The IIA, IIB and eleven dimensional theories and their common $E_{11}$ origin.

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Abstract

We show that the commonly considered half BPS solutions of eleven dimensional supergravity and the ten dimensional type II theories, when expressed in terms of $E_{11}$ group elements, take the universal form $\exp(-\frac{1}{2} \ln N \beta \cdot H) \exp((1 - N)E_\beta)$. Using this formula we find new potential solutions to the $E_{11}$ non-linearly realisations corresponding to active fields which are beyond those in the supergravity approximations. These include the space filling nine brane of the IIB theory. We use $E_{11}$ to give a correspondence between the fields of the eleven dimensional and the IIA and IIB non-linear realisations without assuming any dimensional reduction. As one consequence, we find the eleven dimensional origin of the eight brane solution of the massive IIA theory.
1. Introduction

It is a consequence of supersymmetry that the scalars in supergravity multiplets belong to non-linear realisations. The first such example was in the four dimensional $N = 4$ supergravity theory [1] and perhaps the most celebrated example concerns the four dimensional maximal supergravity where the scalars belong to a non-linear realisation of $E_7$ [2]. A detailed account of the literature on such symmetries can be found in the introduction of reference [14]. The eleven dimensional supergravity theory does not possess any scalars and it was widely believed that these symmetry algebras were not present in this theory. However, it was found that the bosonic sector eleven dimensional supergravity theory could be formulated as a non-linear realisation [3]. The infinite dimensional algebra involved in this construction was the closure of a finite dimensional algebra, denoted $G_{11}$, with the eleven dimensional conformal algebra. The non-linear realisation was carried out by ensuring that the equations of motion were invariant under both finite dimensional algebras, taking into account that some of their generators were in common. The algebra $G_{11}$ involved the space-time translations together with an algebra $\hat{G}_{11}$ which contained $A_{10}$ and the Borel subalgebra of $E_7$ as subalgebras. The algebra $\hat{G}_{11}$ was not a Kac-Moody algebra, however, it was conjectured [4] that the theory could be extended so that the algebra $\hat{G}_{11}$ was promoted to a Kac-Moody algebra. It was shown that this Kac-Moody symmetry would have to contain a certain rank eleven Kac-Moody algebra denoted $E_{11}$ [4].

Consequently, it was argued [4] that an extension of eleven dimensional supergravity should possess an $E_{11}$ symmetry that was non-linearly realised. In particular, the symmetries found when the eleven dimensional supergravity theory was dimensionally reduced would be present in this eleven dimensional theory. One of the advantages of a non-linear realisation is that the dynamics is largely specified by the algebra if the chosen local subalgebra is sufficiently large. This was not the case for the $G_{11}$ considered in [3] when taken in isolation as the local subalgebra was chosen to be just the Lorentz algebra, but it is the case for $E_{11}$ with the local subalgebra that was specified in [4]. The role of space-time and the relation to the later work of [6] are discussed in the very recent paper [9].

A similar picture emerged for the IIA and IIB supergravity theories in ten dimensions and it was conjectured that these theories could be extended such that they were invariant under $E_{11}$ [4,5]. As a result, it became clear that the two different type II theories in ten dimensions arise by taking different $A_9$ subalgebras of $E_{11}$ to correspond to the gravity sector of the theory. [4,5].

Arguments similar to those advocated for eleven dimensional supergravity in [4] were proposed to apply to gravity [12] in D dimensions and the effective action of the closed bosonic string [4] generalised to D dimensions and the underlying Kac-Moody algebras were identified. It was realised that the algebras that arose in all these theories were of a special kind and were called very extended Kac-Moody algebras [13]. Indeed, for any finite dimensional semi-simple Lie algebra $\mathcal{G}$ one can systematically extend its Dynkin diagram by adding three more nodes to obtain an indefinite Kac-Moody algebra denoted $\mathcal{G}^{+++}$. In this notation $E_{11}$ is written as $E_8^{+++}$. The algebras for gravity and the closed bosonic string being $A_{D-3}^{+++}$ [12] and $D_{D-2}^{+++}$ [4] respectively.

It was proposed in [4,12,13] and [14,15,16,10], that the non-linear realisation of any very extended algebra $\mathcal{G}^{+++}$ leads to a theory, called $\mathcal{V}_\mathcal{G}$ in [16], that at low levels includes
gravity and the other fields and it was hoped that this non-linear realisation contains an infinite number of propagating fields that ensures its consistency. Indeed, it was shown [16] that the low level content of the adjoint representation of $G^{+++}$ predicted a field content for a non-linear realisation of $G^{+++}$ which was in agreement with the oxidation theory associated with algebra $G$.

Some papers have uncovered relationships between the solutions in the oxidised theories and the $G^{+++}$ symmetry conjectured to be present in their extension. In reference [14], the non-linear realisation of $G^{+++}$ restricted to its Cartan subalgebra was constructed and the resulting Weyl transformations were shown to transform the moduli of the Kasner solutions into each other. Furthermore, for $E_8^{+++} = E_{11}$ and $D_2^{+++}$ these Weyl transformations were shown to be the U duality transformations in the corresponding string theories. Furthermore in [15] it was shown that the theories associated with $G^{+++}$ at low levels, i.e. the corresponding oxidised theories, admit BPS intersecting solutions. The results of [14,15] were generalised in [10] to an alternative theory; as in [4] the theory was assumed to be a non-linear realisation of $E_{11}$, but the fields were assumed to depend on an auxiliary parameter. The symmetries of the eleven dimensional theory reduced on spheres solutions was discussed from the $E_8^{+++}$ perspective in [20].

The aim of the work of [4] was to deduce the underlying symmetries of M theory in the hope that they would provide a understanding of what M theory actually is. However, as we will see in this paper, the presence of such symmetries may be useful to elucidate some unresolved questions about the relations between the eleven dimensional theory and the IIA and IIB theories and the branes that occur in them. As such, we restrict our attention to the case of $E_8^{+++}$ and investigate the relationship between the solutions of eleven dimensional supergravity and IIA and IIB supergravities and their underlying $E_8^{+++}$ symmetry. One advantage of a non-linear realisation is that the field content, and in essence the dynamics, is determined just by group theory. In particular, the fields of the non-linear realisation $V_{E_8}$ are contained in the group element of $E_8^{+++}$ when modded out by the action of the local subgroup. Consequently, given any solution we can write down the corresponding $E_8^{+++}$ group element. In sections three and four we carry out this process for the most common half BPS solutions of eleven dimensional supergravity [32] and the IIA [33] and IIB [34] supergravity theories in ten dimensional and find that the corresponding group element has a particularly elegant form, namely

$$g = \exp\left(-\frac{1}{2}\ln N\beta \cdot H\right)\exp((1 - N)E_\beta)$$

(1.1)

where $N$ is a harmonic function and $\beta$ is the root of $E_8^{+++}$ whose corresponding generator is $E_\beta$.

In section five, we assume that equation (1.1) also leads to solutions of the non-linearly realised theory $V_{E_8^{+++}}$ and derive such solutions for roots which correspond to fields that are beyond the supergravity theory. In particular, we find the space filling nine brane of the IIB theory. In section six, we exploit the fact that the eleven dimensional theory and the IIA and IIB theories have a common $E_8^{+++}$ origin. In particular, we use this to systematically find relations between the fields and coordinates of the three theories. Finally, in section seven we discuss the consequences of this work.
2. Kac-Moody algebras and their non-linear realisations

In this section we recall some of the basic properties of Kac-Moody algebras \[18\] and their non-linear realisations. We will illustrate the general discussion for the case of \(E_8^{+++}\).

A Kac-Moody algebra is defined by its Cartan matrix \(A_{ab}\) which by definition satisfies the following properties:

\[
A_{aa} = 2, \tag{2.1}
\]
\[
A_{ab} \text{ for } a \neq b \text{ are negative integers or zero,} \tag{2.2}
\]
and
\[
A_{ab} = 0 \text{ implies } A_{ba} = 0. \tag{2.3}
\]

The Kac-Moody algebra is formulated in terms of its Chevalley generators which consist of the generators of the commuting Cartan subalgebra, denoted by \(H_a\), as well as the generators of the positive and negative simple roots, denoted by \(E_a\) and \(F_a\) respectively. The Chevalley generators are taken to obey

\[
[H_a, H_b] = 0, \tag{2.4}
\]
\[
[H_a, E_b] = A_{ab} E_b, \quad [H_a, F_b] = -A_{ab} F_b, \tag{2.5}
\]
\[
[E_a, F_b] = \delta_{ab} H_a, \tag{2.6}
\]

as well as the Serre relation

\[
[E_a, \ldots [E_a, E_b] \ldots] = 0, \quad [F_a, \ldots [F_a, F_b] \ldots] = 0 \tag{2.7}
\]

In equation (1.7) there are \(1 - A_{ab}\) number of \(E_a\)’s in the first equation and the same number of \(F_a\)’s in the second equation. Given the generalised Cartan matrix \(A_{ab}\), one can uniquely reconstruct the entire Kac-Moody algebra by taking the multiple commutators of the simple root generators subject to the above Serre relations. In particular one can find, at least as a matter of principle, the generators and roots of the Kac-Moody algebra.

The Cartan matrix is given in terms of the simple roots \(\alpha_a\) by

\[
A_{ab} = 2 \frac{(\alpha_a, \alpha_b)}{(\alpha_a, \alpha_a)} \tag{2.8}
\]

In the Cartan-Weyl basis, the Kac-Moody algebra is generated by \(H_i\) and \(E_a\) and \(F_a\) where \(H_i\) and \(H_a\) are related by \(H_a = 2 \frac{\alpha_a^i H_i}{(\alpha_a, \alpha_a)}\) where \(\alpha_a^i\) are the components of the simple root \(\alpha_a\). The commutator of \(H_i\) with the generators \(E_a\) and \(F_a\) is given by

\[
[H_i, E_a] = \alpha_a^i E_a, \quad [H_i, F_a] = -\alpha_a^i F_a. \tag{2.9}
\]

It follows that a generator \(E_\alpha\) associated with root \(\alpha\) obeys the commutator

\[
[H_i, E_\alpha] = \alpha^i E_\alpha \tag{2.10}
\]
and as a result

\[ [H_c, E_\alpha] = 2 \frac{(\alpha_c, \alpha)}{\alpha_c} E_\alpha \]  

(2.11)

A result that is obvious with out ever leaving the original Chevalley basis by considering \( E_\alpha \) as a multiple commutator of simple roots and then taking its commutator with \( H_c \).

Associated with the generator \( E_\alpha \) we can associate in a natural way an element of the Cartan subalgebra which is given by

\[ \alpha^i H_i \]  

(2.12)

It is the Borel sub-algebra of this \( A_1 \) which will play such a central role in this paper.

It is well known, and easy to check, that the above Serre relations, and so the Kac-Moody algebra, are invariant under the involution which acts on the Chevalley generators as

\[ E_a \to \eta_a F_a, \quad F_a \to \eta_a E_a, \quad H_a \to -H_a \]  

(2.13)

where \( \eta_a = \mp 1 \) for any positive simple root.

The so called Cartan involution invariant subalgebra is that given by taking all minus signs. For any combination of signs \( \eta_a \) one can consider the corresponding invariant generators which generate a sub-algebra.

When \( \det A_{ab} > 0 \) the Kac-Moody algebra is one of the finite dimensional semi-simple Lie algebras classified by Cartan. When \( \det A_{ab} = 0 \) we find the well known affine Lie algebras. However, when \( \det A_{ab} < 0 \) very little is known about these algebras. Indeed, apart from a few exceptional cases, an explicit formulation of the generators, or even their number, is known.

As we explained in the introduction the Kac-Moody algebras that are of most interest are the very extended Kac-Moody algebras [13]. Given any finite-dimensional simple Lie algebra \( \mathcal{G} \), there is a well-known procedure for constructing a corresponding affine algebra \( \mathcal{G}^+ \) by adding a node to the Dynkin diagram in a certain way which is related to the properties of the highest root of \( \mathcal{G} \). One may also further increase by one the rank of the algebra \( \mathcal{G}^+ \) by adding to the Dynkin diagram a further node that is attached to the affine node by a single line [21]. This is called the overextension \( \mathcal{G}^{++} \). The very extension, denoted \( \mathcal{G}^{+++} \), is found by adding yet another node to the Dynkin diagram that is attached to the overextended node by one line [13]. The rank of very extended exceeds by three the rank of the finite-dimensional simple Lie algebra \( \mathcal{G} \) from which one started.

The \( E_{11} \), or \( E_{8}^{++} \), algebra contains the generators \( K^{a,b} \) at level 0, corresponding to the \( A_{10} \) subalgebra, and the generators

\[ R^{a_1 a_2 a_3}, R^{a_1 a_2 \ldots a_6}, R^{a_1 a_2 \ldots a_8, b} \]  

(2.14)

at levels zero, 1, 2 and 3 respectively [4]. The generators of \( E_{11} \) at higher levels are listed in references [17, 11].

The corresponding Borel sub-algebra up to, and including, level 3 obeys the commutation relations [4]

\[ [K^{a,b}, K^c] = \delta_b^c K^{a,d} - \delta_d^a K^{c,b}, \]  

(2.16)

\[ [K^{a,b}, R^{c_1 \ldots c_6}] = \delta_{c_1}^b R^{a_{c_2 \ldots c_6} + \ldots}, \quad [K^{a,b}, R^{c_1 \ldots c_3}] = \delta_{c_1}^b R^{a_{c_2 c_3} + \ldots}, \]  

(2.17)
\[
[K^a_{\ b}, R^{c_1\ldots c_8\ d}] = (\delta^c_b R^{ac_2\ldots c_8\ d} + \cdots) + \delta^d_b R^{c_1\ldots c_8\ a}, \tag{2.18}
\]
\[
[R^{c_1\ldots c_3}, R^{c_4\ldots c_6}] = 2R^{c_1\ldots c_6}, \quad [R^{a_1\ldots a_6}, R^{b_1\ldots b_3}] = 3R^{a_1\ldots a_6[b_1 b_2 b_3]}, \tag{2.19}
\]
where \(+\ldots\) means the appropriate anti-symmetrisation. The above commutators can be deduced, using the Serre relations and from the identification of the Chevalley generators of \(E_{11}\) which are given by [4]
\[
E_a = K^a_{\ a+1}, a = 1, \ldots, 10, \quad E_{11} = R^{91011}, \tag{2.20}
\]
and
\[
H_a = K^a_{\ a} - K^a_{\ a+1}, a = 1, \ldots, 10, \quad H_{11} = -\frac{1}{3}(K^1_{\ 1} + \cdots + K^8_{\ 8}) + \frac{2}{3}(K^9_{\ 9} + K^{10}_{\ 10} + K^{11}_{\ 11}). \tag{2.21}
\]
The commutators involving the analogous negative level generators are given in [35].

Non-linear realisation of a algebra \(G\) with respect to a subalgebra \(H\) is just a theory which is built from the group elements \(g\) of \(G\) such that it is invariant under the symmetry
\[
g \rightarrow g_0 g h \tag{2.22}
\]
where \(g_0\) are constant group elements and \(h\) are depend on the same variables as \(g\). On of the advantages of a non-linear realisation is that the dynamics is largely specified by the algebra if the local subgroup \(H\) is sufficiently large. In [4] the local subalgebra was chosen to be the so called Cartan involution invariant subalgebra. Although one can use this, it requires a Wick rotation to get to get a space-time with a signature with one minus sign. The idea of different signs was discussed in [10] and the choice \(\eta_1 = +1\), all the remaining \(\eta\)'s all negative was advocated in [10,9] to get a space-time with a signature with one minus sign directly. However, as observed in equation (2.13), we find that one can take several possible choices of signs for the \(\eta\)'s resulting in theories with different space-time signatures. This idea has also occurred to the author of the very recent paper [19] who has explored its consequences in detail. Since one is interested in equations which are, after possible eliminations, second order in derivatives, we believe that the dynamics is essentially determined if any of the above local subalgebra is chosen.

For the case of \(E_{8+++}\), the local sub-algebra was used in reference [3,4] to express the group element in the form
\[
g = \exp(\sum_{a \leq b} \hat{h}^a_{\ b} K^a_{\ b}) \exp(\frac{A_{c_1\ldots c_3} R^{c_1\ldots c_3}}{3!}) \exp(\frac{A_{c_1\ldots c_6} R^{c_1\ldots c_6}}{6!}) \exp(\frac{h_{c_1\ldots c_8\ d} R^{c_1\ldots c_8\ d}}{8!}) \cdots, \tag{2.23}
\]
where the fields \(\hat{h}^a_{\ b}, A_{c_1\ldots c_3}\) and \(A_{c_1\ldots c_6}\) depend on \(x^\mu\).

The fields \(\hat{h}^a_{\ b}\) encode the gravitational degrees of freedom and are related to the vierbein by the equation [3]
\[
e^a_{\mu} = (e^{\hat{h}})^a_{\mu} \tag{2.24}
\]
From the non-linear realisation of [3,4], we can identify \(A_{c_1\ldots c_3}\) and \(A_{c_1\ldots c_6}\) as the fields of the three form and its dual of eleven dimensional supergravity. The field \(h_{c_1\ldots c_8\ d}\) plays the
role of the dual field of gravity [4]. It is important for what follows to realise that the gauge fields that occur in the group element are referred to the tangent space. Given a solution it is then straightforward using the above relations to construct the corresponding \(E_{11}\) group element. Although one can choose whatever parameterisation of the group element one likes, the brane solution will take on a more universal form if we treat the off-diagonal components of \(h_{a}^{b}\) in the same way as the gauge fields and adopt the parameterisation
\[
g = g_{h}g_{A} \tag{2.25}
\]
where
\[
g_{h} = \exp\left(\sum_{a} h_{a}^{a}K_{a}\right)\exp\left(\sum_{a < b} h_{a}^{b}K_{a}^{b}\right) \tag{2.26}
\]
and
\[
g_{A} = \exp(A_{c_{1}...c_{3}}\frac{R^{c_{1}...c_{3}}}{3!})\exp(A_{c_{1}...c_{6}}\frac{R^{c_{1}...c_{6}}}{6!})\exp(h_{c_{1}...c_{8},d}\frac{R^{c_{1}...c_{8},d}}{8!}) \ldots, \tag{2.27}
\]
The vierbein is then given by
\[
e_{\mu}^{a} = e^{h_{a}}(e^{\tilde{h}})_{\mu}^{a} \tag{2.28}
\]
where \(\tilde{h}_{a}^{b} = h_{a}^{b} - h_{a}^{a}\delta_{a}^{b}\) is the off-diagonal part of \(h_{a}^{b}\). This new parameterisation will only affect solutions with an off diagonal component to the metric such as the pp-wave.

Although we have reviewed the construction of the non-linear realisation for \(E_{8}^{++}+\) it is straightforward to generalise to any \(\mathcal{G}^{+++}\) once one has identified the preferred \(SL(D)\) sub-algebra associated with gravity.

### 3. The half BPS solutions of eleven dimensional supergravity

In this section we will cast half BPS solutions of eleven dimensional supergravity as \(E_{11}\) group elements. We show that they have a very special form given in equation (1.1). We begin by giving a detailed discussion of the case of the M2 brane.

#### 3.1 The M2 brane

This solution [22] has a line element given by
\[
ds^{2} = N_{2}^{-\frac{4}{3}}(-(dx_{1})^{2} + (dx_{2})^{2} + (dx_{3})^{2}) + N_{2}^{\frac{1}{3}}((dx_{4})^{2} + \ldots + (dx_{11})^{2}), \tag{3.1}
\]

Together with a four form field strength
\[
F_{123i} = \pm \frac{\partial}{\partial x_{i}}N_{2}^{-1}, \quad i = 4, \ldots, 11. \tag{3.2}
\]

In these equations \(N_{2}\) is a harmonic function of the form \(N_{2} = 1 + \frac{k}{r^{2}}\) where \(r^{2} = (x_{4})^{2} + \ldots + (x_{11})^{2}\).

Using equation (2.28), we conclude that
\[
(e^{h})_{1}^{1} = (e^{h})_{2}^{2} = (e^{h})_{3}^{3} = N_{2}^{-\frac{4}{3}}, \quad (e^{h})_{4}^{4} = \ldots = (e^{h})_{11}^{11} = N_{2}^{\frac{1}{3}} \tag{3.3}
\]
while, taking the upper sign, the three form gauge field is given by

\[ A_{123} = N_2^{-1} - 1, \quad A_{123}^T = 1 - N_2. \]  

In this equation the superscript T makes it clear that we are dealing with the gauge field expressed with respect to the tangent space. We do not use this symbol if the nature of the indices already makes it clear we are dealing with a quantity referred to the tangent space, i.e. \( A_{a_1a_2a_3} \).

Substituting these values into the \( E_8^{+++} \) group element of equation (2.25) we find that the M2 brane corresponds to

\[ g = \exp \left( -\frac{1}{2} \ln N_2 \left( \frac{2}{3} (K^1_1 + K^2_2 + K^3_3) - \frac{1}{3} (K^4_4 + \ldots + K^{11}_{11}) \right) \right) \exp \left( (1 - N_2) R^{123} \right) \]  

Examining equations (2.20) and (2.21) we recognise that the first factor contains \( H_{11} \) and the second factor \( E_{11} \) if we were to shift the indices by +8 mod 8. We can change \( E_{11} \) to \( R_{123} \) by taking the commutator of the former with the following elements

\[ K^1_2, \ 2K^2_2, \ 3K^3_3, \ldots, \ 3K^8_9, \ 2K^9_{10}, \ K^{10}_{11} \]  

in an appropriate order. Using equation (2.20) we find that the root corresponding to \( R^{123} \) is given by

\[ \beta_{M2} = \alpha_{11} + \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + 3\alpha_8 + 2\alpha_9 + \alpha_{10} \]  

The Cartan sub-algebra element in \( E_8^{+++} \) corresponding to \( R^{123} \) is

\[ \beta_{M2}^2 H_i = H_{11} + H_1 + 2H_2 + 3H_3 + \ldots + 3H_9 + 2H_9 + H_{10} \]

\[ = \frac{2}{3} (K^1_1 + K^2_2 + K^3_3) - \frac{1}{3} (K^4_4 + \ldots + K^{11}_{11}) \]  

Hence, we find that the M2 brane corresponds to the \( E_8^{+++} \) group element

\[ g = \exp \left( -\frac{1}{2} \beta_{M2} \cdot H \ln N_2 \right) \exp \left( (1 - N_2) E_{\beta_{M2}} \right) \]  

3.2 The M5 brane

This solution [23] has the line element

\[ ds^2 = N_5^{-\frac{2}{3}} \left( -(dx_1)^2 + (dx_2)^2 + \ldots + (dx_6)^2 \right) + N_5^{\frac{2}{3}} \left( (dx_7)^2 + \ldots + (dx_{11})^2 \right). \]  

where \( N_5 = 1 + \frac{k}{r^2} \) and \( r^2 = (x_7)^2 + \ldots + (x_{11})^2 \). While the four form field strength is given by

\[ F_{ijklm} = \pm \epsilon_{ijklm} \frac{\partial}{\partial x^n} N_5, \ \ i, j, k, l, m = 7, \ldots, 11 \]
From the viewpoint of this field strength the M5 is a magnetic brane, but in this paper we view all branes as electric branes and so consider it as arising from the dual gauge field $A_{\mu_1...\mu_6}$ whose dual field strength in tangent space is given by

$$F_{c_1...c_7} = -\frac{1}{4!} \epsilon_{c_1...c_7b_1...b_4} F^{b_1...b_4} \tag{3.12}$$

Taking the lower sign in equation (3.11), we find that $F_{1...6k} = \partial_k N_5^{-1}$ and so

$$A_{1...6}^T = 1 - N_5 \tag{3.13}$$

The corresponding $E_{++}^+++$ group element is given by

$$g = \exp\left( -\frac{1}{2} \ln N_5 \left( \frac{1}{3} (K^1_1 + \ldots + K^6_6) - \frac{2}{3} (K^7_7 + \ldots + K^{11}_{11}) \right) \right) \exp((1 - N_5) R^{1...6}) \tag{3.14}$$

We now wish to express the above group element in a more elegant form. Equation (2.19) expresses $R^{6...11}$ as the commutator of $R^{678}$ with $E_{11} = R^{01011}$ and, taking into account the action of $K^{a \beta}$’s to change the former generator from $E_{11}$, we find the root corresponding to $R^{6...11}$ is $2\alpha_1 + \alpha_6 + 2\alpha_7 + 3\alpha_8 + 2\alpha_9 + \alpha_{10}$. However, the gauge field that occurs in the solution corresponds to $R^{1...6}$ and its associated root $\beta_{M5}$ is found to be

$$\beta_{M5} = (2\alpha_1 + \alpha_6 + 2\alpha_7 + 3\alpha_8 + 2\alpha_9 + \alpha_{10})$$

$$+ (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 5\alpha_6 + 4\alpha_7 + 3\alpha_8 + 2\alpha_9 + \alpha_{10}) \tag{3.15}$$

Using equation (2.21) we find that

$$\beta_{M5} \cdot H = \frac{1}{3} (K^1_1 + \ldots + K^6_6) - \frac{2}{3} (K^7_7 + \ldots + K^{11}_{11}) \tag{3.16}$$

It is then obvious that the group element corresponding to the M5 brane of equation (3.13) can be written as

$$g = \exp\left( -\frac{1}{2} \beta_{M5} \cdot H \ln N_5 \right) \exp((1 - N_5) E_{\beta_{M5}}) \tag{3.17}$$

### 3.3 The M Wave

The pp wave solution is given by [24]

$$ds^2 = -(1 - K)(dx_1)^2 + (1 + K)(dx_2)^2 - 2Kdx_1dx_2 + ((dx_3)^2 + \ldots + (dx_{11})^2) \tag{3.18}$$

where $N_{pp} = 1 + K = 1 + \frac{k}{r}$. The three and six form gauge fields vanish and this solution is purely gravitational. The non-trivial components of the vierbein are given by

$$e_1^1 = \frac{1}{\sqrt{1 + K}}, \quad e_2^2 = \sqrt{1 + K}, \quad e_1^2 = -\frac{K}{\sqrt{1 + K}}, \quad e_2^1 = 0 \tag{3.19}$$
Using equation (2.28) we find that

\[ h_{1}^{1} = -\frac{1}{2} \ln(1 + K), \quad h_{2}^{2} = +\frac{1}{2} \ln(1 + K), \quad h_{1}^{2} = -K \]  

(3.20)

The associated group element of \( E_{8}^{+++} \) is

\[ g = \exp\left(-\frac{1}{2} \ln N_{pp} \beta_{pp} \cdot H\right) \exp(1 - N_{pp}) E_{\beta_{pp}} \]  

(3.21)

since the roots associated to the generator \( E_{1} = K^{1}_{2} \) is \( \alpha_{1} = \beta_{pp} \) and \( \beta_{pp} \cdot H = K^{1}_{1} - K^{2}_{2} = H_{1} \).

Hence, the above half BPS solutions of M theory when expressed in terms of \( E_{8}^{+++} \) group elements have a universal form which given by equation (1.1).

The M Monopole of M theory involves the imperfectly understood dual graviton as the active field and we will treat it in the context of the IIA theory below.

### 4. Type II theories

It has been argued that both IIA supergravity and IIB supergravity when suitably extended possess an \( E_{8}^{+++} \) symmetry \([4,5]\). The difference between the two theories arises from the two different ways the \( A_{9} \) sub-algebra associated with gravity, and so space-time, is selected from \( E_{8}^{+++} \). Given the Dynkin diagram of \( E_{8}^{+++} \) starting with the very extended node one can find a \( A_{9} \) sub-Dynkin diagram in only two ways. When we get to the junction of \( E_{8}^{+++} \) Dynkin diagram, situated at the node labeled 8, we can continue along the line with two further nodes taking only the first node to belong to \( A_{9} \) or we can find the final \( A_{9} \) node by taking it to be the only one in the other choice of direction at the junction. These corresponding theories are IIA and IIB theories respectively.

In this section, we will consider group elements of the form of the universal formula (1.1) and find what are the metric and gauge fields. We will find that we recover all the well known half BPS solutions of the type II theories including the eight brane of the massive IIA theory.

#### 4.1 IIA

The low level generators of \( E_{8}^{+++} \) when decomposed to the \( A_{9} \) sub-algebra relevant to the IIA theory are given by \([4]\)

\[ K^{a_{1}b}, R^{a_{1}...a_{q}}, q = 0, 1, \ldots, 8 \]  

(4.1)

In the non-linear realisation this implies a set of gauge fields which is precisely in agreement with the field content of IIA supergravity. The NS-NS sector corresponds to \( q - 1 = -1, 1, 5, 7 \) and the R-R sector to \( q - 1 = 0, 2, 4, 6 \). The \( q = 0 \) generator, denoted \( R \), corresponding to the dilaton. In addition to those listed in equation (4.1) one also finds a nine form generator which leads to nine form gauge field that is associated with the massive IIA theory \([16]\). We will consider this generator in section 4.3. The generators at higher levels are tabulated in \([16]\). These include generators whose corresponding fields
are the dual of gravity which will also lead to solutions as well as an infinite number of further generators.

The generators of equation (4.1) obey the usual commutators with the $K^a_b$ generators of $A_{10}$ as well as

$$[R, R^{a_1 \ldots a_q}] = c_q R^{a_1 \ldots a_q}, \quad [R^{a_1 \ldots a_p}, R^{a_1 \ldots a_q}] = c_{p,q} R^{a_1 \ldots a_{(p+q)}}$$  \hspace{1cm} (4.2)

In these relations

$$c_{p+1} = \frac{\eta (p - 3)}{4}, \quad \eta = \left\{ \begin{array}{ll}
1, & R - R \\
-1, & NS - NS
\end{array} \right.$$ 

and

$$c_{1,2} = -c_{2,3} = -c_{3,3} = c_{2,5} = c_{1,5} = 2, \quad c_{1,7} = 3, \quad c_{2,6} = 2, \quad c_{3,5} = 1. \hspace{1cm} (4.3)$$

Commutators involving higher level generators on the right-hand side than those in equation (4.1) are not shown, but some of these are given later.

The $E_a$ Chevalley generators of $E_8^{+++}$ are given by [4]

$$E_a = K^a a_{a+1}, \quad a = 1, \ldots, 9, \quad E_{10} = R^{10}, \quad E_{11} = R^{910}; \hspace{1cm} (4.4)$$

while the Cartan sub-algebra generators are

$$H_a = K^a_a - K^{a+1}_{a+1}, \quad a = 1, \ldots, 9, \quad H_{10} = -\frac{1}{8}(K^1_1 + \ldots + K^9_9) + \frac{7}{8} K^{10}_{10} - \frac{3}{2} R,$$

$$H_{11} = -\frac{1}{4}(K^1_1 + \ldots + K^8_8) + \frac{3}{4}(K^9_9 + K^{10}_{10}) + R. \hspace{1cm} (4.5)$$

The non-linear realisation of $E_8^{+++}$ with the appropriate $A_9$ subalgebra relevant to IIA supergravity leads to at low levels to the IIA supergravity action [3]. This calculation should be viewed with the hindsight provided by the relations to the $E_{11}$ generators given in [4]. The group element was parameterised by $g = g_h g_A$ where $g_h$ was given in equation (2.28) with the indices restricted to be ten or less and

$$g_A = e^{(1/9!)\Lambda_1 \ldots \Lambda_9 R^{a_1 \ldots a_9}} e^{(1/8!)\Lambda_1 \Lambda_8 R^{a_1 \ldots a_8}} e^{(1/7!)\Lambda_1 \Lambda_7 R^{a_1 \ldots a_7}} \times$$

$$e^{(1/6!)\Lambda_1 \Lambda_6 R^{a_1 \ldots a_6}} e^{(1/5!)\Lambda_1 \Lambda_5 R^{a_1 \ldots a_5}} e^{(1/3!)\Lambda_1 \Lambda_3 R^{a_1 \ldots a_3}} e^{(1/2!)\Lambda_1 \Lambda_2 R^{a_1 \ldots a_2}} e^{A_R R} e^{A_R R} \ldots. \hspace{1cm} (4.6)$$

With this parameterisation the fields which appear in the group element are those found in the IIA supergravity theory as formulated in [3]. We note that the dilaton occurs at the end of the expression.

For each generator in equation (4.1) we wish to construct the group element corresponding to equation (1.1) and then read off the corresponding solution. The generators of equation (4.1) which corresponding to the highest weight states of the $A_9$ representations are $R^{10-p...910}$. We use the commutators of equation (4.2) to construct these generators
in terms of multiple commutators of the Chevalley generators and we then find that the corresponding roots are given by

\[
\hat{\beta}_{p+1} = -\frac{(p+1)}{8}(K^1_1 + \ldots + K^{10-p-1}_10-p-1) + \frac{(7-p)}{8}(K^{10-p}_{10-p} + \ldots + K^{10}_{10}) + b_p R
\]

(4.7)

where

\[
b_p = \begin{cases} 
   \eta \frac{(p-3)}{2}, & p \leq 6 \\
   0, & p = 7 
\end{cases}
\]

(4.8)

We want to consider the root \(\beta_{p+1}\) associated with the generator \(R^{1\ldots p+1} = E_{\beta_{p+1}}\), which is a lowest weight representation of \(A_9\). Taking account of the required commutators with the \(A_9\) generators we find that

\[
\beta_{p+1} \cdot H = \frac{(7-p)}{8}(K^1_1 + \ldots + K^{p+1}_{p+1}) - \frac{(p+1)}{8}(K^{p+2}_{p+2} + \ldots + K^{10}_{10}) + b_p R
\]

(4.9)

We now consider the group element of equation (1.1);

\[
g = \exp(-\frac{1}{2}\ln N_p \beta_{p+1} \cdot H) \exp((1 - N_p) E_{\beta_{p+1}})
\]

(4.10)

To read off the values of the fields that occur in the ten dimensional IIA supergravity theory we must cast the group element in the form of equation (4.6). In particular, we must put the dilaton factor at the end of the expression. The dilaton-gauge parts of the group element of equation (4.10) are of the form

\[
\exp(-\frac{b_p}{2}\ln N_p R) \exp((1 - N_p) E_{\beta_{p+1}})
\]

(4.11)

which is equal to

\[
\exp(N_p^{-\frac{b_p}{2}(1 - N_p) E_{\beta_{p+1}}}) \exp(-\frac{b_p}{2}\ln N_p R)
\]

(4.12)

Hence, the solution corresponding to the root \(\beta_{p+1}\) has a metric given by

\[
d s^2 = N_p^{-(7-p)} (-dx_1)^2 + (dx_2)^2 + \ldots + (dx_{p+1})^2 + N_p^{-\frac{(p+1)}{2}} ((dx_{p+2})^2 + \ldots + (dx_{10})^2),
\]

(4.13)

a dilaton given by

\[
e^A = (N_p)^{-\frac{b_p}{2}},
\]

(4.14)

and a gauge field, with world indices, is given by

\[
A_{1\ldots p+1} = N_p^{-1} - 1.
\]

(4.15)

In this last expression one must take account of the change from tangent to world indices. Thus from equation (1.1) we recover the half BPS solutions for the F1 string and the NS5 brane, both in the NS-NS sector, as well as the \(p = 0, 2, 4, 6\) D-branes in the R-R sector.
sector [40]. We do take $p = -1$ as the corresponding generator is just the dilaton generator $R$ which is in the Cartan subalgebra and so falls outside the scheme we have considered. We find a solution for $p = 7$, but it is just that for Minkowski space-time with trivial dilaton.

4.2 IIB

The low level generators of $E_{8}^{++}$ decomposed with respect to the $A_{9}$ sub-algebra relevant to the IIB theory are given by [5]

$$K^{\alpha}_{\beta}, R_{s}, R_{s}^{c_{1}c_{2}}, R_{s}^{c_{1}...c_{4}}, R_{s}^{c_{1}...c_{6}}, R_{s}^{c_{1}...c_{8}}, \quad s = 1, 2$$

(4.16)

where $s$ can take values 1 or 2 corresponding to the NS-NS and R-R sectors respectively. The corresponding Goldstone fields are in agreement with the field content of the IIB supergravity theory provided one adds the appropriate dual fields. As for the case of the IIA theory we do not add the dual of gravity although they are present and would lead to new solutions. The higher order generators are tabulated in [16].

The generators in equation (4.16) obey the usual relations with the $A_{9}$ generators $K^{\alpha}_{\beta}$ as well as the relations

$$[R_{s_{1}}^{c_{1}...c_{p}}, R_{s_{2}}^{c_{1}...c_{q}}] = c_{p,q}^{s_{1},s_{2}} R_{s(s_{1},s_{2})}^{c_{1}...c_{p+q}},$$

$$[R_{1}, R_{s}^{c_{1}...c_{q}}] = d_{q}^{s} R_{s}^{c_{1}...c_{q}}, \quad [R_{2}, R_{s}^{c_{1}...c_{q}}] = \tilde{d}_{q}^{s} R_{s(2,s_{1})}^{c_{1}...c_{q}}.$$  

(4.17)

In the last line, we have separately written the commutators for the dilaton generator $R_{1}$ with coefficient $d_{q}^{s}$, and the axion generator $R_{2}$ with coefficient $\tilde{d}_{q}^{s}$. The superscript $s = s(s_{1},s_{2})$ depends on the fields in the commutator and it satisfies the properties $s(1, 1) = s(2, 2) = 1$, $s(1, 2) = s(2, 1) = 2$. The constants in the above commutation relations are given by

$$d_{p+1}^{s} = \eta^{s} \frac{(p - 3)}{4}, \quad \eta^{1} = -1, \eta^{2} = 1$$

(4.18)

and

$$c_{2,2}^{1,2} = -c_{2,2}^{2,1} = -1, \quad c_{2,4}^{1,2} = -c_{2,4}^{2,1} = 4, \quad c_{2,6}^{1,2} = 1, \quad c_{2,6}^{1,1} = -c_{2,6}^{2,2} = \frac{1}{2}$$

$$\tilde{d}_{2}^{1} = -\tilde{d}_{6}^{2} = -\tilde{d}_{8}^{2} = 1, \quad \tilde{d}_{2}^{1} = \tilde{d}_{6}^{1} = \tilde{d}_{8}^{1} = 0$$

(4.19)

Commutators involving generators at higher levels than those in equation (4.16) on the right-hand side are not given, but are discussed later on.

The Chevalley generators of $E_{8}^{++}$, as it appears in IIB, are given by [5]

$$E_{a} = K^{a}_{a+1}, a = 1, \ldots 8, \quad E_{9} = R^{910}_{1}, \quad E_{10} = R_{2}, \quad E_{11} = K^{9}_{10}.$$  

(4.20)

as well as

$$H_{a} = K^{a}_{a} - K^{a+1}_{a+1}, a = 1, \ldots, 8, \quad H_{9} = K^{9}_{9} + K^{10}_{10} + R_{1} - \frac{1}{4} \sum_{a=1}^{10} K^{a}_{a},$$

$$H_{10} = -2 R_{1}, \quad H_{11} = K^{9}_{9} - K^{10}_{10}.$$  

(4.21)
The group element of $E_8^{++}$ can be written, taking account of the local subalgebra, as

$$g = g_h g_A,$$  \hspace{1cm} (4.22)

where $g_h$ is as in equation (2.28), but with indices restricted to be ten or less and

$$g_A = e^{(1/8!)}A^2_{a_1 \cdots a_8} \varepsilon^{a_1 \cdots a_8} e^{(1/8!)}A^1_{a_1 \cdots a_8} \varepsilon^{a_1 \cdots a_8} (A_2^{a_1 \cdots a_6} + A_1^{a_1 \cdots a_6} R_1^{a_1 \cdots a_6})$$

$$\times e^{(1/6!)}(A^2_{a_1 a_2} R_2^{a_1 a_2} + A_1^{a_1 a_2} R_1^{a_1 a_2}) e^{A_2} R_2 e^{A_1 R_1} \ldots .$$  \hspace{1cm} (4.23)

Carrying out the non-linear realisation of $E_8^{++}$ at low levels one finds [5] the IIB supergravity theory with the above parameterisation of fields. The comparison with much of the known literature is facilitated by relabel $A^1 = \sigma$ and $A^2 = \chi$.

We now wish to calculate for each generator of the $E_8^{++}$ algebra in equation (4.16) the corresponding group element of the form of equation (1.1) and then find what solution it corresponds to. Using the above equations and following the same steps as for the IIA case, we find the roots $\beta_{p+1}^s$ corresponding to the lowest weight $A_9$ states in the representations of equation (4.16), i.e. $R_{s}^{12}$, $R_{2}^{1234}$, ... The corresponding elements in the Cartan sub-algebra are given by

$$\beta_{p+1}^s \cdot H = \frac{(7-p)}{8}(K_1^{1} + \ldots + K_{p+1}^{p+1}) - \frac{(p+1)}{8}(K_{p+2}^{p+2} + \ldots + K_{10}^{10}) + b_p^s R_1$$  \hspace{1cm} (4.24)

where

$$b_p^s = \begin{cases} \eta_s^{(p-3)} & \text{otherwise} \\ 0 & \text{if } p = 7, s = 1 \end{cases}.$$  \hspace{1cm} (4.25)

We now consider the group element of equation (1.1), namely

$$g = \exp(-\frac{1}{2} \ln N_p^s \beta_{p+1}^s \cdot H) \exp((1 - N_p^s)E_{\beta_{p+1}^s})$$  \hspace{1cm} (4.26)

To read off the values of the fields that occur in the ten dimensional IIB supergravity theory we must reorder the dilaton and gauge parts of the group element to bring it into the form of equation (4.23). The solution corresponding to the root $\beta_{p+1}^s$ has a metric given by

$$ds^2 = (N_p^s)^{-\frac{p+1}{8}} \left(-(dx_1)^2 + (dx_2)^2 + \ldots + (dx_{p+1})^2\right) + (N_p^s)^{\frac{(p+1)}{8}} \left((dx_{p+2})^2 + \ldots + (dx_{10})^2\right)$$  \hspace{1cm} (4.27)

a dilaton given by

$$e^A = (N_p^s)^{-\eta_s^{(p-3)}} ,$$  \hspace{1cm} (4.28)

and a gauge field given by

$$A_{1 \ldots p+1}^T = (N_p^s)^{-1} - 1$$  \hspace{1cm} (4.29)

As before we have made the change to express the gauge field with respect to the tangent space.
Equation (4.27-29) are just the half BPS solutions of the IIB supergravity theory. In particular the \( p = 1, 5, 7 \) branes of the NS-NS sector and the \( p = -1, 1, 3, 5, 7 \) branes of the R-R sector [40]. We note that these include the instanton \( p = -1 \) solution of reference [27]. We also find two brane solutions, although the one in the NS-NS sector is just Minkowski space-time with trivial dilaton and so its interpretation as a seven brane is rather degenerate. In section 5.2 we will find another seven brane solution corresponding to a higher level generator.

4.3 Massive IIA

Remarkably, the low level generators of \( E_8^{+++} \) when decomposed with respect to \( A_9 \) subalgebra relevant to the IIA theory contain the generators of equation (4.1) as well as the generator \( R^{a_1...a_9} \) [16]. This corresponds in the non-linear realisation to a rank nine anti-symmetric tensor gauge field \( A_{a_1...a_9} \). Such a field has previously proved useful in reformulating the massive IIA theory in such a way that it has an eight brane solution [30,28]. This latter formulation can be described as a non-linear realisation [31] which involves the fields corresponding to the generators of equation (4.1) as well as \( A_{a_1...a_9} \). In this theory, the nine form generator arises from the commutator of lower rank generators as [31]

\[
[R^{a_1a_2}, R^{a_3...a_9}] = -4R^{a_1...a_9}
\]

(4.30)

This relation also follows from the \( E_8^{+++} \) algebra as is easily seen using the roots given in [16]. Using this equation and equation (4.7), we find that the root \( \beta_9 \) corresponding to the \( A_9 \) lowest weight generator \( R^{12...9} \) gives rise to the Cartan sub-algebra element

\[
\beta_9 \cdot H = -\frac{1}{8}(K^1 + \ldots + K^9) - \frac{9}{8}K^{10}_{10} + \frac{5}{2}R
\]

(4.31)

The solution corresponding to the root \( \beta_9 \) has a metric given by

\[
ds^2 = (N_8)^{\frac{2}{9}}((-dx_1)^2 + (dx_2)^2 + \ldots + (dx_9)^2) + (N_8)^{\frac{2}{9}}(dx_{10})^2),
\]

(4.27)

a dilaton given by

\[
e^A = (N_8)^{-\frac{3}{4}},
\]

(4.28)

and a gauge field given by

\[
A_{1...9} = (N_8)^{-1} - 1
\]

(4.29)

This agrees with putting \( p = 8 \) in the general formulae of equations (3.13-15). It is precisely the eight brane solution found in reference [30,28]. We will discuss the eleven dimensional origin of this solution in the next section.

5 Higher level branes

\[1\] The underlying Kac-Moody algebra was not identified for the massive IIA theory described as a non-linear realisation in [31], but if one excludes the space-time translation generator then it is clear that it is also \( E_8^{+++} \).
Clearly, we can construct a group element of the form of equation (1.1) for any root of the $E_{8}^{+++}$ algebra. However, we do not have an explicit expression for the $E_{8}^{+++}$ non-linear realisation at higher levels than that considered in the above sections and so we can not be sure that such group elements will correspond to solutions of this theory. Nonetheless, given the universal form of equation (1.1) for the half BPS solutions it is encouraging to think that this formula provides solutions in general. In this section, we will find explicit expressions for the field configuration corresponding equation (1.1) for certain of the higher order generators of $E_{8}^{+++}$ in eleven dimensions. We do this for the eleven dimensional theory and the IIB theory.

5.1 Branes in Eleven dimensions at level four

To find the roots associated with the branes at level four it will first prove useful to find the roots associated to the generators of $E_{8}^{+++}$ at level three. The only such generators are $R^{a_{1}...a_{5},b}$ and they arise [4] in terms of lower level generators as the commutator $[R^{a_{1}...a_{5}},R^{b_{1}...b_{5}}] = 3R^{a_{1}...a_{5}}[b_{1}b_{2},b_{3}]$. The $A_{10}$ highest weight is the generator $R^{4...1111}$ and its associated root $\beta_{8,1}$ is given by $\beta_{8,1} = \alpha_{11} + (2\alpha_{11} + \alpha_{5} + 2\alpha_{7} + 3\alpha_{8} + 2\alpha_{9} + \alpha_{10}) + \alpha_{4} + 2\alpha_{5} + \ldots + 2\alpha_{8} + \alpha_{9}$. The corresponding Cartan subalgebra element is $\beta_{8,1} \cdot H = -(K^{11} + K^{22} + K^{33}) + K^{1111}$. We note, using equation (1.1), that the corresponding solution has the correct metric to be Taub-Nut except for the off diagonal components of the metric which presumably arise from the dual gravity field.

At level four $E_{8}^{+++}$ contains the generators [17,11]

$$R^{a} (1, 2, 3, 4, 5, 6, 7, 8, 5, 2, 4), \quad R^{(ab)}_{c}, \quad (0, 1, 2, 3, 4, 5, 6, 7, 4, 1, 4)$$

and

$$R^{a_{1}a_{2}a_{3}}_{b_{1}b_{2}} (0, 0, 1, 2, 3, 4, 5, 6, 4, 2, 4) \quad (5.1)$$

where the numbers in brackets are the positive integers $n_{a}$ for the corresponding root, i.e. $\alpha = \sum a n_{a} \alpha_{a}$, of the $A_{10}$ highest weight components, $R^{11}, R^{1111}$ and $R^{121011}$. The above generators satisfy the constraints $R^{(ab)}_{b} = 0$, $R^{a_{1}a_{2}c}_{b_{1}c} = 0$. Taking into account these conditions, we find that the $E_{8}^{+++}$ algebra at level four has the commutation relation $^{2}$

$$[R^{a_{1}a_{2}a_{3}}, R^{b_{1}...b_{5},c}] = \epsilon^{b_{1}...b_{5}[a_{1}a_{2}a_{3}]} R^{c} - \frac{1}{(D - 1)} \epsilon^{b_{1}...b_{5}[a_{1}a_{2}a_{3}]} R^{c} + \frac{3}{2} \epsilon^{b_{1}...b_{5}[a_{1}a_{2}a_{3}]} R^{c} R^{a} R^{c} R^{a}$$

$$+ \frac{3}{2} \left( \epsilon^{b_{1}...b_{5}[a_{1}a_{2}a_{3}]} R^{c} R^{a} R^{c} R^{a} \right)$$

The highest weight components of the above generators arise from the following commutators

$$[R^{123}, R^{4...1111}] \propto R^{11} + \ldots, \quad [R^{2311}, R^{4...1111}] \propto R^{1111}, \quad [R^{3910}, R^{4...1111}] \propto R^{91011} \quad (5.2)$$

$^{2}$ We note in passing that the one generator of SL(32) in the local subalgebra not identified precisely in [35] is $R^{a} - R_{a}$.
We can now read off the roots corresponding to these level four generators. For example, $R_{11}^{11}$ corresponds to the root

$$\beta_1 = \beta_{8,1} + \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + 3\alpha_8 + 2\alpha_9 + \alpha_{10} + \alpha_{11}$$  \hspace{1cm} (5.3)

We are interested in the electric branes and so we consider the lowest weight generators $R_1^1$, $R_{11}^{11}$ and $R_{1_01}^{123}$ with corresponding roots $\beta_1$, $\beta_{2,9}$ and $\beta_{3,7}$ respectively. The Cartan subalgebra generator associated with $R_1^1$ is given by

$$\beta_1 \cdot H = \frac{2}{3}K_{11}^1 - \frac{1}{3}(K_{12}^2 + \ldots + K_{11}^{11})$$  \hspace{1cm} (5.4)

and the corresponding solution, using equation (1.1), is

$$ds^2 = (N_1)^{-\frac{2}{3}}(-(dx_1)^2) + (N_1)^{\frac{2}{3}}((dx_2)^2 + \ldots + dx_{11})^2)$$  \hspace{1cm} (5.5)

This has the form of a D0 brane.

The electric brane corresponding to the generator $R_{c}^{(ab)}$ has a Cartan subalgebra element

$$\beta_{2,9} \cdot H = \frac{5}{3}K_{11}^1 - \frac{1}{3}(K_{21}^2 + \ldots + K_{10}^{10}) - \frac{4}{3}K_{11}^{11},$$  \hspace{1cm} (5.6)

and the corresponding metric is

$$ds^2 = (N_{2,9})^{-\frac{2}{3}}(-(dx_1)^2) + (N_{2,9})^{\frac{2}{3}}((dx_2)^2 + \ldots + dx_{10})^2) + (N_{2,9})^{\frac{4}{3}}(dx_{11})^2$$  \hspace{1cm} (5.7)

Finally, the electric brane corresponding to the generator $R_{b_1b_2}^{a_1a_2a_3}$ has a Cartan subalgebra element

$$\beta_{3,7} \cdot H = \frac{2}{3}(K_{11}^1 + K_{12}^2 + K_{13}^3) - \frac{1}{3}(K_{4}^{(4)} + \ldots + K_{9}^{9}) - \frac{4}{3}(K_{10}^{10} + K_{11}^{11}),$$  \hspace{1cm} (5.8)

and the corresponding metric is

$$ds^2 = (N_{3,7})^{-\frac{2}{3}}(-(dx_1)^2 + (dx_2)^2 + (dx_3)^2) + (N_{3,7})^{\frac{2}{3}}((dx_4)^2 + \ldots + (dx_9)^2)$$

$$+ (N_{3,7})^{\frac{4}{3}}(dx_{10})^2 + (dx_{11})^2)$$  \hspace{1cm} (5.9)

As might be expected the potential solutions corresponding to $E_8^{+++}$ generators that have a more complicated structure than just a set of anti-symmetric indices also have more complicated form. There is an obvious correspondence between the indices on the generator and the form of the solution. For example, $R_1^1$ has its world volume in the 1 direction and the omitted indices are transverse and $R_{1011}^{123}$ has a ”world volume” in the two separate parts in the 1, 2 and 3 directions and the 10 and 11 directions with the rest being ”transverse”. This applies to all the branes in the previous sections which have anti-symmetrised indices. Where symmetrised indices occur, such as in equation (5.7) the situation seems a bit more complicated.

5.2 Higher branes in IIB
In the IIB theory, the generators at the next levels beyond those of equation (4.16) are [16]

\[ R_{1}^{a_{1}\ldots a_{7},b} \quad (0,0,0,1,2,3,4,5,4,2,2); \quad S_{2}^{a_{1}\ldots a_{8}} \quad (0,0,1,2,3,4,5,6,4,3,3); \]

\[ R_{2}^{a_{1}\ldots a_{10}} \quad (1,2,3,4,5,6,7,8,5,1,4) \]

Clearly, one could also write the last generator as an \( A_{9} \) scalar. These arise as the commutators

\[ [R_{1}^{a_{1}\ldots a_{6}}, R_{1}^{b_{1}b_{2}}] = -\frac{1}{2} R_{1}^{a_{1}\ldots a_{6}} - R_{1}^{a_{1}\ldots a_{6}[b_{1},b_{2}]}, \quad [R_{2}^{a_{1}\ldots a_{6}}, R_{2}^{b_{1}b_{2}}] = \frac{1}{2} R_{1}^{a_{1}\ldots a_{6}} + R_{1}^{a_{1}\ldots a_{6}[b_{1},b_{2}]}, \]

\[ [R_{2}^{a_{1}\ldots a_{4}, R_{2}^{b_{1}b_{4}}] = -8 R_{1}^{a_{1}\ldots a_{4}[b_{1}b_{2}b_{3},b_{4}]}, \]

\[ [R_{2}^{a_{1}a_{2}}, R_{1}^{a_{3}\ldots a_{8}}] = S_{2}^{a_{1}\ldots a_{8}}, \quad [R_{1}^{a_{1}a_{2}}, R_{2}^{a_{3}\ldots a_{10}}] = R_{2}^{a_{1}\ldots a_{10}} \]

The subscripts 1 and 2 correspond to the assignment to generalised R-R and NS-NS sectors which are defined by the generalisation of the rule for the commutators that is used for the supergravity fields, namely the commutator of a R-R generator with a NS-NS generator gives a R-R generator and all other commutators give a NS-NS generators.

The generator \( R_{1}^{a_{1}\ldots a_{7},b} \) leads to the dual graviton field. Its corresponding solution is an analogue of Taub-Nut and we will not consider it further here. The generator \( R_{2}^{a_{1}\ldots a_{10}} \) leads to a rank ten gauge field which was conjectured in [16] to be the gauge field associated with the space filling nine brane. The root corresponding to the generator \( R_{2}^{a_{1}\ldots a_{10}} \) is

\[ \beta_{10}^{2} = \beta_{8}^{2} + \beta_{1}^{1} + K_{1}^{1} + K_{2}^{2} - K_{9}^{9} - K_{10}^{10} = -(K_{1}^{1} + \ldots + K_{10}^{10}) + 3R_{1} \]

The corresponding solution is

\[ ds^{2} = N(-(dx_{1})^{2} + (dx_{2})^{2} + \ldots + (dx_{10})^{2}), \quad e^{A} = N^{-\frac{3}{2}} \]

This has the form of a space filling nine brane and as there are no transverse coordinates we expect that \( N \) is a constant. This brane belongs to the R-R sector and must be the space filling nine brane anticipated using world sheet arguments in [29]. It would also be interesting to make the connection between this work and the IIB supersymmetry algebra of reference [44] which incorporates a ten form field.

Finally, we find the solution corresponding to the lowest weight generator \( S_{2}^{1\ldots 8} \) whose corresponding root is given by

\[ \beta_{8}^{2} = -(K_{9}^{9} + K_{10}^{10}) - 2R_{1}. \]

The metric part of the solution being

\[ ds^{2} = -(dx_{1})^{2} + (dx_{2})^{2} \ldots (dx_{8})^{2} + N((dx_{9})^{2} + (dx_{10})^{2}), \]

which we recognise as a seven brane. In addition to the solution given in section 4.2 we now have two seven brane solutions in the R-R sector. The one in this latter section
is associated with the dual of the axion and should be related to that given in reference [27,28]. However, more seven branes were found in reference [42] and it would be interesting to establish a precise comparison.

6 Relations between the eleven dimensional, IIA and IIB theories

The fact that the eleven dimensional theory, the IIA and IIB theories all have an underlying non-linear $E_8^{+++}$ symmetry is consistent with the general belief that they are all limits of some underlying theory often referred to as M theory. In a non-linear realisation there is a one to one correspondence between the fields and the generators in the algebra outside the local subgroup. Hence, given a field in for example eleven dimensional theory we can identify its $E_8^{+++}$ root and find in the IIA or IIB theory the field which corresponds to that root. As such, we find a correspondence between the fields in the eleven dimensional theory and those in the IIA and IIB theories. In this section, we will give these correspondences at low levels and in particular use it to find how the massive IIA theory and its eight brane are related to the eleven dimensional theory.

As such, viewing M theory as having a $E_8^{+++}$ symmetry allows a more precise understanding of what M theory is. Indeed, it is clear from the $E_8^{+++}$ perspective that the underlying theory does not have a specified dimension. An established piece of M theory dogma is that it is a theory in eleven dimensions and so to distinguish the approaches one might call the underlying theory the $E_8^{+++}$ theory, or E-theory for short.

6.1 Correspondence between the eleven dimensional and IIA theories

The correspondence between the generators of the eleven dimensional formulation of $E_8^{+++}$ and the $E_8^{+++}$ formulation appropriate to the IIA theory in ten dimensions was given at low levels in [4]. Demanding equivalence of the Cartan subalgebra generators of the IIA theory in equations (4.5) and the eleven dimensional theory in equation (2.21) one finds the relations

$$\tilde{\epsilon}_a^a = \epsilon_a^a, \quad a = 1 \ldots 10, \quad \tilde{R} = \frac{1}{12}(-\sum_{a=1}^{10} \epsilon_a^a + 8 \epsilon_{11}^{11}).$$

(6.1)

In this equation and all the equations in this subsection we treat the eleventh index, denoted 11, as special while the indices $a, b, \ldots$ take the range $a, b, \ldots = 1, \ldots, 10$. We also denote the generators of the IIA theory with a $\tilde{\cdot}$. Equating the simple root generators of equation (2.20) and equation (4.4) we find that

$$\tilde{K}_{a+1} = K_{a+1}, \quad a = 1 \ldots 9, \quad \tilde{\epsilon}^{10} = \epsilon_{11}, \quad \tilde{R}^{910} = R^{91011}$$

(6.2)

Since by definition in a Kac-Moody algebra all generators are formed from the commutators of the simple root generators, the above equation fixes the correspondence for all generators. Comparing the resulting commutators in the two theories we find that [4]

$$\tilde{\epsilon}_b^a = \epsilon_b^a, \quad \tilde{\epsilon}^{a} = \epsilon_{11}^{a}, \quad \tilde{\epsilon}^{a_1a_2} = \epsilon^{a_1a_211}, \quad \tilde{\epsilon}^{a_2a_3a_4} = \epsilon^{a_2a_3a_4}$$

$$\tilde{\epsilon}^{a_1 \ldots a_6} = \epsilon^{a_1 \ldots a_511}, \tilde{\epsilon}^{a_1 \ldots a_6} = -\epsilon^{a_1 \ldots a_6}.$$
\[
\tilde{R}^{a_1...a_7} = \frac{1}{2} R^{a_1...a_711,11}, \quad \tilde{R}^{a_1...a_8} = \frac{3}{8} R^{a_1...a_8,11} \tag{6.3}
\]

The relations between the fields can be trivially read off from the relations between the generators, for example \( \tilde{A}_a = h_a^{11}, \tilde{A}_{a_1a_2} = A_{a_1a_211}, \tilde{A}_{a_1a_2a_3} = A_{a_1a_2a_3} \) etc. These are of course consistent at low levels with the relations that relate the IIA theory to the eleven dimensional supergravity theory by dimensional reduction on a circle \([33]\). However, we would stress that we are not regarding the eleven dimensional theory as dimensionally reduced on a circle. Substituting these replacements into the eleven dimensional group element of equation (2.23) we would find, after a suitable rearrangement, the IIA group element of equation (4.6). The rearrangement just leads to a set of field redefinitions.

Some higher level generators and their corresponding roots are given in \([17,11]\) and \([16]\) for the eleven dimensional and IIA theories respectively. In particular, examining these tables one finds that the nine form generator \( R^{a_1...a_9} \) of the \( E_8^{+++} \) that arises in the IIA theory has the root \((0,1,2,3,4,5,6,7,4,1,4)\) which corresponds to the highest weight component \( R_3^{1...11} \). On the other hand, in the non-linear realisation of \( E_8^{+++} \) with the \( A_{10} \) subalgebra that leads to the eleven dimensional theory, the root \((0,1,2,3,4,5,6,7,4,1,4)\) is the highest weight component of the level four generator \( R_c^{(ab)} \), that is \( R^{(1111)} \) or raising the lowered index with epsilon \( R^{(1111)1...11} \). Since the \( A_9 \) subalgebra of \( A_{10} \) is in common we can identify all the components of the IIA nine form generator from their highest weight components in the obvious way. Performing such identifications for other generators we find that

\[
\tilde{R}^{a_1...a_9} = \tilde{R}^{(1111)a_1...a_911}, \quad \tilde{R}^{a_1...a_7,b} = R^{a_1...a_711,b}, \quad \tilde{R}^{b,a_1...a_9} = R^{(b11)a_1...a_911},
\]

\[
\tilde{R}_{b_1b_2}^{a_1a_2} = \tilde{R}_{b_1b_2}^{a_1a_211}, \quad \tilde{R} = R^{11}, \ldots \tag{6.4}
\]

The precise coefficients cannot be deduced by identifying the generators from their roots and are not necessarily one as shown. They can be determined by comparison of the commutators in the two theories as will be done elsewhere.

We can now trace the eleven dimensional origin of the eight brane of the massive IIA theory, considered in section (4.3). As explained above, the IIA generator \( R^{a_1...a_9} \), which is associated with the massive IIA theory, corresponds in the eleven dimensional theory to the level four generators \( R_c^{(ab)} \), or equivalently \( R^{(ab)c_1...c_{10}} \), and so to the eleven dimensional field \( A_{(ab)}^c \). This strongly suggests that the eleven dimensional theory contains not only the ten dimensional IIA supergravity \([33]\), but also the massive IIA supergravity theory \([43]\) when suitably truncated. However, to see this one must include in the non-linear realisation the fields corresponding to the generators of level four. This theory will be an extension of eleven dimensional supergravity that has so far not been constructed and so it is not surprising that this relation between the two theories has not have been noticed so far. Clearly, it would also be interesting to establish this connection in detail and find the relation between the solution of equation (5.6) and the eight brane solution of equation (4.27). This differs from the eleven dimensional interpretation of the eight brane of \([39]\).

The above discussion side stepped the issue of space-time. In \([35]\) space-time was introduced into the non-linear realisation by considering the fundamental representation, denoted \( l_1 \) associated with the very extended node and taking the semi-direct product with
$E_{8}^{++}$. However, as discussed in [35], the $l_1$ representation not only introduces the usual space-time coordinates, but central charge coordinates and an infinite number of other coordinates arising from generators at higher levels. These are listed for low levels for the eleven dimensional theory in [35,9].

The correspondence between the generalised coordinates in the two theories can be found in much the same way as above. The highest weight state in the $l_1$ representation corresponds to the space-time generator $P_1$. However, this is the same in both theories. The identification of higher order generators in the two theories can then be found by the action of the $E_{8}^{++}$ generators in each theory together with a knowledge of their identification. We find that

$$\hat{P}_a = P_a, \quad \hat{Z} = [\hat{P}_{10}, \hat{K}_{10}^{10}] = [P_{10}, K_{11}^{10}] = P_{11},$$

and similarly at higher levels. In fact, the generator $\hat{Z}$ occurs as a central charge in the IIA supersymmetry algebra. We note that both theories contain fields that depend on the same number of generalised coordinates and so from this viewpoint neither theory is preferred.

6.2 Correspondence between the eleven dimensional and IIB theories

It is well known that if one reduces the ten dimensional IIA and IIB supergravities on a circle to nine dimensions the resulting supergravities coincide and the reduced IIA and IIB string theories are related by T duality. However, it is generally not expected that there exist explicit relations between the IIA and IIB theories in ten dimensions, that is without any dimensional reduction. However, as explained at the beginning of this section, as both theories are based on $E_{8}^{++}$, we can relate their generators and hence their fields. In this case, the algebras corresponding to their gravity sectors only overlap on a $A_8$ subalgebra and so they have the index ranges $a, b, c, \ldots = 1, 2, \ldots, 9$ in common. In what follows in this subsection $a, b, c, \ldots$ are assumed to take this range. Equating the Cartan subalgebra elements in equation (2.21) and in equation (4.21) and putting a $\hat{}$ on all the IIB generators we find that

$$\hat{K}^a_a = K^a_a, \quad a = 1 \ldots 9, \quad \hat{K}^{10}_{10} = \frac{1}{3} \sum_{a=1}^{9} K^a_a - \frac{2}{3}(K^{10}_{10} + K^{11}_{11})$$

$$\hat{R}_1 = -\frac{1}{2}(K^{10}_{10} - K^{11}_{11})$$

Identifying the simple roots of equations (2.20) and (4.20) we find that

$$\hat{K}^a_{a+1} = K^a_{a+1}, \quad a = 1 \ldots 8, \quad \hat{K}^9_{10} = R^{91011}, \quad \hat{R}_2 = K^{10}_{11}, \quad \hat{R}^{910}_{11} = K^{9}_{10}$$

This completely fixes the identification of the generators in the two theories. To find the higher level identifications we use these relations in conjunction with the commutators in the two theories. One finds that

$$\hat{R}^{10}_{11} = K^{a}_{10}, \quad \hat{R}^{ab}_{1} = R^{ab11}, \quad \hat{R}^{a10}_{2} = -K^a_{11}, \quad \hat{R}^{ab}_{2} = -R^{ab10},$$

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\[
\hat{R}^a_{1\ldots a_3 a_{10}} = -R^{a_1\ldots a_3}, \quad \hat{R}^a_{2\ldots a_4} = 2R^{a_1\ldots a_4 1011}
\]
\[
\hat{R}^a_{2\ldots a_5 a_{10}} = -\frac{1}{2}R^{a_1\ldots a_5 11}, \quad \hat{R}^a_{2\ldots a_6} = \frac{1}{2}R^{a_1\ldots a_6 1011, 11},
\]
\[
\hat{R}^a_{1\ldots a_5 a_{10}} = \frac{1}{2}R^{a_1\ldots a_5 10}, \quad \hat{R}^a_{1\ldots a_6} = -\frac{1}{2}R^{a_1\ldots a_6 1011, 10},
\]
\[
\hat{R}^a_{2\ldots a_7 a_{10}} = \frac{1}{2}R^{a_1\ldots a_7 11, 11}, \quad \hat{R}^a_{1\ldots a_7} = -\frac{1}{2}(R^{a_1\ldots a_7 10, 11} + R^{a_1\ldots a_7 11, 10}),
\]
\[
\hat{R}^a_{1\ldots a_7 10, 10} = R^{a_1\ldots a_6}, \quad \hat{S}^a_{a_1\ldots a_7} = -\frac{1}{2}\hat{R}^a_{1\ldots a_6 10, 10}
\]
\[
\hat{R}^a_{1\ldots a_7 10, a} = -\frac{3}{4}(R^{a_1\ldots a_7 10, 11} - R^{a_1\ldots a_7 11, 10})
\]
\[
\hat{R}^a_{1\ldots a_6 10, b} = \frac{1}{4}(R^{a_1\ldots a_6 b 11, 10} - R^{a_1\ldots a_6 b 10, 11}) - R^{a_1\ldots a_6 1011, b}
\]

The coefficients are deduced by comparison of the commutators of the two theories. We have omitted field comparisons that involve level four generators in the eleven dimensional theory, but it is interesting to observe that some generators that are associated with the IIB supergravity, including the dual fields, correspond to eleven dimensional generators whose associated fields are beyond those found in the eleven dimensional supergravity approximation. The field correspondence can be read off i.e. \( \hat{A}^{a}_{10} = h^{a}_{10}, \hat{A}^{a}_{1b} = A_{abb}, \ldots \) We note that the IIB ten form generator \( R^{a_1\ldots a_{10}} \) considered in equation (5.10), which gives rise to the space filling nine brane, corresponds to the generator \( R^{1111}_{10} \), or equivalently \( R^{1111111\ldots 911} \), of the eleven dimensional theory. Hence, the eight brane of the massive IIA theory and the space filling nine brane of the IIB theory have a common eleven dimensional origin in different components of the field \( A_{ab} \).

We can also establish the correspondence between the generalised coordinates as for the IIA case above. Again the highest weight states of the two \( l_1 \) representations corresponding to the two space-time generators \( P_1 \) can be identified. The identification for all the other generators in the two \( l_1 \) representations then follows. We find that
\[
\hat{P}_a = P_a, \quad \hat{P}_{10} = [\hat{P}_9, \hat{K}^9_{10}] = [P_9, R^{91011}] = Z^{1011}, \quad \hat{Z}^{10} = [\hat{P}_9, \hat{R}^9_{10}] = [P_9, K^9_{10}] = P_{10}
\]
and similarly at higher levels. In the last step we used the relation \([P_a, R^{c_1\ldots c_3}]) = 3\delta^a_{[c} Z^{c_2 c_3]} \) [35]. Hence we must identify the generalised coordinates as
\[
\hat{x}^a = x^a, \quad \hat{x}^{10} = z_{1011}, \quad \hat{z}_{10} = x_{10}, \ldots
\]

Thus the equivalence of the IIB theory with the eleven dimensional theory requires an exchange of space-time coordinates with central charges which is beyond the scope of the more usual supergravity considerations. We note that the two theories have a dependence on the generalised coordinates that is in a one to one correspondence and so the IIB theory does not depend on less coordinates than the eleven dimensional theory.

7 Discussion
In this paper we have concentrated on single brane solutions, but multi-brane solutions can be formed by taking superpositions of single brane solutions [41]. In the framework of this paper one can multiply two group elements and find a third which we may try to interpret as a solution. Indeed, given two branes whose corresponding group elements \( g_1 \) and \( g_2 \) are of the form of equation (1.1) with roots \( \beta_1 \) and \( \beta_2 \) their product is given by

\[
g_1 g_2 = \exp(-\frac{1}{2} \ln N_1 \beta_1 \cdot H) \exp(1 - N_1) E_{\beta_1} \exp(-\frac{1}{2} \ln N_2 \beta_2 \cdot H) \exp(1 - N_2) E_{\beta_2}
\]

\[
= \exp(-\frac{1}{2} \ln N_1 \beta_1 \cdot H - \frac{1}{2} \ln N_2 \beta_2 \cdot H) \exp\{(1 - N_1)(1 - N_2)\over\beta_1 \cdot \beta_2 \} \exp(1 - N_2) E_{\beta_2}
\]  

(7.1)

For example, one can consider the case of the M2 brane of equation (3.9) with the M-wave of equation (3.21). In this case, \( \beta_1 = \beta_{M_2} \) and \( \beta_2 = \beta_{pp} \). One finds that \( \beta_{M_2} \cdot \beta_{pp} = 0 \) and the resulting solution takes the form

\[
d s^2 = N_2^{-\frac{2}{3}} (- (1 - K) (dx_1)^2 + (1 + K) (dx_2)^2 - 2K dx_1 dx_2 + (dx_3)^2
\]

\[
+ N_2^{\frac{1}{3}} ((dx_4)^2 + \ldots + (dx_{11})^2).
\]

(7.2)

The gauge field is given by \((N_2)^{-1} - 1\). This is indeed the known solution for a M2 brane superimposed with a M wave. Similarly, we can find the known solution for two M2 branes in the 123 and 145 directions by multiplying the corresponding group elements. In this case, \( \beta_1 = \beta_{M_2} \) and \( \beta_2 = \beta_{M_2} - (\alpha_2 + 2\alpha_3 + \alpha_4) \). One finds these roots are orthogonal and the solution derived from the product of the group elements is again the known solution. One can also add yet another M2 brane in the 167 direction and the group multiplication also gives the known solution for three M2 branes. All these branes have \( \frac{1}{4} \) supersymmetry. It would be interesting to find the \( E_8^{+++} \) group element for the more general branes that preserve \( \frac{1}{4} \) supersymmetry and the general rules for constructing branes by group multiplication of their corresponding group elements.

In fact, there do exist other half BPS solution than those considered in this paper, such as when the M2 brane lies within the M5 brane as considered in reference [36]. It would be interesting to find the \( E_8^{+++} \) form of such solutions as well as those that preserve all 32 supersymmetries. The elegant form of the group element for the usual half BPS branes leads one to think that \( E_8^{+++} \) may also be useful for classifying solutions with a given amount of supersymmetry.

The work in this paper also gives a new framework for considering the idea [37] that the scattering of BPS branes defines some kind of algebra. Given a BPS brane we can define its \( E_8^{+++} \) group element and so the scattering of branes can be interpreted as an operation on several copies of \( E_8^{+++} \). An elementary example can be thought of as the above group multiplication to form composite branes. It would be interesting to find what this operation is in general. Clearly, this is related to the dynamics of the underlying non-linear realisation which we have yet to bring into play.

In any non-linear realisation the fields are encoded in a group element and their transformations under the symmetries is given in equation (2.22). In the more familiar non-linear realisations used in particle physics, space-time is introduced in an adhoc way.
and \( g_0 \) does not depend on space-time. As a result, a group transformation does not change the space-time dependence of any solution. In the case of the non-linear realisations considered here, the group element in one way or another involves space-time and so one can expect that the possible group transformations that one can carry out on any solutions are much more extensive. As such, one can expect that \( E_8^{+++} \) transformations will relate very large areas of the moduli space of solutions. Certainly, the way the expression of solutions in terms of group elements makes it particularly easy to carry out Weyl and other transformations on the solutions.

The algebra \( E_8^{+++} \) contains generators corresponding to the supergravity fields and their dual. However, in this paper we have regarded all branes as electric branes. As a result, when dealing with a brane that is usually regarded as a magnetic brane, such as the M5 brane, we take the corresponding six rank dual gauge field to be the active field. Nonetheless, one might expect that the gauge fields and their dual are related by a duality condition expressed through their field strengths and so one can wonder if both gauge fields should be active. The resolution of this paradox probably relates to the fact that most brane solutions are not solutions of pure supergravity, but require external sources which are also outside the non-linearly realised theory considered here. It is to be hoped that the incorporation of these sources allows the purely electric choice we have taken.

One could also consider equation (1.1) for generators none of whose indices are in the 1, or time, direction. The resulting potential brane solutions would have a world volume that is Euclidean and are likely to be related to S branes. We will report on this possibility elsewhere. It is straightforward to extend the considerations in this paper to any \( G^{+++} \) [38].

In section six, we explained that the common \( E_8^{+++} \) origin of the eleven dimensional theory and the IIA and IIB theories when viewed as non-linear realisations allows us to find explicit one to one correspondences between any two of these three theories. We did this by relating the IIA and IIB theories to the eleven dimensional theory, but this implies a similar correspondence between the IIA and IIB theories which we could have found directly without involving the eleven dimensional theory. It is important to stress that this correspondence does not require a compactification to nine dimensions, nor does the one between the eleven dimensional and the IIA theory require a reduction on a circle. The correspondences allows us to find explicit relations between the fields of the three theories. We also found that if we regard space-time to arise as part of the fundamental representation \( l_1 \) associated with the very extended node, as advocated in [35], then one can also find the correspondences between the generalised coordinates of the three theories. This would also be the case if one adopted the view point, as advocated as in [6] and [10], that space-time is in some way contained in the Kac-Moody algebra. We note that it was observed [9] that if this was the case then it was likely that the Kac-Moody algebra also encoded the central charge coordinates and other generalised coordinates. We note that all three theories depend on the same number of generalised coordinates which are in a one to one correspondence.

Given the non-linear realisation for one theory, one can turn it into the non-linear realisation of one of the other theories by carrying out the changes of fields, and if needed the coordinates, in the group element and then rearranging it to be of the desired form.
Thus, the three theories seem to be much more closely related than previously thought. It is important to note that some of the dual fields associated with IIB supergravity and the nine form in the massive IIA theory correspond to fields in the eleven dimensional theory that are at levels higher than that which appear in the eleven dimensional supergravity approximation. Also, the space-time coordinate $\hat{x}^{10}$ of the ten dimensional IIB theory becomes a component of the central charge of the eleven dimensional theory as well other unusual changes. This is beyond the approximations of eleven dimensional supergravity [32] and the IIA [33] and IIB supergravity [34] theories that have formed the basis for so much of our knowledge over recent years and it also involves effects not found in the context of string theory alone.

At the very least, it is encouraging to see that the $E_8^{+++}$ algebra contains generators leading to the eight brane of the massive IIA theory and the space filling nine brane of the IIB theory and we are able to find how these objects are related to the eleven dimensional theory.

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