Abstract

We consider the maximum bipartite matching problem in stochastic settings, namely the query-commit and price-of-information models. In the query-commit model, an edge \( e \) independently exists with probability \( p_e \). We can query whether an edge exists or not, but if it does exist, then we have to take it into our solution. In the unweighted case, one can query edges in the order given by the classical online algorithm of Karp, Vazirani, and Vazirani \([KV\,90]\) to get a \((1 - 1/e)\)-approximation. In contrast, the previously best known algorithm in the weighted case is the \((1/2)\)-approximation achieved by the greedy algorithm that sorts the edges according to their weights and queries in that order.

Improving upon the basic greedy, we give a \((1 - 1/e)\)-approximation algorithm in the weighted query-commit model. We use a linear program (LP) to upper bound the optimum achieved by any strategy. The proposed LP admits several structural properties that play a crucial role in the design and analysis of our algorithm. We also extend these techniques to get a \((1 - 1/e)\)-approximation algorithm for maximum bipartite matching in the price-of-information model introduced by Singla \([Si\,18]\), who also used the basic greedy algorithm to give a \((1/2)\)-approximation.

1 Introduction

Maximum matching is an important problem in theoretical computer science. We consider it in the query-commit and price-of-information models. These settings model the situation where the input is random, but we can know a specific part of the input by incurring some cost, which is explicit in the price-of-information setting, i.e., pay \( \pi_e \) to know the weight of the edge \( e \), and implicit in the query-commit, i.e., the algorithm queries existence of \( e \), and if \( e \) exists, the algorithm has to take \( e \) into its solution. We formalize this now. In the query-commit model, an edge \( e \) independently exists with probability \( p_e \). We can query whether an edge exists or not, but if it does exist, then we have to take it into our solution (hence the name query-commit). So, specifically, for the matching problem, if \( M \) is our current matching, then we cannot query an edge \( e \) if it intersects with \( M \). In the price-of-information model introduced by Singla \([Si\,18]\), edge weights are random variables and we can query for the weight \( W_e \) of an edge \( e \) by paying a (fixed) cost \( \pi_e \). We want to query a set \( Q \) of edges and output a matching \( M \subseteq Q \) such that \( W(M) - \sum_{e \in Q} \pi_e \) is maximized.

In the query-commit model, for bipartite graphs, one can query edges in the order given by the classical algorithm of Karp, Vazirani, and Vazirani \([KV\,90]\) to get a \((1-1/e)\) approximation. However, in the more general weighted query-commit setting where an edge \( e \) exists with weight \( w_e \) with probability \( p_e \) and does not exist with probability \( 1 - p_e \), it is not clear how to use such a strategy. In fact, prior to our work, the best algorithm for the weighted setting was the basic greedy that sorts the edges by weight \( w_e \) and then queries in that order to get a
(1/2)-approximate matching. Similarly, in the price-of-information setting, Singla gave a (1/2)-approximation algorithm based on the greedy approach. In this work, we beat the greedy algorithm using new techniques and give clean algorithms that achieve an approximation ratio of $1 - 1/e$ (improving from $1/2$) in the settings of weighted query-commit as well as price-of-information. A key component of our approach is to upper bound the optimum achieved by any strategy using a linear program (LP), and exploit several structural properties of this LP in the design of our algorithms. We now give a high level description of these techniques.

**Techniques**

We first solve the weighted query-commit version, and this part of the paper contains the essential ideas. We then expand on these ideas to solve the price-of-information version. Let us first focus on the weighted query-commit setting. Here, the input is a bipartite graph $G = (A, B, E)$, and for each $e \in E$, its existence probability $p_e$ and weight $w_e$. Our goal is to design a polynomial-time algorithm that gives a sequence of edges to query such that after the last query, we end up with a matching of large weight. First, to get a handle on the expected value of an optimum strategy (OPT), consider the linear program (LP) below. For a vertex $u$, let $E_u$ be the set of edges incident to $u$.

Maximize $\sum_{e \in E} x_e \cdot w_e$,

subject to $\sum_{e \in F} x_e \leq \Pr[\text{an edge in } F \text{ exists}], \quad \text{ for all } u \in A \cup B \text{ and for all } F \subseteq E_u,$

$x_e \geq 0, \quad \text{ for all } e \in E.$

Let $x'_e$ be the probability that OPT solution contains $e$. Since OPT can have at most one edge incident to $u$, for any $F \subseteq E_u$, the events in $\{\text{OPT contains } e : e \in F\}$ are disjoint. Hence, $\sum_{e \in F} x'_e$ is the probability that OPT contains an edge in $F$, which must be at most the probability that at least one edge in $F$ exists. Therefore, $(x'_e)_{e \in E}$ has to satisfy the above LP.

We can solve this LP in polynomial time using a submodular-function-minimization algorithm as a separation oracle for the ellipsoid algorithm. Let $x^*$ be the solution. For a $u \in A$, if we restrict $x^*$ to $E_u$ to get, say, $x^*_u$, we can write $x^*_u$ as a convex combination of extreme points of the polytope

$$\{x \in \mathbb{R}_{+}^{E_u} : \sum_{e \in F} x_e \leq \Pr[\text{an edge in } F \text{ exists}] \quad \forall F \subseteq E_u\}.$$ (1)

A key part of our approach is the nice structural properties of extreme points: the subsets corresponding to the tight constraints for an extreme point $y$ of the above polytope form a chain over a subset of $E_u$, because the right hand side of the constraints is strictly submodular. For a $y$, if we query the edges in the order given by its chain then it can be proved that we commit to an edge $e \in E_u$ with probability $y_e$. Since $x^*_u$ can be written as a convex combination of such extreme points, if we select an extreme point with probability equal to its coefficient in the convex combination and query the edges in the order given by its chain, we commit to an edge $e \in E_u$ with probability $x^*_e$. See Figure 1. But if we do this independently for each vertex in $A$, then we end up with collisions on $B$, so we have to do contention resolution there.

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1Actually, we implicitly use “multiple-weight query-commit” as an intermediate step, where an edge has nonnegative random weight, and an algorithm can ask queries of the form “is weight of $e$ greater than $c?$” and if it is, then the algorithm has to add $e$ to its solution. Then we use a reduction by Singla to reduce from price-of-information.
Figure 1: Here, we have $E_u = \{\{u, b_2\}, \{u, b_3\}\}$. We write $x^*_u = (4/9, 2/9)$ as a convex combination of extreme points of the polytope described in Equation (1): $x^*_u = (2/3)(1/2, 1/6) + (1/3)(1/3, 1/3)$. For $(1/2, 1/6)$, the inequalities corresponding to sets $\{\{u, b_2\}\}$ and $\{\{u, b_2\}, \{u, b_3\}\}$ are tight, and for $(1/3, 1/3)$, the inequalities corresponding to sets $\{\{u, b_3\}\}$ and $\{\{u, b_2\}, \{u, b_3\}\}$ are tight. Observe that these form a chain. We note that it is possible that an inequality corresponding to a nonnegativity constraint is tight. Now, say we query the edges in the order given by the chain. For $(1/2, 1/6)$, we first query $\{u, b_2\}$ then $\{u, b_3\}$, so we select $\{u, b_2\}$ with probability $p_{\{u, b_2\}} = 1/2$ and $\{u, b_3\}$ with probability $(1 - p_{\{u, b_2\}}) = 1/6$, which does indeed correspond to the extreme point $(1/2, 1/6)$.

Effectively, for each vertex in $v \in B$, its neighbor $u$ arrives independently with probability $x^*_{uv}$ and weight $w_{uv}$, and it has to pick one neighbor so that its expected utility is close to $\sum_{uv} x^*_{uv} \cdot w_{uv}$. This setting is similar to the prophet secretary problem, and we extend the ideas of Ehsani et al. [EHKS18] to achieve this.

To generalize this from two-point distributions to multi-point distributions, we think of an edge $e$ to have as many copies as the values its weight can take. But now instead of being independent, the existence of these copies is correlated. To handle this, we write a more general LP with constraints corresponding to sets that are from a lattice family. We can solve submodular-function-minimization over a lattice family in polynomial time [GLS81], which gives us a separation oracle for this LP. Also, the extreme points of the polytope defined by constraints for a left-hand-side vertex correspond to a chain with properties similar to the two-point-distribution case (see Lemma 12).

Once we have a query-commit algorithm for multi-point distributions, we can basically get a price-of-information algorithm by a clean reduction [Sin18]: for each edge $e$ with weight given by the random variable $X_e$ and probing cost $\pi_e$, let $\tau_e$ be the solution to the equation $E[\max\{(X_e - \tau_e), 0\}] = \pi_e$, and let $Y_e = \min(X_e, \tau_e)$ be a new random variable. Now run the query-commit algorithm with $Y_e$, and whenever the algorithm queries any copy of the edge $e$, we probe $e$, and we only pay $\pi_e$ the first time we probe that edge.

Organization of the Paper

Next, we review the related work. In Section 2, we see the weighted query-commit algorithm; we then extend it to the price-of-information setting in Section 3.

Related Work

As mentioned earlier, the algorithm of Karp et al. gives a $(1 - 1/e) = 0.632$ approximation in the unweighted query-commit model for bipartite graphs. Costello et al. [CTT12] give a
Motivated by applications in kidney exchange and online dating, Chen et al. [CIK+09] consider the matching problem in the query-commit model with further constraint that for each vertex \( v \), the algorithm can query at most \( t_v \) edges incident to it (\( t_v \) is a part of the input) and give a \( (1/4) \)-approximation algorithm. Bansal et al. [BGL+12] improve it to \( (1/3) \) for bipartite graphs and to \( (1/3.46) \) for general graphs, and also give a \( (1/4) \)-approximation in the weighted query-commit (for general graphs); both of these ratios for the unweighted case are further improved by Adamczyk et al. [AGM15], who give a \( (1/3.709) \)-approximation for general graphs. Baveja et al. [BCN+17] improve this to \( (1/3.224) \). We mention that this setting is more general than the setting we consider, because \( t_v = \deg(v) \) for us, i.e., we do not put any restriction on the number of edges incident to a vertex that we can query.

Molinaro and Ravi [MR11] give an optimal algorithm for a very special class of sparse graphs in the unweighted query-commit setting.

Blum et al. [BDH+15] consider the maximum matching problem, where, in the input graph, an edge \( e \) exists with probability \( p_e \), the algorithm can query the existence of an edge and does not have to commit, but needs to minimize the number of queries subject to outputting a good approximation. This model is considered in several follow-up works [AKL16, AKL17, MY18, BR18].

Further Related Work in Other Stochastic Models

Feldman et al. [FMMM09] consider an online variant of stochastic matching where the algorithm gets as input a bipartite graph \( G = (A, B, E) \), and a distribution \( \mathcal{D} \) over \( B \), and \( n \) elements are drawn i.i.d. from \( B \) according to \( \mathcal{D} \) (so there may be repetitions) that the algorithm accesses online. When a copy \( v \sim \mathcal{D} \) arrives online, we have to match it to an unmatched vertex \( u \) in \( A \) such that \( \{u, v\} \in E \). Note that here the existence of an edge is not random in itself but that of a vertex is. Again, Karp et al.'s algorithm gives a \( (1 - 1/e) \)-approximation, and Feldman et al. give a 0.67-approximation. Significant amount of work has been done on this variant as well [MGS11, HMZ11, BGL+12, JL14, ACM15, BSSX16].

We would like to mention the works of Dean et al. [DGV04, DGV05], where they consider stochastic problems where cost of an input unit is only known as a probability distribution that is instantiated after the algorithm commits to including the item in the solution. Charikar et al. [CCP05] and Katriel et al. [KKMU07] consider two stage optimization problems, where the first stage is stochastic with a lower cost for decisions, and in the second stage, with an increase in the decision cost, the actual input is known.

2 A Weighted Query-Commit Algorithm

In this section, we present a \( (1-1/e) \)-approximation algorithm for the maximum weight bipartite matching problem (MWBM) in the query-commit model.

Let \( G = (A, B, E, w) \) be a weighted bipartite graph with \( |A| = |B| = n \), where each edge \( e \in E \) has weight \( w(e) \) and exists independently with probability \( p_e \). Let \( \mathbb{I}_e \) be the indicator variable for the event that edge \( e \) exists (i.e., \( \mathbb{I}_e = 1 \) if and only if edge \( e \) exists). The actual weight of an edge \( e \) is chosen as a probability distribution that takes value \( w(e) \) with probability \( p_e \) and value zero with probability \( 1 - p_e \). We consider a more general distribution of edge weights in Section [5].
value of \( \mathbb{I}_e \) can only be determined by querying edge \( e \) to check whether it exists or not. Given such a graph \( G \), a query-commit algorithm for MWBM (adaptively) queries a sequence of edges \( Q = (e_1, \ldots, e_m) \) and outputs a valid matching \( M \subseteq Q \). Since a query-commit algorithm is committed to add any queried edge that exists to its output, \( M = \{ e \in Q : \mathbb{I}_e = 1 \} \).

Let \( \mathcal{A} \) be a query-commit algorithm for MWBM and let \( M(\mathcal{A}) \) denote its output on input \( G \). We define the expected utility of \( \mathcal{A} \), \( U(\mathcal{A}) := \mathbb{E}[\sum_{e \in M(\mathcal{A})} w(e)] \) to be the expected weight of its output matching, where the expectation is over the randomness of the existence of edges and any internal randomness of the algorithm. Let \( \text{OPT} = \max_{\mathcal{A}} U(\mathcal{A}) \), where the maximum is taken over all query-commit algorithms,\footnote{We remark that our approximation guarantee in the query-commit model also works with respect to the stronger offline adversary that knows all outcomes before selecting the matching. We have presented it in this way as the guarantees in the price-of-information model are of this type.} \( \text{OPT} \) be the optimal expected query-commit utility. We give a query-commit algorithm \text{Approx-QC} whose expected query-commit utility \( U(\text{Approx-QC}) \) is at least \((1 - 1/e)\text{OPT}\).

As described in the introduction, our approach consists of two stages. First, we solve a linear program to bound the optimal expected query-commit utility \( \text{OPT} \). For each edge \( e \in E \) we associate a variable \( x_e \), which we think of as the probability that an optimal algorithm adds edge \( e \) to its output. We set our LP constraints on \( x_e \) variables to be those that must be satisfied by any algorithm that outputs a valid matching formed only of edges that exist. Then the optimal expected utility is upper bounded by the maximum of the weighted sum \( \sum_{e \in E} x_e \cdot w(e) \) subject to our LP constraints.

Then, we use the structural properties of our LP polytope to define a distribution over the permutations of edges and use this distribution to set the query order in our algorithm. For this, we first define distributions \( \mathcal{D}_{a}^{\text{QC}} \) over the permutations of neighboring edges for each vertex \( a \in A \) such that the following holds: If we draw a random permutation \( \sigma_a \) from \( \mathcal{D}_{a}^{\text{QC}} \) for each \( a \in A \), query edges in the order of \( \sigma_a \), and finally add to our output the first edge in \( \sigma_a \) that exists, then the expected utility is optimal. However, the output is not guaranteed to be a matching, because it might have more than one edge incident to a vertex in \( B \) (i.e., collisions).

To deal with the issue of collisions in \( B \), we view the problem as a collection of prophet secretary problem instances that has an instance for each vertex \( b \in B \). In the prophet secretary problem, we have one item to sell (position to be filled) and a set of buyers (secretaries) arrive in a random order. Each buyer makes a take it or leave it offer to buy the item at some random price (or each secretary has some random skill level) whose distribution we know beforehand, and we are interested in maximizing the profit (or hiring the best secretary).

In their recent work, Ehsani et al. \cite{EHKS18} gave a \((1 - 1/e)\)-competitive algorithm for the prophet secretary problem. Inspired by this result, we design our algorithm so that for each \( b \in B \), it recovers at least \((1 - 1/e)\)-fraction of the expected weight of the edge incident to \( b \) in an optimal algorithm’s output. To elaborate, we pick a uniformly random permutation of the vertices in \( A \) and treat them as buyers arriving at uniformly random times. Then, as seen from the perspective of a fixed vertex (i.e. an item) \( b \in B \), buyers make a one-time offer with random values (corresponding to weights of edges incident to \( b \)), and our algorithm uses dynamic thresholds that depend on arrival times to make the decision of whether to sell the item at the offered price or not. One may view our algorithm as running the \((1 - 1/e)\)-competitive prophet secretary algorithm in parallel for each of the vertices in \( B \), but the way we set the dynamic thresholds is different.

The rest of this section is organized as follows: In Section 2.1 we describe how to upper bound \( \text{OPT} \) using an LP. In Section 2.2 we analyze the structure of the LP polytope and show how to construct the distributions \( \mathcal{D}_{a}^{\text{QC}} \) for each \( a \in A \). Finally in Section 2.3 we present our
proposed algorithm and adapt the ideas of Ehsani et al. [EHKS18] to analyze its approximation guarantee.

2.1 Upper-bounding the optimal expected utility

For each vertex \( u \in A \cup B \), let \( \delta(u) \) denote the set of edges incident to \( u \). For a subset of edges \( F \subseteq E \), let \( f(F) \) be the probability that at least one edge in \( F \) exists, then \( f(F) = 1 - \prod_{e \in F} (1 - p_e) \), because each edge \( e \) exists independently with probability \( p_e \).

Fix any query-commit algorithm \( A \). For each edge \( e \in E \), let \( x_e \) be the probability that \( A \) outputs \( e \). Consider a vertex \( u \in A \cup B \) and a subset \( F \subseteq \delta(u) \). Since \( A \) outputs a valid matching, the events that edge \( e \) being added to the output of \( A \) for each \( e \in F \) are disjoint, and hence the probability that \( A \) adds one of the edges in \( F \) to its output is \( \sum_{e \in F} x_e \). But, for an edge to be added to the output, it must exist in the first place, and thus it must be the case that \( \sum_{e \in F} x_e \leq f(F) \) (because \( f(F) \) is the probability that at least one edge in \( F \) exists). Therefore, \( x = (x_e)_{e \in E} \) is a feasible solution to the following linear program, which we call LP\(_{QC}\). So, the expected utility \( U(A) \) of \( A \) is at most the value of LP\(_{QC}\), which implies Lemma 1 below.

\[
\text{Maximize} \quad \sum_{e \in E} x_e \cdot w(e),
\]

subject to \( \sum_{e \in F} x_e \leq f(F), \) for all \( u \in A \cup B \) and for all \( F \subseteq \delta(u) \),

\( x_e \geq 0, \) for all \( e \in E \).

**Lemma 1.** The optimal expected query-commit utility, OPT, is upper bounded by the value of LP\(_{QC}\).

2.2 Structure of the LP and its implications

Although LP\(_{QC}\) has exponentially many constraints, we can solve it in polynomial time.

**Lemma 2.** The linear program LP\(_{QC}\) is polynomial-time solvable.

**Proof.** Observe that for a fixed vertex \( u \in A \cup B \), the constraints \( \sum_{e \in F} x_e \leq f(F) \) for all \( F \subseteq \delta(u) \), can be re-written as \( 0 \leq g_u(F) \) for all \( F \subseteq \delta(u) \), where \( g_u(F) = f(F) - \sum_{e \in F} x_e \) is a submodular function (notice that \( f \) is submodular because it is a coverage function while \( \sum_{e \in F} x_e \) is clearly modular). Thus we can minimize \( g_u \) over all subsets of \( \delta(u) \) for all \( u \in A \cup B \) in polynomial time using \( O(n) \) submodular minimizations to find a violating constraint. If none of the minimizations gives a negative value and if \( x_e \geq 0 \) for all \( e \in E \), then the solution is feasible. Thus we can solve LP\(_{QC}\) in polynomial-time using the ellipsoid method. \( \square \)

For the rest of this section, we assume that \( 0 < p_e < 1 \) for all \( e \in E \). We can safely ignore those edges \( e \in E \) for which \( p_e = 0 \), and for those with \( p_e = 1 \), we can scale down the probabilities (at a small loss in the objective value) due to the following lemma.

**Lemma 3.** Let \( \tilde{p}_e = (1 - \gamma)p_e \) for all \( e \in E \), and for a subset \( F \subseteq E \), let \( \tilde{f}(F) = 1 - \prod_{e \in F} (1 - \tilde{p}_e) \) be the probability that at least one edge in \( F \) exists under the scaled down probabilities \( \tilde{p}_e \). If we replace \( f(F) \) in LP\(_{QC}\) by \( \tilde{f}(F) \), the value of the resulting LP is at least \( (1 - \gamma) \) times the value of LP\(_{QC}\).
Proof. Fix a set $F \subseteq E$ and label the edges in $F$ from 1 through $|F|$. For $i = 1, \ldots, |F|$, let $Q_i = 1 - \prod_{e=1}^{i}(1-p_e)$ and let $\tilde{Q}_i = 1 - \prod_{e=1}^{i}(1-p_e)$. Then, for $i > 1$ we have that $Q_i = Q_{i-1} + p_i(1 - Q_{i-1})$, and similarly, $\tilde{Q}_i = \tilde{Q}_{i-1} + \tilde{p}_i(1 - \tilde{Q}_{i-1})$. By definition, we have $f(F) = Q_{|F|}$ and $\tilde{f}(F) = \tilde{Q}_{|F|}$. We now prove that $\tilde{Q}_i \geq (1 - \gamma)Q_i$ for $i = 1, \ldots, |F|$ by induction.

For the base case, we have $\tilde{Q}_1 = (1 - \gamma)Q_1$. Notice that, by the definition of $\tilde{p}_i$’s, we have $Q_i \geq \tilde{Q}_i$. Thus, for $i > 1$ we have that

$$\tilde{Q}_i = \tilde{Q}_{i-1} + \tilde{p}_i(1 - \tilde{Q}_{i-1}) \geq (1 - \gamma)Q_{i-1} + (1 - \gamma)p_i(1 - Q_{i-1}) \geq (1 - \gamma)Q_{i-1} + (1 - \gamma)p_i(1 - Q_{i-1}) = (1 - \gamma)(Q_i + p_i(1 - Q_{i-1})) = (1 - \gamma)Q_i.$$

Thus if we scale down the polytope defined by the constraints of LPQC by a factor of $(1 - \gamma)$, the resulting polytope is contained inside the polytope defined by $\tilde{f}(F)$ constraints. Moreover, all extreme points of both the polytopes have non-negative coordinates and the objective function has non-negative coefficients. Hence the claim of Lemma 3 follows.

Remark. The expected utility of our proposed algorithm is $(1 - 1/e) \cdot \text{OPT}^* \geq (1 - 1/e) \cdot \text{OPT}$, where $\text{OPT}^*$ is the optimal LP value of LPQC. Thus, in the cases where the assumption $p_e < 1$ for all $e \in E$ does not hold, we can scale the probabilities down by $(1 - \gamma)$, and consequently the guarantee on expected utility will at least be $(1 - \gamma)(1 - 1/e) \cdot \text{OPT}$ due to Lemma 3. We can choose $\gamma$ to be arbitrarily small. To implement the scaling down operation, we can simply replace each query made by an algorithm with a function that only queries with probability $(1 - \gamma)$.

The assumption that $0 < p_e < 1$ for all $e \in E$ yields the following lemma on the function $f$.

Lemma 4. Fix a vertex $u \in A \cup B$, and suppose that $0 < p_e < 1$ for all $e \in \delta(u)$. Then the function $f$ is strictly submodular and strictly increasing on subsets of $\delta(u)$. That is:

1. For all subsets $A, B \subseteq \delta(u)$ such that $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$, $f(A) + f(B) > f(A \cup B) + f(A \cap B)$.
2. For all $A \subseteq B \subseteq \delta(u)$, $f(A) < f(B)$.

Proof. Let $g(F) = 1 - f(F) = \prod_{e \in F}(1 - p_e)$ (note that $g(\emptyset) = 1$). Notice that for $F_1, F_2 \subseteq F$ such that $F_1 \cap F_2 = \emptyset$ and $F_1 \cup F_2 = F$, it holds that $g(F) = g(F_1) \cdot g(F_2)$. Let $A, B \subseteq \delta(u)$ be two sets such that $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$. It is sufficient to show that $g(A) + g(B) < g(A \cup B) + g(A \cap B)$. We have

$$g(A) + g(B) = g(A \cap B) \left( g(A \setminus B) + g(B \setminus A) \right),$$

and

$$g(A \cup B) + g(A \cap B) = g(A \cap B) \left( g(A \setminus B) \cdot g(B \setminus A) + 1 \right).$$
Since $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$, and $0 < p_e < 1$ for all $e \in E$, we have $g(A \cap B) > 0$ and both $a, b < 1$. Thus $a + b < 1 + a \cdot b$, because $(1 - a)(1 - b) > 0$. This combined with Equations (2) and (3) yields Property [1].

Now consider $A \subseteq \delta(u)$ and any edge $e \in \delta(u) \setminus A$. To prove Property [2], it is sufficient to show that $f(A \cup \{e\}) > f(A)$, or equivalently, $g(A \cup \{e\}) < g(A)$. This is straightforward since $g(A \cup \{e\})/g(A) = 1 - p_e < 1$, because $p_e > 0$.

Let $x^* = (x^*_e)_{e \in E}$ be an optimal solution to $LP_{QC}$. Fix a vertex $a \in A$. Then, $x_a = (x^*_e)_{e \in \delta(a)}$, which is $x^*$ restricted only to those coordinates that correspond to edges in $\delta(a)$, satisfy the following constraints:

$$
\sum_{e \in F} x_e \leq f(F), \quad \text{for all } F \subseteq \delta(a),
$$

$$
\text{for all } e \in \delta(a).
$$

Notice that these constraints are only a subset of the constraints of $LP_{QC}$.

Let $P^\text{QC}_a$ denote the polytope defined by the above constrains. The extreme points of $P^\text{QC}_a$ have a nice structure that becomes crucial when designing the probability distribution $P^\text{QC}_a$ over permutations of edges. Namely, for any extreme point, the sets for which Constraint [4] is tight form a chain. Moreover, each set in the chain has exactly one more element than its predecessor and this element is non-zero coordinate of the extreme point. Formally, we have Lemma 5 below.

**Lemma 5.** Let $y = (y_e)_{e \in \delta(a)}$ be an extreme point of $P^\text{QC}_a$ and let $Y = \{e \in \delta(a) : y_e > 0\}$ be the set of edges that correspond to the non-zero coordinates of $y$. Then there exist $|Y|$ subsets $S_1, \ldots, S_{|Y|}$ of $\delta(a)$ such that $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{|Y|}$ with the following properties:

1. Constraint [4] is tight for all $S_1, S_2, \ldots, S_{|Y|}$. That is $\sum_{e \in S_i} y_e = f(S_i)$ for all $i = 1, \ldots, |Y|$.

2. For each $i = 1, \ldots, |Y|$, the set $S_i \setminus S_{i-1}$ contains exactly one element $e_i$, and $y_{e_i}$ is non-zero (i.e., $e_i \in Y$).

**Proof.** It is clear that at least $|Y|$ constraints in [4] are tight. If $|Y| = 1$, the claim of the lemma is obviously true. Suppose that $|Y| > 1$. Now let $A, B \subseteq \delta(a)$ be two different sets for which Constraint [4] is tight. Then we have

$$
f(A) + f(B) = \sum_{e \in A} y_e + \sum_{e \in B} y_e = \sum_{e \in A \cup B} y_e + \sum_{e \in A \cap B} y_e \leq f(A \cup B) + f(A \cap B),
$$

where the last inequality follows because $y$ satisfies Constraint [4].

Observe that, if $A \nsubseteq B$ and $B \nsubseteq A$, then by Lemma 4, $f(A) + f(B) > f(A \cup B) + f(A \cap B)$. Thus, it must be the case that either $A \nsubseteq B$ or $B \nsubseteq A$, and consequently there exist $|Y|$ sets $S_1, S_2, \ldots, S_{|Y|}$ such that $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{|Y|}$, for which Constraint [4] is tight.

For each $i = 1, 2, \ldots, |Y|$, we thus have that $\sum_{e \in S_i \setminus S_{i-1}} y_e = f(S_i) - f(S_{i-1}) > 0$, where the inequality is due to the strictly increasing property of $f$. This implies that each $S_i \setminus S_{i-1}$ must contain at least one edge $e_i$ such that $y_{e_i} > 0$, and since there are only $|Y|$ non-zero coordinates in $y$, each $S_i \setminus S_{i-1}$ must contain exactly one such $e_i$. Now, suppose that some $S_i \setminus S_{i-1}$ contains some $e'_i$ such that $y_{e'_i} = 0$. Then $f(S_i) = \sum_{e \in S_i} y_e = \sum_{e \in S_i \setminus e'_i} y_e \leq f(S_i \setminus \{e'_i\}) < f(S_i)$ yields a contradiction. Here, the first inequality is due to Constraint [4] whereas the last inequality is due the strictly increasing property of $f$. 

\Box
Now fix a vertex \(a \in A\) and consider the simple query algorithm given in Algorithm 1, which outputs at most one edge adjacent to vertex \(a\). In Algorithm 1, \(D_a^{QC}\) is a distribution over the permutations of edges in some subsets of \(\delta(a)\) that, by Lemma 6, can be found in polynomial time. We have the following lemma considering Algorithm 1.

1. Draw a permutation \(\sigma\) from \(D_a^{QC}\) for some fixed vertex \(a \in A\).
2. foreach edge \(e\) in the order of \(\sigma\) do
   3. Query edge \(e\) to check whether it exists.
   4. If edge \(e\) exists, output \(e\) and terminate.

Algorithm 1: Query algorithm for selecting an edge adjacent to a fixed vertex \(a \in A\).

Lemma 6. Let \(x^*\) be an optimal solution to LP\(_{QC}\) and let \(x^*_a = (x^*_a)^{e \in \delta(a)}\) be its restriction to the coordinates that corresponds to edges in \(\delta(a)\). Then there exists a distribution \(D_a^{QC}\) over the permutations of subsets of \(\delta(a)\) with the following property: When the permutation \(\sigma\) is drawn from \(D_a^{QC}\) in Algorithm 1, the probability that the algorithm outputs the edge \(e\) is \(x^*_a\); hence, the expected weight of the edge output by Algorithm 1 is \(\sum_{e \in E_a} x^*_a \cdot w(e)\). Moreover, a permutation of edges from \(D_a^{QC}\) can be sampled in polynomial time.

Proof. Let \(y = (y_e)_{e \in \delta(a)}\) be an extreme point of \(P_a^{QC}\) and let \(|Y|\) be set of non-zero coordinates of \(y\). Let \(\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_{|Y|}\) be the chain of sets (for which Constraint \(4\) is tight) guaranteed by Lemma 5 for the extreme point \(y\). We can efficiently find the chain by first setting \(S_Y = Y\), and iteratively recovering \(S_{i-1}\) from \(S_i\) by trying all possible \(S_i \setminus \{e\}\) for \(e \in S_i\) to check whether Constraint \(4\) is tight. For each \(i = 1, \ldots, |Y|\), let \(e_i\) be the unique element in \(S_i \setminus S_{i-1}\) and let \(\sigma_y = (e_1, \ldots, e_{|Y|})\). Notice that \(S_i = \{e_1, \ldots, e_i\}\), and hence \(y_{e_i} = \sum_{e \in S_i} y_e - \sum_{e \in S_{i-1}} y_e = f(S_i) - f(S_{i-1})\). Thus if we select \(\sigma_y\) as the permutation in Algorithm 1 and query according to that order, the probability that it outputs the edge \(e_i\) is exactly

\[
Pr[\text{some edge in } S_i \text{ appears}] - Pr[\text{some edge in } S_{i-1} \text{ appears}] = f(S_i) - f(S_{i-1}) = y_{e_i}.
\]

Note that the point \(x^*_a\) is contained in polytope \(P_a^{QC}\). Thus, using the constructive version of Caratheodory’s theorem, we can efficiently find a convex combination \(x^*_a = \sum_{i \in [k]} a_i \cdot y^{(i)}\), where \(a_i \geq 0\) for all \(i \in [k]\), \(y^{(i)}\) is an extreme point of \(P_a^{QC}\) for all \(i \in [k]\), \(\sum_{i \in [k]} a_i = 1\), and \(k = poly(|E_a|)\). This is because we can optimize a linear function over \(P_a^{QC}\) in polynomial time using submodular minimization as a separation oracle, and for such polytopes, the constructive version of Caratheodory’s theorem holds (See Theorem 6.5.11 of [GLSSS88]).

Define the distribution \(D_a^{QC}\) such that it gives permutation \(\sigma^{y^{(i)}}\) with probability \(a_i\). If follows that, if we sample according to this distribution in Algorithm 1, then for any fixed edge \(e\), the probability that the algorithm outputs the edge \(e\) is \(\sum_{i \in [k]} a_i \cdot y^{(i)} = x^*_a\). Consequently, the expected weight of the output of Algorithm 1 is \(\sum_{e \in E_a} x^*_a \cdot w(e)\).

2.3 Proposed algorithm and analysis

Suppose that we run Algorithm 1 for all vertices \(a \in A\) and let \(M'\) be the set of all output edges. Then we have

\[
\mathbb{E} \left[ \sum_{e \in M'} w(e) \right] = \sum_{a \in A} \sum_{e \in \delta(a)} x^*_e \cdot w(e) = \sum_{e \in E} x^*_e \cdot w(e) \geq \text{OPT}.
\]
Furthermore, $M'$ contains at most one adjacent edge per each vertex $a \in A$. However $M'$ may contain more than one adjacent edge for some vertices $b \in B$, and hence it may not be valid matching.

Now again suppose that we run Algorithm 1 as described above for all vertices $a \in A$ in some arbitrary order. Consider some fixed vertex $b \in B$. By Lemma 6, from the perspective of $b$, an edge $e \in \delta(b)$ appears with probability $x^*_e$ (when we say an edge $e = (a, b)$ appears, it means that Algorithm 1, when run on vertex $a$, outputs the edge $e$). Viewing the vertices in $A$ as buyers, we think of the appearance of an edge $e = (a, b)$ as a buyer $a$ making a take-it or leave-it offer of value $w_e$ for item $b$. Thus if we use a uniformly random order of vertices in $A$, picking an edge adjacent to the fixed vertex $b$ can be viewed as an instance of the prophet secretary problem.

The $(1 - 1/e)$-competitive algorithm algorithm given by Ehsani et al. \cite{EHS18} for the prophet secretary problem first sets a base price for the item. If some buyer comes at time $t \in [0, 1]$, and if the item is not already sold, then the algorithm sells the item to this buyer if the offered price is at least $(1 - e^{t-1})$ times the base price. Since the prophet secretary problem deals with a single item, the goal is to choose the buyer with highest offer, and hence they set base price of the item as the expected value of the maximum offer.

However, rather than picking the maximum weighted edge adjacent to each $b$, we want to maximize the total weight of the matching constructed. Thus, we set the base price $c_b$ for each $b$, not as the the expectation of the offline secretary problem, but as the expected weight of the edge adjacent to $b$ in some optimal offline maximum-weight bipartite matching. To be concrete, we set $c_b = \sum_{e \in E_b} x^*_e \cdot w(e)$ (recall that we can think of $x^*_e$ as the probability that some fixed optimal algorithm for maximum weighted bipartite matching in query-commit model adds edge $e$ to its output).

1. Solve LP$_{QC}$ to get $x^*$ and find the permutation distributions $D_{QC}^a$ for all $a \in A$.
2. For each vertex $a \in A$, select $t_a \in [0, 1]$ (arrival time) independently and uniformly at random.
3. For each vertex $b \in B$, set the base price $c_b = \sum_{e \in E_b} x^*_e \cdot w(e)$.
4. Let $M$ be an empty matching.
5. **foreach** vertex $a \in A$ in the increasing order of $t_a$ **do**
   6. Draw a permutation $\sigma$ of edges from $D_{QC}^a$.
   7. **foreach** $e = (a, b)$ in the order of $\sigma$ **do**
      8. if $w(e) \geq (1 - e^{t_a-1}) \cdot c_b$ and $b$ is not matched then
         9. Query edge $e$ to check whether it exists.
         10. If it exists, add it to $M$ and continue to next vertex in $A$.
      else
         11. Flip a coin that give HEADS with probability $p_e$.
         12. If HEADS, continue to next vertex in $A$.
8. return the matching $M$.

**Algorithm 2: Outline of Approx-QC.**

We present the pseudo-code of our algorithm Approx-QC in Algorithm 2. We start by independently assigning each $a \in A$ a uniformly random arrival time $t_a \in [0, 1]$, and then for each vertex $a \in A$ in the order of the arrival time, we run a slightly modified version of the query algorithm given in Algorithm 1. For each $b \in B$, we pick an edge $e = (a, b)$ if it appears and if
its weight exceeds the threshold \((1 - e^{a-1}) \cdot c_b\). Since a query-commit algorithm is committed to adding any queried edge that exists, we query an edge \(e\) only if \(e \cap B\) is not already assigned to some other vertex \(a \in A\) and its weight \(w(e)\) exceeds the threshold. But we still need to make sure that, for a fixed \(b\), edges \(e \in \delta(b)\) appears (in the sense that if we run Algorithm \(\Pi\) it outputs the edge \(e\)) with probability \(x_e^*\). Hence we have the else clause of the conditional in Algorithm 2 that simulates the behavior of Algorithm \(\Pi\) in the cases we decide not to actually query an edge.

We conclude this section with Theorem 7 which shows that our algorithm \(\text{Approx-QC}\) is \((1 - 1/e)\)-approximate. The proof follows exactly the same lines (except for the definition of base price \(c_b\)) as in Ehsani et al. \[EHKS18\] to show that the expected weight of the edge adjacent to a fixed vertex \(b \in B\) in the output of \(\text{Approx-QC}\) is at least \((1 - 1/e) \cdot c_b\). Then by the linearity of expectation, the expected utility of \(\text{Approx-QC}\) is at least \((1 - 1/e) \cdot \sum_{b \in B} c_b = (1 - 1/e) \cdot \sum_{e \in E} x_e^* \cdot w(e) \geq \text{OPT}\) (recall that \(c_b = \sum_{e \in \delta(b)} x_e^* \cdot w(e)\)).

**Theorem 7.** The expected utility \(\mathcal{U}(\text{Approx-QC})\) of the algorithm \(\text{Approx-QC}\) is at least \((1 - 1/e) \cdot \text{OPT}\).

**Remark.** For the sake of completeness, we reproduce the analysis of Ehsani et al. \[EHKS18\] in Section 3 for our more general algorithm in the price of information model (see Theorem 17).

## 3 Extension to the Price of Information Model

In this section, we present a \((1 - 1/e)\)-approximation algorithm for the maximum weight bipartite matching problem (MWBM) in the price of information (PoI) model introduced by Singla \[Sin18\]. Our strategy is essentially the same as that used in Section 2 for the query-commit model except for a few enhancements.

Let \(G = (A \cup B, E)\) be a bipartite graph where each edge \(e \in E\) independently takes some random weight \(X_e\) from a known probability distribution. The weight distributions can be different for different edges and are independent. To find the realization of \(X_e\) (i.e., the actual weight of the edge \(e\)), we have to query the edge \(e\) at a cost of \(\pi_e\). Consider an algorithm \(\mathcal{A}\) that queries a subset \(Q\) of edges \(E\) and outputs a valid matching \(M \subseteq Q\). We call such an algorithm a PoI algorithm for MWBM. We define the expected PoI utility of Algorithm \(\mathcal{A}\) as

\[
\mathcal{U}(\mathcal{A}) := \mathbb{E} \left[ \sum_{e \in M} X_e - \sum_{e \in Q} \pi_e \right],
\]

where the expectation is taken over the randomness of \(X_e\)’s and any internal randomness of the algorithm. The goal is to design a polynomial-time PoI algorithm \(\mathcal{A}\) for MWBM such that its expected PoI utility \(\mathcal{U}(\mathcal{A})\) is maximized. Let \(\text{OPT} = \max \mathcal{U}(\mathcal{A})\) denote the optimal expected PoI utility, where the maximization is over all PoI algorithms.

Let \(E = \{e_1, \ldots, e_m\}\), and let \(X = (X_{e_1}, \ldots, X_{e_m})\). Let \(\mathcal{M}\) be the collection of all valid bipartite matchings in \(G\). The following lemma is due to Singla \[Sin18\].

**Lemma 8.** For each edge \(e \in E\), let \(\tau_e\) be the solution to the equation \(\mathbb{E}[\max\{X_e - \tau_e, 0\}] = \pi_e\) and let \(Y_e = \min(X_e, \tau_e)\). Then the optimal expected PoI utility \(\text{OPT}\) is upper bounded by \(\mathbb{E}X [\max_{M \in \mathcal{M}} \sum_{e \in M} Y_e]\).

To derive our algorithm, we go through the same two stages as in Section 2. We first construct a linear program (LP), this times defining the constraints using the probability distributions of \(Y_e\)’s (that were defined in Lemma 8), and use its value together with Lemma 8.
to upper bound OPT. For this, we discretize the distributions of \( Y_e \)'s, and in contrast to the query-commit setting, we now define variables \( x_{e,v} \) for each edge-value pair \((e,v)\). We think of \( x_{e,v} \) as the joint probability that \( Y_e = v \) and some fixed optimal PoI algorithm includes edge \( e \) in its output. The objective value of this LP upper bounds the quantity \( \mathbb{E}_X [ \max_{M \in \mathcal{M}} \sum_{e \in M} Y_e ] \), which in turn is an upper bound of the optimal expected PoI utility as stated in Lemma 8.

We describe the construction of our LP in Section 3.1. Next, in Section 3.2, we analyze the structure of our new LP as we did in the previous section and use it to define analogous probability distributions over subsets of edge-value pairs. Finally, in Section 3.3 we put everything together to construct our \((1 - 1/e)\)-approximate PoI algorithm for MWBM.

### 3.1 Upper-bounding the optimal expected utility.

Assume that the distributions of \( Y_e \) are discrete.\footnote{We can e.g. achieve this by geometric grouping into polynomially many classes.} For each \( e \in E \), let \( V_e \) denote the set of possible values of \( Y_e \). For each vertex \( u \in A \cup B \), let \( E_u = \{(e, v) : e \in \delta(u), v \in V_e \} \) be the set of all edge-value pairs for all edges incident to \( u \). Let \( E_{all} = \bigcup_{u \in A} E_u \) be the set of all edge-value pairs. For each edge \( e \in E \) and value \( v \in V_e \), let \( p_{e,v} \) be the probability that \( Y_e = v \), and for a set \( F \subseteq E_{all} \), let \( f(F) \) be the probability that \( Y_e = v \) for at least one edge-value pair \((e,v) \in F \).

Fix any Pol algorithm \( A \) for MWBM. For each edge-value pair \((e,v) \in E_{all} \), let \( A_{e,v} \) be the event that \( Y_e = v \) and \( A \) includes edge \( e \) in its output, and let \( x_{e,v} = \Pr[A_{e,v}] \). Now fix a vertex \( u \in A \cup B \) and a set \( F \subseteq E_u \). Then \( \Pr[\cup_{(e,v) \in F} A_{e,v}] \leq \Pr[Y_e = v \text{ for some } (e,v) \in F] = f(F) \).

But since all events \( A_{e,v} \) for \((e,v) \in E_u \) are mutually disjoint (as with the query-commit setting, the algorithm \( A \) outputs a valid matching and thus the output has at most one edge incident to vertex \( u \)), \( \Pr[\cup_{(e,v) \in F} A_{e,v}] = \sum_{(e,v) \in F} \Pr[A_{e,v}] = \sum_{(e,v) \in F} x_{e,v} \). Thus we have \( \sum_{(e,v) \in F} x_{e,v} \leq f(F) \), and this must be true for all \( u \in A \cup B \) and \( F \subseteq E_u \).

Now consider the following LP, which we call \( \text{LP}_{\text{PoI}} \):

\[
\begin{align*}
\text{Maximize} & \quad \sum_{(e,v) \in E_{all}} x_{e,v} \cdot v, \\
\text{subject to} & \quad \sum_{(e,v) \in F} x_{e,v} \leq f(F) \quad \text{for all } F \subseteq E_u \text{ for all } u \in A \cup B, \\
& \quad x_{e,v} \geq 0 \quad \text{for all } (u,v) \in E_{all}.
\end{align*}
\]

We have the following lemma concerning \( \text{LP}_{\text{PoI}} \).

**Lemma 9.** The optimal expected PoI utility OPT is upper bounded by the value of \( \text{LP}_{\text{PoI}} \).

**Proof.** By Lemma 8 we have that \( OPT \leq \mathbb{E}_X [\max_{M \in \mathcal{M}} \sum_{e \in M} Y_e] \). Now consider an algorithm \( \mathcal{A} \) that queries \( Y_e \) for all edges \( e \in E \) and outputs a maximum weighted bipartite matching \( M \) of \( G \). Setting \( x_{e,v} \) to be the joint probability that \( Y_e = v \) and \( e \in M \) for each edge-value pair \((e,v) \in E_{all} \) gives a feasible solution to \( \text{LP}_{\text{PoI}} \). Hence \( \mathcal{U}(\mathcal{A}) = \mathbb{E}_X [\max_{M \in \mathcal{M}} \sum_{e \in M} Y_e] \leq \sum_{(e,v) \in E_{all}} x^*_{e,v} \cdot v \), where \( x^* = (x^*_{e,v})_{(e,v) \in E_{all}} \) is an optimal solution of \( \text{LP}_{\text{PoI}} \). \( \square \)

### 3.2 Structure of the LP and its implications.

Now we analyze the structure of \( \text{LP}_{\text{PoI}} \). Our analysis closely follows that of Section 2.2.

As usual, \( f \) is a coverage function and hence it is submodular. Thus, for each vertex \( u \in A \cup B \), we can use submodular minimization to check whether any constraint of the form \( \sum_{(e,v) \in F} x_{e,v} \leq f(F) \) is violated for any subset \( F \subseteq E_u \). This yields Lemma 10 below.
Lemma 10. The linear program $LP_{\text{Pol}}$ is solvable in polynomial-time.

We proceed as follows. Consider the query strategy given in Algorithm 3 that queries edges incident to a fixed vertex $a \in A$ in some random order. This is the PoI version of the Algorithm 1 given for the query-commit setting. Following (almost) the same procedure as in Section 2.2, we find distributions $D_{\text{Pol}}^a$ that makes Algorithm 3 pick an edge $e$ that has value $v$ with probability $x_{e,v}^a$, and then use those to construct a PoI algorithm for MWBM that gives $(1 - 1/e)$ approximation guarantee. But the issue here is that Algorithm 3 considers the distributions of $Y_e$’s and does not pay query costs whereas our final approximate PoI algorithm needs to consider the distributions of $X_e$’s and has to pay query costs.

1. Let $z_e = \text{Null}$ for all $e \in \delta(a)$.
2. Draw a permutation $\sigma$ from $D_{\text{Pol}}^a$.
3. \textbf{foreach} $(e, v)$ in the order of $\sigma$ \textbf{do}
4. \hspace{1em} If $z_e = \text{Null}$, draw $z_e$ from a distribution identical to that of $Y_e$.
5. \hspace{1em} If $z_e = v$, output $e$ and terminate.

\textbf{Algorithm 3:} Query algorithm for selecting an edge incident to a fixed vertex $a \in A$.

Now consider the way we defined $\tau_e$ (which we used to define $Y_e$’s), and observe that the values of $X_e$ above the threshold $\tau_e$, on expectation, covers the cost $\pi_e$ of querying it. Thus, if we can make sure that the first time we query an edge $e$ (i.e., the time where we pay the price $\pi_e$) in Algorithm 3 is for the value $\tau_e$, then we can still use it to construct our final PoI matching algorithm (where we actually query $X_e$ values, and when an edge $e$ is queried for the first time for value $\tau_e$, in expectation we actually get a net value of $\tau_e$ with probability $x_{e,\tau_e}^a$ after paying $\pi_e$). Using a careful construction, we make sure that distributions $D_{\text{Pol}}^a$ only gives those permutations where for any edge $e$, the pair $(e, \tau_e)$ appears before any other pair $(e, v)$.

For such a construction, we consider a slightly different polytope $P_{\text{Pol}}^a$ (as opposed to how we defined $P_{\text{QC}}^a$) for each $a \in A$. Fix a vertex $a \in A$, and consider the family $E_a$ of subsets of $E_a$ defined as follows:

$$E_a := \{ F \subseteq E_a : (e, v) \in F \Rightarrow (e, v') \in F \text{ for all } v' \geq v \text{ such that } (e, v') \in E_a \}.$$ 

I.e., $E_a$ is a family of subsets of $E_a$ that satisfy the following: If a set $F$ of edge-value pairs is in $E_a$ and an edge-value pair $(e, v)$ is in $F$, then $F$ also contains all edge-value pairs for the same edge $e$ having values greater than $v$. It is easy to verify that if $A, B \in E_a$ then both $A \cup B \in E_a$ and $A \cap B \in E_a$, which makes $E_a$ a lattice family. (I.e., the sets in $E_a$ forms a lattice where intersection and union serve as $\text{meet}$ and $\text{join}$ operations respectively.)

We define below the polytope $P_{\text{Pol}}^a$ using a constraint for each set in the family $E_a$.

$$\sum_{(e, v) \in F} x_{e,v} \leq f(F) \quad \text{for all } F \in E_a \quad (5)$$

$$x_{e,v} \geq 0 \quad \text{for all } (e, v) \in E_a.$$ 

Analogous to our assumption $0 < p_e < 1$ for all $e \in E$ for the query-commit setting, we now assume that each $p_{e,v} > 0$ and for each edge $e$, $\sum_{v \in V_e} p_{e,v} < 1$. (I.e., we can assume that with some small probability $p_{e,*}$ the edge $e$ does not exist or equivalently, we can also assume $p_{e,0} > 0$ and $0 \notin V_e$. We omit the details, but one can use the same argument of re-scaling the probabilities to justify this assumption.) Under these assumptions on $p_{e,v}$’s, we have the following lemma. The proof resembles that of Lemma 4 from Section 2.2, and we defer it to Appendix A.
Lemma 11. Fix a vertex \( a \in A \). If \( p_{e,v} > 0 \) for all \((e,v) \in E_a\) and \( \sum_{v \in V_e} p_{e,v} < 1 \) for all \( e \in \delta(a) \), the function \( f \) is strictly submodular and strictly increasing on the lattice family \( \mathcal{E}_a \). Formally,

1. For any \( A, B \in \mathcal{E}_a \) such that \( A \setminus B \neq \emptyset \) and \( B \setminus A \neq \emptyset \), \( f(A) + f(B) > f(A \cap B) + f(A \cup B) \), and

2. For any \( A \subseteq B \subseteq E_a \), \( f(B) > f(A) \).

Similarly to the query-commit setting, we now analyze the structure of the extreme points of polytope \( P^\text{PoI}_a \). We have the following lemma, which is a slightly different version of Lemma \( \text{[5]} \) from Section \( \text{[2]} \).

**Lemma 12.** Let \( \mathbf{y} = (y_{e,v})_{(e,v) \in E_a} \) be an extreme point of \( P^\text{PoI}_a \) and let \( Y = \{ e \in E_a : y_{e,v} > 0 \} \) be the set of non-zero coordinates of \( \mathbf{y} \). Then there exist \( |Y| \) subsets \( S_1, \ldots, S_{|Y|} \) of \( E_a \) such that \( S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{|Y|} \) with the following properties:

1. Constraint \( \text{[3]} \) is tight for all \( S_1, S_2, \ldots, S_{|Y|} \). That is \( \sum_{(e,v) \in S_i} y_{e,v} = f(S_i) \) for all \( i = 1, \ldots, |Y| \).

2. For each \( i = 1, \ldots, |Y| \), the set \( (S_i \setminus S_{i-1}) \cap Y \) contains exactly one element \((e_i, v_i)\).

Moreover, for any other \((e, w) \in S_i \setminus S_{i-1} \), we have \( e = e_i \) and \( w \geq v_i \).

**Proof.** Property \( \text{[1]} \) and the fact that each \((S_i \setminus S_{i-1}) \cap Y \) contains exactly one pair \((e_i, v_i)\) follows from the proof of Lemma \( \text{[5]} \). It remains to show that each \( S_i \setminus S_{i-1} \) additionally contains only those edge-value pairs \((e_i, w)\) for which \( w \geq v_i \).

Suppose to the contrary that there is some \( S_i \setminus S_{i-1} \) that contains at least one other pair \((e', v')\) that violates this property. Let \( S'_i = S_{i-1} \cup \{(e_i, w) : w \geq v_i \} \). Then \( S_{i-1} \subseteq S'_i \subseteq S_i \) and \( S'_i \) is also in the family \( \mathcal{E}_a \). Thus we have that \( f(S'_i) \geq \sum_{(e,v) \in S'_i} y_{e,v} = \sum_{(e,v) \in S_i} y_{e,v} = f(S_i) \), which is a contradiction because \( f \) is strictly increasing and \( S'_i \subseteq S_i \). Here the first inequality holds because \( \mathbf{y} \) is in \( P^\text{PoI}_a \) and the first equality holds because \( S'_i \) contains all coordinates in \( S_i \) for which \( \mathbf{y} \) is non-zero. The last equality is true because \( S_i \) corresponds to a tight constraint for the extreme point \( \mathbf{y} \).

As in the previous section, we are now ready to construct the distribution \( D^\text{PoI}_a \). We present this explicit construction in the proof of Lemma \( \text{[13]} \) stated below, which is the counterpart of Lemma \( \text{[6]} \).

**Lemma 13.** Let \( \mathbf{x}^* \) be an optimal solution of LP\(_{\text{PoI}}\). For each vertex \( a \in A \), there exist a distribution \( D^\text{PoI}_a \) over the permutations of (subsets of) edge-value pairs in \( E_a \) that satisfies the following properties:

1. For each permutation \( \sigma \) drawn from \( D^\text{PoI}_a \), if edge-value pair \((e, v)\) appears in \( \sigma \), then the edge-value pair \((e, w)\) appears before \((e, v)\) in \( \sigma \) for all \( w \in V_e \) such that \( w \geq v \),

2. \( \Pr[\text{Algorithm \[3\] outputs } e] = \sum_{v \in V_e} x^*_{e,v} \) for all \( e \in \delta(a) \), and

3. \( \sum_{v \in V_e : v \geq w} \Pr[\text{Algorithm \[3\] outputs } e \text{ with value } v] \cdot v \geq \sum_{v \in V_e : v \geq w} x^*_{e,v} \cdot v \) for all \( e \in \delta(a) \) and \( w \in \mathbb{R}^+ \).

Moreover, a permutation of edge-value pairs from \( D^\text{PoI}_a \) can be sampled in polynomial time.
Proof. As with the case of Lemma 6, we first associate a permutation of edge-value pairs with each extreme point of $P_a^{\text{Pol}}$.

For an extreme point $y$, let $Y$ and $S_1, \ldots, S_{|Y|}$ be as defined in Lemma 12. Consider the permutation $\sigma_y$ of elements in $S_{|Y|}$ that is defined as follows: Start with $\sigma_y = []$ and for each $i = 1, \ldots, |Y|$, append to it the edge-value pairs in $S_i \setminus S_{i-1}$ in the decreasing order of value. Recall that for all $i = 1, \ldots, |Y|$, all edge-value pairs in $S_i \setminus S_{i-1}$ corresponds to a single edge. Let $(e, v) \in S_i$. Then, by the definition of the family $E_u$, $(e, v') \in S_i$ for all $v' \in E_e$ such that $v' \geq v$. Thus none of the sets $S_{i+1} \setminus S_i, S_{i+2} \setminus S_{i+1}, \ldots, S_{|Y|} \setminus S_{|Y|-1}$ can contain an edge-value pair $(e, v')$ such that $v' > v$. Also, since the elements in $S_i \setminus S_{i-1}$ are appended to $\sigma_y$ in decreasing order of values, $\sigma_y$ has the following property: If at any point the edge-value pair $(e, v) \in \sigma_y$ appears in $\sigma_y$, then $(e, w)$ appears in $\sigma_y$ before $(e, v)$ for all $w \in E_e$ such that $w > v$.

Let $\sigma_y = (e_1, v_1), \ldots, (e_i, u_i)$ be the permutation of edge value pairs associated with the extreme-point $y$. Let $T_i := \{(e_1, v_1), \ldots, (e_i, u_i)\}$ denote the set of first $i$ edge-value pairs in $\sigma_y$. If we select permutation $\sigma_y$ in Line 2 in Algorithm 3, the probability of it picking edge $e_i$ with value $v_i$ is exactly $f(T_i) - f(T_{i-1})$. Now define a new vector $y'$ with the same indices as $y$ as follows: For each $(e_j, v_j) \in \sigma_y$, $y'_e = f(T_j) - f(T_{j-1})$, and all the other coordinates of $y'$ are 0. Notice that $y'$ and $y$ satisfy the following:

1. $\sum_{e \in V_e} y'_{e,v} = \sum_{e \in V_e} y_{e,v}$ for all $e \in \delta(a)$, and
2. $\sum_{e \in V_e: v \geq w} y'_{e,v} \cdot v \geq \sum_{e \in V_e: v \geq w} y_{e,v} \cdot v$ for all $e \in \delta(a)$ and $w \in \mathbb{R}^+$.

To see this fix some set $S_i$ and let $(e_j, v_j), (e_{j+1}, v_{j+1}), \ldots, (e_k, v_k)$ be all the edge-value pairs in $S_i - S_{i-1}$. Then we know that $e_j = e_{j+1} = \cdots = e_k$, $v_j > v_{j+1} > \cdots > v_k$, and $y_{e_k,v_k} \neq 0$. We thus have that $\sum_{j \leq j} y'_{e_j,v_j} = f(T_k) - f(T_{j-1}) = f(S_i) - f(S_{i-1}) = y_{e_k,v_k}$ (recall that $(e_k, v_k)$ is the unique element in $S_i \setminus S_{i-1}$ for which $y_{e_k,v_k}$ is non-zero). This holds for elements in $S_i \setminus S_{i-1}$ in all $i = 1, \ldots, |Y|$, and yields the Property 1 above. To see Property 2 notice that within each $S_i \setminus S_{i-1}$, the weight of the non-zero coordinate $y_{e_k,v_k}$ is re-distributed among $y_{e_j,v_j}, \ldots, y_{e_k,v_k}$, and that $v_{j'} \geq v_k$ for $j' = j, j + 1, \ldots, k$.

Let $\sigma$ be the random variable that denotes the permutation picked by Algorithm 3. Then we have that, for all $e \in \delta(u)$,

$\Pr[\text{Algorithm 3 picks } e | \sigma = \sigma_y] = \sum_{e \in V_e} y'_{e,v} = \sum_{e \in V_e} y_{e,v},$

and for all $(e, w) \in E_u$,

$\sum_{e, w \in V_e: v \geq w} \Pr[\text{Algorithm 3 picks } e \text{ with value } v | \sigma = \sigma_y] \cdot v = \sum_{e, w \in V_e: v \geq w} y'_{e,v} \cdot v \geq \sum_{e, w \in V_e: v \geq w} y_{e,v} \cdot v.$

Now let $x^*_a$ be the restriction of the optimal solution the coordinates in $E_a$. Since the constraints that define $P_a^{\text{Pol}}$ are only a subset of the constraints of LP$_{\text{Pol}}$, $x^*_a$ lies in $P_a^{\text{Pol}}$. If we can optimize a linear function of $P_a^{\text{Pol}}$ in polynomial time, then we can follow the same lines of the proof of Lemma 9 and use the constructive version of Caratheodory’s theorem to find a convex combination $x^*_a = \sum_{i \in [k]} a_i \cdot y^{(i)}$, where $k = \text{poly}(|E_a|)$ and for each $i \in [k]$, $y^{(i)}$ is an extreme point of $P_a^{\text{Pol}}$. However, unlike in the query-commit case, the polytope $P_a^{\text{Pol}}$ only has constraints for sets of a lattice family, and as a result, we cannot use the usual submodular minimization as a separation oracle for LP$_{\text{Pol}}$. But luckily, Grötschel et al. [GLSS1] showed that we can minimize any submodular function over a lattice family in polynomial time. Thus we can efficiently find such a convex combination.
Once we have the convex combination, the rest is exactly the same as the query-commit setting. The distribution $\mathcal{D}_a^{PoI}$ returns the permutation $\sigma_{y(i)}$ with probability $a_i$ for all $i \in [k]$. One can easily verify that Properties 1-3 hold for this distribution.

3.3 Proposed algorithm and analysis.

We now present a $(1 - 1/e)$-approximate PoI algorithm Approx-PoI for MWBM.

The algorithm closely resembles the algorithm Approx-QC we presented for the query-commit model, but has two key differences. First, we now have to pay a price for querying edges. But, as we prove later, this price is already taken care of by the way we defined $Y_e$ variables. The second difference is that for each edge $e \in E$, we now have multiple values to consider, but regardless, with respect to a fixed vertex (i.e. an item) $b \in B$, the vertices $a \in A$ can still be viewed as buyers; The appearance of multiple edge-value pairs for the same edge can be interpreted as a distribution over values that the buyer $a$ offers for the item $b$.

The outline of our algorithm is given in Algorithm 4. Note that, we again have the else clause in the conditional to make sure that we do not change the probability distributions of the appearances of edge-value pairs even if we decide not to query some edges for certain values.

```plaintext
1 Solve $LP^{*}$ to get $x^*$ and find the permutation distributions $\mathcal{D}_a^{PoI}$ for all $a \in A$.
2 For each vertex $a \in A$, select $t_a \in [0, 1]$ (arrival time) independently and uniformly at random.
3 For each vertex $b \in B$, let $c_b = \sum_{(e, v) \in E_b} x^*_e \cdot v$.
4 Let $z_e = \text{Null}$ for all $e \in E$.
5 Let $M$ be an empty matching.
6 foreach vertex $a \in A$ in the increasing order of $t_a$ do
   7 Draw a permutation $\sigma$ from $\mathcal{D}_a^{PoI}$.
   8 foreach $(e = (a, b), v)$ in the order of $\sigma$ do
      9 if $v \geq (1 - e^{t_a - 1}) \cdot c_b$ and $b$ is not matched then
         10 If $z_e = \text{Null}$, pay $\pi_e$ and query edge $e$ to find its actual value. Let $z_e$ be this value.
         11 If $\min(z_e, \tau_e) = v$, add $e$ to $M$ and continue to next vertex in $A$.
      else
         12 If $z_e = \text{Null}$, draw $z_e$ from a distribution identical to that of $X_e$.
         13 If $\min(z_e, \tau_e) = v$, continue to next vertex in $A$.
   14 return the matching $M$.
```

Algorithm 4: Outline of Approx-PoI.

The analysis of the expected PoI utility of Approx-PoI consists of two parts. First we use the same technique used by Singla [Sin18] to show that we can analyze the expected utility using $Y_e$ variables and no query costs instead of using $X_e$ variables with query costs. Next we reproduce almost the same analysis by Ehsani et al. [EHKS18] to prove the approximation guarantee.

Let $Z$ be the value we get from Algorithm 4. Then $Z = \sum_{e \in E} \left( I^\text{pick}_e X_e - I^\text{query}_e \pi_e \right)$, where $I^\text{pick}_e$ and $I^\text{query}_e$ are the indicator variables for the events that edge $e$ is picked in Line 11 and it is queried in Line 10 respectively.

Now suppose that we run an identical copy of Algorithm 4 in parallel but we do not pay for querying in Line 10, but instead of gaining $X_e$, we only get $Y_e = \min(X_e, \tau_e)$. We say that this latter execution in the “free-information” world whereas the original algorithm runs in the
“price-of-information” world or PoI world. Let $Z'$ be the value we get in the free information world. We have the following lemma.

**Lemma 14.** The expected utility $\mathbb{E}[Z]$ of Approx-PoI (which is the PoI world) is equal to the expected utility of its counterpart in the free information world.

**Proof.** Consider a case where the algorithm picks an edge that is already queried before. In this case, both algorithms get the same value, so the expected increase to $Z$ and $Z'$ are the same.

Now consider the case where both algorithms query for some edge $e$. Notice that if an edge $e$ is queried at any point, it is queried for the edge-value pair $(e, \tau_e)$. This is because $\tau_e$ is the maximum possible value for an edge $e$, and as a result, $(e, \tau_e)$ appears before any other $(e, v)$ for $v < \tau_e$ (if it appears at all) in the permutations chosen in Line 7 of Algorithm 4. In this case, the expected increase to $Z'$ in the free information world is $\tau_e \cdot \Pr[X_e \geq \tau_e]$. The expected increase to $Z$ in the PoI world is

$$\begin{align*}
-\pi_e + \int_{\tau_e}^{\infty} t \cdot p_e(t) dt &= -\pi_e + \int_{\tau_e}^{\infty} (t - \tau_e) \cdot p_e(t) dt + \int_{\tau_e}^{\infty} \tau_e \cdot p_e(t) dt \\
&= -\pi_e + \mathbb{E}[(X_e - \tau_e)^+] + \tau_e \cdot \Pr[X_e \geq \tau_e].
\end{align*}$$

\[\square\]

To conclude our analysis, we now present following theorem on the approximation guarantee.

**Theorem 15.** The expected PoI utility $U(Approx-PoI)$ of Approx-PoI is at least $(1 - 1/e) \cdot \text{OPT}$. 

**Proof.** By Lemma [14] we have $U(Approx-PoI) = \mathbb{E}[Z] = \mathbb{E}[Z']$. Since the optimal value of LP_Pol is an upper bound on the optimal value OPT (by Lemma [9], it is sufficient to show that

$$\mathbb{E}[Z] \geq (1 - 1/e) \cdot \sum_{(e, v) \in E_{\text{all}}} x_{e,v}^* v = (1 - 1/e) \cdot \sum_{b \in B} c_b.$$ 

(Recall that $c_b = \sum_{(e, v) \in E_b} x_{e,v}^* v$ as defined on Line 3 in Algorithm 4). Let $Z'_b = \sum_{a \in \delta(b)} \pi_{a,b} Y_{a,b}$, so that $Z' = \sum_{b \in B} Z'_b$. We show that for any $b \in B$, $\mathbb{E}[Z'_b] \geq (1 - 1/e) \cdot c_b$. Then the theorem follows from the linearity of expectation. The following calculations now follow the analysis in [EHKS18] and are included for completeness.

We proceed by splitting $Z'_{a,b}$ into two parts:

$$Z'_{a,b} = \pi_{a,b}^\text{pick} (Y_{a,b} - (1 - e^{t_a - 1} \cdot c_b) + \pi_{a,b}^\text{pick} (1 - e^{t_a - 1} \cdot c_b).$$

Define $r(t) := \Pr[\text{no edge incident to } b \text{ is picked before time } t]$ and $\alpha(t) := 1 - e^{-t_a - 1}$. Notice that $r(t)$ is decreasing, and since our $t_a$’s are from a continuous distribution, $r(t)$ is a differentiable function. Thus we have

$$\mathbb{E} \left[ \sum_{a \in \delta(b)} N_{a,b} \right] = -\int_0^1 r'(t) \cdot (1 - 1/e^{t-1}) \cdot c_b dt = -c_b \int_0^1 r'(t) \cdot \alpha(t) dt.$$
By applying integration by parts,

\[
E \left[ \sum_{a \in \delta(b)} N_{a,b} \right] = -c_b \left[ (r(t) \cdot \alpha(t))_0^1 - \int_0^1 r(t) \cdot \alpha'(t) \, dt \right]
\]

\[
= c_b \left( (1 - 1/e) + \int_0^1 r(t) \cdot \alpha'(t) \, dt \right). \tag{6}
\]

Now we consider the expectation of \( M_{a,b} \). Note that the inequality below is due to the third property of the distributions \( D^\text{Pol}_a \) of edge-value pairs we used in the algorithm (see Lemma 13).

\[
E[M_{a,b}|t_a = t] \geq \Pr[\text{no edge incident to } b \text{ is picked before } t | t_a = t] \sum_{v \in V_{a,b}} \sum_{v \geq \alpha(t) \cdot c_b} x^*_v(v - \alpha(t) \cdot c_b).
\]

If \( t_a = t \), this means that \( t_a \) could not have arrived before \( t \). Hence \( \Pr[\text{no edge incident to } b \text{ is picked before } t | t_a = t] \geq \Pr[\text{no edge incident to } b \text{ is picked before } t] \), and thus we have

\[
\sum_{a \in \delta(b)} E[M_{a,b}|t_a = t] \geq \Pr[\text{no edge incident to } b \text{ is picked before } t | t_a = t] \sum_{a \in \delta(b)} \sum_{v \in V_{a,b}} \sum_{v \geq \alpha(t) \cdot c_b} x^*_v(v - \alpha(t) \cdot c_b)
\]

\[
\geq \Pr[\text{no edge incident to } b \text{ is picked before } t] \sum_{a \in \delta(b)} \sum_{v \in V_{a,b}} x^*_v(v - \alpha(t) \cdot c_b)
\]

\[
\geq r(t) \sum_{a \in \delta(b)} \sum_{v \in V_{a,b}} x^*_v(v - \alpha(t) \cdot c_b)
\]

\[
= r(t) \left( c_b - \alpha(t) \cdot c_b \sum_{a \in \delta(b)} \sum_{v \in V_{a,b}} x^*_v \right)
\]

\[
\geq r(t)(1 - \alpha(t)) \cdot c_b.
\]

Since \( t_a \) is uniformly distributed over \([0,1]\) for each \( a \in A \), it follows that

\[
E \left[ \sum_{a \in \delta(b)} M_{a,b} \right] = \int_0^1 \sum_{a \in \delta(b)} E[M_{a,b}|t_a = t] \, dt \geq c_b \int_0^1 r(t) \cdot \alpha(t) \, dt. \tag{7}
\]

Now (6) + (7) yields

\[
E \left[ \sum_{a \in \delta(b)} Z'_{a,b} \right] = E \left[ \sum_{a \in \delta(b)} N_{a,b} \right] + E \left[ \sum_{a \in \delta(b)} M_{ij} \right]
\]

\[
\geq c_b \left( (1 - 1/e) + \int_0^1 r(t) \cdot \alpha'(t) \, dt \right) + c_b \int_0^1 r(t)(1 - \alpha(t)) \, dt
\]

\[
= c_b(1 - 1/e) + c_b \int_0^1 r(t) \left( 1 - \alpha(t) + \alpha'(t) \right) \, dt
\]

\[
= (1 - 1/e) \cdot c_b.
\]
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A Proofs of Some Supplementary Results

Lemma 16 (Lemma 11). Fix a vertex \( a \in A \). If \( p_{e,v} > 0 \) for all \((e,v) \in E_a \) and \( \sum_{v \in V_e} p_{e,v} < 1 \) for all \( e \in \delta(a) \), the function \( f \) is strictly submodular and strictly increasing on the lattice family \( \mathcal{E}_a \). Formally,

1. For any \( A, B \in \mathcal{E}_a \) such that \( A \setminus B \neq \emptyset \) and \( B \setminus A \neq \emptyset \), \( f(A) + f(B) > f(A \cap B) + f(A \cup B) \), and

2. For any \( A \subsetneq B \subseteq E_a \), \( f(B) > f(A) \).

Proof. It is easy to see that \( f(F) = 1 - \prod_{e \in F}(1 - \sum_{v \in V_e:(e,v) \in F}p_{e,v}) \). Consider two sets \( A, B \in \mathcal{E}_a \) such that \( A \setminus B \neq \emptyset \) and \( B \setminus A \neq \emptyset \). For each \( e \in E \), let \( a_e = 1 - \sum_{v \in V_e:(e,v) \in A}p_{e,v} \) and \( b_e = 1 - \sum_{v \in V_e:(e,v) \in B}p_{e,v} \). Thus we have that \( f(A) = 1 - \prod_{e \in A} a_e \), \( f(B) = 1 - \prod_{e \in B} b_e \), \( f(A \cup B) = 1 - \prod_{e \in E} \min(a_e,b_e) \), and \( f(A \cap B) = 1 - \prod_{e \in E} \max(a_e,b_e) \). The last two equations follow from the definition of the family \( \mathcal{E}_a \).

Now we have

\[
\begin{align*}
\quad f(A) + f(B) - f(A \cup B) - f(A \cap B) &= \prod_{e \in E} \min(a_e,b_e) + \prod_{e \in E} \max(a_e,b_e) - \prod_{e \in E} a_e - \prod_{e \in E} b_e. \\
&= \prod_{e \in E} \min(a_e,b_e) + \prod_{e \in E} \max(a_e,b_e) - \prod_{e \in E} \prod_{e \in E} a_e - \prod_{e \in E} b_e.
\end{align*}
\]

Thus to prove Property 1 it is sufficient to prove that the right hand side of (8) is strictly greater than zero, which is equivalent to showing that \( \prod_{e \in E} \min(a_e,b_e) + \prod_{e \in E} \max(a_e,b_e) > \prod_{e \in E} a_e + \prod_{e \in E} b_e \).

Since \( A \setminus B \neq \emptyset \), we have \( a_e < b_e \) for at least one edge \( e_1 \in E \). To see this, let \((e,v) \in A \setminus B \). Then for such edge \( e \), it follows from the definition of set family \( \mathcal{E}_a \) that the set \( A_e = A \cap \{ (e,w) : w \in V_e \} \) contains all the elements in the set \( B_e = B \cup \{ (e,w) : w \in V_e \} \), and in addition, it also contains at least one more element, namely \((e,v)\). Since \( p_{e,v} > 0 \), \( a_e < b_e \). Similarly, we have \( a_e > b_e \) for at least one edge \( e_2 \in E \) (since we assumed that \( p_{e,v} > 0 \) for all \((e,v) \in E_a \)). Let \( E_1 \) be the (nonempty) set of edges for which \( a_e < b_e \), and let \( E_2 \) be the (nonempty) set of edges for which \( a_e > b_e \). Without loss of generality we assume that \( E_1 \cup E_2 = E \) (if \( a_e = b_e \) for some \( e \), then we can divide (8) by \( a_e \) since \( \sum_{v \in V_e} p_{e,v} < 1 \), \( a_e > 0 \)). We then have

\[
\begin{align*}
\prod_{e \in E} \min(a_e,b_e) + \prod_{e \in E} \max(a_e,b_e) &= \prod_{e \in E_1} b_e \prod_{e \in E_2} a_e + \prod_{e \in E_2} a_e \prod_{e \in E_1} b_e \\
&> \prod_{e \in E} a_e \prod_{e \in E} b_e \\
&= \prod_{e \in E} a_e \prod_{e \in E} b_e
\end{align*}
\]

as required. The inequality above follows from the rearrangement inequality as \( \prod_{e \in E_1} b_e > \prod_{e \in E_1} a_e \) and \( \prod_{e \in E_2} a_e > \prod_{e \in E_2} b_e \).

As for Property 2 suppose that \( A \subseteq B \). Then for all \( e \in E \), \( a_e \geq b_e \), and for at least one \( e \in E \), \( a_e > b_e \). Hence \( f(B) - f(A) = \prod_{e \in E} a_e - \prod_{e \in E} b_e > 0 \). \( \square \)