Duality between the massive sine-Gordon and the massive Schwinger models at finite temperature

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Abstract

The massive Schwinger and the massive sine-Gordon models are proved to be equivalent at finite temperature, using the path-integral framework. The well known relations among the parameters of these models to establish the duality at $T = 0$, also remain valid at non zero temperature.

1 Introduction

Dualities in 1+1 dimensional quantum field theories are very well explored\cite{1} since Coleman first reported the duality between the sine-Gordon and the Thirring model \cite{2}. Under the duality map, the soliton of the sine-Gordon theory is mapped to the fundamental fermion of the Thirring model and the meson states of the SG theory to the fermion anti-fermion bound states. This duality has been extended for the finite temperature case using the path integral approach developed by Naon and others\cite{3,4,5,6}. Likewise, there is also a duality in between the massive Schwinger model and the massive sine-Gordon model\cite{3} but the finite temperature extension of this duality has not been reported yet. It would be intriguing to see if the compactification of the time variable into a circle of radius $\beta = 1/T$ preserves the equivalence between the massive sine-Gordon and the massive Schwinger models at a fixed radius $\beta$ (or at a fixed temperature).

The path integral approach to study 1+1 dimensional field theories developed by Naon and others has been very useful to understand dualities\cite{3,4}. Note, neither the massive Schwinger nor the massive Thirring models are exactly solvable. Both of them can be solved using this path integral method by treating the massless part exactly while doing a perturbative expansion over the mass parameter\cite{3}. The partition function of the massive Thirring (massive Schwinger) is then identified term by term with the sine-Gordon model (massive sine-Gordon) with appropriate duality transformation first suggested by Coleman\cite{2}. The finite temperature extension of such a duality has been successfully reported for the massive Thirring/sine-Gordon\cite{4} but not yet for the massive Schwinger/massive sine-Gordon. One of the main reasons why it is not yet reported is due to the confusion with the finite temperature spectrum of exactly solvable sector (massless Schwinger model). According to the path integral approach of M. V. Manias, C. M. Naon and M. L. Trobo\cite{5}, the thermodynamic partition function of the massless Schwinger model consists of a massive boson, a massless fermion along with zero-mass gauge excitation. But the study of S. T. Love\cite{7} in the

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\textsuperscript{1}Massive/massless sine-Gordon models are also not exactly solvable. Same treatment is also done in this side where we treat the potential term of the sine-Gordon perturbatively.
operator approach concludes that this same theory is completely equivalent to an ensemble of noninteracting, neutral, massive, Bose particles of mass \( \frac{e}{\sqrt{\pi}} \). This discrepancy can be taken care of by following some very simple steps (see Appendix) and it is found that the finite temperature particle spectrum of this sector is nothing but a massive free boson of mass \( \frac{e}{\sqrt{\pi}} \). Therefore, the duality between massive Schwinger and massive sine-Gordon models can hopefully be extended using the path integral approach[4, 3, 5] at finite temperature background. That would be the main goal of this paper.

2 The massive Schwinger and massive sine-Gordon models at finite temperature

We start with the partition function of the massive Schwinger model in the imaginary time formalism using the path integral approach at finite temperature[8, 9]. The Lagrangian density of the well-known massive Schwinger model,

\[
\mathcal{L}_S = -\bar{\psi}(i\partial + eA)\psi + \text{im}_0\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F_{\mu\nu},
\]

where,

\[
\gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \gamma_1 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.
\]

The partition function in Euclidean (1+1)-dimensional space-time is given,

\[
Z_S = N_0N_\beta \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\int d^2x \mathcal{L}_s},
\]

where \( \int d^2x = \int_0^\beta dx_0 \int_{-\infty}^{\infty} dx_1 \) with \( \beta = \frac{1}{T} \). Here \( N_0 \) denotes an infinite constant which does not depend on temperature whereas \( N_\beta \) is a temperature-dependent constant which is to be determined from the free partition function\[2\]. The functional integral is carried over fermionic fields obeying antiperiodic boundary conditions in the time direction,

\[
\psi(x_0, x_1) = -\psi(x_0 + \beta, x_1).
\]

Now doing the decoupling transformation,

\[
\psi = e^{\gamma_5\phi} \chi,
\]

\[
\bar{\psi} = \bar{\chi} e^{\gamma_5\phi},
\]

\[
A_\mu = -\frac{1}{e} \epsilon_{\mu\nu} \partial_\nu \phi + \partial_\mu \eta.
\]

Here the bosonic and fermionic fields obey periodic and anti-periodic boundary conditions in the time direction respectively,

\[
\phi(x_0 + \beta, x_1) = \phi(x_0, x_1),
\]

\[
\chi(x_0 + \beta, x_1) = -\chi(x_0, x_1).
\]

While doing the decoupling transformation one has to take into account the appropriate Jacobian.

\[
\mathcal{D}\psi \mathcal{D}\bar{\psi} = J_F \mathcal{D}\chi \mathcal{D}\bar{\chi},
\]

\[
\mathcal{D}A_\mu = J_A \mathcal{D}\phi \mathcal{D}\eta.
\]

\[2\] see for instance eq. 2.2 of ref. [5]
The first Jacobian $J_F$ is not trivial due to the anomaly and the fact that we perform a chiral transformation. It is computed for finite temperature in Fujikawa method\[10\] and is given by
\[
J_F = e^{-\frac{1}{2\pi} \int d^2x (\partial_t \phi)^2}.
\] (12)

And the bosonic Jacobian is is given by,
\[
J_A = \det(-\nabla^2 g),
\] (13)

and working on a Lorentz Gauge we find out\[3, 5\],
\[
Z_S = N_0 N_\beta \int \mathcal{D}\phi \mathcal{D}\bar{\chi} e^{-\int d^2x \mathcal{L}_{eff}},
\] (14)

where,
\[
\mathcal{L}_{eff} = -i\bar{\chi} \gamma^\mu \partial_\mu \chi + \frac{1}{2e^2} \phi \Box \Box \phi - \frac{1}{2\pi} \phi \Box \phi + im_0 \bar{\chi} e^{2\gamma^5 \phi} \chi.
\] (15)

Now defining the finite temperature bosonic propagator $\Delta'_F(x)$,
\[
\frac{1}{e^2} \Box \Box - \frac{1}{\pi} \Box \Delta'_F(x) = \delta^2(x).
\] (16)

Solving,
\[
\Delta'_F(x) = \pi(\Delta_F(0, x) - \Delta_F\left(\frac{e}{\sqrt{\pi}}, x\right)),
\] (17)

here, $\Delta_F(m_0, x)$ denotes a free scalar propagator with mass $m_0$ at finite temperature in two dimensions. And it can be written as\[4\],
\[
\Delta_F(m_0, x) = \frac{1}{2\pi} K_0(m_0\beta|Q(x)|),
\] (18)

whereas,
\[
\Delta_F(0, x) = -\ln\left(\frac{\beta c |Q(x)|}{\sqrt{\pi}}\right),
\] (19)

where $c$ is a numerical constant (related to Euler’s constant). Here, the dimensionless “generalized coordinates” $Q \equiv (Q_0, Q_1)$ are\[4\],
\[
Q_0 = -\cosh\left(\frac{x_1\pi}{\beta}\right) \sin(\frac{x_0\pi}{\beta})
\] (20)
\[
Q_1 = -\sinh\left(\frac{x_1\pi}{\beta}\right) \cos(\frac{x_0\pi}{\beta})
\] (21)

And the massless fermionic propagator at finite temperature in terms of generalized coordinates is,
\[
S(x) = \frac{i}{\beta} \frac{Q(x)}{Q^2(x)}
\] (22)

Now the partition function can be written as,
\[
Z_S = CZ_{BE} \sum_{n=0}^{\infty} \frac{(-im_0)^n}{n!} \left(\prod_{j=1}^n \int d^2x_j \bar{\chi}(x_j) e^{2\gamma^5 \phi} \chi(x_j)\right)_T.
\] (23)
Here, \((.\ldots)_T\) denotes the thermal average over unperturbed ensemble and \(Z_{BE}\) is the Bose-Einstein distribution for massive boson\(^{[11]}\) of mass \(\frac{e}{\sqrt{\pi}}\). This is the contribution from the massless part of the Lagrangian (see appendix to find out why it is equal to \(Z_{BE}\)). \(C\) is an irrelevant constant related to the zero-point energies which can be absorbed in normalization constant. \(Z_{BE}\) is defined as\(^{[11]}\),

\[
\ln Z_{BE} = - \int dk \ln (1 - e^{-\beta \sqrt{k^2 + \frac{e^2}{\pi}}}),
\]

(24)

In order to compute the partition function, one has to separate the boson factor from the free fermionic part by using the identity,

\[
\bar{\chi} e^{2\gamma_5 \phi} \chi = e^{2\phi} \bar{\chi} \frac{1 + \gamma_5}{2} \chi + e^{-2\phi} \bar{\chi} \frac{1 - \gamma_5}{2} \chi.
\]

(25)

Therefore, the partition function now becomes (ignoring constant zero point energy),

\[
Z_S = Z_{BE} \sum_{k=0}^{\infty} \frac{(-im_0)^{2k}}{k!^2} \prod_{j=1}^{k} \int d^2x_j d^2y_j \mathcal{I}_{1 > \text{boson} < I_2 > \text{fermion}},
\]

(26)

where \(I_1\) and \(I_2\) are defined as,

\[
I_1 = e^{2\sum_i (\phi(x_i) - \phi(y_i))}
\]

(27)

\[
I_2 = \prod_{j=1}^{k} \bar{\chi}(x_j) \frac{1 + \gamma_5}{2} \chi(x_j) \bar{\chi}(y_j) \frac{1 - \gamma_5}{2} \chi(y_j),
\]

(28)

Now using this well-known identity,

\[
\langle e^{-\sum_j \nu_j \phi(x_j)} \rangle_{\text{bosonic}} = e^{\frac{1}{2} \sum_{i,j} \nu_i \nu_j [\Delta^2 Q(x_i - x_j)]},
\]

(29)

we can solve the bosonic part of the thermal average and using the well known properties of \(\gamma_5\) we can rewrite eq. (28) as,

\[
I_2 = \prod_{j=1}^{k} \bar{\chi}_1(x_j) \chi_1(x_j) \bar{\chi}_2(y_j) \chi_2(y_j),
\]

(30)

where,

\[
\chi = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}.
\]

(31)

This quantity \(\prod_{j=1}^{k} \langle \bar{\chi}_1(x_j) \chi_1(x_j) \bar{\chi}_2(y_j) \chi_2(y_j) \rangle\), i.e. the fermionic part of thermal average can be easily calculated for massless fermions\(^{[11]}\),

\[
\prod_{j=1}^{k} \langle \bar{\chi}_1(x_j) \chi_1(x_j) \bar{\chi}_2(y_j) \chi_2(y_j) \rangle = (-1)^k \prod_{i>j}^{k} (\beta |Q(x_i - x_j)|^2 (\beta |Q(y_i - y_j)|)^2) /
\]

\[
\prod_{i,j}^{k} (\beta |Q(x_i - y_j)|^2).
\]

(32)
As a result, the partition function becomes,

\[
Z_S = Z_{BE} \sum_{k=0}^{\infty} \frac{(\frac{\text{mec}}{\sqrt{\pi}})^{2k}}{(k!)^2} \prod_{j=1}^{k} \int d^2 x_j d^2 y_j \left[ \sum_{i>j} \beta Q(x_i - x_j))^2 | \beta Q(y_i - y_j))|^2 \sum_{i<j} | \beta Q(x_i - x_j)|^{-2} | \beta Q(y_i - y_j)|^{-2} \right] \\
\exp\left\{ -2 \sum_{i>j} K_0(\frac{\beta e}{\sqrt{\pi}} | Q(x_i - x_j))|) + K_0(\frac{\beta e}{\sqrt{\pi}} | Q(y_i - y_j))| - K_0(\frac{\beta e}{\sqrt{\pi}} | Q(x_i - y_j))| \right\} \\
- K_0(\frac{\beta e}{\sqrt{\pi}} | Q(x_i - y_j))| \right\},
\]

\[ (33) \]

Here, \( m = \frac{m_0}{2\pi} \). We can see that the massless bosonic excitation cancels out the fermionic contribution to \( Z_S \). Turning our attention towards the Lagrangian of the massive sine-Gordon model,

\[
\mathcal{L}_{SG} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m_{SG} \phi^2 - \frac{\alpha}{\lambda^2} \cos(\lambda \phi)
\]

(34)

The partition function for this model,

\[
Z_{SG} = N_0 N_\beta \int \mathcal{D}\phi e^{-\int d^2 x \mathcal{L}_{SG}}.
\]

(35)

where the integration runs over scalar fields is periodic in the time direction,

\[
\phi(x_0 + \beta, x_1) = \phi(x_0, x_1)
\]

(36)

With the help of bosonic identity eq. (29), the partition function of massive sine Gordon model (after identifying \( m_{SG} = \frac{m_0}{\sqrt{\pi}} \)),

\[
Z_{SG} = Z_{BE} \sum_{k=0}^{\infty} \frac{\alpha}{\lambda^2}^{2k} \frac{1}{(1!^2)^{k}} \prod_{j=1}^{k} \int d^2 x_j d^2 y_j \left[ -\frac{\lambda^2}{2\pi} \sum_{i>j} K_0(\frac{\beta e}{\sqrt{\pi}} | Q(x_i - x_j))|) + K_0(\frac{\beta e}{\sqrt{\pi}} | Q(y_i - y_j))|) \right] \\
- K_0(\frac{\beta e}{\sqrt{\pi}} | Q(x_i - y_j))| \right\}
\]

(37)

Comparing \( Z_S \) and \( Z_{SG} \), we see that the two partition functions are identical provided the relations,

\[
\frac{\alpha}{\lambda^2} = \frac{\text{mec}}{\sqrt{\pi}}
\]

(38)

\[
\lambda^2 = 4\pi
\]

(39)

These are the exact duality transformation for the zero temperature case studied by Naon. The perturbative series in the mass term of the massive Schwinger model is found to be term-by-term identical with the perturbative series in \( \alpha \) for the massive sine-Gordon model, provided the relations among the parameters of two models given in Eqs. (38)-(39) are taken into account. Finally, we have proved using the path integral framework that compactification of the time variable into a circle with radius \( \beta = 1/T \) conserve the equivalence between the massive sine-Gordon and the massive Schwinger models at fixed radius (i.e. fixed temperature) \( \beta \), just like \( T = 0 \). We can go into several directions based on the result of this work. For instance, following the recent development of calculating entanglement entropy at zero temperature vacuum for Sine Gordon and Schwinger models, it would be intriguing to calculate the entanglement entropy of these models at finite temperature, given the duality is now established at finite temperature. Besides
that, It has been already proved that this duality holds even for curved space background at zero temperature. We would like to check the status of the duality for curved space at finite temperature.

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3 Appendix: Finite temperature spectrum of Massless Schwinger model

The Lagrangian density of the massless Schwinger model in Euclidean (1+1)-dimensional space-time is given as,

\[ \mathcal{L} = -\bar{\psi}(i\partial + eA)\psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \]  

(40)

The partition function can be written as,

\[ Z = N_0 N_\beta \int DA_\mu D\psi D\bar{\psi} e^{-\int_{\beta} d x L_s}, \]

(41)

where, \( \int_{\beta} d^2 x = \int_{0}^{\beta} dx_0 \int_{-\infty}^{\infty} dx_1 \) with, \( \beta = \frac{1}{T} \). Also, the functional integral is performed over fermionic fields satisfying antiperiodic boundary conditions in the time direction,

\[ \psi(x_0, x_1) = -\psi(x_0 + \beta, x_1). \]

(42)

Now, according to ref. [7], the partition function at finite temperature can be written as,

\[ \ln Z = 2 \int \frac{dk}{2\pi} \left\{ \frac{k\beta}{2} + \ln(1 + e^{-\beta k}) \right\} + \int \frac{dk}{2\pi} \left\{ \frac{\beta}{2} (k - k') + \ln \frac{1 - e^{-k\beta}}{1 - e^{-k'\beta}} \right\}, \]

(43)

where, \( k'^2 = k^2 + \frac{e^2}{\pi} \). It has been quoted in reference [5] that the finite temperature partition function of massless Schwinger model is not equal to corresponding to the massive boson times free massless fermions, as one could have naively expected. But rather in this partition function there is also a factor associated to the zero-mass gauge excitation which appears in the Lowenstein-Swieca solution for the massless Schwinger model. This is in disagreement with the result of Love[7]. Because according to ref. [7] the finite temperature particle content of the theory contains only a massive non interacting boson of mass \( \frac{e}{\sqrt{\pi}} \). They showed it by simply taking the thermal ensemble average of the Schwinger model Hamiltonian. And it is then shown to be equivalent to an ensemble of neutral, massive, noninteracting Bose particles with all massless excitations absent. This is clearly not the conclusion of ref. [5]. But final form of the partition function of ref. [5] (i.e. eq. 42) actually agrees secretly with Love[7]. It can be verified in some very simple steps. Rewriting eq. (42),

\[ \ln Z = \ln Z_{\text{vac}} + 2 \ln Z_F - \ln Z_{BE} + \ln Z'_{BE}, \]

(44)

3see text after equation 3.27
where,

\[ \ln Z_{\text{vac}} = \int \frac{dk}{2\pi} \left( \frac{3}{2} k - \frac{1}{2} k' \right) \beta, \]

(45)

\[ \ln Z_F = \int \frac{dk}{2\pi} \ln(1 + e^{-\beta k}), \]

(46)

\[ \ln Z_{BE} = - \int \frac{dk}{2\pi} \ln(1 - e^{-\beta k}), \]

(47)

\[ \ln Z'_{BE} = - \int \frac{dk}{2\pi} \ln(1 - e^{-\beta k'}). \]

(48)

Here, \( \ln Z_{\text{vac}} \), \( \ln Z_F \), \( \ln Z_{BE} \), \( \ln Z'_{BE} \) denote vacuum energy, free massless fermionic, free massless bosonic and free massive bosonic contribution (with mass \( \frac{e}{\sqrt{\pi}} \)) to the the partition function. Now using the following integral formulas[11, 12]4,

\[ - \int \frac{dk}{2\pi} \ln(1 - e^{-\beta k}) = \frac{1}{\beta} \frac{\pi^2}{6}, \]

(49)

\[ 2 \int \frac{dk}{2\pi} \ln(1 + e^{-\beta k}) = \frac{1}{\beta} \frac{\pi^2}{6}, \]

(50)

we can rewrite the partition function. Due to the existence of minus sign in front of \( \ln Z_B \) in eq. (6), massless fermionic and massless bosonic contributions cancel each other and we are left with,

\[ \ln Z = \ln Z_{\text{vac}} + \ln Z'_B. \]

(51)

Using the usual definition, \( \langle H \rangle = -\frac{\partial}{\partial \beta} \ln Z \) we easily find out from eq. (6),

\[ \langle H \rangle_T = \langle H \rangle_{T=0} + \int \frac{dk}{2\pi} \frac{k}{e^{k^2 + \frac{e^2}{\pi}} - 1}, \]

(52)

which is the final result of Love (eq. 45 of ref. [7]). Therefore, it can now be concluded that finite temperature particle spectrum in both of the approaches match with each other, predicting finite temperature Schwinger model contains only a massive non interacting boson of mass \( \frac{e}{\sqrt{\pi}} \).

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