On the number of waves arriving at the vertices of Platonic solids

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Abstract

The problem of counting the number of waves arriving at the vertex of a polyhedron is motivated by study of wave propagation on the manifolds with flat metric and finitely many singular points. In the article it was solved for the case of Platonic solids using three nontrivial results from number theory. This growth turns out to be subexponential.

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Let us consider a polyhedron. At the initial moment of time, a concentric wave emerges from the selected vertex $S$ along the surface of the polyhedron with a unit velocity. When it comes to any vertex, a new concentric wave appears. The problem is to understand how many waves will arrive at the selected vertex-recorder $A$ at a given time $t$ (see [4] for details of the same dynamical system on a hybrid Riemannian manifolds). Earlier questions that are related to the properties of the Laplace operator were considered (see [12], [13] for details and references).

Obviously the motion of waves on such a surface is determined by geodesics that emerges from the vertex $S$ and arrive at the vertex $A$. We have a two-dimensional flat surface with a finite number of conical singularities and we are considering geodesics joining two points. Such objects were studied in the well-known papers [1], [7]. Recent results of [2], [3], [10] allow us to solve this problem for regular tetrahedron, octahedron, icosahedron and cube. The number of waves in the vertex is determined by the number of broken geodesic lines, the length of which does not exceed the given parameter. If we know the distribution of components of these broken lines, then using the results from analytical number theory (theorems on the distribution of abstract primes, see [5], [6] for details), we can find the growth of the number of waves.

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1. Geodesics on cube and the regular tetrahedron/octahedron/icosahedron

It is well known that the geodesics between the vertices of cube or regular tetrahedron correspond to segments between lattice points in the plane for square or triangular lattice.

Let us call a point \( p \) of square (respectively triangular) lattice irreducible if the segment from the origin to \( p \) does not contain any other lattice point. For square lattice the equivalent condition is that the coordinates of \( p \) are coprime integers.

**Proposition 1.1.** Fix some vertex \( v \) of the tetrahedron. Then all geodesics from \( v \) to the vertices of the tetrahedron (including \( v \) itself) are in one-to-one correspondence with the set of irreducible points of triangular lattice inside a sector of angle \( \frac{4\pi}{3} \).

Fix some vertex \( v \) of the cube. Then all geodesics from \( v \) to the vertices of the cube (including \( v \) itself) are in one-to-one correspondence with the set of irreducible points of square lattice inside a sector of angle \( \frac{3\pi}{2} \).

Thus we know the asymptotics as \( l \to \infty \) of the number of geodesics from one vertex to others with length less than \( l \) – it coincides with the asymptotics of irreducible lattice points in the sector of fixed angle inside the circle of radius \( l \), which is \( \frac{3\pi}{4} l^2 \) for cube and \( \frac{2\pi}{3} l^2 \) for tetrahedron. The next natural question here is to understand how much of these geodesics connect \( v \) to each of the vertices individually rather than all together. This question is answered by [2] for tetrahedron and cube and by [3] for all platonic solids. It is shown that each of the vertices gets asymptotically a fixed fraction of all the geodesics from \( v \).

But we need to find asymptotics of the number of lengths of geodesics, not of the number of geodesics. These can be different since different geodesics may have equal length, e.g. points with coordinates \((11, 3)\) and \((9, 7)\) on a square lattice have equal distance \( \sqrt{130} \) from the origin. Therefore, our question is to find the set of all possible norms of irreducible lattice vectors. Here again arises a question how much of these norms correspond to geodesics from \( v \) to some individual vertex, but we do not know the answer. We will only be interested in the asymptotics of these norms for geodesics to any vertex of our polyhedron.

Such norms are exactly the numbers of the form \( \sqrt{x^2 + y^2} \) for square lattice and of the form \( \sqrt{x^2 + xy + y^2} \) for triangular one with integer \( x, y \). Denote these sets of numbers by \( A_\triangle \) and \( A_\Box \).

**Proposition 1.2.** The set of lengths of geodesics between vertices of cube is exactly the set \( A_\Box \) of lengths of vectors from square lattice. The set of lengths of geodesics between vertices of tetrahedron (or octahedron, or icosahedron) is exactly the set \( A_\triangle \) of lengths of vectors from triangular lattice.

The asymptotics of such numbers is credited to Landau [8] and Ramanujan [9]. From their results (see [10]) follows that

**Proposition 1.3.** The number \( a_\Box(l) = \#(A_\Box \cap (0, l)) \) of different lengths of vectors from square lattice not exceeding \( l \) grows as \( \frac{\sqrt{\gamma_\Box l^2}}{2\sqrt{\ln l}} \) where \( \gamma_\Box \) is called the Landau—Ramanujan constant and equals approximately 0.76422.
Analogously for triangular lattice: $a_{\triangle}(l) \simeq \frac{\gamma_{\triangle} l^2}{2\sqrt{\ln l}}$ with $\gamma_{\triangle}$ approximately 0.64.

The constants $\gamma_{\square}$ and $\gamma_{\triangle}$ are defined by

$$
\gamma_{\square} = \left( \frac{1}{2} \prod_{p \equiv 3 \pmod{4}} \frac{p^2}{p^2 - 1} \right)^{1/2}, \quad 
\gamma_{\triangle} = \left( \frac{1}{2\sqrt{3}} \prod_{p \equiv 2 \pmod{3}} \frac{p^2}{p^2 - 1} \right)^{1/2},
$$

where product is over the set of primes.

Then the lengths of broken geodesics with vertices at the vertices of regular tetrahedron/octahedron/icosahedron (respectively cube) are exactly the sums of the numbers from $A_{\triangle}$ (respectively $A_{\square}$) with non-negative integer coefficients. Note that different such sums may have equal values, even sum numbers from $A_{\triangle}$ and $A_{\square}$ may be integer multiples of others. Let us drop these values and consider $B_{\triangle}$ and $B_{\square}$ to be the sets of values from $A_{\triangle}$ and $A_{\square}$ which are not positive integer multiples of other values. It is clear that squares of values from $B_{\triangle}$ and $B_{\square}$ are exactly square-free positive integers of the form $x^2 + xy + x^2$ and $x^2 + y^2$ respectively.

The asymptotics of these numbers can be shown [11] to be $\frac{\pi^2}{6}$ times less than the previous asymptotics:

**Proposition 1.4.** The number $b_{\square}(l) = \#(B_{\square} \cap (0, l))$ grows as $\frac{3\gamma_{\square} l^2}{\pi^2 \sqrt{\ln l}}$.

Analogously for tetrahedron: $b_{\triangle}(l) \simeq \frac{3\gamma_{\triangle} l^2}{\pi^2 \sqrt{\ln l}}$.

The lengths of broken geodesics with vertices at the vertices of cube (respectively tetrahedron, or octahedron, or icosahedron) are still the sums of the numbers from $B_{\square}$ (resp. $B_{\triangle}$) with non-negative integer coefficients. Any two such sums are different since the set $\{\sqrt{n} \mid n \text{ is a square-free positive integer}\}$ is linearly independent over $\mathbb{Q}$.

**2. Asymptotics for the number of waves**

**2.1. Abstract prime number theorem**

We will use the abstract additive prime number theorem. Let us consider an arithmetical semigroup $G$ generated by products of the sequence $p_1, p_2, \ldots$ of distinct elements of a semigroup (called abstract primes). If we have a norm on a semigroup then we could introduce two functions: counting function of the elements of a semigroup (i.e. abstract integers) $N_G(x)$ that are smaller than $x$ and a counting function of abstract primes $\pi_G(x)$. Now we could investigate how assumptions about the asymptotic behaviour of one of these functions as $x \to \infty$ influences that of the other. In the general setting, any theorem of this kind could be called an “abstract prime number theorem” (see [5]).

John Knopfmacher (see [5]) wrote: “The abstract additive prime number theorem is closely related to classical theorems of additive analytic number theory due to Hardy and Ramanujan, and concerning arithmetical functions
of the same general type as the classical partition function \( p(n) \). In fact, the
theorem itself may to a large extent be regarded as a reformulation of a certain
well-known ‘Tauberian’ theorem of Hardy and Ramanujan”.

**Theorem 2.1** (Additive Abstract Prime Number Theorem). Let \( G \) denote an
additive arithmetical semigroup such that there exist positive constants \( C \) and
\( \kappa \), and a real constant \( \nu \), such that

\[
\pi^\#_G(x) \sim Cx^\kappa (\ln x)^\nu \text{ as } x \to \infty.
\]

Then, as \( x \to \infty \),

\[
N^\#_G = \exp \{ [c_G + o(1)]x^{\kappa/(\kappa+1)}(\ln x)^{\nu/(\kappa+1)} \},
\]

where

\[
c_G = \kappa^{-1}(\kappa + 1)^{-(\kappa-\nu)/\kappa}[\kappa C \Gamma(\kappa + 1) \zeta(\kappa + 1)]^{1/\kappa}.
\]

2.2. Waves on regular tetrahedron and cube

From the results of first section we see that for regular tetrahedron and cube
we have \( \kappa = 2 \) and \( \nu = -1/2 \).

Let us substitute this to the theorem 2.1. We obtain that

\[
\pi^\#_G(x) \sim Cx^2 (\ln x)^{-1/2} \text{ as } x \to \infty.
\]

Then, as \( x \to \infty \),

\[
N^\#_G = \exp \{ [c_G + o(1)]x^{\pi/2}(\ln x)^{-1/2} \},
\]

where

\[
c_G = \frac{37}{2} (2\zeta(3))^2.
\]

Here \( \zeta(3) \approx 1.202056903159594285399738 \) is the Apery’s constant and \( \Gamma(3) = 2! = 2 \).

So we have

\[
c_G = \frac{37}{2} (4\zeta(3))^2.
\]

Now we are ready to formulate the theorem for cube and regular tetrahedron.

**Theorem 2.2** (Asymptotics of Waves in Vertices for tetrahedron, octahedron,
icosahedron and cube). The asymptotics of the number of waves that arrive up
to the time \( t \) at all vertices of

- **regular tetrahedron/octahedron/icosahedron:**

\[
N_\Delta = \exp \left( \left[ \frac{\sqrt{27}}{3\sqrt{2}} \sqrt{\frac{\zeta(3)\gamma_\Delta}{\pi^2}} + o(1) \right] \frac{t^{\pi/2}}{\sqrt{(\ln t)}} \right),
\]

- **cube:**

\[
N_\Box = \exp \left( \left[ \frac{\sqrt{27}}{\sqrt{2}} \sqrt{\frac{\zeta(3)\gamma_\Box}{\pi^2}} + o(1) \right] \frac{t^{\pi/2}}{\sqrt{(\ln t)}} \right).
\]
Here $\gamma$ is the Landau—Ramanujan constant and $\zeta(s)$ is the Riemann zeta function.

Note that for the case of a cube we have $c_G \simeq 1.8690$ and for the case of a regular tetrahedron we have $c_G \simeq 1.7617$.

We see that the number of arriving waves grows slower than the exponent which is in perfectly fits the results of computer experiments.

3. Geodesics on dodecahedron and other pentagonal surfaces

For pentagonal surfaces – the ones which can be glued from unit regular pentagons – the situation is a bit more complicated. That is because there is no plane lattice containing all vertices of the pentagon. However the trajectory unfolding construction (see Fig. 1) yields that the vector corresponding to a geodesic connecting the vertices of our surface can be represented as integer linear combination of sides of the regular pentagon. Moreover, we may restrict to only positive combinations:

**Lemma 3.1.** Let $v$ be an unfolded trajectory vector for a geodesic connecting the vertices of pentagonal surface and let $e_1, \ldots, e_5$ be vectors of the sides of a pentagon with orientation chosen such that the dot products $(v, e_i)$ are positive. Then there exist non-negative integers $n_1, \ldots, n_5$ such that $v = n_1 e_1 + \cdots + n_5 e_5$.

**Proof.** Let us unfold the metric along $v$. We obtain a sequence of pentagons of which any consecutive ones share a common edge. Now we choose a broken line path from the beginning of $v$ to its end going along the edges of our sequence of pentagons. This path can be chosen such that its projection to $v$ goes monotonously (see Fig. 2). We just show step-by-step that in each pentagon the
vertices of edges intersecting with $v$ are reachable by such monotonous broken lines. The existence of such broken line implies that the desired decomposition of vector $v$ exists.

Figure 2: Decomposition of an unfolded trajectory.

**Corollary 3.2.** The number $a_{O}(l)$ of lengths of geodesics between vertices of pentagonal surface with length less than $l$ satisfies $a_{O}(l) < \gamma l^5$ for some $\gamma$.

**Proof.** There is a the finite number of orientations $e_1, \ldots, e_5$ of edges. Let us fix any of them and estimate the number $c(l)$ of vectors $v$ with length less than $l$ decomposable as $v = n_1 e_1 + \cdots + n_5 e_5$ with non-negative $n_i$ with extra condition that the dot products $(v, e_i)$ are positive. Consider some $v_0$ from these vectors. Then

$$ (v, v) \geq \frac{1}{(v_0, v_0)(v_0, v_0)} (\sum_i n_i (e_i, v_0))^2 > \epsilon \sum_i n_i^2 $$

for some $\epsilon > 0$ independent of $v$. Therefore the number of $v$’s of length less than $l$ is less than the number of sequences $(n_1, \ldots, n_5)$ such that $\sum_i n_i^2 < \frac{1}{\epsilon} l^2$. Obviously, this is less than $\gamma l^5$ for some $\gamma$.

From Corollary 3.2 together with Theorem 2.1 we get

**Theorem 3.3** (Asymptotics of Waves in Vertices for dodecahedron). The asymptotics of the number of waves that arrive up to the time $t$ at all vertices of a dodecahedron

$$ N_O < \exp \left( c t^{\frac{5}{2}} \right) $$

for some $c$.

4. Further work

The upper bound for the dodecahedron could be improved and the asymptotics could be obtained, as for other Platonic solids. For polyhedra with “rational” angles one could use the covers by translation surfaces and Siegel-Veech
constants (see [1] for details). Also it is interesting to consider wave propagation on polyhedra in non-Euclidean spaces.

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References

[1] A. Eskin, H. Masur, A.Zorich Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel–Veech constants, Publications Mathématiques de l’IHÉS 97 (1), 61-179.

[2] Fuchs, D. Geodesics on Regular Polyhedra with Endpoints at the Vertices. Arnold Math J. 2, 201–211 (2016). https://doi.org/10.1007/s40598-016-0040-z.

[3] Diana Davis, Victor Dods, Cynthia Traub, Jed Yang, Geodesics on the regular tetrahedron and the cube, Discrete Mathematics, Volume 340, Issue 1, 2017, Pages 3183-3196.

[4] Chernyshev V.L, Tolchennikov A.A. Asymptotic estimate for the counting problems corresponding to the dynamical system on some decorated graphs. Ergodic Theory and Dynamical Systems. Cambridge University Press, Volume 38, Issue 5, 2018. pp. 1697-1708. doi:10.1017/etds.2016.102.

[5] Knopfmacher, J.. Abstract Analytic Number Theory, 2nd edn. Dover Publishing, New York, 1990.

[6] Nazaikinskii, V. E.. On the entropy of the Bose–Maslov gas. Dokl. Akad. Nauk 448(3) (2013), 266–268.

[7] Jayadev S. Athreya, David Aulicino, W. Patrick Hooper with an appendix by Anja Randecker (2020) Platonic Solids and High Genus Covers of Lattice Surfaces, Experimental Mathematics, DOI: 10.1080/10586458.2020.1712564.

[8] Edmund Landau. Über die Einteilung der positive ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate, Archiv der Mathematik und Physik (3) 13 (1908), 305-312.

[9] S. Ramanujan. letter to G.H. Hardy, 16 January, 1913.

[10] Balaji, V., Edwards, O., Loftin, A. M., Mcharo, S., Phillips, L., Rice, A., Tsegaye, B. (2019). Lattice Configurations Determining Few Distances. arXiv preprint arXiv:1911.11688
[11] Cohen E. Arithmetical functions associated with arbitrary sets of integers // Acta Arithmetica. — 1959. — Vol. 5. — P. 407—415.

[12] Kokotov, A., Lagota, K. (2020). Green Function and Self-adjoint Laplacians on Polyhedral Surfaces. Canadian Journal of Mathematics, 72(5), 1324-1351. doi:10.4153/S0008414X19000336

[13] Lukzen, E.N., Shafarevich, A.I. On the kernel of the Laplace operator on two-dimensional polyhedra. Russ. J. Math. Phys. 24, 488-493 (2017). https://doi.org/10.1134/S1061920817040070