EXISTENCE OF EXTREMAL FUNCTIONS FOR THE STEIN-WEISS INEQUALITIES ON THE HEISENBERG GROUP

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ABSTRACT. In this paper, we establish the existence of extremals for two kinds of Stein-Weiss inequalities on the Heisenberg group. More precisely, we prove the existence of extremals for the \(|u|\) weighted Stein-Weiss inequalities in Theorem 1.1 and the \(|z|\) weighted Stein-Weiss inequalities in Theorem 1.4. Different from the proof of the analogous inequality in Euclidean spaces given by Lieb [18] using Riesz rearrangement inequality which is not available on the Heisenberg group, we employ the concentration compactness principle to obtain the existence of the maximizers on the Heisenberg group. Our result is also even new in the Euclidean case because we don’t assume that the exponents \(\alpha\) and \(\beta\) of the double weights in the Stein-Weiss inequality (1.1) are both nonnegative (see Theorem 1.3 and more generally Theorem 1.5). Therefore, we extend Lieb’s result of existence of extremal functions of the Stein-Weiss inequality in the Euclidean space to the case where \(\alpha\) and \(\beta\) are not necessarily both nonnegative (see Theorem 1.3). Furthermore, since the absence of translation invariance of the Stein-Weiss inequalities, additional difficulty presents and one cannot simply follow the same line of Lions’ idea to obtain our desired result. Our methods can also be used to obtain the existence of optimizers for several other weighted integral inequalities (Theorem 1.5).

Keywords: Concentration compactness principle; Existence of extremal functions; Heisenberg group, Stein-Weiss inequalities.

1. Introduction

We recall the classical Stein-Weiss inequality on \(\mathbb{R}^n\):

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x|^{-\alpha} |x-y|^{-\lambda} f(x) g(y)|^{-\beta} dx dy \leq C_{n,\alpha,\beta,p,q'} \|f\|_{L^{q'}(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)},
\]

where \(p, q', \alpha, \beta\) and \(\lambda\) satisfy the following conditions,

\[
\frac{1}{q'} + \frac{1}{p} + \frac{\alpha + \beta + \lambda}{n} = 2, \quad \frac{1}{q'} + \frac{1}{p} \geq 1,
\]

\[
1 < p, q < \infty, \quad \alpha + \beta \geq 0, \quad \alpha < \frac{n}{q}, \quad \beta < \frac{n}{p'}, \quad 0 < \lambda < n.
\]

Lieb [18] applied rearrangement inequalities to establish the existence of extremals for inequality (1.1) in the case \(p < q\) and \(\alpha, \beta \geq 0\). Furthermore, he also proved that inequality (1.1) doesn’t have extremal functions in the case of \(p = q\). Beckner ([1, 2]) gave the sharp constants of inequality (1.1) when \(p = q = 2\). The authors of [24] obtained the sharp constant \(C_{n,\alpha,\beta,p,q'}\) in
the case of $p = q > 1$. It should also be noted that the sharp constants for inequality (1.1) in the case of $1 < p, q' < \infty$ and $1 < \frac{1}{p} + \frac{1}{q'} < 2$ are still unknown.

When $\alpha = \beta = 0$, the Stein-Weiss inequality (1.1) reduces to the following Hardy-Littlewood-Sobolev inequality (15, 23),

\begin{equation}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x - y|^{-\lambda} f(x) g(y) dxdy \leq C_{n,p,q'} \|f\|_{L^{q'}(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)},
\end{equation}

where $1 < q', p < \infty$, $0 < \lambda < n$ and $\frac{1}{q'} + \frac{1}{p} + \frac{\lambda}{n} = 2$. Lieb and Loss [19] used the layer cake representation to prove that the sharp constants $C_{n,p,q'}$ satisfy the following estimate

\[ C_{n,p,q'} \leq \frac{n}{n - \lambda} \left( \frac{\pi^{\frac{n}{2}}}{\Gamma(1 + \frac{n}{2})} \right)^{\frac{\lambda}{n}} \left( \frac{\lambda q'}{n(q' - 1)} \right)^{\frac{\lambda}{n}} + \left( \frac{\lambda p}{n(p - 1)} \right)^{\frac{\lambda}{n}}. \]

Furthermore, when $p = q' = \frac{2n}{2n - \lambda}$, Lieb [18] also gave the explicit formula of sharp constants and maximizers. More precisely,

**Theorem A.** For $1 < q', p < \infty$, $0 < \lambda < n$ and $\frac{1}{q'} + \frac{1}{p} + \frac{\lambda}{n} = 2$, there exists sharp constants $C_{n,p,q'}$ and maximizers of $f \in L^{q'}(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, such that

\begin{equation}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) |x - y|^{-\lambda} g(y) dxdy \leq C_{n,p,q'} \|f\|_{L^{q'}(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.
\end{equation}

If $p = q' = \frac{2n}{2n - \lambda}$, then

\[ C_{n,p,q'} = \pi^{\frac{n}{2}} \frac{\Gamma(\frac{n}{2} - \lambda)}{\Gamma(n - \lambda)} \left( \frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \right)^{\frac{\lambda}{n}}. \]

In this case, equality (1.3) holds if and only if

\[ f = \frac{c_1}{(d + |x - x_0|^2)^{\frac{2n - \lambda}{2}}}, \]

for some $x_0 \in \mathbb{R}^n$ and $d > 0$.

Another natural question is whether there exist some similar inequalities such as (1.1) and (1.2) on the Heisenberg group? Folland and Stein first in [9] give a positive answer in terms of fractional integral operator. For simplicity, we introduce some background knowledge about Heisenberg group. The $n$-dimensional Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is a Lie group with group structure given by

\[ uv = (z, t)(z', t') = (z + z', t + t' + 2Im(z \cdot \overline{z'}) + 2Im(z \cdot \overline{z'})) \]

for any two points $u = (z, t)$ and $v = (z', t')$ in $\mathbb{H}^n$. Haar measure on $\mathbb{H}^n$ is the usual Lebesgue measure $du = dxdzdt$. The Lie algebra of $\mathbb{H}^n$ is generated by the left invariant vector fields

\[ T = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + 2x_j \frac{\partial}{\partial t}. \]

For each real number $\lambda \in \mathbb{R}$ and $u = (z, t) \in \mathbb{H}^n$, we denote the dilation $\delta_\lambda(u) = (\lambda z, \lambda^2 t)$, the homogenous norm on $\mathbb{H}^n$ as $|u| = (|z|^4 + t^2)^{\frac{1}{4}}$, and $Q = 2n + 2$ as the homogenous dimension. With this norm, a Heisenberg ball centered at $u = (z, t) \in \mathbb{H}^n$ with radius $R$ is given by $B_{\mathbb{H}^n}(u, R) = \{ v \in \mathbb{H} : d(u^{-1}v) < R \}$, where $v = (z', t')$, $u^{-1} = (-z, -t)$, and $d(u, v) := |u^{-1}v| = |uv^{-1}|$ denotes left-invariant quasi-metric on Heisenberg group.
Folland and Stein first [9] established the following Hardy-Littlewood-Sobolev inequality on the Heisenberg group.

**Theorem B.** For $1 < q', p < \infty$, $0 < \lambda < Q$ and $\frac{1}{q'} + \frac{1}{p} + \frac{\lambda}{Q} = 2$, there exists some constant $C_{Q,p,q'}$ such that for any $f \in L^{q'}(\mathbb{H}^n)$, $g \in L^p(\mathbb{H}^n)$, there holds

$$
\left(1.4\right) \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} f(u) |u^{-1} v|^{-\lambda} g(v) dudv \leq C_{Q,p,q'} \|f\|_{L^{q'}(\mathbb{H}^n)} \|g\|_{L^p(\mathbb{H}^n)}.
$$

In conjunction with the CR Yamabe problem on the CR manifolds (see [16, 17]), Jerison and Lee [17] proved the sharp version and gave the optimizer of the $L^2(\mathbb{H}^n)$ to $L^{2Q-2}(\mathbb{H}^n)$ Sobolev inequality on the Heisenberg group which is equivalent to the Hardy-Littlewood-Sobolev inequality (1.4) for $\lambda = Q - 2$ and $p = q' = \frac{2Q}{Q+2}$. Frank and Lieb [12] established the best constants and extremal functions for this inequality (1.4) in the case of $p = q' = \frac{2Q}{Q-\lambda}$. More precisely, they obtained the following result.

**Theorem C.** For $0 < \lambda < Q$ and $p = q' = \frac{2Q}{Q-\lambda}$, then for any $f \in L^{p}(\mathbb{H}^n)$ and $g \in L^{p}(\mathbb{H}^n)$, there holds

$$
\left(1.5\right) \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} f(u) |u^{-1} v|^{-\lambda} g(v) dudv \leq \left(\frac{\pi^{n+1}}{2^{n-1}n!}\right)^{\frac{1}{Q}} \frac{n! \Gamma(\frac{Q-\lambda}{2})}{\Gamma^2(\frac{2Q-\lambda}{4})} \lambda^Q \|f\|_{L^{p}(\mathbb{H}^n)} \|g\|_{L^{p}(\mathbb{H}^n)}
$$

with equality if and only if

$$
f(u) = c_1 H(\delta(a^{-1} u)), \quad g(u) = c_2 H(\delta(a^{-1} u))
$$

for some $c_1, c_2 \in \mathbb{R}$, $\delta > 0$ and $a \in \mathbb{H}^n$. Here $H$ is the function given by $H(u) = \left(1 + |z|^2\right)^2 + t^4 \frac{2Q-\lambda}{4}$.\[13\]

In the Euclidean space, Carlen, Carillo and Loss gave a simple proof of the sharp Hardy-Littlewood-Sobolev inequality when $\lambda = n - 2$ for $n \geq 3$ via a monotone flow governed by the fast diffusion equation [14]. Frank and Lieb [10, 11] further employed a rearrangement-free technique developed in [12] to recapture the best constant of inequality (1.4). This method has also been successfully applied to obtain sharp constants for similar inequalities on quaternionic and octonionic Heisenberg groups (see [15, 16, 17]). For Hardy-Littlewood-Sobolev inequality (1.4) on the Heisenberg group, little was known about maximizers and sharp constants except the case $p = q' = \frac{2Q}{Q-\lambda}$. Han [13] proved the existence of extremals of inequality (1.4) for all the case of $q'$ and $p$ through the concentration compactness principle of Lions [20, 21].

Han, Lu and Zhu in [14] utilized the theory of weighted inequalities for integral operators to establish the Stein-Weiss inequalities (1.11) on the Heisenberg group.

**Theorem D.** For $1 < p < \infty$, $1 < q' < \infty$, $0 < \lambda < Q = 2n + 2$ and $\alpha + \beta \geq 0$ such that $\lambda + \alpha + \beta \leq Q$, $\beta < Q_p$, $\alpha < Q_q$ and $\frac{1}{p} + \frac{1}{q'} + \frac{\lambda + \alpha + \beta}{Q} = 2$, there exists some constant $C_{Q,\alpha,\beta,p,q'} > 0$ such that for any functions $f \in L^{q'}(\mathbb{H}^n)$ and $g \in L^p(\mathbb{H}^n)$, there holds

$$
\left(1.6\right) \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{f(u)g(v)}{|u|^\alpha |v|^{\lambda}|u^{-1} v|^{\beta}} dudv \leq C_{Q,\alpha,\beta,p,q'} \|f\|_{L^{q'}(\mathbb{H}^n)} \|g\|_{L^p(\mathbb{H}^n)}.
$$
Furthermore, they also studied the weighted Hardy-Littlewood-Sobolev inequalities with different weights, i.e., $|z|$ weights. More precisely, they obtained

**Theorem E.** For $1 < p < \infty$, $1 < q' < \infty$, $0 < \lambda < \mathcal{Q} = 2n + 2$ and $\alpha + \beta \geq 0$ such that $\lambda + \alpha + \beta \leq \mathcal{Q}$, $\beta < \frac{2n}{p}$, $\alpha < \frac{2n}{q}$, and $\frac{1}{p} + \frac{1}{q} + \frac{\lambda + \alpha + \beta}{\mathcal{Q}} = 2$, there exists some nonnegative function $g$ satisfying

(1.7) \[
\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{f(u)g(v)}{|z|^\alpha |u^{-1}v|^{\lambda} |z'|^\beta} du dv \leq C_{Q,\alpha,\beta,p,q'} \|f\|_{L^{q'}(\mathbb{H}^n)} \|g\|_{L^p(\mathbb{H}^n)},
\]

where $u = (z, t)$ and $v = (z', t')$.

It is natural to ask whether there exist extremal functions for the Stein-Weiss inequalities. Due to the absence of Riesz rearrangement inequalities on the Heisenberg group, one cannot simply follow the same line of Lieb [13] to establish the existence of extremals for the Stein-Weiss inequality on the Heisenberg group. On the other hand, it is well known that the concentration compactness principle is an essential tool in dealing with the existence of extremal functions for geometrical inequalities. Han [13] applied the concentration compactness argument, which was originally used by Lions to solve the existence of extremals for Sobolev inequalities in Euclidean spaces, to obtain the existence of extremals for the Hardy-Littlewood-Sobolev inequality on the Heisenberg group. For the Stein-Weiss inequality, this problem is highly non-trivial because one cannot apply the same method of Han [13] due to the loss of translation invariance of the Stein-Weiss inequality. We overcome this difficulty by using the method of combining the concentration compactness argument and dilation invariance to establish the attainability of the best constant of the Stein-Weiss inequality (1.6) on the Heisenberg group. For the above Stein-Weiss inequality with $|z|$ weights on the Heisenberg group, Beckner [1] gave the sharp constant when $\alpha = \beta = \frac{Q - \lambda}{2}$, $p = q = 2$ and proved the nonexistence of optimizers. However, it is completely open in other general cases including existence question. We verify that there exist extremal functions for the $|z|$ weighted Stein-Weiss inequalities in Theorem [1.4] in general case. Define the operator

$$I_{\lambda}(g)(u) = \int_{\mathbb{H}^n} \frac{g(v)}{|u^{-1}v|^{\lambda}} dv.$$  

By duality, we can see that inequality (1.6) is equivalent to the following double weighted inequality

(1.8) \[
\|I_{\lambda}(g)|u|^{-\alpha}\|_{L^q(\mathbb{H}^n)} \leq C_{n,\alpha,\beta,p,q'} \|g|v|^{\beta}\|_{L^p(\mathbb{H}^n)}.
\]

Consider the following maximizing problem

(1.9) \[
C_{Q,\alpha,\beta,p,q'} := \sup \{\|I_{\lambda}(g)|u|^{-\alpha}\|_{L^q(\mathbb{H}^n)} : g \geq 0, \|g|v|^{\beta}\|_{L^p(\mathbb{H}^n)} = 1\},
\]

it is easy to verify that the extremals for the Stein-Weiss inequality on the Heisenberg group are those solving the maximizing problem (1.9).

**Theorem 1.1.** Under assumptions of Theorem D, if we furthermore assume that $q > p$, then there exists some nonnegative function $g$ satisfying $\|g|v|^{\beta}\|_{L^p(\mathbb{H}^n)} = 1$ and $\|I_{\lambda}(g)|u|^{-\alpha}\|_{L^q(\mathbb{H}^n)} = C_{Q,\alpha,\beta,p,q'}$.

**Remark 1.2.** Lieb [13] employed the Riesz rearrangement inequalities to establish the existence of extremals for inequality (1.1). Hence they must assume that both $\alpha$ and $\beta$ are nonnegative. In our proof, we remove this assumption. Our method can be applied to Euclidean space without
any change. Hence, our existence results of extremal functions are also new in $\mathbb{R}^n$. We state our new result in Euclidean space as follows and its proof can be given identically as on the Hieisenberg group and therefore we needn't give a separate proof.

**Theorem 1.3.** For $1 < p < \infty$, $1 < q' < \infty$, $0 < \lambda < n$ and $\alpha + \beta \geq 0$ such that $\lambda + \alpha + \beta \leq n$, $\beta < \frac{m}{p'}$, $\alpha < \frac{m}{q'}$ and $\frac{1}{p} + \frac{1}{q} + \frac{\lambda + \alpha + \beta}{n} = 2$, there exists some constant $C_{n,\alpha,\beta,p,q'} > 0$ such that for any functions $f \in L^{q'}(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, there holds

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha|x - y|^{\lambda}|y'|^{\beta}} \, dxdy \leq C_{n,\alpha,\beta,p,q'} \|f\|_{L^{q'}(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.
$$

Furthermore, if we assume that $q > p$, then the best constant $C_{n,\alpha,\beta,p,q'}$ could be achieved.

Similar to Theorem 1.1, we also obtain the existence result involved with the Stein-Weiss inequalities with $|z|$ weights on the Heisenberg group.

**Theorem 1.4.** Under the assumptions of Theorem E, if we furthermore assume that $q > p$, then there exists some nonnegative function $g$ satisfying $\|g|z'|^{\beta}\|_{L^p(\mathbb{R}^n)} = 1$ and $\|I_\lambda(g)|z|^{-\alpha}\|_{L^q(\mathbb{R}^n)} = C_{q,\alpha,\beta,p,q'}$.

Next, we will prove a more general result.

**Theorem 1.5.** For $1 < p < \infty$, $1 < q' < \infty$, $0 < \lambda < n$ and $\alpha + \beta \geq 0$ such that $\lambda + \alpha + \beta \leq n + m$, $\beta < \frac{m}{p'}$, $\alpha < \frac{m}{q'}$ and $\frac{1}{p} + \frac{1}{q} + \frac{\lambda + \alpha + \beta}{n + m} = 2$, there exists some constant $C_{n,m,\alpha,\beta,p,q'} > 0$ such that for any functions $f \in L^{q'}(\mathbb{R}^{n+m})$ and $g \in L^p(\mathbb{R}^{n+m})$, there holds

$$
\int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} \frac{f(x)g(y)}{|x'|^{\alpha}|x - y|^{\lambda}|y'|^{\beta}} \, dxdy \leq C_{n,m,\alpha,\beta,p,q'} \|f\|_{L^{q'}(\mathbb{R}^{n+m})} \|g\|_{L^p(\mathbb{R}^{n+m})},
$$

where $x = (x', x'')$, $y = (y', y'') \in \mathbb{R}^m \times \mathbb{R}^n$. Furthermore, if we assume that $q > p$, then the best constant $C_{n,m,\alpha,\beta,p,q'}$ could be achieved.

Once we establish the existence of extremals for the inequality (1.11), we naturally are concerned about some properties such as radial symmetry for their extremals. In order to achieve this purpose, one can maximize the functional

$$
J(f, g) = \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} \frac{f(x)g(y)}{|x'|^{\alpha}|x - y|^{\lambda}|y'|^{\beta}} \, dxdy
$$

under the constraint $\|f\|_{L^{q'}} = \|g\|_{L^p} = 1$. Assume that $f$ and $g$ is a pair of extremal functions to the inequality (1.11). According to Euler-Lagrange multipliers theorem, we obtain

$$
\begin{cases}
J(f, g)f(x)'^{q'-1} = \int_{\mathbb{R}^{n+m}} |x'|^{-\alpha}|x - y|^{-\lambda}g(y)|y'|^{-\beta} \, dy,
J(f, g)g(x)'^{p-1} = \int_{\mathbb{R}^{n+m}} |x'|^{-\beta}|x - y|^{-\lambda}f(y)|y'|^{-\alpha} \, dy.
\end{cases}
$$

Set $u = c_1f'^{-1}$, $v = c_2g'^{-1}$, $\frac{1}{q'-1} = p_1$ and $\frac{1}{p-1} = p_2$, then for a proper choice of constants $c_1$ and $c_2$, system (1.13) becomes

$$
\begin{cases}
u(x) = \int_{\mathbb{R}^{n+m}} |x'|^{-\alpha}|x - y|^{-\lambda}v^{p_2}(y)|y'|^{-\beta} \, dy,
\mu(x) = \int_{\mathbb{R}^{n+m}} |x'|^{-\beta}|x - y|^{-\lambda}u^{p_1}(y)|y'|^{-\alpha} \, dy,
\end{cases}
$$

where $\frac{1}{p_1-1} + \frac{1}{p_2-1} = \frac{\alpha + \beta + \lambda}{n + m}$. Applying the methods of moving plane in integral forms [6], we obtain symmetry results for positive solutions of the system (1.14).
Theorem 1.6. Assume that \((u, v) \in L^{p+1}(\mathbb{R}^{m+n}) \times L^{p+1}(\mathbb{R}^{m+n})\) is a pair of positive solutions of the integral system (1.14), then \(u(x)|_{\mathbb{R}^m \times \{0\}}\) and \(v(x)|_{\mathbb{R}^m \times \{0\}}\) are radially symmetric and monotone decreasing about the origin in \(\mathbb{R}^m\), \(u(x)|_{\{0\} \times \mathbb{R}^n}\) and \(v(x)|_{\{0\} \times \mathbb{R}^n}\) are radially symmetric and monotone decreasing about some \(x_0 \in \mathbb{R}^n\).

2. The Proof of Theorem 1.1

In this section, we will give the proof of attainability of the best constant for the inequality (1.6). Though our proof is inspired by Lions’ proof of the existence of extremals for the classical Sobolev inequalities, the loss of translation invariance for the Stein-Weiss inequalities (namely, doubly weighted Hardy-Littlewood-Sobolev inequalities) presents additional difficulty because of the presence of weights. Moreover, lack of the analogous compact imbedding associated with the integral inequalities produces more difficulties in applying directly the Lions concentration compactness principle.

Assume that \(\{\tilde{g}_i\}\) is a maximizing sequence for problem (1.9), namely \(\|\tilde{g}_i\|_{L^p(\mathbb{H}^n)} = 1\) and \(\lim_{i \to +\infty} \|I_{\lambda}(\tilde{g}_i)|u|^{-\alpha}\|_{L^q(\mathbb{H}^n)} = C_{Q,\alpha,\beta,p,q'}\). Since \(\int_{\mathbb{H}^n} |\tilde{g}_i|^p |v|^\beta dv = 1\) for all \(i \in \mathbb{N}\), there exists \(r_i > 0\) such that
\[
\int_{B_{ri}(0,1)} |\tilde{g}_i|^p |v|^{\beta p} dv = 1/2.
\]
We define a new sequence \(\{g_i\}\) given by
\[
g_i(v) := r_i^{(Q+\beta p)/p} \tilde{g}_i(r_i^\gamma v)
\]
for all \(i \in \mathbb{N}\) and \(v = (z, t) \in \mathbb{H}^n\). Clearly \(\{g_i\}\) is also a maximizing sequence for problem (1.9). Moreover, we also have that
\[
\int_{B_{ri}(0,1)} |g_i|^p |v|^{\beta p} dv = 1/2
\]
for all \(i \in \mathbb{N}\). We define the measure:
\[
\mu_i := |g_i|^p |v|^{\beta p} dv \quad \text{and} \quad \nu_i := |I_{\lambda}(g_i)|^q |u|^{-\alpha q} du.
\]
Then it follows that there exist two bounded measures \(\mu\) and \(\nu\) such that
\[
\mu_i \to \mu \quad \text{and} \quad \nu_i \to \nu \quad \text{weakly in the sense of measures as } i \to \infty.
\]
Furthermore, by the lower semi-continuity of the measure, one can get
\[
\int_{\mathbb{H}^n} d\mu \leq \lim_{i \to +\infty} \int_{\mathbb{H}^n} d\mu_i = 1 \quad \text{and} \quad \int_{\mathbb{H}^n} d\nu \leq \lim_{i \to +\infty} \int_{\mathbb{H}^n} d\nu_i = C_{Q,\alpha,\beta,p,q'}.
\]
We now apply Lions’ first concentration compactness (see [20, 21]) to the sequence of measure \(\{\mu_i\}\).

Lemma 2.1. There exists a subsequence of \(\{\mu_i\}\) such that one of the following condition holds:

(a) (Compactness) There exists \(\{\nu_i\}\) in \(\mathbb{H}^n\) such that for each \(\varepsilon > 0\) small enough, we can find \(R_\varepsilon > 0\) with
\[
\int_{B_{R_\varepsilon}(v_i,R_\varepsilon)} d\mu_i \geq 1 - \varepsilon \quad \text{for all } i.
\]
(b)(Vanishing) For all \( R > 0 \), there holds:

\[
\lim_{i \to \infty} \left( \sup_{v \in \mathbb{H}^n} \int_{B_{\mathbb{H}}(v, R)} d\mu_i \right) = 0.
\]

(c)(Dichotomy) There exist \( k \in (0, 1) \) such that for any \( \varepsilon > 0 \), there exists \( R_\varepsilon > 0 \) and a sequence \( \{v_i^\varepsilon\}_{i \in \mathbb{N}} \), with the following property: given \( R' > R_\varepsilon \), there are non-negative measures \( \mu_i^1 \) and \( \mu_i^2 \) such that

\[
0 \leq \mu_i^1 + \mu_i^2 \leq \mu_i, \quad \text{Supp}(\mu_i^1) \subset B_{\mathbb{H}}(v_i^\varepsilon, R_\varepsilon), \quad \text{Supp}(\mu_i^2) \subset \mathbb{H}^n \setminus B_{\mathbb{H}}(v_i^\varepsilon, R'),
\]

\[
\mu_i^1 = \mu_i|_{B_{\mathbb{H}}(v_i^\varepsilon, R_\varepsilon)}, \quad \mu_i^2 = \mu_i|_{\mathbb{H}^n \setminus B_{\mathbb{H}}(v_i^\varepsilon, R')},
\]

\[
\limsup_{i \to \infty} \left( |k - \int_{\mathbb{H}^n} d\mu_i^1| + |(1 - k) - \int_{\mathbb{H}^n} d\mu_i^2| \right) \leq \varepsilon.
\]

**Lemma 2.2.** Let \( \{g_i\}_{i} \) is a maximizing sequence for problem (1.9), then compactness (a) holds. In particular, we also have that \( \int_{\mathbb{H}^n} d\mu_i = 1 \).

**Proof.** Clearly, (b) can not occur because \( \int_{B_{\mathbb{H}}(0,1)} |g_i(v)|^p |v|^{\beta p} dv = \frac{1}{2} \). Next, it suffices to show that (c) also does not happen. We argue this by contradiction. Suppose that (c) occurs, then for any \( \varepsilon > 0 \), we can find \( R_0 > 0 \) and \( \{v_i\} \subset \mathbb{H}^n \) such that for any given \( R \geq R_0 \), there holds

\[
\|g_i|v|^{\beta} \chi_{B_{\mathbb{H}}(v, R)}\|_p^p = k + O(\varepsilon) \quad \text{and} \quad \|g_i|v|^{\beta} \chi_{B_{\mathbb{H}}(v, R)}\|_p^p = 1 - k + O(\varepsilon).
\]

We first claim that

\[
\lim_{R \to +\infty} I_\lambda(g_i \chi_{B_{\mathbb{H}}(v, R)})(u) = I_\lambda(g_i)(u) \quad \text{for} \quad u \in \mathbb{H}^n.
\]

In fact,

\[
\begin{aligned}
|I_\lambda(g_i)(u) - I_\lambda(g_i \chi_{B_{\mathbb{H}}(v, R)})(u)| &= |I_\lambda(g_i \chi_{B_{\mathbb{H}}(v, R)})(u)| \\
&= \left| \int_{\mathbb{H}^n} \left( g_i \chi_{B_{\mathbb{H}}(v, R)}(v) \right) \frac{1}{|u^{-1}v|^\lambda} dv \right| \\
&\leq \left( \int_{B_{\mathbb{H}}(v, R)} |g_i(v)| |v|^{\beta p} dv \right)^{\frac{1}{p'}} \left( \int_{B_{\mathbb{H}}(v, R)} |u^{-1}v|^{-\lambda p'} |v|^{-\beta p'} dv \right)^{\frac{1}{p'}} \\
&\to 0
\end{aligned}
\]

as \( R \to +\infty \). Now we can apply the Brezis-Lieb lemma (3) to obtain

\[
\|I_\lambda(g_i)|u|^{-\alpha}\|_q^q = \|I_\lambda(g_i \chi_{B_{\mathbb{H}}(v, R)})|u|^{-\alpha}\|_q^q + \|I_\lambda(g_i \chi_{B_{\mathbb{H}}(v, R)})|u|^{-\alpha}\|_q^q + o(1).
\]

In light of the Stein-Weiss inequality and condition (2.1), we have

\[
\begin{aligned}
\|I_\lambda(g_i \chi_{B_{\mathbb{H}}(v, R)})|u|^{-\alpha}\|_q^q + \|I_\lambda(g_i \chi_{B_{\mathbb{H}}(v, R)})|u|^{-\alpha}\|_q^q + o(1)
&\leq C_{Q,\alpha,\beta,p,q}^q \|g_i \chi_{B_{\mathbb{H}}(v, R)}|v|^{\beta p} \|_q^p + C_{Q,\alpha,\beta,p,q}^q \|g_i \chi_{B_{\mathbb{H}}(v, R)}|v|^{\beta p} \|_q^p + o(1) \\
&\leq C_{Q,\alpha,\beta,p,q}^q (k + O(\varepsilon))^{\frac{q}{p'}} + \leq C_{Q,\alpha,\beta,p,q}^q (1 - k + O(\varepsilon))^{\frac{q}{p'}} + o(1) \\
&\leq C_{Q,\alpha,\beta,p,q}^q (k^{\frac{q}{p'}} + (1 - k)^{\frac{q}{p'}}) + O(\varepsilon) + o(1) \\
&< C_{Q,\alpha,\beta,p,q}^q,
\end{aligned}
\]

which is a contradiction with \( \|I_\lambda(g_i)|u|^{-\alpha}\|_q^q \to C_{Q,\alpha,\beta,p,q}^q \). Hence compactness condition (a) must holds.
Now, we turn to the proof of $\int_{\mathbb{H}^n} d\mu = 1$. According to $\int_{B_\mathbb{H}(0,1)} |g_i|^p|v|^\beta p\,dv = \frac{1}{2}$, we claim that \{v_i\} must be a bounded sequence in $\mathbb{H}^n$. We argue this by contradiction. If \{v_i\} is unbounded, then we can find subsequence $v_i$ satisfying $|v_i| \to \infty$. In view of (a), we obtain for any sufficiently small $\varepsilon > 0$, there exist $R_0, R > R$ such that

$$\int_{B_\mathbb{H}(v_i,R)} |g_i|^p|v|^\beta p\,dv \leq \varepsilon.$$ 

On the other hand, for any $v \in B_\mathbb{H}(0,1)$, there holds $|v - v_i| \geq |v_i| - |v| \to \infty$. Then it follows that for sufficiently large $i$, there holds

$$\int_{B_\mathbb{H}(0,1)} |g_i|^p|v|^\beta p\,dv \leq \int_{B_\mathbb{H}(0,1)} |g_i|^p|v|^\beta p\,dv \leq \varepsilon,$$

which is a contradiction. Then $v_i$ must be a bounded sequence in $\mathbb{H}^n$. Hence, we may assume for any $\varepsilon > 0$, there exist $R_0$ such that for any $R \geq R_0$, there holds

$$\int_{B_\mathbb{H}(0,R)} d\mu_i \geq 1 - \varepsilon.$$

Then for any $R > R_0$ and $\phi \in C_c^\infty(\mathbb{H}^n)$ with $\phi|_{B_\mathbb{H}(0,R)} = 1$, then by the weak convergence $\mu_i$ in the sense of measure, we obtain

$$\int_{\mathbb{H}^n} d\mu \geq \int_{\mathbb{H}^n} \phi(v)\,d\mu = \lim_{i \to \infty} \int_{\mathbb{H}^n} |g_i|^p|v|^\beta p\phi(v)\,dv \geq \lim_{i \to \infty} \int_{B_\mathbb{H}(0,R)} |g_i|^p|v|^\beta p\,dv \geq \lim_{i \to \infty} \int_{B_\mathbb{H}(0,R_0)} d\mu_i \geq 1 - \varepsilon.$$

Let $\varepsilon \to 0$, we derive that $\int_{\mathbb{H}^n} d\mu \geq 1$. Then we accomplish the proof of Lemma 2.2. \hfill $\square$

**Lemma 2.3.** Let $\{g_i\}_i$ be a maximizing sequence satisfying

$$\|g_i(v)|v|^\beta\|_{L^p(\mathbb{H}^n)} = 1 \quad \text{and} \quad \int_{B_\mathbb{H}(0,R)} |g_i(v)|^p|v|^\beta p\,dv \geq 1 - \varepsilon(R).$$

we may assume that $g_i(v)|v|^\beta \to g(v)|v|^\beta$ weakly in $L^p(\mathbb{H}^n)$ (by passing to a subsequence if necessary). Then, by passing to a subsequence again if necessary,

$$I_\lambda(g_i)(u)|u|^{-\alpha} \to I_\lambda(g)(u)|u|^{-\alpha} \quad \text{a.e..}$$

**Proof.** We show that $I_\lambda(g_i)(u)|u|^{-\alpha} \to I_\lambda(g)(u)|u|^{-\alpha}$ in measure to ensure the existence of a pointwisely convergent subsequence of $\{g_i\}$. Observe that for $M > R$, applying Stein-Weiss inequalities on Heisenberg group and (2.5), we derive that

$$\|I_\lambda(g_i)(u)|u|^{-\alpha} \chi_{B_\mathbb{H}(0,M)}\|_{L^q(\mathbb{H}^n)} \leq \|I_\lambda(g_i\chi_{B_\mathbb{H}(0,R)}(u)|u|^{-\alpha} \chi_{B_\mathbb{H}(0,M)}\|_{L^q(\mathbb{H}^n)} + \|I_\lambda(g_i\chi_{B_\mathbb{H}(0,R)}(u)|u|^{-\alpha} \chi_{B_\mathbb{H}(0,M)}\|_{L^q(\mathbb{H}^n)} \leq \|I_\lambda(g_i\chi_{B_\mathbb{H}(0,R)}(u)|u|^{-\alpha} \chi_{B_\mathbb{H}(0,M)}\|_{L^q(\mathbb{H}^n)} + \|I_\lambda(g_i\chi_{B_\mathbb{H}(0,R)}(u)|u|^{-\alpha} \chi_{B_\mathbb{H}(0,M)}\|_{L^q(\mathbb{H}^n)} = \|I_\lambda(g_i\chi_{B_\mathbb{H}(0,R)}(u)|u|^{-\alpha} \chi_{B_\mathbb{H}(0,M)}\|_{L^q(\mathbb{H}^n)} \leq \|I_\lambda(g_i\chi_{B_\mathbb{H}(0,R)}(u)|u|^{-\alpha} \chi_{B_\mathbb{H}(0,M)}\|_{L^q(\mathbb{H}^n)} + \epsilon(R).$$


Since $\frac{1}{p} + \frac{1}{q} + \frac{\lambda + \alpha + \beta}{Q} = 2$ and $\beta < \frac{Q}{p}$, we get $Q < (\alpha + \lambda)q$ and
\[
\|g_i \chi_{B_{0}(R)}(0,0,0)|L^1(\mathbb{H}^n)\| \leq \|g_i(v)|v|^\beta|L^p(\mathbb{H}^n)\| \frac{1}{|v|^\beta \chi_{B_{0}(0,R)}|L^p(\mathbb{H}^n)} < \infty.
\]
Using Minkowski's inequality, we obtain
\[
\|I_\lambda(g_i|u|^{-\alpha}\chi_{B_{0}(0,M)}|L^q(\mathbb{H}^n)}\]
\[
\leq \|g_i \chi_{B_{0}(0,R)}|L^1(\mathbb{H}^n)}\left(\int_{|u| \geq M} \frac{1}{|u| - R} \chi_{|u|^{-\alpha}du}\right)^\frac{1}{q}
\]
\[
\leq \|g_i \chi_{B_{0}(0,R)}|L^p(\mathbb{H}^n)}\| |v|^{-\beta}|L^{p'}(\mathbb{H}^n)} \frac{1}{(M - R)^{\alpha + \lambda - \frac{Q}{q}}} \to 0
\]
for every fixed $R$ as $M \to \infty$. Therefore, we have
\[
\|I_\lambda(g_i|u|^{-\alpha}\chi_{B_{0}(0,M)}|L^q(\mathbb{H}^n)} \leq \epsilon(M),
\]
and that is,
\[
\|I_\lambda(g_i|u|^{-\alpha} - I_\lambda(g_i|u|^{-\alpha}\chi_{B_{0}(0,M)}|L^q(\mathbb{H}^n)} \leq \epsilon(M).
\]
Since $g_i(v)|v|^\beta \to g(v)|v|^\beta$ weakly in $L^p(\mathbb{H}^n)$, we have
\[
\|g(v)|v|^\beta\chi_{B_{0}(0,R)}|L^p(\mathbb{H}^n)} \leq \liminf_{j \to \infty} \|g_i(v)|v|^\beta\chi_{B_{0}(0,R)}|L^p(\mathbb{H}^n)} \leq \epsilon(R).
\]
Similarly, for $g$ one can derive,
\[
\|I_\lambda(g|u|^{-\alpha} - I_\lambda(g|u|^{-\alpha}\chi_{B_{0}(0,M)}|L^q(\mathbb{H}^n)} \leq \epsilon(M).
\]
Therefore, given $k > 0$,
\[
\{|I_\lambda(g_i|u|^{-\alpha} - I_\lambda(g|u|^{-\alpha}|B_{0}(0,M)\{u| \geq 15k\} + \}
\]
\[
\{|I_\lambda(g_i|u|^{-\alpha} - I_\lambda(g|u|^{-\alpha}\chi_{B_{0}(0,M)}(u) \geq 5k\}) + \}
\]
\[
\{|I_\lambda(g_i|u|^{-\alpha}\chi_{B_{0}(0,M)}(u) - I_\lambda(g|u|^{-\alpha}\chi_{B_{0}(0,M)}(u)|B_{0}(0,M)\{u| \geq 5k\}| + \}
\]
\[
\{I_\lambda(g|u|^{-\alpha}\chi_{B_{0}(0,M)}(u) - I_\lambda(g|u|^{-\alpha}|B_{0}(0,M)\{u| \geq 5k\}| \leq 2\frac{\epsilon(M)}{5R} + \{|I_\lambda(g_i|u|^{-\alpha} - I_\lambda(g|u|^{-\alpha}) \geq 5k\} \cap B_{0}(0,M)\}
\]
Thus, it remains to estimate the second term above. Denote
\[
I_\lambda^g(g|(u) = \int_{B_{0}(u,\eta)} g(v)|u^{-1}\chi dv,
\]
then
\[
I_\lambda^g(g_i \chi_{B_{0}(0,R)}(u)|u|^{-\alpha} \to I_\lambda^g(g \chi_{B_{0}(0,R)}(u)|u|^{-\alpha}
\]
for all $u \in \mathbb{H}^n$ because
\[
|u^{-1}\chi|^{-\lambda} \chi_{B_{0}(0,R)}|L^p(\mathbb{H}^n)} \chi_{B_{0}(u,\eta)} \in L^{p'}(\mathbb{H}^n)
\]
for any fixed $u \in \mathbb{H}^n$ and $\eta > 0$. Therefore, $I_\lambda^g(g_i \chi_{B_{0}(R)}(u)|u|^{-\alpha} \to I_\lambda^g(g \chi_{B_{0}(R)}(u)|u|^{-\alpha}$ locally in measure, which means,
\[
\{|I_\lambda^g(g_i \chi_{B_{0}(0,R)}(u)|u|^{-\alpha} - I_\lambda^g(g \chi_{B_{0}(0,R)}(u)|u|^{-\alpha}) \geq k\} \cap B_{0}(0,M) = o(1).
\]

On the other hand, since \( \lambda + \alpha + \beta \leq Q \), we can derive that
\[
\| I_\lambda (g \chi_{B_{\mathbb{H}}(0,R)}) (u) \| u \|^{-\alpha} - I_\lambda^q (g \chi_{B_{\mathbb{H}}(0,R)}) (u) \| u \|^{-\alpha} \|_{L^1(\mathbb{H}^n)} \\
= \int_{\mathbb{H}^n} \int_{B_{\mathbb{H}}(u,\eta)} \frac{g_i(v) \chi_{B_{\mathbb{H}}(0,R)}}{|u^{-1}v|^\lambda} du \ | u \|^{-\alpha} \|_{L^1(\mathbb{H}^n)} \\
\leq \| g_i(v) \chi_{B_{\mathbb{H}}(0,R)} \|_{L^1(\mathbb{H}^n)} \cdot \left( \int_{B_{\mathbb{H}}(u,\eta)} \frac{1}{|u^{-1}v|^\lambda} du \right) \ | u \|^{-\alpha} \|_{L^1(\mathbb{H}^n)} \\
\leq C \eta^{Q-\lambda-\alpha} \rightarrow 0
\]
for every fixed \( R \) as \( \eta \rightarrow 0 \), where we use the fact
\[
\int_{B_{\mathbb{H}}(v,\eta)} \frac{1}{|u^{-1}v|^\lambda} du \leq \int_{B_{\mathbb{H}}(v,\eta)} \frac{1}{|u|^{\lambda+\alpha}} + \int_{B_{\mathbb{H}}(v,\eta)} \frac{1}{|u^{-1}v|^\lambda} du.
\]
That is,
\[
|I_\lambda (g_i \chi_{B_{\mathbb{H}}(0,R)}) - I_\lambda^q (g_i \chi_{B_{\mathbb{H}}(0,R)})|_{L^q(\mathbb{H}^n)} \leq O(\eta).
\]
Similarly, we can derive the analogous statement for \( f \),
\[
|I_\lambda (g \chi_{B_{\mathbb{H}}(0,R)}) - I_\lambda^q (g \chi_{B_{\mathbb{H}}(0,R)})|_{L^q(\mathbb{H}^n)} \leq O(\eta).
\]
Also notice that
\[
\| I_\lambda (g_i) \| u \|^{-\alpha} - I_\lambda (g_i \chi_{B_{\mathbb{H}}(0,R)}) \| u \|^{-\alpha} \|_{L^q(\mathbb{H}^n)} \leq C n, \alpha, p, q, \| g \chi_{B_{\mathbb{H}}(0,R)} \| \leq \epsilon(R)
\]
and
\[
\| I_\lambda (g_i) \| u \|^{-\alpha} - I_\lambda (g_i \chi_{B_{\mathbb{H}}(0,R)}) \| u \|^{-\alpha} \|_{L^q(\mathbb{H}^n)} \leq C n, \alpha, p, q, \| g \chi_{B_{\mathbb{H}}(0,R)} \| \leq \epsilon(R)
\]
Combining (2.9), (2.14)
\[
\left| \{ I_\lambda (g_i)(u) \| u \|^{-\alpha} - I_\lambda (g)(u) \| u \|^{-\alpha} \geq 5k \} \cap B_M(0) \right| \\
\leq 2 \left( \frac{\epsilon(R)}{k} \right)^q + 2 \frac{O(\eta)}{k} + o(1).
\]

**Lemma 2.4.** There exist \( I \subset \mathbb{N} \) at most countable and a family \( \{ u_i \}_{i \in I} \) in \( \mathbb{H}^n \) such that
\[
\nu = |I_\lambda (g)| \| u \|^{-\alpha q} du + \sum_{i \in I} v_i \delta_{u_i}
\]
and
\[
\mu \geq |g|^p \| v \|^{\beta p} dv + \sum_{i \in I} \mu_i \delta_{u_i}.
\]
where \( \nu_i = \nu(u_i), \mu_i = \mu(u_i) \) for all \( i \in I \). Furthermore, we also have Moreover \( \nu_i \leq C_{Q,\alpha,\beta,p,q}^q i^{\frac{2}{\beta}}. \)

The original version of the above Lemma is on \( \mathbb{R}^n \), but it can also be done on \( \mathbb{H}^n \) because the crucial ingredient as Lemma 1.2 in [20] still holds in \( \mathbb{H}^n \). Thus we omit the detailed proof.

**The Proof of Theorem 1.1.** Assumptions of Theorem 1.1 imply that one of \( \alpha \) and \( \beta \) is nonnegative. Without the loss of generality, we may assume that \( \alpha > 0 \). Once we obtain \( \|g(v)|v|^\beta\|_p = 1 \), according to Stein-weiss inequality and \( \|g_i(v)|v|^\beta \rightarrow g(v)|v|^\beta \) weakly in \( L^p(\mathbb{H}^n) \), then \( g_i(v)|v|^\beta \rightarrow g(v)|v|^\beta \) strongly in \( L^p(\mathbb{H}^n) \) and \( \|I_\lambda(g)|u|^{-\alpha}\|_q = C_{Q,\alpha,\beta,p,q} \). Hence \( g \) is an extremal function for the Stein-Weiss inequality on Heisenberg group \( \mathbb{H}^n \).

Now, we turn to the proof of \( \|f(g)|v|^\beta\|_p = 1 \). we argue this by contradiction. If

\[
\|g(v)|v|^\beta\|_p = k < 1,
\]

and observe that \( \mu(\mathbb{H}^n) = 1 \) and \( \nu(\mathbb{H}^n) = C_{Q,\alpha,\beta,p,q} \), then in light of Lemma 2.4, we derive that

\[
\nu(\mathbb{H}^n) = \|I_\lambda(g)|u|^{-\alpha}\|_q + \sum_{i \in I} \nu_i
\]

\[
\leq C_{Q,\alpha,\beta,p,q}^q \|g(v)|v|^\beta\|_p^{q} + \sum_{i \in I} C_{Q,\alpha,\beta,p,q}^q i^{\frac{2}{\beta}}
\]

\[
\leq C_{Q,\alpha,\beta,p,q}^q k^{q} + C_{Q,\alpha,\beta,p,q}^q (\sum_{i \in I} \mu_i)^{\frac{2}{\beta}}
\]

\[
\leq C_{Q,\alpha,\beta,p,q}^q.
\]

Hence the above inequality must be equality. Since \( q > p \), we must have \( \mu = \delta_{u_0}, \nu = C_{Q,\alpha,\beta,p,q}^q \delta_{u_0} \) and \( g = 0 \). Next, we claim this is impossible to happen. We discuss this by distinguishing two cases.

**Case 1.** If \( u_0 = 0 \), then \( \int_{B_{3}(0,1)} d\mu = 1 \), which arrives at a contradiction with the initial hypotheses

\[
\int_{B_{3}(0,1)} |g_i(v)|p|v|^\beta d\nu = \frac{1}{2}.
\]

**Case 2.** if \( u_0 \neq 0 \), we will show that there exists some \( \delta > 0 \) such that

\[
0 \notin B_{3}(u_0, \delta) \text{ and } \lim_{i \to \infty} \int_{B_{3}(u_0, \delta)} |I_\lambda(g_i)|q|u|^{-\alpha q} du = 0,
\]

which arrive at a contradiction with \( \int_{B_{3}(u_0, \delta)} d\nu = C_{Q,\alpha,\beta,p,q}^q \). In fact, since

\[
I_\lambda(g_i)(u) \rightarrow I_\lambda(g)(u) = 0 \forall u \in \mathbb{H}^n,
\]

in light of Stein-Weiss inequality (1.6), we have

\[
\int_{B_{3}(u_0, \delta)} |I_\lambda(g_i)|t|u|^{-\alpha t} du \leq \int_{B_{3}(u_0, \delta)} |I_\lambda(g)|t|u|^t du
\]

\[
\leq \int_{\mathbb{H}^n} |I_\lambda(g)|t|u|^t du
\]

\[
\leq \int_{\mathbb{H}^n} |g_i|^p|v|^\beta d\nu,
\]

(2.17)
where \( \frac{1}{p} = \frac{1}{q} + \frac{\lambda + \beta}{Q} - 1 \). This together with the fact \( \alpha > 0 \), \( \frac{1}{q} = \frac{1}{p} + \frac{\lambda + \alpha + \beta}{Q} - 1 \) leads to \( t > q \).

Then it follows from inequality (2.17) and \( I_\lambda(g_i)(u) \to 0 \) a.e. \( u \in \mathbb{H}^n \) that
\[
\lim_{i \to \infty} \int_{B_\delta(u, \delta)} |I_\lambda(g_i)|^q |u|^{-\alpha q} du = 0.
\]

Then we accomplish the proof of Theorem 1.1.

3. The Proof of Theorem 1.4

In this section, we will give the existence results about the Stein-Weiss inequalities with \( |z| \) weights on the Heisenberg group. Namely, we will give the proof of Theorem 1.4. Since the idea of proof is similar to that of Theorem 1.1 in principle, we will give an approximately outline without detailed explanation.

It is not hard to check that extremals of the inequality (1.7) are those solving maximizing problem
\[
C_{Q, \alpha, \beta, p, q} := \sup \{ \| I_\lambda(g) |z|^{-\alpha} \|_{L^p(\mathbb{H}^n)} : f \geq 0, \| |g|^{\beta} \|_{L^p(\mathbb{H}^n)} = 1 \}.
\]

Since \( \sup \{ \| I_\lambda(g) |z|^{-\alpha} \|_{L^p(\mathbb{H}^n)} : f \geq 0, \| |g|^{\beta} \|_{L^p(\mathbb{H}^n)} = 1 \} \) is dilation-invariant and translation-invariant about \( v_0 = (0, t_0) \), hence we may suppose that \( \{g_i\}_i \) is also a maximizing sequence for problem (3.1) with
\[
\int_{B_\delta(0,1)} |g_i|^p |z|^{\beta p} dv = \sup_{t \in \mathbb{R}} \int_{B_\delta((0,t),1)} |g_i|^p |z|^{\beta p} dv = \frac{1}{2}
\]
for all \( i \in \mathbb{N} \). We define the measure:
\[
\mu_i := |g_i|^p |z|^{\beta p} dv \text{ and } \nu_i := |I_\lambda(g_i)|^q |z|^{-\alpha q} du.
\]

Then there exist two bounded measures \( \mu \) and \( \nu \) such that
\[
\mu_i \rightharpoonup \mu \text{ and } \nu_i \rightharpoonup \nu \text{ weakly in the sense of measures as } i \to \infty.
\]

Easily, it follows from lower semi-continuity of the measure that
\[
\int_{\mathbb{H}^n} d\mu \leq \lim_{i \to +\infty} \int_{\mathbb{H}^n} d\mu_i = 1 \text{ and } \int_{\mathbb{H}^n} d\nu \leq \lim_{i \to +\infty} \int_{\mathbb{H}^n} d\nu_i = C_{Q, \alpha, \beta, p, q}.
\]

We now apply Lions’ first concentration compactness to the sequence of measure \( \{\mu_i\} \).

**Lemma 3.1.** There exists a subsequence of \( \{\mu_i\} \) such that one of the following condition holds:

(a) (Compactness) There exists \( \{v_i\} \) in \( \mathbb{H}^n \) such that for each \( \epsilon > 0 \) small enough, we can find \( R_\epsilon > 0 \) with
\[
\int_{B_\delta(v_i, R_\epsilon)} d\mu_i \geq 1 - \epsilon \text{ for all } i.
\]

(b) (Vanishing) For all \( R > 0 \), there holds:
\[
\lim_{i \to \infty} \left( \sup_{v \in \mathbb{H}^n} \int_{B_\delta(v, R)} d\mu_i \right) = 0.
\]

(c) (Dichotomy) There exist \( k \in (0,1) \) such that for any \( \epsilon > 0 \), there exists \( R_\epsilon > 0 \) and a sequence \( \{v_i^\epsilon\}_{i \in \mathbb{N}} \) with the following property: given \( R' > R_\epsilon \), there are nonnegative measures \( \mu_1^\epsilon \) and \( \mu_2^\epsilon \) such that
\[
0 \leq \mu_1^\epsilon + \mu_2^\epsilon \leq \mu_i, \quad \text{Supp}(\mu_1^\epsilon) \subset B_\epsilon(v_i^\epsilon, R_\epsilon), \quad \text{Supp}(\mu_2^\epsilon) \subset \mathbb{H}^n \setminus B_\epsilon(v_i^\epsilon, R'),
\]

where
\[
\frac{1}{p} = \frac{1}{q} + \frac{\lambda + \beta}{Q} - 1, \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} + \frac{\lambda + \alpha + \beta}{Q} - 1.
\]
Lemma 3.2. Suppose that \{g_i\}_1 is a maximizing sequence for problem (3.1), then compactness (a) holds. In particular, we also have that \(\int_{\mathbb{H}^n} d\mu = 1\).

Proof. Clearly, (b) can not occur because \(\int_{B_{\mathbb{H}^n}(0,1)} |g_i|^p z'^\beta p dz = \frac{1}{2}\). Next, we only need to show that (c) also does not happen. We argue this by contradiction. Suppose that (c) occurs, then for any \(\varepsilon > 0\), we can find \(R_0 > 0\) and \(\{v_i\} \subset \mathbb{H}^n\) such that for any given \(R \geq R_0\), there holds

\begin{equation}
\|g_i|z'|^\beta \chi_{B_{\mathbb{H}^n}(v_i, R)}\|_p = \alpha + O(\varepsilon)\quad \text{and} \quad \|g_i|z'|^\beta \chi_{B_{\mathbb{H}^n}(v_i, R)}\|_p = 1 - \alpha + O(\varepsilon).
\end{equation}

Direct computations yield that

\begin{equation}
|I_{\lambda}(g_i)(u) - I_{\lambda}(g_i \chi_{B_{\mathbb{H}^n}(v_i, R)})(u)|
= \left| \int_{\mathbb{H}^n} \frac{(g_i \chi_{B_{\mathbb{H}^n}(v_i, R)})(v)}{|u^{-1}v|^\lambda} dv \right|
\leq \left( \int_{B_{\mathbb{H}^n}(v_i, R)} |g_i(v)|z'|^\beta p dv \right)^{\frac{1}{p}} \left( \int_{B_{\mathbb{H}^n}(v_i, R)} |u^{-1}v|^{-\lambda p'}|z|^{-\beta p'} dv \right)^{\frac{1}{p'}}
\end{equation}

here \(v = (z, t)\). under the assumptions of Theorem 1.4, we obtain \((\lambda + \beta)p' > Q\) and \(\beta p' < 2n\). On the other hand

\begin{equation}
\left( \int_{B_{\mathbb{H}^n}(v_i, R)} |u^{-1}v|^{-\lambda p'}|z|^{-\beta p'} dv \right)^{\frac{1}{p'}} \leq \int_{R} r^{Q-1-(\lambda+\beta)p'} dr \int_{\Sigma} |z'|^{-\beta p'} d\mu(\xi^*),
\end{equation}

where \(\Sigma = \{u \in \mathbb{H}^n : |u| = 1\}\) and \(\xi^* = (z^*, t^*) \in \Sigma\). Hence,

\begin{equation}
\lim_{R \to \infty} I_{\lambda}(g_i \chi_{B_{\mathbb{H}^n}(v_i, R)})(u) = I_{\lambda}(g_i)(u) \quad \text{for} \quad u \in \mathbb{H}^n.
\end{equation}

Now we can apply the Brezis-Lieb lemma to obtain

\begin{equation}
\|I_{\lambda}(g_i)|z|^{-\alpha}\|_q^q = \|I_{\lambda}(g_i \chi_{B_{\mathbb{H}^n}(v_i, R)})|z|^{-\alpha}\|_q^q + \|I_{\lambda}(g_i \chi_{B_{\mathbb{H}^n}(v_i, R)})|z|^{-\alpha}\|_q^q + o(1).
\end{equation}

Combining Stein-Weiss inequality and condition (3.3), we conclude that

\begin{equation}
\|I_{\lambda}(g_i \chi_{B_{\mathbb{H}^n}(v_i, R)})|z|^{-\alpha}\|_q^q \leq C_{Q, \alpha, \beta, p, q}^q \|g_i \chi_{B_{\mathbb{H}^n}(v_i, R)}|z'|^\beta p\|_p^p + C_{Q, \alpha, \beta, p, q}^q \|g_i \chi_{B_{\mathbb{H}^n}(v_i, R)}|z'|^\beta p\|_p^p + o(1)
\end{equation}

\begin{equation}
< C_{Q, \alpha, \beta, p, q}^q \left( k^{\frac{p}{q}} + (1 - k) k^{\frac{p}{q}}\right) + O(\varepsilon) + o(1)
\end{equation}

which arrives at a contradiction with \(\|I_{\lambda}(g_i)|z|^{-\alpha}\|_q^q \to C_{Q, \alpha, \beta, p, q}^q\).

Now, we claim that \(\int_{\mathbb{H}^n} d\mu = 1\). It follows from \(\int_{B_{\mathbb{H}^n}(0,1)} |g_i|^p |z'|^\beta p dz = \frac{1}{2}\) that \(\{v_i\}\) must be a bounded sequence in \(\mathbb{H}^n\). Hence, we may assume for any \(\varepsilon > 0\), there exist \(R_0\) such that for any \(R \geq R_0\), there holds

\begin{equation}
\int_{B_{\mathbb{H}^n}(0, R)} d\mu_i \geq 1 - \varepsilon.
\end{equation}
Then choose $\phi \in C^\infty_c(\mathbb{H}^n)$ with $\phi|_{B_R(0, R)} = 1$, then by the weak convergence in the sense of measure, we have
\[
\int_{\mathbb{H}^n} d\mu \geq \int_{\mathbb{H}^n} \phi(u) d\mu \\
\geq \lim_{i \to \infty} \int_{B_R(0, R)} |g_i|^p |z'|^\beta p du \\
\geq \lim_{i \to \infty} \int_{B_R(0, R)} d\mu_i \geq 1 - \varepsilon.
\]
(3.6)

Since $\varepsilon$ is arbitrarily small, we derive that $\int_{\mathbb{H}^n} d\mu \geq 1$. This accomplishes the proof of Lemma 3.2. \qed

The following lemma states that
\[
I_\lambda(g_i)(u)|z|^{-\alpha} \to I_\lambda(g)(u)|z|^{-\alpha} \text{ a.e.}
\]
Since the proof is similar to that of Lemma 2.3, we omit the detailed proof.

**Lemma 3.3.** Let $\{g_i\}$ be a maximizing sequence satisfying
\[
\|g_i(v)|z'|^\beta\|_{L^p(\mathbb{H}^n)} = 1 \quad \text{and} \quad \int_{B_R(0, R)} |g_i(v)|^p |z'|^\beta p dv \geq 1 - \varepsilon(R).
\]
we may assume that $g_i(v)|z'|^\beta \to g(v)|z'|^\beta$ weakly in $L^p(\mathbb{H}^n)$ (by passing to a subsequence if necessary). Then, by passing to a subsequence again if necessary,
\[
I_\lambda(g_i)(u)|z|^{-\alpha} \to I_\lambda(g)(u)|z|^{-\alpha} \text{ a.e.}
\]

**Lemma 3.4.** There exist $I \subset \mathbb{N}$ at most countable and a family $\{u_i\}_{i \in I}$ in $\mathbb{H}^n$ such that
\[
\nu = |I_\lambda(g)(u)|^q|z|^{\alpha q} du + \sum_{i \in I} \nu_i \delta_{u_i}
\]
and
\[
\mu \geq |g|^p |z'|^\beta p dv + \sum_{i \in I} \mu_i \delta_{u_i}
\]
where $\nu_i = \nu(u_i)$, $\mu_i = \mu(u_i)$ for all $i \in I$. Furthermore, we also have Moreover $\nu_i \leq C_{Q, \alpha, \beta, p, q} \mu_i^{q/2}$.

**The Proof of Theorem 1.4:** Without loss of generality, we may assume that $\alpha > 0$. Once we obtain $\|g|z'|^\beta\|_p = 1$, according to Stein-Weiss inequality and $g_i|z'|^\beta \to g|z'|^\beta$ weakly in $L^p(\mathbb{H}^n)$, then $g_i|z'|^\beta \to g|z'|^\beta$ strongly in $L^p(\mathbb{H}^n)$ and $\|I_\lambda(g)|z|^{-\alpha}\|_q = C_{Q, \alpha, \beta, p, q'}$. Hence $g$ is a extremal function for Stein-Weiss inequality on Heisenberg group $\mathbb{H}^n$. Now, we turn to the proof of $\|g|z'|^\beta\|_p = 1$. we argue this by contradiction. If
\[
\|g|z'|^\beta\|_p = k < 1,
\]
and observe that \( \mu(\mathbb{H}^n) = 1 \) and \( \nu(\mathbb{H}^n) = C_{Q, \alpha, \beta, p, q} \), then in light of Lemma 3.4, we derive that
\[
\nu(\mathbb{H}^n) = \|I_\lambda(g)|z|^{-\alpha}\|_q^q + \sum_{i \in I} \nu_i
\leq C_{Q, \alpha, \beta, p, q}^q \|g|z'|^\beta\|_p^q + \sum_{i \in I} C_{Q, \alpha, \beta, p, q}^q \mu_i^\beta
\leq C_{Q, \alpha, \beta, p, q}^q k^\beta + C_{Q, \alpha, \beta, p, q}^q (\sum_{i \in I} \mu_i)^\beta
\leq C_{Q, \alpha, \beta, p, q}^q.
\]
(3.7)

Hence the above inequality must be equality. Since \( q > p \), we must have \( \mu = \delta_{u_0} \), \( \nu = C_{Q, \alpha, \beta, p, q}^q \delta_{u_0} \) and \( g = 0 \). Next, we claim this is impossible to happen. We discuss this by distinguishing two case.

Case 1. If \( u_0 = (z_0, t_0) \) with \( z_0 = 0 \), then \( \int_{B_{\mathbb{H}}((0,t_0),1)} d\mu = 1 \), which arrives at a contradiction with the initial hypotheses
\[
\sup_{t \in \mathbb{R}} \int_{B_{\mathbb{H}}((0,t),1)} |g_i|^p |z'|^\beta p du = \frac{1}{2}.
\]

Case 2. if \( u_0 = (z_0, t_0) \) with \( z_0 \neq 0 \), we will show that there exists some \( \delta > 0 \) such that
\[
(0,t) \notin B_{\mathbb{H}}(u_0, \delta) \quad \text{and} \quad \lim_{t \to \infty} \int_{B_{\mathbb{H}}(u_0, \delta)} |I_\lambda(g_i)|^q |z|^{-\alpha q} du = 0,
\]
which arrive at a contradiction with \( \int_{B_{\mathbb{H}}(u_0, \delta)} dv = C_{Q, \alpha, \beta, p, q}^q \). In fact, since
\[
I_\lambda(g_i)(u) \to I_\lambda(g)(u) = 0, \; \forall \; u \in \mathbb{H}^n,
\]
in light of Stein-Weiss inequality in case of \( \beta = 0 \), we have
\[
\int_{B_{\mathbb{H}}(u_0, \delta)} |I_\lambda(g_i)|^f |z|^{-\alpha t} du \lesssim \int_{B_{\mathbb{H}}(u_0, \delta)} |I_\lambda(g_i)|^t du
\lesssim \int_{\mathbb{H}^n} |I_\lambda(g_i)|^t du
\lesssim \int_{\mathbb{H}^n} |g_i|^p |z'|^\beta p dv,
\]
where \( \frac{1}{t} = \frac{1}{p} + \frac{\lambda + \beta}{Q} - 1 \). This together with the fact \( \alpha > 0 \), \( \frac{1}{q} = \frac{1}{p} - \frac{\lambda + \alpha + \beta}{Q} - 1 \) and \( I_\lambda(g_i)(u) \to 0 \) a.e. \( u \in \mathbb{H}^n \) yields
\[
\lim_{t \to \infty} \int_{B_{\mathbb{H}}(u_0, \delta)} |I_\lambda(g_i)|^q |z|^{-\alpha q} du = 0.
\]

Then we accomplish the proof of Theorem 1.4

4. The Proof of Theorem 1.5

In this section, we will use Sawyer and Wheeden condition on weighted inequalities for integral operators in [22] and concentration compactness principle to establish inequality (1.11) and existence of their extremal functions. Observe that inequality (1.11) is equivalent to the boundedness of the following weighted fractional integral operator:
\[
\|T(g)|z'|^{-\alpha}\|_{L^q(\mathbb{R}^{m+n})} \leq C \|g|y'|^\beta\|_{L^p(\mathbb{R}^{m+n})},
\]
where $T(g)$ is defined as 
\[
T(g) = \int_{\mathbb{R}^{n+m}} |x - y|^{-\lambda} g(y) dy.
\]

In order to prove that operator $T$ is bounded from $L^p(|y'|^\beta dy, \mathbb{R}^{m+n})$ to $L^q(|x'|^{-\alpha} dx, \mathbb{R}^{m+n})$, according to Sawyer and Wheeden’s result [22], we only need to verify that the following two conditions hold.

**Lemma 4.1.** The operator $T$ is bounded from $L^p(|y'|^\beta dy, \mathbb{R}^{m+n})$ to $L^q(|x'|^{-\alpha} dx, \mathbb{R}^{m+n})$ if and only if the following two conditions hold.

1. There exists $\varepsilon > 0$ such that for any pair of balls $B$ and $B'$ with radius $r$ and $r'$ satisfying $B' \subset 4B$,
\[
\left(\frac{r'}{r}\right)^{Q-\varepsilon} \left(\frac{\phi(B')}{\phi(B)}\right) \leq C_\varepsilon.
\]

2. There exists $t > 1$ such that for any ball $B \subset \mathbb{R}^{n+m}$,
\[
\phi(B)|B|^{\frac{1}{\nu} + \frac{\beta}{p}} \left(\frac{1}{|B|} \int_B |x'|^{-\alpha q t} dx\right)^{\frac{1}{q}} \left(\frac{1}{|B|} \int_B |y'|^{-\beta p t} dy\right)^{\frac{1}{p}} \leq C_t.
\]

Here $\phi(B) = 2^{4\lambda} r^{-\lambda}$ and $B, B' \in \mathbb{R}^{m+n}$. According to the assumptions of Theorem 1.5, we see that there exists $t > 1$ such that $\alpha q t < m$, $\beta p t < m$. Then it follows from that integrals $\int_B |x'|^{-\alpha q t} dx$ and $\int_B |x'|^{-\beta p t} dx$ are finite.

5. **The Proof of Theorem 1.6**

In this section, we will use the method of moving plane in integral forms introduced by Chen, Li and Ou [6] to prove that symmetry results for each pair of solutions $(u, v)$ of integral system (1.4). In order to prove our theorem, we first introduce some notations. For $\lambda \in \mathbb{R}$, denote
\[
H_\nu = \{x \in \mathbb{R}^{m+n} : x_1 > \nu\}.
\]

For $x = (x_1, \hat{x}) \in \mathbb{R}^{m+n}$, we denote $x_\lambda = (2\nu - x_1, \hat{x})$. Write $u_\lambda(x) = u(x^\nu)$ and $v_\nu(x) = v(x^\nu)$. Assume that $(u, v) \in L^{p_1+1}(\mathbb{R}^{m+n}) \times L^{p_2+1}(\mathbb{R}^{n+m})$ is a pair of positive solutions of the following integral system
\[
\begin{cases}
\int_{\mathbb{R}^{n+m}} \frac{1}{|x'|^\alpha |x - y|^{-\lambda} v^p(y) |y'|^{-\beta}} dy = 0, \\
\int_{\mathbb{R}^{n+m}} \frac{1}{|x'|^{-\beta} |x - y|^{-\lambda} u^q(y) |y|^{-\alpha}} dy = 0,
\end{cases}
\]

where $\frac{1}{p_1-1} + \frac{1}{p_2-1} = \frac{\alpha + \beta + \lambda}{n+m}$. We only prove that $u(x)|_{\mathbb{R}^m}$ and $v(x)|_{\mathbb{R}^m}$ are radially symmetric and monotonously decreasing about origin. We need the following lemma whose proof can be seen in [5].

**Lemma 5.1.** If $(u, v)$ is a pair of nonnegative solutions of \((5.1)\), then for any $x \in \mathbb{R}^{m+n}$, there holds
\[
\begin{align*}
\frac{1}{|x'|^\alpha |x - y|^{-\lambda} v^p(y)} dy \\
+ \int_{H_\nu} \left(\frac{1}{|x'|^{-\beta} |x - y|^{-\lambda} u^q(y) |y|^{-\alpha}} - \frac{1}{|x'|^\alpha |x - y|^{-\lambda} v^p(y)}\right) dy \\
+ \int_{H_\nu} \left(\frac{1}{|x'|^\alpha |x - y|^{-\lambda} v^p(y) |y|^{-\alpha}} - \frac{1}{|x'|^{-\beta} |x - y|^{-\lambda} v^p(y)}\right) dy
\end{align*}
\]
and
\[ v(x) - v(x') = \int_{H_v} \left( \frac{1}{|x'|^{\beta}|x' - y|^{\gamma}} \frac{1}{|x|^{\alpha}|x - y|^{\lambda}} \right) \left( \frac{1}{(|y'|)^{\alpha}} - \frac{1}{|y|^{\alpha}} \right) u^p(y) dy \]
\[ + \int_{H_v} \left( \frac{1}{|x'|^{\beta}|x' - y|^{\gamma}} - \frac{1}{|x|^{\alpha}|x - y|^{\lambda}} \frac{1}{|y'|^{\alpha}} \frac{1}{|x - y|^{\lambda}} \right) u^p(y) dy \]
\[ + \int_{H_v} \left( \frac{1}{|x'|^{\beta}|x' - y|^{\gamma}} - \frac{1}{|x|^{\alpha}} \frac{1}{|x - y|^{\lambda}} \frac{1}{|y'|^{\alpha}} \right) (u^p(y) - u^p_1(y)) dy. \]

Now we are in a position to prove Theorem 1.6, the proof will be divided into two steps.

**Step 1.** We show that for sufficiently negative \( \nu \),
\[ u_\nu(x) - u(x) \leq 0, \quad v_\nu(x) - v(x) \leq 0, \quad \forall x \in H_\nu. \]
Define
\[ H_\nu^u = \{ \xi \in H_\nu | u_\nu(x) - u(x) > 0 \}, \quad H_\nu^v = \{ x \in H_\nu | v_\nu(x) - v(x) > 0 \}. \]
We only need to verify that \( H_\nu^u \) and \( Q_\nu^v \) must be empty for sufficiently negative \( \nu \).

Applying Mean Value Theorem and Lemma 5.1, we obtain that for any \( x \in H_\nu^u \), there holds
\[ u_\nu(x) - u(x) \leq \int_{H_\nu} \left( |x|^{\alpha}|x - y|^{\lambda}|(y')^{\beta} - |(x')^{\beta}|^{\alpha}|x - y|^{\lambda}|(y')^{\beta} - |(x')^{\beta}|^{\alpha} \right) \left( u^p(y) - u^p_2(y) \right) dy \]
\[ \leq q_0 \int_{H_\nu} \left| x \right|^{\alpha}|x - y|^{\lambda}|(y')_\beta - |(x')_\beta|^{\alpha} \right| v^p_0(y) v^p_0(y) - v(y) dy \]
and
\[ v_\lambda(x) - v(x) \leq \int_{H_\nu} \left( |x|^{\beta}|x - y|^{\lambda}|(y')^{\alpha} - |(x')^{\alpha}|^{\beta}|x - y|^{\lambda}|(y')^{\alpha} - |(x')^{\alpha}|^{\beta} \right) \left( u^p(y) - u^p_1(y) \right) dy \]
\[ \leq p_0 \int_{H_\nu} \left| x \right|^{\beta}|x - y|^{\lambda}|(y')^{\alpha} - |(x')^{\alpha}|^{\beta} \right| u^p_0(y) u^p_0(y) - u(y) dy. \]

Thanks to \( (u, v) \in L^{p+1}(\mathbb{R}^m) \times L^{p+1}(\mathbb{R}^m) \) and integral inequality (1.11), we derive that
\[ \| u_\lambda - u \|_{L^{p+1}(H_\nu)} \leq p_2 \| v_\nu \|_{L^{p+1}(Q_\nu)} \| v_\nu - v \|_{L^{p+1}(H_\nu)} \]
and
\[ \| v_\nu - v \|_{L^{p+1}(H_\nu)} \leq p_1 \| u_\nu \|_{L^{p+1}(H_\nu)} \| u_\nu - u \|_{L^{p+1}(H_\nu)}. \]
This together with (5.3) and (5.4) yields that
\[ \| u_\nu - u \|_{L^{p+1}(H_\nu)} \leq p_1 p_2 \| v_\nu \|_{L^{p+1}(H_\nu)} \| u_\nu \|_{L^{p+1}(H_\nu)} \| v_\nu - v \|_{L^{p+1}(H_\nu)} \]
and
\[ \| v_\nu - v \|_{L^{p+1}(H_\nu)} \leq p_1 q_1 \| u_\nu \|_{L^{p+1}(H_\nu)} \| u_\nu \|_{L^{p+1}(H_\nu)} \| v_\nu - v \|_{L^{p+1}(H_\nu)} \]
In virtue of the conditions $(u, v) \in L^{p_1+1}(\mathbb{R}^{m+n}) \times L^{p_2+1}(\mathbb{R}^{m+n})$, we can choose sufficiently negative $\nu$ such that

$$
(5.7) \quad \|u_\nu - u\|_{L^{p_1+1}(H_2^\nu)} \leq \frac{1}{2}\|u_\nu - u\|_{L^{p_1+1}(H_2^\nu)}, \quad \|v_\nu - v\|_{L^{p_2+1}(H_2^\nu)} \leq \frac{1}{2}\|v_\nu - v\|_{L^{p_2+1}(H_2^\nu)},
$$

which implies that $H_\nu^u$ and $H_\nu^v$ must be empty sets.

**Step 2.** The inequality (5.2) provides a starting point for moving the plane. Now we start from the negative infinity of $x_1$-axis and move the plane to the right as long as (5.2) holds. Define

$$
\nu_0 = \sup\{\nu \mid u_\nu(x) \leq u(x), \quad v_\nu(x) \leq v(x), \quad \mu \leq \nu, \quad \forall \, x \in H_\mu\}.
$$

We will show that $nu_0 = 0$. Suppose on the contrary that $\nu_0 < 0$, we will show that $u$ and $v$ must be symmetric about the plane $x_1 = \nu_0$, namely

$$
(5.8) \quad u_{\nu_0}(x) \equiv u(x), \quad v_{\nu_0}(x) \equiv v(x), \quad \forall \, x \in H_{\nu_0}.
$$

Otherwise, we may assume on $H_{\nu_0}$,

$$
u_0(x) \leq u(x), \quad v_0(x) \leq v(x), \quad \text{but} \quad u_{\nu_0}(x) \neq u(x) \quad \text{or} \quad v_{\nu_0}(x) \neq v(x).
$$

In the case of $u_{\nu_0}(\xi) \neq u(x)$ on $H_{\nu_0}$, one can employ Lemma 5.1 to obtain $u_{\nu_0}(x) < u(x)$ and $v_{\nu_0}(x) < v(x)$ in the interior of $H_{\nu_0}$.

Next, we will show that the plane can be moved further to the right. That is to say, there exists an $\varepsilon > 0$ such that for any $\nu \in [\nu_0, \nu_0 + \varepsilon)$,

$$
(5.9) \quad u_\nu(x) \leq u(x), \quad v_\nu(x) \leq v(x), \quad \forall \, x \in H_\nu.
$$

Set

$$
\overline{H}_{\nu_0}^u = \{x \in H_{\nu_0} \mid u(x) \geq u_{\nu_0}(\xi)\}, \quad \overline{Q}_{\nu_0}^v = \{x \in H_{\nu_0} \mid v(x) \geq v_{\nu_0}(x)\}.
$$

It is clear that both $\overline{H}_{\nu_0}^u$ and $\overline{Q}_{\nu_0}^v$ must be measure zero and

$$
\lim_{\nu \to \nu_0} H_\nu^u \subset \overline{H}_{\nu_0}^u, \quad \lim_{\nu \to \nu_0} Q_\nu^v \subset \overline{Q}_{\nu_0}^v.
$$

Combining this and integrability conditions $(u, v) \in L^{p_1+1}(\mathbb{R}^{m+n}) \times L^{p_2+1}(\mathbb{R}^{m+n})$, one can choose $\varepsilon$ small enough such that for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$, there holds

$$p_1 p_2 \|u\|_{L^{p_1+1}(H_2^\nu)} \|v\|_{L^{p_2+1}(H_2^\nu)} \leq \frac{1}{2}.
$$

As what we did in the estimates of (5.1), one can obtain

$$
\|u - u_\nu\|_{L^{p_1+1}(H_2^\nu)} = \|v - v_\nu\|_{L^{p_2+1}(H_2^\nu)} = 0,
$$

which implies that $H_\nu^u$ and $H_\nu^v$ must be measure zero.

Finally, we show that the plane cannot stop before touching the origin. We argue this by contradiction. Suppose the plane stops at $x_1 = \nu_0 < 0$, according to the above argument, we know that $u$ and $v$ must be symmetric about the plane $x_1 = \nu_0$, that is to say

$$
(5.8) \quad u_{\nu_0}(x) \equiv u(x), \quad v_{\nu_0}(x) \equiv v(x), \quad \forall \, x \in H_{\nu_0}.
$$
Since $|x - y^0|^{-\lambda} < |x - y|^{-\lambda}$, $|x^0 - y|^{-\lambda} < |x - y|^{-\lambda}$, $|(x^0)'| > |x'|$ and $|(y^0)'| > |y'|$, we derive that
\[
\begin{align*}
u_0(x) - u(x) &= \int_{H_{y_0}} \left((|x^0|')^{-\alpha}|x^0 - y|^{-\lambda} - |x'|^{-\alpha}|x - y|^{-\lambda})u^2(y)|y'|^{-\beta} dy \right. \\
&\quad+ \int_{H_{y_0}} \left((|x^0|')^{-\alpha}|x^0 - y^0|^{-\lambda} - |x'|^{-\alpha}|x - y^0|^{-\lambda})v^2(y)|y'|^{-\beta} dx \\
&\quad< \int_{H_{y_0}} \left(|x'|^{-\alpha}|x - y|^{-\lambda} - |x'|^{-\alpha}|x^0 - y|^{-\lambda})u^2(y)|y'|^{-\beta} dy \\
&\quad+ \int_{H_{y_0}} \left(|x'|^{-\alpha}|x - y|^{-\lambda} - |x'|^{-\alpha}|x^0 - y|^{-\lambda})v^2(y)|y'|^{-\beta} dy \\
&\quad= \int_{H_{y_0}} \left(|x'|^{-\alpha}(|x - y|^{-\lambda} - |x^0 - y|^{-\lambda})v^2(y)|(|y^0|')^{-\beta} - |y|^{-\beta})dy \\
&\quad< 0,
\end{align*}
\] which leads to a contradiction. Then it follows that $v_0 = 0$, $u_0(x) \leq u(x)$ and $v_0(x) \leq v(x)$. Carrying out the above procedure in the opposite direction, one can also obtain the moving plane must stop before the origin. Hence, we have $u_0(x) \geq u(x)$ and $v_0(x) \geq v(x)$. Since the $x_1$ direction can be replace by the $x_i$ direction for $i = 1, 2, \cdots, m$, we deduce that $u(x)|_{\mathbb{R}^m}$ and $v(x)|_{\mathbb{R}^m}$ must be radially symmetric and monotone decreasing about the origin.

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