On source parameters from particle correlations and spectra

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Abstract Analytic and numeric approximations are studied in detail for a hydrodynamic parameterization of single-particle spectra and two-particle correlation functions in high energy hadron-proton and heavy ion reactions. Two very different sets of model parameters are shown to result in similarly shaped correlation functions and single particle spectra in a rather large region of the momentum space. However, the absolute normalization of the single-particle spectra is found to be highly sensitive to the choice of the model parameters. For data fitting the analytic formulas are re-phrased in terms of parameters of direct physical meaning, like mean transverse flow. The difference between the analytic and numeric approximations are determined as an analytic function of source parameters.

1 Introduction

In 1994-95, a series of papers were written by the Buda-Lund collaboration on the study of particle correlations and single-particle spectra for non-relativistic, three-dimensionally expanding as well as for relativistic, one-dimensionally expanding or three-dimensionally expanding finite systems [1, 2, 3]. In these papers, it has been emphasized for the first time, that observation of the “true” sizes of particle sources is possible only if the single-particle spectra and the two-particle correlation functions are simultaneously analyzed. The reason was also given: the HBT radii (effective sizes measured by correlation techniques) were found to be dominated by the shorter of the geometrical and the thermally induced length-scales, while the

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width of the rapidity distribution or the slope of the transverse mass distribution is found to be dominated by the longer of the geometrical and the thermal scales. The appearance of the thermal length-scales is related to flow and temperature gradients, i.e. to the change of the mean momentum of the emitted particles with changing the coordinates of the particle emission. Within a thermal radius, these changes are not bigger than the width of the local momentum distribution.

This important effect is presently being re-discovered by various other groups, as a consequence of the emerging simultaneous analysis of particle correlations and spectra, proposed first in 1994 by the Buda-Lund collaboration. The question arises: Is it possible to uniquely determine the geometrical source radii (“true” source sizes) from a simultaneous analysis of particle correlations and spectra? This question is basically the same as the analogous question in the momentum space: Is it possible to uniquely determine the freeze-out temperature and the transverse flow from a simultaneous analysis of particle spectra and correlations?

We prove by an example that if the absolute normalization of particle spectra is not given, then it may be impossible to select from among different minima based only on the shape of the single-particle momentum-distribution and on the two-particle correlation functions.

We perform the analysis with the help of an analytically as well as numerically well studied model, the hydrodynamical parameterization of the Buda-Lund collaboration [3]. The domain of applicability of the analytical approximations is determined numerically in ref. [6] for this model. In ref. [7], this model is shown to describe the single-particle spectra and the two-particle correlations at $(\pi/K) + p$ reactions at CERN SPS simultaneously.

The same model was tested in refs. [4, 8] against the preliminary NA44 data on $S + Pb$ reactions at CERN SPS, however, it was shown there that it is difficult to find unique, reliable values of the fit parameters. The sources of the difficulty are the lack of absolute normalization of spectra and the experimental difficulty of proper estimate of systematic errors. We know from earlier fitting of this model to spectra without absolute normalization [6], that the final results are rather sensitive to errors and normalizations. In fact, we compare here the two physically different minima, found by fitting the Buda-Lund hydro model of ref. [3] to NA44 preliminary data on $S + Pb$ central reactions at 200 AGeV. In ref. [8] results were reported on fitting simultaneously the recently obtained absolutely normalized but still preliminary particle spectra and final correlations data for the same reaction as analysed in ref. [6].

In the next section the hydrodynamic model is presented, along with a new reparameterization of the basic formulas. An approximate analytic solution to this model is formulated in a new manner. For a comparison, a numerical approximation method is also schemed up. In the subsequent section, radius parameters and single particle spectra are calculated using the analytic and numeric methods. The transverse mass and the rapidity dependence of the results are shown for a substantial range of momentum space. The results are also used to make estimations
to the systematic errors introduced by the particular approximations. Finally, we summarize and emphasize the importance of the experimental determination of the absolutely normalized single particle spectra.

2 The model and its re-parameterization

The hydrodynamic model of ref. [3] is briefly recapitulated below, in a general form. The analytic results are then reformulated with new notation. A numerical evaluation scheme is also summarized afterwards.

2.1 The model

The Buda-Lund model [3] model makes a difference between the central (core) and the outskirts (halo) regions of high energy reactions. The pions that are emitted from the core consist of two types: a) They could be emitted directly from the hadronization of wounded, string-like nucleons, rescattering with a typical 1 fm/c scattering time as they flow outwards. b) Alternatively, they could be produced from the decays of short-lived resonances such as $\rho$, $N^*$, $\Delta$ or $K^*$, whose decay time is also of the order of 1-2 fm/c. This core region of the particle source is resolvable by Bose-Einstein correlation measurements. In contrast, the halo region consists of decay products of long-lived resonances such as the $\omega$, $\eta$, $\eta'$ and $K^0_L$, whose decay time is greater than 20 fm/c. This halo is not resolvable by Bose-Einstein measurements with the present techniques, however, it is affecting the Bose-Einstein correlation functions by suppressing their strength.

In general, the following emission function $S_c(x, p)$ applies to a hydrodynamically evolving core of particle source:

$$S_c(x, p)\, d^4x = \frac{g}{(2\pi)^3} \frac{d^4\Sigma^\mu(x)p_\mu}{\exp\left(\frac{u^\mu(x)p_\mu}{T(x)} - \frac{\mu(x)}{T(x)}\right) + s},$$

(1)

where the subscript $c$ refers to the core, the factor $d^4\Sigma^\mu(x)p_\mu$ describes the flux of particles through a finite, narrow layer of freeze-out hypersurfaces. The statistics is encoded by $s$, Bose-Einstein statistics corresponds to $s = -1$, Boltzmann approximation to $s = 0$ while the Fermi-Dirac statistics corresponds to $s = +1$. The four-momentum reads as $p = p^\mu = (E_p, \mathbf{p})$. The four-coordinate vector reads as $x = x^\mu = (t, r_x, r_y, r_z)$. For cylindrically symmetric, three-dimensionally expanding, finite systems it is assumed that any of these layers can be labelled by a unique value of $\tau = \sqrt{t^2 - r_z^2}$, and the random variable $\tau$ is characterized by a probability distribution, such that

$$d^4\Sigma^\mu(x)p_\mu = m_4 \cosh[\eta - \eta] H(\tau)\, d\tau\, d\eta\, dr_x\, dr_y.$$
Here \( m_t = \sqrt{m^2 + p_x^2 + p_y^2} \) stands for the transverse mass, the rapidity \( y \) and the space-time rapidity \( \eta \) are defined as \( y = 0.5 \log \left( (E + p_z)/(E - p_z) \right) \) and \( \eta = 0.5 \log \left( (t + r_z)/(t - r_z) \right) \) and the duration of particle emission is characterized by \( H(\tau) \propto \exp(-((\tau - \tau_0)^2)/(2\Delta r^2)) \). Here \( \tau_0 \) is the mean emission time, \( \Delta r \) is the duration of the emission in (proper) time. The four-velocity and the local temperature and density profile of the expanding matter is given by

\[
 u^\mu(x) = \left( \cosh[\eta] \cosh[\eta_t], \sinh[\eta_t] \frac{r_x}{r_t}, \sinh[\eta_t] \frac{r_y}{r_t}, \sinh[\eta] \cosh[\eta_t] \right),
\]

assuming a linear transverse flow profile. The inverse temperature profile is characterized by the central value and its variance in transverse and temporal direction, and we assume a Gaussian shape of the local density distribution:

\[
 T(x) = \frac{1}{T_0} \left( 1 + \frac{a^2}{2\tau_0^2} \right) \left( 1 + \frac{(\tau - \tau_0)^2}{2\tau_0^2} \right),
\]

\[
 \frac{\mu(x)}{T(x)} = \frac{\mu_0}{T_0} - \frac{r_x^2 + r_y^2}{2R_G^2} - \frac{(\eta - \eta_0)^2}{2\Delta \eta^2},
\]

where \( \mu(x) \) is the chemical potential and \( T(x) \) is the local temperature characterizing the particle emission.

### 2.2 Core/halo correction

The effective intercept parameter \( \lambda^*(y, m_t) \) of the Bose-Einstein correlation function measures the fraction of pions from the core versus the total number of pions at a given value of \( p \), when interpreted in the core/halo picture [10, 11, 12]. With this factor the total invariant spectrum in \( y \) rapidity and transverse mass \( m_t \) follows as

\[
 \frac{d^2n}{dy dm_t^2} = \frac{1}{\sqrt{\lambda^*}} \frac{d^2n_c}{dy dm_t^2} = \frac{1}{\pi \sqrt{\lambda^*}} \int S_c(x, p) d^4x.
\]

The momentum dependence of \( \lambda^* \) parameter is to be measured by the experimental collaborations. The experimental determination of \( \lambda^*(p) \) is very important not only because it gives a measure of the contribution of the core to the total number of particles at a given momentum, but also as it provides a measure of the mean transverse flow and a new signal of partial \( U_A(1) \) symmetry restoration [11]. Therefore it is strongly recommended that experiments report this \( \lambda^*(p) \) parameter of the Bose-Einstein correlation function and not just present partial fit results, like the momentum dependence of the radius parameters.
2.3 Re-parameterization

The original version of the core model contains 3 dimensionless parameters, $a$, $b$ and $d$, that control the transverse decrease of the temperature field, the strength of the (linear) transverse flow profile and the temporal changes of the temperature field, respectively, keeping only the mean and the variances of the inverse temperature distributions. These are very useful in obtaining simple formulas, however, they make the interpretation of the fit results less transparent. Hence we re-express them with new parameters with more direct physical meaning.

The surface temperature is introduced as $T_r = T(r_x = r_y = R_G, \tau = \tau_0)$ and the “post-freeze-out” temperature denotes the local temperature after most of the freeze-out process is over, $T_t = T(r_x = r_y = 0; \tau = \tau_0 + \sqrt{2}\Delta \tau)$. Here $R_G$ stands for the transverse geometrical radius of the source, $\tau_0$ denotes the mean freeze-out time, $\Delta \tau$ is the duration of the particle emission and we denote the temperature field by $T(x)$. The central temperature at mean freeze-out time is denoted by $T_0 = T(r_x = r_y = 0; \tau = \tau_0)$.

Then the relative transverse and temporal temperature decrease can be introduced as

$$\langle \Delta T/T \rangle_r = \frac{T_0 - T_r}{T_r}, \quad (8)$$
$$\langle \Delta T/T \rangle_t = \frac{T_0 - T_t}{T_t}, \quad (9)$$

and it is worthwhile to introduce the mean transverse flow as the transverse flow at the geometrical radius as

$$\langle u_t \rangle = b \frac{R_G}{\tau_0}. \quad (10)$$

The dimensionless model parameters can thus be expressed with these new, physically more straightforward parameters as

$$a^2 = \frac{\tau_0^2}{R_G^2} \langle \Delta T/T \rangle_r, \quad (11)$$
$$b = \frac{\tau_0}{R_G} \langle u_t \rangle, \quad (12)$$
$$d^2 = \frac{\tau_0^2}{\Delta \tau^2} \langle \Delta T/T \rangle_t. \quad (13)$$

Note, that eqs. (8) and (10) were introduced earlier in ref. [7] also to simplify the interpretation of data fitting. We present the complete re-parameterization herewith, including eqs. (11) and the re-parameterization of both the radius parameters and single-particle spectra.
2.4 Analytic approximations

In Ref. [3], the Boltzmann approximation to the above emission function was evaluated in an analytical manner, applying approximations around the saddle point of the emission function. The resulting formulas express the Invariant Momentum Distribution (IMD) and the Bose-Einstein correlation function (BECF). Now we re-express the formulas given in ref. [3] with the help of our new parameters.

The particle spectra can be expressed in the following simple form:

\[ N(p) = \frac{g}{(2\pi)^3} E V C \exp \left( -\frac{p \cdot u(\tau) - \mu(\tau)}{T(\tau)} \right), \]  

(14)

\[ E = m_t \cosh(\eta), \]  

(15)

\[ V = (2\pi)^{(3/2)} R_{\parallel} R_{\perp}^2 \Delta \tau / \Delta \tau, \]  

(16)

\[ C = \frac{1}{\sqrt{\lambda_e}} \exp \left( -\frac{\Delta \eta^2}{2} \right). \]  

(17)

Here the quantity \( \tau \) stands for the average value of the space-time four-vector parameterized by \((\tau, \eta, r_x, r_y)\), denoting longitudinal proper-time, space-time rapidity and transverse directions. These values are given as

\[ \tau = \tau_0, \]  

(18)

\[ \eta = \frac{y_0 - y}{1 + \Delta \eta^2 m_t T_0}, \]  

(19)

\[ r_x = \langle u_t \rangle R_G \frac{p_t}{T_0 + E} \left( \langle u_t \rangle + \langle \Delta T/T \rangle r \right), \]  

(20)

\[ r_y = 0. \]  

(21)

\( E \) denotes the energy of a particle from the center of particle emission, measured in the Longitudinal Center of Mass System (LCMS) frame. The effective volume is denoted by \( V \), see below for details, and the correction factor \( C \) takes into account the effects of long-lived resonances and the deviation of the saddle-point result from the more possible naive expectation, which would be the same expression with \( C = 1 \). The notation \( \tau \) denotes an invariant quantity \( a \), that depends on \( y - y_0, m_t, T \) and the other parameters of the model in a boost-invariant manner. In ref. [3] this was denoted by \( \tau^* \), for example, \( \Delta \eta^* \) in the present paper was denoted by \( \Delta \eta \). The average invariant volume of particle emission in eqs. (14,16) is given by \( V \) that is in our case the product of the average (momentum-dependent) transverse area \((2\pi R_{\perp}^2)\), the average (momentum-dependent) longitudinal source size \((2\pi)^{1/2} R_{\parallel} \). The averaging of the size of the effective volume over the duration
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of particle emission is expressed by the factor \( \frac{\Delta \tau}{\Delta \tau} \). These quantities read as

\[
\begin{align*}
\Delta \tau^2 &= \frac{\Delta \tau^2}{1 + \langle \Delta T/T \rangle r E/T_0}, \\
\Delta \eta^2 &= \frac{\Delta \eta^2}{1 + \langle \Delta \eta \rangle} E/T_0, \\
R_{||}^2 &= \frac{\tau^2 \Delta \eta^2}{1 + \langle \Delta T/T \rangle r E/T_0}, \\
R_{\perp}^2 &= \frac{R_{||}^2}{1 + \langle (\Delta t)^2 + \langle \Delta T/T \rangle r \rangle E/T_0}.
\end{align*}
\]

(22) (23) (24) (25) (26)

This completes the specification of the shape of particle spectrum. These results correspond to the equations given in ref. [3] although they are re-expressed with new combination of the variables. Please note that the Boltzmann-factor can be expressed approximately as

\[
\exp \left( -\frac{p \cdot u(x) - \mu(x)}{T(x)} \right) \simeq \exp \left[ \frac{\mu_0}{T_0} - \frac{(y - y_0)^2}{2(\Delta \eta^2 + T_0/m_t)} \right] \exp \left[ -\frac{m_t}{T_0} \right] \times \exp \left[ \frac{(\Delta t)^2(m_t^2 - m^2)}{2T_0 \left( T_0 + m_t(\langle u_t \rangle^2 + \langle \Delta T/T \rangle r) \right)} \right].
\]

(27)

The HBT radius parameters were evaluated in ref. [3] in the following way: the space-time rapidity \( \eta_s \) was defined as the solution of the equation

\[
\frac{\partial S(\eta_s)}{\partial \eta} = 0.
\]

(28)

The Longitudinal Saddle Point System (LSPS) was introduced as the frame where \( \eta_s = 0 \). At the so-called saddle-point one has

\[
\frac{\partial S}{\partial \tau} = \frac{\partial S}{\partial \eta} = \frac{\partial S}{\partial r_x} = \frac{\partial S}{\partial r_y} = 0.
\]

(29)

This resulted in a set of transcendental equations for the position of the saddle-point. These equations for the positions were solved approximately with the help of an expansion of the transcendental equation in terms of small parameters, like the deviation of \( \eta_s \) from the mean rapidity of the pair. The calculation resulted in
the following value for $\eta_s$:

$$\eta_s = y + \frac{y_0 - y}{1 + \Delta \eta^2 (\frac{m_t}{T_0} - 1)}.$$  

(30)

Hence the relative rapidity of the LSPS frame as compared to the LCMS frame is

$$\eta_s^L = \frac{y_0 - y}{1 + \Delta \eta^2 (\frac{m_t}{T_0} - 1)}.$$  

(31)

Note that LCMS is the frame where the rapidity belonging to the mean momentum of the pair vanishes, $y = 0$ [13]. In ref. [3] this quantity was denoted by the slightly more complicated notation $\eta_{s}^{LCMS} = \eta_s^L$, and the maximum of the Boltzmann factor, which in our present notation reads as $\bar{\eta} = \eta_s$ and stands for a modified saddle-point, used for the calculation of the particle spectra in ref. [3]. Note that $\eta_s^L$ may deviate from $\eta$ substantially at low values of $m_t$, especially if $T > m$.

This frame, defined by the maximum of the Boltzmann factor, given by $\bar{\eta}$ in the LCMS, plays a key role in the calculations. Hence this frame, introduced in ref. [3] without a name, deserves a name. We suggest Longitudinal Boltzmann Center System (LBCS). In general, this LBCS frame is defined by the solution of the

$$\frac{\partial f_B(\eta)}{\partial \eta} = 0$$  

(32)

equation, where $f_B = \exp (-[pu(x) - \mu(x)]/T(x))$ stands for the Boltzmann factor only, but does not include the Cooper-Frye flux term. A more detailed comparison between the LBCS and LSPS frames is in preparation for a separate publication [14].

In ref. [3] the spectrum was evaluated in the LBCS frame, while the correlation functions in the LSPS frame. Let us recapitulate the results for the two-particle correlation functions: The mean momentum is denoted by $K = (p_1 + p_2)/2$, its components are $(K_0, K_L, K_t, 0)$ in the $(t, r_z, r_x, r_y)$ reference frame. The transverse velocity $\beta_t$ reads as

$$\beta_t = \frac{K_t}{K_0} = \frac{K_t}{M_t \cosh(y)} = \frac{V_t}{\cosh(y)}.$$  

(33)

The effective source parameters are obtained as

$$\frac{1}{\Delta \eta_s^2} = \frac{1}{\Delta \eta^2 (\eta \rightarrow \eta_s^L)} - \frac{1}{\cosh^2(\eta_s^L)},$$  

(34)

$$R^2_\parallel = \tau_s \Delta \eta_s^2 = \tau_s^2 \Delta \eta^2 (\eta \rightarrow \eta_s^L),$$  

(35)

$$R^2_\perp = R^2 = \overline{\tau_s^2 (\eta \rightarrow \eta_s^L)},$$  

(36)

$$R^2_t = \Delta \tau_s^2 = \overline{\tau^2 (\eta \rightarrow \eta_s^L)}.$$  

(37)
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The modified position of the maximal emissivity is given in the LCMS frame by \((\tau_s, \eta, r_{x,s}, 0)\), where

\[
\begin{align*}
    r_{x,s} &= r_{x}(\eta \rightarrow \eta^L_s), \\
    \tau_s &= \tau = \tau_0.
\end{align*}
\]

(38) (39)

This notation means that, for example, \(\Delta \eta^*\) corresponds to the function \(\Delta \eta\) when the variable \(\eta\) is replaced by the quantity \(\eta^L_s\). Note, that \(\Delta \eta^*\) corresponds to the effective space-time rapidity width of the particle emission function in an LSPS calculation, while \(\Delta \eta\) corresponds to the width of the Boltzmann-factor only in the LBCS frame. Similar relations hold for \(R\) and \(R^*, \Delta \tau\) and \(\Delta \tau^*\). The Bose-Einstein correlation functions can be written in the so-called Bertsch-Pratt side-out-long ref. frame as

\[
C(\Delta k, K) = 1 + \lambda_\ast \exp \left( -R^2_{\text{side}}Q^2_{\text{side}} - R^2_{\text{out}}Q^2_{\text{out}} - R^2_LQ^2_L - R^2_{\text{out},L}Q_LQ_{\text{out}} \right),
\]

(40)

\[
\begin{align*}
    R^2_{\text{side}} &= R^2_s = R^2_s, \\
    R^2_{\text{out}} &= R^2_{\text{side}} + \delta R^2_{\text{out}}, \\
    \delta R^2_{\text{out}} &= \frac{V_\ast^2}{\cosh^2(y)} \left[ \cosh^2(\eta_s)R^2_\tau + \sinh^2(\eta_s)R^2_{\parallel} \right], \\
    R^2_L &= \frac{1}{\cosh^2(y)} \left[ \cosh^2(\eta^L_s)R^2_\tau + \sinh^2(\eta^L_s)R^2_{\parallel} \right], \\
    R^2_{\text{out},L} &= -\frac{V_\ast}{\cosh^2(y)} \left[ \cosh(\eta_s) \sinh(\eta^L_s)R^2_\tau + \sinh(\eta_s) \cosh(\eta^L_s)R^2_{\parallel} \right].
\end{align*}
\]

(41) (42) (43) (44) (45)

Here we have utilized the form of equations given in ref. [3] and the transformation of \(K_L\) to \(M_t\ cosh(y)\) as introduced in ref. [10].

Although the two-particle Bose-Einstein correlation function is manifestly co-variant [11], its Bertsch-Pratt parameterization is frame-dependent. A possible covariant parameterization [17, 18], applied recently to the particle interferometry by the Regensburg group [19, 20]. We find that the simplest possible covariant generalization is not exactly the YKP parameterization, but the formulation given by the Buda-Lund collaboration in ref. [3], see especially eqs. (44) and (21-26) of ref. [3]. Here we repeat only the results after Gaussian approximation to the source function:

\[
C(\Delta k, K) = 1 + \lambda_\ast \exp(-Q^2_0R^2_\tau - Q^2_0R^2_{\parallel} - Q^2_0R^2_z),
\]

(46)

\[
\begin{align*}
    Q_\tau &= Q_{\text{side}} + Q_{\text{out}}, \\
    Q_\parallel &= Q \cdot n(x^L_\tau) = Q_0 \cosh(\eta^L_\tau) - Q_z \sinh(\eta^L_\tau), \\
    Q_z &= \sqrt{Q \cdot Q - (Q \cdot n(x^L_\tau))^2 - Q^2_\tau} = Q_0 \sinh(\eta^L_\tau) - Q_z \cosh(\eta^L_\tau).
\end{align*}
\]

(47) (48) (49)
where \( n(x_s^L) = (\cosh(\eta_s^L), 0, 0, \sinh(\eta_s^L)) \) is a normal-vector at \( x_s \) in LCMS. Note, that this formulation is equivalent with the YKP formulation, however the correlation function is given by a simple purely quadratic form in the present formulation, in contrast to the YKP expression, where the invariant relative momentum combination \( Q_\eta \) is written out explicitly in terms of its non-invariant components. A more detailed comparison between the YKP and the Buda-Lund parameterization is being discussed in ref. [14].

2.5 Numeric approximations

Approximate single particle spectra and Bose-Einstein correlation functions can be calculated by numerical integration of equation (1), as well, expressing the means and the variances of the hydrodynamically evolving core of particle emission. The original method of 'means and variances' was proposed first by the Regensburg group in ref. [15]. We shall utilize herewith the core/halo corrected version of these relations as given recently in ref. [21]. The limitation of these approximations is discussed in refs. [3, 21]. For example, possible double-Gaussian structures or non-Gaussian features are neglected in this approximation. The analytic approximation yields Gaussian functions in proper time \( \tau \) and space-time rapidity \( \eta \), hence includes a deviation from Gaussian shape in \( t \) and \( z \). The numerical approximation assumes Gaussian forms in \( t \) and \( z \) that is not so well suited to the kinematics of ultra-relativistic reactions as Gaussians in \( \tau \) and \( \eta \). The spectra and the HBT radius parameters are defined in the Gaussian core/halo model approximation as follows:

\[
C(K, \Delta k) = 1 + \lambda_s(K) \exp\left(-R_{i,j}^2(K)\Delta k_i \Delta k_j\right),
\]

\[
\lambda_s(K) = \frac{[N_c(K)/N(K)]^2}{},
\]

\[
R_{i,j}^2(K) = \langle (x_i - \beta_i t) (x_j - \beta_j t) \rangle_c - \langle (x_i - \beta_i t) \rangle_c \langle (x_j - \beta_j t) \rangle_c,
\]

\[
\langle f(x, K) \rangle_c = \int d^4x f(x, K) S_c(x, K),
\]

where \( i, j = \text{side}, \text{out} \) or \( \text{long} \) as before, and \( S_c(x, K) \) is the emission function that characterizes the central core, as given by eq. 1. The mean momentum is defined as \( K = 0.5(p_1 + p_2) \), the relative momentum is given by \( \Delta k = p_1 - p_2 \). The spectra of all the bosons and the spectra of the bosons from the core described as

\[
N(p) = \langle 1 \rangle = \int d^4x [S_c(x, p) + S_{\text{halo}}(x, p)],
\]

\[
N(p)_c = \langle 1 \rangle_c = \int d^4x S_c(x, p).
\]

In this picture, the reduction of the intercept parameter is the only effect on the correlation function that stems from the halo, the variances of the core correspond to the Gaussian core/halo model radii of the measured correlation function. Although
the above expressions are formally similar to the original version of Gaussian model-independent radii of ref. [13], they cannot be obtained with an expansion around $Q = 0$, as they correspond to a large $Q$ expansion of the Bose-Einstein correlation function $[22, 24]$.

3 Calculating observables

Both analytic and numeric approximations were used to calculate and show the momentum dependence of the observables from the hydrodynamical model of ref. [3]. These observables are the effective radius parameters (HBT radii), the shape and the slope of the single-particle spectra. Figures are drawn for two very different sets of source parameters (or model parameters). To recall, the particle source is characterized by the means and the variances of the density distribution, the inverse temperature distribution and a linear flow. This yields 9 free parameters, $\mu_0, T_0, \tau_0, R_G, \Delta \tau, \Delta \eta, \langle \Delta T/T \rangle_r, \langle u_t \rangle, \langle \Delta T/T \rangle_t$, respectively.

The examined momentum space is divided into 40x40 sub-intervals in $m_t$ and $y$ dimensions, respectively, that allowed for fine resolution of the distributions. As a drawback, with such a resolution the numeric integration version takes much longer time than the analytic one. In the presented case generation of data took 10 hours for one run. Scales on the pictures are kept the same for the same kind of distributions except for the average emission rate that differs remarkably for the two sets of model parameters. The actual parameter set values (with $\mu_0 = 0$) used in the particular calculations are indicated below each drawing and they are denoted by names Source Parameters Set 1 and Source Parameter Set 2. Parameter Set 1 was obtained from a ref. [9], fitting absolutely normalized spectra in the NA44 acceptance, while Parameter Set 2 was obtained in ref. [8], fitting unnormalized preliminary particle spectra together with correlation data. Note that one can distinguish between different model parameters only if the value of $\mu_0$ is known from other observations than the ones already exploited in the present analysis.

3.1 Analytic results

The two model parameter sets mentioned above were applied to the analytic expressions formulated in the previous section by eqs. (14) to (45). See Figures 1 and 2 for details of the momentum space distributions of the observables. Notice the substantial difference of the particle spectra for Parameter Set 1 and Parameter Set 2. Also notice the deviations of the radius parameters at small $m_t$ and at large relative rapidities to midrapidity for the two different source parameter sets. Along with the comparison to the numeric results later this reflects the limitations of this kind of approximation. See ref. [14] for an improved treatment.
3.2 **Numeric results**

The numerically evaluated HBT radius parameters and single particle spectra are obtained from the eqs. (50,52,54,55), utilizing the Boltzmann approximation to the source function of eq. (1). Note that this scheme is not an exact calculation, but an approximation in a different way than the analytic approach, therefore it is suitable to estimate the systematic errors of the model parameters and to cross-check the uniqueness of the minimum in fitting the model to experimental data. See Figures 3 and 4 for details of HBT radius parameter distributions as well as single particle spectra in this approximation scheme.

3.3 **Differences between the analytic and the numeric results**

The differences between the observables as calculated from the two sorts of model approximations are presented on Figures 5 and 6. From these drawings one can learn the critical ranges where the two approximation schemes differ from each other beyond a given tolerance level.

Figures 5 and 6 show the rapidity and transverse mass dependence of the relative deviations between the analytic and the numeric approximations. For example, let us consider the top left panel of Figure 5. On this panel, the relative deviation of the numerical and analytic approximations for the side radius component is evaluated in the following manner: The analytical result for a \((y, m_t)\) bin \(i\) is denoted by \(a_i\), the numerical result is denoted by \(n_i\). Then the relative deviation between the two approximation schemes is defined as

\[
\delta^2 a_i = \frac{(a_i - n_i)^2}{a_i^2},
\]

which is plotted on the top left panel for the side radius parameter and in subsequent panels for the out, long and cross term, the spectra and the slope of the spectra on the subsequent panels. This quantity will be understood and estimated in the next subsection as a function of some small expansion parameters, that are analytically obtainable for any set of source parameters. In turn, this result can be utilized to improve the fits of the analytic expressions to measured data.

3.4 **Estimating systematic errors of approximations**

The aim of the present subsection is to analytically understand the systematic errors on the analytic approximations that we utilize to evaluate the spectra and the HBT radius parameters.

In a fit with the analytic approximations, the \(\chi^2\) of the fit is given as

\[
\chi^2_a = \sum_i \frac{(d_i - a_i)^2}{e_i^2}.
\]

(57)
where \( d_i \) denotes the measured data point at a given bin, for example, \( R_{\text{side}}(y_i, m_{t,i}) \), and the experimental error on this quantity is given by \( e_i \). A numeric fit to the same data minimizes the following numeric \( \chi^2 \) distribution. This may have different minima for the model parameters from the minima of \( \chi^2 \). However, the \( \chi^2 \) distribution can be approximately reconstructed from an analytic fit, as follows:

\[
\chi^2_n = \sum_i \frac{(d_i - n_i)^2}{e_i^2},
\]

\[ \chi^2_n \approx \chi^2_a + \delta \chi^2_a, \tag{58} \]

\[ \delta \chi^2_a = \sum_i \frac{\delta^2 a_i}{e_i^2} = \sum_i \delta^2 A_i \frac{a_i^2}{e_i^2}, \tag{59} \]

where \( \delta^2 a_i = (a_i - n_i)^2 \) is the difference between the analytic and numeric result, and can be regarded as the systematic error of the analytic approximation, while the relative systematic error of the analytic calculation is \( \delta^2 A_i = \frac{\delta^2 a_i}{a_i^2} \). (Keep in mind that \( a_i \) can be any of the analytically evaluated radius parameters or analytic result for the particle spectra). Our purpose is to obtain approximate analytical expressions for the relative error of the analytical approximations, \( \delta^2 A_i \). These quantities are shown in Figures 5 and 6.

Figures 5 and 6 indicate large relative errors in certain regions of the \((y, m_t)\) plane. These regions coincide with the regions where the so called small expansion parameters of the model start to reach values close to 1. The analytic expressions for the observables (HBT radius parameters and single-particle spectra) were obtained in ref. [3] under the condition that the parameters \((\eta^L_s, \Delta \eta_s, r_{x,s}/\tau_0)\) are all much less than 1. This was due to the approximate nature of the solution of the saddle-point equations, and the expansion of the transcendental equations in terms of these small parameters. Figures 7 and 8 show the \((y, m_t)\) dependence of these small parameters and their squares. We observe the expected similarities to the distributions of the relative errors \( \delta^2 A_i \), Figure 5 and 6. The different small parameters become large in well separated domains of the momentum space (typically below 100MeV and above 1 GeV), e.g. \( \eta^L_s \) at small \( m_t \) and large \(|y - y_0|\), \( \Delta \eta_s \) at small \( m_t \) and small \(|y - y_0|\), \( r_{x,s}/\tau_0 \) at large \( m_t \) independently of \( y \). As a consequence, the relative errors of the analytic approximations for any radius parameter or momentum distribution can be parameterized as a linear combination of the squared small parameters:

\[
\delta^2 A = c_1^A (\eta^L_s)^2 + c_2^A \Delta \eta_s^2 + c_3^A (r_{x,s}/\tau_0)^2. \tag{61} \]

The three constants \((c_1^A, c_2^A, c_3^A)\) are determined for the observables \( A = R_{\text{side}}, R_{\text{out}}, R_{\text{long}}, R_{\text{out-long}}^2, N(p), T_{\text{eff}}(y, m_t) \). Since the small parameters are expressed as
a function of the model parameters or “true” source parameters, eqs. (35-37), the desired analytical formula for all the systematic errors of the evaluation of all these 6 observables is given in the form of eq. (61).

Note that the region, where the analytic expressions are most precise, corresponds to a curved region in the \((y, m_t)\) plane, that at low \(m_t\) starts off-mid-rapidity and at high \(m_t\) shifts to mid-rapidity, in case of pions. This region almost exactly coincides with NA44 acceptance for pions. For heavier particles all the 3 small parameters decrease substantially. The analytic calculation is thus more precise for heavier particles than for pions.

For determining the numerical coefficients \(c_i^A\), the distributions of the differences on Figures 5 and 6 the CERN optimizing package MINUIT was used [23]. MINUIT finds the minimum value of a multi-parameter function and it analyzes the shape of the function around the minimum. The fitted error distributions in terms of the small parameters are shown on Figures 9 to 12 together with the best estimates of the systematic errors with the help of eq. (4). The coefficients \(c_i^A\) of the parameterized systematic error distributions along with their errors are shown in Table 1 and Table 2.

Note that the coefficients \(c_i^A\) are found to be only weakly dependent on the source parameter sets for most of the cases, they are all smaller then the coefficients for Parameter Set 1 multiplied by 2. On Figure 11, the fit of \((\delta N(p)/N(p))^2\) reflects the effect that above 0.5 MeV the calculation was forced to take the amplitudes with low weight due to the fact that the absolute values of \(N(p)\) are very small in this range.

4 Conclusion

The first combined HBT and spectrum analysis, reported by the Buda-Lund collaboration in 1994-95, is re-visited here for the purpose of a systematic numerical and analytical evaluation of the model. We identified the regions in the rapidity-transverse mass plane where the analytic approximations and the numerical ones deviate from each other. These regions were found to coincide with the regions where the small expansion parameters of the analytic approximation start to grow significantly. The deviation between the analytical and the numerical results is characterized by positive definite quadratic polinomials built up from the small parameters. We find that the NA44 acceptance is ideally suited for the precise evaluation of the Buda-Lund model.

As a by-product we find that the parameters of the Bose-Einstein correlation functions as well as the shape and the slope parameters of the double differential invariant momentum distribution are similar within 10% for two physically very different model parameter value sets in the momentum space domain where the approximations are valid. However, these sets result in a factor of 7 - 10 difference in the absolute normalization of the single-particle spectra. Hence, it is strongly
recommended to publish the experimentally measured single-particle spectra with their absolute normalization for future CERN and RHIC heavy ion experiments, for as many type of particles as possible, as a two-dimensional function of $y, m_t$. Figures 1 to 4 illustrate that the $y$ and the $m_t$ dependence of $N(y, m_t)$ can not be factorized.

Of course, our example of two physically different parameter sets resulting in similar correlations and unnormalized spectra does not imply that such similarity is achieved for any two different parameter sets. The role of each parameter can be investigated, for instance, analytically like in ref. [3, 6]. The typical behaviour that one expects is that for different values of the model parameters the particle correlations and spectra are different. We would like to remind the readers that the existence of certain scaling limiting cases was pointed out also in ref. [3], where the dependence of the HBT radii on some of the model parameters was analytically shown to vanish in certain domains of the parameter space.

Note that the intercept parameter of the correlation function is included in the core/halo correction factor, hence it is also strongly recommended to publish the experimental HBT results including a momentum-dependent determination of $\lambda^*$, too.

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Table captions

Table 1. Coefficients $c_i^A$ of the parameterized systematic error distributions for Source Parameter Set 1.

Table 2. Coefficients $c_i^A$ of the parameterized systematic error distributions for Source Parameter Set 2.

Figure captions

Fig. 1. Simultaneous results for particle spectra and HBT radius parameters. The analytic approximations were utilized to evaluate the model for Parameter Set 1.

Fig. 2. Simultaneous results for particle spectra and HBT radius parameters. The analytic approximations were utilized to evaluate the model for Parameter Set 2.

Fig. 3. Simultaneous results for particle spectra and HBT radius parameters. The numeric approximations were utilized to evaluate the model for Parameter Set 1.

Fig. 4. Simultaneous results for particle spectra and HBT radius parameters. The numeric approximations were utilized to evaluate the model for Parameter Set 2.

Fig. 5. Relative deviations between the analytic and the numeric approximations to the HBT parameters and particle spectra for Parameter Set 1.

Fig. 6. Relative deviations between the analytic and the numeric approximations to the HBT parameters and particle spectra for Parameter Set 2.

Fig. 7. Small expansion parameters of the hydrodynamic core model for Parameter Set 1.

Fig. 8. Small expansion parameters of the hydrodynamic core model for Parameter Set 2.

Fig. 9. Parametrization of the relative deviations between the analytic and the numeric approximations to the HBT parameters and particle spectra for Parameter Set 1.

Fig. 10. Parametrization of the relative deviations between the analytic and the numeric approximations to the HBT parameters and particle spectra for Parameter Set 2.

Fig. 11. Parametrization of the relative deviations between the analytic and the numeric approximations to the HBT parameters and particle spectra for Parameter Set 1.
Fig. 12. Parametrization of the relative deviations between the analytic and the numeric approximations to the HBT parameters and particle spectra for Parameter Set 2.
On source parameters

| $\delta^2 A$ | $c_1^A$ | $c_2^A$ | $c_3^A$ |
|--------------|---------|---------|---------|
| $(\delta R_{\text{side}}/R_{\text{side}})^2$ | $0.015 \pm 0.001$ | $0.003 \pm 0.001$ | $0.002 \pm 0.001$ |
| $(\delta R_{\text{out}}/R_{\text{out}})^2$ | $0.013 \pm 0.001$ | $0.001 \pm 0.001$ | $0.039 \pm 0.001$ |
| $(\delta R_{\text{long}}/R_{\text{long}})^2$ | $0.027 \pm 0.001$ | $0.014 \pm 0.001$ | $0.032 \pm 0.001$ |
| $(\delta R_{\text{out}}^2/R_{\text{out}}^2)^2$ | $0.130 \pm 0.001$ | $0.208 \pm 0.001$ | $0.001 \pm 0.001$ |
| $(\delta N(p)/N(p))^2$ | $0.014 \pm 0.001$ | $0.044 \pm 0.001$ | $0.015 \pm 0.001$ |
| $(\delta T_{\text{eff}}/T_{\text{eff}})^2$ | $0.006 \pm 0.001$ | $0.042 \pm 0.001$ | $0.001 \pm 0.001$ |

Table 1:

| $\delta^2 A$ | $c_1^A$ | $c_2^A$ | $c_3^A$ |
|--------------|---------|---------|---------|
| $(\delta R_{\text{side}}/R_{\text{side}})^2$ | $0.001 \pm 0.001$ | $0.007 \pm 0.001$ | $0.003 \pm 0.001$ |
| $(\delta R_{\text{out}}/R_{\text{out}})^2$ | $0.008 \pm 0.001$ | $0.002 \pm 0.001$ | $0.001 \pm 0.001$ |
| $(\delta R_{\text{long}}/R_{\text{long}})^2$ | $0.018 \pm 0.001$ | $0.024 \pm 0.001$ | $0.031 \pm 0.001$ |
| $(\delta R_{\text{out}}^2/R_{\text{out}}^2)^2$ | $0.154 \pm 0.003$ | $0.398 \pm 0.002$ | $0.001 \pm 0.001$ |
| $(\delta N(p)/N(p))^2$ | $0.035 \pm 0.001$ | $0.052 \pm 0.001$ | $0.001 \pm 0.001$ |
| $(\delta T_{\text{eff}}/T_{\text{eff}})^2$ | $0.012 \pm 0.001$ | $0.042 \pm 0.001$ | $0.001 \pm 0.001$ |

Table 2:
Fig. 1.
On source parameters

Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
On source parameters

Fig. 8.
Fig. 9.
Fig. 10.
On source parameters

Fig. 12.
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