On the Quasi-fixed Point in the Running of CP-violating Phases of Majorana Neutrinos

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Abstract

Taking the standard parametrization of three-flavor neutrino mixing, we carefully examine the evolution of three CP-violating phases ($\delta, \alpha_1, \alpha_2$) with energy scales in the realistic limit $\theta_{13} \to 0$. If $m_3$ vanishes, we find that the one-loop renormalization-group equation (RGE) of $\delta$ does not diverge and its running has no quasi-fixed point. When $m_3 \neq 0$ holds, we show that the continuity condition derived by Antusch et al is always valid, no matter whether the $\tau$-dominance approximation is taken or not. The RGE running of $\delta$ undergoes a quasi-fixed point determined by a nontrivial input of $\alpha_2$ in the limit $m_1 \to 0$. If three neutrino masses are nearly degenerate, it is also possible to arrive at a quasi-fixed point in the RGE evolution of $\delta$ from the electroweak scale to the seesaw scale or vice versa. Furthermore, the continuity condition and the quasi-fixed point of CP-violating phases in another useful parametrization are briefly discussed.

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Recent solar [1], atmospheric [2], reactor [3] and accelerator [4] neutrino oscillation experiments have provided us with very robust evidence that neutrinos are massive and lepton flavors are mixed. The phenomenon of lepton flavor mixing is described by a $3 \times 3$ unitary matrix $V$. A particular parametrization of $V$ has been advocated by the Particle Data Group [5]:

$$
V = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}c_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{12}c_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}
\begin{pmatrix}
    e^{i\alpha_1/2} & 0 & 0 \\
    0 & e^{i\alpha_2/2} & 0 \\
    0 & 0 & 1
\end{pmatrix},
$$

(1)

where $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$ (for $ij = 12, 23$ and 13). The phase parameters $\alpha_1$ and $\alpha_2$ are commonly referred to as the Majorana CP-violating phases, because they are only physical for Majorana neutrinos and have nothing to do with CP violation in the neutrino-neutrino and antineutrino-antineutrino oscillations. A global analysis of current experimental data yields [6] $30^\circ < \theta_{12} < 38^\circ$, $36^\circ < \theta_{23} < 54^\circ$ and $\theta_{13} < 10^\circ$ at the 99% confidence level. In addition, the neutrino mass-squared differences $\Delta m_{21}^2 \equiv m_2^2 - m_1^2 = (7.2 \cdots 8.9) \times 10^{-5}$ eV$^2$ and $\Delta m_{32}^2 \equiv m_3^2 - m_2^2 = \pm (1.7 \cdots 3.3) \times 10^{-3}$ eV$^2$ have been extracted from solar and atmospheric neutrino oscillations at the same confidence level [6]. The sign of $\Delta m_{32}^2$ remains undetermined and three CP-violating phases of $V$ are entirely unrestricted.

Note that $\theta_{13} = 0$, which may naturally arise from an underlying flavor symmetry (e.g., $S_3$ [7] or $A_4$ [8]), is absolutely allowed by the present experimental data. Note also that either $m_1 = 0$ or $m_3 = 0$, which can be obtained from a specific neutrino mass model (e.g., the minimal seesaw model [9]), is absolutely consistent with current neutrino oscillation data. These interesting limits deserve some careful consideration in the study of neutrino phenomenology. For instance, the one-loop renormalization-group equation (RGE) of $\delta$ includes the $1/\sin \theta_{13}$ term which is very dangerous in the limit $\theta_{13} \to 0$ [10]. It has been noticed by Antusch et al [11] that the derivative of $\delta$ can keep finite when $\theta_{13}$ approaches zero, if $\delta, \alpha_1$ and $\alpha_2$ satisfy a novel continuity condition in the $\tau$-dominance approximation (in which the small contributions of electron and muon Yukawa couplings to the RGEs are safely neglected). It has also been noticed by us [12] that the RGE running of $\delta$ may undergo a nontrivial quasi-fixed point driven by the nontrivial inputs of $\alpha_1$ and $\alpha_2$ in the tri-bimaximal neutrino mixing scenario [13] with a near mass degeneracy of three neutrinos.

We find it desirable to examine the continuity condition obtained by Antusch et al [11] without taking the $y_\tau^2$-dominance approximation, where $y_\tau$ denotes the tau-lepton Yukawa coupling eigenvalue. The reason is simply that the $y_\tau^2$ and $y_\mu^2$ contributions to the RGE of $\delta$ may also involve the $1/\sin \theta_{13}$ terms and become dangerous in the limit $\theta_{13} \to 0$. On the other hand, it is desirable to look at possible quasi-fixed points in the RGE running of $\delta$ by choosing more generic neutrino mixing scenarios with vanishing (or vanishingly small) $\theta_{13}$ and considering different patterns of the neutrino mass spectrum.

The main purpose of this paper is just to carry out a careful analysis of the RGE evolution of three CP-violating phases ($\delta, \alpha_1, \alpha_2$) in the realistic limit $\theta_{13} \to 0$ from the electroweak scale $\Lambda_{\text{EW}} \sim 10^2$ GeV to the typical seesaw scale $\Lambda_{\text{SS}} \sim 10^{14}$ GeV. If $m_3$ vanishes, we find that the RGE of $\delta$ does not diverge and its running has no quasi-fixed point. This new observation demonstrates that our previous understanding of the running behaviors of $\delta$ is more or less incomplete. When $m_3 \neq 0$ holds, we show that the continuity condition derived
by Antusch et al can be rediscovered even though the $y_e^2$ and $y_\mu^2$ contributions to the RGE of $\delta$ are not neglected. The RGE running of $\delta$ undergoes a quasi-fixed point determined by a nontrivial input of $\alpha_2$ in the limit $m_1 \to 0$. If three neutrino masses are nearly degenerate (either $\Delta m_{32}^2 > 0$ or $\Delta m_{32}^2 < 0$), a quasi-fixed point may also show up in the RGE evolution of $\delta$ from the electroweak scale to the seesaw scale (or vice versa). Finally we give some brief comments on the continuity condition and the quasi-fixed point of CP-violating phases in another useful parametrization of $V$.

The exact one-loop RGEs of three neutrino masses ($m_1, m_2, m_3$), three mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$) and three CP-violating phases ($\delta, \alpha_1, \alpha_2$) have already been derived by Antusch et al [11] and can be found from the web page [14]. Their results, which have been confirmed by Mei and Zhang independently [15], clearly show that only the RGE of $\delta$ contains the $1/\sin \theta_{13}$ term. For simplicity, here we only write out the derivative of $\delta$ in an exact but compact way:

$$\frac{d\delta}{dt} = C \left( y_e^2 - y_\mu^2 \right) \cdot \frac{m_3 \chi}{\Delta m_{31}^2 \Delta m_{32}^2} \cdot \frac{\sin 2\theta_{12} \sin 2\theta_{23}}{\sin \theta_{13}} + \text{other terms} ,$$

where $t \equiv \ln(\mu/\Lambda_{SS})$ with $\mu$ being an arbitrary renormalization scale below $\Lambda_{SS}$ but above $\Lambda_{EW}$, $C = -3/2$ in the standard model (SM) or $C = 1$ in the minimal supersymmetric standard model (MSSM),

$$\chi = m_3 \Delta m_{21}^2 \sin \delta + m_2 \Delta m_{31}^2 \sin (\delta + \alpha_2) - m_1 \Delta m_{32}^2 \sin (\delta + \alpha_1) ,$$

and “other terms” stand for those terms which do not include the $1/\sin \theta_{13}$ factor. We find that the $y_e^2$ contribution to $d\delta/dt$ does not involve $1/\sin \theta_{13}$ at all, while the $1/\sin \theta_{13}$ terms associated with $y_\mu^2$ and $y_\tau^2$ contributions to $d\delta/dt$ are identical in magnitude but have the opposite sign. When the $\tau$-dominance approximation is taken (i.e., neglecting the $y_e^2$ and $y_\mu^2$ contributions in the RGEs), Eq. (2) reproduces the approximate $1/\sin \theta_{13}$ term of $d\delta/dt$ given in Ref. [11].

In the limit $\theta_{13} \to 0$, which is allowed (and even favored [6]) by current neutrino oscillation data, the $1/\sin \theta_{13}$ term in $d\delta/dt$ diverges. To keep $d\delta/dt$ finite, the divergence of $1/\sin \theta_{13}$ has to be cancelled by its associate factor. Eq. (2) indicates that $m_3 \chi = 0$ needs to be satisfied, in order to cancel the divergence induced by $1/\sin \theta_{13}$ in the limit $\theta_{13} \to 0$. There are two separate possibilities:

1. $m_3 = 0$. This special but interesting possibility was not mentioned in Ref. [11]. In this case, the derivative of $\delta$ is apparently finite for vanishing or vanishingly small $\theta_{13}$. Hence the corresponding RGE running of $\delta$ is expected to be mild and have no quasi-fixed point. Note that only the difference between $\alpha_1$ and $\alpha_2$ has physical significance in the limit $m_3 \to 0$, just like the instructive case in the minimal seesaw model with

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Note that the Majorana phases $\varphi_1$ and $\varphi_2$ defined in Refs. [11] and [14] are equivalent to $\alpha_1$ and $\alpha_2$ defined in Eq. (1): $\varphi_i = -\alpha_i$ (for $i = 1, 2$).
two heavy right-handed Majorana neutrinos [16]. If both $m_3 = 0$ and $\theta_{13} = 0$ hold at a given energy scale, one can easily show that $m_3$ and $\theta_{13}$ will keep vanishing at any scale between $\Lambda_{\text{EW}}$ and $\Lambda_{\text{SS}}$ [17]. In this particular case, $\delta$ is not well-defined and has no physical meaning at all energy scales.

2. $\chi = 0$. In this case, one may arrive at the continuity condition from Eq. (3):

$$\cot \delta = \frac{m_1 \cos \alpha_1 - (1 + \zeta) m_2 \cos \alpha_2 - \zeta m_3}{(1 + \zeta) m_2 \sin \alpha_2 - m_1 \sin \alpha_1},$$

(4)

where $\zeta \equiv \Delta m^2_{21}/\Delta m^2_{32} \approx \pm (2.2 \cdots 5.2) \times 10^{-2}$. Although this result is equivalent to the one derived by Antusch et al in the $\tau$-dominance approximation [11], it is now obtained by us in no special assumption or approximation. Given $\alpha_1 = \alpha_2 = 0$, the resultant quasi-fixed point is trivially $\delta = 0$ or $\delta = \pi$. A nontrivial quasi-fixed point in the RGE running of $\delta$ can in general result from the nontrivial inputs of $\alpha_1$ and $\alpha_2$.

In the following, we shall take some typical examples to illustrate the quasi-fixed point in the RGE evolution of $\delta$ from $\Lambda_{\text{EW}}$ to $\Lambda_{\text{SS}}$ either in the SM or in the MSSM.

3 In view of the fact that the absolute mass scale of three light neutrinos and the sign of $\Delta m^2_{32}$ remain unknown, let us consider four possible patterns of the neutrino mass spectrum to simplify the continuity condition in Eq. (4) and discuss the quasi-fixed point in the RGE running of $\delta$ by taking a few typical numerical examples.

(1) Normal hierarchy: $m_1 \ll m_2 \ll m_3$. For simplicity, we typically take $m_1 = 0$ at $\Lambda_{\text{EW}}$ in our analysis. Then $m_2 = \sqrt{\Delta m^2_{21}} \approx 8.9 \times 10^{-3}$ eV and $m_3 = \sqrt{|\Delta m^2_{32}| + \Delta m^2_{21}} \approx 5.1 \times 10^{-2}$ eV can be obtained from the best-fit values $\Delta m^2_{21} = 8.0 \times 10^{-5}$ eV and $|\Delta m^2_{32}| = 2.5 \times 10^{-3}$ eV [6]. In this special case, the phase parameter $\alpha_1$ has no physical significance and Eq. (4) can easily be simplified to

$$m_2 \sin \delta + m_3 \sin (\delta + \alpha_2) = 0.$$

(5)

Once the initial value of $\alpha_2$ is fixed at $\Lambda_{\text{EW}}$, the value of $\delta$ at its quasi-fixed point can be determined.

\[2\] Note that the one-loop RGEs of $m_i$ (for $i = 1, 2, 3$) have the form $\text{d}m_i/\text{d}t \propto m_i$ [14]. Hence $m_i = 0$ keeps unchanged if it is initially given at one energy scale.

\[3\] Our numerical calculations follow a “running and diagonalizing” procedure [11]: we first compute the RGE evolution of lepton mass matrices starting from $\Lambda_{\text{EW}}$, and then extract their mass eigenvalues and flavor mixing parameters at $\Lambda_{\text{SS}}$. This approach itself is independent of our analytical derivation of the RGEs for three neutrino masses $(m_1, m_2, m_3)$, three mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$ and three CP-violating phases $(\delta, \alpha_1, \alpha_2)$. If the limit $\theta_{13} \to 0$ is taken at $\Lambda_{\text{EW}}$, any initial input of $\delta$ is allowed but it does not take any effect in the RGE running. The finite running result of $\delta$ is actually attributed to the initial values of two Majorana phases $\alpha_1$ and $\alpha_2$. 


To illustrate, we present our typical numerical examples in Table I. Because \( m_1 = 0 \) and \( \theta_{13} = 0 \) are taken at \( \Lambda_{EW} \), the corresponding phase parameters \( \alpha_3 \) and \( \delta \) are not well-defined and their initial values at \( \Lambda_{EW} \) have no physical meaning. While \( m_1 = 0 \) keeps unchanged up to the seesaw scale, \( \theta_{13} \) will become non-vanishing due to the radiative correction. It is also the radiative correction that gives rise to nontrivial values of \( \delta \) at \( \mu > \Lambda_{EW} \), but \( \alpha_1 \) remains arbitrary and has no physical significance as a direct consequence of \( \Lambda_{EW} \) in Table II. In this case, the sum of \( m_1 \) takes only negligible small, as one can see from Table I. The nontrivial output of \( \delta \) comes from the nontrivial input of \( \alpha_2 \) via the RGE running effects. Of course, larger values of \( \tan \beta \) will in general lead to larger radiative corrections to the neutrino mass and mixing parameters in the MSSM.

(2) Inverted hierarchy: \( m_3 \ll m_1 \ll m_2 \). We have pointed out that the derivative of \( \delta \) must be finite when \( m_3 = 0 \) holds; i.e., there is no quasi-fixed point in the RGE running of \( \delta \) in the limit \( m_3 \to 0 \). If \( m_3 \) is very small but non-vanishing, however, the evolution of \( \delta \) may undergo a quasi-fixed point in the limit \( \theta_{13} \to 0 \). For illustration, we typically take \( m_3 = 10^{-5} \text{ eV} \) at \( \Lambda_{EW} \) in our analysis. Then we obtain \( m_2 = \sqrt{|\Delta m_{32}^2| + m_3^2} \approx 5.0 \times 10^{-2} \text{ eV} \) and \( m_1 \approx \sqrt{|\Delta m_{32}^2| - \Delta m_{21}^2 + m_3^2} \approx 4.9 \times 10^{-2} \text{ eV} \) from the best-fit values \( \Delta m_{21}^2 = 8.0 \times 10^{-5} \text{ eV} \) and \( |\Delta m_{32}^2| = 2.5 \times 10^{-3} \text{ eV} \) [6]. The first term on the right-hand side of Eq. (3) is safely negligible. As a result, \( \chi = 0 \) leads to an approximate relation for three phase parameters at the quasi-fixed point of \( \delta \):

\[
\alpha \sin (\delta + \alpha_2) \approx m_2 \sin (\delta + \alpha_1) .
\]

The value of \( \delta \) can therefore be determined from this equation together with the initial inputs of \( \alpha_3 \) and \( \alpha_2 \) at \( \Lambda_{EW} \).

We present a few numerical examples in Table II for illustration. Although \( \theta_{13} = 0 \) is taken at \( \Lambda_{EW} \), it will become non-vanishing at \( \mu > \Lambda_{EW} \) due to the radiative correction. The phase parameter \( \delta \), whose RGE running is driven by the initial values of two Majorana phases \( \alpha_1 \) and \( \alpha_2 \), turns out to be nontrivial at \( \mu > \Lambda_{EW} \) and keeps on staying at its quasi-fixed point. It is easy to check that Eq. (6) is well satisfied by the values of three CP-violating phases at \( \Lambda_{SS} \) as shown in Table II. A particularly interesting possibility is \( \alpha_1 \approx \alpha_2 \) (Case III in Table II). In this case, the sum of \( \delta \) and \( \alpha_1 \) (or \( \alpha_2 \)) is close to \( n\pi \) (for \( n = 0, \pm 1, \pm 2, \cdots \)), which can also be seen from Eq. (6).

(3) Near degeneracy with \( \Delta m_{32}^2 > 0 \): \( m_1 \ll m_2 \ll m_3 \). For simplicity, we typically take \( m_1 = 0.2 \text{ eV} \) at \( \Lambda_{EW} \) in our numerical calculations. Then \( m_1 \approx m_2 \approx m_3 \) automatically holds, and the continuity condition \( \chi = 0 \) for the RGE running of \( \delta \) in the limit \( \theta_{13} \to 0 \) can approximately be simplified to

\[
\Delta m_{32}^2 [\sin (\delta + \alpha_1) - \sin (\delta + \alpha_2)] \approx \Delta m_{21}^2 [\sin \delta + \sin (\delta + \alpha_2)] .
\]

This equation may further be simplified in two different cases:

- If \( \alpha_1 \neq \alpha_2 \) holds, the right-hand side of Eq. (7) is strongly suppressed by the smallness of \( \Delta m_{21}^2 \), leading to

\[
\Delta m_{32}^2 (\delta + \alpha_1) \approx \Delta m_{21}^2 (\delta + \alpha_2)
\]
\[
\sin (\delta + \alpha_1) \approx \sin (\delta + \alpha_2) ;
\]

namely, \(\delta \approx -(\alpha_1 + \alpha_2)/2 + (n + 1/2)\pi\) (for \(n = 0, \pm 1, \pm 2, \cdots\)) can be achieved.

- If \(\alpha_1 \approx \alpha_2\) holds, the left-hand side of Eq. (7) is either vanishing or vanishingly small, leading to

\[
\sin \delta + \sin (\delta + \alpha_2) \approx 0 ;
\]

namely, \(\delta \approx -\alpha_2/2 + n\pi\) (for \(n = 0, \pm 1, \pm 2, \cdots\)) can be obtained.

These two simple relations have been found by us in Ref. [12], where radiative corrections to the generalized tri-bimaximal neutrino mixing pattern is discussed in detail from the seesaw scale down to the electroweak scale.

A few numerical examples are given in Table III for illustration. One can see that \(\delta = 0\) at \(\mu > \Lambda_{EW}\) is a straightforward consequence of \(\alpha_1 = \alpha_2 = 0\) at \(\mu = \Lambda_{EW}\) (Case I), implying that CP keeps to be a good symmetry in the RGE evolution. This result illustrates that the nontrivial running of \(\delta\) must be driven by the nontrivial inputs of \(\alpha_1\) and \(\alpha_2\), if the limit \(\theta_{13} \to 0\) is taken at \(\Lambda_{EW}\). Note that the outputs of \(\delta, \alpha_1\) and \(\alpha_2\) in Case II of Table III, where the inputs of \(\alpha_1\) and \(\alpha_2\) are different from each other, satisfy the analytical approximation in Eq. (8) very well. On the other hand, the numerical example shown in Case III of Table III (\(\alpha_1 = \alpha_2\) at \(\Lambda_{EW}\)) is consistent with the analytical approximation in Eq. (9).

(4) Near degeneracy with \(\Delta m_{32}^2 < 0\): \(m_3 \lesssim m_1 \lesssim m_2\). For simplicity, we typically take \(m_1 = 0.2\) eV at \(\Lambda_{EW}\) in our analysis. We find that the analytical approximations obtained in Eqs. (7), (8) and (9) for the continuity condition of \(d\delta/dt\) in the limit \(\theta_{13} \to 0\) are also applicable for the case of \(\Delta m_{32}^2 < 0\). Three numerical examples are presented in Table IV to illustrate the RGE evolution of nine neutrino mixing parameters. Comparing between Tables III and IV, one can see that the sign flip of \(\Delta m_{32}^2\) only causes very mild changes of the RGE running behaviors of relevant quantities. When \(\alpha_1 \neq \alpha_2\) holds (Case II of Tables III and IV), flipping the sign of \(\Delta m_{32}^2\) is simply equivalent to shifting the output value of \(\delta\) by \(\pi\). The latter is certainly compatible with the solution to Eq. (8), which has the ambiguities of \(n\pi\) (for \(n = 0, \pm 1, \pm 2, \cdots\)). When \(\alpha_1 \approx \alpha_2\) holds (Case III of Tables III and IV), the quasi-fixed point of \(\delta\) is almost insensitive to the sign flip of \(\Delta m_{32}^2\).

Because three neutrino masses are nearly degenerate, the relevant mixing angles and CP-violating phases may get significant radiative corrections [18]. This well-known feature can be seen either for \(\Delta m_{32}^2 > 0\) or for \(\Delta m_{32}^2 < 0\) and either in the SM or in the MSSM.

We have examined possible quasi-fixed points in the RGE evolution of three leptonic CP-violating phases \((\delta, \alpha_1, \alpha_2)\) by taking the realistic limit \(\theta_{13} \to 0\) in the standard parametrization of three-flavor neutrino mixing. While the Majorana phases \(\alpha_1\) and \(\alpha_2\) do not undergo any divergence in their RGE running from the electroweak scale to the seesaw scale, the phase parameter \(\delta\) may in general have a quasi-fixed point. This interesting point has essentially been observed in Refs. [11] and [12], where the RGE of \(\delta\) is derived in the \(\tau\)-dominance approximation. Our present analysis, which is based on the exact RGE of \(\delta\), has more generally demonstrated that there may only exist a single nontrivial quasi-fixed point
and it exactly obeys the continuity condition obtained before. We have also noticed that the derivative of $\delta$ will not diverge and its running has no quasi-fixed point, if $m_3$ vanishes. This new observation allows us to have a complete insight into the correlation between the neutrino mass spectrum and the RGE running of $\delta$. Taking four typical patterns of the neutrino mass spectrum, we have explicitly illustrated how the evolution of $\delta$ undergoes a quasi-fixed point determined by the nontrivial inputs of $\alpha_1$ and $\alpha_2$ in the limit $\theta_{13} \to 0$.

Let us remark that our physical understanding of the quasi-fixed point(s) in the RGE evolution of leptonic CP-violating phases is not subject to any concrete parametrization. Nevertheless, a good parametrization may simplify our analytical calculations and make the underlying physics more transparent [19]. As recently pointed out in Ref. [20], the one-loop RGEs of neutrino mixing parameters will take very simple and instructive forms in the $\tau$-dominance approximation, if the lepton mixing matrix $V$ is parametrized as 4:

$$V = \begin{pmatrix} c_\ell & s_\ell e^{-i\phi} & 0 \\ -s_\ell e^{i\phi} & c_\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ -s & c & 0 \end{pmatrix} \begin{pmatrix} c_\nu & s_\nu & 0 \\ -s_\nu & c_\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\rho} & 0 & 0 \\ 0 & e^{i\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (10)$$

where $c_\ell \equiv \cos \theta_\ell$, $s_\ell \equiv \sin \theta_\ell$, $c_\nu \equiv \cos \theta_\nu$, $s_\nu \equiv \sin \theta_\nu$, $c \equiv \cos \theta$ and $s \equiv \sin \theta$. In order to keep the RGE of $\phi$ finite in the realistic limit $\theta_\ell \to 0$ (equivalent to $\theta_{13} \to 0$ in the standard parametrization), three CP-violating phases must satisfy the continuity condition

$$\frac{(m_1^2 + m_3^2) \sin \phi + 2m_1m_3 \sin (\phi + 2\rho)}{\Delta m_{31}^2} = \frac{(m_2^2 + m_3^2) \sin \phi + 2m_2m_3 \sin (\phi + 2\sigma)}{\Delta m_{32}^2}. \quad (11)$$

It is straightforward to check that Eq. (11) holds trivially for arbitrary values of $\phi$, $\rho$ and $\sigma$ in the limit $m_3 \to 0$, implying that there must be no nontrivial fixed point in the evolution of $\phi$ from the electroweak scale to the seesaw scale (or vice versa). This continuity condition is therefore equivalent to $m_3 \chi = 0$ (namely, $m_3 = 0$ or $\chi = 0$) which can be extracted from Eq. (2) in the standard parametrization. But similar to $\alpha_1$ and $\alpha_2$, the Majorana phases $\rho$ and $\sigma$ do not undergo any quasi-fixed point in their RGE running.

We conclude that radiative corrections to a specific neutrino mass model have to be taken into account in a very careful way, in particular when there exists the quasi-fixed point in the running of its phase parameters. To test a theoretical model or a phenomenological ansatz, it is crucial to measure the smallest mixing angle $\theta_{13}$ (or $\theta_\ell$) and the CP-violating phase $\delta$

4Note that the phase convention of $V$ in Eq. (10) is slightly different from that taken in Ref. [20]. The present choice allows the phase factor $e^{\pm i\phi}$ to automatically vanish in the limit $\theta_\ell \to 0$, thus it is more convenient for a numerical analysis of the quasi-fixed point in the RGE running of $\phi$. To leading order, we have $\theta_{12} \approx \theta_\nu$, $\theta_{23} \approx \theta$, $\theta_{13} \approx \theta_3 \sin \theta$, $\delta \approx \phi$, $\alpha_1 \approx 2\rho$ and $\alpha_2 \approx 2\sigma$ for the relations of two parametrizations.
(or $\phi$) in the future neutrino oscillation experiments. Any experimental information about the Majorana phases $\alpha_1$ and $\alpha_2$ (or $\rho$ and $\sigma$) is extremely useful in order to distinguish one model from another, e.g., by looking at their different sensitivities to radiative corrections.

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TABLE I. **Normal hierarchy** with $m_1 = 0$: numerical examples for radiative corrections to neutrino masses, lepton flavor mixing angles and CP-violating phases from $\Lambda_{\text{EW}} \sim 10^2$ GeV to $\Lambda_{\text{SS}} \sim 10^{14}$ GeV. The Higgs mass $m_H = 140$ GeV (SM) or $\tan \beta = 10$ (MSSM) has typically been input in our calculation.

| | SM Case I | SM Case II | SM Case III | MSSM Case I | MSSM Case II | MSSM Case III |
|---|---|---|---|---|---|---|
| **$m_1$ (eV)** | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ |
| | 0 | 0 | 0 | 0 | 0 | 0 |
| **$\Delta m^2_{21}$ ($10^{-5}$ eV$^2$)** | 8.0 | 15.63 | 8.0 | 15.63 | 8.0 | 15.63 |
| **$\Delta m^2_{32}$ ($10^{-3}$ eV$^2$)** | 2.5 | 4.89 | 2.5 | 4.89 | 2.5 | 4.89 |
| **$\theta_{12}$ ($^\circ$)** | 34 | 34.00 | 34 | 34.00 | 34 | 34.00 |
| **$\theta_{23}$ ($^\circ$)** | 45 | 45.00 | 45 | 45.00 | 45 | 45.00 |
| **$\theta_{13}$ ($^\circ$)** | 0 | $1.59 \times 10^{-4}$ | 0 | $1.51 \times 10^{-4}$ | 0 | $1.40 \times 10^{-4}$ |
| **$\delta$ ($^\circ$)** | – | 334.37 | – | 307.98 | – | 279.99 |
| **$\alpha_1$ ($^\circ$)** | – | – | – | – | – | – |
| **$\alpha_2$ ($^\circ$)** | 30 | 30.00 | 60 | 60.00 | 90 | 90.00 |
| **$m_1$ (eV)** | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ |
| | 0 | 0 | 0 | 0 | 0 | 0 |
| **$\Delta m^2_{21}$ ($10^{-5}$ eV$^2$)** | 8.0 | 11.59 | 8.0 | 11.59 | 8.0 | 11.59 |
| **$\Delta m^2_{32}$ ($10^{-3}$ eV$^2$)** | 2.5 | 3.63 | 2.5 | 3.63 | 2.5 | 3.63 |
| **$\theta_{12}$ ($^\circ$)** | 34 | 33.98 | 34 | 33.98 | 34 | 33.98 |
| **$\theta_{23}$ ($^\circ$)** | 45 | 44.95 | 45 | 44.95 | 45 | 44.95 |
| **$\theta_{13}$ ($^\circ$)** | 0 | $7.51 \times 10^{-3}$ | 0 | $7.14 \times 10^{-3}$ | 0 | $6.60 \times 10^{-3}$ |
| **$\delta$ ($^\circ$)** | – | 154.37 | – | 127.98 | – | 99.98 |
| **$\alpha_1$ ($^\circ$)** | – | – | – | – | – | – |
| **$\alpha_2$ ($^\circ$)** | 30 | 30.00 | 60 | 60.00 | 90 | 90.00 |
TABLE II. **Inverted hierarchy** with $m_3 = 10^{-5}$ eV: numerical examples for radiative corrections to neutrino masses, lepton flavor mixing angles and CP-violating phases from $\Lambda_{EW} \sim 10^2$ GeV to $\Lambda_{SS} \sim 10^{14}$ GeV. The Higgs mass $m_H = 140$ GeV (SM) or $\tan \beta = 10$ (MSSM) has typically been input in our calculation.

|          | Case I          | Case II         | Case III         |
|----------|-----------------|-----------------|------------------|
| **SM**   |                 |                 |                  |
| $m_3$ (eV) | $10^{-5}$       | $10^{-5}$       | $10^{-5}$        |
| $\Delta m^2_{21}$ (10^{-5} eV^2) | 8.0            | 8.0             | 8.0              |
| $\Delta m^2_{32}$ (10^{-3} eV^2) | -2.5           | -2.5            | -2.5             |
| $\theta_{12}$ (°) | 34             | 34              | 34               |
| $\theta_{23}$ (°) | 45             | 45              | 45               |
| $\theta_{13}$ (°) | 0              | 1.53 x 10^{-7} | 7.90 x 10^{-8}  |
| $\delta$ (°) | -240.85        | -241.77         | -330.01          |
| $\alpha_1$ (°) | 60             | 59.94           | 45               |
| $\alpha_2$ (°) | 0              | -0.03           | 15               |
| **MSSM** |                 |                 |                  |
| $m_3$ (eV) | $10^{-5}$       | $10^{-5}$       | $10^{-5}$        |
| $\Delta m^2_{21}$ (10^{-5} eV^2) | 8.0            | 8.0             | 8.0              |
| $\Delta m^2_{32}$ (10^{-3} eV^2) | -2.5           | -2.5            | -2.5             |
| $\theta_{12}$ (°) | 34             | 34              | 34               |
| $\theta_{23}$ (°) | 45             | 45              | 45               |
| $\theta_{13}$ (°) | 0              | 7.20 x 10^{-6} | 3.73 x 10^{-6}  |
| $\delta$ (°) | -58.71         | -60.48          | -149.94          |
| $\alpha_1$ (°) | 60             | 62.87           | 45               |
| $\alpha_2$ (°) | 0              | 1.23            | 15               |
TABLE III. Near degeneracy with $\Delta m^2_{32} > 0$: numerical examples for radiative corrections to neutrino masses, lepton flavor mixing angles and CP-violating phases from $\Lambda_{\text{EW}} \sim 10^2$ GeV to $\Lambda_{\text{SS}} \sim 10^{14}$ GeV. The Higgs mass $m_H = 140$ GeV (SM) or $\tan \beta = 10$ (MSSM) has typically been input in our calculation.

|       | Case I | Case II | Case III |
|-------|--------|---------|----------|
|       | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ |
| $m_1$ (eV) | 0.2 | 0.28 | 0.2 | 0.28 | 0.2 | 0.28 |
| $\Delta m^2_{21}$ (10^{-5} eV^2) | 8.0 | 15.47 | 8.0 | 15.47 | 8.0 | 15.47 |
| $\Delta m^2_{32}$ (10^{-3} eV^2) | 2.5 | 4.88 | 2.5 | 4.89 | 2.5 | 4.88 |
| $\theta_{12}$ (°) | 34 | 34.77 | 34 | 34.02 | 34 | 34.77 |
| $\theta_{23}$ (°) | 45 | 45.05 | 45 | 45.02 | 45 | 45.03 |
| $\theta_{13}$ (°) | 0 | 8.00 x 10^{-4} | 0 | 2.43 x 10^{-2} | 0 | 6.13 x 10^{-4} |
| $\delta$ (°) | – | 0 | – | 271.06 | – | 319.99 |
| $\alpha_1$ (°) | 0 | 0 | 260 | 259.61 | 80 | 80.00 |
| $\alpha_2$ (°) | 0 | 0 | 100 | 99.82 | 80 | 80.00 |

MSSM

|       | Case I | Case II | Case III |
|-------|--------|---------|----------|
|       | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ | $\Lambda_{\text{EW}}$ | $\Lambda_{\text{SS}}$ |
| $m_1$ (eV) | 0.2 | 0.24 | 0.2 | 0.24 | 0.2 | 0.24 |
| $\Delta m^2_{21}$ (10^{-5} eV^2) | 8.0 | 22.15 | 8.0 | 17.59 | 8.0 | 22.40 |
| $\Delta m^2_{32}$ (10^{-3} eV^2) | 2.5 | 3.66 | 2.5 | 3.68 | 2.5 | 3.66 |
| $\theta_{12}$ (°) | 34 | 14.48 | 34 | 33.35 | 34 | 14.31 |
| $\theta_{23}$ (°) | 45 | 42.48 | 45 | 43.98 | 45 | 43.52 |
| $\theta_{13}$ (°) | 0 | 3.60 x 10^{-2} | 0 | 1.12 | 0 | 2.77 x 10^{-2} |
| $\delta$ (°) | – | 0 | – | 81.41 | – | 139.92 |
| $\alpha_1$ (°) | 0 | 0 | 260 | 273.63 | 80 | 80.13 |
| $\alpha_2$ (°) | 0 | 0 | 100 | 106.03 | 80 | 80.13 |
TABLE IV. Near degeneracy with $\Delta m^2_{32} < 0$: numerical examples for radiative corrections to neutrino masses, lepton flavor mixing angles and CP-violating phases from $\Lambda_{EW} \sim 10^2$ GeV to $\Lambda_{SS} \sim 10^{14}$ GeV. The Higgs mass $m_H = 140$ GeV (SM) or $\tan \beta = 10$ (MSSM) has typically been input in our calculation.

|                  | Case I       | Case II       | Case III      |
|------------------|--------------|---------------|---------------|
| $m_1$ (eV)       | $\Lambda_{EW}$ | $\Lambda_{SS}$ | $\Lambda_{EW}$ | $\Lambda_{SS}$ | $\Lambda_{EW}$ | $\Lambda_{SS}$ |
| $\Delta m^2_{21}$ (10$^{-5}$ eV$^2$) | 8.0          | 15.28         | 8.0           | 15.27         | 8.0           | 15.28         |
| $\Delta m^2_{32}$ (10$^{-3}$ eV$^2$) | $-2.5$       | $-4.89$       | $-2.5$        | $-4.89$       | $-2.5$        | $-4.89$       |
| $\theta_{12}$ (°) | 34           | 34.78         | 34            | 34.02         | 34            | 34.78         |
| $\theta_{23}$ (°) | 45           | 44.95         | 45            | 44.98         | 45            | 44.97         |
| $\theta_{13}$ (°) | 0            | $7.42 \times 10^{-4}$ | 0 | $2.35 \times 10^{-2}$ | 0 | $5.68 \times 10^{-4}$ |
| $\delta$ (°)    | -            | 0             | -             | 89.55         | -             | 319.99        |
| $\alpha_1$ (°)  | 0            | 0             | 260           | 259.60        | 80            | 80.00         |
| $\alpha_2$ (°)  | 0            | 0             | 100           | 99.82         | 80            | 80.00         |

|                  | Case I       | Case II       | Case III      |
|------------------|--------------|---------------|---------------|
| $m_1$ (eV)       | $\Lambda_{EW}$ | $\Lambda_{SS}$ | $\Lambda_{EW}$ | $\Lambda_{SS}$ | $\Lambda_{EW}$ | $\Lambda_{SS}$ |
| $\Delta m^2_{21}$ (10$^{-5}$ eV$^2$) | 8.0          | 23.34         | 8.0           | 17.72         | 8.0           | 23.31         |
| $\Delta m^2_{32}$ (10$^{-3}$ eV$^2$) | $-2.5$       | $-3.61$       | $-2.5$        | $-3.58$       | $-2.5$        | $-3.61$       |
| $\theta_{12}$ (°) | 34           | 13.69         | 34            | 33.45         | 34            | 13.85         |
| $\theta_{23}$ (°) | 45           | 47.42         | 45            | 46.02         | 45            | 46.42         |
| $\theta_{13}$ (°) | 0            | $3.70 \times 10^{-2}$ | 0 | 1.14         | 0            | $2.84 \times 10^{-2}$ |
| $\delta$ (°)    | -            | 0             | -             | 260.86        | -             | 139.93        |
| $\alpha_1$ (°)  | 0            | 0             | 260           | 270.98        | 80            | 80.12         |
| $\alpha_2$ (°)  | 0            | 0             | 100           | 104.88        | 80            | 80.12         |