Annihilation Rate of Heavy $0^{-+}$ Quarkonium in Relativistic Salpeter Method

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Abstract

Two-photon and two-gluon annihilation rates of $\eta_c$, $\eta'_c$, $\eta_b$ and $\eta'_b$ are estimated in the relativistic Salpeter method. By solving the full Salpeter equation with a well defined relativistic wave function, we estimate $M_{\eta_c} = 2.979 \pm 0.432$ GeV, $M_{\eta'_c} = 3.566 \pm 0.437$ GeV, $M_{\eta_b} = 9.364 \pm 1.120$ GeV and $M_{\eta'_b} = 9.941 \pm 1.112$ GeV. We calculated the transition amplitude using the Mandelstam formalism and estimate the decay widths: $\Gamma(\eta_c \to 2\gamma) = 7.14 \pm 0.95$ KeV, $\Gamma(\eta'_c \to 2\gamma) = 4.44 \pm 0.48$ KeV, $\Gamma(\eta_b \to 2\gamma) = 0.384 \pm 0.047$ KeV and $\Gamma(\eta'_b \to 2\gamma) = 0.191 \pm 0.025$ KeV. We also give estimates of total widths by the two-gluon decay rates: $\Gamma_{\text{tot}}(\eta_c) = 19.6 \pm 2.6$ MeV, $\Gamma_{\text{tot}}(\eta'_c) = 12.1 \pm 1.3$ MeV, $\Gamma_{\text{tot}}(\eta_b) = 6.98 \pm 0.85$ MeV and $\Gamma_{\text{tot}}(\eta'_b) = 3.47 \pm 0.45$ MeV.
I. INTRODUCTION

It is well known that two-photon or two-gluon annihilation rate of heavy $0^{-+}$ quarkonium $c\bar{c}$ or $b\bar{b}$ is related to the wave function, so this process will be helpful to understand the formalism of inter-quark interactions, and can be a sensitive test of the potential model. With the replacement of the photons by gluons, the finial state becomes two gluon state, which will be helpful to give information on the total width of the corresponding quarkonium.

Experimentally there are quite many results for the decay width $\Gamma_{tot}(\eta_c)$ with a wide range of values and uncertainties by different collaborations; for example, in recent experiment of Barbar they give $34.3 (2.3) (0.9)$ MeV, much larger than the cited value $16.0^{+3.6}_{-3.2}$ MeV by Particle Data Group. However, $\eta'_c$ has been just declared observed by Belle and by Barbar; $\eta_b$ and $\eta'_b$ have not been observed yet, even though there were some experiments to search for $\eta_b$, e.g., the ALEPH collaboration. In short, unlike the corresponding vector $1^{--}$ quarkonium which can be produced directly by $e^+e^-$ annihilation, experiments on $0^{-+}$ quarkonium have just begun, even for $\eta_c$. This due to the small cross section; presently there are $57.7 \times 10^6$ $J/\Psi$ events collected with the BES-II detector, but there are only $2547 \pm 90 \eta_c$ events collected by the Barbar detector.

For the theoretical estimates of the annihilation rate for $\eta_c^{(s)}$ and $\eta_b^{(s)}$, we have various methods readily available in hand. First was the non-relativistic calculation, then the relativistic corrections were found to be important especially for $c\bar{c}$ states. In recent years, many authors try to focus on the relativistic corrections and there are already some versions of relativistic calculation, and they give improved results over the non-relativistic methods. In this letter, we give yet another relativistic calculation by the instantaneous Bethe-Salpeter method, which is a full relativistic method with a well defined relativistic form of the wave function.

There are two sources of relativistic corrections; one is the correction in relativistic kinematics which appears in the decay amplitudes through a well defined form of relativistic wave function (i.e. not merely through the wave function at origin); the other relativistic correction comes via the relativistic inter-quark dynamics, which requires not only a well defined relativistic wave function but also a good relativistic formalism to describe the interactions among quarks.

The Bethe-Salpeter equation is a well-known tool to describe a relativistic bound state. And the Salpeter equation is the special case of Bethe-Salpeter equation when the interaction is instantaneous. It has been shown that the instantaneous approach is a good approximation.
in heavy mesons, especially for the equal-mass quarkonium, since the non-instantaneous correction was found to be very small in equal mass system[8]. The full Salpeter equation includes two parts of the wave function, the positive and negative energy part. In the case of heavy mesons the negative energy part usually gives a smaller contribution than the positive energy part, and therefore, to simplify the calculation for the heavy mesons authors like to make a further approximation to the Salpeter equation by ignoring the negative part contribution. However, since we are considering full relativistic calculation, and the negative energy contribution was found to be not very small for some cases[9], in this letter we will solve the full Salpeter equation including the negative contribution, and use the full Salpeter wave function to estimate the annihilation decay width of quarkonium.

We note that the form of the wave function is also important in the calculation, since the corrections of the relativistic kinetics come mainly through it. We begin from the quantum field theory, analyze the parity and charge conjugation of bound state, and give a formula for the wave function that is in a relativistic form with definite parity and charge conjugation symmetry. Another important thing is how to use the relativistic wave function of bound state to obtain a relativistic transition amplitude, since a non-relativistic transition amplitude even with a relativistic wave function will lose the benefit of relativistic effects caused by the relativistic wave function. The Mandelstam formalism is well suited for the computation of relativistic transition amplitude, and we begin with this formulism to give a formula of the transition amplitude.

In Sec. II, we give theoretical details for the transition amplitude in Mandelstam formalism and the corresponding wave function with a well defined relativistic form. In Sec. III, the full Salpeter equation is solved, and the mass spectra and numerical value of the wave function are obtained. Then the two-photon decay width and full width of heavy $0^{-+}$ quarkonium are estimated. In Sec. III, short discussions and a summary are also given.

**II. THEORETICAL DETAILS**

According to the Mandelstam formalism, the relativistic transition amplitude of a quarkonium decaying into two photons (see figure 1) can be written as:

$$T = i\sqrt{3} (ie_q)^2 \int \frac{d^4q}{(2\pi)^4} \text{tr} \left\{ \chi(q) \left[ \hat{\gamma}_2 S(p_1 - k_1) \hat{\gamma}_1 + \hat{\gamma}_1 S(p_1 - k_2) \hat{\gamma}_2 \right] \right\} ,$$

where $k_1, k_2; \varepsilon_1, \varepsilon_2$ are the momenta and polarization vectors of photons; $e_q = \frac{2}{3}$ for charm quark and $e_q = \frac{1}{3}$ for bottom quark; $p_1$ and $p_2$ are the momenta of constitute quark and
antiquark; \( \chi(q) \) is the quarkonium Bethe-Salpeter wave function with the total momentum \( P \) and relative momentum \( q \), related by

\[
p_1 = \alpha_1 P + q, \quad \alpha_1 \equiv \frac{m_1}{m_1 + m_2},
\]
\[
p_2 = \alpha_2 P - q, \quad \alpha_2 \equiv \frac{m_2}{m_1 + m_2}.
\]

Since \( p_{10} + p_{20} = M \), the approximation \( p_{10} = p_{20} = \frac{M}{2} \) is a good choice for the equal mass system \([11, 12, 13]\). Having this approximation, we can perform the integration over \( q \) to reduce the expression, with the notation for the Salpeter wave function \( \Psi(q) = \int \frac{d\vec{q}}{(2\pi)^3} \chi(q) \), to

\[
T = \sqrt{\frac{3}{2}} (ee_q)^2 \int \frac{d\vec{q}}{(2\pi)^3} \text{tr} \left\{ \Psi(\vec{q}) \left[ \frac{1}{p_1 - k_1} \dot{\epsilon}_1 + \frac{1}{p_1 - k_2} \dot{\epsilon}_2 \right] \right\},
\]

(2)

Here the relativistic Salpeter wave function \( \Psi(\vec{q}) \) of \( 0^{-+} \) state with a definite parity (–) and charge conjugation (+) can be written as \([14, 15]\):

\[
\Psi(\vec{q}) = \left[ \gamma_0 \varphi_1(\vec{q}) + \varphi_2(\vec{q}) + \frac{\slashed{q} \gamma_0}{m_1} \varphi_1(\vec{q}) \right] \gamma_5,
\]

(3)

where \( \omega_1 = \sqrt{m_1^2 + q^2} \) and \( \omega_2 = \sqrt{m_2^2 + q^2} \). The wave function \( \varphi_1(\vec{q}), \varphi_2(\vec{q}) \) and bound state mass \( M \) can be obtained by solving the full Salpeter equation with the constituent quark mass as input, and they should satisfy the normalization condition:

\[
\int \frac{d\vec{q}}{(2\pi)^3} \frac{8\omega_1}{m_1} \varphi_1(\vec{q}) \varphi_2(\vec{q}) = 2M.
\]

(4)

Putting wave function \( \Psi(\vec{q}) \) into the amplitude Eq. (2) and performing the trace, the amplitude becomes

\[
T = \sqrt{\frac{3}{2}} (ee_q)^2 \int \frac{d\vec{q}}{(2\pi)^3} 4 \varphi_1(\vec{q}) \epsilon_{\mu \nu \alpha \beta} P^\mu \epsilon_1^\alpha k_1^\beta \left[ \frac{1}{(p_1 - k_1)^2} + \frac{1}{(p_1 - k_2)^2} \right].
\]

(5)

With this relativistic amplitude, the two photon decay width can be written as

\[
\Gamma(0^{-+} \to \gamma \gamma) = 12\pi \alpha^2 e_Q^4 M \left\{ \int \frac{d\vec{q}}{(2\pi)^3} \varphi_1(\vec{q}) \left[ \frac{1}{(p_1 - k_1)^2} + \frac{1}{(p_1 - k_2)^2} \right] \right\}^2,
\]

(6)

where \( \alpha = \frac{e^2}{4\pi} \). One can easily check that in the non-relativistic limit (by removing the dependence on the relative momentum \( q \)) the decay width depends on the wave function at the origin.
The two gluon decay width of quarkonium can be easily obtained from the two photon decay width, with a simple replacement in the photon decay width formula

\[ E_q^4 \alpha^2 \rightarrow \frac{2}{9} \alpha_s^2. \] 

That is:

\[ \Gamma(0^{-+} \rightarrow gg) = \frac{8}{3} \pi \alpha_s^2 M \left\{ \int \frac{d\vec{q}}{(2\pi)^3} \phi_1(\vec{q}) \left[ \frac{1}{(p_1 - k_1)^2} + \frac{1}{(p_1 - k_2)^2} \right] \right\}^2. \] 

### III. NUMERICAL RESULTS AND DISCUSSIONS

In our previous works \[14, 15\], the full Salpeter equation has been solved and the corresponding eigenvalue and the wave function Eq. \[3\] have been obtained numerically. We will not show the details of the calculation here, but only give the final results; interested readers can find them in Refs. \[14, 15\].

When solving the full Salpeter equation, we choose a phenomenological Cornell potential. There are some parameters in this potential including the constituent quark mass and one loop running coupling constant. In previous paper Ref. \[14\], the following best-fit values of input parameters were obtained by fitting the mass spectra for heavy meson 0\(^{-}\) states:

\[ a = e = 2.7183, \alpha = 0.06 \text{ GeV}, V_0 = -0.60 \text{ GeV}, \lambda = 0.2 \text{ GeV}^2, \Lambda_{QCD} = 0.26 \text{ GeV} \quad \text{and} \quad m_c = 1.7553 \text{ GeV}. \]

With this parameter set, we solve the full Salpeter equation and obtain the mass spectra and wave functions of quarkonium. We first fit the mass of \(M_{\eta_c} = 2.979 \text{ GeV}\) and then get the mass of \(\eta'_c\):

\[ M_{\eta'_c} = 3.566 \text{ GeV}, \] which is a little lower than the recent experimental data \(M_{\eta'_c} = 3.6308 \pm 0.0034 \pm 0.0010 \text{ GeV} \[1\] and \(M_{\eta'_c} = 3.654 \pm 0.006 \pm 0.008 \text{ GeV} \[3\]. With the
obtained wave function and Eq. 5, we calculate the decay width of $\eta_c \to 2\gamma$ and $\eta'_c \to 2\gamma$, with the result:

$$\Gamma(\eta_c \to \gamma\gamma) = 7.14 \text{ KeV},$$  \hspace{1cm} (9)

$$\Gamma(\eta'_c \to \gamma\gamma) = 4.44 \text{ KeV}.$$  \hspace{1cm} (10)

To give the numerical analysis of two–gluon decays, we need to fix the value of the renormalization scale $\mu$ in $\alpha_s(\mu)$. In the case of $\eta_c$ we choose the charm quark mass $m_c$ as the energy scale and obtain the coupling constant $\alpha_s(m_c) = 0.36^{14}$. The corresponding two–gluon annihilation rates of $\eta_c$ and $\eta'_c$ are:

$$\Gamma(\eta_c \to gg) = 19.6 \text{ MeV},$$  \hspace{1cm} (11)

$$\Gamma(\eta'_c \to gg) = 12.1 \text{ MeV}.$$  \hspace{1cm} (12)

For the case of $\eta_b$ and $\eta'_b$, our previously derived input parameters in the potential should not work, because they were obtained from fitting data of heavy-light mesons. $\eta_b$ and $\eta'_b$ being heavy-heavy mesons, we change the previous scale parameters to $m_b = 5.13 \text{ GeV}$, $\Lambda_{\text{QCD}} = 0.2 \text{ GeV}$, and other parameters are not changed. With this set of parameters, the mass of $\eta_b$ is obtained as $M_{\eta_b} = 9.364 \text{ GeV}$, about 100 MeV lower than the $\Upsilon$ mass, and $\eta'_b$ mass as $M_{\eta'_b} = 9.941 \text{ GeV}$. Now the coupling constant at the scale of bottom quark mass is $\alpha_s(m_b) = 0.232$. The corresponding decay widths are:

$$\Gamma(\eta_b \to \gamma\gamma) = 0.384 \text{ KeV},$$  \hspace{1cm} (13)

$$\Gamma(\eta'_b \to \gamma\gamma) = 0.191 \text{ KeV},$$  \hspace{1cm} (14)

$$\Gamma(\eta_b \to gg) = 6.98 \text{ MeV},$$  \hspace{1cm} (15)

$$\Gamma(\eta'_b \to gg) = 3.47 \text{ MeV}.$$  \hspace{1cm} (16)

In Table I we list our results with theoretical uncertainties, which are obtained by varying all the input parameters simultaneously within $\pm 10\%$ of the central values, and taking the largest variation of the results. In this table, we assume the total width of heavy quarkonium is dominated by its two-gluon decay rate, $\Gamma_{\text{tot}} \simeq \Gamma_{2g}$. The most recent theoretical predictions and experimental data are also shown in the same table.

From the tables, we can see that our results of $\Gamma_{2\gamma}^{\eta_c}$ agree well with other theoretical estimates of Refs. 13, 16, 17, and $\Gamma_{\text{tot}}^{\eta_c}$ with Refs. 13, 18; our results of $\Gamma_{2\gamma}^{\eta_b}$ and $\Gamma_{2\gamma}^{\eta'_b}$ agree with Refs. 13, 16, 17, 18; but our results of $\eta'_c$ are larger than the theoretical predictions by others, but consistent with the recent experiment data 1.
We comment that in this work we did not include the QCD radiative correction because we focus mainly on the relativistic corrections, though there is no doubt that the QCD correction is very important and an interesting topic. We have shown the uncertainties of our theoretical estimates by varying all the input parameters simultaneously within ±10% of the central values. It should also be pointed out that within these parameter ranges the uncertainty caused by the value of $\alpha_s(\mu)$ is very important because when we determine the total widths in our calculation we need the precise value of $\alpha_s^2(\mu)$.

In summary, by solving the relativistic full Salpeter equation with a well defined form of the wave function, we obtain the mass spectra: $M_{\eta_c} = 2.979 \pm 0.432$ GeV, $M_{\eta'_c} = 3.566 \pm 0.437$ GeV, $M_{\eta_b} = 9.364 \pm 1.120$ GeV and $M_{\eta'_b} = 9.941 \pm 1.112$ GeV. With the help of Mandelstam formalism for the transition amplitude, we estimate two-photon decay rates: $\Gamma(\eta_c \rightarrow 2\gamma) = 7.14 \pm 0.95$ KeV, $\Gamma(\eta'_c \rightarrow 2\gamma) = 4.44 \pm 0.48$ KeV, $\Gamma(\eta_b \rightarrow 2\gamma) = 0.384 \pm 0.047$ KeV and $\Gamma(\eta'_b \rightarrow 2\gamma) = 0.191 \pm 0.025$ KeV, and the total decay widths: $\Gamma_{tot}(\eta_c) = 19.6 \pm 2.6$ MeV, $\Gamma_{tot}(\eta'_c) = 12.1 \pm 1.3$ MeV, $\Gamma_{tot}(\eta_b) = 6.98 \pm 0.85$ MeV and $\Gamma_{tot}(\eta'_b) = 3.47 \pm 0.45$ MeV.

### Table I: Recent theoretical and experimental results of two-photon decay width and total width.

|               | $\Gamma_{\eta_c}^{2\gamma}$, KeV | $\Gamma_{\eta_c}^{2\gamma}$, MeV | $\Gamma_{\eta'_c}^{2\gamma}$, KeV | $\Gamma_{\eta'_c}^{2\gamma}$, MeV | $\Gamma_{\eta_b}^{2\gamma}$, KeV | $\Gamma_{\eta'_b}^{2\gamma}$, KeV |
|---------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| Ours          | 7.14±0.95                        | 19.6±2.6                         | 4.44±0.48                        | 12.1±1.3                         | 0.384±0.047                      | 0.191±0.025                      |
| Münz [9]      | 3.50±0.40                        | 1.38±0.30                        | 0.22±0.04                        | 0.11±0.02                        |
| Chao [13]     | 6-7                              | 17-23                            | 2                                | 5-7                              | 0.46                             | 0.21                             |
| Ebert [16]    | 5.5                              | 1.8                              | 0.35                             | 0.15                             |
| Fabiano [17]  | 7.6±1.5                          |                                  |                                  | 0.466±101                        |
| Gupta [18]    | 10.94                            | 23.03                            |                                  |                                  |
| PDG [2]       | 7.2 ± 1.2                        | 16.1±3.1                         | 0.9                              |                                  |
| BABAR [1]     | 34.3(2.3)(0.9)                   |                                  |                                  | 17(8.3)(2.5)                     |
| BES [19]      | 17(3.7)(7.4)                     |                                  |                                  |                                  |
| CLEO [20]     | 7.60(0.8)(2.3)                   | 27.0(5.8)(1.4)                   |                                  |                                  |
| L3 [21]       | 6.9(1.7)(2.1)                    |                                  |                                  |                                  |
| AMY [22]      | 27(16)(10)                       |                                  |                                  |                                  |
| E760 [23]     | 6.7±2.1                          | 23.9±12.6                        | 0.1                              |                                  |
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