ON COMPACTNESS AND $L^p$-REGULARITY IN THE $\bar{\partial}$-NEUMANN PROBLEM

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ABSTRACT. Let $\Omega$ be a $C^4$-smooth bounded pseudoconvex domain in $\mathbb{C}^2$. We show that if the $\bar{\partial}$-Neumann operator $N_1$ is compact on $L^2_{(0,1)}(\Omega)$ then the embedding operator $\mathcal{J} : \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \to L^2_{(0,1)}(\Omega)$ is $L^p$-regular for all $2 \leq p < \infty$.

1. INTRODUCTION

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \leq q \leq n$, and let $\text{Dom}^2(\bar{\partial})$ and $\text{Dom}^2(\bar{\partial}^*)$ denote the domains of the densely defined operators $\bar{\partial}$ and $\bar{\partial}^*$ in $L^2_{(0,q)}(\Omega)$, respectively. On bounded pseudoconvex domains, Hörmander in [Hör65] proved the following basic estimate,

$$\|f\|_{L^2} \lesssim \|\bar{\partial}f\|_{L^2} + \|\bar{\partial}^*f\|_{L^2}$$

for all $(0,q)$-forms $f \in \text{Dom}^2(\bar{\partial}) \cap \text{Dom}^2(\bar{\partial}^*) \subset L^2_{(0,q)}(\Omega)$. The sum on the right hand side is called the $L^2$-graph norm of the $(0,q)$-form $f$. In other words, the embedding operator

$$\mathcal{J} : \text{Dom}^2(\bar{\partial}) \cap \text{Dom}^2(\bar{\partial}^*) \to L^2_{(0,q)}(\Omega)$$

is bounded, where the space on the left hand side is endowed with the graph norm.

Let $1 < p, \tilde{p} < \infty$ such that $p^{-1} + \tilde{p}^{-1} = 1$. We define $\text{Dom}^p(\bar{\partial}) = \{f \in L^p_{(0,q)}(\Omega) : \bar{\partial}f \in L^p_{(0,q+1)}(\Omega)\}$. We define $\text{Dom}^p(\bar{\partial}^*)$ as follows: we say $f \in \text{Dom}^p(\bar{\partial}^*)$ if $f \in L^p_{(0,q)}(\Omega)$ and there exists $C > 0$ such that

$$|\langle f, \bar{\partial}g \rangle| \leq C\|g\|_{L^{\tilde{p}}_{(0,q)}}$$

for all $g \in L^{\tilde{p}}_{(0,q-1)}(\Omega)$ with $\bar{\partial}g \in L^{\tilde{p}}_{(0,q)}(\Omega)$. Finally, we define the space

$$D^p_{(0,q)}(\Omega) = \text{Dom}^p(\bar{\partial}) \cap \text{Dom}^p(\bar{\partial}^*) \subset L^p_{(0,q)}(\Omega)$$

and endow it with the $L^p$-graph norm $\|\cdot\|_{G^p}$ defined as

$$\|f\|_{G^p} = \|\bar{\partial}f\|_{L^p} + \|\bar{\partial}^*f\|_{L^p}$$

for $f \in D^p_{(0,q)}(\Omega)$. We note that on bounded pseudoconvex domains, $\|\cdot\|_{G^p}$ is a norm because $\bar{\partial}f = 0$ and $\bar{\partial}^*f = 0$ imply that $f = 0$ for $1 \leq q \leq n$ (see, for example, [CS01, (4.4.2) in section 4.4]).

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Definition 1. We say that the operator $\mathcal{J}$ is $L^p$-regular on $D^p_{(0,q)}(\Omega)$ if there exists $C > 0$ such that
\[
\|\mathcal{J} f\|_{L^p} = \|f\|_{L^p} \leq C \|f\|_{G^p} = C \left( \|\overline{\partial} f\|_{L^p} + \|\overline{\partial}^* f\|_{L^p} \right)
\]
for all $f \in D^p_{(0,q)}(\Omega)$.

That is, whenever $\mathcal{J} : D^p_{(0,q)}(\Omega) \to L^p_{(0,q)}(\Omega)$ is a bounded embedding we say that it is $L^p$-regular. In particular, by Hörmander’s basic estimate above, $\mathcal{J}$ is $L^2$-regular on bounded pseudoconvex domains. We note that $D^p_{(0,q)}(\Omega)$ is a Banach space (for $1 \leq q \leq n$ with the graph norm $\|\cdot\|_{G^p}$) when $\mathcal{J}$ is $L^p$-regular.

The operator $\mathcal{J}$ is related to the $\overline{\partial}$-Neumann operator $N$, the bounded inverse of the complex Laplacian $\overline{\partial} \overline{\partial} + \partial \partial^*$ on $L^2_{(0,q)}(\Omega)$, as $N = \mathcal{J} \mathcal{J}^*$ (see, for example, [Str10, Proof of Theorem 2.9]). Hence, $N$ is compact if and only if $\mathcal{J}$ is compact. In this note, we show that compactness of $N$ implies $L^p$-regularity of $\mathcal{J}$ for $2 \leq p < \infty$. We also note that it is not yet clear if $\mathcal{J}$ is $L^p$-regular for $1 < p < 2$ under the compactness assumption. We further note that the question of whether the $\overline{\partial}$-Neumann operator or the Bergman projection are bounded in $L^p$-norm whenever $\mathcal{J}$ is compact is open as well.

Although the mapping properties of the canonical operators relate well in the $L^2$-Sobolev setting, similar equivalences in the $L^p$ setting are less clear. In [BS91], Bonami and Sibony obtained $L^p$ estimates for the solutions of $\overline{\partial}$-problem under some Sobolev estimates. In [HZ19] Harrington and Zeytuncu obtained some $L^p$ estimates on the canonical operators under the assumption of the existence of good weight functions. Both assumptions are more stringent than the compactness of $N$ and hence the $L^p$ estimates are more general. Also, recently, Haslinger in [Has16, Theorem 2.2] showed that if $\mathcal{J}$ gains regularity in the $L^p$ scale then $N$ is compact. In this paper, we observe a property that is less general than the ones in [BS91, HZ19] under a weaker assumption, and that is in the converse direction of the result in [Has16]. Namely, in Theorem 1 below, we show that compactness of $N_1$ (at the $L^2$ level) induces $L^p$-regularity of $\mathcal{J}$ for $2 \leq p < \infty$.

**Theorem 1.** Let $\Omega$ be a $C^4$-smooth bounded pseudoconvex domain in $\mathbb{C}^2$. Assume that $N_1$ is compact on $L^2_{(0,1)}(\Omega)$ (or, equivalently, $\mathcal{J}$ is compact on $D^2_{(0,1)}(\Omega)$). Then $\mathcal{J}$ is $L^p$-regular on $D^p_{(0,1)}(\Omega)$ for all $2 \leq p < \infty$.

We note that the $L^p$ boundedness is not an automatic consequence of compactness on $L^2$; as we demonstrate with Example 1, in which we present an operator that is compact on the $L^2$ space but unbounded on any $L^p$ spaces for $p \neq 2$.

In the rest of the paper, we use the symbol $x \lesssim y$ to mean that there exists $C > 0$ such that $x \leq Cy$. Furthermore, when we write a family of inequalities depending on a parameter $\varepsilon$
\[
x \lesssim \varepsilon y,
\]
we mean that there exists $C > 0$ that is independent of $\varepsilon$ such that $x \leq C\varepsilon y$.

2. Proof of Theorem 1

One can prove the following density lemma similarly as in [CS01, Lemma 4.3.2] (see also [Str10, Proposition 2.3]) using an $L^p$ version of Friedrichs Lemma (see, for instance, [BLD01, Lemma 3.1]).

**Lemma 1.** Let $\Omega$ be a $C^{k+1}$-smooth bounded domain in $\mathbb{C}^n$, $1 \leq q \leq n$, and $1 < p < \infty$. Then $C^k(\partial \Omega) \cap \text{Dom}(\overline{\partial}^*)$ is dense in $D^p_0(\Omega)$ in the graph norm $f \to \|f\|_{L^p} + \|\overline{\partial} f\|_{L^p} + \|\overline{\partial}^* f\|_{L^p}$. The statement also holds with $k$ and $k + 1$ replaced with $\infty$.

We will need the following lemma which is a corollary of [JK95, Theorem 1.1].

**Lemma 2** (Jerison-Kenig). Let $\Omega$ be a $C^1$-smooth bounded domain in $\mathbb{R}^n$ and $1 < p < \infty$. Then there exists $C > 0$ such that

\[(1) \quad \|u\|_{W^{1,p}} \leq C\|\Delta u\|_{W^{-1,p}}\]

for all $u \in W^{1,p}_0(\Omega)$.

Using the lemmas above together with the proof of [Str10, Lemma 2.12] one can prove the following estimate on the normal component of forms. We note that, in the lemma below, $f_{\text{norm}}$ denotes the normal component of $f$ (see (2.86) in [Str10]).

**Lemma 3.** Let $\Omega$ be a $C^4$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \leq q \leq n$, and $1 < p < \infty$. There exists $C > 0$ such that if $f \in D^p_0(\Omega)$ then $f_{\text{norm}} \in W^{1,p}_{0,\Omega}(\Omega)$ and

\[
\|f_{\text{norm}}\|_{W^{1,p}} \leq C \left(\|\overline{\partial} f\|_{L^p} + \|\overline{\partial}^* f\|_{L^p} + \|f\|_{L^p}\right).
\]

We will use Lemma 2 to also prove the following $L^p$ version of [CS01, Proposition 5.1.1].

**Proposition 1.** Let $\Omega$ be a $C^2$-smooth bounded domain in $\mathbb{C}^n$, $1 < p < \infty$, $1 \leq q \leq n$, and $\phi \in C^1(\overline{\Omega})$ such that $\phi = 0$ on $\partial \Omega$. Then there exists $C > 0$ such that

\[
\|\phi f\|_{W^{1,p}} \leq C \left(\|\overline{\partial} f\|_{L^p} + \|\overline{\partial}^* f\|_{L^p} + \|f\|_{L^p}\right)
\]

for $f \in D^p_0(\Omega)$.

**Proof.** First we assume that $g \in D^p_0(\Omega)$ with coefficient functions in $W^{1,p}_0(\Omega)$. Then we have

\[
\|g\|_{W^{1,p}} \leq \|\Delta g\|_{W^{-1,p}} \leq \|\overline{\partial} g\|_{L^p} + \|\overline{\partial}^* g\|_{L^p}.
\]

Then we substitute $g = \phi f$ for $f \in C^1(\overline{\Omega}) \cap \text{Dom}(\overline{\partial}^*)$ in the inequality above

\[
\|\phi f\|_{W^{1,p}} \leq \|\overline{\partial}(\phi f)\|_{L^p} + \|\overline{\partial}^*(\phi f)\|_{L^p} \leq \|\overline{\partial} f\|_{L^p} + \|\overline{\partial}^* f\|_{L^p} + \|f\|_{L^p}.
\]
Then we use Lemma 1 to conclude the proof.

The interpolation inequality for Sobolev spaces together with Proposition 1 imply the following corollary.

**Corollary 1.** Let \( \Omega \) be a \( C^2 \)-smooth bounded domain in \( \mathbb{C}^n \), \( 1 < p < \infty \), \( 1 \leq q \leq n \), and \( \phi \in C(\overline{\Omega}) \) such that \( \phi = 0 \) on \( b\Omega \). Then the multiplication operator \( M_{\phi} : D^{p}_{(0,q)}(\Omega) \rightarrow L^p_{(0,q)}(\Omega) \) is compact.

In other words, in the terminology of [CS09], continuous functions on \( \overline{\Omega} \) that vanish on the boundary are compactness multipliers.

We note that even though [Str10, Lemma 4.3] is stated for Hilbert spaces the proof works for Banach spaces as well. In the proof of i) of Lemma 4 below one uses the facts that on reflective Banach spaces bounded sequences have weakly convergent subsequences (see [Yos95, Theorem 1 on pg 126]) as well as compact operators map weakly convergent sequences to convergent sequences. Therefore, proof of [Str10, Lemma 4.3] (see also exercise 6.13 in [Bre11]) implies the following lemma.

**Lemma 4.** Let \( T : X \rightarrow Y \) be a bounded linear map where \( X \) is a normed linear space and \( Y \) is a Banach space.

1. Assume that for all \( \epsilon > 0 \) there exist a Banach space \( Z_\epsilon \) and a compact linear map \( K_\epsilon : X \rightarrow Z_\epsilon \) such that

\[
\|Tx\|_Y \leq \epsilon \|x\|_X + \|K_\epsilon x\|_{Z_\epsilon}
\]

for all \( x \in X \). Then \( T \) is compact.

2. Assume that \( X \) is reflexive Banach space, \( T \) is compact and \( K : X \rightarrow Z \) is an injective bounded linear map. Then for any \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) such that

\[
\|Tx\|_Y \leq \epsilon \|x\|_X + C_\epsilon \|Kx\|_Z
\]

for all \( x \in X \).

**Proof of Theorem 1.** We define \( K : D^p_{(0,1)}(\Omega) \rightarrow L^p_{(0,1)}(\Omega) \) as \( Kf = \rho f \) where \( \rho(z) = dist(z, b\Omega) \) is the distance of \( z \) to the boundary of \( \Omega \). Then Corollary 1 implies that \( K \) is compact for all \( 1 < p < \infty \). We note that \( K \) is an injection as well.

Since \( \Omega \) is a bounded pseudoconvex domain, \( D^2_{(0,1)}(\Omega) \) is a Hilbert space. Then we use ii. in Lemma 4 to get the following estimates: for any \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) such that

\[
\|f\|_{L^2} \leq \epsilon (\|\overline{\partial} f\|_{L^2} + \|\overline{\partial}^* f\|_{L^2}) + C_\epsilon \|Kf\|_{L^2}
\]

for all \( f \in D^2_{(0,1)}(\Omega) \).

First we show how to get \( L^4 \)-regularity. Let \( F = f_1 \overline{w}_1 + f_2 \overline{w}_2 \) be in \( D^4_{(0,1)}(\Omega) \subset L^4_{(0,1)}(\Omega) \) such that \( f_2 \) is the normal component. Because of Lemma 1, without loss of generality, we may assume that \( f_1 \) and \( f_2 \) are \( C^3 \)-smooth on \( \overline{\Omega} \). We denote \( F_2 = f_1^2 \overline{w}_1 + f_2^2 \overline{w}_2 \). Since \( f_2 \) vanishes
on the boundary we have $F_2 \in D^2_{(0,1)}(\Omega)$. Then $\|F\|_{L^4}^2 \approx \|F_2\|_{L^2}^2 < \infty$ and

$$\|F_2\|_{L^2} \lesssim \varepsilon (\|\overline{\partial}F_2\|_{L^2} + \|\overline{\partial}^* F_2\|_{L^2}) + C_\varepsilon \|K F_2\|_{L^2}$$

$$\lesssim \varepsilon (\|f_1 \overline{L}_2 f_1 - f_2 \overline{L}_1 f_2\|_{L^2} + \|f_1 L_1 f_1 + f_2 L_2 f_2\|_{L^2} + \|f_2\|_{L^2})$$

$$+ C_\varepsilon \|K F_2\|_{L^2}$$

$$\lesssim \varepsilon (\|f_1 \overline{L}_2 f_1 - f_1 \overline{L}_1 f_2 + f_2 \overline{L}_1 f_2\|_{L^2})$$

$$+ \varepsilon (\|f_1 L_1 f_1 + f_1 L_2 f_2\|_{L^2} + \|f_1 L_2 f_2 - f_2 L_2 f_2\|_{L^2} + \|f_2\|_{L^2})$$

$$+ C_\varepsilon \|K F_2\|_{L^2}$$

By absorbing the terms that are multiple of $\|F_2\|_{L^2}$ into the left hand side we get

$$\|F_2\|_{L^2} \lesssim \varepsilon \left(\|f_1 \overline{\partial}F\|_{L^2} + \|f_1 - f_2\|\overline{\partial}_1 f_2\|_{L^2} + \|f_1 \overline{\partial}^* F\|_{L^2} + \|(f_1 - f_2) L_2 f_2\|_{L^2}\right)$$

$$+ C_\varepsilon \|K F_2\|_{L^2}.$$  

Using the facts that $\|F_2\|_{L^2} \approx \|F\|_{L^4}^2 < \infty$, $K F_2 = \rho F_2$, and the Cauchy-Schwarz inequality we get

$$\|F\|_{L^4}^2 \lesssim \|\overline{\partial}F\|_{L^4}^2 + \|\overline{\partial}^* F\|_{L^4}^2 + \|f_2\|_{W^{1,4}}^2 + \|f_2\|_{W^{1,4}}^2 + C_\varepsilon \|\rho F\|_{L^4}^2.$$  

Using the inequality $2|xy| \leq |x|^2 + |y|^2$ on right hand side we can absorb $\|F\|_{L^4}$ into the left hand side and get (C_\varepsilon below is different from its previous values, but it still depends on \varepsilon only)

$$\|F\|_{L^4}^2 \lesssim \varepsilon \left(\|f_1 \overline{\partial}F\|_{L^2}^2 + \|\overline{\partial}^* F\|_{L^4}^2 + \|f_2\|_{W^{1,4}}^2\right) + C_\varepsilon \|\rho F\|_{L^4}^2.$$  

Now we will concentrate on $\|f_2\|_{W^{1,4}}$. Using Lemma 2 and Lemma 3 we get

$$\|f_2\|_{W^{1,4}}^2 \lesssim \|\overline{\partial}F\|_{L^4}^2 + \|\overline{\partial}^* F\|_{L^4}^2 + \|F\|_{L^4}^2.$$  

Then the inequality (3) turns into

$$\|F\|_{L^4}^2 \lesssim \varepsilon \left(\|\overline{\partial}F\|_{L^2}^2 + \|\overline{\partial}^* F\|_{L^4}^2\right) + C_\varepsilon \|\rho F\|_{L^4}^2.$$  

That is, we showed that for \varepsilon > 0 given there exists C_\varepsilon > 0 such that

$$\|\mathcal{J} F\|_{L^4} \leq \varepsilon \left(\|\overline{\partial}F\|_{L^4}^4 + \|\overline{\partial}^* F\|_{L^4}^4\right) + C_\varepsilon \|K F\|_{L^4}$$  

for $F \in D^2_{(0,1)}(\Omega)$. Therefore, $\mathcal{J} : D^2_{(0,1)}(\Omega) \to L^4_{(0,1)}(\Omega)$ is a compact operator. Furthermore, since $\mathcal{J}$ is $L^4$-regular, one can show that $D^2_{(0,1)}(\Omega)$ is a Banach space.
In a similar fashion, we use estimates (4) to show that for every \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that
\[
\| JF \|_{L^8} \leq \varepsilon \left( \| \bar{\partial}F \|_{L^8} + \| \bar{\partial}^+F \|_{L^8} \right) + C_\varepsilon \| KF \|_{L^8}
\]
for \( F \in D^8_\Omega \). That is, \( J : D^8_\Omega \to L^8_\Omega \) is a compact linear map (by Lemma 4) and \( D^8_\Omega \) is a Banach space. Inductively, we show that \( J : D^{2^p}_\Omega \to L^{2^p}_\Omega \) is a compact linear map and \( D^{2^p}_\Omega \) is a Banach space for \( p \in \mathbb{Z}^+ \).

Note that for any \( p \in \mathbb{Z}^+ \), we have \( D^{2^p}_\Omega \cap D^{2^{p+1}}_\Omega = D^{2^{p+1}}_\Omega \) and \( D^{2^p}_\Omega + D^{2^{p+1}}_\Omega \subset D^{2^{p+1}}_\Omega \). In other words, for \( 2^p < q < 2^{p+1} \) we get
\[
D^{2^p}_\Omega \subset D^{q}_\Omega \subset D^{2^{p+1}}_\Omega
\]
and since the graph norm is the sum of \( L^p \) norms we conclude that \( D^q_\Omega \) is an intermediate space ([BL76, Definition 2.4.1]) for two Banach spaces \( D^{2^p}_\Omega \) and \( D^{2^{p+1}}_\Omega \). Now, by the complex interpolation theorem ([BL76, Chapter 4]) we conclude that \( J : D^{p}_\Omega \to L^{p}_\Omega \) is \( L^p \)-regular and \( D^p_\Omega \) is a Banach space for all \( 2 \leq p < \infty \). \( \square \)

**Remark 1.** The proof of Theorem 1 shows that we have the same result for \((p, n - 1)\)-forms on \( C^4\)-smooth bounded pseudoconvex domains in \( \mathbb{C}^n \).

We note that \( \text{Ker}(\bar{\partial}) \) and \( A^2(\Omega) \perp \) denote the set of \( \bar{\partial} \)-closed forms and the orthogonal complement of the Bergman space \( A^2(\Omega) \subset L^2(\Omega) \), respectively.

**Proposition 2.** Let \( \Omega \) be a bounded pseudoconvex domain in \( \mathbb{C}^n, n \geq 2 \), and \( 1 < p \leq 2 \). Assume that \( J \) is \( L^p \)-regular on \( D^p_\Omega \). Then the following operators are bounded
\[
\begin{align*}
&\text{i. } \bar{\partial}N_2 : L^p_{\Omega,2}(\Omega) \cap L^2_{\Omega,2}(\Omega) \cap \text{Ker}(\bar{\partial}) \to L^p_{\Omega,1}(\Omega), \\
&\text{ii. } \bar{\partial}N_0 : L^p(\Omega) \cap L^2(\Omega) \cap A^2(\Omega) \perp \to L^p_{\Omega,1}(\Omega).
\end{align*}
\]

**Proof.** Since \( J \) is \( L^p \)-regular and there exists \( C > 0 \) such that
\[
\| f \|_{L^p} \leq C \left( \| \bar{\partial}f \|_{L^p} + \| \bar{\partial}^+f \|_{L^p} \right)
\]
for \( f \in D^p_\Omega \). Note that \( \bar{\partial}N_0g \in \text{Dom}^\perp(\bar{\partial}^+) \subset \text{Dom}^\perp(\bar{\partial}^+) \) for \( g \in L^p(\Omega) \cap L^2(\Omega) \cap A^2(\Omega) \perp \) and \( p \leq 2 \). Then applying the estimate (5) to \( \bar{\partial}N_0g \) we get
\[
\| \bar{\partial}N_0g \|_{L^p} \leq C \| \bar{\partial}^+ \bar{\partial}N_0g \|_{L^p} = C \| g \|_{L^p}
\]
for \( g \in L^2(\Omega) \cap A^2(\Omega) \perp \).

Similarly, if we apply (5) to \( \bar{\partial}^+N_2h \) with \( h \in L^p_{\Omega,2}(\Omega) \cap L^2_{\Omega,2}(\Omega) \cap \text{Ker}(\bar{\partial}) \) we get
\[
\| \bar{\partial}^+N_2h \|_{L^p} \leq C \| \bar{\partial}^+ \bar{\partial}^+N_2h \|_{L^p} = C \| h \|_{L^p}.
\]
Hence the proof of the proposition is complete. \[\square\]

The following example shows that the $L^p$ boundedness of an operator $T$ is not an automatic consequence of the compactness of $T$ on $L^2$.

**Example 1.** Set

$$\phi(z) = \exp \left( \frac{-1}{1 - |z|} \right)$$

and consider the weighted Bergman space $A^2(\mathbb{D}, \phi)$ on the unit disc. The weighted Bergman projection $B_\phi$ is studied in [Dos04, Dos07, Zey13], and it was noted that $B_\phi$ is unbounded on $L^p(\mathbb{D}, \phi)$ for any $p \neq 2$.

We define an operator $T$ on $L^2(\mathbb{D}, \phi)$ by

$$T : L^2(\mathbb{D}, \phi) \to L^2(\mathbb{D}, \phi)$$

$$Tf(z) = B_\phi(f)(z)(1 - |z|^2)^2.$$  

The operator $T$ is bounded, linear and self-adjoint. Furthermore, we show that $T$ is compact on $L^2(\mathbb{D}, \phi)$ yet it is unbounded on $L^p(\mathbb{D}, \phi)$ for any $p \neq 2$.

First we show that $T$ is compact. For $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \mathbb{D}$ such that $(1 - |z|^2)^2 < \varepsilon$ on $\mathbb{D} \setminus K$. We define $S_\varepsilon f = \chi_{K_\varepsilon} Tf$ where $\chi_{K_\varepsilon}$ is the characteristic function of $K_\varepsilon$. Montel’s theorem implies that $S_\varepsilon$ is compact.

$$\|Tf\|^2 = \|Tf\|^2_{L^2(\mathbb{D}\setminus K_\varepsilon, \phi)} + \|Tf\|^2_{L^2(K_\varepsilon, \phi)} \leq \varepsilon \|B_\phi f\|^2 + \|S_\varepsilon f\|^2 \leq \varepsilon \|f\|^2 + \|S_\varepsilon f\|^2.$$  

That is, $T$ satisfies compactness estimates and by Lemma 4 it is a compact operator on $L^2(\mathbb{D}, \phi)$ (see also [D’A02, Proposition V.2.3] or [Str10, Lemma 4.3]).

Next we show that $T$ is unbounded on $L^p(\mathbb{D}, \phi)$ for any $p \neq 2$. Let $0 < p < 2$ and

$$f_n(z) = z^{kn}$$

where $k$ is a positive integer to be determined later. Then one can compute that

$$Tf_n(z) = a_n z^{kn-n} (1 - |z|^2)^2$$

where

$$a_n = \frac{\int_{\mathbb{D}} |z|^{2kn} \phi(z) dA(z)}{\int_{\mathbb{D}} |z|^{2(k-1)n} \phi(z) dA(z)}.$$  

Furthermore,

$$\|Tf_n\|^p_p \leq \left( \frac{\int_{\mathbb{D}} |z|^{2kn} \phi(z) dA(z)}{\int_{\mathbb{D}} |z|^{2(k-1)n} \phi(z) dA(z)} \right)^p \frac{\int_{\mathbb{D}} |z|^{pkn-pn} (1 - |z|^2)^2 \phi(z) dA(z)}{\int_{\mathbb{D}} |z|^{pkn+pn} \phi(z) dA(z)}.$$  

We need the following asymptotic [Dos07, Lemma 1]

$$\int_{\mathbb{D}} |z|^t (1 - |z|^2)^{2s} \phi(z) dA(z) \sim (t + 1)^{-\frac{4s-3}{4}} \exp(-2\sqrt{t+1})$$
as $t \to \infty$.

We have the following asymptotic computations

$$
\frac{\|Tf_n\|_p^p}{\|f_n\|_p^p} = \left( \frac{\int_D |z|^{p_k n} \phi(z) dA(z)}{\int_D |z|^{2(p_k - 1)} \phi(z) dA(z)} \right)^p \frac{\int_D |z|^{|p_k n - p_n| (1 - |z|^2)^{2p} \phi(z) dA(z)}}{\int_D |z|^{|p_k n + p_n| \phi(z) dA(z)}}
$$

$$
\sim \frac{(2kn + 1)^{-3p/4} \exp(-2p\sqrt{2kn + 1})}{(2kn - 2n + 1)^{-3p/4} \exp(-2p\sqrt{2kn - 2n + 1})}
\times \frac{(pkn - pn + 1)^{-4p/3} \exp(-2\sqrt{pkn - pn + 1})}{(pkn + pn + 1)^{-3/4} \exp(-2\sqrt{pkn + pn + 1})}.
$$

$$
\sim C_{k, p} n^{-p} \exp(2D_{k, p, n})
$$

as $n \to \infty$ where

$$
C_{k, p} = \frac{(k + 1)^{3/4}}{p^p k^{3p/4} (k - 1)^{(3 + p)/4}}
$$

and

$$
D_{k, p, n} = -p\sqrt{2kn + 1} + p\sqrt{2kn - 2n + 1} - \sqrt{pkn - pn + 1} + \sqrt{pkn + pn + 1}
$$

$$
= \frac{-2pn}{\sqrt{2kn + 1} + \sqrt{2kn - 2n + 1}} + \frac{2pn}{\sqrt{pkn - pn + 1} + \sqrt{pkn + pn + 1}}
$$

$$
\geq \frac{pn}{\sqrt{pkn + pn + 1}} - \frac{pn}{\sqrt{2kn - 2n + 1}}
$$

$$
\geq p\sqrt{n} \left( \frac{1}{\sqrt{p} + 1} - \frac{1}{\sqrt{2k - 2}} \right).
$$

Then one can show that for large $k$ we have

$$
\frac{1}{\sqrt{p} + 1} - \frac{1}{\sqrt{2k - 2}} \geq \frac{1}{2} \left( \frac{1}{\sqrt{p}} - \frac{1}{\sqrt{2}} \right) > 0.
$$

Therefore, for large $k$ we have

$$
D_{k, p, n} \geq \frac{\sqrt{np}}{2} \left( \frac{1}{\sqrt{p}} - \frac{1}{\sqrt{2}} \right).
$$

Therefore, for $k$ large enough we have $C_{k, p} n^{-p} \exp(2D_{k, p, n}) \to \infty$ as $n \to \infty$. Then we conclude that $\frac{\|Tf_n\|_p}{\|f_n\|_p} \to \infty$ as $n \to \infty$. Hence $T$ is not bounded on $L^p(D, \phi)$ for any $p < 2$. Furthermore, the fact that $T$ is self-adjoint implies that $T$ is unbounded on $L^p(D, \phi)$ for any $p \neq 2$.

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