Quantum communication in a superposition of causal orders

Sina Salek,1 Daniel Ebler,2 and Giulio Chiribella1,2,3

1Department of Computer Science, University of Oxford, Wolfson Building, Parks Road, United Kingdom
2Department of Computer Science, The University of Hong Kong, Pokfulam Road, Hong Kong
3Canadian Institute for Advanced Research, CIFAR Program in Quantum Information Science, Toronto, ON M5G 1Z8

Quantum mechanics allows for situations where the relative order between two processes is entangled with a quantum degree of freedom. Here we show that such entanglement can enhance the ability to transmit quantum information over noisy communication channels. We consider two completely dephasing channels, which in normal conditions are unable to transmit any quantum information. We show that, when the two channels are traversed in an indefinite order, a quantum bit sent through them has a 25% probability to reach the receiver without any error. For partially dephasing channels, a similar advantage takes place deterministically: the amount of quantum information that can travel through two channels in a superposition of orders can be larger than the amount of quantum information that can travel through each channel individually.

Introduction. At its core, information theory is a theory of resources. In Shannon’s information theory, the resources are classical: the carriers of information are classical systems, travelling along well-defined trajectories. In quantum Shannon theory, the carriers of information are quantum particles, whose internal state can be in a superposition of the basic classical states. Still, all information carriers are assumed to travel along well-defined trajectories. In particular, when a message is sent through a sequence of communication channels, the order in which the channels are traversed is assumed to be well-defined.

In principle, quantum mechanics allows particles to experience multiple processes in a superposition of different orders [1]. In these situations, the order of the processes becomes entangled with a quantum degree of freedom. This type of entanglement, called causal non-separability [2, 3], is a new resource, which provides advantages in quantum computation [4, 5], nonlocal games [2], and communication complexity [6]. Recently, entanglement between the order of two processes and a quantum bit has been experimentally realized with photons, using polarization [7, 8] and orbital angular momentum degrees of freedom [9]. An even more radical realization could arise in a quantum gravity scenario, where the order of two processes could be entangled with the configuration of the masses in the universe [10].

Recently, we addressed the extension of quantum Shannon theory to scenarios where the order of the communication channels can be in a superposition [11]. We found out that the correlations between the message and the quantum system controlling the order have dramatic consequences on the ability to transmit classical data. Specifically, a receiver with access to such entanglement can decode classical messages even if they have travelled through two completely depolarizing channels, each of which replaces the input state with white noise. This effect has been tested experimentally in a new experiment [12].

For classical communication, it turns out that the advantage of superposing orders is a generic phenomenon. Every two channels with constant output, when traversed in a superposition of two orders, enable the transmission of some classical information. In contrast, advantages in quantum communication are more elusive. For instance, the example of Ref. [11] does not apply to quantum communication: no quantum bit can be sent reliably through two completely depolarizing channels, even if their order is indefinite. It is then natural to ask whether the superposition of orders may offer any advantage in quantum communication, or whether instead there exist some fundamental reason why the advantages are limited to classical communication.

In this Letter we answer the question, showing that two noisy channels in a superposition of two different orders can transmit more quantum information than each channel individually. This situation is illustrated pictorially in Figure 1. The simplest example involves two completely dephasing channels, which collapse quantum states on the two complementary bases \{\lvert0\rangle, \lvert1\rangle\} and \{\lvert+\rangle, \lvert-\rangle\}, respectively. In normal conditions, no quantum information can go through either of the two channels. Instead, as we demonstrate below, when the two channels act in a superposition of two orders, a quantum bit travelling through them has a 25% probability to reach the receiver without any error. The successful transmission is heralded by the outcome of a measurement on the order qubit, which allows the receiver to know whether the transmission has been successful or not. The availability of a noiseless heralded channel can be used to implement the BB84 cryptographic protocol [13], or to establish entanglement in the E91 protocol [14], or, more generally, to implement any other protocol that only requires the transmission of individual qubits uncorrelated with each other.

The combination of two channels in indefinite order also gives rise to deterministic advantages. Explicitly, we show that for partially dephasing channels there exists a parameter region where the quantum capacity of two channels in a superposition of orders is larger than

\[\text{amount of quantum information that can travel through each channel individually.}\]
definite causal order.

of quantum information beyond what was possible in a

can increase the entanglement and enable transmission

techniques are available. Hereafter, we will call such
degree of freedom the order qubit. Mathematically, the
quantum SWITCH is a higher-order transformation [15,16] that takes in input two channels $\mathcal{E}$ and $\mathcal{F}$, along with a
state $\omega$ of the order qubit, and returns in output the
channel $S_\omega(\mathcal{E}, \mathcal{F})$ defined as

$$S_\omega(\mathcal{E}, \mathcal{F})(\rho) := \sum_{i,j} W_{ij} (\rho \otimes \omega) W_{ij}^\dagger,$$

with Kraus operators

$$W_{ij} := E_i F_j \otimes |0\rangle \langle 0| + F_j E_i \otimes |1\rangle \langle 1|,$$

where $\{E_i\}$ and $\{F_j\}$ are the Kraus operators of $\mathcal{E}$ and $\mathcal{F}$, respectively. Intuitively, the Kraus operators $W_{ij}$ can be interpreted as describing a particle experiencing the elementary processes $E_i$ and $F_j$ either in the order $E_i F_j$ or in the order $F_j E_i$, depending on whether the order qubit is in the state $|0\rangle$ or $|1\rangle$. It is important to stress, however, that the definition of the channel $S_\omega(\mathcal{E}, \mathcal{F})$ is independent of the choice of Kraus operators for $\{E_i\}$ and $\{F_j\}$, and instead depends solely on the channels $\mathcal{E}$ and $\mathcal{F}$ [11].

In the following we will consider the use of the channel $S_\omega(\mathcal{E}, \mathcal{F})$ for quantum communication between a sender and a receiver. It is important to stress that the sender can only encode information in the particle, and not in the order qubit. The order qubit is part of the definition of the communication channel, and is only available to the receiver. Physically, this situation would arise naturally in the gravitational realization of the quantum SWITCH, where the order qubit corresponds to two alternative configurations of the masses in the universe [10]. In this case it is natural to assume that the sender cannot modify the configuration of the masses, but the receiver can observe it in order to better decode the sender’s message.

The ability of a channel $\mathcal{N}$ to transmit quantum information is quantified by its quantum capacity $Q(\mathcal{N})$, defined as the number of qubits transmitted per channel use in the limit of asymptotically many uses [17,19]. A lower bound to the capacity is given by the coherent information, defined as

$$Q^{(1)}(\mathcal{N}) := \max_{\phi} \langle A | B \rangle_\sigma$$

$$:= \max_{\phi} H(B)_\sigma - H(AB)_\sigma,$$

where $A$ and $B$ are the input and output systems of channel $\mathcal{N}$, respectively, $H(X)_\rho := -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy of the system $X$ in the state $\rho$, with $X \in \{B, AB\}$, the optimisation is with respect to all pure bipartite states $\phi_{AA'}$, involving the input system $A$ and a reference system isomorphic to $A$, and $\sigma := (\mathcal{I}_A \otimes N_{A' \rightarrow B})(\phi_{AA'})$. Since the coherent information is

\[\begin{align*}
\rho &\quad \mathcal{E}(\rho) \\
\mathcal{F}(\rho) &\quad \mathcal{E}(\mathcal{F}(\rho))
\end{align*}\]

FIG. 1: On the top: pictorial representation of the history of a quantum particle experiencing two noisy channels $\mathcal{F}$ and $\mathcal{E}$ in a superposition of the two alternative orders $\mathcal{EF}$ and $\mathcal{FE}$. Each frame shows the state of the particle’s internal degree of freedom at a given moment of time. On the bottom: history of particles travelling through only one of the channels $\mathcal{F}$ or $\mathcal{E}$.

Our findings highlight a new link between quantum Shannon theory and indefinite causal order. In quantum Shannon theory, the ability to transmit quantum information depends on the ability to establish entanglement between the sender’s and receiver’s labs. Here we see that, when the order of the communication channels in between them is a quantum degree of freedom, it can increase the entanglement and enable transmission of quantum information beyond what was possible in a definite causal order.

Preliminaries. A quantum channel $\mathcal{E}$ acting on a state $\rho$ can be expressed in the Kraus representation as $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$, where $\{E_i\}$ are suitable operators. Intuitively, each operator $E_i$ can be thought as representing an elementary process, whereby the state $\rho$ is transformed into the state $\rho_i \propto E_i \rho E_i^\dagger$ with probability $p_i = \text{Tr}[E_i^\dagger E_i \rho]$. The quantum SWITCH [11] describes the mechanism whereby the particle experiences two quantum channels $\mathcal{E}$ and $\mathcal{F}$ in a superposition of two different orders, $\mathcal{EF}$ and $\mathcal{FE}$. The order depends on the state of a quantum degree of freedom, which can be taken to be two-dimensional without loss of generality. Hereafter, we will call such degree of freedom the order qubit. Mathematically, the quantum SWITCH is a higher-order transformation [15,16] that takes in input two channels $\mathcal{E}$ and $\mathcal{F}$, along with a state $\omega$ of the order qubit, and returns in output the channel $S_\omega(\mathcal{E}, \mathcal{F})$ defined as

$$S_\omega(\mathcal{E}, \mathcal{F})(\rho) := \sum_{i,j} W_{ij} (\rho \otimes \omega) W_{ij}^\dagger,$$

with Kraus operators

$$W_{ij} := E_i F_j \otimes |0\rangle \langle 0| + F_j E_i \otimes |1\rangle \langle 1|.$$
generally non-additive [20], the bound \( Q(N) \geq Q^{(1)}(N) \) can be a strict inequality, indicating that the channel \( N \) allows for higher capacity if entangled inputs are used.

Heralded noiseless communication through dephasing channels. Let \( E \) be the bit flip channel \( E(\rho) = (1 - p)\rho + pX\rho X \) and let \( F \) be the phase flip channel \( F(\rho) = (1 - q)\rho + qZ\rho Z \), where \( X \) and \( Z \) are the Pauli matrices \( X = |1\rangle\langle 1| + |0\rangle\langle 0| \) and \( Z = |0\rangle\langle 0| - |1\rangle\langle 1| \), and \( p \) and \( q \) are probabilities. The bit flip and phase flip channels belong to the family of degradable channels, whose quantum capacity is exactly equal to the coherent information, namely \( Q(N) = Q^{(1)}(N) \) [21]. Specifically, their capacities are [22]

\[
Q(E) = 1 - H_2(p) \\
Q(F) = 1 - H_2(q),
\]

where \( H_2(x) := -x \log x - (1 - x) \log(1 - x) \) is the binary entropy.

When two noisy channels are combined in a definite order, the transmission of quantum information satisfies a bottleneck inequality [23], stating that the overall quantum capacity is smaller than the minimum among \( Q(E) \) and \( Q(F) \). We now show that, when the two channels are combined in a superposition of orders, the bottleneck inequality can be violated. To this purpose, insert the bit flip and phase flip channels into the quantum SWITCH, initializing the order qubit in the state \( \omega = |+\rangle\langle +| \). The resulting channel \( S_{|+\rangle\langle+|}(E, F) \) can be computed by separately evaluating the four terms corresponding to the operator basis \( \{|0\rangle, |1\rangle\} \) in the space of the order qubit. The diagonal terms are

\[
S_{|\rangle\langle k|}(E, F)(\rho) = \left[ (1 - p)(1 - q)\rho + p(1 - q)X\rho X + q(1 - p)Z\rho Z \right] \otimes |k\rangle\langle k|,
\]

where \( Y \) is the Pauli matrix \( Y = i(|1\rangle\langle 0| - |0\rangle\langle 1|) \) and \( k \in \{0, 1\} \). The off-diagonal terms, representing the interference between the two causal orders, are

\[
S_{|\rangle\langle k\oplus 1|}(E, F)(\rho) = \left[ (1 - p)(1 - q)\rho + p(1 - q)X\rho X + q(1 - p)Z\rho Z \right. \\
\left. - pq Y\rho Y \right] \otimes |k\rangle\langle k\oplus 1|,
\]

where \( \oplus \) denotes addition modulo 2. On the particle’s Hilbert space, the only difference between diagonal and off-diagonal terms is the minus sign appearing in front of the term \( Y\rho Y \). This change of sign arises from the non-commutativity of the Pauli operators, expressed by the condition \( ZX = -XZ = iY \).

Let us now combine the diagonal and off-diagonal terms into the final state of the particle and the order qubit. Using Equations (5) and (6) one obtains the expression

\[
S_{|+\rangle\langle+|}(E, F)(\rho) = \left[ (1 - p)(1 - q)\rho + p(1 - q)X\rho X + q(1 - p)Z\rho Z \right] \otimes |+\rangle\langle +| + \left[ pq Y\rho Y \right] \otimes |\rangle\langle -|.
\]

The above state is a mixture of two states of the quantum particle, labelled by two orthogonal states \( \{|+\rangle, |-\rangle\} \) of the order qubit. If the order qubit is measured in the Fourier basis, the outcome corresponding to the state \( |-\rangle \) will herald the presence of the state \( \rho_- = Y\rho Y \). In this case, happening with probability \( pq \), the receiver can decode the sender’s message without any error, just by applying the correction operation \( Y \).

The above discussion shows that two dephasing channels combined in the quantum SWITCH yield a heralded noiseless communication channel with probability \( pq \). This is particularly remarkable when \( p = q = 1/2 \), because in this case the two channels \( E \) and \( F \) are completely dephasing, and therefore no quantum information can be sent through either of them individually. In stark contrast, a particle that traverses the two channels in a superposition of two orders can deliver a qubit to the receiver with probability 25%. The ability to transmit single qubits probabilistically could be used to implement the BB84 protocol [13], the distribution of entanglement in the E91 protocol [14], or any other cryptographic protocol that does not require the transmission of correlated qubits. The probabilistic nature of the transmission would reduce the key rate by a factor 4 compared to the rate noiseless protocols, but this would still be infinitely better than communicating in a definite causal order, where no key could be established at all.

Notice that the possibility of heralded noiseless communication is not limited to the bit flip and phase flip channels, but it applies more generally to every pair of qubit dephasing channels, i.e. every pair of channels that collapse the qubit’s state in some pair of bases. Since all these channels are unitarily equivalent to a bit flip and a phase flip, heralded noiseless communication can be ob-
tained by adjusting the bases with suitable pre-processing and post-processing operations.

The probabilistic removal of the noise, observed when two dephasing channels are combined in the quantum SWITCH, bears some similarity with the technique of error filtration proposed by Gisin et al in Ref. [24]. Error filtration concerns the propagation of a particle a superposition of paths, each of which is subject to an independent noise. By performing an interference measurement in the output, it is then possible to discard events where the noise is more severe, thereby extracting a cleaner signal. The noise removal implemented by the quantum SWITCH is a similar phenomenon, except that the noise in the two paths is not independent: a particle traveling through the channel $S_\omega(E,F)$ will experience either the elementary process $E_iF_j$ or the elementary process $F_jE_i$, which are correlated with one another. For dephasing channels, the presence of such correlation enables a complete removal of the noise, an effect that cannot be observed if the processes along the different paths are independent [see our appendix].

Violation of the bottleneck inequality. We now show that indefinite causal order also offers a deterministic advantage. To this purpose, we evaluate the coherent information that indefinite causal order also offers a deterministic advantage. To this purpose, we evaluate the coherent information of the channel $$H(\rho^{(1)}(S_{++})(E,F)) = 1 + H_2(p^2) - 2H_2(p),$$ for the values of $p$ where the expression in Eq. (8) is non-negative, and $H_2(p^2) - 2H_2(p)$ otherwise.

Since the coherent information is a lower bound to the capacity, every region where $H_2(p^2) - 2H_2(p)$ exceeds $Q(\varepsilon)$ or $Q(F)$ will be a region where indefinite causal order offers an advantage. The comparison is made in Figure 2, which demonstrates the existence of an advantage for all values of $p \geq 0.62$. In this parameter region, the quantum capacity of channels $E$ and $F$ in indefinite causal order is larger than the capacity of each channel individually. By the bottleneck inequality, this also means that the capacity in indefinite causal order is larger than the capacity of the channels $E F$ and $F E$, corresponding to definite causal orders. In fact, the bottleneck inequality also implies an advantage over every channel of the form $AE\overrightarrow{BFC}$, where $A$, $B$, and $C$ are arbitrary quantum channels matching the inputs and outputs of $E$ and $F$.

Conclusions. We have shown that the possibility of indefinite causal order in quantum mechanics has a striking impact on quantum communication. A particle experiencing two processes in a quantum superposition of two alternative orders can sometimes carry more quantum information than than a particle travelling through each process individually. This holds even if both processes completely block quantum information. When the processes take place in a superposition of different orders, the blockage can be lifted, allowing quantum bits to arrive intact to a receiver with a finite probability of success.

The origin of the advantage is that the quantum SWITCH transfers information from the initial state of the particle to the final state of the particle and the quantum degree of freedom that controlled the order. When this happens, the quantum information is locked in the correlations: no quantum information can be retrieved from the particle alone, nor from the order-controlling system alone. In particular, if the order-controlling system is discarded, or projected in the basis associated to the two classical orders, then the quantum information carried by the particle becomes inaccessible. Despite the correlations in the output state are of classical nature, they are made possible by the quantum entanglement between the order and the order-controlling quubit, also referred to as causal non-separability [2, 8]. Our work demonstrates that physical scenarios exhibiting causal non-separability offer a resource for quantum communication, which could be exploited in key distribution protocols and other communication protocols where quantum information is encoded in single qubits.

Acknowledgments. This work is supported by the National Natural Science Foundation of China through grant 11675136, the Croucher Foundation, the Canadian Institute for Advanced Research (CIFAR), the Hong Research Grant Council through grant 17326616, and the Foundational Questions Institute through grant FQXi-
Comparison with error filtration

Here we show a difference between the quantum SWITCH mechanism and the technique or error filtration proposed in Ref. [24]. Specifically, we show that when two random unitary channels are combined together, error filtration can reduce the error, but cannot eliminate it completely.

Suppose that a quantum particle can travel through one of two regions, \( R_0 \) and \( R_1 \), in which it undergoes one of the two unitary gates \( U_0 \) or \( U_1 \), respectively. The choice of the region is determined by the states \(|0\rangle\) and \(|1\rangle\) of a quantum degree of freedom, e.g. the path of the particle, acting as a control qubit. The internal degree of freedom of the particle and the control qubit evolve through the gate \( W = U_0 \otimes |0\rangle\langle 0| + U_1 \otimes |1\rangle\langle 1| \). If the control qubit is initially in the state \(|\alpha\rangle = |0\rangle + |1\rangle\) and is finally postselected in the state \(|\beta\rangle = |0\rangle + |\beta_1\rangle |1\rangle \), then the internal degree of freedom will experience the process described by the operator \( U_{\alpha,\beta} \).

\[
U_{\alpha,\beta} = \alpha \beta_0^* U_0 + \alpha_1 \beta_1 U_1. \tag{9}
\]

Now, suppose that the unitary gates \( U_0 \) and \( U_1 \) are random, with probability distributions \( p(U_0) \) and \( q(U_1) \). If the initial state of the particle is \( \rho \), then the final state will be

\[
\rho' \propto \sum_{U_0, U_1} p(U_0) q(U_1) A_{U_0, U_1} \rho A_{U_0, U_1}^\dagger, \tag{10}
\]

where the sum runs over the possible values of \( U_0 \) and \( U_1 \), assumed to be a finite set for simplicity.

One may ask under which conditions the postselected evolution is unitary, meaning that the postselected state \(|\alpha\rangle\rangle|\beta\rangle\rangle \) is of the form \( U \rho U^\dagger \) for some unitary operator \( U \). We have the following

**Proposition 1.** Suppose that the unitary \( U_0 \) can take (at least) two distinct values, \( V \) and \( W \), and the unitary \( U_1 \) can take (at least) two distinct values, \( S \) and \( T \). If at least three of the unitaries \( \{V, W, S, T\} \) are linearly independent, then there exist no pair of states \(|\alpha\rangle\rangle|\beta\rangle\rangle \) such that the evolution \(|\alpha\rangle\rangle|\beta\rangle\rangle \) is unitary.

**Proof.** Since equation (10) is a convex decomposition of \( \rho' \), the condition \( \rho' = U \rho U^\dagger \) implies

\[
A_{U_0, U_1} \rho A_{U_0, U_1}^\dagger \propto U \rho U^\dagger, \tag{11}
\]

for every \( U_0 \) and \( U_1 \). Since \( \rho \) is an arbitrary density matrix, the above condition holds if and only if

\[
A_{U_0, U_1} \propto U. \tag{12}
\]

Without loss of generality, we assume that \( U_0 \) can take the values \( V \) and \( W \), and that \( U_1 \) can take the value \( S \).
where \( \{V, W, S\} \) are linearly independent. Now, condition (12) implies
\[
A_{V,S} = \lambda A_{W,S}
\]
for some proportionality constant \( \lambda \in \mathbb{C} \). Using the definition (9), the above condition can be rephrased as
\[
\alpha_0 \beta_0 V + (\alpha_1 \beta_1)(1 - \lambda) S = \alpha_0 \beta_0 W.
\]
Since \( V, W, \) and \( S \) are linearly independent, one must have \( \alpha_0 \beta_0 = 0 \). From Equation (9), it follows that the output state is
\[
\rho' \propto \sum_{U_1} q(U_1) U_1 \rho U_1^\dagger.
\]
Since the unitary \( U_1 \) can take two distinct values, the evolution \( \rho \rightarrow \rho' \) cannot be unitary.

Proposition 1 can be applied to the random-unitary channels \( \mathcal{E} \) and \( \mathcal{F} \), both of which are random-unitary, and have the Pauli matrices \( \mathbf{Pauli} = \{I, X, Y, Z\} \) as possible unitaries. According to Proposition 1 a particle that experiences two independent unitary gates \( U_0 \in \mathbf{Pauli} \) and \( U_1 \in \mathbf{Pauli} \) will be subject to an unavoidable noise, even in postselection. This fact is not in contradiction with our result on the noiseless heralded communication enabled by the quantum SWITCH: in the quantum SWITCH, the two alternative processes \( U_0 \) and \( U_1 \) are of the form
\[
U_0 = X^i Z^j \quad \text{and} \quad U_1 = Z^j X^i,
\]
for \( i \) and \( j \) chosen in \( \{0, 1\} \) with suitable probabilities. In this case, the unitaries \( U_0 \) and \( U_1 \) are not independent, and therefore Proposition 1 does not apply. This observation shows that the heralded noiseless communication is a consequence of the superposition of alternative orders, which correlates the elementary processes experienced by the quantum particle in its two alternative histories.