RELATIONS FOR QUADRATIC HODGE INTEGRALS VIA STABLE MAPS

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Abstract. Following Faber-Pandharipande, we use the virtual localization formula for the moduli space of stable maps to \( \mathbb{P}^1 \) to compute relations between Hodge integrals. We prove that certain generating series of these integrals are polynomials.

Let \( \overline{M}_{g,n} \) be the moduli space of \( n \)-pointed genus \( g \) stable curves. It is a proper smooth Deligne Mumford (DM) stack of dimension \( 3g - 3 + n \). We denote by \( \pi : \mathcal{U}_{g,n} \rightarrow \overline{M}_{g,n} \) the universal curve and by \( \sigma_i : \overline{M}_{g,n} \rightarrow \mathcal{U}_{g,n} \) the sections associated to the marking \( i \) for all \( 1 \leq i \leq n \). We denote by \( \omega_{\mathcal{U}_{g,n}/\overline{M}_{g,n}} \) the relative dualizing sheaf of \( \pi \). We will consider the following classes in \( A^*(\overline{M}_{g,n}) \):

- For all \( 0 \leq i \leq g \), \( \lambda_i \) stands for the \( i \)-th Chern class of the Hodge bundle, i.e. the vector bundle \( E = \pi_* \omega_{\mathcal{U}_{g,n}/\overline{M}_{g,n}} \). For all \( \alpha \in \mathbb{C} \), we denote \( \Lambda_g(\alpha) = \sum_{j=0}^{g} \alpha^{g-j} \lambda_j \), and \( \Lambda_g(\alpha) = (-1)^g \Lambda_g(-\alpha) \).

- For all \( 1 \leq i \leq n \), we denote \( \psi_i \) the Chern class of the cotangent line at the \( i \)th marking \( \mathcal{L}_i = \sigma_i^* (\omega_{\mathcal{U}_{g,n}/\overline{M}_{g,n}}) \).

A Hodge integral is an intersection number of the form:

\[
\int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \Lambda_g(t_1) \cdots \Lambda_g(t_m),
\]

where \( k_1, \ldots, k_n \) are non-negative integers and \( t_1, \ldots, t_m \) are complex numbers. If \( m = 1, 2, \) or \( 3 \), then the above integral is called a linear, double, or triple Hodge integrals respectively. Relations between linear Hodge integrals where proved in [FP00a] using the Gromov-Witten theory of \( \mathbb{P}^1 \) and the localization formula of [GP99]. This approach was also used in [FP00b] and [TZ03] to prove certain properties of triple Hodge integrals. Linear and triple Hodge integrals naturally appeared in the GW-theory of Calabi-Yau 3-folds, thus explaining a more abundant literature on the topic. However, double Hodge integrals have appeared recently in the Quantization of Witten-Kontsevich generating series (see [Blo20]), in the theory of spin Hurwitz numbers (see [GKL21]), and in the GW theory of blow-ups of smooth surfaces (see [GKLS22]).

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\(^{1}\)Here we use the convention of [FP00a] for \( \Lambda_g(\alpha) \) and \( \Lambda_g(\alpha) \)
In the present note, we consider the following power series in $\mathbb{C}[\alpha][t]$ defined using double Hodge integrals

\begin{equation}
    P_a(\alpha,t) = \sum_{g \geq 0} t^g \left( \int_{\mathcal{M}_{g,n+1}} \frac{\Lambda_g^V(1)\Lambda_g^V(\alpha)}{1 - \psi_0} \prod_{i=1}^{n}(2a_i + 1)!(-4\psi_i)^{a_i} \right) \exp\left( \frac{t}{24} \right)
\end{equation}

where $a = (a_1, \ldots, a_n)$ is a vector of non-negative integers. If $n = 1$, we use the convention: $\int_{\mathcal{M}_{0,2}} \psi_1 \frac{\Lambda_1^V(1)\Lambda_1^V(\alpha)}{1 - \psi_2} = (-1)^a$.

**Theorem 0.1.** $P_a(\alpha, t)$ is a monic polynomial in $\mathbb{C}[\alpha][t]$ of degree $|a|$ in $t$.

Here we provide the first values of $P_a(-\alpha - 1, t)$. In the list below we omit the variables $-\alpha - 1$ and $t$ in the notation:

$P(\_\_) = 1$

$P(1) = t + 12$

$P(2) = t^2 - 10\alpha t + 240$

$P(1,1) = t^2 - 12t$

$P(3) = t^3 + (-77/3\alpha - 28)t^2 + 280t + 6720$

$P(2,1) = t^3 + (-10\alpha - 48)t^2 + (240\alpha + 240)t$

$P(1,1,1) = t^3 - 72t^2 + 432t$

$P(4) = t^4 + (-43\alpha - 72)t^3 + (126\alpha^2 + 756\alpha + 840)t^2 + 10080t + 241920$

$P(3,1) = t^4 + (-77/3\alpha - 100)t^3 + (1232\alpha + 1624)t^2$

$P(2,2) = t^4 + (20\alpha + 100)t^3 + (-100\alpha^2 - 1360\alpha - 1680)t^2$

$P(2,1,1) = t^4 + (-10\alpha - 132)t^3 + (840\alpha + 3120)t^2 + (-8640\alpha - 8640)t$

$P(1,1,1,1) = t^4 - 168t^3 + 5616t^2 - 20736t$.

Considering these first values, we conjecture that $P_a$ is a polynomial of total degree $|a|$ in both variables $t$ and $\alpha$.

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1. Preliminaries

We denote by $\mathcal{M}_{g,n}(\mathbb{P}^1, 1)$ the moduli space of stable maps of degree 1 to $\mathbb{P}^1$. It is a proper DM stack of virtual dimension $2g + n$. Here we can define in an analogous way the Hodge bundle $\mathcal{E}$, the cotangent line bundles $\mathcal{L}_i$ and we denote again $\lambda_i$ and $\psi_i$ the respective Chern classes. We also have the forgetful and evaluation maps

$$\pi: \mathcal{M}_{g,n+1}(\mathbb{P}^1, 1) \to \mathcal{M}_{g,n}(\mathbb{P}^1, 1), \quad \text{and} \quad ev_i: \mathcal{M}_{g,n+1}(\mathbb{P}^1, 1) \to \mathbb{P}^1.$$

Throughout this note the enumeration of markings starts from 0. Furthermore, $\pi$ is the morphism that forgets the marking $p_0$ and $ev_i$ is the evaluation of a stable map to the $i$-th marked point. The vector bundle $T := R^1\pi_*(ev_0^*\mathcal{O}_{\mathbb{P}^1}(-1))$ is of rank $g$ and we denote by $y$ its top Chern class. We will denote:

$$\langle n \prod_{i=0}^{n-1} \tau_{a_i}(\omega) | y \rangle_{\mathbb{P}^1_{g,1}} := \int_{[\mathcal{M}_{g,n}(\mathbb{P}^1, 1)]^{vir}} n \prod_{i=0}^{n-1} \psi_i^{a_i} ev_i^*(\omega) y$$

where $\omega$ denotes the class of a point in $\mathbb{P}^1$.

**Theorem 1.1** (Localization Formula, [GP99], [FP00a]). Let $g \in \mathbb{Z}_{\geq 0}$, let $a \in \mathbb{Z}_{\geq 0}^n$ such that $|a| \leq g$. Then, for all complex numbers $\alpha$, and $t \in \mathbb{C}^*$, we have

$$\langle \prod_{i=1}^{n} \tau_{a_i}(\omega) | y \rangle_{\mathbb{P}^1_{g,1}} = \sum_{g_1+g_2=g} \int_{\mathcal{M}_{g_1,n+1}} t^n \prod_{i=1}^{n} \psi_i^{a_i} \frac{\Lambda^\vee_1(t) \Lambda^\vee_2(\alpha t)}{t(t-\psi_0)} \times \int_{\mathcal{M}_{g_2,1}} \frac{\Lambda^\vee_2(-t) \Lambda^\vee_2((\alpha + 1)t)}{t(-t-\psi_0)}.$$

Here we use the convention $\int_{\mathcal{M}_{0,1}} \psi_0^a = 1$.

**Proposition 1.2** (4.1 of [TZ03]). For all complex numbers $\alpha$ we have

$$F(\alpha, t) = 1 + \sum_{g>0} t^{2g} \int_{\mathcal{M}_{g,1}} \frac{\Lambda^\vee_2(1) \Lambda^\vee_2(\alpha)}{1-\psi_0} = \exp \left( -\frac{t^2}{24} \right).$$

Besides, we have the String and Dilaton equation for Hodge integrals.

**Proposition 1.3.** Let $g, n \in \mathbb{Z}_{\geq 0}$ such that $2g - 2 + n > 0$.

(i) [Dilaton equation for Hodge integrals] Let $(a_1, ..., a_n) \in \mathbb{Z}_{\geq 0}^n$ and assume that there exist $i_0$ such that $a_{i_0} = 1$. Then

$$\int_{\mathcal{M}_{0,n+1}} \frac{\psi_{i_0} \prod_{i \neq i_0} \psi_i^{a_i} \prod_{j=1}^g \lambda_k^b}{1-\psi_0} = (2g - 2 + n) \int_{\mathcal{M}_{g,n}} \frac{\prod_{i=1}^{n-1} \psi_i^{a_i} \prod_{j=1}^g \lambda_k^b}{1-\psi_0}.$$
(ii) [String equation for Hodge integrals] Let \((a_1, ..., a_n) \in \mathbb{Z}_{\geq 0}^n\) and assume that there exist \(i_0\) such that \(a_{i_0} = 0\). Then we have

\[
\int_{\mathcal{M}_{g,n+1}} \frac{\prod_{i=1}^n \psi_i^{a_i} \prod_{j=1}^g \lambda_k^{b_j}}{1 - \psi_0} = \int_{\mathcal{M}_{g,n}} \frac{\prod_{i=1}^{n-1} \psi_i^{a_i} \prod_{j=1}^g \lambda_k^{b_j}}{1 - \psi_0} + \sum_{j=1}^n \int_{\mathcal{M}_{g,n}} \frac{\psi_j^{a_j-1} \prod_{i \neq j} \psi_i^{a_i} \prod_{k=1}^g \lambda_k^{b_k}}{1 - \psi_0}.
\]

2. The calculation

Note that the GW-invariant \(\langle \prod_{i=1}^n \tau_{a_i}(\omega) \rangle_{g,1}^{p_1}\) is 0 unless \(|a| = g\) for dimensional reasons. Indeed, \(\dim_{\mathbb{C}}[\mathcal{M}_{g,n}(\mathbb{P}^1, 1)]^{\text{vir}} = 2g + n\) and the cycle we are integrating is in codimension \(g + |a| + n\). Using the above localization formula, and Lemma 2.1 of [TZ03] the intersection number \(\langle \prod_{i=1}^n \tau_{a_i}(\omega) \rangle_{g,1}^{p_1}\) is expressed as:

\[
\sum_{g_1 + g_2 = g} t^{g_1} \int_{\mathcal{M}_{g_1,n+1}} t^{n} \prod_{i=1}^n \psi_i^{a_i} \Lambda_{g_1}^V(t) \Lambda_{g_2}^V(\alpha t) \cdot \int_{\mathcal{M}_{g_2,1}} \frac{\Lambda_{g_2}^V(-t) \Lambda_{g_2}^V((\alpha + 1)t)}{-t(-t - \psi_0)}
= \sum_{g_1 + g_2 = g} \sum_{\alpha | \alpha = g} \psi_0^{3g_2-2}.
\]

In the last equation we used Proposition 1.2 in order to replace \(\int_{\mathcal{M}_{g_2,1}} \frac{\Lambda_{g_2}^V(1) \Lambda_{g_2}^V(-\alpha + 1)}{-1 - \psi_0}\) with \((-1)^{g_2} \int_{\mathcal{M}_{g_2,1}} \psi_0^{3g_2-2} - 1\).

We define

\[
A_{g,a}(\alpha) = \sum_{g_1 + g_2 = g} \int_{\mathcal{M}_{g_1,n+1}} \prod_{i=1}^n \psi_i^{a_i} \Lambda_{g_1}^V(1) \Lambda_{g_2}^V(\alpha) \cdot \int_{\mathcal{M}_{g_2,1}} \psi_0^{3g_2-2}.
\]

Then, we have

\[
A_{g,a}(\alpha) = \begin{cases} 0 & |a| < g, \\ \langle \prod_{i=1}^n \tau_{a_i}(\omega) \rangle_{g,1}^{p_1} & |a| = g. \end{cases}
\]

By the definition of \(\Lambda_{g}^V(t)\) we see that \(\Lambda_{g}^V(1) \Lambda_{g}^V(-\alpha + 1)\) is a polynomial in \(\alpha\) of degree \(g\), which actually determines the degree of \(A_{g}(\alpha)\).
We now present a proof for the main result.

Proof. [of Theorem 0.1] We begin by stating the well known fact

\[ 1 + \sum_{g \geq 0} t^g \int_{\mathcal{M}_{g,1}} \psi_0^{3g-2} = \exp \left( \frac{t}{24} \right) \]

proven in section 3.1 of [FP00a]. Now, we consider the product of \( \exp \left( \frac{t}{24} \right) \) and

\[ \sum_{g \geq 0} t^g \left( \int_{\mathcal{M}_{g,n+4}} - \psi_0 \prod_{i=1}^{n} (2a_i + 1)!!(-4\psi_i)^{a_i} \right) \]

to obtain a new power series whose coefficients in degree \( g \) are given by

\[ \sum_{g_1 + g_2 = g} \int_{\mathcal{M}_{g_1,n+1}} \prod_{i=1}^{n} (2a_i + 1)!!(-4)^a \prod_{i=1}^{n} \psi_i^{a_i} \Lambda_{g_1}^\vee(1)\Lambda_{g_2}^\vee(\alpha) \cdot \int_{\mathcal{M}_{g_2,1}} \psi_0^{3g_2-2} \]

This is exactly \( A_{g,a}(\alpha) \cdot \prod_{i=1}^{n} (2a_i + 1)!!(-4)^a \). Hence, we can rewrite the power series \( P_a(\alpha, t) \) in the form

\[ P_a(\alpha, t) = \sum_{g \geq 0} t^g A_g(\alpha) \]

As it is computed in the start of Section 2 we have that the numbers \( A_{g,a}(\alpha) \) vanish when \( g > |a| \). Hence, we get that all coefficients of the power series \( P_a(\alpha, t) \) vanish when \( g > |a| \), i.e. \( P_a(\alpha, t) \) is a polynomial of degree \( |a| \). Furthermore, the top coefficient of \( P_a(\alpha, t) \), i.e. the coefficient of \( t^{|a|} \) is given by

\[ \langle \prod_{i=1}^{n} (2a_i + 1)!! \cdot \tau_a(\omega)|y|^{|a|} \rangle_{|a|} \]

This value is computed in [KL11] and is actually equal to 1. In particular, the number \( \prod_{i=1}^{n} (2a_i + 1)!! \) is here to make the polynomial monic. \( \square \)

We now prove several other properties of the polynomials \( P_a \).

**Proposition 2.1.** The constant term \( c_0 \) of \( P_a(\alpha, t) \) is non zero if and only if \( n = 1 \) where then \( c_0 = (-1)^a \prod_{i=1}^{n} (2a_i + 1)!! \) or if \( n > 1 \) and \( \sum_{i=1}^{n} a_i \leq n - 2 \) where then

\[ c_0 = \prod_{i=1}^{n} (2a_i + 1)!! \frac{(n-2)!}{a_1! \cdots (n-2-\sum a_i)!} \]

**Proof.** We only compute the integrals appearing in the constant term of this polynomial since then we only have to multiply with \( \prod_{i=1}^{n} (2a_i + 1)!!(-4)^a \). The integral in the constant term of \( P_a(\alpha, t) \) is given by \( \int_{\mathcal{M}_{0,n+4}} \prod_{i=1}^{n} \psi_i^{a_i} \). When \( n = 1 \), using the convention \( \int_{\mathcal{M}_{0,2}} \frac{\psi^{a}}{1-\psi_0} = (-1)^a \) we get that

\[ c_0 = (-1)^a \prod_{i=1}^{n} (2a_i + 1)!! \]
When \( n > 1 \), if \( \sum_{i=1}^{n} a_i > n - 2 \), then \( c_0 \) is zero for dimensional reasons. Otherwise, we have

\[
\int_{\mathcal{M}_{0,n+1}} \prod_{i=1}^{n} \frac{\psi_i a_i}{1 - \psi_0} = \int_{\mathcal{M}_{0,n+1}} \psi_0^{n-2 - \sum a_i} \prod_{i=1}^{n} \psi_i a_i = \frac{(n - 2)!}{a_1! \cdots (n - 2 - \sum a_i)!}
\]

and so we obtain the desired result.

\[\square\]

**Proposition 2.2.** Let \( n \geq 3 \). Then we have the following rules:

(i) [String equation]

\[
P_{(a_1,\ldots,a_{n-1},0)}(\alpha,t) = P_{(a_1,\ldots,a_{n-1})}(\alpha,t) - \sum_{i=1}^{n} (8a_i + 4) P_{(a_1,\ldots,a_{i-1},a_i-1,\ldots,a_{n-1})}(\alpha,t)
\]

(ii) [Dilaton equation]

\[
P_{(a_1,\ldots,a_{n-1},1)}(\alpha,t) = (t - 12n + 24) P_{(a_1,\ldots,a_{n-1})}(\alpha,t) - 24tP'_{(a_1,\ldots,a_{n-1})}(\alpha,t)
\]

**Proof.** We define the power series

\[
\tilde{P}_a(\alpha,t) = \sum_{g \geq 0} t^g \left( \int_{\mathcal{M}_{g,n+1}} \prod_{i=1}^{n} \psi_i a_i \Lambda^Y(1) \Lambda^Y(\alpha) \right)
\]

Note that the following equation holds.

\[
P_a(\alpha,t) = \prod_{i=1}^{n} (2a_i + 1)!! (-4)^a \tilde{P}_a(\alpha,t) \exp \left( \frac{t}{24} \right)
\]

We can rewrite the coefficients of \( \tilde{P}_a(\alpha,t) \) as

\[
\sum_{k=0}^{g} \sum_{j=0}^{g} (-1)^{g+k} (a + 1)^{g-j} \int_{\mathcal{M}_{g,n+1}} \prod_{i=1}^{n} \psi_i a_i \lambda_k \lambda_j \frac{\Lambda^Y(1) \Lambda^Y(\alpha)}{1 - \psi_0}
\]

(i) Applying the String equation for Hodge integrals we obtain the following formula

\[
\tilde{P}_{(a_1,\ldots,a_{n-1},0)}(\alpha,t) = \tilde{P}_{(a_1,\ldots,a_{n-1})}(\alpha,t) + \sum_{i=1}^{n} \tilde{P}_{(a_1,\ldots,a_i-1,\ldots,a_{n-1})}(\alpha,t)
\]

Hence, multiplying with \( \prod_{i=1}^{n-1} (2a_i + 1)!! (-4)^a \exp \left( \frac{t}{24} \right) \) we obtain the desired result after a straightforward calculation.

(ii) Applying Dilaton equation for Hodge integrals we obtain the following formula

\[
\tilde{P}_{(a_1,\ldots,a_{n-1},1)}(\alpha,t) = 2 \sum_{g \geq 0} gt^g \int_{\mathcal{M}_{g,n+1}} \prod_{i=1}^{n-1} \psi_i a_i \Lambda^Y(1) \Lambda^Y(\alpha) \frac{\lambda_k \lambda_j}{1 - \psi_0} + (n - 2) \tilde{P}_{(a_1,\ldots,a_{n-1})}(\alpha,t)
\]
Note that the first term of the sum is equal to $2t\tilde{P}'_{(a_1,\ldots,a_{n-1})}(\alpha,t)$.

Now, multiplying both sides of the equation above with

$$\prod_{i=1}^{n-1} (2a_i + 1)(-4)^a_i \exp \left( \frac{t}{24} \right)$$

we have

$$\frac{-1}{12} P_{(a_1,\ldots,a_{n-1},1)}(\alpha,t) = (n-2)P_{(a_1,\ldots,a_{n-1})}(\alpha,t)$$

$$+ 2t \left( \prod_{i=1}^{n-1} (-4)^{a_i} (2a_i + 1)!! \right) \tilde{P}'_{(a_1,\ldots,a_{n-1})}(\alpha,t) e^{t/24}$$

$$= (n-2)P_{(a_1,\ldots,a_{n-1})}(\alpha,t)$$

$$+ 2t (P'_{(a_1,\ldots,a_{n-1})}(\alpha,t) - \frac{1}{24} P_{(a_1,\ldots,a_{n-1})}(\alpha,t)).$$

Finally clearing the denominators we obtain the desired result.

We recall Mumford’s relation $\Lambda^\vee_g(1) \cdot \Lambda^\vee_g(-1) = 1$ (see [Mum83]). In particular, $P_a(-1,t)$ is defined by integrals of $\psi$-classes.

**Corollary 2.3.** For any vector $a \in \mathbb{Z}_{\geq 0}^n$, the power series

$$P_a(-1,t) = \prod_{i=1}^{n} (2a_i + 1)!! (-4)^{a_i} \exp \left( \frac{t}{24} \right) \cdot \sum_{g \geq 0} (-t)^g \int_{\overline{M}_{g,n+1}} \prod_{i=1}^{n} \psi_i^{a_i} \frac{\prod_{i=1}^{n} \psi_i^{a_i}}{1 - \psi_0}$$

is a polynomial of degree $|a|$.

In this case the polynomiality as well as a closed expression were proved in [LX11].

**REFERENCES**

[Blot20] X. Blot. The quantum Witten-Kontsevich series and one-part double Hurwitz numbers. 2020, arXiv:2004.07581.

[FP00a] C. Faber and R. Pandharipande. Hodge integrals and Gromov-Witten theory. *Invent. Math.*, 139(1):173–199, 2000.

[FP00b] C. Faber and R. Pandharipande. Logarithmic series and Hodge integrals in the tautological ring. *Michigan Math. J.*, 48:215–252, 2000. With an appendix by Don Zagier, Dedicated to William Fulton on the occasion of his 60th birthday.

[GKL21] A. Giacchetto, R. Kramer, and D. Lewański. A new spin on Hurwitz theory and els via theta characteristics. 2021, arXiv:2104.05697.

[GKLS22] Alesandro Giacchetto, Reinier Kramer, Danilo Lewański, and Adrien Sauvaget. The spin gromov-witten/hurwitz correspondence for $\mathbb{P}^1$. 2022.

[GP99] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999.

[KL11] Y.-H. Kiem and J. Li. Low degree GW invariants of spin surfaces. *Pure Appl. Math. Q.*, 7(4, Special Issue: In memory of Eckart Viehweg):1449–1475, 2011.

[LX11] K. Liu and H. Xu. The $n$-point functions for intersection numbers on moduli spaces of curves. *Adv. Theor. Math. Phys.*, 15(5):1201–1236, 2011.
[Mum83] D. Mumford. Towards an enumerative geometry of the moduli space of curves. In Arithmetic and geometry, Vol. II, volume 36 of Progr. Math., pages 271–328. Birkhäuser Boston, Boston, MA, 1983.

[TZ03] G. Tian and J. Zhou. Quadratic recursion relations of Hodge integrals via localization. Acta Math. Sin. (Engl. Ser.), 19(2):209–232, 2003.

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