AF-embeddability for Lie groups with $T_1$ primitive ideal spaces

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Abstract

We study simply connected Lie groups $G$ for which the hull-kernel topology of the primitive ideal space $\text{Prim}(G)$ of the group $C^*$-algebra $C^*(G)$ is $T_1$, that is, the finite subsets of $\text{Prim}(G)$ are closed. Thus, we prove that $C^*(G)$ is AF-embeddable. To this end, we show that if $G$ is solvable and its action on the centre of $[G,G]$ has at least one imaginary weight, then $\text{Prim}(G)$ has no nonempty quasi-compact open subsets. We prove in addition that connected locally compact groups with $T_1$ ideal spaces are strongly quasi-diagonal.

1. Introduction

Considerable attention has been recently paid to finite-dimensional approximation properties of $C^*$-algebras, such as quasi-diagonality or embeddability into a $C^*$-algebra that is an inductive limit of finite-dimensional $C^*$-algebras — for short, AF-embeddability.

In the present paper, we study these properties for $C^*$-algebras that occur in the non-commutative harmonic analysis, that is, the $C^*$-algebras of connected Lie groups. In sharp contrast to the $C^*$-algebras of countable discrete groups, we have already found in [3] that there exist solvable (hence amenable) connected Lie groups whose corresponding $C^*$-algebras are not quasi-diagonal, and in particular they are not AF-embeddable. It turns out, however, that affirmative results can be obtained under natural topological assumptions on the primitive ideal space, as described below.

The primitive ideal space $\text{Prim}(G)$ of a locally compact group $G$ is the primitive ideal space of the group $C^*$-algebra $C^*(G)$, endowed with its hull-kernel topology. One of the main results of this paper is: If $G$ is a simply connected Lie group for which $\text{Prim}(G)$ is $T_1$, then $C^*(G)$ is AF-embeddable (Theorem 5.3). We recall that the class of connected Lie groups $G$ for which $\text{Prim}(G)$ is $T_1$ was characterized in [19] and [24] in Lie algebraic terms. Using that characterization, it turns out that it simultaneously contains many differing classes of solvable Lie groups that are not necessarily of type I, such as the celebrated Mautner groups, Dixmier groups, the diamond groups, the rigid motion groups, nilpotent Lie groups and compact groups.

If the topological condition $T_1$ is not required, then $C^*(G)$ is still AF-embeddable whenever $G$ is a connected and simply connected solvable Lie group and its action on the centre of the commutator group $[G,G]$ (which is nilpotent) has a purely imaginary weight (Corollary 4.5). We recall that, without the weight conditions, connected, simply connected and solvable Lie groups may not be AF-embeddable (see [3, Theorem 2.15]).

Our approach to the above AF-embeddability result is based on a preliminary investigation of the dual topology of the solvable Lie groups, a topic that, despite some remarkable works, remains notoriously difficult even for some type I solvable Lie groups. Our key technical result shows that if $G$ is a second countable locally compact group which has a closed normal subgroup $L$ such that $A := G/L$ is abelian and the closure of every orbit of $G$ in $L$ is the spectrum of

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the restriction of a unitary representation of $G$, then there is a homeomorphism of quasi-orbit spaces

$$(Prim(G)/\hat{A})^\sim \simeq (Prim(L)/G)^\sim.$$  

(See Theorem 3.6.) Using the results of [23], we show that the above conditions on $G$ are satisfied for arbitrary connected and simply connected solvable Lie groups, with $L = [G,G]$. What the above homeomorphism facilitates for us is to check that when the action of $G$ on the centre of $L$ has at least one purely imaginary weight, $Prim(G)$ has no nonempty quasi-compact open subsets (Theorem 4.1), which further allows an application of a result of [13]. As an application of Theorem 5.3, we finally prove that if $A$ is the $C^*$-algebra of a simply connected solvable Lie group, then the following implication holds true: if the image of $A$ in every irreducible representation is AF-embeddable, then $A$ itself is AF-embeddable (Corollary 6.7). Apart from the well-known examples of that implication, we obtain absence of nonempty open quasi-compact subsets of $Prim(G)$ if $G$ is a simply connected solvable Lie group such that the action of $G$ action on the centre of $[G,G]$ has at least one purely imaginary weight. As a consequence, we obtain absence of nonempty open quasi-compact subsets of $Prim(G)$ if $G$ is a simply connected solvable Lie group of type $R$. Earlier results of this type can be found in [9, §4.1] in the special case of Lie groups whose centres are nondiscrete. That condition, however, is too restrictive even for some simple and yet quite important examples of solvable Lie groups such as the Mautner group or its generalized versions from [1]. Section 5 contains our main result on AF-embeddability of $C^*(G)$ if $G$ is a simply connected Lie group for which $Prim(G)$ is $T_1$. Finally, in Section 6 we show that if $G$ is a simply connected solvable Lie group, then the $T_1$ condition on $Prim(G)$ is equivalent to the condition that the image of $C^*(G)$ in every irreducible representation is AF-embeddable, and this hence implies that $C^*(G)$ is itself AF-embeddable by the results of Section 5. In addition, we prove that every connected, locally compact group with $T_1$ primitive ideal space is strongly quasi-diagonal (Corollary 6.6).

2. Preliminaries

We prove here some preliminary results for later use. For the notions and notation in $C^*$-algebras and topology, we refer the reader to [2, 6 10, 30].

2.1. The space of closed subsets of a topological space

We first recall the topology defined on the set of all closed subsets of a topological space, seen as a set with a partial ordering given by inclusion. For details, we refer the reader to [14].

**Definition 2.1.** Let $X$ be a topological space. For $F \subseteq X$ closed, we define

$$\downarrow F := \{ F' \mid F' \subseteq F, F' \text{ closed in } X \}.$$

We denote by $Cl(X)$ the space of all closed subsets of $X$ endowed with its upper topology, that is, the topology for which a sub-base of closed sets consists of $X$ and sets of the form $\downarrow F$, with $F \subseteq X$ closed. See [14, Definition O-5.4].
One of the main objects in our paper is the space of quasi-orbits of an action of a topological group on a topological space. In these spaces, the points are not necessarily closed, so we need to briefly recall the notion of $T_0$-ization of a topological space. See [30, Chapter 6] for more details.

**Definition 2.2.** Let $X$ be topological space, and consider the equivalence relation

$$x \sim y \iff \{x\} = \{y\}.$$ 

Then the $T_0$-ization of $X$ is the quotient space

$$X^\sim = X/\sim,$$

with the corresponding quotient topology. The quotient map $q: X \to X/\sim$ is called the $T_0$-ization map.

Part of the following lemma is known in some special cases (see [30, Lemma 6.8]), but we give the proofs here for the sake of completeness.

**Lemma 2.3.** Let $X$ be a topological space, and $G$ a topological group with a continuous action $G \times X \to X$, $(g, x) \mapsto g \cdot x$. Denote by $r: X \to X/G$ the canonical quotient map, by $q: X/G \to (X/G)^\sim$ the $T_0$-ization map, and by $Q: X \to (X/G)^\sim$, $Q := q \circ r$ the quasi-orbit map. Then we have:

(i) the map $r$ is continuous, open and surjective, and $r^{-1}(\{r(x)\}) = \overline{G \cdot x}$ for every $x \in X$;

(ii) for every $x, y \in X$, $\{r(x)\} = \{r(y)\}$ if and only if $G \cdot x = G \cdot y$;

(iii) $q$ and $Q$ are continuous, open and surjective;

(iv) the map $\iota: (X/G)^\sim \to \text{Cl}(X)$, given by

$$\iota(Q(x)) = \overline{G \cdot x}$$

is well defined, and a homeomorphism onto its image.

**Proof.** (i) The maps $r$ and $\iota$ are continuous and surjective since it is a quotient map. The fact that $r$ is open is well known: For every $D$ open in $X$, $r^{-1}(r(D)) = G \cdot D \subseteq X$ is open, hence $r(D)$ is open in the quotient topology.

We have that $G \cdot x = r^{-1}(\{r(x)\}) \subseteq r^{-1}(\{r(x)\})$, and this last set is closed since $r$ is continuous, hence $\overline{G \cdot x} \subseteq r^{-1}(\{r(x)\})$.

It remains to prove the converse inclusion. We note that, since $r$ is an open map, the set $\overline{G \cdot x}$ is $G$-invariant, that is, $\overline{G \cdot x} = r^{-1}(r(\overline{G \cdot x}))$. Therefore its complement $X \setminus \overline{G \cdot x}$ is also $G$-invariant, thus $r^{-1}(r(X \setminus \overline{G \cdot x})) = X \setminus \overline{G \cdot x}$. Since $(X \setminus \overline{G \cdot x}) \cap G \cdot x = \emptyset$, it follows then that

$$r(x) \notin r(X \setminus \overline{G \cdot x}). \quad (2.1)$$

On the other hand, using that $r$ is an open mapping, the subset $r(X \setminus \overline{G \cdot x}) \subseteq X/G$ is open. Therefore (2.1) implies $\{r(x)\} \cap r(X \setminus \overline{G \cdot x}) = \emptyset$, hence

$$r^{-1}(\{r(x)\}) \cap r^{-1}(r(X \setminus \overline{G \cdot x})) = \emptyset.$$

Using again that $X \setminus \overline{G \cdot x}$ is $G$-invariant, it follows that $r^{-1}(\{r(x)\}) \cap (X \setminus \overline{G \cdot x}) = \emptyset$. Thus we get the inclusion $r^{-1}(\{r(x)\}) \subseteq \overline{G \cdot x}$.

(ii) The assertion follows immediately from (i) for the implication ‘⇐’ using that the map $r$ is surjective.
(iii) The map $q$ is continuous and surjective since it is a quotient map. To prove that $q$ is open, it suffices to show that $q^{-1}(q(r(D)))$ is open for every $D$ open in $X$. We have that
\[
q^{-1}(q(r(D))) = \{ r(x) \mid \exists y \in D, q(r(x)) = q(r(y)) \}
\]
\[
= \{ r(x) \mid \exists y \in D, \{ r(x) \} = \{ r(y) \} \}
\]
\[
= \{ r(x) \mid \exists y \in D, \mathcal{G} \cdot x = \mathcal{G} \cdot y \},
\]
where in the last line we have used (ii). If there is $y \in D$ such that $\mathcal{G} \cdot x = \mathcal{G} \cdot y$, then $D \cap \mathcal{G} \cdot x \neq \emptyset$, hence $D \cap \mathcal{G} \cdot x \neq \emptyset$. On the other hand, if $D \cap \mathcal{G} \cdot x \neq \emptyset$, then for $y \in D \cap \mathcal{G} \cdot x$ we have that $\mathcal{G} \cdot x = \mathcal{G} \cdot y$, hence $\mathcal{G} \cdot x = \mathcal{G} \cdot y$. Summing up, we have obtained that
\[
q^{-1}(q(r(D))) = \{ r(x) \mid D \cap \mathcal{G} \cdot x \neq \emptyset \} = r(\mathcal{G} \cdot D),
\]
therefore $q^{-1}(q(r(D)))$ is open.

The fact that $Q$ is a continuous, open and surjective map follows immediately.

(iv) The map $\iota$ is well defined and injective since $Q(x) = Q(y)$ if and only if $\{ r(x) \} = \{ r(y) \}$ and if and only if $\mathcal{G} \cdot x = \mathcal{G} \cdot y$, by (ii).

A sub-basis of open sets for the topology on $\Cl(X)$ is given by the sets of the form
\[
U(D) = \{ F \in \Cl(X) \mid F \cap D \neq \emptyset \},
\]
where $D$ are open sets in $X$. For $D \subseteq X$ open,
\[
\iota^{-1}(U(D)) = \{ Q(x) \mid \iota(Q(x)) \in U(D) \}
\]
\[
= \{ Q(x) \mid \mathcal{G} \cdot x \cap D \neq \emptyset \}
\]
\[
= \{ Q(x) \mid \mathcal{G} \cdot x \cap D \neq \emptyset \} = Q(\mathcal{G} \cdot D),
\]
hence $\iota^{-1}(U(D))$ is open. It follows that $\iota$ is continuous. On the other hand, using the arguments above,
\[
\iota(Q(D)) = \{ \mathcal{G} \cdot x \mid x \in D \}
\]
\[
= \{ \mathcal{G} \cdot x \mid \mathcal{G} \cdot x \cap D \neq \emptyset \}
\]
\[
= \iota((X/G)^\sim) \cap U(D).
\]
It follows that $\iota : (X/G)^\sim \to \iota((X/G)^\sim)$ is open as well, hence $\iota$ is a homeomorphism onto its image (see [5, p. TGI.30]).

**Definition 2.4.** Whenever the conditions Lemma 2.3 are satisfied, we denote
\[
(X/G)^\sim := \iota((X/G)^\sim) \subseteq \Cl(X)
\]
with the induced topology. Hence $(X/G)^\sim$ is a homeomorphic copy of $(X/G)^\sim$.

### 2.2. Upper topology and inner hull-kernel topology

Let $G$ be a second countable locally compact group $G$, with its $C^*$-algebra $C^*(G)$. We denote by $\Prim(G)$ the space of primitive ideals in $C^*(G)$ with the hull-kernel topology. We denote by $\widehat{G}$ the space of equivalence classes $[\pi]$ of irreducible unitary representations $\pi$ of $G$ endowed with the hull-kernel topology, that is, the inverse image topology for the canonical map $\kappa : \widehat{G} \to \Prim(G)$, $\kappa([\pi]) = \text{Ker} \pi$. We also note that, by the definition of the topology of $\widehat{G}$, the surjective mapping $\kappa$ is both open and closed. Here and throughout this paper, for every continuous unitary representation $\pi$ of $G$, we denote also by $\pi$ its corresponding nondegenerate $*$-representation of $C^*(G)$. In particular, we identify $\widehat{G} = C^*_\text{red}(G)$. 

We denote by $\mathcal{T}(G)$ the set of all equivalence classes of unitary representations of $G$ in separable complex Hilbert spaces. For any $S_1, S_2 \subseteq \mathcal{T}(G)$, we write $S_1 \preceq S_2$ ($S_1$ is weakly contained in $S_2$) if $\bigcap_{\pi \in S_1} \text{Ker} \pi \supseteq \bigcap_{\pi \in S_2} \text{Ker} \pi$, and $S_1 \approx S_2$ ($S_1$ is weakly equivalent to $S_2$) if $S_1 \preceq S_2$ and $S_2 \preceq S_1$.

On $\mathcal{T}(G)$, we consider the inner hull-kernel topology, which restricted to $\hat{G}$ is the hull-kernel topology. (See [12, ].) We recall that for every $*$-representation $T$ of the $C^*$-algebra $C^*(G)$ there exists a unique closed subset $\text{spec} (T) \subseteq \hat{G}$ — the spectrum of the unitary equivalence class of $*$-representations $[T]$ — which is weakly equivalent to $[T]$, and

$$\text{spec} (T) = \{ [\tau] \in \hat{G} \mid \tau \preceq T \}.$$ 

(See [10, Definition 3.4.6].)

We note that for simplicity, for a unitary representation $T$ of $G$, we sometimes write $T$ instead of $[T]$, and $T \approx T_0$ when $\{ [T] \} \approx \{ [T_0] \}$, where $T_0$ is another unitary representation of $G$.

**Remark 2.5.** The inner hull-kernel topology of $\mathcal{T}(G)$ is the coarsest topology for which the mapping

$$\text{spec} : \mathcal{T}(G) \to \text{Cl} (\hat{G})$$

is continuous.

**Lemma 2.6.** Let $G$ be a second countable locally compact group. Define the map

$$\hat{\kappa} : \text{Cl} (\hat{G}) \to \text{Cl} (\text{Prim}(G)),$$

$$\hat{\kappa} (\mathcal{F}) := \{ \kappa ([\pi]) \mid [\pi] \in \mathcal{F} \} \text{ for all } \mathcal{F} \in \text{Cl} (\hat{G}).$$

Then $\hat{\kappa}$ is continuous.

**Proof.** The map $\hat{\kappa}$ takes values in $\text{Cl} (\text{Prim}(G))$ since $\kappa : \hat{G} \to \text{Prim}(G)$ is a closed map. To prove that $\hat{\kappa}$ is continuous, it is enough to show that for every $\mathcal{S} \in \text{Cl} (\text{Prim}(G))$, the set $\hat{\kappa}^{-1} (\downarrow \mathcal{S})$ is closed in $\text{Cl} (\hat{G})$. By a simple computation, we see that $\hat{\kappa}^{-1} (\downarrow \mathcal{S}) = \downarrow \kappa^{-1} (\mathcal{S})$, and the assertion above follows since $\kappa^{-1} (\mathcal{S})$ is a closed subset of $\hat{G}$. \qed

**Definition 2.7.** We keep the notation above. For every $*$-representation $T$ of the $C^*$-algebra $C^*(G)$, we define the support of the unitary equivalence class $[T] \in \mathcal{T}(G)$ by

$$\text{supp} (T) := \hat{\kappa} (\text{spec} (T)) \in \text{Cl} (\text{Prim}(G)).$$

**Remark 2.8.** The map $\text{supp} : \mathcal{T}(G) \to \text{Cl} (\text{Prim}(G))$ is continuous. This is a consequence of the definition, Remark 2.5 and Lemma 2.6.

**Lemma 2.9.** Let $L$ be a locally compact group, and let $\kappa : \hat{L} \to \text{Prim}(L)$, $\pi \mapsto \text{Ker} \pi$ be its canonical map. Then for $T \in \mathcal{T}(L)$ and $A \in \text{Cl} (\hat{L})$, $\text{spec} (T) = A$ if and only if $\text{supp} (T) = \kappa (A)$.

**Proof.** This follows from the fact that, since $\kappa$ is an closed and open surjective map, $\kappa^{-1} (\kappa (A)) = A$, and $\text{supp} (T) = \kappa (\text{spec} (T))$ for every $A \in \text{Cl} (\hat{L})$. \qed

**Lemma 2.10.** Let $G$, $L$ be locally compact groups such that $G$ has a continuous action $G \times C^*(L) \to C^*(L)$ by automorphisms of $C^*(L)$, and let $\kappa : \hat{L} \to \text{Prim}(L)$, $\pi \mapsto \text{Ker} \pi$ be the canonical map corresponding to $L$. Then the map

$$\kappa^n : (\hat{L}/G)^\infty \to (\text{Prim}(L)/G)^\infty, \ G \cdot \pi \mapsto G \cdot \kappa (\pi)$$

is a homeomorphism.
Proof. We recall the canonical identification \( \widehat{C^*(L)} = \widehat{L} \), therefore \( G \) acts continuously on \( \widehat{L} \) and \( \text{Prim}(L) \). Then for every \( A \subseteq \widehat{L} \), \( A = \kappa^{-1}(\kappa(A)) \), since \( \kappa \) is open, closed and surjective. It follows that for \( A_1, A_2 \subseteq \widehat{L} \), \( A_1 = A_2 \) if and only if \( \kappa(A_1) = \kappa(A_2) \). Thus for \( \pi_1, \pi_2 \in \widehat{L} \), we have that

\[
G \cdot \pi_1 = G \cdot \pi_2 \Leftrightarrow \kappa(G \cdot \pi_1) = \kappa(G \cdot \pi_2) \Leftrightarrow \kappa(\pi_1) = \kappa(\pi_2).
\]

Therefore the map \( \kappa^\infty \) is well defined and injective. Since \( \kappa \) is surjective, \( \kappa^\infty \) is surjective as well. Hence \( \kappa^\infty \) is bijective.

The map of \( \kappa^\infty \) is open and continuous since the diagram

\[
\begin{array}{ccc}
\widehat{L} & \xrightarrow{\kappa} & \text{Prim}(L) \\
\downarrow & & \downarrow \\
(\widehat{L}/G)^\infty & \xrightarrow{\kappa^\infty} & (\text{Prim}(L)/G)^\infty
\end{array}
\]

is commutative, where the down-arrows are the maps \( \widehat{L} \rightarrow (\widehat{L}/G)^\infty \), \( \pi \mapsto \overline{G \cdot \pi} \), and \( \text{Prim}(L) \rightarrow (\text{Prim}(L)/G)^\infty \), \( P \mapsto \overline{G \cdot P} \), which are open and continuous, while \( \kappa \) is continuous, closed and open. Hence \( \kappa^\infty \) is a homeomorphism. \( \square \)

2.3. Actions of Lie groups on abelian Lie groups, and the absence of quasi-compact open sets in the space of quasi-orbits

Let \( G \) be a topological group with its centre \( Z \). For every unitary irreducible representation \( \pi: G \rightarrow \mathcal{B}(\mathcal{H}_\pi) \), there exists a character \( \chi_\pi: Z \rightarrow \mathbb{T} \) with \( \pi(g) = \chi_\pi(g) \text{id}_{\mathcal{H}_\pi} \) for every \( g \in Z \). The character \( \chi_\pi \) actually depends on the unitary equivalence class of \( \pi \) rather than on \( \pi \) itself, hence we obtain a well-defined mapping

\[
R^G: \hat{G} \rightarrow \hat{Z}, \quad [\pi] \mapsto \chi_\pi.
\] (2.2)

**Lemma 2.11.** Let \( G \) be a separable locally compact group with its centre \( Z \). Then the following assertions hold.

(i) The mapping \( R^G: \hat{G} \rightarrow \hat{Z} \) is surjective and continuous.

(ii) There exists a surjective continuous mapping \( \text{Res}^G_Z: \text{Prim}(G) \rightarrow \hat{Z} \) satisfying \( \text{Res}^G_Z(\ker \pi) = R^G([\pi]) \) for every \( [\pi] \in \hat{G} \).

(iii) If \( G \) is amenable, then \( R^G: \hat{G} \rightarrow \hat{Z} \) and \( \text{Res}^G_Z: \text{Prim}(G) \rightarrow \hat{Z} \) are open maps.

**Proof.** (i) By [10, Proposition 18.1.5], we obtain that, for every subset \( S \subseteq \hat{G} \), \( R^G \) maps the closure of \( S \) into the closure of \( R^G(S) \), hence \( R^G \) is continuous.

To prove that \( R^G \) is surjective, let \( \chi \in \hat{Z} \) be arbitrary and let \( \pi_\chi: G \rightarrow \mathcal{B}(\mathcal{H}) \) be the unitary representation induced from \( \chi \). Since \( G \) is separable, it follows by [12, Theorem 4.5] that \( \pi|_Z \) is weakly equivalent to the orbit of \( \chi \) under the natural action of \( G \) on \( Z \). However, since \( Z \) is the centre of \( G \), that \( G \)-orbit of \( \chi \) is the singleton \( \{\chi\} \subseteq \hat{Z} \), and therefore \( R^G([\pi_\chi]) = \chi \).

(ii) The mapping \( \text{Res}^G_Z \) exists and is continuous by (i), using [30, Lemma C.6] or [30, Corollary 6.15].

(iii) As noted above, the \( G \)-orbit of any \( \chi \in \hat{Z} \) is \( \{\chi\} \subseteq \hat{Z} \) hence, since \( \hat{Z} \) is Hausdorff, it follows that \( G \) acts minimally on \( \hat{Z} \). Then, since \( G \) is amenable, for any \( \chi \in \hat{G} \) and \( \pi \in \hat{G} \), one has \( \pi \leq \text{Ind}^G_Z(\chi) \) if and only if \( \chi \leq \pi|_Z \) by [15, Theorem 3.3]. Taking into account the definition of \( R^G: \hat{G} \rightarrow \hat{Z} \), we then obtain

\[
(R^G)^{-1}(\chi) = \{\pi \in \hat{G} \mid \chi \leq \pi|_Z \} = \{\pi \in \hat{G} \mid \pi \leq \text{Ind}^G_Z(\chi) \} = \text{spec}(\text{Ind}^G_Z(\chi)).
\]
Therefore

$$(\text{Res}_Z^G)^{-1}(\chi) = \text{supp} (\text{Ind}_Z^G(\chi)) =: S_{00}(\chi)$$

for arbitrary $\chi \in \hat{Z}$. The mapping $S_{00}: \hat{Z} \to \text{Cl}(\text{Prim}(G))$ defined this way is continuous as a composition of continuous maps. For every $C \in \text{Cl}(\text{Prim}(G))$, we have

$$(\text{Res}_Z^G)^{\mathfrak{t}}(C) := \{\chi \in \hat{Z} \mid (R^G)^{-1}(\chi) \subseteq C\} = S_{00}^{-1}(\downarrow C),$$

hence $(\text{Res}_Z^G)^{\mathfrak{t}}(C) \subseteq \hat{Z}$ is a closed subset since $\downarrow C \subseteq \text{Cl}(\text{Prim}(G))$ is a closed subset and the map $S_{00}$ is continuous. Since $\text{Res}_Z^G: \text{Prim}(G) \to \hat{Z}$ is surjective, it then follows that $\text{Res}_Z^G$ is an open mapping. Finally, $R^G = (\text{Res}_Z^G) \circ \kappa$ is open since $\kappa: \hat{G} \to \text{Prim}(G)$ is an open map. \hfill \Box

**Remark 2.12.** Lemma 2.11 provides an alternative approach to [9, Proposition 4.1] that does not need continuous fields of $C^*$-algebras. The connection with the earlier approach is that, if the locally compact group $G$ is amenable, then one has the natural open continuous surjective mapping $\text{Res}_Z^G: \text{Prim}(G) \to \hat{Z}$ by Lemma 2.11, hence $C^*(G)$ has the structure of a continuous $C^*$-bundle on $\hat{Z}$ by [30, Theorem C.26].

Before proceeding, we need a definition.

**Definition 2.13.** Let $K$ be a connected Lie group with its Lie algebra $\mathfrak{k}$, $V$ a finite-dimensional real vector space and $\rho: K \to \text{End}(V)$ a representation of $K$ on $V$. By a slight abuse of notation, for every $T \in \text{End}(V)$, we denote by again by $T$ its corresponding extension to a $C^\infty$-linear operator on the complexification $V_\mathbb{C} := \mathbb{C} \otimes_{\mathbb{R}} V$, that is, the operator $\text{id}_{\mathbb{C}} \otimes T$ is denoted again by $T$. Then an $\mathbb{R}$-linear functional $\lambda: \mathfrak{k} \to \mathbb{C}$ is called a weight of the representation $\rho$ if there exists $w \in V_\mathbb{C} \setminus \{0\}$ for which $d\rho(X)w = \lambda(X)w$ for every $X \in \mathfrak{k}$. We say that $\lambda$ is a purely imaginary weight if $\lambda(\mathfrak{k}) \subseteq i\mathbb{R}$.

For instance, if $K$ is a solvable Lie group and the spectrum of $d\rho(X)$ is contained in $i\mathbb{R}$ for every $X \in \mathfrak{k}$, then there exists a purely imaginary weight of $\rho$ as a direct consequence of Lie’s theorem on representations of solvable Lie algebras. If $K$ is actually a nilpotent Lie group, then every weight of $\rho$ is purely imaginary if and only if the spectrum of $d\rho(X)$ is contained in $i\mathbb{R}$ for every $X \in \mathfrak{k}$, by the weight-space decomposition of representations of nilpotent Lie algebras.

**Remark 2.14.** Let $Z$ be an abelian Lie group, connected and simply connected. We may then assume that $Z$ is a finite-dimensional real vector space. Let $K$ a locally compact group, and $\alpha: K \to \text{End}(Z)$ be a continuous representation; it induces a continuous action $K \times \hat{Z} \to \hat{Z}$. On the other hand, $\hat{Z}$ can be identified with $Z^*$, by $\xi \mapsto \chi_\xi$, with $\chi_\xi(z) = e^{i(\xi,z)}$, where $\langle \cdot, \cdot \rangle: Z^* \times Z \to \mathbb{R}$ is the duality bracket. With this identification, $\alpha$ induces a representation $\alpha^*: K \to \text{End}(Z^*)$, which is fact the contragredient of $\alpha$.

**Lemma 2.15.** Let $Z$ be an abelian Lie group, connected and simply connected, $K$ a connected and simply connected Lie group with its Lie algebra $\mathfrak{k}$, and a continuous representation $\alpha: K \to \text{End}(Z)$. Assume that the action $d\alpha: \mathfrak{k} \to \text{End}(Z)$ has at least a purely imaginary weight. Then $(Z^*/K)^\sim$ contains no non-empty open quasi-compact subsets.

**Proof.** With the notations in Remark 2.14, $Z$ is a vector space, $\hat{Z} \simeq Z^*$, and the hypothesis implies that the action $d\alpha^*: \mathfrak{k} \to \text{End}(Z^*)$ has at least a purely imaginary weight.
Let $\lambda_0 : \mathfrak{k} \to i\mathbb{R}$ be a purely imaginary weight of $d\alpha^*$. By the weight space decomposition of $Z^*$ with respect to the Lie algebra representation $d\alpha^* : \mathfrak{k} \to \text{End}(Z^*)$, as discussed in [4, §2.2], it follows that there exists a linear subspace $Z^*_0 \subseteq Z^*$ satisfying the following conditions.

- $d\alpha^*(\mathfrak{k})Z^*_0 \subseteq Z^*_0$ and $\dim_{\mathbb{R}} Z^*/Z^*_0 = 2$ when $\lambda_0 \neq 0$, or $\dim_{\mathbb{R}} Z^*/Z^*_0 = 1$ when $\lambda_0 \equiv 0$.
- When $\lambda_0 \neq 0$, the space $Z^*_1 := Z^*/Z^*_0$ has the structure of a $\mathbb{C}$-vector space for which

$$\left(\forall y \in \mathfrak{k}\right)\left(\forall x \in Z^*\right) \quad d\alpha^*(y)x \in \lambda_0(y)x + Z^*_0.$$  

Then there is the commutative diagram

$$
\begin{array}{ccc}
Z^* & \xrightarrow{f} & Z^*_1 \\
\downarrow{q} & & \downarrow{q_0} \\
(Z^*/K)^\sim & \xrightarrow{\tilde{f}} & Z^*_1/\exp\lambda_0(\mathfrak{k})
\end{array}
$$

where

$$f : Z^* \to Z^*_1, \quad x \mapsto x + Z^*_0,$$

$$q : Z^* \to (Z^*/K)^\sim, \quad q(x) := Kx,$$

$$q_0 : Z^*_1 \to Z^*_1/\exp\lambda_0(\mathfrak{k}), \quad q_0(x + Z^*_0) := (\exp\lambda_0(\mathfrak{k}))x + Z^*_0,$$

and

$$\tilde{f} : (Z^*/K)^\sim \to Z^*_1/\exp\lambda_0(\mathfrak{k}), \quad \tilde{f}(Kx) := (\exp\lambda_0(\mathfrak{k}))x.$$ 

Here $\lambda_0(\mathfrak{k}) = i\mathbb{R}$ when $\lambda_0 \neq 0$, hence $\exp\lambda_0(\mathfrak{k}) = \mathbb{T}$; otherwise $\lambda_0(\mathfrak{k}) = 0$ and $\exp\lambda_0(\mathfrak{k}) = \{1\}$. Then the mappings $q$ and $q_0$ are surjective, continuous, and open by [30, Lemma 6.12], while $f$ is clearly surjective, continuous, and open. It then follows by the above commutative diagram that $\tilde{f}$ is surjective, continuous, and open. Moreover, when $\lambda_0 \neq 0$, $Z^*_1$ is a 1-dimensional complex vector space, and there is a homeomorphism $\Psi_0 : Z^*_1/\exp\lambda_0(\mathfrak{k}) \to [0, \infty)$; otherwise $\lambda_0(\mathfrak{k}) = 0$ and we obtain a homeomorphism $\Psi_0 : Z^*_1/\exp\lambda_0(\mathfrak{k}) \to \mathbb{R}$. Neither $[0, \infty)$ nor $\mathbb{R}$ have nonempty open quasi-compact subsets. Thus, we obtain a surjective, continuous and open map $\Psi_0 \circ \tilde{f}$ from $(Z^*/K)^\sim$ onto a topological space that has no nonempty open quasi-compact subsets. Since the image of any open quasi-compact set through a continuous open mapping is again open and quasi-compact ([5, Theorem 2, TG I 62]), there can be no nonempty open and quasi-compact subsets in $(Z^*/K)^\sim$. This concludes the proof. \hfill \Box

### 3. Quasi-orbit spaces

In the present section, unless otherwise mentioned, we keep the following notation and assumption.

**Setting 3.1.** Let $G$ be a fixed second countable locally compact group and assume that the following conditions hold.

1. There is a closed normal subgroup $L$ of $G$ such that $G/L$ is abelian.
2. For every $\mathcal{P} \in \text{Prim}(L)$, there is $T \in \hat{G}$ such that

$$\text{supp} (T|_L) = \overline{G \cdot \mathcal{P}}.$$

**Remark 3.2.** Condition (2) in Setting 3.1 is equivalent to the following condition.

2'. For every $\pi \in \hat{L}$, there is $T \in \hat{G}$ such that

$$\text{spec} (T|_L) = \overline{G \cdot \pi}.$$ 

This follows from Lemma 2.9.
Lemma 3.3. Let $G$ be as above.

(i) For every $T \in \hat{G}$, there is $\pi \in \hat{L}$ such that

$$\text{supp}(T|_L) = G \cdot \text{Ker} \pi.$$

(ii) For $\pi_1, \pi_2 \in \hat{L}$, $G \cdot \pi_1 = G \cdot \pi_2$ if and only if $G \cdot \text{Ker} \pi_1 = G \cdot \text{Ker} \pi_2$, and if and only if the representations $\text{Ind}^G_G(\pi_1)$ and $\text{Ind}^G_G(\pi_2)$ are weakly equivalent.

Proof. The assertion (i) is immediate from [15, Theorem 2.1].

(ii) By Lemma 2.10, $G \cdot \pi_1 = G \cdot \pi_2$ if and only if $G \cdot \text{Ker} \pi_1 = G \cdot \text{Ker} \pi_2$.

It is thus enough to show that $G \cdot \pi_1 = G \cdot \pi_2$ if and only if the representations $\text{Ind}^G_G(\pi_1)$ and $\text{Ind}^G_G(\pi_2)$ are weakly equivalent. If $\pi_1 \in G \cdot \pi_2$, then $\pi_1 \preceq G \cdot \pi_2$, by [10, Theorem 3.4.10]. Thus, $\text{Ind}^G_G(\pi_1) \preceq \text{Ind}^G_G(\pi_2)$ by [12, Theorem 4.2], since $\text{Ind}^G_G(g \cdot \pi_2) = \text{Ind}^G_G(\pi_2)$ for every $g \in G$. (See [7, Lemma 2.1.3].) The converse implication follows directly from [12, Theorem 4.5].

For the next results, we use the notation and definitions in Subsection 2.2.

Lemma 3.4. The map

$$S_0 : (\text{Prim}(L)/G) \to \text{Cl}(\text{Prim}(G)), \quad G \cdot \text{Ker} \pi \mapsto \text{supp}(\text{Ind}^G_G(\pi))$$

is continuous.

Proof. We first remark that $S_0$ is well defined by Lemma 3.3(ii). Denote by $Q_0$ the map $\hat{L} \to (\text{Prim}(L)/G) \cong$, $Q_0(\pi) = G \cdot \text{Ker} \pi$, that is continuous, surjective and open by Lemma 2.3(iii), (iv) and Lemma 2.10. We have the commutative diagram

$$
\begin{array}{ccc}
\hat{L} & \xrightarrow{\text{Ind}^G_G} & T(\hat{G}) \\
\downarrow Q_0 & & \downarrow \text{spec} \\
(\text{Prim}(L)/G) & \xrightarrow{S_0} & \text{Cl}(\text{Prim}(G)) \\
\end{array}
$$

where the mapping supp is continuous by Remark 2.8, while $\text{Ind}^G_G$ is continuous by [11, Theorem 4.1].

If $D$ is an open subset in $\text{Cl}(\text{Prim}(G))$, then $(S_0 \circ Q_0)^{-1}(D)$ is open in $\hat{L}$, hence $Q_0((S_0 \circ Q_0)^{-1}(D))$ is open in $(\text{Prim}(L)/G) \cong$. On the other hand, since $Q_0$ is surjective we have that $Q_0((S_0 \circ Q_0)^{-1}(D)) = S_0^{-1}(D)$. We have thus obtained that $S_0^{-1}(D)$ is open in $(\text{Prim}(L)/G) \cong$ whenever $D$ is open in $\text{Cl}(\text{Prim}(G))$, hence $S_0$ is continuous.

Lemma 3.5. The map

$$R : \text{Prim}(G) \to (\text{Prim}(L)/G) \cong, \quad \text{Ker} T \mapsto \text{supp}(T|_L)$$

is well defined, continuous and surjective.

Proof. From Lemma 3.3(i), it follows that for every $T \in \hat{G}$, there is $\pi \in \hat{L}$ such that $\text{supp}(T|_L) = G \cdot \text{Ker} \pi$. Hence the map $R_0 : \hat{G} \to (\text{Prim}(L)/G) \cong$, $T \mapsto \text{supp}(T|_L)$ is well defined. It is also continuous, since the restriction map $\hat{G} \to T(L)$, $\hat{G} \to T|_L$ is continuous, and $\text{supp} : T(L) \to \text{Cl}(\text{Prim}(L))$ is continuous. On the other hand, if $T_1, T_2 \in \hat{G}$ are such that
Ker $T_1 = \text{Ker} T_2$, then $R_0(T_1) = R_0(T_2)$, therefore the map $R$ is well defined and we have the commutative diagram

$$
\begin{array}{ccc}
\hat{G} & \xrightarrow{\kappa} & \text{Prim}(G) \\
\downarrow R_0 & & \downarrow R \\
(\text{Prim}(L)/G)^\infty & & \\
\end{array}
$$

Since the canonical map $\kappa: \hat{G} \to \text{Prim}(G)$, $\kappa([T]) = \text{Ker} T$, is continuous and open, it follows that $R$ is continuous as well.

Surjectivity follows by the assumption (2) in Setting 3.1.

We now introduce some notation needed in Theorem 3.6. Set $A := G/L$ and let $p: G \to A$ be the canonical quotient homomorphism. The character group $\hat{A}$ has a continuous action by automorphisms of $C^*(G)$,

$$\hat{A} \times C^*(G) \to C^*(G), \quad (\chi, \varphi) \mapsto \chi \cdot \varphi,$$

where $(\chi \cdot \varphi)(g) := \chi(p(g))\varphi(g)$ if $\chi \in \hat{A}$, $\varphi \in L^1(G)$, and $g \in G$. This gives rise to continuous actions of $\hat{A}$ on $\text{Prim}(G)$ and on $C^*(\hat{G})$, respectively. Taking into account the canonical identification $C^*(G) \simeq \hat{G}$, the corresponding action of $\hat{A}$ on classes of irreducible representation of $G$ is given by

$$\hat{A} \times \hat{G} \to \hat{G}, \quad (\chi, T) \mapsto \chi \cdot T = (\chi \circ p) \otimes T,$$

where, for any unitary irreducible representation $T: G \to \mathcal{B}(\mathcal{H})$ and any $\chi \in \hat{A}$, one defines $\chi \cdot T: G \to \mathcal{B}(\mathcal{H})$. $(\chi \cdot T)(g) := \chi(p(g))T(g)$. As usual, we denote by $(\text{Prim}(G)/\hat{A})^\sim$ and $(\hat{G}/\hat{A})^\sim$ the quasi-orbit spaces corresponding to the above actions of $\hat{A}$ on $\text{Prim}(G)$ and on $\hat{G}$, respectively, and by $(\text{Prim}(G)/\hat{A})^\sim$ and $(\hat{G}/\hat{A})^\sim$ their homeomorphic copies in $\text{Cl}(\text{Prim}(G))$ and $\text{Cl}(\hat{G})$, respectively.

**Theorem 3.6.** Let $G$ be a second countable locally compact group. Assume that there exists a closed normal subgroup $L$ of $G$ such that $G/L$ is abelian, and for every $P \in \text{Prim}(L)$ there is $T \in \hat{G}$ such that

$$\text{supp}(T|_L) = \overline{G \cdot P}.$$

Then there is a homeomorphism of quasi-orbit spaces

$$\text{(Prim}(L)/G)^\sim \to (\text{Prim}(G)/\hat{A})^\sim. \quad (3.1)$$

**Proof.** We prove in fact that the map

$$S: (\text{Prim}(L)/G)^\sim \to (\text{Prim}(G)/\hat{A})^\sim, \quad \overline{G \cdot \text{Ker} \pi} \mapsto \text{supp}(\text{Ind}^G_L(\pi)). \quad (3.2)$$

is a well-defined homeomorphism.

We first show that the image of $S$ is contained in $(\text{Prim}(G)/\hat{A})^\sim$. To this end, using Remark 3.2 and Lemma 2.9, it suffices to prove the following:

$$\pi \in \hat{L}, \ T \in \hat{G}, \ \text{spec}(T|_L) = \overline{G \cdot \pi} \Rightarrow \text{spec}(\text{Ind}^G_L(\pi)) = \overline{A \cdot T} \in (\hat{G}/\hat{A})^\sim. \quad (3.3)$$

To prove (3.3), let $\lambda: A \to \mathcal{B}(L^2(A))$ be the regular representation of $A$. Since $A$ is a locally compact abelian group, the Fourier transform $L^2(A) \to L^2(\hat{A})$ is a unitary equivalence between $\lambda$ and the unitary representation by multiplication operators,

$$\hat{\lambda}: A \to \mathcal{B}(L^2(\hat{A})), \quad (\hat{\lambda}(a)\psi)(\chi) := \chi(a)\psi(\chi).$$
Using the direct integral decomposition \( \hat{\Lambda} = \int \hat{\Lambda} \chi d\chi \) with respect to a Haar measure on \( \hat{\Lambda} \), we obtain a unitary equivalence

\[
\int \chi \cdot T d\chi = \int (\chi \circ p) \otimes T d\chi \simeq \int (\chi \circ p) d\chi \otimes T = (\hat{\Lambda} \circ p) \otimes T.
\]

Denoting by \( \tau : L \to \mathbb{T} \) the trivial representation of \( L \), we have \( \lambda \circ p = \text{Ind}_L^G(\tau) \), hence there is a unitary equivalence \( \hat{\Lambda} \circ p \simeq \text{Ind}_L^G(\tau) \).

On the other hand, by [12, Lemma 4.2], there is a unitary equivalence

\[
\text{Ind}_L^G(\tau) \otimes T \simeq \text{Ind}_L^G(\tau \otimes T|_L) = \text{Ind}_L^G(T|_L).
\]

Since the support of the Haar measure of \( \hat{\Lambda} \) is equal to \( \hat{\Lambda} \), it then follows by [12, Theorem 3.1] that the set \( \hat{\Lambda} \cdot T = \{ \chi \cdot T \mid \chi \in \hat{\Lambda} \} \subseteq \hat{G} \) is weakly equivalent to the representation \( \text{Ind}_L^G(T|_L) \). Since \( T|_L \) is weakly equivalent to \( G \cdot \pi \), we obtain (3.3), by using [12, Theorem 4.2] and the same argument as in Lemma 3.3(ii) above.

We now construct a continuous inverse of the mapping (3.1). To this end we note that for arbitrary \( T \in \hat{G} \) and \( \chi \in \hat{\Lambda} \), we have \( (\chi \cdot T)|_L = T|_L \), hence \( R(\text{Ker} (\chi \cdot T)) = R(\text{Ker} T) \). Since the mapping \( R : \text{Prim}(G) \to (\text{Prim}(L)/G)^\approx \) is continuous and surjective by Lemma 3.5, while the topological space \( (\text{Prim}(L)/G)^\approx \) is \( T_0 \), it follows by [30, Lemma 6.10] and Lemma 2.3(iv) that there exists a continuous surjective mapping \( R' : (\text{Prim}(G)/\hat{\Lambda})^\approx \to (\hat{L}/G)^\approx \) for which the diagram

\[
\begin{array}{ccc}
\text{Prim}(G) & \xrightarrow{\iota \circ Q} & \text{(Prim}(G)/\hat{\Lambda})^\approx \\
\text{Prim}(G) \xrightarrow{\iota \circ Q} & R & \xrightarrow{R'} \text{(Prim}(L)/G)^\approx \\
\end{array}
\]

is commutative, where \( Q, \iota \) are maps given by Lemma 2.3 for \( X \) replaced by \( \text{Prim}(G) \) and \( G \) replaced by \( \hat{\Lambda} \).

To see that the continuous mappings \( R' \) and (3.1) are inverse to each other, we recall (Lemma 3.3(i)) that for every \( T \in \hat{G} \) there is \( \pi \in \hat{L} \) such that \( \text{supp} (T|_L) = \hat{G} \cdot \text{Ker} \pi \). Then, by (3.4), we have

\[
R'(\hat{\Lambda} \cdot \text{Ker} T) = R'((\iota \circ Q)(\text{Ker} T)) = R(\text{Ker} T)
\]

\[
= \text{supp} (T|_L) = \hat{G} \cdot \text{Ker} \pi.
\]

Hence

\[
(S \circ R')(\hat{\Lambda} \cdot \text{Ker} T) = S(\hat{G} \cdot \text{Ker} \pi) = \text{supp} (\text{Ind}_{\hat{L}}^G(\pi))
\]

\[
= \kappa_G(\hat{\Lambda} \cdot T) = \hat{\Lambda} \cdot \text{Ker} T,
\]

where the last equality holds true since the kernel mapping \( \kappa_G : \hat{G} \to \text{Prim}(G) \) is continuous, closed, and equivariant with respect to the action of the automorphism group of \( C^*(G) \). (See also Definition 2.7.) On the other hand, for \( \pi \in \hat{L} \), there is \( T \in \hat{G} \) such that \( \text{supp} (T|_L) = \hat{G} \cdot \text{Ker} \pi \) (Setting 3.1(2)). Then by (3.3) and the properties of \( \kappa_G \), as above, we have

\[
(R' \circ S)(\hat{G} \cdot \text{Ker} \pi) = R'(\hat{\Lambda} \cdot \text{Ker} T)
\]

\[
= R'((\iota \circ Q)(\text{Ker} T)) = R(\text{Ker} T) = \text{supp} (T|_L) = \hat{G} \cdot \text{Ker} \pi.
\]

Thus \( S \circ R' = \text{id} \) and \( R' \circ S = \text{id} \), hence \( S^{-1} = R' \), which is a continuous mapping. This completes the proof of the fact that \( S \) in (3.6) is a homeomorphism. \( \square \)
4. **AF-embeddable solvable Lie groups**

In the present section, we assume that $G$ is a connected simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Then the commutator subgroup $L := [G, G]$ is nilpotent, and a connected, simply connected, closed normal subgroup of $G$. We denote by $Z$ the centre of $L$. The Lie algebra of $L$ is $[\mathfrak{g}, \mathfrak{g}]$, and we denote by $\mathfrak{z}$ the Lie algebra of $Z$.

The main result of this section is the following theorem.

**Theorem 4.1.** Let $G$ be a connected and simply connected solvable Lie group, $L = [G, G]$ and let be $Z$ the centre of $L$. Assume that the action of $G$ on $Z$ has at least one purely imaginary weight. Then Prim$(G)$ has no nonempty open quasi-compact subsets.

Before proving this theorem, we give some important consequences. We recall the following definition (see [1]).

**Definition 4.2.** Let $G$ be a Lie group. Then $G$ is said to be of type $R$ provided the spectra of the operators of the adjoint action of $G$ on its Lie algebra $\mathfrak{g}$ are contained in the unit circle. Equivalently, for every $X \in \mathfrak{g}$ the spectrum of the operator $\text{ad}_\mathfrak{g}X : \mathfrak{g} \to \mathfrak{g}$ should be contained in $i\mathbb{R}$.

**Corollary 4.3.** Let $G$ be a connected and simply connected solvable Lie group of type $R$. Then Prim$(G)$ has no nonempty open quasi-compact subsets.

Proof. The condition that $G$ is of type $R$ implies that the action of $G$ on $Z$ has purely imaginary weights. Therefore the result is a consequence of Theorem 4.1. □

We recall that if $G$ is a connected simply connected solvable Lie group, the primitive ideals of $C^*(G)$ are maximal, or equivalently, Prim$(G)$ is $T_1$, if and only if $G$ is of type $R$; see [24, Theorem 2]. Therefore the previous corollary may be re-written as follows.

**Corollary 4.4.** Let $G$ be a connected and simply connected solvable Lie group such that Prim$(G)$ is $T_1$. Then Prim$(G)$ has no nonempty open quasi-compact subsets.

**Corollary 4.5.** Let $G$ be a connected and simply connected solvable Lie group, $L = [G, G]$ and let be $Z$ the centre of $L$. Assume that the action of $G$ on $Z$ has at least one purely imaginary weight. Then $C^*(G)$ is AF-embeddable.

Proof. The result is a consequence of Theorem 4.1 and [13, Corollary B]. □

**Corollary 4.6.** Let $G$ be a connected and simply connected solvable Lie group such that Prim$(G)$ is $T_1$. Then $C^*(G)$ is AF-embeddable.

Proof. The result is a consequence of Corollary 4.4 and [13, Corollary B]. □

To prove Theorem 4.1, we need the following technical tool.

**Proposition 4.7.** Let $G$ be a connected and simply connected solvable Lie group, $L = [G, G]$ and let be $Z$ the centre of $L$. Then there is a continuous, surjective and open map

$$\Phi : \text{Prim}(G) \to (\hat{Z}/G)^\sim.$$
Proof. We show first that the group \( G \) and its closed normal subgroup \( L \) that satisfy the conditions is Theorem 3.6. We note that the Lie group \( L \) is nilpotent of type I (even liminary). The quotient \( A = G/L \) is abelian since \( L = [G, G] \), so it remains to show that for every \( \pi \in \hat{L} \) there exists \( T \in \hat{G} \) such that

\[
\text{supp}(T|_L) = \overline{G \cdot \text{Ker} \pi}.
\]

(4.1)

Since \( G \) is a connected simply connected solvable Lie group, for every \( \pi \in \hat{L} \) there is a closed normal subgroup \( K \) of \( G \), \( L \subseteq K \), a representation \( \rho \in \hat{K} \) with \( \rho|_L = \pi \) such that \( T_0 = \text{Ind}^G_K \rho \) is a factor representation. (See [24, p. 83–83] and the references therein.) Let \( T \in \hat{G} \) be such that

\[
\text{supp} (T|_L) = \overline{G \cdot \text{Ker} \pi}.
\]

(4.1)

Hence

\[
T|_K \approx T_0|_K \approx G \cdot \rho \subseteq \hat{K}.
\]

By Theorem 3.6 and the fact that \( L \) is of type I, we obtain a homeomorphism

\[
R : (\text{Prim}(G)/\hat{A})^\sim \rightarrow (\hat{L}/G)^\sim.
\]

Denote by \( Q : \text{Prim}(G) \rightarrow (\text{Prim}(G)/\hat{A})^\sim \) the quasi-orbit map (see Lemma 2.3); it is a continuous, open and surjective map. Thus, if we set \( R := R' \circ Q \), we obtain a continuous, open and surjective map

\[
R : \text{Prim}(G) \rightarrow (\hat{L}/G)^\sim.
\]

On the other hand, by Lemma 2.11, there is a continuous, surjective and open map \( R^\sim : \hat{L} \rightarrow \hat{Z} \). It is easy to see that \( R^\sim \) is also \( G \)-equivariant, hence it gives rise to a continuous, surjective and open mapping \( \overline{R^\sim} : \hat{L}/G \rightarrow \hat{Z}/G \).

Let \( r_L : \hat{L}/G \rightarrow (\hat{L}/G)^\sim \) be the canonical continuous mapping from \( \hat{L}/G \) onto its \( T_0 \)-ization (in the sense of Definition 2.2), and let \( r_Z : \hat{Z}/G \rightarrow (\hat{Z}/G)^\sim \) be its analogous map for the group \( Z \). Then \( r_Z \circ \overline{R^\sim} : \hat{L}/G \rightarrow (\hat{Z}/G)^\sim \) is continuous. Hence there is a continuous mapping \( \overline{R^\sim} : (\hat{L}/G)^\sim \rightarrow (\hat{Z}/G)^\sim \) with \( \overline{R^\sim} \circ r_L = r_Z \circ \overline{R^\sim} \), by [30, Lemma 6.10].

We thus obtain the commutative diagram

\[
\begin{array}{ccc}
\hat{L} & \xrightarrow{R^L} & \hat{Z} \\
\downarrow{q_0} & & \downarrow{q_Z} \\
\hat{L}/G & \xrightarrow{\overline{R^\sim}} & \hat{Z}/G \\
\downarrow{r_L} & & \downarrow{r_Z} \\
(\hat{L}/G)^\sim & \xrightarrow{\overline{R^\sim}} & (\hat{Z}/G)^\sim
\end{array}
\]

where the surjective mappings \( \kappa_L := r_L \circ q_L \) and \( \kappa_Z := r_Z \circ q_Z \) are continuous and open by Lemma 2.3(iii). Thus \( \kappa_Z \circ R^L = \overline{R^\sim} \circ \kappa_L \), where \( \kappa_Z \circ R^L \) and \( \kappa_L \) are surjective open mappings. Therefore, by [5, Chapter 3, §4, Propositions 2–3] again, the continuous mapping \( \overline{R^L} \) is open.

The map

\[
\Phi := \overline{R^L} \circ R : \text{Prim}(G) \rightarrow (\hat{Z}/G)^\sim
\]

is a continuous, open and surjective map, thus it satisfies the properties in the statement. \( \square \)
Proof of Theorem 4.1. The closed, normal subgroup $L$ is connected and simply connected, thus the quotient $A := G/L$ is an abelian Lie group, connected and simply connected, and its Lie algebra is $\mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

The action of $G$ on $Z$, $\alpha: G \to \text{End}(Z)$, is trivial on $L$, therefore there is a group homomorphism $\alpha_Z: A \to \text{End}(Z)$ such that $\alpha_Z \circ \alpha = \alpha$, where $Q$ is the quotient map $Q: G \to A$. Then $\alpha_Z$ induces a natural action of $A$ on $Z' \simeq \hat{G}$, and $(Z'/G)^\sim = (Z'/A)^\sim$.

Since the action of $G$ on $Z$ has at least one purely imaginary weight, the action $\alpha_Z: \mathfrak{a} \to \text{End}(Z)$ has at least one purely imaginary weight. Then the assertion in the statement follows from Proposition 4.7 and Lemma 2.15.

Example 4.8 (Generalized $ax + b$-groups). Let $\mathcal{V}$ be a finite-dimensional real vector space, $D \in \text{End}(\mathcal{V})$, and $G_D := \mathcal{V} \rtimes_{\alpha_D} \mathbb{R}$ their corresponding generalized $ax + b$-group. Then we claim that the following assertions are equivalent.

(i) Either $\text{Re } z > 0$ for every $z \in \text{spec } (D)$ or $\text{Re } z < 0$ for every $z \in \text{spec } (D)$.
(ii) The $C^*$-algebra $C^*(G_D)$ is not quasidiagonal.
(iii) The $C^*$-algebra $C^*(G_D)$ is not AF-embeddable.
(iv) There exists a nonempty quasi-compact open subset of $\widehat{G_D}$.
(v) There exists a nonempty quasi-compact open subset of $\text{Prim}(G_D)$.
(vi) The set $\widehat{G_D} \setminus \text{Hom}(G_D, T)$ is a nonempty quasi-compact open subset of $\widehat{G_D}$.
(vii) There exist nonzero self-adjoint idempotent elements of $C^*(G_D)$.

Proof of claim. (i) $\iff$ (ii) $\iff$ (iii): See [3, Theorem 2.15].
(i) $\iff$ (iv)$\implies$(vi): See [17, Theorem 1.1].
(vi)$\implies$(iv): Obvious.
(iv)$\implies$(v): The canonical mapping $\widehat{G_D} \to \text{Prim}(G_D)$, $[\pi] \mapsto \text{Ker } \pi$, is continuous and open, hence it maps any nonempty quasi-compact open subset of $\widehat{G_D}$ onto a nonempty quasi-compact open subset of $\text{Prim}(G_D)$.
(v) $\implies$ (iv): Since there exists a nonempty quasi-compact open subset of $\text{Prim}(G_D)$, it follows by Theorem 4.1 that $\text{Re } z \neq 0$ for every $z \in \text{spec } (D)$, hence $G_D$ is an exponential Lie group. In particular $G_D$ is type I, hence the canonical mapping $\widehat{G_D} \to \text{Prim}(G_D)$, $[\pi] \mapsto \text{Ker } \pi$, is a homeomorphism. This shows that (iv) follows by the hypothesis (v).
(vii)$\implies$(iv): We actually note a more general fact: If $\mathcal{A}$ is a $C^*$-algebra and $0 \neq p = p^*= p^2 \in \mathcal{A}$, then the set $Z_p := \{[\pi] \in \hat{\mathcal{A}} \mid \pi(p) \neq 0\}$ is a nonempty quasi-compact open subset of $\hat{\mathcal{A}}$.

In fact, for every $*$-representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$, the operator $\pi(p) \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection hence the condition $\pi(p) \neq 0$ is equivalent to $\|\pi(p)\| \geq 1$. Hence $Z_p = \{[\pi] \in \hat{\mathcal{A}} \mid \|\pi(p)\| > 0\}$, and then $Z_p$ is open since the function $[\pi] \mapsto \|\pi(p)\|$ is lower semicontinuous on $\hat{\mathcal{A}}$ by [10, Proposition 3.3.2]. On the other hand, $Z_p = \{[\pi] \in \hat{\mathcal{A}} \mid \|\pi(p)\| \geq 1\}$, hence $Z_p$ is quasi-compact by [10, Proposition 3.3.7]. And finally, $Z_p \neq \emptyset$ since $p \neq 0$.

(i)$\implies$(vii): See [17, Lemma 2.3].

5. AF-embeddability of simply connected Lie groups with $T_1$ primitive ideal spaces

We prove here the main result of the paper, Theorem 5.3. Along with the results of the preceding sections, its proof requires the following two lemmas.

Lemma 5.1. Let $K$ be a compact group that acts on a topological space $X$ by a continuous action $K \times X \to X$, and let $q: X \to X/K$ be the corresponding quotient map. Then for $C \subseteq X/K$, we have that $C$ is quasi-compact if and only if $q^{-1}(C)$ is quasi-compact.
Proof. Assume first that $q^{-1}(C)$ is quasi-compact. Since $q$ is surjective and continuous, $C = q(q^{-1}(C))$ is quasi-compact.

For the direct implication, let $\{x_j\}_{j \in J}$ be any net in $q^{-1}(C)$. Then $\{q(x_j)\}$ is a net in $C$. Since $C$ is quasi-compact, selecting a suitable subnet, we may assume that there is $c \in X/K$ such that $q(x_j) \to c$ in $X/K$. Hence there exist $k_j, j \in J$, and $x \in q^{-1}(c)$ such that $k_j \cdot x_j \to x$.

The group $K$ is compact, therefore, by selecting a suitable subnet, we may assume that there is $k \in K$ such that $k_j \to k$, and thus $k_j^{-1} \cdot k_j \cdot x_j \to k^{-1} \cdot x$. On the other hand, $q(k^{-1} \cdot x) = q(x) = c$, hence $k^{-1} \cdot x \in q^{-1}(c) \subseteq q^{-1}(C)$.

Thus every net in $q^{-1}(C)$ has a subnet that converges to some point in $q^{-1}(C)$, hence $q^{-1}(C)$ is quasi-compact.

Lemma 5.2. Let $G_2$ a locally compact group, $K$ a compact group that acts continuously on $G_2$ and consider $G_1 = K \ltimes G_2$. Assume that $\text{Prim}(G_2)$ is $T_1$. If $\text{Prim}(G_2)$ has no nonempty open quasi-compact subsets, then $\text{Prim}(G_1)$ has no nonempty open quasi-compact subsets.

Proof. We have that $C^\ast(G_1) = K \ltimes C^\ast(G_2)$ ([30, Proposition 3.11]). Then the restriction of ideals gives a mapping $R: \text{Prim}(G_1) \to (\text{Prim}(G_2)/K)^\ast$, which is continuous, surjective and open. (See [16, Theorem 4.8].) Since the action of $K$ on $\text{Prim}(G_2)$ is continuous, $K$ is compact, and $\text{Prim}(G_2)$ is $T_1$, it follows by [19, Corollary, p. 213] that the orbits of $K$ in $\text{Prim}(G_2)$ are closed, hence $(\text{Prim}(G_2)/K)^\ast = \text{Prim}(G_2)/K$. Thus we get that $R: \text{Prim}(G_1) \to \text{Prim}(G_2)/K$ is continuous, surjective and open. The quotient map $q: \text{Prim}(G_2) \to \text{Prim}(G_2)/K$ is continuous, open and surjective, as well.

Assume now that there is $C$ an open quasi-compact subset of $\text{Prim}(G_1)$. Then $R(C)$ is quasi-compact and open in $\text{Prim}(G_2)/K$, and by Lemma 5.1, $q^{-1}(R(C))$ is a quasi-compact and open subset of $\text{Prim}(G_2)$. By hypothesis, this implies that $q^{-1}(R(C)) = \emptyset$, hence $C = \emptyset$. This completes the proof of the lemma.

Theorem 5.3. Let $G$ be a simply connected Lie group such that $\text{Prim}(G)$ is $T_1$. Then both $C^\ast(G)$ and the reduced $C^\ast$-algebra $C^\ast_r(G)$ are AF-embeddable.

Proof. We can write $G = S_1 \times G_1$, where both $S_1$ and $G_1$ are simply connected Lie groups, $S_1$ is semisimple, and $G_1$ has no semisimple factor. (See proof of [25, Theorem 2, p. 47].)

Since $S_1$ is liminary, $C^\ast(S_1)$ and $C^\ast_r(S_1)$ are AF-embeddable (see Remark 6.2). On the other hand, $C^\ast(S_1)$ is nuclear by [6, Proposition 2.7.4], since it is liminary hence type I. Therefore we have that

$$C^\ast (G) \simeq C^\ast (S_1) \otimes C^\ast(G_1),$$

(5.1)

$$C^\ast_r (G) \simeq C^\ast_r (S_1) \otimes C^\ast_r(G_1).$$

(5.2)

Thus it suffices to prove that $C^\ast(G_1)$ and $C^\ast_r(G_1)$ are AF-embeddable.

From (5.1), it follows that there is a homeomorphism

$$\text{Prim}(G) \simeq \text{Prim}(S_1) \times \text{Prim}(G_1).$$

(5.3)

It is straightforward to check that if the product of two topological spaces is $T_1$, then each of them is $T_1$. Hence, by (5.3), $\text{Prim}(G_1)$ is $T_1$.

Since $G_1$ has no semisimple factor, we then have by [25, Proposition 3, p. 47] that $G_1 = K \ltimes G_2$, where $K$ and $G_2$ are simply connected Lie groups, $K$ is compact, while $G_2$ is solvable of type $R$. It follows that $G_1$ is amenable (see [20, Proposition 11.13]), hence $C^\ast_r(G_1) = C^\ast(G_1)$. By Theorem 4.3, we have that either $G_2 = \{1\}$ or there are no nonempty quasi-compact open subsets of $\text{Prim}(G_2)$. Using Lemma 5.2, we get that either $G_1 = K$ or
Prim(G1) has no nonempty quasi-compact and open subsets. Thus by [13, Corollary B] $C^*_r(G_1) = C^*(G_1)$ is AF-embeddable.

**Example 5.4 (Generalized Mautner groups).** Let $V_1$ and $V_2$ be finite-dimensional real vector spaces, regarded as abelian additive groups $(V_j, +)$ for $j = 1, 2$. Then for any compact abelian connected subgroup $T \subseteq GL(V_2) = Aut(V_2)$ and any continuous group morphism $\beta: V_1 \to T$, $v \mapsto \beta_t$, the corresponding semidirect product group $V_2 \rtimes_{\beta} V_1$ is a connected simply connected solvable Lie group of type R. In fact, for every $v_1 \in V_1$, the one-parameter group $\{\beta_t v_1 = e^{t \beta(v_1)} \mid t \in \mathbb{R}\}$ is contained in $Aut(V_2)$, and this directly implies that all the eigenvalues of $d\beta(v_1)$ are contained in $i\mathbb{R}$. On the other hand, the adjoint representation of the Lie group $V_2 \rtimes_{\beta} V_1$ is given by $(Ad(v_2, v_1))(w_2, w_1) = (d\beta(v_1)w_2, 0)$ for all $v_j, w_j \in V_j$ for $j = 1, 2$, hence the spectrum of $Ad(v_2, v_1)$ is contained in $i\mathbb{R}$.

On the other hand, the commutant $T^\prime := \{g \in GL(V_2) \mid (\forall h \in T) \; gh = hg\}$ is a closed subgroup of $GL(V_2)$ with $T \subseteq T^\prime$ since $T$ is abelian. For any compact connected subgroup $K \subseteq T^\prime$, we define

$$\alpha: K \to GL(V_2 \times V_1), \; k \mapsto \alpha_k := k \times id_{V_1}.$$  

A simple computation shows that $\alpha(K) \subseteq Aut(V_2 \rtimes_{\beta} V_1)$ hence, we may define the semidirect product group $(V_2 \rtimes_{\beta} V_1) \rtimes_{\alpha} K$.

It is often the case that $K$ can be selected to be simply connected. For instance, when $V_2 = V_{2a} \otimes_C V_{2b}$ is a tensor product of complex vector spaces with dim$_C V_{2b} \geq 2$ and the image of $\beta$ consists of $C$-linear maps contained in $End_C(V_{2a}) \otimes id_{V_{2b}}$. Then $T^\prime$ contains $id_{V_{2a}} \otimes End_C(V_{2b})$ hence, since dim$_C V_{2b} \geq 2$, we may select $K \subseteq T^\prime$ with $K$ isomorphic to the special unitary group $SU(n)$ for suitable $n \geq 2$, and then $K$ is simply connected. In that case, the group $(V_2 \rtimes_{\beta} V_1) \rtimes_{\alpha} K$ satisfies the hypothesis of Theorem 5.3.

**Example 5.5 (Automorphisms of Heisenberg groups).** For $n \geq 1$, consider the Heisenberg group $H_{2n+1} = (\mathbb{R} \times \mathbb{C}^n, \cdot)$. We simultaneously regard $\mathbb{C}^n$ as a complex Hilbert space with its usual scalar product $\langle \cdot, \cdot \rangle$ and as a real vector space endowed with the symplectic form $\omega(\cdot, \cdot) := \text{Im}(\langle \cdot, \cdot \rangle)$ and with its corresponding symplectic group

$$Sp(2n, \mathbb{R}) = \{g \in End_{\mathbb{R}}(\mathbb{C}^n) \mid (\forall v, w \in \mathbb{C}^n) \; \omega(gv, gw) = \omega(v, w)\}.$$  

There is natural injective morphism of Lie groups

$$\alpha: Sp(2n, \mathbb{R}) \to Aut(H_{2n+1}), \; g \mapsto \alpha_g = id_{\mathbb{R}} \times g.$$  

The unitary group $U(n)$ is a maximal compact subgroup of the simple Lie group $Sp(2n, \mathbb{R})$, therefore $SU(n)$ is a simply connected compact subgroup of $Sp(2n, \mathbb{R})$. Hence the semidirect product group $H_{2n+1} \rtimes_{\alpha} SU(n)$ is a simply connected Lie group that satisfies the hypothesis of Theorem 5.3.

**Example 5.6 (Free nilpotent Lie algebras).** Let $V$ be a finite-dimensional real vector space. For any integer $r \geq 1$, let $n_r(V)$ be the corresponding free $r$-step nilpotent Lie algebra generated by $V$, and let $N_r(V)$ be the connected simply connected nilpotent Lie group whose Lie algebra is $n_r(V)$. The correspondence $V \mapsto n_r(V)$ gives a functor from the category of finite-dimensional real vector spaces to the category of $r$-step nilpotent real Lie algebras. The action of that functor on morphisms gives a natural injective morphism of Lie groups $\alpha: GL(V) \to Aut(N_r(V))$. Then for any simply connected compact group $K \subseteq GL(V)$, one obtains the semidirect product group $N_r(V) \rtimes_{\alpha} K$, which is a simply connected Lie group that satisfies the hypothesis of Theorem 5.3.
6. On primitively AF-embeddable $C^*$-algebras

**Definition 6.1.** A $C^*$-algebra $\mathcal{A}$ is called primitively AF-embeddable if its primitive quotients $\mathcal{A}/\mathcal{P}$ for arbitrary $\mathcal{P} \in \text{Prim}(\mathcal{A})$ are AF-embeddable $C^*$-algebras.

**Remark 6.2.** We recall that for a type I, separable $C^*$-algebra $\mathcal{A}$, if $\mathcal{A}$ is primitively AF-embeddable, then it is AF-embeddable. Indeed, since $\mathcal{A}$ is primitively AF-embeddable, it follows that for every irreducible $*$-representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$ the $C^*$-algebra $\pi(\mathcal{A}) \cong \mathcal{A}/\text{Ker} \pi$ is AF-embeddable, hence $\pi(\mathcal{A})$ is stably finite by [29, Lemma 1.3]. Therefore $\mathcal{A}$ is residually finite by [29, Proposition 3.2], and [29, Theorem 3.6] implies that $\mathcal{A}$ is AF-embeddable.

It is not clear to what extent the above Remark 6.2 carries over beyond the type I $C^*$-algebras, that is, if every primitively AF-embeddable $C^*$-algebra is AF-embeddable. (See also [8,].) We prove, however, that this is the case for all $C^*$-algebras of connected, simply connected solvable Lie groups, irrespectively of whether they are type I or not. More specifically, we prove that if $\mathcal{A}$ is the $C^*$-algebra of a connected, simply connected solvable Lie group, then the following implications hold true:

$$\mathcal{A} \text{ is primitively AF-embeddable} \quad (1) \quad \iff \quad \mathcal{A} \text{ is strongly quasi-diagonal} \quad (2) \quad \iff \quad \text{Prim}(\mathcal{A}) \text{ is } T_1 \quad (3) \quad \implies \quad \mathcal{A} \text{ is AF-embeddable}.$$

Specifically, we prove the equivalence (1) in Corollary 6.7, (2) is a consequence of the same corollary and of [19], while the implication (3) follows from Theorem 5.3.

The following fact is related to [3, Theorem 1.1] and is applicable to Lie groups that need not be simply connected or solvable.

**Lemma 6.3.** Let $G$ be a connected Lie group. If $\text{Prim}(G)$ is $T_1$, then the following assertions hold.

(i) The $C^*$-algebra $C^*(G)$ is primitively AF-embeddable.

(ii) The Lie group $G$ is type I if and only if it is liminary.

**Proof.** (i) The hypothesis that $\text{Prim}(G)$ is $T_1$ is equivalent to the fact that every primitive ideal of $C^*(G)$ is maximal, which is further equivalent to the fact that every primitive quotient of $C^*(G)$ is a simple $C^*$-algebra. Then, by [22, Theorem 2, p. 161], for every $\mathcal{P} \in \text{Prim}(G)$ we have that $C^*(G)/\mathcal{P} \cong \mathcal{A}_{\mathcal{P}} \otimes K(\mathcal{H}_{\mathcal{P}})$, where $\mathcal{A}_{\mathcal{P}}$ is a simple $C^*$-algebra which is either 1-dimensional or a noncommutative torus, and $\mathcal{H}_{\mathcal{P}}$ is a suitable Hilbert space. To conclude the proof, we must show that if a noncommutative torus is simple, then it is AF-embeddable. One way to obtain that conclusion is to combine the fact that a noncommutative torus is a simple $C^*$-algebra, then it is an approximately homogeneous $C^*$-algebra by [21, Theorem 3.8], and on the other hand, every approximately sub-homogeneous $C^*$-algebra is AF-embeddable by [26, Proposition 4.1]. Alternatively, one can reason as follows in order to obtain the even stronger property that $\mathcal{A}_{\mathcal{P}}$ embeds into a unital simple AF-algebra: We recall that if $\mathcal{A}_{\mathcal{P}}$ is a noncommutative torus and if $\mathcal{A}_{\mathcal{P}}$ is moreover a simple $C^*$-algebra, then $\mathcal{A}_{\mathcal{P}}$ has a tracial state $\tau$. (See, for instance, [21, Theorem 1.9] for $\tau$ is a tracial state, the set $N_{\tau} := \{a \in \mathcal{A}_{\mathcal{P}} \mid \tau(a^*a) = 0\}$ is a closed 2-sided ideal of $\mathcal{A}_{\mathcal{P}}$, which is a simple $C^*$-algebra, hence $N_{\tau} = \{0\}$, that is, the tracial state $\tau$ is faithful. Moreover, $\mathcal{A}_{\mathcal{P}}$ is separable, nuclear, and satisfies the Universal Coefficient Theorem of [27]. (See, for instance, the proof of [21, Theorem 3.8].) On the other hand, every trace on a nuclear $C^*$-algebra is amenable by [6, Proposition 6.3.4], hence $\tau$ is a faithful amenable trace on $\mathcal{A}_{\mathcal{P}}$. Since $\mathcal{A}_{\mathcal{P}}$ is nuclear, hence exact, it then follows by [28, Theorem A] that $\mathcal{A}_{\mathcal{P}}$ embeds into a simple AF-algebra. We thus see that $C^*(G)/\mathcal{P}$ is AF-embeddable for arbitrary $\mathcal{P} \in \text{Prim}(G)$. 

(ii) The group $G$ is type I if and only if every primitive quotient is type I, and then the assertion follows as a by-product of the above reasoning, since no simple noncommutative torus is type I, as seen, for instance, by using [3, Lemma 4.2(ii)] for the aforementioned tracial state $\tau$. □

For the next result (Theorem 6.5), the connected locally compact group $G$ does not have to be a Lie group. This is not entirely surprising, due to the Lie theoretic characterization [19, Theorem 1] of connected locally compact groups whose primitive ideal space is $T_1$.

We start by proving the next lemma, which is essentially contained in proof of [19, Theorem 4 and Corollary 3, p. 212], at least in the special case of connected groups. However, we give here a more direct proof that does not use the above mentioned [19, Theorem 1]. We recall that a topological group $G$ is called almost connected if the quotient group $G/G_0$ is compact, where $G_0 \subseteq G$ is the connected component that contains the unit element $1 \in G$.

**Lemma 6.4.** Let $G$ be an almost connected, locally compact group, and denote by $L(G)$ the set of all compact normal subgroups $H \subseteq G$ for which $G/H$ is a Lie group. Then $\text{Prim}(G)$ is $T_1$ if and only if $\text{Prim}(G/H)$ is $T_1$ for every $H \in L(G)$.

**Proof.** For every $H \in L(G)$, we denote by $j_H : G \to G/H$ its corresponding quotient map. We define

$$\widehat{j_H} : \widehat{G/H} \to \widehat{G}, \quad [\sigma] \mapsto [\sigma \circ j_H]$$

and

$$(j_H)^* : \text{Prim}(G/H) \to \text{Prim}(G), \quad \text{Ker} \sigma \mapsto \text{Ker} (\sigma \circ j_H).$$

By [2, Proposition 8.C.8], there exists a surjective $*$-morphism

$$(j_H)_* : C^*(G) \to C^*(G/H)$$

satisfying $\sigma \circ (j_H)_* = \sigma \circ j_H$ for every $[\sigma] \in G/H \simeq C^*(G/H)$, where $[\sigma \circ j_H] \in \widehat{G} \simeq C^*(\widehat{G})$. Therefore

$$(j_H)^*(\mathcal{P}) = ((j_H)_*)^{-1}(\mathcal{P}) \text{ for every } \mathcal{P} \in \text{Prim}(C^*(G/H)). \quad (6.1)$$

The mapping $\widehat{j_H}$ is a homeomorphism of $\widehat{G/H}$ onto an open-closed subset of $\widehat{G}$ by [18, Theorem 5.4] or [18, Theorem 5.2(i)]. Similarly, the mapping $(j_H)^*$ is a homeomorphism of $\text{Prim}(G/H)$ onto a closed subset of $\text{Prim}(G)$ by (6.1) and [10, Proposition 3.2.1].

Since the maps $\kappa_G : \widehat{G} \to \text{Prim}(G), [\pi] \mapsto \text{Ker} \pi$ and $\kappa_{G/H} : \widehat{G/H} \to \text{Prim}(G/H), [\sigma] \mapsto \text{Ker} \sigma$ are open, continuous, and surjective, while the diagram

$$\begin{array}{ccc}
\widehat{G/H} & \xrightarrow{\widehat{j_H}} & \widehat{G} \\
\downarrow \kappa_{G/H} & & \downarrow \kappa_G \\
\text{Prim}(G/H) & \xrightarrow{(j_H)^*} & \text{Prim}(G)
\end{array}$$

is commutative, we obtain that the image of the mapping $(j_H)^*$ is also open, hence is an open-closed subset of $\text{Prim}(G)$.

Finally, one has

$$\hat{G} = \bigcup_{H \in L(G)} \widehat{j_H}(\widehat{G/H})$$
by [18, Theorem 5.4], since $G$ is almost connected, hence also

$$\text{Prim}(G) = \bigcup_{H \in \mathcal{L}(G)} (j_H)^* (\text{Prim}(G/H))$$

and then the assertion follows at once since we have already seen that every set $(j_H)^* (\text{Prim}(G/H))$ is an open-closed subset of $\text{Prim}(G)$ for $H \in \mathcal{L}(G)$. □

**Theorem 6.5.** Let $G$ be a connected, locally compact group. If $\text{Prim}(G)$ is $T_1$, then $C^*(G)$ is primitively AF-embeddable.

**Proof.** We use the notation of Lemma 6.4 and its proof. Let $\pi: G \to \mathcal{B}(\mathcal{H})$ be an arbitrary unitary irreducible representation. It follows by [18, Theorem 3.1] that there exist $H \in \mathcal{L}(G)$ and an unitary irreducible representation $\pi_0: G/H \to \mathcal{B}(\mathcal{H})$ with $\pi = \pi_0 \circ j_H$. Hence, when we extend $\pi$ and $\pi_0$ to irreducible $*$-representations of $C^*(G)$ and $C^*(G/H)$, respectively, we have

$$\pi = \pi_0 \circ (j_H)_*,$$

where $(j_H)_*: C^*(G) \to C^*(G/H)$ is a surjective $*$-morphism. (See [2, Proposition 8.C.8].)

Since $\text{Prim}(G)$ is $T_1$, it follows by Lemma 6.4 that $\text{Prim}(G/H)$ is $T_1$. On the other hand, $G$ is connected, hence $G/H$ is a connected Lie group, hence $C^*(G/H)$ is primitively AF-embeddable by Lemma 6.3(i). In particular, $\pi_0(C^*(G/H))$ is an AF-embeddable $C^*$-algebra. Since $\pi(C^*(G)) = \pi_0((j_H)_*(C^*(G))) = \pi_0(C^*(G/H))$, it follows that $\pi(C^*(G))$ is AF-embeddable. This completes the proof. □

**Corollary 6.6.** Let $G$ be a connected, locally compact group. If $\text{Prim}(G)$ is $T_1$, then $C^*(G)$ is strongly quasi-diagonal.

**Proof.** Use Theorem 6.5 and the fact that every AF-embeddable $C^*$-algebra is quasi-diagonal. □

In the case of connected and simply connected solvable Lie groups, the above results go in the reverse direction, as well; we have the following corollary.

**Corollary 6.7.** Let $G$ be a connected simply connected solvable Lie group. Then the following assertions are equivalent.

(i) $G$ is of type $R$.

(ii) $C^*(G)$ is primitively AF-embeddable.

(iii) $C^*(G)$ is strongly quasi-diagonal.

**Proof.** The implication (i) $\Rightarrow$ (ii) follows from Lemma 6.3 and [24, Theorem 2, p. 161]. On the other hand, (ii) clearly implies (iii).

We now prove that (iii) $\Rightarrow$ (i), by showing that if $G$ is connected simply connected solvable and not of the type $R$, then it cannot be strongly quasi-diagonal. To this end, we show that there exists a closed 2-sided ideal $\mathcal{J} \subseteq C^*(G)$ such that the quotient $C^*(G)/\mathcal{J}$ is not strongly quasi-diagonal.

It follows by [1, Proposition 2.2, Chapter V] that if $G$ is a connected simply connected solvable Lie group and $G$ is not of type $R$, then there exists a connected simply connected closed normal subgroup $H \subseteq G$ for which the quotient Lie group $G/H$ is isomorphic to one of the following Lie groups.
(1) $S_2 := \mathbb{R} \times \mathbb{R}$, the connected real $ax + b$-group, defined via
$$\alpha : (\mathbb{R}, +) \to \text{Aut}(\mathbb{R}, +), \quad \alpha(t)s = e^t s.$$ 

(2) $S_3^\sigma := \mathbb{R}^2 \rtimes_\sigma \mathbb{R}$, defined for $\sigma \in \mathbb{R} \setminus \{0\}$ via
$$\alpha^\sigma : (\mathbb{R}, +) \to \text{Aut}(\mathbb{R}^2, +), \quad \alpha^\sigma(t) = e^{\sigma t} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$ 

(3) $S_4 := \mathbb{R}^2 \rtimes_\beta \mathbb{R}^2$, defined via
$$\beta : (\mathbb{R}^2, +) \to \text{Aut}(\mathbb{R}^2, +), \quad \beta(t, s) = e^t \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}.$$ 

We get thus short exact sequence of amenable locally compact groups
$$1 \to H \to G \to G/H \to 1$$
that leads to a short exact sequence of $C^*$-algebras
$$0 \to \mathcal{J} \to C^*(G) \to C^*(G/H) \to 0,$$
for a suitable closed 2-sided ideal $\mathcal{J}$ of $C^*(G)$. Therefore, in order to show that $C^*(G)$ is not strongly quasi-diagonal, it suffices to check that the $C^*$-algebra of the above groups $S_2$, $S_3^\sigma$ with $\sigma \in \mathbb{R} \setminus \{0\}$, and $S_4$ are not strongly quasi-diagonal, and this fact was established in the proof of [3, Theorem 1.1, (i) $\Rightarrow$ (iii)].

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