CAP-PRODUCT STRUCTURES
ON THE FINTUSHEL-STERN SPECTRAL SEQUENCE

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Abstract. We show that there is a well-defined cap-product structure on the Fintushel-Stern spectral sequence. Hence we obtain the induced cap-product structure on the \( \mathbb{Z}_8 \)-graded instanton Floer homology. The cap-product structure provides an essentially new property of the instanton Floer homology, from a topological point of view, which multiplies a finite dimensional cohomology class by an infinite dimensional homology class (Floer cycles) to get another infinite dimensional homology class.

1. Introduction

In [2], p298, Atiyah stated that In the product we then have four types of cycle

\[
\text{finite} \times \text{finite}, \quad \text{cofinite} \times \text{cofinite},
\]

\[
\text{cofinite} \times \text{finite}, \quad \text{finite} \times \text{cofinite}.
\]

The first two give ordinary homology and cohomology respectively. The other two are quite different from these and give an obvious sense “middle-dimensional” homology, one “positive” and one “negative.” It is the purpose of this paper to elaborate on this comment and to show this point rigorously. A. Floer [5] introduced a novel homology theory for oriented closed 3-manifolds with the homology groups of \( S^3 \). Subsequently, Donaldson [1] discovered that the instanton Floer homology has an intimate relationship to his gauge theoretic 4-manifold invariants [5]. More precisely, if \( X = X_1 \cup_Y X_2 \), where \( X_i(i = 1, 2) \) is a 4-manifold with homology 3-sphere \( Y \) boundary, then the relative Donaldson invariants \( \Phi(X_1) : H_2(X_1; \mathbb{Z}) \to HF_*(Y) \) and \( \Phi(X_2) : H_2(X_2; \mathbb{Z}) \to HF_*(\overline{Y}) \) lie in the Floer homology cycles \( \overline{Y} \) has opposite orientation of \( Y \). The natural pairing in the Floer theory provides \( \Phi(X) = \langle \Phi(X_1), \Phi(X_2) \rangle \) (see [1, 3]). The pairing of the Floer cycles defines 4-dimensional Donaldson invariant.

In a formal way, the Floer homology can be viewed as the “middle-dimensional” homology of the infinite dimensional space \( B_Y \) of gauge equivalence classes of \( SU(2) \)-connections on \( Y \times SU(2) \). Let \( Z(X_i)(i = 1, 2) \) be the set of boundary values of instantons over \( X_i \). The \( Z(X_i) \) represents a cycle of homology class \( \Phi(X_i) \) which is independent of metrics. The pairing \( \langle \Phi(X_1), \Phi(X_2) \rangle \) can be identified with the intersection \( Z(X_1) \cap Z(X_2) \) in \( B_Y \). The formalization alone might suggest that there should be a product from \( H^*(B_Y) \otimes HF_*(Y) \) to \( HF_*(Y) \) (see [1] §4.2), as multiplying a finite dimensional cohomology class by an infinite dimensional class to get another infinite dimensional class.

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class (see [1] p.298). Such a cap-product structure on the instanton Floer homology shows that the Floer homology groups are, from a topological point of view, something essentially new [1] p298. Dually, this is equivalent to a cup-product structure on the instanton Floer cohomology $H^*(B_Y) \otimes HF^*(Y) \to HF^*(Y)$ (see [1, 4]).

In this paper, we show that there is a well-defined cap-product structure of $H^*(B_Y, \mathbb{Q})$ on the $\mathbb{Z}_8$-graded instanton Floer homology. For a single 3-dimensional cohomology class in $H^*(B_Y, \mathbb{Q})$, this cap-product is known [3, 4]. In order to obtain the cap-product structure, we have to overcome the main difficulty arising from the non-compactness of the moduli space $\mathcal{M}_{Y \times \mathbb{R}}(a, b)$ with flat connections $a$ and $b$ over $Y$. For a cohomology class $\omega \in H^*(B_Y, \mathbb{Q})$ with degree $\geq 8$, the pairing $\langle \omega, \mathcal{M}_{Y \times \mathbb{R}}(a, b) \rangle$ in general is not defined, since $\mathcal{M}_{Y \times \mathbb{R}}(a, b)$ has three possible non-compactness cases from (1) the $\mathbb{R}$-action, (2) the concentrated instanton bubbling on $Y \times \mathbb{R}$ and (3) the chain anti-self-dual connection splitting the moduli space into codimension one (or $\geq 1$) pieces. The non-compactness of the $\mathbb{R}$-action is not serious. We can work on the space $\mathcal{M}_{Y \times \mathbb{R}}(a, b)/\mathbb{R} = \hat{\mathcal{M}}_{Y \times \mathbb{R}}(a, b)$. The non-compactness of the bubbling type (2) is problematic. We use the $\mathbb{Z}$-graded instanton Floer homology defined by Fintushel-Stern [6] to fix Chern-Simons values in one of the energy band $(r, r + 1)$, where $r$ is a regular value of the Chern-Simons functional. Any bubbling required energy at least 1 forces the corresponding moduli space to have different lifting for the generators $a$ and $b$. So the moduli space with fixed lifting $a^{(r)}$ and $b^{(r)}$ contains no bubbling limit point. The chain anti-self-dual connections also provide the codimension one components in the compactification. The evaluation of cohomology class on the compactification does not make sense. This is why we restrict ourself to the rational cohomology group $H^*(B_Y, \mathbb{Q})$. The rational cohomology group $H^*(B_Y, \mathbb{Q})$ is generated by the $\mu$-map construction. Thus we can represent each cohomology class by its Poincaré dual as a divisor from the homology cycle of the 3-manifold $Y$ itself. Thus adapting the 4-dimensional technique (dimension counting replaced by spectral flow counting), we can show that the set $P.D(\omega) \cap \hat{\mathcal{M}}_{Y \times \mathbb{R}}(a, b)$ is compact by ruling out those three non-compact parts. In turn, there is a spectral sequence which starts from the $\mathbb{Z}$-graded Floer homology and converges to the $\mathbb{Z}_8$-graded Floer homology. Thus the cap-product structure on the $\mathbb{Z}$-graded Floer homology induces the cap-product structure on the spectral sequence, hence on the $\mathbb{Z}_8$-graded Floer homology. Our main theorem is following.

**Theorem** [Theorem 4.3] There is a well-defined cap-product action of $H^*(B_Y; \mathbb{Q})$ on the Fintushel-Stern spectral sequence $(E^n_{i,j}(Y), d^k)$.

In particular, Theorem 4.3 shows that there is a well-defined cap-product (cup-product) structure on the instanton Floer homology (cohomology). There are some cross-product structure on the $SO(3)$ Floer homology which proved by Braam and Donaldson for rational coefficients and by the author for integral coefficients (see [8] for more references). Note that the space $B_Y$ is not globally a product but only infinitesimally (because its tangent bundle decomposes). Thus its middle-dimensional homology cannot be reduced to ordinary homology and cohomology by a factorization.
Thus the Floer homology groups are, from a topological point of view, something essentially new (quoted from [1] page 298).

The paper is organized as follows. In §2, we give a review on the instanton Floer theory of integral homology 3-spheres. We start the rational cohomology group and ring of $\mathcal{B}_Y$ in §3.1, and show that there is a well-defined cap-product action on the $\mathbb{Z}$-graded instanton Floer homology in §3.2. Using the commutative property of the cap-product action with higher differentials, we prove our main theorem in §4.

2. The instanton Floer homology of integral homology 3-spheres

In this section, we give a description of the gauge theory on 3-manifolds and review the definition of the instanton Floer homology (see [3, 6, 7]).

Let $Y$ be an oriented closed 3-dimensional smooth manifold with $H_1(Y, \mathbb{Z}) = 0$, and let $P \to Y$ be a smooth principal $SU(2)$-bundle (always trivial). Fix a trivialization $Y \times SU(2)$ of $P$ and let $\theta$ be the associated trivial connection. Denote the Sobolev $L^k_p$ space of connections on $P$ by $\mathcal{A}_Y$. This space has a natural affine structure with underlying vector space $\Omega^1(Y, adP)$ where $adP$ is the adjoint bundle. $\mathcal{A}_Y$ is acted upon by the gauge group $G = L^k_p(\Omega^0_0(Y, adP))$ of bundle automorphisms of $P$, and the orbit space $\mathcal{B}_Y = \mathcal{A}_Y/G$ is well-defined when $k + 1 > 3/p$. The universal cover $\tilde{\mathcal{B}}_Y$ of $\mathcal{B}_Y$ defined as $\tilde{\mathcal{B}}_Y = \mathcal{A}_Y/G_0$, where $G_0 \subset G$ is the group of degree zero gauge transformations. The irreducible connections form an open and dense subspace $\mathcal{B}^*_Y$ of $\mathcal{B}_Y$ which is a Banach manifold with

$$T_a \mathcal{B}^*_Y \equiv \{ b \in L^k_p(\Omega^1(Y, adP)) \mid d^*_a b = 0 \}$$

where $d^*_a$ is the $L^2$-adjoint of $d_a$ (covariant derivative on sections of $adP$) with respect to some metric on $Y$.

The Chern-Simons functional $cs : \mathcal{A}_Y \to \mathbb{R}$ is defined as

$$cs(a) = \frac{1}{8\pi^2} \int_Y \text{tr}(a \wedge da + \frac{2}{3} a \wedge a \wedge a).$$

and satisfies $cs(g \cdot a) = cs(a) + \text{deg}(g)$ for gauge transformations $g : Y \to SU(2)$. Thus $cs$ is well-defined on $\tilde{\mathcal{B}}(P) = \mathcal{A}(P)/\{ g \in \mathcal{G} : \text{deg}(g) = 0 \}$ and it descends to a function $cs : \mathcal{B}(P) \to \mathbb{R}/\mathbb{Z}$ which plays the role of a Morse function in defining Floer homology. The differential of $cs$ is

$$dcs(a)(\alpha) = \int_Y \text{tr}(F_a \wedge \alpha),$$

hence its critical set consists of the flat connections $\mathcal{R}(\mathcal{B}(P)) = \{ a \in \mathcal{B}(P) \mid F_a = 0 \}$ (i.e., $F_a$ is the curvature 2-form on $Y$). It is well-known that the elements of $\mathcal{R}(\mathcal{B}(P))$ are in 1-1 correspondence with those of

$$\mathcal{R}(Y) = \text{Hom}(\pi_1(Y), SU(2))/adSU(2),$$

the $SU(2)$-representations of $\pi_1(Y)$ modulo conjugacy. Let $\mathcal{R}^*(Y) = \mathcal{R}(Y) \setminus \{ \theta \}$ be the space of irreducible $SU(2)$ representations.
Define the weighted Sobolev space $L^p_{k,\delta}$ on sections $\xi$ of a bundle over $Y \times \mathbb{R}$ to be the space of $\xi$ for which $\epsilon_{\delta} \cdot \xi$ is in $L^p_{k}$, where $\epsilon_{\delta}(y, t) = e^{\delta |t|}$ for $|t| \geq 1$.

For any $\delta > 0$ and any $SO(3)$ anti-self-dual connection $A$ on the bundle over $Y \times \mathbb{R}$, the anti-self-duality operator

$$d_A^* + d_A^+ : L^p_{k+1,\delta}(\Omega^1(Y \times \mathbb{R}, adP)) \to L^p_{k,\delta}(\Omega^0 \oplus \Omega^2_{\mathbb{Z}})(Y \times \mathbb{R}, adP)$$

is Fredholm. We call $A$ regular if $d_A^* + d_A^+$ is surjective and we call the moduli space $\mathcal{M}_{Y \times \mathbb{R}}$ of perturbed anti-self-dual connections with finite energy regular if it contains only orbits of regular $A$'s.

The spectral flow is $SF(a_\alpha, a_\beta) = \text{Index}(d_A^* + d_A^+)(a_\alpha, a_\beta)$ (mod $2p_1(A)$), where $p_1(A)$ is the Pontryagin form. The $\text{Index}(d_A^* + d_A^+)(a_\alpha, a_\beta)$ is given by

$$\text{Index}(d_A^* + d_A^+)(a_\alpha, a_\beta) = -2 \int_{Y \times \mathbb{R}} p_1(A) - \frac{h_\beta + \rho_\beta(0)}{2} + \frac{-h_\alpha + \rho_\alpha(0)}{2},$$

where $h_\alpha$ is the sum of the dimensions of $H^i(Y, ada_\alpha)(i = 0, 1)$, and $\rho_\alpha(0)$ is the $\rho$-invariant of the signature operator $*d_{a_\alpha} - d_{a_\alpha}*$ over $Y$ restricted to even forms (see [3, 4]). We use the trivial index $\text{Index}(d_A^* + d_A^+)(a_\alpha, a_\beta)$ (mod 8).

Floer’s chain group $C_j(Y)$ is defined to be the free module generated by irreducible flat connections $a_\alpha \in f^{-1}(0)$ with $\mu(a_\alpha) \equiv j$ (mod 8).

The Chern-Simons and spectral flow functionals on $\mathcal{A}_Y$ descends to functionals $cs$ and $SF : \tilde{\mathcal{B}}_Y \to \mathbb{R}$. Let $cs(\mathcal{R}_Y^\ast)$ be the image values of Chern-Simons functional in $\mathbb{R}$. So $cs(\mathcal{R}_Y^\ast)$ (mod 1) is a finite set and independent of the choice of $\theta$. The set $\mathcal{R}_Y = \mathbb{R} \setminus cs(\mathcal{R}_Y^\ast)$ consists of all the regular values of the Chern-Simons functional.

Given $a \in \mathcal{R}_Y \subset B_Y$, let $a^{(r)} \in \mathcal{R}_Y \subset \tilde{\mathcal{B}}_Y$ be the unique lift of $a^{(r)}$ (mod 1) for $r \in \mathbb{R}$. Note that $SF(a^{(r)}) = SF(a^{(r)}, \theta)$ is an integer.

For $a, b \in B_Y$, choose any smooth representatives $a, b \in \mathcal{A}_Y$ and a connection $A$ on $Y \times \mathbb{R}$ whose $t$-component vanishes and which equals $a$ for large negative $t$ and equals $b$ for large positive $t$. Set $A_\delta(a, b) = A + L^4_{1,\delta}(\Omega^1(Y \times \mathbb{R}, adSU(2)))$. It is acted smoothly upon by the gauge group

$$G_\delta = \{g \in L^4_{2,\text{loc}}(Y \times \mathbb{R}, adSU(2))| \exists \: T > 0, \xi \in L^4_{2,\text{loc}}(\Omega^0(Y \times \mathbb{R}, adSU(2))) \text{ such that } g = \exp(\xi) \text{ for } |t| \geq T\}.$$

Then the quotient space $B_\delta(a, b) = \mathcal{A}_\delta(a, b)/G_\delta$ is a smooth Banach manifold. Note that the space $B_\delta(a, b)$ depends on the homotopy class of path $A$ of connections between $a$ and $b$. Each $A \in \mathcal{A}_Y(a, b)$ is (temporal) gauge equivalent to a connection whose component in the $\mathbb{R}$-direction vanishes. Floer’s crucial observation is that trajectories of the vector field $f$, i.e., the flow lines of

$$\frac{\partial a}{\partial t} + f(a(t)) = 0, \quad \text{or} \quad \frac{\partial a}{\partial t} = *F(a(t)),$$

can be identified with instantons $A$ on $Y \times \mathbb{R}$ and $A|_{Y \times \{t\}} = a(t)$. Let $\mathcal{M}(a_\alpha, a_\beta)$ be the space of (perturbed) anti-self-dual connections $A$ with asymptotics flat connections $a_\alpha$ and $a_\beta$. The space $\mathcal{M}(a_\alpha, a_\beta)$ consists of infinitely many smooth manifold components, the dimensions of various components differ by a multiple of 8 and all equal to $\mu(a_\alpha) - \mu(a_\beta)$ (mod 8). Let $\mathcal{M}^1(a_\alpha, a_\beta)$ be the
union of all the 1-dimensional components in $\mathcal{M}(a, b)$. Then $\mathcal{M}^1(a, b)$ has a canonical orientation together with a proper, free $\mathbb{R}$-action induced by the translations on $Y \times \mathbb{R}$. It follows that $\mathcal{M}^1(a, b)/\mathbb{R} = \hat{\mathcal{M}}(a, b)$ is a compact oriented 0-manifold (a finite set of signed points) by the Proposition 1c.2 in [7]. The differential $\partial_Y : C_\cdot(Y) \to C_{\cdot-1}(Y)$ of the Floer chain complex is defined by

$$\partial_Y a_\alpha = \sum_{\beta \in \mathcal{C}_{j-1}(Y)} \# \hat{\mathcal{M}}(a_\alpha, a_\beta) a_\beta,$$

where $\# \hat{\mathcal{M}}(a_\alpha, a_\beta)$ is the algebraic number of points, and the sign is given by the spectral flow. Floer has shown that $\partial^2 = 0$, hence the homology of this complex $\{C_j(Y), \partial_Y\}_{j \in \mathbb{Z}}$ is the (instanton) Floer homology $HF_\ast(Y)(\ast \in \mathbb{Z}_8)$. It is shown that $HF_\ast(Y)$ is independent of the choice of metric $\sigma$ on $Y$ and of regular perturbations by the Theorem 2 in [7].

For lifts $\hat{a}$ and $\hat{b}$ in $\hat{\mathcal{B}}_Y$, denote $\hat{\mathcal{B}}_\delta(\hat{a}, \hat{b})$ for lifts of connections in $\mathcal{B}_\delta(a, b)$. If $A \in \mathcal{B}_\delta(a, b)$ for $a, b \in \mathcal{R}^+_Y$ and $\hat{A}$ is any lift to $\hat{\mathcal{B}}_\delta(\hat{a}, \hat{b})$, then $\text{ind}D_{\hat{A}} = SF(\hat{a}) - SF(\hat{b})$. For $\mu(a) \equiv j \pmod{8}$, $b = \theta$ and $r \in \mathbb{R}_Y$, the spectral flow $SF(a^{(r)})$ is independent of the choice of trivial connection $\theta$ used to define Chern-Simons functional on $\mathcal{A}_\delta(a, \theta)$.

**Definition 2.1.** Define the chain groups $C_n^{(r)}(Y) = \{a \in \mathcal{R}^+_Y \mid SF(a^{(r)}) = n \in \mathbb{Z}\}$ and $\partial^{(r)} : C_n^{(r)}(Y) \to C_{n-1}^{(r)}(Y)$ by

$$\partial^{(r)} a = \sum_{b \in C_{n-1}^{(r)}(Y)} \# \hat{\mathcal{M}}_Y(a, b) \cdot b.$$

Note that $\partial^{(r)} a$ defined in [7] can be identified with $\partial^{(r)} a = \sum_{b \in C_{n-1}^{(r)}(Y)} \# \hat{\mathcal{M}}_Y(a^{(r)}, b^{(r)}) \cdot b$. For any $A \in \hat{\mathcal{M}}_Y(a, b)$, there is a unique lift $\hat{A} \in \hat{\mathcal{M}}_Y(a^{(r)}, b^{(r)})$ (similarly for $\hat{A} \in \hat{\mathcal{M}}_Y(a^{(r)}, b^{(r)})$). Since $\text{Ind}_{\hat{A}} = SF(a^{(r)}) - SF(b^{(r)}) = 1$, we have $SF(\hat{b}^{(r)}) = n - 1$ and $\hat{b} = b^{(r)}$ the preferred lift. Hence $\hat{A} \in \hat{\mathcal{M}}_Y(a^{(r)}, b^{(r)})$. The orientation of $A$ and $\hat{A}$ are same, so the counting $\# \hat{\mathcal{M}}_Y(a, b)$ can be also expressed as $\# \hat{\mathcal{M}}_Y(a^{(r)}, b^{(r)})$.

The key observation is that the Chern-Simons functional is non-decreasing along the gradient trajectories in $\hat{\mathcal{B}}_Y$. By the similar argument of Floer [7], Fintushel and Stern [7] obtained:

1. $\partial^{(r)}_{n+1} \circ \partial^{(r)}_n = 0$ for $n \in \mathbb{Z}$. The homology of $\{C_n^{(r)}(Y), \partial^{(r)}_n\}$, denoted by $H_\ast(C_n^{(r)}(Y), \partial^{(r)}_n) = I^{(r)}_\ast(Y), \ast \in \mathbb{Z}$, is the $\mathbb{Z}$-graded Floer homology which is independent of metrics and perturbations.

2. If $[r, s] \subset \mathbb{R}_Y$, then $I^{(r)}(Y) = I^{(s)}(Y)$ and $I^{(r+1)}(Y) = I^{(r)}_{s+8}(Y)$.

3. The $\mathbb{Z}$-graded Floer homology $\{I^{(r)}_n(Y), n \in \mathbb{Z}\}$ determines the $\mathbb{Z}_8$-graded Floer homology groups by filtering the Floer chain complex.

For $r \in \mathbb{R}_Y, j \in \mathbb{Z}_8, n \in \mathbb{Z}$ and $n \equiv j \pmod{8}$, define the free abelian groups

$$F_n^{(r)} C_j(Y) = \sum_{k \geq 0} C_{n+8k}^{(r)}(Y). \quad (2.2)$$

Then there is a finite length decreasing filtration of $C_j(Y)(j \in \mathbb{Z}_8)$:

$$\cdots \subset F_{p+8}^{(r)} C_j(Y) \subset F_p^{(r)} C_j(Y) \subset F_{p-8}^{(r)} C_j(Y) \subset \cdots \subset C_j(Y),$$

in the preferred lift. Hence

$$\sum_{k \geq 0} C_{n+8k}^{(r)}(Y).$$

This is a natural representation of the CAP-product structure.
Theorem 2.2. [Theorem 5.1 in 3] Theorem 2.2. \[ C_j(Y) = \bigcup_{k \in \mathbb{Z}} F^{(r)}_{j+8k} C_j(Y). \]

Since the perturbed Chern-Simons functional is non-decreasing along the gradient flows, it follows that Floer’s boundary map \( \partial_Y : F^{(r)}_n C_j(Y) \to F^{(r)}_{n-1} C_{j-1}(Y) \) preserves the filtration

\[
\begin{array}{cccccccc}
\cdots & \subset F^{(r)}_{n+8} C_j(Y) & \subset F^{(r)}_n C_j(Y) & \subset \cdots & \subset C_j(Y) \\
\downarrow & \downarrow \partial_Y & \downarrow \partial_Y & \downarrow \\
\cdots & \subset F^{(r)}_{n+7} C_{j-1}(Y) & \subset F^{(r)}_{n-1} C_{j-1}(Y) & \subset \cdots & \subset C_{j-1}(Y)
\end{array}
\]

So the \( \mathbb{Z}_8 \)-graded Floer chain complex has a decreasing bounded filtration \( (F^{(r)}_n C_*(Y), * \in \mathbb{Z}_8) \). The homology of the vertical chain subcomplex in the above filtration gives \( F^{(r)}_n I_j(Y) \),

\[
\begin{array}{cccccccc}
\cdots & \subset F^{(r)}_{n+8} I_j(Y) & \subset F^{(r)}_n I_j(Y) & \subset \cdots & \subset HF_j(Y) \\
\downarrow & \downarrow \partial_Y & \downarrow \partial_Y & \downarrow \\
\cdots & \subset F^{(r)}_{n+7} I_{j-1}(Y) & \subset F^{(r)}_{n-1} I_{j-1}(Y) & \subset \cdots & \subset HF_{j-1}(Y)
\end{array}
\]

This is a bounded filtration for the \( \mathbb{Z}_8 \) graded Floer homology groups \( \{HF_j(Y)\}_{j \in \mathbb{Z}_8} \):

\[
\cdots \subset F^{(r)}_{n+8} HF_j(Y) \subset F^{(r)}_n HF_j(Y) \subset F^{(r)}_{n-8} HF_j(Y) \subset \cdots \subset HF_j(Y),
\]

where \( F^{(r)}_n HF_j(Y) = \text{Im}[F^{(r)}_n I_j(Y) \to HF_j(Y)] \). By the definition of the filtration (2.2),

\[ F^{(r)}_n C_j(Y)/F^{(r)}_{n+8} C_j(Y) = C^{(r)}_n(Y), \]

and the induced chain map is precisely \( \partial^{(r)}_n \).

**Theorem 2.2.** [Theorem 5.1 in 3] (i) There is a spectral sequence \( (E^{k}_{n,j}, d^k) \) (periodic in \( j \) with period 8) with

\[
E^{k}_{n,j}(Y) \cong I^{(r)}_n(Y), \quad d^k : E^{k}_{n,j}(Y) \to E^{k}_{n+8k-1,j-1}(Y),
\]

\[ E^{\infty}_{n,j}(Y) = F^{(r)}_n I_j(Y)/F^{(r)}_{n+8} I_j(Y), \quad (j \in \mathbb{Z}_8, n \in \mathbb{Z}, n \equiv j \pmod{8}). \]

Furthermore, the groups \( E^{k}_{n,j}(Y) \) are topological invariants.

(ii) Any one period of the spectral sequence \( (E^{k}_{n,j}(Y), d^k) \) converges to \( E^{\infty}_{\mathbb{Z}_8}(Y)(j \in \mathbb{Z}_8) \) which is isomorphic to the bigraded module associated to the filtration \( F^{(r)}_n \) of \( I_j(Y) \) \( (j \in \mathbb{Z}_8) \):

\[ HF_j(Y) \cong \bigoplus_{k \in \mathbb{Z}} E^{\infty}_{n+8k,j}(Y). \]

\[ \square \]

Note that our formulation of the spectral sequence is slightly different from the one in 3. We span the spectral sequence formulated in 3 periodically in \( j \)-direction. So all higher differentials \( d^k(k \geq 1) \) maps into the same direction with degree \((8k - 1, -1)\).

**Corollary 2.3.** For \( j \in \mathbb{Z}_8 \), \( \bigoplus_{k \in \mathbb{Z}} I^{(r)}_{j+8k}(Y) = HF_j(Y) \) if and only if \( d^k = 0(k \geq 1) \) in the spectral sequence.
Proof: Since the following is true for one period of the $E_{s,*}^k$:

$$HF_j(Y) = \bigoplus_{k \in \mathbb{Z}} E_{n+sk,j}^\infty(Y),$$

and $E_{n,j}^3(Y)$ is given by the $\mathbb{Z}$-graded Floer homology, so the result follows from the fact that $(E_{s,*}^k(Y), d^k)$ collapses at the $E^1$-term.

\[ \square \]

3. A CAP-PRODUCT STRUCTURE ON $I_s^{(r)}(Y, \mathbb{Z})$

In this section, we show that the $\mathbb{Z}$-graded Floer homology $I_s^{(r)}(Y, \mathbb{Z})$ admits a $H^*(\mathcal{B}_Y, \mathbb{Q})$-cap-product structure: $H^*(\mathcal{B}_Y, \mathbb{Q}) \otimes I_s^{(r)}(Y, \mathbb{Z}) \to I_s^{(r)}(Y, \mathbb{Z})$. The main task is to overcome the non-compactness when one evaluates cohomology classes on the moduli spaces. This is the essential point that we use the $\mathbb{Z}$-graded Floer homology to filter out the ideal instantons over $Y \times \mathbb{R}$, and use the rational cohomology classes to rule out the chain connections (codimension one components).

3.1. The cohomology ring $H^*(\mathcal{B}_Y, \mathbb{Q})$. In this subsection we describe that the cohomology ring (group) of $\mathcal{B}_Y$ with rational coefficients is generated by two special classes. Both classes can be obtained by the $\mu$-map procedure with divisors realized by time-translation invariant divisors.

**Proposition 3.1.** For an integral homology 3-sphere $Y$ and $a, b \in \mathcal{R}^*(Y)$, the rational cohomology ring $H^*(\tilde{\mathcal{B}}_{Y \times \mathbb{R}}(a, b); \mathbb{Q})$ is the tensor product of a polynomial algebra on the four-dimensional generator $\nu$ and an exterior algebra on the one-dimensional generator $\mu(Y)$:

$$H^*(\tilde{\mathcal{B}}_{Y \times \mathbb{R}}(a, b); \mathbb{Q}) = \text{Sym}^*(H_0(Y \times \mathbb{R})) \otimes \Lambda^*(H_3(Y \times \mathbb{R})).$$

Proof: Since $H_*(Y, \mathbb{Z}) = H_*(S^3, \mathbb{Z})$, we have $H_1(Y \times \mathbb{R}) = H_2(Y \times \mathbb{R}) = H_4(Y \times \mathbb{R}) = 0$; $H_0(Y \times \mathbb{R}) = H_3(Y \times \mathbb{R}) = \mathbb{Z}$. The rational cohomology of $\tilde{\mathcal{B}}_{Y \times \mathbb{R}}(a, b)$ is the product of a polynomial algebra on even-dimensional generators and an exterior algebra on odd-dimensional ones. The generators arise from the $\mu_{-4\mathbb{R}}$ construction for $\tilde{\mathcal{B}}_{Y \times \mathbb{R}}(a, b)$ (see [1] Chapter 5 and [2]). Here $p_1$ is the first Pontryagin class of the base-point fibration $\tilde{\mathcal{B}}_{Y \times \mathbb{R}}(a, b) \to \mathcal{B}_{Y \times \mathbb{R}}(a, b)$. So the result follows.

These two cohomology classes $\nu = \mu((y_0, t_0))$ and $\mu(Y)$ are obtained through the general universal $\mu$-map construction. We give their geometric construction to pass down to the cohomology classes in $\mathcal{B}_Y$.

For the purpose of finding the cap-product structures on the instanton Floer homology, we can fix our asymptotic values as irreducible flat connections corresponding elements in $\mathcal{R}^*(Y)$. For $a$ and $b$ in $\mathcal{R}^*(Y)$, $\mathcal{B}_{Y \times \mathbb{R}}^*(a, b)$ is the equivalence classes of irreducible connections over $Y \times \mathbb{R}$ with asymptotic values $a$ and $b$. Pick a point in $Y$, call $y = \{y_0\}$, $\mathcal{G}_0$ is gauge transformation group of $SU(2)$-vector bundle which acts as identity between the fibers over $\{y, +\infty\}$ and $\{y, -\infty\}$ in $Y \times \mathbb{R}$. This is a normal subgroup of $\mathcal{G}_0$, thus we can form $\tilde{\mathcal{B}}_{Y \times \mathbb{R}}^*(a, b) = \mathcal{A}_Y^3 / \mathcal{G}_0$. Let us choose a path $\{y\} \times \mathbb{R}$ in $Y \times \mathbb{R}$ representing $[\mathbb{R}]$. For each connection $A \in \mathcal{A}_Y^3(a, b)$, we first fix the trivializations at $P_{(y, \pm\infty)}$ and identify them. Choose $p \in P_{(y, \pm\infty)}$ lies over $\{y, -\infty\}$, then there is a unique path that is
always horizontal with respect to the connection $A$. So the parallel transport along $\{y\} \times \mathbb{R}$ gives an automorphism of the fibers between $P_{(y, -\infty)}$ and $P_{(y, +\infty)}$ and after a choice of $p$ this determines an element of $SO(3)$ the homology of the path. Let $h_{[R]}(A)$ be the holonomy of the connection along the path. This homomorphism of the fiber $P_{(y, +\infty)}$ and $P_{(y, -\infty)}$ depends on the equivalence class of $A$, so the construction defines a map

$$h_{[R]} : \overline{B}_Y \times \mathbb{R}(a, b) \rightarrow \text{Hom}(P_{(y, -\infty)}, P_{(y, +\infty)}) \equiv SO(3).$$

This map is independent of the time-translation since $h_{[R]}(A) = h_{[R]}(A_0)$ for the instanton $A_0 \in \overline{B}_Y \times \mathbb{R}(a, b)$. Thus the map factor through $\overline{B}_Y^*$ (identified with $\overline{B}_Y^* \times \mathbb{R}(a, b)$), so we obtain $h_{[R]} : \overline{B}_Y^* \rightarrow SO(3)$. Then we pull-back the fundamental class $[SO(3)] \in H^3(SO(3), \mathbb{Z})$ to get a cohomology class $h_{[R]}([SO(3)]) \in H^3(\overline{B}_Y^*, \mathbb{Z})$. By the slant product in $\mathbb{R}$ §5.1.2, $\mu(y_0 \times \mathbb{R}) = \mu(y_0, t_0) = \nu$. See [4] §4.2 for different, but equivalent, constructions of this class.

Next we consider the map $\tilde{\mu}_{-\frac{1}{2}p_1} : H_3(Y \times \mathbb{R}) \rightarrow H^1(\overline{B}_Y \times \mathbb{R}(a, b))$. For the unique fundamental class $[Y]$ of $Y$, each connection $A$ over $Y \times \mathbb{R}$ can be used to calculate the Chern-Simons invariant of $A_{Y \times \{t\}}$. This invariant takes $S^1$-value and is independent of the $t$-variable. Therefore it defines a map $cs_Y : \overline{B}_Y \times \mathbb{R}(a, b) \rightarrow S^1$. Hence $cs_Y([S^1])$ is an element of $H^1(\overline{B}_Y \times \mathbb{R}(a, b))$. By the construction $cs_Y^*([S^1]) \in H^1(\overline{B}_Y \times \mathbb{R}(a, b)/\mathbb{R}) = H^1(\overline{B}_Y)$. From the Chern-Weil theory of the second Chern class, $cs_Y^*([S^1])$ coincides with $\tilde{\mu}_{-\frac{1}{2}p_1}([Y]) = \mu(Y)$. So we have the following.

**Proposition 3.2.** For an integral homology 3-sphere $Y$,

$$H^*(B_Y; \mathbb{Q}) = H^*(\overline{B}_Y; \mathbb{Q}) = \text{Sym}^*(\nu) \otimes \Lambda^*(\mu(Y)).$$

**Proof:** By Proposition [3] and the discussion above, we have

$$H^*(\overline{B}_Y; \mathbb{Q}) = \text{Sym}^*(\nu) \otimes \Lambda^*(\mu(Y)).$$

The total space $\overline{B}_Y^*$ of the base-point fibration has the same weak homotopy type as $\overline{B}_Y$. By the Gysin sequence, we obtain

$$H^*(\overline{B}_Y^*; \mathbb{Q}) = \oplus \nu^i \cup H^{*-i}(\overline{B}_Y; \mathbb{Q}).$$

So $\overline{B}_Y^*$ has the same cohomology ring of $\overline{B}_Y$. The structure of $\overline{B}_Y$ normal to the singular strata space $\overline{B}_Y \setminus \overline{B}_Y^*$ is a cone on an infinite dimensional space. The result follows.

Since both classes $\nu$ and $\mu(Y)$ are arisen from the $\mu$-map construction, we also obtain their Poincaré dual $V_{y_0} = P.D(\nu)$ and $V_Y = P.D(\mu(Y))$ in $\overline{B}_Y(a, b)$. The divisors $V_{y_0}$ and $V_Y$ are $t$-independent so that they also represent two divisors in the space $\overline{B}_Y$.

3.2. The well-defined action of $H^*(B_Y, \mathbb{Q})$ on $I^*_r(Y, \mathbb{Z})$. Let us first recall the Floer-Uhlenbeck compactness on $Y \times \mathbb{R}$.

**Definition 3.3.** An ideal anti-self-dual connection (trajectory) over $Y \times \mathbb{R}$, of Chern number $k$, is a pair

$$(A; (x_1, ..., x_l)) \in M_{Y \times \mathbb{R}}^{k-l}(a, b) \times S^l(Y \times \mathbb{R})$$

where $A$ is a point in $M_{Y \times \mathbb{R}}^{k-l}(a, b) \cap \mathcal{B}_\delta$ and $(x_1, ..., x_l)$ is an unordered $l$-tuple of points of $Y \times \mathbb{R}$.
Let \( \{A_n\}, n \in \mathbb{N} \), be a sequence of connections of charge \( k \) on the \( SU(2) \) bundle \( P \) over \( Y \times \mathbb{R} \). We say that the gauge equivalence classes \( \{A_n\} \) converge weakly to a limiting ideal anti-self-dual connection \( (A; (x_1, ..., x_1)) \) if

(i): The action densities converge as measures, i.e. for any continuous function \( f \) on \( Y \times \mathbb{R} \),

\[
\int_{Y \times \mathbb{R}} f|F(A_n)|^2 d\mu \rightarrow \int_{Y \times \mathbb{R}} f|F(A)|^2 d\mu + 8\pi^2 \sum_{i=1}^l f(x_i).
\]

(ii): There are bundle maps

\[
\rho_n : P|_{Y \times \mathbb{R}\setminus\{x_1, ..., x_1\}} \rightarrow P|_{Y \times \mathbb{R}\setminus\{x_1, ..., x_1\}}
\]

such that \( \rho_n^*(A_n) \) converges to \( A \) in \( C^\infty \) on compact subsets of the punctured manifold.

**Definition 3.4.** Let \( a \) and \( b \) be flat \( SU(2) \) connections over \( Y \). A chain of connections \( (B_1, ..., B_n) \) from \( a \) to \( b \) is a finite set of connections over \( Y \times \mathbb{R} \) which converge to flat connections \( c_{i-1}, c_i \) as \( t \rightarrow \mp \infty \) such that \( c = c_0, c_n = b, \) and \( B_i \) connects \( c_{i-1}, c_i \) for \( 0 \leq i \leq n \).

We say that the sequence \( \{A_n\} \in \mathcal{M}_{Y \times \mathbb{R}}^k(a, b) \) is (weakly) convergent to the chain of connections \( (B_1, ..., B_n) \) if there is a sequence of \( n \)-tuples of real numbers \( \{t_{\alpha,1} \leq \cdots \leq t_{\alpha,n}\}_\alpha \), such that \( t_{\alpha,i} - t_{\alpha,i-1} \rightarrow \infty \) as \( \alpha \rightarrow \infty \), and if, for each \( i \), the translates \( t_{\alpha,i}^*A_\alpha = A_\alpha(\circ - t_{\alpha,i}) \) converge weakly to \( B_i \).

**Definition 3.5.** An “ideal chain connection” joining flat connections \( a \) and \( b \) over \( Y \) is a set \( (A_j; x_{j1}, ..., x_{jl_j})_{1 \leq j \leq J} \), where \( (A_j)_{1 \leq j \leq J} \) is a chain connection and for each \( j \), \( (A_j; x_{j1}, ..., x_{jl_j}) \) is an ideal instanton.

In this setup, a version of the Uhlenbeck compactness theorem holds. We state it in a form proved by Floer in \( \mathbb{F} \) (see also in \( \mathbb{S} \)).

**Theorem 3.6.** [Floer-Uhlenbeck compactness on \( Y \times \mathbb{R} \)] Let \( A_\alpha \in \mathcal{M}_{Y \times \mathbb{R}}^k \cap B_\delta(a_\alpha, b_\alpha) \) be a sequence of anti-self-dual connections with uniformly bounded action. Then there exists a subsequence converging to an ideal chain connection \( (A_j; x_{j1}, ..., x_{jl_j})_{1 \leq j \leq J} \). Moreover, one has

\[
\sum_{j=1}^J (k_j + l_j) = k, \quad c_2(A_j) = k_j \text{ (not necessarily an integer)}.
\]

In order to define a cohomology class evaluation on moduli spaces over \( Y \times \mathbb{R} \), we need to rule out the ideal chain connections over \( Y \times \mathbb{R} \).

**Definition 3.7.** For a cohomology class \( \omega \in H^p(B_Y; \mathbb{Q}) \), the action of \( \omega \) is given by

\[
\omega \cap : \quad C_n^{(p)}(Y) \rightarrow C_{n-p-1}^{(p)}(Y)
\]

\[
a \mapsto \sum_{SF(a^{(p)}) = SF(b^{(p)})} \#(\tilde{\mathcal{M}}_{Y \times \mathbb{R}}(a) \cap P.D(\omega)) \cdot b,
\]
where \( \#(\hat{M}_{Y \times \mathbb{R}}(a, b) \cap P.D(\omega)) \) is the algebraic number of \( \omega \) evaluating on the \( p \)-dimensional moduli space \( \hat{M}_{Y \times \mathbb{R}}(a, b) \) in \( B_Y \).

Note that every rational cohomology class in \( H^*(B_Y; \mathbb{Q}) \) can be represented as \( \omega = \nu^k \cup \mu(Y)^l \). So the action of \( \omega \) defined above is equivalent to the following:

\[
(\omega \cap)(a) = \begin{cases} 
\#(\hat{M}_{Y \times \mathbb{R}}(a, b) \cap V_{y_1} \cap \cdots V_{y_k} \cap \cdots V_{r^1} \cap \cdots V_{r^l}) \cdot b & \text{if } SF(a^{(r)}) - SF(b^{(r)}) - 1 = 3k + l \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( \nu \) and \( \mu(Y) \) are induced from cohomology classes on 4-manifold \( Y \times \mathbb{R} \), we can use the same construction in \( B \) to show that the intersection \( \hat{M}_{Y \times \mathbb{R}}(a, b) \) with \( V_y \) and \( V_r \) is transversal. Using the translation invariant, we pass the transversality for the intersection in \( B_Y \). By Theorem 3.6, we see that \( \hat{M}_{Y \times \mathbb{R}}(a, b) \) in general does not represent a genuine cycle in \( B_Y \), so the pairing between \( \hat{M}_{Y \times \mathbb{R}}(a, b) \) with cohomology classes in Definition 3.7 needs to be justified.

**Lemma 3.8.** For \( a \in C_n^{(r)}(Y) \) and \( b \in C_{n-p-1}^{(r)}(Y) \), the compactification of \( \hat{M}_{Y \times \mathbb{R}}(a^{(r)}, b^{(r)}) \) does not contain any ideal anti-self-dual connection over \( Y \times \mathbb{R} \).

Proof: Suppose not. There is a sequence \( \{A_n \in \hat{M}_{Y \times \mathbb{R}}(a^{(r)}, b^{(r)})\} \) which converges to an ideal anti-self-dual connection \( (A; (x_1, \cdots, x_i)) \in \hat{M}_{Y \times \mathbb{R}}^{p+1-n}(\alpha, \beta) \times S^4 \). By the definition, we have

\[
\text{cs}(a^{(r)}) - \text{cs}(b^{(r)}) = \frac{1}{8\pi^2} \int_{Y \times \mathbb{R}} Tr F_{A_n} \wedge F_{A_n},
\]

\[
\text{ind}_{A_n} = SF(a^{(r)}) - SF(b^{(r)}) = p + 1.
\]

The Chern-Simons is continuous in the sense of weak convergence. So we get

\[
\lim_{n \to \infty} \frac{1}{8\pi^2} \int_{Y \times \mathbb{R}} Tr F_{A_n} \wedge F_{A_n} = \frac{1}{8\pi^2} \int_{Y \times \mathbb{R}} Tr F_A \wedge F_A + \sum_{j=1}^i n_j,
\]

where \( n_j \) is the multiplicity of \( x_j \), and \( \sum_{j=1}^i n_j \neq 0 \) by the hypothesis. For the asymptotic values \( \alpha \) and \( \beta \) of \( A \), we have

\[
\text{cs}(\alpha) - \text{cs}(\beta) + \sum_{j=1}^i n_j = \text{cs}(a^{(r)}) - \text{cs}(b^{(r)}).
\]

If \( \sum_{j=1}^i n_j \neq 0 \), then \( \alpha \neq a^{(r)} \) or \( \beta \neq b^{(r)} \) since \( \text{ind}_A = p + 1 - \sum_{j=1}^i n_j \). Thus for the fixed \( a^{(r)} \) and \( b^{(r)} \), the compactification of \( \hat{M}_{Y \times \mathbb{R}}(a^{(r)}, b^{(r)}) \) does not contain any ideal anti-self-dual connection.

Note that the Chern-Simons value is proportional to the spectral flow by Atiyah-Patodi-Singer’s formula. Any ideal anti-self-dual connection at a point in \( Y \times \mathbb{R} \) requires at least energy of spectral flow \( 8 \), in which the Chern-Simons value decreases its value by 1. This changes the elements for the different regular values. So our \( Z \)-graded Floer chain complex rules out this situation.

**Lemma 3.9.** For \( \omega \in H^p(B_Y; \mathbb{Q}) \), \( a \in C_n^{(r)} \) and \( b \in C_{n-p-1}^{(r)} \), the intersection \( \hat{M}_{Y \times \mathbb{R}}(a, b) \cap P.D(\omega) \) is compact.
Proof: Note that $\omega = \nu^k \cup \mu(Y)^l$, so it suffices to show that the intersection $\hat{M}_{Y \times \mathbb{R}}(a, b) \cap V_{y_1} \cap \cdots V_{y_k} \cap V_Y^1 \cap \cdots V_Y^l$ is compact.

Suppose $\{A_n\}$ is a sequence in the intersection $\hat{M}_{Y \times \mathbb{R}}(a, b) \cap V_{y_1} \cdots V_Y^l$. There exists a unique lift of the sequence $\hat{A}_n \in \hat{M}_{Y \times \mathbb{R}}(a^{(r)}, b^{(r)})$. By Lemma 3.8, there exists a subsequence converging to an ideal chain connection. By Lemma 3.11, we know that the ideal anti-self-dual connection does not occur. So the sequence $A_n$ does not have ideal anti-self-dual connection. For the ideal chain connections $(B_1, \ldots, B_n)$ such that

$$B_i \in \hat{M}_{Y \times \mathbb{R}}(c_{i-1}^{(r)}, c_i^{(r)}) \cap V_{y_1} \cap \cdots V_{y_k} \cap V_Y \cap \cdots V_Y^l,$$

$$\sum_{i=1}^n k_i = k, \sum_{i=1}^n l_i = l.$$

If $SF(c_{i-1}^{(r)}) - SF(c_i^{(r)}) - 1 < 3k_i + l_i$, by the dimension counting and the general position of the divisors, then

$$\hat{M}_{Y \times \mathbb{R}}(c_{i-1}^{(r)}, c_i^{(r)}) \cap V_{y_1} \cap \cdots V_{y_k} \cap V_Y \cap \cdots V_Y^l = \emptyset.$$

So we may have the situation

$$SF(c_{i-1}^{(r)}) - SF(c_i^{(r)}) - 1 \geq 3k_i + l_i. \quad (3.1)$$

Also we know that

$$\sum_{i=1}^n (SF(c_{i-1}^{(r)}) - SF(c_i^{(r)})) = SF(a^{(r)}) - SF(b^{(r)}) = 3k + l + 1. \quad (3.2)$$

Adding (3.1) for $i$, we have

$$\sum_{i=1}^n (SF(c_{i-1}^{(r)}) - SF(c_i^{(r)})) - 1 = \sum_{i=1}^n (SF(c_{i-1}^{(r)}) - SF(c_i^{(r)})) - n \geq 3 \sum_{i=1}^n k_i + \sum_{i=1}^n l_i \geq 3k + l$$

Combining with (3.2), we obtain $n \leq 1$. So there is no ideal chain connection. The result follows. \hfill \Box

**Proposition 3.10.** The action $\omega \cap$ is well-defined for $\omega \in H^*(By; \mathbb{Q})$.

Proof: By Lemma 3.8, we have $\hat{M}_{Y \times \mathbb{R}}(a, b) \cap V_{y_1} \cap \cdots V_{y_k} \cap V_Y^1 \cap \cdots V_Y^l$ is a compact, 0-dimensional, oriented manifold. So the algebraic number in the expression $(\omega \cap)(a)$ is finite. It is easy to check that $\omega \cap$ is independent of the choices of the representatives $V_{y_i}(i = 1, \ldots, k)$ and $V_Y^j(j = 1, \ldots, l)$ (same as in \Box). Therefore the action $\omega \cap$ is well-defined. \hfill \Box

**Lemma 3.11.** The action $\omega \cap$ commutes with $\partial^{(r)}$, i.e., $\partial_{r-1}^{(r)} \circ (\omega \cap) = (\omega \cap) \circ \partial_{r+1}^{(r)}$.

Proof: For $a \in C_n^{(r)}(Y)$ and $c \in C_{n-1}^{(r)}(Y)$, the space

$$K = \hat{M}_{Y \times \mathbb{R}}(a,c) \cap V_{y_1} \cap \cdots V_{y_k} \cap V_Y \cap \cdots V_Y^l$$

is a compact, 0-dimensional, oriented manifold. So the algebraic number in the expression $(\omega \cap)(a)$ is finite. It is easy to check that $\omega \cap$ is independent of the choices of the representatives $V_{y_i}(i = 1, \ldots, k)$ and $V_Y^j(j = 1, \ldots, l)$ (same as in \Box). Therefore the action $\omega \cap$ is well-defined. \hfill \Box
is a 1-dimensional manifold in $B_Y$. The boundary components of $K$ gives
\[(\hat{\mathcal{M}}_{Y \times \mathbb{R}}(a, b) \cap P.D(\omega)) \times \hat{\mathcal{M}}_{Y \times \mathbb{R}}(b, c) \bigcap \hat{\mathcal{M}}_{Y \times \mathbb{R}}(a, b_1) \times (\hat{\mathcal{M}}_{Y \times \mathbb{R}}(b_1, c) \cap P.D(\omega)).\]

Again the ideal anti-self-dual connections are ruled out by the same method in Lemma 3.8. By Lemma 3.9, the counting argument gives
\[SF(c_1^{(r)} - SF(c_1^{(r)})) - 1 \geq 3k_i + l_i, \quad 3 \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} l_i = p,\]
for the ideal chain connection $(\hat{B}_i)_{1 \leq i \leq n}$. Thus adding the above inequality we obtain $n \leq 2$. So there is a possible chain connection $(B_1, B_2) \in \hat{\mathcal{M}}_{Y \times \mathbb{R}}(a, b) \times \hat{\mathcal{M}}_{Y \times \mathbb{R}}(b, c)$ boundary component with $SF(a^{(r)}) - SF(b^{(r)}) = i$ and $SF(b^{(r)}) - SF(c^{(r)}) = p + 2 - i$, where $1 \leq i \leq p + 1$. Let $d : B_Y \to B_Y \times B_Y$ be the diagonal map. So $d^*(\omega) \in H^*(B_Y \times B_Y; \mathbb{Q})$ and $P.D(d^*(\omega)) = P.D(\omega) \times P.D(\omega) \in H_*(B_Y \times B_Y; \mathbb{Q})$. Thus we have
\[\#(\hat{\mathcal{M}}_{Y \times \mathbb{R}}(a, b) \times \hat{\mathcal{M}}_{Y \times \mathbb{R}}(b, c)) \cap P.D(d^*(\omega)) = \#(\hat{\mathcal{M}}_{Y \times \mathbb{R}}(a, b) \cap P.D(\omega)) \times \#(\hat{\mathcal{M}}_{Y \times \mathbb{R}}(b, c) \cap P.D(\omega)) = 0,\]
by the dimension reason from transversality. Even though there are boundary components other than we expect, they contribute zero. Hence the result follows.

The map $\omega \cap$ in Definition 3.7 defines a $\mathbb{Z}$-graded chain map, and induces a map (still denoted by $\omega \cap$) on the $\mathbb{Z}$-graded Floer homology:
\[\omega \cap : I_n^{(r)}(Y, \mathbb{Z}) \to I_n^{(r)}_{n-\deg(\omega)-1}(Y, \mathbb{Z}).\]
Now for any $\omega \in H^*(B_Y; \mathbb{Q})$, there is a well-defined action on the $\mathbb{Z}$-graded Floer homology. Thus we obtain the desired cap-product structure
\[H^*(B_Y; \mathbb{Q}) \otimes I_n^{(r)}(Y, \mathbb{Z}) \to I_n^{(r)}(Y, \mathbb{Z})\]
on the $\mathbb{Z}$-graded Floer homology.

**Remark:** It would be nice to show that the cap-product structure induces a $H^*(B_Y; \mathbb{Q})$-module structure on $I_n^{(r)}(Y, \mathbb{Z})$. Clearly, we have $(\omega_1 + \omega_2) \cap (a) = (\omega_1 \cap)(a) + (\omega_2 \cap)(a)$, $(1\cap)(a) = a$ and $(\omega \cap)(a_1 + a_2) = (\omega \cap)(a_1) + (\omega \cap)(a_2)$. The only thing left is to check the ring homomorphism from $H^*(B_Y; \mathbb{Q})$ to End($I_n^{(r)}(Y, \mathbb{Z})$). This is equivalent to $(\omega_1 \cup \omega_2) \cap = (\omega_1 \cap) \circ (\omega_2 \cap)$. Note that $(\omega_1 \cup \omega_2) \cap$ acts on $I_n^{(r)}(Y, \mathbb{Z}) \to I_{n-\deg(\omega_1)-\deg(\omega_2)}^{(r)}(Y, \mathbb{Z})$, but the composition $(\omega_1 \cap) \circ (\omega_2 \cap)$ acts on $I_n^{(r)}(Y, \mathbb{Z}) \to I_{n-\deg(\omega_1)-\deg(\omega_2)-2}^{(r)}(Y, \mathbb{Z})$. It is impossible to have $(\omega_1 \cup \omega_2) \cap = (\omega_1 \cap) \circ (\omega_2 \cap)$. The ring structure on $H^*(B_Y; \mathbb{Q})$ is not preserved by the cap-product in Proposition 3.10. The $-1$ shift is due to the noncompact $\mathbb{R}$-action on the moduli spaces.
4. CAP-PRODUCT STRUCTURES ON THE SPECTRAL SEQUENCE $E_{n,j}^k(Y)$

In this section, we are going to extend the cap-product structure to the Fintushel-Stern spectral sequence. The cap-product structure on $I_n(Y, \mathbb{Z})$ serves as an initial step as the cap-product on $E_1^{s,s}(Y)$ in Theorem 2.2. Then we show that the cap-product structure can be deduced on $E_{n,j}^k(Y)$, in particular on the $Z_\ast$-graded instanton Floer homology.

**Lemma 4.1.** For $\omega \in H^n(B_Y; \mathbb{Q})$, the action $\omega \cap$ in Definition 3.4 is a filtration preserving homomorphism.

**Proof:** Recall that the filtration defined in the Fintushel-Stern spectral sequence is given by

$$F_r^{(n)}C_j(Y) = \sum_{k \geq 0} C_{n+sk}^{(r)}(Y).$$

Thus the map in Lemma 3.10 gives a well-defined map

$$\omega \cap : C_{n+sk}^{(r)}(Y) \rightarrow C_{n+sk-\deg(\omega)-1}^{(r)}(Y),$$

which induces a map $\omega \cap : F_r^{(n)}C_j(Y) \rightarrow F_r^{(n)}C_{j-\deg(\omega)-1}(Y)$. Here $j_\omega \equiv n - \deg(\omega) - 1 \pmod{8}$. Thus we obtain a filtration preserving map $\omega \cap$ with degree $-\deg(\omega) - 1$.

Note that $\omega \cap$ does not send $C_{n+sk}^{(r)}(Y)$ to $C_{n+8(k+1)-\deg(\omega)-1}^{(r)}(Y)$. By Lemma 3.11 and Theorem 2.2 $d^0 = 0^{(r)}$ in the spectral sequence. So

$$\omega \cap : E_{0}^{n,j} \rightarrow E_{0}^{n-\deg(\omega)-1, j-\deg(\omega)-1}(Y)$$

commutes with $d^0$ by Lemma 3.11. There is an induced action (still denoted by $\omega \cap$)

$$\omega \cap : E_{1}^{n,j} \rightarrow E_{1}^{n-\deg(\omega)-1, j-\deg(\omega)-1}(Y).$$

Our spectral sequence is periodical in $j$-direction, so the same map appears infinitely many times at $j \pmod{8}$. So the rational cohomology ring $H^\ast(B_Y; \mathbb{Q})$ acts diagonally on $E_1^{1,s}(Y)$. The task is to show that this action commutes with all higher differentials $d^k(k \geq 1)$.

**Lemma 4.2.** The map $\omega \cap$ commutes with $d^k(k \geq 1)$.

**Proof:** Note that each element of $E_{n,j}^k(Y)$ is a survivor from previous differentials, and we only consider the rational cohomology class $\omega$. So $a \in E_{n,j}^k(Y)$ is an element in $F_n^{(r)}C_j(Y)$ with $\partial a \in F_{n-1+sk}^{(r)}C_{j-1}(Y)$. The higher differential $d^k$ is again the algebraic counting of 1-dimensional moduli space from $a \in E_{n,j}^k(Y)$ to $b \in E_{n-1+8k,j-1}^k(Y)$. The following diagram

$$\begin{array}{ccc}
E_{n,j}^k & \xrightarrow{d^k} & E_{n+8k-1,j-1}^k \\
| \omega \cap | & & | \omega \cap | \\
E_{n-\deg(\omega)-1,j-\deg(\omega)-1}^k & \xrightarrow{d^k} & E_{n-\deg(\omega)+8k-2,j-\deg(\omega)-2}^k
\end{array}$$

is commutative by the same method of proof in Lemma 3.11. \qed
Theorem 4.3. There is a well-defined cap-product action of $H^*(B_Y;\mathbb{Q})$ on the Fintushel-Stern spectral sequence $(E^k_{n,j}(Y), d^k)$.

Proof: There is a well-defined induced action $H^*(B_Y;\mathbb{Q}) \otimes E^1_{n,j}(Y) \to E^1_{n-deg(\omega)-1,j-deg(\omega)-1}(Y)$ by Lemma 4.1. Inductively, we obtain the cap-production $\omega \cap$ on $E^k_{n,j}(Y)$ by Lemma 4.2 for all $k \geq 1$.

Corollary 4.4. The $\mathbb{Z}_8$-graded instanton Floer homology admits a $H^*(B_Y;\mathbb{Q})$-group action from the cap-product structure.

Proof: By Theorem 2.2, any one period of the Fintushel-Stern spectral sequence converges to the $\mathbb{Z}_8$-graded instanton Floer homology. So the result follows from Theorem 4.3 for the term $E^\infty_{n,j}(Y)$.

Remark: Note that $H^*(B_Y;\mathbb{Q})$ acts on $E^k_{*,*}(Y)$ as a group, not ring. So there is no $H^*(B_Y;\mathbb{Q})$-module structure from our definition of the action.

Example: Let $Y = \Sigma(2, 3, 5)$ be the Poincaré 3-sphere. By a result of Fintushel-Stern [3],

$$I^0_i(Y, \mathbb{Z}) = \mathbb{Z}(a_\alpha); I^0_5(Y, \mathbb{Z}) = \mathbb{Z}(a_\beta); I^0_i(Y, \mathbb{Z}) = 0, \quad i \neq 1, 5.$$ 

By a result of Kronheimer (see also [2] Lemma 5.1), we have

$$\nu \cap : C^0_5(Y, \mathbb{Z}) \to C^0_1(Y, \mathbb{Z})$$

given by $(\nu \cap)(a_\beta) = 2a_\alpha$, and $\mu(Y) \cap$ acts trivially by the degree reason. Any other cohomology class $\nu^k \cup \mu(Y)^l$ ($l \neq 0$) acts trivially on $I^0_*(Y, \mathbb{Z})$. In this case, the Fintushel Stern spectral sequence collapses at $E^3_{*,*}(Y)$. So the cap-product structure on the $\mathbb{Z}_8$-graded instanton Floer homology has only one nontrivial action

$$\nu \cap : HF_5(Y, \mathbb{Z}) \to HF_1(Y, \mathbb{Z}).$$

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