New approximate analytical solutions for the nonlinear fractional Schrödinger equation with second-order spatio-temporal dispersion via double Laplace transform method

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In this paper, a modified nonlinear Schrödinger equation with spatiotemporal dispersion is formulated in the senses of Caputo fractional derivative and conformable derivative. A new generalized double Laplace transform coupled with Adomian decomposition method has been defined and applied to solve the newly formulated nonlinear Schrödinger equation with spatiotemporal dispersion. The approximate analytical solutions are obtained and compared with each other graphically.

KEYWORDS
Caputo fractional derivative, conformable derivative, double Laplace transform, nonlinear fractional Schrödinger equation

MSC CLASSIFICATION
35Q55; 26A33

1 | INTRODUCTION

Fractional derivatives (FDs), which are generalized forms of fractional calculus (FC), have been recently applied in various modeling scenarios arising from phenomena in science and engineering. FDs are used particularly in modeling various complex engineering systems in mechanics. Various systems in science and engineering have been investigated in several
works\textsuperscript{1–6} using analytical and approximate analytical methods to find solutions to the fractional equations in the sense of FDs and fractional integrals such as Caputo, Riemann–Liouville, and Grünwald–Letnikov.\textsuperscript{7} All FDs such as Caputo, Riemann–Liouville, and Grünwald–Letnikov have shown the property of linearity, but the usual derivative properties such as the derivatives of constant, quotient rule, product rule, and chain rule cannot be satisfied using any of those FDs.\textsuperscript{8}

Khalil et al. introduced a new generalized local fractional derivative known as conformable derivative (CD), which is basically an extension of the usual limit-based derivative.\textsuperscript{9,10} CD occurs naturally and satisfies all properties of usual derivatives.\textsuperscript{9,10} The first author discussed in\textsuperscript{11} three different analytical methods for solving two-dimensional wave equation involving CD. Introducing CD formulations to (linear/nonlinear) partial differential equations (PDEs) can easily transform them into simpler versions\textsuperscript{10} than other commonly used FD formulations such as Riemann–Liouville, Caputo, and Grünwald–Letnikov because the difficulty of finding analytical solutions using the classical fractional definitions makes CD formulation a good option for certain cases. The physical interpretation for CD can be simply presented as a modified version of the usual derivative in magnitude and direction\textsuperscript{12} (see also Zhao and Luo\textsuperscript{13}). Khalil et al\textsuperscript{14} described the geometrical meaning of CD using the concept of fractional cords. For a comparative review and analysis of all definitions of FDs and fractional operators, we refer to Teodoru et al.\textsuperscript{15}

From this newly defined derivative, many essential elements of the mathematical analysis of functions of a real variable have been successfully developed, among which we can mention: mean value theorem, Rolle’s theorem, chain rule, conformable integration by parts formulas, conformable power series expansion, CDs, and integrals of complex-valued functions of a real variable, conformable boundary value problems: Sturm’s theorems and Green’s function, complex conformable integral (ComI), and conformable single and double Laplace transform (DLTr) definitions.\textsuperscript{9,11,16–21} The conformable partial derivative of the order $\gamma \in (0, 1]$ of the real-valued functions of several variables and conformable gradient vector are defined, and a conformable Clairaut’s theorem for partial derivatives is proven in Atangana et al.\textsuperscript{22} In Gözütok,\textsuperscript{23} the conformable Jacobian matrix is introduced; chain rule for multivariable CD is defined; and the relation between conformable Jacobian matrix and conformable partial derivatives is investigated. In Martinez et al.\textsuperscript{24} two new results on homogeneous functions involving their conformable partial derivatives are introduced, specifically the homogeneity of the conformable partial derivatives of a homogeneous function and conformable Euler’s theorem.

The PDEs, particularly the Schrödinger equation, have been applied in several applications in physics and engineering due to the importance of this equation in nonlinear optics, which can successfully explain the dynamics of optical soliton propagation in optical fibers. Optical solutions for the complex Ginzburg–Landau equation with Kerr law nonlinearity formulated in the sense of truncated M-fractional derivative and beta derivative were investigated in Khalil et al.\textsuperscript{25} In addition, optical solitons for Kundu–Eckhaus equation were obtained via the methods of modified tanh coth and extended Jacobi elliptic function expansion\textsuperscript{26} and the first integral method,\textsuperscript{27} respectively. Optical solitons were also studied for various other models in nonlinear optics (see several works\textsuperscript{28–33}). Various fractional formulations have been introduced in Ghanbari et al. and Yaşar and Yaşar,\textsuperscript{34,35} to obtain exact optical soliton solutions for modified nonlinear Schrödinger equation (MNLSE) with spatiotemporal dispersion. Finding analytical and approximate analytical solutions for the modified forms of nonlinear fractional Schrödinger equation has become a common research interest for physicists and applied mathematicians in the field of optical soliton propagation because of the applications of this equation in plasma, optics, electromagnetism, fluid dynamics, and optical communication.\textsuperscript{34,35} The dynamics of optical soliton propagation in optical fiber can be interpreted from the MNLSE with second-order spatiotemporal dispersion and group velocity dispersion coefficients.\textsuperscript{34} Given $\Psi(x, t)$ as a complex-valued wave function that represents the macroscopic property of wave profile of the spatial and temporal variables, which are expressed as $x$ and $t$, respectively, then MNLSE can be written as\textsuperscript{34}

$$i \left( \frac{\partial \Psi}{\partial x} + \omega_1 \frac{\partial \Psi}{\partial t} \right) + \alpha_2 \frac{\partial^2 \Psi}{\partial t^2} + \omega_3 \frac{\partial^2 \Psi}{\partial x^2} + |\Psi|^2 \Psi = 0,$$

where\textsuperscript{34} $\omega_1$ is proportional to the ratio of group speed; $\omega_2$ is a group velocity dispersion coefficient; and $\omega_3$ is a spatial dispersion coefficient.

To formulate (1) in the sense of FDs, let us first define Caputo fractional derivative (CpFD) as follows:

**Definition 1.** For $\xi, \gamma > 0$, given two functions: $h(x)$ and $h(t)$, such that for $x, t > 0$, the CpFD of $h$ of order $\xi$ and $\gamma$, denoted by $D^\xi_x (h)(x)$ and $D^\gamma_t (h)(t)$, respectively, where $D^\xi_x$ and $D^\gamma_t$ are Caputo derivative operators, which can be simply expressed as\textsuperscript{36}
\[ D^\xi_h(x) = \frac{1}{\Gamma(\Omega - \xi)} \int_0^x (x - \eta)^{\Omega - \xi - 1} h^{(\xi)}(\eta)d\eta; \Omega - 1 < \xi \leq \Omega \text{ for } \Omega \in \mathbb{N}, \]  
(2)

\[ D^\mu_t h(t) = \frac{1}{\Gamma(w - \gamma)} \int_0^t (t - \mu)^{w - \gamma - 1} h^{(w)}(\mu)d\mu; w - 1 < \gamma \leq w \text{ for } w \in \mathbb{N}. \]  
(3)

If \( \xi = \Omega \) and \( \gamma = w \), where \( \Omega, w \in \mathbb{N} \), then \( D^\xi_h(x) = \frac{d^\xi}{dx^\xi} h(x) \) and \( D^\mu_t h(t) = \frac{d^\mu}{dt^\mu} h(t) \). CpFD is very useful in science and engineering due to their important properties such as the inclusion of initial and boundary conditions in the fractional formulation of CpFD.\(^2,37\) Let us now define the Mittag–Leffler function:

**Definition 2.** The Mittag–Leffler function, denoted by \( E_{\xi, \zeta}(t) \), can be expressed as follows:\(^38\):

\[ E_{\xi, \zeta}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\xi m + \zeta)}, \text{ where } t, \xi \in \mathbb{C} \text{ and } \Re(\xi) > 0. \]  
(4)

From the above definition, the Mittag–Leffler function, denoted by \( E(t, h, c) \), can be written\(^36\) as \( E(t, h, c) = t^\delta E_1, h + 1(ct) \), and the fractional derivative of Mittag–Leffler function can also be expressed\(^36\) as \( D^\delta_t t^\delta E_{\xi, \zeta}(ct^\delta) = t^{\delta - 1} E_{\xi, \zeta - \delta}(ct^\delta) \), where \( \delta \geq 0 \). Let us now define the CD as follows:

**Definition 3.** Given a function \( h : [0, \infty) \to \mathbb{R} \), such that for all \( t > 0 \), the CD of order \( \gamma \in (0, 1] \) of \( h \), denoted by \( M_\gamma(h) \), can be represented as

\[ h^{(\gamma)}(t) = M_\gamma(h)(t) = \lim_{\xi \to 0} \frac{h(t + \xi t^{1-\gamma}) - h(t)}{\xi}. \]  
(5)

Suppose that \( h \) is \( \gamma \)-differentiable in some \( (0, d) \), \( d > 0 \), and the limit of \( h^{(\gamma)}(t) \) exists as \( t \to 0^+ \), then from CD definition, the following is obtained:

\[ h^{(\gamma)}(0) = M_\gamma(h)(0) = \lim_{t \to 0^+} h^{(\gamma)}(t). \]  
(6)

It is also important to define here the ComI\(^12\) as follows:

**Definition 4.** For \( \gamma \in (0, 1] \), given a function \( h : [0, \infty) \to \mathbb{R} \), such that for all \( t \geq 0 \), the \( \gamma \)-order ComI of \( h \) from 0 to \( t \) can be expressed as

\[ I_\gamma(h)(t) = \int_0^t h(\psi)d\psi = \int_0^t h(\psi)\psi^{\gamma-1}d\psi. \]  
(7)

If we suppose \( \gamma = 1 \), then we have \( I_1(h)(t) = I_{\gamma=1}(t^{-1}h)(t) \), which represents the classical improper Riemann integral of a function \( h(t) \). Given a continuous function, \( h \), on \( (0, \infty) \) and for \( \gamma \in (0, 1) \), then \( M_\gamma(h)(t)[I_\gamma(h)(t)] = h(t) \).

**Lemma 1 (Abdeljawad\(^16\)).** Given a function \( h : (c, d) \to \mathbb{R} \) as a differentiable function and \( \gamma \in (0, 1] \), then for all \( c > 0 \), we have the following:

\[ I_\gamma^c M_\gamma^c(h)(t) = h(t) - h(c). \]

In addition, the following theorem\(^8,9\) shows that \( M_\gamma \) satisfies all the standard properties of basic limit-based derivative as follows:

**Theorem 2.** For \( \gamma \in (0, 1] \), given two functions say: \( h \) and \( v \) to be assumed \( \gamma \)-differentiable at a point \( t \), then the following is obtained:

1. \( M_\gamma(ch + ev) = cM_\gamma(h) + eM_\gamma(v) \), for all \( c, e \in \mathbb{R} \).
(2) \( M_r(t) = rt^{-r} \), for all \( r \in \mathbb{R} \).
(3) \( M_r(hv) = hM_r(v) + vM_r(h) \).
(4) \( M_{\frac{h}{v}} \frac{vM_r(h) - hM_r(v)}{v^r} \)
(5) \( M_r(\lambda) = 0 \), for all constant functions \( h(t) = \lambda \).
(6) If \( h \) is assumed to be a differentiable function, then \( M_r(h)(t) = t^{1-r} \frac{dh}{dt} \).

The conformable partial derivative of a real-valued function with several variables is defined in Atangana et al. and Gözü tok22,23 as follows:

**Definition 5.** Let \( h \) be a real-valued function with \( n \) variables and \( c = (c_1, c_2, ..., c_n) \in \mathbb{R}^n \) be a point whose \( i \)th component is positive. Then, we have

\[
\lim_{\xi \to 0} h(c_1, c_2, ..., c_i + \xi c_i^{1-\gamma}, ..., c_n) - h(c_1, c_2, ..., c_n).
\]

If the limit exists, the \( i^{th} \) conformable partial derivative of \( h \) of the order \( \gamma \in (0, 1] \) at \( c \) is denoted by \( \frac{\partial}{\partial c_i} h(c) \).

**Remark 3.** Let \( \gamma \in (0, \frac{1}{\Lambda}] \), \( \Lambda \in \mathbb{Z}^+ \) and \( h \) be a real-valued function with \( n \) variables defined on an open set \( D \), such that for all \( (x_1, x_2, ..., x_n) \in D \), each \( x_i > 0 \). The function, \( h \), is said to be \( C_0^\gamma(D, \mathbb{R}) \) if all its conformable partial derivatives of order less than or equal to \( \Lambda \) exist and are continuous on \( D \).24

For more information about other related FDs' definitions and their physical and geometrical interpretations, we refer to Kaabar.11 The main goal of this paper is to obtain approximate analytical solutions for MNLSE in (1) using double Laplace transform method in the sense of Caputo derivative and CD. From Definition 3, let us first formulate the MNLSE in (1) in the sense of CD as follows:

\[
\begin{align*}
 i \left( D_x^\gamma \Psi \left( \frac{\chi^\gamma}{\gamma^\gamma} \frac{t^\gamma}{\gamma^\gamma} \right) + \omega_1 D_x^2 \Psi \left( \frac{\chi^\gamma}{\gamma^\gamma} \frac{t^\gamma}{\gamma^\gamma} \right) + \omega_2 D_x D_t \Psi \left( \frac{\chi^\gamma}{\gamma^\gamma} \frac{t^\gamma}{\gamma^\gamma} \right) + \omega_3 D_t^3 \Psi \left( \frac{\chi^\gamma}{\gamma^\gamma} \frac{t^\gamma}{\gamma^\gamma} \right) + |\Psi|^2 \Psi = 0; \\
 i D_t^\gamma \Psi \left( \frac{\chi^\gamma}{\gamma^\gamma} \frac{t^\gamma}{\gamma^\gamma} \right) + \omega_1 D_t^2 \Psi \left( \frac{\chi^\gamma}{\gamma^\gamma} \frac{t^\gamma}{\gamma^\gamma} \right) + \omega_2 D_t D_x \Psi \left( \frac{\chi^\gamma}{\gamma^\gamma} \frac{t^\gamma}{\gamma^\gamma} \right) + \omega_3 D_x^3 \Psi \left( \frac{\chi^\gamma}{\gamma^\gamma} \frac{t^\gamma}{\gamma^\gamma} \right) + |\Psi|^2 \Psi = 0,
\end{align*}
\]

(8)

where \( i = \sqrt{-1} \), \( 0 < \gamma, \delta \leq 1 \), \( t, x > 0 \).

From Definition 1, MNLSE in (1) can be similarly formulated in the sense of CpFD as follows:

\[
\begin{align*}
 i \left( \frac{\partial \Psi(x, t)}{\partial \chi^\gamma} + \omega_1 \frac{\partial \Psi(x, t)}{\partial t^\gamma} \right) + \omega_2 \frac{\partial^2 \Psi(x, t)}{\partial t^\gamma \partial \chi^\gamma} + \omega_3 \frac{\partial^3 \Psi(x, t)}{\partial \chi^2 \partial t^\gamma} + |\Psi|^2 \Psi = 0, \\
 i \frac{\partial \Psi(x, t)}{\partial t^\gamma} + \omega_1 i \frac{\partial^2 \Psi(x, t)}{\partial t^\gamma} + \omega_2 \frac{\partial^2 \Psi(x, t)}{\partial t^\gamma \partial \chi^\gamma} + \omega_3 \frac{\partial^3 \Psi(x, t)}{\partial \chi^2 \partial t^\gamma} + |\Psi|^2 \Psi = 0,
\end{align*}
\]

(9)

where \( i = \sqrt{-1} \), \( 0 < \gamma, \delta \leq 1 \), \( t, x > 0 \).

## 2 THE ANALYTICAL SOLUTIONS OF NONLINEAR FRACTIONAL SCHröDINGER EQUATION

Many differential equations can be easily solved by applying the method of Laplace transform (DT) for a single-variable function. New generalized forms of the classical Laplace transform methods such as DLTr and multiple Laplace transform have been first introduced in Estrin and Higgins39 to solve PDEs. Recently, DLTr has become an interesting topic of research for many mathematicians and researchers40-42 because not many research studies have been done on this topic43 and because of the need to find an efficient method for solving PDEs. Applying the DLTr in the sense of FDs has been
rarely discussed, and it is considered as an open problem.\textsuperscript{44}DLTr has been successfully introduced in solving some fractional differential equations (FDEs) such as the fractional heat equation and the fractional telegraph equation via the definition of C\textsuperscript{p}FD\textsuperscript{,44} According to our knowledge, the generalized DLTr method has never been applied before for solving the MNLSE in (1) in the senses of C\textsuperscript{p}FD and CD. Therefore, the results in this work are new and worthy.

The first author has defined\textsuperscript{11} the conformable DLTr (CmDLTr) as follows:

**Definition 6.** Given a function, $\Psi\left(\frac{x^\gamma}{y^\delta}\right) : [0, \infty) \to \Re$, such that for all $x, t > 0$, the CmDLTr of order $\gamma, \delta \in (0, 1]$ of $\Psi\left(\frac{x^\gamma}{y^\delta}\right)$, denoted by $\mathcal{E}_{\gamma, \delta}^x \Psi\left(\frac{x^\gamma}{y^\delta}\right)$, starting from 0 can be expressed as follows:

$$\mathcal{E}_{\gamma, \delta}^x \Psi\left(\frac{x^\gamma}{y^\delta}\right) = \mathcal{E}_{\gamma}^x \mathcal{E}_{\delta}^y \Psi\left(\frac{x^\gamma}{y^\delta}\right) = \mathcal{E}_{\gamma}^x \mathcal{E}_{\delta}^y \Psi_{\gamma, \delta}(s_1, s_2),$$

where $s_1, s_2 \in \mathbb{C}$. If the integral in the above definition exists, then this definition holds true.

To define the DLTr in the sense of Caputo partial FDS, let us assume that $\Psi^\gamma_t (s_1, s_2) = \int_0^\infty \int_0^\infty e^{-(s_1 + s_2)t} \Psi\left(\frac{x^\gamma}{y^\delta}\right)x^{\gamma-1}t^{\delta-1}dt.$ From Dhunde and Waghmare,\textsuperscript{38} theorems 3.1 and 3.3 in Anwar et al,\textsuperscript{44} and theorem 2 in Khan et al,\textsuperscript{45} the Caputo DLTr can be defined as follows:

**Definition 7.** Given a function, $\Psi(x, t) : [0, \infty) \to \Re$, such that for all $x, t > 0$, the DLTr of the Caputo partial FDS (CpDLTr) of $\Psi(x, t)$ of orders $\xi$ and $\gamma$ where $\xi \in (j - 1, j]$ and $\gamma \in (b - 1, b]$ such that $\xi, \gamma > 0$ and $j, b \in \mathbb{N}$, denoted by $\frac{\partial^\xi}{\partial x^\xi} \Psi(x, t)$ and $\frac{\partial^\gamma}{\partial t^\gamma} \Psi(x, t)$, respectively, can be expressed as

$$\mathcal{E}^\xi_{\xi} \left[ \mathcal{E}^\gamma_{\gamma} \Psi(x, t) \right] = s_1^\xi \Psi^\xi_{\xi}(s_1, s_2) - \sum_{i=0}^{\xi-1} s_1^{\xi-i-1} \mathcal{E}^i_{\xi} \left[ \frac{\partial\Psi(0, t)}{\partial x^i} \right],$$

$$\mathcal{E}^\gamma_{\gamma} \left[ \mathcal{E}^\xi_{\xi} \Psi(x, t) \right] = s_2^\xi \Psi^\gamma_{\gamma}(s_1, s_2) - \sum_{a=0}^{\gamma-1} s_2^{\gamma-a} \mathcal{E}^a_{\gamma} \left[ \frac{\partial\Psi(x, 0)}{\partial t^a} \right].$$

The DLTr in the sense of conformable partial FDS can be similarly defined\textsuperscript{11} as follows:

**Definition 8.** Given a function, $\Psi\left(\frac{x^\gamma}{y^\delta}\right) : [0, \infty) \to \Re$, such that for all $x, t > 0$, the DLTr of the conformable partial FDS (CmDLTr) of $\Psi\left(\frac{x^\gamma}{y^\delta}\right)$ of orders $\gamma$ and $\delta$ where $\gamma, \delta \in (0, 1]$, denoted by $\frac{\partial^\gamma}{\partial x^\gamma} \Psi\left(\frac{x^\gamma}{y^\delta}\right)$ and $\frac{\partial^\delta}{\partial t^\delta} \Psi\left(\frac{x^\gamma}{y^\delta}\right)$, respectively, can be written as

$$\mathcal{E}^\gamma_{\gamma} \left[ \mathcal{E}^\delta_{\delta} \Psi\left(\frac{x^\gamma}{y^\delta}\right) \right] = s_1^\gamma \Psi^{\gamma}_{\gamma}(s_1, s_2) - \sum_{i=0}^{\gamma-1} s_1^{\gamma-i-1} \mathcal{E}^i_{\gamma} \left[ \frac{\partial\Psi(0, t)}{\partial x^i} \right],$$

$$\mathcal{E}^\delta_{\delta} \left[ \mathcal{E}^\gamma_{\gamma} \Psi\left(\frac{x^\gamma}{y^\delta}\right) \right] = s_2^\delta \Psi^{\delta}_{\delta}(s_1, s_2) - \sum_{a=0}^{\delta-1} s_2^{\delta-a} \mathcal{E}^a_{\delta} \left[ \frac{\partial\Psi(x, 0)}{\partial t^a} \right].$$

The first author has proved the existence and uniqueness of CmDLTr in Kaabar,\textsuperscript{11} while the existence and uniqueness of CpDLTr have been discussed in Anwar et al.\textsuperscript{44} It is obvious that the formulas of the DLTr in Definitions 7 and 8 are the same when $j, b = 1$. Therefore, the general definition of CmDLTr coincides with the general definition of CmDLTr when $j, b = 1$. The properties of CmDLTr and CpDLTr have been discussed in Özkan and Ali\textsuperscript{46} and Omran and Kilicman,\textsuperscript{47} respectively. For $\gamma, \delta \in (0, 1]$, let us now define the formula of the inverse fractional DLTr\textsuperscript{38,43} for both conformable and Caputo FDS, denoted by $(\mathcal{E}^\gamma_{\gamma})^{-1} \Psi^{\gamma}_{\gamma}(s_1, s_2)$, as follows:

**Definition 9.** Given an analytic function: $\Psi^{\gamma}_{\gamma}(s_1, s_2)$, for all $s_1, s_2 \in \mathbb{C}$ and for $\gamma, \delta \in (0, 1]$, such that $Re[s_1 \geq \eta]$ and $Re\{s_2 \geq \sigma\}$, where $\eta, \sigma \in \Re$, then the inverse fractional DLTr (IFDLTr) can be expressed\textsuperscript{11} as follows:
To solve the MNLSE in (1) in the senses of CpFD and CD (see Equations 9 and 8) by the methods of CpDLTr and CmDLTr, respectively, let us first rewrite both Equations (9) and (8) as follows:

\[
\begin{aligned}
\frac{\partial^2 \Psi(x, t)}{\partial x^2} &= -\frac{a_2 D_{\gamma}^{2\delta} \Psi(x, t)}{a_3} - i \frac{\partial^2 \Psi(x, t)}{\partial x \partial t^2} - \frac{\partial^2 \Psi(x, t)}{\partial t^2} - \frac{1}{a_3} |\Psi|^2 \Psi, \\
\text{subject to the following initial and boundary conditions:}
\end{aligned}
\]

\[
\Psi(x, 0) = a_0(x) \quad \text{and} \quad \frac{\partial \Psi(x, 0)}{\partial t} = a_1(x),
\]

\[
\Psi(0, t) = b_0(t) \quad \text{and} \quad \frac{\partial \Psi(0, t)}{\partial x} = b_1(t),
\]

where \( i = \sqrt{-1}, 0 < \gamma, \delta \leq 1, t, x > 0; x, t \in \mathbb{R}^+, \) and \( a_0, a_1, b_0, b_1 \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^+). \)

By applying the single Laplace transform to initial and boundary conditions in (16) and (17), respectively, we obtain the following:

\[
\begin{aligned}
\mathcal{L} \left[ \Psi(x, 0) \right] &= \mathcal{L} \left[ a_0(x) \right] = \mathcal{L} \left[ a_0(s_1) \right] = \mathcal{L} \left[ \frac{\partial \Psi(x, 0)}{\partial t} \right] = \mathcal{L} \left[ a_1(s_1) \right], \\
\mathcal{L} \left[ \Psi(0, t) \right] &= \mathcal{L} \left[ b_0(t) \right] = \mathcal{L} \left[ b_0(s_2) \right] = \mathcal{L} \left[ \frac{\partial \Psi(0, t)}{\partial x} \right] = \mathcal{L} \left[ b_1(s_2) \right], \\
\mathcal{L} \left[ \Psi(x, 0) \right] &= \mathcal{L} \left[ a_0(x) \right] = \mathcal{L} \left[ n_0(s_1) \right] = \mathcal{L} \left[ D_x \Psi \left( \frac{x^\ell}{\delta}, 0 \right) \right] = \mathcal{L} \left[ n_1(s_1) \right], \\
\mathcal{L} \left[ \Psi(0, t) \right] &= \mathcal{L} \left[ m_0(t) \right] = \mathcal{L} \left[ m_0(s_2) \right] = \mathcal{L} \left[ D_x \Psi \left( 0, \frac{t^\ell}{\delta} \right) \right] = \mathcal{L} \left[ m_1(s_2) \right].
\end{aligned}
\]

Let us now apply the CpDLTr (Definition 7) to both left-hand and right-hand sides of Equation (16), we obtain

\[
\begin{aligned}
\mathcal{L}^\ell \mathcal{L}^\ell \left[ \frac{\partial^2 \Psi(x, t)}{\partial x^2} \right] &= \mathcal{L}^\ell \mathcal{L}^\ell \left[ \Psi(s_1, s_2) \right] = \frac{\mathcal{L}^\ell \mathcal{L}^\ell \left[ a_0(s_1) \right]}{s_1} + \frac{\mathcal{L}^\ell \mathcal{L}^\ell \left[ a_1(s_1) \right]}{s_1} \\
&= -\frac{1}{s_1^2} \left[ \mathcal{L}^\ell \mathcal{L}^\ell \left[ a_2 \frac{D_{\gamma}^{2\delta} \Psi(x, t)}{a_3} \right] + \mathcal{L}^\ell \mathcal{L}^\ell \left[ i \frac{\partial^2 \Psi(x, t)}{\partial x \partial t^2} \right] + \mathcal{L}^\ell \mathcal{L}^\ell \left[ \frac{\partial^2 \Psi(x, t)}{\partial t^2} \right] + \mathcal{L}^\ell \mathcal{L}^\ell \left[ \frac{1}{a_3} |\Psi|^2 \Psi \right] \right].
\end{aligned}
\]
where $|\Psi|^2\Psi = \Psi^2\Psi^*$ such that $\Psi^*$ is the conjugate of $\Psi$.

From the Adomian decomposition method (ADcM) (see Gündoğdu and Gözükızılı and Nuruddeen et al\textsuperscript{48,49} for more information about ADcM), Equation (20) is written according to the following standard operator form for nonlinear PDEs (NPDEs): $L\Psi(x,t) + R\Psi(x,t) + N\Psi(x,t) = S(x,t)$, where $N$ represents the nonlinear differential operator, $L$ represents the second-order partial differential operator, $R$ represents the remaining linear operator, and $S(x,t)$ represents a source term. This method was first introduced by G. Adomian in 1980s where ADcM is well known for obtaining a series solution whose each term is obtained recursively.\textsuperscript{49} Therefore, by applying the method of CpDLTr coupled with ADcM, the decomposition infinite series can be expressed for both linear and nonlinear terms in Equation (2), respectively, as follows:

$$\Psi(x,t) = \sum_{i=0}^{\infty} \Psi_i(x,t)$$

$$N(\Psi(x,t)) = \sum_{i=0}^{\infty} \phi_i(\Psi(x,t)),$$

where the above nonlinear term, denoted by $N(\Psi(x,t))$, is represented by infinite series of the Adomian polynomials, denoted by $\phi_i$, which can be expressed\textsuperscript{49} as follows:

$$\phi_i = \frac{1}{i!} \frac{d^i}{dt^i} \left[ N\left( \sum_{j=0}^{\infty} \Omega_j \phi_j \right) \right] \bigg|_{\Omega_j=0}, i = 0, 1, 2, 3, \ldots,$$

so we can write some of those terms as follows:

$$\phi_0 = N(\Psi_0);\phi_1 = \Psi_1N'(\Psi_0);\phi_2 = \Psi_2N'(\Psi_0) + \frac{1}{2!} \Psi^2_1N''(\Psi_0).$$

By applying the standard NPDE operator form and (21) to Equation (20), we obtain the following:

$$\ell^x\ell^t \left[ \sum_{i=0}^{\infty} \Psi_i(x,t) \right] = \frac{1}{s_1} \ell^x\ell^t [b_0(t)] + \frac{1}{s_1} \ell^x\ell^t [b_1(t)]$$

$$- \frac{1}{s_1^2} \left[ \ell^x\ell^t \left[ R \left( \sum_{i=0}^{\infty} \Psi_i(x,t) \right) \right] + \ell^x\ell^t \left[ \sum_{i=0}^{\infty} \phi_i \right] \right],$$

where $R[\Psi(x,t)] = \frac{\omega_2}{\omega_3} \frac{\partial^{25}\Psi(x,t)}{\partial t^{25}} + \frac{i}{\omega_3} \frac{\partial^{\omega_2}\Psi(x,t)}{\partial x^{\omega_2}} + \frac{\omega_1}{\omega_3} \frac{\partial^{\omega_3}\Psi(x,t)}{\partial t^{\omega_3}}$

and $\phi_i[\Psi(x,t)] = \frac{1}{\omega_3} \Psi^2_i\Psi^*.$

Let us now write some of the Adomian polynomials, $\phi_i$, using the formula in (22) as follows:

$$\phi_0 = \frac{1}{\omega_3} \Psi^2_0\Psi^*,$$

$$\phi_1 = \frac{2}{\omega_3} \Psi_0\Psi_1\Psi^* + \frac{1}{\omega_3} \Psi^2_0\Psi^*,$$

$$\phi_2 = \frac{2}{\omega_3} \Psi_0\Psi_2\Psi^* + \frac{1}{\omega_3} \Psi^2_1\Psi^* + \frac{2}{\omega_3} \Psi_0\Psi_1\Psi^* + \frac{1}{\omega_3} \Psi^2_1\Psi^*.$$
\[\Psi_0(x, t) = b_0(t) + x b_1(t), \]
\[\Psi_1(x, t) = -\left(\epsilon^\gamma\right)^{-1}(\epsilon^t)^{-1} \left[ \frac{1}{s^1} \left( \epsilon^\gamma \ell \left[ \frac{\omega_2 \partial^{2\gamma} \Psi_0(x, t)}{\omega_3} \frac{\partial}{\partial t^{2\gamma}} + i \frac{\partial}{\partial x^\gamma} \Psi_0(x, t) + i \frac{\omega_3 \partial^{2\gamma} \Psi_0(x, t)}{\omega_3} \right] + \epsilon^\gamma \ell \left[ \psi_0(\Psi(x, t)) \right] \right) \right], \quad (24)\]

Similarly, we can apply the CmDLTr (Definition 8) to both sides of Equation (17), we have

\[\ell^\gamma \ell \left[ D^{2\gamma}_s \Psi \left( \frac{x^\gamma}{\gamma}, \frac{t^\gamma}{\delta} \right) \right] = \Psi(s_1, s_2) = \frac{s^2_2 - 1}{s^2_1} \ell^\gamma \ell \left[ m_0(t) \right] + \frac{s^2_1}{s^2_2} \ell^\gamma \ell \left[ m_1 \left( \frac{t^\gamma}{\delta} \right) \right] \]

\[\ell^\gamma \ell \left[ D^{2\gamma}_s \Psi \left( \frac{x^\gamma}{\gamma}, \frac{t^\gamma}{\delta} \right) \right] = \Psi(s_1, s_2) = \frac{1}{s^2_1} \ell^\gamma \ell \left[ m_0(t) \right] + \frac{1}{s^2_1} \ell^\gamma \ell \left[ m_1 \left( \frac{t^\gamma}{\delta} \right) \right] \]

where \(|\Psi|^2 \Psi = \Psi^2 \Psi^*\) such that \(\Psi^*\) is the conjugate of \(\Psi\).

Let us now apply the standard NPDE operator form and (21) to Equation (25), we have

\[\ell^\gamma \ell \left[ \sum_{i=0}^{\infty} \Psi_i(x, t) \right] = \frac{1}{s^1} \ell^\gamma \ell \left[ m_0(t) \right] + \frac{1}{s^1} \ell^\gamma \ell \left[ m_1 \left( \frac{t^\gamma}{\delta} \right) \right] \]

\[\ell^\gamma \ell \left[ R \left( \sum_{i=0}^{\infty} \Psi_i(x, t) \right) \right] + \ell^\gamma \ell \left[ \sum_{i=0}^{\infty} \psi_i \right] \]

where \(R[\Psi(x, t)] = \left[ \frac{\omega_2}{\omega_3} D^{2\gamma}_s \Psi \left( \frac{x^\gamma}{\gamma}, \frac{t^\gamma}{\delta} \right) + \frac{i}{\omega_3} D^\gamma_x \Psi \left( \frac{x^\gamma}{\gamma}, \frac{t^\gamma}{\delta} \right) + \frac{i \omega_3}{\omega_3} D^\gamma_y \Psi \left( \frac{x^\gamma}{\gamma}, \frac{t^\gamma}{\delta} \right) + \epsilon^\gamma \ell \left[ \psi_0(\Psi(x, t)) \right] \right], \)

and \(\psi_i[\Psi(x, t)] = \frac{1}{\omega_3} \psi^2 \psi^*\).

We apply the inverse DLTr to both sides of Equation (26) to obtain the general solution to Equation (17), recursively as follows:

\[\Psi_0(x, t) = m_0 \left( \frac{t^\gamma}{\gamma} \right) + \frac{x^\gamma}{\gamma} m_1 \left( \frac{t^\gamma}{\delta} \right), \]
\[\Psi_1(x, t) = -\left(\epsilon^\gamma\right)^{-1}(\epsilon^t)^{-1} \left[ \frac{1}{s^1} \left( \epsilon^\gamma \ell \left[ \frac{\omega_2 \partial^{2\gamma} \Psi_0(x, t)}{\omega_3} \frac{\partial}{\partial t^{2\gamma}} + i \frac{\partial}{\partial x^\gamma} \Psi_0(x, t) + i \frac{\omega_3 \partial^{2\gamma} \Psi_0(x, t)}{\omega_3} \right] + \epsilon^\gamma \ell \left[ \psi_0(\Psi(x, t)) \right] \right) \right], \quad (27)\]

\[\Psi_{i+1}(x, t) = -\left(\epsilon^\gamma\right)^{-1}(\epsilon^t)^{-1} \left[ \frac{1}{s^1} \left( \epsilon^\gamma \ell \left[ R[\Psi_i(x, t)] \right] + \epsilon^\gamma \ell \left[ \psi_i(\Psi(x, t)) \right] \right) \right], \quad \text{for } i \geq 0.\]
2.1 Numerical Experiment 1

By applying definitions and properties of CpFD and DLTr, the following numerical experiment will solve Equation (16) analytically: let \( \omega_1 = \omega_2 = \omega_3 = 1 \), and \( b_0(t) = e^\beta; b_1(t) = 0; a_0(x) = a_1(x) = 0 \) in (16), we have

\[
\frac{\partial^2 \Psi(x,t)}{\partial x^2} = -\frac{\partial^2 \Psi(x,t)}{\partial t^2} - i \frac{\partial \Psi(x,t)}{\partial x} - i \frac{\partial \Psi(x,t)}{\partial t} - |\Psi|^2 \Psi,
\]

subject to the following initial and boundary conditions:

\[
\Psi(x,0) = 0 \quad \text{and} \quad \frac{\partial \Psi(x,0)}{\partial t} = 0,
\]

\[
\Psi(0,t) = e^\alpha \quad \text{and} \quad \frac{\partial \Psi(0,t)}{\partial x} = 0,
\]

where \( i = \sqrt{-1} \), \( 0 < \gamma, \delta \leq 1 \), \( t, x > 0; x, t \in \mathbb{R}^+ \).

To solve Equation (28), we use our result in (24) as follows:

\[
\Psi_0(x,t) = e^\alpha,
\]

\[
\Psi_1(x,t) = -(\epsilon^\alpha)^{-1} (e^\alpha)^{-1} \left[ \frac{1}{S^\gamma_1} \left( e^\alpha E \left[ \frac{\partial^2 \Psi_0(x,t)}{\partial t^2} + i \frac{\partial \Psi_0(x,t)}{\partial x} + i \frac{\partial \Psi_0(x,t)}{\partial t} \right] + e^\alpha e^t [\phi_0(\Psi(x,t))] \right) \right]
\]

\[
= -(\epsilon^\alpha)^{-1} (e^\alpha)^{-1} \left[ \frac{1}{S^\gamma_1} \left( e^\alpha E \left[ \frac{\partial^2 \Psi_0(x,t)}{\partial t^2} + i \frac{\partial \Psi_0(x,t)}{\partial x} + i \frac{\partial \Psi_0(x,t)}{\partial t} \right] + e^\alpha e^t [\phi_0^2(\Psi_0)] \right) \right]
\]

\[
= -\frac{x^{2\gamma}}{\Gamma(2\gamma+1)} \left[ [t^{1-2\delta} E_{1,2-2\delta}(it) - t^{1-\delta} E_{1,1-\delta}(it)] + e^\alpha \right],
\]

\[
\Psi_2(x,t) = -(\epsilon^\alpha)^{-1} (e^\alpha)^{-1} \left[ \frac{1}{S^\gamma_1} \left( e^\alpha E \left[ \frac{\partial^2 \Psi_1(x,t)}{\partial t^2} + i \frac{\partial \Psi_1(x,t)}{\partial x} + i \frac{\partial \Psi_1(x,t)}{\partial t} \right] + e^\alpha e^t [\phi_1(\Psi(x,t))] \right) \right]
\]

\[
= -(\epsilon^\alpha)^{-1} (e^\alpha)^{-1} \left[ \frac{1}{S^\gamma_1} \left( e^\alpha E \left[ \frac{\partial^2 \Psi_1(x,t)}{\partial t^2} + i \frac{\partial \Psi_1(x,t)}{\partial x} + i \frac{\partial \Psi_1(x,t)}{\partial t} \right] + e^\alpha e^t [2\Psi_0 \Psi_1^* + \Psi_0^2 \Psi_1^*] \right) \right]
\]

\[
= -\frac{x^{2\gamma}}{\Gamma(2\gamma+1)} \left[ \frac{2 x^{2\gamma}}{\Gamma(2\gamma+1)} \frac{\partial \Psi}{\partial t^2} \left[ \left[ i E(t,1-2\delta,i) - E(t,1-\delta,i) \right] + e^\alpha \right] \right]
\]

\[
-\frac{x^{2\gamma}}{\Gamma(2\gamma+1)} \left[ \frac{\partial}{\partial \alpha} \left[ \left[ i E(t,1-2\delta,i) - E(t,1-\delta,i) \right] + e^\alpha \right] \right]
\]

\[
= \frac{x^{2\gamma}}{\Gamma(4\gamma+1)} \left[ i E(t,1-4\delta,i) - E(t,1-3\delta,i) + i E(t,1-2\delta,i) \right]
\]
By applying the definitions and properties of CD in Silva et al. and Abdeljawad12,16 and DLTr, the following numerical experiment solves Equation (17) analytically: let \( \alpha_1 = \alpha_2 = \alpha_3 = 1 \), and \( m_0(t) = e^{\delta t}; m_1(t) = 0; n_0(x) = n_1(x) = 0 \) in (17), we have

\[
\begin{align*}
D_x^{\alpha_1} \Psi \left( \frac{x'}{\gamma}, \frac{t'}{\delta} \right) &= -D_t^{\alpha_2} \Psi \left( \frac{x'}{\gamma}, \frac{t'}{\delta} \right) - iD_x^{\alpha_3} \Psi \left( \frac{x'}{\gamma}, \frac{t'}{\delta} \right) - iD_t^{\alpha_4} \Psi \left( \frac{x'}{\gamma}, \frac{t'}{\delta} \right) - |\Psi|^2 \Psi, \\
\text{subject to the following initial and boundary conditions:} & \quad \Psi \left( \frac{x'}{\gamma}, 0 \right) = 0 \text{ and } D_t \Psi \left( \frac{x'}{\gamma}, 0 \right) = 0,
\end{align*}
\]

\[
\Psi \left( 0, \frac{t'}{\delta} \right) = e^{\delta t} \text{ and } D_x \Psi \left( 0, \frac{t'}{\delta} \right) = 0,
\]

where \( i = \sqrt{-1}, 0 < \gamma, \delta \leq 1, t, x > 0; x, t \in \mathbb{R}^+ \).

To solve Equation (30), we use our result in (27) as follows:

By using all above obtained results, the general approximate analytical solution to Equation (28) can be written as follows:

\[
\Psi(x,t) = e^{\delta t} - \frac{x^{2\gamma}}{\Gamma(2\gamma+1)} \left[ |E(t,1-2\delta,i) - E(t,1-\delta,i)| + e^{\delta t} \right] + \frac{x^{2\gamma}}{\Gamma(3\gamma+1)} \left[ iE(t,1-3\delta,i) - E(t,1-2\delta,i) + iE(t,1-\delta,i) - E(t,1-\delta,i) \right] + \ldots
\]

Hence, the approximate analytical solution for the MNLSE in (1) in the sense of CpFD has been easily obtained via the DLTr coupled with the ADcM.
FIGURE 1 3D plot of the real part (A) with its contour plot (B) and imaginary part (C) with its contour plot (D) of the approximate analytical solution in (29) for $\gamma = \delta = 0.25$ [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 2 3D plot of the real part (A) with its contour plot (B) and imaginary part (C) with its contour plot (D) of the approximate analytical solution in (29) for $\gamma = \delta = 0.75$ [Colour figure can be viewed at wileyonlinelibrary.com]
FIGURE 3 3D plot of the real part (A) with its contour plot (B) and imaginary part (C) with its contour plot (D) of the approximate analytical solution in (29) for $\gamma = 0.50$ and $\delta = 0.85$ [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 4 3D plot of the real part (A) with its contour plot (B) and imaginary part (C) with its contour plot (D) of the approximate analytical solution in (29) for $\gamma = 0.75$ and $\delta = 0.85$ [Colour figure can be viewed at wileyonlinelibrary.com]
FIGURE 5 3D plot of the real part (A) with its contour plot (B) and imaginary part (C) with its contour plot (D) of the approximate analytical solution in (29) for $\gamma = \delta = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 6 3D plot of the real part (A) with its contour plot (B) and imaginary part (C) with its contour plot (D) of the approximate analytical solution in (31) for $\gamma = \delta = 0.25$ [Colour figure can be viewed at wileyonlinelibrary.com]
FIGURE 7  3D plot of the real part (A) with its contour plot (B) and imaginary part (C) with its contour plot (D) of the approximate analytical solution in (31) for $\gamma = \delta = 0.75$ [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 8  3D plot of the real part (A) with its contour plot (B) and imaginary part (C) with its contour plot (D) of the approximate analytical solution in (31) for $\gamma = 0.50$ and $\delta = 0.85$ [Colour figure can be viewed at wileyonlinelibrary.com]
FIGURE 9 3D plot of the real part (A) with its contour plot (B) and imaginary part (C) with its contour plot (D) of the approximate analytical solution in (31) for $\gamma = 0.75$ and $\delta = 0.85$ [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 10 3D plot of the real part (A) with its contour plot (B) and imaginary part (C) with its contour plot (D) of the approximate analytical solution in (31) for $\gamma = \delta = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]
\[ \Psi_0(x,t) = e^{it}, \]
\[ \Psi_1(x,t) = - (e^{it})^{-1} (e^{it})^{-1} \]
\[ \times \left[ \frac{1}{t^2} \left[ e^{it} \left[ D_{x}^{2} \Psi_0 \left( \frac{x}{\gamma}, \frac{t}{\delta} \right) + iD_{x}^{\delta} \Psi_0 \left( \frac{x}{\gamma}, \frac{t}{\delta} \right) + iD_{x}^{1} \Psi_0 \left( \frac{x}{\gamma}, \frac{t}{\delta} \right) \right] + e^{it} \left[ \phi_0(\Psi(x,t)) \right] \right] \right] \]
\[ = - (e^{it})^{-1} (e^{it})^{-1} \]
\[ \times \left[ \frac{1}{t^2} \left[ e^{it} \left[ D_{x}^{2} \Psi_0 \left( \frac{x}{\gamma}, \frac{t}{\delta} \right) + iD_{x}^{\delta} \Psi_0 \left( \frac{x}{\gamma}, \frac{t}{\delta} \right) + iD_{x}^{1} \Psi_0 \left( \frac{x}{\gamma}, \frac{t}{\delta} \right) \right] + e^{it} \left[ \phi_0(\Psi(x,t)) \right] \right] \right] \]
\[ = - \frac{x^{2y-1}}{\gamma^{2y-1} \Gamma(2\gamma)} \left[ -e^{it} - e^{-it} \right] \]
\[ \Psi_2(x,t) = - (e^{it})^{-1} (e^{it})^{-1} \]
\[ \times \left[ \frac{1}{t^2} \left[ e^{it} \left[ D_{x}^{2} \Psi_1 \left( \frac{x}{\gamma}, \frac{t}{\delta} \right) + iD_{x}^{\delta} \Psi_1 \left( \frac{x}{\gamma}, \frac{t}{\delta} \right) + iD_{x}^{1} \Psi_1 \left( \frac{x}{\gamma}, \frac{t}{\delta} \right) \right] + e^{it} \left[ \phi_1(\Psi(x,t)) \right] \right] \right] \]
\[ = - \frac{x^{2y-1}}{\gamma^{2y-1} \Gamma(2\gamma)} \left( \frac{x^{2y-1}}{\gamma^{2y-1} \Gamma(2\gamma)} e^{it} + \frac{x^{2y-2}}{\gamma^{2y-2} \Gamma(2\gamma)} e^{-it} \right) \]
\[ \times \left[ \frac{2x^{2y-1}}{\gamma^{2y-1} \Gamma(2\gamma)} e^{-it} + \frac{x^{2y-1}}{\gamma^{2y-1} \Gamma(2\gamma)} e^{it} \right] \]
\[ = \frac{x^{4y-2}}{\gamma^{4y-2} \Gamma(4\gamma)} e^{it} - \frac{i(2\gamma - 1)x^{3y-2}}{\gamma^{4y-2} \Gamma(4\gamma)} e^{it} + \frac{x^{4y-2}}{\gamma^{4y-2} \Gamma(4\gamma)} e^{-it} - \frac{2x^{4y-2}}{\gamma^{4y-2} \Gamma(4\gamma)} e^{-it} \]
\[ = -i \frac{(2\gamma - 1)x^{3y-2}}{\gamma^{4y-2} \Gamma(4\gamma)} e^{it} + \frac{x^{4y-2}}{\gamma^{4y-2} \Gamma(4\gamma)} e^{it} + 2e^{-it} - e^{-it} \]
\[ = -i \frac{(2\gamma - 1)x^{3y-2}}{\gamma^{4y-2} \Gamma(4\gamma)} e^{it} + \frac{x^{4y-2}}{\gamma^{4y-2} \Gamma(4\gamma)} e^{it} + 2e^{-it}, \]
and so on.

By using the above obtained results, the general approximate analytical solution to Equation (30) can be written as follows:

\[ \Psi(x,t) = e^{it} - \frac{x^{2y-1}}{\gamma^{2y-1} \Gamma(2\gamma)} \left[ -e^{it} - e^{-it} \right] - \frac{i(2\gamma - 1)x^{3y-2}}{\gamma^{4y-2} \Gamma(4\gamma)} e^{it} + \frac{x^{4y-2}}{\gamma^{4y-2} \Gamma(4\gamma)} e^{it} + 2e^{-it} \]
\[ + \ldots \quad (31) \]

Hence, the approximate analytical solution for the MNLSE in (1) in the sense of CD has also been easily obtained via the DLTr coupled with the ADcM.

## 3 The Graphical Comparisons of Solutions

In this section, the obtained approximate solutions in both (29) and (31) have been graphically compared for various values of \( \gamma \) and \( \delta \) (see Figures 1–10) where each graph shows both real and imaginary parts of solution. In addition, Table 1 provides a comparison of absolute approximate solutions from CpdLTr in (29) and from CmDLTr in (31).
at $\gamma = \delta = 0.25$, $\gamma = \delta = 0.75$, and $\gamma = \delta = 1$ with the exact solution from Example 9 in Hamed et al.\textsuperscript{36} According to Table 1, at $x = t = 0.1$, $x = t = 0.3$, and $x = t = 0.5$, the absolute error value from exact and the approximate solution from CpdDLTr is less than the absolute error value from exact and the approximate solution from CmDLTr. At $x = t = 0.7$, for $\gamma = \delta = 0.25$ and 0.75, the absolute error value from exact and the approximate solution from CmDLTr is less than the absolute error value from exact and the approximate solution from CpdDLTr, while at $x = t = 0.7$ for $\gamma = \delta = 1$, the approximate solution from CpdDLTr converges to the exact solution better than the one from CmDLTr. Similarly, at $x = t = 0.9$ for $\gamma = \delta = 1$, the approximate solution from CpdDLTr converges to the exact solution better than the one from CmDLTr. Therefore, the obtained approximate solution in the sense of Caputo FDs is much better than the obtained solution in the sense of CDs. On one hand, the Caputo FD is a nonlocal fractional operator, which provides a good interpretation to the physical behavior of systems, while the CD is a type of local fractional derivative, which is basically a generalized form of usual limit-based derivative that lacks some of the important properties to be classified as a fractional derivative. As a result, solving systems of NPDEs in the sense of Caputo FDs is highly recommended. However, exploring the definition of CD is also interesting because as the authors believe that any new mathematical definition deserves to be explored and investigated.

4 | CONCLUSION

Nonlinear Schrödinger equation has been an interesting field of research for many mathematicians and scientists due to the important applications of this equation in physics and engineering. This research study provides a powerful mathematical tool to solve the nonlinear Schrödinger equation involving both Caputo FDs and CD. Therefore, the generalized DLTr method can be efficiently applied in solving nonlinear fractional Schrödinger equation and all other nonlinear fractional PDEs.

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CONFLICT OF INTERESTS

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