Relativistic two fermion treatment of hyperfine transitions

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Abstract

A system of two fermions with different masses and interacting by the Coulomb potential is presented in a completely covariant framework. The spin–spin interaction, including the anomalous magnetic moments of the two fermions, is added by means of a Breit term. We solve the resulting fourth order differential system by evaluating the spectrum and the eigenfunctions. The interaction vertex with an external electromagnetic field is then determined. The relativistic eigenfunctions are used to study the photon emission from a hyperfine transition and are checked for the calculation of the Lamb shift due to the electron vacuum polarization in the muonic hydrogen.

Keywords: two fermion relativistic wave equation, bound state spectrum, electromagnetic transitions

1. Introduction

The precision physics of simple atomic systems has become a highly developed discipline both on the experimental and theoretical sides. Many new papers appear in each issue of the most important physics journals and a simple search on the web gives millions of records. The accuracy by which theoretical calculations reproduce the measured quantities is really remarkable and many effects of different types are being constantly added to improve the agreement with experiments and to give a much better determination of some fundamental physical constants. This, of course, is very demanding on the amount of work to be done, as the approach generally used is a perturbation expansion in the fine c. It also reproduces the one particle Dirac equation when one of the two masses tends to infinity, overcoming some of the difficulties of the Bethe–Salpeter approach (see, eg. [5]). An evident drawback is that some systems, such as deuterium, are excluded by our treatment which, however, includes most of the interesting simple atoms. More recently the numerical aspects of our scheme were reconsidered and improved, especially in connection with the application of the two fermion equation to the study of meson spectra by means of the Cornell potential and of a Breit term [6]. The results have shown that a completely covariant description can produce a unified framework for the case of light mesons too, where non relativistic approaches had failed; it was also made clear, contrary to common wisdom, that a complete covariance was highly desirable in order to deal with mesons composed of quarks with different
masses [7], even when a non relativistic approximation seemed to have a good chance of being successful.

Just as the Dirac equation for a single fermion in a central potential is written in terms of a pair of first order differential equations, the two interacting fermions unavoidably give rise to a differential system of order four. This can cause some annoyance, not only because the most immediate insight into physical problems has been developed in terms of second order equations, but also since most of the analytic and numeric approximations have been mainly adapted to the Schrödinger equation. At present we have not found any really sensible analytical method to discuss our fourth order Schrödinger equation. At present we have not found any really sensible analytical method to discuss our fourth order Schrödinger equation.

We then consider in more detail the elementary process of a photon emission from a hyperfine transition of the relativistic two fermion system. From the coupling of each component particle with an external electromagnetic field, we determine the vertex in the global/relative coordinates and we calculate the transition rates. We need, for this, the wave functions of the hyperfine states perturbatively corrected in the spin–spin interaction. They are found by evaluating the first order term of the Taylor series of the eigenspinors with respect to an external parameter $\epsilon$, namely by the same Hellmann–Feynman procedure used for the spectrum. In such a way, within the numerical accuracy, we take into account the complete sum over all the intermediate states of the usual expansion [4]. We then carry on the elementary field theoretical calculation without any further approximation avoiding, in particular, the evaluation of the (usually non relativistic) eigenfunctions at the origin. We recall that in the case of the quarkonium physics, this last procedure has been the source of an animated debate whether a smoothing procedure at the origin was necessary or not [6, 9]. Apart from this last minor point, however, the correct covariant treatment of the two body fermion system naturally points out some features with a clear physical explanation. In the first place the frequency of the emitted photon, determined by the four momentum conservation, shows the recoil correction due to the finite mass of both components. Next, the matrix element of the process is given by the sum of two contributions that involve the kinematical properties of each particle separately whose ratio turns out to be the square of the ratio of the two component masses. Finally a new factor slightly modifying the transition rate is generated by covariance. Due to the values of the masses involved in the systems under investigation, all the previous corrections are indeed very small.

In the last section we briefly summarize our results and we add a possible use of the explicit eigenfunctions to calculate the one and two loops approximations of the electron vacuum polarization for the muonic hydrogen, expressed by means of the Uheling and Källen-Sabry potentials [10, 11]. It is well known that they constitute the dominant contributions to the Lamb shift for this atomic species [2]. We do not repeat, in this paper, the derivation of the two fermion equation (2.1), already exposed in [3, 4]. However, since the framework is more general and changes have been brought with respect to what was presented in [3, 4], we produce all the necessary ingredients that make it possible to determine the actual system of equations and to calculate its solutions. In order to maintain a plain exposition of the results they are gathered in the appendixes. In these we thus explain the relationship between the two particle coordinates and the global/relative ones. We give the explicit '16-dim spherical spinors’ obtained by diagonalizing angular momentum and parity of the two fermion system. We then show the resulting eight independent first order equations and their reduction to the fourth order system (2.4), (2.5) due to the existence of four algebraic relations among the unknown functions. This procedure is completely analogous to the reduction of the Dirac equation in a central field. To conclude, we notice that the charted course is absolutely straightforward and physically transparent, although its implementation has required a certain deal of computer algebra, which also turned out to be rather useful for simplifying some thorny problems connected with numerical precision.
2. The spectral problem for the HFS

We denote by $\gamma_{fi}$ the Dirac gamma matrices acting in the spinor space of the $ith$ fermion of mass $m_{fi}$. Assuming $m_{f1} > m_{f2}$, we put $M = m_{f1} + m_{f2}$ and $\rho = (m_{f1} - m_{f2})/M$. In a system of units with $\hbar = c = 1$, we use coordinates and momenta given by the Wigner vectors $\mathbf{r}$ and $\mathbf{q}$ defined by means of a canonical transformation on the global and relative variables of the two fermions. We also denote by $Z^e$ the canonical global position four-vector. The details are given in appendix A. The two fermion wave equation reads [3, 4]

$$\left[ \left( \gamma^0, i \gamma^k \delta_{ik} - \gamma^k \gamma^0 \right) q_{de} + \frac{1}{2} \left( \gamma^0 \gamma^0 + \gamma^0 \gamma^0 \right) M \right] \psi(r) = 0. $$

(2.1)

The eigenvalue $\lambda$ is the square root of the squared total momentum, $\alpha = e^2/\hbar c \equiv e^2$ is the fine coupling constant, $\alpha/r$ the Coulomb interaction. Finally

$$V_B(r) = k_1 k_2 \frac{\alpha}{2r^2} \gamma^0 \gamma^1 \gamma^k \gamma^0 \gamma_2 \left( \delta_{iab} + \frac{\delta_{iab}}{r^2} \right)$$

is the Breit term which describes the spin–spin interactions and generates the hyperfine splitting, $k_1$ and $k_2$ being the factors accounting for the anomalous magnetic moments of the two fermions. As in [4], the first order perturbative correction to the eigenvalues is evaluated by multiplying $V_B(r)$ times a parameter $\epsilon$ in (2.1) and taking the first term of the Taylor expansion of the eigenvalues with respect to $\epsilon$ from the numerical solutions of the differential equations. The radial system was obtained in [3, 4]. Introducing the dimensionless variables $s, w$ by

$$s = m_2 r, \quad \lambda = m_2 \left( \frac{1 + \rho}{1 - \rho} \right) + 1 + \frac{1 + \rho}{2} \alpha^2 w $$

(2.2)

we obtain for $m_2^{-1}(\lambda + \alpha/r)$ the dimensionless expression

$$h(s) = \frac{2}{1 - \rho} + \frac{1 + \rho}{2} \alpha^2 w + \frac{\alpha}{s}. $$

(2.3)

In analogy to what is done for the single fermion Dirac equation, the explicit system of differential equations corresponding to the Hamiltonian (2.1) is obtained by introducing the spherical 16-components spinors of definite energy, angular momentum and parity, as given in appendix B. A more detailed derivation of the system is briefly summarized in appendix C. Its general form reads

$$\frac{d}{ds} u_i(s) = \sum_{j=1}^4 M_{ij}(s) u_j(s) $$

(2.4)

and the non vanishing elements of the matrix $M_{ij}(s)$ for the system with even parity are

$$M_{12}(s) = M_{33}(s) = \frac{2\rho \sqrt{j(j + 1)}}{(1 - \rho) s h(s)} $$

$$M_{13}(s) = -\frac{1}{2\hbar(s)} \left( h^2(s) - \frac{4\rho^2}{(1 - \rho)^2} \right) $$

$$M_{21}(s) = M_{34}(s) = \frac{2\rho \sqrt{j(j + 1)}}{(1 - \rho) s h(s) - 2 \varepsilon \alpha} $$

$$M_{22}(s) = M_{44}(s) = \frac{s^2 h^2(s) - \frac{4\rho^2 s^2}{(1 - \rho)^2} - 4 \alpha^2 \varepsilon^2}{(1 - \rho)^2} $$

$$M_{31}(s) = \frac{h(s)}{2s} + \frac{1}{2s} \left( 4 \varepsilon \alpha - \frac{4j(j + 1)}{s h(s) - 2 \varepsilon \alpha} \left( \frac{4s^2}{(1 - \rho)^2 (s h(s) - 4 \varepsilon \alpha)} \right) \right) $$

$$M_{33}(s) = 2/s $$

$$M_{42}(s) = -M_{24} + \frac{2\rho \sqrt{j(j + 1)}}{s^2 h(s)} $$

(2.5)

In appendix C it is also explained how to obtain the coefficients of the odd system from (2.5). We report in table 1 the results of our calculation of the hyperfine splittings $\Delta_{HFS}$ of the 1s and 2s levels, together with the values of $D_{21} = 8\Delta_{HFS}(2s) - \Delta_{HFS}(1s)$ as already defined in the introduction. In table 2 the values of $\Delta_{HFS}$ for the 3s, 4s, $2p^{1/2}$ and $2p^{3/2}$ levels are presented. We give some details concerning the numerical method we have used for the solution. We are dealing with a boundary value problem for a fourth order differential system having two singular points, at the origin and at infinity. No further singularities arise from the
mainly due to the initial contribution of the fourth order in the fine structure constant [13]. An analogous situation is met for the 3He$^+$ ion [13], in contrast with the electronic Helium ion, for which we reproduce quite well the HFS given in literature [2] and the $D_{21}$ factor with an acceptable agreement. We will try to get a deeper insight into this problem in the future.

3. The transition probabilities

We retrace the general quantum mechanical procedure to calculate the transition probability between the hyperfine split $s$-states. We consider in the Coulomb gauge the wave function of a photon with polarization $e_\sigma$ [14]:

$$A(k, \sigma) = \frac{\sqrt{4\pi}}{2\omega} e_\sigma e^{-i\mathbf{k}\cdot\mathbf{r}}.$$  

The interaction Hamiltonian reads

$$H_{\text{int}} = -e z \alpha_{(1)} : \mathcal{A}^{(1)} - \alpha_{(2)} : \mathcal{A}^{(2)} :$$

where $\mathcal{A}^{(i)} = A(x_{(i)})$ and $\alpha_{(i)} = y_{(i)} \gamma_0$ are the Dirac $\alpha$-matrices acting in the spinor space of the $i$th fermion. The charge of the lighter fermion (electron or $\mu$) is $-e$ and $z$ is the atomic number of the heavier fermion. We choose the $P = 0$ frame and we use the coordinates $(Z, r)$ with the relations (4.9) as explained in appendix A. We factorize the wave functions $\psi_{\ell}'(Z, r)$ and $\psi_{\ell}'(Z, r)$ of initial and final atomic states, normalized in the box, into

$$\psi_{\ell}'(Z, r) = V^{-1/2} e^{-\mathbf{1}/2Z \psi_{\ell}'(r), \quad \ell = i, f. \quad (3.6)}$$

where $\psi_{\ell}(r)$ and $\psi_{\ell}(r)$ are the 16-component spinors of appendix B corresponding to initial and final energies, angular momenta and parities. After some straightforward calculation and using the definition of $\Delta$ given in appendix A, the first perturbation order of the $S$-matrix element in the finite volume $V$ normalization reads

$$S_i = -\frac{(2\pi)^4}{\sqrt{2\omega V}} \delta^{4}(P_j + k - P_i) \int d^4 \mathbf{r} \frac{\sqrt{4\pi}}{\sqrt{V^2}} \psi_{j}'(r) e_\sigma \left( \bar{\alpha}_{(1)} e^{-i\mathbf{1}/2Z \mathbf{r}} - \bar{\alpha}_{(2)} e^{i\mathbf{1}/2Z \mathbf{r}} \right) \psi_{\ell}(r).$$

Here $\bar{\alpha}_{(i)}$ are the matrices obtained from $\alpha_{(i)}$ by applying the unitary transformation generated by the change of basis we have made so to give the spinor components the order shown in appendix B.

| Atom | $\Delta_{1S}(3s)$ | $\Delta_{1S}(4s)$ | $\Delta_{1S}(2p^{1/2})$ | $\Delta_{1S}(2p^{3/2})$ |
|------|----------------|----------------|----------------|----------------|
| $(p, e)$ | 52.617 | 22.198 | 59.196 | 59.221$^{[14]}$ |
| $(\mu^*, e)$ | 165.357 | 69.760 | 186.252 | 187.1$^{[12]}$ |
| $(^3\text{He}^*, e)$ | -320.993 | -135.418 | -361.100 | - |
| $(p, \mu)$ | 6.764 | 2.854 | 7.682 | 7.820$^{[13]}$ |
| $(^3\text{He}^*, \mu)$ | -50.828 | -21.443 | -57.028 | 58.713$^{[13]}$ |

Table 2. The $3s, 4s, 2p^{1/2}, 2p^{3/2}$ HFS for some hydrogenic atoms. Units are as in table 1. To our knowledge no data for the $3s$ and $4s$ levels are reported in literature.
The \( \delta^2 \)-function gives the energy-momentum conservation, 
\[
P_i^0 = P_j^0 + \omega, \quad P = P_f + k
\]
and contains the recoil of the atom due to the radiation emission. When the radiation wavelength \( 2\pi/\omega \) is much larger than the characteristic scale length of the atomic system, we can consider the first order expansion
\[
e^{-i(\pm \omega)k \cdot r} = 1 \pm (1/2 \pm \Delta) k \cdot r + o((k \cdot r)^2).
\]
The angular properties of the spinors imply that
\[
\int d^3r \psi^*_j(r) \hat{a}_{ij}(r) \psi_j(r) = 0, \quad j = 1, 2.
\]
We then let \( k = \omega n \) with \( n^2 = 1 \) and, taking into account the \( \delta^2 \)-function and the expression of \( \Omega \) given in appendix A, we introduce the quantity
\[
d_{ji} = -ie \int d^3r (n \cdot r) \psi^*_j(r) \\
\times \left[ \hat{a}_{(1)}(1) \left( \frac{1}{2} - \frac{m_i^2 - m_j^2}{2\lambda_i} \right) \\
+ \hat{a}_{(2)}(1) \left( \frac{1}{2} + \frac{m_i^2 - m_j^2}{2\lambda_i} \right) \right] \psi_j(r).
\]
The \( S \)-matrix element then reads
\[
S_{ji} = \left( 4\pi\alpha/V^3 \right)^{1/2} (2\pi)^3 \delta^3(P_f + k - P)(\epsilon^*_j \cdot d_{ji})
\]
and, as usual, we get a transition rate
\[
dw = \frac{\omega}{2\pi} \delta^3(P_f + k - P) \sum_{\sigma} |\epsilon^*_j \cdot d_{ji}|^2 d^3k d^3P_f
\]
As \( \int d^3P/2P^0 = \int d^4P \theta(P^0) \delta(P^2 - \lambda^2) \) we next integrate over the final global momentum, finding
\[
dw = \frac{\omega^3}{2\pi\lambda}(\lambda_i - \omega)\delta\left(\omega - \frac{\lambda^2 - \lambda_i^2}{2\lambda_i}\right) \sum_{\sigma} |\epsilon^*_j \cdot d_{ji}|^2
\]
where \( d\Omega_n \) is the unit solid angle in the direction \( n \). Reinserting the \( h \) and \( c \) factors, the final integration over the solid angle gives the total transition rate
\[
w = \frac{4}{3} \frac{\omega^3}{\hbar c^3} \Lambda_{ji}^2 |\mu_{ji}|^2
\]
where
\[
\hbar\omega = \frac{\lambda_i + \lambda_f}{2\lambda_i} (\lambda_i - \lambda_f), \quad \Lambda_{ji}^2 = \frac{\lambda_i^2 + \lambda_j^2}{2\lambda_i^2}
\]
and \( |\mu_{ji}|^2 \) is the common value of \( |\epsilon^*_j \cdot d_{ji}|^2 \) for each of the two independent circular polarizations \( (\epsilon_1 \pm i\epsilon_2)/\sqrt{2} \).

Observe that the two different contributions present in the expression (3.7)—and consequently in the transition rate \( w \)—clearly reflect the fact that we are dealing with a genuine two body problem. Indeed, loosely speaking, the two terms can be attributed to the two different fermions, whose velocities are represented by the \( \alpha \) matrices of the corresponding tensor component. The numerical results confirm this interpretation showing that the ratio of the two contribution turns out to be practically coincident with the ratio of the square of the fermion masses.

Let us now consider in more detail the \( d_{ji} \) for the \( s \) hyperfine transitions. We have taken, for computational convenience, the spinors \( \psi_i(r) \) referring to the \( n1s0 \) states and the spinors \( \psi_j(r) \) to the \( n3s1 \) states, \( n \) denoting the level quantum number. For the \( n3s1 \) states, having \( j = 1 \), the value of \( m = -1, 0, 1 \) has also to be specified. Recalling that
\[
n \cdot s = \frac{2\pi}{3} s \left( Y_{1j}^m(\theta, \phi)n_+ - Y_{1j}^m(\theta, \phi)n_- + \sqrt{2} Y_{1j}^0(\theta, \phi)n_3 \right),
\]
in the dimensionless variables (2.2), the matrix element (3.7) is given by combinations of the integrals
\[
D_{ji}^{(1)} = \int d^3\psi_i n_{3s1,n}(s) Y_{1j}^m(\theta, \phi)n_+ \hat{a}_{ij}(s) \psi_{10}(s)
\]
with \( m, a = -1, 0, 1; j = 1, 2; b = 1, 2, 3 \). For a given \( m \) only few components of the \( \hat{a} \)-matrices give non vanishing contribution. Moreover, integrating over the angular variables, the spherical symmetry gives rise to identities reducing the problem to the calculation of a single integral for each tensor type of \( \hat{a} \)-matrix. We have indeed
\[
D_{ji}^{(1)} = \int d^3\psi_i n_{3s1,n}(s) Y_{1j}^m(\theta, \phi)n_+ \hat{a}_{ij}(s)
\]
and all other choices of \( m, a, b \) giving a vanishing result.

After the integration over the angular variables, we are finally left with
\[
D_{ji}^{(1)} = -\frac{1}{12\sqrt{2}} \int_0^\infty ds s^3 \left[ \sqrt{6} \left( d_1^{(i0)} c_1^{(i1)} + d_1^{(i0)} c_1^{(i1)} \right) \right. \\
+ \sqrt{3} \left( d_0^{(i0)} d_1^{(i1)} + a_1^{(i0)} b_0^{(i1)} \right) \\
+ 3 \left( a_0^{(i0)} b_1^{(i1)} + a_1^{(i0)} b_0^{(i1)} \right) \left. \right]\],
\[
D_{ji}^{(2)} = -\frac{1}{12\sqrt{2}} \int_0^\infty ds s^3 \left[ \sqrt{6} \left( d_0^{(i0)} c_1^{(i1)} + d_1^{(i0)} c_1^{(i1)} \right) \right. \\
+ \sqrt{3} \left( d_0^{(i0)} d_1^{(i1)} + a_1^{(i0)} b_0^{(i1)} \right) \\
- 3 \left( a_0^{(i0)} b_1^{(i1)} + a_1^{(i0)} b_0^{(i1)} \right) \left. \right]\]
where we have denoted by \( a_i^{(i0)}, d_i^{(i0)} \), \( i = 0, 1 \), the coefficients \( a_i(s), d_i(s) \) of the spinors of the states \( n1s0 \) normalized to unity and by \( b_i^{(i1)}, c_i^{(i1)}, d_i^{(i1)} \), \( i = 0, 1 \), the coefficients \( b_i(s), c_i(s), d_i(s) \) of the spinors of the states \( n3s1 \) with \( m = 1 \) normalized to unity (see appendix B). Since the hyperfine splitting is due to the Breit term, the eigenfunctions of the split hyperfine states make sense only if they include the first perturbative correction due to the Breit interaction. It is equally evident that in practice the correction cannot be calculated by an expansion over a complete set of states. As we said in the introduction, instead, we adopt for the eigenfunctions the same procedure used for the spectrum: namely, we calculate the solutions of the systems (2.4) for
\( e V_{\|}(r), \) for different values of \( e, \) at the corresponding eigenvalues, we construct the spinors of appendix B and we find the corrected eigenstates by means of a first order expansion in \( e. \) A detailed proof of the procedure is given in [4]. In table 3 we report the numerical results for the \( D_{[n]}^{(j)} \), \( n, j = 1, 2, \) of the electronic and muonic Hydrogen. Finally, after the insertion of the appropriate \( h \) and \( e \) factors, we find that \( \mu_B = 9.273845 \text{erg Gauss}^{-1}, \) a value very close to the Bohr magneton commonly used for this type of calculation. The resulting transition rate turns out to be \( \approx 2.87 \times 10^{-15} \text{sec}^{-1}. \)

Let us focus on some features cleanly pointed out by the completely covariant treatment of the photon emission by a two fermion system that we have presented. In the first place we look at the emitted radiation and we find separate contributions due to the two particles, corresponding to the terms \( \mathcal{a}_{1,2}(1/2 \pm (m_1^2 - m_2^2)/(2\lambda_1^2)). \) From table 3 we see that \( D_{[n]}^{(2)}/D_{[n]}^{(1)} = 1836.04 \) and \( D_{[n]}^{(2)}/D_{[n]}^{(1)} = 8.88 \) for the \( n = 1 \) hyperfine transition of the electronic and muonic hydrogen respectively, reproducing very accurately the ratios of the proton mass to the electron and muon masses. On the other hand \( \Delta \approx (m_1^2 - m_2^2)/(2m_1 m_2) \approx 1/2 - m_2/m_1, \) so that \( 1/2 + \Delta \approx 1 \) and \( 1/2 - \Delta \approx m_2/m_1, \) giving a ratio of the two amplitudes very closely proportional to \( (m_1/m_2)^2. \)

Next we observe that the emission process determines the photon frequency carrying a correction \( (\lambda_1 + \lambda_2)/2\lambda_1 \) to the pure spectroscopic difference of the energy of the levels. Its origin is evident from the dynamics of the process. In the cases we have dealt with, this correction is very tiny; even in the muonic Hydrogen, where the effect is larger, the only appreciable change is for the level \( 21\ell_0 \) which passes from \( 182.621 \text{ to } 182.614 \text{ meV}. \) Finally, a further factor \( \Lambda_\| \) correcting almost imperceptibly the transition rate arises from the covariance.

4. Conclusions

In this final section we summarize the results we have found so far and we add some comments on the points we think to be the most relevant. Firstly we stress again that the covariant treatment of two relativistic interacting fermions necessarily implies a fourth order system. The attempt of using second order differential equations that occurs, for instance, for a non relativistic initial description of the atom invariably results in a perturbation series not very easy to deal concurrently with the series of the radiative corrections. On the one hand, the analytic discussion of the equations becomes exceedingly hard and we have to resort to a numerical treatment if we want a non perturbative solution. The method applies uniformly to all the atomic states so that we can reasonably expect the same order of accuracy for the spectral levels provided that the numerical precision is maintained sufficiently high. On the other hand, our approach also permits a straightforward determination of the wave functions. When it is required, the first order perturbation correction due to the spin–spin interaction is calculated easily enough by a numerical treatment. The eigenfunctions have been used to find in a completely covariant way the emission rate of a photon in a hyperfine transition. To conclude, we want to present the results obtained by using our relativistic purely Coulomb eigenfunctions to determine the corrections to the \( 2s \) and \( 2p \) levels of the muonic hydrogen due to the vacuum polarization by electrons [2, 10, 11] that constitute the leading contribution to the Lamb shift for the muonic hydrogen. We refer to [2] for the Feynman diagrams they are related to. The shifts to the energy we have found are given in table 4 and they are in complete agreement with the leading terms contribution reported in literature [10]. We finally notice that in the hyperfine transition we have, in a certain sense, determined the interaction vertex of a photon with the composite two fermion system. We would naturally carry on with the electrodynamical radiative corrections in our covariant framework. For this purpose the propagator of the two interacting fermions by the Coulomb potential is required. Obviously its determination cannot be other than numerical and work is in progress in this direction.

### Appendix A. Coordinates

Denote by \( x_{(1)}^\mu \) and \( p_{(1)}^\mu \) coordinates and momentum of the two fermions. We indicate in boldface the spatial 3-vectors. Denote also by \( y_{(1)}^\mu = \gamma^\nu \otimes I \) and \( \gamma^\nu = I \otimes \gamma^\nu \) the Dirac \( \gamma \)-matrices acting on the spinor components of the first and second particle respectively. The same notation is used for the
Dirac matrices $\alpha = \gamma_0 \gamma_r$. Let $\eta_{\mu}$ be the Minkowski metric and
\[
X^\mu = (1/2) \left( x^{\mu}_{(1)} + x^{\mu}_{(2)} \right) \quad P^\mu = p^{\mu}_{(1)} + p^{\mu}_{(2)}
\]
\[
x^\mu = x^{\mu}_{(1)} - x^{\mu}_{(2)} = (1/2) \left( p^{\mu}_{(1)} - p^{\mu}_{(2)} \right).
\]
We define the tensor
\[
e^\mu_{\beta}(P) = \frac{P^\mu}{\sqrt{P^2}}, \quad e^\mu_\beta(P) = \eta_{\mu} - \frac{P^\mu}{\sqrt{P^2}} \left[ P_\beta + \eta_{\beta} \sqrt{P^2} \right] \frac{1}{\sqrt{P^2}} \left[ P_\beta + \sqrt{P^2} \right]
\]
($\alpha = 1, 2, 3$). It represents a Lorentz transformation to the $P = 0$ frame as it satisfies the identities
\[
\eta_{\mu} e^\mu_{\beta}(P) e^\nu_\beta(P) = \eta_{\nu}, \quad \eta_{\nu} e^\mu_{\beta}(P) e^\nu_\beta(P) = \eta_{\mu}
\]
We construct the canonical variables
\[
\bar{q} = e^\mu_{\beta}(P) q_\beta, \quad \bar{r} = e^\mu_{\beta}(P) r_\beta, \quad q_\beta = e^\mu_{\beta}(P) x_\mu
\]
\[
Z^\mu = X^\mu + \frac{\epsilon_{abc} P^a \eta^b \eta^c}{\sqrt{P^2}} L_0
\]
\[
+ \frac{\epsilon^\mu_\alpha \left( q_\beta - e^\mu_\beta \bar{q} \right)}{\sqrt{P^2}} + \frac{P^\mu}{\sqrt{P^2}} \bar{r}.
\]
where $r$ and $q$ are Wigner vectors of spin one and $Z$ is a Newton-Wigner position vector for a particle with angular momentum $L_0 = \epsilon_{abc} q_\beta$. In [3, 4] a canonical reduction of the phase space was obtained by $\bar{q} = \sqrt{\bar{q}^2} \Delta$, with $\Delta = (m_1^2 - m_2^2)/(2n^2)$, corresponding to the Lorentz invariant cyclic relative time coordinate $\bar{r}$. This is the phase space where interactions are defined. We also denote by $r^2 = e^\mu_\beta e^\mu_\beta$ the Lorentz invariant squared modulus of the Wigner vector $r$.

In the $P = 0$ frame we then have
\[
x_{(1)} = Z + (1/2 - \Delta) r, \quad x_{(2)} = Z - (1/2 + \Delta) r.
\]
where now $\Delta = (m_1^2 - m_2^2)/2n^2$.

**Appendix B. Spherical 16-dim spinors**

For the sake of completeness we report here the state vectors $\Psi_+^+, \Psi_-^-$ of definite energy, angular momentum $(j, m)$, even and odd parity $(-)^j$ and $(-)^{j+1}$ with respect to the angular momentum. The even state is given the order
\[
\Psi_+^+ = \left( \Psi_+^{(M)}, \Psi_+^{(-M)}, \Psi_+^{(-), \mu}, \Psi_+^{(\mu)} \right)
\]
where $\Psi_+^{(Q)} = \left( \Psi_+^{(Q) +}, \Psi_+^{(Q) -}, \Psi_+^{(Q) \mu}, \Psi_+^{(Q) \mu} \right)$
with $Q = \pm M, \mp \mu$.

The explicit expressions of the components read:
\[
\Psi_+^{(M)} = Y^{(M)}_{m}(\theta, \phi) a_0(s)
\]
\[
\Psi_+^{(-M)} = -\sqrt{j - m + 1 \sqrt{j + m}} \sqrt{j + 1} Y^{-M}_{m-1}(\theta, \phi) b_0(s)
\]
\[
\Psi_+^{(-)} = \sqrt{j - m + 1 \sqrt{j + m}} \sqrt{j + 1} Y^{+M}_{m-1}(\theta, \phi) b_0(s)
\]
\[
\Psi_+^{(\mu)} = \sqrt{j + 1 \sqrt{j + m}} \sqrt{j - m + 1} Y^{+M}_{m+1}(\theta, \phi) b_1(s)
\]
\[
\Psi_+^{(-\mu)} = \sqrt{j - m + 1 \sqrt{j + m}} \sqrt{j + 1} Y^{-M}_{m+1}(\theta, \phi) b_1(s)
\]
\[
\Psi_+^{(J)} = \frac{m}{\sqrt{j + 1}} Y^{(M)}_{m}(\theta, \phi) a_0(s)
\]
\[
\Psi_+^{(-J)} = \frac{\sqrt{j + m}}{\sqrt{j + 1}} Y^{(-M)}_{m}(\theta, \phi) a_0(s)
\]
\[
\Psi_+^{(J \mu)} = \frac{\sqrt{j + m}}{\sqrt{j + 1}} Y^{(-M \mu)}_{m}(\theta, \phi) a_0(s)
\]
\[
\Psi_+^{(-J \mu)} = \frac{\sqrt{j + m}}{\sqrt{j + 1}} Y^{(-M - \mu)}_{m}(\theta, \phi) a_0(s)
\]

We get the odd state $\Psi_-$ by the parity transformation
\[
\Psi_- = \left( \begin{bmatrix} 0 & I_8 \\ I_8 & 0 \end{bmatrix} \right) \Psi_+
\]
This amounts to changing the sign of the mass $m_1$. 

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Appendix C. Equations

The radial differential equations are obtained by applying the Hamiltonian operator (2.1) to the even and odd states and requiring the vanishing of the coefficients of the different spherical harmonics in each component of the resulting vector. For each parity we get eight independent equations. Let \( D^{(s)} \) be the operator \( d/ds + \kappa/s \) and \( f_\pm(s) = f_0(s) \pm f_1(s) \), \( f = a, b, c, d \).

The even system is:

\[
\sqrt{J} D^{(j+1)} a_+(s) = - \sqrt{J} + 1 D^{(j+1)} b_-(s) + 2 \sqrt{J} + 1 c_0(s) \left( \frac{2 \rho}{1 - \rho} + h(s) \right)
\]

\[
- \frac{2 \alpha (j + 1) c_1(s)}{\sqrt{2j + 1}} - \frac{2 \alpha \sqrt{J} (j + 1) d_1(s)}{s \sqrt{2j + 1}} = 0
\]

\[
\sqrt{J} D^{(j+1)} a_-(s) + \sqrt{J} + 1 D^{(j+1)} b_-(s) - \sqrt{J} + 1 c_1(s) \left( \frac{2 \rho}{1 - \rho} - h(s) \right) - \frac{2 \alpha (j + 1) c_0(s)}{\sqrt{2j + 1}} = 0
\]

\[
\sqrt{J} + 1 D^{(j+1)} a_+(s) - \sqrt{J} D^{(j)} b_-(s) + 2 \sqrt{J} + 1 d_0(s) \left( \frac{2 \rho}{1 - \rho} - h(s) \right) + \frac{2 \alpha j d_1(s)}{\sqrt{2j + 1}} + \frac{2 \alpha \sqrt{J} (j + 1) c_1(s)}{s \sqrt{2j + 1}} = 0
\]

\[
\sqrt{J} + 1 D^{(j)} a_+(s) - \sqrt{J} D^{(j+1)} b_-(s) + 2 \sqrt{J} + 1 d_1(s) \left( \frac{2 \rho}{1 - \rho} - h(s) \right) + \frac{2 \alpha jd_0(s)}{\sqrt{2j + 1}} + \frac{2 \alpha \sqrt{J} (j + 1) c_0(s)}{s \sqrt{2j + 1}} = 0
\]

\[
\sqrt{J} D^{(j+1)} c_+(s) = \sqrt{J} + 1 D^{(j+2)} d_+(s) + \frac{4 \alpha \sqrt{J} (j + 1) a_1(s)}{s} - \frac{2 \alpha \sqrt{J} + 1 a_0(s)}{s} = 0
\]

\[
\sqrt{J} D^{(j+1)} c_-(s) = \sqrt{J} + 1 D^{(j+2)} d_-(s) - \sqrt{J} + 1 d_0(s) \left( \frac{2 \rho}{1 - \rho} + h(s) \right) - \frac{4 \alpha \sqrt{J} + 1 a_1(s)}{s} = 0
\]

\[
\sqrt{J} + 1 D^{(j+1)} c_-(s) + \sqrt{J} D^{(j+2)} d_-(s) - \sqrt{J} + 1 b_0(s) \left( \frac{2 \rho}{1 - \rho} - h(s) \right) - \frac{2 \alpha \sqrt{J} + 1 a_0(s)}{s} = 0
\]

\[
\sqrt{J} + 1 D^{(j+1)} c_-(s) + \sqrt{J} D^{(j+2)} d_-(s) - \sqrt{J} + 1 b_0(s) \left( \frac{2 \rho}{1 - \rho} - h(s) \right) - \frac{2 \alpha \sqrt{J} + 1 a_0(s)}{s} = 0
\]

From it we get four algebraic equation. Defining

\[
u_\pm(s) = - \sqrt{J} c_\pm(s) - \sqrt{J + 1} d_\pm(s) = 0
\]

and introducing \( u_i(s), i = 1, ..., 4 \), with \( u_1(s) = a_+(s), u_2(s) = b_-(s), u_3(s) = u_+(s), u_4(s) = v_-(s) \), they read

\[
a_+(s) = \frac{2s u_1(s)}{(1 - \rho)(s h(s) - 4 \alpha \varepsilon)}
\]

\[
b_+(s) = \frac{2s u_2(s)}{(1 - \rho)(s h(s) - 2 \alpha \varepsilon)}
\]

\[
u_+(s) = \frac{2\sqrt{J}(j + 1) u_1(s) - 2\rho s u_4(s)}{s h(s) - 2 \alpha \varepsilon - (1 - \rho)(s h(s) - 2 \alpha \varepsilon)}
\]

We are eventually left with the system (2.4), (2.5). By the final observation of appendix B, the odd system and the corresponding algebraic relations are obtained by letting

\[
\frac{2}{1 - \rho} \rightarrow \frac{2 \rho}{1 - \rho} \quad \text{and} \quad \frac{2}{1 - \rho} \rightarrow -\frac{2}{1 - \rho}
\]

Appendix D. Electron vacuum polarization potentials

We report here the potentials describing the vacuum polarization by electrons, firstly calculated by Källen and Sabry and later on redetermined by other authors [10, 11]. For the sake of completeness we report here the expressions we have used to calculate the muonic hydrogen Lamb shift, given in the last of [10, 11].

\[
V_e = -\alpha \int_0^1 dv \, \rho(v) \exp(-\lambda v)/v
\]

where \( \lambda = 4m_e^2/\sqrt{1 - v^2} \), \( m_e \) being the electron mass and where the densities \( \rho(v) \) are defined by

\[
\rho_1(v) = \frac{\alpha}{\pi} \frac{v^2(1 - 1/3 \, v^2)}{(-v^2 + 1)}
\]

\[
\rho_{11}(v) = -\alpha^2 \frac{v^2(1 - 1/3 \, v^2)[(16 - 6v^2 + 3 \, v(3 - v^2)\ln(1 - v)/(1 + v)]}{(9\pi^2(1 - v^2))^{\frac{1}{2}}}
\]
\[ \rho_2(v) = 2a^2v \left( 3a^2(1 - v^2) \right)^{-1} \left\{ \left( 3 - v^2 \right)(1 + v^2) \right. \\
\left. \times \left[ \text{Li}_2\left( \frac{1 - v}{1 + v} \right) + 2 \text{Li}_2\left( \frac{1 - v}{1 + v} \right) \right] \right. \\
+ \ln \left( \frac{1 + v}{1 - v} \right) \left[ \frac{3}{2} \ln \left( \frac{1}{1 + v} + 1/2v \right) - \ln(v) \right] \right. \\
+ \ln \left( \frac{1 + v}{1 - v} \right) \left[ \frac{11}{16} (3 - v^2)(1 + v^2) + \frac{1}{4} v^4 \right] \\
+ \frac{3}{2} v (3 - v^2) \ln \left( \frac{1 - v^2}{4} \right) - 2 v (3 - v^2) \\
\ln(v) + \frac{3}{8} v (5 - 3 v^2) \right\} \]

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