POSITIVE VISCOSITY SOLUTIONS OF A THIRD DEGREE HOMOGENEOUS PARABOLIC INFINITY LAPLACE EQUATION

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Abstract. In this paper, we investigate positive viscosity solutions of a third degree homogeneous parabolic equation $u^2 u_t = \Delta \infty u$. We prove a comparison principle, existence and uniqueness of continuous positive viscosity solutions.

1. Introduction. In this paper, we study a nonlinear degenerate parabolic equation

$$u^2 u_t = \Delta \infty u,$$

where $u(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is the unknown function, $u_t = \frac{\partial u}{\partial t}$,

$$\Delta \infty u := (D^2 u Du) \cdot Du = \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i} u_{x_j}$$

(2)

denotes the 1-homogeneous of the very popular infinity Laplace operator, $Du$ and $D^2 u$ denote the spatial gradient vector and Hessian matrix of $u$ respectively, $u_{x_i} = \frac{\partial u}{\partial x_i}$, $u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. Our goal is to establish the comparison principle, existence and uniqueness of positive viscosity solutions.

The infinity Laplace equation $\Delta \infty u = 0$ is the Euler-Lagrange equation associated with $L^\infty$-variational problem related to absolutely minimizing Lipschitz extensions of functions defined on the boundary of a bounded domain $\Omega \subset \mathbb{R}^n$. See for details [4, 5, 6, 7, 10, 13] and the reference therein. The parabolic equation involving the infinity Laplacian has received a lot of attention in the last decade, notably due to its application to image processing, the main usage being in the reconstructions of damaged digital images [8]-[9]. For numerical purpose it has been necessary to consider also the evolution equation corresponding to the infinity Laplace operator.

Three types of parabolic infinity Laplace equations were described by Lindqvist (page 6 in [16]), that is

$$\frac{\partial u}{\partial t} = \Delta \infty u, \quad |\nabla u|^2 \frac{\partial u}{\partial t} = \Delta \infty u, \quad u^2 \frac{\partial u}{\partial t} = \Delta \infty u.$$
Theorem 1.1. Let problems regarding existence and uniqueness have been attracting great interest. degenerate, but also has no variational structure and divergence form. In particular, third degree homogeneous equation (1) is of intrinsic interest, because it is not only the comparison principle for positive viscosity solutions of equation (1). In order to obtain the existence result with the aid of the uniform continuity estimates. We equality, and also provided numerous explicit solutions. Due to the degeneracy and problems, established interior and boundary Lipshitz estimates and a Harnack inequality, and also provided numerous explicit solutions. Due to the degeneracy and the singularity of the equation, they introduced the approximating equations to obtain the existence result with the aid of the uniform continuity estimates. We direct the reader to these works for a more in-depth discussion.

Here we are interested in the last equation in (3), namely equation (1). The third degree homogeneous equation (1) is of intrinsic interest, because it is not only degenerate, but also has no variational structure and divergence form. In particular, problems regarding existence and uniqueness have been attracting great interest.

We now state our main theorem as follows.

**Theorem 1.1.** Let \( Q_T = U \times (0, T) \), where \( U \subseteq \mathbb{R}^n \) is a bounded domain, and let \( \psi \in C(\mathbb{R}^{n+1}) \) and \( \psi \geq c > 0 \) on \( \partial_p Q_T \) for some positive constant \( c \). Then there exists a unique positive viscosity solution \( u \in C(Q_T \cup \partial_p Q_T) \) of the problem

\[
\begin{align*}
    u^2 u_t &= \Delta_\infty u, & \text{in } Q_T, \\
    u &= \psi, & \text{on } \partial_p Q_T.
\end{align*}
\]

In Theorem 1.1, \( \partial_p Q_T = (\partial U \times [0, T]) \cup (U \times \{0\}) \) denotes the parabolic boundary of \( Q_T \). Here, a viscosity solution \( u \) of the problem (4) is a viscosity solution \( u \) of the equation (1) satisfying the Dirichlet boundary condition \( u = \psi \) on \( \partial_p Q_T \) pointwise. The definition of a viscosity solution of equation (1) will be given in Section 2.

Note that if \( \psi \leq c' < 0 \) on \( \partial_p Q_T \) for some negative constant \( c' \) in Theorem 1.1, then there exists a unique negative viscosity solution \( u \in C(Q_T \cup \partial_p Q_T) \) of the problem (4). We state this result as the first corollary of Theorem 1.1.

**Corollary 1.1.** Under the assumptions of Theorem 1.1 with \( \psi \in C(\mathbb{R}^{n+1}) \) and \( \psi \geq c > 0 \) on \( \partial_p Q_T \) for some positive constant \( c \), replaced by \( \psi \leq c' < 0 \) on \( \partial_p Q_T \) for some negative constant \( c' \), then there exists a unique negative viscosity solution \( u \in C(Q_T \cup \partial_p Q_T) \) of the problem (4).

Corollary 1.1 follows directly by letting \( v = -u \) in Theorem 1.1 and using the third degree homogeneity of equation (1).

The second corollary of Theorem 1.1 is the existence of nonnegative viscosity solutions of the problem (4) when \( \psi \geq 0 \) on \( \partial_p Q_T \).

**Corollary 1.2.** Under the assumptions of Theorem 1.1 with \( \psi \in C(\mathbb{R}^{n+1}) \) and \( \psi \geq c > 0 \) on \( \partial_p Q_T \) for some positive constant \( c \), replaced by \( \psi \in C(\mathbb{R}^{n+1}) \) and \( \psi \geq 0 \) on \( \partial_p Q_T \), then there exists a nonnegative viscosity solution \( u \in C(Q_T \cup \partial_p Q_T) \) of the problem (4). Moreover, if \( \psi \equiv 0 \) on \( \partial_p Q_T \), then \( u \equiv 0 \) is the unique viscosity solution of (4).

In this work, there are two main difficulties in the equation (1). One is the coefficient \( u^2 \) of the term \( u_t \), the other is the degeneracy of the infinity Laplace operator \( \Delta_\infty u \) itself. Since \( u^2 \) is the coefficient of \( u_t \), it is difficult to study the comparison principle for positive viscosity solutions of equation (1). In order to
overcome this difficulty, we take $\rho = \log u$ to transform equation (1) to the equation
\begin{equation}
\rho_t = \Delta_\infty \rho + |D\rho|^4,
\end{equation}
so that Jensen’s method [13] of the comparison principle can be carried out in the usual way. The degeneracy of $\Delta_\infty u$ causes trouble in the proof of the existence for positive viscosity solutions. In order to overcome this difficulty, we introduce the approximating equations of equation (1), that is,
\begin{equation}
u^2 u_t = \varepsilon (1 + |Du|^2) \Delta u + \Delta_\infty u,
\end{equation}
where $\varepsilon \in (0, 1]$ is a constant. Note that such approximations in (6) are different from those in [6, 2, 18, 19]. Then the existence of positive viscosity solutions with positive continuous boundary and initial data is established with the aid of the approximating equations (6) and the uniform estimates. These uniform estimates are derived by using the barrier argument.

We remark that it is also standard to prove the existence of viscosity solutions of the Dirichlet problem (4) by Perron’s method as in [11]. However, it is not clear the viscosity solution obtained by the Perron’s method can satisfy the Dirichlet boundary condition pointwise. This is the reason why we choose the method of elliptic approximations in this paper to prove the existence of viscosity solutions of the Dirichlet problem (4).

Note that it is not clear whether the existence of viscosity solutions to (4) holds for sign changing $\psi$. It would be interesting to consider the problem (4) with sign changing $\psi$ and study its sign changing viscosity solutions.

The organization of this paper is as follows. In Section 2, we give the notations, definitions of positive (negative, or nonnegative) viscosity supersolutions and viscosity subsolutions of equation (1). We also prove the comparison principle of the positive viscosity solutions. In Section 3, we introduce uniform parabolic approximatons of the problem (4) and prove a weak maximum principle. We derive various uniform estimates including a uniform Lipschitz boundary estimate at $t = 0$, a full uniform Lipschitz estimate in time and a uniform Hölder estimate at the lateral boundary. Then the main existence result, Theorem 1.1, is proved by the approximating procedure. Finally, the proofs of Corollaries 1.1, 1.2, and Theorem 3.2 (which is a byproduct for equation (5)) are presented.

2. Viscosity solutions and comparison principle. In this section, we give the definition of positive (negative, or nonnegative) viscosity solutions of equation (1). A transformation of equation (1) is introduced together with its viscosity solutions. Finally, a comparison principle for the positive viscosity solutions of equation (1) is established in Theorem 2.2.

**Definition 2.1.** Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. A positive upper (lower) semicontinuous function $u : \Omega \to \mathbb{R}$ is a positive viscosity subsolution (supersolution) of equation (1) in $\Omega$ if, whenever $(\bar{x}, \bar{t}) \in \Omega$ and test function $\varphi \in C^2(\Omega)$ are such that
\begin{enumerate}
  \item[(i)] $u(\bar{x}, \bar{t}) = \varphi(\bar{x}, \bar{t})$,
  \item[(ii)] $u(x, t) < (>) \varphi(x, t)$ for all $(x, t) \in \Omega$, $(x, t) \neq (\bar{x}, \bar{t})$,
\end{enumerate}
then
\begin{equation}
\varphi^2(\bar{x}, \bar{t}) \varphi_t(\bar{x}, \bar{t}) \leq (\geq) \Delta_\infty \varphi(\bar{x}, \bar{t}).
\end{equation}
A continuous positive function $u : \Omega \to \mathbb{R}$ is called a positive viscosity solution of (1), if $u$ is both a positive viscosity subsolution and a positive viscosity supersolution.
If the viscosity solution (supersolution, or subsolution) \( u \) in Definition 2.1 is negative (nonnegative), we call it negative (nonnegative) viscosity solution (supersolution, or subsolution).

**Remark 2.1.** Let \( Q_T = U \times (0, T) \) and suppose that a function \( u : Q_T \to \mathbb{R} \) can be written as \( u(x, t) = v(x) \) for some upper semicontinuous function \( v : U \to \mathbb{R} \). Then \( u \) is a viscosity subsolution of equation (1) if and only if

\[
- \lim_{z \to x, t \to t_0} \frac{u(z, t_0) - u(x, t)}{z - x} \leq 0 \quad \text{in the viscosity sense.}
\]

Before proving the comparison principle, we introduce a transformation of the original equation (1), namely, if \( u \) solves the differential equation in (1) and \( \rho = \log u \), then a simple calculation yields

\[
\rho_t = \Delta \rho + |D\rho|^4. \tag{8}
\]

In Definition 2.1, discarding “positive” and replacing equations (1) and (7) by equations (8) and

\[
\phi(x, t) \leq (\geq), \Delta \phi(x, t) + |D\phi(x, t)|^4
\]

respectively, we get the definition of viscosity solutions of equation (8). Note that the definitions of viscosity solutions for equations (1) and (8) correspond to the standard definitions of viscosity solutions in [11].

When \( u > 0 \), it is easy to check that \( u \) is a positive viscosity solution of equation (1) in \( Q_T \) if and only if \( \rho \) is a viscosity solution of equation (8) in \( Q_T \).

For a cylinder \( Q_T = U \times (0, T) \), where \( U \subset \mathbb{R}^n \) is a bounded domain, we denote the lateral boundary by

\[
S_T = \partial U \times [0, T]
\]

and the parabolic boundary by

\[
\partial_p Q_T = S_T \cup (U \times \{0\}).
\]

Notice that both \( S_T \) and \( \partial_p Q_T \) are compact sets. Next, we shall prove a comparison principle of positive viscosity solutions to the initial-boundary problem of equation (1) by using the perturbation argument of Jensen [13].

**Theorem 2.2 (Comparison principle).** Suppose \( Q_T = U \times (0, T) \), where \( U \subset \mathbb{R}^n \) is a bounded domain. Let \( u \) and \( v \) be a positive viscosity subsolution and a positive viscosity supersolution of equation (1) in \( Q_T \), respectively, such that

\[
0 < \limsup_{(x, t) \to (z, s)} u(x, t) \leq \liminf_{(x, t) \to (z, s)} v(x, t) < +\infty, \tag{9}
\]

for all \((z, s) \in \partial_p Q_T\). Then

\[
u(x, t) \leq v(x, t) \quad \text{for all } (x, t) \in Q_T. \tag{10}
\]

**Proof.** Since (9) holds for all \((z, s) \in \partial_p Q_T\), then \( u \) and \( v \) have positive bounds in \( \overline{Q_T} \) from below and above. Such boundedness of \( u \) and \( v \) can be derived by the maximum principle, (see the case when \( \varepsilon = 0 \) in Lemma 3.1 in Section 3). Letting \( \rho(x, t) = \log u(x, t) \) and \( \zeta(x, t) = \log v(x, t) \), then

\[
\rho_t \leq \Delta \rho + |D\rho|^4, \quad \zeta_t \geq \Delta \zeta + |D\zeta|^4
\]

hold in the viscosity sense in \( Q_T \), and \( \rho \leq \zeta \) on \( \partial_p Q_T \). To prove (10), it is enough to prove \( \rho \leq \zeta \) for all \((x, t) \in Q_T\). For completeness, we give the detailed proof.

We argue by contradiction. By replacing \( \zeta \) with \( \zeta(x, t) = \zeta(x, t) + \frac{\delta}{T-t} \), where \( \delta \) is a positive constant, we may obtain that \( \zeta \) is a strict supersolution and \( \zeta(x, t) \to \infty \) uniformly in \( x \) as \( t \to T \). Indeed, since \( \zeta(x, t) \) is a viscosity supersolution of equation

\[
\rho_t \leq \Delta \rho + |D\rho|^4, \tag{11}
\]

and
Let \( \delta > 0 \), we have \((\varphi + \frac{\delta}{T-t})_t - \Delta_\infty(\varphi + \frac{\delta}{T-t}) - |D(\varphi + \frac{\delta}{T-t})|^4 \geq \frac{\delta}{|T-t|^2} > 0 \). Suppose that
\[
\sup_{Q_T}(\rho(x,t) - \zeta(x,t)) > 0 \tag{11}
\]
and let
\[
\omega_j(x,t,y,s) = \rho(x,t) - \zeta(y,s) - \frac{j}{4}|x-y|^4 - \frac{j}{2}(t-s)^2.
\]
Denote by \((x_j, t_j, y_j, s_j)\) the maximum point of \(\omega\) relative to \(U \times [0, T] \times U \times [0, T]\). It follows from (11) and the fact that \(\rho < \zeta\) on \(\partial_\rho Q_T\) that for large enough \(x_j, y_j \in U\) and \(t_j, s_j \in [0, T]\), \(\zeta\) achieves its local maximum at \((x_j, t_j, y_j, s_j)\). Therefore, as in Corollary 1.2, besides the homogeneous boundary condition case \(\psi \equiv 0\) on \(\partial_\rho Q_T\), it is not clear that the uniqueness result holds for \(\nabla x\) solutions of the initial and boundary value problem to equation (1) can only apply to positive viscosity solutions. Therefore, as in Corollary 1.2, besides the homogeneous boundary condition case \(\psi \equiv 0\) on \(\partial_\rho Q_T\), it is not clear that the uniqueness result can hold for general nonnegative viscosity solutions.

Remark 2.4. Let \(u : \Omega \to \mathbb{R}\) be a positive viscosity subsolution of equation (1) in \(\Omega\). If \((\hat{x}, \hat{t}) \in \Omega\) and \(\varphi \in C^2(\Omega)\) are such that (i) \(u(\hat{x}, \hat{t}) = \varphi(\hat{x}, \hat{t})\), (ii) \(u(x,t) < \varphi(x,t)\) for all \((x,t) \in \Omega \cap \{t \leq \hat{t}\}\), \((x,t) \neq (\hat{x}, \hat{t})\), we have \(\varphi^2 \varphi_t(\hat{x}, \hat{t}) \leq \Delta_\infty \varphi(\hat{x}, \hat{t})\). Indeed, assuming that there exists \((\hat{x}, \hat{t}) \in \Omega\) and \(\varphi \in C^2(\Omega)\) such that (i) and (ii) hold, and \(\varphi^2 \varphi_t(\hat{x}, \hat{t}) > \Delta_\infty \varphi(\hat{x}, \hat{t})\), we imply that \(\varphi\) is a strict viscosity supersolution of
equation (1) in \( Q_\varepsilon := B_\varepsilon(\hat{x}) \times (\hat{t} - \varepsilon, \hat{t}) \) for some \( \varepsilon > 0 \). Since \( \sup_{\partial_\varepsilon Q_\varepsilon}(\varphi - u) > 0 \), by the comparison principle in Theorem 2.2, we have \( \sup_{Q_\varepsilon}(\varphi - u) > 0 \). This contradicts with the fact that \( u(\hat{x}, \hat{t}) = \varphi(\hat{x}, \hat{t}) \).

3. **Existence theorem.** In this section, we introduce a uniform elliptic regularization for the initial boundary value problem of equation (1). By establishing the uniform estimates for the solutions of the regularized problems, we prove the existence and uniqueness of positive viscosity solutions to equation (1) with the initial boundary data \( \psi \geq c > 0 \). Then the proofs of Corollaries 1.1 and 1.2 are given. At the end, we also prove the existence and uniqueness of viscosity solutions to the initial boundary value problem of the equation (5).

We consider the following approximating problem of (4),

\[
\begin{cases}
\mathcal{L}_\varepsilon u := 0, & \text{in } Q_T, \\
u = \psi, & \text{on } \partial_\varepsilon Q_T.
\end{cases}
\]

where \( \psi \in C^2(\mathbb{R}^{n+1}) \), \( \psi \geq c > 0 \) for some positive constant \( c \), and

\[
\mathcal{L}_\varepsilon u := u^2 u_t - [\varepsilon(1 + |Du|^2)\Delta u + \Delta_\infty u] = u^2 u_t - \sum_{i,j=1}^n a_{ij}(D u) u_{ij}
\]

with

\[a_{ij}(D u) = \varepsilon(1 + |D u|^2)\delta_{ij} + u_i u_j, \quad 0 < \varepsilon \leq 1,\]

where \( a_{ij} \in C^\infty(\mathbb{R}^n) \), \( u_i = u_{x_i} \) and \( u_{ij} = u_{x_i x_j} \). Similar to Definition 2.1, we can also define the viscosity solution (supersolution, or subsolution) of the equation

\[
\mathcal{L}_\varepsilon u = 0.
\]

We first establish the weak maximum principle for viscosity solutions of the equation (15) in the following lemma.

**Lemma 3.1.** Let \( Q_T = U \times (0, T) \), where \( U \subseteq \mathbb{R}^n \) is a bounded domain. If \( u = u_\varepsilon \in \text{usc}(Q_T \cup \partial_\varepsilon Q_T) \) and is a viscosity subsolution of equation (15), then \( \sup_{Q_T} u_\varepsilon \leq \sup_{\partial_\varepsilon Q_T} u_\varepsilon \). If \( u = u_\varepsilon \in \text{lsc}(Q_T \cup \partial_\varepsilon Q_T) \) and is a viscosity supersolution of equation (15), then \( \inf_{Q_T} u_\varepsilon \geq \inf_{\partial_\varepsilon Q_T} u_\varepsilon \).

**Proof.** First, we choose \( \tau \) close to \( T \) with \( \tau < T \). We shall show the weak maximum principle holds in \( Q_\varepsilon \) for any \( \tau < T \). Note that \( \overline{Q}_\varepsilon \) is compactly contained in \( Q_T \cup \partial_\varepsilon Q_T \). Let \( \delta = \sup_{Q_\varepsilon} u_\varepsilon - \sup_{Q_\varepsilon} u_\varepsilon \), we assume that \( \delta > 0 \) and obtain a contradiction.

Since \( \sup_{Q_\varepsilon} u_\varepsilon > \sup_{\partial_\varepsilon Q_\varepsilon} u_\varepsilon \), it follows that either (i) there is a point \( (y, s) \in Q_\varepsilon \) such that \( u_\varepsilon(y, s) = M \), or (ii) \( u_\varepsilon(x, t) < M \), for all \( (x, t) \in Q_\varepsilon \), and there is sequence \( (y_k, s_k) \in Q_\varepsilon \) such that \( s_k \to \tau \) (since \( u_\varepsilon \) is a upper semicontinuous function in \( \overline{Q}_\varepsilon \) and \( \lim_{k \to \infty} u_\varepsilon(y_k, s_k) = M \). In any case, there is a point \( (y, s) \in Q_\varepsilon \) such that \( u_\varepsilon(y, s) > \sup_{\partial_\varepsilon Q_\varepsilon} u_\varepsilon + \frac{3\delta}{4} \) and \( 0 < s < \tau \).

For \( 0 < \epsilon_0 \leq \min(\frac{\delta}{4}, \frac{1}{4}) \), we construct the function

\[
\varphi(x, t) = \sup_{\partial_\varepsilon Q_\varepsilon} u_\varepsilon + B g(t) + \frac{\epsilon_0}{16} \left[ 4 - \left( \frac{\pi_1(x) - \varrho}{3nR} \right)^2 \right], \quad \text{for } (x, t) \in \overline{Q}_\varepsilon,
\]
where \( l > 3 \) is a constant, \( B = \max(\delta, \sup_{\Omega} u_{\varepsilon}(x, \tau) - \sup_{\Omega} u_{\varepsilon}) \), \( g(t) \) is a function satisfying \( g(t) \in C^2 \), \( g(t) \geq 0 \) and \( g'(t) \geq 0 \), (for instance, \( g(t) = 0 \) for \( 0 \leq t < s \), and \( g(t) = \frac{(t-s)^2}{(t-s)^2 + 1} \) for \( s \leq t \leq \tau \)), \( \tau_1(x) = x_1 \in (\rho, \bar{\rho}) \) for all \( x \in U \), \( R \) is constant satisfying \( R > \bar{\rho} - \rho \). By calculations, we have \( \varphi(x, t) \geq \sup_{\partial_{\rho} Q,} u_{\varepsilon} + \frac{\bar{\rho} \varepsilon}{8} \) for all \((x, t) \in \Omega\), \( \varphi(x, t) \geq \sup_{\partial_{\rho} Q,} u_{\varepsilon}(x, \tau) + \frac{\bar{\rho} \varepsilon}{8} \) for all \( x \in \Omega \), and \( u_{\varepsilon}(y, s) - \varphi(y, s) > 0 \). By direct calculations, we have \( \varphi(t, z, \theta) \geq 0 \), \( \Delta_{\infty} \varphi(z, \theta) < 0 \), and \( \Delta \varphi(z, \theta) < 0 \). Therefore, we have
\[
\varepsilon (1 + |D\varphi(z, \theta)|^2) \Delta \varphi(z, \theta) + \Delta_{\infty} \varphi(z, \theta) < 0 \leq \varphi^2(z, \theta) \varphi_{\varepsilon}(z, \theta),
\]for \( 0 < \varepsilon \leq 1 \). We obtain a contradiction and our assertion holds in \( \Omega \) for any \( \tau < T \).

If \( \sup_{Q,} u_{\varepsilon} > \sup_{\partial_{\rho} Q,} u_{\varepsilon} \), then there is a point \((y, s) \in Q_T \) (with \( 0 < s < T \)) such that \( u_{\varepsilon}(y, s) > \sup_{\partial_{\rho} Q,} u_{\varepsilon} \). Selecting \( s < \bar{s} < T \), then \( \sup_{\partial_{\rho} Q,} u_{\varepsilon} < u_{\varepsilon}(y, s) \leq \sup_{\partial_{\rho} Q,} u_{\varepsilon} \leq \sup_{\partial_{\rho} Q,} u_{\varepsilon} \). This is a contradiction and the first assertion of the lemma holds. The second assertion can be proved similarly by the above argument. We omit its detailed proof.

Note that the viscosity sub and super solutions in Lemma 3.1 has no sign restrictions. We remark that Lemma 3.1 can also hold when \( \varepsilon = 0 \), since (16) still holds in this case.

From Lemma 3.1, the viscosity solution \( u = u_{\varepsilon} \) of the problem (13) has lower and upper bounds and satisfies
\[
0 < c \leq u \leq \sup_{\partial_{\rho} Q,} \psi < +\infty.
\]

Then the problem (13) is equivalent to
\[
\begin{align*}
\hat{L}^\varepsilon u := 0, & \quad \text{in } Q_T, \\
\hat{u} = \psi, & \quad \text{on } \partial_{\rho} Q_T,
\end{align*}
\]where
\[
\hat{L}^\varepsilon u := u_t - \sum_{i,j=1}^n \tilde{a}_{ij}(Du)u_{ij} = u_t - \sum_{i,j=1}^n \frac{a_{ij}(Du)u_{ij}}{u^2}.
\]

Moreover, for \( \tilde{a}_{ij} \) in (19), we observe that
\[
0 < \lambda(u, Du)|\xi|^2 \leq \tilde{a}_{ij}(Du)|\xi|^2 \leq \Lambda(u, Du)|\xi|^2
\]for all \( \xi \in \mathbb{R}^n \setminus \{0\} \), where \( \lambda(u, Du) = \frac{\varepsilon (1 + |Du|^2)}{u^2} \) and \( \Lambda(u, Du) = \varepsilon (1 + |Du|^2) + |Du|^2 \).

By calculations, we have
\[
\frac{\Lambda(u, Du)}{\lambda(u, Du)} = \frac{\varepsilon (1 + |Du|^2) + |Du|^2}{\varepsilon (1 + |Du|^2)} = 1 + \frac{|Du|^2}{\varepsilon (1 + |Du|^2)} \leq 1 + \frac{1}{\varepsilon},
\]where \( 0 < \varepsilon \leq 1 \). From (20) and (21), the equation \( \hat{L}^\varepsilon u := 0 \) in problem (18) is uniformly parabolic. Then for any fixed constant \( \varepsilon \in (0, 1] \), the existence of a smooth solution \( u_{\varepsilon} \) of the problem (18) is guaranteed by the classical existence theory in [15]. Since problems (13) and (18) are equivalent, \( u_{\varepsilon} \) is also a smooth solution of the problem (13).
Next, we aim to obtain a solution of the problem (4) as a limit of the functions $u_\varepsilon$ as $\varepsilon \to 0$. This amounts to proving uniform estimates for $u_\varepsilon$ that are independent of $0 < \varepsilon \leq 1$. These uniform estimates can be obtained by using the standard barrier argument.

**Proposition 3.1** (Lipschitz boundary regularity at $t = 0$). Let $Q_T = U \times (0, T)$, where $U \subseteq \mathbb{R}^n$ is a bounded domain, and suppose that $u = u_\varepsilon$ is a positive smooth function satisfying the problem (13). If $\psi \in C^2(\mathbb{R}^{n+1})$ and $\psi \geq c > 0$ for some positive constant $c$, then there exists $C \geq 0$ with $C = C(\|D^2\psi\|_\infty, \|\psi_t\|_\infty, \|D\psi\|_\infty)$, but independent of $0 < \varepsilon \leq 1$ such that

$$|u(x, t) - \psi(x, 0)| \leq Ct$$

for all $(x, t) \in Q_T$. Moreover, if $\psi$ is only continuous in $x$ (and possibly discontinuous in $t$), then the modulus of continuity of $u$ on $U \times \{0\}$ can be estimated in terms of $\|\psi\|_\infty$ and the modulus of continuity of $\psi$ in $x$.

**Proof.** Suppose first that $\psi \in C^2(\mathbb{R}^{n+1})$, and let $\omega(x, t) = \psi(x, 0) + \lambda t$, where $\lambda > 0$ is a constant to be determined. For $(x, t) \in Q_T$, we have

\[
L\omega = \omega_t - \varepsilon (1 + |D\omega|^2) \Delta \omega + \Delta_\infty \omega \\
= (\psi(x, 0) + \lambda t)^2 \lambda - \varepsilon (1 + |D\omega|^2) \Delta \omega + \Delta_\infty \omega \\
\geq \psi^2 \lambda - \varepsilon (1 + |D\psi|^2) \Delta \psi + D^2\psi \cdot D\psi \\
\geq \psi^2 \lambda - (n\varepsilon \|D^2\psi\|_\infty + n\varepsilon |D\psi|^2 \|D^2\psi\|_\infty + |D\psi|^2 \|D^2\psi\|_\infty) \\
\geq \psi^2 \lambda - (n\varepsilon + (1 + n\varepsilon)|D\psi|^2 \|D^2\psi\|_\infty \geq 0, \tag{22}
\]

if $\lambda \geq \frac{n + (1+n)\|D\psi\|_\infty^2}{\varepsilon^2 \|D^2\psi\|_\infty}$. Therefore $\omega$ is a supersolution of the equation (15). Clearly, $\omega(x, 0) = u(x, 0)$ for all $x \in U$. Moreover, by using the mean value theorem,

$$\omega(x, t) = \psi(x, 0) + \lambda t \geq \psi(x, 0) + \|\psi_t\|_\infty t \geq \psi(x, t) = u(x, t)$$

holds for all $x \in \partial U$ and $0 < t < T$, if $\lambda \geq \|\psi_t\|_\infty$. Therefore, by the comparison principle,

$$u(x, t) \leq \omega(x, t) = \psi(x, 0) + \lambda t$$

for all $(x, t) \in Q_T$. By considering also the lower barrier $(x, t) \mapsto \psi(x, 0) - \lambda t$, we have the Lipschitz estimate

$$|u(x, t) - \psi(x, 0)| \leq Ct, \tag{23}$$

where

$$C = \max \left\{ \frac{n + (1+n)\|D\psi\|_\infty^2}{\varepsilon^2 \|D^2\psi\|_\infty, \|\psi_t\|_\infty} \right\}. \tag{24}$$

Next, we suppose that $\psi$ is only continuous, and fix $x_0 \in U$. For a given $\mu > 0$, we choose $0 < \tau < \text{dist}(x_0, \partial U)$ such that $|\psi(x, 0) - \psi(x_0, 0)| < \mu$ whenever $|x - x_0| < \tau$, and consider the smooth functions

$$\psi_{\pm}(x, t) = \psi(x_0, 0) \pm \mu \pm \frac{2\|\psi\|_\infty}{\tau^2} |x - x_0|^2. \tag{25}$$

It is readily checked that $\psi_- \leq \psi \leq \psi_+$ on $\partial_\mu Q_T$. Thus if $u_{\pm}$ are the unique solutions to equation (15) with initial and boundary value $\psi_{\pm}$ of class $C^2(\mathbb{R}^{n+1})$, respectively, by the comparison principle, we have

$$u_- \leq u \leq u_+ \text{ in } Q_T. \tag{26}$$
Applying the estimate (23) for \( u_\pm \) yields
\[
|u_\pm(x_0, t) - \psi_\pm(x_0, 0)| \leq C t,
\]
where \( C \) is the constant in (24) with \( \psi \) replaced by \( \psi_\pm \). Since
\[
(\psi_\pm)_t = 0, \quad \frac{n + (1 + n)}{c^2} \|D\psi_\pm\|_\infty^2 \|D^2\psi_\pm\|_\infty \geq 0,
\]
we have
\[
\max \left\{ \frac{n + (1 + n)}{c^2} \|D\psi_\pm\|_\infty^2 \|D^2\psi_\pm\|_\infty, \|((\psi_\pm)_t)\|_\infty \right\}
\]
\[
= \frac{n + (1 + n)}{c^2} \|D\psi_\pm\|_\infty^2 \|D^2\psi_\pm\|_\infty
\]
\[
\leq \left[ (1 + n) \left( \frac{4\|\psi\|_\infty (x - x_0)}{\tau^2} \right)^2 + n \right] \frac{4\|\psi\|_\infty}{c^2 \tau^2}. \tag{28}
\]
Combining (27) and (28), we have
\[
|u_\pm(x_0, t) - \psi_\pm(x_0, 0)| \leq t \left[ (1 + n) \left( \frac{4\|\psi\|_\infty}{\tau} \right)^2 + n \right] \frac{4\|\psi\|_\infty}{c^2 \tau^2}. \tag{29}
\]
By (25), (26) and (29), we have
\[
|u(x_0, t) - \psi(x_0, 0)| \leq \mu + t \left[ (1 + n) \left( \frac{4\|\psi\|_\infty}{\tau} \right)^2 + n \right] \frac{4\|\psi\|_\infty}{c^2 \tau^2},
\]
and complete the proof of Proposition 3.1. \( \square \)

The full Lipschitz estimate in time now follows easily with the aid of Proposition 3.1 and the comparison principle.

**Corollary 3.1.** Let \( Q_T = U \times (0, T) \) and \( u = u_\varepsilon \) be as in Proposition 3.1. If \( \psi \in C^2(\mathbb{R}^{n+1}) \) and \( \psi \geq c > 0 \) for some positive constant \( c \), then there exists \( C \geq 0 \) with \( C = C(\|D^2\psi\|_\infty, \|\psi\|_\infty, \|D\psi\|_\infty) \), but independent of \( 0 < \varepsilon \leq 1 \) such that
\[
|u(x, t) - u(x, s)| \leq C|t-s| \quad \text{for all} \quad (x, t) \in Q_T \quad \text{and} \quad (x, s) \in Q_T. \tag{30}
\]
Moreover, if \( \psi \) is only continuous, then the modulus of continuity of \( u \) in \( U \times (0, T) \) can be estimated in terms of \( \|\psi\|_\infty \) and the modulus of continuity of \( \psi \) in \( x \) and \( t \).

**Proof.** Let \( v(x, t) = u(x, t + \tau), \tau > 0 \), then both \( u \) and \( v \) are solutions to (15) in \( Q_\tau := U \times (0, T - \tau) \). If \( \psi \in C^2(\mathbb{R}^{n+1}) \) with \( \psi \geq c > 0 \), by the classic comparison principle and Proposition 3.1, we have
\[
\sup_{Q_\tau} |u - v| = \sup_{\partial Q_\tau} |u - v|
\]
\[
\leq \max \left\{ \|u(\cdot, \tau) - \psi(\cdot, 0)\|_{\infty, U}, \sup_{x \in \partial U} \left( \|u(x, \cdot) - u(x, \cdot + \tau)\|_{\infty, (0, T)} \right) \right\}
\]
\[
\leq \max \{ C\tau, \|\psi\|_\infty \tau \} = \tilde{C}\tau,
\]
for some positive constant \( \tilde{C} \), which implies the Lipschitz estimate (30). The proof for the case when \( \psi \) is only continuous is similar. To avoid the repetitions, we omit its proof. \( \square \)
We now discuss the Hölder regularity of \( u = u_\varepsilon \) at the lateral boundary.

**Proposition 3.2.** (Hölder regularity at the lateral boundary) Let \( Q_T = U \times (0, T) \), where \( U \subseteq \mathbb{R}^n \) is a bounded domain, and suppose that \( u = u_\varepsilon \) is a positive smooth function satisfying the problem (3.1). If \( \psi \in C^2(\mathbb{R}^{n+1}) \) and \( \psi \geq c > 0 \) for some positive constant \( c \), then for each \( 0 < \alpha < 1 \), there exists a constant \( C \geq 2\|\psi\|_\infty \geq 1 \) depending on \( \alpha \), \( \|\psi\|_\infty \), \( ||D\psi||_\infty \), \( ||\psi_t||_\infty \) but independent of \( \varepsilon \) such that

\[
|u(x, t) - \psi(x_0, t_0)| \leq C|x - x_0|^\alpha
\]

for all \( (x_0, t_0) \in \partial U \times (0, T), \ x \in U \cap B_\gamma(x_0) \) (where \( \gamma \in (0, 1) \)) and \( \varepsilon > 0 \) sufficiently small (depending on \( \alpha \)). Moreover, if \( \psi \) is only continuous, then the modulus of continuity of \( u \) in \( x \) can be estimated in terms of \( \|\psi\|_\infty \) and the modulus of continuity of \( \psi \) in \( x \).

**Proof.** We first consider the case when \( \psi \in C^2(\mathbb{R}^{n+1}) \). Let

\[
\omega(x, t) = u(x_0, t_0) + C|x - x_0|^\alpha - M(t - t_0)
\]

where \( (x_0, t_0) \in \partial U \times (0, T), \ t_0 > 0 \) and \( 0 < \alpha < 1 \). By a direct calculation, we have

\[
L^\omega \omega = \frac{\partial^2}{\partial t^2} - [\alpha(1 + |D\omega|^2)]D\omega + \Delta_\infty \omega
\]

\[
= -M\omega^2 - [\alpha(1 + C^2\omega^2)|x - x_0|^{2\alpha - 2}]C\alpha(n + \alpha - 2)|x - x_0|^\alpha - 2
\]

\[
+ C^3\alpha^3(\alpha - 1)|x - x_0|^{3\alpha - 4}
\]

\[
= -M\omega^2 + C\alpha|x - x_0|^\alpha - 2C^2\omega^2(1 - \alpha)|x - x_0|^{2\alpha - 2}
\]

\[
- \varepsilon(n + \alpha - 2) - \varepsilon C^2\omega^2(n + \alpha - 2)|x - x_0|^{2\alpha - 2}.
\]

If \( C \geq 2\|\psi\|_\infty \geq 1 \), there exists \( \gamma \in (0, 1) \) such that \( C\gamma^\alpha = 2\|\psi\|_\infty \), we have for \( x \in U \cap B_\gamma(x_0) \),

\[
C^2\alpha^2(1 - \alpha)|x - x_0|^{2\alpha - 2} - \varepsilon(n + \alpha - 2) - \varepsilon C^2\omega^2(n + \alpha - 2)|x - x_0|^{2\alpha - 2}
\]

\[
= C^2\alpha^2|x - x_0|^{2\alpha - 2}[1 - \varepsilon(n + \alpha - 2)] - \varepsilon(n + \alpha - 2)
\]

\[
\geq C^2\alpha^2|x - x_0|^{2\alpha - 2}[1 - \varepsilon(n + \alpha - 2)] - \frac{\alpha^2}{10}(1 - \alpha)
\]

\[
= (\frac{C}{|x - x_0|^{1 - \alpha}})^2\alpha^2(1 - \alpha)(1 - \frac{\alpha^2}{10}) - \frac{\alpha^2}{10}(1 - \alpha)
\]

\[
\geq \alpha^2(1 - \alpha)(1 - \frac{\alpha^2}{10}) - \frac{\alpha^2}{10}(1 - \alpha)
\]

\[
= \frac{9 - \alpha^2}{10}\alpha^2(1 - \alpha),
\]

for \( \varepsilon \) satisfying

\[
\varepsilon \in \left\{ \begin{array}{ll} \left(0, \frac{\alpha^2(1 - \alpha)}{10(n + \alpha - 2)}\right], & \text{if } n > 1, \\ \left(0, +\infty\right), & \text{if } n = 1. \end{array} \right.
\]
Using Cauchy’s inequality and $C \gamma^\alpha = 2\|\psi\|_\infty$, we can estimate $\omega^2$ for $(x, t) \in Q_T \cap (B_r(x_0) \times (t_0 - 1, t_0))$,

$$
\begin{align*}
\omega^2 &= [u(x_0, t_0) + C|x - x_0|^\alpha + M(t_0 - t)]^2 \\
&= [\psi(x_0, t_0) + C|x - x_0|^\alpha + M(t_0 - t)]^2 \\
&= 3\psi^2(x_0, t_0) + 3C^2|x - x_0|^{2\alpha} + 3M^2(t_0 - t)^2 \\
&= 3\left[\psi^2 + C^2|x - x_0|^{2\alpha} + M^2(t_0 - t)^2\right] \\
&\leq 3\left(5\|\psi\|^2_\infty + M^2\right).
\end{align*}
$$

By (31), (32) and (34), we have

$$
\mathcal{L}^\omega \geq -3M(5\|\psi\|^2_\infty + M^2) + \frac{\alpha^3(9 - \alpha^2)(1 - \alpha)}{10} C \geq 0,
$$

for $(x, t) \in Q_T \cap (B_r(x_0) \times (t_0 - 1, t_0))$, provided that $\varepsilon$ is in the range of (33) and

$$
C \geq \max \left\{2\|\psi\|_\infty, \frac{30M(5\|\psi\|^2_\infty + M^2)}{\alpha^3(9 - \alpha^2)(1 - \alpha)}\right\}.
$$

Next we will show that $M$ and $C$ can be chosen so that $\omega \geq u$ on the parabolic boundary of $Q_T \cap (B_r(x_0) \times (t_0 - 1, t_0))$.

**Case 1.** If $t_0 > 1$, we consider a point $(x, t)$ such that $x \in \partial U \cap B_r(x_0)$ and $t_0 - 1 < t \leq t_0$. Since $|x - x_0| < \gamma \leq 1$ and $u = \psi$ on $\partial U \cap B_r(x_0)$, we have

$$
\begin{align*}
|u(x, t)| &\leq |u(x_0, t_0)| + \|D\psi\|_\infty|x - x_0| + \|\psi_t\|_\infty(t_0 - t) \\
&\leq u(x_0, t_0) + C|x - x_0|^\alpha + M(t_0 - t) = \omega(x, t),
\end{align*}
$$

provided $C \geq \|D\psi\|_\infty$ and $M \geq \|\psi_t\|_\infty$, for $x \in \partial U \cap B_r(x_0)$ and $t_0 - 1 < t \leq t_0$. On the other hand, if $x \in U \cap \partial B_r(x_0)$ and $t_0 - 1 < t \leq t_0$, we have

$$
\begin{align*}
\omega(x, t) &= u(x_0, t_0) + C|x - x_0|^\alpha + M(t_0 - t) \\
&= u(x_0, t_0) + C\gamma^\alpha + M(t_0 - t) \\
&\geq 2\|\psi\|_\infty \geq u(x, t),
\end{align*}
$$

where the positivity of $u$ and $C \gamma^\alpha = 2\|\psi\|_\infty$ are used. Finally, we consider the bottom of the cylinder. For $x \in U \cap B_r(x_0)$ and $t = t_0 - 1$, by the positivity of $u$, we have

$$
\omega(x, t) = u(x_0, t_0) + C|x - x_0|^\alpha + M \geq \|\psi\|_\infty \geq u(x, t)
$$

if $M \geq \|\psi\|_\infty$.

In conclusion, we have now shown that if we choose $M \geq \max \{\|\psi_t\|_\infty, \|\psi\|_\infty\}$ and $C \geq \max \{\|D\psi\|_\infty, \|\psi_t\|_\infty, \frac{30M(5\|\psi\|^2_\infty + M^2)}{\alpha^3(9 - \alpha^2)(1 - \alpha)}\}$, then $\omega \geq u$ for $(x, t) \in Q_T \cap (B_r(x_0) \times (t_0 - 1, t_0))$ by the comparison principle. In particular,

$$
\begin{align*}
|u(x, t_0)| &\leq \omega(x, t_0) = \psi(x_0, t_0) + C|x - x_0|^\alpha
\end{align*}
$$

for $x \in U \cap B_r(x_0)$. The other half of the estimate claimed follows by considering the lower barrier $(x, t) \mapsto u(x_0, t_0) - C|x - x_0|^\alpha + M(t - t_0)$.

**Case 2.** When $t_0 < 1$, we consider the cylinder $Q_T \cap (B_r(x_0) \times (0, t_0))$, and notice that since $u = \psi$ on the bottom of this cylinder,

$$
\begin{align*}
u(x, 0) &= \psi(x, 0) \\
&\leq \|D\psi\|_\infty|x - x_0| + \|\psi_t\|_\infty t_0 + u(x_0, t_0) \\
&\leq C|x - x_0|^\alpha + Mt_0 + u(x_0, t_0) = \omega(x, 0)
\end{align*}
$$
for \( x \in U \cap B_{\gamma}(x_0) \) if \( C \geq \|D\psi\|_\infty \) and \( M \geq \|\psi\|_\infty \). The rest of the argument is analogous to the previous case.

For the case when \( \psi \) is only continuous, the argument is similar. We omit its proof.

Hence, we have completed the proof of Proposition 3.2. \( \square \)

With the uniform estimates in Corollary 3.1 and Proposition 3.2, we can now give the proof of the main result, Theorem 1.1.

**Proof of Theorem 1.1.** If \( \psi \in C(\mathbb{R}^{n+1}) \) and \( \psi \geq c > 0 \) for some positive constant \( c \), \( u_c \) is the unique smooth solution to the problem (13). Corollary 3.1, Proposition 3.2 and the comparison principle imply that the family \( \{u_c\} \) is equi-continuous and uniformly bounded. Hence, the Ascoli-Arzela compactness theorem, up to a subsequence, \( u_c \to u \) uniformly, and \( u \) is a unique viscosity solution to equation (1) with the initial and boundary data \( \psi \) by the stability properties of viscosity solutions. The existence for a general continuous data \( \psi \) follows by approximating the data by smooth functions and using Corollary 3.1 and Proposition 3.2. The uniqueness follows from the comparison principle, Theorem 2.2. In addition, we obtain that the viscosity solution \( u \) of (4) is Lipschitz continuous with respect to the time variable \( t \) and Hölder continuous in the space variable \( x \). \( \square \)

Corollaries 1.1 and 1.2 are direct consequences of Theorem 1.1. For completeness, we present these proofs successively.

**Proof of Corollary 1.1.** Letting \( \tilde{\psi} := -\psi \geq -c' > 0 \), by Theorem 1.1, there exists a unique positive viscosity solution \( v \in C(Q_T \cup \partial_p Q_T) \) of the problem

\[
\left\{ \begin{array}{ll}
v^2 v_t = \Delta_\infty v, & \text{in } Q_T, \\
v = \tilde{\psi}, & \text{on } \partial_p Q_T.
\end{array} \right.
\tag{36}
\]

Hence, by the third degree homogeneity of equation in (36), \( u := -v \in C(Q_T \cup \partial_p Q_T) \) is a negative viscosity solution of the problem (4). The uniqueness of negative viscosity solution of the problem (4) follows from Remark 2.3. \( \square \)

**Proof of Corollary 1.2.** Since \( \psi \in C(\mathbb{R}^{n+1}) \) and \( \psi \geq 0 \), let

\[
\psi_n(x, t) = \psi(x, t) + \frac{1}{n} \quad \text{for } n \in N^+,
\]

where \( N^+ \) denotes the set of positive integers. For a given \( n \), we have \( \psi_n(x, t) \geq \frac{1}{n} > 0 \). By Theorem 1.1, there exists \( u_n \in C(Q_T \cup \partial_p Q_T) \) is the unique positive viscosity solution to

\[
\left\{ \begin{array}{ll}
u^2 u_t = \Delta_\infty u & \text{in } Q_T, \\
u(x, t) = \psi_n(x, t) & \text{on } \partial_p Q_T.
\end{array} \right.
\]

Since \( \lim_{n \to \infty} \sup_{(x, t) \in \partial_p Q_T} |\psi_n(x, t) - \psi(x, t)| = 0 \), we have \( \psi_n(x, t) \) converges uniformly to \( \psi(x, t) \). By Lemma 3.1, we have \( \inf_{\partial_p Q_T} \psi_n \leq u_n \leq \sup_{\partial_p Q_T} \psi_n \), \( \inf_{\partial_p Q_T} \psi \leq u \leq \sup_{\partial_p Q_T} \psi \), thus we have \( \inf_{\partial_p Q_T} \psi_n \leq u_n \leq \sup_{\partial_p Q_T} \psi_n \), \( \inf_{\partial_p Q_T} \psi \leq u \leq \sup_{\partial_p Q_T} \psi \), that is

\[
|u_n - u| \leq \sup_{\partial_p Q_T} (\psi_n - \psi) \leq \sup_{\partial_p Q_T} |\psi_n - \psi| = \frac{1}{n}.
\]
Therefore, $u_n(x,t)$ uniformly converges to a nonnegative $u(x,t)$ as $n \to +\infty$, and $u(x,t) \in C(Q_T \cup \partial_p Q_T)$. We know from the stability properties of viscosity solutions that $u$ is a nonnegative viscosity solution of the problem (4).

Moreover, when $\psi \equiv 0$ on $\partial_p Q_T$, by $\varepsilon = 0$ case of Lemma 3.1, we have $u \equiv 0$ on $Q_T \cup \partial_p Q_T$. \hfill \Box

As a byproduct, we give the existence and uniqueness of the viscosity solution to the initial and boundary value problem of the equation (5).

**Theorem 3.2.** Let $Q_T = U \times (0,T)$, where $U \subseteq \mathbb{R}^n$ is a bounded domain. Suppose that $\psi \in C(\mathbb{R}^{n+1})$ and $\psi$ is bounded on $\partial_p Q_T$, then there exists a unique viscosity solution $\rho \in C(Q_T \cup \partial_p Q_T)$ of the problem

\[
\begin{aligned}
\rho_t &= \Delta_\infty \rho + |D\rho|^4, & \text{in } Q_T, \\
\rho &= \psi, & \text{on } \partial_p Q_T.
\end{aligned}
\]  

(37)

**Proof.** Let $\tilde{\psi} := e^\psi$, by Theorem 1.1, there exists a unique positive viscosity solution $u \in C(Q_T \cup \partial_p Q_T)$ of the problem

\[
\begin{aligned}
u u_t &= \Delta_\infty u, & \text{in } Q_T, \\
u &= \tilde{\psi}, & \text{on } \partial_p Q_T.
\end{aligned}
\]  

(38)

Hence, $\rho := \log u \in C(Q_T \cup \partial_p Q_T)$ is a viscosity solution of the problem (4). The uniqueness of viscosity solution of the problem (37) follows form the proof of Theorem 2.2. \hfill \Box

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