Triebel–Lizorkin spaces with general weights

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Abstract
In this paper, the author introduces Triebel–Lizorkin spaces with general smoothness. We present the \( \varphi \)-transform characterization of these spaces in the sense of Frazier and Jawerth and we prove their Sobolev embeddings. Also, we establish the smooth atomic and molecular decomposition of these function spaces. To do these we need a generalization of some maximal inequality to the case of general weights.

Keywords Atom · Molecule · Triebel–Lizorkin space · Embedding · Muckenhoupt class

Mathematics Subject Classification 42B25 · 42B35 · 46E35

1 Introduction
This paper is a continuation of [19], where the author introduced Besov spaces with general smoothness and presented some their properties, such as the \( \varphi \)-transform characterization in the sense of Frazier and Jawerth, the smooth atomic, molecular and wavelet decomposition, and the characterization of these function spaces in terms of the difference relations.

The spaces of generalized smoothness are have been introduced by several authors. We refer, for instance, to Bownik [10], Cobos and Fernandez [14], Goldman [33, 34], and Kalyabin [40]; see also Besov [7, 8], and Kalyabin and Lizorkin [41]. More general Besov spaces with variable smoothness were explicitly studied by Ansorena and Blasco [4, 5], including characterizations by differences
and atomic decomposition. The wavelet decomposition (with respect to a compactly supported wavelet basis of Daubechies type) of nonhomogeneous Besov spaces of generalized smoothness was achieved by Almeida [2]. The theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the study of trace spaces on fractals, see Edmunds and Triebel [23, 24], were they introduced the spaces $B^{\alpha;\Psi}_{p,q}$, where $\Psi$ is a so-called admissable function, typically of log-type near 0. For a complete treatment of these spaces we refer the reader the work of Moura [48]. Further results on Besov spaces of variable smoothness are given in [1].

More general function spaces of generalized smoothness can be found in Caetano and Leopold [12], Farkas and Leopold [25], and reference therein.

Recently, Dominguez and Tikhonov [21] gave a treatment of function spaces with logarithmic smoothness (Besov, Sobolev, Triebel–Lizorkin), including various new characterizations for Besov norms in terms of different, sharp estimates for Besov norms of derivatives and potential operators (Riesz and Bessel potentials) in terms of norms of functions themselves and sharp embeddings between the Besov spaces defined by differences and by Fourier-analytical decompositions as well as between Besov and Sobolev/Triebel–Lizorkin spaces.

Tyulenev introduced in [65–67] a new family of Besov spaces of variable smoothness which cover many classes of Besov spaces, where the norm on these spaces was defined with the help of classical differences. Based on this weighted class and [19] we introduce Triebel–Lizorkin spaces of variable smoothness, is defined as follows. Let $\mathcal{S}(\mathbb{R}^n)$ be the set of all Schwartz functions $\varphi$ on $\mathbb{R}^n$, i.e., $\varphi$ is infinitely differentiable and

$$
\left\| \varphi \right\|_M = \sup_{\beta \in \mathbb{N}_0^n, |\beta| \leq M} \sup_{x \in \mathbb{R}^n} |\partial^\beta \varphi(x)| (1 + |x|)^{n+M+|\beta|} < \infty
$$

for all $M \in \mathbb{N}$. Select a Schwartz function $\varphi$ such that

$$
\text{supp}(\mathcal{F}(\varphi)) \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \}
$$

and

$$
|\mathcal{F}(\varphi)(\xi)| \geq c \quad \text{if} \quad 3/5 \leq |\xi| \leq 5/3,
$$

where $c > 0$ and we put $\varphi_k = 2^{kn} \varphi(2^k \cdot)$, $k \in \mathbb{Z}$. Here $\mathcal{F}(\varphi)$ denotes the Fourier transform of $\varphi$, defined by

$$
\mathcal{F}(\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x) dx, \quad \xi \in \mathbb{R}^n.
$$

Let

$$
\mathcal{S}_\infty(\mathbb{R}^n) = \{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} x^\beta \varphi(x) dx = 0 \text{ for all multi-indices } \beta \in \mathbb{N}_0^n \}.
$$

Following Triebel [59], we consider $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of $\mathcal{S}(\mathbb{R}^n)$, including the
topology. Thus, $S_1(\mathbb{R}^n)$ is a complete metric space. Equivalently, $S_1(\mathbb{R}^n)$ can be defined as a collection of all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that semi-norms
\[
\| \varphi \|_M = \sup_{|\beta| \leq M} \sup_{\xi \in \mathbb{R}^n} |\partial^\beta \varphi(\xi)| (|\xi|^M + |\xi|^{-M}) < \infty
\]
for all $M \in \mathbb{N}_0$, see [9, Section 3]. Let $S'_\infty(\mathbb{R}^n)$ be the topological dual of $S_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functionals on $S_\infty(\mathbb{R}^n)$. Let $0 < p < \infty$ and $0 < q \leq \infty$. Let $\{t_k\}$ be a $p$-admissible sequence, i.e. $t_k \in L^{\text{loc}}_p(\mathbb{R}^n)$, $k \in \mathbb{Z}$. The Triebel–Lizorkin space $F_{p,q}(\mathbb{R}^n, \{t_k\})$ is the collection of all $f \in S'_\infty(\mathbb{R}^n)$ such that
\[
\|f|F_{p,q}(\mathbb{R}^n, \{t_k\})\| = \left\| \left( \sum_{k = -\infty}^{\infty} t_k^{q} |\varphi_k \ast f|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} < \infty
\]
with the usual modifications if $q = \infty$.

We have organized the article in three sections. First we give some preliminaries and recall some basic facts on the Muckenhoupt classes and the weighted class of Tyulenev. Also we give some key technical lemmas needed in the proofs of the main statements. Especially, we present new version of weighted vector-valued maximal inequality of Fefferman and Stein. In Sect. 3 several basic properties such as the $\varphi$-transform characterization are obtained. We extend the well-known Sobolev embeddings to these function spaces and we give the atomic and molecular decomposition of these function spaces.

In Sect. 4 we study the inhomogeneous spaces $F_{p,q}(\mathbb{R}^n, \{t_k\})$, and we outline analogous results for these spaces. Other properties of these function spaces are given in [20].

\section{Maximal inequalities}

Our arguments of this paper essentially rely on the weighted boundedness of Hardy–Littlewood maximal function. In this paper we will assume that the weight sequence $\{t_k\}$ used to define the space $F_{p,q}(\mathbb{R}^n, \{t_k\})$ lies in the new weighted class $\check{X}_{2,\sigma,p}$ (see Definition 2.5). Therefore we need a new version of Hardy–Littlewood maximal inequality. Throughout this paper, we make some notation and conventions.

\subsection{Notation and conventions}

We denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. The expression $f \lesssim g$ means that $f \leq c g$ for some independent constant $c$ (and non-negative functions $f$ and $g$), and $f \approx g$ means $f \lesssim g \lesssim f$.

By $\text{supp}(f)$ we denote the support of the function $f$, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of $E$ and $1_E$ denotes its characteristic function. By $c$ we denote generic positive constants, which may have different values at different occurrences.
A weight is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere. For measurable set $E \subset \mathbb{R}^n$ and a weight $\gamma$, $\gamma(E)$ denotes

$$\int_E \gamma(x) \, dx.$$  

Given a measurable set $E \subset \mathbb{R}^n$ and $0 < p \leq \infty$, we denote by $L_p(E)$ the space of all functions $f : E \to \mathbb{C}$ equipped with the quasi-norm

$$\|f|L_p(E)\| = \left( \int_E |f(x)|^p \, dx \right)^{1/p} < \infty,$$

with $0 < p < \infty$ and

$$\|f|L_\infty(E)\| = \text{ess-sup}_{x \in E} |f(x)| < \infty.$$

For a function $f$ in $L^1_{\text{loc}}(\mathbb{R}^n)$, we set

$$M_A(f) = \frac{1}{|A|} \int_A |f(x)| \, dx$$

for any $A \subset \mathbb{R}^n$. Furthermore, we put

$$M_{A,p}(f) = \left( \frac{1}{|A|} \int_A |f(x)|^p \, dx \right)^{1/p},$$

with $0 < p < \infty$. Further, given a measurable set $E \subset \mathbb{R}^n$ and a weight $\gamma$, we denote the space of all functions $f : \mathbb{R}^n \to \mathbb{C}$ with finite quasi-norm

$$\|f|L_p(\mathbb{R}^n, \gamma)\| = \|f|L_p(\mathbb{R}^n)\|$$

by $L_p(\mathbb{R}^n, \gamma)$.

If $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$, then $p'$ is called the conjugate exponent of $p$.

Let $0 < p, q \leq \infty$. The space $L_p(\ell_q)$ is defined to be the set of all sequences $\{f_k\}$ of functions such that

$$\|\{f_k\}|L_p(\ell_q)\| = \left( \sum_{k=-\infty}^{\infty} |f_k|^q \right)^{1/q} |L_p(\mathbb{R}^n)| < \infty$$

with the usual modifications if $q = \infty$ and if $\{t_k\}$ is a sequence of functions then

$$\|\{f_k\}|L_p(\ell_q, \{t_k\})\| = \|\{t_kf_k\}|L_p(\ell_q)\|.$$

In what follows, $Q$ will denote a cube in the space $\mathbb{R}^n$ with sides parallel to the coordinate axes and $l(Q)$ will denote the side length of the cube $Q$. For all cubes $Q$ and $r > 0$, let $rQ$ be the cube concentric with $Q$ having side length $rl(Q)$. For $v \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$, denote by $Q_{v,m}$ the dyadic cube,
\[ Q_{v,m} = 2^{-v}([0, 1]^n + m). \]

For the collection of all such cubes we use \( Q = \{ Q_{v,m} : v \in \mathbb{Z}, m \in \mathbb{Z}^n \} \). For each cube \( Q \), we denote by \( x_{v,m} \) the lower left-corner \( 2^{-v}m \) of \( Q = Q_{v,m} \).

### 2.2 Muckenhoupt weights

The purpose of this subsection is to review some known properties of Muckenhoupt class.

**Definition 2.1** Let \( 1 \leq p < \infty \). We say that a weight \( \gamma \) belongs to the Muckenhoupt class \( A_p(\mathbb{R}^n) \) if there exists a constant \( C > 0 \) such that for every cube \( Q \) the following inequality holds

\[
M_Q(\gamma)M_{Q,p/p}(\gamma^{-1}) \leq C. \tag{2.1}
\]

The smallest constant \( C \) for which (2.1) holds is denoted by \( A_p(\gamma) \). As an example, we can take

\[ \gamma(x) = |x|^2, \quad x \in \mathbb{R}. \]

Then \( \gamma \in A_p(\mathbb{R}^n), 1 < p < \infty \), if and only if \( -n < \gamma < n(p - 1) \).

For \( p = 1 \) we rewrite the above definition in the following way.

**Definition 2.2** We say that a weight \( \gamma \) belongs to the Muckenhoupt class \( A_1(\mathbb{R}^n) \) if there exists a constant \( C > 0 \) such that for every cube \( Q \) and for a.e. \( y \in Q \) the following inequality holds:

\[
M_Q(\gamma) \leq C\gamma(y). \tag{2.2}
\]

The smallest constant \( C \) for which (2.2) holds is denoted by \( A_1(\gamma) \). The above classes have been first studied by Muckenhoupt [49] and use to characterize the boundedness of the Hardy–Littlewood maximal function on \( L^p(\gamma) \), see the monographs [31, 35] for a complete account on the theory of Muckenhoupt weights.

We recall a few basic properties of the class of \( A_p(\mathbb{R}^n) \) weights, see [22, Chapter 7], [35, Chapter 7] and [58, Chapter 5].

**Lemma 2.3** Let \( 1 \leq p < \infty \).

(i) If \( \gamma \in A_p(\mathbb{R}^n) \), then for any \( 1 \leq p < q \), \( \gamma \in A_q(\mathbb{R}^n) \).

(ii) Let \( 1 < p < \infty \). \( \gamma \in A_p(\mathbb{R}^n) \) if and only if \( \gamma^{1-p'} \in A_p(\mathbb{R}^n) \).

(iii) Let \( \gamma \in A_p(\mathbb{R}^n) \). There is \( C > 0 \) such that for any cube \( Q \) and a measurable subset \( E \subset Q \).
\[
\left( \frac{|E|}{|Q|} \right)^{p-1} M_Q(\gamma) \leq CM_E(\gamma).
\]

(iv) Suppose that \( \gamma \in A_p(\mathbb{R}^n) \) for some \( 1 < p < \infty \). Then there exists a \( 1 < p_1 < p < \infty \) such that \( \gamma \in A_{p_1}(\mathbb{R}^n) \).

(v) If \( \gamma \in A_p(\mathbb{R}^n) \), then for any \( 0 < \varepsilon < 1, \gamma^\varepsilon \in A_p(\mathbb{R}^n) \).

(vi) Let \( 1 \leq p < \infty \) and \( \gamma \in A_p(\mathbb{R}^n) \). Then there exist \( \delta \in (0, 1) \) and \( C > 0 \) depending only on \( n, p, \) and \( A_p(\gamma) \) such that for any cube \( Q \) and any measurable subset \( S \) of \( Q \) we have

\[
\frac{M_S(\gamma)}{M_Q(\gamma)} \leq C \left( \frac{|S|}{|Q|} \right)^{\delta-1}.
\]

The following theorem gives a useful property of \( A_p(\mathbb{R}^n) \) weights (reverse Hölder inequality), see [35, Chapter 7] or [47, Chapter 1].

**Theorem 2.4** Let \( 1 \leq p < \infty \) and \( \gamma \in A_p(\mathbb{R}^n) \). Then there exist a constants \( C > 0 \) and \( \varepsilon > 0 \) depending only on \( p \) and the \( A_p(\mathbb{R}^n) \) constant of \( \gamma \), such that for every cube \( Q \),

\[
M_{Q,1+\varepsilon}(\gamma) \leq CM_Q(\gamma).
\]

### 2.3 The weight class \( \mathcal{X}_{a,p} \)

Let \( 0 < p \leq \infty \). A weight sequence \( \{t_k\} \) is called \( p \)-admissible if \( t_k \in L^\text{loc}_p(\mathbb{R}^n) \) for all \( k \in \mathbb{Z} \). We mention here that

\[
\int_E t_k^p(x) \, dx < c(k)
\]

for any compact set \( E \subseteq \mathbb{R}^n \). For a \( p \)-admissible weight sequence \( \{t_k\} \) we set

\[
t_{k,m} = \|t_k|L_p(Q_{k,m})\|, \quad k, m \in \mathbb{Z}^n.
\]

Tyulenev [64] introduced the following new weighted class and used it to study Besov spaces of variable smoothness.

**Definition 2.5** Let \( \alpha_1, \alpha_2 \in \mathbb{R}, p, \sigma_1, \sigma_2 \in (0, +\infty), \alpha = (\alpha_1, \alpha_2) \) and let \( \sigma = (\sigma_1, \sigma_2) \). We let \( \mathcal{X}_{\alpha,\sigma,p} = \mathcal{X}_{\alpha,\sigma,p}(\mathbb{R}^n) \) denote the set of \( p \)-admissible weight sequences \( \{t_k\} \) satisfying the following conditions. There exist numbers \( C_1, C_2 > 0 \) such that for any \( k \leq j \) and every cube \( Q \),

\[
M_{Q,p}(t_k)M_{Q,\sigma_1}(t_j^{-1}) \leq C_1 2^{\alpha_1(k-j)}, \tag{2.3}
\]
\[
M_{Q,p}^{-1}(t_k)M_{Q,\sigma_1}(t_j) \leq C_22^{\sigma_2(j-k)}. \tag{2.4}
\]

The constants \(C_1, C_2 > 0\) are independent of both the indexes \(k\) and \(j\).

**Remark 2.6**

(i) We would like to mention that if \(\{t_k\}\) satisfies (2.3) with \(\sigma_1 = r(p/r)'\) and \(0 < r < p < \infty\), then \(t_k^p \in A_{p/r}(\mathbb{R}^n)\) for any \(k \in \mathbb{Z}\).

(ii) We say that \(t_k \in A_p(\mathbb{R}^n), k \in \mathbb{Z}, 1 < p < \infty\), have the same Muckenhoupt constant if

\[
A_p(t_k) = c, \quad k \in \mathbb{Z},
\]

where \(c\) is independent of \(k\).

(iii) Definition 2.5 is different from the one used in [64, Definition 2.1] and Definition 2.7 in [65], because we used the boundedness of the maximal function on weighted Lebesgue spaces.

**Example** Let \(0 < r < p < \infty\), a weight \(\omega^p \in A_\Psi(\mathbb{R}^n)\) and

\[
\{s_k\} = \{2^k\omega^p\}_{k \in \mathbb{Z}}, \quad s \in \mathbb{R}.
\]

Clearly, \(\{s_k\}_{k \in \mathbb{Z}}\) lies in \(\dot{X}_{\alpha,\sigma,p}\) for \(\alpha_1 = \alpha_2 = s\), \(\sigma = (r(p/r)', p)\).

**Remark 2.7** Let \(0 < \theta \leq p < \infty\). Let \(\alpha_1, \alpha_2 \in \mathbb{R}, \sigma_1, \sigma_2 \in (0, +\infty)\), \(\sigma_2 \geq p\), \(\alpha = (\alpha_1, \alpha_2)\) and let \(\sigma = (\sigma_1 = \theta(\frac{p}{\theta})', \sigma_2)\). Let a \(p\)-admissible weight sequence \(\{t_k\} \in \dot{X}_{\alpha,\sigma,p}\). Then

\[
\alpha_2 \geq \alpha_1,
\]

see [19].

Further notation will be properly introduced whenever needed.

**2.4 Auxiliary results**

In this subsection we present some results which are useful for us. Let recall the vector-valued maximal inequality of Fefferman and Stein [26]. As usual, we put

\[
\mathcal{M}(f)(x) = \sup_Q M_Q(f), \quad f \in L^1_{\text{loc}}(\mathbb{R}^n),
\]

where the supremum is taken over all cubes with sides parallel to the axis and \(x \in Q\). Also we set

\[
\mathcal{M}_\sigma(f) = \sup_Q M_{Q,\sigma}(f), \quad 0 < \sigma < \infty.
\]

Observe that \(\mathcal{M}_\sigma(f)\) can be rewritten as
\[ M_\sigma(f) = (M(|f|^\sigma))^{1/\sigma}, \quad 0 < \sigma < \infty. \]

**Theorem 2.8** Let \( 1 < p \leq \infty \). Then
\[
\|M(f)|L_p(\mathbb{R}^n)\| \lesssim \|f|L_p(\mathbb{R}^n)\|
\]
holds for all \( f \in L_p(\mathbb{R}^n) \).

For the proof see [35, Chapter 7]. Now, we state the vector-valued maximal inequality of Fefferman and Stein [26].

**Theorem 2.9** Let \( 0 < p < \infty, 0 < q \leq \infty \) and \( 0 < \sigma < \min(p, q) \). Then
\[
\left\| \left( \sum_{k=-\infty}^{\infty} (M_\sigma(f_k))^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \lesssim \left\| \left( \sum_{k=-\infty}^{\infty} |f_k|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}
\]
(2.5)
holds for all sequence of functions \( \{f_k\} \in L_p(\ell_q) \).

We shall require the following theorem, that is the Fefferman–Stein’s inequality, see [26].

**Lemma 2.10** Let \( 1 < p < \infty \). Given a non-negative real valued functions \( f \) and \( g \). We have
\[
\int_{\mathbb{R}^n} (M(f)(x))^p g(x)dx \leq c \int_{\mathbb{R}^n} (f(x))^p M(g)(x)dx,
\]
with \( c \) independent of \( f \) and \( g \).

For the proof see [35, Chapter 7]. We need the following version of the Calderón–Zygmund covering lemma, see [15, Lemma 3.3], [16, Appendix A] and [51, Chapter 7].

**Lemma 2.11** Let \( f \) be a measurable function such that \( M_Q(f) \to 0 \) as \( |Q| \to \infty \) and given a positive number \( a \) such that \( a > 2^{n+1} \). For each \( i \in \mathbb{Z} \) there exists a disjoint collection of maximal dyadic cubes \( \{Q^{i,h}\}_h \) such that for each \( h \)
\[
a^i \leq M_{Q^{i,h}}(f) \leq 2^n a^i
\]
and
\[
\Omega_i = \{x \in \mathbb{R}^n : M(f)(x) > 4^n a^i\} \subset \bigcup_h 3Q^{i,h}.
\]
Let
\[
E^i = \bigcup_h Q^{i,h}
\]
and
\[ E^{i,h} = Q^{i,h} \setminus (Q^{i,h} \cap E^{i+1,h}). \]

Then \( E^{i,h} \subset Q^{i,h} \), there exists a constant \( \beta > 1 \), depending only on \( a \), such \( |Q^{i,h}| \leq \beta |E^{i,h}| \) and the sets \( E^{i,h} \) are pairwise disjoint for all \( i \) and \( h \).

Next, we recall the following Hadamard’s three line theorem for subharmonic functions, see [50, Theorem 14.15].

**Theorem 2.12** Let \( f \) be a nonnegative, bounded function on the strip \( 0 \leq \Re(z) \leq 1 \) of the complex plane such that \( \log f(z) \) is subharmonic in the open strip \( 0 < \Re(z) < 1 \) and continuous on the strip \( 0 \leq \Re(z) \leq 1 \). If there are positive constants \( M_1, M_2 \), such that \( f(0 + iy) \leq M_1 \) and \( f(1 + iy) \leq M_2 \) for every real \( y \), then

\[ f(\theta + iy) \leq M_1^{1-\theta} M_2^\theta \]

for every \( \theta \in [0, 1] \) and any real \( y \).

Let \( S \) be the linear space consisting of all sequences \( \{f_k\} \) with \( f_k \) simple function on \( \mathbb{R}^n \) for each \( k \), and \( f_k = 0 \) for \( |k| \) large enough. Then \( S \) is dense in \( L^p(\ell_q, \{t_k\}) \), see [6].

The main aim of the following lemma is to extend an interpolation result for sublinear operators on Lebesgue spaces obtained by Calderón and Zygmund in [13] to similar operators acting on \( L^p(\ell_q, \{t_k\}) \) spaces.

Consider a mapping function \( T \), which maps measurable functions on \( \mathbb{R}^n \) into measurable functions on \( \mathbb{R}^n \). We say that \( T \) is sublinear if it satisfies (i) \( T(f) \) is defined (uniquely) if \( f = f_1 + f_2 \), \( T(f_1) \) and \( T(f_2) \) are defined, and

\[ |T(f_1 + f_2)| \leq |T(f_1)| + |T(f_2)|. \]

(ii) For any constant \( c \), \( T(cf) \) is defined if \( T(f) \) is defined and \( |T(cf)| = |c||T(f)| \).

**Lemma 2.13** Let \( t_k, k \in \mathbb{Z} \) be locally integrable functions on \( \mathbb{R}^n \) and \( 1 < q_i < \infty \), \( 1 < p_i < \infty \), \( M_i > 0 \), \( i = 0, 1 \). Suppose that \( T \) is a sublinear operator satisfying

\[ \| \{t_k T(f_k)\} \|_{L^p(\ell_q, \{t_k\})} \leq M_i \| \{t_k f_k\} \|_{L^p(\ell_q, \{t_k\})} \]

for any \( \{t_k f_k\} \in L^p(\ell_q, \{t_k\}), i = 0, 1 \). Then \( T \) can be extended to a bounded operator:

\[ \| \{t_k T(f_k)\} \|_{L^p(\ell_q)} \leq M_1^{1-\theta} M_2^\theta \| \{t_k f_k\} \|_{L^p(\ell_q)} \] (2.6)

for any \( \{t_k f_k\} \in L^p(\ell_q) \), where

\[ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 0 \leq \theta \leq 1. \]
Proof We will do the proof into two steps.

Step 1. We prove that (2.6) holds for any sequence of simple functions \( \{t_{k}f_{k}\} \in S \).

Substep 1.1. Preparation. Assume that

\[
\| \{t_{k}f_{k}\} \|_{L_{p}(\ell_{q})} = 1
\]

and we put

\[
I = \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^{\mathbb{n}}} t_{k}(x)|T(f_{k})(x)|g_{k}(x)\,dx,
\]

where \( \{g_{k}\} \in S \) such that

\[
\| \{g_{k}\} \|_{L_{p'}(\ell_{q'})} \leq 1.
\]

From our assumption on the sequence of simple functions \( \{t_{k}f_{k}\} \) and \( \{g_{k}\} \), we clearly have that

\[
t_{k}f_{k} = \sum_{l=1}^{N_{1}} |a_{l,k}|c_{l,k} \mathbb{1}_{E_{l,k}}, \quad \text{and} \quad g_{k} = \sum_{j=1}^{N_{2}} b_{j,k} \mathbb{1}_{E'_{j,k}},
\]

where \( |c_{l,k}| = 1, b_{j,k} > 0, l = 1, \ldots, N_{1}, j = 1, \ldots, N_{2}, N_{1}, N_{2} \in \mathbb{N} \) and

\[
t_{k}f_{k} = g_{k} = 0
\]

for sufficiently large \( k, |k| \geq N, N \in \mathbb{N} \). For \( k \) fixed, the sets \( E_{l,k} \) are disjoint. The same property holds for the sets \( E'_{j,k} \). For any \( z \in \mathbb{C} \) we set

\[
\frac{1}{q(z)} = \frac{1 - z}{q_{0}} + \frac{z}{q_{1}}, \quad \frac{1}{p(z)} = \frac{1 - z}{p_{0}} + \frac{z}{p_{1}}
\]

and

\[
\Phi(z) = \sum_{|k| \leq N} \int_{\mathbb{R}^{\mathbb{n}}} t_{k}(x)|T(o(\cdot, z)|f_{k}|^{q/q(z)} \text{sign} f_{k})(x)|(g_{k}(x))^{\beta(\Re(z))} \vartheta(x, \Re(z))\,dx,
\]

where

\[
o(\cdot, z) = A(z)(\cdot) t_{k}^{-\alpha(z)}(\cdot), \quad A(\cdot) = \| \{t_{k}f_{k}\}_{|k| \leq N} \|_{\ell_{q}}
\]

and

\[
\vartheta(\cdot, z) = B^{\kappa(z)-\beta(z)}(\cdot), \quad B(\cdot) = \| \{g_{k}\}_{|k| \leq N} \|_{\ell_{q}}
\]

with
\[
\tau(z) = \frac{p}{q(z)} - \frac{q(z)}{p}, \quad \alpha(z) = 1 - \frac{q(z)}{p}, \quad \kappa(z) = \frac{1 - \frac{p(z)}{1 - \frac{q(z)}}}{1 - \frac{1}{q(z)}}.
\]

The non-negative function \( \Phi \) reduces to \( I \) for \( z = \theta \). Using the fact that \( \{g_k\} \) is a sequence of simple functions we have

\[
\Phi(z) = \sum_{j=1}^{N_2} \sum_{|k| \leq N} \int_{E_{j,k}} \vartheta(x, \Re(z)) t_k(x) \left| T(\psi_{z,j,k}) (x) \right| \, dx,
\]

where

\[
\psi_{z,j,k} = \omega(\cdot, z)(b_{j,k})^{\beta(\Re(z))} \sum_{l=1}^{N_1} t_{l,k}^{-1} a_{l,k} \left| \frac{q(z)}{q_{l,k}} \right| x_{E_{j,k}}.
\]

We put

\[
\Psi_{j,k}(z) = \int_{E_{j,k}} \vartheta(x, \Re(z)) t_k(x) \left| T(\psi_{z,j,k}) (x) \right| \, dx, \quad |k| \leq N, j = 1, \ldots, N_2.
\]

**Substep 1.2.** We prove that \( \Phi \) is continuous in the strip \( 0 \leq \Re(z) \leq 1 \). First we prove that \( \Psi_{j,k} \) is well defined. Let us consider the integrals

\[
I_{z,j,k}^1 = \int_{E_{j,k}} \vartheta(x, \Re(z)) t_k(x) \left| T(\omega(\cdot, z) f_k^{1/q(z)} \text{sign} f_k) (x) \right|^{p_0} \, dx
\]

and

\[
I_{z,j,k}^2 = \int_{E_{j,k}} \left| \vartheta(x, \Re(z)) \right|^{p_0} (g_k(x))^{p_0 \beta(\Re(z))} \, dx.
\]

An easy calculation shows that the integral

\[
J = \int_{\mathbb{R}^n} \left( \sum_{|k| \leq N} t_{k}^{p_0} (x) \left| \omega(x, z) f_k (x) \right|^{q/z(q(z))} \right)^{p_0/q_0} \, dx
\]

is just

\[
\int_{\mathbb{R}^n} A^{p_0 \tau(\Re(z))} (x) \left( \sum_{|k| \leq N} \left( t_k (x) \right)^{(1-\tau(\Re(z)))q_0} \right)^{p_0/q_0} \, dx.
\]

We may assume without loss of generality that \( q_1 < q_0 \). As a consequence we see that

\[
\frac{1}{q} (1 - \alpha(\Re(z))) = \frac{1 - \Re(z)}{q_0} + \frac{\Re(z)}{q_1} > \frac{1}{q_0}.
\]

This yields
\[
\left( \sum_{|k| \leq N} t_k^{1-\alpha(\Re(z))q_0} |f_k(x)|^{1-\alpha(\Re(z))q_0} \right)^{1/q_0} \\
\leq \left( \sum_{|k| \leq N} t_k^q |f_k(x)|^q \right)^{(1-\alpha(\Re(z)))/q},
\]

since \( \ell_r \leq \ell_s, r \leq s \). Hence

\[
J \leq \left\| \{t_k f_k\} \right\|_{L_{p(\Re(z))}(\ell_q)} \left\| p_{0(\Re(z))} \right\| \leq \infty,
\]

because of \( \{t_k f_k\} \) is a sequence of simple functions. Consequently, by Hölder’s inequality,

\[
\Psi_{j,k}(z) \leq \left( \sum_{|j| \leq N} t_{j,k}^1 \right)^{1/p_0} \left( \sum_{|j| \leq N} t_{j,k}^2 \right)^{1/p_0'} \\
\leq \left\| \{t_k T(\alpha(\cdot, z)|f_k|^{q'/q(z)} \text{sign} f_k)\} \right\|_{L_{p_0}(\ell_{q_0})} \left( \sum_{|j| \leq N} t_{j,k}^2 \right)^{1/p_0'}.
\] (2.7)

Our assumption on \( T \) yields that the first term in (2.7) is finite. \( \sum_{|j| \leq N} t_{j,k}^2 \) is finite since \( \{g_k\} \in S \). Hence \( \Phi(z) \) exists for each \( z \) in the strip \( 0 \leq \Re(z) \leq 1 \).

Now we prove the continuity of \( \Phi \) in the strip \( 0 \leq \Re(z) \leq 1 \). We clearly have that

\[
\Psi_{j,k}(z + \Delta z) - \Psi_{j,k}(z) = J_{1,j,k}(z, \Delta z) + J_{2,j,k}(z, \Delta z),
\] (2.8)

where

\[
J_{1,j,k}(z, \Delta z) = \int_{E_{j,k}} \left( \vartheta(x, \Re(z + \Delta z)) - \vartheta(x, \Re(z)) \right) t_k(x) |T(\psi_{j,k}(x))| \, dx
\]

and

\[
J_{2,j,k}(z, \Delta z) = \int_{E_{j,k}} \vartheta(x, \Re(z + \Delta z)) t_k(x) \left( |T(\psi_{j,z,j,k}(x))| - |T(\psi_{j,k}(x))| \right) \, dx.
\]

We want to find the limit of (2.8) as \( \Delta z \) approaches 0, so we may assume that \( |\Delta z = \Re(\Delta z) + i\Im(\Delta z)| \leq 1 \). After a simple calculation we find that

\[
|\vartheta(x, \Re(z + \Delta z)) - \vartheta(x, \Re(z))| \\
= |B^{\kappa(\Re(z)) - \beta(\Re(z))}(x) \left( B^{d+\Re(\Delta z)}(x) - 1 \right)| \\
\leq B^{\kappa(\Re(z)) - \beta(\Re(z))}(x) \left( B^{d+\Re(\Delta z)}(x) + 1 \right) \\
\leq B^{\kappa(\Re(z)) - \beta(\Re(z))}(x) \left( B^d(x) \max(1, B^h(x), B^{-h}(x)) + 1 \right),
\]

where
h = \frac{1 - \frac{1}{p_1}}{1 - \frac{1}{q}} + \frac{1 - \frac{1}{p_0}}{1 - \frac{1}{q_0}}.
\quad d = \frac{1 - \frac{1}{p_0}}{1 - \frac{1}{q}} - \frac{1 - \frac{1}{q_0}}{1 - \frac{1}{q}}.

Recall that

\[ |T(\psi_{z,j,k})(x)| = (b_{j,k})^{\beta(|z|)} |T(\omega(\cdot, z)|f_k|^\frac{p}{q} \text{sign} f_k)(x)|.\]

Therefore, the function

\[ x \rightarrow |\vartheta(x, \Re(z + \Delta z)) - \vartheta(x, \Re(z))| t_k(x) |T(\psi_{z,j,k})(x)| \]

is integrable. Dominated convergence theorem yields that \( J_{1,j,k}(z, \Delta z) \) tends to zero as \( \Delta z \) tends to 0. Now

\[ |J_{2,j,k}(z, \Delta z)| \leq \int_{E_{j,k}^z} |\vartheta(x, \Re(z + \Delta z))| t_k(x) |T(\psi_{z+\Delta z,j,k} - \psi_{z,j,k})(x)| dx \]

since

\[ |T(\psi_{z+\Delta z,j,k})| - |T(\psi_{z,j,k})| \leq |T(\psi_{z+\Delta z,j,k} - \psi_{z,j,k})|, \quad j \in \{1, \ldots, N_2\}, |k| \leq N. \]

Hence,

\[ |J_{2,j,k}(z, \Delta z)| \leq \Lambda_{j,k} \times B_{j,k}, \quad j \in \{1, \ldots, N_2\}, |k| \leq N, \]

where

\[ \Lambda_{j,k} = (b_{j,k})^{\beta(|z|)} \left( \int_{E_{j,k}^z} \vartheta^{p_0}(x, \Re(z + \Delta z)) dx \right)^{1/p_0} \]

and

\[ B_{j,k} = \left( \int_{E_{j,k}^z} t_k^{p_0}(x) |T((b_{j,k})^{-\beta(|z|)}(\psi_{z+\Delta z,j,k} - \psi_{z,j,k}))(x)|^{p_0} dx \right)^{1/p_0}. \]

Clearly, \( \Lambda_{j,k}, j \in \{1, \ldots, N_2\}, |k| \leq N \) is bounded. To estimate \( B_{j,k} \) we have

\[ |\psi_{z+\Delta z,j,k}| \]

\[ = |\omega(\cdot, z + \Delta z)| (b_{j,k})^{\beta(|z+\Delta z|)} \sum_{l=1}^{N_1} t_k^{-1} a_{l,k} |^{q/|q(z+\Delta z)|} \lambda_{E_{l,k}} \]

\[ = |\psi_{z,j,k}| A^{\sigma(|\Delta z|)} (b_{j,k})^{q(1/q_0-1/q_1)|\Re(\Delta z)|} \sum_{l=1}^{N_1} a_{l,k} |^{q(1/q_1-1/q_0)|\Re(\Delta z)|} \lambda_{E_{l,k}}, \]

where
\[
\sigma = p \left( \frac{1}{p_1} - \frac{1}{p_0} \right) + q \left( \frac{1}{q_0} - \frac{1}{q_1} \right).
\]

There are two cases: \( 0 \leq \Re(\Delta z) \leq 1 \) and \(-1 \leq \Re(\Delta z) < 0 \). In the first case we obtain the estimates

\[
\sum_{l=1}^{N_1} |a_{l,k}|q^{(1/q_1-1/q_0)}\Re(\Delta z) \psi_{E_{l,k}} = A^q(1/q_1-1/q_0)\Re(\Delta z)
\]

and

\[
(b_{j,k})^q^{(1/q_0-1/q_1)}\Re(\Delta z) \leq \max \left( 1, (b_{j,k})^q(1/q_0-1/q_1) \right).
\]

Indeed, the left-hand side of (2.10) is equal to

\[
\left( \sum_{l=1}^{N_1} |a_{l,k}|c_{l,k}^q \right)^{(1/q_1-1/q_0)}\Re(\Delta z) = A^q(1/q_1-1/q_0)\Re(\Delta z).
\]

Now, if \( b_{j,k} \geq 1, j \in \{1, \ldots, N_2\}, |k| \leq N \), then

\[
(b_{j,k})^q(1/q_0-1/q_1)\Re(\Delta z) \leq 1,
\]

since \( q_1 < q_0 \) and \( 0 \leq \Re(\Delta z) \leq 1 \). While if \( 0 < b_{j,k} < 1 \), then

\[
(b_{j,k})^q(1/q_0-1/q_1)\Re(\Delta z) = (b_{j,k})^q(1/q_0-1/q_1)(\Re(\Delta z)-1)(b_{j,k})^q(1/q_0-1/q_1) \\
\leq (b_{j,k})^q(1/q_0-1/q_1).
\]

Thus, we obtain (2.11). Assume that \(-1 \leq \Re(\Delta z) < 0 \). We obtain

\[
(b_{j,k})^q(1/q_0-1/q_1)\Re(\Delta z) \leq \max \left( 1, (b_{j,k})^q(1/q_0-1/q_0) \right),
\]

\[
|a_{l,k}|^{q(1/q_1-1/q_0)}\Re(\Delta z) \leq \max(1, |a_{l,k}|^{q(1/q_0-1/q_1)})
\]

and

\[
A^{q(1/q_0-1/q_1)}\Re(\Delta z) \leq \max(1, A^{q(1/q_1-1/q_0)}),
\]

\[
A^{q\Re(\Delta z)} < \max(1, A^{q(1/q_1-1/q_0)}).
\]

Substituting these estimations in (2.9) we find that

\[
|\psi_{z+\Delta z,j,k}| \\
\leq |\psi_{z,j,k}| \max \left( 1, \max(1, A^{q(1/q_1-1/q_0)}), \max(1, A^\sigma, A^{-\sigma}) \right) \mu_{j,k},
\]

where \( \{\mu_{j,k}\} \in S \). This estimate together with
\[ \{ \mathcal{I}_k(b_{j,k})^{-\beta(R(z))}\psi_{z,j,k} \}_{|z| \leq N} \in \mathcal{L}_{p_0}(\mathcal{E}_{q_0}) \]
guarantees that the function
\[
x \mapsto \mathcal{I}_k(b_{j,k})^{-\beta(R(z))}|\psi_{z,j,k}(x)|
\]
\[
\times \max \left(1, \max(1, A^{q(1/q_1-1/q_0)}(x)), \max(1, A^\sigma(x), A^{-\sigma}(x)) \right) \mu_{j,k}(x)
\]
belongs to \( \mathcal{L}_{p_0}(\mathcal{E}_{q_0}), \ |k| \leq N, \ j \in \{1,\ldots, N_2\} \). Hence,
\[
\left( \int_{E_{j,k}} \mathcal{I}_k^{p_0}(x)|T((b_{j,k})^{-\beta(R(z))}(\psi_{z+\Delta z,j,k} - \psi_{z,j,k}))(x)|^{p_0} \, dx \right)^{1/p_0} \leq \left( \int_{\mathbb{R}^n} \sum_{|z| \leq N} \mathcal{I}_k^{q_0}(x)|(b_{j,k})^{-\beta(R(z))}(\psi_{z+\Delta z,j,k}(x) - \psi_{z,j,k}(x))|^{q_0} \, dx \right)^{p_0/q_0} \}
\]
Dominated convergence theorem yields that \( J_{z,j,k}(z, \Delta z) \) tends to zero as \( \Delta z \) tends to 0 and \( \Psi_{j,k} \) is continuous at \( z \). Hence \( \Phi \) is continuous in the strip \( 0 \leq R(z) \leq 1 \).

**Substep 1.3.** Here we prove that \( \Phi \) is bounded in the strip \( 0 \leq R(z) \leq 1 \). From Substep 1.2, we need only to prove that \( I_{z,j,k}^2 \leq M \), for a suitable constant \( M \) independent of \( z \). Since
\[
B^{(R(z))-\beta(R(z))}(x) \leq \max(1, B^\rho(x))(g_k(x))^{-\beta(R(z))}
\]
for any \( x \in \mathbb{R}^n \) and \( |k| \leq N \), we find that
\[
I_{z,j,k}^2 \leq \int_{E_{j,k}} \max(1, B^\rho(x)) \, dx \leq M,
\]
we are done.

**Substep 1.4.** In this step we prove that \( \log \Psi_{j,k}(z) \) is subharmonic. We fix a harmonic function \( h(z) \), and denote by \( H(z) \) the analytic function whose real part is \( h(z) \). We need only to prove that \( \Psi_{j,k}(z)e^{h(z)} \) is subharmonic for every harmonic \( h(z) \). Since the problem is local, we may consider \( h \) and \( H \) in a given circle. Put
\[
\psi_{z,j,k}^* = \psi_{z,j,k}e^{H(z)} \quad \text{and} \quad \Psi_{j,k}^*(z) = \Psi_{j,k}(z)e^{H(z)}.
\]
We fix \( z \), take a \( q > 0 \), and denote by \( z_1, \ldots, z_p \) be a system of points equally spaced over the circumference of the circle with center \( z \) and radius \( q \). Our estimate use partially some techniques already used in [43]. First we prove that \( \log |T(\psi_{z,j,k})| \) is subharmonic. We want to show that
\[
|T(\psi_{z,j,k}^*)| = e^{h(z)}|T(\psi_{z,j,k})| \leq \frac{1}{2\pi} \int_0^{2\pi} |T(\psi_{z+te^{i\theta},j,k})| \, dt.
\]
Let us calculate the limit of
\[ \left\| \frac{1}{r} \sum_{v=1}^{r} \psi_{z_v,j,k}^* \right\|_{p_1} \tag{2.12} \]

as \( r \) tends to infinity. We estimate \( |\psi_{z_v,j,k}^*|, v = 1, \ldots, r \). Clearly

\[ |\psi_{z_v,j,k}^*| \leq C_2 t_k^{-1} A^{\tau(\Re(z_v))} \sum_{l=1}^{N_1} |\mathcal{X}_{E_{l,k}}|. \]

We consider separately the possibilities \( \tau(\Re(z_v)) \geq 0 \) and \( \tau(\Re(z_v)) < 0 \). In the first case

\[ A^{\tau(\Re(z_v))} \leq C_3 1 \leq N_1 \| \ell_q \| \leq C_4 \left\| \sum_{l=1}^{N_1} |\mathcal{X}_{E_{l,k}}| \right\|_{|l| \leq N} = C_4 K. \]

If \( \tau(\Re(z_v)) < 0 \), then

\[ A^{\tau(\Re(z_v))} \leq |t_0 f_0|^{\tau(\Re(z_v))} \leq C_5 \sum_{l=1}^{N_1} |\mathcal{X}_{E_{l,0}}|. \]

Therefore,

\[ |\psi_{z_v,j,k}^*| \lesssim \max(1, K) t_k^{-1} g_{j,k} \]

where \( \{g_{j,k}\}_{k \in \mathbb{Z}} \in S \). Notice that the implicit constant independent of \( r, v, j \) and \( k \). This guarantees that (2.12) can be estimated by

\[ C \left\| \psi_{z_v,j,k}^* - \frac{1}{r} \sum_{v=1}^{r} \psi_{z_v,j,k}^* \right\|_{p_1}. \]

In view of the fact that

\[ \psi_{z,j,k}^*(x) = \lim_{r \to \infty} \frac{1}{r} \sum_{v=1}^{r} \psi_{z_v,j,k}^*(x), \]

which is valid since \( \psi_{z,j,k}^* \) is analytic for each \( x \), dominated convergence theorem yields that the last norm tends to zero as \( r \) tends to infinity. Therefore, (2.12) tends to zero as \( r \) tends to infinity. There exist subsequences

\[ \left\{ T \left( \psi_{z,j,k}^* - \frac{1}{r_l} \sum_{v=1}^{r_l} \psi_{z_v,j,k}^* \right) \right\}_l \]

converging to zero as \( r_l \) tends to infinity. Hence,
\[ |T(\psi_{z,j,k}^*)| \leq \lim_{r_i \to \infty} \left| T \left( \frac{1}{r_i} \sum_{v=1}^n \psi_{z,v,j,k}^* \right) \right| \]

\[ = \lim_{r_i \to \infty} \frac{1}{r_i} \sum_{v=1}^n |T(\psi_{z,v,j,k}^*)| \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} |T(\psi_{z,qe^{it},j,k})| \, dt. \]

In addition, the mapping \( z \to \log(\vartheta(\cdot, \Re(z))t_k(\cdot)) \) is subharmonic. Consequently, \( z \to \log(\vartheta(\cdot, \Re(z))t_k(\cdot)|T(\psi_{z,j,k})|) \) is subharmonic. Then

\[ e^{h(z)} \vartheta(x, \Re(z))t_k(x)|T(\psi_{z,j,k}^*)(x)| \]

\[ \leq \frac{1}{2\pi} \int_0^{2\pi} \vartheta(x, \Re(z + qe^{it}))t_k(x) |T(\psi_{z,qe^{it},j,k}^*)(x)| \, dt. \]

We integrate with respect to \( x \) over the set \( E'_{j,k} \), we apply Fubini’s theorem and observe that

\[ \Psi_{j,k}(z)e^{h(z)} = \int_{E'_{j,k}} \vartheta(x, \Re(z))t_k(x) |T(\psi_{z,j,k}^*)(x)| \, dx, \]

we obtain

\[ \Psi_{j,k}^*(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \Psi_{j,k}^*(z + qe^{it}) \, dt. \]

Hence, \( \Psi_{j,k}^* \) is subharmonic in the strip \( 0 \leq \Re(z) \leq 1 \).

**Substep 1.5.** We prove that (2.6) holds for any sequence of simple functions \( \{t_{k_i}f_i\}_{k_i \in \mathbb{Z}} \in S \). Let \( z = \Re(z) + i\Im(z), 0 \leq \Re(z) \leq 1 \) and \( \Im(z) \in \mathbb{R} \). From Substeps 1.2–1.4, \( \Phi \) is continuous, bounded its logarithm is subharmonic in the strip \( 0 \leq \Re(z) \leq 1 \). Let us show that \( \Phi(i\Im(z)) \leq M_1 \) for any \( \Im(z) \in \mathbb{R} \). We have

\[ D^{\alpha(0)} = D^{\beta(0)} = D^{\beta(0) - \beta(0)} = D^{\beta(0)/p_0' - q_0}/q_0 \]

\[ |D^{q/(\Im(z))}| = D^{1-q}/q_0 \]

for any \( D \geq 0 \). By Hölder’s inequality \( \Phi(i\Im(z)) \) can be estimated by

\[ \left( \int_{\mathbb{R}^n} \left( \sum_{|k| \leq N} t_{k_i}^{\alpha(0)}(x) |T(\omega(\cdot, i\Im(z))f_i^{q/(\Im(z))}\text{sign}f_i)(x)|^{q_0} \right)^{p_0'/q_0} \, dx \right)^{1/p_0} \]

\[ \times \left( \int_{\mathbb{R}^n} \left( \sum_{|k| \leq N} (G_k(x))^{\beta(0)}(x, 0)^{q_0} \right)^{p_0'/q_0} \, dx \right)^{1/p_0'}. \]

The second integral is just
\[
\left( \int_{\mathbb{R}^n} \left( \sum_{|k| \leq N} g_k^p(x) \right)^{p'/q'} \, dx \right)^{1/p'_0} \leq 1,
\]
since \(\kappa(0) - \beta(0) = p'/p'_0 - q'/q'_0\) and \(\beta(0) = q'/q'_0\). The boundedness of \(T\) on \(L_{p_0}(\ell_{q_0}, \{t_k\})\) yields that the first integral does not exceed \(M_1\left( \int_{\mathbb{R}^n} \left( \sum_{|k| \leq N} t_k^{q_0}(x) \right) \omega(x, i\Im(z)) |f_k(x)|^{q/(q_i(z))} \text{sign}_k |q_0| \, dx \right)^{1/p_0}\), \((2.13)\)

but \((2.13)\) is just
\[
M_1 \| \{t_k f_k\} |L_p(\ell_q)| \|^{p/p_0} = M_1.
\]

Similarly, we obtain
\[
\Phi(1 + i\Im(z)) \leq M_2.
\]

From our previous substeps \(\log \Phi(z)\) is continuous bounded above and subharmonic in the strip \(0 \leq \Re(z) \leq 1\). In addition, \(\log \Phi(z)\) does not exceed \(\log M_1\) and \(\log M_2\) on the lines \(\Re(z) = 0\) and \(\Re(z) = 1\), respectively. Therefore, we apply the Hadamard three-line theorem for subharmonic functions, see Theorem 2.12, we obtain that
\[
\log \Phi(z) \leq (1 - \theta) \log M_1 + \theta \log M_2, \quad 0 < \Re(z) < 1.
\]

In particular,
\[
I = \Phi(\theta) \leq M_1^{1-\theta} M_2^\theta.
\]

Finally, we have proved that
\[
\| \{t_k T(f_k)\} |L_p(\ell_q)| \| \leq M_1^{1-\theta} M_2^\theta \| \{t_k f_k\} |L_p(\ell_q)| \|
\]
for each sequence of simple functions \(\{t_k f_k\}\).

Step 2. We prove \((2.6)\). Assume \(q_1 < q_0\) and let \(\{f_k\} \in L_p(\ell_q, \{t_k\})\). Then \(\{t_k f_k\} \in L_p(\ell_q)\). There exists a sequence \(\{g_n\}_{n \in \mathbb{N}} = \{\{g^k_n\}_{k \in \mathbb{Z}}\}_{n \in \mathbb{N}} \subset S \subset L_p(\ell_q)\) such that \(\{g_n\}_{n \in \mathbb{N}}\) converges to \(\{t_k f_k\}\) in \(L_p(\ell_q)\). Hence,
\[
\| \{t_k^{-1} g_n^k - f_k\} |L_p(\ell_q, \{t_k\}) \|
\]
tends to zero as \(n\) tends to infinity. Therefore,
\[
\| \{t_k^{-1} g_n^k - f_k\} |L_p(\ell_{q_0}, \{t_k\}) \|
\]
tends to zero as \(n\) tends to infinity, since \(L_p(\ell_q) \hookrightarrow L_p(\ell_{q_0})\). Observe that
\[
\| T(f_k) - |T(t_k^{-1} g^k_n)| \leq |T(f_k - t_k^{-1} g^k_n)|
\]
for any \(k \in \mathbb{Z}\) and any \(n \in \mathbb{N}\). Therefore,
\[
\left\| \{ |T(f_k)| \} - \{ |T(t_k^{-1} g_n)| \} \right\|_{L^p(\ell_q, \{ t_k \})}
\]
can be estimated by
\[
\left\| \{ T(f_k - t_k^{-1} g_n) \} \right\|_{L^p(\ell_q, \{ t_k \})} \lesssim \left\| \{ t_k^{-1} g_n - f_k \} \right\|_{L^p(\ell_q, \{ t_k \})},
\]
where this inequality follows by the fact that \( \{ t_k^{-1} g_n \} \in L^p(\ell_q, \{ t_k \}) \) since
\[
\left\| \{ t_k^{-1} g_n \} \right\|_{L^p(\ell_q, \{ t_k \})} = \left\| \{ g_n \} \right\|_{L^p(\ell_q)} < \infty.
\]
This yields that
\[
\left\| \{ T(f_k - t_k^{-1} g_n) \} \right\|_{L^p(\ell_q, \{ t_k \})}
\]
tends to zero as \( n \) tends to infinity. Therefore, \( \{ \{ T(t_k^{-1} g_n) \} \}_{n \in \mathbb{N}} \) converges to \( \{ |T(f_k)| \} \) in \( L^p(\ell_q, \{ t_k \}) \)-norm, which yields that \( \{ t_k |T(t_k^{-1} g_n)| \} \}_{n \in \mathbb{N}} \) converges to \( t_k |T(t_k^{-1} g_n)| \) in \( L^p(\ell_q) \)-norm. Hence, \( \{ t_k |T(t_k^{-1} g_n)| \} \}_{n \in \mathbb{N}} \) converges to \( t_k |T(f_k)| \) in \( L^p \) for every \( k \in \mathbb{Z} \). The Cantor diagonal technique gives an increasing sequence \( \{ n_k(n) \}_{n \in \mathbb{N}} \) in \( \mathbb{N} \) such that \( \{ t_k |T(t_k^{-1} g_{n_k(n)})| \} \}_{n \in \mathbb{N}} \) converges to \( t_k |T(f_k)| \) for every \( k \in \mathbb{Z} \). We have
\[
\left\| \{ T(f_k) \} |L^p(\ell_q, \{ t_k \}) \right\| = \left\| \{ t_k T(f_k) \} |L^p(\ell_q) \right\|
\]
\[
= \left\| \left( \sum_{k=-\infty}^{\infty} t_k^q |T(f_k)|^q \right)^{1/q} \right\|_p
\]
\[
= \left\| \left( \sum_{k=-\infty}^{\infty} \lim_{n \to \infty} (t_k |T(t_k^{-1} g_{n_k(n)})|)^q \right)^{1/q} \right\|_p
\]
\[
\leq \liminf_{n \to \infty} \left( \sum_{k=-\infty}^{\infty} \left( t_k |T(t_k^{-1} g_{n_k(n)})| \right)^q \right)^{1/q}
\]
by Fatou Lemma. We have proved that the last norm is bounded by
\[
M_1^{1-\theta} M_2^\theta \left\| \{ t_k^{-1} g_{n_k(n)} \} \right\|_{L^p(\ell_q, \{ t_k \})} \]
\[
\leq M_1^{1-\theta} M_2^\theta \left\| \{ t_k^{-1} g_{n_k(n)} - f_k \} \right\|_{L^p(\ell_q, \{ t_k \})} + M_1^{1-\theta} M_2^\theta \left\| \{ f_k \} \right\|_{L^p(\ell_q, \{ t_k \})}.
\]
Consequently,
\[
\left\| \{ T(f_k) \} \right\|_{L^p(\ell_q, \{ t_k \})} \leq M_1^{1-\theta} M_2^\theta \left\| \{ f_k \} \right\|_{L^p(\ell_q, \{ t_k \})}
\]
for any \( \{ t_k f_k \} \in L^p(\ell_q) \) and the lemma is proved. \( \square \)

Now we state the main result of this subsection.
Lemma 2.14  Let $1 < \theta \leq p < \infty$ and $1 < q < \infty$. Let $\{t_k\}$ be a $p$-admissible weight sequence such that $p^j_k \in A_p(\mathbb{R}^n)$, $k \in \mathbb{Z}$. Assume that $p^j_k$, $k \in \mathbb{Z}$ have the same Muckenhoupt constant, $A_q(t_k) = c$, $k \in \mathbb{Z}$. Then

$$\left\| \left( \sum_{k=-\infty}^{\infty} t^q_k(\mathcal{M}(f_k))^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \leq \left\| \left( \sum_{k=-\infty}^{\infty} t^q_k|f_k|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}$$

holds for all sequences of functions $\{t_kf_k\} \in L_p(\ell_q)$.

**Proof**  We will do the proof into two steps.

*Step 1.* We prove (2.14) with $1 < q \leq p < \infty$. The case $p = q$ follows, e.g., by [19], so we assume that $1 < q < p < \infty$. Let $j \in \mathbb{N}$ be such that $q^j \leq p < q^{j+1}$.

*Substep 1.1.* We consider the case $q^j < p < q^{j+1}$. To prove we additionally do it into the two steps Substeps 1.1.1 and 1.1.2.

*Substep 1.1.1.* We consider the case $1 < q \leq \theta^j$. From Lemma 2.3/(i), $t^j_k \in A_{p/q}(\mathbb{R}^n)$. By duality the left-hand side of (2.14) is bounded by

$$\sup \sum_{k=-\infty}^{\infty} \int t_k(x)\mathcal{M}(f_k)(x)|g_k(x)|dx = \sup \sum_{k=-\infty}^{\infty} T_k,$$

where the supremum is taken over all sequence of functions $\{g_k\} \in L_{p'}(\ell_{q'})$ with

$$\left\| \{g_k\}|L_{p'}(\ell_{q'})\right\| \leq 1,$$

where $p'$ and $q'$ are the conjugate exponent of $p$ and $q$, respectively. Let $Q$ be a cube. By Hölder’s inequality,

$$M_Q(f_k) \leq \frac{1}{|Q|} \left\| t_kf_k|L_p(Q)\right\| \left\| t_k^{-1}|L_{p'}(Q)\right\| \leq \frac{c}{|Q|} \left\| t_k^{-1}|L_{p'}(Q)\right\|, \quad k \in \mathbb{Z}$$

with $c > 0$ is independent of $k$. Since $t^j_k \in A_{p/q}(\mathbb{R}^n)$, $k \in \mathbb{Z}$, by Lemma 2.3/(i), $t^j_k \in A_p(\mathbb{R}^n)$, $k \in \mathbb{Z}$ and

$$\frac{1}{|Q|} \left\| t_k^{-1}|L_{p'}(Q)\right\| \leq c \left\| t_k|L_p(Q)\right\|^{-1}.$$

Moreover, $\left\| t_k|L_p(Q)\right\| \rightarrow \infty$ as $|Q| \rightarrow \infty$ for any $k \in \mathbb{Z}$. Hence we can apply Lemma 2.11. Let

$$\Omega^j_k = \{x \in \mathbb{R}^n : \mathcal{M}(f_k)(x) > 4^n \lambda^j\}, \quad k, i \in \mathbb{Z}$$

with $\lambda > 2^{n+1}$ and

$$H^j_k = \{x \in \mathbb{R}^n : 4^n \lambda^j < \mathcal{M}(f_k)(x) \leq 4^n \lambda^{j+1}\}, \quad k, i \in \mathbb{Z}.$$

We have
\[
T_k = \sum_{i=-\infty}^{\infty} \int_{H_k} t_k(x) \mathcal{M}(f_k(x)) |g_k(x)| \, dx \leq 4^n \sum_{i=-\infty}^{\infty} \lambda_i^{i+1} \int_{Q^c_k} t_k(x) |g_k(x)| \, dx.
\]

Let \( \{Q^j_k\}_h \) be the collection of maximal dyadic cubes as in Lemma 2.11 with
\[
Q_k^i \subset \bigcup_h 3Q^j_k,
\]
which implies that
\[
T_k \leq 4^n \sum_{i=-\infty}^{\infty} \sum_{h=0}^{\infty} \lambda_i^{i+1} \int_{3Q^j_k} t_k(x) |g_k(x)| \, dx, \quad k \in \mathbb{Z}.
\]

(2.15)

Applying Hölder’s inequality,
\[
\int_{3Q^j_k} t_k(x) |g_k(x)| \, dx \leq \left( \int_{3Q^j_k} t_k^r(x) \, dx \right)^{1/\tau} \left( \int_{3Q^j_k} |g_k(x)|^{\tau'} \, dx \right)^{1/\tau'}
= |3Q^j_k| M_{3Q^j_k, \tau}(t_k) M_{3Q^j_k, \tau'}(g_k)
\]
with \( \tau > 1 \). Put \( \tau = p(1+\varepsilon) \) with \( \varepsilon \) as in Theorem 2.4, which is possible since \( t_k^p \in A_{p/\theta}(\mathbb{R}^n) \), \( k \in \mathbb{Z} \). Obviously, we have
\[
M_{3Q^j_k, \tau}(t_k) = M_{3Q^j_k, p(1+\varepsilon)}(t_k) \leq c M_{3Q^j_k, p}(t_k), \quad k \in \mathbb{Z}.
\]

Since \( t_k^p \), \( k \in \mathbb{Z} \) have the same Muckenhoupt constant and from the proof of Theorem 7.2.2 in [35] the constant \( c \) is independent of \( k \). Therefore,
\[
\int_{3Q^j_k} t_k(x) |g_k(x)| \, dx \leq |Q^j_k| M_{3Q^j_k, p}(t_k) M_{3Q^{j'}_k, \tau'}(g_k).
\]

We deduce from the above that
\[
\lambda_i^i \int_{3Q^j_k} t_k(x) |g_k(x)| \, dx \leq |Q^j_k| M_{3Q^j_k, p}(t_k) M_{3Q^{j'}_k, \tau'}(g_k).
\]

By Hölder’s inequality,
\[
M_{Q^{j'}_k}(f_k) \leq M_{3Q^{j'}_k, \tau'}(t_k^{-1}) M_{3Q^{j'}_k, \lambda}(t_k f_k), \quad s = \frac{p}{q_i}
\]
and, with the help of the fact that \( q_i > 1 \),
\[
M_{3Q^{j'}_k, \tau'}(t_k^{-1}) \leq \left( M_{3Q^{j'}_k, \tau}(t_k^{-p}) \right)^{1/p}.
\]

Hence,
\[ \lambda^i \int_{3Q_k^h} t_k(b(x)g_k(x))dx \leq |Q_k^h| M_{3Q_k^h}^{i,h}(t_kf_k)M_{3Q_k^h}^{i,h}(g_k), \]

because of \( t_k^h \in A_s(\mathbb{R}^n), \; k \in \mathbb{Z} \). Since \(|Q_k^h| \leq \beta |E_k^h| \), with \( E_k^h = Q_k^h \setminus (\bigcup_b Q_{k+b}^{i,h}) \) and the family \( E_k^h \) are pairwise disjoint, the last expression is bounded by

\[ c \int_{E_k^h} M_{3Q_k^h}^{i,h}(t_kf_k)M_{3Q_k^h}^{i,h}(g_k)dx \]

\[ \lesssim \int_{\mathbb{R}^n} \mathcal{M}_s(t_kf_k)(x)\mathcal{M}_e(g_k)(x)1_{E_k^h}(x)dx. \]

Therefore, the right-hand side of (2.15) does not exceed

\[ c \sum_{i=-\infty}^{\infty} \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \mathcal{M}_s(t_kf_k)(x)\mathcal{M}_e(g_k)(x)1_{E_k^h}(x)dx \]

\[ \lesssim \int_{\mathbb{R}^n} \mathcal{M}_s(t_kf_k)(x)\mathcal{M}_e(g_k)(x)dx. \]

This implies that

\[ \sum_{k=-\infty}^{\infty} T_k \lesssim \int_{\mathbb{R}^n} \sum_{k=-\infty}^{\infty} \mathcal{M}_s(t_kf_k)(x)\mathcal{M}_e(g_k)(x)dx. \]

By Hölder’s inequality the term inside the integral is bounded by

\[ \left( \sum_{k=-\infty}^{\infty} \left( \mathcal{M}_s(t_kf_k)(x) \right)^q \right)^{1/q} \left( \sum_{k=-\infty}^{\infty} \left( \mathcal{M}_e(g_k)(x) \right)^q \right)^{1/q'} \]

for any \( x \in \mathbb{R}^n \). Again by Hölder’s inequality \( \sum_{k=-\infty}^{\infty} T_k \) can be estimated by

\[ c \left\| \left( \sum_{k=-\infty}^{\infty} \left( \mathcal{M}_s(t_kf_k) \right)^q \right)^{1/q} |L_p(\mathbb{R}^n)| \right\| \left\| \left( \sum_{k=-\infty}^{\infty} \left( \mathcal{M}_e(g_k) \right)^q \right)^{1/q'} |L_{p'}(\mathbb{R}^n)| \right\|. \]

where in the second inequality we used the vector-valued maximal inequality of Fefferman and Stein (2.5), because of \( 0 < s < q < p \), and the fact that
\[
\left\| \left( \sum_{k=-\infty}^{\infty} (M_{\tau}(g_k))^q \right)^{1/q'} \right\|_{L^{q'}(\mathbb{R}^n)} \lesssim \left\| \{g_k\}L^{q'}(\ell_q^p) \right\| \lesssim 1,
\]

since \( \tau = p(1 + \varepsilon) > p > q \).

**Substep 1.1.2.** We consider the case \( \theta < q < p \). Let

\[
\frac{1}{q} = \frac{1 - \lambda}{p} + \frac{\lambda}{p'}, \quad 0 < \lambda < 1.
\]

Applying Lemma 2.13 we obtain the desired estimate.

**Substep 1.2.** We have proved (2.14) in case of \( q_j < p < q_j + 1 \). Now we study the case \( p = q_j, j \in \mathbb{N} \). Then \( j \geq 2 \). Since \( t_k^p \in A_{p}(\mathbb{R}^n), k \in \mathbb{Z} \), from Theorem 2.4 there exists a number \( \gamma_1 > 0 \) such

\[
M_{Q,1+\gamma_1}(t_k^p) \lesssim M_Q(t_k^p), \quad k \in \mathbb{Z},
\]

where the implicit constant is independent of \( k \), see, e.g., [35, Theorem 7.2.5 and Corollary 7.2.6]. In addition by Lemma 2.3/(i), \( t_k^{-q'} \in A_{p'}(\mathbb{R}^n) \) and again by Theorem 2.4 there exists a number \( \gamma_2 > 0 \) such

\[
M_{Q,1+\gamma_2}(t_k^{-q'}) \lesssim M_Q(t_k^{-q'}), \quad k \in \mathbb{Z}.
\]

Let

\[
0 < \gamma < \min \left( \gamma_1, \frac{p - p/q}{p/q - 1} \right).
\]

Then

\[
M_{Q,1+\gamma}(t_k^p) \left( M_{Q,1+\gamma}(t_k^{-q'}) \right)^{\gamma} \lesssim 1, \quad k \in \mathbb{Z},
\]

where the implicit constant is independent of \( k \). Hence, \( t_k^{p(1+\gamma)} \in A_{p}(\mathbb{R}^n), k \in \mathbb{Z} \). Therefore, \( t_k^p \in A_{p_1}(\mathbb{R}^n), k \in \mathbb{Z} \), where

\[
p_1 = \frac{p + \gamma}{1 + \gamma},
\]

see again [35, Exercise 7.1.3]. Observe that

\[
q^{1-1} < p_1 < p = q^j
\]

and \( t_k^{p_1} \in A_{p_1}(\mathbb{R}^n), k \in \mathbb{Z} \), see Lemma 2.3/(v). Substep 1.1 gives
The same procedure yields that
\[ t_k^{p(1+\nu)} \in A_p(\mathbb{R}^n) \subset A_{p(1+\nu)}(\mathbb{R}^n), \quad k \in \mathbb{Z} \text{ with} \]
\[ 0 < \nu < \min \left( \gamma_1, \gamma_2, q - 1 \right) \]
and then \( q^j = p < p(1 + \nu) < q^{j+1} \). Also, Substep 1.1. gives
\[
\left\| \left( \sum_{k=-\infty}^{\infty} t_k^q (\mathcal{M}(f_k))^q \right)^{1/q} \right\|_{L_{p(1+\nu)}(\mathbb{R}^n)} \leq \left\| \left( \sum_{k=-\infty}^{\infty} t_k^q |f_k|^q \right)^{1/q} \right\|_{L_{p(1+\nu)}(\mathbb{R}^n)}.
\]
Again, by Lemma 2.13 we obtain the desired estimate.

**Step 2.** We shall prove (2.14) with \( 1 < p < q < \infty \). Let \( 1 < q < \theta < \infty \). Again, by duality the left-hand side of (2.14), raised to the power \( \theta \), is just
\[
\sup_{x \in \mathbb{R}^n} \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} t_k^q(x)(\mathcal{M}(f_k)(x))^q |g_k(x)| \, dx = \sup V,
\]
where the supremum is taken over all sequence of functions \( \{g_k\} \) such that
\[
\{g_k\} \in L_{(p/\theta)'(\ell_{(q/\theta)'})}, \quad \|\{g_k\}|L_{(p/\theta)'(\ell_{(q/\theta)'})}\| \leq 1. \tag{2.16}
\]
By Lemma 2.10 and Hölder’s inequality, \( V \) is bounded by
\[
c \int_{\mathbb{R}^n} \sum_{k=-\infty}^{\infty} |f_k(x)|^q \mathcal{M}(t_k^q g_k)(x) \, dx
\leq \left\| \left( \sum_{k=-\infty}^{\infty} t_k^q |f_k|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}^q
\times \left\| \left( \sum_{k=-\infty}^{\infty} t_k^{q(q/\theta)'} (\mathcal{M}(t_k^q g_k))^{(q/\theta)'}) \right)^{1/(q/\theta)'} \right\|_{L_{(p/\theta)'(\ell_{(q/\theta)')}}(\mathbb{R}^n)}.
\]
By Step 1, the second term is bounded because of \( (q/\theta)' < (p/\theta)' \),
\[
t_k^{-(p/\theta)'} \in A_{(p/\theta)'}(\mathbb{R}^n), \quad k \in \mathbb{Z},
\]
and from Lemma 2.3/(iv) there exists a \( 1 < \delta < (p/\theta)' < \infty \) such that
\[
t_k^{-(p/\theta)'} \in A_{(p/\theta)'}/\delta(\mathbb{R}^n), \quad k \in \mathbb{Z}.
\]
Thus, the desired estimate follows by Step 1 and (2.16). The proof is complete. \( \square \)
Remark 2.15

(i) We would like to mention that the result of this lemma is true if we assume that \( t_k \in A_{p/0}(\mathbb{R}^n) \), \( k \in \mathbb{Z} \), \( 1 < p < \infty \) with

\[
A_{p/0}(t_k) \leq c, \quad k \in \mathbb{Z},
\]

where \( c > 0 \) independent of \( k \).

(ii) The proof of Lemma 2.14 for \( t_k \equiv \omega, \ k \in \mathbb{Z} \) is given in [3, 42].

(iii) Lemma 2.14 with \( t_k \equiv \omega, \ k \in \mathbb{Z} \), see, e.g., [52], can be obtained using the extrapolation theory of Garcia-Cuerva and Rubio de Francia [31] or by the theory of vector-valued singular integral with operator-valued kernel, see [54].

(iv) To circumvent the drawbacks of dealing with general weights, we use different techniques than those using the papers [3, 31, 42, 54].

(v) In view of Lemma 2.3/(iv) we can assume that \( t_k \in A_{p}(\mathbb{R}^n) \), \( k \in \mathbb{Z} \), \( 1 < p < \infty \) with

\[
A_p(t_k) \leq c, \quad k \in \mathbb{Z},
\]

where \( c > 0 \) independent of \( k \).

We need the following lemma, which is a discrete convolution inequality.

**Lemma 2.16** Let \( 0 < a < 1, 1 \leq p \leq \infty, 1 \leq r \leq \infty \) and \( 0 < q < \infty \). Let \( \{f_k\} \) and \( \{g_k\} \) be two sequences of positive real functions and denote

\[
\delta_k = \sum_{j=-\infty}^{k} d^{k-j} \| g_k f_j \|_{L^1(\mathbb{R}^n)}^{1/q}, \quad k, v \in \mathbb{Z}
\]

and

\[
\eta_k = \sum_{j=k+v}^{\infty} d^{j-k} \| g_k f_j \|_{L^1(\mathbb{R}^n)}^{1/q}, \quad k, v \in \mathbb{Z}.
\]

Then there exists a constant \( c > 0 \) depending only on \( a \) and \( q \) such that

\[
\sum_{k=-\infty}^{\infty} \delta_k^q + \sum_{k=-\infty}^{\infty} \eta_k^q \leq c \left( \sum_{k=-\infty}^{\infty} f_k^r \right)^{1/r} \| L_p(\mathbb{R}^n) \| \left( \sum_{k=-\infty}^{\infty} g_k^r \right)^{1/r} \| L_{p'}(\mathbb{R}^n) \|.
\]

(2.17)

**Proof** As the proof for \( \{\eta_k\} \) is similar, we only consider \( \{\delta_k\} \). We will do the proof in two steps.

**Step 1.** We prove our estimate under the restriction \( 0 < q \leq 1 \). We have
Applying Hölder’s inequality to estimate
\[
\sum_{k=-\infty}^{\infty} \left\| g_k f_{k-i} \right\|_{L^1(\mathbb{R}^n)}
\]
by
\[
\left\| \left( \sum_{k=-\infty}^{\infty} f_k \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)} \left( \sum_{k=-\infty}^{\infty} \left\| g_k \right\|_{L^{p'}(\mathbb{R}^n)} \right)^{1/r'}.
\]
By the fact that \( \sum_{i=-\infty}^{\infty} a^{i} \leq 1 \) we obtain the desired estimate.

**Step 2.** We consider the case \( 1 < q < \infty \). By duality,
\[
\left( \sum_{k=-\infty}^{\infty} \delta_k^{q} \right)^{1/q} = \sup \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k+v} a^{k-j} \left\| g_k f_j \right\|_{L^1(\mathbb{R}^n)} \left. \right|^{1/q} h_k = \sup T,
\]
where the supremum is taken over all sequence of positive real numbers \( \{h_k\} \in \ell_q \)
with
\[
\left( \sum_{k=-\infty}^{\infty} h_k^q \right)^{1/q} \leq 1.
\]
Again by Hölder’s inequality,
\[
T = \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} a^i \left\| g_k f_{k-i} \right\|_{L^1(\mathbb{R}^n)} \left. \right|^{1/q} h_k
\]
\[
\leq \sum_{i=-\infty}^{\infty} a^i \left( \sum_{k=-\infty}^{\infty} \left\| g_k f_{k-i} \right\|_{L^1(\mathbb{R}^n)} \right)^{1/q}.
\]
As in Step 1, we obtain the desired estimate. The proof is complete. \( \square \)

Using the same type of arguments as in Lemma 2.16 it is easy to prove the following lemma.
Lemma 2.17 Let \( a > 0, 1 \leq p \leq \infty, 1 \leq r \leq \infty \) and \( 0 < q < \infty \). Let \( \{f_k\} \) and \( \{g_k\} \) be two sequences of positive real functions and denote

\[
\delta_k = \sum_{j=k}^{k+v} a^{j-k} \left\| g_{k+j} |L_1(\mathbb{R}^n) \right\|^{1/q}, \quad k \in \mathbb{Z}, v \in \mathbb{N}
\]

and

\[
\eta_k = \sum_{j=k+1}^{k} a^{j-k} \left\| g_{k+j} |L_1(\mathbb{R}^n) \right\|^{1/q}, \quad k \in \mathbb{Z}, l \leq 0.
\]

Then there exists a constant \( c > 0 \) depending only on \( a, v, l \) and \( q \) such that (2.17) holds.

The next lemmas are important for the study of our function spaces.

Lemma 2.18 Let \( v \in \mathbb{Z}, K \geq 0, 1 < \theta \leq p < \infty, 1 < q < \infty \) and \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \). Let \( \{t_k\} \in \tilde{X}_{\alpha, \sigma, p} \) be a p-admissible weight sequence with \( \sigma = (\sigma_1 = \theta(p/\theta)', \sigma_2 \geq p) \). Then for all sequence of functions \( \{t_k f_k\} \in L_p(\ell_q) \),

\[
\left\| \left( \sum_{k=-\infty}^{\infty} t_k^q \left( \sum_{j=-\infty}^{k+v} 2^{(j-k)K} M(f_j) \right)^q \right) \right\|_{L_p(\mathbb{R}^n)}^{1/q}
\]

(2.18)

if \( K > \alpha_2 \) and

\[
\leq \left\| \left( \sum_{k=-\infty}^{\infty} t_k^q \left( \sum_{j=k+1}^{k+v} 2^{(j-k)K} M(f_j) \right)^q \right) \right\|_{L_p(\mathbb{R}^n)}^{1/q}
\]

(2.19)

if \( K < \alpha_1 \).

Proof We divide the proof into two steps.

Step 1. We will present the proof of (2.18). We separate this step into two distinct cases: \( 1 < q \leq p < \infty \) and \( 1 < p < q < \infty \).

Substep 1.1. We consider the case \( 1 < q \leq p < \infty \). By duality the left-hand side of (2.18) is just

\[
\sup \sum_{k=-\infty}^{\infty} \left( \int_{\mathbb{R}^n} t_k(x) \sum_{j=-\infty}^{k+v} 2^{(j-k)K} M(f_j)(x)g_k(x) \right) dx = \sup \sum_{k=-\infty}^{\infty} S_k,
\]

where the supremum is taking over all sequence of functions \( \{g_k\} \in L_{q'}(\ell_q) \) with
We easily find that
\[
S_k = \sum_{j=-\infty}^{k+v} 2^{(j-k)K} \int_{\mathbb{R}^n} \mathcal{M}(f_j(x)) t_k(x) |g_k(x)| \, dx = \sum_{j=-\infty}^{k+v} 2^{(j-k)K} D_{k,j}
\]
for any \( k \in \mathbb{Z} \). As in Lemma 2.14 we find that \( M_Q(f_j) \to 0 \) as \( |Q| \to \infty \) for any \( j \in \mathbb{Z} \). Therefore, we can apply Lemma 2.11. Let
\[
\Omega^i_j = \{ x \in \mathbb{R}^n : \mathcal{M}(f_j(x)) > 4^n \lambda^i \}, \quad j, i \in \mathbb{Z}
\]
with \( \lambda > 2^{n+1} \) and
\[
H^i_j = \{ x \in \mathbb{R}^n : 4^n \lambda^i < \mathcal{M}(f_j(x)) \leq 4^n \lambda^{i+1} \}, \quad j, i \in \mathbb{Z}.
\]
Let \( \{ Q_j^{i,h} \}_h \) be the collection of maximal dyadic cubes as in Lemma 2.11 with
\[
\Omega^i_j \subset \cup_h 3Q_j^{i,h}.
\]
We find that
\[
D_{k,j} = \sum_{i=-\infty}^{\infty} \int_{H^i_j} t_k(x) \mathcal{M}(f_j(x)) |g_k(x)| \, dx \\
\leq \sum_{i=-\infty}^{\infty} \frac{\lambda^i}{2} \int_{\Omega^i_j} t_k(x) |g_k(x)| \, dx \\
\leq \sum_{i=-\infty}^{\infty} \sum_{h=0}^{\infty} \frac{\lambda^i}{2} \int_{3Q_j^{i,h}} t_k(x) |g_k(x)| \, dx \\
\leq \sum_{i=-\infty}^{\infty} \sum_{h=0}^{\infty} \frac{1}{|Q_j^{i,h}|} \int_{Q_j^{i,h}} f_j(x) \, dx \int_{3Q_j^{i,h}} t_k(x) |g_k(x)| \, dx
\]
for any \( j \leq k + v \). Notice that, by the Hölder inequality, we find that for all \( j \leq k + v \),
\[
\frac{1}{|Q_j^{i,h}|} \int_{3Q_j^{i,h}} t_k(x) |g_k(x)| \, dx \leq M_{3Q_j^{i,h}, \delta}(g_k) M_{3Q_j^{i,h}, \delta}(t_k),
\]
with \( \delta = p(1+\varepsilon) \) and \( \varepsilon \) as in Theorem 2.4, which is possible since \( t_k^p \in A_{p/\theta}(\mathbb{R}^n) \) for any \( k \in \mathbb{Z} \). We shall distinguish two cases. Let \( j \in \mathbb{Z} \) be such that \( j \leq k \). We obtain by (2.4)
\[
M_{3Q_j^{i,h}, \delta}(t_k) \lesssim M_{3Q_j^{i,h}, p}(t_k) \lesssim 2^{2q(k-j)} M_{3Q_j^{i,h}, p}(t_j).
\]
By Hölder’s inequality,
\[ 1 = \left( \frac{1}{|3Q_j^h|} \int_{3Q_j^h} t_j^{-\eta}(x)t_j^\eta(x)dx \right)^{1/\eta} \leq M_{3Q_j^h,\varphi}(t_j) \]

for any \( \eta > 0 \) and any \( \varphi, \tau > 0 \) with \( 1/\eta = 1/\varphi + 1/\tau \). Taking any \( 0 < \varphi < \sigma_1 \) and any \( 0 < \tau < q < \infty \) we obtain

\[ 1 \leq M_{3Q_j^h,\varphi}(t_j^{-1})M_{3Q_j^h,\tau}(t_j), \]

which together with (2.3) implies that

\[ M_{3Q_j^h,\delta}(t_k) \lesssim 2^{\omega_1(k-j)}M_{3Q_j^h,\tau}(t_j), \]

which further implies that

\[ M_{3Q_j^h,\delta}(t_k) \lesssim \int_{Q_j^h} \frac{|f_j(x)|}{|3Q_j^h|} dx \lesssim 2^{\omega_1(k-j)} M_{3Q_j^h,\tau}(t_j) \int_{3Q_j^h} |f_j(x)| dx \lesssim 2^{\omega_1(k-j)} |Q_j^{j,h}| M_{3Q_j^h,\tau}(t_j M(f_j)). \]

Thus,

\[ \frac{1}{|3Q_j^h|} \int_{Q_j^h} |f_j(x)| dx \int_{3Q_j^h} t_k(x) |g_k(x)| dx \]

is bounded by

\[ c \cdot 2^{\omega_1(k-j)} |Q_j^{j,h}| M_{3Q_j^h,\delta}(g_k) M_{3Q_j^h,\tau}(t_j M(f_j)), \]

where the positive constant \( c \) is independent of \( j, h \) and \( k \). Let \( j \in \mathbb{Z} \) be such that \( k < j \leq k + v \). By (2.21) and (2.3) we obtain

\[ M_{3Q_j^h,\delta}(t_k) \lesssim M_{3Q_j^h,\varphi}(t_k) \lesssim 2^{\omega_1(k-j)} M_{3Q_j^h,\varphi}(t_j). \]

Consequently,

\[ D_{k,i} \leq \Lambda_{k,j} \sum_{i=-\infty}^{\infty} \sum_{h=0}^{\infty} |Q_j^{j,h}| M_{3Q_j^h,\delta}(g_k) M_{3Q_j^h,\tau}(t_j M(f_j)) \]

for any \( j \leq k + v \), where

\[ \Lambda_{k,j} = \begin{cases} 2^{\omega_1(k-j)} & \text{if } j \leq k, \\ 2^{\omega_1(k-j)} & \text{if } k < j \leq k + v. \end{cases} \]

Since \( |Q_j^{j,h}| \leq \beta |E_j^{j,h}| \), with \( E_j^{j,h} = Q_j^{j,h} \setminus (Q_j^{j,h} \cap (\cup_h Q_j^{j+1,h})) \) and the family \( E_j^{j,h} \) are pairwise, we find that
\[ D_{k,j} \leq \mathcal{L}_{k,j} \sum_{i=-\infty}^{\infty} \sum_{h=0}^{\infty} |E_{j,h}^i| M_{3Q_{j,h}^2}(g_k) M_{3Q_{j,h}^2}(t_j \mathcal{M}(f_j)) \]
\[ = c \mathcal{L}_{k,j} \sum_{i=-\infty}^{\infty} \sum_{h=0}^{\infty} \int_{E_{j,h}^i} M_{3Q_{j,h}^2}(g_k) M_{3Q_{j,h}^2}(t_j \mathcal{M}(f_j)) \, dx \]
\[ \leq \mathcal{L}_{k,j} \sum_{i=-\infty}^{\infty} \sum_{h=0}^{\infty} \int_{E_{j,h}^i} \mathcal{M}_{\sigma}(g_k)(x) M_{\tau}(t_j \mathcal{M}(f_j))(x) \, dx \]

for any \( j \leq k + v \). Since \( K > \alpha_2 \), applying Lemmas \ref{lem:2.16} and \ref{lem:2.17}, \( \sum_{k=-\infty}^{\infty} S_k \) can be estimated by

\[ c \left\| \left( \sum_{k=-\infty}^{\infty} (\mathcal{M}(t_k \mathcal{M}(f_k)))^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \left\| \left( \sum_{k=-\infty}^{\infty} (\mathcal{M}_{\sigma}(g_k))^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \]
\[ \leq \left\| \left( \sum_{k=-\infty}^{\infty} t_k^\theta |f_k|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}, \]

where we used the vector-valued maximal inequality of Fefferman and Stein (2.5), Lemma \ref{lem:2.14} and (2.20).

**Substep 1.2.** We consider the case \( 1 < p < q < \infty \). Again by duality the left-hand side of (2.18), raised to the power \( \theta \), is just

\[ \sup_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} t_k(x) \left( \sum_{j=-\infty}^{k+v} 2^{(j-k)K} \mathcal{M}(f_j)(x) \right)^{\theta} |g_k(x)| \, dx = \sup_{k=-\infty}^{\infty} S_k, \]

where the supremum is taking over all sequence of functions \( \{g_k\} \) such that

\[ \{g_k\} \in L_{(p/\theta)'}(\ell_{(q/\theta)'}, \mathbb{R}^n), \quad \|\{g_k\}\|_{L_{(p/\theta)'}(\ell_{(q/\theta)'})} \leq 1. \quad (2.22) \]

Notice that, by the Minkowski inequality, we obtain that

\[ S_k = \left\| t_k \sum_{j=-\infty}^{k+v} 2^{(j-k)K} \mathcal{M}(f_j)|g_k|^{1/\theta} |L_{\theta}(\mathbb{R}^n)| \right\|^\theta \]
\[ \leq \left( \sum_{j=-\infty}^{k+v} 2^{(j-k)K} \left( \int_{\mathbb{R}^n} t_k(x) (\mathcal{M}(f_j)(x))^{\theta} |g_k(x)| \, dx \right)^{1/\theta} \right)^{\theta} \]

for any \( k \in \mathbb{Z} \). Using Lemma \ref{lem:2.10}, we deduce that, for any \( j \leq k + v \),
\[ \int_{\mathbb{R}^n} t_k^\theta(x)(\mathcal{M}(f_j(x)))^\theta |g_k(x)| \, dx \leq \int_{\mathbb{R}^n} |f_j(x)|^\theta \mathcal{M}(t_k^\theta g_k)(x) \, dx = c \, D_{k,j}, \]

where the positive constant \( c \) is independent of \( k \) and \( j \). Let \( Q \) be a cube. By Hölder’s inequality,

\[ \frac{1}{|Q|} \int_Q t_k^\theta(x)|g_k(x)| \, dx \leq \frac{1}{|Q|} \| g_k \|_{L_p(Q)} \| \mathcal{M}(t_k^\theta)|L_p(Q)\| \leq \frac{1}{|Q|} \| t_k \|_{L_p(Q)} \|^\theta, \]

with \( c > 0 \) is independent of \( k \). Since \( \{t_k\} \) is a \( p \)-admissible sequence satisfying (2.3) with \( \sigma_1 = \theta(p/\theta)' \), we find that

\[ \frac{1}{|Q|^\frac{1}{\theta}} \| t_k \|_{L_p(Q)} \leq C \| Q \|^\frac{1}{\theta+1/\sigma_1-1/\theta} \| t_k^{-1} \|_{L_{\sigma_1}(Q)} \|^\frac{1}{\theta-1} = C \| t_k^{-1} \|_{L_{\sigma_1}(Q)} \|^\frac{1}{\theta-1}, \]

with \( C > 0 \) is independent of \( Q, \theta \) and \( k \). Since \( t_k^\theta \in A_{p/\theta}(\mathbb{R}^n) \) it follows by Lemma 2.3/(ii),

\[ t_k^{-\sigma_1} \in A_{(p/\theta)'}(\mathbb{R}^n), \quad k \in \mathbb{Z}. \]

Hence \( \| t_k^{-1} \|_{L_{\sigma_1}(Q)} \|^{-1} \to 0 \), \( |Q| \to \infty \), \( k \in \mathbb{Z} \). Therefore, we can apply Lemma 2.11. Using the same arguments as in proof of Lemma 2.14, we get

\[ D_{k,j} \leq \sum_{i=-\infty}^{\infty} \sum_{h=0}^{\infty} i^i \int_{3Q_k^h} |f_j(x)|^\theta \, dx \]

\[ \leq \sum_{i=-\infty}^{\infty} \sum_{h=0}^{\infty} \frac{1}{|Q_k^h|^\frac{i}{\theta}} \int_{3Q_k^h} |f_j(x)|^\theta \, dx \int_{Q_k^h} t_k^\theta(x)|g_k(x)| \, dx, \]

where we have used the inequality

\[ i^i \leq \frac{1}{|Q_k^h|^\frac{i}{\theta}} \int_{Q_k^h} t_k^\theta(x)|g_k(x)| \, dx \leq 2^a i^i. \]

On the other hand, by Hölder’s inequality, we obtain

\[ \int_{3Q_k^h} |f_j(x)|^\theta \, dx \leq |Q_k^h|^\frac{\theta}{p} M_{3Q_k^h,v}(t_j^{-\theta}) M_{3Q_k^h,v}(t_j^\theta |f_j|^\theta) \]

with \( v' = (p/\theta)'(1+\varepsilon) \) and \( \varepsilon \) as in Theorem 2.4 since, again, \( t_k^{-\sigma_1} \in A_{(p/\theta)'}(\mathbb{R}^n) \) for any \( k \in \mathbb{Z} \). Therefore,

\[ M_{3Q_k^h,v}(t_j^{-\theta}) = (M_{3Q_k^{h,\sigma_1}(1+\varepsilon)}(t_j^{-1}))^\theta \lesssim (M_{3Q_k^{h,\sigma_1}(1+\varepsilon)}(t_j^{-1}))^\theta. \]

As before, by Hölder’s inequality,
\[
1 = \left( \frac{1}{|3Q_k^h|} \int_{3Q_k^h} t_k^0(x) t_k^0(x) \, dx \right)^{1/q} \leq M_{3Q_k^h,0}(t_k^{-1}) M_{3Q_k^h,p}(t_k) \quad (2.23)
\]
for any \( q > 0 \) with \( 1/q = 1/\theta + 1/p \). Again, we distinguish two cases. Let \( j \in \mathbb{Z} \) be such that \( j \leq k \). From (2.4) we obtain
\[
M_{3Q_k^h,p}(t_k) \leq 2^{2z(k-j)} M_{3Q_k^h,p}(t_j), \quad j \leq k.
\]
Therefore,
\[
1 \leq 2^{2z(k-j)} M_{3Q_k^h,0}(t_k^{-1}) M_{3Q_k^h,p}(t_j), \quad j \leq k.
\]
Multiplying by \( M_{3Q_k^h,\sigma_1}(t_j^{-1}) \) and using (2.3) we get
\[
M_{3Q_k^h,\sigma_1}(t_j^{-1}) \leq 2^{2z(k-j)} M_{3Q_k^h,0}(t_k^{-1}).
\]
Now, let \( j \in \mathbb{Z} \) be such that \( k < j \leq k + v \). From (2.23) and (2.3) we obtain
\[
M_{3Q_k^h,\sigma_1}(t_j^{-1}) \leq 2^{2z(k-j)} M_{3Q_k^h,\sigma_1}(t_k^{-1}).
\]
Hence,
\[
|Q_k^i| M_{3Q_k^h}(|f_j|_t^0) M_{Q_k^h}(t_k^0 g_k),
\]
can be estimated by
\[
c^{\Lambda^0}_{k,j} |Q_k^i| M_{Q_k^h}(t_k^0 g_k) M_{Q_k^h,v}(t_j^0 |f_j|_t^0) M_{3Q_k^h}(t_k^{-0}) \leq \Lambda^0_{k,j} |Q_k^i| M_{Q_k^h}(t_k^{-0} \mathcal{M}(t_k^0 g_k)) M_{3Q_k^h,v}(t_j^0 |f_j|_t^0).
\]
Consequently,
\[
D_{k,j} \leq \Lambda^0_{k,j} \sum_{i=-\infty}^{\infty} \sum_{h=0}^{\infty} M_{3Q_k^h,v}(t_j^0 |f_j|_t^0) \int_{3Q_k^h} t_k^0(x) \mathcal{M}(t_k^0 g_k)(x) \, dx
\]
for any \( j \leq k + v \). Since \( |Q_k^i| \leq \beta |E_k^i| \), with \( E_k^i = Q_k^i \setminus (\bigcup_{h} Q_k^{i-1} h) \) and of the family \( E_k^i \) are pairwise, as before we find that
\[
D_{k,j} \leq \Lambda^0_{k,j} \int_{\mathbb{R}^n} \mathcal{M}(t_k^{-0} \mathcal{M}(t_k^0 g_k))(x) \mathcal{M}(t_j^0 |f_j|_t^0)(x) \, dx.
\]
Since \( K > \kappa_2 \) and \( v < p/\theta \leq q/\theta \), applying Lemma 2.16, we obtain
\[
\sum_{k=-\infty}^{\infty} S_k \text{ can be estimated by}
\]
where we used the vector-valued maximal inequality of Fefferman and Stein (2.5), Lemma 2.14 and (2.22).

This proves the first part of the lemma.

Step 2. We will present the proof of (2.19). To prove we additionally do it into the two steps Substeps 2.1 and 2.2.

Substep 2.1. We consider the case $1 < q \leq 2$. We employ the same notation as in Substep 1.1. We obtain

\[
D_{k,j} \lesssim \sum_{l=-\infty}^{\infty} \sum_{h=0}^{\infty} j^l \int_{Q_j^{h+1}} t_k(x) |g_k(x)| \, dx \lesssim \sum_{l=-\infty}^{\infty} \sum_{h=0}^{\infty} \frac{1}{|Q_j^{h+1}|} \int_{Q_j^{h+1}} |f_j(x)| \, dx \int_{Q_j^{h+1}} t_k(x) |g_k(x)| \, dx.
\]

Let $j \geq k + \nu$. Recall that

\[
M_{3Q_j^{h+1},\delta}(t_k) \leq M_{3Q_j^{h+1},p}(t_k) \quad \text{and} \quad 1 \leq M_{3Q_j^{h+1},\sigma_{1}(t_j^{-1})M_{3Q_j^{h},(t_j)}.
\]

Using (2.3), we find

\[
M_{3Q_j^{h+1},\delta}(t_k) \lesssim 2^{2^l(k-l)} M_{3Q_j^{h+1},p}(t_j), \quad j \geq k.
\]

Now, assume that $k + \nu \leq j < k$. From (2.4), we get

\[
M_{3Q_j^{h},p}(t_k) \lesssim 2^{2^l(k-l)} M_{3Q_j^{h},p}(t_j).
\]

Hence
\[ M_{3Q^h,t}(t_k) \int_{Q^h} |f_j(x)| \, dx \lesssim F_{k,j} M_{3Q^h,t}(t_j) \int_{Q^h} |f_j(x)| \, dx \lesssim F_{k,j} |Q_j| M_{3Q^h,t}(t_j, M(f_j)), \]

where

\[ F_{k,j} = \begin{cases} 2^{a_1(k-j)}, & \text{if } j \geq k, \\ 2^{a_2(k-j)} & \text{if } k + \nu \leq j < k. \end{cases} \]

Repeating the same arguments of Substep 1.1, we obtain the desired estimate.

**Substep 2.2.** We show (2.19) under the assumption 1 < p < q < \infty. We employ the same notation as in Substep 1.2. From (2.23) we get

\[ M_{3Q^h_{\sigma},(t_j^{-1})} \lesssim 2^{a_1(k-j)} M_{3Q^h_{\sigma},(t_k^{-1})}, \quad j \geq k. \]

We omit the proof since it is essentially similar to the Substep 2.1 and Substep 1.2, respectively.

The proof of lemma is complete. □

**Remark 2.19** Let \( i \in \mathbb{Z}, 1 < \theta \leq p < \infty, 1 < q < \infty \) and \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \). Let \( \{t_k\} \in \hat{X}_{\alpha,p} \) be a \( p \)-admissible weight sequence with \( \sigma = (\sigma_1 = \theta(p/\theta)', \sigma_2 \geq p) \). From Lemma 2.18 we easily obtain

\[ \left\| \left( \sum_{k=-\infty}^{\infty} t_k^q (\mathcal{M}(f_{k+i}))^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \lesssim \left\| \left( \sum_{k=-\infty}^{\infty} t_k^q |f_k|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \]

holds for all sequence of functions \( \{t_k f_k\} \in L_p(\ell_q) \), where the implicit constant depends on \( i \). Indeed, we have

\[ \mathcal{M}(f_{k+i}) \leq \sum_{j=-\infty}^{k+i} 2^{(j-k-i)M} \mathcal{M}(f_j), \quad M > \alpha_2, k \in \mathbb{Z}. \]

Lemma 2.18 yields the desired result.

### 3 The spaces \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \)

In this section we present the Fourier analytical definition of Triebel–Lizorkin spaces of variable smoothness and we prove their basic properties in analogy to the classical Triebel–Lizorkin spaces.

#### 3.1 The \( \varphi \)-transform characterization

Select a pair of Schwartz functions \( \varphi \) and \( \psi \) satisfy
\[
\text{supp}(\mathcal{F}(\varphi)) \cup \text{supp}(\mathcal{F}(\psi)) \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \},
\]
(3.1)

\[
|\mathcal{F}(\varphi)(\xi)|, |\mathcal{F}(\psi)(\xi)| \geq c \quad \text{if} \quad 3/5 \leq |\xi| \leq 5/3
\]
(3.2)

and

\[
\sum_{k=-\infty}^{\infty} \mathcal{F}(\varphi)(2^{-k}\xi) \mathcal{F}(\psi)(2^{-k}\xi) = 1 \quad \text{if} \quad \xi \neq 0,
\]
(3.3)

where \(c > 0\). Throughout the paper we put \(\tilde{\varphi}(x) = \overline{\varphi(-x)}, x \in \mathbb{R}^n\). Let \(\varphi \in \mathcal{S}(\mathbb{R}^n)\) be a function satisfying (3.1)–(3.2). We recall that there exists a function \(\psi \in \mathcal{S}(\mathbb{R}^n)\) satisfying (3.1)–(3.3), see [30, Lemma (6.9)].

**Remark 3.1** Let \(\dot{F}_{p,q}^{s}(\mathbb{R}^n, \{t_k\})\) the spaces under consideration. We would like to mention that the elements of the spaces \(\dot{F}_{p,q}^{s}(\mathbb{R}^n, \{t_k\})\) are not distributions but equivalence classes of distributions. Observe that \(\dot{F}_{p,p}^{s}(\mathbb{R}^n, \{t_k\})\) is just the space \(\dot{B}_{p,p}^{s}(\mathbb{R}^n, \{t_k\})\), where the space \(\dot{B}_{p,q}^{s}(\mathbb{R}^n, \{t_k\})\), \(0 < p, q \leq \infty\), is defined to be the set of all \(f \in \mathcal{S}'_{\infty}(\mathbb{R}^n)\) such that

\[
\|f|\dot{B}_{p,q}^{s}(\mathbb{R}^n, \{t_k\})\| = \left( \sum_{k=-\infty}^{\infty} \|t_k(\varphi_k * f)\|_{L^p(\mathbb{R}^n)}^{q} \right)^{1/q} < \infty,
\]

which studied in detail in [19].

Using the system \(\{\varphi_k\}\) we can define the quasi-norms

\[
\|f|\dot{F}_{p,q}^{s}(\mathbb{R}^n)\| = \left( \sum_{k=-\infty}^{\infty} 2^{ksq} |\varphi_k * f|^{q} \right)^{1/q} \|L^p(\mathbb{R}^n)\|
\]

for constants \(s \in \mathbb{R}, 0 < p < \infty\) and \(0 < q \leq \infty\). The Triebel–Lizorkin space \(\dot{F}_{p,q}^{s}(\mathbb{R}^n)\) consist of all distributions \(f \in \mathcal{S}'_{\infty}(\mathbb{R}^n)\) for which

\[
\|f|\dot{F}_{p,q}^{s}(\mathbb{R}^n)\| < \infty.
\]

It is well-known that these spaces do not depend on the choice of the system \(\{\varphi_k\}\) (up to equivalence of quasi-norms). Further details on the classical theory of these spaces, included the nonhomogeneous case, can be found in [27, 29, 30, 55, 59–61].

One recognizes immediately that if \(\{t_k\} = \{2^{sk}\}, s \in \mathbb{R}\), then

\[
\dot{F}_{p,q}(\mathbb{R}^n, \{2^{sk}\}) = \dot{F}_{p,q}^{s}(\mathbb{R}^n).
\]
(3.4)

Moreover, for \(\{t_k\} = \{2^{sk}w\}, s \in \mathbb{R}\) with a weight \(w\) we re-obtain the weighted Triebel–Lizorkin spaces; we refer, in particular, to the papers [11, 38, 53, 56, 57] for a comprehensive treatment of the weighted spaces.

A basic tool to study the above function spaces is the following Calderón reproducing formula, see [69, Lemma 2.1].

\[\text{Birkhäuser}\]
Lemma 3.2 Suppose that \( \varphi, \psi \in S(\mathbb{R}^n) \) satisfy (3.1) through (3.3). If \( f \in \mathcal{S}'(\mathbb{R}^n) \), then
\[
\forall = \sum_{k=-\infty}^{\infty} 2^{-kn} \sum_{m \in \mathbb{Z}^n} \tilde{\varphi}_k \ast f(2^{-k}m)\psi_k(\cdot - 2^{-k}m).
\] (3.5)

Let \( \varphi, \psi \in S(\mathbb{R}^n) \) satisfying (3.1) through (3.3). Recall that the \( \varphi \)-transform \( S_\varphi \) is defined by setting
\[
(S_\varphi f)_{k,m} = \langle f, \varphi_{k,m} \rangle,
\]
where \( \varphi_{k,m}(x) = 2^{k^2} \varphi(2^k x - m), m \in \mathbb{Z}^n \) and \( k \in \mathbb{Z} \). The inverse \( \varphi \)-transform \( T_\varphi \) is defined by
\[
T_\varphi \lambda = \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \hat{\lambda}_{k,m} \psi_{k,m},
\]
where \( \lambda = \{ \lambda_{k,m} \}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \subset \mathbb{C} \), see [29].

Now we introduce the corresponding sequence spaces of \( \mathcal{F}_{p,q}(\mathbb{R}^n, \{ t_k \}) \).

Definition 3.3 Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \). Let \( \{ t_k \} \) be a \( p \)-admissible weight sequence. Then for all complex valued sequences \( \lambda = \{ \lambda_{k,m} \}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \subset \mathbb{C} \) we define
\[
\mathcal{F}_{p,q}(\mathbb{R}^n, \{ t_k \}) = \left\{ \lambda : \| \lambda \|_{\mathcal{F}_{p,q}(\mathbb{R}^n, \{ t_k \})} < \infty \right\},
\]
where
\[
\| \lambda \|_{\mathcal{F}_{p,q}(\mathbb{R}^n, \{ t_k \})} = \left\| \left( \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq/2} t_k^q |\lambda_{k,m}|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}^{1/q},
\]
with the usual modifications if \( q = \infty \).

Allowing the smoothness \( t_k, k \in \mathbb{Z} \) to vary from point to point will raise extra difficulties to study these function spaces. But by the following lemma the problem can be reduced to the case of fixed smoothness, see [20].

Proposition 3.4 Let \( 0 < \theta < p < \infty \), \( 0 < q < \infty \) and \( 0 < \delta \leq 1 \). Assume that \( \{ t_k \} \) satisfying (2.3) with \( \sigma_1 = \theta(p^*)' \) and \( j = k \). Then
\[
\| \lambda \|_{\mathcal{F}_{p,q,\delta}(\mathbb{R}^n, \{ t_k \})} = \left\| \left( \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq(1/2 + 1/\delta p)} t_k^q |\lambda_{k,m}|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}^{1/q},
\]
is an equivalent quasi-norm in \( \mathcal{F}_{p,q}(\mathbb{R}^n, \{ t_k \}) \), where

\[ \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq(1/2 + 1/\delta p)} t_k^q |\lambda_{k,m}|^q \]

is an equivalent quasi-norm in \( \mathcal{F}_{p,q,\delta}(\mathbb{R}^n, \{ t_k \}) \), where
\[ t_{k,m,\delta} = \| t_k |L_{\delta p}(Q_{k,m}) \|, \quad k \in \mathbb{Z}, m \in \mathbb{Z}^n. \]

The following important properties of the sequence spaces will be required in what follows.

**Lemma 3.5** Let \( 0 < \theta \leq p < \infty \) and \( 0 < q < \infty \). Let \( \{t_k\} \) be a \( p \)-admissible weight sequence satisfying (2.3) with \( \sigma_1 = 0(p/\theta)' \) and \( j = k \). Let \( k \in \mathbb{Z}, m \in \mathbb{Z}^n \) and \( \lambda \in \dot{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \). Then there exists \( c > 0 \) independent of \( k \) and \( m \) such that
\[
|\lambda_{k,m}| \leq c 2^{-kn/2} t_{k,m} \| \lambda \|_{\dot{f}_{p,q}(\mathbb{R}^n, \{t_k\})}.
\]

**Proof** Let \( \lambda \in \dot{f}_{p,q}(\mathbb{R}^n, \{t_k\}), k \in \mathbb{Z} \) and \( m \in \mathbb{Z}^n \). Since \( \{t_k\} \) is a \( p \)-admissible sequence satisfying (2.3) with \( \sigma_1 = 0(p/\theta)' \), we get by Hölder’s inequality
\[
|\lambda_{k,m}| = \left( \frac{1}{|Q_{k,m}|} \int_{Q_{k,m}} |\lambda_{k,m}|^\theta dy \right)^{1/\theta}
\leq M_{Q_{k,m},p}(\lambda_{k,m,tk}) M_{Q_{k,m},\sigma_1}(t_k^{-1})
\leq c 2^{-kn/2} t_{k,m} \| \lambda \|_{\dot{f}_{p,q}(\mathbb{R}^n, \{t_k\})},
\]
where \( c > 0 \) is independent of \( k \in \mathbb{Z} \) and \( m \in \mathbb{Z}^n \). \( \square \)

The following lemma is a slight variant of [19]. For the convenience of the reader, we give some details.

**Lemma 3.6** Let \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \), \( 0 < \theta \leq p < \infty \) and \( 0 < q \leq \infty \). Let \( \{t_k\} \in \hat{X}_{\alpha,p} \) be a \( p \)-admissible weight sequence with \( \sigma = (\sigma_1 = 0(p/\theta)', \sigma_2 = p) \). Let \( \psi \in \mathcal{S}(\mathbb{R}^n) \) satisfying (3.1) and (3.2). Then for all \( \lambda \in \dot{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \)
\[
T_\psi \lambda = \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{k,m} \psi_{k,m},
\]
converges in \( \mathcal{S}'(\mathbb{R}^n) \); moreover, \( T_\psi : \dot{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \to \mathcal{S}'(\mathbb{R}^n) \) is continuous.

**Proof** Let \( \lambda \in \dot{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \) and \( \varphi \in \mathcal{S}(\mathbb{R}^n) \). We see that
\[
\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{k,m}| \langle \psi_{k,m}, \varphi \rangle = I_1 + I_2,
\]
where
\[
I_1 = \sum_{k=-\infty}^{0} \sum_{m \in \mathbb{Z}^n} |\lambda_{k,m}| \langle \psi_{k,m}, \varphi \rangle \quad \text{and} \quad I_2 = \sum_{k=1}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{k,m}| \langle \psi_{k,m}, \varphi \rangle.
\]
It suffices to show that both \( I_1 \) and \( I_2 \) are dominated by
Estimate of $I_1$. Let us recall the following estimate, see (3.18) in [10]. For any $L > 0$, there exists a positive constant $M \in \mathbb{N}$ such that for all $\varphi, \psi \in S_\infty(\mathbb{R}^n)$, $i, k \in \mathbb{Z}$ and $m, h \in \mathbb{Z}^n$,

$$
\left| \left\langle \varphi_{k,m}, \psi_{i,h} \right\rangle \right| \leq \| \varphi \|_{S_{M+1}} \| \psi \|_{S_{M+1}} \left( 1 + \frac{|2^{-k}m - 2^{-i}h|^n}{\max(2^{-kn}, 2^{-in})} \right)^{-L} \min\left( 2^{(i-k)nL}, 2^{(k-i)nL} \right).
$$

Therefore,

$$
\left| \left\langle \psi_{k,m}, \varphi \right\rangle \right| \leq \| \varphi \|_{S_{M+1}} \| \psi \|_{S_{M+1}} \left( 1 + \frac{|2^{-k}m|^n}{\max(1, 2^{-kn})} \right)^{-L} 2^{-|k|nL}.
$$

Our estimate employs partially some decomposition techniques already used in [29, 44]. For each $j \in \mathbb{N}$ we define

$$
\Omega_j = \{ m \in \mathbb{Z}^n : 2^{j-1} < |m| \leq 2^j \} \quad \text{and} \quad \Omega_0 = \{ m \in \mathbb{Z}^n : |m| \leq 1 \}.
$$

Thus,

$$
I_1 \lesssim \sum_{k=-\infty}^{0} 2^{knL} \sum_{m \in \mathbb{Z}^n} \frac{|\hat{\varphi}_{k,m}|}{(1 + |m|)^{nL}} \leq c \sum_{k=-\infty}^{0} 2^{knL} \sum_{m \in \Omega_j} \frac{|\hat{\varphi}_{k,m}|}{(1 + |m|)^{nL}} \lesssim \sum_{k=-\infty}^{0} 2^{knL} \sum_{j=0}^{\infty} \sum_{m \in \Omega_j} 2^{-nLj} \sum_{m \in \Omega_j} |\hat{\varphi}_{k,m}|.
$$

Let $0 < \varrho < \min(1, \theta)$ be such that $1/\varrho = 1/\tau + 1/\sigma_1$ with $0 < \tau < \min\left( \frac{1}{1 - 1/\sigma_1}, p \right)$. We have

$$
I_1 \lesssim \sum_{k=-\infty}^{0} \sum_{j=0}^{\infty} 2^{-nL(j-k)} \left( \sum_{m \in \Omega_j} |\hat{\varphi}_{k,m}|^\varrho \right)^{1/\varrho} = c \sum_{k=-\infty}^{0} \sum_{j=0}^{\infty} 2^{(1/\varrho - L)nLj + knL} \left( \frac{2^{(k-j)n}}{\max(1, 2^{-kn})} \right)^{1/\varrho} \| \varphi \|_{S_{M+1}} \| \psi \|_{S_{M+1}} \left( 1 + \frac{|2^{-k}m|^n}{\max(1, 2^{-kn})} \right)^{-L} 2^{-|k|nL} \sum_{m \in \Omega_j} |\hat{\varphi}_{k,m}|^{1/\varrho} \hat{\varphi}_{k,m}(y)dy \right)^{1/\varrho}.
$$

Let $y \in \bigcup_{z \in \Omega_j} Q_{k,z}$ and $x \in Q_{0,0}$. Then $y \in Q_{k,z}$ for some $z \in \Omega_j$ and $2^{j-1} < |z| \leq 2^j$. From this it follows that
which implies that $y$ is located in the ball $B(x, 2^{j-k+\delta_n})$. In addition, from the fact that

$$|y| \leq |y - x| + |x| \leq 2^{j-k+\delta_n} + 1 \leq 2^{j-k+c_n}, \quad c_n \in \mathbb{N},$$

we have that $y$ is located in the ball $B(0, 2^{j-k+c_n})$. Therefore, by Hölder’s inequality

$$\left(2^{(k-j)n} \int_{\cup_{l \in \mathbb{Z}} Q_{k,l} \sum_{m \in \Omega_i} |\lambda_{k,m}|^q \chi_{k,m}(y) dy \right)^{1/q} \leq \left(2^{(k-j)n} \int_{B(x, 2^{j-k+c_n})} \sum_{m \in \Omega_i} |\lambda_{k,m}|^r t^l_k \chi_{k,m}(y) dy \right)^{1/r} M_{B(0, 2^{j-k+c_n}), \sigma_1}(t_k^{-1}) \lesssim M_\tau \left( \sum_{m \in \mathbb{Z}^n} t_k |\lambda_{k,m}| \chi_{k,m}(x) \right) M_{B(0, 2^{j-k+c_n}), \sigma_1}(t_k^{-1}).$$

Since $t_k^{-\sigma_1} \in A_{(p/q)'}(\mathbb{R}^n)$, $k \in \mathbb{Z}$, by Lemma 2.3/(iii), (2.3) and (2.4) we obtain

$$M_{B(0, 2^{j-k+c_n}), \sigma_1}(t_k^{-1}) \lesssim 2^{(j-k)p} M_{B(0, 1), \sigma_1}(t_k^{-1}) \lesssim 2^{(j-k)p} \left( M_{B(0, 1), \rho}(t_k) \right)^{-1} \lesssim 2^{(j-k)p-2k^2} \left( M_{B(0, 1), \sigma_2}(t_0) \right)^{-1}$$

for any $k \leq 0$ and any $j \in \mathbb{N}_0$. Hence, for any $L$ large enough,

$$I_1 \lesssim \sum_{k=-\infty}^{0} 2^{(nL-2k-n/p)} M_\tau \left( \sum_{m \in \mathbb{Z}^n} t_k |\lambda_{k,m}| \chi_{k,m}(x) \right), \quad x \in Q_{0,0}.$$ 

The last term is bounded in the $L_p(Q_{0,0})$-quasi-norm by $c \|\lambda\|_{L_p(\mathbb{R}^n, \{t_k\})}$ with the help of Theorem 2.8.

**Estimate of $I_2$.** We have

$$\|\psi_{k,m}, \varphi\| \lesssim 2^{-knL} \|\varphi\|_{S_{m+1}} \|\psi\|_{S_{m+1}} \left( 1 + 2^{-kn} |m|^n \right)^{-L}, \quad k \geq 1.$$ 

For each $j, k \in \mathbb{N}$, define

$$\Omega_{j,k} = \{m \in \mathbb{Z}^n : 2^{j+k-1} < |m| \leq 2^{j+k} \} \quad \text{and} \quad \Omega_{0,k} = \{m \in \mathbb{Z}^n : |m| \leq 2^k \}.$$

Then we find
which is just the term
\[ I_2 \lesssim \sum_{k=1}^{\infty} 2^{-knL} \sum_{m \in \mathbb{Z}^n} |\lambda_{k,m}| \left( 1 + 2^{-k|m|} \right)^{nL} \]
\[ = c \sum_{k=1}^{\infty} 2^{-knL} \sum_{j=0}^{\infty} \sum_{m \in \Omega_{j,k}} |\lambda_{k,m}| \left( 1 + 2^{-k|m|} \right)^{nL} \]
\[ \leq c \sum_{k=1}^{\infty} 2^{-knL} \sum_{j=0}^{\infty} 2^{-nj} \sum_{m \in \Omega_{j,k}} |\lambda_{k,m}|. \]

Let \( 0 < q < \min(1, \theta) \) be such that \( 1/q = 1/\tau + 1/\sigma_1 \) with \( 0 < \tau < p \). Using the embedding \( \ell_q \hookrightarrow \ell_1 \) we find that \( I_2 \) does not exceed
\[ c \sum_{k=1}^{\infty} 2^{-knL} \sum_{j=0}^{\infty} 2^{-nj} \left( \sum_{m \in \Omega_{j,k}} |\lambda_{k,m}|^q \right)^{1/q} \]
which is just the term
\[ c \sum_{k=1}^{\infty} 2^{-knL} \sum_{j=0}^{\infty} 2^n(\frac{q}{q-nL})j \left( \int_{\cup_{z \in \Omega_{j,k}} Q_{k,z}} \sum_{m \in \Omega_{j,k}} |\lambda_{k,m}|^q |Z_{k,m}(y)| dy \right)^{1/q}. \]

Let \( y \in \bigcup_{z \in \Omega_{j,k}} Q_{k,z} \) and \( x \in Q_{0,0} \). Then \( y \in Q_{k,z} \) for some \( z \in \Omega_{j,k} \) and \( 2^{j-1} < 2^{-k}|z| \leq 2^j \). From this it follows that
\[ |y - x| \leq |y - 2^{-k}z| + |x - 2^{-k}| \leq \sqrt{n} 2^{-k} + |x| + 2^{-k}|z| \leq 2^{j+\delta_n}, \quad \delta_n \in \mathbb{N}, \]
which implies that \( y \) is located in the ball \( B(x, 2^{j+\delta_n}) \). In addition, from the fact that
\[ |y| \leq |y - x| + |x| \leq 2^{j+\delta_n} + 1 \leq 2^{j+c_n}, \quad c_n \in \mathbb{N}, \]
we have that \( y \) is located in the ball \( B(0, 2^{j+c_n}) \). Therefore,
\[ \left( \int_{\bigcup_{z \in \Omega_{j,k}} Q_{k,z}} \sum_{m \in \Omega_{j,k}} |\lambda_{k,m}|^q |Z_{k,m}(y)| dy \right)^{1/q} \]
\[ \leq 2^kn/q \left( \int_{B(x, 2^{j+c_n})} \sum_{m \in \Omega_{j,k}} |\lambda_{k,m}|^q |\tilde{T}_k^z Z_{k,m}(y)| dy \right)^{1/\tau} M_{B(0, 2^{j+c_n}), \sigma_1}(I_k^{-1}) \]
\[ \lesssim 2^kn/q M_\tau \left( \sum_{m \in \mathbb{Z}^n} \tilde{T}_k^z \lambda_{k,m} Z_{k,m} \right)(x) M_{B(0, 2^{j+c_n}), \sigma_1}(I_k^{-1}). \]

By (2.3) and Lemma 2.3/(vi) we obtain
\[
M_{B(0,2^{(1/c_0)})}(t_k^{-1}) 
\leq 2^{-kz_1} (M_{B(0,2^{(1/c_0)})}(t_0))^{-1} 
\leq 2^{(n/p-n\delta/p)-kz_1} (M_{B(0,1)}(t_0))^{-1} .
\]

Therefore,
\[
I_2 \leq \sum_{k=1}^{\infty} 2^{k(nL-n/q+x_1)} \mathcal{M}_r \left( I_k \sum_{m, \sigma} \lambda_{k,m} \lambda_{k,m}(x), \quad x \in Q_{0,0} \right) (3.6)
\]

for any \( L \) large enough. Now we take the \( L_p(Q_{0,0}) \)-quasi-norm of both sides of (3.6) and then use Theorem 2.8, we obtain
\[
I_2 \leq \left\| \lambda_{f,p,q}(\mathbb{R}^n, \{ t_k \}) \right\| .
\]

The proof is complete. \( \Box \)

For a sequence \( \lambda = \{ \lambda_{k,m} \}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \subset \mathbb{C}, 0 < r \leq \infty \) and a fixed \( d > 0 \), set
\[
\lambda_{k,m,r,d}^* = \left( \sum_{t \in \mathbb{Z}^n} \frac{|\lambda_{k,m}|^r}{\left( 1 + 2^k |2^{-k}h - 2^{-k}m| \right)^d} \right)^{1/r}
\]
and \( \lambda_{r,d}^* := \{ \lambda_{k,m,r,d}^* \}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \subset \mathbb{C}.

**Lemma 3.7** Let \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2, 0 < \theta < p < \infty, 0 < q < \infty, \gamma \in \mathbb{Z} \) and \( d > n \). Let \( \{ t_k \} \) be a \( p \)-admissible weight sequence satisfying (2.3) with \( \alpha_1 = \theta(p/\theta)' \) and \( \alpha_2 \in \mathbb{R} \). Then
\[
\left\| \lambda_{\min(p,q),d}^* \tilde{f}_{p,q}(\mathbb{R}^n, \{ t_k \}) \right\| \approx \left\| \lambda_{f,p,q}(\mathbb{R}^n, \{ t_k \}) \right\|. \quad (3.7)
\]

In addition if \( \{ t_k \} \) satisfies (2.4) with \( \alpha_2 \geq p \) and \( \alpha_2 \in \mathbb{R} \), then
\[
\left\| \lambda_{\min(p,q),d}^* \tilde{f}_{p,q}(\mathbb{R}^n, \{ t_k \}) \right\| \leq \left\| \lambda_{f,p,q}(\mathbb{R}^n, \{ t_k \}) \right\|. \quad (3.8)
\]

**Proof** The proof is similar to that given in [19]. First we prove (3.7). Obviously,
\[
\left\| \lambda_{f,p,q}(\mathbb{R}^n, \{ t_k \}) \right\| \leq \left\| \lambda_{d}^* \tilde{f}_{\min(p,q),d}(\mathbb{R}^n, \{ t_k \}) \right\| .
\]

Let \( n \min(p,q)/d < a < \min(p,q), j \in \mathbb{N} \) and \( m \in \mathbb{Z}^n \). Define
\[
\Omega_{j,m} = \{ h \in \mathbb{Z}^n : 2^{j-1} < |h-m| \leq 2^j \}, \quad \Omega_{0,m} = \{ h \in \mathbb{Z}^n : |h-m| \leq 1 \}.
\]

Then
The last expression can be rewritten as

\[
\sum_{j=0}^{\infty} 2^{(a \min(p,q)/a-d)j} \left( 2^{(k-j)n} \int_{\cup_{\Omega_{j,m}} \mathcal{Q}_{k,z}} \sum_{h \in \Omega_{j,m}} |\hat{\lambda}_{k,h}| \chi_{k,h}(y) \, dy \right)^{\min(p,q)/a}.
\]

Let \( y \in \cup_{\Omega_{j,m}} \mathcal{Q}_{k,z} \) and \( x \in \mathcal{Q}_{k,m} \). Then \( y \in \mathcal{Q}_{k,z} \) for some \( z \in \Omega_{j,m} \) which implies that \( 2^{j-1} \leq |z - m| \leq 2^j \). From this it follows that

\[
|y - x| \leq |y - 2^{-k} z| + |x - 2^{-k} z| \\
\leq \sqrt{n} \cdot 2^{-k} + |x - 2^{-k} m| + 2^{-k} |z - m| \\
\leq 2^j - k + \delta_n, \quad \delta_n \in \mathbb{N},
\]

which implies that \( y \) is located in the ball \( B(x, 2^{-k+\delta_n}) \). Therefore, (3.9) can be estimated by

\[
c \mathcal{M}_a \left( \sum_{h \in \mathbb{Z}^n} \hat{\lambda}_{k,h} \chi_{k,h} \right)(x),
\]

where the positive constant \( c \) is independent of \( k \). Consequently

\[
\| \hat{\lambda}_{\min(p,q),d} \|_{p,q, \mathbb{R}^n, \{t_{k-}\}} \leq c \mathcal{M}_a \left( \sum_{h \in \mathbb{Z}^n} \hat{\lambda}_{k,h} \chi_{k,h} \right)^{1/q}.
\]

Applying Lemma 2.14 we estimate (3.10) by

\[
c \left\| \left( \sum_{k=-\infty}^{\infty} 2^{knq/2} \left( \mathcal{M}_a \left( \sum_{h \in \mathbb{Z}^n} \hat{\lambda}_{k,h} \chi_{k,h} \right) \right)^{q} \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}.
\]

To prove (3.8) we use again Lemma 2.14 combined with Remark 2.19. □
Now we have the following result which is called the $\varphi$-transform characterization in the sense of Frazier and Jawerth. It will play an important role in the rest of the paper.

**Theorem 3.8** Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$, $0 < p \leq \infty$ and $0 < q < \infty$. Let $\{t_k\} \in \mathcal{X}_{\mathbf{a}, \sigma, p}$ be a $p$-admissible weight sequence with $\sigma = (\sigma_1 = \theta(p/\theta), \sigma_2 \geq p)$. Let $\varphi, \psi \in S(\mathbb{R}^n)$ satisfying (3.1) through (3.3). The operators

\[ S_\varphi : \mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \to \mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \]

and

\[ T_\psi : \mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \to \mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \]

are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})$.

**Proof** The proof is a straightforward adaptation of [29, Theorem 2.2] and [19]. For any $f \in S_0'((\mathbb{R}^n))$ we put $\sup(f) := \{\sup_{k,m}(f)\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n}$ where

\[ \sup(x) = 2^{-kn/2} \sup_{y \in Q_0} |\hat{\varphi}_k * f(y)|, \quad k \in \mathbb{Z}, m \in \mathbb{Z}^n. \]

For any $\gamma \in \mathbb{N}_0$, we define the sequence $\inf_{\gamma}(f) = \{\inf_{k,m,\gamma}(f)\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n}$ by setting

\[ \inf_{k,m,\gamma}(f) = 2^{-kn/2} \max \left\{ \inf_{y \in Q} |\hat{\varphi}_k * f(y)| : \tilde{Q} \subset Q_{k,m}, l(\tilde{Q}) = 2^{-\gamma} \right\}, \]

where $k \in \mathbb{Z}, m \in \mathbb{Z}^n$ and $\hat{\varphi}_k = 2^{kn} \varphi(2^{-k} \cdot)$, $k \in \mathbb{Z}$.

**Step 1.** In this step we prove that

\[ \| \inf_{\gamma}(f) \mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \| \lesssim \| f \mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \|. \]

Define a sequence $\lambda = \{\lambda_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^n}$ by

\[ \lambda_{j,k} = 2^{-jn/2} \inf_{y \in Q_j} |\hat{\varphi}_{j-k} * f(y)|, \quad j \in \mathbb{Z}, k \in \mathbb{Z}^n. \]

Then for all $0 < r < \infty$, any $k \in \mathbb{Z}, m \in \mathbb{Z}^n$ and a fixed $\lambda > n$, we have

\[ \inf_{k,m,\gamma}(f) \mathcal{F}_{k,m} \lesssim 2^{\gamma(d/r+n/2)} \sum_{h \in \mathbb{Z}^n : Q_{k+\gamma,h} \subset Q_{k,m}} \lambda_{k+\gamma,h}^{\gamma + \gamma}. \]

Picking $r = \min(p, q)$, we obtain
Step 2. We will prove that
\[ c H \]
Hence for \( \gamma \) sufficiently large we obtain by applying Lemma 3.7,
\[ \inf_\gamma (f)^* \approx \sup_\gamma (f)^* \]
Hence for \( \gamma > 0 \) sufficiently large we obtain by applying Lemma 3.7,
\[ \inf_\gamma (f)^* \approx \sup_\gamma (f)^* \]
and
\[ \| \inf_\gamma (f)^* \| \approx \| \sup_\gamma (f)^* \| \]
Therefore,
\[ \| \inf_\gamma (f)^* \| \approx \| \sup_\gamma (f)^* \| \]
From the definition of the spaces \( \tilde{F}_{p,q} (\mathbb{R}^n, \{ t_k \}) \) it follows that
\[ \| f \| \tilde{F}_{p,q} (\mathbb{R}^n, \{ t_k \}) \approx \| \sup_\gamma (f)^* \| \tilde{F}_{p,q} (\mathbb{R}^n, \{ t_k \}) \]
Consequently (3.12) and Step 1 yield (3.11).

Step 3. In this step we prove the boundedness of \( S_\psi \) and \( T_\psi \). We have

\[ \| \inf_\gamma (f) \| \tilde{F}_{p,q} (\mathbb{R}^n, \{ t_k \}) \approx \| \sup_\gamma (f)^* \| \tilde{F}_{p,q} (\mathbb{R}^n, \{ t_k \}) \]

We apply Lemma 3.7 to estimate the last expression by
\[ c 2^{2d/r} \left| \left( \sum_{k=-\infty}^{\infty} 2^{knq/2} \sum_{h \in \mathbb{Z}^n} (t_{k-\gamma} \hat{\lambda}_{k,h}^a)^q \chi_{k,h} \right)^{1/q} \right| L_p (\mathbb{R}^n) \]

Step 2. We will prove that
\[ \| \inf_\gamma (f) \| \tilde{F}_{p,q} (\mathbb{R}^n, \{ t_k \}) \| \approx \| f \| \tilde{F}_{p,q} (\mathbb{R}^n, \{ t_k \}) \| \approx \| \sup_\gamma (f) \| \tilde{F}_{p,q} (\mathbb{R}^n, \{ t_k \}) \|. \]
\(|(S_{\alpha f})_{k,m}| = |(f, \varphi_{k,m})| = 2^{-kn/2} |f \ast \hat{\varphi}_k(2^{-k}m)| \leq \sup_k (f) .\)

Step 2 yields that
\[
\|S_{\alpha f}[\hat{f}_{p,q}(\mathbb{R}^n, \{t_k\})] \| \leq \| f \|_{\hat{F}_{p,q}(\mathbb{R}^n, \{t_k\})} .
\]

To prove the boundedness of \(T_\psi\) suppose \(\lambda = \{\lambda_j, h\}_{j \in \mathbb{Z}, h \in \mathbb{Z}^n}\) and
\[
T_\psi \lambda = \sum_{j=-\infty}^{\infty} \sum_{h \in \mathbb{Z}^n} \lambda_{j, h} \psi_{j, h} .
\]

Obviously
\[
\hat{\varphi}_k \ast T_\psi \lambda = \sum_{j=-k}^{k+1} \sum_{h \in \mathbb{Z}^n} \lambda_{j, h} \hat{\varphi}_k \ast \psi_{j, h} .
\]

Since \(\varphi\) and \(\psi\) belong to \(\mathcal{S}(\mathbb{R}^n)\) we obtain
\[
|\hat{\varphi}_k \ast \psi_{j, h}(x)| \lesssim 2^{j(n/2)} (1 + 2^j |x - 2^{-j}h|)^{-d/\min(1, \min(p,q))} , \quad d > n ,
\]
where the implicit constant is independent of \(j, k, h\) and \(x\). Therefore, if \(x \in Q_{k+1, z} \subset Q_{k, m} \subset Q_{k-1, l}, \ z, l \in \mathbb{Z}^n\), then we obtain
\[
|\hat{\varphi}_k \ast T_\psi \lambda(x)| \lesssim 2^{kn/2} \sum_{j=k-1}^{k+1} \sum_{h \in \mathbb{Z}^n} \frac{|\lambda_{j, h}|}{(1 + 2^j |x - 2^{-j}h|)^{d/\min(1, \min(p,q))}} .
\]

Assume that \(0 < \min(p, q) \leq 1\). Using the inequality
\[
\left( \sum_{h \in \mathbb{Z}^n} |a_h| \right)^{\min(p,q)} \leq \sum_{h \in \mathbb{Z}^n} |a_h|^{\min(p,q)} , \quad \{a_h\}_{h \in \mathbb{Z}^n} \subset \mathbb{C} ,
\]
we obtain
\[
|\hat{\varphi}_k \ast T_\psi \lambda(x)| \lesssim 2^{kn/2} \sum_{j=k-1}^{k+1} \left( \sum_{h \in \mathbb{Z}^n} \frac{|\lambda_{j, h}|^{\min(p,q)}}{(1 + 2^j |x - 2^{-j}h|)^{d}} \right)^{1/\min(p,q)} . \tag{3.13}
\]

Now if \(\min(p, q) > 1\), then by the Hölder inequality and the fact that
\[
\sum_{h \in \mathbb{Z}^n} 1 \lesssim 1 , \quad (1 + 2^j |x - 2^{-j}h|)^d \lesssim 1 ,
\]
we also have (3.13) with \(\min(p, q) > 1\). Hence, if \(x \in Q_{k+1, z} \subset Q_{k, m} \subset Q_{k-1, l}\), then we have
\[
|\hat{\varphi}_k \ast T_\psi \lambda(x)| \lesssim 2^{kn/2} (\lambda_{k-1, l, \min(p,q), d}^* + \lambda_{k, m, \min(p,q), d}^* + \lambda_{k+1, z, \min(p,q), d}^*) .
\]

Consequently
\[ \| T_{\psi} \tilde{\lambda} \tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \| \leq \sum_{i=-1}^{1} I_i, \]

where

\[ I_i = \left\| \left( \sum_{k=-\infty}^{\infty} 2^{knq/2} \left\| \sum_{h \in \mathbb{Z}^n} (t_{k+i} \tilde{\lambda}_{k,h,\min(p,q),d} Z_{k,h})^q \right\|_{L_p(\mathbb{R}^n)} \right)^{1/q} \right\|, \]

\( i = -1, 0, 1. \) Applying (3.8) we obtain

\[ \| T_{\psi} \tilde{\lambda} \tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \| \leq \| \tilde{\lambda} \tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \|. \]

The proof is complete. \( \square \)

**Remark 3.9** This theorem can then be exploited to obtain a variety of results for the \( \tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \) spaces, where arguments can be equivalently transferred to the sequence space, which is often more convenient to handle. More precisely, under the same hypothesis of the last theorem,

\[ \left\{ f, \varphi_{k,m} \right\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \]

is quasi-Banach spaces. They are Banach spaces if \( 1/p < \infty \) and \( 1/q < \infty. \)

**Lemma 3.11** Let \( 0 < \theta \leq p < \infty \) and \( 0 < q < \infty. \) Let \( \{t_k\} \in \tilde{\mathfrak{T}}_{x,\sigma,p} \) be a \( p \)-admissible weight sequence with \( \sigma = (\sigma_1 = 0(p/\theta)', \sigma_2 \geq p). \) The definition of the spaces \( \tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \) is independent of the choices of \( \varphi \in S(\mathbb{R}^n) \) satisfying (3.1) and (3.2).
it follows that $\sum_{i=0}^{\infty} A^{(i)}$ is a is absolutely convergent in $L_p(\ell_q)$, so the sequence $\{A^{(i)}\}_{i \in \mathbb{N}_0}$ converges in $L_p(\ell_q)$ and

$$\left\| \sum_{i=0}^{\infty} A^{(i)} \right\|_{L_p(\ell_q)} < \infty.$$ 

Then

$$\sum_{m \in \mathbb{Z}^n} t_k \mathcal{I}_{k,m} \sum_{i=0}^{\infty} |\mathcal{I}_{k,m}^{(i)}| = \sum_{i=0}^{\infty} \sum_{m \in \mathbb{Z}^n} t_k |\mathcal{I}_{k,m}^{(i)}| \mathcal{I}_{k,m} < \infty, \quad k \in \mathbb{Z}, \text{a.e.,}$$

which yields that

$$\sum_{i=0}^{\infty} \mathcal{I}_{k,m}^{(i)} < \infty, \quad k \in \mathbb{Z}, m \in \mathbb{Z}^n, \text{a.e.}$$

Therefore,

$$\left\| \sum_{i=0}^{\infty} \mathcal{I}_{k,m}^{(i)} \right\|_{\overset{\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})}} \leq \sum_{i=0}^{\infty} \left\| A^{(i)} \right\|_{L_p(\ell_q)} < \infty.$$ 

This completes the proof. $\square$

Applying this lemma and Theorem 3.8 we obtain the following useful properties of these function spaces, see [32].

**Theorem 3.12** Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$, $0 < \theta \leq p < \infty$ and $0 < q < \infty$. Let $\{t_k\} \in \mathcal{X}_{\mathbf{a},p}$ be a $p$-admissible weight sequence with $\sigma = (\sigma_1 = \theta (p/\theta)', \sigma_2 \geq p)$. $\overset{\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})}$ are quasi-Banach spaces. They are Banach spaces if $1 \leq p < \infty$ and $1 \leq q < \infty$.

**Theorem 3.13** Let $0 \leq \theta \leq p < \infty$ and $0 < q < \infty$. Let $\{t_k\} \in \mathcal{X}_{\mathbf{a},p}$ be a $p$-admissible weight sequence with $\sigma = (\sigma_1 = \theta (p/\theta)', \sigma_2 \geq p)$ and $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$.

(i) We have the embedding

$$\mathcal{S}_\infty(\mathbb{R}^n) \hookrightarrow \overset{\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})}. \quad \text{(3.14)}$$

In addition $\mathcal{S}_\infty(\mathbb{R}^n)$ is dense in $\overset{\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})}$. 

(ii) We have the embedding

$$\overset{\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})} \hookrightarrow \mathcal{S}_\infty'(\mathbb{R}^n).$$

**Proof** The proof is a variant of that given for Besov spaces in [19]. For the convenience of the reader, we give some details. The embedding (3.14) follows by
\( S_\infty(\mathbb{R}^n) \hookrightarrow \tilde{B}_{p,\min(p,q)}(\mathbb{R}^n, \{t_k\}) \hookrightarrow \tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \),

see [19] for the first embedding. Now, we prove the density of \( S_\infty(\mathbb{R}^n) \) in \( \tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \). Let \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \) satisfying (3.1) through (3.3) and \( f \in \tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \). Let

\[
f_N = \sum_{k=-N}^{N} \tilde{\psi}_k \ast \varphi_k \ast f, \quad N \in \mathbb{N}.
\]

Observe that

\[
\varphi_j \ast \tilde{\psi}_k = 0, \quad \text{if } k \notin \{j-1, j, j+1\}.
\]

Then, by Lemma 2.14,

\[
\|f_N|_{\tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\})} \| = \left\| \left( \sum_{|k| \leq N+1} |t_k(\varphi_k \ast \tilde{\psi}_k \ast \bar{\varphi}_k \ast f)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}
\]

\[
\lesssim \left\| \left( \sum_{|k| \leq N+1} |t_k \mathcal{M}_\tau(\varphi_k \ast f)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}
\]

\[
\lesssim \|f|_{\tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\})} \| < \infty
\]

for any \( N \in \mathbb{N} \) where \( \varphi_k = \varphi_{k-1} \ast \varphi_k + \varphi_{k+1}, k \in \mathbb{Z} \) and \( 0 < \tau < \min(1, p, q) \). The first inequality follows by Lemma 2.4 of [17]. Consequently,

\[
\|f - f_N|_{\tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\})} \| \leq \left\| \left( \sum_{|k| \geq N+1} |t_k(\varphi_k \ast \tilde{\psi}_k \ast \bar{\varphi}_k \ast f)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}
\]

\[
\lesssim \left\| \left( \sum_{|k| \geq N+1} |t_k \mathcal{M}_\tau(\varphi_k \ast f)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}
\]

\[
\lesssim \left\| \left( \sum_{|k| \geq N+1} |t_k(\varphi_k \ast f)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)},
\]

where we used again Lemma 2.14. The dominated convergence theorem implies that \( f_N \) approximate \( f \) in \( \tilde{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \). But \( f_N, N \in \mathbb{N} \) is not necessary an element of \( S_\infty(\mathbb{R}^n) \), so we need to approximate \( f_N \) in \( S_\infty(\mathbb{R}^n) \). Let \( \omega \in \mathcal{S}(\mathbb{R}^n) \) with \( \omega(0) = 1 \) and \( \text{supp}(\mathcal{F}(\omega)) \subseteq \{ \xi : |\xi| \leq 1 \} \). Put

\[
f_{N, \delta} := f_N \omega(\delta \cdot), \quad 0 < \delta < 1.
\]

We have \( f_{N, \delta} \in S_\infty(\mathbb{R}^n) \) see [68, Lemma 5.3], and
After simple calculation, we obtain

\[
\varphi_j \left[ (\tilde{\psi}_k \ast \varphi_k \ast f)(\omega(\delta \cdot)) \right](x) = \int_{\mathbb{R}^n} \varphi_k \ast f(y) \varphi_j \ast (\tilde{\psi}_k \omega(\delta \cdot + y))(x - y) \, dy, \quad x \in \mathbb{R}^n,
\]

which together with the fact that

\[
\text{supp}(\mathcal{F}(\tilde{\psi}_k \omega(\delta \cdot + y))) \subset \{ \xi : 2^{k-2} \leq |\xi| \leq 2^{k+1} \}, \quad y \in \mathbb{R}^n, \quad |k| \leq N
\]

if \(0 < \delta < 2^{-N-3}\) yield that

\[
\varphi_j \left[ (\tilde{\psi}_k \ast \varphi_k \ast f)(\omega(\delta \cdot)) \right] = 0 \quad \text{if} \quad |j - k| \geq 2.
\]

Therefore, we obtain that

\[
\|f_N - f_{N,\delta}\|_{\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})}
\]

can be estimated by

\[
\left( \sum_{|k| \leq N+2} |t_k(\varphi_k \ast \sum_{i=-2}^{2} [\tilde{\psi}_{k+i} \ast \varphi_{k+i} \ast f](1 - \omega(\delta \cdot))]|^q \right)^{1/q} \|L_p(\mathbb{R}^n)\|.
\]

Again, by Lebesgue’s dominated convergence theorem \(f_{N,\delta}\) approximate \(f_N\) in the spaces \(\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})\). This prove that \(\mathcal{S}_{\infty}(\mathbb{R}^n)\) is dense in \(\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})\).

The proof of (ii) follows by the embedding

\[
\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \hookrightarrow \mathcal{B}_{p,\max(p,q)}(\mathbb{R}^n, \{t_k\}) \hookrightarrow \mathcal{S}_{\infty}(\mathbb{R}^n),
\]

where the second embedding is proved in [19].

### 3.2 Embeddings

For our spaces introduced above we want to show Sobolev embedding theorems. We recall that a quasi-Banach space \(A_1\) is continuously embedded in another quasi-Banach space \(A_2, A_1 \hookrightarrow A_2\), if \(A_1 \subset A_2\) and there is a \(c > 0\) such that

\[
\|f|A_2\| \leq c\|f|A_1\|
\]

for all \(f \in A_1\). We begin with the following elementary embeddings.

**Theorem 3.14** Let \(0 < p < \infty\) and \(0 < q \leq r < \infty\). Let \(\{t_k\} \in \check{X}_{2, \sigma, p}\) be a \(p\)-admissible weight sequence with \(\sigma = (\sigma_1 = \theta((\frac{r}{p})', \sigma_2 \geq p)\). We have
\[ \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \hookrightarrow \hat{F}_{p,r}(\mathbb{R}^n, \{t_k\}). \]

**Proof** It is a ready consequence of the embeddings between Lebesgue sequence spaces. □

The main result of this subsection is the following Sobolev-type embedding. In the classical setting this was done in [39, 59]. We set

\[ w_k(Q) = \left( \int_Q W_k^p(x) \, dx \right)^{1/p_1} \quad \text{and} \quad t_k(Q) = \left( \int_Q T_k^p(x) \, dx \right)^{1/p_0}, \]

where \( Q \in \mathcal{Q} \) with \( \ell(Q) = 2^{-k}, k \in \mathbb{Z} \).

**Theorem 3.15** Let \( 0 < \theta \leq p_0 \leq p_1 < \infty \) and \( 0 < q, r < \infty \). Let \( \{t_k\} \) be a \( p_0 \)-admissible weight sequence satisfying (2.3) with \( p = p_0, \sigma_1 = \theta(p_0/\theta)' \) and \( j = k \). Let \( \{w_k\} \) be a \( p_1 \)-admissible weight sequence satisfying (2.3) with \( p = p_1, \sigma_1 = \theta(p_1/\theta)' \) and \( j = k \). If \( w_k(Q) \leq t_k(Q)(p_0) \) for all \( Q \in \mathcal{Q} \) with \( \ell(Q) = 2^{-k}, k \in \mathbb{Z} \), then we have

\[ \hat{f}_{p_0,q}(\mathbb{R}^n, \{t_k\}) \hookrightarrow \hat{f}_{p_1,r}(\mathbb{R}^n, \{w_k\}). \]

**Proof** Let \( f \in \hat{f}_{p_0,q}(\mathbb{R}^n, \{t_k\}) \). Without loss of generality, we may assume that

\[ \| \hat{f}_{p_0,q}(\mathbb{R}^n, \{t_k\}) \| = 1. \]

We set

\[ f_k(x) = \sum_{m \in \mathbb{Z}^n} 2^{kn/2} |Q_{k,m}|^{-1/p_1} t_{k,m}(p_0) |\hat{f}_{k,m}|_L_{p_1} |x|_m(x), \quad x \in \mathbb{R}^n, k \in \mathbb{Z}. \]

Using Proposition 3.4 and the fact that

\[ w_{k,m}(p_1) \leq t_{k,m}(p_0), \quad k \in \mathbb{Z}, m \in \mathbb{Z}^n, \]

we obtain

\[ \| \hat{f}_{p_1,r}(\mathbb{R}^n, \{w_k\}) \| \leq \left( \sum_{k=-\infty}^{\infty} f_k^r \right)^{1/r} \| L_{p_1}(\mathbb{R}^n) \|. \]

Now we prove our embedding. Let \( K \in \mathbb{Z} \). By Lemma 3.5,

\[ |\hat{f}_{k,m}| \leq 2^{-kn/2} t_{k,m}^{-1}(p_0) \]

for any \( k \in \mathbb{Z} \) and \( m \in \mathbb{Z}^n \). Therefore,
\[
\sum_{k=-\infty}^{K} f_k^r(x) \lesssim \sum_{k=-\infty}^{K} 2^{knr/p_1} \leq C 2^{Knr/p_1}.
\] (3.15)

On the other hand, it follows that
\[
\sum_{k=K+1}^{\infty} f_k^r(x) \lesssim \sup_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}^n} 2^{kn/2} |Q_{k,m}|^{-1/p_0} t_{k,m}(p_0) |\lambda_{k,m}| x_{k,m}(x) \right)^r \sum_{k=K+1}^{\infty} 2^{Knr(1/p_1 - 1/p_0)} \] (3.16)

The identity
\[
\| g \|_{L_{p_1}(\mathbb{R}^n)} \|_{p_1} = p_1 \int_0^{\infty} y^{p_1-1} \| \{ x \in \mathbb{R}^n : |g(x)| > y \} \| dy,
\]
justifies the estimate
\[
\| \lambda f_{p_1,r}(\mathbb{R}^n, \{ w_k \}) \|_{p_1} \lesssim \int_0^{\infty} y^{p_1-1} \| \{ x \in \mathbb{R}^n : \left( \sum_{k=-\infty}^{\infty} f_k^r(x) \right)^{1/r} > y \} \| dy.
\]

We use (3.15) with \( K \) the largest integer such that
\[
C 2^{Knr/p_1} \leq \frac{y^r}{2}.
\]

Since \( 2^{kn(1/p_0 - 1/p_1)} y \geq c y^{p_1/p_0} \), using (3.16), we obtain
\[
\left\{ x \in \mathbb{R}^n : \sum_{k=-\infty}^{\infty} f_k^r(x) > y^r \right\} \lesssim \left\{ x \in \mathbb{R}^n : \sum_{k=K+1}^{\infty} f_k^r(x) > \frac{y^r}{2} \right\},
\]
and hence
\[
\left\{ x \in \mathbb{R}^n : \sum_{k=-\infty}^{\infty} f_k^r(x) > y^r \right\}
\]
do not exceed
\[
c \left\{ x \in \mathbb{R}^n : \sup_{k \in \mathbb{Z}} (2^{kn(1/p_0 - 1/p_1)} f_k(x)) > c 2^{Knr(1/p_0 - 1/p_1)} y \right\}.
\]

Therefore,
Theorem 3.16 Let the matrix

\[ f(k) = 2^{kn(1/p_0 - 1/p_1)} f_k(x) > c \]

Hence, the theorem is proved.

After a simple change of variable, we estimate the last term by

\[
c \int_0^\infty y^{p_0 - 1} \left\{ x \in \mathbb{R}^n : \sup_{k \in \mathbb{Z}} \left( 2^{kn(1/p_0 - 1/p_1)} f_k(x) \right) > y \right\} dy
\]

where we have used again Proposition 3.4. Hence, the theorem is proved. \( \square \)

From Theorems 3.8 and 3.15, we infer the following Sobolev-type embedding for \( \dot{F}_{p,q}^r(\mathbb{R}^n, \{ t_k \}) \).

**Theorem 3.16** Let \( 0 < \theta \leq p_0 < p_1 < \infty \) and \( 0 < q, r < \infty \). Let \( \{ t_k \} \in \dot{X}_{2_0, \sigma, p_0} \) be a \( p_0 \)-admissible weight sequence with \( \sigma = (\sigma_1 = \theta(p_0/\theta'), \sigma_2 \geq p_0) \) and \( x_0 = (x_{1,0}, x_{2,0}) \in \mathbb{R}^2 \). Let \( \{ w_k \} \in \dot{X}_{2_1, \sigma, p_1} \) be a \( p_1 \)-admissible weight sequence with \( \sigma = (\sigma_1 = \theta(p_1/\theta'), \sigma_2 \geq p_1) \) and \( x_1 = (x_{1,1}, x_{2,1}) \in \mathbb{R}^2 \). Then

\[
\dot{F}_{p_0, q}^r(\mathbb{R}^n, \{ t_k \}) \hookrightarrow \dot{F}_{p_1, r}^r(\mathbb{R}^n, \{ w_k \}),
\]

hold if

\[
w_{k,Q}(p_1) \leq t_{k,Q}(p_0)
\]

for all \( Q \in Q \) and all \( k \in \mathbb{Z} \).

### 3.3 Atomic and molecular decompositions

We will use the notation of [29]. We shall say that an operator \( A \) is associated with the matrix \( \{a_{k,m} \}_{k,m} \in \mathbb{Z}, m \in \mathbb{Z}^n \), if for all sequences \( \lambda = \{ \lambda_{k,m} \}_{k,m} \in \mathbb{Z} \subset \mathbb{C} \),

\[
A \lambda = \{ (A \lambda)_{k,m} \}_{k,m} \in \mathbb{Z} = \left\{ \sum_{v=\infty}^{\infty} \sum_{h \in \mathbb{Z}^n} a_{k,v}^{m} \lambda_{v,h} \right\}_{k,m} \in \mathbb{Z}^n
\]

We will use the notation

\[
J = \frac{n}{\min(1, p, q)}.
\]

We say that \( A \), with associated matrix \( \{a_{k,m} \}_{k,m} \in \mathbb{Z}, m \in \mathbb{Z}^n \), is almost diagonal on \( \dot{F}_{p,q}^r(\mathbb{R}^n, \{ t_k \}) \) if there exists \( \varepsilon > 0 \) such that

\[
\sup_{k,m} \frac{|a_{k,m}^{p_0}|}{o_{Q_{k,m}^{p_0}}(\varepsilon)} < \infty,
\]

where
\[ \omega_{Q_{k,m}}^{r,v,h}(e) = \left(1 + \frac{|x_{Q_{k,m}} - x_{P_{r,h}}|}{\max(2^{-k}, 2^{-v})}\right)^{-J-\varepsilon} \begin{cases} \frac{2^{(v-k)(z_2 + (n+\varepsilon)/2)}}{2^{(v-k)(z_1 - (n+\varepsilon)/2 - J+n)}}, & \text{if } v \leq k, \\ \frac{2^{(v-k)(z_2 + (n+\varepsilon)/2)}}{2^{(v-k)(z_1 - (n+\varepsilon)/2 - J+n)}}, & \text{if } v > k. \end{cases} \] (3.17)

Using Lemma 2.18 the following theorem is a generalization of [29, Theorem 3.3].

**Theorem 3.17** Let \( x_1, x_2 \in \mathbb{R}, 0 < \theta \leq p < \infty \) and \( 0 < q < \infty \). Let \( \{t_k\}_k \in \dot{X}_{\sigma,p} \) be a \( p \)-admissible weight sequence with \( \sigma_1 = \theta(p/\theta)' \) and \( \sigma_2 \geq p \). Any almost diagonal operator \( A \) on \( \dot{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \) is bounded.

**Proof** We write \( A \equiv A_0 + A_1 \) with

\[
(A_0 \lambda)_{k,m} = \sum_{v=-\infty}^{k} \sum_{h \in \mathbb{Z}^n} a_{Q_{k,m}}^{r,v,h} \lambda_{v,h}, \quad k \in \mathbb{Z}, m \in \mathbb{Z}^n
\]

and

\[
(A_1 \lambda)_{k,m} = \sum_{v=k+1}^{\infty} \sum_{h \in \mathbb{Z}^n} a_{Q_{k,m}}^{r,v,h} \lambda_{v,h}, \quad k \in \mathbb{Z}, m \in \mathbb{Z}^n.
\]

**Estimate of \( A_0 \).** From (3.17), we obtain

\[
| (A_0 \lambda)_{k,m} | \leq \sum_{v=-\infty}^{k} \sum_{h \in \mathbb{Z}^n} 2^{(v-k)(z_2 + (n+\varepsilon)/2)} \frac{|\lambda_{v,h}|}{\left(1 + 2^v |x_{k,m} - x_{v,h}| \right)^{J+\varepsilon}}
\]

\[
= \sum_{v=-\infty}^{k} 2^{(v-k)(z_2 + (n+\varepsilon)/2)} S_{k,v,m}.
\]

For each \( j \in \mathbb{N}, k \in \mathbb{Z} \) and \( m \in \mathbb{Z}^n \) we define

\[
\Omega_{j,k,m} = \{ h \in \mathbb{Z}^n : 2^{j-1} < 2^v |x_{k,m} - x_{v,h}| \leq 2^j \}
\]

and

\[
\Omega_{0,k,m} = \{ h \in \mathbb{Z}^n : 2^v |x_{k,m} - x_{v,h}| \leq 1 \}.
\]

Let \( n/(J + \frac{\varepsilon}{2}) < \tau < \min(1, p, q) \). We rewrite \( S_{k,v,m} \) as follows

\[
S_{k,v,m} = \sum_{j=0}^{\infty} \sum_{h \in \Omega_{j,k,m}} \frac{|\lambda_{v,h}|}{\left(1 + 2^v |x_{k,m} - x_{v,h}| \right)^{J+\varepsilon}}
\]

\[
\leq \sum_{j=0}^{\infty} 2^{-(J+\varepsilon)j} \sum_{h \in \Omega_{j,k,m}} |\lambda_{v,h}|.
\]

By the embedding \( \ell_\tau \hookrightarrow \ell_1 \) we deduce that
\[
S_{k,v,m} \leq \sum_{j=0}^{\infty} 2^{-(J+\varepsilon)j} \left( \sum_{h \in \Omega_{k,m}} |\lambda_{v,h}|^\tau \right)^{1/\tau} \\
= \sum_{j=0}^{\infty} 2^{(n/\tau-j-\varepsilon)j} \left( \int_{\cup \Omega_{k,m}Q_v} \sum_{h \in \Omega_{k,m}} |\lambda_{v,h}|^{\frac{\tau}{\tau-1}} Z_{v,h}(y)dy \right)^{1/\tau}.
\]

Let \( y \in \cup z \in \Omega_{j,k,m}Q_v, z \) and \( x \in Q_{k,m} \). It follows that \( y \in Q_v, z \) for some \( z \in \Omega_{j,k,m} \) and \( 2j-1 < 2v |2^{-k}m - 2^{-v}z| \leq 2j \). From this we obtain that

\[
|y - x| \leq |y - 2^{-k}m| + |x - 2^{-k}m| \\
\leq 2^{-v} + 2^{j-v} + 2^{-k} \\
\leq 2^{j-v+\delta_n}, \quad \delta_n \in \mathbb{N},
\]

which implies that \( y \) is located in the ball \( B(x, 2^{j-\nu+\delta_n}) \). Consequently

\[
S_{k,v,m} \lesssim \mathcal{M}_c \left( \sum_{h \in \mathbb{Z}^n} \lambda_{v,h} Z_{v,h} \right)(x)
\]

for any \( x \in Q_{k,m} \) and any \( k \leq v \). Applying Lemma 2.18, we obtain that

\[
\|A_0\lambda|f_{p,q}(\mathbb{R}^n, \{t_k\})\|
\]

is bounded by

\[
c\|\lambda|f_{p,q}(\mathbb{R}^n, \{t_k\})\|.
\]

**Estimate of \( A_1 \).** Again from (3.17), we see that

\[
|(A_1\lambda)_{v,h}| \leq \sum_{v=k+1}^{\infty} \sum_{h \in \mathbb{Z}^n} 2^{(v-k)(x_1 - \varepsilon/2 - J + n/2)} |\lambda_{v,h}| \left( 1 + 2^k |x_{k,m} - x_{v,h}| \right)^{J+\varepsilon} \\
= \sum_{v=k+1}^{\infty} 2^{(v-k)(x_1 - \varepsilon/2 - J + n/2)} T_{k,v,m},
\]

We proceed as in the estimate of \( A_0 \) we can prove that

\[
T_{k,v,m} \leq c 2^{(v-k)n/\tau} \mathcal{M}_c \left( \sum_{h \in \mathbb{Z}^n} \lambda_{v,h} Z_{v,h} \right)(x), \quad v > k, x \in Q_{k,m},
\]

where \( n/(J + \frac{\varepsilon}{2}) < \tau < \min(1, p, q) \) and the positive constant \( c \) is independent of \( v, k \) and \( m \). Again applying Lemma 2.18 we obtain

\[
\|A_1\lambda|f_{p,q}(\mathbb{R}^n, \{t_k\})\|
\]

is bounded by
\[ c \left\| \lambda \mathcal{J}_{p,q}^{\hat{\lambda}}(\mathbb{R}^n, \{t_k\}) \right\|. \]

Hence, the theorem is proved. \(\square\)

**Definition 3.18** Let \( \alpha_1, \alpha_2 \in \mathbb{R}, 0 < p < \infty \) and \( 0 < q < \infty \). Let \( \{t_k\} \) be a \( p \)-admissible weight sequence. Let \( N = \max\{J - n - \alpha_1, -1\} \) and \( \alpha_2^* = \alpha_2 - [\alpha_2] \).

(i) Let \( k \in \mathbb{Z} \) and \( m \in \mathbb{Z}^n \). A function \( \varrho_{Q_{k,m}} \) is called an homogeneous smooth synthesis molecule for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \) supported near \( Q_{k,m} \) if there exist a real number \( \delta \in (\alpha_2^*, 1] \) and a real number \( M \in (J, \infty) \) such that

\[ \int_{\mathbb{R}^n} x^\beta \varrho_{Q_{k,m}}(x)dx = 0 \quad \text{if} \quad 0 \leq |\beta| \leq N, \quad (3.18) \]

\[ |\varrho_{Q_{k,m}}(x)| \leq 2^\frac{\delta}{m}(1 + 2^k|x - x_{Q_{k,m}}|)^{-\max(M,M - \alpha_1)}, \quad (3.19) \]

\[ |\partial^\beta \varrho_{Q_{k,m}}(x)| \leq 2^{k(|\beta| + \delta)}(1 + 2^k|x - x_{Q_{k,m}}|)^{-M} \quad \text{if} \quad |\beta| \leq \alpha_2 \quad (3.20) \]

and

\[ |\partial^\beta \varrho_{Q_{k,m}}(x) - \partial^\beta \varrho_{Q_{k,m}}(y)| \leq 2^{k(|\beta| + \delta)}|x - y|^\delta \sup_{|z| \leq |x - y|} \left(1 + 2^k|x - z - x_{Q_{k,m}}|\right)^{-M} \quad \text{if} \quad |\beta| = \alpha_2. \quad (3.21) \]

A collection \( \{\varrho_{Q_{k,m}}\}_{k \in \mathbb{Z},m \in \mathbb{Z}^n} \) is called a family of homogeneous smooth synthesis molecules for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \), if each \( \varrho_{Q_{k,m}}, k \in \mathbb{Z}, m \in \mathbb{Z}^n \), is an homogeneous smooth synthesis molecule for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \) supported near \( Q_{k,m} \).

(ii) Let \( k \in \mathbb{Z} \) and \( m \in \mathbb{Z}^n \). A function \( b_{Q_{k,m}} \) is called an homogeneous smooth analysis molecule for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \) supported near \( Q_{k,m} \) if there exist a \( \kappa \in ((J - \alpha_2)^*, 1] \) and an \( M \in (J, \infty) \) such that

\[ \int_{\mathbb{R}^n} x^\beta b_{Q_{k,m}}(x)dx = 0 \quad \text{if} \quad 0 \leq |\beta| \leq \alpha_2, \quad (3.22) \]

\[ |b_{Q_{k,m}}(x)| \leq 2^\frac{\delta}{m}(1 + 2^k|x - x_{Q_{k,m}}|)^{-\max(M,M + n + \alpha_2 - J)}, \quad (3.23) \]

\[ |\partial^\beta b_{Q_{k,m}}(x)| \leq 2^{k(|\beta| + \delta)}(1 + 2^k|x - x_{Q_{k,m}}|)^{-M} \quad \text{if} \quad |\beta| \leq N \quad (3.24) \]

and

\[ |\partial^\beta b_{Q_{k,m}}(x) - \partial^\beta b_{Q_{k,m}}(y)| \quad (3.25) \]
\[ 2^k(\beta + \frac{1}{2} + \kappa) |x - y|^\kappa \sup_{|z| \leq |x|} (1 + 2^k |x - z - x_{Q_{k,m}}|)^{-M} \text{ if } |\beta| = N. \]

(3.26)

A collection \( \{b_{Q_{k,m}}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \) is called a family of homogeneous smooth analysis molecules for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \), if each \( b_{Q_{k,m}} \), \( k \in \mathbb{Z}, m \in \mathbb{Z}^n \), is an homogeneous smooth synthesis molecule for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \) supported near \( Q_{k,m} \).

We will use the notation \( \{b_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \) instead of \( \{b_{Q_{k,m}}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \). The proof of the following lemma is given in [19].

**Lemma 3.19** Let \( x_1, x_2, J, M, N, \delta, \kappa, p \) and \( q \) be as in Definition 3.18. Let \( \{t_k\} \) be a \( p \)-admissible weight sequence. Suppose \( \{q_{v,h}\}_{v \in \mathbb{Z}, h \in \mathbb{Z}^n} \) is a family of smooth synthesis molecules for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \) and \( \{b_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \) is a family of homogeneous smooth analysis molecules for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \). Then there exist a positive real number \( \varepsilon_1 \) and a positive constant \( c \) such that

\[ |\langle q_{v,h}, b_{k,m} \rangle| \leq c \omega_{Q_{k,m}, p, h}(\varepsilon), \quad k, v \in \mathbb{Z}, h, m \in \mathbb{Z}^n \]

if \( \varepsilon \leq \varepsilon_1 \).

As an immediate consequence, we have the following analogues of the corresponding results on [29, Corollary B.3].

**Corollary 3.20** Let \( x_1, x_2, J, M, N, \delta, \kappa, p \) and \( q \) be as in Definition 3.18. Let \( \{t_k\} \) be a \( p \)-admissible weight sequence. Let \( \Phi \) and \( \varphi \) satisfy, respectively (3.1) and (3.2).

(i) If \( \{\varphi_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \) is a family of homogeneous synthesis molecules for the Triebel–Lizorkin spaces \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \), then the operator \( A \) with matrix \( a_{Q_{k,m}, p, h} = \langle q_{v,h}, \varphi_{k,m} \rangle, k, v \in \mathbb{Z}, m, h \in \mathbb{Z}^n \), is almost diagonal.

(ii) If \( \{b_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \) is a family of homogeneous smooth analysis molecules for the Triebel–Lizorkin spaces \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \), then the operator \( A \), with matrix \( a_{Q_{k,m}, p, h} = \langle q_{v,h}, b_{k,m} \rangle, k, v \in \mathbb{Z}, m, h \in \mathbb{Z}^n \), is almost diagonal.

Let \( f \in \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \) and \( \{b_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \) be a family of homogeneous smooth analysis molecules. To prove that \( \langle f, b_{Q_{k,m}} \rangle, k \in \mathbb{Z}, m \in \mathbb{Z}^n \), is well defined for all homogeneous smooth analysis molecules for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \), we need the following result, which proved in [9, Lemma 5.4]. Suppose that \( \Phi \) is a smooth analysis (or synthesis) molecule supported near \( Q \in Q \). Then there exists a sequence \( \{\varphi_k\}_{k \in \mathbb{N}} \subset S(\mathbb{R}^n) \) and \( c > 0 \) such that \( c \varphi_k \) is a smooth analysis (or synthesis) molecule supported near \( Q \) for every \( k \), and \( \varphi_k(x) \rightarrow \Phi(x) \) uniformly on \( \mathbb{R}^n \) as \( k \rightarrow \infty \).

Now we have the following smooth molecular characterization of the spaces \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \). We refer the reader to [19] for the corresponding result for Besov spaces.
Theorem 3.21 \ Let $a_1, a_2 \in \mathbb{R}$, $0 < \theta \leq p < \infty$ and $0 < q < \infty$. Let $\{t_k\}_k \in \mathcal{X}_{a_1,a_2}$ be a $p$-admissible weight sequence with $\sigma_1 = \theta (p/\theta)^\gamma$ and $\sigma_2 \geq p$. Let $J, M, N, \delta$ and $\kappa$ be as in Definition 3.18.

(i) \ If $f = \sum_{v=-\infty}^{\infty} \sum_{h \in \mathbb{Z}^n} \varphi_{v,h} \lambda_{v,h}$, where $\{\varphi_{v,h}\}_{v \in \mathbb{Z}, h \in \mathbb{Z}^n}$ is a family of homogeneous smooth synthesis molecules for $\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})$, then for all $\lambda \in j_{\mathbb{F}_{p,q}}(\mathbb{R}^n, \{t_k\})$

$$\|f|\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})\| \lesssim \|\lambda|j_{\mathbb{F}_{p,q}}(\mathbb{R}^n, \{t_k\})\|.$$ 

(ii) \ Let $\{b_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n}$ be a family of homogeneous smooth analysis molecules. Then for all $f \in \mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})$

$$\|\{f, b_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n}|j_{\mathbb{F}_{p,q}}(\mathbb{R}^n, \{t_k\})\| \lesssim \|f|\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})\|.$$ 

Proof \ The proof is a slight variant of [19, 29]. For the convenience of the reader, we give some details.

Step 1. Proof of (i). By (3.5) we can write

$$\varphi_{v,h} = \sum_{k=-\infty}^{\infty} 2^{-kn} \sum_{m \in \mathbb{Z}^n} \tilde{\varphi}_k * \varphi_{v,h}(2^{-k}m) \psi_k(\cdot - 2^{-k}m)$$

for any $v \in \mathbb{Z}, h \in \mathbb{Z}^n$. Therefore,

$$f = \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} S_{k,m} \psi_{k,m} = T \psi S,$$

where $S = \{S_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n}$, with

$$S_{k,m} = 2^{-kn/2} \sum_{v=-\infty}^{\infty} \sum_{h \in \mathbb{Z}^n} \tilde{\varphi}_k * \varphi_{v,h}(2^{-k}m) \lambda_{v,h}.$$ 

From Theorem 3.8, we have

$$\|f|\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})\| = \|T \psi S|\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})\| \lesssim \|S|\mathcal{F}_{p,q}(\mathbb{R}^n, \{t_k\})\|.$$ 

But

$$S_{k,m} = \sum_{v=-\infty}^{\infty} \sum_{h \in \mathbb{Z}^n} a_{Q_{k,m}} \lambda_{v,h},$$

with

$$a_{Q_{k,m}} = \langle \varphi_{v,h}, \tilde{\varphi}_{k,m} \rangle, \quad k, v \in \mathbb{Z}, m, h \in \mathbb{Z}^n.$$ 

Applying Lemma 3.19 and Theorem 3.17 we find that
\[ \| S[\hat{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \| \leq \| \lambda \hat{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \|. \]

**Step 2. Proof of (ii).** We have

\[
\langle f, b_{k,m} \rangle = \sum_{v=-\infty}^{\infty} 2^{-vn} \sum_{m \in \mathbb{Z}^n} \langle \psi_v(\cdot - 2^{-v} h), b_{k,m} \rangle \hat{\varphi}_v * f(2^{-v} h) \\
= \sum_{v=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \langle \psi_v, b_{k,m} \rangle \hat{\lambda}_{v,h} \\
= \sum_{v=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^n} a_{Q_v,m,v,h} \hat{\lambda}_{v,h},
\]

where

\[ a_{Q_v,m,v,h} = \langle \psi_v, b_{k,m} \rangle, \quad \hat{\lambda}_{v,h} = 2^{-vn/2} \hat{\varphi}_v * f(2^{-v} h). \]

Again by Lemma 3.19 and Theorem 3.17 we find that

\[
\| \{ \langle f, b_{k,m} \rangle \}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} | \hat{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \| \leq \| \{ \lambda_{v,h} \}_{v \in \mathbb{Z}, h \in \mathbb{Z}^n} | \hat{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \| \\
= c \| \{ (S_{\varphi})_{v,h} \}_{v \in \mathbb{Z}, h \in \mathbb{Z}^n} | \hat{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \| .
\]

Applying Theorem 3.8 we find that

\[
\| \{ \langle f, b_{k,m} \rangle \}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} | \hat{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \| \leq \| f | \hat{f}_{p,q}(\mathbb{R}^n, \{t_k\}) \| .
\]

The proof is complete. \( \square \)

**Definition 3.22** Let \( \alpha_1, \alpha_2 \in \mathbb{R}, 0 < p < \infty, 0 < q < \infty \) and \( N = \max \{ J - n - \alpha_1, -1 \} \). Let \( \{ t_k \} \) be a p-admissible weight sequence. A function \( a_{Q_v,m} \) is called an homogeneous smooth atom for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \) supported near \( Q_{k,m}, k \in \mathbb{Z} \) and \( m \in \mathbb{Z}^n \), if

\[
\text{supp}( a_{Q_v,m} ) \subseteq 3Q_{k,m} \quad (3.27)
\]

\[
| \hat{\varphi} a_{Q_v,m}(x) | \leq 2^{2n(|\beta|+1/2)} \quad \text{if} \ 0 \leq |\beta| \leq \max(0, 1 + |\alpha_2|), \ x \in \mathbb{R}^n \quad (3.28)
\]

and if

\[
\int_{\mathbb{R}^n} x^\beta a_{Q_v,m}(x) dx = 0 \quad \text{if} \ 0 \leq |\beta| \leq N \text{ and } k \in \mathbb{Z}. \quad (3.29)
\]

A collection \( \{ a_{Q_v,m} \}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \) is called a family of homogeneous smooth atoms for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \), if each \( a_{Q_v,m} \) is an homogeneous smooth atom for \( \hat{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \) supported near \( Q_{v,m} \). We point out that in the moment condition (3.29) can be strengthened into that...
\[
\int_{\mathbb{R}^n} x^\beta a_{Q,m}(x) \, dx = 0 \quad \text{if } 0 \leq |\beta| \leq \tilde{N} \text{ and } k \in \mathbb{Z}
\]
and the regularity condition (3.28) can be strengthened into that
\[
|\partial^\beta a_{Q,m}(x)| \leq 2^{|\beta|+1/2} \quad \text{if } 0 \leq |\beta| \leq K, \quad x \in \mathbb{R}^n,
\]
where \(K\) and \(\tilde{N}\) are arbitrary fixed integer satisfying \(K \geq \max(0, 1 + |x_2|)\) and \(\tilde{N} \geq \max\{J - n - x_1, -1\}\). If an atom \(a\) is supported near \(Q_v, m\), then we denote it by \(a_{v,m}\).

Now we come to the atomic decomposition theorem, see [19] for Besov spaces and the same arguments are true for Triebel–Lizorkin spaces.

**Theorem 3.23** Let \(x_1, x_2 \in \mathbb{R}, \ 0 < \theta \leq p < \infty, \ 0 < q < \infty. \) Let \(\{t_k\} \in \tilde{X}_{x,\sigma,p}\) be a \(p\)-admissible weight sequence with \(\sigma_1 = 0(p/\theta)'\) and \(\sigma_2 \geq p\). Then for each \(f \in \dot{F}_{p,q}(\mathbb{R}^n, \{t_k\})\), there exist a family \(\{Q_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n}\) of homogeneous smooth atoms for \(\dot{F}_{p,q}(\mathbb{R}^n, \{t_k\})\) and \(\lambda = \{\lambda_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \hat{H}_{p,q}(\mathbb{R}^n, \{t_k\})\) such that
\[
f = \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{k,m} Q_{k,m}, \quad \text{converging in } S'_\infty(\mathbb{R}^n)
\]
and
\[
\| \{\lambda_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \|_{\dot{F}_{p,q}(\mathbb{R}^n, \{t_k\})} \leq \|f\|_{\dot{F}_{p,q}(\mathbb{R}^n, \{t_k\})}.
\]
Conversely, for any family of homogeneous smooth atoms for \(\dot{F}_{p,q}(\mathbb{R}^n, \{t_k\})\) and \(\lambda = \{\lambda_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \hat{H}_{p,q}(\mathbb{R}^n, \{t_k\})\)
\[
\| \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{k,m} Q_{k,m} \|_{\dot{F}_{p,q}(\mathbb{R}^n, \{t_k\})} \leq \| \{\lambda_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \|_{\hat{H}_{p,q}(\mathbb{R}^n, \{t_k\})}.
\]

**Remark 3.24**

(i) We mention that the techniques of [37] are incapable of dealing with spaces of variable smoothness. Also our assumptions on the weight \(\{t_k\}\) play an exceptional role in the paper.

(ii) We draw the reader’s attention to paper [46] where generalized Besov-type and Triebel–Lizorkin-type spaces are studied. They assumed that the weight sequence \(\{t_k\}\) lies in some class which different from the class \(\tilde{X}_{x,\sigma,p}\).
4 The non-homogeneous space $F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})$

In this section, we present the inhomogeneous version of our results given above. Let $\Phi, \psi, \varphi$ and $\Psi$ satisfy

$$\Phi, \Psi, \varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$$

$$\text{supp}(\mathcal{F}(\Phi)) \cup \text{supp}(\mathcal{F}(\Psi)) \subset B(0, 2), \quad |\mathcal{F}(\Phi)(\xi)|, |\mathcal{F}(\Psi)(\xi)| \geq c,$$  \hspace{1cm} (4.1)

if $|\xi| \leq 5/3$ and

$$\text{supp}(\mathcal{F}(\varphi)) \cup \text{supp}(\mathcal{F}(\psi)) \subset B(0, 2) \setminus B(0, 1/2), \quad |\mathcal{F}(\varphi)(\xi)|, |\mathcal{F}(\psi)(\xi)| \geq c,$$  \hspace{1cm} (4.2)

if $3/5 \leq |\xi| \leq 5/3$, such that

$$\mathcal{F}(\Phi)(\xi)\mathcal{F}(\Psi)(\xi) + \sum_{k=1}^{\infty} \mathcal{F}(\varphi)(2^{-k}\xi)\mathcal{F}(\psi)(2^{-k}\xi) = 1, \quad \xi \in \mathbb{R}^n,$$  \hspace{1cm} (4.3)

where $c > 0$. Let $\Phi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy, respectively, (4.2) and (4.3). We recall that by [29, pp. 130–131] or [30, Lemma 6.9], there exist functions $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (4.2) and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (4.3) such that (4.4) holds.

The $\varphi$-transform $S_\varphi$ is defined by setting

$$(S_\varphi f)_{0,m} = \langle f, \Psi_m \rangle,$$

with $\Psi_m(x) = \Psi(x - m)$ and

$$(S_\varphi f)_{k,m} = \langle f, \varphi_{k,m} \rangle,$$

where $\varphi_{k,m}(x) = 2^{kn/2} \varphi(2^kx - m), k \in \mathbb{N}$ and $m \in \mathbb{Z}^n$. The inverse $\varphi$-transform $T_\psi$ is defined by

$$T_\psi \lambda = \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \Psi_m + \sum_{k=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{k,m} \psi_{k,m},$$

with $\lambda = \{\lambda_{k,m}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \subset \mathbb{C}$, see again [29].

Now we present the inhomogenous version of Definition 2.5.

**Definition 4.1** Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 \in (0, +\infty]$, $\alpha = (\alpha_1, \alpha_2)$ and let $\sigma = (\sigma_1, \sigma_2)$. We let $X_{\alpha,\sigma,p} = X_{\alpha,\sigma,p}(\mathbb{R}^n)$ denote the set of $p$-admissible weight sequences $\{t_k\}_{k \in \mathbb{N}_0}$ satisfying (2.3) and (2.4) for any $0 \leq k \leq j$, with constants $C_1, C_2 > 0$ are independent of both the indexes $k$ and $j$.

**Example** A sequence $\{\gamma_j\}_{j \in \mathbb{N}_0}$ of positive real numbers is said to be admissible if there exist two positive constants $d_0$ and $d_1$ such that
For an admissible sequence \( \{ \gamma_j \}_{j \in \mathbb{N}_0} \), let
\[
\gamma_j = \inf_{k \geq 0} \frac{\gamma_{j+k}}{\gamma_k} \quad \text{and} \quad \bar{\gamma}_j = \sup_{k \geq 0} \frac{\gamma_{j+k}}{\gamma_k}, \quad j \in \mathbb{N}_0.
\]
Let
\[
\alpha_j = \lim_{j \to \infty} \frac{\log \gamma_j}{j} \quad \text{and} \quad \beta_j = \lim_{j \to \infty} \frac{\log \bar{\gamma}_j}{j},
\]
be the upper and lower Boyd index of the given sequence \( \{ \gamma_j \}_{j \in \mathbb{N}_0} \), respectively. Then
\[
\gamma_j \gamma_k \leq \gamma_{j+k} \leq \bar{\gamma}_j \bar{\gamma}_k, \quad j, k \in \mathbb{N}_0
\]
and for each \( \varepsilon > 0 \),
\[
c_1 2^{(\beta_j - \varepsilon)j} \leq \gamma_j \leq c_2 2^{(\alpha_j + \varepsilon)j}, \quad j \in \mathbb{N}_0
\]
for some constants \( c_1 = c_1(\varepsilon) > 0 \) and \( c_2 = c_2(\varepsilon) > 0 \). Also, \( \gamma_1 \) and \( \bar{\gamma}_1 \) are the best possible constants \( d_0 \) and \( d_1 \) in (4.5), respectively. Clearly the sequence \( \{ \gamma_j \}_{j \in \mathbb{N}_0} \) lies in \( X_{\sigma, p} \) for \( \alpha_1 = \beta_1 = \gamma_1 = \varepsilon \) and \( 0 < p, \sigma_1, \sigma_2 \leq \infty \).

These type of admissible sequences are used in [25] to study Besov and Triebel–Lizorkin spaces in terms of a generalized smoothness, see also [36].

Let us consider some examples of admissible sequences. The sequence \( \{ \gamma_j \}_{j \in \mathbb{N}_0} \),
\[
\gamma_j = 2^{sj}(1+j)^b(1+\log(1+j))^c, \quad j \in \mathbb{N}_0
\]
with arbitrary fixed real numbers \( s, b \) and \( c \) is a an admissible sequence with
\[
\beta_j = \alpha_j = s.
\]

**Example** Let \( 0 < r < p < \infty \), a weight \( \omega^p \in A_c(\mathbb{R}^n) \) and
\[
\{ s_k \} = \{ 2^{ks} \omega^p \}_{k \in \mathbb{N}_0}, \quad s \in \mathbb{R}.
\]
Obviously, \( \{ s_k \}_{k \in \mathbb{N}_0} \) lies in \( X_{\sigma, p} \) for \( \alpha_1 = \alpha_2 = s \), \( \sigma = (r/p/r', p) \).

Now, we define the spaces under consideration.

**Definition 4.2** Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \). Let \( \{ t_k \}_{k \in \mathbb{N}_0} \) be a \( p \)-admissible weight sequence. Let \( \Phi, \phi \in S(\mathbb{R}^n) \) satisfy (4.2) and (4.3), respectively, and we put \( \varphi_k = 2^{kn} \Phi(2^k \cdot) \), \( k \in \mathbb{N}_0 \). The Triebel–Lizorkin space \( F_{p,q}(\mathbb{R}^n); \{ t_k \}_{k \in \mathbb{N}_0} \) is the collection of all \( f \in S'(\mathbb{R}^n) \) such that
\[ \|f \|_{F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})} = \left\| \left( \sum_{k=0}^{\infty} t_k^q |\varphi_k * f|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} < \infty, \]

with the usual modifications if \( q = \infty \), where \( \varphi_0 \) is replaced by \( \Phi \).

Now we introduce the inhomogeneous sequence spaces \( F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \). Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \) be a \( p \)-admissible weight sequence. Then for all complex valued sequences \( \lambda = \{ \lambda_{k,m} \}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \subset \mathbb{C} \) we define

\[ F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) = \left\{ \lambda : \| \lambda_{F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})} \| < \infty \right\}, \]

where

\[ \| \lambda_{F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})} \| = \left\| \left( \sum_{k=0}^{\infty} 2^{knq/2} \sum_{m \in \mathbb{Z}^n} t_k^q |\lambda_{k,m}|^q \lambda_{k,m} \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}. \]

We have the following analogue of Theorem 3.8.

**Theorem 4.3** Let \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2, 0 < \theta \leq p < \infty \) and \( 0 < q < \infty \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \in X_{\alpha,\sigma,p} \) be a \( p \)-admissible weight sequence with \( \sigma = (\sigma_1 = 0(p/\theta)', \sigma_2 \geq p) \). Let \( \varphi, \psi \) satisfying (4.1) through (4.4). The operators

\[ S_\varphi : F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \to F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \]

and

\[ T_\psi : F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \to F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \]

are bounded. Furthermore, \( T_\psi \circ S_\varphi \) is the identity on \( F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \).

As a consequence the analogues of Corollary 3.10 are now clear. We obtain the following useful properties of these function spaces.

**Theorem 4.4** Let \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2, 0 < \theta \leq p < \infty \) and \( 0 < q < \infty \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \in X_{\alpha,\sigma,p} \) be a \( p \)-admissible weight sequence with \( \sigma = (\sigma_1 = 0(p/\theta)', \sigma_2 \geq p) \). \( F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) are quasi-Banach spaces. They are Banach spaces if \( 1 \leq p < \infty \) and \( 1 \leq q < \infty \).

Let \( 0 < \theta \leq p < \infty \) and \( 0 < q < \infty \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \in X_{\alpha,\sigma,p} \) be a \( p \)-admissible weight sequence with \( \sigma = (\sigma_1 = 0(p/\theta)', \sigma_2 \geq p) \) and \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \). As in Theorem 3.13 we have the embedding

\[ S(\mathbb{R}^n) \to F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}). \]

In addition \( S(\mathbb{R}^n) \) is dense in \( F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \). Also if \( 0 < \theta \leq p < \infty \) and \( 0 < q < \infty \), then
Theorem 4.5  Let \(0 < p < \infty\) and \(0 < q \leq r < \infty\). Let \(\{t_k\}_{k \in \mathbb{N}_0} \in X_{\mathbb{C}, p}\) be a \(p\)-admissible weight sequence with \(\sigma = (\sigma_1 = (p(0/\theta))^j, \sigma_2 \geq p)\). We have

\[
F_{p, q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \hookrightarrow F_{p, r}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}).
\]

All the results in Sect. 3.2 are true for the inhomogeneous case.

We begin with the following elementary embeddings, where the proof can be obtained using the properties of sequence Lebesgue spaces.

Theorem 4.6  Let \(0 < \theta \leq p < 0 < q \leq r < \infty\). Let \(\{t_k\}_{k \in \mathbb{N}_0} \in X_{\mathbb{C}, \sigma_p}\) be a \(p\)-admissible weight sequence satisfying (2.3) with \(p = p_0\), \(\sigma_1 = (p_0/\theta)^j\) and \(j = k \geq 0\). Let \(\{w_k\}\) be a \(p_0\)-admissible weight sequence satisfying (2.3) with \(p = p_1\), \(\sigma_1 = (p_1/\theta)^j\) and \(j = k \geq 0\). If \(w_{k, Q}(p_1) \leq t_{k, Q}(p_0)\) for all \(Q \in \mathbb{Q}\) with \(\ell(Q) = 2^{-k}, k \in \mathbb{N}_0\), then we have

\[
F_{p_0, q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \hookrightarrow F_{p_1, r}(\mathbb{R}^n, \{w_k\}_{k \in \mathbb{N}_0}).
\]

From Theorems 4.3 and 4.6, we have the following Sobolev-type embedding conclusions for \(F_{p, q}(\mathbb{R}^n, \{t_k\})\).

Theorem 4.7  Let \(0 < \theta \leq p_0 < p_1 < \infty\) and \(0 < q < r < \infty\). Let \(\{t_k\}_{k \in \mathbb{N}_0} \in X_{\mathbb{C}, \sigma_{p_0}}\) be a \(p_0\)-admissible weight sequence with \(\sigma = (\sigma_1 = (p_0/\theta)^j, \sigma_2 \geq p_0)\) and \(\mathbb{C} = (\mathbb{C}_{1, 0}, \mathbb{C}_{2, 0}) \subset \mathbb{R}^2\). Let \(\{w_k\}_{k \in \mathbb{N}_0} \in X_{\mathbb{C}, \sigma_{p_1}}\) be a \(p_1\)-admissible weight sequence with \(\sigma = (\sigma_1 = (p_1/\theta)^j, \sigma_2 \geq p_1)\) and \(\mathbb{C} = (\mathbb{C}_{1, 1}, \mathbb{C}_{2, 1}) \subset \mathbb{R}^2\). Then

\[
F_{p_0, q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \hookrightarrow F_{p_1, r}(\mathbb{R}^n, \{w_k\}_{k \in \mathbb{N}_0}),
\]

hold if

\[
w_{k, Q}(p_1) \leq t_{k, Q}(p_0)
\]

for all \(Q \in \mathbb{Q}\) and all \(k \in \mathbb{N}_0\) with \(\ell(Q) = 2^{-k}, k \in \mathbb{N}_0\).

In the sequel, we shall say that an operator \(A\) is associated with the matrix \(\{a_{\lambda, m, n, h}\}_{k, \ell, m, h \in \mathbb{Z}^n, l \in \mathbb{C}}\), if for all sequences \(\lambda = \{\lambda_{k, m}\}_{k \in \mathbb{N}_0} \subset \mathbb{C}, \lambda_{k, m} \in \mathbb{C}.\)
Theorem 4.9
Let \( A \), with associated matrix \( \{a_{Q_{k,m}}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \), is almost diagonal on \( f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) if there exists \( \varepsilon > 0 \) such that

\[
\sup_{k, v \in \mathbb{N}_0, m, h \in \mathbb{Z}^n, \alpha \in (0, \varepsilon)} |a_{Q_{k,m}}(h, \alpha)| < \infty,
\]

where \( \alpha_{Q_{k,m}}(\varepsilon) \) as in Sect. 5. Let \( x_1, x_2 \in \mathbb{R}, 0 < \theta \leq p < \infty \) and \( 0 < q \leq \infty \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \subset X_{x,\sigma} \) be a \( p \)-admissible weight sequence with \( \sigma_1 = \theta(\theta/0)' \) and \( \sigma_2 \geq p \). It is obvious that an operator \( A \) on \( f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) given by an almost diagonal matrix is bounded.

Let \( J \) be defined as in Sect. 3. We present the inhomogeneous versions of Definition 3.18.

**Definition 4.8** Let \( x_1, x_2 \in \mathbb{R}, 0 < p < \infty \) and \( 0 < q \leq \infty \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \) be a \( p \)-admissible weight sequence. Let \( N = \max\{J - n - x_1, -1\} \) and \( z_2^* = x_2 - \lfloor z_2 \rfloor \).

(i) We say that \( \varrho_{Q_{k,m}}, k \in \mathbb{N}_0, m \in \mathbb{Z}^n \), is an inhomogeneous smooth synthesis molecule for \( f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) supported near \( Q_{k,m} \) if it satisfies, for some real number \( \delta \in (z_2^*, 1] \) and a real number \( M \in (J, \infty) \), (3.18), (3.19), (3.20) and (3.21) if \( k \in \mathbb{N} \). If \( k = 0 \) we assume (3.20), (3.21) and

\[
|\varrho_{Q_{k,m}}(x)| \leq (1 + |x - x_{Q_{k,m}}|)^{-M}.
\]

A collection \( \{\varrho_{Q_{k,m}}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \) is called a family of inhomogeneous smooth synthesis molecules for \( f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \), if each \( \varrho_{Q_{k,m}} \) is an inhomogeneous smooth synthesis molecule for \( f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) supported near \( Q_{k,m} \).

(ii) We say that \( b_{Q_{k,m}}, k \in \mathbb{N}_0, m \in \mathbb{Z}^n \), is an inhomogeneous smooth analysis molecule for \( f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) supported near \( Q_{k,m} \) if it satisfies, for some \( \kappa \in ((J - z_2)^*, 1] \) and an \( M \in (J, \infty) \), (3.22), (3.23), (3.24) and (3.25) if \( k \in \mathbb{N} \). If \( k = 0 \) we assume (3.24), (3.25) and

\[
|b_{Q_{k,m}}(x)| \leq (1 + |x - x_{Q_{k,m}}|)^{-M}.
\]

A collection \( \{b_{Q_{k,m}}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \) is called a family of inhomogeneous smooth analysis molecules for \( f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \), if each \( b_{Q_{k,m}} \) is an inhomogeneous smooth synthesis molecule for \( f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) supported near \( Q_{k,m} \).

As a consequence, we formulate the inhomogeneous counterpart of Theorem 3.21.

**Theorem 4.9** Let \( x_1, x_2 \in \mathbb{R}, 0 \leq \theta \leq p < \infty \) and \( 0 < q < \infty \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \subset X_{x,\sigma,p} \) be a \( p \)-admissible weight sequence with \( \sigma_1 = \theta(\theta/0)' \) and \( \sigma_2 \geq p \). Let \( J, M, N, \delta \) and \( \kappa \) be as in Definition 4.8.
(i) If \( f = \sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}^n} q_{k,m} \lambda_{k,m}, \) where \( \{q_{k,m}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \) is a family of inhomogeneous smooth synthesis molecules for \( F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \), then for all \( \lambda \in f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \)

\[
\|f|F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})\| \lesssim \|\lambda|f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})\|.
\]

(ii) Let \( \{b_{k,m}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \) be a family of inhomogeneous smooth analysis molecules. Then for all \( f \in f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \)

\[
\|\{f, b_{k,m}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n}|f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})\| \lesssim \|f|F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})\|.
\]

Now we present the analogue of smooth atomic decomposition. First we need the definition of inhomogeneous smooth.

**Definition 4.10** Let \( \alpha_1, \alpha_2 \in \mathbb{R}, 0 < p < \infty, 0 < q \leq \infty \) and \( N = \max\{J - n - \alpha_1, -1\} \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \) be a \( p \)-admissible weight sequence. A function \( a_{Q,k,m} \) is called an inhomogeneous smooth atom for \( F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) supported near \( Q_{k,m}, k \in \mathbb{N}_0 \) and \( m \in \mathbb{Z}^n \), if it is satisfies (3.27), (3.28) and (3.29) if \( k \in \mathbb{N} \). If \( k = 0 \) we assume (3.27) and (3.28).

A collection \( \{a_{Q,k,m}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \) is called a family of inhomogeneous smooth atoms for \( F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \), if each \( a_{Q,k,m} \) is an inhomogeneous smooth atom for \( F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) supported near \( Q_{k,m} \).

Now we come to the atomic decomposition theorem.

**Theorem 4.11** Let \( \alpha_1, \alpha_2 \in \mathbb{R}, 0 < \theta \leq p < \infty, 0 < q < \infty \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \in X_{\alpha_2, \theta, p} \) be a \( p \)-admissible weight sequence with \( \alpha_1 = 0(p/\theta)' \) and \( \alpha_2 \geq p \). Then for each \( f \in f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \), there exist a family \( \{q_{k,m}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \) of inhomogeneous smooth atoms for the spaces \( F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) and \( \lambda = \{\lambda_{k,m}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) such that

\[
f = \sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{k,m} q_{k,m}, \quad \text{converging in } S'(\mathbb{R}^n),
\]

and

\[
\|\{\lambda_{k,m}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n}|f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})\| \lesssim \|f|F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})\|.
\]

Conversely, for any family of inhomogeneous smooth atoms for the spaces \( F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) and \( \lambda = \{\lambda_{k,m}\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) we have
\[ \left\| \sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \hat{\lambda}_{k,m} q_{k,m} |F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})| \right\|_{L_q} \lesssim \left\| \left\{ \hat{\lambda}_{k,m} \right\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} \right\|_{L_q} \left\| f_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \right\|_{L_q}. \]

**Remark 4.12** One of the applications of atomic and molecular decompositions of the spaces \( \dot{F}_{p,q}(\mathbb{R}^n, \{t_k\}) \) and \( F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) is studying the continuity of singular integral operators of non convolution type in such function spaces, where it is enough to show that it maps every family of smooth atoms into a family of smooth molecules. For classical Triebel–Lizorkin spaces we refer the reader to, e.g., [28, 63].

It is also interesting to study other properties and characterizations of these function spaces such as the wavelet characterization, see, e.g., Lemarié and Meyer [45], and Triebel [62].

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