Research article

The sine-Gordon expansion method for higher-dimensional NLEEs and parametric analysis

Purobi Rani Kundu a, Md. Rezwan Ahamed Fahim a, Md. Ekramul Islam a, M. Ali Akbar b, c*

a Department of Mathematics, Pabna University of Science and Technology, Bangladesh
b Department of Applied Mathematics, University of Rajshahi, Bangladesh

ARTICLE INFO

Keywords:
The sine-Gordon expansion method
Estevez-Mansfield-Clarkson equation
Riemann wave equation
Soliton solutions

ABSTRACT

The Estevez-Mansfield-Clarkson (EMC) equation and the (2+1)-dimensional Riemann wave (RW) equation are important mathematical models in nonlinear science, engineering and mathematical physics which have remarkable applications in the field of plasma physics, fluid dynamics, optics, image processing etc. Generally, through the sine-Gordon expansion (SGE) method only the lower-dimensional nonlinear evolution equations (NLEEs) are examined. However, the method has not yet been extended of finding solutions to the higher-dimensional NLEEs. In this article, the SGE method has been developed to rummage the higher-dimensional NLEEs and established steady soliton solutions to the earlier stated NLEEs by putting in use the extended higher-dimensional sine-Gordon expansion method. Scores of soliton solutions are figure out which confirms the compatibility of the extended SGE method. The solutions are analyzed for both lower and higher-dimensional nonlinear evolution equations through sketching graphs for alternative values of the associated parameters. From the figures it is notable to perceive that the characteristic of the solutions depend upon the choice of the parameters. This study might play an impactful role in analyzing higher-dimensional NLEEs through the extended SGE approach.

1. Introduction

In the scientific field many phenomena are described through nonlinear evolution equations (NLEEs) which are a special sort of partial differential equations. It is not easy to examine a nonlinear model of real-life problems theoretically or numerically. In recent years much attention is paid by the researchers to establish better and efficient methods for determining solutions approximate or exact, analytical or numerical to nonlinear models [1, 2, 3]. For large scale studies in different sectors of physical sciences and engineering, NLEEs have become quite popular in the recent times. Since the success rate of NLEEs is high in illustrating versatile problems in different sectors, searching solitary wave solutions has gained popularity to the researchers. Thus, a number of methods have been developed by various researchers to carry out exact and explicit stable soliton solutions of nonlinear physical models, such as, the tanh-function expansion and its various modifications [4], the exp-function method [5], the ansatz method [6], the sine-cosine method [7], the F-expansion method [8], the complex hyperbolic-function method [9], the variational iteration method [10], the \((G'/G)\)-expansion method [11], the Jacobi elliptic function method [12], the improved Bernoulli sub-equation function method [13], the homotopy analysis method [14], the Adomian decomposition method [15], the modified extended tanh method [16], the \((G'/G)\)-expansion method [17], the finite element method [18], the first integral method [19], the alternative \((G'/G)\) expansion method [20], the modified simple equation method [21], the modified two-component Dullin-Gottwald-Holm (mDGH2) system [22], the Riemann-Hilbert method [23, 24, 25], the Lie symmetry method [26], the long wave limit method [27], the truncated Painlevé expansion method [28], the sine-Gordon expansion (SGE) method [29, 30, 31, 32, 33, 34] and several type of soliton [35, 36] process. The reputed sine-Gordon equation method was developed based on the wave transformation and it functions only for lower-dimensional NLEEs. There are many higher-dimensional NLEEs concerning real life problems and to interpret them explicitly further soliton solutions are needed. However, new solutions of higher-dimensional NLEEs have not yet been investigated by extending the sine-Gordon expansion method. Therefore, the objective of this article is to extend the SGE method for higher-dimensional NLEEs and make use of this method to establish
broad-ranging stable soliton solutions to the Estevez-Mansfield-Clarkson equation [37] and the Riemann wave equation [38].

In section 2, we describe the SGE method to analyze the (1+1)-dimensional equation and the (2+1)-dimensional equation. In section 3, we put in use the SGE method to the earlier stated equations. The results and discussion and physical explanations are provided in section 4 and finally the conclusion is given in section 5.

2. Methodology

In this section, we give a brief description of the sine-Gordon expansion method. In the first part, we discuss about SGE method for the (1+1)-dimensional NLEEs and in the second part, we narrate the method for the (2+1)-dimensional NLEEs.

2.1. The sine-Gordon expansion method for (1+1)-dimensional equation

First, we consider the general form of the sine-Gordon equation of two variables $x$ and $t$ as follows:

\[
\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = \alpha^2 \sin(V).
\]  

(2.1.1)

Here $V = V(x,t)$ is any arbitrary function and $\alpha \neq 0$ is a real constant.

Now, let us consider the traveling wave variable

\[
V(x,t) = V(\xi), \quad \xi = \lambda(x - mt),
\]

(2.1.2)

where $\lambda$ is the wave number and $m$ is the velocity of the travelling wave.

With the knowledge of (2.1.2), we deduce an ordinary differential equation (ODE) as following, from Eq. (2.1.1):

\[
\frac{d^2 V}{d\xi^2} = \frac{\alpha^2}{\lambda^2(1 - m^2)} \sin(V).
\]

(2.1.3)

It is possible to transform Eq. (2.1.3) as

\[
\left( \frac{d}{d\xi} \sqrt{\frac{V}{\xi}} \right)^2 = \frac{\alpha^2}{\lambda^2(1 - m^2)} \sin\left( \frac{V}{\xi} \right) + \alpha_1,
\]

(2.1.4)

where $\alpha_1$ is the integral constant.

If we consider $\alpha_1 = 0$, $f(\xi) = \left( \frac{V}{\xi} \right)$ and $\rho^2 = \frac{\alpha^2}{\lambda^2(1 - m^2)}$ and putting in use these values into Eq. (2.1.4), we achieve

\[
\frac{df}{d\xi} = \rho \sin(f).
\]

(2.1.5)

If we set $\rho = 1$ in Eq. (2.1.5), we gain

\[
\frac{df}{d\xi} = \sin(f).
\]

(2.1.6)

Now, taking the assistance of the variable separation principle, we acquire the following relations

\[
\sin(f) = \sin(f(\xi)) = \frac{2\rho^2 \exp(\xi)}{1 + \rho^2 \exp(2\xi)} = \text{sech}(\xi), \text{ for } \beta = 1,
\]

(2.1.7)

\[
\cos(f) = \cos(f(\xi)) = \frac{-1 + \rho^2 \exp(2\xi)}{1 + \rho^2 \exp(2\xi)} = \tanh(\xi), \text{ for } \beta = 1,
\]

(2.1.8)

wherein $\beta$ is the constant of integration.

Let us consider a NLEE with two variables $x$ and $t$ as follows

\[
\varphi\left(V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial t}, \frac{\partial^2 V}{\partial x^2}, \frac{\partial^2 V}{\partial t^2}, \ldots \right) = 0,
\]

(2.1.9)

wherein $V = V(x,t)$ is an unidentified function, $\varphi$ is a polynomial of the variable $V$ and its derivatives. Here $x$ is the spatial variable, $t$ is the temporal variable and its partial derivatives with respect to $t, x$ respectively are $\frac{\partial V}{\partial t} \frac{\partial V}{\partial x}$ etc.

Following with the sine-Gordon expansion method, we might take for the solution of Eq. (2.1.9) as

\[
V(\xi) = A_0 + \sum_{n=0}^{N} \tanh^{-1}(\xi) \left[ B_n \text{sech}(\xi) + A_n \tanh(\xi) \right].
\]

(2.1.10)

Using identities prescribed in (2.1.7) and (2.1.8) into solution (2.1.10), we establish

\[
V(f(\xi)) = A_0 + \sum_{n=0}^{N} \cos^{n-1}(\xi) \left[ B_n \sin(f(\xi)) + A_n \cos(f(\xi)) \right].
\]

(2.1.11)

In order to determine the value of $N$ we use balancing principle by considering the highest power nonlinear term and the higher derivative in the obtained NODE. Equalizing each coefficient of $[\sin^i(f(\xi)) \cos^j(f(\xi))]$ to zero yields a system of algebraic equations. Resolving this system of algebraic equations provides the values of $A_n, B_n, \lambda$ and $m$. Finally, plugging the values of $A_n, B_n, \lambda$ and $m$ into (2.1.10), we accomplish the solution to the NLEE Eq. (2.1.9).

2.2. The sine-Gordon expansion method for (2+1)-dimensional equation

In this section, we consider the (2+1)-dimensional sine-Gordon equation of three variables $x, y, t$ as follows:

\[
\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} = \sin(V).
\]

(2.2.1)

Here $V$ is an unidentified function $V = V(x,y,t)$. Now, we consider a wave variable in the form

\[
V(x, y, t) = V(\xi), \quad \xi = \mu x + ay - \sigma t,
\]

(2.2.2)

wherein $\mu$ and $\alpha$ are wave numbers and $c$ is the velocity of the travelling wave.

Using (2.2.2), the (2+1)-dimensional sine-Gordon Eq. (2.2.1) can be transformed to get an ordinary differential equation (ODE) of the following form

\[
\frac{d^2 V}{d\xi^2} = \frac{1}{(\mu - \alpha)(\mu + \sigma)} \sin(V).
\]

(2.2.3)

We can modify Eq. (2.2.3) as follows

\[
\left( \frac{d}{d\xi} \sqrt{\frac{V}{\xi}} \right)^2 = \frac{1}{(\mu - \alpha)(\mu + \sigma)} \sin\left( \frac{V}{\xi} \right) + m,
\]

(2.2.4)

wherein $m$ is the constant of integration.

Let us set $m = 0$, $f(\xi) = \left( \frac{V}{\xi} \right)$ and $\rho^2 = \frac{1}{(\alpha + \mu)(\mu + \sigma)}$ into Eq. (2.2.4), thus we found

\[
\frac{df}{d\xi} = \rho \sin(f).
\]

(2.2.5)

If we assign $\rho = 1$, Eq. (2.2.4) becomes

\[
\frac{df}{d\xi} = \sin(f).
\]

(2.2.6)
Now, taking the assistance of the variable separation principle, we ascertain the following relations
\[
\sin(f) = \sin(f(\xi)) = \frac{2\beta \exp(\xi)}{1 + \beta^2 \exp(2\xi)} = \text{sech}(\xi), \text{ for } \beta = 1, \tag{2.2.7}
\]
\[
\cos(f) = \cos(f(\xi)) = \frac{-1 + \beta^2 \exp(2\xi)}{1 + \beta^2 \exp(2\xi)} = \tanh(\xi), \text{ for } \beta = 1, \tag{2.2.8}
\]
wherein $\beta$ is the constant of integration.

At the stage, we consider a $(2+1)$-dimensional NLEE with three variables $x$, $y$ and $t$ as follows:
\[
\phi\left(V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial t}\right) = 0, \tag{2.2.9}
\]
wherein $V = V(x,y,t)$ is an unidentified function, $\phi$ is a polynomial of the variable $V$. Here $x$ and $y$ are the spatial variables, $t$ is the time variable and its partial derivatives with respect to $t$, $x$ and $y$ respectively are $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ etc.

Following with the $(1+1)$-dimensional sine-Gordon expansion method, we might take the solution of Eq. (2.2.9) as
\[
V(\xi) = A_0 + \sum_{n=1}^{N} \tan^{-1}(\xi)B_n \text{sech}(\xi) + A_n \tanh[\xi]. \tag{2.2.10}
\]

Using the identities (2.2.7) and (2.2.8), from solution (2.2.10) we derive
\[
V(f(\xi)) = A_0 + \sum_{n=1}^{N} \cos^{-1}(f(\xi))B_n \sin(f(\xi)) + A_n \cos(f(\xi)). \tag{2.2.11}
\]

To determine the value of $N$, we use balancing principle by considering the highest power nonlinear term and the higher derivative in the obtained NODE. Equalizing the coefficients of $[\sin(f(\xi))]\cos(f(\xi))]$ to be zero, yields a system of algebraic equations. Resolving this system of algebraic equations provides the values of $A_0$, $B_n$, $\alpha$, $\beta$ and $\epsilon$. Finally, using the values of $A_0$, $B_n$, $\alpha$, $\beta$ and $\epsilon$ in Eq. (2.2.10), we can accomplish the required solution to the $(2+1)$-dimensional NLEE (2.2.9).

3. Exact solution using the sine-Gordon expansion method

In this module, we have extracted the general and broad-ranging closed form stable travelling wave solutions to the Estevez-Mansfield-Clarkson (EMC) equation and the Riemann wave (RW) equation.

3.1. The Estevez-Mansfield-Clarkson equation

We consider the Estevez-Mansfield-Clarkson equation of the following form [28]:
\[
\frac{\partial^2 U}{\partial x^2} + \beta \left( \frac{\partial^2 U}{\partial x \partial t} + \frac{\partial^3 U}{\partial t^2} \right) + \frac{\partial^3 U}{\partial x^3} = 0, \tag{3.1.1}
\]
where $\beta$ is a constant. For the EMC equation, we consider the following wave transformation
\[
U(x,t) = U(\xi), \xi = \sqrt{k}(x - ct), \tag{3.1.2}
\]
wherein $c$ is the wave velocity.

By means of the wave transformation (3.1.2), Eq. (3.1.1) converts into the under mentioned NODE:
\[
c \frac{\partial^2 u}{\partial \xi^2} + \beta c \sqrt{k} U^\prime - U'' = 0. \tag{3.1.3}
\]

Integrating (3.1.3) once and neglecting the integral constant, we ascertain
\[
c \frac{\partial^2 u}{\partial \xi^2} + \beta c \sqrt{k} U'' = 0. \tag{3.1.4}
\]

According to the principle of balancing, from Eq. (3.1.4) we obtain $n = 3$. Therefore, the structure of the solution of Eq. (3.1.4) is
\[
U(w) = A_0 + B_1 \sin(w) + A_1 \cos(w) + B_2 \sin(w)\cos(w) + A_2 \cos^2(w) \tag{3.1.5}
\]

It is evident to figure out diverse derivatives of the solution (3.1.5) which are suggested in the underneath
\[
U(w) = B_1 \sin(w) + B_2 \cos^2(w) - B_3 \sin^2(w) - 2A_1 \cos^3(w) - B_1 \cos^3(w)\sin(w) - 2B_3 \sin^2(w) - 3A_1 \cos^2(w) - 2B_1 \sin(w), \tag{3.1.6}
\]
and
\[
U''(w) = -5B_1 \sin(w)\cos(w) + B_1 \cos^3(w)\sin(w) - 4A_1 \sin^2(w)\cos^2(w) + 2A_1 \sin^4(w) + 2B_2 \sin^2(w) - 18B_3 \sin^2(w)\cos^2(w) + B_3 \sin(w)\cos^4(w) + 16A_2 \sin^4(w)\cos^2(w) - 8A_2 \sin^2(w)\cos^3(w) - 28B_3 \sin^2(w)\cos(w) - 31B_3 \sin^4(w)\cos(w) + B_3 \sin(w)\cos(w) - 6A_1 \sin^4(w) + 42A_1 \sin^2(w)\cos^2(w) - 12A_3 \sin^2(w)\cos^4(w). \tag{3.1.7}
\]

By means of the derivatives assembled in (3.1.6) and (3.1.7), from (3.1.4), we find the following result:
Now, if we equate the coefficients of like power of $\sin(w)$ and $\cos(w)$ to zero, from Eq. (3.1.8) we will find an algebraic system of equations and solving these equations with the help of Maple, we get the following solution sets:

**Family 1** : 
\[
\mu \frac{\partial^2 A_1^2}{\partial t^2} = \frac{\mu^2 A_1^4}{36}, \quad A_1 = A_1, B_1 = 0, A_2 = 0, B_2 = 0, \quad c = 0.
\]

\[
(3.1.9)
\]

**Family 2** : 
\[
\frac{\partial^2 B_1}{\partial t^2} = -\frac{\partial^2 B_1}{\partial t^2}, \quad A_1 = iB_1, B_1 = B_1, A_2 = A_2 = 0, B_2 = B_2 = 0.
\]

\[
(3.1.10)
\]

**Case I.** Substituting the values of parameters arranged in (3.1.9) along with (3.1.2) into solution (3.1.5), we accomplish

\[
\begin{align*}
\mu \alpha A_1^2 + \mu \alpha B_1^2 - 2\alpha \beta + 4\alpha^2 \beta A_1 + \mu \alpha B_1^2 + \alpha \beta + 2\alpha \beta (f) (-2 \mu \alpha \beta B_1 - 2 \beta B_1 + 2 \mu \alpha \beta B_1 + 2 \alpha \beta) \cos(f) + \cos(2f) \sin(f) \\
\cos(
\begin{align*}
&+ \left( -2 \alpha \beta - 4 \alpha^2 \beta A_1 + 4 \alpha^2 \beta A_1 + 4 \beta \alpha \beta B_1 + 2 \alpha \beta (f) \cos(f) + \cos(2f) \sin(f) \right) (2 \mu \alpha \beta B_1 + 2 \alpha \beta) \\
&- 2 \mu \alpha \beta B_1 - 2 \beta B_1 + 2 \mu \alpha \beta B_1 + 2 \alpha \beta) \cos(f) + \cos(2f) \sin(f) \\
&+ \mu \alpha A_1^2 + \mu \alpha B_1^2 - \mu \alpha B_1^2 \cos(4f) + (4 \alpha^2 \beta a_1 - 2 \mu \alpha \beta B_1 + 2 \alpha \beta + 2 \mu \alpha \beta B_1 - 2 \beta B_1 - 2 \mu \alpha \beta B_1 + 2 \alpha \beta) \cos(f) + \cos(2f) \sin(f)
\end{align*}
\right)
\end{align*}
\]

\[
(3.2.8)
\]

\[
U_1(x,t) = A_0 + A_1 \tanh \left( \frac{\beta A_1}{36} \left( x - \frac{9t}{\beta A_1^2} \right) \right).
\]

\[
(3.1.11)
\]

**Case II.** Inserting the values of coefficients organized in (3.1.10) and using (3.1.2) into solution (3.1.5), we attain

\[
U_2(x,t) = A_0 + B_1 \sec \left( \frac{\beta B_1}{3} \left( x + \frac{9t}{\beta B_1^2} \right) \right) + B_1 \tan \left( \frac{\beta B_1}{3} \left( x + \frac{9t}{\beta B_1^2} \right) \right).
\]

\[
(3.1.12)
\]

The results derived here are important findings for the research of the physical applications of discrete nonlinear dynamics, study for shallow water wave etc.

### 3.2. The Riemann wave equation

We consider Riemann wave (RW) equation of following form \[29]\ :

\[
\frac{\partial V}{\partial t} + m \frac{\partial V}{\partial x} + n \frac{\partial W}{\partial x} + p \frac{\partial V}{\partial x} = 0,
\]

\[
(3.2.1)
\]

\[
\frac{\partial V}{\partial x} = \frac{\partial W}{\partial x}.
\]

\[
(3.2.2)
\]

where $n, m, l$ are constants. To investigate the Riemann wave equation, we relate the following wave transformation

\[
V(x,t) = V(\xi), \quad \xi = mx + ny - ct
\]

\[
(3.2.3)
\]

wherein $c$ is the wave velocity.

Substituting (3.2.3) into (3.2.1) and using (3.2.2), we reach to the subsequent NODE as follows:

\[
-cV' + \mu q V'' + (\mu + 2\nu) VV' = 0.
\]

\[
(3.2.4)
\]

Integrating once and neglecting the integral constants we get

\[
-2cV + 2\mu q V' + (\mu + \nu + 2\nu) V = 0.
\]

\[
(3.2.5)
\]

According to the principle of balancing on Eq. (3.2.5), we obtain $n = 2$.

Using solution (2.2.11) together with $n = 2$, we bring out

\[
V'(f) = A_0 + B_1 \sin(f) + A_1 \cos(f) + B_2 \sin(f) \cos(f) + A_3 \cos^2(f).
\]

\[
(3.2.6)
\]

It is simple to extract various derivatives of the solution (3.2.6) which are demonstrated below

\[
V''(f) = -B_1 \sin^2(f) + B_1 \cos^2(f) - 2A_3 \sin(f) \cos(f) + 2A_2 \sin^2(f) - 4A_3 \cos^2(f) \sin^2(f).
\]

\[
(3.2.7)
\]

Substituting solution (3.2.6) and (3.2.7) in Eq. (3.2.5), we accomplish

\[
\begin{align*}
\mu \alpha A_1^2 + \mu \alpha B_1^2 - 2\alpha \beta + 4\alpha^2 \beta A_1 + \mu \alpha B_1^2 + \alpha \beta + 2\alpha \beta (f) (-2 \mu \alpha \beta B_1 - 2 \beta B_1 + 2 \mu \alpha \beta B_1 + 2 \alpha \beta) \cos(f) + \cos(2f) \sin(f) \\
\cos(
\begin{align*}
&+ \left( -2 \alpha \beta - 4 \alpha^2 \beta A_1 + 4 \alpha^2 \beta A_1 + 4 \beta \alpha \beta B_1 + 2 \alpha \beta (f) \cos(f) + \cos(2f) \sin(f) \right) (2 \mu \alpha \beta B_1 + 2 \alpha \beta) \\
&- 2 \mu \alpha \beta B_1 - 2 \beta B_1 + 2 \mu \alpha \beta B_1 + 2 \alpha \beta) \cos(f) + \cos(2f) \sin(f) \\
&+ \mu \alpha A_1^2 + \mu \alpha B_1^2 - \mu \alpha B_1^2 \cos(4f) + (4 \alpha^2 \beta a_1 - 2 \mu \alpha \beta B_1 + 2 \alpha \beta + 2 \mu \alpha \beta B_1 - 2 \beta B_1 - 2 \mu \alpha \beta B_1 + 2 \alpha \beta) \cos(f) + \cos(2f) \sin(f)
\end{align*}
\right)
\end{align*}
\]

\[
(3.2.8)
\]

We obtain the algebraic system of equations from Eq. (3.2.8) by equating the coefficients of like power of $\sin(f)$ and $\cos(f)$ to zero and solving these equations, we attain the following values of the unknown parameters with the help of Maple:

**Family 1** : 
\[
\alpha = \frac{\mu \alpha A_0}{4 \mu^2 \beta - n \beta}, \quad A_0 = A_1 = 0, A_2 = -3A_0, B_1 = 0, B_2 = 0.
\]

\[
(3.2.9)
\]

**Family 2** : 
\[
\alpha = \frac{\mu \alpha A_0}{4 \mu^2 \beta - n \beta}, \quad A_0 = A_1 = 0, A_2 = -A_0, B_1 = 0, B_2 = 0.
\]

\[
(3.2.10)
\]

**Family 3** : 
\[
\alpha = \frac{-\mu \alpha A_0}{4 \mu^2 \beta - n \beta}, \quad A_0 = A_1 = 0, A_2 = -3A_0, B_1 = 0, B_2 = 0.
\]

\[
(3.2.11)
\]

**Family 4** : 
\[
\alpha = \frac{-\mu \alpha A_0}{6 \mu^2 \beta - n \beta}, \quad A_0 = A_1 = 0, A_2 = -A_0, B_1 = 0, B_2 = 0.
\]

\[
(3.2.12)
\]

**Case I.** Substituting the values of parameter arranged in (3.2.9) along with (3.2.3) into solution (3.2.6), we accomplish
with Eq. (3.2.3) into solution (3.2.6), we derive

\[ V_1(x, y, t) = A_0 - 3A_0 \tanh^2 \left( -\mu x + \frac{\mu A_0 y}{4\mu^2 I - nA_0} + \frac{4\mu^3 mA_0 t}{4\mu^2 I - nA_0} \right) \]  
(3.2.13)

**Case II.** Putting the values of coefficients organized in (3.2.10) along with Eq. (3.2.3) into solution (3.2.6), we attain

\[ V_2(x, y, t) = A_0 - A_0 \tanh^2 \left( -\mu x + \frac{\mu A_0 y}{12\mu^2 I - nA_0} + \frac{4\mu^3 mA_0 t}{12\mu^2 I - nA_0} \right) \]  
(3.2.14)

**Case III.** Embedding the values of parameters prescribed in (3.2.11) along with Eq. (3.2.3) into solution (3.2.6), we achieve

\[ V_3(x, y, t) = A_0 + \frac{3}{2}iA_0 \sech \left( \mu x + \frac{\mu A_0 y}{4\mu^2 I - nA_0} + \frac{\mu^3 mA_0 t}{4\mu^2 I - nA_0} \right) \tanh \left( \mu x + \frac{\mu A_0 y}{4\mu^2 I - nA_0} + \frac{\mu^3 mA_0 t}{4\mu^2 I - nA_0} \right) - \frac{3}{2}A_0 \tanh^3 \left( \mu x + \frac{\mu A_0 y}{4\mu^2 I - nA_0} + \frac{\mu^3 mA_0 t}{4\mu^2 I - nA_0} \right). \]  
(3.2.15)

**Case IV.** Again for the values organized in (3.2.12) and Eq. (3.2.3) into solution (3.2.6), we derive

\[ V_4(x, y, t) = A_0 - A_0 \sech \left( -\mu x + \frac{\mu A_0 y}{6\mu^2 I - nA_0} + \frac{\mu^3 mA_0 t}{6\mu^2 I - nA_0} \right) \tanh \left( -\mu x + \frac{\mu A_0 y}{6\mu^2 I - nA_0} + \frac{\mu^3 mA_0 t}{6\mu^2 I - nA_0} \right) - A_0 \tanh^3 \left( -\mu x + \frac{\mu A_0 y}{6\mu^2 I - nA_0} + \frac{\mu^3 mA_0 t}{6\mu^2 I - nA_0} \right). \]  
(3.2.16)

It is important to note that the wave solutions of the Riemann wave equation found here are effective, resourceful and were not established in the earlier research. The above solutions might be fruitful for research in using optics instead of electronics in digital image processing, to investigate the unidirectional wave propagation in nonlinear media with dispersion relativistic one-particle theory etc.
4. Results and discussion and graphical representations

In this section, we analyze the solutions using the graphical representations. In first part, we discuss the solutions of Riemann wave equation and in final part we examine the solutions of the Estevez-Mansfield-Clarkson equation. We have sketched graph for each of the solution and their nature has been described.

4.1. Figures of the solutions to the Estevez-Mansfield-Clarkson equation

In this section, the obtained stable soliton solutions to the Estevez-Mansfield-Clarkson equation are shown in graphs for different values of the parameters using Matlab software. It is noteworthy to turn up that the characteristic of the wave profile depend on the values of the associated parameters. To clarify the fact, we have sketched different graphs of the solution (3.1.11) for different values of the free parameters $A_0, A_1, \beta$. The solution function $U_1$ depicts simple soliton wave shown in Figure 1(a) for the values of the parameter $A_0 = 1, A_1 = 1, \beta = 1$. If we increase the values of $A_1$ and $\beta$ by 1 as well as increases the speed velocity $c$ and wave parameter $k$, specifically for $A_0 = 1, A_1 = 2, \beta = 2$, the solution $U_1$ turns into a smooth kink type soliton displayed in Figure 1(b), whereas for the values $A_0 = 1, A_1 = 4$, $\beta = 5$, the same solution $U_1$ represents an ideal kink-shape soliton demonstrated in Figure 1(c). Kink waves are traveling waves which incline or upsurge from one asymptotic position to another and meet to constant at infinity. The contour figures are depicted within the interval $-5 \leq x \leq 5$ and $0 \leq t \leq 3$.

Besides, solution (3.1.12) gives different types of figures when the values of the parameters change along with directions. For the values $A_0 = 1, B_1 = 2, \beta = 3$, the solution function $U_2$ is spike type singular periodic wave asserted in Figure 2(a) and if we shrunk the direction (x) and increasing the values of parameters as $A_0 = 2, B_1 = 3, \beta = 5$, then $U_2$...
clearly display the singular point presented in Figure 2(b). On the other hand, when we increasing the value of parameters by $A_0 = 5, \beta_1 = 6, \beta = 8$, the same solution represents a completely different type wave which is periodic shown in Figure 2(c). For the values $A_0 = 6, \beta_1 = 8, \beta = 5$ also depicts irregular periodic type wave shown in Figure 2(d).

Similarly, the solution $U_2$ exhibited in Figure 2(e) for the values of the parameters $A_0 = 7, \beta_1 = 11, \beta = 14$ within the interval $-2 \leq x \leq 2$ and $0 \leq t \leq 1$.

It is remarkable that in this study, we establish different type of soliton, such as, kink, singular kink, and periodic wave to the Estevez-Mansfield-Clarkson equation for different values of the parameters from a broad-ranging solution in a unique way.

4.2. Figures of the solutions of the Riemann wave equation

In this section, the obtained stable wave solutions to the Riemann wave equation are illustrated through figures and speculated the nature of these waves for dissimilar values of the parameters using the MATLAB software.

The solution function (3.2.13) represents bell-shape soliton wave portrayed in Figure 3(a) for the values of the free parameters $A_0 = 1, \mu = 1, I = 1, m = 1, n = 1$. Furthermore, after increasing the value of parameters by $A_0 = 2, \mu = 2, I = 3, m = 3, n = 5$, the same solution $V_1$ displays the singular bell-shape soliton depicted in Figure 3(b). With the change of value of only one constant, videlicet $A_0$ in the general solution by a negative value as $A_0 = -2$, then Figure 3(b) converse to singular anti-bell shape soliton shown in Figure 3(c). The contour figure is depicted within the interval $-10 \leq x \leq 10$ and $0 \leq t \leq 1$.

Solutions (3.2.14) also represent are alike figures and thus have not been sketched to avoiding repetition.

For the solution (3.2.15), we attain different values for the real part of the solution and also for the imaginary part. For the values of the parameters $A_0 = 1, \mu = 1, I = 1, m = 1, n = 1$, the real part of solution $V_3$ represents bell-shape soliton and the imaginary part displays kink shape wave shown in Figure 4(a). After changing the value of constant of principle solution by $A_0 = -1$, the solution $V_3$ shows anti-bell shape soliton for real part but unchanged for imaginary part illustrated in Figure 4(b). We achieved the singular bell-shape soliton for real part and singular kink soliton for imaginary part of solution $V_3$ by increasing the values of free parameters by $A_0 = 2, \mu = 3, I = 4, m = 5, n = 6$ and shown in Figure 4(c). Only change the value of parameter as $A_0 = -2$, the real part of the solution $V_3$ gives singular anti-bell shape wave but the imaginary part is likely singular kink type wave (see Figure 4(d)). The contour figures are depicted within the interval $-10 \leq x \leq 10$ and $0 \leq t \leq 2$.

Solutions (3.2.16) also spectacles are analogous figure and thus to avoiding repetition we have not been noted here.

The results and graph, it is clear that, both the equations we have analyzed in this article give hyperbolic and trigonometric solutions and the structure of the solutions are: bell-soliton soliton, singular bell-shape soliton, anti-bell shape soliton, kink soliton wherein the figures rich in verities.

5. Conclusions

In this article, the SGE method has successfully been extended to establish the stable solitary wave solutions to the higher-dimensional Riemann wave equation and the basic SGE method has been put through to the Estevez-Mansfield-Clarkson equation. It is affirmed that, now the sine-Gordon expansion method can be used to find solutions for both lower and higher-dimensional nonlinear evolution equations. This analysis shows that the devised algorithm might be an effective tool to mathematicians and theoretical physicists for physical applications of discrete nonlinear dynamics with special emphasis on the systems that can be integrated by analytic methods or at least admit special explicit solutions. The research in this volume will also be of interest to engineers working in discrete dynamics as well as to theoretical biologists.

Declarations

Author contribution statement

Purobi Rani Kundu: Conceived and designed the experiments; Wrote the paper.

Md. Rezwan Ahamed Fahim: Performed the experiments; Analyzed and interpreted the data; Wrote the paper.

Md. Ekramul Islam: Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

M. Ali Akbar: Analyzed and interpreted the data.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data availability statement

No data was used for the research described in the article.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

Acknowledgements

The authors express their sincere thanks to the anonymous referee(s) and the editor for their valuable suggestions and comments that help to improve the article.

References

[1] X.Y. Wang, Exact and explicit solitary wave solution for the generalized Fisher equation, J. Phys., Lett. 131 (4-5) (1988) 277–279.
[2] A. Jeffrey, M.N.B. Mohamad, Exact solutions to the k Dodd–Bessel equation, Wave Motion 14 (1991) 369–375.
[3] M. Wadati, The exact solution of the modified Korteweg-de Vries equation, J. Phys. Soc. Jpn. 32 (1972) 1681.
[4] S.A. Elwakil, S.K. El-banky, M.A. Zahran, R. Sabry, Modified extended tanh-function method for solving nonlinear partial differential equations, Phys. Lett. 299 (2-3) (2002) 179–186.
[5] M.A. Akbar, N.H.M. Ali, Exp-function method for Duffing equation and new solutions of (2+1)-dimensional dispersive long wave equations, Prog. Appl. Math. 1 (2) (201) 30–42.
[6] H. Triki, A. Yıldırım, T.O.M. Aldonsary, A. Biswas, Shock wave solution of the Benney-Luke equation, Rom. J. Phys. 57 (2012) 1029–1034.
[7] A.M. Wazwaz, Sine-cosine method for handling nonlinear wave equations, Math. Comput. Model. 40 (2004) 499–508.
[8] M.A. Abdou, The extended F-expansion method and its application for a class of nonlinear evolution equations, Chaos, Solit. Fractals 31 (2007) 95–104.
[9] C. Bai, H. Zhou, Complex hyperbolic-function method and its applications to nonlinear equations, Phys. Lett. 355 (2006) 22–30.
[10] J.H. He, Variational iteration method for delay differential equations, Commun. Nonlinear Sci. Numer. Simulat. 2 (1997) 230–235.
[11] M. Wang, X. Li, J. Zhang, The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. 372 (2008) 417–423.
[12] E.N. Aksan, H. Bulut, M. Kayhan, Some wave simulation properties of the (2+1)-dimensional breaking solution equation, ITM (conferences) 13 (2017), 01014.
[13] T. Islam, M.A. Akbar, A.K. Azad, Traveling wave solutions to some nonlinear fractional partial differential equations through the rational (G'/G)-expansion method, J. Ocean Engg. Sci. 3 (2018) 76–81.
[14] T. Hayat, M. Sajid, Homotopy analysis of MHD boundary layer flow of an upper-convected Maxwell fluid, Int. J. Eng. Sci. 45 (2007) 393–401.
[15] M. Inc, The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method, J. Math. Anal. Appl. 345 (2008) 476–484.

[16] M.M.A. Khater, Exact traveling wave solutions for an important mathematical physics model, J. Appl. Math. Bioinf. 6 (2016) 37–48.

[17] A. Iftikhar, A. Ghafoor, T. Zubair, S. Firdous, S.T. Mohyud-Din, (G/G,1/G)-expansion method for traveling wave solutions of (2+1)-dimensional generalized KdV, Sine-Gordon and Landau-Ginzburg-Higgs equations, Sci. Res. Essays 8 (2013) 1349–1359.

[18] W. Dan, Finite element method for the space and time fractional Fokker-Planck equation, SIAM J. Numer. Anal. 47 (1) (2008) 204–226.

[19] A. Iftikhar, A. Ghafoor, T. Zubair, S. Firdous, S.T. Mohyud-Din, (G/G,1/G)-expansion method for traveling wave solutions of (2+1)-dimensional generalized KdV, Sine-Gordon and Landau-Ginzburg-Higgs equations, Sci. Res. Essays 8 (2013) 1349–1359.

[20] W. Peng, Finite element method for the space and time fractional Fokker-Planck equation, SIAM J. Numer. Anal. 47 (1) (2008) 204–226.

[21] M.A. Akbar, N.H.M. Ali, Exact and solitary wave solutions to the (2+1)-dimensional Chaffee-Infante equation and the dimensionless form of the Zakharov equation, Adv. Diff. Eqn. 2019 (2019) 446.

[22] M.A. Akbar, N.H.M. Ali, The alternative (G/G)-expansion method and its applications to nonlinear partial differential equations, Int. J. Phys. Sci. 6 (35) (2011) 7910–7920.

[23] M.A. Akbar, N.H.M. Ali, Exact and solitary wave solutions for the Tzitzeica-Dodd-Bullough and the modified KdV-Zakharov-Kuznetsov equations using the modified simple equation method, Ain Shams Engr. J. 4 (4) (2013) 903–909.

[24] S.F. Tian, J.J. Yang, Z.Q. Li, Y.R. Chen, Blow-up phenomena of a weakly dissipative modified two-component Dullin-Gottwald-Holm system, Appl. Math. Lett. 106 (2020) 106578.

[25] W.Q. Peng, S.F. Tian, T.T. Zhang, Initial value problem for the pair transition coupled nonlinear Schrödinger equations via the Riemann-Hilbert method, Compl. Anal. Operator Theory 14 (2020) 38.

[26] W.Q. Peng, S.F. Tian, X.B. Wang, T.T. Zhang, Y. Fang, Riemann-Hilbert method and multi-soliton solutions for three-component coupled nonlinear Schrödinger equations, J. Geom. Phys. 146 (2019) 103508.

[27] T.Y. Xu, S.F. Tian, W.Q. Peng, Riemann-Hilbert approach for multi-soliton solutions of generalized coupled fourth-order nonlinear Schrödinger equations, Math. Methods Appl. Sci. 43 (2) (2002) 865–880.

[28] S.F. Tian, Lie symmetry analysis, conservation laws and solitary wave solutions to a fourth-order nonlinear generalized Boussinesq water wave equation, Appl. Math. Lett. 100 (2020) 106056.