Global regularity for minimal sets near a $T$-set and counterexamples

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Abstract. We discuss the global regularity of 2-dimensional minimal sets that are near a $T$-set (i.e., the cone over the 1-skeleton of a regular tetrahedron centered at the origin), that is, whether every global minimal set in $\mathbb{R}^n$ that looks like a $T$-set at infinity is a $T$-set or not. The main point is to use the topological properties of a minimal set at a large scale to control its topology at smaller scales. This is how one proves that all 1-dimensional Almgren-minimal sets in $\mathbb{R}^n$ and all 2-dimensional Mumford–Shah-minimal sets in $\mathbb{R}^3$ are cones. In this article we discuss two types of 2-dimensional minimal sets: Almgren-minimal sets in $\mathbb{R}^3$ whose blow-in limits are $T$-sets, and topological minimal sets in $\mathbb{R}^4$ whose blow-in limits are $T$-sets. For the former we eliminate a potential counterexample that was proposed by several people, and show that a genuine counterexample should have a more complicated topological structure; for the latter we construct a potential example using a Klein bottle.

0. Introduction

This paper deals with the global regularity of 2-dimensional minimal sets in $\mathbb{R}^3$ and $\mathbb{R}^4$ that look like a $T$-set at infinity in $\mathbb{R}^3$ and $\mathbb{R}^4$. The motivation is that we want to decide whether all global minimal sets in $\mathbb{R}^n$ are cones.

This Bernstein type of problem is of interest for all kinds of minimizing problems in geometric measure theory and calculus of variations. It is natural to ask what a global minimizer looks like, once we know local regularity for minimizers. Well known examples are global regularity for complete 2-dimensional minimal surfaces in $\mathbb{R}^3$, area or size minimizing currents in $\mathbb{R}^n$, or global minimizers for the Mumford–Shah functional. Some of them admit very nice descriptions. See [2], [16], [15], and [3] for further information.

Now let us say something more precise about minimal sets. Briefly, a minimal set is a closed set which minimizes the Hausdorff measure among a certain class

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of competitors. Different choices of classes of competitors give different kinds of minimal sets. So we have the following general definition.

**Definition 0.1** (Minimal sets). Let $0 < d < n$ be integers. A closed set $E$ in $\mathbb{R}^n$ is said to be minimal of dimension $d$ in $\mathbb{R}^n$ if

$$H^d(E \cap B) < \infty$$

for every compact ball $B \subset \mathbb{R}^n$,

and

$$H^d(E \setminus F) \leq H^d(F \setminus E)$$

for any competitor $F$ for $E$.

**Remark 0.2.** We can of course give the definition of locally minimal sets (and the definitions of Almgren and topological competitors that will appear later) where we replace $\mathbb{R}^n$ in Definition 0.1 by any open set $U \subset \mathbb{R}^n$. This makes no difference when we discuss local regularity, but for global regularity, the ambient space $\mathbb{R}^n$ always plays an important role.

In this paper we will discuss the following two kinds of minimal sets, that is, sets that minimize the Hausdorff measure among two classes of competitors.

**Definition 0.3** (Almgren competitor, Al-competitor for short). Let $E$ be a closed set in $\mathbb{R}^n$. An Almgren competitor for $E$ is a closed set $F \subset \mathbb{R}^n$ that can be written as $F = \varphi(E)$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz map such that there exists a compact ball $B \subset \mathbb{R}^n$ such that

$$\varphi|_{B^c} = \text{id} \quad \text{and} \quad \varphi(B) \subset B.$$

Such a $\varphi$ is called a deformation in $B$, and $F$ is also called a deformation of $E$ in $B$.

Roughly speaking, we say that $E$ is Almgren-minimal when there is no deformation $F = \varphi(E)$, where $\varphi$ is Lipschitz and $\varphi(x) - x$ is compactly supported, for which the Hausdorff measure $H^d(F)$ is smaller than $H^d(E)$ in large balls. The definition of Almgren minimal sets was invented by Almgren [1] to describe the behaviors of physical objects that span a given boundary with as little surface area as possible, such as soap films.

The second type of competitors was introduced by the author in [10] and [12], where she tried to generalize the definition of Mumford–Shah minimal sets (MS-minimal for short) to higher codimensions. In both definitions, for MS competitors and topological competitors, we ask that a competitor has certain topological properties of the initial set. Sometimes this condition is easier to handle than the deformation condition that is imposed for Al-competitors.

**Definition 0.4** (Topological competitor). Let $E$ be a closed set in $\mathbb{R}^n$. We say that a closed set $F$ is a topological competitor of dimension $d$ ($d < n$) of $E$, if there exists a ball $B \subset \mathbb{R}^n$ such that

1) $F \setminus B = E \setminus B$;
2) For every Euclidean \((n - d - 1)\)-sphere \(S \subset \mathbb{R}^n \setminus (B \cup E)\), if \(S\) represents a nonzero element in the singular homology group \(H_{n-d-1}(\mathbb{R}^n \setminus E; \mathbb{Z})\), then it is also nonzero in \(H_{n-d-1}(\mathbb{R}^n \setminus F; \mathbb{Z})\).

**Remark 0.5.** When \(d = n - 1\), this is the definition of a MS-competitor, where we impose a separation condition on the complement of the set.

The so-defined class of topological minimizers is contained in the class of Almgren minimal sets (see [12], Corollary 3.17), and admits some good properties that we are not able to prove for Almgren minimal sets.

Our goal is to show that a minimal set in \(\mathbb{R}^n\) is a cone. Topological minimal sets are automatically Almgren minimal, hence we start with Almgren minimal sets, knowing that what we shall say below will also hold for topological minimal sets.

Let \(E\) be a \(d\)-dimensional reduced Almgren minimal set in \(\mathbb{R}^n\). Reduced means that there are no unnecessary points. More precisely, we say that \(E\) is reduced if

\[
H^d(E \cap B(x,r)) > 0 \quad \text{for } x \in E \text{ and } r > 0.
\]

Recall that the definition of minimal set is invariant modulo sets of measure zero, and it is not hard to see that for each Almgren (resp. topological) minimal set \(E\), its closed support \(E^*\) (the reduced set \(E^* \subset E\) with \(H^d(E \setminus E^*) = 0\)) is a reduced Almgren (resp. topological) minimal set. Hence we can restrict ourselves to discussing only reduced minimal sets.

Now fix any \(x \in E\), and set

\[
\theta_x(r) = r^{-d}H^d(E \cap B(x,r)).
\]

This density function \(\theta_x\) is nondecreasing for \(r \in (0, \infty)\) (see Proposition 5.16 in [4]). In particular the two values

\[
\theta(x) = \lim_{t \to 0^+} \theta_x(t) \quad \text{and} \quad \theta_\infty(x) = \lim_{t \to \infty} \theta_x(t)
\]

exist, and are called the density of \(E\) at \(x\), and the density of \(E\) at infinity, respectively. It is easy to see that \(\theta_\infty(x)\) does not depend on \(x\), hence we shall denote it by \(\theta_\infty\).

Theorem 6.2 of [4] says that if \(E\) is a minimal set, \(x \in E\), and \(\theta_x(r)\) is a constant function of \(r\), then \(E\) is a minimal cone centered on \(x\). Thus, by the monotonicity of the density functions \(\theta_x(r)\) for any \(x \in E\), if we can find a point \(x \in E\) such that \(\theta(x) = \theta_\infty\), then \(E\) is a cone and we are done.

On the other hand, the possible values for \(\theta(x)\) and \(\theta_\infty\) for any \(E\) and \(x \in E\) are not arbitrary. By Proposition 7.31 of [4], for each \(x\), \(\theta(x)\) is equal to the density at the origin of a \(d\)-dimensional Al-minimal cone in \(\mathbb{R}^n\). The argument given near equation (18.33) of [4], which is similar to the proof of Proposition 7.31 of [4], gives that \(\theta(x)\) is also equal to the density at the origin of a \(d\)-dimensional Al-minimal cone in \(\mathbb{R}^n\). In other words, if we denote by \(\Theta_{d,n}\) the set of all possible numbers that can be the density at the origin of a \(d\)-dimensional Almgren-minimal cone in \(\mathbb{R}^n\), then \(\theta_\infty \in \Theta_{d,n}\), and, for any \(x \in E\), \(\theta(x) \in \Theta_{d,n}\).
Figure 1. A $\mathcal{Y}$-set (left); a $\mathcal{T}$-set (right).

Thus we restrict the range of $\theta_{\infty}$ and $\theta(x)$. Recall that the set $\Theta_{d,n}$ is possibly very small for any $d$ and $n$. For example, $\Theta_{2,3}$ contains only three values: 1 (the density of a plane), 1.5 (the density of a $\mathcal{Y}$-set, which is the union of three closed half planes with a common boundary $L$, and that meet along the line $L$ with 120° angles), and $dT$ (the density of a $\mathcal{T}$-set, i.e., the cone over the 1-skeleton of a regular tetrahedron centered at 0). See Figure 1.

Recall that the reason why $\theta_{\infty}$ has to lie in $\Theta_{d,n}$ is that, for any $\mathcal{Al}$-minimal set $E$, all its blow-in limits have to be $\mathcal{Al}$-minimal cones (see the argument near equation (18.33) of [4]). A blow-in limit of $E$ is the limit of any converging (for the Hausdorff distance) subsequence of $E_r = r^{-1}E$ as $r \to \infty$.

Hence the value of $\theta_{\infty}$ implies that at sufficiently large scales, $E$ looks like an $\mathcal{Al}$-minimal cone of density $\theta_{\infty}$.

This is the same reason why $\theta(x) \in \Theta_{d,n}$. Here we look at the behavior of $E_r$ when $r \to 0$, and the limit of any converging subsequence is called a blow-up limit (this might not be unique!). Such a limit is also an $\mathcal{Al}$-minimal cone $C$ ([4], Proposition 7.31). This means that, at some very small scales around each $x$, $E$ looks like an $\mathcal{Al}$-minimal cone $C$ of density $\theta(x)$. In this case we call $x$ a $C$ type point of $E$.

After the discussion above, our problem will be solved if we can prove that every minimal cone $C$ satisfies the following property:

\[ d_{x,r}(E,F) = \frac{1}{r} \max \{ \sup \{ d(y,F) : y \in E \cap B(x,r) \}, \sup \{ d(y,E) : y \in F \cap B(x,r) \} \} \]
Remark 0.6. Note that $C_x$ should also be a minimal cone. It is natural to ask whether two minimal cones that admit the same density at the origin should be the same cone (modulo isometry). This is too much to hope, because in [11] the author gave a continuous family of minimal cones having the same density at the origin, but for which any two cones in the family are nonisometric. However, the cones in this family admit the same topology. We do not know whether two minimal cones with the same density at the origin must admit the same topology.

Besides the global regularity, the property (0.8) helps also to control the relative distances $d_{x,r}$ between a minimal set and minimal cones in the balls $B(x,r)$ and the local speed of decay of the density function $\theta_x(r)$, because this property gives a lower bound on $\theta_x(r)$. When we prove (0.8) for a minimal cone $C$, we can get nicer local regularity results, that is, if a minimal set is very near $C$ in a ball, then it should be equivalent to $C$ in a smaller ball via a bi-Hölder homeomorphism (a $C^1$-diffeomorphism in good cases). See [5] for details.

So far we know many minimal cones that satisfy the property (0.8). For a plane it is easily derived from the rectifiability of minimal sets; for a $\mathbb{Y}$-set, the proof is based on a topological argument (see [4], Proposition 16.24); for the unions of two almost orthogonal planes in $\mathbb{R}^4$, the author proved in [14] the property (0.8) for them, by constructing competitors with minimal graphs and using some regularity results for solutions of elliptic systems.

We do not know any minimal cone that does not satisfy the property (0.8), but there are at least two minimal cones for which we do not know whether (0.8) holds: the $T$-set, and the sets $Y \times Y \in \mathbb{R}^4$, whose minimality has recently been proved in [13]. The topology of the set $Y \times Y$ is more complicated than that of $T$-sets, but as we will see soon, the situation of $T$-sets is already tricky.

In this paper we will treat the property (0.8) for the minimal cones $T$ under the two types of definitions for “minimal”.

In Section 1, we discuss (0.8) for $T$-sets in $\mathbb{R}^3$, where the set $E$ in (0.8) is an Almgren-minimal set. Notice that for the $T$-set, no topological argument is enough. There is an example $E_0 \subset \overline{B}(0,1)$ proposed by several people (see [17], page 110, or [4], section 19), which is such that

$$E_0 \cap \partial B(0,1) = T \cap \partial B(0,1),$$

where $T$ is a $T$-set centered at the origin, and $E_0$ satisfies all the known local regularity properties for Al-minimal sets, but $E_0$ contains no $T$-point (see [4], Section 19, for a description for $E_0$). We will eliminate this potential counterexample $E_0$, and give some descriptions for real potential counterexamples if they exist. In fact, the topological structure of $E_0$ is too simple; the set of its $\mathbb{Y}$ type points is a union of unknotted $C^1$ curves. In this case we can easily deform $E$ along one of these unknotted curves to another set with less measure. Notice that, for every minimal set, the set of type $\mathbb{Y}$ points is a union of $C^1$ curves, hence a potential counterexample should contain a knotted curve in the set of its type $\mathbb{Y}$ points. See Proposition 1.2 and Corollary 1.4 for details.

From the author’s point of view, this complicated topology condition contradicts the spirit of minimal sets. However, we will still give an example of a set
that admits this complicated topology. Hence we are still not able to prove (0.8) for \( T \)-sets and Almgren minimal sets.

In Section 2 we discuss the property (0.8) for \( T \)-sets in \( \mathbb{R}^4 \), with topological minimal sets. Recall that in \( \mathbb{R}^3 \) this property has already been proved in [4]). Topological minimality seems to be stronger than Al-minimality, and it is proved in Section 18 of [4] that in \( \mathbb{R}^3 \) the property (0.8) holds for \( T \)-sets. In particular Theorem 1.9 in [4] says that all the 2-dimensional topological minimal sets in \( \mathbb{R}^3 \) are cones. However, in \( \mathbb{R}^4 \), when the codimension is 2, things are complicated. We do not know the list of minimal cones in this case. And even if we make some additional assumption (see (2.1), which says that in \( \mathbb{R}^4 \) there is no 2-dimensional minimal cone whose density is less than the density of a \( T \)-set other than a plane or a \( \gamma \)-set), we still end up with a topological counterexample that satisfies all the known local regularity properties.

Some of the results attributed in the present article to [4] can be found in other (earlier) references, e.g. [19], but here, for simplicity, [4] is cited systematically.

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Some useful notation

In all that follows, minimal set means Almgren minimal set;
\([a,b]\) is the line segment with end points \( a \) and \( b \);
\([a,b)\) is the half-line with initial point \( a \) and passing through \( b \);
\( B(x,r) \) is the open ball with radius \( r \) and centered on \( x \);
\( \overline{B}(x,r) \) is the closed ball with radius \( r \) and center \( x \);
\( \overrightarrow{ab} \) is the vector \( b-a \);
\( H^d \) is the Hausdorff measure of dimension \( d \);
\( d_H(E,F) = \max\{\sup\{d(y,F) : y \in E, \sup\{d(y,E) : y \in F\}\}\} \) is the Hausdorff distance between two sets \( E \) and \( F \).

\[ d_{x,r}(E,F) = \frac{1}{r} \max\{\sup\{d(y,F) : y \in E \cap B(x,r)\}, \sup\{d(y,E) : y \in F \cap B(x,r)\}\} \].

1. Existence of a point of type \( T \) for a 2-dimensional Al-minimal set in \( \mathbb{R}^3 \)

1.1. Introduction

In this section we treat the old problem of the characterization of 2-dimensional Al-minimal sets in \( \mathbb{R}^3 \), and restrict the class of potential Al-minimal sets that are not cones.

Recall that this problem for 2-dimensional topological minimal sets in \( \mathbb{R}^3 \) (which coincide with MS-minimal sets in this case) has been solved positively in [4], where Theorem 1.9 says that all 2-dimensional MS-minimal sets in \( \mathbb{R}^3 \) are cones.
Global regularity for $T$ and counterexamples

The proof of this theorem consists essentially in proving the property (0.8) for all 2-dimensional MS-minimal cones in $\mathbb{R}^3$. There are only three types of minimal cones in this case, namely, planes, $Y$-sets, and $T$-sets. In [4], (0.8) has been proved for planes and $Y$-sets, only under the assumption of Almgren minimality. MS-minimality is used to prove (0.8) for $T$-sets, for which Al-minimality seems to be less powerful.

However, in Subsection 1.2 we are going to eliminate the well-known potential counterexample (see [4], Section 19). Topologically this example satisfies all known local regularity properties for Al-minimal sets, but we will still manage to give another topological criterion (Proposition 1.2 and Corollary 1.4) for minimal sets that look like a $T$-set at infinity, and use this property to prove that the potential counterexample, as well as some other similar sets, cannot be Almgren-minimal. This topological criterion seems to be really strange, and, intuitively, cannot be satisfied by any global minimal set. However, topologically, sets that admit such a property exist, and we construct such an example in Subsection 1.3.

In Subsection 1.4 we will treat another similar problem, that is, for a $T$-set $T$, is the set $T \cap B(0,1)$ the only minimal set $E$ in $B(0,1)$ such that $E \cap \partial B(0,1) = T \cap \partial B(0,1)$? While all the above arguments give some methods for controlling the measure of a set by topology, in Subsection 1.4 we will give some way to control the topology of a set by its measure.

1.2. A topological criterion for potential counterexamples

In this section we given a topological condition that must be satisfied by any 2-dimensional non-conical Almgren minimal set in $\mathbb{R}^3$.

First let us recall some facts about such sets. Let $E$ be such a set. We look at the sets

$$E(r,x) = \frac{1}{r}(E-x)$$

where $r$ tends to infinity.

For every sequence $\{t_k\}_{k \in \mathbb{N}}$ which tends to infinity and such that $E(t_k,x)$ converges (in all compact sets, for the Hausdorff distance), the limit (called a blow-in limit) should be a minimal cone $C$ (see the arguments in [4] near equation (18.33)). Now by the classification of singularities in [19], $C$ should be a plane, a $Y$-set, or a $T$-set. By [4], $C$ cannot be a plane or a $Y$-set. Hence $C$ is a $T$-set. Thus there exists a $T$-set $T$ centered at the origin, and a sequence $\{t_k\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \to \infty} t_k = \infty \quad \text{and} \quad \lim_{t_k \to \infty} d_{0,t_k}(E,T) = 0.$$ (1.2)

Denote the unit ball by $B = B(0,1)$. Denote by $y_i$, $1 \leq i \leq 4$, the 4 points of type $Y$ of $T \cap \partial B$. Denote by $C$ the convex hull of $\{y_i, 1 \leq i \leq 4\}$, which is a regular tetrahedron inscribed in $B$. Set $T_C = T \cap C$. A simple calculation gives

$$\frac{1}{2} H^2(\partial C) = \frac{4}{3} \sqrt{3} < 2\sqrt{2} = H^2(T_C).$$ (1.3)

Set $\delta = \frac{1}{2}(H^2(T_C) - \frac{1}{2}H^2(\partial C))$. Then a minor modification of the proof of Lemma 16.43 of [4] gives:
Lemma 1.1. There exists $\epsilon_1 > 0$ such that, if $d_{0,2}(E, T) < \epsilon_1$, then

\begin{equation}
H^2(E \cap C) > H^2(T_C) - \delta.
\end{equation}

On the other hand, there exists $\epsilon_2 > 0$ such that if $d_{0,1}(E, T) < \epsilon_2$, then in the annulus $B(0, 3/2) \setminus B(0, 1/2)$, $E$ is a $C^1$ version of $T$ (see [4], Section 18). More precisely, in $B(0, 3/2) \setminus B(0, 1/2)$, the set $E_Y$ of points of type $\mathcal{Y}$ in $E$ is the union of four $C^1$ curves $\eta_i, 1 \leq i \leq 4$. Each $\eta_i$ is very near the half-line $[0, y_i)$, and among each $\eta_i$, there exists a tubular neighborhood $T_\eta$ of $\eta_i$, which contains $B([0, y_i), r)$ for some $r > 0$, such that $E$ is a $C^1$ version of a $\mathcal{Y}$-set in $T_\eta$. And for the part of $E \setminus E_Y$, $E \cap B(0, 3/2) \setminus B(0, 1/2)$ is composed of 6 flat surfaces $E_{ij}, 1 \leq i < j \leq 4$. Each $E_{ij}$ is very near $T_{ij}$, where $T_{ij}$ is the cone over the great arc $l_{ij}$, which is the great arc on $\partial B$ that connects $y_i$ and $y_j$. Thus each $E_{ij}$ is a locally minimal set that is near a plane. Then by an argument similar to the proof of Proposition 6.14 of [11], outside $\bigcup_{1 \leq i \leq 4} T_\eta$, $E_{ij}$ is the graph of a $C^1$ function of $T_{ij}$. Hence, in $B(0, 3/2) \setminus B(0, 1/2)$, $E$ is the image of $T$ by a $C^1$ diffeomorphism $\varphi$, whose derivative is very near the identity.

Thus by (1.2), and possibly modulo a dilation, we can suppose that for $t_k = 2$,

\begin{equation}
d_{0,2}(E, T) < \min\{\epsilon_1, \epsilon_2\},
\end{equation}

which gives (1.4), and that in $B(0, 3/2) \setminus B(0, 1/2)$, $E$ is a $C^1$ version of $T$.

In particular, on the boundary of $C$, $E \cap \partial C$ admits the same topology as $T \cap \partial C$. That is, $E \cap \partial C$ is composed of six piecewise $C^1$ curves $w_{ij}, 1 \leq i < j \leq 4$. The end points of each $w_{ij}$ are $b_i$ and $b_j$. In other words, for each $1 \leq i \leq 4$, the three curves $w_{ij}, j \neq i$ meet at their common end point $b_i$. Each $b_i$ is very near $y_i$, where $y_1, 1 \leq i \leq 4$ are the 4 points of type $\mathcal{Y}$ of $T \cap \partial C$. Then $w_{ij}$ is very near $[b_i, b_j]$. Moreover, if we denote by $\Omega_i, 1 \leq i \leq 4$, the connected component of $\partial C \cap E$ which is opposite to $b_i$, bounded by the $w_{kl}, k, l \neq i$, then we can ask that $\epsilon$ is small enough so that

\begin{equation}
\frac{1}{4}H^2(\Omega_i) > H^2(\partial C) - \delta,
\end{equation}

where $\frac{1}{4}H^2(\partial C)$ is the measure of a face of $\partial C$ (see Figure 2).

Now suppose that there is no point of type $\mathcal{T}$ in $E \cap C$. Recall that $E_Y$ is the set of all points of type $\mathcal{Y}$ in $E$. Then by the $C^1$ regularity around points of type $\mathcal{Y}$ (see [5], Theorem 1.15 and Lemma 14.6), $E_Y \cap \partial C$ is composed of $C^1$ curves, whose endpoints are $b_i, 1 \leq i \leq 4$. Then there exists two curves $\gamma_1, \gamma_2 \subset E_Y$ whose endpoints are the $b_i$. Suppose, for example, that $\gamma_1 \cap \partial C = \{b_1, b_2\}$, and $\gamma_2 \cap \partial D = \{b_3, b_4\}$.

Now by the $C^1$ regularity for points of type $\mathcal{Y}$ (see [5], Theorem 1.15 and Lemma 14.6), for each $x \in \gamma_1$, there exists a neighborhood $B(x, r)$ such that in $B(x, r)$, $E$ is a $C^1$ version of $Y + x$ which cuts $B(x, r)$ into 3 connected components. Then by the compactness of $\gamma_1$, there exists $r > 0$ such that in the tubular neighborhood $B(\gamma_1, r)$ of $\gamma_1$, $E$ is a distorted $Y$ set, whose singular set is $\gamma_1$, and $E$ divides $B(\gamma_1, r)$ into three connected components. Each component is a long tube that joins one of the three $\Omega_i$ near $b_1$ to one of the three $\Omega_i$ near $b_2$. Notice that if,
for $i \neq j$, $\Omega_i$ and $\Omega_j$ are connected by one of these long tubes, then they lie in the same connected component of $B\setminus E$. As a result, there exist $1 \leq i, j \leq 4, i \neq j$ such that $\Omega_i$ and $\Omega_j$ are in the same connected component of $B\setminus E$, and there exists a long tube $T$ along $\gamma_1$ which connects $\Omega_i$ and $\Omega_j$.

Now suppose that there exists a deformation $f$ in $C$ (see Definition 0.3), two indices $1 \leq i \neq j \leq 4$, and two points $x \in \Omega_i, y \in \Omega_j$, such that

$$f(E) \subset C\setminus[x, y].$$

It is then not hard to find a Lipschitz deformation $g: C\setminus B([x, y], r) \to G := \partial C\setminus(\Omega_i \cup \Omega_j)$ such that $g = \text{id}$ on $G$. To construct of such a $g$, we can imagine that we enlarge the “hole” $B([x, y], r)$ and push every point in $C\setminus B([x, y], r)$ towards the set $G$. For example we give, in Figure 3, a sketch illustrating what happens when $E \cap \partial C = T \cap \partial C$. For any set $E$ we have only to make some tiny modification, since $E\cap \partial C$ is a $C^1$ version of $T\cap \partial C$. Here for each half-plane $D$ that is bounded by the line containing $[x, y]$, we just map $D \cap C\setminus B([x, y], r)$ to the boundary $G\cap D$ (the thicker segments or point in the figure).

Then the function $h := g \circ f$ sends $E$ to a subset of $G$ for $t$ large, and moreover, $g$ does not move $E \cap \partial C = \cup_{k=1}^4 \partial \Omega_k$.

The above argument implies that in $C$ we can deform $E$ to a subset of $G$.

Now by (1.4) and (1.6),

$$H^2(h(E)) \leq H^2(G) = H^2(\partial C) - H^2(\Omega_i) - H^2(\Omega_j)$$

$$< \frac{1}{2} H^2(\partial C) + 2\delta = H^2(T_C) - 2\delta < H^2(E \cap C),$$

which contradicts the fact that $E$ is minimal. As a result, if $E$ does not contain any $T$ type point, then there is no deformation $f$ of $E$ in $C$ such that $C\setminus f(E)$ contains a segment that connects two different $\Omega_i$. On the other hand, if $E$ contains
a T-point, then by the argument surrounding (0.6), E is in fact the T centered at this T-point. In this case there is no such deformation f, either. We have therefore:

**Proposition 1.2.** Let E be a 2-dimensional Almgren-minimal set in $\mathbb{R}^3$ such that

1) $d_{0,2}(E,T) < \min\{\epsilon_1, \epsilon_2\};$

2) E does not contain any T-point.

Let $C$ and $\Omega_i$ be as above. Then there exists no deformation f of E in C such that $C \setminus f(E)$ contains a segment that connects two different $\Omega_i$, $1 \leq i \leq 4$.

**Remark 1.3.** By Proposition 1.2, the tube T along $\gamma_1$ cannot be too simple. For example if there exists a Lipschitz homeomorphism f which is a deformation in C such that

\begin{equation}
 f(\gamma_1) = [b_1, b_2],
\end{equation}

(in this case we say that $\gamma_1$ is not “knotted”), then

\begin{equation}
 C \setminus f(E) = f(C \setminus E) \supset f(\gamma_1) = [b_1, b_2],
\end{equation}

which contradicts Proposition 1.2. Thus we get the following:

**Corollary 1.4.** If E contains no T-point, then $\gamma_1$ and $\gamma_2$ are “knotted”.

Because of this corollary, the potential counterexample $E_0$ proposed in [4] is not a real counterexample, since neither $\gamma_i$, $i = 1, 2$ in this example is knotted (in the next section we shall explain what $E_0$ looks like topologically). Thus we have:

**Corollary 1.5.** The set $E_0$ given in Section 19 of [4] is not Almgren-minimal.

To sum up, if a minimal set E satisfies (1.2), then both $\gamma_1$ and $\gamma_2$ are knotted. It is not easy to imagine how to knot a $\gamma$-set without producing new singularities. However this kind of set does exist. We will construct an example in the next section.
1.3. A set that admits two knotted \( \gamma \)-curves

The purpose of this subsection is to give a topological example of an \( E \) for which both \( \gamma_1 \) and \( \gamma_2 \) are knotted. First we look at the well-known example \( E_0 \), because such an example already requires a certain imagination. In this example both \( \gamma_i, i = 1, 2 \) are not knotted.

We take a torus \( T_0 \) (see Figure 4). Denote by \( C_0 \) (the green circle in the figure) the longest horizontal circle (the equator), and fix any vertical circle \( L_0 \) in \( T_0 \) (the red circle in the figure). Denote by \( x_0 \) their intersection. Take \( r_0 > 0 \) such that \( B_0 = B(x_0, r_0) \cap T_0 \) (the blue circle) is a non-degenerate topological disc. Denote by \( a_1 \) and \( a_2 \) the intersection of \( \partial B_0 \) and \( C_0 \), and by \( b_1 \) and \( b_2 \) the intersection of \( \partial B_0 \) and \( L_0 \).

Denote by \( \widetilde{a}_1 \widetilde{a}_2 = C_0 \setminus B_0 \) the arc between \( a_1 \) and \( a_2 \), and by \( \widetilde{b}_1 \widetilde{b}_2 = L_0 \setminus B_0 \) the arc between \( b_1 \) and \( b_2 \). Next denote by \( S_1 \) the vertical planar domain bounded by \( [b_1, b_2] \cup \widetilde{b}_1 \widetilde{b}_2 \). On the other hand, denote by \( P \) the plane containing \( C_0 \), and take a closed disk \( B_1 \subset P \) which contains \( \widetilde{a}_1 \widetilde{a}_2 \) and whose boundary contains \( a_1 \) and \( a_2 \). Now denote by \( \widetilde{a}_1 \widetilde{a}_2 = \partial B_1 \setminus B_0 \) the larger arc of \( \partial B_1 \) between \( a_1 \) and \( a_2 \), and denote by \( S_2 \subset P \) the part between \( \widetilde{a}_1 \widetilde{a}_2 \) and \( \widetilde{a}_1 \widetilde{a}_2 \).

Now we happily claim that the set \((T_0 \setminus B_0) \cup S_1 \cup S_2\) is topologically the example \( E_0 \) given in Section 19 of [4]. Here \( a_1, a_2, b_1, b_2 \) are the four \( \gamma \)-points which correspond to the four \( \gamma \)-points in \( E_0 \cap \partial B(0, 1) \), \( \widetilde{a}_1 \widetilde{a}_2 \) and \( \widetilde{b}_1 \widetilde{b}_2 \) correspond to \( \gamma_1 \) and \( \gamma_2 \) respectively; \( \widetilde{a}_1 \widetilde{a}_2 \) and \( [b_1, b_2] \), together with the four arcs on \( \partial B_0 \) between \( a_i \) and \( b_j \), \( i, j = 1, 2 \), correspond to the six curves of \( E_0 \cap \partial B(0, 1) \) (but of course we need to deform this picture a lot). If we were to modify our topological example so that the surfaces meet each other with \( 120^\circ \) angles along the curves \( \gamma_i \), then we would get \( E_0 \).

After the above discussion, we are now ready to construct (in \( \mathbb{R}^3 \)) our example \( E_1 \), where \( \gamma_1 \) and \( \gamma_2 \) are both knotted. Moreover, in \( \mathbb{R}^3 \setminus E_1 \) there is no non-knotted curve that connects \( \Omega_1 \) to \( \Omega_2 \), or \( \Omega_3 \) to \( \Omega_4 \). The idea is to replace the \( \gamma_1 \) and \( \gamma_2 \) in \( E_0 \), which are a pair of cogenerators of \( \pi_1(T_0) \), by another pair of knotted representatives of cogenerators of \( \pi_1(T_0) \) of the torus \( T_0 \).
Let us first point out that the following example $E_1$ is just a topological one, and it is not very likely that $E_1$ is minimal.

Still, take our torus $T_0$. Let the notation be as before; that is, $C_0$ denotes the longest horizontal circle (the equator) and $L_0$ is a vertical circle in $T_0$. Denote by $x_0$ their intersection. Take $r_0 > 0$ such that $B_0 = B(x_0, r_0) \cap T_0$ (the blue circle) is a non-degenerate topological disc. Denote by $a_1$ and $a_2$ the intersection of $\partial B_0$ and $C_0$, and by $b_1$ and $b_2$ the intersection of $\partial B_0$ and $L_0$.

Denote by $\Gamma = \mathbb{Z}^2$ the integer lattice in $\mathbb{R}^2$. We identify $T_0$ with the image of $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$. For any two integers $m, n \in \mathbb{Z}$, denote by $d(m, n) = [(0, 0), (m, n)]$ the segments with endpoints $(0, 0)$ and $(m, n)$. Denote by $K(m, n) = \pi(d(m, n))$. Then $K(m, n)$ is a simple closed curve (that is, $\pi$ is injective on $d(m, n)$) if and only if the greatest common divisor $(m, n)$ of $m$ and $n$ equals 1. For any integers $m, n, a, b$ with $(m, n) = (a, b) = 1$, $K(m, n)$ and $K(a, b)$ represent a pair of cogenerators of $\pi_1(T_0)$ if and only if $|\det(\begin{pmatrix} a & b \\ m & n \end{pmatrix})| = 1$ (see [18]). Without loss of generality, suppose that $K(1, 0) = L_0$, and $K(0, 1) = C_0$.

Take a pair of knotted curves $K(2, 3)$ and $K(3, 4)$ which represent a pair of cogenerators of $\pi_1(T_0)$. Then the two curves intersect at one point. Without loss of generality, suppose this point of intersection is $x_0$. Denote by $\text{Int}(T_0)$ and $\text{Ext}(T_0)$ the two connected components of $S^3 \setminus T_0$.

First we want to construct two surfaces $S_1$ and $S_2$, such that $S_1 \subset \text{Ext}(T_0)$, $S_2 \subset \text{Int}(T_0)$, $\partial S_1 = K(2, 3)$, and $\partial S_2 = K(3, 4)$.

Notice that the torus knot $K(3, 2)$ is a trefoil knot (see Figure 5, left), which bounds a nonorientable surface $S_1' \subset \text{Int}(T_0)$ (see Figure 5, right). The pair of topological spaces $(\text{Int}(T_0) \cup T_0, T_0)$ is homeomorphic to $(\text{Ext}(T_0) \cup T_0, T_0)$, by some homeomorphism $\varphi$ or $S^3$ that sends the point $\infty$ to a point in $\text{Int}(T_0)$, and $\varphi(K(3, 2)) = K(2, 3)$. Thus $K(2, 3)$ bounds a surface $S_1 = \varphi(S_1') \subset \text{Ext}(T_0)$.

The curve $K(3, 4)$ intersects with the vertical circle $L_0$ at four points $p_0, p_1, p_2$ and $p_3$ in clockwise order. Denote by $s_0 \subset \text{Int}(T_0)$ the vertical planar disk whose boundary is $L_0$, and for $\theta \in [0, 2\pi)$, let $s_\theta$ denote the vertical section disk of $\text{Int}(T_0)$ with polar angle $\theta$ (see Figure 6). Then $s_\theta$ also intersects $K(3, 4)$ at four points, and when $\theta < 2\pi$, the intersection of $K(3, 4)$ with the tube $\cup_{0 \leq \theta < \theta \leq 3} s_\theta$ is the disjoint union of four curves. Then for each $0 \leq i \leq 3$, there is a point on $K(3, 4) \cap s_\theta$ that is connected to $p_i$ by one of these four curves. Denote by this point $p_i(\theta)$. Notice that...
at the angle $2\pi$, we have $s_0 = s_{2\pi}$, hence for each $0 \leq i \leq 3$, $p_i(2\pi)$ is one of the two points on $s_0$ that is adjacent to $p_i$, and \( \{p_i(2\pi), 0 \leq i \leq 3\} = \{p_i(0), 0 \leq i \leq 3\} \). Moreover, since $K(3,4) = \cup_{0 \leq i \leq 3} p_i([0,2\pi))$ is connected, if $p_i(2\pi) = p_j$, then $p_j(2\pi) \neq p_i$. Under all these conditions, there are only two possibilities: either $p_i(2\pi) = p_{i+1}$ for $i = 0, 1, 2$ and $p_3(2\pi) = p_0$, or $p_i(2\pi) = p_{i-1}$ for $i = 1, 2, 3$ and $p_0(2\pi) = p_4$. Without loss of generality, suppose that $p_i(2\pi) = p_{i+1}$ for $i = 0, 1, 2$ and $p_3(2\pi) = p_0$.

![Figure 6.](image)

Take an isotopy $f : [0,2\pi] \times s_0 \to \text{Int}(T_0) \cup T_0$ such that $f_0 = \text{id}$, $f_0(s_0) = f(0,s_0) = s_0$, $f_0(\partial s_0) = \partial s_0$, and $f_0(p_i) = p_i(\theta)$. Then the image $S'_2 = f([0,3\pi/2] \times ([p_0,p_1] \cup [p_2,p_3]))$ is a surface inside $T_0$, whose boundary is the curve

\[
\{ K(3,4) \cap \big( \bigcup_{0 \leq \theta \leq 3\pi/2} s_\theta \big) \} \cup \{ [p_0,p_1] \cup [p_2,p_3] \} \cup f_{3\pi/2}([p_0,p_1] \cup [p_2,p_3]).
\]

Now we have to find a surface in the remainder, i.e., in $\cup_{3\pi/2 < \theta < 2\pi} s_\theta$, whose boundary is $K(3,4) \cap \big( \cup_{3\pi/2 < \theta < 2\pi} s_\theta \big) \cup \{ [p_0,p_1] \cup [p_2,p_3] \} \cup f_{3\pi/2}([p_0,p_1] \cup [p_2,p_3])$. Notice that we cannot continue to use the image under $f_{3\pi/2}$, $3\pi/2 < \theta < 2\pi$, because $f_{3\pi/2}(p_0,p_1)$ will be something that connects $p_1$ and $p_2$, rather than a curve that connects $p_0$ to $p_1$ or $p_2$ to $p_3$. However, we can find the solution by a saddle surface $S''_2$. Refer to Figure 7, where $a_i$ denotes $f_{3\pi/2}(p_i)$.

Denote by $S_2 = S'_2 \cup S''_2 \subset \text{Int}(T_0)$, then $\partial S_2 = K(3,4)$.

Now to sum up, we have found two surfaces $S_1 \subset \text{Ext}(T_0)$ and $S_2 \subset \text{Int}(T_0)$, with $\partial S_1 = K(2,3)$ and $\partial S_2 = K(3,4)$.

Now we take a diffeomorphism of $S^3$, which maps $T_0$ to $T_0$, $\text{Int}(T_0)$ to $\text{Int}(T_0)$, and $\text{Ext}(T_0)$ to $\text{Ext}(T_0)$. Moreover we ask that the images $l_1$ of $K(2,3)$ and $l_2$ of $K(3,4)$ satisfy that $l_1 \cap l_2 = x_0$; $l_1 \cap B_0 = C_0 \cap B_0$, the shorter arc of $C_0$ between $a_1$ and $a_2$; and $l_2 \cap B_0 = L_0 \cap B_0$ the arc of $L_0$ between $b_1$ and $b_2$ that passes through $x_0$. Then the images of $S_1$ and $S_2$ are still two surfaces $S_3$ and $S_4$, with $\partial S_3 = l_1$, $\partial S_4 = l_2$, $S_3 \subset \text{Ext}(T_0)$ and $S_4 \subset \text{Int}(T_0)$.

We still need to make some modifications, because the two surfaces $S_3$ and $S_4$ meet each other at the boundary. So we take a homeomorphism $\varphi$ of $S^3$, which fixes $T_0 \setminus B_0$, and satisfies $\varphi(l_1 \cap B_0) = \overline{a_1,a_2}$ and $\varphi(l_2 \cap B_0) = [b_1,b_2]$. Then
Figure 7.

\[ S_5 = \varphi(S_3) \subset \text{Ext}(T_0), \quad \text{and} \quad S_6 = \varphi(S_4) \text{ is contained in } \text{Int}(T_0) \setminus C, \] where \( C \) denotes the convex hull of \( \{a_1, a_2, b_1, b_2, x_0\} \).

Let \( E_1 = (T_0 \setminus B_0) \cup S_5 \cup S_6 \). Let \( \gamma_1 = l_1 \setminus B_0, \) and \( \gamma_2 = l_2 \setminus B_0 \). These are two knotted \( \mathcal{Y} \) curves of \( E_1 \), because as in \( E_0 \), we have two surfaces, \( S_5 \) and \( S_6 \), such that \( \partial S_5 = \gamma_1 \cup \hat{a}_1, a_2 \) and \( \partial S_6 = \gamma_2 \cup [b_1, b_2] \). We can deform \( E_1 \) into \( B(0, 1) \), such that \( E_1 \cap \partial B(0, 1) = T \cap \partial B(0, 1) \) for some \( \mathcal{T} \)-set \( T \). Here \( a_1, a_2, b_1, b_2 \) are the four \( \mathcal{Y} \)-points which correspond to the four \( \mathcal{Y} \)-points in \( E_1 \cap \partial B(0, 1) \), while \( \hat{a}_1a_2 \) and \( [b_1, b_2] \), together with the four arcs on \( \partial B_0 \) between \( a_i \) and \( b_j, i, j = 1, 2 \) correspond to the six curves of \( E_1 \cap \partial B(0, 1) \).

Thus we have constructed an example whose set of \( \mathcal{Y} \)-points is the union of two knotted curves. Moreover, we cannot find any non-knotted curve in \( B(0, 1) \setminus E_1 \) that connects \( a_1 \) to \( a_2 \), or \( b_1 \) to \( b_2 \). That is, there is no deformation \( f \) of \( E_1 \) in \( B(0, 1) \) such that \( B(0, 1) \setminus f(E_1) \) contains a segment that connects two different \( \Omega_i \). While this example \( E_1 \) seems too complicated to be minimal, we do not know how to prove this.

However, for another closely related problem, we can prove that a minimal set does not admit a knotted \( \mathcal{Y} \) curve. See the next section.

1.4. Another related problem

We take a \( \mathcal{T} \)-set \( T \) centered at the origin. That is, \( T \) is the cone over the 1-skeleton of a regular tetrahedron \( C \) centered at the origin and inscribed in the unit ball.

In this section we will discuss whether there exists a set \( E \subset \overline{B}(0, 1) \) different from \( T \cap \overline{B}(0, 1) \), that is minimal in \( B(0, 1) \), and such that \( E \cap \partial B(0, 1) = T \cap \partial B(0, 1) \).

Denote by \( B = B(0, 1) \subset \mathbb{R}^3 \), and by \( \overline{B} \) its closure. Then \( T \) divides the sphere \( \partial B \) into four equal triangular open regions \( \{S_i\}_{1 \leq i \leq 4} \), with

\[
\bigcup_{i=1}^4 S_i = \partial B \quad \text{and} \quad \bigcup_{i=1}^4 S_i = \partial B \setminus T.
\]
Recall that $T$ divides $\partial C$ into four equal open planar triangles $\{\Omega_i\}_{1 \leq i \leq 4}$. For notational convenience we ask that for each $i$, $S_i$ and $\Omega_i$ share the same three vertices.

Denote by $a_j$, $1 \leq j \leq 4$, the four vertices of $T \cap \partial B$, where $a_j = \cap_{i \neq j} \overline{S_i} \cap \partial B$ is the point opposite to $S_j$.

**Proposition 1.6.** Let $E \subset \overline{B} \cap \mathbb{R}^3$ be a closed, 2-rectifiable, locally Ahlfors regular set with

$$E \cap \partial B = T \cap \partial B.$$  

Then:

1) If $H^2(E) < H^2(T \cap B)$, 

there exists $1 \leq i < j \leq 4$, and four points $a, b, c$ and $d$ that lie in a common plane such that $a \in S_i$, $d \in S_j$, $b, c \in B \setminus E$, $\angle abc > \pi/2$, $\angle bed > \pi/2$ and $[a, b] \cup [b, c] \cup [c, d] \subset \overline{E}$. 

Here $[x, y]$ denotes the segment with endpoints $x$ and $y$, and $\angle abc \in [0, \pi]$ denotes the angle of the smaller sector bounded by $ba$ and $bc$.

2) If $E$ is a reduced minimal set in $B$ and satisfies (1.12), then

$E = T \cap \overline{B}$, or (1.13) is true.

Before we prove Proposition 1.6, we first give a corollary.

**Corollary 1.7.** Let $E \subset \overline{B}$ be a reduced minimal set in $B$ and satisfying (1.12). Then if $E \neq T \cap \overline{B}$, we have

$$H^2(E) \leq H^2((T \cap \overline{B}) \setminus (C \cup G)) = H^2(T \cap \overline{B}) - (2\sqrt{2} - 4\sqrt{3}/3) \approx 0.519.$$  

**Proof.** Let $E$ be such a set. Then by (1.14), (1.13) is true. Since (1.13) gives the existence of a deformation $f$ in $B$ such that $f(E) \subset B \setminus [a, d]$, we can deform $E$ on a subset of $(T \cap B \setminus C) \cup G$ ($C$ denotes the interior of $C$), where $G = \partial C \setminus (\Omega_i \cup \Omega_j)$ (recall that $\Omega_i$ and $\Omega_j$ are the two faces of $C$ corresponding to the two faces $S_i$ and $S_j$ of $\partial B$, where $S_i$ and $S_j$ contain the points $a$ and $d$).

Thus by (2.4),

$$H^2(E) \leq H^2((T \cap \overline{B}) \setminus (C \cup G)) = H^2(T \cap \overline{B}) - (2\sqrt{2} - 4\sqrt{3}/3).$$  

**Proof of Proposition 1.6.** We are going to prove 1) by contraposition. Suppose that (1.13) is not true.

Denote by $P_j$ the plane orthogonal to $\overrightarrow{ao_j}$ and tangent to the unit sphere, and denote by $p_j$ the orthogonal projection to $P_j$. Set $R_j = p_j(\overline{S_j}) \subset p_j(\cup_{i \neq j} \overline{S_i}) \subset P_j$. Then for each $1 \leq j \leq 4$ and each $x \in R_j$,

$$p_j^{-1}(x) \cap E \neq \emptyset.$$
In fact if (1.17) is not true for some $j$, that is, $R_j \backslash p_j(E) \neq \emptyset$. As the projection of a compact set, $p_j(E)$ is compact. Thus $R_j \backslash p_j(E)$ is a nonempty open set. Note that $R_j \backslash \bigcup_{i \neq j} p_j(S_i)$ has measure zero, therefore

\[(R_j \backslash p_j(E)) \cap \bigcup_{i \neq j} p_j(S_i) \neq \emptyset.\]

Take $x \in (R_j \backslash p_j(E)) \cap \bigcup_{i \neq j} p_j(S_i)$. Then $x \notin \partial R_j$, because $\partial S_j \subseteq E$ and hence $\partial R_j = \partial p_j(S_j) = p_j(\partial S_j) \subseteq p_j(E)$. As a result, $p_j^{-1}(x) \cap B$ is a segment $[a, d]$ perpendicular to $P_j$, with $a \neq d$, $a \in S_j^a$ and $d \in \bigcup_{i \neq j} S_i^d$. Take $b, c \in [a, d]$ such that $a, b, c$ and $d$ are different. Then (1.13) holds, which contradicts our hypothesis.

Hence (1.17) holds. Now for each $x \in R_j$, denote by $f_j(x)$ the point in $p_j^{-1}(x) \cap E$ which is the nearest to $R_j$. In other words, $f_j(x)$ is the first point in $E$ whose projection is $x$. This point exists by (1.17), and is unique, since $p_j^{-1}(x)$ is a line orthogonal to $R_j$.

Let $A_j = f_j(R_j)$. Then $A_j$ is measurable. In fact,

\[(1.19) \quad A_j = \{x \in E : \forall y \in E \text{ such that } d(y, P_j) < d(x, P_j), |p_j(y) - p_j(x)| > 0\}
\[
= \bigcap_{p,q} \{x \in E : \forall y \in E \text{ such that } d(y, P_j) < d(x, P_j) - 2^{-p}, |p_j(y) - p_j(x)| > 2^{-q}\}.
\]

Now $E$ is rectifiable, hence $A_j \subseteq E$ is also rectifiable. Therefore for almost all $x \in A_j$, the approximate tangent plane $T_x A_j$ of $A_j$ at $x$ exists. Denote by $v_j = \overrightarrow{oa_j}/|oa_j|$ the unit exterior normal vector of $P_j$, and denote by $w_j(x)$ the unit vector orthogonal to $T_x A_j$ such that $\langle v_j, w_j(x) \rangle \geq 0$. Then $w_j(x)$ is well defined for every $x \in A_j$ with $T_x A_j \perp P_j$.

Denote by

\[(1.20) \quad E_j = \{x \in A_j : T_x A_j \perp P_j\}.
\]

Then $w_j$ is a measurable vector field on $E_j$. On the other hand, by Sard’s theorem, $H^2(p_j(A_j \backslash E_j)) = 0$. Since $p_j$ is injective on $A_j$, $p_j(A_j \backslash E_j) = p_j(A_j \backslash p_j(E_j)) = R_j \backslash p_j(E_j)$, and thus

\[(1.21) \quad H^2(R_j \backslash p_j(E_j)) = 0.
\]

Moreover, for almost all $x \in E_j$, $T_x A_j = T_x E_j$.

We are going to show that

\[(1.22) \quad \int_{E_j} \langle v_j, w_j(x) \rangle \, dx = H^2(R_j).
\]

First, we apply the area formula for Lipschitz maps between rectifiable sets (see 3.2.20 of [7]), with $m = \nu = 2$, $W = E_j$, $f = p_j$, $g = 1_{R_j}$, and we get

\[(1.23) \quad \int_{E_j} \| \Lambda_2 apDp_j(x) \| \, dH^2 x = \int_{R_j} N(p_j, z) \, dH^2 z.
\]
Moreover, by (1.17) and (1.21), \( N(p_j, z) \geq 1 \) for almost all \( z \in R_j \). On the other hand, \( N(p_j, z) \leq 1 \) since \( E_j \) is contained in the set \( A_j \) on which \( p_j \) is injective. Hence \( N(p_j, z) = 1 \) for almost all \( z \in R_j \). Therefore

\[
(1.24) \quad \int_{R_j} N(p_j, z) \, dH^2 z = H^2(R_j).
\]

For the left side of (1.23), take \( w_j^1(x) \) to be a unit vector in \( T_x E_j \) such that \( w_j^1(x) \parallel R_j \), and \( w_j^2(x) \) to be the unit vector in \( T_x E_j \) orthogonal to \( w_j^1(x) \). Then

\[
(1.25) \quad || \wedge_2 a D p_j (x) || = || p_j (w_j^1(x)) \wedge p_j^2 (w_2(x)) || = | p_j (w_j^1(x)) | | p_j (w_j^2(x)) |
\]

The first inequality is because \( p_j \) is a linear map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), the second inequality is because \( p_j (w_j^1(x)) \perp p_j (w_j^2(x)) \), and the last is because \( w_j^1(x) \parallel S_j \).

Now set \( v_j^1(x) = p_j (w_j^2(x)) / | p_j (w_j^2(x)) | \in P_j \). This is well defined because \( T_x E_j \not\parallel P_j \) and hence \( | p_j (w_j^2(x)) | > 0 \). Then \( w_j(x), w_j^2(x), v_j, v_j^2(x) \) are all orthogonal to \( w_j^1(x) \), and hence belong to a single plane, with \( w_j(x) \perp w_j^2(x), v_j \perp v_j^2(x) \). Therefore

\[
(1.26) \quad | \langle v_j(x), v_j \rangle | = | \langle w_j^2(x), v_j^2(x) \rangle | = | p_j (w_j^2(x)) |.
\]

Since by definition \( \langle w_j(x), v_j \rangle \geq 0 \),

\[
(1.27) \quad \langle w_j(x), v_j \rangle = | p_j (w_j^2(x)) | = || \wedge_2 a D p_j (x) ||,
\]

by (1.25). Combining (1.23), (1.24) and (1.27), we get (1.22). Note that \( v_j \) does not depend on \( E \).

Now for \( x \in A_j \setminus E_j \), we define a measurable vector field \( w_j(x) \) such that \( w_j(x) \perp T_x A_j \). Then \( \langle w_j(x), v_j \rangle = 0 \) for almost all \( x \in A_j \setminus E_j \). Hence we have

\[
(1.28) \quad \int_{A_j} \langle v_j, w_j(x) \rangle dx = H^2(R_j).
\]

We sum over \( j \), and get

\[
(1.29) \quad \sum_{j=1}^{4} \int_{A_j} \langle v_j, w_j(x) \rangle \, dx = \sum_{j=1}^{4} \int_{E_j} \langle v_j, w_j(x) \rangle \, dx = \sum_{i=1}^{4} H^2(R_j).
\]

Next, set \( E_j^0 = E_j \setminus \cup_{i \neq j} E_i, E_{ij} = (E_i \cap E_j) \setminus \cup_{k \neq i, j} E_k \) for \( i \neq j \). We claim that

\[
(1.30) \quad E_j \setminus (E_j^0 \cup \bigcup_{i \neq j} E_{ij}) \text{ is of measure zero for all } j.
\]

Suppose (1.30) is not true. Then there exist three different \( i, j \) and \( k \) such that \( E_i \cap E_j \cap E_k \) has positive measure. Suppose for example that \( i = 1, j = 2 \) and \( k = 3 \), and set \( E_{123} = E_1 \cap E_2 \cap E_3 \). Now since \( E_{123} \) is a measurable rectifiable set of positive measure, and \( E_{123} \subset E \), for almost all \( x \in E_{123} \), the approximate tangent
plane $T_xE_{123}$ of $E_{123}$ at $x$ exists and equals $T_xE$. Moreover, since $E$ is locally Ahlfors regular, $T_xE$ is a real tangent plane (see, for example, [3], Exercise 41.21, page 277). We choose and fix such an $x \in E_{123}$.

By definition of $A_j$, for $j = 1, 2, 3$, the segment $[x, p_j(x)] \cap E = \{x\}$. And by definition of $E_j$, $T_xE \not\subset P_j$, and hence $[x, p_j(x)] \cap (T_xE + x) = \{x\}$, since $[x, p_j(x)] \perp P_j$. The affine subspace $T_xE + x$ separates $\mathbb{R}^3$ into two half-spaces, and since for $j = 1, 2, 3$, $(x, p_j(x)) \cap (T_xE + x) = \emptyset$, there exist $1 \leq i < j \leq 3$ such that $(x, p_i(x))$ and $(x, p_j(x))$ are on the same side of $T_xE + x$. Suppose for example that $i = 1$ and $j = 2$.

For $i = 1, 2$, denote by $\alpha_i$ the angle between $[x, p_i(x)]$ and $T_xE + x$. Set $\alpha = \min\{\alpha_1, \alpha_2\}$. Then since $T_xE$ is a real tangent plane, there exists $r > 0$ such that for all $y \in E \cap B(x, r)$,\n\begin{equation}
(1.31) \quad d(y, T_xE + x) < \frac{r}{2} \sin \alpha.
\end{equation}

Set $b = [x, p_1(x)] \cap \partial B(x, r)$ and $c = [x, p_2(x)] \cap \partial B(x, r)$. Then by definition of $\alpha$, $d(b, T_xE + x) \geq r \sin \alpha$, and $d(c, T_xE + x) \geq r \sin \alpha$. Since $b$ and $c$ are on the same side of $T_xE + x$, for all $y \in [b, c]$, $d(y, T_xE + x) \geq r \sin \alpha$, and hence $[b, c] \cap E = \emptyset$, because of (1.31).

Now set $a = p_1(x)$ and $d = p_2(x)$. Note that in the triangle $\Delta xbc$, $|xb| = |xc|$, which gives that $\angle xbc = \angle xcb$. But $\angle xbc + \angle xcb + \angle bxc = \pi$, $\angle bxc > 0$, Hence $\angle xbc = \angle xcb < \pi/2$. As a result, $\angle abc = \pi - \angle xbc > \pi/2$ and $\angle bcd = \pi - \angle xcb > \pi/2$. Thus we have found four points $a, b, c$ and $d$ such that (1.13) is true, which contradicts our hypothesis.

Thus we get (1.30). And consequently we have\n\begin{equation}
(1.32) \quad H^2(\cup_{j=1}^4 E_j) = \sum_{j=1}^4 H^2(E_j^0) + \sum_{1 \leq i < j \leq 4} H^2(E_{ij}).
\end{equation}

For estimating the measure, we are going to use the paired calibration method (introduced in [9]). Recall that $v_j$ is the unit exterior normal vector of $P_j$. Thus by (1.29),\n\begin{equation}
\sum_{i=1}^4 H^2(R_i) = \sum_{j=1}^4 \int_{E_j} \langle v_j, w_j(x) \rangle \, dx
\end{equation}
\begin{equation}
= \sum_{j=1}^4 \int_{E_j^0} \langle v_j, w_j(x) \rangle \, dx + \sum_{1 \leq i < j \leq 4} \int_{E_{ij}} \langle v_i, w_i(x) \rangle + \langle v_j, w_j(x) \rangle \, dx.
\end{equation}

For the first term,\n\begin{equation}
\left| \int_{E_j^0} \langle v_j, w_j(x) \rangle \, dx \right| \leq \int_{E_j^0} |\langle v_j, w_j(x) \rangle| \, dx \leq \int_{E_j^0} |v_j| |w_j(x)| \, dx = H^2(E_j^0),
\end{equation}
and hence\n\begin{equation}
\sum_{j=1}^4 \int_{E_j^0} \langle v_j, w_j(x) \rangle \, dx \leq \sum_{j=1}^4 \int_{E_j^0} |\langle v_j, w_j(x) \rangle| \, dx \leq \sum_{j=1}^4 H^2(E_j^0).
\end{equation}
For the second term, observe that \( w_i(x) = \pm w_j(x) \) for \( x \in E_{ij} \), hence we set \( \epsilon_x = w_i(x)/w_j(x) \). Then

\[
|\langle v_i, w_i(x) \rangle + \langle v_j, w_j(x) \rangle| = |(v_i + \epsilon(x)v_j, w_i(x))| \\
\leq |v_i + \epsilon(x)v_j| |w_i(x)| = |v_i + \epsilon(x)v_j| \leq \max\{|v_i + v_j|, |v_i - v_j|\}.
\]

By definition of \( v_i \), the angle between \( v_i \) and \( v_j \) is the supplementary angle of the angle \( \theta_{ij} \) between \( P_i \) and \( P_j \). A simple calculus gives

\[
|v_i + v_j| = \frac{2}{\sqrt{3}} < 1, \quad |v_i - v_j| = \frac{2\sqrt{2}}{\sqrt{3}} > 1.
\]

Hence \( \max\{|v_i + v_j|, |v_i - v_j|\} = |v_i - v_j| > 1 \). Denote by \( D \) this value. By (1.36),

\[
|\int_{E_{ij}} \langle v_i, w_i(x) \rangle + \langle v_j, w_j(x) \rangle \, dx| \\
\leq \int_{E_{ij}} |\langle v_i, w_i(x) \rangle + \langle v_j, w_j(x) \rangle| \, dx \leq DH^2(E_{ij}),
\]

and hence

\[
\left| \sum_{1 \leq i < j \leq 4} \int_{E_{ij}} \langle v_i, w_i(x) \rangle + \langle v_j, w_j(x) \rangle \, dx \right| \\
\leq \sum_{1 \leq i < j \leq 4} \left| \int_{E_{ij}} \langle v_i, w_i(x) \rangle + \langle v_j, w_j(x) \rangle \, dx \right| = D \sum_{1 \leq i < j \leq 4} H^2(E_{ij}).
\]

Combining (1.33), (1.35), and (1.39), we get

\[
\sum_{i=1}^{4} H^2(R_j) \leq \sum_{j=1}^{4} H^2(E_j^0) + D \sum_{1 \leq i < j \leq 4} H^2(E_{ij}) \\
\leq D \left[ \sum_{j=1}^{4} H^2(E_j^0) + \sum_{1 \leq i < j \leq 4} H^2(E_{ij}) \right] \quad \text{(since } D > 1) \\
= DH^2(\cup_{j=1}^{4} E_j) \leq DH^2(E).
\]

On the other hand, we can do the same thing for \( T \), the cone over the 1-skeleton of the regular tetrahedron \( C \). Since \( T \) separates the four faces of \( C \), (1.13) is automatically false for \( T \). Then, by the foregoing, we can see that \( T_i^0 = \emptyset \) for all \( i \), \( \epsilon_{ij} = -1 \) for all \( i \neq j \), and \( (v_i - v_j) \perp T_xT \) for almost all \( x \in T_{ij} \), which implies that

\[
\langle v_i, w_i(x) \rangle + \langle v_j, w_j(x) \rangle = D
\]

for all \( x \in T_{ij} \). So briefly, the inequalities throughout the argument above are equalities for \( T \cap \overline{T} \). As a result,

\[
DH^2(E) \geq \sum_{i=1}^{4} H^2(R_j) = DH^2(T \cap \overline{T}),
\]
and hence

\[(1.43) \quad H^2(E) \geq H^2(T \cap \overline{B})\]

for all \(E\) that do not satisfy (1.13).

Now we prove 2). Let \(E\) be a reduced minimal set. Then it is rectifiable and locally Ahlfors regular in \(B\) ([6]).

First note that \(H^2(E) \leq H^2(T \cap \overline{B})\). In fact, for each \(x \in \overline{B} \setminus T\), there exists 1 \(\leq i \leq 4\), such that \(x\) and \(S_i\) belong to the same connected component of \(B \setminus T\). Denote by \(f(x)\) the first intersection of \(x + [0, a_i)\) with \(T\). Then \(f: \overline{B} \to T\) is a 2-Lipschitz retraction (see Figure 8). Now if \(E\) is a minimal set that satisfies (1.12), for each \(\epsilon > 0\), we define \(g_\epsilon: \partial B \cup E \to T \cup \partial B\) by \(g_\epsilon(x) = f(x)\) for \(x \in E \cap \overline{B}(0, 1 - \epsilon)\), \(g_\epsilon(x) = x\) for \(x \in \partial B\). Then we can extend \(g_\epsilon\) to a 2-Lipschitz map which sends \(\overline{B}\) to \(\overline{B}\) by Kirszbraun’s Theorem ([7], Thm. 2.10.43), see Figure 8. Thus \(g_\epsilon\) deforms \(E \cap \overline{B}(0, 1 - \epsilon)\) to a subset of \(T \cap \overline{B}(0, 1 - \epsilon)\). Thus we have

\[(1.44) \quad H^2(g_\epsilon(E)) = H^2(g_\epsilon(E \cap \overline{B}(0, 1 - \epsilon))) + H^2(g_\epsilon(E \setminus \overline{B}(0, 1 - \epsilon)))
\leq H^2(g_\epsilon(T \cap \overline{B}(0, 1 - \epsilon))) + \text{Lip}(g_\epsilon)^2 H^2(E \setminus \overline{B}(0, 1 - \epsilon))
= H^2(T \cap \overline{B}(0, 1 - \epsilon)) + 4H^2(E \setminus \overline{B}(0, 1 - \epsilon))
< H^2(T \cap \overline{B}) + 4H^2(E \setminus \overline{B}(0, 1 - \epsilon))\]

The second term tends to 0 when \(\epsilon\) tends to 0. That is, for any \(\delta > 0\), there exists \(\epsilon(\delta) > 0\) such that

\[(1.45) \quad H^2(g_\epsilon(E)) < H^2(T \cap \overline{B}) + \delta.\]

Now \(E\) is minimal, hence for any \(\delta > 0\),

\[(1.46) \quad H^2(E) \leq H^2(g_\epsilon(\delta)(E)) \leq H^2(T \cap \overline{B}) + \delta,\]

therefore

\[(1.47) \quad H^2(E) \leq H^2(T \cap \overline{B}).\]
Hence to prove 2), it is enough to prove that if (1.13) does not hold, and 
$H^2(E) = H^2(T \cap \overline{B})$, then $E = T \cap \overline{B}$. In particular $E$ contains a point of type $T$.

By the arguments in 1), if (1.13) is not true, and $H^2(E) = H^2(T \cap \overline{B})$, then the inequalities (1.34)–(1.36) and (1.38)–(1.40) are all equalities. Thus we have:

1) For almost all $x \in E_{ij}$, $T_x E_{ij} \perp v_i - v_j$. Denote by $P_{ij}$ the plane perpendicular to $v_i - v_j$. Then for almost all $x \in E_{ij}$, $T_x E_{ij} = P_{ij}$.

2) For all $j$, $H^2(E_j^0) = 0$, since $D > 1$.

3) For all $j$, $H^2(A_j \setminus E_j) = 0$.

4) For all $j$, $P_j(E) = P_j(E_j) = R_j$.

Thus for almost all $x \in E$, $T_x E$ exists and is one of the $P_{ij}$. If $x$ is a point such that $T_x E$ exists, by the $C^1$ regularity ([5], Theorem 1.15 and Lemma 14.4), there exists $r = r(x) > 0$ such that in $B(x, r)$, $E$ is the graph of a $C^1$ function from $T_x E$ to $T_x E^\perp$, which implies that in $B(x, r)$, the function $f : E \cap B(x, r) \to G(3, 2)$, $f(y) = T_y E$ is continuous. But for $T_y E$ we have only six choices $P_{ij}, 1 \leq i < j \leq 4$, which are isolated points in $G(3, 2)$, and so $T_y E = T_x E$ for all $y \in B(x, r) \cap E$. As a result $E \cap B(x, r) = (T_x E + x) \cap B(x, r)$, a disk parallel to $P_{ij}$.

Thus by the $C^1$ regularity, the set $E_P = \{x \in E \cap B : T_x E$ exists $\}$ is a $C^1$ manifold, and is open in $E$. Thus we deduce that

\[(1.48)\]

each connected component of $E_P$ is part of a plane that is parallel to one of the $P_{ij}$.

Set $E_Y = \{x \in E : x$ is of type $Y\}$. Then $E_Y \neq \emptyset$, because otherwise by (1.48), $E \cap B$ is the intersection of $B$ with a translation of one of the $P_{ij}$, but then $E \cap \partial B$ is surely not $T \cap \partial B$.

Now if $x \in E_Y$, by the $C^1$ regularity around points of type $Y$ ([5], Theorem 1.15 and Lemma 14.6), there exists $r = r(x) > 0$ such that in $B(x, r)$, $E$ is $C^1$ equivalent to a $Y$-set $Y$. Denote by $L_Y$ the spine of $Y$, and by $S_1$, $S_2$ and $S_3$ the three open half-planes of $Y$. Then if we denote by $\varphi$ the $C^1$ diffeomorphism which sends $Y$ onto $E$ in $B(x, r)$, the $\varphi(S_i) \cap B(x, r)$, $1 \leq i \leq 3$, are connected $C^1$ manifolds, and hence each of them is a part of a plane parallel to $P_{ij}$. Consequently, $\varphi(L_y) \cap B(x, r)$ is an open segment passing through $x$ and parallel to one of the $D_j$, $1 \leq j \leq 4$, where $D_j = P_{ij} \cap P_{jk}$.

Hence $E_Y \cap B$ is a union of open segments $I_1, I_2, \ldots$, each of which is parallel to one of the $D_j$, and every endpoint is either a point in the sphere $\partial B$, or a point of type $T$. Moreover,

\[(1.49)\]

for each $x \in E_Y$ such that $T_x E_Y = D_j$, there exists $r > 0$ such that, in $B(x, r)$, $E$ is a $Y$-set whose spine is $x + D_j$.

Now if $x \in E$ is a $T$-point, then by the arguments above, the blow-up limit $C_x E$ of $E$ at $x$ is the set $T$ (the set $T$ defined at the very beginning of this section). As a result, for each segment $I_i$, at least one of its endpoints is in the unit sphere. In fact, if both of the endpoints $x$ and $y$ of $I_i$ are of type $T$, then at least one of the two blow-up limits $C_x E$ and $C_y E$ is not the set $T$, because two parallel $T$-sets cannot be connected by a common spine.

Hence all the segments $I_i$ touch the boundary.
Lemma 1.8. If $x$ is a $T$-point (and hence $C_xE = T$), then $(T + x) \cap B \subset E$.

Proof. By the $C^1$ regularity around points of type $T$, there exists $r > 0$ such that in $B(x, r)$, $E$ is a $C^1$ version of $T + x$. Then by (1.48) and (1.49), $E \cap B(x, r) = (T + x) \cap B(x, r)$. Denote by $L_i$, $1 \leq i \leq 4$, the four spines of $T + x$. Then $L_i \cap B \subset E_Y$, because $L_i \cap B(x, r)$ is part of a segment $I_i \subset E_Y$, which has already an endpoint $x$ that does not belong to the unit sphere, hence the other endpoint must be in the sphere, which yields $I_1 = L_1 \cap B(0, 1)$.

Now we take a one parameter family of open balls $B_s$ with radii $r \leq s \leq 1$, with $B_s = B(x, r)$ and $B_1 = B(0, 1)$, such that
\begin{enumerate}
  \item $B_s \subset B_{s'}$ for all $s < s'$;
  \item $\cap_{1 > s > 0} B_t = B_s$ and $\cup_{t < s} B_t = B_s$ for all $r \leq s \leq 1$.
\end{enumerate}

Set $R = \inf\{s > 0, (T + x) \cap B_s \not\subset E\}$. We claim that $R = 1$.

Suppose this is not true. By definition of $B_s$, the four spines and the six faces of $T + x$ are never tangent to $\partial B_s$, $r < s < 1$, since $B(x, r) \subset B_s$.

Now for each $y \in \partial B_R \cap (T + x)$, $y$ is not a $T$-point. In fact, if $y$ belongs to one of the $L_i$, then $y$ is a $Y$ point, since $L_i \setminus \{x\} \subset E_Y$ and $L_i \cap B \subset E_Y$; if $y$ is not a $Y$ point, then there exist $i$ and $j$ such that $y \in x + P_{ij}$. Thus there exists $r_y > 0$ such that $B(y, r_y) \cap (x + T)$ is a disk $D_y$ centered at $y$. Now by definition of $R$, for all $s < r$, $B_s \cap (T + x) \subset E$, and hence $B_R \cap (T + x) \subset E$. Hence $D \cap B_R \cap B(y, r_y) \subset E$, which means that $y$ cannot be a point of type $T$.

If $y$ is a point of type $P$ (i.e., a planar point), suppose for example that $y \in P_{ij} + x$. Then $T_y E = P_{ij}$. By (1.48), and since $R < 1$, there exists $r_y > 0$ such that $E \cap B(y, r_y) = (P_{ij} + y) \cap B(y, r_y)$. In other words,

\begin{enumerate}
  \item there exists $r_y > 0$ such that $E$ coincides with $T + x$ in $B_R \cup B(y, r_y)$.
\end{enumerate}

If $y$ is a point of type $Y$, then it is in one of the $L_i$. By the same argument as above, using (1.49), we get also (1.50).

Hence (1.50) is true for all $y \in \partial B_R \cap (T + x)$. As $\partial B_R \cap (T + x)$ is compact, we have thus a uniform $r > 0$ such that for each $y$, (1.50) is true if we set $r_y = r$. However, this contradicts the definition of $R$.

Hence $R = 1$, but $B_1 \subset B$ is of radius 1, so $B_1 = B$. Then by definition of $R$ we get the conclusion of Lemma 1.8. \hfill \qed

By Lemma 1.8, we know that if $x$ is a $T$-point, then $x$ has to be the origin, because of (1.12). Hence $T \cap B \subset E$. In this case, we have $E = T \cap \overline{B}$, because $H^2(E) = H^2(T \cap \overline{B})$.

We still have to discuss the case when there is no point of type $T$. In this case, the same kind of argument as in Lemma 1.8 gives the following.

Lemma 1.9. Let $x$ be a $Y$ point in $E$, and $T_y E_Y = D_y$. Denote by $Y_j$ the $Y$ whose spine is $D_j$. Then

\begin{enumerate}
  \item $Y_j + x \cap B \subset E$.
\end{enumerate}

But this is impossible, because $E \cap \partial B = T \cap \partial B$ contains no full part of $(Y_j + x) \cap \partial B(0, 1)$ for any $x$ and $j$.

Hence we have $E = T \cap \overline{B}$, and thus (1.14). \hfill \qed
2. Existence of a point of type $T$ for a 2-dimensional topological minimal set in $\mathbb{R}^4$

2.1. Introduction

In this section we discuss the property (0.8) for 2-dimensional topological minimal sets in $\mathbb{R}^4$ whose blow-in limits are $T$-sets. This kind of set exists trivially because a $T$-cone is topological minimal in $\mathbb{R}^3$, and by Proposition 3.18 of [11], it is topological minimal in any $\mathbb{R}^n$ for $n \geq 3$.

One wonders whether there is any other type of topological minimal sets in $\mathbb{R}^4$ that look at infinity like $T$-sets without themselves being $T$-sets. Recall that in $\mathbb{R}^3$ there are no such sets (see Proposition 18.1 of [4]). An important and useful property of $\mathbb{R}^3$ is that in $\mathbb{R}^3$ there are only two kinds of minimal cones whose densities are less than that of a $T$-set: the planes and the $Y$-sets. Hence if the blow-in limits of a non-conical minimal set are $T$-sets, then by the monotonicity of density, in this minimal set all points are of type $P$ or $Y$, and hence there can hold the same properties as those stated in (1.4)-(1.6) and Figure 2, and by the same argument.

We do not know if there exists a minimal cone in $\mathbb{R}^4$ whose density is between those of $Y$-sets and $T$-sets. However $T$-sets are the only minimal cones that admit the simplest topology except for planes and $Y$-sets. Hence it is likely that in $\mathbb{R}^4$ there are no minimal cones between $Y$-sets and $T$-sets.

Consequently we make the following additional assumption. Denote by $d_T$ the density of $T$-sets, and suppose that

\[(2.1) \quad \text{the only minimal cones in } \mathbb{R}^4 \text{ whose densities are less than } d_T \text{ are the planes and the } Y \text{-sets.}\]

We are going to discuss, under the assumption (2.1), the Bernstein type property for topological minimal sets in $\mathbb{R}^4$ that look like a $T$-set at infinity.

2.2. A topological criterion for potential counterexamples

Throughout this subsection, we assume that (2.1) is true.

Let $E$ be a 2-dimensional topological minimal set in $\mathbb{R}^4$ that looks like a $T$-set at infinity. That is, there exists a $T$-set $T$ centered at the origin, and a sequence $\{r_k\} \in \mathbb{N}$ such that

\[(2.2) \quad \lim_{k \to \infty} r_k \to \infty \quad \text{and} \quad \lim_{k \to \infty} d_{0,r_k}(E,T) = 0.\]

We want to find a type $T$ point in the set $E$.

Now the set $E$ is of codimension 2, hence the topological condition is imposed on the group $H_1(\mathbb{R}^4 \setminus E, \mathbb{Z})$.

Denote by $\{y_i\}_{1 \leq i \leq 4}$ the four $Y$-points in $T \cap \partial B(0,1)$. Denote by $l_{ij} \subset T \cap \partial B(0,1)$ the great arc on the sphere that connects $y_i$ and $y_j$. The cone $T$ is composed of 6 closed sectors $\{T_{ij}\}_{1 \leq i \neq j \leq 4}$, where $T_{ij}$ is the cone over $l_{ij}$. Denote by $x_{ij}, 1 \leq i \neq j \leq 4$, the middle point of $l_{ij}$. Denote by $P_{ij}$ the 2-plane orthogonal
Figure 9.

to \(T_{ij}\) and passing through \(x_{ij}\). Set \(B_{ij} = B(x_{ij}, 1/10) \cap P_{ij}\), and denote by \(s_{ij}\) the boundary of \(B_{ij}\). Then \(s_{ij}\) is a circle, that does not touch \(T\), and \(B_{ij} \cap T = B_{ij} \cap T_{ij} = x_{ij}\).

Fix an orthonormal basis \(\{e_i\}_{1 \leq i \leq 4}\) of \(\mathbb{R}^4\). We are going to give an orientation to each \(s_{ij}\), and denote these oriented circles by \(\vec{s}_{ij}\).

For each \(B_{ij}\), there are two orientations \(\sigma_1 = x \wedge y\) and \(\sigma_2 = -x \wedge y\), where \(x\) and \(y\) are two mutually orthogonal unit vectors that belong to the plane containing \(B_{ij}\). Take the \(k \in \{1, 2\}\) such that \(\det_{1 \leq i \leq 4} \mathbf{e}_i \wedge y_i y_j \wedge \sigma_k > 0\), and denote by \(\vec{B}_{ij}\) the oriented disk \(B_{ij}\) with this orientation. Denote by \(\vec{s}_{ij} = \partial \vec{B}_{ij}\) the oriented circle, and by \([\vec{s}_{ij}]\) the element in \(H_1(\mathbb{R}^4 \setminus T; \mathbb{Z})\) represented by \(\vec{s}_{ij}\). The six \([\vec{s}_{ij}]\), \(1 \leq i < j \leq 4\), are all different, however they are algebraically dependent.

Figure 9 gives an idea of the above definition (although it is drawn in \(\mathbb{R}^3\)). Since \(T\) is contained in \(\mathbb{R}^3\), if we fix an orientation of the complementary dimension in \(\mathbb{R}^4\), the orientation of \(B_{ij}\) defined before corresponds to one of the orientation of the line orthogonal to \(T_{ij}\) in \(\mathbb{R}^3\). And this orientation of the line corresponds to the orientations of \(l_{ij}\) by the right-hand rule. Hence in Figure 9 we indicate the orientation of \(l_{ij}\) with arrows to express the orientation of \([\vec{s}_{ij}]\). In the figure, the orientation \(\tilde{S}_{ij}\) means \([\vec{s}_{ij}]\).

Thus we have

\[
(2.3) \quad \vec{s}_{ij} = -\vec{s}_{ji}, [\vec{s}_{ij}] = -[\vec{s}_{ji}].
\]

Note that \(\{[\vec{s}_{ij}]\}, 1 \leq i, j \leq 4\) is a set of generators of the group \(H_1(\mathbb{R}^4 \setminus T; \mathbb{Z})\).

We say that \([s_{ij}]\) and \([s_{kl}]\) (without the vector arrows) are different (in a homology group) if

\[
(2.4) \quad [\vec{s}_{ij}] \neq \pm [\vec{s}_{kl}],
\]

and write \(s_{ij} \sim s_{kl}\) if \([\vec{s}_{ij}] = \pm [\vec{s}_{kl}]\).
Global regularity for $T$ and counterexamples

Return to the set $E$. Without loss of generality, we can suppose (modulo replacing $E$ by $E/r_k$ for some $k$ large) that $d_{k,3}(E,T)$ is small enough (for example less than a certain $\epsilon_0$). Then (by the argument between (1.4) and (1.5)) in $B(0,5/2)\setminus B(0,1/2)$, $E$ is a $C^1$ version of $T$. Therefore in $B(0,5/2)\setminus B(0,1/2)$, $E$ is composed of six $C^1$ faces $E_{ij}$, that are very close to the $T_{ij}$. The $E_{ij}$ meet in threes, on four $C^1$ curves $\eta_i$, $1 \leq i \leq 4$, each $\eta_i$ is very near the half-line $[a_i,y_i)$, and near each $\eta_i$, there exists a tubular neighborhood $T_i$ of $\eta_i$, which contains $B([a_i,y_i), r)$ for some $r > 0$, in which $E$ is a $C^1$ version of a $\mathcal{Y}$-set. See Section 18 of [4] for details. In total, there is a $C^1$ diffeomorphism $\varphi$, which is very near the identity, such that in $B(0,5/2)\setminus B(0,1/2)$, $E$ coincides with $\varphi(T)$. $E_{ij}$ corresponds to $\varphi(T_{ij})$, and $\eta_i$ corresponds to $\varphi([0,y_i))$.

In particular, since $E$ is very near $T$ in $B(0,5/2)\setminus B(0,1/2)$, $s_{ij} \cap E = \emptyset$, and $B_{ij} \cap E = B_{ij} \cap E_{ij}$ is a one point set, so that locally each $s_{ij}$ links $E_{ij}$, and hence is an element (possibly zero) in $H_1(\mathbb{R}^4 \setminus E,Z)$, too.

Now we discuss the values in $H_1(\mathbb{R}^4 \setminus E,Z)$ for these $s_{ij}$.

**Lemma 2.1.** Let $E$ be an $\mathcal{A}$-minimal set that satisfies (2.2). Let the notation be as above. Then if

\[(2.5) \quad \text{for all } 1 \leq i < j \leq 4, \quad [s_{ij}^+] \neq 0 \text{ in } H_1(\mathbb{R}^4 \setminus E,Z),\]

and

\[(2.6) \quad \text{at least 5 of the } [s_{ij}] \text{ are mutually different in } H_1(\mathbb{R}^4 \setminus E,Z),\]

then $E$ contains at least one point of type other than $\mathbb{P}$ and $\mathcal{Y}$.

**Proof.** We prove this by contradiction. Suppose that there are only $\mathbb{P}$ and $\mathcal{Y}$-points. Then for all $x \in E$, the density $\theta(x) = \lim_{r \to 0} \frac{1}{r^2} H^2(B(x,r) \cap E)/r^2$ of $E$ at $x$ is either $3/2$ or $1$. In other words, all singular points in $E$ are of type $\mathcal{Y}$.

Denote by $E_Y$ the set of all the $\mathcal{Y}$-points of $E$. Then $E_Y \cap B(0,2)$ are composed of $C^1$ curves, whose endpoints belong to $\partial B(0,2)$ (see [4], Lemma 18.11, and for the $C^1$ regularity around $\mathcal{Y}$-points, see [5] Theorem 1.15 and Lemma 14.6).

The following argument is the same as following Lemma 18.11 in Section 18 of [4] (where the reader can find more details). Here we only sketch the argument.

Since $E$ looks very much like $T$ in $B(0,5/2)\setminus B(0,1/2)$, we have $E_Y \cap \partial B(0,2) = \{a_1, a_2, a_3, a_4\}$, $E_Y \cap \partial B(0,1) = \{b_1, b_2, b_3, b_4\}$, where $b_j$ is the point among the $b_j$, $1 \leq j \leq 4$, nearest to $a_i$. Then through each $a_i$ there passes a curve in $E_Y$, and hence locally $a_i$ lies in the intersection of three half surfaces $E_{ij}$, $j \neq i, 1 \leq j \leq 4$.

But on the sphere $\partial B(0,2)$ we have four $\mathcal{Y}$-points, hence, without loss of generality, we can suppose that there is a curve $\gamma_1$ of $E_Y$ that enters the ball $B(0,2)$ at $a_1$ and leaves the ball at $a_2$, and another curve $\gamma_2$ which enters the ball at $a_3$ and leaves it at $a_4$ (see Figure 10, where the green curves represent the $\gamma_i, i = 1, 2$, and we do not know much about the structure of $E$ in $B_{1/2} = B(0,1/2)$). Near each point $x$ of $\gamma_1$, there exists a $C^1$-ball $B(x,r_x)$ of $x$, in which $E$ is the image of a $\mathcal{Y}$ set under a $C^1$ diffeomorphism. By compactness of the curve $\gamma_1$, there exists a tubular neighborhood $I_1$ of $\gamma_1$ such that $E \cap I_1$ is composed exactly of three surfaces that meet along the curve $\gamma_1$. 
Since $\gamma_1$ connects $a_1$ and $a_2$, it passes through $b_1$ and $b_2$. Near each $b_i$, the set $E$ is composed of three half surfaces $E_{ij}$, $j \neq i$. Then since $E$ is locally composed of three half surfaces all along $\gamma_1$, these three half surfaces connect $E_{12}$, $E_{13}$ and $E_{14}$ to $E_{21}$, $E_{23}$ and $E_{24}$. Hence we know that $s_{12}$, $s_{13}$ and $s_{14}$ are homotopic to $s_{21}$, $s_{23}$ and $s_{24}$ (but we do not know which is homotopic to which). A similar argument gives also that $s_{31}$, $s_{32}$ and $s_{34}$ are homotopic to $s_{41}$, $s_{42}$ and $s_{43}$.

For the part $\gamma_1$ we have the following six cases:

\begin{align}
  s_{12} &\sim s_{21}, \quad s_{13} \sim s_{23}, \quad s_{14} \sim s_{24}; \\
  s_{12} &\sim s_{21}, \quad s_{13} \sim s_{24}, \quad s_{14} \sim s_{23}; \\
  s_{12} &\sim s_{23}, \quad s_{13} \sim s_{21}, \quad s_{14} \sim s_{24}; \\
  s_{12} &\sim s_{23}, \quad s_{13} \sim s_{24}, \quad s_{14} \sim s_{21}; \\
  s_{12} &\sim s_{24}, \quad s_{13} \sim s_{21}, \quad s_{14} \sim s_{23}; \\
  s_{12} &\sim s_{24}, \quad s_{13} \sim s_{23}, \quad s_{14} \sim s_{21}.
\end{align}

(2.7)

Note also that automatically $s_{12} \sim s_{21}$, hence the six cases reduce to the following four (modulo the symmetry between the indices 3 and 4):

\begin{align}
  s_{13} &\sim s_{23}, \quad s_{14} \sim s_{24}; \\
  s_{13} &\sim s_{24}, \quad s_{14} \sim s_{23}; \\
  s_{12} &\sim s_{23} \sim s_{13}, \quad s_{14} \sim s_{24}; \\
  s_{12} &\sim s_{23} \sim s_{14}, \quad s_{13} \sim s_{24}.
\end{align}

(2.8)

Similarly, for the part $\gamma_2$, we have the following four cases:

\begin{align}
  s_{31} &\sim s_{41}, \quad s_{32} \sim s_{42}; \\
  s_{31} &\sim s_{42}, \quad s_{32} \sim s_{41}; \\
  s_{34} &\sim s_{41} \sim s_{31}, \quad s_{32} \sim s_{42}; \\
  s_{34} &\sim s_{41} \sim s_{32}, \quad s_{31} \sim s_{42}.
\end{align}

(2.9)
Combining (2.8) and (2.9), we have the eight cases

\begin{equation}
(s_1 \sim s_2 \sim s_3 \sim s_4; \quad s_1 \sim s_2 \sim s_3 \sim s_4; \quad s_1 \sim s_2 \sim s_3 \sim s_4; \quad s_1 \sim s_4; \quad s_1 \sim s_4; \quad s_1 \sim s_4; \quad s_1 \sim s_4; \quad s_1 \sim s_4;)
\end{equation}

(2.10)

In particular, at most four of the \([s_{ij}], 1 \leq i < j \leq 4\), are different, which contradicts our hypothesis that at least five of the \({[s_{ij}], 1 \leq i < j \leq 4}\) are different in \(H^1(\mathbb{R}^4\setminus E; \mathbb{Z})\),

\[\Box\]

**Corollary 2.2.** Let \(E\) be a 2-dimensional reduced Almgren-minimal set in \(\mathbb{R}^4\) such that (2.2), (2.5) and (2.6) hold. Suppose also that (2.1) holds. Then \(E\) is a \(\mathbb{T}\)-set parallel to \(T\).

**Proof.** By Lemma 2.1, \(E\) contains a point \(x\) of type other than \(\mathbb{P}\) and \(\mathbb{Y}\), hence by (2.1), the density \(\theta(x)\) of \(E\) at \(x\) is larger than or equal to \(d_T\). Define \(\theta(t) = t^{-2}H^2(E \cap B(x, t))\) the density function of \(E\) at \(x\). By Proposition 5.16 of [4], \(\theta(t)\) is nondecreasing on \(t\). Then (2.2) and Lemma 16.43 of [4] give that \(\lim_{t \to 0} \theta_t = d_T\). Since we already know that \(\theta(x) = \lim_{t \to 0} \theta(t) \geq d_T\), the monotonicity of \(\theta\) yields that \(\theta(t) > d_T\) for all \(t > 0\). By Theorem 6.2 of [4], the set \(E\) is a minimal cone centered at \(x\), with density \(d_T\). Thus, by (2.2), \(E\) is a \(\mathbb{T}\)-set centered at \(x\) and parallel to the set \(T\). 

After Corollary 2.2, there remains only to discuss the case where \(E\) is topologically minimal, and no more than 4 of the \([s_{ij}], 1 \leq i < j \leq 4\) are different.

First we prove some properties of these \(s_{ij}\).

**Lemma 2.3.** 1) \(\sum_{j \neq i} [s_{ij}] = 0\) for all \(1 \leq i \leq 4\).

2) For each \(i \neq j \neq k\), \([s_{ij}] \neq 0\), and \([s_{ij}] \neq [s_{jk}]\).

**Proof.** 1) Fix \(1 \leq i \leq 4\). We write \(\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}\), where \(T \subset \mathbb{R}^3\).

Recall that \(y_i, 1 \leq i \leq 4\), are the four \(\mathbb{Y}\)-points of \(T \cap \partial B(0, 1); T_{ij}\) is the sector of \(T\) passing through the origin and \(y_i, y_j; x_{ij}\) is the middle point of the great arc passing through \(y_i, y_j; P_{ij}\) is the plane passing containing \(x_{ij}\) and orthogonal to \(T_{ij}\); and \(s_{ij} = \partial B_{ij}\), where \(B_{ij} = B(x_{ij}, 1/10) \cap P_{ij}\).
Denote by $Y_i$ the cone over $Z_i := \cup_{j \neq i} \hat{y}_i x_{ij}$, where $\hat{y}_i x_{ij}$ denotes the great arc connecting $y_i$ and $x_{ij}$ (see Figure 11 of $Y_1 \subset \mathbb{R}^3$ below), and by $C_T$ the convex hull of $Y_i$. Set $C = C_T \times \mathbb{R}$. Since $C$ is a cone, $C \setminus T$ is also a cone. Note that $Z_i \subset S^3 \cap C$ is a spherical $\mathcal{Y}$-set of dimension 1. We want to show that $\sum_{j \neq i} \hat{s}_{ij} = 0$ in $H_1(C \setminus T, \mathbb{Z})$.

Note that $\hat{s}_{ij}$ is homotopic in $C \setminus T$ to its radial projection $\hat{s}'_{ij}$ on $S^3$ (the orientation of $\hat{s}'_{ij}$ is induced by $\hat{s}_{ij}$ on the sphere $S^3$). In fact, denoting by $\pi_S$ the radial projection of $\mathbb{R}^4 \setminus \{0\}$ to $S^3$, for each $x \in s_{ij}$, the segment $[x, \pi_S(x)]$ belongs to a radial half-line that does not meet any other radial half-lines. In particular, since $x \in \mathbb{R}^4 \setminus T$, where $T$ is a union of radial half-lines, $[x, \pi_X(x)] \cap T = \emptyset$. Hence if we set $f_t(x) = (1 - t)x + t\pi_S(x), 0 \leq t \leq 1$, then $f_t$ is a homotopy between $\hat{s}_{ij}$ and $\hat{s}'_{ij} = \pi_S(\hat{s}_{ij})$.

Therefore on the sphere, in $C \cap S^3$, the $s_{ij}, j \neq i$, are topologically three circles that link respectively the three branches of $Z_i$. Recall that the pair of topological spaces $(C \cap S^3, Z_i)$ is homotopic to $(\mathbb{R}^3, Y)$ where $Y$ is a 1-dimensional $\mathcal{Y}$-set. However in $(\mathbb{R}^3, Y)$, the union of the three oriented circles that link the three branches of $Y$ is the boundary of an oriented manifold with boundary contained in $\mathbb{R}^3 \setminus Y$. Hence, similarly, there exists an oriented manifold with two-dimensional boundary $\Sigma \subset C \cap S^3 \setminus Z_i$ such that $\partial \Sigma = \cup_{j \neq i} \hat{s}'_{ij}$ (see Figure 12, where $s_{ij}$ denotes the oriented circle $\hat{s}_{ij}$, and the orientation of $\Sigma$ is indicated by the exterior normal vector $\hat{n}$). Therefore, after a smooth triangulation under which $\Gamma$ and $s_{ij}$ are all smooth chains, we have $\partial[\Sigma] = \cup_{j \neq i} \hat{s}'_{ij}$. Since $\Sigma \subset C \cap S^3 \setminus Z_i \subset \mathbb{R}^4 \setminus T$, $\sum_{j \neq i} \hat{s}'_{ij} = 0$ in $H_1(\mathbb{R}^4 \setminus T, \mathbb{Z})$. Then since $\hat{s}'_{ij}$ is homotopic to $\hat{s}_{ij}$,

$$\sum_{j \neq i} \hat{s}_{ij} = 0 \quad \text{in} \quad H_1(\mathbb{R}^4 \setminus T, \mathbb{Z}).$$

Now since $E$ is as near as we like to $T$, we can suppose that $\Sigma$ and the $f_t(s_{ij})$ do not touch $E$. Thus we get (2.11).
2) Without loss of generality, suppose for example that $i = 1$, $j = 2$ and $k = 3$. If $[S_{12}] = [S_{23}]$, then, by (2.11),

$$(2.15) \ [\v s_{24}] = 0.$$  

Hence we have only to prove (2.12).

Suppose for example that $i = 2$ and $j = 4$. Then (2.15) means that there exists a smooth simplicial 2-chain $\Gamma$ in $\mathbb{R}^4 \setminus E$ such that $\partial \Gamma = \v s_{24}$. Since $E$ is closed, there exists a neighborhood $U$ of $\Gamma$ such that $U \cap E = \emptyset$. In particular, $s_{24} \subset U$.

Set $D = E \cap B(x_{24}, 1/10)$. Then by the regularity of the minimal set $E$ which is very near $T$, in $B(x_{24}, 1/8)$, $E$ is a piece of very flat surface that is almost a disc. Therefore $D$ is a surface with positive measure.

Set $F = E \setminus D$, so $F \setminus B(0, 2) = E \setminus B(0, 2)$. We want to show that $F$ is a topological competitor of $E$ with respect to the ball $B(0, 2)$.

Suppose that $\gamma \subset \mathbb{R}^4 \setminus (B(0, 2) \cup E)$ is an oriented circle. We have to show that if $[\gamma]$ is zero in $H_1(\mathbb{R}^4 \setminus F, \mathbb{Z})$, then it is zero in $H^1(\mathbb{R}^4 \setminus E, \mathbb{Z})$.

Now if $[\gamma]$ is zero in $H_1(\mathbb{R}^4 \setminus F, \mathbb{Z})$, then there exists a smooth simplicial 2-chain $\Sigma \subset \mathbb{R}^4 \setminus F$ such that $\partial \Sigma = \gamma$. By the transversality theorem (see, for example, Theorem 2.1 in Chapter 3 of [8]), we can require that $\Sigma$ is transversal to $\partial B(x_{24}, 1/10)$.

If $\Sigma \cap B(x_{24}, 1/10) = \emptyset$, then $\Sigma \subset \mathbb{R}^4 \setminus E$ too, and hence $[\gamma] = 0 \in H_1(\mathbb{R}^4 \setminus E, \mathbb{Z})$. If $\Sigma \cap B(x_{24}, 1/10) \neq \emptyset$, then, by the transversality of $\Sigma$ and $\partial B(x_{24}, 1/10)$, and by Proposition 2.36 of [12], their intersection is a closed smooth simplicial 1-chain $s \subset \partial B(x_{24}, 1/10)$.

Now we work in $B_1 := \overline{B}(x_{24}, 1/10)$. Since $D$ is a very flat topological disc,

$$(2.16) \ H_1(\overline{B}_1 \setminus D) = \mathbb{Z},$$

whose generator is $[\v s_{24}]$. As a result, there exists $n \in \mathbb{Z}$ such that $[s] = n[\v s_{24}]$. Hence there exists a smooth simplicial 1-chain $R \subset \overline{B}_1 \setminus D$ such that $\partial R = s - n\v s_{24}$.

Recall that $\Gamma \subset \mathbb{R}^4 \setminus E$ is such that $\partial \Gamma = \v s_{24}$. As a result, $\Sigma' = \Sigma \setminus \overline{B}_1 + n\Gamma + R$ is a 2-chain satisfying $\partial [\Sigma'] = [\gamma]$. Moreover $\Sigma' \subset \mathbb{R}^4 \setminus E$. Hence $[\gamma]$ is also zero in $H^1(\mathbb{R}^4 \setminus E, \mathbb{Z})$.  

![Figure 12](image-url)
Thus, \( F = E \setminus D \) is a topological competitor of \( E \).

However, \( D \) has positive measure, hence

\[
H^2(F) < H^2(E),
\]

which contradicts the fact that \( E \) is topologically minimal.

Thus we have proved (2.12), and hence (2.13), and so we have completed the proof of Lemma 2.3. \( \square \)

Now we return to our discussion of the case where \( E \) is very near a \( T \)-set \( T \) at scale 1, but contains no point of type other than \( \mathbb{P} \) and \( T \). After a consideration (see Section 19 of [10] for details) of the eight cases in (2.10), using Lemma 2.3, the only possibility for \([\vec{s}_{ij}]\) in \( H_1(\mathbb{R}^4 \setminus E, \mathbb{Z})\) is:

\[
(2.18) \quad [\vec{s}_{13}] = -[\vec{s}_{24}] = \alpha, [\vec{s}_{14}] = -[\vec{s}_{23}] = \beta, [\vec{s}_{34}] = \alpha - \beta, [\vec{s}_{12}] = -\alpha - \beta.
\]

Thus we get the following proposition.

**Proposition 2.4.** Let \( E \) be a reduced topological minimal set of dimension 2 in \( \mathbb{R}^4 \), that verifies (2.2). Let the conventions and notation be as at the beginning of this section, and suppose also that \( \gamma_1 \) connects \( a_1 \) and \( a_2 \), while \( \gamma_2 \) connects \( a_3 \) and \( a_4 \). Then if there exists \( r > 0 \) such that \( d_{0,3r}(E,T) < \epsilon_0 \) (where \( \epsilon_0 \) is the one defined in the paragraph below (2.4)), but the \( s_{ij} \) do not satisfy (2.18) with respect to \( \frac{1}{r}E \), then \( E \) is a \( T \)-set parallel to \( T \).

### 2.3. An example

In this section, the notation and conventions are as in Subsection 2.2. We give an example of a set that satisfies (2.18).

Set \( w_{ij} = E_{ij} \cap \partial B(0,1) \) (see Figure 13, where \( \vec{w}_{ij} \) is denoted \( \vec{w}_{ij} \)). Then the \( w_{ij} \) are \( C^1 \) curves. Denote also by \( \vec{w}_{ij} \) the oriented curve from \( b_i \) to \( b_j \).

Now suppose that \( E_Y = \gamma_1 \cup \gamma_2 \). In other words, all points in \( E \) are \( \mathbb{P} \) points, except for the two curves. (For the case where \( E_Y \neq \gamma_1 \cup \gamma_2 \), we know that \( E_Y \setminus (\gamma_1 \cup \gamma_2) \) is a union of closed curves, because the only endpoints of \( E_Y \cap \partial B(0,1) \) are \( \{b_i\}_{1 \leq i \leq 4} \). This is thus a more complicated case.)

**Lemma 2.5.** \( \gamma_1 \cup \gamma_2 \cup w_{12} \cup w_{34} \) is the boundary of a \( C^1 \) surface \( S_0 \subset E \), and \( S_0 \) contains only points of type \( \mathbb{P} \).

**Proof.** By the \( C^1 \) regularity of minimal sets, the part of \( E \) in \( B(0,1) \) is composed of \( C^1 \) manifolds \( S_1, S_2, \ldots \) whose boundaries are unions of curves in the set \( Bd = \{w_{ij}, \gamma_1, \gamma_2\} \). Thus there exists \( k \in \mathbb{N} \) such that \( w_{12} \) is part of the boundary of the manifold \( S_k \). But \( \partial S_k \) is a union of several closed curves, while \( w_{12} \) is not closed. Hence there exists a curve \( \gamma \in Bd \) that touches \( w_{12} \) and such that \( \gamma \) is also part of \( \partial S_k \). If one of the \( w_{14} \) (resp. one of the \( w_{2j} \)) is part of \( \partial S_k \) and touches \( w_{12} \), we have \([\vec{s}_{12}] = [\vec{s}_{14}] \) (with orientation) (resp. \([\vec{s}_{12}] = [\vec{s}_{2j}] \)), which contradicts (2.13).

Hence the only possibility for \( \gamma \) is \( \gamma_1 \). This means the union of \( w_{12} \) and \( \gamma_1 \) is part of the boundary of a manifold \( S_k \), and except for \( w_{34} \), the boundary of \( S_k \) contains no other \( w_{ij} \). A similar argument gives also that the union of \( w_{34} \) and \( \gamma_2 \)
is part of the boundary of a manifold $S_1$, and the boundary of $S_1$ contains no other $w_{ij}$, except perhaps for $w_{12}$.

Thus, either the union of these four curves $w_{12}$, $w_{34}$, $\gamma_1$ and $\gamma_2$ is the boundary of a surface $S_k$, or the union of $w_{12}$ and $\gamma_1$ and the union of $w_{34}$ and $\gamma_2$ are the boundaries of two surfaces $S_k$ and $S_l$. In any case, the union of the four curves is the boundary of a $C^1$ surface $S_0 \subset E$, which is not necessarily connected.

By Lemma 2.5, if we excise the surface $S_0$ from $E$, then $E \setminus S$ is composed of a union of $C^1$ surfaces, whose boundaries are unions of curves belonging to $\text{Bd} E = \{w_{13}, w_{14}, w_{23}, w_{24}, \gamma_1, \gamma_2\}$. By the same argument above, there are two surfaces $S_1$ and $S_2$, with $\partial S_1 = w_{13} \cup \gamma_2 \cup w_{24} \cup \gamma_1$, and the other one $\partial S_2 = w_{23} \cup \gamma_2 \cup w_{14} \cup \gamma_1$. Moreover $S_1 \cup S_2$ is also a connected topological manifold, for which we can define a local orientation, even near $\partial S_1$ and $\partial S_2$.

Thus topologically the boundaries of the two surfaces $S_1$ and $S_2$ are like the boundaries of two squares, one with the four vertices $b_1$, $b_3$, $b_4$ and $b_2$ (we write
them in the order of adjacency), the other with four vertices $b_1, b_4, b_3$ and $b_2$. Moreover, we have to glue the two $\overrightarrow{b_1b_2}$ in $S_1$ and $S_2$ together, and the same for $\overrightarrow{b_3b_4}$. Note that these two gluings have different directions (see Figure 14).

Notice also that after the gluing, $S_1 \cup S_2$ cannot be orientable.

**Remark 2.6.** Since $S_1 \cup S_2$ is not orientable, $[\vec{s}_{13}], [\vec{s}_{14}], [\vec{s}_{24}]$ and $[\vec{s}_{23}]$ are all of order 2 in $H_1(\mathbb{R}^3 \setminus E, \mathbb{Z})$.

In fact for a connected surface $S$, the non-orientability means that for each point $x \in S$ we can find a path $\gamma : [0, 1] \to S$ such that $\gamma(0) = \gamma(1) = x$, and if we denote by $n(t) = x(t) \wedge y(t) \in \wedge_2 N_{\gamma(t)}S$ a continuous unit normal 2-vector field on $\gamma$, where $x(t), y(t) \in N_{\gamma(t)}S$ are unit normal vector fields, and $n, x$ and $y$ are continuous with respect to $t$, then $n(0) = -n(1)$. Note that $n(t)$ can also represent the oriented plane in $\mathbb{R}^3$. Define, for each $r > 0$, $s_r(t) : T = \mathbb{R}/\mathbb{Z} \to P_t = P(x(t) \wedge y(t))$ and $\theta \mapsto r[\cos(2\pi \theta)x(t) + \sin(2\pi \theta)y(t)]$. Then the images of $s_r(0)$ and $s_r(1)$ are the same circle, but with opposite orientations: $s_r(0)(t) = s_r(1)(-t)$.

Let $Q_t = \gamma(t) + P(t)$. Fix $r > 0$ sufficiently small such that for each $t \in [0, 1]$, $B(\gamma(t), r) \cap Q_t \cap S = \{\gamma(t)\}$.

Define $G : T \times [0, 1] \to \mathbb{R}^4$ by $G(\theta, t) = s_r(t)(\theta) + \gamma(t)$. This is a continuous map, with $G(T \times \{0\}) = s_r(0)$ and $G(T \times \{1\}) = s_r(1) = -s_r(0)$. As a result, the oriented circle $s_r(0)$ is homotopic to $-s_r(0)$, and is hence of order 2.

Now for each $s \in \{[\vec{s}_{13}], [\vec{s}_{14}], [\vec{s}_{24}], [\vec{s}_{23}]\}$, we can first find a circle $s'$ homotopic to $s$, such that there exists $x$ and $\gamma$ as before, and that there exists $R > 0$ such that $s_R(0) = s'$. We can find $r > 0$ as above. Then $s_r(0)$ is homotopic to $s'$, and hence $s$. Therefore $[s] = [s_r(0)]$ is of order 2.

We will construct a set $E \subset \mathbb{R}^4$, with all the above properties. That is, in $B(0, 1)$, the set $E$ is the union of $S_0$ and $S_1 \cup S_2$ as above, $S_1 \cup S_2$ is a non-orientable topological manifold, and $S_0$ has two connected components, that meet $S_1 \cup S_2$ at $\gamma_1$ and $\gamma_2$ respectively. Outside the ball $B(0, 1)$, $E$ is a $C^1$ version of $T$, and it looks like $T$ at infinity. Moreover, $H_1(\mathbb{R}^4 \setminus E)$ and the $[\vec{s}_{ij}]$ satisfy (2.18).

Take two copies of squares (see Figure 14), one with vertices (written in the clockwise order) $b_1, b_3, b_4$ and $b_2$, the other with vertices $b_1, b_4, b_3$ and $b_2$. We glue the two sides $\overrightarrow{b_1b_4}$ in $S_1$ and $S_2$ together, and we do the same for $\overrightarrow{b_3b_2}$. Thus we get a Möbius band in $\mathbb{R}^3$ (see Figure 15).

Next, take a very big regular tetrahedron centered at the origin (with vertices $y_i$, $1 \leq i \leq 4$) which contains the Möbius band constructed before. For each $i$, take a smooth curve $L_i$ issuing from $b_i$ and going to infinity, such that $L_i$ tends to $[0, y_i]$ (see Figure 15).

Then take, for each $1 \leq i \neq j \leq 4$, a $C^1$ surface $E_{ij}$, homeomorphic to $\mathbb{R}^2$, whose boundary is $L_i \cup L_j \cup [b_i b_j]$. Note that all $E_{ij}$ go to infinity, hence in $\mathbb{R}^3$, $E_{23}$ and $E_{14}$, or $E_{13}$ and $E_{24}$ must meet each other. We work in $\mathbb{R}^4$ to avoid this.

Thus we get a set that looks like a $T$ at infinity, and we cannot give any simple reason why a set with such a topology cannot be topologically minimal.
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