NON-LINEARIZABLE ACTIONS OF COMMUTATIVE REDUCTIVE GROUPS.

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ABSTRACT. We generalize a construction of Freudenburg and Moser-Jauslin in order to obtain an example of a non-linearizable action of a commutative reductive group on the affine space for every field $k$ of characteristic zero which admits a quadratic extension.

1. Introduction and Summary

Let $G$ be an algebraic group acting on the affine space $\mathbb{A}^n$. If $G$ is reductive it is not so easy to find an example of such an action which is not obviously linearizable. At a time it was even conjectured that at least over an algebraically closed field $k$ every action of a reductive group is linearizable ([1]).

But then Schwarz found a non-linearizable such action ([7]). His example is an action of the complex group $O_2(\mathbb{C})$ which is a semi-direct product of $\mathbb{Z}/2\mathbb{Z}$ and the multiplicative group $G_m(\mathbb{C}) \simeq \mathbb{C}^*$. Related examples have been deduced, but results about commutative reductive groups remain rare. See [2] for more about these topics.

Recently, Freudenburg and Moser-Jauslin constructed an example of a non-linearizable real algebraic action of $S^1$ ([5]). $S^1$ is a non-standard real form of the multiplicative group $G_m$. In this note we will go through the arguments of Freudenburg and Moser-Jauslin in order to verify that in their construction the field of real numbers can be replaced by an arbitrary field of characteristic zero with a non-trivial form of the multiplicative group.

Theorem 1. Let $k$ be a field of characteristic zero, $\bar{k}$ an algebraic closure and $H^0$ a $k$-group. We assume that $H^0$ is isomorphic to the multiplicative group $G_m$ over $\bar{k}$, but not over $k$. (Such $k$-groups are often called non-split $k$-forms of the multiplicative group $G_m$.)

Then there is an action of $H^0$ on the affine space $\mathbb{A}^4$, defined over $k$, which is not linearizable over $k$.

As usual, an action of a $k$-group $H^0$ on the affine space $\mathbb{A}^n$ given by a morphism $\mu : H^0 \times \mathbb{A}^n \rightarrow \mathbb{A}^n$ is called linearizable over $k$ if and
only if there exists an automorphism \( \phi \) of \( \mathbb{A}^n \) as a \( k \)-variety such that \( v \mapsto \phi(\mu(g, \phi^{-1}(v))) \) is a linear map for every \( g \in H^0 \).

**Corollary 1.** Let \( k \) be a field of characteristic zero which contains a non-square \( \alpha \in k^* \setminus (k^*)^2 \).

Then there is a one-dimensional commutative connected reductive \( k \)-group \( G \) with a non-linearizable action on \( \mathbb{A}^4 \).

**Proof.** As discussed in §2 below the existence of a quadratic field extension implies the existence of a non-split \( k \)-form of the multiplicative group \( G_m \). \( \square \)

**Corollary 2.** Let \( k \) be one of the following:

1. the field \( \mathbb{R} \) of real numbers,
2. the field \( \mathbb{Q}_p \) of \( p \)-adic numbers,
3. a number field,
4. a finitely generated extension of \( \mathbb{Q} \),
5. the field of rational functions of a positive-dimensional variety defined over some field \( k_0 \) of characteristic zero,
6. the field of meromorphic functions of a complex manifold which admits a non-constant meromorphic function,
7. \( \mathbb{Q}^{ab} \), i.e. the maximal abelian field extension of \( \mathbb{Q} \).

Then there is a one-dimensional commutative connected reductive \( k \)-group \( G \) with a non-linearizable action on \( \mathbb{A}^4 \).

**Proof.** In view of the preceding results it suffices to show that each of these fields admits a non-trivial quadratic extension.

For \( \mathbb{R} \) one takes \( \mathbb{C} = \mathbb{R}[i] \). For \( k = \mathbb{Q}^{ab} \) we recall that there exists a number field \( E \) with the quaternion group \( \Gamma \) as Galois group (e.g. \( \mathbb{Q} \left[ \sqrt{2} + 2)(\sqrt{3} + 3) \right] \), see [4]). The quaternion group \( \Gamma \) is two-step nilpotent. Hence its commutator group is a non-trivial commutative subgroup \( \Gamma' \). Let \( L \) denote the intermediate field corresponding to \( \Gamma' \). Then \( E/L \) is a quadratic extension. Furthermore, \( L \subset \mathbb{Q}^{ab} \) (because \( \text{Gal}(L/\mathbb{Q}) \cong \Gamma' \) is abelian) and \( E \not\subset \mathbb{Q}^{ab} \) (because \( \text{Gal}(E/\mathbb{Q}) \cong \Gamma \) is not abelian). Thus we obtain a quadratic extension \( EQ^{ab}/\mathbb{Q}^{ab} \).

For all the other fields \( k \) listed above there is a discrete valuation. A discrete valuation gives in particular a group homomorphism \( v \) from the multiplicative group \( k^* \) to the additive group \( (\mathbb{Z}, +) \). Evidently an element \( x \in k^* \) can not be a square unless \( v(x) \) is even. Therefore the existence of a discrete valuation implies that there exists a non-trivial quadratic extension. \( \square \)

Thus most well-known fields are either algebraically closed or admit a quadratic extension. But of course there are also fields which are
neither algebraically closed nor admit a quadratic extension. For instance, let \( \Gamma \) be the absolute Galois group of \( \mathbb{Q} \) and consider all group homomorphisms \( \rho \) from \( \Gamma \) to 2-groups (=finite groups whose order is a power of 2). Then the intersection of the kernels \( \cap \rho \ker \rho \) defines a closed subgroup \( H \subset \Gamma \) such that the corresponding (infinite) extension field \( L \) of \( \mathbb{Q} \) is neither algebraically closed nor admits any quadratic extension.

2. \( k \)-forms

Here we recall some basic facts about \( k \)-forms, see \[\text{[S]}, \text{Ch. 3, \S1}.\]

Let \( k \) be a field of characteristic zero, \( K/k \) be a Galois extension and \( G \) a \( k \)-group. One is interested in classifying all \( k \)-groups \( H \) which are \( K \)-isomorphic to \( G \). It is a standard result that these “\( k \)-forms of \( G \)” are classified by the Galois cohomology set

\[
H^1(\text{Gal}(K/k), \text{Aut}_k(G)).
\]

Here we are interested in the special case where \( G = G_m \) is the multiplicative group. Then \( \text{Aut}_k(G) \simeq \mathbb{Z}/2\mathbb{Z} \), because \( G_m \) admits only two automorphisms: the identity and \( z \mapsto \frac{1}{z} \). Thus the action of \( \text{Gal}(K/k) \) on \( \text{Aut}_k(G_m) \) is necessarily trivial and consequently

\[
H^1(\text{Gal}(K/k), \text{Aut}_k(G_m)) \simeq \text{Hom}(\text{Gal}(K/k), \mathbb{Z}/2\mathbb{Z}).
\]

Hence \( k \)-forms of \( G_m \) are in one-to-one correspondence with group homomorphisms from \( \text{Gal}(K/k) \) to \( \mathbb{Z}/2\mathbb{Z} \) and therefore in one-to-one correspondence to quadratic field extensions of \( k \) which are contained in \( K \).

In concrete terms these \( k \)-forms of \( G_m \) can be described as follows:

Let \( K/k \) be a quadratic extension of fields in characteristic zero. Then there is a non-square \( \alpha \in k \) such that \( K = k[X]/(X^2 - \alpha) \). Each element \( z \in K \) can be written in unique way as \( z = x + ty \) where \( t \) is a fixed element with \( t^2 = \alpha \) and \( x, y \in k \). A quadratic extension is necessarily Galois and the non-trivial element of \( \text{Gal}(K/k) \) acts by \( z = x + ty \mapsto x - ty \). Therefore the Norm homomorphism

\[
N_{K/k} : K^* \rightarrow k^* \]

is given by \( N_{K/k}(x + ty) = (x + ty)(x - ty) = x^2 - \alpha y^2 \). The group of all elements \( z \in K^* \) with \( N_{K/k}(z) = 1 \) can now be identified with the \( k \)-rational points of the following \( k \)-group:

\[
H = \left\{ \begin{pmatrix} x & \alpha y \\ y & x \end{pmatrix} : x^2 - \alpha y^2 = 1 \right\}.
\]

In the special case \( \mathbb{C}/\mathbb{R} \) this yields the real one-dimensional compact torus \( S^1 \).
Let us explain the connection with Galois cohomology. Let $K/k$ be a quadratic extension and let $\sigma$ be the non-trivial element of $\text{Gal}(K/k)$. Then the $k$-form described above corresponds to the cocycle mapping $\sigma$ to the automorphism $z \mapsto z^{-1}$ of $G_m$.

More generally, if $X$ is a variety defined over $K$ with an involution $\tau$ defined over $k$, there is a $k$-form $Y$ of $X$ associated to the cocycle $\sigma \mapsto \tau$ and there is a bijection $Y(k) \simeq \{ p \in X(K) : \sigma p = \tau(p) \}$.

3. The criterion of Masuda-Petrie

Let $G$ be a $k$-group and let $V, W$ be $G$-modules and let $\text{Vec}_G(V, W)$ be the set of $G$-vector bundles over the $G$-variety $V$ such that the fiber over $0_V$ is isomorphic to $W$.

Let us assume that there exists a subgroup $I \subset G$ such that $(V \oplus W)^I = V \oplus \{0\}$ where $(V \oplus W)^I$ denotes the set of fixed points for the $I$-action on $W \oplus V$.

Let $E \in \text{Vec}_G(V, W)$. Apriori, triviality of $E$ as $G$-vector bundle is a stronger condition than linearizability of the $G$-action on the total space: In the second condition all isomorphisms from $E$ to $V \oplus W$ as variety may be used to linearize the $G$-action on $E$ while in the first condition only those isomorphisms are admitted which preserve the vector bundle structure, i.e., are compatible with the projection on $V$ and linear on each fiber. However, Masuda and Petrie showed in [3] that under the above mentioned additional assumption both conditions are equivalent.

Let us recall their arguments, for the convenience of the reader as well as in order to convince us that this works over any base field in characteristic zero. Assume that there is a $G$-isomorphism of varieties $\phi : E \to V \oplus W$. Then $E^I$ is mapped onto $V \oplus \{0\}$. It follows easily that the zero-section of $E \to V$ equals the fixed point set $E^I$. Moreover it follows that $\phi$ induces an isomorphism between the respective normal bundles. But the normal bundle for the zero-section in a vector bundle is isomorphic to the given vector bundle itself. In this way we obtain a trivialization of $E$ as a $G$-vector bundle.

4. The Schwarz action

Let $G$ be the non-trivial semi-direct product of $\mathbb{Z}/2\mathbb{Z} = \{e, \tau\}$ with the multiplicative group $G_m$. Then $G$ is an algebraic group defined over $\mathbb{Q}$ (and therefore defined over every field $K$ of characteristic zero).

Schwarz introduced in [7] an action of $G$ on $\mathbb{A}_4$ which is defined over $\mathbb{Q}$ and is not linearizable over $\mathbb{C}$ (and therefore a fortiori not linearizable
over any field $k$ of characteristic zero). In concrete terms it is given as follows:

We write $\mathbb{Z}/2\mathbb{Z} = \{e, \tau\}$ and define the action on $E = \mathbb{A}^4 = \mathbb{A}^2 \times \mathbb{A}^2$ by

$$
\lambda : \left( \begin{array}{c} a \\ b \\ x \\ y \end{array} \right) \longrightarrow \left( \begin{array}{c} \lambda^2 a \\ \lambda^{-2} b \\ \lambda^3 x \\ \lambda^{-3} y \end{array} \right)
$$

for $\lambda \in G_m(\mathbb{C}) = \mathbb{C}^*$ and

$$
\tau : \left( \begin{array}{c} a \\ b \\ x \\ y \end{array} \right) \longrightarrow \left( \begin{array}{c} b \\ a \\ 1 + ab + (ab)^2 \\ a^3 1 - ab \end{array} \right) \cdot \left( \begin{array}{c} y \\ x \end{array} \right).
$$

Moser-Jauslin showed that the action of $\mathbb{Z}/2\mathbb{Z} = \{e, \tau\}$ can be linearized over $\mathbb{Q}$ (see [6]). In concrete terms this linearization as obtained in [6] can be described as follows: $\phi \circ \tau \circ \phi^{-1}$ is a diagonal linear endomorphism where $\phi$ is the automorphism given as:

$$
\phi : \left( \begin{array}{c} a \\ b \\ x \\ y \end{array} \right) \longrightarrow \left( \begin{array}{c} a \\ b \\ c_{11} c_{12} \\ c_{21} c_{22} \end{array} \right) \cdot \left( \begin{array}{c} x \\ y \end{array} \right)
$$

with

$$
c_{11} = 2(1 + a - 2b - a^2 b + 2ab^2 - b^3),
$$
$$
c_{12} = 2(1 - 2a + b + ab + a^4 - 2a^3 b + a^2 b^2),
$$
$$
c_{21} = -2 - a - b + ab - b^2 + a^2 b - b^3,
$$
$$
c_{22} = 2 + b + a + a^2 + ab - a^3 + a^2 b - a^4 + a^2 b^2.
$$

Note that $\phi$ is defined over $\mathbb{Q}$.

Note furthermore that the map

$$
\left( \begin{array}{c} a \\ b \\ x \\ y \end{array} \right) \longrightarrow \left( \begin{array}{c} a \\ b \end{array} \right)
$$

realizes $E$ as $G$-vector bundle of rank 2 over $\mathbb{A}^2$.

5. A non split form of the Schwarz action

The algebraic group $G$ and the space $E = \mathbb{A}^4$ both admit an involution: On $G$ we take conjugation by $\tau$ and on $E$ we take $\tau$ acting as described in the preceding section. If $\mu : G \times E \to E$ denotes the morphism defining the Schwarz action, then

$$
\mu(g^\tau, \tau v) = \tau \cdot g \cdot \tau^{-1} \cdot \tau v = \tau \mu(g, v).
$$
Thus this pair of involutions on $G$ and $E$ is compatible with the Schwarz action. Furthermore, both involutions are defined over $\mathbb{Q}$.

We fix a quadratic extension $K/k$ of fields of characteristic zero and let $\sigma$ denote the non-trivial element of $\text{Gal}(K/k)$. We fix an element $j \in K$ such that $\sigma j = -j$. Then $K = k \oplus jk$ and $K = k[j] \simeq k[X]/(X^2 - \alpha)$ where $\alpha = j^2$. Now we can obtain $k$-forms of $G$, $E$ and the action $\mu$ associated to the Galois cocycle which maps the non-trivial element $\sigma$ of $\text{Gal}(K/k)$ to the respective involution chosen above. We call these $k$-forms $H$, $E_0$ and $\mu_0$ respectively.

In concrete terms this works as follows:

$$H(k) = \{ g \in G(K) : \sigma g = \tau g \tau^{-1} \}.$$  

Since the involution $\tau$ is defined over $\mathbb{Q}$, it commutes with the Galois action of any Galois group for any Galois extension $K/k$. We define

$$E_0(k) = \{ v \in E(K) \simeq K^4 : \sigma v = \tau(v) \}$$

Using the linearization $\phi$ of the involution $\tau$, we see that a vector $(z_1, \ldots, z_4) \in K^4 \simeq \phi(E)(K)$ is contained in $\phi(E_0(k))$ if $jz_1, z_2, z_3, z_4 \in k$. Therefore $E_0(k) \simeq k^4$ and $E_0 \otimes_k K = E$. Now $H(k)$ is generated by $\tau$ and those $\lambda \in G_m(K) = K^*$ for which $\sigma \lambda \lambda = 1$. Since $\tau$ commutes with $\sigma \circ \tau$ and $E_0(k)$ is the fixed point set of $\sigma \circ \tau$, the set $E_0(k)$ is stabilized by $\tau$. Next consider $\lambda \in (H(k))^0$ and $v \in E_0(k)$. Then $\sigma \lambda \lambda = 1$ and

$$\sigma \circ \tau(\lambda v) = \tau \circ \sigma(\lambda v) = \tau(\sigma \lambda \lambda v) = \tau(\lambda^{-1} \sigma v) = \lambda v.$$  

Hence $\lambda v \in E_0(k)$. Thus $H(k)$ stabilizes $E_0(k)$.

We underline that for the Schwarz action of $G$ on $\mathbb{A}^4$ there is one set of coordinates for which the connected component $G^0$ containing the neutral element acts linearly and a second, different set of coordinates for which $\tau$ acts linearly, but no coordinate set for which the whole group $G$ acts linearly.

The $k$-form $E_0$ of $E = \mathbb{A}^4$ is isomorphic to $\mathbb{A}^4$ as a $k$-variety. We deduced this using the set of coordinates in which $\tau$ is linear. In these coordinates the $G^0$-action is not linear. Therefore the $H^0$-action on $E_0$ need not be linearizable (over $k$) although the $G^0$-action on $E$ is.

We also remark that there is a unique fixed point $0$ for $H^0$ on $E_0$. It follows that the $H^0$-action in $E_0$ is either not linearizable at all or is isomorphic to the induced $H^0$-action on the tangent space $T_0E_0$ of this fixed point.

Furthermore, the involution $\tau$ as well as the automorphism of $E$ linearizing $\tau$ are compatible with the structure of $E$ as a $G$-vector
bundle, which implies that $E_0$ inherits this structure, i.e. is an $H$-vector bundle of rank 2 over $\mathbb{A}^2$.

6. Reduction from $H$ to $H^0$

**Proposition 1.** Let $K/k$ be a quadratic extension of fields of characteristic zero, $\text{Gal}(K/k) = \{\text{id}_K, \sigma\}$, $S = \{x \in K^*: N_{K/k}(x) = x^{\sigma} x = 1\}$ and $j \in K \setminus k$. For $\lambda \in S$ define a morphism $\mu_\lambda: \mathbb{A}^2 \rightarrow \mathbb{A}^2$ by

$$\mu_\lambda(x, y) = (\lambda^2 x, \lambda^3 y)$$

and let $\tau: K^2 \rightarrow K^2$ be a map with the following properties:

(i) $\tau(x, y) = (\sigma x, \phi(x, y)) \ \forall x \in K$ for some map $\phi: K^2 \rightarrow K$.

(ii) The map $y \mapsto \phi(x, y)$ is $k$-linear in $y$ for every fixed $x \in K$.

(iii) There is a $k$-morphism $\tau_0: \mathbb{A}^4 \rightarrow \mathbb{A}^4$ such that

$$\tau_0(z, w, u, v) = (a, b, c, d)$$

whenever

$$\tau(z + jw, u + jv) = (a + jb, c + jd) \quad (z, w, u, v, a, b, c, d \in k)$$

(iv) $\tau \circ \tau = \text{id}_{K^2}$.

(v) $\tau \circ \mu_\lambda = \mu_\lambda \circ \tau$ for all $\lambda \in S$.

Then there exists an element $\alpha \in S$ such that $\tau(x, y) = (\sigma x, \alpha^{\sigma} y)$ for all $x, y \in K$.

In particular, the map $\tau: K^2 \rightarrow K^2$ is $k$-linear.

**Remark.** In the formulation of the above proposition $S$ is a group acting by $k$-linear transformations on $K^2 \simeq k^4$. The group $S$ and the map $\tau$ give a semi-direct product of $\mathbb{Z}/2\mathbb{Z}$ and $S$ acting on $K^2 \simeq k^4$ and the proposition says that under certain assumptions the action of this semi-direct product on $K^2 \simeq k^4$ is necessarily also by $k$-linear transformations.

**Proof.** Condition (iv) implies that $\tau$ is bijective.

Any $k$-linear map between $K$ vector spaces can be written as a sum of a $K$-linear and a $K$-antilinear map. Therefore condition (ii) implies that there are maps $\phi', \phi'': K \rightarrow K$ such that

$$\phi(x, y) = \phi'(x)y + \phi''(x)^{\sigma} y$$

for all $x, y \in K$.

Then

$$\tau(x, y) = (\sigma x, \phi'(x)y + \phi''(x)^{\sigma} y) \quad \forall x, y \in K.$$
The condition $\tau \circ \tau = id_{K^2}$ translates into:

(1) \[ \phi'(\sigma x)\phi'(x) + \phi''(\sigma x)\phi''(x) = 1 \quad \forall x \in K \]

(2) \[ \phi'(\sigma x)\phi''(x) + \phi''(\sigma x)\phi'(x) = 0 \quad \forall x \in K \]

Condition (v) yields: It follows that

(3) \[ \phi'(x) = \lambda^6 \phi'((\lambda^2)x) \quad \forall x \in K, \lambda \in S \]

(4) \[ \phi''(x) = \phi''((\lambda^2)x) \quad \forall x \in K, \lambda \in S \]

Condition (iii) implies that $\phi'$ can be written as a finite sum

\[ \phi'(x) = \sum_{k,l} a_{k,l} x^k (\sigma x)^l \quad (a_{k,l} \in K) \]

Then

\[ \lambda^6 \phi'((\lambda^2)x) = \sum_{k,l} a_{k,l} \lambda^{6+2k-2l} x^k (\sigma x)^l \quad \forall x \in K, \lambda \in S \]

because $\lambda = \lambda^{-1}$ for $\lambda \in S$. Hence equation (3) implies that $a_{k,l} = 0$ unless $3 + k = l$. It follows that

\[ \phi'(x) = (\sigma x)^3 R(x\sigma x) \quad \forall x \in K \]

for some polynomial $R \in K[T]$.

Similarly the equation (4) implies that

\[ \phi''(x) = Q(x\sigma x) \quad \forall x \in K \]

for some polynomial $Q \in K[T]$.

It follows that

\[ \phi''(\sigma x) = \phi''(x) = Q(x\sigma x) \]

and

\[ \phi'(\sigma x) = x^3 R(x\sigma x) \]

for all $x \in K$.

Using these identities, equation (1) transforms into

(5) \[ (x\sigma x)^3 (R(x\sigma x))^2 + Q(x\sigma x)\sigma Q(x\sigma x) = 1 \]

If

\[ Q(z) = \sum_{k=0}^{d} c_k z^k \]

with $c_d \neq 0$, then the map

\[ z \mapsto Q(z)\sigma Q(z) \quad (z \in k) \]

is given by a polynomial $P \in k[T]$ of degree $2d$ with top coefficient $c_d \sigma c_d \neq 0$. 
Hence equation (5) implies
\[ z^3(R(z))^2 + P(z) = 1 \]
for all \( z \in k \) which can be written in the form \( z = x^\sigma x = N_{K/k}(x) \).
The above equation thus holds for infinitely many \( z \in k \) and therefore
implies an identity of polynomials.
However, the first term \( (z^3(R(z))^2) \) is a polynomial in \( z \) of odd degree
(unless \( R \equiv 0 \)) while the second term \( (P(z)) \) is a polynomial of even
degree.
From this it follows that this identity can hold only in the degenerate
case where \( R \equiv 0 \) and \( P \equiv 1 \).
Then \( Q \) is a constant polynomial taking an element of \( S \) as value. □

7. Proof of the theorem

Proof. Due to the classification of \( k \)-forms of \( G_m \) it follows that there
is a quadratic extension of fields \( K/k \) with \( \text{Gal}(K/k) = \{id_K, \sigma\} \) such
that \( H^0 \) arises as described above in §5. We consider \( G \) and \( H \) and
their actions \( \mu \) resp. \( \mu_0 \) on \( \mathbb{A}^4 \) as defined in §5 above. We claim that
the \( H^0 \)-action \( \mu_0 \) on \( E_0 \simeq \mathbb{A}^4 \) is not linearizable. To see this, assume
the contrary. Recall that \( E_0 \) is a \( H \)-vector bundle over \( \mathbb{A}^2 \). Observe
that \((-1)^\sigma(-1) = 1 \) and that therefore \( I = \{1, -1\} \subset S = \{z \in K^*: \)
\( z^\sigma z = 1\} \). Regarded as subgroup of \( H^0(K) \simeq S \), the group \( I \) acts on \( E_0 \)
fixing precisely the zero-section. Therefore we can employ the criterion
of Masuda-Petrie (§3) and deduce that \( E_0 \) must be trivial as \( H^0 \)-vector
bundle over \( \mathbb{A}^2 \). Hence, after applying a suitable \( k \)-automorphism of
\( E_0 \), the \( H^0 \)-action on \( E_0 \) becomes linear. Now \( H \) is the semi-direct
product of \( \mathbb{Z}/2\mathbb{Z} = \{e, \tau\} \) and \( H_0 \) with
\[
H_0(k) = \{g \in G(K) : \sigma g = (g)^{-1}\}\]
\[= S = \{\lambda \in G_m(K) = K^*: N_{K/k}(\lambda) = 1\} \].
Linearity of the \( H^0 \)-action on \( E_0 = k^4 \) implies that there is \( H^0 \)-equivariant
\( k \)-isomorphism between \( E_0 \) and the tangent space at the fixed point
\( 0 \in E_0 \). Therefore we can identify \( E_0 = k^4 \) with \( K^2 \) in such a way that
the \( H^0 \)-action takes the form
\[ \lambda : (x, y) \mapsto (\lambda^2 x, \lambda^3 y) \quad \forall \lambda \in S. \]
Because \( E_0 \) is a \( H \)-vector bundle over \( k^2 \), conditions \((i) - (iii)\) of the
proposition in §6 are fulfilled. The group structure of \( H = \{e, \tau\} \rtimes H^0 \)
implies that conditions \((iv)\) and \((v)\) are fulfilled as well.
Hence we may invoke this proposition and deduce that the action of
\( \tau \) on \( E_0 \) is already be given by a \( k \)-linear map. Since \( H \) is generated
by \( \tau \) and \( H^0 \), it follows that the \( H \)-action on \( E_0 \) is necessarily \( k \)-linear.
as soon as the $H^0$-action is $k$-linear. However, the action of $H$ is a $k$-form of the $G$-action on $E$. It follows that the $G$-action on $E$ must be linearizable. This is a contradiction, because the latter action is not linearizable over $\mathbb{C}$. □

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