On the Pólya conjecture for circular sectors and for balls

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Abstract

In 1954, G. Pólya conjectured that the counting function $N(\Omega, \Lambda)$ of the eigenvalues of the Laplace operator of the Dirichlet (resp. Neumann) boundary value problem in a bounded set $\Omega \subset \mathbb{R}^d$ is lesser (resp. greater) than $(2\pi)^{-d} \omega_d |\Omega| \Lambda^{d/2}$. Here $\Lambda$ is the spectral parameter, and $\omega_d$ is the volume of the unit ball. We prove this conjecture for both Dirichlet and Neumann boundary problems for any circular sector, and for the Dirichlet problem for a ball of arbitrary dimension. We heavily use the ideas from [10].

Introduction

0.1 Formulation of the results

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Denote the eigenvalues of the Laplace operator of the Dirichlet problem in $\Omega$ by

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots, \quad \lambda_k \to +\infty,$$

and of the Neumann problem in $\Omega$ by

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \ldots, \quad \mu_k \to +\infty,$$

taking multiplicity into account. In the book [12, Chapter XIII] Pólya conjectured that the inequalities

$$\lambda_k(\Omega) \geq \frac{4\pi^2}{(\omega_d |\Omega|)^{2/d}} k^{2/d} \quad (0.1)$$

$$\mu_{k+1}(\Omega) \leq \frac{4\pi^2}{(\omega_d |\Omega|)^{2/d}} k^{2/d} \quad (0.2)$$

hold for all open bounded sets $\Omega$ and for all natural numbers $k$. Here $|\Omega|$ is the Lebesgue measure of the set $\Omega$, and $\omega_d = |B_1|$ is the volume of the unit ball in $\mathbb{R}^d$. In terms of counting functions

$$N_D(\Omega, \Lambda) := \# \{ k : \lambda_k \leq \Lambda \}, \quad N_N(\Omega, \Lambda) := \# \{ k : \mu_k \leq \Lambda \},$$

the inequalities (0.1), (0.2) are equivalent to

$$N_D(\Omega, \Lambda) \leq \frac{\omega_d |\Omega|}{(2\pi)^d} \Lambda^{d/2}, \quad (0.3)$$

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\[ N_N(\Omega, \Lambda) \geq \frac{\omega_d |\Omega|}{(2\pi)^d} \Lambda^{d/2} \quad (0.4) \]

for all \( \Lambda \geq 0 \). Note that the coefficient coincides with the coefficient in the Weyl asymptotics for counting functions. Note also that the formulation of the hypothesis by Pólya himself in the book [12] slightly differs from the formulation above, see the comments below. The generally accepted name of the Pólya conjecture is fixed for the inequalities (0.1)–(0.4).

In 1961, Pólya proved [13] the estimate (0.1) for tiling sets and the inequality (0.2) for regularly tiling sets. The set \( \Omega \) is called tiling if the whole space \( \mathbb{R}^d \) can be covered by non-intersected copies of \( \Omega \) up to the set of measure zero. The set \( \Omega \) is regularly tiling if the corresponding covering is periodic. In 1966, Kellner proved [6] the inequality (0.2) for all tiling domains. (Note in the brackets that it is non-trivial question if there exists a tiling set that is not regularly tiling. An example of such set was constructed [16] in 2011 only.)

No counterexample to Pólya conjecture is known. In 2022, the first proof of the conjecture for a non-tiling domain was done by Levitin, Polterovich and Sher. In a breakthrough paper [10] they proved the Pólya conjecture for the disk both for Dirichlet and Neumann problems. Their proof of (0.3) and (0.4) for sufficiently large \( \Lambda \) (for \( \sqrt{\Lambda} \geq 84123 \)) is theoretical, and it is computer verification for \( \Lambda \leq (84123)^2 \). As a simple corollary (see Lemma 4.2 below) the conjecture holds for circular sectors with angle \( 2\pi/l \), \( l \in \mathbb{N} \), also for Dirichlet and for Neumann problems.

We develop the ideas from [10]. We justify the Pólya conjecture for Dirichlet and Neumann problems for circular sectors with arbitrary angle. We prove also the conjecture for Dirichlet case for balls in all dimensions. Let us formulate the results. Let \( 0 < \alpha \leq 2\pi \). Denote by \( \Omega_\alpha \) the circular sector with angle \( \alpha \),

\[ \Omega_\alpha = \{ x = (r, \varphi) : 0 < r < 1, 0 < \varphi < \alpha \} \quad (0.5) \]

in polar coordinates on the plane. If \( \alpha = 2\pi \) we get a circle with a cut along a radius. Clearly, \( |\Omega_\alpha| = \frac{\alpha}{2} \). The Weyl coefficient for \( \Omega_\alpha \) is

\[ \frac{\omega_2 |\Omega|}{(2\pi)^2} = \frac{\alpha}{8\pi}. \]

**Theorem 0.1.** Let \( 0 < \alpha \leq 2\pi, \Lambda > 0 \). We have

\[ N_D(\Omega_\alpha, \Lambda) < \frac{\alpha \Lambda}{8\pi}. \]

**Theorem 0.2.** Let \( 0 < \alpha \leq 2\pi, \Lambda \geq 0 \). We have

\[ N_N(\Omega_\alpha, \Lambda) > \frac{\alpha \Lambda}{8\pi}. \]

Let \( B_1 \) be a unit ball in \( \mathbb{R}^d \). Its measure is

\[ |B_1| = \frac{\pi^{d/2}}{\Gamma \left( \frac{d+2}{2} \right)}, \]

and the Weyl coefficient for \( B_1 \) is

\[ \frac{|B_1|^2}{(2\pi)^d} = \frac{1}{2^d \Gamma \left( \frac{d+2}{2} \right)^2}. \]

Here \( \Gamma \) is the Euler gamma-function.
Theorem 0.3. Let $\Lambda > 0$. For Dirichlet problem in a ball $B_1 \subset \mathbb{R}^d$ we have

$$N_D(B_1, \Lambda) < \frac{\Lambda^{d/2}}{2^d \Gamma (\frac{d+2}{2})^2}.$$  

In particular, we give a purely theoretical proof of the Dirichlet Pólya conjecture for a disk, see Theorem 4.4 below. For the Neumann problem in a disk we give a theoretical proof of (0.4) for $\Lambda \geq 531$, see Theorem 8.1 below. We give Theorems 4.4 and 8.1 for the sake of completeness.

0.2 About proof

The problem in a disk admits separating of variables. The eigenvalues of the Dirichlet (resp. Neumann) Laplace operator are the squared zeros (resp. squared zeros of the first derivative) of Bessel function $J_m$ with integer indices. This leads to two-term asymptotics for the Dirichlet problem

$$N_D(B_1, \Lambda) = \frac{\Lambda}{4} - \frac{\sqrt{\Lambda}}{2} + R_D(\Lambda), \quad \Lambda \to \infty,$$  

see [8] or [2] with $R_D(\Lambda) = O(\Lambda^{1/3})$, and [3] with $R_D(\Lambda) = O(\Lambda^{131/416} + \varepsilon)$. Analogue of this formula for the Neumann problem is

$$N_N(B_1, \Lambda) = \frac{\Lambda}{4} + \frac{\sqrt{\Lambda}}{2} + o(\Lambda), \quad \Lambda \to \infty.$$  

The asymptotics (0.6), (0.7) mean that the inequalities (0.3), (0.4) hold for $\Lambda \geq \Lambda_0$ for some $\Lambda_0$, but the value of this $\Lambda_0$ is poorly controlled.

Levitin, Polterovich and Sher instead of asymptotics of Bessel’s zeros proved the uniform bounds for them

$$\# \{k \in \mathbb{N} : j_{\nu,k} \leq x\} \leq \left[ \frac{1}{\pi} \left( x^2 - \nu^2 - \nu \arccos \frac{\nu}{x} \right) + \frac{1}{4} \right]$$  

(0.8)

and

$$\# \{k \in \mathbb{N} : j'_{\nu,k} \leq x\} \geq \left[ \frac{1}{\pi} \left( x^2 - \nu^2 - \nu \arccos \frac{\nu}{x} \right) + \frac{3}{4} \right],$$  

see Theorems 1.1 and 7.1 below. Here $j_{\nu,k}$ (resp. $j'_{\nu,k}$) is the $k$-th zero of Bessel function $J_\nu$ (resp. of the function $J'_\nu$); $[x]$ is the integer part of $x$. Thus, it is enough to consider the sums of the following type:

$$\sum_{m \leq \sqrt{\Lambda}} \left[ \frac{1}{\pi} \left( \sqrt{\Lambda - m^2} - m \arccos \frac{m}{\sqrt{\Lambda}} \right) + s \right]$$  

(0.9)

with $s = \frac{1}{4}$ for the Dirichlet case and $s = \frac{3}{4}$ for the Neumann case. These sums coincide with the numbers of lattice points under a curve. Note that the area under the curve under consideration is just the principal term $\frac{3}{4}$ in (0.6), (0.7), see Lemma 2.1 below. The methods of counting the number of lattice points under a curve are well developed in the analytic number theory (such as van der Corput Lemma and so on). In this way the authors of [10] obtained the formulas (0.6), (0.7) with a well controlled remainder, and the formulas (0.3), (0.4) for the disk for large but concrete $\Lambda$.

We also study the sums of type (0.9). A new point is the uniform bounds (instead of asymptotics) for such sums via the area under the curve (from above for $s = 1/4$ and from below for $s = 3/4$), see Theorems 3.3, 3.6, 6.4 and 6.6 below.
0.3 Remarks

1. There are different definitions of the counting function in the literature:

\[ N(\Lambda) = \# \{ k : \lambda_k < \Lambda \} \quad \text{or} \quad N(\Lambda) = \# \{ k : \lambda_k \leq \Lambda \}. \]

These two functions differ from each other in a countable set of points (the eigenvalues themselves). The choice of definition has no importance for the hypotheses (0.3), (0.4) because of continuity of the function in the RHS.

2. It is easy to see that the inequalities (0.1)–(0.4) are fulfilled in 1D.

3. The inequalities (0.1)–(0.4) are invariant under rescaling. So, we consider domains with a characteristic size 1 (unit ball and sectors of unit disk).

4. The initial formulation of the conjecture in the book [12] differs from the formulation given above. First, Pólya has formulated his conjecture and later proved it for tiling sets in [13] in 2D case only. It is more natural to formulate the conjectures in arbitrary dimension \( d \). Pólya’s proofs for tiling sets can be repeated literally in any dimension.

Second, Pólya conjectured the inequalities

\[ \lambda_k > \frac{4\pi k}{|\Omega|}, \quad \mu_k < \frac{4\pi k}{|\Omega|} \]

(in 2D). There are two distinctions here:

- the inequalities are strict, and this is stronger than (0.1), (0.2);
- the second inequality is for \( \mu_k \), and not for \( \mu_{k+1} \), and this assumption is weaker.

It seems to be more natural to formulate the conjectures as in (0.1), (0.2). Note also that in [13] Pólya proved just the inequalities

\[ \lambda_k \geq \frac{4\pi k}{|\Omega|} \geq \mu_{k+1}. \]

5. The constants in the inequalities (0.1)–(0.4) can not be improved as they coincide with the constants in the Weyl law

\[ N_{D,N}(\Omega, \Lambda) \sim \frac{\omega_d |\Omega|}{(2\pi)^d} \Lambda^{d/2}, \quad \Lambda \to +\infty, \]

see for example [1], [14].

6. The following inequalities are known to hold for all open sets \( \Omega \) and for all \( \Lambda \):

\[ N_D(\Omega, \Lambda) \leq \left( \frac{d+2}{d} \right)^{d/2} \frac{\omega_d |\Omega|}{(2\pi)^d} \Lambda^{d/2}, \]

see [11], and

\[ N_{N}(\Omega, \Lambda) \geq \frac{2}{d+2} \frac{\omega_d |\Omega|}{(2\pi)^d} \Lambda^{d/2}, \]

see [7]. These estimates are similar to (0.3), (0.4) but the constants are worse.

In [9] the following is proved.

**Theorem 0.4.** Let \( \Omega = \Omega_1 \times \Omega_2, \Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}, \Omega_1 \) and \( \Omega_2 \) are bounded. If \( d_1 \geq 2 \) and the Dirichlet Pólya conjecture holds for \( \Omega_1 \), then the same is true for \( \Omega \).

This theorem and the results of the present paper yield

**Corollary 0.5.** Let \( \Omega = \Omega_1 \times \Omega_2, \Omega_1 \) be a circular sector or a ball of arbitrary dimension \( d_1 \geq 2 \), and \( \Omega_2 \) be a bounded set. Then (0.3) holds.
0.4 Plan of the paper
In §1 we give an alternative proof of the estimate (0.8). In §2 we introduce a function $G$ which plays an important role in the sequel. In §3 we establish some estimates from above for some of type (0.9) with $s = 1/4$. In §4 and §5 we prove the Dirichlet Pólya conjecture for circular sectors, for the disk and for balls. In §6 we establish some estimates from below for some of type (0.9) with $s = 3/4$. Finally, in §7 and §8 we prove the Neumann Pólya conjecture for circular sectors, and for the disk for sufficiently large $\Lambda$.

1 Zeros of Bessel functions
If $\nu \to +\infty$ and the ratio $\frac{\nu}{\pi} > 1$ is fixed then the asymptotics for the Bessel function

$$J_\nu(x) \sim \left(\frac{2}{\pi \sqrt{x^2 - \nu^2}}\right)^{1/2} \cos \left(\sqrt{x^2 - \nu^2} - \nu \arccos \frac{\nu}{x} - \frac{\pi}{4}\right)$$

is known, see [17, §8.41]. This asymptotics hints that the number of zeros of $J_\nu$ can be estimated via the number of zeros of the function in the RHS. For $\nu \geq 0$ introduce the function

$$F_\nu(x) = \sqrt{x^2 - \nu^2} - \nu \arccos \frac{\nu}{x}, \quad x \in [\nu, +\infty).$$

We have

$$F_\nu(\nu) = 0, \quad F'_\nu(x) = \frac{\sqrt{x^2 - \nu^2}}{x},$$

so the function $F$ is strictly increasing on $[\nu, +\infty)$.

**Theorem 1.1.** Let $\nu \geq 0$. Denote by $a_k$ the zeros of the function

$$\sin \left(\frac{\pi}{4} + F_\nu(x)\right) = \cos \left(\frac{\pi}{4} - F_\nu(x)\right)$$

enumerated in the increasing order. Then

1) $j_{\nu,k} \geq a_k$ for all $k \in \mathbb{N}$, where $j_{\nu,k}$ is the $k$-th zero of $J_\nu$;

2) for $\Lambda \geq 0$ we have

$$\#\{k \in \mathbb{N} : j_{\nu,k} \leq \sqrt{\Lambda}\} \leq \left[\frac{1}{\pi} \left(\sqrt{\Lambda - \nu^2} - \nu \arccos \frac{\nu}{\sqrt{\Lambda}}\right) + \frac{1}{4}\right].$$

**Remark 1.2.** a) The claims 1) and 2) are equivalent due to the obvious identity

$$\#\{k \in \mathbb{N} : a_k \leq \sqrt{\Lambda}\} = \left[\frac{1}{\pi} \left(F_\nu(\sqrt{\Lambda}) + \frac{\pi}{4}\right)\right].$$

b) The inequality 2) is proved in [15, Prop. 2.20]. The proof is based on the study of the phase function $\theta_\nu(x) = \arctan(Y_\nu(x)/J_\nu(x))$ and its asymptotics, see also [5]. We suggest another proof of 1) using the known asymptotics of zeros $j_{\nu,k}$ only. Roughly speaking we apply the Sturm comparison theorem to two functions having common zero at infinity. This idea is borrowed from [4]. Such proof seems to be simpler.

c) For $\nu = 0$ the claim 1) means $j_{0,k} \geq \pi k - \pi/4$. This inequality was proved in [4].
Proof of Theorem 1.1. The function \( u(x) := \sqrt{x}J_\nu(x) \) satisfies the equation

\[
    u''(x) + \left(1 - \frac{\nu^2 - 1/4}{x^2}\right)u(x) = 0. \tag{1.1}
\]

Consider the function

\[
    v(x) = \frac{\sqrt{x}}{(x^2 - \nu^2)^{1/4}} \sin (F_\nu(x) + b)
\]

with some \( b \in (\pi/4, \pi/2) \). Then

\[
    v'(x) = -\frac{\nu^2}{2\sqrt{x}(x^2 - \nu^2)^{5/4}} \sin (F_\nu(x) + b) + \frac{(x^2 - \nu^2)^{1/4}}{\sqrt{x}} \cos (F_\nu(x) + b),
\]

\[
    v''(x) = \left(\frac{6\nu^2x^2 - \nu^4}{4x^{3/2}(x^2 - \nu^2)^{9/4}} - \frac{(x^2 - \nu^2)^3}{x^3}\right) \sin (F_\nu(x) + b).
\]

Therefore,

\[
    v''(x) + \left(1 - \frac{\nu^2}{x^2} - \frac{\nu^2(6x^2 - \nu^2)}{4x^2(x^2 - \nu^2)^2}\right)v(x) = 0 \quad \text{on} \quad (\nu, +\infty). \tag{1.2}
\]

Denote by \( v_k = v_k(b) \) the zeros of the function \( v \). By definition,

\[
    \sqrt{v_k^2 - \nu^2} - \nu \arccos \frac{\nu}{v_k} + b = \pi k, \quad k \in \mathbb{N}.
\]

Clearly,

\[
    \sqrt{v_k^2 - \nu^2} = v_k + O(v_k^{-1}), \quad \arccos \frac{\nu}{v_k} = \frac{\pi}{2} + O(v_k^{-1}), \quad k \to \infty.
\]

Therefore,

\[
    v_k(b) = \pi \left(k + \frac{\nu}{2}\right) - b + O(k^{-1}), \quad k \to \infty.
\]

On the other hand the asymptotics

\[
    j_{\nu,k} = \pi \left(k + \frac{\nu}{2} - \frac{1}{4}\right) + O(k^{-1}), \quad k \to \infty,
\]

is well known, see for example [17, §15.53]. As \( b > \pi/4 \) we have

\[
    j_{\nu,m} > v_m(b) \tag{1.3}
\]

starting from some number \( m \). The coefficient in front of \( u \) in (1.1) is greater than the coefficient in front of \( v \) in (1.2):

\[
1 - \frac{\nu^2 - 1/4}{x^2} > 1 - \frac{\nu^2}{x^2} > 1 - \frac{\nu^2}{x^2} - \frac{\nu^2(6x^2 - \nu^2)}{4x^2(x^2 - \nu^2)^2} \quad \forall x \in [\nu, +\infty).
\]

By Strum’s comparison theorem there is a zero of \( u \) between \( v_k(b) \) and \( v_{k+1}(b) \). So, if \( j_{\nu,k} \leq v_k(b) \) for some number \( k \), then \( j_{\nu,k+1} \leq v_{k+1}(b) \), and by induction \( j_{\nu,m} \leq v_m(b) \) for all \( m \geq k \) which contradicts (1.3). Therefore, (1.3) is fulfilled for all natural \( m \).

Finally, each function \( v_k \) is continuous in \( b \). Thus,

\[
    j_{\nu,m} \geq v_m(\pi/4) = a_m. \quad \blacksquare
\]
2 Function $G$

Fix $\Lambda > 0$. The function

$$G(t) := \frac{1}{\pi} \left( \sqrt{\Lambda - t^2} - t \arccos \frac{t}{\sqrt{\Lambda}} \right), \quad t \in [0, \sqrt{\Lambda}],$$

(2.1)

plays the crucial role in the sequel. Clearly,

$$G(0) = \frac{\sqrt{\Lambda}}{\pi}, \quad G(\sqrt{\Lambda}) = 0,$$

$$G'(t) = -\frac{1}{\pi} \arccos \frac{t}{\sqrt{\Lambda}}, \quad G'(0) = -\frac{1}{2}, \quad G'(\sqrt{\Lambda}) = 0,$$

$$G''(t) = \frac{1}{\pi \sqrt{\Lambda - t^2}} > 0 \quad \text{for all} \quad t \in [0, \sqrt{\Lambda}].$$

Thus, $G$ is a non-negative strictly decreasing convex function.

Lemma 2.1. Let $m \geq 0$. Then

$$\int_0^{\sqrt{\Lambda}} t^m G(t) \, dt = \frac{\Gamma \left( \frac{m+1}{2} \right) \Lambda^{\frac{m+2}{2}}}{4 \sqrt{\pi} (m+2) \Gamma \left( \frac{m+4}{2} \right)}.$$  

In particular,

$$\int_0^{\sqrt{\Lambda}} G(t) \, dt = \frac{\Lambda}{8}.$$  

Proof. Changing the variables $t = \sqrt{\Lambda} \cos \tau$, we obtain

$$\int_0^{\sqrt{\Lambda}} t^m G(t) \, dt = \frac{\Lambda^{\frac{m+2}{2}}}{\pi} \int_0^{\pi/2} (\cos \tau)^m (\sin \tau - \tau \cos \tau) \sin \tau \, d\tau.$$  

(2.2)

It is well known that

$$\int_0^{\pi/2} (\cos \tau)^\rho (\sin \tau)^\sigma \, d\tau = \frac{1}{2} B \left( \frac{\rho + 1}{2}, \frac{\sigma + 1}{2} \right) = \frac{\Gamma \left( \frac{\rho+1}{2} \right) \Gamma \left( \frac{\sigma+1}{2} \right)}{2 \Gamma \left( \frac{\rho+\sigma+2}{2} \right)}$$

for non-negative $\rho, \sigma$. Therefore,

$$\int_0^{\pi/2} (\cos \tau)^m (\sin \tau)^2 \, d\tau = \frac{\sqrt{\pi} \Gamma \left( \frac{m+1}{2} \right)}{4 \Gamma \left( \frac{m+4}{2} \right)},$$

$$\int_0^{\pi/2} \tau (\cos \tau)^{m+1} \sin \tau \, d\tau = -\frac{\tau (\cos \tau)^{m+2}}{m+2} \bigg|_0^{\pi/2} + \frac{1}{m+2} \int_0^{\pi/2} (\cos \tau)^{m+2} \, d\tau = \frac{\sqrt{\pi} \Gamma \left( \frac{m+3}{2} \right)}{2 (m+2) \Gamma \left( \frac{m+4}{2} \right)}.$$  

So,

$$\int_0^{\pi/2} (\cos \tau)^m (\sin \tau - \tau \cos \tau) \sin \tau \, d\tau = \frac{\sqrt{\pi} \Gamma \left( \frac{m+1}{2} \right)}{4 (m+2) \Gamma \left( \frac{m+4}{2} \right)}.$$  

Substituting this into (2.2) we get the result. ☐
3 Estimates of sums of type $\sum [g(m) + \frac{1}{4}]$

**Lemma 3.1.** Let $A, B \in \mathbb{Z}$, $A < B$. Let $g$ be a decreasing convex function on $[A-1/2, B-1/2]$, and $g(B-1) \geq 0$. Then

$$\int_C^{B-1/2} g(t) \, dt \geq 0 \quad \text{for any } C \in [A-1/2, B-3/2]. \quad (3.1)$$

The equality is possible only if either $g$ is identically zero, or $C = B - 3/2$, $g(B-1) = 0$ and $g$ is a linear function.

**Proof.** The function $g$ is convex, so

$$\int_{B-3/2}^{B-1/2} g(t) \, dt \geq g(B - 1) \geq 0.$$ 

Next, $g(t) \geq 0$ for $t \leq B - 3/2$, which implies (3.1). \qed

**Lemma 3.2.** Let $A, B \in \mathbb{Z}$, $A < B$. Let $g \in C^1[A - 1/2, B]$ be a decreasing convex function. Assume that

$$g'(t) \in \left[\frac{1}{2}, 0\right] \quad \text{for all } t. \quad (3.2)$$

Assume moreover that

$$n + 1 \geq g(A) \geq \ldots \geq g(B - 1) \geq n \geq g(B)$$

for some $n \in \mathbb{Z}$. Then

$$\sum_{m=A}^{B-1} \left[ g(m) + \frac{1}{4} \right] \leq \int_{A-1/2}^{B-1/2} g(t) \, dt. \quad (3.3)$$

The equality is possible only if $g$ is a linear function.

**Proof.** The claim does not change if we add an integer number to the function $g$. So, without loss of generality we can assume $n = 0$.

We denote

$$k := \# \{m : 3/4 \leq g(m) \leq 1\},$$

and consider three cases.

First case $k = 0$. The sum in the LHS of (3.3) is equal to zero, so (3.3) follows from (3.1).

Second case $k = 1$. Here

$$1 \geq g(A) \geq \frac{3}{4} > g(A + 1) \geq \ldots \geq g(B - 1) \geq 0 \geq g(B).$$

The sum in the LHS of (3.3) is equal to 1. The assumption (3.2) yields $g(A + 1) \geq 1/4$.

Furthermore, by convexity of $g$

$$\int_{A-1/2}^{A+3/2} g(t) \, dt \geq g(A) + g(A + 1) \geq 1.$$

If $B = A + 2$ then (3.3) is proved. If $B \geq A + 3$ then (3.3) follows from (3.1) with $C = A + 3/2$. 
Third case $k \geq 2$. Here

$$1 \geq g(A) \geq \ldots \geq g(A+k-1) \geq \frac{3}{4} > g(A+k) \geq \ldots \geq g(B-1) \geq 0 \geq g(B). \quad (3.4)$$

The sum in the LHS of (3.3) is equal to $k$. By convexity of the function $g$

$$g(A) + g(A+2k-2) \geq 2g(A+k-1) \geq \frac{3}{2} \Rightarrow g(A+2k-2) \geq \frac{1}{2} \Rightarrow B \geq A+2k-1. \quad (3.5)$$

Next,

$$\int_{A-1/2}^{A+2k-3/2} g(t) \, dt \geq (2k-1)g(A+k-1) \geq \frac{6k-3}{4} > k,$$

because $k \geq 2$. If $B = A + 2k - 1$ then (3.3) is proved. If $B \geq A + 2k$ then (3.3) follows from (3.1) with $C = A + 2k - 3/2$. ■

**Theorem 3.3.** Let $a, b \in \mathbb{Z}$, $a < b$. Let $g \in C^1[a-1/2, b]$ be a non-negative decreasing convex function. Assume that (3.2) holds true and that $g(b) = 0$. Then

$$\sum_{m=a}^{b} \left[ g(m) + \frac{1}{4} \right] \leq \int_{a-1/2}^{b} g(t) \, dt.$$

**Proof.** Let

$$N = \lfloor g(a) \rfloor + 1 \geq 1.$$ 

For $n = 0, 1, \ldots, N$, denote

$$m_n = \min \{ m \in \mathbb{Z} : g(m) \leq n \}.$$

In particular,

$$m_N = a, \quad m_0 \leq b \quad \text{and} \quad g(t) = 0 \text{ if } t \geq m_0.$$

Therefore,

$$\sum_{m=a}^{b} \left[ g(m) + \frac{1}{4} \right] = \sum_{n=0}^{N-1} \sum_{m=m_{n+1}}^{m_n} \left[ g(m) + \frac{1}{4} \right].$$

By virtue of the precedent Lemma

$$\sum_{m=m_{n+1}}^{m_n} \left[ g(m) + \frac{1}{4} \right] \leq \int_{m_{n+1}-1/2}^{m_n-1/2} g(t) \, dt.$$

Therefore,

$$\sum_{m=a}^{b} \left[ g(m) + \frac{1}{4} \right] \leq \int_{a-1/2}^{m_0-1/2} g(t) \, dt \leq \int_{a-1/2}^{b} g(t) \, dt,$$

because the function $g$ is non-negative. ■

**Lemma 3.4.** Let $A, B \in \mathbb{Z}$, $A < B$. Let $g \in C^1[A, B]$ be a decreasing convex function. Let (3.2) hold and assume that

$$n + 1 \geq g(A) \geq n + \frac{1}{8} \quad \text{and} \quad g(B-1) \geq n \geq g(B)$$
for some \( n \in \mathbb{Z} \). Then

\[
\frac{1}{2} \left[ g(A) + \frac{1}{4} \right] + \sum_{m=A+1}^{B-1} \left[ g(m) + \frac{1}{4} \right] \leq \int_A^{B-1/2} g(t) \, dt. \tag{3.6}
\]

If \( B = A+1 \) the sum in the LHS is absent. The equality is possible only if \( g \) is a linear function.

**Proof.** Again without loss of generality one can assume \( n = 0 \). Denote

\[
k := \# \{ m : 3/4 \leq g(m) \leq 1 \},
\]

and consider three cases.

First case \( k = 0 \). The LHS of (3.6) is equal to zero. The assumptions (3.2) and \( g(A) \geq 1/8 \) yield

\[
\int_A^{A+1/2} g(t) \, dt \geq \int_0^{1/2} \left( \frac{1}{8} - \frac{t}{2} \right) \, dt = 0.
\]

Now, (3.6) follows from (3.1).

Second case \( k = 1 \). The LHS of (3.6) is equal to \( 1/2 \). The assumptions (3.2) and \( g(A) \geq 3/4 \) yield \( B \geq A + 2 \) and

\[
\int_A^{A+3/2} g(t) \, dt \geq \int_0^{3/2} \left( \frac{3}{4} - \frac{t}{2} \right) \, dt = \frac{9}{16} > \frac{1}{2}.
\]

Now, (3.6) follows from (3.1).

Third case \( k \geq 2 \). Here (3.4) is fulfilled. The LHS of (3.6) is equal to \( (k - 1/2) \). Convexity of \( g \) yields (3.5). Furthermore,

\[
\int_A^{A+2k-2} g(t) \, dt \geq (2k-2)g(A+k-1) \geq \frac{3k-3}{2} \geq k - \frac{1}{2},
\]

because \( k \geq 2 \). Again, (3.6) follows from (3.1) with \( C = A + 2k - 2 \). \( \blacksquare \)

**Lemma 3.5.** Let \( A, B \in \mathbb{Z}, A < B \). Let \( g \in C^1[A,B] \) be a decreasing convex function. Let (3.2) hold and assume that

\[
n + 2 \geq g(A) \geq n + 1 \quad \text{and} \quad g(B-1) \geq n \geq g(B)
\]

for some \( n \in \mathbb{Z} \). Then the inequality (3.6) holds true. If \( g \) is not linear then the inequality is strict.

**Proof.** If \( g(A) \geq n + 9/8 \) the claim follows from Lemma 3.2 and Lemma 3.4.

Let \( n + 9/8 \geq g(A) \geq n + 1 \). Again one can assume \( n = 0 \). Denote

\[
k := \# \{ m : 3/4 \leq g(m) \leq 9/8 \},
\]

and consider two cases.

First case \( k = 1 \). The LHS of (3.6) is equal to \( 1/2 \). The assumptions (3.2) and \( g(A) \geq 1 \) yield \( B \geq A + 2 \) and

\[
\int_A^{A+3/2} g(t) \, dt \geq \int_0^{3/2} \left( 1 - \frac{t}{2} \right) \, dt = \frac{15}{16} > \frac{1}{2},
\]

and consider two cases.

First case \( k = 0 \). The LHS of (3.6) is equal to zero. The assumptions (3.2) and \( g(A) \geq 1/8 \) yield

\[
\int_A^{A+1/2} g(t) \, dt \geq \int_0^{1/2} \left( \frac{1}{8} - \frac{t}{2} \right) \, dt = 0.
\]

Now, (3.6) follows from (3.1).

Second case \( k = 1 \). The LHS of (3.6) is equal to \( 1/2 \). The assumptions (3.2) and \( g(A) \geq 3/4 \) yield \( B \geq A + 2 \) and

\[
\int_A^{A+3/2} g(t) \, dt \geq \int_0^{3/2} \left( \frac{3}{4} - \frac{t}{2} \right) \, dt = \frac{9}{16} > \frac{1}{2}.
\]

Now, (3.6) follows from (3.1).

Third case \( k \geq 2 \). Here (3.4) is fulfilled. The LHS of (3.6) is equal to \( (k - 1/2) \). Convexity of \( g \) yields (3.5). Furthermore,

\[
\int_A^{A+2k-2} g(t) \, dt \geq (2k-2)g(A+k-1) \geq \frac{3k-3}{2} \geq k - \frac{1}{2},
\]

because \( k \geq 2 \). Again, (3.6) follows from (3.1) with \( C = A + 2k - 2 \). \( \blacksquare \)

**Lemma 3.5.** Let \( A, B \in \mathbb{Z}, A < B \). Let \( g \in C^1[A,B] \) be a decreasing convex function. Let (3.2) hold and assume that

\[
n + 2 \geq g(A) \geq n + 1 \quad \text{and} \quad g(B-1) \geq n \geq g(B)
\]

for some \( n \in \mathbb{Z} \). Then the inequality (3.6) holds true. If \( g \) is not linear then the inequality is strict.

**Proof.** If \( g(A) \geq n + 9/8 \) the claim follows from Lemma 3.2 and Lemma 3.4.

Let \( n + 9/8 \geq g(A) \geq n + 1 \). Again one can assume \( n = 0 \). Denote

\[
k := \# \{ m : 3/4 \leq g(m) \leq 9/8 \},
\]

and consider two cases.

First case \( k = 0 \). The LHS of (3.6) is equal to zero. The assumptions (3.2) and \( g(A) \geq 1/8 \) yield

\[
\int_A^{A+1/2} g(t) \, dt \geq \int_0^{1/2} \left( \frac{1}{8} - \frac{t}{2} \right) \, dt = 0.
\]

Now, (3.6) follows from (3.1).

Second case \( k = 1 \). The LHS of (3.6) is equal to \( 1/2 \). The assumptions (3.2) and \( g(A) \geq 3/4 \) yield \( B \geq A + 2 \) and

\[
\int_A^{A+3/2} g(t) \, dt \geq \int_0^{3/2} \left( \frac{3}{4} - \frac{t}{2} \right) \, dt = \frac{9}{16} > \frac{1}{2}.
\]

Now, (3.6) follows from (3.1).

Third case \( k \geq 2 \). Here (3.4) is fulfilled. The LHS of (3.6) is equal to \( (k - 1/2) \). Convexity of \( g \) yields (3.5). Furthermore,

\[
\int_A^{A+2k-2} g(t) \, dt \geq (2k-2)g(A+k-1) \geq \frac{3k-3}{2} \geq k - \frac{1}{2},
\]

because \( k \geq 2 \). Again, (3.6) follows from (3.1) with \( C = A + 2k - 2 \). \( \blacksquare \)
and (3.6) follows from (3.1).

Second case $k \geq 2$. The LHS of (3.6) is equal to $(k - 1/2)$. We have

\[ g(A) \leq \frac{9}{8}, \quad g(A + k - 1) \geq \frac{3}{4}. \]

By convexity, $g(A + 2k - 2) \geq 3/8$ and $B \geq A + 2k - 1$. Furthermore,

\[ \int_{A}^{A+2k-2} g(t) \, dt \geq (2k-2)g(A + k - 1) \geq \frac{3k-3}{2} \geq k - \frac{1}{2}, \]

because $k \geq 2$. Again, (3.6) follows from (3.1).

**Theorem 3.6.** Let $a, b \in \mathbb{Z}, a < b$. Let $g \in C^{1}[a,b]$ be a non-negative decreasing convex function. Assume that (3.2) holds true and that $g(b) = 0$. Then

\[ \frac{1}{2} \left[ g(a) + \frac{1}{4} \right] + \sum_{m=a+1}^{b} \left[ g(m) + \frac{1}{4} \right] \leq \int_{a}^{b} g(t) \, dt. \]

The equality is possible only if $g$ is a linear function.

**Proof.** If $g(a) \leq 1/8$ then LHS vanishes, and RHS is non-negative. If $1/8 \leq g(a) \leq 1$ the claim follows from Lemma 3.4.

Let $g(a) \geq 1$,

\[ N = \lfloor g(a) \rfloor + 1 \geq 2. \]

For $n = 0, 1, \ldots, N$, denote

\[ m_n = \min \{ m \in \mathbb{Z} : g(m) \leq n \}. \]

In particular,

\[ m_N = a, \quad m_0 \leq b \quad \text{and} \quad g(t) = 0 \text{ if } t \geq m_0. \]

Therefore,

\[ \frac{1}{2} \left[ g(a) + \frac{1}{4} \right] + \sum_{m=a+1}^{b} \left[ g(m) + \frac{1}{4} \right] = \frac{1}{2} \left[ g(a) + \frac{1}{4} \right] + \sum_{m=m_N}^{m_{N-1}} \left[ g(m) + \frac{1}{4} \right] + \sum_{n=0}^{N-3} \sum_{m=m_n+1}^{m_{n+1}} \left[ g(m) + \frac{1}{4} \right], \]

where the last sum is absent if $N = 2$. We have the inequalities

\[ \frac{1}{2} \left[ g(a) + \frac{1}{4} \right] + \sum_{m=m_N}^{m_{N-1}} \left[ g(m) + \frac{1}{4} \right] \leq \int_{a}^{m_{N-2}} g(t) \, dt \]

due to Lemma 3.5, and

\[ \sum_{m=m_{n+1}}^{m_n} \left[ g(m) + \frac{1}{4} \right] \leq \int_{m_{n+1}}^{m_{n}} g(t) \, dt \]
due to Lemma 3.2. Therefore,

$$\frac{1}{2} \left[ g(a) + \frac{1}{4} \right] + \sum_{m=a+1}^{b} \left[ g(m) + \frac{1}{4} \right] \leq \int_{a}^{\sqrt{N \pi}} d(t) \, dt + \sum_{n=0}^{N-3} \int_{m_{n}+1/2}^{m_{n}-1/2} g(t) \, dt$$

$$= \int_{a}^{\sqrt{N \pi}} g(t) \, dt \leq \int_{a}^{b} g(t) \, dt$$

because $g(t)$ is non-negative.

Remark 3.7. One can see from the proofs that in Lemma 3.5 the inequality is always strict, and in Theorem 3.6 the equality is possible only if $g \equiv 0$.

4 Dirichlet problem in a circular sector and in the disk

First, we prove Pólya’s conjecture for the sectors with angle greater than $\pi$.

Lemma 4.1. Let $\Omega_{\alpha}$ be the sector defined in (0.5) with $\pi \leq \alpha \leq 2 \pi$. Then

$$N_{D}(\Omega_{\alpha}, \Lambda) < \frac{\alpha \Lambda}{8 \pi}.$$

Proof. It is well known that the eigenfunctions of the Laplace operator of the Dirichlet problem in $\Omega_{\alpha}$ are

$$J_{m \alpha} \left( j_{m \alpha, k} r \right) \sin \frac{m \pi \varphi}{\alpha}; \quad m, k \in \mathbb{N}.$$

Here $(r, \varphi)$ are polar coordinates, $J_{\nu}$ are Bessel functions, $j_{\nu, k}$ is the $k$-th zero of the function $J_{\nu}$. The corresponding eigenvalues are

$$j_{m \alpha, k}^{2}, \quad m, k \in \mathbb{N}.$$

Thus,

$$N_{D}(\Omega_{\alpha}, \Lambda) = \sum_{m=1}^{\infty} \# \left\{ k \in \mathbb{N} : j_{m \alpha, k} \leq \sqrt{\Lambda} \right\}.$$

As $j_{\nu, 1} > \nu$, see [17, §15.3], all terms in the RHS with $m \geq \alpha \sqrt{\Lambda} / \pi$ vanish. Due to Theorem 1.1

$$N_{D}(\Omega_{\alpha}, \Lambda) \leq \sum_{m=1}^{\left\lfloor \frac{\alpha \sqrt{\Lambda}}{\pi} \right\rfloor} \left[ \frac{1}{\pi} \left( \sqrt{\Lambda - \frac{m^{2} \pi^{2}}{\alpha^{2}}} - \frac{m \pi}{\alpha} \arccos \frac{m \pi}{\alpha \sqrt{\Lambda}} \right) + \frac{1}{4} \right].$$

(4.1)

where the function $G$ is defined in (2.1). Introduce the function

$$g(t) = G \left( \frac{\pi t}{\alpha} \right) \quad \text{on} \quad \left[ 0, \frac{\alpha \sqrt{\Lambda}}{\pi} \right]$$

and extend it by zero on $\left[ \frac{\alpha \sqrt{\Lambda}}{\pi}, b \right]$, where $b := \left[ \frac{\alpha \sqrt{\Lambda}}{\pi} \right] + 1$. Then $g$ is the non-negative decreasing convex function. Moreover,

$$g'(t) = \frac{\pi}{\alpha} G' \left( \frac{\pi t}{\alpha} \right) = -\frac{1}{\alpha} \arccos \frac{\pi t}{\alpha \sqrt{\Lambda}} \in \left[ -\frac{\pi}{2 \alpha}, 0 \right] \quad \text{on} \quad \left[ 0, \frac{\alpha \sqrt{\Lambda}}{\pi} \right].$$
The relation (3.2) is fulfilled because \( \alpha \geq \pi \) by assumption. Thus, we can apply Theorem 3.3:

\[
\sum_{m=1}^{[\alpha \sqrt{\pi}]} \left( g(m) + \frac{1}{4} \right) \leq \int_{1/2}^{b} g(t) \, dt < \int_{0}^{\alpha \sqrt{\pi}} g(t) \, dt. \tag{4.2}
\]

Next,

\[
\int_{0}^{\alpha \sqrt{\pi}} g(t) \, dt = \frac{\alpha}{\pi} \int_{0}^{\sqrt{\Lambda}} G(s) \, ds = \frac{\alpha \Lambda}{8\pi} \tag{4.3}
\]

by virtue of Lemma 2.1. Formulas (4.1), (4.2) and (4.3) imply the claim. 

The following fact is proved in [10, Theorem 1.7].

**Lemma 4.2.** Let Pólya’s conjecture for the Dirichlet (resp. Neumann) problem in \( \Omega \subset \mathbb{R}^d \) hold. Let \( \Omega' \) be a domain which tiles \( \Omega \). Then Pólya’s conjecture for the Dirichlet (resp. Neumann) problem in \( \Omega' \) holds as well.

**Lemma 4.3.** If Pólya’s conjecture holds for Dirichlet or Neumann problem for sectors \( \Omega_\alpha \) for all \( \alpha \in [\pi, 2\pi] \), then the same is true for all \( \alpha \in (0, \pi) \).

**Proof.** Let \( 0 < \alpha < \pi \). Let \( N \) be a natural number such that

\[
\pi < N\alpha \leq \pi + \alpha < 2\pi.
\]

Clearly, the sector \( \Omega_\alpha \) tiles the sector \( \Omega_{N\alpha} \). Now, the claim follows from the precedent Lemma. 

**Proof of Theorem 0.1.** Follows from Lemma 4.1 and Lemma 4.3. 

**Theorem 4.4.** Let \( B_1 \subset \mathbb{R}^2 \) be the unit disk, \( \Lambda > 0 \). Then

\[
N_D(B_1, \Lambda) < \frac{\Lambda}{4}.
\]

**Proof.** The eigenfunctions of the Laplace operator of the Dirichlet problem in the disk are

\[
J_{|m|} (j_{|m|,k} r) e^{im\phi}, \quad m \in \mathbb{Z}, \ k \in \mathbb{N}.
\]

The corresponding eigenvalues are

\[
j_{|m|,k}^2, \quad m \in \mathbb{Z}, \ k \in \mathbb{N},
\]

and therefore

\[
N_D(B_1, \Lambda) = \# \left\{ k \in \mathbb{N} : j_{0,k} \leq \sqrt{\Lambda} \right\} + 2 \sum_{m=1}^{[\sqrt{\Lambda}]} \# \left\{ k \in \mathbb{N} : j_{m,k} \leq \sqrt{\Lambda} \right\}.
\]

Due to Theorem 1.1

\[
N_D(B_1, \Lambda) \leq \left[ G(0) + \frac{1}{4} \right] + 2 \sum_{m=1}^{[\sqrt{\Lambda}]} \left[ G(m) + \frac{1}{4} \right],
\]

where the function \( G \) is defined in (2.1). Put \( b = \left[ \sqrt{\Lambda} \right] + 1 \), and extend the function \( G \) by zero to \([\sqrt{\Lambda}, b]\). This function satisfies all conditions of Theorem 3.6. Now, Theorem 3.6 and Lemma 2.1 imply

\[
N_D(B_1, \Lambda) < 2 \int_{0}^{\sqrt{\Lambda}} G(t) \, dt = \frac{\Lambda}{4}. \quad \blacksquare
\]

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5 Dirichlet problem in a ball

5.1 Auxiliary facts

Lemma 5.1. Let

\[ n \in \mathbb{N}, \ n \geq 2, \quad a_1, \ldots, a_{n-1}, t \geq 0. \]

Then

\[ na_1 \ldots a_{n-1}t \leq a_1 \ldots a_{n-1}(a_1 + \cdots + a_{n-1}) + t^n. \]

Proof. The well-known inequality

\[ (b_1 \ldots b_n)^{1/n} \leq \frac{b_1 + \cdots + b_n}{n}, \quad b_j \geq 0, \]

with

\[ b_j = (a_1 \ldots a_{n-1})a_j \quad \text{if} \quad j < n, \quad b_n = t^n \]

gives the claim. ■

Let \( d \geq 3. \) Introduce the piecewise constant function \( f \) on \([0, \infty),\)

\[ f(t) = \left( \left\lfloor \frac{t + \frac{d}{2} - 1}{d - 2} \right\rfloor \right). \tag{5.1} \]

Here we used the notation

\[ \begin{pmatrix} n \\ k \end{pmatrix} = \begin{cases} \frac{n!}{k!(n-k)!}, & \text{if} \ n \geq k, \\ 0, & \text{if} \ n < k. \end{cases} \]

Lemma 5.2. We have

\[ \int_0^t f(s) \, ds \leq \frac{t^{d-1}}{(d-1)!} \quad \text{for all} \ t \in [0, \infty). \]

Proof. If \( t \leq \frac{d}{2} - 1 \) then \( f(s) = 0 \) for \( s < t, \) and there is nothing to prove. Let

\[ m + \frac{d}{2} - 1 \leq t < m + \frac{d}{2}, \quad m \in \mathbb{N}_0. \]

Then

\[ \int_0^t f(s) \, ds = \sum_{k=0}^{m-1} \int_{k+\frac{d}{2}-1}^{k+\frac{d}{2}} f(s) \, ds + \int_{m+\frac{d}{2}-1}^t f(s) \, ds \]
\[ = \sum_{k=0}^{m-1} \left( \frac{k + d - 2}{d - 2} \right) + \left( t - m - \frac{d}{2} + 1 \right) \left( \frac{m + d - 2}{d - 2} \right), \]

the first sum is absent here if \( m = 0. \) We will use the well-known properties

\[ \begin{pmatrix} j \\ l \end{pmatrix} = \begin{pmatrix} j + 1 \\ l + 1 \end{pmatrix} - \begin{pmatrix} j \\ l + 1 \end{pmatrix}, \quad j \geq l, \tag{5.2} \]
\[ \sum_{j=l}^{r} \begin{pmatrix} j \\ l \end{pmatrix} = \begin{pmatrix} r + 1 \\ l + 1 \end{pmatrix}, \quad r \geq l, \tag{5.3} \]
the second is the direct corollary of the first one. Therefore,
\[
\int_0^t f(s) \, ds = \left( \frac{m + d - 2}{d - 1} \right) + \left( t - m - \frac{d}{2} + 1 \right) \left( \frac{m + d - 2}{d - 2} \right)
= \frac{(m + d - 2)!}{(d - 1)! m!} \left( m + \left( t - m - \frac{d}{2} + 1 \right) (d - 1) \right)
= \frac{(m + 1) \ldots (m + d - 2)}{(d - 1)!} \left( (d - 1) t - (d - 2) m - \frac{(d - 1)(d - 2)}{2} \right).
\] (5.4)

Lemma 5.1 with \( n = d - 1, \ a_j = m + j \) implies
\[
(d - 1)(m + 1) \ldots (m + d - 2) t \leq (m + 1) \ldots (m + d - 2) \left( (d - 2)m + \frac{(d - 1)(d - 2)}{2} \right) + t^{d-1}.
\]

Now, (5.4) yields
\[
\int_0^t f(s) \, ds \leq \frac{t^{d-1}}{(d - 1)!}. \quad \blacksquare
\]

**Lemma 5.3.** Let \( d \geq 3, \ \Lambda > 0 \). Let the functions \( f \) and \( G \) be defined by (5.1) and (2.1). Then
\[
\int_0^{\sqrt{\Lambda}} f(t)G(t) \, dt \leq \frac{1}{(d - 2)!} \int_0^{\sqrt{\Lambda}} t^{d-2}G(t) \, dt.
\]

**Remark 5.4.** If \( d = 2 \) then \( f(t) \equiv 1 \), and Lemma 5.2 and Lemma 5.3 are true, the inequalities become obvious equalities. We will not use this fact.

**Proof.** Introduce the function
\[
F(t) = \int_0^t \left( \frac{s^{d-2}}{(d - 2)!} - f(s) \right) \, ds, \quad t \in [0, \infty).
\]

By the precedent Lemma \( F(t) \geq 0 \) for all \( t \). Integrating by parts we obtain
\[
\int_0^{\sqrt{\Lambda}} \left( \frac{t^{d-2}}{(d - 2)!} - f(t) \right) G(t) \, dt = F(t)G(t)|_{0}^{\sqrt{\Lambda}} - \int_0^{\sqrt{\Lambda}} F(t)G'(t) \, dt.
\]

Here the first term in the RHS vanishes due to \( F(0) = 0, \ G(\sqrt{\Lambda}) = 0 \), and the last integral is non-positive due to \( G'(t) \leq 0 \) for all \( t \). Thus,
\[
\int_0^{\sqrt{\Lambda}} \left( \frac{t^{d-2}}{(d - 2)!} - f(t) \right) G(t) \, dt \geq 0. \quad \blacksquare
\]

### 5.2 Dirichlet Pólya’s conjecture for a ball

Let \( d \geq 3 \). The eigenfunctions of the Laplace operator of the Dirichlet problem in the unit ball in \( \mathbb{R}^d \) are
\[
r^{1-\frac{d}{2}} j_{m+\frac{d}{2}-1} \left( j_{m+\frac{d}{2}-1, k} r \right) Y_m(\omega), \quad r \in [0, 1), \ \omega \in S^{d-1}.
\]

Here \((r, \omega)\) are spherical coordinates, \( Y_m \) are spherical harmonics, id est eigenfunctions of the Laplace-Beltrami operator on the unit sphere \( S^{d-1} \),
\[
-\Delta_{S^{d-1}} Y_m = m(m + d - 2) Y_m, \quad m \in \mathbb{N}_0.
\]
The dimension of the corresponding eigenspace is
\[ \kappa_m := \binom{m + d - 1}{d - 1} - \binom{m + d - 3}{d - 1}. \]
So, the eigenvalues of the Dirichlet Laplace operator in the ball are
\[ j_{m+d-1,k}^2, \quad m \in \mathbb{N}_0, \ k \in \mathbb{N}, \]
with multiplicity \( \kappa_m \). Thus,
\[ N_D(B_1, \Lambda) = \sum_{m=0}^{\infty} \kappa_m \# \{ k \in \mathbb{N} : j_{m+d-1,k} \leq \sqrt{\Lambda} \}. \]
The terms in the RHS with \( m + \frac{d}{2} - 1 \geq \sqrt{\Lambda} \) vanish. Theorem 1.1 yields
\[ N_D(B_1, \Lambda) \leq \sum_{m=0}^{\left\lfloor \sqrt{\Lambda} - \frac{d}{2} \right\rfloor} \kappa_m \left[ G \left( m + \frac{d}{2} - 1 \right) + \frac{1}{4} \right]. \quad (5.5) \]

**Lemma 5.5.** The identity
\[ \sum_{m=0}^{\left\lfloor \sqrt{\Lambda} - \frac{d}{2} \right\rfloor} \kappa_m \left[ G \left( m + \frac{d}{2} - 1 \right) + \frac{1}{4} \right] = \sum_{m=0}^{\left\lfloor \sqrt{\Lambda} - \frac{d}{2} \right\rfloor} \left( m + d - 3 \right) \times \]
\[ \times \left[ G \left( m + \frac{d}{2} - 1 \right) + \frac{1}{4} \right] + 2 \sum_{n=0}^{\left\lfloor \sqrt{\Lambda} - \frac{d}{2} \right\rfloor} \left[ G \left( m + \frac{d}{2} + n \right) + \frac{1}{4} \right]. \]
holds true.

**Proof.** The factor in front of \( G \left( k + \frac{d}{2} - 1 \right) + \frac{1}{4} \) in the RHS is equal to
\[ \left( k + d - 3 \right) + 2 \sum_{m=0}^{k-1} \left( m + d - 3 \right) = \sum_{m=0}^{k-1} \left( m + d - 3 \right) + \sum_{m=0}^{k-1} \left( m + d - 3 \right) \]
\[ = \left( k + d - 2 \right) + \left( k + d - 3 \right), \]
where we used (5.3). Due to (5.2)
\[ \left( k + d - 2 \right) + \left( k + d - 3 \right) = \left( k + d - 1 \right) - \left( k + d - 3 \right) = \kappa_k \]
by definition of \( \kappa_k \). \( \blacksquare \)

**Proof of Theorem 0.3.** Formula (5.5) and Lemma 5.5 imply the estimate
\[ N_D(B_1, \Lambda) \leq \sum_{m=0}^{\left\lfloor \sqrt{\Lambda} - \frac{d}{2} \right\rfloor} \left( m + d - 3 \right) \times \]
\[ \times \left[ G \left( m + \frac{d}{2} - 1 \right) + \frac{1}{4} \right] + 2 \sum_{n=0}^{\left\lfloor \sqrt{\Lambda} - \frac{d}{2} \right\rfloor} \left[ G \left( m + \frac{d}{2} + n \right) + \frac{1}{4} \right]. \]
By virtue of Theorem 3.6 with $a = m + \frac{d}{2} - 1$, $b = \left[ \sqrt{\Lambda} \right] + 1$, and $g = G$ extended by zero to $[\sqrt{\Lambda}, b]$, we have

$$
\left[ G \left( m + \frac{d}{2} - 1 \right) + \frac{1}{4} \right] + 2 \sum_{n=0}^{[\sqrt{\Lambda} - m - \frac{4}{d}]} \left[ G \left( m + \frac{d}{2} + n \right) + \frac{1}{4} \right] < 2 \int_{m+\frac{d}{2}-1}^{\sqrt{\Lambda}} G(t) \, dt.
$$

Therefore,

$$
N_D(B_1, \Lambda) < 2 \sum_{m=0}^{[\sqrt{\Lambda} - \frac{4}{d} + 1]} \left( m + d - 3 \atop d - 3 \right) \int_{m+\frac{d}{2}-1}^{\sqrt{\Lambda}} G(t) \, dt = 2 \int_{0}^{\sqrt{\Lambda}} f(t) G(t) \, dt.
$$

Here $f(t) = 0$ if $t < \frac{d}{2} - 1$; if $m + \frac{d}{2} - 1 \leq t < m + \frac{d}{2}$ then

$$
f(t) = \sum_{k=0}^{m} \left( k + d - 3 \atop d - 3 \right) = \left( m + d - 2 \atop d - 2 \right)
$$
due to (5.3). Thus, this function $f$ coincides with the function defined in (5.1). Using Lemma 5.3 we get

$$
N_D(B_1, \Lambda) < \frac{2}{(d-2)!} \int_{0}^{\sqrt{\Lambda}} t^{d-2} G(t) \, dt. \tag{5.6}
$$

By virtue of Lemma 2.1

$$
\frac{2}{(d-2)!} \int_{0}^{\sqrt{\Lambda}} t^{d-2} G(t) \, dt = \frac{\Gamma \left( \frac{d-1}{2} \right) \Lambda^{d/2}}{2d \sqrt{\pi} (d-2)! \Gamma \left( \frac{d+2}{2} \right)} \tag{5.7}
$$

The duplication formula

$$
\sqrt{\pi} (d-2)! = \sqrt{\pi} \Gamma(d-1) = 2^{d-2} \Gamma \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d}{2} \right)
$$

implies

$$
\frac{\Gamma \left( \frac{d-1}{2} \right) \Lambda^{d/2}}{2d \sqrt{\pi} (d-2)! \Gamma \left( \frac{d+2}{2} \right)} = \frac{\Lambda^{d/2}}{2^d \Gamma \left( \frac{d+2}{2} \right)^2}. \tag{5.8}
$$

Now, (5.6), (5.7) and (5.8) imply

$$
N_D(B_1, \Lambda) < \frac{\Lambda^{d/2}}{2^d \Gamma \left( \frac{d+2}{2} \right)^2}. \tag*{\blacksquare}
$$

Theorem 0.3 and Lemma 4.2 imply

**Corollary 5.6.** The Dirichlet Pólya conjecture holds for and domain which tiles a ball.
6 Estimates of sums of type $\sum[g(m) + \frac{3}{4}]$

Lemma 6.1. Let $A, B \in \mathbb{Z}$, $A < B$. Let $g$ be a convex function on $[A, B]$. Then

$$
\sum_{m=A}^{B-1} g(m) \leq \frac{B - A + 1}{2} g(A) + \frac{B - A - 1}{2} g(B).
$$

Proof. We have

$$
g(A + k) \leq \left(1 - \frac{k}{B - A}\right) g(A) + \frac{k}{B - A} g(B), \quad 0 \leq k \leq B - A,
$$

by convexity of $g$. Adding these inequalities for $k = 0, \ldots, B - A - 1$, we obtain the claim. $$
\square
$$

Lemma 6.2. Let $a, b \in \mathbb{Z}$, $a < b$. Let $g$ be a non-negative convex function on $[a, b]$. Let $g(a) \geq \frac{1}{4}$, $g(b) = 0$. Denote

$$
m_0 := \min \left\{ m \in \mathbb{Z} : g(m) < \frac{1}{4} \right\}, \quad b_0 := \min \{ t \in \mathbb{R} : g(t) = 0 \}. \quad (6.1)
$$

If $m_0 \geq b_0$ then

$$
\frac{1}{2} g(a) + \sum_{m=a+1}^{m_0-1} g(m) \geq \int_a^{b_0} g(t) \, dt. \quad (6.2)
$$

If $m_0 < b_0$ then

$$
\frac{1}{2} g(a) + \sum_{m=a+1}^{m_0-1} g(m) \geq \int_a^{b_0} g(t) \, dt - \frac{1}{8} (b_0 - m_0 + 1). \quad (6.3)
$$

If $m_0 = a + 1$ the sums in the LHS are absent.

Proof. By convexity of $g$

$$
\frac{g(c) + g(c + 1)}{2} \geq \int_c^{c+1} g(t) \, dt.
$$

Therefore,

$$
\frac{1}{2} g(a) + \sum_{m=a+1}^{m_0-1} g(m) + \frac{1}{2} g(m_0) \geq \int_a^{m_0} g(t) \, dt.
$$

If $m_0 \geq b_0$ then $g(m_0) = 0$ and (6.2) is proved.

Assume that $m_0 < b_0$. By definition $g(m_0) < 1/4$, so

$$
\frac{1}{2} g(a) + \sum_{m=a+1}^{m_0-1} g(m) \geq \int_a^{m_0} g(t) \, dt - \frac{1}{8}. \quad (6.4)
$$

Finally, again by convexity of $g$

$$
\int_{m_0}^{b_0} g(t) \, dt \leq \frac{1}{2} g(m_0)(b_0 - m_0) \leq \frac{b_0 - m_0}{8}. \quad (6.5)
$$

The inequalities (6.4) and (6.5) imply (6.3). $$
\square$$
Lemma 6.3. Let $A, B \in \mathbb{Z}$, $A < B$. Let $g \in C^1[A, B]$ be a decreasing convex function. Assume that
\[ n + \frac{1}{4} > g(A) \geq \ldots \geq g(B-1) \geq n - \frac{3}{4} \geq g(B) \] (6.6)
for some $n \in \mathbb{Z}$. If (3.2) is satisfied then
\[ \sum_{m=A}^{B-1} \left[ g(m) + \frac{3}{4} \right] \geq \sum_{m=A}^{B-1} g(m). \] (6.7)

If
\[ g'(t) \in \left[-\frac{1}{3}, 0\right] \quad \text{for all } t, \] (6.8)
then
\[ \sum_{m=A}^{B-1} \left[ g(m) + \frac{3}{4} \right] \geq \frac{1}{4} + \sum_{m=A}^{B-1} g(m). \] (6.9)

Proof. By assumption,
\[ \left[ g(m) + \frac{3}{4} \right] = n \quad \text{for } m = A, \ldots, B-1. \]

If $B = A + 1$ then the LHS equals to $n$. Due to (3.2)
\[ g(A) \leq g(B) + \frac{1}{2} \leq n - \frac{1}{4}. \]

So, (6.7) and (6.9) are fulfilled in this case.

Let $B \geq A + 2$. The LHS equals to $(B - A)n$. Estimate the sum in the RHS by Lemma 6.1:
\[ \sum_{m=A}^{B-1} g(m) \leq \frac{B - A + 1}{2} \left( n + \frac{1}{4} \right) + \frac{B - A - 1}{2} \left( n - \frac{3}{4} \right) = (B - A) n - \frac{1}{4} (B - A - 2). \]

This implies (6.7). Moreover, if $B \geq A + 3$ then (6.9) holds true.

It remains to consider the case where $B = A + 2$ and (6.8) is fulfilled. In this case
\[ g(A + 1) = g(B - 1) \leq g(B) + \frac{1}{3} \leq n - \frac{5}{12}, \quad g(A) \leq g(A + 1) + \frac{1}{3} \leq n - \frac{1}{12}. \]

Therefore, the sum in the RHS
\[ g(A) + g(A + 1) \leq 2n - \frac{1}{2} < 2n - \frac{1}{4}. \]

\[ \square \]

Theorem 6.4. Let $a, b \in \mathbb{Z}$, $a < b$. Let $g \in C^1[a, b]$ be a non-negative decreasing convex function. Let $g(a) \geq 1/4$, $g(b) = 0$. Let $m_0, b_0$ be defined in (6.1).

a) Let (3.2) be satisfied. If $m_0 \geq b_0$ then
\[ \sum_{m=a}^{b} \left[ g(m) + \frac{3}{4} \right] \geq \frac{1}{2} g(a) + \int_{a}^{b_0} g(t) \, dt. \] (6.10)
If \( m_0 < b_0 \) then
\[
\sum_{m=a}^{b} \left[ g(m) + \frac{3}{4} \right] \geq \frac{1}{2} g(a) + \int_{a}^{b_0} g(t) \, dt - \frac{1}{8} (b_0 - m_0 + 1).
\] (6.11)

b) Let (6.8) be satisfied. If \( m_0 \geq b_0 \) then
\[
\sum_{m=a}^{b} \left[ g(m) + \frac{3}{4} \right] \geq \frac{3}{4} g(a) + \int_{a}^{b_0} g(t) \, dt - \frac{1}{16}.
\] (6.12)

If \( m_0 < b_0 \) then
\[
\sum_{m=a}^{b} \left[ g(m) + \frac{3}{4} \right] \geq \frac{3}{4} g(a) + \int_{a}^{b_0} g(t) \, dt - \frac{1}{16} (2b_0 - 2m_0 + 3).
\] (6.13)

Proof. Let
\[
N = \left[ g(a) + \frac{3}{4} \right] \geq 1.
\] (6.14)

For \( n = 1, \ldots, N \) denote
\[
m_n = \min \left\{ m \in \mathbb{Z} : g(m) < n + \frac{1}{4} \right\},
\] (6.15)
in particular, \( m_N = a \). We have
\[
\sum_{m=a}^{b} \left[ g(m) + \frac{3}{4} \right] = \sum_{m=a}^{m_n-1} \left[ g(m) + \frac{3}{4} \right] = \sum_{n=0}^{N-1} \sum_{m=m_n+1}^{m_n-1} \left[ g(m) + \frac{3}{4} \right].
\] (6.16)

a) If (3.2) is satisfied Lemma 6.3 yields
\[
\sum_{m=m_n+1}^{m_n-1} \left[ g(m) + \frac{3}{4} \right] \geq \sum_{m=m_n+1}^{m_n-1} g(m),
\]
and therefore,
\[
\sum_{m=a}^{b} \left[ g(m) + \frac{3}{4} \right] \geq \sum_{m=a}^{m_n-1} g(m).
\]

Now, (6.10) and (6.11) follows from Lemma 6.2.

b) If (6.8) is satisfied Lemma 6.3 yields
\[
\sum_{m=m_n+1}^{m_n-1} \left[ g(m) + \frac{3}{4} \right] \geq \frac{1}{4} + \sum_{m=m_n+1}^{m_n-1} g(m),
\]
and therefore,
\[
\sum_{m=a}^{b} \left[ g(m) + \frac{3}{4} \right] \geq \frac{N}{4} + \sum_{m=a}^{m_n-1} g(m).
\]

Next,
\[
N = \left[ g(a) + \frac{3}{4} \right] \geq g(a) - \frac{1}{4}.
\]

Now, (6.12) and (6.13) follows from Lemma 6.2. ■
Lemma 6.5. Let $A, B \in \mathbb{Z}$, $A < B$. Let $g \in C^1[A, B]$ be a decreasing convex function. Assume that the conditions (6.6) and (3.2) are fulfilled. Then
\[
\frac{1}{2} \left[ g(A) + \frac{3}{4} \right] + \sum_{m=1}^{B} \left[ g(m) + \frac{3}{4} \right] \geq \frac{1}{2} g(A) + \sum_{m=1}^{B} g(m) + \frac{1}{8}.
\]

Proof. If $B = A + 1$ the sums in both sides are absent. The LHS is equal to $n/2$. The assumptions (3.2) and (6.6) implies
\[
g(A) \leq g(B) + \frac{1}{2} \leq \frac{n}{4},
\]
and therefore the RHS
\[
\frac{1}{2} g(A) + \frac{1}{8} \leq \frac{n}{2}.
\]

Let $B \geq A + 2$. The LHS is equal to $(B - A - 1/2)n$. Due to Lemma 6.1
\[
\frac{1}{2} g(A) + \sum_{m=1}^{B} g(m) + \frac{1}{8} \leq \frac{B - A}{2} g(A) + \frac{B - A - 1}{2} g(B) + \frac{1}{8}
\]
\[
\leq \frac{B - A}{2} \left( n + \frac{1}{4} \right) + \frac{B - A - 1}{2} \left( n - \frac{3}{4} \right) + \frac{1}{8}
\]
\[
= \left( B - A - \frac{1}{2} \right) n - \frac{1}{4} (B - A - 2) \leq \left( B - A - \frac{1}{2} \right) n
\]
because $B - A \geq 2$. 

Theorem 6.6. Let $a, b \in \mathbb{Z}$, $a < b$. Let $g \in C^1[a, b]$ be a non-negative decreasing convex function satisfying (3.2). Assume that
\[
g(a) \geq \frac{1}{4}, \quad g'(a) < -\frac{1}{3}, \quad g(b) = 0,
\]
and that $m_0 < b_0$ where these quantities are defined in (6.1). Let $t_*$ be such a point that $g'(t_*) = -1/3$. Then
\[
\frac{1}{2} \left[ g(a) + \frac{3}{4} \right] + \sum_{m=a+1}^{b} \left[ g(m) + \frac{3}{4} \right] \geq \int_{a}^{b_0} g(t) \, dt - \frac{1}{8} (b_0 - m_0) + \frac{1}{4} \max \left( g(t_*) - \frac{5}{4}; 0 \right).
\]

Proof. Introduce the numbers $N, m_1, \ldots, m_N$ by the formulas (6.14), (6.15). We have
\[
\frac{1}{2} \left[ g(a) + \frac{3}{4} \right] + \sum_{m=m_{N+1}}^{b} \left[ g(m) + \frac{3}{4} \right] = \frac{1}{2} \left[ g(a) + \frac{3}{4} \right] + \sum_{m=m_{N+1}}^{b} \left[ g(m) + \frac{3}{4} \right] + \sum_{n=0}^{N-2} \sum_{m=m_{n+1}}^{m_{n-1}} \left[ g(m) + \frac{3}{4} \right],
\]
the last sum is absent if $N = 1$. By Lemma 6.5
\[
\frac{1}{2} \left[ g(a) + \frac{3}{4} \right] + \sum_{m=m_{N+1}}^{b} \left[ g(m) + \frac{3}{4} \right] \geq \frac{1}{2} g(a) + \sum_{m=m_{N+1}}^{b} g(m) + \frac{1}{8}.
\]
By Lemma 6.3
\[ \sum_{n=0}^{N-2} \sum_{m=m_{n+1}}^{m_{n-1}} \left[ g(m) + \frac{3}{4} \right] \geq \sum_{n=0}^{N-2} \sum_{m=m_{n+1}}^{m_{n-1}} g(m) + \frac{K}{4}, \]
where
\[ K = \max \left( \left[ g(t_*) - \frac{1}{4} \right]; 0 \right). \]

Therefore,
\[ \frac{1}{2} \left[ g(a) + \frac{3}{4} \right] + \sum_{m=a+1}^{b} \left[ g(m) + \frac{3}{4} \right] \geq \frac{1}{2} g(a) + \sum_{m=a+1}^{m_0-1} g(m) + \frac{K}{4} + \frac{1}{8}. \]

Finally, Lemma 6.2 implies
\[ \frac{1}{2} \left[ g(a) + \frac{3}{4} \right] + \sum_{m=a+1}^{b} \left[ g(m) + \frac{3}{4} \right] \geq \int_{a}^{b_0} g(t) \, dt - \frac{1}{8} (b_0 - m_0) + \frac{K}{4}. \]

7 Neumann problem in a circular sector

Eigenfunctions of the Laplace operator of the Neumann problem in the circular sector \( \Omega_\alpha \) are
\[ J_{\frac{m\pi}{\alpha}} \left( j_{\frac{m\pi}{\alpha}, k} r \right) \cos \frac{m\pi \varphi}{\alpha}; \quad m \in \mathbb{N}_0, k \in \mathbb{N}. \]

Here \( j_{\nu,k} \) is the \( k \)-th positive zero of the function \( J_{\nu} \) if \( \nu > 0 \), and \( k \)-th non-negative zero of \( J_{0}^{\prime} \) if \( \nu = 0 \); it means \( j_{0,1} = 0 \) and the corresponding eigenfunction for \( m = 0, k = 1 \) is a constant function. The corresponding eigenvalues are
\[ \left( j_{\frac{m\pi}{\alpha}, k} \right)^2, \quad m \in \mathbb{N}_0, k \in \mathbb{N}. \]

Thus,
\[ N_N(\Omega_\alpha, \Lambda) = \sum_{m=0}^{\infty} \# \left\{ k \in \mathbb{N} : j_{\frac{m\pi}{\alpha}, k} \leq \sqrt{\Lambda} \right\}. \quad (7.1) \]

As \( j_{\nu,1} \geq \nu \), see [17, §15.3], all terms in the RHS with \( m \geq \alpha \sqrt{\Lambda} / \pi \) vanish.

The following estimate is proved in [10, Lemma 3.4].

**Theorem 7.1.** Let \( \nu \geq 0, \Lambda \geq 0 \). Then
\[ \# \left\{ k \in \mathbb{N} : j_{\nu, k} \leq \sqrt{\Lambda} \right\} \geq \frac{1}{\pi} \left( \sqrt{\Lambda - \nu^2} - \nu \arccos \frac{\nu}{\sqrt{\Lambda}} \right) + \frac{3}{4}. \]

**Remark 7.2.** a) Formally, Theorem 7.1 is proved in [10] for integer \( \nu \geq 0 \). The proof for arbitrary \( \nu \geq 0 \) can be repeated literally.

b) We have not found a simple proof of this estimate.
The relation (7.1) and Theorem 7.1 imply the inequality

$$N_{N}(\Omega_{\alpha}, \Lambda) \geq \sum_{m=0}^{[\frac{\alpha \sqrt{\Lambda}}{\pi}]} \left[ G \left( \frac{\pi m}{\alpha} \right) + \frac{3}{4} \right],$$  

(7.2)

where the function $G$ is defined in (2.1).

First, we prove Pólya’s conjecture for the sectors with angle greater than $\pi$.

**Lemma 7.3.** Let $\pi \leq \alpha \leq 2\pi$. Then

$$N_{N}(\Omega_{\alpha}, \Lambda) > \frac{\alpha \Lambda}{8\pi}. \quad (7.3)$$

**Proof.** We have $\mu_1 = 0$, therefore

$$N_{N}(\Omega_{\alpha}, \Lambda) \geq 1 > \frac{\Lambda}{4} \geq \frac{\alpha \Lambda}{8\pi} \quad \text{if} \quad \Lambda < 4.$$

The zeros $j_{\nu,k}'$ of $J_{\nu}'$ increase in $\nu$, see for example [17, §15.6], so

$$j_{\frac{\pi}{\alpha},1}' \leq j_{1,1}' .$$

Next, $j_{1,1}' < 2$, see [17, §15.3], so $\mu_2(\Omega_{\alpha}) < 4$. Therefore,

$$N_{N}(\Omega_{\alpha}, \Lambda) \geq 2 > \frac{\Lambda}{4} \geq \frac{\alpha \Lambda}{8\pi} \quad \text{if} \quad 4 \leq \Lambda < 8.$$

Thus, we can assume $\Lambda \geq 8$. Put $b = \left[ \frac{\alpha \sqrt{\Lambda}}{\pi} \right] + 1$, and define

$$g(t) = G \left( \frac{\pi t}{\alpha} \right) \quad \text{on} \quad \left[ 0, \frac{\alpha \sqrt{\Lambda}}{\pi} \right],$$

and extend it by zero on $\left[ \frac{\alpha \sqrt{\Lambda}}{\pi}, b \right]$. Then $g$ is a non-negative decreasing convex function,

$$g(0) = \frac{\sqrt{\Lambda}}{\pi} > \frac{1}{4}, \quad g'(t) = -\frac{1}{\alpha} \arccos \frac{\pi t}{\alpha \sqrt{\Lambda}}, \quad (7.4)$$

and (3.2) is fulfilled due to $\alpha \geq \pi$. We can apply Theorem 6.4 a). Recall that

$$\int_{0}^{\frac{\alpha \sqrt{\Lambda}}{\pi}} g(t) \, dt = \frac{\alpha \Lambda}{8\pi},$$

see (4.3). If $m_0 \geq \alpha \sqrt{\Lambda}/\pi$ where $m_0$ defined in (6.1), then (7.2) and (6.10) imply

$$N_{N}(\Omega_{\alpha}, \Lambda) \geq \sum_{m=0}^{[\frac{\alpha \sqrt{\Lambda}}{\pi}]} \left[ g(m) + \frac{3}{4} \right] \geq \frac{\sqrt{\Lambda}}{2\pi} + \int_{0}^{\frac{\alpha \sqrt{\Lambda}}{\pi}} g(t) \, dt > \frac{\alpha \Lambda}{8\pi},$$

and the claim is proved.
Assume 

\[ m_0 < \alpha \sqrt{\Lambda}/\pi. \]

Then (7.2) and (6.11) imply

\[ N_N(\Omega_{\alpha}, \Lambda) \geq \frac{\sqrt{\Lambda}}{2\pi} + \int_0^{\frac{\alpha \sqrt{\Lambda}}{\pi}} g(t) \, dt - \frac{1}{8} \left( \frac{\alpha \sqrt{\Lambda}}{\pi} - m_0 + 1 \right). \]  

(7.5)

We have

\[ g \left( \frac{\alpha \sqrt{\Lambda}}{2\pi} \right) = G \left( \frac{\sqrt{\Lambda}}{2} \right) = \frac{3\sqrt{3} - \pi}{6\pi} \sqrt{\Lambda} > \frac{1}{4} \]

as \( \Lambda \geq 8 \). It means

\[ m_0 > \frac{\alpha \sqrt{\Lambda}}{2\pi}. \]  

(7.6)

Now, (7.5) implies

\[ N_N(\Omega_{\alpha}, \Lambda) \geq \frac{\alpha \Lambda}{8\pi} + \frac{\sqrt{\Lambda}}{2\pi} - \frac{\alpha \sqrt{\Lambda}}{16\pi} - \frac{1}{8}. \]

If \( \alpha \leq \frac{3\pi}{2} < 8 - \pi \) then

\[ \frac{\sqrt{\Lambda}}{2\pi} - \frac{\alpha \sqrt{\Lambda}}{16\pi} = \frac{(8 - \alpha) \sqrt{\Lambda}}{16\pi} > \frac{\sqrt{\Lambda}}{16} > \frac{1}{8}, \]

and (7.3) is satisfied.

Finally, let \( \alpha \geq \frac{3\pi}{2} \). Then (7.4) yields (6.8), and we can apply Theorem 6.4 b). So, (7.2), (6.13) and (4.3) imply

\[ N_N(\Omega_{\alpha}, \Lambda) \geq \frac{3g(0)}{4} + \int_0^{\frac{\alpha \sqrt{\Lambda}}{\pi}} g(t) \, dt - \frac{1}{16} \left( \frac{2\alpha \sqrt{\Lambda}}{\pi} - 2m_0 + 3 \right) \]

\[ = \frac{\alpha \Lambda}{8\pi} + \frac{3\sqrt{\Lambda}}{4\pi} - \frac{1}{16} \left( \frac{2\alpha \sqrt{\Lambda}}{\pi} - 2m_0 + 3 \right). \]

Now, (7.6) yields

\[ N_N(\Omega_{\alpha}, \Lambda) \geq \frac{\alpha \Lambda}{8\pi} + \frac{3\sqrt{\Lambda}}{4\pi} - \frac{\alpha \sqrt{\Lambda}}{16\pi} - \frac{3}{16}. \]

Finally,

\[ \frac{3\sqrt{\Lambda}}{4\pi} - \frac{\alpha \sqrt{\Lambda}}{16\pi} \geq \frac{(6 - \pi) \sqrt{\Lambda}}{8\pi} > \frac{6 - \pi}{4\pi} > \frac{3}{16} \]

where we used \( \alpha \leq 2\pi \) and \( \sqrt{\Lambda} > 2 \). Therefore, in this case (7.3) holds true as well.

Proof of Theorem 0.2. Follows from Lemma 7.3 and Lemma 4.3.

8 Neumann problem in the disk

Theorem 8.1. Let \( B_1 \subset \mathbb{R}^2 \) be the unit disk. Let \( \Lambda \geq 531 \). Then

\[ N_N(B_1, \Lambda) > \frac{\Lambda}{4}. \]
Proof. The eigenfunctions of the Laplace operator of the Neumann problem in the disk are
\[ J_{|m|} (j_{|m|, k} r) e^{im \varphi}, \quad m \in \mathbb{Z}, \ k \in \mathbb{N}. \]
The corresponding eigenvalues are
\[ \left( j_{|m|, k} \right)^2, \quad m \in \mathbb{Z}, \ k \in \mathbb{N}, \]
and therefore
\[ N_{\nu}(B_1, \Lambda) = \# \left\{ k \in \mathbb{N} : j_{0,k} \leq \sqrt{\Lambda} \right\} + 2 \sum_{m=1}^{[\sqrt{\Lambda}]} \# \left\{ k \in \mathbb{N} : j_{m,k} \leq \sqrt{\Lambda} \right\}. \]
Due to Theorem 7.1
\[ N_{\nu}(B_1, \Lambda) \geq \left[ G(0) + \frac{3}{4} \right] + 2 \sum_{m=1}^{[\sqrt{\Lambda}]} G(m) + \frac{3}{4}, \]
where the function $G$ is defined in (2.1). Put $b = \left[ \sqrt{\Lambda} \right] + 1$, and extend the function $G$ by zero to $[\sqrt{\Lambda}, b]$. We apply Theorem 6.6 with $g = G, a = 0, b_0 = \sqrt{\Lambda}$. The number $t_*$ is defined by the equality $G'(t_*) = -1/3$. It means
\[ t_* = \frac{\sqrt{\Lambda}}{2}, \quad G(t_*) = \frac{3 \sqrt{3} - \pi}{6\pi} \sqrt{\Lambda}. \]
Therefore,
\[ N_{\nu}(B_1, \Lambda) \geq 2 \int_0^{\sqrt{\Lambda}} G(t) \, dt - \frac{1}{4} \left( \sqrt{\Lambda} - m_0 \right) + \frac{1}{2} \left( G(t_*) - \frac{5}{4} \right), \]
where $m_0$ is defined by (6.1).
Show that
\[ m_0 > \sqrt{\Lambda} \cos \sigma \quad \text{with} \quad \sigma := \frac{3\pi}{20}. \]
Indeed,
\[ G\left( \sqrt{\Lambda} \cos \sigma \right) = \frac{\sqrt{\Lambda}}{\pi} \left( \sin \sigma - \sigma \cos \sigma \right) > \frac{1}{4}, \]
whenever
\[ \sqrt{\Lambda} \geq \frac{\pi}{4 \left( \sin \sigma - \sigma \cos \sigma \right)} \approx 23,023. \quad (8.1) \]
Now,
\[ N_{\nu}(B_1, \Lambda) \geq \frac{\Lambda}{4} - \frac{1 - \cos \sigma}{4} \sqrt{\Lambda} + \frac{3 \sqrt{3} - \pi}{12\pi} \sqrt{\Lambda} - \frac{5}{8}. \]
Furthermore,
\[ \left( \frac{3 \sqrt{3} - \pi}{12\pi} - \frac{1 - \cos \sigma}{4} \right) \sqrt{\Lambda} > \frac{5}{8} \]
whenever
\[ \sqrt{\Lambda} > \frac{15\pi}{6 \sqrt{3} - 8\pi + 6\pi \cos \sigma} \approx 22,935. \quad (8.2) \]
The estimates (8.1) and (8.2) are provided by the condition $\Lambda \geq 531$. \( \blacksquare \)
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