Crystal mean field based trial wave functions for the FQHE ground states

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Employing the Haldane-Rezayi periodic representation, the crystalline determinantal Hall crystal mean field solutions derived in previous works are used to construct variational wave functions for the FQHE at \( \nu = 1/q \). The proposed states optimize the short range correlations in a similar measure as the Laughlin ones, since the zero of the states when the coordinates of two particles join is of order \( q \). However, the proposed wave functions also incorporate the crystalline correlations of the mean field problem, through a determinantal mean field function entering their construction. The above properties, lead to the expectation that the considered states can be competitive in energy per particle with the Laughlin ones. Their similar structure also could explain why the breaking of the translation invariance in the FQHE ground states can result to be a weak one, which after disregarded, produce the Laughlin states as good approximations. Calculation for checking these possibilities are under consideration.

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I. INTRODUCTION

In a previous work (See Ref. [1]) analytical solutions of the Hartree-Fock problem in a magnetic field corresponding to filling factors \( \nu = 1/q \ (q \text{ odd}) \) and a number of particles per cell \( \gamma = 1, 1/2 \), were determined. The form of the obtained wave functions directly suggested a possible way of constructing variational states, being able to reduce the energy through optimizing both: short range as well as crystalline correlations. The appropriate picture for this purpose appeared to be the procedure to impose periodic boundary conditions in a magnetic field developed by Haldane and Rezayi in Ref. [2].

This paper intends to construct the mentioned states. The main idea is the following: the mean field one particle states after written in the Haldane-Rezayi picture, have a structure which is given by the product of a determinantal function which contains all the dependence of the quantum numbers of the HF states, times a factor which have a large number of zeroes which are completely attached to the periodicity lattice of the HF states. The position of those zeroes have no dependence of the set of the quantum numbers of the filled mean field states at all. Moreover, the number of zeroes of those kinematical factors in the HF determinant as a function of any one of the identical particle coordinates, are just \((q-1)N_e\) were \(N_e\) is the number of particles. This property means that the mentioned number of zeroes is identical to the one which \(q\)-power defines the variational Laughlin states in the Haldane-Rezayi scheme. Therefore, substituting the crystalline like factors in the mean field determinantal state by the liquid like ones appearing in the Laughlin wave function, leads to the same short range behavior (a zero of order \( q \)) when any two particles join one another. However, in addition the presence of the determinantal functions remains representing the crystalline information associated with the optimization of the mean field problem. Therefore, the proposed state have the chance of slightly improving the energies per particle of the Laughlin ones. This possibility opens a way of checking whether or not the long standing claims about the breaking of the translation invariance in QHE ground state can have a confirmation which in addition retains the demonstrated robustness of the Laughlin or Jain descriptions.

The correctness of such results can support the relevance of the mean field solutions, long time ago proposed by one of the present authors (F.C.), as correct precursor states within a perturbative description of the FQHE ground states. In it, the inclusion of correlations in each order could approach the results in the real FQHE ground states. Evaluations of the energy per particle in the examined states are planned to be considered.

In the first Section, the single particle orbitals solving the HF problem for the two considered classes of states are explicitly expressed in the Haldane-Rezayi scheme for the periodic reduction of the \( HF \) problem. These states were derived in Ref. [1] and the notation and definitions here closely follows the ones in that work. Thus, they should be considered as complementary articles. The next section aboard the main subject of the paper, the construction of the variational states, which are motivated by the special structure of the one mean field one particle orbitals.
II. THE HF SINGLE PARTICLE STATES AT \( \nu = 1/q \) IN THE HALDANE-REZAYI SCHEME

Let us consider in this first section the determination of the analytic form of the one particle mean field orbitals for filling factors \( \nu = 1/q \). Expression for these orbitals were obtained in the previous paper [1]. The new representation will be more appropriate (but not essentially needed) for the discussion below, in which we want to consider in periodic boundary conditions.

A formula given in the periodic boundary conditions scheme developed by Haldane and Rezayi in Ref. [2], will be helpful. It expresses any wave-function in the first Landau level which satisfy periodic boundary conditions, in terms of its zeros laying inside the first region of periodicity. The expression is

\[
\varphi_L(x) = \exp(\tilde{k} \, z^* - \frac{y^2}{2r_0^2}) \prod_{i=1}^{N_1} \vartheta_1(\frac{\pi}{L_1}(z^* - z_\nu) - \tau^*).
\]

The constant \( \tilde{k} \) is purely imaginary, and for large \( A \) as in our case is almost continuous. The \( z_\nu \) are the complex coordinates of the zeros of the wave-functions, which should be a total number equal to the number of flux quanta \( N_\nu \) piercing the area \( A \). The periodic conditions are imposed on the boundary of this zone (that we will call first periodicity zone). In our situation, having a sample of area \( A \) with large sides of length \( L_1 \) and \( L_2 \) and forming between them an angle of \( \frac{2\pi}{6} \), the parameter \( \tau^* \) has the form

\[
\tau^* = \frac{L_1}{L_2} \exp(-\frac{2\pi i}{6}).
\]

We will, also assume that \( L_1 = L_2 \). In the present case the complex conjugate variable \( z^* \) appears in place of \( z \) because the difference in conventions with reference [2]. Also, \( \varphi_L(x) \) is written in the Landau gauge. After performing the necessary gauge transformation, the corresponding functions in the axial gauge take the forms

\[
\varphi(x) = \exp(k \, z^* + \frac{z^*}{4r_0^2} - \frac{x^2}{4r_0^2}) \prod_{i=1}^{N_1} \vartheta_1(\frac{\pi}{L_1}(z^* - z_\nu) - \tau^*).
\]

Let us consider below the determination of the functions for each of the two cases of particle number per unit cell \( \gamma = 1 \) and \( \gamma = 1/2 \).

A. Case \( \gamma = 1 \)

First we fix \((q - 1)\) zeros to each of the points of the lattice of periodicity of the density \( R \) being inside the first periodicity zone. Also one zero will be chosen for each cell at positions differing from each point \( R \) by a constant \( C_k \) to be determined. The first condition assures that the wave functions have zero of order \((q - 1)\) at any point of the lattice \( R \). Then, the functions have the form

\[
\varphi^{(0)}_k(x) = \exp(k \, z^* + \frac{z^*}{4r_0^2} - \frac{x^2}{4r_0^2}) \times \prod_{R} \left\{ \left( \vartheta_1(\frac{\pi}{L_1}(z^* - R^*) - \tau^*) \right)^{(q-1)} \vartheta_1(\frac{\pi}{L_1}(z^* - R^* - C_k) - \tau^*) \right\},
\]

where according with the conventions employed the complex coordinates for the vectors of the lattice \( R \) are defined by the integers \( n_1 \) and \( n_2 \) as

\[
R = n_1a + n_2a_2, \quad R^* = n_1a + n_2a_2^*.
\]

in which \( a \) is the size of the unit cell side defined in Ref. [1], and for any vector \( d = (d_1, d_2, 0) \) we write

\[
d = d_1 + d_2i, \quad d^* = d_1 - d_2i.
\]

The number of cells in the first periodicity region and their sizes are related by

\[
L_1 = a \, N_1, \quad L_2 = a \, N_2, \quad N_1 = N_2, \quad N_1 = \text{even number},
\]

\[
C_k = 0.
\]
and the allowed values of \( n_1 \) and \( n_2 \) in the first periodicity region are

\[
\begin{align*}
n_1 &\in \{-\frac{N_1}{2}, -\frac{N_1}{2} + 1, \ldots, 0, \ldots, \frac{N_1}{2}, -1\}, \\
n_2 &\in \{-\frac{N_2}{2}, -\frac{N_2}{2} + 1, \ldots, 0, \ldots, \frac{N_2}{2}, -1\}.
\end{align*}
\]

Let us now impose the condition on the functions to pertain to a given \( q \)-multiplet which is characterized by the eigenvalues \( \exp(-i\mathbf{k}.\mathbf{R}) \) of the translation operators \( T_R \) in the lattice vectors \( \mathbf{R} \). For this purpose it is sufficient to consider the translations under the unit cell vectors \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \). The action of a general translation operator over a function of the first Landau level having an analytic part defined in Ref.\cite{1} can be written as

\[
T_R F(z^*) \exp(-\frac{x^2}{4r_o^2}) = \exp(-\frac{RR^*}{4r_o^2} + \frac{R}{2r_o^2}) F(z^* - R^*) \exp(-\frac{x^2}{4r_o^2}).
\]

Employing this relation in performing a translation in \( \mathbf{a}_1 \) we have

\[
T_{\mathbf{a}_1} \varphi^{(0)}_{\mathbf{k}}(x) = \exp(\bar{k} z^* + \frac{z^*z}{4r_o^2} - \frac{x^2}{4r_o^2}) \times \exp(-\bar{k} a)
\]

\[
\prod_R \left\{ \left( \vartheta \left( \frac{1}{L_1} (z^* - R^* - a) - \tau^* \right) \right)^{(q-1)} \vartheta \left( \frac{1}{L_1} (z^* - R^* - a - C_k) \right) \right\} | - \tau^* \right).
\]

But, under the shift \( R^* \to R^* + a \), almost all the appearing \( \vartheta \) functions transform one in another. Only the ones having \( n_1 = (\frac{N_2}{2} - 1) \) for all values of \( n_2 \), transform in different ones not appearing and \( n_1 = \frac{N_2}{2} \). But, representing \( \frac{N_2}{2} \) as \(-\frac{N_2}{2} + N_1\) and using the translation property of the elliptic function

\[
\vartheta \left( w^* - \pi | - \tau^* \right) = -\vartheta \left( w^* | - \tau^* \right),
\]

allows to write

\[
T_{\mathbf{a}_1} \varphi^{(0)}_{\mathbf{k}}(x) = (-1)^{qN_2} \exp(-\bar{k} a) \varphi^{(0)}_{\mathbf{k}}(x).
\]

Remembering that \( N_2 \) was selected as an even number we have \((-1)^{qN_2} = 1\). Furthermore, after choosing the imaginary parameter \( \bar{k} \) as \( \bar{k} = \mathbf{k}.\mathbf{a}_1 i \) it follows

\[
T_{\mathbf{a}_1} \varphi^{(0)}_{\mathbf{k}}(x) = \exp(-\mathbf{k}.\mathbf{a}_1 i) \varphi^{(0)}_{\mathbf{k}}(x),
\]

which is one of the eigenvalue conditions obeyed by the \( q \) multiplets. Next, it rests to impose on \( \varphi^{(0)}_{\mathbf{k}} \) the character of being an eigen-function of \( T_{\mathbf{a}_2} \). Performing the translation in \( \mathbf{a}_2 \) follows

\[
T_{\mathbf{a}_2} \varphi^{(0)}_{\mathbf{k}}(x) = \exp(\bar{k} z^* + \frac{z^*z}{4r_o^2} - \frac{x^2}{4r_o^2}) \times \exp\left(\frac{a^2}{4r_o^2} + \frac{z^*}{2r_o^2} (a_2 - a_2^*) - \bar{k}a_2^*\right)
\]

\[
\prod_R \left\{ \left( \vartheta \left( \frac{1}{L_1} (z^* - R^* - a_2^*) \right)^{-1} \right) \vartheta \left( \frac{1}{L_1} (z^* - R^* - a_2^* - C_k) \right) \right\} | - \tau^* \right),
\]

where as before, almost all the \( \vartheta \) transform among themselves and solely the ones having \( n_2 = (\frac{N_2}{2} - 1) \) pass to have \( n_2 = \frac{N_2}{2} \). Thus, again representing \( \frac{N_2}{2} = -\frac{N_2}{2} + N_2 \) and making use the other translation property of the elliptic function \( \vartheta \)

\[
\vartheta \left( w^* - \pi \tau^* | - \tau^* \right) = -\exp(-i(2w^* - \pi \tau^*)) \vartheta \left( w^* | - \tau^* \right)
\]

the relation \( \cite{3} \) takes the form

\[
T_{\mathbf{a}_2} \varphi^{(0)}_{\mathbf{k}}(x) = (-1)^{qN_1} \exp(D_{\mathbf{k}}) \varphi^{(0)}_{\mathbf{k}}(x)
\]

\[
D_{\mathbf{k}} = \frac{2\pi i}{a} C_k - ik.\mathbf{a}_1 \tau^* - \frac{a^2}{4r_o^2} (1 - \tau^2) - i\pi q.
\]
which after imposing the condition on the function of being an eigen-function of $T_{a_2}$ with eigenvalue $\exp(-i k \cdot a_1)$, that is $D_k = -i k \cdot a_1$, the constant $C_k$ is fixed to be

$$C_k = \frac{a}{\pi} (k \cdot a_2 - k \cdot a_1 \tau^*) + \frac{ia}{2\sqrt{3}} (1 - \tau^{2*}) - \frac{q}{2},$$

which assures the resting eigenvalue condition

$$T_{a_2} \varphi_k^{(0)}(x) = \exp(-i k \cdot a_1) \varphi_k^{(0)}(x).$$

Finally, let us consider that the function inside each $q$ multiplet showing a zero of order $(q - 1)$ should be unique. However, each multiplet is also uniquely determined by the value of the quasi-momentum $k$ defining its eigenvalues $\exp(-i k \cdot a_1)$ and $\exp(-i k \cdot a_2)$ under the translations $T_{a_1}$ and $T_{a_2}$, respectively. Therefore, the just constructed function $\varphi_k^{(0)}(x)$ should be unique one showing zeros of order $(q - 1)$ laying in the multiplet indexed by $k$.

### B. Case $\gamma = 1/2$

Since the case $\gamma = 1/2$, has some additional subtleties, let us also consider it in detail. In this case we fix $\frac{(q - 1)}{2}$ zeros to each point of the lattice $\mathbf{R}$ and again shift the zero of the other factor in the constant to be determined $C_{[k, \sigma]}$. Then,

$$\varphi_{(k, \sigma)}^{(0)}(x) = \exp(k \cdot z^* + \frac{z^{*2}}{4r^2_o} - \frac{x^2}{4r^2_o}) \times$$

$$\prod_{n \in \mathbb{Z}} \left\{ \vartheta_1 \left( \frac{\pi}{L} (z^* - R^*) - \tau^* \right) \right\}^{(q-1)} \prod_{R_n} \left\{ \vartheta_1 \left( \frac{\pi}{L} (z^* - R^*_n - C_{[k, \sigma]} \tau^*) \right) \right\}.$$

The position of the zero of order $\frac{(q - 1)}{2}$ are at all the points $R^*_n = n_1 a + n_2 a_2^*$, being inside the first periodicity zone. However, in order to be able of generating the doublet of functions for the two values of $\sigma$, the arguments of the functions not showing zeros exactly at the lattice points $R$ will be defined by the two set of lattice vectors

$$R_{\sigma} = n_1^* a + n_2 a_2, R^* = [2m_1 + \frac{(1 - \sigma)}{2}] a + n_2 a_2^*,$$

$$m_1 = 0, \pm 1, \pm 2, ..., \sigma = \pm 1.$$

where $a$ is the size of the unit cell defined in Ref. [1] for $\phi/\phi_o = q/2$. The union of the two lattices gives the larger lattice $R$.

In this case the sizes of the first periodicity zone and the number of cells per side satisfy

$$L_1 = a N_1, L_2 = a N_2,$$

$$N_1 = N_2^* = 4t + 2, t = 1, 2, ...,$$

and now the range of allowed values for the indices defining the lattices $R_{\sigma}$ are

$$n_1^* \in \{-\left(\frac{N_1}{2} - 1\right), \cdots, \left(-\frac{N_1}{2} - 1\right), \cdots, 0, ..., \left\lfloor \frac{N_1}{2} - 3 \right\rfloor \},$$

$$n_2 \in \{-\left(\frac{N_2}{2} - 1\right), \cdots, \left(-\frac{N_2}{2} - 1\right), \cdots, 0, ..., \left\lfloor \frac{N_2}{2} - 1 \right\rfloor \}.$$

Let us now impose the condition that the functions pertains to the space formed by union of the two $q$-multiplets associated to $\sigma = \pm 1$. This set is fixed by the momenta $k$ in Brillouin cell of the reciprocal lattice associated to the
lattice vectors $2\mathbf{R}$. The action of $T_{a_1}$ can be written as

$$T_{a_1} \varphi^{(0)}_{(k, \sigma)}(x) = \exp(\tilde{k} \cdot z^* + \frac{z^2}{2r^2} - \frac{x^2}{2r^2}) \times \exp(-\tilde{k} \cdot a) \times$$

$$\prod_{R} \left\{ \vartheta_1\left(\frac{\pi}{L_1}(z^* - |n_1 + 1|a + n_2a_2^e)| - \tau^*\right)^{(q-1)/2} \right\} \times$$

$$\prod_{R^c} \left\{ \vartheta_1\left(\frac{\pi}{L_1}(z^* - |n_1^e a + n_2 a_2^e| - C_k)| - \tau^*\right)^{(q-1)/2} \right\} .$$

Note, that for $\sigma = -1$ all the lattice $\mathbf{R}_{-1}$ transforms in the lattice $\mathbf{R}_{1}$. For the factor showing the zeros exactly at the lattice $\mathbf{R}$, only the $\vartheta_1$ functions having $n_1 = \left(\frac{N_1}{2} - 1\right)$ for all values of $n_2$ do not transform in other functions appearing in the product. But, representing for them $\frac{N_1}{2}$ as $-\frac{N_1}{2} + N_1$ and employing the symmetry property (2), it is possible to write

$$T_{a_1} \varphi^{(0)}_{(k, \sigma)}(x) = \exp(-\tilde{k} \cdot a) \varphi^{(0)}_{(k, -\sigma)}(x),$$

a relation which indicates that the translation in $a_1$ interchanges the values of $\sigma$. Fixing the imaginary parameter $\tilde{k}$ as $\tilde{k} = k \cdot a_1 i$ allows to fix the transformation property

$$T_{a_1} \varphi^{(0)}_{(k, \sigma)}(x) = \exp(-k \cdot a_1 i) \varphi^{(0)}_{(k, -\sigma)}(x).$$

Thus, the functions for the two values of $\sigma$ transform among themselves. This property, in turns implies that the functions are eigen-functions of the lattice formed by the vectors $2\mathbf{R}$. Let us inspect finally the action on $\varphi^{(0)}_k$ of the operator $T_{a_2}$.

$$T_{a_2} \varphi^{(0)}_{(k, \sigma)}(x) = \exp(\tilde{k} \cdot (z^* - a_2^e) + \frac{z^2 - a_2^e}{2r^2} - \frac{x^2}{2r^2}) \times$$

$$\exp(-\frac{a_2 a_2^e}{2r^2}) \times$$

$$\prod_{R} \left\{ \vartheta_1\left(\frac{\pi}{L_1}(z^* - |n_1 + 1|a + n_2 + 1) a_2^e)| - \tau^*\right)^{(q-1)/2} \right\} \times$$

$$\prod_{R^c} \left\{ \vartheta_1\left(\frac{\pi}{L_1}(z^* - |n_1^e a + n_2 a_2^e| - C_k)| - \tau^*\right)^{(q-1)/2} \right\} .$$

In this case, only the functions for the indices $n_2 = \left(\frac{N_2}{2} - 1\right)$ for any value of $n_1$ transform in other $\vartheta_1$ terms being absent in the original product. After again representing $\frac{N_2}{2} = -\frac{N_2}{2} + N_2$ and using (1) it is possible to impose the eigenvalue relation

$$T_{a_2} \varphi^{(0)}_{(k, \sigma)}(x) = \exp(-i k \cdot a_2) \varphi^{(0)}_{(k, \sigma)}(x),$$

after selecting the constant $C_{(k, \sigma)}$ as given by

$$C_{(k, \sigma)} = \frac{a}{\pi} (k \cdot a_2 - k \cdot a_1 \tau^*) + \frac{ia q}{2\sqrt{3}} (1 - \tau^{2*}) - \frac{(q - \sigma) a}{2} + a q \alpha$$

where $\alpha$ is equal to 0 or 1 if $\frac{N_2}{2}$ is even or odd respectively.

In this case it happens that the function inside each $q$ multiplet constructed in Ref. [1] and showing a zero of order $(\frac{q}{2} - 1)$ should be unique for each of the two values of $\sigma$ if the set of associated equations fix them. Therefore, there is a doublet of functions satisfying the vanishing conditions. Then, the two functions $\varphi^{(0)}_{(k, \sigma)}(x)$ and $\varphi^{(0)}_{(k, -\sigma)}(x)$ determined here, uniquely expands this doublet for a given value of $k$. However, which linear combination defining the specific functions fixed in each of the multiplets formed by the functions $\chi^{(r, \sigma)}_k$ defined in Ref. [1] could be easily determined.
III. TRIAL STATES INCLUDING SHORT RANGE AS WELL AS CRYSTALLINE CORRELATIONS

A. Case $\gamma = 1/2$

Changing the order with respect to the previous section, let us consider first the Slater determinant related with the HF problem in the case $\gamma = \frac{1}{2}$ for which the HF single particle orbitals are $\varphi_{(k,\sigma)}^{(0)}(x)$. The mean field determinantal state in this case has the form

$$\Psi_{hf}^{(k)} = \Psi_{hf}[z_1^*, z_2^*, \ldots z_{N_e}^*, z_1, z_2, \ldots z_{N_e}] = Det [\varphi_{(k,\sigma)}^{(0)}(x_j)],$$

where the wavevectors $k_i$ pertain to the two degenerate bands (for both values of $\sigma$), defining the ground state of the HF problem. These momenta satisfy the periodicity condition of the Haldane-Rezayi procedure and their number is equal to one half of the number of electrons in the problem. As shown in the previous section the orbitals have the structure

$$\varphi_{(k,\sigma)}^{(0)}(x) = \prod_{R} \left( \vartheta_1 \left( \frac{2}{L_1} |z^* - R^*| - \tau^* \right) \right) \chi_{(k,\sigma)}^{(0)}(z^*) \exp(\frac{z^*2}{4r_o^2} - \frac{x^2}{4r_o^2}) = P(z^*) \chi_{(k,\sigma)}^{(0)}(z^*) \exp(k \cdot z^* + \frac{z^*2}{4r_o^2} - \frac{x^2}{4r_o^2}),$$

$$P(z^*) = \prod_{R} \left( \vartheta_1 \left( \frac{2}{L_1} |z^* - R^*| - \tau^* \right) \right),$$

$$\chi_{(k,\sigma)}^{(0)}(z^*) = \exp(k \cdot z^*) \prod_{R_o} \left( \vartheta_1 \left( \frac{2}{L_1} |z^* - R_o^* - C_{(k,\sigma)}| - \tau^* \right) \right),$$

where all the dependences of the momentum quantum numbers and the index $\sigma$ are contained in the new function $\chi$.

The product of $\vartheta_1$ functions: $P(z^*)$ is an analytical function of $z^*$ showing zeroes at all the points of the lattice $R$. Its number of zeroes as a function of $z^*$ is given by $\frac{(q-1)}{2}$ times the number of points of the lattice $R$. But, this last value is equal to twice the number of particles of the system $N_e$ (recall that each cell has $q/2$ flux quantum this $\gamma = 1/2$ case). The just described structure of the single particle HF orbitals leads to the following form for the Slater determinant

$$\Psi_{hf}^{(k)} = Det [\varphi_{(k,\sigma)}^{(0)}(x_j)] = \prod_{k=1,2,..N_e} P(z_k^*) Det [\chi_{(k,\sigma)}^{(0)}(z_k^*)] \exp(\sum_{i=1,2,..N_e} (k \cdot z_i^* + \frac{z_i^*2}{4r_o^2} - \frac{x^2}{4r_o^2})),$$

$$\Phi(z_1^*, z_2^*, \ldots z_{N_e}^*) D[z_1^*, z_2^*, \ldots z_{N_e}^*] \exp(\sum_{i=1,2,..N_e} (\frac{z_i^*2}{4r_o^2} - \frac{x^2}{4r_o^2})), \quad \text{(5)}$$

It can be noticed that the functions $\Phi$, when considered as depending of any of the electron coordinates $z^*$, show a number of zeroes equal to $(q - 1)N_e$. Since the determinant $Det [\chi_{(k,\sigma)}^{(0)}(z_i^*)]$ in addition is showing a number $N_e$ zeroes, the total number of zeroes of the function $\Psi_{hf}^{(k)}$ with respect to any variable is equal to the number of flux quanta $qN_e$ passing through the periodicity area. Therefore, this basic requirement of the periodic representation of the space of single particle wave functions is satisfied.

Thus, the above property directly suggests the main idea addressed in this paper: to propose variational states which are close connected with the states being the counterpart of the homogeneous Laughlin in the Haldane-Rezayi
scheme. After transforming to the axial gauge the form of these periodic Laughlin states (which were determined in the Landau gauge in Ref [onlinecite{halerez}]) is

\[
\Psi_L = \Psi_L[z_1, z_2, ... z_{N_e}, z_1, z_2, ... z_{N_e}] = \exp\left( \sum_{i=1,2...N_e} \left( \frac{\bar{z}_i^2}{4\bar{r}_o^2} - \frac{x_i^2}{4r_o^2} \right) \right) \times \prod_{i,j=1,2...N_e}^{q-1} \left\{ \vartheta_1\left( \frac{\pi}{L_1}(z_i^* - z_j^*) - \tau^* \right) \right\}
\]

\[
= \Phi_L(z_1, z_2, ... z_{N_e})D_L[z_1, z_2, ... z_{N_e}] \exp\left( \sum_{i=1,2...N_e} \left( \frac{\bar{z}_i^2}{4\bar{r}_o^2} - \frac{x_i^2}{4r_o^2} \right) \right)
\]

\[
\Phi_L(z_1^*, z_2^*, ... z_{N_e}^*) = \left\{ \vartheta_1\left( \frac{\pi}{L_1}(Z^* - \bar{R}^*) - \tau^* \right) \right\}^{q-1} \prod_{i,j=1,2...N_e}^{q-1} \left\{ \vartheta_1\left( \frac{\pi}{L_1}(z_i^* - z_j^*) - \tau^* \right) \right\} = D_L^{-1}(z_1^*, z_2^*, ... z_{N_e}^*) \quad (6)
\]

\[
D_L[z_1^*, z_2^*, ... z_{N_e}^*] = \vartheta_1\left( \frac{\pi}{L_1}(Z^* - \bar{R}^*) - \tau^* \right) \prod_{i,j=1,2...N_e}^{q-1} \vartheta_1\left( \frac{\pi}{L_1}(z_i^* - z_j^*) - \tau^* \right)
\]

\[
Z = \sum_{i=1,2...N_e} z_i, \quad \bar{R} = \sum_{R} \bar{R}
\]

where the center of mass wave function defined in [2] has been chosen to have its \( q \) zeroes at the point \( Z = \bar{R} \). That is, at the point given by the sum of all the lattice \( R \) coordinates. The center of mass momenta, also defined in [2] is also taken to vanish. The above representation, indicates that the factors \( \Phi \) in (5), and \( \Phi_L \) in (6), when considered as functions of the coordinates \( z_i^* \) of any particular electron, have the same number of zeroes equal to \( (q-1)N_e \). The same equality in the number of zeroes for any particular coordinate \( z_i^* \) also show the functions \( D \) and \( D_L \). Therefore, the noticed identity in the number of zeroes of the functions \( \Phi \) and \( \Phi_L \), directly leads to the idea of substituting the function \( \Phi \) in (5) by the function \( \Phi_L \), in order to construct new variational states. These new wave functions will show the same order of the zeroes when any two particle equalize their coordinates, but in addition they are expected to also retain the sort crystalline correlations which optimized the mean field problem.

Considering the above remarks, the new variational states are proposed in the form

\[
\Psi_N^{(\Phi')} = \Psi_N[z_1^*, z_2^*, ... z_{N_e}^*, z_1, z_2, ... z_{N_e}] = \Phi_L(z_1^*, z_2^*, ... z_{N_e}^*)D_L^{(\Phi')}[z_1^*, z_2^*, ... z_{N_e}^*] \exp\left( \sum_{i=1,2...N_e} \left( \frac{\bar{z}_i^2}{4\bar{r}_o^2} - \frac{x_i^2}{4r_o^2} \right) \right),
\]

\[
\Phi_L(z_1^*, z_2^*, ... z_{N_e}^*) = \left\{ \vartheta_1\left( \frac{\pi}{L_1}(Z^* - \bar{R}^*) - \tau^* \right) \right\}^{q-1} \prod_{i,j=1,2...N_e}^{q-1} \left\{ \vartheta_1\left( \frac{\pi}{L_1}(z_i^* - z_j^*) - \tau^* \right) \right\} = D_L^{(\Phi')}^{-1}(z_1^*, z_2^*, ... z_{N_e}^*)
\]

\[
D_L^{(\Phi')}[(z_1^*, z_2^*, ... z_{N_e}^*)] = \vartheta_1\left( \frac{\pi}{L_1}(Z^* - \bar{R}^*) - \tau^* \right) \prod_{i,j=1,2...N_e}^{q-1} \vartheta_1\left( \frac{\pi}{L_1}(z_i^* - z_j^*) - \tau^* \right)
\]

Let us now argue that this many particle variational state when considered as a function of any of its coordinates satisfies the periodicity condition of the problem. For this purpose it is useful to write the HF determinantal estate in the form

\[
\Psi_N^{(\Phi')} = \frac{\Phi_L(z_1^*, z_2^*, ... z_{N_e}^*)}{\Phi(z_1, z_2, ... z_{N_e})} \Psi_h^{(\Phi')}[z_1^*, z_2^*, ... z_{N_e}^*, z_1, z_2, ... z_{N_e}],
\]

\[
= \left\{ \frac{F_L(z_1^*, z_2^*, ... z_{N_e}^*)}{F(z_1, z_2, ... z_{N_e})} \right\}^{q-1} \Psi_h^{(\Phi')}[z_1^*, z_2^*, ... z_{N_e}^*, z_1, z_2, ... z_{N_e}],
\]

\[
= \left\{ \eta[z_1^*, z_2^*, ... z_{N_e}^*] \right\}^{q-1} \Psi_h^{(\Phi')}[z_1^*, z_2^*, ... z_{N_e}^*, z_1, z_2, ... z_{N_e}],
\]

\[
F(z_1^*, z_2^*, ... z_{N_e}^*) = \prod_j \prod_{\bar{R}} \left\{ \vartheta_1\left( \frac{\pi}{L_1}(z_j^* - \bar{R}^*) - \tau^* \right) \right\}^{q-1}
\]

\[
F_L(z_1^*, z_2^*, ... z_{N_e}^*) = \left\{ \vartheta_1\left( \frac{\pi}{L_1}(Z^* - \bar{R}^*) - \tau^* \right) \right\}^{q-1} \prod_{i,j=1,2...N_e}^{q-1} \left\{ \vartheta_1\left( \frac{\pi}{L_1}(z_i^* - z_j^*) - \tau^* \right) \right\}.
\]
But, the form of the magnetic translations in the axial gauge (as defined by its action on the analytic factors $A$ of the exponential $\exp(-\frac{x^2}{4r_o^2})$) is

$$T_R A(z^*) = \exp(-\frac{RR^*}{4r_o^2} + \frac{Rz^*}{2r_o}) A(z^* - R^*).$$

and $\Psi_{hf}$ by construction, is invariant under the magnetic translations of any of $z_i^*$ variables in the $L_1$ and $L_1\tau^*$, corresponding to the elementary shifts in the lattice in which periodicity conditions are imposed. Therefore, the variation of the wave function $\Psi^{(N)}$ will be determined by the corresponding change in the analytic factor $\eta$, under the shifts $\eta(z_1^*, z_2^*...z_N^* - \pi, ..., z_N^*)$ and $\eta(z_1^*, z_2^*...z_N^* - \pi\tau^*, ..., z_N^*)$ in any of the particular coordinates $z_i^*$.

Let us argue below that under those changes, both of the functions $F_L$ and $F$ becomes themselves times a common factor function which cancels in the ratio defining $\eta$. This property implies that the proposed trial states also satisfies the periodic boundary conditions in each of the particle coordinates.

For this purpose the transformations (1) allow to write

$$F(z_1^*, z_2^*...z_i^* - \pi\tau^*, ..., z_N^*) = (-1)^N \exp\left(-i \sum_{R} \left(\frac{2\pi}{L_1}(z_i^* - R^*) - \pi\tau^*\right)\right) F(z_1^*, z_2^*...z_i^*, ..., z_N^*),$$

$$= (-1)^N \exp\left(-2\pi N_e \frac{z_i^*}{L_1} + \frac{2\pi i}{L_1} \bar{R} + i\pi N_e \tau^*\right) F(z_1^*, z_2^*...z_i^*, ..., z_N^*),$$

where $\bar{R} = \sum_R R$ is the sum of all the complex coordinates of the lattice $R$ which are in the first periodicity zone. For the factor $F_L$, on another hand, it follows

$$F_L(z_1^*, z_2^*...z_i^* - \pi\tau^*, ..., z_N^*) = (-1)^N \exp\left(-i \sum_{j \neq i} \left(\frac{2\pi}{L_1}(z_i^* - z_j^*) - \pi\tau^*\right) - \frac{2\pi}{L_1}(Z^* - \bar{R}) + i\pi \tau^*\right) F_L(z_1^*, z_2^*...z_i^*, ..., z_N^*),$$

$$= (-1)^N \exp\left(-2\pi N_e \frac{z_i^*}{L_1} + \frac{2\pi i}{L_1} \bar{R} + i\pi N_e \tau^*\right) F_L(z_1^*, z_2^*...z_i^*, ..., z_N^*).$$

Therefore, the similar way of transformation of the functions $F_L$ and $F$ implies the invariance of the factor $\eta$ under the translations in $L_1\tau^*$. The invariance under the shifts in $L_1$ also is directly obtained, thanks to the simplicity in the transformations (2). The validity of these two properties determines that the proposed states also satisfy the periodicity conditions imposed over each of the single particle variables. Thus, they are allowed states in the periodic quantization of the electron system in the magnetic field.

B. Case $\gamma = 1$

In this case the construction is simpler since the index $\sigma$ is absent. The $HF$ single particle orbitals are $\varphi_{k}^{(0)}(x)$ where the only quantum number is now the momentum

$$\Psi_{hf}^{(1)} = \Psi_{hf}^{(1)}[z_1^*, z_2^*, ..., z_N^*, z_1, z_2, ..., z_N] = Det[\varphi_{k}^{(0)}(x_j)],$$

in which the wave vectors $k_i$ pertain to the lower non degenerate band defining the $HF$ state. Again, these momenta satisfy the periodicity condition of the Haldane-Rezayi problem and their number in this case is equal to the number of electrons in the problem $N_e$. As shown in the previous section the orbitals have the structure

$$\varphi_{k}^{(0)}(x) = \prod_{R} \left\{ \left( \vartheta_1(\frac{\pi}{L_1}(z^* - R^*)) - \tau^* \right)^{(q-1)} \right\} \chi_{k}(z^*) \exp(k z^* + \frac{z^2}{4r_o^2} - \frac{x^2}{4r_o^2}),$$

$$= P(z^*) \chi_{k}(z^*) \exp(k z^* + \frac{z^2}{4r_o^2} - \frac{x^2}{4r_o^2}),$$

$$P(z^*) = \prod_{R} \left\{ \left( \vartheta_1(\frac{\pi}{L_1}(z^* - R^*)) - \tau^* \right)^{(q-1)} \right\},$$

$$\chi_{k}(z^*) = \prod_{R} \left\{ \vartheta_1(\frac{\pi}{L_1}(z^* - R^* - C_k)) - \tau^* \right\}.$$
where all the dependence of the momentum quantum numbers is contained in the new function \( \chi \) and the exponential factor \( \exp(kz^*) \).

Similarly as in the last subsection the Slater determinant of the HF solution can be represented as follows

\[
\Psi_{hf}^{(1)} = \text{Det} [\varphi_{k_i}^{(0)}(x_j)],
\]

\[
= \left( \prod_{k=1,2,..N_e} P(z_k^*) \right) \text{Det} [\chi_{k_i}^{(0)}(z_k^*)] \exp\left( \sum_{i=1,2,..N_e} (k z_i^* + z_i^2/4r_o^2 - x_i^2/4r_o^2) \right),
\]

\[
= \Phi(z_1^*, z_2^*, ..., z_N_e^*) D[z_1^*, z_2^*, ..., z_N_e^*] \exp\left( \sum_{i=1,2,..N_e} (z_i^2/4r_o^2 - x_i^2/4r_o^2) \right),
\]

\[
\Phi(z_1^*, z_2^*, ..., z_N_e^*) = \prod_{R} \left\{ \left( \vartheta_1 \left( \frac{\pi}{L_1} \left| z^* - R^* \right| - \tau^* \right) \right)^{-1} \right\},
\]

\[
D[z_1^*, z_2^*, ..., z_N_e^*] = \exp\left( \sum_{i=1,2,..N_e} k z_i^* \right) \text{Det} [\chi_{k_i}^{(0)}(z_k^*)].
\]

Again, the new factors \( \Phi \), when considered as function of any of the electron coordinates \( z^* \), have \((q-1)N_e\) zeroes. Since the determinant \( \text{Det} [\chi_{k_i}^{(0)}(z_k^*)] \) also shows a number \( N_e \) of zeroes, the total number of vanishing points of \( \Psi_{hf}^{(1)} \) (with respect to any particular variable) is again equal to the number of flux quanta.

Thus, following the same steps as in the last section, in this case, the variational states for this case are proposed in the form

\[
\Psi_N^{(1)} = \Phi_L(z_1^*, z_2^*, ..., z_N_e^*) D^{(1)}[z_1^*, z_2^*, ..., z_N_e^*] \exp\left( \sum_{i=1,2,..N_e} (z_i^2/4r_o^2 - x_i^2/4r_o^2) \right),
\]

\[
\Phi_L(z_1^*, z_2^*, ..., z_N_e^*) = \left\{ \left( \vartheta_1 \left( \frac{\pi}{L_1} \left| Z^* - R^* \right| - \tau^* \right) \right)^{-1} \right\} \prod_{i,j=1,2,..N_e} \left\{ \vartheta_1 \left( \frac{\pi}{L_1} \left| z_i^* - z_j^* \right| - \tau^* \right) \right\}^{-1},
\]

\[
D^{(1)}(z_1^*, z_2^*, ..., z_N_e^*) = \exp\left( \sum_{i=1,2,..N_e} k z_i^* \right) \text{Det} [\chi_{k_i}^{(0)}(z_k^*)].
\]

In an almost identical way, as in the previous section it can be argued that these functions satisfy the periodic boundary conditions.

### IV. SUMMARY

The Haldane-Rezayi procedure for describing periodic boundary conditions in presence of a magnetic field is employed for constructing new variational states which are expected to optimize short range correlations in a similar form as the Laughlin states. However, their construction also include crystalline correlations which were shown to optimize the mean field HF energies. Two classes of states are proposed which described by the number of electrons per unit cell \( \gamma \) which defines the associated HF solution. The cases \( \gamma = 1, 1/2 \) are considered for any filling factor of the forms \( 1/q \). They determine two classes of alternative states: one showing \( q \) flux quanta per periodicity cell and the other having \( q/2 \) quanta per cell. The states are shown to satisfy the periodic boundary conditions. Also, as it should be, the wave functions when considered as functions of any of the identical particle coordinates, they show a number of zeroes which is equal to the number of flux quanta piercing the periodicity area of the system. For the considered cases of filling factors \( \nu = 1/q \) the number of flux quanta is \( qN_e \), where \( N_e \) is the number of electrons in the sample.

The main elements of the proposal can be schematically resumed as follows. Let us consider in particular the HF Slater determinant of single particle orbitals described Ref. [1] for the case \( \gamma = 1 \). In these determinant \( (q-1)N_e \) of its zeros as a function of the coordinates of any given particle are fixed at the points of the periodicity lattice. These zeroes appear in a factor function \( \Phi \) which is completely independent of the quantum numbers of the filled orbitals. Then, this basic property allows to transform the mean field Slater determinant by substituting the factor function \( \Phi \) by a factor function \( \Phi_L \) having the same number of zeroes that \( \Phi \) but only depending on the differences of the coordinates in the usual Jastrow way. This procedure suggests a possibility for a lowering of the energies per particle,
by reducing short range Coulomb repulsion. This is related with the fact that the zeroes of the proposed states, when the coordinates of two particles equalize, is of the same order $q$ as in the Laughlin states. It should be noticed that the $N_e$ zeroes associated to the functions $D^{(1)}$ and $D^{(\frac{1}{2})}$ have spatial locations depending on the quantum number $k$, which label each occupied orbital in the factor having the form of an Slater determinant. It could be the case that the presence of remaining crystalline mean field information in the determinantal factors $D^{(1)}$ and $D^{(\frac{1}{2})}$ could lower the energy per particle of the Laughlin anstaz. The evaluation of the correlation energy for these states is under consideration. We hope that this proposal can stimulate further research directed to check whether the corrections to the mean field approach early introduced by one of the authors (F.C), could furnish a general theory for the FQHE, showing the relevant Laughlin or Jain procedures as closely approximate limits.

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