Cross-over from BCS superconductivity to Bose condensation and high-$T_c$ superconductors

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Abstract

We consider the Eliashberg theory in the coupling region where some fundamental qualitative deviations from the conventional BCS-like behaviour begin to appear. These deviations are identified as the onset of a cross-over from BCS superconductivity to Bose condensation. We point out that the beginning of this cross-over occurs when the gap $\Delta_g$ becomes comparable to the boson energies $\Omega_{ph}$. This condition is equivalent to the condition of Ref. [20] $k_F \xi \approx 2\pi$ and traduces the physical constraint that the distance the paired electron covers during the absorption of the virtual boson, cannot be larger than the coherence length. The frontier region of couplings is of the order of $\lambda \approx 3$, and high-$T_c$ materials are concerned. A clear qualitative indication of the occurrence of a cross-over regime should be a dip structure above the gap in the density of states of excitations. Comparing our results with tunneling and photoemission experiments we conclude that high-$T_c$ materials (cuprates and fullerides) are indeed at the beginning of a cross-over from BCS superconductivity to Bose condensation, even though the fermionic nature still prevails. Taking into account the analysis of Ref. [20], we predict a dip structure in heavy fermion and organic superconductors. Non-adiabatic effects beyond Migdal’s theory are considered and give insight on the robustness of Eliashberg theory in describing qualitatively this cross-over regime, although for the quantitative interpretation of the results the inclusion of non-adiabatic corrections can be important.

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I. INTRODUCTION

Since the discovery of high-$T_c$ superconductivity a large effort has been made to understand its origin but the question remains unsolved and highly controversial. Although attempts have been made to fit the high-$T_c$ superconducting properties within the conventional Eliashberg theory (ET) framework \([1, 2, 3, 4, 5]\), many people believe that the relevant mechanism might be new based also on the idea that the critical temperatures accessible by conventional theories should exceed $\approx 20K$ \([6]\). There is not rigorous justification for this idea which is due principally to an erroneous conclusion based on partial numerical results by McMillan \([7]\), corrected a long time ago by Allen and Dynes \([8]\).

Actually, there is not formal limitation on the critical temperatures one can reach within ET provided the electron-boson coupling strength or mass enhancement parameter $\lambda$ is sufficiently strong. However it is not clear which are the maximal couplings $\lambda$ for which ET is valid. Indeed as the coupling strength grows the system should leave the BCS pairing regime and switch to a Bose condensation regime \([9, 10, 11]\). This last regime is considered by several authors as relevant in relation to high temperature superconductivity \([12, 13, 14]\).

Little is known on the conditions under which this cross-over occurs. Several authors have investigated the intermediate regime between BCS pairing and Bose condensation for particular models \([15, 16, 17, 18, 19, 20]\). For a most general treatment of this very important intermediate regime it is necessary to extend Eliashberg theory beyond Migdal’s theorem. Although significant progress have been achieved in this last direction \([21, 22, 23, 24]\) the problem remains extremely complicate.

It is surprising that a study of ET at $\lambda \to \infty$ (where of coarse ET is not valid) leads to a regime with local pairs \([25]\), however quantitatively this regime is very different from the realistic Bose condensation regime \([11]\). This is an important remark that points to
an unexpected robustness of ET. Here we investigate the question of the cross-over from BCS superconductivity to Bose condensation from the ET point of view \cite{26}. We will be concentrated in the region of couplings where we first identify signs of a cross-over regime. We will see in particular which are the couplings for which the need to go beyond Migdal’s theorem is manifest within ET. To this fundamental question there is not clear answer up to now. We will try also to locate the high-$T_c$ materials with respect to this cross-over.

This paper is organized as follows. In section II we study the structure of the gap function and point out that the cross-over should occur when the gap $\Delta_g$ becomes of the order of the phonon frequency $\Omega_D$. In section III we give a simple physical interpretation of this condition and show its equivalence to the condition derived in Ref. \cite{20} from a different perspective. In section IV we discuss the location of high-$T_c$ SC with respect to this crossover. In particular we analyze the properties of the density of states and we conclude that they are indeed close to this crossover but still preserve a fermionic character. In section V we discuss the robustness of Eliashberg theory by including corrections beyond Migdal’s theorem. Finally in section VI we discuss the conclusions of this work and outline the possible developments.

II. CROSS-OVER AND THE GAP FUNCTION

When the characteristic energies of the superconducting state (gap, $T_c$) are not completely negligible compared to the boson energies (which are the characteristic energies for the variations of the propagators in the coupled electron-boson problem) for the correct description of the BCS-type superconductivity it is necessary to consider the retarded nature of the effective interaction and to use the more elaborate ET framework. Although initially this framework was developed for the treatment of phonon mediated super-
conductivity, its applicability does not depend on the nature of the exchanged bosons, provided the energies of those bosons are low compared to the electronic energies and the applicability of Migdal’s theorem is preserved. That is why in the following we will discuss about phonons or bosons indifferently.

The gap function is proportional to the anomalous Green’s function and therefore it is intimately related with the order parameter and the nature of the superconducting state. Within the Eliashberg theory framework this function is the solution of the Eliashberg equations \[27, 28\]. For the solution of these equations we follow the Marsiglio Shossmann and Carbotte method \[29\]. We first solve the Eliashberg equations for imaginary Matsubara energies

\[
\Delta(i\omega_n)Z(i\omega_n) = \pi T \sum_m \left[ \int_0^\infty d\Omega \frac{2\Omega \alpha^2(\Omega) F(\Omega)}{\Omega^2 + (\omega_n - \omega_m)^2} - \mu_c \Theta(\omega_c - |\omega_m|) \right] \frac{\Delta(i\omega_m)}{\sqrt{\omega_m^2 + \Delta^2(i\omega_m)}} \tag{1a}
\]

\[
Z(i\omega_n) = 1 + \frac{\pi T}{\omega_n} \sum_m \int_0^\infty d\Omega \frac{2\Omega \alpha^2(\Omega) F(\Omega)}{\Omega^2 + (\omega_n - \omega_m)^2} \frac{\omega_m}{\sqrt{\omega_m^2 + \Delta^2(i\omega_m)}} \tag{1b}
\]

and then we continue analytically to real energies by solving iteratively the following system of equations

\[
\Delta(\omega, T)Z(\omega, T) = \pi T \sum_m \left[ \int_0^\infty d\Omega \frac{2\Omega \alpha^2(\Omega) F(\Omega)}{\Omega^2 - (\omega - i\omega_m)^2} - \right.

- \mu^*(\omega_c) \Theta(\omega_c - |\omega_m|) \left] \frac{\Delta(i\omega_m, T)}{\sqrt{\omega_m^2 + \Delta^2(i\omega_m, T)}} \right. +

+i \pi \int_0^\infty d\Omega \alpha^2(\Omega) F(\Omega) \left\{ \left[ N(\Omega, T) + f(\Omega - \omega, T) \right] \frac{\Delta(\omega - \Omega, T)}{\sqrt{(\omega - \Omega)^2 - \Delta^2(\omega - \Omega, T)}} \right.

+ \left[ N(\Omega, T) + f(\Omega + \omega, T) \right] \frac{\Delta(\omega + \Omega, T)}{\sqrt{(\omega + \Omega)^2 - \Delta^2(\omega + \Omega, T)}} \right\} \tag{2a}
\]
\[ Z(\omega, T) = 1 + \frac{i\pi T}{\omega} \sum_m \int_0^\infty d\Omega \frac{2\Omega \alpha^2(\Omega) F(\Omega)}{\Omega^2 - (\omega - i\omega_m)^2} \frac{\omega_m}{\sqrt{\omega_m^2 + \Delta^2(i\omega_m, T)}} + \]

\[ + \frac{i\pi}{\omega} \int_0^\infty d\Omega \alpha^2(\Omega) F(\Omega) \left\{ \left[ N(\Omega, T) + f(\Omega - \omega, T) \right] \frac{\omega - \Omega}{\sqrt{(\omega - \Omega)^2 - \Delta^2(\omega - \Omega, T)}} + \right. \]

\[ \left. + \left[ N(\Omega, T) + f(\Omega + \omega, T) \right] \frac{\omega + \Omega}{\sqrt{(\omega + \Omega)^2 - \Delta^2(\omega + \Omega, T)}} \right\} \]  

(2b)

where the functions \( N(\Omega, T) \) and \( f(\Omega, T) \) are the statistical functions of bosons and fermions respectively and \( \Delta(i\omega_m, T) \) is the solution of equations (1). The electron-boson coupling spectral function, or Eliashberg function \( \alpha^2(\Omega) F(\Omega) \), is related to the coupling strength \( \lambda \) by

\[ \lambda = \int_0^\infty \frac{2\alpha^2(\Omega) F(\Omega) d\Omega}{\Omega}. \]  

(3)

The solution of equations (1) and (2) provides the complex gap function

\[ \Delta(\omega, T) = \Delta_1(\omega, T) + i\Delta_2(\omega, T) \]  

(4)

with excellent precision at any temperature regime \[28\]. This allows a detailed analysis of the structure of the gap functions. The superconducting gap \( \Delta_g \) is related to the real part of the gap function by

\[ \Delta_g(T) = \text{Re}\{\Delta(\omega = \Delta_g(T), T)\} \]  

(5)

In order to isolate the coupling effects and avoid interference with spectral effects we begin with the analysis of a single Einstein frequency for the boson spectrum defined by

\[ \alpha^2(\Omega) F(\Omega) = \frac{\lambda \Omega_E}{2} \delta(\Omega - \Omega_E) \]  

(6)
We then consider the solutions of Eliashberg equations for different couplings and temperature regimes, with emphasis on the region of coupling where qualitative deviations from the conventional BCS behavior begin to appear.

II.a The zero temperature behavior

Some of our resulting complex gap functions at the zero temperature regime are shown in figure (1). We can clearly identify the sharp structures associated with the Einstein boson peak. The real part of the gap function \( \Delta_1(\omega, T) \) is a measure of the effective boson mediated quasiparticle-quasiparticle interaction. The behavior of this function for \( \lambda = 1 \) (figure 1a) is essentially conventional. For frequencies lower than \( \Omega_E + \Delta_g \) it is positive indicating that the effective electron interaction is attractive, while for higher energies it is negative indicating a repulsive effective interaction. If one considers that the mediating bosons are phonons, as it is the case in conventional low-\( T_c \) superconductors, this behavior is associated to the fact that the lattice can only be polarized in phase by charge fluctuations with energies lower or of the order of its characteristic energies. The pronounced peak at \( \omega = \Omega_E + \Delta_g \) is due to the resonant exchange of phonons at this frequency. It is interesting to note that multiphonon processes are also visible.

For larger couplings (figures 1b and 1c) the remarkable point on the \( \Delta_1(\omega, T = 0) \) behavior is that the multiphonon processes become dominant for superconductivity. In fact, the largest value of \( \Delta_1 \) corresponds to such processes. Associated to this is the fact that \( \Delta_1(\omega, T) \) remains positive at energies which are several times higher than the characteristic boson energies \( \Omega_E \). In the case of phonon mediated superconductivity this should indicate that for sufficiently strong couplings the lattice is polarized in phase even if the energies of the charge fluctuations are much higher than its characteristic energies.

Another important point is that for energies lower than \( \Omega_E + \Delta_g \) the gap function is essentially structureless. This is a fundamental characteristic of Eliashberg equations and
it is closely related to the s-wave symmetry of the considered pairing. In fact, considering the imaginary part of the gap function $\Delta_2(\omega, T)$ one can see that this function is zero in this region of frequencies. Since $\Delta_2$ is associated with the damping effects, this means that there are no relaxation processes in this region of frequencies or, in other terms, the gap is sharply defined and there are no available states inside the gap. It is not difficult to see from Eliashberg equations that when $\Delta_2 = 0$ the real part $\Delta_1$ is structureless.

When the coupling grows, $\Delta_g$ becomes comparable to the phonon (or other boson) energies $\Omega_E$. Since the real part of the gap function $\Delta_1(\omega, T \approx 0)$ is structureless for frequencies lower than $\Omega_E + \Delta_g$, it remains positive at least in this energy range, even if the characteristic phonon frequencies are much lower. A new scale for the variations of $\Delta_1(\omega, T \approx 0)$, independent from the phonon frequency scale $\Omega_E$, is introduced naturally in this way. The dominance of multiboson processes is closely related with the considered s-wave symmetry of the pairing.

The importance of multiphonon processes is also clear in the imaginary part $\Delta_2(\omega, T \approx 0)$ of the gap function. One can see that the maximal values of $\Delta_2(\omega, T \approx 0)$ are obtained for energies equal to the gap plus several times the Einstein energy $\Omega_E$, indicating that the most important virtual phonon (or other boson) emission takes place at those energies. The dominance of multiboson processes induces a broadenning of $\Delta_2(\omega, T \approx 0)$. The broadenning of $\Delta_2(\omega, T \approx 0)$ is associated with the strong breakdown of Landau Fermi liquid picture for the virtually excited states occupied by the coupled electrons.

When the effective interaction responsible for superconductivity originates from the exchange of virtual bosons, this interaction is more important if the energy is almost conserved during the emission or the absorption of the boson. As a result the most important configurations are those with excitations (from the Fermi level) of the order of $\Omega_E$. The validity of the quasiparticle picture for those virtually excited states is not
necessary within the conventional Eliashberg framework. Provided Migdal’s theorem is valid, the breakdown of the quasiparticle picture for the virtually excited states of the coupled electrons does not imply a non Fermi liquid behaviour in the normal state. In fact when electron-boson vertex corrections are neglected within Migdal’s theorem, the self-energy effects in the off-diagonal sector of the theory are not at all related with the self-energy effects in the diagonal sector of the theory. The question of the validity of Migdal’s theorem and of the occurrence or not of a cross-over from BCS superconductivity to Bose condensation, is equivalent here with the question of whether one can admit so important self-energy effects in the off-diagonal sector of the theory, without consider any influence on the diagonal sector.

It is not clear whether the dominance of multiboson processes characterising those important self-energy effects in the off-diagonal sector, is conceptually acceptable or if it is simply due to an arbitrary extension of Eliashberg theory beyond the limits of its validity. In fact the dominance of multiphonon processes implies that the lattice can be polarized in phase by charge fluctuations with much higher frequencies than its characteristic frequencies and this result seems rather unphysical. For the moment we just remark this point as one of the anomalies with respect to the conventional weak coupling behavior. We remark also that this anomaly manifest when the gap $\Delta_g$ becomes comparable to the boson energies $\Omega_E$ at sufficiently strong couplings ($\lambda > 2$).

II.b The finite temperature behavior

We have seen in the previous subsection that for couplings of the order of $\lambda \approx 3$, some important qualitative deviations from the conventional low-temperature behavior begin to appear. The nature of those deviations casts doubts on the validity of ET at those coupling regimes. We investigate here the behavior at finite temperatures, and study in particular if analogous anomalies occur in the temperature dependence of the gap
function. We give in figures 2 and 3 some of our results at two characteristic temperature regimes and for the same couplings as in figure 1. At finite temperatures, the qualitative behavior of the gap function changes dramatically when the coupling grows. Since we use in all cases an Einstein boson spectrum those anomalies are not associated with the spectral structure and the following discussion is of general validity. Those changes are related to the changes in the zero temperature regime since they appear simultaneously as the coupling grows.

Let us examine more precisely these finite temperature anomalies. At finite temperature the structure of the gap function is complex. We examine first the weak coupling calculations. (figures 2a and 3a). In addition to the structures at $\Delta_g + n\Omega_E$ present also at low temperatures, new structures appear at energies $n\Omega_E - \Delta_g(T)$. Those structures were identified a longtime ago [27] as due to the recombination processes. In those processes the injected electron combines with a thermally excited quasiparticle and forms a bound pair by emitting $n$ bosons. The signature of those processes appears at energies of the order of $n\Omega_E - \Delta_g(T)$ since adding the energy of the injected electron $\Delta_g$ and subtracting the energy of the $n$ emitted bosons $n\Omega_E$ gives zero, namely the energy of the bound pair. Those processes are not present in the zero temperature regime because thermally excited quasiparticles above the gap are necessary. The absence of recombination processes at low temperatures is therefore related with the s-wave symmetry of the considered pairing. As we will see in the following those processes play a very important role at higher couplings, and are in fact responsible for the anomalous (with respect to the weak coupling BCS-like regime of ET) temperature dependence of the gap function.

The first anomaly at higher couplings with respect to the $\lambda = 1$ regime appears in the low energy behavior of $\Delta_1(\omega, T)$. The low energy behaviour is very important since it is directly related to the superconducting gap by equation (5). We remark that when $\lambda = 1$
(figs. 2a, 3a) the real part of the gap function is almost constant at low energies. Then the superconducting gap $\Delta_g$ obtained from equation (5) is almost identical to the zero energy solution of Eliashberg equations which is of course the same in real and imaginary energies. We can consider therefore the gap as an equilibrium quantity since Eliashberg equations for imaginary energies (1) are sufficient to define it. The fact that $\Delta_1(\omega, T)$ is constant at low energies is associated with the fact that $\Delta_2(\omega, T)$ is zero at those energies, and therefore there are no available states inside the gap $\Delta_g(T)$ even at finite temperatures.

At stronger couplings (figs. 2b, 2c, 3b, 3c) $\Delta_1(\omega, T)$ is no more constant at low energies. In fact we see an important enhancement of $\Delta_1(\omega, T)$ at low energies. This behaviour is accompanied by a non zero value of $\Delta_2(\omega, T)$ in this region of energies. The superconducting gap $\Delta_g$ is no more completely determined by the imaginary axis Eliashberg equations and it becomes a dynamic quantity.

The non-zero value of $\Delta_2(\omega, T)$ and the rapid variation of $\Delta_1(\omega, T)$ at low energies are principally due to the one boson recombination processes. In fact one can follow the evolution with the coupling of the peak at $\Omega_E - \Delta_g(T)$ associated with the one boson recombination processes. When the coupling grows the value of $\Delta_g(T)$ is more important and therefore the recombination processes appear at lower energies in the $\Omega_E$ units. When the coupling is sufficiently large and the gap $\Delta_g(T)$ becomes of the order of $\Omega_E/2$ or greater the recombination processes influence the relevent $\omega \leq \Delta_g(T)$ region of the gap function. The rapid enhancement of $\Delta_1(\omega, T)$ at low energies and the negative value of $\Delta_2(\omega, T)$ can be associated with the peaks of the one praticle recombination processes at $\Omega_E - \Delta_g(T)$. Of course when the peak due to the recombination processes appears in the $\omega \leq \Delta_g(T)$ region of $\Delta_1(\omega, T)$, the gap $\Delta_g(T)$ given from equation (5) becomes larger and therefore the influence of recombination processes becomes even larger and so on.
In addition the recombination processes can also be considered as partially responsible for the general smoothing of the gap function for large couplings. Looking for example in figure (2b) one can see that the gap function is surprisingly smooth given the fact that we use an Einstein spectrum. Thermal smearing is not sufficient to explain this fact since for $\lambda = 2$ and $T = 0.7 T_c$ (fig. 2b) the temperature is always low compared to the boson energies ($T \approx 0.15 \Omega_E$). In fact if $\Delta_g(T) \geq \Omega_E/2$ we have a superposition of normal processes which appear at $n \Omega_E + \Delta_g(T)$ and recombination processes which appear at $(n + 1) \Omega_E - \Delta_g(T)$. This superposition together with the thermal smearing explains the general smoothness of the gap function.

On the other hand, the imaginary axis solutions of the Eliashberg equations remain close to the BCS behavior even for strong couplings. We can see this in figure (4) where we plot the temperature dependence of $[\Delta(i\omega_n = 0, T)]^2$ solution of equations (1) for $\lambda = 3$. For weak couplings we expect this function to vary linearly with temperature near $T_c$. This is due to the fact that when $\lambda \to 0$ we have $\Delta_g(T) \approx \Delta(i\omega_n = 0, T)$ and in the BCS regime as well as in the moderate coupling regime of ET, the order parameter $\Delta_g(T)$ has a critical exponent $1/2$. This exponent is associated with the mean field nature of those situations. We remark in figure 4 that this exponent is valid even for $\lambda = 3$, since $[\Delta(i\omega_n = 0, T)]^2$ has indeed a linear temperature dependence near $T_c$.

The significance of the observation of the $1/2$ exponent near $T_c$ is clear. It is associated with the fact that in the BCS regime (as well as in the conventional moderate coupling regime of ET) the range of the effective interaction or the diameter of the pairs or the coherence length $\xi$ are infinite or very large. As a consequence there is not any visible critical regime and the mean field treatment, which implies the $1/2$ exponent for the order parameter, is valid even near $T_c$. When the gap $\Delta_g$ remains an equilibrium property for weak couplings, this standard behavior is recovered. But, as the coupling grows, the
recombination processes influence $\Delta_g$ which becomes now a dynamical property. However, it has been shown that ET describes with excellent precision the $T \to T_c$ behavior of the high-$T_c$ cuprates [30] and that in those materials we are precisely in a regime where the gap $\Delta_g$ associated with the energy at which the experimental density of states is maximal $\Delta_{\text{exp}}$ [30] is not an equilibrium property but a dynamic one.

The fact that the recombination processes induce this deviation from the mean field behavior is significant. This clearly indicates that the system enters into a fluctuation critical regime at strong couplings with continuous thermal pair breaking and pair recombination. However, by constitution, Eliashberg theory cannot treat correctly this fluctuation critical regime given its mean field nature. We expect for example the existence of preformed pairs above $T_c$, which is impossible within ET. It is already remarkable that this theory gives such a precise indication for the onset of this fluctuation regime. When the gap becomes a dynamic quantity it is necessary to go beyond Migdal’s theorem in order to treat correctly the resulting critical fluctuation regime.

In the $T \to 0$ regime we remarked that the multiboson processes become dominant for superconductivity when, at sufficiently strong couplings, the gap $\Delta_g(T \approx 0)$ becomes comparable to the boson characteristic energies $\Omega_E$. This reflects extremely important self-energy effects in the off-diagonal sector inducing a strong breakdown of the quasiparticle picture. The question is whether one can admit so important self-energy effects in the off-diagonal sector without any influence on the diagonal sector. This question is equivalent to the question of the validity of Migdal’s theorem. Here, in the finite temperature regime, we remarked that when $\Delta_g(T) \geq \Omega_E/2$ the recombination processes influence the gap $\Delta_g(T)$ which becomes a dynamic quantity indicating the development of a critical fluctuation regime. For the correct treatment of this regime it is necessary in principle to go beyond Migdal’s theorem. In both temperature regimes the need to go beyond Migdal’s
Theorem reflects the cross-over from the BCS pairing to the Bose condensation regime. We conclude therefore that the cross-over from BCS superconductivity to Bose condensation occurs when the gap $\Delta_g$ becomes comparable to the characteristic boson energies. In other terms, Eliashberg theory is clearly sufficient only for couplings lower or of the order of $\lambda = 3$.

III. PHYSICAL MEANING OF THE CROSS-OVER CONDITION $\Delta_g \approx \Omega_{ph}$

From the analysis of the behavior of the gap function at $T \to 0$ and $T \to T_c$ we conclude that the cross-over from BCS to Bose condensation starts when the gap $\Delta_g$ becomes of the order of the phonon energies $\Omega_{ph}$. Here we will arrive to the same conclusion by a simple physical argument. In this way we will give a simple physical interpretation to the condition for the cross-over $\Delta_g \approx \Omega_{ph}$.

The basic difference between the BCS and the Bose condensation regimes relies on the way the electrons absorb or emit the virtual phonons which carry the attractive interaction. In the case of BCS condensation the electrons travel during the absorption or emission of the virtual phonons while in the case of Bose condensation the electrons are immobile. When we start from the BCS limit and we enhance the coupling through Eliashberg theory we suppose that electrons are mobile during absorption or emission of the virtual phonons.

The characteristic energy of the exchanged phonons is $\Omega_{ph}$. The characteristic time for the absorption or emission of the virtual phonon $\tau$ is related to the phonon energies $\tau \approx \Omega_{ph}^{-1}$. During this time the paired electron travel with the fermi velocity $v_F$ and covers a distance $L = v_F \tau$. This distance might be related to the coherence length $\xi$ which in the case of a BCS-like regime characterizes the range of the attractive interaction. The gap $\Delta_g$ is directly related to the coherence length $\xi$ by $\xi = v_F/(\pi \Delta_g)$. Physically,
the condition $\Delta_g \leq \Omega_{ph}$ is equivalent with the condition $L \leq \xi$. If the distance the electron covers during the absorption or emission of the virtual phonon is larger than $\xi$ the BCS concept of the coherence length has no more physical meaning, and this puts the limits of validity for Eliashberg theory. It is clear that retardation effects are fundamental for the correct description of the intermediate regime, in agreement with what has been argued by Zheng, Avignon and Bennemann [19].

Recently Pistolesi and Strinati [20], studying the model fermionic hamiltonian introduced by Nozières and Schmitt-Rink [15], arrived to the conclusion that the natural variable for the cross-over from BCS superconductivity to Bose condensation is the product $k_F\xi$ and that Cooper-pair-based superconductivity is stable against bosonization down to $k_F\xi \approx 2\pi$. Our results are in remarkable agreements with their conclusions. In particular their condition for the beginning of the cross-over from BCS superconductivity to Bose condensation $k_F\xi \approx 2\pi$ is equivalent with our condition $\Delta_g \approx \Omega_{ph}$ as can be easily seen from the previous discussion.

The interpretation given in Ref. [20] of the Uemura plot [31] can be transposed in terms of the relationship between $\Delta_g$ and $\Omega_{ph}$ directly associated with the coupling strength $\lambda$ within the Eliashberg framework. Within our analysis, the materials which are close to the dashed line $T = T_B$ in the Uemura plot (see Fig. 5) are the materials in which the gap $\Delta_g$ is of the order of the relevant phonon frequencies for superconductivity $\Omega_{ph}$. Within the conventional Eliashberg framework this means that these materials have a coupling strength $\lambda \approx 3$. The closer the materials are to the $T = T_B$ line the higher is the coupling. It is well known in particular that the coupling in Nb is higher than the coupling in Al, and that the coupling in the Chevrel phases is even higher [28].

The conclusion of Ref. [20] that the line $T = T_B$ in the Uemura plot (dashed line in Fig. 5) is associated with the condition $k_F\xi = 2\pi$ is equivalent with the condition $\Delta_g \approx \Omega_{ph}$
within our analysis and can be verified by independent experiments in the superconducting state. In particular exploiting the gap ratio measurements one can obtain information on the relative importance of the gap $\Delta_g$ and the relevant phonon frequencies $\Omega_{ph}$. A systematic study of the gap ratio spectral dependence leads to the conclusion that in the cuprates as well as in the fullerides the gap is precisely of the order of the relevant phonon energies and that within Eliashberg theory in both materials the couplings are strong $\lambda \approx 3$. Cuprates and fullerides are precisely close to the $T = T_B$ line in figure 5. The interpretation of Ref. [20] to the Uemura plot and its equivalent translation given here are therefore confirmed by the independent analysis of Ref. [32].

In the following we will see that some characteristic structures in the tunneling and photoemission spectra of the oxides and the fullerides (dip-like and second peak structures above the gap), will give further support to the previous analysis. In fact, a dip structure structure above the gap has been observed in the density of states of cuprates and fullerides. As we will see in the next section, this structure arises naturally when the gap is of the order of the relevant for superconductivity phonon energies, and its observation in cuprates and fullerides indicates that these materials are at the beginning of cross-over from BCS to Bose, although of course the fermionic nature clearly prevails. A final justification of the interpretation of Ref. [20] to the Uemura plot (and its traduction given here), will be the eventual experimental observation of the dip structure (and the other anomalies described in the next section) on all the materials which are close to the $T = T_B$ line in figure Fig. 5. In other terms we predict the existence of a dip structure above the gap in the density of states of organic, heavy fermion and chevrel superconductors.

**IV. DENSITY OF STATES AND HIGH-TEMPERATURE SUPERCONDUCTORS**
In the previous sections we localized the cross-over from BCS superconductivity to Bose condensation. We pointed out that retardation effects are fundamental for the occurrence of the cross-over and we gave a physical interpretation to the condition we derived $\Delta_g \geq \Omega_{ph}$ which is in fact equivalent to that derived in Ref. [20]. From the interpretation of Ref. [20] to the Uemura plot [31] we obtained some evidence that the high-$T_c$ materials are close to this cross-over. In this section we will try to situate more clearly the high-$T_c$ materials with respect to this cross-over, and statuate on the necessity or not to go beyond Migdal’s theorem in order to understand completely the high-$T_c$ phenomenology. In that way we will test independently the interpretation of the Uemura plot given in Ref. [20] taking advantage from the equivalent image of this interpretation we proposed in section 3.

The anomalies discussed in sections II.a and II.b can be visible in the one particle excitations spectrum. The density of states of the one particle excitations is of fundamental importance since it is an experimentally accessible quantity. Tunneling and photoemission experiments may give direct information on the quasiparticle spectrum of a superconductor, and thus they are considered to be of great importance for the theoretical understanding of High-$T_c$ superconductivity [34].

Those experiments are strongly dependent on eventual surface inhomogeneities. In particular for $YBa_2Cu_3O_7$ there are great uncertainties on the surface stoichiometry and the bulk superconducting properties are not easily accessible by a surface technique. Those problems are amplified by the fact that the coherence length $\xi$ is very short in these materials. Those experimental difficulties generated great controversies at the beginning of high-$T_c$ superconductivity in particular concerning the existence or not of a Fermi surface or of a gap. Now the experiments are improved and the existence of a Fermi surface is clearly established by photoemission. On the other hand, the surface problems
which are very important for the study of the superconducting behavior are very greatly reduced if one replace $YBa_2Cu_3O_7$ by $Bi_2Sr_2CaCu_2O_8$.

The more precise angle resolved photoemission (ARPES) experiments on $Bi_2Sr_2CaCu_2O_8$\textsuperscript{35, 36} show clearly, with the onset of superconductivity, a transfer of spectral weight from the gap region to a peak at approximately $45 \text{meV}$ below the Fermi energy, which indicates the opening of the superconducting gap $\Delta_g$. However at higher energies they show some structures which appear to be anomalous. They show in particular a dip-like structure which appears for energies of the order of $80 - 90 \text{meV}$ below $E_F$, (in all directions in [36] and mostly in the $\Gamma - M$ direction in [35]). Data from reference [36] also indicate the onset of a broad band or second peak at higher energies, just after the dip-like structure.

On the other hand, completely analogous features have been reported from tunneling experiments \textsuperscript{37, 38}. Furthermore, by comparing ARPES data in the $\Gamma - M$ direction from [35] and tunneling data from [37], one can see that these features have not only almost the same structure, but they also have almost the same temperature dependence. In particular their energetic position appears to be almost temperature independent. This gives support to the idea that these are not experimental artifacts, but might represent specific structures of the quasiparticle density of states.

Up to now these structures have been considered to represent the non-conventional nature of $Bi_2Sr_2CaCu_2O_8$ superconductivity, and have been analyzed by Littlewood and Varma in the context of the Marginal Fermi Liquid \textsuperscript{40}, by Anderson \textsuperscript{41} in the context of his model of interlayer tunneling superconductivity, by Alexandrov and Ranninger \textsuperscript{42} in the context of a polaronic system, and by D.Coffey and L.Coffey \textsuperscript{43} as evidence of d-wave pairing. As we will see in the following, all these exotic features can also be reproduced and understood in the context of s-wave, conventional, strong coupling, Eliashberg theory.
of superconductivity \[14, 15\].

Within our framework, the density of states of quasiparticles in the superconducting state is related to that in the normal state by

\[
\frac{N_s(\omega, T)}{N_n(\omega, T)} = \text{Re} \left\{ \frac{\omega}{\sqrt{\omega^2 - \Delta^2(\omega, T)}} \right\}
\]

where \(\Delta(\omega, T) = \Delta_1(\omega, T) + i\Delta_2(\omega, T)\) is the gap function, solution of the Eliashberg equations.

Since we will discuss now qualitatively the experimentally reported anomalous structures we will use a more realistic spectral function for our calculations. We will consider the Eliashberg function of \(Pb\), provided by inversion of tunneling data \[46\] (shown in figure (6)) multiplied by a constant factor \(a\). The coupling strength \(\lambda\) defined by

\[
\lambda = a \int_{0}^{\infty} d\Omega \alpha^2(\Omega) F(\Omega)/\Omega
\]

adjusting the parameter \(a\), will enable us to consider three coupling regimes, \(\lambda = 1\), \(\lambda = 3\) and \(\lambda = 5\). As will be shown later, the choice of the spectral function will not have much effect since the interesting structures are not associated with spectral structures, and our qualitative discussion is generic. The coulomb pseudopotential will be taken equal to zero, bearing in mind that a nonzero value of \(\mu^*\) reproduces the same phenomenology but for slightly higher values of the coupling, which depend on the exact value of \(\mu^*\).

The calculated density of states for different temperatures, and for the cited values of \(\lambda\), is reported in figure (7). For \(\lambda = 1\), (Fig. 7a), the density of states has the standard BCS behaviour. The peak indicating the gap \(\Delta_g(T)\) is clear, and the density of states is zero for \(\omega < \Delta_g(T)\). The temperature dependence of the peak position, indicating the temperature dependence of the gap, follows quite well the BCS law \(\Delta(T) \approx 3.06 T_c \sqrt{1 - T/T_c}\).

As we already noticed the experimental behaviour of \(Bi_2Sr_2CaCu_2O_8\) is different from this behaviour on four counts. First, the density of states inside the gap is nonzero for \(T \neq 0\), and furthermore it rises with rising \(T\). Second, the peak position does not seem
to move to zero as \( T \to T_c \), third, after the peak there is a dip-like structure already mentioned, and fourth, there is a second peak structure.

Looking now to the stronger coupling calculations, \( \lambda = 3 \) and \( \lambda = 5 \), (Figures 7b and 7c), one can see that all four above-mentioned anomalies with respect to the BCS phenomenology are present. The finite density of states inside the gap at \( T \neq 0 \) and the anomalous temperature dependence of the peak position for strong couplings were first reported numerically by Allen and Rainer [47] together with the absence of the Hebel-Slichter peak in the NMR relaxation at the same couplings. We report here that this behavior is accompanied, at higher and temperature independent frequencies, by a dip-like and a second-peak structure. The form and temperature dependence of these structures are very similar to the behavior of the experimentally reported anomalous structures [35, 36, 37, 38]. We insist to the fact that the temperature independence of the energetic position of those structures reported experimentally in Refs. [37, 38] is perfectly reproduced in figure (7). One can conclude already that the quasiparticle spectrum provided by tunneling and photoemission on \( Bi_2Sr_2CaCu_2O_8 \), is consistent with strong coupling s-wave boson exchange superconductivity.

Before going further in analyzing the physical meaning of those structures, we can make in figure (7) another important remark. The exact frequency position (in \( \Delta_g(T = 0) \) units) of the dip-like structure and of the second peak structure are coupling strength dependent. The higher are the couplings the lower are the energies (in \( \Delta_g(T = 0) \) units) at which those structures appear. The origin of this behavior will be discussed later. This provides us with a fundamentally new tool in extracting the coupling strength from experiments. To exploit this new tool it is necessary to study systematically the influence of the spectral structure. The systematic exploitation of this new tool will be left for a future publication. For the moment we just precise here that a crude comparison of experimental data with
the present calculations leads to couplings of the order of \( \lambda \approx 2.5 - 4.0 \) although the experimental data seem not sufficiently precise. Therefore, our analysis is quantitatively compatible with the experimentally reported structures.

Let us analyze now carefully what is the physical meaning of those anomalies. The first point we remark is that those structures are not associated with spectral structures. To illustrate this we display in figure (8) the density of states obtained using an Einstein spectrum with \( \lambda = 3 \) at \( T = 0.3T_c \). We can clearly identify the dip-like structure and the second peak structure to be completely independent from the spectrum which causes sharp structures in the shorter phonon frequency \( \Omega_E \) scale. In fact as we will point out in the following those anomalous structures are associated with the anomalies in the gap function localized in section II, and indicate that \textit{high-}T_c \textit{materials are at the beginning of a cross-over from BCS superconductivity to Bose condensation, the fermionic character being clearly dominant.} Notice that in the same conclusion arrived Rietveld Chen and Van der Marel [48] from a completely independent experimental analysis.

To understand the anomalous dip-like and second peak structures which appear at energies higher than the gap, it is useful to make a \( \Delta(\omega, T) / \omega \ll 1 \) expansion of equation (7), as it has been done in the pioneering work of Scalapino Schrieffer and Wilkins [49]. The reduced density of states can be written as follows

\[
\frac{N_n(\omega, T)}{N_n(\omega, T)} \approx 1 + \frac{1}{2} \left[ \left( \frac{\Delta_1(\omega, T)}{\omega} \right)^2 - \left( \frac{\Delta_2(\omega, T)}{\omega} \right)^2 \right] + \ldots
\] (8)

The dip-like structure appears when \( \Delta_1(\omega, T) \) drops down and \( \Delta_2(\omega, T) \) becomes dominant. In the conventional weak coupling regime the energetic scale of the variations of the gap function is determined by the boson energies. Therefore, equation (8) illustrates why the boson spectrum of the conventional superconductors is visible in the tunneling experiments on conventional superconductors.

However, as we have seen in section II.a, at strong couplings, the scale of the variations
of the gap function is no more associated with the boson energies. This is the reason why
the dip-like and second peak structures are not associated with the spectral structure. As
we discussed in section II.a, this new scale of variations of the gap function reflects very
important self-energy effects in the off-diagonal sector. The dip-like structure indicates,
therefore, the breakdown of the Landau Fermi-liquid picture because of the strong coupling
to bosons.

The second peak structure is also not associated with spectral structures and follows
the development of the dip-like structure. Looking in figure (1c), one can see that
\( \Delta_1(\omega, T) \) is positive at low energies, then changes sign at approximately \( \Delta_g + 3\Omega_E \)
and then it changes for a second time sign at approximately \( \Delta_g + 6\Omega_E \). The second
peak structure is associated precisely with this second sign change of \( \Delta_1(\omega, T) \). This
is obvious if we compare figure (1c) with figure (8). We have seen that \( \Delta_1(\omega, T) \) is
positive at low energies because the lattice is polarized in phase by charge fluctuations
with energies lower or of the order of its characteristic energies and the effective electron
interaction reflected by \( \Delta_1(\omega, T) \) is attractive at those energies. It is unclear why the
effective interaction after being repulsive at higher energies, becomes once more attractive
at even higher energies. However this behavior is clearly associated to the fact that the
gap function has now a new scale of variations.

In the \( \lambda \to \infty \) and \( \lambda = 100 \) calculations presented in reference [25], the density
of states has a multipeaked structure with narrow peaks (delta peaks for \( \lambda \to \infty \)) at
energies which are multiples of the gap energy. It has been suggested by S. Kivelson (see
Ref. 18 in reference [25]) that those multiple peak structures might indicate the local
nature of the pairs in this regime. However this \( \lambda \to \infty \) regime of Eliashberg theory is
quantitatively different from a Bose condensation regime [11]. This is natural since the
Bose condensation regime corresponds to a zero Fermi velocity and therefore Migdal’s
Theorem has no more any sense.

Although Eliashberg theory is certainly not valid in the $\lambda \to \infty$ regime considered in [23], it is absolutely not clear if it is so in the much more moderate regime $\lambda \approx 3$ considered here. We think nevertheless, that the picture proposed in Ref. [23] for the multipeaked structure might be relevant for our second peak structure. Indeed the second peak structure might be associated with the presence at $\lambda \approx 3$ of almost localized pairs. Within this picture our results indicate the coexistence in the high-$T_c$ materials of BCS pairing and almost localized pairs. Apparently it is possible to have both BCS excitations breaking the superconducting order and “individual” excitations of almost localized pairs.

The association of the second peak structure with the existence of almost localized pairs is in agreement with the presence of this peak just after the dip-like structure. We have seen in fact that the dip-like structure indicates the strong breakdown of quasiparticle picture for the states occupied by the paired electrons. In the BCS limit where the states occupied by the coupled electrons are free electron states (plane waves), the pairs are extended to infinity ($\xi \to \infty$). The important self energy effects reflected by the dip-like structure result to an important limitation of the coherence length $\xi$. In the high-$T_c$ materials in particular, the coherence length $\xi$ is of the order of the lattice spacing [341], and thus the presence of almost localized pairs is not unreasonnable. Therefore, dip-like structure and second peak structure are perfectly compatible and both reflect the very important self-energy effects in the off-diagonal sector due to the strong electron-boson coupling.

Associating the second peak with the presence of almost localized pairs, we are also able to understand why this structure appears at lower energies (in $\Delta_g(T = 0)$ units) when the coupling grows. In fact the second peak structure must appear at energies higher than $3\Delta_g$, because $\Delta_g$ is the energy of the injected electron and for the excitation
of a completely localized pair (with $\xi \to 0$) one needs an additional energy of the order of $2\Delta_g$. However the pairs are not completely localized in the coupling regime that we consider here ($\xi \neq 0$) that is why we call them almost localized pairs. For the excitation of an almost localized pair we need an energy higher than $2\Delta_g$ since one has to take into account the superposition with the other almost localized pairs. When the coupling grows the spatial extension of the pairs and also the superposition of the pairs is reduced. As a result, when the coupling grows, the second peak appears at lower energies in $\Delta_g$ units reflecting precisely the reduction of the superposition of the almost localized pairs. We understand, therefore, why the energetic position (in $\Delta_g(T = 0)$ units) of the anomalous dip-like and second-peak structures can be an interesting new tool for extracting the coupling strength from experiment.

Concerning the temperature independence of the energetic position of those structures, we can understand its origin looking in figures (1c) and (3c). Comparing those last figures one can see that for couplings of the order of $\lambda \approx 3$ the gap function has the same energetic scale for its variations at $T = 0.3T_c$ (figure 1c) and $T = 0.9T_c$ (figure 3c). The temperature independence of the energetic position of the dip-like and second peak structures, reflects the temperature independence of the energetic scale of the variations of the gap function.

We already remarked that the new scale of the variations of the gap function appears when the gap $\Delta_g$ becomes comparable to the boson energies. Since the scale of variations of the gap function is then influenced by the gap $\Delta_g$ its temperature independence might be associated with the anomalous temperature behaviour of $\Delta_g(T)$ reflected by the first peak in the density of states. Therefore, the anomalous temperature behaviour at low energies ($\omega \leq \Delta_g$) of the density of states first reported in [47] is intimately related to the temperature independence of the energetic position of the "anomalous" dip-like and second-peak structures at higher energies.
We have seen previously that the whole "anomalous" behavior at high energies of the density of states can be associated with the high energy behavior of the gap function at those strong couplings. It is natural to expect that the anomalous temperature dependence of the density of states at low energies might reflect the anomalous temperature behavior of the gap function at lower energies. We have discussed this behavior in section II.b. We have seen in particular that when $\Delta_g(T) \geq \Omega_E/2$ the recombination processes influence the relevant $\omega \leq \Delta_g$ region in the density of states indicating that the system develops a critical fluctuation regime. As a result the gap is no more a thermodynamic quantity but becomes a dynamic quantity.

The anomalous temperature dependence of the first peak in the density of states (indicating the anomalous temperature dependence of $\Delta_g(T)$ discussed in section II.b) and the finite density of states inside the gap are due to the recombination processes and they also indicate that high-$T_c$ materials are at the beginning of a cross-over from BCS superconductivity to Bose condensation. We remark at that point that using a phenomenological parametric approach, Dynes et al. [51] identified recombination processes as responsible for the much weaker damping effects in the density of states of low temperature superconductors. As we discussed in section II.b, the dominance of those processes is evidence of a cross-over only when it concerns the low energy region (gap region $\omega \leq \Delta_g(T)$). This occurs at sufficiently strong couplings ($\lambda > 2$) and in the high-$T_c$ superconducting state.

V. UNDERSTANDING THE ROBUSTNESS OF ELIASHBERG THEORY BY INCLUDING CORRECTIONS BEYOND MIGDAL'S THEOREM

In the previous section we have seen that high-$T_c$ materials are at the beginning of a cross-over from BCS superconductivity to Bose condensation. However Eliashberg theory appears surprisingly robust in the description of the phenomenology of those materials and
therefore appears quantitatively relevant in a regime where in principle a generalization beyond Migdal’s theorem should be in principle necessary.

Recently there have been detailed studies of the first order non-adiabatic corrections to Migdal’s theorem \[21, 22\]. In particular the Eliashberg equations can be generalized near \(T_c\) in the following way \[52\]:

\[
\Delta(i\omega_n)Z(i\omega_n) = \pi T_c \sum_m \frac{\lambda_\Delta(i\omega_n, i\omega_m; Q_c) \Omega_E^2}{(\omega_n - \omega_m)^2 + \Omega_E^2} \frac{\Delta(i\omega_m) 2\pi \arctg \left(\frac{E}{2Z(i\omega_m)|\omega_m|}\right)}{\arctg \left(\frac{E}{2Z(i\omega_m)|\omega_m|}\right)} \tag{9}
\]

\[
Z(i\omega_n) = 1 + \frac{\pi T_c}{\omega_n} \sum_m \frac{\lambda_Z(i\omega_n, i\omega_m; Q_c) \Omega_E^2}{(\omega_n - \omega_m)^2 + \Omega_E^2} \frac{\omega_m 2\pi \arctg \left(\frac{E}{2Z(i\omega_m)|\omega_m|}\right)}{\arctg \left(\frac{E}{2Z(i\omega_m)|\omega_m|}\right)} \tag{10}
\]

where \(\Omega_E\) is the energy of the considered phonon Einstein spectrum, and the adiabatic corrections were included in the effective coupling functions \(\lambda_\Delta\) and \(\lambda_Z\) defined as follows

\[
\lambda_\Delta(i\omega_n, i\omega_m; Q_c) = \lambda [1 + 2\lambda P_V(i\omega_n, i\omega_m; Q_c) + \lambda P_c(i\omega_n, i\omega_m; Q_c)] \tag{11}
\]

\[
\lambda_Z(i\omega_n, i\omega_m; Q_c) = \lambda [1 + \lambda P_V(i\omega_n, i\omega_m; Q_c)] \tag{12}
\]

The functions \(P_V\) and \(P_c\) refer to the first order non-adiabatic corrections in the diagonal and off-diagonal sectors respectively and \(\lambda\) is the electron-phonon coupling in the conventional Eliashberg theory framework. It is supposed a momentum independent scattering up to a momentum cut-off \(Q_c = q_c/2k_F\) and in the case of half-filling \(E = 2E_F\). Details on the considered model and the approximations made to obtain the first non-adiabatic corrections \(P_V\) and \(P_c\) and on their detailed structure can be found in Ref. \[22\].

A very important remark has been done on the behavior of the physical solutions of equations (9-12). It has been pointed out in Ref. \[52\] that the frequency structure of \(P_V\) and \(P_c\) is marginal in particular with the momentum cut-off dependence of \(T_c\) obtained solving equations (9-12). Neglecting the dependence of the non-adiabatic corrections \(P_V\) and \(P_c\) on the Matsubara frequencies and replacing them by their average over those frequencies, leads with a good precision to the same \(T_c\) as that obtained if one take into account the full frequency dependence \[52\].
When the $\omega_m$ dependence is neglected, equations (9-12) have a form very similar to that of the conventional Eliashberg equations (Eqs. 1a and 1b) for $T \to T_c$. This is more clear for weak couplings where the renormalization function $Z(i\omega_n)$ can be replaced by the unity in the module of arctg in equations (9) and (10). Then the non-adiabatic corrections play the role of effective couplings, which nevertheless have a strong dependence on the momentum cut-off $Q_c$ and on the Migdal parameter $m = \Omega_E/E_F$, and maybe different for different properties [52]. Unfortunately it is rather difficult to write generalized Eliashberg equations at $T \to 0$. This should be very interesting since one could maybe give a more complete interpretation to the condition for the cross-over $\Delta_g(T = 0) \approx \Omega_{ph}$, introducing both momentum and adiabaticity effects. Work in this direction is in progress.

The fact that the first order non-adiabatic corrections act as an effective coupling, gives a clue to understand the robustness of Eliashberg theory in the description of the superconducting state even when the cross-over from Eliashberg theory to Bose condensation has started. This indicates that the corrections beyond Migdal’s theorem should have more qualitative effects in the normal state than in the superconducting state. If we examine the high-$T_c$ phenomenology we remark that this is precisely the case, and therefore the detailed study of such corrections might be an important step for the understanding of some anomalies in the normal state phenomenology.

VI. CONCLUSIONS

We studied carefully the Eliashberg theory behavior in the coupling region where some important qualitative deviations from the BCS behavior appear. Such deviations are:

- Dominance of multiphonon processes for superconductivity;

- Dissociation of the scale of variations of the gap function from the phonon energetic scale;

26
The superconducting gap $\Delta_g$ is no longer a thermodynamic quantity but becomes a dynamic one;

- The energetic scale of the variations of the gap function is temperature independent;

We showed that all these qualitative deviations occur when $\Delta_g$ becomes of the same order with the characteristic energies of the phonons (or other bosons in “exotic” theories) which mediate the attractive interaction $\Omega_{ph}$, and indicate the beginning of a cross-over from BCS superconductivity to Bose condensation. We pointed out that the condition $\Delta_g \approx \Omega_{ph}$ is in fact equivalent to the condition $k_F\xi \approx 2\pi$ derived in Ref. [20] and also to the physical constraint $L \approx \xi$ where $L$ is the distance the paired electron covers during the absorption of the virtual phonon and $\xi$ the superconducting coherence length.

The previous anomalies are reflected by anomalies in the density of states of excitations which is an experimentally accessible quantity. Such anomalies are:

- Finite density of states inside the gap at finite temperatures;
- Enhancement of the density of states inside the gap by enhancing the temperature;
- The peak indicating the experimental gap do not move to zero when $T \to T_c$;
- Dip structure above the gap at temperature independent but coupling strength dependent energies;
- Second peak structure above the dip which has the same temperature and coupling dependence with the dip structure;

We showed that the finite density of states inside the gap and the whole anomalous temperature behavior of the density of states is due to the recombination processes. The anomalous dip and second peak structures indicate the breakdown of Fermi liquid picture for the virtually excited states occupied by the paired electrons, because of the strong electron-phonon coupling.

The interpretation done in Ref. [20] of the Uemura plot [31] is confirmed from our
analysis. Taking into account the analysis of Ref. [20] we predict the presence of a dip structures (and all the previously cited anomalies) in all the materials that are close to the \( T = T_B \) line in figure 5 (cuprates, fullerides, organic superconductors, heavy fermion compounds, chevrel phases etc.) Since within Eliashberg theory the cross-over starts at couplings of the order of \( \lambda > 2 \) (for which \( \Delta_g \approx \Omega_{ph} \)), if we refer to the experimental gap ratios \( 2\Delta_g(T \approx 0)/T_c \) of cuprates and fullerides [32, 33], we have another independent evidence that the high-\( T_c \) materials might be concerned from the previously cited anomalies. Indeed, tunneling and photoemission experiments on cuprates [35, 36, 37, 38, 53] and fullerides [39] appear in perfect agreement with the previous predictions. \emph{We predict a dip structure in organic and heavy fermion superconductors}. The eventual observation of the dip in these last materials will definitely confirm our analysis.

Another important point emerging from the comparison of our analysis with the experimental behavior of cuprates, is the surprising robustness of Eliashberg theory at least concerning the one-particle excitation spectrum considered here. In fact the critical fluctuation regime [30] and the anomalously large self-energy effects in the off-diagonal sector resulting in the coexistence of BCS pairing and almost localized pairs, are situations for which the generalisation of Eliashberg theory beyond Migdal’s theorem appears in-avoidable. The robustness of our framework is understood considering a generalization of Eliashberg equations to include first order non-adiabatic corrections [21, 22, 52]. The frequency structure of the aconsidered corrections, is for some properties irrelevant, in which case the generalized Eliashberg equations have a similar structure with the bare-ones, except that the effective couplings have an internal momentum structure and can be different for different properties. Since the first non adiabatic corrections do not have significant influence on the frequency structure of Eliashberg equations, they do not introduce significant qualitative effects on the resulting phenomenology. Therefore, the bare
equations remain qualitatively relevant at the beginning of the cross-over regime.

On the other hand, non-adiabatic corrections can have important even qualitative implications on other parts of the phenomenology. This is more probable in the dynamical properties and especially in the normal state infrared conductivity. In fact while the low energy dynamic behaviour in the high-$T_c$ superconducting state appears to be evidence for the relevance of Eliashberg theory [54] and the behaviour near $T_c$ quantitatively understood within ET [30], the temperature dependence of the infrared conductivity in the normal state is difficult to accomodate within this framework [55]. In general we expect the deviations from ET induced by those corrections more important in the normal state. If we consider for example the NMR relaxation rate measurements, although the absence of the Hebel-Slichter peak in the superconducting state due to a development of a critical fluctuation regime [30] can be understood within ET [17, 30, 56, 54], for the understanding of deviations from the Korringa behavior in the normal state (which should be due to the existence of preformed pairs above $T_c$) it is necessary to consider non-adiabatic vertex corrections.

In conclusion our results indicate that high-$T_c$ materials are at the beginning of a cross-over from BCS superconductivity to Bose condensation, the BCS component being dominant. Retardation effects are fundamental for the occurrence of this cross-over and therefore might be included in any attempt to describe the intermediate regime [19]. The Eliashberg conventional framework appears to be rather robust especially concerning the one particle behavior as well the low energy dynamics in the superconducting state. The eventual deviations of the high-$T_c$ phenomenology from the Eliashberg behavior (especially in the normal state) might be due to the impossibility of this theory to describe completely this cross-over regime. The study of non adiabatic vertex corrections might be fundamental for the understanding of those deviations.
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FIGURE CAPTIONS

**Figure 1:** The real part (full lines) and the imaginary part (dashed lines) of the gap function solution of the Eliashberg equations using an Einstein spectrum at the “zero” temperature regime $T = 0.3T_c$ and for couplings: (a) $\lambda = 1$, (b) $\lambda = 2$, and (c) $\lambda = 3$.

**Figure 2:** Same as in figure 1 for the finite temperature regime $T = 0.7T_c$. The couplings considered are: (a) $\lambda = 1$, (b) $\lambda = 2$, and (c) $\lambda = 3$.

**Figure 3:** Same as in figure 2 for $T = 0.9T_c$.

**Figure 4:** The temperature dependence of the square of the gap function solution of equations (1) for a zero imaginary energy. Notice the linear temperature dependence near $T_c$.

**Figure 5:** The Uemura plot (fig. 3 in Ref. [31]). We predict a dip structure above the gap (and various other anomalies described in section IV) in the density of states of all the materials that are close to the $T = T_B$ line. In these materials, the superconducting gap $\Delta_g$ is of the same order with the characteristic energies of the boson mediators of superconductivity (in conventional theories the relevant phonon energies).

**Figure 6:** The Eliashberg function of Pb used in the calculations of figure 7.

**Figure 7:** The reduced density of states $N_s(\omega, T)/N_n(\omega, T)$ for the Pb spectrum (shown in figure 5), at different coupling regimes, and for temperatures $T = 0.3T_c$ (full),
$T = 0.7T_c$ (dotted), $T = 0.9T_c$ (dashed), $T = 0.95T_c$ (dot-dashed), and $T = 0.975T_c$ (triple-dot-dashed).

**Figure 8:** The reduced density of states obtained using an Einstein spectrum with $T = 0.3T_c$ and $\lambda = 3$. The corresponding gap function is shown in figure 1c.