Axisymmetric Stationary Space-Times of Constant Scalar Curvature in Four Dimensions

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Abstract—We construct a special class of four-dimensional axisymmetric stationary space-times whose Ricci scalar is constant but are not Einstein space-times. We find that this solution has a ring singularity. At the end, we discuss some numerical results for these space-times.

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1. INTRODUCTION

A family of four-dimensional stationary axisymmetric space-times was first derived by R. Kerr in 1963, describing uncharged rotating black holes, which is a natural extension of static Schwarzschild solutions; for a review see, e.g., [1], and for a review with some recent developments see e.g., [2]. So far, this class of solutions was used in astrophysics explaining such phenomena as quasars and accreting stellar-mass black hole systems, see, e.g., [2].

Therefore, it is of interest to study stationary axisymmetric space-times, which is the aim of this paper. Here, we construct a special class of 4D axisymmetric stationary space-times of constant scalar curvature in the Boyer–Lindquist coordinates. First, we discuss the construction of Einstein space-times (known to be Kerr–Einstein space-times) with a nonzero cosmological constant by solving a modified Ernst equation for a nonzero cosmological constant. In other words, we rederive Carter’s result of [3]. Then, we proceed to construct spaces of constant scalar curvature which are not Einstein by modifying the previous result, namely, we add two additional functions to the metric functions to have a more general form of the metric, but it has the structure of a polynomial, namely, a quartic polynomial with five independent constants. This new family of axisymmetric stationary space-times of constant scalar curvature still admits a ring singularity.

The structure of the paper can be mentioned as follows. In Section 2 we give a quick review on axisymmetric stationary space-times. Then, we begin our construction of Kerr–Einstein space-times in Section 3. We discuss the construction of axisymmetric stationary space-times of constant scalar curvature in Section 4 and study its singularity structure. In Subsection 4.2 we show the numerical result for a singularity discussed in Section 4. Finally, we conclude our main results in Section 5.

2. AXISYMMETRIC STATIONARY SPACE-TIMES: A BRIEF REVIEW

Suppose we have a metric in the general form
\[ ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \] (1)
defined on a 4D space-time $M^4$, where $x^\mu$ parameterizes a local chart on $M^4$, and $\mu, \nu = 0, ..., 3$. Then, we simplify the case as follows. In a stationary axisymmetric space-time, the time coordinate $t$ and the azimuthal angle $\varphi$ are considered to be $x^0$ and $x^1$, respectively. A stationary axisymmetric metric is invariant under simultaneous transformations $t \to -t$ and $\varphi \to -\varphi$, which yields
\[ g_{00} = g_{03} = g_{12} = g_{13} = 0, \] (2)
and moreover, all nonzero metric components depend only on \( x^2 \equiv r \) and \( x^3 \equiv \theta \). The latter condition implies \( g_{23} = 0 \), and the metric (1) can be simplified into [1, 4]

\[
\begin{align*}
    ds^2 &= -e^{2\nu}dt^2 + e^{2\psi}(d\varphi - \omega dt)^2 \\
    &\quad + e^{2\mu_2}dr^2 + e^{2\mu_3}d\theta^2, \\
\end{align*}
\]

where \((\nu, \psi, \omega, \mu_2, \mu_3) \equiv (\nu(r, \theta), \psi(r, \theta), \omega(r, \theta), \mu_2(r, \theta), \mu_3(r, \theta))\). The Christoffel symbols and components of the Ricci tensor are given in detail in the Appendix. The Ricci scalar of the above metric is

\[
\begin{align*}
    -R &= 2e^{-2\mu_2}\left(\psi_{,2,2} + \psi_{,2}(\psi - \mu_2 + \mu_3)_{,2}
    \right)
    + \psi_{,2}\nu_{,2} + \nu_{,2}(\nu - \mu_2 + \mu_3)_{,2} + \mu_{3,2}^2
    \mu_{3,2}^2 - \frac{1}{4}\omega^2_{,2}e^{2(\psi - \nu)}
    \right)
    + 2e^{-2\mu_3}\left(\psi_{,3,3} + \psi_{,3}(\psi + \mu_2 - \mu_3)_{,3}
    \right)
    + \psi_{,3}\nu_{,3} + \nu_{,3}(\nu + \mu_2 - \mu_3)_{,3} + \mu_{2,3,3}
    + \mu_{2,3}(\mu_2 - \mu_3)_{,3} - \frac{1}{4}\omega^2_{,3}e^{2(\psi - \nu)}],
\end{align*}
\]

where we have defined

\[
    f_{,\mu} \equiv \frac{\partial f}{\partial x^\mu}, \quad f_{,\mu\nu} \equiv \frac{\partial^2 f}{\partial x^\mu \partial x^\nu}.
\]

3. EINSTEIN SPACE-TIMES

In this section, we construct a class of axisymmetric space-times satisfying the Einstein condition

\[
    R_{\mu\nu} = \Lambda g_{\mu\nu},
\]

with \( \Lambda \) the cosmological constant, yielding the following coupled nonlinear equations:

\[
\begin{align*}
    \left( e^{3\psi - \nu - \mu_2 + \mu_3} \omega_{,2} \right)_{,2}
    - \left( e^{3\psi - \nu + \mu_2 - \mu_3} \omega_{,3} \right)_{,3} &= 0, \\
    \left( e^{\psi + \nu} \right)_{,2,3} - \left( \psi + \nu \right)_{,2} \mu_{2,3} - \left( \psi + \nu \right)_{,3} \mu_{3,2}
    + \psi_{,2} \psi_{,3} + \nu_{,2} \nu_{,3} &= \frac{1}{2} e^{2(\psi - \nu)} \omega_{,2} \omega_{,3}, \\
    \left( e^{\mu_{3,2} - \mu_2} \left( e^{\beta} \right)_{,2} \right)_{,2}
    + \left( e^{\mu_2 - \mu_3} \left( e^{\beta} \right)_{,3} \right)_{,3}
    &= -2\Lambda e^{\beta + \mu_2 + \mu_3}, \\
    \left( e^{\beta - \mu_2 + \mu_3} \left( \psi - \nu \right)_{,2} \right)_{,2}
    + \left( e^{\beta + \mu_2 - \mu_3} \left( \psi - \nu \right)_{,3} \right)_{,3}
    &= -3e^{\psi - \nu} \left( e^{\mu_2 - \mu_3} \omega_{,2} \omega_{,2} + e^{\mu_2 - \mu_3} \omega_{,3} \omega_{,3} \right) - 4e^{\mu_3 - \mu_2} \left( \beta_{,2} \mu_{2,3} + \psi_{,2} \nu_{,2} \right)
    - 4e^{\mu_2 - \mu_3} \left( \beta_{,3} \mu_{3,2} + \psi_{,3} \nu_{,3} \right)
    - 2e^{-\beta} \left[ \left( e^{\mu_3 - \mu_2} \left( e^{\beta} \right)_{,2} \right)_{,2}
    + \left( e^{\mu_2 - \mu_3} \left( e^{\beta} \right)_{,3} \right)_{,3}
    - e^{2(\psi - \nu)} \left( e^{\mu_3 - \mu_2} \omega_{,2} \omega_{,2} + e^{\mu_2 - \mu_3} \omega_{,3} \omega_{,3} \right) \right].
\end{align*}
\]

where we have defined

\[
    \beta \equiv \psi + \nu.
\]

This class of solutions is called Kerr-(anti-)de Sitter solutions.

3.1. The Functions \( \mu_2 \) and \( \mu_3 \)

First of all, we simply take \( e^{\mu_2} \) as

\[
    e^{\mu_2} = \frac{(r^2 + a^2 \cos^2 \theta)^{1/2}}{\Delta_r^{(0)}},
\]

where \( \Delta_r^{(0)} \equiv \Delta_r^{(0)}(r) \), and \( a \) is a constant related to the angular momentum of a black hole [1]. Next, we assume that the function \( e^{2(\mu_3 - \mu_2)} \) and \( e^{2\beta} \) are separable as

\[
\begin{align*}
    e^{2(\mu_3 - \mu_2)} &= \Delta_r^{(0)} \sin^2 \theta, \\
    e^{2\beta} &= \Delta_\theta^{(0)} \Delta_\theta^{(0)}
\end{align*}
\]

with \( \Delta_\theta^{(0)} \equiv \Delta_\theta^{(0)}(\theta) \). Thus Eq. (9) can be cast into the form

\[
\begin{align*}
    &\left[ \Delta_r^{(0)} \Delta_\theta^{(0)} \right]_{,2} + \frac{1}{\sin \theta} \left[ \Delta_\theta^{(0)} \Delta_\theta^{(0)} \right]_{,3}
    &= -2\Lambda (r^2 + a^2 \cos^2 \theta)
\end{align*}
\]

Employing the variable separation method, we then obtain

\[
\begin{align*}
    \Delta_r^{(0)} &= -\frac{\Lambda}{3} r^4 + c_1 r^2 + c_2 r + c_3, \\
    \Delta_\theta^{(0)} &= -\frac{\Lambda}{3} a^2 \cos^4 \theta - c_1 \cos^2 \theta \\
    &\quad - c_4 \cos \theta + c_5,
\end{align*}
\]

where \( c_i \) for \( i = 1, \ldots, 5 \), are real constants. To make a contact with [3], one has to set \( c_i \) to be

\[
\begin{align*}
    c_1 &= -\frac{\Lambda}{3} a^2, \quad c_2 = -2M, \quad c_3 = a^2, \\
    c_4 &= 0, \quad c_5 = 1,
\end{align*}
\]

such that we have

\[
\begin{align*}
    \Delta_r^{(0)} &= -\frac{\Lambda}{3} r^2 \left( r^2 + a^2 \right) + r^2 - 2Mr + a^2, \\
    \Delta_\theta^{(0)} &= \left( 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \right) \sin^2 \theta.
\end{align*}
\]
3.2. The Functions $\omega$, $\nu$, $\psi$ and Ernst Equation

To obtain an explicit form of $\omega$, $\nu$, $\psi$, we have to transform (7) and (10) into a so-called Ernst equation with nonzero $\Lambda$ using (18). This can be structured as follows.

First, we introduce a pair of functions $(\Phi, \Psi)$ via

$$
\Phi_2 = e^{2(\psi-\nu)} \Delta_\theta^{(0)} \omega_p,
$$

$$
\Phi_p = -e^{2(\psi-\nu)} \Delta_\theta^{(0)} \omega_2,
$$

$$
\Psi \equiv e^{\psi-\nu} \Delta_\theta^{(0)} \Delta_\theta^{(0)} \theta,
$$

where $p \equiv \cos \theta$. Then (7) and (10) can be cast as

$$
\Psi \left( \Delta_\theta^{(0)} \Phi_2, \omega_p \right) = 2 \Delta_\theta^{(0)} \omega_p \Phi_2, 2 \Delta_\theta^{(0)} \Phi_2 \Phi_p,
$$

$$
\Psi \left( \Delta_\theta^{(0)} \Phi_2, \omega_p \right) = \Delta_\theta^{(0)} \left( (\omega_2)^2 - (\Phi_2)^2 \right) - \Delta_\theta^{(0)} \left( (\omega_p)^2 - (\Phi_p)^2 \right),
$$

(20)

respectively. Defining the complex function $Z \equiv \Psi + i\Phi$, Eqs. (20) can be rewritten in the Ernst form

$$
\Re Z \left( \Delta_\theta^{(0)} \Phi_2, \omega_p \right) = \Delta_\theta^{(0)} \omega_p Z_2 + \Delta_\theta^{(0)} Z_2 \Phi_p.
$$

(21)

Note that one could obtain another solution of (21), say, $\tilde{Z} \equiv \Psi + i\tilde{\Phi}$ by the conjugate transformation

$$
\tilde{\Psi} = \Delta_\theta^{(0)} \Phi_3,
$$

$$
\tilde{\Phi}_2 = \frac{\Delta_\theta^{(0)}}{\chi^2} \omega_3,
$$

$$
\tilde{\Phi}_3 = -\frac{\Delta_\theta^{(0)}}{\chi^2} \omega_2,
$$

(22)

where

$$
\chi \equiv e^{2(\nu-\psi)} - \omega_2^2,
$$

$$
\bar{\omega} \equiv e^{2(\nu-\psi)} - \omega_2^2.
$$

(23)

In the latter basis, we find

$$
\tilde{\Psi} = \Delta_\theta^{(0)} - a^2 \Delta_\theta^{(0)}
$$

$$
\tilde{\Phi}_2 = \frac{2aM \cos \theta}{r^2 + a^2 \cos^2 \theta} + \frac{2\Lambda}{3} \cos \theta.
$$

(24)

After some computation, we conclude that [3]

$$
e^{2\psi} = \frac{(r^2 + a^2)^2 \Delta_\theta^{(0)}}{r^2 + a^2 \sin^2 \theta},$$

$$
e^{2\nu} = \frac{(r^2 + a^2 \cos^2 \theta) \Delta_\theta^{(0)}}{(r^2 + a^2)^2 \Delta_\theta^{(0)} - \Delta_\theta^{(0)} a^2 \sin^2 \theta},$$

$$
\omega = \frac{a(r^2 + a^2) \Delta_\theta^{(0)} - a \sin^2 \theta \Delta_\theta^{(0)}}{(r^2 + a^2)^2 \Delta_\theta^{(0)} - \Delta_\theta^{(0)} a^2 \sin^4 \theta}.
$$

(25)

4. Space-Times of Constant Ricci Scalar

In this section we extend the previous results to the case of spaces of constant Ricci scalar, namely,

$$
R = g^{\mu\nu} R_{\mu\nu} = k,
$$

(26)

where $k$ is a constant. To have an explicit solution, we simply replace $\Delta_\theta^{(0)}$ and $\Delta_\theta^{(0)}$ in (14), (18), and (25) with

$$
\Delta_\theta = \Delta_\theta^{(0)} + f(r),
$$

$$
\Delta_\theta = \Delta_\theta^{(0)} + h(\theta).
$$

(27)

Then, inserting these modified functions mentioned above to (26), we simply have

$$
-k(r^2 + a^2 \cos^2 \theta) = (\Delta_\theta^{(0)}),
$$

$$
\frac{1}{\sin \theta} \left[ \frac{1}{\sin \theta} \right]_{,3},
$$

(28)

which gives

$$
-(k - 4\Lambda)(r^2 + a^2 \cos^2 \theta) = 2 \left[ f^{1/2} \left( f^{1/2} \right) \right]_{,2},
$$

$$
+ \frac{2}{\sin \theta} \left[ \frac{1}{\sin \theta} \right]_{,3}.
$$

(29)

The solution of (29) is given by

$$
f(r) = -\frac{1}{12} (k - 4\Lambda) r^4 + \frac{1}{2} C_1 r^2 + C_2 r + C_3,
$$

$$
h(\theta) = -\frac{1}{12} (k - 4\Lambda) a^2 \cos^4 \theta - \frac{1}{2} C_1 \cos^2 \theta - C_4 \cos \theta + C_5,
$$

(30)

where $C_i$, $i = 1, ..., 5$ are real constants. Some remarks are in order. First, the functions $f(r)$ and $g(\theta)$ have the same structure as in (16), namely, they are quartic polynomials with respect to $r$ and $\cos \theta$, respectively. Second, the constant $\Lambda$ here is no longer the cosmological constant. Finally, Einstein space-times can be obtained by setting $k = 4\Lambda$ and $C_3 = a^2 C_5$ with other $C_i$, $i = 1, 2, 4$ being free constants.

Now we can state our main result as follows.

Theorem 1. Suppose we have an axisymmetric space-time $M^4$ endowed with metric

$$
ds^2 = -e^{2\nu} dt^2 + e^{2\psi} (d\varphi - \omega dt)^2 + e^{2\mu_2} dr^2 + e^{2\mu_3} d\theta^2,
$$

(31)

satisfying

$$
e^{2(\mu_3 - \mu_2)} = \Delta_\theta \frac{\sin^2 \theta}{\Delta_\theta},
$$

$$
e^{2\beta} = \Delta_\theta \Delta_\theta.
$$

(32)
\[
e^{2\psi} = \frac{(r^2 + a^2)^2 \Delta_{\theta} - \Delta_r a^2 \sin^4 \theta}{r^2 + a^2 \cos^2 \theta},
\]
\[
e^{2\nu} = \frac{(r^2 + a^2 \cos^2 \theta) \Delta_{\theta} \Delta_r}{(r^2 + a^2)^2 \Delta_{\theta} - \Delta_r a^2 \sin^4 \theta},
\]
\[
\omega = \frac{a(r^2 + a^2) \Delta_{\theta} - a \sin^2 \theta \Delta_r}{(r^2 + a^2)^2 \Delta_{\theta} - \Delta_r a^2 \sin^4 \theta},
\]
where \( \Delta_r \) and \( \Delta_{\theta} \) are given by (27). Then, there exists a family of space-times of constant scalar curvature with
\[
\Delta_r = -\frac{1}{3} a r^2 a^2 + r^2 - 2 M r + a^2 - \frac{k}{12} r^4
+ \frac{1}{2} C_1 r^2 + C_2 r + C_3,
\]
\[
\Delta_{\theta} = -\cos^2 \theta + \frac{\Lambda}{3} a^2 \cos^2 \theta - \frac{k}{12} r^2 \cos^2 \theta
- \frac{1}{2} C_1 \cos^2 \theta - C_4 \cos \theta + C_5 + 1,
\]
where \( C_i, i = 1, \ldots, 5 \), are real constants. The metric (31) becomes Einstein if \( k = 4 \Lambda \) and \( C_3 = a^2 C_5 \).

**Proof.** Suppose \( f_{\mu \nu} = R_{\mu \nu} - \Delta g_{\mu \nu} \), then for the metric (31) we have
\[
f_{00} = \frac{1}{4} g_{00}(k - 4 \Lambda) + \frac{(C_3 - a^2 C_5)(a^2 \Delta_{\theta} + \Delta_r)}{\rho^6},
\]
\[
f_{10} = \frac{1}{4} g_{10}(k - 4 \Lambda) - \frac{a((a^2 + r^2) \Delta_{\theta} + \sin^2 \theta(\Delta_r))}{\rho^6} \times (C_3 - a^2 C_5),
\]
\[
f_{11} = \frac{1}{4} g_{11}(k - 4 \Lambda) + \frac{(a^2 + r^2)^2 \Delta_{\theta} + a^2 \sin^4 \theta(\Delta_r)}{\rho^6} \times (C_3 - a^2 C_5),
\]
\[
f_{22} = \frac{1}{4} g_{22}(k - 4 \Lambda) - \frac{(C_3 - a^2 C_5)}{\rho^2 \Delta_r},
\]
\[
f_{33} = \frac{1}{4} g_{33}(k - 4 \Lambda) + \frac{(C_3 - a^2 C_5) \sin^2 \theta}{\rho^2 \Delta_{\theta}},
\]
whereas the other components vanish. Then, the trace of \( f_{\mu \nu} \) is given by
\[
g^{\mu \nu} f_{\mu \nu} = k - 4 \Lambda,
\]
which implies that \( R = k \).

The norm of the Riemann tensor for the case at hand, in general, has the form
\[
R_{\mu \nu \alpha \beta} R_{\mu \nu \alpha \beta} = \frac{384 r^4}{(r^2 + a^2 \cos^2 \theta)^6} \times \left[ 2 a^2 C_4 \cos \theta \left( -a^2 C_5 + r(C_2 - 2 M) + C_3 \right) + \left( -a^2 C_5 + r(aC_4 + C_2 - 2 M) + C_3 \right) \right. \\
+ \left( -C_5 a^2 + r(-aC_4 + C_2 - 2 M) + C_3 \right)
\]
\[
+ \frac{192 r^2}{(r^2 + a^2 \cos^2 \theta)^5} \left[ a^2 C_4 \cos \theta \left( 3a^2 C_5 - 4C_2 r - 3C_3 + 8 M r \right) - 5r(C_2 - 2 M) \right. \\
\left. \left( C_3 - a^2 C_5 \right) - 2 \left( C_3 - a^2 C_5 \right)^2 - 3r^2 \right. \\
\times \left. \left( -aC_4 + C_2 - 2 M \right) \left( aC_4 + C_2 - 2 M \right) \right]
\]
\[
+ \frac{8}{(r^2 + a^2 \cos^2 \theta)^4} \left[ 6a^2 C_4 \cos \theta \left( -a^2 C_5 + 3r(C_2 - 2 M) + C_3 \right) + 30r(C_2 - 2 M) \right. \\
\left. \left( C_3 - a^2 C_5 \right) + 7 \left( C_3 - a^2 C_5 \right)^2 + 27r^2 \right. \\
\times \left. \left( aC_4 + C_2 - 2 M \right) \left( -aC_4 + C_2 - 2 M \right) \right]
\]
\[
- \frac{12}{(r^2 + a^2 \cos^2 \theta)^3} \left( -aC_4 + C_2 - 2 M \right) + \frac{k^2}{6},
\]
which might have a negative value, as observed in [5] for the Kerr–Newman metric with \( k = 0 \). This is so because the space-time metric is indefinite. The norm (36) shows that the space-time has a real ring singularity at \( r = 0 \) and \( \theta = \pi/2 \), with radius \( a \).

Now, we would like to find some geometrical invariants of this solution by showing the explicit expression of Komar integrals such as the Komar mass and Komar angular momentum. The definition of the Komar integral is
\[
Q = \frac{1}{N} \int dS_{\mu \nu} (\nabla^\nu \zeta^\mu + \omega^\mu_\nu),
\]
where the antisymmetric tensor \( \omega^\mu_\nu \) is a solution of
\[
\nabla_\mu \omega^\mu_\nu = R^\nu_\rho \zeta^\rho,
\]
and \( \zeta^\mu \) is the Killing vector corresponding to the symmetry. \( N \) is any suitable normalization constant whose exact value does not have any significance.
Since (37) is an integral on the boundary of a spacelike hypersurface $\Sigma$, we can transform the expression into a volume integral on $\Sigma$ by using the Gauss theorem. As such, we have

$$ Q = \frac{1}{N} \int_{\partial \Sigma} dS_{\mu \nu} \nabla^{\mu} \zeta^{\nu} + \frac{1}{N} \int_{\Sigma} \sqrt{\gamma} \sigma_{\mu} R_{\mu \nu} \zeta^{\nu} d^3 x, \quad (39) $$

where $\sigma_{\mu}$ is a unit vector normal to $\Sigma$, and $\gamma = \det[\gamma_{ij}]$, where $\gamma_{ij}$ are metric tensor components of the hypersurface $\Sigma$.

Now, let us calculate the Komar mass that corresponds to the timelike Killing vector $\zeta(t) = \partial_t$. The explicit integral expression of this mass is

$$ \mathcal{M} = \frac{1}{4\pi} \int_{\partial \Sigma} dS_{01} \nabla^{0} \zeta^{1} 
= \frac{1}{4\pi} \int_{\Sigma} \sqrt{\gamma} \sigma_{0} R_{00} \zeta^{0} d^3 x. \quad (40) $$

Firstly, we calculate $\nabla^{0} \zeta^{1}$ as follows:

$$ \nabla^{0} \zeta^{1} = g^{00} \Gamma^{1}_{\mu \nu} \zeta^\nu = g^{00} \Gamma^{1}_{00} + g^{30} \Gamma^{1}_{30} 
= - \frac{2(a^2 + r^2)}{(a^2 \cos(2\theta) + a^2 + 2r^2)^3} \times \left[ \Delta_\theta(ab^2 + a^2 + 2r^2) + 4a^2 r \Delta \theta - 4r \right] 
= \frac{k}{12} r - \frac{2M - C_2}{2r^2} + O(r^{-4}). \quad (41) $$

Since $dS_{01} \approx -r^2 \sin \theta d\theta d\varphi$ for $r \gg a$, we have

$$ \frac{1}{4\pi} \int_{\partial \Sigma} dS_{01} \nabla^{0} \zeta^{1} = M - \frac{C_2}{2} - \frac{k}{12} r^3 \quad (42) $$

for the first term evaluated at large $r$. Now, consider $R_{00}$ that behaves like

$$ R_{00} = \frac{1}{4} k + O(r^{-4}) \quad (43) $$

at large $r$. As such, we can expect that performing volume integration in the second term up to an arbitrary large $r$ gives

$$ \frac{1}{4\pi} \int_{\Sigma} \sqrt{\gamma} \sigma_{0} R_{00} \zeta^{0} d^3 x \approx \frac{k}{12} r^3. \quad (44) $$

This term exactly cancels out the third term of the previous surface integral. Thus we can conclude that the Komar mass of our system is given by

$$ \mathcal{M} = M - \frac{C_2}{2}. \quad (45) $$

As a conclusion, the constant $C_2$ is insignificant in our solution, hence, we can take it to be zero without any loss of generality since the mass parameter, $M$, can be redefined to absorb $C_2$.

Now, for the Komar angular momentum, the spacelike Killing vector is $\zeta(\varphi) = \partial_{\varphi}$. As such, the integral expression of it is

$$ J = -\frac{1}{\pi} \int_{\partial \Sigma} dS_{01} \nabla^{0} \zeta^{1} - \frac{1}{\pi} \int_{\Sigma} \sqrt{\gamma} \sigma_{0} R_{00} \zeta^{0} d^3 x. \quad (46) $$

Again, we calculate $\nabla^{0} \zeta^{1}$ for the new Killing vector as follows:

$$ \nabla^{0} \zeta^{1} = g^{00} \Gamma^{1}_{\mu \nu} \zeta^\nu 
= g^{00} \Gamma^{1}_{00} + g^{30} \Gamma^{1}_{30} 
= -a \left( \frac{(a^2 - 4k) + 6C_1 + 12C_4 \cos \theta - 12C_5}{12}\right) 
- \frac{3a(C_2 - 2M) \sin \theta}{2r^2} + O(r^{-3}). \quad (47) $$

Thus the first term is given by

$$ \int_{\partial \Sigma} dS_{01} \nabla^{0} \zeta^{1} = -4r \left[ a^2 \frac{k - 4\Lambda}{12} + a \left( \frac{C_1 - C_5}{2}\right) \right] 
+ a \left( M - \frac{C_2}{2}\right) + O(r^{-1}). \quad (48) $$

The corresponding Ricci tensor component satisfies

$$ R_{03} = 2a \frac{C_3 - a^2 C_5}{r^4} \sin \theta + O(r^{-6}). \quad (49) $$

Thus we expect that the second term tends to zero at large $r$. As a conclusion, the Komar angular momentum of this system is given by

$$ J = a \left[ M - \frac{C_2}{2}\right] - 4r \left[ a^2 \frac{k - 4\Lambda}{12} \right] 
+ a \left( \frac{C_1 - C_5}{2}\right). \quad (50) $$

We can observe that the second term diverges as $r \to \infty$. To avoid this problem, we can choose $C_1 = 2C_5$ and $k = 4\Lambda$. Thus, in order to have a physical angular momentum, we need to take our solution back to the Einstein limit, which gives us the following expression for the Komar mass and angular momentum:

$$ \mathcal{M} = M, \quad J = a M. \quad (51) $$

We can see that the parameters $M$ and $a$ are, indeed, similar to the mass and angular momentum of a classical rotating black hole. The term linear in $k - 4\Lambda$ and $C_3/2 - C_5$ that make $J$ diverge for an asymptotically flat or asymptotically AdS solution came from the statement that Ricci scalar equal to a constant, $R = k$, which, in general, does not obey Einstein’s equation.
4.1. Regularity and Horizons

Since an axisymmetric configuration is invariant under rotation along its azimuthal coordinate (SO(2) group on the azimuthal plane), we need to know the behavior of the solution on fixed points of SO(2). In 4D space-time, the standard one-parameter SO(2) transformation, in Cartesian coordinates, is given by

\[ O_2(\phi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos \tilde{\phi} & \sin \tilde{\phi} & 0 \\ 0 & -\sin \tilde{\phi} & \cos \tilde{\phi} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \] (52)

with \( \tilde{\phi} \) the transformation parameter. Fixed points, \( X_i \), of this symmetry group satisfy \( X_i = O_2 X \). The solution of such an equation is

\[ X = \begin{bmatrix} t \\ 0 \\ 0 \\ z \end{bmatrix}, \] (53)

where \( t \) and \( z \) are arbitrary. Transforming these points into spherical coordinates gives us arbitrary \( t \), \( r \), while \( \phi \) must be fixed at \( \theta = 0 \) or \( \pi \). This set of fixed points is known as the rotation axis.

Evaluating (31) at \( \theta = \pi \) or \( 0 \) gives

\[ ds^2 = -e^{2\nu} dt^2 + e^{2\psi} (d\phi - \omega dt)^2 + e^{2\mu_2} dr^2 \] (54)

with

\[ e^{2\nu} = \frac{(r^2 + a^2)}{\Delta_r}, \quad e^{2(\psi + \nu)} = \Delta_r \Delta_\theta, \]
\[ e^{2\psi} = (r^2 + a^2) \Delta_\theta, \quad e^{2\nu} = \frac{\Delta_r}{(r^2 + a^2)}, \]
\[ \omega = \frac{a}{(r^2 + a^2)}. \] (55)

The quantity \( \Delta_\theta \) is now given by

\[ \Delta_\theta = \frac{1}{3} [a^2 - a^2 - \frac{1}{2} C_1 \pm C_4 + C_5]. \]

Transforming \( r \) to Cartesian coordinate gives the following metric:

\[ ds^2 = -\frac{\Delta_r - a^2 \Delta_\theta}{z^2 + a^2} dt^2 + \frac{z^2 + a^2}{\Delta_r} dz^2. \] (56)

The resulting 2D manifold on the rotation axis has a similar coordinate singularity (back-hole horizon) with the full rotating solution, but the true singularity is only one, in contrast to the full solution that possesses a ring singularity. We can observe that taking \( a = 0 \), implying a zero angular momentum, gives us the Reissner–Nordström–de Sitter solution with an extra term containing \( C_1 \) and \( C_2 \).

There is a special region where the metric changes its signature, that is, where \( \Delta_r < a^2 \Delta_\theta \). Thus for every \( z \) that solves \( G(z) > 0 \), where

\[ G(z) = a^2 \Delta_\theta - \Delta_r \]
\[ = \frac{\Lambda}{3} a^2 (a^2 + z^2) - \frac{k}{12} (a^4 - z^4) \]
\[ - (z^2 + a^2) + 2Mz - \frac{1}{2} C_1 (a^2 + z^2) \]
\[ - C_2 z - C_3 \pm C_4 a^2 + C_5 a^2 \] (57)

and the space-time is locally Euclidean. This region is located near the black-hole horizon where \( \Delta_r \) is close to zero. The existence of these locally Euclidean regions is a problem because the boundary between locally Minkowskian and locally Euclidean regions is irregular. We should again argue that this irregularity comes from the fact that our formulation is more general than Einstein’s (which is always assumed to be locally Minkowskian everywhere), and such an extension allows irregularities to show up in our solution.

There are actually some ways to prove that this irregularity is not pathological, for example, by showing that these locally Euclidean regions are hidden behind black-hole horizons, which will be discussed below.

Consider the mean extrinsic curvature of a 2D spatial hypersurface constructed from an normal timelike vector that is orthogonal to the 3D hypersurface and an outward (radial) pointing vector for this space-time, given by

\[ K = \frac{\Delta_r}{2 \sqrt{2} (a^2 \cos^2 \theta + r^2) \sqrt{\frac{\Delta_r}{a^2 \cos^2 \theta + a^2 + 2r^2}}} \]
\[ \times \frac{4r (a^2 + r^2) \Delta_\theta - a^2 \sin^4 \theta \Delta_r'}{((a^2 + r^2)^2 \Delta_\theta - a^2 \sin^4 \theta \Delta_r')}. \] (58)

The apparent horizons are solutions of \( K = 0 \), and we can directly see that the event horizons, where \( \Delta_r = 0 \), coincide with some of the apparent horizons.

Firstly, let us consider the locations of event horizons by solving \( \Delta_r = 0 \). The problem of finding event horizons in this space-time can be reduced to the problem of solving a quartic equation given by

\[ -A r^4 + B r^2 + C r + D = 0, \] (59)

where we have defined

\[ A \equiv \frac{k}{12}, \quad B \equiv 1 + \frac{C_1}{2} - \frac{a^2 \Lambda}{3}, \]
\[ C \equiv -2M, \quad D \equiv a^2 + C_3. \] (60)
The quartic equation (59) has four possible solutions where one solution is negative-definite, hence, we are left with only three possible solutions for the horizons.

The two smaller solutions should be considered as black-hole horizons, they are

\[
\begin{align*}
    r_+ &= \frac{1}{2} \sqrt{\phi_1 - \frac{1}{2} \sqrt{\phi_2 + \frac{2C}{A\phi_3}}}, \\
    r_- &= -\frac{1}{2} \sqrt{\phi_1 + \frac{1}{2} \sqrt{\phi_2 - \frac{2C}{A\phi_3}}},
\end{align*}
\]

such that \( r_+ \geq r_- \), and we have defined the new functions \( \phi_1 \equiv \phi_1(A, B, C, D) \), \( \phi_2 \equiv \phi_2(A, B, C, D) \), and \( \phi_3 \equiv \phi_3(A, B, C, D) \), explicitly given in Appendix B in Eqs. (76), (77), and (78).

The largest solution of (59) is a cosmological horizon, given by

\[
    r_c = \frac{1}{2} \sqrt{\phi_1 + \frac{1}{2} \sqrt{\phi_2 + \frac{2C}{A\phi_3}}},
\]

From here, we can see that the physical region lies in \( r_+ < r < r_c \), where the metric is timelike. The regions \( r_- \leq r \leq r_+ \) and \( r \geq r_c \) have a spacelike metric, thus, the qualitative features of this space-time related to the location of timelike region are similar to those we find in the Einstein limit, except for the fact that this solution possesses some locally Euclidean regions. Since these locally Euclidean regions are located behind the horizons, these regions are not pathological and do not have any physical significance.

The number of apparent horizons of this space-time is actually larger than the number of event horizons since taking (58) equal to zero can also be done by taking

\[
    4r(a^2 + r^2)\Delta_\theta - a^2 \sin^4 \theta \Delta_\theta = 0,
\]

that possess at most three different solutions for \( r \). This proves that there exist trapped regions that do not coincide with event horizons at their boundaries.

Since we have two black-hole horizons, it is interesting to consider a “critical” case where the “discriminant” of the quartic polynomial (59) vanishes, namely,

\[
-256A^3D^3 - 128A^2B^2D^2 + 144A^2BC^2D \\
- 27A^2C^4 - 16AB^4D + 4AB^3C^2 = 0,
\]

with \( k \neq 0 \). The roots of (65) have the form

\[
2M = \sqrt{\frac{2}{27}} \times \sqrt[3]{\frac{B^3}{A} + 36BD \pm \sqrt{(B^2 - 12AD)^3}} \times \frac{A}{A},
\]

which gives a “critical” mass of a black hole describing a situation where some roots of (59) coincide at \( k > 0 \). In addition, the term inside the square root must be positive in order to have a physical solution.

Generally, the metric described in Theorem 1 may not be related to Einstein’s general relativity since our method described above does not use the notion of energy-momentum tensor. To make a contact with general relativity, we could simply set some constants, for example,

\[
C_1 = C_2 = C_4 = C_5 = 0, \quad C_3 = q^2 + g^2,
\]

with \( k = 4\Lambda \), where \( q \) and \( g \) are the electric and magnetic charges, respectively. This setup gives the Kerr–Newman–Einstein metric describing a dyonic rotating black hole with a non-zero cosmological constant [6]. For \( \Lambda > 0 \), the “critical” mass of a black hole in this case is simply

\[
M = \sqrt{\frac{1}{54}} \frac{3}{\Lambda} \left( 1 - \frac{a^2\Lambda}{3} \right)^3 \\
+ 36 \left( 1 - \frac{a^2\Lambda}{3} \right) (a^2 + q^2 + g^2) \\
- \frac{3}{\Lambda} \left( 1 - \frac{a^2\Lambda}{3} \right)^2 - 4\Lambda(a^2 + q^2 + g^2)^{3/2} \right)^{1/2},
\]

where the inner horizon and the event horizon coincide. In this critical case, the inner timelike region \( 0 \leq r \leq R_- \) and the physical region \( r_+ \leq r \leq r_c \) are connected, which leads to a naked singularity at the origin.

4.2. Numerical Results

With the Kretschmann scalar (36) at hand, we can identify the true singular points within our metric solutions because this scalar become singular at singular points. Some black hole classes might have different true singular points, which depends on their parameters. We conclude the results as follows.

1. For every static black-hole solution, \( r = 0 \) is a true singularity.

2. For every stationary axisymmetric black-hole solutions, \( r^2 + a^2 \cos^2 \theta = 0 \) is a true singularity.

To see this clearer, we plot some profiles of the Kretschmann scalar for Kerr and Kerr–Newman black holes by tweaking our parameters to reproduce those three black-hole solutions. These plots are given in Figs. 1, 2 with \( M = 1 \), \( a = 0.8 \) for the Kerr
solution, and \( a = 0.8 \) and \( C_3 = q^2 + g^2 = 0.64 \) for the Kerr–Newman solution.

From Figs. 1 and 2 we observe the real ring singularity at \( r = 0 \) and \( \theta = \pi/2 \) with radius \( a \), which are mentioned in the previous section. The locations of singular points are not affected by the asymptotic structure of space-time, hence the value of the constant Ricci scalar does not alter the singularity. We can see that for stationary cases, there exist a region where the Kretschmann scalar takes a negative value near the singularity. One interesting fact is that the singularity can be avoided for inward radial motion through the black-hole north and south poles. This indicates a fundamentally different structure between those space-time manifolds. The Kerr and Kerr–Newman solutions are connected, i.e., taking a chargeless limit of the Kerr–Newman solution reproduces the Kerr one, which is demonstrated in Fig. 3.

The special results in (68) for a critical black hole in which all horizons coincide can be generalized to cases where parameters other than \( M, k, \Lambda \) and \( a \) are taken into account. Firstly, we introduce another reduced parameter \( A, B, C \) for Eq. (59) which satisfy \( A = B/A, B = C/A, \) and \( C = D/A \). The corresponding discriminant equation, \( D = 0 \), becomes

\[
16A^4C - 4A^3B^2 - 128A^2C^2 + 144AB^2C
\]

\[
-27B^4 + 256C^3 = 0.
\]  

Equation (69) represents surfaces in 3D space spanned by \( A, B, C \). These surfaces divide the space into three regions: Region I which has 4 real solutions, region II (see Fig. 4). The surfaces that emerge from the discriminant equation can be divided into three independent ones. We can identify the three surfaces by solving Eq. (69) as a cubic equation for \( C \), which gives three solutions, given explicitly in Appendix B in Eqs. (79)–(81). Each solutions represents an independent surface, they are depicted in Fig. 5. By identifying all curves which arise from intersections between the surfaces, we conclude that there are five different characteristics of critical points, see Table 1.

Because we know that the parameters \( A, B, \) and \( C \) are functions of the black hole parameters \( M, k, \Lambda \),

| Critical points       | \( A = B = C = 0 \)                                      |
|-----------------------|---------------------------------------------------------|
| Quadruple point       | \( A = 2i\sqrt{3}\sqrt{C} \)                           |
|                       | \( B = \pm\frac{8\sqrt{3-1C^{3/4}}}{3^{3/4}} \)          |
|                       | \( C < 0 \)                                              |
| Triple point          | \( A = -2\sqrt{C} \)                                    |
|                       | \( B = 0 \)                                              |
|                       | \( C > 0 \)                                              |
| 2 double points       | \( A < 0 \), \( B \) indetermined, \( \)                |
|                       | and 2 real                                              |
|                       | \( A = C_{II} \) or \( C = C_{III} \)                   |
| Double points         | \( A, B \) indetermined\( \)                           |
| and 2 complex         | \( C = C_I \) or \( C = \) upper \( C_{II} \)            |
Fig. 3. Demonstration of the connection between Kerr and Kerr–Newman solutions by increasing the black-hole charge,\[ C_3 = q^2 + g^2 = 0.16, 0.36, 0.64, 0.81 \text{ (upside down)}, \]for \( a = 0.8 \).

Fig. 4. Three-parameter space spanned by \( A, B, C \) where region I (left) has 4 real solutions, region II (middle) has 2 real solutions, and region III (right) has no real solutions.
Table 2. Comparison of some well-known black-hole solutions. (Produced from the Einstein limit which also has a constant scalar curvature, but is constrained by the Einstein equations)

|                      | $M$ | $a$ | $A$ | $k$ | $C_1$ | $C_3$ | $C_4$ | $C_5$ |
|----------------------|-----|-----|-----|-----|-------|-------|-------|-------|
| Schwarzschild        | ✓   | 0   | 0   | 0   | 0     | 0     | 0     | 0     |
| Kerr                 | ✓   | ✓   | 0   | 0   | 0     | 0     | 0     | 0     |
| Reissner–Nordström   | ✓   | 0   | 0   | 0   | $q^2 + g^2$ | 0     | 0     | 0     |
| Kerr–Newman          | ✓   | ✓   | 0   | 0   | $q^2 + g^2$ | 0     | 0     | 0     |
| Schwarzschild–(A)dS | ✓   | 0   | ✓   | 4$A$ | 0     | 0     | 0     | 0     |
| Kerr–(A)dS           | ✓   | ✓   | ✓   | 4$A$ | 0     | 0     | 0     | 0     |
| Reissner–Nordström–(A)dS | ✓   | 0   | ✓   | 4$A$ | 0     | $q^2 + g^2$ | 0     | 0     |
| Kerr–Newman–(A)dS    | ✓   | ✓   | ✓   | 4$A$ | 0     | $q^2 + g^2$ | 0     | 0     |
| Kerr–Newman–(A)dS*   | ✓   | ✓   | ✓   | 4$A$ | ✓     | ✓     | ✓     | ✓     |
| This paper           | ✓   | ✓   | ✓   | ✓   | ✓     | ✓     | ✓     | ✓     |

It is found that static and stationary solutions have different Kretschmann scalar profile characteristics. We found that four coordinate singularities are generally present, and there are five different classes of critical conditions on black hole parameters where the horizons coincide.

5. CONCLUSIONS

We have shown in Theorem 1 that there exists a family of constant-curvature axisymmetric stationary space-times characterized by the metric solution (33). The newly found solution generalized the well-known solutions, such as the Kerr and Kerr–Newman ones, by introducing four new parameters (see Table 2).

The true singularities are identified by utilizing the Kretschmann scalar, as given in Figs. 1, 2, and it is found that static and stationary solutions have different Kretschmann scalar profile characteristics. We found that four coordinate singularities are generally present, and there are five different classes of critical conditions on black hole parameters where the horizons coincide.

Appendix A

SPACETIME CONVENTION

In Appendix A we collect some space-time quantities which are useful for the analysis in the paper.
Christoffel symbols:

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) . \]  \hfill (70)

Riemann curvature tensor:

\[ -R^\rho_{\mu\sigma\nu} = \partial_\sigma R^\rho_{\mu\nu} - \partial_\nu R^\rho_{\mu\sigma} + \Gamma^\lambda_{\mu\sigma} \Gamma^\rho_{\lambda\nu} - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} . \]  \hfill (71)

Ricci tensor:

\[ R_{\mu\nu} = R^\rho_{\mu\rho\nu} = \partial_\rho R^\rho_{\mu\nu} - \partial_\nu R^\rho_{\mu\rho} + \Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\lambda\nu} - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\rho} . \]  \hfill (72)

Ricci scalar:

\[ R = g^{\mu\nu} R_{\mu\nu} . \]  \hfill (73)

Nonzero Christoffel symbols for the metric (3):

\[ \Gamma^0_{02} = \nu_2 - \frac{1}{2} \omega \psi_2 e^{2(\psi - \nu)} , \]

\[ \Gamma^0_{03} = \nu_3 - \frac{1}{2} \omega \psi_3 e^{2(\psi - \nu)} , \]

\[ \Gamma^0_{12} = \frac{1}{2} \omega \psi_2 e^{2(\psi - \nu)} , \]

\[ \Gamma^0_{13} = \frac{1}{2} \omega \psi_3 e^{2(\psi - \nu)} , \]

\[ \Gamma^1_{20} = -\omega (\psi_2 - \nu_2) - \frac{1}{2} \omega \psi_2 (1 + \omega^2 e^{2(\psi - \nu)}) , \]

\[ \Gamma^1_{12} = \psi_2 + \frac{1}{2} \omega \psi_2 e^{2(\psi - \nu)} , \]

\[ \Gamma^1_{13} = \psi_3 + \frac{1}{2} \omega \psi_3 e^{2(\psi - \nu)} , \]

\[ \Gamma^2_{00} = \nu_2 e^{2(\psi - \nu)} - \omega (\psi_2 + \omega \psi_2) e^{2(\psi - \nu)} , \]

\[ \Gamma^2_{01} = \left( \frac{1}{2} \omega \psi_2 + \omega \psi_2 \right) e^{2(\psi - \nu)} , \]

\[ \Gamma^2_{11} = -\psi_2 e^{2(\psi - \nu)} , \]

\[ \Gamma^2_{22} = \mu_2 + 2 , \]

\[ \Gamma^2_{23} = \mu_3 - 1 , \]

\[ \Gamma^2_{33} = -\mu_3 e^{2(\mu_2 - \mu_3)} , \]

\[ \Gamma^3_{00} = \nu_3 e^{2(\psi - \nu)} - \omega (\psi_3 + \omega \psi_3) e^{2(\psi - \nu)} , \]

\[ \Gamma^3_{01} = \left( \frac{1}{2} \omega \psi_3 + \omega \psi_3 \right) e^{2(\psi - \nu)} , \]

\[ \Gamma^3_{11} = -\psi_3 e^{2(\psi - \nu)} , \]

\[ \Gamma^3_{22} = -\mu_2 e^{2(\mu_2 - \mu_3)} , \]

\[ \Gamma^3_{23} = \mu_3 + 2 , \]

\[ \Gamma^3_{33} = \mu_3 - 1 . \]  \hfill (74)

Nonzero components of the Ricci tensor:

\[ R_{00} = e^{2(\psi - \nu)} \left( \nu_2 + v_2 (\psi + \nu - \mu_2 + \mu_3) \right) . \]  \hfill (75)
DEFINITIONS OF $\phi_1$, $\phi_2$, $\phi_3$, $C_1$, $C_{II}$ AND $C_{III}$

The functions $\phi_1 \equiv \phi_1(A, B, C, D)$, $\phi_2 \equiv \phi_2(A, B, C, D)$, and $\phi_3 \equiv \phi_3(A, B, C, D)$ are defined by

\[
\phi_1 \equiv \frac{2B}{3A} - \frac{3}{\sqrt{2}A} \sqrt[3]{\frac{\sqrt{72ABD - 27AC^2 + 2B^3} - 4(B^2 - 12AD)^3 + 72ABD - 27AC^2 + 2B^3}{2(B^2 - 12AD)}}
\]

\[
\phi_2 \equiv \frac{4B}{3A} + \frac{3}{\sqrt{2}A} \sqrt[3]{\frac{\sqrt{72ABD - 27AC^2 + 2B^3} - 4(B^2 - 12AD)^3 + 72ABD - 27AC^2 + 2B^3}{2(B^2 - 12AD)}}
\]

\[
\phi_3 \equiv \frac{2B}{3A} - \frac{3}{\sqrt{2}A} \sqrt[3]{\frac{\sqrt{72ABD - 27AC^2 + 2B^3} - 4(B^2 - 12AD)^3 + 72ABD - 27AC^2 + 2B^3}{2(B^2 - 12AD)}}
\]

The solutions of zero discriminant equation, (69), are as follows:

\[
C_1 = \frac{A^2}{6} + \frac{1}{24} \left[ -8A^6 - 540A^3B^2 + 3\sqrt{3}B(8A^3 + 27B^2)^3 + 729B^4 \right]
\]

\[
C_{II} = \frac{A^2}{6} + \frac{1}{48i} \left[ \sqrt{3} + i \right] \left[ -8A^6 - 540A^3B^2 + 3\sqrt{3}B(8A^3 + 27B^2)^3 + 729B^4 \right]
\]

\[
C_{III} = \frac{A^2}{6} - \frac{1}{48} \left[ \sqrt{3} - i \right] \left[ -8A^6 - 540A^3B^2 + 3\sqrt{3}B(8A^3 + 27B^2)^3 + 729B^4 \right]
\]
To visualize clearer the shape of the horizon that occurs, we give here two examples of event horizons for Kerr–de Sitter and Kerr–Newman–de Sitter black holes, shown in Figs. 6 and 7. The value of $\Lambda$ is chosen to be large enough, such that the cosmological horizon can be observed.

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**CONFLICT OF INTEREST**

The authors declare that they have no conflicts of interest.

**REFERENCES**

1. S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon, Oxford, UK, 1985).

2. S. A. Teukolsky, “The Kerr Metric,” Class. Quantum Grav. 32, 124006 (2015); arXiv:1410.2130.

3. B. Carter, “Black holes equilibrium states,” in *Proceedings, Ecole d’Ete de Physique Theorique: Les Astres Occlus Les Houches* (Ed. B. DeWitt, C. M. DeWitt, Gordon and Breach, New York, 1973).

4. S. Chandrasekhar, “The Kerr metric and stationary axis-symmetric gravitational field,” Proc. Roy. Soc. Lond. A 358, 405 (1978).

5. R. C. Henry, “Kretschmann scalar for a Kerr–Newman black hole,” Astrophys. J. 535, 350 (2000); astro-ph/9912320.

6. M. R. Setare and M. B. Altaie, “The Cardy-Verlinde formula and entropy of topological Kerr–Newman black holes in de Sitter spaces,” Eur. Phys. J. C 30, 273 (2003); hep-th/0304072.