HARMONIC MORPHISMS OF ALLOF-WALLACH SPACES OF POSITIVE CURVATURE

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Abstract. An infinite family of distinct harmonic morphisms with minimal circle fibers from the 7-dimensional homogeneous Allof-Wallach spaces of positive curvature onto the 6-dimensional flag manifolds is given.

1. Introduction

In 1965, Eells and Sampson [13] initiated a theory of harmonic maps in which variational problems play central roles in geometry; Harmonic map is one of solutions of important variational problems which is a critical point of the energy functional $E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g$ for smooth maps $\varphi$ of $(M, g)$ into $(N, h)$. The Euler-Lagrange equations are given by the vanishing of the tension filed $\tau(\varphi)$. On the other hand, Fuglede [14] in 1978 and Ishihara [22] in 1979, introduced independently the alternative notion of harmonic morphism which preserves harmonic functions (see [4]). Harmonic morphisms are one of important examples of harmonic maps.

In this paper, we give new examples of harmonic morphisms, indeed, an infinite family of distinct harmonic morphisms from the 7 dimensional homogeneous space of positive sectional curvature into the 6 dimensional flag manifold. Namely, we show the following theorem.

**Theorem** (cf. Theorem 4.2) Let $(P, g) = (M_{k, \ell}, g_t) = (SU(3)/T_{k, \ell}, g_t)$, $k, \ell \in \mathbb{Z}$, $(k, \ell) = 1; -1 < t < 0$, or $0 < t < \frac{1}{3}$, be infinitely many distinct homogeneous the 7-dimensional Allof-Wallach spaces of positive sectional curvature, and let $(M, h)$, the 6-dimensional flag manifold $(SU(3)/T, h)$. Then, the Riemannian submersions with circle fibers,
\( \pi : (P, g) = (M_{k, \ell}, g_t) \rightarrow (M, h) = (SU(3)/T, h) \), are all harmonic morphisms with minimal fibers.

Here, the subgroups \( T_{k, \ell} \) and \( T \) of \( SU(3) \) and the homogeneous space \( M_{k, \ell} \) are given as follows.

\[
T_{k, \ell} = \left\{ \begin{pmatrix} e^{2\pi i \theta} & 0 & 0 \\ 0 & e^{2\pi i \theta} & 0 \\ 0 & 0 & e^{-2\pi i (k+\ell)} \end{pmatrix} \mid \theta \in \mathbb{R} \right\}
\]

\[
T = \left\{ \begin{pmatrix} e^{2\pi i \theta_1} & 0 & 0 \\ 0 & e^{2\pi i \theta_2} & 0 \\ 0 & 0 & e^{-2\pi i (\theta_1+\theta_2)} \end{pmatrix} \mid \theta_1, \theta_2 \in \mathbb{R} \right\} \subset G = SU(3),
\]

and \( M_{k, \ell} = G/T_{k, \ell} = SU(3)/T_{k, \ell} \).

2. Preliminaries

2.1. Harmonic maps. We first prepare the materials for the first and second variational formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map \( \varphi : (M, g) \rightarrow (N, h) \), of a compact Riemannian manifold \( (M, g) \) into another Riemannian manifold \( (N, h) \), which is an extremal of the energy functional defined by

\[
E(\varphi) = \int_M e(\varphi) v_g,
\]

where \( e(\varphi) := \frac{1}{2} |d\varphi|^2 \) is called the energy density of \( \varphi \). That is, for any variation \( \{ \varphi_t \} \) of \( \varphi \) with \( \varphi_0 = \varphi \),

\[
\left. \frac{d}{dt} \right|_{t=0} E(\varphi_t) = - \int_M h(\tau(\varphi), V) v_g = 0, \quad (2.1)
\]

where \( V \in \Gamma(\varphi^{-1}TN) \) is a variation vector field along \( \varphi \) which is given by \( V(x) = \frac{d}{dt} \bigg|_{t=0} \varphi_t(x) \in T_{\varphi(x)}N, \ (x \in M) \), and the tension field is given by \( \tau(\varphi) = \sum_{i=1}^{m} B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN) \), where \( \{ e_i \}_{i=1}^{m} \) is a locally defined orthonormal frame field on \( (M, g) \), and \( B(\varphi) \) is the second fundamental form of \( \varphi \) defined by

\[
B(\varphi)(X, Y) = (\nabla d\varphi)(X, Y)
\]

\[
= (\nabla_X d\varphi)(Y) = \nabla_X (d\varphi(Y)) - d\varphi(\nabla_X Y), \quad (2.2)
\]
for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, $\nabla$, and $\nabla^h$, are Levi-Civita connections on $TM$, $TN$ of $(M, g)$, $(N, h)$, respectively, and $\tilde{\nabla}$, and $\nabla$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2.1), $\varphi$ is harmonic if and only if $\tau(\varphi) = 0$.

2.2. **Riemannian submersions.** We prepare with several notions on the Riemannian submersions. A $C^\infty$ mapping $\pi$ of a $C^\infty$ Riemannian manifold $(P, g)$ into another $C^\infty$ Riemannian manifold $(M, h)$ is called a Riemannia submersion if (0) $\pi$ is surjective, (1) the differential $d\pi = \pi_* : T_uP \rightarrow T_{\pi(u)}M$ ($u \in P$) of $\pi : P \rightarrow M$ is surjective for each $u \in P$, and (2) each tangent space $T_uP$ at $u \in P$ has the direct decomposition:

$$T_uP = V_u \oplus H_u, \quad (u \in P),$$

which is orthogonal decomposition with respect to $g$ such that $V = \text{Ker}(\pi_*u) \subset T_uP$ and (3) the restriction of the differential $\pi_* = d\pi_u$ to $H_u$ is a surjective isometry, $\pi_*(H_u, g_u) \rightarrow (T_{\pi(u)}M, h_{\pi(u)})$ for each $u \in P$ (cf. [4]). A manifold $P$ is the total space of a Riemannian submersion over $M$ with the projection $\pi : P \rightarrow M$ onto $M$, where $p = \dim P = k + m$, $m = \dim M$, and $k = \dim \pi^{-1}(x)$, $(x \in M)$. A Riemannian metric $g$ on $P$, called adapted metric on $P$ which satisfies

$$g = \pi^*h + k \quad (2.3)$$

where $k$ is the Riemannian metric on each fiber $\pi^{-1}(x)$, $(x \in M)$. Then, $T_uP$ has the orthogonal direct decomposition of the tangent space $T_uP$,

$$T_uP = V_u \oplus H_u, \quad u \in P, \quad (2.4)$$

where the subspace $V_u = \text{Ker}(\pi_*u)$ at $u \in P$, the vertical subspace, and the subspace $H_u$ of $P_u$ is called horizontal subspace at $u \in P$ which is the orthogonal complement of $V_u$ in $T_uP$ with respect to $g$.

In the following, we fix a locally defined orthonormal frame field, called adapted local orthonormal frame field to the projection $\pi : P \rightarrow M$, $\{e_i\}_{i=1}^p$ corresponding to (2.9) in such a way that

- $\{e_i\}_{i=1}^m$ is a locally defined orthonormal basis of the horizontal subspace $H_u$ ($u \in P$), and
- $\{e_i\}_{i=1}^k$ is a locally defined orthonormal basis of the vertical subspace $V_u$ ($u \in P$).

Corresponding to the decomposition (2.9), the tangent vectors $X_u$, and $Y_u$ in $T_uP$ can be defined by

$$X_u = X^V_u + X^H_u, \quad Y_u = Y^V_u + Y^H_u, \quad (2.5)$$

$$X^V_u, \quad Y^V_u \in V_u, \quad X^H_u, \quad Y^H_u \in H_u \quad (2.6)$$

for $u \in P$. 
Then, there exist a unique decomposition such that
\[ g(X_u, Y_u) = h(\pi_*X_u, \pi_*Y_u) + k(X^V_u, Y^V_u), \quad X_u, Y_u \in T_uP, \ u \in P. \]

2.3. The reduction of the harmonic map equation. Hereafter, we treat with the above problem more precisely in the case \( \dim(\pi^{-1}(x)) = 1, \ (u \in P, \pi(u) = x) \). Let \( \{e_1, e_1, \ldots, e_m\} \) be an adapted local orthonormal frame field being \( e_{n+1} = e_m \), vertical. The frame fields \( \{e_i : i = 1, 2, \ldots, n\} \) are the basic orthonormal frame field on \((P, g)\) corresponds to an orthonormal frame field \( \{e_1, e_2, \ldots, e_n\} \) on \((M, g)\).

In this section, we determine the biharmonic equation precisely in the case that \( p = m + 1 = \dim P, \ m = \dim M, \) and \( k = \dim \pi^{-1}(x) = 1 \) \((x \in M)\). Since \([V, Z] \) is a vertical field on \( P \) if \( Z \) is basic and \( V \) is vertical (cf. [36], p. 461). Therefore, for each \( i = 1, \ldots, n, [e_i, e_{n+1}] \) is vertical, so we can write as follows.

\[ [e_i, e_{n+1}] = \kappa_i e_{n+1}, \quad i = 1, \ldots, n \quad (2.7) \]

where \( \kappa_i \in C^\infty(P) \) \((i = 1, \ldots, n)\). For two vector fields \( X, Y \) on \( M \), let \( X^*, Y^* \) be the horizontal vector fields on \( P \). Then, \([X^*, Y^*] \) is a vector field on \( P \) which is \( \pi \)-related to a vector field \([X, Y]\) on \( M \) (for instance, [49], p. 143). Thus, for \( i, j = 1, \ldots, n, [e_i, e_j] \) is \( \pi \)-related to \([e_i, e_j]\), and we may write as

\[ [e_i, e_j] = \sum_{k=1}^{n+1} D^k_{ij} e_k, \quad (2.8) \]

where \( D^k_{ij} \in C^\infty(P) \) \((1 \leq i, j \leq n; 1 \leq k \leq n + 1)\).

2.4. The tension field. In this subsection, we calculate the tension field \( \tau(\pi) \). We show that

\[ \tau(\pi) = -d\pi \left( \nabla_{e_{n+1}} e_{n+1} \right) = -\sum_{i=1}^{n} \kappa_i e_i. \quad (2.9) \]
Indeed, we have
\[
\tau(\pi) = \sum_{i=1}^{m} \left\{ \nabla_{e_i}^* d\pi(e_i) - d\pi (\nabla_{e_i} e_i) \right\} \\
= \sum_{i=1}^{n} \left\{ \nabla_{e_i}^* d\pi(e_i) - d\pi (\nabla_{e_i} e_i) \right\} + \nabla_{e_{n+1}}^* d\pi(e_{n+1}) - d\pi (\nabla_{e_{n+1}} e_{n+1}) \\
= -d\pi \left( \nabla_{e_{n+1}} e_{n+1} \right) \\
= -\sum_{i=1}^{n} \kappa_i e_i.
\]
Because, for \( i, j = 1, \ldots, n \),
\[
d\pi (\nabla_{e_{i}} e_{j}) = \nabla_{e_{i}}^* h_{j}, \text{ and } \nabla_{e_{i}}^* d\pi(e_{i}) = \nabla_{d\pi(e_{i})}^* d\pi(e_{i}) = \nabla_{e_{i}}^* h_{i}.
\]
Thus, we have
\[
\sum_{i=1}^{n} \left\{ \nabla_{e_i}^* d\pi(e_i) - d\pi (\nabla_{e_i} e_i) \right\} = 0. \quad (2.10)
\]
Since \( e_{n+1} = e_m \) is vertical, \( d\pi(e_{n+1}) = 0 \), so that \( \nabla_{e_{n+1}}^* d\pi(e_{n+1}) = 0 \).
Furthermore, we have, by definition of the Levi-Civita connection, we have, for \( i = 1, \ldots, n \),
\[
2g(\nabla_{e_{n+1}} e_{n+1}, e_i) = 2g(e_{n+1}, [e_i, e_{n+1}]) = 2\kappa_i,
\]
and \( 2g(\nabla_{e_{n+1}} e_{n+1}, e_{n+1}) = 0 \). Therefore, we have
\[
\nabla_{e_{n+1}} e_{n+1} = \sum_{i=1}^{n} \kappa_i e_i,
\]
and then,
\[
d\pi \left( \nabla_{e_{n+1}} e_{n+1} \right) = \sum_{i=1}^{n} \kappa_i e_i. \quad (2.11)
\]
Thus, we obtain (2.9). \( \square \)

Thus, we obtain the following theorem:

**Theorem 2.1.** Let \( \pi : (P, g) \to (M, h) \) be a Riemannian submersion over \( (M, h) \). Then,

The tension field \( \tau(\pi) \) of \( \pi \) is given by
\[
\tau(\pi) = -d\pi \left( \nabla_{e_{n+1}} e_{n+1} \right) = -\sum_{i=1}^{n} \kappa_i e_i, \quad (2.12)
\]
where \( \kappa_i \in C^\infty(P), (i = 1, \ldots, n) \).
3. HARMONIC MORPHISMS

Definition 3.1. (1) A smooth map \( \pi : (P, g) \to (M, h) \) is harmonic if the tension field vanishes, \( \tau(\pi) = 0 \), and

(2) \( \pi : (P, g) \to (M, h) \) is a harmonic morphism (cf. [4], p. 106) if, for every harmonic function \( f : (M, h) \to \mathbb{R} \), the composition \( f \circ \pi : (P, g) \to \mathbb{R} \) is also harmonic.

(3) \( \pi : (P, g) \to (M, h) \) is horizontally weakly conformal (cf. [4], p. 46) if, the differential \( \pi_* : T_pP \to T_{\pi(p)}M \) is surjective, and

\[
(\pi^*h)(X,Y) = \Lambda(p) g(X,Y) \quad (X, Y \in \mathcal{H}_p),
\]

for some non-zero number \( \Lambda(p) \neq 0 (p \in P) \). Here, \( \mathcal{H}_p \) is the horizontal subspace of \( T_pP \) for the Riemannian submersion \( \pi : (P, g) \to (M, h) \) satisfying that

\[
\begin{cases}
T_pP = V_p \oplus \mathcal{H}_p, \\
V_p = \ker(\pi_*),
\end{cases}
\]

It is well known (cf. [4], p. 108) that

Theorem 3.2. (Fuglede, 1978 and Ishihara, 1979) Let \( \varphi : (P, g) \to (M, g) \) be a Riemannian submersion. Then, it is harmonic morphism if and only if the following two conditions hold:

(i) \( \varphi : (P, g) \to (M, h) \) is harmonic and

(ii) \( \varphi : (P, g) \to (M, h) \) is horizontally weakly conformal.

Corollary 3.3. (cf. [4], p. 123) A Riemannian submersion \( \varphi : (P, g) \to (M, g) \) is a harmonic morphism if and only if \( \varphi \) has minimal fibers.

Example. Let \( G \) be a compact Lie group, and \( K \subset H \subset G \), closed subgroups of \( G \). It is well known ([4], p. 123) that, the natural projection \( \pi : (G/K, g) \to (G/H, h) \) is a harmonic Riemannian submersion with totally geodesic fibers. Notice that both the Riemannian metrics \( g \) on \( G/K \) and \( h \) on \( G/H \) are the invariant Riemannian metrics on \( G/K \) and \( G/H \) are induced from the \( \text{Ad}(G) \)-invariant inner product \( \langle \, , \rangle \) on the Lie algebra \( g \) of \( G \). The Allot-Wallach’s Riemannian metrics of positive sectional curvature in our main theorem, Theorem 4.2, are invariant Riemannian metrics, but are different from the ones induced from the \( \text{Ad}(G) \)-invariant inner product \( \langle \, , \rangle \) on the Lie algebra \( g \) of \( G \).
4. Statement of main theorem

We first prepare the setting of the Allof-Wallach theorem [2]. Let $G = SU(3)$, and

$$T_{k, \ell} = \left\{ \begin{pmatrix} e^{2\pi i k \theta} & 0 & 0 \\ 0 & e^{2\pi i \ell \theta} & 0 \\ 0 & 0 & e^{-2\pi i (k+\ell)} \end{pmatrix} \bigg| \theta \in \mathbb{R} \right\} \subset T = \left\{ \begin{pmatrix} e^{2\pi i \theta_1} & 0 & 0 \\ 0 & e^{2\pi i \theta_2} & 0 \\ 0 & 0 & e^{-2\pi i (\theta_1+\theta_2)} \end{pmatrix} \bigg| \theta_1, \theta_2 \in \mathbb{R} \right\} \subset G_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & \det(x^{-1}) \end{pmatrix} \bigg| x \in U(2) \right\} \subset G = SU(3),$$

and the Lie algebras of $G, T_{k, \ell}, T, G_1$ by $\mathfrak{g}, \mathfrak{t}_{k, \ell}, \mathfrak{t}, \mathfrak{g}_1$, respectively. Let the Ad($G$)-invariant inner product $\langle , \rangle_0$ by

$$\langle X, Y \rangle_0 := -\text{Re}(\text{Tr}(XY)), \quad X, Y \in \mathfrak{g},$$

$$\mathfrak{m} = \mathfrak{g}_1^\perp := \left\{ \begin{pmatrix} 0 & 0 & z_2 \\ 0 & 0 & z_1 \\ -\overline{z}_2 & -\overline{z}_1 & 0 \end{pmatrix} \bigg| z_1, z_2 \in \mathbb{C} \right\},$$

$$\mathfrak{t}_{k, \ell} := \left\{ \begin{pmatrix} 2\pi i k \theta & 0 & 0 \\ 0 & 2\pi \ell \theta & 0 \\ 0 & 0 & -2\pi i (k+\ell) \theta \end{pmatrix} \bigg| \theta \in \mathbb{R} \right\},$$

$$V_1 := \mathfrak{t}_{k, \ell}^\perp \cap \mathfrak{g}_1, \quad V_2 := \mathfrak{g}_1^\perp = \mathfrak{m},$$

and let

$$\mathfrak{g} = \mathfrak{su}(3) = \mathfrak{t}_{k, \ell} \oplus V_1 \oplus V_2,$$

the orthogonal direct decomposition of $\mathfrak{g}$ with respect to the inner product $\langle , \rangle_0$. For $-1 < t < \infty$, let the new inner product $\langle , \rangle_t$ by

$$\langle x_1 + x_2, y_1 + y_2 \rangle_t := (1 + t)\langle x_1, y_1 \rangle_0 + \langle x_2, y_2 \rangle_0,$$

(4.1)

where $x_i, y_i \in V_i \ (i = 1, 2)$, and let $g_t$, the corresponding $G$-invariant Riemannian metric on the homogeneous space $G/T_{k, \ell}$. Then,

**Theorem 4.1.** (Allof and Wallach [2]) The homogeneous space $(G/T_{k, \ell}, g_t)$ corresponding to (4.1) with $(-1 < t < 0)$ or $(0 < t < \frac{1}{3})$ have strictly positive sectional curvature.

We state our main theorem as follows:
Theorem 4.2. Let $\pi$ be the Riemannian submersion of $(SU(3)/T_{k,\ell}, g_t)$ onto $(SU(3)/T, h)$, where $(SU(3)/T, h)$ is a flag manifold with the SU(3)-invariant Riemannian metric $h$ corresponding the inner product $\langle , \rangle_0$ on $g$. Then, it is a harmonic morphism, i.e., for every harmonic function on a neighborhood $V$ in $SU(3)/T$, the composition $f \circ \pi$ is harmonic on a neighborhood $\pi^{-1}(V)$ in $SU(3)/T_{k,\ell}$, and also it has minimal fibers.

5. Proof of main theorem

Here, in this section, we give a proof of Theorem 4.2. We take a basis $\{X_0, X_1, X_2\}$ of $V_1$ and the one $\{X_3, X_4, X_5, X_6\}$ of $V_2$ as follows:

$$X_0 = \frac{i}{\sqrt{5\Gamma}} \begin{pmatrix} 2k + \ell & 0 & 0 \\ 0 & 2m + \ell & 0 \\ 0 & 0 & 2k + m \end{pmatrix},$$

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$X_5 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}. $$

Here $\Gamma := k^2 + \ell^2 + k\ell$, $m := -k - \ell$, and $\{X_0, X_1, X_2, X_3, X_4, X_5, X_6\}$ is an orthonormal basis of $V_1 \oplus V_2$ with respect to $\langle , \rangle_0$, and $g = su(3) = t_{k,\ell} \oplus V_1 \oplus V_2$.

Then, the basis of $V_1 \oplus V_2$

$$\left\{ \frac{1}{\sqrt{1+t}} X_0, \frac{1}{\sqrt{1+t}} X_1, \frac{1}{\sqrt{1+t}} X_2, X_3, X_4, X_5, X_6 \right\} \quad (5.1)$$

is orthonormal with respect to the inner product $\langle , \rangle_t$, $(-1 < t < \infty)$ in (4.1). We denote $M_{k,\ell} := SU(3)/T_{k,\ell}$ with $k$ and $\ell \in \mathbb{Z}$ with $(k, \ell) = 1$, and the corresponding local unit orthonormal vector fields on $M_{k,\ell} := SU(3)/T_{k,\ell}$ by

$$\{ e_{i}^t, e_{1}^t, e_{2}^t, e_{3}^t, e_{4}^t, e_{5}^t, e_{6}^t \} \quad (5.2)$$

For the projection $\pi : M_{k,\ell} = SU(3)/T_{k,\ell} \to M = SU(3)/T$, each element $e_i^t$ ($i = 0, 1, \ldots, 6$) in (5.2) corresponds by $\pi_*$ (the differential
Lemma 5.1. We have
\[ \left\{ 0, \frac{1}{\sqrt{1+t}} e'_1, \frac{1}{\sqrt{1+t}} e'_2, e'_3, e'_4, e'_5, e'_6 \right\}, \]
where \( \{e'_1, e'_2, e'_3, e'_4, e'_5, e'_6\} \) is an orthonormal frame field on \((SU(3)/T, h)\).

By definition of the Levi-Civita connection of a Riemannian metric \( g_t \), for every vector field \( X \) on \( P = M_{k,\ell} \),
\[
2 g_t(X, \nabla^{g_t}_{e'_0} e'_0) = e'_0 g_t(X, e'_0) + e'_0 g_t(X, e'_0) - X g_t(e'_0, e'_0) \\
\quad + g_t(e'_0, [X, e'_0]) + g_t(e'_0, [X, e'_0]) - g_t(X, [e'_0, e'_0]) \\
= 2 \left\{ e'_0 g_t(X, e'_0) + g_t(e'_0, [X, e'_0]) \right\}. \tag{5.3}
\]
Thus, we have
\[
g_t(X, \nabla^{g_t}_{e'_0} e'_0) = e'_0 g_t(X, e'_0) + g_t(e'_0, [X, e'_0]). \tag{5.3}
\]
On the other hand, we have
\[
e'_0 g_t(e'_i, e'_0) = 0 \quad (i = 0, 1, \ldots, 6) \tag{5.4}
\]
Indeed,\[
e'_0 g_t(e'_i, e'_0) = 0 \quad (i = 0, 1, \ldots, 6) \tag{5.5}
\]
and by a straightforward computation, we have the following Lemma:

**Lemma 5.1.** We have
\[
\begin{align*}
\left[ \frac{1}{\sqrt{1+t}} X_1, \frac{1}{\sqrt{1+t}} X_0 \right] &= -\frac{3(k + \ell)}{(1+t)\sqrt{5}} X_2, \\
\left[ \frac{1}{\sqrt{1+t}} X_2, \frac{1}{\sqrt{1+t}} X_0 \right] &= -\frac{3(k + \ell)}{(1+t)\sqrt{5}} X_1, \\
\left[ X_3, \frac{1}{\sqrt{1+t}} X_0 \right] &= \frac{-3\ell}{\sqrt{1+t}\sqrt{5}} X_4, \left[ X_4, \frac{1}{\sqrt{1+t}} X_0 \right] = \frac{3\ell}{\sqrt{1+t}\sqrt{5}} X_3, \\
\left[ X_5, \frac{1}{\sqrt{1+t}} X_0 \right] &= \frac{3k}{\sqrt{1+t}\sqrt{5}} X_6, \left[ X_6, \frac{1}{\sqrt{1+t}} X_0 \right] = \frac{-3k}{\sqrt{1+t}\sqrt{5}} X_5.
\end{align*}
\]

By Lemma 5.1, we have
\[
ge_t(e'_0, [X, e'_0]) = 0 \quad (\forall X = X_i \ (i = 0, 1, \ldots, 6)). \tag{5.6}
\]
By (5.3), (5.4), (5.5), we have
\[
ge_t(X, \nabla^{g_t}_{e'_0} e'_0) = 0 \quad (\forall X \in \mathcal{X}(M_{k,\ell})). \tag{5.7}
\]
which implies that
\[ \nabla g \tau e_0^t = 0. \] (5.8)

Then, we have
\[ \tau(\pi) = -d\pi(\nabla g \tau e_0^t) = 0. \] (5.9)

Therefore, by (5.8) and (5.9), the submersion \( \pi \) is a harmonic map with minimal fibers. Due to Corollary 3.3, we have Theorem 4.2. \( \square \)

**Remark 5.2.**

(1) In our main theorem, Theorem 4.2, since \((G/T, h)\) is a flag manifold, so a Kähler manifold, it admits a lot of harmonic function on an open subset \( V \) in \( G/T \). For a harmonic function \( f \) on an open subset \( V \subset G/T \), then \( f \circ \pi \) is harmonic on \( \pi^{-1}(V) \).

(2) Our fibration \( \pi : (SU(3)/T_k, \ell, g) \to (SU(3)/T, h) \) has close similarities to the Hopf fibration \( \pi' : (S^{2n+1}, g_0) \to (\mathbb{C}P^n, h_0) \). Both the total spaces have positive sectional curvature, and both the base spaces are Kähler manifolds.

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