Equivalence of Lattice Orbit Polytopes

FRIEDER LADISCH AND ACHILL SCHÜRMANN

Dedicated to Jörg M. Wills on the occasion of his 80th birthday

Abstract. Let $G$ be a finite permutation group acting on $\mathbb{R}^d$ by permuting coordinates. A core point (for $G$) is an integral vector $z \in \mathbb{Z}^d$ such that the convex hull of the orbit $Gz$ contains no other integral vectors but those in the orbit $Gz$. Herr, Rehn and Schürmann considered the question for which groups there are infinitely many core points up to translation equivalence, that is, up to translation by vectors fixed by the group. In the present paper, we propose a coarser equivalence relation for core points called normalizer equivalence. These equivalence classes often contain infinitely many vectors up to translation, for example when the group admits an irrational invariant subspace or an invariant irreducible subspace occurring with multiplicity greater than 1. We also show that the number of core points up to normalizer equivalence is finite if $G$ is a so-called QI-group. These groups include all transitive permutation groups of prime degree. We give an example to show how the concept of normalizer equivalence can be used to simplify integer convex optimization problems.

1. Introduction

Let $G \leq \text{GL}(d, \mathbb{Z})$ be a finite group. We consider orbit polytopes $\text{conv}(Gz)$ of integral vectors $z \in \mathbb{Z}^d$, that is, the convex hull of an orbit of a point $z$ with integer coordinates. We call $z$ a core point for $G$ when the vertices are the only integral vectors in the orbit polytope $\text{conv}(Gz)$. Core points were introduced in [2, 17] in the context of convex integer optimization, in order to develop new techniques to exploit symmetries. Core points are relevant to symmetric convex integer optimization, since a $G$-symmetric convex set contains an integer vector if and only if it contains a core point for $G$. So when a $G$-invariant convex integer optimization problem has a solution, then there is a core point attaining the optimal value. As the set of core points is itself $G$-symmetric, it even suffices to consider only one core point from each $G$-orbit. In this way, solving a $G$-invariant convex integer optimization problem can be reduced to a considerably smaller set of integral vectors. Furthermore, since core points are known to be close to some $G$-invariant subspace [34, Theorem 3.13]

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[18, Theorem 9], one can for example use them to add additional linear or quadratic constraints to a given symmetric problem (see, for instance, [34, Section 7.3]).

Previous work on core points mainly considered groups for which there are only finitely many core points up to a natural equivalence relation called translation equivalence [2, 17, 18]. For some of these groups, even a naive enumeration approach is sufficient to beat state-of-the-art commercial solvers. Moreover, when there are only finitely many core points up to translation equivalence, then one can parametrize the core points of $G$ in a natural way, and thereby obtain a beneficial reformulation of a $G$-invariant problem [17]. This technique was used to solve a previously open problem from the MIPLIB 2010 [24] (see [17]). More elaborate algorithms taking advantage of core points, possibly combined with classical techniques from integer optimization, have yet to be developed.

In this paper we extend the list of groups for which core points can be parametrized. This is achieved by introducing a new equivalence relation for core points, which is coarser than the translation equivalence previously used. It turns out that this new equivalence relation not only helps to classify core points, but also suggests a new way to transform $G$-invariant convex integer optimization problems in a natural way into possibly simpler equivalent ones. Knowing a group $G$ of symmetries, elements of its normalizer in $\text{GL}(d, \mathbb{Z})$ can be used to transform a $G$-invariant convex integer optimization problem linearly into an equivalent $G$-invariant problem. As we show in Section 6 for the case of integer linear problem instances, the transformed optimization problems are sometimes substantially easier to solve. To apply this technique in general, one needs to know how to find a good transformation from the normalizer which yields an easy-to-solve transformed problem. While this is easy in some cases as in our examples, we do not yet understand satisfactorily how to find good transformations from the normalizer in general.

In the following, we write

$$\text{Fix}(G) = \{ v \in \mathbb{R}^d \mid gv = v \text{ for all } g \in G \}$$

for the fixed space of $G$ in $\mathbb{R}^d$. When $z$ is a core point and $t \in \text{Fix}(G) \cap \mathbb{Z}^d$, then $z + t$ is another core point. We call the core points $z$ and $z + t$ translation equivalent. Herr, Rehn and Schürmann [18] consider the question of whether there are finitely or infinitely many core points up to translation equivalence in the case where $G$ is a permutation group acting by permuting coordinates. Their methods can be used to show that there are only finitely many core points up to translation when $\mathbb{R}^d / \text{Fix}(G)$ has no $G$-invariant subspaces other than the trivial ones [34, Theorem 3.13]. They conjectured that in all other cases, there are infinitely many core points up to translation. This has been proved in special cases but is open in general.

In this paper, we consider a coarser equivalence relation, where we allow to multiply core points with invertible integer matrices $S \in \text{GL}(d, \mathbb{Z})$ which normalize the subgroup $G$. Thus two points $z$ and $w$ are called normalizer equivalent, when
$w = Sz + t$, where $S$ is an element of the normalizer of $G$ in $\text{GL}(d, \mathbb{Z})$ (in other words, $S^{-1}GS = G$), and $t \in \text{Fix}(G) \cap \mathbb{Z}^d$. In Theorem 4.1, we will determine when these coarser equivalence classes contain infinitely many points up to translation equivalence, in terms of the decomposition into irreducible invariant subspaces. For example, if $\mathbb{R}^d$ has an irrational invariant subspace $U \leq \mathbb{R}^d$ (that is, a subspace $\{0\} \neq U \leq \mathbb{R}^d$ such that $U \cap \mathbb{Z}^d = \{0\}$), then each integer point $z$ with nonzero projection onto $U$ is normalizer equivalent to infinitely many points, which are not translation equivalent. This yields another proof of the result of Herr, Rehn and Schürmann [18, Theorem 32] that there are infinitely many core points up to translation, when there is an irrational invariant subspace.

In Theorem 5.1, we prove the following: Suppose that $G \leq S_d$ is a transitive permutation group acting on $\mathbb{R}^d$ by permuting coordinates. Suppose that $\text{Fix}(G)^\perp$ contains no rational $G$-invariant subspace other than $\{0\}$ and $\text{Fix}(G)^\perp$ itself. (A subspace of $\mathbb{R}^d$ is rational if it has a basis contained in $\mathbb{Q}^d$.) Such a group $G$ is called a QI-group. Then there are only finitely many core points up to normalizer equivalence.

For example, this is the case when $d = p$ is a prime number (and $G \leq S_p$ is transitive). In the case that the group is not 2-homogeneous, there are infinitely many core points up to translation, but these can be obtained from finitely many by multiplying with invertible integer matrices from the normalizer.

The paper is organized as follows. In Section 2, we introduce different equivalence relations for core points and make some elementary observations. Section 3 collects some elementary properties of orders in semisimple algebras. In Section 4, we determine when the normalizer equivalence classes contain infinitely many points up to translation equivalence. In Section 5, we prove the aforementioned result on QI-groups. Sections 4 and 5 can mostly be read independently from one another. Finally, in Section 6 we give an example to show how normalizer equivalence can be applied to integer convex optimization problems with suitable symmetries.

### 2. Equivalence for core points

Let $V$ be a finite-dimensional vector space over the real numbers $\mathbb{R}$ and $G$ a finite group acting linearly on $V$.

**2.1. Definition.** An orbit polytope (for $G$) is the convex hull of the $G$-orbit of a point $v \in V$. It is denoted by

$$P(G, v) = \text{conv}\{gv \mid g \in G\}.$$ 

Let $\Lambda \subseteq V$ be a full $\mathbb{Z}$-lattice in $V$, that is, the $\mathbb{Z}$-span of an $\mathbb{R}$-basis of $V$. Assume that $G$ maps $\Lambda$ onto itself.

**2.2. Definition.** [17] An element $z \in \Lambda$ is called a core point (for $G$ and $\Lambda$) if the only lattice points in $P(G, z)$ are its vertices, that is, the elements in the orbit $Gz$. 
In other words, $z$ is a core point if
\[ P(G, z) \cap \Lambda = Gz. \]

The barycenter
\[ \frac{1}{|G|} \sum_{g \in G} gv \in P(G, v) \]
is always fixed by $G$. If $Fix_V(G)$, the set of vectors in $V$ fixed by all $g \in G$, consists only of 0, then the barycenter of each orbit polytope is the zero vector. In this case, only the zero vector is a core point [34, Remark 3.7, Lemma 3.8].

More generally, the map
\[ e_1 = \frac{1}{|G|} \sum_{g \in G} g \]
gives the projection from $V$ onto the fixed space $Fix_V(G)$ [37, §2.6, Theorem 8], and thus yields a decomposition $V = Fix_V(G) \oplus \ker(e_1)$ into $G$-invariant subspaces. If $Fix_V(G)$ consists only of 0, then the barycenter of each orbit polytope is the zero vector. In this case, only the zero vector may not be contained in $\Lambda$.

An important class of examples where $e_1 \Lambda \not\subseteq \Lambda$ is transitive permutation groups $G \leq S_d$, acting on $V = \mathbb{R}^d$ by permuting coordinates, and where $\Lambda = \mathbb{Z}^d$. The fixed space consists of the vectors with all entries equal and is thus generated by the all-ones vector $1 := (1, 1, \ldots, 1)^t$. For $v = (v_1, \ldots, v_d)^t$ we have $e_1v = (\sum_i v_i)/d \cdot 1$. In particular, we see that $e_1\Lambda$ contains all integer multiples of $(1/d)1$. We can think of $\Lambda$ as being partitioned into layers, where a layer consists of all $z \in \Lambda$ with $z^t 1 = k$ (equivalently, $e_1z = (k/d)1$), for a fixed integer $k$.

Returning to general groups of integer matrices, we claim that for each $v \in e_1\Lambda$, there are core points $z$ with $e_1z = v$. Namely, among all $z \in \Lambda$ with $e_1z = v$, there are elements such that $\sum_g \|gz\|^2$ is minimal, and these are core points.

If $z$ is a core point and $b \in Fix_\Lambda(G)$, then $z + b$ is a core point, too, because $P(G, z + b) = P(G, z) + b$. Such core points should be considered as equivalent. This viewpoint was adopted by Herr, Rehn and Schürmann [17, 18]. In the present paper, we consider a coarser equivalence relation. We write $GL(\Lambda)$ for the invertible $\mathbb{Z}$-linear maps $\Lambda \to \Lambda$. Since $\Lambda$ contains a basis of $V$, we may view $GL(\Lambda)$ as a subgroup of $GL(V)$. (If $V = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$, then we can identify $GL(\Lambda)$ with $GL(d, \mathbb{Z})$, the set of matrices over $\mathbb{Z}$ with determinant $\pm 1$.)

By assumption, $G$ is a subgroup of $GL(\Lambda)$. We use the following notation from group theory: The normalizer of a finite group $G$ in $GL(\Lambda)$ is the set
\[ N_{GL(\Lambda)}(G) := \{ S \in GL(\Lambda) \mid \forall g \in G: S^{-1} g S \in G \}. \]
The centralizer of $G$ in $GL(\Lambda)$ is the set
\[ C_{GL(\Lambda)}(G) := \{ S \in GL(\Lambda) \mid \forall g \in G: S^{-1} g S = g \}. \]
2.3. Definition. Two points \( z \) and \( w \) are called \textit{normalizer equivalent} if there is a point \( b \in \text{Fix}_A(G) \) and an element \( S \) in the normalizer \( N_{GL(\Lambda)}(G) \) of \( G \) in \( GL(\Lambda) \) such that \( w = Sz + b \). The points are called \textit{centralizer equivalent} if \( w = Sz + b \) with \( S \in C_{GL(\Lambda)}(G) \) and \( b \in \text{Fix}_A(G) \). Finally, we call two points \( z \) and \( w \) \textit{translation equivalent} when \( w - z \in \text{Fix}_A(G) \).

Since \( C_{GL(\Lambda)}(G) \subseteq N_{GL(\Lambda)}(G) \), each normalizer equivalence class is a union of centralizer equivalence classes, and obviously each centralizer equivalence class is a union of translation equivalence classes. The definition is motivated by the following simple observation:

2.4. Lemma. If
\[
   w = Sz + b \quad \text{with} \quad S \in N_{GL(\Lambda)}(G) \quad \text{and} \quad b \in \text{Fix}_A(G),
\]
then \( x \mapsto Sx + b \) defines a bijection between
\[
P(G, z) \cap \Lambda \quad \text{and} \quad P(G, w) \cap \Lambda.
\]

In particular, \( z \) is a core point for \( G \) if and only if \( w \) is a core point for \( G \).

Proof. The affine bijection \( x \mapsto Sx + b \) maps the orbit polytope \( P(G, z) \) to another polytope. The vertex \( gz \) is mapped to the vertex
\[
   Sgz + b = (SgS^{-1})Sz + b = hSz + b = h(Sz + b) = hw,
\]
where \( h = SgS^{-1} \in G \) (since \( S \) normalizes \( G \)). The second to last equality follows as \( b \) is fixed by \( G \). As \( SgS^{-1} \) runs through \( G \) as \( g \) does, it follows that \( x \mapsto Sx + b \) maps the orbit \( Gz \) to the orbit \( Gw \) and thus maps the orbit polytope \( P(G, z) \) to the orbit polytope \( P(G, w) \). Since \( x \mapsto Sx + b \) also maps \( \Lambda \) onto itself, the result follows.

Notice that a point \( w \) is equivalent to \( z = 0 \) (for any of the equivalence relations in Definition 2.3) if and only if \( w = S \cdot 0 + b = b \in \text{Fix}_A(G) \). Any \( w \in \text{Fix}_A(G) \) is a core point. We call these points the \textit{trivial core points}.

In the important example of transitive permutation groups, the fixed space is one-dimensional. More generally, when \( V \) is spanned linearly by some orbit \( Gz \), then \( \text{Fix}_V(G) \) is spanned by \( e_1z \) and thus \( \dim(\text{Fix}_V(G)) \leq 1 \).

2.5. Remark. Suppose that \( \text{Fix}_V(G) \) has dimension 1. Then there is at most one \( w \in N_{GL(\Lambda)}(G)z \) such that \( w \neq z \) and \( w \) is translation equivalent to \( z \).

Proof. The elements of \( N_{GL(\Lambda)}(G) \) map \( \text{Fix}_A(G) \) onto itself and thus act on \( \text{Fix}_V(G) \) as \( \pm 1 \). Let \( S \in N_{GL(\Lambda)}(G) \). Suppose \( w = Sz \) and \( z \) are translation equivalent, so that \( Sz - z = b \in \text{Fix}_A(G) \). Then \( b = e_1b = e_1Sz - e_1z = Se_1z - e_1z = \pm e_1z + e_1z \) and thus either \( Sz = z \) or \( Sz = z - 2e_1z \). (The latter case can only occur when \( 2e_1z \in \Lambda \).) \( \square \)
In particular, if the orbit $N_{\text{GL}(\Lambda)}(G)z$ is infinite, then the normalizer equivalence class of a nontrivial core point $z$ contains infinitely many translation equivalence classes.

Also notice that when $z$ is a nontrivial core point, then $e_1z$ must not be a lattice point.

Herr, Rehn, and Schürmann [18, 16, 34] considered the question of whether the set of core points up to translation is finite or infinite (in the case where $G$ acts by permuting coordinates). We might ask the same question about core points up to normalizer equivalence as defined here. Also, it is of interest whether our bigger equivalence classes contain finitely or infinitely many points up to translation.

2.6. Example. Let $G = S_d$, the symmetric group on $d$ elements, acting on $\mathbb{R}^d$ by permuting coordinates, and $\Lambda = \mathbb{Z}^d$. We identify $G$ with the group of all permutation matrices. For this group, Bödi, Herr and Joswig [2] have shown that every core point is translation equivalent to a vector with all entries 0 or 1. (Conversely, these vectors are obviously core points.) One can show that the normalizer of the group $G$ of all permutation matrices in $\text{GL}(d, \mathbb{Z})$ is generated by $-I$ and the group $G$ itself. As $G$ is transitive on the subsets of $\{1, \ldots, d\}$ of size $k$, all 0/1-vectors with fixed number $k$ of 1’s are normalizer equivalent. A vector $z$ with $k$ ones and $d - k$ zeros is also normalizer equivalent to the vector $-z + 1$ with $d - k$ ones and $k$ zeros. Thus up to normalizer equivalence, there are only $\lfloor d/2 \rfloor + 1$ core points.

2.7. Example. Let $G = C_d = \langle (1, 2, \ldots, d) \rangle$ be a cyclic group, again identified with a matrix group which acts on $\mathbb{R}^d$ by permuting the coordinates cyclically. For $d = 4$ we have a finite normalizer (as we will see in Section 4) but infinitely many core points up to normalizer or translation equivalence: for example, all the points $(1 + m, -m, m, -m)^t$, $m \in \mathbb{Z}$, are core points for $C_4$ [18, Example 26].

If $d = p$ is prime, then we will see that there are only finitely many core points up to normalizer equivalence, but for $p \geq 5$ the normalizer is infinite and there are infinitely many core points up to translation equivalence. (See Example 5.9 below.) For $d = 8$ (say), the normalizer is infinite and there are infinitely many core points up to normalizer equivalence. Namely, let $b_1 \in \mathbb{R}^8$ be the first standard basis vector and let $v \in \mathbb{R}^8$ be the vector with entries alternating between 1 and -1. Then the points $b_1 + mv$ for $m \in \mathbb{Z}$ are core points [18, Theorem 30] (the construction principle here is the same as above in the case $d = 4$). The circulant $8 \times 8$-matrix $S$ with first row $(2, 1, 0, -1, -1, -1, 0, 1)$ is contained in the centralizer of $G$ and has infinite order. Since $S$ is symmetric and $Sv = v$, we have $v^tS^kb_1 = v^tb_1 = 1$ for all $k \in \mathbb{Z}$ and thus the vectors $S^kb_1 + mv$ are all different for different pairs $(k, m) \in \mathbb{Z}^2$. And since we also have $S\mathbf{1} = \mathbf{1}$, where $\mathbf{1} = (1, 1, \ldots, 1)^t$ spans the fixed space, we also see that different vectors of the form $S^kb_1 + mv$ can not be translation equivalent. Finally, one can show that the subgroup generated by $S$ has finite index in the normalizer.
\[ N_{GL(8, \mathbb{Z})}(C_8) \]. Thus at most finitely many of the points \( b_1 + mv \) can be normalizer equivalent to each other.

It is sometimes easier to work with the centralizer \( C_{GL(\Lambda)}(G) \) instead of the normalizer \( N_{GL(\Lambda)}(G) \), which yields a slightly finer equivalence relation. By the following simple observation, the \( C_{GL(\Lambda)}(G) \)-equivalence classes can not be much smaller than the \( N_{GL(\Lambda)}(G) \)-equivalence classes:

2.8. Lemma. \( |N_{GL(\Lambda)}(G) : C_{GL(\Lambda)}(G)| \) is finite.

Proof. The factor group \( N_{GL(\Lambda)}(G)/C_{GL(\Lambda)}(G) \) is isomorphic to a subgroup of \( \text{Aut}(G) \) [21, Corollary X.19], and \( \text{Aut}(G) \) is finite, since \( G \) itself is finite by assumption. \( \square \)

3. Preliminaries on Orders

In this section, we collect some simple properties of orders in semisimple algebras over \( \mathbb{Q} \). Orders are relevant for us since the centralizer \( C_{GL(\Lambda)}(G) \) can be identified with the unit group of such an order, as we explain below.

Recall the following definition [35]: Let \( A \) be a finite-dimensional algebra over \( \mathbb{Q} \) (associative, with one). An order (or \( \mathbb{Z} \)-order) in \( A \) is a subring \( R \subset A \) which is finitely generated as a \( \mathbb{Z} \)-module and contains a \( \mathbb{Q} \)-basis of \( A \). (Here, “subring” means in particular that \( R \) and \( A \) have the same multiplicative identity.) In other words, an order is a full \( \mathbb{Z} \)-lattice in \( A \) which is at the same time a subring of \( A \).

For the moment, assume that \( W \) is a finite-dimensional vector space over the rational numbers \( \mathbb{Q} \), and let \( \Lambda \) be a full \( \mathbb{Z} \)-lattice in \( W \), that is, the \( \mathbb{Z} \)-span of a \( \mathbb{Q} \)-basis of \( W \). Let \( A := \text{End}_{\mathbb{Q}G}(W) \) be the ring of \( \mathbb{Q}G \)-module endomorphisms of \( W \), that is, the set of linear maps \( \alpha : W \rightarrow W \) such that \( \alpha(gv) = g\alpha(v) \) for all \( v \in W \) and \( g \in G \). This is just the centralizer of \( G \) in the ring of all \( \mathbb{Q} \)-linear endomorphisms of \( W \).

We claim that

\[ R := \{ \alpha \in A \mid \alpha(\Lambda) \subseteq \Lambda \} \]

is an order in \( A \). Namely, choose a \( \mathbb{Z} \)-basis of \( \Lambda \). This basis is also a \( \mathbb{Q} \)-basis of \( W \). By identifying linear maps with matrices with respect to the chosen basis, \( A \) gets identified with the centralizer of \( G \) in the set of all \( d \times d \) matrices over \( \mathbb{Q} \), and \( R \) gets identified with the centralizer of \( G \) in the set of \( d \times d \) matrices with entries in \( \mathbb{Z} \). It follows that \( R \) is finitely generated as a \( \mathbb{Z} \)-module, and for every \( \alpha \in A \) there is an \( m \in \mathbb{Z} \) such that \( m\alpha \in R \). Thus \( R \) is an order of \( A \). (Also, \( R \cong \text{End}_{\mathbb{Z}G}(\Lambda) \) naturally.)

Moreover, the centralizer \( C_{GL(\Lambda)}(G) \) is exactly the set of invertible elements of \( R \), that is, the unit group \( U(R) \) of \( R \). For this reason, it is somewhat easier to work with \( C_{GL(\Lambda)}(G) \) instead of the normalizer \( N_{GL(\Lambda)}(G) \). The unit group \( U(R) \)
of an order $R$ is a finitely generated (even finitely presented) group [23, Section 3]. Finding explicit generators of $\mathbf{U}(R)$ (and relations between them) is in general a difficult task, but there do exist algorithms for this purpose [3]. The situation is somewhat better when $R$ is commutative, for example when $R \cong \mathbb{Z}A$, where $A$ is a finite abelian group [9]. Moreover, it is quite easy to give generators of a subgroup of $\mathbf{U}(\mathbb{Z}A)$ which has finite index in $\mathbf{U}(\mathbb{Z}A)$ [19, 28].

We now collect some general elementary facts about orders that we need. (For a comprehensive treatment of orders (not only over $\mathbb{Z}$), we refer the reader to Reiner’s book on maximal orders [35]. For unit groups of orders, see the survey article by Kleinert [23].)

3.1. **Lemma.** Let $R_1$ and $R_2$ be two orders in the $\mathbb{Q}$-algebra $A$. Then $R_1 \cap R_2$ is also an order in $A$.

**Proof.** Clearly, $R_1 \cap R_2$ is a subring.

Since $R_2$ is finitely generated over $\mathbb{Z}$ and $\mathbb{Q}R_1 = A$, there is a non-zero integer $m \in \mathbb{Z}$ with $mR_2 \subseteq R_1$. Thus $mR_2 \subseteq R_1 \cap R_2$. Since $mR_2$ contains a $\mathbb{Q}$-basis of $A$, it follows that $R_1 \cap R_2$ contains such a basis. As a submodule of a finitely generated $\mathbb{Z}$-module, $R_1 \cap R_2$ is again finitely generated. Thus $R_1 \cap R_2$ is an order of $A$. \[\square\]

3.2. **Lemma.** Let $R_1$ and $R_2$ be orders in the $\mathbb{Q}$-algebra $A$ with $R_1 \subseteq R_2$. Then $|\mathbf{U}(R_2) : \mathbf{U}(R_1)|$ is finite.

**Proof.** There exists a non-zero integer $m$ such that $mR_2 \subseteq R_1$. Suppose that $u$, $v \in \mathbf{U}(R_2)$ are such that $u - v \in mR_2$. Then $u \in v + mR_2$ and thus $uv^{-1} \in 1 + mR_2 \subseteq R_1$. Similarly, $vu^{-1} \in 1 + mR_2 \subseteq R_1$. Thus $uv^{-1} \in \mathbf{U}(R_1)$. This shows $|\mathbf{U}(R_2) : \mathbf{U}(R_1)| \leq |R_2 : mR_2| < \infty$, as claimed. \[\square\]

3.3. **Corollary.** Let $R_1$ and $R_2$ be two orders in the $\mathbb{Q}$-algebra $A$. Then $\mathbf{U}(R_1)$ is finite if and only if $\mathbf{U}(R_2)$ is finite.

**Proof.** By Lemma 3.1, $R_1 \cap R_2$ is an order. By Lemma 3.2, the index $|\mathbf{U}(R_i) : \mathbf{U}(R_1 \cap R_2)|$ is finite for $i = 1, 2$. The result follows. \[\square\]

4. **Finiteness of equivalence classes**

In this section we determine for which groups $G$ the normalizer equivalence classes are finite or not. We use the notation introduced in Section 2. Thus $G$ is a finite group acting on the finite-dimensional, real vector space $V$, and $\Lambda \subseteq V$ is a full $\mathbb{Z}$-lattice in $V$ which is stabilized by $G$. A subspace $U \subseteq V$ is called $\Lambda$-rational if $U \cap \Lambda$ contains a basis of $U$, and $\Lambda$-irrational if $U \cap \Lambda = \{0\}$. If $U$ is an irreducible $\mathbb{R}G$-submodule, then $U$ is either $\Lambda$-rational or $\Lambda$-irrational.

4.1. **Theorem.** Let

$$V = U_1 \oplus \cdots \oplus U_r$$


be a decomposition of $V$ into irreducible $RG$-subspaces. Then $N_{GL(A)}(G)$ has finite order if and only if all the $U_i$'s are $\Lambda$-rational and pairwise non-isomorphic.

The proof of Theorem 4.1 involves some non-trivial representation and number theory. By Lemma 2.8, the normalizer $N_{GL(A)}(G)$ is finite if and only if all the $U_i$'s are $\Lambda$-rational and pairwise non-isomorphic. For this reason, it is more convenient to work with the $Q$-vector space $W := Q\Lambda$. (We get back our $V$ from $W$ by scalar extension, that is, $V \cong R \otimes_Q W$.)

Fix a decomposition of $W = \bigoplus_{i=1}^r m_i S_i$, where we assume that $S_i \not\sim S_j$ for $i \neq j$. Set $D_i := \text{End}_{QG}(S_i)$, which is by Schur’s lemma [25, (3.6)] a division ring, and finite-dimensional over $Q$.

4.2. Lemma. With the above notation, we have

$$\text{End}_{QG}(W) \cong \bigotimes_{i=1}^r M_{m_i}(D_i),$$

where $M_{m}(D)$ denotes the ring of $m \times m$ matrices with entries in $D$. If $R_i$ is an order in $D_i$ for each $i$, then

$$R := \bigotimes_{i=1}^r M_{m_i}(R_i)$$

is an order in $\text{End}_{QG}(W)$.

Proof. The first assertion is a standard observation, used, for example, in one proof of the Wedderburn-Artin structure theorem for semisimple rings [25, Thm. 3.5 and proof]. The assertion on orders is then easy.

In particular, the group of units of $R$ is then isomorphic to the direct product of groups of the form $GL(m_i, R_i)$. To prove Theorem 4.1, in view of Corollary 3.3, it suffices to determine when all these groups are finite. The following is a first step toward the proof of the theorem:

4.3. Corollary. If some $m_i > 1$, then $U(R)$ (and thus $N_{GL(A)}(G)$) is infinite.

Proof. $U(R)$ contains a subgroup isomorphic to $GL(m_i, R_i)$, which contains the group $GL(m_i, \mathbb{Z})$. This group is infinite if $m_i > 1$.

To continue with the proof of Theorem 4.1, we have to look at the units of an order $R_i$ in $D_i$. We will need extension of scalars for algebras over a field via tensor products, as explained in [10, Chapter 3]. Thus for a $Q$-algebra $A$, we get an $R$-algebra denoted by $R \otimes_Q A$. We use the following theorem of Käte Hey which can be seen as a generalization of Dirichlet’s unit theorem:
4.4. Theorem. [23, Theorem 1] Let $D$ be a finite-dimensional division algebra over $\mathbb{Q}$, and let $R$ be an order of $D$ with unit group $U(R)$. Set

$$S = \{ d \in \mathbb{R} \otimes \mathbb{Q} D \mid (\det d)^2 = 1 \}.$$ 

Then $S/ U(R)$ is compact. (Here $\det d$ refers to the action of $d$ as linear operator on $\mathbb{R} \otimes \mathbb{Q} D$. One can also use the reduced norm, of course.)

From this, we can derive the following result (probably well known):

4.5. Lemma. Let $D$ be a finite-dimensional division algebra over $\mathbb{Q}$ and $R$ an order of $D$. Then $|U(R)| < \infty$ if and only if $\mathbb{R} \otimes \mathbb{Q} D$ is a division ring.

Proof. Suppose $D_R := \mathbb{R} \otimes \mathbb{Q} D$ is a division ring. By Frobenius’s theorem [10, Theorem 3.20], we have $D_R \cong \mathbb{R}, \mathbb{C},$ or $\mathbb{H}$. In each case, one checks that the set $S$ defined in Theorem 4.4 is finite. Thus the discrete group $U(R) \subseteq S$ must be finite. (Notice that we did not use Theorem 4.4 here—only that $U(R) \subseteq S$.)

Conversely, suppose that $D_R$ is not a division ring. Then there is some non-trivial idempotent $e \in D_R$, that is, $e^2 = e$, but $e \neq 0, 1$. (This follows since $D_R$ is semisimple.) Set $f = 1 - e$. Then for $\lambda, \mu \in \mathbb{R}$, we have $\det(\lambda e + \mu f) = \lambda^{k_1} \mu^{k_2}$ with $k_1 = \dim(D_R e)$ and $k_2 = \dim(D_R f)$. In particular, for every $\lambda \neq 0$ there is some $\mu$ such that $\lambda e + \mu f \in S$. This means that $S$ is unbounded, and thus not compact. It follows from Theorem 4.4 that $U(R)$ can not be finite. \hfill $\Box$

Proof of Theorem 4.1. First, assume that we are given a decomposition $V = U_1 \oplus \cdots \oplus U_r$ as in the theorem. Then $S_i := U_i \cap \mathbb{Q} \Lambda$ contains a basis of $U_i$ and thus is non-zero and necessarily simple as a $\mathbb{Q} G$-module. Thus

$$W = V \cap \mathbb{Q} \Lambda = S_1 \oplus \cdots \oplus S_r$$

is a decomposition of $W$ into simple $\mathbb{Q} G$-modules, which are pairwise non-isomorphic. It follows that

$$\text{End}_{\mathbb{Q}}(W) \cong D_1 \times \cdots \times D_r,$$

where $D_i = \text{End}_{\mathbb{Q} G}(S_i)$. Since $\mathbb{R} \otimes \mathbb{Q} D_i \cong \text{End}_{\mathbb{R} G}(U_i)$ is a division ring, too, it follows that the orders of each $D_i$ have a finite unit group, by Lemma 4.5. Thus $\mathbb{C}_{GL(\Lambda)}(G)$ is finite.

Conversely, assume that $N_{GL(\Lambda)}(G)$ is finite. It follows from Corollary 4.3 that $m_i = 1$ for all $i$ (in the notation introduced before Lemma 4.2). Thus $W$ has a decomposition into simple summands which are pairwise non-isomorphic:

$$W = S_1 \oplus \cdots \oplus S_r.$$ 

Let $D_i = \text{End}_{\mathbb{R} G}(S_i)$. Then Lemma 4.5 yields that $\mathbb{R} \otimes \mathbb{Q} D_i$ is a division ring, too. Since $\mathbb{R} \otimes \mathbb{Q} D_i \cong \text{End}_{\mathbb{R} G}(\mathbb{R} S_i)$, it follows that $U_i := \mathbb{R} S_i$ is simple. (Otherwise, the projection to a nontrivial invariant submodule would be a zero-divisor in $\text{End}_{\mathbb{R} G}(U_i)$.) For $i \neq j$, we have $U_i \nRightarrow U_j$ by the Noether-Deuring theorem [25, Theorem 19.25]. Thus $V$ has a decomposition $V = U_1 \oplus \cdots \oplus U_r$ as required. \hfill $\Box$
4.6. **Remark.** Let \( z \in V \) be an element such that the orbit \( Gz \) linearly spans \( V \). Then the normalizer equivalence class of \( z \) contains infinitely many translation equivalence classes if (and only if) \( N_{\text{GL}(\Lambda)}(G) \) has infinite order.

**Proof.** The “only if” part is clear, so assume that \( N_{\text{GL}(\Lambda)}(G) \) has infinite order. By Remark 2.5, it suffices to show that the orbit \( N_{\text{GL}(\Lambda)}(G)z \) has infinite size. By Lemma 2.8, the centralizer \( C_{\text{GL}(\Lambda)}(G) \) has also infinite order. If \( cz = z \) for \( c \in C_{\text{GL}(\Lambda)}(G) \), then \( cgz = gcz = gz \) for all \( g \in G \) and thus \( c = 1 \). Thus

\[
\infty = |C_{\text{GL}(\Lambda)}(G)| = |C_{\text{GL}(\Lambda)}(G)z| \leq |N_{\text{GL}(\Lambda)}(G)z|.
\]

\[\square\]

So when \( N_{\text{GL}(\Lambda)}(G) \) is infinite, only elements contained in proper invariant subspaces can have finite orbits under the normalizer. (Notice that the linear span of an orbit \( Gz \) is always a \( G \)-invariant subspace of \( V \).) If \( G \) is a transitive permutation group acting on the coordinates, then there are always points \( z \) such that the orbit \( Gz \) spans the ambient space—for example, \( z = (1, 0, \ldots, 0)^t \).

When \( V \) has an irrational invariant subspace, then \( N_{\text{GL}(\Lambda)}(G) \) is infinite, by Theorem 4.1. Thus if \( z \) is a core point for \( G \) such that its orbit spans the ambient space, then there are infinitely many core points, even up to translation. This was first proved by Rehn [34, 18] for permutation groups.

Another consequence of Theorem 4.1 and the remark above is that there are infinitely many core points for transitive permutations groups \( G \) acting on \( V = \mathbb{R}^d \) such that \( V \) is not multiplicity-free (as an \( \mathbb{R}G \)-module).

4.7. **Example.** Consider the regular representation of a group \( G \), that is, \( G \) acts on \( \mathbb{Q}G \) by left multiplication, so it permutes the canonical basis \( G \). As a lattice, we choose the group ring \( \mathbb{Z}G \), the vectors with integer coordinates. Then \( \text{End}_{\mathbb{Z}G}(\mathbb{Z}G) \cong \mathbb{Z}G \).

Units of group rings are a much studied problem. A theorem of Higman says that \( U(\mathbb{Z}G) \) is finite if and only if \( G \) is abelian of exponent 1, 2, 3, 4 or 6, or \( G \cong Q_8 \times E \) with \( E^2 = \{1\} \). This can also be derived from Theorem 4.1.

In Example 2.7, we described some core points in the cases \( G = C_4 \) and \( C_8 \). In the case of \( C_8 \), the decomposition of \( \mathbb{Q}C_8 \) into simple modules is given by

\[
\mathbb{Q}C_8 \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}[\varepsilon] \oplus \mathbb{Q}[e^{2\pi i/8}].
\]

Over \( \mathbb{R} \), the last summand decomposes into two invariant, irrational subspaces of dimension 2. The normalizer of \( C_8 \) is infinite because of this last summand. Of course, any \( z \) contained in the sum of the first three summands has only a finite orbit under the normalizer, for example \( z = (1, 0, 0, 0, 1, 0, 0, 0)^t \).

When \( p \) is prime and \( p \geq 5 \), then \( U(\mathbb{Z}C_p) \) is infinite, but there are only finitely many core points up to normalizer equivalence in \( \mathbb{Z}C_p \), by Theorem 5.1 below.
5. Rationally irreducible

Suppose that $\Lambda = \mathbb{Z}^d$, and assume that $G$ acts on $\mathbb{R}^d$ by matrices in $\text{GL}(d, \mathbb{Z})$. A subspace $U \subseteq \mathbb{R}^d$ is called irrational if $U \cap \mathbb{Q}^d = \{0\}$ and rational if $U$ has a basis contained in $\mathbb{Q}^d$. If $U$ is an irreducible $\mathbb{R}G$-submodule, then $U$ is either rational or irrational.

In this section, we consider permutation groups acting on $\mathbb{R}^d$ by permuting coordinates. (We conjecture that a version of the main result remains true more generally for finite matrix groups $G \subseteq \text{GL}(d, \mathbb{Z})$, but we are not able to prove it yet. One problem is that we can not extend Lemma 5.2 below to this more general setting.)

Since permutation matrices are orthogonal, it follows that the orthogonal complement $U^\perp$ of any $G$-invariant subspace is itself $G$-invariant. Following Dixon [8], we call a transitive permutation group $G$ a QI-group, when $\text{Fix}(G)^\perp$ does not contain any rational $G$-invariant subspace other than $\{0\}$ and $\text{Fix}(G)^\perp$ itself. Notice that $\text{Fix}(G)^\perp$ contains no non-trivial rational invariant subspaces if and only if $\text{Fix}(G)^\perp \cap \mathbb{Q}^d$ contains no proper $G$-invariant subspace other than $\{0\}$. In algebraic language, this means that $\text{Fix}(G)^\perp \cap \mathbb{Q}^d$ is a simple module over $\mathbb{Q}G$.

Let us emphasize that by definition, QI-groups are transitive. Thus the fixed space $\text{Fix}(G)$ is generated by the all ones vector $(1, 1, \ldots, 1)^t$, and so $\dim \text{Fix}(G) = 1$.

5.1. Theorem. Let $G \leq S_d$ be a QI-group. Then there is a constant $M$ depending only on the group $G$ such that every core point is normalizer equivalent to a core point $w$ with $\|w\|^2 \leq M$. In particular, there are only finitely many core points for $G$ up to normalizer equivalence.

We divide the proof of Theorem 5.1 into a number of lemmas. The idea is the following: We show that for any vector $z \in \mathbb{Z}^d$ there is some $c \in C_{\text{GL}(d, \mathbb{Z})}(G)$ such that the projections of $cz$ to the different irreducible real subspaces of $\text{Fix}(G)^\perp$ have approximately the same norm. (At the same time, this point $cz$ is one with minimal norm in the orbit $C_{\text{GL}(\Lambda)}(G)z$.) When $z$ is a core point, at least one of these norms must be “small” by a fundamental result of Herr, Rehn, and Schürmann [18, Theorem 9] (Theorem 5.8 below).

We begin with a short reminder of some character theory. The facts we need can be found in any basic text on representations of finite groups, for example Serre’s text [37]. Saying that a group $G$ acts linearly on a (finite-dimensional) vector space $V$ over some field $K$ is equivalent to having a representation $R: G \to \text{GL}(V)$ (or even $R: G \to \text{GL}(d, K)$ when $V = K^d$). The character $\chi$ of $R$ (or $V$) is the function defined by $\chi(g) = \text{tr}(R(g))$. An irreducible character is the trace of an irreducible representation $R: G \to \text{GL}(d, \mathbb{C})$ over the field of complex numbers $\mathbb{C}$. The set of irreducible characters of the group $G$ (over the complex numbers) is denoted by $\text{Irr}(G)$. For finite groups $G$, this is a finite set. Indeed, by the orthogonality relations,
the set \( \text{Irr}(G) \) is orthonormal with respect to a certain inner product on the space of all functions \( G \to \mathbb{C} \) [37, Section 2.3, Thm. 3].

Every character of a finite group can be written uniquely as a nonnegative integer linear combination of irreducible characters. This corresponds to the fact that for each representation \( G \to \text{GL}(V) \) on some vector space \( V \) over \( \mathbb{C} \), we can write \( V \) as a direct sum of irreducible, \( G \)-invariant subspaces [37, §1.4, Thm. 2, §2.3, Thm. 4].

Suppose \( \chi \) is the character of some representation \( R \) of the finite group \( G \). Then the eigenvalues of \( R(g) \), where \( g \in G \), must be \(|G|\)th roots of unity. Thus the values of \( \chi \) are contained in the field generated by the \(|G|\)th roots of unity. We write \( \mathbb{Q}(\chi) \) for the field generated by all values of \( \chi \). It follows that \( \mathbb{Q}(\chi) \) is a finite Galois extension of \( \mathbb{Q} \), with abelian Galois group \( \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}) \).

The following lemma appears in Dixon’s paper [8, Lemma 6(b)].

5.2. Lemma (Dixon [8]). Let \( G \) be a QI-group and let \( \pi \) be the character of the corresponding permutation representation of \( G \). Let \( \chi \in \text{Irr} G \) be an irreducible constituent of \( \pi^{-1} \) (the character of \( G \) on \( \text{Fix}(G)^{\perp} \)). Then

\[
\pi = 1 + \sum_{\alpha \in \Gamma} \chi^\alpha, \quad \text{where} \quad \Gamma = \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}).
\]

For the moment, we work with the complex space \( \mathbb{C}^d \), on which \( G \) acts by permuting coordinates. Recall that to each \( \chi \in \text{Irr} G \) there corresponds a central primitive idempotent of the group algebra \( \mathbb{C}G \), namely

\[
e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g \in Z(\mathbb{C}G).
\]

If \( V \) is any \( \mathbb{C}G \)-module, then \( e_\chi \) acts on \( V \) as the projection onto its \( \chi \)-homogeneous component. So the image \( e_\chi(V) \) coincides with the set \( \{v \in V \mid e_\chi v = v\} \), and the character of \( e_\chi(V) \) is an integer multiple of \( \chi \) [37, §2.6]. In the present situation, it follows from Lemma 5.2 that

\[
U := e_\chi(\mathbb{C}^d) = \{v \in \mathbb{C}^d \mid e_\chi v = v\}
\]
is itself an irreducible module affording the character \( \chi \). The projection \( e_\chi \) maps the standard basis of \( \mathbb{C}^d \) to vectors contained in \( K^d \), where \( K := \mathbb{Q}(\chi) \). Thus \( U \) has a basis contained in \( K^d \). (This means that the representation corresponding to the linear action of \( G \) on \( U \) can be described by matrices with all entries in \( K \). Thus \( \chi \) is the character of a representation where all matrices have entries in \( K = \mathbb{Q}(\chi) \).)

Another consequence of Lemma 5.2 is that we have the decomposition

\[
\mathbb{C}^d = \text{Fix}(G) \oplus \bigoplus_{\gamma \in \Gamma} U^\gamma.
\]

Here \( U^\gamma \) means this: Since \( U \) has a basis in \( \mathbb{Q}(\chi)^d \), we can apply \( \gamma \) to the coordinates of the vectors in such a basis. The linear span of the result is denoted by \( U^\gamma \). This is independent of the chosen basis.
5.3. Lemma. Set $A := \mathbf{C}_{\mathbf{M}_d(\mathbb{Q})}(G) = \{a \in \mathbf{M}_d(\mathbb{Q}) \mid \forall g \in G: ag = ga\}$, the full centralizer of $G$ in the ring of $d \times d$-matrices over $\mathbb{Q}$. There is an algebra homomorphism $\lambda: A \to \mathbb{Q}(\chi)$ such that each $a \in A$ acts on $U^\gamma$ by multiplication with $\lambda(a)^\gamma$, and such that $\lambda(a^\gamma) = \overline{\lambda(a)}$. There is another homomorphism $m: A \to \mathbb{Q}$ such that

$$A \cong \mathbb{Q} \times \mathbb{Q}(\chi) \quad \text{via} \quad a \mapsto (m(a), \lambda(a)).$$

The isomorphism $A \cong \mathbb{Q} \times \mathbb{Q}(\chi)$ appears in Dixon’s paper [8, Lemma 6(d)] and follows from Lemma 5.2 together with general results in representation theory. But as we need the specific properties of the map $\lambda$ from the lemma, we give a detailed proof.

Proof of Lemma 5.3. Suppose the matrix $a$ centralizes $G$, and let $\lambda(a) \in \mathbb{C}$ be an eigenvalue of $a$ on $U$. The corresponding eigenspace is $G$-invariant since $a$ centralizes $G$. Since $U$ is irreducible, $U$ is contained in the eigenspace of $\lambda(a)$.

When $a \in A \subseteq \mathbf{M}_d(\mathbb{Q})$, then $a$ maps $U \cap \mathbb{Q}(\chi)^d \neq \{0\}$ to itself, and thus $\lambda(a) \in \mathbb{Q}(\chi)$. This defines the algebra homomorphism $\lambda: A \to \mathbb{Q}(\chi)$.

When $u \in U \cap \mathbb{Q}(\chi)^d$, $\gamma \in \Gamma$, and $a \in A$, then $au^\gamma = (au)^\gamma = \lambda(a)^\gamma u^\gamma$. Thus $a$ acts as $\lambda(a)^\gamma$ on $U^\gamma$.

Each $a \in A$ acts also on the one-dimensional fixed space by multiplication with some $m(a) \in \mathbb{Q}$. As

$$\mathbb{C}^d = \text{Fix}(G) \oplus \bigoplus_{\gamma \in \Gamma} U^\gamma,$$

we see that the space $\mathbb{C}^d$ has a basis of common eigenvectors for all $a \in A$. With respect to this basis, each $a$ is a diagonal matrix, where $m(a)$ appears once and $\lambda(a)^\gamma$ appears $\chi(1)$-times for each $\gamma \in \Gamma$. In particular, the map $A \ni a \mapsto (m(a), \lambda(a))$ is injective.

Since $G$ acts orthogonally with respect to the standard inner product on $\mathbb{C}^d$, the above decomposition into irreducible subspaces is orthogonal and we can find an orthonormal basis of common eigenvectors of all $a \in A$. From this, it is clear that $\lambda(a^\gamma) = \lambda(a)^\gamma = \overline{\lambda(a)}$.

To see that $a \mapsto (m(a), \lambda(a))$ is onto, let $(q, \mu) \in \mathbb{Q} \times \mathbb{Q}(\chi)$. Define

$$\varphi(q, \mu) := q e_1 + \sum_{\gamma \in \Gamma} (\mu e_\chi)^\gamma$$

$$= q \frac{1}{|G|} \sum_{g \in G} q + \frac{\chi(1)}{|G|} \sum_{g \in G} \left( \sum_{\gamma \in \Gamma} (\mu e_{\chi^{-1}})^\gamma \right) g$$

$$\in \mathbb{Z}(\mathbb{Q}G).$$

Then the corresponding map $v \mapsto \varphi(q, \mu)v$ is in $A$, and from $\varphi(q, \mu)e_1 = q e_1$ and $\varphi(q, \mu)e_\chi = \mu e_\chi$ we see that $m(\varphi(q, \mu)) = q$ and $\lambda(\varphi(q, \mu)) = \mu$. This finishes the proof that $A \cong \mathbb{Q} \times \mathbb{Q}(\chi)$.

$\square$
5.4. **Lemma.** Set \( W := (U + U) \cap \mathbb{R}^d \). Then the decomposition of \( \mathbb{R}^d \) into irreducible \( \mathbb{R}G \)-modules is given by
\[
\mathbb{R}^d = \text{Fix}(G) \oplus \bigoplus_{\alpha \in \Gamma_0} W^\alpha, \quad \Gamma_0 = \text{Gal}(\mathbb{Q}(\chi) \cap \mathbb{R})/\mathbb{Q}).
\]
(In particular, \( W \) is irreducible as an \( \mathbb{R}G \)-module.) For \( w \in W^\alpha \) and \( a \in A \), we have
\[
\|aw\|^2 = \left(\lambda(a)\lambda(a)\right)\alpha |w|^2.
\]
**Proof.** When \( \mathbb{Q}(\chi) \subseteq \mathbb{R} \), then \( U = U \) and \( W = U \cap \mathbb{R}^d \). The result is clear in this case.
Otherwise, we have \( U \cap \mathbb{R}^d = \{0\} \) and \( U \cap U = \{0\} \), and so \( W = (U \oplus U) \cap \mathbb{R}^d \neq \{0\} \), and thus again \( W \) is simple over \( \mathbb{R}G \).

The extension \( \mathbb{Q}(\chi)/\mathbb{Q} \) has an abelian Galois group, and thus \( \mathbb{Q}(\chi) \cap \mathbb{R} \) is also Galois over \( \mathbb{Q} \). The Galois group \( \Gamma_0 \) is isomorphic to the factor group \( \Gamma / \{1d, \kappa\} \), where \( \kappa \) denotes complex conjugation. Suppose \( \alpha \in \Gamma_0 \) is the restriction of \( \gamma \in \Gamma \) to \( \mathbb{Q}(\chi) \cap \mathbb{R} \). Then
\[
W^\alpha = \left((U + U) \cap \mathbb{R}^d\right)^\alpha = (U^\gamma + \overline{U}^\gamma) \cap \mathbb{R}^d = (U^\gamma + U^{\kappa\gamma}) \cap \mathbb{R}^d.
\]
The statement about the decomposition follows.

The last statement is immediate from Lemma 5.3. \( \square \)

5.5. **Lemma.** Let \( C := C_{\text{GL}(d,\mathbb{Z})}(G) \), and define
\[
L: C \rightarrow \mathbb{R}^{\Gamma_0}, \quad L(c) := \left(\log(|\lambda(c)|^\alpha)\right)_{\alpha \in \Gamma_0},
\]
Then the image \( L(C) \) of \( C \) under this map is a full lattice in the hyperplane
\[
H = \left\{ (x_\alpha)_{\alpha \in \Gamma_0} \mid \sum_{\alpha \in \Gamma_0} x_\alpha = 0 \right\}.
\]
We will derive this lemma from the following version of Dirichlet’s unit theorem [31, Satz 1.7.3]:

5.6. **Lemma.** Let \( K \) be a finite field extension over \( \mathbb{Q} \), let \( \alpha_1, \ldots, \alpha_r: K \rightarrow \mathbb{R} \) be the different real field embeddings of \( K \), and let \( \beta_1, \beta_2, \ldots, \beta_s, \beta_s: K \rightarrow \mathbb{C} \) be the different complex embeddings of \( K \), whose image is not contained in \( \mathbb{R} \). Let \( O_K \) be the ring of algebraic integers in \( K \) and \( l: K^* \rightarrow \mathbb{R}^{r+s} \) the map
\[
z \mapsto l(z) = (\log|z^{\alpha_1}|, \ldots, \log|z^{\alpha_r}|, \log|z^{\beta_1}|, \ldots, \log|z^{\beta_s}|) \in \mathbb{R}^{r+s}.
\]
Then the image \( l(U(O_K)) \) of the unit group of \( O_K \) under \( l \) is a full lattice in the hyperplane
\[
H = \left\{ x \in \mathbb{R}^{r+s} \mid \sum_{i=1}^{r+s} x_i = 0 \right\}.
\]
In the proof of Lemma 5.5, we will apply this result to \( K = \mathbb{Q}(\chi) \). Set \( F = K \cap \mathbb{R} \), \( \Gamma_0 = \text{Gal}(F/\mathbb{Q}) \) and \( \Gamma = \text{Gal}(K/\mathbb{Q}) \). If \( F = K \subseteq \mathbb{R} \), then \( r = |K : \mathbb{Q}| \) and \( s = 0 \). In this case, \( \{\alpha_1, \ldots, \alpha_r\} = \Gamma = \Gamma_0 \). If \( K \nsubseteq \mathbb{R} \), then \( |K : F| = 2 \), \( r = 0 \), and \( s = |F : \mathbb{Q}| \). In this case, we may identify the set \( \{\beta_1, \ldots, \beta_s\} \) with the Galois group \( \Gamma_0 \); for each \( \alpha \in \Gamma_0 \), there are two extensions of \( \alpha \) to the field \( K \), and these are complex conjugates of each other. Thus we get a set \( \{\beta_1, \ldots, \beta_s\} \) as in Lemma 5.6 by choosing exactly one extension for each \( \alpha \in \Gamma \). The map \( l \) is independent of this choice anyway.

It follows that in both cases, we may rewrite the map \( l \) (somewhat imprecisely) as

\[
\ell(z) = \left( \log|z^\alpha| \right)_{\alpha \in \Gamma_0}.
\]

**Proof of Lemma 5.5.** First notice that the entries of \( L(c) \) can be written as

\[
\log \left( \frac{\alpha(c) \lambda(c)}{\lambda(c)^\alpha} \right)^\alpha = \log \left( \frac{\alpha(c)^\alpha \lambda(c)^\alpha}{\lambda(c)^{\alpha^2}} \right) = \log|\lambda(c)^\alpha|^2 = 2 \log|\lambda(c)^\alpha|,
\]

where we tacitly replaced \( \alpha \) by an extension to \( \mathbb{Q}(\chi) \) when \( \mathbb{Q}(\chi) \nsubseteq \mathbb{R} \). Thus \( L(c) = 2l(\lambda(c)) \) for all \( c \in C \), with \( l \) as in Lemma 5.6.

In view of Lemma 5.6, it remains to show that the group \( \lambda(C) \) has finite index in \( U(O_K) \). We know that \( C \) is the group of units in \( C_{M_d(\mathbb{Z})}(G) \cong \text{End}_{\mathbb{Z}^d}(\mathbb{Z}^d) \), which is an order in \( A \cong \mathbb{Q} \times K \). Another order in \( \mathbb{Q} \times K \) (in fact, the unique maximal order) is \( \mathbb{Z} \times O_K \) with unit group \( \{\pm1\} \times U(O_K) \). By Lemma 3.2, it follows that \( C \) has finite index in \( \{\pm1\} \times U(O_K) \). Thus \( \lambda(C) \) has finite index in \( U(O_K) \) and the result follows.

For each \( v \in \mathbb{R}^d \), let \( v_\alpha \) be the orthogonal projection of \( v \) onto the simple subspace \( W^\alpha \).

**5.7. Lemma.** There is a constant \( D \), depending only on the group \( G \), such that for every \( v \in \mathbb{R}^d \) with \( v_\alpha \neq 0 \) for all \( \alpha \in \Gamma_0 \), there is a \( c \in C \) with

\[
\frac{\|c(c_\alpha)\|^2}{\|c(c_\beta)\|^2} \leq D
\]

for all \( \alpha, \beta \in \Gamma_0 \).

As \( \text{Fix}(G)^\perp \cap \mathbb{Q}^d \) is a simple module, the assumption \( v_\alpha \neq 0 \) for all \( \alpha \) holds in particular for all \( v \in \mathbb{Q}^d \setminus \text{Fix}(G) \).

**Proof of Lemma 5.7.** By Lemma 5.5, there is a compact set \( T \),

\[
T \subset H = \left\{(x_\alpha) \in \mathbb{R}^{\Gamma_0} \mid \sum_\alpha x_\alpha = 0 \right\},
\]

such that \( H = T + L(C) \). (For example, we can choose \( T \) as a fundamental parallelepiped of the full lattice \( L(C) \) in \( H \).)

For \( v \in \mathbb{R}^d \) as in the statement of the lemma, define

\[
N(v) = \left( \log\|v_\alpha\|^2 \right)_\alpha \in \mathbb{R}^{\Gamma_0}.
\]
Let \( S \in \mathbb{R}^{\Gamma_0} \) be the vector having all entries equal to
\[
s := \frac{1}{|\Gamma_0|} \sum_{\alpha} \log \| v_\alpha \|^2.
\]
This \( s \) is chosen such that \( N(v) - S \in H \). Thus there is \( c \in C \) such that \( L(c) + N(v) - S = t = (t_\alpha) \).

As
\[
\| (cv)_\alpha \|^2 = \| cv_\alpha \|^2 = (\lambda(c) \lambda(c))^{\alpha} \| v_\alpha \|^2,
\]
it follows that
\[
N(cv) = L(c) + N(v)
\]
in general. Thus
\[
\log \| cv_\alpha \|^2 - \log \| cv_\beta \|^2 = (\log \| cv_\alpha \|^2 - s) - (\log \| cv_\beta \|^2 - s)
\]
\[
= (N(cv) - S)_\alpha - (N(cv) - S)_\beta
\]
\[
= t_\alpha - t_\beta
\]
\[
\leq \max_{\alpha,t} t_\alpha - \min_{\beta,t} t_\beta =: D_0.
\]

This maximum and minimum exist since \( T \) is compact. The number \( D_0 \) may depend on the choice of the set \( T \), but not on \( v \) or \( c \). Thus \( \| cv_\alpha \|^2/\| cv_\beta \|^2 \) is bounded by \( D := e^{D_0} \).

We see from the proof that we get a bound whenever we have a subgroup \( C_0 \) of \( C_{\text{GL}(d,\mathbb{Z})}(G) \) such that \( L(C_0) \) is a full lattice in the hyperplane \( H \). Of course, we do not get the optimal bound then, but in practice it may be difficult to compute the full centralizer.

We will prove Theorem 5.1 by combining the last lemma with the following fundamental result [18, Theorem 9] (which is actually true for arbitrary matrix groups [34, Theorem 3.13]).

5.8. Theorem. Let \( G \leq S_d \) be a transitive permutation group. Then there is a constant \( C \) (depending only on \( d \)) such that for each core point \( z \), there is a non-zero invariant subspace \( U \leq \text{Fix}(G)^\perp \) over \( \mathbb{R} \) such that \( \| z |_U \|^2 \leq C \).

In our situation, the \( W^\alpha \) from Lemma 5.4 are the only irreducible subspaces, and thus for every core point \( z \) there is some \( \alpha \in \Gamma_0 \) with \( \| z_\alpha \|^2 \leq C \).

Proof of Theorem 5.1. Let \( z \) be a core point with \( z \notin \text{Fix}(G) \). We want to show that there is a \( c \in C_{\text{GL}(d,\mathbb{Z})}(G) \) and a vector \( b \in \text{Fix}(G) \cap \mathbb{Z}^d \), such that \( \| cz + b \| \leq M \), where \( M \) is a constant depending only on \( G \) and not on \( z \). By Lemma 5.7, there is \( c \in C_{\text{GL}(d,\mathbb{Z})}(G) \) such that \( \| cz_\alpha \|^2 \leq D \| cz_\beta \|^2 \) for all \( \alpha, \beta \in \Gamma \), where \( D \) is some constant depending only on \( G \) and not on \( z \).

Since \( y = cz \) is also a core point (Lemma 2.4), Theorem 5.8 yields that there is a \( \beta \in \Gamma \) with \( \| y_\beta \|^2 \leq C \) (where, again, the constant \( C \) depends only on the group,
not on \( z \). It follows that the squared norms of the other projections \( y_\alpha \) are bounded by \( CD \). Thus

\[
\|y|_{\text{Fix}(G)\perp}\|^2 \leq C + (|\Gamma| - 1)CD
\]

is bounded.

Since the projection to the fixed space can be bounded by translating with some \( b \in \text{Fix}(G) \cap \mathbb{Z}^d \), the theorem follows. \( \square \)

5.9. Example. Let \( p \) be a prime, and let \( G = C_p \leq S_p \) be generated by a \( p \)-cycle acting on \( \mathbb{R}^p \) by (cyclically) permuting coordinates. Then \( G \) is a QI-group. (Of course, every transitive group of prime degree is a QI-group.) For \( p \) odd, \( \mathbb{R}^p \) decomposes into \( \text{Fix}(G) \) and \( (p - 1)/2 \) irreducible subspaces of dimension 2. Here the lattice can be identified with the group ring \( \mathbb{Z}G \), and thus \( C_{\text{GL}(\mathbb{Z}^5)}(G) \cong U(\mathbb{Z}G) \). The torsion free part of this unit group is a free abelian group of rank \( (p - 3)/2 \).

Let us see what constant we can derive for \( p = 5 \). For concreteness, let \( g = (1, 2, 3, 4, 5) \) and \( G = \langle g \rangle \). We have the decomposition

\[
\mathbb{R}^5 = \text{Fix}(G) \oplus W \oplus W'.
\]

The projections from \( \mathbb{R}^5 \) onto \( W \) and \( W' \) are given by

\[
e_W = \frac{1}{5}(2 + ag + bg^2 + bg^3 + ag^4), \quad a = \frac{-1 + \sqrt{5}}{2},
\]

\[
e_{W'} = \frac{1}{5}(2 + bg + ag^2 + ag^3 + bg^4), \quad b = \frac{-1 - \sqrt{5}}{2}.
\]

The centralizer of \( G \) has the form

\[
C_{\text{GL}(\mathbb{Z}^5)}(G) = \{\pm I\} \times G \times \langle u \rangle,
\]

where \( u \) is a unit of infinite order. Here we can choose \( u = -1 + g + g^4 \) with inverse \(-1 + g^2 + g^3 \). To \( u \) corresponds the matrix

\[
(1)
\begin{pmatrix}
-1 & 1 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & -1
\end{pmatrix}
\in \text{GL}(5, \mathbb{Z}).
\]

This unit acts on \( W \) as \(-1 + a\) and on \( W' \) as \(-1 + b\). For the constant \( D \) of Lemma 5.7, we get \( D = (b - 1)^2 = 2 - 3b = (7 + 3\sqrt{5})/2 \). For the constant \( C \) in Theorem 5.8, we get a bound \( C = 48/5 \) (from the proof). We can conclude that every core point is equivalent to one with squared norm smaller than \( M = (2/5) + (48/5)(1 + 2 - 3b) \approx 50.6 \).

We can get somewhat better bounds by applying Theorem 5.8 “layerwise”. The \( k \)-layer is, by definition, the set of all \( z \in \mathbb{Z}^d \) with \( \sum z_i = k \). In our example, every lattice point is equivalent to one in layer 1 or layer 2.
For example, it can be shown that each core point in the 1-layer is equivalent to a point \( z \) with \( \|z\|^2 \leq 31 \). However, this bound is still far from optimal. Using the computer algebra system GAP [13], we found that the only core points of \( C_5 \) in the 1-layer up to normalizer equivalence are just
\[
(1, 0, 0, 0, 0)^t, \quad (1, 1, 0, 0, -1)^t, \quad (1, 1, 1, 0, -2)^t,
(2, 1, 0, -1, -1)^t, \quad (2, 1, -2, 0, 0)^t.
\]
(The normalizer \( N_{\text{GL}(5,\mathbb{Z})}(G) \) is generated by the centralizer and the permutation matrix corresponding to the permutation \((2, 3, 5, 4)\).) For completeness, we also give a list of core points up to normalizer equivalence in the 2-layer:
\[
(1, 1, 0, 0, 0)^t, \quad (1, 1, 1, 0, -1)^t, \quad (2, 1, 0, 0, -1)^t,
(2, 1, 1, -1, -1)^t, \quad (2, 1, -2, 0)^t.
\]
Every nontrivial core point for \( C_5 \) is normalizer equivalent to exactly one of these ten core points.

For this example, an infinite series of core points of the form
\[
(f_{j+1}, 0, f_j, f_j, 0)^t,
\]
where \( f_j \) is the \( j \)th Fibonacci number, was found by Rehn [34, 5.2.2]. Each point in this series is normalizer equivalent to one of the two obvious core points \((1, 0, 0, 0, 0)^t\) and \((1, 0, 1, 1, 0)^t\). This follows from
\[
(1 - g - g^4)(f_{j+1}, 0, f_j, f_j, 0)^t = (f_{j+1}, -f_{j+2}, 0, 0, -f_{j+2})^t
\]
and thus
\[
(1 - g - g^4)(f_{j+1}, 0, f_j, f_j, 0)^t + f_{j+2}(1, 1, 1, 1, 1)^t = (f_{j+3}, 0, f_{j+2}, f_{j+2}, 0)^t.
\]

5.10. Example. Now set
\[
G = \langle (1, 2, 3, 4, 5), (1, 4)(2, 3) \rangle \cong D_5,
\]
the dihedral group of order 10. Then
\[
C_{\text{GL}(5,\mathbb{Z})}(G) = \{ \pm I \} \langle u \rangle,
\]
where \( u \) is as in the previous example. The normalizer of \( G \) is the same as that of the cyclic group \( C_5 = \langle (1, 2, 3, 4, 5) \rangle \). In particular, normalizer equivalence for \( D_5 \) and \( C_5 \) is the same equivalence relation. Of the core points from the last example, only \((1, 0, 0, 0, 0)^t\) and \((1, 1, 0, 0, 0)^t\) are also core points for \( D_5 \). (In fact, for most of the other points, we have some lattice point on an interval between two vertices—for example \((1, 0, 0, 0, 0)^t = (1/2)((1, 1, 0, 0, -1)^t + (2, 5)(3, 4)(1, 1, 0, 0, -1)^t) \). Thus there are only two core points up to normalizer equivalence in this example.

5.11. Remark. The number of core points up to normalizer equivalence seems to grow quickly for cyclic groups of prime order. For \( p = 7 \), we get 515 core points up to normalizer equivalence.
Herr, Rehn, and Schürmann [18] conjectured that a finite transitive permutation group $G$ has infinitely many core points up to translation equivalence if the group is not 2-homogeneous. This conjecture is still open but is known to be true in a number of special cases, including imprimitive permutation groups and all groups of degree $d \leq 127$.

It is known that a permutation group $G \leq S_d$ is 2-homogeneous if and only if $\text{Fix}_{S_d}(G)^\perp$ is irreducible [6, Lemma 2(iii)]. In this case, there are only finitely many core points up to translation equivalence [18, Corollary 10].

We propose the following conjecture, which is the converse of Theorem 5.1:

5.12. Conjecture. Let $G \leq S_d$ be a transitive permutation group such that $\text{Fix}(G)^\perp$ contains a rational $G$-invariant subspace other than $\{0\}$ and $\text{Fix}(G)^\perp$ itself. Then there are infinitely many core points up to normalizer equivalence.

This can be seen as a generalization of the Herr-Rehn-Schürmann conjecture, since translation equivalence refines normalizer equivalence, and since whenever $\text{Fix}(G)^\perp$ contains a nontrivial irrational $G$-invariant subspace, there are infinitely many core points up to translation equivalence by Theorem 4.1 (or [18, Theorem 32]).

6. Application to integer linear optimization

In this last section we describe a possible application of the concept of normalizer equivalence to symmetric integer linear optimization problems. For many years it has been known that symmetry often leads to difficult problem instances in integer optimization. Standard approaches like branching usually work particularly poorly when large symmetry groups are present, since a lot of equivalent subproblems have to be dealt with in such cases. Therefore, in recent years several new methods for exploiting symmetries in integer linear programming have been developed. See, for example, [29, 12, 5, 22, 27, 32, 14, 11, 20] and the surveys by Margot [30] and Pfetsch and Rehn [33] for an overview. These methods (with the exception of [11]) fall broadly into two classes: Either they modify the standard branching approach, using isomorphism tests or isomorphism free generation to avoid solving equivalent subproblems, or they use techniques to cut down the original symmetric problem to a less symmetric one, which contains at least one element of each orbit of solutions. By now, many of the leading commercial solvers, like CPLEX [7], Gurobi [15], and XPRESS [38], have included some techniques to detect and exploit special types of symmetries. Accompanying their computational survey [33], Pfetsch and Rehn also published implementations of some symmetry exploiting algorithms for SCIP [36], like isomorphism pruning and orbital branching.

Core points were introduced as an additional tool to deal with symmetries in integer convex optimization problems. Knowing the core points for a given symmetry group allows one to restrict the search for optima to this subset of the integer vectors [17]. There are many possible ways how core points could be used. For
instance, one could use the fact that core points are close to invariant subspaces, by adding additional quadratic constraints (second order cone constraints). In the case of QI-groups, hence with finitely many core points up to normalizer equivalence (Theorem 5.1), one could try to systematically run through core points satisfying the problem constraints.

In contrast to the aforementioned approaches, we here propose natural reformulations of symmetric integer optimization problems using the normalizer of the symmetry group. Recall that a general standard form of an integer linear optimization problem is

\[
\text{max } c^T x \text{ such that } Ax \leq b, \ x \in \mathbb{Z}^d,
\]

for some given matrix \( A \) and vectors \( b \) and \( c \), all of them usually rational. If \( c = 0 \), then we have a so-called feasibility problem, asking simply whether or not there is an integral solution to a given system of linear inequalities. Geometrically, we are asking whether some polyhedral set (a polytope, if bounded) contains an integral point.

A group \( G \subseteq \text{GL}(d, \mathbb{Z}) \) is called a group of symmetries of problem (2) if the constraints \( Ax \leq b \) and the linear objective function \( c^T x \) are invariant under the action of \( G \) on \( \mathbb{R}^d \), that is, if \( c^T (gx) = c^T x \) and \( A(gx) \leq b \) for all \( g \in G \) whenever \( Ax \leq b \). The first condition is, for instance, satisfied if \( c \) is in the fixed space \( \text{Fix}(G) \).

Practically, computing a group of symmetries for a given problem is usually reduced to the problem of finding symmetries of a suitable colored graph \([4, 33]\). Quite often in optimization, attention is restricted to groups \( G \subseteq S_d \) acting on \( \mathbb{R}^d \) by permuting coordinates.

Generally, a linear reformulation of a problem as in (2) can be obtained by an integral linear substitution \( x \mapsto Sx \) for some matrix \( S \in \text{GL}(d, \mathbb{Z}) \):

\[
\text{max}(c^T S)x \text{ such that } (AS)x \leq b, \ x \in \mathbb{Z}^d.
\]

(More generally, one can use integral affine substitutions \( x \mapsto Sx + t \) with \( S \in \text{GL}(d, \mathbb{Z}) \) and \( t \in \mathbb{Z}^d \). For simplicity, we assume \( t = 0 \) in the discussion to follow.) We remark that reformulations as in (3) with a matrix \( S \in \text{GL}(d, \mathbb{Z}) \) can of course be applied to any linear integer optimization problem. In fact, this is a key idea of Lenstra’s famous polynomial time algorithm in fixed dimension \( d \) \([26]\). In Lenstra’s algorithm, the transformation matrix \( S \) is chosen to correspond to a suitable LLL-reduction of the lattice, such that the transformed polyhedral set \{ \( x \in \mathbb{R}^d \mid (AS)x \leq b \} \) is sufficiently round. This idea has successfully been used for different problem classes of integer linear optimization problems (for an overview see \([1]\)). The main difficulty is the choice of an appropriate unimodular matrix \( S \) which simplifies the optimization problem.

If the symmetry group of an optimization problem contains the group \( G \), then it is natural to choose matrices \( S \) which keep the problem \( G \)-invariant. When \( S \) is an
element of the normalizer $N_{GL(d, \mathbb{Z})}(G)$, problem (2) is $G$-invariant if and only if (3) is $G$-invariant. Note also that then $(e^t)S^t$ is in $\text{Fix}(G)$.

We illustrate the idea with a small concrete problem instance of (2) which is invariant under the cyclic group $C_5$. In particular, using core points, we construct $C_5$-invariant integral optimization problems that are quite hard or even impossible to solve for state-of-the-art commercial solvers like CPLEX or Gurobi. For instance, this is often the case when the constraints $Ax \leq b$ can be satisfied by real vectors $x$, but not by integral ones.

6.1. Example. The orbit polytope $P(C_5, z)$ of some integral point $z$ has a description with linear inequalities of the form $x_1 + \cdots + x_5 = k$ and $Ax \leq b$, where $A$ is a circulant $5 \times 5$-matrix

$$A = \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 \\
a_2 & a_3 & a_4 & a_5 & a_1 \\
a_3 & a_4 & a_5 & a_1 & a_2 \\
a_4 & a_5 & a_1 & a_2 & a_3 \\
a_5 & a_1 & a_2 & a_3 & a_4
\end{pmatrix}$$

with integral entries $a_1, \ldots, a_5$, and $b \in \mathbb{Z}^5$ satisfies $b_1 = \cdots = b_5$. If $z$ is a core point and if we replace $b_i$ by $b'_i := b_i - 1$, then we get a system of inequalities having no integral solution.

Applying this construction to the core point

$$z = U^{10} \cdot (1, 1, 1, 0, -2)^t,$$

where $U$ is the matrix from (1) in Example 5.9, we get parameters

$$a_1 = 515161, \quad a_2 = 18376, \quad a_3 = -503804, \quad a_4 = -329744, \quad a_5 = 300011, \quad b'_1 = 60.$$ 

We can vary the values of $k \equiv 1 \mod 5$ (geometrically, this corresponds to translating the polytope by some integral multiple of the all-ones vector). This gives a series of problem instances on which the commercial solvers very often not finish within a time limit of 10000 seconds on a usual desktop computer. For $k = 1$, which seems computationally the easiest case, a solution still always takes more than 4000 seconds.

However, knowing that a given problem such as the above is $C_5$-invariant, we can try to find an easier reformulation (3) by using matrices from the centralizer. As a rule of thumb, we assume that a transformed problem with smaller coefficients is “easier.” Here, the torsion free part of the centralizer is generated by the matrix $U$ from (1) in Example 5.9, and so the only possible choices for $S$ are $U$ and $U^{-1}$. (A matrix of finite order will probably not simplify a problem significantly.) Here, applying $S = U$ yields an easier problem, and one quickly finds that after applying $S$ ten times, the problem is not simplified further by applying $U$ (or $U^{-1}$). In other words, we transform the original problem instance with $U^{10}$. This yields an
equivalent $C_5$-invariant feasibility problem, which is basically instantly solved by the commercial solvers (finding that there is no integral solution).

As far as we know, this approach is in particular by far better than any previously known one that uses the symmetries of a cyclic group. One standard approach is, for example, to add symmetry-breaking inequalities $x_1 \leq x_2, \ldots, x_1 \leq x_5$. This yields an improved performance in some cases but is far from the order of computational gain that is possible with our proposed reformulations.

In general, when an integer linear program (2) is invariant under a QI-group $G$, and when it has any solutions at all, then Theorem 5.1 tells us that there is a transformation $x \mapsto Sx + t$ with $S \in N_{GL(d,\mathbb{Z})}(G)$ such that the reformulated problem has a feasible solution in a given finite set (a set of representatives of core points under normalizer equivalence). Heuristically, this means that we should be able to transform any $G$-invariant problem into one of bounded difficulty: By Lemma 5.7, for any vector $x \in \mathbb{R}^d$, there is an element $S \in C_{GL(d,\mathbb{Z})}(G)$ such that the projections of $Sx$ to the different $G$-invariant subspaces have approximately the same norm. This means that the orbit polytope of $Sx$ is “round.”

Our approach is particularly straightforward when the torsion free part of the centralizer $C_{GL(d,\mathbb{Z})}(G)$ has just rank 1, as in the example with $G = C_5$ above. When the centralizer contains a free abelian group of some larger rank, then it is less clear how to reduce the problem efficiently. A possible heuristic is as follows: Recall that in Lemma 5.5, we described a map $L$ which maps the centralizer, and thus its torsion-free part of rank $r$ (say), onto a certain lattice in $\mathbb{R}^{r+1}$. This maps the problem of finding a reformulation (3) with “small” $AS$ to a minimization problem on a certain lattice. For example, when we minimize $\|AS\|$, this translates to minimizing a convex function on a lattice. So we can find a good reformulation by finding a lattice point in $\mathbb{R}^{r+1}$ which is close to the minimum, using, for instance, LLL-reduction. This will be further studied in a forthcoming paper.

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**Universität Rostock, Institut für Mathematik, Ulmenstr. 69, Haus 3, 18057 Rostock, Germany**

E-mail address: frieder.ladisch@uni-rostock.de

E-mail address: achill.schuermann@uni-rostock.de