Novel quantum phases of two-component bosons with pair hopping in synthetic dimension

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We study two-component (or pseudospin-1/2) bosons with pair hopping interactions in synthetic dimension, for which a feasible experimental scheme on a square optical lattice is also presented. Previous studies have shown that two-component bosons with on-site interspecies interaction can only generate nontrivial interspecies paired superfluid (super-counter-fluidity or pair-superfluid) state. In contrast, apart from interspecies paired superfluid, we reveal two new phases by considering this additional pair hopping interaction. These novel phases are intraspecies paired superfluid (molecular superfluid) and an exotic non-integer Mott insulator which shows a non-integer atom number at each site for each species, but an integer for total atom number.

Ultracold quantum gases are highly controllable systems, in which various novel interaction and detection techniques can be realized, and the extreme physical parameter regimes can be reached [1–6]. Thus, ultracold quantum gas systems have been used to simulate quantum many-body systems and provide an ideal platform to discover novel quantum states. In bosonic systems, there are two kinds of boson pair condensation states with either the intraspecies pairing [7] or the interspecies pairing [8]. The interspecies paired superfluid state has been proposed in a two-component Bose-Hubbard model with on-site interspecies interaction [8–17]. Moreover, the intraspecies paired superfluid or molecular superfluid (MSF) has also been predicted in three different single-component bosonic systems, i.e., an atomic Bose gas with a Feshbach Resonance [18–20], attractive Bose-Hubbard model with three-body on-site constraint [21, 22] and extended Bose-Hubbard model (EBHM) with pair hopping [23–26].

Unfortunately, the MSF in single-component bosonic systems has not been observed experimentally. One reason is the short lifetime of molecular condensates by using the Feshbach resonance technique [20]. Besides, it is quite difficult to realize the attractive Bose-Hubbard model with a three-body constraint. Moreover, MSF is predicted in EBHM (when \( V \neq 0 \)) under large value of pair hopping \( P \) and nearest-neighbor interaction \( V \) [23–26], but it is hard to reach this parameter region in experiment. In a real experimental system, \( P \) and \( V \) are much smaller than normal hopping and on-site interaction by 3–4 orders of magnitude [27]. Indeed, the calculation in EBHM ignores the effect of an important term, i.e., density-induced tunneling \( T \), which could be much larger than \( V \) and \( P \). Thus, alternative feasible experimental schemes such as implementing a feasible scheme in the interacting two-component bosonic systems, are imperiously needed to observe this fascinating MSF state. Meanwhile, there is still a lack of a study on the exotic Mott insulator (MI) phase in the interacting two-component bosons. On the whole, two-component bosons with novel interaction may provide an opportunity for discovering the novel phases.

On the other hand, by periodically shaking optical lattice [6, 28–31] or modulating interaction strength [32, 33], Floquet technique has shown its ability to engineer the form and intensity of interactions in various experiments. So far, Floquet engineering is mainly focused on manipulating the ‘single-particle hopping’ processes [34–46], where the hopping amplitude or hopping phase (Peierls phase) depends on the occupation numbers of the sites relevant to hopping processes. The internal atomic degrees of freedom, e.g., pseudospin, can be considered as the synthetic “dimensions” [47]. By coupling to a periodically modulating radio-frequency field, a new type of two-particle hopping process with pair hopping interaction along a synthetic dimension or synthetic pair hopping (SPH) interaction (see Fig. 2) can be realized in a two-component boson system.

In this letter, we propose a Floquet engineering scheme in two-component boson system to generate such a new two-particle hopping process with SPH interaction. Two novel quantum states of matter may emerge, including the molecular superfluid (MSF) state and the non-integer Mott insulator (NMI) state. The NMI state displays that the number of the total atoms of two-component at each site is an integer, but each-component is non-integer. This NMI phase may provide a possible platform to discover the exotic magnetic phase. Furthermore, the detection of these two novel states has been addressed. The realization of our scheme provides a basis for further exploration of the exciting many-body phases in synthetic dimensions.

The effective Hamiltonian.— We now turn to the realization of SPH interaction for two-component bosons on square optical lattice, by using periodic modulating radio-frequency field. We firstly introduce the time-dependent Hamiltonian which is used to describe the physics of this periodic modulated two-component boson
system. In order to illustrated conveniently and vividly, the relevant physical processes of this time-dependent systems have shown in one-dimensional (1D) systems (see Fig. 1). Then the corresponding time-dependent Hamiltonian reads $\hat{H}(t) = \hat{H}_{\text{Kin}} + \hat{H}_{\text{rf}}(t) + \hat{H}_U$, where the on-site interaction contains three terms $\hat{H}_U = \hat{H}_{aa}^U + \hat{H}_{bb}^U + \hat{H}_{ab}^U$. Here $\hat{H}_{\text{Kin}}$ describes normal hopping terms between nearest neighbour site for each spin and chemical potential, which have the form $\hat{H}_{\text{Kin}} = -J \sum_s (\hat{A}_s^\dagger \hat{A}_{s+1} + H.C.) - \mu \sum_s \hat{A}_s^\dagger \hat{A}_s$, where $J$ is the spin-independent hopping amplitude, $\mu$ is spin-independent chemical potential and $\hat{A}_s = (\hat{a}_s, \hat{b}_s)^T$ are vector field with annihilation operators $\hat{a}_s$ ($\hat{b}_s$) on lattice site $s$ for spin-down (spin-up) component. Two spin states coupled by periodic radio-frequency field and the corresponding hamiltonian reads $\hat{H}_{\text{rf}}(t) = (\hbar \Delta/2) \sum_s \hat{A}_s^\dagger \hat{A}_s - \hbar \Omega(t)/2 \sum_s \hat{A}_s^\dagger \hat{A}_s$, where $\Delta = \omega_{\text{res}} - \omega_{\text{rf}}$ is the detuning of the radio wave ($\omega_{\text{res}}$) from the atomic resonance ($\omega_{\text{rf}}$), $\Omega(t) = \Omega \sin(\omega t)$ is Rabi frequency, and $\hat{A}_s$ and $\hat{A}_s^\dagger$ are pauli matrices. Intraspecies and interspecies on-site interactions are denoted by $\hat{H}_{aa}^U = (U_{aa}/2) \sum_s n_{aa}(n_{aa} - 1)$, $\hat{H}_{bb}^U = (U_{bb}/2) \sum_s n_{bb}(n_{bb} - 1)$, $\hat{H}_{ab}^U = U_{ab} \sum_s n_{aa}n_{bb}$, where $U_{aa}$, $U_{bb}$, and $U_{ab}$ labels the strength of the on-site repulsive interactions.

Then we obtain the effective Hamiltonian (see a derivation in the Supplemental Material (SM) [50].)

$$\hat{H}_{\text{eff}} = -J \sum_s (\hat{a}_s^\dagger \hat{a}_{s+1} + \hat{b}_s^\dagger \hat{b}_{s+1} + H.C.) - \mu \sum_s (\hat{n}_{aa} + \hat{n}_{bb})$$

$$+ U_{aa}^\text{eff} \sum_s \hat{n}_{aa}^s(n_{aa} - 1) + U_{bb}^\text{eff} \sum_s \hat{n}_{bb}^s(n_{bb} - 1)$$

$$+ U_{ab}^\text{eff} \sum_s \hat{n}_{aa}^s \hat{n}_{bb}^s + W \sum_s \left( \hat{a}_s^\dagger \hat{b}_s^\dagger \hat{a}_s \hat{b}_s + \hat{b}_s^\dagger \hat{a}_s^\dagger \hat{b}_s \hat{a}_s \right),$$

where the preceding five terms describe two-component Bose-Hubbard model [8–10] and the $W$ term represents the processes of SPH along a synthetic dimension. Here, the effective on-site interaction strength and SPH interaction in Eq. (1) are given by $U_{aa}^\text{eff} = U_{aa} - [\Omega/(2\omega)]^2(U_{aa} - U_{ab})$, $U_{bb}^\text{eff} = U_{bb} + 2\Delta U[\Omega/(2\omega)]^2$, $U_{ab}^\text{eff} = U_{ab} - [\Omega/(2\omega)]^2(U_{bb} - U_{ab})$, $W = -(\Delta U/2) [\Omega/(2\omega)]^2$, $\Delta U = (U_{aa} + U_{bb})/2 - U_{ab}$.

To reveal the relevant physical processes of this effective Hamiltonian more clearly, we choose a 1D system as an example, where it can be mapped to a coupled two-spin chain (synthetic chain) systems, and every single chain represents one specie of boson. The relevant processes are shown in Fig. 2. Although this interesting Hamiltonian in Eq. (1) are obtained with detuning $\Delta = 0$, we can also obtain it with an effective detuning $h\Delta_{\text{eff}} = h\Delta - (\mu_a - \mu_b) = 0$ even if detuning $\Delta \neq 0$. This condition can be satisfied by tuning $\mu_a$ and $\mu_b$ via changing fillings $n_a$ and $n_b$.

The phase diagrams.— At below, the phase diagrams will be numerically studied by the Gutzwiller method that has been successfully used to study various phenomena such as stationary states [51–53], time evolution [54–56] and excitation dynamics [57]. We will use the cluster Gutzwiller method [58], which can well capture the quantum fluctuations for a larger cluster to obtain the phase diagrams of the two-component boson gases with SPH interaction on a square optical lattice. We can naively assume that there exist the nontrivial molecule superfluid state ($\langle \hat{a}_i \hat{a}_j \rangle = 0$ but $\phi_{\text{Da}} = (\langle \hat{a}_i \hat{a}_i \rangle \neq 0$) apart from the phases which has been found in the two-component Bose systems with $W = 0$. The previous research on the two-component boson system with zero SPH interaction reveals that the asymmetric case ($U_{aa} \neq U_{bb}$) shows rich phases than the symmetric one ($U_{aa} = U_{bb}$) [10]. Thus, we study the phase diagrams for the asymmetric case of two-component boson system with finite SPH interaction. We have chosen a typical asymmetric case.
Fig. 3, where we choose the cluster as $1 \times 2$.

There are five phases, i.e., 2MI, SCF, 2SF NMI, and $SF_b+MSF_a$ ($\psi_a=0, \psi_b \neq 0, \phi_Da \neq 0$, and $\phi_Db \neq 0$). The 2SF, 2MI, and SCF phases have been discussed [8, 10], but NMI phase and $SF_b+MSF_a$ are nontrivial phases which have rarely been predicted in two-component boson systems. Surprisingly, there is no $SF_b+MI_a$ phase which usually exists in two-component Bose-Hubbard models for the asymmetric case [10]. Transiting from the NMI phase by increasing the value of tunneling amplitude $J$, systems go into an intriguing $SF_b+MSF_a$ phase which can exist in the larger parameter regions of phase diagrams (see Fig. 3). In this parameter region, if we switch off the SPH interaction ($W = 0$), the $SF_b+MSF_a$ phase will become $SF_b+MI_a$ phase. In this sense, the $SF_b+MI_a$ phase can be considered as the matrix phase of $SF_b+MSF_a$ phase. In brief, NMI and $MSF_a$ phases are induced by the intriguing SPH interaction $W$. At below, we will analyse the property of NMI and $MSF_a$ phases, respectively.

This nontrivial NMI phase is incompressible, and has nontrivial density distribution feature which shows an integer total atom number at each site while a non-integer atom number for each species. This distribution feature of NMI phase is significantly different from the atom distribution of 2MI phase, and the atom distribution of each site as a function of variation $\mu$ with hopping amplitude $J$ fixed is presented in Fig. 4(a). The reason why there exist such intriguing NMI phase is that in the limit $J=0$ (NMI phase), the total number $\hat{n}_a$ is a good quantum number but $\hat{n}_a$ and $\hat{n}_b$ are not, since the Hamiltonian $H_{J=0}$ commutes with $\hat{n}_a$ but does not commute with $\hat{n}_b$ or $\hat{n}_a$. For the $J<J_{\text{critical}}$ case, the property of ground state is unchanged, but parameter region is shrunk, thus the ground state is also the NMI phase. Furthermore, the intriguing non-integer feature of NMI phase provides a possible platform to discover a variety of the interesting magnetic phases.

![Phase Diagram](image)

**FIG. 3:** The phase diagram of two species Bose gases with SPH interaction $W$ in square optical lattice. The interaction parameters are $U_{aa} = 1.0$, $U_{bb} = 0.7$, $U_{ab} = 0.5$, and $W = -0.1$. There are five phases, moreover a novel NMI phase has not been researched up to now.

![Graph](image)

**FIG. 4:** (a) The total particle number $n_a+n_b$, the number of spin-down (spin-up) component $n_a$ ($n_b$) as a function of chemical potential $\mu$ with $J=0$, 0.002, 0.016. (b) $n_a+n_b$, $n_a$, and $n_b$ as a function of $W$ with $J=0.002$ and $\mu=2.0$, meanwhile the superfluid order parameter ($\langle \hat{a} \hat{a} \rangle$) and ($\hat{b} \hat{b}$) as a function of $W$ are also shown, where the pink (black) vertical axis is indicated the value of superfluid order parameter (particle number). Here the interaction parameters are $U_{aa} = 1.0$, $U_{bb} = 0.7$, $U_{ab} = 0.5$ for figure (a) and (b).

By changing the value of $W$ and keeping intensity of the other interactions, the systems can evolve from the NMI phase into the novel $SF_b+MSF_a$ phase [see Fig. 4(b)], where the $SF_b+MSF_a$ phase is characterized by normal superfluid of $b$ (spin-up) component and nontrivial $MSF_a$ (spin-down) component. The characteristics of the $MSF_a$ can partly be understood via the coherent state. It's well known that the coherent state satisfies the condition $\psi_a \neq 0$ and $\psi_Db \neq 0$, and even or odd coherent state [59, 60] satisfies the condition $\psi_a = 0$ and $\psi_Db \neq 0$, where even and odd coherent state read $|0\rangle+\cdots+\alpha^{2n}|2n\rangle/\sqrt{(2n)!}|/\cosh|\alpha|^2$ and $|\alpha|1+\cdots+\alpha^{2n+1}(2n+1)/\sqrt{(2n+1)!}|/\sinh|\alpha|^2$, respectively. As is well known, the perfect superfluid phase (the ground state of the Bose-Hubbard model for non-interaction limit $U=0$) is the coherent state, but the superfluid phase (in the case of $U \neq 0$) is not the coherent state [2]. Thus, the perfect $MSF_a$ can be considered as an odd or even coherent state, but $MSF_a$ state is no longer an even or odd coherent state for interacting systems.

**Symmetry analysis.**— Here we analyse the general symmetry feature of the phases and transitions between them. It is obvious that a finite SPH interaction $W$ breaks $U(1) \times U(1)$ symmetry of the trivial two-component boson Hamiltonian $(W=0)$ down to $U(1) \times Z_2$ symmetry (Under the phase transformations $\hat{b}_i \rightarrow \hat{b}_i e^{i\theta}$ and $\hat{a}_i \rightarrow \hat{a}_i e^{i\theta}$ or $\hat{a}_i \rightarrow \hat{a}_i e^{i(\theta+\pi)}$, the Hamiltonian in Eq. (1) keep unchanged). Here 2MI and NMI phases break no symmetry, but the SCF, $SF_b+MSF_a$, and 2SF phases are related to different ways that the $U(1) \times Z_2$ symmetry is broken. More specifically, the SCF phase
breaks discrete $Z_2$ subgroup but the $U(1)$ symmetry is remaining. The $\text{SF}_b + \text{MSF}_a$ phase breaks $U(1) \times Z_2$ symmetry except for the special point $\theta = \pi$, where $\text{SF}_b$ order changes sign ($\langle \hat{b} \rangle \to \langle \hat{b} \rangle e^{i\pi}$) and $\text{MSF}_a$ order keeps unchanged ($\langle \phi_{\text{Da}} \rangle \to \phi_{\text{Da}} e^{i2\pi}$ and $\phi_{\text{Da}} \to \phi_{\text{Da}} e^{i2\pi}$). This type of symmetry breaking is rarely revealed in natural condensed-matter systems. The $2\text{SF}$ phase totally breaks the $U(1) \times Z_2$ symmetry.

The effective-field analysis of the possible phases. — We will qualitatively analyze the reason why such rich phases can exist in two-component bosons with SPH interaction. In a $W = 0$ case, the mean-field phase diagrams can be obtained by minimizing the free energy $\mathcal{F}_0$ of two-component Bose-Hubbard model [10]. The corresponding phase diagrams can be divided into two typical cases: if the interaction is symmetric, there are three phases, i.e., $2\text{SF}$, $2\text{MI}$ and $\text{SCF}$ ($U_{ab} > 0$) [8]; if the interaction is asymmetric, the possible phases are $2\text{SF}$, $2\text{MI}$, $\text{SCF}$ and $\text{SF}_b + \text{MSF}_a$ [10]. For the $W \neq 0$ case, we can also use the effective field theory to analyze the possible phases of this system. We can assume the free energy $\mathcal{F}$ has the form (see SM) [50]

$$\mathcal{F} = \mathcal{F}_0 + \frac{1}{2} \left[ r_{\text{Da}} |\phi_{\text{Da}}|^2 + r_{\text{Db}} |\phi_{\text{Db}}|^2 + r_{\text{DD}} |\phi_{\text{Da}} \phi_{\text{Db}} + H.c.| \right] + \frac{1}{4} \left[ g_{\text{Da}} |\phi_{\text{Da}}|^4 + g_{\text{Db}} |\phi_{\text{Db}}|^4 \right] - \frac{1}{2} g_{\text{SCF}} \phi_{\text{Da}}^* \phi_{\text{Db}}^* \psi_A^\dagger \psi_B + H.c.$$  

(2)

with the condition $r_{\text{Da}} > 0$, $r_{\text{Db}} > 0$, $g_{\text{Da}} > 0$, $g_{\text{Db}} > 0$. Here the notation $\phi_{\text{Da}}$ ($\phi_{\text{Db}}$) is MSF order of the spin-down (spin-up) component. For the asymmetric case ($U_{aa} > U_{bb}$), there are four phases, i.e., $2\text{SF}$, $2\text{MI}$, $\text{SCF}$ and $\text{SF}_b + \text{MSF}_a$ which satisfy the corresponding saddle point equations [50]. Three of them ($2\text{SF}$, $2\text{MI}$, $\text{SCF}$) have been predicted in a two-component boson system without SPH interaction. Surprisingly, the phase $\text{SF}_b + \text{MSF}_a$ can not exist in this two-component boson system with SPH interaction, and it is replaced by the interesting phase $\text{SF}_b + \text{MSF}_a$ which has not been predicted in two-component boson system without SPH interaction. This conclusion is in good agreement with numerical calculation. Still, the reason for the existence of NMI can not be revealed by the effective-field analysis (EFS), owing to EFS unable to capture the information of the atom distribution.

Experimental realization and detection. — If we choose the $[\Omega/(2\omega)]^2 = 0.05 \ll 1$, only the interesting SPH interaction is important and the high-order terms ($O[f^4(t)/h^3]$) are ignored (see SM) [50], then Hamiltonian in Eq. (1) can adequately describe all relevant processes of this periodic driving system. If we want to find NMI and MSF$_a$ phase in a real experimental system, $W \propto [\Omega/(2\omega)]^2$ must be far less than on-site interaction. By choosing $U_{aa} = 2939/2850$, $U_{bb} = 2039/2850$, $U_{ab} = 61/150$ (can be realized via a Feshbach resonance) and $[\Omega/(2\omega)]^2 = 0.05$, the effective on-site interactions have the same vales as the vales presented in Fig. (5), while the $W = -0.0117$ is far less than on-site interaction. In this feasible region, NMI and MSF$_a$ phases can also occupy a larger region in the phase diagram, thus the prospects of observing NMI and MSF$_a$ states within this interesting driving system is much larger. Moreover, the nontrivial feature of the number distribution of NMI state can be directly detected by combining spin-removal technique [61, 62] and in-situ imaging techniques [63] which are been successfully to detect the bosonic MI [64–66] and fermionic MI [67, 68] with single-atom and single-site resolution. The previous research has shown that the MSF and SF phases are distinguished via time-of-flight (TOF) shadow images [20], thus SF$_b$ + MSF$_a$ can be directly detected by spin-resolved TOF images [69].

Discussion and Conclusions. — We have theoretically proposed to engineer a new two-particle hopping process with an SPH interaction in the two-component boson system via periodically modulating the radio-frequency field. This intriguing SPH interaction can lead to two interesting phases, i.e., NMI and MSF$_a$. The NMI state is a new type of Mott insulator, in which the total number at each site is an integer, but each component is a non-integer. The MSF$_a$ state has been proposed for some years, and no much progress has been made to host such a state in a realistic system. The region of NMI and MSF$_a$ states are small shrunken with rapidly decreasing the SPH interaction [see Figs. (3, 5)]. Thus, the prospects of observing the interesting NMI and MSF$_a$ states are optimistic in a realistic system. Furthermore, the detection schemes of these two novel phases are also addressed. The realization of our scheme provides a possible platform for further exploration of intriguing magnetic phases and interesting many-body phases in synthetic dimensions.

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