Weighted Grassmannians

Alessio Corti Miles Reid

In memory of Paolo Francia

Abstract

Many classes of projective algebraic varieties can be studied in terms of graded rings. Gorenstein graded rings in small codimension have been studied recently from an algebraic point of view, but the geometric meaning of the resulting structures is still relatively poorly understood. We discuss here the weighted projective analogs of homogeneous spaces such as the Grassmannian $\text{Gr}(2,5)$ and orthogonal Grassmannian $\text{OGr}(5,10)$ appearing in Mukai’s linear section theorem for Fano 3-folds, and show how to use these as ambient spaces for weighted projective constructions. This is a first sketch of a subject that we expect to have many interesting future applications.

1 Introduction

We are interested in describing algebraic varieties explicitly in terms of graded rings and, conversely, in algebraic varieties for which such an explicit description is possible. Our varieties $X$ always come with a polarisation $A$, usually the canonical class or an integer submultiple of it. Our favourites include the following:

(1) canonical curves, K3 surfaces, Fano 3-folds. A Fano 3-fold $V$ is canonically polarised by its anticanonical class $A = -K_V$. We consider K3 surfaces with Du Val singularities polarised by a Weil divisor. A canonical curve $C$ is a curve of genus $g \geq 2$ with its canonical polarisation by $K_C$; but we will more often be concerned with subcanonical

*For the proceedings of Paolo Francia memorial conference, Genova, Sep 2001, edited by Mauro Beltrametti, to appear de Gruyter 2002
curves, polarised by a divisor $A$ that is a submultiple of $K_C = kA$, and variants on orbifold curves also occur naturally (see Altmok, Brown and Reid [ABR]).

(2) Regular canonical surfaces, Calabi–Yau 3-folds.

(3) Regular canonical 3-folds.

If $X, O(1)$ is a polarised $n$-fold with $K_X = O(k)$, Mukai defines the coindex of $X$ to be $n + 1 + k$; the above are varieties of coindex 3, 4 and 5.

**Remark 1.1.** (a) Describing a variety explicitly means embedding it into a suitable ambient space and writing down its equations. This is closely related to the problem of finding generators and relations for the graded ring

$$R(X, A) = \bigoplus_{n=0}^{\infty} H^0(X, nA).$$

In all the examples we consider, this is a Gorenstein ring; this property is one of the most powerful general tools we have in studying $X$ and its deformations. It seems to us that this point is not adequately appreciated.

(b) Varieties often come in ladders of successive hyperplane sections. For example, in good situations, a general elephant $S \in |-K_V|$ on a non-singular Fano 3-fold $V$ is a K3 surface polarised by $A = -K_V|_S$, and a general $C \in |A|_S$ a canonical curve. Finding the equations of $V$ is closely related to finding the equations of $S$ or $C$, and often practically equivalent to it.

(c) The natural context to study Fano 3-folds is the Mori category of projective varieties with terminal singularities. The key examples of these are cyclic quotient singularities

$$\frac{1}{r}(1, a, -a) = \mathbb{C}^3/(\mathbb{Z}/r\mathbb{Z}),$$

where the notation signifies that the cyclic group $\mathbb{Z}/r\mathbb{Z}$ acts diagonally with weights $1, a, -a$, and $\text{hcf}(a, r) = 1$. We are thus led to consider K3 surfaces with singularities $\frac{1}{r}(a, -a)$ polarised by ample Weil divisors, and, one further step down the ladder, *orbifold* canonical curves; compare [ABR].
Our original motivation is Mukai’s description of a prime Gorenstein Fano 3-fold of genus $7 \leq g \leq 10$ as a linear section of a special projective homogeneous space, that is, the quotient $G/P$ of a (semisimple) Lie group $G$ by a maximal parabolic subgroup $P$. For example, consider $V = \mathbb{C}^{2n}$ endowed with a complex symmetric quadratic form $q$; it is traditional to take

$$q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \text{where } I = I_{n \times n},$$

so that $V = U \oplus U^\vee$ where $U = \langle e_1, \ldots, e_n \rangle$. It is well known that the space of isotropic $n$-dimensional vector subspaces of $V$ splits into two components for reasons of “spin”. The O’Grassmann or orthogonal Grassmann variety $\text{OGr}(n, 2n)$ is one connected component; we take the component containing the reference subspace $U$. It is a homogeneous space for the group $G = \text{SO}(2n, \mathbb{C})$, and has a natural Plücker style spinor embedding in the projective space $\mathbb{P}(S^+) = \mathbb{P}^{15}$ of the spinor representation $S^+ = \wedge^{\text{even}} U$. Mukai proves the following result:

**Theorem 1.2 (Mukai [Mu]).** A prime Gorenstein Fano 3-fold of genus 7 is a linear section of $\text{OGr}(5, 10)$ in its spinor embedding. In other words, there are 7 hyperplanes $H_1, \ldots, H_7$ of $\mathbb{P}(S^+) = \mathbb{P}^{15}$ such that

$$(V_{12} \subset \mathbb{P}^8) = \text{OGr}(5, 10) \cap H_1 \cap \cdots \cap H_7.$$

We wanted to see how far these ideas of Mukai generalise. In this note, we define weighted Grassmann and orthogonal Grassmann varieties, and study some examples of their linear sections.

**Acknowledgements**

The treatment of weighted projective homogeneous spaces in Section 3 is based on notes of Ian Grojnowski [G], whom we had hoped to involve as coauthor. We have also benefitted from discussion with Jorge Neves, whose forthcoming paper [N] takes these ideas further. Gavin Brown and Roberto Pignatelli helped us with computer algebra calculations.

**2 Weighted $\text{Gr}(2, 5)$**

**The affine Grassmannian $\text{aGr}(2, 5)$**

Weighted versions of a projective homogeneous variety $\Sigma$ arise on dividing out the affine cone over $\Sigma$ by different $\mathbb{C}^\times$ actions. The construction is
particularly transparent for $\text{Gr}(2,5)$. Set $V = \mathbb{C}^5$; the affine Grassmann variety

$$a\text{Gr}(2,5) \subset \bigwedge^2 V$$

can be defined in any of the following equivalent ways:

(i) The variety of skew tensors of rank $\leq 2$, that is, the image of $V \times V \to \bigwedge^2 V$ given by $(a, b) \mapsto a \wedge b$. In coordinates, we write

$$\bigwedge^2 \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix} = \begin{pmatrix} c_{12} & c_{13} & c_{14} & c_{15} \\ c_{23} & c_{24} & c_{25} & \vdots \\ c_{34} & c_{35} & \vdots & \ddots \\ c_{45} & \vdots & \ddots & \ddots \end{pmatrix}, \quad (2.1)$$

where $c_{ij} = \det \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$. Our convention is to write out only the upper diagonal entries of the $5 \times 5$ skew matrix $(c_{ij})$.

(ii) The closed orbit of the highest weight vector $e_{12} = (1, 0, \ldots) \in \bigwedge^2 V$ under the action of $\text{GL}(5) = \text{GL}(V)$. In other words, any tensor of rank 2 is in the $\text{GL}(5)$ orbit of $(1, 0, 0, 0) \wedge (0, 1, 0, 0, 0)$.

(iii) The quotient of the variety $M(2,5)$ of $2 \times 5$ matrices by $\text{SL}(2)$ acting on the left: indeed, the ring of invariant functions is generated by the $2 \times 2$ minors $c_{ij} = \det \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$.

(iv) The variety defined by the $4 \times 4$ Pfaffians of the generic $5 \times 5$ skew matrix, that is, the Plücker equations

$$\text{Pf}_{ijkl} = x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk} = 0,$$

where $x_{ij}$ for $1 \leq i < j \leq 5$ are coordinates on $\bigwedge^2 V$. The point is just that setting the Pfaffians of a skew matrix $(x_{ij})$ equal to zero enforces rank $\leq 2$.

(v) In other words, $a\text{Gr}(2,5)$ has affine coordinate ring

$$R = \mathbb{C}[a\text{Gr}(2,5)] = \mathbb{C}[\langle x_{ij} \rangle]/I, \quad (2.2)$$

where $I$ is the ideal $I = \langle \text{Pf}_1, \ldots, \text{Pf}_5 \rangle$, and $a\text{Gr}(2,5) = \text{Spec} R$. 


Equivariant resolution

As a prelude to introducing weights and defining $w\text{Gr}$, it is convenient to explain the symmetry group of $a\text{Gr}(2,5) \subset \bigwedge^2 V$ and to write its equations and syzygies in their full symmetry. Under the induced action of $\text{GL}(V)$ on $\bigwedge^2 V$, the scalar matrices $\lambda I$ act by $\lambda^2$. However, the straight Grassmannian $\text{Gr}(2,5) \subset \mathbb{P}^9$ is the quotient of $a\text{Gr}(2,5)$ by $\mathbb{C}^\times$ acting on $\bigwedge^2 V$ by overall scalar multiplication by $\mu \in \mathbb{C}^\times$, and this is not covered by the $\text{GL}(5)$ action; the full symmetry group is thus a double cover of $\text{GL}(V)$ (an index 2 central extension). Rather than introducing notation for the double cover, we write $L = \mathbb{C}$ with the usual action of $\mathbb{C}^\times$, and view the Plücker embedding as $a\text{Gr}(2,5) \hookrightarrow \bigwedge^2 V \otimes L$, where $\text{GL}(5)$ acts on the first factor and $\mathbb{C}^\times$ on the second. We also write

$$D = \text{det} V \otimes L^2 = \bigwedge^5 V \otimes L^2, \quad (2.3)$$

a 1-dimensional representation of $\text{GL}(V) \times \mathbb{C}^\times$. It is useful to bear in mind the straight homogeneous case, when $D$ pushed forward to $\mathbb{P}^9$ corresponds to $O(-2)$, and $L$ to $O(-1)$.

**Proposition 2.1.** There are universal maps of vector bundles over the affine space $\mathbb{A}^{10} = \bigwedge^2 V \otimes L$

$$M: \bigwedge^2 V \otimes L \to \mathbb{C} \quad (\text{that is, } V \otimes L \to V^\vee)$$

and

$$\text{Pf} = \text{Pf} M: V^\vee \otimes D = \bigwedge^4 V \otimes L^2 \to \mathbb{C}. \quad (2.4)$$

Note that interpreted intrinsically, $\text{Pf}$ is the second wedge of $M: V \otimes L \to V^\vee$.

Now write $\mathcal{O}$ for the structure sheaf of $\mathbb{A}^{10}$, and $M: \mathcal{O} \otimes \bigwedge^2 V \otimes L \to \mathcal{O}$, etc., for the above universal maps viewed as sheaf homomorphisms. Then the structure sheaf $\mathcal{O}_{a\text{Gr}}$ of $a\text{Gr}(2,5)$ has a $\text{GL}(5) \times \mathbb{C}^\times$ equivariant projective resolution of the form

$$0 \leftarrow \mathcal{O} \leftarrow \mathcal{O} \otimes V^\vee \otimes D \leftarrow M \mathcal{O} \otimes V \otimes L \otimes D \leftarrow \text{Pf} \mathcal{O} \otimes L \otimes D^2 \leftarrow 0$$

$$\downarrow \mathcal{O}_{a\text{Gr}}.$$
Proof In coordinates \( x_{ij} \) on \( \bigwedge^2 V \otimes L \), the map \( M \) is the generic \( 5 \times 5 \) skew matrix \( (x_{ij}) \) and \( \text{Pf} = (\text{Pf}_1, \ldots, \text{Pf}_5) \) its vector of Pfaffians. Thus (2.4) follows at once from the well known fact that the ideal of \( \text{aGr}(2,5) \) is generated by the 5 Pfaffians, and \( M \) is the matrix of syzygies between them.

Remark 2.2. Each term in (2.4) is a \( G \)-equivariant bundle, where \( G = \text{GL}(5) \times \mathbb{C}^\times \), and the complex gives the projective resolution of \( \mathcal{O}_{\text{aGr}} \) in terms of \( G \)-equivariant vector bundles on the ambient space \( \bigwedge^2 V \otimes L \). We write out the definitions for completeness.

Let \( G \) be a group and \( Y \) a space with a left \( G \)-action \( G \times Y \to Y \); write \( l_g: Y \to Y \) for the action of \( g \in G \). A \( G \)-equivariant sheaf is a sheaf \( F \) on \( Y \), together with isomorphisms \( \alpha_g: F \to l_g^* F \) satisfying \( \alpha_{g_2g_1} = l_{g_2}^* (\alpha_{g_1}) \circ \alpha_{g_2} \). (2.5)

The collection of maps \( \{\alpha_g\} \) is called a \( G \)-linearisation or descent data for \( F \); the cocycle condition in (2.5) ensures that \( G \) acts on the pushforward \( \pi_* F \), where \( \pi: Y \to X = Y/G \) is the quotient morphism. A quasicoherent sheaf \( F \) over an affine scheme \( Y = \text{Spec} A \) is the associated sheaf \( F = \tilde{\mathcal{P}} \) for an \( A \)-module \( F = \Gamma(Y, F) \); a \( G \)-equivariant sheaf arises in the same way from a module \( F \) over the twisted group ring \( A \ast G \). That is, \( F \) is an \( A \)-module with a representation of \( G \) such that \( g(am) = g(a)g(m) \) for all \( a \in A \) and \( m \in F \), where \( G \) has the left action on \( A \) by \( g(a) = l_g^g(a) \).

If \( G \) acts freely with quotient \( X = Y/G \), taking pushforward and invariant sections identifies a \( G \)-equivariant sheaf with a sheaf on \( X \); if \( G \) has fixed points, the same construction only gives an orbisheaf (or a sheaf on the quotient stack \( [Y/G] \) of which \( X \) is the coarse moduli space). With a little common sense, we can mostly ignore this point, and pretend that we get a genuine sheaf on the space \( Y/G \).

Remark 2.3. The \( \bigwedge^2 V \otimes L \) occurring here tells us how to define \( \text{Gr}(2,5) \)-bundles over an arbitrary base scheme \( S \), more or less as for conic bundles: choose a rank 5 vector bundle \( V \), a line bundle \( \mathcal{L} \) and a morphism \( \mu: \bigwedge^2 V \otimes \mathcal{L} \to \mathcal{O}_S \), and take the locus \( \mu \leq 2 \) defined by the relative equations \( \text{Pf}(\mu) = \bigwedge^2 \mu = 0 \). If \( \mathcal{L} \) is a square, say \( \mathcal{L} = \mathcal{L}_0^2 \), we can get rid of it by replacing \( V \mapsto V \otimes \mathcal{L}_0 \).

The definition of \( \text{wGr}(2,5) \)

Choosing weights on \( \text{aGr}(2,5) \) is equivalent to specifying a 1-parameter subgroup \( \mathbb{C}^\times \hookrightarrow \text{GL}(5) \times \mathbb{C}^\times \). Up to conjugacy, we can choose it in the maximal
torus, that is, diagonal of the form
\begin{equation*}
\left( \text{diag}(\lambda^{w_1}, \lambda^{w_2}, \lambda^{w_3}, \lambda^{w_4}, \lambda^{w_5}); \lambda^u \right) \subset \GL(5) \times \C^\times.
\end{equation*}

In order to put weights on aGr(2, 5) \subset \wedge^2 V, we thus specify integer weights \((w_1, \ldots, w_5)\) on \(V\), and a separate overall weight \(u\) on \(\wedge^2 V\). The ambient space \(\wedge^2 V\) thus has coordinates
\[x_{ij} \quad \text{with} \quad \wt x_{ij} = w_i + w_j + u.\]

Replacing \(w_i \mapsto w_i - \left\lfloor \frac{u}{2} \right\rfloor\), we can always take \(u = 0\) or \(1\). In fact, for brevity in calculations, we usually use the trick of absorbing the weight \(u\) into the \(w_i\) by \(w_i \mapsto w_i + u\), at the cost of working with half-integers \(w_i\). For odd \(u\) this is formally incorrect, but completely harmless, and hardly ever leads to confusion.

**Definition 2.4.** Let \(w = (w_1, \ldots, w_5)\) and \(u\) be weights such that \(w_i + w_j + u > 0\) for all \(i, j\). We define
\[\wGr(2, 5) = \left( \text{aGr}(2, 5) \setminus 0 \right) / \C^\times,\]
where \(\C^\times\) acts on aGr(2, 5) \(\subset \wedge^2 V\) by \(x_{ij} \mapsto \lambda^{w_i + w_j + u} x_{ij}\). Clearly
\[\wGr(2, 5) = \text{Proj} R\]
where \(R = \C[aGr(2, 5)]\) is the affine coordinate ring as in (v) above, graded by \(\wt x_{ij} = w_i + w_j + u\). By definition, \(\wGr(2, 5)\) comes with a Plücker embedding in weighted projective space (wps) \(\P^9(\{w_i + w_j + u\})\), and is defined by the usual Plücker equations, the \(4 \times 4\) Pfaffians of the generic \(5 \times 5\) skew matrix \((x_{ij})\).

The elementary properties of \(\wGr(2, 5)\) are easy enough to figure out. We get affine charts by setting \(x_{ij} \neq 0\), where (say)
\[x_{12} = \det \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.\]

This chart is the quotient \(\C^6/(\Z/\wt x_{12})\) of \(\C^6\) by the cyclic group of order \(\wt x_{12} = w_1 + w_2 + u\) acting on coordinates \(b_3, b_4, a_3, a_4, a_5\) with weights
\[
\begin{align*}
w_1 + w_3 + u, & \quad w_1 + w_4 + u, & \quad w_1 + w_5 + u, \\
w_2 + w_3 + u, & \quad w_2 + w_4 + u, & \quad w_2 + w_5 + u.
\end{align*}
\]
This formula shows the point of our shorthand setting $u = 0$, allowing the $w_i$ to be half-integers. As with weighted projective spaces, we usually impose “well formed” conditions to ensure that the cyclic group acts effectively and without ramification in codimension 1. We omit the details, but compare [F], Definition 6.9.

The Hilbert numerator of $w\text{Gr}(2, 5) \subset \mathbb{P}(\{w_i + w_j + u\})$ is

$$\prod_{i,j}(1 - t^{w_i + w_j + u})P(t) = 1 - \sum_{i=1}^{5} t^{d - w_i} + \sum_{j=1}^{5} t^{d + w_j + u} - t^{2d + u},$$

where $d = \sum w_i + 2u$. This formula is a numerical version of (2.4), and essentially equivalent to it by the splitting principle. Multiplying by $(1 - t)^3$, we deduce that

$$\deg w\text{Gr} = \frac{\sum (d - w_i^3) - \sum (d + w_j + u)}{\prod (w_i + w_j + u)}.$$ 

If $w\text{Gr}(2, 5)$ is well formed, its canonical class is $K_{w\text{Gr}(2, 5)} = \mathcal{O}(-2d - u)$. In fact the wps has $K = -\det(\bigwedge^2 V \otimes L)$, which has degree

$$-\sum \text{wt} x_{ij} = -4 \sum w_i - 10u = -4d - 2u,$$

and $w\text{Gr}(2, 5) \subset \mathbb{P}(\bigwedge^2 V \otimes L)$ has the adjunction number $\deg(L \otimes D^2) = 2d + u$ by (2.4).

**Tautological sequences**

Tautological vector bundles over $a\text{Gr}(2, 5)$ can be discussed in several ways, parallel to the different treatments of $a\text{Gr}(2, 5)$. Taking invariants of the $\mathbb{C}^\times$ action gives rise to tautological (orbi-)bundles on $w\text{Gr}(2, 5)$, as in the case of the straight Grassmannian. These sheaves on $w\text{Gr}(2, 5)$ can also be understood in terms of the well known Serre correspondence

$$\mathcal{E} \mapsto E_\ast = \bigoplus_{k \geq 0} H^0(\mathcal{E}(k)).$$

We describe the Serre module of the tautological bundles explicitly as modules over the affine coordinate ring of $a\text{Gr}$.

First, $a\text{Gr}$ is locally a codimension 3 complete intersection wherever the matrix of syzygies $M$ has rank 2, that is, at every point of $a\text{Gr} \setminus 0$. At any
such point, we can use two rows of $M$ to express 2 of the 5 Pfaffians as linear combinations of the others, so that the ideal sheaf $\mathcal{I}_{\text{aGr}}$ is locally generated by 3 Pfaffians. Thus the conormal sheaf to $\text{aGr}(2,5)$ is a vector bundle of rank 3 outside the origin with 5 sections. In more detail, consider (2.4) as a resolution of the ideal sheaf $\mathcal{I}_{\text{aGr}}$:

$$0 \leftarrow \mathcal{I}_{\text{aGr}} \xleftarrow{\mathcal{P}} \mathcal{O} \otimes V^\vee \otimes D \xleftarrow{M} \mathcal{O} \otimes V \otimes L \otimes D \leftarrow \cdots.$$ 

Tensoring with $\mathcal{O}_{\text{aGr}} = \mathcal{O}/\mathcal{I}_{\text{aGr}}$ gives the exact sequence

$$0 \leftarrow \mathcal{I}/\mathcal{T} \xleftarrow{\mathcal{P}} \mathcal{O}_{\text{aGr}} \otimes V^\vee \otimes D \xleftarrow{M} \mathcal{O}_{\text{aGr}} \otimes V \otimes L \otimes D \leftarrow \cdots ; \quad (2.6)$$

twisting back by $D^{-1}$ gives a tautological exact sequence

$$0 \leftarrow \mathcal{F} \xleftarrow{\mathcal{P}} \mathcal{O}_{\text{aGr}} \otimes V^\vee \leftarrow \mathcal{E} \otimes L \leftarrow 0 \quad (2.7)$$
of vector bundles over $\text{aGr} \setminus 0$, where $\mathcal{F} = \mathcal{I}/\mathcal{T} \otimes D^{-1}$ and the identification $\ker \text{Pf} = \text{im} M = \mathcal{E} \otimes L$ is justified below.

Next, $M$ has rank 2 at every point of $\text{aGr}(2,5) \setminus 0$, so that if we restrict the sheaf homomorphism $M: \mathcal{O} \otimes V \otimes L \otimes D \rightarrow \mathcal{O} \otimes V^\vee \otimes D$ to $\text{aGr}(2,5)$, this restriction maps onto a $\text{GL}(5) \times \mathbb{C}^\times$ equivariant sheaf over $\text{aGr}(2,5)$ that is a rank 2 vector bundle on $\text{aGr}(2,5) \setminus 0$. We twist it back by $L^{-1}D^{-1}$ for convenience, obtaining a second tautological exact sequence:

$$0 \leftarrow \mathcal{E} \leftarrow \mathcal{O}_{\text{aGr}} \otimes V \leftarrow \mathcal{K} \leftarrow 0. \quad (2.8)$$

Here $\mathcal{E}$ is the same sheaf as in (2.7), up to the indicated twist, because the sequence in (2.6) is exact. By playing with determinant bundles in (2.7) and (2.8) one sees that $\det \mathcal{E} = L = \det \mathcal{F}$ so that $\mathcal{E}^\vee = \mathcal{E} \otimes L$, and then the two sequences are dual, which determines the kernel in (2.8):

$$0 \leftarrow \mathcal{E} \leftarrow \mathcal{O}_{\text{aGr}} \otimes V \leftarrow \mathcal{F}^\vee \leftarrow 0. \quad (2.9)$$

We can concatenate the exact sequences (2.9) and (2.7) to obtain the following explicit description of the module $E_s = H^0(\text{aGr}(2,5), \mathcal{E})$ over the affine coordinate ring $R = \mathbb{C}[(x_{ij})]/I = \mathbb{C}[\text{aGr}(2,5)]$: it is generated by 5 sections $s_1, \ldots, s_5$ that one identifies either with the columns of $M$, or with the 5 columns $s_i = \binom{a_i}{b_i}$ subject to the 10 relations

$$x_{ij} s_k - x_{ik} s_j + x_{jk} s_i \quad \text{for } 1 \leq i < j < k \leq 5. \quad (2.10)$$
We can say the same thing in invariant terms by taking global sections in the exact sequence

\[ 0 \to E \to O_{aGr} \otimes V \to O_{aGr} \otimes \bigwedge^2 V^\vee. \]

Our 4th and final treatment of the bundle \( E \) is intrinsic and starts from the model \((aGr(2,5) \setminus 0) = M(2,5)^*/\mathrm{SL}(2)\). Consider the given representation of \( \mathrm{SL}(2) \) on \( \mathbb{C}^2 \) and the diagonal action of \( \mathrm{SL}(2) \) on the trivial bundle \( M(2,5) \times \mathbb{C}^2 \); the quotient is the total space of a rank 2 vector bundle \( E \) on \( aGr(2,5) \setminus 0 \). The sections of \( E \) are functions \( f: M(2,5)^* \to \mathbb{C}^2 \) that transform as

\[ f(gM) = gf(M) \quad \text{for all} \quad g \in \mathrm{SL}(2) \quad \text{and} \quad M \in M(2,5). \]

We can identify this bundle \( E \) with any of the above constructions: the 5 columns of \( M \) give global sections of \( E \), and they satisfy the same relations as in (2.10), leading to the same presentation of \( E \) by \( \mathrm{GL}(5) \times \mathbb{C}^\times \)-equivariant free sheaves on \( aGr(2,5) \setminus 0 \). The advantage of this construction is that, since it involves the 2-planes parametrised by points of \( aGr(2,5) \), it really relates to the functor represented by \( aGr(2,5) \), and thus to the traditional tautological bundle of a Grassmannian.

**Examples**

*Example 2.5.* Take \( w = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) \); then \( wGr(2,5) \subset \mathbb{P}(1^6, 2^4) \) is given by a skew matrix

\[
M = \begin{pmatrix}
x_{12} & x_{13} & x_{14} & y_1 \\
x_{23} & x_{24} & y_2 \\
x_{34} & y_3 \\
y_4
\end{pmatrix}
\]

with weights \( \begin{pmatrix}
1 & 1 & 1 & 2 \\
1 & 1 & 2 \\
1 & 2 \\
2
\end{pmatrix} \).

This has 5 Pfaffians of degrees 2, 3, 3, 3, 3. The section \( V = wGr(2,5) \cap (2)^3 \) by 3 general forms of weight 2 is a Fano 3-fold with

\[ h^0(V, -K) = 6, \quad -K^3 = 6 + \frac{1}{2}; \]

that is, \( g = 4 \) and \( V \) has in general a singular point \( \frac{1}{2}(1,1,1) \); the corresponding family of K3 surfaces is No. 2 in Altinok’s list, Altinok3(2) in the Magma database.
Because the 3 equations of degree 2 are general, they involve 3 of the weight 2 coordinates \( y_1, y_2, y_3 \) with nonzero coefficients, and we can use them to eliminate \( y_i \) as generators. Thus we say that \( V \) is a quasilinear section of \( \text{wGr}(2,5) \). By analogy with Mukai’s results, we want to call this a linear section theorem, but we keep the “quasi” for the moment to keep away the unclean spirit.

The section \( S = \text{wGr}(2,5) \cap (2)^4 \) by 4 forms of weight 2 has been studied in detail by Neves [N]; we can use the 4 equations to write \( y_i = q_i(x) \) for \( i = 1, \ldots, 4 \), giving a canonical surface \( S \subset \mathbb{P}^5 \) with \( p_g = 6, K^2 = 13 \) defined by the Pfaffians of

\[
\begin{pmatrix}
  l_{12} & l_{13} & l_{14} & q_1 \\
  l_{23} & l_{24} & q_2 \\
  l_{34} & q_3 \\
  q_4 \\
\end{pmatrix}
\]

with \( l_{ij} \) linear and \( q_i \) quadratic forms on \( \mathbb{P}(1^6, 2^4) \). Conversely (and slightly more generally), Neves [N] shows that a surface \( S \) with \( p_g = 6, K^2 = 13 \) satisfying appropriate generality assumptions has a nongeneral canonical curve \( C \in |K_S| \) for which the restricted linear system splits as \( |K_S|_C = g^1_6 + g^1_7 \). Following Mukai’s strategy, Neves shows how to derive the “tautological” rank 2 vector bundle \( E \) over \( S \) and the embedding of \( S \) into \( \text{wGr}(2,5) \) or a cone over it from this Brill–Noether data on \( C \). The linear entries \( l_{ij} \) of \( M \) may be linearly dependent, corresponding to a model of \( S \) as a section of a cone over \( \text{wGr}(2,5) \).

**Example 2.6.** Taking \( w = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}) \) gives \( \text{wGr}(2,5) \subset \mathbb{P}(1^3, 2^6, 3) \) defined by a matrix with weights

\[
\begin{pmatrix}
  1 & 1 & 2 & 2 \\
  1 & 2 & 2 \\
  2 & 2 \\
  3 \\
\end{pmatrix},
\]

having Pfaffians of degrees 3, 3, 4, 4, 4.

(a) Write \( \mathcal{C} \text{wGr}(2,5) \subset \mathbb{P}(1^4, 2^6, 3) \) for the projective cone over \( \text{wGr}(2,5) \); this means that we add one extra variable of degree 1 to the homogeneous coordinate ring, not involved in any relation. Then a general quasilinear section \( S = \mathcal{C} \text{wGr}(2,5) \cap (2)^5 \) of the cone by 5 general forms of degree 2 is a K3 surface with ample Weil divisor \( D \) satisfying

\[
h^0(S, D) = 4, \quad D^2 = 4 + \frac{2}{3}
\]
that is, \( g = 3 \) and \( S \) has a singular point \( \frac{1}{3}(1,2) \). This family of K3 surfaces is Altinok3(3).

The new phenomenon in this example is that \( S \) has \( h^0(S,D) = 4 \), so that the graded ring \( R(S,D) \) has 4 generators \( x_1, \ldots, x_4 \) of degree 1. On the other hand, the matrix only has 3 entries of weight 1, so that not all the \( x_i \) can appear as degree 1 terms. Thus \( S \) is obtained from the cone over \( w\text{Gr}(2,5) \).

(b) A general quasilinear section \( S = w\text{Gr}(2,5) \cap (2)^3 \cap (3) \) is a K3 with ample \( D \) such that

\[
h^0(S,D) = 3 \quad \text{and} \quad D^2 = 2 + 3 \times \frac{1}{2}
\]

that is, \( g = 2 \) and \( S \) has 3 singular points \( \frac{1}{2}(1,1) \); this is Altinok3(5).

The following is due to Selma Altınok.

**Theorem 2.7 (Altınok [Al]).** There are precisely 69 families of K3 surfaces with cyclic singularities \( \frac{1}{r}(a,-a) \) whose general element is a codimension 3 subvariety in weighted projective space given by the \( 4 \times 4 \) Pfaffians of a skew \( 5 \times 5 \) matrix.

The next result is a nice structural description of these surfaces; unfortunately, we don’t know how to prove it in an entirely conceptual way.

**Proposition 2.8.** All K3 surfaces of Altınok are quasilinear sections of a weighted Grassmannian \( w\text{Gr}(2,5) \) or a cone over \( w\text{Gr}(2,5) \).

**Proof** Ultimately, this is based on a case by case check against Altınok’s list. By Buchsbaum–Eisenbud, \( S \subset \mathbb{P}(a_1, \ldots, a_6) \) is defined (scheme theoretically) by the \( 4 \times 4 \) Pfaffians of a \( 5 \times 5 \) skew matrix

\[
\begin{pmatrix}
f_{12} & f_{13} & f_{14} & f_{15} \\
f_{23} & f_{24} & f_{25} \\
f_{34} & f_{35} \\
f_{45}
\end{pmatrix}
\]

where the entry \( f_{ij}(y_1, \ldots, y_6) \) is a weighted homogeneous form of degree \( d_{ij} \) in the coordinates \( y_i \) (the condition for \( S \) to be a K3 implies that, if \( b_i = \deg Pf_i \) is the degree of the \( i \)th Pfaffian, then \( \sum b_i = 2 \sum a_i \)). Now an easy combinatorial argument shows that the 5 Pfaffians are weighted homogeneous, if and only if \( d_{ij} = w_i + w_j \) for some \( w_i, i = 1, \ldots, 5 \).
idea is to map $S$ to the weighted Grassmannian $w\text{Gr}(2,5)$ with weights $w = (w_1, \ldots, w_5)$, immersed in $\mathbb{P}(x_{ij})$, by setting

$$f_{ij} = x_{ij}$$

and check that this is an embedding and maps to a quasilinear section. As far as we can tell, these must be checked explicitly on each of the 69 families of Altınok. Intuitively, the key point is that, for $S$ to be a K3, the degrees $f_{ij}$ must be “small”. Slightly more precisely, the formula $\sum b_i = 2 \sum a_i$ implies in practice that many of the $d_{ij}$ equal an $a_i$, which is to say that $f_{ij}$ is linear in one of the variables.

Remark 2.9. If $S \subset w\text{Gr}(2,5)$ is a K3 quasilinear section, it is tempting to try to reconstruct the embedding from intrinsic data on $S$, by analogy with Mukai’s constructions and Neves [N]. It is easy to see that $E|_S$ is a rigid simple vector bundle, hence stable and uniquely characterised by its Chern classes and local nature at the singularities. We know many ad hoc constructions but no unified way to produce the bundle directly on $S$, and no a priori reason why it must exist. In the case of an (orbifold) canonical curve $C$, the vector bundles $E|_C$ arising from embeddings in $w\text{Gr}(2,5)$ are often interesting and rather exceptional from the point of view of higher rank Brill–Noether theory.

3 Weighted homogeneous spaces

This section is based on a close reading of part of Ian Grojnowski’s notes [G]. We give the general definition of weighted projective homogeneous spaces under an algebraic group $G$ and describe an explicit atlas of coordinate charts on them. A homogeneous variety that is projective is of course homogeneous under a semisimple group $G$; however, weighted homogeneous spaces always involve central extensions, as we saw with $\text{SL}(5)$, $\text{GL}(5)$ and $\text{GL}(5) \times \mathbb{C}^\times$ in the preceding section. Thus we work from now on with a reductive group $G$.

Notation

Let $G$ be a reductive complex algebraic group. We fix a maximal torus and Borel subgroup $T \subset B \subset G$ and write $\mathfrak{B} = G/B$ for the maximal flag variety. Let $X = \text{Hom}(T, \mathbb{C}^\times)$ be the lattice of weights (or characters), and $Y = \text{Hom}(\mathbb{C}^\times, T)$ the dual lattice of 1-parameter subgroups, with the perfect pairing $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$. 

13
Recall that the roots of $G$ are defined as the weights of $T$ appearing in the adjoint action of $T$ on the Lie algebra $\mathfrak{g}$. We write $\Delta \subset X$ for the set of these. A root $\alpha \in \Delta$ determines an involution of the maximal torus $T$, and hence a reflection $r_\alpha$ of $X$; these reflections generate the Weyl group $W(G)$. The negative roots $-\Delta^+$ are the roots appearing in $b/t$. Let $S \subset \Delta^+$ be the set of simple roots.

**Projective homogeneous spaces and parabolic subgroups**

A projective homogeneous space under $G$ is a quotient space $\Sigma = G/P$ by a parabolic subgroup $P$. Every such is conjugate to a standard parabolic subgroup, that is, one containing $B$. A standard parabolic subgroup $P$ corresponds to the subset of simple roots

$$I = \{ \alpha \in S \mid r_\alpha(B) \subset P \} \subset S.$$

We recover $P$ as follows: let $W_I \subset W(G)$ be the subgroup generated by $r_\alpha$ for $\alpha \in I$; then $P = P_I = BW_IB$. We write $\Sigma_I = G/P_I$ for the corresponding projective homogeneous space (or generalised flag variety).

**Dominant weights**

A weight $\chi \in X$ extends to a unique character $B \to \mathbb{C}^*$, and hence gives rise to a line bundle $\mathcal{O}(\chi)$ on $\mathfrak{B}$. A weight $\chi$ is dominant if $V_\chi = H^0(\mathfrak{B}, \mathcal{O}(\chi)) \neq 0$; thus the cone $X^+$ of dominant weights is the effective cone of $\mathfrak{B}$. If $\chi$ is a dominant weight, $V_\chi$ is an irreducible representation of $G$ with highest weight vector $v_\chi$. The linear system $|V_\chi|$ is free, and defines an equivariant morphism $G \to \mathbb{P}(V_\chi)$ whose image $\Sigma = G \cdot v_\chi$ is the orbit of the highest weight line $\mathbb{C}v_\chi$. Therefore $\Sigma = G/P$ is a projective homogeneous space, where $P = \text{Stab}(\mathbb{C}v_\chi)$ is a parabolic subgroup. Then $P = P_I$ as above and $\Sigma = \Sigma_I \subset \mathbb{P}(V_\chi)$ is a generalised Plücker embedding.

**Definition of weighted homogeneous spaces**

Let $\rho \in Y = \text{Hom}(\mathbb{C}^*, G)$ be a 1-parameter subgroup, and $u \geq 0$ an overall weight. We use $\rho$ and $u$ to make $V_\chi$ into a representation of $G \times \mathbb{C}^*$, with the second factor acting by

$$\lambda : v \mapsto \lambda^u \rho(\lambda) \cdot v.$$

We assume from now on that this action has only positive weights.
Remark 3.1. This never happens if, say, $G$ is semisimple and $u = 0$. However, we can always make it happen by taking $u$ large enough; more precisely, the weights are all positive if and only if

$$N_w = \langle \chi, wp \rangle + u > 0 \quad \text{for every } w \in W(G).$$

Here we assume that this condition is satisfied.

Then the quotient $\mathbb{P}(V_{\chi})(\rho, u) = (V_{\chi} \setminus 0)/\mathbb{C}^\times$ is a weighted projective space. From the description $a\Sigma_I \subset V_{\chi} = G \cdot v_{\chi}$, we see that $a\Sigma_I$ is invariant under the $\mathbb{C}^\times$-action.

Definition 3.2. The weighted homogeneous variety associated to this data is the quotient

$$w\Sigma_I = \left(a\Sigma_I \setminus 0\right)/\mathbb{C}^\times \subset \mathbb{P}(V_{\chi})(\rho, u).$$

To stress the choices of the data $\chi, \rho, u$, we write $w\Sigma_I = w\Sigma_I(\chi, \rho, u)$.

Lemma 3.3. We have $w\Sigma_I = w\Sigma_I(\chi, \rho, u) = w\Sigma_I(\chi, wp, u)$ for all $w \in W(G)$.

Proof Almost obvious, but see the explicit coordinatisation given below. \qed

Coordinate charts

We write down explicit $T$-invariant coordinate charts on weighted homogeneous varieties as quotients of affine spaces by a cyclic group. This explicit coordinate atlas is useful in studying various properties of $w\Sigma_I$.

Let $U^-$ be the unipotent radical of the opposite Borel subgroup $B^-$. Choose a $T$-equivariant isomorphism $\mathbb{C}^{\Delta^+} \cong U^-$, where $T$ acts on $\mathbb{C}^{\Delta^+}$ by $x \cdot s_\alpha = \alpha(x^{-1})s_\alpha$. Thus a 1-parameter subgroup $\rho: \mathbb{C}^\times \to G$ gives rise to an action of $\mathbb{C}^\times$ on $\mathbb{C}^{\Delta^+}$ by

$$\lambda \cdot s_\alpha = \lambda^{-\langle \alpha, \rho \rangle} s_\alpha.$$

As a warm up, we start with the maximal flag variety $\mathcal{B} = G/B$. Then for each $w \in W(G)$, the image of $wU^-v_{\chi}$ in $\mathbb{P}(V_{\chi})$ is an open set of $\mathcal{B}$, isomorphic to the affine space $\mathbb{C}^{\Delta^+}$ with $T$-action twisted by $w^{-1}$. Moreover, the union of these open sets is all of $\mathcal{B}$. Thus we get a covering of
the weighted flag variety \( w\Sigma \) by \(|W(G)| \) open subsets, each isomorphic to \( \mathbb{C}^{\Delta^+}/\mu_{Nw} \), where \( N_w = u + \langle \chi, w\mu \rangle \) as before, and \( \lambda \in \mu_{Nw} \) acts by
\[
\lambda \cdot s_\alpha = \lambda^{-\langle \alpha, u\rho \rangle}.
\]

We now treat the general case of \( \Sigma_I = G/P_I \), for \( I \subset S \). Write \( \Delta^I \) for the roots that can be written as linear combinations of the roots in \( I \), and \( \Delta^I_+ = \Delta^I \cap \Delta^+ \). We set
\[
U^- = U^J \times U^-_J \quad \text{where} \quad U^-_J \cong \mathbb{C}^{\Delta^-_J} \quad \text{and} \quad U^J \cong \mathbb{C}^{\Delta^+_J \setminus \Delta^+_I}
\]
\((T\text{-equivariantly})\). Then the weighted homogeneous space \( w\Sigma_I(\chi, \rho, k) \) admits a cover by \(|W(G)/W_I| \) open charts, each a cyclic quotient of affine space. The chart corresponding to \( w \) is the image of \( wU^J(v_\chi) \) in \( \mathbb{P}(V_\chi)(\rho, u) \); it is isomorphic to \( U^J/\mu_{Nw} \) where \( \lambda \in \rho_{Nw} \) acts by
\[
\lambda \cdot s_\alpha = \lambda^{-\langle \alpha, u\rho \rangle}.
\]

**Problem 3.4.** As with weighted projective spaces, to use weighted homogeneous spaces \( w\Sigma \) as ambient spaces in which to construct varieties, we need to study questions such as when a subvariety \( X \subset \Sigma \) is well formed (that is, no orbifold behaviour in codimension 0 or 1, no quasi-reflections), or quasismooth (that is, the affine cone over \( X \) is nonsingular); for \( w\Sigma \) itself, it seems reasonable to expect that the \( \mathbb{C}^\times \) action on \( a\Sigma_I \) is well formed if and only if its action on \( a\Sigma_I \) is. By analogy with the toric case, there must be straightforward adjunction formulas for the canonical class of weighted homogeneous spaces \( w\Sigma \), together with criteria to determine whether the affine cone \( a\Sigma \) is Gorenstein or \( \mathbb{Q} \)-Gorenstein, and has terminal or canonical singularities. The results of the preceding section on \( \text{Gr}(2,5) \) raise the interesting question of writing down the projective resolution of \( a\Sigma_I \subset V_\chi \) in equivariant terms; Lascoux \[\text{[La]}\] has related results in some important cases that might serve as a model. Since \( \Sigma_I \) has the status of a generalised flag variety, it is also interesting to study the corresponding tautological structures over \( a\Sigma_I \).

### 4 Weighted orthogonal Grassmannian \( O\text{Gr}(5,10) \)

As in the introduction, let \( V = \mathbb{C}^{10} \) with a nondegenerate quadratic form \( q \); a change of basis puts \( q \) in the normal form
\[
q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \text{that is,} \quad V = U \oplus U^\vee, \text{where} \ U = \langle e_1, \ldots, e_5 \rangle.
\]
We write \( f_1, \ldots, f_5 \) for the dual basis of \( U^\vee \). A vector subspace \( F \subset V \) is isotropic if \( q \) is identically zero on \( F \). For example, \( U \) is an isotropic 5-space. Since \( q \) is nondegenerate, it is clear that an isotropic subspace \( F \subset V \) has dimension \( \leq 5 \). We say that a maximal isotropic subspace is a generator of \( q \), or of the quadric hypersurface \( Q : (q = 0) \subset \mathbb{P}(V) \). The parity \( \dim F \cap U \mod 2 \) is known to be locally constant in a continuous family of generators \( F_\lambda \). Thus parity splits the generators into two connected components. We choose the component containing the reference subspace \( U \). Thus we define the orthogonal Grassmann or O'Grassmann variety \( \text{OGr}(5,10) \) by

\[
\text{OGr}(5,10) = \left\{ F \in \text{Gr}(5,V) \left| \begin{array}{c}
F \text{ is isotropic for } q \\
\text{and } \dim F \cap U \text{ is odd}
\end{array} \right. \right\}
\]

**The Weyl group** \( W(D_5) \)

The study of the algebraic group \( \text{SO}(10,\mathbb{C}) \), its double cover \( \text{Spin}(10) \to \text{SO}(10,\mathbb{C}) \), and their representations is governed by the Weyl group \( W(D_5) \), which acts as a permutation group on every combinatoric set in the theory. We are particularly interested in two permutation representations of \( W(D_5) \) that base respectively the given representation \( V \) of \( \text{SO}(10) \), and the space of spinor \( S^+ = \bigwedge^{\text{even}} U \), which is a representation of \( \text{Spin}(10) \).

The given representation \( V = U \oplus U^\vee \) has basis \( e_1, \ldots, e_5, f_1, \ldots, f_5 \), and \( W(D_5) \) acts by permuting the indices \( \{1, \ldots, 5\} \) on the \( e_i \) and \( f_i \) simultaneously, and by swapping evenly many \( e_i \) with \( f_i \). For example, the permutations

\[(e_1 f_1)(e_2 f_2) \quad \text{and} \quad (e_1 f_1)(e_2 f_2)(e_3 f_3)(e_4 f_4)\]

are elements of \( W(D_5) \). One checks that in this permutation representation, \( W(D_5) \) is the Coxeter group generated by the 5 involutions

\[
(12) \quad (23) \quad (34) \quad (45)
\]

with the Coxeter relations indicated by the Dynkin diagram.

The spinor representation \( S^+ \) has basis the 16 nodes of the graph

\[
\Gamma = 5\text{-cube modulo antipodal identification.}
\]

The action of \( W(D_5) \) on \( \Gamma \) has 5 involutions parallel to the facets of the 5-cube, whose product is the antipodal involution, and thus acts trivially on
\( \Gamma \). These define a normal subgroup \((\mathbb{Z}/2)^5/(\text{diag}) \triangleleft W(D_5)\), the quotient by which is the symmetric group \(S_5\) permuting the 5 orthogonal directions of the 5-cube. Thus \(W(D_5)\) is the extension \((\mathbb{Z}/2)^4 \triangleleft W(D_5) \twoheadrightarrow S_5\).

To introduce notation for the nodes of \(\Gamma\), we break the symmetry by choosing a preferred node \(x = x_0 \in \Gamma\) and an order 1, 2, 3, 4, 5 on the 5 edges \(xx_1, \ldots, xx_5\) out of \(x\). Then \(\Gamma\) consists of \(x_I\), where \(I \subset \{1, 2, 3, 4, 5\}\), and \(x_I = x_{CI}\) (where \(CI\) is the set complement \(CI = \{1, 2, 3, 4, 5\} \setminus I\)). The short representatives are \(x, x_i, x_{ij}\), with \(x = x_0 = x_{12345}, x_1 = x_{2345}, x_2 = x_{345}, \) etc. We use this below to work out the equations and syzygies of \(OGr(5, 10)\).

The symmetry here is the same as that of the 16 lines on the del Pezzo surface of degree 4, see Reid \(R\).

**Notation**

We take the construction \(S^+ = \mathbb{C} \oplus \wedge^2 U \oplus \wedge^4 U\) as the definition of \(S^+\), without attempting to deal with it intrinsically (which can be done in terms of the even Clifford algebra). This construction depends on the choice of \(U\) or of the decomposition \(V = U \oplus U^\vee\), and \(S^+\) is a representation of the double cover \(\text{Spin}(10)\), not of \(\text{SO}(10)\) itself.

We write \((e, M, P) \in S^+\) for an element of \(S^+\), where \(e \in \mathbb{C}\), \(M = (x_{ij})\) is a skew 5 \(\times\) 5 matrix and \(P\) a 5 \(\times\) 1 column vector. If \(M = (x_{ij})\) is a skew 5 \(\times\) 5 matrix then \(\text{Pf} M\) is the column vector of its Pfaffians, that is,

\[
\text{Pf} M = \begin{pmatrix}
x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} \\
-x_{13}x_{45} + x_{14}x_{35} - x_{15}x_{34} \\
x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24} \\
-x_{12}x_{35} + x_{13}x_{25} - x_{15}x_{23} \\
x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}
\end{pmatrix}
\]

**Affine cover of \(OGr(5, 10)\)**

Since \(V = U \oplus U^\vee\), a 5-plane \(F \subset V\) near \(U\) is the graph of a linear map \(\varphi: U \rightarrow U^\vee\), so that \(U \in \text{Gr}(5, 10)\) has an affine neighbourhood parametrised by \(\text{Hom}(U, U^\vee)\): in other words, \(F\) has a basis of 5 vectors in \(V = \mathbb{C}^5\) that we can write as a matrix \((I, M)\) with \(I = I_5\) and \(M\) a 5 \(\times\) 5 matrix. One sees that \(F\) is isotropic for \(q\) if and only if the linear map \(\varphi\) or the matrix \(M\) is skew. Thus an affine neighbourhood of \(U\) in \(OGr(5, 10)\) is given by \((I, M)\) with \(M\) a skew 5 \(\times\) 5 matrix.

There are 16 standard affine pieces of \(OGr(5, 10)\). Each is obtained from this one by acting on the basis of \(V\) by a permutation of \(W(D_5)\). That
is, take matrices \((I, M)\) with \(M\) skew, and swap evenly many of the first 5 columns with the corresponding columns from the last 5.

**The spinor embedding** \(\text{aOGr}(5, 10) \subset S^+\)

Corresponding to the different treatment of tensors of rank 2 in \(\bigwedge^2 V\) in Section 3, we can write down 4 characterisations of *simple* spinors. For a spinor \(s \in S^+\), we have the following equivalent conditions:

(i) Explicit: \(s\) is in the \(W(D_5)\)-orbit of a spinor of the form \(e(1, M, \text{Pf}M)\) with \(e \in \mathbb{C}\) and \(M\) a skew \(5 \times 5\) matrix.

(ii) Orbit of highest weight vector: \(s\) is in the \(\text{Spin}(10)\)-orbit of the spinor \((1, 0, \ldots, 0)\) \(\in S^+\).

(iii) Quotient by \(\text{SL}(5)\): consider all \(5 \times 10\) matrices \(N\) whose rows base a generator \(\Pi \in \text{OGr}(5, 10)\) of \(q\), and take the quotient by \(\text{SL}(5)\) acting by left multiplication. To explain this briefly, suppose that \(N = (A, B)\) with nonsingular first \(5 \times 5\) block \(A\). Then up to the \(\text{SL}(5)\) action, \(N\) is of the form \(e(I, M)\). The affine embedding into \(S^+\) is then given by \(e(1, M, \text{Pf}M)\). Every other \(N\) is of this form up to the action of the Weyl group \(W(D_5)\).

(iv) Equations: \(s = (e, M, P) \in S^+\) satisfies the 10 equations

\[ eP = \text{Pf}M, \quad MP = 0 \]

(see below). The first set of 5 equations with \(e \neq 0\) describes the embedding of the first affine open in spinor space. On the other hand, as we discuss next, this set of equations is \(W(D_5)\)-invariant.

**Remark 4.1.** We use the following point of view in (iii): it is well known that the spinor embedding \(\text{OGr} \hookrightarrow \mathbb{P}(S^+)\) is the Veronese square root of the Plücker embedding \(\text{OGr} \hookrightarrow \text{Gr}(5, 10) \hookrightarrow \mathbb{P}(\bigwedge^5 \mathbb{C}^{10})\). In other words, up to a straightforward (!) change of coordinates, the set of \(5 \times 5\) minors of \(N\) is the second symmetric power of the set of spinor coordinate functions \(e, x_{ij}, \text{Pf}_k\).
Equations of $\text{aOGr}(5,10)$

As described above, $S^+$ has a basis indexed by the graph $\Gamma$. A pair $x_I, x_J$ is an edge of $\Gamma$ (that is, $x_I$ is joined to $x_J$) if and only if $I$ and $J$ or $I$ and $CJ$ differ by one element. Because of this definition, edges of $\Gamma$ fall into 5 sets of 8 parallel edges, with directions given by adding the same $i$: for example, the 8 edges

$$xx_1, \quad x_ix_{i+1}, \quad x_{ij}x_{kl} \text{ with } \{i, j, k, l\} = \{2,3,4,5\}$$

are all of the form $x_Ix_{I+1}$, so in the 1 direction. Two edges of $\Gamma$ are remote if no edge of $\Gamma$ joins either end of one to either end of the other. Any two parallel edges either form two sides of a square, or are remote. Each set of 8 parallel edges breaks up into two remote quads. For example, the 8 edges in the 1 direction give

$$xx_1, x_{23}x_{45}, x_{24}x_{35}, x_{25}x_{34} \quad \text{and} \quad x_{12}x_{13}, x_{14}x_{15}, x_{15}x_{12},$$

The 10 equations of $\text{OGr}(5,10)$ in (iv) are sums of these quads with appropriate choice of signs:

$$xx_1 - x_{23}x_{45} + x_{24}x_{35} - x_{25}x_{34} = 0 \quad \text{and} \quad x_{12}x_{13} + x_{14}x_{15} - x_{15}x_{12} = 0,$$

and permutations.

The 10 equations centred at $x$ are written in terms of the matrices

$$M = \begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ x_{23} & x_{24} & x_{25} \\ x_{34} & x_{35} \\ -\text{sym} & x_{45} \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$  

They take the form

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \end{pmatrix} = xv - \text{Pf}M = 0 \quad \text{and} \quad \begin{pmatrix} N_{-1} \\ N_{-2} \\ N_{-3} \\ N_{-4} \\ N_{-5} \end{pmatrix} = Mv = 0.$$

The 16 first syzygies are also indexed by the 16 vertices of $\Gamma$. Each is a
5 term syzygy involving the 5 neighbouring vertices:

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The 2nd syzygies are likewise indexed by the 16 monomials; they form a 16 × 16 symmetric matrix with typical columns

\[
S(x) = \begin{pmatrix}
S(x_1) & S(x_{12}) \\
x^2 & xx_1 - 2N_1 & xx_{12} \\
x_1 - 2N_1 & x_1^2 & x_1x_{12} + 2N_{-2} \\
x_2 - 2N_1 & x_1x_2 & x_2x_{12} - 2N_{-1} \\
x_3 - 2N_1 & x_1x_3 & x_3x_{12} \\
x_4 - 2N_1 & x_1x_4 & x_4x_{12} \\
x_5 - 2N_1 & x_1x_5 & x_5x_{12} \\
x_{12} & x_1x_{12} + 2N_{-2} & x_2^2 \\
x_{13} & x_1x_{13} + 2N_{-3} & x_{12}x_{13} \\
x_{14} & x_1x_{14} + 2N_{-4} & x_{12}x_{14} \\
x_{15} & x_1x_{15} + 2N_{-5} & x_{12}x_{15} \\
x_{23} & x_1x_{23} & x_{12}x_{23} \\
x_{24} & x_1x_{24} & x_{12}x_{24} \\
x_{25} & x_1x_{25} & x_{12}x_{25} \\
x_{34} & x_1x_{34} & x_{12}x_{34} + 2N_5 \\
x_{35} & x_1x_{35} & x_{12}x_{35} - 2N_4 \\
x_{45} & x_1x_{45} & x_{12}x_{45} + 2N_3 
\end{pmatrix}
\]

**Numerology**

From this we get the following numerology and representation theory. Write \( U = \mathbb{C}^5 \) with weights \( w_1, \ldots, w_5 \), where \( SO(10) \) acts on \( V = U \oplus U^\vee \). As in Section 1, to ensure that all the weights are positive, we introduce a further overall weight \( u \) on \( S^+ \). To keep track of this, we introduce the bigger group
$G = \text{Spin}(10) \times \mathbb{C}^\times$, and replace $S^+$ by $S^+ \otimes L$, where Spin(10) acts on the first factor in the usual way, and $\mathbb{C}^\times$ acts on $L = \mathbb{C}$ with weight $u$. Set

$$S^+ = \bigwedge^{\text{even}} U \otimes L = (\mathbb{C} \oplus \bigwedge^2 U \oplus \bigwedge^4 U) \otimes L$$

for the 16 dimensional spinor space. The only representations we need are $V, S^+$, its dual $S^-$ and their twists by line bundles. By analogy with Proposition 2.1 and (2.3), we define

$$D = \bigwedge^5 U \otimes L^2, \quad s = \sum w_i = \text{wt} \bigwedge^5 U \quad \text{and} \quad d = \text{wt} D = s + 2u,$$

Then the generators have weights

- $\text{wt} x = u,$
- $\text{wt} x_{ij} = u + w_i + w_j,$
- $\text{wt} x_i = u + w_j + w_k + w_l + w_m = u + s - w_i.$

The average of the 16 weights is $\frac{1}{2}d$.

The 10 relations with their representative terms have weights

$$N_i = xx_i + \cdots \mapsto 2u + w_j + w_k + w_l + w_m = d - w_i,$$
$$N_{-i} = \sum x_jx_{ij} \mapsto d + w_i,$$

which have average $d$. The 16 first syzygies have weights

$$T(x) = \sum x_iN_{-i} \mapsto 2d - u,$$
$$T(x_i) = xN_{-1} + \cdots \mapsto 2d - u - s + w_i,$$
$$T(x_{ij}) = x_iN_j + \cdots \mapsto 2d - u - w_i - w_j,$$

which have average $\frac{3}{2}d$, and the 16 second syzygies

$$S(x) = x^2T(x) + \cdots \mapsto 2d + u,$$
$$S(x_i) = xx_iT(x) + \cdots \mapsto 2d + u + s - w_i,$$
$$S(x_{ij}) = xx_{ij}T(x) + \cdots \mapsto 2d + u + w_i + w_j,$$

which have average $\frac{5}{2}s$.

\footnote{As in (2.3–2.4), this move cleans up the formulas below in a most miraculously way. However, to be quite honest, at the time of writing, we have absolutely no idea what representation $D$ is, or for what group it is supposed to be an equivariant line bundle on $\mathbb{C}^{16} = S^+$. Cf. Problem 4.2.}
To write out the Hilbert series of \( \text{wOGr}(5,10) \) with the above weights, introduce the Laurent polynomials

\[
Q_V = \sum_i t^{w_i} + \sum_i t^{w_i-1}
\]

\[
Q_{S^+} = 1 + \sum_{i,j} t^{w_i+w_j} + \sum_i t^{s-w_i}
\]

\[
Q_{S^-} = 1 + \sum_{i,j} t^{-w_i-w_j} + \sum_i t^{-s+w_i}
\]

Then \( \text{wOGr}(5,10) \) has Hilbert series

\[
P(t) = 1 - t^d Q_V + t^{2d-u} Q_{S^-} - t^{2d+u} Q_{S^+} + t^{3d} Q_V - t^{4d}.
\]

This numerology implies that the spaces of relations, first syzygies, etc., in the resolution are the following representations of \( \text{Spin}(10) \):

\[
O \leftarrow V \otimes D \leftarrow S^- \otimes D^2 \otimes L^{-1} \leftarrow S^+ \otimes D^2 \otimes L \leftarrow V \otimes D^3 \leftarrow D^4 \leftarrow 0.
\]

Providing it is well formed, \( \text{wOGr}(5,10) \) has canonical class \( K_{\text{wGr}} = O(-4d) \). In fact the wps \( \mathbb{P}(S^+ \otimes L) \) has \( K_{\mathbb{P}} = -8d \) (the sum of weights of the coordinates), and by (4.1), the adjunction number of \( \text{wGr} \subset \mathbb{P} \) equals \( 4d \).

**Problem 4.2.** We believe that the affine O’Grassmannian \( a\text{OGr}(5,10) \) has an equivariant resolution of the form (4.1), in complete analogy with Proposition 2.1. We have written out the maps in this sequence in explicit coordinate expressions in our treatment, with the right \( W(D_5) \) symmetry and weights. It should be possible to specify them intrinsically in terms of Clifford multiplication.

## 5 Examples

We have searched in vain for examples of Fano 3-folds, K3 surfaces or canonical surfaces as quasilinear sections of \( \text{wGr}(5,10) \), and we believe that there are very few, or even none, apart from the well known straight cases. In this section, we construct nice examples of a canonical 3-fold and a Calabi–Yau 3-fold having isolated cyclic quotient singularities.

**Example 5.1.** Let \( V \) be a regular 3-fold of general type with \( p_g = 7, K^3 = 21 \) and \( 2 \times \frac{1}{6}(1,1,1) \) singularities. The plurigenus formula of Fletcher and Reid [YPG] states that

\[
p_n = \begin{cases} 
1, & n = 1 \\
p_g, & n = 2 \\
n(n-1)(2n-1)K^3 + (2n-1)(p_g-1) + l(n), \quad & n \geq 2,
\end{cases}
\]

where \( l(n) \) is the degree of the linear system of hypersurfaces of degree \( n \) in \( \mathbb{P}^3 \).
where $l(n)$ is a sum of the local orbifold contributions

$$ l(n) = \begin{cases} \frac{n}{4} & \text{if } n \text{ is even} \\ \frac{n-1}{4} & \text{if } n \text{ is odd} \end{cases} $$

from each of the $\frac{1}{2}(1,1,1)$ singularities. One easily calculates the Hilbert function $H(t) = \sum p_n t^n$ from this:

$$ H(t) = 1 + t + \frac{t + t^2}{(1-t)^2} \cdot (p_g - 1) + \frac{t^2 + t^3}{(1-t)^4} \cdot \frac{K^3}{2} + \frac{1}{4} \times \frac{t^2}{(1-t)(1-t^2)} $$

$$ = 1 + 7t + 29t^2 + 83t^3 + 190t^4 + 370t^5 + 645t^6 + 1035t^7 + 1562t^8 \cdots $$

$$ = 1 + 4t + 10t^2 + 12t^3 + 10t^4 + 4t^5 + t^6 $$

$$ = \frac{(1-t)^7}{(1-t)^3(1-t^2)}. $$

We need seven generators in degree 1 (since $p_g = 7$) and at least two in degree 2 to accommodate the two $\frac{1}{2}(1,1,1)$ singularities; the simplest possibility is that $V$ has codimension 5 in $\mathbb{P}(1^7, 2^2)$, with Hilbert numerator

$$ (1-t)^7(1-t^2)^2 H(t) = 1 - t^2 - 8t^3 + 7t^4 + 8t^5 - 8t^7 - \cdots $$

We easily recognise this as the Hilbert numerator of the weighted orthogonal Grassmannian $wOG(5, 10)$ with weights $w = (0, 0, 0, 0, 1)$, $u = 1$ and $s = 1$, therefore $d = 3$. With this choice of weights, the coordinates $x_0$, $x_{ij}$ for $1 \notin \{i,j\}$ and $x_{2345}$ have weight 1; all other coordinates have weight 2. The spinor embedding takes $wOG(5, 10)$ into $\mathbb{P}(1^8, 2^8)$ and we construct $V$ as a general quasilinear section

$$ V = wOG(5, 10) \cap (1) \cap (2)^6. $$

We check that the canonical class adds up: either $V \subset \mathbb{P}(1^7, 2^2)$, with adjunction number $4d = 12$ gives $-7 \times 1 - 2 \times 2 + 12 = 1$, or $V = (1) \cap (2)^6 \subset wOG$ has $K_V = O(-4d + 1 + 6 \times 2) = O(1)$.

**Example 5.2.** Let $V$ be a Calabi–Yau 3-fold polarised by a divisor $A$ with $A^3 = \frac{6}{5}$ and $A \cdot c_2 = \frac{108}{5}$, and having singular points $P' = \frac{1}{3}(1,1,1)$, $P'' = \frac{1}{3}(2,2,2)$, and $Q = \frac{1}{7}(3,3,4)$ (we are writing these so that $A = O(1)$).

The orbifold Riemann–Roch formula of Fletcher and Reid [YPG] states that

$$ p_n = \frac{A^3}{6} n^3 + \frac{A \cdot c_2}{12} n + c_{P'}(n) + c_{P''}(n) + c_Q(n) $$

24
where \( c_\bullet(n) \) is a local contribution from the singularity that can be calculated explicitly using the instructions in [YPG]. Following the instructions, we discover that \( c_P'(n) + c_P''(n) = 0 \) for all \( n \), and

\[
c_Q(n) = \begin{cases} 
0 & \text{if } n \equiv 0 \text{ mod } 5; \\
0 & \text{if } n \equiv 1 \text{ mod } 5; \\
\frac{-1}{5} & \text{if } n \equiv 2 \text{ mod } 5; \\
\frac{1}{5} & \text{if } n \equiv 3 \text{ mod } 5; \\
0 & \text{if } n \equiv 4 \text{ mod } 5.
\end{cases}
\]

From this it is easy to calculate the Hilbert function

\[
H(t) = 1 + \frac{A^3}{6} \times \frac{(1 + 4t + t^2)t}{(1-t)^4} + \frac{A \cdot c_2}{12} \times \frac{t}{(1-t)^2} + \frac{1}{5} \times \frac{-t^2 + t^3}{1-t^5}
\]

\[
= \frac{1 - 2t + 3t^2 - t^3 - t^4 + t^5 + t^6 - 3t^7 + 2t^8 - t^9}{(1-t)^4(1-t^5)}
\]

\[
= 1 + 2t + 5t^2 + 11t^3 + 20t^4 + 34t^5 + 54t^6 + 81t^7 + 117t^8 + \cdots
\]

We see that we need to multiply by \((1 - t)^2(1 - t^2)^2\):

\[
(1 - t)^2(1 - t^2)^2(1 - t^5)H(t) = 1 + 3t^3 - 2t^5 + 2t^6 - 3t^8 - t^{11}
\]

Then we need three generators in degree 3:

\[
(1 - t)^2(1 - t^2)^3(1 - t^5)H(t) = 1 - 2t^5 - 4t^6 + 3t^8 + 2t^9 + 2t^{11} + \cdots
\]

At first sight this looks like a plausible \(6 \times 10\) codimension 4 format; the typical example of this is a nonspecial canonical curve \( C \) of genus 6, that is known to be a quadric section of a cone over \( \text{Gr}(2, 5) \). We might hope to find \( V \) as a nonlinear section of a cone over a weighted \( \text{wGr}(2, 5) \). Indeed the polynomial in the last displayed equation is the Hilbert numerator of \( C \cap (6) \subset \mathbb{P}(1, 2^3, 3^6, 4) \). However, this is a mirage of a fairly typical type: although it would have the correct Hilbert function, a quasilinear section of this variety can’t have a \( \frac{1}{5}(3, 3, 4) \) singularity. The simplest assumption is that \( V \) is codimension 5; the easiest guess is that there is an additional generator (and relation) in degree 4, giving

\[
(1 - t)^2(1 - t^2)^3(1 - t^4)(1 - t^5)H(t) =
\]

\[
1 - t^4 - 2t^5 - 4t^6 + 3t^8 + 4t^9 + 4t^{10} + 2t^{11} - 2t^{13} - \cdots
\]
We easily recognise this as the Hilbert numerator of the weighted orthogonal Grassmannian \( \text{wOGr}(5,10) \) with weights \( \mathbf{w} = (0,0,1,1,2) \), \( u = 1 \) and \( s = 4 \), therefore \( d = 6 \), embedded in \( \mathbb{P}(1^2, 2^4, 3^4, 4^4, 5^2) \), with canonical class \( \mathcal{O}(-4d) = \mathcal{O}(-24) \). We can construct \( V \) as a general quasilinear section
\[
V = \text{wOGr}(5,10) \cap (2)^2 \cap (3) \cap (4)^3 \cap (5)
\]
(the calculation that \( V \) has the correct singularities is a bit tedious but can be done by hand).

References

[Al] S. Altmok, *Graded rings corresponding to polarised K3 surfaces and \( \mathbb{Q} \)-Fano 3-folds*, Univ. of Warwick Ph.D. thesis, Sep. 1998, vii+93 pp., see www.maths.warwick.ac.uk/~miles/doctors/Selma

[ABR] S. Altmok, G. Brown and M. Reid, *Fano 3-folds, K3 surfaces and graded rings*, in Singapore International Symposium in Topology and Geometry (NUS, 2001), Edited by: A. J. Berrick, M. C. Leung and X. W. Xu, to appear Contemp. Math. AMS, 2002, math.AG/0202092, 29 pp.

[FH] W. Fulton and J. Harris, *Representation theory. A first course*, Springer, 1991

[Fl] A. R. Iano-Fletcher, *Working with weighted complete intersections*, in Explicit birational geometry of 3-folds, C.U.P., 2000, pp. 101–173

[G] Ian Grojnowski, *Weighted flag varieties*, will be available from the URL www.maths.warwick.ac.uk/~miles/3folds/wfv.ps

[La] Alain Lascoux, *Syzygies des variétés déterminantales*, Adv. in Math. 30 (1978) 202–237

[Mu] MUKAI Shigeru, *Curves and symmetric spaces. I*, Amer. J. Math. 117 (1995) 1627–1644

[N] J. Neves, *A note on regular surfaces of general type with \( K^2 = 13 \) and \( p_g = 6 \)*, work in progress

[PR] Stavros Papadakis and Miles Reid, *Kustin–Miller unprojection without complexes*, preprint math.AG/0011094, 18 pp., to appear in J. Alg. Geom.
[R] Miles Reid, *The complete intersection of two or more quadrics*, Cambridge Ph.D. thesis, Jun 1972, 84 pp., available from the URL www.maths.warwick.ac.uk/~miles/3folds/qu.ps

[YPG] Miles Reid, *Young person’s guide to canonical singularities*, in Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., 46, Part 1, AMS, 1987, pp. 345–414

[Ki] Miles Reid, *Graded rings and birational geometry*, in Proc. of algebraic geometry symposium (Kinosaki, Oct 2000), K. Ohno (Ed.), 1–72, www.maths.warwick.ac.uk/~miles/3folds/Ki/Ki.ps

Alessio Corti,
DPMMS, University of Cambridge,
Centre for Mathematical Sciences,
Wilberforce Road, Cambridge CB3 0WB, U.K.
e-mail: a.corti@dpmms.cam.ac.uk
tel: can.dpmms.cam.ac.uk/~corti

Miles Reid,
Math Inst., Univ. of Warwick,
Coventry CV4 7AL, England
e-mail: miles@maths.warwick.ac.uk
tel: www.maths.warwick.ac.uk/~miles