Simple solutions to the Einstein Equations in spaces with unusual topology

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Abstract

We discuss simple vacuum solutions to the Einstein Equations in five dimensional space-times compactified in two different ways. In such spaces, one black hole phase and more then one black string phase may exist. Several old metrics are adapted to new background topologies to yield new solutions to the Einstein Equations. We then briefly talk about the angular momentum they may carry, the horizon topology and phase transitions that may occur.

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1 Introduction

Higher dimensional gravity has been a very active area of theoretical physics lately. Long ago, the Schwarzschild solution has been generalized to higher dimensions by Tangherlini \cite{1}. Higher dimensional metrics that satisfy the Einstein Equations can be obtained from lower dimensional ones by tensoring a Ricci-Flat space to a known vacuum solution. In this fashion, one can build things like black strings or black branes by tensoring some flat euclidean space to a black hole solution. A black string living in 5 dimensions would then have the metric:

\[
 ds_{BS}^2 = ds_{BH}^2 + dx_4^2
\]  

where \( ds_{BH}^2 \) is any black hole metric.

The uniform black strings have been shown to be unstable when not thick enough \cite{2,3} and stationary non-uniform solutions have been found \cite{4,5}. Numerical evolutions of black strings have also shown them to be unstable and suggest that thin black strings decay into black holes \cite{6}. There is still some controversy over the end point of the black string instability \cite{7} as in classical General Relativity the horizon topology cannot easily change.

Black ring solutions that have horizon topology \( S^2 \times S^1 \) in asymptotically flat five dimensional space-time and violate black hole uniqueness theorem were recently discovered \cite{8,9,10,11,12}.

So far, solutions of the Einstein Equations like black holes, black strings and branes or gravitational waves have only been studied in space-times that are either non-compact or periodically compactified. In this essay I will take a few baby steps away from the beaten path and explore a few simple solutions of Einstein Equations in spaces with antiperiodic compactification and spaces where compactification is done by identifying parallel hyperplanes after rotating them by some angle \( \alpha \).

We’ll explore solutions with simple metrics and distinct topologies and discuss phase transitions that are likely to occur between these black objects.

The spaces discussed here are not very much like the world we live in. In the limit where the length of the compact dimension becomes zero, the five dimensional space-time antiperiodically compactified becomes an orbifold. The same is true for the space with twisted compactification.

Throughout this essay, \( x_0 \) stands for time and \( x_4 \) is the compact direction.
Figure 1: An observer near a black hole in an antiperiodically compactified space will see an array of infinitely many black hole images arranged like above. This solution is neither stationary, nor exact as the holes feel an attractive force $F$ toward the $x_1 = 0$ hyperplane.

2 Antiperiodic compactification

Antiperiodic compactification of five dimensional space-time is given by the following identification:

$$ (x_0, x_1, x_2, x_3, x_4) = (x_0, -x_1, x_2, x_3, x_4 + L) \quad (2) $$

The resulting space is flat but its non-trivial topology has the following implications

- The space is not orientable. Therefore one cannot have a globally-defined volume form.
- The hyperplane $x_1 = 0$, although non-singular is singled out as ”special” as it is invariant under the identification (2).
- There are no globally defined continuous vector fields with nowhere vanishing $x_1$ component.
- The $x_1$ component of the momentum of a particle moving in this space is not conserved.

Just like in the case of periodic compactification [3], exact solutions for
Figure 2: Type A and Type B black strings in antiperiodically compactified background. Picture depicts covering space. The Type A string is both stationary and exact, but the horizon has topology of a higher dimensional Klein bottle. Type B is neither stationary nor exact but the horizon has the "nice" $S^2 \times S^1$ topology.

five dimensional black holes and black rings cannot be written with our present knowledge. One non-rotating, uncharged black hole in this background will be a static solution only if its center lies in the $x_1 = 0$ hyperplane. In this case, the metric is actually identical to the one of a black hole sitting in periodically compactified space because of its reflection symmetry.

If it wanders away, the black hole will be attracted toward $x_1 = 0$ by its image and it will undergo an oscillatory motion. (Fig. 1)

The antiperiodic compactification allows two types of black strings to exist. We’ll call them Type A and Type B.

Uncharged uniform non-rotating black string solutions will be stationary only if they lie on the $x_1 = 0$ hyperplane. In this case (Type A, Fig. 2), the metric is identical to that of the uniform black string living in periodically compactified space:

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \frac{1}{1 - \frac{r_0}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 + dx_4^2$$  \hspace{1cm} (3)

This metric is invariant under (2). To see this clearly we will write (2) in $(t, r, \theta, \phi, x_4)$ coordinates. The transformation from spherical to Cartesian coordinates is

$$x = r \sin \theta \cos \phi$$  \hspace{1cm} (4)
\[ y = r \sin \theta \sin \phi \]  
\[ z = r \cos \theta \]  

Depending on which of \( x, y, z \) we pick to be \( x_1 \), Eq. (2) becomes one of the following:

\[ (t, r, \theta, \phi, x_4) = (t, r, \theta, \pi - \phi, x_4 + L) \]  
\[ (t, r, \theta, \phi, x_4) = (t, r, \theta, -\phi, x_4 + L) \]  
\[ (t, r, \theta, \phi, x_4) = (t, r, \pi - \theta, \phi, x_4 + L) \]  

The metric (3) is invariant under (7, 8, 9). Actually, due to spherical symmetry, only one of (7), (8), (9) would have been enough but we’ll need them all a little later when we discuss rotating black strings.

Unlike its periodically compactified counterpart the horizon topology of this string is that of a three dimensional Klein bottle. The surface has no singularities but it is non-orientable.

An infinite rotating black string in uncompactified or periodically compactified space-time will have the metric of a Kerr black hole tensored with an extra flat direction:

\[ ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr\sin^2 \theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2\Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2 \theta}{\Sigma}\right)\sin^2 \theta d\phi^2 + dx_4^2 \]  

where

\[ a = \frac{J}{M} \]  
\[ \Delta = r^2 - 2Mr + a^2 \]  
\[ \Sigma = r^2 + a^2 \cos^2 \theta \]
One can regard $M$ and $J$ as mass and angular momentum per unit length in the appropriate units.

The metric (10) is only invariant under (2) and therefore the only uniformly rotating black strings that can survive in antiperiodically compactified background are those rotating in the $(x_2, x_3)$ plane.

Perturbations that do not alter the horizon topology can move parts of the Type A string away from $x_1 = 0$ hyperplane, but the string horizon will always intersect $x_1 = 0$.

A topologically different black string (Fig. 2) Type B) is obtained by tensoring a flat direction to the space-time containing two Schwarzschild black holes at rest with respect to each other and then doing the identification (2). The original metric has the right symmetries to be invariant under the identification (2). The solution has the following properties:

- The horizon topology is $S^2 \times S^1$, it is orientable and may rotate in the planes $(x_1, x_2)$, $(x_1, x_3)$ and $(x_2, x_3)$. The length of the string loop is $2L$, twice that of the compact direction.
- Although the string may rotate, the total angular momentum in the $(x_1, x_2)$ and $(x_2, x_3)$ planes will be zero as an observer living in the antiperiodically compactified space will see the two equal sides of the string rotating in opposite directions. Only rotation in the $(x_2, x_3)$ plane will yield non-zero total angular momentum.
- One cannot write an exact solution for the Type B black string right now because no exact solution exists for two black holes colliding head on or at rest with respect to each other in 4D.
- It is not stationary. Black holes will plunge toward each other and coalesce in a finite amount of time. One could ”fix” this by adding the right cosmological constant or some charge.

2.1 Phase transitions

When pulled toward $x_1 = \pm \infty$ by an external agent the Type A string will deform somewhat like in Fig 3. At some point this string will become more massive than a Type B string of the same thickness. Then it becomes energetically favorable for the Type A string to decay either in a Type B string or - if light enough - to Gregory-Laflamme decay into a black hole.

Moving the hole or the Type B string away from the $x_1 = 0$ hyperplane is much easier than moving the Type A string away. If we assume the Type A string remains of constant thickness and its mass is proportional to its length
Figure 3: A perturbed Type A black string in a world with space compactified as \((x_0, x_1, x_2, x_3, x_4) = (x_0, -x_1, x_2, x_3, x_4 + L)\) may undergo a phase transition onto either a Type B string or a black hole, both of which can move away easier from the \(x_1 = 0\) hyperplane.

then for large distances from \(x_1 = 0\) the agent pulling this string away will work against a potential

\[
V \sim \sqrt{x_1^2 + \left(\frac{L}{2}\right)^2}
\]  

(14)

which, for large enough \(x_1\) becomes

\[
V \sim |x_1|
\]  

(15)

while moving a black hole or a Type B string away from \(x_1 = 0\) when \(x_1\) is large enough means only working against a potential

\[
V \sim \frac{1}{|x_1|}
\]  

(16)

which obviously goes to zero for large \(x_1\).
3 Compactification with a twist

In previous section we looked at antiperiodically compactified space. In this section we’ll identify hyperplanes after performing a rotation, namely

\[(x_0, x_1, x_2, x_3, x_4) = (x_0, x_1 \cos(\alpha) + x_2 \sin(\alpha), x_2 \cos(\alpha) - x_1 \sin(\alpha), x_3, x_4 + L)\] (17)

These spaces have been studied in [13] and found to be related to magnetic fields in the Kaluza-Klein Theory.

Although we will deal with arbitrary \( \alpha \) here, the particular case of (17) when \( \alpha = \pi \) is similar to (2) when \( \alpha = \pi \) is similar to (2)

\[(x_0, x_1, x_2, x_3, x_4) = (x_0, -x_1, -x_2, x_3, x_4 + L)\] (18)

The transformation (17) leaves points on the line \( x_1 = x_2 = 0 \) unchanged. Therefore, a Schwarzschild or Kerr-Newmann black hole rotating in the \((x_1, x_2)\) plane centered on this line will have identical metric with its counterpart living in periodically compactified space.

Axisymmetric black strings rotating in the \((x_1, x_2)\) plane or not rotating at all will have metrics identical with their counterparts living in periodically compactified space (10) and respectively (3) just because their metrics are invariant under (17) which becomes

\[(t, r, \theta, \phi, x_4) = (t, r, \theta, \phi + \alpha, x_4)\] (19)

If \( \alpha = \frac{m}{n} \pi, m, n \in \mathbb{Z}, n \neq 0 \) one can come up with a seemingly odd solution to the Einstein Equations - a set of \( n \) uniform black strings equally spaced on a large enough circle. Due to the compactification (17) the end of one string will be identified with the beginning of another and the final solution will have no "free ends".

Since the space is orientable, one can have the whole solution rotate at the exact speed necessary to keep the strings at a constant distance from the origin.

If one neglects gravitational radiation, this becomes a rather strange solution, even stationary in the corotating frame.

When the string (1) is taken away from the origin, its length must increase. Neglecting the perturbations the string causes to the background geometry, the minimum length of the string situated a distance \( r \) away from
the origin is $\sqrt{(2r \sin(\frac{\alpha}{2}))^2 + L^2}$. For large $r$ and $\alpha \neq 2k\pi$, $k \in \mathbb{Z}$ the string length and hence its mass becomes proportional to $r$ and it becomes energetically favorable for the string to either Gregory-Laflamme decay into a black hole or, if massive enough, a string that winds around the compact direction $n$ times. This phase transition will be rather interesting and unusual for large $n$.

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