On Sharp Constants for Dual Segal–Bargmann $L^p$ Spaces

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Abstract

We study dilated holomorphic $L^p$ space of Gaussian measures over $\mathbb{C}^n$, denoted $\mathcal{H}_{p,\alpha}^n$, with variance scaling parameter $\alpha > 0$. The duality relations $(\mathcal{H}_{p,\alpha}^n)^* \cong \mathcal{H}_{p',\alpha}^n$ hold with $\frac{1}{p} + \frac{1}{p'} = 1$, but not isometrically. We identify the sharp lower constant comparing the norms on $\mathcal{H}_{p',\alpha}^n$ and $(\mathcal{H}_{p,\alpha}^n)^*$, and provide upper and lower bounds on the sharp upper constant. We prove a local version of the sharpness of the upper constant.

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1 Introduction

This paper is concerned with the holomorphic $L^p$ spaces associated to Gaussian measures on $\mathbb{C}^n$. In the case $p = 2$, such spaces are often called Segal–Bargmann spaces [3] or Fock spaces [7]. They are core examples in the theory of holomorphic reproducing kernel Hilbert spaces, with connections to quantum field theory, stochastic analysis, and beyond. The scaling of duality between these holomorphic $L^p$-spaces is still not fully understood; this paper presents some new sharp results, and new puzzles about these dual norms.

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To fix notation, let $\alpha > 0$, $n \in \mathbb{N}$, and let $\gamma^n_\alpha$ denote the following Gaussian probability measure on $\mathbb{C}^n$:

$$\gamma^n_\alpha(dz) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2} \lambda^n(dz),$$

where $\lambda^n$ is the Lebesgue measure on $\mathbb{C}^n$. The spaces considered in this paper are of the form $L^p_{hol}(\gamma^n_\alpha)$ for $1 \leq p < \infty$ and some $\alpha > 0$, the subspaces of the full $L^p(\gamma^n_\alpha)$-spaces consisting of holomorphic functions. These are Banach spaces in the usual $L^p$-norm. However, as discovered by Sjörgen [6] and proved as [2 Proposition 1.5], in this scaling, with $\frac{1}{p'} + \frac{1}{p} = 1$ as usual, $L^p_{hol}(\gamma^n_\alpha)$ and $L^{p'}(\gamma^n_\alpha)$ are not dual to each other when $p \neq 2$. It was shown by Janson, Peetre, and Rochberg [4] that the correct scaling requires the parameter $\alpha$ to dilate with $p$. That is, we define the **dilated holomorphic** $L^p$ **space** as

$$\mathcal{H}^n_{p,\alpha} \equiv L^p_{hol}(\gamma^n_{\alpha/p/2}) = \left\{ f \in \text{Hol}(\mathbb{C}^n) : \int |f(z) e^{-\alpha|z|^2/2}|^p \lambda^n(dz) < \infty \right\},$$

with norm

$$\|h\|_{p,\alpha} = \|h\|_{\mathcal{H}^n_{p,\alpha}} \equiv \left(\int_{\mathbb{C}^n} |h|^p d\gamma^n_{\alpha/p/2}\right)^{1/p}. $$

Similarly, if $\Lambda \in (\mathcal{H}^n_{p,\alpha})^*$ is a bounded linear functional, denote its dual norm by

$$\|\Lambda\|_{p,\alpha} = \|\Lambda\|_{(\mathcal{H}^n_{p,\alpha})^*} \equiv \sup_{g \in \mathcal{H}^n_{p,\alpha} \setminus \{0\}} \frac{|\Lambda(g)|}{\|g\|_{p,\alpha}}. $$

(We de-emphasize the $n$-dependence of the norms $\| \cdot \|_{p,\alpha}$ and $\| \cdot \|_{p,\alpha}^*$; it will always be clear from context.) It was shown in [4] that $\mathcal{H}^n_{p,\alpha}$ and $\mathcal{H}^n_{p',\alpha}$ are **dual** spaces for $1 < p < \infty$. One of the two main theorems of the present authors’ paper [2] was the following estimate on the sharp constants of comparison for the dual norms.

**Theorem 1.1** (Theorem 1.2 in [2]). Let $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{p'} = 1$. Define the constant $C_p$ by

$$C_p \equiv 2 \frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}. $$

Let $n \in \mathbb{N}$ and $\alpha > 0$. Define $\langle f, g \rangle = \int_{\mathbb{C}^n} f \overline{g} d\gamma^n_\alpha$. Then for any $h \in \mathcal{H}^n_{p,\alpha}$,

$$\|h\|_{p',\alpha} \leq \|\langle \cdot, h \rangle \|_{p,\alpha} \leq C^n_p \|h\|_{p,\alpha}. $$

Presently, we are interested in the sharpness of the inequalities in (1.5). In fact, the first inequality is sharp, and this yields a new concise proof of a pointwise bound for the space $\mathcal{H}^n_{p,\alpha}$.

**Theorem 1.2.** Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $\alpha > 0$. Then

$$\inf_{h \in \mathcal{H}^n_{p,\alpha} \setminus \{0\}} \frac{\|\langle \cdot, h \rangle \|_{p,\alpha}}{\|h\|_{p',\alpha}} = 1. $$

It follows that, for any $z \in \mathbb{C}^n$, and any $g \in \mathcal{H}^n_{p,\alpha}$,

$$|g(z)| \leq e^{-\frac{\alpha}{2}|z|^2} \|g\|_{p,\alpha}. $$

(1.6)

**Remark 1.3.** The bound (1.6) is well-known; it can be found, for example, as [7, Theorem 2.8]. In fact, it is common to define a supremum norm on holomorphic functions $g$ (see, for example, [7]) as

$$\|g\|_{\infty,\alpha} = \sup_{z \in \mathbb{C}^n} g(z) e^{-\frac{\alpha}{2}|z|^2},$$

in which case (1.6) can be elegantly rewritten as $\|g\|_{\infty,\alpha} \leq \|g\|_{p,\alpha}$ for $1 < p < \infty$. 

That the first inequality in (1.5) should hold sharply is natural to expect from the method of proof given in [2]. Indeed, it is instructive to write (1.5) in the alternate form proven in our first paper. Note that \( \mathcal{H}_{p,\alpha}^n \) is a closed subspace of \( L^2(\gamma_{\alpha}^n) \); let \( P_{\alpha}^n : L^2(\gamma_{\alpha}^n) \to \mathcal{H}_{p,\alpha}^n \) denote the orthogonal projection. In fact, \( P_{\alpha}^n \) is an integral operator that is bounded from \( L^p(\gamma_{\alpha\beta}^n) \) to \( \mathcal{H}_{p,\alpha}^n \) for all \( 1 < p < \infty \), as was originally shown in [4]. Denote by \( \|P_{\alpha}^n\|_{p \to p} = \|P_{\alpha}^n : L^p(\gamma_{\alpha}^n) \to \mathcal{H}_{p,\alpha}^n\| \). In [2] Lemma 1.18, we proved that

\[
\frac{1}{\|P_{\alpha}^n\|_{p \to p}} \|h\|_{p',\alpha} \leq \frac{1}{C_p^n}\|\langle \cdot, h \rangle_\alpha\|_{p,\alpha} \leq \|h\|_{p',\alpha},
\]

(1.7)

The \( 1/C_p^n \) in the middle term comes from the global geometry underlying these spaces. Note from \( \|P_{\alpha}^n\|_{p \to p} \) that \( \mathcal{H}_{p,\alpha}^n \) can be thought of as consisting of “holomorphic sections”; functions \( F \) of the form \( F(z) = f(z)e^{-\alpha |z|^2/2} \) for some holomorphic \( f \); the integrability condition for containment in \( \mathcal{H}_{p,\alpha}^n \) is then simply that \( F \in L^p(\mathbb{C}^n, \lambda^n) \). The factor \( 1/C_p^n \) then arises from the constants relating the norms \( \| \cdot \|_{p,\alpha} \) and \( \| \cdot \|_{p',\alpha} \) to the \( L^p(\mathbb{C}^n, \lambda^n) \)- and \( L^{p'}(\mathbb{C}^n, \lambda^n) \)-norms, yielding \( p/1/p \) and \( p'/1/p' \) factors from the normalization coefficients of the measures \( \gamma_{\alpha\beta}^n/2 \). The first inequality in (1.7) then simplifies due to the first main theorem [2] Theorem 1.1], which states that \( \|P_{\alpha}^n\|_{p \to p} = C_p^n \). The sharpness of the first inequality in (1.5) is indicative of the fact that the orthogonal projection \( P_{\alpha}^n \) controls the geometry of the spaces \( \mathcal{H}_{p,\alpha}^n \).

In this context, the second inequality in (1.7), and hence in (1.5), is simply Hölder’s inequality. In the larger spaces \( L^p(\mathbb{C}^n, \lambda^n) \) and \( L^{p'}(\mathbb{C}^n, \lambda^n) \) where the section spaces \( \mathcal{H}_{p,\alpha}^n \) live, Hölder’s inequality is, of course, sharp: if \( F \in L^p(\mathbb{C}^n, \lambda^n) \), then the function \( G = |F|^{p-2}F \) is in \( L^{p'}(\mathbb{C}^n, \lambda^n) \) and \( \|G\|_{p'} = \|F\|_p^{p'/p} \), so that \( \langle F, G \rangle = \|F\|^p_2 = \|F\|_p\|G\|_{p'} \). However, the function \( G \) is typically not a holomorphic section, and so it is not a surprise that the same saturation argument fails in the spaces \( \mathcal{H}_{p,\alpha}^n \). In fact, we can say more.

**Theorem 1.4.** Let \( 1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1, \) and \( \alpha > 0 \). If \( g \in \mathcal{H}_{p,\alpha}^n \) and \( h \in \mathcal{H}_{p',\alpha}^n \) are non-zero, then

\[
|\langle g, h \rangle_\alpha| < C_p^n \|g\|_{p,\alpha} \|h\|_{p',\alpha}.
\]

(1.8)

Theorem 1.4 asserts that Hölder’s inequality is a strict inequality in the Segal-Bargmann spaces. It is a priori possible that the inequality is nevertheless saturated by a sequence in \( \mathcal{H}_{p,\alpha}^n \times \mathcal{H}_{p',\alpha}^n \), but we believe this is not the case. Indeed, we conjecture that the second inequality in (1.5) is not sharp. To the question of the sharp constant, we prove the following.

**Theorem 1.5.** Let \( 1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1, n \in \mathbb{N}, \) and \( \alpha > 0 \). For \( g \in \mathcal{H}_{p,\alpha}^n \) and \( h \in \mathcal{H}_{p',\alpha}^n \) nonzero, define

\[
\mathcal{R}_{p,\alpha}(g, h) = \frac{|\langle g, h \rangle_\alpha|}{\|g\|_{p,\alpha} \|h\|_{p',\alpha}}.
\]

Then

\[
C_p^n \leq \sup_{g \in \mathcal{H}_{p,\alpha}^n \setminus \{0\}} \sup_{h \in \mathcal{H}_{p',\alpha}^n \setminus \{0\}} \mathcal{R}_{p,\alpha}(g, h) \leq C_p^n.
\]

(1.9)

We can exhibit sequences in \( \mathcal{H}_{p,\alpha}^n \times \mathcal{H}_{p',\alpha}^n \) that saturate (1.8) with \( C_p^{n/2} \) in place of \( C_p^n \), as will be demonstrated in the proof of Theorem 1.5 (cf. (3.10)); indeed, such a saturating sequence can be built from monomials. If the same bound could be shown to hold not only for monomials but all holomorphic polynomials, this would prove the sharpness of the \( C_p^{n/2} \)-bound in general (since holomorphic polynomials are dense in \( \mathcal{H}_{p,\alpha}^n \)). While we have not yet proven this conjecture, we can prove a local version of it (in the case \( n = 1 \)), which is our final theorem.

**Theorem 1.6.** For \( N \in \mathbb{N} \), let \( \mathcal{A}^N \) denote the affine space of monic holomorphic polynomials \( q(z) = z^N + O(z^{N-1}) \) of degree \( N \). There is a neighborhood \( \mathcal{B}^N \subseteq \mathcal{A}^N \) of \( z^N \) such that

\[
\sup_{q_1, q_2 \in \mathcal{B}^N} \mathcal{R}_{p,\alpha}(q_1, q_2) < C_p^{1/2}.
\]
The supremum over $\mathcal{B}^N$ converges to $C_p^{1/2}$ as $N \to \infty$.

This is a local version of the desired sharp inequality in the sense that, if the neighborhood $\mathcal{B}^N$ could be shown to be all of $\mathcal{A}^N$, this would prove the conjectured sharp version of (1.5) in the case $n = 1$; namely

$$\|h\|_{p',\alpha} \leq \|\langle \cdot \rangle_{p,\alpha}\|_{p,\alpha}^{*} \leq C_p^{1/2} \|h\|_{p',\alpha}.$$ 

2 The Sharp Lower Constant

Our overall goal in this section is to prove Theorem 1.2. To prove this theorem and others in the paper, many integrals involving Gaussian and exponential functions will be calculated. Often the details of these calculations will be omitted, but are based on the following formula (cf. 5):

**Lemma 2.1.** Let $A$ be a complex symmetric matrix, $v$ a vector in $\mathbb{R}^k$, and let $(\cdot, \cdot)$ denote the standard inner product on $\mathbb{R}^k$. Define the function $f(x) = \exp(-(x, Ax) + 2(v, x))$. Then $f \in L^1(\mathbb{R}^k)$ if and only if $\Re(A)$ is positive definite, and in this case,

$$\int_{\mathbb{R}^k} e^{-(x, Ax)+2(v,x)} \, dx = \frac{\pi^{k/2}}{\sqrt{\det(A)}} e^{(v, A^{-1}v)}.$$ 

(2.1)

Theorem 1.2 has two claims: first, that

$$\inf_{h \in \mathcal{G}_{p',\alpha}(\mathbb{C}^n) \setminus \{0\}} \frac{\|\langle \cdot \rangle_{p,\alpha}\|_{p,\alpha}}{\|h\|_{p',\alpha}} = 1$$ 

(2.2)

and the infimum is achieved on functions of the form $h_{\alpha}^{\gamma}(w) = e^{\alpha(w,z)}$ for any $z \in \mathbb{C}^n$. Secondly, for any $z \in \mathbb{C}^n$, and any $g \in \mathcal{G}_{p,\alpha}$,

$$|g(z)| \leq e^{-\frac{\gamma}{2} |z|^2} \|g\|_{p,\alpha}$$ 

(2.3)

and the above equation is sharp.

As we stated in the introduction, the equation (2.3) is well-known. In fact one can use it to prove the first part of the Theorem 1.2. To see how, we first record a fact (first proved in 4) that will be useful in the following arguments as well. The projection $P_{\alpha}^n : L^2(\gamma_{\alpha}^n) \to \mathcal{G}_{p,\alpha}$ is given by the integral operator

$$(P_{\alpha}^n g)(z) = \int_{\mathbb{C}^n} e^{\alpha(z,w)} g(w) \gamma_{\alpha}^n(w) \, dw = \langle g, h_{\alpha}^{\gamma} \rangle_{\alpha}.$$ 

(2.4)

Since polynomials are dense in $L^2(\gamma_{\alpha}^n)$, we may extend this integral operator to act densely on any space in which polynomials are dense. The first main theorem of 2 shows that $P_{\alpha}^n$ is, in fact, bounded on $L^p(\gamma_{\alpha_p}^n)$, with image in $\mathcal{G}_{p,\alpha}$. Now, any holomorphic polynomial $g$ over $\mathbb{C}^n$ is in $\mathcal{G}_{p,\alpha}^n$ and so $P_{\alpha}^n g = g$; since holomorphic polynomials are dense in $\mathcal{G}_{p,\alpha}$, it therefore follows that

$$(P_{\alpha} g)(z) = \langle g, h_{\alpha}^{\gamma} \rangle_{\alpha} = g(z), \quad \text{for all} \quad z \in \mathbb{C}^n, \ g \in \mathcal{G}_{p,\alpha}.$$ 

(2.5)

**Remark 2.2.** Since $\mathcal{G}_{p,\alpha} \subset L^p(\gamma_{\alpha_p}^n)$, it might seem more natural to expect that the reproducing formula for functions in $\mathcal{G}_{p,\alpha}$ should involve the reproducing kernel $h_{\alpha}^{\gamma_p/2}(w) = e^{\alpha_p(z,w)}$. This would be true if the inner product used was $\langle \cdot, \cdot \rangle_{\alpha_p/2}$; it is a remarkable and useful fact that, using the fixed inner product $\langle \cdot, \cdot \rangle_{\alpha}$ for all $\mathcal{G}_{p,\alpha}$ spaces gives a consistent reproducing kernel for all of them.
Now we can state the following lemma:

**Lemma 2.3.** Let \(1 < p < \infty\). If \((2.3)\) holds for all \(g \in \mathcal{H}_{p,\alpha}^n\) and \(z \in \mathbb{C}^n\), then \((2.2)\) holds.

**Proof.** Assume \((2.3)\) holds for all \(g \in \mathcal{H}_{p,\alpha}^n\). Note that by \((1.5)\) we know that
\[
\inf_{h \in \mathcal{H}_{p',\alpha}} \frac{\|\langle \cdot, h \rangle_{\alpha} \|^*_{p,\alpha}}{\|h\|_{p',\alpha}} \geq 1
\]
(2.6)

Assume that \((2.3)\) holds for functions \(g\) in \(\mathcal{H}_{p,\alpha}^n\). For any fixed \(z \in \mathbb{C}^n\), the functional \(\langle \cdot, h \rangle_{\alpha}\) is pointwise evaluation at \(z\), cf. \((2.5)\). Thus, by \((2.3)\), we know that
\[
\|\langle \cdot, h \rangle_{\alpha}\|_{p,\alpha}^* \leq e^{-\frac{\alpha}{2}|z|^2}
\]
However, one can easily calculate using Lemma 2.1 that
\[
\|h\|_{p',\alpha} = e^{-\frac{\alpha}{2}|z|^2},
\]
proving that \(\|\langle \cdot, h \rangle_{\alpha}\|_{p,\alpha}^* \leq \|h\|_{p',\alpha}\). Thus, by \((2.6)\) we have proven \((2.2)\) and shown that this infimum is achieved at each \(h\), as desired. \(\square\)

Note that Lemma 2.3 actually proves Theorem 1.2 since \((2.3)\) is known to hold for all \(g \in \mathcal{H}_{p,\alpha}^n\) and \(z \in \mathbb{C}^n\). However, we have an alternate proof of Theorem 1.2 that proves the result independently of the a priori truth of \((2.3)\). That is, without assuming \((2.3)\) is true, we can prove both \((2.2)\) and \((2.3)\). This proof is based on the following lemma:

**Lemma 2.4.** Let \(n \in \mathbb{N}, \alpha > 0\), and \(1 < p < \infty\) with \(\frac{1}{p} + \frac{1}{p'} = 1\). Let \(h \in \mathcal{H}_{p,\alpha}^n\). Then
\[
\|\langle \cdot, h \rangle_{\alpha}\|_{p,\alpha}^* = C_p^n \inf_{f \in P_{\alpha,\lambda}^{-1}h} \|f\|_{p',\alpha}.
\]
Furthermore, there exists a function \(\tilde{f} \in P_{\alpha,\lambda}^{-1}h\) where the infimum is achieved.

The rest of this section is devoted to proving the Lemma 2.4 and using it to prove Theorem 1.2.

### 2.1 A Relationship Between the Norm, the Dual Norm, and the Projection \(P_{\alpha}\)

Before we prove Lemma 2.4 we need some preliminary results. While we refer to the mapping \(P_{\alpha} : L^p(\gamma_{\alpha p/2}) \to \mathcal{H}_{p,\alpha}^n\) as a “projection,” it is of course not a true orthogonal projection for \(p \neq 2\) as in this case \(\mathcal{H}_{p,\alpha}^n\) is not a Hilbert space. However, it acts like a projection in the following ways (as proven in \([4]\)): \(P_{\alpha}\) is the identity on elements in \(\mathcal{H}_{p,\alpha}^n\) (this was actually shown in \((2.3)\)) and \(P_{\alpha}\) is “self-adjoint” in the following sense:
\[
\langle P_{\alpha} g, h \rangle_{\alpha} = \langle g, P_{\alpha} h \rangle_{\alpha} \text{ for all } (g, h) \in L^p(\gamma_{\alpha}^n) \times L^{p'}(\gamma_{\alpha}^n). \tag{2.7}
\]
As we alluded to in the Introduction, to prove a statement about the spaces \(\mathcal{H}_{p,\alpha}^n\) it can be useful to prove an analogous statement in a corresponding Lebesgue measure setting. Indeed, the differing measures of \(\gamma_{\alpha p/2}\) and \(\gamma_{p/2}\) preclude us from using some basic results of duality in \(L^p\) spaces. To remove this complication, we define a mapping \(g_{\alpha,p} : L^p(\gamma_{\alpha p/2}) \to L^p(\mathbb{C}^n, \lambda^n)\) as
\[
(g_{\alpha,p} f)(z) = \left(\frac{p \alpha}{2 \pi}\right)^{n/p} e^{-\frac{\alpha}{2} |z|^2} f(z).
\]
It is easy to check that $g_{\alpha,p}$ is an isometric isomorphism. Furthermore, define the set $S^p_{\alpha}$ as the image of $\mathcal{H}^n_{p,\alpha}$ under $g_{\alpha,p}$ above. That is,

$$S^p_{\alpha} = \{ F : \| F \|_{p,\lambda} < \infty, \ z \mapsto F(z) e^{i\alpha z^2} \text{ is holomorphic} \}$$

where $\| F \|_{p,\lambda}$ is the $L^p(\mathbb{C}^n, \lambda^n)$ norm of $F$. The space $S^p_{\alpha}$ is the set of so-called “holomorphic sections” mentioned in the introduction. Using the isomorphism $g_{\alpha,p}$ one can see that $(S^p_{\alpha})^* = S^p_{\alpha}'$ as identified using the usual Lebesgue integral pairing $(G,H)_\lambda = \int_{\mathbb{C}^n} G \overline{H} \ d\lambda^n$.

Define a new operator $Q_{\alpha} : L^p(\mathbb{C}^n, \lambda^n) \to L^p(\mathbb{C}^n, \lambda^n)$ as

$$Q_{\alpha} = g_{\alpha,p} P_{\alpha} g_{\alpha,p}^{-1}.$$

Note that $Q_{\alpha}$ does not actually depend on $p$. Indeed, $g_{\alpha,p}$ only depends on $p$ through multiplication by $p$-dependent constant. From this fact, it is easy to see that

$$Q_{\alpha} = g_{\alpha,2} P_{\alpha} g_{\alpha,2}^{-1},$$

justifying the notation. By definition, the following diagram commutes:

Denote by $\| Q_{\alpha} \|_{p \to p}$ the norm of $Q_{\alpha}$ as an operator on $L^p(\mathbb{C}^n, \lambda^n)$. Since $g_{\alpha,p}$ and its inverse are isometric, it is not difficult to show that $P_{\alpha}$ and $Q_{\alpha}$ share many similar properties. Specifically,

1. $\| Q_{\alpha} \|_{p \to p} = \| P_{\alpha} \|_{p \to p},$

2. $Q_{\alpha}$ is the identity on $S^p_{\alpha}$ and maps onto $S^p_{\alpha}$ for $1 < p < \infty$, and

3. $Q_{\alpha}$ is “self-adjoint” in the sense of (2.7) in the pairing $(G,H)_\lambda = \int_{\mathbb{C}^n} G \overline{H} \ d\lambda^n$.

To precisely state the third fact above, we write

$$(Q_{\alpha} G, H)_\lambda = (G, Q_{\alpha} H)_\lambda \quad \text{for all} \quad (G, H) \in L^p(\mathbb{C}^n, \lambda^n) \times L^{p'}(\mathbb{C}^n, \lambda^n). \quad (2.8)$$

We can now state and prove a result analogous to Lemma 2.4 for the space of holomorphic sections. Below $\| \cdot \|_{(S^p_{\alpha})^*}$ denotes the dual norm of $S^p_{\alpha}$.

**Lemma 2.5.** Let $H \in S^p_{\alpha}'$. Then

$$\| (\cdot, H)_\lambda \|_{(S^p_{\alpha})^*} = \inf_{F \in Q^{-1}_{\alpha} H} \| F \|_{p',\lambda}.$$

Furthermore, there exists some $\tilde{F} \in Q^{-1}_{\alpha} H \subseteq L^{p'}(\mathbb{C}^n, \lambda^n)$ such that

$$\inf_{F \in Q^{-1}_{\alpha} H} \| F \|_{p',\lambda} = \| \tilde{F} \|_{p',\lambda}.$$
Proof of Lemma 2.5. Let $H \in \mathbb{S}_p^\alpha$ be arbitrary. We prove the first equation of the lemma by showing that

\[
\|\langle \cdot, H \rangle_\alpha \|_{(\mathbb{S}_p^\alpha)^*} \leq \inf_{F \in \mathbb{Q}_p^\alpha} \|F\|_{p', \lambda}, \quad \text{and}
\]
\[
\|\langle \cdot, H \rangle_\alpha \|_{(\mathbb{S}_p^\alpha)^*} \geq \inf_{F \in \mathbb{Q}_p^\alpha} \|F\|_{p', \lambda}.
\]

We first prove (2.9). Let $F \in \mathbb{Q}_p^\alpha$ be arbitrary. Then

\[
\|\langle \cdot, H \rangle_\alpha \|_{(\mathbb{S}_p^\alpha)^*} = \sup_{G \in \mathbb{S}_p^\alpha} \frac{|\langle G, H \rangle_\alpha|}{\|G\|_{p, \lambda}} = \sup_{G \in \mathbb{S}_p^\alpha} \frac{|\langle G, Q_\alpha F \rangle_\alpha|}{\|G\|_{p, \lambda}} = \sup_{G \in \mathbb{S}_p^\alpha} \frac{|\langle Q_\alpha G, F \rangle_\lambda|}{\|G\|_{p, \lambda}} = \sup_{G \in \mathbb{S}_p^\alpha} \frac{|\langle G, F \rangle_\lambda|}{\|G\|_{p, \lambda}}
\]

Since $F \in \mathbb{Q}_p^\alpha$ was arbitrary, we have proven (2.9).

To prove (2.10), define the linear functional $\Lambda : \mathbb{S}_p^\alpha \to \mathbb{C}$ as

\[
\Lambda(G) = \langle G, H \rangle_\lambda.
\]

Note that $\|\langle \cdot, H \rangle_\alpha \|_{(\mathbb{S}_p^\alpha)^*} = \|\Lambda\|$, so that $\Lambda$ is bounded. By the Hahn-Banach theorem there is a linear functional $\tilde{\Lambda} : L^p(\mathbb{C}^n, \lambda_n) \to \mathbb{C}$ that extends $\Lambda$ without increasing its norm. As $\tilde{\Lambda} \in (L^p(\mathbb{C}^n, \lambda_n))^*$, there exists a function $\tilde{F} \in L^{p'}(\mathbb{C}^n, \lambda_n)$ such that

\[
\tilde{\Lambda}(G) = \langle G, \tilde{F} \rangle_\lambda.
\]

First note that for any $G \in L^p(\lambda)$ we have

\[
\langle G, Q_\alpha \tilde{F} \rangle_\lambda = \langle Q_\alpha G, \tilde{F} \rangle_\lambda = \tilde{\Lambda}(Q_\alpha G) = \Lambda(Q_\alpha G) = \langle Q_\alpha G, H \rangle_\lambda = \langle G, H \rangle_\lambda,
\]

proving $\tilde{F} \in Q_p^\alpha$. Then

\[
\|\langle \cdot, H \rangle_\alpha \|_{(\mathbb{S}_p^\alpha)^*} = \|\tilde{\Lambda}\| = \sup_{G \in L^p(\lambda)} \frac{|\langle G, \tilde{F} \rangle_\lambda|}{\|G\|_{p, \lambda}} = \|\tilde{F}\|_{p', \lambda} \geq \inf_{F \in \mathbb{Q}_p^\alpha} \|F\|_{p', \lambda},
\]

proving (2.10). Combining (2.9) and the preceding inequality, we see that $\|\tilde{F}\|_{p', \lambda} = \inf_{F \in \mathbb{Q}_p^\alpha} \|F\|_{p', \lambda}$, completing the lemma.

We can now provide a proof for Lemma 2.4.

Proof of Lemma 2.4 For $g \in L^p(\gamma_{\alpha p}/2)$ and $h \in L^{p'}(\gamma_{\alpha p}/2)$, a straightforward calculation reveals that

\[
\langle g, h \rangle_\alpha = C^\alpha_p \cdot (g_{\alpha, p}, g_{\alpha, p'}) \cdot h_\lambda.
\]

Note that the constant $C^\alpha_p$ pops up above since we are combining two different isometries: $g_{\alpha, p}$ and $g_{\alpha, p'}$. A straightforward combination of (2.11) and Lemma 2.5 completes the proof.

Remark 2.6. Before moving on to a proof of Theorem 1.2, we note here that we can use Lemma 2.4 to rederive (1.5). That is, the inequality

\[
\|h\|_{p', \alpha} \leq \|\langle \cdot, h \rangle_\alpha\|_{p', \lambda} \leq C^\alpha_p \|h\|_{p', \alpha}.
\]

Let $h \in \mathcal{H}_{p, \alpha}$ be arbitrary. For the first inequality, note that for any $f \in P_p^{-1} h$

\[
\frac{\|h\|_{p', \alpha}}{\|P_p\|_{p' \to p'}} \leq \|f\|_{p', \alpha}.
\]

(2.12)
Thus,
\[ \frac{\|h\|_{p',\alpha}}{\|P_\alpha\|_{p'} - p'} \leq \inf_{f \in P_\alpha^{-1} h} \|f\|_{p',\alpha}. \]

Also, \( h \in P_\alpha^{-1} h \), so that
\[ \inf_{f \in P_\alpha^{-1} h} \|f\|_{p',\alpha} \leq \|h\|_{p',\alpha}. \]

Putting these inequalities together gives us
\[ \frac{\|h\|_{p',\alpha}}{\|P_\alpha\|_{p'} - p'} \leq \inf_{f \in P_\alpha^{-1} h} \|f\|_{p',\alpha} \leq \|h\|_{p',\alpha}. \]

Using Lemma 2.4 and the fact that \( \|P_\alpha\|_{p'} = C^n_\alpha \) (from [2]) in the above equation gives us
\[ \frac{\|h\|_{p',\alpha}}{C^n_\alpha} \leq \frac{\|\langle \cdot, h \rangle \|_{p,\alpha}}{C^n_\alpha} \leq \|h\|_{p',\alpha}. \]

Multiplying the above by \( C^n_\alpha \) gives us a proof of (1.5).

### 2.2 Proof of Theorem 1.2 Using Lemma 2.4

We are now ready to prove Theorem 1.2. We first prove
\[ \inf_{h \in \mathcal{H}^n_{p,\alpha} \setminus \{0\}} \frac{\|\langle \cdot, h \rangle \|_{p,\alpha}}{\|h\|_{p',\alpha}} = 1 \] (2.13)

and that this infimum is achieved. Note that by the proof of (1.5) in Remark 2.6, to prove equality in (2.13) it suffices to show that there exists some \( h \in \mathcal{H}^n_{p,\alpha} \) and a \( f \in P_\alpha^{-1} h \) such that equality in (2.12). That is
\[ \frac{\|h\|_{p',\alpha}}{C^n_\alpha} = \|f\|_{p',\alpha}. \] (2.14)

Let \( f(z) = \frac{p_\alpha}{2\pi} e^{-\frac{z^2}{2}} \); then \( P_\alpha f \equiv 1 \equiv h_0^n \) (the \( z = 0 \) case of the function \( h_0^n(w) = e^{\alpha(w,z)} \)). A straightforward computation shows that \( h = h_0^n \) and \( f \) satisfy (2.14). This proves (2.13).

Now, using (2.5), we have \( \langle g, h_0^n \rangle_{\alpha} = g(0) \). We just showed that \( \|\langle \cdot, h_0^n \rangle_{\alpha} \|_{p,\alpha} = \|h_0^n\|_{p',\alpha} \), which means that the following inequality is sharp:
\[ |g(0)| \leq \|g\|_{p,\alpha} \quad \text{for all} \quad g \in \mathcal{H}^n_{p,\alpha}. \] (2.15)

Let \( z \in \mathbb{C}^n \) be arbitrary. Let \( g \in \mathcal{H}^n_{p,\alpha} \) be arbitrary. Define a new function \( g_z(w) = g(z + w)e^{-\alpha(w,z)} \). Note that \( g_z \) is holomorphic and
\[ \|g_z\|_{p,\alpha} = \left( \frac{\alpha_p}{2\pi} \right)^n \int_{\mathbb{C}^n} |g(z + w)|e^{-\alpha(w,z)}|p_\alpha e^{-\alpha(p|w|^2/2} \lambda^n (dw) \]
\[ = \left( \frac{\alpha_p}{2\pi} \right)^n \int_{\mathbb{C}^n} |g(y)|e^{-\alpha(y,z)}|p_\alpha e^{-\alpha(p|y-z|^2/2} \lambda^n (dy) \]
\[ = e^{-\alpha|z|^2/2} \int_{\mathbb{C}^n} |g(y)|p_\alpha e^{-\alpha|y-z|^2/2} \lambda^n (dy) = e^{-\alpha|z|^2/2} \|g\|_{p,\alpha} < \infty, \]
proving that \( g_z \in \mathcal{H}^n_{p,\alpha} \). Applying (2.15) to \( g_z \) yields the inequality
\[ |g(z)| \leq e^{\alpha|z|^2/2} \|g\|_{p,\alpha} \text{ for all } g \in \mathcal{H}^n_{p,\alpha}. \] (2.16)

A straightforward calculation shows that the inequality (2.16) is an equality when \( g = h_0^n \), proving the inequality sharp. The sharpness of (2.16) proves that \( \|\langle \cdot, h_0^n \rangle_{\alpha} \|_{p,\alpha} = e^{\alpha|z|^2/2} = \|h_0^n\|_{p',\alpha} \), proving the infimum (2.13) is achieved at each \( h_0^n \) and completing the proof.


3 The Strictness of Hölder’s Inequality and a Lower Bound for the Sharp Upper Constant

As Theorem 1.2 is proven, we know that the left-hand inequality of (1.5) is sharp. For the remainder of the paper, we will consider the right-hand inequality, that is

\[ \| \langle \cdot, h \rangle \|_{\alpha} = C_{p,\alpha}^n \| h \|_{p',\alpha}. \]  

(3.1)

As we stated in the introduction, we do not know whether (3.1) is sharp, but Theorems 1.4, 1.5, and 1.6 suggest that it is not sharp. We presently prove Theorems 1.4 and 1.5.

3.1 The Proof of Theorem 1.4

Here will prove that Hölder’s inequality is not sharp in the Segal-Bargmann spaces. Let \( 1 < p < \infty, g \in \mathcal{H}_{p,\alpha}^n \), and \( h \in \mathcal{H}_{p',\alpha}^n \), neither identically 0. We will proceed by contradiction. That is, suppose that \( g \) and \( h \) give equality in Hölder’s inequality (modified by the constant \( C_{p,\alpha}^n \) to account for the scaling of the spaces \( \mathcal{H}_{p,\alpha}^n \)). Thus,

\[
\langle g, h \rangle_{\alpha} = \left| \int_{\mathbb{C}^n} g(z) \overline{h(z)} \gamma_n^\alpha(dz) \right| = \left| \int_{\mathbb{C}^n} \left| g(z) \overline{h(z)} \right| \gamma_n^\alpha(dz) \right|
\]

(3.2)

\[
\leq \left( \frac{\alpha}{\pi} \right)^n \left( \int_{\mathbb{C}^n} |g(z)| e^{-\alpha|z|^2/2} |h(z)| e^{-\alpha|z|^2/2} \lambda^n(dz) \right)^{1/p} \left( \int_{\mathbb{C}^n} |h(z)| e^{-\alpha|z|^2/2} \lambda^n(dz) \right)^{1/p'}
\]

(3.3)

\[
= \left( \frac{\alpha}{\pi} \right)^n \left( \frac{2\pi}{p\alpha} \right)^{n/p} \left( \frac{2\pi}{p'\alpha} \right)^{n/p'} \|g\|_{L^p(\gamma_{\alpha/2})} \|h\|_{L^{p'}(\gamma_{\alpha/2})} = \|g, h\|_{\alpha},
\]

proving that both (3.2) and (3.3) are actually equalities. For equality in (3.3), we must have

\[
|g(z)| e^{-\alpha|z|^2/2} = \beta \|g\|_{L^p(\gamma_{\alpha/2})} e^{-\alpha|z|^2/2} |h(z)| e^{-\alpha|z|^2/2} |p'
\]

for some \( \beta > 0 \). Rearranging the above gives us

\[
|g(z)| = \beta |h(z)| |p'/p \ e^{-\alpha|z|^2/2 |p'.
\]

(3.4)

For (3.2) to be an equality, we must have

\[
g(z) \overline{h(z)} = e^{i\theta_0} f(z),
\]

(3.5)

where \( \theta_0 \in [0, 2\pi] \) and \( f \) is a nonnegative real-valued function. By replacing \( g(z) \) with \( \beta^{-1} e^{-i\theta_0} g(z) \), we preserve holomorphicity and the finiteness of the \( \| \cdot \|_{p,\alpha} - \text{norm} \). Thus, without loss of generality, we may assume that \( e^{i\theta_0} = \beta = 1 \), and replace Equations (3.4) and (3.5) with

\[
|g(z)| = |h(z)| |p'/p \ e^{-\alpha|z|^2/2 |p'.
\]

(3.6)

and

\[
g(z) \overline{h(z)} = f(z), \text{ where } f \text{ is non-negative real-valued.}
\]

(3.7)
Thus, we need only prove the left-hand inequality of (1.9). To that end, we will consider the case where \( h \) shows that this ratio. Theorem 1.5 concerns bounds on this ratio; namely that (1.9), reproduced below, holds:

\[
\frac{g(z)}{h(z)} = \frac{g(z)\overline{h(z)}}{|h(z)|^2} = \frac{f(z)}{|h(z)|^2} > 0 \quad \text{for} \quad z \in U.
\]

Thus, \( g/h \) is a positive holomorphic function, and so it is equal to a positive constant \( c \) on \( U \). Eq. (3.6) then shows that

\[
c|h(z)| = |h(z)|^{p'/p}e^{-\frac{\alpha p}{p-1}|z|^2},
\]

and rearranging this gives

\[
|h(z)| = c_1 e^{\frac{\alpha}{2}|z|^2}, \quad c_1 = e^{\frac{\alpha}{p(p-1)}}.
\]

Fix any point \( z = (z_1, \ldots, z_n) \in U \); then there is some disk \( D \subset \mathbb{C} \) such that \( \{(\zeta, z_2, \ldots, z_n) : \zeta \in D\} \subset U \). Thus the function \( h_1(\zeta) = h(\zeta, z_2, \ldots, z_n) \) is holomorphic and non-vanishing on \( D \), and we have

\[
|h_1(\zeta)| = c_1 e^{\frac{\alpha}{2}|\zeta|^2+|z_2|^2+\cdots+|z_n|^2}.
\]

The function \( h_2(\zeta) = c_1^{-1}e^{-\frac{\alpha}{2}(|\zeta|^2+|z_2|^2+\cdots+|z_n|^2)}h_1(\zeta) \) is therefore holomorphic and non-vanishing on \( D \), and \( |h_2(\zeta)| = e^{\frac{\alpha}{2}|\zeta|^2} \). It follows that \( h_1 \) has a holomorphic logarithm \( \ell \) on \( D \), so

\[
e^{\frac{\alpha}{2}|\zeta|^2} = |h_2(\zeta)| = |e^{\ell(\zeta)}| = e^{\Re(\zeta)}, \quad z \in D.
\]

As \( \exp \) is one-to-one on \( \mathbb{R} \), it follows that \( \Re(\zeta) = \frac{\alpha}{2}|\zeta|^2 \) for \( \zeta \in D \). This is impossible, since \( \ell \) is holomorphic, but \( \zeta \mapsto \frac{\alpha}{2}|\zeta|^2 \) is not harmonic. This concludes the proof.

### 3.2 The Proof of Theorem 1.5

As in the Introduction, define \( R_{p,\alpha}(g, h) \) as

\[
R_{p,\alpha}(g, h) = \frac{|\langle g, h \rangle_\alpha|}{\|g\|_p \|h\|_{p',\alpha}}.
\]

Note that the sharp constant for (3.1) is equal to \( \sup_{g \in \mathcal{H}_{p,\alpha}^n \setminus \{0\}} \sup_{h \in \mathcal{H}_{p',\alpha}^n \setminus \{0\}} R_{p,\alpha}(g, h) \), hence our interest in this ratio. Theorem 1.5 concerns bounds on this ratio; namely that (1.9), reproduced below, holds:

\[
C_p^n/2 \leq \sup_{g \in \mathcal{H}_{p,\alpha}^n \setminus \{0\}} \sup_{h \in \mathcal{H}_{p',\alpha}^n \setminus \{0\}} R_{p,\alpha}(g, h) \leq C_p^n.
\]

There are many ways to prove the right-hand side of (1.9). In particular, we can rewrite Theorem 1.4 in terms of \( R_{p,\alpha}(g, h) \) to say that for any \( g \in \mathcal{H}_{p,\alpha}^n \) and \( h \in \mathcal{H}_{p',\alpha}^n \), we have

\[
R_{p,\alpha}(g, h) < C_p^n.
\]

By the above, we have

\[
\sup_{g \in \mathcal{H}_{p,\alpha}^n \setminus \{0\}} \sup_{h \in \mathcal{H}_{p',\alpha}^n \setminus \{0\}} R_{p,\alpha}(g, h) \leq C_p^n. \tag{3.8}
\]

Thus, we need only prove the left-hand inequality of (1.9). To that end, we will consider the case where \( g \) and \( h \) are monomials. Note that (by the rotational invariance of \( \gamma_{\alpha}^n \)) distinct monomials are orthogonal, so we will
consider only $g = h$. For $k_1, \ldots, k_n \in \mathbb{N}$, define the $g_{k_1, k_2, \ldots, k_n}(z) \equiv z_1^{k_1}z_2^{k_2} \ldots z_n^{k_n}$. Note that

$$
\|g_{k_1, k_2, \ldots, k_n}\|_{p, \alpha}^p = \left(\frac{\alpha p}{2 \pi}\right)^n \int_{\mathbb{C}} |z_1|^{k_1 p} |z_2|^{k_2 p} \ldots |z_n|^{k_n p} e^{-(\alpha p/2)(|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2)} \lambda^n(dz)
$$

$$
= \left(\frac{\alpha p}{2 \pi}\right)^n \prod_{j=1}^n \int_{\mathbb{C}} |z_j|^{k_j p} e^{-(\alpha p/2)|z_j|^2} \lambda(dz)
$$

$$
= \prod_{j=1}^n \|g_{k_j}\|_{p, \alpha}^p
$$

where $g_k : \mathbb{C} \to \mathbb{C}$ is given by $g_k(z) = z^k$. Note, then, that

$$
\mathcal{R}_{p, \alpha}(g_{k_1, \ldots, k_n}, g_{k_1, \ldots, k_n}) = \prod_{j=1}^n \mathcal{R}_{p, \alpha}(g_{k_j}, g_{k_j}). \tag{3.9}
$$

Hence, to prove the left-hand side of (3.9), it suffices to show that

$$
\sup_{k \in \mathbb{N}} \mathcal{R}_{p, \alpha}(g_k, g_k) = C_p^{1/2}. \tag{3.10}
$$

As usual, denote the Gamma function $\Gamma(z)$ as

$$
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re z > 0.
$$

Then, using polar coordinates, we have

$$
\|g_k\|_{p, \alpha}^p = \left(\frac{\alpha p}{2 \pi}\right)^n \int_{\mathbb{C}} |z|^{k p} e^{-(\alpha p/2)|z|^2} \lambda(dz) = \left(\frac{\alpha p}{2 \pi}\right)^n \int_0^\infty r^{k p} e^{-(\alpha p/2)r^2} dr.
$$

Using the substitution $u = \frac{\alpha p}{2} r^2$ yields

$$
\|g_k\|_{p, \alpha}^p = \int_0^\infty (r^2)^{k p/2} e^{-(\alpha p/2)r^2} (\alpha p)rdr = \left(\frac{2}{\alpha p}\right)^{k p/2} \int_0^\infty u^{k p/2} e^{-u} du = \left(\frac{2}{\alpha p}\right)^{k p/2} \Gamma(kp/2 + 1).
$$

Thus, we have

$$
\mathcal{R}_{p, \alpha}(g_k, g_k) = \frac{|\langle g_k, g_k \rangle|}{\|g_k\|_{p, \alpha} \|g_k\|_{p', \alpha}} = \frac{\|g_k\|_{2, \alpha}^2}{\|g_k\|_{p, \alpha} \|g_k\|_{p', \alpha}} = \left(\frac{\alpha p}{4}\right)^{k p/2} \frac{\Gamma(kp/2 + 1)}{\Gamma(kp/2 + 1)^{1/p} \Gamma(kp'/2 + 1)^{1/p'}}. \tag{3.11}
$$

Using the Gamma function relation $\Gamma(z + 1) = z \Gamma(z)$, it is convenient to express this ratio as

$$
\frac{\Gamma(k+1)}{\Gamma(kp/2 + 1)^{1/p} \Gamma(kp'/2 + 1)^{1/p'}} = \frac{k}{(kp/2)^{1/p} (kp'/2)^{1/p'}} \cdot \frac{\Gamma(k)}{(kp/2)^{1/p} \Gamma(kp'/2)^{1/p'}} = C_p \cdot \frac{\Gamma(k)}{(kp/2)^{1/p} \Gamma(kp'/2)^{1/p'}}. \tag{3.12}
$$
Moreover, the limit of this expression as R → ∞, we have
\[ S(z) \equiv \ln \left( \sqrt{\frac{z}{2\pi}} \left( \frac{e}{z} \right)^z \Gamma(z) \right) = \int_0^\infty \frac{2 \arctan(t/z)}{e^{2\pi t} - 1} dt. \] (3.13)

See, for example, [1, (6.1.50)]. Thus, we can express the Gamma function precisely as
\[ \Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-S(z)} z^{-\frac{1}{2}}. \] (3.14)

With this in hand, together with (3.11) and (3.12), we have the following expression for R_{p,\alpha}(g_k, g_k).
\[
R_{p,\alpha}(g_k, g_k) = C_p \left( \frac{pp'}{4} \right)^{k/2} \frac{\Gamma(k)}{\Gamma(kp/2)^{1/p} \Gamma(kp'/2)^{1/p'}} \left( \sqrt{2\pi} k^{k-\frac{1}{2}} e^{S(k)-k} \right)^{1/p'} \left( \sqrt{2\pi} (kp/2)^{k-1/2} e^{S(kp/2)-kp/2} \right)^{1/p}.
\]
\[
= C_p \left( \frac{pp'}{4} \right)^{k/2} \frac{k^{k-\frac{1}{2}}}{(kp/2)^{k-1/2}} e^{S(k)-kp/2} (kp/2)^{-1/2} (p/2)^{1/2}.
\]
\[ = C_p^{1/2} \cdot e^{S(k)-\frac{1}{2} S(kp/2)-\frac{1}{2} S(kp'/2)}. \]

Thus, to prove (3.10) and thus Theorem 1.5, it suffices to prove the following proposition.

**Proposition 3.1.** For any p ∈ (1, ∞) \ {2} and any k ∈ \mathbb{N},
\[ S(k) - \frac{1}{p} S(kp/2) - \frac{1}{p'} S(kp'/2) < 0. \]

Moreover, the limit of this expression as k → ∞ is 0.

**Proof.** Denote the integrand of S(x) as s(t, x):
\[ s(t, x) = \frac{2 \arctan(t/x)}{e^{2\pi t} - 1}. \]

Note that s ∈ C^∞((0, ∞)^2); the first two x derivatives are as follows:
\[
\frac{\partial s}{\partial x}(t, x) = -\frac{1}{t^2 + x^2} \frac{2t}{e^{2\pi t} - 1}, \quad \frac{\partial^2 s}{\partial x^2}(t, x) = \frac{x}{(t^2 + x^2)^2} \frac{4t}{e^{2\pi t} - 1}.
\]

Thus, for each t > 0, x → s(t, x) is strictly convex on (0, ∞). In particular, since \frac{1}{p} + \frac{1}{p'} = 1 and \frac{1}{p}, \frac{1}{p'} ∈ (0, 1), and since p ≠ p', we have
\[ s(t, x) = s \left( t, \frac{xp}{2} \right) + \frac{1}{p'} s \left( t, \frac{xp'}{2} \right). \]

Since t → s(t, x) is strictly positive, upon integration this inequality remains strict, and so
\[ S(x) = \int_0^\infty s(t, x) dt < \int_0^\infty \frac{1}{p} s \left( t, \frac{xp}{2} \right) dt + \int_0^\infty \frac{1}{p'} s \left( t, \frac{xp'}{2} \right) dt = \frac{1}{p} S(xp/2) + \frac{1}{p'} S(xp'/2). \]

Taking x = k ∈ \mathbb{N} proves the first statement of the proposition.

For the second statement, it suffices to show that lim_{x→∞} S(x) = 0. As computed above, \frac{\partial s}{\partial x}(t, x) < 0, and so x → s(t, x) is decreasing; in particular, for x ≥ 1 the integrand is ≤ \frac{2 \arctan(t)}{e^{2\pi t} - 1}, which is an L^1(0, ∞) function. Since \lim_{x→∞} \arctan(t/x) = 0 for each fixed t, it follows from the Dominated Convergence Theorem that lim_{x→∞} S(x) = 0, completing the proof. □
Remark 3.2. (1) Note, from (3.13), the statement \( \lim_{x \to \infty} S(x) = 0 \) is (up to a logarithm) precisely the usual statement of Stirling’s approximation:

\[
1 = \lim_{x \to \infty} \frac{\Gamma(x)}{\sqrt{2\pi x} \left( \frac{x}{e} \right)^x} = \lim_{x \to \infty} e^{S(x)}.
\]

We include the Dominated Convergence Theorem proof above just for completeness.

(2) The above computations are only valid for \( k > 0 \). However, it is easy to check that \( R_{p,\alpha}(g_0, g_0) = 1 < C_p^{1/2} \), since \( g_0 = 1 \).

Thus, we have completed the proof of Theorem 1.5. Let us also note, for use in the next section, that Proposition 3.1 actually shows that, for each \( k \in \mathbb{N} \),

\[
R_{p,\alpha}(g_k, g_k) < C_p^{1/2}.
\] (3.15)

4 Local Sharpness of the Upper Constant

In this section, we prove Theorem 1.6. To begin, we give a slight variant of the definition for the affine space \( A^N_\alpha \) in the statement of the theorem. First, some useful notation: we rescale the monomial functions \( g_k(z) = z^k \) so they are normalized in \( L^2(\mathbb{C}, \gamma_\alpha) \).

**Notation 4.1.** Fix \( \alpha > 0 \). For \( k \geq 0 \), denote by \( \psi_{\alpha,k} \) the function

\[
\psi_{\alpha,k}(z) = \sqrt{\frac{\alpha^k}{k!}} z^k \in L^2(\mathbb{C}, \gamma_\alpha).
\] (4.1)

The functions \( \{\psi_{\alpha,k}\}_{k \in \mathbb{N}} \) form an orthonormal basis for \( \mathcal{H}_{2,\alpha} \). It will be convenient to expand all polynomials in this basis – i.e. with appropriate normalization.

**Definition 4.2.** Fix \( N \in \mathbb{N} \) and \( \alpha > 0 \). Let \( A^N_\alpha \) denote the following affine space of holomorphic polynomials of degree \( N \):

\[
A^N_\alpha = \left\{ q = \sum_{k=0}^{N-1} (a_k + ib_k) \psi_{\alpha,k} + \psi_{\alpha,N} : a_k, b_k \in \mathbb{R} \text{ for } 0 \leq k \leq N - 1 \right\}.
\]

We have chosen to normalize the polynomials in the space so that the leading term is unit length in \( \mathcal{H}_{2,\alpha} \), rather than to make the polynomials monic. This makes no difference to any of the inequalities we presently consider, since the functions \( R_{p,\alpha} \) are scale invariant, and it will simplify the following computations. The normalization coefficients in all but the \( z^N \) term make no difference to the definition, but will be notationally handy in the use of the coefficient names \( a_k \) and \( b_k \). Indeed, we may now think of the restriction of \( R_{p,\alpha} \) to \( A^N_\alpha \times A^N_\alpha \) as a function of four \( \mathbb{R}^N \) variables. Given \( a = (a_0, \ldots, a_{N-1}) \) and \( b = (b_0, \ldots, b_{N-1}) \), let

\[
q_{a,b} = \sum_{k=0}^{N-1} (a_k + ib_k) \psi_{\alpha,k} + \psi_{\alpha,N} \in A^N_\alpha
\] (4.2)

where, for readability, we have suppressed the \( (\alpha, N) \)-dependence of \( q_{a,b} \). Then \( A^N_\alpha = \{ q_{a,b} : a, b \in \mathbb{R}^N \} \). Accordingly, define the function \( R^N_{p,\alpha} : (\mathbb{R}^N)^4 \to \mathbb{R}_+ \) by

\[
R^N_{p,\alpha}(a, b, c, d) = R_{p,\alpha}(q_{a,b}, q_{c,d}).
\] (4.3)
One obvious way to prove Theorem 1.6 would be to show that the function $R_{p,\alpha}^N$ has a local maximum at $0 \in (\mathbb{R}^N)^4$, meaning that $\mathcal{R}_{p,\alpha}|_{\partial N}$ achieves a local maximum at $(\psi_{\alpha,N}, \psi_{\alpha,N})$, where we have computed in (3.15) that the value is less than $C_p^{1/2}$ and converges to $C_p^{1/2}$ as $N \to \infty$, as desired. In fact, this approach needs a slight modification, since the function $R_{p,\alpha}^N$ is not quite amenable to the necessary elementary calculus techniques: it is not $C^2$ for any $p \neq 2$. To account for this, we will use the following cutoff approximation.

**Definition 4.3.** For $\epsilon > 0$, denote by $\mathbb{C}_\epsilon = \mathbb{C} - \overline{\mathbb{D}(0, \epsilon)}$, the complex plane with the centered disk of radius $\epsilon$ removed. For $N \in \mathbb{N}$, $p > 1$, and $\alpha > 0$, define a function $R_{p,\alpha}^{N,\epsilon}: (\mathbb{R}^N)^4 \to \mathbb{R}_+$ as follows: given $a, b, c, d \in \mathbb{R}^N$, using the notation of (4.2),

$$R_{p,\alpha}^{N,\epsilon}(a, b, c, d) = \frac{|\langle q_{a,b,c,d}, a \rangle|}{\|1_{\mathbb{C}_\epsilon} q_{a,b} p,\alpha \|_{p,\alpha} \|1_{\mathbb{C}_\epsilon} q_{c,d} \|_{p',\alpha}}.$$  \hfill (4.4)

Elementary properties of the functions $R_{p,\alpha}^{N,\epsilon}$ are contained in the following lemma. For convenience, we denote a vector in $(\mathbb{R}^N)^4$ simply as $x$ when convenient.

**Lemma 4.4.** For any $\epsilon > 0$ and $x \in (\mathbb{R}^N)^4$, $R_{p,\alpha}^{N,\epsilon}(x) \geq R_{\alpha,p}(x)$, and $\lim_{\epsilon \downarrow 0} R_{p,\alpha}^{N,\epsilon}(x) = R_{p,\alpha}^N(x)$. Moreover, for all sufficiently small $\epsilon > 0$, $R_{p,\alpha}^{N,\epsilon}(0) < C_p^{1/2}$.

**Proof.** Since $|1_{\mathbb{C}_\epsilon} f| \leq |f|$ for any function $f$, we have $\|1_{\mathbb{C}_\epsilon} f\|_{p,\alpha} \leq \|f\|_{p,\alpha}$, so the first inequality follows from the definitions. Similarly, since $|1_{\mathbb{C}_\epsilon} f| \| f \|_{p,\alpha}$, the limit statement follows immediately from the Monotone Convergence Theorem. Now, by (3.15), $R_{p,\alpha}^N(0) = \mathcal{R}_{p,\alpha}(\psi_{\alpha,N}, \psi_{\alpha,N}) = \mathcal{R}_{p,\alpha}(g_N, g_N) < C_p^{1/2}$. Since $R_{p,\alpha}^{N,\epsilon}(0)$ converges to $R_{p,\alpha}^N(0)$ as $\epsilon \downarrow 0$, it follows that, for all sufficiently small $\epsilon > 0$, $R_{p,\alpha}^{N,\epsilon}(0) < C_p^{1/2}$, as desired. \hfill \Box

We now proceed to show that, for each fixed $N, \epsilon, p, \alpha$, the function $R_{p,\alpha}^{N,\epsilon}$ has a local maximum at $0 \in (\mathbb{R}^N)^4$. The following two sections give the proofs of criticality and negative definiteness of the Hessian of the function at $0$.

### 4.1 Criticality

To begin, to simplify notation somewhat, we suppress the explicit dependence on the variables $a, b, c, d \in \mathbb{R}^N$, and denote

$$f_\epsilon = 1_{\mathbb{C}_\epsilon} q_{a,b}, \quad g_\epsilon = 1_{\mathbb{C}_\epsilon} q_{c,d}.$$  \hfill (4.5)

When convenient, we will also let $f = f_0 = q_{a,b}$ and $g = g_0 = q_{c,d}$.

The following lemma gives the computation of the derivatives of the terms in the denominator of (4.4).

**Lemma 4.5.** Let $\epsilon > 0$, $\alpha > 0$, and $p > 1$. The functions $\|f_\epsilon\|_{p,\alpha}, \|g_\epsilon\|_{p',\alpha}: (\mathbb{R}^N)^2 \to \mathbb{R}_+$ are $C^1$ in a neighborhood of $0$, and their partial derivatives are given by

$$\left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) \|f_\epsilon\|_{p,\alpha} = \frac{p}{2} \|f_\epsilon\|_{p,\alpha}^{1-p} \langle G_{p,\alpha}(f_\epsilon), \psi_{\alpha,k} \rangle_\alpha$$  \hfill (4.6)

$$\left( \frac{\partial}{\partial c_k} + i \frac{\partial}{\partial d_k} \right) \|g_\epsilon\|_{p',\alpha} = \frac{p'}{2} \|g_\epsilon\|_{p',\alpha}^{1-p'} \langle G_{p',\alpha}(g_\epsilon), \psi_{\alpha,k} \rangle_\alpha$$  \hfill (4.7)

for $0 \leq k \leq N - 1$, where, for $f \in \mathcal{H}_{p,\alpha}^1$, the function $G_{p,\alpha}(f) \in L^{p'}(\mathbb{C}, \gamma_{\alpha p'})/2)$ is given by

$$G_{p,\alpha}(f)(z) = |f(z)|^{p-2} f(z) e^{-\frac{\gamma_{\alpha p'}}{2} |z|^2}.$$  \hfill (4.7)

**Remark 4.6.** The functional $G_{p,\alpha}$ is a modification of the Hölder maximizer functional $G = |F|^{p-2} F$ which maps $L^p$ to $L^{p'}$ and satisfies $\langle F, G \rangle = \|F\|_p \|G\|_{p'}$; $G_{p,\alpha}$ accomplishes the same task in the dilated spaces $\mathcal{H}_{p,\alpha}$.
Proof. The \( p \)th power of the function being differentiated in (4.6) is
\[
\|f_\epsilon\|_{L^p(\mathbb{R}^N)}^p = \frac{\alpha p}{2\pi} \int_{\mathbb{R}^N} |f_\epsilon(z)|^p e^{-\frac{\alpha p}{2}|z|^2} \, dz = \frac{\alpha p}{2\pi} \int_{\mathbb{R}^N} |f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} \, dz.
\]
If \( h \) is a holomorphic function of a complex variable \( \zeta \), then \( |h|^p \) is \( C^\infty \) at all points other than the zeroes of \( h \), in which case
\[
\frac{d}{d\zeta} |h|^p = \frac{p}{2} |h|^{p-2} \frac{dh}{d\zeta},
\]
\[
\frac{d}{d\zeta} |h|^p = \frac{p}{2} |h|^{p-2} \frac{dh}{d\zeta}.
\]
Applying this with the complex variable \( \zeta = a_k + ib_k \), so that \( \frac{d}{da_k} = \frac{1}{2} (\frac{\partial}{\partial a_k} - i \frac{\partial}{\partial b_k}) \) and \( \frac{d}{db_k} = \frac{1}{2} (\frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k}) \), the partial derivatives of the integrand are
\[
\left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) |f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} = p |f(z)|^{p-2} f(z) \bar{\psi}_{\alpha,k}(z) e^{-\frac{\alpha p}{2}|z|^2},
\]
which exist except at the zeroes \( f \). The function \( q_{a,b}(z) = \psi_{\alpha,N}(z) = \sqrt{\alpha^{iN}/N!} z^k \) has all zeroes at \( z = 0 \). It follows that, for all \( a, b \in (\mathbb{R}^N)^2 \) sufficiently close to \((0,0)\), all zeroes of the polynomial \( q_{a,b} \) are contained in \( \mathbb{D}(0, \epsilon) \). So, for \((a, b)\) in this neighborhood, the integrand is differentiable on \( \mathbb{C}_\epsilon \), and its partial derivative is bounded by
\[
p \sqrt{\frac{\alpha^k}{k!}} |f(z)|^{p-1} |z|^k e^{-\frac{\alpha p}{2}|z|^2}.
\]
Given any compact subset \( K \) of \((\mathbb{R}^N)^2 \), there is a constant \( C_K < \infty \) so that \( |f(z)| = |q_{a,b}(z)| \leq C_K (1 + |z|^N) \) for all \((a, b)\) in \( K \). Hence, for such \( a, b \) the derivative (4.9) is uniformly bounded by a constant times
\[
(1 + |z|^N)^{p-1} |z|^k e^{-\frac{\alpha p}{2}|z|^2},
\]
and this function is \( L^1 \). The integrand is also manifestly continuous in \( a, b \) in the neighborhood where \( f(z) \neq 0 \). Hence, by the Dominated Convergence Theorem, \( \|f_\epsilon\|_{L^p(\mathbb{R}^N)} \) is \( C^1 \), and its derivative is given by differentiating under the integral sign:
\[
\left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) \|f_\epsilon\|_{L^p(\mathbb{R}^N)} = \frac{\alpha p}{2\pi} \int_{\mathbb{R}^N} p |f(z)|^{p-2} f(z) \bar{\psi}_{\alpha,k}(z) e^{-\frac{\alpha p}{2}|z|^2} \, dz.
\]
Taking \( p \)th power and applying the chain rule, it follows that \( \|f_\epsilon\|_{L^p(\mathbb{R}^N)} \) is differentiable, and
\[
\left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) \|f_\epsilon\|_{L^p(\mathbb{R}^N)} = \frac{1}{p} \left( \|f_\epsilon\|_{L^p(\mathbb{R}^N)} \right)^{1/p-1} \left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) \|f_\epsilon\|_{L^p(\mathbb{R}^N)}
= \frac{\alpha p}{2\pi} \int_{\mathbb{R}^N} |f(z)|^{p-2} f(z) \bar{\psi}_{\alpha,k}(z) e^{-\frac{\alpha p}{2}|z|^2} \, dz
= \frac{p}{2} \|f_\epsilon\|_{L^p(\mathbb{R}^N)}^{1-p} \int_{\mathbb{R}^N} |f(z)|^{p-2} f(z) e^{-\frac{\alpha}{2}(p-2)|z|^2} \bar{\psi}_{\alpha,k}(z) \frac{\alpha}{\pi} e^{-\alpha|z|^2} \, dz,
\]
which establishes the validity of (4.6), as desired. Eq. (4.7) follows identically, replacing \( p \) with \( p' \) and \((a, b)\) with \((c, d)\).
Corollary 4.7. Let $\epsilon, \alpha > 0$ and $p > 1$. The function $R_{p,\alpha}^{N,\epsilon}: (\mathbb{R}^N)^4 \to \mathbb{R}_+$ is differentiable in a neighborhood of 0, and its partial derivatives for $0 \leq k \leq N - 1$ are given by

\[
\left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) R_{p,\alpha}^{N,\epsilon} = \frac{\text{sgn}\left( \langle f, g \rangle_\alpha \right) \langle g, \psi_{\alpha,k} \rangle_\alpha - \frac{p}{2} \langle f, g \rangle_\alpha (\cdot)_{p,\alpha} (G_{p,\alpha}(f_\epsilon), \psi_{\alpha,k} \rangle_\alpha}{\| f \|_{p,\alpha} \cdot \| g \|_{p',\alpha}}
\]

and

\[
\left( \frac{\partial}{\partial c_k} + i \frac{\partial}{\partial d_k} \right) R_{p,\alpha}^{N,\epsilon} = \frac{\text{sgn}\left( \langle f, g \rangle_\alpha \right) \langle f, \psi_{\alpha,k} \rangle_\alpha - \frac{p'}{2} \langle f, g \rangle_\alpha (\cdot)_{p',\alpha} (G_{p',\alpha}(g_\epsilon), \psi_{\alpha,k} \rangle_\alpha}{\| f \|_{p,\alpha} \cdot \| g \|_{p',\alpha}}
\]

where $\text{sgn}(w) = \frac{w}{|w|} w \neq 0$.

Proof. By our choice of normalization of the coefficient variables $a, b, c, d$, we have

\[
\langle f, g \rangle_\alpha = \sum_{k=0}^{N-1} (a_k + ib_k)(c_k - id_k) + 1.
\]

The value of this polynomial at 0 is 1, and so there is a neighborhood of 0 on which it is non-zero. Thus, on this neighborhood, the modulus $|\langle f, g \rangle_\alpha|$ is differentiable. From (4.8) with $p = 1$, we therefore have

\[
\left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) (\langle f, g \rangle_\alpha) = \text{sgn}\left( \langle f, g \rangle_\alpha \right) \langle g, \psi_{\alpha,k} \rangle_\alpha
\]

and

\[
\left( \frac{\partial}{\partial c_k} + i \frac{\partial}{\partial d_k} \right) (\langle f, g \rangle_\alpha) = \text{sgn}\left( \langle f, g \rangle_\alpha \right) \langle f, \psi_{\alpha,k} \rangle_\alpha
\]

where in (4.13) we have applied (4.8) to the holomorphic function $h(\zeta) = \langle f, g \rangle_\alpha$ with $\zeta = c_k + id_k$. Hence, using the quotient rule and using the fact that $g_\epsilon$ is independent of $a, b$, we have

\[
\left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) R_{p,\alpha}^{N,\epsilon} = \frac{1}{\| g \|_{p',\alpha} \| f \|_{p,\alpha}} \left[ \langle f, g \rangle_\alpha \left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) - \langle f, g \rangle_\alpha \left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) \right]
\]

and now applying (4.6) and (4.12) yields (4.10). Eq. (4.11) follows similarly from (4.7) and (4.13).

This brings us to the main result of this section.

Proposition 4.8. Let $\epsilon, \alpha > 0$, $p > 1$, and $N \in \mathbb{N}$. Then the function $R_{p,\alpha}^{N,\epsilon}: (\mathbb{R}^N)^4 \to \mathbb{R}_+$ of (4.4) is differentiable in a neighborhood of 0, and has a critical point at 0.

Proof. When $a = b = c = d$, both $f = g = \psi_{\alpha,N}$; hence the inner products $\langle f, \psi_{\alpha,k} \rangle$ and $\langle g, \psi_{\alpha,k} \rangle$ are 0 for $k < N$. Similarly, we may compute that

\[
\langle G_{p,\alpha}(f_\epsilon), \psi_{\alpha,k} \rangle_\alpha = \frac{\alpha}{\pi} \int_{\mathbb{C}_+} |z|^N |p-2z|^N e^{-\frac{\alpha}{2}|p-2z|^2} \frac{1}{\psi_{\alpha,k}(z)} e^{-\alpha |z|^2} dz
\]

\[
= \frac{\alpha}{\pi} \sqrt{\frac{\alpha^k}{k!}} \left( \frac{\alpha^N}{N!} \right)^{\frac{N+1}{2}} \int_{\mathbb{C}_+} \frac{z^N z^k}{z^N} e^{-\frac{\alpha}{2}|z|^2} e^{-\alpha |z|^2} dz.
\]

The measure $\mu(dz) = 1_{\mathbb{C}_+}(z) \frac{1}{z^N} e^{-\frac{\alpha}{2}|z|^2} e^{-\alpha |z|^2} dz$ is rotationally-invariant, and the functions $z^k$ are in $L^2(\mu)$ for all $k$; thus $z^N$ and $z^k$ are $\mu$-orthogonal for $k \neq N$ (as is easily seen by computing in polar coordinates), and the inner product $\langle G_{p,\alpha}(f_\epsilon), \psi_{\alpha,k} \rangle_\alpha$ is 0.

Hence, we have computed that both inner products in (4.10) are 0 for each $k < N$, which shows that all the $a, b$ partial derivatives of $R_{p,\alpha}^{N,\epsilon}$ are 0 at 0. An identical computation with (4.11) shows that the $c, d$ partial derivatives are also 0 at 0. Corollary 4.7 shows that $R_{p,\alpha}^{N,\epsilon}$ is differentiable in a neighborhood of 0. This proves the proposition. □
4.2 The Hessian

We now proceed to compute the second derivatives of $R_{p,\alpha}^{N}\epsilon$ in a neighborhood of 0. From the formulas for the partial derivatives in (4.10) and (4.11), it is clear that this is a somewhat involved task. The most complicated calculations are the partial derivatives of the inner products with $G_{p,\alpha}(f_{e})$ and $G_{p',\alpha}(g_{e})$, and so we begin there.

Lemma 4.9. For $\sigma > 0$ and $x \in \mathbb{R}$, let

$$\Gamma_{\sigma}(x) = \int_{\sigma}^{\infty} u^{x-1}e^{-u}du.$$  

(4.14)

Let $\alpha, \epsilon > 0$, $p > 1$, and $N \in \mathbb{N}$, and define

$$\Phi_{k,\alpha}^{N,\epsilon}(p) = \frac{1}{k!(N!)^{p/2-1}} \left(\frac{2}{p}\right)^{N(p/2-1)+k} \frac{\Gamma_{\alpha p e^{\epsilon/2}}}{(N(p/2-1) + k + 1)}.$$  

(4.15)

Fix $k, \ell < N$. For $z \in \mathbb{C}$, let $f_{e}(z), g_{e}(z) : (\mathbb{R}^{N})^{2} \rightarrow \mathbb{C}$ be the functions given in (4.5). Then the functions $\langle G_{p,\alpha}(f_{e}), \psi_{\alpha,k}\rangle_{\alpha}$ and $\langle G_{p',\alpha}(g_{e}), \psi_{\alpha,k}\rangle_{\alpha}$ are $C^{1}$ in a neighborhood of 0, and the partial derivatives at 0 are

$$\frac{\partial}{\partial a_{\ell}} \langle G_{p,\alpha}(f_{e}), \psi_{\alpha,k}\rangle_{\alpha} \bigg|_{a=b=0} = \delta_{k\ell} \Phi_{k,\alpha}^{N,\epsilon}(p)$$  

(4.16)

$$\frac{\partial}{\partial b_{\ell}} \langle G_{p,\alpha}(f_{e}), \psi_{\alpha,k}\rangle_{\alpha} \bigg|_{a=b=0} = i\delta_{k\ell} \Phi_{k,\alpha}^{N,\epsilon}(p)$$  

(4.17)

$$\frac{\partial}{\partial c_{\ell}} \langle G_{p',\alpha}(g_{e}), \psi_{\alpha,k}\rangle_{\alpha} \bigg|_{c=d=0} = \delta_{k\ell} \Phi_{k,\alpha}^{N,\epsilon}(p')$$  

(4.18)

$$\frac{\partial}{\partial d_{\ell}} \langle G_{p',\alpha}(g_{e}), \psi_{\alpha,k}\rangle_{\alpha} \bigg|_{c=d=0} = i\delta_{k\ell} \Phi_{k,\alpha}^{N,\epsilon}(p').$$  

(4.19)

Remark 4.10. From the Monotone Convergence Theorem, $\lim_{\sigma \downarrow 0} \Gamma_{\sigma}(x)$ is equal to the usual Gamma function $\Gamma(x)$ when $x > 0$; for $x \leq 0$, the limit is $+\infty$. Since at least one of the arguments $N(p/2-1) + k + 1$ and $N(p'/2-1) + k + 1$ is negative for some $k$, Lemma 4.9 suggests that the second derivatives of $R_{p,\alpha}^{N}$ blow up to infinity. This is one of the primary reasons we use the cutoff approximation of all integrals over $\mathbb{C}_{\epsilon}$, to avoid such singularities.

Proof. We will deal only with $\langle G_{p,\alpha}(f_{e}), \psi_{\alpha,k}\rangle_{\alpha}$, as the calculations for $\langle G_{p',\alpha}(g_{e}), \psi_{\alpha,k}\rangle_{\alpha}$ are essentially the same. The function in question is

$$\langle G_{p,\alpha}(f_{e}), \psi_{\alpha,k}\rangle_{\alpha} = \frac{\alpha}{\pi} \int_{\mathbb{C}_{\epsilon}} |f(z)|^{p-2} \frac{f(z)e^{-\frac{a}{2}(p-2)\overline{z}z}}{\psi_{\alpha,k}(z)} \frac{e^{-\alpha|z|^{2}}dz}{\overline{\psi_{\alpha,k}(z)}e^{-\frac{a}{2}|z|^{2}} dz}.$$  

(4.20)

The integrand $|f(z)|^{p-2} \frac{f(z)}{\overline{\psi_{\alpha,k}(z)}e^{-\frac{a}{2}|z|^{2}}}e^{-\frac{a}{2}|z|^{2}}$ is $C^{\infty}$ on the same (e-dependent) neighborhood of $(a, b) = (0, 0)$ described following (4.9). We compute the partial derivatives $\frac{\partial}{\partial a_{\ell}}$ and $\frac{\partial}{\partial b_{\ell}}$ utilizing (4.8) with $\zeta = a_{\ell} + ib_{\ell}$. Since $\overline{\psi_{\alpha,k}(z)}e^{-\frac{a}{2}|z|^{2}}$ is independent of $a, b$, we only differentiate $|f(z)|^{p-2} f(z)$. We have

$$\frac{1}{2} \left( \frac{\partial}{\partial a_{\ell}} + i \frac{\partial}{\partial b_{\ell}} \right) \left[ |f(z)|^{p-2} f(z) \right] = \frac{d}{d\zeta} \left[ |f(z)|^{p-2} f(z) \right] = \frac{p-2}{2} |f(z)|^{p-4} f(z)^{2} \frac{df(z)}{d\zeta}$$

$$= \frac{p-2}{2} |f(z)|^{p-4} f(z)^{2} \psi_{\alpha,\ell}(z).$$  

(4.21)
where we have used the fact that \( f(z) \) is a holomorphic function of \( \zeta \) so \( \frac{df}{d\zeta} f(z) = 0 \). Similarly

\[
\frac{1}{2} \left( \frac{\partial}{\partial a_{\ell}} - i \frac{\partial}{\partial b_{\ell}} \right) [ |f(z)|^{p-2} f(z)] = \frac{d}{d\zeta} [ |f(z)|^{p-2} f(z)] = \left( \frac{d}{d\zeta} |f(z)|^{p-2} \right) f(z) + |f(z)|^{p-2} \frac{df(z)}{d\zeta} = \frac{p-2}{2} |f(z)|^{p-4} f(z) \frac{df(z)}{d\zeta} \cdot f(z) + |f(z)|^{p-2} \frac{df(z)}{d\zeta} = \frac{p}{2} |f(z)|^{p-2} \psi_{a,\ell}(z). \tag{4.22}
\]

Adding and subtracting (4.21) and (4.22) gives us the partial derivatives:

\[
\frac{\partial}{\partial a_{\ell}} [ |f(z)|^{p-2} f(z)] = \frac{p-2}{2} |f(z)|^{p-4} f(z)^2 \psi_{a,\ell}(z) + \frac{p}{2} |f(z)|^{p-2} \psi_{a,\ell}(z) \tag{4.23}
\]

\[
i \frac{\partial}{\partial b_{\ell}} [ |f(z)|^{p-2} f(z)] = \frac{p-2}{2} |f(z)|^{p-4} f(z)^2 \psi_{a,\ell}(z) - \frac{p}{2} |f(z)|^{p-2} \psi_{a,\ell}(z). \tag{4.24}
\]

It follows that the \( \frac{\partial}{\partial a_{\ell}} \) and \( \frac{\partial}{\partial b_{\ell}} \) partial derivatives of the integrand in (4.20) are both bounded in modulus by

\[
\frac{|p-2|+p}{2} |f(z)|^{p-2} |\psi_{a,\ell}(z)| |\psi_{a,k}(z)| e^{-\frac{\alpha}{2} |z|^2} \leq (p+1) \sqrt{\frac{\alpha^{k+\ell}}{k!}} |f(z)|^{p-2} |z|^{k+\ell} e^{-\frac{\alpha}{2} |z|^2}.
\]

The same argument given in the proof of Lemma 4.2 now shows that the partial derivatives are locally uniformly bounded by \( L^1 \) functions, and so by the Dominated Convergence Theorem the function in question is differentiable. We may therefore differentiate under the integral. The expressions in (4.23) and (4.24) are manifestly continuous functions of \( a, b, c, d \) near \( 0 \), and so the Dominated Convergence Theorem also shows that \( \langle G_{p,a}(f), \psi_{a,k} \rangle = C^1 \), as claimed.

Evaluating at \( a = b = 0 \), we have \( f = \psi_{a,N} \), and so (4.23) and (4.24) yield

\[
\frac{\partial}{\partial a_{\ell}} [ |f(z)|^{p-2} f(z)] \bigg|_{a=b=0} = \left( \frac{\alpha^N}{N!} \right) \frac{\alpha^{k+\ell}}{k!} \int_{\mathbb{C}} \left( \frac{p-2}{2} |z|^N |p-4 |z^{2N}\zeta^\ell + \frac{p}{2} |z|^p |p-2 \zeta^\ell \right) e^{-\frac{\alpha}{2} |z|^2} dz.
\]

and hence

\[
\frac{\partial}{\partial a_{\ell}} (G_{p,a}(f), \psi_{a,k}) \bigg|_{a=b=0} = \frac{\alpha}{\pi} \left( \frac{\alpha^N}{N!} \right) \frac{\alpha^{k+\ell}}{k!} \int_{\mathbb{C}} \left( \frac{p-2}{2} |z|^N |p-4 |z^{2N}\zeta^\ell + \frac{p}{2} |z|^p |p-2 \zeta^\ell \right) e^{-\frac{\alpha}{2} |z|^2} dz.
\]

We evaluate these integrals in polar coordinates.

\[
\int_{\mathbb{C}} \left( \frac{p-2}{2} |z|^N |p-4 |z^{2N}\zeta^\ell + \frac{p}{2} |z|^p |p-2 \zeta^\ell \right) e^{-\frac{\alpha}{2} |z|^2} dz
\]

\[= \int_0^\infty r dr \int_0^{2\pi} d\theta \left( \frac{p-2}{2} r^N |p-4 \zeta^\ell + \frac{p}{2} r^p |p-2 \zeta^\ell \right) e^{-\frac{\alpha}{2} r^2} = \int_0^\infty r^{N(p-4)+k} e^{-\frac{\alpha}{2} r^2} r dr \int_0^{2\pi} \left( \frac{p-2}{2} e^{i(2N-k)\theta} + \frac{p}{2} e^{i(k-\ell)\theta} \right) d\theta.
\]

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Since \( k, \ell < N, 2N - \ell - k > 0 \), and so the first term in the polar integral is 0. The second contributes a factor of \( \frac{1}{2} \cdot 2\pi \delta_{kl} \). As for the radial integral, now taking \( k = \ell \), we change variables \( u = \frac{\rho}{2} r^2 \) to find
\[
\int_0^\infty \rho^{N(p-2)+2k} e^{-\frac{\rho^2}{2}} \, d\rho = \left( \frac{2}{\alpha p} \right)^{N(p/2-1)+k} \int_0^\infty u^{N(p/2-1)+k+1} e^{-u} \, du = \frac{1}{2} \left( \frac{2}{\alpha p} \right)^{N(p/2-1)+k+1} \Gamma_{\alpha p/2}^2 (N(p/2 - 1) + k + 1).
\]
Combining and simplifying yields (4.16) and (4.17). The derivation of (4.18) and (4.19) is very similar. \( \square \)

We can now compute all of the second partial derivatives of the function \( R_{p, \alpha}^N \). The result is as follows.

**Proposition 4.11.** For \( \alpha, \epsilon > 0, p > 1 \), and \( N \in \mathbb{N} \), the function \( R_{p, \alpha}^N : (\mathbb{R}^N)^4 \to \mathbb{R}_+ \) is \( C^2 \) in a neighborhood of 0. For \( 0 \leq k, \ell \leq N - 1 \), its second partial derivatives at 0 are as follows. Define
\[
\Omega_{k, \alpha}^{N, \epsilon} (p) = \left( N! \right)^2 \frac{(p!)^{N/2}}{2^N} \left( \frac{p}{2} \right)^{N-k+1} \Gamma_{\alpha p/2}^2 \left( N(p/2 - 1) + k + 1 \right)
\]
\[
\Omega_{\alpha}^{N, \epsilon} (p) = \left( N! \right)^2 \frac{(p!)^{N/2}}{2^N} \Gamma_{\alpha p/2}^2 \left( N(p/2 - 1) + k + 1 \right).
\]

First, the \( a, b \) derivatives and \( c, d \) derivatives are
\[
\frac{\partial}{\partial a_\ell} \left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) R_{p, \alpha}^{N, \epsilon} \bigg|_0 = -\frac{\delta_{k\ell}}{k!} \Omega_{k, \alpha}^{N, \epsilon} (p)
\]
(4.29)
\[
\frac{\partial}{\partial b_\ell} \left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) R_{p, \alpha}^{N, \epsilon} \bigg|_0 = -i \frac{\delta_{k\ell}}{k!} \Omega_{k, \alpha}^{N, \epsilon} (p)
\]
(4.30)
\[
\frac{\partial}{\partial c_\ell} \left( \frac{\partial}{\partial c_k} + i \frac{\partial}{\partial d_k} \right) R_{p, \alpha}^{N, \epsilon} \bigg|_0 = -\frac{\delta_{k\ell}}{k!} \Omega_{k, \alpha}^{N, \epsilon} (p)
\]
(4.31)
\[
\frac{\partial}{\partial d_\ell} \left( \frac{\partial}{\partial c_k} + i \frac{\partial}{\partial d_k} \right) R_{p, \alpha}^{N, \epsilon} \bigg|_0 = -i \frac{\delta_{k\ell}}{k!} \Omega_{k, \alpha}^{N, \epsilon} (p)
\]
(4.32)

The mixed \( a, b \) and \( c, d \) derivatives are
\[
\frac{\partial}{\partial a_\ell} \left( \frac{\partial}{\partial c_k} + i \frac{\partial}{\partial d_k} \right) R_{p, \alpha}^{N, \epsilon} \bigg|_0 = \delta_{k\ell} \Omega_{\alpha}^{N, \epsilon} (p)
\]
(4.33)
\[
\frac{\partial}{\partial b_\ell} \left( \frac{\partial}{\partial c_k} + i \frac{\partial}{\partial d_k} \right) R_{p, \alpha}^{N, \epsilon} \bigg|_0 = i \delta_{k\ell} \Omega_{\alpha}^{N, \epsilon} (p)
\]
(4.34)
\[
\frac{\partial}{\partial c_\ell} \left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) R_{p, \alpha}^{N, \epsilon} \bigg|_0 = \delta_{k\ell} \Omega_{\alpha}^{N, \epsilon} (p)
\]
(4.35)
\[
\frac{\partial}{\partial d_\ell} \left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) R_{p, \alpha}^{N, \epsilon} \bigg|_0 = i \delta_{k\ell} \Omega_{\alpha}^{N, \epsilon} (p)
\]
(4.36)

(Of course \( \Omega_{\alpha}^{N, \epsilon} (p) = \Omega_{\alpha}^{N, \epsilon} (p') \); we denote them asymmetrically in (4.33)-(4.36) purely for aesthetics.)

**Proof.** This is a laborious but elementary calculation; we outline the derivations of (4.29) and (4.33), and leave the very similar cases (4.30)-(4.32) and (4.34)-(4.36) to the reader. From (4.10), we have
\[
\left( \frac{\partial}{\partial a_k} + i \frac{\partial}{\partial b_k} \right) R_{p, \alpha}^{N, \epsilon} = \frac{\langle g, \psi_{a,k}, \gamma \rangle}{\| g_e \|_{p', \alpha}} \frac{\operatorname{sgn} \langle f, g \rangle_\alpha}{\| f_e \|_{p, \alpha}} \frac{1}{2} \frac{1}{\| g_e \|_{p', \alpha}} \| f_e \|_{p', \alpha}^{1 - p}.
\]
All of the functions present are differentiable in a neighborhood of \(0\), as established throughout the previous section. Since the terms involving only \(g\) are independent of \(a, b\), we have
\[
\frac{\partial}{\partial a} \left( \frac{\partial}{\partial a} + i \frac{\partial}{\partial b} \right) R_{p,\alpha}^{N,\ell} = \left\langle g, \psi_{a,k} \right\rangle \frac{1}{\|g\|_{p',\alpha}} \cdot \frac{\partial}{\partial a} \|f\|_{p,\alpha} \cdot \frac{\partial}{\partial a} \left\langle G_{p,\alpha}(f), \psi_{a,k} \right\rangle \frac{1}{\|f\|_{p,\alpha}}.
\]
From (4.12), the derivatives of \(|\langle f, g \rangle\|_p\) are continuous where they exist, and by the quotient rule this also applies to the derivatives of \(\text{sgn}(\langle f, g \rangle)\) (since \(\langle f, g \rangle\) is differentiable everywhere). Lemmas 4.3 and 4.9 establish that \(\|f\|_{p,\alpha}\) and \(\langle G_{p,\alpha}(f), \psi_{a,k} \rangle\) are \(C^1\). Hence, using the quotient rule, we see that the derivatives above are continuous on the neighborhood of \(0\) where the denominators are differentiable. Thus, we see that \(R_{p,\alpha}^{N,\ell}\) is \(C^2\), as claimed.

Note that, at \(a = b = 0\), \(g = \psi_{a,N}\), and so \(\left\langle g, \psi_{a,k} \right\rangle = 0\) for all \(k < N\). Thus, the first term vanishes, and we have
\[
\frac{\partial}{\partial a} \left( \frac{\partial}{\partial a} + i \frac{\partial}{\partial b} \right) R_{p,\alpha}^{N,\ell} \bigg|_0 = -\frac{p}{2} \|f\|_{p,\alpha} \frac{\partial}{\partial a} \left\langle G_{p,\alpha}(f), \psi_{a,k} \right\rangle \frac{1}{\|f\|_{p,\alpha}}.
\]
From (4.12), we have
\[
\frac{\partial}{\partial a} \langle f, g \rangle \|_{p,\alpha} = \Re \left[ \text{sgn}(\langle f, g \rangle)(c_{\ell} + i\ell) \right]_0 = 0
\]
since we are evaluating all variables including \(\ell\) at \(0\). Employing the chain rule and (4.6), we have
\[
\frac{\partial}{\partial a} \|f\|_{p,\alpha} = (1 + p) \|f\|_{p,\alpha} \frac{\partial}{\partial a} \|f\|_{p,\alpha} = (1 + p) \|f\|_{p,\alpha} \frac{p}{2} \|f\|_{p,\alpha}^{-p} \Re \langle \langle G_{p,\alpha}(f), \psi_{a,k} \rangle \rangle.
\]
Evaluating at \(0\), since \(f = \psi_{a,N}\), as shown in the proof of Proposition 4.8, this inner product is 0, and so we have also shown that
\[
\frac{\partial}{\partial a} \|f\|_{p,\alpha} \bigg|_0 = 0.
\]
Combining (4.38) and (4.39) with product rule, we have
\[
\frac{\partial}{\partial a} \langle f, g \rangle \|_{p,\alpha} \bigg|_0 = \frac{1}{\|1 \geq a, \psi_{a,N}\|_{p,\alpha}^{1+p}} \frac{\partial}{\partial a} \langle G_{p,\alpha}(f), \psi_{a,k} \rangle \bigg|_0.
\]
Combining (4.40) with (4.37) gives
\[
\frac{\partial}{\partial a} \left( \frac{\partial}{\partial a} + i \frac{\partial}{\partial b} \right) R_{p,\alpha}^{N,\ell} \bigg|_0 = -\frac{p}{2} \|1 \geq a, \psi_{a,N}\|_{p,\alpha}^{1+p} \frac{\partial}{\partial a} \langle G_{p,\alpha}(f), \psi_{a,k} \rangle \bigg|_0.
\]
We can evaluate the norms of \(1 \geq a, \psi_{a,N}\) as we did in the proof of Lemma 4.9
\[
\|1 \geq a, \psi_{a,N}\|_{p,\alpha} = \left( \frac{\alpha N}{N!} \right)^{p/2} \int_{\Sigma_{a}} \left| z \right|^{p-1} e^{-\frac{\alpha}{2}|z|^2} dz = \left( \frac{\alpha N}{N!} \right)^{p/2} \frac{\alpha p}{2\pi} \cdot 2\pi \cdot \int_{\ell}^{\infty} r^N e^{-\frac{\alpha}{2}r^2} rdr,
\]
and changing variables \(u = \frac{\alpha}{2}r^2\) and taking \(p\)th roots yields
\[
\|1 \geq a, \psi_{a,N}\|_{p,\alpha} = \frac{\sqrt{\alpha N}}{\sqrt{N!}} (\alpha p)^{1/p} \cdot \frac{2^{N/2}}{(\alpha p)^{N/2}} \Gamma_{\alpha p^{2}/2}(Np/2 + 1)^{1/p} \cdot \frac{1}{\sqrt{N!}} \left( \frac{2}{p} \right)^{N/2} \Gamma_{\alpha p^{2}/2}(Np/2 + 1)^{1/p}.
\]
Combining (4.41) and (4.42) with (4.16) and simplifying yields (4.29).

We now proceed to outline the similar derivation of (4.33). This time starting with (4.11), we have

\[
\left( \frac{\partial}{\partial c_k} + i \frac{\partial}{\partial d_k} \right) R_{p,\alpha}^{N,\epsilon} = \frac{1}{\|g_k\|_{p',\alpha}} \cdot \frac{\langle f, \psi_{\alpha,k} \rangle_\alpha \text{sgn}(\langle f, g \rangle_\alpha)}{\|f_k\|_{p,\alpha}} - \frac{p'}{2} \frac{\langle G_{p',\alpha}(g_k), \psi_{\alpha,k} \rangle_\alpha}{\|g_k\|_{p',\alpha}} \cdot \|f_k\|_{p,\alpha}.
\]

Since the terms involving only \( g \) are independent of \( a, b \), we have

\[
\frac{\partial}{\partial a_\ell} \left( \frac{\partial}{\partial c_k} + i \frac{\partial}{\partial d_k} \right) R_{p,\alpha}^{N,\epsilon} = \frac{1}{\|g_k\|_{p',\alpha}} \cdot \frac{\partial}{\partial a_\ell} \frac{\langle f, \psi_{\alpha,k} \rangle_\alpha \text{sgn}(\langle f, g \rangle_\alpha)}{\|f_k\|_{p,\alpha}} - \frac{p'}{2} \frac{\partial}{\partial a_\ell} \frac{\langle G_{p',\alpha}(g_k), \psi_{\alpha,k} \rangle_\alpha}{\|g_k\|_{p',\alpha}} \cdot \|f_k\|_{p,\alpha}.
\]

In the first term, \( \langle f, \psi_{\alpha,k} \rangle_\alpha = a_k + ib_k \) and so the \( \frac{\partial}{\partial a_\ell} \) derivative of this term is \( \delta_{k\ell} \). Equations (4.38) and (4.39) show that, at \( 0 \), both quantities \( \text{sgn}(\langle f, g \rangle_\alpha) \) and \( \|f_k\|_{p,\alpha} \) have \( \frac{\partial}{\partial a_\ell} \) derivative 0. Thus, the first term above is

\[
\frac{1}{\|g_k\|_{p',\alpha}} \frac{\partial}{\partial a_\ell} \frac{\langle f, \psi_{\alpha,k} \rangle_\alpha \text{sgn}(\langle f, g \rangle_\alpha)}{\|f_k\|_{p,\alpha}} \bigg|_0 = \delta_{k\ell} \frac{\text{sgn}(\langle f, g \rangle_\alpha)}{\|f_k\|_{p,\alpha} \|g_k\|_{p',\alpha}} \bigg|_0 = \delta_{k\ell} \frac{1}{\|1_\alpha \psi_{\alpha,N}\|_{p,\alpha} \|1_\alpha \psi_{\alpha,N}\|_{p',\alpha}}.
\]

For the second term, the derivative at 0 is again by (4.38) and (4.39). Thus, we have

\[
\frac{\partial}{\partial a_\ell} \left( \frac{\partial}{\partial c_k} + i \frac{\partial}{\partial d_k} \right) R_{p,\alpha}^{N,\epsilon} \bigg|_0 = \delta_{k\ell} \frac{1}{\|1_\alpha \psi_{\alpha,N}\|_{p,\alpha} \|1_\alpha \psi_{\alpha,N}\|_{p',\alpha}} = \delta_{k\ell} \frac{\sqrt{N!} (p/2)^{N/2} \cdot \sqrt{N!} (p'/2)^{N/2}}{\Gamma_{op}^{2/2} (Np/2 + 1)^{1/p} \Gamma_{op'}^{2/2} (Np'/2 + 1)^{1/p'}}
\]

using (4.42): simplifying yields (4.33). The other six equations are derived similarly.

We can now analyze the Hessian matrix \( H = H_{p,\alpha}^{N,\epsilon} \) of the function \( R = R_{p,\alpha}^{N,\epsilon} \) at 0. This involves choosing an order for the \( 4N \) variables \( \{a_k, b_k, c_k, d_k\}_{0 \leq k \leq N-1} \). For convenience, we order them as follows:

\[(a_0, c_0, b_0, d_0, a_1, c_1, b_1, d_1, \ldots, a_{N-1}, c_{N-1}, b_{N-1}, d_{N-1}).\]

This means that the Hessian is a \( 4N \times 4N \) block matrix

\[
H = \begin{bmatrix}
H_{0,0} & H_{0,1} & \cdots & H_{0,N-1} \\
H_{1,0} & H_{1,1} & \cdots & H_{1,N-1} \\
\vdots & \vdots & \ddots & \vdots \\
H_{N-1,0} & H_{N-1,1} & \cdots & H_{N-1,N-1}
\end{bmatrix},
\]

where

\[
H_{k,\ell} = \begin{bmatrix}
\frac{\partial^2 R}{\partial a_k \partial a_\ell} & \frac{\partial^2 R}{\partial a_k \partial c_\ell} & \frac{\partial^2 R}{\partial a_k \partial d_\ell} & \frac{\partial^2 R}{\partial a_k \partial \ell} \\
\frac{\partial^2 R}{\partial c_k \partial a_\ell} & \frac{\partial^2 R}{\partial c_k \partial c_\ell} & \frac{\partial^2 R}{\partial c_k \partial d_\ell} & \frac{\partial^2 R}{\partial c_k \partial \ell} \\
\frac{\partial^2 R}{\partial d_k \partial a_\ell} & \frac{\partial^2 R}{\partial d_k \partial c_\ell} & \frac{\partial^2 R}{\partial d_k \partial d_\ell} & \frac{\partial^2 R}{\partial d_k \partial \ell} \\
\frac{\partial^2 R}{\partial \ell \partial a_\ell} & \frac{\partial^2 R}{\partial \ell \partial c_\ell} & \frac{\partial^2 R}{\partial \ell \partial d_\ell} & \frac{\partial^2 R}{\partial \ell \partial \ell}
\end{bmatrix}.
\]

We now translate Proposition 3.11 into a characterization of the block matrices \( H_{k,\ell} \). The verification is bookkeeping, and is left to the reader to check.

**Corollary 4.12.** For \( 0 \leq k, \ell \leq N - 1 \),

\[
H_{k,\ell} = \delta_{k\ell} \begin{bmatrix}
H_k & 0_{2 \times 2} \\
0_{2 \times 2} & H_k
\end{bmatrix}, \quad \text{where} \quad H_k = \begin{bmatrix}
-\frac{1}{k} \Omega_{k,\alpha}^{N,\epsilon}(p) & \bar{\Omega}_{\alpha,\ell}^{N,\epsilon}(p') \\
\bar{\Omega}_{\alpha,\ell}^{N,\epsilon}(p) & -\frac{1}{k} \Omega_{k,\alpha}^{N,\epsilon}(p')
\end{bmatrix},
\]

where \( \Omega_{k,\alpha}^{N,\epsilon} \) and \( \bar{\Omega}_{\alpha,\ell}^{N,\epsilon} \) are the functions in (4.27)–(4.28).
We make one further simplification. Since \( \tilde{u}^{N,e}_{\alpha}(p) = \tilde{u}^{N,e}_{\alpha}(p') \) is independent of \( k \), Corollary 4.12 shows that the Hessian has the block-diagonal form

\[
H = \tilde{u}^{N,e}_{\alpha}(p) \cdot \text{diag}[\hat{H}_0, \hat{H}_0, \hat{H}_1, \hat{H}_1, \ldots, \hat{H}_{N-1}, \hat{H}_{N-1}]
\]  

(4.44)

where

\[
\hat{H}_k = \begin{bmatrix}
-\gamma^{N,e}_{k,\alpha}(p) & 1 \\
1 & -\gamma^{N,e}_{k,\alpha}(p')
\end{bmatrix},
\]

and

\[
\gamma^{N,e}_{k,\alpha}(p) = \frac{1}{k!} \frac{\Omega^{N,e}_{k,\alpha}(p)}{\tilde{u}^{N,e}_{\alpha}(p)} = \frac{N!}{k!} \left( p/2 \right)^{-k+1} \frac{\Gamma_0^{p/2}(N(p/2 - 1) + k + 1)}{\Gamma_0^{p/2}(N(p/2 + 1) + k + 1)}.
\]  

(4.45)

**Proposition 4.13.** Given \( p > 1, N \in \mathbb{N}, \) and \( \alpha > 0 \), for all sufficiently small \( \epsilon > 0 \), the Hessian matrix \( H \) of the function \( \tilde{H}^{N,e}_{p,\alpha} \) at 0 is negative definite.

**Proof.** It is straightforward to verify that a block diagonal matrix is negative definite if and only if each of its blocks is negative definite. Note from (4.14) that \( \Gamma_0(x) \) is the integral of a positive function, and so is positive; it follows that the constants \( \Omega^{N,e}_{k,\alpha}(p) \) and \( \tilde{u}^{N,e}_{\alpha}(p) \) of (4.28) are positive. Thus, to prove the proposition, it suffices to show that the \( 2 \times 2 \) matrices \( \hat{H}_k \) are each negative definite for \( 0 \leq k \leq N - 1 \). Since \( -\gamma^{N,e}_{k,\alpha}(p) < 0 \), we are left only to compute \( \det \hat{H}_k \) and show that it is positive:

\[
\det \hat{H}_k = \frac{\gamma^{N,e}_{k,\alpha}(p)}{\gamma^{N,e}_{k,\alpha}(p)} = \left( \frac{N!}{k!} \right)^2 \left( \frac{pp'}{4} \right)^{N-k+1} \frac{\Gamma_0^{p/2}(N(p/2 - 1) + k + 1)}{\Gamma_0^{p/2}(N(p/2 + 1) + k + 1)} - 1.
\]

Note that this quantity is symmetric in \( p, p' \), and so without loss of generality we assume \( p < 2 \) so that \( p' > 2 \). Thus \( N(p'/2 - 1) + k + 1 > 0 \), and the denominator in the fraction is \( > 0 \). We consider two cases now.

- Suppose that \( k \) is small enough that \( N(p/2 - 1) + k + 1 \leq 0 \). Then \( \Gamma_0^{p/2}(N(p/2 - 1) + k + 1) \to +\infty \) as \( \epsilon \downarrow 0 \), while the other terms in the expression converge to a finite ratio of evaluations of the Gamma function. It follows that, for all sufficiently small \( \epsilon > 0 \), \( \det \hat{H}_k > 0 \) for such \( k \).

- Suppose that \( k \) is large enough that \( N(p/2 - 1) + k + 1 > 0 \). In this case all four terms in the expression converge to (finite) evaluations of the Gamma function:

\[
\lim_{\epsilon \downarrow 0} \det \hat{H}_k = \left( \frac{N!}{k!} \right)^2 \left( \frac{pp'}{4} \right)^{N-k+1} \frac{\Gamma_0^{p/2}(N(p/2 - 1) + k + 1)}{\Gamma_0^{p/2}(N(p/2 + 1) + k + 1)} - 1.
\]

Using the defining Gamma relation \( \Gamma(x + 1) = x\Gamma(x) \) for \( x > 0 \), the ratio simplifies

\[
\frac{\Gamma_0^{p/2}(N(p/2 - 1) + k + 1)}{\Gamma_0^{p/2}(N(p/2 + 1) + k + 1)} = \frac{1}{(Np/2)(Np/2 - 1) \cdots (Np/2 - (N - k) + 1)}
\]

and so

\[
\frac{N! \Gamma_0^{p/2}(N(p/2 - 1) + k + 1)}{\Gamma_0^{p/2}(N(p/2 + 1) + k + 1)} = \frac{N(N - 1) \cdots (k+1)}{(Np/2)(Np/2 - 1) \cdots (Np/2 - (N - k) + 1)}
\]

\[
= \prod_{j=0}^{N-k-1} \frac{N-j}{Np/2-j} = \left( \frac{2}{p} \right)^{N-k} \prod_{j=0}^{N-2j/p} \frac{N-j}{N-2j/p}
\]
Thus
\[
\lim_{\epsilon\downarrow 0} \det \hat{H}_k = \frac{pp'}{4} \prod_{j=0}^{N-k-1} \frac{(N-j)^2}{(N-2j/p)(N-2j/p') - 1}.
\]

The denominator simplifies to give
\[
N^2 - 2j + 4j^2/pp' = (N-j)^2 - (1-4/pp')j^2 < (N-j)^2.
\]
Since \(pp'/4 > 1\), each term in the product is greater than 1, and so this determinant is greater than 0. It follows that, for sufficiently small \(\epsilon > 0\), \(\det \hat{H}_k > 0\) for such \(k\).

Since there are only finitely many \(k\) in the statement, this shows there is a sufficiently small \(\epsilon > 0\) to verify the proposition.

This finally brings us to the proof of our main theorem.

**Proof of Theorem 1.6.** From Lemma 4.4, we may choose \(\epsilon_0 > 0\) small enough that, for \(0 < \epsilon < \epsilon_0\), \(R_{p,0}^{N,\epsilon}(0) < C_p^{1/2}\). Proposition 4.13 shows that there is \(\epsilon_1\) so that the matrix \(H = H^*\) of (4.44) is negative definite for \(0 < \epsilon < \epsilon_1\). Thus, fix some \(\epsilon \in (0, \min\{\epsilon_0, \epsilon_1\})\). Propositions 4.8 and 4.13 show that there is a neighborhood \(B_0^N \subset A_0^N\) of \(0 \in (\mathbb{R}^N)^4\) where \(R_{p,0}^{N,\epsilon}\) is \(C^2\), and possesses a local maximum at \(0\) on this neighborhood. Now utilizing Lemma 4.4, it follows that \(R_{0,0}^N \leq R_{0,p}^{N,\epsilon} \leq R_{p,0}^{N,\epsilon}(0) < C_p^{1/2}\) on \(B_0^N\). Scaling and using (4.3), we find the requisite neighborhood \(B_0^N\) of \(z^N \in A_0^N\). Finally, note that the supremum over \(B_0^N\) is greater than or equal to \(R_{p,0}(z^N, z^N)\) since \(z^N \in B_0^N\), and is also \(\leq C_p^{1/2}\); since \(\lim_{N \to \infty} R_{p,0}(z^N, z^N) = C_p^{1/2}\) by Proposition 3.1, it follows from the Squeeze Theorem that the supremum over \(B_0^N\) converges to \(C_p^{1/2}\) as claimed.

**Remark 4.14.** We have restricted our attention in this section to the 1-dimensional case. It is possible that a similar (though necessarily much more complicated) calculus approach could be used to extend this result to higher dimensions. As the result is local and only suggestive of the conjectured sharp inequality, we satisfy ourselves here with the 1-dimensional case. Note: in [2], our approach was to prove the sharp bound in the case \(n = 1\), and then generalize this to all \(n\) with a tensorial argument that goes colloquially under the name “Segal’s Lemma”. In the present context, this does not produce the desired result, as the approach requires Hölder’s inequality which (cf. Theorem 1.4) is not sharp enough here.

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