Non-Abelian Gravity and
Antisymmetric Tensor Gauge Theory

C.M. Hull

Physics Department, Queen Mary and Westfield College,
Mile End Road, London E1 4NS, U.K.

ABSTRACT

A non-abelian generalisation of a theory of gravity coupled to a 2-form gauge field and a dilaton is found, in which the metric and 3-form field strength are Lie algebra-valued. In the abelian limit, the curvature with torsion is self-dual in four dimensions, or has $SU(n)$ holonomy in $2n$ dimensions. The coupling to self-dual Yang-Mills fields in 4 dimensions, or their higher dimensional generalisation, is discussed. The abelian theory is the effective action for (2,1) strings, and the non-abelian generalisation is relevant to the study of coincident branes in the (2,1) string approach to M-theory. The theory is local when expressed in terms of a vector pre-potential.
1. Introduction

Maxwell theory has a straightforward non-abelian extension to Yang-Mills theory, but it has so far proved impossible to generalise this to obtain non-abelian anti-symmetric gauge theories. The abelian case involves an n-form gauge field $b$ with abelian transformation $\delta b = d\alpha$ and field strength $H = db$, but it seems impossible to find a non-abelian generalisation that is local in terms of $b$ unless $n = 1$. However, it is now believed that there should be a supersymmetric theory in 5 + 1 dimensions that includes a self-dual 2-form gauge field $b$, with $H = *H$ (plus corrections involving scalar fields) in the abelian limit, but which has a non-abelian generalisation that dimensionally reduces to Yang-Mills theory in 4 + 1 dimensions [1-5]. Such theories arise from the world-volume theory of the M-theory 5-brane, which becomes non-abelian in the limit of coincident 5-branes [4,5], or from the low-energy-limit of the type IIB string compactified on $K3$, which has enhanced non-abelian gauge symmetry when two-cycles of the $K3$ degenerate [1]. It has been argued that this theory should exist as a consistent quantum theory [2]. Although we will not have anything to say about this case here, we will be able to find a non-abelian version of a related theory, which may have implications for the six-dimensional tensor theory.

Martinec and Kutasov [6-9] have argued that the superstring with (2,1) worldsheet supersymmetry gives rise to the various branes and vacua of M-theory and string theory. The (2,1) string is a theory of gravity plus an anti-symmetric tensor gauge field and a Yang-Mills field in 2+2 dimensions, with a generalised self-duality condition on both the curvature with torsion and the Yang-Mills field strength. The field equations were obtained in [13] and the action was given in [9,10], and a similar action can be used to describe fields in 10 + 2 dimensions, before null reduction. This gives rise to the various branes [6-9]; for example reducing to a 1 + 1 dimensional real subspace of the 2 + 2 dimensional space gives a Born-Infeld type string action, which can be associated with that of a D-string [8,9]. As coincident branes have non-abelian gauge symmetries, it should be the case that
the (2,1) string action should also have a non-abelian generalisation, which would involve a non-abelian generalisation of gravity plus an anti-symmetric tensor gauge field. We will show that this is indeed the case, giving a theory in which the metric and 2-form gauge field take values in a Lie group. This may be of import for the geometry of matrix theory, especially in view of the proposed relation between matrix theory and the (2,1) string [11].

In (2,1) geometry, the metric and antisymmetric tensor are given in terms of a vector $B$, with

$$g_{\alpha \bar{\beta}} = i(\partial_{\alpha} \bar{B}_{\bar{\beta}} - \bar{\partial}_{\bar{\beta}} B_{\alpha}) \quad (1.1)$$

in complex coordinates. This has the obvious gauge symmetry $\delta B = d\alpha$. It has a natural non-abelian generalisation in which $B$ becomes a non-abelian gauge field with $\delta B = d\alpha + [B, \alpha]$ and field strength $\mathcal{F} = dB + B^2$, so that (1.1) is replaced by

$$g_{\alpha \bar{\beta}} = i\mathcal{F}_{\alpha \bar{\beta}}. \quad (1.2)$$

The 2-form gauge field is also given in terms of $B$ and has a non-abelian generalisation. A gauge covariant theory can then be written in terms of the prepotential $B$, which can be used to define a gauge-covariant derivative etc. In particular, the real reduction of the 2 + 2 dimensional theory gives a non-abelian action of the D-string type. The resulting theory is local in terms of the pre-potential $B$.

The Lie-algebra-valued metric can be used to define a line-element for a manifold whose coordinates are G-valued matrices; this may have applications to matrix theory. However, the theories described here are non-abelian generalisations of the metric and 2-form gauge field in a standard space-time with commuting coordinates. They are invariant under the standard general coordinate transformations.
2. The (2,1) Supersymmetric Sigma-Model

The (1,1) supersymmetric sigma-model defined in a background with metric $g_{ij}$, anti-symmetric tensor $b_{ij}$ and dilaton $\Phi$ is conformally invariant at one-loop if the background fields satisfy

$$R_{ij}^{(+)} - \nabla_i(\nabla_j)\Phi - H_{ik}^j \nabla_k \Phi = 0 \quad (2.1)$$

where $R_{ij}^{(+)}$ is the Ricci tensor for a connection with torsion. We define the connections with torsion

$$\Gamma_{jkl}^{(\pm)i} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \pm H_{jkl}^{(\pm)} \quad (2.2)$$

where $\left\{ \begin{array}{c} i \\ jk \end{array} \right\}$ is the Christoffel connection and the torsion tensor is

$$H_{ijk} = \frac{3}{2} \partial_i b_{jk}. \quad (2.3)$$

The curvature and Ricci tensors with torsion are

$$R_{ij}^{(\pm)} = \partial_i \Gamma_{jkl}^{(\pm)k} - \partial_j \Gamma_{ikl}^{(\pm)k} + \Gamma_{im}^{(\pm)k} \Gamma_{jkl}^{(\pm)m} - \Gamma_{jm}^{(\pm)k} \Gamma_{ikl}^{(\pm)m}, \quad R_{ij}^{(\pm)} = R_{ikl}^{(\pm)} \quad (2.4)$$

The equation (2.1) can be obtained from varying the action

$$S = \int d^D x e^{-2\Phi} \sqrt{|g|} \left( R - \frac{1}{3} H^2 + 4(\nabla \Phi)^2 \right). \quad (2.5)$$

The sigma model is invariant under (2,1) supersymmetry [12-16] if the target space is even dimensional ($D = 2n$) with a complex structure $J_{ij}$ which is covariantly constant

$$\nabla_k^{(\pm)} J_{ij}^{(\pm)} = 0 \quad (2.6)$$

with respect to the connection $\Gamma^{(\pm)}$ defined in (2.2), and with respect to which the metric is hermitian, so that $J_{ij} \equiv g_{ik} J_{kj}$ is antisymmetric.
It is useful to introduce complex coordinates \( z^\alpha, \bar{z}^{\bar{\beta}} \) in which the line element is \( ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^{\bar{\beta}} \) and the exterior derivative decomposes as \( d = \partial + \bar{\partial} \). The conditions for (2,1) supersymmetry imply that the fundamental 2-form

\[
J = \frac{1}{2} J_{ij} d\phi^i d\phi^j = ig_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^{\bar{\beta}}
\]

satisfies

\[
i\partial \bar{\partial} J = 0.
\]

Then

\[
H = i(\partial - \bar{\partial})J
\]

defines a 3-form \( H \) for which the (0,3) and (3,0) parts vanish, and which is closed, \( dH = 0 \), so that locally there is a 2-form \( b \) with \( H = db \). Then

\[
H_{\alpha\beta\bar{\gamma}} = \partial_{[\alpha} g_{\beta]\bar{\gamma}}
\]

Furthermore, (2.8) implies that locally there is a (1,0) form \( k = k_\alpha dz^\alpha \) such that

\[
J = i(\partial \bar{k} + \bar{\partial} k)
\]

The metric and torsion potential are then given, in a suitable gauge, by

\[
g_{\alpha\bar{\beta}} = \partial_\alpha \bar{k}_{\bar{\beta}} + \bar{\partial}_{\bar{\beta}} k_\alpha
\]

\[
b_{\alpha\bar{\beta}} = \partial_\alpha \bar{k}_{\bar{\beta}} - \bar{\partial}_{\bar{\beta}} k_\alpha
\]

Setting \( k_\alpha = iB_\alpha \) recovers (1.2). If \( k_\alpha = \partial_\alpha K \) for some \( K \), then the torsion vanishes and the manifold is Kahler with Kahler potential \( K \), but if \( dk \neq 0 \) then the space is a hermitian manifold of the type introduced in [12]. The metric and torsion are
invariant under [22]
\[ \delta k_\alpha = i \partial_\alpha \chi + \theta_\alpha \] (2.13)
where \( \chi \) is real and \( \theta_\alpha \) is holomorphic, \( \partial_\beta \theta_\alpha = 0 \). Under these transformations, \( g_{\alpha \bar{\beta}} \) is invariant, while \( b_{\alpha \bar{\beta}} \) as defined in (2.12) transforms as
\[ \delta b_{\alpha \bar{\beta}} = -2i \partial_\alpha \bar{\partial}_\beta \chi \] (2.14)
which is an antisymmetric gauge transformation
\[ \delta b_{ij} = \partial_{[i} \lambda_{j]} \] (2.15)
with parameter \( \lambda_\alpha = 2i \partial_\alpha \chi \). The equation (2.9) only defines \( b \) up to a gauge transformation \( b \rightarrow b + d\lambda \). Such a gauge transformation can be used to chose \( b \) to be a (1,1) form as in (2.12), or a (2,0) form plus a (0,2) form given by
\[ b_{\alpha \beta} = \partial_{[\alpha} k_{\beta]}, \quad \bar{b}_{\bar{\alpha} \bar{\beta}} = \partial_{[\bar{\alpha}} \bar{k}_{\bar{\beta}]} \] (2.16)
with \( H_{\alpha \beta \gamma} \) given by
\[ H_{\alpha \beta \gamma} = \partial_\gamma b_{\alpha \beta} \] (2.17)
Under (2.13),
\[ \delta b_{\alpha \beta} = \partial_{[\alpha} \theta_{\beta]} \] (2.18)
which is an antisymmetric gauge transformation (2.15) with parameter \( \lambda_\alpha = \theta_\alpha \).

It will be useful to define the vector
\[ v^i = H_{jkl} J^{ij} J^{kl} \] (2.19)
Together with the \( U(1) \) part of the curvature
\[ C_{ij}^{(+)} = J^l_k R^{(+)} k_{li j} \] (2.20)
and the $U(1)$ part of the connection (2.2)

$$
\Gamma^{(+)}_i = J^k \Gamma^{(+)}_{ik} = i(\Gamma^{(+)}_{i\alpha} - \Gamma^{(+)}_{i\bar{\alpha}})
$$

(2.21)

In a complex coordinate system, (2.20) can be written as $C^{(+)}_{ij} = \partial_i \Gamma^{(+)}_j - \partial_j \Gamma^{(+)}_i$.

If the metric has Euclidean signature, then the holonomy of any metric connection (including $\Gamma^{(\pm)}$) is contained in $O(2n)$, while if it has signature $(2n_1, 2n_2)$ where $n_1 + n_2 = n$, it will be in $O(2n_1, 2n_2)$. The holonomy $\mathcal{H}(\Gamma^{(+)})$ of the connection $\Gamma^{(+)}$ is contained in $U(n_1, n_2)$. It will be contained in $SU(n_1, n_2)$ if in addition

$$
C^{(+)}_{ij} = 0
$$

(2.22)

where the $U(1)$ part of the curvature is given by (2.20). As $C_{ij}$ is a representative of the first Chern class, a necessary condition for this is the vanishing of the first Chern class.

It was shown in [13] that geometries for which

$$
\Gamma^{(+)}_i = 0
$$

(2.23)

in some suitable choice of coordinate system will satisfy the one-loop conditions (2.1), provided the dilaton is chosen as

$$
\Phi = -\frac{1}{2} \log |det g_{\alpha\bar{\beta}}|
$$

(2.24)

which implies

$$
\partial_i \Phi = v_i
$$

(2.25)

Moreover, the one-loop dilaton field equation is also satisfied for compact manifolds, or for non-compact ones in which $\nabla \Phi$ falls off sufficiently fast [10]. This implies that $\mathcal{H}(\Gamma^{(+)}) \subseteq SU(n_1, n_2)$ and these geometries generalise the Kahler Ricci-flat or Calabi-Yau geometries, and reduce to these in the special case in which $H = 0$. These are not the most general solutions of (2.1) [10].
The equation (2.23) can be viewed as a field equation for the potential \( k_\alpha \). It can be obtained by varying the action [9,8,10]

\[
S = \int d^D x \sqrt{|\det g_{\alpha\bar{\beta}}|} \tag{2.26}
\]

where \( g_{\alpha\bar{\beta}} \) is given in terms of \( k_\alpha \) by (2.12). It can be rewritten as

\[
S = \int d^D x |\det g_{ij}|^{1/4} \tag{2.27}
\]

which is non-covariant, as the field equation (2.23) was obtained in a particular coordinate system. However, it is invariant under volume-preserving diffeomorphisms.

3. Coupling to Yang-Mills Fields

This theory of gravity can be coupled to (conventional) Yang-Mills fields [9,10]. If \( A \) is a connection on a holomorphic vector bundle with structure group \( H \) over such a hermitian geometry, then we can define the Bott-Chern form \( \Upsilon \) [18] (constructed in [19,20,21]) by

\[
\text{tr}(F^2) = i\partial\bar{\partial}\Upsilon \tag{3.1}
\]

If \( H \) is abelian, then \( F^m = dA^m \) and there are real scalars \( \phi^m, \theta^m \) \((m = 1, \ldots, \text{rank}(H))\) such that

\[
A^m = d\theta^m + i(\partial - \bar{\partial})\phi^m \tag{3.2}
\]

and the Bott-Chern form can be chosen to be

\[
\Upsilon = -4i\partial\phi^m \bar{\partial}\phi^m
\]

This can be used to define a sigma-model with (2,1) supersymmetry. The
3-form field strength receives a Chern-Simons correction

\[ H = \frac{1}{2} db + \Omega(A) \]  

(3.3)

where \( d\Omega = tr F^2 \). Now

\[ i\bar{\partial} \partial J = tr(F)^2 \]  

(3.4)

and there is (1,0) form \( k \) such that

\[ J = \Upsilon + i(\partial \bar{k} + \bar{\partial} k) \]  

(3.5)

and the metric and torsion potential are given by

\[
\begin{align*}
    g_{\alpha\bar{\beta}} &= i\Upsilon_{\alpha\bar{\beta}} + \partial_\alpha \bar{k}_{\bar{\beta}} + \bar{\partial}_{\bar{\beta}} k_\alpha \\
    b_{\alpha\bar{\beta}} &= i\chi_{\alpha\bar{\beta}} + \partial_\alpha \bar{k}_{\bar{\beta}} - \bar{\partial}_{\bar{\beta}} k_\alpha
\end{align*}
\]  

(3.6)

where \( \chi \) is defined by

\[ \Omega(A) = i(\partial - \bar{\partial})\Upsilon + d\chi \]  

(3.7)

The field equations can be obtained by varying the action (2.27). The Yang-Mills equation is

\[ J_{ij}^i F_{ij} = 0 \]  

(3.8)

This can be generalised to the case of (2,0) geometry, but requires the introduction of gravitational Chern-Simons and Bott-Chern terms [10].

It is sometimes useful to write the metric in terms of a fixed background metric \( \hat{g}_{\alpha\bar{\beta}} \) (e.g. a flat metric) which is given in terms of a potential \( \hat{k} \) by \( \hat{g}_{\alpha\bar{\beta}} = \partial_\alpha \hat{k}_{\bar{\beta}} + \).
\[ \partial_\beta \hat{k}_\alpha, \] and a fluctuation given in terms of a vector field \( B_i \) defined by

\[
B_\alpha = -i(k_\alpha - \hat{k}_\alpha), \quad B_\bar{\alpha} = i(\bar{k}_\bar{\alpha} - \hat{k}^{*}_{\bar{\alpha}})
\] (3.9)

with field strength \( \mathcal{F} = dB \). Then

\[
g_{\alpha\bar{\beta}} = \hat{g}_{\alpha\bar{\beta}} + iF_{\alpha\bar{\beta}} + \Upsilon_{\alpha\bar{\beta}}
\] (3.10)

The gauge symmetry (2.13) has become the usual gauge transformation of an abelian gauge field

\[
\delta B_i = \partial_i \chi
\] (3.11)

and the action (2.26) becomes

\[
S = \int d^Dx \sqrt{|\text{det}(\hat{g}_{\alpha\bar{\beta}} + iF_{\alpha\bar{\beta}} + \Upsilon_{\alpha\bar{\beta}})|}
\] (3.12)

which is similar to a Born-Infeld action. Note that the (2,0) part of \( \mathcal{F} \) is non-zero.

### 4. Non-Abelian Geometry

A Born-Infeld action for an abelian gauge field

\[
S = \int d^Dx \sqrt{|\text{det}(G_{ij} + F_{ij})|}
\] (4.1)

can be generalised to the non-abelian case in which \( F \) takes values in some Lie algebra to give the action

\[
S = \int d^Dx \text{Str} \sqrt{|\text{det}(G_{ij} + F_{ij})|}
\] (4.2)

where \( \text{Str} \) denotes the symmetrised trace. For D-brane actions, it was argued in [17] that this is the appropriate prescription, although in what follows it could be replaced by any other suitable trace prescription.
The similarity between (4.1) and (3.12) suggests that (3.12) should also have a non-abelian generalisation in which $B_\alpha$ becomes a non-abelian gauge field. The vector potential $k_\alpha$ will now be supposed to take values in some Lie algebra $G$, so that $k_\alpha = k^a_\alpha t_a$, where $t_a$ are Lie algebra generators. The abelian transformation (2.13)

$$\delta k_\alpha = i \partial_\alpha \chi$$

(4.3)

will now be generalised to the non-abelian transformation

$$\delta k_\alpha = i \partial_\alpha \chi + [k_\alpha, \chi]$$

(4.4)

where $\chi = \chi^a t_a$. A gauge-covariant $G$-valued generalisation of the metric, $g_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}^{a\bar{a}} t_a$, is given by

$$g_{\alpha\bar{\beta}} = \partial_\alpha \bar{k}_\bar{\beta} + \bar{\partial}_\bar{\beta} k_\alpha - i [k_\alpha, \bar{k}_\bar{\beta}]$$

(4.5)

The natural generalisation of the action (2.26) is then

$$S = \int d^D x \text{Str} \sqrt{\text{det} g_{\alpha\bar{\beta}}}$$

(4.6)

Defining

$$B_\alpha = -i k_\alpha, \quad B_{\bar{\alpha}} = i \bar{k}_{\bar{\alpha}}$$

(4.7)

gives a potential in which (4.3) becomes the usual gauge transformation

$$\delta B_i = \partial_i \chi + [B_i, \chi]$$

(4.8)

with field strength $F = dB + \frac{1}{2} [B, B]$. Then the non-abelian metric is given by the (1,1) part of the field strength

$$g_{\alpha\bar{\beta}} = -i F_{\alpha\bar{\beta}}$$

(4.9)

This can be generalised to include the coupling to Yang-Mills fields $A$ taking
values in a group $H$, which are taken as singlets of $G$. Then

$$g_{\alpha\bar{\beta}} = \Upsilon_{\alpha\bar{\beta}}1 + i\mathcal{F}_{\alpha\bar{\beta}}$$

(4.10)

where $1$ is the identity of $G$ and the action (4.6) becomes

$$S = \int d^Dx \, Str \sqrt{|\det(\Upsilon_{\alpha\bar{\beta}}1 + i\mathcal{F}_{\alpha\bar{\beta}})|}$$

(4.11)

Let $D_i$ be the covariant derivative constructed using the connection $B_i = J_i^j k_j$, so that

$$\mathcal{F}_{ij} = [D_i, D_j]$$

(4.12)

Then the expression (2.10) for $H$ in the abelian case can be generalised to a gauge-covariant field strength

$$H_{\alpha\bar{\beta}\gamma} = D_{[\alpha} g_{\beta\gamma]} = \partial_{[\alpha} g_{\beta\gamma]} + [B_{[\alpha}, g_{\beta\gamma]}]$$

(4.13)

Using the Bianchi identity $D_{[\alpha} \mathcal{F}_{\beta\gamma]} = 0$, this can be rewritten as

$$H_{\alpha\beta\bar{\gamma}} = iD_{[\alpha} \mathcal{F}_{\beta\gamma]} = iD_{\gamma} \mathcal{F}_{\alpha\beta}$$

(4.14)

This can be thought of as a covariantisation of (2.17), so that the antisymmetric tensor gauge field is now given by

$$b_{\alpha\bar{\beta}} = i\mathcal{F}_{\alpha\beta}$$

(4.15)

with $b_{\alpha\bar{\beta}} = 0$. Then both $g_{ij}$ and $b_{ij}$ are invariant under (4.5) or (4.8). The $\theta$
transformation in (2.13),

\[ \delta k_\alpha = \theta_\alpha \quad (4.16) \]

with \( \theta \) holomorphic remains a symmetry if it is now covariantly holomorphic,

\[ D\bar{\beta} \theta_\alpha = 0 \quad (4.17) \]

Then \( g_{\alpha\bar{\beta}} \) and \( H_{\alpha\beta\gamma} \) are invariant under this, while \( b \) transforms as

\[ \delta b_{\alpha\beta} = D[\alpha \theta_\beta] \quad (4.18) \]

This is the non-abelian generalisation of the antisymmetric tensor gauge symmetry \( \delta b = d\lambda \). The gauge-invariant field equation for \( H \) now takes the form \( D^i H_{ijk} = 0 + \ldots \).

Consider now the special case in which the original geometry is Kahler, so that

\[ B_\alpha = -i \partial_\alpha K \quad (4.19) \]

in the abelian case. This implies that \( B_\alpha \) is a holomorphic connection, with \( F_{\alpha\beta} = 0 \). The natural non-abelian generalisation is to take \( B \) to be a connection on a holomorphic vector bundle with structure group \( G \), so that \( F_{\alpha\beta} = 0 \) and there is locally a complex Lie-algebra-valued prepotential \( V \) such that

\[ B_\alpha = V^{-1} \partial_\alpha V \quad (4.20) \]

A gauge transformation can be used to set the real part of \( V \) to zero, so that in the abelian limit \( V = \exp(-iK) \) and (4.19) is recovered. The prepotential \( V \) transforms as

\[ V \rightarrow \bar{\rho} V g \quad (4.21) \]

under a gauge transformation parameterised by the group element \( g \) and under a pregauge transformation with holomorphic group-valued parameter \( \rho(z) \), which reduces to the Kahler gauge transformation \( \delta K = f(z) + \bar{f}(\bar{z}) \) in the abelian limit with \( \rho = \exp(2if) \). In this case, \( H \) is zero.
Acknowledgements

I would like to thank Mohab Abou Zeid for useful discussions.

REFERENCES

1. E. Witten, “Some Comments on String Dynamics”, [hep-th/9507121]. Contributed to STRINGS 95: Future Perspectives in String Theory, Los Angeles, CA, 13-18 Mar 1995.

2. N. Seiberg, “New Theories in Six Dimensions and Matrix Description of M-theory on $T^5$ and $T^5/Z_2$”, [hep-th/9705221].

3. O. Aharony, M. Berkooz, S. Kachru, N. Seiberg, and E. Silverstein, “Matrix description of interacting theories in six dimensions,” [hep-th/9707079].

4. A. Strominger, “Open p-branes,” [hep-th/9512059].

5. P.K. Townsend, Phys. Lett. B373 (1996) 68

6. D. Kutasov and E. Martinec, Nucl. Phys. B477 (1996) 652; [hep-th/9602043].

7. D. Kutasov and E. Martinec and M. O’Loughlin, Nucl. Phys. B477 (1996) 675; [hep-th/9603116].

8. E. Martinec, [hep-th/9608017].

9. D. Kutasov and E. Martinec, [hep-th/9612102].

10. C.M. Hull, [hep-th 9702067].

11. E. Martinec, [hep-th/9706194].

12. C.M. Hull and E. Witten, Phys. Lett. 160B (1985) 398.

13. C.M. Hull, Nucl. Phys. B267 (1986) 266.

14. C.M. Hull, Phys. Lett. 178B (1986) 357.

15. C.M. Hull, in the Proceedings of the First Torino Meeting on Superunification and Extra Dimensions, edited by R. D’Auria and P. Fré, (World Scientific, Singapore, 1986).
16. C.M. Hull, in Super Field Theories (Plenum, New York, 1988), edited by H. Lee and G. Kunstatter.

17. A. Tseytlin, hep-th/9701128.

18. R. Bott and S-S. Chern, Acta Math 114 (1965) 71.

19. R. Bott and S-S. Chern, Essays on Topology and Related Topics, Springer-Verlag (1970) 48.

20. S. Donaldson, Proc. Lond. Math. Soc., 50 (1985) 1; V. Nair and J. Shiff, Nucl. Phys. B371 (1992) 329; Phys. Lett. 246B (1990) 423.

21. P.S. Howe and G. Papadopoulos, Nucl. Phys. B289 (1987) 264.

22. M. Abou Zeid and C.M. Hull, hep-th/9612208.