ONE PROPERTY OF ZERO SET OF FUNCTION INVERTIBLE IN THE SENSE OF EHRENPREIS IN THE SCHWARTZ ALGEBRA

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Key words: Schwartz algebra, entire function, distribution of zero sets, slowly decreasing function.

AMS Mathematics Subject Classification: 30D15, 30E5, 42A38, 46F05

Abstract.
We consider those elements of the Schwartz algebra of entire functions which are Fourier-Laplace transforms of invertible distributions with compact supports on the real line. These functions are called invertible in the sense of Ehrenpreis. The presented result concerns with the properties of zero subsets of invertible in the sense of Ehrenpreis function $f$. Namely, we establish some properties of the zero subset formed by zeros of $f$ laying not far from the real axis.

1 Introduction

Let $\mathcal{E}'$ denote the strong dual to the Fréchet space $\mathcal{E} := C^\infty(\mathbb{R})$. Recall that Fourier-Laplace transform operator acting in $\mathcal{E}'$ is defined by the formula
$$\psi(z) = S(e^{-zt}), \quad S \in \mathcal{E'},$$
and the image $\mathcal{P}$ of $\mathcal{E}'$ under this transform becomes the topological algebra (Schwartz algebra) if we equip it with the topology and the algebraical structure induced from $\mathcal{E}'$. It is well-known that $\mathcal{P}$ consists of all entire functions of exponential type having at most polynomial growth along the real axis [7, Theorem 7.3.1].

The division theorem is valid for $\varphi \in \mathcal{P}$ if the following implication holds:
$$\Phi \in \mathcal{P}, \quad \Phi/\varphi \in Hol(\mathbb{C}) \implies \Phi/\varphi \in \mathcal{P}.$$

Now, we explain why having this property for $\varphi \in \mathcal{P}$ is important for the applications.

In [6] L. Ehrenpreis establishes that the validity of the division theorem for $\varphi \in \mathcal{P}$ is equivalent to the invertibility in the spaces $\mathcal{E}$ and $\mathcal{D}' = (C^\infty_0(\mathbb{R}))'$ of the distribution $S = \mathcal{F}^{-1}(\varphi)$, which means
$$S * \mathcal{E} = \mathcal{E},$$
$$S * \mathcal{D}' = \mathcal{D'},$$
where the symbol $*$ denotes the convolution.

We say that $\varphi \in \mathcal{P}$ is invertible in the sense of Ehrenpreis if the division theorem is valid for it (see [1], [2]).

Below, we will use the following analytical criterion due to L. Ehrenpreis [6, Theorems I, 2.2, Proposition 2.7]: $\varphi \in \mathcal{P}$ is invertible (in the sense of Ehrenpreis) if and
only if it is \textit{slowly decreasing}, i.e. there exists $a > 0$ such that
\[\forall x \in \mathbb{R} \exists x' \in \mathbb{R} : |x - x'| \leq a \ln (2 + |x|), \quad |\varphi(x')| \geq (a + |x'|)^{-a}. \quad (1.1)\]

Each $S \in \mathcal{E}'$ generates the convolution operator $M_S$ acting in $\mathcal{E}$:
\[M_S(f) = S * f, \quad f \in \mathcal{E}.\]

It is easy to check that $\varphi \in \mathcal{P}$ is invertible in the sense of Ehrenpreis if and only if the convolution operator $M_S$ generated by $S = \mathcal{F}^{-1}(\varphi)$ is surjective.

Let $\varphi \in \mathcal{P}$ be invertible in the sense of Ehrenpreis, $\Lambda$ be its zero set, $\Lambda' \subset \Lambda$. Then, $(i\Lambda')$ coincides with the spectrum of the differentiation-invariant subspace $W \subset \mathcal{E}$ which has the following property: each $f \in W$ is represented as a series with grouping of exponential monomials contained in $W$. This fact is established in [6], [4], [3] for the subspaces of the form
\[W = \{f \in \mathcal{E} : S * f = 0\},\]
where $S \in \mathcal{E}'$ is fixed. And it is also true for general subspaces admitting (weak) spectral synthesis with respect to the differentiation operator.

Summarizing the above, we may conclude that there are enough reasons to study the behavior and zero subsets of functions $\varphi \in \mathcal{P}$ which are invertible in the sense of Ehrenpreis.

We establish some restrictions on the distribution of real parts of zeros lying not far from the real axis (Theorem 2.1). The result generalizes [2, Lemma 2] and [6, Proposition 6.1]).

2 Zero sets

Let $\mathcal{M} = \{\mu_j\}$, $\mu_j = \alpha_j + i\beta_j$,
\[0 < |\mu_1| \leq |\mu_2| \leq \ldots,\]
be such that $\beta_j = O(\ln |\mu_j|)$ as $j \to \infty$, and the formula
\[\psi(z) = \lim_{R \to \infty} \prod_{|\mu_j| \leq R} \left(1 - \frac{z}{\mu_j}\right) \quad (2.1)\]
defines entire function of exponential type.

In [2] Lemma 1, we established the following fact.

\textbf{Lemma A.} The function $\psi$ defined by (2.1) is invertible in the sense of Ehrenpreis element of the Schwartz algebra $\mathcal{P}$ if and only if the same is true about the function
\[\psi_1(z) = \lim_{R \to \infty} \prod_{|\alpha_j| \leq R} \left(1 - \frac{z}{\alpha_j}\right). \quad (2.2)\]
Taking into account Lemma A, we give another formulation of [2, Lemma 2].

**Lemma B** Let $\psi \in \mathcal{P}$ be invertible in the sense of Ehrenpreis, $\mathcal{M} = \{\mu_k\} \subset \mathbb{C}$ be its zero set.

Then, for the subset $\mathcal{M}' \subset \mathcal{M}$ defined by

$$\mu_k \in \mathcal{M}' \iff |\text{Im} \mu_k| \leq M_0 \ln |\text{Re} \mu_k|$$

with $M_0 > 0$ fixed,

$$\lim_{|x| \to \infty} \frac{m_{\text{Re}}(x, 1)}{\ln |x|} < \infty,$$

where $m_{\text{Re}}(x, 1)$ denotes the number of points of the sequence

$$\text{Re} \mathcal{M}' = \{\text{Re} \mu_k : \mu_k \in \mathcal{M}'\}$$

contained in the segment $[x - 1; x + 1]$.

Consider non-decreasing function $l : [0; +\infty) \to [1; +\infty)$ satisfying

$$\ln t = O(l(t)), \quad t \to \infty, \quad (2.3)$$

$$\lim_{t \to +\infty} \frac{\ln l(t)}{\ln t} < \frac{1}{2}, \quad (2.4)$$

and

$$\lim_{t \to +\infty} \frac{l(Kt)}{l(t)} < +\infty \quad (2.5)$$

for some $K > 1$.

The following theorem generalizes Lemma B.

**Theorem 2.1.** Let $\psi \in \mathcal{P}$ be invertible in the sense of Ehrenpreis with the zero set $\mathcal{M} = \{\mu_k\}$, and $\mathcal{M}' \subset \mathcal{M}$ be defined by

$$\mu_k \in \mathcal{M}' \iff |\text{Im} \mu_k| \leq M_0 \cdot l(|\text{Re} \mu_k|)$$

for a fixed $M_0 > 0$.

Then,

$$\lim_{|x| \to \infty} \frac{m_{\text{Re}}(x, 1)}{l(|x|)} < \infty, \quad (2.6)$$

where $m_{\text{Re}}(x, 1)$ denotes the number of points of the sequence

$$\text{Re} \mathcal{M}' = \{\text{Re} \mu_k : \mu_k \in \mathcal{M}'\}$$

contained in the segment $[x - 1; x + 1]$.

First, we prove the following auxiliary proposition.
Proposition 2.1. Let \( \psi, \mathcal{M}, \mathcal{M}' \) be the same as in Theorem 2.1,
\[ \mathcal{M}'' = \mathcal{M} \setminus \mathcal{M}', \quad \alpha_j = \text{Re} \mu_j. \]

Then, the function
\[ \psi_1(z) = \lim_{R \to \infty} \left( \prod_{|\mu_j| \leq R} \left( 1 - \frac{z}{\alpha_j} \right) \prod_{|\mu_j| \leq R} \left( 1 - \frac{z}{\mu_j} \right) \right) \quad (2.7) \]
belongs to the algebra \( \mathcal{P} \), and there exists \( M_1 > 0 \) such that
\[ \forall x \in \mathbb{R}, |x| > 2, \exists z' \in \mathbb{C}: |z' - x| \leq M_1 l(|x|) \quad \text{and} \quad \ln |\psi_1(z')| \geq -M_1 l(|z'|). \quad (2.8) \]

Proof. We start with estimating the single multiplier \( 1 - \frac{x}{\alpha_j} \), \( x \in \mathbb{R} \), where \( \alpha_j = \text{Re} \mu_j, \mu_j \in \mathcal{M}' \). It is easy to see that
\[ \left| 1 - \frac{x}{\alpha_j} \right| \leq \left| 1 - \frac{x}{\mu_j} \right| \left( 1 + \frac{M_2^2 \beta^2(\alpha_j)}{\alpha_j^2} \right)^{1/2} \leq \left| 1 - \frac{x}{\mu_j} \right| \left( 1 + O \left( \frac{\beta^2(\alpha_j)}{\alpha_j^2} \right) \right). \]

Taking into account (2.5), we get
\[ \ln |\psi_1(x)| \leq \text{const ln} |\psi(x)|, \quad x \in \mathbb{R}. \quad (2.9) \]

Hence, \( \psi_1 \in \mathcal{P} \).

To estimate \( |\psi_1| \) from below, we consider the auxiliary function
\[ \psi^+(z) = \lim_{R \to \infty} \left( \prod_{|\mu_j| \leq R} \left( 1 - \frac{z}{\mu_j} \right) \prod_{|\mu_j| \leq R} \left( 1 - \frac{z}{\alpha_j} \right) \right), \]
where
\[ \mathcal{M}'_+ = \{ \mu_j \in \mathcal{M}' : \beta_j \geq 0 \}, \quad \mathcal{M}'_\times = \mathcal{M}' \setminus \mathcal{M}'_+. \]

Because of (2.5), we have
\[ l(Kt) \leq C_0 l(t), \quad t \geq 0, \quad (2.10) \]
for some \( C_0 > 0 \). Notice that
\[ \left| 1 - \frac{z}{\mu_j} \right| \leq \frac{|\mu_j - z|}{|\alpha_j|} = \frac{((\alpha_j - x)^2 + (\beta_j - y)^2)^{1/2}}{|\alpha_j|}, \]
where \( \mu_j = \alpha_j + i\beta_j \in \mathcal{M}_+ \), \( z = x + iy \). Together with (2.10), it gives us the inequality
\[
\left| 1 - \frac{z}{\mu_j} \right| \leq \left| 1 - \frac{z}{\alpha_j} \right| \tag{2.11}
\]
if \( z = x + 2iC_0M_0l(|x|), \mu_j \in \mathcal{M}_+ \) and \( |\text{Re}\mu_j| = |\alpha_j| \leq 4K|x| \).

From the other hand, for \( z = x + 2iC_0M_0l(|x|), \mu_j = \alpha_j + i\beta_j \in \mathcal{M}_+ \) we have
\[
\left| 1 - \frac{z}{\mu_j} \right| \leq \left| 1 - \frac{z}{\alpha_j} \right| \cdot \left( 1 + \frac{C_1l^2(\alpha_j)}{\alpha_j^2} \right)^{1/2} \tag{2.12}
\]
if \( |\alpha_j| > 4K|x| \), where the constant \( C_1 > 0 \) depends only on \( l \) and \( M_0 \).

The relations (2.11), (2.11), (2.12) lead to the estimate
\[
\ln |\psi(z)| \leq \text{const} \ln |\psi^+(z)| + O(1) \quad \text{as} \quad |x| \to \infty \tag{2.13}
\]
if \( z = x + 2iC_0M_0l(|x|) \).

By the similar way, we get that
\[
\ln |\psi^+(z)| \leq \text{const} \ln |\psi(z)| + O(1), \quad \text{as} \quad |x| \to \infty, \tag{2.14}
\]
where \( z = x - 2iC_0M_0l(|x|) \).

Applying the analytical criterion \( l \) and the minimum modulus theorem \([8, \text{Ch. 1, Sec. 8, Th. 11}]\), we arrive to the estimate
\[
\ln |\psi(z)| \geq -C_2 \ln |x|
\]
for all \( z : |z - x| = C_2 \ln |x|, x \in \mathbb{R}, |x| > 2 \) and some \( C_2 > 0 \).

Notice that \( \psi \in \mathcal{P} \) implies the inequality
\[
|\psi(z)| \leq C_\psi(2 + |z|)^{C_\psi} e^{C_\psi|\text{Im}z|} \tag{2.15}
\]
with some \( C_\psi > 0 \).

Fix \( x \in \mathbb{R}, |x| > 2 \), set \( z_x = x + iC_2 \ln |x| \) and then apply the minimum modulus theorem to the function \( \frac{\psi}{\psi(z_x)} \) in the disc \( |z - z_x| \leq 4C_0M_0l(|x|) \). Taking into account (2.3), (2.5) and (2.15), we find \( M_2 > 0 \) and \( \theta \in (2; 4) \) such that
\[
\ln |\psi(z)| \geq -M_2 l(|x|) \quad \text{if} \quad |z - z_x| = \theta C_0M_0l(|x|). \tag{2.16}
\]

Further, from (2.13), (2.16) and (2.4)–(2.5), it follows that there exists \( M_3 > 0 \) with the property:
\[
\forall x \in \mathbb{R}, |x| > 2, \exists w_x \in \mathbb{C} \quad \text{such that} \quad |w_x - x| \leq M_3 l(|x|) \quad \text{and} \quad \ln |\psi^+(w_x)| \geq -M_3 l(|w_x|).
\]

Now, we apply the above argument including the minimum modulus theorem to the function \( \frac{\psi^+}{\psi^+(w_x)} \) and the disc \( |z - w_x| \leq R \), where
\[
R = \max\{2M_3 l(|x|), 4C_0M_0l(|x|)\}.
\]

It gives us the estimate for \( \ln |\psi^+(z)| \) which is similar to (2.16). This estimate and (2.14), together with (2.3)–(2.5), lead us to the assertion. \( \square \)
Remark 1. It is not difficult to see that applying the minimum modulus theorem one more time, to the function $\frac{\psi_1}{\psi(x')}$, we obtain the following version of Proposition 2.1: there exists $M_1 > 0$ such that
\[
\exists M_1 > 0 : \forall x \in \mathbb{R}, |x| > 2, \exists x' \in \mathbb{R} : \\
|x' - x| \leq M_1 l(|x|) \quad \text{and} \quad \ln |\psi_1(x')| \geq -M_1 l(|x'|).
\] (2.17)

Proof of Theorem 2.1.
Wlog, we assume that $\psi$ is bounded on the real axis and its exponential type equals 1. Because of (2.9), we may also assume that the same is true for the function $\psi_1$ defined by (2.7). Let $\lambda_k \in \mathcal{M}$ and $\alpha_k = \Re \lambda_k$.

Further in the proof, we use symbol $\psi$ to denote the function $\psi_1$ defined in (2.7).

If (2.6) fails then
\[
\lim_{j \to \infty} \frac{m_{\Re(x_j, 1)}}{l(|x_j|)} = \infty
\] (2.18)
for some $x_j, |x_j| \to \infty$. For clarity, we assume that $x_j > 0$.

Consider entire functions
\[
\psi_j(z) = \psi(z)(z - x_j)^{m_j} \cdot \prod_{k: |\alpha_k - x_j| \leq 1} (z - \alpha_k)^{-1},
\]
where $m_j = m(x_j, 1), j = 1, 2, \ldots$ It is easy to check that the estimates
\[
\sup_{x \in \mathbb{R}} |\psi_j(x)| \leq C_0 2^{m_j}, \quad j = 1, 2, \ldots,
\]
hold with $C_0 = \max\{\sup_{t \in \mathbb{R}} |\psi(t)|, 1\}$. By Bernstein’s theorem [5, Chapter 11], the inequalities
\[
\sup_{x \in \mathbb{R}} |\psi_j^{(n)}(x)| \leq C_0 2^{m_j}
\] (2.19)
are also valid for all $n, j \in \mathbb{N}$.

From the Taylor expansion of $\psi_j$ at $x_j$ and the estimates (2.19), it follows that
\[
|\psi_j(z)| \leq C_0 2^{m_j} (m_j!)^{-1} |z - x_j|^{m_j} e^{\|z - x_j\|}, \quad z \in \mathbb{C}.
\]

Hence, for all $x \in \mathbb{R}$ satisfying the condition
\[
\ln C_0 + |x - x_j| + m_j \ln |x - x_j| - \ln (m_j!) \leq -n l(x_j) - m_j \ln 2,
\] (2.20)
where $n \in \mathbb{N}$, we have the inequality
\[
|\psi_j(x)| \leq x_j^{-n} \cdot 2^{-m_j}.
\] (2.21)

By Stirling’s formula, the relation (2.20) will follow from the inequality
\[
|x - x_j| + m_j \ln |x - x_j| - m_j \ln m_j \leq -n l(x_j) - C_1 m_j,
\]
where $C_1$ is an absolute constant.
Because of (2.18), for each \( n \in \mathbb{N} \) we can find \( j_n \) such that
\[
- n l(x_j) \geq -m_j, \quad j = j_n, j_n + 1, \ldots
\] (2.22)

Fix \( b \in (0; e^{-C_1-2}) \). The estimates (2.21) are valid for \( j \geq j_n \) and all \( x \in \mathbb{R} \) such that \( |x - x_j| \leq bm_j \). This fact, inequalities (2.22) and the relations
\[
|\psi(z)| \leq 2^{m_j}|\psi_j(z)|, \quad z \in \mathbb{C}, \quad j = 1, 2, \ldots,
\]
imply
\[
|\psi(x)| \leq e^{-nl(x_j)}
\]
for \( x \in \mathbb{R} \) satisfying \( |x - x_j| \leq bn l(x_j), \quad j \geq j_n, \quad n \in \mathbb{N} \). It means that \( \psi \) does not satisfy (2.17), i.e. the assumption (2.18) leads to the contradiction.

Q.E.D.

Acknowledgments

This work is supported by Ministry of Science and Higher education of Russian Federation (code of scientific theme FZWU-2020-0027).

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