Exact Combinatorics of Bern-Kosower-type Amplitudes for Two-Loop $\Phi^3$ Theory

Haru-Tada Sato and Michael G. Schmidt

Institut für Theoretische Physik, Universität Heidelberg
Philosophenweg 16, D-69120 Heidelberg, Germany

Abstract

Counting the contribution rate of a world-line formula to Feynman diagrams in $\phi^3$ theory, we explain the idea how to determine precise combinatorics of Bern-Kosower-like amplitudes derived from a bosonic string theory for $N$-point two-loop Feynman amplitudes. In this connection we also present a method to derive simple and compact world-line forms for the effective action.

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1h.sato@thphys.uni-heidelberg.de
2m.g.schmidt@thphys.uni-heidelberg.de
1 Introduction

Field theory can be understood as a singular limit of string theory, and the relation between both theories has been investigated intensively for the purpose of obtaining field theory scattering amplitudes in a remarkably simple way [1]-[6]. String theory organizes scattering amplitudes in a compact form because of conformal symmetry on the world-sheet, and field theory would inherit this useful feature, by which summation of Feynman diagrams is already installed. In particular, Bern and Kosower derived a set of simple rules for one-loop gluon scattering amplitudes through analyzing the field theory limit of a heterotic string theory [1]. Later on, it was realized that these rules can also be derived directly in the world-line approach to quantum field theory [7] and that effective actions can be evaluated most conveniently with this method [8]. There are also applications to gravity [4] and super Yang-Mills theories [5].

It is also interesting to find a multi-loop generalization of Bern-Kosower rules, and various steps toward this direction were made in recent years [9]-[18]. The most well-understood theory is the $\phi^3$ theory [1-13]. Universal expressions (master formulae) for proper $N$-point functions were derived from field theory (world-line approach) [9, 10] and bosonic string theories [11]-[13]; the correspondence of corners of moduli to Feynman diagrams [11, 12], the field theory limits of world-sheet Green function [11], and the determinant factor for moduli integrals [13] were examined in detail. The results in both cases coincide with each other up to a combinatorial problem, which should finally be solved to construct the complete $N$-point functions. The master formulae contain various Feynman integrals labeled by a set of integers which represent the numbers of external legs inserted in internal lines as basic parts of world-line parametrization, and the problem is how to combine the formulas of different set of integers for fixed $N$ in order to make up the desired result.

In this paper we solve this problem in the two-loop case for $\phi^3$ theory, for the purpose of a suggestion to multi-loop generalization. The basic idea to obtain exact $N$-point proper functions from the master formula is the following. From either string or world-line theories, we can derive the $N$-point master formula, which takes, for example, the following form in the one-loop case

$$
\Gamma_1^{\text{1-loop}} = (-g)^N \int_0^\infty dT (4\pi T)^{-D/2} \int_M \prod_{n=1}^{N-1} d\tau_n \exp \left[ \sum_{i<j}^N p_i \cdot p_j G_B(\tau_i, \tau_j) \right] \bigg|_{\tau_N = T} ,
$$

(1.1)
where $G_B$ is the one-loop bosonic Green function

$$G_B(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T},$$

and $M = \{0 \leq \tau_n \leq T; n = 1, \ldots, N - 1\}$. (Here, we have removed the pre-factor $1/2$ on RHS in (1.1) as a matter of convenience for generic arguments). Let us divide the whole region $M$ into $(N - 1)!$ sub-regions according to the orderings of $\tau_n, n = 1, \ldots, N - 1$. One can re-arrange a $N$-point Feynman integral in the same form as (1.1) restricted to a certain sub-region of the master region $M$. In this way the master formula comprises the $N$-point Feynman integral and its topologically independent diagrams at least once. The essential quantity which we are going to discuss in this paper is the ratio (covering multiplicity) between the number of independent Feynman diagrams and the number of corresponding sub-regions. In order to extract each Feynman diagram’s contribution exactly once, we have to determine this ratio. In the one-loop case, this ratio is just 2, and we have only to put the inverse of this number in front of the master formula (1.1).

In the two-loop case, we, in principle, follow the same procedure. However we need more careful treatment than in the one-loop case. We have complications for the following reasons: (i) The manner of dividing a master integration region depends on how to parametrize the 1PI vacuum diagram part. In other words, it depends on the definition of $M$, in which regions each $\tau_n$ runs. In this way, the numbers of sub-regions and their diagram multiplicities also depend on the parametrizations. We have no general idea which choice of parametrization is the most convenient one. (ii) We also have to split $N$, the number of external legs, into a sum of integers like $N = N_1 + N_2 + N_3$ or $N = N' + N_3$ in accordance with the choice of parametrizations. This delivers all Feynman diagrams into a certain number of diagram classes, and each diagram class possesses its own number of Feynman diagrams, sub-regions, and thus its own covering multiplicity. (iii) Since covering multiplicities are different class by class, the pre-factor corrections for master formulas are no longer the simple inverse of the multiplicity. Taking account of these covering multiplicities for each class, we gather them all to make a correct proper $N$-point function expressed in terms of a combination of master formulas. In the world-line formalism, the effective action is a starting point, and in fact the 1PI $N$-point functions are nothing but the Fourier transforms (plane wave expansions) of effective action. This viewpoint is certainly helpful to understand a combinatorial structure of a master formula,
and we will also see that the effective action can be written in the world-line language most conveniently.

This paper is organized as follows. In sect. 2, we explain our notational settings. In sect. 3, we propose a method how to determine the number of topologically independent Feynman diagrams. This is based on the two-loop effective action extracted from an auxiliary field formalism combined with the background field method. Then we analyze two parametrizations case by case in the order of simplicity. The first one is a loop type parametrization which regards \( N' \) legs are on a circle (fundamental loop) and \( N_3 \) legs are on the remaining internal line (which we simply call middle line). In sect. 4, we confine to the \( N_3 = 0 \) cases. These cases are useful to determine the precise normalization of the photon scattering master formula in QED. In sect. 5, we discuss the symmetric parametrization which regards \( N_a, a = 1, 2, 3 \) legs are on the three internal lines. In sect. 6, we also discuss a few sample cases for \( N_3 \neq 0 \) in the loop type parametrization. We shall not give a general prescription in these cases. Some technical details are available in the appendices: In Appendix A we explain how the combinatorics of the symmetric master formula emerges from the effective action in \( \phi^3 \)-theory, and in Appendix B we remark on the translational invariance along a super world-line fundamental loop, on which the invariance is necessary to connect field theory limits of string amplitudes with world-line formulations.

2 Notations for Feynman amplitudes and world-line formulae

Let us classify all two-loop (proper) Feynman diagrams in accord with \( N_a, a = 1, 2, 3 \), the numbers of external legs which are inserted in each of the three internal lines labeled by \( T_a, a = 1, 2, 3 \). Ignoring the ordering of external legs, we define a representative symbol diagram \( F_i \) \( (N_1 \geq N_2 \geq N_3) \), where the label \( i \) (which we refer to as class) may be chosen as \( (N_1, N_2, N_3) \). This diagram \( F_i \) is actually an element of its diagram set \( T_i \) which consists of all topologically inequivalent diagrams \( d_n, n = 1, \ldots, n_i, \) obtained by shuffling all leg orderings of \( F_i \). The \( N \)-point Feynman amplitudes \( \Gamma_N \) are obtained by the following three steps: (i) For a diagram \( d_n \) of a class \( i \), write down the Feynman amplitude \( \Gamma[d_n] \) including its symmetry factor \( S_i \), which
is common for all $d_n$ in the set $\mathcal{T}_i$. (ii) Sum up all $\Gamma[d_n]$ over the set $\mathcal{T}_i$, i.e.

$$\tilde{\Gamma}_F = \sum_{d_n \in \mathcal{T}_i} \Gamma[d_n]. \quad (2.1)$$

Obviously, we simply have $\Gamma_F \equiv \tilde{\Gamma}_F = \Gamma[d_1]$, if a class $i$ is composed of only one diagram, $n_i^T = 1$. (iii) Finally sum up the $\tilde{\Gamma}_F$ of all classes to obtain

$$\Gamma_N = \sum_i \tilde{\Gamma}_F. \quad (2.2)$$

If we exclude from $\mathcal{T}_i$ the diagrams where external legs are inserted in the middle line propagator, in our notation $\Gamma$ will be denoted by $\tilde{\Gamma}$

$$\tilde{\Gamma}_F, \quad \tilde{\Gamma}_N, \quad \text{etc.} \quad (2.3)$$

The symmetry factors of the theory are generally given by (see [19])

$$S = \left[ p_1 (2!)^2 p_2 (3!)^3 \right]^{-1}, \quad (2.4)$$

where $p_n$, $n = 2, 3$, are the numbers of vertex pairs connected directly by $n$ lines, and $p_1$ is the number of vertex permutations which leave the diagram unchanged with external legs held fixed. (For a tadpole part, take $p_2 = 1$).

In the world-line formalism, we have two ways of parametrizing the 1PI vacuum diagram [1]. The first one is the symmetric parametrization, in which we deal with each $T_i$ on equal ground and $N_a$ external legs are inserted in each line $T_a$. The other is the loop-type parametrization, where two lines of $T_a$ are combined to form a fundamental loop parameter $T \ (= T_1 + T_2)$ and $N' = N_1 + N_2$ legs are inserted in the fundamental loop. (Obviously we do not restrict to $N_1 \leq N_2 \leq N_3$ at this stage.) Thus we have the following two representations of a master formula for the two-loop $N$-point amplitudes (see [11] for notational details). The loop type master formula is given by [8, 10, 11]

$$\Gamma^{(N', N_3)}_{M} = \Gamma^{(N', N_3)}_{M}(p_1, \cdots, p_{N'}, p_1^{(3)}, \cdots, p_{N_3}^{(3)})$$

$$= \left(-g\right)^{N+2} \left(\frac{4\pi}{D}\right)^D \int_0^\infty dt d\bar{T} e^{-m^2(T+\bar{T})} \int_{M} \prod_{n=1}^{N} d\tau_{\alpha} d\tau_{\beta} |T\bar{T} + T G_B(\tau_{\alpha}, \tau_{\beta})|^{-D/2} \cdot \quad (2.5)$$

$$\times \exp\left[ \frac{1}{2} \sum_{j<k}^{N'} p_j p_k C^{(1)}_{11}(\tau_j, \tau_k) + \frac{1}{2} \sum_{j<k}^{N_3} p_j^{(3)} p_k^{(3)} C^{(1)}_{33}(\tau_j^{(3)}, \tau_k^{(3)}) + \sum_{j}^{N_4} \sum_{k}^{N_4} p_j p_k^{(3)} C^{(1)}_{13}(\tau_j^{(3)}, \tau_k^{(3)}) \right]_{\tau_{\alpha}=0}$$

$$\times \exp\left[ \frac{1}{2} \sum_{j<k}^{N'} p_j p_k C^{(1)}_{11}(\tau_j, \tau_k) + \frac{1}{2} \sum_{j<k}^{N_3} p_j^{(3)} p_k^{(3)} C^{(1)}_{33}(\tau_j^{(3)}, \tau_k^{(3)}) + \sum_{j}^{N_4} \sum_{k}^{N_4} p_j p_k^{(3)} C^{(1)}_{13}(\tau_j^{(3)}, \tau_k^{(3)}) \right]_{\tau_{\alpha}=0}$$
where $\bar{T} \equiv T_3$. The normalization $(4\pi)^{-D}$ also follows from string theory \[3\]. Introducing obvious cyclic notations such as $p^{(4)} = p^{(1)}$ etc., the symmetric master formula is given by \[3, 10\]

$$
\Gamma^{(N_1, N_2, N_3)} = \Gamma^{(N_1, N_2, N_3)}(p_1^{(1)}, \ldots, p_{N_1}^{(1)}, \ldots, p_{N_3}^{(2)}, \ldots, p_{N_1}^{(2)}, \ldots, p_{N_3}^{(3)}, \ldots, p_{N_3}^{(3)}) = \frac{(-g)^{N_1 + 2}}{(4\pi)^D} \cdot \prod_{a=1}^{3} \int_0^\infty \sum_{j,k} p_j^{(a)} p_k^{(a)} G_{a}^{\text{sym}}(\tau_j^{(a)}, \tau_k^{(a)}) + \sum_{a=1}^{N_a + 1} \sum_{j,k} p_j^{(a)} p_k^{(a+1)} G_{a}^{\text{sym}}(\tau_j^{(a)}, \tau_k^{(a+1)})] .
$$

In both formulae, the subscript $M$ stands for the full integration regions of all $\tau$-parameters

$$
M = \begin{cases} 
0 \leq \tau_n \leq T, 0 \leq \tau_m^{(3)} \leq T_3 & | n = 1, \ldots, N', \beta ; m = 1, \ldots, N_3 \} \quad \text{for} \quad (2.5) \\
0 \leq \tau_n^{(a)} \leq T^{(a)} & | n_a = 1, \ldots, N_a ; a = 1, 2, 3 \} \quad \text{for} \quad (2.6) .
\end{cases}
$$

Since the splitting of the integrations over $M$ depends on the choice of parametrization, these two master formulae possess different diagram contents, to be more precise, different covering multiplicities for Feynman diagrams. It means that \[2.3\] and \[2.6\] do not by themselves coincide with each other. However there is a simple relation between them focusing on a divided (ordered $\tau$-) integration region in each formula \[10\]

$$
\Gamma[d_n] = S \Gamma^{(N', N_3)}_{D_k} = S \Gamma^{(N_1, N_2, N_3)}_{D_k'} ,
$$

where $D_k$ and $D_k'$ express regions obtained by splitting the respective master regions $M$.

To obtain the $N$-point function $\Gamma_N$, we certainly have to use the master formula to sum up over all possible sets of $(N_1, N_2, N_3)$ or of $(N', N_3)$ with some constant weights in each case. The main question is which factors should appear in front of these master formulae in order to really obtain $\Gamma_N$ or $\mathcal{F}_N$.

### 3 Background field plus auxiliary field method

In order to obtain the covering multiplicities, it is necessary to know $n_i^T$ the number of topologies for each diagram class $F_i$. In this section \[3\], we present a method to determine the value of $n_i^T$. In principle, this information is contained in the generating functional. Let us consider the generating functional for the Euclidean Lagrangian

$$
\mathcal{L} = \frac{1}{2} (\partial \phi \partial \phi + m^2 \phi^2) + \frac{g}{3!} \phi^3 ,
$$

\[3\]We appreciate a contribution of M. Reuter to this section \[20\].
in the background field method decomposing $\phi = \varphi + \bar{\phi}$, where $\varphi$ is a quantum field while $\bar{\phi}$ is a classical field. This produces the generating functional

$$Z[\bar{\phi}] = Z_0 \int \mathcal{D}\varphi \exp \left[ - \int \left\{ \frac{1}{2} \varphi(\Delta^{-1} + g\bar{\phi})\varphi + \frac{g}{3!} \varphi^3 \right\} dx \right] ,$$

(3.2)

where $\Delta^{-1}$ is the free inverse propagator $\Delta^{-1} = -\partial^2 + m^2$, and $Z_0$ consists of classical (tree) terms

$$Z_0 = \exp \left[ \int dx \left\{ -\frac{1}{2} \bar{\phi}\Delta^{-1}\bar{\phi} - \frac{g}{3!} \bar{\phi}^3 \right\} \right] .$$

(3.3)

In order to perform the $\varphi$ integration, we further introduce the auxiliary field $B$ which represents $\varphi^2$ by insertion of a delta function

$$\delta(B - \varphi^2) = \int \mathcal{D}\alpha \exp \left[ i \int \alpha(B - \varphi^2) dx \right] .$$

(3.4)

The quantum part of the Lagrangian in (3.2) then reads

$$\mathcal{L}^{new} = \frac{1}{2} \varphi(\Delta^{-1} + i2\alpha)\varphi + \lambda B\varphi + 3\lambda\bar{\phi}B - iB\alpha ,$$

(3.5)

where

$$\lambda = \frac{1}{3!} g .$$

(3.6)

The new expression for $Z[\bar{\phi}]$ is

$$Z[\bar{\phi}] = Z_0 \int \mathcal{D}\alpha \mathcal{D}B \text{Det}^{-1/2}(\Delta^{-1} + i2\alpha) \exp \left[ \frac{\lambda^2}{2} B(\Delta^{-1} + i2\alpha)^{-1} B - 3\lambda\bar{\phi}B + iB\alpha \right] ,$$

(3.7)

where an integration over space-time is understood in the exponent. Applying the following formula for a function of $\alpha$

$$\int \mathcal{D}\alpha \mathcal{D}B f(i\alpha)e^{iB\alpha} = \int \mathcal{D}\alpha \mathcal{D}B f \left( \frac{\partial}{\partial B} \right) e^{iB\alpha} = \int \mathcal{D}B f \left( \frac{\partial}{\partial B} \right) \delta(B) ,$$

(3.8)

we rewrite

$$Z[\bar{\phi}] = Z_0 \cdot \exp \left[ -\frac{1}{2} \text{Tr} \ln(\Delta^{-1} - 2\partial_B) \right] \exp \left[ \frac{\lambda^2}{2} B(\Delta^{-1} - 2\partial_B)^{-1} B - 3\lambda\bar{\phi}B \right] \bigg|_{B=0} .$$

(3.9)

Note that the sign of $\partial_B$ is reversed because of a partial integration for the delta function $\delta(B)$. 

6
Here we put a remark on an alternative calculation. We can also apply the formula similar to (3.8) by exchanging $\alpha$ and $B$. In this case the last term in the exponent of (3.9) becomes $-i3\lambda\bar{\phi}\partial_\alpha$, which provides a simultaneous translation of all $\alpha$'s, and then

$$Z[\bar{\phi}] = Z_0 \cdot \exp\left[ -\frac{1}{2} \text{Tr} \ln(\Delta^{-1} + g\bar{\phi} + 2i\alpha) \right] \exp\left[ -\frac{\lambda^2}{2} \partial_\alpha(\Delta^{-1} + g\bar{\phi} + 2i\alpha)^{-1} \partial_\alpha \right] \bigg|_{\alpha = 0} .$$  

(3.10)

As to two-loop contributions, we may pick up second order terms in $\partial_\alpha$, which act on the $\alpha$ field in the Tr ln loop term as well. Applying a path integral representation for (open/closed) propagator (q.v. (A.4)), we have

$$Z^{2-loop} = Z_0 \lambda^2 (2I_1 + I_2)$$  

(3.11)

with

$$I_1 = I_1[0 \leq \tau_1, \tau_2 \leq S] = \int_0^\infty dS \int_0^S d\tau_1 \int_0^\infty d\tau_2 \int_{y(0)=y(\tau_1)}^{y(S)=y(\tau_2)} \mathcal{D}y e^{-A(y,S)}$$

(3.12)

$$I_2 = \int_0^\infty \frac{dT}{T} \int \mathcal{D}x e^{-A(x,T)} \int_0^T d\tau_1 \int_0^T d\tau_2 \int_0^\infty d\bar{T} \int_{y(0)=x(\tau_1)}^{y(\bar{T})=x(\tau_2)} \mathcal{D}y e^{-A(y,\bar{T})}$$

(3.13)

where

$$A(x,T) = \int_0^T \left( \frac{1}{4} x^2(\tau) + g\bar{\phi}(x) \right) d\tau .$$

(3.14)

Note that $I_2$ and $I_1$ correspond to (a) and (b) in Fig. 2 respectively. An interesting observation is that $I_1$ can be reduced to $I_2$ if we confine to the 1PI parts. Restricting to the region $0 \leq \tau_2 \leq \tau_1$ (outer region corresponds to 1PR parts) and changing variables in integrations $\tau_1 = T$, $S = T + \bar{T}$, we find out

$$I_1^{PI} = I_1[0 \leq \tau_2 \leq \tau_1 \leq S] = \int_0^\infty d\bar{T} \int_0^\infty dT \int_0^T d\tau_2 \int \mathcal{D}x \int_{y(0)=x(T)}^{y(\bar{T})=x(\tau_2)} \mathcal{D}y e^{-A(x,T) - A(y,\bar{T})} .$$

(3.15)

Fixing $\tau_1 = T$ in $I_2$ (q.v. translational invariance in (2.5)), we see $I_1^{PI} = I_2$ and therefore

$$\Gamma^{2-loop} = \frac{g^2}{2 \cdot 3!} I_2 = \frac{g^2}{2 \cdot 3!} I_1^{PI} .$$

(3.16)

If one further changes integration variables as $T_1 = \tau_2$, $T_2 = \tau_1 - \tau_2$, $T_3 = S - \tau_1$ ($S = T_1 + T_2 + T_3$), one can reproduce the symmetric three-propagator expression shown in Appendix A.

Now let us go back to the main purpose mentioned in the heading of the section. Eq. (3.10) can be expanded in a perturbative expansion form

$$Z[\phi] = \sum_{n=0}^\infty (-g)^n z_n[\phi] ,$$

(3.17)
and the function $z_n[\bar{\phi}]$ possesses the following structure: Let $F_i$ be a representative symbol diagram of order $g^n$ without having any tree vertex, and $F_i[\bar{\phi}]$ be the coordinate space representation of $F_i$ in terms of $\Delta$ and $\Delta\bar{\phi}$. Then $z_n[\bar{\phi}]$ is given by a sum of $F_i[\bar{\phi}]$ with certain weights $w_i$;

$$z_n[\bar{\phi}] = \sum_i w_i F_i[\bar{\phi}] .$$

(3.18)

To obtain these weights, one may just calculate $Z[\bar{\phi}]$ term by term at each order of $g$. For example $z_1$ and $z_2$ are given by

$$z_1[\bar{\phi}] = \frac{1}{2} \int d^D x_1 \Delta_{11}(\Delta\bar{\phi})_{11} ,$$

(3.19)

$$z_2[\bar{\phi}] = \frac{1}{2(3!)^2} \int d^D x_1 d^D x_2 \left[ 6\Delta_{12}^3 + 9\Delta_{11}\Delta_{12}\Delta_{22} + 9\Delta_{11}\Delta_{22}(\Delta\bar{\phi})_{11}(\Delta\bar{\phi})_{22} + 18\Delta_{12}^2(\Delta\bar{\phi})_{11}(\Delta\bar{\phi})_{22} \right] ,$$

(3.20)

where

$$\Delta_{ij} = \Delta(x_i, x_j) = (-\partial^2 + m^2)^{-1} = \int \frac{d^D k}{(2\pi)^D} \frac{e^{-ik(x_i-x_j)}}{m^2 + k^2} ,$$

(3.21)

$$(\Delta\bar{\phi})_{ij} = \int d^D y_j \Delta(x_i, y_j)\bar{\phi}(y_j) .$$

(3.22)

The general rules how to determine the coefficients $w_i$ are the following: (i) Consider the symbol diagram $F_i$ possessing $N$ external legs, and compose the same diagram from sewing the diagram parts depicted in Fig. 1. To sew the parts, one must insert some $\partial_B$’s on a $<BB>$ line or on a loop as many as $n$ the order of $g$ (Do not insert on a $<B\bar{\phi}>$ line). Then join the $\partial_B$-crosses and the $B$-dots. (ii) Assign the following numerical factors in each sewn diagram

\[
\begin{align*}
2 & \quad \text{for } \partial_B \\
\left(\frac{1}{2} \lambda^2\right)^n / n! & \quad \text{for } n \text{ propagators } <BB>^n \\
(3\lambda)^n / n! & \quad \text{for } n \text{ external legs } <B\bar{\phi}>^n \\
(2^L L! \prod_{i=1}^L n_i)^{-1} & \quad \text{for } L \text{ loops from Trln part}
\end{align*}
\]

(3.23)

where $n_i$ is the number of vertices on the $i$-th loop. (iii) Finally summing up the factors of all possible sewing diagrams for the $F_i$, we obtain the coefficient $w_i$ for $F_i$ (setting the coupling $\lambda \to 1/3!$). An example, in the case of $F_i[\bar{\phi}] = \Delta_{12}^3$, is shown in Fig. 2, where 3 sewing possibilities exist. The two possibilities are represented by the diagram (a) which also includes
an upside-down attachment of the $<BB>$-line to the loop, and the remaining one possibility is the diagram (b).

\[ \phi \quad B \]
\[ < B\phi > \quad < BB > \]

**Figure 1:** The parts for sewing procedure.

\[ \times \]
\[ \partial_B \quad < B\bar{\phi} > \quad < BB > \]

**Figure 2:** The example of sewing diagrams for $\Delta_{12}^3$.

Now the coefficient $w_i$ can easily be obtained by following the above rules, and we remind ourselves of the important fact that $w_i$ contains the information on the number of topologies of the class in question. Roughly speaking, $w_i$ should be the number of topologically independent diagrams of a class $i$, multiplied by its symmetry factor. However this is over-counting by $N!$ because external legs are not fixed in (3.18) yet. For example, in the case of the 4-point one-loop diagram, $w = 1/8$, $S = 1$, and $n^T = 3$ corresponding to $s$-, $t$-, $u$-channels. Therefore the correct
relation is

\[ w_i = \frac{S_i n_T^T}{N!}. \]  

(3.24)

As can easily be seen from the above rules, the \( w_i \)'s for one-loop \( N \)-point diagrams are universal, namely they lead to

\[ w_{N}^{1\text{-loop}} = \frac{1}{2N}, \]  

(3.25)

and the number of topologies can be read from (3.24). The two loop cases are more complicated, and we discuss them case by case in the following sections.

4 The covering multiplicity for \( N_3 = 0 \) case

In this section, we restrict ourselves to the loop-type parametrization with no leg insertions in the middle line \( T_3 \). The purpose of this section is to give a prescription how to determine the covering multiplicity which is the number of times that a world-line master integration \( \int_M \prod d\tau \) covers all of Feynman diagrams \( d_n \) within a fixed class \( i \).

Let us begin with the one-loop cases for transparency of discussions. Obviously, a symbol diagram \( F_i \) for \( N \)-point diagrams is unique for each \( N \), and the diagram class may rather be labeled by \( N \), instead of the original definition of \( i \). In the master formula (1.1), the \( \tau \)-integration \( \int_M \) can be divided into \((N-1)!\) integration regions \( D_k \), \( k = 1, \ldots, (N-1)! \),

\[ \Gamma_{M}^{1\text{-loop}} = \sum_{k=1}^{(N-1)!} \Gamma_{D_k}^{1\text{-loop}}. \]  

(4.1)

Remember the Feynman integral for a certain graph \( d_{n'} \) can be organized in the following form for some values \( k' \) of \( k \)

\[ \Gamma[d_{n'}] = S \Gamma_{D_k}^{1\text{-loop}}. \]  

(4.2)

(The number of such \( k' \)-values is equal to the covering multiplicity.) Since we must pick up relevant \( D_k \)'s only one time for each topologically different Feynman diagram \( d_n \), \( n = 1, \ldots, n^T \), the covering multiplicity in this case is simply given by the ratio between the number of the domains and the number \( n^T \)

\[ C_N = \frac{(N-1)!}{n^T}. \]  

(4.3)
Taking account of this covering multiplicity, the naive sum over \( k = 1, \ldots, (N - 1)! \) can be reduced to the sum only over a topologically independent \( k \)'s subset identical to \( T_N \)

\[
\Gamma_M^{1-loop} = C_N \sum_{k \in T_N} \Gamma_D^{1-loop} = C_N S_N^{-1} \sum_{n' \in T_N} \Gamma[d_{n'}] = C_N S_N^{-1} \bar{\Gamma}_F ,
\]

where we have used the relation (4.2) at the second equality. From (3.24) and (3.25), we have

\[
n^T = \frac{(N - 1)!}{2S_N} ,
\]

and inserting this into (4.3)

\[
C_N S_N^{-1} = 2 ,
\]

we conclude

\[
\Gamma_N = \bar{\Gamma}_F = \frac{1}{2} \Gamma_M^{1-loop} .
\]

Remember that this factor actually coincides with the one we ignored in (1.1).

In the two-loop cases, we have to remember that another \( \tau \)-integration, \( \int d\tau_3 \), exists on the fundamental loop, when dividing integration regions. Thus the total number of integration domains is \( N_M \equiv (N + 1)! \), and we have

\[
\Gamma_M^{(N,0)} = \sum_{k=1}^{N_M} \Gamma_D^{(N,0)} , \quad \text{where} \quad N = N_1 + N_2 .
\]

Depending on the position of \( \tau_3 \), several different diagram classes may appear, and we must count the covering multiplicities \( C_i \) for the respective diagram classes. To this end, we have to classify the \( N_M \) regions into \( N_{D_i} \) regions for each diagram class \( i \).

Suppose \( \tau_3 \) is in such a position that \( N_1 \) legs are positioned on the left hand side (positive \( \tau \)-direction) of \( \tau_3 \), and \( N_2 \) legs are on the right hand side. Then the class label \( i \) can be chosen as a set of two integers like \( F(N_1, N_2) \), \( N_1 \geq N_2 \), and the number of diagram classes is given by

\[
N_c = \left\lfloor \frac{N}{2} \right\rfloor + 1 .
\]

The number \( N_{D_i} \) of integration regions for a class \( i = (N_1, N_2) \) is equal to

\[
N_{D_i} = N! N_p^{(N_1, N_2)} ,
\]

where \( N_p^{(N_1, N_2)} \) is the number to deliver the two objects \( N_1 \) and \( N_2 \) in the two 'post boxes' (\( \ast | \ast \)); i.e.

\[
N_p^{(N_1, N_2)} = \begin{cases} 
2 & \text{for } N_1 \neq N_2 \\
1 & \text{for } N_1 = N_2 .
\end{cases}
\]
Eq. (4.10) may also be understood as the number of possible constructions of the symbol diagram $F(N_1, N_2)$, namely $N_{D_i} = P_{N_1} P_{N_2} N_p^{(N_1, N_2)}$ where

$$P_{N_1} = N C_{N_1} \times N_1!, \quad P_{N_2} = N - N_1 C_{N_2} \times N_2!, \quad N C_r \equiv \frac{N!}{r! (N - r)!}$$ (4.12)

$P_{N_1}$ ($P_{N_2}$) being the number of possibilities to have $N_1$ ($N_2$) $\tau$-parameters on the left (right) side of $\tau_\beta$ (with over-counting leg orderings). For completeness of arguments, we remark that the total sum of $N_{D_i}$ is equal to $N_M$

$$\sum_{i=1}^{N_c} N_{D_i} = N! \sum_{i=1}^{N_c} N_p^{(N_1, N_2)} = (N + 1)!,$$ (4.13)

where we have used the fact that the number of the classes with $N_1 > N_2$ is given by $\left[\frac{N}{2}\right]/\left[\frac{N+1}{2}\right]$ and the number of the class with $N_1 = N_2$ is 1/0 for $N$ even/odd respectively. After all, the covering multiplicity $C_i$ for the set $T_i$ with $n_i^T$ diagrams is given by the ratio between $N_{D_i}$ and $n_i^T$, thus

$$C_i = \frac{N! N_p^{(N_1, N_2)}}{n_i^T}.$$ (4.14)

Now, taking account of the multiplicity $C_i$, we rewrite (4.8) in the same way as (4.4) for each class

$$\Gamma_{(N,0)}^{(N_c)} = \sum_{i=1}^{N_c} C_i \sum_{k \in T_i^k} \Gamma_{D_k}^{(N,0)} = \sum_{i=1}^{N_c} C_i S_i^{-1} \sum_{n \in T_i} \Gamma_{[d_n]} = \sum_{i=1}^{N_c} C_i S_i^{-1} \tilde{\Gamma}_{F_i},$$ (4.15)

and an important question here is whether the values of $C_i S_i^{-1}$, $i = 1, \ldots, N_c$, are all the same or not. If they are all the same, the unique value gives nothing but a normalization constant for $\Gamma_{(N,0)}^{(N_c)}$ just like seen in (4.7). On the other hand, if some of $C_i S_i^{-1}$ take a different value, we have to add a correction to make the summation weights over $i$ equal. Eliminating $n_i^T$ from the $C_i$ (4.14) by using (3.24), we derive

$$C_i S_i^{-1} = \frac{N_p^{(N_1, N_2)}}{w_i},$$ (4.16)

and we have verified (so far $N \leq 5$) that its explicit values are

$$C_i S_i^{-1} = \begin{cases} 8 & \text{for } (N_1, N_2) = (N, 0) \\ 4 & \text{otherwise} \end{cases}$$ (4.17)

Here, a few remarks are in order: (i) On the ground that graphical symmetries are getting less as $N$ is increasing, we believe that the data obtained above should be valid for larger $N$-values.
as well. (ii) The main reason why the $CS^{-1}$ value for the $F(N,0)$ class is twice that for the rest is the fact that the symmetry factor of $F(N,0)$ is nothing but $1/2$. (The $N = 2$ case is the only exception, where both symmetry factors are $1/2$, but $C$’s are different). (iii) If the fundamental loop has an orientation like a fermion loop in QED, the symmetry factor for $F(N,0)$ actually turns to be $1$, and hence $CS^{-1} = C = 2$ for all classes, where we have divided $C$ by $2$ because of distinguishing different directions of the orientated loop. In other words, $n_j^T$ of the denominator in (4.14) is twice as large as the one for the $\phi^3$ theory. (The $N = 2$ case is again exceptional.)

Now applying the results (4.17) to the formula (4.15), we find

$$\Gamma_{M}^{(N,0)} = 4 \sum_{i=1}^{N_c} \hat{\Gamma}_{F_i} + 4 \hat{\Gamma}_{F(N,0)}.$$  \hspace{1cm} (4.18)

The quantity appearing in the first term of the above equation is nothing but the $N$-point function in question

$$\Psi_{N} = \sum_{i=1}^{N_c} \hat{\Gamma}_{F_i}, \hspace{1cm} (4.19)$$

and the second term is given by the following master formula relation, which will be proven in the next section

$$\hat{\Gamma}_{F(N,0)} = \hat{\Gamma}_{F(N,0,0)} = \frac{1}{4} \Gamma_{M}^{(N,0,0)}.$$  \hspace{1cm} (4.20)

Therefore we arrive at

$$\Psi_{N} = \frac{1}{4} (\Gamma_{M}^{(N,0)} - \Gamma_{M}^{(N,0,0)}).$$  \hspace{1cm} (4.21)

In view of the remark (iii) below (4.17), the $N$-point amplitude of (two-loop) QED photon scatterings is simply given by

$$\Gamma_{N} = \frac{1}{2} \Gamma_{M}^{QED},$$  \hspace{1cm} (4.22)

with

$$\Gamma_{M}^{QED} = -4 \cdot \frac{e^{N+2}}{(4\pi)^D} \int_{0}^{\infty} \frac{dT}{T} \int_{0}^{\infty} \frac{d\tilde{T}}{\tilde{T}} \int \prod_{n=1}^{N} d\tilde{\tau}_{\alpha} d\tilde{\tau}_{\beta} [T \tilde{T} + T \tilde{G}(\tilde{\tau}_{\alpha}, \tilde{\tau}_{\beta})]^{-D/2}$$

$$\times < D \hat{X}(\tilde{\tau}_{\alpha}) \cdot D \hat{X}(\tilde{\tau}_{\beta}) \prod_{n=1}^{N} D \hat{X}(\tilde{\tau}_{n}) \cdot \epsilon_n \exp[ip_n \cdot \hat{X}(\tilde{\tau}_{n})]> \hspace{1cm} (4.23)$$

where $D = \partial_{\theta} - \theta \partial_{\tau}$, and $\hat{X}_{\mu}(\tilde{\tau}_{n})$, $n = 1, \cdots, N, \alpha, \beta$ have to be contracted with the Green function $\hat{G}_{11}^{(1)}(\tilde{\tau}_{j}, \tilde{\tau}_{k})$ obtained from the $G_{11}^{(1)}(\tau_{j}, \tau_{k})$ in (2.3) by substitution of $G_{B}$ with a super-Green function $14$ $\hat{G}(\tilde{\tau}_{1}, \tilde{\tau}_{2}) = G_{B}(\tau_{1}, \tau_{2}) + \theta_{1} \theta_{2} G_{F}(\tau_{1}, \tau_{2})$, where $G_{F}(\tau_{1}, \tau_{2}) = \text{sign}(\tau_{1} - \tau_{2})$. 

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The factor $-\frac{1}{4}$ appearing in $\Gamma_{QED}^M$ corresponds to the fermion degrees of freedom. For scalar QED, just reduce this number to 2, also reduce super variables to bosonic ones: $D\hat{X} \rightarrow \dot{x}$, $\hat{\tau} \rightarrow \tau$ [14, 15].

As shown in Appendix B, one may fix one of the $\tau$-integrations by using the translational invariance to perform the $\tau$-integration trivially. If one wants to fix $\tau_\alpha = 0$, then we replace

$$\int_0^T d\tau_\alpha \rightarrow T \int d\theta_\alpha .$$

(4.24)

5 Combinatorics in the symmetric master formula

In the symmetric parametrization, the following proper $N$-point function can be derived from $Z[\bar{\phi}]$ (see Appendix A)

$$\Gamma_N = \frac{1}{2 \cdot 3!} \sum_{N_1,N_2,N_3=0}^{N} \sum_{\sigma(N_1,N_2,N_3)} \Gamma_{M}^{(N_1,N_2,N_3)}, \quad (N = N_1 + N_2 + N_3)$$

(5.1)

where $\sigma$ stands for possible ways of inserting $N_a$ particles (momenta) in the internal three lines ignoring leg orderings on each line. To perform these summations efficiently, we restrict $N_1 \geq N_2 \geq N_3$ and take account of the multiplicity $N_{p}^{(N_1,N_2,N_3)}$;

$$\sum_{N_1,N_2,N_3}^{N} \rightarrow \sum_{N_1\geq N_2\geq N_3} N_{p}^{(N_1,N_2,N_3)}$$

(5.2)

where

$$N_{p}^{(N_1,N_2,N_3)} = \begin{cases} 1 & \text{if all of } N_a \text{ are the same} \\ 3 & \text{if two of } N_a \text{ are the same} \\ 6 & \text{if none of } N_a \text{ is the same} \end{cases}$$

(5.3)

However this is not the end of the story. There comes another kind of multiplicity from the summation in $\sigma$. The number of $\sigma$’s is given by

$$N_\sigma = NC_{N_1} \cdot N_{-N_1}C_{N_2} \cdot N_{-N_1-N_2}C_{N_3} = \frac{N!}{N_1!N_2!N_3!},$$

(5.4)

and we show some examples of $\sigma(N_1,N_2,N_3)$ with $N_1 \geq N_2 \geq N_3$,

$$\sigma(1,0,0) = (p_1|0|0), \quad \sigma(2,0,0) = (p_1,p_2|0|0), \quad \sigma(3,0,0) = (p_1,p_2,p_3|0|0)$$

(5.5)

$$\sigma(1,1,0) = (p_1|p_2|0), \quad (p_2|p_1|0)$$

(5.6)

$$\sigma(2,1,0) = (p_1,p_2|p_3|0), \quad (p_1,p_3|p_2|0), \quad (p_2,p_3|p_1|0)$$

(5.7)

$$\sigma(1,1,1) = (p_1|p_2|p_3), \quad (p_1|p_3|p_2), \quad (p_2|p_1|p_3), \quad (p_2|p_3|p_1), \ldots \quad \text{etc.},$$

(5.8)
where a momentum $p_k$ delivered in the $a$-th place of $(*|*|*)$ is $p_k^{(a)}$ appearing in the master formula (2.6). Because of (5.2), we find out the multiplicities 2 for $\sigma(1,1,0)$, 6 for $\sigma(1,1,1)$ and 1 for the remainders in these examples (5.7),(5.8). A more general argument for these multiplicities $C_{N_1N_2N_3}$ leads us to the following relations

\[
\begin{align*}
N_p^{(N_1N_2N_3)}C_{N_1N_2N_3} &= 6 \quad \text{for } N_3 \neq 0, \\
N_p^{(N_1N_2)}C_{N_1N_2N_3} &= 2 \quad \text{for } N_3 = 0.
\end{align*}
\]

Noticing the relation

\[
\sum N_p^{(N_1N_2)} = C_iS_i^{-1} \quad \text{for } N_3 = 0, \quad i = (N_1, N_2)
\]

we have

\[
N_p^{(N_1N_2N_3)}C_{N_1N_2N_3} = 6(1 - \delta_{N_3,0}) + \frac{24}{C_iS_i^{-1}}\delta_{N_3,0}.
\]

Then (5.11) turns out to be

\[
\Gamma_N = \frac{1}{2 \cdot 3!} \sum_{N_1 \geq N_2 \geq N_3} N_p^{(N_1N_2N_3)}C_{N_1N_2N_3} \bar{\Gamma}_M^{(N_1N_2N_3)},
\]

where

\[
\bar{\Gamma}_M^{(N_1N_2N_3)} = \sum_{\sigma}^\prime \bar{\Gamma}_M^{(N_1N_2N_3)},
\]

and $\sum'$ means that redundant elements of $\sigma$ have been subtracted according to the multiplicities $C_{N_1N_2N_3}$. First few examples are

\[
\begin{align*}
\bar{\Gamma}_M^{(1,1,0)} &= \Gamma_M^{(1,1,0)}(p_1|p_2|0), \\
\bar{\Gamma}_M^{(1,1,1)} &= \Gamma_M^{(1,1,1)}(p_1|p_2|p_3), \\
\bar{\Gamma}_M^{(2,1,1)} &= \sum_{i<j} \Gamma_M^{(2,1,1)}(p_1,\ldots,\tilde{p}_i,\ldots,\tilde{p}_j,\ldots,p_4|p_3|p_j),
\end{align*}
\]

where $\tilde{p}_j$ means an exclusion of $p_j$.

Now, let us prove (4.20). Recalling that the $\Gamma_N$ should be a sum over all classes labeled by $(N_1, N_2, N_3)$, $N_1 \geq N_2 \geq N_3$,

\[
\Gamma_N = \sum_i \bar{F}_i = \sum_{N_1 \geq N_2 \geq N_3} \bar{F}_{(N_1N_2N_3)},
\]
and comparing this with (5.13), we find
\[ \tilde{\Gamma}_{F(N_1,N_2,N_3)} = \frac{1}{2 \cdot 3!} \cdot N_p^{(N_1N_2N_3)} C_{N_1N_2N_3} \cdot \tilde{\Gamma}_{M}^{(N_1,N_2,N_3)}. \] (5.19)

In the case of \((N,0,0)\), noticing \(\tilde{\Gamma}_{M}^{(N,0,0)} = \Gamma_{M}^{(N,0,0)}\), we obtain (4.20) as
\[ \tilde{\Gamma}_{F(N,0,0)} = \frac{1}{2 \cdot 3!} \cdot \frac{24}{8} \cdot \Gamma_{M}^{(N,0,0)} = \frac{1}{4} \Gamma_{M}^{(N,0,0)}. \] (5.20)

Here we also derive several interesting formulae. Since we already know the result (4.21)
which contains every class such as \((N_1,N_2,0)\), we can partly perform the summation over \(N_1\) and \(N_2\) in (5.13). Using (4.17) and (5.12), we have
\[ \Gamma_N = \Phi_N + \frac{1}{2} \sum_{N_1 \geq N_2 \geq N_3 > 0} \tilde{\Gamma}_{M}^{(N_1,N_2,N_3)}, \] (5.21)
where we see that \(\Phi_N\) is given by
\[ \Phi_N = \sum_{N_1 \geq N_2} \frac{2}{C S^{-1}} \Gamma_{M}^{(N_1,N_2,0)}. \] (5.22)

Comparing this \(\Phi_N\) with (4.21), we can derive a decomposition formula of \(\Gamma_{M}^{(N,0)}\) in terms of \(\tilde{\Gamma}^{(N_1,N_2,0)}\)
\[ \frac{1}{2} \Gamma_{M}^{(N,0)} = \sum_{N_1 \geq N_2} \tilde{\Gamma}^{(N_1,N_2,0)}. \] (5.23)

As a closing remark, we suggest a simple relation between \(w_i\) and \(N_p^{(N_1N_2N_3)}\). Eqs. (4.16) and (5.11) are defined only for \(N_3 = 0\); however eliminating \(N_p^{(N_1N_2)}\) from (4.16) with \(N_3\) being kept finite formally, we find
\[ w_i = \frac{N_p^{(N_1N_2N_3)}}{2 \cdot 3!}. \] (5.24)
We verified, up to \(N = 6\), that this result is correct also for non zero \(N_3\) values.

6 Non zero \(N_3\) case in the loop type formulae

We, in principle, expect a generic method to evaluate combinatorics like discussed in sect.4 in the non zero \(N_3\) case of the loop type master formula. Here we confine ourselves to a few examples as a first step.

The simplest case is \(F(1,1,1)\), which can be covered by the master formula \(\Gamma_{M}^{(2,1)}\). This case does not give rise to any complication from sect. 4, because of only one insertion to the middle
line propagator. Since $\Gamma^{(2,1)}_M$ covers the two classes $F(2,1,0)$ having $C = 4$, $S = 1$ and $F(1,1,1)$ having $C = 2$, $S = 1/2$, we get

$$\Gamma^{(2,1)}_M = 4\Gamma_{F(2,1,0)} + 4\Gamma_{F(1,1,1)}, \quad (6.1)$$

where we have used $\bar{\Gamma}_{F(1,1,1)} = \Gamma_{F(1,1,1)}$. One may obtain the connection of $\Gamma^{(2,1)}_M$ to the symmetric master formula by applying (5.19) to the above equation

$$\frac{1}{2}\Gamma^{(2,1)}_M = \bar{\Gamma}^{(2,1,0)}_M + \Gamma^{(1,1,1)}_M. \quad (6.2)$$

From (5.23) we also have

$$\frac{1}{2}\Gamma^{(3,0)}_M = \bar{\Gamma}^{(3,0,0)}_M + \Gamma^{(2,1,0)}_M. \quad (6.3)$$

These two formulas give a transformation from (5.13), the sum of symmetric master formulae, to a loop type formula for $\Gamma_3$ Writing down (5.13) for $N = 3$

$$\Gamma_3 = \frac{1}{4}\Gamma^{(3,0,0)}_M + \frac{1}{2}\bar{\Gamma}^{(2,1,0)}_M + \frac{1}{2}\Gamma^{(1,1,1)}_M, \quad (6.4)$$

and taking a linear combination of (6.2) and (6.3), we can rewrite (6.4) in the following form

$$\Gamma_3 = \frac{1}{8}\Gamma^{(3,0)}_M + \frac{1}{8}\Gamma^{(2,1)}_M + \frac{1}{4}\Gamma^{(1,1,1)}_M. \quad (6.5)$$

This formula is an interesting form; notice that it does not contain any summation in momentum permutations. (This is also true for (5.21).)

Now, the second simplest example is $F(2,1,1)$, which contains 6 topologically different Feynman diagrams in the class. This class is expected to be covered by $\Gamma^{(3,1)}_M$, and $N_3$ is again equal to one, thus the situation may be similar to the first case. The number of sub-regions is determined by orderings of $\tau_\beta$ and $\tau_n$ on the fundamental loop, i.e. $(N_1 + N_2 + 1)! = 4!$. However these sub-regions can not cover all 6 diagrams of $T(2,1,1)$. In fact, only 3 diagrams among them are covered with the covering multiplicity 4, and the remaining 12 regions cover 3 diagrams belonging to $T(3,1,0)$ with multiplicity 4. In order to cover all necessary diagrams of $T(2,1,1)$, we just permute the momentum $p_j$ inserted in the middle line and gather them all for $j = 1, 2, 3, 4$. Thus we obtain

$$\bar{\Gamma}^{(3,1)}_M = \sum_{j=1}^{4} \Gamma^{(3,1)}_M(p_1, \ldots, \tilde{p}_j, \ldots, p_4|p_j) = 8\bar{\Gamma}_{F(2,1,1)} + 4\bar{\Gamma}_{F(3,1,0)} \quad (6.6)$$

$$= 2\bar{\Gamma}^{(3,1,0)}_M + 4\bar{\Gamma}^{(2,1,1)}_M, \quad (6.7)$$

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where we have used (5.19) at the second equality. From (5.23) and (5.13), we have
\[ \frac{1}{2} \Gamma^{(4,0)}_M = \Gamma^{(4,0,0)}_M + \bar{\Gamma}^{(3,1,0)}_M + \bar{\Gamma}^{(2,2,0)}_M \] (6.8)
and
\[ \Gamma_4 = \frac{1}{4} \Gamma^{(4,0,0)}_M + \frac{1}{2} \bar{\Gamma}^{(3,1,0)}_M + \frac{1}{2} \Gamma^{(2,2,0)}_M + \frac{1}{2} \bar{\Gamma}^{(2,1,1)}_M. \] (6.9)
We can rewrite this \( \Gamma_4 \) into the following form by taking a linear combination of (6.7) and (6.8)
\[ \Gamma_4 = \frac{1}{8} \Gamma^{(4,0)}_M + \frac{1}{8} \bar{\Gamma}^{(3,1)}_M + \frac{1}{4} \bar{\Gamma}^{(2,2,0)}_M. \] (6.10)
This result contains summations in external momenta’s permutations, because of (6.6), the definition of \( \bar{\Gamma}^{(3,1)}_M \), and
\[ \bar{\Gamma}^{(2,2,0)}_M = \Gamma^{(2,2,0)}_M (p_1,p_2|p_3,p_4|0) + \Gamma^{(2,2,0)}_M (p_2,p_4|p_1,p_3|0) + \Gamma^{(2,2,0)}_M (p_2,p_3|p_1,p_4|0). \] (6.11)
However the number of summations is much less than in the standard Feynman rule calculation, which needs a sum of 36 diagrams. The contents of 36 diagrams are 12, 12, 6, 6 for \( \mathcal{T}(4,0,0) \), \( \mathcal{T}(3,1,0) \), \( \mathcal{T}(2,2,0) \), \( \mathcal{T}(2,1,1) \) respectively.

In summary we have obtained the simple expressions for \( \Gamma_3 \) and \( \Gamma_4 \). \( \Gamma_3 \) consists of only three \( \Gamma_M \)'s in (6.3), while the Feynman rules need 3 diagrams (permutations) for \( \mathcal{T}(3,0,0) \) and \( \mathcal{T}(2,1,0) \) each, and one diagram for \( \mathcal{T}(1,1,1) \). For \( \Gamma_4 \) in (6.10) we only need the summation of \( \Gamma^{(4,0)}_M \), 4 permutations of \( \Gamma^{(3,1)}_M \), and 3 permutations of \( \Gamma^{(2,2,0)}_M \). These simplifications can be seen in (5.21) as well.

7 Conclusions

In this paper we determined the correct normalizations and combinatorics of two-loop world-line formulae for proper \( N \)-point Feynman amplitudes in \( \phi^3 \)-theory and QED photon scatterings. For \( \phi^3 \)-theory the full result is given by (5.21), where the first term \( \mathcal{F}_N \) contains the sum over all Feynman diagrams for \( N_3 = 0 \) and can be obtained by only two quantities as shown in (1.21). The second term of (5.21) is also a sum over the remaining Feynman diagrams, where basically we have only to calculate one master formula for each class, then gather all permutations defined in (5.14).

Let us see how simple these results are in the cases of \( N = 4 \) and 5, where the numbers of Feynman diagrams are 36 and 240 (These numbers can be obtained as a sum of all \( n_T \) given
by (2.4), (3.24) and (5.24)). Among those diagrams, 30 and 18 diagrams are encapsulated in the $\mathcal{F}_N$ respectively. Then for $N = 4$ we are left with 6 permutations of $\Gamma^{(2,1,1)}_M$, and for $N = 5$ there are 15 permutations of $\Gamma^{(2,2,1)}_M$, 10 permutations of $\Gamma^{(3,1,1)}_M$ and no more. One might think that these are still many summations.

However this problem is unavoidable as far as we adopt the world-line parametrizations which do not really parametrize two loop cycles. For example, one can see that our loop type formulae (6.5) and (6.10) were able to absorb symmetric type quantities to some extent, but some momentum permutations still remain in (6.10). This is caused by the fact that a vertex on the middle line can not move beyond the joining point along the second loop cycle, while a vertex on the fundamental loop (first loop cycle) can move in the entire loop. Actually from a string theoretical viewpoint, the middle line proper-time variable is defined by the difference between two points $\tau^{(\alpha)}$ and $\tau^{(3)}$ along the second loop cycle [11]. In order to solve this problem, one should really formulate the middle line as a loop, where the new formulation might look alike a string theory more than ever.

Fortunately in the case of QED photon scatterings, our loop parametrization is sufficient to gather all the Feynman diagrams in the single master formula (4.22) [14]. Therefore a multi-loop $N$-point formula will also be obtainable in the same way as discussed in sect. 4. If the covering multiplicity will be given by $C_h$ for the $(h+1)$-loop master integration region $M_h = \{0 \leq \tau_a \leq T | a = 1, \ldots, N, \alpha_i, \beta_j; i = 2, \ldots, h; j = 1, \ldots, h\}$, the $(h+1)$-loop $N$-point Feynman amplitudes are given by

$$\Gamma^{(h+1)}_N = -4C_h^{-1}(4\pi)^{-\frac{D}{2}(h+1)}e^{N+2h} \int_0^\infty dTT^{-D/2} \prod_{i=1}^h \int_0^\infty d\bar{T}_i \cdot \int d\theta_{\alpha_1}$$

$$\times \int_{M_h} d\bar{\tau}_{\alpha_1} \prod_{i=2}^h d\bar{\tau}_{\alpha_i} d\bar{\tau}_{\beta_1} \prod_{n=1}^N d\bar{\tau}_n \cdot (\text{det} \hat{A})^{-D/2}$$

$$\times < \prod_{i=1}^h D\bar{X}(\bar{\tau}_{\alpha_i}) \cdot D\bar{X}(\bar{\tau}_{\beta_1}) \prod_{n=1}^N D\bar{X}(\bar{\tau}_n) \cdot \epsilon_n \exp[ip_n \cdot \hat{X}(\bar{\tau}_n)] > \bigg|_{\tau_{\alpha_1} = 0}$$

where $\text{det} \hat{A}$ is the determinant defined by switching to super Green functions [14] in the determinant factor $\text{det} A$ appearing in the multi-loop $\phi^3$-theory formula [3, 11].

It is valuable to notice in the world-line formulation that the generating functional of $N$-point 1PI amplitudes takes a very simple and compact form, where the fundamental loop and all internal propagators are naturally joined by an auxiliary field (either $B$ or $\alpha$) at an arbitrary
loop order — though we only demonstrated the two-loop cases in sect. 3. In the two-loop cases, this may rather be trivial from the viewpoint of three propagators convolution in a background; however the other version with a circle world-line is non-trivial in the spirit of multi-loop generalization [9].

It will also be interesting to apply our methods to a non-abelian gauge theory and $\phi^4$ theory [20]. As discussed in sect.3, we should first derive a simple expression for the (two-loop) effective action based on world-line expressions for the propagator and fundamental loop in a background. Writing the background field as a sum of plane waves and expanding it to an appropriate order one will obtain the $N$-point amplitudes expressed in terms of a master formula with correct combinatorics (as advocated in sect. 5 and Appendix A). Thus one would implement a compact form for 1PI amplitudes.

**Appendix A. Derivation of (5.1) from $Z[\bar{\phi}]$**

Showing a derivation of (5.1) from (3.2), we briefly explain the origin of the overall factor and combinatorics in (5.1). Since we discuss the proper diagram parts, we may omit the tree part $Z_0$. The quadratic terms in $\varphi$ in (3.2) can be read as the one-loop effective action

$$\Gamma^{1-\text{loop}} = -\frac{1}{2} \ln \text{Det}(-\partial^2 + m^2 + g\bar{\varphi}) ,$$

and the remainder in (3.2) can be interpreted as the internal $\varphi^3$ vertex. We have only to pick up the second order in this vertex in order to make two-loop diagrams. Therefore the desired two-loop contributions are given by

$$\Gamma^{2-\text{loop}} = \frac{g^2}{2 \cdot (3!)^2} \int \mathcal{D}\varphi d^Dx_1 d^Dx_2 \varphi^3(x_1)\varphi^3(x_2) \exp \left[ -\frac{1}{2} \int \varphi(\Delta^{-1} + g\bar{\varphi})\varphi d^Dx \right] .$$

(A.2)

Applying Wick contractions we rewrite

$$\Gamma^{2-\text{loop}} = \frac{g^2}{2 \cdot 3!} \int d^Dx_1 d^Dx_2 < x_1 |(\Delta^{-1} + g\bar{\varphi})^{-1}|x_2 >^3 .$$

(A.3)

We also know the path integral expression for the propagator

$$< x_1 |(\Delta^{-1} + g\bar{\varphi})^{-1}|x_2 > = \int_0^\infty dT \int_{y(0) = x_2} y(T) = x_1 \mathcal{D}y(t) \exp \left[ -\int_0^T \frac{1}{4} \dot{y}^2(\tau) + m^2 + g\bar{\varphi}(y(\tau)) \right] ,$$

(A.4)
where the path integral normalization is given by

\[ \int_{y(0)=x_2}^{y(T)=x_1} \mathcal{D}y(\tau) \exp \left[ - \int_0^T d\tau \frac{1}{4} \dot{y}^2(\tau) \right] = (4\pi T)^{-D/2} \exp \left[ -\frac{(x_1-x_2)^2}{4T} \right]. \] (A.5)

Substituting this into (A.4) and expanding the background field, we have

\[ \Gamma^{2-\text{loop}} = \frac{g^2}{2 \cdot 3!} \sum_{N_1,N_2,N_3=0}^{\infty} \frac{(-g)^{N_1+N_2+N_3}}{N_1!N_2!N_3!} \int d^Dx_1 d^Dx_2 \prod_{a=1}^{3} \int_0^\infty dT_a e^{-m^2 T_a} \] (A.6)

\[ \times \int_{y_a(0)=x_2}^{y_a(T_a)=x_1} \mathcal{D}y_a(\tau) \exp \left[ - \int_0^{T_a} \frac{1}{4} y_a^2(\tau^{(a)}) d\tau^{(a)} \right] \left[ \int_0^{T_a} \bar{\phi}(y_a(\tau^{(a)})) d\tau^{(a)} \right]^{N_a}. \]

Now let us consider plane wave expansions of the background scalar vertex operators

\[ \phi(y) = \sum_{k=1}^{N} e^{ip_k y}. \] (A.7)

If we naively insert this expansion, various terms will appear. As shall be seen after performing the coordinate integrations, we implicitly have a delta function for the total momentum conservation in (A.6). From this reason, we ignore the terms which include the same momentum twice after the plane wave substitutions. Introducing the following notation

\[ V_k^{(a)} = \int_0^{T_a} d\tau^{(a)} \exp \left[ ip_k y_a(\tau^{(a)}) \right]. \] (A.8)

we are thus allowed to perform the replacement

\[ \left[ \int_0^{T_a} \bar{\phi}(y_a(\tau^{(a)})) d\tau^{(a)} \right]^{N_a} \rightarrow N_a! \sum_{i_1<i_2<\cdots<i_{N_a}} V_{i_1}^{(a)} \cdots V_{i_{N_a}}^{(a)}, \] (A.9)

where every \( i_k, k=1, \ldots, N_a \) runs from 1 to \( N_a \) as far as the ordering restriction is satisfied. The number of such terms is \( N_a \)\( C_{N_a} \). Suppose we performed (A.9) for \( a=1 \) and are on the verge of applying (A.3) to the next \( a=2 \), then we should note that \( i_k \) should run among \( N-N_1 \) integers this time. Thus the number of terms is \( N-N_1 \)\( C_{N_2} \). These numbering for external momenta is nothing but the distribution \( \sigma(N_1,N_2,N_3) \) defined at (5.1), and we realize that (A.10) for fixed \( N \) turns out to be

\[ \Gamma^{2-\text{loop}}_N = \frac{(-g)^{N+2}}{2 \cdot 3!} \sum_{N_1,N_2,N_3}^{N} \sum_{\sigma} \int d^Dx_1 d^Dx_2 \prod_{a=1}^{3} \int_0^\infty dT_a e^{-m^2 T_a} \] (A.10)

\[ \times \int_{y_a(0)=x_2}^{y_a(T_a)=x_1} \mathcal{D}y_a(\tau) \exp \left[ - \int_0^{T_a} \frac{1}{4} y_a^2(\tau^{(a)}) d\tau^{(a)} \right] \prod_{n=1}^{N_a} \int_0^{T_a} d\tau^{(a)} e^{i\bar{p}_{k_n} y_a(\tau^{(a)})}. \]
First performing the $y$ integrations for example putting
\[ y_a(\tau) = x_1 + \frac{\tau}{T_a}(x_2 - x_1) + \sum_{m=1}^{\infty} y_m \sin \left( \frac{m\pi \tau}{T_a} \right), \quad (A.11) \]
and secondly performing $x$ integrations, we finally obtain
\[ \Gamma^{2-\text{loop}}_N = \frac{1}{2 \cdot 3!} \sum_{N_1,N_2,N_3=0}^{N} (2\pi)^D \delta(\sum_{a=1}^{3} \sum_{n=1}^{N_a} p_n^{(a)}) \Gamma^{(N_1,N_2,N_3)}_M, \quad (A.12) \]
and this coincides with (5.1) up to the $(2\pi)^D \delta(\sum p_n)$.

**Appendix B. Translational invariance along the fundamental loop**

In this appendix, we show how to fix one of the super world-line $\hat{\tau}$-parameters in view of the invariance of the integrand in (4.23) under the translation
\[ \tau_n \to \tau_n + c \quad \text{for} \quad n = 1, \ldots, N, \alpha, \beta. \quad (B.1) \]
This translation simply follows from the property
\[ \hat{G}(\hat{x}_a, \hat{x}_b) = \hat{G}(\hat{x}_a + c, \hat{x}_b + c), \quad (B.2) \]
since $\hat{G}_{11}$ is made only of $\hat{G}$.

If we introduce a simplified notation, where $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_n$ denote the $n + 1 = N + 2$ super world-line $\hat{\tau}$-parameters of the fundamental loop, and the integrand of the formula (4.23) is called $\hat{f}(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_n)$, then the invariance (B.1) allows us to rewrite the amplitude (4.23) as
\[
\begin{align*}
\mathcal{A}_N &= \int_0^T d\hat{x}_0 \int_0^T d\hat{x}_1 \ldots \int_0^T d\hat{x}_n \hat{f}(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_n) \\
&= \int_0^T d\hat{x}_0 \int_0^T d\hat{x}_1 \ldots \int_0^T d\hat{x}_n \hat{f}(\hat{0}, \hat{x}_1 - x_0, \ldots, \hat{x}_n - x_0) \\
&= \int_0^T d\hat{x}_0 \int_{-x_0}^{-x_0 + T} d\hat{x}_1 \ldots \int_{-x_0}^{-x_0 + T} d\hat{x}_n \hat{f}(\hat{0}, \hat{x}_1, \ldots, \hat{x}_n),
\end{align*}
\]
where we have denoted the following fact by $\hat{0} = (0, \theta_0)$ that the dependence on $x_0$ has disappeared from the integrand, however the dependence on $\theta_0$ still remains in $\hat{f}$.

If $\hat{f}$ was periodic with period $T$ in each variable, we could just replace the integration region
\[ \int_{-x_0}^{-x_0 + T} d\hat{x}_i \to \int_0^T d\hat{x}_i \quad (B.4) \]
for each \( i = 1, \ldots, n \). However the rigorous situation is not straightforward because of the following reasons. First, the super Green function \( \hat{G} \), from which \( \hat{f} \) is constructed, is not periodic. Rather, it satisfies

\[
\hat{G}(\hat{\tau}_1 + \hat{T}, \hat{\tau}_2) = \hat{G}(\hat{\tau}_1, \hat{\tau}_2) \quad \text{if} \quad \tau_1 < \tau_2 \quad \text{(B.5)}
\]

\[
\hat{G}(\hat{\tau}_1 - \hat{T}, \hat{\tau}_2) = \hat{G}(\hat{\tau}_1, \hat{\tau}_2) \quad \text{if} \quad \tau_1 > \tau_2 \quad \text{(B.6)}
\]

under the shift of a super period \( \pm \hat{T} \)

\[
\hat{\tau} = (\tau, \theta) \rightarrow \hat{\tau} \pm \hat{T} = (\tau \pm T, -\theta) \quad \text{(B.7)}
\]

provided that \( |\tau_1 - \tau_2| < T \). Secondly, we have to take account of the similar restricted periodicity for super derivatives of the super Green functions. Namely those are 'anti-periodic' if differentiating the shifting argument, and 'periodic' if differentiating the one not shifting

\[
D_1 \hat{G}(\hat{\tau}_1 \pm \hat{T}, \hat{\tau}_2) = -D_1 \hat{G}(\hat{\tau}_1, \hat{\tau}_2) \quad \text{(B.8)}
\]

\[
D_2 \hat{G}(\hat{\tau}_1 \pm \hat{T}, \hat{\tau}_2) = D_2 \hat{G}(\hat{\tau}_1, \hat{\tau}_2) \quad \text{(B.9)}
\]

where \( \pm \) is understood as the same ordering as in (B.5) and (B.6). Thirdly, we have to notice the following structure of \( \hat{f} \). After the Wick contractions, \( \hat{f} \) becomes a polynomial such that every term contains all \( D_i \), \( i = 1, \ldots, n \), only once for each. According to this, though same arguments may appear some times, however the differentiated one appears exactly once for each \( \hat{x}_i \). Therefore \( \hat{f} \) behaves as if anti-periodic when one of the arguments is shifted by \( \pm \hat{T} \) because of (B.8) and (B.9). Note that \( \hat{f} \) in the bosonic case behaves as if periodic.

Let us demonstrate how these things work in the case of \( n = 1 \) (vacuum diagram).

\[
\mathcal{A}_0 = \int_0^T d\hat{x}_0 \int_{-x_0}^{-x_0+T} d\hat{x}_1 \hat{f}(\hat{0}, \hat{x}_1) \quad \text{(B.10)}
\]

\[
= \int_0^T d\hat{x}_0 \left( \int_{-x_0}^0 d\hat{x}_1 \hat{f}(\hat{0}, \hat{x}_1) + \int_0^{-x_0+T} d\hat{x}_1 \hat{f}(\hat{0}, \hat{x}_1) \right) .
\]

In the first term, \( x_1 \) is the smallest of the two arguments, since \( x_1 < 0 \) and the other is zero. Owing to eqs. (B.5), (B.8) and (B.9) (for \( \tau_1 < \tau_2 \)), we can rewrite

\[
\hat{f}(\hat{0}, \hat{x}_1) = -\hat{f}(\hat{0}, \hat{x}_1 + \hat{T}) .
\]
If we change integration variables from $\hat{x}_1$ to $\hat{x}'_1 = (x'_1, \theta'_1) = (x_1 + T, -\theta_1)$ we then obtain
\[ A_0 = \int_0^T d\hat{x}_0 \left( \int_{-x_0+T}^T d\hat{x}'_1 \hat{f}(0, \hat{x}_1) + \int_{x_0}^{-x_0+T} d\hat{x}'_1 \hat{f}(0, \hat{x}_1) \right) \] 
(B.12)
\[ = \int_0^T d\hat{x}_0 \int_0^T d\hat{x}'_1 \hat{f}(0, \hat{x}_1) = T \int_0^T \theta_0 \int_0^T d\hat{x}_1 \hat{f}(0, \hat{x}_1). \]

In the general cases, we just repeat the procedure as discussed in [15], and finally we conclude
\[ A_N = \int_0^T d\hat{x}_0 \int_0^T d\hat{x}_1 \ldots \int_0^T d\hat{x}_n \hat{f}(0, \hat{x}_1, \ldots, \hat{x}_n) \] 
(B.13)
\[ = T \int d\theta_0 \int_0^T d\hat{x}_1 \ldots \int_0^T d\hat{x}_n \hat{f}(0, \hat{x}_1, \ldots, \hat{x}_n). \]

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