Balanced Combinations of Solutions
in Multi-Objective Optimization

Christian Glaßer       Christian Reitwießner       Maximilian Witek
Julius-Maximilians-Universität Würzburg, Germany
{glasser, reitwiessner, witek}@informatik.uni-wuerzburg.de

Abstract
For every list of integers \(x_1, \ldots, x_m\) there is some \(j\) such that
\[ x_1 + \cdots + x_j - x_{j+1} - \cdots - x_m \approx 0. \]
So the list can be nearly balanced and for this we only need one alternation between addition and subtraction. But what if the \(x_i\) are \(k\)-dimensional integer vectors? Using results from topological degree theory we show that balancing is still possible, now with \(k\) alternations.

This result is useful in multi-objective optimization, as it allows a polynomial-time computable balance of two alternatives with conflicting costs. The application to two multi-objective optimization problems yields the following results:

- A randomized \(1/2\)-approximation for multi-objective maximum asymmetric traveling salesman, which improves and simplifies the best known approximation for this problem.
- A deterministic \(1/2\)-approximation for multi-objective maximum weighted satisfiability.

1 Introduction

Balancing Sums of Vectors. Suppose we are given a sequence of goods \(g_1, \ldots, g_m\), each of which has a value, a weight, and a size. Is it possible to distribute the goods on two trucks such that the loads are nearly the same with respect to value, weight, and size? We show that this is always possible by a very easy partition: For suitable indices \(i, j, k, l\), assign \(g_i, g_{i+1}, \ldots, g_j, g_k, g_{k+1}, \ldots, g_l\) to the first truck and the remaining goods to the second one. In general, if the goods have \(2k\) criteria (value, weight, size, \ldots), then there exist \(k\) intervals of goods such that the goods inside and the goods outside of the intervals are nearly equivalent with respect to all criteria.

More formally, let \(x_1, \ldots, x_m \in \mathbb{N}^{2k}\) be vectors of natural numbers that represent the criteria of each good, and let \(z \in \mathbb{N}^{2k}\) be an upper bound for these vectors (i.e., \(x_i \leq z\) for all \(i\)). Lemma 2.6 provides intervals \(I_1, \ldots, I_k \subseteq \mathbb{N}\) such that for \(I = \bigcup_{i=1}^{k} I_i\),

\[-4kz \leq \sum_{i \in I} x_i - \sum_{i \notin I} x_i \leq 4kz,\]

where the \(\leq\) hold with respect to each component. The same is true if \(x_1, \ldots, x_m \in \mathbb{Z}^{2k}\) are vectors of integers, where \(-z \leq x_i \leq z\) for all \(i\) (Corollary 2.7). The proofs of these balancing results are
based on the Odd Mapping Theorem, a result from topological degree theory, which we apply in a discrete setting. The discretization is responsible for the term $4kz$, which is caused by a rounding error that unavoidably occurs at the boundaries of the intervals $I_1, \ldots, I_k$.

The simplicity of the desired partition (i.e., a union of $k$ intervals) is important for the application of our balancing results. Algorithmically, it means that for fixed dimension $2k$, the right choice for the intervals $I_1, \ldots, I_k$ can be found by exhaustive search in time polynomial in $m$.

**Multi-Objective Optimization.** Many real-life optimization problems have multiple objectives that cannot be easily combined into a single value. Thus, one is interested in solutions that are good with respect to all objectives at the same time. For conflicting objectives we cannot hope for a single optimal solution, but there will be trade-offs. The Pareto set captures the notion of optimality in this setting. It consists of all solutions that are optimal in the sense that there is no solution that is at least as good in all objectives and better in at least one objective. So the Pareto set contains all optimal decisions for a given situation. For a general introduction to multi-objective optimization we refer to the survey by Ehrgott and Gandibleux [EG00] and the textbook by Ehrgott [Ehr05].

For many problems, the Pareto set has exponential size and hence cannot be computed in polynomial time. Regarding the approximability of Pareto sets, Papadimitriou and Yannakakis [PY00] show that every Pareto set has a $(1 - \varepsilon)$-approximation of size polynomial in the size of the instance and $1/\varepsilon$ (for the formal definition of approximation see section 3.1). Hence, even though a Pareto set might be an exponentially large object, there always exists a polynomial-size approximation. This clears the way for a general investigation of the approximability of Pareto sets of multi-objective optimization problems.

In general, inapproximability and hardness results directly translate from single-objective optimization problems to their multi-objective variants. On the other hand, existing approximation algorithms for single-objective problems can not always be used for multi-objective approximation. Using our balancing results we demonstrate a translation of single-objective approximation ideas to the multi-objective case: We obtain a randomized $1/2$-approximation for multi-objective maximum asymmetric TSP and a deterministic $1/2$-approximation for multi-objective maximum weighted satisfiability.

**Traveling Salesman Problem.** The (single-objective) maximum asymmetric traveling salesman problem (MaxATSP, for short) is the optimization problem where on input of a complete directed graph with edge weights from $\mathbb{N}$ the goal is to find a Hamiltonian cycle of maximum weight. Engebretsen and Karpinski [EK01] show that MaxATSP cannot be $(319/320 + \varepsilon)$-approximated (unless P=NP). In 1979, Fisher, Nemhauser and Wolsey [FNW79] gave a $1/2$-approximation algorithm for MaxATSP (remove the lightest edge from each cycle of a maximum cycle cover and connect the remaining parts to a Hamiltonian cycle). Since then, many improvements were achieved and the currently best known approximation ratio of $2/3$ for MaxATSP is given by Kaplan et al. [KLS05].

The $k$-objective variant $k$-MaxATSP is defined analogously with edge weights from $\mathbb{N}^k$. The hardness results for MaxATSP directly translate to its multi-objective variant (just set all but one
component of the edge weights to a constant), but algorithms have to be newly designed. Bläser et al. [BMP08] show that \( k \)-MaxATSP is randomized \( \left( \frac{1}{k+1} - \varepsilon \right) \)-approximable. This was improved by Manthey [Man09] to a randomized \( \left( \frac{1}{2} - \varepsilon \right) \)-approximation for all (fixed) numbers of criteria. Both algorithms extend the cycle cover idea to multiple objectives. With a surprisingly simple algorithm we improve the approximation ratio to \( 1/2 \).

Satisfiability. Given a formula in conjunctive normal form and a non-negative weight in \( \mathbb{N} \) for each clause, the maximum weighted satisfiability problem (MaxSAT, for short) aims to find a truth assignment such that the sum of the weights of all satisfied clauses is maximal. The first approximation algorithm for MaxSAT is due to Johnson [Joh74]. He proved an approximation ratio of \( (2^r-1)/(2^r) \) for formulas where each clause has at least \( r \) literals. His work showed that the general MaxSAT problem is \( 1/2 \)-approximable. Yannakakis [Yan94] improved the approximation ratio of MaxSAT to \( 3/4 \), and Goemans and Williamson [GW94] subsequently gave a simpler algorithm with essentially the same approximation ratio, and later [GW95] improved the approximation ratio to 0.758. Further improvements followed, and the currently best known approximation ratio of 0.7846 is due to Asano and Williamson [AW02]. Regarding lower bounds, Papadimitriou and Yannakakis [PY91] show that MaxSAT is APX-complete. Furthermore, by Hästad [Hås97], MaxSAT cannot be approximated better than \( 7/8 \), unless \( P=NP \).

Only little is known about the multi-objective maximum weighted satisfiability problem (\( k \)-MaxSAT, for short), where each clause has a non-negative weight in \( \mathbb{N}^k \) for some fixed \( k \geq 1 \) and where we wish to maximize the weight of the satisfied clauses. Santana et. al. [SBLL09] apply genetic algorithms to a version of the problem that is equivalent to \( k \)-MaxSAT with polynomially bounded weights. To our knowledge, the approximability of \( k \)-MaxSAT has not been investigated so far.

Using our balancing results, we can transfer a simple idea from single-objective optimization to the multi-objective world: For any truth assignment, the assignment itself or its complementary assignment satisfies at least one half of all clauses. We obtain a (deterministic) \( 1/2 \)-approximation for \( k \)-MaxSAT, independent of \( k \).

2 Balancing Results

2.1 Preliminaries

Let \( a, b \in \mathbb{R} \). We call a function \( f: [a, b] \to \mathbb{R} \) integrable, if it is Lebesgue-integrable on \([a, b]\). This is especially the case for bounded functions \( f \) with only finitely many points of discontinuity. A function \( g: [a, b] \to \mathbb{R}^n \) is componentwise integrable, if all projections \( g_i \) are integrable and in this case we write \( \int_a^b g(x) \, dx \) as abbreviation for the tuple \((\int_a^b g_1(x) \, dx, \ldots, \int_a^b g_n(x) \, dx)\). For \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \) we write \( x \leq y \) if \( x_i \leq y_i \) for all \( i \in \{1, 2, \ldots, n\} \). For a set \( A \subseteq \mathbb{R}^n \), \( \overline{A} \) denotes the (topological) closure of \( A \), and \( \partial A \) denotes the boundary of \( A \). The set \( A \subseteq \mathbb{R}^n \) is symmetric if \( x \in A \iff -x \in A \) for all \( x \in \mathbb{R}^n \).
For bounded, open sets \( D \subseteq \mathbb{R}^n \), continuous functions \( \varphi: D \to \mathbb{R}^n \) and points \( p \in \mathbb{R}^n \setminus \varphi(\partial D) \) the integer \( d(\varphi, D, p) \) is called the Brouwer degree of \( \varphi \) and \( D \) at the point \( p \). We will not define it here, but we note that it captures how often \( p \) is “covered” by \( \varphi(D) \), counting “inverse” covers negatively, and that it generalizes the winding number in complex analysis.

### 2.2 Analytical Version

We apply the following theorems from topological degree theory to get the analytical version of our balancing results.

**Theorem 2.1** ([Llo78, Theorem 2.1]). If \( D \subseteq \mathbb{R}^n \) is bounded and open, \( \varphi: \overline{D} \to \mathbb{R}^n \) is continuous, \( p \notin \varphi(\partial D) \) and \( d(\varphi, D, p) \neq 0 \), then \( p \notin \varphi(D) \).

**Theorem 2.2** (Odd Mapping Theorem, [Llo78, Theorem 3.2.6]). Let \( D \) be a bounded, open, symmetric subset of \( \mathbb{R}^n \) containing the origin. If \( \varphi: \overline{D} \to \mathbb{R}^n \) is continuous, \( 0 \notin \varphi(\partial D) \), and for all \( x \in \partial D \) it holds that \( \frac{\varphi(x)}{|\varphi(x)|} \neq \frac{\varphi(-x)}{|\varphi(-x)|} \), then \( d(\varphi, D, 0) \) is an odd number (and in particular not zero).

**Corollary 2.3.** Let \( D \) be a bounded, open, symmetric subset of \( \mathbb{R}^n \) containing the origin. If \( \varphi: \overline{D} \to \mathbb{R}^n \) is continuous and for all \( x \in \partial D \) it holds that \( \varphi(-x) = -\varphi(x) \), then \( 0 \notin \varphi(D) \).

**Proof.** Assume that \( 0 \notin \varphi(\overline{D}) \). From \( \varphi(-x) = -\varphi(x) \) for \( x \in \partial D \) it follows that the inequality condition of Theorem [2.2] is fulfilled (note that \( 0 \notin \varphi(\partial D) \)) and thus \( d(\varphi, D, 0) \neq 0 \) and by Theorem [2.1] \( 0 \notin \varphi(\overline{D}) \). This is a contradiction. \( \square \)

**Lemma 2.4.** Let \( n \geq 1 \), \( a, b \in \mathbb{R} \), and \( h: [a, b] \to \mathbb{R}^{2n} \) be componentwise integrable. There exist \( n \) closed intervals \( I_1, \ldots, I_n \subseteq [a, b] \) such that for \( I = I_1 \cup \cdots \cup I_n \),

\[
\int_{I} h(x) \, dx = \int_{[a, b] \setminus I} h(x) \, dx.
\]

**Proof.** Observe that it suffices to show this for \( [a, b] = [0, 1] \). Define \( T = \{(t_1, t_2, \ldots, t_{2n}) \in \mathbb{R}^{2n} \mid \sum_{i=1}^{2n} |t_i| \leq 1\} \) and for every \( t = (t_1, \ldots, t_{2n}) \in T \), let

\[
I_t = \bigcup_{1 \leq k \leq 2n \atop t_k > 0} \left[ \sum_{i=1}^{k-1} |t_i|, \sum_{i=1}^{k} |t_i| \right]
\]

and

\[
f: T \to \mathbb{R}^{2n}, \quad f(t) = \int_{I_t} h(x) \, dx - \int_{[0,1] \setminus I_t} h(x) \, dx.
\]

By the formal definition, \( I_t \) is a union of (at most) \( 2n \) closed intervals. However, it can always be written as a union of at most \( n \) closed intervals, by merging adjacent intervals.

We now want to show that \( 0 \in f(T) \) by applying Corollary [2.3] to \( \varphi = f \) and \( D \) being the interior of \( T \). \( D \) is obviously a bounded, open, and symmetric subset of \( \mathbb{R}^{2n} \) containing the origin. The
Figure 1: Illustration of the set $I_t$ for some value of $t = (t_1, \ldots, t_8)$, where $t_1, t_2, t_4$ and $t_8$ are positive and $t_3, t_5, t_6$ and $t_7$ are negative.

function $f$ is continuous because of the fundamental theorem of calculus for the Lebesgue integral and the fact that the endpoints of the intervals in $I_t$ depend continuously on $t$. Furthermore, for any $t \in \partial D$ we get that there are only finitely many points in $[0,1]$ which are not in exactly one of the sets $I_{-t}$ and $I_t$ and thus $f(-t) = -f(t)$ since these finitely many points have no influence on the values of the integrals. Since all preconditions of the corollary are fulfilled, we get $0 \in f(T)$ and thus there exists some $t \in T$ such that

$$\int_{I_t} h(x) \, dx = \int_{[0,1] \setminus I_t} h(x) \, dx.$$ 

As already noted, $I_t$ can be written as a union of at most $n$ closed intervals. We obtain a union of exactly $n$ intervals by adding intervals $[a,a]$.

Lemma 2.5. Let $n \geq 1$, $a, b \in \mathbb{R}$, and $f, g: [a, b] \rightarrow \mathbb{R}^n$ be componentwise integrable. There exist $n$ closed intervals $I_1, \ldots, I_n \subseteq [a, b]$ such that for $I = I_1 \cup \cdots \cup I_n$,

$$\int_I f(x) \, dx + \int_{[a,b] \setminus I} g(x) \, dx = \frac{1}{2} \int_{[a,b]} f(x) + g(x) \, dx.$$ 

Proof. Applying Lemma 2.4 to $h(x) = f(x) - g(x)$ yields some $I \subseteq [a, b]$ that is the union of $n$ closed intervals in $[a, b]$ such that

$$\int_I h(x) \, dx = \int_{[a,b] \setminus I} h(x) \, dx$$

$$\iff \int_I f(x) - g(x) \, dx = \int_{[a,b] \setminus I} f(x) - g(x) \, dx$$

$$\iff \int_I f(x) - g(x) \, dx + \int_{[a,b] \setminus I} g(x) - f(x) \, dx = 0$$

$$\iff (\ast) 2 \int_I f(x) \, dx + 2 \int_{[a,b] \setminus I} g(x) \, dx = \int_{[a,b]} f(x) + g(x) \, dx$$

$$\iff \int_I f(x) \, dx + \int_{[a,b] \setminus I} g(x) \, dx = \frac{1}{2} \int_{[a,b]} f(x) + g(x) \, dx.$$ 

Note that $(\ast)$ is obtained by adding $\int_{[a,b]} f(x) + g(x) \, dx$ to both sides.
2.3 Discretization of the Analytical Results

Now we discretize the analytical results which causes a rounding error that cannot be avoided.

**Lemma 2.6.** Let \( n, m \geq 1 \) and \( x_1, \ldots, x_m, y_1, \ldots, y_m, z \in \mathbb{N}^{2n} \) such that \( x_i \leq z \) and \( y_i \leq z \) for all \( i \). There exist natural numbers \( 1 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_n \leq b_n \leq m \) such that for \( I = \bigcup_{i=1}^n \{a_i, a_i+1, \ldots, b_i-1\} \),

\[
-2nz + \frac{1}{2} \sum_{i=1}^m (x_i + y_i) \leq \sum_{i \in I} x_i + \sum_{i \notin I} y_i \leq 2nz + \frac{1}{2} \sum_{i=1}^m (x_i + y_i).
\]

**Proof.** For the proof it is advantageous to start the indices of \( x_i \) and \( y_i \) at 0. We first define two functions \( f \) and \( g \) that distribute the values \( x_0, \ldots, x_{m-1}, y_0, \ldots, y_{m-1} \in \mathbb{N}^{2n} \) equally over the interval \([0, m)\), and then we apply Lemma 2.5. Let \( f, g : [0, m] \to \mathbb{R}^{2n} \) such that

\[
f(t) = \begin{cases} 
2x_i & \text{if } t \in [i, i + 1/2) \\
(0, \ldots, 0) & \text{otherwise}
\end{cases}
\]

and

\[
g(t) = \begin{cases} 
2y_i & \text{if } t \in [i + 1/2, i + 1) \\
(0, \ldots, 0) & \text{otherwise}
\end{cases}
\]

Figure 2 shows the graph of \( f \) and \( g \). Note that both functions are componentwise integrable. Moreover, for \( i \in \{0, \ldots, m-1\} \),

\[
\int_i^{i+1} f(t) \, dt = x_i \quad \text{and} \quad \int_i^{i+1} g(t) \, dt = y_i.
\]

By Lemma 2.5 there exist closed intervals \( I_i = [a_i, b_i] \subseteq [0, m] \) where \( 1 \leq i \leq n \) such that for \( I = \bigcup_{i=1}^n [a_i, b_i] \) it holds that

\[
\int_I f(t) \, dt + \int_{[0,m]\setminus I} g(t) \, dt = \frac{1}{2} \int_{[0,m]} f(t) + g(t) \, dt.
\]
We may assume $0 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_n \leq b_n \leq m$. For $1 \leq i \leq n$ let

\[ a'_i := \lfloor a_i + 1/2 \rfloor \quad \text{and} \quad b'_i := \lceil b_i + 1/2 \rceil. \]

Note that the $a'_i, b'_i$ are natural numbers such that $0 \leq a'_1 \leq b'_1 \leq a'_2 \leq b'_2 \leq \cdots \leq a'_n \leq b'_n \leq m$. By the definition of $f$ and $g$, for $1 \leq i \leq n$ it holds that

\[
\left| \int_{a_i}^{a'_i} f(t) \, dt \right| + \left| \int_{a_i}^{a'_i} g(t) \, dt \right| \leq z \quad \text{and} \quad \left| \int_{b_i}^{b'_i} f(t) \, dt \right| + \left| \int_{b_i}^{b'_i} g(t) \, dt \right| \leq z,
\]

where $|(v_1, \ldots, v_{2n})| := |(v_1, \ldots, v_{2n})|$ for $v_1, \ldots, v_{2n} \in \mathbb{R}$. So if some $a_i$ (resp., $b_i$) is replaced by $a'_i$ (resp., $b'_i$), then the left-hand side of (2) changes at most by $z$. Hence, for $I' = \bigcup_{i=1}^{n} \{a'_i, b'_i\}$ it holds that

\[
-2nz + \frac{1}{2} \int_{[0,m]} f(t) + g(t) \, dt \leq -2nz + \int_{I'} f(t) + g(t) \, dt \leq -2nz + \frac{1}{2} \int_{[0,m]} f(t) + g(t) \, dt. \quad (3)
\]

Let $I'' = \bigcup_{i=1}^{n} \{a'_i, a'_i + 1, \ldots, b'_i - 1\}$. From (1) and (3) we obtain

\[
-2nz + \frac{1}{2} \sum_{i=0}^{m-1} (x_i + y_i) \leq \sum_{i \in I''} x_i + \sum_{i \notin I''} y_i \leq -2nz + \frac{1}{2} \sum_{i=0}^{m-1} (x_i + y_i).
\]

Next we state the integer variant of Lemma [2.6]

**Corollary 2.7.** Let $n, m \geq 1, x_1, \ldots, x_m \in \mathbb{Z}^{2n}$, and $z \in \mathbb{N}^{2n}$ such that $-z \leq x_i \leq z$ for all $i$. There exist natural numbers $1 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_n \leq b_n \leq m$ such that for $I = \bigcup_{i=1}^{n} \{a_i, a_i + 1, \ldots, b_i - 1\}$,

\[
-4nz \leq \sum_{i \in I} x_i - \sum_{i \notin I} x_i \leq 4nz.
\]

**Proof.** Let $x'_i := z + x_i$ and $y'_i := z - x_i$. Thus $x'_i, y'_i \in \mathbb{Z}^{2n}$ and $x'_i, y'_i \leq 2z$. Lemma [2.6] applied to $x'_i$ and $y'_i$ provides natural numbers $1 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_n \leq b_n \leq m$ such that for $I = \bigcup_{i=1}^{n} \{a_i, a_i + 1, \ldots, b_i - 1\}$,

\[
-4nz + \frac{1}{2} \sum_{i=1}^{m} (x'_i + y'_i) \leq \sum_{i \in I} x'_i + \sum_{i \notin I} y'_i \leq 4nz + \frac{1}{2} \sum_{i=1}^{m} (x'_i + y'_i).
\]

Therefore,

\[
-4nz + \frac{2mz}{2} + \frac{1}{2} \sum_{i=1}^{m} (x_i - x_i) \leq m z + \sum_{i \in I} x_i - \sum_{i \notin I} x_i \leq 4nz + \frac{2mz}{2} + \frac{1}{2} \sum_{i=1}^{m} (x_i - x_i).
\]

\[
\square
\]
For the applications to maximum asymmetric traveling salesman and maximum weighted satisfiability we need the following variant of Lemma 2.8. While providing only a lower bound for the balanced sum, it estimates the rounding error more precisely.

**Lemma 2.8.** Let \( n, m \geq 1 \) and \( x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathbb{N}^{2n} \). There exists an \( n' \in \{0, \ldots, n\} \) and natural numbers \( 1 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_{n'} \leq b_{n'} \leq m \) such that for \( I = \bigcup_{i=1}^{n'} \{a_i, a_i+1, \ldots, b_i\} \),

\[
y_{b_1} + y_{b_2} + \cdots + y_{b_{n'}} + \sum_{i \in I} x_i + \sum_{i \notin I} y_i \geq \frac{1}{2} \sum_{i=1}^{m} (x_i + y_i).
\]

**Proof.** Again let the indices of \( x_i \) and \( y_i \) start at 0 and define the componentwise integrable functions \( f \) and \( g \) as in the proof of Lemma 2.6. So for \( i \in \{0, \ldots, m-1\} \),

\[
\int_{i}^{i+1/2} f(t) \, dt = x_i \quad \text{and} \quad \int_{i+1/2}^{i+1} g(t) \, dt = y_i.
\]  

(4)

By Lemma 2.5 there exists an \( n' \in \{0, \ldots, n\} \) and closed intervals \( I_i = [a_i, b_i] \subseteq [0, m] \) where \( 1 \leq i \leq n' \) such that for \( I = \bigcup_{i=1}^{n'} [a_i, b_i] \) it holds that

\[
\int_I f(t) \, dt + \int_{[0,m] \setminus I} g(t) \, dt \geq \frac{1}{2} \int_{[0,m]} f(t) + g(t) \, dt.
\]  

(5)

Here we only need the inequality even though Lemma 2.5 states an equality. We may assume

\[
0 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_{n'} \leq b_{n'} \leq m.
\]  

(6)

By the definition of \( f \) and \( g \), the following holds for every \( i \in \{0, \ldots, m-1\} \):

\[
t \in [i, i+1/2) \implies g(t) = (0, \ldots, 0)
\]

\[
t \in [i+1/2, i+1) \implies f(t) = (0, \ldots, 0)
\]

**Claim 2.9.** We may assume that \( \{a_j, b_j\} \not\subseteq [i + 1/2, i + 1] \) and \( \{b_j, a_{j+1}\} \not\subseteq [i, i + 1/2] \) for all \( j \in \{1, \ldots, n'\} \) and \( i \in \{0, \ldots, m-1\} \).

**Proof.** If \( a_j, b_j \in [i + 1/2, i + 1] \), then \( f \) is 0 on \([a_j, b_j]\) and hence \( \int_{a_j}^{b_j} f(t) \, dt = 0 \). Thus the left-hand side of (5) does not decrease if we remove the interval \([a_j, b_j]\) from \( I \). Similarly, if \( b_j, a_{j+1} \in [i, i+1/2] \), then \( g \) is 0 on \([b_j, a_{j+1}]\) and hence \( \int_{b_j}^{a_{j+1}} g(t) \, dt = 0 \). Thus the left-hand side of (5) does not decrease if we replace the intervals \([a_j, b_j]\) and \([a_{j+1}, b_{j+1}]\) by the interval \([a_j, b_{j+1}]\). Note that after these changes (which include a decrement of \( n' \)), (5) still holds.

**Claim 2.10.** We may assume that \( a_1, \ldots, a_{n'} \in \mathbb{N} \) and \( b_1 + 1/2, \ldots, b_{n'} + 1/2 \in \mathbb{N} \).

**Proof.** Assume \( a_j \in [i + 1/2, i + 1] \). By Claim 2.9, \( b_j \not\in [i + 1/2, i + 1] \) and hence \( b_j > i + 1 \). Since \( f \) is 0 on \([i + 1/2, i + 1]\), the left-hand side of (5) does not decrease if we let \( a_j := i + 1 \). After this change, (5) still holds.
Assume \( a_j \in [i, i + 1/2) \). By Claim 2.9, \( b_{j-1} \notin [i, i + 1/2) \) and hence \( b_{j-1} < i \) (for \( j \geq 2 \)). Since \( g \) is 0 on \([i, i + 1/2)\), the left-hand side of (5) does not decrease if we let \( a_j := i \). After this change, (6) still holds.

Assume \( b_j \in [i + 1/2, i + 1) \). By Claim 2.9, \( a_j \notin [i + 1/2, i + 1] \) and hence \( a_j < i + 1/2 \). Since \( f \) is 0 on \([i + 1/2, i + 1)\), the left-hand side of (5) does not decrease if we let \( b_j := i + 1/2 \). After this change, (6) still holds.

Assume \( b_j \in [i, i + 1/2) \) and \( i < m \). By Claim 2.9, \( a_{j+1} \notin [i, i + 1/2) \) and hence \( a_{j+1} > i + 1/2 \) (for \( j < n' \)). Since \( g \) is 0 on \([i, i + 1/2)\), the left-hand side of (5) does not decrease if we let \( b_j := i + 1/2 \). After this change, (6) still holds.

It remains to argue for the special case \( b_j = m \). By Claim 2.9, \( a_j \notin [m - 1/2, m] \) and hence \( a_j < m - 1/2 \). Since \( f \) is 0 on \([m - 1/2, m)\), the left-hand side of (5) does not decrease if we let \( b_j := m - 1/2 \). After this change, (6) still holds.

If we split the integrals on the left-hand side of (5) according to \( I = \bigcup_{i=1}^{n'} [a_i, b_i] \), we obtain
\[
\int_0^{a_1} g(t) \, dt + \sum_{i=1}^{n'-1} \left( \int_{a_i}^{b_i} f(t) \, dt + \int_{b_i}^{a_{i+1}} g(t) \, dt \right) + \int_{b_{n'}}^m g(t) \, dt \geq \frac{1}{2} \int_{[0,m]} f(t) + g(t) \, dt. \tag{7}
\]

For \( i \in \{1, \ldots, n'\} \) let \( c_i = b_i - 1/2 \). From Claim 2.10 and (6) it follows that
\[
0 \leq a_1 \leq c_1 < a_2 \leq c_2 < \cdots < a_{n'} \leq c_{n'} \leq m - 1.
\]

Together with (4) we obtain:
\[
\int_0^{a_1} g(t) \, dt = y_0 + y_1 + \cdots + y_{a_1-1} \\
\int_{a_i}^{b_i} f(t) \, dt = x_{a_i} + x_{a_i+1} + \cdots + x_{c_i} \\
\int_{b_i}^{a_{i+1}} g(t) \, dt = y_{c_i} + y_{c_i+1} + \cdots + y_{a_{i+1}-1} \\
\int_{b_{n'}}^m g(t) \, dt = y_{c_{n'}} + y_{c_{n'}+1} + \cdots + y_{m-1}
\]

In these sums, each index \( j \in \{c_1, c_2, \ldots, c_{n'}\} \) appears exactly twice, once as \( x_j \) and once as \( y_j \). All remaining indices \( j \in \{0, \ldots, m-1\} \setminus \{c_1, c_2, \ldots, c_{n'}\} \) appear exactly once, either as \( x_j \) or as \( y_j \). Therefore, with \( I' = \bigcup_{i \in \{1, \ldots, n'\}} \{a_i, a_i+1, \ldots, c_i\} \) the left-hand side of (7) is equal to
\[
y_{c_1} + y_{c_2} + \cdots + y_{c_{n'}} + \sum_{i \in I'} x_i + \sum_{i \notin I'} y_i. \tag{8}
\]
We say that some algorithm is an $\alpha$-Pareto set of $x$ if it is allowed to fail with probability at most $\frac{1}{2}\sum_{i=0}^{m-1}(x_i + y_i)$.

Corollary 2.11. Let $n, m \geq 1$ and $x_1, \ldots, x_m, y_1, \ldots, y_m, z \in \mathbb{N}^n$ such that $y_i \leq z$ for all $i$. There exist $n' \leq \min(n, m)$ disjoint, nonempty intervals $I_1, \ldots, I_{n'} \subseteq \{1, \ldots, m\}$ such that for $I = I_1 \cup \cdots \cup I_{n'}$,

$$n' \cdot z + \sum_{i \in I} x_i + \sum_{i \notin I} y_i \geq \frac{1}{2}\sum_{i=1}^{m}(x_i + y_i).$$

3 Applications to Multi-Objective Optimization

3.1 Preliminaries

Consider some multi-objective maximization problem $O$ that consists of a set of instances $I$, a set of solutions $S(x)$ for each instance $x \in I$, and a function $w$ assigning a $k$-dimensional weight $w(x, s) \in \mathbb{N}^k$ to each solution $s \in S(x)$ depending also on the instance $x \in I$. If the instance $x$ is clear from the context, we also write $w(s) = w(x, s)$. The components of $w$ are written as $w_i$ for $i \in \{1, 2, \ldots, k\}$. For weights $a = (a_1, \ldots, a_k), b = (b_1, \ldots, b_k) \in \mathbb{N}^k$ we write $a \geq b$ if $a_i \geq b_i$ for all $i \in \{1, 2, \ldots, k\}$.

Let $x \in I$. The Pareto set of $x$, the set of optimal solutions, is the set $\{s \in S(x) \mid \neg \exists s' \in S(x) (w(x, s') \geq w(x, s) \text{ and } w(x, s') \neq w(x, s))\}$. For solutions $s, s' \in S(x)$ and $\alpha < 1$ we say $s$ is $\alpha$-approximated by $s'$ if $w_i(s') \geq \alpha \cdot w_i(s)$ for all $i$. We call a set of solutions $\alpha$-approximate Pareto set of $x$ if every solution $s \in S(x)$ (or equivalently, every solution from the Pareto set) is $\alpha$-approximated by some $s'$ contained in the set.

We say that some algorithm is an $\alpha$-approximation algorithm for $O$ if it runs in polynomial time and returns an $\alpha$-approximate Pareto set of $x$ for all input instances $x \in I$. We call it randomized if it is allowed to fail with probability at most $1/2$ over all of its executions. An algorithm is an FPTAS (fully polynomial-time approximation scheme) for a given optimization problem, if on input $x$ and $0 < \varepsilon < 1$ it computes a $(1 - \varepsilon)$-approximate Pareto set of $x$ in time polynomial in $|x| + 1/\varepsilon$. If the algorithm is randomized it is called FPRAS (fully polynomial-time randomized approximation scheme).

3.2 $k$-Objective Maximum Asymmetric Traveling Salesman Problem

Definitions. Let $k \geq 1$. An $\mathbb{N}^k$-labeled directed graph is a tuple $G = (V, E, w)$, where $V$ is some finite set of vertices, $E \subseteq V \times V$ is a set of edges, and $w: E \rightarrow \mathbb{N}^k$ is a $k$-dimensional weight function. We denote the $i$-th component of $w$ by $w_i$ and extend $w$ to sets of edges by taking the sum over the weights of all edges in the set. A set of edges $M \subseteq E$ is called matching in $G$ if no two edges in $M$ share a common vertex. A walk in $G$ is an alternating sequence of vertices and edges.
Given some \(N^k\)-labeled directed graph as input, our goal is to find a maximum Hamiltonian cycle. We will also use the multi-objective version of the maximum matching problem. These two maximization problems are defined as follows:

**k-Objective Maximum Asymmetric Traveling Salesman Problem**

\((k-\text{MaxATSP})\)

Instance: \(N^k\)-labeled directed complete graph \((V, E, w)\)

Solution: Hamiltonian cycle \(C\)

Weight: \(w(C)\)

**k-Objective Maximum Matching (k-MM)\)**

Instance: \(N^k\)-labeled directed graph \((V, E, w)\)

Solution: Matching \(M\)

Weight: \(w(M)\)

Papadimitriou and Yannakakis [PY00] give an FPRAS for \(k-MM\), which we will denote by \(k-MM-\text{Approx}^R\) and use as a black box in our algorithm. Since \(k-MM-\text{Approx}^R\) will be called multiple times, we assume that its success probability is amplified in a way such that the probability that all calls to the FPRAS succeed is at least \(1/2\).

**High-Level Explanation of the Algorithm.** We apply the balancing results to the multi-objective maximum asymmetric traveling salesman problem and obtain a short algorithm that provides a randomized \(1/2\)-approximation. This improves and simplifies the \((1/2-\varepsilon)\)-approximation that was given by Manthey [Man09]. Essentially our algorithm contracts a small number of edges, then computes a maximum matching, adds the contracted edges to the matching, and extends the result in an arbitrary way to a Hamiltonian cycle.

The argument for the correctness of the algorithm is as follows: Each Hamiltonian cycle \(H\) induces two perfect matchings (the edges with odd and the edges with even sequence number in the cycle). For each objective \(i\), the weight of one of the matchings is at least \(1/2 \cdot w_i(H)\). The balancing results assure the existence of a matching \(M\) such that for all objectives the inequality \(w_i(M) \geq 1/2 \cdot w_i(H)\) holds up to a small error. This matching can be approximated with the known FPRAS for multi-objective maximum matching. Moreover, by guessing and contracting a constant number of heavy edges in \(H\) our algorithm can compensate the errors caused by the balancing and by the FPRAS.
Figure 3: Contracting the edge $e = (u, v)$ deletes all edges incident to $v$ and sets the weights of every edge $(u, x)$ to the weights of the edge $(v, x)$ for $x \in V \setminus \{u, v\}$. Any Hamiltonian cycle passes through some edge $(u, x)$ and hence can be expanded to a Hamiltonian cycle through $e$ by replacing $(u, x)$ with the detour $(u, v), (v, x)$.

Contraction and Expansion of Paths. Suppose that for a given $\mathbb{N}^k$-labeled complete directed graph $G = (V, E, w)$ we wish to find some Hamiltonian cycle that contains a particular edge $e = (u, v)$. This reduces to the problem of finding some Hamiltonian cycle in the $\mathbb{N}^k$-labeled complete directed graph $G' = (V', E', w')$ where the edge $e$ is contracted by combining the nodes $u$ and $v$ into a single node while retaining the ingoing edges of $u$ and the outgoing edges of $v$. More formally, we remove $v$ and all incident edges from $G$ and set $w'(u, x) = w(v, x)$ for every $x \in V \setminus \{u, v\}$ (Figure 3). Now suppose we find a Hamiltonian cycle $C'$ in $G'$. Then there exists some $x$ such that $(u, x) \in C'$. Note that $w'(u, x) = w(v, x)$. We replace the edge $(u, x)$ in $C'$ with the detour $(u, v), (v, x)$ and obtain a Hamiltonian cycle $C$ in $G$ passing through $e$. Moreover, $C$ preserves the weights of $C'$ in the sense that $w(C) = w'(C') + w(e)$.

The notion of edge contractions and expansions can easily be extended to sets of pairwise vertex disjoint paths (Figure 4), where each path is contracted edge-by-edge starting at the last edge of the path, and different paths can be contracted in an arbitrary order. We make this precise with the following definitions.

**Definition 3.1.** Let $G = (V, E, w)$ be some $\mathbb{N}^k$-labeled complete directed graph, let $(u, v) \in E$, let $P \subseteq E$ be a path $u_0, e_1, u_1, e_2, u_2, \ldots, e_r, u_r$, and let $Q \subseteq E$ be a set of pairwise vertex disjoint paths $P_1, P_2, \ldots, P_r \subseteq E$.

1. $\text{contract}_{(u,v)}(G) = (V \setminus \{v\}, \{e \in E \mid v \text{ is not incident to } e\}, w')$, where $w'(x, y) = w(x, y)$ if $x \neq u$ and $w'(u, z) = w(v, z)$.
2. $\text{contract}_P(G) = \text{contract}_{e_1}(\text{contract}_{e_2}(\ldots \text{contract}_{e_r}(G) \ldots ))$
3. $\text{contract}_Q(G) = \text{contract}_{P_1}(\text{contract}_{P_2}(\ldots \text{contract}_{P_r}(G) \ldots ))$

We sometimes identify a graph with its edge set and apply contract directly to sets of edges and not to graphs. In this case, we also interpret the value of contract as an edge set.

Observe that the result of contracting several pairwise vertex disjoint paths does not depend on the order in which the paths are contracted.

We define edge expansion in a similar manner. Note that Definition 3.2 becomes essential if $G'$ is obtained from $G$ by a contraction of some set $Q$ of pairwise vertex disjoint paths in $G$. 

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Figure 4: Example for the contraction of the path \{(u, v), (v, y)\} in the graph G resulting in the graph G'' and the following expansion of the tour \{(u, x), (x, u)\} in G''. The final tour in G includes the contracted path.

Definition 3.2. Let G = (V, E, w) and G' = (V', E', w') be two N^k-labeled complete directed graphs, let T ⊆ E' be a Hamiltonian cycle of G', let (u, v) ∈ E, let P ⊆ E be a path \(u_0, e_1, u_1, e_2, u_2, \ldots, e_r, u_r\) and let Q ⊆ E be a set of pairwise vertex disjoint paths \(P_1, P_2, \ldots, P_r\) ⊆ E.

1. expand\(_{(u, v)}\)(T) = \{(x, y) ∈ T | x ≠ u\} ∪ \{(u, v)\} ∪ \{(v, x) | (u, x) ∈ T\}
2. expand\(_P\)(T) = expand\(_{e_r}\)(expand\(_{e_{r-1}}\)(\ldots expand\(_{e_1}\)(T)\ldots))
3. expand\(_Q\)(T) = expand\(_{P_r}\)(expand\(_{P_{r-1}}\)(\ldots expand\(_{P_1}\)(T)\ldots))

Again observe that the result of expanding several pairwise vertex disjoint paths does not depend on the order in which the paths are expanded.

Proposition 3.3. Let G = (V, E, w) be some N^k-labeled complete directed graph, Q ⊆ E be a set of pairwise vertex disjoint paths, and G' = (V', E', w') = contract\(_Q\)(G). For any Hamiltonian cycle T' ⊆ E' of G', the edges T = expand\(_Q\)(T') form a Hamiltonian cycle of G with \(w(T) = w'(T') + w(Q)\).

Approximation Algorithm. First we prove that the following algorithm computes a \((1/2 - \varepsilon)\)-approximation for k-MaxATSP. Then Theorem 3.6 shows that a modification of the algorithm provides a 1/2-approximation.
Algorithm: 2k-MaxATSP-Approx\(_8\)(V, E, w, \(\varepsilon\))

| Input  | \(\mathbb{N}^{2k}\)-labeled complete directed graph \(G = (V, E, w)\) and even \#V |
|--------|--------------------------------------------------------------------------|
| Output | set of Hamiltonian cycles of \(G\)                                      |

1. foreach \(F \subseteq E\) with \(\#F \leq 2k\) that is a set of vertex disjoint paths do
2. \(G' := \text{contract}_F(G)\);
3. \(\mathcal{M} := 2k-\text{MM-Approx}_8(G', \varepsilon)\);
4. foreach \(M \in \mathcal{M}\) do
5. extend \(M\) in an arbitrary way to a Hamiltonian cycle \(T'\) in \(G'\);
6. output \(\text{expand}_F(T')\);
7. end
8. end

Lemma 3.4. Let \(G = (V, E, w)\) be an \(\mathbb{N}^{2k}\)-labeled complete directed graph with an even number of vertices, \(\varepsilon > 0\), and \(T \subseteq E\) some Hamiltonian cycle in \(G\). With probability at least \(1/2\), 2k-MaxATSP-Approx\(_8\)(V, E, w, \(\varepsilon\)) outputs a \((1/2 - \varepsilon)\)-approximation of \(T\) within time polynomial in \(|(V, E, w)| + 1/\varepsilon\).

Proof. Let \(G = (V, E, w)\) be an \(\mathbb{N}^{2k}\)-labeled complete directed graph with even \(m = \#V\), and let \(T\) be some arbitrary Hamiltonian cycle in \(G\).

Claim 3.5. There is a set \(F\) of vertex disjoint paths in \(T\) with \(\#F \leq 2k\) such that there is a matching \(M'\) in the graph \((V', E', w') = \text{contract}_F(G)\) with \(w'(M') \geq \frac{1}{2}w(T) - w(F)\).

Proof. We apply Lemma 2.8 to the sequence of edge weights of \(T\). Having an even number of edges, we can write \(T\) sequentially as

\[
T = u_1, e_1, v_1, f_1, u_2, e_2, v_2, f_2, \ldots, u_p, e_p, v_p, f_p, u_1
\]

where \(u_i, v_i \in V\) and \(e_i, f_i \in T\).

Since \(w(e_i), w(f_i) \in \mathbb{N}^{2k}\), Lemma 2.8 shows that there exist \(k' \in \{0, \ldots, k\}\) and natural numbers \(1 \leq a_1 \leq b_1 < a_2 \leq b_2 < a_3 \leq \cdots < a_{k'} \leq b_{k'} \leq p\) such that for \(I = \bigcup_{i \in \{1, \ldots, k\}} \{a_i, a_i + 1, \ldots, b_i\}\),

\[
w(f_{b_1}) + w(f_{b_2}) + \cdots + w(f_{b_{k'}}) + \sum_{i \in I} w(e_i) + \sum_{i \not\in I} w(f_i) \geq \frac{1}{2} \sum_{i=1}^p (w(e_i) + w(f_i)). \tag{9}
\]

Let \(S = \{f_{b_1}, f_{b_2}, \ldots, f_{b_{k'}}\} \cup \{e_i \mid i \in I\} \cup \{f_i \mid i \not\in I\}\). Observe that it is possible that adjacent edges are contained in \(S\). Figure 5 gives an example.

Observe that for \(1 \leq j \leq p\) and \(f_0 = f_p\) the following holds.

\[
f_{j-1}, e_j \in S \iff \exists i \in \{1, \ldots, k'\}, j = a_i \tag{10}
\]
\[
e_j, f_j \in S \iff \exists i \in \{1, \ldots, k'\}, j = b_i \tag{11}
\]
Figure 5: Some part of the cycle T, where S ⊆ T contains the depicted edges and is partially defined by bi = j, a_i+1 = j + 2, b_i+1 = j + 3, and a_i+2 = b_i+2 = j + 4.

Let \( F = \{e_{a_1}, f_{b_1}, e_{a_2}, f_{b_2}, \ldots, e_{a_k'}, f_{b_k'}\} \) and note that \( \#F = 2k' \). We argue that contracting \( F \) will transform any path in \( S \) into a single edge such that the resulting edge set is a matching:

Suppose \( S \) contains some path \( P = \{e_r, f_r, e_{r+1}, f_{r+1}, \ldots, e_s, f_s\} \), where we assume \( P \) to be maximal (i.e., \( f_{r-1}, e_{s+1} \notin S \)). From (10), (11), and \( a_1 \leq b_1 < a_2 \leq b_2 \ldots < a_{k'} \leq b_{k'} \) we can draw the following conclusions:

- \( e_r, f_r \in P \) yields \( r = b_i \) for some \( 1 \leq i \leq k' \)
- \( f_r, e_{r+1}, f_{r+1} \in P \) yields \( r + 1 = a_{i+1} = b_{i+1} \)
- \( f_{r+1}, e_{r+2}, f_{r+2} \in P \) yields \( r + 2 = a_{i+2} = b_{i+2} \)
- \( \vdots \)
- \( f_{s-1}, e_s, f_s \in P \) yields \( s = a_{s-r+1} = b_{s-r+1} \)

Note that \( e_r \notin F \), since otherwise \( e_{b_i} \in F \), hence \( a_i = r \), and by (10), \( f_{r-1} \in S \), which contradicts the maximality of \( P \). Therefore, contracting \( \{f_{b_i}, e_{a_{i+1}}, f_{b_{i+1}}, \ldots, f_{b_{k'}}\} \subseteq F \) transforms \( P \) into the single edge \( e_r \). A similar argumentation shows the same result for paths that start with some edge \( f_r \) or end with some edge \( e_s \). Hence, contracting \( F \) transforms every path in \( S \) into a single edge, and \( M' = \text{contract}_F(S) \) is a matching in the graph \( (V', E', w') = \text{contract}_F(G) \). We further obtain

\[
\begin{align*}
w'(M') &= w(S) - w(F) \\
&= w(f_{b_1}) + w(f_{b_2}) + \cdots + w(f_{b_{k'}}) + \sum_{i \in I} w(e_i) + \sum_{i \notin I} w(f_i) - w(F) \\
&\geq \frac{1}{2} \sum_{i=1}^{p} (w(e_i) + w(f_i)) - w(F) \\
&= \frac{1}{2} w(T) - w(F)
\end{align*}
\]

which proves the claim.

We fix the iteration of \( 2k\text{-MaxATSP-Approx} \) where the algorithm chooses \( F \) as in the claim. By Claim 3.5 we know that there is a matching \( M' \) of \( G' = (V', E', w') \) with

\[
\begin{align*}
w'(M') &\geq \frac{1}{2} w(T) - w(F).
\end{align*}
\]
Hence, with probability at least $1/2$ the set $M$ contains some matching $M$ of $G'$ such that

$$w'(M) \geq (1 - \varepsilon)w'(M') \geq (1 - \varepsilon) \left( \frac{1}{2}w(T) - w(F) \right).$$

We extend $M$ to some Hamiltonian cycle $T'$ of $G'$ in an arbitrary way without losing weights. By Proposition 3.3 we can expand $T'$ with $F$ and obtain a Hamiltonian cycle $\tilde{T}$ in $G$ with

$$w(\tilde{T}) = w'(T') + w(F) \geq w'(M) + w(F) \geq (1 - \varepsilon) \left( \frac{1}{2}w(T) - w(F) \right) + w(F) \geq (1 - \varepsilon) \frac{1}{2}w(T) + \varepsilon w(F) \geq (1 - \varepsilon) \frac{1}{2}w(T).$$

Moreover, the running time of every operation of the algorithm, including the execution of the randomized maximum matching algorithm, and the number of iterations of the loops are polynomial in the length of the input and $1/\varepsilon$, which completes the proof of the lemma.

**Theorem 3.6.** $k$-MaxATSP is randomized $1/2$-approximable.

**Proof.** Let $G = (V, E, w)$ be an $\mathbb{N}^{2k}$-labeled complete directed graph with even $m = \#V$, and let $T$ be some arbitrary Hamiltonian cycle in $G$. The proof can easily be extended to graphs with an odd number of vertices or objectives.

For each $1 \leq i \leq 2k$ we choose an $f_i \in T$ with $w(f_i) \geq \frac{1}{m}w(T)$ (i.e., $f_i$ is a heaviest edge of $T$ with respect to component $i$). We let $F \subseteq T$ be a smallest set with even cardinality containing all the $f_i$. We get $\#F \leq 2k$ and

$$w(F) \geq \frac{1}{m}w(T). \quad (13)$$

$F$ is a set of vertex disjoint paths in $T$ and hence can be used to contract edges in $G$ and $T$. Let $G' = (V', E', w') = \text{contract}_F(G)$ and $T' = \text{contract}_F(T)$. Clearly, $T'$ is a Hamiltonian cycle in $G'$. Moreover, $G'$ has an even number of vertices. By Lemma 3.4 we can find in polynomial time a Hamiltonian cycle $\tilde{T}$ in $G'$ such that $w'(\tilde{T}) \geq (1 - \varepsilon) \frac{1}{2}w'(T')$, where $\varepsilon = 1/m$. Moreover, we can expand $\tilde{T}$ with $F$ to obtain a Hamiltonian cycle $\hat{T}$ in $G$ such that

$$w(\hat{T}) = w'(\hat{T}) + w(F) \geq (1 - \varepsilon) \frac{1}{2}w'(T') + w(F) \geq \left( \frac{1}{2}w'(T') + \frac{1}{2}w(F) \right) + \frac{1}{2}w(F) - \varepsilon w'(T') \geq \frac{1}{2}w(T) + \frac{1}{2}(w(F) - \varepsilon w'(T'))$$

$$\geq \frac{1}{2}w(T),$$

$$\geq (\ast) \frac{1}{2}w(T),$$
where (∗) follows from

\[ w(F) - \varepsilon w'(T') = w(F) - \frac{1}{m}w'(T') \geq w(F) - \frac{1}{m}w(T) \geq \frac{1}{m}w(T) - \frac{1}{m}w(T) = 0. \]

Note that, although we do not know the set \( F \) of heaviest edges in the Hamiltonian cycle, we can simply try all possible sets of heaviest edges, since the number of objectives \( 2k \) is constant. \( \square \)

### 3.3 \( k \)-Objective Maximum Weighted Satisfiability

#### Definitions.
We consider formulas over a finite set of propositional variables \( V \). A literal is a propositional variable \( v \in V \) or its negation \( \overline{v} \), a clause is a finite, nonempty set of literals, and a formula in conjunctive normal form (CNF, for short) is a finite set of clauses. A truth assignment is a mapping \( I : V \rightarrow \{0, 1\} \). For some \( v \in V \), we say that \( I \) satisfies the literal \( v \) if \( I(v) = 1 \), and \( I \) satisfies the literal \( \overline{v} \) if \( I(v) = 0 \). We further say that \( I \) satisfies the clause \( C \) and write \( I(C) = 1 \) if there is some literal \( l \in C \) that is satisfied by \( I \), and \( I \) satisfies a formula in CNF if \( I \) satisfies all of its clauses. For a set of clauses \( \hat{H} \) and a variable \( v \) let \( \hat{H}[v] = \{ C \in \hat{H} \mid v \in C \} \) be the set of clauses that are satisfied if this variable is set to one, and analogously \( \hat{H}[\overline{v}] = \{ C \in \hat{H} \mid \overline{v} \in C \} \) be the set of clauses that are satisfied if this variable is set to zero.

Given a formula in CNF and a \( k \)-objective weight function that maps each clause to a \( k \)-objective weight, our goal is to find truth assignments that maximize the sum of the weights of all satisfied clauses.

**\( k \)-Objective Maximum Weighted Satisfiability (\( k \)-MaxSAT)**

Instance: Formula \( H \) in CNF over a set of variables \( V \), weight function \( w : H \rightarrow \mathbb{N}^k \)

Solution: Truth assignment \( I : V \rightarrow \{0, 1\} \)

Weight: Sum of the weights of all clauses satisfied by \( I \), i.e., \( w(I) = \sum_{C \in \hat{H}, I(C) = 1} w(C) \)

**High-Level Explanation of the Algorithm.** We apply the balancing results to \( k \)-MaxSAT. For a given formula \( H \) in CNF over the variables \( V \), the strategy is as follows: Start with a list of the variables \( V \) and guess a partition of this list into \( 2k \) consecutive intervals. Assign 1 to the variables in every second interval and 0 to the remaining variables. The balancing results assure the existence of a partition that yields an assignment whose weights are approximately one half of the total weights of \( H \), up to an error induced by the variables at the boundaries of the partition. The error can be removed by first guessing a satisfying assignment for several influential variables \( V^0 \) of the formula. This results in a \( 1/2 \)-approximation for \( k \)-MaxSAT.
Approximation Algorithm. We show that the following algorithm computes a $1/2$-approximation for $2k$-MaxSAT.

Algorithm: $2k$-MaxSAT-Approx$(H, w)$

| Line | Description |
|------|-------------|
| 1    | foreach $V^0 \subseteq V$ with $\#V^0 \leq (2k)^2$ do |
| 2    | let $I(v) := 0$ for all $v \in V^0$; |
| 3    | $G := \{C \in H \mid \neg \exists v \in V^0 (\overline{v} \in C)\}$; |
| 4    | $V^1 := \{v \in V \setminus V^0 \mid 2k \cdot w(G[\overline{v}]) \leq w(H \setminus G)\}$; |
| 5    | set $I(v) := 1$ for all $v \in V^1$; |
| 6    | $V' := V \setminus (V^0 \cup V^1)$; |
| 7    | foreach $a_1, b_1, a_2, b_2, \ldots, a_k, b_k \in \{i \mid v_i \in V'\}$ do |
| 8    | foreach $v_i \in V'$ do |
| 9    | if $\exists j(a_j \leq i \leq b_j)$ then $I(v_i) := 1$ else $I(v_i) := 0$ |
| 10   | end |
| 11   | output $I$ |
| 12   | end |
| 13   | end |

Theorem 3.7. $k$-MaxSAT is $1/2$-approximable.

Proof. In the following, we assume without loss of generality that the number of objectives $2k$ is even. We show that the approximation is realized by the algorithm $2k$-MaxSAT-Approx. First note that this algorithm runs in polynomial time since $k$ is constant. For the correctness, let $(H, w)$ be the input where $H$ is a formula over the variables $V = \{v_1, \ldots, v_m\}$ and $w: H \to \mathbb{N}^{2k}$ is the $2k$-objective weight function. Let $I_o: V \to \{0, 1\}$ be an optimal truth assignment. We show that there is an iteration of the loops of $2k$-MaxSAT-Approx$(H, w)$ that outputs a truth assignment $I$ such that $w(I) \geq w(I_o)/2$. First we show that there is an iteration of the first loop that uses a suitable set $V^0$.

Claim 3.8. There is some set $V^0 \subseteq \{v \in V \mid I_o(v) = 0\}$ with $\#V^0 \leq (2k)^2$ such that for $G = \{C \in H \mid \neg \exists v \in V^0 (\overline{v} \in C)\}$ and any $v \in V \setminus V^0$ it holds that

$$2k \cdot w(G[\overline{v}]) \leq w(H \setminus G) \implies I_o(v) = 1.$$ 

Proof. As a special case, if $\#\{v \in V \mid I_o(v) = 0\} < (2k)^2$, the assertion obviously holds for $V^0 = \{v \in V \mid I_o(v) = 0\}$, since $I_o(v) = 1$ for all $v \in V \setminus V^0$.

Otherwise, let $V^0 = \{u_{2kt+r} \mid r = 1, 2, \ldots, 2k \text{ and } t = 0, 1, \ldots, 2k - 1\}$, where the $u_{2kt+r} \in V$ are defined inductively in the following way:

(II) $H_0 := H$
(IS) $2kt + r - 1 \rightarrow 2kt + r$:

- choose $v \in V \setminus \{u_1, \ldots, u_{2kt+r-1}\}$ such that $I_o(v) = 0$ and $w_r(H_{2kt+r-1[v]})$ is maximal
- $u_{2kt+r} := v$
- $H_{2kt+r} := H_{2kt+r-1} \setminus H_{2kt+r-1[v]}$
- $\alpha_{2kt+r} := w(H_{2kt+r-1[v]})$

We now show that the stated implication holds, so let $v \in V \setminus V^0$ and $j \in \{1, 2, \ldots, 2k\}$ such that $2k \cdot w_j(G[\overline{v}]) > w_j(H \setminus G)$. Because the union $\bigcup_{i=1}^{4k} H_{i-1[\overline{v}_i]} = H \setminus G$ is disjoint, we get

$$w(H \setminus G) = \sum_{r=1}^{2k} \sum_{t=0}^{2k-1} \alpha_{2kt+r} \geq \sum_{t=0}^{2k-1} \alpha_{2kt+j}$$

and thus

$$w_j(G[\overline{v}]) > \frac{1}{2k} \sum_{t=0}^{2k-1} (\alpha_{2kt+j})_j.$$  

Hence, by a pigeonhole argument, there must be some $t \in \{0, 1, \ldots, 2k-1\}$ such that $w_j(G[\overline{v}]) > (\alpha_{2kt+j})_j$. But since $G \subseteq H_{2kt+j-1}$ and thus even $w_j(H_{2kt+j-1[\overline{v}]}) > w_j(G[\overline{v}]) > (\alpha_{2kt+j})_j$, the only reason we did not choose $v$ in the iteration $2kt + j$ (or even earlier) is that $I_o(v) = 1$. 

We choose the iteration of the algorithm where $V^0$ equals the set whose existence is guaranteed by Claim [3.8]. Furthermore let $G$ and $V^1$ be defined as in the algorithm and observe that by the claim it holds that $I_o(v) = 1$ for all $v \in V^1$. Since $I_o(v) = 0$ for all $v \in V^0$, the truth assignment $I$ defined in the algorithm coincides with $I_o$ on $V^0 \cup V^1$.

Let further $V' = V \setminus (V^0 \cup V^1)$ and $G' = \{C \in G \mid \neg \exists v \in V^0 (\overline{v} \in C) \land \neg \exists v \in V^1 (v \in C) \land \exists v \in V' (v \in C \lor \overline{v} \in C)\}$ be the set of clauses that are not yet satisfied by $I$ but that could be satisfied by further extending $I$.

Now we apply the balancing result. Let $L' = V' \cup \{\overline{v} \mid v \in V'\}$. For $v_i \in V'$ let

$$x_i = \sum_{C \in G'[v_i]} \frac{w(C)}{\#(C \cap L')} \quad \text{and} \quad y_i = \sum_{C \in G'[\overline{v}_i]} \frac{w(C)}{\#(C \cap L')}$$

and for $v_i \in V^0 \cup V^1$ let

$$x_i = y_i = 0.$$

It holds that

$$\sum_{v_i \in V} x_i + y_i = \sum_{v_i \in V'} x_i + y_i = w(G').$$

Note that for all $v_i \in V'$, we have the bound $y_i \leq w(G'[\overline{v}_i]) \leq w(G[\overline{v}]) \leq \frac{1}{2k} w(H \setminus G)$ because of the definition of $V'$ and $V^1$. Hence, for all $v_i \in V$,

$$y_i \leq \frac{1}{2k} w(H \setminus G).$$
If we scale all values $x_i$ and $y_i$ to natural numbers, then by Corollary 2.11, there exist $k' \leq k$ disjoint, nonempty intervals $J_1, \ldots, J_{k'} \subseteq \{1, \ldots, m\}$ such that for $J = J_1 \cup \cdots \cup J_{k'}$ it holds that
\[
\sum_{i \in J} x_i + \sum_{i \notin J} y_i \geq \frac{1}{2} w(G') - k' \frac{1}{2k} w(H \setminus G) \geq \frac{1}{2} (w(G') - w(H \setminus G)).
\]

The algorithm tries all combinations of $k$ (possibly empty) intervals $J_1 = [a_1, b_1], \ldots, J_k = [a_k, b_k]$. In particular, it will test the combination of the $k'$ nonempty intervals mentioned in Corollary 2.11.

For $I$ being the truth assignment generated in this iteration it holds that
\[
\w(\{C \in G' \mid I(C) = 1\}) \geq \sum_{i \in J} x_i + \sum_{i \notin J} y_i \geq \frac{1}{2} (w(G') - w(H \setminus G)). \tag{14}
\]

Furthermore, since $I$ and $I_o$ coincide on $V \setminus V'$, we have
\[
\w(\{C \in H \setminus G' \mid I(C) = 1\}) = \w(\{C \in H \setminus G' \mid I_o(C) = 1\}) \geq \w(\{C \in H \setminus G \mid I_o(C) = 1\}) = \w(\{H \setminus G\}). \tag{15}
\]

Thus we finally obtain
\[
\w(I) = \w(\{C \in H \setminus G' \mid I(C) = 1\}) + \w(\{C \in G' \mid I(C) = 1\}) \geq \w(\{C \in H \setminus G' \mid I(C) = 1\}) + \frac{1}{2} (w(G') - w(H \setminus G)) \geq \w(I_o) + \frac{1}{2} w(G'). \tag{16}
\]

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