Selective Credit Assignment

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Efficient credit assignment is essential for reinforcement learning algorithms in both prediction and control settings. We describe a unified view on temporal-difference algorithms for selective credit assignment. These selective algorithms apply weightings to quantify the contribution of learning updates. We present insights into applying weightings to value-based learning and planning algorithms, and describe their role in mediating the backward credit distribution in prediction and control. Within this space, we identify some existing online learning algorithms that can assign credit selectively as special cases, as well as add new algorithms that assign credit backward in time counterfactually, allowing credit to be assigned off-trajectory and off-policy.

1. Introduction

In reinforcement learning (RL) (Sutton and Barto, 2018) an agent must assign credit or blame for the rewards it obtains to past states and actions. This problem is difficult because rewards may be sparse and may occur much later than the events that helped cause them. Moreover, the agent’s observations are typically noisy or aliased, further complicating its reasoning about the root causes of observed rewards. Effective credit assignment across long stretches of time in complex environments remains a largely unsolved and actively pursued research problem (Arjona-Medina et al., 2019; Chelu et al., 2020; Harutyunyan et al., 2019b; Hung et al., 2019; Ke et al., 2018; Mesnard et al., 2020).

In this paper, we describe a generic way of adding selectivity in online credit assignment, leading to a unified view of the space of algorithms available. We present insights into the effect of non-uniformly weighting the learning updates of value-based algorithms to improve credit assignment.

As an example, consider Fig. 1, which contains results of agents playing the Atari game of Ms. Pac-Man. The baseline performance (brown) is due to an agent that updates its action values for each transition using a form of Q-learning (Watkins and Dayan, 1992) with expected eligibility traces (QET) (algorithm details are in later sections). We then consider a modified version of the game, where some of the observations are very noisy—the idea is that this is similar to a hardware camera on a robot that occasionally adds substantial noise, for instance due to a faulty cable. The same algorithm performs far worse on this new version of the game (yellow line at the bottom). However, we recover the baseline performance if we weight the updates (and appropriately modify the algorithm), despite the impoverished input signal (turquoise line).

\textbf{Figure 1 | Atari Ms.Pac-Man:} The curves show mean returns on Ms. Pac-Man for Q-learning with expected eligibility traces (QET), in different settings and with different weightings. The \textit{baseline} QET algorithm updates its values using a uniform weight on each transition. In the \textit{noisy} runs, Gaussian noise is added to the observations with probability $\varepsilon = 0.5$. The \textit{weighted} algorithm down-weights transitions with noisy inputs, and adapts the trace parameter $\lambda$ to be aware of these non-uniform weightings. Shaded areas denote standard error over 5 seeds. These experiments demonstrate that uniform weightings fail in the noisy setting, and performance can be recovered with more careful weighting.
The Q-learning algorithms illustrated in Fig. 1 use a form of temporal difference (TD) learning (Sutton, 1988a) to learn predictions online from sampled experience by bootstrapping on other predictions (Sutton and Barto, 2018). In such algorithms, credit is assigned online via an eligibility trace (Peng and Williams, 1996; Sutton, 1988a; Sutton and Barto, 2018; van Hasselt and Sutton, 2015), which keeps track of how the parameters of the value function estimator should be adjusted when a prediction error occurs to appropriately correct predictions made on earlier time steps. Canonical TD algorithms use recency and frequency to determine appropriate credit assignment (Sutton, 1984); frequent states are updated more, and preceding states that are temporally closer to a prediction error receive more blame for this error.

Our first contribution is an analysis of stability of arbitrary weightings for on-policy algorithms, showing how to ensure stable convergent learning. Previous work has mainly focused on weightings for the off-policy case, or overlooked stability, perhaps because it is not broadly acknowledged that even on-policy algorithms can diverge with a non-uniform weighting. We discuss a simple weighting that ensures convergence and discuss how this on-policy weighting is connected to the idea of emphasis (Sutton et al., 2016) which was proposed to stabilize off-policy TD.

Second, we provide examples of weightings that greatly enhance learning, e.g., in the face of noisy observations, and show that the analysis mentioned above informs concrete algorithms with improved performance.

In RL, we have learning algorithms which use experiential data, and planning algorithms which are driven by a model. We consider the online planning algorithm proposed by van Hasselt et al. (2021), using on-policy expected eligibility traces (ET) in place of the instantaneous traces of standard TD, to propagate credit backward not just to the states that occur on the current trajectory but also to other possible trajectories leading to the current state. This allows counterfactual credit assignment to states that could have happened, but did not occur recently on the current trajectory. Our third contribution, is to provide insights on using selectivity with planning algorithms based on expected traces to improve stability, data efficiency and knowledge reuse. Specifically, we show how selectivity can be used to learn off-policy expected traces, online from a single stream of experience and, separately, how it can be used for sparse credit assignment which of interest for hierarchical learning. We provide concrete examples of weightings for expected traces that enhance planning, e.g., in the face of noisy observations. As an additional contribution, we provide a more computationally efficient version of the value-based control planning algorithm QET (van Hasselt et al., 2021) (the planning counterpart of the learning algorithm $Q(\lambda)$), saving a factor of $|\mathcal{A}|$ — the number of actions, true for all implementations, which can be significant in practice.

2. Background and preliminaries

We denote random variables with uppercase (e.g., $S$) and the obtained values with lowercase letters (e.g., $s$). Multi-dimensional functions or vectors are bolded (e.g., $b$), as are matrices (e.g., $A$). For state-dependent functions, we also allow time-dependent shorthands (e.g., $y_{t} = y(S_t)$).

2.1. Reinforcement learning problem setup

We consider the usual RL setting of an agent interacting with an environment, modelled as an infinite horizon Markov Decision Process (MDP) $(S, A, P, r)$, with a finite state space $S$, a finite action space $A$, a state-transition distribution $P : S \times A \rightarrow \mathcal{P}(S)$—with $\mathcal{P}(S)$ the set of probability distributions on $S$ and $P(s'|s, a)$ the probability of transitioning to state $s'$ from $s$ by choosing action $a$, and a reward function $r : S \times A \rightarrow \mathbb{R}$. A policy $\pi : S \rightarrow \mathcal{P}(A)$ maps states to distributions over actions; $\pi(a|s)$ is the probability of choosing action $a$ in state $s$ and $\pi(s)$ is the probability distribution of actions in state $s$. Let $S_t, A_t, R_t$ denote the random variables of state, action and reward at time $t$, respectively.

The goal of value function $V_{\pi}$, defined as the expectation of
we update the parameters \( \theta \) of a function \( f \) evaluated at \( \mathbf{w} = \mathbf{w}_t \).

For off-policy policy evaluation, the goal is to estimate \( V_{\pi} \) whilst interacting with the MDP by sampling actions according to a different behaviour policy \( \mu \). In control, the learner’s goal is to find a policy \( \pi \) that maximizes the value \( V \). Value-based methods for control (e.g., Q-learning; Watkins and Dayan, 1992) use state-action value functions to learn implicit (e.g., greedy) policies.

2.2. Online credit assignment
We start with credit assignment algorithms used for learning value functions of a given policy—the policy evaluation setting—after which we look at methods that adapt to maximize performance—the control setting.

2.2.1. Learning and planning for evaluation

\( \text{TD}(\lambda) \) A popular and effective algorithm to learn \( V_{\lambda} \approx V_{\pi} \) online and on-trajectory, is \( \text{TD}(\lambda) \) Sutton (1988a):

\[
\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha^w \Delta^V_t, \quad \text{with} \quad \Delta^V_t = \mathbf{e}_t \delta_t, \quad \text{and} \quad \mathbf{e}_t = y \gamma \Delta^V_t \mathbf{e}_{t-1} + \nabla V_{\mathbf{w}}(S_t), \quad (2)
\]

where \( \mathbf{w} \in \mathbb{R}^d \) are parameters of \( V_{\mathbf{w}} \) to be updated, \( \Delta^V_t = \mathbf{e}_t \delta_t \) is an update with \( \text{TD error} \) \( \delta_t = R_{t+1} + y \gamma V_{\mathbf{w}}(S_{t+1}) - V_{\mathbf{w}}(S_t) \) and accumulating eligibility trace \( \mathbf{e}_t \), and \( \alpha^w \in (0,1) \) is a (possibly time-varying) step-size parameter. For instance, \( \mathbf{w} \) could be the weights of a neural network, or of a linear function of a feature mapping \( \mathbf{x}(s) \), s.t. \( V_{\mathbf{w}}(s) = \mathbf{w}^T \mathbf{x}(s) \). The trace-decay parameter \( \lambda_t \in [0,1] \) interpolates between one-step TD learning and Monte-Carlo methods. Several variations exist (e.g., Maei, 2011; Sutton et al., 2014; van Hasselt and Sutton, 2015; van Seijen and Sutton, 2014). For clarity, we focus on the canonical variant above.

\( \text{ETD}(\lambda) \) The ETD(\( \lambda \)) algorithm Sutton et al. (2016) was introduced for correcting TD algorithms when learning is off-policy, i.e. when the learning distribution differs from the sampling distribution due to a discrepancy between the behaviour and the target policy. This solution involves weighting the trace using a history-dependent function:

\[
\mathbf{e}_t = y \gamma \lambda_t \mathbf{e}_{t-1} + \rho_t M_t \nabla V_{\mathbf{w}}(S_t), \quad \text{with} \quad M_t = \lambda_t i_t + (1 - \lambda_t) F_t, \quad \text{and} \quad F_t = y \gamma \rho_t F_{t-1} + i_t,
\]

where \( M \) is the emphasis, \( F \) is the follow-on trace, \( i \) is a non-negative arbitrary interest function, originally introduced to focus learning Mahmood et al. (2015); Sutton et al. (2016), and \( \rho_t = \frac{\pi(A_t|S_t)}{\mu(A_t|S_t)} \) is an importance sampling ratio between the target policy \( \pi \) and the behaviour policy \( \mu \). ETD algorithms optimize the excursion objective, defined as the value error under the stationary distribution of the behaviour policy \( \rho \). Zhang et al. (2020) consider learning the expectation of the follow-on trace \( \mathbb{E} f(s) = \mathbb{E}_\pi[f(S|S_t = s)] \), and using it directly in place of the history-dependent weighting. Similarly, Jiang et al. (2021) use an expectation of the \( n \)-step follow-on trace.

\( \text{ET}(\lambda) \) Expected eligibility trace (ET) algorithms (van Hasselt et al., 2021) have been introduced for off-trajectory, on-policy value learning, replacing the instantaneous trace \( \mathbf{e}_t \) in Eq. (2) with an estimated expectation:

\[
\mathbf{z}_{\theta}(s) = \mathbb{E}_\pi[\mathbf{e}_t | S_t = s]. \quad (5)
\]

Expected traces can be thought of as a true expectation model, and approximations thereof can be learned by regressing on the instantaneous eligibility trace \( \mathbf{e}_t \):

\[
\mathbf{\Theta}_{t+1} = \mathbf{\Theta}_t + \alpha^\theta \Delta^z_t, \quad \text{with} \quad \Delta^z_t = \frac{\partial \mathbb{E}_{\mathbf{\theta}}(S_t)}{\partial \mathbf{\theta}} (\mathbf{e}_t - \mathbf{z}_{\theta_0}(S_t)), \quad (6)
\]

with step-size parameter \( \alpha^\theta \). Expected traces can also be learned by backward or time-reversed TD, leading to multi-step updates similar to the TD(\( \lambda \)) version of TD (see Appendix A.1 for details).
2.2.2. Learning and planning for control

For value-based control, we consider multi-step value-based analogs of TD($\lambda$) and ET($\lambda$). The goal is to learn action-values $Q_w(s, a)$, rather than state-values $V_w(s)$ we used thus far in our exposition, so that we can then derive a greedy policy $\arg\max_a Q_w(s, a)$ with respect to those values.

Q($\lambda$) The Q($\lambda$) algorithm (Peng and Williams, 1996) is the analog of TD($\lambda$) for control. With $w$ now representing the parameters of the action-value function $Q_w$, the corresponding action-value update replaces $\Delta_t^V$ in Eq. (2) with

$$\Delta_t^Q \equiv \left( G_t^\lambda - Q_w(S_t, A_t) \right) \nabla Q_w(S_t, A_t), \text{ with}$$

$$G_t^\lambda = R_{t+1} + \gamma t+1(1 - \lambda t+1) \max_a Q_w(S_{t+1}, a)$$

$$+ \gamma t+1 \lambda t+1 G_{t+1}^\lambda.$$

Standard Q-learning corresponds to $\lambda_t = 0, \forall t$. An alternative derivation of the Q($\lambda$) algorithm (see Appendix B.1), yields:

$$\Delta_t^Q = e_{t-1} R_t^\lambda - Q_w(S_t, A_t) \nabla Q_w(S_t, A_t), \text{ with} \tag{7}$$

$$R_t^\lambda = R_{t+1} + \gamma t+1(1 - \lambda t+1) \max_a Q_w(S_{t+1}, a), \text{ and}$$

$$e_t = \gamma t \lambda_t e_{t-1} + \nabla Q_w(S_t, A_t),$$

where $R_t^\lambda$ is a fixed-horizon one-step target, composed of the reward at the next timestep and the bootstrapped value using a modified discount $\gamma t+1(1 - \lambda t+1)$, $e_t$ is an accumulating eligibility trace, and $\alpha_t^w$ is a step-size parameter.

QET($\lambda$) The QET($\lambda$) algorithm (van Hasselt et al., 2021) is analogous to Q($\lambda$), but using learned models of expected traces in place of the standard instantaneous traces.

3. Selective credit assignment

All the aforementioned algorithms apply some sort of selection mechanism over which samples are used for learning. This choice can be implicit—e.g., in TD($\lambda$) it is determined by the behaviour distribution, or explicit—e.g., in ETD($\lambda$).

We take a unifying view over all the previous algorithms by separating the learning algorithm from this selection mechanism, now captured explicitly in a function $\omega : \mathbb{S} \rightarrow [0, \infty)$. We will mostly consider selectivity as a function of state; other extensions are possible, such as allowing it to be a function of history $\omega : \mathcal{H} \rightarrow [0, \infty)$, with $\mathcal{H}$ the space of histories, or a function over some feature representation.

TD($\lambda, \omega$) We call selective TD($\lambda, \omega$) the generic algorithm that uses a weighting function for the value update, replacing the standard trace with a weighted or selective eligibility trace (henceforth using the $\sim$ superscript to denote explicit selectivity):

$$\bar{e}_t = \gamma t \lambda_t \bar{e}_{t-1} + \omega(S_t) \nabla V_w(S_t). \tag{8}$$

Generally, the weighting function controls how much credit is received by a state. For instance, if $\omega(s) = 0$, contributions in the trace from $s$ are dropped. Hence, predictions at $s$ are then not corrected, and are learned solely via generalization, if at all. The backward view (8) can be equivalently expressed in a forward view as

$$\bar{\Delta}_t^V = \omega_t (G_t^\lambda - V_w(S_t)) \nabla V_w, \text{ with} \tag{9}$$

$$G_t^\lambda = R_{t+1} + \gamma t+1(1 - \lambda t+1) V_w(S_{t+1}) \max_a Q_w(S_{t+1}, a) + \gamma t+1 \lambda t+1 G_{t+1}^\lambda.$$

Fig. 2 (Left) illustrates how state weightings can impact value learning.

It is well known that in the off-policy setting, discrepancies between the behaviour policy used to sample experience and the target policy whose value is being learned can destabilize learning, even with posterior corrections in the form of importance sampling weights Tsitsiklis and Van Roy (1997). Moreover, even when off-policy TD does converge, its solution may be arbitrarily far from the optimal representable value Kolter (2011).

3.1. On-policy TD can diverge

Semi-gradient TD algorithms are known to converge in the on-policy setting, as long as experience is sampled uniformly (Sutton and Barto, 2018; Tsitsiklis and Van Roy, 1997). However, divergence can still happen on-policy when not carefully considering how experience is weighted. Naively using an arbitrary weighting
omega in TD(lambda, omega) can lead to divergence. Consider using TD(lambda, omega) on the problem in Fig. 2 (Right), in which we weight experience non-uniformly by putting a non-zero weight only on the first state (omega(s) = 1) and using a constant trace-decay lambda(s) = 0, ∀s. If gamma > 0.5 and, initially, omega_0 = 1, then omega increases without bounds because we repeatedly update for the transition omega → 2omega, while ignoring the transition 2omega → omega, causing divergence (Sutton and Barto, 2018; van Hasselt et al., 2018).

Similarly, on-policy TD(lambda, omega(·)) = 1) with uniform state weighting and non-uniform bootstrapping (e.g., lambda(s) varies across states) can diverge (White, 2017). In Fig. 2 (Right), this occurs if the weighting is non-zero in both states but we bootstrap only on the second state. In the next section we describe several algorithms that can address this issue.

3.2. Selectivity through emphasis for stability

The aforementioned issues can arise when there is an imbalance between how often a state is used to compute update targets, and how often it is updated itself. Two kinds of weightings have been proposed to correct for this imbalance—emphasis and distribution ratios. Both can be seen as instances of TD(lambda, omega) for different weighting functions omega. Because distribution ratios are generally difficult to estimate, we focus on emphatic weightings. We previously described the emphatic algorithm ETD(lambda), which weights experience using a history-dependent weighting omega_t = rho_t * M_t.

A different instance of TD(lambda, omega) can be obtained by learning the expectation of the follow-on weighting f(s) = E_µ[F_t|S т = s], similar to Zhang et al. (2020) and Jiang et al. (2021):

omega_t = rho_t * m_t, with

m_t = lambda_t + gamma_t (1 - lambda_t) f(s), and f(s) = E[F_t|S т = s],

and using Eq. (10) in Eq. (8). The expected follow-on can be estimated with a function f_varphi ≈ f, with learnable parameters varphi. Canonical learning methods for learning value functions can be applied, by reversing the direction of time in the learning update, similarly to expected eligibility traces\(^1\), e.g., Monte-Carlo regression on the instantaneous follow-on trace, or backward TD (see Appendix B.3). Because of its similarity to the n-step algorithm X-ETD(n) proposed by Jiang et al. (2021) we will refer to this algorithm as X-ETD(lambda) with X(0)-ETD(lambda) denoting the variant where the expected follow-on is learned with backward TD and X(1)-ETD(lambda) the variant where it is learned with regression to the full Monte Carlo follow-on trace.

Empirical illustration on emphatic algorithms. We illustrate these policy-evaluation algorithms on the 5-state MRP introduced by Sutton et al. (2016) (also depicted in Fig. 8 in Appendix C.1). In Fig. 3, we observe that the emphatic algorithms using expected emphasis have lower variance (see Appendix C.1 for more details on the experimental setup).

3.3. Selectivity for on-policy TD

We now describe selectivity functions which make on-policy learning stable. For constant interest

\(^1\)Estimating a single scalar, instead of a d-dimensional vector.
Selective Credit Assignment

Figure 3 | Five-state MRP: (Left): Value error under the stationary distribution of the behaviour policy for off-policy TD(0), ETD(0), X(1)-ETD(0)—which learns emphasis by regressing on the instantaneous trace, and X(0)-ETD(0)—which learns emphasis by backward TD. Off-policy TD and X(1)-ETD visually overlap on the left, as do X(0)-ETD and X(1)-ETD on the right. The higher error of ETD is due to higher variance, caused by the follow-on trace. (Right): Variance of the follow-on (taken to be 0 for off-policy TD, which does not have this trace). Details on the MRP are in Fig. 8, Appendix C.1 and in Sutton et al. (2016), but do not affect this conclusion: follow-on traces typically add variance to the updates.

\[ Q(\lambda, \omega) \] Analogous to TD(\lambda, \omega), we use \( Q(\lambda, \omega) \) to refer to the generic algorithm that adds an explicit weighting function \( \omega \) to the trace of the action-value function:

\[ \tilde{e}_t = y_t \lambda_t \tilde{e}_{t-1} + \omega_t \nabla Q_{\omega_t}(s_t, a_t), \] (13)

to be used in Eq. (7) in place of \( e_t \).

Empirical illustration in deep reinforcement learning To illustrate the importance of this novel connection between weightings and bootstrapping, we used Ms. Pac-Man, a canonical Atari game. We designed the following experimental setup to test the selective \( Q(\lambda_t, \omega_t) \) (with the “t” subscript denoting state or time-dependence), using the trace-decay correction rules in Eq. (11) and Eq. (12). With probability \( \epsilon \), the agent’s observation is replaced with random Gaussian noise, to mimic a noisy observation sensor. To simulate access to a module that detects such noisy observations, we provide access to a time-dependent interest \( i_t \), capturing whether an observation is noisy or not, s.t. \( i_t = 0 \) if the observation at time step \( t \) is noisy, and \( i_t = 1 \) otherwise. Selectivity is entirely dictated by interest, with no other corrections: \( \omega_t \equiv i_t \). For the state-dependent trace-decay function \( \lambda_t \), we use Eq. (12). In Fig. 4–Top-Left & Top-Right we observed that coupling the weighting and the trace-decay function in Eq. (12) recovers the baseline’s performance, displaying robustness to observation noise. The caption and Appendix C.2 contain further details.

4. Planning selectively

We now provide insights on explicitly adding selectivity to planning algorithms. Expected eligibility traces attempt to capture all possible trajectories coalescing into a state. This allows credit to flow more broadly, not just to states that have
happened, but also to states that could have happened, under the same policy. These methods can be interpreted as planning backwards, while their standard backward view counterparts—TD(λ) and Q(λ)—do not plan and only use the current sampled trajectory.
Selective Credit Assignment

An important difference to prior algorithms is that we estimate the decayed previous expected trace $y_t \lambda_t \hat{e}_t$ in Eq. (14), instead of $\hat{e}_t$. We then use $Z_\Theta(s) + \nabla Q_\omega(s, a)$ as the trace for action $a$ to be used in Eq. (7) in place of $\hat{e}_t$. This avoids having to condition $Z_\Theta(s)$ on the action $a$, significantly reducing computation in settings with many actions—for instance in Atari this saves a factor $|\mathcal{A}| = 18$, and in many domains $|\mathcal{A}|$ is higher. In addition we get more data per state than for each, more specific, state-action pair, thereby potentially facilitating learning the traces accurately.

Empirical illustration of learning counterfactuals (models of expected traces) online off-policy Consider the Open World environment illustrated in Fig. 5-Left. The precise setup is described in Appendix C.3. In short, the agent's behaviour $\mu$ is uniformly random and we consider learning about two stochastic policies: one that tends up and right and another that tends down and left. We consider learning two expectation models for the traces associated with those two policies, and then use those expected traces to learn to predict their values. Rewards are, noisily, obtained when bumping into one of the goals (denoted $G$). Bumping into a goal ends the episode; new episodes start at a random location. In Fig. 6, we illustrate the effect of increasing the sparsity of the reward signal, with $\epsilon_r$ indicating the probability that the agent receives a reward of $r = 10/\epsilon_r$ in each goal. The reward is zero otherwise. Fig. 6 shows off-policy expected traces effectively reduce variance.

4.2. Selectively using and learning models of traces

To learn expected traces, van Hasselt et al. (2021) propose a mechanism similar to value bootstrapping by introducing mixture traces $e^\eta_t$ (analogous to $\lambda$-returns, with $\eta$ the counterpart of $\lambda$):

$$e^\eta_t = (1-\eta)Z_\Theta(S_t) + \eta(y_t \lambda_t e^\eta_{t-1} + \nabla V_\omega(S_t)).$$

(15)

The trace-bootstrapping parameter $\eta$ allows us to smoothly interpolate between using expected or instantaneous traces. Using $e^{\eta=0}_t = Z_\Theta(S_t)$ results in counterfactual credit assignment based on expected traces, while using $e^{\eta=1}_t = y_t \lambda_t e^{\eta=1}_{t-1} + \nabla V_\omega(S_t)$ results in trajectory-based learning, relying fully on instantaneous traces. The generic expected trace algorithm ET($\lambda, \eta$) is then defined by $\Delta^\eta = \delta_t e^\eta_t$, and smoothly interpolates between these extremes for $\eta \in [0,1]$.

We now consider how selectivity could influence using expected traces for value learning. It is reasonable to rely on expected traces more in states where they are more accurate, and more on the instantaneous traces otherwise. Assuming an explicit selectivity mechanism, we can constrain the trace-bootstrapping parameter $\eta$:

$$\eta_t = \beta_\eta \hat{\omega}_t + (1 - \hat{\omega}_t),$$

(16)

where $\beta_\eta \in [0,1]$ allows for partial trace-bootstrapping, and we used $\hat{\omega}$ to distinguish this from the $\omega$ used in the value learning process, which could be different. For instance, consider the special case in which selectivity captures partial observability. The value learning could then rely on estimated expected traces more for states that are less aliased, where $\hat{\omega}_t > 0$, and on the instantaneous eligibility trace otherwise, where $\hat{\omega}_t = 0$.

So far, we discussed using selectivity for learn-
ing the value function, either through selective value updates or by adapting the mixing parameter $\eta$ of the mixture trace. But the learning of the expected trace is itself subject to a sampling procedure, in which we can inject selectivity. We can apply the same procedure we did for value learning in Eq. (9):

$$\tilde{\Delta}_t^z \equiv \tilde{\omega}_t \frac{\partial \tilde{z}_t(S_t)}{\partial \Theta} (\tilde{e}_t - z_{\Theta_t}(S_t)),$$

and then using $\tilde{\Delta}_t^z$ in place of $\Delta_t^z$ in Eq. (6).

When the value learning uses mixture traces, the trace learning process can generally save function approximation resources by also focusing learning the expected trace only in those states in which the expected trace is used. If a mixture trace $\tilde{e}_t^\eta$ (similar to Eq. (15)) is used in Eq. (17) in place of $\tilde{e}_t$, we now have a multi-step trace learning process similar to the value learning process in TD($\lambda$), so coupling the dynamic trace-bootstrapping $\eta$ and the weighting $\tilde{\omega}$ analogously ensures stable learning of the expected traces.

**Empirical illustration in deep reinforcement learning.** We again consider Ms. Pac-Man to illustrate the effectiveness of using selective expected eligibility traces, and the importance of coupling the trace-bootstrapping function $\eta$ with the weightings $\tilde{\omega}$, and focusing function approximation resources when learning the model for the trace. We use the same experimental setup as before. Fig. 4–Bottom-Left illustrates the baseline runs of the algorithms without observational noise, whereas Fig. 4–Bottom-Right shows the effect of adding noise to the observations for all the algorithms. We found the algorithms using Eq.(16) and Eq.(17) to recover the original performance of the baselines, despite needing to rely on substantially noisier observations.

### 4.3. Sparse expected eligibility traces

Interestingly, learning expectation models of selective traces (c.f. (8)) with binary weighting functions $\omega : S \rightarrow \{0,1\}$, results in sparse expected eligibility traces. These models are equivalent to expected temporally-extended backward models (Chelu et al., 2020) (proof in Appendix B.2).

In general, backward planning (Chelu et al., 2020; McMahan and Gordon, 2005; Moore and Atkeson, 2004; Peng and Williams, 1993; Sutton et al., 2008; van Hasselt et al., 2019) propagates credit to events possibly responsible for the current outcome, typically using explicit backward transition models. Option models Sutton et al. (1999) describe the long-term effects of temporally-abstract actions (options).

Sparse expected eligibility traces, used in planning algorithms, e.g., selective ET or QET, assign credit in ways akin to jumpy backward planning (cf. Chelu et al., 2020), but without learning an explicit dynamics model. Specifically, for binary weighting functions, the selective planning algorithm QET($\lambda$, $\omega$) is similar to planning with backward option models (Chelu et al., 2020), skipping parts of the state space as dictated by the weighting function.

Credit assignment using sparse expected traces then happens in a sub-MDP within the original one, where the state space now contains only the states captured by the binary weighting function, and the action space is given by an induced option space (containing all the actions/sub-policies in-between these states). In terms of learning, this leads to a way of doing temporally-extended credit assignment in the original MDP. One can even go a step further, and apply selective temporal-discounting, similarly to Harutyunyan et al. (2019c), e.g., using different discounting parameters for the intra-option expected eligibility trace that flows credit inside an option, and the sparse expected trace that flows credit over longer time-spans using options.

**Empirical evaluation of sparse expected traces** In the Four Rooms domain Sutton et al. (1999) (Fig. 5-Right), the agent aims to navigate to a goal location via options that take it from inside each room to its hallways, as shown in the illustration. The reward is 0 everywhere, except at the goal, where it is $10/\epsilon_r$ with probability $\epsilon_r$, and 0 otherwise. The option policies are pre-learned, and illustrated in Fig. 5-Right. Fig. 7 shows the results of using sparse traces for a policy over options, compared with using only primitive actions (see Appendix C.4 for more...
Selective Credit Assignment

Figure 7 | Number of steps per episode. (Left) $Q_\lambda$: $Q(\lambda = 0.9)$ with dense (standard) eligibility traces over primitive actions; $Q_0$: $Q(\lambda, \omega)$ using a sparse eligibility trace for a policy over options (with pre-learned options as illustrated in Fig. 5-Right, $\lambda$ using Eq.(12), $\beta_\lambda = 0.9$ and $\omega_t = 1$ in hallways, and 0 otherwise). (Right) $QET_\lambda$: $QET(\eta = 0, \lambda = 0.9)$ using dense expected eligibility traces over primitive actions; $QET_0$: $QET(\eta_t, \lambda_t, \omega_t)$ using sparse expected eligibility traces for a policy over pre-learned options (with $\eta_t$ using Eq.(16), $\beta_\eta = 0$, $\lambda_t$ using Eq.(12), $\beta_\lambda = 0.9$, and $\omega_t = 1$ only in hallways, and 0 otherwise). (Top): $r = 10$ at the goal. (Bottom): sparse reward signals, $r = 20$, with probability $\epsilon_r = 0.5$, and $r = 0$ otherwise. Sparse expected traces were learned faster and coped better with sparse rewards.

5. Discussion

We discussed the use of weighting functions to stabilize, and focus resources in value-based credit assignment, introducing selective traces. We illustrated the importance of trace corrections even for the on-policy case, as well as the significance of adding corrections for off-policy learning. In the context of expected traces, weightings can act as a guide for when to rely on the learned expected traces. It can also be applied to selectively learn the traces themselves.

We identified potential issues with naively combining weightings with instantaneous and expected traces. Using these as motivation, we proposed and investigated different modifications that allow for safe, selective credit assignment, identifying sufficient conditions linking the credit assignment parameters to ensure stable learning.

Our computational examples illustrate the potential benefits of adding weightings to the credit assignment problem, and show that selective learning can substantially improve performance in some settings.

Although specific weightings have been discussed for off-policy learning before, our definition is generic and does not restrict to a specific weighting, or to off-policy evaluation. Specifically, we consider weightings to focus function approximation resources, and for sparse or jumpy backward planning in the control setting. Emphatic weightings are one instance of selective traces. Other, perhaps more effective, choices are possible. Finding the “best” selection function for specific problems remains an intriguing open problem for future work, and could be domain-specific.

Anand and Precup (2021) proposed a trace correction similar to the closed-form expected emphasis correction we derived, without recognizing that this is related to expected emphasis, and employing it differently. Particularly, they replace the trace-decay $\lambda$ with a preference function $\beta$, and allow zero-step returns rather than the typical $n$-step return, $n \geq 1$. Our formulation is more generic, subsuming TD, ETD, and X-ETD (expected-emphasis ETD) (Section 3.2), and recognizing the coupling between the weighting $\omega$ and the trace-decay $\lambda$ is strongly related to expected emphasis.

Future work Our examples consider the weighting given, or use emphatic weightings to stabilize learning. One direction for future work is adapting the weighting over time, based on experience. One potential approach is (meta) learning $\omega, \lambda, \text{and/or } \gamma$ Sutton and Singh (1994); White and White (2016); Xu et al. (2018); Zahavy et al. (2020), based on variance or bias minimization, or other proxy objectives Kumar et al. (2020). Our results can be used to derive stable convergent updates, by respecting the coupling of these parameters.

A second potential approach is to use ideas for...
learning option terminations (Bacon et al., 2017; Harutyunyan et al., 2019a), which are akin to the credit assignment functions we considered. Inferring controllability Harutyunyan et al. (2019b) could inform weightings over states and actions, e.g., some states are irrelevant if all actions have the same consequences, and some actions could be irrelevant if they are unlikely to be selected. Weightings can be learned based on these intuitions. Hindsight conditioning Harutyunyan et al. (2019b) can also help infer policy-related weightings.

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A. Background and preliminaries (details)

A.1. Expected eligibility traces

Expected eligibility trace (ET) algorithms (van Hasselt et al., 2021) have been introduced for off-trajectory, on-policy value learning, replacing the instantaneous trace $e_t$ with an estimated expectation:

$$z_{\theta}(s) \approx \mathbb{E}_x [e_t | S_t = s] ,$$

with

$$e_t = y_t \lambda_t e_{t-1}^\theta + \nabla V_w(S_t).$$

We can approximate the expected traces by regressing on the instantaneous eligibility trace $e_t$, or by a mechanism similar to value bootstrapping using the mixture trace $e_t^\eta$:

$$e_t^\eta = (1 - \eta) z_{\theta}(S_t) + \eta [y_t \lambda_t e_{t-1}^\theta + \nabla V_w(S_t)].$$

The trace-bootstrapping parameter $\eta$ specifies the credit assignment mechanism with $e_t^{\eta=0} = z_{\theta}(S_t)$ resulting in counterfactual credit assignment (fully relying on the expected traces), while $e_t^{\eta=1} = y_t \lambda_t e_{t-1}^\theta + \nabla V_w(S_t)$ uses factual or trajectory-based learning (reverting fully to instantaneous traces).

The expected trace algorithm ET($\lambda$, $\eta$) is then defined by:

$$w_{i+1} = w_i + \alpha_t w_i \Delta_t^V ,$$

with

$$\Delta_t^V = \delta_t e_t^\eta ,$$

and smoothly interpolates between these two end points (using expected or instantaneous traces).

The expected trace $z_{\theta}(S_t)$ can be learned by regressing on $y_t \lambda_t e_{t-1}^\theta + \nabla V_w(S_t)$, s.t.:

$$\Delta_t^\theta = \frac{\partial z_{\theta}(S_t)}{\partial \Theta_t} \left( y_t \lambda_t e_{t-1}^\theta + \nabla V_w(S_t) - z_{\theta}(S_t) \right) ,$$

$$\Theta_{t+1} = \Theta_t + \alpha_t \Delta_t^\theta ,$$

(18)

or using a different mixture parameter $\tilde{\eta}$ for learning the trace:

$$e_t^{\tilde{\eta}} = (1 - \tilde{\eta}) z_{\theta}(S_t) + \tilde{\eta} [y_t \lambda_t e_{t-1}^\theta + \nabla V_w(S_t)] ,$$

keeping $\eta$ just for determining when to use the model. So updating $\Theta$ with

$$\Delta_t^{\tilde{\eta}} = \frac{\partial z_{\theta}(S_t)}{\partial \Theta_t} \left( y_t \lambda_t e_{t-1}^{\tilde{\eta}} + \nabla V_w(S_t) - z_{\theta}(S_t) \right) ,$$

$$\Theta_{t+1} = \Theta_t + \alpha_t \Delta_t^{\tilde{\eta}} ,$$

where $\alpha_t^{\tilde{\eta}} \in (0, 1)$ is a (possibly time-varying) step-size parameter and $\tilde{\eta}$ is analogous to $\lambda$, but for learning traces rather than values, using TD methods when $\tilde{\eta} = 0$, and Monte-Carlo regression when $\tilde{\eta} = 1$. Note that, like TD(0), we always include at least one sampled transition, even if $\tilde{\eta} = 0$.

B. Proofs and derivations

B.1. New derivation of off-policy Q($\lambda$)

Let $\rho(s, a) = \frac{\pi(a | s)}{\mu(a | s)}$ be the importance sampling ratio between the target policy $\pi$ and the behaviour $\mu$.

In the trajectory context, we may write for any $k$ and $t$: $\rho_k^{\hat{t}} = \prod_{j=k}^t \rho_j$, with $\rho_j = \rho(S_j, A_j)$, with the convention that $\rho_k^{\hat{t}} = 1$ for $t < k$.
With this notation, the off-policy importance sampled \(\lambda\)-return for state-dependent \(\lambda\) and \(\gamma\) can be written as:

\[
G^\lambda_t,\pi = (1 - \lambda_{t+1})G^\pi_{t+1} + \lambda_{t+1}(1 - \lambda_{t+2})G^\pi_{t+2} + \ldots
\]

\[
= (1 - \lambda_{t+1})\rho_t[R_{t+1} + y_{t+1}V(S_{t+1})] + (1 - \lambda_{t+2})\lambda_{t+1}\rho_t[R_{t+1} + y_{t+1}\rho_{t+1}R_{t+2} + y_{t+1}\rho_{t+1}y_{t+2}V(S_{t+1})] + \ldots
\]

\[
= \rho_t(R_{t+1} + (1 - \lambda_{t+1})y_{t+1}V(S_{t+1})) + \lambda_{t+1}y_{t+1}\rho_{t+1}[R_{t+2} + (1 - \lambda_{t+2})y_{t+2}V(S_{t+2})] + \ldots
\]

\[
= \sum_{k=0}^{\infty} \left[ \prod_{j=0}^{k-1} y_{t+j+1}\lambda_{t+j+1}\rho_{t+j} \right] \rho_{t+k}[R_{t+k+1} + (1 - \lambda_{t+k+1})y_{t+k+1}V(S_{t+k+1})]
\]

This is the return from some state \(S_t\) onward. For the return from some state \(S_t\) and action \(A_t\), we can just drop the final importance sampling ratio, yielding:

\[
G^\lambda_t = \sum_{k=0}^{\infty} \left[ \prod_{j=0}^{k-1} y_{t+j+1}\lambda_{t+j+1}\rho_{t+j} \right] [R_{t+k+1} + (1 - \lambda_{t+k+1})y_{t+k+1}V(S_{t+k+1})].
\]

For Q-learning the bootstrapped target is: \(V(S_{t+k+1}) \equiv \max_a Q(S_{t+k+1}, a)\), whereas for SARSA we have:

\(V(S_{t+k+1}) \equiv \mathbb{E}_r [Q(S_{t+k+1}, A)]\).

Let

\[
\Delta_t^Q = (G^\lambda_t - Q_w(S_t, A_t))\nabla Q_w(S_t, A_t)
\]

\[
= G^\lambda_t \nabla Q_w(S_t, A_t) - Q_w(S_t, A_t)\nabla Q_w(S_t, A_t),
\]

be the update to the parameters \(w\) at time step \(t\).

We look at the first part of the update, which can be rewritten as:

\[
\sum_{t=0}^{\infty} G^\lambda_t \nabla Q_w(S_t, A_t) = \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \left[ \prod_{j=0}^{k-1} y_{t+j+1}\lambda_{t+j+1}\rho_{t+j} \right] [R_{t+k+1} + (1 - \lambda_{t+k+1})y_{t+k+1}V(S_{t+k+1})] \nabla Q_w(S_t, A_t)
\]

\[
= \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \left[ \prod_{j=0}^{k-1} y_{t+j+1}\lambda_{t+j+1}\rho_{t+j} \right] [R_{k+1} + (1 - \lambda_{k+1})y_{k+1}V(S_{k+1})] \nabla Q_w(S_t, A_t)
\]

\[
= \sum_{k=0}^{\infty} \sum_{t=0}^{k} \left[ \prod_{j=0}^{k-1} y_{t+j+1}\lambda_{t+j+1}\rho_{t+j} \right] [R_{k+1} + (1 - \lambda_{k+1})y_{k+1}V(S_{k+1})] \nabla Q_w(S_t, A_t)
\]

\[
= \sum_{t=0}^{\infty} \sum_{k=0}^{t} \left[ y_{t+1}(1 - \lambda_{t+1})V(S_{t+1}) \right] \sum_{k=0}^{t} \left[ \prod_{j=k}^{t-1} y_{j+1}\lambda_{j+1}\rho_{j} \right] \nabla Q_w(S_k, A_k)
\]

with

\[
\mathbf{e}_t = \sum_{k=0}^{t} \left[ \prod_{j=k}^{t-1} y_{j+1}\lambda_{j+1}\rho_{j} \right] \nabla Q_w(S_k, A_k)
\]

\[
= \nabla Q_w(S_t, A_t) + y_t\lambda_t \rho_{t-1} \nabla Q_w(S_{t-1}, A_{t-1}) + \ldots
\]

\[
= y_t\lambda_t \rho_{t-1} \mathbf{e}_{t-1} + \rho_t \nabla Q_w(S_t, A_t)
\]
Then, the full update is:

$$
\sum_{t=0}^{\infty} \Delta_t^Q = \sum_{t=0}^{\infty} \left[ R_{t+1} + y_{t+1}(1 - \lambda_{t+1})V(S_{t+1}) \right] e_t - Q_w(S_t, A_t) \nabla Q_w(S_t, A_t),
$$

which is what we use in the new versions of Q(\(\lambda\)) and QET(\(\lambda, \eta\)) algorithms.

When using this definition with the QET algorithm, one important difference to the prior algorithm introduced by van Hasselt et al. (2020) is that we estimate the decayed previous expected trace, so approximating \(z_\Theta^s \approx \mathbb{E}_\pi [y_t \lambda_t e_{t-1} | S_t = s]\), and then using \(z_\Theta^s + \nabla Q_w(s, a)\) as the trace for action \(a\). This avoids having to condition the expected trace on the action, which can significantly reduce computation in some implementations.

### B.2. Sparse expected eligibility traces

**Proposition B.1.** Sparse accumulating traces with binary state weighting functions are equivalent to temporally extended backward models.

**Proof.** From the definition of the selective expected trace, we have

$$
\tilde{z}(s) = \mathbb{E}_\pi [\tilde{e}_t | S_t = s] \\
= \mathbb{E} [y_t \lambda_t \tilde{e}_{t-1} + \nabla V_w(S_t) \omega_t | S_t = s] \\
= \nabla V_w(s) \omega(s) + y(s) \lambda(s) \sum_{\tilde{s}} P_\pi(s|\tilde{s}) \tilde{z}(\tilde{s})
$$

If \(\omega(s) = 1 - y(s) \lambda(s)\) (cf. Eq. (12)), then

$$
\tilde{z}(s) = \omega(s) \nabla V_w(s) + (1 - \omega(s)) \sum_{\tilde{s}} P_\pi(s|\tilde{s}) \tilde{z}(\tilde{s}) \tag{19}
$$

In Eq. (19), considering \(c(s) \equiv \nabla V_w(s)\) a multi-dimensional cumulant and \(\beta(s) \equiv 1 - \omega(s)\) the probability of termination at state \(s\), then \(\tilde{z}(s)\) can be interpreted as a temporally-extended option model for an option defined as: \(o = (\pi, \beta)\), where \(\pi\) is the option’s policy and \(\beta\) is the corresponding binary termination function.

\[ \square \]

### B.3. Weightings for distribution correction: off-policy expected emphasis

We consider methods that learn the expectation of the follow-on weighting \(f(s) = \mathbb{E}_\mu [F_t | S_t = s]\), resulting in a state-dependent selectivity function to be used in place of the history-dependent weighting \(\omega_t\). We call the emphatic algorithm, analogous to \(\lambda\)-discounted TD, resulting from the aforementioned approach, X-ETD(\(\lambda\)) (where “X” comes from “expected”):

$$
\omega_t = \rho_t m_t, \text{ with} \\
m_t = \lambda_t i_t + y_t (1 - \lambda_t) f(s),
$$

The expected follow-on can be estimated using a function \(f_\varphi \approx f\), with learnable parameters \(\varphi\). Learning methods, typically used to learn value functions, can be applied, by reversing the direction...
of time in the learning update, similarly to expected eligibility traces, e.g., Monte-Carlo regression on the instantaneous follow-on trace

\[ \wp_{t+1} = \wp_t + \alpha_t^\wp (F_t - f_{\wp_t}(S_t)) \nabla_{\wp_t} f_{\wp_t}(S_t), \]

or backward TD:

\[ \wp_{t+1} = \wp_t + \alpha_t^\wp (f_{\wp_t}(S_{t-1}) + i_t - f_{\wp_t}(S_t)) \nabla_{\wp_t} f_{\wp_t}(S_t), \]

with \( \alpha_t^f \) a (possibly time-varying) step-size.

Unfortunately, both methods can be problematic. The Monte-Carlo regression problem has targets with infinite variance, so it is not guaranteed to converge, whereas the backward TD method suffers from “off-policiness”, same as value learning.

Similarly to the case of expected eligibility traces, an interpolation between the instantaneous follow-on trace and its estimated expectation is possible using a mixture trace:

\[ F^\eta_t = (1 - \eta^\eta) f_{\wp_t}(S_t) + \eta^\eta y_t \rho_{t-1} f_{\eta_{t-1}} + i_t, \]

with \( \eta^\eta \) the mixing parameter, interpolating between using the expected follow-on \( f(s) \) or the history-dependent instantaneous follow-on trace \( F_t \). The mixture trace \( F^\eta_t \) is then used in the selectivity function \( \omega \) as:

\[ \omega_t = \rho_t \left( \lambda_t i_t + (1 - \lambda_t) F^\eta_t \right), \]

with \( m_t \) the mixed emphasis. We would then use \( \omega \) to learn the value parameters \( w \).

A second mixture trace \( e^\eta_t \) can be used as target in the estimation of the expected follow-on \( f_{\wp} \) (same as for expected traces), one that uses a different mixture parameter \( \eta^\eta \) (possibly different from \( \eta^\eta \), distinguishing the way the trace is learned from how it is used:

\[ e^\eta_t = (1 - \eta^\eta) f_{\wp_t}(S_t) + \eta^\eta y_t \rho_{t-1} F_{t-1} + i_t, \]

The expected mixture trace is then used as target for an estimated model s.t. \( f_{\wp} \approx \mathbb{E}[y_t \rho_{t-1} e^\eta_{t-1} | S_t = s] \), with \( \eta^\eta \) interpolating between Monte-Carlo regression on the instantaneous trace (\( \eta^\eta = 1 \)), and backward TD by bootstrapping on the expected trace (\( \eta^\eta = 0 \)). The follow-on trace learning process updates the trace parameters \( \wp \) with:

\[ \wp_{t+1} = \wp_t + \alpha_t^\wp \left( y_t \rho_{t-1} e^\eta_{t-1} - f_{\wp_t}(S_t) \right) \nabla_{\wp_t} f_{\wp_t}(S_t), \]

with \( \alpha_t^\wp \)—the step size. We use \( X(\eta^\eta)-\text{ETD}(\lambda) \) to explicitly denote the mechanism used to learn the expected follow-on trace, with \( \eta^\eta = 1 \) for Monte-Carlo regression, and \( \eta^\eta = 0 \) for backward TD(0).

B.4. Weightings for on-policy learning: on-policy expected emphasis

Emphatic TD uses a state weighting function of the form:

\[ \omega_t = M_t \rho_t = \rho_t \lambda_t i_t + \rho_t (1 - \lambda_t) F_t, \]

with \( F_t = y_t \rho_{t-1} F_{t-1} + i_t \)

\[ = \rho_t i_t + \rho_t y_t (1 - \lambda_t) \sum_{k=1}^{t} \prod_{j=k}^{t} \rho_{t-j+1} y_{t-j} i_{t-k} \]

\[ = \rho_t i_t + \rho_t y_t (1 - \lambda_t) \rho_{t-1} i_{t-1} + \rho_t y_t (1 - \lambda_t) \rho_{t-1} y_{t-1} \rho_{t-2} i_{t-2} \ldots, \]
For the on-policy learning, we have $\rho_t = 1$, $\forall t$. For on-policy learning and uniform interest $i_t = 1$, $\forall t$, then, the state weighting becomes:

$$
\omega_t = M_t = \lambda_t + (1 - \lambda_t)F_t, \text{ with } F_t = y_tF_{t-1} + 1
$$

$$
= 1 + y_t(1 - \lambda_t)\sum_{k=1}^{t} \left( \prod_{j=k}^{t-1} y_{t-j} \right)
$$

The expected emphasis in this setting is:

$$
f(s) = \lim_{t \to \infty} \mathbb{E}_\pi [F_t | S_t = s] = \left[ 1 + \sum_{k=1}^{t} \prod_{j=k}^{t-1} y_{t-j} \right]_s
$$

**Constant $\gamma$** Furthermore, for constant $\gamma$ we have:

$$
f(s) = 1 + \gamma + \gamma^2 + \cdots = \frac{1}{1 - \gamma}
$$

Replacing the expected emphasis in the definition of the weighting $\omega$:

$$
\omega_t = \lambda_t + (1 - \lambda_t)/(1 - \gamma)
$$

$$
= (\lambda_t - \lambda_t\gamma + 1 - \lambda_t)/(1 - \gamma)
$$

$$
= (1 - \lambda_t\gamma)/(1 - \gamma)
$$

$$
\implies \lambda_t = (\gamma \omega_t + (1 - \omega_t))/\gamma
$$

For constant $\gamma$, we can omit the denominator, since it would be just re-scaling the update by a constant factor which can be folded into the learning rate, yielding:

$$
\omega_t = 1 - \gamma \lambda_t \implies \lambda_t = \gamma \omega_t + (1 - \omega_t)
$$

**Adaptive $\gamma$** Let $P_\pi$ be the transition matrix induced by following policy $\pi$, with $[P_\pi]_{s,s'} = P(s'|s, a)\pi(a|s)$, and $[P_\pi^\top]_{s,s} = P(s|s, a)\pi(a|s)$ the vector corresponding to all entries of the successor states of $s, a$. Let $\Gamma$, and $\Lambda$ be diagonal matrices, the former representing the discount matrix—with diagonal entries $\gamma(s)$, and the latter the trace-decay matrix—using $\lambda(s)$ on its diagonal. Let $d_\pi$ denote the vector, with entries corresponding to the stationary distributions $d_\pi(s)$. Then, in matrix notation, the following hold:

$$
P_\pi^\top d_\pi = \sum_s d_\pi(s)[P_\pi^\top]_{s,s} = d_\pi \implies \Gamma P_\pi^\top d_\pi = \Gamma d_\pi
$$

$$
P_\pi^\top \Gamma d_\pi = \sum_s \gamma(s)d_\pi(s)[P_\pi^\top]_{s,s}.
$$

Furthermore, we can write the stationary distribution under $\pi$ reweighted by the follow-on weighting, in matrix notation:

$$
d_\pi^f = (I - \Gamma P_\pi^\top)^{-1}d_\pi
$$

$$
= d_\pi + \Gamma P_\pi^\top d_\pi + ((\Gamma P_\pi^\top)^2) d_\pi + \cdots
$$

$$
= d_\pi + \Gamma P_\pi^\top d_\pi + \Gamma P_\pi^\top \Gamma P_\pi^\top d_\pi + \cdots
$$
Assuming we can bound the discount factor $\gamma(s)$ with $\beta_\lambda$, $\forall s \in S$, s.t. $\gamma(s) \leq \beta_\lambda$, then:

$$P_\pi^T \Gamma d_s \leq \beta_\lambda \sum_s d_s(s) [P_\pi^T]_{s,s} = \beta_\lambda d_\pi \implies \Gamma P_\pi^T \Gamma d_\pi < \beta_\lambda^2 d_\pi \quad (23)$$

Using the assumption in Eq. 23, we obtain the expected follow-on trace:

$$d'_s = d_s + \Gamma d_s + \Gamma P_\pi^T \Gamma d_s + \ldots$$

$$= (1 + \beta_\lambda + \beta_\lambda^2 + \ldots) d_\pi$$

$$= (1 - \beta_\lambda)^{-1} d_\pi \quad (24)$$

Let $\Omega$ be a weighting matrix, subject to the constraint:

$$\Omega = \Lambda + (I - \Lambda)(I - \Gamma P_\pi^T)^{-1}$$

$$= (I - \Lambda \Gamma P_\pi^T)(I - \Gamma P_\pi^T)^{-1}$$

$$\implies \Lambda = I - \Omega (I - \Lambda \Gamma P_\pi^T)(I - \Gamma P_\pi^T)^{-1}$$

$$= (\Omega \Gamma P_\pi^T + (I - \Omega))(\Gamma P_\pi^T)^{-1} \quad (25)$$

Inserting the result from Eq. (24) in Eq. (25), we obtain:

$$\omega_t = \frac{1 - \gamma_t \lambda_t}{1 - \beta_\lambda}$$

$$\implies \lambda_t = \frac{\beta_\lambda \omega_t + (1 - \omega_t)}{\gamma_t} = \frac{1 - \omega_t (1 - \beta_\lambda) \gamma_t}{\gamma_t} \frac{1 - \omega_t + \omega_t \beta_\lambda}{\gamma_t}$$

The constant discount weighting can be recovered by making $\beta_\lambda = \gamma$ for constant $\gamma$.

### B.4.1. Stability & convergence

For the on-policy case we can show that this coupling between the weighting and the trace decay parameter is sufficient for stability. Moreover, because the traces are on-policy and they do not have importance sampling ratios, the variance is always finite, so the coupling also ensures convergence, not just stability of the value learning process.

Let the value function $V_w$ be a linear function of the form: $V_w(s) = x^T(s)$ with $w \in \mathbb{R}^n$ learnable parameters, where $x : S \rightarrow \mathbb{R}^n$ is a feature mapping. Let $X \in \mathbb{R}^{|S| \times n}$ be the feature matrix whose rows are the vectors $x(s)$ for different states $s$.

Consider the semi-gradient learning update for the selective TD($\lambda$, $\omega$) algorithm:

$$w_{t+1} = w_t + \alpha_t \hat{\varepsilon}_t (R_{t+1} + \gamma_t V_w(S_{t+1}) - V_w(S_t)) \quad (26)$$

with $\hat{\varepsilon}_t = \gamma_t \lambda_t \hat{\varepsilon}_{t-1} + \omega_t \nabla V_w(S_t)$, and $\delta_t$ shorthand for the TD error, $\hat{\varepsilon}_t$ the selective instantaneous eligibility trace, $\alpha_t$ is the step size parameter, $\omega : S \rightarrow [0, 1]$ is the weighting function, $\lambda : S \rightarrow [0, 1]$ is the trace-decay function, $\gamma : S \rightarrow [0, 1]$ is the temporal discounting function. Additionally, we assume the following mild conditions:

1. the state space is finite
2. the feature function $x : S \rightarrow \mathbb{R}^n$ s.t. the $X \in \mathbb{R}^{[S] \times n}$ has linearly independent columns, with bounded variance;
3. the rewards are bounded; 
4. the step-size sequence satisfies the Robbins-Monro conditions Robbins and Monro (1951): 
\[ \sum_{t=0}^{\infty} \alpha_t = \infty \text{ and } \sum_{t=0}^{\infty} \alpha_t^2 < \infty; \]
5. \( \gamma : S \to [0, 1] \) s.t. \( \prod_{k=1}^{\infty} \gamma(S_{t+k}) = 0 \), w.p. 1, \( \forall t > 0; \)
6. \( \omega : S \to [0, 1] \) s.t. \( \prod_{k=1}^{\infty} \omega(S_{t+k}) \neq 0 \), w.p. 1, \( \forall t > 0; \)
7. experience is sampled on-policy from the Markov chain \( (S_t, A_t, R_{t+1}, S_{t+1}) \sim d_x(S_t)\pi(A_t|S_t)P(S_{t+1}|S_t, A_t) \) with stationary distribution \( d_x \).

**Proposition B.2.** For \( V_{w}(s) = w^\top x(s) \), the semi-gradient update of selective TD\((\lambda, \omega)\) in Eq. (26) with \( \omega(s) = 1 - \gamma(s)\lambda(s) \) converges to the fixed point:

\[
\begin{align*}
    \mathbf{w}^* = \mathbb{E} \left[ \sum_{k=0}^{t} y_t^{(k)} (y_{t-k} \omega_{t-k} x_{t-k}) (y_{t+1} x_{t+1} - x_t)^\top \right]^{-1} \mathbb{E} \left[ \sum_{k=0}^{t} y_t^{(k)} (y_{t-k} \omega_{t-k} x_{t-k}) R_{t+1} \right],
\end{align*}
\]

with \( y_t^{(k)} = \prod_{j=t+1}^{t+k} \gamma_j \lambda_j = \prod_{j=t+1}^{t+k} \lambda_j \).

**Proof.** Firstly, under the assumptions 1-4 above, the stochastic algorithm TD\((\lambda, \omega)\) (Eq. (26)) behaves like the corresponding expected update equation under the on-policy stationary distribution \( d_x \):

\[
\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha_t \left[ \mathbb{E}_{d_x} [\mathbf{e}_t R_{t+1}] - \mathbb{E}_{d_x} [\mathbf{e}_t (x_t - \gamma_t x_{t+1})^\top] \mathbf{w}_t \right],
\]

and \( \mathbf{w}_t \) converges with probability 1 to \( \mathbf{w}^* \), the solution to the expected update equation (28).

**Stability** We begin by showing stability of TD\((\lambda, \omega)\). Let \( Z_t = (S_t, A_t, \tilde{e}_t) \) for \( t \geq 0 \) be the Markov chain resulting from adding \( \tilde{e} \) to the stationary Markov chain \( \{ (S_t, A_t) \}_{t=0}^{\infty} \) with transition probabilities given by \( P_\pi \), s.t.:

\[
\tilde{e}_t = \sum_{k=0}^{t-1} \left( \prod_{j=0}^{k-1} y_{t-j} \lambda_{t-j} \right) \omega_t x_{t-k}.
\]

Since \( \tilde{e}_t \) and \( x_{t+1} \) are deterministic functions of \( (S_t, A_t) \) and the distribution of \( s_{t+1} \) only depends of \( s_t \), the resulting chain \( Z_t \) is Markov. Let \( \mathbb{E}_{d_x} [\cdot] \) denote the expectation with respect to the steady state distribution \( d_x \).

Let \( \mathbf{A}(Z_t) = \tilde{e}_t (x_t - \gamma_t x_{t+1})^\top, \mathbf{b}(Z_t) = \tilde{e}_t R_{t+1} \) and \( \mathbf{A} = \mathbb{E}_{d_x} [\tilde{e}_t (x_t - \gamma_t x_{t+1})^\top], \mathbf{b} = \mathbb{E}_{d_x} [\tilde{e}_t R_{t+1}] \). The fixed point equation of the deterministic system is:

\[
\mathbf{w}_{t+1} = (\mathbf{I} - \alpha \mathbf{A}) \mathbf{w}_t + \alpha \mathbf{b}.
\]

Since these methods are not true gradient methods, the asymptotic behaviour of any TD algorithm generally depends on a stability criteria that requires the eigenvalues of the iteration matrix \( \mathbf{A} \) have positive real components Sutton et al. (2016). The stochastic algorithm converges if and only if the deterministic algorithm converges Sutton (1988b) and if both algorithms converge, they converge to the same fixed point.

Let \( \mathbf{A} \) and \( \Gamma \) be matrices with diagonal entries corresponding to the functions \( \lambda \) and \( \gamma \). Let

\[
\mathbf{P}_\pi^\lambda = (\mathbf{I} - P_\pi \Gamma \Lambda)^{-1} \Gamma (\mathbf{I} - \Lambda) P_\pi,
\]
such that:

\[
(I - P_\pi \Gamma \Lambda)^{-1}(I - P_\pi \Gamma) = \\
= (I - P_\pi \Gamma \Lambda)^{-1}(I - P_\pi \Gamma \Lambda + P_\pi \Gamma \Lambda - P_\pi \Gamma) \\
= (I - P_\pi \Gamma)^{-1}(I - P_\pi \Gamma \Lambda + P_\pi \Gamma (\Lambda - I)) \\
= I - (I - P_\pi \Gamma \Lambda)^{-1}\Gamma(I - \Lambda)P_\pi \\
= I - P_\pi^d .
\]

Expanding the A and b matrices we have:

\[
A = X^\top \tilde{D}_\pi (I - P_\pi \Gamma \Lambda)^{-1}(I - P_\pi \Gamma) X, \\
b = X^\top \tilde{D}_\pi (I - P_\pi \Gamma \Lambda)^{-1} r,
\]

where \(\tilde{D}_\pi\) is a diagonal matrix with elements \(d_\pi(s)\omega(s)\) on the diagonal. Following Sutton et al. (2016), we refer to \(K\) as the “key matrix”.

To ensure convergence regardless of the representation function, we use the assumption that \(X\) is full rank and require cf. Sutton et al., 2016 that:

1. the diagonal entries of \(K\) are non-negative
2. the off-diagonal entries are non-positive
3. the row sums are non-negative
4. the column sums are positive

Conditions (1-3) follow from Lemma 4 of White (2017) and the fact that \(\tilde{D}_\pi\) is a non-negative diagonal weighting matrix.

For the last condition, if we assume \(\omega(s) > 0, \forall s \in S\), then, similarly to (Sutton et al., 2016), we have:

\[
1^\top K = 1^\top \tilde{D}_\pi (I - P_\pi \Gamma \Lambda)^{-1}(I - P_\pi \Gamma) \\
(\text{using } \Omega = I - \Lambda \Gamma) \\
= 1^\top \tilde{D}_\pi (I - \Gamma \Lambda) (I - P_\pi \Gamma \Lambda)^{-1}(I - P_\pi \Gamma) \\
= (d_\pi^\Gamma - d_\pi^\pi \Gamma \Lambda) (I - P_\pi \Gamma \Lambda)^{-1}(I - P_\pi \Gamma) \\
(\text{using } d_\pi^\pi = d_\pi^\pi P_\pi) \\
= (d_\pi^\Gamma - d_\pi^\pi P_\pi \Gamma \Lambda) (I - P_\pi \Gamma \Lambda)^{-1}(I - P_\pi \Gamma) \\
= d_\pi^\Gamma (I - P_\pi \Gamma \Lambda) (I - P_\pi \Gamma \Lambda)^{-1}(I - P_\pi \Gamma) \\
= d_\pi^\Gamma (I - P_\pi \Gamma) \\
(\text{using } d_\pi^\pi = d_\pi^\pi P_\pi) \\
= d_\pi^\Gamma (I - \Gamma) > 0
\]

Using \(\omega(s) = 1 - \gamma(s)\lambda(s)\), all components of the column sums become positive. Thus, the key matrix \(A\) is positive definite and the selective TD(\(\lambda, \omega\)) algorithm is stable.

If \(\exists s \in S \text{ s.t. } \omega(s) = 0\), then we can set \(\gamma(s) = \lambda(s) = 1\), which induces a new super-imposed MDP, with temporally-extended dynamics comprising of multi-step transition dynamics and multi-step
cumulated rewards of the original MDP. The new MDP is then stable by invoking the argument proved above.

**Convergence of TD(\(\lambda_t, \omega_t\)).** Stability is a prerequisite for full convergence of the stochastic algorithm. For full convergence, we can apply Theorem 2 from Tsitsiklis and Van Roy (1997), adapted and restated below.

**Theorem 2 from Tsitsiklis and Van Roy (1997)** Consider an iterative algorithm of the form:

\[ w_{t+1} = w_t + \alpha_t ( -A(Z_t)w_t + b(Z_t)) \]

where:

1. the (predetermined) step-size sequence \(\alpha_t\) is positive, non-increasing, and satisfies \(\sum_{t=0}^{\infty} \alpha_t = \infty\) and \(\sum_{t=0}^{\infty} \alpha_t < \infty\);
2. \(Z_t\) is a Markov process with a unique invariant distribution, and there exists a mapping \(h\) from the states of the Markov process to the positive reals, satisfying the remaining conditions. Let \(\mathbb{E}_{d_z}[\cdot]\) stand for the expectation with respect to this invariant distribution;
3. \(A(\cdot)\) and \(b(\cdot)\) are matrix and vector valued functions, respectively, for which \(A = \mathbb{E}_{d_z}[A(Z_t)]\) and \(b = \mathbb{E}_{d_z}[b(Z_t)]\) are well-defined and finite;
4. the matrix \(A\) is positive definite;
5. there exist constants \(C\) and \(q\) such that for all \(Z\):

\[
\sum_{t=0}^{\infty} ||\mathbb{E}[A(Z_t)|Z_0 = Z] - A|| \leq C(1 + h^q(Z)) \quad \text{and} \quad \\
\sum_{t=0}^{\infty} ||\mathbb{E}[b(Z_t)|Z_0 = Z] - b|| \leq C(1 + h^q(Z)) ;
\]
6. for any \(q > 1\) there exists a constant \(\mu_q\) such that for all \(Z, t\)

\[ \mathbb{E}[h^q(Z_t)|Z_0 = Z] \leq \mu_q(1 + h^q(Z)). \]

Then, \(w_t\) converges to \(w^\ast\), with probability \(1\), where \(w^\ast\) is the unique vector that satisfies \(Aw^\ast = b\).

The assumptions of Theorem 2 hold in our case since the last two remaining conditions (v) and (vi), stating that the dependence of \(A(Z_t)\) and \(b(Z_t)\) on \(Z_k, \forall k \leq t\) is exponentially decreasing, are satisfied by the fact that the trace iterates have bounded variance and the fact that \(\{Z_t\}_{t=0}^{\infty}\) is Markov, cf. Assumption 3, Tsitsiklis and Van Roy, 1997. The first follows from the fact that the range of \(\omega\) and \(\lambda\) is \([0, 1]\), and the second by definition of the trace.

**Fixed point.** We now examine the fixed point of the system:

\[ \mathbb{E}[\delta_t \tilde{e}_t] = \mathbb{E}[\tilde{e}_t (R_{t+1} + y_t x^T_{t+1} w - x^T_t w)] = 0. \]

Unfolding the trace, we have:

\[
\tilde{e}_t = y_t \lambda_t \tilde{e}_{t-1} + \omega_t x_t \\
= \omega_t x_t + y_t \lambda_t (y_{t-1} \lambda_{t-1} \tilde{e}_{t-2} + \omega_{t-1} x_{t-1})
\]

\[
= \sum_{k=0}^{t} \prod_{j=0}^{k-1} y_{t-j} \lambda_{t-j} \omega_{t-k} x_{t-k}
\]
The 5-state chain MDP (cf. Sutton et al. (2016)): The states shown as square cause soft termination, with \( \gamma(s) = 0 \), but they do not restart the agent. There are two actions, “left” and “right”, which deterministically cause transitions to the left or right except at the ends of the chain, where there may be a self-transition. The reward on all transitions is \( \lambda_1 \). The behaviour policy selects “left” 2/3rds of the time in all states, which causes more time to be spent in states on the left than on the right. The true value function \( V_x \) is depicted above each state. The notation \( w_i \) describes state aliasing in the observations, denoting that the \( i \)-th component of the current parameter vector is \( w_i \). Since there are five states and only three parameters, it is impossible, to represent the true \( V_x \) exactly.

which results in:

\[
 w^* = E \left( \sum_{k=0}^{t} y_{t-k}^{(k)} \lambda_t^{(k)} \omega_{t-k} x_{t-k} \right) (y_{t+1} x_{t+1} - x_t)_{\top}^{-1} \left( \sum_{k=0}^{t} y_{t-k}^{(k)} \lambda_t^{(k)} \omega_{t-k} x_{t-k} \right) R_{t+1} \right)
\]

with \( y_t^{(k)} = \prod_{j=t+1}^{t+k} y_j \), \( \lambda_t^{(k)} = \prod_{j=t+1}^{t+k} \lambda_j \).

C. Details on empirical illustrations

C.1. Weightings for off-policy distribution correction

Experimental setup We used the 5-state MRP denoted in Fig. 8 to illustrate the following policy-evaluation algorithms: Off-policy TD, ETD, X(1)-ETD, and X(0)-ETD, described next.

Algorithms & hyperparameters The Off-policy TD algorithm is the canonical off-policy evaluation algorithm Precup et al. (2001), without any prior distribution corrections in the form of stationary distribution ratios between the target policy and the behaviour policy; it only uses importance sampling ratios \( \rho \).

The ETD algorithm is the one introduced in Sutton et al. (2016).

For the two emphatic algorithms that use expected traces, we differentiate with \( \eta_f \) (cf. Appendix B.3) the following instances of the generic algorithm \( X(\eta_f) \)-ETD:

- X(1)-ETD – learns the follow-on trace with Monte-Carlo regression on the follow-on trace, cf. Eq. (22) with \( \eta_f = 1 \);
- X(0)-ETD – learns the follow-on by backward TD, i.e. the follow-on trace model bootstraps on itself, cf. Eq. (22) with \( \eta_f = 0 \).

All the algorithms are compared for \( \lambda = 0 \). We use no other internal discount factor beside the chain’s own termination function \( \gamma \). The step-sizes for the value function, for all algorithms, are decayed with \( 1/t^d \), where \( t \) is the time-step, and \( d \) is chosen from \{0.4, 0.5, 0.8, 0.9\}, with the best values obtained...
through hyperparameter search: (i) Off-policy TD: $d = 0.5$, (ii) ETD: $d = 0.9$, (iii) X(1)-ETD: $d = 0.9$, (iv) X(0)-ETD: $d = 0.9$.

For the two algorithms that use expected emphasis, learning rates for the model $\hat{f}_\phi$ that estimates the follow-on trace $\hat{r}$, use the same step-size decay schedule $1/t$, with values for $d_f$ chosen from the same interval, and the best value obtained for both algorithms $d_f = 0.5$. The model is a linear function approximator on the observations, followed by a ReLU non-linearity to keep the output positive (the latter is not important, as similar results can be obtained without it).

C.2. Weightings for on-policy distribution correction – Atari Ms.Pac-Man experiments

For our deep reinforcement learning experiments on Atari games, we used the canonical Ms.Pac-Man to illustrate the importance of the connection between selectivity $\omega$, trace-decay $\lambda$ and trace-bootstrapping $\eta$. We start by describing the experimental setup, followed by the online selective $Q(\lambda, \omega)$ algorithms, after which we move on to the expected traces versions, namely $QET(\lambda, \eta, \omega)$.

**Experimental setup** All the Atari experiments were run with the ALE Bellemare et al. (2012), similarly to van Hasselt et al. (2020), including using action repeats (4x), but without downsampling (or framestacking, or pooling) the observation stream. With probability $\epsilon$, the agent's observation is replaced with random standard Gaussian noise, to mimic a noisy observation sensor. To simulate access to a module that detects such noisy observations, we provide access to a time-dependent interest $i_t$, capturing whether an observation is noisy or not, s.t. $i_t = 0$ if the observation at time step $t$ is noisy, and $i_t = 1$ otherwise. We use no other corrections, e.g. corrections to the discrepancy between the behaviour $\epsilon$-greedy and the target greedy policy, as we found those to not be useful in this setting, resulting in $\omega_t = i_t$.

We keep the discount factor $\gamma = 0.99$ constant. We report the mean return while training on 20M frames.

In all cases, we used $\epsilon$-greedy exploration (cf. Sutton and Barto (2018)), with an $\epsilon = 1e - 2$, which we do not decay.

We use a similar setting to van Hasselt et al. (2020), with the following modifications. We apply a different feature extraction network, cf. Hessel et al. (2019); particularly, we use 2 convolutional layers, with $5 \times 5$ kernels, stride 5, and 64 channels, followed by ReLU nonlinearities. The output is then passed through a 512 linear layer, with ReLU activation.

These experiments were conducted using Jax Bradbury et al. (2018), Haiku Hennigan et al. (2020) and Optax Hessel et al. (2020).

**Algorithms evaluated** We start by describing the baseline algorithm $Q(\lambda)$, followed by the algorithms that specifically apply selectivity.

$Q(\lambda)$ In the algorithm 1, we have $\eta_t = 1\nu t$, i.e. we only use accumulating traces, so we omit line 5, since this algorithm does not use the expected traces $\tilde{z}_\theta$. For each transition, we first decay the trace $\hat{e}$ and then update it using line 6. We further compute the finite-horizon one-step return $R^\lambda$ (line 7), where $\gamma = 0$ on termination (and then $S'$ is the first observation of the next episode). Instead of the usual SGD algorithm illustrated for simplicity in line 10, for training, we use ADAM Kingma and Ba (2015); we learn the value function with momentum= 0.9, which simulates soft-batching; we set the other parameters from ADAM to $b_1 = 0.9$ (the exponential decay rate to track the first moment of past gradients) and $b_2 = 0.9999$ (the exponential decay rate to track the second moment
of past gradients), \(\epsilon = 1e-4\) (the small constant applied to denominator outside the square root–as in Kingma and Ba (2015), to avoid dividing by zero when rescaling). We use a step-size of \(\alpha^w = 1e-5\) for learning the value function.

**Variations of \(Q(\lambda, \omega)\)** The algorithms used in the experiments are all variations of \(Q(\lambda, \omega)\). We label \(Q(\lambda)\), the default baseline algorithm that uses \(\lambda = 0.9\) and uniform weightings over the state space: \(\omega = \omega_1 = 1, \forall t\). We use \(Q(\lambda, \omega_t)\) (with the “\(t\)” subscript denoting state or time-dependence) for the algorithm that uses \(\lambda = 0.9\), but uses the ground-truth interest to set the weighting \(\omega_t\) (1 for non-noisy states, and 0 otherwise). Lastly, \(Q(\lambda_t, \omega_t)\) denotes the algorithm using, in addition to the ground truth interest in setting \(\omega_t\), also Eq. (12) to set \(\lambda_t\).

**QET(\(\lambda, \eta, \omega)\)** The expected-traces algorithm is similar to \(Q(\lambda, \omega)\), except now we use the expected traces \(\hat{z}_s\) in place of the instantaneous traces, so we update the parameters \(\Theta\) as well, in addition to \(w\), using ADAM, with the same hyperparameters as for the value function: momentum, \(b_1, b_2\), and \(\epsilon\). We use the step size \(\alpha^\Theta = 1e-2\) for learning the expected traces.

Similarly to van Hasselt et al. (2020), we also split the computation of \(Q(s, a)\) into two separate parts, such that \(Q(\omega, \xi)(s, a) = \omega'_s x_\xi(s)\). This separation is just so that we can keep labeled separate subsets of parameters as \((\omega, \xi)\) rather than merging all of them into a single vector \(\omega\), using \(x(s)\) to denote the last hidden layer of the feature extraction part of the network, on top of which the last linear layer of the q-function is applied. We keep separate traces for these subsets, and we just apply accumulating instantaneous traces to the feature extraction network, similarly to van Hasselt et al. (2020). This separation is equivalent to keeping one big trace for the combined set. We refer the reader to van Hasselt et al. (2020) for more details on this particularity. The motivation for this split in parameters is to avoid learning an expected trace for the full trace, which has millions of elements. Instead, in practice, we only learn expectations for traces corresponding to the last layer.

The difference from how this algorithm is presented in van Hasselt et al. (2020) is that we only condition the function \(z_\Theta(s)\) on the state, and not the action, due to our new derivation of the algorithm in Appendix B.1.

As customary, we do not backpropagate the gradient coming from the expected traces’ loss further into the feature representation.

**Variations of QET(\(\lambda, \eta, \omega)\)** We now describe the variations of QET compared in the experiment. For the first versions of the algorithms, we use \(\eta = 0, \forall t\), i.e. using expected traces everywhere, instead of the instantaneous counterparts. The algorithms QET(\(\lambda, \eta\))-baseline, QET(\(\eta, \lambda, \omega_t\)), QET(\(\eta, \lambda_t, \omega_t\))
are analogous to $Q(\lambda)$-baseline, $Q(\lambda, \omega)$, and $QET(\lambda, \omega)$, respectively, which were described in the previous section. The algorithm $QET(\eta, \lambda, \omega)$, in addition to using Eq. (12) for setting $\lambda$, and the ground truth interest for setting $\omega$, also uses Eq. (16) to set $\eta$, i.e. it uses the expected trace more in states where the selectivity weighting $\omega$ is higher, and the instantaneous traces more when the weighting is lower. Lastly, $QET(\eta, \lambda, \omega, \Delta^z)$, in addition, uses the coupling Eq. (17) to learn expected traces constrained by selectivity.

**C.3. Off-policy counterfactual evaluation**

**Discussion on learning with function approximation** With function approximation, learning expected traces off-policy can be problematic. Particularly, one can learn selective expected eligibility traces $\tilde{e}_t^\theta$ using Monte-Carlo methods, by regressing on $\gamma_t \rho_t \lambda_t \tilde{e}_{t-1} + \omega_t \nabla V_w(S_t)$, but the traces can have very high variance resulting from the product of importance sampling ratios. On the other hand, learning the traces with backward TD can easily diverge due to “off-policiness”, from the same reasons the value learning process can diverge. We can interpolate between Monte-Carlo methods and backward TD using selective mixture traces, similarly to regular mixture traces, using a different mixture parameter $\tilde{\eta}$, s.t.:

\[
\tilde{e}_t^\theta = (1 - \tilde{\eta})\tilde{e}_t^\theta(S_t) + \tilde{\eta}(\rho_t \gamma_t \lambda_t \tilde{e}_{t-1} + \omega_t \nabla V_w(S_t)) \tag{34}
\]

and updating $\Theta$ with

\[
\Delta_t^z \equiv \frac{\partial \tilde{e}_t^\theta(S_t)}{\partial \Theta_t} \left( \rho_t \gamma_t \lambda_t \tilde{e}_{t-1} + \omega_t \nabla V_w(S_t) - \tilde{z}_\theta(S_t) \right),
\]

\[
\Theta_{t+1} = \Theta_t + \alpha_t^\theta \Delta_t^z,
\]

where $\alpha_t^\theta \in (0, 1)$ is a step-size parameter and $\tilde{\eta}$ is analogous to $\lambda$, but for learning selective traces here, rather than values, using TD methods when $\tilde{\eta} = 0$, and Monte-Carlo regression when $\tilde{\eta} = 1$.

To stabilize learning, we could choose an intermediary value for $\tilde{\eta}$, that achieves an optimal balance between bias and variance. Moreover, we can also stabilize the learning process by instantiating $\omega$ using emphatic weightings, which can guarantee convergence for linear function approximation.

We now discuss the details regarding the empirical illustration in the Open world gridworld domain we used. This illustration is meant to illustrate the learning process, so we do not use function approximation, therefore we omit any non-uniform weightings.

**Experimental setup** In Fig. 5-Left, there are two goals depicted with “G”, giving the same reward of $10/\epsilon_r$, with probability $\epsilon_r$, otherwise 0. All other rewards are 0. When the agent reaches the goal, the episode restarts with a random initial location sampled from the state space.

The agent learns different expected traces for the two policies transitioning it to each of the two corners of the world: top-right and bottom-left (we may call those loosely options), while following a random behaviour policy $\mu$.

To increase the difficulty in learning, we let each option’s policy take a random action with probability $\epsilon_o = 0.2$, and let the environment transition the agent randomly in a cardinal direction with probability $\epsilon_p = 0.05$. We use one-hot state representations, so we omit emphatic weightings, and just use importance sampling ratios between each option’s policy $\pi_0$ and the behaviour policy $\mu$, when learning the expected trace of each option.
Figure 9 | Open World: Value error for policy evaluation using TD($\lambda = 0.98$) and ET($\lambda = 0.98$), for option $o_2$. Increasing the sparsity of the reward signal (left vs right plot), we observe policy-conditioned expected traces coped better with sparse rewards. Shaded areas show standard errors over 20 seeds.

**Algorithms & hyperparameters** We compare the off-policy TD($\lambda$) and ET($\lambda, \eta$) with uniform weightings. We clip the importance sampling ratios to 1. We learn the traces with Monte-Carlo regression, using $\tilde{\eta} = 1$ in Eq. (34). The discount is $\gamma = 0.99$, except at the goals, where it is 0. We set $\lambda = 0.98$. We decay step-sizes with $1/t^d$, with $d \in \{0.5, 0.7, 0.9, 1.\}$, the best values obtained through hyperparameter search: $d = 0.5$, for $\epsilon = 1.$, and $d = 0.9$, for $\epsilon = 0.001$. For the expected traces algorithm, we similarly decay the step size for the trace learning process, using $d_\eta = 0.001$, searched over values $\{0.1, 0.01, 0.001\}$. We learn both the value function and the traces with SGD, starting from a step size of 1.

**Additional results** Fig. 9 illustrates the performance for the option with policy transitioning toward the bottom corner—$o_2$ (with similar results illustrated for the other policy—$o_1$, in the main text).

**C.4. Sparse expected eligibility traces**

To illustrate the idea of sparse expected traces that can flow credit over the state space using temporally-extended actions, we use the following experimental setup.

**Experimental setup** We use the Four Rooms domain from figure Fig. 5-Right. We pre-learn the option policies using pre-defined interest and sub-goal functions, such that the options learn policies as illustrated in Fig. 5-Right. The agent receives a reward of $r = 10.$ with probability $\epsilon$ at the goal, depicted with “G” in Fig. 5-Right. The discount is $\gamma = 0.98$ everywhere, except at the goal where it is 0, after which the episode restarts and the first observation of the new episode is sampled from one of the hallways.

**Algorithms & hyperparameters** The baselines we compare against are the default $Q_A(\lambda = 0.9)$ and $QET_A(\lambda = 0.9, \eta = 0)$, which are defined over the primitive action-space.

For the sparse learning algorithms, we assume the policy over actions is defined using a pre-specified weighting function $\omega$, s.t. $\omega(s) = 1$, if $s$ is a hallway, and 0 otherwise. The policy over options uses only the pre-learned option space, without any primitive actions, which is sufficient for this illustration since the start states are in hallway locations, and so is the goal.

For $Q_O(\lambda_t, \omega_t)$ and $QET_A(\lambda_t, \eta_t, \omega_t)$ (with the “t” subscript indicating state or time-dependence), we use Eq. (12) for setting $\lambda_t$, with $\beta_\lambda = 0.9$, and similarly Eq. (16) for setting $\eta_t$, with $\beta_\eta = 0.9$. We
also learn the traces themselves using Eq. (17). We decay all learning rates for the value function using $1/t^d$, searching over $d \in \{0.1, 0.2, 0.5, 0.7, 0.9, 1.\}$. We find the best parameters are $d = 0.2$ for all algorithms. For the expected traces algorithms, we use the same schedule $1/t^{d_z}$, with $d_z \in \{1., 0.1, 0.01, 0.01\}$, finding the best value is $d_z = 0.1$, for both algorithms. We learn both the value function and the traces with SGD, starting from a step size of 1. For exploration we use $\epsilon$-greedy policies with $\epsilon = 0.1$. The options’ policies are learned with intra-option discount factor $\gamma_O = 0.9$ and without randomness $\epsilon_O = 0.$, i.e. greedy policies. The trace learning process uses Monte-Carlo regression, i.e. $\tilde{\eta} = 1$ in Eq. (34). We do not use importance sampling ratios, or any other corrections.