Mutual Coinduction

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April 9, 2019

Abstract

In this paper we present mutual coinduction as a dual of mutual induction and also as a generalization of standard coinduction. In particular, we present a precise formal definition of mutual induction and mutual coinduction. In the process we present the associated mutual induction and mutual coinduction proof principles, and we present the conditions under which these principles hold.

In spite of some mention of mutual (co)induction in research literature, but the formal definition of mutual (co)induction and the proof of the mutual (co)induction proof principles we present here seem to be the first such definition and proof. As such, to the best of our knowledge, it seems our work is the first to point out that, unlike the case for standard (co)induction, monotonicity of generators is not sufficient for guaranteeing the existence of least and greatest simultaneous fixed points in complete lattices, and that continuity on the other hand is sufficient for guaranteeing their existence.

In the course of our presentation of mutual coinduction we also discuss some concepts related to standard (also called direct) induction and standard coinduction, as well as ones related to mutual (also called simultaneous or indirect) induction. During the presentation we purposely discuss particular standard concepts (namely, fixed points, least and greatest fixed points, pre-fixed points, post-fixed points, least pre-fixed points, and greatest post-fixed points) so as to help motivate the definitions of their more general counterparts for mutual/simultaneous/indirect (co)induction (namely, simultaneous fixed points, least simultaneous fixed points, greatest simultaneous fixed points, least simultaneous pre-fixed points and greatest simultaneous post-fixed points). Greatest simultaneous post-fixed points, in particular, will be abstractions and models of mathematical objects (e.g., points, sets, types, predicates, etc.) that are defined mutually-coinductively.

1 Introduction

Induction and coinduction, henceforth called standard (co)induction, present mathematical models for (standard) recursive definitions (also called direct recursive or self-recursive definitions). In the same way, mutual induction and mutual coinduction, henceforth called mutual (co)induction, present mathematical models for mutually-recursive definitions, which are sometimes also called indirect recursive definitions or simultaneous recursive definitions.

In other concurrent work we present some practical motivations (from programming languages theory) for defining mutual (co)induction [9] (temporarily, included in Appendix B on page 16 of this paper), some intuitions for their mathematical definitions as well as examples of their use [8], and possible formulations of mutual (co)induction in other mathematical disciplines [7]. In that work we conclude that: if mutually-recursive functional programs can be reasoned about mathematically, then also imperative and object-oriented programs (even the worst such programs) can be reasoned about mathematically. Interested readers are invited to check this concurrent work.

This paper is structured as follows. We first motivate the formal definition of mutual (co)induction by presenting in §2 the formal definitions of standard (co)induction, in the context of order theory. Then we present in §3 the order-theoretic definitions of mutual (co)induction and we present the mutual (co)induction proof principles. (We present proofs for the lemmas and theorems of §3 in
Appendix A.) We briefly discuss some work related to ours in §4, then we conclude and discuss some possible future work in §5.

Our work here is a followup on our earlier introductory work in [2, 6, 5]. In addition to the practical motivations mentioned above (whose interest in we started in [1]), the work presented here has also been motivated by that earlier introductory work.

2 Standard Induction and Standard Coinduction

The formulation of standard induction and standard coinduction, and related concepts, that we present here is a summary of the formulation presented in [2, §2.1].

Let $\leq$ (‘is less than or equal to’) be an ordering relation over a set $\mathbb{O}$ and let $F : \mathbb{O} \to \mathbb{O}$ be an endofunction over $\mathbb{O}$ (also called a self-map over $\mathbb{O}$, i.e., a function whose domain and codomain are the same set, thus mapping a set into itself).

Given a point $P \in \mathbb{O}$, the point $F(P)$ is called the $F$-image of $P$.

A point $P \in \mathbb{O}$ is called a pre-fixed point of $F$ if its $F$-image is less than or equal to it, i.e., if $F(P) \leq P$.

Dually, a point $P \in \mathbb{O}$ is called a post-fixed point of $F$ if it is less than or equal to its $F$-image, i.e., if $P \leq F(P)$.

A point $P \in \mathbb{O}$ is called a fixed point of $F$ if it is equal to its $F$-image, i.e., if $P = F(P)$.

(A fixed point of $F$ is simultaneously a pre-fixed point of $F$ and a post-fixed point of $F$.)

Now, if $\leq$ is a complete lattice over $\mathbb{O}$ and if $F$ is a monotonic endofunction over $\mathbb{O}$ (then called a generator), then the least pre-fixed point of $F$, called $\mu_F$, exists in $\mathbb{O}$ and $\mu_F$ is also the least fixed point (lfp) of $F$, and the greatest post-fixed point of $F$, called $\nu_F$, exists in $\mathbb{O}$ and $\nu_F$ is also the greatest fixed point (gfp) of $F$.

Further, for any element $P \in \mathbb{O}$ we have:

- **(standard induction)** if $F(P) \leq P$, then $\mu_F \leq P$,

  which, in words, means that if $P$ is a pre-fixed/inductive point of $F$, then point $\mu_F$ is less than or equal to $P$ (since $\mu_F$, by definition, is the least $F$-inductive point), and,

- **(standard coinduction)** if $P \leq F(P)$, then $P \leq \nu_F$,

  which, in words, means that if $P$ is a post-fixed/coinductive point of $F$, then point $P$ is less than or equal to point $\nu_F$ (since $\nu_F$, by definition, is the greatest $F$-coinductive point).

References  See [15, 33, 2].

3 Mutual Induction and Mutual Coinduction

Intuition

Intuitively, compared to standard (co)induction which involves one self-map from a poset to itself, mutual (co)induction involves two or more mappings between two or more ordered sets (i.e., posets). That’s all. In this paper, for simplicity, we focus only on the case involving just two mappings (also called generators) between two posets. As we define it below, our definition of mutual induction can be extended easily to involve more than two orderings, more than two underlying sets, and more than two mappings between the ordered sets. Also, it should be noted that in some practical applications of mutual (co)induction the two orderings, and their underlying sets, may
Figure 3.1: Illustrating simultaneous pre-fixed points, e.g., points \( O \) and \( P \).

happen to be the same ordering and set. For proper mutual (co)induction, however, there has to be at least two mutual generators/mappings between the two ordered sets.\(^1\)

Also intuitively, the mutual induction and mutual coinduction proof principles are expressions of the properties of two points (i.e., elements of the two ordered sets) that are together (i.e., simultaneously) least pre-fixed points of each of the two generators and of two points that are together greatest post-fixed points of each of the two generators. As such mutual induction and mutual coinduction generalize the standard induction and standard coinduction proof principles. Further, in case the two orderings are complete lattices and the mappings are continuous (thereby called generators or mutual generators), the two least simultaneous pre-fixed points and the two greatest simultaneous post-fixed points will also be mutual fixed points (sometimes also called simultaneous or reciprocal fixed points) of the generators. (For a glimpse, see Figures 3.1, 3.2, and 3.3.)

Formulation

Let \( \leq \) be an ordering relation over a set \( O \) and let \( \sqsubseteq \) be a second ordering relation over a second set \( P \). Further, let \( F : O \rightarrow P \) and \( G : P \rightarrow O \) be two mutual endofunctions over \( O \) and \( P \) (also called indirect or reciprocal self-maps over \( O \) and \( P \), i.e., two functions where the domain of the first is the same as the codomain of the second and vice versa, such that each of the two posets is mapped into the other).\(^2\) Note that given two mutual endofunctions \( F \) and \( G \) we can always compose \( F \) and \( G \) to get the (standard) endofunctions \( G \circ F : O \rightarrow O \) and \( F \circ G : P \rightarrow P \).\(^3,4\)

Given points \( O \in O \) and \( P \in P \), the point \( F(O) \in P \) and the point \( G(P) \in O \) are called the \( F \)-image of \( O \) and the \( G \)-image of \( P \), respectively.

\(^1\)Standard induction is then obtained as a special case of mutual induction by having one ordering and one underlying set and also having one of the two generators—or, more accurately, all but one of the generators—be the identity function. See also Footnote 4.

\(^2\)As in [2], we are focused on unary functions in this paper because we are interested in discussing fixed points and closely-related concepts such as induction and coinduction, to which multi-arity seems to make little difference. For more details, see the brief discussion on arity and currying in [2].

\(^3\)Upon seeing these compositions, readers familiar with category theory may immediately suspect a possible connection with adjunctions. That possibility of a connection will increase when readers check the definitions below, of simultaneous pre-fixed points and simultaneous post-fixed points. We intend to dwell on the possibility of such a connection in future work (or in later versions of this paper).

\(^4\)Note that if we have \( P = O \) (the two underlying sets are the same), \( \sqsubseteq \leq \) (with the same orderings), and if \( G = 1 \) (or \( F = 1 \), where \( 1 \) is the identity function), then we obtain standard (co)induction as a special case of mutual (co)induction. In particular, all definitions presented below will smoothly degenerate to their standard counterparts (i.e., will correspond to ones for standard (co)induction. See §2).
Figure 3.2: The points $O$ and $P$, $O$ and $Q$, and $N$ and $P$ illustrate having multiple simultaneous pre-fixed points that share some of their component points.

Figure 3.3: (Meet-)Continuous mutual generators $F$ and $G$. 
Two points \( O \in \mathcal{O} \), \( P \in \mathcal{P} \) are called *simultaneous* (or *mutual* or *reciprocal*) pre-fixed points of \( F \) and \( G \) if the \( F \)-image of \( O \) is less than or equal to \( P \) and the \( G \)-image of \( P \) is less than or equal to \( O \) (that is, intuitively, if “the images of the two points are less than the two points themselves”), i.e., if

\[
F(O) \sqsubseteq P \text{ and } G(P) \leq O.
\]

Simultaneous pre-fixed points are also called *mutually-inductive points* of \( F \) and \( G \). See Figure 3.1 on page 3 for an illustration of simultaneous pre-fixed points.

- It should be immediately noted from the definition of simultaneous pre-fixed points that, generally-speaking, a single point \( O \in \mathcal{O} \) can be paired with *more than one* point \( P \in \mathcal{P} \) such that \( O \) and \( P \) form a single pair of simultaneous pre-fixed points of \( F \) and \( G \). Symmetrically, a single point \( P \in \mathcal{P} \) can also be paired with more than one point \( O \in \mathcal{O} \) to form such a pair. See Figure 3.2 on the preceding page for an illustration.

- As such, two functions \( \text{PreFP}_{F,G} : \mathcal{O} \to \wp(\mathcal{P}) \) and \( \text{PreFP}_{G,F} : \mathcal{P} \to \wp(\mathcal{O}) \) that compute these sets of points in \( \mathcal{P} \) and \( \mathcal{O} \), respectively, can be derived from \( F \) and \( G \). In particular, we have

\[
\text{PreFP}_{F,G}(O) = \{ P \in \mathcal{P} | F(O) \sqsubseteq P \text{ and } G(P) \leq O \}, \quad \text{and}
\]

\[
\text{PreFP}_{G,F}(P) = \{ O \in \mathcal{O} | F(O) \sqsubseteq P \text{ and } G(P) \leq O \}.
\]

Note that for some points \( O \in \mathcal{O} \), the sets \( \text{PreFP}_{F,G}(O) \) can be the empty set \( \emptyset \), meaning that such points \( O \) are not paired with any points in \( \mathcal{P} \) so as to form simultaneous pre-fixed points of \( F \) and \( G \). Symmetrically, the same observation holds for some points \( P \in \mathcal{P} \) and their images \( \text{PreFP}_{G,F}(P) \).

Dually, two points \( O \in \mathcal{O} \), \( P \in \mathcal{P} \) are called *simultaneous* (or *mutual* or *reciprocal*) post-fixed points of \( F \) and \( G \) if \( P \) is less than or equal to the \( F \)-image of \( O \) and \( O \) is less than or equal to the \( G \)-image of \( P \) (that is, intuitively, if “the two points are less than their two own images”), i.e., if

\[
P \sqsubseteq F(O) \text{ and } O \leq G(P).
\]

Simultaneous post-fixed points are also called *mutually-coinductive points* of \( F \) and \( G \).

- Like for simultaneous pre-fixed points, a point \( O \in \mathcal{O} \) or \( P \in \mathcal{P} \) can be paired with *more than one* point of the other poset to form a single pair of simultaneous post-fixed points of \( F \) and \( G \). (Similar to \( \text{PreFP}_{F,G} \) and \( \text{PreFP}_{G,F} \), two functions \( \text{PostFP}_{F,G} : \mathcal{O} \to \wp(\mathcal{P}) \) and \( \text{PostFP}_{G,F} : \mathcal{P} \to \wp(\mathcal{O}) \) that compute these sets can be derived from \( F \) and \( G \).)

Two points \( O \in \mathcal{O} \), \( P \in \mathcal{P} \) are called *simultaneous* (or *mutual* or *reciprocal*) fixed points of \( F \) and \( G \) if the \( F \)-image of \( O \) is equal to \( P \) and the \( G \)-image of \( P \) is equal to \( O \), i.e., if

\[
F(O) = P \text{ and } G(P) = O.
\]

(As such, two simultaneous fixed points of \( F \) and \( G \) are, simultaneously, simultaneous pre-fixed points of \( F \) and \( G \) and simultaneous post-fixed points of \( F \) and \( G \).)

- Unlike for simultaneous pre-fixed and post-fixed points, a point in \( \mathcal{O} \) or in \( \mathcal{P} \) can be paired with *only one* point of the other poset to form a pair of simultaneous fixed points of \( F \) and \( G \).

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5Note that in standard induction—or, more accurately, in the encoding of standard induction using mutual induction—(ultimately due to the transitivity of \( \leq \) and \( \sqsubseteq \)) it can be said that a standard pre-fixed point \( O \in \mathcal{O} \) is “paired” with one point, namely itself, to form a pair of simultaneous pre-fixed points that encodes the standard pre-fixed point. In other words, standard pre-fixed points in standard induction, when encoded as mutual induction, correspond to simultaneous pre-fixed points (e.g., using the encoding of we mentioned in Footnote 4), and vice versa (i.e., simultaneous pre-fixed points, in such an encoding, will correspond to standard pre-fixed points).
Continuity Unlike the case for standard (co)induction, where the monotonicity of generators is enough to guarantee the existence of least and greatest standard fixed points, due to the possibility of multiple pairings in PreFP and PostFP, the monotonicity of two mutual endofunctions \( F \) and \( G \) is not enough for proving the existence of least and greatest simultaneous fixed points of \( F \) and \( G \). Rather (as can be seen in the proofs in Appendix A), it is the continuity of the mutual endofunctions \( F \) and \( G \) that guarantees the existence of such fixed points. Hence, before proceeding with the formulation of mutual (co)induction we now introduce the important and useful concept of continuity.

**Definition 1** (Continuous Mutual Endofunctions). Two mutual endofunctions \( F : \mathbb{O} \to \mathbb{P} \) and \( G : \mathbb{P} \to \mathbb{O} \) defined over two posets \((\mathbb{O}, \leq)\) and \((\mathbb{P}, \subseteq)\) are continuous mutual endofunctions if and only if for all subsets \( M \subseteq \mathbb{O} \) and \( N \subseteq \mathbb{P} \) whenever greatest lower bounds (also called glbs) \( \land M \in \mathbb{O} \) and \( \cap N \in \mathbb{P} \) and least upper bounds (also called lubs) \( \lor M \in \mathbb{O} \) and \( \cup N \in \mathbb{P} \) exist (in \( \mathbb{O} \) and in \( \mathbb{P} \)) then the corresponding points \( \land F(M) \in \mathbb{P}, \land G(N) \in \mathbb{O}, \lor F(M) \in \mathbb{P}, \) and \( \lor G(N) \in \mathbb{O} \) also exist (in \( \mathbb{O} \) and in \( \mathbb{P} \)), and further, more significantly, we have

\[
\land F(M) = F(\land M) \quad \text{and} \quad \land G(N) = G(\cap N),
\]

\[
\lor F(M) = F(\lor M) \quad \text{and} \quad \lor G(N) = G(\cup N).
\]

As such, if two mutual endofunctions \( F : \mathbb{O} \to \mathbb{P} \) and \( G : \mathbb{P} \to \mathbb{O} \) are continuous then they are said to preserve the glbs and lubs of \( \mathbb{O} \) and \( \mathbb{P} \), whenever these points exist.\(^6\) (Even though somewhat similar, but the notion of continuity we use here is stricter and more uniform than the notion of Scott-continuity used in domain theory, which asserts that mappings preserve only lubs/joins of only directed subsets \([34, 19, 10, 17, 14]\).)

Figure 3.3 on page 4 illustrates meet-continuous (also called semi-continuous) mutual endofunctions \( F \) and \( G \), which preserve only the glbs in \( \mathbb{O} \) and \( \mathbb{P} \) whenever they exist. Dually, join-continuous mutual endofunctions preserve only the lubs in \( \mathbb{O} \) and \( \mathbb{P} \) whenever they exist. As such, mutual endofunctions are (fully) continuous if and only if they are meet-continuous and join-continuous.

The continuity of two mutual endofunctions has a number of useful implications, some of which we use to prove that continuity guarantees the existence of least and greatest simultaneous fixed points.

- First, if \( F \) and \( G \) are continuous mutual endofunctions then \( F \) and \( G \) are also monotonic mutual endofunctions (Lemma 1 in Appendix A).

  - Note that if \( F \) and \( G \) are monotonic but not continuous then only we can assert that the points \( F(\land M) \) and \( G(\cap N) \) are lower bounds of the sets \( F(M) \) and \( G(N) \), but not that they are necessarily the greatest lower bounds of these sets, and we can also assert that the elements \( F(\lor M) \) and \( G(\cup N) \) are upper bounds of the two sets, respectively, but not that they are necessarily the least upper bounds of these sets. More precisely, if \( F \) and \( G \) are monotonic, but not necessarily continuous, then we only have

  \[
  F(\land M) \subseteq \land F(M) \quad \text{and} \quad G(\cap N) \leq \land G(N),
  \]
  \[
  \lor F(M) \subseteq F(\lor M) \quad \text{and} \quad \lor G(N) \leq G(\cup N).
  \]

  Compared to monotonicity, as such, the continuity of \( F \) and \( G \) can be seen as requiring or asserting the equality of these points whenever they exist, intuitively thereby “allowing no elements in between”. Continuity, thus, is said to allow functions \( F \) and \( G \) that “have no jumps” or “have no surprises”. (See [8] for intuitions on mutual continuity.)

- Second, if \( F \) and \( G \) are continuous mutual endofunctions, then the compositions \( G \circ F \) and \( F \circ G \) are (standard) continuous endofunctions, and they are monotonic endofunctions too (Lemma 2 in Appendix A).

\(^6\) In category theory jargon, if \( F \) and \( G \) are continuous then \( F \) and \( G \) are said to commute with the meet/glb \((\land/\cap)\) and join/lub \((\lor/\cup)\) operations.
Further, if \( \mathcal{O} \) and \( \mathcal{P} \) are complete lattices, then the composition functions \( G \circ F : \mathcal{O} \to \mathcal{O} \) and \( F \circ G : \mathcal{P} \to \mathcal{P} \) (like any standard monotonic endofunctions over complete lattices) have standard pre-fixed points and standard post-fixed points in \( \mathcal{O} \) and \( \mathcal{P} \) respectively. Even further, the components \( O \in \mathcal{O} \) and \( P \in \mathcal{P} \) of simultaneous fixed points of \( F \) and \( G \) are always, each component individually, among the standard fixed points of the compositions \( G \circ F \) and \( F \circ G \) (Lemma 3 in Appendix A). The converse, however, does not necessarily hold.

- Third, if \( F \) and \( G \) are continuous mutual endofunctions then \( F \) and \( G \) (and their compositions) map complete lattices to complete lattices (note that, by definition, the empty set is not a complete lattice, and that a singleton poset is a trivial complete lattice). In other words, if subsets \( M \subseteq \mathcal{O} \) and \( N \subseteq \mathcal{P} \) happen to be complete sublattices of \( \mathcal{O} \) and \( \mathcal{P} \) then \( F(M) \) and \( G(N) \) are either empty or are complete sublattices of \( \mathcal{P} \) and \( \mathcal{O} \) respectively (Lemma 4 in Appendix A).

- Fourth and finally, if \( F \) and \( G \) are continuous mutual endofunctions, then for all \( O \in \mathcal{O} \) and \( P \in \mathcal{P} \) the posets \( \text{PreFP}_{F,G}(O) \) and \( \text{PreFP}_{G,F}(P) \) are complete lattices (Lemma 5 in Appendix A). Continuity also guarantees the existence of least simultaneous pre-fixed points (Lemma 6 in Appendix A). We use continuity to prove that the least simultaneous pre-fixed points are also the least simultaneous fixed points (The Simultaneous Fixed Points Theorem in Appendix A).

Having made a digression to discuss the continuity of mutual endofunctions, and some of its implications, we now resume our formulation of mutual (co)induction.

Now, if \( \leq \) is a complete lattice over \( \mathcal{O} \) and \( \subseteq \) is a complete lattice over \( \mathcal{P} \) (i.e., if \( \leq \) is an ordering relation where meets \( \land \) and joins \( \lor \) of all subsets of \( \mathcal{O} \) are guaranteed to exist in \( \mathcal{O} \), and similarly for \( \subseteq, \cap, \) and \( \cup \) in \( \mathcal{P} \)) and if \( F \) and \( G \) are continuous mutual endofunctions over \( \mathcal{O} \) and \( \mathcal{P} \) then we have the following:

- \( F \) and \( G \) are called simultaneous generating functions (or simultaneous generators or mutual generators or reciprocal generators),
- the least simultaneous pre-fixed points of \( F \) and \( G \), called \( \mu_F \) and \( \mu_G \), exist in \( \mathcal{O} \) and \( \mathcal{P} \),
- together, the points \( \mu_F \) and \( \mu_G \) are also the least simultaneous fixed points of \( F \) and \( G \) (as we prove in The Simultaneous Fixed Points Theorem),
- the greatest simultaneous post-fixed points of \( F \) and \( G \), called \( \nu_F \) and \( \nu_G \), exist in \( \mathcal{O} \) and \( \mathcal{P} \), and
- together, the points \( \nu_F \) and \( \nu_G \) are also the greatest simultaneous fixed points of \( F \) and \( G \) (as we prove in The Simultaneous Fixed Points Theorem).

Further, given that \( \mu_F \) and \( \mu_G \) are the least simultaneous pre-fixed points of \( F \) and \( G \) and \( \nu_F \) and \( \nu_G \) are the greatest simultaneous post-fixed points of \( F \) and \( G \), for any element \( O \in \mathcal{O} \) and \( P \in \mathcal{P} \) we have:

- (mutual induction) if \( F(O) \subseteq P \) and \( G(P) \leq O \), then \( \mu_F \leq O \) and \( \mu_G \subseteq P \),

which, in words, means that if \( O \) and \( P \) are simultaneous pre-fixed/inductive/large points of \( F \) and \( G \), then points \( \mu_F \) and \( \mu_G \) are less than or equal to \( O \) and \( P \) (i.e., \( \mu_F \) and \( \mu_G \) are the smallest simultaneously-large points of \( F \) and \( G \)), and,

\[ \text{Note that singleton posets are trivial complete lattices. As such, this condition is already satisfied in an encoding of standard (co)induction using mutual (co)induction. That is because in such an encoding a given pre-fixed point is an element that is paired with precisely one element, namely itself, to form a pair of simultaneous pre-fixed points. Thus, in mutual (co)induction encodings of standard (co)induction, the sets \( \text{PreFP}_{F,G}(O) \), \( \text{PreFP}_{G,F}(P) \), \( \text{PostFP}_{F,G}(O) \), and \( \text{PostFP}_{G,F}(P) \) are always either \( \phi \) or singleton posets for all \( O \in \mathcal{O} \) and all \( P \in \mathcal{P} \).} \]
• (mutual coinduction) if \( P \sqsubseteq F(O) \) and \( O \leq G(P) \), then \( O \leq \nu_F \) and \( P \sqsubseteq \nu_G \),

which, in words, means that if \( O \) and \( P \) are simultaneous post-fixed/coinductive/small points of \( F \) and \( G \), then points \( O \) and \( P \) are less than or equal to points \( \nu_F \) and \( \nu_G \) (i.e., \( \nu_F \) and \( \nu_G \) are the largest simultaneously-small points of \( F \) and \( G \)).

4 Related Work

The work closest to the one we present here seems to be that of Paulson, presented e.g. in [28], to support the development of the Isabelle proof assistant [25]. We already mentioned in §1 the influence of Paulson’s work on motivating our definition of mutual coinduction (we discuss this motivation, and others, in more detail in [9]). Due to Paulson’s interest in making some form of coinduction available in systems such as Isabelle [29], Paulson was interested only in the set-theoretic definition of standard (co)induction and mutual (co)induction [26] (the set-theoretic definition, according to [29], “was definitely the easiest to develop, especially during the 1990s, when no general mechanisation of lattice theory was even available”).

More technically, it seems Paulson was interested in requiring generators to be monotonic (as opposed to requiring their continuity, which is sometimes viewed as an undesirably strong assumption [32, p.72]). As such, Paulson used monotonic generators over the powerset of a disjoint sum domain so as to define (or, rather, encode) mutual set-theoretic (co)induction using standard set-theoretic (co)induction. Additionally, in [26, p.33] Paulson stated that the standard Fixed Point Theorem has been proven in Isabelle ‘only for a simple powerset lattice,’ which made Paulson limit his interests to such “simple powerset lattices,” even when the theorem applies to any complete lattice [15, 11], not just to a particular instance (i.e., not only to powerset lattices). As such, in summary, it seems to us Paulson was not interested in considering the continuity of generators in his work for considerations related to his interests in automated theorem proving.

While not particularly aimed at semantics, Paulson’s work on mutual (co)induction, indirectly, provided semantics for mutual (co)inductive datatypes in functional programming languages (e.g. ML) where mutual datatype constructors were modeled by mutual generators. Functional programming languages, however, are typically structurally-typed and structurally-subtyped. Given that we have interest, rather, in providing semantics for datatypes in mainstream object-oriented programming languages (such as Java, C#, Scala, and Kotlin), which are typically nominally-typed and nominally-subtyped programming languages, our interest is more in the order-theoretic formulation of mutual (co)induction.

Given the difference between our work and Paulson’s work regarding how mutual (co)induction is technically formulated, motivated by the different goals behind the two formulations, we anticipate that results in both works (particularly regarding the definition of simultaneous fixed points) are not in one-to-one correspondence with each other. We, however, keep a more detailed comparison of the formulation of mutual coinduction we present here to the formulation of Paulson—which may reveal more similarities and resolve some of the differences between the two formalizations—for future work.

Another work that is related to ours is the work presented in [12]. Since the work in [12] builds on that of Paulson, and has similar aims in supporting the development of Isabelle, the work in [12] adopts the same specific set-theoretic view of mutual (co)induction as that of Paulson.

5 Conclusion and Future Work

Standard induction (which includes the standard notions of mathematical induction and structural induction) is well-known, and it is relatively easy reason about. Standard coinduction is also

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8 Via noting an order-isomorphism between \( \wp \left( \sum_i D_i \right) \) (the powerset of a disjoint sum) and \( \prod_i \wp(D_i) \) (the product of powersets). In our opinion the use of the disjoint sum in Paulson’s definition of mutual (co)induction, while technically clever, is unnatural and unintuitive (as demonstrated, e.g., in the examples of [26, §4.5]).

9 See [7] and also [6, 5, 9] for a discussion of why order-theory and category-theory seem to be more suited than set-theory for modeling nominally-typed and nominally-subtyped programming languages.
known, but it is a bit shrouded in mystery and unreasonability\textsuperscript{10}. Mutual induction is also known, if somewhat lesser than standard induction. Mutual induction is a bit harder to reason about than standard induction however. Mutual coinduction—our main interest in this paper—is, however, almost unknown, and has (so far) been perceived as being both mysterious and hard to reason about. We hope that this paper, via presenting the definition of mutual coinduction as a simple generalization of the order-theoretic definition of standard coinduction, has put mutual coinduction into more familiar light, and that, by presenting a proof of a related proof principle, it has also made mutual coinduction simpler to reason about.

While the continuity condition on generators in our formulation of mutual (co)induction is sufficient for proving the existence of least and greatest simultaneous fixed points in complete lattices (while monotonicity is insufficient), yet it is not clear to us whether (full) continuity is necessary for such a proof. It may be useful to consider, in some future work, the possibility of relaxing the continuity condition, while still guaranteeing the existence of simultaneous fixed points. It may be useful to also consider the effect of having other more liberal continuity conditions (such as Scott-continuity) on the existence of simultaneous fixed points, or to study simultaneous prefix-fixed points and simultaneous post-fixed points that are not necessarily fixed points, as is done, for example, in the study of algebras and coalgebras in category theory.

As another possible future work that can build on the definition of mutual coinduction we present here, it may be useful to consider defining \textit{infinite mutual coinduction}, which, as we perceive it, generalizes mutual coinduction to involve an infinite (countable, or even uncountable!) number of orderings and generators. As of the time of this writing, we are not aware of immediate applications of infinite mutual coinduction. Given the mystery surrounding both coinduction and some particular areas of science, though, we conjecture that infinite mutual coinduction (if it is indeed reasonably definable) may have applications in areas of science such as quantum physics, \textit{e.g.}, by it offering mathematical models of quantum phenomena such as superposition, entanglement, and/or interference\textsuperscript{11}. Inline with this conjecture, we also intuit and conjecture that infinite mutual coinduction may have an impact on quantum computing, including reasoning about quantum programs and quantum software.

References

\begin{itemize}
\item {\textsuperscript{10}}Likely due to its strong connections with negation \cite{22, 6}.
\item {\textsuperscript{11}}For example, Penrose calls in \cite{30} for some new kind of mathematics for having an accurate understanding of our universe. Perhaps infinite mutual coinduction is a piece of such “new math.”
\end{itemize}
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[26] Lawrence C. Paulson. Set theory for verification: II induction and recursion. *Journal of Automated Reasoning*, 15(2):167–215, 1995. 8
A Lemmas, Theorems and Proofs

In this appendix we present proofs for the lemmas and theorems of §3.

A.1 Supporting Lemmas

Lemma 1 (Continuous Functions are Monotonic). If \( F : \mathbb{O} \to \mathbb{P} \) and \( G : \mathbb{P} \to \mathbb{O} \) are continuous mutual endofunctions over posets \((\mathbb{O}, \leq)\) and \((\mathbb{P}, \sqsubseteq)\) then \( F \) and \( G \) are also monotonic, i.e., for all \( O_1, O_2 \in \mathbb{O} \)

\[
O_1 \leq O_2 \implies F(O_1) \sqsubseteq F(O_2)
\]

and for all \( P_1, P_2 \in \mathbb{P} \)

\[
P_1 \sqsubseteq P_2 \implies G(P_1) \leq G(P_2).
\]

Proof. Let \( O_1, O_2 \in \mathbb{O} \) such that \( O_1 \leq O_2 \) (i.e., in Definition 1, take \( M = \{O_1, O_2\} \) where \( O_1 \leq O_2 \)). From the definition of \( \sqcap \) (as a greatest lower bound), we particularly have

\[
\bigsqcup F(\{O_1, O_2\}) = \bigsqcup \{F(O_1) \cdot F(O_2)\} \sqsubseteq F(O_2).
\]

By the continuity of \( F \), we also have \( \bigsqcup F(\{O_1, O_2\}) = F(\bigwedge \{O_1, O_2\}) \). Given that \( O_1 \leq O_2 \), we have \( \bigwedge \{O_1, O_2\} = O_1 \wedge O_2 = O_1 \) and thus, further, we have \( F(\bigwedge \{O_1, O_2\}) = F(O_1) \). As such, we have \( \bigsqcup F(\{O_1, O_2\}) = F(O_1) \). Substituting for \( \bigsqcup F(\{O_1, O_2\}) \) (the l.h.s.) in (A.1), we get \( F(O_1) \sqsubseteq F(O_2) \), as required.

Similarly for \( G \).

Lemma 2 (Composition Preserves Monotonicity and Continuity). If \( F : \mathbb{O} \to \mathbb{P} \) and \( G : \mathbb{P} \to \mathbb{O} \) are monotonic (continuous) mutual endofunctions over posets \( \mathbb{O} \) and \( \mathbb{P} \), then the compositions \( G \circ F : \mathbb{O} \to \mathbb{O} \) and \( F \circ G : \mathbb{P} \to \mathbb{P} \) are monotonic (continuous) endofunctions over \( \mathbb{O} \) and \( \mathbb{P} \) respectively.
Proof. By substitution, since \( F(O_1) \in \mathcal{P} \) and \( F(O_2) \in \mathcal{P} \) for all \( O_1, O_2 \in \emptyset \), then, by the monotonicity of \( F \) then that of \( G \), we have

\[
O_1 \leq O_2 \implies F(O_1) \sqsubseteq F(O_2) \implies G(F(O_1)) \leq G(F(O_2)),
\]

i.e., that \( G \circ F \) is monotonic.

Similarly, for all \( P_1, P_2 \in \mathcal{P} \), the monotonicity of \( G \) then of \( F \) implies

\[
P_1 \sqsubseteq P_2 \implies G(P_1) \leq G(P_2) \implies F(G(P_1)) \sqsubseteq F(G(P_2)),
\]

i.e., that \( F \circ G \) is monotonic.

Using a very similar but much more tedious argument we can prove that composition preserves continuity too.

**Lemma 3 (Components of Simultaneous Fixed Points are Standard Fixed Points of Compositions).** If \( F : \emptyset \to \mathcal{P} \) and \( G : \mathcal{P} \to \emptyset \) are monotonic mutual endofunctions over posets \( \emptyset \) and \( \mathcal{P} \), then the components of their simultaneous pre-/post-/fixed points are standard pre-/post-/fixed points of the compositions \( G \circ F : \emptyset \to \emptyset \) and \( F \circ G : \mathcal{P} \to \mathcal{P} \).

**Proof.** For each pair of points \( O \in \emptyset \) and \( P \in \mathcal{P} \) of simultaneous pre-fixed points of \( F \) and \( G \), by the definition of simultaneous pre-fixed points we have both inequalities

\[
F(O) \sqsubseteq P \quad \text{and} \quad G(P) \leq O.
\]

Applying \( G \) to both sides of the first inequality, the monotonicity of \( G \) implies that \( G(F(O)) \leq G(P) \).\(^{12}\) Combining this with the second inequality via the common expression \( G(P) \), we have

\[
G(F(O)) \leq G(P) \leq O.
\]

Then, by the transitivity of \( \leq \),

\[
G(F(O)) \leq O
\]

i.e., point \( O \) is a standard pre-fixed point of the composition \( G \circ F \).

Symmetrically, by applying \( F \) to both sides of the second inequality, the monotonicity of \( F \) implies that

\[
F(G(P)) \sqsubseteq F(O) \sqsubseteq P.
\]

and thus, by the transitivity of \( \sqsubseteq \), point \( P \) is a standard pre-fixed point of the composition \( F \circ G \).

(See Figure A.1 for illustration.)

A dual argument implies that components of simultaneous post-fixed points of \( F \) and \( G \) are standard post-fixed points of \( G \circ F \) and \( F \circ G \) respectively.

Combining both results implies that components of simultaneous fixed points of \( F \) and \( G \) are also standard fixed points of \( G \circ F \) and \( F \circ G \).

**Lemma 4 (Continuous Functions Preserve Complete Lattices).** If \( F : \emptyset \to \mathcal{P} \) and \( G : \mathcal{P} \to \emptyset \) are continuous mutual endofunctions over posets \( \emptyset \) and \( \mathcal{P} \), then for all subsets \( M \subseteq \emptyset \) and \( N \subseteq \mathcal{P} \), if \( M \) and \( N \) are complete sublattices of \( \emptyset \) and \( \mathcal{P} \) then the images \( F(M) \) and \( G(N) \) are also complete sublattices of \( \mathcal{P} \) and \( \emptyset \), respectively.

**Proof.** First, let’s consider the case where \( M \subseteq \emptyset \) is a complete lattice. By the definition of a complete lattice, the points \( \sqcap A \) and \( \sqcup A \) exist in \( M \) for all subsets \( A \subseteq M \). As such, the points \( F(\sqcap A) \) and \( F(\sqcup A) \) exist in \( \mathcal{P} \), and, accordingly, by the definition of \( F(M) \), are also members of \( F(M) \). By the continuity of \( F \), we also have \( F(\sqcap A) = \sqcap F(A) \) and \( F(\sqcup A) = \sqcup F(A) \). As such, for any image set \( F(A) \subseteq \mathcal{P} \) a greatest lower bound, \( \sqcap F(A) \), and a least upper bound, \( \sqcup F(A) \), exist.

Further, because of the continuity of \( F \), these two points are members of \( F(M) \). Thus, all subsets of \( M \) have images of their glbs and lubs in \( F(M) \). That by itself does not, however, prove that the set \( F(M) \) is a complete lattice, yet.

\(^{12}\)Note that continuity is not needed here.
To prove that \( F(M) \) is a complete lattice, we have to prove that the points \( \cap B \) and \( \cup B \) exist in \( F(M) \) for all sets \( B \subseteq F(M) \subseteq P \). Given that \( B \) is subset of \( F(M) \), the image of \( M \), then there exists some (one or more) set \( A \subseteq M \) such that \( F(A) = B \). Pick one such set \( A \). Then, for that particular \( A \), we have \( \cap B = \cap F(A) \) and \( \cup B = \cup F(A) \). Since we proved that for all sets \( A \subseteq M \) the points \( \cap F(A) \) and \( \cup F(A) \) are members of \( F(M) \), we conclude that \( \cap B \) and \( \cup B \) are members of \( F(M) \) for all sets \( B \subseteq F(M) \). As such, the set \( F(M) \) is a complete lattice.

Next, by a symmetric argument, if \( N \subseteq P \) is a complete lattice then, by the continuity of \( G \), the set \( G(N) \) is also a complete lattice, as required. \( \square \)

**Lemma 5 (Component Images of Simultaneous Pre-/Post-Fixed Points form Complete Lattices).**

If \( F : \emptyset \rightarrow P \) and \( G : P \rightarrow \emptyset \) are continuous mutual endofunctions over complete lattices \( \emptyset \) and \( P \), then, for all \( O \in \emptyset \) and \( P \in P \), the component image sets \( PreFP_{F,G}(O) \), \( PreFP_{G,F}(P) \), \( PostFP_{F,G}(O) \), and \( PostFP_{G,F}(P) \), as defined in §3, are either empty or are complete lattices.

**Proof.** For a point \( O \in \emptyset \) such that \( PreFP_{F,G}(O) \) is nonempty, define

\[
\prod_O = PreFP_{F,G}(O) = \{ P \in P | F(O) \subseteq P \text{ and } G(P) \leq O \}.
\]

Since \( P \) is a complete lattice, the set \( \prod_O \) (as a subset of \( P \)) has a greatest lower bound \( \cap \prod_O \) and a least upper bound \( \cup \prod_O \) that are members of \( P \). To prove that \( \prod_O \) itself is a complete lattice, first we prove that these two points (i.e., the glb and lub of \( \prod_O \)) are members of \( \prod_O \).

Let’s note that for all points \( O \) where \( \prod_O \) is non-empty we always have the point \( F(O) \) as a member of \( \prod_O \). That’s because, by the reflexivity of \( \subseteq \), we have \( F(O) \subseteq F(O) \). Further, using Lemma 3, we have \( G(F(O)) \leq O \). As such, by the definition of simultaneous pre-fixed points, the points \( O \in \emptyset \) and \( F(O) \in P \) are simultaneous pre-fixed points of \( F \) and \( G \). Hence, \( F(O) \in \prod_O \).

Given that, by the definition of \( \prod_O \), the point \( F(O) \) is less than or equal to all members of \( \prod_O \), we have

\[
\cap \prod_O = F(O),
\]

and, as such, the greatest lower bound of \( \prod_O \) is a member of \( \prod_O \), i.e., we have \( \cap \prod_O \in \prod_O \) as needed.

For \( \cup \prod_O \), we prove that it is a member of \( \prod_O \) more directly, using the definition of \( \prod_O \) and the continuity of \( G \). As for any member of \( P \), for \( \cup \prod_O \) to be a member of \( \prod_O \) we must have \( F(O) \subseteq \cup \prod_O \) and \( G(\cup \prod_O) \leq O \). The first condition is satisfied since we just proved that \( F(O) \) is exactly \( \cap \prod_O \) and, as for any set, we have \( \cap \prod_O \subseteq \cap \prod_O \) whenever such points exist, and as such we have \( F(O) \subseteq \cup \prod_O \). For the second condition (this is where continuity is needed), we have \( G(\cup \prod_O) = \vee G(\prod_O) \) by continuity. Since, by the definition of \( \prod_O \), all members of \( G(\prod_O) \) are less than or equal to \( O \), then \( O \) is an upper bound of \( G(\prod_O) \). As such, we have \( \vee G(\prod_O) \leq O \). Hence, for \( \cup \prod_O \), we have \( G(\cup \prod_O) = \vee G(\prod_O) \leq O \), as required. Hence, \( \cup \prod_O \in \prod_O \) as needed.

Since the greatest lower bound of \( \prod_O \) and the least upper bound of \( \prod_O \) are members of \( \prod_O \), so far we can assert that the set \( \prod_O = PreFP_{F,G}(O) \) is a bounded poset.

To prove that \( \prod_O \) is not only a bounded poset but, rather, that it is a complete lattice, we have to also consider proper subsets of \( \prod_O \) and \( \prod_O \). The argument for proper subsets of \( \prod_O \) is very similar to the one we just used for \( \prod_O \) (as an improper subset of itself). In particular, let \( N \subseteq \prod_O \) be some proper subset of \( \prod_O \) (i.e., is some set of points of \( P \) that, paired with \( O \), are simultaneous pre-fixed points

\[\text{Sounds like the Axiom of Choice is needed here.}\]

\[\text{This brings one of the most delicate points in proving the mutual (co)induction principles. Note that O and F(O) are not necessarily simultaneous pre-fixed points for all O \in \emptyset, but that, according to Lemma 3, points O and F(O) are simultaneous pre-fixed points only whenever there is some P \in P (possibly equal to F(O), and possibly not) such that O and P are simultaneous pre-fixed points of F and G, i.e., such that P witnesses, via G(P), that G(F(O)) \leq O. In particular, it is not necessarily true that for all O \in \emptyset we have G(F(O)) \leq O (Otherwise, all elements O \in \emptyset would have formed simultaneous pre-fixed points of F and G, simply by pairing each O with F(O). This goes counter to intuitions about mutual (co)induction, since it would eventually lead to concluding that all points of \emptyset—and similarly of P—are simultaneous fixed points of F and G!).}\]

Readers should be aware of this delicate and tricky point specific to mutual (co)induction. That’s because this point has no counterpart in standard (co)induction (or, at least, has no obvious counterpart, since in an encoding of standard (co)induction using mutual (co)induction, where say G = 1, we will have G(F(O)) = F(O) \leq O only if F(O) \leq O).
of $F$ and $G$. Again, since $P$ is a complete lattice, the elements $\cap P$ and $\cup P$ exist. We proceed to prove that these points are also members of $P$.

In particular, to be members of $P$ the two points $\cap P$ and $\cup P$ have to satisfy the membership condition of $P$, i.e., they have to form, when paired with $O$, simultaneous pre-fixed points of $F$ and $G$. Again using the continuity of $G$, we can see that this is true since, like we had for $\cap P$ and $\cup P$, we have

$F (O) \sqsubseteq \cap P$ and $G (\cap P) = \wedge G (P) \leq O$, and

$F (O) \sqsubseteq \cup P$ and $G (\cup P) = \vee G (P) \leq O$.

As such, points $\cap P$ and $\cup P$ are members of $P$. Thus, $P$ is a complete lattice.

Using a symmetric argument and the continuity of $F$, we can also prove that the set $\cap P = \text{PreFP}_{F,G} (P)$ is either empty or is a complete lattice for all $P \in P$. (Figure A.1 illustrates the sets $\cap P$ and $\cup P$.)

**Lemma 6** (Components of Pre-/Post-Fixed Points form Complete Lattices). If $F : \mathcal{O} \to P$ and $G : P \to \mathcal{O}$ are continuous mutual endofunctions over complete lattices $\mathcal{O}$ and $P$, then the sets

$$C = \{ O \in \mathcal{O} \mid \exists P \in P, F (O) \sqsubseteq P \text{ and } G (P) \leq O \}, \text{ and}$$

$$D = \{ P \in P \mid \exists O \in \mathcal{O}, F (O) \sqsubseteq P \text{ and } G (P) \leq O \}$$

of all components of simultaneous pre-fixed points are complete sublattices of $\mathcal{O}$ and $P$, respectively. Similarly for simultaneous post-fixed points.

**Proof.** First, let’s note that the definitions of $C$ and $D$ mean that $C$ is the set of all $O \in \mathcal{O}$ where there is some $P \in P$ such that the $F$-image of $O$ is less than $P$ and the $G$-image of $P$ is less than $O$ and, symmetrically, that $D$ is the set of all $P \in P$ where there is some $O \in \mathcal{O}$ such that the $F$-image of $O$ is less than $P$ and the $G$-image of $P$ is less than $O$. (As such, the variables $O$ and $P$ in Equations (A.2) and (A.3) range over the set of all simultaneous pre-fixed points of $F$ and $G$.)

Note also that $C = \bigcup_{P \in P} \text{PreFP}_{G,F} (P)$ and $D = \bigcup_{O \in \mathcal{O}} \text{PostFP}_{F,G} (O)$, but that Lemma 5 does not (by itself) imply that $C$ and $D$ are complete lattices, since the union of complete lattices is not necessarily a complete lattice.

We do proceed, though, similarly to first prove that $C$ is a meet-complete lattice. Particularly, assuming $A \subseteq C$, we prove that $\bigwedge A \in C$. Since $\mathcal{O}$ is a complete lattice, the point $\bigwedge A$ exists in $\mathcal{O}$, and

![Figure A.1: Illustrating Lemma 3 and Lemma 5.](image)
since \( A \) is a subset of \( C \) then the image set \( F(A) \) is a subset of \( D \) (by the definition of \( D \)). By continuity, we have \( F(\bigwedge A) = \bigcap F(A) \). Also, let the set \( B = \{ \exists A \in A \cdot F(A) \sqsubseteq B \land G(B) \leq A \} \) be the set of all points in \( P \) that form simultaneous pre-fixed points when paired with some point in \( A \). Given that for each point \( B \in B \) there exists a point \( A \in A \) such that \( F(A) \in F(A) \) and \( F(A) \sqsubseteq B \) (by the definition of simultaneous fixed points), then the meet of all points \( F(A) \) is less than or equal to the meet of all points \( B \), i.e., we have \( \bigcap F(A) \sqsubseteq \bigcap B \). Using a similar argument, we also have \( \bigwedge G(B) \leq \bigwedge A \). By continuity, substituting for \( \bigcap F(A) \) and \( \bigwedge G(B) \) we have

\[
F\left(\bigwedge A\right) \sqsubseteq \bigcap B \quad \text{and} \quad G\left(\bigcap B\right) \leq \bigwedge A.
\]

As such, for all \( A \subseteq C \), the point \( \bigwedge A \), when paired with the point \( \bigcap B \), forms a pair of simultaneous pre-fixed points of \( F \) and \( G \), and is thus a member of \( C \). As such, \( C \) is a meet-complete lattice. (Figure A.2 illustrates the proof for subsets of \( C \) and \( D \) that have two elements.) Dually, we can prove that set \( C \) is also a join-complete lattice (with the point \( \top_{\mathcal{O}} \), the top element of \( \mathcal{O} \), at its top). Hence, set \( C \) is a meet-complete lattice and a join-complete lattice, i.e., is a complete lattice.

Symmetrically, or by using Lemma 4, we can prove that set \( D \) also is a complete lattice (with the point \( \top_{\mathcal{P}} \), the top element of \( \mathcal{P} \), at its top). As such, the sets of all components of simultaneous pre-fixed points are complete lattices.

Dually, we can also prove that the two sets \( E \) and \( F \) of all components of simultaneous post-fixed points—i.e., duals of sets \( C \) and \( D \)—are complete lattices (with the points \( \bot_{\mathcal{O}} \) and \( \bot_{\mathcal{P}} \) at their bottom), as required.

\[ \square \]

A.2 The Simultaneous Fixed Points Theorem

The following theorem, asserting the existence of least and greatest simultaneous fixed points, is the central theorem of this paper.\(^{15}\)

**Theorem 1** (The Simultaneous Fixed Points Theorem). If \((\mathcal{O}, \leq, \land, \lor)\) and \((\mathcal{P}, \sqsubseteq, \sqcap, \sqcup)\) are two complete lattices and \( F : \mathcal{O} \rightarrow \mathcal{P} \) and \( G : \mathcal{P} \rightarrow \mathcal{O} \) are two continuous mutual endofunctions (i.e., two simultaneous generators) over \( \mathcal{O} \) and \( \mathcal{P} \) then we have the following:

- the least simultaneous pre-fixed points of \( F \) and \( G \), called \( \mu_F \) and \( \mu_G \), exist in \( \mathcal{O} \) and \( \mathcal{P} \),

---

\(^{15}\)To the best of our knowledge, neither mutual (co)induction as we define it in this paper nor a proof of the Simultaneous Fixed Points Theorem have been presented formally before.
• $\mu_F$ and $\mu_G$ are also the least simultaneous fixed points of $F$ and $G$,
• the greatest simultaneous post-fixed points of $F$ and $G$, called $\nu_F$ and $\nu_G$, exist in $\mathcal{O}$ and $\mathcal{P}$, and
• $\nu_F$ and $\nu_G$ are also the greatest simultaneous fixed points of $F$ and $G$.

Proof. Let the set
\[
\mathcal{C} = \{ O \in \mathcal{O} | \exists P \in \mathcal{P}, F(O) \sqsubseteq P \text{ and } G(P) \leq O \}
\]
and the set
\[
\mathcal{D} = \{ P \in \mathcal{P} | \exists O \in \mathcal{O}, F(O) \sqsubseteq P \text{ and } G(P) \leq O \}
\]
be the sets of all components of simultaneous pre-fixed points of $F$ and $G$, and let points $\mu_F$ and $\mu_G$ be defined as
\[
\mu_F = \bigwedge_{F(O) \sqsubseteq P \text{ and } G(P) \leq O} O = \bigwedge \mathcal{C} \tag{A.4}
\]
\[
\mu_G = \bigcap_{F(O) \sqsubseteq P \text{ and } G(P) \leq O} P = \bigcap \mathcal{D}. \tag{A.5}
\]

By Lemma 6, points $\mu_F$ and $\mu_G$ are guaranteed to exist as the least elements of $\mathcal{C}$ and $\mathcal{D}$. By the antisymmetry of $\leq$ and $\sqsubseteq$, we can conclude that
\[
F(\mu_F) = \mu_G \text{ and } G(\mu_G) = \mu_F. \tag{A.6}
\]
First, we note that $F(\mu_F) \sqsubseteq \mu_G$ because $\mu_F$ is the least element of $\mathcal{C}$ and thus, according to the definition of simultaneous pre-fixed points (and as noted in the proof of Lemma 5), its image $F(\mu_F)$ is less than any element of $\mathcal{D}$, including the point $\mu_G$, but, second, also we note that $\mu_G \subseteq F(\mu_F)$ since $\mu_G$ is the least element of $\mathcal{D}$ and thus is less than any point in $\mathcal{D}$, including the point $F(\mu_F)$. By the antisymmetry of $\sqsubseteq$, we conclude that $F(\mu_F) = \mu_G$.

Symmetrically, we also have $G(\mu_G) = \mu_F$.

As such, the points $\mu_F$ and $\mu_G$ are simultaneous fixed points of $F$ and $G$. They are also the least simultaneous pre-fixed points of $F$ and $G$ since, by Equation (A.6), less-demandingly we have
\[
F(\mu_F) \sqsubseteq \mu_G \text{ and } G(\mu_G) \leq \mu_F,
\]
meaning that points $\mu_F$ and $\mu_G$ are simultaneous pre-fixed points of $F$ and $G$, and, by the individual uniqueness and minimality of each of $\mu_F$ and $\mu_G$ (as the meets of the complete lattices $\mathcal{C}$ and $\mathcal{D}$), points $\mu_F$ and $\mu_G$ are the least such points.

Now we have established both that $\mu_F$ and $\mu_G$ form the least simultaneous pre-fixed points of $F$ and $G$, and that $\mu_F$ and $\mu_G$ are simultaneous fixed points of $F$ and $G$, so $\mu_F$ and $\mu_G$ are the least simultaneous fixed points of $F$ and $G$.

Using a dual argument, we can also prove that $\nu_F = \bigvee \mathcal{E}$ and $\nu_G = \bigcup \mathcal{F}$, where sets $\mathcal{E}$ and $\mathcal{F}$ are the duals of sets $\mathcal{C}$ and $\mathcal{D}$ (see proof of Lemma 6), are the greatest simultaneous fixed points of $F$ and $G$, as required.

B Motivations from PL Theory

The significance of mutually-recursive definitions in programming languages (PL) semantics and PL type theory is illustrated in the following examples.
B.1 Mutual Recursion at The Level of Data Values

Lawrence Paulson, in his well-known book ‘ML for the Working Programmer’ [27, p.58], made some intriguing assertions, and presented an intriguing code example. According to Paulson, “Functional programming and procedural programming are more alike than you may imagine”—a statement that some functional programmers today are either unaware of, may oppose, or may silently ignore. Paulson further states, verbatim, that “Any combination of goto and assignment statements — the worst of procedural code — can be translated to a set of mutually recursive functions.”

Then Paulson presents a simple example of imperative code. Here it is.

```plaintext
var x := 0; y := 0; z := 0;
F: x := x + 1; goto G
G: if y < z then goto F else (y := x + y; goto H)
H: if z > 0 then (z := z - x; goto F) else stop
```

To convert this imperative code into pure functional code, Paulson then suggests: “For each of the labels, F, G, and H, declare mutually recursive functions. The argument of each function is a tuple holding all of the variables.”

Here’s the result when the method is applied to the imperative code above:

```plaintext
fun F(x,y,z) = G(x+1,y,z)
and G(x,y,z) = if y<z then F(x,y,z) else H(x,x+y,z)
and H(x,y,z) = if z>0 then F(x,y,z-x) else (x,y,z);
```

We can also introduce object-oriented programming (OOP) in this discussion. In particular, a possible translation of Paulson’s imperative code to (non-imperative) OO code is as follows (in [1] we present a translation to a slightly more-succinct imperative OO code):

```plaintext
class C {
    final x, y, z: int

    // constructor
    constructor C(xx,yy,zz: int) { x = xx; y = yy; z = zz }

    C F() { new C(x+1,y,z).G() }

    C G() {
        if y < z then this.F()
        else new C(x,x+y,z).H()
    }

    C H() {
        if z > 0 then new C(x, y, z-x).F()
        else this
    }
}
```
Now, similar to Paulson’s functional program, calling \texttt{new C(0,0,0).F()} gives the fields x, y, and z their initial values for execution, and returns an object equivalent to \texttt{new C(1,1,0)}—i.e., equivalent to the result of the imperative code.

Pondering a little over some of the “worst” imperative code and over its translations to mutually-recursive functional and object-oriented code suggests a strong similarity—if not equivalence—between OOP, mutually-recursive FP, and procedural/imperative programming. Further, this discussion implies that the mathematical-reasoning benefits of functional programming—particularly the relative simplicity of such reasoning—seem to crucially depend on not heavily using mutually-recursive function definitions (Paulson’s concluding warning can be read as an explicit warning against writing heavily mutually-recursive functional code). As the imperative code and the OOP translation illustrate, and as is commonly known among mainstream and industrial-strength software developers, however, heavily mutually recursive definitions seem to be an essential and natural feature of real-world/industrial-strength programming.

More significantly, the above translation between imperative, functional and object-oriented code seems to also tell us that:

\[
\text{if mutually-recursive functional programs can be reasoned about mathematically,} \\
\text{then also imperative and object-oriented programs (even the worst such programs)} \\
\text{can be reasoned about mathematically.}
\]

A main objective behind the formal definition of mutual (co)induction in this paper is to help in reasoning about mutually-recursive functional programs mathematically (possibly making it even as simple as reasoning about standard recursive functional programs is, based on using the standard (co)induction principles), and, ultimately, to thereby possibly help in reasoning about (even the worst?) imperative and object-oriented programs mathematically too.

It should be noted that sometimes it is possible to reexpress mutual recursion (also called indirect recursion) or convert it into standard recursion (also called direct recursion), e.g., using the inlining conversion method presented in [21, 36] or ‘with the help of an additional argument’ as suggested by Paulson also in [27, p.58]. From the results in [21, 36], however, not all mutual recursion can be converted to standard recursion. Extending from these results, while it is possible that some mutual (co)inductive definitions can be converted to or encoded using standard (co)inductive definitions, we conjecture that not all mutual (co)induction can be translated into standard (co)induction. Hence, the need arises for a genuine formal definition of mutual (co)induction that does not involve an encoding or translation of it into terms of standard (co)induction.

### B.2 Mutual Recursion at The Level of Data Types

The previous section discussed mutual recursion at the level of values (i.e., mutually-recursive data values, functions, or methods). As is well-known in PL type theory, mutually-recursive types are essential for typing mutually-recursive data values [24, 31]. Given the ubiquity of mutually-recursive data values in OOP (via the special variable \texttt{this/self}), mutually-recursive data \textit{types} and mutually-recursive data \textit{type constructors} (e.g., \texttt{classes}, \texttt{interfaces}, \texttt{traits}, ... etc.) are ubiquitous in industrial-strength statically-typed OOP as well (e.g., in Java, C\#, C++, Kotlin, and Scala).

Further, in generic OOP languages such as Java, variance annotations (such as wildcard types in Java) can be modeled by and generalized to interval types [4]. As presented in [4], the definition of ground types in Java depends on the definition of interval types, whose definition, circularly (i.e., mutually-recursively), depends on the definition of ground types. Further, the subtyping relation between ground types depends on the containment (also called \textit{subintervaling} or \textit{interval inclusion}) relation between interval types, and vice versa. As such, the set of ground types and the set of interval types in Java are examples of mutual recursive sets, and the subtyping and the containment relations over these sets, respectively, are mutually recursive relations too. (See [3, 4] for how, under an inductive interpretation, both sets—types and interval types—and both relations—subtyping and containment—can be iteratively constructed from the given set of classes of a Java program together with the subclassing relation between these classes).
To illustrate all aspects of mutual recursion in OOP (i.e., at the level of type, at the level of values, and in defining types/subtyping and interval types/containment), the following OO code, written in an imaginary Java-like language, presents a simple set of mutually recursive classes to model the mutual recursion between in the definition of types and interval types.

```java
class Class {
    String name; // holds the name of the class
}

class Type {
    Class c;
    Interval i; // the type arg of c. null if c is not generic
}

class Interval {
    Type UBnd; // the upper bound of the interval type
    Type LBnd; // its lower bound
}

// Note the mutual recursion (at the level of types) between
// the definitions of classes Type and Interval

// And the following code adds to the code above a simple set of mutually recursive methods to model
// the mutual recursion in the definitions of the subtyping and containment relations.

class Class {
    String name;

    bool isSubclassOf(Class c) {
        // Handle special classes Null and Object
        if(this == NullCls || c == ObjCls) return true;
        // Else check if this class inherits from class c
        return inher_table.lookup(this, c);
    }
}

class Type {
    Class c;
    Interval i;

    bool isSubtypeOf(Type t) {
        // assuming that i and t.i are not null,
        // i.e., that c and t.c are generic
        return c.isSubclassOf(t.c) && i.isSubintOf(t.i);
    }
}

class Interval {
    Type UBnd;
    Type LBnd;

    bool isSubint(Interval i) {
        // covar in upperbound, contravar in lower bound
        return UBnd.isSubtypeOf(i.UBnd) && i.LBnd.isSubtypeOf(LBnd)
    }
}
```
B.3 Mutual Coinduction in OOP

As discussed in detail in §B.2, mutually recursively definitions exist in OOP at two levels: at the level of values (via `this`) and at the level of types (i.e., between classes, and in the definition of the subtyping and containment relations). It should be noted that the subtyping relation is covariant/monotonic w.r.t. containment, and that containment is covariant w.r.t. subtyping in the first argument (i.e., w.r.t. the upper bound of an interval type) and is contravariant w.r.t. the second argument (i.e., w.r.t. the lower bound of an interval type). As such, in Java with interval types, subintervals generates subtypes, and subtypes in the upper bounds of interval types generates subintervals, while subtypes in the lower bounds of interval types generates superintervals.

Just as for standard (co)induction, where the notions of (least) pre-fixed points and (greatest) post-fixed points have relevance and practical value even when such points do not correspond to fixed points, e.g., when the underlying posets are not complete lattices or when the generators are not monotonic (see, for example, [5, 2]), we expect that the notions of (least) simultaneous pre-fixed points and (greatest) post-fixed points to have relevance and practical value even when these points do not correspond to simultaneous fixed points, e.g., when the underlying posets are not complete lattices or when the generators are not continuous (and may not be even monotonic).