Symmetric Shannon capacity is the independence number minus 1

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Abstract
A symmetric variant of the Shannon capacity of graphs is defined and computed.

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Shannon capacity · Independence number · Graph product

Mathematics Subject Classification
05C69 · 05C76

1 Introduction and motivation

The Shannon capacity $\Theta(G)$ (introduced in 1956 by Shannon [5]) is a notoriously hard-to-compute graph parameter defined as the asymptotic growth rate of the independence number of strong direct powers of a graph $G$:

$$\Theta(G) = \lim_{k \to \infty} \left( \frac{\alpha(G \Box_k)}{k} \right)^{1/k}$$

(the limit always exists due to the supermultiplicativity of $\alpha(G \Box_k)$). Even for the 5-cycle the exact value was only determined in 1979 by Lovász [4], and the exact value for the 7-cycle is unknown (asymptotic results exist for large odd cycles, see Bohman [2,3]). As to approximation, Alon and Lubetzky [1] show that the naïve strategy of estimating $\Theta(G)$ by $\alpha(G \Box_k)$ for any fixed $k$ can produce unboundedly high relative error; in fact, there may be arbitrarily high polynomial jumps in the value of $\alpha(G \Box_k)$ at any finite set of $k$’s.

In this paper, we use a modification of Shannon capacity where instead of the strong direct power $G \Box_k$ we consider its quotient $G[k] = G \Box_k / S_k$ by the natural index-permuting action of the symmetric group $S_k$. Since the size of $G[k]$ (and consequently of $\alpha(G[k])$ as well) grows only polynomially, we define

$$F(G) = \lim_{k \to \infty} \frac{\log \alpha(G[k])}{\log k}$$

as a measure of asymptotic growth of $\alpha(G[k])$. Our main result is the following:

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Theorem For any finite graph $G$ we have $F(G) = \alpha(G) - 1$.

Section 2 fixes our notation and Section 4 contains the proof of the Theorem. Section 3 contains a treatment of the particular case of the 5-cycle in a way that does not seem to generalize to all graphs but yields more precise intermediate estimates.

2 Setup

Let $G$ be a simple graph on the vertex set $V = \{v_1, \ldots, v_n\}$. Then the vertex set $V[k]$ of the graph $G[k] = G \Box k / S_k$ consists of partitions of $k$ into $n$ labeled nonnegative integers:

$$V[k] = \{ f : V \rightarrow \mathbb{Z}_{\geq 0} : f(v_1) + \cdots + f(v_n) = k \}.$$

We will consider the elements of $V[k]$ also as equivalence classes of maps from a set of cardinality $k$ to $V$ with two maps being equivalent if they only differ in a permutation of the $k$ elements—configurations of $k$ identical pebbles in the vertices of $G$. Two vertices $f, g \in V[k]$ are connected if one can move some pebbles in the configuration corresponding to $f$ to neighbours in $G$ to obtain the configuration corresponding to $g$ (were the pebbles distinguishable, this construction would give the strong direct product $G \Box k$).

A priori it is not clear whether the limit defining $F(G)$ exists at all: in contrast to the case of distinguishable pebbles, there is no obvious superadditivity property for $\log \alpha(G[k])$ that would ensure convergence. It is however easy to see that

$$\alpha(G[k]) \geq \binom{k + \alpha(G)}{\alpha(G) - 1},$$

—the configurations supported on a fixed independent set $W \subset V$ are clearly independent in $G[k]$—and that

$$\alpha(G[k]) \leq \binom{k + \theta(G)}{\theta(G) - 1},$$

where $\theta(G)$, the clique covering number of $G$, is the minimal number of cliques covering the vertices of $G$—for any clique covering $V = V_1 \cup \cdots \cup V_\theta$ whenever the configurations $f, g \in V[k]$ have the same number of pebbles on the vertices of all $V_j$, they can be moved into one another by using only the edges within the $V_j$. This argument shows that if $\theta(G) = \alpha(G)$, then $\alpha(G[k]) = \binom{k + \alpha(G)}{\alpha(G) - 1}$ and $F(G) = \alpha(G) - 1$ does indeed hold.

The simplest graph for which $\theta(G) > \alpha(G)$ (and thus the estimates above do not determine $\alpha(G[k])$) is the 5-cycle $C_5$; the following section discusses this particular graph.

3 The case of $C_5$

It can be checked that for $k \leq 9$ we have $\alpha(C_5[k]) = k + 1$, but for general $k$ we can only prove the following, weaker result:

Proposition 3.1 For any natural number $k$ we have $\alpha(C_5[k]) \leq \left\lfloor \frac{5(k+2)(k+1)}{2(k+5)} \right\rfloor$.

Proof For simplicity let us index the vertices of $C_5$ by natural numbers from 1 to 5 in such a way that vertices $v_j$ and $v_{j+1}$ are connected by an edge labelled $e_j$ for $j = 1, \ldots, 4$, and the vertices $v_1$ and $v_5$ are also connected by the edge $e_5$. 

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Consider edge configurations of weight $k$ on $C_5$: functions $\psi : E(C_5) \rightarrow \mathbb{Z}_{\geq 0}$ such that $\psi(e_1) + \cdots + \psi(e_5) = k$, also considered as ways to distribute $k$ indistinguishable pebbles among the edges of $C_5$. We declare an edge configuration $\psi$ and a vertex configuration $f$ to be adjacent if one can obtain $\psi$ from $f$ by moving each pebble to an adjacent edge; this defines a (bipartite) graph structure on the set of vertex and edge configurations. Note that if $f$ and $g$ are vertex configurations, then they are neighbours in $C_5[k]$ exactly if there exists an edge configuration $\psi$ adjacent to both of them. As a consequence, if $A \subseteq V[k]$ is an independent set, then the sets of edge configurations adjacent to the elements of $A$ are disjoint.

From this fact we get a bound on $|A|$: if every vertex configuration is adjacent to at least $m$ edge configurations, and their total size cannot exceed $\binom{k+4}{4}$, the total number of edge configurations of weight $k$, then $|A| \leq \frac{(k+4)}{m}$. Unfortunately, there are vertex configurations that are adjacent to few edge configurations: for example, only $k+1$ edge configurations are adjacent to the vertex configuration where all the pebbles are assigned to the same vertex, and this makes the obtained upper bound on $|A|$ too high. To circumvent this problem, we count edge configurations with weight. Firstly, we count only edge configurations that assign zero weight to at least one pair of non-adjacent edges. Let $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ denote the set of such edge configurations, where $S_j$ is the set of those edge configurations that assign zero weight to edges $e_{j-1}$ and $e_{j+1}$ (with indices considered modulo 5). Finally, as the sets $S_j$ are not disjoint, we shall consider each edge configuration weighted by the number of sets $S_j$ that contain it.

Now let $f \in V[k]$ be an arbitrary vertex configuration. Edge configurations in $S_1$ that are adjacent to $f$ have no pebbles on edges $e_2$ and $e_5$ and hence must be formed by moving all pebbles in $v_1$ and $v_2$ to the edge $e_1$, all pebbles in $v_3$ to the edge $e_3$ and all pebbles in $v_5$ to the edge $e_4$. The only freedom left is the distribution of the pebbles in $v_4$ to $e_3$ and $e_4$; the number of such distributions is $f(v_4) + 1$. Repeating this argument for $S_2$ to $S_5$, we count $f(v_1) + 1 + \cdots + f(v_5) + 1 = k + 5$ edge configurations adjacent to $f$, each as many times as there are sets $S_j$ that contain it. The total weight of the edge configurations in $S$ is $|S_1| + \cdots + |S_5| = 5^{(k+2)}$, hence there cannot be more than $\left\lfloor \frac{5^{(k+2)}}{k+5} \right\rfloor = \left\lfloor \frac{5(k+2)(k+1)}{2(k+5)} \right\rfloor$ pairwise non-neighbouring vertex configurations, as claimed.

As an immediate corollary, we get

**Corollary 3.2** $F(C_5) = 1$.

**Open question 3.1** Does $\alpha(C_5[k]) = k + 1$ hold for all $k$?
4 General case

The main Theorem is the immediate consequence of the trivial lower bound and the following estimate:

Proposition 4.1 For any finite graph \( G \) we have \( \alpha(G[k]) = O(k^{\alpha(G) - 1}) \).

Proof We prove our statement by induction on the cardinality of \( V \). For \( |V| = 1 \) the claim is trivial. Assume now that the statement holds for all graphs with less vertices than \( |V| \), and let \( H \subset V[k] \) be an arbitrary independent vertex set in \( G[k] \). Partition \( H \) into pieces \( H_1, \ldots, H_n \), such that in \( H_j \) the vertex \( v_j \) carries the greatest weight; in particular, this weight is at least \( \frac{k}{n} \).

Denote by \( N(v_j) \subset V \) the union of the vertex \( v_j \) and its neighbours in \( G \). Each \( H_j \) we further divide into chunks \( H_{j;m,b_1,...,b_n} \), where the chunk \( H_{j;m,b_1,...,b_n} \) contains those elements of \( H_j \) in which the total weight of \( v_j \) and its neighbours is \( m \) and for all \( i \leq n \) the weight of the vertex \( v_i \) falls into the interval \( \left[ b_i \frac{k}{n}, (b_i + 1) \frac{k}{n} \right) \). The index \( j \) can take \( n \) different values, \( m \) can take \( k + 1 \) different values, and each \( b_i \) can be assumed to be between 0 and \( n^2 \) (the rest are always empty), so altogether we get at most \( n(k + 1)(n^2 + 1)^n = O(k) \) chunks (with an implied dependence on \( n \)). Pick now an arbitrary chunk \( H_{j;m,b_1,...,b_n} \) and consider its elements as configurations on \( G \setminus N(v_j) \) (with the weights on the elements of \( N(v_j) \) omitted). These configurations all have weight \( k - m \) and we claim that they form an independent set in \( (G \setminus N(v_j))[k - m] \). Indeed, if two configurations \( f \) and \( g \) in \( H_{j;m,b_1,...,b_n} \) either coincide or are neighbours in \( (G \setminus N(v_j))[k - m] \), then there exists a pebble transport from \( f \) to \( g \) on \( G \setminus N(v_j) \). On the other hand, there exists a pebble transport from \( f \) to \( g \) on \( N(v_j) \) as well: for each vertex \( v_i \) that is a neighbour of \( v_j \) we send \( f(v_i) - g(v_i) \) pebbles from \( v_i \) to \( v_j \) if \( f(v_i) > g(v_i) \), and we send \( g(v_i) - f(v_i) \) pebbles from \( v_j \) to \( v_i \) if \( g(v_i) > f(v_i) \). Combining these two pebble transports yields a pebble transport from \( f \) to \( g \).

This means that the chunk \( H_{j;m,b_1,...,b_n} \) has size at most \( O(k^{\alpha(G \setminus N(v_j)) - 1}) \) by the induction hypothesis. But any independent set in \( G \setminus N(v_j) \) can be extended by \( v_j \) to form an independent set in \( G \), hence \( \alpha(G \setminus N(v_j)) \leq \alpha(G) - 1 \). Summing this estimate for all the \( O(k) \) chunks we get that \( |H| = O(k^{\alpha(G) - 1}) \), proving the induction step. \( \square \)

Open question 4.1 Does \( \alpha(G[k]) = \left( \frac{k^{\alpha(G) - 1}}{\alpha(G) - 1} \right) \) or at least

\[
\lim_{k \to \infty} \frac{\alpha(G[k])}{k^{\alpha(G) - 1}} = \frac{1}{\alpha(G) - 1}!
\]

hold?

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