Hamiltonian approach to the problem of modeling the hybrid system’s dynamics in air flow

D V Tymoshenko1*, A A Ilyukhin2, A G Klovo1

1Southern Federal University 44, Nekrasovsky lane, Taganrog, 347928, Russia
2Taganrog Institute named after A.P. Chekhov, 48, Initiative str., Taganrog, 347900, Russia

E-mail: dmitrytim@sfedu.ru

Abstract. The dynamics problem of a hybrid mechanical system modeling the behavior of a wing in an aerodynamic flow is investigated. The research method is based on a kinetic analogy between the elastic rod pendulum oscillation and plane deformation problems, which makes it possible to bring the deformation equations system to a Hamiltonian form.

1. Introduction

The problem of experimental determination of the relationship between the final displacements of the points of the elastic rod and its aerodynamic parameters is considered.

It is assumed that the rod is located in the frontal air flow. An absolutely solid plate is rigidly fixed at the upper end. The lower end of the rod is fixed (Figure 1). The flow direction is oriented appropriately oriented to concentrate the effects on the plate’s surface. It is also assumed that bending rod deformations occur in the same plane. The initial position conditions assume the coincidence of the flow velocity with the rod inclination angle. The study of such systems is very relevant from the point of view of interpreting the data obtained during the field experiments on wind-tunnel tests, since the mechanical processes that arise in such experiments are often associated with the elastic elements’ finite nonlinear deformations [1, 2].

Since the plate has finite dimensions, the direction of the acting force and flow rates may vary. Accordingly, it is possible to consider the following representation for the resulting force of flow action

\[ \vec{R} = \vec{S} + \vec{P}, \]

where the resistance force to flow from the plate side and the lifting force of the air flow are indicated by \( \vec{S} \) and \( \vec{P} \) respectively, in which connection \( \vec{S} \parallel \vec{V}, \ \vec{P} \perp \vec{V} \).

The aerodynamic forces’ dependence on the flow velocity is determined by the following experimentally obtained relations:

\[ \vec{S} = s(\alpha)V\vec{V}\rho/2, \ \vec{P} = p(\alpha)V(\vec{i} \times \vec{V})\rho/2, \]

where the air flow incidence and the air flow density are indicated as \( \alpha \) and \( \rho \) respectively, the
The purpose of further research will be a model description of the plate-rod system wind-tunnel test process in terms of identifying the aerodynamic forces’ dependence on the plate incidence on the flow side.

![Figure 1. Rod-plate system](image)

Under the aerodynamic flow action, the moment is additionally applied to the rod-plate system

\[ M = \frac{\rho}{2} d(\alpha) V^2 (p \cos(\alpha) + s \sin(\alpha)), \]

where \( d(\alpha) \) – is the distance between the rod’s attachment points with the plate and the flow pressure application.

Let us consider a rod with a rectilinear initial state, assuming that during the experiment it undergoes a flat bend. The cross-sectional area of the rod is considered as constant.

Then, as a result of the corresponding expressions’ substitution for the projections, we obtain the equilibrium equation of the system in question in the Kirchhoff form [3]:

\[
B \frac{d^2 \theta}{dl^2} + \frac{\rho}{2} s V^2 \sin(\theta) + \frac{\rho}{2} p V^2 \cos(\theta) = 0. \tag{1}
\]

By lowering the resulting equation order, we can find the function \( l(\theta) \) as an integral

\[
B \left( \frac{d\theta}{dl} \right)^2 - \rho s V^2 \cos(\theta) + \rho p V^2 \sin(\theta) - C = 0, \tag{2}
\]

\[
l = \sqrt{B} \int_{\psi}^{\theta} \left( \rho s V^2 \cos(\theta) - \rho p V^2 \sin(\theta) + C \right)^{-1/2} d\theta. \tag{3}
\]

the integration constant value in relation (3) is determined using the initial conditions.

2. Analysis of the mathematical model

It can be assumed that the inflection point of the cane segment coincides with the fixing point \( O \).

This situation can be considered as a separating case for various forms of the rod’s convex state.

The latter means the appearance of an additional boundary condition at the fixation point \( l = 0 \) and the additional boundary condition \( d\theta/dl = 0 \) appears.

According to the equation (2) we obtain:
\[ C = \rho p V^2 \sin(\psi) - \rho s V^2 \cos(\psi). \]  

(4)

Transformation of integral (3) with the help of an additional change of variables \( z = \tan \theta / 2 \) gives the expression:

\[ l = 2B \int_0^1 \left[ \left( 1 + z^2 \right) \left( \rho s V^2 (1 - z^2) - \rho p V^2 2z + C(1 + z^2) \right) \right]^{-1/2} dz. \]  

(5)

A detailed analysis of the parameters included in relation (5) was carried out in [4].

Using the standard trigonometric transformations, we can obtain the following integral:

\[ l = 2B \int_0^1 \sqrt{1 + \tan^2 \theta / 2} \left[ \rho \frac{p^2(\alpha_T)}{s^2(\alpha_T)} \left( \frac{3}{4} - \frac{1}{2 \tan \frac{\theta}{2} - \tan \frac{\psi}{2}} - \frac{1}{4} \tan^2 \frac{\psi}{2} \right) \right]^{-1/2} d\theta, \]  

(6)

allowing to find the dependence \( l(\theta) : \)

\[ l = W \left[ \ln \left( \frac{\cos \left( \frac{\theta - \psi}{2} \right) - 1}{\cos \left( \frac{\theta - \psi}{2} \right) + 1} \right) \right]_0^\theta = W \ln \left( \tan \frac{\theta - \psi}{2} \right)_{\psi = \infty}, \]

where \( W = 2 \times \text{sign} \sin \left( \frac{\theta - \psi}{2} \right) \sqrt{\frac{B s^2(\alpha_T)}{\rho p^2(\alpha_T)}} \left( \frac{3}{4} - \frac{1}{2 \tan \frac{\theta}{2} - \tan \frac{\psi}{2}} - \frac{1}{4} \tan^2 \frac{\psi}{2} \right)^{-1/2} \cos \left( \frac{\psi}{2} \right). \)

Since this integral is divergent at \( \theta = \psi \), it can be concluded that the inflection point at the rod attachment point is impossible.

The established property is essential for the rod’s elastic (axial) line equilibrium possible forms analysis in the deformation process under the flow action.

The idea of further transforming the mathematical model of the hybrid system under consideration is to use the Kirchhoff kinetic analogy between the vibrational movements of the pendulum and the deformation process in the elastic rod, which is known to theoretical mechanics, which manifests itself in the structure coincidence of differential equations describing these processes. To implement this idea, it is necessary to take into account the non-conservative nature of the aerodynamic forces, which is necessary for the full implementation of the Kirchhoff kinetic analogy.

Equation (1) can be transformed to a Hamiltonian system of two equations by choosing the angle as the generalized coordinate \( \theta \) and the momentum \( p_\theta \) as the conjugate coordinate. As a result, we get the equivalent system:

\[ \frac{d\theta}{dt} = \frac{dH}{dp_\theta} = \frac{1}{B} p_\theta, \]

\[ \frac{dp_\theta}{dt} = \frac{dH}{d\theta} = -\frac{1}{2} \rho s V^2 \sin(\theta) - \frac{1}{2} \rho p V^2 \cos(\theta), \]

(7)

for which the Hamilton function can be represented as:
\[ H = \frac{1}{2B} p_\theta^2 - \frac{1}{2} \rho sV^2 \cos(\theta) + \frac{1}{2} \rho pV^2 \sin(\theta). \]  

(8)

We rewrite the system (7) in canonical variables \( \zeta = \theta + \delta, p_\zeta = p_\theta \):

\[
\begin{align*}
\frac{d\zeta}{dl} &= \frac{p_\zeta}{B}, \\
\frac{dp_\zeta}{dl} &= -\frac{1}{2} R \sin(\zeta),
\end{align*}
\]

(9)

where \( R = \rho V^2 \sqrt{p^2 + s^2} \).

The system (9) in the new variables corresponds to the Hamilton function:

\[ H = \frac{p_\zeta^2}{2B} - \frac{1}{2} R \cos(\zeta). \]

3. Integration of the equilibrium equation

The resulting system of differential equations has a solution in the form \( \zeta = 0, p_\zeta = 0 \). We pose the problem of finding a nonzero solution to the resulting system. This problem can be solved using the normalization method proposed by Birkhoff in [5], and developed for the rod systems in [3]. The essence of this method is to represent the function \( H \) in the form of a power series with respect to canonical variables.

We will consider this representation up to the fourth order. The normalization of the Hamilton function is conveniently carried out in relatively complex conjugate variables [3] \( p = p_\zeta + i\zeta, q = p_\zeta - i\zeta \), with respect to which the Hamilton function is written in the form:

\[
\overline{H} = -i \left[ pq - \frac{1}{96} \left( p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4 \right) \right].
\]

(10)

where \( \overline{H} = -2iH \).

Next, the obtained Hamiltonian function in new variables using the Birkhoff transform should be brought to normal form. As a generating function for the indicated transformation, we choose the following:

\[
p = u + \frac{\partial S_4(u, v)}{\partial v}, q = v - \frac{\partial S_4(u, v)}{\partial u}; S_4 = S_{04}v^4 + S_{13}uv^3 + S_{22}u^2v^2 + S_{31}u^3v + S_{40}u^4.
\]

We achieve the form normalization of the function \( \overline{H} \) relatively new variables \( u \) and \( v \) by the appropriate selection of function coefficients \( S_4 \):

\[
H_+ = -i \left[ uv - \frac{1}{16} u^2v^2 \right].
\]

(11)

The transformations performed make it possible to integrate the Hamiltonian system (9) with the corresponding function (11), using the presence of the first integral in this system \( r = uv = \text{const} \).

The corresponding solution has the form:

\[
v = (a + ib)e^{iml}, v = (a - ib)e^{-iml}, \text{ where } m = i \frac{\partial H_+}{\partial (uv)} = \text{const}.
\]
The main variables as the arc coordinate functions are obtained in the form:
\[
p'_{\zeta} = (a - \frac{1}{16} (a^2 + b^2)a)\cos(ml) + \frac{1}{16} (a^2 + b^2)b)\sin(ml) + \frac{1}{32} a(a^2 - 3b^2)\cos(3ml) + \frac{1}{32} b(b^2 - 3a^2)\sin(3ml),
\]
\[
\zeta' = (-b - \frac{1}{16} (a^2 + b^2)b)\cos(ml) + \frac{1}{16} (a^2 + b^2)a)\sin(ml) + \frac{1}{96} b(b^2 - 3a^2)\cos(3ml) + \frac{1}{96} a(a^2 - 3b^2)\sin(3ml).
\]

The arbitrary constants included in the resulting solution are found using the boundary conditions of the original problem, which are relative to the variables \( p'_{\zeta} \) and \( \zeta' \) take the form:
\[
\zeta' = \frac{\sqrt{2}}{2} (\psi + \delta), \quad p'_{\zeta} = \frac{\sqrt{2}}{2} \int \frac{d}{L} Y \sin(\alpha_0 - \psi + \sqrt{2}\zeta'), \text{ where } Y = L\sqrt{\frac{R}{2B}}.
\]  

Substitution of the found integral relations for the functions \( p'_{\zeta} \) and \( \zeta' \) in the boundary conditions (12), allows us to solve the problem of determining the considered rod-plate system’s aerodynamic parameters dependence on the flow velocity and the incidence and to obtain these dependencies in an analytical form.

4. Summary

As a result of the study, the possibilities of applying the Hamiltonian approach to the problem of modeling the behavior of a mechanical system under the action of forces that initially did not have a conservative character are shown.

The considered dynamic system’s feature was possible to be compensated by a corresponding transformation of dynamic variables using the Birkhoff method, which, in turn, made it possible to obtain an analytical solution to the problem in the form of the aerodynamic parameters’ dependences on the arc coordinate and the air flow parameters.

References
[1] Lokshin B Ya, Privalov V A, Samsonov V A 1986 An introduction to the problem of the motion of a body in a resisting medium (Moscow State University Publishing House, Moscow).
[2] Lyons Zh L 1972 Optimal control of systems described by partial differential equations (Mir, Moscow).
[3] Ilyukhin A A 1979 Spatial problems of the nonlinear theory of elastic rods Naukova Dumka, Kiev).
[4] Ilyukhin A A, Stupko S A 2000 An approximate solution to the problem of equilibrium of a plate on an elastic rod in an air stream Solid Mechanics 30 242–246.
[5] Birkhoff G D 1999 Dynamical systems (Publishing house Regular and chaotic dynamics, Moscow).