Explicit Relation Between Two Resolvent Matrices of the Truncated Hausdorff Matrix Moment Problem

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Abstract
We consider the explicit relation between two resolvent matrices related to the truncated Hausdorff matrix moment problem (THMM) in the case of an even and odd number of moments. This relation is described with the help of four families of orthogonal matrix polynomials on the finite interval \([a, b]\) and their associated second kind polynomials.

Keywords Hausdorff matrix moment problem · Resolvent matrix · Orthogonal matrix polynomials

Mathematics Subject Classification 30E05 · 42C05 · 47A56

1 Introduction
In the classical scalar Hamburger moment problem, a real sequence \((s_j)_{j=0}^{\infty}\) is given and the problem is to find a Borel measure \(\sigma\) such that

\[ s_j = \int_{\mathbb{R}} t^j \sigma(dt), \quad j \in \mathbb{N}_0. \]

The problem is called determine if there is exactly one solution \(\sigma\); it is called indeterminate if there is more than one solution. In the indeterminate case, all solutions

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can be described by a $2 \times 2$ Nevanlinna matrix [1, Definition 2.4.3] whose elements are transcendental functions, see [1, Sec. 3.4, p. 113].

In this paper we are concerned with the truncated Hausdorff matrix moment (THMM) problem. Let $a < b$ be real numbers, $q, m \in \mathbb{N}$ and let $(s_j)_{j=0}^m$ be hermitian $q \times q$ matrices. The truncated Hausdorff matrix moment on $[a, b]$ problem consists in finding all positive matrix measures $\sigma$ whose $j$th moment is equal to $s_j$, that is,

$$s_j = \int_a^b t^j \sigma(dt), \quad j = 0, \ldots, m.$$  

We denote the set of all solutions $\sigma$ of the THMM by

$$\mathcal{M}_q^q[[a, b]; (s_j)_{j=0}^m]$$  \hfill (1)

The Stieltjes transform

$$s(z) = \int_a^b \frac{1}{t - z} \sigma(dt)$$

shows that the set (1) is equivalent to the so-called associated solution set

$$\mathcal{G}_q^q[[a, b]; (s_j)_{j=0}^m] := \left\{ s(z) = \int_{[a,b]} \frac{d\sigma(t)}{t - z}, \sigma \in \mathcal{M}_q^q[[a, b], \mathcal{B} \cap [a, b]; (s_j)_{j=0}^m] \right\}.$$  \hfill (2)

As in the scalar case, it can be shown that the set (2) is given by all functions $s$ of the form

$$s(z) = \left[ \alpha^{(m)}(z) p(z) + \beta^{(m)}(z) q(z) \right] \left[ \gamma^{(m)}(z) p(z) + \delta^{(m)}(z) q(z) \right]^{-1},$$  \hfill (3)

where $p$ and $q$ are $q \times q$ matrix-valued functions of $z$ which are meromorphic in $\mathbb{C} \setminus [a, b]$ and satisfy certain positivity conditions, see for instance [14–16]. The coefficient matrix

$$\begin{pmatrix} \alpha^{(m)}(z) & \beta^{(m)}(z) \\ \gamma^{(m)}(z) & \delta^{(m)}(z) \end{pmatrix}$$  \hfill (4)

is called a resolvent matrix of the THMM problem. Note that this matrix is not unique. Indeed, the first resolvent matrix was found in 2001 in [14, Equality (10) and Equality (30)] under the assumption that certain matrices constructed from the given moments are strictly positive. We will call it the resolvent matrix with respect to the point 0. Later, in 2006 and 2007, other resolvent matrices were found in [15, Equality (6.20)] for even $m$ and in [16, Equalities (6.54) through (6.57)] for odd $m$. We will call these matrices the resolvent matrices with respect to the point $a$, see (99) and (129). The papers [15] (2006), Equality (6.20) and [16] (2007), Equalities (6.54) through (6.57) contain
necessary and sufficient conditions for the THMM problem to be indeterminate. In these three works the entries of the resolvent matrices are polynomials in \( z \). In 2015, the resolvent matrices from \([15, 16]\) were expressed in terms of four families of orthogonal polynomials and their associated second kind polynomials in \([7, \text{Equalities (3.24)-odd, (3.37)-even}]\).

We remark that the scalar version of the Hausdorff moment problem was studied by Krein and Nudel’man in their book \([36, \text{Page 115}]\).

**Matrices of Moments**

Let \((s_j^m)_{j=0}^m\) be a sequence of hermitian \( q \times q \) matrices and let \( \sigma \in \mathcal{M}^q_{\geq \mathbb{B} \cap [a, b]; (s_j^m)_{j=0}^m} \) be a solution of the THMM problem. We will need the following perturbed measures defined on Borel sets \( B \) by

\[
\sigma_2(B) := \int_B (b - t)(t - a) \ d\sigma(t), \\
\sigma_3(B) := \int_B (b - t) \ d\sigma(t), \\
\sigma_4(B) := \int_B (t - a) \ d\sigma(t)
\]

and their corresponding sequences of moments \((s_j^{(r)})_{j=0}^m, r = 1, 2, 3, 4.\) Clearly they satisfy

\[
\begin{align*}
s_j^{(1)} &= s_j, \\
s_j^{(2)} &= -abs_j + (a + b)s_{j+1} - s_{j+2}, \\
s_j^{(3)} &= bs_j - s_{j+1}, \\
s_j^{(4)} &= -as_j + s_{j+1}.
\end{align*}
\]

Note that \(s_j^{(2)} = bs_j^{(4)} - s_{j+1}^{(4)} = -as_j^{(3)} + s_{j+1}^{(3)}\).

For \( r = 1, 2, 3, 4, \) we define the block Hankel matrices \(H_{r,j}\) by

\[
H_{r,j} := (s_j^{(r)})_{k+l}^{j} = \begin{pmatrix}
s_0^{(r)} & s_1^{(r)} & \cdots & s_j^{(r)} \\
\vdots & \vdots & \ddots & \vdots \\
s_j^{(r)} & s_{j+1}^{(r)} & \cdots & s_{2j}^{(r)}
\end{pmatrix}.
\]

In the case of an odd number of moments, i.e., if \(m = 2n\), the THMM has a solution if the block matrices \(H_{1,n}\) and \(H_{2,n-1}\) are both nonnegative. See \([16, \text{Theorem 1.3}]\). In the case of an even number of moments, i.e., if \(m = 2n + 1\), the THMM has a solution if the block matrices \(H_{3,n}\) and \(H_{4,n}\) are both nonnegative. See \([15, \text{Theorem 1.3}]\).
We will also need the following matrices.

\[
\tilde{H}_{r,j} := (s_{k+\ell+1}^{(r)})_{k,\ell=0}^{j} = \begin{pmatrix}
\begin{array}{ccc}
s_1^{(r)} & s_2^{(r)} & \cdots & s_{j+1}^{(r)} \\
s_2^{(r)} & s_3^{(r)} & \cdots & s_{j+2}^{(r)} \\
\vdots & \vdots & \ddots & \vdots \\
s_{j+1}^{(r)} & s_{j+2}^{(r)} & \cdots & s_{2j+1}^{(r)}
\end{array}
\end{pmatrix}
\]

obtained from \( H_{r,j+1} \) by deleting the first row and the last column (or alternatively, by deleting the first column and the last row).

**Definition 1.1** Let \([a, b]\) be a finite interval on the real axis \(\mathbb{R}\). The sequence of \(q \times q\) hermitian matrices \((s_k)_{k=0}^{2n}\) (resp. \((s_k)_{k=0}^{2n+1}\)) is called a Hausdorff positive definite sequence on \([a, b]\) if the block Hankel matrices \(H_{1,n}\) and \(H_{2,n-1}\) (resp. \(H_{3,n}\) and \(H_{4,n}\)) are both positive definite matrices. If the sequence \((s_k)_{k=0}^{m}\) is a positive definite sequence, then the THMM problem is called non degenerate.

In the present work, we deal with the non degenerate case. In this case, it can be shown that a holomorphic function \(s\) defined on the upper half plane is an associated solution of the THMM if and only if certain block operator matrices involving \(s\) and the given moments are nonnegative, see [14–16]. Factorization of these matrices shows that this is the case if and only if \(s\) is of the form (3). From (3), we clearly see that we obtain the set all of solutions of the THMM problem if and only if we know the so-called resolvent matrix (4) together with the allowed pairs \(p\) and \(q\). See [15, Definition 5.2] and [16, Definition 5.2]. Different representations of the resolvent matrix are available.

**Resolvent Matrix with Respect to the Point 0**

The techniques employed in [14] and in [15, 16] show that the resolvent matrix can be written as (97) for an even number of moments and as (128) for an odd number of moments of the resolvent matrix. We call them the resolvent matrices with respect to the point 0 and we will denote them by \(V^{(m)}\). Its entries \(\alpha^{(m)}(z), \beta^{(m)}(z), \gamma^{(m)}(z), \) and \(\delta^{(m)}(z)\) are matrix valued polynomial functions of \(z\) which are uniquely constructed from the input set of moments \((s_j)_{j=0}^{m}\).

Its expansion in powers of \(z\) is

\[
V^{(2n+1)} = \tilde{A}_0 + z\tilde{A}_1 + \cdots + z^{n+1}\tilde{A}_{n+1} + z^{n+2}\tilde{A}_{n+2}, \quad \text{if } m = 2n + 1, \quad (7)
\]

\[
V^{(2n)} = \tilde{B}_0 + z\tilde{B}_1 + \cdots + z^n\tilde{B}_n + z^{n+1}\tilde{B}_{n+1}, \quad \text{if } m = 2n. \quad (8)
\]

The coefficients \(\tilde{A}_j\) and \(\tilde{B}_j\) are rational functions of \(a\) and \(b\), see Remarks A.1 and A.2 for explicit formulas.
Resolvent Matrix with Respect to the Point $a$

In [7] a different approach was used to describe the resolvent matrix. The first author used the resolvent matrix from [4] and expressed it in terms of orthogonal polynomials of the first and the second kind. This leads to the resolvent matrix (99) in the case of an even number of moments and to (129) in the case of an odd number of moments denoted by $U^{(m)}$. We call them the resolvent matrices with respect to the point $a$.

For $m = 2n + 1$ the resolvent matrix $U^{(2n+1)}$ can be written as

$$U^{(2n+1)}(z) = \tilde{C}_0 + (z - a)\tilde{C}_1 + \cdots + (z - a)^{n+1}\tilde{C}_{n+1} + (z - a)^{n+2}\tilde{C}_{n+2}. \quad (9)$$

The coefficients $\tilde{C}_j$ for $0 \leq j \leq n+2$ are rational functions of $a$ and $b$; see Remark A.3 for explicit formulas.

For $m = 2n$ the resolvent matrix $U^{(2n)}$ can be expressed in the following form

$$U^{(2n)}(z) = \tilde{D}_0 + (z - a)\tilde{D}_1 + \cdots + (z - a)^{n+1}\tilde{D}_{n+1}. \quad (10)$$

The coefficients $\tilde{D}_j$ for $0 \leq j \leq n+1$ are rational functions of $a$ and $b$; see Remark A.4 for explicit formulas.

The particular representation of the resolvent matrix of the THMM problem plays a crucial role in its factorization. Different factorizations of the mentioned resolvent matrices lead to four families of orthogonal matrix polynomials, the Dyukarev-Stieltjes parameters, continued fractions and the coefficients of the three term recurrence relation. Consequently, the explicit relation between the resolvent matrices given in [7, 14] is relevant.

To the best of the knowledge of the authors, the matrix moment problem was first studied in [35, 37]. In [21, 22, 24–27, 32] orthogonal matrix polynomials were considered to solve matrix moment problems or structural formulas and differential relations. In [23, 28–30] V.P. Potapov’s method of matrix inequalities was used to solve interpolation problems in certain classes of functions. In [31], the THMM problem was recently studied via a Schur-Nevanlinna type algorithm. The operator approach was applied to solve the THMM problem in [20, 39].

Main Result of this Work

In this work we give an explicit relation between the two different types of resolvent matrices in the form

$$U^{(m)}(z) = CV^{(m)}(z)D.$$
the point of view of applications, a relation between two resolvent matrices would allow to rewrite and develop the representation of the set of admissible controls for bounded control systems; see [11, 17, 18].

In our work we have to distinguish the cases when \( m \) is even or odd. We will present various explicit formulas and identities which exhibit interactions between the matrices corresponding to odd and even values of \( m \); see Sect. 3.

2 Matrices of Moments and Orthogonal Matrix Polynomials

In this section, we reproduce some notation from [13] that appear throughout this work. In particular, we recall the definition of the orthogonal matrix polynomials (OMP) \( P_{k,j} \) on \([a, b]\) and their second kind polynomials \( Q_{k,j} \).

2.1 Shifts and Truncations

We start with some auxiliary matrices which do not depend on the moments. Let \( q \in \mathbb{N} \) and \( \mathbb{M}_q(k \times \ell) \) be the set of all \( k \times \ell \) block matrices whose entries are \( q \times q \) matrices. We define the “block down shift” on the vector space \( \mathbb{C}_q^{j+1} \)

\[
T_0 := 0, \quad T_j := \begin{pmatrix}
0 & 0 \\
I & 0 \\
0 & I & 0
\end{pmatrix} \in \mathbb{M}_q((j+1) \times (j+1)), \quad j \geq 1.
\] (11)

Note that all entries in \( T_j \) are \( q \times q \) matrices.

Clearly, every \( T_j \) is nilpotent, and therefore the matrix valued function

\[
R_j : \mathbb{C} \rightarrow \mathbb{M}_q((j+1) \times (j+1)), \quad R_j(z) := (I_{(j+1)q} - zT_j)^{-1}, \quad j \geq 0,
\] (12)

is well-defined and satisfies \( R_j(z) = \sum_{\ell=0}^{j} z^\ell T^\ell \).

We will also make use of the following \( (j+1) \times j \) block matrices

\[
L_{1,j} := \begin{pmatrix}
0 & \cdots & 0 \\
I_{j \times jq} & \cdots & 0 \\
0 & \cdots & I
\end{pmatrix},
\]

\[
L_{2,j} := \begin{pmatrix}
I_{j \times jq} & 0 \\
0 & \cdots & I \\
0 & \cdots & 0
\end{pmatrix} \in \mathbb{M}_q((j+1) \times j)
\] (13)
and the block vectors
\[ v_0 := I, \quad v_j := L_{2,j-1}v_{j-1} = \begin{pmatrix} I \\ 0_{jq \times q} \end{pmatrix}, \quad j \geq 1. \quad (14) \]

Clearly, the following relations hold.

\[
\begin{align*}
L_{1,j} \begin{pmatrix} a_1 \\ \vdots \\ a_j \end{pmatrix} &= \begin{pmatrix} 0 \\ a_1 \\ \vdots \\ a_j \end{pmatrix}, \\
L_{2,j} \begin{pmatrix} a_1 \\ \vdots \\ a_j \end{pmatrix} &= \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ 0 \end{pmatrix}, \\
T_j \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_j \end{pmatrix} &= \begin{pmatrix} 0 \\ a_0 \\ \vdots \\ a_{j-1} \end{pmatrix}, \\
L_{1,j}^* \begin{pmatrix} a_0 \\ \vdots \\ a_j \end{pmatrix} &= \begin{pmatrix} a_1 \\ \vdots \\ a_j \end{pmatrix}, \\
L_{2,j}^* \begin{pmatrix} a_0 \\ \vdots \\ a_j \end{pmatrix} &= \begin{pmatrix} a_0 \\ \vdots \\ a_{j-1} \end{pmatrix}, \\
T_j^* \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_j \end{pmatrix} &= \begin{pmatrix} a_1 \\ \vdots \\ a_j \end{pmatrix},
\end{align*}
\]

where the \( a_\ell \) can be \( q \times q \) matrices or row vectors consisting of \( q \times q \) matrices. By transposition, we obtain

\[
(a_0, \ldots, a_j)L_{1,j} = (a_1, \ldots, a_j), \quad (a_0, \ldots, a_{j-1})L_{1,j}^* = (0, a_0, \ldots, a_{j-1}), \\
(a_0, \ldots, a_j)L_{2,j} = (a_0, \ldots, a_{j-1}), \quad (a_0, \ldots, a_{j-1})L_{2,j}^* = (a_0, \ldots, a_{j-1}, 0), \\
(a_0, \ldots, a_j)T_j = (a_1, \ldots, a_j, 0), \quad (a_0, \ldots, a_j)T_j^* = (0, a_0, \ldots, a_{j-1}).
\]

It follows that

\[
\begin{align*}
L_{1,j}^*L_{1,j} &= L_{2,j}^*L_{2,j} = I, \quad (15) \\
L_{1,j}L_{2,j}^* &= T_j, \quad L_{2,j}L_{1,j}^* = T_{j-1}. \quad (16)
\end{align*}
\]

If we multiply the first equality in (16) by \( L_{1,j}^* \) from the left and by \( L_{2,j} \) from the right, we obtain with (15)

\[
L_{1,j}^*T_j = L_{2,j}^*, \quad T_jL_{2,j} = L_{1,j}, \quad L_{1,j}^*T_jL_{2,j} = I. \quad (17)
\]

If we multiply the first equality in (17) by \( L_{1,j} \) from the right and the second equality by \( L_{2,j}^* \) from the left, we obtain with (16)

\[
L_{1,j}^*T_jL_{1,j} = T_{j-1}, \quad L_{2,j}^*T_jL_{2,j} = T_{j-1}. \quad (18)
\]

If we multiply the first equality in (16) by \( L_{1,j} \) from the right and the second equality from the left by \( L_{1,j} \), we obtain

\[
T_jL_{1,j} = L_{1,j}L_{2,j}^*L_{1,j} = L_{1,j}T_{j-1}. \quad (19)
\]
Similarly, using the transposed equations in (16) we obtain
\[ T_j^* L_{2,j} = L_{2,j} L_{1,j}^* L_{2,j} = L_{2,j} T_{j-1}^*, \]  
(20)

hence also, for every \( z \in \mathbb{C} \),
\[ R_j(z) L_{1,j} = L_{1,j} R_{j-1}(z), \]  
(21)
\[ R_j(z)^* L_{2,j} = L_{2,j} R_{j-1}(z)^*. \]  
(22)

2.2 Vectors and Matrices Involving the Moments

Let \((s_k)_{k=0}^m\) be a Hausdorff positive definite sequence, see Definition 1.1. Hence, by definition, \(H_{1,n}\) and \(H_{2,n}\) are positive definite if \(m = 2n\) and \(H_{3,n}\) and \(H_{4,n}\) are positive definite if \(m = 2n + 1\). Then the matrices \(H_{1,n-1}\), \(H_{2,n-1}\) respectively \(H_{3,n-1}\), \(H_{3,n-1}\) are also strictly positive, in particular they are invertible. If we set
\[ Y_{r,j} := \begin{pmatrix} s_{(r)}^{(r)} & \vdots & s_{2j-1}^{(r)} \end{pmatrix}, \quad 1 \leq j \leq n \text{ if } r = 1, 3, 4 \text{ and } 1 \leq j \leq n - 1 \text{ if } r = 2, \]  
(23)
we can write \(H_{r,j}\) as block matrix and obtain the following factorization
\[ H_{r,j} = \begin{pmatrix} H_{r,j-1} & Y_{r,j} \\ Y_{r,j}^* & s_{2j}^{(r)} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -Y_{r,j}^* H_{r,j-1}^{-1} I \end{pmatrix} \begin{pmatrix} H_{r,j-1} & 0 \\ 0 & \hat{H}_{r,j} \end{pmatrix} \begin{pmatrix} I & 0 \\ -H_{r,j-1}^{-1} Y_{r,j} \end{pmatrix}, \]  
(24)
where
\[ \hat{H}_{r,j} := s_{2j}^{(r)} - Y_{r,j}^* H_{r,j-1}^{-1} Y_{r,j} \]  
(25)
is the so-called Schur complement of the block \(s_{2j}^{(r)}\). It has been used in [22, Equalities 2.34 and 2.35] for \(a = 0\) and \(b = 1\).

Remark 2.1 Recall that \(H_{r,j}\) is strictly positive. Since the matrices on the left and right in the above factorization are adjoint to each other, \(\hat{H}_{r,j}\) is strictly positive too.

Solving for the diagonal matrix in (24) gives
\[ \begin{pmatrix} H_{r,j-1} & 0 \\ 0 & \hat{H}_{r,j} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -Y_{r,j}^* H_{r,j-1}^{-1} I \end{pmatrix} H_{r,j} \begin{pmatrix} I & 0 \\ -H_{r,j-1}^{-1} Y_{r,j} \end{pmatrix}. \]  
(26)
For \( r = 1, 2, 3, 4 \) we define the \((j + 1)q \times q\) matrices
\[
\Sigma_{1,j} := \begin{pmatrix} -H_{1,j-1}^{-1}Y_{1,j} \\ I \end{pmatrix}, \quad \Sigma_{2,j} := \begin{pmatrix} -H_{2,j-1}^{-1}Y_{2,j} \\ I \end{pmatrix},
\]
\[
\Sigma_{3,j} := \begin{pmatrix} -H_{3,j-1}^{-1}Y_{3,j} \\ I \end{pmatrix}, \quad \Sigma_{4,j} := \begin{pmatrix} -H_{4,j-1}^{-1}Y_{4,j} \\ I \end{pmatrix}.
\]

**Corollary 2.2** We have that \( \hat{H}_{r,j} = (Y_{r,j}^*, s_2^{(r)}) \Sigma_{r,j} \) and
\[
\begin{align*}
T_j H_{r,j} \Sigma_{r,j} = 0 \quad \text{and} \quad L_2^* H_{r,j} \Sigma_{r,j} = 0.
\end{align*}
\]

**Proof** The first formula follows directly from (25) and the definition of the \( \Sigma_{r,j} \). The factorization (26) shows that
\[
\begin{bmatrix}
I \\
Y_{r,j}^* H_{r,j-1}^{-1} \\
\hat{H}_{r,j}
\end{bmatrix}
\begin{bmatrix}
H_{r,j-1}^{-1} \\
0 \\
\hat{H}_{r,j}
\end{bmatrix}
= H_{r,j}
\begin{bmatrix}
I \\
\Sigma_{r,j}
\end{bmatrix},
\]
hence, comparing the last column on both sides, we find that
\[
\begin{bmatrix}
0_{nq} \\
\hat{H}_{r,j}
\end{bmatrix} = H_{r,j} \Sigma_{r,j}
\]
from which (29) is an immediate consequence. \(\square\)

Now from the negative first column of \( H_{1,j} \)
\[
u_j := -\begin{pmatrix} s_0 \\ \vdots \\ s_j \end{pmatrix}
\]
we construct the block vectors \( u_{r,j} \) by
\[
u_{1,j} := T_j u_j, \quad \hat{u}_{2,j} := -L_1^* u_{j+1}(I - bT_{j+1})(I - aT_{j+1})u_{j+1},
\]
\[
u_{3,j} := -v_j u_j, \quad \nu_{4,j} := (I - aT_j)u_j
\]
for \( 0 \leq j \leq n - 1 \) for \( r = 2 \) and \( 0 \leq j \leq n \) in the other cases. Moreover, we set
\[
u_{2,0} := -(a + b)s_0 + s_1, \quad \nu_{2,j} := \hat{u}_{2,j} + zv_j s_0, \quad 1 \leq j \leq n - 1.
\]
Note that if we delete the first block entry in each of these vectors, we obtain the negative first column of the block matrix $H_{r,j-1}$. Therefore it is clear that

$$H_{r,n} - T_n \tilde{H}_{r,n} = v_n u_{r,n+1}^n L_{2n}.$$ 

It follows directly from the definition of $u_{1,j}$ and $u_j$ that

$$u_n = L_{1,n+1}^* u_{1,n+1}, \quad u_{1,n+1} = L_{1,n+1} u_n.$$

Moreover, note that

$$\tilde{u}_{2,j} = \begin{pmatrix} u_{2,0} \\ -s_0^{(2)} \\ \vdots \\ -s_{j-1}^{(2)} \end{pmatrix} = -(a + b) \begin{pmatrix} s_0 \\ \vdots \\ s_j \end{pmatrix} + ab \begin{pmatrix} 0 \\ s_0 \\ \vdots \\ s_{j-1} \end{pmatrix} + \begin{pmatrix} s_1 \\ \vdots \\ s_{j+1} \end{pmatrix} = bu_{4,j} + \begin{pmatrix} s_0^{(4)} \\ \vdots \\ s_j^{(4)} \end{pmatrix}$$

$$= (a + b)u_j - abT_n u_j + L_{1,j+1}^* u_{j+1}, \quad 1 \leq j \leq n - 1.$$ (39)

### 2.3 Orthogonal Matrix Polynomials on $[a, b]$

Let $\mathcal{P}$ be the set of matrix polynomials $P(t) = C_n t^n + \ldots + C_0$ with $q \times q$ matrix coefficients $C_n, \ldots, C_0$. We denote by $\deg P := \sup\{j \in \mathbb{N} \cup \{0\} : C_j \neq 0\}$ the degree of $P$. Note that the polynomial $P$ can be written as

$$P(z) = \sum_{j=0}^{n} z^j a_j = (a_0, a_1, \ldots, a_n) \begin{pmatrix} I \\ z \\ \vdots \\ z^n \end{pmatrix} = (a_0, a_1, \ldots, a_n) R_n(z)v_n.$$ (40)

Let $\sigma$ be a $q \times q$ positive measure on $[a, b]$. We define a matrix inner product on the space $\mathcal{P}$:

$$\langle P, Q \rangle_\sigma := \int_{[a,b]} P(t) \sigma(dt) Q^*(t).$$ (41)

For details on matrix valued positive measures and the matrix inner product we refer to [21].
Let us define the moments $s_j^{[\sigma]} := \int_{[a,b]} t^{j} \sigma (dt)$ and the matrix of moments $H_n^{[\sigma]} := (s_j^{[\sigma]})_{j,k=0}^{n}$. Then for all matrix polynomials $P(z) = \sum_{j=0}^{n} z^j a_j$ and $Q(z) = \sum_{j=0}^{n} z^j b_j$ we have that

$$\langle P, Q \rangle_\sigma = (a_0, a_1, \ldots, a_n) \left( \begin{array}{cccc}
s_0 & s_1 & \cdots & s_n \\
s_1 & s_2 & \cdots & s_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
s_n & s_{n+1} & \cdots & s_{2n} \\
\end{array} \right) \left( \begin{array}{c}
b_0^* \\
b_1^* \\
\vdots \\
b_n^* \\
\end{array} \right) = (a_0, a_1, \ldots, a_n) H_n \left( \begin{array}{c}
b_0^* \\
b_1^* \\
\vdots \\
b_n^* \\
\end{array} \right). \quad (42)$$

The sequence $(P_j)_{j=0}^{n}$ is called a finite sequence of orthogonal matrix polynomials with respect to $\sigma$ if $\deg P_k = k$ and

$$\langle P_j, P_\ell \rangle_\sigma = 0, \quad j \neq \ell.$$

**Lemma 2.3** If the matrix of moments $H_j$ is strictly positive, then there exists exactly one finite sequence of orthogonal matrix polynomials $(P_k)_{k=0}^{j}$ with leading coefficient $I$. They are $P_k(z) = (-Y_k^*, I) R_k(z) v_k$.

**Proof** If the matrix of moments $H_j$ is strictly positive, then so are all smaller matrices $H_{j'}$ for $0 \leq j' \leq j$. We construct the orthogonal polynomials $P_k$ inductively. Clearly, $P_0 = I$. Now assume that we already have the polynomials $P_0, \ldots, P_{\ell-1}$. If $P_{\ell}$ is orthogonal to their span, it is also orthogonal to the monomials $I, zI, \ldots, z^{\ell-1}I$. Let $P_{\ell}(z) = z^{\ell} I + \sum_{j=0}^{\ell-1} z^j a_j$. Then, using (42), the equations $\langle P_{\ell}, I \rangle_\sigma = \cdots = \langle P_{\ell}, \ell I \rangle_\sigma = \cdots = (P_{\ell}, z^{\ell-1}I) = 0$ can be written as

$$0 = (a_0, \ldots, a_{\ell-1}, I) \left( \begin{array}{cccc}
s_0 & s_1 & \cdots & s_\ell \\
s_1 & s_2 & \cdots & s_{\ell+1} \\
\vdots & \vdots & \ddots & \vdots \\
s_\ell & s_{\ell+1} & \cdots & s_{2\ell} \\
\end{array} \right) \left( \begin{array}{c}
I & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & I \\
0 & \cdots & 0 \\
\end{array} \right) = (a_0, \ldots, a_{\ell-1}, I) \left( \begin{array}{c}
H_{\ell-1} \\
\vdots \\
Y_\ell^* \\
\end{array} \right)$$

which implies

$$(a_0, \ldots, a_{\ell-1}) = -Y_\ell^* H_{\ell-1}^{-1}, \quad \text{hence} \quad (a_0, \ldots, a_{\ell-1}, I) = (-Y_\ell^* H_{\ell-1}^{-1}, I)$$

and therefore $P_{\ell}(z) = (-Y_\ell^* H_{\ell-1}^{-1}, I) R_{\ell}(z) v_{\ell}$. \qed

Now we recall definitions of the OMP on $[a, b]$. Note that for $r = 1, 2$, the polynomials $P_{r,j}$ and $Q_{r,j}$ (resp. $P_{r+2,j}$ and $Q_{r+2,j}$) were first introduced in [4, page 936] (resp. [38, page 87]).

**Definition 2.4** For $k = 1, 2$, let $(s_j)_{j=0}^{2n+k-1}$ be a Hausdorff positive definite sequence on $[a, b]$. Furthermore, let $H_{r,j}, u_{r,j}, Y_{r,j}$ for $r = 1, 2, 3, 4, R_j$ and $v_j$ be as in (5),
depend on (33), (39), (37), (35), (36), (23), (12) and (14), respectively. For \( r = 1, 2, 3, 4 \), we define the polynomials of the first kind
\[
P_{r,0}(z) := I, \quad P_{r,j}(z) := \Sigma_{r,j}^{\ast} R_j(z) v_j.
\] (43)

Moreover, for \( j \geq 1 \), let
\[
Q_{1,0}(z) := 0, \quad Q_{1,j}(z) := -\Sigma_{1,j}^{\ast} R_j(z) u_{1,j}, \quad (44)
\]
\[
Q_{2,0}(z) := -(u_{2,0} + z s_0), \quad Q_{2,j}(z) := -\Sigma_{2,j}^{\ast} R_j(z) u_{2,j}, \quad (45)
\]
\[
Q_{3,0}(z) := s_0, \quad Q_{3,j}(z) := \Sigma_{3,j}^{\ast} R_j(z) u_{3,j}, \quad (46)
\]
\[
Q_{4,0}(z) := -s_0, \quad Q_{4,j}(z) := \Sigma_{4,j}^{\ast} R_j(z) u_{4,j}. \quad (47)
\]

The matrix polynomials \( Q_{r,j} \) are called polynomials of the second kind.

Note that \((Q_{r,j})_{j=0}^{n}\) is not an orthogonal sequence with respect to \( \sigma \). For more details we refer the reader to [2, Sec. VII.6], [4, Remark 4.2] and [38, Theorem 2.12]. Note that in [4, 12] the polynomials \( P_{r,j} \) (resp. \( Q_{r,j} \)) are denoted by \( \Gamma_{r,j} \) (resp. \( \Theta_{r,j} \)) for \( r = 3, 4 \).

**Remark 2.5** Observe that \( \Sigma_{3,j} \) depends on \( b \) while \( \Sigma_{4,j} \) depends on \( a \). If we write \( \Sigma_{3,j}(b) \) and \( \Sigma_{4,j}(a) \) to make this dependence explicit, we easily see that \( \Sigma_{3,j}(a) = -\Sigma_{4,j}(a) \). Hence also the polynomials \( P_{3,j} \) and \( Q_{3,j} \) respectively \( P_{4,j} \) and \( Q_{4,j} \) depend on \( b \) respectively \( a \). If we denote them by \( P_{3,j}(b, z) \), \( Q_{3,j}(b, z) \) and \( P_{4,j}(a, z) \), \( Q_{4,j}(a, z) \) we see that
\[
P_{3,n}(b, z) = P_{4,n}(b, z), \quad P_{3,n}(a, z) = P_{4,n}(a, z), \quad (48)
\]
\[
Q_{3,n}(b, z) = -Q_{4,n}(b, z), \quad Q_{3,n}(a, z) = -Q_{4,n}(a, z). \quad (49)
\]

In the following we will suppress the parameters \( a \) and \( b \) in the matrices \( \Sigma_{r,j} \) and the polynomials \( P_{r,j} \) and \( Q_{r,j} \).

**Remark 2.6** For \( r = 1, 2, 3, 4 \) and \( z \in \mathbb{C} \) we have that \( Q_{r,n}(z) P_{r,n}^{\ast}(\bar{z}) = P_{r,n}(z) Q_{r,n}(\bar{z})^{\ast} \).

**Proof** In [7, Remark 4.7] it is shown that there are solutions \( \sigma_r \) to the moment problem such that \( s_r(z) := \int_{[a,b]} \frac{1}{1-z} \sigma_r(dt) = \frac{1}{p_r(z)} Q_{r,n}(z) P_{r,n}^{\ast-1}(\bar{z}) \) for \( z \in \mathbb{C} \setminus [a, b] \) and \( p_1(z) = I, \quad p_2(z) = (z-a)(b-z), \quad p_3(z) = (b-z), \quad p_4(z) = (z-a) \). Clearly, \( s_r(z) = s_r(\bar{z})^{\ast} \), hence we obtain \( \frac{1}{p_r(z)} Q_{r,n}(z) P_{r,n}^{\ast-1}(\bar{z}) = \frac{1}{p_r(z)} P_{r,n}^{-1}(z) Q_{r,n}(z) \) and therefore \( P_{r,n}(z) Q_{r,n}(\bar{z}) = Q_{r,n}(z) P_{r,n}^{\ast}(\bar{z}) \). Since both products are polynomials, the equality holds for all \( z \in \mathbb{C} \). \( \square \)

In the following remark we give explicit relations between the Schur complements \( \tilde{H}_{k,j} \) and the polynomials \( P_{1,j}, Q_{2,j}, P_{3,j}, Q_{4,j} \) considered in Corollary 3.4 and Corollary 3.10 in [7].
Remark 2.7 Let $\tilde{H}_{r,j}$, for $r = 1, 2, 3, 4$ be as in (25). Furthermore, let $P_{1,j}, Q_{2,j}, P_{3,j}$ and $Q_{4,j}$ be as in Definition 2.4. The following equalities then hold:

$$
\begin{align*}
\tilde{H}_{1,j} &= - P_{1,j}(a) Q_{4,j}^*(a), \\
\tilde{H}_{2,j-1} &= - Q_{2,j-1}(a) P_{3,j}^*(a), \\
\tilde{H}_{3,j} &= P_{3,j}(a) Q_{2,j}^*(a), \\
\tilde{H}_{4,j} &= Q_{4,j}(a) P_{1,j+1}^*(a).
\end{align*}
$$

Since the positive definiteness of the block Hankel matrix $H_{r,n}$ implies that also $\tilde{H}_{r,n}$ is positive definite, (50) and (51) show the following fact.

Remark 2.8 For $k = 1, 2$, let $(s_j)_{j=0}^{2n+k-1}$ be a Hausdorff positive definite sequence on $[a, b]$. Then the matrices $P_{1,j}(a), Q_{4,j}(a), Q_{2,j}(a)$ and $P_{3,j}(a)$ are invertible.

3 Coupling Identities

We will use a number of identities to describe the explicit relation between the two resolvent matrices $V^{(2n+1)}$ from (98) and $U^{(2n+1)}$ from (99) if $m$ is even and between $V^{(2n)}$ from (128) and $U^{(2n)}$ from (129) if $m$ is odd.

For $r = 1, 2, 3, 4$, the following Ljapunov type identities

$$H_{r,j} T_j^* - T_j H_{r,j} = u_{r,j} v_j^* - v_j u_{r,j}^*$$

are called the fundamental identities of the THMM problem. They are crucial for proving that the associated solution $s(z)$ to a given THMM problem is a solution of a system of two matrix inequalities [14–16].

Recall the definition of $H_{r,j}$ and $\tilde{H}_{r,j}$ from (5) and (6). Then we have that

$$\tilde{H}_{r,j} = L_{2,j+1}^* H_{r,j+1} L_{1,j+1} = L_{1,j+1}^* H_{r,j+1} L_{2,j+1}$$

and

$$H_{1,j} L_{1,j+1}^* - L_{2,j+1}^* H_{1,j+1} T_{j+1}^* = 0,$$

Remark 3.1 Let $H_{r,j}, \tilde{H}_{1,j}, T_j, L_{2,j}, L_{1,j}, v_j, u_{j}, u_1, j, u_3, j, u_4, j$, be as in (5), (6), (11), (13), (14), (32), (33), (36), respectively. Let us give some relations between the matrices of moments. Clearly

$$v_j u_j^* - T_j \tilde{H}_{1,j} + H_{1,j} = 0,$$  
$$u_j v_j^* - T_j \tilde{H}_{1,j} T_j^* + H_{1,j} = 0.$$

Applying $L_{2,j}^*$ from the left to (54b) and using that $L_{2,j}^* \tilde{H}_{1,j} T_j^* = \tilde{H}_{1,j-1} L_{1,j}^*$ we obtain

$$u_{j-1} v_j^* - \tilde{H}_{1,j-1} L_{1,j}^* + L_{2,j}^* H_{1,j} = 0.$$
The following relations between the $H_{r,j}$ are easy to see:

$$H_{3,j} = bH_{1,j} - \tilde{H}_{1,j} \quad \text{and} \quad H_{4,j} = -aH_{1,j} + \tilde{H}_{1,j},$$  (56)

and

$$H_{2,j-1} = -abH_{1,j-1} + (a + b)\tilde{H}_{1,j-1} - L_{2,j}^* \tilde{H}_{1,j} L_{1,j}$$

$$= bH_{4,j-1} - \tilde{H}_{4,j-1} = -aH_{3,j-1} + \tilde{H}_{3,j-1}. \quad (57)$$

Multiplication of (54b) by $a$ respectively $b$ together with (56) yields

$$au_jv_j^* + \tilde{H}_{1,j}(I - aT_j^*) - H_{4,j} = 0, \quad (58)$$

$$bu_jv_j^* + \tilde{H}_{1,j}(I - bT_j^*) + H_{3,j} = 0. \quad (59)$$

Moreover, (56) implies that

$$H_{3,j}^{-1}\tilde{H}_{1,j}H_{3,j}^{-1} = H_{3,j}^{-1}\tilde{H}_{1,j}H_{4,j}^{-1} \quad (60)$$

as follows from

$$H_{4,j}^{-1}\tilde{H}_{1,j}H_{3,j} = \left[\tilde{H}_{1,j} - aH_{1,j}\right]\tilde{H}_{1,j}^{-1}\left[-\tilde{H}_{1,j} + bH_{1,j}\right]$$

$$= \left[-\tilde{H}_{1,j} + bH_{1,j}\right]\tilde{H}_{1,j}^{-1}H_{3,j}\left[\tilde{H}_{1,j} - aH_{1,j}\right] = H_{3,j}^{-1}\tilde{H}_{1,j}^{-1}H_{4,j}. \quad (61)$$

Inserting the expressions for $\tilde{H}_{1,n}$ from (56) in (54a) and (54b), we obtain

$$T_jH_{3,j} = -v_ju_j^* - (I - bT_j)H_{1,j}, \quad (61a)$$

$$H_{3,j}T_j^* = -u_jv_j^* - H_{1,j}(I - bT_j^*), \quad (61b)$$

$$T_jH_{4,j} = v_ju_j^* + (I - aT_j)H_{1,j}, \quad (62a)$$

$$H_{4,j}T_j^* = u_jv_j^* + H_{1,j}(I - aT_j^*). \quad (62b)$$

An analogous equation for $H_{2,j}$ is

$$T_jH_{2,j} = -bu_{4,j}v_j^* + v_j\tilde{u}_{2,j}^* - H_{4,j}(I - bT_j^*) \quad (63)$$

because by (57) and (52) for $r = 4$ we have that

$$T_jH_{2,j} = bT_jH_{4,j} - T_j\tilde{H}_{4,j} = bH_{4,j}T_j - bu_{4,j}v_j^* + bv_ju_{4,j}$$

$$- H_{4,j} + v_j(s_0^{(4)}, \ldots, s_j^{(4)})$$

$$= -bu_{4,j}v_j^* - H_{4,j}(I - bT_j) + v_j(bu_{4,j}^* + (s_0^{(4)}, \ldots, s_j^{(4)}))$$

$$= -bu_{4,j}v_j^* - H_{4,j}(I - bT_j) + v_j\tilde{u}_{2,j}^*. \quad \text{In the last step we used} \quad (39).$$
Remark 3.2 Let $H_{r,j}$, $\tilde{H}_{1,j}$, $T_j$, $L_{2,j+1}$, $L_{1,j}$, $v_j$, $u_{1,j}$, $\tilde{u}_{2,j}$ and $u_{4,j}$, be as in (5), (6), (11), (13), (14), (33), (39) and (36), respectively.

The next two equalities are obtained from (61b) and (62b) by multiplication from the left by $-(I-bT_j)$ and $(I-aT_j)$ respectively.

\begin{align*}
    u_{3,j}v_j^* - (I-bT_j)[H_{3,j}T_{j}^* + H_{1,j}(I-bT_j^*)] &= 0, \\
    u_{4,j}v_j^* - (I-aT_j)[H_{4,j}T_{j}^* - H_{1,j}(I-aT_j^*)] &= 0.
\end{align*}

(64) (65)

Equations (61b) and (62b) together with the fundamental identity (52) for $r = 1$ and $-u_jv_j^* + bu_{1,j}v_j^* = u_{3,j}v_j^*$ and $-u_jv_j^* - au_{1,j}v_j^* = u_{4,j}v_j^*$ yield

\begin{align*}
    H_{3,j}T_{j}^* &= u_{3,j}v_j^* - bv_ju_{1,j} - (I-bT_j)H_{1,j}, \\
    H_{4,j}T_{j}^* &= u_{4,j}v_j^* + av_ju_{1,j} + (I-aT_j)H_{1,j}.
\end{align*}

(66) (67)

Taking the adjoint of (58) and multiplying from the right by $L_{1,j}^*$ gives

\[
av_ju_{1,j+1}^* - H_{4,j}L_{1,j+1}^* + (I-aT_j)\tilde{H}_{1,j}L_{1,j+1}^* = 0.
\] (68)

Using that $\tilde{H}_{1,j}L_{1,j}^* = L_{2,j+1}^*H_{1,j+1} - u_jv_j^*_{j+1}$ we obtain from (68)

\[
u_{4,j}v_j^*_{j+1} + av_ju_{1,j+1}^* - H_{4,j}L_{1,j+1}^* + (I-aT_j)L_{2,j+1}^*H_{1,j+1} = 0.
\] (69)

The identities of the next lemma are used to prove our main results Theorem 4.10 and Theorem 4.18.

Lemma 3.3 Let $T_j$, $L_{2,j}$, $L_{1,j}$, $v_j$, $u_{r,j}$ for $r = 1, 3, 4$, $\tilde{u}_{2,j}$, $\Sigma_{r,j}$ for $r = 1, 3, 4$, be as in (11), (13), (14), (33), (36), (39), (27) and (28), respectively. Then the next equalities hold.

\[
    \begin{bmatrix} v_ju_j^* + (I-bT_j)H_{1,j} \end{bmatrix} \Sigma_{3,j} = 0,
\] (70)

\[
    \begin{bmatrix} bu_{4,j}v_j^* - v_j\tilde{u}_{2,j}^* + H_{4,j}(I-bT_j^*) \end{bmatrix} \Sigma_{2,j} = 0,
\] (71)

\[
    \begin{bmatrix} v_ju_{3,j}^* - bu_{1,j}v_j^* - H_{1,j}(I-bT_j^*) \end{bmatrix} \Sigma_{3,j} = 0,
\] (72)

\[
    \begin{bmatrix} v_ju_{4,j}^* + au_{1,j}v_j^* + H_{1,j}(I-aT_j^*) \end{bmatrix} \Sigma_{4,j} = 0,
\] (73)

\[
    \begin{bmatrix} u_{4,j}v_j^*_{j+1} + av_ju_{3,j+1}^* - H_{4,j}L_{1,j+1}^* \end{bmatrix} \Sigma_{1,j+1} = 0,
\] (74)

\[
    \begin{bmatrix} -H_{4,n}R_n^*(a)L_{1,n+1}^* + R_n(a)u_{4,n}v_j^*_{n+1}R_{n+1}^*(a) \end{bmatrix} \Sigma_{1,n+1} = 0,
\] (75)

\[
    \begin{bmatrix} \Sigma_4^* \begin{bmatrix} bu_{4,j}v_j^* + av_ju_{3,j}^* - (b-a)R_j(a)u_{4,j}v_j^*R_j(a) \end{bmatrix} \Sigma_{3,j} = 0.
\] (76)

Proof Recall that $L_{2,j+1}^*H_{1,j+1} \Sigma_{1,j+1} = 0$ by (29). The equalities (70) and (71) follow from (29) because the left hand sides are equal to $-T_j$ $H_{3,j} \Sigma_{3,j}$ by (61a) and $-T_jH_{2,j} \Sigma_{2,j}$ by (63) respectively. Equalities (72) and (73) are a consequence of (52)
and (70) and (71) respectively. Equality (74) follows from (69). To prove Equality (76), note that by definition of $u_{3,j}$ and $u_{4,j}$ and by (62b) and (61a), we have

\[ bu_{4,j}v_j^* + av_ju_{3,j}^* = bu_jv_j^* - av_ju_j^* + ab(v_ju_j^*T_j^* - T_ju_jv_j^*) \]
\[ = bH_{4,j}T_j^* - bH_{1,j}(I - aT_j^*) + aT_jH_{3,j} + a(I - bT_j)H_{1,j} + ab(v_ju_j^*T_j^* - T_ju_jv_j^*) \]
\[ = bH_{4,j}T_j^* + aT_jH_{3,j} - (b - a)H_{1,j} + ab(v_ju_j^*T_j^* - T_ju_jv_j^* - T_jH_{1,j} + H_{1,j}T_j^*) \]
\[ = bH_{4,j}T_j^* + aT_jH_{3,j} - (b - a)H_{1,j} \]

where in the last step we used (52). Inserting this in (76) and using (29) for $r = 3$ and $r = 4$, we obtain

\[ b_4^*J_{3,j}^* - b_3^*J_{3,j}^* - (b - a)R_j(a)u_{4,j}v_j^*R_j^*(a) \Sigma_{3,j} \]
\[ = -(b - a)\Sigma_{4,j}^*[H_{1,j} + R_j(a)u_{4,j}v_j^*R_j^*(a)] \Sigma_{3,j} \]
\[ = -(b - a)\Sigma_{4,j}^*[H_{1,j}(I - aT_j^*) + u_jv_j^*]R_j^*(a) \Sigma_{3,j} \]
\[ = -(b - a)\Sigma_{4,j}^*H_{4,j}T_j^*R_j(a) \Sigma_{3,j} = 0. \]

In the second to last equality we employed (62a).

Finally we prove Equality (75). We have

\[ - H_{4,n}R_n^*(a)L_{1,n+1}^* + R_n(a)u_{4,n}v_{n+1}^*R_{n+1}^*(a) \Sigma_{1,n+1} \]
\[ = - H_{4,n}R_n^*(a)L_{1,n+1}^* + u_{n}v_{n+1}^*R_{n+1}^*(a) \Sigma_{1,n+1} \]
\[ = - (aH_{1,n} + \tilde{H}_{1,n})L_{1,n+1}^* + u_{n}v_{n+1}^*R_{n+1}^*(a) \Sigma_{1,n+1} \]
\[ = aL_{2,n+1}^*H_{1,n+1}T_{n+1}^* - L_{2,n+1}^*H_{1,n+1}R_{n+1}^*(a) \Sigma_{1,n+1} \]
\[ = - L_{2,n+1}^*H_{1,n+1}R_{n+1}^*(a) \Sigma_{1,n+1} = 0. \]

In the first equality we used (21). The second equality follows from the second equality of (56) and the third equality follows from (55). In the last step we used (31).

**Remark 3.4** Let $\tilde{H}_{1,j}$ be as in (25) and let $Q_{4,j}$ be as in Definition 2.4. Then

\[ v_j^*R_j^*(\tilde{z})H_{1,j}^{-1}R_j(b)v_jQ_{3,j}^*(b) = P_{3,j}^*(\tilde{z}), \quad (77) \]
\[ \tilde{H}_{1,j} = P_{1,j}(b)Q_{3,j}^*(b), \quad (78) \]
\[ I - (z - a)u_{1,j}^*R_j^*(\tilde{z})H_{1,j}^{-1}R_j(a)v_j = Q_{4,j}(z)Q_{4,j}^{-1}(a), \quad (79) \]
\[ I - (z - b)u_{1,j}^*R_j^*(\tilde{z})H_{1,j}^{-1}R_j(b)v_j = Q_{3,j}(z)Q_{3,j}^{-1}(b). \quad (80) \]

**Proof** First of all note that the right hand sides of (79) and (80) make sense because $Q_{4,j}(a)$ is invertible by Remark 2.8 and $Q_{3,j}(b)$ is invertible by (78).

We prove (77) by using (70) in the third step:

\[ v_j^*R_j^*(\tilde{z})H_{1,j}^{-1}R_j(b)v_jQ_{3,j}^*(b) = -v_j^*R_j^*(\tilde{z})H_{1,j}^{-1}R_j(b)v_ju_{3,j}^*R_j^*(b) \Sigma_{3,j} \]
\[ = -v_j^*R_j^*(\tilde{z})H_{1,j}^{-1}R_j(b)v_ju_j^* \Sigma_{3,j} \]
\[ = v_j^* R_j^* (\bar{z}) H_{1,j}^{-1} R_j (b) (I - b T_j) H_{1,j} \Sigma_3,j \]
\[ = v_j^* R_j^* (\bar{z}) \Sigma_3,j = P_{3,j}^* (\bar{z}). \]

To prove (78), we calculate the inverse of the matrix \( H_{1,j} \) using the factorization (24), see also [4, Equality (3.8)):
\[
H_{1,j}^{-1} = \begin{pmatrix} H_{1,j-1}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -H_{1,j-1}^{-1} Y_{1,j} \\ I \end{pmatrix} \tilde{H}_{1,j}^{-1} (-Y_{1,j} H_{1,j-1}^{-1} I). \tag{81}
\]

Inserting (81) in Equality (77) and using the definition 2.4 for \( P_{1,j} \), we find that
\[
P_{3,j}^* (\bar{z}) = v_j^* R_j^* (\bar{z}) H_{1,j}^{-1} R_j (b) v_j Q_{3,j}^* (b)
\[ = v_{j-1}^* R_{j-1}^* (\bar{z}) H_{1,j-1}^{-1} R_{j-1} (b) v_{j-1} Q_{3,j}^* (b) + P_{1,j}^* (\bar{z}) \tilde{H}_{1,j}^{-1} P_{1,j} (b) Q_{3,j}^* (b). \]

The first term in the sum above is a polynomial of degree \( j - 1 \) in \( z \). Since the leading term in the polynomials \( P_{1,j}^* \) and \( P_{3,j}^* \) is \( I \), equating coefficients for \( z^i \), shows that
\[
\tilde{H}_{1,j}^{-1} P_{1,j} (b) Q_{3,j}^* (b) = I
\]
which is equivalent to (78).

The Equality (79) is proved in Equality (3.6) of [7, Lemma 3.2] where the notation \( \Theta_{2,j} \) is used instead of \( Q_{4,j} \).

To prove (80), we verify the equivalent equality
\[
\left[ I - (z - b) u_{1,j}^* R_j^* (\bar{z}) H_{1,j}^{-1} R_j (b) v_j \right] Q_{3,j}^* (b) - Q_{3,j}^* (z) = 0.
\]
Using (48) and (49) and the Definition 2.4 for \( Q_{3,j} \), we obtain
\[
\left[ I - (z - b) u_{1,j}^* R_j^* (\bar{z}) H_{1,j}^{-1} R_j (b) v_j \right] Q_{3,j}^* (b) - Q_{3,j}^* (z)
\[ = -\left[ u_{3,j}^* R_j^* (b) - (z - b) u_{1,j}^* R_j^* (\bar{z}) H_{1,j}^{-1} R_j (b) v_j u_{3,j}^* R_j^* (b) - u_{3,j}^* R_j^* (\bar{z}) \right] \Sigma_3,j. \tag{82}
\]

Using that \( u_{1,j}^* = u_j^* T_j^* \), \( u_{3,j}^* R_j (b) = u_j^* \) and that
\[
\begin{align*}
\left( u_{3,j}^* R_j^* (b) - u_{3,j}^* R_j^* (\bar{z}) \right) &= \left( u_{3,j}^* (b - z) R_j^* (b) T_j^* R_j^* (\bar{z}) = (b - z) u_j^* T_j^* R_j^* (\bar{z}), \right. \\
\end{align*}
\]
the right hand side of (82) becomes
\[
(z - b) \left[ u_j^* T_j^* R_j^* (\bar{z}) + u_j^* T_j^* R_j^* (\bar{z}) H_{1,j}^{-1} R_j (b) v_j u_j^* \right] \Sigma_3,j
\[ = (z - b) u_j^* T_j^* R_j^* (\bar{z}) H_{1,j}^{-1} R_j (b) \left[ (I - b T_j) H_{1,j} + v_j u_j^* \right] \Sigma_3,j = 0.
\]

The last equality is obtained from (70). \( \square \)
**Remark 3.5** Let \((s_j)^{2n}\) be a Hausdorff positive definite sequence on \([a, b]\). Furthermore, let \(P_{1,j}\) and \(Q_{4,j}\) be as in Definition 2.4. Then the matrix \(P_{1,j}(b)\) is invertible by (78).

### 4 Explicit Relation Between Two Resolvent Matrices of the THMM Problem Via OMP

In this section we give a representation of the resolvent matrix \(U^{(m)}\) of the THMM problem in terms of orthogonal matrix polynomials both for an an odd and even number of given moments. Let

\[
J_q := \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \quad (83)
\]

and

\[
\tilde{J}_q := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (84)
\]

where the entries \(I\) and \(0\) are \(q \times q\) matrices. The matrices \(J_q\) and \(\tilde{J}_q\) satisfy

\[
J_q = J_q^*, \quad J_q^2 = I, \quad (85)
\]

\[
\tilde{J}_q = \tilde{J}_q^*, \quad \tilde{J}_q^2 = I. \quad (86)
\]

The properties of the matrix \(J_q\) were used to explain relevant results concerning the next four matrices:

**Definition 4.1** Let \((s_k)^{2n}\) \((s_k)^{2n+1}\) be a Hausdorff positive definite sequence on \([a, b]\). Let \(H_{r,n}, \ u_{r,n}, \ v_n\) and \(J_q\) be defined as in (5), (33), (39), (36), (12), (14) and (83), respectively. For \(r = 1, 2, 3, 4\), the Kovalishina resolvent matrix of the THMM problem is

\[
\tilde{V}_r^{(n)}(z) := I - iz \left( \begin{vmatrix} v_n^* \\ u_{r,n}^* \end{vmatrix} \right) R_n^*(\bar{z}) H_{r,n}^{-1}(v_n, \ u_{r,n}) J_q. \quad (87)
\]

The name Kovalishina resolvent matrix was suggested by Yu. Dyukarev as the work [14] was being prepared. Irina Kovalishina introduced and solved interpolation problems in the Nevanlinna class of functions as well as the matrix moment problem on the real axis; see [33, 34].

Two relevant properties [14] distinguish the matrix \(\tilde{V}_r^{(n)}\). Firstly, we mention the inequality

\[
J_q - \tilde{V}_r^{(n)}(z) J_q (\tilde{V}_r^{(n)})^*(z) \leq 0 \quad (88)
\]
for $z$ to the upper half plane of $\mathbb{C}$. Secondly, the inverse matrix of $\tilde{V}_r^{(n)}$ can be expressed as follows:

$$\left[ \tilde{V}_r^{(n)}(z) \right]^{-1} = J_q \tilde{V}_r^{(n)*}(\tilde{z}) J_q.$$  \hfill (89)

The relations (88) and (89) were used to find the solutions the THMM problem as well as a factorization of the resolvent matrix; see [3, 5, 6] and [10]. The formulas in the next remark can be readily shown by direct calculations.

**Remark 4.2** The matrix defined in (87) can be written in the following form

$$\tilde{V}_r^{(n)}(z) := \begin{pmatrix} \tilde{\alpha}_r^{(n)}(z) & \tilde{\beta}_r^{(n)}(z) \\ \tilde{\gamma}_r^{(n)}(z) & \tilde{\delta}_r^{(n)}(z) \end{pmatrix}, \quad z \in \mathbb{C}, \tag{90}$$

where

$$\tilde{\alpha}_r^{(n)}(z) := I + z v_n^* R_n^*(\tilde{z}) H_{r,n}^{-1} u_n, \quad \tag{91}$$

$$\tilde{\beta}_r^{(n)}(z) := -z v_n^* R_n^*(\tilde{z}) H_{r,n}^{-1} v_n, \quad \tag{92}$$

$$\tilde{\gamma}_r^{(n)}(z) := z u_n^* R_n^*(\tilde{z}) H_{r,n}^{-1} u_n, \quad \tag{93}$$

$$\tilde{\delta}_r^{(n)}(z) := I - z u_n^* R_n^*(\tilde{z}) H_{r,n}^{-1} v_n. \quad \tag{94}$$

### 4.1 Case of an Even Number of Moments

In this section, we consider the explicit relation between the resolvent matrix of the THMM problem for the case of an even number of moments introduced in [14] and the resolvent matrix presented in [7] and in [15].

**Assumption 4.3** In this section we assume that $\tilde{H}_{1,n}$ and $\tilde{H}_{1,n-1}$ are invertible.

**Remark 4.4** Note that

$$\tilde{H}_{1,n} = (b - a)^{-1}(a H_{3,n} + b H_{4,n}). \quad \tag{95}$$

Hence, if $0 \leq a < b$ or $a < b \leq 0$, then $\tilde{H}_{1,n}$ is strictly positive or negative, hence so is $\tilde{H}_{1,n-1}$ and Assumption 4.3 is satisfied, see [19, Prop. 1].

**Definition 4.5** Let $(s_k)_{k=0}^{2n+1}$ be a Hausdorff positive definite sequence on $[a, b]$. Let $H_{3,n}$ and $H_{4,n}$, $\tilde{H}_{1,n}$, $v_n$ and $u_n$ as in (5), (6), (14) and (32), respectively, and define

$$M^{(2n+1)} := -a u_n^* \tilde{H}_{1,n}^{-1} u_n, \quad N^{(2n+1)} := -b v_n^* H_{4,n}^{-1} \tilde{H}_{1,n} H_{3,n}^{-1} v_n \quad \tag{96}$$
with
\[ C^{(2n+1)} := \begin{pmatrix} I & 0 \\ M^{(2n+1)} & I \end{pmatrix}, \quad D^{(2n+1)} := \begin{pmatrix} I & N^{(2n+1)} \\ 0 & I \end{pmatrix} \]

and
\[ V_4^{(2n+1)}(z) := \tilde{V}_4(z)C^{(2n+1)}D^{(2n+1)}. \tag{97} \]

The matrix (97) is called the first auxiliary resolvent matrix of the THMM problem in the case of an even number of moments. In [14] it is denoted by \( U_r(z) \).

Finally, the matrix
\[ V^{(2n+1)}(z) := \begin{pmatrix} I & 0 \\ 0 & (z - a)^{-1}I \end{pmatrix} V_4^{(2n+1)}(z) \begin{pmatrix} I & 0 \\ 0 & (z - a)I \end{pmatrix} \tag{98} \]
is called the resolvent matrix of the THMM problem for the case of an even number of moments. This matrix is defined for \( z \in \mathbb{C} \setminus \{a\} \).

Note that the point \( z = a \) in (98) is a removable singularity; see [14, Theorem 4].

**Definition 4.6** Let \((s_j)_{j=0}^{2n+1}\) be a Hausdorff positive definite sequence on \([a, b]\). Let \(P_{1,n+1}, Q_{1,n+1}, P_{2,n},\) and \(Q_{2,n}\) be as in Definition 2.4. The \(2q \times 2q\) matrix polynomial
\[ U^{(2n+1)}(z) := \begin{pmatrix} \alpha^{(2n+1)}(z) & \beta^{(2n+1)}(z) \\ \gamma^{(2n+1)}(z) & \delta^{(2n+1)}(z) \end{pmatrix}, \quad z \in \mathbb{C}, \tag{99} \]
with
\[ \alpha^{(2n+1)}(z) := Q_{2,n}^*(\bar{z})Q_{2,n}^{*-1}(a), \tag{100} \]
\[ \beta^{(2n+1)}(z) := -Q_{1,n+1}^*(\bar{z})P_{1,n+1}^{*-1}(a), \tag{101} \]
\[ \gamma^{(2n+1)}(z) := -(b - z)(z - a)P_{2,n}^*(\bar{z})Q_{2,n}^{*-1}(a), \tag{102} \]
\[ \delta^{(2n+1)}(z) := P_{1,n+1}^*(\bar{z})P_{1,n+1}^{*-1}(a) \tag{103} \]
is called the resolvent matrix of the THMM problem in the point \(a\) in the case of an even number of moments.

**Remark 4.7**

a) By Assumption 4.3, \(\tilde{H}_{1,n}\) is strictly positive, hence its Schur complement \(\tilde{H}_{1,n}\) is well-defined and can be expressed with the help of the polynomials \(P_{1,n+1}(0)\) and \(\tilde{Q}_{2,n}(0)\) [8, Equality (4.54)]
\[ \tilde{H}_{1,n} = -\tilde{Q}_{2,n}(0)P_{1,n+1}^*(0), \tag{104} \]
where
\[ \tilde{Q}_{2,n}(z) = -\begin{pmatrix} -s_{n+1}, \ldots, s_{2n} \end{pmatrix}^{*} \tilde{H}_{1,n-1}^{-1}, \quad I) R_n(z)u_n. \]
Explicit Relation Between Two Resolvent Matrices...

Note that \( \tilde{Q}_{2,n} \) appears in [8, Definition 4.1] and in [9, Definition 5.1], where it was called \( Q_{2,n} \).

b) If \( \tilde{H}_{1,n} \) is invertible so is the Schur complement \( \tilde{H}_{1,n} \). Consequently, the polynomial \( P_{1,n+1}(0) \) is invertible if \( \tilde{H}_{1,n} \) is invertible.

**Lemma 4.8** Let \( P_{1,n} \) be as in (43) and set

\[
d^{(2n+1)} := P_{1,n+1}^*(0)P_{1,n+1}^{-1}(a).
\]

Furthermore, let \((s_j)_{j=0}^{2n+1} \) be a Hausdorff positive sequence on \([a, b]\) and let \( H_{4,n}, R_n, v_n, u_n, u_{n,A}, P_{2,n}, Q_{1,n}, Q_{2,n} M^{(2n+1)} \) and \( N^{(2n+1)} \) be as in (5), (12), (14), (32), (36), (43), (44), (45) and (96), respectively. Recall that by Assumption 4.3 the matrix \( \tilde{H}_{1,n} \) is invertible. Then the following equalities hold.

\[
d^{(2n+1)} = I - av_n^*H_{4,n}^{-1}R_n(a)u_{4,n},  
\]

(106)

\[
d^{(2n+1)*-1} = I + au_n^*\tilde{H}_{1,n}^{-1}R_n(a)v_n,  
\]

(107)

\[
M^{(2n+1)} = aQ_{1,n+1}^*(0)P_{1,n+1}^{-1}(0)  
\]

(108)

\[
M^{(2n+1)}d^{(2n+1)} = aQ_{1,n+1}^*(0)P_{1,n+1}^{-1}(a),  
\]

(109)

\[
N^{(2n+1)}d^{(2n+1)*-1} = -b v_n^*H_{3,n}^{-1}R_n(a)v_n,  
\]

(110)

\[
N^{(2n+1)}d^{(2n+1)*-1} = -b P_{2,n}^*(0)Q_{2,n}^{-1}(a),  
\]

(111)

\[
(I + M^{(2n+1)}N^{(2n+1)})d^{(2n+1)*-1} = Q_{2,n}^*(0)Q_{2,n}^{-1}(a).  
\]

(112)

(113)

**Proof** First, we observe that

\[
P_{1,n+1}(0)^* - P_{1,n+1}(a) = v_n^*[I - R_{n+1}^*(a)]\Sigma_{1,n+1} = v_n^*[aT_{n+1}^* R_{n+1}^*(a) \Sigma_{1,n+1}  
= -av_n^*L_{2,n+1}^*T_{n+1}^* R_{n+1}^*(a) \Sigma_{1,n+1}  
= -av_n^*R_n^*(a)L_{1,n+1}^* \Sigma_{1,n+1}  
\]

(114)

where we used that \( L_{2,n+1}^*T_{n+1}^* R_{n+1}^*(a) = L_{1,n+1}^* R_{n+1}^*(a) = R_n^*(a)L_{1,n+1}^* \) by (17) and (21).

To show (106) we use (114) and (75) to obtain

\[
P_{1,n+1}^*(0) - P_{1,n+1}(a) = -av_n^*H_{4,n}^{-1}R_n(a)u_4,n \Sigma_{1,n+1}  
= -av_n^*H_{4,n}^{-1}R_n(a)u_4,n v_{n+1}^*R_{n+1}^*(a) \Sigma_{1,n+1}  
= -av_n^*H_{4,n}^{-1}u_n P_{1,n+1}^*(a),  
\]

hence \( d^{(2n+1)} = P_{1,n+1}^*(0)P_{1,n+1}^{-1}(a) = I - av_n^*H_{4,n}^{-1}u_n P_{1,n+1}^*(a).  
\)
To prove (107), we note that by (55) and (31)
\[
\Sigma_{1,n+1}^* L_{1,n+1} = \Sigma_{1,n+1}^* \left[ v_{n+1} u_n^* + H_{1,n+1} L_{2,n+1} \right] = \Sigma_{1,n+1}^* v_{n+1} u_n^* \tilde{H}_{1,n}^{-1}.
\]

Hence (107) follows from (114) because
\[
P_{1,n+1}(0) - P_{1,n+1}(a) = -a \Sigma_{1,n+1}^* L_{1,n+1} R_n(a) v_n
= -a \Sigma_{1,n+1}^* v_{n+1} u_n^* \tilde{H}_{1,n}^{-1} R_n(a) v_n
= -a P_{1,n+1}(0) u_n^* \tilde{H}_{1,n}^{-1} R_n(a) v_n.
\]

Formula (108) follows from (115) and
\[
a Q_{1,n+1}^* P_{1,n+1}^* = -au_{1,n+1}^* \Sigma_{1,n+1}^* \left[ v_{n+1}^* \Sigma_{1,n+1}^* \right]^{-1}
= -au_n^* L_{1,n+1}^* \Sigma_{1,n+1}^* \left[ v_{n+1}^* \Sigma_{1,n+1}^* \right]^{-1}
= -au_{1,n+1}^* \tilde{H}_{1,n}^{-1} R_n(a) v_n
= -au_{1,n+1}^* \tilde{H}_{1,n}^{-1} u_n = M^{(2n+1)}.
\]

Equality (109) readily follows from (108) and (105).
Next we prove (110) using (60) and (107):
\[
N^{(2n+1)} d^{(2n+1)^*} = -bv_{1,n}^* H_{3,n}^{-1} \tilde{H}_{1,n} H_{4,n}^{-1} v_n \left[ I + au_{1,n}^* \tilde{H}_{1,n}^{-1} R_n(a) v_n \right]
= -bv_{1,n}^* H_{3,n}^{-1} \tilde{H}_{1,n} H_{4,n}^{-1} \left[ I + au_n^* u_n^* \tilde{H}_{1,n}^{-1} R_n(a) v_n \right] v_n
= -bv_{1,n}^* H_{3,n}^{-1} \tilde{H}_{1,n} H_{4,n}^{-1} \left[ (I - T_n(a)) \tilde{H}_{1,n} + au_n^* u_n^* \tilde{H}_{1,n}^{-1} R_n(a) v_n \right] v_n
= -bv_{1,n}^* H_{3,n}^{-1} \tilde{H}_{1,n} H_{4,n}^{-1} \tilde{H}_{1,n}^{-1} R_n(a) v_n
= -bv_{1,n}^* H_{3,n}^{-1} R_n(a) v_n
\]

where we used (58) in the second to last step.

Equality (111) follows from (110) and [7, Equation (3.28)] and Equality (112) is an immediate consequence of (111) and the definition of \(d^{(2n+2)}\).

Finally, we prove equality (113) using (110) and (107):
\[
(I + M^{(2n+1)} N^{(2n+1)^*}) d^{(2n+1)^*} = I + au_{1,n}^* \tilde{H}_{1,n}^{-1} R_n(a) v_n + abu_{1,n}^* \tilde{H}_{1,n}^{-1} u_n v_n^* H_{3,n}^{-1} R_n(a) v_n
= I + au_{1,n}^* \tilde{H}_{1,n}^{-1} \left[ H_{3,n} + bu_n v_n^* \right] H_{3,n}^{-1} R_n(a) v_n
= I + au_{1,n}^* \left( 1 - b T_n^* \right) H_{3,n}^{-1} R_n(a) v_n
= I + au_{1,n}^* H_{3,n}^{-1} R_n(a) v_n
= Q_{2,n}^* (0) Q_{2,n}^{-1} (a)
\]

where we used that \(H_{3,n} + bu_n v_n^* = \tilde{H}_{1,n} (I - b T_n^*)\) by (59) and the last equality is a consequence of (100) and [7, Equation (3.26)].
Expressing the above equalities as orthogonal polynomials, we obtain the next identities.

**Corollary 4.9** Let \( P_{r,n} \) and \( Q_{r,n} \) be as in Definition 2.4. Then the following equality holds

\[
P_{1,n+1}(a)Q_{1,n+1}^*(a) - abQ_{1,n+1}(0)P_{2,n}^*(0) - P_{1,n+1}(0)Q_{2,n}^*(0) = 0, \tag{116}
\]
\[
(b - a)Q_{4,n}(a) P_{3,n}^*(a) - bQ_{4,n}(0)P_{3,n}^*(0) - aP_{4,n}(0)Q_{3,n}^*(0) = 0. \tag{117}
\]

**Proof** Equation (116) follows from (113), (105), (108) and (111) and Remark 2.6, while (117) is a direct consequence of (76).

Let

\[
\mathcal{D}^{(2n+1)} := \begin{pmatrix} d^{(2n+1)^*} & 0 \\ 0 & d^{(2n+1)} \end{pmatrix} \tag{118}
\]

where \( d^{(2n+1)} \) is defined in (105).

**Theorem 4.10** Let \( U^{(2n+1)}(z), V^{(2n+1)}, \tilde{Z}_q, \) and \( \mathcal{D}^{(2n+1)} \) be matrices as in Definitions 4.6, 4.5, (84) and (118), respectively. Then the following equality holds:

\[
U^{(2n+1)}(z) = \tilde{Z}_q V^{(2n+1)}(z) \tilde{Z}_q \mathcal{D}^{(2n+1)}. \tag{119}
\]

**Proof** Using the block entries (100)–(103), (91)–(94), (97) and (98), we see that Equality (119) is equivalent to the following four equalities:

\[
\alpha^{(2n+1)}(z) = \left[ \tilde{\gamma}_4^{(n)}(z) N^{2n+1} + \tilde{\delta}_4^{(n)}(z)(I + M^{(2n+1)} N^{(2n+1)}) \right] d^{(2n+1)^*} - 1, \tag{120}
\]
\[
\beta^{(2n+1)}(z) = (z - a)^{-1} \left[ \tilde{\gamma}_4^{(n)}(z) + \tilde{\delta}_4^{(n)}(z) M^{(2n+1)} \right] d^{(2n+1)} - 1, \tag{121}
\]
\[
\gamma^{(2n+1)}(z) = (z - a) \left[ \tilde{\alpha}_4^{(n)}(z) N^{(2n+1)} + \tilde{\beta}_4^{(n)}(z)(I + M^{(2n+1)} N^{(2n+1)}) \right] d^{(2n+1)^*} - 1, \tag{122}
\]
\[
\delta^{(2n+1)}(z) = \left[ \tilde{\alpha}_4^{(n)}(z) + \tilde{\beta}^{(2n+1)}(z) M^{(2n+1)} \right] d^{(2n+1)}. \tag{123}
\]

For the proof of (120) and (122) we observe that, by (71),

\[
R_n^s(\tilde{z}) H_{4,n}^{-1} \left\{ -bu_{4,n} v_n^s + v_n u_{2,n}^s \right\}
\]
\[
\Sigma_{2,n} = R_n^s(\tilde{z})(I - b T_n^s) \Sigma_{2,n} = (I - b T_n^s) R_n^s(\tilde{z}) \Sigma_{2,n}.
\]

Hence it follows from (111) and (113) that
Moreover, \( \alpha(n) = u_* \) by (74), which proves (120). The proof of (122) is similar:

\[
\begin{align*}
(z - a) \left[ \alpha(n) N^{2(n+1)} + \delta_4(n) M^{2(n+1)} \right] d^{(2n+1)} + \delta_4(n) M^{2(n+1)} d^{(2n+1)} = 0
\end{align*}
\]

To prove (121) and (123) we first note that, by (74),

\[
\begin{align*}
R_n^*(\tilde{z}) H_n^1 \left\{ u_{n+1}^* + a v_n u_{n+1}^* \right\} = u_{n+1}^* L_{n+1}^* \Sigma_{n+1} = L_{n+1}^* R_{n+1}^* (\tilde{z}) \Sigma_{n+1}.
\end{align*}
\]

Moreover, \( u_4^* L_{1,n+1} = u_n^*(I - a T_n^*) L_{1,n+1} = u_n^* L_{1,n+1}^* (I - a T_n^*) = u_{1,n+1}^* (I - a T_n^*) \) by (19) and (21). Hence we obtain (121), using (105) and (109), as follows:

\[
\begin{align*}
(z - a)^{-1} \left[ \tilde{\alpha}(n) d^{(2n+1)} \right] &= (z - a)^{-1} \left[ \tilde{\alpha}(n) P_{n+1}^* (0) + a \delta_4(n) Q_{n+1}^* (0) \right] P_{n+1}^* (a)
\end{align*}
\]
In the third equality, we employed (93) and (94). The proof of (123) is similar:

\[
\tilde{\alpha}_4^{(n)}(z)d^{(2n+1)} - \tilde{\beta}_4^{(n)}(z)M^{(2n+1)} d^{(2n+1)} \\
= \left[ \tilde{\alpha}_4^{(n)}(z)P_{1,n+1}^*(0) - a\tilde{\beta}_4^{(n)}(z)Q_{1,n+1}^*(0) \right] P_{1,n+1}^{*\perp}(a) \\
= \left[ v_{n+1}^* R_n^*(\bar{z}) H_{1,n}^{-1} \left\{ u_{4,n} v_{n+1}^* + av_n u_{1,n+1}^* \right\} \right] \Sigma_{1,n+1} P_{1,n+1}^{*\perp}(a) \\
= \left[ v_{n+1}^* L_{1,n+1}^* R_{n+1}^*(\bar{z}) \right] \Sigma_{1,n+1} P_{1,n+1}^{*\perp}(a) \\
= \left[ v_{n+1}^* (I - zT_{n+1}^*) + zv_n^* L_{1,n+1}^* \right] R_{n+1}^*(\bar{z}) \Sigma_{1,n+1} P_{1,n+1}^{*\perp}(a) \\
= v_{n+1}^* R_{n+1}(\bar{z}) \Sigma_{1,n+1} P_{1,n+1}^{*\perp}(a) = P_{1,n+1}(\bar{z}) P_{1,n+1}^{*\perp}(a).
\]

\[
\square
\]

### 4.2 Case of an Odd Number of Moments

In this subsection, we prove the relation between the resolvent matrix proposed in [14] and [16] in the case of an odd number of moments.

First, we reproduce the resolvent matrix from [16] for the case of an odd number of moments.

**Assumption 4.11** In this section we will always assume that

\[
I + av_n^* R_n^*(a) H_{1,n}^{-1} u_{1,n} \text{ is invertible. (Γ)}
\]

**Definition 4.12** Let us define the \( q \times q \) matrices

\[
\Gamma_a := \left( I + av_n^* R_n^*(a) H_{1,n}^{-1} u_{1,n} \right)^{-1} v_n^* R_n^*(a) H_{1,n}^{-1} v_n, \tag{124}
\]

\[
\Gamma_b := \left( I + bv_n^* R_n^*(b) H_{1,n}^{-1} u_{1,n} \right)^{-1} v_n^* R_n^*(b) H_{1,n}^{-1} v_n \tag{125}
\]

and

\[
M^{(2n)} := a\Gamma_a, \quad N^{(2n)} := (b\Gamma_b - a\Gamma_a)^{-1}. \tag{126}
\]

Moreover, we define the \( 2q \times 2q \) matrices

\[
C_1^{(2n)} := \begin{pmatrix} I & M^{(2n)} \\ 0 & I \end{pmatrix}, \quad D_1^{(2n)} := \begin{pmatrix} I & 0 \\ N^{(2n)} & I \end{pmatrix} \tag{127}
\]

and

\[
V^{(2n)}(z) := \tilde{V}_1^{(n)}(z) C_1^{(2n)} D_1^{(2n)}. \tag{128}
\]

The matrix (128) is called the first auxiliary resolvent matrix in the point 0 of the THMM problem in the case of an even number of moments.
Remark 4.13  
(i) Note that $a \Gamma_a = \bar{\alpha}_1^n(a)^{-1} \bar{\beta}_1^n(a)$ and $b \Gamma_b = \bar{\alpha}_1^n(b)^{-1} \bar{\beta}_1^n(b)$.
(ii) We will show in Lemma 4.17 that $b \Gamma_b - a \Gamma_a$ is invertible, hence $N^{(2n)}$ is well-defined.

Now we recall the resolvent matrix of the THMM problem for the case of an odd number of moments given in terms of orthogonal matrix polynomials [7].

Definition 4.14  [7, Theorem 3.5] Let $(s_j)^{2n}_{j=0}$ be a Hausdorff positive definite sequence on $[a, b]$. Let the matrices $P_{3,n}, P_{4,n}, Q_{3,n}$ and $Q_{4,n}$ be as in Definition 2.4. The $2q \times 2q$ matrix polynomial

$$U^{(2n)}(z) := \begin{pmatrix} \alpha^{(2n)}(z) & \beta^{(2n)}(z) \\ \gamma^{(2n)}(z) & \delta^{(2n)}(z) \end{pmatrix}, \quad z \in \mathbb{C},$$

(129)

with

$$\alpha^{(2n)}(z) := Q_{4,n}(z)Q_{4,n}^{-1}(a),$$

(130)

$$\beta^{(2n)}(z) := \frac{1}{b-a} Q_{3,n}^*(\bar{z})P_{3,n}^{-1}(a),$$

(131)

$$\gamma^{(2n)}(z) := (z-a)P_{4,n}^*(\bar{z})Q_{4,n}^{-1}(a),$$

(132)

$$\delta^{(2n)}(z) := \frac{b-z}{b-a} P_{3,n}^*(\bar{z})P_{3,n}^{-1}(a).$$

(133)

is called the resolvent matrix of the THMM problem in the point $a$ in the case of an odd number of moments.

Remark 4.15  Let $Q_{4,n}$ be defined as in (47). Then the matrix $Q_{4,n}(0)$ is invertible.

Proof  Recall that the matrix $Q_{4,n}(a)$ is invertible by Remark 2.8 and that equality (79) gives for $z = 0$

$$I + av_n^* R_n^*(a) H^{-1}_{1,n} u_{1,n} = Q_{4,n}^{-1}(a) Q_{4,n}(0).$$

(134)

Since by Assumption 4.11 the left hand side of (134) is invertible, the matrix $Q_{4,n}(0)$ is invertible.

Definition 4.16  Let $Q_{4,n}$ be as in (47). We set

$$d^{(2n)} := Q_{4,n}^*(0)Q_{4,n}^{-1}(a).$$

(135)

Lemma 4.17  Let $P_{k,n}, Q_{k,n}$ for $k = 3, 4$, $\Gamma_a, \Gamma_b, M^{(2n)}$ and $N^{(2n)}$ be as in Definition 2.4, (124), (125) and (126), respectively. Then the following equalities are valid.

$$\Gamma_a = -P_{4,n}^*(0)Q_{4,n}^{-1}(0),$$

(136)

$$\Gamma_b = P_{3,n}^*(0)Q_{3,n}^{-1}(0),$$

(137)

$$\Gamma_a = \Gamma_a^*.$$

(138)
\[ \Gamma_b = \Gamma_b^*, \] (139)
\[ M^{(2n)} = -a P_{4,n}^*(0) Q_{4,n}^{*-1}(0), \] (140)
\[ N^{(2n)} = (b - a)^{-1} Q_{3,n}^*(0) P_{3,n}^{*-1}(a) Q_{4,n}(0), \] (141)
\[ N^{(2n)} d^{(2n)*-1} = (b - a)^{-1} Q_{3,n}(0)^* P_{3,n}^{*-1}(a), \] (142)
\[ (I + M^{(2n)} N^{(2n)}) d^{(2n)*-1} = (b - a)^{-1} b P_{3,n}^*(0) P_{3,n}^{*-1}(a). \] (143)

**Proof** The equality (136) was shown in the proof of Remark 4.15. To prove (138) and (139) it suffices to note that \( P_{r,n}(0)^* [Q_{r,n}(0)^*]^{-1} = [Q_{r,n}(0)]^{-1} P_{r,n}(0) \) which is an immediate consequence of Remark 2.6. Equation (140) follows directly from the definition of \( M^{(2n)} \) and (136). For the proof of (141) note that by (136), (137) and the selfadjointness of \( \Gamma_a \) we have that

\[
a \Gamma_a - b \Gamma_b = -a Q_{4,n}(0)^{-1} P_{4,n}(0) - b P_{3,n}(0)^* Q_{3,n}(0)^*^{-1}
\]

\[
= -Q_{4,n}(0)^{-1} \left[a P_{4,n}(0) Q_{3,n}(0)^* - b Q_{4,n}(0) P_{3,n}(0)^* \right] Q_{3,n}(0)^*^{-1}
\]

\[
= -Q_{4,n}(0)^{-1} \Sigma_{4,n} \left[a v_n^* u_{3,n}^* - b u_{4,n} v_n^* \right] \Sigma_{3,n} Q_{3,n}(0)^*^{-1}
\]

\[
= - (a - b) (Q_{4,n}(0)^{-1} \Sigma_{4,n} R_n(a) u_{4,n} v_n^* R_n(a)^* \Sigma_{3,n} Q_{3,n}(0)^*^{-1}
\]

\[
= (b - a) (Q_{4,n}(0)^{-1} Q_{4,n}(a)^* P_{3,n}(a)^* Q_{3,n}(0)^*^{-1}.
\]

By Remark 2.8, the matrices \( Q_{4,n}(a)^* \) and \( P_{3,n}(a)^* \) are invertible, hence \( N^{(2n)} \) is well-defined and (141) holds. Now also (142) is clear. Finally, we show (143).

\[(I + M^{(2n)} N^{(2n)}) d^{(2n)*-1} = \left[ [N^{(2n)}]^{-1} + M^{(2n)} \right] N^{(2n)} d^{(2n)*-1} = b \Gamma_b N^{(2n)} d^{(2n)*-1} \]

\[
= (b - a)^{-1} b P_{3,n}^*(0) Q_{3,n}^{*-1}(0) Q_{3,n}(0)^* P_{3,n}^{*-1}(a) = (b - a)^{-1} b P_{3,n}^*(0) P_{3,n}^{*-1}(a).
\]

With \( d^{(2n)} \) as in (135), we define

\[ \mathcal{D}^{(2n)} := \begin{pmatrix} d^{(2n)} & 0 \\ 0 & d^{(2n)*-1} \end{pmatrix}. \] (144)

Now we formulate and prove the main result of this subsection.

**Theorem 4.18** Let \( U^{(2n)}(z), \tilde{\zeta}_q, V_1^{(2n)} \) and \( \mathcal{D}^{(2n)} \) be matrices as in (129), (84), (128) and (144), respectively. Then the following equality holds:

\[ U^{(2n)}(z) = \tilde{\zeta}_q V^{(2n)}(z) \tilde{\zeta}_q \mathcal{D}^{(2n)}. \] (145)

**Proof** Using the block entries (130)–(133), (91)–(94), (128) and (126), we see that (145) is equivalent to the following four equalities:

\[ \alpha^{(2n)}(z) = \left[ \tilde{\gamma}_1^{(n)}(z) M^{(2n)} + \tilde{\delta}_1^{(n)}(z) \right] d^{(2n)}, \] (146)
\[ \beta^{(2n)}(z) = \left[ \gamma_1^{(n)}(z)(I + M^{(2n)}N^{(2n)}) + \tilde{\gamma}_1^{(n)}(z)N^{(2n)} \right] d^{(2n)+1}, \quad (147) \]
\[ \gamma^{(2n)}(z) = \left[ \alpha_1^{(n)}(z)M^{(2n)} + \tilde{\alpha}_1^{(n)}(z) \right] d^{(2n)}, \quad (148) \]
\[ \delta^{(2n)}(z) = \left[ \tilde{\alpha}_1^{(n)}(z)(I + M^{(2n)}N^{(2n)}) + \tilde{\beta}_1^{(n)}(z)N^{(2n)} \right] d^{(2n)+1}. \quad (149) \]

For the proof of (146) and (148) we will use that, by (73),
\[
R_n(\tilde{z})^* H_{1,n}^{-1} \left\{ -au_{1,n}v_n - v_n u_{4,n} \right\} \\
\Sigma_{4,n} = R_n(\tilde{z})^* \left( 1 - aT_n^* \right) \Sigma_{4,n} = (I - aT_n^*) R_n(\tilde{z})^* \Sigma_{4,n}.
\]

Therefore, by (130), (140) and (135), we have:
\[
\left[ \tilde{\gamma}_1^{(n)}(z)M^{(2n)} + \tilde{\beta}_1^{(n)}(z) \right] d^{(2n)} = \left[ -a \tilde{\gamma}_1^{(n)}(z) P_{4,n}(0)^* Q_{4,n}(0)^{*-1} + \tilde{\gamma}_1^{(n)}(z) \right] Q_{4,n}(0)^* Q_{4,n}(0)^{*-1} \\
= \left[ z u_{1,n} R_n(\tilde{z})^* H_{1,n}^{-1} \left\{ -au_{1,n}v_n - v_n u_{4,n} \right\} + u_{4,n}^* \right] \Sigma_{4,n} Q_{4,n}(0)^{*-1} \\
= \left[ z u_{1,n} (1 - aT_n^*) + u_{4,n}^* (1 - zT_n^*) \right] R_n(\tilde{z})^* \Sigma_{4,n} Q_{4,n}(0)^{*-1} \\
= \left[ z u_{1,n}^* R_n(\tilde{z})^* + u_{4,n}^* (I - zT_n^*) \right] R_n(\tilde{z})^* \Sigma_{4,n} Q_{4,n}(0)^{*-1} \\
= u_{4,n}^* R_n(\tilde{z})^* \Sigma_{4,n} Q_{4,n}(a)^{*-1} \\
= Q_{4,n}(\tilde{z})^* Q_{4,n}(a)^{*-1} = \alpha^{(2n)}(z)
\]

where we used that \( u_{1,n}^* (I - aT_n^*) = u_{4,n}^* (I - aT_n^*) T_n^* = u_{4,n}^* T_n^* \). Hence (146) is proved. For the proof of (146), we calculate
\[
\left[ \tilde{\alpha}_1^{(n)}(z)M^{(2n)} + \tilde{\beta}_1^{(n)}(z) \right] d^{(2n)} \\
= \left[ -a \tilde{\alpha}_1^{(n)}(z) P_{4,n}(0)^* Q_{4,n}(0)^{*-1} + \tilde{\beta}_1^{(n)}(z) \right] Q_{4,n}(0)^* Q_{4,n}(0)^{*-1} \\
= \left[ -av_n^* + zv_n^* R_n(\tilde{z})^* H_{1,n}^{-1} \left\{ -au_{1,n}v_n - v_n u_{4,n} \right\} \right] \Sigma_{4,n} Q_{4,n}(a)^{*-1} \\
= \left[ -av_n^* (1 - zT_n^*) + zv_n^* (I - aT_n^*) \right] R_n(\tilde{z})^* \Sigma_{4,n} Q_{4,n}(a)^{*-1} \\
= (z - a) v_n^* R_n(\tilde{z})^* \Sigma_{4,n} Q_{4,n}(a)^{*-1} = (z - a) P_{4,n}(z)^* Q_{4,n}(a)^{*-1} \\
= \gamma^{(2n)}(z).
\]

For the proof of (147) and (147), we will use that, by (72), the following holds:
\[
R_n(\tilde{z})^* H_{1,n}^{-1} \left\{ bu_{1,n}v_n^* - v_n^* u_{3,n}^* \right\} \\
\Sigma_{3,n} = -R_n(\tilde{z})^* (I - bT_n^*) \Sigma_{3,n} = -(I - bT_n^*) R_n(\tilde{z})^* \Sigma_{3,n}.
\]

Therefore, by (143) and (142) we have that
\[
\left[ \gamma_1^{(n)}(z)(I + M^{(2n)}N^{(2n)}) + \tilde{\gamma}_1^{(n)}(z)N^{(2n)} \right] d^{(2n)+1}
Furthermore, assume that

$$A.1.\text{ Expansion in } z$$

Appendix A: Power Series Expansion of the Resolvent Matrices

The authors declare no competing interests.

Declarations

Conflict of interest The authors declare no competing interests.

Appendix A: Power Series Expansion of the Resolvent Matrices

A.1. Expansion in $z = 0$

Let $(s_k)_{k=0}^{2n+1}$ be a Hausdorff positive definite sequence on $[a, b]$. Let $H_{r,n}$, $u_{r,n}$ for $r = 3, 4, R_n$ and $v_n$ be defined as in (5), (33), (34), (35), (36), (12) and (14), respectively. Furthermore, assume that $H_{1,n}$ from (6) is an invertible matrix.

According to Theorem 4 of [14] the entries of the matrix

$$V^{(2n+1)} = \begin{pmatrix} Q_{2n+1}(n)(z) \bar{Q}_{2n+1}(n)(z) \\ \bar{Q}_{2n+1}(n)(z) \bar{Q}_{2n+1}(n)(z) \end{pmatrix}$$

where we used that $u_{1,n}^*(I - bT_n^*) = u_n^*T_n^*(I - bT_n^*) = -u_{3,n}^* T_n^*$. Finally, we prove (149). Again we use (143) and (142) to obtain

$$\mathcal{Q}_1^{(n)}(z)(I + M^{(2n)} N^{(2n)}) + \bar{Q}_1^{(n)}(z) N^{(2n)} d^{(2n)^{-1}}$$

$$= (b - a)^{-1} \left[ b\tilde{\alpha}_1^{(n)}(z) P_{3,n}^*(0) P_{3,n}(a)^{-1} + \bar{\alpha}_1^{(n)}(z) Q_{3,n}(0)^* P_{3,n}(a)^{-1} \right]$$

$$= (b - a)^{-1} \left[ b\tilde{\alpha}_1^{(n)}(z) v_n^* + \bar{\alpha}_1^{(n)}(z) u_{3,n}^* \right] \Sigma_{3,n} P_{3,n}(a)^{-1}$$

$$= (b - a)^{-1} \left[ b\tilde{v}_n^* + z u_n^* R_n(\tilde{z})^* H_{1,n}^{-1} \left[ b u_{1,n} v_n^* - v_n u_{3,n}^* \right] \right] \Sigma_{3,n} P_{3,n}(a)^{-1}$$

$$= (b - a)^{-1} \left[ b\tilde{v}_n^* - z u_n^* (1 - b T_n^*) R_n(\tilde{z})^* \right] \Sigma_{3,n} P_{3,n}(a)^{-1}$$

$$= (b - a)^{-1} \left[ b\tilde{v}_n^* - z u_n^* (1 - b T_n^*) R_n(\tilde{z})^* \right] \Sigma_{3,n} P_{3,n}(a)^{-1}$$

$$= (b - a)^{-1} \left[ b\tilde{v}_n^* - z u_n^* (1 - b T_n^*) R_n(\tilde{z})^* \right] \Sigma_{3,n} P_{3,n}(a)^{-1}$$

$$= \delta_{2n}(z).$$

\[\square\]

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from (98) are

\[
\tilde{\alpha}^{(2n+1)}(z) := I + z v_n^* R_n^*(\tilde{z}) \left( \frac{b H_{4,n} + a H_{3,n}}{b - a} \right)^{-1} u_n, \tag{A1}
\]

\[
\tilde{\gamma}^{(2n+1)}(z) := u_n^* R_n^*(\tilde{z}) \left( \frac{b H_{4,n} + a H_{3,n}}{b - a} \right)^{-1} u_n, \tag{A2}
\]

\[
\tilde{\beta}^{(2n+1)}(z) := (z - b)(z - a) v_n^* R_n^*(\tilde{z}) \frac{a H_{4,n}^{-1} + b H_{3,n}^{-1}}{b - a} v_n, \tag{A3}
\]

\[
\tilde{\delta}^{(2n+1)}(z) := I + u_n^* R_n^*(\tilde{z}) \frac{a(z - b) R_n^{-1}(a) H_{4,n}^{-1} + b(z - a) R_n^{-1}(b) H_{3,n}^{-1}}{b - a} v_n. \tag{A4}
\]

**Remark A.1** Using the Equalities (95) and (56) we find that

\[
a H_{4,n}^{-1} + b H_{3,n}^{-1} = (b - a) H_{4,n}^{-1} \tilde{H}_{1,n} H_{3,n}^{-1} \tag{A5}
\]

and

\[
a(z - b) R_n^{-1}(a) H_{4,n}^{-1} + b(z - a) R_n^{-1}(b) H_{3,n}^{-1} = (b - a) a b T_n^* H_{1,n} - H_{4,n}^{-1} H_{1,n} H_{3,n}^{-1}
\]

\[
+ z(b - a) \left[ H_{4,n}^{-1} \tilde{H}_{1,n} + T_n^* H_{4,n}^{-1}(a b H_{1,n} - (a + b) \tilde{H}_{1,n}) \right] H_{3,n}^{-1}.
\]

Therefore the entries in the matrix \(V^{(2n+1)}\) can be expanded in powers of \(z\) as follows:

\[
\tilde{\alpha}^{2n+1}(z) = I + v_n^* \sum_{j=1}^{n-1} z^j T_n^*(j-1) \tilde{H}_{1,n}^{-1} u_n,
\]

\[
\tilde{\gamma}^{2n+1}(z) = u_n^* \sum_{j=0}^{n} z^j T_n^* \tilde{H}_{1,n}^{-1} u_n,
\]

\[
\tilde{\beta}^{2n+1}(z) = v_n^* \left\{ a b + z \left[ - (a + b) + a b T_n^* \right] + \sum_{j=2}^{n+2} z^j T_n^*(j-2) \left[ I - (a + b) T_n^* + a b T_n^{*2} \right] \right\} \times \tilde{H}_{1,n}^{-1} H_{3,n}^{-1} v_n,
\]

\[
\tilde{\delta}^{2n+1}(z) = u_n^* \left\{ a b + z \left[ (a + b) + a b T_n^* \right] + \sum_{j=1}^{n+1} z^j T_n^*(j-1) \left[ I - (a + b) T_n^* + a b T_n^{*2} \right] \right\} \times \tilde{H}_{1,n}^{-1} H_{3,n}^{-1} v_n
\]

\[
+ u_n^* \left\{ \sum_{j=1}^{n+1} z^j T_n^*(j-1) \left[ I - (a + b) T_n^* + a b T_n^{*2} \right] \right\} H_{4,n}^{-1} \tilde{H}_{1,n} H_{3,n}^{-1} v_n.
\]
and the coefficients of the expansion in (7) are for \(2 \leq j \leq n - 1\)

\[
\tilde{A}_0 = \begin{pmatrix} I & abu_n^*H_{4,n}^{-1} \tilde{H}_{1,n}H_{3,n}^{-1}v_n \\ u_n^*H_{1,n}^{-1}u_n & I + abu_n^*(T_n^*H_{4,n}^{-1}\tilde{H}_{1,n} - H_{4,n}^{-1}H_{1,n}H_{3,n}^{-1}v_n) \end{pmatrix},
\]

\[
\tilde{A}_1 = \begin{pmatrix} v_n^*H_{1,n}^{-1}u_n & v_n^*[abT_n^* - (a + b)I]H_{4,n}^{-1}\tilde{H}_{1,n}H_{3,n}^{-1}v_n \\ u_n^*T_n^*H_{1,n}^{-1}u_n & u_n^*[I - (a + b)T_n^* + ab(T_n^*)^2H_{4,n}^{-1}\tilde{H}_{1,n}H_{3,n}^{-1}v_n] \end{pmatrix},
\]

\[
\tilde{A}_j = \begin{pmatrix} v_n^*T_n^*H_{4,n}^{-1}\tilde{H}_{1,n}^{-1}u_n & v_n^*T_n^*(j-2I - (a + b)T_n^* + abT_n^{*2})H_{4,n}^{-1}\tilde{H}_{1,n}H_{3,n}^{-1}v_n \\ u_n^*T_n^*H_{4,n}^{-1}\tilde{H}_{1,n}^{-1}u_n & u_n^*[I - (a + b)T_n^* + abT_n^{*2}]H_{4,n}^{-1}\tilde{H}_{1,n}H_{3,n}^{-1}v_n \end{pmatrix}.
\]

\[
\tilde{A}_{n+1} = \begin{pmatrix} v_n^*T_n^*H_{4,n}^{-1}\tilde{H}_{1,n}^{-1}u_n & v_n^*T_n^*(n-1I - (a + b)T_n^*)H_{4,n}^{-1}\tilde{H}_{1,n}H_{3,n}^{-1}v_n \\ u_n^*T_n^*H_{4,n}^{-1}\tilde{H}_{1,n}^{-1}u_n & u_n^*[I - (a + b)T_n^* + abT_n^{*2}]H_{4,n}^{-1}\tilde{H}_{1,n}H_{3,n}^{-1}v_n \end{pmatrix},
\]

\[
\tilde{A}_{n+2} = \begin{pmatrix} 0 & v_n^*T_n^*H_{4,n}^{-1}\tilde{H}_{1,n}H_{3,n}^{-1}v_n \\ 0 & 0 \end{pmatrix}.
\]

**Remark A.2** Let the matrices \(C, D\) be as in (127) and let \(\tilde{v}_r^{(n)}\) be as in (90) for \(r = 1\). Then the matrices \(\tilde{B}_0, \tilde{B}_1, \tilde{B}_n\) and \(\tilde{B}_{n+1}\) in the representation (8) are

\[
\tilde{B}_0 = CD,
\]

\[
\tilde{B}_j = \begin{pmatrix} v_n^*T_n^*(j-1I - (a + b)T_n^*)H_{4,n}^{-1}\tilde{H}_{1,n}H_{3,n}^{-1}v_n \\ u_n^*T_n^*(j-1I - (a + b)T_n^*)H_{4,n}^{-1}\tilde{H}_{1,n}H_{3,n}^{-1}v_n \end{pmatrix}.
\]

because

\[
\tilde{\alpha}_r^{(n)}(z) = I + zv_n^*R_n^*(\bar{z})H_{r,n}^{-1}u_{r,n} = I + v_n^*\sum_{j=1}^{n+1} z^j T_n^{*(j-1)}H_{r,n}^{-1}u_{r,n},
\]

\[
\tilde{\beta}_r^{(n)}(z) = -zv_n^*R_n^*(\bar{z})H_{r,n}^{-1}v_n = -v_n^*\sum_{j=1}^{n+1} z^j T_n^{*(j-1)}H_{r,n}^{-1}v_n,
\]

\[
\tilde{\gamma}_r^{(n)}(z) = zu_n^*R_n^*(\bar{z})H_{r,n}^{-1}u_{r,n} = u_n^*\sum_{j=1}^{n+1} z^j T_n^{*(j-1)}H_{r,n}^{-1}u_{r,n},
\]

\[
\tilde{\delta}_r^{(n)}(z) = I - zu_n^*R_n^*(\bar{z})H_{r,n}^{-1}v_n = I - u_n^*\sum_{j=1}^{n+1} z^j T_n^{*(j-1)}H_{r,n}^{-1}v_n.
\]
A.2. Expansion in $z = a$

**Remark A.3** The resolvent matrix $U^{(2n+1)}$ from (99) can be written as in (9) where the coefficient matrices are for $2 \leq j \leq n$ are

$$\tilde{C}_0 = \begin{pmatrix} I & -Q_{1,n+1}^* P_{1,n+1}^* \\ 0 & I \end{pmatrix},$$

$$\tilde{C}_1 = \begin{pmatrix} Q_{2,n}^* (a) Q_{2,n}^* (a) & -Q_{1,n+1}^* P_{1,n+1}^* (a) \\ -(b-a) P_{2,n}^* (a) Q_{2,n}^* (a) & P_{1,n+1}^* (a) P_{1,n+1}^* (a) \end{pmatrix},$$

$$\tilde{C}_j = \frac{1}{j!} \left( j[(j-1) P_{2,n}^* (a) - (b-a) P_{2,n}^* (a)] Q_{2,n}^* (a) - Q_{1,n+1}^* (a) P_{1,n+1}^* (a) \right)$$

$$= \frac{1}{j!} \left( j[(j-1) P_{2,n}^* (a) - (b-a) P_{2,n}^* (a)] Q_{2,n}^* (a) - Q_{1,n+1}^* (a) P_{1,n+1}^* (a) \right)$$

$$\tilde{C}_{n+1} = \frac{1}{(n+1)!} \left( (n+1)[n P_{2,n}^* (a) - (b-a) P_{2,n}^* (a)] Q_{2,n}^* (a) - Q_{1,n+1}^* (a) P_{1,n+1}^* (a) \right)$$

$$\tilde{C}_{n+2} = \frac{1}{n!} \left( P_{2,n}^* (a) Q_{2,n}^* (a) 0 \right).$$

Here $Q_{2,n}^* (a)$ (resp. $Q_{2,n}^* (a)$) denotes the first derivative (resp. $j$-th derivative) with respect to $z$.

**Remark A.4** The resolvent matrix $U^{(2n)}$ from (129) can be written as in (10) where the coefficient matrices are for $1 \leq j \leq n$ are

$$\tilde{D}_0 = \begin{pmatrix} I & \frac{1}{b-a} Q_{3,n}^* (a) P_{3,n}^* (a) \\ 0 & I \end{pmatrix},$$

$$\tilde{D}_j = \frac{1}{j!} \left( Q_{4,n}^* (a) Q_{4,n}^* (a) \right)$$

$$\tilde{D}_{n+1} = \frac{1}{n!} \left( P_{4,n}^* (a) Q_{4,n}^* (a) - \frac{1}{b-a} P_{3,n}^* (a) P_{3,n}^* (a) \right).$$

Here $Q_{4,n}^* (a)$ (resp. $Q_{4,n}^* (a)$) means the first derivative (resp. $j$-th derivative) with respect to $z$. 
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