UNIQUENESS OF QUASI–COXETER STRUCTURES ON KAC–MOODY ALGEBRAS

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Abstract. Let \( g \) be a symmetrisable Kac–Moody algebra, and \( U_t g \) the corresponding quantum group. We showed in [1] that the braided quasi–Coxeter structure on integrable, category \( \mathcal{O} \) representations of \( U_t g \) which underlies the \( R \)–matrix actions arising from the Levi subalgebras of \( U_t g \) and the quantum Weyl group action of the generalised braid group \( B_g \) can be transferred to integrable, category \( \mathcal{O} \) representations of \( g \). We prove in this paper that, up to unique equivalence, there is a unique such structure on the latter category with prescribed \( R \)–matrices and local monodromies. This extends, simplifies and strengthens a similar result of the second author valid when \( g \) is semisimple, and will be used in [2] to describe the monodromy of the rational Casimir connection of an affine Kac–Moody algebra in terms of the corresponding quantum Weyl group operators. Our main tool is a refinement of Enriquez’s universal algebras, which is adapted to the PROP describing a Lie bialgebra endowed with a collection of subalgebras labelled by the subdiagrams of the Dynkin diagram of \( g \).

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1. INTRODUCTION

1.1. This is the second of three papers whose goal is to extend the description of the monodromy of the rational Casimir connection in terms of quantum Weyl group operators given in [13, 14, 15, 16] to the case of an affine Kac–Moody algebra.

In [1], we defined the notion of braided quasi–Coxeter category, which is informally a tensor category carrying commuting actions of Artin’s braid groups and a given generalised braid group on the tensor product of its objects. We showed that such a structure arises from the quantum group $U_\hbar g$ of a symmetrisable Kac–Moody algebra $g$, specifically on the category $O^\text{int}_\hbar$ of integrable, highest weight representations of $U_\hbar g$. The corresponding Artin group actions are given by the universal $R$–matrices of the Levi subalgebras of $U_\hbar g$, and the action of the generalised braid group by the quantum Weyl group operators of $U_\hbar g$. The main result of [1] is that this structure can be transferred to the category $O^\text{int}$ of integrable, highest weight modules for $g$. The proof of this fact relies on the construction of a relative version of the Etingof–Kazhdan quantisation functor which takes as input an inclusion of Lie bialgebras $a \subset b$, and allows in particular to construct an equivalence $O^\text{int}_\hbar \cong O^\text{int}$ which is compatible with a given chain of Levi subalgebras of $g$.

1.2. The goal of the present paper is to prove that $O^\text{int}$ possesses, up to unique equivalence, a unique braided quasi–Coxeter structure with prescribed $R$–matrices and local monodromies. This will be used in [2] to prove that the monodromy of the rational Casimir connection of an affine Kac–Moody algebra $g$ is described by the quantum Weyl group operators of $U_\hbar g$, by showing that the monodromy of the rational KZ and Casimir connections of $g$ arise from a braided quasi–Coxeter structure on $O^\text{int}$.

1.3. The uniqueness of braided quasi–Coxeter structures on $O^\text{int}$ is obtained from a cohomological rigidity result, as was done in the case of a semisimple Lie algebra in [14, 15]. The proof of this result, however, differs significantly from that given in [14, 15]. Indeed, the latter relies on the well–known computation of the Hochschild (coalgebra) cohomology of the enveloping algebra $Ug$ in terms of the exterior algebra $\wedge g$. For an arbitrary Kac–Moody algebra $g$, the tensor powers of $Ug$ need to be replaced by their completion $\hat{Ug}^\otimes n$ with respect to category $O$, since $Ug$ and $Ug^{\otimes 2}$ do not contain the Casimir operator $C$ and the invariant tensor $\Omega = \Delta(C) - C \otimes 1 - 1 \otimes C$ respectively, and are therefore not an appropriate receptacle for the coefficients of the Casimir and KZ connections. While the computation of the Hochschild cohomology of $Ug$ holds for an arbitrary Lie algebra, it is not known to do so, and may in fact fail, for the topological Hopf algebra $\hat{Ug}$, which seems to have a rather unwieldy cohomology.

1.4. Instead of using $\hat{Ug}$, we rely on a refinement of the universal algebra $Ug_{\text{univ}}$ defined by Enriquez. The algebra $Ug_{\text{univ}}$ was introduced in [6] to obtain an alternative cohomological construction of quantisation functors for Lie bialgebras [8], and is obtained from the PROP of Lie bialgebras $\text{LBA}$. It is universal in that it maps to a completion $\hat{Ug}_a$ of the enveloping algebra of the double $g_a = a \oplus a^*$ of any Lie bialgebra $a$. Its image in $\hat{Ug}_a$ is the subalgebra generated by the interlaced powers of the normally ordered Casimir operator of $g_a$, that is consists of all elements of
the form
\[ \sum_{i_1, \ldots, i_N} a_{i_1} a_{i_2} \cdots a_{i_N} : b^\sigma(1) b^\sigma(2) \cdots b^\sigma(N) \]
where \( \{a_i\}, \{b^j\} \) are dual bases of \( \mathfrak{a} \) and \( \mathfrak{a}^* \), \( N \) is an arbitrary integer, and \( \sigma \) a permutation in \( S_N \). The relevant completion of \( U_{\mathfrak{g}_a} \) is with respect to the category of equicontinuous \( \mathfrak{g}_a \)-modules, that is modules on which the action of \( \mathfrak{a}^* \) is locally finite, which is the appropriate generalisation of category \( \mathcal{O} \) for any Drinfeld double.

The crucial insight of [6, 8] is that the algebra \( U_{\mathfrak{g}_{univ}} \) contains enough elements to allow for the construction of quantisation functors, and yet has a computable Hochschild cohomology, which is given by a universal version of the exterior algebra \( \wedge \mathfrak{g}_a \).

1.5. We give in this paper an alternative, and perhaps more natural construction of \( U_{\mathfrak{g}_{univ}} \), by considering the colored PROP \( DY \) which describes a Lie bialgebra \( \mathfrak{a} \), together with a Drinfeld–Yetter module \( V \), that is a vector space \( V \) endowed with an action and coaction of \( \mathfrak{a} \) satisfying an appropriate compatibility condition (the category of such modules is easily seen to be equivalent to that of equicontinuous modules over the double \( \mathfrak{g}_a \) of \( \mathfrak{a} \), but the former is more amenable to a description in terms of PROPs). We then consider the algebra \( U_{\mathfrak{q}_{univ}} = \text{End}_{\mathfrak{DY}}(V) \), and show it to be isomorphic to \( U_{\mathfrak{g}_{univ}} \). This alternative construction makes the algebra structure on \( U_{\mathfrak{g}_{univ}} \), and its action on equicontinuous \( \mathfrak{g}_a \)-modules far more transparent.

We then introduce two refinements of \( U_{\mathfrak{q}_{univ}} \). The first one, \( U_{\mathfrak{q}_{PDY}} \), is obtained from the PROP describing an inclusion of Lie bialgebras \( \mathfrak{a} \subset \mathfrak{b} \), together with a Drinfeld–Yetter module over \( \mathfrak{b} \). Its image in \( \hat{U}_{\mathfrak{g}_b} \) contains the interlaced powers of the normally ordered Casimir operators of the doubles \( \mathfrak{g}_a, \mathfrak{g}_b \) of \( \mathfrak{a} \) and \( \mathfrak{b} \), and its Hochschild cohomology can be computed using the calculus of Schur functor developed by Enriquez [8]. This allows in particular to show that the relative quantisation functor constructed in [1] is unique up to isomorphism.

The second refinement \( U_{\mathfrak{q}} \) is obtained in a similar way from a PROP describing a Drinfeld–Yetter module over a Lie bialgebra \( \mathfrak{b} \) endowed with a collection of subalgebras \( \mathfrak{a}_B \) labelled by the subdiagrams \( B \subset D \) of the Dynkin diagram \( D \) of a given Kac–Moody algebra \( \mathfrak{g} \) with root lattice \( Q \). The image of \( U_{\mathfrak{q}} \) in \( \hat{U}_{\mathfrak{g}_b} \) contains the interlaced powers of the normally ordered Casimir operators of the doubles \( \{\mathfrak{g}_{a_B}\}_{B \subset D} \), which makes it an appropriate receptacle for the coefficients of the Casimir connection of \( \mathfrak{g} \). The computation of the Hochschild cohomology of \( U_{\mathfrak{q}} \) yields the required rigidity result.

1.6. The use of the algebra \( U_{\mathfrak{q}} \) leads to far stronger uniqueness results than had been obtained in [14, 15] for a semisimple Lie algebra \( \mathfrak{g} \). Indeed, as in the case of the universal algebra \( U_{\mathfrak{g}_{univ}} \), \( U_{\mathfrak{q}} \) has trivial first Hochschild cohomology, which implies that the isomorphism of two braided, quasi–Coxeter structures on \( \mathcal{O}^{un} \) is unique up to a unique gauge. This raises the hope that the equivalences we construct may be convergent as series in the deformation parameter \( \hbar \), and could in particular be specialised to non–rational values of \( \hbar \).
1.7. Outline of the paper. In Section 2, we review the Etingof–Kazhdan quantisation of Lie bialgebras and its description in terms of the PROP $\text{LBA}$ of Lie bialgebras. In Section 3, we introduce the PROP $\text{DY}$ describing a Lie bialgebra $\mathfrak{a}$ and a Drinfeld–Yetter module $V$ over it, and the algebra $\mathcal{U}_{\text{DY}} = \text{End}_{\text{DY}}(V)$. We construct a natural basis of this algebra, prove a PBW Theorem for $\mathcal{U}_{\text{DY}}$, and show that it is isomorphic to Enriquez’s universal algebra $\mathcal{U}_{\mathfrak{g} \text{univ}}$. Section 4 carries over a number of properties of $\mathcal{U}_{\mathfrak{g} \text{univ}}$ to $\mathcal{U}_{\text{DY}}$, in particular the computation of its Hochschild cohomology. In Section 5, we introduce a refinement $\text{PLBA}$ of the PROP $\text{LBA}$, which describes an inclusion $\mathfrak{a} \subset \mathfrak{b}$ of Lie bialgebras. We then extend $\text{PLBA}$ by including a Drinfeld–Yetter module $V$ over $\mathfrak{b}$ and consider the corresponding algebra $\mathcal{U}_{\text{PDY}} = \text{End}_{\text{PDY}}(V)$, for which we prove a number of results analogous to those obtained for $\mathcal{U}_{\text{DY}}$. We use these in Section 6 to prove the uniqueness, up to a unique gauge transformation, of the relative quantisation functor constructed in [1]. In Section 7, we further refine $\text{LBA}$ to a PROP $\text{LBA}_Q$ by adding a complete family of orthogonal idempotents corresponding to the non–negative roots of a given Kac–Moody algebra with root lattice $Q$. We also study the algebra $\mathcal{U}_Q = \text{End}_{\text{DY}}(V)$, and compute in particular its Hochschild cohomology. Finally, in Section 8, we use these results to prove the uniqueness of braided quasi–Coxeter structures on the category $O^{\text{int}}$ of integrable, highest weight modules for an arbitrary Kac–Moody algebra.

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2. PROPs and Lie bialgebras

In this section, we review the notion of product-permutation category (PROP), the Etingof–Kazhdan quantisation of Lie bialgebras, and its propic description. For more details, we refer the reader to [9, 10].

2.1. Let $k$ be a field of characteristic zero. A PROP $(\mathcal{C}, S)$ is the datum of

- a strict, symmetric monoidal $k$–linear category $\mathcal{C}$ whose objects are the non–negative integers, such that $[n] \otimes [m] = [n + m]$. In particular, $[n] = 1^{\otimes n}$ and $[0]$ is the unit object;
- a bigraded set $S = \bigsqcup_{m, n \in \mathbb{Z}_{\geq 0}} S_{nm}$ of morphism of $\mathcal{C}$, with $S_{nm} \subset \mathcal{C}([n], [m])$ such that any morphism in $\mathcal{C}$ can be obtained by composition, tensor product or linear combination over $k$ of the morphisms in $S$ and the permutation maps $kS_n \subset \mathcal{C}([n], [n])$.

Every PROP $(\mathcal{C}, S)$ has a presentation in terms of generators and relations. Let $\mathcal{F}_S$ be the PROP freely generated over $\mathcal{C}$. There is a unique symmetric tensor functor $\mathcal{F}_S \to \mathcal{C}$, and $\mathcal{C}$ has the form $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is a tensor ideal in $\mathcal{F}_S$ (i.e., a collection of subspaces $\mathcal{I}_{nm} \subset \mathcal{C}([n], [m])$, such that composition or tensor product (in any order) of any morphism in $\mathcal{C}$ with any morphism in $\mathcal{I}$ is still in $\mathcal{I}$).
Let $\mathcal{C}$ be a PROP, $\mathcal{N}$ a symmetric monoidal $k$-linear category, and $X$ an object in $\mathcal{N}$. A linear algebraic structure of type $\mathcal{C}$ on $X$ is a symmetric tensor functor $\mathcal{G}_X : \mathcal{C} \to \mathcal{N}$ such that $\mathcal{G}_X([1]) = X$. A $\mathcal{C}$-module in $\mathcal{N}$ is a pair $(X, \mathcal{G}_X)$, where $X \in \mathcal{N}$ and $\mathcal{G}_X$ is a linear algebraic structure of type $\mathcal{C}$ on $X$.

2.2. Examples.

- **Associative algebras.** Let $\text{Alg}$ be the PROP $\mathcal{F}_S/I$, where the set $S$ consist of two elements $e \in S_{0,1}$ (the unit) and $m \in S_{2,1}$ (the multiplication), and $I$ is the ideal generated by the relations

\[
m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m) \\
m \circ (e \otimes \text{id}) = \text{id} = m \circ (e \otimes e)
\]

If $\mathcal{N} = \text{Vect}_k$, a linear algebraic structure of type $\text{Alg}$ on $V \in \mathcal{N}$ is an associative algebra structure on $V$.

- **Lie algebras.** Let $\text{LA}$ be the PROP generated by the set $S$ consisting of one element $\mu \in S_{2,1}$ (the bracket) subject to the relations

\[
\mu + \mu \circ (21) = 0 \\
\mu \circ (\mu \otimes \text{id}) \circ ((123) + (312) + (231)) = 0 \quad (2.1)
\]

If $\mathcal{N} = \text{Vect}_k$, a linear algebraic structure of type $\text{LA}$ on $V \in \mathcal{N}$ is a Lie algebra structure on $V$.

2.3. Universal constructions. The functor which assigns to an associative algebra $A$ the Lie algebra $L(A) = A$ with bracket $[a, b] = ab - ba$ can be lifted to a morphism of PROPs $L : \text{LA} \to \text{Alg}$ which maps $[1]_{\text{LA}} \mapsto [1]_{\text{Alg}}$ and $\mu \mapsto m - m \circ (21)$.

The above definition of PROP is, however, not well-suited for the description of the adjoint functor, which assigns to any Lie algebra $\mathfrak{g}$ its universal enveloping algebra $U\mathfrak{g}$. If $\mathfrak{g}$ is a Lie algebra in $\text{Vect}_k$, $U\mathfrak{g}$ is isomorphic to the symmetric algebra $S\mathfrak{g}$ as a vector space, with identification given by the symmetrisation map $\xi : U\mathfrak{g} \to S\mathfrak{g}$. The multiplication $m$ in $U\mathfrak{g}$ is realised on $S\mathfrak{g}$ via multiplication maps $m^{k}_{i,j} : S^i\mathfrak{g} \otimes S^j\mathfrak{g} \to S^k\mathfrak{g}$, satisfying

\[
\xi \circ m \circ (\xi^{-1} \otimes \xi^{-1}) = \sum_{k \leq i+j} m^{k}_{i,j}
\]

The maps $m^{k}_{i,j}$ are given by the Campbell–Hausdorff series, and can be expressed in terms of the bracket on $\mathfrak{g}$ (see for example [3]).

2.4. The Karoubi envelope. In order to describe the above procedure in propic terms, we need to construct the subobjects $S^k[1] \subset [k]$ in $\text{LA}$. This requires replacing $\text{LA}$ with its Karoubi envelope $\hat{\text{LA}}$.

The Karoubi envelope of a category $\mathcal{C}$ is the category $\hat{\mathcal{C}}$ whose objects are pairs $(X, \pi)$, where $X \in \mathcal{C}$ and $\pi : X \to X$ is such that $\pi^2 = \pi$. The morphisms in $\hat{\mathcal{C}}$ are defined to be

\[
\hat{\mathcal{C}}((X, \pi), (Y, \rho)) = \{ f \in \mathcal{C}(X,Y) \mid \rho \circ f = f = f \circ \pi \}
\]

In particular,

\[
\hat{\mathcal{C}}((X, \text{id}), (Y, \text{id})) = \mathcal{C}(X,Y)
\]

and the canonical functor $\mathcal{C} \to \hat{\mathcal{C}}$, mapping $X \mapsto (X, \text{id})$, $f \mapsto f$, is fully faithful. Every idempotent in $\hat{\mathcal{C}}$ splits. Namely, given $(X, \pi) \in \hat{\mathcal{C}}$, there are maps

\[
i = \pi : (X, \pi) \to (X, \text{id}) \quad \text{and} \quad p = \pi : (X, \text{id}) \to (X, \pi)
\]
such that $p \circ i = \text{id}_{(X, \pi)}$.

If $\mathcal{C}$ is a PROP, the Karoubi envelope $\mathcal{C}$ contains the image of all idempotents in $k \mathcal{G}_a$. This allows, for example, to consider the objects $S^n[1]$, for any $n \geq 0$. Finally, one takes the closure under, possibly infinite, inductive limits $S(\mathcal{C})$. In particular, $S(\mathcal{C})$ contains the symmetric algebra

$$S[1] = \bigoplus_{n \geq 0} S^n[1]$$

The construction of the universal enveloping algebra of a Lie algebra corresponds to a functor $U : \text{Alg} \to S(LA)$ mapping $[1]_{\text{Alg}} \mapsto S[1]_{\text{LA}}$.

If $\mathcal{N}$ is closed under inductive limits, then any linear algebraic structure of type $\mathcal{C}$ in $\mathcal{N}$ uniquely extends to an additive symmetric tensor functor

$$\mathcal{G}_X : S(\mathcal{C}) \to \mathcal{N}$$

2.5. The PROPs $\text{LCA}$ and $\text{LBA}$. The PROP of Lie coalgebras $\text{LCA}$ is generated by a morphism $\delta$ in bidegree $(1, 2)$ with relations dual to (2.1), namely

$$\delta \circ (21) \circ \delta = 0 \quad \text{and} \quad ((123) + (312) + (231)) \circ (\delta \otimes \text{id}) \circ \delta = 0 \quad (2.2)$$

The PROP of Lie bialgebras $\text{LBA}$ is generated by $\mu$ in bidegree $(2, 1)$ and $\delta$ in bidegree $(1, 2)$ satisfying (2.1), (2.2), and the coycle condition

$$\delta \circ \mu = (\text{id} - (21)) \circ \text{id} \otimes \mu \circ \delta \otimes \text{id} \circ (\text{id} - (21)) \quad (2.3)$$

2.6. Drinfeld doubles. Let $(a, [\cdot, \cdot]_a, \delta_a)$ be a Lie bialgebra over $k$. The Drinfeld double $g_a$ of $a$ is the Lie algebra defined as follows. As a vector space, $g_a = a \oplus a^*$. The pairing $\langle \cdot, \cdot \rangle : a \otimes a^* \to k$ extends uniquely to a symmetric, nondegenerate bilinear form on $g_a$, such that $a, a^*$ are isotropic subspaces. The Lie bracket on $g_a$ is then defined as the unique bracket compatible with $\langle \cdot, \cdot \rangle$, i.e., such that

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle$$

for all $x, y, z \in g_a$. It coincides with $[\cdot, \cdot]_a$ on $a$, and with the bracket induced by $\delta_a$ on $a^*$. The mixed bracket for $a \in a, b \in a^*$ is equal to

$$[a, b] = ad^*(b)(a) - ad^*(a)(b)$$

where $ad^*$ denotes the coadjoint action of $a^*$ on $a$ and of $a$ on $a^*$, respectively.

The Lie algebra $g_a$ is a (topological) quasitriangular Lie bialgebra, with cobracket $\delta = \delta_a \oplus (\check{-}\delta_a)$, where $\delta_a$ is the (topological) cobracket on $a^*$ induced by $[\cdot, \cdot]_a$, and $r$–matrix $r \in g_a \otimes g_a$ corresponding to the identity in $\text{End}(a) \simeq a \otimes a^* \subseteq g_a \otimes g_a$. Explicitly, if $\{a_i\}_{i \in I}, \{b^i\}_{i \in I}$ are dual bases of $a$ and $a^*$ respectively, then $r = \sum_{i \in I} a_i \otimes b^i \in a \otimes a^*$.

2.7. Drinfeld–Yetter modules. A Lie bialgebra $(a, [\cdot, \cdot]_a, \delta_a)$ has a natural category of representations called Drinfeld–Yetter modules, and denoted $DY_a$. A triple $(V, \pi, \pi^*)$ is a Drinfeld–Yetter module on $a$ if $(V, \pi)$ is an $a$–comodule, that is the map $\pi : a \otimes V \to V$ satisfies

$$\pi \circ \mu = \pi \circ (\text{id} \otimes \pi) - \pi \circ (\text{id} \otimes \pi) \circ (21) \quad (2.4)$$

$(V, \pi)$ is an $a$–comodule, that is the map $\pi^* : V \to a \otimes V$ satisfies

$$\delta \circ \pi^* = (21) \circ (\text{id} \otimes \pi^*) \circ \pi^* - (\text{id} \otimes \pi^*) \circ \pi^* \quad (2.5)$$

and the maps $\pi, \pi^*$ satisfy the following compatibility condition in $\text{End}(a \otimes V)$:

$$\pi^* \circ \pi - \text{id} \otimes \pi \circ (12) \circ \text{id} \otimes \pi^* = [\cdot, \cdot]_a \otimes \text{id} \circ \text{id} \otimes \pi^* - \text{id} \otimes \pi \circ \delta_a \otimes \text{id} \quad (2.6)$$
The category $DY_\mathfrak{g}$ is a symmetric tensor category, and is equivalent to the category $E_{\mathfrak{g}_a}$ of equicontinuous $\mathfrak{g}_a$–modules [9]. A $\mathfrak{g}_a$–module is equicontinuous if the action of the $b^i$'s is locally finite, i.e., for every $v \in V$, $b^i \cdot v = 0$ for all but finitely many $i \in I$. In particular, given $(V, \pi) \in E_{\mathfrak{g}_a}$, the coaction of $\mathfrak{a}$ on $V$ is given by

$$\pi^*(v) = \sum_i a_i \otimes b^i \cdot v \in \mathfrak{a} \otimes V$$

where the equicontinuity condition ensures that the sum is finite. The action of the $r$–matrix of $\mathfrak{g}_a$ on the tensor product $V \otimes W \in E_{\mathfrak{g}_a}$ corresponds, under the identification $E_{\mathfrak{g}_a} \simeq DY_\mathfrak{g}$ with the map $r_{VW} : V \otimes W \to V \otimes W$ given by

$$r_{VW} = \pi_V \otimes \text{id} \circ (12) \circ \text{id} \otimes \pi_W \quad (2.7)$$

**2.8. Etingof–Kazhdan quantisation.** In [9], Etingof and Kazhdan give an explicit procedure to construct a quantisation of $\mathfrak{a}$, that is a Hopf algebra $U_h^\text{EK} \mathfrak{a}$ over $k[[\hbar]]$ endowed with an isomorphism

$$U_h^\text{EK} \mathfrak{a}/hU_h^\text{EK} \mathfrak{a} \simeq U \mathfrak{a}$$

of Hopf algebras, which induces the cobracket $\delta_\mathfrak{a}$ on $\mathfrak{a}$.

The construction proceeds as follows. One considers the Drinfeld category $E_{\mathfrak{g}_a}$, which is a braided deformation of $E_{\mathfrak{g}_a}$, with associativity and commutativity constraints given by

$$\Phi_{U,V,W} = \Phi(h\Omega_{12}, h\Omega_{23}) \quad \text{and} \quad \beta_{V,W} = (12) \circ e^{\Omega/2}$$

where $U, V, W \in E_{\mathfrak{g}_a}$, $\Omega = r + r^{21}$, and $\Phi$ is a fixed Lie associator. Let $f : E_{\mathfrak{g}_a} \to \text{Vect}_{k[[\hbar]]}$ be the forgetful functor. Etingof and Kazhdan construct an explicit tensor structure on $f$, i.e., a collection of natural isomorphisms

$$J_{V,W}^\text{EK} : f(V) \otimes f(W) \to f(V \otimes W)$$

which are the identity modulo $\hbar$ and satisfy the relation

$$f(\Phi) \circ J_{V,W}^\text{EK} \circ (J_{U,V}^\text{EK} \otimes \text{id}) = J_{U,V}^\text{EK} \otimes \text{id} \circ (J_{V,W}^\text{EK}) \quad (2.8)$$

in $\text{Hom}(f(U) \otimes f(V) \otimes f(W), f(U \otimes (V \otimes W)))$.

The algebra $U^\text{EK} \mathfrak{g}_a = \text{End} (f)$ is a topological Hopf algebra, with coproduct induced by the tensor product in $E_{\mathfrak{g}_a}$. Twisting $U^\text{EK} \mathfrak{g}_a$ by $J_{V,W}^\text{EK}$ produces a new Hopf algebra, with a coassociative deformation coproduct $\Delta_f$. In order to produce a quantisation of $\mathfrak{a}$, one considers the Verma module

$$M_\mathfrak{a} = \text{Ind}_{\mathfrak{g}_a}^{\mathfrak{g}_a} C \simeq U \mathfrak{a}$$

and shows that there is a natural embedding $f(M_\mathfrak{a}) \subset \text{End} (f)$. The coproduct $\Delta_f$ induces a coproduct on $f(M_\mathfrak{a})$ which can explicitly computed as the composition

$$f(M_\mathfrak{a}) \xrightarrow{f(\Delta_f)} f(M_\mathfrak{a} \otimes M_\mathfrak{a}) \xrightarrow{(J_{M_\mathfrak{a},M_\mathfrak{a}}^\text{EK})^{-1}} f(M_\mathfrak{a}) \otimes f(M_\mathfrak{a})$$

This induces a Hopf algebra structure on the vector space $f(M_\mathfrak{a}) \simeq U \mathfrak{a}[[\hbar]]$, which quantizes the Lie bialgebra $\mathfrak{a}$. 
2.9. Etingof–Kazhdan quantisation in LBA. In a subsequent paper \[10\], Etingof and Kazhdan showed that the construction of \( J = J_{EK}^M \) is universal, i.e., it can be realized in the PROP LBA. To this end, one first replaces the \( g_a \)-module \( M \) with a Drinfeld–Yetter module in LBA by constructing an action and a coaction of the Lie bialgebra \([1] \in LBA\) on \( M := S[1] \). The twist \( J \) is then defined, using the same formulae as in \([9]\), as an element of
\[
J \in LBA(M \otimes M, M \otimes M)[[\hbar]]
\]
It induces a universal quantisation functor, that is a functor \( Q \) from the PROP of Hopf algebras \( HA \) to \( S(LBA) \), which maps \([1]_{HA} \) to \( S[1]_{LBA} \).

A universal interpretation of the fiber functor \((f, J_{EK})\), rather than of the Hopf algebra \((f(M), J^{-1} f(\Delta_0))\) only, will be given in the next section, by using the PROP of universal Drinfeld–Yetter modules.

3. Universal Drinfeld–Yetter modules

We define in this section the PROP \( DY_n \) describing \( n \) Drinfeld–Yetter modules \( \mathcal{V}_1, \ldots, \mathcal{V}_n \) over a universal Lie bialgebra \( a \). The algebra \( \mathcal{U}_{DV}^n = \text{End}_{DY_n} (\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n) \) is endowed with a canonical morphism
\[
\mathcal{U}_{DV}^n \rightarrow U_{g_a}^{\otimes n}
\]
for any Lie bialgebra \( a \), where \( g_a \) is the double of \( a \) and \( U_{g_a}^{\otimes n} \) the completion with respect to the category of equicontinuous \( g_a \)-modules. We show that the tower of algebras \( \{ \mathcal{U}_{DV}^n \} \) has a cosimplicial structure which is compatible with that on \( \{ U_{g_a}^{\otimes n} \} \), construct a basis of it, and prove a PBW theorem for \( \mathcal{U}_{DV}^n \). We also show that \( \mathcal{U}_{DV}^n \) is isomorphic to Enriquez’s universal algebra \( (U_{g_a}^{\otimes n})_{\text{univ}} \).

3.1. The PROP of universal Drinfeld–Yetter modules. Let \( n \geq 1 \).

Definition. \( DY_n \) is the multicolored PROP generated by \( n + 1 \) objects \([1]\) and \( \{ \mathcal{V}_k \}_{k=1}^n \), and morphisms
\[
\mu : [2] \rightarrow [1] \quad \delta : [1] \rightarrow [2] \\
\pi_k : [1] \otimes \mathcal{V}_k \rightarrow \mathcal{V}_k \\
\pi_k^* : \mathcal{V}_k \rightarrow [1] \otimes \mathcal{V}_k
\]
such that \(([1], \mu, \delta)\) is a Lie bialgebra in \( DY_n \), and every \((\mathcal{V}_k, \pi_k, \pi_k^*)\) is a Drinfeld–Yetter module over \([1]\).

If \((a, [\cdot, \cdot], \delta) \in \text{Vect}_k\) is a Lie bialgebra, then for any \( n \)-tuple \( \{ V_k, \pi_k, \pi_k^* \}_{k=1}^n \) of Drinfeld–Yetter modules over \( a \), there is a unique tensor functor
\[
\mathcal{G}_{(a, V_1, \ldots, V_n)} : DY_n \rightarrow \text{Vect}_k
\]
such that \([1] \mapsto a, \mathcal{V}_k \mapsto V_k \). Set
\[
\mathcal{U}_{DV}^n = \text{End}_{DY_n} \left( \bigotimes_{k=1}^n \mathcal{V}_k \right)
\]
Proposition. Let $f : \text{DY}_a \to \text{Vect}_k$ be the forgetful functor, and $\mathcal{U}^\Omega_{\text{DY}_a} = \text{End} (f^\Omega_a)$. The functors $G_{a, V_1, \ldots, V_n}$ induce an algebra homomorphism

$$\rho^a_n : \mathcal{U}^\Omega_{\text{DY}_a} \to \mathcal{U}^\Omega_{\text{DY}_a}$$

PROOF. We have to show that if $\phi \in \text{End}_{\text{DY}_a} (\bigotimes_{k=1}^n V_k)$ and $f_k \in \text{Hom}_{\text{DY}_a} (V_k, W_k)$, $k = 1, \ldots, n$, one has

$$\rho^a_n (\phi) w_1 \otimes \cdots \otimes w_n \circ f_1 \otimes \cdots \otimes f_n = f_1 \otimes \cdots \otimes f_n \circ \rho^a_n (\phi) v_1 \otimes \cdots \otimes v_n \quad (3.2)$$

For any $p \in \mathbb{N}$ and $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$ such that $|p| = p_1 + \cdots + p_n = p$, define the maps

$$\pi^{(p)} : [p] \otimes \bigotimes_{k=1}^n V_k \to \bigotimes_{k=1}^n V_k$$

(3.3)

as the ordered composition of $p_i$ actions on $V_i$. Similarly for

$$\pi^{**(p)} : [p] \otimes \bigotimes_{k=1}^n V_k \to [p] \otimes \bigotimes_{k=1}^n V_k$$

(3.4)

By the compatibility relation (2.6), we may assume that the morphism $\phi$ is the form

$$\phi = \pi^{(p)} \circ \psi \otimes \text{id}^{\otimes n} \circ \pi^{**(q)}$$

where $p, q \in \mathbb{N}^n$ are such that $|p| = p$, $|q| = q$, and $\psi \in \text{LBA} ([q], [p])$. Set $f = f_1 \otimes \cdots \otimes f_n$. Then

$$\pi^{(p)} \circ \psi \otimes \text{id}^{\otimes n} \circ \pi^{**(q)} \circ f = \pi^{(p)} \circ \psi \otimes \text{id}^{\otimes n} \circ \text{id}^{\otimes q} \circ f \circ \pi^{**(q)}$$

$$= \pi^{(p)} \circ \text{id}^{\otimes p} \circ f \circ \psi \otimes \text{id}^{\otimes n} \circ \pi^{**(q)}$$

$$= f \circ \pi^{(p)} \circ \psi \otimes \text{id}^{\otimes n} \circ \pi^{**(q)}$$

□

3.2. Universal twists in $\text{DY}_a$. The associativity relation (2.8) admits a natural lift to the PROPS $\text{DY}_a$. Note first that the functors

$$G_{(i|j|k)}, G_{(i|j \otimes k)}, G_{(i|j \otimes k \otimes l)} : \text{DY}_a \to \text{DY}_a$$

for distinct $i, j, k \in \{1, 2, 3\}$ induce algebra homomorphisms

$$\text{End}_{\text{DY}_a} (V_1 \otimes V_2) \to \text{End}_{\text{DY}_a} (V_1 \otimes V_2 \otimes V_3)$$

If $T \in \text{End}_{\text{DY}_a} (V_1 \otimes V_2)$, we denote its images under these by $T_{ij}, T_{ij,k}$ and $T_{ijk}$ respectively.

Define the $r$–matrix $r_{V_1, V_2} \in \text{End}_{\text{DY}_a} (V_1 \otimes V_2)$ by (2.7), and set

$$\Omega = r_{V_1, V_2} + (12) \circ r_{V_1, V_2} \circ (12) \in \text{End}_{\text{DY}_a} (V_1 \otimes V_2)$$

Proposition. Let

$$\Phi = \Phi (h \Omega_{12}, h \Omega_{23}) \in \text{End}_{\text{DY}_a} (V_1 \otimes V_2 \otimes V_3) \ [h]$$

be given by a Lie associator, and $J \in \text{End}_{\text{DY}_a} (V_1 \otimes V_2) \ [h]$ be such that

$$\Phi \circ J_{12,3} = J_{1,23} \circ J_{23} \quad (3.5)$$

Then, for any Lie bialgebra $a$, the element $\rho^a_n (J) \in \mathcal{U}^\Omega_{\text{DY}_a} \ [h]$ defines a tensor structure on the forgetful functor $f : \text{DY}_a^\Phi \to \text{Vect}_k [h]$. 


A simple argument, along the lines of [10, §1.4], shows that the Etingof–Kazhdan tensor structure \( J^{\otimes k}_{k W} \) can be lifted to the PROP \( \text{DY} \).

### 3.3. Cosimplicial structure of \( \text{DY}_a \)

Let \( \mathfrak{a} \) be a Lie bialgebra over \( k \). The tower \( \mathcal{U}^{\otimes n}_{DY_a} = \text{End}(f^{\otimes n}) \) is a cosimplicial complex of algebras

\[
\begin{array}{ccc}
  k & \xrightarrow{\text{End}(f)} & \text{End}(f^{\otimes 2}) & \xrightarrow{\text{End}(f^{\otimes 3})} & \cdots
\end{array}
\]

with face morphisms \( d_i^n : \text{End}(f^{\otimes n}) \to \text{End}(f^{\otimes n+1}) \), \( i = 0, \ldots, n+1 \), given by

\[
d_i^n \varphi_{X_1,\ldots,X_{n+1}} = \begin{cases}
  \text{id} \otimes \varphi_{X_2,\ldots,X_{n+1}} & i = 0 \\
  \varphi_{X_1,\ldots,X_i \otimes X_{i+1},\ldots,X_{n+1}} & 1 \leq i \leq n \\
  \varphi_{X_1,\ldots,X_n} \otimes \text{id} & i = n+1
\end{cases}
\]

for \( \varphi \in \text{End}(f^{\otimes n}) \), \( X_i \in \text{DY}_a \), \( i = 1,\ldots,n+1 \). These give rise to the Hochschild differential

\[
d_n = \sum_{i=0}^{n+1} (-1)^i d_i^n : \text{End}(f^{\otimes n}) \to \text{End}(f^{\otimes n+1})
\]

### 3.4. Cosimplicial structure of \( \mathcal{U}^{\otimes n}_{DY} \)

The above construction can be lifted to the PROPs \( \text{DY}_n \). For every \( n \geq 1 \) and \( i = 0,1,\ldots,n+1 \), there are faithful functors

\[
\mathcal{D}_n^{(i)} : \text{DY}_n \to \text{DY}_{n+1}
\]

given by

\[
\mathcal{D}_n^{(i)} = \begin{cases}
  \mathcal{G}_{(i)}(X_1,\ldots,X_{n+1}) & i = 0 \\
  \mathcal{G}_{(i)}(X_1,\ldots,X_i,\otimes X_{i+1},\ldots,X_{n+1}) & 1 \leq i \leq n \\
  \mathcal{G}_{(i)}(X_1,\ldots,X_n) & i = n+1
\end{cases}
\]

These induce algebra homomorphisms

\[
\Delta_n^{(i)} : \mathcal{U}^{\otimes n}_{DY} \to \mathcal{U}^{\otimes n+1}_{DY}
\]

which are universal analogues of the insertion/coproduct maps on \( U_{\mathfrak{g}^{\otimes n}} \). They give the tower of algebras \( \{\mathcal{U}^{\otimes n}_{DY}\}_{n \geq 0} \) the structure of a cosimplicial complex, with Hochschild differential \( d_n = \sum_{i=0}^{n+1} (-1)^i \Delta_n^{(i)} : \mathcal{U}^{\otimes n}_{DY} \to \mathcal{U}^{\otimes n+1}_{DY} \). The morphisms \( \rho^{\otimes n}_{DY} : \mathcal{U}^{\otimes n}_{DY} \to \mathcal{U}^{\otimes n}_{DY} \) defined in 3.1 are compatible with the face morphisms, and therefore with the differentials \( d_n \).

### 3.5. Factorisation of morphisms in \( \text{LBA} \)

We review in 3.5–3.7 the polarised structure of morphisms in the PROP \( \text{LBA} \) and their relation to Lie polynomials obtained in [6, 12]. We include proofs for the reader’s convenience, and because they will be adapted to the refinements of \( \text{LBA} \) introduced in Sections 5 and 7.

The inclusions \( \text{LCA}, \mathbf{LA} \subset \text{LBA} \) induce maps

\[
i_{p,q}^N : \text{LCA}([p],[N]) \otimes \text{LA}([N],[q]) \to \text{LBA}([p],[N]) \otimes \text{LBA}([N],[q]) \to \text{LBA}([p],[q])
\]

given by the composition of morphisms in \( \text{LBA} \).
Proposition. The maps \( \{ i_N^{q, p} \}_{N \geq 0} \) induce an isomorphism
\[
\text{LBA}(p, [q]) \simeq \bigoplus_{N \geq 0} \text{LCA}([p], [N]) \otimes_{\mathfrak{S}_N} \text{LA}([N], [q])
\]

Proof. Morphisms in \( \text{LBA} \) can be represented as linear combinations of oriented graphs with no loops or multiple edges, obtained by (horizontal) composition
\[
\text{LBA}(p, [q]) \otimes \text{LBA}(q, [s]) \overset{\text{comp}}{\rightarrow} \text{LBA}(p, [s])
\]
or tensor product (vertical composition)
\[
\text{LBA}(p, [q]) \otimes \text{LBA}(p', [q']) \overset{\otimes}{\rightarrow} \text{LBA}(p + p', [q + q'])
\]
The cocycle condition \( (2.3) \) allows to reorder every morphism as a linear combination of diagrams where the cobrackets horizontally precede the brackets. Finally, all permutations can be moved after the cobrackets and before the brackets, and identified with elements in \( \mathfrak{S}_N \).

The decomposition in terms of the morphisms in the PROP \( \text{LA} \) and \( \text{LCA} \) follows, and the tensor product in the proposition should be interpreted as horizontal composition of graphs. The natural map to \( \text{LBA} \) factors through the simultaneous action of \( \mathfrak{S}_N \), and provides a surjective map.

In order to check its injectivity, consider the following situation. Let \((\epsilon, \delta)\) be a Lie coalgebra and let \( F(\epsilon) \) be the corresponding Lie bialgebra with free \( \text{LA} \) structure. Let now \( p_N \in ((FL_N)_{\delta N} \otimes (FL_N)_{\delta N})_{\mathfrak{S}_N} \), \( N \geq 0 \), such that \( \sum_N p_N = 0 \) in \( \text{LBA} \).

Given the free algebra structure on \( F(\epsilon) \), there exists a collection of Lie polynomials \( P_N \in FL_N \) such that
\[
p_N = x_1 \ldots x_N \otimes P_N(y_1, \ldots, y_N)
\]
on \( F(\epsilon) \). The restriction of \( p_N \) to \( \epsilon \subset F(\epsilon) \) is equal to the coproduct map \( \delta(P_N) \).

Therefore the map
\[
\sum_{N \geq 0} \delta(P_N) : \epsilon \rightarrow \bigoplus_{N \geq 0} \epsilon^{\otimes N}
\]
is zero. It follows that each \( \delta(P_N) \) is zero. Since the map \( P \mapsto \delta(P) \) is injective, we get \( \overline{p}_N = 0 \), and finally \( p_N = 0 \) for all \( N \). \( \square \)

3.6. Morphisms in \( \text{LA} \) and \( \text{LCA} \) and Lie polynomials.

Lemma. There are natural isomorphisms, compatible with the action of \( \mathfrak{S}_N \) and \( \mathfrak{S}_n \)
\[
\text{LA}([N], [n]) \simeq (FL_N^{\otimes n})_{\delta_n} \simeq \text{LCA}([n], [N])
\]
where \( FL_N \) is the free Lie algebra on \( N \) generators, and \((FL_N^{\otimes n})_{\delta_n}\) is the subspace of its \( n \)-fold tensor product spanned by homogeneous elements of degree one in each variable.

Proof. The identification with \( \text{LA}([N], [n]) \) is obvious, while the identification with \( \text{LCA}([n], [N]) \) is obtained by duality, replacing \( \mu \) with \( \delta \) and reversing the order.

Specifically, assume first that \( n = 1 \) and let \( Q \in (FL_N)_{\delta N} \). Under the embedding \( FL_N \subset FA_N \) (the free algebra on \( N \) generators), we can write
\[
Q = \sum_{\sigma \in \mathfrak{S}_N} Q_\sigma x_{\sigma(1)} \ldots x_{\sigma(N)}
\]
and define $\delta_Q \in \text{LCA}([1], [N])$ by

$$\delta_Q = \sum_{\sigma \in S_N} Q_{\sigma} \circ (\text{id} \otimes N - 2 \otimes \delta) \circ \cdots \circ \delta$$

The definition of $\delta_Q$ is such that if $a$ is a Lie coalgebra and $b$ is a Lie algebra with a pairing $\langle \cdot, \cdot \rangle : a \otimes b \to k$ such that $\langle [a, a'], b \rangle = \langle a \otimes a', \delta(b) \rangle$, then

$$\langle \delta_Q(x), y_1 \otimes \cdots \otimes y_N \rangle = \langle x, Q(y_1, \ldots, y_N) \rangle \quad (3.6)$$

This yields an isomorphism $\text{LCA}([1], [N]) \simeq (FL_N)_{\delta_N}$. For $n \geq 1$, one has

$$\text{LCA}([n], [N]) = \bigoplus_{\sigma \in S_N, |N| = N} \bigotimes_{k=1}^n (FL_{N_k})_{\delta_{N_k}} \simeq (FL_N^\otimes n)_{\delta_N}$$

3.7. Morphisms in $\text{LBA}$ and Lie polynomials.

**Proposition.**

(i) There is an isomorphism of $(\mathfrak{S}_q, \mathfrak{S}_p)$–bimodules

$$\text{LBA}([p], [q]) \simeq \bigoplus_{N \geq 1} ((FL_N^\otimes p)_{\delta_N} \otimes (FL_N^\otimes q)_{\delta_N})_{\mathfrak{S}_N}$$

(ii) Let $F \in k\mathfrak{S}_p$ and $G \in k\mathfrak{S}_q$ be idempotents, and $F[p] = ([p], F)$, $G[q] = ([q], G)$ the corresponding objects in $\text{LBA}$. Then one has

$$\text{LBA}(F[p], G[q]) \simeq \bigoplus_{N \geq 0} (F(FL_N^\otimes p)_{\delta_N} \otimes G(FL_N^\otimes q)_{\delta_N})_{\mathfrak{S}_N}$$

**Proof.** (i) follows from Proposition 3.5 and Lemma 3.6. (ii) Normal ordering in $\text{LBA}$ gives

$$\text{LBA}(F[p], G[q]) \simeq \bigoplus_{N \geq 0} (\text{LCA}(F[p], [N]) \otimes \text{LA}([N], G[q]))_{\mathfrak{S}_N}$$

By 3.6 and 2.4, $\text{LCA}(F[p], [N]) \simeq F(FL_N^\otimes p)_{\delta_N}$ and $\text{LA}([N], G[q]) \simeq G(FL_N^\otimes q)_{\delta_N}$. □

3.8. The tensor and symmetric algebras $T[1]$ and $S[1]$. The following objects of $\text{LBA}$ play an important role in the structure of the algebras $\Omega^n_{br}$:

$$T[1] = \bigoplus_{p \geq 0} [p] \quad \text{and} \quad S[1] = \bigoplus_{p \geq 0} S^p [1]$$
3.8.1. Algebra and simplicial structure. $T[1]$ and $S[1]$ are graded algebra objects in $\mathbf{LBA}$. The product on $T[1]$ is defined on homogeneous components as the identification $[p_1] \otimes [p_2] = [p_1 + p_2]$, and that on $S[1]$ through the composition

$$S^{p_1}[1] \otimes S^{p_2}[1] \hookrightarrow [p_1] \otimes [p_2] = [p_1 + p_2] \rightarrow S^{p_1+p_2}[1]$$

The tower $\{A^{\otimes n}\}$, where $A$ is either $T[1]$ or $S[1]$, is a simplicial complex in $\mathbf{LBA}$ with boundary maps $\{\partial_n^i\}_{i=0}^n : A^{\otimes n} \rightarrow A^{\otimes (n-1)}$ given by

$$\partial_n^i(a_1 \otimes \cdots \otimes a_n) = \begin{cases} 
\varepsilon(a_1) a_2 \otimes \cdots \otimes a_n & i = 0 \\
(\varepsilon(a_i) a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n) & 1 \leq i \leq n-1 \\
\varepsilon(a_n) a_1 \otimes \cdots \otimes a_{n-1} & i = n 
\end{cases}$$

where the augmentation $\varepsilon : A \rightarrow [0]$ is projection onto the degree zero component. These define the Hochchild boundary

$$\partial_n = \sum_{i=0}^n (-1)^i \partial_n^i : A^{\otimes n} \rightarrow A^{\otimes (n-1)}$$

The natural quotient map $T[1] \rightarrow S[1]$ is an augmented algebra homomorphism, and therefore induces a morphism of complexes $(T[1]^{\otimes \bullet}, \partial_{\bullet}) \rightarrow (S[1]^{\otimes \bullet}, \partial_{\bullet})$.

3.8.2. Coalgebra and cosimplicial structure. $T[1]$ and $S[1]$ are also graded coalgebra objects. The coproduct on $T[1]$ is the shuffle coproduct $\Delta$ given on the homogeneous component $[p] \rightarrow \bigoplus_{p_1+p_2=p} [p_1] \otimes [p_2]$ as the sum of the identifications $[p] = [p_1] \otimes [p_2]$. The coproduct on $S[1]$ is defined as the composition

$$S^p[1] \hookrightarrow [p] \rightarrow \bigoplus_{p_1+p_2=p} [p_1] \otimes [p_2] \rightarrow S^{p_1}[1] \otimes S^{p_2}[1]$$

The towers $\{A^{\otimes n}\}$, where $A$ is either $T[1]$ or $S[1]$, are cosimplicial complexes in $\mathbf{LBA}$ with face maps $\{d_n^i\}_{i=0}^{n+1} : A^{\otimes n} \rightarrow A^{\otimes (n+1)}$ given by

$$d_n^i(a_1 \otimes \cdots \otimes a_n) = \begin{cases} 
1 \otimes a_1 \otimes \cdots \otimes a_n & i = 0 \\
a_1 \otimes \cdots \otimes a_{i-1} \otimes \Delta(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_n & 1 \leq i \leq n-1 \\
a_1 \otimes \cdots \otimes a_n \otimes 1 & i = n+1 
\end{cases}$$

where the coaugmentation $1 : [0] \rightarrow A$ is the inclusion of the degree zero component. These define the Hochchild differential

$$d_n = \sum_{i=0}^n (-1)^i d_n^i : A^{\otimes n} \rightarrow A^{\otimes (n+1)}$$

The natural inclusion $S[1] \rightarrow T[1]$ is a morphism of coaugmented coalgebras, and therefore induces a morphism of complexes $(S[1]^{\otimes \bullet}, d_{\bullet}) \rightarrow (T[1]^{\otimes \bullet}, d_{\bullet})$.

3.8.3. The cosimplicial space $\mathbf{LBA}(S[1]^{\otimes n}, S[1]^{\otimes n})^{\text{fin}}$. Let $A = \bigoplus_{p \geq 0} A^p \in \mathbf{LBA}$ be either $T[1]$ or $S[1]$, where the superscript in $A^p$ denotes the grading. Then, $A^{\otimes n} = \bigoplus_{p \in \mathbb{N}^n} A^p$, where $A^p = A^{p_1} \otimes \cdots \otimes A^{p_n}$. Set

$$\text{LBA}(A^{\otimes n}, A^{\otimes n})^{\text{fin}} = \bigoplus_{p,q \in \mathbb{N}^n} \text{LBA}(A^p, A^q)$$

If follows from 3.8.1 and 3.8.2 that the tower $\text{LBA}(A^{\otimes n}, A^{\otimes n})^{\text{fin}}$ has a cosimplicial structure, with differential given by $\phi \rightarrow d_n \circ \phi \circ \partial_{n+1}$. Moreover, the map

$$\text{LBA}(S[1]^{\otimes n}, S[1]^{\otimes n})^{\text{fin}} \rightarrow \text{LBA}(T[1]^{\otimes n}, T[1]^{\otimes n})^{\text{fin}}$$

(3.7)
obtained by combining the natural projection $T[1] \to S[1]$ and injection $S[1] \to T[1]$ is a morphism of cosimplicial spaces.

The following result relates this structure to the standard cosimplicial structure on the tensor and symmetric algebras of the free Lie algebras $FL_N$ via the identifications provided by Proposition 3.7.

**Lemma.** Let $\text{Sym} : SFL_N \to TFL_N$ be the standard symmetrisation map. The following is a commutative diagram of cosimplicial spaces

\[
\begin{array}{ccc}
\text{LBA}(T[1] \otimes^n, T[1] \otimes^n)_{\text{fin}} & \xrightarrow{\text{Sym}} & \bigoplus_{N \geq 0} (TFL_N \otimes^n)^{\delta_N} \otimes (TFL_N \otimes^n)^{\delta_N} \\
\text{LBA}(S[1] \otimes^n, S[1] \otimes^n)_{\text{fin}} & \xrightarrow{\text{Sym} \otimes \text{Sym}} & \bigoplus_{N \geq 0} (SFL_N \otimes^n)^{\delta_N} \otimes (SFL_N \otimes^n)^{\delta_N}
\end{array}
\]

3.9. **Morphisms in $\text{DY}_n$, $n = 1$.**

**Proposition.** The endomorphisms of $V_1 \in \text{DY}_1$ given by

\[ r_{N,N}^\sigma = \pi(N) \circ \sigma \otimes \text{id} \circ \pi^*(N) \]

for $N \geq 0$ and $\sigma \in \mathfrak{S}_N$ are a basis of $\text{End}_{\text{DY}}(V_1)$.

**Proof.** We represent the morphisms $\mu, \delta, \pi, \pi^*$ in $\text{DY}_1$ with the oriented diagrams

\[
\begin{align*}
\mu : \quad & \quad \quad \quad \\
\delta : \quad & \quad \quad \quad \\
\pi : \quad & \quad \quad \quad \\
\pi^* : \quad & \quad \quad \quad
\end{align*}
\]

A non–trivial endomorphism of $V_1$ is represented as a linear combination of oriented diagrams, necessarily starting with a coaction and ending with an action. The compatibility relation (2.6)

\[
\begin{array}{ccc}
\quad \quad \quad \quad \quad \quad \quad \quad & = & + & - \\
\end{array}
\]

allows to reorder $\pi$ and $\pi^*$. The cocycle condition (2.3) allows to reorder brackets and cobrackets as in $\text{LBA}$. Finally, the relations (2.4), (2.5)

\[
\begin{array}{ccc}
\quad \quad \quad \quad \quad \quad \quad \quad & = & + \\
\end{array}
\]

\[
\begin{array}{ccc}
\quad \quad \quad \quad \quad \quad \quad \quad & = & - \\
\end{array}
\]
allow to remove from the graph every \( \mu \) and every \( \delta \) involved. It follows that every endomorphism of \( \mathcal{V}_1 \) is a linear combination of the elements \( r_{N,N}^\sigma \)

for some \( N \geq 0 \) and \( \sigma \in \mathcal{S}_N \). These are linearly independent in \( \mathbf{D} \mathcal{Y}_1 \), and therefore form a basis for \( \text{End}_{\mathbf{D} \mathcal{Y}_1}(\mathcal{V}_1) \).

Proposition 3.9 yields an isomorphism of vector spaces

\[
\text{End}_{\mathbf{D} \mathcal{Y}_1}(\mathcal{V}_1) \simeq \bigoplus_{N \geq 0} k \mathcal{S}_N
\]

It is clear from the description above that the normal ordering on the product of two elements of the basis preserves the total number of strings. Namely, for any \( N,M > 0 \), \( \sigma \in \mathcal{S}_N \), \( \tau \in \mathcal{S}_M \), one gets

\[
\sum_{\rho \in \mathcal{S}_{N+M}} C_{N,M}^\rho \rho_{N,M} = \mathcal{S}_N \mathcal{S}_M
\]

for some \( C_{N,M}^\rho \in k \), i.e., the multiplication in \( \text{End}_{\mathbf{D} \mathcal{Y}_1}(\mathcal{V}_1) \) is \( \mathbb{N} \)-graded.

3.10. Morphisms in \( \mathbf{D} \mathcal{Y}_n \), \( n > 1 \). The description of the morphisms in \( \mathbf{D} \mathcal{Y}_n \) is similar to the case \( n = 1 \). For any \( N \in \mathbb{N} \) and \( N = (N_1, \ldots, N_n) \in \mathbb{N}^n \) such that \( |N| = N \), let

\[
\pi_{N} : a^\otimes N \otimes \bigotimes_{k=1}^n \mathcal{V}_k \to \bigotimes_{k=1}^n \mathcal{V}_k \quad \text{and} \quad \pi^*_{N} : \bigotimes_{k=1}^n \mathcal{V}_k \to a^\otimes N \otimes \bigotimes_{k=1}^n \mathcal{V}_k
\]

be defined by (3.3)–(3.4).

Proposition. The endomorphisms of \( \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n \) given by

\[
r_{N,N'}^\sigma = \pi_{N} \circ \sigma \otimes \text{id} \circ \pi^*_{N'}
\]

where \( N \geq 0 \), \( N, N' \in \mathbb{N}^n \) are such that \( |N| = N \) and \( \sigma \in \mathcal{S}_N \), are a basis of \( \mathfrak{U}_n^{\otimes \mathcal{V}} = \text{End}_{\mathbf{D} \mathcal{Y}_n}(\otimes_{k=1}^n \mathcal{V}_k) \).

3.11. Identification with free algebras. Let \( \mathcal{A}_N \) be the free associative algebra in \( N \) variables, and \( (\mathcal{A}_N^{\otimes N})_{\mathcal{S}_N} \) the subspace spanned by polynomials of degree one in each variable. The symmetric group \( \mathcal{S}_N \) acts diagonally on \( (\mathcal{A}_N^{\otimes N})_{\mathcal{S}_N} \) by simultaneous permutation of the variables.

Corollary. There is an isomorphism of vector spaces

\[
\xi_{\mathcal{D} \mathcal{Y}_n} : \mathfrak{U}_n^{\otimes \mathcal{V}} \to \bigoplus_{N \geq 0} \left( (\mathcal{A}_N^{\otimes N})_{\mathcal{S}_N} \otimes (\mathcal{A}_N^{\otimes N})_{\mathcal{S}_N} \right)_{\mathcal{S}_N}
\]

given by

\[
\xi_{\mathcal{D} \mathcal{Y}_n}(r_{N,N'}^\sigma) = x_{N} \otimes y_{\mathcal{S}(N')}
\]

---

1 Linear independence in \( \mathbf{D} \mathcal{Y}_n \) is easily checked on the free Drinfeld–Yetter module constructed over the comodule \( \mathcal{V}_1 \), following an argument similar to 3.5.
where $\overline{\sigma} = \sigma^{-1} \circ \tau$ and $\tau \in \mathfrak{S}_N$, such that $\tau(i) = N - i$.\footnote{The involution $\tau_N$ is required because of the contravariance of the expression (3.6) with respect to the Lie polynomial $Q$, and to ensure the commutativity of the diagram in Theorem 3.14.}

3.12. **Action of morphisms in $LBA$ on $\otimes_{k=1}^n V_k$.** Any element $\phi_{p,q} \in LBA([p],[q])$ gives rise to a morphism $\tilde{a}(\phi_{p,q})$ of $V_k \in DY_k$ by

$$a(\phi_{p,q}) = \pi([q]) \circ \phi_{p,q} \circ \pi^*([p])$$

If $a$ is a Lie bialgebra and $\{a_i\}, \{b^i\}$ are dual bases of $a$ and $a^*$, the morphism $a(id_{[1]}) = \pi \circ \pi^*$ corresponds, via Proposition 3.1, to the action of the normally ordered Casimir $\sum b_i a^i$ on equicontinuous $g_a$-modules.

More generally, if $T[1] = \bigoplus_{p \geq 0} [p]$, there is a natural map

$$a : LBA(T[1] \otimes^n, T[1] \otimes^n)_{\text{fin}} \to DY_n(\otimes_{k=1}^n V_k, \otimes_{k=1}^n V_k)$$

defined on $\phi_{p,q} \in LBA(T[1], T[1])$, by

$$\phi_{p,q} \mapsto \pi([q]) \circ \phi_{p,q} \circ \pi^*([p])$$

Note, however, that $a$ is not an algebra homomorphism (and, in fact, that the algebra $LBA(T[1] \otimes^n, T[1] \otimes^n)_{\text{fin}}$ is not unital).

3.13. **Identification with Lie polynomials.** Let

$$\text{Sym} : LBA(S[1] \otimes^n, S[1] \otimes^n) \to LBA(T[1] \otimes^n, T[1] \otimes^n)_{\text{fin}}$$

be the map (3.7). The following result shows that the composition $\tilde{a} \circ \text{Sym}$ can be thought of as the symmetrisation map $SI \to UI$ for a Lie algebra $I$.

**Proposition.** The following is a commutative diagram.

$$
\begin{array}{ccc}
\text{DY}_n(\otimes_{k=1}^n V_k, \otimes_{k=1}^n V_k) & \longrightarrow & \bigoplus_{N \geq 0} ((FA_N^\otimes)^\delta_N \otimes (FA_N^\otimes)^\delta_N)_{\mathfrak{S}_N} \\
\downarrow & & \downarrow \\
LBA(T[1] \otimes^n, T[1] \otimes^n)_{\text{fin}} & \longrightarrow & \bigoplus_{N \geq 0} ((TFL_N^\otimes)^\delta_N \otimes (TFL_N^\otimes)^\delta_N)_{\mathfrak{S}_N} \\
\downarrow \text{Sym} & & \downarrow \text{Sym} \circ \text{Sym} \\
LBA(S[1] \otimes^n, S[1] \otimes^n)_{\text{fin}} & \longrightarrow & \bigoplus_{N \geq 0} ((SFL_N^\otimes)^\delta_N \otimes (SFL_N^\otimes)^\delta_N)_{\mathfrak{S}_N}
\end{array}
$$

where the right vertical arrows are the symmetrisation map $SFL_N \to TFL_N$ and quotient map $TFL_N \to UFL_N = FA_N$ for the Lie algebra $FL_N$.

**Proof.** The commutativity of the lower part of the diagram is given by Lemma 3.8.3. To prove the commutativity of the upper part, assume for simplicity that $n = 1$. The proof for $n > 1$ is identical.

Let $P_1 \otimes \cdots \otimes P_p \in FL_N^\otimes$ be an element of degree $\delta_N$, and $\mu_{P_1 \otimes \cdots \otimes P_p} \in LA([N],[p])$ the element corresponding to it by Lemma 3.6. In $FA_N = UFL_N$, the product $P_1 \cdots P_p$ corresponds to an element

$$\sigma_{P_1 \cdots P_p} \in k\mathfrak{S}_N \simeq (FA_N)^{\delta_N}$$

Moreover, the relation (2.4) implies that the following holds in $DY_n([N] \otimes V_1, V_1)$

$$\pi^{(p)} \circ \mu_{P_1 \otimes \cdots \otimes P_p} \otimes \text{id} = \pi^{(N)} \circ \sigma_{P_1 \cdots P_p}$$

(3.8)
Dually, for any $Q_1 \otimes \cdots \otimes Q_n \in FL_N$ of degree $\delta_N$, there are elements
\[
\delta_{Q_1 \otimes \cdots \otimes Q_n} \in \mathcal{LCA}([q], [N]) \quad \text{and} \quad \sigma_{Q_1 \cdots Q_n} \in k \mathcal{E}_N
\]
such that
\[
\delta_{Q_1 \otimes \cdots \otimes Q_n} \otimes \id \circ \pi^*(q) = (\sigma_{Q_1 \cdots Q_n} \circ \tau_N) \otimes \id \circ \pi^*(N)
\]
in $DY_1([V_1, [N] \otimes V_1])$, where $\tau_N \in \mathcal{E}_N$, $\tau_N(i) = N - i$.
This yields the commutativity of the diagram, since by (3.8) and (3.9)
\[
a(\delta_{Q_1 \otimes \cdots \otimes Q_n} \circ \mu_{P_1 \otimes \cdots \otimes P_p}) = \pi^{(p)} \circ \mu_{P_1 \otimes \cdots \otimes P_p} \circ \delta_{Q_1 \otimes \cdots \otimes Q_n} \otimes \id \circ \pi^*(q)
\]
which corresponds exactly to the element
\[
(Q_1 \cdots Q_n) \otimes (P_1 \cdots P_p) \in ((FA_N)_{\delta_N} \otimes (FA_N)_{\delta_N})_{\mathcal{E}_N}
\]
\[\square\]

3.14. PBW theorem for $\mathfrak{U}_n^{Fr}$.

**Theorem.** The map
\[
a \circ \text{Sym} : \mathcal{LBA}(S[1]^\otimes, S[1]^\otimes)^{fn} \to DY_1((\otimes_{k=1}^n V_k, \otimes_{k=1}^n V_k))
\]
is an isomorphism of cosimplicial spaces.

**Proof.** The fact that $a \circ \text{Sym}$ is an isomorphism follows from Proposition 3.13 and the PBW Theorem for the Lie algebra $FL_N$.
Since $\text{Sym}$ is a morphism of cosimplicial spaces, it suffices to prove that
\[
a : \mathcal{LBA}(T[1]^\otimes, T[1]^\otimes)^{fn} \to DY_1((\otimes_{k=1}^n V_k, \otimes_{k=1}^n V_k))
\]
is compatible with the face maps $\{d_i^N\}_{i=0}^{n+1}$. The case $i = 0, n + 1$ is easily checked.
To check the compatibility with $d_i^N$, it suffices to consider the case $n = 1$. Let $\phi \in \mathcal{LBA}([p], [q])$. We need to check the equality of
\[
d_1^1 \circ a(\phi) = \pi^{(q)}_{V_1 \otimes V_1} \circ \phi \circ \id_{V_1} \otimes \pi^{(p)}_{V_1} \otimes \id_{V_1}
\]
and
\[
a \circ d_1^1(\phi) = \sum_{P,Q \in H^2} \pi^{(q)}_{V_1 \otimes V_1} \circ d_1^1(\phi)_{P,Q} \otimes \id_{V_1} \otimes \pi^{(p)}_{V_1} \otimes \id_{V_1}
\]
where $d_1^1(\phi)_{P,Q} \in \mathcal{LBA}(T^P[1], T^Q[1])$ are the homogeneous components of $d_1^1(\phi) = \Delta \circ \phi \circ m$, and $m, \Delta$ are the multiplication and comultiplication of $T[1]$.

The equality now follows from the identities
\[
\bigoplus_{P:[p]=p} m \otimes \id_{V_1} \otimes \pi^{(p)}_{V_1} \otimes \id_{V_1} = \pi^{(p)}_{V_1} \otimes \id_{V_1}
\]
\[
\bigoplus_{Q:[q]=q} \pi^{(q)}_{V_1 \otimes V_1} \circ \Delta \otimes \id_{V_1} \otimes \id_{V_1} = \pi^{(q)}_{V_1} \otimes \id_{V_1}
\]
of maps $V_1 \otimes V_1 \to [p] \otimes V_1 \otimes V_1$ and $[q] \otimes V_1 \otimes V_1 \to V_1 \otimes V_1$ respectively. The first (resp. second) one holds because both sides are the components of the coaction (resp. action) of $T[1]$ on $V_1 \otimes V_1$. \[\square\]
3.15. **Enriquez’s universal algebras.** Let \( \mathfrak{a} \) be a Lie bialgebra, and \( \mathfrak{g}_a = \mathfrak{a} \oplus \mathfrak{a}^* \) its Drinfeld double with \( r \)-matrix \( r_{\mathfrak{g}_a} = \sum_{i \in I} a_i \otimes b^i \in \mathfrak{a} \oplus \mathfrak{a}^* \). In [7, 8], Enriquez introduced the universal algebras \( (U_{\mathfrak{g}} \otimes \mathfrak{a})_{\text{univ}} \) which are associated to the PROP LBA. As \( k \)-vector space

\[
(U_{\mathfrak{g}} \otimes \mathfrak{a})_{\text{univ}} = \bigoplus_{N \geq 0} \left((FA_N^\otimes)_{\delta_N} \otimes (FA_N^\otimes)_{\delta_N}^\ast \right)_{\mathfrak{S}_N}
\]

where \( \mathfrak{S}_N \) acts diagonally by permutation of the variables. \( (U_{\mathfrak{g}})_{\text{univ}} \) has a basis given by the element \( x_1 \cdots x_N \otimes y_{\sigma(1)} \cdots y_{\sigma(N)} \) where \( N \geq 0 \) and \( \sigma \in \mathfrak{S}_N \). It is endowed with an algebra structure such that the linear map \( \rho_{\mathfrak{g}_a} : (U_{\mathfrak{g}})_{\text{univ}} \to \hat{U}_{\mathfrak{g}_a} \)

\[
\rho_{\mathfrak{g}_a}(x_1 \cdots x_N \otimes y_{\sigma(1)} \cdots y_{\sigma(N)}) = \sum_{i \in I_N} a_i x_i \cdots b_{\sigma(N)}^i
\]

is an algebra homomorphism.

There are similar formulae for \( n \geq 2 \). \( (U_{\mathfrak{g}} \otimes \mathfrak{a})_{\text{univ}} \) has a basis given by the elements \( x_N \otimes y_{\sigma(N)} \), where \( \sigma \in \mathfrak{S}_N \), \( N, N' \in \mathbb{N}^n \) are such that \( |N| = N = |N'| \),

\[
x_N = x_{1} \cdots x_{N_1} \otimes x_{N_1+1} \cdots x_{N_1+N_2} \otimes \cdots \otimes x_{N_1+\cdots+N_{k-1}+1} \cdots x_{N_N}
\]

\[
y_{\sigma(N')} = y_{\sigma(1)} \cdots y_{\sigma(N_1')} \otimes \cdots \cdots \otimes y_{\sigma(N_1'+\cdots+N_{k-1}'+1) \cdots y_N}
\]

It is endowed with an algebra structure such that the map

\[
\rho_{\mathfrak{g}_a}^n : (U_{\mathfrak{g}} \otimes \mathfrak{a})_{\text{univ}} \to \hat{U}_{\mathfrak{g}_a}^\otimes \quad \rho_{\mathfrak{g}_a}^n(x_N \otimes y_{\sigma(N)}) = \sum_{i \in I_N} a_N(i) \cdot b_{\sigma(N)}(i)
\]

where

\[
a_N(i) = \bigotimes_{k=1}^{N} a_i \otimes a_{i+N_1+\cdots+N_{k-1}+1} \cdots a_{i+N_{k-1}+\cdots+N_k}
\]

\[
b_{\sigma(N)}(i) = \bigotimes_{k=1}^{N} b_{\sigma(N_1'+\cdots+N_{k-1}'+1)} \cdots b_{\sigma(N_1'+\cdots+N_k')}
\]

is an algebra homomorphism.

3.16. **The isomorphism** \( \mathfrak{U}_{\mathfrak{g}_a} \simeq (U_{\mathfrak{g}} \otimes \mathfrak{a})_{\text{univ}} \). The following result identifies the algebra \( (U_{\mathfrak{g}} \otimes \mathfrak{a})_{\text{univ}} \) with \( \mathfrak{U}_{\mathfrak{g}_a} \), thereby considerably simplifying the proof of the existence of an algebra structure on \( (U_{\mathfrak{g}} \otimes \mathfrak{a})_{\text{univ}} \) given in [7, 8].

Let \( \varepsilon_{\mathfrak{g}_a} : \mathfrak{U}_{\mathfrak{g}_a} \to (U_{\mathfrak{g}} \otimes \mathfrak{a})_{\text{univ}} \) be the map defined in 3.11.

**Proposition.**

(i) \( \varepsilon_{\mathfrak{g}_a} \) is an isomorphism of cosimplicial spaces.

(ii) There is a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{U}_{\mathfrak{g}_a} & \xrightarrow{\rho_{\mathfrak{g}_a}^n} & \hat{U}_{\mathfrak{g}_a} \\
\varepsilon_{\mathfrak{g}_a} \downarrow & \simeq & \downarrow \\
(U_{\mathfrak{g}} \otimes \mathfrak{a})_{\text{univ}} & \xrightarrow{\rho_{\mathfrak{g}_a}^n} & \hat{U}_{\mathfrak{g}_a} \otimes \mathfrak{a}
\end{array}
\]

where the right vertical isomorphism is induced by the equivalence \( \varepsilon_{\mathfrak{g}_a} \simeq \mathfrak{D}_{\mathfrak{g}_a} \).
Proof. (i) The fact that $\xi^n_0$ is an isomorphism was proved in 3.11, and its compatibility with the cosimplicial structure in Lemma 3.8.3. (ii) The commutativity of the diagram follows by direct inspection. □

Remark. It seems very likely that the map $\xi^n_0$ is an algebra homomorphism. This would follow from a detailed inspection of the algebra structure on $(Ug^{\otimes n})$, or from the commutativity of the above diagram if the collection of maps $\rho^n_a$ were known to be bijective. In any event, the above proposition shows that $(Ug^{\otimes n})$ is an isomorphic replacement of $(Ug^{\otimes n})_{univ}$ with a more naturally defined multiplication.

3.17. The $r$–matrix of $\mathcal{U}_0$. The algebra $\mathcal{U}_0$ has a canonical $r$–matrix, i.e., the element $r_{1,1} = \pi_1 \otimes \text{id} \circ (12) \circ \text{id} \circ \pi_1'$. Under $\xi^n_0$, it corresponds to

$$r = x^{(1)} \otimes y^{(2)} \in (FA_1^{\otimes 2})_{\delta_1} \otimes (FA_1^{\otimes 2})_{\delta_1} \subset \mathcal{U}_0^2$$

which is sent to the canonical element of $g_a$ via $\rho_n^2:

$$\rho_n^2(r) = \sum_i a_i \otimes b_i =: r_{ga}$$

We notice that the algebra $\mathcal{U}_0^2$ contains the elements

$$r^{21} = x^{(2)} \otimes y^{(1)} \quad \kappa \otimes 1 = x^{(1)} \otimes y^{(1)} \quad 1 \otimes \kappa = x^{(2)} \otimes y^{(2)}$$

corresponding respectively to the elements in $Ug_a^{\otimes 2}$

$$r_{ga}^{21} = \sum_i b^i \otimes a_i \quad m(r_{ga}) \otimes 1 = \sum_i a_i b^i \otimes 1 \quad 1 \otimes m(r_{ga}) = \sum_i 1 \otimes a_i b^i$$

but there is no analogue of the not–naturally ordered product $\sum_i b^i a_i$.

4. Further properties of $\mathcal{U}_0^n$

In this section, we review some basic facts about the cohomology of Schur functors which are due to Enriquez [8, Sec. 1], and will be used repeatedly. We then carry over several properties of the universal algebras $(Ug^{\otimes n})_{univ}$ proved in [8], to the algebras $\mathcal{U}_0^n$.

4.1. Schur functors. Recall that the set of irreducible representations of the symmetric group $S_N$ is in bijection with minimal idempotents in $kS_N$ modulo the equivalence relation $p \sim upu^{-1}, u \in kS_N$. We denote the set of equivalence classes by $\Pi_N$, and regard it as a subset of $kS_N$ by choosing a representative for each class.

A Schur functor is a functor $F : \text{Vect}_k \to \text{Vect}_k$ of the form

$$F(V) = \bigoplus_{N \geq 0, \pi \in \Pi_N} \pi(V^{\otimes N}) \otimes M_{(N, \pi)}$$

for some multiplicity spaces $M_{(N, \pi)} \in \text{Vect}_k$. A morphism $f : F \to F'$ of Schur functors is a collection of linear maps $f_{(N, \pi)} : M_{(N, \pi)} \to M'_{(N, \pi)}$. If $F_1, F_2$ are Schur functors, the Schur functors $F_1 \oplus F_2$ and $F_1 \otimes F_2$ are defined by

$$F_1 \oplus F_2(V) = F_1(V) \oplus F_2(V) \quad \text{and} \quad F_1 \otimes F_2(V) = F_1(V) \otimes F_2(V)$$

\footnote{By convention, $\Pi_0 = \{\pi_0\}$ and $\pi_0(V^{\otimes 0}) = k.$
The \( k \)-linear, additive, abelian tensor category of Schur functors will be denoted by \( \text{Sch} \). It is endowed with a contravariant involutive functor \( F \to F^* \) given by

\[
F^*(V) = \bigoplus_{N, \pi} \pi(V^\otimes N) \otimes M^*_N
\]

A Schur functor \( F \) is \textit{trivialised} if its multiplicity spaces are all of the form \( M_{(N, \pi)} = k^{d_{(N, \pi)}} \), for some \( d_{(N, \pi)} \in \mathbb{N} \). This is the case for example if \( F \) is \textit{multiplicity free}, that is of the form

\[
F(V) = \bigoplus_{N \geq 0, \pi \in \mathbb{P}_N} \pi(V^\otimes N)
\]

for some subsets \( \mathbb{P}_N \subseteq \Pi_N \). Examples of multiplicity functors are the tensor functors \( T^n(V) = V^\otimes n \), and therefore for their quotient and subfunctors, in particular the exterior powers \( \wedge^n V \) and the multisymmetric functors

\[
S^p V = S^p V \otimes \ldots \otimes S^{p_n} V \subset T^\otimes V
\]

where \( p \in \mathbb{N}^n \) and \( |p| = p_1 + \cdots + p_n \).

4.2. \textbf{Schur bifunctors}. A \textit{Schur bifunctor} is a functor \( F : \text{Vect}_k \times \text{Vect}_k \to \text{Vect}_k \) of the form

\[
F(V_1, V_2) = \bigoplus_{N_1 \geq 0, \pi_1 \in \Pi_{N_1}, N_2 \geq 0, \pi_2 \in \Pi_{N_2}} \pi_1(V_1^\otimes N_1) \otimes \pi_2(V_2^\otimes N_2) \otimes M_{(N_1, \pi_1),(N_2, \pi_2)}
\]

for some multiplicity spaces \( M_{(N_1, \pi_1),(N_2, \pi_2)} \in \text{Vect}_k \). If \( F_1, F_2 \) are Schur functors with multiplicity spaces \( M^1_{(N, \pi)}, M^2_{(N, \pi)} \), the exterior product \( F_1 \boxtimes F_2 \) is the Schur bifunctor with multiplicity spaces \( M_{(N_1, \pi_1),(N_2, \pi_2)} = M^1_{(N_1, \pi_1)} \otimes M^2_{(N_2, \pi_2)} \).

The category \( \text{Sch} \) of Schur functors is endowed with a coproduct functor \( \Delta : \text{Sch} \to \text{Sch}_2 \) given by

\[
\Delta(F)(V_1, V_2) = F(V_1 \oplus V_2)
\]

For example, if \( S = \bigoplus_{n \geq 0} S^n \), \( \wedge = \bigoplus_{n \geq 0} \wedge^n \) are the symmetric and exterior algebra functors, then

\[
\Delta(S) = S \boxtimes S \quad \text{and} \quad \Delta(\wedge) = \wedge \boxtimes \wedge
\]

4.3. \textbf{Cohomology of Schur functors}. Let \( (F^n, d_n)_{n \geq 0} \) be a complex in \( \text{Sch} \).

**Proposition.** \textit{There is a canonical isomorphism of Schur bifunctors}

\[
H^i(\Delta(F^*), \Delta(d_*)) \simeq \Delta(H^i(F^*, d_*))
\]

4.4. \textbf{The Hochschild complex}. The Hochschild complex \( (SV^\otimes n, d_H) \) of a symmetric coalgebra \( SV \) can be interpreted as a complex of Schur functors as follows. For any \( n \geq 1 \), there is a functor \( \Sigma_n : \text{Vect} \to \text{Vect} \) given by \( V \to V^\otimes n = V \otimes k^n \), and natural transformations \( \{ \delta^n_i \}_{i=0}^{n+1} : \Sigma_n \to \Sigma_{n+1} \) induced by the maps \( k^n \to k^{n+1} \) given by

\[
(x_1, \ldots, x_n) \to \begin{cases} (0, x_1, \ldots, x_n) & i = 0 \\ (x_1, \ldots, x_{i-1}, x_i, x_1, x_{i+1}, \ldots, x_n) & 1 \leq i \leq n \\ (x_1, \ldots, x_n, 0) & i = n + 1 \end{cases}
\]

These give rise to a cosimplicial structure on the tower of Schur functors \( S^\otimes n = S \circ \Sigma_n \), whose associated differential is the Hochschild differential \( d_H \). The latter
restricts to zero on $T^\bullet \subset S^{\otimes \bullet}$, where $T^n(V) = V^{\otimes n}$, and gives rise to a quasi–isomorphism
\[(\wedge^\bullet, 0) \to (S^{\otimes \bullet}, d_H) \quad (4.2)\]

It follows in particular that
\[H^\bullet(S^{\otimes \bullet} \boxtimes S^{\otimes \bullet}, d_H \boxtimes d_H) \simeq \wedge^\bullet \boxtimes \wedge^\bullet \quad (4.3)\]

4.5. PROP\textsc{s and Schur bifunctors}. A PROP $P$ gives rise to a functor $P_{\text{Sch}}: \text{Sch}_2 \to \text{Vect}$ which maps a bifunctor $F$ with multiplicity spaces $M_{(N_1, \pi_1), (N_2, \pi_2)}$ to
\[P_{\text{Sch}}(F) = \bigoplus_{N_1 \geq 0, \pi_1 \in \Pi_{N_1}} P(\pi_1[N_1], \pi_2[N_2]) \otimes M_{(N_1, \pi_1), (N_2, \pi_2)}\]

If $F \in \text{Sch}$ is trivialised, with multiplicity spaces $M_{(N, \pi)} = k^d(N, \pi)$, one can define $F[1] \in P$ by
\[F[1] = \bigoplus_{N, \pi} \pi[N]^{\otimes d(N, \pi)}\]
The assignment $F \to F[1]$ uniquely extends to a tensor functor from the full subcategory of trivialised Schur functors to $P$. If $F_1, F_2 \in \text{Sch}$ are trivialised, and $F_1 \boxtimes F_2 \in \text{Sch}_2$ is their exterior product, then
\[P_{\text{Sch}}(F_1 \boxtimes F_2) = P(F_1[1], F_2[1])^{\text{fin}} \quad (4.4)\]
where the latter space is the algebraic direct sum $\bigoplus P(\pi_1[N_1], \pi_2[N_2]) \otimes M_{(N_1, \pi_1)} \otimes M_{(N_2, \pi_2)}$.

4.6. PROP\textsc{s and cohomology}. For any complex $(F^\bullet, d_\bullet)$ in $\text{Sch}_2$, $(P_{\text{Sch}}(F^\bullet), P_{\text{Sch}}(d_\bullet))$ is a complex of vector spaces.

Proposition.
\[H^\bullet(P_{\text{Sch}}(F^\bullet), P_{\text{Sch}}(d_\bullet)) \simeq P_{\text{Sch}}(H^\bullet(F^\bullet, d_\bullet))\]

4.7. Hochschild cohomology.

Proposition. For any PROP $P$, the inclusion
\[(P(\wedge^\bullet[1], \wedge^\bullet[1]), 0) \to (P(S[1]^{\otimes \bullet}, S[1]^{\otimes \bullet})^{\text{fin}}, d_H \boxtimes d_H)\]
is a quasi–isomorphism.

Proof. We have
\[H^\bullet(P(S[1]^{\otimes \bullet}, S[1]^{\otimes \bullet})^{\text{fin}}, d_H \boxtimes d_H) = H^\bullet(P_{\text{Sch}}(S^{\otimes \bullet} \boxtimes S^{\otimes \bullet})^{\text{fin}}, P_{\text{Sch}}(d_H \boxtimes d_H)) \]

where the first and last equalities hold by definition of the functor $LBA_{\text{Sch}}$, and the fact that $S^{\otimes \bullet}, \wedge^\bullet$ are multiplicity free, the second one by Proposition 4.6, and the third one by (4.3). \qed
4.8. Let $a \circ \text{Sym} : \text{LBA}(S[1] \otimes n, S[1] \otimes n)^{\text{fin}} \to \Omega_{\nu}^n = \text{DY}_{n}(\otimes_{k=1}^{n} V_{x^{k}}, \otimes_{k=1}^{n} V_{x^{k}})$ be the maps defined in 3.12 and 3.8.3.

**Theorem.** The map $a \circ \text{Sym}$ induces an isomorphism

$$H^{*}(\Omega_{\nu}^{n}, d_H) \cong \text{LBA}(\wedge^{*}[1], \wedge^{*}[1])$$

In particular, $H^{0}(\Omega_{\nu}^{n}, d_H) = H^{1}(\Omega_{\nu}^{n}, d_H) = 0$.

**Proof.** By Theorem 3.14, $a \circ \text{Sym}$ is an isomorphism of cosimplicial spaces. We have

$$\text{LBA}(S[1] \otimes^{*}, S[1] \otimes^{*})^{\text{fin}} = \text{LBA}((S^{*}) \otimes^{*}[1], S \otimes^{*}[1])^{\text{fin}} = \text{LBA}_{\text{Sch}}(S \otimes^{*}, S \otimes^{*})$$

where the first equality relies on the equality of Schur functors $S = S^{*}$, and the fact that the cosimplicial (resp. simplicial) structure on $S[1] \otimes^{*}$ is induced by that on the Schur functors $S \otimes^{*}$ (resp. on $S^{*} \otimes^{*}$), and the second one from (4.4). The result now follows from Proposition 4.7, and the equality of Schur functors $\wedge^{*} = \wedge$. □

**Remark.** In terms of the identification with Lie polynomials given by Proposition 3.7, the above isomorphism yields

$$H^{*}(\Omega_{\nu}^{n}, d_H) \cong \bigoplus_{N \geq 0} \left[ (\wedge^{*}FL_{N})_{\delta_{N}} \otimes (\wedge^{*}FL_{N})_{\delta_{N}} \right]_{\otimes N}$$

4.9. **Holonomy algebra.** The holonomy algebra $T_n$ is generated over $k$ by elements $\Omega_{ij}$, $1 \leq i < j \leq n$, satisfying relations

$$[\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = 0 \quad [\Omega_{ij}, \Omega_{kl}] = 0$$

for $i, j, k, l$ all distinct. Let $r \in \Omega_{\nu}^{n}$ be the element $r = x^{(1)} \otimes y^{(2)}$.

**Proposition.** The linear map $\xi_n : T_n \to \Omega_{\nu}^{n}$ given by

$$\xi_n(\Omega_{ij}) = r^{ij} + r^{ji}$$

is an injective algebra homomorphism. The maps $\xi_n$ commute with the coproduct maps of $\Omega_{\nu}^{n}$ and $T_n$, given by $\Delta^{(k)} : T_n \to T_{n+1}$, $k = 1, \ldots, n$

$$\Delta^{(k)}(\Omega_{ij}) = \delta_{ki}(\Omega_{i,j+1} + \Omega_{i+1,j}) + \delta_{kj}(\Omega_{ij} + \Omega_{i,j+1})$$

Let $\hat{\Omega}_n$ be the completion of $\Omega_{\nu}^{n}$ with respect to the $\mathbb{N}$–grading (this provides a universal analogue of the topological algebra $\hat{U}_{\mathbb{q}}^{\otimes n}([\h])$), and $\hat{\Omega}_n$ the set of invertible elements in $\hat{\Omega}_n$.

The map $\xi_n$ is compatible with the natural $\mathbb{N}$–gradings on $T_n$ and $\Omega_{\nu}^{n}$, and extends to a map

$$\xi_n : \hat{T}_n \to \hat{\Omega}_n$$

4.10. **Drinfeld associators.** An invertible element $\Phi \in \hat{T}_3^\times$ is called a *Lie associator* if the following relations are satisfied (in $\hat{T}_4$ and $\hat{T}_3$ respectively):

- **Pentagon relation**

$$\Phi_{1,2,3,4} \Phi_{12,3,4} = \Phi_{2,3,4} \Phi_{1,23,4} \Phi_{1,2}$$

- **Hexagon relations**

$$e^{\Omega_{12,3}/2} = \Phi_{23,1} e^{\Omega_{1,3}/2} \Phi_{12,3}^{-1} e^{\Omega_{23}/2} \Phi_{1,23}$$

$$e^{\Omega_{12}/2} = \Phi_{23,1} e^{\Omega_{1,3}/2} \Phi_{12}^{-1} e^{\Omega_{12}/2} \Phi_{1,2}$$
• Invertibility

\[ \Phi_{3,2,1} = \Phi_{3,2,1}^{-1} \]

• 2-jets

\[ \Phi = 1 + \frac{1}{24} [\Omega_{12}, \Omega_{23}] \mod (T_3)_{\geq 3} \]

Let \( \text{Assoc} \) be the set of associators. \( \xi_3 \) gives a map \( \text{Assoc} \to \widehat{U}_3^\times \), providing a realisation of the associators in the universal algebra \( \widehat{U}_3 \).

4.11. Universal invariants.

Definition. The space of invariants \( (U_{D^n})_{\text{inv}} \) is the subspace of all \( x \in U_{D^n} \) satisfying

\[ [r_{01} + \cdots + r_{0n}, x^{1\cdots n}] = [r_{10} + \cdots + r_{n0}, x^{1\cdots n}] = 0 \]

in \( U_{D^{n+1}} \), where \( r \) is the universal \( r \)-matrix in \( U_{D^2} \).

The above condition is equivalently stated as follows. For any \( y \in U_{D^n} \),

\[ [y^{(1\cdots n)}, n+1, \ldots, n+p-1, x^{1,2,\ldots,n}] = 0 \]

in \( U_{D^{n+p-1}} \).

Lemma. The image of \( \xi_n \) lies in \( (U_{D^n})_{\text{inv}} \).

Proof. The \( r \)-matrix \( r \in U_{D^2} \) satisfies the CYBE

\[ [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \]

Applying the permutation \( (23) \) we get

\[ [r_{13}, r_{12}] + [r_{13} + r_{12}, r_{32}] = 0 \]

Adding the two equations we obtain

\[ [r_{13} + r_{12}, r_{23} + r_{32}] = 0 \Rightarrow [r_{13} + r_{12}, \Omega_{23}] = 0 \]

Therefore \( \Omega \in (U_{D^2})_{\text{inv}} \). Similarly for \( n > 2 \).

The morphism \( \rho_{g^n} \) defined in 3.16 preserves the invariant elements. Namely,

Proposition. The map

\[ \rho_{g^n} : U_{D^n} \to \widehat{U}_{g^n} \]

restricts to an algebra homomorphism

\[ \rho_{g^n} : (U_{D^n})_{\text{inv}} \to \widehat{U}_{g^n}^{\otimes \text{End}(id^{g^n})} \]

where \( \widehat{U}_{g^n} \) is the algebra of the endomorphisms of the identity functor of the category of equicontinuous \( g^n \)-modules.

Proof. It is enough to show that, in \( U_{g^n} \)

\[ [r, 1 \otimes x] = 0 = [r_{21}, 1 \otimes x] \iff x \in (U_{g^n})_{\text{inv}} \]

where \( r = \sum_{i \in I} a_i \otimes b^i \). Indeed,

\[ \sum_i a_i \otimes [b^i, x] = 0 \Rightarrow [b^i, x] = 0 \quad \forall i \in I \]

and similarly

\[ \sum_i b^i \otimes [a_i, x] = 0 \Rightarrow [a_i, x] = 0 \quad \forall i \in I \]

The reverse implication is obvious. Similarly for \( n > 1 \).
5. The universal algebra of a split pair

In this section, we give a relative version of the results of Sections 3 and 4 by adapting them to the case of a split pair of Lie bialgebras, as defined in [1].

5.1. The PROP PLBA. Let \((b, a)\) be a split pair of Lie bialgebras, i.e., \(b\) and \(a\) are Lie bialgebras endowed with Lie bialgebra morphisms

\[
a \to b \to a
\]

such that \(p \circ i = id_a\). These maps induce an isometric inclusion of the corresponding Drinfeld doubles \(g_a \subset g_b\).

**Definition.** Let \(\text{PLBA}\) be the multicolored PROP generated by two Lie bialgebra objects \(b, a\) related by Lie bialgebra morphisms \(i : a \to b, p : b \to a\) such that \(p \circ i = id_a\).

The kernel \(m\) of the projection \(p\), is an object of \(\text{PLBA}\), and \(b\) decomposes as \(b = a \oplus m\). \(m\) is an ideal in \(b\), and has a Lie algebra structure. It is also a coideal, but has no natural Lie coalgebra structure.

5.2. Universal property of \(\text{PLBA}\).

**Proposition.**

(i) The PROP \(\text{PLBA}\) is endowed with a pair of functors \(\beta, \alpha : LBA \to \text{PLBA}\) given by

\[
\beta[1] = b \quad \text{and} \quad \alpha[1] = a
\]

and natural transformations \(\tilde{i} : \alpha \to \beta, \tilde{p} : \beta \to \alpha\) such that \(\tilde{p} \circ \tilde{i} = id_a\).

(ii) \(\text{PLBA}\) is universal with respect to property (i): for any tensor category \(C\) for which it holds, there is a unique tensor functor \(F : \text{PLBA} \to C\) such that the following diagram commutes.

**Corollary.**

(i) There is a forgetful functor \(\text{PLBA} \to LBA\), mapping \([1]_{\text{PLBA}}\) to \([1]_{LBA}\) and \(\pi_0\) to \(id_{[1]_{LBA}}\).

(ii) There is a forgetful functor \(\text{PLBA} \to LBA\), mapping \([1]_{\text{PLBA}}\) to \([1]_{LBA}\) and \(\pi_0\) to \(0\).
5.3. Alternative presentation of PLBA. The following presentation of PLBA is more convenient for computations. Let LA be the PROP with generating elements $\mu \in S_{21}$ and $\pi_0 \in S_{11}$ satisfying the relations (5.1),
\[
\pi_0^2 = \pi_0 \quad \text{and} \quad \mu \circ (\pi_0 \otimes \pi_0) = \pi_0 \circ \mu \quad (5.1)
\]
Let PLCA be the PROP with generating elements $\delta \in S_{12}$ and $\pi_0 \in S_{11}$ satisfying the relations (5.2),
\[
\pi_0^2 = \pi_0 \quad \text{and} \quad (\pi_0 \otimes \pi_0) \circ \delta = \delta \circ \pi_0 \quad (5.2)
\]
Let PLBA be the PROP with generating elements $\mu \in S_{21}$, $\delta \in S_{12}$, $\pi_0 \in S_{11}$, satisfying the relations (5.1), (5.2). Finally, let PLA, PLCA, PLBA be the Karoubi envelopes of PLA, PLCA, PLBA respectively.

The two presentations of PLBA are canonically equivalent by sending $[1]$ to $b$ and the idempotent $\pi_0$ to the composition $\iota \circ \rho : b \to b$.

5.4. Factorisation of morphisms in PLBA. Set $\pi_1 = \text{id} - \pi_0$, and $I = \{0, 1\}$. The projections
\[
\pi_i = \pi_{i_1} \otimes \cdots \otimes \pi_{i_N}, \quad i = (i_1, \ldots, i_N) \in I^N
\]
are a complete family of idempotents in PLA([N], [N]) and PLCA([N], [N]), i.e.,
\[
\pi_\iota \circ \pi_\iota = \delta_{\iota\iota} \pi_\iota \quad \text{and} \quad \sum_{\iota \in I^N} \pi_\iota = \text{id}_{[N]}
\]
There is a natural right (resp. left) action of $k[I^N]$ on PLA([N], [q]) and PLCA([p], [N]), given by
\[
\phi \cdot f = \sum_{\iota \in I^N} f(i) \phi \circ \pi_\iota \quad \text{and} \quad f \cdot \psi = \sum_{\iota \in I^N} f(i) \pi_\iota \circ \psi
\]
Set $\Gamma_N = k[I^N] \rtimes \mathcal{S}_N$, where $\sigma \in \mathcal{S}_N$ acts on $f \in k[I^N]$ by $\sigma \cdot f = f \circ \sigma^{-1}$.

**Proposition.** The embeddings PLA, PLCA $\to$ PLBA induce an isomorphism of $(\mathcal{S}_q, \mathcal{S}_p)$-bimodules
\[
\text{PLBA}([p], [q]) \simeq \bigoplus_{N \geq 0} \text{PLCA}([p], [N]) \otimes_{\Gamma_N} \text{PLA}([N], [q])
\]

**Proof.** The proof is similar to that of Proposition 3.5. The computation can be carried out with the PROPs LA, LCA, LBA introduced in 5.3 since these contain the objects $[p], [N], [q]$. A morphism in PLBA can be represented as an oriented graph obtained from the composition of brackets, cobrackets, permutations, and idempotents. The compatibility (2.3) between $\delta$ and $\mu$, and the relations
\[
\delta \circ \pi_0 = (\pi_0 \otimes \pi_0) \circ \delta \quad \pi_0 \circ \mu = \mu \circ (\pi_0 \otimes \pi_0)
\]
allow to reorder the morphisms so that the cobrackets precede the brackets, and the idempotent $\pi_0$ occur in between. This yields a surjective map
\[
\bigoplus_{N \geq 0} (\text{PLCA}([p], [N]) \otimes \text{PLA}([N], [q])) \to \text{PLBA}([p], [q])
\]
which factors through the action of $k[I^N] \rtimes \mathcal{S}_N$. The injectivity follows as in 3.5. $\square$
5.5. Morphisms in $\text{PLA}$ and $\text{PLCA}$.

**Lemma.** There are isomorphisms of left (resp. right) $k[\mathcal{I}^N] \times \mathfrak{S}_N$–modules

$$\text{PLA}([N],[p]) \simeq k[\mathcal{I}^N] \otimes \text{LA}([N],[p])$$

$$\text{PLCA}([p],[N]) \simeq \text{LCA}([p],[N]) \otimes k[\mathcal{I}^N]$$

where $\mathfrak{S}_N$ acts diagonally on the right–hand side, which are compatible with the action of $\mathfrak{S}_p$.

**Proof.** We only explain the isomorphism in $\text{PLA}$ since the proof for $\text{PLCA}$ is similar. Every morphism in $\text{PLA}([N],[p])$ is represented by a linear combination of oriented graphs from $N$ sources to $p$ targets. Since $\text{id}_{[1]} = \pi_0 + \pi_1$, all the edges of these graphs can be assumed to be decorated by the idempotents $\pi_0$ or $\pi_1$. The relations

$$\pi_0 \circ \mu = \mu \circ (\pi_0 \otimes \pi_0)$$

$$\pi_1 \circ \mu = \mu \circ (\pi_1 \otimes \pi_1 + \pi_1 \otimes \pi_0 + \pi_0 \otimes \pi_1)$$

allow to move all idempotents to the $N$ sources and yield the surjectivity of the map

$$k[\mathcal{I}^N] \otimes \text{LA}([N],[p]) \to \text{PLA}([N],[p]) \quad f \otimes P \mapsto f \cdot P$$

Its injectivity follows from the canonical embedding $\text{LA} \to \text{PLA}$ and the isomorphism

$$\bigoplus_{\mathcal{I}^N} \pi_i \text{PLA}([N],[p]) \simeq \text{PLA}([N],[p])$$

□

5.6. $\text{PLBA}$ and Lie polynomials. The following is a direct consequence of 5.5, 5.4 and Lemma 3.6.

**Proposition.**

(i) There is an isomorphism of $(\mathfrak{S}_q, \mathfrak{S}_p)$–bimodules

$$\text{PLBA}([p],[q]) \simeq \bigoplus_{N \geq 0} ((FL_N^{\otimes p}) \delta_N \otimes k[\mathcal{I}^N] \otimes (FL_N^{\otimes q}) \delta_N) \mathfrak{S}_N$$

(ii) Let $F \in k\mathfrak{S}_p$ and $G \in k\mathfrak{S}_q$ be idempotents, and $F[p] = ([p],F)$, $G[q] = ([q],G)$ the corresponding objects in $\text{PLBA}$. Then one has

$$\text{PLBA}(F[p],G[q]) \simeq \bigoplus_{N \geq 0} (F(FL_N^{\otimes p}) \delta_N \otimes k[\mathcal{I}^N] \otimes G(FL_N^{\otimes q}) \delta_N) \mathfrak{S}_N$$

**Corollary.** There are natural isomorphisms

$$\text{PLBA}(T[1]^{\otimes n},T[1]^{\otimes n}) \simeq \bigoplus_{N \geq 0} ((TFL_N^{\otimes n}) \delta_N \otimes k[\mathcal{I}^N] \otimes (TFL_N^{\otimes n}) \delta_N) \mathfrak{S}_N$$

$$\text{PLBA}(S[1]^{\otimes n},S[1]^{\otimes n}) \simeq \bigoplus_{N \geq 0} ((SFL_N^{\otimes n}) \delta_N \otimes k[\mathcal{I}^N] \otimes (SFL_N^{\otimes n}) \delta_N) \mathfrak{S}_N$$
5.7. Universal Drinfeld–Yetter modules and PLBA.

**Definition.** The category $PDY_n$, $n \geq 1$, is the multicored PROP generated by $n + 1$ objects $[1]$ and $\{V_k\}_{k=1,\ldots,n}$, and morphisms

\[
\mu : [2] \to [1] \quad \delta : [1] \to [2] \quad \pi : [1] \to [1]
\]

\[
\pi_k : [1] \otimes V_k \to V_k \quad \pi_k^* : V_k \to [1] \otimes V_k
\]
such that $([1], \mu, \delta, \pi)$ is a PLBA–module in $PDY_n$, and, for every $k = 1, \ldots, n$, $(\mathcal{V}_k, \pi_k, \pi_k^*)$ is a Drinfeld–Yetter module over $[1]$.

5.8. Morphisms in $PDY_n$. Set

\[
\Omega^n_{PDY} = \text{End}_{PDY_n} \left( \bigotimes_{k=1}^n V_k \right) \quad (5.3)
\]

The description of the algebras $\Omega^n_{PDY}$ is obtained along the same lines of Propositions 3.1 and 3.10.

**Proposition.** The endomorphisms

\[
\rho_i^N(N', \sigma) = \pi(N) \circ \pi_k \circ \text{id} \circ \sigma \circ \text{id} \circ \pi(N')
\]

where $N \geq 0$, $N, N' \in \mathbb{N}^n$ are such that $|N| = N = |N'|$, $i \in \mathbb{Z}^N$, and $\sigma \in S_N$, are a basis of $\Omega^n_{PDY}$.

**Corollary.** There is an isomorphism of vector spaces

\[
\xi^n_{PDY} : \Omega^n_{PDY} \to \bigoplus_{N \geq 0} \left( (FA^\otimes_n)_N \otimes k[\mathbb{Z}^N] \otimes (FA^\otimes_n)_N \right)_{\mathcal{S}_N}
\]

given by

\[
\xi^n_{PDY}(\rho^N(N', \sigma)) = \sum_{j=(j_1, \ldots, j_N)} a_{i_1} a_{i_2} \cdots a_{i_n} (\mathcal{S}_N)(j)
\]

5.9. Universal property of algebras of $\Omega^n_{PDY}$. The algebras $\Omega^n_{PDY}$ are universal in the following sense. Let $(b, a)$ be a split pair of bialgebras over $k$, $(\mathfrak{g}_b, \mathfrak{g}_a)$ be the corresponding Drinfeld doubles, and $\mathcal{U}_{DY}^{b_a}$ be the algebra of endomorphisms of the fiber functor $\mathfrak{f}^{\mathfrak{g}_a}$ from the category of Drinfeld–Yetter $b$–modules to $\text{Vect}_k$.

**Proposition.** For any $n \geq 1$, the universal algebra $\Omega^n_{PDY}$ is endowed with an algebra homomorphism $\rho_{(\mathfrak{g}_b, \mathfrak{g}_a)} : \Omega^n_{PDY} \to \tilde{\mathcal{U}}^{\mathfrak{g}_b}$ given by

\[
\rho^n_{(\mathfrak{g}_b, \mathfrak{g}_a)}(\rho^N(N', \sigma)) = \sum_{\substack{i=(j_1, \ldots, j_N) \in I_0 \setminus I_a \setminus I_b \setminus I_a}} a_{N(j)} \cdot \tilde{b}^{(\mathcal{S}_N)(j)}
\]

where $I_0 = I_a$ is the set of indices corresponding to the basis of $a$ and $I_1 = I_b \setminus I_a$.

**Proof.** The map $\rho^n_{(\mathfrak{g}_b, \mathfrak{g}_a)}$ is the composition of the algebra homomorphisms

\[
\Omega^n_{PDY} = \text{End}_{PDY_n} (\bigotimes_{k=1}^n V_k) \to \mathcal{U}_{DY}^{b_a} \simeq \tilde{\mathcal{U}}^{\mathfrak{g}_b} \quad \square
\]

The following is a corollary of the proposition above and 5.1.

**Corollary.**
(i) The universal algebra $\mathfrak{U}^n_{PDY}$ is endowed with two algebra homomorphisms

\[
\begin{array}{ccc}
\mathfrak{U}^n_{PDY} & \xrightarrow{\beta} & \mathfrak{U}^n_{PDY} \\
\alpha & & \\
\end{array}
\]

where

\[
\alpha(r^\sigma_{\Delta,N'}) = r^\sigma_{\Delta,N'} \quad \text{and} \quad \beta(r^\sigma_{\Delta,N'}) = \sum_{\ell \in I_N} r^\sigma_{\Delta,N'}
\]

(ii) For any split pair $(g_b, g_a)$ with $i_a : g_a \twoheadrightarrow g_b$, the diagram

\[
\begin{array}{ccc}
\mathfrak{U}^n_{PDY} & \xrightarrow{\rho_{g_a}} & \mathfrak{U}^n_{PDY} \\
\alpha & & \\
\rho_{g_b} & & \beta \\
\end{array}
\]

is commutative.

5.10. PBW theorem for $\mathfrak{U}^n_{PDY}$. Let

\[
a : \mathcal{PLBA}(T[1]^\otimes n, T[1]^\otimes n)_{fin} \rightarrow \mathcal{PDY}_n(\otimes_{k=1}^n V_k, \otimes_{k=1}^n V_k)
\]

be the map given on $\phi_{p,q} \in \mathcal{PLBA}(T[1]^\otimes p, T[1]^\otimes q)$, by

\[
a(\phi_{p,q}) = \pi^{(p)} \circ \phi_{p,q} \circ \pi^{(q)}
\]

Theorem.

(i) The following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{PDY}_n(\otimes_{k=1}^n V_k, \otimes_{k=1}^n V_k) & \xrightarrow{a} & \bigoplus_{N \geq 0} \left((FA_N^{\otimes n})_{\Delta_N} \otimes k[T_N] \otimes (FA_N^{\otimes n})_{\Delta_N}\right)_{\otimes N} \\
\mathcal{PLBA}(T[1]^\otimes n, T[1]^\otimes n)_{fin} & \xrightarrow{\text{Sym}} & \bigoplus_{N \geq 0} \left((T(FL)_N^{\otimes n})_{\Delta_N} \otimes k[T_N] \otimes (T(FL)_N^{\otimes n})_{\Delta_N}\right)_{\otimes N} \\
\mathcal{PLBA}(S[1]^\otimes n, S[1]^\otimes n)_{fin} & \xrightarrow{\text{Sym}} & \bigoplus_{N \geq 0} \left((S(FL)_N^{\otimes n})_{\Delta_N} \otimes k[T_N] \otimes (S(FL)_N^{\otimes n})_{\Delta_N}\right)_{\otimes N}
\end{array}
\]

(ii) The map $a \circ \text{Sym}$ is an isomorphism of cosimplicial spaces.

5.11. Hochschild cohomology. As in the case of $\mathfrak{U}_{DY}$, the algebras $\{\mathfrak{U}^n_{PDY}\}_{n \geq 1}$ are endowed with insertion maps

\[
\Delta^{(k)}_n : \mathfrak{U}^n_{PDY} \rightarrow \mathfrak{U}^{n+1}_{PDY}
\]
for \( k = 0, \ldots, n + 1 \). These give rise to a relative universal Hochschild complex with the differential \( d_n : \Omega^\bullet_{\text{ov}} \rightarrow \Omega^{n+1}_{\text{ov}} \)
\[
d_n = \sum_{k=0}^{n+1} (-1)^k \Delta^k_a
\]
The morphisms \( \{\rho^n_{(g_b, g_a)}\}_{n \geq 1} \) defined in 5.9 induce a chain map between the corresponding Hochschild complexes.

**Theorem.**

(i) The map \( a \circ \text{Sym} \) induces an isomorphism
\[
H^\bullet(\Omega^\bullet_{\text{ov}}, d_H) \cong \text{PLBA}(\wedge^\bullet[1], \wedge^\bullet[1])
\]
In particular, \( H^0(\Omega^\bullet_{\text{ov}}, d_H) = H^1(\Omega^\bullet_{\text{ov}}, d_H) = 0 \).

(ii) The identification in terms of Lie polynomials of Proposition 5.6 yields
\[
H^\bullet(\Omega^\bullet_{\text{ov}}, d_H) \cong \bigoplus_{N \geq 0} ([\wedge^\bullet FL_N]_{\delta_N} \otimes k[I_N] \otimes (\wedge^\bullet FL_N)_{\delta_N})_{S_N}
\]

5.12. Universal invariants. The homomorphisms \( \alpha, \beta : \Omega^\bullet_{\text{ov}} \rightarrow \Omega^\bullet_{\text{ov}} \) define two distinct \( r \)–matrices in \( \Omega^2_{\text{ov}} \). Namely,
\[
\alpha = \alpha(r) \quad \text{and} \quad \beta = \beta(r)
\]
This allows to define the space of invariants with respect to the small \( r \)–matrix \( r_\alpha \).

**Definition.** The space of \( \alpha \)–invariants \( (\Omega^\bullet_{\text{ov}})_{\text{inv} \alpha} \) is defined as the subspace of all \( x \in \Omega^\bullet_{\text{ov}} \) satisfying
\[
[r_\alpha^{01} + \cdots + r_\alpha^{0n}, x^{1\ldots n}] = [r_\alpha^{10} + \cdots + r_\alpha^{n0}, x^{1\ldots n}] = 0 \quad (5.4)
\]
The above condition is equivalently stated as follows. Let \( \alpha : \Omega^\bullet_{\text{ov}} \rightarrow \Omega^\bullet_{\text{ov}} \). For any \( y \in \text{Im}(\alpha) \),
\[
[y^{(1\ldots n), n+1, \ldots, n+p-1}, x^{1, 2, \ldots, n}] = 0
\]
in \( \Omega^{n+p-1} \).

The morphism \( \rho^n_{(g_b, g_a)} \) defined in 5.9 preserves the invariant elements. Namely,

**Proposition.** The map
\[
\rho^n_{(g_b, g_a)} : \Omega^\bullet_{\text{ov}} \rightarrow \hat{U}_{g_b}^\otimes n
\]
restricts to an algebra homomorphism
\[
\rho^n_{(g_b, g_a)} : (\Omega^\bullet_{\text{ov}})_{\text{inv} \alpha} \rightarrow \hat{U}_{g_b}^\otimes n = \text{End} \left( \mathcal{E}_{g_b} \rightarrow \mathcal{E}_{g_a} \right)
\]

**Proof.** The proof is similar to that of Proposition 4.11. It is enough to observe that the condition \( x \in \hat{U}_{g_b}^\otimes n_{\alpha} \) is equivalent to
\[
[r_\alpha^{01} + \cdots + r_\alpha^{0n}, x^{1\ldots n}] = [r_\alpha^{10} + \cdots + r_\alpha^{n0}, x^{1\ldots n}] = 0 \quad (5.5)
\]
where \( r_a = \rho^n_{(g_b, g_a)}(r_\alpha) \).

5.13. Co-Hochschild differential and invariants.

**Lemma.** \( ((\Omega^\bullet_{\text{ov}})_{\text{inv} \alpha}, d_n) \) is a subcomplex of \( (\Omega^\bullet_{\text{ov}}, d_n) \).
Proof. Assume \( u \in (\mathfrak{U}^{n-1}_{\text{poiv}})^{\text{inv}} \). Since
\[
(id \otimes \Delta^{(n)}) (r_\alpha) = \sum_{i=0}^{n} r^{0i}_\alpha
\]
follows that
\[
\left[(id \otimes \Delta^{(n)})(r_\alpha), (d_{n-1} u)^{1...n}\right] = (id \otimes d_\alpha) \left[(id \otimes \Delta^{(n-1)})(r_\alpha), u^{1...n-1}\right] = 0
\]
and \( d_{n-1}(u) \in (\mathfrak{U}^{n}_{\text{poiv}})^{\text{inv}} \).
\[\square\]

Proposition. \( H^n((\mathfrak{U}^*_n)^{\text{inv}}, d_H) \simeq (\Lambda^n_{\text{poiv}})^{\text{inv}} \)

Proof. Let \( f \in \mathfrak{U}^n_{\text{poiv}} \) such that \( d_H(f) = 0 \). Then there are unique \( d_H(u) \in \mathfrak{U}^n_{\text{poiv}} \) and \( v \in \Lambda^n_{\text{poiv}} \) such that \( f = v + d_H(u) \). Namely, let \( f = v' + d_H(u') \). It follows from the quasi–isomorphism \( \Phi \) that
\[
v - v' = d_H(u - u') = 0
\]
Let \( f \in (\mathfrak{U}^n_{\text{poiv}})^{\text{inv}} \). One has
\[
0 = [(id \otimes \Delta^{(n)})(r), f^{1...n}] = [(id \otimes \Delta^{(n)})(r), v^{1...n}] + (id \otimes d_H) [(id \otimes \Delta^{(n-1)})(r), u^{1...n-1}]
\]
It follows that both summands are zero, and \( v \in (\Lambda^n_{\text{poiv}})^{\text{inv}} \) and \( u \in (\mathfrak{U}^{n-1}_{\text{poiv}})^{\text{inv}} \).

6. Universal relative twists

Let \( (\widehat{\mathfrak{U}}^n_{\text{poiv}})^{\text{inv}} \) be completion of \( (\mathfrak{U}^n_{\text{poiv}})^{\text{inv}} \) with respect to its grading. In this section, we discuss the existence and uniqueness of invertible elements \( J \in (\widehat{\mathfrak{U}}^2_{\text{poiv}})^{\text{inv}} \) satisfying the relative twist equation (6.1). This implies, for any split pair of Lie bialgebras, the uniqueness of the tensor structure on the restriction functor at the level of equicontinuous modules.

6.1. Existence of a universal relative twist. Let \( \Phi \in \text{Assoc} \) and let \( (\mathfrak{b}, \mathfrak{a}) \) be a split pair with corresponding Drinfeld doubles \((\mathfrak{g}_\mathfrak{b}, \mathfrak{g}_\mathfrak{a})\). Let \( \Phi_\mathfrak{b}, \Phi_\mathfrak{a} \) be the images of \( \Phi \) in \( \widehat{U}_{\mathfrak{g}_\mathfrak{b}}^\otimes \) and \( \widehat{U}_{\mathfrak{g}_\mathfrak{a}}^\otimes \) respectively.

In [1, Prop. 5.15], we constructed an element \( J_\Phi \in \widehat{U}_{\mathfrak{g}_\mathfrak{a}}^\otimes \), which is invariant under \( \mathfrak{g}_\mathfrak{a} \), \( J_\Phi = 1 \mod \hbar \), and satisfies
\[
J_\Phi^{23,123} \Phi_\mathfrak{a} = \Phi_\mathfrak{b} J_\Phi^{12} J_\Phi^{12,3}
\]
We also showed [1, Sec. 7] that the construction of \( J_\Phi \) is universal \( i.e. \), that it can be realized as an \( \alpha \)-invariant element
\[
J_\Phi \in (\widehat{\mathfrak{U}}^2_{\text{poiv}})^{\text{inv}} = \text{End}_{\text{poiv}} (\mathfrak{V}_1 \otimes \mathfrak{V}_2)
\]
We summarize this in the following

Theorem. There is a map \( \text{Assoc} \rightarrow (\widehat{\mathfrak{U}}^2_{\text{poiv}})^{\text{inv}}, \Phi \rightarrow J_\Phi \) such that the following holds in \( (\widehat{\mathfrak{U}}^2_{\text{poiv}})^{\text{inv}}, J_\Phi = 1 \mod (\widehat{\mathfrak{U}}^2_{\text{poiv}})^{\text{inv}} \) and
\[
(\Phi_\beta)_J_\Phi = \Phi_\alpha \tag{6.1}
\]
where \( \Phi_\beta, \Phi_\alpha \) are the images of \( \Phi \) in \( (\widehat{\mathfrak{U}}^2_{\text{poiv}})^{\text{inv}} \) via \( \alpha \) and \( \beta \), and
\[
\Phi J_\Phi = J_\Phi^{23,123} \Phi (J_\Phi^{12})^{-1} (J_\Phi^{12})^{-1}
\]
6.2. Uniqueness of universal relative twists. We now show the uniqueness of the twist $J_\Phi$ up to a unique gauge transformation.

**Theorem.** For any $\Phi \in \text{Assoc}$,

$$\{ J \in (\hat{\mathcal{U}}_{\text{prov}}^{m_\alpha}) | (\overline{\Phi}_\beta)_J = \Phi_\alpha, J_0 = 1 \} = \{ u \otimes u \cdot J_\Phi \cdot \Delta(u)^{-1} | u \in (\hat{\mathcal{U}}_{\text{prov}}^{m_\alpha}) \}$$

**Proof.** Assume $J^{(i)} = 1 + \sum_{k \geq 1} J^{(i)}_k \in (\hat{\mathcal{U}}_{\text{prov}}^{m_\alpha})$, $i = 1, 2$, and $\Phi_{J^{(i)}} = \Phi_\alpha$.

One checks, by linearisation of (6.1), that $J^{(i)}_1$ is an element in $(\hat{U}_{\text{prov}}^{m_\alpha})$, satisfying $d_H(J^{(i)}_1) = 0$. This implies that

$$\text{Alt}_2(J_1) = \text{Alt}_2(r - r_\alpha) = r - r_\alpha$$

Up to a gauge, we may always assume $J^{(i)}_1 = r - r_\alpha$, $i = 1, 2$. We want to show that there exists an invertible $u \in \hat{U}_1$ such that

$$u \otimes u \cdot J^{(1)} \cdot \Delta(u)^{-1} = J^{(2)}$$

(6.2)

Assume that (6.2) is true modulo $(\hat{U}_{\text{prov}}^{m_\alpha})$, i.e., there exists $u^{(n-1)} \in \hat{U}_1^{m_\alpha}$ such that

$$u^{(n-1)} \otimes u^{(n-1)} \cdot J^{(1)} \cdot \Delta(u^{(n-1)})^{-1} = J^{(2)} \mod (\hat{U}_{\text{prov}}^{m_\alpha})$$

(6.3)

Let now $\tilde{J}^{(1)}$ be the left–hand side of (6.3), and $\eta \in (\hat{U}_{\text{prov}}^{m_\alpha})$ such that

$$J^{(2)} = \tilde{J}^{(1)} + \eta \mod (\hat{U}_{\text{prov}}^{m_\alpha})$$

One checks that $\tilde{J}^{(1)}$ satisfies $\Phi_{J^{(i)}} = \Phi_\alpha$ modulo $(\hat{U}_{\text{prov}}^{m_\alpha})^{2, n+1}$. Comparing with $\Phi_{J^{(2)}} = \Phi_\alpha$ modulo $(\hat{U}_{\text{prov}}^{m_\alpha})^{2, n+1}$, one gets

$$d_H(\eta) = 1 \otimes \eta + (\id \otimes \Delta)(\eta) - (\id \otimes \Delta)(\eta) - \eta \otimes 1 = 0$$

Therefore, by Proposition 5.13, there exist $v \in (\hat{U}_{\text{prov}}^{m_\alpha})_n$ and $\mu \in (\Lambda_{\text{prov}}^{2, n+1})$, such that

$$\eta = d_H(v) + \mu$$

We claim $\mu = 0$. In this case, we may set $u^{(n)} = (1 - v)u^{(n-1)}$, and we get

$$u^{(n)} \otimes u^{(n)} \cdot J^{(1)} \cdot \Delta(u^{(n)})^{-1} = J^{(2)} \mod (\hat{U}_{\text{prov}}^{m_\alpha})^{2, n+1}$$

There remains to prove the claim. Set $\tilde{J}^{(2)} = u^{(n)} \otimes u^{(n)} \cdot J^{(1)} \cdot \Delta(u^{(n)})^{-1}$. Then

$$\tilde{J}^{(2)} = J^{(2)} + \mu \mod (\hat{U}_{\text{prov}}^{m_\alpha})^{2, n+1}$$

Let $J^{(2)}_{[n+1]}$, $\tilde{J}^{(2)}_{[n+1]}$ be the corresponding truncations. We set

$$\xi = (J^{(2)}_{[n+1]})^{23}(J^{(2)}_{[n+1]})^{1, 23}\Phi - \Phi_\alpha(J^{(2)}_{[n+1]})^{12}(J^{(2)}_{[n+1]})^{1, 12, 3} \mod (\hat{U}_{\text{prov}}^{m_\alpha})^{2, n+2}$$

$$\tilde{\xi} = (\tilde{J}^{(2)}_{[n+1]})^{23}(\tilde{J}^{(2)}_{[n+1]})^{1, 23}\Phi - \Phi_\alpha(\tilde{J}^{(2)}_{[n+1]})^{12}(\tilde{J}^{(2)}_{[n+1]})^{1, 12, 3} \mod (\hat{U}_{\text{prov}}^{m_\alpha})^{2, n+2}$$

$\tilde{J}^{(2)}$ and $J^{(2)}$ are both solutions of $\Phi_J = \Phi_\alpha$, therefore

$$\xi = d_H(J^{(2)}_{[n+1]}) \quad \text{and} \quad \tilde{\xi} = d_H(\tilde{J}^{(2)}_{[n+1]})$$

It follows $d_H \xi = d_H \tilde{\xi} = 0$ and $\text{Alt} \xi = \text{Alt} \tilde{\xi} = 0$. We then observe that

$$\tilde{\xi} - \xi = f(\mu)$$

where $f(\mu) = f^{23}(\mu^{12} + \mu^{13}) + \mu^{23}(f^{12} + f^{13}) - f^{12}(\mu^{13} + \mu^{23}) - \mu^{12}(f^{13} + f^{23})$ and $f = r - r_\alpha$. By straightforward computation, one checks

$$\text{Alt} f(\mu) = [r - r_\alpha, \mu]$$
where \([\cdot,\cdot]\) is the Schouten bracket from \(\Lambda^2_{\text{poly}} \to \Lambda^3_{\text{poly}}\). Therefore \([\bar{r} - \bar{r}_\alpha, \mu] = 0\). Since \([\bar{r} - \bar{r}_\alpha, -] = [\bar{r}, -]\) on \((\Lambda^2_{\text{poly}})^{\text{op}}\), one gets \([\bar{r}, \mu] = 0\). It follows from [7, Prop. 2.2] that the map \([\bar{r}, -]\) has a trivial kernel on \(\Lambda^2_{\text{poly}}\) and \(\Lambda^2_{\text{poly}}\). Therefore \(\mu = 0\), and the theorem is proved. \(\square\)

**Remark.** Theorem 6.2 is a generalisation of the case \(\text{LBA}\) proved in [8, Thm. 2.1]. In particular, one recovers the uniqueness of the twist in \(\text{LBA}\) applying the forgetful functor \(\text{PLBA} \to \text{LBA}\).

### 7. The Casimir category

In this section, we further refine the \(\text{PROP LBA}\) by adding a complete family of orthogonal idempotents corresponding to the non–negative roots of a Kac–Moody algebra \(g\). This allows us to construct a universal algebra containing a family of subalgebras labelled by the subdiagrams of the Dynkin diagram of \(g\).

#### 7.1. The Casimir category

Let \(Q\) be a free abelian group of rank \(l\) with a distinguished basis \(\{\alpha_1, \ldots, \alpha_l\}\), and \(Q_+ \subset Q\) the corresponding non–negative cone,

\[
Q = \bigoplus_{k=1}^{l} \mathbb{Z}\alpha_k \quad \text{and} \quad Q_+ = \bigoplus_{k=1}^{l} \mathbb{Z}_{\geq 0}\alpha_k
\]

Let \(\widetilde{L\mathcal{A}}_Q\) be the \(\text{PROP}\) with generators \(\mu \in S_{21}\) and \(\{\pi_\alpha\}_{\alpha \in Q_+} \subset S_{11}\), and relations (2.1) and

\[
\pi_\alpha \circ \pi_\beta = \delta_{\alpha,\beta}\pi_\alpha \quad (7.1)
\]
\[
\pi_\alpha \circ \mu = \sum_{\beta+\gamma = \alpha} \mu \circ (\pi_\beta \otimes \pi_\gamma) \quad (7.2)
\]

We wish to impose the additional completeness relation

\[
\sum_{\alpha \in Q_+} \pi_\alpha = \text{id}_{[1]} \quad (7.3)
\]

To this end, let \(p \in \mathbb{N}\), and denote by \(k[Q^p_+]^{\text{fin}}\) the algebra of functions on \(Q^p_+\) with finite support. The vector space \(\widetilde{L\mathcal{A}}_Q([p],[q])\) is naturally a \((k[Q^p_+]^{\text{fin}}, k[Q^q_+]^{\text{fin}})\)–bimodule. Let \(L\mathcal{A}_Q\) be the (topological) \(\text{PROP}\) with morphisms

\[
L\mathcal{A}_Q([p],[q]) = k[Q^p_+] \otimes_{k[Q^q_+]^{\text{fin}}} L\mathcal{A}_Q([p],[q]) \otimes_{k[Q^q_+]^{\text{fin}}} k[Q^q_+] \quad (7.4)
\]

Then, \(\sum_{\alpha \in Q_+} \delta_\alpha \in k[Q^p_+]\) acts on \([p]\) as \(\text{id}_{[p]}\). Finally, let \(\overline{L\mathcal{A}}_Q\) be the Karoubi envelope of \(L\mathcal{A}_Q\).

Let \(\overline{L\mathcal{C}}_Q\) be similarly defined as the topological \(\text{PROP}\) with generators \(\delta \in S_{12}\) and \(\{\pi_\alpha\}_{\alpha \in Q_+} \subset S_{11}\) and relations (7.1), (7.3), (2.2), and

\[
\delta \circ \pi_\alpha = \sum_{\beta+\gamma = \alpha} (\pi_\beta \otimes \pi_\gamma) \circ \delta \quad (7.5)
\]

**Definition.** The Casimir category \(\text{LBA}_Q\) is the \(\text{PROP}\) generated by the morphisms \(\mu \in S_{21}, \delta \in S_{12}, \pi_\alpha \in S_{11}, \alpha \in Q_+, \) with relations (7.1), (7.3), (2.1), (2.2), (2.3), (7.2), (7.4), and (7.5).

**Example.** Let \(g\) be Kac–Moody algebra of rank \(l\), with root lattice \(Q\), cone of non–negative roots \(Q_+\), and positive Borel subalgebra \(\bigoplus_{\alpha \in Q_+} \mathfrak{g}_\alpha\). Then \(\mathfrak{h}\) is an \(\text{LBA}_Q\)–module.
7.2. Canonical modules in \( \text{LBA}_Q \). Let \( D = \{ 1, \ldots, l \} \) and, for any subset \( B \subseteq D \), set
\[
Q_B = \bigoplus_{k \in B} \mathbb{Z} \alpha_k \quad \text{and} \quad Q_{B,+} = \bigoplus_{k \in B} \mathbb{Z}_{\geq 0} \alpha_k
\]
The following is clear

**Proposition.**

(i) The idempotent
\[
\pi_B = \sum_{\alpha \in Q_{B,+}} \pi_\alpha
\]
satisfies
\[
\pi_B \circ \mu = \mu \circ (\pi_B \otimes \pi_B) \quad \text{and} \quad \delta \circ \pi_B = (\pi_B \otimes \pi_B) \circ \delta
\]
(ii) The object \( ([1]_B, \mu_B, \delta_B) \), where \( [1]_B = ([1], \pi_B) \),
\[
\mu_B = \pi_B \circ \mu \circ (\pi_B \otimes \pi_B) \quad \text{and} \quad \delta_B = (\pi_B \otimes \pi_B) \circ \delta \circ \pi_B
\]
is an \( \text{LBA}_Q \)-module in \( \text{LBA}_Q \).
(iii) For any \( B \subseteq C \subseteq D \), the pair \( ([1]_C, [1]_B) \) is a \( \text{PLBA}_Q \)-module in \( \text{LBA}_Q \).
Equivalently, there is a canonical functor
\[
\rho_{(B,C)} : \text{PLBA}_Q \to \text{LBA}_Q
\]
mapping \( a \) to \([1]_B\) and \( b \) to \([1]_C\).

7.3. Morphisms in \( \text{LBA}_Q \). The projections
\[
\pi_\mathbf{a} = \pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_N}, \quad \mathbf{a} = (\alpha_1, \ldots, \alpha_N) \in Q^+_N
\]
are a complete family of orthogonal idempotents in \( \text{LA}_Q([N], [N]) \) and \( \text{LCA}_Q([N], [N]) \).
By construction, there is a natural right (resp. left) \( S_Q \times k[Q^+_N] \)-action on \( \text{LA}_Q([N], [q]) \) and \( \text{LCA}_Q([p], [N]) \).

**Proposition.**

(i) The embeddings \( \text{LA}_Q \to \text{LCA}_Q \to \text{LBA}_Q \) induce an isomorphism of \( (S_q, S_p) \)-bimodules
\[
\text{LBA}_Q([p], [q]) \cong \bigoplus_{N \geq 0} \text{LCA}_Q([p], [N]) \otimes_{S_N \times k[Q^+_N]} \text{LA}_Q([N], [q])
\]
(ii) The embeddings \( \text{LA}_Q \to \text{LA}_Q \) and \( \text{LCA}_Q \to \text{LCA}_Q \) induce isomorphisms of right (resp. left) \( k[Q^+_N] \times S_N \)-modules
\[
\text{LA}_Q([N], [q]) \cong k[Q^+_N] \otimes \text{LA}([N], [q])
\]
\[
\text{LCA}_Q([p], [N]) \cong \text{LCA}([p], [N]) \otimes k[Q^+_N]
\]
compatible with the left (resp. right) action of \( S_q \) (resp. \( S_p \)).
(iii) There is an isomorphism of \( (S_q, S_p) \)-bimodules
\[
\text{LBA}_Q([p], [q]) \cong \bigoplus_{N \geq 0} ((FL^{op}_N) \delta_N \otimes k[Q^+_N] \otimes (FL^{op}_N) \delta_N)_{S_N}
\]
where the coinvariants are taken with respect to the diagonal action of \( S_N \).
(iv) There are natural isomorphisms
\[
\mathcal{LBA}_Q(T[1] \otimes_n, T[1] \otimes_n) \cong \bigoplus_{N \geq 0} \left( (TFL_N^\otimes)_{\delta_N} \otimes k[Q^+_N] \otimes (TFL_N^\otimes)_{\delta_N} \right)_{\mathcal{S}_N}
\]
\[
\mathcal{LBA}_Q(S[1] \otimes_n, S[1] \otimes_n) \cong \bigoplus_{N \geq 0} \left( (SF_L_N^\otimes)_{\delta_N} \otimes k[Q^+_N] \otimes (SF_L_N^\otimes)_{\delta_N} \right)_{\mathcal{S}_N}
\]

7.4. Universal Drinfeld–Yetter modules and \( \mathcal{LBA}_Q \).

**Definition.** The category \( \text{DY}_{Q,n} \), \( n \geq 1 \), is the multicolored PROP generated by \( n+1 \) objects \([1]\) and \( \{ V_k \}_{k=1}^n \), and morphisms
\[
\mu : [2] \to [1], \quad \delta : [1] \to [2]
\]
\[
\pi_{\alpha} : [1] \to [1], \quad \alpha \in Q^+_\mathcal{S}
\]
\[
\pi_k : [1] \otimes V_k \to V_k, \quad \pi^*_k : V_k \to [1] \otimes V_k
\]
such that \( ([1], \mu, \delta, \{ \pi_{\alpha} \}) \) is an \( \mathcal{LBA}_Q \)-module in \( \text{DY}_{Q,n} \), and every \((V_k, \pi_k, \pi^*_k)\) is a Drinfeld–Yetter module over \([1]\).

7.5. Morphisms in \( \mathcal{LBA}_Q \). Set
\[
\mathcal{U}_Q^n = \text{End}_{\text{DY}_{Q,n}} \left( \bigotimes_{k=1}^n V_k \right)
\]  
(7.6)

We will refer to \( \mathcal{U}_Q^n \) as the universal Casimir algebra. A basis of it is readily obtained along the same lines as Propositions 3.1 and 3.10.

**Proposition.** The endomorphisms of \( V_1 \otimes \cdots \otimes V_n \) given by
\[
r_{\alpha, \sigma}^{N, N'} = \pi^{(N')} \circ \pi^{\alpha} \circ \text{id} \circ \sigma \circ \text{id} \circ \pi^{*(N')}
\]
where \( N \geq 0, N, N' \in \mathbb{N}^n \) are such that \( |N| = N = |N'|, \alpha \in Q^+_\mathcal{S} \), and \( \sigma \in \mathcal{S}_N \), are a basis of \( \mathcal{U}_Q^n \).

**Corollary.** There is an isomorphism of vector spaces
\[
\xi_Q^n : \mathcal{U}_Q^n \longrightarrow \bigoplus_{N \geq 0} \left( (FA_N^\otimes)_{\delta_N} \otimes k[Q^+_N] \otimes (FA_N^\otimes)_{\delta_N} \right)_{\mathcal{S}_N}
\]
given by
\[
\xi_Q^n \left( r_{\alpha, \sigma}^{N, N'} \right) = x^{\alpha}_{N} \otimes \delta_{\sigma} \otimes y_{\sigma}(N')
\]

7.6. PBW theorem for \( \mathcal{U}_Q^n \). Let
\[
a : \mathcal{LBA}_Q(T[1] \otimes_n, T[1] \otimes_n) \longrightarrow \text{DY}^n_n \left( \otimes_{k=1}^n V_k, \otimes_{k=1}^n V_k \right)
\]
be the map given on \( \phi_{p,q} \in \mathcal{LBA}_Q(T[1], T[1]) \), by
\[
a(\phi_{p,q}) = \pi^{(p)} \circ \phi_{p,q} \circ \pi^{*(q)}
\]

**Theorem.**
(i) The following diagram is commutative

\[
\begin{array}{c}
\text{DY}^n(\otimes_{i=1}^n V_i, \otimes_{k=1}^n V_k) \\
\downarrow \alpha \\
\text{LBA}_\alpha(T^{\otimes n}[1], T^{\otimes n}[1])^{\text{fin}} \\
\downarrow \text{Sym} \\
\text{LBA}_\alpha(S[1]^{\otimes n}, S[1]^{\otimes n})^{\text{fin}} \\
\end{array}
\xrightarrow{\text{Sym}\otimes \text{id} \otimes \text{Sym}}
\begin{array}{c}
\bigoplus_{N \geq 0} \left( (FA_N^\otimes)^{\otimes n} \otimes k[Q^N] \otimes (FA_N^\otimes)^{\otimes n} \right)_{\otimes N} \\
\bigoplus_{N \geq 0} \left( (T(FL_N)^{\otimes n})^{\otimes n} \otimes k[Q^N] \otimes (T(FL_N)^{\otimes n})^{\otimes n} \right)_{\otimes N} \\
\bigoplus_{N \geq 0} \left( (S(FL_N)^{\otimes n})^{\otimes n} \otimes k[Q^N] \otimes (S(FL_N)^{\otimes n})^{\otimes n} \right)_{\otimes N}
\end{array}
\]

(ii) The map $\text{Sym}$ is an isomorphism of cosimplicial spaces.

### 7.7. $\mathfrak{U}_Q$ and completions

Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra with root lattice $Q \cong \mathbb{Z}^l$, cone of non–negative roots $Q_+$, Cartan subalgebra $\mathfrak{h}$ and positive Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{g}_\alpha$. Let $\mathfrak{g}_b = \mathfrak{D}b \cong \mathfrak{g} \oplus \mathfrak{h}$ be the double of $\mathfrak{b}$, and $\hat{U}_{\mathfrak{g}_b^{\otimes n}}^h$ be the algebra of endomorphisms of the forgetful functor from the category of equicontinuous $\mathfrak{g}_b$–modules to $\mathfrak{h}$–modules.

**Proposition.** Let $n \geq 1$. The linear map $\rho^n_Q : \mathfrak{U}_Q^n \rightarrow \hat{U}_{\mathfrak{g}_b^{\otimes n}}^h$ given on generators by

\[
\rho^n_Q(r_{\alpha, N, N'}) = \sum_{\sigma = (i_1, \ldots, i_N)} a_{N(i)} \cdot b^{\sigma(N')/\sigma}
\]

where $I_{\alpha_k}$ is the set of indices corresponding to the basis of $\mathfrak{b}^{\alpha_k}$, is an algebra homomorphism.

**Proof.** $\rho^n_Q$ is the algebra homomorphism

\[
\mathfrak{U}_Q^n = \text{End}_{\text{DY}_{\mathfrak{g}_b}}(\otimes_{i=1}^n V_i) \rightarrow \mathfrak{U}_{\text{DY}_a} \simeq \hat{U}_{\mathfrak{g}_b^{\otimes n}}^h
\]

which naturally factors through $\hat{U}_{\mathfrak{g}_b^{\otimes n}}^h$. \hfill \square

### 7.8. Diagrammatic subalgebras of the universal Casimir algebra

Let $D$ be the Dynkin diagram of a symmetrisable Kac–Moody algebra $\mathfrak{g}$ with Borel subalgebra $\mathfrak{b}$. We can associate to any subdiagram $B \subset D$ the sub Lie bialgebra $\mathfrak{g}_B \subset \mathfrak{g}$.

For any pair of subdiagrams $C \subset B \subset D$, let $\rho^n_{(B,C)}$ be the map

\[
\rho^n_{(B,C)} : \mathfrak{U}_B^n \rightarrow \hat{U}_{\mathfrak{g}_B^{\otimes n}}^h
\]

corresponding to the split pair $(\mathfrak{g}_B, \mathfrak{g}_C)$.

**Proposition.**

(i) Let $\bar{\rho}_B^n : \mathfrak{U}_B^n \rightarrow \mathfrak{U}_B^n$ be the linear map given by

\[
\bar{\rho}_B^n(r_{\alpha, N, N'}) = \sum_{\sigma \in Q^N} r_{\alpha, N, N'}^{\alpha, \sigma}
\]

Then $\bar{\rho}_B$ is an injective algebra homomorphism and satisfies $\rho_B^n = \rho_Q^n \circ \bar{\rho}_B^n$.  


(ii) Let $\overline{\rho}^n_{(B,C)} : \Upsilon_{\text{vor}}^n \rightarrow \Upsilon_Q^n$ be the linear map given by

$$
\overline{\rho}^n_{(B,C)}(\overline{v}_{N,N'}) = \sum_{\sigma, N} \alpha_{N,N'}^\sigma \overline{v}_{N,N'}
$$

where $\alpha \in \mathbb{Q}^+$ belongs to $\mathbb{I}_{B,C}$ if $\alpha_k \in \mathbb{Q}_{B,+}$ whenever $i_k = 0$ and $\alpha_k \in \mathbb{Q}_{C,+} \setminus \mathbb{Q}_{B,+}$ otherwise. Then $\overline{\rho}^n_{(B,C)}$ is an injective algebra homomorphism and satisfies $\overline{\rho}^n_{(B,C)} = \overline{\rho}^n_B \circ \overline{\rho}^n_{(B,C)}$.

7.9. Hochschild cohomology. The algebras $\{\Upsilon_Q^n\}_{n \geq 1}$ are endowed with insertion maps

$$
\Delta_n^{(k)} : \Upsilon_Q^n \rightarrow \Upsilon_Q^{n+1}
$$

for $k = 0, \ldots, n+1$, which give rise to a the Hochschild differential $d_n = \sum_{k=0}^{n+1} (-1)^k \Delta_n^{(k)} : \Upsilon_{\text{vor}}^n \rightarrow \Upsilon_{\text{vor}}^{n+1}$.

Theorem.

(i) The map $a \circ \text{Sym}$ induces an isomorphism

$$
H^n(\Upsilon_Q^* , d_H) \cong \bigwedge \text{BA}_Q(\Lambda^*[1], \Lambda^*[1])
$$

In particular, $H^0(\Upsilon_Q^1, d_H) = H^1(\Upsilon_Q^1, d_H) = 0$.

(ii) The identification in terms of Lie polynomials of Proposition 7.3 yields

$$
H^n(\Upsilon_Q^* , d_H) \cong \bigoplus_{N \geq 0} \left( (\Lambda^*F \Lambda N)_{\delta_N} \otimes k[Q]^\times \otimes (\Lambda^*F \Lambda N)_{\delta_N} \right) \otimes \cdots
$$

7.10. Universal twists. Let $\widehat{\Upsilon}_Q^n$ be the completion of $\Upsilon_Q^n$ with respect to its $\mathbb{N}$-grading. Let $\Phi \in \text{Assoc}$ be a fixed Drinfeld associator, and identify it with its image in $\widehat{\Upsilon}_Q$ via the map defined in 4.9. For any subdiagram $B \subseteq D$, denote by $\Phi_B = \overline{\rho}^n_B(\Phi)$ the image of $\Phi$ in $\widehat{\Upsilon}_Q^n$ under the map $\overline{\rho}^n_B : \Upsilon_{\text{vor}}^n \rightarrow \Upsilon_Q^n$.

Let $J_{\text{rel}} \in \widehat{\Upsilon}_Q^n$ be the universal relative twist constructed in [1] (see Theorem 6.1). For a fixed pair of subdiagrams $C \subseteq B \subseteq D$, let

$$
J_{(C,B)} = \overline{\rho}^2_{(C,B)}(J_{\text{rel}}) \in \widehat{\Upsilon}_Q^{2(C,B)}
$$

be the image of $J_{\text{rel}}$ in $\widehat{\Upsilon}_Q^n$. Then, $J_{(C,B)}$ satisfies

$$
(\Phi_B)J_{(C,B)} = \Phi_C
$$

(7.7)

Theorem.

(i) If $J_i \in \widehat{\Upsilon}_Q^{2(C,B)}$, $i = 1, 2$, are solutions of the relative twist equation (7.7),

with $(J_i)_0 = 1$, $i = 1, 2$, there is a gauge transformation $u \in \widehat{\Upsilon}_Q^{x(C,B)}$ , with

$u_0 = 1$, such that $J_2 = u \otimes u \cdot J_1 \cdot \Delta(u)^{-1}$.

(ii) The gauge transformation $u$ in (i) is unique.

Proof. The proof of (i) follows verbatim that of Theorem 6.2. (ii) Assume that

$$
u \in \Upsilon_{(C,B)}$$

is such that

$$
u \otimes u \cdot J = J \cdot \Delta(u)
$$

and $u = 1 \bmod (\Upsilon_{(C,B)})_n$. Let $v \in (\Upsilon_{(C,B)})_n$ such that $u = 1 + v \bmod (\Upsilon_{(C,B)})_n$. Taking the component of degree $n+1$ in the above equation yields

$$
d_H(v) = v \otimes v \otimes 1 - \Delta(v) = 0
$$

which by Theorem 7.9 implies that $v = 0$. \qed
7.11. Commutation statements. Recall that $\kappa = x \otimes y \in U_{\text{ov}}$ is the universal version of $\sum_i a_i b^i$. We use the notation $\kappa_\alpha = x \otimes \delta_\alpha \otimes y \in (U_Q)_{1}$, for any $\alpha \in Q_+$. Therefore, in $U_Q$,

$$\tilde{\rho}_p(\kappa) = \sum_{\alpha \in Q_+} \kappa_\alpha$$

**Proposition.**

(i) The element $\kappa$ is central in $U_{\text{ov}}$.

(ii) For any $n \geq 2$ and $X \in U_{\text{ov}}^n$,

$$[\kappa^1 + \cdots + \kappa^n, X] = 0$$

(iii) In $U_Q$, for any $\alpha \in Q_+$, $[\kappa_0, \kappa_\alpha] = 0$.

(iv) For any $B \subseteq D$, $\alpha \in Q_{B,+}$,

$$[\kappa_\alpha, \sum_{\beta \in Q_{B,+}} \kappa_\beta] = 0$$

(v) For any sublattice $\Psi \subset Q_+$, $\text{rk}(\Psi) = 2$, $\alpha \in \Psi$,

$$[\kappa_\alpha, \sum_{\beta \in \Psi} \kappa_\beta] = 0$$

**Proof.** (i) follows from an explicit computation in $\text{End}_{DY}(\Psi)$. (ii) The proof is in [8, A1]. (iii) It is enough to check, by explicit computation, that, in $U_Q$,

$$(x \otimes \delta_0 \otimes y)(x \otimes \delta_\alpha \otimes y) = x_1 x_2 \otimes \delta(x_\alpha) \otimes y_2 y_1 = (x \otimes \delta_\alpha \otimes y)(x \otimes \delta_0 \otimes y)$$

(iv)–(v) We use the notation $e_\alpha e^\alpha = x \otimes \delta_\alpha \otimes y$, $\alpha \in Q_+$.

For any $\alpha$ and $\beta$,

$$[e_\alpha e^\alpha, e^\beta e_\beta] = e_\alpha e^\beta e^\alpha e^\beta - e_\beta e^\alpha e_\alpha e^\beta + e_\beta e^{l(\beta)} e^\beta e^{l(\beta)}$$

$$- e^{l(\beta)} e_\beta e^\beta e^\alpha + e_\alpha e^{l(\beta)} e^\beta e^{l(\beta)} - e_\alpha e_\beta e^\alpha e^\beta$$

where, if it exists, $u(\beta) \in Q_+$ such that $u(\beta) + \alpha = \beta$, and, if it exists, $l(\beta) \in Q_+$ such that $l(\beta) + \beta = \alpha$. We will show that the sum over $\beta \in Q_+$ (resp. $Q_{B,+}$, $\Psi$) is equal to zero.

First, $\beta = u(\gamma)$ for $\gamma = \alpha + \beta \in Q_+$ (resp. $Q_{B,+}$, $\Psi$). Therefore,

$$(e_\alpha e_\beta e^\alpha e^\beta - e_\beta e_\alpha e^\alpha e^\beta) + (e_{u(\gamma)} e_{u(\gamma)} e^\alpha e^\alpha - e_\alpha e_{u(\gamma)} e^\alpha e^\alpha) = 0$$

If it exists, $l(\beta) = \alpha - \beta = \gamma \in Q_+$ (resp. $Q_{B,+}$, $\Psi$), and $\beta = l(\gamma)$. Therefore

$$(e_\beta e_{l(\beta)} e^\beta e^{l(\beta)} - e_{l(\beta)} e_\beta e^{l(\beta)} e^\beta + e_{l(\gamma)} e^\gamma e^{l(\gamma)} - e_{l(\gamma)} e_\gamma e^{l(\gamma)}) = 0$$

If it exists, $u(\beta) = \beta - \alpha = \gamma \in Q_+$ (resp. $Q_{B,+}$, $\Psi$), therefore

$$(e_\alpha e_\gamma e^\gamma - e_\gamma e_\alpha e^\gamma) + (e_{u(\beta)} e_{u(\beta)} e^\alpha e^\alpha - e_\alpha e_{u(\beta)} e^\alpha e^\beta) = 0$$

The identity follows.

8. Universal Coxeter structures and Kac–Moody algebras

In this section, we prove the rigidity of braided quasi–Coxeter structures on symmetrisable Kac–Moody algebras.
8.1. Diagrams and nested sets. The terminology in 8.1–8.3 is taken from [15, Part I] and [1, Sec. 2], to which we refer for more details.

A diagram is a nonempty undirected graph $D$ with no multiple edges or loops. We denote the set of vertices of $D$ by $V(D)$. A subdiagram $B \subseteq D$ is a full subgraph of $D$, that is, a graph consisting of a (possibly empty) subset of vertices of $D$, together with all edges of $D$ joining any two elements of it. Two subdiagrams $B_1, B_2 \subseteq D$ are orthogonal if they have no vertices in common and no two vertices $i \in B_1$, $j \in B_2$ are joined by an edge in $D$. $B_1$ and $B_2$ are compatible if either one contains the other or they are orthogonal.

A nested set on a diagram $D$ is a collection $\mathcal{H}$ of pairwise compatible, connected subdiagrams of $D$ which contains the connected components $D_1, \ldots, D_r$ of $D$.

Let $\mathcal{N}_D$ be the partially ordered set of nested sets on $D$, ordered by reverse inclusion. $\mathcal{N}_D$ has a unique maximal element $\mathbf{1} = \{D_i\}_{i=1}^r$ and its minimal elements are the maximal nested se. We denote the set of maximal nested sets on $D$ by $\text{Mns}(D)$. Every nested set $\mathcal{H}$ on $D$ is uniquely determined by a collection $\{\mathcal{H}_i\}_{i=1}^r$ of nested sets on the connected components of $D$. We therefore obtain canonical identifications

$$\mathcal{N}_D = \prod_{i=1}^r \mathcal{N}_{D_i} \quad \text{and} \quad \text{Mns}(D) = \prod_{i=1}^r \text{Mns}(D_i)$$

8.2. Quotient diagrams. Let $B \subsetneq D$ be a proper subdiagram with connected components $B_1, \ldots, B_m$.

**Definition.** The set of vertices of the quotient diagram $D/B$ is $V(D) \setminus V(B)$. Two vertices $i \neq j$ of $D/B$ are linked by an edge if and only if the following holds in $D$

$$i \neq j \quad \text{or} \quad i, j \notin B_i \quad \text{for some} \quad i = 1, \ldots, m$$

For any connected subdiagram $C \subseteq D$ not contained in $B$, we denote by $C \subseteq D/B$ the connected subdiagram with vertex set $V(C) \setminus V(B)$.

For any $B \subset B'$, we denote by $\text{Mns}(B', B)$ the collection of maximal nested sets on $B'/B$. For any $B \subset B' \subset B''$, there is an embedding

$$\cup : \text{Mns}(B'', B') \times \text{Mns}(B', B) \to \text{Mns}(B'', B)$$

such that, for any $\mathcal{F} \in \text{Mns}(B'', B')$, $\mathcal{G} \in \text{Mns}(B', B)$,

$$(\mathcal{F} \cup \mathcal{G})_{B'/B} = \mathcal{G}$$

8.3. $D$–algebras. A $D$–algebra is an algebra $A$ with a collection of subalgebras $\{A_B\}$ indexed by subdiagrams of $D$ and satisfying

$$A_{B_1} \subseteq A_{B_2} \quad \text{and} \quad [A_{B_3}, A_{B_4}] = 0$$

for any $B_1 \subseteq B_2$ and $B_3 \perp B_4$.

8.4. Weak Coxeter structures and the Casimir algebra. Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra with Dynkin diagram $D$, root lattice $Q \cong \mathbb{Z}^l$, cone of non–negative roots $Q_+$, root system $R = R_+ \cup -R_+$, Cartan subalgebra $\mathfrak{h}$ and positive Borel subalgebra $\mathfrak{b} = \mathfrak{h} \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$.

**Definition.** Let $\mathfrak{i}_R^n$ be the ideal in $\mathfrak{u}_R^n$ generated by $\{\pi_\alpha\}_{\alpha \in Q_+ \setminus R_+}$, i.e., the subspace spanned by the endomorphisms with at least one occurrence of $\pi_\alpha$, for some $\alpha \in Q_+ \setminus R_+$, $\alpha \neq 0$. Let $\mathfrak{u}_Q^n$ be the quotient $\mathfrak{u}_R^n = \mathfrak{u}_Q^n / \mathfrak{i}_R^n$.
The properties of $\mathcal{U}_Q$ carry over to $\mathcal{U}_R$, in particular Theorem 7.10 and Proposition 7.11. Since $\rho_Q^0(T_B^R) = 0$, the morphisms $\rho_Q^0$ naturally factor through $\mathcal{U}_Q^R$. The collection of subalgebras $\mathcal{U}_{R_B}$, indexed by subdiagrams $B \subseteq D$, endows $\mathcal{U}_R$ with a structure of cosimplicial $D$–bialgebra. For any $C \subseteq B \subseteq D$, let $\mathcal{U}_{R(C,B)}$ be the subalgebra of $[1]_{C}$–invariants in $\mathcal{U}_{R_B}$.

**Definition.** A weak Coxeter structure on $\hat{\mathcal{U}}_R$ is the datum of

(i) for each connected subdiagram $B \subseteq D$, an $R$–matrices $R_B \in \hat{\mathcal{U}}_{R_B}^2$, and associator $\Phi_B \in \hat{\mathcal{U}}_{R_B}^3$, which are of the following form

$$R_B = e^{\Omega_B/2} \quad \text{and} \quad \Phi_B = \Phi_B^0(\Omega_B,12,\Omega_B,23)$$

where $\Omega_B = r_B + r_B^{21}$, and $\Phi_B^0$ is a Lie associator.

(ii) for each pair of subdiagrams $B' \subseteq B \subseteq D$ and maximal nested set $\mathcal{F} \in \text{Mns}(B',B)$, a relative twist $J^{B,B'}_{\mathcal{F}} \in \hat{\mathcal{U}}_{R(B',B)}^2$, satisfying

$$J^{B,B'}_{\mathcal{F}} = 1 \mod (\mathcal{U}_{R(B',B')}^2)_{\geq 1} \quad (8.1)$$

and

$$(\Phi_B)_{J^{B,B'}_{\mathcal{F}}} = \Phi_{B'} \quad (8.2)$$

which is compatible with vertical decomposition

$$J^{B,B''}_{\mathcal{F}_1 \cup \mathcal{F}_2} = J^{B,B'}_{\mathcal{F}_1} \cdot J^{B',B''}_{\mathcal{F}_2}$$

where $B'' \subseteq B' \subseteq B$, $\mathcal{F}_1 \in \text{Mns}(B,B')$ and $\mathcal{F}_2 \in \text{Mns}(B',B'')$.

(iii) for any $B' \subseteq B$ and pair of maximal nested sets $\mathcal{F}, \mathcal{G} \in \text{Mns}(B,B')$, a gauge transformation, referred to as a *De Concini–Procesi associators*, $\Upsilon_{\mathcal{F},\mathcal{G}} \in \hat{\mathcal{U}}_{R(B,B')}^3$, satisfying

$$\Upsilon_{\mathcal{F},\mathcal{G}} = 1 \mod (\mathcal{U}_{R(B,B')}^3)_{\geq 1} \quad (8.3)$$

$$\Upsilon_{\mathcal{F},\mathcal{G}} \cdot \Upsilon_{\mathcal{F},\mathcal{G}} \cdot J_{\mathcal{G}} \cdot (\Delta(\Upsilon_{\mathcal{F},\mathcal{G}}))^{-1} = J_{\mathcal{F}} \quad (8.4)$$

and such that the following holds

- **Orientation:** for any $\mathcal{F}, \mathcal{G} \in \text{Mns}(B,B')$

  $$\Upsilon_{\mathcal{F},\mathcal{G}} = \Upsilon_{\mathcal{G},\mathcal{F}}^{-1}$$

- **Transitivity:** for any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \text{Mns}(B,B')$

  $$\Upsilon_{\mathcal{H},\mathcal{F}} = \Upsilon_{\mathcal{H},\mathcal{G}} \cdot \Upsilon_{\mathcal{F},\mathcal{G}}$$

- **Factorisation:**

  $$\Upsilon_{(\mathcal{F}_1 \cup \mathcal{F}_2) \cup (\mathcal{G}_1 \cup \mathcal{G}_2)} = \Upsilon_{\mathcal{F}_1,\mathcal{G}_1} \cdot \Upsilon_{\mathcal{F}_2,\mathcal{G}_2}$$

  for any $B'' \subseteq B' \subseteq B$, $\mathcal{F}_1, \mathcal{G}_1 \in \text{Mns}(B,B')$ and $\mathcal{F}_2, \mathcal{G}_2 \in \text{Mns}(B',B'')$.

**Remark.** An assignment of relative twists compatible with vertical decomposition is equivalent to assigning such a twist to any pair of subdiagrams $B' \subseteq B$ such that $|B| = |B'| + 1$, and extending the assignment by composition. In the case of $\mathcal{U}_R$, the analogue of Theorem 7.10 implies that the gauge transformations relating $J_{\mathcal{F}}, J_{\mathcal{G}}$ are unique. The properties of orientation, transitivity, and factorisation follow automatically.
8.5. Twisting of weak Coxeter structures.

Definition.

(i) A universal twist is a pair \((u, F)\) where

- \(u = \{u^{BB'}\}\) is a collection of invertible elements of \(\hat{U}_{R(B, B')}\) indexed by \(F \in \text{Mns}(B, B')\), satisfying the factorisation property

\[
u^{BB''} = \nu^{BB'} \nu^{B'B''}
\]

for any \(B'' \subseteq B' \subseteq B\), \(F_1 \in \text{Mns}(B, B')\), and \(F_2 \in \text{Mns}(B', B'')\).

- \(F = \{F_B\}\) is a collection of invertible elements of \(\hat{U}_{2R(B, B)}\).

(ii) The twisting of a weak Coxeter structure \(\text{Cox} = (\{\Phi_B, R_B\}, \{J^{BB'}_F\}, \{\Upsilon_{FG}\})\) on \(\hat{U}_R\) by a twist \((u, F)\) is the weak Coxeter structure

\(\text{Cox}(u, F) = (\{(\Phi_B)_{F_B}, (R_B)_{F_B}\}, \{(J_F)_{(u, F)}\}, \{(\Upsilon_{FG})_u\})\)

given by

\[
(\Phi_B)_{F_B} = F_B^{23} F_B^{123} \Phi_B (F_B^{12} F_B^{12,3})^{-1}
\]
\[
(R_B)_{F_B} = F_B^{21} R_B F_B^{-1}
\]
\[
(J^{BB'}_F)_{(u, F)} = F_B' u_F^2 J^{BB'}_F \Delta(u_F)^{-1} F_B^{-1}
\]
\[
(\Upsilon_{FG})_u = u_F \Upsilon_{FG} u_F^{-1}
\]

Definition.

(i) A universal gauge is a collection \(a = \{a_B\}\) of invertible elements \(a_B \in \hat{U}_{R(B, B)}\) labelled by the subdiagrams \(B \subseteq D\).

(ii) The gauging of a universal twist \((u, F)\) by \(a\) is the twist \((u_a, F_a)\) given by

\[
(u^{BB'})_a = a_B \cdot u_F \cdot a_B^{-1}
\]
\[
(F_B)_a = a_B^2 \cdot F_B \cdot \Delta(a_B)^{-1}
\]

Remark. It is easy to see that if \((u, F)\) is a universal twist, and \(a\) a universal gauge, the twist of a weak Coxeter structure on \(\hat{U}_R\) by \((u, F)\) is the same as that by \((u_a, F_a)\).

8.6. Uniqueness of universal weak Coxeter structures.

Theorem. Let

\(\text{Cox}_k = (\{\Phi^{(k)}_B, R^{(k)}_B\}, \{J^{(k)}_F\}, \{\Upsilon^{(k)}_{FG}\})\) \(k = 1, 2\)

be two weak Coxeter structures on \(\hat{U}_R\). Then

(i) There is a twist \((u, F)\) such that

\(\text{Cox}_2 = (\text{Cox}_1)_{(u, F)}\)

(ii) The twist \((u, F)\) is unique up to a unique gauge \(a\).
Proof. We first match the associators. The proof of Drinfeld's uniqueness theorem [4, Prop. 3.12] is easily adapted to \( \hat{U}_R \). Therefore, given \( \Phi_1, \Phi_2 \in \hat{U}_R^3 \), there is a symmetric, invariant twist \( J \in \hat{U}_R^2 \) such that \( (\Phi_1)_J = \Phi_2 \). \( J \) is uniquely defined up to multiplication with an element of the form \( a_B^{-1} \oplus a_B^{-1} \Delta(a_B) \), where \( a_B \) belongs to the center of \( \hat{U}_R \) such that \( u_0 = 1 \). Further, \( (R_B)_J = R_B \), since \( R_B = \Delta(e^{\kappa B/2}) e^{-\kappa B/2} \oplus e^{-\kappa B/2} \).

We may therefore assume that

\[
\text{Cox}_k = \{\{\Phi_B, R_B\}, \{J^{(k)}_\mathcal{F}\}, \{\Upsilon^{(k)}_{\mathcal{F}, G}\}\} \quad k = 1, 2
\]

for a given \( \Phi \in \text{Assoc} \). We now match the twist. By 7.10, there exists, for any \( \mathcal{F} \in \text{Mns}(B, B') \), a unique \( u_\mathcal{F} \in \hat{U}_R \) satisfying

\[
J^{(2)}_\mathcal{F} = u_\mathcal{F} \otimes u_\mathcal{F} \cdot J^{(1)}_\mathcal{F} \cdot \Delta(u_\mathcal{F})^{-1}
\]

and \( (u_\mathcal{F})_0 = 1 \). Therefore we can assume

\[
\text{Cox}_k = \{\{\Phi_B, R_B\}, \{J_\mathcal{F}\}, \{\Upsilon^{(k)}_{\mathcal{F}, G}\}\} \quad k = 1, 2
\]

where

\[
\Upsilon^{(k)}_{\mathcal{F}, G} \otimes \Upsilon^{(k)}_{\mathcal{F}, G} \cdot J_\mathcal{F} \cdot \Delta(\Upsilon^{(k)}_{\mathcal{F}, G})^{-1} = J_\mathcal{F} \quad k = 1, 2
\]

Again by Theorem 7.10, it follows that

\[
\Upsilon^{(1)}_{\mathcal{F}, G} = \Upsilon^{(2)}_{\mathcal{F}, G}
\]

and

\[
\text{Cox}_2 = (\text{Cox}_1)_u
\]

where \( u = \{u_\mathcal{F}\} \).

\[\text{Corollary.}\] For any fixed Lie associator \( \Phi \in \text{Assoc} \), there exists, up to a unique equivalence, a unique weak Coxeter structure on \( \hat{U}_R \) with associator \( \Phi \).

8.7. Kac–Moody algebras. Set \( I = \{1, \ldots, l\} \) and let \( A = (a_{ij})_{i,j \in I} \in M_l(k) \) be a generalized Cartan matrix. \( A \) is symmetrizable if there exists a diagonal matrix \( D = \text{diag}(D_i)_{i \in I} \) such that \( B = DA \) is symmetric. The condition \( d_i \in \mathbb{Z}_+ \) and pairwise coprimes uniquely determines \( D \). Set \( r = \text{rk}(A) \), \( I' = \{r + 1, \ldots, l\} \), and assume that the first \( r \) rows of \( A \) are linearly independent.

Recall that the Kac–Moody algebra \( g = g(A) \) is the \( K \)-algebra with generators \( \{e_i, f_i, h_i, d_k\}, i \in I, k \in I', \) and defining relations, for \( i, j \in I, k, s \in I' \),

- \( [h_i, h_j] = [d_k, d_s] = [h_i, d_s] = 0 \)
- \( [h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ij}f_j \)
- \( [d_k, e_j] = \delta_{kj} e_j, [d_k, f_j] = -\delta_{kj} f_j \)
- \( \text{ad}(x_i)^{1-a_{ij}}(x_j) = 0 \) for \( i \neq j \) and \( x = e, f \)

There exists a unique non–degenerate symmetric bilinear form \( (,): g \otimes g \to k \) such that for all \( i, j \in I, k, s \in I' \),

- \( (h_i, h_j) = D_j^{-1} a_{ij} \)
- \( (d_k, d_s) = 0 \)
- \( (h_i, d_k) = \delta_{ki} D_i^{-1} \)
- \( (e_i, f_j) = 0 = (f_i, e_j), (e_i, f_j) = \delta_{ij} D_i^{-1} \)
- \( (d_k, e_j) = (d_k, f_j) = (h_i, e_j) = (h_i, f_j) = 0 \)

Its restriction to \( h \) is non–degenerate.
8.8. \textbf{D–algebras and Kac–Moody algebras.} A symmetrisable Kac–Moody algebra $\mathfrak{g}$ is called \textit{diagrammatic} if it admits a structure of $D_\mathfrak{g}$–algebra, where $D_\mathfrak{g}$ is the Dynkin diagram of $\mathfrak{g}$. Kac–Moody algebras of finite, affine, and hyperbolic type are diagrammatic, as is the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ of any Kac–Moody algebra $\mathfrak{g}$. There exist, however, Kac–Moody algebras in rank $\geq 4$ which are not diagrammatic [1]. To remedy this, we follow a suggestion of P. Etingof, and give below a modified definition of $\mathfrak{g}$ along the lines of [11], which has a bigger Cartan subalgebra.

8.9. \textbf{Extended Kac–Moody algebras.}

\textbf{Definition.} The \textit{extended Kac–Moody algebra} of $A$ is the $k$–algebra $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}(A)$ with generators $\{e_i, f_i, h_i, \lambda_i^\vee\}_{i \in I}$, and defining relations

- $[h_i, h_j] = [\lambda_i^\vee, \lambda_j^\vee] = [h_i, \lambda_j^\vee] = 0$
- $[h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ij}f_j$
- $[\lambda_i^\vee, e_j] = \delta_{ij}e_j, [\lambda_i^\vee, f_j] = -\delta_{ij}f_j$
- $\text{ad}(e_i)^{1-a_{ij}}(e_j) = 0 = \text{ad}(f_i)^{1-a_{ij}}(f_j)$ for any $i \neq j$.

There is a unique symmetric non–degenerate bilinear form $(,): \overline{\mathfrak{g}} \otimes \overline{\mathfrak{g}} \to k$ such that

- $(h_i, h_j) = D_j^{-1}a_{ij}$
- $(\lambda_i^\vee, \lambda_j^\vee) = 0$
- $(h_i, \lambda_j^\vee) = \delta_{ij}D_i^{-1}$
- $(e_i, e_j) = 0 = (f_i, f_j), (e_i, f_j) = \delta_{ij}D_i^{-1}$
- $(\lambda_i^\vee, e_j) = (\lambda_i^\vee, f_j) = (h_i, e_j) = (h_i, f_j) = 0$

Its restriction to

$$\overline{\mathfrak{h}} = \bigoplus_{i \in I} \mathbb{C}h_i \oplus \bigoplus_{i \in I} \mathbb{C}\lambda_i^\vee$$

is non–degenerate.

The following is straightforward.

\textbf{Proposition.}

(i) There is a canonical embedding $\mathfrak{g} \subset \overline{\mathfrak{g}}$ mapping $e_i, f_i, h_i, d_j \mapsto e_i, f_i, h_i, \lambda_j^\vee$

$\begin{align*}
i & = 1, \ldots, l, j = r + 1, \ldots, l. \text{ The inclusion is compatible with the invariant bilinear forms on } \mathfrak{g} \text{ and } \overline{\mathfrak{g}}.

(ii) $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}_D$ has a canonical structure of $D_\mathfrak{g}$–algebra, given by the collection of subalgebras $\{\mathfrak{g}_B\}_{B \subset D}$, where

$$\mathfrak{g}_B = \langle e_i, f_i, h_i, \lambda_i^\vee \mid i \in B \rangle$$

(iii) The triple $(\overline{\mathfrak{g}} \oplus \overline{\mathfrak{h}}, \overline{\mathfrak{b}}\pm)$ is a graded Manin triple, with bilinear form $(,): \overline{\mathfrak{g}} \otimes \overline{\mathfrak{h}} \to k$.

(iv) $\mathfrak{g}' = \overline{\mathfrak{g}}$. 

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8.10. Universal weak Coxeter structures on Kac–Moody algebras.

**Definition.** Let \( g \) be an extended Kac–Moody algebra with Borel subalgebra \( b \subset g \), and root system \( R \). A **universal weak Coxeter structure** on the category of deformation Drinfeld–Yetter \( b \)-modules \( \text{DY}_b \) is the image in \( \hat{U}_b^\hbar = \text{End}(\text{DY}_b \to \text{DY}_b) \) of a weak Coxeter structure on \( \hat{U}_R \) under the map

\[
\rho^n_Q : \hat{U}_R^n \to \hat{U}_b^\hbar = \text{End}(\text{DY}_b \to \text{DY}_b)
\]

The rigidity in \( \hat{U}_R \) implies the following

**Theorem.** Let \( g \) be an extended Kac–Moody algebra and \( \Phi \in \text{Assoc} \) a fixed Lie associator. Up to a unique universal equivalence, there exists a unique universal weak Coxeter structure on \( \text{DY}_b \) with associator \( \Phi \).

8.11. The Casimir subalgebra of a Kac–Moody algebra. Let \( R(k) = l \) and \( C^n_g = \rho^n_Q(\hat{U}_R^n) \) be the subalgebra in \( \hat{U}_g^\hbar \) generated by the elements

\[
\sum_{i \in I} x_i \cdots x_i x_i^{\sigma(1)} \cdots x_i^{\sigma(N)}
\]

where \( N \in \mathbb{Z}_{\geq 0} \), \( \{ x_i \}_{i \in I} \) is a basis of \( b^+ = \bigoplus_{\alpha \in R_+ \cup 0} g_\alpha \), and \( \{ x^i \}_{i \in I} \) is the dual basis in \( b^- \). The topological coproduct on \( \hat{U}_g^\hbar \) induces a cosimplicial structure on \( C_g \).

Theorems 7.10 and 8.6 can be proved for \( C_g \) following the same proofs as in the case of \( \hat{U}_R \).

**Theorem.** Let \( \Phi \in \text{Assoc} \) be a fixed Lie associator. Up to a unique universal equivalence, there exists a unique universal weak Coxeter structure on \( C_g \) with associator \( \Phi \).

8.12. Universal Coxeter structures on Kac–Moody algebra. Let \( g \) be an extended Kac–Moody algebra with Borel subalgebra \( b \) and Dynkin diagram \( D \). Let \( \text{DY}_b^{\text{int}} \) be the category of *integrable* deformation Drinfeld–Yetter \( b \)-modules, i.e., \( \hbar \)-diagonalizable with a locally nilpotent \( n \)-action.

**Definition.** A **universal Coxeter structure** on \( \text{DY}_b^{\text{int}} \) is the data of a universal weak Coxeter structure on \( \hat{U}_g^\hbar = \text{End}(\text{DY}_b^{\text{int}} \to \text{Vect}_k[[\hbar]]) \) and a collection of operators, called local monodromies, \( S_i \in \hat{U}_b^\hbar \), \( i \in D \), of the form

\[
S_i = \tilde{s}_i \cdot \underline{S}_i \quad (8.5)
\]

where \( \underline{S}_i \in \hat{U}_b^\hbar \), \( \underline{S}_i = 1 \mod \hbar \), and \( \tilde{s}_i = \exp(e_i) \exp(-f_i) \exp(e_i) \), satisfying the coproduct identity

\[
\Delta_{f_i}(S_i) = (R_i)^{\gamma_i}_{\alpha}(S_i \otimes S_i) \quad (8.6)
\]

and the generalized braid relations of type \( D \). Namely, for any pairs \( i, j \) of distinct vertices of \( B \), such that \( 2 < m_{ij} < \infty \), and elementary pair \((F, G) \) in \( \text{Mns}(B) \) such that \( i \in F, j \in G \), the following relations hold in \( \hat{U}_g^B \):

\[
\text{Ad}(\Upsilon_{FG})(S_i) \cdot S_j \cdots = S_j \cdot \text{Ad}(\Upsilon_{GF})(S_i) \cdots \quad (8.7)
\]

where the number of factors in each side equals \( m_{ij} \).
8.13. **The local monodromies $S_i$.** Let $i \in D$ be a fixed vertex, and $h_i = kh_i$.

The following computation takes place in $\mathfrak{u} \mathfrak{sl}_2^+$ and we will omit the index $i$ unless necessary.

**Lemma.** Let $S_1, S_2$ be two solutions of
\[
\Delta_J(S_k) = (R)_J^{21}(S_k \otimes S_k) \quad k = 1, 2
\]
of the form (8.5). Then, there exists a unique $u \in k[[\hbar]]$ such that
\[
S_2 = e^{uh}S_1e^{-uh}
\]

**Proof.** Let $S_k = \tilde{s} \cdot S_k$, where
\[
S_k = 1 + \sum_{N \geq 0} \hbar^n s_n^{(k)} \quad s_n^{(k)} \in U \mathfrak{sl}_2^b
\]

The coproduct identity (8.6) reads
\[
\Delta(S_k) \cdot J^{-1} = R \cdot (J^{-1})^{21}_J (S_k \otimes S_k)
\]
where $\theta$ is the Chevalley involution, acting as $-1$ on $\mathfrak{h}$.

We construct a sequence
\[
u_n = \sum_{k=0}^n a_k \hbar^k \quad a_k \in k
\]
such that
\[
S_2 = e^{u_n \hbar}S_1e^{-u_n \hbar} \mod \hbar^{n+1}
\]
Since $S_2 = S_1 = \tilde{s}$ modulo $\hbar$, we may assume $a_0 = 0$. Assume therefore $a_k$ defined for $k = 0, 1, \ldots, n$ for some $n \geq 0$. Let $S'_1$ be given by the right–hand side of (8.9), so that
\[
S_2 = S'_1 + \hbar^{n+1} \eta \mod \hbar^{n+2}
\]
for some $\eta \in U \mathfrak{sl}_2^b$. One readily checks that $S'_1$ satisfies (8.8), since $e^{u_n \hbar}$ is a weight zero group–like element. Subtracting from this the coproduct identity for $S_2$, and computing modulo $\hbar^{n+2}$, we find that
\[
d_H(\eta) = 1 \otimes \eta - \Delta(\eta) + \eta \otimes 1 = 0
\]
It follows $\eta = c \cdot h$, for some $c \in k$. Then if we set $a_{n+1} = -c/2$ we get
\[
e^{h a_{n+1} \hbar^{n+1}} S_1' e^{-h a_{n+1} \hbar^{n+1}} = S_2 = S_2 \mod \hbar^{n+2}
\]
By induction, one gets an element $u \in k[[\hbar]]$ such that
\[
S_2 = \text{Ad}(e^{uh})(S_1)
\]

The weak Coxeter structure is of weight zero and therefore fixed by the group–like element $e^{uh}$, since
\[
e^{uh} \otimes e^{uh} \cdot J \cdot \Delta(e^{uh})^{-1} = J \quad \text{and} \quad \text{Ad}(e^{uh}) \Upsilon = \Upsilon
\]

**Corollary.** Up to gauge transformation, a weak Coxeter structure on $D \mathcal{Y}_b^\text{int}$ can be completed to at most one Coxeter structure.
8.14. Uniqueness of quasi–Coxeter structures on Kac–Moody algebras.

Let \( g \) be an extended Kac–Moody algebra with Borel subalgebra \( b \), and \( \text{DY}^\text{int}_b \) be the deformation category of integrable, Drinfeld–Yetter \( b \)-modules.

The following is the main result of this paper.

**Theorem.** Let \( k = 1, 2 \), and 
\[
\text{Cox}_k = (\{\Phi^{(k)}_B, R^{(k)}_B\}, \{J^{(k)}_F\}, \{Y^{(k)}_{\mathcal{F}_B}\}, \{S^{(k)}_i\})
\]
two universal Coxeter structures on \( \text{DY}^\text{int}_b \). Then,

(i) There is a twist \( (u, F) \) such that 
\[
\text{Cox}_2 = (\text{Cox}_1)_{(u, F)}
\]

(ii) The twist \( (u, F) \) is unique up to a unique gauge \( a \).

**Proof.** The existence of the structure follows from [1]. Let \( (\text{Cox}_k, \{S^{(k)}_i\}), k = 1, 2 \), be two universal Coxeter structures on \( \text{DY}^\text{int}_b \). By 8.6, there exists a universal twisting \( (u, F) \) such that 
\[
\text{Cox}_2 = (\text{Cox}_1)_{(u, F)}
\]
where \( u \) is uniquely determined, and \( F \) is uniquely determined up to multiplication with elements of the form \( a_B^{-1} \otimes a_B^{-1} \Delta(a_B) \), where \( a_B \) belongs to the center of \( \hat{U}_g \). Therefore, \( S^{(2)}_i \) and \( (S^{(1)}_i)_a \) are two Coxeter extensions of \( \text{Cox}_2 \). By Lemma 8.13, there is a unique tuple \( v = (v_1, \ldots, v_n), v_i \in k[[\hbar]] \), such that 
\[
\text{Ad}(e^{v_i\hbar})(S^{(1)}_i)_u = S^{(2)}_i
\]
and 
\[
(\text{Cox}_2, \{S^{(2)}_i\}) = (\text{Cox}_1, \{S^{(1)}_i\})_{(uv, F)}
\]
The theorem is proved. \( \square \)

**Corollary.** Let \( \Phi \in \text{Assoc} \) be a fixed Lie associator. Up to a unique equivalence, there exists a unique universal Coxeter structure on \( \text{DY}_b \) with associator \( \Phi \).

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