Stochastic Analysis of Downlink Heterogeneous Cellular Networks: Based on Spatial Hybrid Model

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Abstract. In this paper, a hybrid model for heterogeneous cellular networks (HCNs) is proposed based on the differences in the distribution characteristics, path loss models and transmit power of macro and micro base stations (BSs). The macro base stations in HCNs are modelled as 2-D PPP which obeys certain parameters, and the small base stations are modelled as 3-D PPP. According to the stochastic geometry theory and the relevant knowledge of probability theory, the interference characteristics in HCNs are analysed, and the downlink coverage probability of the target users in the network are derived. Compared the hybrid interference model proposed in this paper with the traditional model, simulation shows that the hybrid interference model can illustrate the distribution characteristics of each kind of base station in HCNs more accurately, and the network performance based on model proposed in this paper is more accurate.

1. Introduction
As cellular network architecture tends to isomerization, the randomness of communication nodes in spatial distribution becomes more significant. Therefore, stochastic geometric theory plays an increasingly important role in modelling and analysing HCNs [1]. In [2], the authors analyse the multi-tier HCNs by assuming that each tier of the BS obeys the 2-D PPP and all BSs share the same spectrum resources. On this basis, the author deduces the probability of users accessing all BSs and the load of each BS, and further estimates the downlink coverage probability and transmission rate. Although this model is relatively simple, it pioneered the stochastic geometric analysis of HCNs, and laid the foundation for further research. Most of the subsequent research results are based on this. The load balancing problem of HCNs is studied in [3]. In [4], the partial frequency reuse technology in HCNs is studied. In [5], the inter tier spectrum allocation scheme and the BS access mechanism in the two level family base station network are studied. In [6], the cell access strategy in HCNs is analysed. In [7], the interference management and space time association in HCNs are studied.

In addition to the above PPP based HCN performance analysis, many scholars adopt non PPP modelling of HCNs in order to obtain more accurate performance analysis results. Starting from the correlation between BSs, the authors describe the distribution of micro BSs for macro BS edge coverage and hot spot coverage with two non PPP respectively [8]. The literature [9-10] uses poisson clustering process to describe HCNs, the macro and micro BSs are assumed to be the parent processes and sub processes in the poisson cluster process respectively. Considering the minimum distance
between adjacent base stations, a cellular network analysis method based on poisson hard core process is proposed [11]. All the above point process analysis methods are more suitable for cellular networks than PPP and grid methods, and can better describe the spatial characteristics of BSs distribution in real networks. However, the biggest weakness of these point processes is that it is difficult to analyse and is difficult to obtain closed instruction with guiding significance, which limits their application in modelling and analysing interference in wireless cellular networks.

In this work, we propose a new interference model on the basis of the different characteristics of macro and micro BSs in HCNs. According to the theories of PPP and probability, the interferences in HCNs are analysed and downlink coverage probability is obtained. Simulation shows that the network performance index estimated by this model is 1-2dB accurate than the traditional method.

2. System Model

In the urban central area with a large number of buildings, macro BSs are usually deployed at the top of the building to achieve wide coverage and plane distribution, while the micro BSs are deployed inside the building to meet the indoor coverage and are stereoscopic, shown in Figure 1. In the hybrid model, the macro BSs are modelled as 2-D PPP $\Phi_1$ with parameter $\lambda_1$, whereas the micro BSs are modelled as the 3-D PPP $\Phi_2$ with parameter $\lambda_2$. It is worth noting that for the different distribution dimension of macro and micro BS, the units of $\lambda_1$ and $\lambda_2$ are different ($m^{-2}$ and $m^{-3}$ respectively). $P_1$ and $P_2$ indicate the transmitting power of macro and micro BS respectively; $\beta_1$ and $\beta_2$ respectively represent the SINR threshold of macro and micro BS.

Due to the deployment of micro BSs indoors, the path loss of the indoor wireless channel fading model $G \cdot x^{-(\alpha+1)}$ described in [12] is adopted, where $G$ is constant. The general path loss model $PL(x) = x^{-\alpha}$ is adopted for macro BS. Without loss of generality, the target user is assumed at the origin of the coordinates [13].

When a target user communicates with a macro BS, its received SINR is

$$\text{SINR}_1(x) = \frac{P_1 h_{i0} x^{-\alpha_1}}{\sum_{i \in \Phi_1 \setminus \Phi_0} P_1 h_i |Y_i|^{\alpha_0} + \sum_{i \in \Phi_2} P_2 G h_{i0} |Y_i|^{-\alpha_2} + \sigma^2},$$

while communicates with a micro BS, the SINR is represented as

$$\text{SINR}_2(x) = \frac{P_2 G h_{i0} x^{-\alpha_1}}{\sum_{i \in \Phi_2} P_2 G h_i |Y_i|^{\alpha_0} + \sum_{i \in \Phi_1 \setminus \Phi_0} P_1 h_i |Y_i|^{-\alpha_2} + \sigma^2},$$

where $h$ represent shadow fading and $Y$ is the distance from the interfering macro or micro BS to the target user.

Figure 1. Two-tier HCN spatial hybrid model
3. Coverage Probability
This section discusses the downlink coverage probability of HCNs based on hybrid distribution model.
In this section, we first discuss the access problem of user in HCNs, including the probability of association between target user and macro or micro BSs, and the average number of associated users per macro or micro BS. Then we derive the PDF of the distance between the target user and its associated BS, and finally deduce the target user's downlink coverage probability.

Lemma 1: The probability that a target user associate with the macro and micro BS is

\[
A_1 = 2\pi\lambda_1 \int_0^\infty r \exp \left\{ -\frac{4}{3} \pi\lambda_2 \left( \frac{P_2 G}{P_1} \right)^{\frac{3}{3\alpha_2+1}} r^{\frac{3\alpha_1}{\alpha_2+1}} - \pi\lambda_1 r^2 \right\} dr,
\]

and

\[
A_2 = 4\pi\lambda_2 \int_0^\infty r^2 \exp \left\{ -\pi\lambda_1 \left( \frac{P_1}{P_2 G} \right)^{\frac{2\alpha_1}{3\alpha_2+1}} r^{\frac{2\alpha_1}{\alpha_2+1} - 4} - \frac{4}{3} \pi\lambda_2 r^3 \right\} dr.
\]

Proof: Denote symbol n as an index of the BS tier that a typical user communicate with, n will be 1 or 2.
If the received power from the nearest macro BS \( P_{1,n} \) is larger than that of the nearest micro BS, the user will communicate with macro BS. In this way \( n = 1 \); otherwise \( n = 2 \).

\[
A_n = \mathbb{P}[n=1] = \mathbb{E}_R \left[ \mathbb{P} \left[ P_1 R_n^{-\alpha_1} > P_2 G R_n^{-(\alpha_2+1)} \right] \right] = \int_0^\infty R > \left( \frac{P_2 G}{P_1} \right)^{\frac{\alpha_1}{\alpha_2+1}} r^{\alpha_1/(\alpha_2+1)} \right] f_R(r) dr,
\]

where

\[
\mathbb{P} \left[ R > \left( \frac{P_2 G}{P_1} \right)^{\frac{\alpha_1}{\alpha_2+1}} r^{\alpha_1/(\alpha_2+1)} \right] = \exp \left\{ -\frac{4}{3} \pi\lambda_2 \left( \frac{P_2 G}{P_1} \right)^{\frac{3}{3\alpha_2+1}} r^{\frac{3\alpha_1}{\alpha_2+1}} \right\}
\]

and

\[
f_R(r) = \frac{d[1-P[R > r]]}{dr} = e^{-\pi\lambda_2 r^2} 2\pi\lambda_2 r.
\]

Combining (5), (6) with (7), we can get (3). Similarly, we get (4).

Lemma 1 reveals a basic rule, that is, users tend to associate with those BSs with larger distribution density and transmit power, and are unwilling to associate with the BSs with sparse distribution and low transmitting power. This indicates that when we deploy HCNs, we need to consider the transmission power and distribution density of different types of BSs. This enables users to choose associated BSs in macro and micro BSs instead of accessing the same type of BSs uniformly, and ultimately achieves the purpose of balancing network load and improving network efficiency.

Lemma 2: When the target user communicated with macro BS, the pdf of the distance between the target user and its associated macro BS will be

\[
f_{X_1}(x) = \frac{2\pi\lambda_1}{A_1} x \exp \left\{ -\frac{4}{3} \pi\lambda_2 \left( \frac{P_2 G}{P_1} \right)^{\frac{3}{\alpha_2}} x^{\frac{3\alpha_1}{\alpha_2+1}} - \pi\lambda_1 x^2 \right\},
\]

and that between the target user and its associated micro BS is

\[
f_{X_2}(x) = \frac{4\pi\lambda_2}{A_2} x^2 \exp \left\{ -\pi\lambda_1 \left( \frac{P_1}{P_2 G} \right)^{\frac{2\alpha_1}{\alpha_2+1}} x^{\frac{2\alpha_1}{\alpha_2+1} - 4} - \frac{4}{3} \pi\lambda_2 x^3 \right\}.
\]

Proof: The probability that \( X_1 > x \) can be given as
the numerator can be expressed as
\[
P[R_i > x | n = 1] = P[R_i > x, P_{s,i}(R_i) > \max P_{r,i}] = \int_{x}^{\infty} P[R_2 > \left( \frac{P_G}{P_1} \right)^{3/(\alpha_2+1)} r^{\alpha_2/\alpha_2+1}] f_{R_i}(r) dr,
\]
while the denominator can be determined by Lemma 1,
\[
P[X_i > x] = \frac{2\pi\lambda_i}{A_i} \int_{x}^{\infty} r \exp \left\{ -\frac{4}{3} \pi\lambda_2 \left( \frac{P_G}{P_1} \right)^{3/(\alpha_2+1)} r^{3\alpha_2/(\alpha_2+1)} - \pi\lambda_i r^2 \right\} dr.
\]

The pdf of \( X_i \) is
\[
f_{X_i}(x) = \frac{d}{dx} \left[ 1 - P[X_i > x] \right] = \frac{2\pi\lambda_i}{A_i} x \exp \left\{ -\frac{4}{3} \pi\lambda_2 \left( \frac{P_G}{P_1} \right)^{3/(\alpha_2+1)} x^{3\alpha_2/(\alpha_2+1)} - \pi\lambda_i x^2 \right\}.
\]

Similar, we can get (9).

In HCNs composed of macro and micro BSs, the downlink coverage probability of target user \( Cov \) is defined as the probability that the target user’s instantaneous SINR in the network is larger than the predefined threshold. Therefore, we can obtain the coverage probability of the whole heterogeneous network through the total probability formula [13]
\[
Cov = \sum_{k=1}^{2} Cov_k A_k
\]
where \( A_k \) indicates the association probability which is determined by lemma 1 and \( cov_k \) is the coverage probability associated with macro or micro BS.

**Theorem 1:** When the target user communicated with macro BS, its coverage probability will be
\[
Cov_1 = \frac{2\pi\lambda_i}{A_i} \int_{0}^{\infty} x \exp \left\{ -\frac{T x^{\alpha_2}}{SNR} - \pi\lambda_i \int_{0}^{\infty} x^2 T^{2\alpha_2} \frac{du}{1 + u^{\alpha_2+1/3}} \right\} dx - \frac{4}{3} \pi\lambda_2 \int_{0}^{\infty} x^2 \exp \left\{ -\frac{T x^{\alpha_2}}{SNR} G^{-1} \left( \frac{P_G}{P_1} \right)^{3/(\alpha_2+1)} x^{3\alpha_2/(\alpha_2+1)} - \pi\lambda_i x^2 \right\} dx
\]
and that with micro BS is
\[
Cov_2 = \frac{4\pi\lambda_2}{A_2} \int_{0}^{\infty} x^2 \exp \left\{ -\frac{T x^{\alpha_2}}{SNR G^{-1}} - \pi\lambda_i \int_{0}^{\infty} x^{3\alpha_2/(\alpha_2+1)} \frac{du}{1 + u^{\alpha_2+1/3}} \right\} dx - \frac{4}{3} \pi\lambda_2 \int_{0}^{\infty} x^2 \exp \left\{ -\frac{T x^{\alpha_2}}{SNR G^{-1}} \left( \frac{P_G}{P_1} \right)^{3/(\alpha_2+1)} x^{3\alpha_2/(\alpha_2+1)} - \pi\lambda_i x^2 \right\} dx
\]
Therefore, the downlink coverage probability of the entire HCN is
\[
Cov = Cov_1 A_1 + Cov_2 A_2.
\]

**Proof:** When communicate with macro BS, the coverage probability is
\[ Cov_1 = E_x \left[ P[\text{SINR}_1(x) > T] \right] = \int_0^\infty P[\text{SINR}_1(x) > T] f_{X_1}(x) dx, \] (18)

where \( f_{X_1}(x) \) is given in Lemma 2.

Rewrite the SINR of target user as \( \text{SINR}_1(x) = \frac{h_{i,0}}{x^\alpha P_1^{-1} (I_x + \sigma^2)} \), where \( I_x = \sum_{i \in \Phi_1 \setminus \Phi_0} P_h_i |Y_{i1}|^{-\alpha_i} + \sum_{i \in \Phi_2} P_i G_{hi} |Y_{i2}|^{-\alpha_{i2}} \) is interference. Therefore, the probability \( \text{P}[\text{SINR}_1(x) > T] \) in (18), that is the CCDF of received SINR, can be expressed as

\[ \text{P}[\text{SINR}_1(x) > T] = \text{P}\left[h_{i,0} > x^\alpha P_1^{-1} T \left(I_x + \sigma^2\right)\right] = e^{-x^\alpha P_1^{-1} T} \mathcal{L}_{I_1} \left(x^\alpha P_1^{-1} T\right). \] (19)

According to the definition of Laplace transform, the Laplace transform in (19) can be expanded as

\[ \mathcal{L}_{I_1} \left(x^\alpha P_1^{-1} T\right) = E \left[ \exp \left(-x^\alpha P_1^{-1} T \sum_{i \in \Phi_1 \setminus \Phi_0} P_h_i |Y_{i1}|^{-\alpha_i} + \sum_{i \in \Phi_2} P_i G_{hi} |Y_{i2}|^{-\alpha_{i2}} \right) \right] = E \left[ \prod_{i \in \Phi_2} \exp \left(-x^\alpha P_1^{-1} P_i G_{hi} |Y_{i2}|^{-\alpha_{i2}} \right) \right] \cdot E \left[ \prod_{i \in \Phi_1 \setminus \Phi_0} \exp \left(-x^\alpha P_1^{-1} P_h_i |Y_{i1}|^{-\alpha_i} \right) \right]. \] (20)

Following the probability generating functional (PGFL) of 2-D and 3-D PPP [13] and \( h \sim \exp(1) \)

\[ \mathcal{L}_{I_1} \left(x^\alpha P_1^{-1} T\right) = \exp \left(-2\pi \lambda_1 \int_0^\infty \left(1 - \frac{1}{1 + x^\alpha T y^{-\alpha_i}}\right) y dy\right) \cdot \exp \left(-4\pi \lambda_2 \int_0^\infty \left(1 - \frac{1}{1 + x^\alpha P_1^{-1} P_i G T y^{-\alpha_{i2}}}\right) y^2 dy\right). \] (21)

Employing variable substitutions \( u = (x^\alpha T)^{-2/\alpha_i} y^2 \) and \( v = (x^\alpha P_1^{-1} P_i G T)^{-3/(\alpha_{i2})} y^3 \) to the two exponentiation expressions in formula results in (21), we can get

\[ \mathcal{L}_{I_1} \left(x^\alpha P_1^{-1} T\right) = \exp \left(-\pi \lambda_1 \int_{\infty}^{\infty} \frac{x^\alpha T y^{-\alpha_i/2} du}{1 + u^{\alpha_i/2}}\right) \exp \left(-\frac{4}{3} \pi \lambda_2 \int_0^{\infty} \frac{(x^\alpha P_1^{-1} P_i G T)^{3/(\alpha_{i2})} du}{1 + v^{(\alpha_{i2})/3}}\right). \] (22)

Combining (22) and (19), setting \( \frac{T}{\text{SNR}} = \frac{T \sigma^2}{P_1} \), we can get (15). In the similar way, formula (16) can be obtained. Furthermore, the downlink coverage probability in the whole network can be obtained by plugging (15) and (16) into (17).

4. Numerical Results

In this section, we simulate and analyse the coverage probability of two-tier HCN downlink system composed of macro and micro BSs. In addition to special instructions, the simulation environment is set as follows: the radius of a central spherical area \( R = 100\) m; the transmission power of macro and micro BS \( P_1 = 63\) dBm, \( P_2 = 33\) dBm, and the density of macro and micro BSs \( \lambda_1 = 10^{-3} \text{m}^{-2}, \lambda_2 = 2 \times 10^{-3} \text{m}^{-3} \); the path loss factor of macro and micro BS \( \alpha_1 = 3.8, \alpha_2 = 4.5 \); the additive Gauss white noise power \( \sigma^2 = 104\) dBm.

Comparisons of the coverage performance between the hybrid interference modelling method proposed in this paper and the traditional analysis method based 2-D PPP in case of noiseless and noisy environments are shown in Figure 2 and Figure 3 respectively. The distribution of macro and
micro BSs in the actual environment is taken as a reference. The abscissa coordinates the SIR/SINR threshold, and the ordinate represents the coverage probability.

From Figure 2 and Figure 3, we can find that the simulation results based on the hybrid model proposed in this paper are very close to the simulation results of the actual base station distribution scenario, which is more accurate than the traditional analysis based on 2-D PPP 1-2dB. Obviously, the hybrid model in this paper is more suitable for describing the distribution of base stations in HCNs, and can make more accurate prediction of network performance indicators. The reason is that, macro and micro BS appear different spatial distribution characteristics and transmission characteristics in actual environment. In the traditional analysis process, they are uniformly described as 2-D PPP distribution, but there are many differences in parameters such as base station density, path loss factor and transmission power. This paper uses different model to describe the spatial distribution of macro and micro BSs respectively, describes the path loss of macro and micro BSs with different fading models, and achieves good results.

Theorem: Plane Model, 2-D for tier 1
Simulation: Actual BS for tier 1, PPP for tier 2
Simulation: 2-D PPP for tier 1, 3-D PPP for tier 2
Theorem: Hybrid model, 2-D for tier 1, 3-D for tier 2

Figure 2. Coverage probability in no noise environment
Figure 3. Coverage probability in noisy environment

Figure 4 presents the coverage probability for different values of $\alpha$ in the no noise and noisy environment. Compared the line of no-noise with noisy environments, we note that the additive noise has little influence on the coverage probability and the heterogeneous cellular networks are interference limited. In addition, we can find that the coverage probability increases when the path loss exponent $\alpha$ increases, whether or not the noise exists. The reason is that, when $\alpha$ increases, the received signal and the interference will be reduced, but the interference decreases more significantly, so the SINR and coverage probability increase.

Figure 4. Coverage probability in a two-tier HCN with and without thermal noise
5. Conclusions
Through analysing the different characteristics of macro and micro BSs in two-tier HCNs, this paper proposes a hybrid model for interference modelling of HCNs based on 2-D and 3-D PPP, and analyses the network performance based on the new model. Simulations show that the proposed new model in this work appears closer to the reality than traditional 2-D PPP in terms of accuracy.

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