Harmonic Space Construction of the Quaternionic Taub-NUT metric

Evgeny Ivanov\textsuperscript{(a)}, Galliano Valent\textsuperscript{(b)}

\textsuperscript{(a)} Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, 141 980 Moscow region, Russia

\textsuperscript{(b)} Laboratoire de Physique Théorique et des Hautes Energies, Unité associée au CNRS URA 280, Université Paris 7 2 Place Jussieu, 75251 Paris Cedex 05, France

Abstract

We present details of the harmonic space construction of a quaternionic extension of the four-dimensional Taub-NUT metric. As the main merit of the harmonic space approach, the metric is obtained in an explicit form following a generic set of rules. It exhibits $SU(2) \times U(1)$ isometry group and depends on two parameters, Taub-NUT ‘mass’ and the cosmological constant. We consider several limiting cases of interest which correspond to special choices of the involved parameters.
1 Introduction

One of the important implications of the harmonic (super)space method [1] - [4] refers to the hyper-Kähler and quaternion-Kähler geometries where it provides an efficient way of explicit construction of the relevant metrics.

This approach was firstly introduced in the context of $N = 2$ supersymmetry as the method of harmonic superspace [1]. Its basic idea is to extend the standard $N = 2$ superspace by a set of internal (‘harmonic’) variables $u^{\pm i}, u^{+i}u^{-i} = 1$, parametrizing the coset space $S^2 \sim SU(2)/U(1)$ of the automorphism group $SU(2)$ of $N = 2$ superalgebra. The main advantage of such a harmonic extension is the opportunity to single out a new subspace in it, the analytic subspace. It is closed under the supersymmetry transformations and contains half the Grassmann coordinates compared to the initial superspace. It was shown in [1] that all $N = 2$ theories admit an off-shell description in terms of unconstrained harmonic analytic superfields having a clear geometric interpretation.

Later on, it was realized that the harmonics can also be helpful in solving some purely bosonic geometric problems. It was shown in [3] that the constraints of the hyper-Kähler (HK) geometry (the vanishing of certain components of the curvature tensor) can be given an interpretation of the integrability conditions for the existence of analytic fields in a $SU(2)$ harmonic extension of the original 4n-dimensional HK manifold $\{x^{\mu}\}, (i = 1, 2; \mu = 1, \ldots 2n)$. This time, the $SU(2)$ to be ‘harmonized’ is an extra $SU(2)$ rotating three complex structures of the HK manifold. A proper change of variables makes the half of covariant derivatives (their $u^{+}$ projections) ‘short’ that means the existence of an analytic subspace in the $SU(2)$-extended HK manifold. It is parametrized by $2n$ coordinates $x^{+\mu}$ and the harmonics $u^{\pm i}$. In the new basis the HK constraints can be solved in terms of unconstrained HK potential $L^{+4}(x^{+\mu}, u^{\pm i})$. It encodes (at least, locally) all the information about the associated HK metric. It should be stressed that it allows one to explicitly compute the HK metric using a set of rules given in [3]. Indeed, using these techniques new derivations of the Taub-NUT metric [2], of the Eguchi-Hanson metric [5] and the full multicentre metric [6] were given.

In [4], a generalization of this approach to the important class of quaternion-Kähler (QK) manifolds was given. These manifolds generalize the HK ones in such a way that the extra $SU(2)$ group transforming complex structures becomes an essential part of the holonomy group. The HK manifolds can be treated as a degenerate case of the QK ones, with the $SU(2)$ part of the curvature tensor vanishing. One of important implications of QK manifolds is in General Relativity where four-dimensional QK manifolds appear as solutions of Einstein equation with non-zero cosmological constant. Besides, the QK manifolds appear as the target spaces of general $N = 2$ supersymmetric sigma models coupled to $N = 2$ supergravity [7], with the $SU(2)$ curvature proportional to the Einstein constant.

It was shown in [4], that, using the concept of harmonic extensions, the QK geometry constraints can be solved quite similar to those of the HK geometry. Once again, the solution is given in terms of some unconstrained potential $L^{+4}$ living on the analytic subspace parametrized by $SU(2)$ harmonics and half of the original space coordinates. The specificity of the QK case manifests itself in the presence of non-zero constant $SU(2)$ curvature on all steps of the road from $L^{+4}$ to the related QK metric. This somewhat obscures the computations, though they remain more or less direct. It is interesting
to consider some examples in order to see in detail how the machinery proposed in [4] works. Only the simplest case of the homogeneous QK manifold $Sp(n+1)/Sp(1) \times Sp(n)$ (corresponding to the flat HK manifold) was considered as an example in ref. [4].

The aim of this paper is to demonstrate the efficiency of the harmonic space approach on the less trivial example of a quaternionic generalization of the four-dimensional Taub-NUT metric (QTN in what follows). In section 2 we give a brief summary of the concepts and relations of ref. [4] adapted to our practical purpose. Then in sections 3 and 4 we present the actual computations. Like in the HK case [2], they are greatly simplified due to the ‘translational’ $U(1)$ isometry of the QTN metric (its full isometry, like in the HK case, is $SU(2) \times U(1)$, with the $SU(2)$ factor being ‘rotational’ isometry). The metric obtained depends on two parameters, the TN ‘mass’ parameter and the constant $SU(2)$ curvature parameter which can be interpreted as the inverse ‘radius’ of the corresponding ‘flat’ QK background $\sim Sp(2)/Sp(1) \times Sp(1)$. In section 5 we perform the identification of the metric obtained from harmonic space with its standard form given in the literature [9] and consider some important limits related to special choices of the parameters.

## 2 Generalities

Here we sketch the basic points of the construction of [4] (with minor deviations in the notation). We refer the reader to [4] for the detailed explanations and proofs.

One starts with a $4n$-dimensional Riemann manifold with local coordinates $\{x^{\mu m}\}, \mu = 1, 2, ..., 2n; m = 1, 2$ and uses a vielbein formalism. The QK geometry can be defined as a restriction of the general Riemannian geometry in $4n$-dimensions, such that the holonomy group of the latter is required to be a subgroup of $Sp(1) \times Sp(n)\dirac{†}$. Thus one can choose the tangent space group from the very beginning to be $Sp(1) \times Sp(n)$ and define the QK geometry via appropriate restrictions on the curvature tensor lifted to the tangent space (taking into account that the holonomy group is generated by this tensor). As explained in [4], for the QK manifold of any dimension the defining constraints can be concisely written as a restriction on the form of the commutator of two covariant derivatives

$$[\mathcal{D}_\alpha(i), \mathcal{D}_\beta(k)] = -2\Omega_{\alpha\beta}R\Gamma_{(ik)}.$$  

(2.1)

Here

$$\mathcal{D}_\alpha = e^{\mu m}_{\alpha i}(x)\nabla_{\mu m} = e^{\mu m}_{\alpha i}(x) \frac{\partial}{\partial x^{\mu m}} + [Sp(1) \times Sp(n) - \text{connections}],$$

(2.2)

e^{\mu m}_{\alpha i}(x) being the relevant $4n \times 4n$ vielbein with the indices $\alpha = 1, 2, ..., 2n$ and $i = 1, 2$ rotated, respectively, by the local tangent $Sp(n)$ and $Sp(1)$ groups, $\Omega_{\alpha\beta}$ is the $Sp(n)$-invariant skew-symmetric tensor serving to raise and lower the $Sp(n)$ indices ($\Omega_{\alpha\beta}\Omega^{\beta\gamma} = \delta_\gamma^\gamma$), $\Gamma_{(ik)}$ are the $Sp(1)$ generators, and $R$ is a constant, remnant of the $Sp(1)$ component of the Riemann tensor (its constancy is a consequence of the QH geometry constraint and Bianchi identities). The scalar curvature coincides with $R$ up to a positive numerical coefficient, so the cases $R > 0$ and $R < 0$ correspond to compact and non-compact

\footnote{For 4-dimensional case this definition ceases to be meaningful, and it is replaced by the requirement that the totally symmetric part of the $Sp(1)$ component of the curvature tensor lifted to the tangent space is vanishing.}

\footnote{It is an extended detailed version of ref. [8].}
manifolds, respectively. In the limit $R = 0$ eq. (2.1) is reduced to the constraint defining the HK geometry [3], in accord with the interpretation of HK manifolds as a degenerate subclass of the QK ones.

Like in the HK case [3], in order to explicitly figure out what kind of restrictions is imposed by (1) on the vielbein $e_{\alpha i}^{\mu m}(x)$ and, hence, on the metric

$$g^{\mu s \nu m} = e^{\mu s}_{\alpha i} e^{\nu m}_{\alpha i}, \quad g_{\mu s \nu m} = e_{\mu s \alpha i} e^{\nu m}_{\alpha i},$$

(2.3)
one should solve the constraints (2.1) by regarding them as an integrability condition along some complex directions in a harmonic extension of the original manifold. Due to the non-vanishing r.h.s. in (1), the road to such an interpretation in the QK case is more tricky and passes through two steps.

First, one introduces a sort of harmonic variables $v^a_i$, $(v^a_i v^b_i = \delta^{ab})$, on the local $Sp(1) \sim SU(2)$ group acting in the tangent space and represents the generators $\Gamma_{(ik)}$ as differential operators in these extra variables. Correspondingly, the $Sp(1)$-parts of the gauge connection acquires the meaning of extra components of vielbein, while the $Sp(1)$-part of the curvature tensor becomes the relevant torsion (of course, no torsion appear in the $x$-directions). At this step, one converts all the tangent $Sp(1)$ indices $i, k, ..$ with the $Sp(1)$ harmonics and so replaces them by the indices $a, b, c ..$ on which some rigid $SU(2)$ is realized (they are analogs of the indices $\pm$ of the standard $SU(2)$ harmonics, while the rigid $SU(2)$ is an analog of the right $SU(2)$ acting on these $\pm$).

Next, one introduces the second set of harmonics on the rigid $SU(2)$, that time the standard $S^2 \sim SU(2)/U(1)$ ones $u^\pm_a, u^+a, u^-a$ with $u^+_a u^-a = 1$. The parametrization of the bi-harmonic space $Sp(1) \times S^2 = \{v^a_i, u^\pm a\}$ which is most convenient for our purposes is as follows [4]

$$Sp(1) \times S^2 = \{z^0, z^{++}, z^{--}, w^+_i - z^{++} u^-_i, w^-_i = u^-_i\}.$$

(2.4)

One can define three independent covariant derivatives $Z^{++}, Z^{--}, Z^0$ on the $Sp(1)$ part of the bi-harmonic space and three standard harmonic derivatives $D^{++}, D^{--}, D^0$ on its $SU(2)$ part. They form a semi-direct sum of two $su(2)$ algebras. The operator $D^0$ is the customary harmonic $U(1)$-charge counting operator, while the operator $Z^0 = z^0 \partial/\partial z^0$ commuting with $D^0$ was called in [4] the ‘$Sp(1)$ weight operator’: it counts the degree of homogeneity in the $Sp(1)$ coordinate $z^0$ (this coordinate is assumed to be 1 at the origin of the $Sp(1)$ group manifold, so, in a vicinity of the origin, one can consider both its positive and negative powers).

Successively projecting the constraint (1) on the first and second sets of harmonics, one puts it into the following equivalent suggestive form [4]

$$[D^+, D^+_{\alpha}] = -2\Omega_{\alpha \beta} R Z^{++}.$$  

(2.5)

In the basis (2.4)

$$D^+_\alpha = z^0 w^{+i} \mathcal{D}_\alpha = z^0 w^{+i} e^{\mu m}_{\alpha i}(x) \partial_{\mu m} + \ldots \equiv z^0 E^{+\mu m} \partial_{\mu m} + \ldots, \quad Z^{++} = (z^0)^2 \frac{\partial}{\partial z^0},$$

(2.6)

where ‘dots’ stand for the $Sp(n)$ connection and new $Sp(1)$ vielbeins terms. Eq. (2.3), together with the evident relation

$$[Z^{++}, D^+_\alpha] = 0,$$

(2.7)
form a closed algebra of integrability conditions implying the existence of analytic fields on the bi-harmonic extension \{x, z, w\} of the QK manifold \{x\}

\[
\mathcal{D}_\alpha^+ \Phi^{(q,p)}(x, z, w) = 0 , \quad \mathcal{D}_\alpha^0 \Phi^{(q,p)}(x, z, w) = 0 . \tag{2.8}
\]

Such fields can have a definite \(U(1)\) charge \(q\) and \(Sp(1)\) weight \(p\)

\[
D^0 \Phi^{(q,p)} = q \Phi^{(q,p)} , \quad Z^0 \Phi^{(q,p)} = p \Phi^{(q,p)} \Rightarrow \Phi^{(q,p)} = (z^0)^p \Phi^q
\tag{2.10}
\]
due to the commutation relations

\[
[D^0, \mathcal{D}_\alpha^+] = [Z^0, \mathcal{D}_\alpha^+] = \mathcal{D}_\alpha^+ , \quad [D^0, Z^{++}] = 2Z^{++} , \quad [Z^0, Z^{++}] = 2Z^{++} , \quad [D^0, Z^0] = 0 . \tag{2.11}
\]

Eqs. (2.8), (2.9) mean that \(\Phi^{(q,p)}(x, z, w)\) effectively depend only on half of the \(x\) variables and do not depend on \(z^{--}\). One can covariantly eliminate the dependence on \(z^{++}\) as well, taking into account the relations (2.12)

\[
D^{--} - Z^{--} = \frac{\partial}{\partial z^{++}} , \quad [D^{--} - Z^{--}, \mathcal{D}_\alpha^+] = [D^{--} - Z^{--}, Z^{++}] = 0 . \tag{2.13}
\]

Due to these relations (and appropriate ones involving \(D^0\) and \(Z^0\)), the additional analyticity condition

\[
(D^{--} - Z^{--}) \Phi^{(q,p)} = \frac{\partial}{\partial z^{++}} \Phi^{(q,p)} = 0 \Rightarrow \Phi^{(q,p)}(x, z, w) = (z^0)^p \Phi^q(x, w) , \tag{2.14}
\]
is compatible with eqs. (2.8), (2.9), (2.10).

Thus, the original constraints (2.1) amount to the existence of some \(z\)-independent invariant subspace \{x, w\} in the bi-harmonic extension \{x, v, u\} = \{x, z, w\} of the original QK manifold \{x\}. Moreover, as was already mentioned, eqs. (2.5), (2.8) imply an opportunity to extract a smaller subspace \{x_{\mu A}^+, w_{A}^{\pm}\} containing only half the original \(x\)-coordinates. Indeed, the covariant derivatives which introduce the QK analyticity via eqs. (2.8), (2.9), (2.10) and (2.14),

\[
\mathcal{D}_\alpha^+ , \quad Z^{++} = (z^0)^2 \frac{\partial}{\partial z^{--}} , \quad D^{--} - Z^{--} = \frac{\partial}{\partial z^{++}} , \quad Z^0 = z^0 \frac{\partial}{\partial z^0} , \tag{2.15}
\]
form a closed algebra. So, by the Frobenius theorem, there should exist a basis in the space \{x, z, w\}, such that these derivatives are reduced to the partial ones in this basis, i.e. become 'short'. The \(Sp(1)\) derivatives are already short, so it remains to make short the derivative \(\mathcal{D}_\alpha^+\). This can be done by the following change of variables (4) (we do not give the change of \(z^{--}\) as irrelevant for our purposes)

\[
z_{A}^{++} = z^{++} + v^{++}(x, w) , \quad z_0^0 = t(x, w)z^0 , \tag{2.16}
\]

\[
w_A^{+i} = w^{+i} - v^{++}(x, w)w^{-i} , \quad w_A^{-i} = w^{-i} , \tag{2.17}
\]

\[
x_{\mu n}^\pm = x_{\mu n}^\pm + v^{+\mu}(x, w) . \tag{2.18}
\]
Here, $v^{++}(x, w)$, $t(x, w)$ and $v^{\pm \mu}(x, w)$ are ‘bridges’ from the ‘$\tau$-world’ where the QK analyticity is implicit, to the ‘$\lambda$-world’ where it is manifest. The bridges can be consistently chosen independent of $z$-coordinates, once again due to the above integrability relations. One should accompany (2.16) - (2.18) by a rotation of all $Sp(n)$ indices with an appropriate matrix bridge removing the $Sp(n)$ connection part from $D^+_\alpha$. The existence of such a bridge is also guaranteed by the aforementioned Frobenius theorem. As a result, $D^+_\alpha$ in the $\lambda$-world is given by

\begin{equation}
(D^+_\alpha)_{\lambda} = z^0_{\alpha} \left( E^\mu_{\alpha}(x, w) \frac{\partial}{\partial x^\mu_{\alpha}} + \rho^-_{\alpha}(x, w) \frac{\partial}{\partial z^-_{\alpha}} \right) , \tag{2.19}
\end{equation}

where the vielbeins are still functions over the whole space $\{x, w\} = \{x_A, w_A\}$. In what follows, we will ignore the terms containing $z^{\pm \mu}$-derivatives, since one can restrict the consideration to $z$-independent functions in view of the $Sp(1)$ analyticity conditions (2.9), (2.14) and their $\lambda$-world counterparts. As far as we are interested in the QK metric, we can also assume these functions to have $p = 0$, i.e. ignore the $Sp(1)$ weight terms.

Before going further, let us briefly explain the motivation of introducing extra $Sp(1)$ bridges $v^{++}(x, w)$, $t(x, w)$ in the QK case compared to the HK case where no such objects appear [3]. The point is that the $Sp(1)$ vielbein part of $D^+_\alpha$ in the original basis contains a term $\sim \partial_{w^-}$ in parallel with the terms entering with $z$-derivatives [4]. The shift of $z^{++}$ and, respectively, of $w^{++}_i$ in (2.16), (2.17) is necessary to ensure such term to be absent in the analytic basis. Simultaneously, the term $\sim \partial/\partial z^{++}$ is removed. Analogously, the bridge $t(x, w)$ is introduced in order to remove the terms $\sim Z^0$ from $D^+_\alpha$.

We wish to stress that the introduction of $z$-coordinates is, to some extent, an auxiliary intermediate step. It is basically intended to consistently define the analyticity underlying the QK case and to arrive at the ‘true’ harmonic variables $w^{++}_i = u^+ + z^{++}_A u^- = w^+ + v^{++}(x, w) w^-$. The only manifestation of $z$ coordinates is the presence of non-trivial harmonic part in the analyticity-preserving diffeomorphism group of $\{x^\mu_A, w^{\pm i}_A\}$, $\delta w^{++}_i = \lambda^{++}_A (x^+_A, w^+_A) w^{--}_A$, along with the analytic diffeomorphisms $\delta x^{++}_A = \lambda^{++}_A (x^+_A, w^+_A).$ The basic relations of the QK geometry in the $\lambda$-world, including the formula for the QK metric, contain no any trace of these auxiliary coordinates [3].

Next steps towards solving the QK constraints mimic those in the HK case [3]. The basic incentive is to show that the bridges, as well as the $\lambda$-world vielbein, can be expressed in terms of some unconstrained analytic object, the QK potential. Afterwards, the $\tau$-world vielbein and, hence, the sought QK metric can be restored from the $\lambda$-world ones by the change of variables inverse to (2.18).

First, one observes that the particular harmonic dependence of the $\tau$-world $D^+_\alpha$, (2.6), amounts to imposing the relation

\begin{equation}
[D^{++}, D^+_\alpha] = 0 , \tag{2.20}
\end{equation}

along with eqs. (2.7), (2.11), (2.13). In the $\tau$-basis $D^{++}$ contains no derivatives with respect to $x$-coordinates, being simply $D^{++} = \partial^{++}_\mu$ in the parametrization $\{x, v, u\}$. After passing to the $\lambda$-basis by eqs. (2.10) - (2.18), it acquires induced vielbeins (together with

\footnote{Quite analogous reasoning is employed while solving the constraints of self-dual Yang-Mills theory in the harmonic space approach [4] and those of $N = 2$ supersymmetric Yang-Mills theory [4].}
some induced gauge $Sp(n)$ connection as a consequence of the rotation by a matrix $Sp(n)$ bridge). Ignoring the $z_n$-terms and $Sp(n)$-connection, it reads

$$(D^{++})_\lambda = \partial^+_A + H^{+3\mu} \partial^-_\mu + x^{+\mu} \partial^+_\mu + H^{+4} \partial^-_A = \partial^{++}_w + v^{++} D^0,$$  \hspace{1cm} (2.21)

where $\partial^{\pm}_{\mu} \equiv \partial / \partial x^{\pm}_{\mu}$ and $H^{++\mu} = x^{+\mu}$ as a result of partial fixing of the $x^{-\mu}$-diffeomorphisms gauge freedom \cite{4} (in what follows, for brevity, we omit the subscript ‘$A$’ where this cannot cause a confusion). The induced vielbeins are related to the bridges as follows

$$(\partial^{++}_{w} + v^{++}) x^{+\mu}_{A} = H^{+3\mu}$$  \hspace{1cm} (2.22)

and

$$(\partial^{++}_{w} + v^{++}) v^{++} = -H^{+4}.$$  \hspace{1cm} (2.23)

Besides, the gauge-fixing just mentioned yields the following equation for $x^{A\mu}_+$

$$(\partial^{++}_{w} - v^{++}) x^{-\mu}_{A} = x^{+\mu}_{A}.$$  \hspace{1cm} (2.24)

These equations are written in the $\tau$-basis \{$x, w$\} with taking account of the definition (2.18).

Further, the commutation relation (2.20) in the $\lambda$-basis implies the analyticity of $H^{+3\mu}, H^{+4}$, i.e. their independence of $x^{-\mu}$:

$$[(D^{++})_{\lambda}, (\mathcal{D}^{++}_{\alpha})_{\lambda}] = 0 \Rightarrow \partial^+_{\mu} H^{+3\mu} = \partial^+_{w} H^{+4} = 0 \Rightarrow H^{+3\mu} = H^{+3\mu}(x^+_A, w_A), \quad H^{+4} = H^{+4}(x^+_A, w_A),$$  \hspace{1cm} (2.25)

where $(\mathcal{D}^{++}_{\alpha})_{\lambda}$ was defined in (2.19).

The analytic harmonic vielbein $H^{+4}$ is basically just the unconstrained QK potential while $H^{+3\mu}$ is expressed in terms of it. To be more precise, the QK potential $\mathcal{L}^{+4}$, as it was defined in \cite{4}, is related to $H^{+4}$ as

$$H^{+4}(x^+_A, w_A) = R \mathcal{L}^{+4}(x^+_A, w_A) + Rx^+_A H^{+3\mu}(x^+_A, w_A)$$  \hspace{1cm} (2.26)

and

$$H^{+3\mu} = \frac{1}{2} \Omega^{\mu\nu} \tilde{\partial}^-_{\mu} \mathcal{L}^{+4},$$  \hspace{1cm} (2.27)

$$\tilde{\partial}^-_{\mu} \equiv \partial^-_{\mu} + Rx^+_A \partial^-_A,$$  \hspace{1cm} (2.28)

where we have explicitly singled out the constant $R$ as a contraction parameter to the corresponding HK manifold. Note that eqs. (2.26), (2.27) were derived in \cite{4} after several gauge-fixings of the $\lambda$-world gauge freedom, including the one soldering analytic diffeomorphisms in $\{x^+_A, w_A\}$ with the tangent space analytic $Sp(n)$ rotations. Henceforth, we deal with the QK geometry relations in such a maximally gauge-fixed framework, with all the pure gauge quantities gauged away.

It can be shown that the only constraint to be satisfied by $\mathcal{L}^{+4}$ is its analyticity, so this object encodes all the information about the relevant QK metric, whence its name ‘QK potential’. Choosing one or another explicit $\mathcal{L}^{+4}$, and substituting (2.26), (2.27) into eqs. (2.22) - (2.24), one can solve the latter for $x^{+\mu}$ and $v^{++}$ as functions of harmonics and the $\tau$-basis coordinates $x^{\mu n}$ which now can be interpreted as the integration constants of
vielbeins

\[ \text{hence the QK metric itself.} \]

metric components which are comparatively simple giving these rather complicated expressions, we give here the expressions for the \( \lambda \) gauge-fixings) the connection terms, it reads \([4]\)

\[ S_{\text{eq. (2.22) considered as a differential equation on the sphere } S^2 \sim w_{\pm 1}} \text{ (in each specific case solving such equations could be a highly non-trivial task). Having the explicit form of the variable change (2.16) - (2.18), it remains to find the appropriate expression of the } \lambda \text{-world vielbeins in terms of } \mathcal{L}^{+4} \text{ in order to be able to compute the } \tau \text{-world vielbein and hence the QK metric itself.} \]

To this end, one should firstly find the \( \lambda \)-basis form of the covariant derivative \( D^{-} \) (it equals to \( \partial_{u}^{-} \) in the \( \tau \)-basis \( \{x, v, u\} \)). Once again, up to the \( z \)-derivative and \( Sp(n) \) connection terms, it reads \([4]\)

\[
(D^{-})_{\lambda} = (z^0)^{-2} (\psi \Delta^{-} + \ldots) , \tag{2.29}
\]

\[
\Delta^{-} = \partial^{A} + H^{-\nu} H^{-\mu} \partial^{-} + H^{-3\mu} \partial^{-} = \frac{1}{1 - \partial^{-} - \partial^{+}} \partial^{-} , \tag{2.30}
\]

\[
H^{-\nu} = \Delta^{-} x_{A}^{-} = \frac{1}{1 - \partial^{-} - \partial^{+}} \partial^{-} x_{A}^{-} ,
\]

\[
H^{-3\mu} = \Delta^{-} x_{A}^{-} = \frac{1}{1 - \partial^{-} - \partial^{+}} \partial^{-} x_{A}^{-} , \tag{2.31}
\]

\[
\psi = t^2 \left( 1 - \partial^{-} - \partial^{+} \right) = \frac{1}{1 - R x_{A}^{+} H^{-\nu} + \mu} \equiv \frac{1}{1 - R(x \cdot H)} . \tag{2.32}
\]

After that one defines the \( \lambda \)-world derivative (\( D_{\lambda}^{-} \)) by

\[
(D_{\lambda}^{-})_{\lambda} = [(D^{-})_{\lambda}, (D_{\lambda}^{+})_{\lambda}] = (z_{\lambda}^0)^{-1} \left( E_{\lambda}^{-\mu}(x, w) \partial_{\mu}^{+} + E_{\lambda}^{-\mu}(x, w) \dot{\partial}_{\mu}^{-} + \ldots \right) , \tag{2.33}
\]

where the explicit form of the vielbeins on the r.h.s. can be found in \([4]\). Let us emphasize that \(2.33\) is simply the \( \lambda \)-basis form of the \( \tau \)-basis relation

\[
[D^{-}, D_{\lambda}^{+}] = D_{\lambda}^{-} = (z_{\lambda}^0)^{-1} \left( w^{-i} e_{\lambda}^{\nu m} \partial_{\mu n} + \ldots \right) . \tag{2.34}
\]

Making use of the basic postulate of Riemann geometry, that is requiring the commutator \( [(D_{\lambda}^{-})_{\lambda}, (D_{\lambda}^{+})_{\lambda}] \) to contain no torsion in \( x \)-directions, one expresses (after appropriate gauge-fixings) the \( \lambda \)-world vielbeins \( E_{\lambda}^{\mu}, E_{\lambda}^{-\mu} \) in \(2.19\), \(2.33\) in terms of the harmonic vielbeins \( H^{-\nu} H^{-\mu}, H^{-3\mu} \) and, by eqs. \(2.31\), in terms of \( v_{\pm}, x_{A}^{\pm} \). Instead of explicitly giving these rather complicated expressions, we give here the expressions for the \( \lambda \)-world metric components which are comparatively simple

\[
g^{M N}_{(\lambda)} = E^{+M \lambda} E^{-N \lambda} - E^{-M \lambda} E^{+N \lambda} , \quad M, N = (\nu, \nu, \nu) \tag{2.35}
\]

\[
g^{\mu_{+} \nu_{-}}_{(\lambda)} = 0 , \quad g^{\mu_{+} \nu_{-}}_{(\lambda)} = g^{\mu_{-} \nu_{+}}_{(\lambda)} = -E^{-\mu_{+}} E^{+\nu} = \Omega^{\mu}_{\nu}(\partial H)^{-1} \partial_{\mu} , \tag{2.36}
\]

Here

\[
(\partial H)^{\mu}_{\nu} \equiv \partial^{\mu}_{\nu} H^{-\nu} + \mu , \tag{2.37}
\]

\[
H^{-\nu} H^{-\mu} \equiv \frac{1}{1 - R(x \cdot H)} H^{-\nu} H^{-\mu} , \quad H^{-3\mu} \equiv \frac{1}{1 - R(x \cdot H)} H^{-3\mu} . \tag{2.38}
\]
Now, making the inverse change of variables $x^{\pm \mu}_A, w^{\pm i}_A \to x^{\mu}, w^{i}$ in $(D^+_\omega)_\lambda$ and comparing the result with (2.6), (2.34), (2.3), taking account of the completeness relation
\[ w^+ w^k - w^- k w^+ - i \in (D^+_\omega)_\lambda \],
one reads off the expression for the $\tau$-world metric. It is as follows
\[ g^{\mu \nu} = g^{\omega - \sigma} \partial^+ x^{\mu \omega} \partial^+ x^{\nu \sigma} + g^{\omega + \sigma} \left( \partial^- x^{\mu \sigma} \partial^+ x^{\nu \omega} + \partial^- x^{\nu \omega} \partial^+ x^{\mu \sigma} \right), \] (2.39)
where $\partial^-_\mu$ was defined in (2.28). All the quantities entering (2.36), (2.39) can be expressed through the $\tau$-basis derivatives of $x^{\pm \mu}_A$ using (2.31), (2.37), (2.38) and the following relations
\[ \partial^\pm_\mu x^{\mu \omega} \nabla^\omega_{\rho m} = 0, \quad \partial^+ \partial^\pm_\mu x^{\rho m} = \delta^\nu_\mu, \quad (2.40) \]
\[ \partial^\pm_\mu x^{\rho m} = -H^{-\nu \mu} \partial^-_\mu x^{\rho m} - H^{-3 \nu \mu} \partial^+ \partial^\pm_\mu x^{\rho m}. \] (2.41)
Note the useful equation
\[ (\partial^+ + v^+ \partial^\pm) \partial^\pm_\rho x^{\rho} = 0, \] (2.43)
which follows from the analyticity-preserving relation $[D^+, \partial^-_\mu] = 0$ and the obvious property $D^+ x^{\rho k} = 0$.

In the case of 4-dimensional QK manifolds we will deal with in the sequel $(\mu, \nu = 1, 2)$, the expression for the $\tau$-basis metric (2.39) can be essentially simplified. After some algebra one gets
\[ g^{\mu \nu} = \frac{1}{\det(\partial H)} \left[ \frac{1}{1 - R(x \cdot H)} \right]^{\frac{1}{2}} \det(\partial H)^{\mu \nu} \] (2.44)
\[ G^{\mu \nu} = \epsilon^{\lambda \rho} \left[ \partial_\rho \partial^+ \partial^- X^{\mu \nu} + (\mu \nu \leftrightarrow \rho \lambda) \right], \] (2.45)
where
\[ X^{\mu \nu} \equiv \partial^+ \partial^- x^{\mu \nu} \] (2.46)
are solutions of the system of algebraic equations
\[ X^{\mu \nu} \nabla_{\mu \nu} x = \delta^\nu_\rho, \quad X^{\mu \nu} \nabla_{\mu \nu} x = 0. \] (2.47)
Note also the useful formula
\[ \det(\partial H) = \frac{1}{[1 - R(x \cdot H)]^3} \det(\partial H) \] (2.48)
which follows from the relation
\[ (\partial \hat{H})^\mu_\nu = (\partial H)^\rho_\nu \left[ \delta^\rho_\mu + \frac{1}{1 - R(x \cdot H)} \right] \left[ x^+ H^{-\rho - \mu} \right]. \] (2.49)

Before closing this Section, we summarize the steps leading from the given QK potential $\mathcal{L}^{+4}$, eq. (2.26), to the QK metric $g^{\mu \nu}$, eqs. (2.36), (2.39), (2.44), (2.43).
A. Using eqs. (2.26), (2.27), one expresses the harmonic vielbeins $H_{+4}$, $H_{+3\mu}$ through $\mathcal{L}^{+4}$ and substitute these expressions into the differential equations for bridges (2.22) - (2.24).

B. One solves (2.22) - (2.24) for $v^{++}$, $x^{\pm\mu}$ and obtains the latter as functions of the $\tau$-basis coordinates $x^{\mu m}$, $w^{\pm i}$ (this is the most difficult step).

C. One computes the vielbeins $H_{-\rightarrow+\mu}$, $H_{-3\mu}$ using the definition (2.31) and convert their $\partial_{\mu}^{+}$-derivatives into $\partial_{\mu m}$ by making use of eq. (2.40).

D. One computes $\nabla_{\mu n}x^{\pm\rho}$ and finds the components of the inverse matrix $\partial_{\mu}^{+}x^{\rho m}$, $\hat{\partial}_{\mu}^{-}x^{\rho m}$ by solving the algebraic eqs. (2.41) or (2.47).

E. One substitutes all that into eqs. (2.36), (2.39) or (2.44), (2.45) and finds the explicit form of the QK metric as a function of the original coordinates $x^{\mu m}$.

We should stress that these steps go along the same line as in the HK case [3]. The differences are, first, the presence of an extra bridge $v^{++}$ and the necessity to solve the corresponding bridge equation (2.23) and, second, the presence of a non-zero constant $R$ in all the relations and equations that essentially complicates the computations as compared to the HK case. Nevertheless, in both cases, the QK and HK ones, the basic geometric object is $\mathcal{L}^{+4}(x_{+}^{A}, w_{A})$. This implies that any HK metric has its QK counterparts, the former following from the latter via contraction $R \rightarrow 0$. In particular, the flat 4n-dimensional HK manifold has as its QK analogous the compact or non-compact homogeneous spaces $Sp(n+1)/Sp(n) \times Sp(1)$ or $Sp(n,1)/Sp(n) \times Sp(1)$, depending on the sign of $R$. This case corresponds to $\mathcal{L}^{+4} = 0$ [1]. In the next Section we consider a 4-dimensional QK manifold with a simplest non-trivial $\mathcal{L}^{+4}$, the QK analog of the well-known Taub-NUT space [9].

3 Quaternionic Taub-NUT: bridges and harmonic vielbeins

The QK counterpart of the Taub-NUT manifold is characterized by the same $\mathcal{L}^{+4}$ [11]

$$\mathcal{L}^{+4} = \left(c_{(\mu\nu)x_{+}^{\mu}x_{+}^{\nu}}\right)^{2} \equiv \left(\phi^{++}\right)^{2}. \quad (3.1)$$

Here $\mu, \nu = (1,2)$ and $c_{\mu\nu} = c_{\nu\mu}$ is a constant 3-vector satisfying the reality condition [1]

$$\overline{(c_{\mu\nu})} \equiv \epsilon^{\mu\nu} = \epsilon^{\mu\rho}\epsilon^{\nu\sigma}c_{\rho\sigma}. \quad (3.2)$$

The conjugation rules for the coordinates $x^{\pm\mu}$ are as follows [1]

$$\overline{(x^{\pm\mu})} = -\epsilon_{\mu\nu}x^{\pm\nu} \rightarrow \overline{\phi^{++}} = \phi^{++}. \quad (3.3)$$

\footnote{We adopt the convention $\epsilon_{12} = -\epsilon^{12} = 1.$}

\footnote{When applied to the harmonic-dependent objects, the complex conjugation is always understood as a generalized one, i.e. the product of the ordinary conjugation and Weyl reflection of harmonics [1].}
They agree with the following reality condition for the $\tau$-basis coordinates
\[ \bar{x}^\mu = \epsilon_{\mu \nu} \epsilon_{\nu k} x^k. \] (3.4)

In what follows we will frequently use the definition
\[ x^+ \equiv x^1, \quad \bar{x}^+ = (x^+) = -x^2, \quad (\bar{x}^+) = -x^+. \] (3.5)

Making use of the freedom of constant $SU(2)$ rotations of $x^+\mu$ with respect to the doublet index $\mu$, one can bring $c_{\mu \nu}$ to the form with one non-zero component
\[ c_{\mu \nu} \rightarrow c_{12} \equiv -i\lambda, \quad \bar{\lambda} = \lambda. \] (3.6)

In this frame
\[ \phi^{++} = -2i\lambda x^1 x^2 = 2i\lambda x^+ \bar{x}^+. \] (3.7)

The remnant of the above $SU(2)$ is the $U(1)$ symmetry
\[ x^+ ' = e^{i\alpha} x^+, \quad \bar{x}^+ ' = e^{-i\alpha} \bar{x}^+, \] (3.8)
which is an obvious invariance of $\phi^{++}$ and $\mathcal{L}^{++}$, and so it is an isometry of the corresponding QK metric (like in the TN case [2]). It appeared first in a $N = 2$ supersymmetry context [3, 4] where it was called the Pauli-Gürsey symmetry. This symmetry, like in the TN case [2], will essentially help in deducing the QK metric within the present approach.

To construct the QK metric related to $\mathcal{L}^{++}$ (3.1) we will follow the steps listed at the end of the preceding Section. We will omit the subscripts ‘A’ of the analytic subspace coordinates and the subscript ‘w’ of the $\tau$-basis partial harmonic derivatives.

At the step A we have
\[ H^{+3\mu} = \frac{1}{2} \epsilon^{\mu \nu} \partial^+ \mathcal{L}^{++} = 2c_\rho^\nu x^\mu \phi^{++}, \quad H^{+4} = R \left( \mathcal{L}^{++} + x^+ \mathcal{L}^{+3\nu} \right) = -R(\phi^{++})^2, \] (3.9)

\[ \partial^{++} v^{++} + R(v^{++})^2 = (\phi^{++})^2, \] (3.10)
\[ (\partial^{++} + Rv^{++}) x^{+\mu} = 2c_\mu^\nu x^\rho \phi^{++}. \] (3.11)
\[ (\partial^{++} - Rv^{++}) x^{-\mu} = x^{+\mu}. \] (3.12)

In order to have a well-defined HK limit, we have rescaled $v^{++}$ as
\[ v^{++} \Rightarrow Rv^{++}. \]

According to the step B we should now solve the system of differential equations (3.10) - (3.12). We will start with (3.10), (3.11). Defining
\[ v^{++} = \partial^{++} v, \quad \omega \equiv e^{Rv}, \quad \hat{x}^{+\mu} \equiv \omega x^{+\mu}, \quad \hat{\phi}^{++} = c_{\mu \nu} \hat{x}^{+\mu} \hat{x}^{+\nu} = \omega^2 \phi^{++}, \] (3.13)
we rewrite (3.10), (3.11) as
\[ (\partial^{++})^2 \omega = R \left( \frac{(\hat{\phi}^{++})^2}{\omega^3} \right), \] (3.14)
\[ \partial^{++} \hat{x}^{+\mu} = 2 \frac{\hat{\phi}^{++}}{\omega^2} c_\rho^\mu \hat{x}^{+\rho} \equiv 2k^{++} c_\rho^\mu \hat{x}^{+\rho}. \] (3.15)
From eq. (3.15) and the definition of \( \hat{\phi}^{++} \) one immediately finds

\[
\partial^{++} \hat{\phi}^{++} = 0 \quad \Rightarrow \quad \hat{\phi}^{++} = \hat{\phi}^{ik}(x)w^+_i w^+_k ,
\]

(3.16)

We observe that eq. (3.14) is none other than the pure harmonic part of the equation defining the Eguchi-Hanson metric in the harmonic superspace approach [3]! Its general solution was given in [5], it depends on four arbitrary integration constants, that is, in our case, on four arbitrary functions of \( x^{\mu i} \). However, these harmonic constants turn out to be unessential due to four hidden gauge symmetries of the set of equations (3.14) - (3.12). One of them is the scale invariance

\[
v' = v + \beta(x) , \quad \omega' = e^{R\beta} \omega , \quad \hat{\phi}^{++'} = e^{2R\beta} \hat{\phi}^{++} , \quad v^{++'} = v^{++} , \quad x^{\pm \mu'} = x^{\pm \mu} .
\]

(3.17)

It is a trivial consequence of the definition of \( v \) in eq. (3.13). Another symmetry is less trivial, this is a hidden gauge \( SU(2) \) symmetry

\[
\delta v = -\frac{1}{R} \alpha^{+-} + \alpha^{--} \partial^{-+} v , \quad \delta v^{++} = -\frac{1}{R} \alpha^{+-} + \alpha^{--} \partial^{++} v^{++} + 2\alpha^{+-} v^{++} ,
\]

(3.18)

\[
\delta x^{\pm \mu} = \alpha^{--} \partial^{+-} x^{\pm \mu} \pm \alpha^{+-} x^{\pm \mu} ,
\]

(3.19)

with

\[
\alpha^{\pm \pm} = \alpha^{(ik)}(x)w^{\pm}_i w^{\pm}_k , \quad \alpha^{+-} = \alpha^{(ik)}(x)w^+_i w^-_k .
\]

Using this gauge freedom one can gauge away four integration constants in \( \omega \) and write a general solution of eq. (3.14) in a fixed gauge in the following simple form

\[
\omega = \sqrt{1 + R\hat{\phi}^2} \quad \Rightarrow \quad v = \frac{1}{2R} \ln(1 + R\hat{\phi}^2) ,
\]

(3.20)

\[
v^{++} = \partial^{++} v = \frac{\hat{\phi} \hat{\phi}^{++}}{1 + R\hat{\phi}^2} , \quad \hat{\phi} \equiv \hat{\phi}^{(ik)}(x)w^+_i w^-_k .
\]

(3.21)

Of course, one can always restore the general form of the solution as it is given in [5], acting on (3.20) by the group (3.17), (3.18), (3.19) (with making use of a finite form of these transformations). Note that only the rigid subgroup \( \alpha^{(ik)} = \text{const} \) of (3.18), (3.19) preserves the analytic subspace \( \{x^{\pm \mu}_A, w^{\pm i}_A\} \), so the \( \tau \)-world metric is expected to be invariant only under this subgroup. Nevertheless, later on we will see that the whole effect of the full gauge \( SU(2) \) transformation is reduced to the rotation of the metric corresponding to the fixed-gauge solution (3.21) by some harmonic-independent non-singular matrix. Thus in what follows we can stick to the solution (3.20), (3.21).

Note that, defining

\[
2\lambda^2 s^2 \equiv \hat{\phi}^{(ik)} \hat{\phi}^{(ik)} ,
\]

(3.22)

and using the completeness relation for harmonics, one gets an important relation

\[
1 - R\partial^{-} v^{++} = \left(1 - R\lambda^2 s^2\right) \frac{1 - R\hat{\phi}^2}{[1 + R\hat{\phi}^2]^2} .
\]

(3.23)

Now we are prepared to solve eqs. (3.11) or (3.15). This can be done in a full analogy with the HK Taub-NUT case [2], based essentially upon the PG invariance (3.8). Using
\[ \kappa^{++} = \partial^{++} \kappa , \]

(1) \( R > 0 \), \( \kappa \equiv \kappa^{(1)} = \frac{1}{\sqrt{R}} \arctan \left( \sqrt{R} \hat{\phi} \right) \); \hspace{1cm} (3.24)

(2) \( R < 0 \), \( \kappa \equiv \kappa^{(2)} = \frac{1}{\sqrt{|R|}} \arctanh \left( \sqrt{|R|} \hat{\phi} \right) \). \hspace{1cm} (3.25)

For definiteness, in what follows we will choose the solution (3.24). Then, passing to the complex notation (3.5), choosing the particular \( SU(2) \) frame (3.6) and making the redefinition

\[ \hat{x}^{i+} = \exp \{ 2i\kappa \} x^{i+} , \hspace{1cm} \bar{x}^{i+} = \exp \{ -2i\kappa \} \bar{x}^{i+} , \]

we reduce (3.15) to

\[ \partial^{++} \hat{x}^{+} = 0 \hspace{1cm} \Rightarrow \hspace{1cm} \hat{x}^{+} = x^{i} w^{i+} , \hspace{1cm} \bar{x}^{+} = \bar{x}^{i} w^{i+} = -x^{i} w^{i+} , \hspace{1cm} \hat{\phi} = -2i \lambda x^{(i} \bar{x}^{k)} w^{i+} w^{k-} , \]

(3.27)

where, in expressing \( \hat{\phi} \), we essentially made use of the PG symmetry (3.8). Note that the quantity \( s \) defined in eq. (3.22) is expressed as follows

\[ s = x^{i} \bar{x}^{i} . \] \hspace{1cm} (3.28)

Combining eqs. (3.13), (3.20), (3.26) and (3.27) we can now write the expressions for \( x^{+} , \bar{x}^{+} \) in the following form

\[ x^{+} = \frac{1}{\sqrt{1 + R \hat{\phi}^{2}}} \exp \{ 2i\kappa \} x^{i} w^{i+} , \hspace{1cm} \bar{x}^{+} = -\frac{1}{\sqrt{1 + R \hat{\phi}^{2}}} \exp \{ -2i\kappa \} \bar{x}^{i} w^{i+} , \]

(3.29)

where \( \kappa \) and \( \hat{\phi} \) are expressed through \( x^{i} , \bar{x}^{i} \) according to eqs. (3.24), (3.27). Comparing (3.29) with the general definition of the \( x^{+} \) -bridges (2.18), we can identify \( x^{i} , \bar{x}^{i} \) with the \( \tau^{i} \) -world coordinates, i.e. with the coordinates of the initial 4-dimensional QK manifold.

We still need to find \( x^{-} , \bar{x}^{-} \) as functions of \( x^{i} , \bar{x}^{i} \) and harmonics \( w^{i+} \) by solving eq. (3.12). By means of the redefinition

\[ x^{-\mu} = \sqrt{1 + R \hat{\phi}^{2}} \hat{x}^{-\mu} \] \hspace{1cm} (3.30)

it is reduced to the form

\[ \partial^{++} \hat{x}^{-} = \frac{1}{2i \lambda} \hat{x}^{+} \partial \hat{\phi} \exp \{ 2i \kappa \hat{\phi} \} \]

(plus a similar equation for \( \bar{x}^{-} \)). After this, choosing the ansatz

\[ \hat{x}^{-} = f(s, \hat{\phi}) \hat{x}^{-} , \hspace{1cm} \bar{x}^{-} = \bar{f}(\bar{x}^{-} , \bar{x}^{-} = -\bar{x}^{-} w^{i-} , \]

where

\[ x^{-} = x^{i} w^{i-} , \hspace{1cm} \bar{x}^{-} = -\bar{x}^{i} w^{i-} , \] \hspace{1cm} (3.32)
and exploiting useful identities

\[
\hat{\phi}^+ x^- = (\hat{\phi} + i\lambda s)\tilde{x}^+ , \quad \hat{\phi}^- x^+ = (\hat{\phi} - i\lambda s)\tilde{x}^- ,
\]

one finds

\[
\tilde{x}^- = \frac{1}{2\lambda} \left[ \frac{1}{(\lambda s) - i\hat{\phi}} \left( e^{-2i\kappa(i\lambda s)} - e^{2i\kappa(\hat{\phi})} \right) \right] \tilde{x}^- ,
\]

\[
\tilde{x}^+ = \frac{1}{2\lambda} \left[ \frac{1}{(\lambda s) + i\hat{\phi}} \left( e^{-2i\kappa(i\lambda s)} - e^{2i\kappa(\hat{\phi})} \right) \right] \tilde{x}^+ .
\]

(3.34)

Here

\[
\kappa(i\lambda s) \equiv \kappa_0 = \frac{i\lambda}{\sqrt{R}} \arctanh \sqrt{R}(\lambda s) .
\]

(3.35)

Now we are ready to fulfill the step C, i.e. to find explicit expressions for the harmonic vielbein components \(H^{-+\mu}, H^{3\mu}\). It is not too illuminating to give here these expressions: given the expressions for \(x^+\) and \(x^-\), eqs. (3.29), (3.34), these objects can be straightforwardly computed using the definition (2.31). Actually, in order to compute the \(\tau\)-basis metric in the 4-dimensional case, we need only \(H^{-+\mu}\), because just this quantity enters the expression (2.44). Moreover, as we will see, it appears only inside the scalar factor (2.32). The latter turns out to be surprisingly simple

\[
\psi = \frac{1}{1 - R(x \cdot H)} = \left( \frac{1 - R\lambda^2 s^2}{1 + Rs + R\lambda^2 s^2} \right) \frac{1 - R\hat{\phi}^2}{1 + R\hat{\phi}^2} .
\]

(3.36)

4 Quaternionic Taub-NUT metric

At the step D we should calculate the transition matrix elements \(\nabla_{\rho k}x^\pm, \nabla_{\rho k}\tilde{x}^\pm\). The computation is straightforward, though tiresome. To simplify the formulas, it is convenient to define

\[
A(s) \equiv 1 - R\lambda^2 s^2 , \quad B(s) \equiv 1 + \lambda^2 s(4 + Rs) , \quad C(s) \equiv 1 + Rs + R\lambda^2 s^2 .
\]

(4.1)

Then the results of computation are as follows

\[
\nabla_{\rho k}x^+ = e^{2i\kappa} \frac{1}{\sqrt{1 + R\hat{\phi}^2}} \left\{ \delta^1_{\rho} \left( w^+_k F + w^-_k F^{++} \right) + \delta^2_{\rho} \left( w^+_k G + w^-_k G^{++} \right) \right\} ,
\]

(4.2)

\[
F = \frac{1}{4A(1 - R\hat{\phi}^2)} \left\{ 3A + B + 2i\lambda(3C - A)\hat{\phi} + 2iR\lambda(A + C)\hat{\phi}^2(\hat{\phi} + i\lambda s) \right\} .
\]

(4.3)

\[
F^{++} = -\lambda^2 \left( 1 + \frac{C}{A} \right) (\tilde{x}^+ \tilde{x}^+) , \quad G^{++} = \lambda^2 \left( 1 + \frac{C}{A} \right) (\tilde{x}^- \tilde{x}^+) ,
\]

(4.4)

\[
G = -\frac{i\lambda}{A(1 - R\hat{\phi}^2)} \left\{ i\lambda(A + C)(1 + R\hat{\phi}^2) + R(A + B)\hat{\phi} \right\} (\tilde{x}^- \tilde{x}^+) ,
\]

(4.5)

\[
\nabla_{\rho k}x^- = -\frac{1}{2i\lambda} \frac{\sqrt{1 + R\hat{\phi}^2}}{\phi + i\lambda s} \left\{ \delta^1_{\rho} \left( w^-_k T + w^+_k T^{-+} \right) + \delta^2_{\rho} \left( w^-_k S + w^+_k S^{-+} \right) \right\} ,
\]

(4.6)
\[ T = \frac{\lambda^2}{A} \{ (4 - A - C)e^{-2i\kappa_0} + (A + C)e^{2i\kappa} \} (\tilde{x}^- \tilde{x}^+) \],
\[ T^- = \frac{i\lambda}{A(1 - R\hat{\phi}^2)} \left\{ R(A + B)\hat{\phi}e^{2i\kappa} - 2R(1 + 2i\lambda\hat{\phi})e^{-2i\kappa_0} \right. \\
\left. + i\lambda(A + C)(1 + R\hat{\phi}^2) \left( e^{-2i\kappa_0} - e^{2i\kappa} \right) \right\} (\tilde{x}^- \tilde{x}^-) \],
\[ S = \lambda^2 \left( 1 + \frac{C}{A} \right) (e^{-2i\kappa_0} - e^{2i\kappa}) (\tilde{x}^+ \tilde{x}^-) \],
\[ S^- = \frac{i\lambda}{2A(1 - R\hat{\phi}^2)} \left\{ \left[ 4 + (A + B)(A - 1) - 6i\lambda(A + C - 2)\hat{\phi} + 2R(A - 1)\hat{\phi}^2 \right. \\
\left. - 2iR\lambda(A + C)\hat{\phi} \right] \left( e^{-2i\kappa_0} - e^{2i\kappa} \right) + 8i\lambda(A - 1)e^{-2i\kappa_0} \right. \\
\left. + 4i\lambda \left[ 2 - A - i\lambda(A + C - 2)\hat{\phi} \right] \left( e^{-2i\kappa_0} + e^{2i\kappa} \right) \right\} (\tilde{x}^-)^2 \].

The conjugate quantities \( \nabla_{\rho k} \tilde{x}^\pm \) follow from the above expressions by generalized conjugation.

Now we should find the entries of the inverse matrix \( X^+_{\nu} \equiv \partial^+_{\nu} x^{\mu i} \) by solving the set of algebraic equations \([2.47]\). In the complex notation, this set is divided into the two mutually conjugated ones, each consisting of four equations. It is clearly enough to consider one such set, e.g.

\[ X^{+\rho k} \nabla_{\rho k} x^- = 1, \quad X^{+\rho k} \nabla_{\rho k} x^+ = 0, \quad X^{+\rho k} \nabla_{\rho k} \tilde{x}^\pm = 0 \],

where \( X^{+\rho k} \equiv X^1_{+\rho k} \), \( (\tilde{x}^{+\rho k} \equiv -X^2_{+\rho k}) \). Let us define

\[ \hat{X}^{+\rho k} = e^{Rv} X^{+\rho k} = \sqrt{1 + R\hat{\phi}^2} X^{+\rho k}, \quad (\hat{\phi} \equiv \tilde{\phi}^+). \]

It satisfies

\[ \partial^{++} \hat{X}^{+\rho k} = 0 \]

as a consequence of eq. \([2.43]\).

Let us also define

\[ \hat{X}^{(1)} = \hat{X}^{+u_1 w_1}, \quad \hat{X}^{+\cdot (1)} = \hat{X}^{+u_1 w_1}, \quad \hat{X}^{(2)} = \hat{X}^{+2u_1 w_1}, \quad \hat{X}^{++(2)} = \hat{X}^{+2u_1 w_1}. \]

Eqs. \([4.13]\) together with the PG \( U(1) \) invariance fix these quantities up to three integration constants which are functions of \( s \equiv x^4 \tilde{x}_1 \):

\[ \hat{X}^{(1)} = \frac{1}{2i\lambda}(f_1 + f_3 \hat{\phi}), \quad \hat{X}^{++(1)} = (\tilde{x}^+ \tilde{x}^+) f_3, \quad \hat{X}^{(2)} = (\tilde{x}^+ \tilde{x}^-) f_4, \quad \hat{X}^{++(2)} = (\tilde{x}^+)^2 f_4. \]

Now eqs. \([4.14]\) serve to compute these harmonic constants. After substitution of the explicit expressions for \( \nabla_{\rho k} x^\pm \nu \) \([4.12] - [4.10]\), one finds that the last three eqs. from the set \([4.11]\) require

\[ (a) \ f_3 = f_4; \quad (b) \ f_3 = -2i\lambda \frac{A + C}{3A + B} f_1, \]

\[ (c) \ f_4 = f_3. \]
Recall that
\[ f_1 = \frac{i}{2} \lambda (3A + B) e^{2i\kappa_0} . \] (4.17)

As a result one gets fairly simple expressions for \( \hat{X}^{(1)}, \hat{X}^{(2)} \)
\[
\hat{X}^{(1)} = \frac{1}{4} \{ (3A + B) - i\lambda (A + C)\phi \} e^{2i\kappa_0}, \\
\hat{X}^{++(1)} = \lambda^2 (A + C) e^{2i\kappa_0} (\tilde{x}^+ \tilde{x}^+) , \\
\hat{X}^{(2)} = \lambda^2 (A + C) e^{2i\kappa_0} (\tilde{x}^+ \tilde{x}^-) , \\
\hat{X}^{++(2)} = \lambda^2 (A + C) e^{2i\kappa_0} (\tilde{x}^+)^2 . \] (4.18)

It should be pointed out that the system (1.11) is overdetermined, so it provides good self-consistency checks of the correctness of the expressions for \( \nabla_{\mu \kappa} \tilde{x}^{\pm \mu} \).

It is easy now to restore \( \hat{X}^{+ \rho \kappa} \):
\[
\hat{X}^{+1k} = (\partial^+ x^k) = \frac{1}{4} \{ (3A + B) e^{kl} - 4\lambda^2 (A + C) x^{(k} x^{l)} \} w_i^+ e^{2i\kappa_0} , \\
\hat{X}^{+2k} = (\partial^+ x^k) = \frac{1}{4} \{ (3A + B) e^{kl} + 4\lambda^2 (A + C) x^{(k} x^{l)} \} w_i^+ e^{2i\kappa_0} , \\
\hat{X}^{+1k} = (\partial^+ x^k) = \lambda^2 (A + C) (x^k x^l) w_i^+ e^{2i\kappa_0} , \\
\hat{X}^{+2k} = (\partial^+ x^k) = \lambda^2 (A + C) (x^k x^l) w_i^+ e^{2i\kappa_0} . \] (4.19)

Recall that
\[ \tilde{x}^\pm = x^k w^\pm_k , \quad \tilde{x}^\pm = -\tilde{x}^k w^\pm_k , \quad \tilde{(x^k)} = \tilde{x}_k , \quad \tilde{(x^k)} = -x_k . \] (4.20)

Now we have all the necessary bricks to fulfill the last step \( \text{E} \): to compute \( \partial \hat{H} \) and, hence, the whole \( \tau \) world metric \( g_{\rho i,\lambda k} \), eq. (2.39).

It will be convenient to rewrite (2.39) through \( X \), using
\[
\frac{1}{(1 - R x^+ H^{-+\mu})(1 - R \partial^{-\hat{\rho}} H^{++})} = \frac{1}{C (1 + R \hat{\phi}^2)} . \] (4.21)
(recall eqs. (3.23), (3.36)). Then one has
\[
g_{\rho i,\lambda k} = \frac{1}{C \det(\partial \hat{H})} \hat{G}^{\rho i,\lambda k} , \] (4.22)
\[
\hat{G}^{\rho i,\lambda k} = (1 + R \hat{\phi}^2) G^{\rho i,\lambda k} = e^{\omega \beta} \{ \partial^{-\hat{\rho}} \hat{X}^{\rho \lambda} + (\rho \hat{\lambda} \leftrightarrow \rho \lambda) \} . \] (4.23)

Next, one uses eq. (2.48) to rewrite \( \det(\partial \hat{H}) \) as follows
\[
\det(\partial \hat{H}) = \frac{1}{(1 - R x^+ H^{-+\mu})^3} \det(\partial H) = \left( \frac{1 - R \hat{\phi}^2}{1 + R \hat{\phi}^2} \right)^3 \frac{A^3}{C} \det(\partial H) . \] (4.24)

After some algebra one can cast \( \det(\partial H) \) into the following convenient form
\[
\det(\partial H) = \frac{1}{2} \left( 1 - R \partial^{-\hat{\rho}} H^{++} \right)^2 (1 + R \hat{\phi}^2) \left[ e^{\omega \beta} \partial^{-\hat{\rho}} \hat{X}^{\rho \lambda} \hat{X}^{+\lambda} \right] \times \\
\times \left[ \nabla_{\mu \kappa} x^+ \nabla_{\lambda \mu} x^+ \epsilon_{\nu \mu} \right] . \] (4.25)
As a result of rather cumbersome, though straightforward computation one eventually gets the simple expression for $\text{det}(\partial \hat{H})$

$$\text{det}(\partial \hat{H}) = A^2 \frac{B}{C^3} e^{4i\kappa_\sigma} (1 - R\lambda^2 s^2)^2 \frac{1 + 2\lambda^2 s + \lambda^2 s(2 + sR)}{(1 + Rs + R\lambda^2 s^2)^3} e^{4i\kappa_\sigma}. \quad (4.26)$$

As is seen, the harmonic dependence disappeared in $\text{det}(\partial \hat{H})$, as it should be.

The calculation of this determinant is the longest part of the whole story. Once this has been done, the computation of the $\tau$ basis metric amounts to the computation of entries of the matrix $\hat{G}_{\rho i,\lambda l}$. This can be done rather straightforwardly, and the final answer for the inverse metric is

$$g_{1k,1t} = -\frac{C^2 D}{A^2 B} (x^k x^t), \quad (4.27)$$

$$g_{2k,2t} = -\frac{C^2 D}{A^2 B} (\bar{x}^k \bar{x}^t), \quad (4.28)$$

$$g_{1k,2t} = \frac{C^2}{A^2 B} [ -\epsilon^{kt} A^2 + D(x^k \bar{x}^t) ], \quad (4.29)$$

where $D \equiv \lambda^2 (A + C)(A + B) = 2\lambda^2 (2 + Rs)(1 + 2\lambda^2 s)$.

The metric tensor is then readily obtained

$$\begin{align*}
\begin{cases}
  g_{1k,1t} &= \frac{D}{C^2 B} (\bar{x}_k \bar{x}_t), \\
  g_{2k,2t} &= \frac{D}{C^2 B} (x_k x_t), \\
  g_{1k,2t} &= \frac{1}{C^2 B} [B^2 \epsilon_{kt} + D(\bar{x}_k x_t)]
\end{cases}
\end{align*} \quad (4.30)$$

thanks to the identity $A^2 + 2sD = B^2$. Note that the final expressions for the metric and its inverse are valid for any sign of the parameter $R$ even if, on the intermediate steps, the choice of sign was essential.

As the last topic of this Section we give how the metric (2.39) is transformed under the gauge SU(2) transformations (3.18), (3.19). Exploiting only the fact that in the present case $L^{+4}, H^{+4}, H^{+3\mu}$ contain no explicit harmonics in the analytic basis, one can show that

$$\delta g_{\mu m \nu k} = \alpha^+ - \partial^+ g_{\mu m \nu k} + g_{\sigma p \nu k} \Lambda_{\sigma p}^{\mu m}(x) + g_{\mu m \sigma p} \Lambda_{\sigma p}^{\nu k}, \quad (4.31)$$

where

$$\Lambda_{\sigma p}^{\nu k}(x) = \partial_{\sigma p} \alpha^{(lt)}(x) (T^A_{(lt)} x^{\nu k}), \quad T^A_{(lt)} \equiv w^+_A(t \partial_{w^+_A}) + w^-_A(t \partial_{w^-_A}). \quad (4.32)$$

The property that $\Lambda_{\sigma p}^{\nu k}$ does not depend on harmonics in the $\tau$-basis, $\partial^+ \Lambda_{\sigma p}^{\nu k} = 0$, is a direct consequence of the relation $[D^{++}, T^A_{(lt)}] = 0$ which follows from the fact that the analytic harmonic vielbeins bears no dependence on $w^\pm$ in the case at hand. The first term in (4.31) vanishes because $\partial^\pm g^{\mu s \nu m} = 0$. Thus in the rigid case ($\partial_{\rho m} \alpha^{(ik)} = 0$) the metric is not changed at all, while in the general case it undergoes some rotation by each of its world indices by a non-singular matrix $\delta_{\mu}^{\nu} \delta_{\kappa}^{\rho} + \Lambda_{\sigma p}^{\nu k}(x) + \ldots$. Note that, similarly
to the HK TN case, the $SU(2)$ factor of the $U(2)$ isometry group of the QTN metric (it rotates $x^i$ and $\bar{x}^i$ by their doublet indices) in the harmonic space formulation originates from a rigid $SU(2)$ that rotates the doublet indices of the harmonics $w^{\pm i}$ and $\bar{w}^{\pm i}$. The analytic coordinates $x^{+ \mu}$ and the bridge $v^{++}$ behave under this $SU(2)$ as scalars of zero weight.

5 Limiting cases and identification with the known metrics

To compare to the results in the literature [9] one has to use [12]

$$dx^i = x^i \left( \frac{ds}{2s} + is_3 \right) - \bar{x}^i \left( \frac{\sigma_2 - i\sigma_1}{2} \right), \quad d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k.$$  

(5.1)

Using the notation $A \cdot B \equiv A^i B_i$ relation (5.1) implies

$$-\bar{x} \cdot dx = \frac{ds}{2} + is_3, \quad dx \cdot d\bar{x} = \frac{ds^2}{4s} + \frac{s}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \quad s = x \cdot \bar{x}.$$  

(5.2)

The metric given by (4.30) writes

$$\frac{D}{C^2B} \left[ (\bar{x} \cdot dx)^2 + (x \cdot d\bar{x})^2 \right] + 2B \frac{D}{C^2B}dx \cdot d\bar{x} + 2\frac{D}{C^2B}(\bar{x} \cdot dx)(x \cdot d\bar{x}),$$

(5.3)

and becomes

$$\frac{1}{2} \left[ \frac{B}{sC^2} ds^2 + \frac{sB}{C^2} (\sigma_1^2 + \sigma_2^2) + \frac{B}{C^2B} \sigma_3^2 \right].$$  

(5.4)

Three limiting cases give metrics of interest [1]:

1. The $\lambda = 0$ limit leads to

$$\frac{1}{(1 + Rs)^2} \left( \frac{ds^2}{4s} + \frac{s}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \right) = \frac{dx \cdot d\bar{x}}{(1 + Rs)^2}.$$  

(5.5)

For $R > 0$ we have the compact coset $HP^1 \sim S^4 \sim SO(5)/SO(4)$ in its conformally flat form (the so-called de Sitter metric). For $R < 0$ we are left with the hyperbolic metric on the unit ball i.e. the non-compact symmetric coset $SO(4,1)/SO(4)$ (the anti-de Sitter metric).

2. The $R = 0$ limit leads to

$$(1 + 4\lambda^2 s) \frac{ds^2}{s} + s(1 + 4\lambda^2 s)(\sigma_1^2 + \sigma_2^2) + \frac{s}{1 + 4\lambda^2 s} \sigma_3^2,$$  

(5.6)

which we recognize as the Taub-NUT metric which is complete for $s \geq 0$ and $4\lambda^2 \geq 0$.

\*\*We omit the overall 1/2 factor.\*\*
3. The \( R = 4\lambda^2 \) limit leads to

\[
\frac{ds^2}{s(1 + Rs/2)^2} + \frac{s}{(1 + Rs/2)^2} \left( \sigma_1^2 + \sigma_2^2 \right) + \frac{s(1 - Rs/2)^2}{(1 + Rs/2)^4} \sigma_3^2, \tag{5.7}
\]

which is not complete for \( R > 0 \). For \( R < 0 \) the change of coordinates \( r^2 = \frac{-Rs}{(1 - Rs/2)^2} \) brings the metric to the form

\[
\frac{2}{(-R)} \left[ \frac{(dr)^2}{(1 - r^2)^2} + \frac{r^2}{1 - r^2} \frac{\sigma_1^2 + \sigma_2^2}{4} + \frac{r^2}{(1 - r^2)^2} \frac{\sigma_3^2}{4} \right]. \tag{5.8}
\]

If we take \( 0 \leq r < 1 \) then this is Bergmann’s metric in the unit ball of \( \mathbb{C}^2 \) which is complete. It does correspond to the non-compact Kähler symmetric space \( SU(2,1)/U(2) \).

The most general Bianchi IX euclidean Einstein metrics can be deduced from Carter’s results \[13\]. A convenient standardization \[14\] is the following

\[
dr^2 = l^2 \left\{ \frac{r^2 - 1}{\Delta(r)} (dr)^2 + 4 \frac{\Delta(r)}{r^2 - 1} \sigma_3^2 + (r^2 - 1)(\sigma_1^2 + \sigma_2^2) \right\}, \tag{5.9}
\]

with

\[
\Delta(r) = -\frac{\Lambda l^2}{3} r^4 + (1 + 2\Lambda l^2)r^2 - 2Mr + 1 + \Lambda l^2. \tag{5.10}
\]

These metrics are Einstein, with Einstein constant \( \Lambda \) and isometry group \( U(2) \).

In dimension 4 a quaternionic metric is defined to be an Einstein metric with self-dual Weyl tensor. A simple computation gives

\[
W^+ = \frac{1}{l^2} \frac{(-1 + M - \frac{4}{3}\Lambda l^2)}{(r - 1)^3} \ Y, \quad \ Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{5.11}
\]

Imposing \( W^+ = 0 \) and getting rid of the parameter \( M \) gives for the metric

\[
dr^2(Q) = l^2 \left\{ \frac{r + 1}{r - 1} \left( \frac{(dr)^2}{\Sigma(r)} + 4 \frac{r - 1}{r + 1} \Sigma(r) \sigma_3^2 + (r^2 - 1)(\sigma_1^2 + \sigma_2^2) \right) \right\}, \tag{5.12}
\]

where now

\[
\Sigma(r) = 1 - \frac{\Lambda l^2}{3} (r - 1)(r + 3). \tag{5.13}
\]

The identifications

\[
\frac{r - 1}{2} = \frac{4\lambda^2 - R}{1 + Rs + R\lambda^2 s^2}, \quad \frac{4}{3}\Lambda l^2 = \frac{R}{4\lambda^2 - R}, \tag{5.14}
\]

give the relation

\[
4(4\lambda^2 - R) \left[ \frac{B}{sC^2} ds^2 + \frac{sB}{C^2} (\sigma_1^2 + \sigma_2^2) + \frac{sA^2}{C^2B} \sigma_3^2 \right] = \frac{dr^2(Q)}{l^2}. \tag{5.15}
\]
Note that the quaternionic metric (5.12) is complete for $\Lambda < 0$ and is asymptotically Anti-de Sitter. It has been recently considered in [10] under the name Taub-NUT-AdS metric and reveals itself a useful background for computing black-holes entropy.

For the sake of completeness let us mention that this (Weyl) self-dual Einstein metric was also derived in [15] in the form

$$d\tau^2 = \frac{1}{(1-t^2)^2} \left\{ 1 + \frac{m^2 t^2}{1 + m^2 t^4} (dt)^2 + \frac{t^2(1 + m^2 t^4)}{1 + m^2 t^2} \frac{\sigma_3^2}{4} + t^2(1 + m^2 t^2) \frac{(\sigma_1^2 + \sigma_2^2)}{4}\right\}.$$ 

The identifications

$$t^2 = \frac{s - 1}{s + 2m^2 + 1}, \quad -\frac{4}{3} \Lambda l^2 = \frac{1}{m^2 + 1},$$

lead to the relation

$$d\tau^2(\mathbb{Q}) = 16(m^2 + 1) \cdot d\tau^2_1.$$ 

6 Conclusion

In this paper we used the harmonic space formulation of the QK geometry [1] to compute the QK metric with a non-trivial quaternionic potential $L^{+4}$, the four-dimensional quaternionic Taub-NUT metric. We found that the harmonic space techniques, like in the HK case [2, 3, 5], allows one to get the explicit form of the QK metric starting from a given QK potential and following a generic set of rules. It would be interesting to apply this approach to find the QK analogs of some other interesting 4- and higher-dimensional HK metrics, in particular, the quaternionic Eguchi-Hanson metric and the quaternionic generalization of the multicentre metrics of Gibbons and Hawking [17].

A first simple generalization of the Taub-NUT potential would be to add the dipolar breaking [6]

$$L^{+4} = \eta^{--}(\phi^{++})^3, \quad \eta^{--} = \eta^{(ik)} u_i^- u_k^-, \quad \eta^{(ik)} = \text{const}.$$ 

When trying to compute the quaternionic metric with the same $L^{+4}$ along the lines of Sect. 3, one encounters a difficulty already at the step of solving the equation for the relevant bridge $v^{++}$. Making the same changes of variables as those leading to eq. (3.14), we get in this case

$$(\partial^{++})^2 \omega = 2R \eta^{--} \frac{(\hat{\phi}^{++})^3}{\omega^5}, \quad \partial^{++} \hat{\phi}^{++} = 0.$$ 

This equation is not so easy to solve as compared to (3.14). Thus, to attack the multicenter case, it will perhaps prove advantageous to develop some ways around, e.g., the harmonic space version of the quaternionic quotient construction [18].

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References

[1] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, *Class. Quantum Grav.* 1 (1984) 469.

[2] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *Commun. Math. Phys.* 103 (1986) 515.

[3] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *Ann. Phys.* (N.Y.) 185 (1988) 22.

[4] A. Galperin, E. Ivanov and O. Ogievetsky, *Ann. Phys.* (N.Y.) 230 (1994) 201.

[5] A. Galperin, E. Ivanov, V. Ogievetsky and P.K. Townsend, *Class. Quantum Grav.*, 7 (1986) 625.

[6] G.W. Gibbons, D. Olivier, P.J. Ruback and G. Valent, *Nucl. Phys.*, B 296 (1988) 679.

[7] J. Bagger and E. Witten, *Nucl. Phys.* B 222 (1983) 1.

[8] E. Ivanov, G. Valent, ‘Quaternionic Taub-NUT from the harmonic space approach’, Preprint LPTHE 98-43, JINR E2-98-248, September 1998; hep-th/9809108.

[9] T. Eguchi, B. Gilkey and J. Hanson, *Physics Reports*, 66, No. 6 (1980) 213.

[10] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *Ann. Phys.* (N.Y.) 185 (1988) 1.

[11] J.A. Bagger, A.S. Galperin, E.A. Ivanov and V.I. Ogievetsky, *Nucl. Phys.* B 303 (1988) 522.

[12] F. Delduc and G. Valent, *Class. Quantum Grav.*, 10 (1993) 1201.

[13] B. Carter, *Commun. Math. Phys.* 10 (1968) 280.

[14] T. Chave and G. Valent, *Class. Quantum Grav.*, 13 (1996) 2097.

[15] H. Pedersen, *Math. Ann.*, 274 (1986) 35.

[16] S.W. Hawking, C.J. Hunter and Don N. Page, ’Nut Charge, Anti-de Sitter Space and Entropy’, hep-th/9809035.

[17] G. Gibbons and S. W. Hawking, *Phys. Lett.* B 78 (1978) 430.

[18] K. Galicki, *Commun. Math. Phys.* 108 (1987) 117; *Class. Quantum Grav.*, 8 (1991) 1529.