BAKRY-EMERY CALCULUS FOR DIFFUSION WITH ADDITIONAL MULTIPlicative TERM

C. ROBERTO, B. ZEGRALINSKI

Abstract. We extend the $\Gamma_2$ calculus of Bakry and Emery to include a Carré du champ operator with multiplicative term, providing results which allow to analyse inhomogeneous diffusions.

The aim of this paper is to extend Bakry-Emery approach [BE85, Bak94] to deal with some quantities involving operators not only of order one, but also including order zero. One of the motivations for that is a possible application to analysis of hypercontractivity properties for some classes of inhomogeneous Markov semi-groups, see e.g. [RZ].

The setting is as follows: $(Q_t)_{t\geq 0}$ is the semi-group associated to a diffusion operator $L = \Delta - \nabla U \cdot \nabla$, on $\mathbb{R}^n$, where the dot sign stands for the Euclidean scalar product. We assume that $U: \mathbb{R}^n \to \mathbb{R}$ is twice differentiable and satisfies $\int e^{-U(x)} dx = 1$ so that $\mu(dx) = e^{-U(x)} dx$ is a probability measure on $\mathbb{R}^n$. By construction $L$ is symmetric in $L^2(\mu)$. Following Bakry-Emery, we denote by $\Gamma$ the carré du champs bilinear form

$$\Gamma(f, g) := \frac{1}{2} (L(fg) - fLg - gLf),$$

and set $\Gamma(f) := \Gamma(f, f)$. For the diffusion $L$ considered here, we have $\Gamma(f, g) = \nabla f \cdot \nabla g$.

The iterated operator $\Gamma_2$ is defined as

$$\Gamma_2(f, g) = \frac{1}{2} (L\Gamma(f, g) - \Gamma(Lf, g) - \Gamma(f, Lg)),$$

and again, for simplicity, we set $\Gamma_2(f) := \Gamma(f, f)$. One can see that $\text{Hess}(U) \geq \rho$ (as a matrix), $\rho \in \mathbb{R}$, implies $\Gamma_2(f) \geq \rho \Gamma(f)$ for all smooth enough $f$ (see e.g. [ABC$^+$00, Chapter 5], [BGL14]).

One fundamental result of Bakry and Emery [BE85] is that $\Gamma_2 \geq \rho \Gamma$ (the so-called $\Gamma_2$-condition) is equivalent to the following commutation property between the semi-group and the gradient operator (equivalently $\Gamma$):

$$\Gamma(Q_t f) \leq e^{-2\rho t} Q_t(\Gamma(f)), \quad t \geq 0$$

for any $f$ for which $\Gamma(f)$ is well defined, see e.g. [ABC$^+$00, Proposition 5.4.1].

We will prove that a similar equivalence holds for an extended operator we introduce now. Let $W: \mathbb{R}^n \to \mathbb{R}_+$ be smooth enough so that in particular $W^2$ belongs to the domain of $L$. Then, for $f, g$ smooth enough, we set

$$\Gamma^W(f, g) := \Gamma(f, g) + W^2 fg$$

which is therefore a positive bilinear form. The operator $\Gamma^W$ acts as a derivative and multiplicatively. In fact, one can see $\Gamma^W$ as two-dimensional operator, call it $D$, that acts as $Df := (\nabla f, Wf)$. With this notation, $\Gamma^W(f) = |Df|^2 = |\nabla f|^2 + W^2 f^2$ is nothing but

---

Date: February 23, 2021.

Key words and phrases. $\Gamma_2$ calculus.

Supported by the grants ANR-15-CE40-0020-03 - LSD - Large Stochastic Dynamics, ANR 11-LBX-0023-01 - Labex MME-DII and Fondation Simone et Cino del Luca in France, and the grant ... in the UK.
the Euclidean norm squared of the 2 dimensional vector $Df$. Similarly, we introduce the iterated operator
\[
\Gamma^W_2(f, g) := \frac{1}{2} \left( L\Gamma^W(f, g) - \Gamma^W(Lf, g) - \Gamma^W(f, Lg) \right)
\]
\[= \Gamma_2(f, g) + \frac{1}{2} fgL(W^2) + W^2\Gamma(f, g) + 2W\nabla W\nabla(fg)
\]
where the last equality follows from some algebra.

It should be clear from the definition that we are not dealing with $\Gamma^W_2$ calculus for the operator $L^W := L-2W^2$, even though by simple algebra we have $\frac{1}{2} \left( L^W(f, g) - fL^W g - gL^W f \right) = \Gamma^W(f, g)$. The point is that we want to derive commutation formulas for the semi-group $(Q_t)_{t \geq 0}$ associated to $L$ and not for the semi-group associated to $L^W$. This also explains why $\Gamma^W_2$ is defined through the operator $L$ and not $L^W$.

Our first main result reads as follows.

**Theorem 1.** Let $\rho \in \mathbb{R}$. The following are equivalent:

(i) for all $f$ smooth enough $\Gamma^W_2(f) \geq \rho \Gamma^W(f)$

(ii) for all $f$ smooth enough and all $t \geq 0$,
\[\Gamma^W(Q_tf) \leq e^{-2\rho t}Q_t(\Gamma^W(f))\]

(iii) for all $f$ smooth enough and all $t \geq 0$,
\[Q_t(f^2) - (Q_t f)^2 + 2 \int_0^t Q_s \left( W^2(Q_{t-s} f)^2 \right) ds \leq \frac{1-e^{-2\rho t}}{\rho} Q_t(\Gamma^W(f)).\]

**Remark 2.** Above, when $\rho = 0$, the ratio $\frac{1-e^{-2\rho t}}{\rho}$ is understood as its limit (i.e. $2t$). Notice that it is always non-negative.

Observe that, applying (ii) to constant functions $f \equiv C$, $C \neq 0$, leads to $W^2 \leq e^{-2\rho t}Q_t(W^2)$. Therefore, if $\int W^2 d\mu < \infty$ and $\rho > 0$, taking the limit $t \to \infty$ and by ergodicity, we would conclude that $W \equiv 0$. Therefore, for the inequality $\Gamma^W_2(f) \geq \rho \Gamma^W(f)$ to hold for a non trivial $W$, either $\rho \leq 0$ or $\int W^2 d\mu = \infty$. But we have no this restriction removing mean value $\mu(f)$ of the function $f$.

Taking the mean with respect to $\mu$ in (iii) and passing to the limit $t \to \infty$, we get by invariance and ergodicity that for $\rho > 0$, it holds
\[\int f^2 d\mu - \left( \int f d\mu \right)^2 - 2 \int W^2 \int_0^\infty (Q_s f)^2 ds d\mu \leq \frac{1}{\rho} \left( \int |\nabla f|^2 d\mu + \int f^2 W^2 d\mu \right).
\]
This is a sort of Poincaré inequality in particular for a function $f$ with mean value zero. Following Bakry-Emery (see e.g. [ABC+00, proposition 5.5.4]), one can actually prove that the latter holds under the weaker assumption that $\int \Gamma^W_2(f) d\mu \geq \rho \int \Gamma^W(f) d\mu$.

**Proof.** The proof mimics the usual case ($W = 0$).

To prove that (i) implies (ii), fix $t > 0$ and consider the following function $\Psi : s \in [0, t] \to \Psi(s) = Q_s \left( \Gamma^W(Q_{t-s} f) \right) = Q_s (\Gamma(Q_{t-s} f) + W^2(Q_{t-s} f)^2)$. Then, setting $g := Q_{t-s} f$, it holds
\[\Psi'(s) = Q_s \left( L\Gamma(g) + L(W^2 g^2) - 2\Gamma(g, Lg) - 2W^2 g Lg \right)
\[= 2Q_s \left( \Gamma^W_2(g) \right) \geq 2\rho Q_s \left( \Gamma^W(g) \right) = 2\rho \Psi(s)
\]
from which the result of Item (ii) follows.
Now we prove that (ii) implies (iii). Let

\[ \Psi(s) := Q_s((Q_{t-s} f)^2) + 2 \int_0^s Q_u \left( W^2(Q_{t-u} f)^2 \right) du, \quad s \in [0, t]. \]

Then, setting again \( g := Q_{t-s} f \), it holds

\[ \Psi'(s) = Q_s \left( L(g^2) - 2g Lg + 2W^2(Q_{t-s} f)^2 \right) = 2Q_s(\Gamma^W(g)). \]

Therefore

\[ \Psi(t) - \Psi(0) = \int_0^t \Psi'(s)ds = 2 \int_0^t Q_s \left( \Gamma^W(Q_{t-s} f) \right) ds \leq \int_0^t 2e^{-2p(t-s)} Q_s(Q_{t-s}(\Gamma^W(f)))ds \]

\[ \leq \frac{1 - e^{-2pt}}{\rho} Q_t(\Gamma^W(f)). \]

This corresponds to the expected result of Item (iii).

Last we prove that (iii) implies (i). We may use the following expansions left to the reader:

\[ Q_{t} f = f + tL f + \frac{t^2}{2} L(L f) + o(t^2) \]

from which we deduce that

\[ Q_{t}(f^2) - (Q_{t} f)^2 = 2t \Gamma(f) + t^2[L(\Gamma(f)) + 2\Gamma(f, L f)] + o(t^2). \]

On the other hand,

\[ 2 \int_0^t Q_s \left( W^2(Q_{t-s} f)^2 \right) ds = 2tW^2 f^2 + t^2[L(W^2 f^2) + 2W^2 f L f] + o(t^2) \]

and

\[ \frac{1 - e^{-2pt}}{\rho} Q_t(\Gamma^W(f)) = 2t \Gamma^W(f) + t^2[-2\rho \Gamma^W(f) + 2L[\Gamma^W(f)]] + o(t^2). \]

Plugging these expansions into (iii) leads precisely to (i). This ends the proof. \( \square \)

In the next result, we give a condition for Inequality (i) of Theorem 1 to hold. Observe first that

\[ \Gamma^W_2(f) = \Gamma_2(f) + \frac{1}{2} f^2L(W^2) + W^2 \Gamma(f) + 2W \nabla W \nabla (f^2) \]

\[ = \Gamma_2(f) + f^2[W \Delta W + |\nabla W|^2 - W \nabla W \cdot \nabla U] + W^2 |\nabla f|^2 + 4f W \nabla W \cdot \nabla f. \]

Set \( \partial^2_{ij} \) for the second order derivative with respect to the variables \( x_i \) and \( x_j \). Since \( \Gamma_2(f) = \sum_{i,j=1}^n (\partial^2_{ij} f)^2 + (\nabla f)^T \text{Hess}(U)(\nabla f) \), we observe that the condition \( \Gamma_2(f) \geq \rho \Gamma(f) \) is satisfied as soon as \( \text{Hess}(U) \geq \rho \).

**Theorem 3.** Assume that \( \Gamma_2 \geq \rho \Gamma \) for some \( \rho \in \mathbb{R} \) and that

\[ \gamma := \inf_{x \in \mathbb{R}^n, W(x) \neq 0} \left( \frac{\Delta W}{W} - 3 \frac{|\nabla W|^2}{W^2} - \frac{\nabla U \cdot \nabla W}{W} \right) > -\infty. \]

Then, we have

\[ \Gamma^W_2(f) \geq \min(\rho, \gamma) \Gamma^W(f) \]

for all \( f \) smooth enough.

In the above, by convention we set \( \inf \emptyset = +\infty. \)
Example 4. Consider $U(x) = c + \frac{(1 + |x|^2)^{p/2}}{p}$ and $W(x) = \frac{(1 + |x|^2)^{q/2}}{q}$, $x \in \mathbb{R}^n$, $p, q \geq 1$ with $c$ so that $\int e^{-U(x)} dx = 1$. Here, as usual, $|x| = (\sum x^2)^{1/2}$ is the Euclidean norm. The (spurious) form of $U$ and $W$ is here to guarantee smoothness (indeed it would have been easier to work with $W(x) = |x|^q$ that is, however, not smooth on the whole Euclidean space).

We observe that $\nabla U(x) = x(1 + |x|^2)^{(p-2)/2}$, $\nabla W(x) = x(1 + |x|^2)^{(q-2)/2}$ and $\Delta W(x) = n(1 + |x|^2)^{(q-2)/2} + (q - 2)|x|^2(1 + |x|^2)^{(q-4)/2}$. Therefore

$$\frac{\Delta W}{W} - 3 \frac{|\nabla W|^2}{W^2} - \nabla U \cdot \nabla W = \frac{2}{1 + |x|^2} - \frac{q}{1 + |x|^2} \left(2 + (1 + |x|^2)^{(p-2)/2}\right)$$

is bounded below if and only if $p \leq 2$, in which case, Theorem 3 applies and leads to a non-trivial statement.

Proof of Theorem 3. Form the expression of $\Gamma_2^W$ above, we infer that

$$\Gamma_2^W(f) \geq \rho |\nabla f|^2 + f^2 |\nabla \Delta W + |\nabla W|^2 - 2 \nabla U \cdot \nabla W| + W^2 |\nabla f|^2 + 4fW \nabla W \cdot \nabla f.$$  

Now $4fW \nabla W \cdot \nabla f \geq -4f^2 |\nabla W|^2 - W^2 |\nabla f|^2$ so that

$$\Gamma_2^W(f) \geq \rho |\nabla f|^2 + W^2 f^2 \left(\frac{\Delta W}{W} - 3 \frac{|\nabla W|^2}{W^2} - \frac{W \nabla W}{W^2}\right).$$

The expected result follows.

As an immediate corollary, we get the following useful result.

Corollary 5. Assume that $\Gamma_2 \geq \rho \Gamma$ for some $\rho \in \mathbb{R}$ and that

$$\gamma := \inf_{x \in \mathbb{R}^n, W(x) \neq 0} \left(\frac{\Delta W}{W} - 3 \frac{|\nabla W|^2}{W^2} - \frac{\nabla U \cdot \nabla W}{W}\right) > -\infty.$$  

Then, for all $f$ smooth enough, it holds

$$\Gamma^W(Q_t f) \leq e^{-2\min(\rho, \gamma) t} \Gamma^W(f), \quad t \geq 0.$$  

(2)

Next, we show that the quantity $\min(\rho, \gamma)$, that appears in Theorem 3 and Corollary 5, is optimal, in the sense that, for some examples of $U$ and $W$, it cannot be improved.

Observe first that, if $W \equiv 0$, then $\gamma = \infty$ and therefore $\Gamma_2^W \geq \min(\rho, \gamma) \Gamma^W$ is equivalent to $\Gamma_2 \geq \rho \Gamma$ which is known to be optimal (for example for the Gaussian potential $U(x) = |x|^2/2$ for which $\rho = 1$).

In fact, consider the Gaussian potential $U(x) = \frac{|x|^2}{2} - \frac{n}{2} \log(2\pi)$ in $\mathbb{R}^n$, and $W(x) = \sqrt{1 + |x|^2}$, $x \in \mathbb{R}^n$. One has

$$\frac{\Delta W}{W} - 3 \frac{|\nabla W|^2}{W^2} - \nabla U \cdot \nabla W = \frac{n - 2}{1 + |x|^2} + \frac{3}{(1 + |x|^2)^2} - 1$$

form which one infers that $\gamma = -13/12$ if $n = 1$ and $\gamma = -1$ when $n \geq 2$. Since in that specific case $\rho = 1$, $\min(\rho, \gamma) = \gamma$ and therefore, Corollary 5 asserts that, in dimension 2 or higher, $\Gamma^W(Q_t f) \leq e^{2t} \Gamma^W(f)$, for $t \geq 0$. We stress that this goes in the opposite direction of (1), which, in the Gaussian setting, can be recast as $|\nabla Q_t f|^2 \leq e^{-t} Q_t(|\nabla f|^2)$ with optimal decay $e^{-t}$. As we may prove now, $e^{2t}$ is also optimal.

For that purpose, consider the following family of functions

$$f_a(x) := e^{a x}, \quad a = (a_1, \ldots, a_n), x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

where as usual $a \cdot x := \sum_i a_i x_i$ is the scalar product in $\mathbb{R}^n$. Optimality can be obtained equivalently (thanks to Theorem 1) either from the bound $\Gamma^W(Q_t f) \leq e^{2t} \Gamma^W(f)$ (using Melcher’s representation formula for the Ornstein-Uhlenbeck semi-group) or in $\Gamma_2^W(f) \geq -\Gamma^W(f)$. We will dig on the latter by computing $\Gamma_2^W(f_a)$ and $\Gamma^W(f_a)$ for all $a, x \in \mathbb{R}^n$. 

On the one hand we have
\[ \Gamma^W(f_a) = |\nabla f_a|^2 + W^2 f_a^2 = \left( |a|^2 + 1 + |x|^2 \right) f_a^2. \]

On the other hand,
\[
\begin{align*}
\Gamma^W_2(f_a) &= \sum_{i,j=1}^n (\partial^2_{ij} f_a)^2 + (\nabla f_a)^T (\text{Hess} U)(\nabla f_a) + f_a^2 |W\Delta W + |\nabla W|^2 - W W \nabla \cdot \nabla U| \\
&+ W^2 |\nabla f_a|^2 + 4 f_a W W \nabla \cdot \nabla f_a \\
&= f_a^2 \left( |a|^4 + 2 |a|^2 + n + |x|^2 (-1 + |a|^2) + 4 x \cdot a \right)
\end{align*}
\]

Therefore,
\[ \lim_{|x| \to \infty} \frac{\Gamma^W_2(f_a)}{\Gamma^W(f_a)} = -1 + |a|^2. \]

Finally, in the limit $|a| \to 0$, we conclude that the biggest constant $\kappa$ satisfying $\Gamma^W_2(f) \geq \kappa \Gamma^W(f)$ for all $f$ must satisfy $\kappa \leq -1$ and therefore, by Theorem 3, $\kappa = -1$ is optimal, as announced.

In some situations it might be useful to deal with $\sqrt{T}$ instead of $\Gamma$. Unfortunately, there is not a clean commutation result, as in the usual Bakry-Emery theory, for $\Gamma^W$. However, we may prove the following proposition, that is already useful for applications. In particular, such a result was used by the authors to deal with hypercontractivity properties for some class of inhomogeneous Markov semi-groups $[RZ]$.

**Proposition 6.** Assume the following:

(i) there exists $\rho \in \mathbb{R}$ such that for all $f$ smooth enough it holds $\Gamma_2(f) \geq \rho \Gamma(f)$;

(ii) $c := \max \left( 2 \|[\nabla W]\|_{\infty}, \sup_{x: W(x) \neq 0} \left( \frac{4W}{W} - \rho \right) \right) < \infty$.

Then, for all $f$ non-negative, it holds
\[ \sqrt{\Gamma(P_t f)} + WP_t f \leq e^{(c-\rho)t} P_t \left( \sqrt{\Gamma(f)} + W f \right). \]

**Proof.** Following Bakry-Emery, see [ABC+00, proof of Proposition 5.4.5], introduce $\Psi(s) = e^{-\rho s} P_s \left( \sqrt{\Gamma(P_{t-s} f)} + WP_{t-s} f \right)$, $s \in [0, t]$, $t$ being fixed. Therefore, setting $g := P_{t-s} f$, one has
\[ \Psi'(s) = -\rho \Psi(s) + e^{-\rho s} P_s L \sqrt{\Gamma(g)} + P_s (L(W g)) - e^{-\rho s} P_s \left( \frac{\Gamma(g, L g)}{\sqrt{\Gamma(g)}} \right) - e^{-\rho s} P_s (W L g). \]

Now
\[ L \sqrt{\Gamma(g)} = \frac{2 \Gamma(g)}{\sqrt{\Gamma(g)}} - \frac{\Gamma'(\Gamma(g))}{4 \Gamma(g)^{3/2}} \]

Hence, after some algebra, we get
\[ \Psi'(s) = \frac{e^{-\rho s}}{4} P_s \left( \frac{4 \Gamma(g)(\Gamma_2(g) - \rho \Gamma(g)) - \Gamma'(\Gamma(g))}{\Gamma(g)^{3/2}} \right) + e^{-\rho s} P_s (L(W g) - W L g - \rho W g). \]

Assumption (i) ensures that the first term of the right hand side of the latter is non-negative (see [ABC+00, Lemma 5.4.4]). On the other hand,
\[ L(W g) - W L g - \rho W g = g(L W - \rho W) + 2 \nabla W \cdot \nabla g \geq -c (|\nabla g| + W g). \]

It follows that $\Psi'(s) \geq -c \Psi(s)$. In turn, $\Psi(t) \geq \Psi(0) e^{-ct}$ from which the desired result follows. \qed
References

[ABC+00] C. Ané, S. Blachère, D. Chafai, P. Fougeres, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer. Sur les inégalités de Sobolev logarithmiques., volume 10 of Panoramas et Synthèses. S.M.F., Paris, 2000. 1, 2, 5

[Bak94] D. Bakry. L’hypercontractivité et son utilisation en théorie des semigroupes. In Lectures on probability theory. École d’été de probabilités de St-Flour 1992, volume 1581 of Lecture Notes in Math., pages 1–114. Springer, Berlin, 1994. 1

[BE85] D. Bakry and M. Emery. Diffusions hypercontractives. In Séminaire de probabilités, XIX, 1983/84, pages 177–206. Springer, Berlin, 1985. 1

[BGL14] D. Bakry, I. Gentil, and M. Ledoux. Analysis and geometry of Markov diffusion operators, volume 348 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Cham, 2014. 1

[RZ] C. Roberto and B. Zegarlinski. Hypercontractivity for markov semi-groups. preprint arXiv:2101.01616. 1, 5

Université Paris Nanterre, Modal’X, FP2M, CNRS FR 2036, 200 avenue de la République 92000 Nanterre, France

Imperial College of London, Faculty of Natural Sciences, Department of Mathematics, Huxley Building, South Kensington Campus, London SW7 2AZ, UK

Email address: croberto@math.cnrs.fr, b.zegarlinski@imperial.ac.uk