On Estimates of the Number of Collisions for Billiards in Polyhedral Angles

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Abstract We obtain an upper bound of the number of collisions of any billiard trajectory in a polyhedral angle in terms of the minimal eigenvalue of a positive definite matrix which characterizes the angle. Elements of the matrix are scalar products between the unit normal vectors of faces of the angle.

Keywords: billiard; polyhedral angle; elastic collision; reflection; MSC 2000: 58F15, (28A65)

1 Introduction

Let $H_1, \ldots, H_n$ be $n$ hyperplanes in $m$-dimensional Euclidean space $\mathbb{R}^m$ passing through the origin and take a unit normal $\alpha_i$ for each $H_i$, $i = 1, \ldots, n$. Assume the hyperplanes are in general position, i.e. $\bigcap_{i=1}^n H_i$ is an $(m-n)$-dimensional plane, or equivalently, $\alpha_1, \ldots, \alpha_n$ are linear independent. The polyhedral cone corresponding to $\alpha_1, \ldots, \alpha_n$ is

$$Q = \left\{ y \in \mathbb{R}^m \mid (y, \alpha_i) \geq 0, \forall i \right\}.$$ 

Our fundamental object of study is a billiard in $Q$, that is, a point particle moves with a uniform motion in the interior of $Q$ and has specular (optical) reflections at walls $B_i = H_i \cap Q$, $i = 1, \ldots, n$. If a billiard trajectory reaches one of the corners $B_i \cap B_j$, $i \neq j$, its further motion is not defined.

The model is interesting partially because any hard ball system on a line is isomorphic to a billiard in an appropriate polyhedral cone—elastic collisions between hard balls correspond to specular reflections made by the billiard at walls. More generally, all hard ball systems are isomorphic to semi-dispersing billiards. Estimates on the number of collisions of billiard trajectories have been studied for a long time, cf. [2]. We present here in a nutshell only a few selected results rather than a comprehensive review.

In 1978, Sinai [6] proved the existence of uniform estimates of the number of collisions of billiard trajectories in a polyhedral angle. At the same time, he also pointed out that the smooth version of the result should also hold, that is, when the polyhedral angle is replaced by a smooth hypersurface with nonnegative

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This work was supported by the Special Funds for Chinese Major State Basic Research Projects "Nonlinear Science".
second fundamental form, uniform estimates still exist in a neighborhood of a point with a condition of linear independence. Using the same method of Sinai, in 1993, Sevryuk gave a uniform estimate for billiards in a polyhedral angle in terms of a geometrical characteristic of the angle. A milestone is established in 1998 by Burago, Ferleger and Kononenko, see also, uniform estimates were obtained for semi-dispersing billiards on arbitrary Riemannian manifolds with boundaries satisfying a nondegenerate condition.

Now we describe the estimates mentioned above in more details. Sevryuk introduced the concepts of charge and capacity of a polyhedral angle in [5]. The charge of the polyhedral cone $Q$

$$S(Q) = \max_{a \subset Q} \min_{1 \leq i \leq n} \angle(a, H_i),$$

where $0 \leq \angle(a, H_i) \leq \frac{\pi}{2}$ denotes the angle between the ray $a$ emanating from the origin and the hyperplane $H_i$ and the maximum is taken over all rays $a$ that pass within $Q$. The charge $\varphi$ of the set of hyperplanes $H_1, \ldots, H_n$ is the minimum of the charges of polyhedral cones determined by the hyperplanes. Then $0 < S(Q) \leq \frac{\pi}{2}$, $0 < \varphi \leq \frac{\pi}{2}$. The capacity $\psi$ of the set of hyperplanes $H_1, \ldots, H_n$ is

$$0 < \min_{a \subset \mathbb{R}^m} \max_{1 \leq i \leq n} \angle(a, H_i) < \frac{\pi}{2},$$

if the hyperplanes do not all pass through a common line, and $\frac{\pi}{2}$ otherwise. Sevryuk proved $\varphi \leq \psi$ and that arbitrary billiard trajectory in $Q$ has no more than $\frac{\sin^2 \varphi}{2} \left( \frac{4}{\sin^2 \varphi} \right)^{2^{n-1}} - 1$ collisions. The estimate of Burago, Ferleger and Kononenko involves a nondegenerate constant. Following [1], the nondegenerate constant of the polyhedral cone $Q$ is

$$C = \min_{y \in Q \setminus (\bigcap_{i=1}^{n} B_i)} \max_{1 \leq i \leq n} \frac{\dist(y, B_i)}{\dist(y, \bigcap_{j=1}^{n} B_j)}.$$  \hspace{1cm} (1)

Then $0 < C \leq 1$. It is proved that any billiard trajectory in $Q$ has no more than $8 \left( \frac{1}{3} + 2 \right)^{2(n-1)}$ collisions.

Consider the positive definite matrix $((\alpha_i, \alpha_j))_{n \times n}$. Let $\lambda_{\text{min}}$ be its minimal eigenvalue. In this paper, we will prove that the number of collisions of a billiard trajectory in $Q$ does not exceed $n! \left( \frac{4}{\lambda_{\text{min}}} \right)^{n-1}$. The case $n = 1$ is obvious and the arguments proceed by induction on $n$—the number of walls—as in every proof, as far as we know, of finiteness of the number of collisions for polyhedral billiards when $n > 2$.

## 2 Estimate by $\lambda_{\text{min}}$

In what follows, we assume the normals $\alpha_1, \ldots, \alpha_n$ span the whole ambient space $\mathbb{R}^m$ (so $m = n$), i.e. $\bigcap_{i=1}^{n} H_i = \{0\}$. Otherwise, we can project the dynamics to the orthocomplementation of $\bigcap_{i=1}^{n} H_i$.

Before establishing our estimate for the general case, we would like to discuss the interesting case $n = 2$.

It is well known that unfolding a billiard trajectory inside a wedge to a straight line yields a sharp bound $\lfloor \frac{\pi}{\theta} \rfloor$ for the number of collisions, where $\theta = \frac{\pi}{2}$.
arccos \((- (\alpha_1, \alpha_2))\) is the angle of the wedge and \([x]\) is the ceiling function, the smallest integer not less than \(x\), cf. [7]. The argument also shows that if the point particle does not hit the corner of the wedge at first collision, it will never hit the corner in the future.

One can take another way as follows in which only the velocity, rather than the position, of the point particle is concerned. Suppose the particle moves with unit speed and has suffered \(N\) collisions. Let \(v_0, v_1, \ldots, v_N\) on the unit circle be the sequence of velocities. For any \(k, 1 \leq k \leq N - 1\), \(v_k - v_{k-1}\) is parallel to some \(\alpha_i\), say \(\alpha_1\), so \(v_k - v_{k-1}\) is parallel to \(\alpha_2\). That is

\[
v_k - v_{k-1} = \|v_k - v_{k-1}\| \alpha_1, \quad v_{k+1} - v_k = \|v_{k+1} - v_k\| \alpha_2.
\]

It is easy to see that the angle \(\angle v_{k-1}v_kv_{k+1}\) is equal to \(\theta\). By elementary geometry, the arc length of the unit sphere be the sequence of velocities. For any \(k, 1 \leq k \leq N - 1\), \(v_k - v_{k-1}\) is parallel to some \(\alpha_i\), say \(\alpha_1\), so \(v_k - v_{k-1}\) is parallel to \(\alpha_2\). That is

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In the general case, we do not intend to find the best estimate for the number of collisions especially by the method of induction. For some special cases, sharp bounds may be found as we have seen for \(n = 2\). For another example, if \((\alpha_i, \alpha_j) = 0\) for \(|i - j| > 1\), and \((\alpha_i, \alpha_{i+1}) \geq 1\) for \(i = 1, \ldots, n - 1\), then the maximal possible number of collisions of a billiard trajectory is \(\frac{n(n + 1)}{2}\), see [4].

Now we proceed to establish our estimate for the general case.

It is convenient to assume the particle moves with unit speed. Suppose a part of its trajectory has undergone \(N\) reflections at the walls. Let \(v_0, v_1, \ldots, v_N\) on the unit sphere be the sequence of velocities.

Since \(\alpha_1, \ldots, \alpha_n\) are linear independent, they, perceived as \(n\) points in \(\mathbb{R}^n\), determine a hyperplane not passing through the origin. Let \(d > 0\) be the distance from the origin to the hyperplane and \(e\) the unit outer normal of the hyperplane. They are characterized by the equations

\[
(e, \alpha_i) = d > 0, \quad \forall i. \quad (2)
\]

It means that \(d\) is the radius of the inscribed ball of the polyhedral cone \(Q\) with the center \(e\) on the unit sphere. The construction proves the existence of the inscribed ball. Define the matrix \(A = (\alpha_1, \ldots, \alpha_n)\), where \(\alpha_i\) are perceived as column vectors in \(\mathbb{R}^n\). Then equation (2) read

\[
e^T A = d(1, \ldots, 1). \quad (3)
\]

Thus

\[
\frac{1}{d} = \|(1, \ldots, 1)A^{-1}\|, \quad e^T = d(1, \ldots, 1)A^{-1}.
\]

**Lemma 2.1.** Let \(L\) be the length of the zigzag line determined by the points \(v_0, v_1, \ldots, v_N\). Then

\[
L = \sum_{k=0}^{N-1} \|v_{k+1} - v_k\| \leq \frac{2}{d}.
\]

**Proof.** From the law of reflection we have

\[
v_{k+1} - v_k = \|v_{k+1} - v_k\| \alpha_i, \quad k = 0, 1, \ldots, N - 1.
\]

Combining with (2) yields

\[
(v_{k+1} - v_k, e) = d \|v_{k+1} - v_k\|, \quad k = 0, 1, \ldots, N - 1.
\]
Taking the sum over $k$, we obtain

$$L = \frac{1}{d}(v_N - v_0, e) \leq \frac{1}{d} \|v_N - v_0\| \cdot \|e\| \leq \frac{2}{d}.$$

Besides $d$, another constant $\delta$ is involved in our proof. It has already appeared in the Sinai’s original proof [6]. When $\bigcap_{i=1}^n H_i = \{0\}$, formula (1) reads

$$C = \min_{y \neq 0} \max_{1 \leq i \leq n} \frac{1}{\|y\|} \text{dist}(y, B_i) = \min_{y \neq 0} \max_{1 \leq i \leq n} \text{dist}(y, B_i).$$

We present the definition of $\delta$ similar to formula (4):

$$\delta = \min_{\|y\|=1} \max_{1 \leq i \leq n} \text{dist}(y, H_i) = \sin \psi.$$

First of all, $\max_{1 \leq i \leq n} \text{dist}(\cdot, H_i)$ is a continuous function. It is positive everywhere on the unit sphere since $\bigcap_{i=1}^n H_i = \{0\}$. By compactness of the unit sphere, $\delta > 0$.

**Theorem 2.2.** The number of reflections of any billiard trajectory in $Q$ does not exceed $n! \left(\frac{1}{\lambda_{\min}}\right)^{n-1}$.

**Proof.** Induction on $n$. The case $n = 1$ is trivial and suppose we have proved the theorem from 1 to $n-1$. Note that in the inductive hypothesis, the number of walls needs not to be the dimension $m$ of the configuration space. Now we proceed to prove the theorem for $n$. At this stage, we may assume $n = m$ as claimed at the beginning of this section. So we have Lemma 2.1.

Set $N' = (n-1)! \left(\frac{1}{\lambda_{\min}}\right)^{n-2}$. We need the fact from linear algebra that the minimal eigenvalue of any principal submatrix of the positive definite matrix $(\alpha_i, \alpha_j)_{n \times n} = A^T A$ is not less than $\lambda_{\min}$. It easily follows from the minimax principle for eigenvalues, particularly for $\lambda_{\min}$:

$$\lambda_{\min} = \min_{\|x\|=1} \|Ax\|^2 = \min_{x_1^2 + \cdots + x_n^2 = 1} \|x_1 \alpha_1 + \cdots + x_n \alpha_n\|^2.$$

And by the inductive hypothesis, if a sequence of consecutive reflections does not involve all the hyperplanes, then the length of this sequence does not exceed $[N']$.

Suppose $[N'] + 1 \leq N$, thus the first $[N'] + 1$ reflections involve all the hyperplanes. Hence, for any $i$, the points $v_0, v_1, \ldots, v_{[N'] + 1}$ do not lie on the same side of the hyperplane $H_i$. Say, $v_0$ and $v_{k_i}$ do not lie on the same side of $H_i$, then $\|v_{k_i} - v_0\| > \text{dist}(v_0, H_i)$. So the length of the $[N'] + 1$ segments

$$\sum_{k=0}^{[N']} \|v_{k+1} - v_k\| > \max_{1 \leq i \leq n} \text{dist}(v_0, H_i) \geq \delta.$$

It shows that the length of any consecutive $[N'] + 1$ segments of the zigzag line determined by the points $v_0, v_1, \ldots, v_N$ is bigger than $\delta$. But Lemma 2.1 says that the length of the whole zigzag line does not exceed $\frac{2}{d}$. One obtains

$$N < \frac{2}{d} ([N'] + 1) \leq \frac{4}{d} N'.$$
It remains to show $\frac{1}{d^2} \leq \frac{n}{\lambda_{\text{min}}}$. In fact, $\frac{1}{d}$ and $\frac{1}{\delta}$ are both not bigger than $\sqrt{\frac{n}{\lambda_{\text{min}}}}$. To this end, we shall use the minimax principle for minimal eigenvalues and the fact that the minimal eigenvalue of $AA^T$ is the same of $A^TA$, which can be seen from the identity

$$
\det \left( \lambda I - AA^T \right) = \det A \det \left( \lambda A^{-1} - A^T \right) \\
= \det \left( \lambda A^{-1} - A^T \right) \det A \\
= \det \left( \lambda I - A^TA \right).
$$

Taking square of the norm of the two sides of equation (3), one obtains

$$d^2n = \left\| e^T A \right\|^2 \geq \lambda_{\text{min}}.$$

On the other hand, $\delta = \max_{1 \leq i \leq n} \text{dist}(y_0, H_i) = \max_{1 \leq i \leq n} \left| (y_0, \alpha_i) \right|$ for some $y_0$ on the unit sphere. Thus

$$\delta^2 = \max_{1 \leq i \leq n} \left| (y_0, \alpha_i) \right|^2 \geq \frac{1}{n} \sum_{i=1}^{n} \left| (y_0, \alpha_i) \right|^2 = \frac{1}{n} \left\| y_0^T A \right\|^2 \geq \frac{\lambda_{\text{min}}}{n}.$$  

\[ \square \]

**Remark 2.3.** The proof also gives another upper bound $\left( \frac{d}{\delta^2} \right)^{n-1}$ by setting $N' = \left( \frac{1}{\delta^2} \right)^{n-2}$.

**Remark 2.4.** Make an observation. Let $B = (-\alpha_1, \ldots, \alpha_n)$. If $\lambda$ is an eigenvalue of $A^T A$ associated with an eigenvector $\xi = (a_1, \ldots, a_n)^T$, i.e. $A^T A \xi = \lambda \xi$. Then $B^T A \xi = \lambda \eta$, where $\eta = (-a_1, \ldots, a_n)^T$. Therefore $\lambda_{\text{min}}$, as $\delta, \varphi$ and $\psi$, is independent of the choice of the polyhedral cone $Q$ and is indeed determined by the hyperplanes $H_1, \ldots, H_n$.

**Acknowledgements**

The author would like to thank professor Gu, C.H. and Zhou, Z.X. for many questions, comments and suggestions. Thanks also go to professor Hu, H.S. for her staunch support.

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