Perturbation Theory for Population Dynamics

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Abstract

We prove that a recently proposed homotopy perturbation method for the treatment of population dynamics is just the Taylor expansion of the population variables about initial time. Our results show that this perturbation method fails to provide the global features of the ecosystem dynamics.

1 Introduction

Recently, Chowdhury et al [1] proposed the application of homotopy–perturbation method (HPM) to simple models of population dynamics and obtained approximate solutions in the form of perturbation series. The authors show that their approximate analytical results agree with the numerical solution of the problem and appear to suggest that the HPM series are convergent.

A straightforward inspection of such series reveals that they are merely Taylor expansions of the time variable. One does not expect such a local approximation to provide a reasonable description of the dynamics of nonlinear systems, except in the neighbourhood of the initial conditions. Singular points appear spontaneously in nonlinear systems and move around the complex plane as the initial conditions vary [2] which makes unlikely that the time series are valid for all time.
The purpose of this letter is to investigate the range of utility of the homotopy time series to provide useful insight on the dynamics of population models. In Section 2 we show that the HPM of Chowdhury et al \cite{1} always leads to a Taylor expansions of the solution of the nonlinear system about initial time. In Section 3 we analyze the exactly solvable one–dimensional problem already considered by Chowdhury et al \cite{1}. In Section 4 we study their population model for two species \cite{1} and an exactly solvable two–dimensional dynamical model. Finally, in Section 5 we summarize our results and draw some conclusions.

2 Perturbation method

Population models give rise to differential equations of the form

\[ \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \]  

where \( x \) is a vector of the \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( f(x) \) is a vector–valued function with components \( f_1(x), f_2(x), \ldots, f_n(x) \). We assume that this vector–valued function is continuously differentiable.

The HPM proposed by Chowdhury et al \cite{1} is equivalent to a straightforward perturbation theory based on the modified equation \( \dot{x}(\lambda, t) = \lambda f(x(\lambda, t)) \), where \( \lambda \) is a dummy perturbation parameter. Notice that \( x(1, t) = x(t) \) so that we set \( \lambda \) equal to unity at the end of the calculation. The HPM proposes a solution in the form of a series

\[ x(\lambda, t) = \sum_{j=0}^{\infty} x^{(j)}(t)\lambda^j \]  

where \( \lambda \) is finally set equal to unity as said above. Notice that since \( \dot{x}^{(0)}(t) = 0 \) the resulting unperturbed or reference model \( x^{(0)}(t) = x_0 \) seems to be quite poor at first sight. Our results clearly show that it is actually the case. The initial conditions for the perturbation corrections are \( x^{(j)}(0) = 0 \) for all \( j > 0 \).

If we define the new time variable \( \tau = \lambda t \), then \( dx/d\tau = f(x) \) from which we conclude that \( x(\lambda, t) = x(\lambda t) \) and that this particular implementation of the HPM becomes the straightforward time series

\[ x(t) = \sum_{j=0}^{\infty} x_j t^j \]  

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in agreement with the particular results derived by Chowdhury et al. For example, the coefficient of first order is simply $x_1 = f(x_0)$.

3 One-dimensional model

We first consider the simple one-dimensional model

$$\dot{x} = x(b + ax)$$

(4)

where $b \geq 0$ and $a < 0$. In this case we have an unstable node at $x = 0$ and a stable one at $x = -b/a$. The exact solution is

$$x(t) = \begin{cases} \frac{bx_0e^{bt}}{b-ax_0(1-e^{bt})}, & b > 0 \\ \frac{x_0}{1-ax_0t}, & b = 0 \end{cases}$$

(5)

As mentioned above, the HPM series agree with the Taylor expansion of the exact solution about $t = 0$:

$$x(t) = x_0 + tx_0(ax_0 + b) + \frac{t^2x_0(2ax_0 + b)(ax_0 + b)}{2} + \frac{t^3x_0(6a^2x_0^2 + 6abx_0 + b^2)(ax_0 + b)}{6} + \frac{t^4x_0(2ax_0 + b)(12a^2x_0^2 + 12abx_0 + b^2)(ax_0 + b)}{24} + \ldots$$

(6)

Equation (5) clearly shows that the solution has a pole at $t_c = b^{-1}\ln[1 + b/(ax_0)]$ and, therefore, the HPM series does not converge for $t > |t_c|$. When $b = 1$, $a = -3$, and $x_0 = 0.1$, then $|t_c| = 3.253846656$ is much larger than the largest time value chosen by Chowdhury et al. This explains why those authors obtained such good results. If, for example, $x_0 = 1$, then $t_c = -\ln(3/2)$ and the HPM series is unsuitable for $t > \ln(3/2)$ as shown in Table I. Notice that these results reflect the fact that the usefulness of the HPM depends on the initial conditions. Besides, the HPM series does not take into consideration the stable node at $x = -b/a$ and is therefore unable to reveal the main features of the dynamical behaviour of the system.

4 Two-dimensional Models

Chowdhury et al. also discussed the simple two-species model

$$\dot{x} = x(b_1 + a_{11}x + a_{12}y)$$

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\[ y = y(b_2 + a_{21}x + a_{22}y) \] (7)

and obtained accurate results for the model parameters \( b_1 = 0.1, a_{11} = -0.0014, a_{12} = -0.0012, b_2 = 0.08, a_{21} = -0.0009, a_{22} = -0.001 \), and initial conditions \( x_0 = 4, y_0 = 10 \). However, the time interval considered by the authors is too small to give any indication of the evolution of this model ecosystem.

By inspection of the first terms of the HPM expansions

\[ x(t) = x_0 + tx_0(a_{11}x_0 + a_{12}y_0 + b_1) + t^2x_0[2a_{11}^2x_0^2 + 3a_{11}x_0(a_{12}y_0 + b_1) + a_{12}^2y_0^2 + a_{12}y_0(a_{21}x_0 + a_{22}y_0 + 2b_1 + b_2) + b_1^2]/2 + \ldots \]

\[ y(t) = y_0 + ty_0(a_{21}x_0 + a_{22}y_0 + b_2) + t^2y_0[a_{11}a_{21}x_0^2 + a_{12}a_{21}x_0y_0 + a_{22}^2y_0^2 + a_{21}x_0(3a_{22}y_0 + b_1 + 2b_2) + 2a_{22}y_0^2 + 3a_{22}b_2y_0 + b_2^2]/2 + \ldots \] (8)

we appreciate that if the model parameters \( b_i \) and \( a_{ij} \) are sufficiently small (as those chosen by Chowdhury et al [1]), then the time series may give accurate results for an apparently large time interval. However, these series do not take into consideration the critical points of the model equations and therefore they cannot reveal the actual dynamics of the system [2, 3].

The nonlinear dynamical system (7) exhibits four critical points in phase space [2, 3]: an unstable node at \((x, y) = (0, 0)\), two saddle points at \((x, y) = (0, 80)\) and \((x, y) = (71.43, 0)\), and a stable node at \((x, y) = (12.5, 68.75)\).

Fig. 1 shows that the population moves in phase space from the initial condition to the stable node and that the time series is unable to take into account this important dynamical behaviour. Increasing the perturbation order from four (the one used by Chowdhury et al [1]) to ten just improves the accuracy for small time but worsens it at larger time which suggests that the convergence radii of the time series are rather too small. In other words, the time series do not allow us to study the important population portrait in phase space.

In order to appreciate a more dramatic failure of the time series consider the following system of nonlinear equations [3]

\[ \dot{x} = -y + ax(x^2 + y^2) \]

\[ \dot{y} = x + ay(x^2 + y^2) \] (9)

It is unsuitable for population dynamics because it allows negative values of \( x(t) \) and \( y(t) \) but has the great advantage of being exactly solvable. Its
solutions are
\[ x(t) = \frac{r_0 \cos(\theta_0 + t)}{\sqrt{1 - 2ar_0^2t}}, \quad y(t) = \frac{r_0 \sin(\theta_0 + t)}{\sqrt{1 - 2ar_0^2t}} \]
\[ r_0 = \sqrt{x_0^2 + y_0^2}, \quad \theta_0 = \arctan\left(\frac{y_0}{x_0}\right) \] (10)

We have an unstable spiral when \( a > 0 \) and a stable one when \( a < 0 \) [3]. We clearly see the pole at \( t_c = 1/(2ar_0^2) \) and realize that the time series will be completely useless for \( t > |t_c| \).

Fig. 2 shows the numerical and approximate solutions for this model when \( a = -0.5 \) and \( x_0 = y_0 = 2 \). We clearly notice that the time series fail completely to provide a qualitative description of the spiral point and thereby of the global details of the system dynamics.

It is worth mentioning that multiple–scale perturbation theory [2, 3] gives the exact answer for this model and therefore appears to be a much more reliable perturbation approach for nonlinear dynamics.

5 Conclusions

We have shown that:

- the homotopy perturbation method proposed by Chowdhury et al [1] is just the Taylor expansion of the solutions of the nonlinear systems about \( t = 0 \).

- the perturbation series, and consequently the HPM, is limited to a neighbourhood of the initial time determined by the singular point closest to the origin of the complex \( t \)-plane. The locations of the singular points of the nonlinear models shift as the initial conditions vary.

- the HPM does not give an acceptable qualitative description of the most important features of the evolution of the dynamical system in phase space. In this sense, the HPM is by far less useful than the standard linearization which is also a local approach [2, 3].

In principle, other implementations of the homotopy perturbation method may be more suitable for nonlinear dynamics. For example, we may choose
the linear approximation about the critical or fixed points as unperturbed or reference model for the subsequent application of perturbation theory.

Since homotopy perturbation methods have become quite popular and are currently being applied to a wide variety of fields (and references therein), present results become important because they suggest that a more careful scrutiny of the approach’s performance is required.

References

[1] M. S. H. Chowdhury, I. Hashim, and O. Abdulaziz, Phys. Lett. A 368 (2007) 251.

[2] C. M. Bender and S. A. Orszag, Advanced mathematical methods for scientists and engineers, (McGraw-Hill, New York, 1978).

[3] S. H. Strogatz, Nonlinear Dynamics and Chaos, with Applications to Physics, Biology, Chemistry, and Engineering, (Perseus Books, Reading, Massachusetts, 1994).

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Table 1: Logarithmic error $\log \left| \frac{(Exact - Approximate)}{Exact} \right|$ for the fourth–order time series for model (4)

| t  | Logarithmic Error |
|----|-------------------|
| 0.1| -3.14             |
| 0.2| -1.66             |
| 0.3| -0.798            |
| 0.4| -0.193            |
| 0.5| 0.273             |
| 0.6| 0.651             |
| 0.7| 0.968             |
| 0.8| 1.24              |
| 0.9| 1.48              |
| 1.0| 1.69              |

Figure 1: Numerical (squares), fourth–order time series (solid) and tenth–order time series (dashed) curves in the phase plane for model (7).
Figure 2: Numerical (solid) and fifth–order time series (dashed) curves in the phase plane for model (9).