Solutions of the Fractional Reaction Equation and the Fractional Diffusion Equation

R.K. Saxena, A.M. Mathai, and H.J. Haubold

Abstract  In view of the role of reaction equations in physical problems, the authors derive the explicit solution of a fractional reaction equation of general character, that unifies and extends earlier results. Further, an alternative shorter method based on a result developed by the authors is given to derive the solution of a fractional diffusion equation. Fox functions and Mittag-Leffler functions are used for closed-form representations of the solutions of the respective differential equations.

Keywords  Fox function · Mittag-Leffler function · Differential equation · Anomalous reaction & diffusion · Asymptotic expansion · Lévy stable density

1 Introduction

Fractional reaction and diffusion equations involve fractional derivatives with respect to time and space and are studied to describe anomalous reaction and diffusion of dynamical systems with chaotic motion. Fractional reaction equation for Hamiltonian chaos is discussed by Zaslavsky (1994). Solutions and applications of reaction equations are studied by Saichev and Zaslavsky (1997). Solutions of a fractional reaction equation is investigated by Haubold and Mathai (2000) for a simple production-destruction mechanism. This equation was generalized by Saxena et al. (2002). In recent articles, Saxena et al. (2002, 2004a, 2004b) discussed the solution of a number of generalized fractional reaction equations. In the present article, we investigate the solution of a unified fractional reaction equation, which provides unification and extension of results on fractional reaction equations given earlier by Haubold and Mathai (2000) and Saxena et al. (2002, 2004a). We also present the solution of a fractional integral equation discussed by Miller and Ross (1993). Further, an alternative proof of the solution of a fractional diffusion equation given earlier by Kochubei (1990) is investigated, which is based upon a result given by Saxena et al. (2006). Most of the results are obtained in terms of generalized...
Mittag-Leffler functions in elegant and compact form, which are also suitable for numerical computation.

The paper is organized as follows. Section 2 provides the solution of a unified fractional reaction equation while Sect. 3 considers special cases of the equation. A shorter alternative method for the solution of a fractional diffusion equation discussed earlier by Kochubei (1990) is presented in Sect. 4. A series representation and asymptotic expansion of the solution are given in Sect. 5. An H-function representation of a one-sided Lévy stable density is also obtained.

2 Fractional Reaction Equation

In this Section, we present a method based on Laplace transform for deriving the solution of the unified fractional reaction equations.

Theorem 2.1. If \( \Re (v_j) > 0, a_j > 0, j \in N, \) and \( f(t) \) be a given function, defined on \( \mathbb{R}_+ \), then the equation

\[
N(t) - N_0 f(t) = - \sum_{j=1}^{n} a_j \, 0D_t^{-v_j} N(t),
\]

is solvable and its particular solution is given by

\[
N(t) = N_0 \sum_{l=0}^{\infty} (-1)^l \sum_{r_1+\ldots+r_{n-1}=l} \frac{(l)!}{(r_1)!\ldots(r_{n-1})!} \left\{ \prod_{\mu=1}^{n-1} (a_{\mu+1})^{r_\mu} \right\} 
\int_0^t f(u)(t-u)^{\sum_{\mu=1}^{n-1} v_{\mu+1} - 1} E^{(l+1)}_{\sum_{\mu=1}^{n-2} v_{\mu+1}} [-a_1 (t-u)^v] du,
\]

where the summation in (2) is taken over all nonnegative integers \( r_1, \ldots, r_n \) such that \( r_1 + \ldots + r_{n-1} = l \), and provided that the series and integral in (2) are convergent. Here \( 0D_t^{-v_j}, j \in N \) are Riemann-Liouville fractional integrals, defined by

\[
0D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-u)^{v-1} f(u) du, \, \Re(v) > 0,
\]

with \( 0D_0^0 f(t) = f(t) \) (Oldham and Spanier, 1974; Miller and Ross, 1993), \( E_\beta^\gamma (z) \) is the generalized Mittag-Leffler function, defined by Prabhakar (1971) in terms of series representation as

\[
E_\beta^\gamma (z) = \sum_{\tau=0}^{\infty} \frac{(\delta)(\tau+1)}{\Gamma(\beta \tau + \gamma)} z^\tau \quad (\beta, \gamma, \delta \in C, \Re(\beta) > 0, \Re(\gamma) > 0).
\]