Research Article

Coefficient Estimate Problem for a New Subclass of Biunivalent Functions

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We introduce a unified subclass of the function class $\Sigma$ of biunivalent functions defined in the open unit disc. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this subclass. In addition, many relevant connections with known or new results are pointed out.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$. Further, by $\mathcal{S}$, we will denote the class of all functions in $\mathcal{A}$ which are univalent in $U$.

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include, for example, the class $\mathcal{S}^* (\alpha)$ of starlike functions of order $\alpha$ in $U$ and the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha$ in $U$.

It is well known that every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, defined by

$$f^{-1} (f(z)) = z \quad (z \in \mathcal{U}),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots.$$  \hspace{1cm} (3)

A function $f \in \mathcal{A}$ is said to be biunivalent in $U$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $U$. Let $\Sigma$ denote the class of biunivalent functions in $U$ given by (1).

In 1967, Lewin [1] investigated the biunivalent function class $\Sigma$ and showed that $|a_2| < 1.51$; on the other hand Brannan and Clunie [2] (see also [3–5]) and Netanyahu [6] made an attempt to introduce various subclasses of biunivalent function class $\Sigma$ and obtained nonsharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} := \{1, 2, 3, \ldots \}$ is still an open problem. In this line, following Brannan and Taha [4], recently, many researchers have introduced and investigated several interesting subclasses of biunivalent function class $\Sigma$ and they have found nonsharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$; for details, one can refer to the works of [7–13].

Now, we define $\mathcal{R}_\Sigma (\alpha, \lambda)$ of function $f \in \mathcal{A}$ satisfying the following conditions:

$$f \in \Sigma, \quad \left| \arg \left( \frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}} \right) \right| < \frac{\alpha \pi}{2},$$

$$\left| \arg \left( \frac{w^{1-\lambda} g'(w)}{(g(w))^{1-\lambda}} \right) \right| < \frac{\alpha \pi}{2} \quad (z, w \in U; \lambda \geq 0)$$

(4)
for some $\alpha$ ($0 < \alpha \leq 1$), where $g(w)$ is the extension of $f^{-1}(w)$ to $U$. Similarly, we say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{R}_\Sigma(\alpha, \lambda)$ if $f(z)$ satisfies the following inequalities:

\[
\begin{align*}
&f \in \Sigma, \quad \Re \left( \frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}} \right) > \beta, \\
&\Re \left( \frac{w^{1-\lambda} g'(w)}{(g(w))^{1-\lambda}} \right) > \beta \quad (z, w \in U; \lambda \geq 0),
\end{align*}
\]

for some $\beta$ ($0 < \beta < 1$), where $g(w)$ is the extension of $f^{-1}(w)$ to $U$. The classes $\mathcal{R}_\Sigma(\alpha, \lambda)$ and $\mathcal{R}_\Sigma(\beta, \lambda)$ were introduced by Prema and Keerthi [14]; furthermore, for these classes, they have found the following estimates on the first two Taylor-Maclaurin coefficients in (1).

**Theorem 1.** If $f \in \mathcal{R}_\Sigma(\alpha, \lambda)$, $0 < \alpha \leq 1$, and $\lambda \geq 0$, then

\[
|a_2| \leq \frac{2\alpha}{\sqrt{\alpha + 1 + \lambda}} (1 + \lambda), \quad |a_3| \leq \frac{4\alpha^2}{(1 + \lambda)^2} + \frac{2\alpha}{2 + \lambda}.
\]

**Theorem 2.** If $f \in \mathcal{R}_\Sigma(\beta, \lambda)$, $0 < \beta < 1$, and $\lambda \geq 0$, then

\[
|a_2| \leq \frac{2(1 - \beta)}{1 + \lambda}, \quad |a_3| \leq \frac{4(1 - \beta)^2}{(1 + \lambda)^2} + \frac{2(1 - \beta)}{2 + \lambda}.
\]

Motivated by the works of Xu et al. [12, 13], we introduce the following generalized subclass $\mathcal{R}_\Sigma(\phi, \psi, \lambda)$ of the analytic function class $\mathcal{A}$.

**Definition 3.** Let $f \in \mathcal{A}$, and let the functions $\phi, \psi : U \to \mathbb{C}$ be so constrained that

\[
\min \{ \Re (\phi(z)), \Re (\psi(z)) \} > 0 \quad (z \in U),
\]

\[
\phi(0) = \psi(0) = 1.
\]

We say that $f \in \mathcal{R}_\Sigma(\phi, \psi, \lambda)$ if the following conditions are satisfied:

\[
\begin{align*}
f \in \Sigma, \quad &\frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}} \in \phi(U), \\
&\frac{w^{1-\lambda} g'(w)}{(g(w))^{1-\lambda}} \in \psi(U) \quad (z, w \in U),
\end{align*}
\]

where $\lambda \geq 0$ and the function $g(w)$ is the extension of $f^{-1}(w)$ to $U$.

We note that by specializing $\lambda, \phi,$ and $\psi$, we get the following interesting subclasses:

1. $\mathcal{R}_\Sigma(\phi, \psi, 1) = \mathcal{R}_\Sigma^{\phi\psi}$; see [12],
2. $\mathcal{R}_\Sigma(((1+z)/(1-z))^{\alpha}, ((1+z)/(1-z))^{\alpha}, \lambda) = \mathcal{R}_\Sigma(\alpha, \lambda)$ ($0 < \alpha \leq 1; \lambda \geq 0$) and $\mathcal{R}_\Sigma((1 + (1 - 2\beta)z)/(1 - z), (1 + (1 - 2\beta)z)/(1 - z), \lambda) = \mathcal{R}_\Sigma(\beta, \lambda)$ ($0 \leq \beta < 1; \lambda \geq 0$); see [14],
3. $\mathcal{R}_\Sigma(((1+z)/(1-z))^{\alpha}, ((1+z)/(1-z))^{\alpha}, 1) = \mathcal{H}_\Sigma^\alpha$ ($0 < \alpha \leq 1$) and $\mathcal{R}_\Sigma((1 + (1 - 2\beta)z)/(1 - z), (1 + (1 - 2\beta)z)/(1 - z), 1) = \mathcal{H}_\Sigma^\beta (0 \leq \beta < 1)$; see [11].

The objective of the present paper is to introduce a new subclass $\mathcal{R}_\Sigma(\phi, \psi, \lambda)$ and to obtain the estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions in the aforementioned class, employing the techniques used earlier by Xu et al. [12, 13].

### 2. Main Result

In this section, we find the estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions in the class $\mathcal{R}_\Sigma(\phi, \psi, \lambda)$.

**Theorem 4.** Let $f(z)$ be of the form (1). If $f \in \mathcal{R}_\Sigma(\phi, \psi, \lambda)$, then

\[
|a_2| \leq \sqrt{\frac{\phi''(0) + \psi''(0)}{8 + 4\lambda}},
\]

\[
|a_3| \leq \frac{\phi''(0)}{4 + 2\lambda}.
\]

**Proof.** Since $f \in \mathcal{R}_\Sigma(\phi, \psi, \lambda)$, from (9), we have,

\[
\begin{align*}
z^{1-\lambda} f'(z) &= \phi(z) \quad (z \in U), \\
w^{1-\lambda} g'(w) &= \psi(w) \quad (w \in U),
\end{align*}
\]

where

\[
\begin{align*}
\phi(z) &= 1 + \phi_1 z + \phi_2 z^2 + \cdots, \\
\psi(z) &= 1 + \psi_1 z + \psi_2 z^2 + \cdots
\end{align*}
\]

satisfy the conditions of Definition 3. Now, equating the coefficients in (12), we get

\[
(1 + \lambda) a_2 = \phi_1, \quad (2 + \lambda) a_3 = \phi_2, \quad - (1 + \lambda) a_2 = \psi_1, \quad (2 + \lambda) (2a_2^2 - a_3) = \psi_2.
\]

From (14) and (16), we get

\[
\phi_1 = -\psi_1, \quad 2(1 + \lambda)^2 a_2^2 = \phi_1^2 + \psi_1^2.
\]

From (15) and (17), we obtain

\[
a_2^2 = \frac{\phi_2 + \psi_2}{2(2 + \lambda)}.
\]

Since $\phi(z) \in \phi(U)$ and $\psi(z) \in \psi(U)$, we immediately have

\[
|a_2| \leq \sqrt{\frac{\phi''(0) + \psi''(0)}{8 + 4\lambda}}.
\]

This gives the bound on $|a_2|$ as asserted in (10).
Next, in order to find the bound on $|a_3|$, by subtracting (17) from (15), we get
\[ 2(2+\lambda)a_3 - 2(2+\lambda)a_2^2 = \varphi_2 - \psi_2. \] (21)

It follows from (19) and (21) that
\[ a_3 = \frac{\varphi_2}{2+\lambda}. \] (22)

Since $\varphi(z) \in \mathcal{U}$ and $\psi(z) \in \mathcal{U}$, we readily get $|a_3| \leq |\varphi''(0)/(4+2\lambda)|$ as asserted in (11). This completes the proof of Theorem 4.

By setting $\varphi(z) = \psi(z) = ((1+Az)/(1+Bz))^\alpha$, where $-1 \leq B < A \leq 1$ and $0 < \alpha \leq 1$, in Theorem 4, we get the following corollary.

**Corollary 5.** Let $f(z)$ be of the form (1) and in the class $\mathcal{R}_2(A,B,\alpha,\lambda)$. Then,
\[ |a_2| \leq \sqrt{\frac{\alpha^2(A-B)^2-\alpha(A^2-B^2)}{4+2\lambda}}, \] (23)
\[ |a_3| \leq \sqrt{\frac{\alpha^2(A-B)^2-\alpha(A^2-B^2)}{4+2\lambda}}. \]

If we choose $A = 1$ and $B = -1$ in Corollary 5, we have the following corollary.

**Corollary 6.** Let $f(z)$ be of the form (1) and in the class $\mathcal{R}_2(\alpha,\lambda)$, $0 < \alpha \leq 1$ and $\lambda \geq 0$. Then,
\[ |a_2| \leq \alpha \sqrt{\frac{2}{2+\lambda}}, \quad |a_3| \leq \frac{2\alpha^2}{2+\lambda}. \] (24)

**Remark 7.** The estimates found in Corollary 6 would improve the estimates obtained in [14, Theorem 2.2].

If we set $A = 1 - 2\beta$, $B = -1$, where $0 \leq \beta < 1$ and $\alpha = 1$ in Corollary 5, we readily have the following corollary.

**Corollary 8.** Let $f(z)$ be of the form (1) and in the class $\mathcal{R}_2(\beta,\lambda)$, $0 \leq \beta < 1$ and $\lambda \geq 0$. Then,
\[ |a_2| \leq \sqrt{\frac{2(1-\beta)}{2+\lambda}}, \quad |a_3| \leq \frac{2(1-\beta)}{2+\lambda}. \] (25)

**Remark 9.** The estimates found in Corollary 8 would improve the estimates obtained in [14, Theorem 3.2].

**Remark 10.** For $\lambda = 1$, the bounds obtained in Theorem 4 are coincident with the outcome of Xu et al. [12]. Taking $\lambda = 0$ in Corollaries 6 and 8, the estimates on the coefficients $|a_2|$ and $|a_3|$, are the improvement of the estimates on the first two Taylor-Maclaurin coefficients obtained in [10, Corollaries 2.3 and 3.3]. Also, for the choices of $\lambda = 1$, the results stated in Corollaries 6 and 8 would improve the bounds stated in [11, Theorems 1 and 2], respectively. Furthermore, various other interesting corollaries and consequences of our main result could be derived similarly by specializing $\varphi$ and $\psi$.

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