Rational invariant tori and band edge spectra for non-selfadjoint operators

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Abstract

We study semiclassical asymptotics for spectra of non-selfadjoint perturbations of selfadjoint analytic $h$-pseudodifferential operators in dimension 2, assuming that the classical flow of the unperturbed part is completely integrable. Complete asymptotic expansions are established for all individual eigenvalues in suitable regions of the complex spectral plane, near the edges of the spectral band, coming from rational flow-invariant Lagrangian tori.

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1 Introduction

Spectra for semiclassical non-selfadjoint operators often display fascinating features, from lattices of low-lying eigenvalues for operators of Kramers-Fokker-Planck type [4], [6] to eigenvalues for operators with analytic coefficients in dimension one, concentrated to unions of curves, [21], [23], [3], [9]. The work [20] has established that for wide and stable classes of non-selfadjoint analytic pseudodifferential operators in dimension two, the individual eigenvalues can be determined accurately in the semiclassical limit, by means of a complex Bohr-Sommerfeld quantization condition, and form a distorted two-dimensional lattice. Now in many natural situations [14], [27], [26], [28], one encounters non-selfadjoint operators of the form

\[ P_\varepsilon = p(x, hD_x) + i\varepsilon q(x, hD_x), \quad 0 \leq \varepsilon \ll 1, \]

considered on \( \mathbb{R}^n \) or a compact real analytic manifold, with \( P_{\varepsilon=0} \) being selfadjoint. Here \( 0 < h \ll 1 \) is the semiclassical parameter and the second small parameter \( \varepsilon \) represents the strength of the non-selfadjoint perturbation. The principal symbol of \( P_\varepsilon \) in (1.1) is of the form \( p_\varepsilon(x, \xi) = p(x, \xi) + i\varepsilon q(x, \xi) \), where \( p \) is real, and let us also assume, to fix the ideas, that \( q \) is real. Both \( p \) and \( q \) are assumed to be analytic, with \( p \) elliptic near infinity. The spectrum of \( P_\varepsilon \) near the origin is confined to a band of width \( O(\varepsilon) \), and the general problem is to understand the distribution of eigenvalues of \( P_\varepsilon \) near 0, in the semiclassical limit \( h \to 0^+ \). To this end, let us assume that 0 is a regular value of \( p \), so that the energy surface \( p^{-1}(0) \) is a smooth compact submanifold of the phase space. We then know [17], [18] that the real parts of the eigenvalues of \( P_\varepsilon \) near 0 are distributed according to the same Weyl law as that for the unperturbed operator \( P_{\varepsilon=0} \). In order to study the distribution of the
imaginary parts of the eigenvalues, following the method of averaging [31], [1], we let $H_p$ be the Hamilton vector field of $p$ and introduce the time averages

$$\langle q \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} q \circ \exp(tH_p) \, dt, \quad T > 0,$$

(1.2)

of $q$ along the $H_p$–trajectories. It follows from [14], [27], [12] that if $z \in \text{Spec}(P_{\varepsilon})$ is such that $|\text{Re} \, z| \leq \delta$, then

$$\lim_{T \to \infty} \inf_{p^{-1}(0)} \langle q \rangle_T - o(1) \leq \frac{\text{Im} \, z}{\varepsilon} \leq \lim_{T \to \infty} \sup_{p^{-1}(0)} \langle q \rangle_T + o(1),$$

(1.3)

as $(\varepsilon, \delta, h) \to 0^+$. The spectral analysis for non-selfadjoint operators of the form (1.1) has been pursued by the authors in the series of papers [7]–[12], the latter work jointly with S. Vũ Ngọc, when the dimension $n = 2$ and the $H_p$–flow is either periodic or completely integrable. Let us focus, from now on, on the completely integrable case, which will be considered also in the present work. In this case, the energy surface $p^{-1}(0)$ is foliated by invariant Lagrangian tori, along with possibly some other more complicated flow-invariant sets. When $\Lambda \subset p^{-1}(0)$ is an invariant torus such that the rotation number of $H_p$ along $\Lambda$ is Diophantine, i.e. poorly approximated by rational numbers, or more generally, irrational, we have that the time averages $\langle q \rangle_T$ along $\Lambda$ converge to the space average $\langle q \rangle(\Lambda)$ of $q$ over $\Lambda$, as $T \to \infty$. When $\Lambda$ is a torus with a rational rotation number, or a singular set in the foliation of $p^{-1}(0)$, then in analogy with (1.3), we introduce the compact interval

$$Q_\infty(\Lambda) = \left[ \lim_{T \to \infty} \inf_{\Lambda} \langle q \rangle_T, \lim_{T \to \infty} \sup_{\Lambda} \langle q \rangle_T \right]$$

(1.4)

of limits of the time averages above.

Let $F_0 \in \mathbb{R}$ be such that $F_0 = \langle q \rangle(\Lambda_d)$ for a single Diophantine Lagrangian torus $\Lambda_d \subset p^{-1}(0)$, and let us assume that

$$F_0 \notin Q_\infty(\Lambda),$$

(1.5)

for any other invariant set $\Lambda \neq \Lambda_d$ in $p^{-1}(0)$. It was then shown in [12] that the spectrum of $P_{\varepsilon}$ can be determined completely, modulo $O(h^\infty)$, in a rectangle of the form $[-h^\delta/C, h^\delta/C] + i\varepsilon[F_0 - h^\delta/C, F_0 + h^\delta/C]$, where $\delta > 0$ and $\varepsilon$ satisfies $h^K < \varepsilon \ll 1$, for $K \gg 1$. Similarly to [20], the spectrum has a structure of a distorted
two-dimensional lattice, with the horizontal spacing $\sim h$ and the vertical one $\sim \varepsilon h$. A closely related result was obtained in [11], giving a Weyl type asymptotic formula for the number of eigenvalues of $P_\varepsilon$ in an intermediate spectral band, bounded from above and from below by Diophantine levels, such as $F_0$ above. It turned out that the distribution of the imaginary parts of the eigenvalues of $P_\varepsilon$ is governed by a Weyl law, expressed in terms of phase space volumes associated to $p$ and the long time averages of $q$.

Having elucidated the role played by flow-invariant Diophantine tori in the spectral analysis of $P_\varepsilon$, let us now turn the attention to spectral contributions of tori that are rational, which constitutes the subject of the present work. Let $F_0 \in \mathbb{R}$ be such that $F_0 = \langle q \rangle(\Lambda_d)$, for a Diophantine torus $\Lambda_d$ as above, and rather than demanding (1.5), let us assume that there exists a rational torus $\Lambda_r \subset p^{-1}(0)$ such that $F_0 \in \mathbb{Q}_\infty(\Lambda_r)$, $F_0 \neq \langle q \rangle(\Lambda_r)$, while $F_0 \notin \mathbb{Q}_\infty(\Lambda)$, for $\Lambda \neq \Lambda_d, \Lambda_r$. An attempt to determine the individual eigenvalues of $P_\varepsilon$ near $i\varepsilon F_0$ was made by the authors in the work [10], by means of the normal form techniques. As a result, the normal forms near $\Lambda_r$ that we obtained were given by a family of one-dimensional "resonant" non-selfadjoint operators, and the possibility of quite serious pseudospectral phenomena for this family [2] prevented us from computing the eigenvalues individually. Correspondingly, the main result of [10] was weaker, establishing that the spectrum of $P_\varepsilon$ near $i\varepsilon F_0$ was of the form $E_d \cup E_r$, where the "Diophantine" contribution $E_d$ is a distorted lattice that can be described explicitly, as in [12], and the cardinality of the "rational" contribution $E_r$ is $\ll$ than that of $E_d$.

Subsequently, in the course of some numerical experiments, the authors have encountered peculiar pictures of the spectra of $P_\varepsilon$, where the eigenvalues had the form of a "centipede", with the body agreeing with the range of torus averages of $q$ — see Section 8 for the illustrations and the details of the numerical computations. The legs of the centipede were more mysterious at first, but things became clearer when we realized that they represented the influence of suitable rational tori. It became then natural to hope that the eigenvalues near the extremities of the legs could be determined asymptotically in a rigorous way, since the pseudospectral effects should become more moderate near the edges of the spectral band, [2]. The main result of the present work, giving a complete asymptotic description of the individual extremal eigenvalues of $P_\varepsilon$, can be considered as a justification of this hope.

Let us conclude the introduction by formulating, in a rough way, the main result of the paper — see Theorem 2.1 below for the precise statement. Let $\Lambda_0 \subset p^{-1}(0)$ be
a rational Lagrangian torus such that
\[
\inf Q_\infty(\Lambda_0) < \inf_{\Lambda \neq \Lambda_0} (\inf Q_\infty(\Lambda)).
\] (1.6)

The restriction of the $H_p$-flow to $\Lambda_0$ is periodic with primitive period $T_0 > 0$, and the time average $\langle q \rangle_{T_0}$ in [12] can naturally be viewed as a function on the space of closed orbits $\Lambda_0/\exp(\mathbb{R}H_p)$. Let us assume that $\langle q \rangle_{T_0}$, viewed as a function on $\Lambda_0/\exp(\mathbb{R}H_p)$, has a unique minimum which is nondegenerate, and restrict $\varepsilon$ to a suitable interval of the form $h^{1+\eta} \leq \varepsilon \leq h^{1-\eta}$, $\eta > 0$. Then for any fixed $C_0 > 0$ the eigenvalues of $P_\varepsilon$ in the region
\[
\left\{ z \in \mathbb{C}; \; \left| \text{Re } z \right| < \frac{h}{C_0 \sqrt{\varepsilon}}, \; \frac{\text{Im } z}{\varepsilon} \leq \inf Q_\infty(\Lambda_0) + C_0 \frac{h}{\sqrt{\varepsilon}} \right\}
\]
can be determined completely, modulo $O(h^\infty)$, and are given by
\[
\lambda_{j,k} = a(\xi_2) + i\varepsilon b(\xi_2) + \varepsilon^{1/2} h A_{j,k}, \quad \xi_2 = h(j - \frac{k_0}{4}) - \frac{S}{2\pi}, \quad j \in \mathbb{Z}, \; k \in \mathbb{N}. \tag{1.7}
\]
Here $a(0) = 0$, $a'(0) > 0$, $b(0) = \inf Q_\infty(\Lambda_0)$, and $S$ and $k_0$ are the classical action and the Maslov index of a primitive closed $H_p$-trajectory in $\Lambda_0$. We have a complete asymptotic expansion for $\Lambda_{j,k}$ in integer powers of $\tilde{h} = h/\sqrt{\varepsilon}$,
\[
\Lambda_{j,k} \sim \sum_{\nu=0}^\infty \tilde{h}^\nu \lambda_k^\nu(\xi_2, \sqrt{\varepsilon}),
\]
where
\[
\lambda_k^0(0, 0) = de^{\pi i/4}(2k + 1), \quad d > 0.
\]

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2 Statement of the main results

2.1 General assumptions

We shall start by describing the general assumptions on our operators, which will be the same as in [10], [12], as well as in the earlier papers mentioned above. Let
$M$ denote either the space $\mathbb{R}^2$ or a real analytic compact manifold of dimension 2. When $M = \mathbb{R}^2$, let

$$P_\varepsilon = P^w(x, hD_x, \varepsilon; h), \quad 0 < h \leq 1,$$

be the $h$–Weyl quantization on $\mathbb{R}^2$ of a symbol $P(x, \xi, \varepsilon; h)$ (i.e. the Weyl quantization of $P(x, h\xi, \varepsilon; h)$), depending smoothly on $\varepsilon \in \text{neigh}(0, \mathbb{R})$ and taking values in the space of holomorphic functions of $(x, \xi)$ in a tubular neighborhood of $\mathbb{R}^4$ in $\mathbb{C}^4$, with

$$|P(x, \xi, \varepsilon; h)| \leq \mathcal{O}(1)m(\text{Re}(x, \xi)), \quad (2.2)$$

there. Here $1 \leq m \in C^\infty(\mathbb{R}^4)$ is an order function, in the sense that

$$m(X) \leq C_0|X - Y|^{N_0}m(Y), \quad X, Y \in \mathbb{R}^4, \quad (2.3)$$

for some $C_0, N_0 > 0$. We shall assume, as we may, that $m$ belongs to its own symbol class, so that $\partial^\alpha m = \mathcal{O}_\alpha(m)$ for each $\alpha \in \mathbb{N}^4$. Then for $h > 0$ small enough and when equipped with the domain $H(m) := (m^w(x, hD))^{-1}(L^2(\mathbb{R}^2))$, $P_\varepsilon$ becomes a closed densely defined operator on $L^2(\mathbb{R}^2)$.

**Remark.** The analyticity assumptions will allow us to treat the case when $\varepsilon \asymp h^\delta$, for $0 < \delta < 1$. When $\varepsilon = \mathcal{O}(h)$, standard $C^\infty$–microlocal analysis would have been sufficient.

Assume furthermore that

$$P(x, \xi, \varepsilon; h) \sim \sum_{j=0}^{\infty} h^j p_{j, \varepsilon}(x, \xi) \quad (2.4)$$

in the space of holomorphic functions depending smoothly on $\varepsilon \in \text{neigh}(0, \mathbb{R})$ and satisfying (2.2) in a fixed tubular neighborhood of $\mathbb{R}^4$. We assume that $p_{0, \varepsilon}$ is elliptic near infinity,

$$|p_{0, \varepsilon}(x, \xi)| \geq \frac{1}{C}m(\text{Re}(x, \xi)), \quad |(x, \xi)| \geq C, \quad (2.5)$$

for some $C > 0$.

When $M$ is a compact manifold, for simplicity we shall take $P_\varepsilon$ to be a differential operator on $M$, such that for every choice of local coordinates, centered at some point of $M$, it takes the form

$$P_\varepsilon = \sum_{|\alpha| \leq m} a_{\alpha, \varepsilon}(x; h)(hD_x)^\alpha, \quad (2.6)$$
where $a_{\alpha,\varepsilon}(x; h)$ is a smooth function of $\varepsilon \in \text{neigh}(0, \mathbb{R})$ with values in the space of bounded holomorphic functions in a complex neighborhood of $x = 0$, independent of $h$ when $|\alpha| = m$. We further assume that

$$a_{\alpha,\varepsilon}(x; h) \sim \sum_{j=0}^{\infty} a_{\alpha,\varepsilon,j}(x) h^j, \quad h \to 0,$$

(2.7)
in the space of such functions. The semiclassical principal symbol $p_{0,\varepsilon}$, defined on $T^*M$, takes the form

$$p_{0,\varepsilon}(x, \xi) = \sum a_{\alpha,\varepsilon,0}(x) \xi^\alpha,$$

(2.8)
if $(x, \xi)$ are the canonical coordinates on $T^*M$. We make the ellipticity assumption,

$$|p_{0,\varepsilon}(x, \xi)| \geq \frac{1}{C} \langle \xi \rangle^m, \quad (x, \xi) \in T^*M, \quad |\xi| \geq C,$$

(2.9)
for some large $C > 0$. Here we assume that $M$ has been equipped with some real analytic Riemannian metric so that $|\xi|$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ are well-defined.

Sometimes, we write $p_\varepsilon$ for $p_{0,\varepsilon}$ and simply $p$ for $p_{0,0}$. We make the assumption that $P_{\varepsilon=0}$ is formally selfadjoint.

In the case when $M$ is compact, we let the underlying Hilbert space be $L^2(M, \mu(dx))$ where $\mu(dx)$ is the Riemannian volume element.

The assumptions above imply that the spectrum of $P_{\varepsilon}$ in a fixed neighborhood of $0 \in \mathbb{C}$ is discrete, when $0 < h \leq h_0$, $0 \leq \varepsilon \leq \varepsilon_0$, with $h_0 > 0$, $\varepsilon_0 > 0$ sufficiently small. Moreover, if $z \in \text{neigh}(0, \mathbb{C})$ is an eigenvalue of $P_{\varepsilon}$ then $\text{Im} \ z = O(\varepsilon)$.

We furthermore assume that the real energy surface $p^{-1}(0) \cap T^*M$ is connected and that

$$dp \neq 0 \quad \text{along} \quad p^{-1}(0) \cap T^*M.$$

In what follows we shall write

$$p_{\varepsilon} = p + i\varepsilon q + O(\varepsilon^2),$$

(2.10)
in a neighborhood of $p^{-1}(0) \cap T^*M$, and for simplicity we shall assume throughout the paper that $q$ is real valued on the real domain. (In the general case, we should simply replace $q$ below by $\text{Re} \ q$.) We let $H_p = p'_\xi \cdot \partial_x - p'_x \cdot \partial_\xi$ be the Hamilton vector field of $p$. 

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2.2 Assumptions related to the complete integrability

As in [12], [10], let us assume that there exists an analytic real valued function $f$ near $p^{-1}(0) \cap T^*M$ such that $H_p f = 0$, with the differentials $df$ and $dp$ being linearly independent on an open and dense set $\subset \text{neigh}(p^{-1}(0) \cap T^*M, T^*M)$. For each $E \in \text{neigh}(0, \mathbb{R})$, the level sets $\Lambda_{a,E} = f^{-1}(a) \cap p^{-1}(E) \cap T^*M$ are invariant under the $H_p$-flow and form a singular foliation of the 3-dimensional hypersurface $p^{-1}(E) \cap T^*M$. At each regular point (i.e. non-critical point for the restriction of $f$ to $p^{-1}(E)$), the leaves of this foliation are 2-dimensional analytic Lagrangian submanifolds, and each regular leaf is a finite union of tori. In what follows we shall use the word “leaf” and notation $\Lambda$ for a connected component of some $\Lambda_{a,E}$. Let $J$ be the set of all leaves in $p^{-1}(0) \cap T^*M$. Then we have a disjoint union decomposition

$$p^{-1}(0) \cap T^*M = \bigsqcup_{\Lambda \in J} \Lambda, \quad (2.11)$$

where $\Lambda$ are compact connected $H_p$-invariant sets. The set $J$ has a natural structure of a graph whose edges correspond to families of regular leaves and the set $S$ of vertices is composed of singular leaves. The union of edges $J \setminus S$ possesses a natural real analytic structure and the corresponding tori depend analytically on $\Lambda \in J \setminus S$ with respect to that structure. See section 7 in [10] for an explicit description of the Lagrangian foliation in the case when $M$ is an analytic surface of revolution in $\mathbb{R}^3$.

In what follows, we shall assume that the graph $J$ is finite. We shall identify each edge of $J$ analytically with a real bounded interval and this determines a distance on $J$ in the natural way. Assume that the following continuity property holds,

For every $\Lambda_0 \in J$ and every $\varepsilon > 0$, there exists $\delta > 0$, such that if

$$\Lambda \in J, \quad \text{dist}_J(\Lambda, \Lambda_0) < \delta, \quad \text{then} \quad \Lambda \subset \{ \rho \in p^{-1}(0) \cap T^*M; \text{dist}(\rho, \Lambda_0) < \varepsilon \}. \quad (2.12)$$

Remark. Let us assume that $f$ is a Morse-Bott function when restricted to $p^{-1}(0) \cap T^*M$, in the sense that the set of critical points of the restriction of $f$ to $p^{-1}(0) \cap T^*M$ is a disjoint union of connected submanifolds, with the transversal Hessian of $f$ being nondegenerate along each of the submanifolds. In this case, the structure of the singular leaves is known [30]. The set $J$ is then a finite connected graph and the property (2.12) holds.

Each torus $\Lambda \in J \setminus S$ carries real analytic coordinates $x_1, x_2$, identifying $\Lambda$ with $T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, so that along $\Lambda$, we have

$$H_p = a_1 \partial_{x_1} + a_2 \partial_{x_2}, \quad (2.13)$$
where \(a_1, a_2 \in \mathbb{R}\). The rotation number is defined as the ratio
\[
\omega(\Lambda) = [a_1 : a_2] \in \mathbb{RP}^1,
\]
and it depends analytically on \(\Lambda \in J\setminus S\). Recall also that the leading perturbation \(q\) has been introduced in (2.10). For each torus \(\Lambda \in J\setminus S\), we define the torus average \(\langle q \rangle(\Lambda)\) obtained by integrating \(q|_{\Lambda}\) with respect to the natural smooth measure on \(\Lambda\).

We introduce the time averages,
\[
\langle q \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} q \circ \exp (tH_{\rho}) \, dt, \quad T > 0, \tag{2.14}
\]
and consider the compact intervals \(Q_\infty(\Lambda) \subset \mathbb{R}, \Lambda \in J\), defined as in [12],
\[
Q_\infty(\Lambda) = \left[ \lim_{T \to \infty} \inf_{\Lambda} \langle q \rangle_T, \lim_{T \to \infty} \sup_{\Lambda} \langle q \rangle_T \right]. \tag{2.15}
\]
Notice that when \(\Lambda \in J\setminus S\) and \(\omega(\Lambda) \notin \mathbb{Q}\) then \(Q_\infty(\Lambda) = \{\langle q \rangle(\Lambda)\}\). In the rational case, we write \(\omega(\Lambda) = \frac{m}{n}\), where \(m \in \mathbb{Z}\) and \(n \in \mathbb{N}\) are relatively prime, and where we may assume that \(m = O(n)\). When \(k(\omega(\Lambda)) := |m| + |n|\) is the height of \(\omega(\Lambda)\), we recall from Proposition 7.1 in [12] that
\[
Q_\infty(\Lambda) \subset \langle q \rangle(\Lambda) + O\left(\frac{1}{k(\omega(\Lambda))}\right) [-1, 1]. \tag{2.16}
\]

Remark. As \(J\setminus S \ni \Lambda \to \Lambda_0 \in S\), the set of all accumulation points of \(\langle q \rangle(\Lambda)\) is contained in the interval \(Q_\infty(\Lambda_0)\). See the related remark in [10], Section 2.

From Theorem 7.6 in [12] we recall that
\[
\frac{1}{\varepsilon} \Im \left( \text{Spec}(P_{\varepsilon}) \cap \{z; |\Re z| \leq \delta\} \right) \subset \left[ \inf_{\Lambda \in J} \bigcup Q_\infty(\Lambda) - o(1), \sup_{\Lambda \in J} \bigcup Q_\infty(\Lambda) + o(1) \right], \tag{2.17}
\]
as \((\varepsilon, h, \delta) \to 0\).

### 2.3 The main result

Let \(\Lambda_0 \in J\setminus S\) be a rational invariant Lagrangian torus, so that as above, \(\omega_0 := \omega(\Lambda_0) = \frac{m}{n} \in \mathbb{Q}, m = O(n)\). Assume that the isoenergetic condition holds,
\[
(d_{\Lambda_0} \omega)(\Lambda_0) \neq 0. \tag{2.18}
\]
We recall from Section 2 of [10] the behavior of the interval $Q_\infty(\Lambda)$ when $\Lambda \neq \Lambda_0$ is a rational torus in a neighborhood of $\Lambda_0$. Writing $\omega(\Lambda) = \frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ are relatively prime, $p = \mathcal{O}(q)$, we get, using that $\omega(\Lambda) \neq \omega_0$,

$$|\omega(\Lambda) - \omega_0| \geq \frac{1}{nq} \geq \frac{1}{nk(\omega(\Lambda))},$$

and therefore, in view of (2.16),

$$Q_\infty(\Lambda) \subset \langle q \rangle(\Lambda) + \mathcal{O}(\text{dist}(\omega(\Lambda), \omega_0)\infty)[-1, 1].$$

(2.20)

This estimate is uniform in $\omega_0$ provided that we have a uniform upper bound on the height of the rotation number $\omega_0 \in \mathbb{Q}$.

Let us assume that the chosen rational torus $\Lambda_0$ is such that

$$\inf Q_\infty(\Lambda_0) < \inf_{\Lambda \in J \setminus \{\Lambda_0\}} \inf Q_\infty(\Lambda).$$

(2.21)

The result below remains valid with the obvious modifications, if we replace (2.21) by the assumption

$$\sup Q_\infty(\Lambda_0) > \sup_{\Lambda \in J \setminus \{\Lambda_0\}} \sup Q_\infty(\Lambda).$$

(2.22)

Let us choose, as we may, action-angle coordinates $(x, \xi)$ near $\Lambda_0$, so that $\Lambda_0$ is given by $\{\xi = 0\}$ in $T^2_x \times \mathbb{R}^2_\xi$, $p = p(\xi)$, and so that

$$p(0) = 0, \partial_\xi p(0) = 0, \partial_{\xi_2} p(0) > 0, \partial^2_{\xi_1} p(0) \neq 0.$$  

(2.23)

Here the last property follows from (2.18), and in order to fix the ideas, we shall assume that $\partial^2_{\xi_1} p(0) > 0$. By the implicit function theorem we have

$$\partial_{\xi_1} p(\xi) = 0 \Leftrightarrow \xi_1 = f(\xi_2),$$

(2.24)

where $f$ is an analytic function with $f(0) = 0$, and we obtain an analytic family of rational Lagrangian tori $\Lambda_E \subset p^{-1}(E)$, $E \in \text{neigh}(0, \mathbb{R})$, given by

$$\xi_2 = \xi_2(E), \quad \xi_1 = f(\xi_2(E)).$$

(2.25)

Here $\xi_2 = \xi_2(E)$ is the unique smooth solution of the equation $p(f(\xi_2), \xi_2) = E$, close to 0, such that $\xi_2(0) = 0$.  

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Writing \( q = q(x, \xi) \) in terms of the action-angle coordinates \((x, \xi)\), let

\[
\langle q \rangle_2(x_1, \xi) = \frac{1}{2\pi} \int_0^{2\pi} q(x, \xi) dx_2, \quad \xi \in \text{neigh}(0, \mathbb{R}^2),
\]  

be the average of \( q \) with respect to \( x_2 \). We assume that

\[
\mathbf{T} \ni x_1 \mapsto \langle q \rangle_2(x_1, 0) \text{ has a unique minimum which is nondegenerate.} \quad (2.27)
\]

In order to give an invariant description of the assumption (2.27), let us notice that when restricted to \( \Lambda_0 \), the Hamilton flow of \( p \) is periodic of primitive period \( T_0 > 0 \) and the average \( \langle q \rangle_2(x_1, 0) \) can naturally be viewed as the flow average \( \langle q \rangle_{T_0} \), defined as in (2.14), considered as a function on the space of closed \( H_p \)-orbits in \( \Lambda_0 \),

\[
\Lambda_0/\exp(\mathbb{R}H_p) \simeq \mathbf{T}.
\]

In its invariant formulation, the assumption (2.27) therefore states that flow average \( \langle q \rangle_{T_0} \), viewed as a function on \( \Lambda_0/\exp(\mathbb{R}H_p) \), has a unique minimum which is nondegenerate.

It follows from (2.27) that the function \( \mathbf{T} \ni x_1 \mapsto \langle q \rangle_2(x_1, \xi) \) has a unique minimum \( x_1 = x_1(\xi) \) which is nondegenerate, for \( \xi \in \text{neigh}(0, \mathbb{R}^2) \). The range of \( \langle q \rangle_2(\cdot, 0) \) is equal to \( Q_\infty(\Lambda_0) \), so the minimal value, \( \langle q \rangle_2(x_1(0), 0) = \inf Q_\infty(\Lambda_0) \) is situated strictly below \( \inf_{\Lambda \in \mathcal{J} \setminus \{\Lambda_0\}} \inf Q_\infty(\Lambda) \).

In this paper, we shall work under the assumption that the subprincipal symbol of the unperturbed operator \( P_{\varepsilon=0} \) vanishes,

\[
p_{1,0}(x, \xi) = 0. \quad (2.28)
\]

The following is the main result of this work.

**Theorem 2.1** We adopt the assumptions above, in particular, (2.18), (2.21), (2.27), and (2.28). Let us put \( x_1(\xi_2) = x_1(f(\xi_2), \xi_2) \). Let \( \delta \in (1/18, 1/9) \) be fixed and assume that

\[
h^{1/(1-\delta)} \ll \varepsilon \ll h^{6/(5+12\delta)}. \quad (2.29)
\]

Set

\[
\tilde{h} = \frac{h}{\sqrt{\varepsilon}}.
\]
Then for every $C_0 > 0$, we have the following description of the eigenvalues of $P_\varepsilon$ in the region

$$z \in \mathbb{C}; \ |\text{Re} \ z| < \frac{h}{C_0 \varepsilon}, \ \frac{\text{Im} \ z}{\varepsilon} \leq \inf Q_\infty(\Lambda_0) + C_0 \frac{h}{\sqrt{\varepsilon}},$$

valid for all $h > 0$ small enough: the eigenvalues are simple and given by

$$\lambda_{j,k} = p(f(h(j - \theta_2)), f(h(j - \theta_2))) + \sqrt{\varepsilon} h(\lambda_{j,k}^0 + \lambda_{j,k}^1 h + \lambda_{j,k}^2 h^2 + \ldots),$$

with $j \in \mathbb{Z}$, $h(j - \theta_2) = O(h/\sqrt{\varepsilon})$, $N \ni k \leq O(1)$, where $\lambda_{j,k}^0 = \lambda_k^0(h(j - \theta_2), \sqrt{\varepsilon})$ is a smooth function of $\xi_2 = h(j - \theta_2) \in \text{neigh}(0, \mathbb{R})$ and $\sqrt{\varepsilon} \in \text{neigh}(0, \mathbb{R}^+)$, and

$$\lambda_k^0(\xi_2, 0) = e^{i\pi/4} \left( \frac{\partial^2_{\xi_1} p(f(\xi_2), \xi_2)}{\partial^2_{\xi_1} q(x_1(\xi_2), f(\xi_2), \xi_2)} \right)^{\frac{1}{2}} \left( k + \frac{1}{2} \right).$$

Here we have written $\theta_2 = k_0(\alpha_2)/4 + S_2/2\pi h$, where $k_0(\alpha_2)$ and $S_2$ are the Maslov index and the classical action, respectively, of the fundamental cycle in $\Lambda_0$, given by a closed $H_p$-trajectory of minimal period.

**Remark.** Choosing $\delta \in (1/18, 1/9)$ in Theorem 2.1 to be close to 1/9, we see from (2.29) that the description of the eigenvalues in Theorem 2.1 in the region (2.30) is valid in the range

$$h^{\frac{2}{\eta} - \eta} \ll \varepsilon \ll h^{\frac{18}{19} + \eta},$$

when $\eta > 0$ is small. In particular, we are able to reach some cases when $\varepsilon \gg h$, and here the analyticity assumptions seem essential.

**Remark.** The result of Theorem 2.1 admits a natural extension to the case when $\text{Re} \ z \in \text{neigh}(0, \mathbb{R})$ varies in a sufficiently small but fixed neighborhood of $0 \in \mathbb{R}$. Indeed, let us recall the family of rational Lagrangian tori $\Lambda_E \subset p^{-1}(E), \ E \in \text{neigh}(0, \mathbb{R})$, introduced in (2.25). A natural analog of the assumption (2.21) is then valid for $\inf Q_\infty(\Lambda_E)$, relative to the Lagrangian foliation in $p^{-1}(E)$, provided that $|E|$ is small enough. It follows therefore from Theorem 2.1 that the description (2.31) of the spectrum of $P_\varepsilon$ remains valid when

$$|\text{Re} \ z - E| \leq \frac{h}{C_0 \sqrt{\varepsilon}}, \ \frac{\text{Im} \ z}{\varepsilon} \leq \inf Q_\infty(\Lambda_E) + C_0 \frac{h}{\sqrt{\varepsilon}}.$$
uniformly in $E \in \text{neigh}(0, \mathbb{R})$. We conclude therefore that the result of Theorem 2.1 extends to the spectral region

$$\left\{ z \in \mathbb{C}; \ |\text{Re} \, z| < \frac{1}{C}, \ \frac{\text{Im} \, z}{\varepsilon} \leq \inf \, Q_\infty(\Lambda_{\text{Re} \, z}) + \mathcal{O} \left( \frac{h}{\sqrt{\varepsilon}} \right) \right\},$$

for $C > 1$ large enough.

The plan of the paper is as follows. Section 3 is devoted to a general outline of the proof. In Section 4 we construct a global compactly supported weight function $G$, such that the leading symbol of $P_\varepsilon$, acting on the weighted space associated to $G$, becomes $\approx p + i\varepsilon \left( q - H \right) G$, with the imaginary part avoiding the value $\varepsilon \inf \, Q_\infty(\Lambda_0)$ on $p^{-1}(0)$, away from the rational torus $\Lambda_0$. This effectively microlocalizes the spectral problem for $P_\varepsilon$ to a small neighborhood of $\Lambda_0$. The quantum normal form construction for $P_\varepsilon$ in the rational region is carried out in Section 5, using the techniques of secular perturbation theory, thereby reducing the analysis to the study of a one-parameter family of non-selfadjoint operators in dimension one, having double characteristics with elliptic quadratic approximations. In Section 6 we recall the computation of low-lying eigenvalues for such operators, following [4] and [6], and extend the results there to the parameter-dependent case. The final step in the proof of Theorem 2.1 is taken in Section 7, where we carry out a pseudospectral analysis for the family of the one-dimensional operators in question, controlling the resolvent norms and obtaining the spectral localization. It then becomes possible to complete the proof by solving a suitable globally well-posed Grushin problem for $P_\varepsilon$ in a weighted space, using the ideas and techniques of [7], [12]. In Section 8 we present the results of numerical computations illustrating Theorem 2.1. The Appendix establishes some subelliptic resolvent bounds for non-selfadjoint operators of Schrödinger type, playing a principal role in the pseudospectral analysis of Section 7 in the main text. These bounds seem to be of some independent interest, and their proofs are very much based on the techniques developed in [4], [6].

3 Outline of the proof

In this section we shall give a general outline of the proof of Theorem 2.1. Some of the techniques come from the previous works [12], [10], and the presentation below will naturally focus on the new difficulties of pseudospectral nature, encountered in the analysis in the rational region. We shall then also describe heuristically some of the essential ideas employed in overcoming those difficulties, referring to Section 7 and to the Appendix for a detailed rigorous discussion.
The principal symbol of the operator $P_\varepsilon$ in (2.1), (2.6) is of the form

$$p_\varepsilon = p + i\varepsilon q + \mathcal{O}(\varepsilon^2),$$

(3.1)
in a neighborhood of $p^{-1}(0) \cap T^*M$, and thanks to the ellipticity assumptions (2.5), (2.9), we observe that it suffices to make a microlocal study in the region where $p$ is small. Recalling the assumption (2.21) and replacing $q$ by $q - \inf Q_\infty(\Lambda_0)$, in the following discussion we shall assume, for notational simplicity only, that $\inf Q_\infty(\Lambda_0) = 0$. The first step in the argument is a construction of a global weight function $G \in C_\infty(T^*M)$ such that away from a small neighborhood of the rational Lagrangian torus $\Lambda_0$ in $p^{-1}(0) \cap T^*M$, we have

$$q - H_p G \geq \frac{1}{\mathcal{O}(1)}. \quad (3.2)$$

Away from $\Lambda_0$, the weight $G$ satisfies

$$H_p G = q - \langle q \rangle_T,$$

where $\langle q \rangle_T$ has been introduced in (2.14), and when constructing $G$ in a neighborhood of $\Lambda_0$, we introduce action-angle coordinates $(x, \xi) \in T^*T^2$, so that $\Lambda_0 = \{\xi = 0\} \subset T^*T^2$, and

$$p_\varepsilon(x, \xi) = p(\xi) + i\varepsilon q(x, \xi) + \mathcal{O}(\varepsilon^2), \quad (3.3)$$

where the frequencies $\partial_{\xi_1} p(0)$ and $\partial_{\xi_2} p(0)$ are commensurable. After a linear change of variables, we get $\partial_{\xi_1} p(0) = 0$, and the isoenergetic condition (2.18) shows that $\partial_{\xi_2}^2 p(0) \neq 0$. In the following discussion, in order to fix the ideas, we shall consider the model case $p(\xi) = \xi_2 + \xi_1^2$, which suffices to illustrate the difficulties. The weight function $G$ near $\xi = 0$ satisfies the cohomological equation

$$H_p G = q - \tilde{q}, \quad (3.4)$$

modulo $\mathcal{O}(\xi^\infty)$, where $\tilde{q} = \tilde{q}(x_1, \xi)$ is independent of $x_2$ and is such that

$$\tilde{q}(x_1, 0) = \frac{1}{2\pi} \int_0^{2\pi} q(x, 0) \, dx_2 \quad (3.5)$$
is the average of $q(x, 0)$ in the $x_2$-direction. From (2.27) we then know that $\tilde{q}(x_1, 0) \geq 0$ and that $x_1 \mapsto \tilde{q}(x_1, 0)$ has a unique minimum which is nondegenerate. The partial Birkhoff normal form construction, utilized in solving (3.4) may be continued, first at the principal symbol level, and then on the level of operators, leading to the
conclusion that microlocally in the rational region, when acting on an exponentially weighted space, the operator $P_\varepsilon$ is unitarily equivalent to an operator of the form

$$P(x_1, hD_{x_1}, \varepsilon; h) + R(x, hD_{x_1}, \varepsilon; h) : L^2(T^2) \rightarrow L^2(T^2).$$

(3.6)

We refer to Proposition 7.1 in Section 7 for the precise statement. Here the full symbol of $P(x_1, hD_{x_1}, \varepsilon; h)$ is independent of $x_2$ and is given by

$$P(x_1, \xi, \varepsilon; h) = p(\xi) + i\varepsilon\tilde{q}(x_1, \xi) + O(\varepsilon^2 + h^2).$$

(3.7)

The contribution $R(x, \xi, \varepsilon; h) = O((\varepsilon, \xi, h)^\infty)$ in (3.6) is a remainder, which becomes $O(h^\infty)$ when restricting the attention to the region $\xi = O(\varepsilon^\delta)$, for a suitable small fixed $\delta > 0$ — as we shall see, understanding this region suffices for the description of the eigenvalues in Theorem 2.1. In particular, since $\xi$ becomes small, in the following heuristic discussion, we shall make a simplification and assume that $\tilde{q}$ in (3.7) is independent of $\xi$ altogether, depending on $x_1$ only. Let us also suppress the error term $O(\varepsilon^2 + h^2)$ in (3.7), for simplicity. When considering it in Section 7, it will be treated entirely as a perturbation.

Taking a Fourier series decomposition in $x_2$, we may view the operator $P$ in (3.6) as a one-parameter family of operators $P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) = P(\xi_2)$, acting on $L^2(T)$, such that

$$P(\xi_2) = \xi_2 + L_\varepsilon, \quad \xi_2 = h j, \quad j \in \mathbb{Z},$$

(3.8)

where

$$L_\varepsilon = (hD_{x_1})^2 + i\varepsilon\tilde{q}(x_1), \quad \tilde{q} \geq 0,$$

(3.9)

is a one-dimensional non-selfadjoint Schrödinger operator with $\varepsilon\tilde{q}$ as a potential. We are interested in the spectrum of the family (3.8) in the region where $\Re z$ is small and $|\Im z| \leq O(h\sqrt{\varepsilon})$, and the first observation is that the eigenvalues of the operator

$$\frac{1}{\varepsilon}L_\varepsilon = \left(\tilde{h}D_{x_1}\right)^2 + i\tilde{q}(x_1), \quad \tilde{h} = \frac{h}{\sqrt{\varepsilon}},$$

can be determined asymptotically in any disc $|w| < C\tilde{h}$, by means of the harmonic approximation, provided that $\tilde{h} \ll 1$. See [1], [2], and the discussion in Section 6 below. The eigenvalues of $\varepsilon^{-1}L_\varepsilon$ in this region are of the form

$$\mu_k(\tilde{h}) \sim \sum_{j=0}^{\infty} \mu_{k,j}\tilde{h}^{j+1}, \quad k \in \mathbb{N},$$

(3.10)
where
\[ \mu_{k,0} = (2\partial_{x_1}\tilde{q}(x_{11}^{\min}))^{1/2} e^{i\pi/4} \left( k + \frac{1}{2} \right), \quad (3.11) \]
are the eigenvalues of the globally elliptic quadratic operator
\[ D_y^2 + \frac{1}{2} \left( \partial_{x_1}^2 \tilde{q}(x_{11}^{\min}) \right) y^2 \]
acting on \( L^2(\mathbb{R}) \). Here \( x_{11}^{\min} \in T \) is the unique point such that \( \tilde{q}(x_{11}^{\min}) = 0 \). The corresponding eigenvalues of \( P(\xi_2) \) in \( (3.8) \) are given by \( \xi_2 + \varepsilon \mu_k(h) \), and from \( [4], [6] \) we also know that
\[ \| (P(\xi_2) - z)^{-1} \|_{L^2, L^2} \leq O \left( \frac{1}{\sqrt{\varepsilon h}} \right), \quad (3.12) \]
provided that \( |z - \xi_2| \leq C h \sqrt{\varepsilon} \) and that \( (z - \xi_2)/h \sqrt{\varepsilon} \) avoids the quadratic eigenvalues \( \mu_{k,0} \) in \( (3.11) \).

Now \( (3.8) \) is only an approximate direct sum decomposition, and in order to be able to absorb the error terms there, when constructing the resolvent of \( P_\varepsilon \) globally, it is of crucial importance to control the resolvent norms of \( L_\varepsilon \) also near the interval \( [C h \sqrt{\varepsilon}, 1/\mathcal{O}(1)] \). To get such a control, since the spectral parameter remains close to the boundary of the range of the symbol of \( L_\varepsilon \), we apply the method of "bounded exponential weights", which in effect consists of replacing \( L_\varepsilon \) by a new operator for which the infimum of the imaginary part is increased in the non-elliptic region for \( \text{Re}(L_\varepsilon - \omega) \). This method has been carried out in closely related situations in \([2], [4], [6]\), and we apply some of those works in the actual proof in the Appendix. Here we shall merely recall the essential ideas. See also \([15], [22]\).

Let \( G(x_1, \xi_1) \in C^\infty \) be real-valued and odd in \( \xi_1 \). Let us consider formally the conjugated operator
\[ \widetilde{L}_\varepsilon = e^{-\varepsilon G(x_1, h D_{x_1})/h} \circ L_\varepsilon \circ e^{\varepsilon G(x_1, h D_{x_1})/h}, \]
acting on \( L^2 \), or equivalently, the operator \( L_\varepsilon \) acting on the weighted Hilbert space \( e^{\varepsilon G(x_1, h D_{x_1})/h} L^2 \). We want this space to be equal to \( L^2 \), with its norm
\[ \| e^{-\varepsilon G(x_1, h D_{x_1})/h} u \|_{L^2} \]
uniformly equivalent to the standard \( L^2 \)-norm. This is the case if the weight function \( G \) satisfies suitable symbol estimates and has the fundamental property
\[ \frac{\varepsilon G(x_1, \xi_1)}{h} = \mathcal{O}(1), \quad (3.13) \]
uniformly with respect to the various parameters involved.

We view \( e^{i\varepsilon G(x_1,hD_{x_1})/h} \) as a Fourier integral operator with the associated canonical transformation \( \exp (i\varepsilon H_G) \), approximately equal to \((x_1,\xi_1) \mapsto (x_1,\xi_1) + i\varepsilon H_G(x_1,\xi_1)\), since \( \varepsilon \) will be small. Here \( H_G = G_{\xi_1} \cdot \partial_{x_1} - G'_{x_1} \cdot \partial_{\xi_1} \) is the Hamilton vector field of \( G \). By Egorov’s theorem we expect \( \tilde{L}_\varepsilon \) to be an \( h \)-pseudodifferential operator with the symbol

\[
\tilde{L}_\varepsilon(x_1,\xi_1) \approx L_\varepsilon \left( (x_1,\xi_1) + i\varepsilon H_G(L_\varepsilon)(x_1,\xi_1) \right) \\
\approx L_\varepsilon(x_1,\xi_1) - i\varepsilon H_\varepsilon(G).
\]

Here \( L_\varepsilon(x_1,\xi_1) = \xi_1^2 + i\varepsilon \tilde{q}(x_1) \) is the symbol of \( L_\varepsilon \) in (3.9). With \( \ell(x_1,\xi_1) = \xi_1^2 \), we get

\[
\tilde{L}_\varepsilon(x_1,\xi_1) \approx \xi_1^2 + i\varepsilon \tilde{q}(x_1) =: \xi_1^2 + i\varepsilon \tilde{q}(x_1,\xi_1).
\]

When considering \( \tilde{L}_\varepsilon - \omega \) for \( \Re \omega \geq h\sqrt{\varepsilon} \), the most critical region is the one where \( \xi_1^2 \approx \Re \omega \) and it is here that we want to increase \( \inf_{x_1} \tilde{q} \) as much as possible. Naturally, that will not be enough for the complete analysis, but in the following heuristic discussion, we shall restrict the attention to the region where \( \xi_1^2 = \Re \omega \).

Here, we get

\[
\tilde{q}(x_1,\xi_1) = \tilde{q}(x_1) - 2\xi_1 \partial_{x_1} G(x_1,\xi_1) \\
= \tilde{q}(x_1) - 2\sqrt{\Re \omega} \partial_{x_1} G(x_1,(\Re \omega)^{1/2}),
\]

where we recall that \( G \) is odd in \( \xi_1 \), so that \( \tilde{q} \) is even in the same variable. Then

\[
\partial_{x_1} G(x_1) = \frac{\tilde{q}(x_1) - \tilde{q}(x_1)}{2\sqrt{\Re \omega}},
\]

omitting \( \xi_1 = \sqrt{\Re \omega} \) in the argument of \( G \). We want

\[
\inf_{x_1} \tilde{q} - \inf_{x_1} \tilde{q} \approx \gamma^2, \tag{3.14}
\]

for a suitable small parameter \( \gamma \), that we wish to have as large as possible, and to achieve this, we clearly have to modify \( \tilde{q} \) in a \( \gamma \)-neighborhood of \( x_1^{\text{min}} \). Since we also wish \( |G| \) to be as small as possible, we require

\[
\text{supp } G \subset [x_1^{\text{min}} - \gamma, x_1^{\text{min}} + \gamma],
\]

and it is not hard to see that we can find such a \( G \) with

\[
\partial_{x_1} G = O\left( \frac{\gamma^2}{\sqrt{\Re \omega}} \right), \quad G = O\left( \frac{\gamma^3}{\sqrt{\Re \omega}} \right)
\]

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The condition (3.13) is fulfilled, provided that
\[ \frac{\varepsilon \gamma^3}{h \sqrt{\text{Re} \omega}} = \mathcal{O}(1) \iff \gamma = \mathcal{O}(1) \frac{h^{\frac{1}{3}}(\text{Re} \omega)^{\frac{1}{3}}}{\varepsilon^{\frac{1}{3}}} . \] (3.15)
Let \( C \gg 1 \) and let us choose
\[ \gamma = \frac{1}{C} \min \left( 1, \frac{h^{\frac{2}{3}}(\text{Re} \omega)^{\frac{1}{3}}}{\varepsilon^{\frac{1}{3}}} \right) . \] (3.16)
It follows from the heuristic discussion above that in the region where \( h \sqrt{\varepsilon} \leq \text{Re} \omega \leq \frac{1}{\mathcal{O}(1)} \) we obtain the spectral gain
\[ \frac{\varepsilon \gamma^2}{\mathcal{O}(1)} \leq \min \left( \varepsilon, h^{\frac{2}{3}}(\text{Re} \omega)^{\frac{1}{3}} \varepsilon^{\frac{1}{3}} \right) \geq h \sqrt{\varepsilon} , \] (3.17)
in the sense that the resolvent \((L_\varepsilon - \omega)^{-1}\) is well defined in the region
\[ h \sqrt{\varepsilon} \leq \text{Re} \omega \leq \frac{1}{\mathcal{O}(1)}, \quad \text{Im} \omega \leq \frac{1}{C} \min \left( h^{\frac{2}{3}}(\text{Re} \omega)^{\frac{1}{3}} \varepsilon^{\frac{1}{3}}, \varepsilon \right) \]
and that in such a region, we have
\[ \| (L_\varepsilon - \omega)^{-1} \|_{L^2(L^2)} \leq \frac{\mathcal{O}(1)}{\min \left( h^{\frac{2}{3}}(\text{Re} \omega)^{\frac{1}{3}} \varepsilon, \varepsilon \right)} . \] (3.18)
The resolvent estimates such as (3.18) are established in the Appendix, using the machinery of bounded exponential weights and relying on the techniques of [4], [6] — see Propositions A.2 and A.4 there, in particular. With the bounds (3.18) available, we get the corresponding pseudospectral control over the family \( P(x_1, hDx_1, \xi_2, \varepsilon; h) \) in (3.13), in the region where \( |\text{Re} z - \xi_2| \geq ch \sqrt{\varepsilon}, \text{Im} z \leq \mathcal{O}(h \sqrt{\varepsilon}) \), and this allows us, eventually, to construct the resolvent of \( P_\varepsilon \) globally in this region. We therefore obtain some crucial spectral localization, making it possible to carry out the spectral analysis of \( P_\varepsilon \) working with one quantum number \( \xi_2 = hj \) at a time, roughly speaking. A globally well-posed Grushin problem for \( P_\varepsilon \) is finally built from the corresponding one-dimensional Grushin problems for the operator \( L_\varepsilon \) in (3.9), and solving it along the same lines as in [7], [12], [4], we complete the proof of Theorem 2.1.

Remark. Our heuristic arguments seem to indicate that the optimal range for the perturbation parameter \( \varepsilon \) could be
\[ h^2 \ll \varepsilon \ll h^{2/3} , \] (3.19)
as we need $\tilde{h} = h/\sqrt{\varepsilon} \ll 1$ and $\varepsilon \ll \tilde{h}$. Due to many technicalities, we get a smaller range of values around $\varepsilon \approx h$, and leave extension to the range (3.19) as an open problem for future works.

4 Secular reduction and the global weight

The purpose of this section is to construct a globally defined compactly supported weight function, which will allow us to microlocalize the spectral problem for $P_\varepsilon$ to a small neighborhood of the rational torus $\Lambda_0$. In doing so, we shall proceed similarly to [12], with the essential difference that when working near the torus, the basic cohomological equation will have quite different properties, compared to the Diophantine analysis of [12], and will be treated using the secular perturbation theory, see [16], [10].

Let us keep all the assumptions of Section 2 and consider the operator $P_\varepsilon$ with the leading symbol $p_\varepsilon$ in (2.10), in a neighborhood of $p^{-1}(0) \cap T^*M$. Let

$$\kappa_0 : \text{neigh}(\Lambda_0, T^*M) \to \text{neigh}(\xi = 0, T^*T^2),$$

be a real analytic canonical transformation, given by the action-angle variables, such that the properties (2.23) hold. By Taylor expansion and (2.23), we have

$$p(\xi) = p(f(\xi_2), \xi_2) + g(\xi) (\xi_1 - f(\xi_2))^2, \quad g(0) > 0,$$

where $f$ is the analytic function introduced in (2.24).

Implementing $\kappa_0$ in (4.1) by means of a microlocally unitary multi-valued $h$–Fourier integral operator with a real phase, as explained in Theorem 2.4 in [17] and conjugating $P_\varepsilon$ by this operator, we obtain a new $h$–pseudodifferential operator, still denoted by $P_\varepsilon$, defined microlocally near $\xi = 0$ in $T^*T^2$. The full symbol of $P_\varepsilon$ is holomorphic in a fixed complex neighborhood of $\xi = 0$, and the leading symbol is given by

$$p_\varepsilon(x, \xi) = p(\xi) + i\varepsilon q(x, \xi) + O(\varepsilon^2),$$

with $p(\xi)$ of the form (4.2). The function $q$ in (4.3) is real on the real domain. On the operator level, $P_\varepsilon$ acts on the space of microlocally defined Floquet periodic functions on $T^2$, $L^2_0(T^2) \subset L^2_\text{loc}(\mathbb{R}^2)$, elements $u$ of which satisfy

$$u(x - \nu) = e^{i\theta \cdot \nu} u(x), \quad \theta = \frac{S}{2\pi h} + \frac{k_0}{4}, \quad \nu \in 2\pi \mathbb{Z}^2.$$
Here \( S = (S_1, S_2) \) is given by the classical actions,

\[
S_j = \int_{\alpha_j} \eta \, dy, \quad j = 1, 2,
\]

with \( \alpha_j \) forming a system of fundamental cycles in \( \Lambda_0 \), such that

\[
\kappa_0(\alpha_j) = \beta_j, \quad j = 1, 2, \quad \beta_j = \{x \in T^2; x_{3-j} = 0\}.
\]

The tuple \( k_0 = (k_0(\alpha_1), k_0(\alpha_2)) \in \mathbb{Z}^2 \) stands for the Maslov indices of the cycles \( \alpha_j, \) \( j = 1, 2 \).

**Remark.** Using (4.2), we see, using the implicit function theorem, that the energy surface \( p(\xi) = E \), for \( E \in \text{neigh}(0, \mathbb{R}) \), is given by

\[
\xi_2 + \ell(\xi_1, E) = 0,
\]

where \( \ell \) is analytic with \( \ell(\xi_1, 0) \sim \xi_1^2, \ell'_E(0, 0) < 0 \).

Working near the zero section \( \xi = 0 \) in \( T^*T^2 \) and following the method of normal forms [12], [10], we shall now discuss the cohomological equation

\[
H_p G = q - \tilde{q},
\]

where we want the remainder \( \tilde{q} \) to be simpler than \( q \). Here we have

\[
H_p = p_x' \cdot \partial_x,
\]

and thus, (4.6) can be written more explicitly as follows,

\[
\partial_{\xi_2} p(\xi) \partial_{x_2} G + \partial_{\xi_1} p(\xi) \partial_{x_1} G = q - \tilde{q}.
\]

To simplify, we divide this equation by \( \partial_{\xi_2} p \). Writing \( u = G, \ v = (\partial_{\xi_2} p)^{-1}q, \tilde{v} = (\partial_{\xi_2} p)^{-1}\tilde{q} \), we get

\[
(\partial_{x_2} + a(\xi) \partial_{x_1}) u = v - \tilde{v},
\]

where \( a(\xi) = \partial_{\xi_1} p(\xi)/\partial_{\xi_2} p(\xi) \). To simplify further, we replace the variables \( \xi \) by

\[
\eta = (\eta_1, \eta_2) = (\xi_1 - f(\xi_2), \xi_2),
\]

and write, abusing the notation slightly, \( u = u(x, \eta), \ v = v(x, \eta), \tilde{v} = \tilde{v}(x, \eta) \). It follows from (4.2) that the Taylor expansion of \( a \) has the form,

\[
a(\eta) = a_1(\eta_2)\eta_1 + a_2(\eta_2)\eta_1^2 + \ldots, \quad a_1(0) \neq 0,
\]

(4.9)
and let us Taylor expand $u$, $v$ and $\tilde{v}$ similarly,

$$u(x, \eta) = \sum_{k=0}^{\infty} u_k(x, \eta_2) \eta_1^k,$$

$$v(x, \eta) = \sum_{k=0}^{\infty} v_k(x, \eta_2) \eta_1^k,$$  \hspace{1cm} (4.10)

$$\tilde{v}(x, \eta) = \sum_{k=0}^{\infty} \tilde{v}_k(x, \eta_2) \eta_1^k.$$

Inserting these equations into (4.7) and identifying the powers of $\eta_1$, we get

$$\partial_{x_2} u_0 = v_0 - \tilde{v}_0,$$  \hspace{1cm} (4.11)

$$\partial_{x_2} u_1 + a_1 \partial_{x_1} u_0 = v_1 - \tilde{v}_1,$$  \hspace{1cm} (4.12)

$$\partial_{x_2} u_2 + a_1 \partial_{x_1} u_1 + a_2 \partial_{x_1} u_0 = v_2 - \tilde{v}_2,$$  \hspace{1cm} (4.13)

and so on. The general equation is of the form

$$\partial_{x_2} u_k + a_1 \partial_{x_1} u_{k-1} + a_2 \partial_{x_1} u_{k-2} + \ldots + a_k \partial_{x_1} u_0 = v_k - \tilde{v}_k.$$  \hspace{1cm} (4.14)

The parameter $\eta_2$ plays no essential role here and we sometimes suppress it from the notation. For a function $u$ on the torus $T^2$, we introduce its averages in $x_k$, $k = 1, 2$, and its total average by

$$\langle u \rangle_k(x_{3-k}) = \frac{1}{2\pi} \int_0^{2\pi} u(x_1, x_2) dx_k,$$

$$\langle \langle u \rangle \rangle = \langle \langle u \rangle \rangle_2 = \frac{1}{(2\pi)^2} \int_{T^2} u(x_1, x_2) dx_1 dx_2.$$

**Proposition 4.1** Let $v_0, v_1, \ldots \in C^\infty(T^2)$ be smooth functions on $T^2$. A necessary and sufficient condition on the smooth functions $\tilde{v}_0, \tilde{v}_1, \ldots \in C^\infty(T^2)$ for the existence of $u_0, u_1, \ldots \in C^\infty(T^2)$ solving (4.11) and (4.14) for $k \geq 1$, is that

$$\langle \langle \tilde{v}_0 \rangle \rangle_2 = \langle \langle v_0 \rangle \rangle_2,$$

$$\langle \langle \tilde{v}_k \rangle \rangle = \langle \langle v_k \rangle \rangle, \ k \geq 1.$$  \hspace{1cm} (4.15)

**Proof:** The necessity of (4.15) follows from taking the $x_2$-mean of (4.11) and the total mean of (4.14).
Assume that the first equation in (4.15) holds so that \( \langle v_0 - \tilde{v}_0 \rangle_2 = 0 \). Then (4.11) has a solution \( u_0 = u_0^0 \in C^\infty(T^2) \), given by

\[
    u_0^0(x) = \int_0^{x_2} (v_0 - \tilde{v}_0)(x_1, t) dt.
\]

The general solution of (4.11) is of the form \( u^0(x) + f_0(x_1) \), where \( f_0(x_1) \) is any smooth periodic function.

We next consider (4.12) (i.e. (4.14) with \( k = 1 \)), which we write as

\[
    \partial_{x_2} u_1 = v_1 - \tilde{v}_1 - a_1 \partial_{x_1} u_0^0 - a_1 \partial_{x_1} f_0(x_1).
\]

Here the total average of \( v_1 - \tilde{v}_1 - a_1 \partial_{x_1} u_0^0 \) vanishes,

\[
    \langle \langle v_1 - \tilde{v}_1 - a_1 \partial_{x_1} u_0^0 \rangle_2 \rangle_1 = 0,
\]

and hence we can find a periodic smooth function \( f_0(x_1) \), unique up to a constant, such that

\[
    \langle v_1 - \tilde{v}_1 - a_1 \partial_{x_1} u_0^0 \rangle_2 - a_1 \partial_{x_1} f_0(x_1) = 0.
\]

Equivalently,

\[
    \langle v_1 - \tilde{v}_1 - a_1 \partial_{x_1} u_0^0 \rangle_2 - a_1 \partial_{x_1} f_0(x_1) = 0,
\]

and we can therefore find a solution \( u_0^0 \in C^\infty(T^2) \) to (4.17), and hence to (4.12).

Assume by induction that we have found \( u_0, u_1, \ldots u_{k-1} \), solving (4.11) and (4.14), with \( k \) there replaced by \( j = 1, 2, \ldots, k - 1 \). We notice that the general solution of (4.14) with \( k \) replaced by \( k - 1 \), is of the form \( u_{k-1} = u_{k-1}^0 + f_{k-1}(x_1) \) for any smooth periodic function \( f_{k-1} \). We rewrite (4.14) as

\[
    \partial_{x_2} u_k = w_k - a_1 \partial_{x_1} f_{k-1}(x_1),
\]

where,

\[
    w_k = v_k - \tilde{v}_k - a_1 \partial_{x_1} u_{k-1}^0 - a_2 \partial_{x_1} u_{k-2} - \cdots - a_k \partial_{x_1} u_0.
\]

and we notice that \( \langle \langle w_k \rangle_2 \rangle_1 = \langle \langle w_k \rangle \rangle = 0 \). Choose \( f_{k-1} \) such that \( \langle w_k \rangle_2 = a_1 \partial_{x_1} f_{k-1}(x_1) \), or equivalently, so that \( \langle w_k - a_1 \partial_{x_1} f_{k-1} \rangle_2 = 0 \). Then there is a smooth periodic solution \( u_k = u_k^0 \) to (4.18) and hence to (4.14).

An application of Proposition 4.1 allows us to conclude that for any fixed \( N \in \mathbb{N} \), there exists an analytic function \( G_0 \), defined in a fixed neighborhood of \( \xi = 0 \), such that

\[
    H_p G_0 = q - \bar{q} + \mathcal{O}((\xi_1 - f(\xi_2))^N),
\]

(4.19)
for any analytic periodic function $\bar{q}$ which satisfies,

$$\langle \bar{q}(\cdot, \xi) \rangle_2 = \langle q(\cdot, \xi) \rangle_2, \text{ when } \xi_1 = f(\xi_2) \tag{4.20}$$

and

$$\langle \langle \bar{q}(\cdot, \xi) \rangle \rangle = \langle \langle q(\cdot, \xi) \rangle \rangle, \xi \in \text{neigh } (0, \mathbb{R}^2). \tag{4.21}$$

The following choice is convenient and will be made in what follows: let $\chi : \mathbb{R} \to \mathbb{R}$ be real analytic such that $\chi(0) = 0$. We can then take

$$\bar{q}(x_1, \xi) = (1 - \chi(\xi_1 - f(\xi_2))) \langle q \rangle_2(x_1, \xi) + \chi(\xi_1 - f(\xi_2))\langle \langle q(\cdot, \xi) \rangle \rangle, \tag{4.22}$$

which is independent of $x_2$.

Let us now restrict the attention to the energy surface $p^{-1}(0)$. According to (4.5), we have $p(\xi) = 0 \iff \xi_2 + \ell(\xi_1, 0) = 0$, $\ell(\xi_1, 0) \sim \xi_1^2$. It follows from (4.22) that when $p(\xi) = 0$, we may write

$$\inf_{x_1} \bar{q}(x_1, \xi) = (1 - \psi(\xi_1)) k(\xi_1) + \psi(\xi_1) g(\xi_1),$$

where

$$k(\xi_1) = \inf_{x_1} \langle q \rangle_2(x_1, \xi_1, -\ell(\xi_1, 0)) = \langle q \rangle_2(x_1(\xi_1, -\ell(\xi_1, 0)), \xi_1, -\ell(\xi_1, 0)),$$

$$g(\xi_1) = \langle \langle q \rangle \rangle(\xi_1, -\ell(\xi_1, 0)),$$

and $\psi$ is an analytic function such that $\psi(0) = 0$, $\psi'(0) = \chi'(0)$. We compute next

$$\partial_{\xi_1} \inf_{x_1} \bar{q}(x_1, \xi) = k'(\xi_1) + \psi'(\xi_1)(g(\xi_1) - k(\xi_1)) + \psi(\xi_1)(g'(\xi_1) - k'(\xi_1)).$$

$$\partial_{\xi_1}^2 \inf_{x_1} \bar{q}(x_1, \xi) = k''(\xi_1) + \psi''(\xi_1)(g(\xi_1) - k(\xi_1)) + 2\psi'(\xi_1)(g'(\xi_1) - k'(\xi_1))$$

$$+ \psi(\xi_1)(g''(\xi_1) - k''(\xi_1)).$$

Using that

$$k(0) = \inf_{x_1 \in T} \langle q \rangle_2(x_1, 0) = \inf Q_\infty(\Lambda_0) < \langle q \rangle(\Lambda_0) = \langle \langle q \rangle \rangle(0) = g(0),$$

we see that the derivatives $\chi'(0)$ and $\chi''(0)$ of the analytic function $\chi$ in (4.22) can be chosen so that when $p(\xi) = 0$, we have

$$\inf_{x_1 \in T} \bar{q}(x_1, \xi) \geq \inf_{x_1 \in T} \langle q \rangle_2(x_1, 0) + C\xi^2, \tag{4.23}$$

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where the constant $C > 0$ is large. In other words, for $\Lambda \in \text{neigh}(\Lambda_0, J)$, we get

$$\inf_{\Lambda} \tilde{q} \geq \inf_{\Lambda_0} Q_\infty + \frac{1}{O(1)} \text{dist}(\Lambda, \Lambda_0)^2. \quad (4.24)$$

**Remark.** In the preceding discussion, we do not have to restrict ourselves to the energy surface $p^{-1}(0)$. Indeed, introducing the variables

$$\eta = (\eta_1, \eta_2) = (\xi_1 - f(\xi_2), \xi_2),$$

as in (4.8), and repeating the computations above we get

$$\inf_{x_1 \in T} \tilde{q}(x_1, \xi) \geq \inf_{x_1 \in T} \langle q \rangle_2(x_1, f(\xi_2), \xi_2) + C (\xi_1 - f(\xi_2))^2,$$

when $p(\xi) = E$, for $E \in \text{neigh}(0, R)$. Introducing the rational Lagrangian tori $\Lambda_E \subset p^{-1}(E)$, defined in (2.25), we therefore obtain on $p^{-1}(E)$,

$$\inf_{x_1 \in T} \tilde{q}(x_1, \xi) \geq \inf_{Q_\infty(\Lambda_E)} + C_1 \text{dist}(\Lambda, \Lambda_E)^2. \quad (4.25)$$

We shall now construct a suitable global weight function. In doing so, let $G_T$ be an analytic function defined in a neighborhood of $p^{-1}(0) \cap T^*M$, such that

$$H_p G_T = q - \langle q \rangle_T, \quad (4.26)$$

where

$$\langle q \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} q \circ \exp (tH_p) dt, \quad T > 0$$

has been introduced in (2.14). An application of Lemma 2.4 of [12] together with the assumption (2.21) allows us to conclude that outside an arbitrarily small neighborhood of $\Lambda_0$ in $p^{-1}(0) \cap T^*M$, we have

$$\inf (q - H_p G_T) \geq \inf_{\Lambda_0} Q_\infty + \frac{1}{C_0}, \quad (4.27)$$

provided that $T$ is taken large enough. Here $C_0 > 0$ is independent of the neighborhood taken. In these considerations, we are allowed to vary the real energy a little, and we conclude that for any fixed neighborhood $W$ of

$$\bigcup_{|E| \leq E_0} \Lambda_E, \quad 0 < E_0 \ll 1, \quad (4.28)$$
in \( p^{-1}([-E_0, E_0]) \) there exists \( T \) large enough such that
\[
\inf_{p^{-1}([-E_0, E_0]) \setminus W} (q - H_p G_T) \geq \inf_{|E| \leq E_0} \inf Q_\infty(\Lambda_E) + \frac{1}{C_0}.
\]
(4.29)

Here \( C_0 > 0 \) is independent of the neighborhood chosen.

The global weight function will be obtained by gluing together the functions \( G_T := G_T \circ \kappa_0^{-1} \) and \( G_0 \), both viewed as analytic functions defined in a neighborhood of the zero section \( \xi = 0 \) in \( T^*T^2 \). Let \( \psi = \psi(\xi) \in C^\infty(\mathbb{R}^2; [0, 1]) \) be constant on each invariant torus \( \xi = \text{Const} \), and assume that \( \psi = 1 \) near the rational region (4.28), and with support in a small neighborhood of that set. Let us set,
\[
G = (1 - \psi) G_T + \psi G_0.
\]
(4.30)

It follows that
\[
q - H_p G = \psi (q - H_p G_0) + (1 - \psi) \langle q \rangle_T.
\]
(4.31)

In a neighborhood of the rational region (4.28), we have
\[
q - H_p G = \tilde{q} + O \left( (\xi_1 - f(\xi_2))^N \right),
\]
with \( \tilde{q} \) given in (4.22), while further away from this set, we have \( q - H_p G = \langle q \rangle_T \).

In order to understand the behavior of \( \langle q \rangle_T \) near \( \xi = 0 \) for \( T \) large, we write
\[
\langle q \rangle_T(x, \xi) = \frac{1}{T} \int_{-T/2}^{T/2} q(x + tp'(\xi), \xi) \, dt,
\]
and expanding \( q(\cdot, \xi) \) in a Fourier series, we obtain that
\[
\langle q \rangle_T(x, \xi) = \sum_{k = (k_1, k_2) \in \mathbb{Z}^2} e^{ik \cdot x} \hat{q}(k, \xi) \hat{K}(Tk \cdot p'(\xi)).
\]
(4.32)

Here \( \hat{K} \) is the Fourier transform of the characteristic function \( K \) of the interval \([-1/2, 1/2]\). Let us decompose,
\[
\langle q \rangle_T(x, \xi) = \sum_{k_2 \neq 0} \hat{K}(Tp'(\xi) \cdot k) \hat{q}(k, \xi) + \sum_{k_2 = 0} \hat{K}(Tp'(\xi) \cdot k) \hat{q}(k, \xi) e^{ix \cdot k} = I + II,
\]
(4.33)

with the natural definitions of I and II. When estimating I, we use (4.2) and notice that when \( k_2 \neq 0 \), we have
\[
|p'(\xi) \cdot k| \geq |p'_{\xi_2}k_2| - \mathcal{O}(1) |\xi_1 - f(\xi_2)| |k_1| \geq 1 - C |\xi_1 - f(\xi_2)| |k|, \quad C > 0.
\]
Here for notational simplicity we assume that the derivative of the function \( \xi_2 \mapsto p(f(\xi_2), \xi_2) \) is \( \geq 1 \) near 0. It follows that

\[
|p'(\xi) \cdot k| \geq \frac{1}{2},
\]

provided that \( 2C |\xi_1 - f(\xi_2)| |k| \leq 1 \). Let now \( 0 \leq \chi \in C_0^\infty((-1,1)) \) be such that \( \chi = 1 \) on \([-1/2, 1/2]\) and write,

\[
I = \sum_{k_2 \neq 0} \chi(2C |\xi_1 - f(\xi_2)| |k|) \widehat{\Theta}(T p'(\xi) \cdot k) \tilde{q}(k, \xi) e^{ix \cdot k} \tag{4.34}
\]

\[
+ \sum_{k_2 \neq 0} (1 - \chi(2C |\xi_1 - f(\xi_2)| |k|)) \widehat{\Theta}(T p'(\xi) \cdot k) \tilde{q}(k, \xi) e^{ix \cdot k}
\]

\[
= \sum_{k_2 \neq 0} \chi(2C |\xi_1 - f(\xi_2)| |k|) \mathcal{O} \left( \frac{1}{T |p'(\xi) \cdot k|} \right) \tilde{q}(k, \xi) e^{ix \cdot k}
\]

\[
+ \sum_{k_2 \neq 0} (1 - \chi(2C |\xi_1 - f(\xi_2)| |k|)) \widehat{\Theta}(T p'(\xi) \cdot k) \tilde{q}(k, \xi) e^{ix \cdot k}.
\]

It is now easy to see, using the smoothness of \( q \), that

\[
I = \mathcal{O} \left( \frac{1}{T} + |\xi_1 - f(\xi_2)| \right), \quad T \geq 1. \tag{4.35}
\]

When considering the contribution coming from \( I \), we notice that

\[
II = \langle q \rangle_2(x_1, \xi) + \sum_{k_2=0, k_1 \neq 0} \left( \widehat{\Theta}(Tp'_{\xi_1}k_1) - 1 \right) e^{ix_1 \cdot k_1} \tilde{q}(k, \xi). \tag{4.36}
\]

Here \( |p'_{\xi_1}| \sim |\xi_1 - f(\xi_2)| \), in view of (4.2), and we conclude that in the rational region where \( \xi_1 = f(\xi_2), \xi_2 \in \text{neigh}(0, \mathbb{R}) \), we get

\[
\langle q \rangle_T(x, \xi) = \langle q \rangle_2(x_1, \xi) + \mathcal{O} \left( \frac{1}{T} \right).
\]

Away from the rational region \( \xi_1 = f(\xi_2) \), we see directly from (4.33) that \( II \) converges to the torus average \( \langle \langle q \rangle \rangle(\xi) \), as \( T \to \infty \).

Combining the equations and estimates (4.19), (4.24), (4.26), (4.29), and (4.30), we may summarize the discussion above in the following proposition.
Proposition 4.2 Let us make the assumption \((2.21)\). Let \(G_0\) be an analytic solution near \(\xi = 0\) of the equation \((4.6)\), with \(\widetilde{q}\) being of the form \((4.22)\), modulo \(O((\xi_1 - f(\xi_2))^N)\), for some \(N\) fixed large enough. There exists a real-valued function \(G \in C_0^\infty(T^*M)\) such that \(G = G_0 \circ \kappa_0\) in a neighborhood of \(\Lambda_0\), and such that away from a small neighborhood of \(\Lambda_0\) in the region \(p^{-1}([-E_0, E_0]), 0 < E_0 \ll 1\), we have

\[ q - H_p G \geq \inf Q_\infty(\Lambda_0) + \frac{1}{C_0}, \quad C_0 > 0. \]  

(4.37)

When \(\Lambda \subset p^{-1}(0), \Lambda \in \text{neigh}(\Lambda_0, J)\), we have furthermore

\[ \inf_\Lambda (q - H_p G) \geq \inf Q_\infty(\Lambda_0) + \frac{1}{C} \text{dist}(\Lambda, \Lambda_0)^2. \]

Associated to the weight function \(G\) defined in Proposition 4.2, we shall now introduce a suitable small but globally defined deformation of the real phase space \(T^*M\) into the complex domain. When doing so, let \(\widetilde{G} \in C_0^\infty(T^*\widetilde{M})\) be an almost holomorphic extension of \(G\), and let us set

\[ \Lambda_{\varepsilon G} = \exp (\varepsilon H^\text{Im }\sigma_{\text{Re }\widetilde{G}})(T^*M) \subset T^*\widetilde{M}. \]  

(4.38)

Here \(\sigma\) is the complex symplectic \((2,0)\)-form on \(T^*\widetilde{M}\) and \(H^\text{Im }\sigma_{\text{Re }\widetilde{G}}\) is the Hamilton vector field of \(\text{Re }\widetilde{G}\) computed with respect to the real symplectic form \(\text{Im }\sigma\) on \(T^*\widetilde{M}\). It follows that the manifold \(\Lambda_{\varepsilon G}\) is I-Lagrangian and being a small deformation of \(T^*M\), it is also R-symplectic, i.e. an IR-manifold. From [19] and [24], we recall the general relation

\[ i\varepsilon H_{\widetilde{G}} = \varepsilon H^\text{Im }\sigma_{\text{Re }\widetilde{G}}, \]

valid to infinite order along the real domain \(T^*M\). Here \(i\varepsilon H_{\widetilde{G}}\) stands for the real vector field in \(T^*\widetilde{M}\), naturally associated to the complex \((1,0)\) vector field,

\[ i\varepsilon H_{\widetilde{G}} = i\varepsilon \sum_{j=1}^2 \left( \frac{\partial \widetilde{G}}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial \widetilde{G}}{\partial x_j} \frac{\partial}{\partial \xi_j} \right). \]

It follows that in the region where \(G\) is analytic, including a sufficiently small but fixed neighborhood of \(\Lambda_0\), we have

\[ \Lambda_{\varepsilon G} = \exp (i\varepsilon H_G)(T^*M), \]

where we write \(G\) also for the holomorphic extension and recall that \(\exp (i\varepsilon H_G)\) is a holomorphic canonical transformation.
Associated to the IR-manifold $\Lambda_{\varepsilon G}$ is the microlocally exponentially weighted Hilbert space $H(\Lambda_{\varepsilon G})$, defined using the FBI–Bargmann approach, by modifying the exponential weight on the FBI transform side. We refer to [20], [11] for the detailed definition of the space $H(\Lambda_{\varepsilon G})$ in the case when $M = \mathbb{R}^2$, and to [26] and the Appendix of [7] for the case when $M$ is compact. Following [19], [20], [26], let us introduce a microlocally unitary $h$–Fourier integral operator

$$U_G : L^2(M) \to H(\Lambda_{\varepsilon G}),$$

defined microlocally near $p^{-1}(0) \cap T^*M$ and associated to a suitable canonical transformation

$$\kappa_G : \text{neigh}(p^{-1}(0), T^*M) \to \text{neigh}(p^{-1}(0), \Lambda_{\varepsilon G}),$$

such that $\kappa_G = \exp(i\varepsilon H_G)$ near $\Lambda_0$. It follows that the operator

$$P_\varepsilon : H(\Lambda_{\varepsilon G}) \to H(\Lambda_{\varepsilon G})$$

(4.40)

is microlocally near $p^{-1}(0)$ unitarily equivalent to the conjugated operator

$$U_G^{-1}P_\varepsilon U_G : L^2 \to L^2,$$

with the leading symbol

$$p_\varepsilon|_{\Lambda_{\varepsilon G}} \simeq p + i\varepsilon (q - H_p G) + O(\varepsilon^2).$$

Letting

$$U_0 : L^2(M) \to L^2_{\theta}(\mathbb{T}^2)$$

be the semiclassical microlocally unitary Fourier integral operator with a real phase associated to the canonical transformation $\kappa_0$ in (4.1) and using the operator $U_0U_G^{-1}$ associated to the canonical transformation

$$\kappa_0 \circ \kappa_G^{-1} : \text{neigh}(\exp(i\varepsilon H_G)(\Lambda_0), \Lambda_{\varepsilon G}) \to \text{neigh}(\xi = 0, T^*\mathbb{T}^2),$$

we get that microlocally near the Lagrangian torus $\exp(i\varepsilon H_G)(\Lambda_0) \subset \Lambda_{\varepsilon G}$, the operator in (4.40) is unitarily equivalent to an operator $\tilde{P}_\varepsilon$, acting on $L^2_{\theta}(\mathbb{T}^2)$, defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2$, given by

$$\tilde{P}_\varepsilon \sim \sum_{\nu = 0}^{\infty} h^\nu \tilde{p}_\nu(x, \xi, \varepsilon).$$

(4.41)
Here $\tilde{p}_\nu$ are holomorphic functions in a fixed complex neighborhood of $\xi = 0$, smooth in $\varepsilon \in \text{neigh}(0, \mathbb{R})$, and

$$\tilde{p}_0 = p(\xi) + i\varepsilon \tilde{q}(x_1, \xi) + \mathcal{O}(\varepsilon^2) + \varepsilon \mathcal{O}((\xi_1 - f(\xi_2))^N),$$

(4.42)

with $\tilde{q}(x_1, \xi)$ independent of $x_2$ and of the form (4.22). Furthermore, the assumption (2.28) implies that

$$\tilde{p}_1(x, \xi, \varepsilon) = \mathcal{O}(\varepsilon).$$

We may illustrate the microlocal unitary equivalence above by the following commutative diagram,

$$
\begin{align*}
P_\varepsilon : H(\Lambda_{\varepsilon G}) & \longrightarrow H(\Lambda_{\varepsilon G}) \\
\downarrow U_0U_1^{1} & \downarrow U_0U_1^{1} \\
\tilde{P}_\varepsilon : L^2_0(T^2) & \longrightarrow L^2_0(T^2)
\end{align*}
$$

(4.43)

In what follows, we shall drop the tildes from the notation in (4.41) and write simply $P_\varepsilon$ and $p_\nu$, $\nu \geq 0$.

## 5 Quantum normal forms near rational tori

In this section, we shall be concerned with a classical $h$–pseudodifferential operator $P_\varepsilon(x, hD_x; h)$, defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2$, given by the expansion (4.41), with the leading symbol of the form (4.42). Our purpose here is to obtain a normal secular reduction of $P_\varepsilon$, also on the level of lower order symbols, and this will be accomplished in a way very similar to [12], [10].

Let us first discuss the normal form construction at the level of principal symbols. In doing so, we let $\tilde{q}_0 := \tilde{q}$ in (4.42), and write

$$p_0(x, \xi, \varepsilon) = p(\xi) + i\varepsilon \tilde{q}_0(x_1, \xi) + i\varepsilon^2 q_1(x, \xi) + \mathcal{O} (\varepsilon^3 + \varepsilon (\xi_1 - f(\xi_2))^N).$$

(5.1)

Arguing as in Section 3, we can construct an analytic function $G_1$, defined near $\xi = 0$, such that modulo $\mathcal{O}((\xi_1 - f(\xi_2))^N)$, we have

$$H_qG_1 = q_1 - \tilde{q}_1,$$

where $\tilde{q}_1$ is any analytic function satisfying (4.20), (4.21), with $q$ replaced there by $q_1$. It follows that

$$p_0(\exp(i\varepsilon^2 H_{G_1})(x, \xi)) = p(\xi) + i\varepsilon \tilde{q}_0(x_1, \xi) + i\varepsilon^2 \tilde{q}_1(x_1, \xi) + \mathcal{O}(\varepsilon^3 + \varepsilon (\xi_1 - f(\xi_2))^N).$$

Continuing this procedure, we get the following result.
Proposition 5.1 Let \( p_0(x, \xi, \varepsilon) = p(\xi) + i\varepsilon \tilde{q}_0(x_1, \xi) + \mathcal{O}(\varepsilon^2) + \varepsilon \mathcal{O}((\xi_1 - f(\xi_2))^2) \) be analytic defined near \( \xi = 0 \), depending smoothly on \( \varepsilon \in \text{neigh}(0, \mathbb{R}) \). Here \( N \geq 2 \) is arbitrarily large but fixed. Assume that

\[
p(\xi) = p(f(\xi_2), \xi_2) + g(\xi)(\xi_1 - f(\xi_2))^2, \quad g(0) > 0, \quad f(0) = 0,
\]

where \( p(f(\xi_2), \xi_2) = \alpha \xi_2 + \mathcal{O}(\xi_2^2), \alpha > 0 \). There exists a holomorphic canonical transformation \( \kappa_\varepsilon^{(N)} \) of the form

\[
\kappa_\varepsilon^{(N)} = \exp(i\varepsilon^2 H_{G_1}) \circ \cdots \circ \exp(i\varepsilon^N H_{G_{N-1}}), \tag{5.2}
\]

with \( G_j \) analytic near \( \xi = 0, 1 \leq j \leq N - 1 \), such that modulo an error term of the form \( \mathcal{O}(\varepsilon^{N+1} + \varepsilon(\xi_1 - f(\xi_2))^N) \), we have

\[
p_0(\kappa_\varepsilon^{(N)}(x, \xi)) \equiv p(\xi) + i\varepsilon \tilde{q}_0(x_1, \xi) + i\varepsilon^2(\tilde{q}_1(x_1, \xi) + \cdots + \varepsilon^{N-2}\tilde{q}_{N-1})
\]
is independent of \( x_2 \). Here, as discussed before, \( \tilde{q}_0 \) is any analytic function satisfying (4.20), (4.21), and inductively \( \tilde{q}_k \) is any analytic function satisfying (4.20), (4.21), with \( q \) there replaced by a certain function \( q_k \) that depends on the previously chosen \( \tilde{q}_0, \ldots, \tilde{q}_{k-1} \).

As will be discussed in Section 7, the complex canonical transformation \( \kappa_\varepsilon^{(N)} \) in (5.2) can be quantized by means of an elliptic classical \( h \)-Fourier integral operator \( U_\varepsilon \) in the complex domain, depending smoothly on \( \varepsilon \in \text{neigh}(0, \mathbb{R}) \), introduced rigorously on the FBI transform side. In this section, we shall proceed formally, and an application of Egorov’s theorem allows us to conclude that the operator

\[
\tilde{P}_\varepsilon(x, hD_x; h) = U_\varepsilon^{-1} P_\varepsilon(x, hD_x; h)U_\varepsilon
\]
is an \( h \)-pseudodifferential operator, defined microlocally near \( \xi = 0 \), whose symbol has a complete asymptotic expansion

\[
\tilde{P}_\varepsilon(x, \xi; h) \sim \tilde{p}_0 + h\tilde{p}_1 + \ldots, \tag{5.3}
\]
with all \( \tilde{p}_j = \tilde{p}_j(x, \xi, \varepsilon) \) being smooth functions of \( \varepsilon \in \text{neigh}(0, \mathbb{R}) \) with values in the space of holomorphic functions in a fixed complex neighborhood of \( \xi = 0 \), such that

\[
\tilde{p}_0(x, \xi, \varepsilon) = p(\xi) + i\varepsilon \tilde{q}_0(x_1, \xi) + \mathcal{O}(\varepsilon^2) + \varepsilon \mathcal{O}(\varepsilon^{N+1} + \varepsilon(\xi_1 - f(\xi_2))^N). \tag{5.4}
\]
Here the \( \mathcal{O}(\varepsilon^2) \)-term is independent of \( x_2 \) and has the properties described in Proposition 5.1. Furthermore, we still have \( \tilde{p}_1(x, \xi, \varepsilon) = \mathcal{O}(\varepsilon) \).
We shall now simplify the lower order terms \( \tilde{p}_j, j \geq 1 \), in (5.3). To that end, let \( A_\varepsilon(x, hD; h) \) be a classical analytic elliptic \( h \)-pseudo-differential operator of order 0, with symbol
\[
A_\varepsilon(x, \xi; h) \sim a_0(x, \xi, \varepsilon) + ha_1(x, \xi, \varepsilon) + \ldots,
\]
depending smoothly on \( \varepsilon \in \text{neigh}(0, \mathbb{R}) \). Then
\[
A_\varepsilon^{-1} U_\varepsilon^{-1} P_\varepsilon U_\varepsilon A_\varepsilon = A_\varepsilon^{-1} \hat{P}_\varepsilon A_\varepsilon =: \hat{P}_\varepsilon(x, hD_x; h),
\]
where
\[
\hat{P}_\varepsilon(x, \xi; h) \sim \tilde{p}_0(x, \xi, \varepsilon) + h\hat{p}_1(x, \xi, \varepsilon) + h^2\hat{p}_2(x, \xi, \varepsilon) + h^3\hat{p}_3(x, \xi, \varepsilon) + \ldots,
\]
with
\[
\hat{p}_1 = \tilde{p}_1 + \frac{1}{\xi} a_0^{-1} H_{\tilde{p}_0} a_0 = \tilde{p}_1 + \frac{1}{\xi} H_{\tilde{p}_0} b_0,
\]
if \( b_0 = \ln a_0 \), well-defined up to a constant. Thus, looking for \( b_0 \) in terms of a formal power series in \( \varepsilon \) and choosing the terms there suitably, we can arrange so that
\[
\hat{p}_1(x, \xi, \varepsilon) = \tilde{p}_{1,0}(x_1, \xi) + \varepsilon \tilde{p}_{1,1}(x_1, \xi) + \ldots + \mathcal{O}(\varepsilon^{N+1} + (\xi_1 - f(\xi_2))^N),
\]
where \( \tilde{p}_{1,0} \) is any analytic function satisfying (4.20), (4.21) with \( q \) replaced by \( \tilde{p}_{1,=0} \), and inductively, \( \tilde{p}_{1,k} \) is any analytic function satisfying (4.20), (4.21), with \( q \) replaced by a function depending on the previously chosen \( \tilde{p}_{1,0}, \ldots, \tilde{p}_{1,k-1} \).

Iterating this procedure, by choosing also the lower order terms in the expansion of \( A_\varepsilon \), we get the following result, giving a quantum secular normal form construction.

**Proposition 5.2** Let
\[
P_\varepsilon \sim p_0 + hp_1 + \ldots \quad (x, \xi) \in \text{neigh}(\xi = 0, T^*T^2),
\]
be such that \( p_0(x, \xi, \varepsilon) \) has the properties stated in Proposition 5.1. Let \( U_\varepsilon \) be a classical analytic elliptic \( h \)-Fourier integral operator of order 0, associated to the canonical transformation \( \kappa^{(N)}_\varepsilon \) in Proposition 5.1. Then we can construct an elliptic classical analytic \( h \)-pseudo-differential operator of order 0 with symbol as in (5.5), such that
\[
A_\varepsilon^{-1} U_\varepsilon^{-1} P_\varepsilon U_\varepsilon A_\varepsilon = \hat{P}_\varepsilon(x, hD_x; h),
\]
where \( \hat{P}_\varepsilon(x, hD_x; h) \) is of the form
\[
\hat{P}_\varepsilon(x, \xi; h) \sim \tilde{p}_0(x, \xi, \varepsilon) + h\hat{p}_1(x, \xi, \varepsilon) + h^2\hat{p}_2(x, \xi, \varepsilon) + \ldots.
\]
Here the leading term \( \tilde{p}_0 \) is as in (5.4),

\[
\tilde{p}_0(x, \xi, \varepsilon) = p(\xi) + i\varepsilon \tilde{q}_0(x_1, \xi) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^{N+1} + \varepsilon(\xi_1 - f(\xi_2))^N),
\]

with the \( \mathcal{O}(\varepsilon^2) \)-term being independent of \( x_2 \). For \( 1 \leq k \leq N \), we have

\[
\hat{p}_k(x, \xi, \varepsilon) = \hat{p}_{k,0}(x_1, \xi) + \varepsilon \hat{p}_{k,1}(x_1, \xi) + \ldots + \mathcal{O}((\varepsilon^{N+1} + (\xi_1 - f(\xi_2))^N),
\]

(5.10)

where \( \hat{p}_{k,\ell} \) is any function satisfying (4.20), (4.21), with \( q \) replaced by \( \hat{p}_{k,\ell} \); a function that depends on \( \hat{p}_{\tilde{k},\tilde{\ell}} \) for all \( \tilde{k} < k \), or \( \tilde{k} = k \) and \( \tilde{\ell} < \ell \). In particular, we can choose \( \hat{p}_k \) independent of \( x_2 \) modulo \( \mathcal{O}(\varepsilon^{N+1} + (\xi_1 - f(\xi_2))^N) \). We also have

\[
\hat{p}_1(x, \xi, \varepsilon) = \mathcal{O}(\varepsilon).
\]

6 Harmonic approximation for non-selfadjoint operators

In the previous section, we have seen how to eliminate the \( x_2 \)-dependence in the complete symbol of our operator, by means of successive averaging procedures, when working in a small neighborhood of the rational torus \( \Lambda_0 = \{ \xi = 0 \} \subset T^*T^2 \). Following Proposition 5.2 and neglecting the remainder terms there, we shall now consider an operator of the form

\[
P_\varepsilon = P_\varepsilon(x_1, hD_{x_1}; h), \quad (6.1)
\]

defined microlocally near \( \xi = 0 \) in \( T^*T^2 \) and acting on \( L^2_\theta(T^2) \), with a complete symbol independent of \( x_2 \). We assume that the leading symbol of \( P_\varepsilon \) is of the form

\[
p_0(x_1, \xi, \varepsilon) = p(\xi) + i\varepsilon \tilde{q}(x_1, \xi) + \mathcal{O}(\varepsilon^2), \quad p(\xi) = p(f(\xi_2), \xi_2) + g(\xi) (\xi_1 - f(\xi_2))^2,
\]

(6.2)

with \( \tilde{q} \) given in (4.22), and let us recall the assumption (2.27) implying that the function \( T \ni x_1 \mapsto \tilde{q}(x_1, f(\xi_2), \xi_2) \) has a unique minimum which is non-degenerate, for \( \xi_2 \in \text{neigh}(0, \mathbb{R}) \). When discussing the spectral analysis of \( P_\varepsilon \), it is natural, in view of its independence of \( x_2 \), to take a Fourier series expansion in \( x_2 \), thereby reducing the problem, at least formally, to a direct sum of one-dimensional operators

\[
P_\varepsilon(x_1, hD_{x_1}, h(j - \theta_2); h), \quad j \in \mathbb{Z}, \quad \theta_2 = \frac{k_0(\alpha_2)}{4} + \frac{S_2}{2\pi h},
\]

considered for those \( j \) for which \( h(j - \theta_2) \in \text{neigh}(0, \mathbb{R}) \).
In what follows, we shall write $\xi_2 = h(j - \theta_2) \in \text{neigh}(0, R)$, and concentrate our attention on the one-dimensional operator

$$P_\varepsilon(x_1, hD_{x_1}, \xi_2; h), \quad (6.3)$$

acting on $L^2_{\tilde{q}_1}(T)$. Modifying the Floquet conditions on $T$, we may replace $(6.3)$ by the conjugated operator

$$e^{-if(\xi_2)x_1/h} P_\varepsilon(x_1, hD_{x_1}, \xi_2; h) e^{if(\xi_2)x_1/h} = P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h).$$

The full symbol of $P_\varepsilon(x_1, hD_{x_1} + f(\xi_2), \xi_2; h)$ is of the form

$$P_\varepsilon(x_1, \xi_1 + f(\xi_2), \xi_2; h) = \sum_{j=0}^{\infty} h^j p_{j,\varepsilon}(x_1, \xi). \quad (6.4)$$

Here

$$p_{0,\varepsilon}(x_1, \xi) = p(\xi_1 + f(\xi_2), \xi_2) + i\varepsilon\tilde{q}(x_1, \xi_1 + f(\xi_2), \xi_2) + O(\varepsilon^2), \quad (6.5)$$

and from Proposition 5.2 we recall that

$$p_{1,\varepsilon}(x_1, \xi) = O(\varepsilon). \quad (6.6)$$

We can then write $p_{1,\varepsilon} = \varepsilon q_{1,\varepsilon}, q_{1,\varepsilon} = O(1)$.

Let us set

$$\tilde{h} = \frac{h}{\sqrt{\varepsilon}}, \quad (6.7)$$

and assume that

$$\tilde{h} \ll 1. \quad (6.8)$$

We have

$$P_\varepsilon(x_1, \xi_1 + f(\xi_2), \xi_2; h)$$

$$= p(\xi_1 + f(\xi_2), \xi_2) + i\varepsilon\tilde{q}(x_1, \xi_1 + f(\xi_2), \xi_2) + O(\varepsilon^2) + h\varepsilon q_{1,\varepsilon}(x_1, \xi) + \sum_{j=2}^{\infty} h^j p_{j,\varepsilon}$$

$$= p(\xi_1 + f(\xi_2), \xi_2) + \varepsilon \left( i\tilde{q}(x_1, \xi_1 + f(\xi_2), \xi_2) + O(\varepsilon) + \tilde{h}\varepsilon^{1/2} q_{1,\varepsilon} + \sum_{j=2}^{\infty} \tilde{h}^{j+1/2} p_{j,\varepsilon} \right).$$

Here, according to $(6.2)$,

$$p(\xi_1 + f(\xi_2), \xi_2) = p(f(\xi_2), \xi_2) + g(\xi_1 + f(\xi_2), \xi_2)\xi_1^2, \quad g(0) > 0.$$
It follows that on the operator level, we have
\[
P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h) = p(f(\xi_2), \xi_2) + g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 \\
+ i\varepsilon \left( \tilde{q}(x_1, f(\xi_2) + hD_{x_1}, \xi_2) + \mathcal{O}(\varepsilon + \tilde{h}\varepsilon^{3/2} + \tilde{h}^2) \right).
\]
(6.9)

We shall be interested in computing eigenvalues of the one-dimensional operator
\[
P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h)
\]
in the region
\[
|\text{Re } z - p(f(\xi_2), \xi_2)| \leq \mathcal{O}(\varepsilon \tilde{h}),
\]
and directly from (6.9), using cut-offs of the form \(\chi(\xi_1/\sqrt{\varepsilon})\), we see that the corresponding eigenfunctions are microlocally concentrated to the region where \(\xi_1 = \mathcal{O}(\sqrt{\varepsilon})\), provided that the smallness condition (6.8) is strengthened to the following one,
\[
\frac{h}{\sqrt{\varepsilon}} \leq h^\eta, \quad \eta > 0.
\]
(6.10)

It will then be convenient to perform a rescaling of the cotangent variable, corresponding to a suitable change of the semiclassical parameter. Let us write
\[
hD_{x_1} = \sqrt{\varepsilon \tilde{h}}D_{x_1},
\]
and if \(\xi_1, \tilde{\xi}_1\) denote the cotangent variables corresponding to \(hD_{x_1}\) and \(\tilde{h}D_{x_1}\), respectively, we have
\[
\xi_1 = \sqrt{\varepsilon \tilde{\xi}_1}.
\]

It follows that
\[
\frac{1}{\varepsilon}P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h)
\]
can be viewed as an \(\tilde{h}\)-pseudodifferential operator of the form
\[
\frac{1}{\varepsilon}P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h) = \frac{p(f(\xi_2), \xi_2)}{\varepsilon} \\
+ g(f(\xi_2) + \sqrt{\varepsilon \tilde{h}}D_{x_1}, \xi_2)(\tilde{h}D_{x_1})^2 + i\tilde{q}(x_1, f(\xi_2) + \sqrt{\varepsilon \tilde{h}}D_{x_1}, \xi_2) + \mathcal{O}(\varepsilon) \\
+ \tilde{h}\mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(\tilde{h}^2).
\]
(6.12)

Ignoring the constant term \(p(f(\xi_2), \xi_2)/\varepsilon\) in the right hand side, we recognize here essentially a one-dimensional Schrödinger operator with a purely imaginary potential, and to be precise, we can write
\[
\frac{1}{\varepsilon}P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h) = \frac{p(f(\xi_2), \xi_2)}{\varepsilon} + A(x_1, \tilde{h}D_{x_1}, \xi_2, \sqrt{\varepsilon} \tilde{\xi}_1),
\]
where \( A(x_1, \tilde{h}D_{x_1}, \xi_2, \sqrt{\varepsilon}; \hbar) \) is a well-behaved \( \tilde{h} \)-pseudodifferential operator, depending smoothly on \( \xi_2 \in \text{neigh}(0, \mathbb{R}) \) and \( \sqrt{\varepsilon} \geq 0 \), with the leading symbol

\[
g(f(\xi_2) + \sqrt{\varepsilon} \xi_1, \xi_2)\xi_1^2 + i\tilde{q}(x_1, f(\xi_2) + \sqrt{\varepsilon} \xi_1, \xi_2) + \mathcal{O}(\varepsilon),
\]

and with a subprincipal symbol which is \( \mathcal{O}(\sqrt{\varepsilon}) \). Here we have dropped the tilde from the notation for the cotangent variable corresponding to \( \tilde{h}D_{x_1} \), and let us also recall that the operator (6.12) is to be considered microlocally in the region where \( \xi_1 = \mathcal{O}(1) \). The function \( g \) in (6.13) satisfies \( g > 0 \).

**Remark.** If (6.6) is no longer assumed, we can write, assuming that \( h/\varepsilon \ll 1 \),

\[
P_\varepsilon(x_1, \xi_1 + f(\xi_2), \xi_2; h) = p(\xi_1 + f(\xi_2), \xi_2) + i\varepsilon \tilde{q}(x_1, \xi_1 + f(\xi_2), \xi_2) + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(h^j p_{j,\varepsilon}(x_1, \xi)) + \sum_{j=2}^{\infty} h^j \tilde{p}_{j,\varepsilon}(x_1, \xi),
\]

and we can then view \( h/\varepsilon \) as an additional small parameter. As will be seen in Section 6, some pseudospectral considerations will force us to assume that \( \varepsilon/h \) should not be too large, and for that reason, in this work we make the assumption (2.28), leading to (6.6).

The discussion pursued in this section so far indicates that the spectral analysis of the original operator \( P_\varepsilon \) should reduce to that for a family of \( \tilde{h} \)-pseudodifferential operators on \( \mathbb{T} \), with leading symbols of the form (6.13). Letting \( \varepsilon = 0 \) in (6.13) for a while and suppressing the parameter \( \xi_2 \) altogether, we shall now pause to make a digression, in order to recall semiclassical asymptotics for the low lying eigenvalues of non-selfadjoint \( h \)-pseudodifferential operators with double characteristics. In doing so, we shall follow the analysis of [6], which in turn follows [4] closely. Let us also remark that in the present one-dimensional case, the quadratic approximations along the double characteristics are elliptic and consequently, our discussion is considerably simplified, when compared with [6], [4].

Let \( P_0(x, hD_x; h) : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T}) \) be such that \( P_0(x, hD_x; h) \in \text{Op}_w^w(S(\langle \xi \rangle^2)) \), and assume that the semiclassical leading symbol of \( P_0 \) is of the form

\[
p_0(x, \xi) = \xi^2 + iV(x),
\]

(6.14)
where \( V \in C^\infty(\mathbb{T}; \mathbb{R}) \). Assume also, for simplicity, that the subprincipal symbol of \( P_0 \) vanishes. Let us assume that if \( a = \min V \) then \( V^{-1}(a) = \{ x_0 \} \) with \( V''(x_0) > 0 \). We are interested in the eigenvalues of \( P_0 \) in an open disc \( \{ z \in \mathbb{C}; |z - ia| < Ch \} \), for some \( C > 0 \) fixed and all \( h > 0 \) small enough, and to that end we consider the operator

\[
P(x, hD_x; h) = (1 - i)(P_0(x, hD_x; h) - ia),
\]

whose leading symbol \( p(x, \xi) = (1 - i)(p_0(x, \xi) - ia) \) is such that

\[
\text{Re } p(x, \xi) = \xi^2 + V(x) - a \geq 0
\]
is elliptic for large \( \xi \) and vanishes precisely at the point \( (x_0, 0) \in T^*\mathbb{T} \). In a neighborhood of \( (x_0, 0) \) we have

\[
p(x + x_0, \xi) = q(x, \xi) + \mathcal{O}((x, \xi)^3), \quad (x, \xi) \to (0, 0),
\]

where \( q \) is a quadratic form, such that \( \text{Re } q > 0 \). When determining the eigenvalues of \( P(x, hD_x; h) \) in an \( \mathcal{O}(h) \)-neighborhood of 0, naturally only the behavior of the operator in a small neighborhood of \( (x_0, 0) \) matters, and by composing \( p \) with an inverse of the translation

\[
\kappa : \text{neigh}((x_0, 0), T^*\mathbb{T}) \to \text{neigh}((0, 0), T^*\mathbb{R}), \quad \kappa((x_0, 0)) = (0, 0),
\]

we obtain an \( h \)-pseudodifferential operator

\[
P(x, \xi; h) \sim \sum_{j=0}^\infty h^j p_j(x, \xi), \quad p_1 = 0,
\]
defined microlocally near \( (0, 0) \in T^*\mathbb{R} \), such that the leading symbol \( p_0 = p \) satisfies

\[
p(x, \xi) = q(x, \xi) + \mathcal{O}((x, \xi)^3),
\]

where \( q \) is quadratic with

\[
\text{Re } q > 0.
\]

We extend \( P(x, \xi; h) \) to be globally defined on \( \mathbb{R}^2 \) as an element of the symbol class \( S(1) \), such that

\[
\text{Re } p(x, \xi) \geq 0, \quad (\text{Re } p)^{-1}(0) = \{ (0, 0) \},
\]

and such that

\[
\text{Re } p(x, \xi) \geq \frac{1}{C}, \quad |(x, \xi)| \geq C > 0.
\]

An application of Theorem 1.1 of [6] allows us to conclude that the following result holds, which we state directly for the operator \( P_0(x, hD_x; h) \). See also [5] for related results in the analytic case.
Theorem 6.1 Let the operator \( P_0(x, hD_x; h) : C^\infty(T) \to C^\infty(T) \) have the principal symbol of the form (6.14), a vanishing subprincipal symbol, and let us assume that if \( a = \min V \) then \( V^{-1}(a) = \{ x_0 \} \) with \( b := V''(x_0) > 0 \). Let \( C > 0 \). Then there exists \( h_0 > 0 \) such that for all \( 0 < h \leq h_0 \), the spectrum of the operator \( P_0(x, hD_x; h) \) in the open disc in the complex plane \( D(ia, Ch) \) is given by the simple eigenvalues of the form,

\[
z_k \sim ia + h \left( \lambda_{k,0} + h \lambda_{k,1} + h^2 \lambda_{k,2} + \ldots \right).
\]

Here \( \lambda_{k,0} \) are the eigenvalues in \( D(0, C) \) of the elliptic quadratic operator

\[
q^w(x, D_x) = D_x^2 + i b^{2} x^2,
\]

acting on \( L^2(\mathbb{R}) \), which are given by

\[
\lambda_{k,0} = \left( \frac{b}{2} \right)^{1/2} e^{i\pi/4} (2k + 1), \quad k \in \mathbb{N}, \quad k = \mathcal{O}(1).
\]

Remark. Theorem 6.1 continues to be valid when the operator \( P_0(x, hD_x; h) \) acts on an \( L^2 \)-space of Floquet periodic functions on \( T \) and indeed, the eigenvalues described in this result do not depend on the Floquet conditions, modulo \( \mathcal{O}(h^\infty) \).

Coming back to the operator in (6.12), with the leading symbol (6.13), we shall next have to extend the result of Theorem 6.1 to the parameter dependent case, and to this end it will be convenient to recall briefly the main steps in the proof of Theorem 6.1. Let \( P = P(x, hD_x; h) \) be an \( h \)-pseudodifferential operator on \( \mathbb{R} \) satisfying (6.18) – (6.22). Following [6], let us recall that the proof of Theorem 6.1 proceeds by constructing a well-posed Grushin problem for the operator \( P \), of the form

\[
(P - hz)u + R_- u_- = v, \quad R_+ u = v, \quad z \in \text{neigh}(\lambda_0, C),
\]

in the space \( L^2(\mathbb{R}) \times \mathbb{C} \). Here \( \lambda_0 \) is an eigenvalue of \( q^w(x, D_x) \) such that \( |\lambda_0| < C \). The operators \( R_- : \mathbb{C} \to L^2 \) and \( R_+ : L^2 \to \mathbb{C} \) are defined as follows,

\[
R_- u_- = u_- e, \quad R_+ u = (u, f)_{L^2},
\]

where \( e \) is an eigenfunction of \( q^w(x, hD_x) \) corresponding to the eigenvalue \( h\lambda_0 \), and \( f \) is an eigenfunction of the adjoint operator \( q^w(x, hD_x) \), corresponding to the eigenvalue \( h\lambda_0 \).

The verification of the well-posedness of (6.24) consists of two steps, both carried out after a metaplectic FBI transform,

\[
T : L^2(\mathbb{R}) \to H_{\phi_0}(\mathbb{C}).
\]
Here

\[ H_{\Phi_0}(C) = \text{Hol}(C) \cap L^2(C; e^{-2\Phi_0/h}L(dx)), \]

and \( \Phi_0 \) is a suitable strictly subharmonic quadratic form. In the first step, we concentrate on the region \( |x| \leq h^\rho, x \in C \), for some \( 1/3 < \rho < 1/2 \). Arguing as in [6], we obtain the following a priori estimate for the problem (6.24), based on the quadratic approximation of \( P \) near the origin — see formula (3.25) in [6],

\[
\begin{align*}
&\| (h + |x|^2)^{1/2} \chi_0 \left( \frac{x}{h^\rho} \right) u \| + h^{-1/2} |u_-| \\
&\leq C \| (h + |x|^2)^{-1/2} \chi_0 \left( \frac{x}{h^\rho} \right) v \| + C \| (h + |x|^2)^{-1/2} \chi_0 \left( \frac{x}{h^\rho} \right) (P - Q)u \| \\
&+ \mathcal{O}(h^{1/2}) |v_+| + C \sqrt{\frac{h}{h^2 \rho}} \| (h + |x|^2)^{1/2} 1_K \left( \frac{x}{h^\rho} \right) u \|. 
\end{align*}
\]

(6.27)

Here \( u, v \in H_{\Phi_0}(C) \), the norms are taken in the space \( L^2(C; e^{-2\Phi_0/h}L(dx)) \), and we have also written \( P \) for the conjugated operator \( TPT^{-1} \). The function \( \chi_0 \in C^\infty_0(C) \) is fixed, with \( \chi_0 = 1 \) near 0, and \( K \) is a fixed compact neighborhood of \( \text{supp}(\nabla \chi_0) \), \( 0 \notin K \). Furthermore, \( Q = Tq^w(x, hD_x)T^{-1} \), and therefore, as explained in [6], we have

\[
\| (h + |x|^2)^{-1/2} \chi_0 \left( \frac{x}{h^\rho} \right) (P - Q)u \| = \mathcal{O} \left( \frac{h^{3\rho}}{h^{1/2}} \right) \| u \|. 
\]

(6.28)

Using (6.27) and (6.28), we obtain

\[
\begin{align*}
&h \| \chi_0 \left( \frac{x}{h^\rho} \right) u \|^2 + h^{-1} |u_-|^2 \\
&\leq \frac{\mathcal{O}(1)}{h} \| v \|^2 + \mathcal{O}(h^{6\rho-1}) \| u \|^2 + \mathcal{O}(h) \| v_+ \|^2 + \mathcal{O}(h) \| 1_K \left( \frac{x}{h^\rho} \right) u \|^2. 
\end{align*}
\]

(6.29)

Notice that the lower bound \( \rho > 1/3 \) implies that here \( h^{6\rho-1} \ll h \).

In the second step of the proof, we consider the exterior region, \( |x| \geq h^\rho \), and here we use that

\[
\text{Re} \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x} (x) \right) \geq \frac{h^{2\rho}}{C}, \quad C > 0.
\]

Exploiting the sharp Gårding inequality in the form of a quantization-multiplication formula, as explained in [6], see also [29], we obtain the following exterior a priori estimate for the problem (6.24),

\[
\begin{align*}
&h^{2\rho} \int \chi \left( \frac{x}{h^\rho} \right) |u(x)|^2 e^{-2\Phi_0(x)/h} L(dx) \leq \mathcal{O}(1) \| v \| \| u \| \\
&+ \mathcal{O}(h^\infty) \| u_- \| \| u \| + \mathcal{O}(h) \| u \|^2.
\end{align*}
\]

(6.30)

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Here $\chi \in C_0^\infty(C; [0,1])$ vanishes near $x = 0$ and $\chi = 1$ for large $x$. Assuming that $1/3 < \rho < 1/2$, the bounds (6.29) and (6.31) can be glued together, and we get the a priori estimate

$$h ||u|| + |u_-| \leq \mathcal{O}(1) (||v|| + h |v_+|),$$

and the consequent well-posedness of the Grushin problem (6.24). Asymptotic expansions for the eigenvalues of $P$ follow exactly as explained in [4], [6]. In the present one-dimensional situation, the eigenvalues are simple and only integer powers of $h$ occur in the expansions (6.23).

Turning the attention to the parameter-dependent case, let $P_{\varepsilon} = P_{\varepsilon}(x, hD_x; h) \in \text{Op}_h^\omega(S(1)), \varepsilon \geq 0$, be an $h$–pseudodifferential operator depending smoothly on $\sqrt{\varepsilon}$, such that $P_{\varepsilon=0} = P$ satisfies (6.18), (6.19), (6.20), (6.21), (6.22). In particular, the leading symbol $p_\varepsilon$ of $P_{\varepsilon}$ satisfies

$$p_\varepsilon(x, \xi) = p(x, \xi) + \mathcal{O}(\sqrt{\varepsilon})$$

in the sense of symbols in $S(1)$, and the subprincipal symbol of $P_{\varepsilon}$ is $\mathcal{O}(\sqrt{\varepsilon})$. Assume also that near $(0, 0)$, (6.32) improves to

$$p_\varepsilon(x, \xi) = p(x, \xi) + \mathcal{O} \left( \sqrt{\varepsilon} |\xi| + \varepsilon \right),$$

see also (6.13). We would like to conclude that the Grushin problem (6.24) with $P$ replaced by $P_{\varepsilon}$ remains well-posed, provided that $\varepsilon > 0$ is not too large, and to that end, we shall simply inspect the two steps above.

In the region $|x| \leq h^\rho$, we argue as above, with $P$ replaced by $P_{\varepsilon}$, and using (6.33), together with the fact that subprincipal symbol of $P_{\varepsilon}$ is $\mathcal{O}(\sqrt{\varepsilon})$, we see that we get an additional term in the right hand side of (6.27) of the form

$$|| (h + |x|^2)^{-1/2} \chi_0 \left( \frac{x}{h^\rho} \right) (P_{\varepsilon} - P) u || = \mathcal{O} \left( \frac{\sqrt{\varepsilon} h^\rho + \varepsilon}{h^{1/2}} \right) || u ||.$$

Here we also assume that we have chosen the FBI transform in (6.26) so that (6.33) holds on the transform side. As for the exterior region $|x| \geq h^\rho$, replacing $P$ by $P_{\varepsilon}$, we get an additional term in the right hand side of (6.30), given by

$$\mathcal{O}(1) \sqrt{\varepsilon} ||u||^2.$$

It follows that to absorb the two extra terms (6.34), (6.35), we need to meet the following conditions,

$$\frac{\varepsilon^{1/2} h^\rho + \varepsilon}{h^{1/2}} \ll h^{1/2},$$

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and
\[ \sqrt{\varepsilon} \ll h^{2\rho}. \]
The first condition is satisfied provided that
\[ \varepsilon \ll h^{2-2\rho}, \]
since \( \rho < 1/2 \), and the second one holds when
\[ \varepsilon \ll h^{4\rho}. \]
We conclude that the Grushin problem (6.24) remains well-posed when \( P \) is replaced by \( P_{\varepsilon} \), provided that
\[ \varepsilon \ll h^{4\rho}, \quad (6.36) \]
since \( 1/3 < \rho < 1/2 \). Combining this observation with the standard perturbation theory for eigenvalues of multiplicity one \( [13] \), we obtain the following result.

**Proposition 6.2** Let \( P_{\varepsilon}(x, hD_x; h) \), \( \varepsilon \geq 0 \), be a smooth function of \( \sqrt{\varepsilon} \) with values in \( \text{Op}_{h}^{w}(S(1)) \), such that when \( \varepsilon = 0 \), we have the properties (6.18)–(6.22). Assume that (6.33) holds. Then for \( \varepsilon \leq h^{4+\eta}, \eta > 0 \), the eigenvalues of \( P_{\varepsilon}(x, hD_x; h) \) in the region \( \{z \in \mathbb{C}; |z| < Ch\} \) are given by the simple eigenvalues of the form
\[
z_k \sim h \left( \lambda_{k,0}(\sqrt{\varepsilon}) + h\lambda_{k,1}(\sqrt{\varepsilon}) + \ldots \right), \quad k \in \mathbb{N}, \quad k = O(1),
\]
where \( \lambda_{k,j}(\sqrt{\varepsilon}) \) are smooth functions of \( \sqrt{\varepsilon} \geq 0 \), \( j \geq 0 \), with \( \lambda_{k,0}(0) \) being the eigenvalues of the quadratic operator \( q_{w}(x, D_x) \), described explicitly in Theorem 6.1. When \( z \in \mathbb{C} \) is such that \( |z| < Ch \) and \( \text{dist}(z, \text{Spec}(P_{\varepsilon})) \geq h/O(1) \), we have
\[
(z - P_{\varepsilon})^{-1} = \frac{O(1)}{h} : L^2 \to L^2. \quad (6.37)
\]
In our considerations, see (6.13), when applying Proposition 6.2 we should replace the semiclassical parameter \( h \) by \( \tilde{h} = h/\sqrt{\varepsilon} \), which in view of (6.36) leads to the condition
\[
\varepsilon \ll \tilde{h}^{4\rho}, \quad 1/3 < \rho < 1/2, \quad (6.38)
\]
so that
\[
\varepsilon \ll h^{\frac{4}{5}+\eta}. \]
When \( \rho = 1/3 \), the power in the right hand side = 4/5, and it follows that we have the well-posedness of the Grushin problem provided that
\[
\varepsilon \leq O(h^{\frac{4}{5}+\eta}), \quad \eta > 0. \quad (6.39)
\]
Remark. In the proof of Proposition 6.2 above, the presence of the parameter $\sqrt{\varepsilon}$ was treated by a direct perturbation argument, leading to the upper bound (6.36). The purpose of this remark is to outline an alternative approach to the parameter-dependent case, leading to sharper bounds on $\varepsilon$. While sharpening the result of Proposition 6.2 below would not lead to an improvement in Theorem 2.1, which is the main result of this work, we believe that the alternative approach sketched below may be of some independent interest. Since its precise realization is likely to demand a greater technical investment, the argument developed in this remark will be quite brief and we hope to be able to develop it further in a future work.

Let $P_\varepsilon(x, \xi; h)$ be a real analytic function of $\varepsilon \in \text{neigh}(0, \mathbf{R})$ with values in the space of bounded holomorphic functions in a tubular neighborhood of $\mathbf{R}^2$, such that as $h \to 0^+$,

$$P_\varepsilon(\rho; h) \sim p_\varepsilon(\rho) + hp_{1,\varepsilon}(\rho) + \ldots, \quad \rho = (x, \xi).$$

For $\varepsilon = 0$, let us assume that the leading symbol $p := p_0$ is such that $\text{Re} p \geq 0$ is elliptic at infinity, vanishing precisely at $\rho = 0$. Assume furthermore that we have, $p(\rho) = q(\rho) + \mathcal{O}(\rho^3), \quad \rho \to 0,$

where $q$ is quadratic with $\text{Re} q$ positive definite. In particular, $\rho = 0$ is a non-degenerate critical point for $p$ and an application of the implicit function theorem shows that for $\varepsilon$ small, $p_\varepsilon$ has a non-degenerate critical point $\rho(\varepsilon)$ in the complex domain, depending analytically on $\varepsilon$, with $\rho(\varepsilon) = \mathcal{O}(\varepsilon)$. Passing to the FBI transform side by means of a metaplectic FBI transform $T$, as in (6.26), let us continue to write $\rho(\varepsilon) = (x(\varepsilon), \xi(\varepsilon)) = \mathcal{O}(\varepsilon)$ for the image of the critical point $\rho(\varepsilon)$ under the complex linear canonical transformation $\kappa_T$ associated to $T$.

We know from [24] that $\kappa_T(\mathbf{R}^2) = \Lambda_{\Phi_0} = \{(x, (2/i)\partial_x \Phi_0(x)); \ x \in \mathbf{C}\}$, where $\Phi_0$ is the strictly subharmonic quadratic form introduced in (6.26). We shall now discuss the problem of constructing a weight function $\Phi_\varepsilon \in C^\infty(\mathbf{C})$ such that

$$\Phi_\varepsilon = \Phi_0 + \mathcal{O}(h), \quad |\nabla^2 (\Phi_\varepsilon - \Phi_0)| \ll 1, \quad (6.40)$$

and with $\rho(\varepsilon) \in \Lambda_{\Phi_\varepsilon} = \{(x, (2/i)\partial_x \Phi_\varepsilon(x)); \ x \in \mathbf{C}\}$. The function $\Phi_\varepsilon$ is then strictly subharmonic and if we set $H_{\Phi_\varepsilon}(\mathbf{C}) = \text{Hol}(\mathbf{C}) \cap L^2(\mathbf{C}; e^{-2\Phi_\varepsilon/h} L(dx))$, then we have $H_{\Phi_\varepsilon} = H_{\Phi_0}$, with uniformly equivalent norms. To get the complete asymptotic expansions of the eigenvalues of $P_\varepsilon$ in $D(p_\varepsilon(\rho_\varepsilon), Ch)$, as in Proposition 6.2, one should then work with the operator $P_\varepsilon$ acting on the space $H_{\Phi_\varepsilon}$. We need

$$\xi(\varepsilon) = \frac{2}{i} \frac{\partial \Phi_\varepsilon}{\partial x}(x(\varepsilon)),$$
and let us notice that
\[ \xi(\varepsilon) - \frac{2}{i} \partial \Phi_0 \frac{\partial}{\partial x}(x(\varepsilon)) = O(\varepsilon). \]

With \( \partial \Phi_\varepsilon(x(\varepsilon)) \) already determined, we try
\[
\Phi_\varepsilon(x) = \Phi_0(x) + 2\text{Re} \left( (\partial_\varepsilon \Phi_\varepsilon(x(\varepsilon)) - \partial_\varepsilon \Phi_0(x(\varepsilon))) \cdot (x - x(\varepsilon)) \right) \chi \left( \frac{x - x(\varepsilon)}{h^\alpha} \right)
\]
\[
= \Phi_0(x) + h^\alpha (\ell_\varepsilon \chi) \left( \frac{x - x(\varepsilon)}{h^\alpha} \right),
\]
where \( \chi \in C_0^\infty(\mathbb{C}) \) is a standard cut-off near 0 and
\[
\ell_\varepsilon(y) = 2\text{Re} \left( (\partial_\varepsilon \Phi_\varepsilon(x(\varepsilon)) - \partial_\varepsilon \Phi_0(x(\varepsilon))) \cdot y \right)
\]
is linear, such that \( \ell_\varepsilon = O(\varepsilon) \) as a linear form. Then
\[
\nabla^k (\Phi_\varepsilon - \Phi_0) = O(\varepsilon) h^{\alpha - k\alpha}, \quad k \geq 0,
\]
and in view of (6.40), we need \( \varepsilon h^\alpha \leq O(h) \), \( \varepsilon / h^\alpha \ll 1 \). We get the conditions \( \varepsilon \leq O(h^{1-\alpha}) \), \( \varepsilon \ll O(h^\alpha) \), and it follows that the optimal choice of \( \alpha \) is given by \( \alpha = 1/2 \). This leads to the condition \( \varepsilon \ll O(\sqrt{h}) \). In our applications, we should replace \( \varepsilon \) by \( \sqrt{\varepsilon} \) and \( h \) by \( \tilde{h} = h / \sqrt{\varepsilon} \), leading to the condition
\[
\varepsilon \ll \tilde{h} = \frac{h}{\sqrt{\varepsilon}},
\]
so that we get
\[
0 \leq \varepsilon \ll O(h^{2/3}), \quad (6.41)
\]
which is sharper than (6.39). One conjectures therefore that the result of Proposition 6.2 extends to this range of \( \varepsilon \) and we hope to return to this observation in a future paper.

We shall finish this section by a formal application of Proposition 6.2 to the microlocally defined operator \( P_\varepsilon(x_1, hD_{x_1}, \xi_2; h) \) in (6.3), acting on \( L^2_{\theta_1}(T) \): assume that \( \varepsilon > 0 \) is such that
\[
\tilde{h} = \frac{h}{\sqrt{\varepsilon}} \leq h^\eta, \quad \eta > 0,
\]
and that (6.39) holds. It follows that the eigenvalues of \( P_\varepsilon(x_1, hD_{x_1}, \xi_2; h) \) in the region
\[
|z - p(f(\xi_2), \xi_2) - i\varepsilon q_2(x_1(\xi_2), f(\xi_2), \xi_2)| \leq O(\sqrt{\varepsilon} h)
\]

are given by
\begin{align}
z_k &= p(f(\xi_2), \xi_2) + i\varepsilon(q)_2(x_1(\xi_2), f(\xi_2), \xi_2) \\
& \quad + \sqrt{\varepsilon} h(\lambda_{k,0} + \lambda_{k,1}\tilde{h} + \lambda_{k,2}\tilde{h}^2 + \ldots), \quad N \ni k \leq \mathcal{O}(1),
\end{align}
where \( \lambda_{k,j} = \lambda_{k,j}(\xi_2, \sqrt{\varepsilon}), j \geq 0 \), is a smooth function of \( \xi_2 \in \text{neigh}(0, \mathbb{R}), \sqrt{\varepsilon} \geq 0 \), with
\begin{align}
\lambda_{k,0}(\xi_2, 0) &= e^{i\pi/4}(\partial_{\xi_2}^2 p(f(\xi_2), \xi_2))^{1/2}(\partial_{x_1}^2 (q)_2(x_1(\xi_2), f(\xi_2), \xi_2))^{1/2}\left(k + \frac{1}{2}\right).
\end{align}
Here we recall from (2.27) that \( x_1(\xi_2) \in T \) is the unique point of minimum of the function \( x_1 \mapsto (q)_2(x_1, f(\xi_2), \xi_2) \).

7 Pseudospectral bounds and the global Grushin problem

The discussion pursued in the previous section shows that we are able to determine the low-lying eigenvalues of suitable localized one-dimensional operators \( P_\varepsilon(x_1, hD_{x_1}, \xi_2; h) \) in (6.3), occurring in the normal form reduction, provided that the perturbative parameter \( \varepsilon \) satisfies
\begin{align}
h^{2-\eta} \leq \varepsilon \leq h^{4/5+\eta}, \quad \eta > 0.
\end{align}
The purpose of this section is to complete the proof of Theorem 2.1 by constructing a global well-posed Grushin problem for \( P_\varepsilon - z \), leading to the description of the eigenvalues in the region described in Theorem 2.1. In doing so, we shall have to strengthen the bounds in (7.1), as a consequence of some precise pseudospectral analysis for the family of the one-dimensional non-selfadjoint operators \( P_\varepsilon(x_1, hD_{x_1}, \xi_2; h) \), with \( \xi_2 \) playing the role of parameters.

Our first task is to give a global definition of the \( h \)-dependent weighted Hilbert space, where the Grushin problem will be studied. Similarly to [12], the weighted space in question will be associated to a globally defined IR-manifold \( \Lambda \subset T^*\tilde{\Lambda} \), which is \( \mathcal{O}(\varepsilon) \)-close to \( T^*M \) and agrees with it outside a compact set. Specifically, the manifold \( \Lambda \) will be obtained as an \( \mathcal{O}(\varepsilon^2) \)-perturbation of the IR-manifold \( \Lambda_{\varepsilon G} \), introduced in (4.38), where the perturbative modification will only take place in a sufficiently small but fixed neighborhood of the rational torus \( \Lambda_0 \).
Let us recall therefore that in Section 4, we have shown that micro locally near the Lagrangian torus \( \exp (i\varepsilon H_G) (\Lambda_0) \subset \Lambda_G \), the operator in (4.40) is unitarily equivalent to an analytic \( h \)-pseudodifferential operator \( P_\varepsilon \), defined microlocally near \( \xi = 0 \) in \( T^*T^2 \) and acting on \( L^2_0(T^2) \), such that the leading symbol of \( P_\varepsilon \) is of the form

\[
p_0(x, \xi, \varepsilon) = p(\xi) + i\varepsilon \tilde{q}(x_1, \xi) + \mathcal{O}(\varepsilon^2) + \varepsilon \mathcal{O}((\xi_1 - f(\xi_2))^N),
\]

(7.2)

where \( p(\xi) \) is given in (4.2) and \( N \geq 2 \) is arbitrarily large but fixed. See also (4.43) for an illustration of the unitary equivalence by means of a commutative diagram.

Let

\[
k_\varepsilon^{(N)} : \text{neigh}(\xi = 0, T^*\tilde{T}^2) \to \text{neigh}(\xi = 0, T^*\tilde{T}^2), \quad \tilde{T}^2 = C^2/2\pi\mathbb{Z}^2
\]

(7.3)

be the holomorphic canonical transformation, introduced in Proposition 5.1. Considering the IR-manifold \( k_\varepsilon^{(N)}(T^*T^2) \subset T^*T^2 \), defined in a complex neighborhood of \( \xi = 0 \), we conclude, arguing as in Section 5 in [12], that there exists a \( C^\infty \) strictly plurisubharmonic function \( \Phi_\varepsilon(x) \), defined for \( x \in C^2/2\pi\mathbb{Z}^2 \), \( |\text{Im}x| \leq 1/O(1) \), such that in the \( C^\infty \)–sense,

\[
\Phi_\varepsilon(x) = \Phi_0(x) + \mathcal{O}(\varepsilon^2), \quad \Phi_0(x) = \frac{1}{2}(\text{Im}x)^2,
\]

and such that the operator

\[
P_\varepsilon = \mathcal{O}(1) : T^{-1}H_{\Phi_\varepsilon}(|\text{Im}x| < 1/C) \to T^{-1}H_{\Phi_\varepsilon}(|\text{Im}x| < 1/C)
\]

(7.4)

is, microlocally near \( k_\varepsilon^{(N)}(T^2 \times \{\xi = 0\}) \), unitarily equivalent to an operator \( \tilde{P}_\varepsilon \), given in (5.3), (5.4), acting on \( L^2_0(T^2) \). Here

\[
T : L^2(T^2) \to H_{\Phi_0}(C^2/2\pi\mathbb{Z}^2)
\]

is the standard unitary FBI–Bargmann transform on the 2-torus, associated to the quadratic phase function \( i(x - y)^2/2 \), as discussed in [10], and we have written

\[
H_{\Phi_\varepsilon}(\Omega) = \text{Hol}(\Omega) \cap L^2(\Omega, e^{-2\Phi_\varepsilon/h L}(dx)),
\]

for \( \Omega \subset C^2/2\pi\mathbb{Z}^2 \) open, including the Floquet periodic versions of the spaces. Let us also point out that the unitary equivalence between the operators \( P_\varepsilon \) in (7.4) and \( \tilde{P}_\varepsilon \) is realized by means of a microlocally unitary \( h \)-Fourier integral operator \( U_\varepsilon \) in
the complex domain, quantizing the canonical transformation in (7.3). Similarly to (4.43), we may illustrate it in a commutative diagram,

\[
P_\varepsilon : T^{-1}H_{\Phi_\varepsilon}(|\text{Im } x| < 1/C) \xrightarrow{U_\varepsilon} T^{-1}H_{\Phi_\varepsilon}(|\text{Im } x| < 1/C)
\]

\[
\tilde{P}_\varepsilon : L^2_\theta(T^2) \xrightarrow{U_\varepsilon} L^2_\theta(T^2)
\]

(7.5)

In particular, according to Proposition 5.1, the leading symbol of \(\tilde{P}_\varepsilon\) is independent of \(x_2\), modulo \(O(\varepsilon^{N+1} + \varepsilon(\xi_1 - f(\xi_2))^N)\). The subprincipal symbol of \(\tilde{P}_\varepsilon\) is \(O(\varepsilon)\).

**Remark.** From [12], we may recall that writing

\[
\Lambda_{\Phi_\varepsilon} : \quad \xi = \frac{2}{i} \frac{\partial \Phi_\varepsilon}{\partial x}(x), \quad |\text{Im } x| \leq \frac{1}{\mathcal{O}(1)},
\]

we have \(\Lambda_{\Phi_\varepsilon} = \kappa_T \circ \kappa_\varepsilon^{(N)}(T^*T^2)\), where the canonical transformation \(\kappa_T\) associated to \(T\) is given by

\[
T^*\tilde{T}^2 \ni (y, \eta) \mapsto (y - i\eta, \eta) = (x, \xi) \in T^*\tilde{T}^2.
\]

Let us also remark that the writing (7.4) is somewhat informal, and a precise statement is obtained by considering the action of the conjugated operator \(TP_\varepsilon T^{-1}\) on the space \(H_{\Phi_\varepsilon}(|\text{Im } x| < 1/C)\), see also [12].

We obtain a globally defined IR-manifold \(\Lambda \subset T^*\tilde{M}\), which is \(\varepsilon\)-close to \(T^*M\) everywhere in the \(C^\infty\)-sense, agrees with that set away from \(p^{-1}(0)\), and in a complex neighborhood of \(\Lambda_0\), it is obtained by replacing

\[
\exp (i\varepsilon H_G) \circ \kappa_0^{-1}(T^*T^2)
\]

by

\[
\exp (i\varepsilon H_G) \circ \kappa_0^{-1} \circ \kappa_\varepsilon^{(N)}(T^*T^2),
\]

(7.6)

which amounts to an \(O(\varepsilon^2)\)-deformation \(\Lambda_{\varepsilon G}\) in a neighborhood of \(\Lambda_0\). Here we recall the holomorphic canonical transformation \(\exp (i\varepsilon H_G)\), identifying \(\Lambda_{\varepsilon G}\) and \(T^*M\) in a neighborhood of \(\Lambda_0\), and the real analytic canonical transformation \(\kappa_0\) in (4.1), given by the action-angle coordinates near \(\Lambda_0\). The spectral analysis required in order to compute the extremal eigenvalues of \(P_\varepsilon\) in Theorem 2.1 will be carried out in the globally defined \(h\)-dependent Hilbert space \(H(\Lambda)\), associated to the IR-manifold \(\Lambda\) by the FBI–Bargmann approach.
Recalling Proposition 4.2 and taking into account also Proposition 5.2, eliminating the $x_2$-dependence in the normal form by means of a pseudodifferential conjugation, we may summarize the discussion so far in the following result.

**Proposition 7.1** There exists a globally defined smooth IR-manifold $\Lambda \subset T^*\tilde{M}$ and a $C^\infty$-Lagrangian torus $\tilde{\Lambda}_0 \subset \Lambda$, which is an $\mathcal{O}(\varepsilon)$–perturbation of the rational torus $\Lambda_0$ in the $C^\infty$–sense, such that when $\rho \in \Lambda$ is away from an $\varepsilon^\delta$-neighborhood of $\tilde{\Lambda}_0$ in $\Lambda$ and

$$|\text{Re } P_\varepsilon(\rho; h)| \leq \frac{\varepsilon^{2\delta}}{C},$$

(7.7)

for $C > 0$ large enough, then we have

$$\text{Im } P_\varepsilon(\rho; h) \geq \varepsilon \inf Q_\infty(\Lambda_0) + \frac{\varepsilon^{2\delta+1}}{\mathcal{O}(1)}.$$  (7.8)

Here $0 < \delta < 1/2$ is so small that $\varepsilon^\delta \gg \max(h^{1/2}, \varepsilon^{1/2})$. The manifold $\Lambda$ is $\mathcal{O}(\varepsilon)$-close to $T^*M$ and agrees with it away from a neighborhood of $p^{-1}(0) \cap T^*M$. We have

$$P_\varepsilon = \mathcal{O}(1) : H(\Lambda, m) \to H(\Lambda).$$

Furthermore, there exists an elliptic $h$–Fourier integral operator with a complex phase

$$U = \mathcal{O}(1) : H(\Lambda) \to L^2_\theta(T^2),$$

such that microlocally near $\tilde{\Lambda}_0$, we have

$$UP_\varepsilon = (P(x_1, hD_{x_1}, \varepsilon; h) + R(x, hD_{x_1}, \varepsilon; h))U.$$

Here $P(x_1, hD_{x_1}, \varepsilon; h) + R(x, hD_{x_1}, \varepsilon; h)$ is defined microlocally near $\xi = 0$ in $T^*T^2$, the full symbol of $P(x_1, hD_{x_1}, \varepsilon; h)$ is independent of $x_2$, and

$$R(x, \xi, \varepsilon; h) = \mathcal{O}(\varepsilon^{N+1} + (\xi_1 - f(\xi_2))^N + h^{N+1}), \quad f(0) = 0.$$  (7.9)

Here $N$ is arbitrarily large but fixed. The leading symbol of $P(x_1, hD_{x_1}, \varepsilon; h)$ is of the form

$$p(\xi) + i\varepsilon \tilde{q}(x_1, \xi) + \mathcal{O}(\varepsilon^2),$$

where

$$p(\xi) = p(f(\xi_2), \xi_2) + g(\xi)(\xi_1 - f(\xi_2))^2, \quad g(0) > 0, \quad f(0) = 0,$$  (7.10)

and $T \ni x_1 \mapsto \tilde{q}(x_1, f(\xi_2), \xi_2)$ has a unique minimum, when $\xi_2 \in \text{neigh}(0, R)$, which is also non-degenerate. The subprincipal symbol of $P(x_1, hD_{x_1}, \varepsilon; h)$ is $\mathcal{O}(\varepsilon)$.  

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Using Proposition 7.1, we shall now discuss a priori estimates for the equation
\[(P_\varepsilon - z)u = v, \tag{7.11}\]
when \(u \in H(\Lambda, m), v \in H(\Lambda),\) and the spectral parameter \(z \in \mathbb{C}\) is confined to the region
\[|\text{Re } z| \leq \frac{\varepsilon^{2\delta}}{\mathcal{O}(1)}, \quad \text{Im } z \leq \varepsilon \inf Q_{\infty}(\Lambda_0) + \mathcal{O}(\sqrt{\varepsilon h}). \tag{7.12}\]

When doing so, following [12], [11], we shall make use of a suitable partition of unity on the manifold \(\Lambda,\) defined using Proposition 7.1 and consisting of smooth functions satisfying slightly degenerate symbolic estimates. Indeed, the presence of such slightly exotic symbols is natural here, as we are dealing with methods based on the techniques of normal forms, introducing error terms vanishing to a high order along the invariant tori. See also [25]. When quantizing the corresponding symbols defined on \(\Lambda,\) in the case when \(M = \mathbb{R}^2,\) we follow [11] and reduce the quantization procedure to that of Weyl on the standard phase space \(T^*\mathbb{R}^2,\) by means of a \(C^\infty-\) canonical transformation
\[\kappa : \text{neigh}(p^{-1}(0), T^*\mathbb{R}^2) \to \text{neigh}(p^{-1}(0), \Lambda),\]
such that
\[\kappa(X) = X + i\varepsilon H_G(x) + \mathcal{O}(\varepsilon^2),\]
and the corresponding unitary Fourier integral operator with a complex phase mapping \(L^2(\mathbb{R}^2)\) to \(H(\Lambda).\) In the case when \(M\) is compact, we use the Toeplitz quantization, following [26].

Let us consider a smooth partition of unity on the manifold \(\Lambda,\)
\[1 = \chi + \psi_1 + \psi_2. \tag{7.13}\]
Here \(\chi \in C_0^\infty(\Lambda), \nabla^m \chi = \mathcal{O}(\varepsilon^{-2\delta m}), m \geq 0,\) is a cutoff function supported in an \(\varepsilon^\delta-\) neighborhood of \(\hat{\Lambda}_0\) intersected with the region where \(|\text{Re } P_\varepsilon| \leq \varepsilon^{2\delta} / C.\) Specifically, we shall obtain \(\chi\) by choosing a suitable function \(\chi_0 \in C_0^\infty(T^*\mathbb{T}^2), \partial^\alpha \chi_0 = \mathcal{O}(\varepsilon^{-2\delta|\alpha|}), |\alpha| \geq 0,\) depending on \(\xi\) only, \(\chi_0 = \chi_0(\xi),\) and conjugating the operator \(\chi_0(hD_x)\) by the microlocal inverse of the operator \(U\) in Proposition 7.1. In particular, we get, using that the subprincipal symbol of \(P_{\varepsilon=0}\) vanishes,
\[[P_\varepsilon, \chi] = \mathcal{O}\left(\frac{\varepsilon^3}{\varepsilon^{2\delta}}\right) + \mathcal{O}\left(\frac{\varepsilon h}{\varepsilon^{2\delta}}\right) : H(\Lambda) \to H(\Lambda). \tag{7.14}\]
The function \( 0 \leq \psi_1 \in C^\infty(\Lambda) \) in (7.13) satisfies
\[
\nabla^m \psi_1 = O_m(\varepsilon^{-2\delta m}), \quad m \geq 0,
\]
and is such that
\[
|\text{Re } P_\varepsilon(\rho; h)| \geq \frac{\varepsilon^{2\delta}}{C}
\]
(7.15)
near the support of \( \psi_1 \). Finally, \( 0 \leq \psi_2 \in C^\infty_0(\Lambda) \) in (7.13) is such that
\[
\nabla^m \psi_2 = O_m(\varepsilon^{-2\delta m}), \quad m \geq 0,
\]
(7.16)
and furthermore, \( \psi_2 \) is supported in a region invariant under the \( H_p \)-flow, where
\[
\text{Im } P_\varepsilon(\rho; h) \geq \varepsilon \inf Q_\infty(\Lambda_0) + \frac{\varepsilon^{1+2\delta}}{\mathcal{O}(1)}.
\]
(7.17)

We also arrange, as we may, so that \( \psi_2 \) Poisson commutes with \( p \), the leading symbol of \( P_{\varepsilon=0} \) acting on \( H(\Lambda) \).

Let us now return to the equation (7.11). Assume that \( \delta \in (0, 1/2) \) is so small that
\[
\varepsilon^{2\delta} \geq h^{\frac{1}{2}-\eta},
\]
(7.18)
for some fixed \( \eta > 0 \). We can then follow the slightly degenerate parametrix construction for \( P_{\varepsilon} - z \), near the support of \( \psi_1 \), described in detail in Section 4 of [11] and obtain that
\[
|| \psi_1 u || \leq \frac{\mathcal{O}(1)}{\varepsilon^{2\delta}} || v || + \mathcal{O}(h^\infty) || u ||.
\]
(7.19)

Here and in what follows the norms are taken in the space \( H(\Lambda) \).

When discussing estimates for \( \psi_2 u \), let us notice that \( \text{Im } P_\varepsilon(\rho; h) = \mathcal{O}(\varepsilon) \) on \( \Lambda \), and near supp \( \psi_2 \), we have, in view of (7.17) and (7.12),
\[
\text{Im } (P_\varepsilon(\rho; h) - z) \geq \frac{\varepsilon^{1+2\delta}}{\mathcal{O}(1)} - \mathcal{O}(\sqrt{\varepsilon} h).
\]
(7.20)

Therefore, with a new implicit constant, we get near the support of \( \psi_2 \),
\[
\frac{1}{\varepsilon} \text{Im } (P_\varepsilon(\rho; h) - z) \geq \frac{\varepsilon^{2\delta}}{\mathcal{O}(1)},
\]
(7.21)
provided that the following lower bound on \( \varepsilon \) holds,
\[
h^{2/(1+4\delta)} \ll \varepsilon.
\]
(7.22)
The lower bound (7.22) is of the same form as (7.1). Using $h/\varepsilon^{4\delta}$ as the natural semiclassical parameter and applying the sharp Gårding inequality, we get in view of (7.21),

$$\frac{1}{\varepsilon} \text{Im} ((P_\varepsilon - z)\psi_2 u, \psi_2 u) \geq \left( \frac{\varepsilon^{2\delta}}{O(1)} - O(1) \right) \| \psi_2 u \|^2 - O(h^\infty) \| u \|^2$$

$$\geq \frac{\varepsilon^{2\delta}}{O(1)} \| \psi_2 u \|^2 - O(h^\infty) \| u \|^2, \quad (7.23)$$

provided that we strengthen (7.18) by assuming that

$$\frac{h}{\varepsilon^{6\delta}} \leq h^\eta, \quad \eta > 0. \quad (7.24)$$

It follows from (7.23) that

$$\frac{\varepsilon^{2\delta+1}}{O(1)} \| \psi_2 u \|^2 \leq O(1) \| v \| \| \psi_2 u \| + \text{Im} ([P_\varepsilon, \psi_2] \tilde{\psi}_2 u, \psi_2 u) + O(h^\infty) \| u \|^2. \quad (7.25)$$

Here $\tilde{\psi}_2 \in C^\infty_0(\Lambda)$ has the same properties as $\psi_2$ and is such that $\tilde{\psi}_2 = 1$ near $\text{supp} (\psi_2)$. When estimating the commutator $[P_\varepsilon, \psi_2]$ in (7.25), we get by the Weyl calculus, using (7.16) together with the fact that the subprincipal symbol of $P_{\varepsilon=0}$ vanishes and $p$ and $\psi_2$ Poisson commute,

$$[P_\varepsilon, \psi_2] = [P_{\varepsilon=0}, \psi_2] + O\left( \frac{\varepsilon h}{\varepsilon^{2\delta}} \right) = O\left( \frac{h^3}{\varepsilon^{6\delta}} \right) + O\left( \frac{\varepsilon h}{\varepsilon^{2\delta}} \right) = O\left( \frac{\varepsilon h}{\varepsilon^{2\delta}} \right). \quad (7.26)$$

Here we have also used that $h^2 \ll \varepsilon^{1+4\delta}$, in view of (7.22). Combining (7.25) and (7.26), we get

$$\frac{\varepsilon^{2\delta+1}}{O(1)} \| \psi_2 u \|^2 \leq O(1) \| v \| \| \psi_2 u \| + O\left( \frac{\varepsilon h}{\varepsilon^{2\delta}} \right) \| \tilde{\psi}_2 u \|^2 + O(h^\infty) \| u \|^2, \quad (7.27)$$

and therefore,

$$\| \psi_2 u \|^2 \leq \frac{O(1)}{\varepsilon^{4\delta+2}} \| v \|^2 + O\left( \frac{h}{\varepsilon^{4\delta}} \right) \| \tilde{\psi}_2 u \|^2 + O(h^\infty) \| u \|^2. \quad (7.28)$$

Combining (7.24), (7.28), and a standard iteration argument, we conclude that

$$\| \psi_2 u \| \leq \frac{O(1)}{\varepsilon^{1+2\delta}} \| v \| + O(h^\infty) \| u \|. \quad (7.29)$$
Using (7.13), (7.19), and (7.29), we obtain the following a priori estimate for the problem (7.11), (7.12),

\[ \| (1 - \chi) u \| \leq \frac{O(1)}{\varepsilon^{1+2\delta}} \| v \| + O(h^\infty) \| u \|, \]

which holds provided that \( \delta \in (0, 1/2) \) and the conditions (7.22), (7.24) are fulfilled. In the subsequent analysis, we may therefore concentrate the attention on the region \( \text{supp}(\chi) \), for the cutoff function \( \chi \) in (7.13).

Let us recall that the function \( \chi \in C_0^\infty(\Lambda) \), \( \nabla^m \chi = O(\varepsilon^{-2\delta m}) \), \( m \geq 0 \), in (7.13) is supported in an \( \varepsilon^\delta \)-neighborhood of \( \hat{\Lambda}_0 \) intersected with the region where \( |\text{Re} P_\varepsilon| \leq \varepsilon^2/\varepsilon^{2\delta}/C \). Writing

\[ (P_\varepsilon - z) \chi u = \chi v + [P_\varepsilon, \chi] u, \]

and applying the Fourier integral operator \( U \) of Proposition 7.1, we get

\[ (P(x_1, hD_x, \varepsilon; h) - z) U \chi u = U \chi v + U [P_\varepsilon, \chi] u + Tu. \]

(7.31)

Here, using (7.23), we see that

\[ T = O(h^M) : H(\Lambda) \rightarrow L^2_\delta(T^2), \]

(7.32)

where \( M \) can be taken as large as we wish, provided that the integer \( N \) in Proposition 7.1 is taken large enough. Furthermore, as discussed above, we may arrange so that

\[ U \chi = \chi_0 U + O(h^\infty) : H(\Lambda) \rightarrow H(\Lambda), \]

where \( \chi_0 = \chi_0(hD_x, \varepsilon) \) is of the form

\[ \chi_0(\xi, \varepsilon) = \chi_1 \left( \frac{\xi_1}{\varepsilon^\delta} \right) \chi_1 \left( \frac{\xi_2}{\varepsilon^{2\delta}} \right), \]

where \( \chi_1 \in C_0^\infty(\mathbb{R}) \) is a standard cutoff to a neighborhood of 0. In particular, using (7.10) we see that the support of \( \chi_0 \) is contained in the region where

\[ |\xi| = O(\varepsilon^\delta), \quad |p(\xi)| \leq O(\varepsilon^{2\delta}). \]

Modifying the operator \( T \) in (7.31) slightly, we get

\[ (P(x_1, hD_x, \varepsilon; h) - z) \chi_1 \left( \frac{hD_{x_1}}{\varepsilon^\delta} \right) \chi_2 \left( \frac{hD_{x_2}}{\varepsilon^{2\delta}} \right) U u = U \chi v + U [P_\varepsilon, \chi] u + Tu \]

(7.33)
In the subsequent analysis we shall therefore be working on the cotangent space \( T^*T^2 \), in the region where
\[
\xi_1 = \mathcal{O}(\varepsilon^\delta),
\] (7.34)
while
\[
\xi_2 = \mathcal{O}(\varepsilon^{2\delta}).
\] (7.35)

Taking a Fourier series expansion in \( x_2 \), we get a direct sum decomposition
\[
P(x_1, hD_x, \varepsilon; h) = \bigoplus_{j \in \mathbb{Z}} P(x_1, hD_{x_1}, \xi_2, \varepsilon; h), \quad \xi_2 = h(j - \theta_2),
\] (7.36)
where, according to (7.35), the summation is restricted only to those \( j \in \mathbb{Z} \) for which \( \xi_2 = \mathcal{O}(\varepsilon^{2\delta}) \). We shall consider the question of inverting the operator
\[
P(x_1, hD_x, \varepsilon; h) - z = \bigoplus_j \left( P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) - z \right),
\] (7.37)
where, compared to (7.12), the real part of \( z \) will be localized further to the region
\[
|\text{Re} \, z| \leq \frac{h}{C \sqrt{\varepsilon}},
\] (7.38)
where \( C > 0 \) is large enough but fixed. Since in Proposition 7.1 we have introduced errors that are \( \mathcal{O}(h^M), \ M \gg 1, \) see (7.32), we would first like to show that the one-dimensional non-selfadjoint operator
\[
P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) - z : L^2_{\theta_1}(T) \to L^2_{\theta_1}(T)
\] (7.39)
is invertible, microlocally in the region where \( \xi_1 = \mathcal{O}(\varepsilon^\delta) \), with an inverse of temperate growth in \( 1/h \), when \( \xi_2 = \mathcal{O}(\varepsilon^{2\delta}) \) is such that \( |\xi_2| \geq h/C_1 \sqrt{\varepsilon} \), for a suitable fixed \( C_1 \), satisfying \( 0 < C_1 < C \). In doing so, it will be convenient to distinguish two cases, depending on the sign of \( \xi_2 \).

**Case 1.** Let us assume first that \( \xi_2 = \mathcal{O}(\varepsilon^{2\delta}) \) is such that
\[
\xi_2 \geq \frac{h}{C_1 \sqrt{\varepsilon}}.
\] (7.40)
Then, after a unitary conjugation, we can write, on the level of operators,
\[
e^{-if(\xi_2)x_1/h} P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) e^{if(\xi_2)x_1/h} - z
\]
\[
= p(f(\xi_2), \xi_2) + g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 + i\varepsilon \tilde{g}(x_1, f(\xi_2) + hD_{x_1}, \xi_2) + \mathcal{O}(\varepsilon^2)
\]
\[
+ h\mathcal{O}(\varepsilon) + \mathcal{O}(h^2) - z.
\] (7.41)
Here the conjugated operator, acting on the space of Floquet periodic functions $L^2_{\theta_1 + f(\xi_2)}(T)$, is still considered microlocally in the region where $\xi_1 = O(\varepsilon^\delta)$, since $f(\xi_2) = O(\xi_2) = O(\varepsilon^{2\delta})$. Recalling that the derivative of the function $\xi_2 \mapsto p(f(\xi_2), \xi_2)$ is strictly positive near $\xi_2 = 0$, we conclude, using (7.38), (7.40), and the positivity of $g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2$, that the real part of the operator in (7.41), which is of the form

$$p(f(\xi_2), \xi_2) + g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 - \text{Re} z + O(\varepsilon^2 + h^2) + hO(\varepsilon),$$

is $\geq h/(\tilde{C}\sqrt{\varepsilon})$, for some $\tilde{C} > 0$, and is therefore invertible, microlocally in the region $\xi_1 = O(\varepsilon^{\delta})$, with the norm of the inverse being $O(\sqrt{\varepsilon}/h)$. Here we also use that $\varepsilon^2 \ll h/\sqrt{\varepsilon}$, in view of (7.1). It is therefore clear that the full operator in (7.41) is invertible, microlocally in the region $\xi_1 = O(\varepsilon^{\delta})$, with a microlocal inverse of the norm $O(\sqrt{\varepsilon}/h)$.

**Case 2.** We assume now that $\xi_2 = O(\varepsilon^{2\delta})$ is such that

$$\xi_2 \leq -\frac{h}{C_1\sqrt{\varepsilon}}. \quad (7.42)$$

Similarly to (7.41), we write

$$e^{-if(\xi_2)x_1/h} P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) e^{if(\xi_2)x_1/h} - z = g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 + i\varepsilon\bar{q}(x_1, f(\xi_2) + hD_{x_1}, \xi_2) + O(\varepsilon^2) + hO(\varepsilon) + O(h^2) - w, \quad (7.43)$$

where

$$w = z - p(f(\xi_2), \xi_2) \quad (7.44)$$

satisfies

$$\text{Re} w \geq \frac{h}{C_2\sqrt{\varepsilon}}, \quad \text{Im} w \leq \varepsilon \inf Q_\infty(\Lambda_0) + O(\sqrt{\varepsilon}h), \quad (7.45)$$

for a suitable $C_2 > 0$. In view of (7.35), we have

$$\bar{q}(x_1, f(\xi_2) + \xi_1, \xi_2) = \bar{q}(x_1, \xi_1, 0) + O(\varepsilon^{2\delta}), \quad (7.46)$$

and therefore, on the operator level we obtain that

$$e^{-if(\xi_2)x_1/h} P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) e^{if(\xi_2)x_1/h} - z = g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 + i\varepsilon\bar{q}(x_1, hD_{x_1}, 0) + R - w, \quad (7.47)$$

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where
\[ R = \mathcal{O}(\varepsilon^{1+2\delta} + \varepsilon h + \varepsilon^2 + h^2) : L^2_{\theta_1+f(\xi_2)}(\mathbb{T}) \to L^2_{\theta_1+f(\xi_2)}(\mathbb{T}). \] (7.48)

We may also assume that in (7.47), the operator \( \tilde{q}(x_1, hD_{x_1}, 0) \) is given by the classical \( h \)-quantization. It follows from (7.33) that thanks to the presence of the cutoff \( \chi_1(hD_x/\varepsilon^\delta) \), to invert the operator in (7.39), microlocally in the region where \( \xi_1 = \mathcal{O}(\varepsilon^\delta) \), we should consider the following equation
\[
\left( g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 + i\varepsilon \tilde{q}(x_1, hD_{x_1}) + R - w_1 \right) \chi_1 \left( \frac{hD_{x_1}}{\varepsilon^\delta} \right) u = v, \]
(7.49)
for \( u, v \in L^2_{\theta_1+f(\xi_2)}(\mathbb{T}) \). Here \( w_1 = w - i\varepsilon \inf Q_\infty(\Lambda_0) \) and
\[
\tilde{q}(x_1, \xi_1) = \tilde{q}(x_1, 0) + k(x_1, \xi_1)\varphi \left( \frac{\xi_1}{\varepsilon^\delta} \right),
\]
where
\[
k(x_1, \xi_1) = \xi_1 \int_0^1 (\partial_{\xi_1} \tilde{q})(x_1, t\xi_1) \, dt.
\]
and \( \varphi \in C_0^\infty(\mathbb{R}) \) is such that \( \varphi = 1 \) near \( \text{supp} (\chi_1) \). In particular, the function \( \tilde{q}(x_1, \xi_1) \) satisfies the assumptions for the function \( \tilde{V} \) in Proposition A.4 in Appendix. Let us set
\[
A(x_1, hD_{x_1}) = g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 + i\varepsilon \tilde{q}(x_1, hD_{x_1}),
\]
We would like to invert the operator \( A(x_1, hD_{x_1}) + R - w_1 \), occurring in the left hand side of (7.49) by an application of Proposition A.4 and to that end, we shall assume that
\[
\varepsilon^{1+\frac{2}{3}} \ll h.
\] (7.50)
Write
\[
A(x_1, hD_{x_1}) - w_1 = \varepsilon \left( \frac{A(x_1, hD_{x_1}) - w_2}{\varepsilon} \right), \quad w_2 = \frac{w_1}{\varepsilon} = \frac{w}{\varepsilon} - i \inf Q_\infty(\Lambda_0). \] (7.51)
It follows from (7.45) that the spectral parameter \( w_2 \) satisfies
\[
\text{Re} \, w_2 \geq \frac{h}{C_2\varepsilon^{3/2}}, \quad \text{Im} \, w_2 \leq \mathcal{O}(\tilde{h}). \] (7.52)
Here we recall that $\tilde{h} = h/\sqrt{\varepsilon}$. In order to be able to apply Proposition A.4 to (7.51) we finally have to impose the smallness condition
\begin{equation}
\tilde{h} |w_2|^{1/2} \ll 1,
\end{equation}
and using (7.35), (7.38), (7.44), and (7.51), we see that (7.53) holds provided that
\begin{equation}
\frac{h}{\varepsilon^\delta} \ll 1.
\end{equation}
We shall therefore require that the condition
\begin{equation}
h \ll \varepsilon^{1-\delta}
\end{equation}
holds. Once the conditions (7.50) and (7.54) both hold, we are in the position to apply Proposition A.4 to (7.51), obtaining that
\begin{equation}
(A(x_1, hD_{x_1}) - w_1)^{-1} = \varepsilon^{-1}O(\tilde{h}^{-2/3} |w_2|^{-1/3}) : L^2_{\theta_1+f(\xi_2)}(T) \rightarrow L^2_{\theta_1+f(\xi_2)}(T).
\end{equation}
We get, using that $|w_2| \geq h/(C_2\varepsilon^{3/2})$,
\begin{equation}
(A(x_1, hD_{x_1}) - w_1)^{-1} = \varepsilon^{-1}O(\tilde{h}^{-1} \varepsilon^{1/3}) = O(h^{-1} \varepsilon^{-1/6}) : L^2_{\theta_1+f(\xi_2)}(T) \rightarrow L^2_{\theta_1+f(\xi_2)}(T).
\end{equation}
Returning to the equation (7.49), we would like to use a standard Neumann series argument to invert the operator $A(x_1, hD_{x_1}) + R - w_1$ in the left hand side of (7.49), and according to (7.56) and (7.48), we know that this is possible provided that
\begin{equation}
h^{-1} \varepsilon^{-1/6} \left(\varepsilon^{2\delta+1} + \varepsilon h + \varepsilon^2 + h^2\right) \ll 1,
\end{equation}
which, in view of (7.1) and (7.22), is equivalent to the condition
\begin{equation}
h^{-1} \varepsilon^{\frac{5}{6}+2\delta} \ll 1.
\end{equation}
Comparing the upper bounds (7.58) and (7.50), we see that the latter is implied by the former, provided that
\begin{equation}
0 < \delta < 1/9.
\end{equation}
In what follows, we shall adopt the smallness condition (7.59). We arrive therefore at the following upper bound on $\varepsilon$,
\begin{equation}
\varepsilon \ll h^{6/(5+12\delta)}.
\end{equation}
which is a strengthening of the upper bound in (7.1).

We shall now also examine the lower bounds on $\varepsilon$ that we have imposed in the course of our argument in this section. To that end, we recall that the lower bounds have been introduced in (7.22), (7.24), and (7.54). Comparing first the lower bounds (7.22) and (7.54), we see that we have

$$h^{2/(1+4\delta)} \ll h^{1/(1-\delta)}$$

when (7.59) holds, and our lower bound on $\varepsilon$ becomes

$$h^{1/(1-\delta)} \ll \varepsilon.$$  
 \hfill (7.61)

We should then check the validity of (7.24) when (7.61) holds, and to that end we observe that indeed,

$$\frac{h}{\varepsilon^{6\delta}} \leq \frac{h}{h^{6\delta}/(1-\delta)} \leq h^\eta, \quad \eta > 0,$$

thanks to (7.59).

Combining the bounds (7.60) and (7.61), we get the permissible range

$$h^{1/(1-\delta)} \ll \varepsilon \ll h^{6/(5+12\delta)},$$  
 \hfill (7.62)

where $\delta \in (0, 1/9)$. The range in (7.62) is non-empty for $\delta \in (0, 1/9)$ precisely when

$$\frac{1}{1 - \delta} > \frac{6}{5 + 12\delta} \iff \delta > \frac{1}{18}.$$

Let us summarize the discussion above in the following result.

**Proposition 7.2** Let us consider the operator

$$P(x_1, hD_x, \varepsilon; h) = \bigoplus_{j \in \mathbb{Z}} P(x_1, hD_{x_1}, \xi_2, \varepsilon; h), \quad \xi_2 = h(j - \theta_2) = \mathcal{O}(\varepsilon^{2\delta}),$$

microlocally in the region $\xi_1 = \mathcal{O}(\varepsilon^\delta), \xi_2 = \mathcal{O}(\varepsilon^{2\delta})$, where $1/18 < \delta < 1/9$. Assume that the spectral parameter $z \in \mathbb{C}$ is such that

$$|\text{Re } z| \leq \frac{h}{C\sqrt{\varepsilon}}, \quad \text{Im } z \leq \varepsilon \inf \mathcal{Q}_\infty(\Lambda_0) + \mathcal{O}(\sqrt{\varepsilon}h),$$  
 \hfill (7.63)
for some constant $C > 0$. Assume furthermore that
\[ h^{1/(1-\delta)} \ll \varepsilon \ll h^{6/(5+12\delta)}, \tag{7.64} \]
and let us assume that the quantum numbers $\xi_2 = \mathcal{O}(\varepsilon^{2\delta})$ satisfy
\[ |\xi_2| \geq \frac{h}{\mathcal{O}(1)\sqrt{\varepsilon}}. \]

Then there exists a family of operators
\[ E(\xi_2, \varepsilon; h) = \mathcal{O}(\varepsilon^{-1/6}h^{-1}) : L^2_{\theta_1} \to L^2_{\theta_1} \]
such that
\[ (E(\xi_2, \varepsilon; h)(P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) - z) - 1) \chi \left( \frac{hD_{x_1}}{\varepsilon^\delta} \right) = \mathcal{O}(h^\infty) : L^2_{\theta_1} \to L^2_{\theta_1} \]
for every $\chi \in C_0^\infty(\mathbb{R})$ with support in a sufficiently small but fixed neighborhood of 0.

**Remark.** Notice that to reach powers of $h$ that are $< 1$ in (7.64), it suffices to take $\delta > 1/12$. To obtain the range in (7.64) that is as large as possible, we should choose $\delta \in (1/18, 1/9)$ to be close to $1/9$.

In what follows, we continue to assume that the spectral parameter $z \in \mathbb{C}$ is confined to the region (7.63), and we shall assume that (7.64) holds, for some $\delta \in (1/18, 1/9)$. It follows therefore from Proposition 7.2 that in the orthogonal sum decomposition (7.37), we can restrict the attention to the quantum numbers $\xi_2 = h(j - \theta_2)$ such that
\[ |\xi_2| \leq \frac{h}{C_1\sqrt{\varepsilon}} = \frac{\tilde{h}}{C_1}, \quad C_1 > 0. \tag{7.65} \]

Using this refined localization in the parameter $\xi_2$, we shall now proceed to show that the spectrum of the operator $P_\varepsilon$ in the region (7.63) is contained in the union of the pairwise disjoint bands of the form
\[ |p(f(\xi_2), \xi_2) - \text{Re} z| \leq C_0\sqrt{\varepsilon}h, \quad \xi_2 = h(j - \theta_2) = \mathcal{O}(\tilde{h}), \tag{7.66} \]
where $C_0 > 1$ is large enough but fixed. When doing so, we shall proceed similarly to the arguments above, relying upon Proposition A.4 and treating the parameter $\xi_2$ in a perturbative way.
Let us assume that $z \in \mathbb{C}$ satisfies (7.63) and is such that for some sufficiently large fixed $C_0 > 1$, we have
\[ |p(f(\xi_2), \xi_2) - \text{Re } z| \geq C_0 \sqrt{\varepsilon h}, \quad (7.67) \]
for all $\xi_2 = h(j - \theta_2) = \mathcal{O}(\tilde{h})$. Similarly to (7.43), we write
\[
e^{-if(\xi_2)x_1/h} P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) e^{if(\xi_2)x_1/h} - z \]
\[ = g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 + i\varepsilon \tilde{q}(x_1, f(\xi_2) + hD_{x_1}, \xi_2) + \mathcal{O}(\varepsilon^2) + h\mathcal{O}(\varepsilon) + \mathcal{O}(h^2) - w, \quad (7.68) \]
where $w = z - p(f(\xi_2), \xi_2)$ satisfies
\[ |\text{Re } w| \geq C_0 \sqrt{\varepsilon h}, \quad \text{Im } w \leq \varepsilon \inf Q_\infty(\Lambda_0) + \mathcal{O}(\sqrt{\varepsilon h}). \quad (7.69) \]
Now, in view of (7.65), we have
\[ \tilde{q}(x_1, f(\xi_2) + hD_{x_1}, \xi_2) = \tilde{q}(x_1, hD_{x_1}, 0) + \mathcal{O}(\tilde{h}), \]
and arguing as in the discussion of "Case 2" above, we see that we have to invert the problem
\[ (g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 + i\varepsilon \tilde{q}(x_1, hD_{x_1}) + R - w_1) \chi_1 \left( \frac{hD_{x_1}}{\varepsilon \delta} \right) u = v, \quad (7.70) \]
where $\tilde{q}(x_1, hD_{x_1})$ satisfies the assumptions in Proposition A.4 and
\[ R = \mathcal{O}(\varepsilon \tilde{h} + \varepsilon^2 + \varepsilon h + h^2) : L^2_{\tilde{\theta}_1 + f(\xi_2)}(\mathbf{T}) \rightarrow L^2_{\tilde{\theta}_1 + f(\xi_2)}(\mathbf{T}). \quad (7.71) \]
The spectral parameter $w_1$ in (7.70) satisfies, in view of (7.69),
\[ \frac{1}{\varepsilon} |\text{Re } w_1| \geq C_0 \tilde{h}, \quad \frac{1}{\varepsilon} \text{Im } w_1 \leq \mathcal{O}(\tilde{h}). \quad (7.72) \]
An application of Proposition A.4 gives, as before,
\[ (g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 + i\varepsilon \tilde{q}(x_1, hD_{x_1}) - w_1)^{-1} \]
\[ = \varepsilon^{-1} \mathcal{O} \left( \tilde{h}^{-2/3} |w_1|^{-1/3} \varepsilon^{1/3} \right) : L^2_{\tilde{\theta}_1 + f(\xi_2)}(\mathbf{T}) \rightarrow L^2_{\tilde{\theta}_1 + f(\xi_2)}(\mathbf{T}), \quad (7.73) \]
and using (7.72), we see that the bound on the operator norm in (7.73) does not exceed
\[ \varepsilon^{-1} \mathcal{O}(C_0^{-1/3} \tilde{h}^{-1}). \quad (7.74) \]
To invert the full operator
\[ g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 + i\varepsilon \tilde{q}(x_1, hD_{x_1}) + R - w_1, \] (7.75)

in the left hand side of (7.70), in view of (7.71) and (7.74), we have to check that
\[ \varepsilon^{-1} \tilde{h}^{-1} C_0^{-1/3} \left( \varepsilon \tilde{h} + \varepsilon^2 + h^2 \right) \ll 1, \] (7.76)

which is satisfied for \( C_0 > 1 \) large enough, since clearly, \( \varepsilon \ll \tilde{h} \), in view of (7.64). The bound on the norm of the inverse of the operator in (7.75) is therefore also given by (7.74).

Combining Proposition 7.2 with the discussion above, including the estimates (7.73), (7.74), we conclude that if \( z \in \mathbb{C} \) satisfies (7.63) and is such that (7.67) holds, then the operator \( P(x_1, hD_x, \varepsilon; h) - z \) is invertible, microlocally in the region where \( \xi_1 = \mathcal{O}(\varepsilon^\delta), \xi_2 = \mathcal{O}(\varepsilon^{2\delta}) \), with a microlocal inverse of the norm
\[ (P(x_1, hD_x, \varepsilon; h) - z)^{-1} = \mathcal{O}(\varepsilon^{-1/2}h^{-1}) : L^2_\partial \to L^2_\partial. \] (7.77)

Coming back to (7.33), we therefore obtain for such \( z \)'s,
\[ ||U\chi u|| \leq \mathcal{O}(\varepsilon^{-1/2}h^{-1})||v|| + \mathcal{O}(\varepsilon^{-1/2}h^{-1})\mathcal{O}(\varepsilon h)\mathcal{O}(\varepsilon^{2\delta})||u|| + \mathcal{O}(h^{M'})||u||. \] (7.78)

where \( M' \gg 1 \). Here we have also used (7.14). Combining (7.78) and (7.30), we obtain the following result.

**Proposition 7.3** Assume that
\[ h^{1/(1-\delta)} \ll \varepsilon \ll h^{6/(5+12\delta)}, \] (7.79)

for some \( \delta \in (1/18, 1/9) \). Then the spectrum of the operator \( P_\varepsilon : H(\Lambda, m) \to H(\Lambda) \)
in the region
\[ |\text{Re } z| \leq \frac{h}{C\sqrt{\varepsilon}}, \quad \text{Im } z \leq \varepsilon \inf Q_\infty(\Lambda_0) + \mathcal{O}(\sqrt{\varepsilon}h). \] (7.80)

is contained in the disjoint union of the bands of the form
\[ |p(f(\xi_2), \xi_2) - \text{Re } z| \leq C_0 \sqrt{\varepsilon}h, \quad \xi_2 = h(j - \theta_2) = \mathcal{O}(\tilde{h}), \] (7.81)

where \( C_0 > 1 \) is large enough but fixed.
We shall finally obtain a precise description of the spectrum of $P_\varepsilon$ in the region $(7.81)$, for a given value of $j \in \mathbb{Z}$, such that $\xi_2 = h(j - \theta_2) = O(h)$. In doing so, in view of the localization for $\Im z$, we may assume that

\[
|z - p(f(\xi_2), \xi_2) - i\varepsilon\langle q \rangle_2(x_1(\xi_2), f(\xi_2), \xi_2)| \leq C_0 \sqrt{\varepsilon} h,
\]

(7.82)

where $\xi_2 = h(j - \theta_2) = O(h)$, and we then know that only the operator

\[
P(x_1, hD_{x_1}, \xi_2; \varepsilon; h) : L^2_{\theta_1} \to L^2_{\theta_1},
\]

(7.83)

in (7.36) contributes to the spectrum in this region. Let us introduce the quadratic elliptic operator

\[
Q(t, D_t; \xi_2) = g(f(\xi_2), \xi_2)D_t^2 + \frac{i}{2} \left( \partial_{x_1}^2 \langle q \rangle_2(x_1(\xi_2), f(\xi_2), \xi_2) \right) t^2,
\]

(7.84)

and let $e_k, \xi_2 \in L^2(\mathbb{R})$, $k \in \mathbb{N}$, be eigenfunctions of $Q(t, D_t; \xi_2)$ corresponding to the eigenvalues $\lambda_k(\xi_2)$ given in (6.43). Let also $f_k, \xi_2$ be eigenfunctions of the adjoint $Q^*(t, D_t; \xi_2)$, corresponding to the eigenvalues $\lambda_k(\xi_2)$. An application of Proposition 6.2 allows us to conclude that if (7.82) holds and the rescaled spectral parameter

\[
\frac{1}{\sqrt{\varepsilon} h} (z - p(f(\xi_2), \xi_2) - i\varepsilon\langle q \rangle_2(x_1(\xi_2), f(\xi_2), \xi_2))
\]

avoids a small but fixed neighborhood of the eigenvalues $\lambda_k(\xi_2)$ in the disc $|z| < C_0$, then $z$ is not in the spectrum of the operator in (7.83), with

\[
(P(x_1, hD_{x_1}, \varepsilon; h) - z)^{-1} = O\left( \frac{1}{\varepsilon^{1/2} h} \right) : L^2_{\theta_1} \to L^2_{\theta_1}.
\]

In view of the analysis above, we conclude that then $z \notin \text{Spec}(P_\varepsilon)$. It remains therefore for us to discuss the setup of the global Grushin problem for $P_\varepsilon$ when the spectral parameter $z \in \mathbb{C}$ is such that

\[
\frac{1}{\sqrt{\varepsilon} h} \left( z - p(f(\xi_2), \xi_2) - i\varepsilon\langle q \rangle_2(x_1(\xi_2), f(\xi_2), \xi_2) \right) \in \text{neigh}(\lambda_k(\xi_2), \mathbb{C}),
\]

(7.85)

for some $k \in \mathbb{N}$ with $k = O(1)$. Using the notation of Proposition 7.1, let us set

\[
R_+ : H(\Lambda) \to \mathbb{C}, \quad R_+ u = R_+ (\xi_2, k) (U \chi u, \varepsilon \xi_2)_{L^2_{\theta_2}},
\]

(7.86)
where \( e^{i\xi_2 x_2/\hbar} \) and \( R_+(\xi_2, k) \) has been introduced in (6.25), using the eigenfunctions \( f_{k, \xi_2} \). Define also
\[
R_- : C \to H(\Lambda), \quad R_- u_- = U^{-1} (R_-(\xi_2, k) u_- \otimes e^{i\xi_2 x_2/\hbar}).
\] (7.87)

Here \( R_-(\xi_2, k) \) has been introduced in (6.25), and \( U^{-1} \) is a microlocal inverse of \( U \). Arguing as in Section 6 of [7], we obtain that when (7.85) holds, the Grushin operator
\[
P(z) = \begin{pmatrix} (P_\varepsilon - z)/\varepsilon & R_- \\ R_+ & 0 \end{pmatrix} : H(\Lambda, m) \times C \to H(\Lambda) \times C
\]
is invertible, and the corresponding effective Hamiltonian \( E_{-+}(z) : C \to C \) vanishes precisely when \( z \) is of the form (6.42), (6.43). This completes the proof of Theorem 2.1.

8 Numerical illustrations of spectra

The purpose of this section is to present the results of numerical computations of the spectra of \( P_\varepsilon \), in the following situation, which is easily implemented: let us consider
\[
P_\varepsilon = -\hbar^2 \Delta_{x,y} + i\varepsilon q(x, y; hD_x, hD_y),
\]
\[
q(x, y; hD_x, hD_y) = q_0(x, y) + q_1(x, y)hD_x + q_2(x, y)hD_y,
\] (8.1)
on the torus \( M = T^2_{x,y} = (\mathbb{R}/2\pi\mathbb{Z})^2 \). Here \( q_0, q_1, q_2 \) are real trigonometric polynomials of degree \( \leq F \in \{1, 2, \ldots\} \). We shall consider the spectrum of this operator near the energy \( E_0 = 1 \).

The general assumptions (2.6), (2.7), (2.9) are fulfilled, the operator \( P_{\varepsilon=0} \) is selfadjoint, and the leading semiclassical symbol is of the form (2.10) with
\[
p = \xi^2 + \eta^2, \quad q(x, y; \xi, \eta) = q_0(x, y) + q_1(x, y)\xi + q_2(x, y)\eta.
\] (8.2)

We also have \( dp \neq 0 \) along \( p^{-1}(1) \cap T^*M \).

The Hamilton flow of \( p \) is completely integrable and we have the decomposition (2.11), for \( p^{-1}(1) \) rather than \( p^{-1}(0) \), where
\[
J = \bigcup_{(\xi, \eta) \in T} \Lambda_{\xi, \eta}, \quad \Lambda_{\xi, \eta} = T^2_{x,y} \times \{ (\xi, \eta) \}.
\]
We have
\[ q_\ell(x, y) = \sum_{|j|, |k| \leq F} \hat{q}_\ell(j, k) e^{i(jx + ky)}, \]
where the reality of \( q_\ell \) is equivalent to the property,
\[ \hat{q}_\ell(-j, -k) = \overline{\hat{q}_\ell(j, k)}. \]
Rather than taking some particular explicit choice of \( q \), we generate \( \hat{q}_\ell \) at random by choosing
\[ \hat{q}_\ell(j, k) = \frac{1}{2} \left( A_\ell(j, k) + \overline{A_\ell(-j, -k)} \right), \quad A_\ell(j, k) = e^{-\kappa |j-k|} \alpha_{j,k}, \]
where \( \alpha_{j,k} \sim \mathcal{N}(0,1) \) are independent Gaussian random variables. The parameter \( \kappa > 0 \) induces an off-diagonal exponential decay, corresponding to the assumption that \( q(x, y; \xi, \eta) \) is analytic in \( (x, y) \). Then,
\[ q(x, y; \xi, \eta) = \sum_{(j,k) \in [-F,F]^2} \hat{q}(j, k; \xi, \eta) e^{i(jx + ky)}, \]
where
\[ \hat{q}(j, k; \xi, \eta) = \hat{q}_0(j, k) + \hat{q}_1(j, k) \xi + \hat{q}_2(j, k) \eta. \]
Here and below it is understood that \([-F,F]\) is an interval in \( \mathbb{Z} \).

Let \( \Lambda_{\xi,\eta} \in J \) be a rational torus, so that \( (\xi, \eta) \in \mathbb{T} \) and \( \xi/\eta \in \mathbb{Q} \cup \{\infty\} \). The \( H_p \)-trajectories in \( \Lambda_{\xi,\eta} \) are of the form
\[ \gamma: \mathbb{R} \ni s \mapsto ((x_0, y_0) + 2s(\xi, \eta), (\xi, \eta)). \]
The restriction of \( q \) to such a trajectory is
\[ q(\gamma(s); \xi, \eta) = \sum_{(j,k) \in [-F,F]^2} \hat{q}(j, k; \xi, \eta) e^{i((x_0, y_0) + 2s(\xi, \eta) \cdot (j,k))}. \quad (8.3) \]
For the corresponding limit of the trajectory average,
\[ \langle q \rangle_\gamma = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} q(\gamma(s); \xi, \eta) ds, \]
the terms in \( (8.3) \) with \( (\xi, \eta) \cdot (j, k) \neq 0 \) give a zero contribution, and we get
\[ \langle q \rangle_\gamma = \sum_{(j,k) \in [-F,F]^2 \cap (\xi,\eta)^\perp} \hat{q}(j, k; \xi, \eta) e^{i((x_0, y_0) \cdot (j,k))}. \]
If we write \((\xi, \eta) = (-n, m)/(m, n)\) with \((n, m) \in \mathbb{Z}^2\), \(\gcd(n, m) = 1\), then \(\mathbb{Z}^2 \cap (\xi, \eta)^\perp = \mathbb{Z}(m, n)\) and the intersection of this set with \([-F, F]^2\) (viewed as a subset of \(\mathbb{Z}^2\)) is equal to

\[
\{\mu(m, n); \mu \in \mathbb{Z}, |\mu| \leq F/\max(|m|, |n|)\}.
\]

This gives,

\[
\langle q \rangle_\gamma = \sum_{[-F/\max(|m|, |n|)]} \hat{q}(\mu(m, n); \xi, \eta)e^{i\mu t(\gamma)},
\]

where \(t(\gamma) = (x_0, y_0) \cdot (m, n)\) varies in \([0, 2\pi]\) and can take any value in that interval and \([\cdot]\) denotes the integer part. It follows that

\[
Q_\infty(\Lambda_{\xi, \eta}) = \left\{ \sum_{[-F/\max(|m|, |n|)]} \hat{q}(\mu(m, n); \xi, \eta)e^{i\mu t}; 0 \leq t \leq 2\pi \right\}.
\]

When \(\max(|m|, |n|) > F\) this interval reduces to the torus average \(\langle q \rangle_{\Lambda_{\xi, \eta}} = \langle q \rangle_{\Lambda_{\xi, \eta}}\), so we get non-trivial intervals only for the finitely many values \((m, n) \in [-F, F]^2\) with \(\gcd(m, n) = 1\).

We have written MatLab programs for the production of \(q\) and for the calculation of \(\langle q \rangle_\Lambda, Q_\infty(\Lambda)\), as well as the supremum and infimum of \(q\) over each torus in \(J\). For the graphics, we parametrize \(J\) by \(\arg(\xi + i\eta)\) and the figure below shows:

- The torus average \(\langle q \rangle_\Lambda\),
- The torus max and min of \(q\),
- \(Q_\infty(\Lambda)\) for each relevant rational torus.
By running the simulation several times we get a series of figures where quite a few exhibit the features above. In order to have a numerical illustration of the main result of this work it is important that some of the vertical segments (corresponding to $Q_\infty(\Lambda)$ for rational tori) reach above the supremum or below the infimum of the curve of torus averages. A larger $F$ will produce a richer picture with more vertical segments, but it will also complicate the numerical calculations of the eigenvalues, so we settled for $F = 2$ as a reasonable choice. We also found that $\kappa = 2$ produces some – not too many – visible vertical segments.

Once an interesting $q$ has been selected, we compute the spectrum numerically by working on the level of Fourier coefficients. Thus, if we are interested in the eigenvalues with real parts in $[E_1, E_2]$, where $E_1 < E_2$ are close to 1, we work with Fourier modes $e^{i(jx + ky)}$ for $(j, k)$ in the set $\mathcal{E}$ of $(j, k) \in \mathbb{Z}^2$, satisfying $(hj)^2 + (hk)^2 \in [E_1, E_2]$, i.e.

$$ |(j, k)| \in [\sqrt{E_1}/h, \sqrt{E_2}/h]. $$

The number $\#\mathcal{E}$ of such modes is $\approx \pi(E_2 - E_1)/h^2$ and we ask MatLab to compute
the spectrum of the $\mathcal{E} \times \mathcal{E}$-matrix $\mathcal{A}_\varepsilon = \left( a_\varepsilon(j, k; \tilde{j}, \tilde{k}) \right)_{(j, k), (\tilde{j}, \tilde{k}) \in \mathcal{E}}$, given by

$$a_\varepsilon(j, k; \tilde{j}, \tilde{k}) = \hbar^2 (j^2 + k^2) \delta_{(j, k), (\tilde{j}, \tilde{k})}$$

$$+ i \varepsilon \left( \tilde{q}_0 (j - \tilde{j}, k - \tilde{k}) + \tilde{q}_1 (j - \tilde{j}, k - \tilde{k}) h \tilde{j} + \tilde{q}_2 (j - \tilde{j}, k - \tilde{k}) h \tilde{k} \right).$$

We cannot let $\#\mathcal{E}$ be larger than a few thousand and still we would like $h$ to be small and the energy shell $[E_1, E_2]$ thick enough so that the eigenvalues with real part inside are not influenced by boundary effects. In the simulations below we have chosen the same $q$ as the one in the figure above and we settled for $E_1 = 0.85$, $E_2 = 1$, $h = 1/100$, leading to $\#\mathcal{E} \approx 5000$. Since the spectra are of width $\varepsilon$, we rescale the imaginary axis and represent graphically the set of $(\text{Re } z, \text{Im } z / \varepsilon)$ for $z$ in the spectrum of $P_\varepsilon$. We let $\varepsilon$ take the values $h/2$, $h$, $2h$, $4h$, $8h$, $16h$, in agreement with Theorem 2.1.
Spectrum of \( p + i\epsilon q \), \( \epsilon = 0.01 \), \( h = 0.01 \), \( \kappa = 2 \), \( F = 2 \)
Spectrum of $p + i\epsilon q$, $\epsilon = 0.02$, $h = 0.01$, $\kappa = 2$, $F = 2$
Spectrum of $p + i\epsilon q$, $\epsilon=0.04$, $\hbar=0.01$, $\kappa=2$, $F=2$
Spectrum of $p + i\epsilon q$, $\epsilon = 0.08$, $h = 0.01$, $\kappa = 2$, $F = 2$
These eigenvalues form a kind of a centipede with legs sticking out from the main body. The majority of the eigenvalues are in the body whose position corresponds nicely to the range of the curve of torus averages on the first picture. The legs reach out to the supremum of the highest and the infimum of the lowest vertical segments corresponding to $Q_\infty(\Lambda)$ for rational tori $\Lambda$.

Undoing the scaling of $\text{Im } z$, the inclination of the legs should theoretically be close to 45 degrees and by measuring this for one of the legs on one of the figures we found some (but not excellent) agreement.

The main result of this work, Theorem 2.1, describes the individual eigenvalues near the extremities of the legs in terms of rational tori. A mathematical treatment of the eigenvalues further inside seems more difficult because of the pseudospectral effects that are likely to get stronger there.

By staring at the pictures directly from the pdf file and creating a movie by switching the pages, we see that most of the (rescaled) eigenvalues remain fixed while those in the legs and some others move. The fixed ones probably correspond to irrational tori and the moving ones to tori that are rational.
A Subelliptic estimates for Schrödinger type operators

The purpose of this appendix is to establish suitable resolvent estimates for some non-selfadjoint operators of Schrödinger type, instrumental in the pseudospectral analysis of Section 7. While in the considerations of Section 7, we are concerned with operators on the one-dimensional torus, it will be convenient to analyze the case of $\mathbb{R}$ first. See also [15].

Let

$$P_0 = g(hD_x)(hD_x)^2 + iV(x), \quad V \in C^\infty(\mathbb{R}; \mathbb{R}).$$

(A.1)

Assume that the function $g \in C^\infty(\mathbb{R}; \mathbb{R})$ is such that

$$g - 1 \in C_0^\infty(\mathbb{R})$$

(A.2)

with

$$g \geq 1, \quad |\xi g'(\xi)| \ll 1.$$  \hspace{1cm} (A.3)

We may notice that the conditions (A.3) are invariant under the scaling $g(\xi) \rightarrow g(\lambda \xi), \lambda > 0.$ We also assume that the potential $V$ is such that

$$0 \leq V, \quad \partial^j_x V \in L^\infty(\mathbb{R}), \quad j \geq 2,$$

(A.4)

and let us make the ellipticity assumption,

$$V(x) \geq \frac{x^2}{C}, \quad |x| \geq C,$$

(A.5)

for some constant $C > 0.$ The semiclassical symbol of $P_0$, $p_0(x, \xi) = g(\xi)\xi^2 + iV(x)$ satisfies $p_0 \in S(m),$ where $m(x, \xi) = 1 + x^2 + \xi^2,$ and when equipped with the domain $H(m)$, the natural Sobolev space associated to the order function $m$, the operator $P_0$ becomes closed densely defined on $L^2(\mathbb{R}).$ The spectrum of $P_0$ is discrete and we notice that

$$\text{Spec } (P_0) \subset p_0(\mathbb{R}^2) = \{ z \in \mathbb{C}; \arg z \in [0, \pi/2] \}.$$  \hspace{1cm} (A.6)

Let us make the basic assumption that

$$V^{-1}(0) = \{ 0 \} \subset \mathbb{R}$$

(A.7)

and that

$$V''(0) > 0.$$  \hspace{1cm} (A.8)
We are interested in estimates for the resolvent of $P_0$,

$$(P_0 - z)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

when the spectral parameter $z \in \mathbb{C}$ is such that $0 \leq \text{Im } z \leq O(h)$ and $|z| \gg h$. When establishing those, we shall combine some of the results and techniques of [4] and [6].

In what follows, rather than working with $P_0$, it will be convenient to consider the operator

$$P = -iP_0 = V(x) - ig(hD_x)(hD_x)^2$$

with the symbol $p = p_1 + ip_2$, where

$$p_1(x, \xi) = V(x) = V_0(x) + O(x^3), \quad V_0(x) = \frac{1}{2} V''(0)x^2 > 0,$$

and

$$p_2(x, \xi) = -g(\xi)\xi^2.$$

It follows from (A.3) that

$$|\partial_\xi p_2| \sim |\xi|,$$

and therefore we obtain the fundamental property

$$V_0(x) + \frac{H^2}{p^2_2}V_0(x, \xi) \sim |(x, \xi)|^2, \quad (x, \xi) \in \mathbb{R}^2.$$  

(A.10)

Following [4], let us set, writing $X = (x, \xi) \in \mathbb{R}^2$,

$$G_0(X; h) = h^{2/3} \frac{H^2}{p^2_2} V_0 \left( M \frac{V_0(x)}{|h|X|^{2/3}} \right), \quad |X| \geq h^{1/2}. \quad (A.11)$$

Here $M \geq 1$ is a constant to be taken large enough and $\psi \in C^\infty_0(\mathbb{R}; [0, 1])$ is such that $\text{supp } (\psi) \subset (-2, 2)$ and $\psi = 1$ on $[-1, 1]$. It is then straightforward to verify that in the region where $|X| \geq h^{1/2}$, we have

$$G_0 = O(h), \quad H G_0 = O(1) \frac{h^{2/3}}{|X|^{1/3}} = O(h^{1/2}), \quad (A.12)$$

and

$$\partial^2 G_0 = O(1) \left( \frac{h^{2/3}}{|X|^{4/3}} + \frac{h^{1/3}}{|X|^{2/3}} + \frac{h^{2/3}}{|X|^{1/3}} \right) = O(1). \quad (A.13)$$
Indeed, the validity of (A.12) and (A.13) follows easily once we observe that in the region where $0 \leq MV_0(x) \leq 2(h|X|)^{2/3}$, and $|X| \geq h^{1/2}$, we have

$$H_{p_2}V_0 = \mathcal{O}(|X| h^{1/3} |X|^{1/3}), \quad \nabla(H_{p_2}V_0) = \mathcal{O}(|X|), \quad \nabla^2(H_{p_2}V_0) = \mathcal{O}(1 + |X|),$$

and

$$\nabla^j \left( \frac{V_0(x)}{(h|X|)^{2/3}} \right) = \mathcal{O}(1) \frac{1}{h^{j/3} |X|^{j/3}}, \quad j = 1, 2.$$

Still working in the region $|X| \geq h^{1/2}$ and following [4] closely, let us obtain a lower bound for the function $V_0 + \varepsilon_0 H_{p_2}G_0$, where $\varepsilon_0 > 0$ is a constant to be chosen. We have, in view of (A.12),

$$H_{p_2}G_0 = \mathcal{O}(h^{2/3} |X|^{2/3}),$$

and therefore, in the region where $MV_0 \geq h^{2/3} |X|^{2/3}$, we get

$$V_0 + \varepsilon_0 H_{p_2}G_0 \geq \left( \frac{1}{M} - \mathcal{O}(\varepsilon_0) \right) h^{2/3} |X|^{2/3} \geq \frac{1}{\mathcal{O}(1)M} h^{2/3} |X|^{2/3},$$

if we choose $\varepsilon_0 > 0$ small enough. In the region where $MV_0 < h^{2/3} |X|^{2/3}$, we have

$$G_0 = h^{2/3} \frac{H_{p_2}V_0}{|X|^{1/3}},$$

and therefore

$$V_0 + \varepsilon_0 H_{p_2}G_0 = V_0 + \varepsilon_0 h^{2/3} |X|^{-4/3} H_{p_2}^2 V_0 + R,$$

where

$$R = \varepsilon_0 h^{2/3}(H_{p_2}V_0)H_{p_2} |X|^{-4/3} = \mathcal{O} \left( \frac{\varepsilon_0 h}{M^{1/2}} \right) = \mathcal{O} \left( \frac{\varepsilon_0}{M^{1/2}} \right) h^{2/3} |X|^{2/3}.$$

Using (A.10), we see therefore that

$$V_0 + \varepsilon_0 H_{p_2}G_0 \geq \varepsilon_0 \frac{h^{2/3} |X|^{2/3}}{\mathcal{O}(1)} - \mathcal{O} \left( \frac{\varepsilon_0}{M^{1/2}} \right) h^{2/3} |X|^{2/3} \geq \varepsilon_0 \frac{h^{2/3} |X|^{2/3}}{\mathcal{O}(1)},$$

provided that we take $M$ sufficiently large but fixed. It follows that in the entire region where $|X| \geq h^{1/2}$, we get

$$V_0 + \varepsilon_0 H_{p_2}G_0 \geq \frac{h^{2/3} |X|^{2/3}}{\mathcal{O}(1)}. \quad (A.14)$$
We shall now extend the definition of $G_0$ to all of $\mathbb{R}^2$, and following [4], let us set
\[ G(X; h) = \left( 1 - \chi \left( \frac{X}{h^{1/2}} \right) \right) G_0(X; h). \] (A.15)
Here $\chi \in C_0^\infty(\mathbb{R}^2; [0, 1])$ is such that $\chi = 1$ when $|X| \leq 1$. It follows from (A.12) and (A.13) that
\[ G = \mathcal{O}(h), \quad H_G = \mathcal{O}(h^{1/2}), \quad \partial^2 G = \mathcal{O}(1). \]
Furthermore, using (A.14) we immediately check that on all of $\mathbb{R}^2$, we have
\[ V_0 + \varepsilon_0 H_{p_2} G \geq \frac{h^{2/3} |X|^{2/3}}{\mathcal{O}(1)} - \mathcal{O}(h). \] (A.16)
We may therefore summarize the discussion above by stating that there exist constants $\varepsilon_0 > 0$ and $c_0 > 0$, such that we have for all $h > 0$ sufficiently small,
\[ V_0(x) + \varepsilon_0 H_{p_2} G(X) + c_0 h \geq \frac{h^{2/3} |X|^{2/3}}{\mathcal{O}(1)}, \quad X = (x, \xi) \in \mathbb{R}^2. \] (A.17)
The estimate (A.17) is analogous to the estimate (4.26) of Section 4 of [6], if we take $k_0 = 1$ there. Taking (A.17) as the starting point and arguing exactly as in that work, we find that everything works without any change, provided that the spectral parameter $z \in \mathbb{C}$ is such that for some fixed $C_0 > 1$, we have
\[ \text{Re} z \leq \mathcal{O}(1) h^{2/3} |z|^{1/3}, \quad C h \leq |z| \leq C_0. \] (A.18)
Here $C \gg 1$ is a constant large enough and the implicit constant in (A.18) does not depend on $C$. We therefore obtain the a priori estimate
\[ h^{2/3} |z|^{1/3} \| u \|_{L^2} \leq \mathcal{O}(1) \| (P - z) u \|_{L^2}, \quad u \in \mathcal{S}(\mathbb{R}^n). \] (A.19)
for $z$ satisfying (A.18).

It therefore remains to discuss the case when $z \in \mathbb{C}$ is such that

$$\text{Re } z \leq O(1) h^{2/3} |z|^{1/3}, \quad |z| \geq C_0. \quad (A.20)$$

Continuing to follow the arguments of Section 4 in [6], we obtain from the equation (4.34) there that there exist positive constants $c_1, c_2$ such that for $z \in \mathbb{C}$ satisfying (A.20), we have

$$|| (P - z)u ||_{L^2} || u ||_{L^2} + c_1 h^{2/3} |z|^{1/3} \left( \varphi \left( \frac{|X|^2}{|z|} \right)^w u, u \right) \geq c_2 h^{2/3} |z|^{1/3} || u ||_{L^2}^2. \quad (A.21)$$

Here $\varphi \in C_0^\infty(\mathbb{R}^{2m}; [0, 1])$ is a cut-off near 0 such that

$$|p(X) - z| \geq \frac{|z|}{2}, \quad (A.22)$$

on the support of $\varphi(|X|^2 / |z|)$. The spectral parameter $z$ in (A.20) can be arbitrarily large and when estimating the scalar product in the left hand side of (A.21), we can apply Lemma 8.2 in [4], exactly as it stands, to conclude that

$$\left( \varphi \left( \frac{|X|^2}{|z|} \right)^w u, u \right) \leq \frac{O(1)}{|z|^2} || (P - z)u ||_{L^2}^2 + O(1) h|| u ||_{L^2}^2. \quad (A.23)$$

Indeed, it is easily seen that the proof of Lemma 8.2 of [4] applies in the present situation, using the ellipticity property (A.22) and the fact that the symbol $p$ satisfies

$$|p(X)| \leq O(1) |X|^2, \quad X \in \mathbb{R}^2, \quad \partial^\alpha p \in L^\infty(\mathbb{R}^2), \quad |\alpha| \geq 2. \quad (A.24)$$

Combining (A.21) and (A.23), we get

$$Z || u ||_{L^2}^2 \leq O(1) \frac{Z}{|z|^2} || (P - z)u ||_{L^2}^2 + O(1)|| (P - z)u ||_{L^2} || u ||_{L^2}. \quad (A.25)$$

Here we have written $Z = h^{2/3} |z|^{1/3}$ for brevity. It follows that

$$Z || u ||_{L^2}^2 \leq O(1) \left( \frac{1}{Z} + \frac{Z}{|z|^2} \right) || (P - z)u ||_{L^2}^2, \quad (A.26)$$

and using also that $Z \leq |z|$, we get

$$h^{2/3} |z|^{1/3} || u ||_{L^2} \leq O(1)|| (P - z)u ||_{L^2}. \quad (A.27)$$

We summarize the discussion above in the following result.
Theorem A.1 Let $P_0 = g(hD_x)(hD_x)^2 + iV(x)$ be such that (A.2), (A.3), (A.4), (A.5), (A.7), (A.8) hold. There exist constants $c_0 > 0$ and $c_1 > 0$ such that for every $C > 1$ large enough, we have for
\[ \text{Im } z \leq c_1 h^{2/3} |z|^{1/3}, \quad |z| \geq Ch, \] (A.28)
the following estimate,
\[ h^{2/3} |z|^{1/3} \| u \|_{L^2} \leq c_1 \| (P_0 - z)u \|_{L^2}, \quad u \in H(m). \] (A.29)
It follows that when $z$ satisfies (A.28) then $z$ is in the resolvent set of $P_0$ and we get the resolvent estimate
\[ (P_0 - z)^{-1} = \mathcal{O} \left( h^{-2/3} |z|^{-1/3} \right) : L^2(\mathbb{R}) \to L^2(\mathbb{R}). \] (A.30)

Remark. The discussion above and the result of Theorem A.1 extends to the case of operators on $\mathbb{R}^n$.

In what follows, the result of Theorem A.1 will only be applied in the case when $\text{Im } z = \mathcal{O}(h)$. Furthermore, in the considerations in Section 7, we are concerned with operators on the one-dimensional torus $T = \mathbb{R}/2\pi \mathbb{Z}$, and our next task is therefore to adapt Theorem A.1 to this setting. Let us consider therefore
\[ P = g(hD_x)(hD_x)^2 + iV(x), \] (A.31)
where $g \in C^\infty(\mathbb{R}; \mathbb{R})$ satisfies (A.2), (A.3), and let $0 \leq V \in C^\infty(T)$ be such that $V^{-1}(0) = \{x_0\}$, with $V''(x_0) > 0$. We may write
\[ P = (hD_x)^2 + \varphi(hD_x) + iV(x), \quad \varphi \in C_0^\infty(\mathbb{R}; \mathbb{R}). \] (A.32)
Let $\chi \in C^\infty(T^n; [0, 1])$ be supported in a small neighborhood of $x_0$, and such that $\chi = 1$ near $x_0$. On the support of $\chi$, the result of Theorem A.1 can be applied and we conclude that
\[ h^{2/3} |z|^{1/3} \| \chi u \|_{L^2} \leq \mathcal{O}(1)\| (P - z)\chi u \|_{L^2} + \mathcal{O}(h^\infty)\| u \|_{L^2}, \quad u \in C^\infty(T^n), \] (A.33)
provided that
\[ |z| \geq Ch, \quad \text{Im } z \leq \mathcal{O}(h). \] (A.34)

Using that on the support of $1 - \chi$, the potential $V$ is bounded from below and $\text{Im } z = \mathcal{O}(h)$, we see that for all $h > 0$ small enough, we have
\[ \text{Im}((P - z)(1 - \chi)u, (1 - \chi)u)_{L^2} \geq \frac{1}{\mathcal{O}(1)}\| (1 - \chi)u \|_{L^2}^2, \] (A.35)
and therefore,
\[ ||(1 - \chi)u||_{L^2} \leq C||(P - z)(1 - \chi)u||_{L^2}. \]  
(A.36)

Combining the estimates (A.33) and (A.36), we see that for all \( h > 0 \) small enough,
\[ Z||\chi u||_{L^2} + ||(1 - \chi)u||_{L^2} \leq \mathcal{O}(1)||P - z||_2 u||_{L^2} + \mathcal{O}(1)||[P, \chi]u||_{L^2}. \]  
(A.37)

Here we continue to write \( Z = h^{2/3} |z|^{1/3} \) and we notice that \( h \ll Z \). We would like to estimate the commutator term in the right hand side of (A.37), and to that end we write, using (A.32),
\[ \mathcal{O}(1)||[P, \chi]u||_{L^2} \leq \mathcal{O}(h)||u||_{L^2} + \mathcal{O}(h)||hu'||_{L^2}. \]  
(A.38)

The first term in the right hand side of (A.38) can be absorbed into the left hand side of (A.37), and we only have to estimate \( \mathcal{O}(h)||hu'||_{L^2} \). Now
\[ (\varphi(hD_x)u, u)_{L^2} + ||hu'||^2_{L^2} = \text{Re} ((P - z)u, u)_{L^2} + \text{Re} z||u||^2_{L^2}, \]
and therefore,
\[ ||hu'||_{L^2} \leq ||(P - z)u||_{L^2}^{1/2} ||u||_{L^2}^{1/2} + |z|^{1/2} ||u||_{L^2} + \mathcal{O}(1)||u||_{L^2}. \]  
(A.39)

We obtain, combining (A.37), (A.38), and (A.39),
\[ Z||\chi u||_{L^2} + ||(1 - \chi)u||_{L^2} \leq \mathcal{O}(1)||P - z||_2 u||_{L^2} + \mathcal{O}(h)|z|^{1/2} ||u||_{L^2} + \mathcal{O}(h)||P - z||_2 ||u||_{L^2}^{1/2} \]  
(A.40)
so that
\[ Z||\chi u||_{L^2} + ||(1 - \chi)u||_{L^2} \leq \mathcal{O}(1)||P - z||_2 u||_{L^2} + \mathcal{O}(Z^{3/2})||u||_{L^2}. \]  
(A.41)

Assuming that
\[ Z = h^{2/3} |z|^{1/3} \ll 1, \]
we may absorb the second term in the right hand side of (A.41) into the left hand side, obtaining that
\[ h^{2/3} |z|^{1/3} ||u||_{L^2} \leq \mathcal{O}(1)||(P - z)u||_{L^2}. \]

We may summarize the discussion above in the following proposition.
Proposition A.2 Let $P = g(hD_x)(hD_x)^2 + iV(x)$ be such that $g - 1 \in C_0^\infty(\mathbb{R}; \mathbb{R})$ satisfies $g \geq 1$, $|\xi g'(\xi)| \ll 1$. Assume furthermore that $0 \leq V \in C^\infty(\mathbb{T})$, with $V^{-1}(0) = \{x_0\}$ and $V''(x_0) > 0$. Then for every $C > 1$ large enough, we have for

$$\text{Im } z = \mathcal{O}(h), \ |z| \geq Ch,$$

satisfying

$$h |z|^{1/2} \ll 1,$$  \hspace{1cm} (A.42)

the following estimate

$$h^{2/3} |z|^{1/3} \|u\|_{L^2} \leq \mathcal{O}(1)\| (P - z)u \|_{L^2}.$$  \hspace{1cm} (A.43)

The a priori estimate (A.43) is equivalent to the corresponding estimate for the resolvent of $P$, which provides a resolvent bound in the model case, fundamental for the analysis in Section 7. Now the operators that one encounters there are somewhat more general than the Schrödinger type operator in (A.31), in that the potential $V(x)$ should be replaced by a more general $h$-pseudodifferential operator, which furthermore is multiplied by a small coupling constant. We shall now proceed to analyze this more general case, essentially by reducing it to the model situation treated above.

Let us first consider the following operator on $\mathbb{R}$,

$$P_\varepsilon(x, hD_x) = g(hD_x)(hD_x)^2 + i\varepsilon \tilde{V}^w(x, hD_x),$$  \hspace{1cm} (A.44)

where $g$ satisfies (A.2), (A.3), and following (7.1), we assume that

$$h^2 \ll \varepsilon \ll h^{4/5}.$$  \hspace{1cm} (A.45)

The function $\tilde{V} \in C^\infty(\mathbb{R}^2)$ in (A.44) is of the form

$$\tilde{V}(x, \xi) = V(x) + k(x, \xi)\phi \left( \frac{\xi}{\varepsilon^\delta} \right), \quad \delta \in (0, 1/2),$$  \hspace{1cm} (A.46)

where $V$ is assumed to satisfy (A.4), (A.5), (A.7), and (A.8), and $\varphi \in C_0^\infty(\mathbb{R}^n)$ is a standard cutoff function near $\xi = 0$. The function $k$ is such that

$$\partial^\alpha k \in L^\infty(\mathbb{R}^2), \quad \alpha \in \mathbb{N}^2, \quad k(x, 0) = 0.$$  \hspace{1cm} (A.47)
We would like to extend Theorem A.1 to the operator \( P_\varepsilon(x,hD_x) \), and to that end we shall simply inspect the arguments above. Writing
\[
\frac{1}{i\varepsilon} P_\varepsilon(x,hD_x) = -ig(\sqrt{\varepsilon}hD_x)(\tilde{h}D_x)^2 + \tilde{V}''(x,\sqrt{\varepsilon}hD_x), \quad \tilde{h} = \frac{h}{\sqrt{\varepsilon}} \ll 1, \tag{A.48}
\]
we shall view \((1/i\varepsilon)P_\varepsilon\) as an \(\tilde{h}\)-pseudodifferential operator with the symbol
\[
p(x,\xi) = p_1 + ip_2, \tag{A.49}
\]
where
\[
p_1(x,\xi) = \tilde{V}(x,\sqrt{\varepsilon}\xi), \quad p_2(x,\xi) = -g(\sqrt{\varepsilon}\xi)\xi^2. \tag{A.50}
\]
Writing
\[
p_1(x,0) = V(x) = V_0(x) + \mathcal{O}(x^3), \quad V_0(x) = \frac{1}{2}V''(0)x^2,
\]
we see that uniformly in \(\varepsilon\), we have
\[
V_0 + H_{p_2}^2 V_0 \sim |X|^2, \quad X \in \mathbb{R}^2. \tag{A.51}
\]
Here we also notice that \(\partial^\alpha p \in L^\infty(\mathbb{R}^2)\) for all \(\alpha \in \mathbb{N}^2\) with \(|\alpha| \geq 2\), uniformly in \(\varepsilon\). Arguing as in (A.11), (A.16), (A.17), we conclude that there exists a real-valued weight function \(G \in C^\infty(\mathbb{R}^2)\) with
\[
G(X) = \mathcal{O}(\tilde{h}),
\]
such that for some constants \(\delta_1 > 0\) and \(c_1 > 0\), we have for \(0 < \tilde{h} \) small enough,
\[
\text{Re} p(x,0) + \delta_1 H_{\text{Im}p} G(X) + c_1\tilde{h} \geq \frac{1}{C}\tilde{h}^{2/3} |X|^{2/3}, \quad X = (x,\xi) \in \mathbb{R}^2. \tag{A.52}
\]
Using (A.46) and (A.52), we obtain that
\[
\text{Re} p(X) + \delta_1 H_{\text{Im}p} G(X) + c_1\tilde{h} \geq \frac{1}{C}\tilde{h}^{2/3} |X|^{2/3} - 1_{\sqrt{\varepsilon}|\xi| \leq \mathcal{O}(\varepsilon^{1/2})} \mathcal{O}(\sqrt{\varepsilon}|\xi|). \tag{A.53}
\]
Here we have used that
\[
k(x,\xi) = \mathcal{O}(|\xi|). \tag{A.54}
\]
When estimating the right hand side in (A.53), we notice that
\[
\frac{1}{2C}\tilde{h}^{2/3} |X|^{2/3} - 1_{\sqrt{\varepsilon}|\xi| \leq \mathcal{O}(\varepsilon^{1/2})} \mathcal{O}(\sqrt{\varepsilon}|\xi|) \geq \frac{1}{2C}|\xi|^{2/3} \left(\tilde{h}^{2/3} - 1_{\sqrt{\varepsilon}|\xi| \leq \mathcal{O}(\varepsilon^{1/2})} \mathcal{O}(\sqrt{\varepsilon}|\xi|^{1/3})\right)
\geq \frac{1}{2C}|\xi|^{2/3} \left(\tilde{h}^{2/3} - \mathcal{O}(\sqrt{\varepsilon}\varepsilon^{5/3-1/6})\right) \geq 0, \tag{A.55}
\]
provided that
\[ \varepsilon^{5/3+1/2-1/6} \ll \tilde{h}^{2/3} = \frac{h^{2/3}}{\varepsilon^{1/3}}. \]  
(A.56)
The latter condition is equivalent to
\[ \varepsilon^{1+\frac{2}{3}} \ll h. \]  
(A.57)
Assuming that (A.57) holds, we conclude that
\[ \text{Re} \ p(X) + \delta_1 H_{1 \text{mp}} G(X) + c_1 \tilde{h} \geq \frac{1}{O(1)} \tilde{h}^{2/3} |X|^{2/3}, \quad X \in \mathbb{R}^2. \]  
(A.58)
In order to be able to apply the general arguments of \cite{6} and \cite{4} to the operator \( P_\varepsilon/i\varepsilon \), similarly to the discussion leading to Theorem A.1 above, we should also observe that for \( |z| \gg \tilde{h} \), we have in the region where
\[ |X|^2 \leq \frac{|z|}{C}, \]
that the ellipticity condition
\[ |p(X) - z| \geq \frac{|z|}{C_1}, \]
holds, uniformly in \( \varepsilon \). Indeed, this follows from (A.46), (A.49), (A.50), (A.54), as well as the fact that \( \varepsilon \ll \tilde{h} \ll |z| \). It is then straightforward to check that the arguments in the beginning of the appendix apply and we obtain the following result.

**Theorem A.3** Let \( P_\varepsilon = g(hD_x)(hD_x)^2 + i\varepsilon \tilde{V}w(x,hD_x) \), where \( h^2 \ll \varepsilon \ll h^{4/5} \). Assume that \( g \in C^\infty(\mathbb{R};\mathbb{R}) \) satisfies (A.2), (A.3), \( \tilde{V} \) is of the form (A.46), (A.47), and that (A.4), (A.5), (A.7), (A.8) hold. Assume that
\[ \varepsilon^{1+\frac{2}{3}} \ll h. \]
Then we have, writing \( \tilde{h} = h/\sqrt{\varepsilon} \),
\[ \left( \frac{P_\varepsilon}{\varepsilon} - z \right)^{-1} = O \left( \tilde{h}^{-2/3} |z|^{-1/3} \right) : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \]  
(A.59)
provided that \( |z| \geq C\tilde{h} \) for \( C > 1 \) sufficiently large, and
\[ \text{Im} \ z \leq O(\tilde{h}). \]
Repeating the arguments leading to Proposition A.2, with Theorem A.1 replaced by Theorem A.3 and with an estimate of the form (A.35) obtained by an application of Gårding’s inequality, we next obtain an adaptation of Theorem A.3 to the setting of the torus.

**Proposition A.4** Let
\[ P_\varepsilon = g(hD_x)(hD_x)^2 + i\varepsilon \tilde{V}^w(x,hD_x), \]
where \( h^2 \ll \varepsilon \ll h^{4/5} \), and \( g \in C^\infty(\mathbb{R}; \mathbb{R}) \) is such that \( g - 1 \in C_0^\infty(\mathbb{R}) \) with
\[ g \geq 1, \quad |\xi g'(\xi)| \ll 1. \]

Assume that \( \tilde{V} \) is of the form
\[ \tilde{V}(x,\xi) = V(x) + k(x,\xi)\varphi \left( \frac{\xi}{\varepsilon^\delta} \right), \quad \delta \in (0, 1/2). \]

Here \( 0 \leq V \in C^\infty(\mathbf{T}), \quad V^{-1}(0) = \{x_0\}, \quad V''(x_0) > 0, \quad k \in S(T^*\mathbf{T}, 1), \quad k(x,0) = 0, \) and \( \varphi \in C_0^\infty(\mathbb{R}) \). Assume that \( \varepsilon^{1+\frac{3}{2}} \ll h \) and let us set
\[ \tilde{h} = \frac{h}{\sqrt{\varepsilon}} \ll 1. \]

Let \( z \in \mathbf{C} \) be such that \( \text{Im} z = \mathcal{O}(\tilde{h}), \quad |z| \geq C\tilde{h}, \) for \( C > 1 \) sufficiently large, and
\[ \tilde{h}|z|^{1/2} \ll 1. \]

Then we have
\[
\left( \frac{P_\varepsilon}{\varepsilon} - z \right)^{-1} = \mathcal{O}\left( \frac{h^{-2/3}|z|^{-1/3}}{\tilde{h}} \right) : L^2(\mathbf{T}) \to L^2(\mathbf{T}). \tag{A.60}
\]

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