A Non-local Quasi-Equilibrium State in the Bhatnagar–Gross–Krook Boltzmann Equation for Thermo-hydrodynamics: Conservation Laws, The Boltzmann H-Theorem, and the Fluctuation-Dissipation Theorem

Hiroshi Otomo

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Abstract

The Bhatnagar–Gross–Krook (BGK) Boltzmann equation with the Maxwellian–Boltzmann-type equilibrium state leads to the set of thermo-hydrodynamic equations such as the continuity, the Navier–Stokes, and the heat-transfer equations in the scaling limit. In this study, using an analytical technique of the kinetic generator, we employ a non-local formulation for the equilibrium state, called a quasi-equilibrium state, to derive the thermo-hydrodynamic equations with flexible choices of transport coefficients and the equation of state (EOS). It is formulated with the non-local macroscopic quantities so that the long-range interaction is explicitly included. According to the new formulation, the consistency with the conservation laws, the Boltzmann H-theorem, and the fluctuation-dissipation theorem is investigated.

Keywords  Bhatnagar–Gross–Krook Boltzmann equation · Thermo-hydrodynamic equations · Quasi-equilibrium state · Boltzmann-H theorem

1 Introduction

The collective motion of dissipative particles in a rarefied gas is usually solved by the Boltzmann equation [1–3]. This innovative mathematical representation, born in the spirit of Boltzmann, has been continuously developed beyond its original scope. The Bhatnagar–Gross–Krook (BGK) collision operator approximates the collision term to avoid the complex collision operator while preserving the conservation laws and the relaxation process to the equilibrium state [4]. Using the Maxwellian distribution as the equilibrium state, one derives the set of thermo-hydrodynamic equations such as the continuity, Navier–Stokes, and heat transfer equations in the scaling limit of the Boltzmann equation. Based on this fact, studies in recent decades have focused on solving the macroscopic fluid flow using the kinetic-
based equation \[5, 6\]. In the derived Navier–Stokes equation and the heat transfer equation, however, the Prandtl number is unity, while that for a monoatomic gas is about \(2/3\) \[1, 2\]. In addition, their transport coefficients, such as viscosity and thermal conductivity, depend linearly on temperature and density.

The ellipsoidal statistical (ES) BGK model \[7\] and the Shaknov BGK model \[8\] have been proposed to solve this unity Prandtl number problem. In the ES BGK model, the equilibrium state is formulated by replacing the temperature in the Maxwellian distribution by a tensorial form. It allows the viscosity to be varied by a model parameter and has been shown to guarantee the non-negative distribution and to obey the Boltzmann H-theorem in a certain range of the parameter \[9\]. In contrast to the ES BGK model, where the viscosity is a variable for the flexible Prandtl number, the Shaknov BGK model introduces an additional factor to the Maxwellian distribution, including the heat flux vector and making the thermal conductivity a variable. Such models have shown promising results in terms of both accuracy and computational efficiency compared to the Direct Simulation Monte Carlo (DSMC) method \[10\]. However, most of these studies focus on the gas flow at a certain high range of Knudsen number, where the transport coefficients are likely to be highly dependent on temperature and density. As a result, such approaches may be inefficient or invalid where the Knudsen number is modest. On the other hand, during these few decades, the other promising formulations are presented in the continuum space. In one of them, a desirable collision operator for the thermohydrodynamic equation was discussed by focusing on the moments of the equilibrium state \[11\]. In the other study, using the multiple relaxation time (MRT) for moments of the nonequilibrium parts projected to the Hermite space, a desirable set of relaxation times for flexible choice of transport coefficients was derived \[12\]. They successfully show the ways to improve the applicable range of the Prandtl number.

In discrete space, the kinetic-based numerical method, the lattice Boltzmann method, has been extensively developed to solve macroscopic thermo-hydrodynamic problems, resulting in several major schools such as the passive-scalar approach \[13, 14\], the two-distribution approach \[15\], and the multispeed approach \[16, 17\]. Unlike the first two approaches, the multispeed approach solves the Boltzmann equation for a distribution function that includes density, momentum, and temperature. Despite its physical soundness, however, the multispeed approach requires a number of discrete particle velocities, which easily leads to instability and complexity. Some of the recent versions have significantly improved the robustness by adding the thermal force to the hydrodynamic equation \[18\], regularizing the collision operator with the Hermite polynomial \[19, 20\], or using the H-theorem together with the quasi-equilibrium model \[21\] and the body-centered cubic lattices \[22\]. For example, in some of these studies \[21, 22\], the equilibrium state was derived from the minimization of the H-function in discrete space. This method is known to have advantages in terms of robustness and has therefore been extensively studied with different forms of the H-function, such as one based on the Tsallis entropy \[23, 24\]. However, in addition to the inevitable complexity of multiple particle velocities, they sometimes suffer from limitations such as temperature variation, automatically adjusted viscosity to satisfy the H-theorem, or the constant Prandtl number. Even if they can be treated numerically, it is not easy to derive the corresponding formulation in continuum space from the models in discrete space.

In another direction, the BGK Boltzmann equation is studied as a kinetic generator for a wide class of nonlinear partial differential equations in the discrete space as well as in the continuum space \[3, 25–27\]. In a study \[3\], after properly choosing the equilibrium state in the continuum space, the BGK Boltzmann equation was solved with the finite element method and successfully showed consistent solutions with the Burgers’, Korteweg-de Vries, and Kuramoto-Sivashinsky equations. It is shown that, since it was first studied a few decades
ago [28], the BGK Boltzmann equation has the property of being a kinetic generator for a large class of nonlinear partial differential equations. The keys to this success are the introduction of scale dependence in the equilibrium state, the efficient decomposition of the scale dependence of the distribution function using the streaming operator, and computational testing.

In this study, using the technique in [3] and considering the BGK Boltzmann equation as a kinetic-based representation for the distribution function of the ordinary fluids, we discuss a mathematical form of the equilibrium state in the continuum space for solving the macroscopic thermo-hydrodynamic equations whose transport coefficients and the EOS can be flexibly set without the density and temperature dependence. Unlike the previous studies in which the collision form was derived for the flexible choice of the Prandtl number [7, 8, 11, 12, 29] or derived from the H-theorem [21, 22], the equilibrium state in this study is formulated so that the nonlocal effects are explicitly included in the equilibrium state in the context of the multi-relaxation processes. According to the model in this study, we discuss the consistency with the conservation laws, the Boltzmann H-theorem and the fluctuation-dissipation theorem. This paper is organized as follows. In Sect. 2, a mathematical form of the equilibrium state in the BGK Boltzmann equation is presented. In Sect. 3, the continuity, the Navier–Stokes, and the heat-transfer equations are derived with the Chapman-Enskog expansion using the nonlocal equilibrium state. In Sect. 4, the consistency with conservation laws, the Boltzmann H-theorem, and the fluctuation-dissipation theorem are discussed. In Sect. 5, the features of the present model, the relation with previous studies, and its possible application are discussed.

2 A Non-local Quasi-equilibrium State for the Thermo-hydrodynamic Equations

The Boltzmann equation for a particle distribution function \( f \) with the BGK collision operator can be written as follows;

\[
\frac{\partial f}{\partial t} + c v_j \frac{\partial f}{\partial x_j} = -\frac{f - f_{eq}}{\tau},
\]

where \( \tau \) is the relaxation time and \( v_j \) is the particle velocity. Throughout this paper, all repeated indices imply summation over them. Here all quantities are dimensionless using the characteristic length \( l_0 \) and time \( t_0 \). For example, in the rarefied gas, \( l_0 \) is chosen as the mean-free path and \( t_0 \) is chosen so that the characteristic particle velocity is equal to \( l_0/t_0 \). When considering the case of ordinary fluids, it is not necessary to choose the characteristic propagation velocity of \( f \) as the characteristic particle velocity. Accordingly, the scaling factor \( c \) is explicitly written in the second term on the left-hand side. Since the propagation velocity of \( f \) cannot exceed the particle velocity, \( c \) should be less than 1.

The Maxwellian-Boltzmann distribution in spatial dimension \( D \) for density \( \rho \) and temperature \( T \) is

\[
f_{eq}(\rho, v, T) = \rho \frac{\rho}{(2\pi T)^{D/2}} \exp \left(-\frac{|v - u|^2}{2T}\right),
\]

where the unit of \( T \) is properly chosen so that the particle mass \( m \) and the Boltzmann constant \( k_B \) are set to unity. The equilibrium state, Eq. (2), is the solution of the Boltzmann equation with the original complex collision operator, whose application is usually justified for the rarefied gas. In previous studies [1, 2], the derived Navier–Stokes and heat transfer equations using Eq. (2) have transport coefficients linearly dependent on density and temperature.
The collision term, the right-hand side in Eq. (1), indicates that the distribution function \( f \) changes to an equilibrium state \( f^{eq} \) with a single relaxation time \( \tau \). Here we try to consider multiple relaxation processes, e.g. a collision process in which \( f \) changes to an equilibrium state \( f^{eq} \) via an equilibrium state \( g^{(eq,1)} \) in a different scale. The relaxation process is assumed to be characterized by two relaxation times \( \tau_A \) and \( \tau_B \), where \( \tau_A < \tau_B \). Then the collision term can be written as

\[
- \frac{f - g^{(eq,1)}}{\tau_A} - \frac{g^{(eq,1)} - f^{eq}}{\tau_B}.
\]

(3)

Similar collision processes have already been discussed in many previous studies, for example [30], with a different context in lattice Boltzmann theory. According to the arrangements, we obtain

\[
\text{Eq. (3)} = - \frac{f - g^{(eq,1)} \left(1 - \frac{\tau_A}{\tau_B}\right) - f^{eq} \frac{\tau_A}{\tau_B}}{\tau_A} = - \frac{f - f^{(eq,\text{new})}}{\tau_A},
\]

(4)

where \( f^{(eq,\text{new})} = g^{(eq,1)} \left(1 - \frac{\tau_A}{\tau_B}\right) + f^{eq} \frac{\tau_A}{\tau_B} \). As a result, the new effective equilibrium state, \( f^{(eq,\text{new})} \), referred to in this paper as the quasi-equilibrium state or the non-local equilibrium state, contains two equilibrium states and has a dependence on the relaxation times of \( \tau_A \) and \( \tau_B \). We can assume that the relaxation process from \( g^{(eq,1)} \) to \( f^{eq} \) is one from a non-equilibrium state slightly deviating from the local equilibrium state to the local equilibrium state, which is relevant to the dynamics related to viscosity and thermal conductivity according to the linear response theory [31]. Then \( f^{(eq,\text{new})} \) should contain the viscosity and thermal conductivity information. The example shown here is about two relaxation processes, but can be easily generalized to any number of multiple relaxation processes. In this way, the multiple relaxation processes can be formulated implicitly with the BGK Boltzmann equation by a proper definition of the equilibrium state. If each of the relaxation processes occurs in a different scale, the equilibrium state should contain the multi-scale information. For such types of equilibrium state, the analysis method in [3, 25–27] is useful. In this previous study, the decomposition of the distribution function for the scaling parameter \( \epsilon \) from the Chapman-Enskog method is efficiently performed using the streaming operator. Furthermore, the \( \epsilon \) dependence in the equilibrium state was introduced. This allowed us to generalize the formulation for a wide class of hydrodynamic partial differential equations. Here, based on the same concept, the moments of the distribution function are decomposed in Sect. 3 and nonlocal elements in the equilibrium state are introduced in the following as one of the possible forms for the equilibrium state.

In this study, a following equilibrium state is used to derive the continuity, Navier–Stokes, and heat transfer equations with flexible options for transport coefficients and the EOS,

\[
\bar{f}^{eq} = \frac{\rho}{(2\pi \bar{T})^{D/2}} \exp\left(-\frac{|v - \bar{u}|^2}{2\bar{T}}\right) + \delta f^{eq},
\]

(5)
where $\tilde{u}$, $\tilde{T}$, and $\delta f^{eq}$ are the renormalized local velocity, temperature, and a Gaussian distribution correction term defined as follows,

$$\rho \tilde{u}_i = \rho u_i + \beta c \tau_1 \frac{\partial E_p}{\partial x_i},$$

(6)

$$\rho \tilde{T} = \rho T + \beta c \tau_2 \left\{ E_p \frac{\partial u_j}{\partial x_j} - \frac{c \tau_1}{2 \rho} \left| \frac{\partial E_p}{\partial x_i} \right|^2 \right\},$$

(7)

$$\delta f^{eq} = - \beta f^{eq} \frac{\tilde{\mu}}{\rho T^2} \left\{ (v_i - u_i) (v_k - u_k) - \frac{(v - u)^2}{D} \frac{\partial E_p}{\partial x_i} \delta_{l,k} \right\} \frac{\partial u_l}{\partial x_k},$$

$$- \frac{\beta f^{eq}}{4 \rho T^3} \left\{ (v - u)^2 - (D + 2)T \right\} (v_k - u_k) \frac{\partial T}{\partial x_k},$$

(8)

where $E_p = \rho T - P$, $\beta$ is a factor for additional terms, $\mu$ is the dynamic viscosity, $\kappa$ is the thermal conductivity, $\tilde{\mu} = \mu - c \tau P$, and $\tilde{\kappa} = \kappa - 2c \tau P$. Here the unit of temperature $T$ is properly chosen so that $m$, $k_B$ and the heat capacity at constant volume $C_v$ are set to unity. By assigning the correct formula to the pressure $P$, the EOS can be implemented. The relaxation times $\tau_1$ and $\tau_2$ are introduced in Sect. 3 with the MRT scheme, but through the derivation we find that $\tau_1 = \tau_2 = \tau$ is desirable to assume. Here $\sigma$ is the normal stress tensor and $\Pi$ is a stress tensor defined as follows,

$$\sigma_{i,j} := \mu \tilde{\sigma}_{i,j} (u) + \lambda_b \frac{2}{D} \frac{\partial u_i}{\partial x_i} \delta_{i,j},$$

(9)

$$\tilde{\sigma}_{i,j} := \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{D} \frac{\partial u_i}{\partial x_i} \delta_{i,j},$$

(10)

$$\Pi_{i,j} := \frac{2}{D} \rho T \tau_1 \tilde{\sigma}_{i,j},$$

(11)

where $\mu$ is the dynamic viscosity and $\lambda_b$ is the bulk viscosity. Obviously, Eq. (5) satisfies Galilean invariance. In addition, with the $\epsilon$ ordering in Sect. 3, it guarantees the non-negativity of the distribution. For example, in eq. (7), the first term is positive in the leading order of $\epsilon$ and should be greater than the absolute value of the remaining terms.

### 3 Derivation of the Continuity, the Navier–Stokes and the Heat-Transfer Equations

In this section, using the nonlocal equilibrium state in Eq. (5), the continuity, the Navier–Stokes, and the heat transfer equations are derived from the BGK-Boltzmann equation. For the sake of simplicity, the case in two spatial dimensions is considered.

To account for different forms of relaxation, the collision term is written with the MRTs $\tau_{m,n}$,

$$\frac{\partial f}{\partial t} + c v_j \frac{\partial f}{\partial x_j} = - \sum_{m \geq 0, n > 0} M_{m,n} \mathcal{H}_{m,n} e^{-v^2/C_{n,m}} \frac{1}{\tau_{m,n}},$$

(12)

where

$$M_{m,n} := \int \int (f - f^{eq}) \mathcal{H}_{m,n} dv_x dv_y,$$

(13)

$$C_{m,n} := \pi 2^{m+n} m! n!,$$

(14)
and $\mathcal{H}$ is the Hermite polynomial. Integrating Eq. (12) for $v_x$ and $v_y$ after multiplying $\mathcal{H}_{l,k}$, one obtains,

$$\frac{\partial d_{l,k}}{\partial t} + c \frac{\partial}{\partial x_j} < \mathcal{H}_{l,k} | v_j f > = - \frac{d_{l,k} - d_{l,k}^{eq}}{\tau_{l+k}},$$

where

$$d_{l,k} = \int \int f \mathcal{H}_{l,k} d v_x d v_y,$$

$$d_{l,k}^{eq} = \int \int f^{eq} \mathcal{H}_{l,k} d v_x d v_y.$$  

(15)

Here, the orthogonality of the Hermite polynomial, $\int \int \mathcal{H}_{l,k} \mathcal{H}_{m,n} e^{-v^2} d v_x d v_y = C_{m,n} \delta_{l,m} \delta_{k,n}$, is used.

Next, the Chapman-Enskog expansion is applied to Eq. (15). Time and space are ordered by a small dimensionless scaling parameter $\epsilon$. The time derivative is ordered as $\partial_t = \sum_{k=1}^{\infty} \epsilon^k \partial_{t_k}$ where $t_k$ denotes the $k$-th order time scale. The spatial derivative is ordered as $\partial_x = \epsilon \partial_{x_1}$. In the Chapman-Enskog method, gradients of the conserved quantities are assumed to be small. In addition to the expansion for $f$,

$$f = \sum_{l=0}^{\infty} \epsilon^l f^{(l)},$$

the equilibrium state $f^{eq}$ itself is assumed to depend on the scaling parameter $\epsilon$ as [3],

$$f^{eq} = \sum_{l=0}^{\infty} \epsilon^l f^{(eq,l)}.$$  

(19)

Accordingly, $d_{n,m}$ and $d_{n,m}^{eq}$ in Eqs. (16) and (17) are expanded as,

$$d_{n,m} = \sum_{l=0}^{\infty} \epsilon^l d_{n,m}^{(l)},$$

$$d_{n,m}^{eq} = \sum_{l=0}^{\infty} \epsilon^l d_{n,m}^{(eq,l)},$$

where $d_{n,m}^{(l)}$ and $d_{n,m}^{(eq,l)}$ are composed of $f^{(l)}$ and $f^{(eq,l)}$, respectively. At the $\epsilon^0$ order, Eq. (15) can be written as,

$$d_{l,k}^{(0)} = d_{l,k}^{eq}.$$  

(22)

At the $\epsilon$ order, Eq. (15) can be written as,

$$\frac{\partial d_{l,k}^{(0)}}{\partial t_1} + c \frac{\partial}{\partial x_{1,j}} < \mathcal{H}_{l,k} | v_j f^{(0)} > = - \frac{d_{l,k}^{(1)} - d_{l,k}^{(eq,1)}}{\tau_{l+k}}.$$  

(23)

Henceforth, the relaxation times are assumed to be identical if the sum of their two indices is identical. With this assumption, the relaxation time is written with a single subscript as $\tau_{l+k}$. Using the recursive relation of the Hermite polynomial,

$$v_j \mathcal{H}_{m,n} = \frac{1}{2} \mathcal{H}_{m+\delta_{y,x},n+\delta_{y,y}} + (m \delta_{y,x} + n \delta_{y,y}) \mathcal{H}_{m-\delta_{y,x},n-\delta_{y,y}},$$

(24)
and notations,
\[ D_{i,j,k}[d] := \delta_{i,x} \left( \frac{d_{l+1,k}}{2} + l d_{l-1,k} \right) + \delta_{i,y} \left( \frac{d_{l,k+1}}{2} + k d_{l,k-1} \right), \] (25)

\[ D_{i,j,k}^{(p)} = D_{i,j,k}[d^{(p)}]. \] (26)

Eq. (23) can be written as,
\[ \frac{\partial d_{l,k}(0)}{\partial t_1} + c \frac{\partial T_{j,l,k}(0)}{\partial x_{1,j}} = - \frac{d_{l,k}(1) - d_{l,k}^{(eq,1)}}{\tau_{l+k}}. \] (27)

After arrangements, it becomes,
\[ d_{l,k}^{(1)} = d_{l,k}^{(eq,1)} - \tau_{l+k} \left( \frac{\partial d_{l,k}(0)}{\partial t_1} + c \frac{\partial T_{j,l,k}(0)}{\partial x_{1,j}} \right). \] (28)

At the \( e^2 \) order, Eq. (15) can be written as,
\[ \frac{\partial d_{l,k}^{(0)}}{\partial t_2} + \frac{\partial d_{l,k}^{(1)}}{\partial t_1} + c \frac{\partial D_{j,l,k}^{(1)}}{\partial x_{1,j}} = - \frac{d_{l,k}^{(2)} - d_{l,k}^{(eq,2)}}{\tau_{l+k}}. \] (29)

Using Eq. (28), \( D_{i,j,k}^{(1)} \) can be written as,
\[ D_{i,j,k}^{(1)} = D_{i,j,k}^{(eq,1)} - \left( \frac{\partial \hat{D}_{i,j,k}[d^{(0)}]}{\partial t_1} + c \frac{\partial \hat{D}_{i,j,k}[D_{j,l,k}^{(0)}]}{\partial x_{1,j}} \right), \] (30)

where
\[ D_{i,j,k}^{(eq,1)} = D_{i,j,k}[d^{(eq,1)}], \] (31)

\[ \hat{D}_{i,j,k}[d] = \delta_{i,x} \left( \frac{d_{l+1,k}}{2} \tau_{l+1+k} + l d_{l-1,k} \tau_{l-1+k} \right) + \delta_{i,y} \left( \frac{d_{l,k+1}}{2} \tau_{l+k+1} + k d_{l,k-1} \tau_{l+k-1} \right). \] (32)

Therefore,
\[ \hat{D}_{i,j,k}[D_{j,l,k}] = \delta_{i,x} \left( \frac{D_{j,l+1,k}}{2} \tau_{l+1+k} + l D_{j,l-1,k} \tau_{l-1+k} \right) + \delta_{i,y} \left( \frac{D_{j,l,k+1}}{2} \tau_{l+k+1} + k D_{j,l,k-1} \tau_{l+k-1} \right). \] (33)

As a result, substituting Eqs. (30) to (29), we obtain,
\[ \frac{\partial d_{l,k}^{(0)}}{\partial t_2} + \frac{\partial d_{l,k}^{(1)}}{\partial t_1} + c \frac{\partial D_{j,l,k}^{(eq,1)}}{\partial x_{1,j}} - c \left( \frac{\partial^2 \hat{D}_{i,j,k}[d^{(0)}]}{\partial t_1 \partial x_{1,j}} + c \frac{\partial^2 \hat{D}_{i,j,k}[D_{j,l,k}^{(0)}]}{\partial x_{1,i} \partial x_{1,j}} \right) = - \frac{d_{l,k}^{(2)} - d_{l,k}^{(eq,2)}}{\tau_{l+k}}. \] (34)

Equations (27) and (34) are consistent with the kinetic generator results, Table 1 in the previous study [3]. Next, these equations are considered at certain \( l \) and \( k \) since they lead to the set of the thermo-hydrodynamic equations. After a careful study of these relations and taking into account the consistency with the physical laws, we have formulated the nonlocal equilibrium Eqs. (5–8) as one of the examples in the choice of \( d_{l,k}^{(eq,1)} \) and \( d_{l,k}^{(eq,2)} \).
For convenience, symbolic forms of $d_{l,k}^{(eq)}$ are derived in Appendix A and summarized in Table 1, in which the second, third, and fourth columns show the derived results for each $\epsilon$ order.

Through the following derivations, the details of which are presented in Appendix B, Appendix C, and Appendix D, we find that $\tau_0 = \tau_1 = \tau_2 = \tau_3 = \tau$ is a desirable condition for deriving the continuity, Navier–Stokes, and heat transfer equations. Therefore, the MRT scheme is not effective for deriving the thermo-hydrodynamic equations in this study. However, it may be useful in future studies of higher order dynamics.

Where $\{l, k\} = \{0, 0\}$, Eqs. (27) and (34) can be written as,

$$\frac{\partial \rho}{\partial t} + c \frac{\partial \rho u_j}{\partial x_{1,j}} = 0,$$

(35)

$$\frac{\partial \rho}{\partial t} = 0,$$

(36)

respectively. Assembling them, we obtain the continuity equation,

$$\frac{\partial \rho}{\partial t} + c \frac{\partial \rho u_j}{\partial x_j} = 0,$$

(37)

with the second $\epsilon$ order accuracy.

Where $\{l, k\} = \{1, 0\}$, Eq. (27) is written as,

$$\frac{\partial \rho \mu_x}{\partial t_1} + c \frac{\partial \rho \mu_x u_j}{\partial x_{1,j}} = -c \frac{\partial}{\partial x_1} \left( \rho T - \beta (\rho T - P) \right).$$

(38)

When $\beta = 1$, the right hand side becomes $-c \frac{\partial P}{\partial x_1}$. On the other hand, in the case of ideal monoatomic gas, namely $P = \rho T$, the right hand side also becomes $-c \frac{\partial P}{\partial x_1}$ when $\beta = 0$. As a result, for both cases, Eq. (38) can be written as,

$$\frac{\partial \rho \mu_x}{\partial t_1} + c \frac{\partial \rho \mu_x u_j}{\partial x_{1,j}} = -c \frac{\partial P}{\partial x_1}.$$

(39)

Where $\{l, k\} = \{1, 0\}$, Eq. (34) is written as,

$$\frac{\partial \rho \mu_x}{\partial t_2} = c \frac{\partial \Sigma_{1,x,j}}{\partial x_{1,j}},$$

(40)

where $\beta = 1$ and,

$$\frac{\partial \rho \mu_x}{\partial t_2} = c \frac{\partial \Pi_{1,x,j}}{\partial x_{1,j}},$$

(41)

in the case of monoatomic ideal gas when $\beta = 0$. Here the first subscript 1 for $\sigma$ and $\Pi$ denotes the subscript for $x$ and $y$ in all spatial derivatives in $\sigma$ and $\Pi$.

Combining Eqs. (39) and (40), we obtain the Navier–Stokes equation,

$$\frac{\partial \rho \mu_x}{\partial t} + c \frac{\partial \rho \mu_x u_j}{\partial x_j} = -c \frac{\partial P}{\partial x} + c \frac{\partial \sigma_{x,j}}{\partial x_j},$$

(42)

with the second $\epsilon$ order accuracy where $\beta = 1$. As shown in Eqs. (39) and (41), in the case of monoatomic ideal gas at $\beta = 0$, Eq. (42) can also be derived but its stress tensor has the shear viscosity of $c \tau P$ and zero bulk viscosity, which are consistent with previous studies [1, 2].
Taking the sum of Eq. (27) at \( \{l, k\} = \{2, 0\} \) and Eq. (27) at \( \{l, k\} = \{0, 2\} \), we obtain,

\[
\frac{\partial \rho T}{\partial t_1} + c \frac{\partial u_j \rho T}{\partial x_{1,j}} = -cP \frac{\partial u_j}{\partial x_{1,j}},
\]

(43)

when \( \beta = 1 \) and in the case of monoatomic ideal gas when \( \beta = 0 \).

Taking the sum of Eq. (34) at \( \{l, k\} = \{2, 0\} \) and Eq. (34) at \( \{l, k\} = \{0, 2\} \), we obtain,

\[
\frac{\partial \rho T}{\partial t_2} = c \left( u_k \frac{\partial \sigma_{1,j,k}}{\partial x_{1,j}} + \frac{\partial}{\partial x_{1,j}} \left( \kappa \frac{\partial T}{\partial x_{1,j}} \right) \right),
\]

(44)

when \( \beta = 1 \),

\[
\frac{\partial \rho T}{\partial t_2} = c \left( -\frac{\partial c u_k \Pi_{1,j,k}}{\partial x_{1,j}} + 2 \frac{\partial}{\partial x_{1,j}} \left( c\tau P \frac{\partial T}{\partial x_{1,j}} \right) \right),
\]

(45)

in the case of monoatomic ideal gas when \( \beta = 0 \). Combining Eqs. (43) and (44), we obtain the heat-transfer equation,

\[
\frac{\partial \rho T}{\partial t} + c \frac{\partial u_j \rho T}{\partial x_j} = -cP \frac{\partial u_j}{\partial x_j} + c \left( u_k \frac{\partial \sigma_{j,k}}{\partial x_j} + \frac{\partial}{\partial x_j} \left( \kappa \frac{\partial T}{\partial x_j} \right) \right),
\]

(46)

with the second \( \epsilon \) order accuracy when \( \beta = 1 \). As shown in Eqs. (43) and (45), in the case of monoatomic ideal gas when \( \beta = 0 \), Eq. (46) can be derived as well but the thermal conductivity \( \kappa \) is 2\( c\tau P \) and the stress tensor has shear viscosity of \( c\tau P \) and zero bulk viscosity, which are consistent with previous studies [1, 2].

4 The Conservation Laws, the Boltzmann H-Theorem, and the Fluctuation-Dissipation Theorem

The conservation laws for mass, momentum, and energy are discussed according to the non-local equilibrium state, Eq. (5). They can be checked by taking moments of the collision term, specifically moments of the difference between \( f \) and \( \tilde{f}^{eq} \). The zeroth moment of \( f \) and \( \tilde{f}^{eq} \) are obviously \( d_{0,0} = \tilde{d}_{0,0} = \rho \) as shown in Appendix A. The first moment of \( f \) and \( \tilde{f}^{eq} \) are \( d_{1,0} = \tilde{d}_{1,0} = 2\rho u_x \) and \( d_{1,0}^{eq} = \tilde{d}_{1,0}^{eq} = 2\beta c\tau_1 \frac{\partial E_p}{\partial x} \) as shown in Appendix A. The second term of \( d_{1,0}^{eq} \) comes from the renormalized velocity in Eq. (6),

\[
\tilde{u}_i = \rho u_i + \beta c\tau_1 \frac{\partial E_p}{\partial x_i}.
\]

The correction term, \( \tilde{u}_{term1} \) contributes to the pressure term in the Navier–Stokes equation as shown in Appendix C, replacing the pressure force of the monoatomic ideal gas with the pressure force of a specified EOS. In addition, \( \tilde{u}_{term1} \) helps to achieve the second-order accuracy in the continuity equation. Since \( \tilde{u}_{term1} \) disappears when it is integrated over the space with the zero boundary contributions, the conservation of momentum is guaranteed for any \( \beta \). The second moment of \( f \) and \( \tilde{f}^{eq} \) are \( d_{2,0} = \tilde{d}_{2,0} = 8E - 4\rho \) and \( d_{2,0}^{eq} + \tilde{d}_{2,0}^{eq} = 8\tilde{E} - 4\rho \) where \( \tilde{E} = \rho \tilde{T} + \frac{1}{2} \rho \tilde{u}^2 \) as shown in Appendix A. Here, the renormalized temperature is
defined in Eq. (7),

\[ \rho \dot{T} = \rho T + \beta c \tau_2 \begin{pmatrix} E_p \frac{\partial u_j}{\partial x_j} - \frac{c \tau_1}{2 \rho} \left| \frac{\partial E_p}{\partial x_i} \right|^2 \end{pmatrix}. \]

The first correction term, ”Term1”, replaces the compression term in Eq. (46) from the ideal monoatomic gas by the compression term from the ordinary fluids with the same motivation as ”Term1”. The last correction term, ”Term2”, removes the unnecessary nonlinear term originated from \( \mu \bar{\theta}^2 \) for the energy conservation. As shown in Appendix A, using Eqs. (6) and (7), \( q_{0,2}^{eq} + q_{0,2}^{eq} = 8 \rho T + 4 \rho u^2 - 4 \rho + 8 c \tau \frac{3 E_p a_{\eta j}}{\partial x_j} \). The last term disappears after the integration over the space. As a result, the spatial integration for \( d_{2,0} + d_{0,2} \) is equivalent to one for \( d_{2,0} + d_{0,2}^{eq} \). It guarantees the conservation of the energy \( E \).

The consistency with the Boltzmann H-theorem brings an advantage for numerical stability [21, 22, 32–34]. It is also well known that the BGK Boltzmann equation with the Maxwellian distribution for the equilibrium state satisfies the H-theorem. Namely, the following functional \( \mathcal{H} \) decreases monotonically,

\[ \mathcal{H} = \int \int f \ln f \, dv \, dx, \]  

and has its minimum where \( f = f^{eq} \). For the non-local equilibrium state, Eq. (5), using the conservation of mass, momentum, and energy, the monotonic behavior of \( \mathcal{H} \) toward \( f^{eq} \) can be shown if the deviation of the pressure formulation from the ideal-gas EOS, \( E_p \), is sufficiently small, \( E_p \leq \mathcal{O}(\varepsilon \cdot \max \{ \mu, \tilde{\kappa} \} / c \tau) \). To show the monotonic behavior of \( \mathcal{H} \), after multiplying \( \ln f + 1 \) by the BGK Boltzmann equation, Eq. (1), and taking integration for \( v \) and \( x \), one can derive,

\[ \frac{d \mathcal{H}}{dt} = -\frac{1}{\tau} \int \int \left( f - f^{eq} \right) \left( \ln f - \ln f^{eq} \right) d v d x - \frac{1}{\tau} \int \int \left( f - f^{eq} \right) \ln f^{eq} d v d x. \]  

Due to the monotonicity of the logarithm, the first kernel on the right-hand side is negative. When the second kernel is expanded by \( \varepsilon \), its leading order is,

\[ -\int \int \left( f - f^{eq} \right) \ln f^{eq} d v d x \approx -\varepsilon^2 \int \int \left( f^{(1)} - f^{(eq,1)} \right) \frac{f^{(eq,1)}}{f^{(eq,0)}} d v d x \leq 0, \]

whose derivation is shown in Appendix E in the case of \( \mu \geq 0 \) and \( \tilde{\kappa} \geq 0 \) which can be obtained by adjusting \( c \) and \( \tau \) if necessary. As a result, \( \frac{d \mathcal{H}}{dt} \leq 0 \). If we consider the following functional \( \mathcal{L} \) for minimizing \( \mathcal{H} \) under conservation of mass, momentum, and energy,

\[ \mathcal{L} = \int \int \left( f \ln f + l_0 f + l_1 f v + l_2 f v^2 \right) d v d x, \]

where \( l_0, l_1, \) and \( l_2 \) are the Lagrange multipliers, then we find that \( f \) satisfying \( \frac{\delta \mathcal{L}}{\delta f} = 0 \) takes the Gaussian form. Therefore, the equilibrium state of Eq. (5) can satisfy this criterion with a proper choice of \( l_0, l_1, \) and \( l_2 \) if \( \delta f^{eq} \) is ignored. Since \( \delta f^{eq} \) contributes to the dissipation where \( \mu \geq 0 \) and \( \tilde{\kappa} \geq 0 \) and its effects decrease with time, the functional \( \mathcal{H}(f) \) is likely to decrease monotonically toward \( \mathcal{H}(f^{eq}) \) via \( \mathcal{H}(f^{eq}) \).

Next, the consistency with the fluctuation-dissipation theorem is examined. In this paragraph, we assume \( E_p \leq \mathcal{O}(\varepsilon) \) and choose an inertial frame in which the local fluid velocity
\( u|_{t} \) is zero. Using the x-y component of a microscopic pressure tensor \( P_{xy} \) and the microscopic heat flux \( S_{i} \), the transport coefficients are formulated in two spatial dimensions as,

\[
\mu = \frac{c}{T} \int_{0}^{\infty} \langle P_{xy}(t) P_{xy}(0) \rangle dt, \quad (51)
\]

\[
\kappa = \frac{c}{2T^{2}} \int_{0}^{\infty} \langle S_{i}(t)S_{i}(0) \rangle dt. \quad (52)
\]

Here the angle bracket denotes the time-correlation function. Assuming that \( f \) is close to \( \tilde{f}^{eq} \), the right-hand side of Eq. (51) is written as;

\[
\text{RHS of Eq. (51)} = \frac{c}{T} \int \int \int_{0}^{\infty} v_{x} v_{y} v'_{x} v'_{y} \tilde{f}^{eq} (v) \mathcal{P} \left( v', t|v, 0 \right) dtdvdv'. \quad (53)
\]

Here \( \mathcal{P} \left( v', t|v, 0 \right) \) is the conditional probability of finding the particle with velocity \( v' \) at time \( t \) where the particle with velocity \( v \) was found at time 0. It can be written with the functional derivative as,

\[
\mathcal{P} \left( v', t|v, 0 \right) = \frac{\delta f \left( v', t \right)}{\delta f \left( v, 0 \right)} = \frac{\delta f |_{v',t}}{\delta f |_{v,0}} + \int_{0}^{t} \int \frac{\delta f |_{v',t}}{\delta \tilde{f}^{eq} |_{v'',t'}} \frac{\delta \tilde{f}^{eq} |_{v'',t'}}{\delta f |_{v,0}} dv''dt' + \cdots . \quad (54)
\]

The first term on the right-hand side represents the correlation between an initial state and a state after advection. It should be the exponential decay in time due to the collision during the advection of the particle. The second term on the right-hand side represents the correlation between an initial state and an advected state after a collision. Specifically, particles collide after advecting from an initial state with velocity \( v \) and then follow the equilibrium state. After that, the equilibrium state advects back to its initial position at time \( t \) with particle velocity \( v' \), which is opposite to \( v \). When both of these two terms are explicitly written, they become

\[
\text{Eq. (54)} = \exp \left( -\frac{t}{\tau} \right) \delta \left( v' - v \right) + \int_{0}^{t} \int \exp \left( -\frac{t-t'}{\tau} \right) \delta \left( v' - v'' \right) \frac{\delta \tilde{f}^{eq} |_{v'',t'}}{\delta f |_{v,0}} \delta \left( t' - \tau \right) dv''dt' + \cdots . \quad (55)
\]

Here, to simplify the time integration for \( t' \), typical time for the collision \( \tau \) is chosen using the Dirac delta function \( \delta \left( t' - \tau \right) \). The initial state \( f |_{v,0} \) contributes to the equilibrium state \( \tilde{f}^{eq} |_{v'',t'} \) via the macroscopic physical quantity \( X |_{t'} \), \( X = \{ \rho, v_{\alpha}, T, \nabla_{\alpha} v_{\beta}, \nabla_{\alpha} T \} \) where \( \alpha \) and \( \beta \) are indices for the spatial direction. Using Eq. (54) and the functional derivative of \( X |_{t'} \) with respect to \( \delta f |_{v,0} \),

\[
\frac{\delta X |_{t'}}{\delta f |_{v,0}} = \left\{ \delta \left( t' \right) + \delta \left( v \right) \exp \left( -\frac{t'}{\tau} \right) \frac{v_{\alpha}}{\rho} \delta \left( t' \right) \frac{v^{2} - 2T}{2\rho} \right\} + \left\{ \delta \left( t' \right) + \delta \left( v \right) \exp \left( -\frac{t'}{\tau} \right) \frac{v_{\mu}}{\rho v_{\alpha} t'} \exp \left( -\frac{t'}{\tau} \right) \frac{-1}{v_{\alpha} \tau} \frac{v^{2} - 2T}{2\rho} \exp \left( -\frac{t'}{\tau} \right) \right\}, \quad (56)
\]
whose derivations are given in Appendix F, Eq. (53) can be written as,

\[ \text{Eq. (53)} \approx \tau c \rho T + \frac{\bar{\mu}}{\rho^2 T^3} \int \int v_y^2 \bar{f}^{eq}(v) v_x^2 v_y^2 f^{eq}(v') dv dv' = \tau c \rho T + \bar{\mu} \approx \mu, \]  

(57)

whose detailed derivations are given in Appendix F. Similarly, the right-hand side of Eq. (52) can be written as,

\[ \text{RHS of Eq. (52)} \approx \frac{c^2}{2T^2} \int \int_0^\infty \left( \frac{v_x^2 + v_y^2}{2} - 2T \right) \left( \frac{v_x'^2 + v_y'^2}{2} - 2T \right) \bar{f}^{eq}(v) \mathcal{P}(v', t|v, 0) dt dv dv', \]

\[ \approx 2c \rho T \tau + \bar{\kappa} \approx \kappa \]  

(58)

using Eqs. (54), (55), and (56). Again the detailed derivation is given in Appendix F. As a result, the non-local equilibrium state shows consistency with the fluctuation-dissipation theorem where \( E_p \leq \mathcal{O}(\epsilon) \).

### 5 Discussion

The nonlocal equilibrium state, Eq. (5), successfully leads to the set of thermo-hydrodynamic equations with flexible choice of transport coefficients and EOS. It includes the long-range interaction with the non-local terms in the renormalized fluid velocity and temperature, Eqs. (6) and (7), and an additional state to the Gaussian distribution, Eq. (8). The BGK Boltzmann equation with this equilibrium state is consistent with the law of mass, momentum, and energy conservation. It also shows consistency with the Boltzmann H-theorem, where \( E_p \leq \mathcal{O}(\epsilon \cdot \max \{\bar{\mu}, \bar{\kappa}\} / c \tau) \), and the fluctuation-dissipation theorem, where \( E_p \leq \mathcal{O}(\epsilon) \).

Such a constraint on \( E_p \), the deviation from the ideal gas EOS, may sound reasonable, since the strong particle interaction can lead to the large modification of the EOS.

In previous studies of multiphase flow [35–38], the body force from the pseudopotential has been taken into account to realize the phase separation, leading to the EOS modification. Such a force term introduces nonlocal information into the collision process, similar to the present nonlocal pseudo-equilibrium state. The velocity modification in Eq. (6) may correspond to some such models, if \( E_p \) is assumed to come from the pseudopotential. In addition, the temperature has been corrected as shown in Eq. (7) for better Galilean invariance and accurate treatment of compression effects. This additional correction may be useful for future improvements of the numerical models.

The ES-BGK and Shaknov-BGK models discussed in Sect. 1 modified the mathematical form of the collision term to allow a flexible choice of the Prandtl number, similar to the present model. Some particle methods based on these models have already shown promising results for the gas flow in the nozzle and channels [10, 39, 40]. They are expected to handle the Knudsen number effects in the slip flow regime and the transition flow regime. However, due to the density and temperature dependence of the transport coefficients, their extensive application to the continuum flow regime is limited. In order to establish a kinetic-based approach that can cover a wide range of Knudsen numbers, the combination between such models and the present model may be promising after the present model is transferred to the discrete space for numerical simulation.

In contrast to the present model, the collision model in the previous studies [7, 8, 12, 19–22] incorporates the nonlocal information using the second and third order moments of \( f \) and their deviation from \( f^{eq} \). That is, they used the following approximate relation obtained...
at the $\epsilon$ order of the Chapman-Enskog expansion under the assumption of negligible time variation,

$$
\int (v_\alpha v_\beta \cdots) c v_j \frac{\partial f^{(0)}}{\partial x_{1,j}} dv \approx - \int (v_\alpha v_\beta \cdots) \left( f^{(1)} - f^{eq,1} \right) dv / \tau, \quad (59)
$$

where $n \geq 2$, which indicates that the second and the third-order moments of $f$ or its deviation from $f^{eq}$ are proportional to the derivatives of the fluid velocity and temperature at equilibrium. In particular, in these previous studies, the distribution function $f$ is considered as,

$$
f = f^{eq}_M + f^{neq}, \quad (60)
$$

where $f^{eq}_M$ is the Maxwellian distribution function and $f^{neq}$ is a state deviating from $f^{eq}_M$ whose moments contain the non-local information of $\rho$, $u$ and $T$. In the multiple-time relaxation time (MRT) scheme, the relaxation time for $f^{neq}$ is defined for each moment respectively so that the correct hydrodynamic equation is derived $[12, 19, 20, 41]$. On the other hand, in this study, $f$ is considered as,

$$
f = f^{eq}_M + \delta \hat{f}^{eq} + \hat{f}^{neq}, \quad (61)
$$

where $\delta \hat{f}^{eq} = \hat{f}^{eq} - f^{eq}_M$ using $\hat{f}^{eq}$ defined in Eq. (5) and $\hat{f}^{neq}$ is the rest of the non-equilibrium state. Following the relaxation process discussed in Sect. 2, we find that the relaxation time $\tau$ in Eq. (1) defines how fast $f^{neq}$ decays and the state $f$ is changed to $f^{eq}_M + \delta \hat{f}^{eq}$, which can be varied in time and space with respect to the transport coefficients. This relaxation scheme seems to be closer to the quasi-equilibrium scheme $[21, 22, 30]$ than to the MRT scheme. The practical role of $\tau$ in Eq. (1) can be understood as follows. If one carefully examines Eqs. (1), (6), and (7), one can see that $\tau$ and $c$ can appear with their products everywhere except for the time derivative term in Eq. (1). It shows that the roles of $c$ and $\tau$ are similar except for effects on the time scale. According to the derived macroscopic equations, Eqs. (37), (42) and (46), the reduction of $c$ leads to the reduction of the time scale. In addition, at higher $\epsilon$ orders than the second order, higher spatial derivative terms should appear, bringing the higher power of $c$ from the advection term in Eq. (1). Therefore, the reduction of $c$ leads to the reduction of the truncation error terms in the macroscopic equations. Consequently, reducing $\tau$ is likely to reduce the effects of the truncation terms without changing the time scale, although the choice of its magnitude is limited compared to $c$. From the point of view of the relaxation processes in Sect. 2, $\tau$ is relevant to the relaxation from $f$ to $g^{eq,1}$, which does not affect the non-equilibrium phenomena related to viscosity and thermal conductivity, but is related to the high $\epsilon$ order physics and stability in the numerical simulation.

A collision operator based on the Enskog theory or the Revised Enskog Theory (RET) has been proposed as an extended Boltzmann collision operator for dense particle systems taking into account the excluded volume interaction and the particles’ correlation $[42]$ and has been widely applied to complex fluids such as the colloidal flow, the active liquids, and the multiphase flows $[43–49]$. Indeed, the rheological properties in colloidal flow $[44, 45]$ and the phase transition in multiphase flow $[47, 49]$ are successfully reproduced in previous studies. Although such a collision operator is expected to describe non-equilibrium physics better than the BGK collision operator, its complicated mathematical structure prohibits many engineering applications at the macroscopic scale. To solve this problem, the RET collision operator is sometimes mapped to the BGK collision operator $[43, 44, 50]$. However, the RET collision operator is nonzero on the Maxwellian distribution in most cases, which is
not realized with the BGK collision operator, even with its MRT scheme version. On the other hand, the present scheme in this study can avoid this discrepancy with the virtue of the nonlocal equilibrium state form in Eq. (5). As a result, once the transport coefficients and the equation of state are estimated, for example, using Grad’s method in the RET collision operator [43–45, 50] and/or the Virial expansion [51], they can be mapped to the extended BGK collision operator in this study in the leading order accuracy of the Navier–Stokes equation. The same strategy is also available when the potential force term is integrated into the kinetic equation, called the Vlasov-Enskog equation [46]. If it is necessary to consider the effects of higher order moments, the MRT scheme can be applied for \( \tilde{f}_{\text{neq}} \) in Eq. (61) and their moments can correspond to the moments of the RET collision operator.

The connection with the extended Boltzmann equation introduced above can help the development of advanced models for the analysis tool of particle image velocimetry (PIV) and particle tracking velocimetry (PTV), popular fluid velocity detection tools in experimental fluid dynamics [52–54]. With the recent advancement of high-speed camera and the discovery of high refractive materials, the tracer particle size can be reduced to a few nanometers, which helps to avoid the disturbance of the flow field and apply to the microfluidic flow [55]. Moreover, the pronounced fluctuation due to the Brownian motion gives us more information such as the temperature using the fluctuation-dissipation theorem [56]. However, its behavior in complex fluids and its dependence on the non-equilibrium states due to, for example, the flow and temperature fields are not yet fully understood. Since the extended Boltzmann equation can produce the Fokker-Plank equation in a scaling limit [57], the drift and diffusion coefficients can be formulated with the microscopic physical quantities such as the particle-particle interaction potential. Higher order terms can also be systematically derived. This new approach will hopefully contribute to the development of advanced analytical tools for the experiments.

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Declarations

Conflict of interest  We have no conflicts of interest to disclose. All authors declare that they have no conflicts of interest.

Appendix A: Moments of \( \tilde{f}_{\text{eq}} \) in Eq. (5)

The non-local equilibrium state \( \tilde{f}_{\text{eq}} \) defined in Eq. (5) is projected into the Hermite space in two spatial dimensions. For convenience, the moments of the Maxwellian-Boltzmann-type distribution \( f_{\text{eq}} \) in Eq. (2) are written in followings,
\[
\left( \begin{array}{c}
1 \\
v_x \\
v_x^2 \\
v_x^3 \\
v_x^4 \\
v_x^5 \\
v_x^6
\end{array} \right) f_{eq} dv_x dv_y = \left( \begin{array}{c}
\rho u_x \\
\rho T + \rho u_x^2 \\
3 \rho T u_x + \rho u_x^3 \\
3 \rho T^2 + 6 \rho u_x^2 T + \rho u_x^4 \\
15 \rho T^2 u_x + 10 \rho u_x^3 T + \rho u_x^5 \\
15 \rho T^3 + 45 \rho T^2 u_x^2 + 15 \rho T u_x^4 + \rho u_x^6
\end{array} \right).
\]  
(A.1)

\[
\left( \begin{array}{c}
v_x v_y \\
v_x v_y^2 \\
v_x^2 v_y \\
v_x^2 v_y^2
\end{array} \right) f_{eq} dv_x dv_y = \left( \begin{array}{c}
\rho u_x u_y \\
\rho u_x (T + u_y^2) \\
\rho (T + u_y^2) (T + u_y^2)
\end{array} \right).
\]  
(A.2)

For \( d_{eq}^{0,0} \), the equilibrium state is integrated as,

\[
d_{eq}^{0,0} = \int f_{eq} dv_x dv_y = \rho.
\]  
(A.3)

It is because

\[
\int \delta f_{eq} dv_x dv_y = -\beta \tilde{\mu} \frac{\partial u_l}{\partial x_k} \int f_{eq} \left( (v_l - u_l) (v_k - u_k) - \frac{(v - u)^2}{2} \delta_{l,k} \right) dv_x dv_y
\]

\[
- \beta \kappa \frac{\partial T}{\partial x_k} \int f_{eq} \left( (v - u)^2 - 4T \right) (v_k - u_k) dv_x dv_y = 0,
\]  
(A.4)

using Eqs. (8), (A.1), and (A.2).

For \( d_{eq}^{1,0} \), using Eqs. (5) and (6), the integration can be done as,

\[
d_{eq}^{1,0} = 2 \int \int f_{eq} v_x dv_x dv_y = 2 \rho \tilde{u}_x = 2 \rho u_x + 2 \beta c T \frac{\partial E_p}{\partial x}.
\]  
(A.5)

It is because

\[
\int \delta f_{eq} v_x dv_x dv_y = -\beta \tilde{\mu} \frac{\partial u_l}{\partial x_k} \int v_x f_{eq} \left( (v_l - u_l) (v_k - u_k) - \frac{(v - u)^2}{2} \delta_{l,k} \right) dv_x dv_y
\]

\[
- \beta \kappa \frac{\partial T}{\partial x_k} \int v_x f_{eq} \left( (v - u)^2 - 4T \right) (v_k - u_k) dv_x dv_y = 0,
\]  
(A.6)

using Eqs. (8), (A.1), and (A.2).
For $d_{2,0}^{eq}$, using Eqs. (6) and (7), we obtain,

$$
d_{2,0}^{eq} = \int \int \tilde{f}^{eq} (4v_x^2 - 2) \, dv_x \, dv_y = 4 \left( T + u_x^2 \right) - 2 \rho - 2 \beta \bar{\mu} \left( \bar{\sigma}_{x,x} - \bar{\sigma}_{y,y} \right)
$$

$$
= 4 (\rho T + \mu u_x^2) - 2 \rho + 4 \beta c \tau_2 \left\{ \int \frac{\partial u_j}{\partial x} \left| \frac{\partial E_p}{\partial x} \right|^2 \right\}
+ 8 \beta c \tau_1 \int \frac{\partial E_p}{\partial x} + 4 \beta^2 c^2 \tau_1 \left( \frac{\partial E_p}{\partial x} \right)^2 - 2 \beta \bar{\mu} \left( \bar{\sigma}_{x,x} - \bar{\sigma}_{y,y} \right)
$$

$$
= 4 (\rho T + \mu u_x^2) - 2 \rho + 4 \beta c \tau_2 \int \frac{\partial u_j}{\partial x} \left| \frac{\partial E_p}{\partial x} \right|^2 + 4 \beta^2 c^2 \tau_1 \left( \frac{\partial E_p}{\partial x} \right)^2 . \quad (A.7)
$$

It is because

$$
\int \delta f^{eq} v_x^2 \, dv_x \, dv_y = - \beta \frac{\bar{\mu}}{\rho T^2} \int \frac{\partial u_j}{\partial x} \left( v_l - u_l \right) \left( v_k - u_k \right) - \frac{(v - u)^2}{2} \delta_{l,k} \, dv_x \, dv_y
$$

$$
- \beta \frac{\bar{\kappa}}{4 \rho T^3} \int \frac{\partial T}{\partial x} \left( v_l - u_l \right) \left( v_k - u_k \right) \, dv_x \, dv_y
$$

$$
= - \frac{\beta}{2} \bar{\mu} \left( \frac{\partial u_x}{\partial y} \right) - \frac{\beta}{2} \bar{\mu} \left( \frac{\partial u_y}{\partial x} \right) = - \frac{\beta}{2} \bar{\mu} \left( \bar{\sigma}_{x,x} - \bar{\sigma}_{y,y} \right) . \quad (A.8)
$$

using Eqs. (8), (A.1), and (A.2). Therefore where $\tau_1 = \tau_2 = \tau$ and $\beta = 1$, the sum of $d_{2,0}^{eq} + d_{0,2}^{eq}$ becomes,

$$
d_{2,0}^{eq} + d_{0,2}^{eq} = 8 \rho T + 4 \rho u_x^2 - 4 \rho + 8 \beta c \tau \frac{\partial E_p u_j}{\partial x} . \quad (A.9)
$$

For $d_{1,1}^{eq}$, using Eq. (6), we obtain,

$$
d_{1,1}^{eq} = 4 \int \int \tilde{f}^{eq} v_x v_y \, dv_x \, dv_y = 4 \rho \bar{u}_x \bar{u}_y - 4 \beta \bar{\mu} \bar{\sigma}_{x,y}
$$

$$
= 4 \rho u_x u_y + 4 \beta c \tau_1 \left\{ \frac{\partial E_p}{\partial y} + \frac{\partial E_p}{\partial x} \right\} - 4 \beta \bar{\mu} \bar{\sigma}_{x,y} . \quad (A.10)
$$

It is because

$$
\int \delta f^{eq} v_x v_y \, dv_x \, dv_y = - \beta \frac{\bar{\mu}}{\rho T^2} \int \frac{\partial u_l}{\partial x} \left( v_l - u_l \right) \left( v_k - u_k \right) - \frac{(v - u)^2}{2} \delta_{l,k} \, dv_x \, dv_y
$$

$$
- \beta \frac{\bar{\kappa}}{4 \rho T^3} \int \frac{\partial T}{\partial x} \left( v_l - u_l \right) \left( v_k - u_k \right) \, dv_x \, dv_y
$$

$$
= - \frac{\beta}{2} \bar{\mu} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) = - \beta \bar{\mu} \bar{\sigma}_{x,y} , \quad (A.11)
$$

using Eqs. (8), (A.1), and (A.2).
For \( d_{3,0}^{eq} \), using Eqs. (6) and (7), we obtain,

\[
d_{3,0}^{eq} = \int \int \tilde{f}^{eq} \left( 8v_3^3 - 12v_3 \right) dv_x dv_y = 8\rho \tilde{u}_3^3 + 24\tilde{u}_x \rho \tilde{T} - 12\rho \tilde{u}_x
\]

\[
- 12\beta \tilde{u}_x (\tilde{\sigma}_{x,x} - \tilde{\sigma}_{y,y}) - 12\beta \tilde{k} \frac{\partial T}{\partial x}
\]

\[
= 8\rho \tilde{u}_3^3 + 24\tilde{u}_x \rho T - 12\rho \tilde{u}_x + (24u_x^2 + 24\tilde{T} - 12) \beta c \tau_1 \frac{\partial E_p}{\partial x} + 24u_x \beta c \tau_2 E_p \frac{\partial u_j}{\partial x_j}
\]

\[
- 12\beta \tilde{u}_x (\tilde{\sigma}_{x,x} - \tilde{\sigma}_{y,y}) - 12\beta \tilde{k} \frac{\partial T}{\partial x} + H.O.
\]  

(A.12)

It is because

\[
\int \delta f^{eq} \left( 8v_3^3 - 12v_3 \right) dv_x dv_y
\]

\[
= -\frac{\beta}{\rho T^2} \frac{\partial u_l}{\partial x_k} \int 8v_3^3 f^{eq} \left\{ (v_l - u_l)(v_k - u_k) - \frac{(v - u)^2}{2}\delta_{l,k} \right\} dv_x dv_y
\]

\[
- \frac{\beta}{4\rho T^3} \frac{\partial T}{\partial x_k} \int 8v_3^3 f^{eq} \left\{ (v - u)^2 - 4T \right\} (v_k - u_k) dv_x dv_y,
\]

\[
= -12\beta \tilde{u}_x (\tilde{\sigma}_{x,x} - \tilde{\sigma}_{y,y}) - 12\beta \tilde{k} \frac{\partial T}{\partial x}.
\]  

(A.13)

using Eqs. (8), (A.1), and (A.2).

For \( d_{2,1}^{eq} \), using Eqs. (6) and (7), we obtain,

\[
d_{2,1}^{eq} = \int \int \tilde{f}^{eq} \left( 8v_2^2 v_y - 4v_y \right) dv_x dv_y
\]

\[
= 8\rho u_y u_y + 4u_x \rho T - 4\rho u_y + 16u_x u_y \beta c \tau_1 \frac{\partial E_p}{\partial x} + (8u_x^2 + 8\tilde{T} - 4) \beta c \tau_1 \frac{\partial E_p}{\partial y}
\]

\[
+ 8u_y \beta c \tau_2 E_p \frac{\partial u_j}{\partial x_j} - 8\beta \tilde{u}_x (u_y \tilde{\sigma}_{x,x} + u_x (\tilde{\sigma}_{x,y} + \tilde{\sigma}_{y,x})) - 4\beta \tilde{k} \frac{\partial T}{\partial y} + H.O.
\]  

(A.14)

It is because

\[
\int \delta f^{eq} \left( 8v_2^2 v_y - 4v_y \right) dv_x dv_y
\]

\[
= -\frac{\beta}{\rho T^2} \frac{\partial u_l}{\partial x_k} \int 8v_2^2 v_y f^{eq} \left\{ (v_l - u_l)(v_k - u_k) - \frac{(v - u)^2}{2}\delta_{l,k} \right\} dv_x dv_y
\]

\[
- \frac{\beta}{4\rho T^3} \frac{\partial T}{\partial x_k} \int 8v_2^2 v_y f^{eq} \left\{ (v - u)^2 - 4T \right\} (v_k - u_k) dv_x dv_y,
\]

\[
= -8\beta \tilde{u}_x (u_y \tilde{\sigma}_{x,x} + u_x (\tilde{\sigma}_{x,y} + \tilde{\sigma}_{y,x}) - 4\beta \tilde{k} \frac{\partial T}{\partial y}
\]  

(A.15)

using Eqs. (8), (A.1), and (A.2).

For \( d_{4,0}^{eq} \), using Eqs. (6) and (7), we obtain,

\[
d_{4,0}^{eq} = \int \int \tilde{f}^{eq} \left( 16u_x^4 - 48u_x^2 + 12 \right) dv_x dv_y = 48\rho (T^2 + 2u_x^2 T + \tilde{u}_x^4) - 48\rho T - 48\rho u_x^2
\]

\[
+ 12\rho + H.O.
\]  

(A.16)
### Table 1  Symbolical forms of the projected equilibrium state in Eq. (5) into the Hermite space for each rank of Hermite polynomials (raw) and each $\epsilon$ order (column)

|      | Zeroth | First | Second |
|------|--------|-------|--------|
| $d^\text{eq}_{0,0}$ | $\rho$ | 0 | 0 |
| $d^\text{eq}_{1,0}$ | $2\rho u_x$ | $2\beta c_1 \frac{\partial E_p}{\partial x}$ | 0 |
| $d^\text{eq}_{2,0}$ | $4\left(\rho T + \rho u_x^2\right) - 2\rho$ | $4\beta c_2 E_p \frac{\partial u_j}{\partial x_j} + 8\beta c_1 u_x \frac{\partial E_p}{\partial x}$ | $-2\beta c_2^2 \frac{\partial E_p}{\partial x_j} \left| \frac{\partial E_p}{\partial x_j} \right|^2$ |
| $d^\text{eq}_{1,1}$ | $4\rho u_x u_y$ | $4\beta c_1 \left( u_x \frac{\partial E_p}{\partial y} + u_y \frac{\partial E_p}{\partial x} \right)$ | $-4\beta \mu \tilde{\sigma}_{x,y}$ |
| $d^\text{eq}_{2,1}$ | $8\rho u_x^2 u_y + 8u_y \rho T - 12\rho u_x$ | $\beta c_1 \left( 24u_x^2 + 24T - 12 \right) \frac{\partial E_p}{\partial x}$ | $-12\beta \mu \tilde{u}_x \left( \tilde{\sigma}_{x,x} - \tilde{\sigma}_{y,y} \right) - 12\beta \tilde{\kappa} \frac{\partial T}{\partial x}$ |
| $d^\text{eq}_{1,2}$ | $8\rho u_x^2 u_y + 8u_y \rho T - 4\rho u_y$ | $16u_x u_y \beta c_1 \frac{\partial E_p}{\partial x}$ | $-8\beta \mu \left( u_y \tilde{\sigma}_{x,x} + u_x \left( \tilde{\sigma}_{x,y} + \tilde{\sigma}_{y,x} \right) \right) - 4\beta \tilde{\kappa} \frac{\partial T}{\partial y}$ |
| $d^\text{eq}_{2,2}$ | $48\rho \left( T^2 + 2u_x^2 T + u_x^4 \right)$ | $-T - u_x^2 + \frac{1}{4}$ | $-T - u_x^2 + \frac{1}{4}$ |
| $d^\text{eq}_{3,1}$ | $16\rho u_y \left( u_x^3 + 3u_x T - \frac{3}{2}u_x \right)$ | $-T - u_x^2 + \frac{1}{4}$ | $-T - u_x^2 + \frac{1}{4}$ |
| $d^\text{eq}_{2,3}$ | $16\rho \left( T + u_x^2 \right) \left( T + u_x^2 \right)$ | $-T - u_x^2 + \frac{1}{4}$ | $-T - u_x^2 + \frac{1}{4}$ |
For $d_{3,1}^{eq}$, using Eqs. (6) and (7), we obtain,
\[
d_{3,1}^{eq} = \int \int \tilde{f}^{eq} (16u_x^3v_y - 24v_xv_y) \, dv_x \, dv_y = 16 (\rho u_x^3 + 3 \rho u_x T) \, u_y - 24 \rho u_x \, u_y + H.O. \tag{A.17}
\]

For $d_{2,2}^{eq}$, using Eqs. (6) and (7), we obtain,
\[
d_{2,2}^{eq} = \int \int \tilde{f}^{eq} (16u_x^2v_y^2 - 8 (v_x^2 + v_y^2) + 4) \, dv_x \, dv_y
= 16 \rho \left( T + u_x^2 \right) \left( T + u_y^2 \right) - 16 \rho T - 8 \rho (u_x^2 + u_y^2) + 4 \rho + H.O. \tag{A.18}
\]

The results are summarized in Table 1.

**Appendix B: Derivation of the Continuity Equation**

Where \{l, k\} = \{0, 0\}, Eq. (27) is
\[
\frac{\partial d_{0,0}^{(0)}}{\partial t_1} + c \frac{\partial D_{0,0}^{(0)}}{\partial x_{1,j}} = - \frac{d_{0,0}^{(1)} - d_{0,0}^{(eq,1)}}{\tau_0}. \tag{B.1}
\]

Because we defined $d_{0,0} = d_{0,0}^{eq}$ and $d_{0,0}^{(eq,p)} = 0$ for $p > 0$, $d_{0,0}^{(p)} = 0$ for $p > 0$. Using this fact and Table 1, Eq. (B.1) becomes,
\[
\frac{\partial \rho}{\partial t_1} + c \frac{\partial \rho u_j}{\partial x_{1,j}} = 0. \tag{B.2}
\]

which is the continuity equation in the $\epsilon$ order. It is because
\[
D_{x,0,0}^{(0)} = \frac{d_{1,0}^{(0)}}{2} = \frac{d_{1,0}^{(eq,0)}}{2},
\]
\[
D_{y,0,0}^{(0)} = \frac{d_{0,1}^{(0)}}{2} = \frac{d_{0,1}^{(eq,0)}}{2}. \tag{B.3}
\]

Where \{l, k\} = \{0, 0\}, Eq. (34) is
\[
\frac{\partial d_{0,0}^{(0)}}{\partial t_2} + \frac{\partial d_{0,0}^{(1)}}{\partial t_1} + c \frac{\partial D_{0,0}^{(eq,1)}}{\partial x_{1,j}} - c \left( \frac{\partial^2 \tilde{D}_{0,0}^{(0)}[d^{(0)}]}{\partial t_1 \partial x_{1,j}} + c \frac{\partial^2 \tilde{D}_{0,0}^{(0)}[D_{0,0}^{(0)}]}{\partial x_{1,j} \partial x_{1,j}} \right) = - \frac{d_{0,0}^{(2)} - d_{0,0}^{(eq,2)}}{\tau_0}. \tag{B.4}
\]

Using $d_{0,0}^{(p)} = d_{0,0}^{(eq,p)} = 0$ for $p > 0$ and Table 1, it can be written as,
\[
\frac{\partial \rho}{\partial t_2} + c \frac{\partial D_{0,0}^{(eq,1)}}{\partial x_{1,j}} - c \left( \frac{\partial^2 \tilde{D}_{0,0}^{(0)}[d^{(0)}]}{\partial t_1 \partial x_{1,j}} + c \frac{\partial^2 \tilde{D}_{0,0}^{(0)}[D_{0,0}^{(0)}]}{\partial x_{1,j} \partial x_{1,j}} \right) = 0. \tag{B.5}
\]

The left second term in Eq. (B.5) can be written as,
\[
\frac{\partial D_{j,0,0}^{(eq,1)}}{\partial x_{1,j}} = \frac{1}{\partial x_1} \left( \frac{d_{j,0}^{(eq,1)}}{2} \right) + \frac{1}{\partial y_1} \left( \frac{d_{0,1}^{(eq,1)}}{2} \right) = \beta c t_1 \frac{\partial^2 E_p}{c t_{term}^1}. \tag{B.6}
\]
The left third term in Eq. (B.5) can be written as,

\[
\frac{\partial^2 D_{i,0,0}[d^{(0)}]}{\partial t_1 \partial x_{1,j}} = \tau_1 \frac{\partial}{\partial t_1} \left( \frac{\partial}{\partial x_1} \left( \frac{d_{1,0}^{(eq,0)}}{2} \right) + \frac{\partial}{\partial y_1} \left( \frac{d_{0,1}^{(eq,0)}}{2} \right) \right),
\]

\[
= \tau_1 c \begin{cases} 
\frac{\partial}{\partial x_{1,l}} \left( \frac{\partial \rho u_j u_l}{\partial x_{1,j}} - \frac{\partial P}{\partial x_{1,l}} \right) \end{cases}.
\]  

(B.7)

Here we used the Navier–Stokes equation in the \( \epsilon \) order, Eq. (C.5). The left fourth term in Eq. (B.5) can be written as,

\[
\frac{\partial^2 D_{i,0,0}[D^{(0)}_{j,0,0}]}{\partial x_{1,i} \partial x_{1,j}} = -c \tau_1 \frac{1}{2} \left( \frac{\partial^2 \rho T}{\partial x_{1,i} \partial x_{1,j}} + \frac{\partial^2 \rho u_x^2}{\partial x_{1,i} \partial y_{1,j}} + \frac{\partial^2 \rho u_y^2}{\partial y_{1,i} \partial x_{1,j}} + 2 \frac{\partial^2 \rho u_x u_y}{\partial x_{1,i} \partial y_{1,j}} \right). 
\]

(C.1)

Combining Eqs. (B.6), (B.7), and (B.8), we see

\[
[cterm1] + [cterm3] + [cterm4] = 0,
\]

(B.9)

where \( \beta = 1 \) or \( \beta = 0 \) in the case of the ideal monoatomic gas. In addition,

\[
[cterm2] + [cterm5] + [cterm6] = 0.
\]

(B.10)

As a result, Eq. (B.5) becomes,

\[
\frac{\partial \rho}{\partial t_2} = 0.
\]

(B.11)

which is the continuity equation in the \( \epsilon^2 \) order.

Assembling the temporal derivatives up to \( \epsilon^2 \) order, we derive the continuity equation,

\[
\frac{\partial \rho}{\partial t} + c \frac{\partial \rho u_j}{\partial x_j} = 0,
\]

(B.12)

with the second-order accuracy of \( \epsilon \).

**Appendix C: Derivation of the Navier–Stokes Equation**

Where \( \{l, k\} = \{1, 0\} \), Eq. (27) is

\[
\frac{\partial d_{1,0}^{(0)}}{\partial t_1} + c \frac{\partial T_{j,1,0}^{(0)}}{\partial x_{1,j}} = -\frac{d_{1,0}^{(1)} - d_{1,0}^{(eq,1)}}{\tau_1}.
\]

(C.1)
Because we define \( d_{1,0} = d_{1,0}^{(eq,0)} = 2 \rho u_x, d_{1,0}^{(p)} = 0 \) for \( p > 0 \). In addition, using Table 1, Eq. (C.1) can be written as,

\[
2 \frac{\partial \rho u_x}{\partial t_1} + c \frac{\partial D_{j,1,0}^{(0)}}{\partial x_{1,j}} = 2 \beta c \frac{\partial E_p}{\partial x_1}. \tag{C.2}
\]

Moreover, because

\[
\frac{\partial D_{j,1,0}^{(0)}}{\partial x_{1,j}} = \frac{\partial}{\partial x_1} \left( \frac{d_{2,0}^{(0)}}{2} + d_{0,0}^{(0)} \right) + \frac{\partial}{\partial y_1} \left( \frac{d_{1,1}^{(0)}}{2} \right),
\]

\[
= \frac{\partial}{\partial x_1} \left( 2 \left( \rho T + \rho u_x^2 \right) \right) + \frac{\partial}{\partial y_1} \left( 2 \rho u_x u_y \right), \tag{C.3}
\]

Eq. (C.2) can be written as,

\[
\frac{\partial \rho u_x}{\partial t_1} + c \frac{\partial \rho u_x u_j}{\partial x_{1,j}} = -c \frac{\partial}{\partial x_1} \left( \rho T - \beta E_p \right). \tag{C.4}
\]

Where \( \beta = 1 \) or in the case of the ideal monoatomic gas at \( \beta = 0 \), it becomes

\[
\frac{\partial \rho u_x}{\partial t_1} + c \frac{\partial \rho u_x u_j}{\partial x_{1,j}} = -c \frac{\partial P}{\partial x_1}. \tag{C.5}
\]

which is the Navier–Stokes equation in the \( \epsilon \) order.

Where \( \{ l, k \} = \{ 1, 0 \}, \) Eq. (34) is

\[
\frac{\partial d_{1,0}^{(0)}}{\partial t_2} + \frac{\partial d_{1,0}^{(1)}}{\partial t_1} + c \frac{\partial D_{j,1,0}^{(eq,1)}}{\partial x_{1,j}} - c \left( \frac{\partial^2 \tilde{D}_{j,1,0} [d_{1,0}^{(0)}]}{\partial t_1 \partial x_{1,j}} + c \frac{\partial^2 \tilde{D}_{j,1,0} [D_{j,1,0}^{(0)}]}{\partial x_{1,j} \partial x_{1,j}} \right) = -\frac{d_{1,0}^{(2)} - d_{1,0}^{(eq,2)}}{\tau_1}. \tag{C.6}
\]

Using \( d_{1,0}^{(p)} = 0 \) for \( p > 0 \) and Table 1, it can be written as,

\[
\frac{2}{c} \frac{\partial \rho u_x}{\partial t_2} = -\frac{\partial D_{j,1,0}^{(eq,1)}}{\partial x_{1,j}} + \frac{\partial^2 \tilde{D}_{j,1,0} [d_{1,0}^{(0)}]}{\partial t_1 \partial x_{1,j}} + c \frac{\partial^2 \tilde{D}_{j,1,0} [D_{j,1,0}^{(0)}]}{\partial x_{1,j} \partial x_{1,j}}. \tag{C.7}
\]

The first term in the right-hand side can be written as,

\[
- \frac{\partial D_{j,1,0}^{(eq,1)}}{\partial x_{1,j}} = - \frac{\partial}{\partial x_1} \left( \frac{d_{2,0}^{(eq,1)}}{2} - d_{0,0}^{(eq,1)} \right) - \frac{\partial}{\partial y_1} \left( \frac{d_{1,1}^{(eq,1)}}{2} \right),
\]

\[
= - \frac{\partial}{\partial x_1} \left\{ 2 \beta c \tau_2 E_p \frac{\partial u_j}{\partial x_{1,j}} + 4 \beta c \tau_1 u_x \frac{\partial E_p}{\partial x_1} \right\} - \frac{\partial}{\partial y_1} \left\{ 2 \beta c \tau_1 \left( u_x \frac{\partial E_p}{\partial y_1} + u_y \frac{\partial E_p}{\partial x_1} \right) \right\}
\]

\[
+ \beta \left( \frac{\partial \tilde{\mu} (\tilde{\sigma}_{x,x} - \tilde{\sigma}_{y,y})}{\partial x_1} + 2 \frac{\partial \tilde{\mu} \tilde{\sigma}_{x,y}}{\partial y_1} \right)
\]

\[
= -2 \beta c \left\{ \frac{\partial}{\partial x_1} \left\{ 2 \frac{\partial E_p u_x}{\partial x_1} - E_p \left( \frac{\partial u_x}{\partial x_1} - \frac{\partial u_y}{\partial y_1} \right) \right\} \right.
\]

\[
+ \frac{\partial}{\partial y_1} \left\{ \rho \frac{T u_x}{\partial y_1} - \rho T \frac{\partial u_x}{\partial y_1} + \rho \frac{T u_y}{\partial x_1} - \rho T \frac{\partial u_y}{\partial x_1} - u_x \frac{\partial P}{\partial y_1} - u_y \frac{\partial P}{\partial x_1} \right\}
\]

\[
+ 2 \beta \left( \frac{\partial \sigma_{1,x,j}}{\partial x_{1,j}} - c \frac{\partial \Pi_{1,x,j}}{\partial x_{1,j}} \right)
\]
\[-2\beta \tau c \left[ 2 \frac{\partial^2 E_p u_x}{\partial x_1^2} + \frac{\partial}{\partial x_1} \left( -\rho T + P \right) \left( \frac{\partial u_x}{\partial x_1} - \frac{\partial u_y}{\partial y_1} \right) + \frac{\partial^2 \rho T u_x}{\partial y_1^2} \right. \\
+ \left. \frac{\partial^2 \rho T u_y}{\partial x_1 \partial y_1} - \frac{\partial}{\partial y_1} \left( \frac{\partial P}{\partial x_1} u_y + \frac{\partial P}{\partial y_1} u_x \right) + \frac{\partial}{\partial y_1} \left( -\rho T \left( \frac{\partial u_x}{\partial y_1} + \frac{\partial u_y}{\partial x_1} \right) \right) \right] + 2\beta \left( \frac{\partial \sigma_{1,j}}{\partial x_1,j} - c \frac{\partial \Pi_{1,j}}{\partial x_1,j} \right) \right]

\[-2\beta \tau c \left[ 2 \frac{\partial^2 E_p u_x}{\partial x_1^2} + \frac{\partial}{\partial x_1} \left( P \left( \frac{\partial u_x}{\partial x_1} - \frac{\partial u_y}{\partial y_1} \right) \right) + \frac{\partial^2 \rho T u_x}{\partial y_1^2} \right] \]

\[
\begin{align*}
\frac{\partial^2 \bar{u}_{0,0}[y_0]}{\partial x_1,j} &= \frac{\partial}{\partial t_1} \left( \frac{\partial}{\partial x_1} \left( \frac{\tau_2 d_{2,0}}{2} + \tau_0 d_{0,0} \right) + \frac{\partial}{\partial y_1} \left( \frac{\tau_2 d_{1,1}}{2} \right) \right), \\
&= \frac{\tau_2}{2} \left( 4(\rho T + \rho u_x^2) - 2\rho \right) + \frac{\tau_2}{2} \frac{\partial^2}{\partial t_1 \partial y_1} + \tau_0 \frac{\partial^2 \rho}{\partial t_1 \partial x_1}. \quad \text{(C.9)}
\end{align*}
\]

If we use the continuity equation in the \( \epsilon \) order, Eq. (B.2), and the Navier–Stokes equation in the \( \epsilon \) order, Eq. (C.5), we can derive,

\[
\rho \frac{\partial u_x}{\partial t_1} = -u_x \frac{\partial \rho}{\partial t_1} - c \frac{\partial \rho u_x u_j}{\partial x_1,j} - c \frac{\partial P}{\partial x_1}, \\
= c u_x \frac{\partial \rho u_j}{\partial x_1,j} - c \frac{\partial \rho u_x u_j}{\partial x_1,j} - c \frac{\partial P}{\partial x_1}, \\
= -c \left( \rho u_j \frac{\partial u_x}{\partial x_1,j} \right) - c \frac{\partial P}{\partial x_1}. \quad \text{(C.10)}
\]

Using it, Eqs. (B.2), and (C.5), we can derive,

\[
\begin{align*}
\frac{\partial \rho u_x^2}{\partial t_1} &= u_x^2 \frac{\partial \rho}{\partial t_1} + 2u_x \frac{\partial u_x}{\partial t_1}, \\
&= -u_x^2 c \left( \frac{\partial \rho u_j}{\partial x_1,j} \right) - 2c \left( \rho u_x u_j \frac{\partial u_x}{\partial x_1,j} \right) - 2c u_x \frac{\partial P}{\partial x_1}, \\
&= -c \left( \frac{\partial \rho u_x^2}{\partial x_1,j} + 2u_x \frac{\partial P}{\partial x_1} \right). \quad \text{(C.11)}
\end{align*}
\]
Here we used Eqs. (B.2), (C.5), the heat-transfer equation in the 

\[ \frac{\partial \rho u_x u_y}{\partial t_1} = \rho u_x \frac{\partial u_y}{\partial t_1} + u_y \frac{\partial \rho u_x}{\partial t_1}, \]

\[ = -c \left( \rho u_x \frac{\partial u_y}{\partial y_1} + \rho u_x \frac{\partial u_y}{\partial x_1} \right) - cu_x \frac{\partial P}{\partial y_1} - c \left( u_y \frac{\partial \rho u_x}{\partial x_1} + u_y \frac{\partial \rho u_x}{\partial y_1} - u_y \frac{\partial P}{\partial x_1} \right), \]

Applying Eqs. (C.11) and (C.12) to (C.9), where \( \tau_0 = \tau_2 = \tau \) we obtain,

\[ \frac{\partial^2 \tilde{D}_{j,0,0}[d^{(0)}]}{\partial t_1 \partial x_{1,j}} = -c \frac{\tau}{2} \left[ \frac{\partial}{\partial x_1} \left( \frac{\partial u_j}{\partial x_1} + \frac{\partial P}{\partial x_1} \right) + \frac{\partial}{\partial y_1} \left( \frac{\partial \rho u_x u_j}{\partial x_1} + u_x \frac{\partial P}{\partial y_1} + u_y \frac{\partial P}{\partial x_1} \right) \right], \]

\[ = -2c \tau \left[ \frac{\partial (\rho T + 2P)}{\partial x_1} \frac{\partial^2 \rho u_x T}{\partial x_1 \partial y_1} + \frac{\partial^2 \rho u_x T}{\partial x_1 \partial y_1} + \frac{\partial}{\partial x_1} \left( P \left( \frac{\partial u_x}{\partial x_1} + \frac{\partial u_y}{\partial y_1} \right) \right) + \frac{\partial}{\partial y_1} \left( u_x \frac{\partial P}{\partial y_1} + u_y \frac{\partial P}{\partial x_1} \right) \right], \]

\[ + \frac{\partial^2 \rho u_x^3}{\partial x_1^2} + 2 \frac{\partial^2 \rho u_x u_y}{\partial x_1 \partial y_1} + \frac{\partial^2 \rho u_x u_y}{\partial y_1^2}. \]  

(C.13)

Here we used Eqs. (B.2), (C.5), the heat-transfer equation in the \( \epsilon \) order, Eq. (D.4).

The third term in the right-hand side of Eq. (C.7) can be written as,

\[ \frac{\partial^2 \tilde{D}_{j,1,0}[D^{(0)}_{j,1,0}]}{\partial x_{1,i} \partial x_j} = c \left[ \frac{\partial^2}{\partial x_1^2} \left( \frac{\tau_2}{2} \left( \frac{d_{3,0}^{(0)}}{2} + 2d_{1,0}^{(0)} \right) + \frac{d_{1,0}^{(0)}}{2} \tau_0 \right) + \frac{\partial^2}{\partial y_1^2} \left( \frac{\tau_2}{2} \left( \frac{d_{2,1}^{(0)}}{2} + 2d_{1,0}^{(0)} \right) \right) \right], \]

\[ = c \tau \left[ \frac{\partial^2}{\partial x_1^2} \left( 2\rho u_x^3 + 6u_x \rho T \right) + \frac{\partial^2}{\partial y_1^2} \left( 2\rho u_x u_y^2 + 2u_x \rho T \right) \right], \]

\[ + \frac{\partial^2}{\partial x_1 \partial y_1} \left( 2\rho u_x^2 u_y + 4u_x \rho T + 2\rho u_x^2 u_y \right), \]

\[ = c \tau \left[ \frac{\partial^2}{\partial x_1^2} \left( 6\rho u_x T \right) + \frac{\partial^2}{\partial y_1^2} \left( 2\rho u_x T \right) + \frac{\partial^2}{\partial x_1 \partial y_1} \left( 4\rho u_x T \right) \right]. \]
Assembling Eqs. (C.8), (C.13), and (C.14), we obtain,

\[
\begin{align*}
\rho \frac{\partial u_x}{\partial t_2} &= \frac{\partial}{\partial x_1} \left( \rho \frac{\partial u_x}{\partial x_1} - \frac{\partial}{\partial y_1} \left( \frac{\partial P u_x}{\partial y_1} + \frac{\partial}{\partial x_1} \left( \frac{\partial P u_y}{\partial x_1} + \frac{\partial}{\partial y_1} \left( u_x \frac{\partial P}{\partial y_1} + u_y \frac{\partial P}{\partial x_1} \right) \right) \right), \\
\rho \frac{\partial u_y}{\partial t_2} &= \frac{\partial}{\partial y_1} \left( \rho \frac{\partial u_y}{\partial y_1} - \frac{\partial}{\partial x_1} \left( \frac{\partial P u_y}{\partial x_1} + \frac{\partial}{\partial y_1} \left( u_x \frac{\partial P}{\partial x_1} + u_y \frac{\partial P}{\partial y_1} \right) \right) \right), \\
\frac{\partial}{\partial x_1} \frac{\partial}{\partial y_1} \left( \frac{\partial P}{\partial x_1} + \frac{\partial P}{\partial y_1} \right) &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}.
\end{align*}
\]

As a result, where \( \beta = 1 \), all of them are canceled or negligible. Only the remaining term in the right-hand side is the last term of Eq. (C.8). Therefore, Eq. (C.7) results in,

\[
\frac{\partial \rho u_x}{\partial t_2} = c \frac{\partial \sigma_{1,x,j}}{\partial x_{1,j}},
\]

which is the Navier–Stokes equation in the \( \epsilon^2 \) order.

On the other hand, in the case of the ideal monoatomic gas, \( P = \rho T \), at \( \beta = 0 \), they becomes

\[
\begin{align*}
\rho \frac{\partial u_x}{\partial t_2} &= \frac{\partial}{\partial x_1} \left( \rho \frac{\partial u_x}{\partial x_1} - \frac{\partial}{\partial y_1} \left( \frac{\partial P u_x}{\partial y_1} + \frac{\partial}{\partial x_1} \left( \frac{\partial P u_y}{\partial x_1} + \frac{\partial}{\partial y_1} \left( u_x \frac{\partial P}{\partial y_1} + u_y \frac{\partial P}{\partial x_1} \right) \right) \right) \right), \\
\rho \frac{\partial u_y}{\partial t_2} &= \frac{\partial}{\partial y_1} \left( \rho \frac{\partial u_y}{\partial y_1} - \frac{\partial}{\partial x_1} \left( \frac{\partial P u_y}{\partial x_1} + \frac{\partial}{\partial y_1} \left( u_x \frac{\partial P}{\partial x_1} + u_y \frac{\partial P}{\partial y_1} \right) \right) \right), \\
\frac{\partial}{\partial x_1} \frac{\partial}{\partial y_1} \left( \frac{\partial P}{\partial x_1} + \frac{\partial P}{\partial y_1} \right) &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}.
\end{align*}
\]

As a result, Eq. (C.7) results in,

\[
\frac{\partial \rho u_x}{\partial t_2} = c^2 \frac{\partial \sigma_{1,x,j}}{\partial x_{1,j}},
\]
which is equivalent to Eq. (C.21) if the stress tensor $\sigma$ has shear viscosity of $cP \tau$ and zero bulk viscosity.

Assembling the temporal derivatives up to $\epsilon^2$ orders, at $\beta = 1$ we derive the Navier–Stokes equation,

$$\frac{\partial \rho u_x}{\partial t} + c \frac{\partial \rho u_x u_j}{\partial x_j} = -c \frac{\partial P}{\partial x} + c \frac{\partial \sigma_{x,j}}{\partial x_j}. \quad \text{(C.25)}$$

with the second-order accuracy of $\epsilon$. In the case of the ideal monoatomic gas at $\beta = 0$, the stress tensor $\sigma$ has shear viscosity of $cP \tau$ and zero bulk viscosity.

### Appendix D: Derivation of the Heat Transfer Equation

The sum of Eq. (27) at $\{l, k\} = \{2, 0\}$ and Eq. (27) at $\{l, k\} = \{0, 2\}$ is,

$$\frac{\partial d_{2,0}^{(0)} + d_{0,2}^{(0)}}{\partial t_1} + c \frac{\partial D_{j,2,0}^{(0)} + D_{j,0,2}^{(0)}}{\partial x_{1,j}} = - \frac{d_{2,0}^{(1)} - d_{2,0}^{(eq,1)}}{\tau_2}. \quad \text{(D.1)}$$

Because we define $d_{2,0} + d_{0,2} = d_{2,0}^{(eq,0)} + d_{0,2}^{(eq,0)} = 8 \rho T + 4 \rho u^2 - 4 \rho, d_{2,0}^{(p)} + d_{0,2}^{(p)} = 0$ for $p > 0$. In addition, using Table 1, Eq. (D.1) becomes,

$$\frac{\partial (8 \rho T + 4 \rho u^2 - 4 \rho)}{\partial t_1} + c \frac{\partial u_j}{\partial x_{1,j}} \left(4 \rho u_j^3 + 16 \rho u_j \rho T - 4 \rho u_j + 4 \rho u_j u_j\right)
= 8 \beta c E_p \frac{\partial u_j}{\partial x_{1,j}} + 8 \beta c \frac{\tau_1}{\tau_2} u_j \frac{\partial E_p}{\partial x_{1,j}}. \quad \text{(D.2)}$$

If we apply the continuity equation in the $\epsilon$ order, Eqs. (B.2), (C.11), and its extension for $\frac{\partial \rho u_x^2}{\partial t_1}$, where $\tau_1 = \tau_2$, it becomes,

$$8 \frac{\partial \rho T}{\partial t_1} - 8c \left( \frac{\partial P u_j}{\partial x_{1,j}} - P \frac{\partial u_j}{\partial x_{1,j}} \right) + 16c \frac{\partial u_j \rho T}{\partial x_{1,j}} = 8 \beta c \frac{\partial E_p u_j}{\partial x_{1,j}}. \quad \text{(D.3)}$$

When $\beta = 1$, Eq. (D.3) can be arranged as,

$$\frac{\partial \rho T}{\partial t_1} + c \frac{\partial u_j \rho T}{\partial x_{1,j}} = -c P \frac{\partial u_j}{\partial x_{1,j}}, \quad \text{(D.4)}$$

which is the heat transfer equation in the $\epsilon$ order.

On the other hand, in the case of the ideal monoatomic gas, namely $P = \rho T$ at $\beta = 0$, Eq. (D.3) becomes,

$$\frac{\partial \rho T}{\partial t_1} + c \frac{\partial u_j \rho T}{\partial x_{1,j}} = -c P \frac{\partial u_j}{\partial x_{1,j}}, \quad \text{(D.5)}$$

which is equivalent to Eq. (D.4) and the heat transfer equation in the $\epsilon$ order.
The sum of Eq. (34) at \( l,k \) = \{2,0\} and Eq. (34) at \( l,k \) = \{0,2\} is,
\[
\frac{\partial \delta T^{(0)}_{l,0} + \delta T^{(0)}_{0,2}}{\partial t} + c \frac{\partial D^{(eq,1)}_{j,2,0} + D^{(eq,1)}_{j,0,2}}{\partial x_{1,j}} - c \left( \frac{\partial^2 D^{(eq,1)}_{j,2,0} + D^{(eq,1)}_{j,0,2}}{\partial x_{1,j} \partial x_{1,j}} \right) + \frac{c^2}{\tau_2} \left( \frac{\partial^2 D^{(eq,1)}_{j,2,0} + D^{(eq,1)}_{j,0,2}}{\partial x_{1,j} \partial x_{1,j}} \right)
\]
\[
- \frac{d_2^{(eq,2)} + d_{0,2}^{(eq,2)}}{\tau_2} = 0.
\]

(D.6)

Here we used \( d_2^{(p)} + d_{0,2}^{(p)} = 0 \) for \( p > 0 \).

For convenience, the following equation is derived using the continuity and Navier–Stokes

\[
\frac{\partial \rho u_i^2}{\partial t} = 2u_i \frac{\partial \rho u_i}{\partial t} = 2c u_i \frac{\partial \sigma_{i,i,j}}{\partial x_{1,j}}.
\]

(D.7)

In addition, using Eqs. (C.10), (C.11), the Navier–Stokes and the heat transfer equation in the \( \epsilon^2 \) order, Eqs. (B.11) and (C.21),
\[
\frac{\partial \rho u_i^3}{\partial t} = u_x \frac{\partial \rho u_i^3}{\partial t} + \rho u_x^2 \frac{\partial u_i}{\partial t} - 3 c u_x^2 \frac{\partial \rho}{\partial x_1} + \mathcal{O} \left( \frac{\partial \rho u_i^4}{\partial x_1} \right),
\]

(D.8)

\[
\frac{\partial \rho u_i u_j}{\partial t} = u_x \frac{\partial \rho u_i u_j}{\partial t} + \rho u_y^2 \frac{\partial u_i}{\partial t} - 2 c u_x u_y \frac{\partial \rho}{\partial y_1} - c u_y \frac{\partial \rho}{\partial x_1} + \mathcal{O} \left( \frac{\partial \rho u_i^4}{\partial x_1} \right),
\]

(D.9)

\[
\frac{\partial \rho T u_i}{\partial t} = u_x \frac{\partial \rho T u_i}{\partial t} + \rho T \frac{\partial u_i}{\partial t} = -c u_x \left( \frac{\partial \rho T u_i}{\partial x_{1,j}} + \rho \frac{\partial u_i}{\partial x_{1,j}} \right) + T c \left( \rho u_j \frac{\partial u_i}{\partial x_{1,j}} + \frac{\partial \rho}{\partial x_1} \right).
\]

(D.10)

Using Table 1, the continuity equation in the \( \epsilon^2 \) order, Eqs. (B.11), and (D.7), the first term in the left-hand side of Eq. (D.6) becomes,
\[
\frac{\partial d_2^{(0)} + d_{0,2}^{(0)}}{\partial t} = 8 \frac{\partial \rho T + 4 \rho u_i^2 - 4 \rho}{\partial t} = 8 \frac{\partial \rho T}{\partial t} + 8 c u_i \frac{\partial \sigma_{i,i,j}}{\partial x_{1,j}}.
\]

(D.11)

Using Table 1, where \( \tau_1 = \tau_2 = \tau \), the second term in the left-hand side of Eq. (D.6) becomes,
\[
c \frac{\partial D^{(eq,1)}_{j,2,0} + D^{(eq,1)}_{j,0,2}}{\partial x_{1,j}} = c \frac{\partial}{\partial x_1} \left( \frac{d_2^{(eq,1)} + 2 d_{1,0}^{(eq,1)} + d_{1,2}^{(eq,1)}}{2} \right) + (x \leftrightarrow y),
\]

\[
= \beta c^2 \left\{ \frac{\partial}{\partial x_1} \left[ 4 u_x^2 + 8 u_x^2 + 16 T - 4 \frac{\partial E_P}{\partial x_1} + 16 u_x E_P \frac{\partial u_j}{\partial x_{1,j}} \right] + \frac{\partial}{\partial x_1} \left[ 8 u_x u_y \frac{\partial E_P}{\partial y_1} \right] \right\}
\]

\[
+ \beta c \frac{\partial}{\partial x_1} \left[ 6 \tilde{\mu} u_x (\tilde{\sigma}_{x,x} - \tilde{\sigma}_{y,y}) - 6 \kappa \frac{\partial T}{\partial x} - 4 \tilde{\mu} \left( u_x \tilde{\sigma}_{y,y} + u_y (\tilde{\sigma}_{x,x} + \tilde{\sigma}_{y,y}) - 2 \kappa \frac{\partial T}{\partial x_1} \right) \right] + (x \leftrightarrow y),
\]

\[
= \beta c^2 \left\{ \frac{\partial}{\partial x_1} \left[ 4 u_x^2 + 8 u_x^2 + 16 T - 4 \frac{\partial E_P}{\partial x_1} + 16 u_x E_P \frac{\partial u_j}{\partial x_{1,j}} \right] + \frac{\partial}{\partial x_1} \left[ 8 u_x u_y \frac{\partial E_P}{\partial y_1} \right] \right\}
\]

\[
+ 8 \beta c \frac{\partial}{\partial x_1} \left( \kappa \frac{\partial T}{\partial x_1} \right) - 8 \beta c \frac{\partial}{\partial x_1} \left( \tilde{\mu} u_j \tilde{\sigma}_{x,x} \right) + (x \leftrightarrow y)
\]

(D.12)
Here, \((x \leftrightarrow y)\) denotes the commuted terms in which \(x\) and \(y\) are exchanged from the entire former terms.

Using Table 1, Eqs. (D.8), (D.9), (D.10), and the Navier–Stokes equation in the \(\epsilon\) order, Eq. (C.5), the third term in the left-hand side of Eq. (D.6) becomes,

\[
-c^2 \frac{\partial^2 \tilde{D}_{j,0}[d^{(0)}]}{\partial t_1 \partial x_{1,j}} + \tilde{D}_{j,0,2}[d^{(0)}]
\]

\[
= -c^2 \frac{\partial}{\partial t_1} \left[ \frac{\partial}{\partial x_1} \left( \frac{1}{2} \tau_3 d_{3,0}^{(0)} + 2 \tau_1 d_{1,0}^{(0)} + \frac{1}{2} \tau_3 d_{1,2}^{(0)} \right) \right] + (x \leftrightarrow y),
\]

\[
= -c^2 \frac{\partial}{\partial t_1} \left[ \frac{\partial}{\partial x_1} \left( 4 \rho u_x^3 + 16 \rho T u_x - 8 \rho u_x + 4 \rho u_x u_y^2 + 4 \tau_1 \rho u_x \right) \right] + (x \leftrightarrow y),
\]

\[
= -4c^2 \frac{\partial}{\partial x_1} \left[ -3 \tau_3 u_x^2 \frac{\partial P}{\partial x_1} - 2 u_x u_y \tau_3 \frac{\partial P}{\partial y_1} - u_y^2 \tau_3 \frac{\partial P}{\partial x_1} \right]
\]

\[
- \left( \frac{\partial P}{\partial x_1} + \frac{\partial \rho u_x^2}{\partial x_1} + \frac{\partial \rho u_x u_y}{\partial y_1} \right) \frac{-8 \tau_3 + 4 \tau_1}{4}
\]

\[
= -4 \tau_3 u_x \left( \frac{\partial \rho T u_j}{\partial x_{1,j}} + P \frac{\partial u_j}{\partial x_{1,j}} \right) - 4 \tau_3 \left( \rho u_j \frac{\partial u_x}{\partial x_{1,j}} + \frac{\partial P}{\partial x_1} \right)
\]

\[
+ \mathcal{O} \left( \frac{\partial^2 \rho u^4}{\partial x_1^2} \right) + (x \leftrightarrow y).
\]

(D.13)

Using Table 1, the fourth term in the left-hand side of Eq. (D.6) becomes,

\[
-c^2 \frac{\partial^2 \tilde{D}_{i,2,0}[D_{j,0}^{(0)}] + \tilde{D}_{i,0,2}[D_{j,0}^{(0)}]}{\partial x_{1,i} \partial x_{1,j}}
\]

\[
= -c^2 \left\{ \frac{\partial^2}{\partial x_1^2} \left[ \frac{3}{4} \tau_3 + \tau_1 \right] d_{2,0}^{(0)} + 2 \tau_1 d_{0,0}^{(0)} + \tau_3 \left( \frac{d_{2,2}^{(0)}}{4} + \frac{d_{0,2}^{(0)}}{2} \right) \right] \right.
\]

\[
+ \frac{\partial}{\partial x_1} \frac{\partial}{\partial y_1} \left[ 2 (\tau_1 + \tau_3) d_{1,0}^{(0)} + 2 \tau_3 d_{1,1}^{(0)} + d_{1,3}^{(0)} \right] \right\} + (x \leftrightarrow y),
\]

\[
= -c^2 \left\{ \frac{\partial^2}{\partial x_1^2} \left[ \frac{\rho T (8 \tau_3 + 4 \tau_1) + \rho u_x^2 (4 \tau_1 - 8 \tau_3) + \tau_3 (28 \rho u_x^2 + 4 \rho T u_x^2 + 16 \rho T^2) }{T H H O - A} \right] \right.
\]

\[
+ \frac{\partial}{\partial x_1} \frac{\partial}{\partial y_1} \left[ \frac{\rho u_x u_y (8 \tau_3 + 4 \tau_1) + \tau_3 (24 \rho u_x T u_y) }{T H H O - C} \right] \right\} + \mathcal{O} \left( \frac{\partial \rho u^4}{\partial x_1^2} \right) + (x \leftrightarrow y) .\]

(D.14)

Using Table 1, the last term in the left-hand side of Eq. (D.6) becomes,

\[
- \frac{d_{2,0}^{(eq.2)} + d_{0,2}^{(eq.2)}}{\tau_2} = 0.
\]

(D.15)
Next, we consider the sum of Eqs. (D.13) and (D.14). Because the sum for $[THHO - B]$ and $[THHO - C]$ are zero and the sum for $[THHO - A]$ is $c^2 \frac{\partial^2 E_p}{\partial x_1^2} (8\tau_3 - 4\tau_1)$, we have,

$$-c^2 \frac{\partial^2 \tilde{D}_{j,2.0}[d^{(0)}]}{\partial \tau_1 \partial x_{1,j}} - c^2 \frac{\partial^2 \tilde{D}_{i,2.0}[D^{(0)}_{j,2.0}] + \tilde{D}_{i,0.2}[D^{(0)}_{j,0.2}]}{\partial x_{1,i} \partial x_{1,j}}$$

$$= -4c^2 \frac{\partial}{\partial x_1} \left[ -3\tau_3 u_x^2 \frac{\partial P}{\partial x_1} - 2u_x u_y \tau_3 \frac{\partial P}{\partial y_1} - u_y^2 \tau_3 \frac{\partial P}{\partial x_1} + \frac{\partial E_p}{\partial x_1} \left( -\frac{8\tau_3 + 4\tau_1}{4} \right) \right]$$

$$-4\tau_3 u_x \left( \frac{\partial P}{\partial x_{1,j}} + P \frac{\partial u_j}{\partial x_{1,j}} \right) - 4T \tau_3 \left( \rho u_j \frac{\partial u_x}{\partial x_{1,j}} + \frac{\partial P}{\partial x_1} \right)$$

$$-c^2 \left\{ \frac{\beta^2}{\partial x_1^2} \left[ \tau_3 \left( 28\rho u_x^2 T + 4\rho T u_y^2 + 16\rho T^2 \right) \right] + \frac{\beta^2}{\partial x_{1,i} \partial y_1} \left[ 24\rho u_x T u_y \right] \right\}$$

$$+ \mathcal{O} \left( \frac{\partial \rho u^4}{\partial x_1^2} \right) + (x \leftrightarrow y),$$

where $\tau_3 = \tau_1 = \tau$.

Where $\beta = 1$, the sum of Eqs. (D.12), (D.15), and (D.16) can be written as,

$$c \frac{\partial D^{(eq,1)}_{j,2.0} + D^{(eq,1)}_{j,0.2}}{\partial x_{1,j}} - c \frac{\partial^2 \tilde{D}_{j,2.0}[d^{(0)}] + \tilde{D}_{j,0.2}[d^{(0)}]}{\partial \tau_1 \partial x_{1,j}}$$

$$= -c^2 \frac{\partial^2 \tilde{D}_{i,2.0}[D^{(0)}_{j,2.0}] + \tilde{D}_{i,0.2}[D^{(0)}_{j,0.2}]}{\partial x_{1,i} \partial x_{1,j}}$$

$$= -c^4 \frac{\partial}{\partial x_1} \left[ +8\rho T \left( u_x \frac{\partial u_x}{\partial x_1} - u_x \frac{\partial u_y}{\partial y_1} + u_y \frac{\partial u_x}{\partial x_1} + u_y \frac{\partial u_y}{\partial y_1} \right) + 16\rho T \frac{\partial T}{\partial x_1} \right]$$

$$= -8c \frac{\partial}{\partial x_1} \left( \frac{\partial T}{\partial x_1} \right) - 8c \frac{\partial \tilde{\mu}_i u_x}{\partial x_1} + \mathcal{O} \left( \frac{\partial \rho u^4}{\partial x_1^2} \right) + (x \leftrightarrow y),$$

$$= -8c \frac{\partial}{\partial x_1} \left( \Pi_{1,i,j} u_i + 2\rho T \tau \frac{\partial T}{\partial x_{1,j}} \right) - 8c \frac{\partial}{\partial x_{1,j}} \left( \frac{\partial T}{\partial x_{1,j}} \right)$$

$$-8c \frac{\partial \tilde{\mu}_i u_{i,j}}{\partial x_{1,j}} + \mathcal{O} \left( \frac{\partial \rho u^4}{\partial x_1^2} \right).$$

(D.17)

In the case of the monoatomic ideal gas at $\beta = 0$, the sum of Eqs. (D.12) and (D.16) can be written as,

$$c \frac{\partial D^{(eq,1)}_{j,2.0} + D^{(eq,1)}_{j,0.2}}{\partial x_{1,j}} - c \frac{\partial^2 \tilde{D}_{j,2.0}[d^{(0)}] + \tilde{D}_{j,0.2}[d^{(0)}]}{\partial \tau_1 \partial x_{1,j}}$$
\[-c^2 \frac{\partial^2 \tilde{D}_{i,0}[D^{(0)}_{j,0}] + \tilde{D}_{i,0}[D^{(0)}_{j,0}]}{\partial x_{1,i} \partial x_{1,j}} - \frac{d_{2,0}^{(eq,2)} + d_{0,2}^{(eq,2)}}{\tau_2} \]

\[= -c^2 \tau \frac{\partial}{\partial x_1} \left[ 8 \rho T \left( u_x \frac{\partial u_x}{\partial x_1} - u_x \frac{\partial u_y}{\partial y_1} + u_y \frac{\partial u_y}{\partial x_1} + u_y \frac{\partial u_x}{\partial y_1} \right) - 16 \rho T \frac{\partial T}{\partial x_1} \right] - 16c^2 \tau \frac{\partial^2 \rho T^2}{\partial x_1^2} + O \left( \frac{\partial u^4}{\partial x_1^2} \right) + (x \leftrightarrow y), \]

which is equivalent to Eq. (D.17) because \( \bar{\kappa} = \bar{\mu} = 0 \) and \( P = \rho T \) in this condition.

Lastly, assembling the above equation and Eq. (D.11), where \( \beta = 1 \), we can write Eq. (D.6) as,

\[8 \frac{\partial \rho T}{\partial t_2} + 8c u_i \frac{\partial \sigma_{i,j}}{\partial x_{1,j}} - 8c^2 \frac{\partial}{\partial x_{1,j}} \left[ \Pi_{1,i,j} u_i + 2P \tau \frac{\partial T}{\partial x_{1,j}} \right] - 8c \frac{\partial}{\partial x_{1,j}} \left( (\kappa - 2c \rho T \tau) \frac{\partial T}{\partial x_{1,j}} \right) - 8 \frac{\partial u_i}{\partial x_{1,j}} (\sigma_{1,i,j} - c \Pi_{1,i,j}) = 0, \quad (D.19) \]

which can be arranged further as,

\[\frac{\partial \rho T}{\partial t_2} = c \sigma_{1,i,j} \frac{\partial u_i}{\partial x_{1,j}} + c \frac{\partial}{\partial x_{1,j}} \left( \kappa \frac{\partial T}{\partial x_{1,j}} \right). \quad (D.20)\]

It is the heat transfer equation in the \( \epsilon^2 \) order.

In the case of the monoatomic ideal gas at \( \beta = 0 \), Eq. (D.6) can be written as,

\[8 \frac{\partial \rho T}{\partial t_2} + 8c u_i \frac{\partial \Pi_{1,i,j}}{\partial x_{1,j}} - 8c^2 \frac{\partial}{\partial x_{1,j}} \left[ \Pi_{1,i,j} u_i + 2P \tau \frac{\partial T}{\partial x_{1,j}} \right] = 0, \quad (D.21) \]

which can be arranged further as,

\[\frac{\partial \rho T}{\partial t_2} = c \Pi_{1,i,j} \frac{\partial u_i}{\partial x_{1,j}} + \frac{\partial}{\partial x_{1,j}} \left( 2c^2 P \tau \frac{\partial T}{\partial x_{1,j}} \right). \quad (D.22)\]

It is equivalent to Eq. (D.20) if the heat conductivity \( \kappa \) is \( 2c \rho T \) and the shear stress tensor has shear viscosity of \( cP \tau \) and zero bulk viscosity.

Assembling the temporal derivatives up to the \( \epsilon^2 \) order, one derives the heat-transfer equation,

\[\frac{\partial \rho T}{\partial t} + c \frac{\partial u_j \rho T}{\partial x_j} = -cP \frac{\partial u_j}{\partial x_j} + c \sigma_{i,j} \frac{\partial u_i}{\partial x_j} + c \frac{\partial}{\partial x_j} \left( \kappa \frac{\partial T}{\partial x_j} \right), \quad (D.23)\]

with the second-order accuracy of \( \epsilon \) at \( \beta = 1 \). In the case of the ideal monoatomic gas at \( \beta = 0 \), the heat conductivity \( \kappa \) is \( 2c \rho T \) and the shear stress tensor has shear viscosity of \( cP \tau \) and zero bulk viscosity.

**Appendix E: Derivation of the H-Theorem**

The functional \( \mathcal{H} \) defined in Eq. (47) is analyzed using the non-local equilibrium state in Eq. (5) with an assumption of \( E_p \leq O (\epsilon \cdot \max \{ \bar{\mu}, \bar{\kappa} \} / c\tau) \). In addition, \( \bar{\mu} \geq 0 \) and \( \bar{\kappa} \geq 0 \),
which can be realized by adjusting $c$ and $\tau$ if necessary, are assumed. To show the monotonic behavior of $\mathcal{H}$, after multiplying $\ln f + 1$ to the BGK Boltzmann equation, Eq. (1), and taking integration for $v$ and $x$, one can derive,

$$\frac{d\mathcal{H}}{dt} = -\frac{1}{\tau} \int \int (f - \tilde{f}^{eq}) \left( \ln f - \ln \tilde{f}^{eq} \right) d\nu dv dx - \frac{1}{\tau} \int \int (f - \tilde{f}^{eq}) \ln \tilde{f}^{eq} d\nu dv dx.$$  

(E.1)

Due to the monotonicity of the logarithm, the first kernel in the right-hand side is obviously negative. If the second kernel is expanded by $\epsilon$ as done in Sect. 3, we obtain,

$$-\int \int (f - \tilde{f}^{eq}) \ln \tilde{f}^{eq} d\nu dv dx = -\int \int \left[ \epsilon (f^{(1)} - \tilde{f}^{(eq,1)}) \ln \tilde{f}^{(eq,0)} - \epsilon^2 \left\{ (f^{(2)} - \tilde{f}^{(eq,2)}) \ln \tilde{f}^{(eq,0)} + (f^{(1)} - \tilde{f}^{(eq,1)}) \frac{\tilde{f}^{(eq,1)}}{\tilde{f}^{(eq,0)}} \right\} \right] d\nu dv dx + \mathcal{O}(\epsilon^3).$$  

(E.2)

The term at $\epsilon$ order and the first term at $\epsilon^2$ order in Eq. (E.2) are zero due to the conservation of mass, momentum, and energy. As a result,

$$-\int \int (f - \tilde{f}^{eq}) \ln \tilde{f}^{eq} d\nu dv dx \approx -\epsilon^2 \int \int (f^{(1)} - \tilde{f}^{(eq,1)}) \frac{\tilde{f}^{(eq,1)}}{\tilde{f}^{(eq,0)}} d\nu dv dx. \quad \text{(E.3)}$$

Using the results of the Chapman-Enskog expansion for the BGK Boltzmann equation, Eq. (1), at $\epsilon$ order,

$$f^{(1)} - \tilde{f}^{(eq,1)} = -\tau \left( \frac{\partial}{\partial \tau} + cv_j \frac{\partial}{\partial x_j} \right) \tilde{f}^{(eq,0)}, \quad \text{(E.4)}$$

it becomes,

$$\text{Eq. (E.3)} = \tau \epsilon^2 \int \int \left[ \left( \frac{\partial}{\partial \tau} + cv_j \frac{\partial}{\partial x_j} \right) \tilde{f}^{(eq,0)} \right] \frac{\tilde{f}^{(eq,1)}}{\tilde{f}^{(eq,0)}} d\nu dv dx. \quad \text{(E.5)}$$

Using

$$\frac{\partial \tilde{f}^{(eq,0)}}{\partial \tau} = \frac{\tilde{f}^{(eq,0)}}{\rho} \frac{\partial \rho}{\partial \tau} + \frac{v_i - u_i}{T} \tilde{f}^{(eq,0)} \frac{\partial u_i}{\partial \tau} + \left( \frac{(v - u)^2}{2T^2} - \frac{1}{T} \right) \tilde{f}^{(eq,0)} \frac{\partial T}{\partial \tau}, \quad \text{(E.6)}$$

$$v_j \frac{\partial \tilde{f}^{(eq,0)}}{\partial x_j} = v_j \tilde{f}^{(eq,0)} \frac{\partial \rho}{\partial x_j} + v_i - u_i \frac{\tilde{f}^{(eq,0)}}{T} \frac{\partial u_j}{\partial x_j} + v_j \left( \frac{(v - u)^2}{2T^2} - \frac{1}{T} \right) \tilde{f}^{(eq,0)} \frac{\partial T}{\partial x_j}, \quad \text{(E.7)}$$

one finds the temporal derivative terms and a few spatial derivative terms in Eq. (E.5) vanish.

It is because $\int \tilde{f}^{(eq,1)} |v|^l dv$ where $l \leq 2$ contains only derivative terms related $E_p$ as shown in 1 and Table 1. Because $E_p \leq \mathcal{O}(\epsilon \cdot \max \{\tilde{\mu}, \tilde{\kappa}\} / \epsilon \tau) \cdot \int \tilde{f}^{(eq,1)} |v|^l dv = 0$ at $\epsilon$ order. As a result,
Eq. (E.5) = \begin{align*}
\frac{c}{T} \int \int \left[ v_l \frac{\partial u_l}{\partial x_{1,j}} + \frac{v^2 - 2v_l u_l}{2T^2} \frac{\partial T}{\partial x_{1,j}} \right] dv dx,
\end{align*}

\begin{align*}
\begin{aligned}
\left(1 \right) & \ 
\left[ -\frac{\beta c}{T} \tilde{\mu} \tilde{\sigma}_{1,i,j} \frac{\partial u_l}{\partial x_{1,j}} - \frac{\beta c \tilde{\kappa}}{T^2} \frac{\partial T}{\partial x_{1,j}} \right]^2, \\
\left(2 \right) & \ 
\frac{c}{T} \frac{\beta c \tilde{\mu}}{T} \left( \tilde{\sigma}_{1,i,j} \tilde{\sigma}_{1,i,j} + \left| \frac{\partial u_l}{\partial x_{1,j}} \right|^2 \right) - \frac{\beta c \tilde{\kappa}}{T^2} \frac{\partial T}{\partial x_{1,j}} \right]^2, \\
\left(3 \right) & \ 
\left( \beta \tilde{\mu} \tilde{\kappa} \left( \tilde{\sigma}_{x,x} - \tilde{\sigma}_{y,y} \right) - \frac{3}{2} \beta \tilde{\kappa} \frac{\partial T}{\partial x} \right) \left( -\tilde{\mu} \left( \tilde{u}_x \tilde{\sigma}_{y,y} + \tilde{u}_y \tilde{\sigma}_{x,x} + \tilde{\sigma}_{x,y} \right) \right) - \frac{3}{2} \beta \tilde{\kappa} \frac{\partial T}{\partial x}
\end{aligned}
\end{align*}

(E.8)

using

\begin{align*}
\int \left( \begin{array}{c}
v_x \ v_y \\
v_x^2 \\
v_y^2
\end{array} \right) f^{(eq.1)} dv_x dv_y = \left( \begin{array}{c}
-\beta \tilde{\mu} \tilde{\sigma}_{x,y}
-\frac{1}{2} \beta \tilde{\mu} \left( \tilde{\sigma}_{x,x} - \tilde{\sigma}_{y,y} \right)
-\frac{3}{2} \beta \tilde{\kappa} \left( \tilde{u}_x \tilde{\sigma}_{y,y} + \tilde{u}_y \tilde{\sigma}_{x,x} + \tilde{\sigma}_{x,y} \right)
\end{array} \right)
\end{align*}

(E.9)

As a result, because \( \beta \geq 0 \), Eq. (E.5) should be zero or negative if \( \tilde{\mu} \) and \( \tilde{\kappa} \) are zero or positive. Note that the leading term of the left-hand side in Eqs. (E.3), (E.5), includes only terms related to \( \tilde{\mu} \) and \( \tilde{\kappa} \) but does not include the terms related to \( E_p \). Since the former terms are always larger than the latter terms because of \( E_p \leq \mathcal{O} (\epsilon \cdot \max \{ \tilde{\mu}, \tilde{\kappa} \} / c\tau) \), it is not necessary to explicitly estimate them here. Accordingly Eq. (E.3) is zero or negative and therefore \( \frac{dH}{dt} \leq 0 \) where \( E_p \leq \mathcal{O} (\epsilon \cdot \max \{ \tilde{\mu}, \tilde{\kappa} \} / c\tau) \), \( \tilde{\mu} \geq 0 \), and \( \tilde{\kappa} \geq 0 \).

Appendix F: Derivation of the Transport Coefficients from the Fluctuation-Dissipation Theorem

The transport coefficients such as the viscosity and the thermal conductivity are derived from the fluctuation-dissipation theorem. In this derivation, we choose an inertial frame so that the local fluid velocity is zero, \( u|_x = 0 \) and assume \( E_p \leq \mathcal{O}(\epsilon) \). According to the fluctuation-dissipation theorem, the following integration for the time correlation are considered,

\begin{align*}
\frac{c}{T} \int_0^\infty \langle P_{xy} (t) P_{xy} (0) \rangle dt, \quad (F.1)
\end{align*}

\begin{align*}
\frac{c}{2T^2} \int_0^\infty \langle S_t (t) S_t (0) \rangle dt, \quad (F.2)
\end{align*}

where \( P_{xy} \) is the x-y component of a microscopic pressure tensor and \( S_t \) is microscopic heat flux. To obtain the shear viscosity, the first correlation is taken for the flow subject to the shear stress, namely where \( |\partial_x u_y| \neq 0 \). To obtain the thermal conductivity, the second correlation is taken for the flow subject to the thermal stress, \( |\partial_x T| \neq 0 \).

Assuming the state is close to the non-local equilibrium state, one writes Eq. (F.1) and Eq. (F.2) as;

\begin{align*}
\text{Eq. (F.1)} & = \frac{c}{T} \int \int v_x v_y v'_x v'_y f^{eq} (v) \mathcal{P} (v', t|v, 0) dtdvdv', \\
\text{Eq. (F.2)} & = \frac{c}{2T^2} \int \int \left( \frac{v_x^2 + v_y^2}{2} - 2\tau \right) \left( \frac{v'_x^2 + v'_y^2}{2} - 2\tau \right) (v_x v'_x + v_y v'_y) f^{eq} (v) \mathcal{P} (v', t|v, 0) dtdvdv'.
\end{align*}

(F.3) (F.4)

\( \mathcal{P} \)
Here $P(v', t|v, 0)$ is the conditional probability of finding the particle of velocity $v'$ at time $t$ where the particle of velocity $v$ was observed at time $0$. It may be written with the functional derivatives as,

$$P(v', t|v, 0) = \frac{\delta f(v', t)}{\delta f(v, 0)} = \frac{\delta f(v', t)}{\delta f(v, 0)} + \int_0^t \int \frac{\delta f(v', t')}{\delta f(v, 0)} \frac{\delta f_{eq}(v', t')}{\delta f(v, 0)} d\nu' dt' + \cdots. \tag{F.5}$$

The first term in the right-hand side shows the correlation between an initial state and a state after advection;

$$\frac{\delta f(v', t)}{\delta f(v, 0)} = \exp \left(-\frac{t}{\tau}\right) \delta(v' - v). \tag{F.6}$$

The exponential decay is due to the collision during the particle’s advection. The second term in the right-hand side of Eq. (F.5) shows the correlation between an initial state and an advected state after a collision. Specifically, particles collide after advecting from an initial state with velocity $v$ and then follow the equilibrium state. After that, the equilibrium state advects back to the original position at time $t$ with velocity $v'$ which is the opposed number to $v$. It may be further written as,

$$\int_0^t \int \frac{\delta f(v', t')}{\delta f(v, 0)} \frac{\delta f_{eq}(v', t')}{\delta f(v, 0)} d\nu' dt' = \int_0^t \int \exp \left(-\frac{t - t'}{\tau}\right) \delta(v' - v'') \frac{\delta f_{eq}(v', t')}{\delta X|v'} \frac{\delta X|v'}{\delta f(v, 0)} \delta(t' - \tau) d\nu'' dt', \tag{F.7}$$

where $X = \{\rho, u, T, \nabla u, \beta, \nabla T\}$, the macroscopic quantities defining the equilibrium state $f_{eq}$, and $\alpha$ and $\beta$ are indices for the spatial direction. In the second equation, to simplify the time integration for $t'$, the collision is assumed to happen at time $\tau$, which results in the Dirac delta function $\delta(t' - \tau)$.

Next, the functional derivative $\frac{\delta X|v'}{\delta f(v, 0)}$ is estimated. With the definition of the functional derivative,

$$\int \frac{\delta X}{\delta f}(v) \phi(v) d\nu = \lim_{\epsilon \rightarrow 0} \frac{X[f + \epsilon \phi] - X[f]}{\epsilon}. \tag{F.8}$$

where $\phi$ is an arbitrary function, one can write the case of $X = \rho$ as,

$$\int \frac{\delta \rho|_{x,t}}{\delta f|_{v,x,0}} \phi(v) d\nu = \lim_{\epsilon \rightarrow 0} \frac{\rho[f + \epsilon \phi|_{x,t}]|_{x,t} - \rho[f]|_{x,t}}{\epsilon}. \tag{F.9}$$

Since the function $\phi$ takes the values along the direction of $f|_{v,x,0}$, the first term of the numerator in the right-hand side can be written as,

$$\rho[f + \epsilon \phi|_{x,t}] = \int \{f(v)|_{x,t} + \epsilon \phi \mathcal{F}(v)|_{x,t}\} d\nu, \tag{F.10}$$

$$\mathcal{F}(v)|_{x,t} = \delta(t) + \delta(v) \exp \left(-\frac{t}{\tau}\right). \tag{F.11}$$

The first term in $\mathcal{F}(v)$ is a correlation between $f|_{v,x,0}$ and $f|_{v,x,t}$ at time 0 and the second term is a correlation between $f|_{v,x,0}$ and $f|_{v,x,t}$ at $v = 0$. Applying it to Eq. (F.9), one derives,

$$\frac{\delta \rho|_{x,t}}{\delta f|_{v,x,0}} = \delta(t) + \delta(v) \exp \left(-\frac{t}{\tau}\right). \tag{F.12}$$
Similarly, in the case of $X = u_\alpha$, because

$$u_\alpha \left[ f + \epsilon \phi |_{x_0} \right] = \frac{\int \{ f |_{x,t} + \epsilon \phi \mathcal{F}(v) |_{x,t} \} \nu_\alpha dv}{\int \{ f |_{x,t} + \epsilon \phi \mathcal{F}(v) |_{x,t} \} dv},$$

$$= \frac{\int f \nu_\alpha dv}{\int f dv} + \epsilon \frac{\int \phi \mathcal{F}(v) |_{x,t} \nu_\alpha dv}{\rho} - \epsilon \frac{\rho u_\alpha}{\rho^2} \int \mathcal{F}(v) |_{x,t} dv + O \left( \epsilon^2 \right),$$

(F.13)

using the Taylor expansion, $\frac{A + \epsilon}{B + \epsilon'} = \frac{A}{B} + \epsilon \frac{1}{B} - \epsilon' \frac{A}{B^2} + \cdots$, one obtains,

$$\frac{\delta u_\alpha |_{x,t}}{\delta f |_{v,x,0}} = \frac{\nu_\alpha}{\rho} \delta (t),$$

(F.14)

if $u_\alpha |_{x} = 0$ is considered.

In the case of $X = T$, using the definition of $T = \int f v^2 dv / (2 \rho)$ where $u |_{x} = 0$ and the Taylor expansion, $\frac{A + \epsilon}{B + \epsilon'} = \frac{A}{B} + \epsilon \frac{1}{B} - \epsilon' \frac{A}{B^2} + \cdots$, because

$$T \left[ f + \epsilon \phi |_{x,0} \right] = T \left[ f \right] + \epsilon \left\{ \frac{\int \phi \mathcal{F}(v) |_{x,t} v^2 dv}{2 \rho} - \frac{\int \phi \mathcal{F}(v) |_{x,t} 2 \rho T dv}{2 \rho^2} \right\} + O \left( \epsilon^2 \right),$$

$$= T \left[ f \right] + \epsilon \int \phi \mathcal{F}(v) \left( \frac{\rho v^2 - 2 \rho T}{2 \rho^2} \right) dv + O \left( \epsilon^2 \right),$$

$$= T \left[ f \right] + \epsilon \int \phi \left\{ \frac{v^2 - 2 T}{2 \rho} \left( \delta (t) + \delta (v) \exp \left( \frac{-t}{\tau} \right) \right) \right\} dv + O \left( \epsilon^2 \right),$$

(F.15)

one obtains,

$$\frac{\delta T |_{x,t}}{\delta f |_{v,x,0}} = \frac{v^2 - 2 T}{2 \rho} \left( \delta (t) + \delta (v) \exp \left( \frac{-t}{\tau} \right) \right).$$

(F.16)

In the case of $X = \partial_\alpha u_\beta$, using $\partial_\alpha u_\beta |_{x,t} \approx \left( \int f |_{x,t} v_\beta dv - \int f |_{x - \Delta x_\alpha,t} v_\beta dv \right) / \Delta x_\alpha$, because

$$\partial_\alpha u_\beta |_{x,t} \left[ f + \epsilon \phi |_{x,0} \right] = \partial_\alpha u_\beta |_{x,t} \left[ f \right] - \epsilon \int \phi \frac{v_\beta}{\rho \Delta x_\alpha} \exp \left( \frac{-t}{\tau} \right) \delta (v_\alpha tc - \Delta x_\alpha) dv + O \left( \epsilon^2 \right),$$

(F.17)

if $u_\alpha |_{x} = 0$ is considered, one obtains,

$$\frac{\delta \partial_\alpha u_\beta |_{x,t}}{\delta f |_{v,x,0}} = - \frac{v_\beta}{\rho v_\alpha tc} \exp \left( \frac{-t}{\tau} \right).$$

(F.18)

In the case of $X = \partial_\alpha T$, using $\partial_\alpha T |_{x,t} \approx \left( \int f v^2 dv / 2 \rho \right) |_{x,t} - \left( \int f v^2 dv / 2 \rho \right) |_{x - \Delta x_\alpha,t} \right\} / \Delta x_\alpha$ where $u |_{x} = 0$, because

$$\partial_\alpha T |_{x,t} \left[ f + \epsilon \phi |_{x,0} \right] = \partial_\alpha T |_{x,t} \left[ f \right] - \epsilon \int \phi \frac{1}{\Delta x_\alpha} \frac{v^2 - 2 T}{2 \rho} \exp \left( \frac{-t}{\tau} \right) \delta (v_\alpha tc - \Delta x_\alpha) dv + O \left( \epsilon^2 \right),$$

(F.19)
one obtains,

\[
\frac{\delta \partial_\alpha T|_{x,t}}{\delta f|_{v,x,0}} = - \frac{1}{v_\alpha t c} \frac{v^2 - 2T}{2\rho} \exp \left( -\frac{t}{\tau} \right). \tag{F.20}
\]

In summary, in the inertial frame of \(u|_x = 0\), the functional derivative for the macroscopic quantities \(X\) can be written as,

\[
\frac{\delta X|_{\nu'}}{\delta f|_{v,0}} = \left\{ \delta \left( t' \right) + \delta \left( v \right) \exp \left( -\frac{t'}{\tau} \right), \frac{v_\alpha}{\rho} \delta \left( v \right), \frac{v^2 - 2T}{2\rho} \left( \delta \left( t' \right) + \delta \left( v \right) \exp \left( -\frac{t'}{\tau} \right) \right), \right. \\
- \frac{v^2 - 2T}{2\rho} \exp \left( -\frac{t'}{\tau} \right) \left. \right\}. \tag{F.21}
\]

If we consider applying Eqs. (F.21) to (F.7), we find that all terms related to \(\delta \left( t' \right)\) in Eq. (F.21) are negligible due to the delta function, \(\delta \left( t' - \tau \right)\) in Eq. (F.7). In addition, if we consider applying Eqs. (F.5) and (F.7) to (F.3), and (F.4) using Eq. (F.21), we find that all terms related to \(\delta \left( v \right)\) in Eq. (F.21) are negligible because both of \(P_{xy}\) and \(S_i\) are zero if particle velocity is zero. As a result, we should consider only the derivatives of Eqs. (F.18) and (F.20).

First, for the flow under the shear stress, using Eqs. (F.18), (F.7) can be written as,

\[
\int_0^t \int \frac{\delta f|_{\nu',t}}{\delta f|_{\nu,0}} \frac{\delta \tilde{f}_{eq}|_{\nu',t'}}{\delta f|_{\nu,0}} d\nu' dt' \\
= \int \exp \left( -\frac{t - t'}{\tau} \right) \beta f_{eq} \frac{\tilde{\mu}}{\rho T^2} v'_a v'_\beta - \frac{v_\beta}{\rho v_\alpha t c} \exp \left( -\frac{t'}{\tau} \right) \delta \left( t' - \tau \right) dt', \\
= \exp \left( -\frac{t}{\tau} \right) \beta f_{eq} \frac{\tilde{\mu}}{\rho T^2} v'_a v'_\beta - \frac{v_\beta}{\rho v_\alpha t c}. \tag{F.22}
\]

Here the following equation derived from Eqs. (5) and (8) was used,

\[
\frac{\delta \tilde{f}_{eq}|_{\nu',t'}}{\delta \partial_\alpha u_\beta|_{\nu'}} = -\beta f_{eq}|_{\nu',t'} \frac{\tilde{\mu}}{\rho T^2} v'_a v'_\beta. \tag{F.23}
\]

As a result, the conditional probability, Eq. (F.5), can be written as,

\[
\mathcal{P} \left( v', t|v, 0 \right) = \exp \left( -\frac{t}{\tau} \right) \delta \left( v' - v \right) + \exp \left( -\frac{t}{\tau} \right) \beta f_{eq} \frac{\tilde{\mu}}{\rho T^2} v'_a v'_\beta - \frac{v_\beta}{\rho v_\alpha t c} + \cdots. \tag{F.24}
\]

If it is substituted to Eq. (F.3), one obtains,

\[
\text{Eq. (F.1)} \approx \frac{c}{T} \int \int \int_0^\infty v_\alpha v'_a v'_\beta \tilde{f}_{eq} \left( v \right) \left\{ \exp \left( -\frac{t}{\tau} \right) \delta \left( v' - v \right) \\
+ \exp \left( -\frac{t}{\tau} \right) \beta f_{eq} \frac{\tilde{\mu}}{\rho T^2} v'_a v'_\beta - \frac{v_\beta}{\rho v_\alpha t c} \right\} dtdvdv', \\
= \frac{c}{T} \left( \tau \int v_\alpha^2 v'_a v'_\beta \tilde{f}_{eq} \left( v \right) d\nu + \frac{\tilde{\mu}}{\rho T^2 c} \int \int v_\alpha^2 v'_a v'_\beta \tilde{f}_{eq} \left( v \right) f_{eq} \left( v' \right) d\nu dv' \right) \approx \tau \rho T c + \tilde{\mu} \approx \mu. \tag{F.25}
\]
Here $E_p \leq \mathcal{O}(\epsilon)$ and the following equations were used,

$$\int \begin{pmatrix} 1 \\ v_x^2 \\ v_y^2 \\ v_x v_y \\ \rho \\ \rho T \\ 3\rho T^2 \\ 15\rho T^3 \\ \rho T^2 \end{pmatrix} f^{\text{eq}} dv_x dv_y = \begin{pmatrix} \rho \\ \rho T \\ 3\rho T^2 \\ 15\rho T^3 \\ \rho T^2 \end{pmatrix}, \quad \text{(F.26)}$$

where $|u| = 0$.

For the flow under the thermal stress, using Eqs. (F.20), (F.7) can be written as,

$$\int_0^t \frac{\delta f}{\delta f^{\text{eq}} | \cdot, v'} dv' dt'$$

$$= \int_0^t \exp \left( \frac{-t-t'}{\tau} \right) \beta f^{\text{eq}} (v') \frac{\tilde{\kappa}}{2\rho T^3} (v'^2 - 4T) v'_a \frac{1}{v_a \tau c} v^2 - 2T \exp \left( \frac{-t'}{\tau} \right) \delta \left( t' - t \right) dt' .$$

Here the following equation derived from Eqs. (5) and (8),

$$\frac{\delta f^{\text{eq}} | v', t'}{\delta f^{\text{eq}} | v, t} = -\beta f^{\text{eq}} | v, t' \cdot \frac{\tilde{\kappa}}{4\rho T^3} (v'^2 - 4T) v'_a , \quad \text{(F.28)}$$

was used. As a result, the conditional probability, Eq. (F.5), can be written as,

$$\mathcal{P}(v', t|v, 0) = \exp \left( \frac{-t}{\tau} \right) \delta \left( v' - v \right)$$

$$+ \exp \left( \frac{-t}{\tau} \right) \beta f^{\text{eq}} (v') \frac{\tilde{\kappa}}{2\rho T^3} (v'^2 - 4T) v'_a \frac{1}{v_a \tau c} v^2 - 2T + \cdots . \quad \text{(F.29)}$$

If it is substituted to Eq. (F.4), using Eq. (F.26) one obtains,

$$\frac{c}{2T^2} \int \int \int_0^\infty \left( \frac{v_x^2 + v_y^2}{2} - 2T \right) \left( \frac{v_x^2 + v_y^2}{2} - 2T \right) f^{\text{eq}} (v) \mathcal{P}(v', t|v, 0) dtdv dv' ,$$

$$= \frac{c}{2T^2} \left\{ \tau \int \left( \frac{v_x^2 + v_y^2}{2} - 2T \right)^2 \left( v_x^2 + v'_y \right) f^{\text{eq}} (v) dv$$

$$+ \beta \frac{\tilde{\kappa}}{4\rho^2 T^3 c} \int \int \left( \frac{v_x^2 + v_y^2}{2} - 2T \right) f^{\text{eq}} (v) \frac{v^2 - 2T}{v_a} \left( v_x v'_x + v_y v'_y \right)$$

$$\left( \frac{v_x^2 + v_y^2}{2} - 2T \right) f^{\text{eq}} (v') \left( v'^2 - 4T \right) v'_a dtdv' \right\} ,$$

$$\approx \frac{\tau c}{2T^2} 4\rho T^3 + \frac{1}{2T^2} \frac{\tilde{\kappa}}{4\rho^2 T^3} \left( 2\rho T^2 \right) \left( 4\rho T^3 \right) = 2\rho T \tau c + \tilde{\kappa} \approx \kappa , \quad \text{(F.30)}$$

where Eq. (F.26) and $E_p \leq \mathcal{O}(\epsilon)$ are used.
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