Asynchronous Fully-Decentralized SGD in the Cluster-Based Model

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This paper presents fault-tolerant asynchronous Stochastic Gradient Descent (SGD) algorithms. SGD is widely used for approximating the minimum of a cost function $Q$, as a core part of optimization and learning algorithms. Our algorithms are designed for the cluster-based model, which combines message-passing and shared-memory communication layers. Processes may fail by crashing, and the algorithm inside each cluster is wait-free, using only reads and writes.

For a strongly convex function $Q$, our algorithm tolerates any number of failures, and provides convergence rate that yields the maximal distributed acceleration over the optimal convergence rate of sequential SGD.

For arbitrary functions, the convergence rate has an additional term that depends on the maximal difference between the parameters at the same iteration. (This holds under standard assumptions on $Q$.) In this case, the algorithm obtains the same convergence rate as sequential SGD, up to a logarithmic factor. This is achieved by using, at each iteration, a multidimensional approximate agreement algorithm, tailored for the cluster-based model.

The algorithm for arbitrary functions requires that at least a majority of the clusters contain at least one non-faulty process. We prove that this condition is necessary when optimizing some non-convex functions.

1 INTRODUCTION

Optimization attempts to minimize the value of a cost function $Q : \mathbb{R}^d \rightarrow \mathbb{R}$, that is, find $x^* \in \arg \min_{x \in \mathbb{R}^d} Q(x)$. Among their many uses, optimization problems play a key role in machine and deep learning [22]. SGD [31] repeatedly applies the update rule $x_{t+1} = x_t - \eta_t G(x_t, z_t)$, in each iteration $t$. The arguments for this rule are the learning parameter $x_t$, the learning rate $\eta_t$, and a random sample $z_t$ from data distribution $D$; $G(x_t, z_t)$ computes the stochastic gradient of $x_t$ and $z_t$, which is a biased estimator of the true gradient $\nabla Q(x_t)$.

In learning applications, SGD is applied to functions $Q$ of high dimension $d$, using a massive number of stochastic gradients [10]. The convergence of the basic SGD algorithm can be improved by mini-batch SGD, which computes $B$ stochastic gradients using $B$ samples, drawn uniformly at random from $D$. The average of these $B$ gradients has variance that is a factor of $B$ smaller than $\sigma^2$, the variance of a single stochastic gradient. Since gradients can be computed independently, (mini-batch) SGD is a prime target for computation in large-scale distributed and parallel systems.

This paper considers completely decentralized and asynchronous SGD in a model that combines both shared memory and message passing. In this cluster-based model [6, 29], processes are partitioned to disjoint clusters, each sharing a memory space. Additionally, all processes can communicate by message passing. (See Figure 1.) Process may fail by crashing; that is, stopping to take steps. This model is interesting from a practical point-of-view, as it captures several system architectures. The model also encapsulates, in a smooth manner, the two major communication models for distributed and parallel computing, and provides a theoretical foundation for both.

In an iteration of a typical distributed SGD algorithm, a worker perform some local computation and then, aggregates the computed values to collectively compute parameters for the next iteration [7]. This can be done in a centralized manner, where a parameter server aggregates all the computation done by the workers (Figure 2 (left)), or in a decentralized manner (Figure 2 (middle)), where each worker holds a copy of the parameters. A straightforward, synchronized implementation of (mini-batch) SGD requires locks or barriers to ensure that workers proceed in lock-step, thereby harming the performance.

Our first main contribution (Section 3) shows that when the function $Q$ is strongly convex, a simple asynchronous algorithm that collects $N$ messages in each iteration, matches the convergence rate of a sequential mini-batch SGD algorithm (for strongly-convex functions) with batch size $N$ [3, 8]. Unlike synchronous mini-batch SGD [11], where synchronization is needed inside multiprocessing workers, our algorithm is fully asynchronous, avoiding
costly synchronization mechanisms. Our analysis of this algorithm is relatively simple, and leverages the strong convexity of $Q$ to prove convergence, despite the fact that each process holds a local copy of the learning parameters.

**Theorem 1 (simplified).** For any smooth strongly-convex function $Q$ with minimum $x^*$, the output $x_i$ of process $i$ from the cluster-based SGD algorithm optimizing $Q$ with parameter $N$ after $T$ iterations satisfies

$$\mathbb{E} \left[ \|x_i - x^*\|_2^2 \right] = O \left( \frac{1}{T} + \frac{\sigma^2}{NT} \right)$$

Somewhat surprisingly, this algorithm makes no assumption about the ratio between the number of failures $f$ and the number of processes $n$ (only that $f < n$). Our second main contribution (Section 5) shows that this does not hold in general, when $Q$ is not strongly convex: for some functions, any SGD algorithm requires that at least a majority of the clusters contain at least one nonfaulty process. In the special case of singleton clusters, i.e., when communication is only through message passing, this reduces to assuming that $n > 2f$.

Our third main contribution (Section 4) is a general SGD algorithm, under the assumption that at least a majority of the clusters contain at least one nonfaulty process. It has a weaker convergence guarantee, with an additional term, denoted $\Delta$, depending on the difference between the learning parameters of the different processes during the algorithm execution.

**Theorem 3 (simplified).** For any smooth function $Q$, the output $x_i$ of process $i$ from the cluster-based SGD algorithm optimizing $Q$ with parameter $N$ for $T$ iterations satisfies

$$\mathbb{E} \left[ \|\nabla Q(x_i)\|_2^2 \right] = O \left( \frac{1}{\sqrt{NT}} + \frac{\sigma^2}{\sqrt{NT}} + T\Delta \right)$$

The first two terms of Theorem 3 match the convergence rate achieved in standard analysis for non-convex objectives [16]. Unlike the strongly convex case, where difference between the learning parameters is intrinsically bounded, here we bound $\Delta$ using *multidimensional approximate agreement (MAA)* [27]. In MAA, processes start with inputs in $\mathbb{R}^d$, and the outputs of nonfaulty processes should be "close together" and in the convex-hull of the their inputs. We use a shared-memory adaptation of [15] to bound the difference between the values sent from the same cluster. (Shared memory replaces the assumption of non-split communication patterns made in [15].) By ensuring every pair of processes communicate with a representative process (not necessarily the same process) from at least one common cluster, we ensure good contraction at each iteration.

MAA causes each iteration to require several *communication rounds*. (A communication round requires each process to send a message and receive responses from a subset of nonfaulty processes; the cost of inter-cluster communication in each round is neglected.) We prove that our general SGD algorithm can match the convergence rate of the sequential algorithm, *up to a logarithmic blowup* in the number of communication rounds.
Theorem 4 (simplified). For any smooth function $Q$, the output $x_i$ of process $i$ from the cluster-based SGD algorithm optimizing $Q$ with parameter $N$ for $R$ communication rounds satisfies

$$\mathbb{E} \left[ \| \nabla Q(x_i) \|^2 \right] = \tilde{O} \left( \frac{1}{\sqrt{NR}} + \frac{\sigma^2}{\sqrt{NR}} \right)$$

Our algorithms are completely decentralized and asynchronous. The intra-cluster algorithms are wait-free and use only read and write operations on the cluster’s shared memory; no locks or barriers are used inside a cluster. Each process computes and sends stochastic gradients independently from its cluster, thereby achieving a speedup in the total number of processes and not clusters. Our algorithms also improve the fault tolerance relative to a pure message passing model, since a process can represent its cluster [6].

Related Work

Elastic consistency [28] is a framework that assumes that the difference between the parameter used to compute the stochastic gradient by a process and the actual global parameter is bounded. SGD converges under this assumption, for both convex and non-convex objective functions. This framework is agnostic to the system model and can be applied to several existing frameworks and algorithms. One example is in shared-memory [5, 30], where processes access the same learning parameter stored in memory and update it, each coordinate at a time, using fetch&add. Another example is with message-passing [23], where the parameter server may receive stale gradients, i.e., gradients computed using old parameters. Both cases assume bounded asynchrony, with a maximum delay $\tau$ on the staleness of gradients. In contrast, our algorithms are completely asynchronous, and do not assume any bound on the difference between the iterations different processes are in at any point in the execution. We achieve this by ignoring stale gradients from past iterations, allowing us to explicitly bound the convergence, without bounding the maximum delay or relying on elastic consistency.

Approximate agreement [12] was defined for one dimensional inputs and outputs. Multidimensional approximate agreement was defined for asynchronous systems with Byzantine failures [27], where it was proved that the optimal fault-tolerance for this problem is $n > f(d + 2)$, for inputs with dimension $d$. They require the correct processes’ outputs to be close together and in the convex-hull of the correct inputs. The lower bound can be circumvented by using averaging agreement [14], where convexity is replaced by the requirement that the Euclidean difference between the average of the nonfaulty inputs and outputs should be proportional to the initial Euclidean distance between the inputs. Using averaging agreement allows [14] to prove the convergence of the average of the outputs of nonfaulty nodes.

The lower bound of $n > f(d + 2)$ [27] does not apply with crash failures, and we solve asynchronous MAA assuming only $n > 2f$. Using convexity of MAA, we prove that the learning parameters at each process converges individually, and not just the average parameters over all processes. Byzantine optimization was introduced in [32]. It was shown that $2f$-redundancy is a necessary and sufficient condition for $f$-resilient Byzantine deterministic optimization, both exact [18] and approximate [24].

In the heterogeneous case, where each process has a different cost function, averaging agreement is equivalent to asynchronous decentralized Byzantine learning [14]. This paper shows two algorithms, one that requires $n \geq 6f + 1$, achieving the best-possible asymptotic averaging constant, and another that requires only $n \geq 3f + 1$, by using improved techniques for approximate agreement [2]. They also prove a lower bound of $n > 3f$ for averaging agreement, implying the same lower bound for Byzantine learning. Our algorithms are similar to the homogeneous-cost algorithm of [14], but their algorithm tolerates Byzantine failures while our algorithm tolerates only crash failures. However, our algorithms are tailored to the cluster-based model, exploiting shared-memory communication, while their algorithms rely only on message passing. In addition, our algorithms improve the convergence rate as a function of the number of participating processes, a feature that is lacking in many
Byzantine-resilient SGD algorithms [25]. An exception is the algorithm of [4], which achieves speedup in the number of processes, but this algorithm is synchronous and centralized.

A hybrid setting, with a set of parameter servers and a set of workers (see Figure 2 (right)), is considered in [13]. They assume that the number of workers is $3f_w + 1$, where at most $f_w$ workers can be Byzantine, and the number of parameter servers is $3f_{ps} + 2$, where at most $f_{ps}$ parameter servers can be Byzantine. They make a non-standard assumption that any communication pattern between non-Byzantine servers eventually holds with probability 1. They consider smooth non-convex cost functions, but do not bound the convergence rate of their algorithm and only show eventual convergence.

**Federated learning** (cf. [20]) considers the centralized setting with synchronous iterations; however, not all the workers participate in each learning iteration. A prominent algorithm is FedAvg [26]. When each worker does one or more SGD iterations and sends its updated parameters to the parameter server, which simply averages them to compute the parameters for the next iteration. It was shown [21] that non-uniform distribution of data across the workers can hinder the convergence of FedAvg, by introducing drift in the updates sent by the workers to the server. The SCAFFOLD [21] algorithm adds a correction term to mitigate the drift introduced by the workers. Despite the differences between our model and the federated learning framework, our analysis utilizes some techniques from [21].

## 2 Preliminaries

### 2.1 Distributed Model

There are $n$ processes, $1, \ldots, n$, which are partitioned to $m$ disjoint clusters, $P_1, \ldots, P_m$. Formally, $P_i \subseteq \{1, \ldots, n\}$, $P_i \cap P_j = \emptyset$ for every $i$ and $j$, $1 \leq i < j \leq m$, and $\bigcup_{i=1}^{m} P_i = \{1, \ldots, n\}$. Given a process $i$, cluster$(i)$ is its cluster, i.e., $i \in P_{\text{cluster}(i)}$. We slightly abuse notation, and define for a set of processes identifies $V \subseteq [n]$, $\text{cluster}(V) = \{\text{cluster}(i) : i \in V\}$.

Any pair of processes can send messages to each other, using an asynchronous and reliable communication link. In addition, a *shared memory* is associated with each cluster. This memory is accessed with read and write operations, only by the processes in the cluster.

Processes may crash and stop taking steps. We say cluster $c$ crashes, if every process $i \in P_c$ has crashed. The maximal number of processes that can crash is denoted $f$, $1 \leq f \leq n$, while $f_c$, $1 \leq f_c \leq m$, denotes the maximal number of clusters that can crash.

### 2.2 Stochastic Gradient Descent

The Euclidean norm of a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is $\|x\|_2 \triangleq \sqrt{\sum_{i=1}^{d} |x_i|^2}$; we use the standard notation, $\|x\|_2^2 \triangleq (\|x\|_2)^2 = \sum_{i=1}^{d} |x_i|^2$. The inner product of $x, y \in \mathbb{R}^d$ is $\langle x, y \rangle \triangleq x^T y = \sum_{i=1}^{d} x_i y_i$. The variance of a random vector $x$ is $\mathbb{V}[x] \triangleq \mathbb{E}\left[\|x - \mathbb{E}[x]\|_2^2\right]$.

Each process can access the same data distribution $D$, and loss function $\ell(x, z)$, which takes a learning parameter $x \in \mathbb{R}^d$ and a data point $z \in D$. Given a learning parameter $x \in \mathbb{R}^d$, the cost function $Q$ is:

$$Q(x) \triangleq \mathbb{E}_{z \sim D}[\ell(x, z)].$$

(1)

A distributed *Stochastic Gradient Descent (SGD)* algorithm collectively minimizes the cost function $Q$, i.e., it finds $x^* \in \text{arg min}_{x \in \mathbb{R}^d} Q(x)$.

$$x^* \in \text{arg min}_{x \in \mathbb{R}^d} Q(x).$$

(2)

The cost function is *differentiable* and *smooth*, i.e., for a positive constant $L \in \mathbb{R}$

$$\forall x, y \in \mathbb{R}^d, \|\nabla Q(x) - \nabla Q(y)\|_2 \leq L\|x - y\|_2,$$

(3)
where $\nabla Q(x) \in \mathbb{R}^d$ is the gradient of $Q$ at $x$.

Smoothness implies the following upper bound on $Q$,

$$\forall x, y \in \mathbb{R}^d, \; Q(y) \leq Q(x) + \langle \nabla Q(x), y - x \rangle + \frac{L}{2} \|y - x\|^2.$$  \hspace{1cm} (4)

The gradient at $x \in \mathbb{R}^d$ can be estimated by the stochastic gradient $G(x, z) = \nabla \ell(x, z) \in \mathbb{R}^d$, calculated at a data point $z$ that is drawn uniformly at random from $D$. The stochastic gradient is an unbiased estimator of the true gradient:

$$E_{z \sim D}[G(x, z)] = \nabla Q(x).$$  \hspace{1cm} (5)

In addition, the estimations have bounded variance, i.e., there is a non-negative constant $\sigma \in \mathbb{R}$ such that

$$\forall z \sim D \left[ \|G(x, z) - \nabla Q(x)\|^2 \right] \leq \sigma^2.$$  \hspace{1cm} (6)

These are standard assumptions in SGD analysis [8, 9, 16].

At the end of the algorithm, each nonfaulty process $i$ outputs an estimate of the learning parameter, $x^i \in \mathbb{R}^d$. We require the algorithm to externally converge with expected error $\epsilon > 0$ (simply called to converge in the optimization literature). In the strongly-convex case, where there is a single minimum $x^*$, the algorithm externally converges if for every nonfaulty process $i$:

$$E \left[ \|x^i - x^*\|^2 \right] \leq \epsilon.$$  \hspace{1cm} (7)

In the non-convex case, we require the algorithm to converge to a point with zero gradient. In this case, the algorithm externally converges if for every nonfaulty process $i$:

$$E \left[ \|\nabla Q(x^i)\|^2 \right] \leq \epsilon$$  \hspace{1cm} (8)

The algorithm internally converge with expected error $\delta > 0$, if for every pair of nonfaulty processes $i$ and $j$:

$$E \left[ \|x^i - x^j\|^2 \right] \leq \delta$$  \hspace{1cm} (9)

3 STRONGLY-CONVEX COST FUNCTIONS

We start by presenting the algorithm when the function $Q$ is strongly convex. In this case, the algorithm is very simple, while enjoying several attractive features. Formally, $Q$ is $\mu$-strongly convex, for $\mu > 0$, if for every $x, y \in \mathbb{R}^d$,

$$Q(y) \geq Q(x) + \langle \nabla Q(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2$$  \hspace{1cm} (10)

If the cost function is strongly convex, then it has a single minimum, denoted $x^*$. Algorithm 1 works in iterations, corresponding to the iterations of a sequential SGD algorithm. A process starts an iteration $t$ with a local learning parameter, and computes a learning parameter for the next iteration. First, the process computes a stochastic gradient using its current learning parameter, performs a local SGD step, and sends the updated learning parameter to all the other processes. After receiving learning parameters from iteration $t$ from $N$ processes, the process averages all the parameters. In the last iteration, each process outputs the learning parameter it has computed. The algorithm works for an arbitrary, but fixed value of $N$, and makes no assumption on the number of process failures $f$ or cluster failures $f_c$. We only require that $N \leq n - f$, to ensure that a process can make progress. As we prove, however, the larger $N$ is, the better the convergence of the algorithm is.
The learning parameter for the first iteration of all processes is \( x_1 \in \mathbb{R}^d \). The learning rate for iteration \( t \) is \( \eta_t \), and it is the same for all processes across all clusters (for a given \( t \)). The learning rate can be constant, i.e., \( \eta_t = \eta \) for every iteration \( t \geq 1 \), or the sequence of learning rates might be decreasing.

Let \( V_t \) be the set of processes that compute learning parameters for iteration \( t + 1 \) in Line 8; \( V_0 \) are the processes that execute the first line. Note that \( V_t \subseteq V_{t-1} \), for \( t \geq 1 \), and that all processes that send a message in iteration \( t \) (Line 6) are in \( V_t \). We use the notation \( \mathbb{E}_t \) for the expectation over iteration \( t \), i.e., the expectation only over the randomness of the stochastic gradients computed in iteration \( t \).

Lemma 1 states that the average of \( B \) stochastic gradients, each computed using different learning parameters, has variance of at most \( \frac{\sigma^2}{B} \).

**Lemma 1.** For any \( x_1, \ldots, x_B \in \mathbb{R}^d \) and \( z_1, \ldots, z_B \) drawn uniformly at random from \( \mathcal{D} \),

\[
\mathbb{V} \left[ \frac{1}{B} \sum_{i=1}^{B} G(x_i, z_i) \right] \leq \frac{\sigma^2}{B}
\]

**Proof.**

\[
\mathbb{V} \left[ \frac{1}{B} \sum_{i=1}^{B} G(x_i, z_i) \right] = \mathbb{E} \left[ \left\| \frac{1}{B} \sum_{i=1}^{B} (G(x_i, z_i) - \mathbb{E}[G(x_i, z_i)]) \right\|^2 \right]
\]

\[
= \frac{1}{B^2} \mathbb{E} \left[ \sum_{i=1}^{B} (G(x_i, z_i) - \mathbb{E}[G(x_i, z_i)]) \right] \sum_{j=1}^{B} (G(x_j, z_j) - \mathbb{E}[G(x_j, z_j)])
\]

\[
= \frac{1}{B^2} \sum_{i=1}^{B} \sum_{j=1}^{B} \mathbb{E} \left[ \left( G(x_i, z_i) - \mathbb{E}[G(x_i, z_i)] \right) \left( G(x_j, z_j) - \mathbb{E}[G(x_j, z_j)] \right) \right]
\]

by (18)

Note that \( z_i \) and \( z_j \) are stochastically independent for \( i \neq j \), and therefore, \( G(x_i, z_i) \) and \( G(x_j, z_j) \) are also stochastically independent and we get that

\[
\mathbb{E} \left[ \left( G(x_i, z_i) - \mathbb{E}[G(x_i, z_i)] \right) \left( G(x_j, z_j) - \mathbb{E}[G(x_j, z_j)] \right) \right] = \mathbb{E}[G(x_i, z_i)] \mathbb{E}[G(x_j, z_j)] - \mathbb{E}[G(x_i, z_i) G(x_j, z_j)]
\]

\[
= 0
\]

Using this in the first equation yields

\[
\mathbb{V} \left[ \frac{1}{B} \sum_{i=1}^{B} G(x_i, z_i) \right] = \frac{1}{B^2} \sum_{i=1}^{B} \sum_{j=1}^{B} \mathbb{E} \left[ \left( G(x_i, z_i) - \mathbb{E}[G(x_i, z_i)] \right) \left( G(x_j, z_j) - \mathbb{E}[G(x_j, z_j)] \right) \right]
\]

\[
= \frac{1}{B^2} \sum_{i=1}^{B} \mathbb{E} \left[ \left( G(x_i, z_i) - \mathbb{E}[G(x_i, z_i)] \right)^2 \right]
\]

\[
= \frac{1}{B} \sum_{i=1}^{B} \mathbb{V}[G(x_i, z_i)] \leq \frac{\sigma^2}{B}
\]

by (6)

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**Algorithm 1** Asynchronous and decentralized cluster-based SGD, for a strongly-convex cost function: code for process \( i \)

**Global input:** initial point \( x_i \)

1: \( x_i^0 \leftarrow x_i \)

2: for \( t = 1 \ldots T \) do

3: draw uniformly at random \( z \in \mathcal{D} \)

4: \( g_i^t \leftarrow G(x_i^t, z) \)

5: \( y_i^t \leftarrow x_i^t - \eta g_i^t \)

6: broadcast \( \langle t, y_i^t \rangle \) to all processes

7: wait to receive exactly \( N \) messages of the form \( \langle t, - \rangle \)

8: \( x_i^{t+1} \leftarrow \text{avg}(\text{received learning parameters}) \)

9: output \( x_i^{T+1} \)
Since Line 8 in Algorithm 1 averages the received parameters, Lemma 1 implies:

**Lemma 2.** For any iteration \( t \geq 1 \) and set of processes \( S \subseteq V_t \),

\[
\forall \left\| \frac{1}{|S|} \sum_{i \in S} g_i^t \right\|_2 \leq \frac{\sigma^2}{|S|}
\]

We first show that the diameter of the parameters at each iteration is contracted compared to the diameter of the previous iteration. Using this, we will show that the algorithm achieves internal convergence. This contraction happens implicitly due to strong convexity, since intuitively, all the processes gravitate to the unique minimum. The next lemma is used in the proof of Lemma 4.

**Lemma 3 ([21, Lemma 6]).** Let \( Q \) be an \( L \)-smooth and \( \mu \)-strongly convex function and let \( \eta \leq \frac{1}{\mu} \). Then for any \( x, y \in \mathbb{R}^d \),

\[
\left\| x - \eta \nabla Q(x) - y + \eta \nabla Q(y) \right\|_2^2 \leq (1 - \eta \mu) \left\| x - y \right\|_2^2
\]

**Lemma 4.** Let \( Q \) be an \( L \)-smooth and \( \mu \)-strongly convex function, then for every iteration \( t \geq 1 \) where \( \eta_t \leq \frac{1}{\mu} \),

\[
\max_{i, j \in V_t} \mathbb{E} \left[ \left\| x_{i, t+1}^j - x_{i, t}^j \right\|_2^2 \right] \leq (1 - \eta_t \mu) \max_{i, j \in V_{t-1}} \mathbb{E} \left[ \left\| x_{i, t}^j - x_{i, t-1}^j \right\|_2^2 \right] + \frac{4\sigma^2 \eta_t^2}{N}
\]

**Proof.** Let two processes \( i, j \in V_t \). Let \( S_1 = \{i_1, ..., i_N\} \) and \( S_2 = \{j_1, ..., j_N\} \) be the sets of processes that were used to compute \( x_{i, t+1}^j \) and \( x_{i, t+1}^j \) in Line 8, respectively.

\[
\left\| x_{i, t+1}^j - x_{i, t}^j \right\|_2^2 = \left\| \frac{1}{N} \sum_{k \in S_1} y_{i, k} - \frac{1}{N} \sum_{k \in S_2} y_{j, k} \right\|_2^2
\]

\[
= \left\| \frac{1}{N} \sum_{k=1}^{N} x_{i, k}^j - \eta_t \sum_{k=1}^{N} g_{i, k} - \frac{1}{N} \sum_{k=1}^{N} x_{j, k}^j + \frac{\eta_t}{N} \sum_{k=1}^{N} g_{j, k} \right\|_2^2
\]

Following Propositions 1 and 2 and Lemma 2,

\[
\mathbb{E} \left[ \left\| x_{i, t+1}^j - x_{i, t}^j \right\|_2^2 \right] \leq \frac{1}{N} \sum_{k=1}^{N} \left\| \left( x_{i, k}^j - \eta_t \nabla Q(x_{i, k}^j) - x_{j, k}^j + \eta_t \nabla Q(x_{j, k}^j) \right) \right\|_2^2 + \frac{4\sigma^2 \eta_t^2}{N}
\]

\[
\leq \frac{1}{N} \sum_{k=1}^{N} \left\| x_{i, k}^j - \eta_t \nabla Q(x_{i, k}^j) - x_{j, k}^j + \eta_t \nabla Q(x_{j, k}^j) \right\|_2^2 + \frac{4\sigma^2 \eta_t^2}{N} \quad \text{by (16)}
\]

Therefore,

\[
\mathbb{E} \left[ \left\| x_{i, t+1}^j - x_{j, t+1}^j \right\|_2^2 \right] \leq \frac{1}{N} \sum_{k=1}^{N} \mathbb{E} \left[ \left\| x_{i, k}^j - \eta_t \nabla Q(x_{i, k}^j) - x_{j, k}^j + \eta_t \nabla Q(x_{j, k}^j) \right\|_2^2 \right] + \frac{4\sigma^2 \eta_t^2}{N}
\]

\[
\leq \frac{1}{N} \sum_{k=1}^{N} (1 - \eta_t \mu) \mathbb{E} \left[ \left\| x_{i, k}^j - x_{j, k}^j \right\|_2^2 \right] + \frac{4\sigma^2 \eta_t^2}{N} \quad \text{by Lemma 3}
\]

\[
\leq (1 - \eta_t \mu) \max_{k, l \in V_t} \mathbb{E} \left[ \left\| x_{i, k}^j - x_{j, l}^j \right\|_2^2 \right] + \frac{4\sigma^2 \eta_t^2}{N}
\]
Lemma 6 shows that the external convergence rate in the strongly-convex case does not depend on the diameter of the learning parameters at the same iteration. It relies on the next lemma.

**Lemma 5 ([28, Lemma 9]).** Let \( Q \) be an \( L \)-smooth and \( \mu \)-strongly convex function with a single minimum \( x^* \), then for any \( x \in \mathbb{R}^d \)

\[
\langle \nabla Q(x), x - x^* \rangle \geq \frac{1}{2L} \|\nabla Q(x)\|^2 + \frac{\mu}{2} \|x - x^*\|^2
\]

**Lemma 6.** Let \( Q \) be an \( L \)-smooth and \( \mu \)-strongly convex function with a single minimum \( x^* \). Then, for every iteration \( t \) where \( \eta_t \leq \frac{1}{L} \),

\[
\max_{i \in V_t} \mathbb{E}_t \left[ \|x^i_{t+1} - x^*\|^2 \right] \leq (1 - \eta_t \mu) \max_{i \in V_{t-1}} \mathbb{E}_t \left[ \|x^i_t - x^*\|^2 \right] + \frac{\sigma^2 \eta_t^2}{N}
\]

**Proof.** Let process \( i \in V_t \) and let \( S \) be the set of \( N \) processes that was used to compute \( x^i_{t+1} \) in Line 8.

\[
\mathbb{E}_t \left[ \|x^i_{t+1} - x^*\|^2 \right] = \mathbb{E}_t \left[ \left\| \frac{1}{N} \sum_{j \in S} y^j_i - x^* \right\|^2 \right]
\]

\[
= \mathbb{E}_t \left[ \left\| \frac{1}{N} \sum_{j \in S} x^j_i - \eta_t \frac{1}{N} \sum_{j \in S} g^j_i - x^* \right\|^2 \right]
\]

\[
= \left\| \frac{1}{N} \sum_{j \in S} x^j_i - \mathbb{E}_t \left[ \eta_t \frac{1}{N} \sum_{j \in S} g^j_i \right] - x^* \right\|^2 + \mathbb{E}_t \left[ \eta_t \frac{1}{N} \sum_{j \in S} g^j_i \right] \quad \text{by Proposition 1}
\]

\[
\leq \left\| \frac{1}{N} \sum_{j \in S} (x^j_i - \eta_t \nabla Q(x^j_i) - x^*) \right\|^2 + \frac{\eta_t^2 \sigma^2}{N} \quad \text{by Lemma 2 and (5)}
\]

\[
\leq \frac{1}{N} \sum_{j \in S} \left( \left\| x^j_i - x^* \right\|^2 - 2 \eta_t \langle \nabla Q(x^j_i), x^j_i - x^* \rangle + \eta_t^2 \left\| \nabla Q(x^j_i) \right\|^2 \right) + \frac{\eta_t^2 \sigma^2}{N} \quad \text{by (16)}
\]

\[
\leq \frac{1}{N} \sum_{j \in S} \left( 1 - \eta_t \mu \right) \left\| x^j_i - x^* \right\|^2 + \left( \eta_t^2 - \frac{\eta_t^2}{L} \right) \left\| \nabla Q(x^j_i) \right\|^2 \quad \text{by Lemma 5}
\]

\[
\leq \frac{1}{N} \sum_{j \in S} (1 - \eta_t \mu) \left\| x^j_i - x^* \right\|^2 + \left( \eta_t^2 - \frac{\eta_t^2}{L} \right) \left\| \nabla Q(x^j_i) \right\|^2 + \frac{\eta_t^2 \sigma^2}{N} \quad \eta_t \leq \frac{1}{L}
\]

Therefore,

\[
\mathbb{E}_t \left[ \|x^i_{t+1} - x^*\|^2 \right] = \mathbb{E}_t \left[ \left\| x^i_{t+1} - x^* \right\|^2 \right] \leq \frac{1}{N} \sum_{j \in S} (1 - \eta_t \mu) \mathbb{E}_t \left[ \left\| x^j_i - x^* \right\|^2 \right] + \frac{\eta_t^2 \sigma^2}{N}
\]

\[
\leq (1 - \eta_t \mu) \max_{j \in V_{t-1}} \mathbb{E}_t \left[ \left\| x^j_i - x^* \right\|^2 \right] + \frac{\eta_t^2 \sigma^2}{N}
\]

\[\square\]
Asynchronous Fully-Decentralized SGD in the Cluster-Based Model

Note that the terms we get in Lemma 4 and Lemma 6 are very similar to the ones obtained in [8, Section 4.2] for the strongly-convex case using mini-batches of size $N$. However, we bound the difference between the learning parameters, while they bound the difference between the function values at the learning parameters. The next lemma is adapted from [8, Theorem 4.7].

**Lemma 7.** If for any $t \geq 1$,

$$a_{t+1} \leq (1 - \eta_t \mu) a_t + c \eta_t^2$$

for some $c, \mu > 0$. Then, for decreasing learning rate $\eta_t$, such that for all $t \geq 1$,

$$\eta_t = \frac{\beta}{\gamma + t}$$

for some $\beta > \frac{1}{\mu}$ and $\gamma > 0$, then,

$$a_t \leq \frac{\nu}{\gamma + t}$$

where $\nu = \max \left\{ \frac{\beta^2 c}{\beta \mu - 1}, (\gamma + 1) a_1 \right\}$.

**Proof.** The proof is by induction. For the base case,

$$a_2 \leq \left( 1 - \frac{\beta \mu}{\gamma + 1} \right) a_1 + \frac{\beta^2 c}{(\gamma + 1)^2}$$

$$= \left( \frac{\gamma}{(\gamma + 1)^2} - \frac{\beta \mu - 1}{(\gamma + 1)^2} \right) (\gamma + 1) a_1 + \frac{\beta \mu - 1}{(\gamma + 1)^2} \frac{\beta^2 c}{\beta \mu - 1}$$

$$\leq \left( \frac{\gamma}{(\gamma + 1)^2} - \frac{\beta \mu - 1}{(\gamma + 1)^2} \right) \nu + \frac{\beta \mu - 1}{(\gamma + 1)^2} \nu$$

$$= \frac{\gamma}{(\gamma + 1)^2} \nu \leq \frac{\nu}{\gamma + 2}$$

For the induction step,

$$a_{t+1} \leq (1 - \eta_t \mu) a_t + c \eta_t^2$$

$$\leq \left( 1 - \frac{\beta \mu}{\gamma + t} \right) \frac{\nu}{\gamma + t} + \frac{\beta^2 c}{(\gamma + t)^2}$$

$$= \frac{\gamma + t - \beta \mu}{(\gamma + t)^2} \nu + \frac{\beta^2 c}{(\gamma + t)^2}$$

$$\leq \frac{\gamma + t - 1}{(\gamma + t)^2} \nu - \left( \frac{\beta \mu - 1}{(\gamma + t)^2} \right) \nu + \frac{\beta^2 c}{(\gamma + t)^2} \leq \frac{\nu}{\gamma + t + 1}$$

$$\leq 0 \text{ by the definition of } \nu$$

Combining Lemma 4 and Lemma 6 with Lemma 7 we get our first main result:

**Theorem 1.** Let $Q$ be an $L$-smooth and $\mu$-strongly convex function with a single minimum $x^*$, then for decreasing learning rate $\eta_t = \frac{\beta}{\gamma + t} \leq \frac{1}{T}$ for some constants $\beta > \frac{1}{\mu}$ and $\gamma > 0$,

$$\max_{i,j \in V_T} \mathbb{E} \left\{ \left\| x_{T+1}^j - x_{T+1}^i \right\|_2^2 \right\} \leq \frac{(\gamma + 1) \left\| x_1 - x^* \right\|^2_2}{\gamma + T + 1} + \frac{4 \beta^2 \sigma^2}{(\gamma + T + 1)(\beta \mu - 1)N}$$

(Internal convergence)
When the function $Q$ is not strongly convex, if processes obtain disjoint estimations at each iteration, they may compute diverging learning parameters in each iteration. For this reason, we need to ensure that processes communicate with intersecting sets of clusters. To further expedite the contraction rate, and reduce the distance between the learning parameters, we end each iteration with multidimensional approximate agreement (MAA).

The parameter $N$ determines how many messages a process waits for in every iteration; to ensure progress, we require that $N \leq n - f$, as in the strongly-convex case. Furthermore, we assume that $f \leq \lceil (m - 1)/2 \rceil$, where $m$ is the number of clusters, to guarantee the convergence of the MAA algorithm. This can be translated to the following requirement on the number of processes failures $f$: any set $P \subseteq [n]$ of size $n - f$ must satisfy that $\text{cluster}(P) \geq \lceil n/2 \rceil + 1$, i.e., $P$ represents a majority of the processes $[6]$.

4 NON-CONVEX COST FUNCTIONS

When the function $Q$ is not strongly convex, if processes obtain disjoint estimations at each iteration, they may compute diverging learning parameters in each iteration. For this reason, we need to ensure that processes communicate with intersecting sets of clusters. To further expedite the contraction rate, and reduce the distance between the learning parameters, we end each iteration with multidimensional approximate agreement (MAA).

The input to MAA is the local learning parameter, and its output serves as the learning parameter for the next iteration.

Formally, in multidimensional approximate agreement, each process starts with input $x_i \in \mathbb{R}^d$ and outputs a value $y_i \in \mathbb{R}^d$, such that:

**Convexity:** The outputs are in the convex hull of the inputs, that is, they are a convex combination of the outputs.

**$q$-Contraction:** The outputs are contacted by a factor of $q$ relative to the inputs, that is, for every pair of nonfaulty processes $i, j$, $\|y_i - y_j\|^2_2 \leq q \max_{k,l \in [n]} \|x_k - x_l\|^2_2$.

Algorithm 2 is the asynchronous, decentralized cluster-based SGD algorithm, for the non-convex case. One difference from Algorithm 1 is in Line 8, calling MAA with parameter $q_i$, which determines its contraction rate for iteration $t$. Another difference is that processes send the stochastic gradients they computed, average the received gradients to a mini-batch stochastic gradient and then use it to perform a local SGD step. Every process outputs the learning parameter of the same iteration $\tau \in [T]$, which is drawn uniformly at random; this is a typical practice in SGD algorithms for non-convex objective functions [8]. Calling MAA means that an iteration consists of several communication rounds.

The parameter $N \in [n]$ determines how many messages a process waits for in every iteration; to ensure progress, we require that $N \leq n - f$, as in the strongly-convex case. Furthermore, we assume that $f \leq [m - 1)/2], where $m$ is the number of clusters, to guarantee the convergence of the MAA algorithm. This can be translated to the following requirement on the number of processes failures $f$: any set $P \subseteq [n]$ of size $n - f$ must satisfy that $\text{cluster}(P) \geq [n/2] + 1$, i.e., $P$ represents a majority of the processes $[6]$.

4.1 Cluster-Based Multidimensional Approximate Agreement Algorithm

Algorithms 3 solves MAA in the cluster-based model. The algorithm leverages inter-cluster communication to increase the number of failures that can be tolerated, and only requires read and write operations. The algorithm
works in rounds, each starting with shared-memory MAA within each cluster (Line 3). This allows processes to wait to receive messages from a majority of the clusters at each round, rather than waiting for a majority of the processes, which is the usual practice in crash-tolerant message-passing algorithms. This guarantees that every pair of processes receive a value from a common cluster. Any two values sent from the same cluster have smaller diameter, compared to diameter of all the processes values, since they are the output of the inter-cluster MAA algorithm. After collecting enough messages, the process computes the next round value using an aggregation rule called MidExtremes \cite{15} (abbreviated \text{MidExt} in Line 8). MidExtremes returns the average of the two values that realize the maximum Euclidean distance among all received vectors. Formally, for \( X \subseteq \mathbb{R}^d \)

\[
\text{MidExt}(X) = (a + b)/2, \text{ where } (a, b) = \arg \max_{(a, b) \in \mathbb{R}^2} \|a - b\|_2.
\]

**Algorithm 3** Multidimensional approximate agreement in the cluster-based model: code for process \( i \) in cluster \( c \)

\[
\text{MAA}(x, q):
\]

1. \( x'_i \leftarrow x \)
2. for \( r = 1 \ldots R = \lceil \log_{23/4} q \rceil \) do
3. \( y'_i \leftarrow \text{SMMAA}_c(x'_i, 1/6) \)
   \( \triangleright \) Shared-memory algorithm inside cluster \( c \)
4. broadcast \( (r, y'_i) \) to all processes
5. wait to receive messages of the form \( (r, -) \)
6. from at least \( \lfloor m/2 \rfloor + 1 \) clusters
7. \( Rcv \leftarrow \) set of received values
8. \( x'_{i+1} \leftarrow \text{MidExt}(Rcv) \)

**Theorem 2** Algorithm 3 satisfies \( q \)-contraction within \( \lceil \log_{23/4} q \rceil \) communication rounds.

**Proof.** Since the value computed in a round is a convex combination of values of the previous round, Algorithm 3 satisfies the convexity property (see \cite{15}). Let \( V_r \) be the processes that compute a value for iteration \( r + 1 \) (Line 8), and let \( V_0 \) be all the processes that execute the first line. In Appendix A.2 (Lemma 19), we prove that for every round \( r \),

\[
\max_{i,j \in V_r} \| x'_{i+1} - x'_j \|_2 \leq 23/24 \max_{i,j \in V_{r-1}} \| x'_i - x'_j \|_2.
\]

By induction, after \( R = \lceil \log_{23/4} q \rceil \) rounds we have

\[
\max_{i,j \in V_R} \| x'_{R+1} - x'_j \|_2 \leq \left( \frac{23}{24} \right)^R \max_{i,j \in V_0} \| x'_i - x'_j \|_2 \leq q \max_{i,j \in V_0} \| x'_i - x'_j \|_2.
\]

\( \square \)

### 4.2 Internal Convergence

As in Section 3, let \( V_t \) be the set of processes that compute learning parameters for iteration \( t + 1 \) (Line 8) and \( V_0 \) be the processes that execute the first line. Similarly to Lemma 2, Lemma 1 implies the next lemma for Algorithm 2.

**Lemma 8.** For any iteration \( t \geq 1 \) and process \( i \in V_t \),

\[
\forall [g'_i] \leq \frac{\sigma^2}{N}
\]
The next lemma bounds the diameter of the learning parameters of iteration $t + 1$ in terms of the diameter of the previous iteration $t$. First, we bound diameter after each process performs a local SGD step using the aggregated mini-batch stochastic gradient. Then, we apply the contraction property of the approximate agreement instance.

**Lemma 9.** For every iteration $t \geq 1$,

$$\max_{i,j \in V_t} \mathbb{E}\left[\left\|\mathbf{x}_{t+1}^i - \mathbf{x}_{t+1}^j\right\|^2\right] \leq q_t (2 + 2L^2 \eta_t^2) \max_{i,j \in V_{t-1}} \mathbb{E}\left[\left\|\mathbf{x}_t^i - \mathbf{x}_t^j\right\|^2\right] + \frac{4\sigma^2 \eta_t^2}{N}$$

**Proof.** Let two processes $i, j \in V_t$. Let $S_1$ be the set of processes that was used to compute $g_t^i$ in Line 6, and let $S_2$ be the set of processes that was used to compute $g_t^j$ in Line 6.

$$\mathbb{E}\left[\left\|\mathbf{y}_t^i - \mathbf{y}_t^j\right\|^2\right] = \mathbb{E}\left[\left\|\mathbf{x}_t^i - \eta_t g_t^i - (\mathbf{x}_t^j - \eta_t g_t^j)\right\|^2\right]$$

by Proposition 1

$$\leq \mathbb{E}\left[\left\|\mathbf{x}_t^i - \eta_t g_t^i\right\|^2 + \mathbb{E}\left[\left\|\mathbf{x}_t^j - \eta_t g_t^j\right\|^2 + \mathbb{E}\left[\eta_t g_t^i - \eta_t g_t^j\right]\right] \right]$$

by Proposition 2

$$\leq \mathbb{E}\left[\left\|\mathbf{x}_t^i - \eta_t \sum_{k \in S_1} \nabla Q(\mathbf{x}_t^i) - \mathbf{x}_t^i + \frac{\eta_t}{|S_1|} \sum_{k \in S_2} \nabla Q(\mathbf{x}_t^k)\right\|^2\right] + \frac{4\sigma^2 \eta_t^2}{N}$$

by Lemma 8 and (5)

$$\leq \frac{2 |S_1| L^2}{|S_1|} \sum_{k \in S_1} \nabla Q(\mathbf{x}_t^k) - \nabla Q(\mathbf{x}_t^k) + \frac{1}{|S_1|} \sum_{k \in S_1} \nabla Q(\mathbf{x}_t^i) \right\|^2\right] + \frac{4\sigma^2 \eta_t^2}{N}$$

by (16)

$$\leq \frac{2 |S_1| L^2}{|S_1|} \sum_{k \in S_1} \nabla Q(\mathbf{x}_t^k) - \nabla Q(\mathbf{x}_t^k) + \frac{4\sigma^2 \eta_t^2}{N}$$

by Proposition 3

$$\leq \frac{2 |S_1| L^2}{|S_1|} \sum_{k \in S_1} \nabla Q(\mathbf{x}_t^k) - \nabla Q(\mathbf{x}_t^k) + \frac{4\sigma^2 \eta_t^2}{N}$$

by (3)

Therefore,

$$\mathbb{E}\left[\left\|\mathbf{y}_t^i - \mathbf{y}_t^j\right\|^2\right] = \mathbb{E}\left[\mathbb{E}\left[\left\|\mathbf{y}_t^i - \mathbf{y}_t^j\right\|^2\right]\right] \leq \left(2 + 2L^2 \eta_t^2\right) \max_{i,j \in V_{t-1}} \mathbb{E}\left[\left\|\mathbf{x}_t^i - \mathbf{x}_t^j\right\|^2\right] + \frac{4\sigma^2 \eta_t^2}{N}$$

Since this is true for any pair of processes $i, j \in V_t$, we get that

$$\max_{i,j \in V_t} \mathbb{E}\left[\left\|\mathbf{y}_t^i - \mathbf{y}_t^j\right\|^2\right] \leq \left(2 + 2L^2 \eta_t^2\right) \max_{i,j \in V_{t-1}} \mathbb{E}\left[\left\|\mathbf{x}_t^i - \mathbf{x}_t^j\right\|^2\right] + \frac{4\sigma^2 \eta_t^2}{N}$$

Finally, using $q_t$-contraction property of MAA in Line 8 implies the lemma. □

### 4.3 External Convergence

In this section, we make a standard assumption that $Q$ is lower bounded by $Q^*$, i.e., for every $x \in \mathbb{R}^d$, $Q(x) \geq Q^*$.

For iteration $t$ and process $i \in V_t$, the **effective gradient** [13] is

$$G_t^i \triangleq \frac{\mathbf{x}_t^i - \mathbf{x}_{t+1}^i}{\eta_t}$$

We bound the difference between the effective gradient and the true gradient in each iteration, depending on the diameter of the learning parameters in the same iteration. This claim will help us prove that the algorithm
eventually converges, as the effective change between two consecutive iterations is close enough to the true gradient. The next lemma, proved in Appendix A.3, is the analog of Lemma 8 for the effective gradient.

**Lemma 10.** For every iteration $t \geq 1$ and process $i \in V_t$,

$$\mathbb{E} \left[ \frac{1}{t} \| G_i^t - \mathbb{E} [ G_i^t ] \|_2^2 \right] \leq \frac{\sigma^2}{N}$$

**Proof.** By the convexity of MAA, $x_{i,t+1} = \sum_{j=1}^k w_j y_{ij}$ for some $\{i_1, \ldots, i_k\} \subseteq V_t$ and weights $\sum_{j=1}^k w_j = 1$.

$$x_{i,t}^t - x_{i,t+1} = x_{i,t}^t - \sum_{j=1}^k w_j y_{ij} = x_{i,t}^t - \sum_{j=1}^k w_j \left( x_{ij}^t - \eta_t g_{ij}^t \right)$$

Hence,

$$\mathbb{E} \left[ \frac{1}{t} \| G_i^t - \mathbb{E} [ G_i^t ] \|_2^2 \right] = \mathbb{E} \left[ \frac{1}{t^2} \| (x_{i,t}^t - x_{i,t+1}) - \mathbb{E} \left[ \frac{1}{t} (x_{i,t}^t - x_{i,t+1}) \right] \|_2^2 \right]$$

$$= \mathbb{E} \left[ \frac{1}{t^2} \left\| \sum_{j=1}^k w_j g_{ij}^t - \sum_{j=1}^k w_j \mathbb{E} \left[ g_{ij}^t \right] \right\|_2^2 \right]$$

$$\leq \sum_{j=1}^k w_j \mathbb{E} \left[ \| g_{ij}^t - \mathbb{E} [ g_{ij}^t ] \|_2^2 \right] \quad \text{by Proposition 4}$$

$$= \sum_{j=1}^k w_j \sqrt{\mathbb{E} \left[ \| g_{ij}^t \|_2^2 \right]} \leq \frac{\sigma^2}{N} \quad \text{by Lemma 8}$$

**Lemma 11.** For every iteration $t \geq 1$ and process $i \in V_t$,

$$\mathbb{E} \left[ \left\| \mathbb{E} [ G_i^t ] - \nabla Q(x_i^t) \right\|_2^2 \right] \leq \left( \frac{2}{\eta_t^2} + 2L^2 \right) \max_{t,i \in V_t} \mathbb{E} \left[ \| x_{i,t}^t - x_{i,t+1} \|_2^2 \right]$$

**Proof.** By convexity of MAA, $x_{i,t+1} = \sum_{j=1}^k w_j y_{ij}$ for some $\{i_1, \ldots, i_k\} \subseteq V_t$ and weights $\sum_{j=1}^k w_j = 1$.

$$\left\| \mathbb{E} [ G_i^t ] - \nabla Q(x_i^t) \right\|_2^2 = \frac{1}{\eta_t} \left\| x_{i,t}^t - \sum_{j=1}^k w_j x_{ij}^t \right\|_2^2 + \sum_{j=1}^k w_j \mathbb{E} \left[ g_{ij}^t - \nabla Q(x_i^t) \right]$$

$$\leq \frac{2}{\eta_t} \left\| x_{i,t}^t - \sum_{j=1}^k w_j x_{ij}^t \right\|_2^2 + 2 \sum_{j=1}^k w_j \mathbb{E} \left[ g_{ij}^t - \nabla Q(x_i^t) \right]$$

$$\leq \frac{2}{\eta_t} \sum_{j=1}^k w_j \left\| x_{i,t}^t - x_{ij}^t \right\|_2^2 + 2 \sum_{j=1}^k w_j \mathbb{E} \left[ g_{ij}^t - \nabla Q(x_i^t) \right]$$

$$\leq \frac{2}{\eta_t} \sum_{j=1}^k w_j \left\| x_{i,t}^t - x_{ij}^t \right\|_2^2 + 2L^2 \sum_{j=1}^k w_j \left\| x_{ij}^t - x_{ij}^t \right\|_2^2$$

by (5) and (16)
Therefore,
\[
    \mathbb{E}\left[ \|G_t^i - \nabla Q(x_t^i)\|_2^2 \right] \leq \left( \frac{2}{\eta_t^2} + 2L^2 \right) \sum_{j=1}^k w_j \mathbb{E}\left[ \|x_t^i - x_{t+1}^j\|_2^2 \right] \leq \left( \frac{2}{\eta_t^2} + 2L^2 \right) \max_{i,j \in V_t} \mathbb{E}\left[ \|x_t^i - x_t^j\|_2^2 \right]
\]

**Lemma 12.** Let \( Q \) be an \( L \)-smooth function. For every iteration \( t \geq 1 \) where \( \eta_t \leq \frac{1}{4L} \) and process \( i \in V_t \),
\[
    \frac{\eta_t}{4} \|\nabla Q(x_t^i)\|_2^2 \leq Q(x_t^i) - \mathbb{E}_t[Q(x_{t+1}^i)] + \eta_t \mathbb{E}_t[\|G_t^i\|_2^2] + \frac{\sigma^2 \eta_t^2 L}{N}
\]

**Proof.** By (11), \( x_{t+1}^i = x_t^i - \eta_t G_t^i \). By using (4)
\[
    \mathbb{E}_t[Q(x_{t+1}^i)] \leq Q(x_t^i) - \eta_t \mathbb{E}_t[\langle G_t^i, \nabla Q(x_t^i) \rangle] + \frac{\eta_t^2 L}{2} \mathbb{E}_t[\|G_t^i\|_2^2]
\]
(12)

For the second term
\[
    - \mathbb{E}_t[\langle G_t^i, \nabla Q(x_t^i) \rangle] = -\mathbb{E}_t[\mathbb{E}_t[\langle G_t^i, \nabla Q(x_t^i) \rangle] \quad \text{by (18)}
\]
(13)

For the third term
\[
    \frac{1}{2} \mathbb{E}_t[\|G_t^i\|_2^2] = \frac{1}{2} \mathbb{E}_t[\|G_t^i - \nabla Q(x_t^i) + \nabla Q(x_t^i)\|_2^2]
\]
\[
    \leq \mathbb{E}_t[\|G_t^i - \nabla Q(x_t^i)\|_2^2] + \|\nabla Q(x_t^i)\|_2^2 \quad \text{by (16)}
\]
(14)

Combining (12), (13) and (14),
\[
    \mathbb{E}_t[Q(x_{t+1}^i)] \leq Q(x_t^i) + \left( \eta_t^2 L - \frac{\eta_t}{4} \right) \|\nabla Q(x_t^i)\|_2^2 + \eta_t \mathbb{E}_t[\|G_t^i\|_2^2] + \frac{\sigma^2 \eta_t^2 L}{N}
\]
\[
    \leq Q(x_t^i) - \frac{\eta_t}{4} \|\nabla Q(x_t^i)\|_2^2 + \eta_t \mathbb{E}_t[\|G_t^i\|_2^2] + \frac{\sigma^2 \eta_t^2 L}{N}
\]
(15)

The lemma follows by rearrangement.

**Theorem 3.** Let \( Q \) be an \( L \)-smooth cost function. Then for \( T \geq 16L^2 N \), constant learning rate \( \eta = \sqrt{\frac{2}{NT}} \) and any process \( i \in V_T \),
\[
    \min_{t \in T} \mathbb{E}\left[ \|\nabla Q(x_t^i)\|_2^2 \right] \leq \frac{4(4Q(x_0) - Q^*)}{\sqrt{NT}} + \frac{4\sigma^2 L}{\sqrt{NT}} + \left( \frac{8T}{N} + 8L^2 \right) \max_{t \in [T], i,j \in V_t} \mathbb{E}\left[ \|x_t^i - x_t^j\|_2^2 \right]
\]

**Proof.** Since \( \eta = \sqrt{\frac{2}{NT}} \leq \frac{1}{4L} \), following Lemma 12 and 11
\[
    \frac{\eta}{4} \mathbb{E}\left[ \|\nabla Q(x_t^i)\|_2^2 \right] \leq \mathbb{E}[Q(x_t^i)] - \mathbb{E}[Q(x_{t+1}^i)] + \left( \frac{2}{\eta} + 2L^2 \right) \max_{i,j \in V_t} \mathbb{E}\left[ \|x_t^i - x_t^j\|_2^2 \right] + \frac{\sigma^2 \eta_t^2 L}{N}
\]
Therefore,

\[
\min_{t \in [T]} \mathbb{E}\left[ \left\| \nabla Q(x_t^i) \right\|_2^2 \right] \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[ \left\| \nabla Q(x_t^i) \right\|_2^2 \right] \leq \frac{4(Q(x_1) - Q^*)}{T \eta} + \frac{4\sigma^2 L}{N} + \frac{8}{\eta^2} + \frac{8L^2}{N} \max_{t \in [T]} \max_{i,j \in V_T} \mathbb{E}\left[ \left\| x_t^i - x_t^j \right\|_2^2 \right] \]

The next theorem proves that using an MAA algorithm with \( O(\log T) \) communication rounds, Algorithm 2 has convergence rate which match the sequential SGD algorithm, up to a logarithmic factor in the number of rounds.

**Theorem 4.** Let \( Q \) be an \( L \)-smooth cost function. Consider Algorithm 2 with \( T \geq \max\{16L^2N, 4N\} \), constant learning rate \( \eta = \frac{\sqrt{N}}{\sqrt{T}} \) and for every iteration \( t \), \( q_t = \frac{\eta}{4} \). Then, after \( R = T \left[ \log_{23/24} \left( \sqrt{N}/4\sqrt{T} \right) \right] \) communication rounds for every process \( i \in V_T \) and some constant \( C > 0 \)

\[
\min_{t \in [T]} \mathbb{E}\left[ \left\| \nabla Q(x_t^i) \right\|_2^2 \right] \leq \frac{C(Q(x_1) - Q^*) \log_2 R}{\sqrt{NR}} + \frac{5C\sigma^2 L \log_2 R}{\sqrt{NR}} + \frac{4C\sigma^2 \log_2 R}{\sqrt{NR}}
\]

**Proof.** Following Lemma 22 (proved in Appendix A.3),

\[
\min_{t \in [T]} \mathbb{E}\left[ \left\| \nabla Q(x_t^i) \right\|_2^2 \right] \leq \frac{4(Q(x_1) - Q^*)}{\sqrt{NT}} + \frac{20\sigma^2 L}{\sqrt{NT}} + \frac{16\sigma^2}{\sqrt{NT}}
\]

The MAA algorithm requires \( \left[ \log_{23/24} \left( \sqrt{N}/4\sqrt{T} \right) \right] \) communication rounds at each iteration. After \( T \) iterations this yields \( R = T \left[ \log_{23/24} \left( \sqrt{N}/4\sqrt{T} \right) \right] \leq TC \log_2 T \) communication rounds, for some constant \( C > 0 \). Hence,

\[
\min_{t \in [T]} \mathbb{E}\left[ \left\| \nabla Q(x_t^i) \right\|_2^2 \right] \leq \frac{4C(Q(x_1) - Q^*) \log_2 T}{\sqrt{NR}} + \frac{20C\sigma^2 L \log_2 T}{\sqrt{NR}} + \frac{16C\sigma^2 \log_2 T}{\sqrt{NR}}
\]

Since processes return the learning parameters of iteration \( \tau \), drawn uniformly at random, Theorem 4 proves that Algorithm 2 converge externally in \( \tilde{O}(\epsilon^{-3}) \) communication rounds. By Lemma 21 (proved in Appendix A.3),

\[
\max_{i,j \in V_{T-1}} \mathbb{E}\left[ \left\| x_t^i - x_t^j \right\|_2^2 \right] = O\left( T^{-3/2} \right) = \tilde{O}\left( R^{-3/2} \right).
\]

This yields internal convergence in \( \tilde{O}(\epsilon^{-3/2}) \) communication rounds. Note that in both terms dependencies on \( (Q(x_1) - Q^*), L, \sigma \) and \( N \) are neglected.
5 IMPOSSIBILITY OF ASYNCHRONOUS SGD IN PARTITIONED SYSTEMS, FOR ARBITRARY FUNCTIONS

To prove the impossibility result, we use a more formal model that captures \textit{probabilistic indistinguishably} \cite{17}. A \textit{configuration} $C$ consists of the local state of each process, pending messages that were not received yet and the shared memory state of each cluster. An \textit{event} is either a delivery of some message by process $i$ or some operation on its cluster shared memory. A \textit{step} consists of some local computation, possibly a \textit{coin flip}, and a single event. By applying an event preformed by process $i$ to configuration $C$, we obtain a new configuration with a new local state for process $i$, possibly removing or adding messages from the pending messages buffer and the updated shared memory state of $i$’s cluster.

Given a configuration $C$, for every process there is a fixed probability for every step it can take from $C$. An \textit{execution tree} $T$ (called an \textit{ensemble} in \cite{17}), is a directed weighted tree where each node is a configuration and the edges are all the possible steps that can be taken from this configuration. The weight on each edge is exactly the probability for the step to be taken from the parent configuration. The root of the execution tree is an \textit{initial configuration}. Any path in the execution tree, beginning from the root, defines a legal \textit{execution}. The probability over the execution tree for an execution to occur is the product of weights along the path that defines the execution. Given an infinite execution tree $T$ and an execution $\alpha$ in $T$, we say that process $i$ \textit{crashed} in $\alpha$ if $i$ stops taking steps from some point in $\alpha$. $P(\alpha)$ is the set of all non-crashed processes in $\alpha$.

We prove the result assuming only a \textit{weak} adversary, which cannot observe the local coin flips, shared memory states and the messages sent during the execution.

The \textit{i-local execution tree} $T_i$ has a node for each of $i$’s local states that appear in execution tree $T$. The node is labeled with the probability of $i$ reaching this state in $T$, which is equal to the probability to reach some node in $T$ that contains this local state. There is an edge from node $u$ to $v$, if there is an execution in $T$ where the local state in $v$ directly follows the local state in $u$. This can be extended to an \textit{S-local execution tree}, for a set of processes $S$. For simplicity, we assume that once a process transitioned to some local state it will never repeat it in the execution. This can be achieved by including the full history of local events in the local state, as done in \cite{17}.

Two execution trees are \textit{probabilistically indistinguishable} to process $i$, if they induce the same $i$-local execution tree. It was shown \cite{17} that if two execution trees are probabilistically indistinguishable to process $i$, then the probability for $i$ to perform some action in these two trees is equal. This easily extends to sets of processes.

Let $A^{seq}(Q, T, x_0)$ be the sequential SGD algorithm optimizing the function $Q$ for $T$ iterations starting from $x_0$. Let $x_{seq}$ be a random variable, corresponding to the output of the sequential algorithm. For $\beta \in \mathbb{R}$ and point $x \in \mathbb{R}^d$, let $E_{seq}(\beta, x)$ be the event that $\|x_{seq} - x\|_2 \leq \beta$. For a set of points $S \subseteq \mathbb{R}^d$, $E_{seq}(\beta, S)$ denotes the event that for some $x \in S$, $\|x_{seq} - x\|_2 \leq \beta$, i.e., $E_{seq}(\beta, S) = \cup_{x \in S} E_{seq}(\beta, x)$.

Let $A$ be an algorithm in the cluster-based model and let $T$ be an execution tree of algorithm $A$. Let $x'(T) \in \mathbb{R}^d$ be a random variable, corresponding to the output of process $i$ from algorithm $A$ over execution tree $T$. If $i$ crashed then this value is $\perp$. For a set of processes $P$, let $E(\beta, x, T, P)$ be the event that for some process $i \in P$, such that $x'(T) \neq \perp$, $\|x'(T) - x\|_2 \leq \beta$. For a set of points $S$, let $E(\beta, S, T, P) = \bigcup_{x \in S} E(\beta, x, T, P)$.

We say that $A$ \textit{preserves the convergence distribution over an execution tree} $T$, if for any $\beta > 0$, set of points $S$ and set of processes $P$,

$$P[E(\beta, S, T, P)] \geq P[E_{seq}(\beta, S)]$$

We say that $A$ \textit{distributively implements} $A^{seq}(Q, T, x_0)$ \textit{over an execution tree} $T$ if it converge externally (8) and internally (9) over the outputs $\{x'(T)\}_{i=1}$, and it preserves the convergence distribution over $T$. The distance between two sets of points $S_1, S_2 \subseteq \mathbb{R}^d$ is defined as $\text{dist}(S_1, S_2) \triangleq \min_{x_1 \in S_1, x_2 \in S_2} \|x_1 - x_2\|_2$.

\textbf{Definition 1.} A function $Q$ is $(y, p)$-split for $y > 0$ and $p \in [0, 1]$, if there are two sets of points $S_1, S_2 \subseteq \mathbb{R}^d$ where $\text{dist}(S_1, S_2) \geq d$ and for some $\alpha, \beta > 0$ such that $(d - \alpha - \beta) \geq y$, $P[E_{seq}(\alpha, S_1)] P[E_{seq}(\beta, S_2)] \geq p$. 

This definition is motivated by the fact that non-convex functions have several stationary points. Although SGD converges to a stationary point (see (8)), it may happen that SGD, starting from the same initial point, converges to different stationary points in different random executions. Recall that a stationary point \( x \in \mathbb{R}^d \) satisfies \( \nabla Q(x) = 0 \). A stationary point can be either global or local minimum or maximum, or a saddle point, which is not a local extremum of the function. We denote the event that SGD converges to a stationary point using the event \( E_{\text{seq}}(\beta, x) \), and for a point in a set of stationary points \( S \) using the event \( E_{\text{seq}}(\beta, S) \), for some small \( \beta \). Note that for a stationary point \( x \), using smoothness (3),

\[
\|\nabla Q(x_{\text{seq}})\|_2^2 = \|\nabla Q(x_{\text{seq}}) - \nabla Q(x)\|_2^2 \leq L^2\|x_{\text{seq}} - x\|_2^2.
\]

This implies that if the algorithm converges near a stationary point, then this point has small squared gradient.

The \((\gamma, p)\)-split definition formalizes the phenomenon where the probability of SGD to converge \( \gamma \) apart in two different random executions is at least \( p \). This intuition is formalized in the next lemma.

**Lemma 13.** Let \( \mathcal{A} \) be a cluster-based algorithm and \( T \) an execution tree of \( \mathcal{A} \). If \( \mathbb{P}[E(\alpha, S_1, T, A) \cap E(\beta, S_2, T, B)] \geq p \) for \( \alpha, \beta > 0 \), sets of processes \( A, B \) and \( S_1, S_2 \subseteq \mathbb{R}^d \) such that \( \text{dist}(S_1, S_2) \geq d \), then there are some \( a \in A, b \in B \) such that

\[
\mathbb{E}\left[\|x^a(T) - x^b(T)\|_2^2\right] \geq \frac{p(d - \alpha - \beta)^2}{n^2}
\]

**Proof.** If both events \( E(\alpha, S_1, T, A) \) and \( E(\beta, S_2, T, B) \) happen in an execution in \( T \), then there are two processes \( i \in A, j \in B \) and points \( x_1 \in S_1, x_2 \in S_2 \) such that, \( \|x^i(T) - x^j(T)\|_2 \leq \alpha \) and \( \|x^i(T) - x^j(T)\|_2 \leq \beta \).

\[
\|x_1 - x_2\|_2 = \|x_1 - x^i(T) + x^i(T) - x^j(T) + x^j(T) - x_2\|_2 \\
\leq \|x_1 - x^i(T)\|_2 + \|x^i(T) - x^j(T)\|_2 + \|x^j(T) - x_2\|_2 \tag{Triangle inequality}
\]

By rearranging the inequality we get

\[
\max_{a \in A, b \in B} \left\|x^a(T) - x^b(T)\right\|_2 \geq \left\|x^i(T) - x^j(T)\right\|_2 \geq \|x_1 - x_2\|_2 - \|x_1 - x^i(T)\|_2 - \|x^j(T) - x_2\|_2 \geq d - \alpha - \beta.
\]

Denote the event \( E = E(\alpha, S_1, T, A) \cap E(\beta, S_2, T, B) \). Using the previous inequality,

\[
\mathbb{E}\left[\max_{a \in A, b \in B} \left\|x^a(T) - x^b(T)\right\|_2^2\right] \geq \mathbb{P}[E] \mathbb{E}\left[\max_{a \in A, b \in B} \left\|x^a(T) - x^b(T)\right\|_2^2\right] \geq p(d - \alpha - \beta)^2.
\]

Therefore,

\[
\frac{1}{|A||B|} \sum_{a \in A, b \in B} \mathbb{E}\left[\left\|x^a(T) - x^b(T)\right\|_2^2\right] \geq \frac{1}{n^2} \mathbb{E}\left[\max_{a \in A, b \in B} \left\|x^a(T) - x^b(T)\right\|_2^2\right] \geq \frac{p(d - \alpha - \beta)^2}{n^2}.
\]

This implies that there is some \( a \in A \) and \( b \in B \) such that \( \mathbb{E}\left[\left\|x^a(T) - x^b(T)\right\|_2^2\right] \geq \frac{p(d - \alpha - \beta)^2}{n^2} \), otherwise, the average must be smaller.

We require the distributed algorithm to preserve the same convergence distribution as the sequential algorithm in order to implement the SGD algorithm. For functions that SGD can converge \( \gamma \) apart with large enough probability and \( \gamma \), we show that there is no distributed implementation that can tolerate partitioning. Intuitively, this is because each partition can converge individually to points that are \( \gamma \) apart from each other.

The proof of the next theorem adapts the impossibility proofs of [6, 19] to probabilistic algorithms.

**Theorem 5.** If function \( Q \) is \((\gamma, p)\)-split for \( p > \frac{\delta \gamma^2}{4} \) and \( \gamma \geq \sqrt{\delta n} \), then no algorithm \( \mathcal{A} \) distributively implements \( \mathcal{A}^{\text{seq}}(Q, T, x_0) \) over all execution trees \( T \) that contain an execution \( a \) such that \( |\text{cluster}(P(a))| \leq |m/2| \).
We present crash-tolerant asynchronous SGD algorithms for the cluster-based model. For strongly convex functions, our algorithm obtains maximal speedup of the convergence rate over the sequential algorithm, and tolerates any number of failures. For other functions, we employ multidimensional approximate agreement to bring parameters close together in each iteration. This algorithm requires that there is a nonfaulty process in a majority of clusters. We prove that this condition is necessary for optimizing certain functions.

Concentrating on crash failures allows to obtain good bounds on the convergence rate, as well as optimal bounds on the ratio of faulty processes. In addition, this leads to simpler and more modular proofs. We believe that the cluster-based model with crash failures can offer a blueprint for designing optimization algorithms for high-performance computing (HPC) systems. The HPC architecture includes many multi-processor computers, each running multiple threads that share a memory space, which are connected by a network. An interesting direction is to extend our algorithms to handle process recoveries, using non-volatile RAM. This makes them a particularly good fit for contemporary exascale systems like the Aurora supercomputer [1], which is expected to feature persistent memory.

Future work could explore the use of specific properties of objective functions, beyond strong convexity, to improve the algorithms. Another direction is to have more cooperative inter-cluster computation, while still avoiding waiting, for example, by using stronger primitive operations, like compare&swap or fetch&add.

**Proof.** Assume there is an algorithm $\mathcal{A}$ that distributively implements $\mathcal{A}^{\text{seq}}(Q, T, x_0)$ over any execution tree $T$ that contains an execution $\alpha$ such that $|\text{cluster}(P(\alpha))| \leq \lfloor m/2 \rfloor$. Partition the processes into two sets of processes $A$ and $B$, such that $\text{cluster}(A) \cap \text{cluster}(B) = \emptyset$ and $|A| + |B| = n$. Note that either $|\text{cluster}(A)| \leq \lfloor m/2 \rfloor$ or $|\text{cluster}(B)| \leq \lfloor m/2 \rfloor$.

Consider the following three execution trees. In execution tree $T_1$, processes in $B$ don’t take any local steps, i.e., processes in $B$ are crashed in all executions in $T_1$. Similarly, in execution tree $T_2$ processes in $A$ don’t take any local steps. Following our assumption, algorithm $\mathcal{A}$ distributively implements $\mathcal{A}^{\text{seq}}(Q, T, x_0)$ over both $T_1$ and $T_2$.

Since $Q$ is $(\gamma, p)$-split, there are two sets of points $S_1, S_2 \subseteq \mathbb{R}^d$ where $\text{dist}(S_1, S_2) \geq d$ and for some $\alpha, \beta > 0$ such that $(d - \alpha - \beta) \geq \gamma$, $\mathbb{P}[E_{\text{seq}}(\alpha, S_1)] \mathbb{P}[E_{\text{seq}}(\beta, S_2)] \geq p$.

The third execution tree $T_3$ is an interleaving of $T_1$ and $T_2$, constructed as follows: all messages between processes in $A$ and $B$ are delayed until all processes finish executing the algorithm. We build the tree such that the even levels are nodes from $T_1$, and the odd levels are nodes from $T_2$. Specifically, nodes in level $2$ of $T_3$ are the nodes from the $l$-th level of $T_1$, and nodes in level $2l + 1$ of $T_3$ are the nodes from the $l$-th level of $T_2$. Since $\text{cluster}(A) \cap \text{cluster}(B) = \emptyset$, processes in $A$ do not share memory with processes in $B$, and vice versa. By construction, processes in one set do not receive messages from the other set until the algorithm terminates. Therefore, processes in one set do not affect the processes in the other set, and this construction yields a legal execution tree.

By construction, $T_1$ and $T_2$ are probabilistically indistinguishable to processes in $A$, and $T_2$ and $T_3$ are probabilistically indistinguishable to processes in $B$. In addition, the events $E(\alpha, S_1, T_5, A)$ and $E(\beta, S_2, T_5, B)$ are independent of each other. Therefore, by (15),

$$\mathbb{P}[E(\alpha, S_1, T_5, A) \cap E(\beta, S_2, T_5, B)] = \mathbb{P}[E(\alpha, S_1, T_5, A)] \mathbb{P}[E(\beta, S_2, T_5, B)] \geq \mathbb{P}[E_{\text{seq}}(\alpha, S_1)] \mathbb{P}[E_{\text{seq}}(\beta, S_2)] \geq p.$$

By Lemma 13, for some $a \in A$ and $b \in B$

$$E\left[\|x^a(T) - x^b(T)\|_2^2\right] \geq \frac{p(d - \alpha - \beta)^2}{n^2} \geq \frac{p\gamma^2}{n^2} \geq \delta.$$

This contradicts internal convergence property (9) of Algorithm $\mathcal{A}$. \hfill $\square$

6 SUMMARY AND DISCUSSION

We present crash-tolerant asynchronous SGD algorithms for the cluster-based model. For strongly convex functions, our algorithm obtains maximal speedup of the convergence rate over the sequential algorithm, and tolerates any number of failures. For other functions, we employ multidimensional approximate agreement to bring parameters close together in each iteration. This algorithm requires that there is a nonfaulty process in a majority of clusters. We prove that this condition is necessary for optimizing certain functions.

Our results assume processes fail only by crashing, which is an adequate model for several computing systems. Concentrating on crash failures allows to obtain good bounds on the convergence rate, as well as optimal bounds on the ratio of faulty processes. In addition, this leads to simpler and more modular proofs. We believe that the cluster-based model with crash failures can offer a blueprint for designing optimization algorithms for high-performance computing (HPC) systems. The HPC architecture includes many multi-processor computers, each running multiple threads that share a memory space, which are connected by a network. An interesting direction is to extend our algorithms to handle process recoveries, using non-volatile RAM. This makes them a particularly good fit for contemporary exascale systems like the Aurora supercomputer [1], which is expected to feature persistent memory.

Future work could explore the use of specific properties of objective functions, beyond strong convexity, to improve the algorithms. Another direction is to have more cooperative inter-cluster computation, while still avoiding waiting, for example, by using stronger primitive operations, like compare&swap or fetch&add.
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A APPENDIX

A.1 Useful Mathematical Propositions

For every set of vectors $v_1, \ldots, v_n \in \mathbb{R}^d$, we use the two following relaxed versions of the triangle inequality [14, 21].

$$\left\| \sum_{i=1}^{n} v_i \right\|_2^2 \leq n \sum_{i=1}^{n} \| v_i \|_2^2$$

(16)

$$\forall a > 0, \left\| v_i + v_j \right\|_2^2 \leq (1 + a) \| v_i \|_2^2 + \left(1 + \frac{1}{a}\right) \| v_j \|_2^2$$

(17)

**Proposition 1.** Let $x \in \mathbb{R}^d$ be a random variable, then

$$\mathbb{E}[\|x\|_2^2] = \|\mathbb{E}[x]\|_2^2 + \mathbb{V}[x]$$

**Proof.** For any random variable $X$, $(\mathbb{E}[X - \mathbb{E}[X]])^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. The lemma directly follows from this equality. □

**Proposition 2.** Let $x, y \in \mathbb{R}^d$ be two random variables, then

$$\mathbb{V}[x - y] \leq 2 \mathbb{V}[x] + 2 \mathbb{V}[y]$$

**Proof.**

$$\mathbb{V}[x - y] = \mathbb{E}[\|x - y - \mathbb{E}[x] + \mathbb{E}[y]\|_2^2]$$

$$\leq 2 \mathbb{E}[\|x - \mathbb{E}[x]\|_2^2] + 2 \mathbb{E}[\|\mathbb{E}[y] - y\|_2^2]$$

by (16)

$$= 2 \mathbb{V}[x] + 2 \mathbb{V}[y]$$

□

**Proposition 3.** Let $S_1, S_2 \subseteq \mathbb{R}^d$, then

$$\left\| \frac{1}{|S_1|} \sum_{y \in S_1} y - \frac{1}{|S_2|} \sum_{y' \in S_2} y' \right\|_2^2 \leq \frac{1}{|S_1||S_2|} \sum_{y \in S_1} \sum_{y' \in S_2} \|y - y'\|_2^2$$

**Proof.**

$$\left\| \frac{1}{|S_1|} \sum_{y \in S_1} y - \frac{1}{|S_2|} \sum_{y' \in S_2} y' \right\|_2^2 = \left\| \frac{1}{|S_1|} \sum_{y \in S_1} y - \frac{1}{|S_1|} \sum_{y' \in S_2} \frac{|S_2|}{|S_1|} y' \right\|_2^2$$

$$= \left\| \frac{1}{|S_1|} \sum_{y \in S_1} y - \frac{1}{|S_1|} \sum_{y \in S_2} \frac{1}{|S_2|} \sum_{y' \in S_2} |S_2| y' \right\|_2^2$$

$$\leq \frac{1}{|S_1|} \sum_{y \in S_1} \left\| y - \frac{1}{|S_2|} \sum_{y' \in S_2} y' \right\|_2^2$$

by (16)

$$\leq \frac{1}{|S_1||S_2|} \sum_{y \in S_1} \sum_{y' \in S_2} \|y - y'\|_2^2$$

by (16)

□

We also need a weighted version of this proposition.

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The next two lemmas are used to prove convergence rates of MidExtremes and ApproachExtreme, when every pair of processes has a common value in each round. The squared Euclidean diameter of a set $A \subseteq \mathbb{R}^d$ is $\mathrm{diam}(A) \triangleq \max_{x,y \in A} \|x - y\|_2^2$. Note that $\sqrt{\mathrm{diam}(A)} = \max_{x,y \in A} \|x - y\|_2$.

### A.2 Cluster-Based Multidimensional Approximate Agreement (Section 4.1)

Two multidimensional approximate agreement algorithms are presented in [15]. The first is MidExtremes, which was defined earlier. The other is ApproachExtreme, which returns the average of the current value and the value that is farthest from it among all received vectors. Formally, for $X \subseteq \mathbb{R}^d$ and $y \in \mathbb{R}^d$

$$\text{ApproachExt}(X, y) = (y + b)/2, \text{ where } b = \arg \max_{b \in X} \|y - b\|_2.$$
LEMMA 14 ([15, LEMMA 4]). Let $a, a', b, b' \in \mathbb{R}^d$ such that $\sqrt{\text{diam}(\{a, a', b, b'\})} \leq \|a - b\|_2 + \|a' - b'\|_2$. Then, setting $m = (a - b)/2$ and $m' = (a' - b')/2$,
$$
\|m - m'\|_2^2 \leq \frac{7}{8} \text{diam}(\{a, a', b, b'\})
$$

LEMMA 15 ([15, LEMMA 5]). Let $a, a', b, b' \in \mathbb{R}^d$ such that $\sqrt{\text{diam}(\{a, a', b, b'\})} \leq 2\|a - b\|_2 + 2\|a' - b'\|_2$. Then, setting $m = (a - b)/2$ and $m' = (a' - b')/2$
$$
\|m - m'\|_2 \leq \frac{31}{32} \text{diam}(\{a, a', b, b'\})
$$

We use these two lemmas to prove the correctness of Algorithm 4. Then, we use techniques from [15] to prove the convergence rate of Algorithm 3, where any pair of processes receive a value from at least one common cluster.

**Shared-memory Multidimensional Approximate Agreement.** Algorithm 4 adapts the algorithms of [15] for approximate and asymptotic consensus to shared memory. The algorithm proceeds in (asynchronous) rounds: in each round, the process writes its value and then collects the values written by other processes for this round; these values are aggregated to compute its value for the next round. The aggregation rule used in line 5 can be MidExtremes or ApproachExtreme.

Let $V_r$ be the processes that write a value to $A_{r+1}$. For process $i \in V_r$, $x_{r+1}^i$ is the value written by $i$ to $A_{r+1}$ and $x_{r+1}$ is the multi-set of all values ever written to $A_{r+1}$.

**Algorithm 4** Shared-memory multidimensional approximate agreement: code for process $i$ in cluster $c$

1. $A_i[1], \ldots, A_{R-1}[i]$, initially ⊥
2. \text{SMMA}(x, q):
3. \text{do}
4. $\text{Collect}(A_{r}[j])$ or $\text{ApproachExt}(X, A_{r}[i])$
5. $A_{r+1} \leftarrow \text{MidExt}(X)$
6. \text{return} $A_{R-1}[i]$

**Lemma 16.** Algorithm 4 satisfies $q$-contraction within $[\log_{7/8} q]$ rounds when using MidExtremes, and $[\log_{31/32} q]$ rounds when using ApproachExtreme.

**Proof.** Let round $r \geq 1$, and two processes $i, j \in V_r$. Let $x_i, x_j$ be the value of $X$ held by $i, j$ after executing Line 4 in round $r$, respectively. Since at each round processes first write to the array and then read it, the first value $c$ written to $A_i$ must appear in the collect of both $i$ and $j$. This means that $c \in X_i \cap X_j$. Let $a, b$ be the values such that $x_{r+1}^i = (a + b)/2$ and $a', b'$ be the values such that $x_{r+1}^j = (a' + b')/2$.

If the aggregation rule used is MidExtremes, since $c \in X_i$ we have that $\|a - c\|_2 \leq \|a - b\|_2$ and $\|b - c\|_2 \leq \|a - b\|_2$. Similarly, as $c \in X_j$, $\|a' - c\|_2 \leq \|a' - b'\|_2$ and $\|b' - c\|_2 \leq \|a' - b'\|_2$. Therefore, using triangle inequality, $\sqrt{\text{diam}(\{a, a', b, b'\})} \leq \|a - b\|_2 + \|a' - b'\|_2$ and following Lemma 14,
$$
\left\|x_{r+1}^i - x_{r+1}^j\right\|_2^2 \leq \frac{7}{8} \text{diam}(\{a, a', b, b'\}) \leq \frac{7}{8} \text{diam}(x_r)
$$

Since this is true for any pair of processes in $V_r$, then $\text{diam}(x_{r+1}) \leq \frac{7}{8} \text{diam}(x_r)$. After $R = [\log_{7/8} q]$ rounds
$$
\text{diam}(x_{R+1}) \leq \left(\frac{7}{8}\right)^R \text{diam}(x_1) = \left(\frac{7}{8}\right)^{\log_{7/8} q} \text{diam}(x_1) \leq q \text{diam}(x_1).
$$

If the aggregation rule used is ApproachExtreme, similarly to the previous case we have that $\|a - c\|_2 \leq \|a - b\|_2$ and $\|a' - c\|_2 \leq \|a' - b'\|_2$. Note that the other two inequalities may not hold in this case. Therefore, using triangle inequality, $\sqrt{\text{diam}(\{a, a', b, b'\})} \leq 2\|a - b\|_2 + 2\|a' - b'\|_2$ and following Lemma 15,
$$
\left\|x_{r+1}^i - x_{r+1}^j\right\|_2^2 \leq \frac{31}{32} \text{diam}(\{a, a', b, b'\}) \leq \frac{31}{32} \text{diam}(x_r).
$$
Similarly to the previous case, after \( R = \lceil \log_{31/32} q \rceil \) rounds we get that \( \text{diam}(x_{R+1}) \leq q \text{diam}(x_1) \). \( \Box \)

Following Lemma 16, Algorithm 4 satisfies 1/6-contraction in 14 rounds with MidExtremes, and 57 rounds with ApproachExtreme.

Proof of Algorithm 3. Let \( V_r \) be the processes that compute a value for iteration \( r + 1 \) (Line 8), and let \( V_b \) be all the processes that execute the first line. Let \( x_r \) be the multi-set of round \( r \) values computed by processes in \( V_{r-1} \) and \( y_r \) be the multi-set of the outputs from the SMMAA algorithm in Line 3 of round \( r \). The convexity property of the shared-memory algorithm implies that for any round \( r \)

\[
\text{diam}(y_r) \leq \text{diam}(x_r)
\]

(20)

Lemma 18 is a generalization of Lemma 14 and Lemma 15. We use the next lemma to prove it.

Lemma 17 ([15, Lemma 3]). Let \( a, b, c \in \mathbb{R}^d \), then setting \( m = (a + b)/2 \)

\[
||m - c||_2^2 \leq \frac{1}{2} ||a - c||_2^2 + \frac{1}{2} ||b - c||_2^2 - \frac{1}{4} ||a - b||_2^2
\]

Lemma 18. Let \( a, a', b, b' \in \mathbb{R}^d \) such that

\[
\text{diam}(\{a, a', b, b'\}) \leq c||a - b||_2^2 + c||a' - b'||_2^2 + d.
\]

(21)

for \( c, d > 0 \). Then, setting \( m = (a - b)/2 \) and \( m' = (a' - b')/2 \)

\[
||m - m'||_2^2 \leq \frac{4c - 1}{4c} \text{diam}(\{a, a', b, b'\}) + \frac{d}{4c}
\]

Proof. By Lemma 17

\[
||a - m'||_2^2 = \frac{1}{2} ||a' - a||_2^2 + \frac{1}{2} ||b' - a||_2^2 - \frac{1}{4} ||a' - b'||_2^2 \leq \text{diam}(\{a, a', b, b'\}) - \frac{1}{4} ||a' - b'||_2^2
\]

\[
||b - m'||_2^2 = \frac{1}{2} ||a' - b||_2^2 + \frac{1}{2} ||b' - b||_2^2 - \frac{1}{4} ||a' - b'||_2^2 \leq \text{diam}(\{a, a', b, b'\}) - \frac{1}{4} ||a' - b'||_2^2
\]

Hence,

\[
||m - m'||_2^2 = \frac{1}{2} ||a - m'||_2^2 + \frac{1}{2} ||b - m'||_2^2 - \frac{1}{4} ||a - b||_2^2 \leq \text{diam}(\{a, a', b, b'\}) - \frac{1}{4} ||a' - b'||_2^2
\]

by Lemma 17

\[
\leq \text{diam}(\{a, a', b, b'\}) - \frac{1}{4} (||a - b||_2^2 + ||a' - b'||_2^2)
\]

\[
\leq \text{diam}(\{a, a', b, b'\}) - \frac{1}{4} \text{diam}(\{a, a', b, b'\}) + \frac{d}{4c}
\]

by (21)

\[
= \frac{4c - 1}{4c} \text{diam}(\{a, a', b, b'\}) + \frac{d}{4c}
\]

\( \Box \)

Lemma 19. For any round \( r \)

\[
\text{diam}(x_{r+1}) \leq \frac{23}{24} \text{diam}(x_r)
\]

Proof. Let processes \( i, j \in V_r \). Let \( a, b \) be the values such that \( x_{r+1} = (a + b)/2 \) and \( a', b' \) be the values such that \( x_{r+1} = (a' + b')/2 \). Since processes \( i \) and \( j \) received messages from a majority of the clusters, then both processes must have received messages from the same cluster \( c \). Thus, process \( i \) received value \( y^k_r \) and process \( j \) received value \( y^l_r \), where \( k \) and \( l \) are part the same cluster. Since both \( y^k_r \) and \( y^l_r \) are the outputs of the
same SMMA algorithm satisfying $1/10$-contraction. $\|y^k - y^l\|_2^2 = d \leq \frac{1}{6} \text{diam}(x_r)$. By using (16), we get that $\text{diam}(\{a, a', b, b'\}) \leq 3\|a - b\|_2^2 + 3\|a' - b'\|_2^2 + 3d$. By Lemma 18
\[ \left\| x_{r+1}^i - x_{r+1}^j \right\|_2^2 \leq \frac{11}{12} \text{diam}(\{a, a', b, b'\}) + \frac{d}{4} \leq \frac{11}{12} \text{diam}(y_r) + \frac{d}{4} \leq \frac{11}{12} \text{diam}(x_r) + \frac{1}{24} \text{diam}(x_r) = \frac{23}{24} \text{diam}(x_r) \text{ by (20)} \]

The lemma follows since the bound holds for every pair of vectors in $x_{r+1}$. □

Next, we show that ApproachExtreme can also be used in Algorithm 3 and the algorithm converges after $\lceil \log_{79/80} q \rceil$ asynchronous rounds, when the SMMA satisfies $1/10$-contraction.

**Lemma 20.** Consider Algorithm 3 with the following changes: (1) ApproachExt$(R_{cv}, y_i^j)$ is used in Line 5, (2) the SMMA in Line 3 must satisfy $1/10$-contraction and (3) $R = \lceil \log_{79/80} q \rceil$ in Line 2. Then, for any round $r$
\[ \text{diam}(x_{r+1}) \leq \frac{79}{80} \text{diam}(x_r) \]

**Proof.** Let processes $i, j \in V_r$. Let $a, b$ be the values such that $x_{r+1}^i = (a + b)/2$ and $a', b'$ be the values such that $x_{r+1}^j = (a' + b')/2$. Similarly to the proof of the previous lemma, process $i$ received value $y_i^k$ and process $j$ received value $y_j^l$ such that $\|y_i^k - y_j^l\|_2^2 = d \leq \frac{1}{79} \text{diam}(x_r)$. By using (16), we get that $\text{diam}(\{a, a', b, b'\}) \leq 10\|a - b\|_2^2 + 10\|a' - b'\|_2^2 + 5d$. By Lemma 18
\[ \left\| x_{r+1}^i - x_{r+1}^j \right\|_2^2 \leq \frac{39}{40} \text{diam}(\{a, a', b, b'\}) + \frac{d}{8} \leq \frac{11}{12} \text{diam}(y_r) + \frac{d}{4} \leq \frac{39}{40} \text{diam}(x_r) + \frac{1}{80} \text{diam}(x_r) = \frac{79}{80} \text{diam}(x_r) \text{ by (20)} \]

ApproachExtreme and MidExtremes offer a tradeoff between local computation and the number of communication rounds. MidExtremes requires $O(n)$ more local computation than ApproachExtreme, but its contraction rate is better, yielding fewer communication rounds.

### A.3 Additional Proofs for Non Strongly Convex Functions (Section 4)

The next two lemmas are used to prove Theorem 4. First, we show values for the parameters of approximate agreement, which yield convergence rate that is comparable to the sequential algorithm with constant learning rate.

**Lemma 21.** Consider Algorithm 2 with constant learning rate $\eta \leq \min\{\frac{1}{2}, \frac{1}{L}\}$ and for every iteration $t$, $q_t = \frac{2}{t}$. Then for any iteration $t \geq 1$,
\[ \max_{i,j \in V_t} \mathbb{E} \left[ \left\| x_{t+1}^i - x_{t+1}^j \right\|_2^2 \right] \leq \frac{2\sigma^2\eta^3}{N} \]

**Proof.** Following Lemma 9
\[ \max_{i,j \in V_t} \mathbb{E} \left[ \left\| x_{t+1}^i - x_{t+1}^j \right\|_2^2 \right] \leq q_t (2 + 2L^2\eta^2) \max_{i,j \in V_{t-1}} \mathbb{E} \left[ \left\| x_t^i - x_t^j \right\|_2^2 \right] + q_t \frac{4\sigma^2\eta^2}{N} \]
\[
\frac{1}{2} (\eta + L^2 \eta^3) \max_{i, j \in V_{t-1}} E \left[ \|x_i - x_i^t\|_2^2 \right] + \frac{\sigma^2 \eta^3}{N} \\
\leq \eta \max_{i, j \in V_{t-1}} E \left[ \|x_i - x_i^t\|_2^2 \right] + \frac{\sigma^2 \eta^3}{N} \quad \eta \leq \frac{1}{L} \\
\leq \frac{1}{2} \max_{i, j \in V_{t-1}} E \left[ \|x_i - x_i^t\|_2^2 \right] + \frac{\sigma^2 \eta^3}{N} \quad \eta \leq \frac{1}{2}
\]

By denoting \( u_t = \max_{i, j \in V_{t-1}} E \left[ \|x_i - x_i^t\|_2^2 \right] \) we can write \( u_{t+1} \leq \frac{1}{2} u_t + \frac{\sigma^2 \eta^3}{N} \). By induction and using the fact that \( u_1 = 0 \) we get that

\[
u_{t+1} \leq \frac{\sigma^2 \eta^3}{N} \sum_{i=0}^{t-1} 2^{-i} \leq \frac{\sigma^2 \eta^3}{N} \sum_{i=0}^{\infty} 2^{-i} \leq \frac{2\sigma^2 \eta^3}{N}
\]

\[
\square
\]

Using the parameters of Lemma 21, Algorithm 3 requires \( \log_{123/(4)} (\eta/4) \) asynchronous rounds. This yields \( T \log_{123/(4)} (\eta/4) \) asynchronous rounds in Algorithm 2.

**Lemma 22.** Let \( Q \) be an \( L \)-smooth cost function. Consider Algorithm 2 with \( T \geq \max \{16L^2 N, 4N\} \), constant learning rate \( \eta = \frac{\sqrt{N}}{\sqrt{T}} \) and parameters set as in Lemma 21, then for every process \( i \in V_T \)

\[
\min_{t \in [T]} E \left[ \|\nabla Q(x_i^t)\|_2^2 \right] \leq \frac{4(Q(x_1)^t - Q^*)}{\sqrt{NT}} + \frac{20\sigma^2 L}{\sqrt{NT}} + \frac{16\sigma^2}{\sqrt{NT}}
\]

**Proof.** Note that \( \eta = \frac{\sqrt{N}}{\sqrt{T}} \leq \min \{\frac{1}{4}, \frac{1}{2}\} \). Following Theorem 3

\[
\min_{t \in T} E \left[ \|\nabla Q(x_i^t)\|_2^2 \right] \leq \frac{4(Q(x_1) - Q^*)}{\sqrt{NT}} + \frac{4\sigma^2 L}{\sqrt{NT}} + \left( \frac{8T}{N} + 8L^2 \right) \max_{t \in [T]} \max_{i, j \in V_t} E \left[ \|x_i - x_i^t\|_2^2 \right] \\
\leq \frac{4(Q(x_1) - Q^*)}{\sqrt{NT}} + \frac{4\sigma^2 L}{\sqrt{NT}} + \left( \frac{16T}{N} + 16L^2 \right) \frac{\sigma^2 \eta^3}{N} \quad \text{by Lemma 21} \\
= \frac{4(Q(x_1) - Q^*)}{\sqrt{NT}} + \frac{4\sigma^2 L}{\sqrt{NT}} + \left( \frac{16T}{N} + 16L^2 \right) \frac{\sqrt{N} \sigma^2}{T^{3/2}} \quad \eta = \frac{\sqrt{N}}{\sqrt{T}} \\
= \frac{4(Q(x_1) - Q^*)}{\sqrt{NT}} + \frac{4\sigma^2 L}{\sqrt{NT}} + \frac{16\sqrt{N} \sigma^2}{\sqrt{NT}} + \frac{16\sqrt{N} \sigma^2}{\sqrt{NT}} \frac{\sigma^2}{T^{3/2}} \quad T \geq N \\
\leq \frac{4(Q(x_1) - Q^*)}{\sqrt{NT}} + \frac{20\sigma^2 L}{\sqrt{NT}} + \frac{16\sigma^2}{\sqrt{NT}}
\]

\[
\square
\]