Dual Aharanov-Bohm Berry Phase and the Electric Vector Potential due to the Generation of Electricity through Permanent Bound and Free Charge Polarization

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To understand the creation of electromagnetic energy (or a photonic degree of freedom) from an external energy source, an electromotive force must be generated, capable of separating positive and negative charges. The separation of charges (free or bound) may be modelled as a permanent polarization, which has a non-zero electric vector curl, created by an external force per unit charge, sometimes referred as an impressed electric field. The resulting system forms a physical dipole in the static case, or a Hertzian dipole in the time dependent case. This system is the electrical dual of the magnetic solenoid described by a magnetic vector potential and excited by an electrical current. Correspondingly, the creation of an electric dipole, from the forceful separation of positive and negative charge, may be described by an electric flux density, which exhibits an electric vector potential and a magnetic current boundary source, within the frame work of two-potential theory without the need for the existence of the magnetic monopole. From this result we derive the Dual electric Aharanov-Bohm (DAB) Berry phase and make the conjecture that it should be equivalent to the geometric phase that is described in modern electric polarization theory, which also describes the nature of the permanent polarization of a ferroelectric. This work gives a formal meaning to the electric vector potential that defines the DAB geometric phase, and determines that a permanent polarization has both a scalar and vector potential component, and we show that it must be considered to fully describe the nature of a physical electric dipole, which inevitably is a generator of electricity. Additionally, we show that Faraday’s and Ampere’s law may be derived from the time rate of change of the Aharanov-Bohm phase and the DAB phase respectively, independent of the electromagnetic gauge.

I. Introduction

The magnetic Aharonov-Bohm (AB) effect is a phenomenon where a charged particle’s wave function is effected by the magnetic vector potential, $\vec{A}$, despite both the electric and magnetic field being zero [1]. Underlying this effect is the general concept of geometric or Berry phase [2] apparent in many areas of physics [3] and not restricted to quantum mechanics, which includes optics [4, 5], condensed matter physics [6, 7], fluid mechanics [8], and so forth. Other related effects includes; 1) The Aharonov-Casher effect [9, 13], which describes the effect of neutral particles with magnetic moments, effected by an isolated static positive or negative electric charge. The isolated electric monopole charge distribution creates an effective charge vector potential experienced by magnetic particles, and has been measured using magnetic flux vorticies [11] or neutrons (with a dipole moment) [10]. Like the AB effect the charge vector vorticies associated with the Aharonov-Casher effect reveals a geometric phase in a charge-vortex interaction [14]; 2) The He-McKellar-Wilkens effect [15, 16], dual to the Aharonov-Casher effect, which looks at the effect of neutral particles with electric dipole moments (EDMs) induced by a magnetic monopole, and; 3) The
Figure 1: Illustration of the electric field generated by an idealised dipole made from two point charges $\pm q_e$ separated by a displacement vector $\vec{L}$. There is an associated external force per unit charge, $\vec{E}^e$, which supplies the energy to separate (and hence polarize) the impressed free charges. The non-conservative nature means an electromotive force (or voltage), is generated, resulting in a voltage output across the dipole.

The dual Aharonov-Bohm (DAB) effect, which associates a Berry phase with an EDM or a permanent polarization (macroscopic collection of EDMs), such as that exhibited by an electret [17–19] or ferroelectret [20] due to an electric vector potential.

In the strict sense of duality, the DAB experiment requires monopoles to measure the DAB effect. However, the DAB geometric phase should be equivalent to the known one discovered in the 1990s, due to the spontaneous permanent polarization of a ferroelectric [20], or the permanent polarization of an electret in general [18, 20–23], and a magnetic monopole was not necessary to prove the existence of this already widely accepted geometric phase.

Classically a permanent polarization consists of equal and opposite charges, $\pm q_e$, displaced by a finite distance, $\vec{L}$, to create a macroscopic EDM, $\vec{d} = q_e \vec{L}$, where the vector direction is defined from $-q_e$ to $+q_e$ as shown in Fig.1 (with net charge = 0). To displace the charges an external impressed force per unit charge is required to separate positive and negative charges in the induction process. This concept is the basis of generating electrical power from an external energy source, which supplies a non-conservative electromotive force [24–26], allowing a voltage to exist across positively and negatively charged terminals. This means an external force in the opposite direction of the Coulomb force is required to keep the charges in static equilibrium, otherwise they will accelerate towards each other. At large distances from the dipole, the electric field appears as an ideal dipole field determined by the EDM, $\vec{d} = q_e \vec{L}$. The ideal dipole exist only in the limit as $\vec{L} \rightarrow 0$ and $q_e \rightarrow \infty$. In contrast, for distances close to the separated charges a real physical dipole has electromagnetic structure.

The ideal oscillating time dependent dipole is commonly known as a Hertzian dipole, and in the quasi static limit, $|\vec{r}| < \frac{\lambda}{6}$ ($\lambda$ is the wavelength of the radiation), the electrostatic near field dominates. Within the dipole, the voltage and current oscillating out of phase as reactive power (no work is done) driven by an electric vector potential [24]. In contrast, external to the dipole, the electric field can be describe by either an electric scalar or vector potential, as the field is capable of doing work on a test charge, but also exist as a reactive near field (or fringing field) due to the unusual boundary condition between the outside and inside of the electric dipole.

In the case that the medium is an insulator, we consider a macroscopic bound charge dipole known as an electret [24]. An electret exhibits a quasi-permanent polarization (or more correctly a metastable state, which can last for years) in the absence of an applied electric field. Such materials are commonly used for energy harvesting and electricity generation [27–32]. Moreover, modern polarization theory introduced in the 1990s [21–23] has shown that the general definition of the polarization was not solely calculable from bulk characteristics of the volume of bound charge, and that a change of polarization only had physical meaning if it was quantified by using a Berry phase. This technique has been very successful in first-principles studies of ferroelectric materials [20, 33].

In this work we use the fact that the per-
manent polarization vector has both a non-zero curl and divergence. Thus, can be defined as a combination of a scalar and vector potential. Importantly, the electric vector potential gives a non-zero tangential surface term, which at the boundary can be viewed as an effective magnetic current \[24\], a entity related to a Berry phase. Furthermore, we find that the time rate of change of the Berry phase leads to the derivation of Ampere’s Law (magnetomotive force), and the rate of change of the AB phase leads to the derivation of Faraday’s law (electromotive force). This is consistent with prior work, which derives motive forces from the Aharonov-Bohm and Aharonov-Casher effects \[34\].

II. Static Macroscopic Electric Dipole

For a dipole, the standard text book example assumes unphysical point charges, shown in fig.1, with a scalar potential given by,

\[ V = \frac{q_i^e}{4\pi\epsilon_0} \left( \frac{1}{r_+} - \frac{1}{r_-} \right) = \frac{q_i^e (r_- - r_+)}{4\pi\epsilon_0 r_+ r_-}, \]  

(1)

where, in spherical coordinates \((r, \phi, \theta)\),

\[ r^2 = r^2 + \frac{L^2}{4} \pm Lr \cos \theta. \]  

(2)

Calculating the potential difference between the two point charges from eqn.1, we derive an infinite voltage! In other words, an infinite impressed force per unit charge \(\vec{E}_i^e\) is required to keep two point charges separated at a finite distance, which highlights the unphysical nature of the point charge dipole.

A better approximation is to assume ideal surface charges, so \(q_i^e = \sigma_i^e \pi a_e^2\) as shown in fig.2, so the force is spread over an area. Such permanent electric dipoles occurs in bound charge (ideal electret) and free charge (battery or electricity generation) systems \[24\]. In this work we analyse the dipoles in terms of potentials, and show that an electric vector potential drives the creation of electricity. We also define the separation of free charge by a polarization vector,

\[ \vec{F}_e = \epsilon_0 \vec{E}_e^i = \sigma_e^i \hat{z}, \]  

(3)

which is related to the impressed force \[24\]. In contrast, for a bound charge system with a permanent dipole is an electret, and assuming a linear electronic susceptibility,

\[ \vec{F}_e = \epsilon \vec{E}_e^i = \sigma_e^i \hat{z}, \]  

(4)

where \(\epsilon_r\) is the dielectric constant, \(\epsilon = \epsilon_0 \epsilon_r\) and \(\sigma_e^i\) represents impressed free or bound charge respectively. In these cases an effective magnetic current surface density exists at the radial boundary of the dipole and acts as a source.
term, which has been shown to be given by [24],
\[ \vec{\kappa}_m = -\frac{\sigma_i}{\epsilon} \hat{\phi}, \] (5)
in the Weber convention. Next we consider the general time dependent case.

### III. Quasi-Static Time Dependent
Macroscopic Hertzian Dipole; Fields and Potentials

Maxwell’s equations for an ideal electricity generator with impressed bound or free charge \( \epsilon = \epsilon_0 \) volume density, \( \rho_i^e \), has been shown to be given by [24] (Weber convention),
\[ \nabla \cdot \vec{E} = \rho_i^e \] and \[ \nabla \cdot \vec{E}_c = -\rho_i^m, \] (6)
\[ \nabla \times \vec{B} - \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 (\vec{J}_e^i + \vec{J}_f); \] \[ \vec{J}_e^i = \epsilon \frac{\partial \vec{E}_c}{\partial t}, \] (7)
\[ \nabla \cdot \vec{B} = 0, \] (8)
\[ \nabla \times \vec{E}_c + \frac{\partial \vec{B}}{\partial t} = 0 \] and \[ \nabla \times \vec{E}_c = -\vec{J}_m. \] (9)
or in terms of the total electric field, \( \vec{E}_T \) by
\[ \nabla \cdot \vec{E}_T = 0, \] (10)
\[ \nabla \times \vec{B} - \epsilon \mu_0 \frac{\partial \vec{E}_T}{\partial t} = \vec{J}_f, \] (11)
\[ \nabla \cdot \vec{B} = 0, \] (12)
\[ \nabla \times \vec{E}_T + \frac{\partial \vec{B}}{\partial t} = -\vec{J}_m, \] (13)
with the following constitutive relations
\[ \vec{E}_T = \vec{E}_c^i + \vec{E} \] (14)
Here, \( \vec{J}_f \) in the lossless case has zero divergence, since \( \rho_f = 0 \), and \( \vec{J}_m \) also has zero divergence since \( \rho_m^i = 0 \). \( \vec{J}_m \) exists on the radial boundary of the dipole, and drives the impressed electric field, \( \vec{E}_c^i \), by the left hand rule and also sets the boundary condition for the parallel components of the fields on the radial boundary. Here the \( \frac{\partial \vec{B}}{\partial t} \) term in eqn. (13) can be identified as a magnetic displacement current and \( \vec{J}_f \) can only exist if an external circuit is coupled to the ideal electricity generator [27–32], or the generator is non-ideal with an effective internal resistance.

The modified form of these equations means in general an electric vector potential, \( \vec{C} \), can be introduced, along with the electric scalar potential, \( V \), and the magnetic vector potential, \( \vec{A} \). The possible existence of an electric vector potential and a magnetic scalar potential has been postulated to exist through the dual of Maxwell’s equations being excited by magnetic monopoles and magnetic currents [35–38] and is known as two-potential theory. Moreover, the electrical engineering community have also shown that the dual of Maxwell’s equation may be excited by non-conservative electromagnetic systems or electricity generators [24–26], without the need for monopoles to exist. Thus, from two-potential theory, and given there is no magnetic scalar field in the system we are describing, we may write the potential of the defined fields in eqns. (6)-(14) as,
\[ \vec{B} = -\frac{1}{\epsilon} \nabla \times \vec{C} + \nabla \times \vec{A} \] (15)
\[ \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \] (16)
\[ \vec{E}_c^i = \frac{\vec{B}_c^i}{\epsilon} = \nabla V - \frac{1}{\epsilon} \nabla \times \vec{C}; \] (17)
\[ \vec{E}_T = \frac{\vec{D}_T}{\epsilon} = -\frac{1}{\epsilon} \nabla \times \vec{C} - \frac{\partial \vec{A}}{\partial t} \] (18)
Note, the field that experiences the “pure” vec-
tor potential is \( \vec{E}_T = \vec{E}_c + \vec{E} \), for both the free and bound system.

Inside the dipole the polarization field, \( \vec{P}_i \), exists without any applied electric field, with both vector and scalar potential components, with the scalar component exactly equal and opposite to the scalar potential of the \( \vec{E} \) field, consistent with eqn. (6). Meanwhile, \( \vec{E}_c \) and \( \vec{E}_T \) have the same vector curl and hence the same component of electric vector potential, while satisfying the constitutive given by eqn. (14).

Outside the both the free and bound charge dipole, \( \vec{E}_c = 0 \), which means from eqn. (17),

\[
\vec{\nabla} V = \frac{1}{\epsilon} \vec{\nabla} \times \vec{C}
\]

This then gives two ways to describe the electric field outside the dipole with either an electric vector or scalar potential. In the quasi static limit the solution is dominated by the electrostatic near field of the dipole, which is reactive with the impressed current and voltage out of phase [24]. Thus, the electric field can be thought as a continuation of the same vector potential within the dipole, with the electric field given by the right hand rule, source from the magnetic current at the boundary, as shown in Fig. 1. On the other hand, the electric field can do work on a test particle, and in this case takes the form of an equivalent scalar potential as given by eqn. (19).

Now by substituting the fields given in eqns. (15) and (18) back into the electric and magnetic Gauss’ law we obtain

\[
\frac{\partial(\vec{\nabla} \cdot \vec{A})}{\partial t} = 0; \quad \frac{\partial(\vec{\nabla} \cdot \vec{C})}{\partial t} = 0,
\]

so the divergence of the vector potentials must be time independent. Then by substituting either (16) or (17) into Gauss’ law, and using (20) we obtain,

\[
\nabla^2 V = \frac{-\rho_e^i}{\epsilon}
\]

Substituting, (15) and (18) into Faraday’s law we obtain,

\[
\vec{\nabla} \times \vec{\nabla} \times \vec{C} + \mu_0 \epsilon \frac{\partial^2 \vec{C}}{\partial t^2} = \epsilon \vec{J}_m \tag{22}
\]

then by substituting, (15) and (18) into Ampere’s law we obtain,

\[
\vec{\nabla} \times \vec{\nabla} \times \vec{A} + \mu_0 \epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}_f \tag{23}
\]

It is well known that there is more than one set of potentials that can generate the same fields, given that \( \vec{\nabla} \times \vec{\nabla} \times \vec{C} = -\nabla^2 \vec{C} + \vec{\nabla}(\vec{\nabla} \cdot \vec{C}) \) to simplify we chose the gauge where the divergence of the vector potentials are zero (Coulomb Gauge), so we obtain,

\[
\Box^2 \vec{C} = -\epsilon \vec{J}_m \tag{24}
\]

and

\[
\Box^2 \vec{A} = -\mu_0 \vec{J}_f \tag{25}
\]

Thus, we have successfully calculated the potentials in terms of the impressed sources, \( \vec{J}_m^i \) and \( \rho_e^i \) as well as any free current in the system, \( \vec{J}_f \). For the lossless system with no load, \( \nabla \cdot \vec{J}_f = 0 \). Note, that the impressed current, \( \vec{J}_e^i = \epsilon \frac{\partial \vec{E}_i}{\partial t} \) in our presentation is not considered a source term, as it is described as a polarization current, which can either be from free or bound charge, impressed by the external force per unit charge, \( \vec{E}_c \).

IV. Berry Phase of a Macroscopic Electric Dipole

Since we have defined a simple macroscopic polarization of bound or free charge with respect to a 3D electric vector potential \( \vec{C} \), we might believe a Berry phase exists similar to modern polarization theory [7, 39]. In this case, the electric vector potential is a 3D Berry connection, and the Berry curvature is the field, \( \vec{D}_T = \epsilon \vec{E}_T = \vec{P}_i + \epsilon \vec{E} \), given by eqn. (18). In fact, the electric dipole is dual to the mag-
nnetic dipole, which was used in the original AB thought experiment, so on this premises a DAB electric effect should exist, and has been considered previously \[17\] 19.

First lets consider semi-classically the well known AB magnetic Berry phase of a long cylindrical electromagnetic solenoid, $\Delta \phi_{AB}$, and with the use of eq. (15) we can show,

$$\phi_{AB} = \frac{q}{\hbar} \oint_{P} \vec{A} \cdot d\vec{l} = \frac{q}{\hbar} \int_{S} \nabla \times \vec{A} \cdot d\vec{S} = \frac{q}{\hbar} \int_{S} \vec{B} \cdot d\vec{S} + \mu_{0} \frac{q}{\hbar} \int_{S} \frac{\partial \vec{C}}{\partial t} \cdot d\vec{S}. \quad (26)$$

Here, the closed path, $P$, of integration of the magnetic vector potential on the LHS of eqn. (26) encloses the surface, $S$, in which the magnetic flux flows, with the first term on the RHS the static contribution to the AB geometric phase, while the second term adds the time dependent term. For the static case if we consider, $P$ as the path at the mid point of the solenoid around the the electric current boundary, the minimum value of enclosed magnetic flux will be given by the flux quantum, $\Phi_{0} = \hbar/(2e)$, so that $\int_{S} \vec{B} \cdot d\vec{S} = n\Phi_{0}$ for a superconducting system with $n$ Cooper pairs ($q = 2e$). In contrast, for a normal conductor with free electrons ($q = e$), $\int_{S} \vec{B} \cdot d\vec{S} = 2n\Phi_{0}$ (measured by Webb et. al. [40]). Thus, in general the static AB phase in both the superconducting and normal conducting case is given by, $\phi_{AB} = 2n\pi$.

Now we consider in analogy the DAB electric phase $\phi_{EAB}$, and with the use of eq. (18) we obtain,

$$\phi_{EAB} = \frac{1}{q} \oint_{P} \vec{C} \cdot d\vec{l} = \frac{1}{q} \int_{S} \nabla \times \vec{C} \cdot d\vec{S} = \frac{1}{q} \int_{S} \vec{D}_{T} \cdot d\vec{S} + e \frac{1}{q} \int_{S} \frac{\partial \vec{A}}{\partial t} \cdot d\vec{S}. \quad (27)$$

Here, the closed path, $P$, of integration of the electric vector potential on the LHS of eqn. (27) encloses the surface, $S$, in which the electric flux flows. Thus, in analogy, the first term on the RHS gives the static DAB geometric phase, while the second gives the general time dependent term. For the static case the geometric phase depends on the enclosed electric flux, $\Phi_{E} = \int_{S} \vec{D}_{T} \cdot d\vec{S}$, which for a path, $P$, at the mid point of the magnetic current boundary, should be equal to the quantum of electric charge, $q = e$, for a single electron system or, $q = 2e$, for a paired electron system. These equations should be valid for both bound-charge and free-charge permanently polarized systems, which generate electricity.

Considering modern polarization theory based on Berry phase, the definition of polarization was developed through the crystal lattice surface and volume charge distributions, which are necessarily related to the electric scalar potential. The theory does not calculate the geometric phase of a permanent polarization through a vector potential related to an effective surface magnetic current, required for non-conservative electrodynamic systems, which generate electrical power, such as an electret [24]. However, because the vector curl of $\vec{P}$, and hence $\vec{D}_{T}$ is non-zero as given by eqn. (17) and (18) an equivalent definition or description should be possible through the electric vector potential, with a magnetic current boundary source.

As discussed by Vanderbuilt [39], modern polarization theory is based on the heuristic replacement of the position vector, $\vec{r} \rightarrow i\hat{k}$, by the $\hat{k}$-derivative operator. Thus, Berry phase is considered in momentum space rather that position space, and the polarization is quantised, so that $\vec{P} \rightarrow \vec{P} + \Delta \vec{P}_{k}$, corresponds to $\phi_{EAB} \rightarrow \phi_{EAB} + 2\pi$ [23, 39]. In contrast, our approach allows us to relate the same quanta of polarization to the DAB electric phase in position space, without the need for considering periodic unit cells (or a supercell) in $\hat{k}$-space. In a similar way, Onoda et.al. [20] have described the topological nature of polarization and charge pumping [41] in ferroelectrics using an analogy to magnetostatics, by introducing a vector field with a Berry phase as a linear response of the covalent part of polarization, which has incorporated a generalization of the Born charge tensor. In principle this type of
description should be equivalent to a polarization with a non-zero curl and an electric vector potential as introduced in this work, so the calculation may be done in position space. A similar strategy has also been previously presented in [42, 43], and suggests the magnetic current boundary source carries the crystal momentum. If we examine the unit cell of a supercell in modern polarization theory, this means the minimum electric flux should be related by

$$
\Delta \Phi_E = \int_S \Delta P^e \cdot d\vec{S},
$$

which is equal to 2e for a spin paired non-magnetic system and e for a spin polarized system [23, 39], so in general the static DAB phase is a Berry phase given by,

$$
\phi_{E_{AB}} = 2n\pi.
$$

V. Motive Force Equations from the Time Dependence of Aharonov-Bohm Phase

Previously an equivalence between the Aharonov-Bohm effect of a solenoid and the Aharonov-Casher effect of a charged rod has been demonstrated, where the time-dependent Aharonov-Casher phase was shown to induce a motive force via the SU(2) spin gauge field [44]. In a similar way to the time dependence Aharonov-Bohm effect derives Faraday’s law, responsible for electromagnetic induction and the electromotive force (emf). Here we show that the time dependence of the DAB phase derives Ampere’s law, the equation responsible for magnetomotive force (mmf).

First we consider the time rate of change of eqn. (27) and combining it with (15) we obtain,

$$
- \mu_0 \oint P \nabla \times \vec{A} \cdot d\vec{l} = \mu_0 \oint \vec{B} \cdot d\vec{l} + \mu_0 \oint P \nabla \times \vec{A} + \mu_0 \oint \frac{\partial^2 \vec{A}}{\partial t^2} \cdot d\vec{S},
$$

which becomes,

$$
F = \frac{1}{\mu_0} \oint \vec{B} \cdot d\vec{l} = \epsilon \frac{\partial}{\partial t} \oint \vec{E}_T \cdot d\vec{S} + \frac{1}{\mu_0} \oint \nabla \times \vec{A} + \mu_0 \oint \frac{\partial^2 \vec{A}}{\partial t^2} \cdot d\vec{S},
$$

which is the integral form of Ampere’s law [24]. Here, \( F \) is defined as the magnetomotive force (mmf), then by rearranging eqn. (32) we obtain,

$$
F_T = I_{enc} = \oint \vec{J}_f \cdot d\vec{S} = F - \frac{\partial \Phi_E}{\partial t}.
$$

Here, \( F_T = I_{enc} = N \times I \), is the enclosed electrical current boundary source of a magnetic dipole or inductor coil with N turns.
Figure 3: Field and potential plots for a cylindrical dipole of either free or bound charge with AR=1. A) 2D vector plot of the normalized electric flux density $\vec{D}$ at $y = 0$, in the $(r - z)$ plane, calculated from eqns. (40) and (41). B) 2D vector plot of the normalized electric field $\vec{E}$ at $y = 0$, in the $(r - z)$ plane, calculated from eqn. (39). C) 2D colour density plot of the normalized electric scalar potential $\vec{V}$ at $y = 0$, in the $(r - z)$ plane, calculated from eqn. (35).

VI. Electronic Properties Macroscopic Cylindrical Dipole

In this section we assume a static (or quasi-static approximation), to analyse the electronic properties of a cylindrical electric dipole of varying aspect ratios ($AR = \frac{2a_e}{L}$), in terms of the electric fields and potentials as described in Section III, where $L$ is the axial length, and $a_e$ the radius of the cylinder. Assuming a macroscopic dipole of the form shown in Fig. 2, we vary the aspect ratio and calculate the electric scalar, $V$, and vector, $\vec{C}$, potentials, as well as the electric field, $\vec{E}$, and electric flux density, $\vec{D}$, ranging from a flat pancake structure ($AR \to \infty$) to a long needle type structure ($AR \to 0$). Vec-

Figure 4: Not to scale field and potential plots for a cylindrical dipole of either free or bound charge. Above AR=10: A) 2D vector plot of the normalized electric flux density $\frac{\vec{D}}{\sigma i_e}$ at $y = 0$, in the $(r - z)$ plane, calculated from eqns. (40) and (41). B) 2D vector plot of the normalized electric field $\frac{\vec{E}}{\sigma i_e}$ at $y = 0$, in the $(r - z)$ plane, calculated from eqn. (39). C) 2D colour density plot of the normalized electric scalar potential $\frac{\chi V}{\sigma i_e}$ at $y = 0$, in the $(r - z)$ plane, calculated from eqn. (35). Below similar plots to above but with AR=0.1: D) $\frac{\vec{D}}{\sigma i_e}$ at $y = 0$, in the $(r - z)$ plane. E) $\frac{\vec{E}}{\sigma i_e}$ at $y = 0$, in the $(r - z)$ plane. F) $\frac{\chi V}{\sigma i_e}$ at $y = 0$, in the $(r - z)$ plane.
Figure 5: Above: Normalized electric vector potential versus normalized radial distance, at \( z = 0 \), from the centre of the electric dipole for various aspect ratios, compared to the infinitely long dipole (\( AR \to 0 \)). Below: Normalized electric scalar potential, versus normalized axial position, at \( r = 0 \), from the centre of the electric dipole for various aspect ratios, compared to the infinitely wide dipole (\( AR \to \infty \)). Here, the length of the dipole is \( L \), where \( AR = \frac{2a_e}{L} \), so the end face of the dipole are at \( z/L = \pm \frac{1}{2} \).

Figure 6: Above: Normalized \( z \) component of the electric field, \( E_z \), versus normalized radial distance, at \( z = 0 \), from the centre of the electric dipole for various aspect ratios. Note for the infinite dipole (\( AR \to 0 \)) the electric field is zero for all \( r \). Below: Normalized \( z \) component of the electric flux density, \( D_{Tz} \), versus normalized axial distance, at \( r = 0 \), from the midpoint of the electric dipole for various aspect ratios. Note, for the infinitely wide dipole (\( AR \to \infty \)) \( D_{Tz} \) is zero for all \( z \). Note the tangential \( E_z \) field across the radial boundary of the dipole, at \( \frac{r}{a_e} = 1 \), is continuous, while the normal \( D_{Tz} \) field is continuous across the axial boundary at, \( \frac{z}{L} = \pm \frac{1}{2} \).

Assuming a constant polarization of \( \vec{P}_i = \sigma_i^z \hat{z} \) within the boundaries of the cylindrical dipole, then there will be a constant impressed surface charge density at each axial end face of \( \pm \sigma_i^z \), with a corresponding impressed surface magnetic current density at the radial boundary \( (r = a_e) \) of value, \( \epsilon \vec{k}_m = -\delta(r - a_e)\sigma_i^z \hat{\phi} \) [24]. The potentials and field can be calculated from the surface charge density and the surface magnetic current density using the following equations:

1) The electric scalar potential,
\[
V(\vec{r}) = \frac{1}{4\pi\epsilon} \int_S \frac{\sigma_i^z(\vec{r'}) dA}{|\vec{r} - \vec{r'}|},
\]
so the normalized value in cylindrical coordi-
nates is given by,
\[
\frac{\epsilon V(\vec{r})}{\sigma_e^i} = \frac{1}{4\pi} \int_0^{a_e} \int_0^{2\pi} \frac{\delta(z' - \frac{L}{2}) - \delta(z' + \frac{L}{2})}{|\vec{r}' - \vec{r}|^2} r' \rho' \rmd \phi' \rmd r'.
\]
(35)

2) The electric vector potential,
\[
\vec{C}(\vec{r}) = \frac{\epsilon}{4\pi} \int_S \vec{E}_m(\vec{r}) \frac{d^2\rho'}{|\vec{r} - \vec{r}'|}.
\]
(36)
so the normalized value in cylindrical coordinates is given by,
\[
\frac{\vec{C}(\vec{r})}{\sigma_e^i} = -\frac{\hat{\phi}}{4\pi} \int_0^{a_e} \int_0^{2\pi} \frac{\delta(r' - a_e)}{|\vec{r} - \vec{r}'|} \rho' \rmd \phi' \rmd z'.
\]
(37)

3) The electric field vector (\(\vec{E} = -\nabla V\) ),
\[
\vec{E}(\vec{r}) = \frac{1}{4\pi \epsilon} \int_S \frac{\sigma_e^i(\vec{r}')}{(|\vec{r} - \vec{r}'|)^2} \vec{z}' \rmd A.
\]
(38)
so the normalized value in cylindrical coordinates is given by,
\[
\frac{\epsilon \vec{E}(\vec{r})}{\sigma_e^i} = \frac{1}{4\pi} \int_0^{a_e} \int_0^{2\pi} \frac{\delta(z' - \frac{L}{2}) - \delta(z' + \frac{L}{2})}{(\vec{r} - \vec{r}')^2} \rho' \rmd \phi' \rmd r'.
\]
(39)

4) The electric flux density may in principle be calculated through integrating over the magnetic current from the relation \(\vec{D} = -\vec{\nabla} \times \vec{C}\) (eqn. 18). However instead, since we have already calculated \(\vec{E}(\vec{r})\), we use the relation, \(\vec{D}_T(\vec{r}) = \epsilon \vec{E}_T(\vec{r}) + \vec{P}_\epsilon\), which leads to the following normalized values,
\[
\frac{\vec{D}(\vec{r})}{\sigma_e^i} = \frac{\epsilon \vec{E}(\vec{r})}{\sigma_e^i} + \hat{z} \text{ inside the dipole}
\]
(40)
\[
\frac{\vec{D}(\vec{r})}{\sigma_e^i} = \frac{\epsilon \vec{E}(\vec{r})}{\sigma_e^i} \text{ outside the dipole}
\]
(41)

Some interesting points come out of these simulations. For small aspect ratios \((AR \rightarrow 0)\) ±\(\sigma_e^i\) charges are separated by large distances with respect to the radius of the charge. In this case the electric field and scalar potential between the charges approaches zero (see Fig. 5 and Fig. 6), in this case the electric vector potential is the main component of emf generation. The opposite occurs for large aspect ratios for pancake like structures. In this case the total electric field, \(\vec{E}_T\) or electric flux density \(\vec{D}_T\) approaches zero. For this case because \(\vec{E}_e^i \approx -\vec{E}\) inside the dipole, the potential difference between the axial end faces due to the electric scalar potential is equivalent to the emf generated across the dipole, and the electric vector potential approaches zero. This finding is consistent with [24], which determined that the magnetic current boundary source best describes the output voltage of an AC or DC generator, rather than the electric field.

VII. Discussion

A macroscopic time independent magnetic dipole can in principle exist without loss as a persistent DC current in a superconducting wire loop or coil not requiring any energy or power. For this situation all parts of Faraday’s law in eqn. (30) are zero, as there is no voltage or emf required. The strength of the magnetic dipole depends on the enclosed electrical current in the loop. For a superconducting coil, a current may be trapped with the use of a persistent switch, and the strength of the magnetic field will depend on the applied mmf, \(\mathcal{F}_T = NI\), as given by Ampere’s law in eqn. (33). Thus, once trapped the mmf exists as stored energy, \(E_m = \frac{1}{2}LI^2\) \((L\) is the inductance of the loop or coil), and no work is required to keep the dipole energised.

The electromagnetic dual of the magnetic dipole is the electric dipole. However, for the electric dipole to exist an emf must be ap-
plied to force the separation of charges, contrary to the magnetic dipole, this charge separation requires work, or an impressed force per unit charge from an external energy source. Thus, it is apparent that a permanent EDM or an ensemble of permanent EDMs is an electricity generator (i.e. an electret). The natural tendency is for the electric dipole to discharge or decay and emit a photon [44], which means the electric dipole is intrinsically metastable and are less common in nature, in the quantum mechanical picture this is related to the fact that EDMs violate both the parity and time-reversal symmetries. We have shown the voltage supplied by the electric dipole is determined by the enclosed effective magnetic current at the tangential boundary given by eqn (30). In this dual system the electric vector potential exists, and has a geometric phase.

The next question is, can we devise an experiment to measure this geometric phase in a similar way to the well-known AB experiment? Any experiment will need a full quantum mechanical description to understand if it would work, and act on the interference fringes of a passing particle such as an electron or a particle with an electric or magnetic dipole moment [17–19]. From fig. 3 we notice the vector potential is maximum just outside the rim of the dipole at the centre, at this same place the electric field is minimum. Passing particles around different directions would be the dual of the original AB experiment. Another way would be to configure an experiment which generates electricity in the regime dominated by the electric vector potential, and confirm the voltage output, this has already been undertaken with energy harvesters, where electricity is generated by a polarization in the absence of an applied electric field [24].

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[1] Y. Aharonov and D. Bohm, “Significance of electromagnetic potentials in the quantum theory,” Phys. Rev. 115, 485–491 (1959).
[2] Michael Victor Berry, “Quantal phase factors accompanying adiabatic changes,” Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 392, 45–57 (1984).
[3] F. Wilczek and A. Shapere, eds., Geometric Phases in Physics (Singapore: World Scientific, 1989).
[4] R.Y. Chiao, Analogies in Optics and Micro Electronics: Berry’s Phases in Optics, edited by van Haeringen W. and Lenstra D. (Springer, Dordrecht., 1990).
[5] S. G. Lipson, “Berry’s phase in optical interferometry: a simple derivation,” Opt. Lett. 15, 154–155 (1990).
[6] Raffaele Resta, “Manifestations of berry’s phase in molecules and condensed matter,” Journal of Physics: Condensed Matter 12, R107–R143 (2000).
[7] Di Xiao, Ming-Che Chang, and Qian Niu, “Berry phase effects on electronic properties,” Rev. Mod. Phys. 82, 1959–2007 (2010).
[8] Manolis Perrot, Pierre Delplace, and Antoine Venaille, “Topological transition in stratified fluids,” Nature Physics 15, 781–784 (2019).
[9] Y. Aharonov and A. Casher, “Topological quantum effects for neutral particles,” Phys. Rev. Lett. 53, 319–321 (1984).
[10] A. Cimmino, G. I. Opat, A. G. Klein, H. Kaiser, S. A. Werner, M. Arif, and R. Clothier, “Observation of the topological aharonov-casher phase shift by neutron interferometry,” Phys. Rev. Lett. 63, 380–383 (1989).
[11] W. J. Elion, J. J. Wachters, L. L. Sohn, and J. E. Mooij, “Observation of the aharonov-casher effect for vortices in josephson-junction arrays,” Phys. Rev. Lett. 71, 2311–2313.
[12] M. König, A. Tschetschetkin, E. M. Hankiewicz, Jairo Sinova, V. Hock, V. Daumer, M. Schäfer, C. R. Becker, H. Buhmann, and L. W. Molenkamp, “Direct observation of the aharonov-casher phase,” Phys. Rev. Lett. 96, 076804 (2006).

[13] Eytan Grosfeld, Babak Seradjeh, and Smitha Vishveshwara, “Proposed aharonov-casher interference measurement of non-abelian vortices in chiral p-wave superconductors,” Phys. Rev. B 83, 104513 (2011).

[14] E. Simánek, “Vortex-charge interaction and aharonov-casher effect in two-dimensional superconductors,” Phys. Rev. B 55, 2772–2775 (1997).

[15] Xiao-Gang He and Bruce H. J. McKellar, “Topological phase due to electric dipole moment and magnetic monopole interaction,” Phys. Rev. A 47, 3424–3425 (1993).

[16] Martin Wilkens, “Quantum phase of a moving dipole,” Phys. Rev. Lett. 72, 5–8 (1994).

[17] Raffaele Resta and D. Vanderbilt, “Physics of Ferroelectrics: A Modern Perspective,” edited by K. Rabe, C. H.Ahn, and J.-M. Triscone (Springer-Verlag, Berlin, 2007).

[18] N. Cabibbo and E. Ferrari, “Quantum electrodynamics with dirac monopoles,” Il Nuovo Cimento 14(1), 1147–1154 (1962).

[19] Daniel Zwanziger, “Local-lagrangian quantum field theory of electric and magnetic charges,” Phys. Rev. D 3, 880–891 (1971).

[20] D. Singleton, “Topological electric charge,” International Journal of Theoretical Physics 34, 2453–2466 (1995).

[21] Constantine A Balanis, Advanced Engineering Electromagnetics (John Wiley., 2012).

[22] Y. Yang, W. Guo, K.C. Pradel, G. Zhu, Y. Zhou, Y. Zhang, Y. Hu, L. Lin, and Z.L. Wang, “Pyroelectric nanogenerators for harvesting thermoelectric energy,” Nano Letters 12, 2833–2838 (2012) cited By 384.

[23] Chikako Sano, Manabu Ataka, Gen Hashiguchi, and Hiroshi Toshiyoshi, “An electret-augmented low-voltage mems electrostatic out-of-plane actuator for acoustic transducer applications,” Micromachines 11 (2020), 10.3390/mi11030267.

[24] R. Resta and D. Vanderbilt, Physics of Ferroelectrics: A Modern Perspective, edited by K. Rabe, C. H.Ahn, and J.-M. Triscone (Springer-Verlag, Berlin, 2007).

[25] R. Resta, “Theory of the electric polarization in crystals,” Ferroelectrics 136, 51–55 (1992).

[26] Raffaele Resta, “Macroscopic polarization in crystalline dielectrics: the geometric phase approach,” Rev. Mod. Phys. 66, 899–915 (1994).

[27] R. D. King-Smith and David Vanderbilt, “Theory of polarization of crystalline solids,” Phys. Rev. B 47, 1651–1654 (1993).

[28] Michael E. Tobar, Ben T. McAllister, and Maxim Goryachev, “The electrodynamics of free and bound charge electricity generators using impressed sources and the modification of maxwell’s equations,” arXiv:1904.05774 [physics.class-ph] (2019).

[29] Roger E. Harrington, Introduction to Electromagnetic Engineering, 2nd ed. (Dover Publications, Inc., 31 East 2nd Street, Mineola, NY 11501, 2012).

[30] Zhong Lin Wang, “On maxwell’s displacement current for energy and sensors: the origin of nanogenerators,” Materials Today 20, 74 – 82 (2017).

[31] Sangchul Oh, Chang-Mo Ryu, and Sung-Ho Suck Salk, “Equivalence between aharonov-bohm and aharonov-casher effects, and motive forces,” Phys. Rev. A 50, 5320–5323 (1994).

[32] N. Cabibbo and E. Ferrari, “Quantum electrodynamics with dirac monopoles,” Il Nuovo Cimento 23, 1147–1154 (1962).

[33] Michael E. Tobar, Ben T. McAllister, and Maxim Goryachev, “The electrodynamics of free and bound charge electricity generators using impressed sources and the modification of maxwell’s equations,” arXiv:1904.05774 [physics.class-ph] (2019).
[39] David Vanderbilt, *Berry Phase in Electronic Structure Theory, Electric Polarization, Orbital Magnetization and Topological Insulators* (Cambridge University Press, 2018).

[40] R. A. Webb, S. Washburn, C. P. Umbach, and R. B. Laibowitz, “Observation of $\frac{e}{2}$ aharonov-bohm oscillations in normal-metal rings,” *Phys. Rev. Lett.* **54**, 2696–2699 (1985).

[41] D. J. Thouless, “Quantization of particle transport,” *Phys. Rev. B* **27**, 6083–6087 (1983).

[42] Zhong Fang, Naoto Nagaosa, Kei S. Takahashi, Atsushi Asamitsu, Roland Mathieu, Takeshi Ogasawara, Hiroyuki Yamada, Masashi Kawasaki, Yoshinori Tokura, and Kiyoyuki Terakura, “The anomalous hall effect and magnetic monopoles in momentum space,” *Science* **302**, 92–95 (2003).

[43] Xue-Yang Song, Yin-Chen He, Ashvin Vishwanath, and Chong Wang, “Electric polarization as a nonquantized topological response and boundary luttinger theorem,” arXiv:1909.08637 (2019).

[44] David J. Griffiths, “Dynamic dipoles,” *American Journal of Physics* **79**, 867–872 (2011) https://doi.org/10.1119/1.3591336