Robust data-driven state-feedback design

Julian Berberich¹, Anne Romer¹, Carsten W. Scherer², and Frank Allgöwer¹

Abstract—We consider the problem of designing robust state-feedback controllers for discrete-time linear time-invariant systems, based directly on measured data. The proposed design procedures require no model knowledge, but only a single open-loop data trajectory, which may be affected by noise. First, a data-driven characterization of the uncertain class of closed-loop matrices under state-feedback is derived. By considering this parametrization in the robust control framework, we design data-driven state-feedback gains with guarantees on stability and performance, containing, e.g., the $H_\infty$-control problem as a special case. Further, we show how the proposed framework can be extended to take partial model knowledge into account. The validity of the proposed approach is illustrated via a numerical example.

I. INTRODUCTION

Recently, the design of controllers directly from measured data has received increasing interest [1], [2]. While established methods, e.g., those based on reinforcement learning, rarely address closed-loop guarantees, there has been a renewed effort to provide such guarantees using novel statistical estimation techniques [3], [4], [5], [6]. A potential alternative is robust control with prior set membership identification [7], which is however well-known to be computationally demanding. In general, providing non-conservative end-to-end guarantees for the closed loop using noisy data of finite length is an open problem, even if the data is generated by a linear time-invariant (LTI) system.

A promising approach towards this goal relies on behavioral systems theory. In [8], it was proven that the vector space of all input-output trajectories of an LTI system is spanned by time-shifts of a single measured trajectory, given that the respective input signal is persistently exciting. Thus, a single data trajectory can be used to characterize an LTI system, without any prior identification steps. Recently, there have been various contributions which consider this result in the context of data-driven system analysis and control, including dissipativity verification from measured data [9] or an extension of [8] to certain classes of nonlinear systems [10]. Moreover, the recent work [11] derives a simple data-dependent closed-loop parametrization of LTI systems under state-feedback. This parametrization is used to solve various control problems from data, including stabilization and linear-quadratic regulation. However, no meaningful guarantees were given in the presence of noisy data.

It is the goal of the present paper to provide non-conservative end-to-end guarantees for data-driven control based on a single noisy data trajectory of finite length. This is achieved by extending the approach of [11] to account for noise and applying robust control techniques to the resulting uncertain system class. Another recent paper [12] considers data-driven analysis and control with not persistently exciting data. In particular, it is shown for noise-free data that certain control problems can be solved directly from data, even if the system cannot be uniquely identified, thus illustrating advantages of direct data-driven control. Similarly, the results of the present paper do not require persistence of excitation explicitly. Moreover, our results lead to simple design procedures for direct data-driven control with desirable closed-loop guarantees, and are thus a promising alternative to identification-based control.

The paper is structured as follows. After stating the problem formulation in Section II, we use noisy data to describe the uncertain closed loop under state-feedback, and we apply known robust control methods to design controllers with stability and performance guarantees in Section III. Moreover, we extend the proposed, purely data-driven approach to systems with mixed data-driven and model-based components. In Section IV we apply the robust state-feedback design techniques successfully to an unstable example system. The paper is concluded in Section V.

II. PRELIMINARIES

We denote the $n \times n$ identity matrix by $I_n$, where the index is omitted if the dimension is clear from the context. For a matrix $A$ with full row rank, we denote by $A^\dagger$ its Moore-Penrose inverse. Further, $A^\perp$ denotes a matrix containing a basis of the kernel of $A$. We write $\ell_2$ for the space of square-summable sequences. In a linear matrix inequality (LMI), $*$ represents blocks, which can be inferred from symmetry. Moreover, we define, for elements $\{x_k\}_{k=1}^{i+L+N-2}$ of a sequence $x$, the Hankel matrix

$$X_{i,L}^N := \begin{bmatrix} x_i & x_{i+1} & \ldots & x_{i+N-1} \\ x_{i+1} & x_{i+2} & \ldots & x_{i+N} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i+L-1} & x_{i+L} & \ldots & x_{i+L+N-2} \end{bmatrix}.$$ 

That is, the matrix $X_{i,L}^N$ starts with the element $x_i$ and has $L$ rows and $N$ columns. As a shorthand notation, we abbreviate
$N$-windows of $x$, starting at $i = 0$ and $i = 1$, by
\[
X = X_{0,1} = \begin{bmatrix} x_0 & x_1 & \ldots & x_{N-1} \end{bmatrix},
\]
\[
X_+ = X_{1,1} = \begin{bmatrix} x_1 & x_2 & \ldots & x_N \end{bmatrix},
\]
respectively. In the present paper, we consider LTI systems of the form
\[
\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} A_{tr} & B_w & B_{tr} \\ C & D_w & D \end{bmatrix} \begin{bmatrix} x_k \\ u_k \\ w_k \end{bmatrix},
\]
where $x_k \in \mathbb{R}^n$ is the state, $w_k \in \mathbb{R}^{m_w}$ is the disturbance, $u_k \in \mathbb{R}^m$ is the control input, and $z_k \in \mathbb{R}^p$ is the performance output. We design state-feedback controllers $u_k = K x_k$ to control the system $\{1\}$. Our design procedures are purely data-driven and do not require knowledge of the true system matrices $A_{tr}, B_{tr}$. We do, however, assume that the matrices $B_w, C, D_w, D$ are known. For our purposes, $B_w$ is essentially a parameter to model the influence of the disturbance, whereas $C, D_w, D$ constitute a user choice for performance.

**Definition 1.** The sequence $\{x_k, u_k\}_{k=0}^{N-1}$ is called persistently exciting if the matrix $\begin{bmatrix} X \\ U \end{bmatrix}$ has full row rank.

Definition $\{1\}$ is a classical definition of persistence of excitation and is well-known to have many important implications, in particular in the area of subspace identification. According to $\{8\}$, controllability and a certain rank property of the input are sufficient for persistence of excitation.

**Theorem 2 (\{8, Corollary 1\}).** If $(A_{tr}, B_{tr}, B_w)$ is controllable and the matrix
\[
\begin{bmatrix} W_{N-N} \\ 0_{n+1} \\ \alpha_{n+1-N} \end{bmatrix}
\]
has full row rank, then $\{x_k, u_k\}_{k=0}^{N-1}$ is persistently exciting.

In $\{11\}$, it is shown how a single, persistently exciting open-loop trajectory can be employed to recover the system matrices of an LTI system. Furthermore, a linear parametrization of the closed loop under state-feedback is derived, depending also on a single open-loop data trajectory. It is the contribution of the present paper to extend the framework of $\{11\}$ in order to provide robust stability and performance guarantees in the presence of noise. In contrast to $\{11\}$, persistence of excitation will generally not be required for our results.

Throughout this paper, we consider the following scenario: From simulation or an experiment, a single open-loop input-state sequence $\{x_k, u_k\}_{k=0}^N$ is obtained as a trajectory of $\{1\}$ for some unknown disturbance $\{\hat{w}_k\}_{k=0}^{N-1}$. This trajectory is used directly for robust controller design, without prior system identification. The only available information on the disturbance realization is the following bound on the matrix
\[
\hat{W} = [\hat{w}_0 \ \hat{w}_1 \ \ldots \ \hat{w}_{N-1}].
\]

**Assumption 3.** The matrix $\hat{W}$ is an element of
\[
\mathcal{W} = \left\{ W \in \mathbb{R}^{m_w \times N} \mid \begin{bmatrix} W^T & Q_w & S_w \\ I & S_w^T & R_w \end{bmatrix} W \succeq 0 \right\},
\]
for some known matrices $Q_w, S_w \in \mathbb{R}^{m_w \times N}$, $R_w \in \mathbb{R}^{N \times N}$ with $R_w \succ 0$.

Through Assumption $\{3\}$ it is assumed that the unknown disturbance realization, which affects the measured data, lies in some known set which is described by a quadratic matrix inequality. Implicitly, $W \in \mathcal{W}$ implies a quadratic bound on the sequence $\{w_k\}_{k=1}^N$ and encompasses many practical bounds as special cases. For instance, if the maximal singular value of $W$ is bounded as $\sigma_{\text{max}}(W) \leq \hat{w}$, then $W \in \mathcal{W}$ holds with $Q_w = -I$, $S_w = 0$, $R_w = \hat{w}^2 I$. More generally, a description of the form $W \in \mathcal{W}$ provides a flexible framework to model general noise signals, in particular when multiple quadratic matrix inequalities are combined. It is an interesting aspect for future research to derive suitable matrices $Q_w, S_w, R_w$ for different, practically relevant scenarios such as norm bounds on the sequence $\{w_k\}_{k=0}^{N-1}$.

### III. Data-driven state-feedback

In this section, we consider the design of state-feedback gains, based directly on measured data which is perturbed by a disturbance satisfying Assumption $\{3\}$ First, we derive a data-driven characterization of the uncertain closed loop, using a single open-loop data trajectory. Thereafter, we apply known robust control methods to this parametrization in order to design state-feedback controllers which guarantee stability and performance for all closed-loop matrices that are consistent with the measured data. Finally, we extend the proposed framework to systems with mixed data-driven and model-based components.

#### A. Uncertain closed-loop parametrization

In the following, we extend $\{11\}$ by characterizing the closed-loop dynamics of $\{1\}$ under state-feedback, using noisy measurements. Let $\{x_k, u_k\}_{k=0}^N$ be a measured trajectory of $\{1\}$, corresponding to an unknown disturbance realization $W$. We define $\Sigma_{A,B}$ as the set of all pairs $(A, B)$ that are consistent with the data $\{x_k, u_k\}_{k=0}^N$ for some noise instance $W \in \mathcal{W}$, i.e.,
\[
\Sigma_{A,B} = \{ (A, B) \mid X_+ = AX + BU + B_w W, W \in \mathcal{W} \}.
\]
Using fixed data matrices $X$ and $U$, $\Sigma_{A,B}$ parametrizes the unknown system matrices $A$ and $B$ via $W$. By assumption, the true disturbance realization $W$ satisfies $X_+ = A_{tr}X + B_{tr}U + B_w W$ and, therefore, the true pair $(A_{tr}, B_{tr})$ is an element of $\Sigma_{A,B}$. Furthermore, for some state-feedback gain $K$, we define the set of closed-loop matrices that are consistent with the data as
\[
\Sigma_{A,B}^K = \{ A_K \mid A_K = A + BK, (A, B) \in \Sigma_{A,B} \}.
\]
In the following, we show that an exact parametrization of $\Sigma_{A,B}^K$ can be constructed directly from open-loop data. To
this end, for some matrix $G \in \mathbb{R}^{N \times n}$, we define $A_G$ as the set of matrices $A_G \in \mathbb{R}^{n \times n}$ such that
\begin{equation}
A_G = (X_+ - B_u W) G,
\end{equation}
for some $W \in \mathcal{W}$ satisfying
\begin{equation}
(X_+ - B_u W) \begin{bmatrix} X \\ U \end{bmatrix} = 0.
\end{equation}

Theorem 4. If $G \in \mathbb{R}^{N \times n}$ and $K \in \mathbb{R}^{m \times n}$ satisfy
\begin{equation}
\begin{bmatrix} X \\ U \end{bmatrix} G = \begin{bmatrix} I \\ K \end{bmatrix},
\end{equation}
then $\Sigma_{A,B}^K = A_G$.

Proof. First, we note that the constraint (3) is equivalent to the implication
\begin{equation}
\begin{bmatrix} X \\ U \end{bmatrix} \tilde{V} = 0 \Rightarrow (X_+ - B_u W) \tilde{V} = 0,
\end{equation}
for any matrix $\tilde{V}$ with $N$ rows. By the Fredholm alternative, this is in turn equivalent to the existence of a solution $V$ to the system of linear equations
\begin{equation}
V \begin{bmatrix} X \\ U \end{bmatrix} = X_+ - B_u W.
\end{equation}

Proof of $\Sigma_{A,B}^K \subseteq A_G$: Let $A_K \in \Sigma_{A,B}^K$, i.e., there exist matrices $A, B$ as well as $W \in \mathcal{W}$ such that
\begin{equation}
A_K = A + BK,
\end{equation}

\begin{equation}
X_+ = AX + BU + B_u W.
\end{equation}

Then, it follows that
\begin{equation}
A_K \begin{bmatrix} A \\ B \end{bmatrix} + BK = \begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} G
\end{equation}
and
\begin{equation}
(\tilde{X}_+ - B_u W) G.
\end{equation}

It remains to show that $W$ satisfies (3) or, equivalently, there exists $X$ such that (5) holds. It follows directly from (7) that $V = [A \\ B]$ solves (5), which thus proves $A_K \in A_G$.

Proof of $A_G \subseteq \Sigma_{A,B}^K$: Let $A_G \in A_G$, i.e., there exists $W \in \mathcal{W}$ such that (2) and (3) hold. We need to show the existence of matrices $A, B$ as well as $W \in \mathcal{W}$ such that
\begin{equation}
A + BK = (X_+ - B_u W) G,
\end{equation}

\begin{equation}
X_+ = AX + BU + B_u W.
\end{equation}

Letting $\tilde{W} = W$, these equations are equivalent to
\begin{equation}
\begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} = (X_+ - B_u W) [I \ G].
\end{equation}

Using (2), this is in turn equivalent to
\begin{equation}
\begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} [I \ G] = (X_+ - B_u W) [I \ G].
\end{equation}

Since $A_G \in A_G$, there exists a solution $V$ to (5). Hence, the choice $[A \\ B] = V$ satisfies (5), which implies $A_G \in \Sigma_{A,B}^K$.

Theorem 4 provides an exact parametrization of the uncertain closed loop under a fixed state-feedback $K$, using a single open-loop trajectory of the unknown system. In particular, no closed-loop measurements and no model knowledge are required to construct the set $A_G$, which parametrizes the uncertain closed loop. This set relies on fixed data matrices $X$ and $U$, which are obtained offline, and is parametrized via the disturbance $W \in \mathcal{W}$ satisfying (3). The equation (3) ensures that the matrices in $A_G$ contain only those matrices $A, B$ satisfying the system dynamics.

In general, the condition (3) only requires that $X$ has full row rank, but not necessarily that the data are persistently exciting. Nevertheless, if $\{x_k, u_k\}_{k=0}^{N-1}$ is persistently exciting, then, for any state-feedback $K$, (4) can be solved for $G$, i.e., any possible closed-loop matrix can be constructed. Equivalently, the set of all $A_G$ with $G \in \mathbb{R}^{N \times n}$ satisfying $XG = I$ is equal to the set of all possible closed-loop matrices under state-feedback.

Corollary 5. If $\{x_k, u_k\}_{k=0}^{N-1}$ is persistently exciting, then it holds that
\begin{equation}
\{A_G \in A_G \mid G \in \mathbb{R}^{N \times n}, XG = I\}
= \{A_K \in \Sigma_{A,B}^K \mid K \in \mathbb{R}^{m \times n}\}.
\end{equation}

Proof. This follows directly from Theorem 4.

If the data are not persistently exciting, then (9) holds with “\subseteq” instead of “=", since only closed-loop dynamics with feedback gains $K$ of the form $K = UG$ are captured in the description. Corollary 5 suggests that the set $A_G$ can be employed to design controllers with robustness guarantees for all closed-loop matrices in $\Sigma_{A,B}^K$, by optimizing over the parameter $G$ instead of the gain $K$. However, the disturbance $W$ is not only restricted by $W \in \mathcal{W}$ but also via the affine constraint (3) and therefore, the construction of $A_G$ requires the computation of the kernel of $[X^T \ U^T]$, which may be undesirable from a numerical viewpoint.

In Sections III-B and III-C, we employ a superset of $A_G$, which neglects the constraint (3), to derive simple robust controller design procedures for closed-loop stability and performance, respectively.

B. Robust state-feedback for stability

In this section, we apply known robust control methods to render all matrices in $A_G$ stable. To facilitate the design, we consider
\begin{equation}
A_G^c = \{A_G \mid A_G = (X_+ - B_u W) G, W \in \mathcal{W}\},
\end{equation}
which is a superset of the uncertain closed loop $A_G$, i.e., $A_G \subseteq A_G^c$. The difference between $A_G^c$ and $A_G$ is that the latter considers only those disturbances $W \in \mathcal{W}$, which satisfy the $n$ constraints defined by (3). Hence, $A_G^c$ is in general larger than $A_G$ and, therefore, controller design based on $A_G^c$ is generally more conservative than design based on $A_G$. Nevertheless, $A_G^c$ admits a simpler parametrization and can be translated directly into a standard robust control format. Further, as we will see in Section IV, considering
instead of $\mathcal{A}_G$ leads to meaningful robust controllers also for practical examples.

In the following, we exploit that the parametrization $\mathcal{A}_G^\circ$ is equivalent to a particular lower linear fractional transformation (LFT) (compare [13, Chapter 10]). To be more precise, the matrices in $\mathcal{A}_G^\circ$ can be described as a lower LFT of a nominal closed-loop system depending on $G$ with the disturbance $W$, i.e.,

\[
\begin{bmatrix}
    x_{k+1} \\
    \tilde{z}_k
\end{bmatrix} =
\begin{bmatrix}
    X + G & B_w \\
    -G & 0
\end{bmatrix}
\begin{bmatrix}
    x_k \\
    \tilde{w}_k
\end{bmatrix},
\]

(11)

where $W \in \mathcal{W}$. It follows from Theorem 3 that, if $G$ satisfies $XG = I$, the above LFT contains all potential closed-loop systems under control with state-feedback $K = UG$. The following result exploits this fact by using robust control methods to design a stabilizing controller parameter $G$ for the LFT (11), which hence stabilizes all elements of $\Sigma^K_{A,B}$.

**Corollary 6.** If there exist $X > 0$, $G \in \mathbb{R}^{N \times n}$ such that

\[
XG = I
\]

(12)
as well as

\[
\begin{bmatrix}
    * & * & * & * \\
    * & * & * & * \\
    * & * & * & *
\end{bmatrix}^T
\begin{bmatrix}
    -X & 0 & 0 & 0 \\
    0 & X & 0 & 0 \\
    0 & 0 & Q_w & S_w \\
    0 & 0 & S_w^\top & R_w
\end{bmatrix}
\begin{bmatrix}
    I & 0 & 0 & 0 \\
    X + G & B_w & 0 & 0 \\
    0 & -G & 0 & 0 \\
    0 & 0 & I & 0
\end{bmatrix} < 0,
\]

(13)

then $A + BK$ with $K = UG$ is stable for all $(A, B) \in \Sigma_{A,B}$.

**Proof.** This follows from an application of known robust control methods to the system (11) (cf. [14], [15]).

Corollary 6 applies robust control methods to robustly stabilize the uncertain system class $\mathcal{A}_G^\circ$, and thereby, according to Theorem 3, all closed-loop matrices that are consistent with the data. Similar to Theorem 3, it does not require persistently exciting data explicitly. Thus, it may be possible to find a controller $K$ which stabilizes all elements of $\Sigma^K_{A,B}$, even if persistence of excitation does not hold, i.e., if the data is not sufficiently rich for system identification. Similar phenomena were analyzed for system analysis and control from noise-free data in [12], where also full row rank of $X$ was sufficient to design stabilizing controllers from data.

Nevertheless, persistence of excitation is required for equality in (9), i.e., to construct any closed-loop system (cf. Corollary 5), and thus, it enhances feasibility of (13). In particular, if the data are persistently exciting and there exists a controller which stabilizes all matrices in $\mathcal{A}_G^\circ$ with a common Lyapunov function, then (12) and (13) are feasible. Hence, Corollary 6 contains two main sources of conservatism: a) the difference between $\mathcal{A}_G^\circ$ and $\Sigma^K_{A,B}$ and b) the fact that a common Lyapunov function is employed for stabilization, similar to simple model-based robust controller design methods. Nevertheless, Corollary 6 provides computationally tractable conditions, based directly on open-loop data, to design controllers with stability guarantees.

**Remark 7.** Although (13) is not an LMI, it is routine to transform it into one following the same steps as in model-based robust state-feedback design (compare [14], [15]). To be more precise, after performing a congruence transformation on (13) with $\text{diag}(X^{-1}, I)$ and applying the Schur complement twice, the nonlinear matrix inequality (13) becomes an LMI in the variables $Y = X^{-1}, M = GX^{-1}$.

Further, multiplying (12) by $Y$ from the right leads to the linear equality constraint $XM = Y$, which can be solved together with the LMI using standard solvers. The stabilizing state-feedback gain can then be recovered as $K = UM^{-1}$.

Corollary 6 suggests a valuable alternative to sequential system identification and stabilizing robust control. In particular, in the presence of deterministic noise, identification-based methods are usually either computationally intractable, overly conservative, or they admit no guarantees from finite data. As an alternative, one may consider a stochastic setting, where recent work has addressed finite-time guarantees on system identification with sequential robust control [3], [4], [5], [6]. These results are based on sophisticated statistical analysis and many of them rely on restrictive assumptions, such as the availability of multiple independent data trajectories, each of which only supplies one data tuple to the estimator. In contrast, our approach relies on simple matrix manipulations combined with existing robust control methods and requires only a single data trajectory.

**Remark 8.** For the state-feedback stabilization problem under additive state measurement noise, [11] provides sufficient conditions for closed-loop stability. However, this result relies on assumptions that cannot be verified from measured data. Moreover, in contrast to the approach of [11], an extension of Corollary 6 to more general (robust) control objectives is straightforward.

C. Robust state-feedback for performance

Next, we consider the system (1) including the performance channel $w \mapsto z$. The goal is to use data $\{x_k, u_k\}_{k=0}^\infty$ of (1), affected by noise satisfying Assumption 3, in order to design $K$ such that the closed-loop matrix $A_K$ is stable and the following quadratic performance specification on $\mathcal{Y}$ is guaranteed for all $A_K \in \Sigma^K_{A,B}$.

**Definition 9.** We say that the closed-loop system (1) with state feedback $u_k = Kz_k$ satisfies quadratic performance with index $P = \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}$, where $R \succeq 0$, if there exists an $\varepsilon > 0$ such that

\[
\sum_{k=0}^\infty \begin{bmatrix} w_k \\ z_k \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} w_k \\ z_k \end{bmatrix} \leq -\varepsilon \sum_{k=0}^\infty w_k^\top w_k
\]

(14)

for all $w \in \ell_2$.

Important special cases of Definition 9 are $Q = -\gamma^2I, S = 0, R = I$ for the $H_\infty$-control problem and $Q = 0, S = -I, R = 0$ for closed-loop strict passivity. Note that the disturbance $w$ enters the present problem setting
in two different ways. First, it perturbs the measured input-state trajectory during the initial data generation for which \( w \) is bounded as \( W \in \mathcal{W} \). Second, it enters the control objective of achieving quadratic performance of the channel \( w \mapsto z \). For instance, a desired \( \mathcal{H}_\infty \)-performance of this channel corresponds to a robustness objective for the closed loop with respect to noise.

Similar to the previous section, the uncertain closed loop of (1), including the performance channel \( w \mapsto z \), can be written as a lower LFT. To be more precise, for a state-feedback gain \( K = UG \), where \( G \) satisfies \( I = XG \), a superset of the uncertain closed loop from \( w \) to \( z \) can be parametrized as

\[
\begin{bmatrix}
\frac{x_{k+1}}{z_k} \\
\tilde{w}_k = W\tilde{z}_k
\end{bmatrix} =
\begin{bmatrix}
X + G \\
C + DG
\end{bmatrix}
\begin{bmatrix}
B_w & B_w & 0 & 0 \\
0 & 0 & 0 & 0 \\
\tilde{z}_k & \tilde{w}_k
\end{bmatrix}
\begin{bmatrix}
x_k \\
\tilde{w}_k
\end{bmatrix},
\]

for \( W \in \mathcal{W} \). The above system contains two disturbance inputs: \( w \) to model the performance channel \( w \mapsto z \), representing the control objective of closed-loop quadratic performance, and \( \tilde{w} \) to model the uncertainty originating from the noisy data, similar to the LFT (11). The following result derives state-feedback controllers with robust performance for (15).

**Corollary 10.** If there exist \( X > 0, G \in \mathbb{R}^{n \times n}, \lambda > 0 \), such that (16) and

\[
XG = I
\]

hold, then, for any \((A, B) \in \Sigma_{A,B}\),

i) \( A + BK \) with \( K = UG \) is stable,

ii) (11) with \( u_k = Kx_k \) satisfies quadratic performance with index \( P \).

**Proof.** The result follows from known robust control methods (cf. [14], [15]). \( \square \)

Corollary 10 applies known robust control methods to guarantee closed-loop quadratic performance for all systems in \( \mathcal{A}_g \). And, therefore, for all systems \( \Sigma_{A,B} \subseteq \mathcal{A}_g \) that are consistent with the measured data. Hence, according to Definition 9, the closed-loop channel \( w \mapsto z \) satisfies quadratic performance over an infinite time-horizon for arbitrary disturbance inputs which are not required to satisfy a bound of the form \( W \in \mathcal{W} \). In order to achieve this goal, a data trajectory of finite length and the (finite-horizon) assumption \( \hat{W} \in \mathcal{W} \) on the disturbance generating the data are sufficient. It is straightforward to extend the above result to exogenous inputs \( w^p \) for the performance channel \( w^p \mapsto z \) which are different from the noise perturbing the initial data trajectory, i.e., \( w^p \neq w \).

If the scalar multiplier \( \lambda \) is fixed, then (16) can be transformed into an LMI, following the same steps as in Remark 7. Thus, the proposed feasibility problem can be solved via a line-search over \( \lambda \).

**D. Systems with partial model knowledge**

We conclude the section by presenting an extension of the proposed framework to systems with mixed data-driven and model-based components. To this end, we consider systems of the form

\[
\begin{bmatrix}
\frac{x_{k+1}}{\tilde{x}_{k+1}} \\
\frac{z_k}{\tilde{z}_k} \\
\frac{\tilde{w}_k}{w_k}
\end{bmatrix} =
\begin{bmatrix}
A_1 & A_2 & B_{w1} & B_1 \\
A_3 & A_4 & B_{w2} & B_2 \\
C_1 & C_2 & D & \\
\end{bmatrix}
\begin{bmatrix}
x_k \\
\tilde{x}_k \\
\tilde{w}_k \\
\end{bmatrix},
\]

where the matrices \( A_1 \) and \( B_1 \) are unknown, but all other matrices occurring in (18) are known. Further, a single openloop data trajectory \( \{x_k, \tilde{x}_k, u_k\}_{k=0}^{N-1} \), which is perturbed by some unknown disturbance realization \( \tilde{W} \in \mathcal{W} \), is available. In the following, we consider the closed loop of (18) under control with state-feedback \( u_k = K_1x_k + K_2\tilde{x}_k \). Suppose there exist matrices \( G_1 \in \mathbb{R}^{n \times n}, G_2 \in \mathbb{R}^{n \times \tilde{n}} \), where \( n \) and \( \tilde{n} \) are the dimensions of \( x_k \) and \( \tilde{x}_k \), respectively, such that

\[
\begin{bmatrix}
I & 0 \\
K_1 & K_2
\end{bmatrix} =
\begin{bmatrix}
X & U \\
\end{bmatrix}
\begin{bmatrix}
G_1 & G_2 \\
\end{bmatrix}.
\]

Multiplying (19) from the left by \([A_1 \ B_1]\), we obtain

\[
A_1 + B_1K_1 = (X + A_2\tilde{x} - B_{w1}W)G_1, \\
B_1K_2 = (X + A_2\tilde{x} - B_{w1}W)G_2.
\]

These relations allow us to replace all occurrences of the unknown matrices \( A_1 \) and \( B_1 \) in the closed-loop dynamics. Thus, following the same steps as in the previous sections, we obtain the LFT (20) with \( W \in \mathcal{W} \), which parametrizes a superset of the uncertain closed loop dynamics of (18) under the above state-feedback. Note that this LFT depends only on known matrices and the open-loop data trajectory \( \{x_k, \tilde{x}_k, u_k\}_{k=0}^{N-1} \). The structure of (20) resembles that of the LFT (13) and therefore, robust controllers for the mixed system (18) can be derived by proceeding as in Section III-C.

In contrast to the previous sections, the condition (19) requires not only that \( X \) has full row rank but also that

\[
\begin{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
X + G & C + DG \\
C + DG & -G
\end{bmatrix}
\begin{bmatrix}
B_w & I \\
I & 0
\end{bmatrix}
\end{bmatrix}
\end{bmatrix} < 0
\]
$N \geq n + \bar{n}$. Moreover, if $[X^T \quad U^T]^T$ has full row rank, i.e., the data-driven component of [18] is persistently exciting, and $N \geq n + \bar{n}$, then, for any matrices $K_1$ and $K_2$, there exist matrices $G_1$ and $G_2$ satisfying [19], i.e., any controller can be constructed. Finally, it can be seen from (20) that a trajectory of the state $\hat{x}$ corresponding to the model-based component of [18] is not required if $A_2 = 0$.

**Remark 11.** Our original motivation for considering the above mixed data-driven and model-based configuration comes from $H_{\infty}$-loop-shaping: The $H_{\infty}$-control problem is usually not solved for the performance channel $w \mapsto z$ directly, but rather for the channel $w \mapsto z^f$, where $z^f$ is the output of a filter with input $z$. In this scenario, the known components of [18] are mainly that of the filter, whereas the unknown matrices $(A_1, B_1)$ are equal to $(A_{tr}, B_{tr})$ from [1]. By iteratively refining the filter dynamics and solving the robust performance design problem for the LFT (20), we can thus systematically perform loop-shaping for the system [1], without knowledge of $(A_{tr}, B_{tr})$.

**IV. Example**

In this section, we apply the results of Section III to the robust $H_{\infty}$-control problem for an unstable example system.

We consider System (1) with

$$A_{tr} = \begin{bmatrix} -0.5 & 1.4 & 0.4 \\ -0.9 & 0.3 & -1.5 \\ 1.1 & 1 & -0.4 \end{bmatrix}, \quad B_{tr} = \begin{bmatrix} 0.1 & -0.3 \\ -0.1 & -0.7 \\ 0.7 & -1 \end{bmatrix},$$

$$B_w = I_3, \quad C_z = I_3, \quad D_z = 0, \quad D_{zw} = 0,$$

where it is assumed that $A_{tr}$ and $B_{tr}$ are not available. We generate data $\{x_k, u_k\}_{k=0}^N$ of length $N = 20$ by sampling the input $u_k$ uniformly from $[-1, 1]^2$ and the disturbance $\tilde{w}$ uniformly from the ball $\|\tilde{w}\|_2 \leq \tilde{w}$, where $\tilde{w} = 0.02$. This implies the disturbance bound $\tilde{W} \in \mathcal{W}$ for $Q_w = -I, S_w = 0, R_w = \tilde{w}^2 I$. In the following, we compute a state-feedback gain via Corollary 10 to achieve robust closed-loop quadratic performance with index $P = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}$ for a possibly small $\gamma > 0$, i.e., a small $H_{\infty}$-norm of $w \mapsto z$.

Following the procedure described in Remark 7, we verify that (16) and (17) are feasible for $\gamma = 2.4$ and we obtain a corresponding controller as $K = \begin{bmatrix} -2.45 & -1.29 & -2.4 \\ -0.61 & -0.03 & -2.18 \end{bmatrix}$, which leads to a closed-loop $H_{\infty}$-norm of $2.3$. In contrast, the minimal achievable $H_{\infty}$-norm using a nominal (model-based) state-feedback is $2.2$. Thus, the proposed approach yields a controller with guaranteed performance close to the ideal case with full model knowledge, despite noisy measurements.

In the following, we analyze the influence of the data length $N$ on the feasibility of (16) and (17) for the above design problem. To keep the signal-to-noise ratio (approximately) constant, we modify the bound $\tilde{w}$ of the noise generating the data linearly with $N$, i.e., $\tilde{w} = \frac{20}{\sqrt{N}}$. For each data horizon $4 \leq N \leq 20$, we perform 100 experiments to generate data for the controller design, each with different (random) inputs $u$ and disturbances $\tilde{w}$ as described above. Figure 1 shows the number of successful designs depending on $N$. It can be observed that the feasibility of (16) and (17) is enhanced if $N$ increases, and $N \geq 15$ suffices to successfully design a controller from 100 out of 100 experiments. Moreover, even for $N$ as low as 4, the design is successful in more than 50% of the scenarios.

![Fig. 1. Number of successful designs for which (16) and (17) are feasible for the present example, depending on the data length $N$. For each horizon $N$, 100 experiments are carried out with varying random inputs and disturbances to generate data for controller design according to Corollary 10.](image)

Finally, we comment on the computational complexity of the feasibility problem stated in Corollary 10. After its reformulation (cf. Remark 7), the problem contains an LMI with $2(n + m_w) + p_z + N$ rows, i.e., it is of size $35 \times 35$ for the above example with $N = 20$, as well as an equality constraint of size $n \times n = 9$. Moreover, the matrix variables $Y$ and $M$ are of size $n \times n = 3 \times 3$ and $N \times n = 20 \times 3$, respectively. The complexity of standard LMI solvers scales cubically with the number of decision variables.

1Note that $Y$ is symmetric and has therefore only $\frac{n(n+1)}{2}$ free decision variables.

\[
\begin{bmatrix}
    x_{k+1} \\
    \tilde{x}_{k+1} \\
    \tilde{z}_k \\
    \tilde{w}_k
\end{bmatrix} =
\begin{bmatrix}
    (X_+ - A_2 \tilde{X})G_1 & A_2 + (X_+ - A_2 \tilde{X})G_2 \\
    A_2 + B_2 \tilde{G}_1 & A_2 + B_2 \tilde{G}_2 \\
    C_1 + D_\tilde{U}G_1 & C_2 + D_\tilde{U}G_2 \\
    -G_1 & -G_2
\end{bmatrix}
\begin{bmatrix}
    B_{w1} & B_{w1} \\
    B_{w2} & 0 \\
    D_w & 0 \\
    0 & 0
\end{bmatrix}
\begin{bmatrix}
    x_k \\
    \tilde{x}_k \\
    \tilde{z}_k \\
    \tilde{w}_k
\end{bmatrix}
\] (20)
variables. Thus, the proposed controller design method scales cubically with the data length $N$ and proportionally to $n^6$ if $n$ is the system dimension, similar as in model-based robust controller design.

V. Conclusion

The present paper provides direct, data-driven design procedures for state-feedback gains, which achieve guaranteed closed-loop stability and performance, using noisy input-state data. Based on a data-driven parametrization of the closed-loop matrices that are consistent with the data, known robust control methods can be applied. The closed-loop parametrization is extended to a setting with partial model knowledge, and the design procedures are applied successfully to an unstable example system. The proposed approach leads to end-to-end guarantees for the closed loop, using a single noisy open-loop data trajectory of finite length, and is thus a promising alternative to sequential system identification and robust control. Future research should extend the results of this paper to robust data-driven output-feedback control.

REFERENCES

[1] Z.-S. Hou and Z. Wang, “From model-based control to data-driven control: Survey, classification and perspective,” Information Sciences, vol. 235, pp. 3–35, 2013.
[2] B. Recht, “A tour of reinforcement learning: The view from continuous control,” Annual Review of Control, Robotics, and Autonomous Systems, 2018.
[3] N. Matni and S. Tu, “A tutorial on concentration bounds for system identification,” arXiv:1906.11395, 2019.
[4] N. Matni, A. Proutiere, A. Rantzer, and S. Tu, “From self-tuning regulators to reinforcement learning and back again,” arXiv:1906.11392, 2019.
[5] R. Boczar, N. Matni, and B. Recht, “Finite-data performance guarantees for the output-feedback control of an unknown system,” in Proc. 57th IEEE Conf. on Decision and Control, 2018, pp. 2994–2999.
[6] S. Dean, H. Mania, N. Matni, B. Recht, and S. Tu, “On the sample complexity of the linear quadratic regulator,” Foundations of Computational Mathematics, 2019, https://doi.org/10.1007/s10208-019-09426-v.
[7] M. Milanese and A. Vicino, “Optimal estimation theory for dynamic systems with set membership uncertainty: an overview,” Automatica, vol. 27, no. 6, pp. 997–1009, 1991.
[8] J. C. Willems, P. Rapisarda, I. Markovsky, and B. De Moor, “A note on persistency of excitation,” Systems & Control Letters, vol. 54, pp. 325–329, 2005.
[9] A. Romer, J. Berberich, J. Köhler, and F. Allgöwer, “One-shot verification of dissipativity properties from input-output data,” IEEE Control Systems Letters, vol. 3, no. 3, pp. 709–714, 2019.
[10] J. Berberich and F. Allgöwer, “A trajectory-based framework for data-driven system analysis and control,” arXiv:1903.10723, 2019.
[11] C. De Persis and P. Tesi, “On persistency of excitation and formulas for data-driven control,” arXiv:1903.0684, 2019.
[12] H. J. van Waarde, J. Eising, H. L. Trentelman, and M. Kanat Camlibel, “Data informativity: a new perspective on data-driven system analysis and control,” arXiv:1908.00468, 2019.
[13] K. Zhou, J. C. Doyle, and K. Glover, Robust and optimal control. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1996.
[14] C. Scherer and S. Weiland, Linear Matrix Inequalities in Control, 3rd ed. New York: Springer-Verlag, 2000.
[15] C. Scherer, “Robust mixed control and linear parameter-varying control with full-block scalings,” in Advances in Linear Matrix Inequality Methods in Control. SIAM: Philadelphia, 2000, pp. 187–207.