2015

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Recommended Citation
Belcastro, Sarah-Marie and Haas, Ruth, "Triangle-free Uniquely 3-Edge Colorable Cubic Graphs" (2015). Faculty Publications. Paper 5.
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TRIANGLE-FREE UNIQUELY 3-EDGE COLORABLE CUBIC GRAPHS

SARAH-MARIE BELCASTRO AND RUTH HAAS

Abstract. This paper presents infinitely many new examples of triangle-free uniquely 3-edge colorable cubic graphs. The only such graph previously known was given by Tutte in 1976.

1. History

Recall that a cubic graph is 3-regular, that a proper 3-edge coloring assigns colors to edges such that no two incident edges receive the same color, that edge-Kempe chains are maximal sequences of edges that alternate between two colors, and that a Hamilton cycle includes all vertices of a graph.

It is well known that a cubic graph with a Hamilton cycle is 3-edge colorable, as the Hamilton cycle is even (and thus 2-edge colorable) and its complement is a matching (that can be monochromatically colored). A uniquely 3-edge colorable cubic graph must have exactly three Hamilton cycles, each an edge-Kempe chain in one of the \( \binom{3}{2} \) pairs of colors. The converse is not true, as a cubic graph may have some colorings with Hamilton edge-Kempe chains and other colorings with non-Hamilton edge-Kempe chains; examples are given in [12].

The literature classifying uniquely 3-edge colorable cubic graphs is sparse; there is no complete characterization [7]. It is well known that the property of being uniquely 3-edge colorable is invariant under application of \( \Delta - Y \) transformations. It was conjectured that every simple planar cubic graph with exactly three Hamilton cycles contains a triangle [13, Cantoni], and also that every simple planar uniquely 3-edge colorable cubic graph contains a triangle [4]. The latter conjecture is proved in [4], where it is also shown that if a simple planar cubic graph has exactly three Hamilton cycles, then it contains a triangle if and only if it is uniquely 3-edge colorable.

Tutte, in a 1976 paper about the average number of Hamilton cycles in a graph [13], offhandedly remarks that one example of a nonplanar triangle-free uniquely 3-edge colorable cubic graph is the generalized Petersen graph \( P(9,2) \), pictured in Figure 1. He describes the graph as two 9-cycles \( a_0 \ldots a_8, b_0 \ldots b_8 \), with additional edges \( a_i b_{2i} \) and index arithmetic done modulo 9.

2010 Mathematics Subject Classification. 05C10, 05C15.

Key words and phrases. edge coloring, unique coloring, triangle-free, cubic graphs.

Ruth Haas’s work partially supported by NSF grant DMS-1143716 and Simons Foundation Award Number 281291.
Figure 1. The generalized Petersen graph \( P(9, 2) \) labeled with Tutte’s indices.

(The generalized Petersen graph \( P(m, 2) \) is defined analogously, and in fact the known cubic graphs with exactly three Hamilton cycles and multiple distinct 3-edge colorings are \( P(6k + 3, 2) \) for \( k > 1 \) [12].) It appears that the search for examples of triangle-free nonplanar uniquely 3-edge colorable cubic graphs ended with Tutte, or at least that any further efforts have been unsuccessful. Multiple sources ([6], [7], [9]) note that Tutte’s example is the only known triangle-free nonplanar example. It has been conjectured [3] that \( P(9, 2) \) is the only example. In Section 2 we give infinitely many such graphs.

2. New examples of triangle-free nonplanar uniquely 3-edge colorable cubic graphs

In [2] the authors introduced the following construction. Consider two cubic graphs \( G_1, G_2 \), and form \( G_1 \triangleright G_2 \) by choosing a vertex \( v_i \) in \( G_i \) (\( i = 1, 2 \)), removing \( v_i \) from \( G_i \) (\( i = 1, 2 \)), and adding a matching of three edges joining the three neighbors of \( v_1 \) with the three neighbors of \( v_2 \). Of course there are many ways to choose \( v_1, v_2 \), and many ways to identify their incident edges, so the construction is not unique. However, it is reversible; given a cubic graph \( G \) with a 3-edge cut, we may decompose \( G = G_1 \triangleright G_2 \). In that paper we proved the following.

**Theorem 2.1.** [3.8 of 2] Let \( G_1, G_2 \) be cubic graphs and \( a_i \) the number of 3-edge colorings of \( G_i \). Then \( G_1 \triangleright G_2 \) has \( a_1 a_2 \) edge colorings.

Define \( G^\triangleright \) to be the infinite family of graphs consisting of all graphs of the form \( G \triangleright G \triangleright \cdots \triangleright G \). This leads to the following corollaries of Theorem 2.1.

**Theorem 2.2.** If \( G \) is a uniquely 3-edge colorable graph, then all graphs in \( G^\triangleright \) are uniquely 3-edge colorable.
Proof. The proof proceeds by induction on the number of copies of $G$. 

Corollary 2.3. All members of the infinite family $P(9, 2)^\uparrow$
are uniquely 3-edge colorable.

Note. In [5], Goldwasser and Zhang proved that if a uniquely 3-edge colorable graph has an edge cut of size 3 or 4 such that each remaining component contains a cycle, then the graph can be decomposed into two smaller uniquely 3-edge colorable graphs. It seems they did not observe the reverse construction.

2.1. Examples and Properties. The smallest member of $P(9, 2)^\uparrow$ is of course $P(9, 2)$, which has 18 vertices. For every integer $k > 1$ there are multiple graphs in $P(9, 2)^\uparrow$ with $16k + 2$ vertices. Nonisomorphic examples with $k = 2$ are shown in Figures 2 and 3.

The graphs in $P(9, 2)^\uparrow$ are clearly all nonplanar. We show next that there are graphs in $P(9, 2)^\uparrow$ of every other orientable and nonorientable genus.
Theorem 2.4. Every graph in $P(9,2)^\gamma$ with $16k + 2$ vertices has orientable and nonorientable genus at most $k$. Further, there is a large subfamily of graphs in $P(9,2)^\gamma$, each of which has $16k + 2$ vertices and orientable and nonorientable genus exactly $k$.

Proof. We will show by induction that any graph $Q_k$ created using the $\gamma$-construction with $k$ copies of $P(9,2)$ has orientable and nonorientable genus at most $k$. The base case holds because $P(9,2)$ embeds on both the torus (see Figure 4 (left)) and on the projective plane (see Figure 4 (right)).

Now consider $Q_k$, a graph created using the $\gamma$-construction with $k$ copies of $P(9,2)$. $Q_k$ was obtained by removing and associating some $v \in Q_{k-1}$ via the $\gamma$-construction, where $Q_{k-1}$ is some graph created using $k-1$ copies of $P(9,2)$ that has genus $k-1$ or less by inductive hypothesis. Let $\hat{Q}_k$ be the graph produced by simply identifying the vertices $v$ and $w$. The graph $\hat{Q}_k$ has two blocks that meet at this vertex and by Theorem 1 of [1], the genus of $\hat{Q}_k$ is the sum of the genera of the blocks (so it is $k$). Replacing the cut vertex by a 3-edge cut to implement the $\gamma$ construction does not increase the genus, which completes the proof of the upper bound on genus.

A copy of a subdivision of $K_{3,3}$ is highlighted in the embedding of $P(9,2)$ shown in Figure 5. There are four vertices $\{t_1,t_2,t_3,t_4\}$ whose edges are not involved in the subdivided $K_{3,3}$. Any (or all) of $\{t_1,t_2,t_3,t_4\}$ can be removed and the resulting graph will still have a $K_{3,3}$ minor. If $Q_k$ is formed such that in each copy of $P(9,2)$ only (some) of vertices $\{t_1,t_2,t_3,t_4\}$ are used in the $\gamma$ construction then there will still be $k$ disjoint copies of subdivisions of $K_{3,3}$ in $Q_k$. The genus of a graph is the sum of the genera of its components [1 Cor. 2], so using this construction $Q_k$ has a minor with orientable (resp.
nonorientable) genus exactly $k$. It is straightforward to draw an embedding of sample $Q_k$ on a surface of orientable or nonorientable genus $k$. \qed

3. Conclusion

While we have provided infinitely many examples of triangle-free nonplanar uniquely 3-edge colorable cubic graphs, it is still unknown whether other examples exist. All our examples support Zhang’s conjecture \cite{goldwasser1996} that every triangle-free uniquely 3-edge colorable cubic graph contains a Petersen graph minor. That conjecture remains open.

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