Axiomatic conformal theory in dimensions > 2 and AdS/CT correspondence

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To Sasha Polyakov with admiration and love

Abstract

We formulate axioms of conformal theory (CT) in dimensions > 2 modifying Segal’s axioms for two-dimensional CFT. (In the definition of higher-dimensional CFT one includes also a condition of existence of energy-momentum tensor.) We use these axioms to derive the AdS/CT correspondence for local theories on AdS. We introduce a notion of weakly local quantum field theory and construct a bijective correspondence between conformal theories on the sphere $S^d$ and weakly local quantum field theories on $H^{d+1}$ that are invariant with respect to isometries. (Here $H^{d+1}$ denotes hyperbolic space= Euclidean AdS space.) We give an expression of AdS correlation functions in terms of CT correlation functions. The conformal theory has conserved energy-momentum tensor iff the AdS theory has graviton in its spectrum.

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1 Introduction

The AdS/CFT correspondence [1], [2], [3] played very important role in the development of quantum field theory and string theory. This correspondence relates string theory on AdS with conformal field theory on the boundary. It was understood very soon [2], [3] that a similar correspondence can be constructed for local quantum theories on AdS and conformal theories (CT) on the boundary (AdS/CT correspondence). The main goal of this paper is to give a very simple rigorous proof of the AdS/CT correspondence for local theories. We show that, for every local quantum field theory on $(d+1)$-dimensional AdS that is invariant with respect to isometries, one can construct $d$-dimensional conformal field theory with the same space of states. Moreover, our construction can be applied also to weakly local theories (see Section 3 for the definition of weak locality.) The CT has a conserved energy-momentum tensor iff the theory on AdS has the graviton in its spectrum. (Notice, however, that our constructions can be applied to quantum gravity only in the framework of perturbation theory.)

Let us emphasize that our statement does not cover the original example of $N = 4$ SYM theory that comes from string theory (not from local quantum field theory).
We did not analyze in detail the relation of our considerations to the existing heuristic constructions (see [5], [6], [7] for review). It seems these constructions do not always lead to genuine conformal theories (Polyakov, private communication); in those cases they definitely differ from our construction. It is clear, however, that our formulas either agree with standard constructions, or constitute a more precise version of these constructions.

We work in the Euclidean setting. Hence our AdS is Euclidean AdS that is hyperbolic space (Lobachevsky space) from the viewpoint of mathematician and our conformal theories are defined on $S^d$ or $\mathbb{R}^d$.

Our proof is based on the axiomatics of conformal theory in dimensions $> 2$. Our axioms modify Segal’s axioms for two-dimensional CFT [9], [10]. (Segal’s papers contain also discussion of axioms of quantum field theory in the general case.) Segal starts with Riemann surfaces (two-dimensional conformal manifolds) having holes with parameterized boundaries. To every boundary he assigns vector space $H$. The holes are divided in two classes (“incoming” and “outgoing”). If we have $m$ incoming holes and $n$ outgoing holes CFT specifies a map $H^\otimes m \to H^\otimes n$. Segal’s axioms describe what happens if we sew two surfaces. Our axioms for higher-dimensional theories are based on the same ideas. We consider the standard $S^d$ of radius 1 with holes, but we allow only round holes. We do not consider two types of holes, but this is irrelevant. We could modify our axioms to consider both types of holes. Instead of talking about sphere with holes we are talking about collections of non-overlapping parameterized round balls. The conformal group acts on these collections; factorizing the space of collections with respect to this action, we obtain the space $\mathcal{M}_n$, an analog of moduli space of Riemann surfaces with holes in our setting. Notice that $\mathcal{M}_n$ is finite-dimensional; this is related to the fact that the conformal group is finite-dimensional in dimensions $> 2$. To specify conformal theory (CT) we assign to every element of $\mathcal{M}_n$ an $n$-linear functional on the space of states $H$ (an element of a tensor power of $H^\ast$). We formulate axioms of CT and analyze their relation to other approaches. Following the suggestion of [5] we reserve the name CFT for CT with conserved energy-momentum tensor.

Axiomatic conformal field theory became very fashionable recently under the funny name “conformal bootstrap”. The renewed interest to conformal bootstrap suggested by A. Polyakov many years ago was generated by papers where it was shown that the axioms of unitary CFT are strong enough to prove very good estimates for anomalous dimensions in 3D Ising model [11], [12].

To derive the AdS/CT correspondence, we notice that one can construct the space $\mathcal{M}_n$ starting with hyperbolic space $H^{d+1}$ (we should consider half-spaces instead of balls). Now having a local quantum field theory on hyperbolic space we can define functionals entering the definition of CT. (If the theory is determined by a local action $S$, we integrate $e^{-S}$ over the complement to half-spaces.)

The paper does not depend on any papers about CFT or about AdS. In Section 2 we formulate our axioms of CT and in Section 4 we relate them to other approaches. In Section 3 we derive the AdS/CT correspondence. In Section 5 we discuss the AdS/CT

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1 A rigorous proof of AdS/CT correspondence was claimed in [5]. However, it seems that Rehren’s construction not necessarily leads to conformal theories with OPE.

2 Segal talks about cobordisms instead of incoming and outgoing holes, but this is only terminological difference.
dictionary. In particular, we express AdS partition functions and AdS correlation functions in terms of CT correlation functions. It is not clear whether our dictionary is completely equivalent to existing ones; however, we show that it is very close both to GKPW dictionary suggested in [2], [4] and to BDHM dictionary suggested in [13] (see [2], [5] for review).

2 Axiomatic conformal theory

The group of conformal transformations of the sphere $S^d$ is denoted by $\text{Conf}_d$; it is generated by inversions. Its connected component is isomorphic to $SO(1, d + 1)$. We define a round ball in $S^d$ as a conformal map of the standard round ball into $S^d$. Notice that this means that we have fixed a conformal parameterization of the boundary of a round ball in $S^d$ (a conformal map of $S^{d-1}$ onto the boundary). Let us consider the space of $n$ non-overlapping round balls on the sphere $S^d$. The conformal transformations act on this space; we denote by $M_n$ the space of conformal classes of ordered collections of $n$ non-overlapping round balls (the space of orbits of $\text{Conf}_d$ in the space of collections of balls). The sphere $S^d$ is conformally equivalent to the Euclidean space $\mathbb{R}^d$; round balls in $S^d$ correspond to round balls, complements to round balls and half-spaces in $\mathbb{R}^d$ with conformal parameterization of boundaries. The space $M_1$ consists of one point, in general the space $M_n$ is a smooth manifold of dimension $(n - 1) \dim \text{Conf}_d = \frac{(n-1)(d+2)(d+1)}{2}$. The group of permutations $S_n$ acts on $M_n$ in an obvious way. One can construct a natural map $\phi_{nm} : M_n \times M_m \rightarrow M_{n+m-2}$. To construct this map we will work in $\mathbb{R}^d$. Then performing a conformal transformation we can consider the last ball in $M_n$ as the half-space $x_d \geq 0$ and the first ball in $M_m$ as the half-space $x_d \leq 0$. The remaining $m+n-2$ balls specify a point in $M_{m+n-2}$. (We can represent a ball as a half-space in many ways. However, we have fixed a conformal parameterization of the ball; this allows us to specify a unique transformation of the ball onto half-space.) Notice that the map $\phi_2$ specifies an associative multiplication on $M_2$; in other words $M_2$ can be considered as semigroup. More generally, the operations $\phi_{nm}$ specify associative multiplication in the union $M$ of spaces $M_n$. The map $\phi_{n,2}$ determines an action of the semigroup $M_2$ on $M_n$. Of course, the construction of the map $\phi_{nm}$ can be given directly in $S^d$. In particular, the action of the semigroup $M_2$ on $M_n$ replaces the last ball in the collection specifying an element of action of the semigroup $M_2$ on $M_n$ by a smaller ball in the interior of the last ball.

To give an axiomatic description of CT we fix a topological vector space $\mathcal{H}$ (the space of states) and an element $a \in \mathcal{H} \otimes \mathcal{H}$. In a basis $e_i$ of $\mathcal{H}$ we can write $a = a^{ik} e_i e_k$. The element $a$ determines an associative multiplication in the direct sum $H$ of vector spaces $(\mathcal{H}^*)^\otimes n$ dual to tensor powers $\mathcal{H}^\otimes n$. In the basis $e_i$ the elements of $H$ can be represented as covariant tensors of various ranks. We can represent the product of a tensor $r_{i_1,...,i_n}$ (= a linear functional on $\mathcal{H}^\otimes n$) and a tensor $s_{k_1,...,k_m}$ (= a linear functional on $\mathcal{H}^\otimes m$) as a tensor of rank $n+m-2$ (= a linear functional on $\mathcal{H}^\otimes (n+m-2)$).

\footnote{Our construction is reminiscent of the definition of little disks operad.}

\footnote{Sometimes it is convenient to consider instead of $\mathcal{H}^*$ a dense subspace of it. We will disregard these subtleties.}
as a contraction of the last index of $r$ with the first index of $s$ by means of the tensor $a^{ik}$. Notice that the tensor $a$ specifies an inner product in $H^*$; the multiplication can be defined in terms of this product.

We assume that for every point of $M_n$ we have a map $\psi_n : H^\otimes n \to \mathbb{C}$ (a multilinear functional $\psi_n(h_1,...,h_n)$ where $h_k \in H$). This functional should depend continuously on the point of $M_n$. If necessary to emphasize the dependence on the point of $M_n$ we will use the notation $\psi_n(B_1,...,B_n,h_1,...,h_n)$ where $B_1,...,B_n$ are balls specifying this point. Together the functionals $\psi_n$ determine a continuous map $\Psi : M \to H$. We assume that this map commutes with the actions of the group of permutations $S_n$, i.e. the functional $\psi_n(B_1,...,B_n,h_1,...,h_n)$ is $S_n$-invariant. The main axiom of CT is the requirement that the map $\Psi$ is a homomorphism (the product in $M$ goes to the product in $H$).

One can reformulate the main axiom in the following way. Let us consider non-overlapping balls $B_1,...,B_{r+s}$ specifying an element of $M_{r+s}$ and corresponding functional $\psi_{r+s}(h_1,...h_{r+s})$. Let us choose a sphere $S^{d-1}$ in such a way that the first $r$ balls are inside the sphere and the last $s$ balls are outside it. This sphere bounds two balls $B_{in}$ and $B_{out}$. The balls $B_1,...,B_r$, $B_{out}$ specify an element of $M_{r+1}$. For fixed $h_1,...,h_r$ the corresponding functional $\psi_{r+1}$ determines an element $\Psi_1 = \psi_1(h_1,...,h_r) \in H^*$. The balls $B_{in}, B_{r+1},...,B_{r+s}$ specify an element of $M_{s+1}$. For fixed $h_{r+1},...,h_{r+s}$ the corresponding functional $\psi_{s+1}$ determines an element $\Psi_2 = \psi_2(h_{r+1},...,h_{r+s}) \in H^*$. An equivalent formulation of the main axiom is the expression of $\psi_{r+s}$ as the inner product of $\Psi_1$ and $\Psi_2$:

$$\psi_{r+s}(h_1,...h_{r+s}) = \langle \Psi_1(h_1,...,h_r), \Psi_2(h_{r+1},...,h_{r+s}) \rangle.$$  \hfill (1)

(Recall that the tensor $a$ specifies an inner product in $H^*$.)

Let us explain the physical origin of these constructions. Let us consider a conformally invariant local action functional $S$ on $\mathbb{R}^d$ or, equivalently, on $S^d$. Let us calculate the corresponding partition function on the domain $V_n$ obtained from $S^d$ by deleting $n$ balls as a functional integral of $e^{-S}$ over the space of fields on $V_n$. This partition function depends on $s$; it should be identified with $\psi_n(h_1,...,h_n)$. (Hence $H$ should be identified with the space of boundary states.) The main axiom of CT comes from the remark that $V_{n+m-2}$ can be represented as a union of $V_m$ and $V_n$ having a common part of boundary that can be identified with $S^{d-1}$. (To calculate $\psi_{n+m-2}$ we do the integral over fields defined on $V_{n+m-2}$. We can do this in two steps. First, we calculate the integrals over the fields defined on $V_n$ and $V_m$, we get $\psi_n$ and $\psi_m$. Second, we paste together these two answers inserting a $\delta$-function that guarantees that the fields on $V_n$ and $V_m$ coincide on the common boundary and integrating over the fields on this boundary. This integration gives us a scalar product on the space $H^*$.)

Let us consider the homomorphism $\psi_2 : M_2 \to H^* \otimes H^*$ in more detail. The multiplication in the space $H^* \otimes H^*$ can be represented in coordinates as an operation

\footnote{In two-dimensional theories the infinite-dimensional conformal Lie algebra has central extension, therefore we should allow projective representations. Conformal Lie algebra in dimension $> 2$ does not have central extensions, but still it is possible that the homomorphism $\Psi$ is multivalued.}

\footnote{Notice, that our considerations did not use conformal invariance in any way, they were based only on locality of action. Moreover, even locality is not quite necessary; see below.}

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transforming a pair of tensors $x_{ik}, y_{ik}$ into the tensor $z_{ik} = x_{il} a^{ls} y_{sk}$. Raising the second index of tensor $x_{ik}$ by means of tensor $a^{kl}$ we obtain a tensor $z^s_i = x_{il} a^{ls}$, that can be considered as an element of the ring $End \mathcal{H}$ of linear operators in $\mathcal{H}$. It is easy to check that $z^k_i = \tilde{x}^s_i y^s_k$. This means that $\psi_2$ specifies a homomorphism of $\mathcal{M}_2 \to End \mathcal{H}$. In other words, the semigroup $\mathcal{M}_2$ acts on $\mathcal{H}$. It is easy to verify that the Lie algebra of the semigroup $\mathcal{M}_2$ coincides with the Lie algebra $so(1, d + 1)$ of the group $SO(1, d + 1)$.

To prove this fact we notice that in $\mathbb{R}^d$ every element of $\mathcal{M}_2$ can be represented as the exterior of the unit ball and a parameterized round ball inside the unit ball. This representation is unique. This remark allows us to identify $\mathcal{M}_2$ with the subsemigroup of $Conf_d$ that consists of elements mapping the unit ball into its interior.) We conclude that this Lie algebra acts on $\mathcal{H}$. An important one-dimensional subsemigroup $\mathcal{L}$ of $\mathcal{M}_2$ corresponds to dilations. An element of $\mathcal{L}$ consists of two balls having centers in the south pole and north pole of $S^d$ respectively (the parameterizations are fixed in such a way that the corresponding points lie at the same great circle). In the $\mathbb{R}^d$ picture we should fix some point and consider the interior of a sphere with a center at this point and the exterior of a larger sphere with the same center. The corresponding element of $\mathcal{L}$ will be denoted by $T_\alpha$ where $\alpha = \log \frac{R}{r}$ where $r$ stands for smaller radius, $R$ for larger radius. It is easy to check that $T_\alpha T_\beta = T_{\alpha + \beta}$. The infinitesimal generator of the subgroup $\mathcal{L}$ will be denoted by $S$; we fix this generator in such a way that $T_\alpha = e^{-\alpha S}$.

In the Lie algebra of the conformal group $Conf_d$, the element $S$ corresponds to dilation.

3 AdS/CT

To derive the AdS/CT correspondence, we interpret the spaces $\mathcal{M}_n$ in terms of Euclidean AdS space. From the viewpoint of mathematics, this is the hyperbolic space (Lobachevsky space) $H^{d+1}$. It can be considered as a connected component of the hyperboloid $x_0^2 - x_1^2 - ... - x_{d+1}^2 = R^2$ in $(d + 2)$-dimensional space. Equivalently, we can consider the space $\mathbb{R}^{1,d+1}$ with indefinite inner product (one positive sign and $d + 1$ negative signs); then the hyperbolic space is singled out by the equation $<x, x> = R^2$ and inequality $x_0 > 0$. (We will fix $R = 1$; in other words we consider hyperbolic space with curvature $K = -1$.) It follows from this representation that the isometry group of hyperbolic space is isomorphic to $Conf_d$ and its connected component is isomorphic to $SO(1, d + 1)$. Applying stereographic projection with the center at the point $(-1, 0, ... 0)$, we obtain the Poincaré ball interpretation of hyperbolic space. (We are projecting into the hyperplane $x_0 = 0$; the hyperbolic space $H^{d+1}$ is identified with the open unit ball $x_1^2 + ... + x_{d+1}^2 < 1$. ) The points of the unit sphere $S^d$ are called boundary points, or ideal points, or points at infinity of the hyperbolic space $H^{d+1}$. The isometries of $H^{d+1}$ induce conformal transformations on $S^d$.

Notice that the ideal points of a hyperplane in $H^{d+1}$ constitute a sphere $S^{d-1}$ conformally embedded into the ideal sphere $S^d$. The group $Conf_d$ acts transitively on the space of hyperplanes, hence it is sufficient to check this statement for one hyperplane. It is obviously true for the hyperplane $x_1 = 0$ in the Poincaré ball. Conversely, taking into account that $Conf_d$ acts transitively on the space of conformal spheres $S^{d-1}$ in $S^d$, we see that every such sphere consists of ideal points of some hyperplane. A hyperplane divides $H^{d+1}$ in two half-spaces; this allows us to analyze ideal points of half-spaces.
Let us consider parameterized half-spaces of $H^{d+1}$ (in other words we consider isometric maps of the standard half-space into hyperbolic space $H^{d+1}$). It follows from the above considerations that parameterized half-spaces are in one-to-one correspondence with conformally parameterized round balls in $S^d$. This allows us to describe spaces $\mathcal{M}_n$ in terms of hyperbolic space. Namely, we should consider the space of ordered collections of $n$ non-overlapping half-spaces $(\Gamma_1, ..., \Gamma_n)$. The group $\text{Conf}_d$ acts on this space; by definition $\mathcal{M}_n$ is the space of orbits of this action. The definition of associative multiplication in the union $\mathcal{M}$ of the spaces $\mathcal{M}_n$ can be given in the following way. Represent an element of $\mathcal{M}_m$ as a collection of $n$ parameterized half-spaces where the last half-space in the Poincaré ball interpretation is $x_1 \geq 0$. Represent an element of $\mathcal{M}_n$ as a collection of $m$ parameterized half-spaces where the first half-space in the Poincaré ball interpretation is $x_1 \leq 0$. Then the first $n-1$ half-spaces in the collection of $n$ half-spaces together with last $m-1$ half-spaces in the collection of $m$ half-spaces specify a product of these two elements as an element of $\mathcal{M}_{n+m-2}$.

Now it is easy to prove that a local quantum field theory on hyperbolic space that is invariant with respect to the isometry group generates $d$-dimensional CT.

If such a theory is specified by a local action functional $S$, we can construct a partition function $\psi_n$ that corresponds to the collection of $n$ half-spaces $(\Gamma_1, ..., \Gamma_n)$ by integrating $e^{-S}$ over the fields defined on the complement to the union of half-spaces. We assume that this integral makes sense. The partition function depends on the choice of boundary conditions that should be specified on the boundary of every half-space (on hyperplane) and at infinity; we assume that the boundary conditions at infinity are $\text{Conf}_d$-invariant. We obtain a symmetric functional $\psi_n(\Gamma_1, ..., \Gamma_n, h_1, ..., h_n)$ where $h_i$ belongs to the space of boundary states $\mathcal{H}$. The functionals $\psi_n(h_1, ..., h_n)$ depend on the point of $\mathcal{M}_n$ (because we have assumed that the action is $\text{Conf}_d$-invariant) and depend continuously on this point. Together they specify a map $\Psi$ of the space $\mathcal{M}$ into the direct sum $H$ of tensor powers of $\mathcal{H}^*$. To prove that the $\text{Conf}_d$-invariant quantum field theory on hyperbolic space $H^{d+1}$ induces CT on $S^d$, we should check that this map is a homomorphism. We can do this using standard manipulations with functional integrals that we repeated already in the case of conformal action functionals.

Notice that it is not necessary to start with action functionals. One can use an axiomatic definition of local Euclidean QFT on a manifold $X$ that takes as a starting point partition functions $Z_U$ on some domains in $X$ depending on some data on boundaries of these domains. It is not clear how to formulate full system of axioms for these partition functions (and it seems that some additional data are needed). However, some requirements are clear. In particular, in the case when two domains $U_1$ and $U_2$ have a common component of boundary we should have an expression of the partition function for $U = U_1 \cup U_2$ in terms of partition functions for $U_1$ and $U_2$. For example, let us suppose that the boundary of $U_1$ has two components $\Sigma_1, \Sigma$ and the boundary of $U_2$ has two components $\Sigma$ and $\Sigma_2$ (here $\Sigma$ is the common component). Then the partition function $Z_{U_1}$ is a linear functional on the spaces of boundary states, i.e. an element of $\mathcal{H}^*_1 \otimes \mathcal{H}^*$, and the partition function $Z_{U_2}$ is an element of $\mathcal{H} \otimes \mathcal{H}^*_2$. (Notice that the $\Sigma$ enters the boundaries of $U_1$ and $U_2$ with opposite orientations, therefore corresponding spaces of boundary states are dual. Using the pairing between dual spaces we obtain $Z_U$ as an element of $\mathcal{H}^*_1 \otimes \mathcal{H}^*_2$. (Here $\mathcal{H}_i$ stands for boundary conditions on $\Sigma_i$.) This statement has an obvious generalization to the case of several
components of boundary. The generalization (gluing axiom) can be used to verify that \( \Psi \) is a homomorphism.

We have proven that the Conf\(_{d}\)-invariant quantum field theory on hyperbolic space \( H^{d+1} \) (on Euclidean AdS) induces CT on \( S^d \). Notice that CT in our definition not necessarily has conserved energy-momentum tensor (is not necessarily a CFT). We will argue that such a tensor does exist iff the corresponding quantum field theory on hyperbolic space has the graviton in its spectrum.

Let us assume now that we have a CT on \( S^d \). Can it be obtained from Conf\(_{d}\)-invariant quantum field theory on hyperbolic space \( H^{d+1} \)? It is easy to see that for some (non-standard!) definition of quantum field theory the answer is positive. We will say that a quantum field theory on \( H^{d+1} \) is specified by a symmetric functional \( \psi_n(\Gamma_1, ..., \Gamma_n, h_1, ..., h_n) \) where \( h_i \) belongs to the space of boundary states \( \mathcal{H} \) and \( \Gamma_i \) are non-overlapping half-spaces; we assume that this functional (the partition function on the complement to half-spaces \( \Gamma_i \)) is Conf\(_{d}\)-invariant. We fix a scalar product on the space \( \mathcal{H}^* \). Using this scalar product we can formulate the gluing axiom; if this axiom is satisfied we say that our quantum field theory is weakly local.\(^7\) It is obvious that conformal field theories on \( S^d \) are in one-to-one correspondence with weakly local Conf\(_{d}\)-invariant quantum field theory on hyperbolic space \( H^{d+1} \) (on Euclidean AdS).

One can try to apply the above considerations to the string theory on AdS (or on a product of AdS and a compact manifold). One can consider the partition function of string on the domain in AdS bounded by hyperplanes. (Hyperplanes should be considered as \( D \)-branes or as stacks of \( D \)-branes. For example, in the case of \( AdS_5 \times S^5 \) one could consider \( D5 \)-branes of the form \( AdS_4 \times S^2 \).\(^8\)) It is natural to conjecture that the string theory is weakly local; this conjecture is supported by some heuristic considerations. However, this conjecture does not lead to AdS/CFT correspondence; it leads to a particular case of so called AdS/dCFT correspondence\(^3\).

4 CT basics

We have used an axiomatic approach to CT. Let us discuss the relation of our approach to standard formalism. As in the standard approach, the Lie algebra \( \text{so}(1, d+1) \) acts on the space of states \( \mathcal{H} \). Eigenvectors of the dilation operator \( S \) are called scaling states, corresponding eigenvalues are called anomalous dimensions and denoted by \( \Delta \). We assume that scaling states form a basis in \( \mathcal{H} \) (i.e. every element of \( \mathcal{H} \) can be presented as a convergent series \( \sum c_n e_n \) where \( e_n \) are linearly independent scaling states). Scaling states that are highest weight vectors are called primary states. (Recall that the Lie algebra \( \text{so}(1, d+1) \) acting on \( \mathbb{R}^d \) is generated by translations \( P_\mu \), orthogonal transformations \( M_{\mu \nu} \), dilation \( S \) and conformal boosts \( K_\mu \). In these notations, a primary

\(^7\) The functionals \( \psi_n \) specify a map of the union \( \mathcal{M} \) of the spaces \( \mathcal{M}_n \) into direct sum of vector spaces \( \mathcal{H}^\otimes n \); the gluing axiom is equivalent to the statement that this map is a homomorphism with respect to operations described above.

\(^8\) We do not consider \( D \)-branes as dynamical objects. However, one can formulate an analog of background independence for \( D \)-branes: a variation of \( D \)-brane can be represented as a variation of a state on the original \( D \)-brane.
state $\omega$ is characterized by the condition $K_\mu \omega = 0.$ Every primary state generates a subrepresentation. Other scaling states belonging to this subrepresentation are called descendants. One can construct descendants using the remark that for scaling state $\rho$ with anomalous dimension $\Delta$ the state $P_\mu \rho$ is a scaling state with anomalous dimension $\Delta + 1.$ (This follows from the commutation relation $[S,P_\mu] = P_\mu$)

To describe correlation functions on $\mathbb{R}^d$ in our approach, we notice first of all that in the construction of the action of the semigroup $\mathcal{M}_2$ on $\mathcal{M}_n$ we have singled out the last ball. We can get $n$ actions of $\mathcal{M}_2$ on $\mathcal{M}_n$ adjoining an element of $\mathcal{M}_2$ to other balls. (To get these $n$ actions, we can also combine the action we started with and the action of permutations.) In particular, the direct product of $n$ copies of the semigroup $\mathcal{L} \subset \mathcal{M}_2$ acts on $\mathcal{M}_n.$ This action changes the radii of the balls, but does not change their centers. All these semigroups act also on $\mathcal{H};$ we use the same notation for generators in both cases. By definition, the functional $\psi_n(B_1, ..., B_n, h_1, ..., h_n)$ is compatible with the action of semigroups, in particular

$$\psi_n(e^{-\alpha_1 S}B_1, ..., e^{-\alpha_n S}B_n, e^{-\alpha_1 S}h_1, ..., e^{-\alpha_n S}h_n) = \psi_n(B_1, ..., B_n, h_1, ..., h_n).$$

Working in $\mathbb{R}^d$ we will introduce notation $B(x,r)$ for the ball of radius $r$ with center at the point $x$ parameterized in the standard way. Then it follows from the above formula that

$$\psi_n(B(x_1, 1), ..., B(x_n, 1), h_1, ..., h_n) = \psi_n(B(x_1, r_1), ..., B(x_n, r_n), r_1^{\Delta_1}h_1, ..., r_n^{\Delta_n}h_n). \quad (2)$$

If $h_1, ..., h_n$ are scaling states with anomalous dimensions $\Delta_1, ..., \Delta_n$ we can rewrite this equation in the form

$$\psi_n(B(x_1, 1), ..., B(x_n, 1), h_1, ..., h_n) = \psi_n(B(x_1, r_1), ..., B(x_n, r_n), r_1^{\Delta_1}h_1, ..., r_n^{\Delta_n}h_n). \quad (3)$$

We will use the notation $<\hat{h}_1(x_1)...\hat{h}_n(x_n)>$ for the LHS of (2). Notice that the LHS sometimes is not well defined because the unit balls overlap; to define $<\hat{h}_1(x_1)...\hat{h}_n(x_n)>$ in this case we should use the RHS for small radii $r_i.$ It is always well defined in the case when the points $x_1, ..., x_n$ are distinct.

In the standard terminology, the functions $<\hat{h}_1(x_1)...\hat{h}_n(x_n)>$ are correlation functions for local fields $\hat{h}_i(x)$ corresponding to states $h_i$ in state -operator correspondence. However, we do not need the notion of local field. Notice that knowing the functions $<\hat{h}_1(x_1)...\hat{h}_n(x_n)>$ and the dilation operator $S,$ we can restore the functions $\psi_n$ using (2). The answer is especially simple in the case when $h_i$ are scaling states with anomalous dimensions $\Delta_i,$ then we can use (3). We obtain

$$\psi_n(B(x_1, r_1), ..., B(x_n, r_n), h_1, ..., h_n) = r_1^{-\Delta_1}...r_n^{-\Delta_n} <\hat{h}_1(x_1)...\hat{h}_n(x_n)> \quad (4)$$

This allows us to derive the axioms we are using starting with any approach to CT (at least formally). For example, we can start with the approach of [14]. From the other side, one can derive the properties of correlation functions used in other approaches from our axioms. In particular, one can derive the transformation rules for correlation functions from (2) taking infinitesimally small radii in the RHS.

Let us discuss, for example, the derivation of OPE (operator product expansion). We assume that $h_1, ..., h_n$ are scaling states with anomalous dimensions $\Delta_1, ..., \Delta_n$ and
that the scaling states $e_\alpha$ with anomalous dimensions $\Delta_\alpha$ form a basis of the space $\mathcal{H}$.

Let us suppose that $||x_2 - x_1|| < R$ where $R = \min_{i>2} ||x_i - x_1||$. Then there exists a convergent expression

$$
\langle \hat{h}_1(x_1)...\hat{h}_n(x_n) \rangle = \sum_\alpha C_\alpha(x_2 - x_1) \langle \hat{e}_\alpha(x_1)\hat{h}_3(x_3)...\hat{h}_n(x_n) \rangle
$$

(5)

where $C_\alpha(x)$ are homogeneous functions of degree $\Delta_1 + \Delta_2 - \Delta_\alpha$ (they depend on states $h_1, h_2, e_\alpha$, but do not depend on $h_3, ..., h_n$).

To prove this statement, we apply (1) to the case when $r = 2, s = n - 2, S^{d-1}$ is a sphere of radius $R - \epsilon$ with the center $x_1, B_i$ stands for a small ball with the center at $x_i$. We decompose the element $\Psi_1$ in a series with respect to the basis $e_\alpha$ and apply (1).

Notice that, knowing coefficients $C_\alpha$ for primary fields, we can express these coefficients for descendants. This allows us to rewrite (5) as a sum over primaries.

We have defined the correlation functions on $\mathbb{R}^d$. In a very similar way, one can define correlation functions on $S^d$ and find their relation to correlation functions on $\mathbb{R}^d$ using the fact that expressions $\psi_n(B_1, ..., B_n, h_1, ..., h_n)$ are conformally invariant. Notice, however, that there exists no standard parameterization of a ball in $S^d$, therefore the correlation functions on $S^d$ depend not only on the points $x_1, ..., x_n \in S^d$, but also on some additional data (for example, one can fix orthogonal frames at these points).

## 5 AdS/CT dictionary

We identified the group of conformal transformations of $S^d$ with the group of isometries of hyperbolic space $H^{d+1}$. The Lie algebra $so(1, d+1)$ of this group acts on the space of boundary states. We identify the spaces of boundary states in CT and in AdS; they carry the same representation of $so(1, d+1)$.

Let us discuss the interpretation of the subsemigroup $\mathcal{L}$ in AdS. One can check directly that the generator of this semigroup, the dilation $S$, in the language of the hyperboloid $x_0^2 - ... - x_{d+1}^2 = 1$ can be interpreted as "rotation" in the plane $(x_0, x_{d+1})$, i.e. as the vector field (infinitesimal transformation)

$$
\hat{S} = x_0 \frac{\partial}{\partial x_{d+1}} + x_{d+1} \frac{\partial}{\partial x_0}.
$$

This can be proven without calculations: we should look at geometric properties of these transformations. In particular, it is clear that $\hat{S}$ transforms into itself the straight line in $H^{d+1}$ specified by the equations $x_1 = ... = x_d = 0$. This means that the corresponding transformation of the ideal sphere should have two fixed points ; this is true for dilation $S$.

One can introduce coordinates $\tau, \rho, \Omega_i$ on hyperbolic space using the formulas

$$
x_0 = \frac{\cosh \tau}{\cos \rho}, \quad x_{d+1} = \frac{\sinh \tau}{\cos \rho}, \quad x_i = \tan \rho \Omega_i.
$$

(6)

In these coordinates $\hat{S} = \frac{\partial}{\partial \tau}$. One can say that $\tau$ plays the role of (imaginary) time and the dilation in CT corresponds to the time translation in AdS. Hence scaling...
states correspond to stationary states in AdS, anomalous dimensions to energy levels. Representations of \(so(1,d+1)\), generated by primary states correspond to particle multiplets. In particular, the conserved energy-momentum tensor corresponds to the graviton, because both of them are related to the same representation of \(so(1,d+1)\). This justifies our statement that CT has conserved energy-momentum tensor (is a CFT) iff the AdS theory has the graviton in its spectrum. Conserved currents correspond to gauge particles. (See [5] for more detail).

Notice that our axioms of CT are not satisfactory in dimension 2. However, if we add to them the existence of conserved energy-momentum tensor we obtain two-dimensional CFT at genus zero (it is not clear whether we have modular invariance). The energy-momentum tensor is not a primary field in the standard definition of two-dimensional CFT, but it is a primary field in our definition; it can be considered as highest weight vector of some representation of \(so(1,3)\). There are no propagating gravitons in three-dimensional gravity, however, we can define a graviton in three dimensions as a state that transforms according to the same representation of \(so(1,3)\) as energy-momentum tensor in two dimensions. Then we can claim that a weakly local field theory on \(H^3\) containing graviton induces genus zero two-dimensional CFT.

Let us give geometric interpretation of the semigroup \(L\) in hyperbolic space. Recall that in \(R^d\) and in \(S^d\) this semigroup is specified by the family of balls sitting inside a fixed ball and having common center. In hyperbolic space we have instead a family of half-spaces sitting inside a fixed half-space and orthogonal to a straight line. (Saying that the half-space is orthogonal to a straight line we have in mind that the bounding hyperplane is orthogonal to this line.) This statement will be used later in the proof of formula (7). To prove the statement, we recall that in coordinates \(\tau, \rho, \Omega\) the transformations of the semigroup \(L\) are imaginary time translations \(\tau \rightarrow \tau + \text{const.}\). This gives us an obvious example of the embedding of \(L\) in the hyperbolic space \(M_2\) by half-spaces \(\tau \leq \text{const}\) embedded in the half-space \(\tau \leq 0\) (such a half-space together with half-space \(\tau \geq 0\) determines a point of \(M_2\).) It is clear that in this example half-spaces are orthogonal to the line \(\rho = 0, \Omega;\) we can say that \(L\) consists of shifts along this line. All other examples are obtained from this one by isometries (the group \(\text{Conf}_q\) acts on the space of straight lines transitively). Notice that to give the geometric interpretation of \(L\) we should fix not only half-spaces, but also their parameterizations; the coordinate description gives us the parameterizations we need.

Let us express the partition functions \(\psi_n(\Gamma_1,\ldots,\Gamma_n, h_1,\ldots, h_n)\) on the AdS side in terms of correlation functions of CT. By definition, these functions coincide with partition functions \(\psi_n(B_1,\ldots, B_n, h_1,\ldots, h_n)\) of CT theory (here \(B_i\) are round balls corresponding to half-spaces \(\Gamma_i\)). Therefore it is clear that the expression in terms of correlation functions exists. To describe this expression in more detail, we fix a point \(O\) of hyperbolic space and draw a straight line starting at \(O\) and going in the direction to \(\Gamma_i\); we assume that this line is orthogonal to the hyperplane bounding \(\Gamma_i\). We denote the ideal point of this line by \(x_i\). Then we can prove that

\[
\psi_n(\Gamma_1,\ldots,\Gamma_n, h_1,\ldots, h_n) = e^{-\sum \rho_i \Delta_i} <\hat{h}_1(x_1)\ldots\hat{h}_n(x_n)>
\]

(7)

where \(<\hat{h}_1(x_1)\ldots\hat{h}_n(x_n)>\) stands for correlation function on the sphere \(S^d\). We assume here that \(h_i\) are scaling states with anomalous dimensions \(\Delta_i\). The distance between \(O\) and the hyperplane bounding \(\Gamma_i\) is denoted by \(\rho_i\); this distance can be positive or
negative.

To prove (7) we use the identification of $L$ with family of half-spaces orthogonal to a fixed straight line. The formula follows immediately from (4) and this identification. One can say it is a hyperbolic version of (4).

Notice that the correlation function on the sphere $S^d$ entering the RHS of (7) depends not only on the points $x_1, ..., x_n$, but also on some additional data (on orthogonal frames at these points); these data are specified by the parameterizations of the half-spaces $\Gamma_i$.

One can drop the assumption that $h_i$ are scaling states, then (7) takes the form

$$
\psi_n(\Gamma_1, ..., \Gamma_n, h_1, ..., h_n) = \langle e^{-\rho_1 S}h_1(x_1) ... e^{-\rho_n S}h_n(x_n) \rangle
$$

Notice that we can take $\rho_i \to \infty$ in (7), then in the functional integral for $\psi_n$ we integrate fields defined on the whole hyperbolic space except ”small” domains around $x_i$. (These domains are small in the Poincaré ball, but in hyperbolic space they are half-spaces.) The elements $h_1, ..., h_n$ specify the boundary conditions on the boundaries of these domains. In this form (7) is close, but not identical, to the formulas in GKPW dictionary [2], [3], [7], [5].

To relate (7) to formulas in BDHM dictionary [13], [7] one should calculate $\langle \phi(z), h; \Gamma \rangle$ defined as a partition function on half-space $\Gamma$ with boundary condition $h$ and with insertion of the field $\phi$ at the point $z \in \Gamma$. Let us assume that the distance of $z$ from the point $O$ is equal to $r = r(z)$, the distance of $\Gamma_\rho$ from the point $O$ is equal to $\rho$ and $\Gamma_\rho$ is obtained from $\Gamma_0$ by means of a shift along the straight line connecting $O$ and $z$. (All $\Gamma_\rho$ are orthogonal to this line). We can consider $\langle \phi(z), h; \Gamma_\rho \rangle$ as an $\mathcal{H}^*$ - valued function of $r$ and $\rho$ (a linear functional on $\mathcal{H}$), but we will consider it as $\mathcal{H}$-valued function $F(r, \rho)$. (An inner product in $\mathcal{H}$ specifies an embedding of $\mathcal{H}$ into $\mathcal{H}^*$; we identify $\mathcal{H}$ with the image of this embedding and assume that $\langle \phi(z), h; \Gamma \rangle$ lies in this image.) One can represent this function in the form

$$
F(r, \rho) = e^{(\rho - r)S}h(\phi).
$$

(Due to invariance with respect to isometries the function $F(r, \rho)$ depends only on the difference $r - \rho$. From the other side it follows from the gluing formula that $F(r, \rho')$ can be obtained from $F(r, \rho)$ by means of action of the operator $e^{(\rho' - \rho)S}$.)

Now we can calculate the correlation function $\langle \phi_1(z_1) ... \phi_n(z_n) \rangle$ obtained by insertion of the fields $\phi_1, ..., \phi_n$ at the points $z_1, ..., z_n$ of hyperbolic space in terms of correlation functions of CT on $S^d$. We assume that there exist non-overlapping half-spaces $\Gamma_i$ such that $z_i \in \Gamma_i$ and $\Gamma_i$ is orthogonal to the straight line connecting $O$ and $z_i$. (It is easy to get rid of this assumption.) Then the application of the gluing formula allows us to express the correlation function in terms of $\langle \phi(z), h_i; \Gamma_i \rangle$ and $\psi_n(\Gamma_1, ..., \Gamma_n, h_1, ..., h_n)$. Using (9) and (8) we obtain

$$
\langle \phi_1(z_1) ... \phi_n(z_n) \rangle = \langle e^{-r_1 S}h(\phi_1)(x_1) ... e^{-r_n S}h(\phi_n)(x_n) \rangle
$$

where $r_i$ is the distance from $O$ to $z_i$. This formula generalizes the formulas of [19].

The BDHM dictionary is based on the consideration of asymptotic behavior of the LHS in (10) as $r_i \to \infty$; we see that this behavior is governed by the homogeneous
part of $h(\phi_i)$ having the minimal anomalous dimension; we denote this field by $h(\phi_i)$ and the corresponding dimension by $\Delta_i$. Then (10) gives the asymptotic behavior of the LHS:

$$<\phi_1(z_1)\ldots\phi_n(z_n)> \approx e^{-\sum r_i \Delta_i} <\hat{h_1}(\phi_1)(x_1)\ldots\hat{h_n}(\phi_n)(x_n)>.$$  

(11)

Formula (7) can be used in both directions: from CT to AdS or from AdS to CT. However, if we want to find the CT corresponding to a given theory on AdS it is better to use different techniques. Namely, one should take the domain bounded by two hyperplanes orthogonal to the fixed straight line and an isometric map of one hyperplane onto another hyperplane. We construct a non-compact hyperbolic manifold using the isometry to identify the hyperplanes. It is easy to express the partition function on this manifold (depending on the distance between the hyperplanes and on the element of $SO(1, d)$ specifying the isometry) in terms of the representation of $so(1, d+1)$ in the space of boundary states. Conversely, knowing the partition function we can get the information about this representation (that is the same in AdS and in CT).

6 Minkowski space

We have worked in Euclidean setting; it is not clear how to formulate similar axioms of CT in Minkowski space (or, better, in its compactification $S^{d-1} \times S^1$ or, even better, in the universal cover of the compactification). This is an interesting problem. However, it is important to emphasize that the formula (10) can be analytically continued to Minkowski setting. (Notice, that the RHS of this formula is expressed in terms correlation functions on the sphere $S^d$; one should express these functions in terms of correlation functions on Euclidean space and then analytically continue to Minkowski space. The LHS will give correlation functions on Lorentzian AdS.)

The formula (10) allows us to apply the general theory of the paper [18] to quantum field theories on Lorentzian $(d+1)$-dimensional AdS.

Notice that the starting point of [18] is an algebra of observables $\mathcal{A}$ equipped with the action of commutative Lie group $T$. The elements of this Lie group are denoted by $(t, \vec{x})$ and the corresponding automorphisms by $\alpha(t, \vec{x})$; if $A \in \mathcal{A}$ we can consider a ”field” $A(t, x) = \alpha(t, \vec{x})A$. (Here $t \in \mathbb{R}, \vec{x} \in \mathbb{R}^{d-1}$, the automorphisms $\alpha$ can be interpreted as time and space translations. The algebra $\mathcal{A}$ should be equipped with involution $^*$, automorphisms commute with the involution.) The state $\omega$ on the algebra $\mathcal{A}$ is defined as a linear functional $\omega$ obeying $\omega(1) = 1, \omega(A^* A) \geq 0$. If the state $\omega$ is translationally invariant we can use the GNS (Gelfand-Naimark-Segal) construction to define a (pre) Hilbert space $\mathcal{H}$ with action of the algebra $\mathcal{A}$ and the group $T$; the vector $\Omega$ corresponding to the state $\omega$ is annihilated by the generators of $T$ (by energy and momentum operators). If the energy operator is non-negative one says that $\Omega$ is a physical vacuum. We say that the theory is asymptotically commutative if the $||[A(t, \vec{x}), B]|$ tends to zero for large $\vec{x}$ and is polynomially bounded with respect to $t$ (More precisely, we should require that $\int d\vec{x} ||[A(t, \vec{x}), B]| < g(t)$ where $g(t)$ is a polynomial. Operators $A, B$ in $\mathcal{H}$ correspond to observables $A, B \in \mathcal{A}$.)
It was proven in \cite{18} that starting with asymptotically commutative theory one can define scattering in $d$-dimensional space-time. For a local field theory on $(d+1)$-dimensional Lorentzian AdS one can construct the algebra of observables $\mathcal{A}$ using smeared fields. The group $\text{Conf}_d$ acting on this algebra contains a $d$-dimensional commutative subgroup $T$ generated by $P_\mu$. (Notice that this subgroup cannot be extended to $(d+1)$-dimensional commutative subgroup, hence from the viewpoint of \cite{18} $(d+1)$-dimensional AdS should be related to $d$-dimensional scattering.) It seems that using \eqref{10} one can prove asymptotic commutativity of the theory at hand; moreover, it seems that the same formula implies the coincidence of the scattering in the asymptotically commutative theory on $(d+1)$-dimensional theory on AdS and the scattering of the corresponding CT.

7 Unitary theories

It is well known that unitarity in Minkowski space is equivalent to reflection positivity in the Euclidean approach \cite{15}. It was proven in \cite{16} that similarly unitarity in AdS is equivalent to reflection positivity in Euclidean AdS (in hyperbolic space). In this section we give a definition of reflection positivity in our setting. The relation between reflection positivity in AdS and in CT follows easily from this definition. Let us fix a conformal $(d-1)$-dimensional sphere $S^{d-1}$ in $S^d$ or in $\mathbb{R}^d$. We say that a conformal map $R$ is a reflection with respect to this sphere if it leaves all points of this sphere intact (if the sphere is a hyperplane in $\mathbb{R}^d$, this is an ordinary reflection, otherwise this is an inversion). The map $R$ induces a transformation $h \mapsto h^*$ of the space of states $\mathcal{H}$ (we use this notation, because in the language of state-operator correspondence the operator $\hat{h}^*(x)$ is adjoint to $\hat{h}(x)$).

The reflection positivity condition can be written in the form
\begin{equation}
\psi_2(R(B), B, h^*, h) \geq 0 \tag{12}
\end{equation}
where $B$ denotes a ball inside the fixed sphere. In more general form this condition can be written in the following way
\begin{equation}
\psi_{2n}(R(B_n), ..., R(B_1), B_1, ..., B_n, h_{n*}, ..., h_1^*, h_1, ..., h_n) \geq 0 \tag{13}
\end{equation}
where $B_1, ..., B_n$ are non-overlapping balls inside the fixed sphere.

It is obvious that the reflection positivity condition in CT is equivalent to a similar condition in Euclidean AdS (in hyperbolic space). Instead of fixed sphere and reflection with respect to this sphere we should talk about fixed hyperplane and reflection with respect to this hyperplane, instead of balls we should consider half-spaces.

It seems that it is possible to check that the correlation functions in CT with reflection positivity property satisfy all axioms for Schwinger functions (Euclidean Green functions) of unitary conformal field theory in the sense of \cite{14}.

8 de Sitter space

One can construct the spaces $\mathcal{M}_n$ in the framework of de Sitter space, however, it is not clear that this construction can be used to derive the dS/CFT correspondence \cite{17}.
Both hyperbolic space and de Sitter space can be defined by the equation

\[ <x, x> = c \]

in the space \( \mathbb{R}^{d+2} \). (Here \( <x, x> = x_0^2 - x_1^2 - \ldots - x_{d+1}^2 \), for hyperbolic space \( c = 1 \), for de Sitter space \( c = -1 \).) Half-spaces in the hyperbolic space can be specified by the formula \( <a, x> \geq 0 \) where \( <a, a> < 0 \). We can use the same formula to define half-spaces in de Sitter space. Then we have one-to-one correspondence between half-spaces in these two spaces; this leads to the construction of \( \mathcal{M}_n \) in terms of de Sitter space. 

(To construct the correspondence between parameterized half-spaces we should notice that the correspondence between half-spaces commutes with the action of \( \text{Conf}_d \).) More precisely, in de Sitter space we should work with collections of half-spaces satisfying the condition that the corresponding hyperplanes have compact intersections pairwise. The space of these collections is not precisely \( \mathcal{M}_n \), but the closure of it is \( \mathcal{M}_n \). This follows from the remark that non-intersecting, but not parallel hyperplanes in hyperbolic space correspond to hyperplanes on de Sitter space that have compact intersections (if the intersection is compact it is homeomorphic to a sphere). Alas, one cannot directly apply the construction of Section 3 to the de Sitter space.

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