Comment on “Analysis of a Charge-Pump PLL: A New Model” by M. van Paemel

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Abstract

In this short communication we comment on the non-linear mathematical model of CP-PLL introduced by V.Paemel. We reveal and obviate some shortcomings in the model.

I. Introduction

M. van Paemel’s article [1] became the first one where a complete nonlinear mathematical model of CP-PLL was derived. The classical models (see e.g. [2–4]) considered approximation of the phase detector dynamics in continuous time and linearization. While approximate models are useful for analysis of small frequency deviations between VCO and Ref signals, Paemel’s models can be also used for studying out-of-lock behaviour.

However, the algorithm suggested in [1] does not always work. Below we reveal and obviate shortcomings in the Paemel’s model and discuss corresponding numerical examples.

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II. Numerical examples

The following examples demonstrate that algorithm and formulas suggested by M. van Paemel should be used carefully for simulation even inside allowed area (see original paper Fig 18 and Fig. 22). While the examples are given for the first time, the main idea of Example 1 was already noticed by P. Acco and O. Feely [5], [6]. P. Acco and O. Feely considered only near-locked state, therefore they didn’t notice problems with out-of-lock behaviour. Example 2 and Example 3 demonstrate problems with out-of-lock behaviour, which was not discovered before.

A. Example 1

Consider the following set of parameters and initial state:

\[ R_2 = 0.2; C = 0.01; K_v = 20; I_p = 0.1; T = 0.125; \]
\[ \tau(0) = 0.0125; v(0) = 1. \]  

(1)

Calculation of normalized parameters (equations (27)-(28) and (44)-(45) in [1])

\[ K_N = I_p R_2 K_v T = 0.05, \]
\[ \tau_{2N} = \frac{R_2 C}{T} = 0.016, \]
\[ F_N = \frac{1}{2\pi} \sqrt{\frac{K_N}{\tau_{2N}}} \approx 0.2813, \]
\[ \zeta = \frac{\sqrt{K_N \tau_{2N}}}{2} \approx 0.0141, \]  

(2)

shows that parameters (1) correspond to allowed area in Fig. 1 (equations (46)–(47), Fig 18 and Fig. 22 in [1]):
Fig. 1: Parameters for Example 1, Example 2, and Example 3 correspond to allowed area (see Fig. 18 and Fig. 22 in [1])

\[
F_N < \frac{\sqrt{1 + \zeta^2 - \zeta}}{\pi} \approx 0.3138, \quad (3)
\]

\[
F_N < \frac{1}{4\pi\zeta} \approx 5.6438. \quad (4)
\]

Now we use the flowchart in Fig. 2 (Fig. 10 in [1]) to compute \(\tau(1)\) and \(v(1)\): since \(\tau(0) > 0\) and \(\tau(0) < T\), we proceed to case 1) and corresponding relation for \(\tau(k+1)\) (equation (7) in [1]):

\[
\tau(k+1) = \frac{-I_pR_2 - v(k) + \sqrt{(I_pR_2 + v(k))^2 - \frac{2I_p}{C}(v(k)(T - \tau(k)) - \frac{1}{K_v})}}{\frac{I_p}{C}}. \quad (4)
\]

However, the expression under the square root in (4) is negative:

\[
(I_pR_2 + v(0))^2 - \frac{2I_p}{C}(v(0)(T - \tau(0)) - \frac{1}{K_v}) = -0.2096 < 0. \quad (5)
\]

Therefore the algorithm is terminated with error.

**B. Example 2**

Consider the same parameters as in Example 1, but \(\tau(0) = -0.098\):

\[
R_2 = 0.2; C = 0.01; K_v = 20; I_p = 0.1; T = 0.125; \quad \tau(0) = -0.098; \quad v(0) = 1. \quad (6)
\]
In this case (2), (3), and Fig. 1 are the same as in Example 1, i.e. we are in the “allowed area”. Now we compute $\tau(1)$ and $v(1)$ following the flowchart in Fig. 2 since $\tau(0) < 0$. 

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Fig. 10. Flowchart of the algorithm.

Fig. 2: Demonstration of Example 1, Example 2, and Example 3 in the flowchart of the algorithm (see Fig. 10 in [1]).
proceed to case 2) and corresponding equation for \( \tau(k+1) \) (equation (9) in [1]):

\[
\tau(1) = \frac{1}{K_v} - \frac{I_p R_2 \tau(0) - \frac{I_p \tau(0)^2}{2C}}{v(0)} - T + \tau(0) = -0.21906,\]

\[-0.2191 < -T = -0.125.\]

This fact indicates cycle-slipping (out of lock). According to the flowchart in Fig. 2 (see Fig. 10 in [1]), we should proceed to case 6) and recalculate \( \tau(1) \). First step of case 6) is to calculate \( t_1, t_2, t_3, \ldots \) (equations (16) and (17) in [1]):

\[
t_n = \frac{v_{n-1} - I_p R_2 - \sqrt{(v_{n-1} - I_p R_2)^2 - 2I_p R_2 \cdot \frac{1}{K_v}}}{\frac{I_p}{C}},
\]

\[
v_n = v_{n-1} - \frac{I_p}{C} t_n,\]

\[
v_0 = v(k - 1).\]

Since \( k = 0 \), then

\[
t_1 = \frac{v_0 - I_p R_2 - \sqrt{(v_0 - I_p R_2)^2 - 2I_p R_2 \cdot \frac{1}{K_v}}}{\frac{I_p}{C}},
\]

\[
v_1 = v_0 - \frac{I_p}{C} t_1,\]

\[
v_0 = v(-1).\]

However, \( v(-1) \) doesn’t make sense and algorithm terminates with error. Even if we suppose that there is a mistype and \( v_0 = v(0) \), then relation under the square root become negative:

\[
(v(0) - I_p R_2)^2 - 2I_p R_2 \cdot \frac{1}{C K_v} = -0.0396 < 0.
\]

In both cases the algorithm is terminated with error. Note, that modification of case 2) corresponding to VCO overload (equation (35) in [1]) can not be applied here, since \( v(0) > I_p R_2 \) (no overload) and \( v(1) \) is not computed yet because of the error.\(^1\)

\(^1\) However, this can be fixed, by setting \( v(-1) = v(0) - \frac{I_p}{C} \tau(0) \).
C. Example 3

Consider parameters:

\[
\tau(0) = -0.123; \quad v(0) = 0.6, \quad (11)
\]

\[
R_2 = 0.2; C = 0.02; K_v = 20; I_p = 0.1; T = 0.125.
\]

Similar to (2) and (3)

\[
K_N = 0.05, \quad \tau_{2N} = 0.032,
\]

\[
F_N \approx 0.1989, \quad \zeta = 0.02,
\]

\[
F_N < \frac{\sqrt{1 + \zeta^2} - \zeta}{\pi} \approx 0.3120,
\]

\[
F_N < \frac{1}{4\pi\zeta} \approx 3.9789,
\]

parameters (11) correspond to allowed area in Fig. 1 (equations (46)-(47), Fig. 18 and Fig. 22 in [1]).

Now we compute \(\tau(1)\) and \(v(1)\) following the flowchart in Fig. 2: since \(\tau(0) < 0\) one proceeds to case 2) and corresponding equation for computing \(\tau(k+1)\) (equation (9) in [1]):

\[
\tau(1) = \frac{1}{R_2} - \frac{I_p R_2 \tau(0)}{v(0)} - \frac{I_p \tau(0)^2}{2C} - T + \tau(0) \approx -0.224,
\]

\[
-0.224 < -T = -0.125.
\]

The last inequality indicates cycle-slipping (out of lock). According to the flowchart in Fig. 2 (see Fig. 10 in [1]), one proceeds to case 6) and recalculates \(\tau(1)\). First step of case 6) is to calculate \(t_1, t_2, t_3, \ldots\) using (8) (see equations (16) and (17) in [1]) until \(t_1 + t_2 + \ldots + t_n > |\tau(0)|\). Even if we suppose \(v(-1) = v(0) - \frac{I_p}{C} \tau(0)\), we get

\[
t_1 = 0.0463, \quad v_1 = 1.215;
\]

\[
t_2 = 0.0618, \quad v_2 = 0.983;
\]

\[
t_1 + t_2 = 0.1081 < |\tau(0)| = 0.123.
\]
However, \( t_3 \) cannot be computed, because the relation under the square root in (8) is negative:

\[
(v_2 - I_p R_2)^2 - 2 \frac{I_p}{C} \cdot \frac{1}{K_v} \approx -0.0726.
\] (16)

### III. Corrected discrete time model of CP-PLL

Below we suggest the corrected *discrete time nonlinear mathematical model of CP-PLL*, in which shortcomings were fixed. The problem with flowchart (see Fig. 2) is that the sign of \( \tau(k+1) \) by case 1) is used to decide that actually case 3) should be used. Similarly case 2) always precede case 4),5),6) which may lead to errors. However, it is possible to use \( \tau(k) \) and \( v(k) \) explicitly to decide which case should be used. This allows one to avoid square roots of negative numbers, reduce number of cases from 6 to 4, and apply methods from theory of discrete time dynamical system (see, e.g. [6]).

Here \( v(0) \) and \( \tau(0) \) are initial conditions and Paemel’s notations are used.

\[
\tau(k+1) = \begin{cases} 
\frac{-b + \sqrt{b^2 - 4ac}}{2a}, & \tau(k) \geq 0, \quad c \leq 0, \\
\frac{1}{\omega_{\text{vco}} + K_v v(k)} - T + (\tau(k) \mod T), & \tau(k) \geq 0, \quad c > 0, \\
l_b - T, & \tau(k) < 0, \quad l_b \leq T, \\
\frac{-b + \sqrt{b^2 - 4ad}}{2a}, & \tau(k) < 0, \quad l_b > T,
\end{cases}
\]

\[
v(k+1) = v(k) + \frac{I_p}{C} \tau(k+1).
\]

(17)
Here VCO frequency is $f_{\text{VCO}} = \omega_{\text{VCO}}^{\text{free}} + K_v v(k)$, and $\omega_{\text{VCO}}^{\text{free}}$ is a free-running (quiescent) frequency (in V.Paemel’s paper $\omega_{\text{VCO}}^{\text{free}} = 0$). If at some point VCO become overloaded one should stop simulation or use another set of equations, based on ideas similar to (34) and (35) in [1]. Overload conditions are

$$
\tau(k) > 0 \text{ and } v(k) + \frac{\omega_{\text{VCO}}^{\text{free}}}{K_{\text{vco}}} - \frac{I_p}{C} \tau_k < 0,
$$

$$
\tau(k) < 0 \text{ and } v(k) + \frac{\omega_{\text{VCO}}^{\text{free}}}{K_{\text{vco}}} - I_p R_2 < 0.
$$

Remark that following the ideas from [5], [7], the number of parameters in (17) can be reduced to just two ($\alpha$ and $\beta$) by the following change of variables

$$
\tau_{\alpha\beta}(k) = \frac{\tau(k)}{T}, \quad \omega_{\alpha\beta}(k) = T \left( \omega_{\text{VCO}}^{\text{free}} + K_v v(k) \right) - 1,
$$

$$
\alpha = K_v I_p T R_2, \quad \beta = \frac{K_v I_p T^2}{2C}.
$$

IV. Numerical examples for corrected model

Let’s apply corrected model (17) to Example 1, Example 2, and Example 3. All three examples assume $\omega_{\text{VCO}}^{\text{free}} = 0.$
A. Example 1

By (17) and (1) we calculate value of $c$:

$$c = (T - (\tau(0) \mod T))K_v v(0) - 1 = 1.2500$$

(20)

and since $\tau(0) \geq 0$ and $c > 0$ we get

$$
\tau(1) = \frac{1}{K_v v(0)} - T + (\tau(0) \mod T) = -0.0625,
$$

(21)

$$v(1) = v(0) + \frac{I_p}{C} \tau(1) = 0.3750.
$$

Illustration of this example is shown in Fig. 3.

Note, that in this case there is no saturation, since the filter output (VCO input) is positive, see Fig. 3.

B. Example 2

By (17) and (6) we have $l_b \approx 0.0059$. Since $l_b \leq T$, then $\tau(1) \approx -0.1191, v(1) \approx -0.1906$.

In this case the VCO is overloaded (see Fig. 4). Model (17) correctly detects overload by (18)

$$v(1) + \frac{\omega_{\text{free}}}{K_{\text{vco}}} - I_p R_2 \approx -0.2106 < 0$$

(22)

and stops simulation.

C. Example 3

Note, that in this case VCO is not overloaded, since the filter output (VCO input) is positive, see Fig. 5. Equations (17) allow to correctly calculate next step:

$$
\tau(1) = -0.0569, \quad v(1) = 0.3153.
$$

(23)

V. Conclusions

There were many attempts to generalize equations derived in [1] for higher-order loops (see, e.g. [8]–[13]), however the resulting transcendental equations can not be solved analytically without using approximations.
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Illustrations for numerical examples

\[ R_2 = 0.2; C = 0.01; K_v = 20; I_p = 0.1; T = 0.125 \]
\[ \tau(0) = 0.0125; v(0) = 1 \]
\[ \tau(1) = -0.0625; v(1) = 0.3750 \]

Fig. 3: Application of corrected model to Example 1: Reference signal, VCO output, PFD output, and filter output.
Fig. 4: Application of corrected model to Example 2: Reference signal, VCO output, PFD output, and filter output.
$R_2 = 0.2; C = 0.02; K_v = 20; I_p = 0.1; T = 0.125$

$\tau(0) = -0.123, \psi(0) = 0.6$

$\tau(1) = -0.0569, \psi(1) = 0.3153$

Fig. 5: Application of corrected model to Example 3: Reference signal, VCO output, PFD output, and filter output.