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The approximation of generalized Log-aesthetic curves using Quintic Bezier curves

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Abstract. Generalized Log Aesthetic Curve segments (GLAC) are aesthetic curves that have monotonic curvature profile and hence they are considered fair. In the field of Computer-Aided Design (CAD), there exists a demand to construct fair curves for various design intent. However, we cannot implement GLAC in CAD system partly due to its transcendental form. A viable solution is to approximate GLACs using a quintic polynomial curve in the form of Bezier using curvature error measure. The problem of this approach is that it requires a formidable size of computations due to arc length reparametrization. In this paper, we introduce a new method of calculating curvature error measure using natural spline interpolation function to minimize computation effort while preserving the accuracy. The final section shows numerical examples depicting the proposed approximation of two types of the GLAC, which clearly indicate the efficiency of proposed method.

Keywords: Log-Aesthetic Curves, approximation, spirals, high quality curves, product design.

1. Introduction

The Generalized Log Aesthetic Curves (GLAC) with its two types of formulations namely $\rho$ -shift and $\kappa$ -shift been established via curve synthesis process. These two types of GLACs are formulated by extending the formulation of Generalized Cornu Spiral (GCS) in a similar manner to Log Aesthetic Curves (LAC) [1]. If we manipulate the radius of curvature of LAC then we may obtain the first formulation of GLAC called $\rho$ -shift GLAC radius of curvature function:

$$
\rho_{\text{GLAC}}(s) = \begin{cases} 
  e^{\alpha s} + \nu, & \alpha = 0 \\
  (\Lambda \alpha s + 1)^{\frac{1}{2}} + \nu, & \text{otherwise}
\end{cases},
$$

hence the curvature function becomes:

$$
\kappa_{\text{GLAC}}(s) = \begin{cases} 
  \frac{1}{e^{\alpha s} + \nu}, & \alpha = 0 \\
  \frac{1}{(\Lambda \alpha s + 1)^{\frac{1}{2}} + \nu}, & \text{otherwise}
\end{cases},
$$

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and the directional angle of $\rho$-shift GLAC is:

$$\theta_{GLAC}(s) = \theta_0 + \int_0^s \kappa_{GLAC}(u) du,$$

finally the parametric form of the $\rho$-shift GLAC:

$$C_{GLAC}(s) = P_0 + \left[ \int_0^s \cos[\theta_{GLAC}(u)] du, \int_0^s \sin[\theta_{GLAC}(u)] du \right].$$

where $P_0$ is the starting point, $\theta_0$ is the angle between the tangent at the beginning point anti-clockwise from the x-axis, and $\{\Lambda, \alpha, v\}$ are real variables related to the shape of the GLAC segment [1]. In similar manner, the second formulation of GLAC which is $\kappa$-shift GLAC with radius of curvature function denoted as follows:

$$\rho_{GLAC}(s) = \begin{cases} 1 & e^{\alpha v} + v, \alpha = 0 \\ 1 & (\Lambda as + 1) \frac{1}{2} + v, otherwise \end{cases}$$

hence, the curvature function becomes:

$$\kappa_{GLAC}(s) = \begin{cases} e^{\alpha v} + v, \alpha = 0 \\ (\Lambda as + 1) \frac{1}{2} + v, otherwise \end{cases}$$

the directional angle of $\kappa$-shift GLAC is then:

$$\theta_{GLAC}(s) = \theta_0 + \int_0^s \kappa_{GLAC}(u) du,$$

the formulation of the direction angle for $\kappa$-shift GLAC can be integrated analytically without the need of numerical integration to calculate unlike $\rho$-shift GLAC, thus the directional angle of $\kappa$-shift GLAC is become as follows [2]:

$$\theta_{GLAC}(s) = \theta_0 + \begin{cases} \frac{1}{\alpha} (1 - e^{-\alpha s}) + vs, \alpha = 0 \\ \frac{1}{\alpha} \log[(\Lambda s + 1)] + vs, \alpha = 1 \\ (\Lambda s + 1) \frac{\alpha}{2} - 1 + vs, otherwise \end{cases}$$

The parametric form of planar $\kappa$-shift GLAC is then given by the following form:

$$C_{GLAC}(s) = P_0 + \left[ \int_0^s \cos[\theta_{GLAC}(u)] du, \int_0^s \sin[\theta_{GLAC}(u)] du \right]$$

where $s \in [0, S]$ [2]. The representation of GLAC is in a transcendental form, which is inconvenient for CAD applications; thus the following section elaborates an approximation technique with quintic Bézier form so that we may directly implement for CAD systems.

In 2012, Cross and Cripps approximated the generalized Cornu spiral (GCS) using a parameterized arc length curvature error measure which required a formidable size of computations of quantic Bezier curves. In order to overcome this problem, Lu in 2013 suggested a modified error measure to reduce computation. However, Cross and Cripps's error measure was still needed in order to assess its approximation quality. In 2016, Albayari et al. proposed a new approach to compute the error measure.
by making a correspondence between the general parameter and arc length parameter between these two curves. Although their measure reduces computation, the accuracy of the results depends on the numbers of selected nodes.

In this paper, we propose a modified algorithm to approximate GLAC with G³ data using quintic Bezier curves. First, we approximate the arc length function of quintic Bezier curve in the form of natural splines of degree three. Using this function, we calculate the quintic Bezier curve error measure to reduce computation and at the same time meets given tolerance for GLAC approximation.

The rest of the paper is arranged as follows. Section 2 shows the proposed approximation method to approximate the GLAC. Section 3 specifies parameterized arc length curvature error measure and it is followed by the introduction of modified error measure. The last section exhibits numerous numerical examples to show the efficiency and accuracy of the proposed error measure for two types of GLACs.

2. The Approximation Method
For simplicity, consider the κ-shift GLAC which has starting point at the origin, thus the parametric representation of GLAC is:

\[ C_{\text{GLAC}}(s) = \{x(s), y(s)\} = \left\{ \int_0^s \cos[\theta(u)] \, du, \int_0^s \sin[\theta(u)] \, du \right\} \quad (10) \]

Then

\[ x'(s) = \cos[\int_0^s k(r) \, dr] \quad (11) \]
\[ x''(s) = -k(s) \sin[\int_0^s k(r) \, dr] \quad (12) \]
\[ x'''(s) = \frac{x''(s)}{k(s)} - x'(s)k^2(s) \quad (13) \]

Similarly for \( y(s) \)

\[ y'(s) = \sin[\int_0^s k(r) \, dr] \quad (14) \]
\[ y''(s) = k(s) \cos[\int_0^s k(r) \, dr] \quad (15) \]
\[ y'''(s) = \frac{y''(s)}{k(s)} - x'(s)k^2(s) \quad (16) \]

So from (13) and (16) we have:

\[ C'''(s) = \frac{C''(s)}{k(s)} - \kappa(s)k^2(s) \quad (17) \]

For simplicity consider \((x, y)\) is the GLAC end point at \( s = 1 \), \( \theta \) is the winding angle at the end point, \( \kappa_n \) is the curvature of GLAC at the start point and \( \kappa_1 \) is the curvature of GLAC at the end point. Then the derivations of \( C(s) \) at the end points become:

\[ C(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad C'(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C''(0) = \begin{pmatrix} 0 \\ 1 + \nu \end{pmatrix}, \quad C'''(0) = \begin{pmatrix} -(1 + \nu)^2 \\ -\lambda \end{pmatrix}, \quad C(1) = \begin{pmatrix} x \\ y \end{pmatrix}, \quad C'(1) = \begin{pmatrix} \cos[\theta] \\ \sin[\theta] \end{pmatrix}, \quad C''(1) = \begin{pmatrix} \cos[\theta] \\ \sin[\theta] \end{pmatrix} \]
Let the jth derivative of GLAC w.r.t. s evaluated at s=a denoted by $C^{(j)}(a) = \left( D_{a,j}^i \right)$, Cross & Cripps, has shown that the quintic Bezier curves can be represented in the following form [5]:

$$V(t) = \sum_{i=0}^{5} \left( \begin{array}{c} 5 \\ i \end{array} \right) (1-t)^{5-i} t^i V_i,$$

where,

$$V_0 = D_0^0, V_1 = \frac{\beta_1}{5} D_0^2 + D_0^0, V_2 = \frac{\beta_1^2}{20} D_0^4 + \left( \frac{\beta_1}{20} + \frac{2\beta_2}{5} \right) D_0^2 + D_0^0,$$

$$V_3 = \frac{\gamma_z^2}{20} D_0^6 + \left( \frac{\gamma_z^2}{20} - \frac{2\gamma_1}{5} \right) D_0^4 + D_0^2, V_4 = \frac{-\gamma_1}{5} D_0^4 + D_0^2, V_5 = D_0^4. \quad (19)$$

From (18) and (19) the control points for GLACs are:

$$V_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad V_1 = \frac{\beta_1}{5} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad V_2 = \left( \frac{\beta_2}{20} + \frac{2\beta_1}{5} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\beta_1^2}{20} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} x \\ y \end{pmatrix} + \left( \frac{\gamma_z^2}{20} - \frac{2\gamma_1}{5} \right) \begin{pmatrix} \cos[\theta] \\ \sin[\theta] \end{pmatrix} + \frac{\beta_1^2}{20} \begin{pmatrix} -\sin[\theta] \\ \cos[\theta] \end{pmatrix}$$

$$V_4 = \begin{pmatrix} x \\ y \end{pmatrix} - \frac{\gamma_1}{5} \begin{pmatrix} \cos[\theta] \\ \sin[\theta] \end{pmatrix}, \quad V_5 = \begin{pmatrix} x \\ y \end{pmatrix} \quad (20)$$

### 3. The Error Measure

The curvature function of GLAC is a monotonic function, which guarantees the fairness of the curve. Hence, an error measure which depends on the curves’ curvatures is suitable for approximation purpose rather than calculating the distance between approximated quintic Bezier curve against GLAC. In 2012, Cross and Cripps approximated Generalized Cornu Spiral (GCS) using curvature error measure as follows [3]:

$$e_{max} = \max_{t \in [0,1]} \frac{|\kappa_r(t) - \kappa_c(t)|}{\max \{ |\kappa_c(t)| \}} \quad (21)$$

where $\kappa_r(t)$ is the curvature function of quintic Bezier curve and $\kappa_c(t)$ is the curvature function of the GCS. The advantage of this measure is it can accurately reflects the curvature profile and produce a smooth approximation results. However, the disadvantage is, it is computationally costly due to the arc length reparameterization of the polynomial curve. In order to reduce high computational cost, Lu proposed modified curvature error measure in 2013 [4]:

$$e_{max} = \max_{t \in [0,1]} \frac{|\kappa_r(t) - \kappa_c(t)|}{\max \{ |\kappa_c(t)| \}} \quad (22)$$

to calculate the error, the curvatures are evaluated in sequence of equally spaced parameter values $\{t_i\}_{i=0}^{N/25}$.
Despite the fact that Lu’s curvature error measure consumes lesser computation power than Cross and Cripps’s curvature measure, Cross and Cripps’s error measure was still used to assess the quality of the final approximation. Hence, one may end up with calculating both $\varepsilon_{\text{max}}$ and $\tilde{\varepsilon}_{\text{max}}$ for a proper GCS approximation with quintic Bezier curves[4].

In 2016, Albayari et. al proposed a to calculate the corresponding error measure at 25 equally spaced points between parameter $t$ and parameter $s$, which guarantees the accuracy of approximation [5]:

$$\tilde{\varepsilon}_{\text{max}} = \max_{i=0,...,n} \left[ \max \left( \kappa_i(t_i) - \kappa_c(s_i) \right) \right],$$

where $\kappa_i(t_i)$ and $\kappa_c(s_i), i = 0, ..., n = 25$ are the correspondence curvature values of GCS and the approximation curve at the same arc length respectively.

Although their error measure reduces the computation and increases accuracy but the accuracy of the results depends on the numbers of selected nodes. There are two problems with this approach, firstly an ideal number of nodes to obtain a proper approximation are unknown and secondly, the maximum curvature error may occur in between any two nodes. Furthermore, the underlying problem calculating the curvature errors of Bezier quintic is the parametrization problem whereby quintic Bezier curves are in the form of a general parameter $t$ whereas GCS and GLAC are in the form of arc length parametrization.

Thus, we first obtain a sample of points from quintic Bezier curves, then approximate with natural spline of order three to represent the arc length function of Bezier curves before calculating the curvatures. Hence, the new curvature error is now as follows:

$$\hat{\varepsilon}_{\text{max}} = \max_{s \in [0,1]} \left[ \max \left( \kappa_i(s) - \kappa_c(s) \right) \right],$$

where $\kappa_i(s)$ is the curvature function of $V(s(t))$ normalized to the domain $[0,1]$, and $\kappa_c(s)$ is the curvature function of the GLAC.

4. The Approximation For The Two Types Of The GLACs

The following algorithm summarizes the approximation steps. We set the tolerance as 0.05 which ensures a high quality definition as proposed in [3]. Take note that the initial values of $\beta_1 = \gamma_1 = 1$ are set to the arc length of the approximated GLAC to represent $C^1$ continuity if the GLAC is normalized, otherwise we set $\beta_1 = \gamma_1 = 1$ to reduce number of iterations.

**Step 1.** Use the initial approximation with $\beta_1 = \gamma_1 = S$.

The approximation considered acceptable if $\hat{\varepsilon}_{\text{max}} \leq \text{tolerance}$ . Otherwise go to the next step.

**Step 2.** Find initial $(\beta_1, \gamma_1$) by solving $\beta_2 = \gamma_2 = 0$ as shown in [3] and choose the values of $(\beta_1, \gamma_1$) which lead a minimum curvature error.

**Step 3.** Search numerically for new values of $\beta_1$ and $\gamma_1$ which minimize the curvature error and re-iterate until given tolerance is satisfied.

**Step 4.** If step 3 fails, either split the curve and approximate each arc separately or increase the tolerance. Re-start approximation process.

Below are some examples on approximating two types of GLACs and the plots related to these approximations. The Figure 1-3 corresponds to numerical set of data in row 3 of Table 1.
Table 1. The errors of two types GLAC tested with new error measure.

| The type of GLAC | S       | $S_1$ | $S_2$ | $\delta_{max}$ | No. of iterations |
|------------------|---------|-------|-------|-----------------|-------------------|
| $\kappa$-shift   | 1       | (1,1) | (1,1) | 0.00014         | 0                 |
| $\kappa$-shift   | 1       | (-1,0) | (1,1) | 0.00198         | 0                 |
| $\kappa$-shift   | 4       | (1,0) | (4.0) | 0.046           | 16                |
| $\kappa$-shift   | 4       | (1.0) | (4.0) | 0.0489          | 20                |
| $\rho$-shift     | 1       | (1,1) | (1,1) | 0.00029         | 0                 |
| $\rho$-shift     | 4       | (1.0) | (4.0) | 0.031           | 0                 |
| $\rho$-shift     | 5       | (-1.0) | (5.0) | 0.0476          | 10                |
| $\rho$-shift     | 8       | (-1.0) | (6.0) | 0.0976          | 2                 |
| $\rho$-shift     | 9       | (-1.0) | (8.0) | 0.0098          | 2                 |

5. Conclusion
A new type of error measure has been proposed to approximate two types of GLACs with given tolerance using quintic Bezier curves. Numerical results indicate the proposed method can effectively approximate GLACs. Future works include a detailed analysis on reducing number of iterations based on the combinations of turning angle, shape parameters and end curvatures.

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