Comment on “Lattice QCD analysis of the strangeness magnetic moment of the nucleon”

Chun Wa Wong

Department of Physics and Astronomy, University of California, Los Angeles, CA 90095-1547
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The recent chirally extrapolated result of Leinweber and Thomas [Phys. Rev. D 62, 074505 (2000), or LT] for the nucleon strangeness form factor $G_{3u}(0) = -0.16 \pm 0.18 \mu_N$ differs markedly from the earlier result $-0.75 \pm 0.30 \mu_N$ obtained by Leinweber [Phys. Rev. D 53, 5115 (1996)] from the same lattice data. An unresolved problem in the LT analysis of lattice data is identified and addressed. A value of $G_{3u}(0) = -0.55 \pm 0.37 \mu_N$ is obtained at $R_d = 0.55$ by extrapolating only the nucleon isoscalar lattice data.

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Past lattice calculations made by Leinweber and collaborators of baryon magnetic moments (MMs) have been restricted to massive quarks equivalent to pion masses $m$ greater than 600 MeV. The calculated moments have to be extrapolated to the physical pion mass $m = 140$ MeV before comparison with experiment. Extrapolating linearly in $m^2$, Leinweber [3] has estimated the nucleon strangeness form factor at momentum transfer $Q^2 = 0$ to be $G_{3u}(0) = -0.75(30)$ units of $\mu_N$. (The number inside parentheses gives the statistical uncertainty in the last digits.) A more sophisticated nonlinear extrapolation of the same lattice data using a chiral extrapolation $(\chi E)$ formula proposed by Leinweber, Lu and Thomas (LLT) [4] has recently been made by Leinweber and Thomas (LT) [5]. The result is -0.16 (18).

This LT estimate contains an unresolved problem caused by the use of an extrapolated valence $u$-quark contribution $u_{\Xi_0}$ for the baryon $\Xi^0$ that disagrees with experiment to renormalize another extrapolated term $u_n$. The purpose of this Comment is to suggest that the lattice data should be renormalized before the $\chi E$, and to point out that $G_{3u}(0)$ can be extracted more readily from the nucleon isoscalar lattice data. The result is -0.55 (37), where the error comes from uncertainties in the extrapolation formula, in the least-square fitting, and in the SU(6) constants $F$ and $D$.

To extrapolate a MM term $\mu(m)$ calculated on a lattice for several large values of the pion mass $m$ to its physical value at $m_\pi$, LT use a Padé approximant given by their Eq. (13) (denoted here LT13) that is the inverse of a quadratic Taylor approximant in $m$. The extrapolated results, needed in this Comment, are read from Figs. 4 and 5 of [5] and given in Table 1. They agree well with experimental values [6] where known, as pointed out in [5] for $\mu_p = p$ and $\mu_n = n$.

For technical reasons, LT use a simple quadratic Taylor approximant in $m$, Eq. (14) in LT and called here LT14, for $u_{\Xi_0}$. (In terms of $s = m^2$, LT14 is the often-used linear approximant improved by a certain leading nonanalytic (LNA) term $\chi m$.) Table 1 shows that the extrapolated value for $u_{\Xi_0}$ differs significantly from the experimental value $\Xi^0 - \Xi^-$. A concise review of these results in greater details has been given by [5].

The discrepancy from experiment is not due to the use of LT14, as I have explicitly obtained an LT13 fit giving the same extrapolated value of $u_{\Xi_0} = -0.37$ (and -0.38 0.3 for LT14). The discrepancy comes instead from the sign of $\chi$ that controls the curvature direction caused by the LNA term. In the LT model, $\chi$ has opposite signs for $u_n$ and $u_{\Xi_0}$, because the dominant pion cloud has opposite signs, namely $n \to p\pi^- \to n$ compared to $\Xi^0 \to \Xi^- \pi^+ \to \Xi^0$. This sign makes it difficult for the $\chi E$ of $u_{\Xi_0}$ to curve towards the experimental point. This is the unresolved problem in LT mentioned at the beginning of this Comment.

To extract $G_M^\Sigma$ (dropping the $Q^2$ argument for notational simplicity), LT use a certain “ratio” method of Leinweber [6] under which the disconnected-loop (DL) contribution $\mu_{DL}^N$ of the nucleon sea is calculated from one of the following formulas:

$$\mu_{DL}^N = \frac{1}{3} (p + 2n - f_n u_n), \quad f_n = \frac{\Xi^0 - \Xi^-}{u_{\Xi_0}}; \quad (1)$$

$$\mu_{DL}^N = \frac{1}{3} (2p + n - f_p u_p), \quad f_p = \frac{\Sigma^+ - \Sigma^-}{u_{\Sigma^+}}. \quad (2)$$

That is, the extrapolated value of $u_n$ is not used directly, but only after renormalization by $f_n$. Whenever $u_{\Xi_0}$ extrapolates correctly to its experimental value $\Xi^0 - \Xi^-$, the correction factor $f_n$ is 1 and therefore has no effect. The correction factor differs from 1 only when the extrapolation is unsuccessful. Then the argument for using the nontrivial correction factor $f_n \neq 1$ is that the ratio $u_n/u_{\Xi_0}$ that appears has smaller systematic errors [5].

The large renormalization $f_n = 1.62$ found for Eq. (1) is the direct consequence of the aforementioned failure of the LT extrapolation for $u_{\Xi_0}$. In contrast, $f_p = 0.973$ needed in Eq. (2) contains little renormalization because LT13 works well there. The LT renormalized values (actually $f_i u_i$) are shown in Table 1 in the column marked LTR with errors from LT.
Is this postulated cancellation of systematic errors after LT extrapolations real? On the lattice, \( u_n \) and \( u_{\Xi^0} \) have very similar values, and their ratios are likely to contain smaller systematic errors. Unfortunately, these ratios cannot be used directly in the \( \chi E \), and the \( u \)’s have to be extrapolated separately for the lattice data used. Hence the correct procedure is to renormalize one set of lattice data to fit an available experimental MM (thus ensuring that \( f_1 = 1 \) always), and to use the same renormalization on the second set of lattice data before the \( \chi E \).

This scheme can readily be executed for \( u_{\Xi^0} \), since the original extrapolation is already close to experiment. With the lattice data for different \( m \) actually correlated, moving up and down together, one can simply take \( \{ u_1 + x \Delta u_1 \} \) as the corrected input with the unknown \( x \) chosen to be -0.23 to make the extrapolated value of \( u_{\Xi^0} \) fit \( \Sigma^+ - \Sigma^- \). The same is then used to renormalize the lattice data for \( u_p \). The LT13 extrapolated result, \( u_p = 4.16 \), agrees with that from Eq. (3).

When the same procedure is applied to \( u_{\Xi^0} \), it reaches a minimum of only -0.52, a little short of the experimental value. (The solution for the input \( g(\{ u_1 \} \) scaled by an adjustable factor \( y \) reaches down to only -0.47.) There are LT14 solutions \( u_p = 2.34, u_n = 1.54 \) that are very different from those of the Leinweber equations. Since LT14 also does very poorly for the nucleon MMs themselves, giving \( p = 3.14 \) and \( n = -2.39 \), it appears to be unreliable.

The lattice data can also be renormalized by tilting and bending. This is conveniently realized by including the experimental point in an LT13 fit and using the resulting set of ratios \( \{ u_i(\text{output})/u_i(\text{input}) \} \) to renormalize the second set of lattice data before its \( \chi E \). The resulting renormalized lattice data (RLD) extrapolate to \( u_n = -0.64(15) \), showing only a quarter of the renormalization effect from Eq. (2), while \( u_p = 4.13(24) \) agrees well with Eq. (3). These results suggest that Eq. (3) is reliable, but Eq. (2) greatly over-estimates the renormalization.

The statistical errors shown are obtained by treating the lattice data as correlated, i.e., calculated from \( \{ u_i \pm p \Delta u_i \} \), where \( p = \sqrt{1 - (N_p/N_d)} = 1/\sqrt{3} \), where \( N_d = 3 \) is the number of lattice data and \( N_p = 2 \) is the number of fitted parameters.

An alternative explanation of the discrepancy in \( u_{\Xi^0} \) that cannot be excluded at the present time is that its \( \chi \) parameter is different from that calculated in LT. For example, the \( \chi \) parameter needed to fit the experimental point together with the original lattice data is \( \chi(\{ u_{\Xi^0} \}) = 0.2 (0.3/0.5) \), showing the change of sign needed for the \( \chi E \) to curve towards the experimental point. In other words, a relatively small change in the LT value of \( \chi(\{ u_{\Xi^0} \}) = -0.4 \) is enough to change its sign. In the rest of this Comment, I shall work only with the LT model.

The value of \( G^*_M = -0.16(18) \) reported by LT is obtained from Eq. (2) using the ratio \( u_n/u_{\Xi^0} = 1.51(37) \). (The reported statistical error of 0.18 for \( G^*_M \) is a misprint. It should read 0.27.) LT has also obtained \( G^*_M = -0.57(42) \) from Eq. (3) using the ratio \( u_p/u_{\Xi^0} = 1.14(11) \), unchanged from the Leinweber linearly extrapolated value of 1.14(8). This is the better estimate because the renormalization involved is so much smaller and better established.

The second issue raised in this Comment is that the nucleon isoscalar quantity \( G^*_M \) can be extracted from nucleon isoscalar lattice data alone without using any information from \( u_{\Xi^0} \) or \( u_{\Xi^-} \). The resulting physical picture for \( G^*_M \) is sufficiently simple to permit a rather unique answer to be obtained for any extrapolation formula used such as LT13. Rough estimates of other uncertainties can then be made readily.

The quantity to be extracted is the isoscalar DL contribution that, under isospin symmetry, is simply

\[
\mu_{DL} = \frac{1}{2} (p + n) - \frac{1}{6} (u_p + u_n).
\]

The numerical answer is \( \mu_{DL} = -0.18(5) \) using the experimental value of \( p + n \) and the unrenormalized LT results shown in Table III. The answer is -0.10(7) with the original LT renormalization, and -0.14(9) using RLD to renormalize \( u_p \) and \( u_n \). The statistical errors shown are obtained from those in \( u_p \) and \( u_n \) treated as independent.

To do better in both value and error, it is necessary to extrapolate \( u_p + u_n \) together. Fortunately, there is enough information in the lattice data tabulated in LWD to extract the covariance \( \langle (\Delta u_p)(\Delta u_n) \rangle \) and to calculate the statistical error for any linear combination \( au_1 + bu_2 \) on the lattice. The combination is then extrapolated to \( m_\pi \) by using the chiral parameter \( a\chi_1 + b\chi_2 \).

The least-square fitting program and input lattice data used here are first validated by checking against published results. The top panel of Table III gives a comparison with the results of LLT for the nucleon MMs \( p \) and \( n \). The data used come from two independent sources. My errors are obtained by assuming that all lattice data are correlated (i.e., using \( N_p = 2 \) and \( N_d = 6 \)). The chiral parameters \( \chi \) are defined in terms of the SU(6) constants \( F \) and \( D \). The one-loop corrected values \( \mu_{DL} \) are used in both LLT and LT. The table shows good agreement in both extrapolated values and fitting errors.

In the second panel of the table, extrapolations are obtained for the correlated lattice data used in LT. Both values and errors agree with their results.

The third panel of Table III shows the best results from Table I obtained by extrapolating \( u_p \pm u_n \) together, using different \( \chi \) parameters for flavor and valence contributions. The quality of this single-step extrapolation is first checked for the nucleon MMs. The isovector moment \( p - n = u_p - u_n \) on the lattice because the isoscalar sea contributes nothing. Table III shows that the \( p - n \) result of 4.65 for LT13 agrees well with the value of 4.68 from separate extrapolations. Both differ somewhat from
the value \( u_p - u_n = 4.81 \) from separate extrapolations. The isoscalar moment extrapolates to \( p + n = 1.03 \), in agreement with the value of 1.10 from separate extrapolations and with experiment. In contrast, LT14 appears inadequate for both nucleon MMs.

The much more successful formula LT13 proposed by LLT has the correct \( 1/m_q \propto 1/m^2 \) behavior for heavy quarks. Its comparative success in chiral extrapolations comes from a reduction of the effect of the LNA term when placed in the denominator. Finally, the importance of having the correct heavy-quark limit is illustrated by using another inverse Taylor approximant M4D where the quadratic term in \( u \) is replaced in the denominator. It works well too for the large isovector moment \( (p-n)/2 \) that dominates the nucleon MMs. This suggests once again that the LNA term should be in the denominator.

The extrapolated value of \( u_p + u_n \) is also needed to calculate \( \mu^N_D \) from Eq. (3). It is obtained from the lattice data for \( u_p + u_n \) by extrapolating with a valence parameter \( \chi^v \) different from the flavor value \( \chi^f = 0 \) used for \( p + n \). The results obtained with different extrapolation formulas are shown in Table II. The very small statistical error found for \( \mu^N_D \), denoted in the table as \( \mu^N_D(1) \), comes from the fact that while each term in Eq. (3) varies substantially as all lattice data move up and down together, the difference between the two terms remains essentially unchanged.

To determine if \( p + n \) should be fitted in extracting \( \mu^N_D \), I renormalize the lattice input data to \( \{u_i + x\Delta u_i\} \) where \( x \) is adjusted to reproduce the experimental value. The changes for all three formulas, also given in the table, are all quite small, showing that the procedure is quite robust.

Unfortunately, there is a problem in the extrapolations for \( p + n \), and also for \( p \) and \( n \) separately. The lattice data used do not contain any DL contribution, which has in fact been assumed negligibly small in all LT extrapolations. Unpublished lattice data from Dong, Liu and Williams(DLW) [11,12] seem to indicate that the DL contribution is not necessarily negligible in the mass range \( m \approx 0.6-0.9 \) GeV of the LWD data. Unfortunately, the covariances for these DL lattice data have not been saved, thus precluding a quantitative estimate of the missing DL contribution for different \( m \)’s. This missing DL term is formally twice as strong and fractionally 4-6 times more important in \( p + n \) as it is in each MM separately. It gives an unknown systematic error to the extrapolated \( p + n \).

Given this unresolved problem in the lattice data, one might want to use the experimental value of \( p + n \) in Eq. (3). The results, shown in the table as \( \mu^N_D(2) \), can differ significantly from \( \mu^N_D(1) \). The large error shown comes from the statistical fitting error of \( u_p + u_n \).

These two estimates of \( \mu^N_D \) agree if the extrapolated \( u_p + u_n \) agrees with experiment. It is therefore interesting to see what happens when the lattice data for \( u_{zz} \) and \( u_{zz} \) are used to renormalize the lattice data for \( u_n \) and \( u_p \) before extrapolating \( u_p \pm u_n \) by LT13 and M4D. The results, given in the lower panel of Table II, show that the extrapolated \( p + n \) by LT13 is now close to the experimental value. I take \( \mu^N_D(2) \) as the better estimate because it contains no systematic error from \( p + n \) and a more conservative error estimate. This is the “best” result shown in Table II. The difference of 0.07 between LT13 and M4D shall be taken to be an additional uncertainty arising from incomplete knowledge of the correct \( \chi^E \) formula. This large uncertainty has not been included in LT.

Errors also arise from the choice of the SU(6) constants \( F \) and \( D \) on which the \( \chi \)’s depend. The \( \chi \) parameter for the isoscalar MM \( p + n \) is always \( \chi^f = 0 \) for any \( F \) and \( D \). The valence parameter \( \chi^v \) for \( u_p + u_n \) is proportional to (and has the same sign as) \( \beta^v = -(a(F + D)^2, \ a = 2/3 \left[ 5 - 6r + 9r^2 \right] / (1 + r^2) \),

\[
\beta^v = -(a(F + D)^2, \ a = 2/3 \left[ 5 - 6r + 9r^2 \right] / (1 + r^2) \),
\]

where \( r \equiv F/D \). The factor \( a \) is always positive, so that both \( \mu^N_D \) and \( G^A_M \) extracted from it are always negative in LT extrapolations. The factor \( a \) has a minimum of 2.39 at \( r = 0.74 \), and the values 2.40 and 2.45, respectively, at the one-loop corrected value of \( r = 0.66 \) and the tree-level value of \( r = 0.58 \) [10]. Hence the error from uncertainties in \( r \) is small and will be neglected.

With \( r = 0.66 \), the uncertainty in \( F + D \) can be determined by treating it as a free parameter in LT13 fitted to both experimental and lattice data for each nucleon MM \( p \) or \( n \). The expanded data set used by LLT then gives

\[
(F + D)_p = 0.96(20), \ (F + D)_n = 1.00(20), \quad \Delta(u_p + u_n) = 0.21, \quad \Delta\mu^N_D = 0.04.
\]

This yields a total error of 0.10 for \( \mu^N_D \) reported in Table II when quadratically combined with the errors of 0.06 (from least-square fitting) and 0.07 (from the \( \chi^E \) formula).

The resulting values of \( G^A_M \) are shown in the lowest panel of Table II for \( R_d^* = 0.55 \) and 0.59 as used by LT and LLT, respectively. The value of -0.55(37) at \( R^* = 0.55 \) agrees with the result -0.36(20) of DLW [11] obtained from a direct evaluation of the DL contribution. The value \( R_d^* = 0.55 \) used here is actually their ratio of directly evaluated strange to \( u/d \) DL contributions.

From the perspective of Eq. (3), the Leinweber Eq. (4) or (5) contains a spurious nucleon isovector term

\[
\Delta\mu^D_N = \frac{1}{6} [p - n - f_i(u_p - u_n)],
\]

where \( f_i = f_p \) or \( f_n \). With \( f_p \) close to 1, this term is very small in Eq. (5). This is another reason why Eq. (4) is better than Eq. (5).
Additional systematic errors not taken into consideration here include effects from finite-volume and finite-spacing errors and from the quenched approximation not already simulated by the $\chi_E$ formula used [4]. They also include uncertainties in the ratio $R_s$ and differences between the LT model and true QCD, such as the contributions of neglected kaon loops and of $m_q \ln m_q$ terms, in as far as they affect the parameter analogous to $\beta^v$ of Eq. (4) that drives the DL contribution in the $\chi_E$.

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### TABLE I. Comparison of extrapolated magnetic moment $\mu_i$ (in units of $\mu_N$) obtained by Leinweber, Lu and Thomas (LT) [4] and by Leinweber and Thomas (LT) [5] without and with LT renormalization (LRT) with experimental moments and with results obtained here using nucleon lattice data only (third panel of the table). The number inside parentheses gives the statistical uncertainty in the last digits. Estimates of $G_{\Sigma i}(R_s)$ in the lowest panel are obtained from $\mu_N^{\Sigma i}$ as defined by Eq. (3) using the experimental value of $p + n$.

| $\mu_i$ | LLT/LT | Here | LTR | Expt |
|---------|--------|------|-----|------|
| $p = \mu_p$ | 2.85(22) | 2.85(21) | 2.7928... |
| $n = \mu_n$ | -1.90(15) | -1.91(15) | -1.9130... |
| $p$ | 2.89(19) | | |
| $n$ | -1.79(20) | | |
| $u_{\Sigma^+}$ | 3.72(27) | 3.73(28) | 3.618(27) |
| $u_p$ | 4.26(25) | 4.26(25) | 4.12(40) |
| $u_n$ | -0.56(15) | -0.55(11) | -0.90(22) |
| $u_{\Xi^0}$ | -0.37(3) | -0.37(3) | -0.599(14) |
| $u_p + u_n$ | 3.53(33) | | |
| $p + n$ | 0.91(10) | | 0.8798... |

### TABLE II. Extrapolated nucleon magnetic moments obtained by using the chiral extrapolation formulas LT14, LT13 and M4D from nucleon lattice data without and with data renormalization to fit $p + n$. The lower panel gives results obtained from renormalized lattice data (RLD) for $u_p$ and $u_n$ that use information from $u_{\Xi^0}$ and $u_{\Sigma^+}$.

| | Expt | LT14 | LT13 | M4D |
|------|------|------|------|------|
| $p - n$ | 4.67(42) | 4.7058... |
| $p + n$ | | | | |
| $u_p + u_n$ | | | | |
| $\mu_N^{DL}(1)$ | 0.8798... | 0.88 | 0.88 |
| $\mu_N^{DL}(2)$ | | | | |

$$
\begin{array}{cccc}
\mu_N^{DL}(1) & \mu_N^{DL}(2) \\
LT14 & 5.50(23) & 3.84(14) & -0.2560(2) & -0.21(2) \\
LT13 & 4.65(40) & 3.89(41) & -0.136(5) & -0.21(7) \\
M4D & 4.43(26) & 3.24(16) & -0.189(5) & -0.10(3) \\
RLD: & & & & \\
LT13 & 4.67(42) & 3.53(33) & -0.134(4) & -0.15(6) \\
M4D & 4.44(26) & 3.12(15) & -0.186(4) & -0.08(3) \\
\end{array}
$$