Effective de Sitter space and large-scale spectral dimension (3+1)

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De Sitter space-time, essentially our own universe, is plagued by problems at the quantum level. It is unstable if its symmetries are not violated, the holographic principle is problematic to implement since the manifold has no boundary but only a cosmological horizon and it is very difficult to obtain as a string theory vacuum. Here we propose that Lorentzian de Sitter space-time is not fundamental but constitutes only an effective description of a more fundamental quantum gravity ground state. This cosmological ground state is a graph, appearing on large scales as a Riemannian manifold of constant negative curvature. We model the behaviour of matter near this equilibrium state as Brownian motion in the effective thermal environment of graph fluctuations, driven by a universal time parameter. We show how negative curvature dynamically induces the asymptotic emergence of relativistic coordinate time and of ballistic motion governed by the isometry group of an “effective Lorentzian manifold” of opposite, positive curvature, i.e. de Sitter space-time: free fall in positive curvature is asymptotically equivalent to Brownian motion in negative curvature. The local limit theorem for negative curvature implies that the large-scale spectral dimension of this “effective de Sitter space-time” is (3+1) independently of its microscopic topological dimension.

Our universe is fast approaching de Sitter space-time (for a review see [1]). However, from a theoretical point of view, de Sitter space-time seems to be plagued by problems [2]. These arise mainly from the incompatibility of the concept of time in general relativity and quantum mechanics. Time, in general relativity and cosmology is associated with the negative signature coordinate of a fundamental Lorentzian manifold. Classical motion on this manifold is “free fall” along geodesic trajectories. Quantum mechanics is predicated on a Hamiltonian, quantum mechanical time is the corresponding Hamiltonian flow, a concept a priori independent of geometry. If we want to define a quantum Hamiltonian flow by using a negative-signature coordinate of a curved Lorentzian manifold we need a time-like Killing vector generating the corresponding translations. On de Sitter space-time this is possible only within a static patch bounded by a cosmological horizon. Unfortunately, the resulting quantum space-time turns out to be unstable [2, 3] if the de Sitter symmetry is not violated [4].

On the other side, the holographic principle [5, 6] (for a review see [7]), relating bulk phenomena to a set of boundary degrees of freedom and considered one of the pillars of quantum gravity, cannot be applied directly to de Sitter space-time simply because this has no boundary. The only possibility is to put the “boundary” degrees of freedom on the horizon, which, however does not yield a clear and appealing picture [4]. Finally, it looks like de Sitter space-time is very difficult to obtain as a ground state of string theory [8]. De Sitter space-time, essentially our own universe, seems fraught by problems.

But is de Sitter space-time really necessary to describe the fate of our universe? To determine the curvature of space-time one needs to measure the trajectories of test objects. We now show that, asymptotically, the sign of the curvature is ambiguous: “free fall” in constant positive curvature becomes equivalent to Brownian motion in constant negative curvature. Therefore, in this asymptotic region, it is not possible to determine from the motion of test particles if one is living on de Sitter space-time or on an “effective de Sitter space-time” which is actually a constant negative curvature Riemannian manifold with emergent coordinate-time.

Recently, we proposed [11–14] a purely combinatorial quantum gravity model based on random graphs governed by the combinatorial Ollivier-Ricci curvature [15–19]. In this model there is no a priori notion of either time or space, only abstract binary relations, as suggested originally by Wheeler [20] (for a review see [21]). Nevertheless, when graph edges are endowed with a length scale, geometry emerges from pure randomness in a second-order phase transition from strong to weak gravity. In the geometric phase the ground state graphs are discretizations of negative-curvature Riemannian manifolds, so-called Cartan-Hadamard manifolds (for a review see [22])

Based on this model, here we propose that de Sitter space-time is not fundamental but only an effective asymptotic description of our universe. The cosmological ground state is, rather a graph appearing, on large distances, as a negative-curvature Riemannian manifold. We model the behaviour of matter particles near this equilibrium state by their interaction with the effective “thermal” environment given by graph fluctuations. Of course, as soon as small deviations from equilibrium are considered, we have to introduce a universal time parameter driving these deviations. We will identify this as the original quantum mechanical time concept. Note that this time has nothing to do with geometry, it is an absolute concept that can be considered as a discrete ordinal variable or represented itself by a graph [23].

The result of environmental thermal fluctuations on matter particles is Brownian motion (for a review see [24]). On flat manifolds this is simply caused by scattering events randomly pushing the particle around. On curved manifolds, however the interplay of scattering with curvature makes Brownian motion look very different [25]. This is particularly true for negative-curvature manifolds [27–30].
Here we show how the negative curvature forces the leading behaviour of Brownian motion to become asymptotically deterministic and ballistic on the geodesics of the corresponding de Sitter space-time of reversed-sign, positive curvature, playing the role of the cosmological constant. Correspondingly, the probability density of the particle distribution is governed by the wave equation on this “effective de Sitter space-time”, after the universal time is dynamically soldered to a manifold coordinate by the diffusion equation itself. Essentially, Brownian scatterings events in this emerging coordinate time are asymptotically “diluted away”, so that only the ballistic motion on times scales shorter than the Brownian momentum relaxation time survives. Since time is a manifold coordinate, in this asymptotic region, the admitted ballistic evolutions must be representations of the isometry group $SO_{+}(D, 1)$ of the ground state manifold. This is the de Sitter group, which replaces the Poincaré group in presence of a cosmological constant [31–33], up to the absence of time-reversal symmetry, which amounts to an emergent arrow of time. The Lorentzian de Sitter picture with the general-relativistic time as a manifold coordinate emerges thus as an asymptotic, effective description of a more fundamental physics.

Brownian motion, generated by the Laplace operator, defines the spectral dimension of a generic metric space, be it a manifold, a fractal or a graph. Essentially, the diffusion process probes the geometric characteristics on a given scale and determines what is the “effective number of directions” seen by the random walker on that scale. On a Euclidean manifold it coincides, of course, with the topological dimension. On a fractal, however, it must not even be an integer. And negative curvature can have a very large effect too. Due to the asymptotic ballistic character of Brownian motion, the diffusion process is far too fast to probe the large-scale geometry. However, a slower diffusion process can be defined that can indeed probe the large-scale spectral properties of the negative-curvature manifold [34] relative to the runaway time dimension. The resulting spectral dimension is always 3, independent of the topological dimension of the manifold. This result is a general consequence of the local limit theorem for negative-curvature manifolds [35] and implies that the large-scale spectral dimension of effective de Sitter space-time is always $(3 + 1)$, independently of its microscopic topological dimension.

Of course, all the problems plaguing fundamental de Sitter space-time are absent in this picture. The fundamental manifold is Riemannian, coordinate time is just an emergent description of the asymptotic behaviour of Brownian motion, so there is no need of a time-like Killing vector and there is no cosmological horizon. Moreover, negative-curvature manifolds of dimension $D$ have a natural boundary $S^{D−1}$ and conformal transformations on this boundary are in one-to-one correspondence with the interior $SO_{+}(D, 1)$ isometries (see, e.g., [36]). The holographic principle thus emerges naturally. All in all, the picture emerging in this model is reminiscent of the Hartle-Hawking no-boundary universe [37, 38] specialized to de Sitter space-time.

Let us consider a Riemannian manifold of constant negative curvature $-H^2$ [22]. In polar coordinates [27–30] the metric is given by

$$\begin{align*}
    ds^2 &= dz^2 + G^2(z) d\Omega^2, \\
    G(z) &= \frac{\sinh Hz}{H}.
\end{align*}$$

(1)

with $d\Omega^2$ the standard metric on the $(D-1)$-dimensional sphere $S^{D-1}$, which is conveniently parametrized by $(D - 1)$ angles,

$$
\begin{align*}
    \omega_1 &= \cos \theta_1, \\
    \omega_2 &= \sin \theta_1 \cos \theta_2, \\
    \vdots
\end{align*}
$$

(2)

with $0 \leq \theta_i < \pi$ for $1 \leq i < D - 1$ and $0 \leq \theta_{D-1} < 2\pi$ and metric given by

$$d\Omega^2 = \sum_{i=1}^{D} d\omega_i^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \ldots + \sin^2 \theta_1 \ldots \sin^2 \theta_{D-2} d\theta_{D-1}^2.
$$

(3)

The manifold can be obtained by an embedding into a $(D + 1)$-dimensional Minkowski space such that

$$
\begin{align*}
    x_0 &= \frac{1}{H} \cosh Hz, \\
    x_i &= \frac{1}{H} \sinh Hz \omega_i, \\
    i &= 1 \ldots D,
\end{align*}
$$

(4)

so that $d^2 = -dx_0^2 + \sum_{i=1}^{D} dx_i^2$ and $x_0^2 - \sum_{i=1}^{D} x_i^2 = 1/H^2$. Note that the manifold is intrinsically Riemannian even if it is embedded in Minkowski space.

Brownian motion on the universal cover of a general negative-curvature manifold is asymptotically ballistic [26]. Let $B(t)$ be the Brownian motion starting from a point $x$ with coordinate $Hz(x) \gg 1$ and $d$ the distance on the manifold. Then the central limit theorem in negative curvature [26] implies that there exist a positive number $w$ such that

$$
\frac{d(x, B(t))}{t} = w + O\left(\frac{1}{\sqrt{t}}\right) N(0, 1).
$$

(5)

where $N(0, 1)$ denotes a Gaussian of mean zero and unit standard deviation. This implies that, asymptotically, where this Gaussian distribution can be neglected, Brownian motion on a negative-curvature manifold is invariant under its isometries. In case of constant negative curvature, the group of isometries is $O_{+}(D, 1)$, or $SO_{+}(D, 1)$ if we impose preservation of orientations. This follows immediately from the fact that the manifold metric is induced by the embedding in a Minkowski space of one higher dimension. The “+” subscript, indicates that the Lorentz group is orthochronous, i.e. that the time-reversal symmetry is absent. This is an expression of the fact that a
constant negative curvature manifold is one sheet of a generalized two-sheeted hyperboloid: time reversal would connect the two sheets but is not a symmetry of a single sheet.

On a constant negative-curvature manifold (1), Brownian motion can be decomposed in a radial component \( z(t) \) and an angular component \( \Omega(t) \) [27–30]. The radial part satisfies

\[
z(t) = z_0 + W(t) + \frac{D - 1}{2} \int_0^t \frac{G'(z(s))}{G(z(s))} \, ds ,
\]

where \( W(t) \) is a 1-dimensional Brownian motion [28, 29] and \( t \) is the absolute Parisi-Wu time previously discussed. Using \( G(z) \) in (1) we obtain

\[
\frac{z(t)}{t} = \frac{r_0 + W(t)}{t} + \frac{D - 1}{2} H \int_0^t \coth H(z(s)) \, ds .
\]

Since the standard deviation of the 1D Brownian motion \( W(t) \) is \( \sigma_W \propto \sqrt{t} \) and \( \coth H \gg 1 \) for positive \( z \), the probabilistic component can be neglected for large \( t \) and \( z(t) \to \infty \). We thus find the leading-order deterministic asymptotic behaviour [28]

\[
\frac{z(t)}{t} = \frac{D - 1}{2} H + O \left( \frac{1}{\sqrt{t}} \right) N(0, 1) , \quad H(z(t)) \gg 1 .
\]

(8)

For test particles reaching the asymptotic region \( H \gg 1 \) the random component becomes negligible and the manifold coordinate \( z \) is dynamically soldered with absolute time and emerges as general-relativistic coordinate time.

At this point we have to pause to discuss dimensions. The manifold coordinate \( z \) has dimension length, as does the quantity \( 1/H \), which is the radius of curvature. The absolute time \( t \) is a diffusion parameter and, as such, has the usual diffusion coefficient \( \mathcal{D} \) with dimension length$^2$/time absorbed in its definition, so that \( t = \mathcal{D} t \) is actually a length$^2$, whereas \( \bar{t} \) is measured in “seconds”. The radius of curvature \( \mathcal{1}/H \) and the diffusion coefficient \( \mathcal{D} \) can be traded for one length scale and a fundamental speed \( c = \mathcal{D} H \), so that \( H t = c \bar{t} \) has dimension length.

The angular process \( \Omega(t) \) converges on the sphere \( S^{D-1} \). It can be transformed into standard Brownian motion \( \Omega(T) \) on \( S^{D-1} \) by the time change [27, 29]

\[
T(t) = \int_0^t \frac{1}{G(z(s))^2} ,
\]

and its quadratic variation is given by [29]

\[
[\Omega^2](t) = (D - 1) T(t) .
\]

(10)

In general, the new time \( T(t) \) depends itself on a random variable. However, this is not so asymptotically. Let us consider Brownian trajectories over the time span \([t, \infty)\) that converge to the same angle \( \Omega_{\infty} \), where \( t \gg 1/H^2 \). Because of continuity, their quadratic variation is

\[
[\Omega_{\infty}^2](t) = (D - 1) \int_t^\infty ds \frac{1}{G(z_s)^2} .
\]

(11)

Since \( H^2 t \gg 1 \) we can substitute \( z(s) \) with its limiting value (8). Then, using (1) we obtain

\[
[\Omega_{\infty}^2](z) = 4 e^{-2Hz} + O \left( e^{-4Hz} \right) = 4 e^{-2H(D-1)w} + O \left( e^{-4HD-1}w \right) ,
\]

where we have introduced the new variable \( w = Ht/2 = c\bar{t}/2 \), such that \( z = (D - 1) w \) and we will henceforth neglect the subdominant exponential.

The quadratic variation of Brownian motion satisfies, on large enough scales, the Einstein relation,

\[
[x^2](t) = t ,
\]

which is equivalent (in 1D) to the diffusion equation

\[
\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u ,
\]

for the probability distribution \( u(t, x) \) of the diffusing particles. On scales shorter than the mean free path \( \ell_p = v \tau_p \), however, Brownian motion is ballistic (see [39] and references therein), with a quadratic variation proportional to the square of time: particles moves along geodesics with constant velocity \( v \),

\[
[x^2](t) = \frac{1}{\ell_p^2} t^2 .
\]

(15)

In particular, Brownian trajectories originating from a given point always start out as ballistic, up to the characteristic time scale \( \tau_p \) [39]. In the present case, however, we have the reversed situation: since the angular Brownian motion converges to a fixed angle, it is the final section of the converging trajectories that becomes ballistic for \( H \gg 1 \). In particular

\[
[\Omega_{\infty}^2](z) = \tau^2 ,
\]

\[
\tau = -2 e^{-H \bar{t}} = -2 e^{-H(D-1)w} ,
\]

(16)

where \( \tau \) will be momentarily identified with what is known as conformal time in expansionary cosmology, specifically in de Sitter space-time. Note that small intervals in \( \tau \) correspond to large intervals in coordinate time \( z \) for \( H \gg 1 \). Essentially, the constant characteristic time of angular Brownian scatterings in the rescaled time \( T(t) \) (or \( \tau(t) \)) gets diluted away in the emerging coordinate time \( z \) and only ballistic motion survives. In this asymptotic regime, the trajectories are concentrated on rays \( [d (\Omega(t) - \Omega_{\infty}) - \tau] \), with \( d \) the geodesic distance on the sphere and the probability distribution of particles with coordinates \( \theta_i \) and \( z \gg 1/H \) is governed by the wave equation

\[
\left( -\frac{\partial^2}{\partial \tau^2} + \Delta_\Omega \right) u(z, \theta_i) = 0 ,
\]

where \( \Delta_\Omega \) is the Laplacian on the sphere.

We would now like to derive the corresponding wave equation in coordinates \( (w, \theta_i) \), instead of \((z, \theta_i)\). To do so we first note that (17) can be written as

\[
\left( -\frac{1}{r_w^2} \frac{\partial^2}{\partial \tau^2} + \frac{1}{r_w^2} \Delta_\Omega \right) u(z, \theta_i) = 0 ,
\]

(18)
where \( r_o = (1/2H) \exp H z \) is the original radius (asymptotically) in (1) and \((1/r_o^2)\Delta_o\), the angular component of the Laplacian, is the generator of angular Brownian motion at fixed radius \( r_o \) in embedding space. We are primarily interested, however, in angular Brownian motion at fixed radius \((D - 1)r_o\) with \( r_o = (1/2H) \exp (H w)\) in coordinates \( w \) and \( \theta \). This is generated by \((1/(D - 1)^2 r_o^2)\Delta_o\). Correspondingly, the wave equation in coordinates \((w, \theta)\) becomes

\[
\left( -\frac{\partial^2}{\partial t^2} + \frac{1}{(D - 1)^2 r_o^2} \Delta_o \right) u(w, \theta) = 0, \tag{19}
\]

where now both \( \theta \) and the radii have all to be considered as functions of the new coordinate \( w \). Expressed through this new coordinate \( w \) we obtain

\[
\left( -\frac{\partial^2}{\partial w^2} - (D - 1)H \frac{\partial}{\partial w} + 4H^2 e^{-2Hw} \Delta_o \right) u(w, \theta) = 0. \tag{20}
\]

In the asymptotic regime \( Hw \gg 1 \), this is the wave equation on a Lorentzian manifold of positive curvature \(+H^2\) with coordinates \( w \) and \( \theta \), obtained by substituting \( \cosh \leftrightarrow \sinh \) in (4). This manifold is exactly de Sitter space-time [1]. We have thus derived that Brownian motion on a Riemannian manifold of negative curvature propagates asymptotically along the null geodesics of the corresponding Lorentzian de Sitter space-time, after the absolute diffusion time is dynamically soldered to the radial coordinate of the manifold. The absolute value of the negative curvature becomes the cosmological constant \([40]\). The original negative-curvature Riemannian manifold is thus asymptotically equivalent to an "emergent, effective de Sitter space-time".

To make contact with our spatially flat universe we shall also consider the flat slicing of the negative-curvature manifold, given by the embedding

\[
\begin{align*}
X_0 &= \frac{1}{H} \cosh Hv + \frac{1}{2H^2} r^2 \exp (Hv), \\
X_i &= \frac{1}{H} \sinh Hv - \frac{1}{2H^2} r^2 \exp (Hv), \\
X_i &= \frac{X_i}{H} e^{Hv}, \quad i = 2 \ldots D, \tag{21}
\end{align*}
\]

with \( r^2 = \sum x_i^2 \) and the metric taking the form \( ds^2 = dv^2 + (1/H^2) \exp (2Hv) dx^2 \) with \( dx^2 = \sum x_i^2 \) the flat Euclidean metric on \( \mathbb{R}^{D-1} \) (with this definition the \( x_i \) are dimensionless, distances are measured in units of \( 1/H \)). Since the two times \( z \) and \( v \) coincide asymptotically we have the same picture: asymptotically, the Lorentzian picture with \( v \) as a time coordinate of the flat slicing of de Sitter space-time with \( \cosh \leftrightarrow \sinh \) in (21) is an effective description of ballistic diffusion on the fundamental negative-curvature manifold.

The dynamical emergence of coordinate time and the dilution of Brownian scatterings mean that, asymptotically, the equations describing the original Brownian motion can be formulated entirely as deterministic, ballistic equations in terms of polar manifold coordinates. This, however, implies further that these equations, as viewed by an observer in another inertial frame of reference, should be covariant under the isometry group of the emergent, effective de Sitter space-time while the original absolute time survives as cosmic time. This covariance is satisfied since the de Sitter isometry group \( SO(D, 1) \) coincides, up to time-reversal symmetry, with the exact ground state isometry group \( SO_+(D, 1) \). The absence of time-reversal symmetry means that emergent coordinate time automatically comes endowed with a forward arrow. Note that time-reversal symmetry violation is one of the three Sakharov conditions needed for a matter-antimatter asymmetry in the universe (for a review see [41]). The de Sitter group \( SO(D, 1) \) replaces the Poincaré group of special relativity when not only a velocity \( c \) is fixed but also a length scale \( 1/H \) [31] (for a review see [32]). In particular, the homogenous Lorentz subgroup still describes rotations and boosts, while the translations of the Poincaré group become combinations of translations and proper conformal transformations [33]. The breakdown of ordinary translation symmetry is the only deviation of de Sitter relativity from ordinary special relativity.

The basic idea of the present model is that, for asymptotic Brownian motion in constant negative curvature, one coordinate emerges dynamically as time, while Brownian scatterings in the orthogonal "spatial" motion are diluted away so that only the ballistic motion below the characteristic scattering time survives. If this short-time dynamics is generalized to include complex probability amplitudes and quantum walks (for a review see e.g. [42]), then the isometry group of the constant negative curvature manifold guarantees that all familiar Lorentz representations are admitted as asymptotic particle excitations, including massive ones. Quantum walks would then describe the microscopic dynamics of matter with respect to the universal time; this is decohered by scatterings on the link fluctuations of the fundamental graph, becoming thus Brownian motion [43]. In the asymptotic region of the manifold where these graph fluctuations and the corresponding Brownian scatterings are diluted away, quantum walks in universal time become the usual quantum behaviour for Lorentz representation with one manifold coordinate as time.

Every geodesically complete, D-dimensional manifold \( M \) of constant negative curvature has a natural geometric boundary manifold \( \partial M \) defined as the locus of all equivalence classes of geodesic rays that remain in bounded distance of each other. If \( M \) is simply connected, the boundary \( \partial M \) is a sphere \( S^{D-1} \) [46] (for a review see e.e. [36]). Moreover, there is a one-to-one correspondence between interior \( SO_+(D, 1) \) isometries on \( M \) and (D-1)-dimensional conformal transformations on the boundary \( \partial M \) [36]. This correspondence is a Riemannian version of the holographic principle on Lorentzian negative-curvature manifolds like anti-de-Sitter (adS) space [5, 6] (for a review see [7]). However, via the emerging time coordinate it defines asymptotically holography on effective de Sitter space-time.

Let us finally focus on the spectral dimension of effective de Sitter space-time. In general, diffusion processes probe the intrinsic geometry of a manifold. In particular, the heat kernel trace (here \( t \) is again the same time as originally introduced in
the Brownian motion equations (6) and following)
\[ K(t) = \text{tr} \, e^{\Delta t}, \]
measures the return probability after time \( t \) and defines the spectral dimension of the manifold via the spectral function (for a review see [47])
\[ d_s(t) = -2 \frac{d \ln K(t)}{d \ln t}. \]
The spectral function \( d_s(t) \) probes the spectral dimension of the manifold in different regimes, on microscopic scales for \( t \to 0 \), on large scales for \( t \to \infty \). It measures the effective number of independent directions available for a random walker. For a Euclidean manifold it coincides with its topological dimension and is independent of \( t \). In general, however, it can differ from the topological dimension and needs not even be an integer for fractals and graphs.

The heat trace kernel for manifolds of constant negative curvature \((-H^2)\) has the uniform continuous estimate [48, 49]
\[ K(t) = \left(1 + H^2 t\right)^{-(D-3)/2} e^{-\frac{(D-1)^2}{2} H^2 t}, \]
which gives the spectral curve
\[ d_s(t) = D - (D - 3) \left( \frac{H^2 t}{1 + H^2 t} + \frac{(D - 1)^2}{2} H^2 t \right). \]
In general, one would expect that, in the limit \( t \to \infty \) of infinite diffusion time, the diffusion process on \( M \) probes the “geometry at infinity”. However, in the present case this is not so since the spectral function diverges for \( t \to \infty \). This is because, on a negative curvature manifold, the Laplacian has a spectral gap
\[ \lambda_0 = -\lim_{t \to \infty} \frac{\ln K(t)}{t} = \frac{(D - 1)^2 H^2}{4}, \]
representing the bottom of the spectrum of the positive operator \(-\Delta\). As a consequence of this spectral gap, the return probabilities \( K(t) \) are dominated by an exponential behaviour at large \( t \), which is equivalent to ballistic diffusion. In other words, because of ballistic diffusion, the process runs away to infinity too quickly to sample the geometric properties of the manifold [34].

If one wants to probe the geometry on large scales one must define a modified, slower diffusion process, dominated by the Laplacian eigenvalues just above \( \lambda_0 \) after the spectral gap, i.e. the runaway deterministic time dimension has been subtracted. This can be done and this slower diffusion process is called the infinite Brownian loop [34]. Let us denote by \( \varphi_0 \) the most symmetric eigenstate of the Laplacian corresponding to the lowest eigenvalue \( \lambda_0 \). The infinite Brownian loop is then the relativized \( \varphi_0 \)-process [50], with generator
\[ \hat{A}(f) = \frac{1}{\varphi_0} \Delta (f \varphi_0) + \lambda_0 f = \hat{A} f + 2 \nabla \ln \varphi_0 \cdot \nabla f. \]
As anticipated, in this process measured relatively to the ground state \( \varphi_0 \), the spectral gap falls out. This corresponds to dropping the exponential in (24), which gives the modified spectral function
\[ \tilde{d}_s(t) = D - (D - 3) \left( \frac{H^2 t}{1 + H^2 t} \right), \]
whose infinite-time limit is
\[ D_{\text{inf}} = \lim_{t \to \infty} \tilde{d}_s(t) = 3, \]
indpendently of \( D \). The quantity \( D_{\text{inf}} \) is called the “pseudo-dimension”, or “dimension at infinity” of a constant negative curvature manifold [34]. It measures the asymptotic spectral dimension of the manifold on large scales, as probed by the slow random process in the radial direction which survives after “subtracting time”. This slow random process always sees three Euclidean dimensions and it is confined to the Weyl chamber [34], which means, in this case, that it is confined to the forward direction, another expression of the emergent arrow of time.

We have derived the spatial spectral dimension three on large scales in the case of constant negative curvature. This result, however, is a general consequence of the local limit theorem [35] and is valid for any manifold of strictly negative curvature. This theorem states that, on the universal cover of any closed and connected manifold of strictly negative curvature, there exists a positive constant \( C \) such that
\[ \lim_{t \to \infty} t^{3/2} e^{-\lambda_0 t} K(t) = C, \]
where \( K(t) \) is the heat kernel trace and \( \lambda_0 \) is the spectral gap. In other words, when the spectral gap (time) is “subtracted” the spectral dimension at infinity is always three, independently of the topological dimension \( D \). On effective de Sitter space-time the spectral dimension changes thus from \( D \) Riemannian dimensions and a universal time on microscopic graph scales to \((3+1)\) Lorentzian dimensions on large scales. Perhaps the most interesting case is \( D = 2 \). In this case the cosmological ground state is a fluctuating negative-curvature surface (note that this is a well defined surface with Hausdorff dimension 2 [14]) and the flow from 2 Euclidean spectral dimensions to \((3+1)\) Lorentzian dimension is reminiscent of causal dynamical triangulations [51], with the difference that here coordinate time and manifold causality are emergent.

**CONCLUSION**

The picture emerging from the present model is that the asymptotic appearance of de Sitter space-time with a cosmological constant is only an effective description of a different fundamental ground state. Going backwards in time, ever more frequent stochastic scatterings with the fundamental graph link fluctuations decohere quantum mechanics and turn the coordinate time of de Sitter space-time into a random variable on a Riemannian negative curvature manifold.
There are no singularities, no horizon and the holographic principle appears naturally due to the fundamental negative curvature. The spectral dimension of the resulting of the asymptotic manifold is $(3+1)$ independent of its topological dimension.

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