ON A POLYNOMIAL ZETA FUNCTION

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Abstract. We introduce a polynomial zeta function \( \zeta_{P_n}^{(p)} \), related to certain problems of mathematical physics, and compute its value and the value of its first derivative at the origin \( s = 0 \), by means of a very simple technique. As an application, we compute the determinant of the Dirac operator on quaternionic vector spaces.

1. Introduction

The aim of this paper is to look at few simple properties of a particular class of zeta functions, which we dub polynomial zeta functions because they are associated with a polynomial \( P_n \) of degree \( n \):

\[
\zeta_{P_n}^{(p)}(s) = \sum_{k=0}^{\infty} \frac{k^p}{P_n(k)^s}.
\]

Our motivations arise from physical questions, where, substantially, \( \zeta_{P_n}^{(p)}(0) \) defines the determinant of an operator \( O \) having \( \lambda_k = P_n(k) \) as eigenvalues each one having degeneration \( k^p \):

\[
\det O = e^{-\zeta_{P_n}^{(p)}(0)}
\]

When \( O \) represent the Hamiltonian operator of a field theory, this provides the (Euclidean) effective action

\[
S_{\text{eff}} = -\log(\det O) = \zeta_{P_n}^{(p)}(0).
\]

Thus, exact calculations of non perturbative quantum field theories effects can be performed, as for example the Schwinger effect on flat or curved background, see for example [BVW], [AZ], or discharge effects on certain charged black hole backgrounds, [BCDa], [BCDb]. However, we think that a general investigation of the properties of the polynomial zeta functions could be of a certain interest for pure mathematics also.

Our strategy is very simple and is essentially based on a direct application of the Abel-Plana formula.

2. The polynomial zeta function

Let us consider the polynomial

\[
P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0
\]

with \( a_0, a_n \neq 0 \). Suppose that the zeros of \( P_n \) are not in \( \mathbb{N} \). We define the polynomial zeta function

\[
\zeta_{P_n}^{(p)}(s) := \sum_{k=0}^{\infty} \frac{k^p}{P_n(k)^s}, \quad p \in \mathbb{N}.
\]

We are interested to compute \( \zeta_{P_n}^{(p)}(0) \) and \( \zeta_{P_n}^{(p)}(0) \).

To this hand, let us first assume that all zeros \( x_i, i = 1, \ldots, n, \) of \( P_n \) have negative real part: \( \Re x_i < 0 \). With these hypothesis, the function

\[
f(z) = \frac{z^p}{P_n(z)^s}
\]

for \( \Re s > 0 \) is regular on \( \Re z \geq 0 \) and satisfies the conditions

1. \( \lim_{y \to -\infty} e^{-2\pi|y|} f(x + iy) = 0 \),
2. \( \lim_{x \to +\infty} \int_{-\infty}^{\infty} e^{-2\pi|y|} f(x + iy) dy = 0 \).

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So, we can apply the Abel-Plana formula [WW] to get

\[ \zeta_{p,n}(s) = \frac{1}{2a_0^s} \delta_{p,0} + \int_0^\infty \frac{x^p}{P_n(x)^s} dx + i \int_0^\infty \left[ \frac{(ix)^p}{P_n(ix)^s} - \frac{(-ix)^p}{P_n(-ix)^s} \right] \frac{1}{e^{2\pi x} - 1} dx. \quad (2.3) \]

We write the polynomial \( P_n \) as

\[ P_n(x) = a_n \prod_{i=1}^n (x - x_i) \equiv a_n P_0^n(x), \]

so that \( P_0^n \) is monic. We also define

\[ \tilde{P}_n(x) = x^n P_0^n(1/x) = \prod_{i=1}^n (1 - xix). \]

To perform our computations we need two lemmas.

**Lemma 1.** Let \( P_0^n \) be a monic polynomial of degree \( n \) and having all zeros \( x_i, i = 1, \ldots, n \) with negative real part, and \( p \in \mathbb{N} \). Then

\[ \lim_{s \to 0} \int_0^\infty x^p \left\{ \frac{1}{P_0^n(x)^s} - \frac{1}{n} \sum_{i=1}^n \frac{1}{(x - x_i)^s} \right\} dx = 0. \quad (2.4) \]

Here we mean the limit of the analytic continuation \( f(s) \) of the expression defined by the integral.

**Proof.** We have

\[ f(s) = \int_0^1 x^p \left\{ \frac{1}{P_0^n(x)^s} - \frac{1}{n} \sum_{i=1}^n \frac{1}{(x - x_i)^s} \right\} dx + \int_1^\infty x^p \left\{ \frac{1}{P_0^n(x)^s} - \frac{1}{n} \sum_{i=1}^n \frac{1}{(x - x_i)^s} \right\} dx. \]

For the first integral we can exchange the limit with the integral, so that it vanishes when \( s \to 0 \). For the second integral, it is convenient to take the change of integration variable \( x \mapsto 1/x \) so that it becomes

\[ I(s) = \int_0^1 \left[ \frac{x^{ns-p-2}}{P_n(x)^s} - \frac{1}{n} \sum_{i=1}^n \frac{x^{s-p-2}}{(1 - xix)^s} \right] dx. \]

The integrand has a branch point in \( x = 0 \), so that we introduce a cut on the positive real semiaxis. Next, we define a path \( \Gamma \), starting from the point \( x = 1 \equiv e^0 \), going around \( x = 0 \) and then ending to the point \( x = 1^+ \equiv e^{2\pi i} \). See the figure 1. This is not a closed path, but in the usual way it is easy to show that, when \( I \)
is well defined, we obtain

\[ I(s) = \frac{1}{e^{2\pi ins} - 1} \int_{\Gamma} x^{ns-p-2} \frac{dx}{P_n(x)^s} - \frac{1}{e^{2\pi is} - 1} \frac{1}{n} \int_{\Gamma} x^{s-p-2} \frac{dx}{(1-xi)^s}. \]

As \( \Gamma \) does not pass through \( x = 0 \), this integral are well defined in a neighborhood of \( s = 0 \) and we can take the limit or differentiate under the integral sign. Note that for \( s = 0 \) the monodromy becomes trivial and the path closes, so that both integrals vanish. Thus we can use the de l’Hospital rule to get

\[
\lim_{s \to 0} I(s) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{x^{p+2}} \left[ (\log x - \frac{1}{n} \log P_n(x)) - \frac{1}{n} (n \log x - \sum_{i=1}^{n} \log(1-x_i)) \right] dx = 0.
\]

A second important step is the following:

**Lemma 2.** Let \( p \) a positive integer and \( F_{p_n}^{(p)}(s) \) the analytic continuation on the complex plane of the function

\[ F_{p_n}^{(p)}(s) = \int_{0}^{\infty} x^p \left\{ \frac{\log P_n(x)}{P_n(x)^s} - \sum_{i=1}^{n} \frac{\log(x-x_i)}{(x-x_i)^s} \right\} dx, \quad \Re(s) > p + 1. \]

Then

\[ F_{p_n}^{(p)}(0) = \frac{1}{2n} \sum_{l=1}^{p} \frac{1}{l(p+1-l)} \left( \sum_{i=1}^{n} x_i \right) \left( \sum_{j=1}^{n} x_j^{p-l+1} \right) + \frac{H_p}{p+1} \sum_{i=1}^{n} x_i^{p+1}, \quad (2.5) \]

where \( H_p = \sum_{i=1}^{p} \frac{1}{i} \) are the harmonic numbers. In particular \( F_{p_n}^{(0)}(0) = 0 \).

**Proof.** The proof is very similar to the previous one so that we will be essential. As before we first obtain

\[
\lim_{s \to 0} F_{p_n}^{(p)}(s) = \lim_{s \to 0} [J(s) - \sum_{i=1}^{n} J_i(s)],
\]

\[
J(s) := \int_{0}^{1} x^{ns-p-2} \frac{\log P_n(x) - n \log x}{P_n(x)^s} dx,
\]

\[
J_i(s) := \int_{0}^{1} x^{s-p-2} \frac{\log(1-x_i) - \log x}{(1-x_i)^s} dx
\]

Taking \( \Gamma \) as above, and noting the monodromy of the logarithm, we have

\[
\int_{\Gamma} x^{ns-p-2} \frac{\log P_n(x) - n \log x}{P_n(x)^s} dx = (e^{2\pi ins} - 1) J(s) - \frac{2\pi i e^{2\pi ins}}{e^{2\pi ins} - 1} \int_{\Gamma} x^{ns-p-2} \frac{dx}{P_n(x)^s}
\]

so that

\[
J(s) = \frac{1}{e^{2\pi is} - 1} \int_{\Gamma} x^{ns-p-2} \frac{\log P_n(x) - n \log x}{P_n(x)^s} dx + \frac{2\pi i e^{2\pi is}}{(e^{2\pi is} - 1)^2} \int_{\Gamma} x^{ns-p-2} \frac{dx}{P_n(x)^s}
\]

and similarly

\[
J_i(s) = \frac{1}{e^{2\pi is} - 1} \int_{\Gamma} x^{s-p-2} \frac{\log(1-x_i) - \log x}{(1-x_i)^s} dx + \frac{2\pi i e^{2\pi is}}{(e^{2\pi is} - 1)^2} \int_{\Gamma} x^{s-p-2} \frac{dx}{(1-x_i)^s}.
\]

Again, we can use the de l’Hospital rule to obtain

\[
F_{p_n}^{(p)}(0) = -\frac{1}{2} \frac{1}{(p+1)!} \frac{d^{p+1}}{dx^{p+1}} \left[ \frac{1}{n} \left( \sum_{i=1}^{n} \log(1-x_i) \right)^2 - \sum_{i=1}^{n} (\log(1-x_i))^2 \right]_{x=0}.
\]
Note that for \( p = 0 \) this vanishes. For \( p \geq 1 \), using the Leibnitz rule for derivation, we obtain

\[
F^{(p)}_{P_n}(0) = -\frac{1}{2(p+1)!} \sum_{l=1}^{p} \binom{p+1}{l} \left[ \frac{1}{n} \left( \sum_{i=1}^{n} \frac{d^{l-1}}{dx^{l-1}} \frac{x_i}{1-x_i x} \right) \left( \sum_{j=1}^{n} \frac{d^{p-l}}{dx^{p-l}} \frac{x_j}{1-x_j x} \right) \right]
\]

\[
-\frac{n}{l(p-l+1)} \left[ \frac{1}{n} \left( \sum_{i=1}^{n} x_i^l \right) \left( \sum_{j=1}^{n} x_j^{p-l} \right) - \sum_{i=1}^{n} x_i^{p+1} \right],
\]

from which the thesis follows.

\[
\square
\]

2.1. \( \zeta_{P_n}^{(2m)}(0) \). From (2.3) and using Lemma 1, if \( p = 2m \), we get

\[
\zeta_{P_n}^{(2m)}(0) = \frac{1}{2} \delta_{m,0} + \lim_{s \to 0} \int_{0}^{\infty} \frac{x^{p}}{P_n(x)^{s}} \, dx = \frac{1}{2} \delta_{m,0} + \lim_{s \to 0} \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{\infty} \frac{x^p}{(x - x_i)^s} \, dx.
\]

Using

\[
\int_{0}^{\infty} \frac{x^p}{(x - x_i)^s} \, dx = \frac{\Gamma(s - p - 1)p!}{\Gamma(s)} (-x_i)^{p+1-s}
\]

we get

\[
\zeta_{P_n}^{(2m)}(0) = \frac{1}{2} \delta_{m,0} - \frac{1}{2m+1} \sum_{i=1}^{n} (-x_i)^{2m+1}.
\]

For example, using \( x_1 + \ldots + x_n = -a_{n-1}/a_n \), we get

\[
\zeta_{P_n}(0) = \zeta_{P_n}^{(0)}(0) = \frac{1}{2} - \frac{1}{n} \frac{a_{n-1}}{a_n}.
\]

2.2. \( \zeta_{P_n}^{(2m+1)}(0) \). For the case \( p = 2m + 1 \), we get

\[
\zeta_{P_n}^{(2m+1)}(0) = \lim_{s \to 0} \int_{0}^{\infty} \frac{x^{p}}{P_n(x)^{s}} \, dx - (-1)^{m+2} \int_{0}^{\infty} \frac{x^{2m+1}}{e^{2\pi x} - 1} \, dx.
\]

From Lemma 1 and (2.6) we get

\[
\zeta_{P_n}^{(2m+1)}(0) = \frac{1}{2m+2} \sum_{i=1}^{n} x_i^{2m+2} - (-1)^{m+2} \frac{(2m+1)!}{(2\pi)^{2m+2}} \zeta(2m).
\]

Using

\[
\sum_{i=1}^{n} x_i^2 = \frac{a_{n-1}^2}{a_n^2} - 2 \frac{a_{n-2}}{a_n},
\]

we find for example

\[
\zeta_{P_n}^{(1)}(0) = \frac{1}{2} \frac{a_{n-1}^2}{a_n^2} - \frac{a_{n-2}}{a_n} - \frac{1}{12}.
\]
2.3. $\zeta_{P_n}^{(p)}(0)$. We first note that

$$\zeta_{P_n}^{(p)}(0) = -\zeta_{P_n}^{(0)}(0) \log a_n + \zeta_{P_n}^{(p)}(0).$$

(2.11)

Differentiating (2.3) with respect to $s$, for $P_n^0$, we obtain

$$\zeta_{P_n}^{(p)}(s) = \frac{-1}{2} \sum_{i=1}^{n} \log(-x_i) \delta_{P_0} - \int_{0}^{\infty} x^p \left\{ \log P_n(x) \frac{\log P_n(x)}{P_n(x)^s} + i \left[ \log P_n(ix) \frac{\log P_n(-ix)}{P_n(-ix)^s} \right] \frac{1}{e^{2\pi x} - 1} \right\} dx.$$ 

Now

$$\lim_{s \to 0} i \int_{0}^{\infty} \left[ (ix)^p \log P_n(ix) \frac{\log P_n(-ix)}{P_n(-ix)^s} \right] \frac{dx}{e^{2\pi x} - 1} = i \sum_{i=1}^{n} \left[ (ix)^p \log(ix - x_i) - (ix)^p \log(-ix - x_i) \right] \frac{dx}{e^{2\pi x} - 1}$$

$$= -i \frac{d}{ds} \sum_{i=1}^{n} \int_{0}^{\infty} [(ix)^p(ix - x_i)^{-s} - (ix)^p(ix - x_i)^{-s}] \frac{dx}{e^{2\pi x} - 1}.$$

We can apply the Abel-Plana formula to this expression to obtain

$$\zeta_{P_n}^{(p)}(0) = \lim_{s \to 0} \int_{0}^{\infty} x^p \left\{ \log P_n(x) \frac{\log P_n(x)}{P_n(x)^s} - \sum_{i=1}^{n} \log(x - x_i) \right\} dx + \sum_{i=0}^{n} \zeta_{P_n}^{(p)}(-x_i, 0),$$

where

$$\zeta_{H}(a, s) = \sum_{k=0}^{\infty} k^p \frac{k^p}{(k + a)^s}.$$ 

Writing

$$k^p = [(k + a) - a]^p = \sum_{l=0}^{p} \binom{p}{l} (k + a)^l (-a)^{p-l},$$

we see that

$$\zeta_{H}(a, s) = \sum_{l=0}^{p} \binom{p}{l} (-a)^{p-l} \zeta_{H}(a, s - l)$$

where

$$\zeta_{H}(a, s) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}$$

is the classical Hurwitz zeta function [WW]. From Lemma 2 and (2.11) we finally get

$$\zeta_{P_n}^{(p)}(0) = -\zeta_{P_n}^{(p)}(0) \log a_n + \frac{1}{2n} \sum_{l=1}^{p} \left[ \frac{1}{l(p + 1 - l)} \sum_{i=1}^{n} x_i^l \left( \sum_{j=1}^{n} x_j^{p-l+1} \right) \right] - \frac{H_p}{p+1} \sum_{i=1}^{n} x_i^{p+1}$$

$$+ \sum_{i=1}^{n} \sum_{l=0}^{p} \binom{p}{l} x_i^{-l} \zeta_{H}(-x_i, -l).$$

(2.12)

For example, for $p = 0$ and $p = 1$ we find

$$\zeta_{P_n}^{(0)}(0) = \zeta_{P_n}^{(0)}(0) = -\left( \frac{1}{2} - \frac{1}{n} \right) \log a_n + \log \prod_{i=1}^{n} \Gamma(-x_i) \frac{1}{(2\pi)^{\frac{p}{2}}},$$

(2.13)

$$\zeta_{P_n}^{(1)}(0) = -\left( \frac{1}{2} a_{n-1}^2 - \frac{a_{n-2}}{a_n} - \frac{1}{12} \right) \log a_n + \frac{1}{2} \left[ a_{n-1}^2 \left( \frac{1}{n} - 1 \right) + \frac{a_{n-2}}{a_n} \right]$$

$$+ \sum_{i=1}^{n} \left( x_i \log \Gamma(-x_i) + \zeta_{H}(-x_i, -1) \right),$$

(2.14)

where we used the identity

$$\zeta_{H}(a, 0) = \log \frac{\Gamma(a)}{\sqrt{2\pi}}.$$
24. Remark. We determined formulas (2.7), (2.9) and (2.12) with the assumption that all zeros of the polynomial have negative real part. However, it is easy to see that the same formulas hold true for the general case, the only assumption being $x_i \notin \mathbb{N}$ for any $i = 1, \ldots, n$. To prove this it suffices to note that

$$\zeta^{(p)}_{P_n}(s) = \sum_{k=0}^{N-1} \frac{k^p}{P_n(k)^s} + \sum_{k=0}^{\infty} \frac{(k + N)^p}{P_n(k)^s} = \sum_{k=0}^{N-1} \frac{k^p}{P_n(k)^s} + \sum_{l=0}^{p} \binom{p}{l} N^{p-l} \zeta^{(l)}_{P_n(N)}(s) \quad (2.15)$$

where $P_n^{(N)}(x) := P_n(x + N)$ has zeros in $x_i - N$. If we choose $N > \max_{i=1, \ldots, n}(\Re x_i)$, then we can differentiate (2.15) and use our previous results to $\zeta^{(l)}_{P_n(N)}(s)$ to show that they hold true for $\zeta^{(p)}_{P_n}(s)$ also.

3. Final Remarks

We studied the polynomial zeta function $\zeta^{(p)}_{P_n}$ limiting ourself to the problem of compute its value and the value of its first derivative at $s = 0$. This is because our interest in physical applications. For example, our results can be used to compute the Dirac operator on a quaternionic projective space $\mathbb{H}P^n$. Its eigenvalues are

$$\pm \sqrt{\lambda_{m,k}}, \quad \pm \sqrt{\mu_{m,k}}$$

with

$$\lambda_{m,k} = 4m^2 + 4m(2n + k) + 4n + k + k - 4(1 + k), \quad k \in \{1, \ldots, n - 1\}, \quad m \in \mathbb{N}$$

$$\mu_{m,k} = \lambda_{m,k+1} + 8(k + 1), \quad k \in \{1, \ldots, n - 1\}, \quad m \in \mathbb{N} \quad \text{if} \quad k = n$$

The logarithm of the determinant of the Dirac operator is

$$\zeta'(0) = \frac{1}{2} \zeta''(0) - \frac{1}{4} \zeta(0) \log(-1).$$

Thus we can apply our results to get

$$\det(\theta)_{\mathbb{H}P^n} = e^{\zeta'(0)} = (-1)^{\frac{2}{3}(5n+3)} \frac{\pi^{2n-1}}{2^{6n^2-2n-2}(n-1)![(n-1)!](n+1)!} \prod_{k=1}^{n-1} (n + k - 1)!(n + k + 1)!.$$

Another very interesting physical application is to the problem of describing the discharging of a charged black hole. In the case when this is described by a Nariai solution, both the Dirac and the Klein-Gordon equations can be explicitly solved and the effective action describing the Schwinger pair production can be computed exactly. This phenomenon will be described in details elsewhere [BCDa], [BCDa], so that we only note that, in that case, it arises the problem to compute the derivative in $s = 0$ of multiple sums of the form (see also [E], [SZb])

$$\zeta^{(m)}_{P_n}(s) = \sum_{m \in \mathbb{Z}^n} \frac{1}{P_n(m_1 + \ldots + m_m)^s} = \sum_{k=0}^{\infty} \frac{d^{(m)}_k}{P_n(k)^s}, \quad (3.1)$$

where $d^{(m)}_k$ is the cardinality of the set of partitions $k = m_1 + \ldots + m_m$ of $k$. Now, using the recurrence relation $d^{(m)}_k = \sum_{l=0}^{n} d^{(m-1)}_l$ one can easily expand $d^{(m)}_k$ as a polynomial

$$d^{(m)}_k = \sum_{l=0}^{m} c_l k^l.$$

For example

$$d^{(2)}_k = k + 1, \quad d^{(3)}_k = \frac{1}{2} k^2 + \frac{3}{2} k + 1, \quad d^{(4)}_k = \frac{1}{6} k^3 + k^2 + \frac{11}{6} k + 1.$$

Then

$$\zeta^{(m)}_{P_n}(0) = \sum_{l=0}^{m} c_l \zeta^{(l)}_{P_n}(0). \quad (3.2)$$
Finally, going beyond the physical applications, we think that the polynomial zeta function is interesting by itself so that its properties deserve a deeper investigation [SZa], [SZc].

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