M \setminus I\), by repeating steps (26)-(30), we conclude that there are at most finitely many times at which \(I \cap J^1(n) = \emptyset\) and \(I \cap J^1(n + 1) \neq \emptyset\). This together with the fact that \(I \cap J^1(n) = \emptyset\) happens infinite times, shows that there exists \(T_1 \geq 0\) such that \(I \cap J^1(n) = \emptyset\), \(\forall n \geq T_1\).
\( k - 1 \rightarrow k \ (1 < k \leq c) \): Assume that for a fixed \( n \geq \max\{T_l | 1 \leq l < k\} \), we have \( I \cap J^k(n) = \emptyset \) and \( I \cap J^k(n + 1) \neq \emptyset \). Thus, there exists \( i \in I \) such that \( i \in J^k(n + 1) \setminus J^k(n) \). Therefore,

\[
\sum_{j \in \bigcup_{l=1}^k J^l(n)} |A_{ij}(n)(X_j(n) - X_i(n))| \geq \epsilon
\]  

(32)

Once again, we note that \( \bigcup_{l=1}^k J^l(n) \subseteq \bar{I} \), and repeat steps (26)-(30) to show that there exists \( T_k \geq 0 \) such that \( I \cap J^k(n) = \emptyset \), \( \forall n \geq T_k \).

\textbf{Corollary 2:} A chain \( \{A(n)\} \in \mathcal{P}^* \) is ergodic if and only if it has the infinite jet-flow property.

\textbf{B. Weakly Aperiodic Chains \cite{19}}

In this part, we see how the weak aperiodicity property, as defined in \cite{19}, guarantees the infinite jet-flow property to hold over each island. In accordance with \cite{19}, weak aperiodicity of a chain is defined as follows:

\textbf{Definition 8:} A chain \( \{A(n)\} \) of stochastic matrices is said to be \textit{weakly aperiodic} if there exists \( \gamma > 0 \) such that for every distinct \( i, j \in M \) and each \( n \geq 0 \), there exists \( l \in M \) such that

\[
A_{li}(n).A_{lj}(n) \geq \gamma A_{ij}(n).
\]  

(33)

\textbf{Lemma 2:} Let \( \{A(n)\} \) be a chain of stochastic matrices that is weakly aperiodic. Then, the infinite jet-flow property holds over each island of \( \{A(n)\} \).

\textbf{Proof:} Let \( \{A(n)\} \) be weakly aperiodic, \( I \) be an arbitrary island of \( \{A(n)\} \), and \( J \) be an arbitrary jet in \( I \). If jet-limit \( J^* \) exists, since \( I \) is a connected component of the infinite flow graph, \( U(J^*, I \setminus J^*) \) is unbounded. Consequently, \( U(J, I \setminus J) \) is unbounded. Thus instead, assume that for jet \( J \), the jet-limit does not exist. Therefore, for infinitely many times \( n \), we must have: \( J(n + 1) \not\subseteq J(n) \). Let \( n \) be fixed and \( J(n + 1) \not\subseteq J(n) \). Thus, there exists \( i \in J(n + 1) \setminus J(n) \). From the weakly aperiodicity property of \( \{A(n)\} \), for every \( j \in J(n) \), there exists \( l \in M \) such that:

\[
\gamma A_{ij}(n) \leq A_{li}(n).A_{lj}(n) \leq \min\{A_{li}(n), A_{lj}(n)\} \leq U_n(J, M \setminus J)
\]  

(34)

where \( U_n \) is defined in Eq. (9). The reason for the last inequality is that, no matter \( l \in J(n + 1) \) or \( l \not\in J(n + 1) \), one of \( A_{li}(n), A_{lj}(n) \) appears in \( U_n(J, M \setminus J) \). Hence,

\[
\sum_{j \in J(n)} \gamma A_{ij}(n) \leq |J(n)|U_n(J, M \setminus J).
\]  

(35)
On the other hand,
\[ \sum_{j \in J(n)} \gamma A_{ij}(n) = \gamma \sum_{j \in J(n)} A_{ij}(n) = \gamma \left( |J(n)| - \sum_{j \notin J(n)} A_{ij}(n) \right) \geq \gamma \left( |J(n)| - U_n(J, M \setminus J) \right) \]
(36)

Relations (35) and (36) imply:
\[ U_n(J, M \setminus J) \geq \gamma |J(n)| / (\gamma + |J(n)|) \geq \gamma / (1 + \gamma) \]
(37)

Since inequality (37) holds for infinite times \( n \), \( U(J, M \setminus J) = \sum_{n=0}^{\infty} U_n(J, M \setminus J) \) is unbounded.

Theorem 5 and Lemma 2 immediately imply the following corollary which is the deterministic counterpart of Theorem 4 of [19].

**Corollary 3:** every weakly aperiodic chain in class \( \mathcal{P}^* \) is class-ergodic.

C. Self-Confident and Cut-Balanced Chains [9]

**Definition 9:** [4] A chain \( \{ A(n) \} \) of stochastic matrices is **self-confident** with bound \( \delta \) if \( A_{ii}(n) \geq \delta, \forall i \in M, \forall n \geq 0 \).

**Definition 10:** [9] A chain \( \{ A(n) \} \) of stochastic matrices is **cut-balanced** with bound \( \Psi \) if for every \( M_1 \subseteq M \) and \( n \geq 0 \):
\[
\sum_{i \in M_1} \sum_{j \in M_1} A_{ij}(n) \leq \Psi \sum_{i \in M_1} \sum_{j \notin M_1} A_{ij}(n)
\]
(38)

**Theorem 6:** [4], [19] If chain \( \{ A(n) \} \) is self-confident and cut-balanced, then it is class-ergodic and the islands form the ergodic classes of \( \{ A(n) \} \).

**Proof:** Assume that \( \{ A(n) \} \) has self-confidence and cut-balance properties with bounds \( \delta \) and \( \Psi \) respectively. Since \( \{ A(n) \} \) is in class \( \mathcal{P}^* \) ([19], Theorem 7), from Theorem 5 it is sufficient to show that for an arbitrary island \( I \) and an arbitrary proper jet \( J \) in \( I \), we have \( U(J, I \setminus J) = \infty \). Indeed, if jet-limit \( J^* \) exists, unboundedness of \( U(J, I \setminus J) \) is immediately implied from unboundedness of \( U(J^*, I \setminus J^*) \). Otherwise, there are infinite times \( n \) such that \( J(n) \neq J(n+1) \). At every such \( n \), there exists \( i \in I \) such that \( i \in (J(n) \setminus J(n+1)) \cup (J(n+1) \setminus J(n)) \). Therefore, \( U_n(J, I \setminus J) \geq A_{ii}(n) \geq \delta \). Since there are infinite such times, \( U(J, I \setminus J) \) is unbounded.
D. Balanced Asymmetric Chains [4]

Definition 11: A chain \( \{A(n)\} \) of stochastic matrices is said to be balanced asymmetric with bound \( \Psi \), if for every subsets \( M_1, M_2 \subseteq M \) of the same cardinality, and for every \( n \geq 0 \):

\[
\sum_{i \notin M_1} \sum_{j \in M_2} A_{ij}(n) \leq \Psi \sum_{i \in M_1} \sum_{j \notin M_2} A_{ij}(n)
\]  

(39)

Theorem 7: Every balanced asymmetric chain is in class \( \mathcal{P}^* \).

To prove Theorem 7, we first state the following lemma.

Lemma 3: Let \( A_{N \times N} \) be a balanced asymmetric matrix with bound \( \Psi \). Then, there exists a permutation matrix \( P_{N \times N} \) such that the product \( PA \) is self-confident with bound \( \delta = 4/(\Psi N^2 + 4N - 4) \).

Proof: Form a bipartite-graph \( \mathcal{H}(\mathcal{V}, \mathcal{E}) \) from \( A \) with \( N \) nodes in each part. Let \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \), each a copy of \( M \), be sets of nodes of the two parts of \( \mathcal{H} \). For every \( i \in \mathcal{V}_1 \) and \( j \in \mathcal{V}_2 \), connect \( i \) to \( j \) if \( A_{ij} \geq \delta = 4/(\Psi N^2 + 4N - 4) \). We wish to show that \( \mathcal{H} \) has a perfect matching. By Hall’s Marriage Theorem, it suffices to show that for every subset \( K \subseteq \mathcal{V}_1 \), we have \( |D(K)| \geq |K| \) where

\[
D(K) = \{j \in \mathcal{V}_2 \mid \exists i \in K \text{ s.t. } (i, j) \in \mathcal{E}\}
\]  

(40)

Assume that on the contrary, there exists \( K \subseteq \mathcal{V}_1 \) such that \( k' = |D(K)| < |K| = k \). Let \( K = \{c_1, \ldots, c_k\} \) and \( D(K) = \{d_1, \ldots, d_{k'}\} \). Define \( K' \subsetneq K \) by \( K' = \{c_1, \ldots, c_{k'}\} \). We now have:

\[
\sum_{i \in K'} \sum_{j \notin D(K)} A_{ij} < k'(N - k')\delta \leq \delta N^2/4.
\]  

(41)

On the other hand,

\[
\sum_{i \notin K'} \sum_{j \in D(K)} A_{ij} \geq \sum_{i \in K \setminus K'} \sum_{j \in D(K)} A_{ij} = (k - k') - \sum_{i \in K \setminus K'} \sum_{j \notin D(K)} A_{ij} \\
\geq (k - k') - (k - k')(N - k')\delta \geq 1 - (N - 1)\delta.
\]  

(42)

Since \( K', D(K) \subsetneq M \) are of identical cardinalities, the balanced asymmetry property of \( A \) together with inequalities (41) and (42) imply that

\[
1 - (N - 1)\delta < \delta \Psi N^2/4
\]  

(43)

Thus, \( \delta > 4/(\Psi N^2 + 4N - 4) \), which is a contradiction. Therefore, \( \mathcal{H} \) has a perfect matching and consequently, there exists a permutation \( \tau \) such that \( A_{\tau(i), i} \geq \delta, \forall i \). Consequently, the permutation
matrix $P$ with $e_{r(i)}$ as its $i$th row, where $e_j$ denotes a row vector of length $N$ with 1 in the $j$th position and 0 in every other position, is such that the product $PA$ is self-confident with $\delta$. 

**Proof of Theorem 7**: We now prove Theorem 7. Let $\{A(n)\}$ be a balanced asymmetric chain with bound $\Psi$. Set: $\delta = 4/(\Psi N^2 + 4N - 4)$. We recursively define sequence $\{P(n)\}$ of permutation matrices as follows: From Lemma 3, we know that there exists a permutation matrix $P(0)$ such that the product $P(0)A(0)$ is self-confident with $\delta$. Find permutation matrix $P(n)$, $n \geq 1$, such that the product $P(n)A(n)P^t(n-1)$ is self-confident with $\delta$. Note that existence of $P(n)$ is implied by Lemma 3, taking into account that the product $A(n)P^t(n-1)$ is balanced asymmetric with bound $\Psi$, since the columns of the product are a permutation of the columns of $A(n)$, itself a balanced asymmetric matrix with bound $\Psi$. Hence, if we define chain $\{B(n)\}$ by:

$$B(n) = P(n)A(n)P^t(n-1),$$

then, $\{B(n)\}$ has both self-confident and balanced asymmetry properties. Since balanced asymmetry is stronger than cut-balance, from [19] we conclude that chain $\{B(n)\}$ belongs to the set $P^*$. Furthermore, it is straightforward to show that if $\{\pi(n)\}$ is an absolute probability sequence adapted to chain $\{B(n)\}$, then $\{\pi(n)P(n-1)\}$, where $P(-1) = I_{N \times N}$, is an absolute probability sequence adapted to chain $\{A(n)\}$. This immediately implies that $\{A(n)\} \in P^*$. 

From Theorem 7, we know that, as for the class $P^*$, the components of the absolute probability process are uniformly bounded away from zero, and also there is no $J^0$ in the jet decomposition of the D-S Theorem. Therefore, we again consider $J^1, \ldots, J^c$ as the jet decomposition.

**Theorem 8**: If $\{A(n)\}$ is balanced asymmetric, then the cardinality of each jet in the jet decomposition of the D-S Theorem, becomes time-invariant after a finite time.

**Proof**: Let $\{A(n)\}$ be balanced asymmetric with bound $\Psi$. It suffices to show that there are finite times in which cardinality of a jet, in the jet decomposition of the D-S Theorem, increases by at least 1. In the following, we see what happens when the cardinality of a jet, say $J^k$, increases. Assume that for a fixed $n \geq 0$, we have $|J^k(n+1)| > |J^k(n)|$. For an arbitrary $i \in J^k(n+1)$, let $T \subseteq J^k(n+1)$ be such that $i \not\in T$ and $|T| = |J^k(n)|$. Thus by the balanced asymmetry property,

$$\sum_{j \in J^k(n)} a_{ij}(n) \leq \sum_{i \in T} \sum_{j \in J^k(n)} a_{ij}(n) \leq \Psi \sum_{i \in T} \sum_{j \not\in J^k(n)} a_{ij}(n) \leq \Psi \sum_{i \in J^k(n+1)} \sum_{j \not\in J^k(n)} a_{ij}(n)$$

(45)
Therefore,
\[
\sum_{i \in J^k(n+1)} \sum_{j \in J^k(n)} a_{ij}(n) \leq |J^k(n + 1)| \cdot \Psi \sum_{i \in J^k(n+1)} \sum_{j \not\in J^k(n)} a_{ij}(n) \tag{46}
\]

On the other hand,
\[
\sum_{i \in J^k(n+1)} \sum_{j \in J^k(n)} a_{ij}(n) = |J^k(n + 1)| - \sum_{i \in J^k(n+1)} \sum_{j \not\in J^k(n)} a_{ij}(n) \tag{47}
\]

Eqs. (46) and (47) together imply:
\[
\sum_{i \in J^k(n+1)} \sum_{j \not\in J^k(n)} a_{ij}(n) \geq \frac{|J^k(n + 1)|}{1 + \Psi |J^k(n + 1)|} \geq \frac{1}{1 + \Psi} \tag{48}
\]

Once again since the total interactions between \( J^k \) and \( \overline{J}^k \) is finite over the infinite time interval, inequalities (48) can occur for finitely many times \( n \), and this completes the proof.

An immediate corollary of Theorem 8 as follows.

**Corollary 4:** Consider a multi-agent system with dynamics (1), where \( \{A(n)\} \) is balanced asymmetric. Then, \( z_i(n) \) converges for every \( i \in M \), as \( n \) goes to infinity, where \( z_i(n) \) is the \( i \)th least value among \( X_1(n), \ldots, X_N(n) \).

**Definition 12:** A chain \( \{A(n)\} \) of stochastic matrices is said to have the absolute infinite flow property, if for every jet \( J \) in \( M \) with a time-invariant size, \( U(J, M \setminus J) \) is unbounded.

**Theorem 9:** If \( \{A(n)\} \) is balanced asymmetric, then, \( \{A(n)\} \) is class-ergodic if and only if the absolute infinity property holds over each island of \( \{A(n)\} \). Furthermore, in case of class-ergodicity, islands are ergodic classes are \( \{A(n)\} \).

**Proof:** From Theorem 7, we know that \( \{A(n)\} \in \mathcal{P}^* \). Therefore, taking advantage of Theorem 5, it suffices to show that absolute infinite flow and infinite jet-flow properties are equivalent on each island. Obviously, the former is implied from the latter. We prove the converse as follows: Let the absolute infinite flow property holds over each island. Assume that \( I \) is an arbitrary island of \( \{A(n)\} \) and \( J \) is an arbitrary jet in \( I \). If the cardinality of jet \( J \) becomes time-invariant after a finite time, unboundedness of \( U(J, I \setminus J) \) is immediately implied from the absolute infinite flow property over \( I \). Otherwise, the cardinality of \( J \) increases infinitely many times by at least 1. In this case, from the proof of Theorem 8, we know that \( V(J, M \setminus J) \) is unbounded, and consequently \( U(J, M \setminus J) \) is unbounded following Lemma 1. Moreover,
\[
U(J, M \setminus J) + U(I \setminus J, M \setminus I) = U(J, I \setminus J) + U(I, M \setminus I) \tag{49}
\]
Since $U(I, M \setminus I)$ is bounded, unboundedness of $U(J, M \setminus J)$ implies that $U(J, I \setminus J) = \infty$. This completes the proof.

**Corollary 5:** If chain $\{A(n)\}$ is balanced asymmetric, then it is ergodic if and only if it has the absolute infinite flow property.

**V. CONCLUSION**

We considered a general linear distributed averaging algorithm in discrete time. Following the work by Touri and Nedić [18], and recalling the notion of jets from [2], we introduced a property of chains of stochastic matrices, i.e., the infinite jet-flow property, that happens to be a fairly strong necessary condition for ergodicity of the chain. Moreover, for the chain to be class-ergodic, the infinite jet-flow property must hold over each connected component of the infinite flow graph, as defined in [19].

We then illustrated the close relationship between the D-S Theorem by Sonin [15] and convergence properties of consensus algorithms. By employing the D-S Theorem, we showed that the necessary conditions found earlier, are also sufficient in case the chain is of the class $P^*$ [19]. We argued that the previous related results in the literature, [4], [9], [19] in particular, can be simply implied by the equivalent conditions that we obtained. In future work, we shall extend our results to the case when the number of agents increases to infinity, although the D-S Theorem holds only if $N$ is finite.

**REFERENCES**

[1] D.P. Bertsekas and J.N. Tsitsiklis. Parallel and distributed computation: numerical methods, *Englewood Cliffs, NJ: Prentice-Hall*, 1989.
[2] D. Blackwell. Finite Non-homogeneous Markov Chains, *IMS Collections*, Vol. 46, pages 594–599, 1945.
[3] V.D. Blondel, J.M. Hendrickx, A. Olshevsky, and J.N. Tsitsiklis. Convergence in multi-agent coordination, consensus, and flocking, in proceedings of the *44th IEEE Conference on Decision and Control (CDC 2005)*, Seville, Spain, pages 2996–3000, 2005.
[4] S. Bolouki, R.P. Malhamé. Ergodicity and Class-Ergodicity of Balanced Asymmetric Stochastic Chains, *GERAD Technical Report*, G–2012–93, 2012.
[5] S. Chatterjee and E. Senata. Towards consensus: some convergence theorems on repeated averaging, *Journal of Applied Probability*, Vol. 14, No. 345, pages 89–97, 1977.
[6] M. DeGroot. Reaching a consensus, *Journal of the American Statistical Association*, Vol. 69, No. 345, pages 118–121, 1974.
[7] J.M. Hendrickx and V.D. Blondel. Convergence of different linear and nonlinear Vicsek models, in proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2006), Kyoto, Japan, pages 1229–1240, 2006.

[8] J.M. Hendrickx. Graphs and networks for the analysis of autonomous agent systems, Ph.D Dissertation, Université Catolique de Louvain, 2008.

[9] J.M. Hendrickx, J.N. Tsitsiklis. Convergence of type-symmetric and cut-balanced consensus seeking systems, IEEE Transactions on Automatic Control, Vol. PP, No. 99, 2012.

[10] A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules, IEEE Transactions on Automatic Control, Vol. 48, No. 6, pages 988–1001, 2003.

[11] A. Kolmogorov. Zur Theorie der Markoffschen Ketten, Mathematische Annalen, Vol. 112, No. 1, pages 155–160, 1936.

[12] S. Li and H. Wang. Multi-agent coordination using nearest neighbor rules: a revisit to Vicsek model, arXiv:cs/0407021v2 [cs.MA], 2004.

[13] J. Lorenz. A stabilization theorem for continuous opinions dynamics, Physica A, Vol. 355, No. 1, pages 217–233, 2005.

[14] L. Moreau. Stability of multi-agent systems with time-dependent communication links, IEEE Transactions on Automatic Control, Vol. 50, No. 2, pages 169–182, 2005.

[15] I.M. Sonin. The Decomposition-Separation Theorem for Finite Nonhomogeneous Markov Chains and Related Problems, IMS Collections, Vol. 4, pages 1–15, 2008.

[16] B. Touri and A. Nedić. On ergodicity, infinite flow and consensus in random models, IEEE Transactions on Automatic Control, Vol. 56, No. 7, pages 1593–1605, 2011.

[17] B. Touri and A. Nedić. On approximations and ergodicity classes in random chains, IEEE Transactions on Automatic Control, Vol. 57, No. 11, pages 2718–2730, 2012.

[18] B. Touri and A. Nedić. On Backward Product of Stochastic Matrices, IFAC Journal of Automatica, Vol. 48, No. 8, pages 1477–1488, 2012.

[19] B. Touri and A. Nedić. Product of Random Stochastic Matrices, arXiv:1110.1751 [math.PR], 2011.

[20] J.N. Tsitsiklis, D.P. Bertsekas, and M. Athans. Distributed asynchronous deterministic and stochastic gradient optimization algorithms, IEEE Transactions on Automatic Control, Vol. AC-31, No. 9, pages 803–812, 1986.

[21] J.N. Tsitsiklis. Problems in decentralized decision making and computation, Ph.D Dissertation, Department of EECS, MIT, Cambridge, 1984.