DUALIZING COMPLEXES OF SEMINORMAL AFFINE SEMIGROUP RINGS AND TORIC FACE RINGS

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Abstract. We characterize the seminormality of an affine semigroup ring in terms of the dualizing complex, and the normality of a Cohen-Macaulay semigroup ring by the “shape” of the canonical module. We also characterize the seminormality of a toric face ring in terms of the dualizing complex. A toric face ring is a simultaneous generalization of Stanley-Reisner rings and affine semigroups.

1. Introduction

Let \( M \) be a finitely generated additive submonoid of \( \mathbb{Z}^d \) (i.e., \( M \) is an affine semigroup) with \( \mathbb{Z}M \cong \mathbb{Z}^d \), and \( C(M) := \mathbb{R}_{\geq 0} M \subset \mathbb{Z}^d \otimes \mathbb{R} \cong \mathbb{R}^d \) the polyhedral cone spanned by \( M \). Set \( \overline{M} := \mathbb{Z}M \cap C(M) \). Throughout the paper, we assume that \( M \) is positive, that is, \( M \) has no invertible element except 0.

In the former half of the present paper, we study the affine semigroup ring \( k[M] = \bigoplus_{a \in M} k x^a \) of \( M \) over a field \( k \). Now we have \( \dim k[M] = d \). It is a classical result that if \( R = k[M] \) is normal (equivalently, \( M = \overline{M} \)), then \( R \) is Cohen-Macaulay and the canonical module \( \omega_R \) has an easy description. On the other hand, the behavior of non-normal affine semigroup rings is delicate and complicated, and many works have been done on this subject.

Definition 1.1. Let \( A \) be a reduced noetherian commutative ring, and \( Q(A) \) its total quotient ring. We say \( A \) is seminormal, if \( a \in Q(A) \) and \( a^2, a^3 \in A \) imply \( a \in A \).

This notion is much more natural than it seems. In fact, it is known that \( R \) is seminormal if and only if \( \text{Pic} R \cong \text{Pic}(R[x]) \). See [15] and the references cited therein.

The seminormality of an affine semigroup ring \( R = k[M] \) is characterized in a combinatorial (resp. homological) way by Reid and Roberts [13] (resp. Bruns, Li and Römer [5]). In the present paper, we will give a new characterization using the dualizing complex. Our characterization is relatively closer to that in [5]. However, contrary to their result, ours does not use the \( \mathbb{Z}^d \)-grading of the local cohomology modules (or the dualizing complex). To introduce our result, we need preparation.

For a face \( F \) of the cone \( C(M) \), \( M_F := M \cap F \) is a submonoid of \( M \). The semigroup ring \( k[M_F] \) can be seen as a quotient ring of \( R \), and its normalization \( k[\overline{M}_F] \) has the natural \( R \)-module structure. Then we have the following complex.

\[
\begin{align*}
+I_R^* : 0 & \longrightarrow +I_R^{-d} \longrightarrow +I_R^{-d+1} \longrightarrow \cdots \longrightarrow +I_R^0 \longrightarrow 0, \\
+I_R^{-i} = \bigoplus_{F: \text{a face of } C(M)} k[\overline{M}_F].
\end{align*}
\]
The differential map $\partial : +I^i_R \to +I^{i+1}_R$ is the combination of the natural surjections $k[\tilde{M}_F] \twoheadrightarrow k[\tilde{M}_G]$ for faces $F, G$ with $F \supset G$ and $\dim F = \dim G + 1$.

**Proposition 2.3.** For a semigroup ring $R = k[M]$, it is seminormal if and only if $+I^*_R$ is quasi-isomorphic to the dualizing complex $D^*_R$.

We can characterize the normality of $k[M]$ using the dualizing complex in a similar way. As a byproduct of this observation, we have the following (strange?) result.

**Theorem 3.1** For $R = k[M]$, the following are equivalent.

(a) $R$ is normal.

(b) $R$ is Cohen-Macaulay and the canonical module $\omega_R$ is isomorphic to the ideal $(x^a | a \in M \cap \text{int}(\mathcal{C}(M)))$ of $R$ as (graded or nongraded) $R$-modules.

The implication (a) $\Rightarrow$ (b) is a classical result due to Hochster, Stanley and Danilov.

Stanley-Reisner rings and affine semigroup rings are important subjects of combinatorial commutative algebra. The notion of *toric face rings*, which originated in an earlier work of Stanley [14], generalizes both of them, and has been studied by Bruns, Römer, and their coauthors (e.g. [2, 4, 8]). Roughly speaking, to make a toric face ring $k[M]$ from a (locally) polyhedral CW complex $\mathcal{X}$, we assign each cell $\sigma \in \mathcal{X}$ an affine semigroup $M_\sigma \subset \mathbb{Z}^{\dim \sigma + 1}$, and "glue" their semigroup rings $k[M_\sigma]$ along with $\mathcal{X}$.

Recently, Nguyen [11] studied seminormal toric face rings mainly focusing on the local cohomology modules, but he also remarked that $k[M]$ is seminormal if and only if $k[M_\sigma]$ is seminormal for all $\sigma$. In this sense, the seminormality is a natural condition for toric face rings.

Generalizing the construction for affine semigroup rings, a toric face ring $k[M]$ of dimension $d$ admits the cochain complex $+I^*_R$ of the form

$$0 \to +I^{-d}_R \to +I^{-d+1}_R \to \cdots \to +I^0_R \to 0$$

with

$$+I^{-i}_R := \bigoplus_{\sigma \in \mathcal{X}, \dim \sigma = i-1} k[M_\sigma],$$

where $k[M_\sigma]$ is the normalization of $k[M_\sigma]$.

**Theorem 5.2** If a toric face ring $R = k[M]$ is seminormal, then $+I^*_R$ is quasi-isomorphic to the dualizing complex $D^*_R$. (The converse is also true. See Proposition 5.12)

Under the assumption that each $k[M_\sigma]$ is normal (of course, $+I^{-i}_R = \bigoplus_{\dim \sigma = i-1} k[M_\sigma]$), in this case), the above theorem was proved by the present author and Okazaki ([12, Theorem 5.2]). Even in this case, the proof requires quite technical argument, since $R$ is not a graded ring in the usual sense. The proof of Theorem 5.2 heavily depends on [12, Theorem 5.2], but we have to make more effort.

Finally, for an arbitrary toric face ring $R = k[M]$, we study the local cohomology modules $H^*_m(R)$ at the "graded" maximal ideal $m$. Let $+R$ (resp. $\tilde{R}$) be the seminormalization (resp. cone-wise normalization) of $R$. Both of them are toric face rings supported by the same CW complex $\mathcal{X}$ as $R$, but the construction of the latter is not straightforward (see Example 5.3). In §6, we show that we show
that \( H^i_m(\mathcal{R}) \subset H^i_m(\widetilde{R}) \), and \( H^i_m(\widetilde{R}) \neq 0 \) implies \( H^i_m(R) \neq 0 \). Hence we have; \( R \) is Cohen-Macaulay \( \Rightarrow +R \) is Cohen-Macaulay \( \Rightarrow \widetilde{R} \) is Cohen-Macaulay. We remark that the Cohen-Macaulay property of \( \widetilde{R} \) only depends on the topology of the underlying space of \( \mathcal{X} \) (and \( \text{char}(k) \)).

**Convention.** In this paper, we use the following notation: For a commutative ring \( A \), \( \text{Mod} A \) denotes the category of \( A \)-modules.

For cochain complexes \( M^\bullet \) and \( N^\bullet \), \( M^\bullet \cong N^\bullet \) means that two complexes are isomorphic in the derived category, and \( M^\bullet = N^\bullet \) means that these are isomorphic as (explicit) complexes. If \( M^\bullet \cong N^\bullet \), we say these two complexes are quasi-isomorphic (especially when a direct quasi-isomorphism \( M^\bullet \to N^\bullet \) or \( N^\bullet \to M^\bullet \) exists).

While the word “dualizing complex” sometimes means its isomorphism class in the derived category, we use the convention that the dualizing complex \( D_A^\bullet \) of a noetherian ring \( A \) is a dualizing complex of the form

\[
0 \to D_A^{-\dim A} \to \cdots \to D_A^{-1} \to D_A^0 \to 0
\]

with

\[
D_A^{-i} = \bigoplus_{p \in \text{Spec } A} E(A/p),
\]

(1.1)

where \( E(A/p) \) is the injective envelope of \( A/p \).

2. Dualizing complexes of seminormal affine semigroup rings

For the convention and notation about an affine semigroup \( M \subset \mathbb{Z}^d \) and the cone \( C(M) \subset \mathbb{R}^d \) spanned by \( M \), see the previous section.

Let

\[
\mathbb{k}[M] := \bigoplus_{a \in M} \mathbb{k} x^a \subset \mathbb{k}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]
\]

be the semigroup ring of \( M \) over a field \( \mathbb{k} \). Here, for \( a = (a_1, \ldots, a_d) \in \mathbb{Z}^d \), \( x^a \) denotes the monomial \( \prod_{i=1}^d x_i^{a_i} \). Clearly, \( R := \mathbb{k}[M] \) is a \( \mathbb{Z}^d \)-graded ring, and \( \text{*Mod } R \) denotes the category of \( \mathbb{Z}^d \)-graded \( R \)-modules.

For \( M = \bigoplus_{a \in \mathbb{Z}^d} M_a \in \text{*Mod } R \), set

\[
M_{C(M)} := \bigoplus_{a \in \mathbb{Z}^d \cap C(M)} M_a.
\]

It is clear that \( M_{C(M)} \) is a \( \mathbb{Z}^d \)-graded \( R \)-submodule of \( M \), and we call it the \( C(M) \)-graded part of \( M \). Similarly, for a cochain complex \( M^\bullet \) in \( \text{*Mod } R \), we can defined a subcomplex \( (M^\bullet)_{C(M)} \).

For a face \( F \) of \( C(M) \),

\[
M_F := M \cap F
\]

is a submonoid of \( M \). Consider the monomial ideal (i.e., \( \mathbb{Z}^d \)-graded ideal)

\[
p_F := (x^a \mid a \in M \setminus M_F)
\]

of \( R \). Since \( R/p_F \) is isomorphic to the affine semigroup ring \( \mathbb{k}[M_F] \) of \( M_F \), \( p_F \) is a prime ideal. Conversely, any monomial prime ideal coincide with \( p_F \) for some \( F \). We regard \( \mathbb{k}[M_F] \) as an \( R \)-module through \( R/p_F \cong \mathbb{k}[M_F] \).
For a face $F$ of $\mathcal{C}(M)$, $T_F := \{x^a \mid a \in M_F\} \subset R$ is a multiplicatively closed subset. So we have the localization $T_F^{-1}R$ of $R$ by $T_F$. The Čech complex $\check{C}^\bullet_R$ is defined as follows:

$$\check{C}^\bullet_R : 0 \rightarrow \check{C}_R^0 \rightarrow \check{C}_R^1 \rightarrow \cdots \rightarrow \check{C}_R^d \rightarrow 0,$$

where

$$\check{C}_R^i := \bigoplus_{\text{dim } F = i} T_F^{-1}R.$$

The differential map $\partial : \check{C}_R^i \rightarrow \check{C}_R^{i+1}$ is given by

$$\partial(x) = \sum_{G \supset F, \dim G = i+1} \varepsilon(G, F) \cdot \iota_{G,F}(x),$$

where $\iota_{G,F}$ is the natural injection $T_F^{-1}R \rightarrow T_G^{-1}R$ for $G \supset F$, and $\varepsilon(G, F)$ is the incidence function of the regular CW complex given by a cross section of $\mathcal{C}(M)$. The precise information on $\varepsilon(G, F)$ is found in [3, §6.2], and we will use this function later in a more general situation. Here we just remark that $\varepsilon(G, F) = \pm 1$ for all $F, G$ with $G \supset F$ and $\dim G = \dim F + 1$, and this signature makes $\check{C}^\bullet_R$ a cochain complex.

As shown in [3 Theorem 6.2.5], the local cohomology module $H^*_m(R)$ at the graded maximal ideal $m := \{x^a \mid 0 \neq a \in M\}$ is isomorphic to $H^4(\check{C}^\bullet_R)$ in $\text{Mod } R$. Moreover, $\check{C}^\bullet_R$ is a $(\Z^d\text{-graded})$ flat resolution of $R\Gamma_m R$.

The $\Z^d\text{-graded}$ Matlis dual $(T_F^{-1}R)^\vee$ of $T_F^{-1}R$ is of the form

$$(T_F^{-1}R)^\vee = \bigoplus_{a \in M_F - M} \k e_a,$$

where $e_a$ is a basis element with the degree $a$, and

$$M_F - M = \{b - c \mid b \in M_F \text{ and } c \in M\}.$$
For a face $F$ of the polyhedral cone $C(M)$, we regard
\[ k[\mathbb{Z}M_F \cap F] := \bigoplus_{b \in \mathbb{Z}M_F \cap F} k x^b \]
as a \( \mathbb{Z}^d \)-graded \( k[M] \)-module by
\[ x^a x^b = \begin{cases} x^{a+b} & \text{if } a \in M_F, \\ 0 & \text{otherwise,} \end{cases} \]
for \( x^a \in k[M] \) and \( x^b \in k[\mathbb{Z}M_F \cap F] \). Note that \( k[\mathbb{Z}M_F \cap F] \) is the normalization of \( k[M_F] \), and
\[ \ast E(k[M_F])_{C(M)} \cong k[\mathbb{Z}M_F \cap F] \]
as \( k[M] \)-modules. Let \( F, G \) be faces of \( C(M) \) with \( F \supset G \). It is easy to see that \( k[\mathbb{Z}M_G \cap G] \) is a quotient module of \( k[\mathbb{Z}M_F \cap F] \) (note that \( \mathbb{Z}M_G \) is a sublattice of \( \mathbb{Z}M_F \cap G \)). Hence there is the \( \mathbb{Z}^d \)-graded surjection
\[ \pi_{G,F} : k[\mathbb{Z}M_F \cap F] \twoheadrightarrow k[\mathbb{Z}M_G \cap G], \]
which is the \( C(M) \)-graded part of \( p_{G,F} \) (if \( \dim G = \dim F - 1 \)).

Hence the \( C(M) \)-graded part
\[ +I^*_R := (J^*_R)_{C(M)} \]
of the complex \( J^*_R \) is of the form
\[ +I^*_R : 0 \rightarrow +I^*_R -d \rightarrow +I^*_R -d+1 \rightarrow \cdots \rightarrow +I^*_R 0 \rightarrow 0, \]
\[ +I^*_R = \bigoplus_{F: \text{a face of } C(M)} \mathbb{Z}^d \mathbb{Z}M_F \cap F. \]
The differential map \( \partial : +I^*_R -i \rightarrow +I^*_R -i+1 \) is given by
\[ \partial(x) = \sum_{G \subset F, \dim G = i-1} \varepsilon(F,G) \cdot \pi_{G,F}(x), \]
for \( x \in k[M_F] \subset +I^*_R -i \).

As is well-known, \( R = k[M] \) is normal if and only if \( M = \overline{M} := \mathbb{Z}M \cap C(M) \).
We can characterize the seminormality of \( R \) in a similar way. For a face \( F \) of \( C(M) \), \( \text{int}(F) \) denotes its relative interior. Clearly,
\[ C(M) = \bigsqcup_{F: \text{a face of } C(M)} \text{int}(F). \]
Set
\[ +M := \bigsqcup_{F: \text{a face of } C(M)} \mathbb{Z}M_F \cap \text{int}(F). \tag{2.1} \]
Then \( +M \) is an affine semigroup with \( M \subseteq +M \subseteq \overline{M} \) and \( +(+M) = +M \).

**Theorem 2.1** (L. Reid and L.G. Roberts [13], Bruns, Li and Römer [5]). For an affine semigroup ring \( R = k[M] \), the following are equivalent.

(i) \( R \) is seminormal.
(ii) \( M = +M \).
(iii) \( H_m^i(R)a \neq 0 \) for \( a \in \mathbb{Z}^d \) implies \( -a \in C(M) \).

Hence \( +R := k[+M] \) is the seminormalization of \( R = k[M] \).
In the above theorem, the equivalence between (1) and (ii) (resp. (i) and (iii)) is [13, Theorem 4.3] (resp. [5, Theorem 4.7]).

Example 2.2. For the additive submonoid
\[ M = \{(m, n) \mid m \geq 0, n \geq 1\} \cup \{(2m, 0) \mid m \geq 0\} \]
of \(\mathbb{N}^2\), \(k[M]\) is seminormal, but not normal.

Proposition 2.3. If \(R = k[M]\) is seminormal, then \(+I^*_R\) is isomorphic to the \(\mathbb{Z}^d\)-graded dualizing complex \(J^*_R\) in the derived category \(D^b(\text{Mod } R)\), hence \(+I^*_R \cong D^*_R\) in \(D^b(\text{Mod } R)\). Conversely, if \(+I^*_R \cong D^*_R\) in \(D^b(\text{Mod } R)\) then \(R\) is seminormal.

Proof. We start from the proof of the first assertion. Since \(H^i_m(R^\vee) \cong H^{-i}(J^*_R)_a \neq 0\) implies \(a \in \mathcal{C}(M)\) by Theorem 2.1. Hence the \(\mathcal{C}(M)\)-graded part \(+I^*_R\) of \(J^*_R\) is quasi-isomorphic to \(J^*_R\) itself.

Next, we show the last assertion. For the seminormalization \(+R\) of \(R\), the explicit computation gives the isomorphism \(+I^*_R = +I^*_+R\) as cochain complexes of \(R\)-modules. We just shown that \(+I^*_R \cong D^*_R\) also in \(D^b(\text{Mod } R)\). Since \(+R\) is a finitely generated \(R\)-module, \(\text{Hom}^*(+I^*_R, D^*_R) \cong (+R)\) in \(D^b(\text{Mod } R)\). Clearly, we also have \(\text{Hom}^*(+I^*_R, D^*_R) \cong R\). So taking the functor \(\text{Hom}^*(-, D^*_R)\) to \(+I^*_R = +I^*_+R\), we have \(R \cong +R\) as \(R\)-modules. It means that \(R = +R\), and hence \(R\) is seminormal. \(\square\)

3. The normality and the canonical module of an affine semigroup ring

Consider the following subcomplex of \(+I^*_R\):
\[
I^*_R : 0 \longrightarrow I^*_R \longrightarrow I^*_R \longrightarrow \cdots \longrightarrow I^*_R \longrightarrow 0,
\]
\[
I^*_R = \bigoplus_{\dim F = 1} \mathbb{Z}^F.\]

If \(R\) is normal, then \(k[M_F]\) is normal for all \(F\) and \(I^*_R = +I^*_R\). Hence, in this case, \(I^*_R\) is quasi-isomorphic to the dualizing complex \(D^*_R\). This is a well-known result essentially appears in [3]. The next result states that the converse also holds.

Theorem 3.1. For an affine semigroup ring \(R = k[M]\), the following are equivalent.

(i) \(R\) is normal.

(ii) The complex \(I^*_R\) is quasi-isomorphic to the dualizing complex \(D^*_R\).
(iii) \( R \) is Cohen-Macaulay and the canonical module \( \omega_R \) is isomorphic to the ideal \( W_R := ( x^a \mid a \in M \cap \text{int}(C(M)) ) \) of \( R \) in \( \text{Mod} \ R \).

The implication (i) \( \Rightarrow \) (iii) is a classical result due to Hochster, Stanley and Danilov (c.f. [3] Theorem 6.3.5]). Note that if \( R \) is normal then \( \omega_R \cong W_R \) even in \( \text{Mod}^\ast R \).

**Proof.** (i) \( \Rightarrow \) (ii): We have mentioned above.

(ii) \( \Rightarrow \) (iii): The assertion follows from direct computation similar to the proof of [3] Theorem 6.3.4 [but we have to take the \( \mathbb{Z}^d \)-graded Matlis dual].

(iii) \( \Rightarrow \) (i): Since \( W_R \) and \( \omega_R \) are \( \mathbb{Z}^d \)-graded modules, \( \text{Hom}_R(W_R, \omega_R) \) has the natural \( \mathbb{Z}^d \)-grading. On the other hand, since \( W_R \cong \omega_R \) in \( \text{Mod} R \) now, we have \( \text{Hom}_R(W_R, \omega_R) \cong R \) in \( \text{Mod} R \). Since the unit group of \( R \) is \( k \setminus \{0\} \), the way to equip the (ungraded) module \( R \) with a \( \mathbb{Z}^d \)-grading is unique up to a shift. Hence there is \( a \in \mathbb{Z}^d \) such that \( \text{Hom}_R(W_R, \omega_R) \cong R(-a) \) in \( \text{Mod}^\ast R \). We use \( a \) in this meaning throughout this proof.

By [3] Proposition 3.3.18], \( R/W_R \) is a Gorenstein ring of dimension \( d-1 \) and \( \text{Ext}^1_R(R/W_R, \omega_R) \cong R/W_R \) in \( \text{Mod} R \). By an argument similar to the above, these are isomorphic even in \( \text{Mod}^\ast R \) up to a degree shift. Since \( \text{Hom}_R(W_R, \omega_R) \cong R(-a) \) in \( \text{Mod}^\ast R \), the short exact sequence \( 0 \rightarrow W_R \rightarrow R \rightarrow R/W_R \rightarrow 0 \) yields

\[
\text{Ext}^1_R(R/W_R, \omega_R) \cong R/W_R(-a). \tag{3.1}
\]

Note that \( J^*_{R/W_R} := \text{Hom}_R(R/W_R, J^*_{R}) \) is the \( \mathbb{Z}^d \)-graded dualizing complex of \( R/W_R \), and

\[
\text{H}^{-d+1}(J^*_{R/W_R}) \cong \text{Ext}^1_R(R/W_R, \omega_R) \tag{3.2}
\]

in \( \text{Mod}^\ast R \). Since

\[
\text{Hom}_R(R/W_R, ^*E(k[M_F])) = \begin{cases} 0 & \text{if } F = C(M), \\ ^*E(k[M_F]) & \text{if } F \text{ is a proper face of } C(M), \end{cases}
\]

\( J^*_{R/W_R} \) coincides with the brutal truncation \( J^>_R \) of \( J^*_{R} \) (for this assertion, we do not use any assumption on \( R = k[M] \)).

Let \( ^*R = k[^*M] \) be the seminormalization of \( R \). Since

\[
(J^*_{R/W_R})_{C(M)} = (J^*_{R})_{C(M)} = ^*I^+_R
\]

for all \( i > -d \), we have

\[
(J^*_{R/W_{R^+R}})_{C(M)} = +I^+_{R^+R} = (J^*_{R/W_R})_{C(M)}
\]

where \( J^*_{R/W_R} \) is the \( \mathbb{Z}^d \)-graded dualizing complex of \( R/W_R \). Hence we have

\[
[H^{-d+1}(J^*_{R/W_R})]_{C(M)} \cong [H^{-d+1}(J^*_{R/W_{R^+R}})]_{C(M)} \cong [\text{Ext}^1_R(R/W_R, \omega_R)]_{C(M)}.
\]

If \( +R \) is normal, then \( W_{R^+} \) is its canonical module, and

\[
[H^{-d+1}(J^*_{R/W_R})]_{C(M)} \cong \text{Ext}^1_R(R/W_{R^+R}, \omega_R) \cong +R/W_{R^+R}.
\]

In general, there might be gap between \( [H^{-d+1}(J^*_{R/W_R})]_{C(M)} \) and \( +R/W_{R^+R} \), but easy computation shows that \( H^{-d+1}(J^*_{R/W_R}) \) still contains a submodule which is isomorphic to \( +R/W_{R^+R} \) in \( \text{Mod} R \). (Note that \( [H^{-d+1}(J^*_{R/W_R})]_{C(M)} \) is isomorphic to the kernel of \( \partial : +I^+_{R^+R} \rightarrow +I^+_{R^+R} \).) Combining this fact with (3.1) and (3.2), we have a \( \mathbb{Z}^d \)-graded injection

\[
+R/W_{R^+R} \hookrightarrow (R/W_R)(-a).
\]
It implies that \( a = 0 \), and hence \( W_R \cong \omega_R \) in \(^*\text{Mod} R \). Since \( H^i_R(R)_b = \omega_R \) \((-b = (W_R)_{-b}) \neq 0 \) implies \( b \in -C(M) \), \( R \) is seminormal by Theorem 2.1.

Since \( R \) is seminormal, we have

\[
M \cap \text{int}(C(M)) = ZM \cap \text{int}(C(M)) = \overline{M} \cap \text{int}(C(M)),
\]

and \( W_R \) coincides with the canonical module \( \omega_{\overline{R}} (= W_{\overline{R}}) \) of \( \overline{R} \), where \( \overline{R} = k[M] \) with \( \overline{M} = ZM \cap C(M) \) is the normalization of \( R \). Hence we have

\[
\overline{R} \cong \text{Hom}(\omega_{\overline{R}}, \omega_R) = \text{Hom}(W_R, \omega_R) \cong \text{Hom}(\omega_R, \omega_R) \cong R
\]

in \( \text{Mod} R \). Hence \( \overline{R} \cong R \) and \( R \) is normal. \( \square \)

**Remark 3.2.** Let \( \overline{R} = k[\overline{M}] \) be the normalization of \( R = k[M] \). For a face \( F \) of \( C(M) \), \( ZM_F \) is a sublattice of \( Z\overline{M}_F \), and hence \( k[ZM_F \cap F] \) is a direct summand of \( k[\overline{M}_F] \) as an \( R \)-module. So \( ^*I^R \) is a submodule (actually, a direct summand) of \( I^R \) for each \( i \), but it does not mean \( ^*I^R \) is a subcomplex of \( I^R \).

For example, consider the seminormal semigroup \( M \) given in Example 2.2. Then \( R \) is of the form \( k[x^2, y, xy] \). In this case, \( ^*I^R = k[x, y] \), \( ^*I^R = k[x^2] \oplus k[y] \), and the degree \((1, 0)\) component of \( \partial : ^*I^R \to ^*I^R \) is the zero map. On the other hand, the normalization \( \overline{R} \) of \( R \) is \( k[x, y] \). Hence \( ^*I^\overline{R} = k[x, y] \), \( ^*I^\overline{R} = k[x] \oplus k[y] \), and the degree \((1, 0)\) component of \( \partial : ^*I^\overline{R} \to ^*I^\overline{R} \) is non-zero.

Anyway, this phenomena makes the proof of Theorem 5.2 below complicated.

4. Preliminaries on toric face rings

Let \( \mathcal{X} \) be a finite regular CW complex with the intersection property, and \( X \) its underlying topological space. More precisely, the following conditions are satisfied.

1. \( \emptyset \in \mathcal{X} \) (for the convenience, we set \( \dim \emptyset = -1 \), \( X = \bigcup_{\sigma \in \mathcal{X}} \sigma \), and the cells \( \sigma \in \mathcal{X} \) are pairwise disjoint.

2. If \( \emptyset \neq \sigma \in \mathcal{X} \), then, for some \( i \in \mathbb{N} \), there exists a homeomorphism from the \( i \)-dimensional ball \( \{ x \in \mathbb{R}^i \mid ||x|| \leq 1 \} \) to the closure \( \overline{\sigma} \) of \( \sigma \) which maps \( \{ x \in \mathbb{R}^i \mid ||x|| < 1 \} \) onto \( \sigma \).

3. For \( \sigma \in \mathcal{X} \), the closure \( \overline{\sigma} \) is the union of some cells in \( \mathcal{X} \).

4. For \( \sigma, \tau \in \mathcal{X} \), there is a cell \( v \in \mathcal{X} \) such that \( \overline{\sigma} = \overline{\tau} \cap \overline{v} \) (here \( v \) can be \( \emptyset \)).

We regard \( \mathcal{X} \) as a partially ordered set (\emph{poset} for short) by \( \sigma \geq \tau \) if \( \overline{\sigma} \supset \overline{\tau} \).

The following definitions of conical complexes and monoidal complexes are taken from [12], and equivalent to the original ones in Bruns, Koch and Römer [4] under the assumption that the cones \( C_\sigma \) contain no line (equivalently, the semigroups \( \mathcal{M}_\sigma \) are all positive). However, the notation has been changed a little from that of [12] for the usages in the present paper.

**Definition 4.1.** A conical complex \((\Sigma, \mathcal{X}; \{ \iota_{\sigma, \tau} \}) \) on \( \mathcal{X} \) consists of the following data.

1. To each \( \sigma \in \mathcal{X} \), we assign an Euclidean space \( E_\sigma = \mathbb{R}^{\dim \sigma + 1} \).

2. The injection \( \iota_{\sigma, \tau} : C_\tau \to C_\sigma \) for \( \sigma, \tau \in \mathcal{X} \) with \( \sigma \geq \tau \) satisfying the following.
   a. \( \iota_{\sigma, \tau} \) can be lifted to a linear map \( \tilde{i}_{\sigma, \tau} : E_\tau \to E_\sigma \).
   b. The image \( \iota_{\sigma, \tau}(C_\tau) \) is a face of \( C_\sigma \). Conversely, for a face \( C' \) of \( C_\sigma \), there is a sole cell \( \tau \) with \( \tau \leq \sigma \) such that \( \iota_{\sigma, \tau}(C_\tau) = C' \).
(c) \( \iota_{\sigma,\sigma} = \text{Id}_{C_\sigma} \) and \( \iota_{\sigma,\tau} \circ \iota_{\tau,\nu} = \iota_{\sigma,\nu} \) for \( \sigma, \tau, \nu \in \mathcal{X} \) with \( \sigma \geq \tau \geq \nu \).

A polyhedral fan \( \Sigma \) in \( \mathbb{R}^n \) gives a conical complex. In this case, as an underlying CW complex, we can take \( \{ \text{int}(C \cap S^{n-1}) \mid C \in \Sigma \} \), where \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \), and the injections \( \iota_{\sigma,\tau} \) are inclusion maps.

**Example 4.2.** Consider the following cell decomposition of a Möbius strip. Regarding each rectangles as the cross-sections of 3-dimensional cones, we have a conical complex that is not a fan (see [2]).

Let \( L_\sigma \) be the set of lattice points \( \mathbb{Z}^{\dim \sigma + 1} \) of \( E_\sigma = \mathbb{R}^{\dim \sigma + 1} \). Assume that \( \tilde{\iota}_{\sigma,\tau}(L_\tau) = \tilde{\iota}_{\sigma,\tau}(E_\tau) \cap L_\sigma \) for all \( \sigma, \tau \in \mathcal{X} \) with \( \sigma \geq \tau \).

**Definition 4.3.** A monoidal complex supported by a conical complex \( (\Sigma, \mathcal{X}, \{\iota_{\sigma,\tau}\}) \) is a set of monoids \( \mathcal{M} = \{M_\sigma\}_{\sigma \in \mathcal{X}} \) with the following conditions:

1. \( M_\sigma \subset L_\sigma = \mathbb{Z}^{\dim \sigma + 1} \) for each \( \sigma \in \mathcal{X} \), and it is a finitely generated additive submonoid (so \( M_\sigma \) is an affine semigroup);
2. \( M_\sigma \subset C_\sigma \) and \( \mathbb{R}_{\geq 0}M_\sigma = C_\sigma \) for each \( \sigma \in \mathcal{X} \);
3. for \( \sigma, \tau \in \mathcal{X} \) with \( \sigma \geq \tau \), the map \( \iota_{\sigma,\tau} : C_\tau \to C_\sigma \) induces an isomorphism \( M_\tau \cong M_\sigma \cap \iota_{\sigma,\tau}(C_\tau) \) of monoids.

If \( \Sigma \) is a rational fan in \( \mathbb{R}^n \), then \( \{ C \cap \mathbb{Z}^n \mid C \in \Sigma \} \) gives a monoidal complex. More generally, taking submonoids of \( C \cap \mathbb{Z}^n \) carefully, we can get a “non-normal” monoidal complex.

For a monoidal complex \( \mathcal{M} = \{M_\sigma\}_{\sigma \in \mathcal{X}} \), set

\[
|\mathcal{M}| := \lim_{\sigma \in \mathcal{X}} M_\sigma,
\]

where the direct limit is taken with respect to \( \iota_{\sigma,\tau} : M_\tau \to M_\sigma \) for \( \sigma, \tau \in \mathcal{X} \) with \( \sigma \geq \tau \). Note that \( |\mathcal{M}| \) is just a set and no longer a monoid in general. Since all \( \iota_{\sigma,\tau} \) is injective, we can regard \( M_\sigma \) as a subset of \( |\mathcal{M}| \). For example, if \( \{M_\sigma\}_{\sigma \in \mathcal{X}} \) comes from a fan in \( \mathbb{R}^n \), then \( |\mathcal{M}| = \bigcup_{\sigma \in \mathcal{X}} M_\sigma \subset \mathbb{Z}^n \).

Let \( a, b \in |\mathcal{M}| \). If there is some \( \sigma \in \mathcal{X} \) with \( a, b \in C_\sigma \), there is a unique minimal cell among these \( \sigma \)'s. (In fact, if \( C_{\sigma_1}, C_{\sigma_2} \in \mathcal{X} \) contain both \( a \) and \( b \), there is a cell \( \tau \in \mathcal{X} \) with \( \tau = \overline{\sigma_1} \cap \overline{\sigma_2} \) by our assumption on \( \mathcal{X} \), and \( C_{\tau} \) contains both \( a \) and \( b \).) If \( \sigma \) is the minimal one with this property, we have \( a, b \in M_\sigma \) and we can define \( a + b \in M_\sigma \subset |\mathcal{M}| \). If there is no \( \sigma \in \mathcal{X} \) with \( a, b \in C_\sigma \), then \( a + b \) does not exist.
Definition 4.4 (\cite{4}). Let \( \{M_\sigma\}_{\sigma \in \mathcal{X}} \) be a monoidal complex with \( |M| := \varprojlim M_\sigma \), and \( k \) a field. Then the \( k \)-vector space
\[ k[\mathcal{X}] := \bigoplus_{a \in |\mathcal{X}|} k \cdot x^a, \]
where \( x \) is a variable, equipped with the following multiplication
\[ x^a \cdot x^b = \begin{cases} x^{a+b} & \text{if } a + b \text{ exists}, \\ 0 & \text{otherwise}, \end{cases} \]
has a \( k \)-algebra structure. We call \( k[\mathcal{X}] \) the toric face ring of \( \mathcal{X} \) over \( k \).

Clearly, \( \dim k[\mathcal{X}] = \dim \mathcal{X} + 1 \). In the rest of this paper, we set \( d := \dim k[\mathcal{X}] \).

Stanley-Reisner rings and affine semigroup rings (of positive semigroups) can be established as toric face rings. If \( \mathcal{X} \) comes from a fan in \( \mathbb{R}^n \), then \( k[\mathcal{X}] \) admits a \( \mathbb{Z}^n \)-grading with \( \dim_k k[\mathcal{X}]_a \leq 1 \) for all \( a \in \mathbb{Z}^n \). But this is not true in general.

Example 4.5 (\cite{4, Example 4.6}). Consider the conical complex in Example 4.2. Assigning normal semigroup rings of the form \( k[a, b, c, d]/(ac - bd) \) to each rectangles, we have a toric face ring of the form
\[ k[x, y, z, u, v, w]/(xv - uy, vz - yw, xz - uw, uvw, uvz), \]
which does not admit a nice multi-grading. We can also get a similar example whose \( k[M_\sigma] \) are not normal.

We say a toric face ring \( R = k[\mathcal{X}] \) is cone-wise normal, if \( k[M_\sigma] \) is normal for all \( \sigma \in \mathcal{X} \). The notion of cone-wise normal toric face rings coincides with that of the ring \( k[\mathcal{W}_F] \) associated with a weak fan \( \mathcal{W}_F \) introduced in Bruns and Gubeladze \cite{4}. They gave an example of a cone-wise normal toric face ring which does not admit a \( \mathbb{Z} \)-grading with \( R_0 = k \) (\cite{4} Example 2.7).

For \( \sigma \in \mathcal{X} \), a monomial ideal \( p_\sigma := (x^a \mid a \in |\mathcal{X}| \setminus M_\sigma) \) of \( R \) is prime. In fact, the quotient ring \( R/p_\sigma \) is isomorphic to the affine semigroup ring \( k[M_\sigma] \). We regard \( k[M_\sigma] \) as an \( R \)-module, through \( R/p_\sigma \cong k[M_\sigma] \).

Set
\[ I^{-i}_R := \bigoplus_{\dim \sigma = i-1} k[M_\sigma] \]
for \( i = 0, \ldots, d \), and define \( \partial : I^{-i}_R \to I^{-i+1}_R \) by
\[ \partial(y) = \sum_{\dim \tau = i-2} \varepsilon(\sigma, \tau) \cdot \pi_{\tau, \sigma}(y) \]
for \( y \in k[M_\sigma] \subset I^{-i}_R \), where \( \pi_{\tau, \sigma} \) is the natural surjection \( k[M_\sigma] \to k[M_\tau] \) (note that if \( \tau \leq \sigma \) then \( p_\tau \subset p_\sigma \)) and \( \varepsilon \) is an incidence function of \( \mathcal{X} \). Then
\[ I^{-i}_R : 0 \to I^{-d}_R \to I^{-d+1}_R \to \cdots \to I^0_R \to 0 \]
is a cochain complex of finitely generated \( R \)-modules. The following is the main result of \cite{12}.

Theorem 4.6 (\cite{12} Theorem 5.2). If \( R \) is cone-wise normal, then \( I^*_R \) is quasi-isomorphic to the dualizing complex \( D^*_R \) of \( R \).
The proof of the main result in the next section largely depends on (the proof of) Theorem 4.6, but the proof in [12] is long and technical. So we summarize it here for the reader’s convenience. See [12] for detail.

An outline of the proof of Theorem 4.6. To prove the theorem, we realize $I_{R}^{*}$ as a subcomplex of $D_{R}^{*}$. Set $c(\sigma) := \dim \sigma + 1 = \dim k[M_{\sigma}]$ for a cell $\sigma$. The proof is divided into three steps.

Step 1. We have a canonical injection $i_{\sigma} : k[M_{\sigma}] \hookrightarrow D_{R}^{-c(\sigma)}$.

We fix a cell $\sigma$, and set $c := c(\sigma)$. Since $k[M_{\sigma}]$ is normal, it is Cohen-Macaulay and admits the canonical module simply denoted by $\omega_{\sigma}$. Note that

$$H^{-c}(\text{Hom}_{R}(\omega_{\sigma}, D_{R}^{*})) = \text{Ext}^{-c}_{R}(\omega_{\sigma}, D_{R}^{*}) \cong k[M_{\sigma}].$$

Since $\text{Hom}_{R}(\omega_{\sigma}, D_{R}^{-c(\sigma) - 1}) = 0$, the cohomology $H^{-c}(\text{Hom}_{R}(\omega_{\sigma}, D_{R}^{*}))$ is the kernel of the map

$$\text{Hom}_{R}(\omega_{\sigma}, \partial D_{R}^{*}) : \text{Hom}_{R}(\omega_{\sigma}, D_{R}^{*}) \longrightarrow \text{Hom}_{R}(\omega_{\sigma}, D_{R}^{-c(\sigma) + 1}).$$

(4.1)

Through the identification,

$$\text{Hom}_{R}(\omega_{\sigma}, D_{R}^{-c}) = \text{Hom}_{R}(k[M_{\sigma}], D_{R}^{-c}) \cong \{ y \in D_{R}^{-c} \mid p_{\sigma}y = 0 \},$$

the kernel of the map (4.1) is

$$i_{\sigma}(k[M_{\sigma}]) := \{ y \in D_{R}^{-c} \mid p_{\sigma}y = 0 \text{ and } \partial D_{R}(q_{\sigma}y) = 0 \},$$

where $q_{\sigma}$ is the set \{ $a$ \in $M_{\sigma}$ \rotatebox{90}{$\cap$} int($C(M_{\sigma})$) \}. (Note that $\omega_{\sigma}$ is the ideal of $k[M_{\sigma}]$ generated by $q_{\sigma}$.) Clearly, $i_{\sigma}(k[M_{\sigma}]) \cong k[M_{\sigma}].$

Of course, we just chose the subset $i_{\sigma}(k[M_{\sigma}])$ of $D_{R}^{-c}$, not an injection $i_{\sigma} : k[M_{\sigma}] \hookrightarrow D_{R}^{-c}$. However, the $R$-module $k[M_{\sigma}]$ is generated by a single element, and the choice of a generator (i.e., the choice of $i_{\sigma}$) is unique up to constant multiplication. This small ambiguity does not affect the argument below.

Step 2. $\bigoplus_{\sigma \in X} i_{\sigma}(k[M_{\sigma}])$ is a subcomplex of $D_{R}^{*}$.

The dualizing complex $D_{\sigma}^{*} := D_{k[M_{\sigma}]}(k[M_{\sigma}], D_{R}^{*})$ of $k[M_{\sigma}]$ coincides with $\text{Hom}_{R}(k[M_{\sigma}], D_{R}^{*})$, which can be seen as a subcomplex of $D_{R}^{*}$. Since $k[M_{\sigma}]$ is $\mathbb{Z}^{c(\sigma)}$-graded, we have the $\mathbb{Z}^{c(\sigma)}$-graded dualizing complex $J_{\sigma}^{*} := J_{k[M_{\sigma}]}^{*}$, and a quasi-isomorphism $J_{\sigma}^{*} \rightarrow D_{\sigma}^{*}$. Composing this morphism with $D_{\sigma}^{*} \rightarrow D_{R}^{*}$, we get a chain map $h_{\sigma} : J_{\sigma}^{*} \rightarrow D_{R}^{*}$ which induces

$$H^{i}(\text{Hom}_{R}(\omega_{\sigma}, J_{\sigma}^{*})) \cong H^{i}(\text{Hom}_{R}(\omega_{\sigma}, D_{R}^{*})).$$

(4.2)

Applying the same argument as Step 1, we have an injection $^{*}i_{\sigma,\tau} : k[M_{\tau}] \hookrightarrow J_{\sigma}^{-c(\tau)}$ for a cell $\tau$ with $\tau \leq \sigma$. By (4.2), it is easy to see that

$$i_{\tau}(k[M_{\tau}]) = h_{\sigma} \circ ^{*}i_{\sigma,\tau}(k[M_{\tau}]).$$

On the other hand, we have that

$$(J_{\sigma}^{*})_{C_{\sigma}} = \bigoplus_{\tau \leq \sigma} ^{*}i_{\sigma,\tau}(k[M_{\tau}]),$$

(4.3)

where $C_{\sigma}$ is the polyhedral cone spanned by $M_{\sigma}$. Since $J_{\sigma}^{*}$ is a $\mathbb{Z}^{c(\sigma)}$-graded complex, the right side of (4.3) is a subcomplex of $J_{\sigma}^{*}$. Since $h_{\sigma}$ is a chain map, $\bigoplus_{\tau \leq \sigma} i_{\sigma}(k[M_{\tau}])$ forms a subcomplex of $D_{R}^{*}$. It implies that $\bigoplus_{\sigma \in X} i_{\sigma}(k[M_{\sigma}])$ is also a subcomplex of $D_{R}^{*}$. \hfill \Box
Since $\bigoplus_{\sigma \in \mathcal{X}} i_\sigma(\mathbb{k}[M_\sigma])$ is isomorphic to $I_R^\bullet$, it suffices to show the following.

**Step 3.** $D_R^\bullet$ is quasi-isomorphic to its subcomplex $\bigoplus_{\sigma \in \mathcal{X}} i_\sigma(\mathbb{k}[M_\sigma])$.

The argument for this step will be used around the proof of Theorem 5.11 after a slight generalization. There, we explain this idea in detail, so we do not give a summary here. \(\square\)

5. **Dualizing complexes of seminormal toric face rings**

We start from the following fact pointed out by Nguyen [11].

**Proposition 5.1** ([11] Proposition 3.5). For a toric face ring $\mathbb{k}[M]$, the following are equivalent.

(i) $\mathbb{k}[M]$ is seminormal.

(ii) $\mathbb{k}[M_\sigma]$ is seminormal for all $\sigma \in \mathcal{X}$.

Recall the precise definition of a monoidal complex $\mathcal{M}$ given in the previous section. For each $\sigma \in \mathcal{X}$, let $^{\ast}M_\sigma \subset L_\sigma$ be the monoid constructed from $M_\sigma$ by the operation in [2,1], that is, $\mathbb{k}[^{\ast}M_\sigma]$ is the seminormalization of $\mathbb{k}[M_\sigma]$. Then $^{\ast}\mathcal{M} := \{^{\ast}M_\sigma\}_{\sigma \in \mathcal{X}}$ forms a monoidal complex, and $^{\ast}R := \mathbb{k}[^{\ast}\mathcal{M}]$ is the seminormalization of $R := \mathbb{k}[\mathcal{M}]$. In particular, $R$ is seminormal if and only if $\mathcal{M} = ^{\ast}\mathcal{M}$.

On the other hand, $\mathbb{k}[\mathbb{Z}M_\sigma \cap C_\sigma]$ is the normalization of $\mathbb{k}[M_\sigma]$ (since we do not assume that $\mathbb{Z}M_\sigma = L_\sigma$, we have $\mathbb{Z}M_\sigma \cap C_\sigma \neq L_\sigma \cap C_\sigma$ in general), but $\{\mathbb{Z}M_\sigma \cap C_\sigma\}_{\sigma \in \mathcal{X}}$ does not form a monoidal complex. The monoidal complex $\mathcal{M}$ of Example 5.3 below gives a counter example.

We consider the following cochain complex

$$+I_R^\bullet : 0 \rightarrow +I_R^{-d} \rightarrow +I_R^{-d+1} \rightarrow \cdots \rightarrow +I_R^0 \rightarrow 0$$

with

$$+I_R^{-i} := \bigoplus_{\sigma \in \mathcal{X}} \mathbb{k}[\mathbb{Z}M_\sigma \cap C_\sigma].$$

The differential map $\partial$ is given by

$$\partial(y) = \sum_{\dim \tau = i-2} \varepsilon(\sigma, \tau) \cdot \pi_{\tau, \sigma}(y)$$

for $y \in \mathbb{k}[\mathbb{Z}M_\sigma \cap C_\sigma] \subset I_R^{-i}$, where $\pi_{\tau, \sigma}$ is the natural surjection $\mathbb{k}[\mathbb{Z}M_\sigma \cap C_\sigma] \rightarrow \mathbb{k}[\mathbb{Z}M_\tau \cap C_\tau]$. Clearly, $+I_R^\bullet$ is a cochain complex of finitely generated $R$-modules.

**Theorem 5.2.** If a toric face ring $R = \mathbb{k}[\mathcal{M}]$ is seminormal, then $+I_R^\bullet$ is quasi-isomorphic to the dualizing complex $D_R^\bullet$.

To prove the theorem, we need preparation. For each $\sigma \in \mathcal{X}$, set $\tilde{M}_\sigma := L_\sigma \cap C_\sigma$. Then $\{\tilde{M}_\sigma\}_{\sigma \in \mathcal{X}}$ is a monoidal complex again. We can regard that $|\mathcal{M}| := \lim_{\sigma} \tilde{M}_\sigma$ contains $|\mathcal{M}|$ as a subset.

**Example 5.3.** While $\mathbb{k}[\tilde{M}_\sigma]$ is always a normal semigroup ring, it is not the normalization of $\mathbb{k}[M_\sigma]$. For example, consider the monoidal complex illustrated below. Let $M_\sigma$ be the monoid corresponding to the first quadrant, then $\mathbb{k}[M_\sigma] = \mathbb{k}[x^2, y]$ is normal, but we have $\mathbb{k}[\tilde{M}_\sigma] = \mathbb{k}[x, y] \supset \mathbb{k}[M_\sigma]$. 
Set $\tilde{R} := \mathbb{k}[\tilde{M}]$. The next result holds, even if $\mathbb{k}[\mathcal{M}]$ is not seminormal.

**Lemma 5.4.** For any $\mathcal{M}$, $\tilde{R} = \mathbb{k}[\tilde{M}]$ is a finitely generated module over $R = \mathbb{k}[\mathcal{M}]$.

*Proof.* It suffices to show that $\mathbb{k}[\tilde{M}_\sigma]$ is finitely generated as a $\mathbb{k}[\mathcal{M}_\sigma]$-module for each $\sigma \in \mathcal{X}$. This must be a well-known result, but we give a proof here for the reader’s convenience. If $\dim \sigma = 0$, then the assertion is clear (in fact, $\mathbb{k}[\mathcal{M}_\sigma]$ is a polynomial ring with one variable in this case). If $\dim \mathbb{k}[\mathcal{M}_\sigma] > 1$, set $A := \mathbb{k}[\mathcal{M}_\sigma]$, and let $A'$ be the $A$-subalgebra of $\mathbb{k}[\mathcal{M}_\sigma]$ generated by $\{ \mathbb{k}[\mathcal{M}_\tau] \mid \tau < \sigma, \dim \tau = 0 \}$. By the above remark, $A'$ is a finitely generated $A$-module. Since $\mathbb{k}[\tilde{M}_\sigma]$ is the normalization of $A'$, it is a finitely generated as an $A'$-module, hence also as an $A$-module. \hfill $\square$

We regard $\mathbb{k}[\tilde{M}_\sigma]$ as an $R$-module by the compositions of the ring homomorphisms $R \to R/p_\sigma(\cong \mathbb{k}[\mathcal{M}_\sigma]) \hookrightarrow \mathbb{k}[\tilde{M}_\sigma]$, which is the same thing as $R \hookrightarrow R \to \mathbb{k}[\tilde{M}_\sigma]$.

As in the previous section, we set $c(\sigma) := \dim \sigma + 1 = \dim \mathbb{k}[\tilde{M}_\sigma]$. For the simplicity, the dualizing complexes $D^*_\mathbb{k}[\tilde{M}_\sigma]$ (resp. $D^*_{\mathbb{k}[\tilde{M}_\sigma]}$) of $\mathbb{k}[\mathcal{M}_\sigma]$ (resp. $\mathbb{k}[\tilde{M}_\sigma]$) is denoted by $D^*_\sigma$ (resp. $D^*_{\sigma}$). Since both $\mathbb{k}[\tilde{M}_\sigma]$ and $\mathbb{k}[\tilde{M}_\sigma]$ are $\mathbb{Z}^{(\sigma)}$-graded, they admit the $\mathbb{Z}^{(\sigma)}$-graded dualizing complexes $J^*_\sigma := J^*_{\mathbb{k}[\tilde{M}_\sigma]}$ and $J^*_\sigma := J^*_{\mathbb{k}[\tilde{M}_\sigma]}$ respectively. Similarly, we also set $+I^*_\sigma := +I^*_{\mathbb{k}[\tilde{M}_\sigma]}$ and $I^*_\sigma := I^*_{\mathbb{k}[\tilde{M}_\sigma]}(= +I^*_{\mathbb{k}[\tilde{M}_\sigma]}(\mathcal{M}_\sigma))$ for the simplicity.

Since $\tilde{R}$ is cone-wise normal, $I^*_R$ is quasi-isomorphic to $D^*_R$ by Theorem 4.6. Moreover, we have the following.

**Lemma 5.5.** There is a quasi-isomorphism $\psi : I^*_R \to D^*_R$ such that the induced map $\psi_* := \text{Hom}_R^*(\mathbb{k}[\tilde{M}_\sigma], \psi) : I^*_\sigma \to D^*_\sigma$ is a quasi-isomorphism for all $\sigma \in \mathcal{X}$.

*Proof.* This fact has been shown in the proof of [12, Theorem 5.2] (Theorem 4.6 of the present paper). Recall the outline of the proof introduced in the previous section. \hfill $\square$

Since $\tilde{R}$ is finitely generated as an $R$-module by Lemma 5.3, we have $D^*_R = \text{Hom}_R^*(\tilde{R}, D^*_R)$. Via the canonical injection $R \hookrightarrow \tilde{R}$, we have a chain map $\lambda : D^*_R = \text{Hom}_R^*(\tilde{R}, D^*_R) \longrightarrow \text{Hom}_R^*(R, D^*_R) = D^*_R$.
Similarly, for each $\sigma$, the injection $\mathbb{k}[\mathbb{M}_\sigma] \hookrightarrow \mathbb{k}[\widetilde{\mathbb{M}}_\sigma]$ induces a chain map $\lambda_\sigma : D^*_\sigma \rightarrow D^*_\sigma$. Since $\mathbb{k}[\widetilde{\mathbb{M}}_\sigma]$ is a finitely generated $\mathbb{Z}^{c(\sigma)}$-graded module over $\mathbb{k}[\mathbb{M}_\sigma]$ and $J^*_\sigma$ is the dualizing complex in the $\mathbb{Z}^{c(\sigma)}$-graded context, we have $\text{Hom}_{\mathbb{k}[\mathbb{M}_\sigma]}(\mathbb{k}[\widetilde{\mathbb{M}}_\sigma], J^*_\sigma) = J^*_\sigma$. The injection $\mathbb{k}[\mathbb{M}_\sigma] \hookrightarrow \mathbb{k}[\widetilde{\mathbb{M}}_\sigma]$ induces the $\mathbb{Z}^{c(\sigma)}$-graded chain map $\mu_\sigma : J^*_\sigma \longrightarrow J^*_\sigma$.

Note that $\mathbb{M}_\sigma$ and $\widetilde{\mathbb{M}}_\sigma$ span the same polyhedral cone $C_\sigma$. Since $\mathbb{k}[\mathbb{M}_\sigma]$ is seminormal and $\mathbb{k}[\widetilde{\mathbb{M}}_\sigma]$ is normal, we have $J^*_\sigma \cong (J^*_\sigma)_{C_\sigma} = +I^*_\sigma$ and $J^*_\sigma \cong (J^*_\sigma)_{C_\sigma} = +I^*_\sigma = I^*_\sigma$ as shown in the proof of Theorem 2.3. Taking the $C_\sigma$-graded part of $\mu_\sigma$, we have the chain map

$$\mu_\sigma : I^*_\sigma \longrightarrow +I^*_\sigma.$$  

**Lemma 5.6.** For the quasi-isomorphism $\psi_\sigma : I^*_\sigma \rightarrow D^*_\sigma$ of Lemma 5.3, we have a quasi-isomorphism $\phi_\sigma : +I^*_\sigma \rightarrow D^*_\sigma$ which makes the following diagram commutative.

$$
\begin{array}{ccc}
I^*_\sigma & \xrightarrow{\psi_\sigma} & D^*_\sigma \\
\mu_\sigma \downarrow & & \downarrow \lambda_\sigma \\
+I^*_\sigma & \xrightarrow{\phi_\sigma} & D^*_\sigma 
\end{array}
$$

**Proof.** It is easy to see that there exist quasi-isomorphisms $\psi'_\sigma : J^*_\sigma \rightarrow D^*_\sigma$ and $\phi'_\sigma : +I^*_\sigma \rightarrow D^*_\sigma$ such that $\psi_\sigma$ and $\phi_\sigma$ are their restrictions to $I^*_\sigma$ and $+I^*_\sigma$ respectively. Since $\mu_\sigma$ is the restriction of $\mu'_\sigma : J^*_\sigma \longrightarrow J^*_\sigma$, it suffices to construct a quasi-isomorphism $\phi'_\sigma : J^*_\sigma \rightarrow D^*_\sigma$ with

$$
\begin{array}{ccc}
J^*_\sigma & \xrightarrow{\psi'_\sigma} & D^*_\sigma \\
\mu'_\sigma \downarrow & & \downarrow \lambda_\sigma \\
J^*_\sigma & \xrightarrow{\phi'_\sigma} & D^*_\sigma 
\end{array}
$$

Since $J^*_\sigma \cong D^*_\sigma$ in $D^0(\text{Mod} \mathbb{k}[\mathbb{M}_\sigma])$, we have a quasi-isomorphism $\xi : J^*_\sigma \rightarrow D^*_\sigma$. Taking $\text{Hom}_{\mathbb{k}[\mathbb{M}_\sigma]}(\mathbb{k}[\widetilde{\mathbb{M}}_\sigma], -)$, we get a chain map

$$\xi_* : J^*_\sigma = \text{Hom}_{\mathbb{k}[\mathbb{M}_\sigma]}(\mathbb{k}[\widetilde{\mathbb{M}}_\sigma], J^*_\sigma) \longrightarrow \text{Hom}_{\mathbb{k}[\mathbb{M}_\sigma]}(\mathbb{k}[\widetilde{\mathbb{M}}_\sigma], D^*_\sigma) = D^*_\sigma.$$  

Note that $J^*_\sigma$ is a cochain complex of injective objects in the category $\ast\text{Mod}(\mathbb{k}[\mathbb{M}_\sigma])$ of $\mathbb{Z}^{c(\sigma)}$-graded $\mathbb{k}[\mathbb{M}_\sigma]$ modules, and $\mathbb{k}[\widetilde{\mathbb{M}}_\sigma] \in \ast\text{Mod}(\mathbb{k}[\mathbb{M}_\sigma])$. Hence $\xi_*$ is a quasi-isomorphism.

Clearly, $\xi_*$ is $\mathbb{k}[\widetilde{\mathbb{M}}_\sigma]$-linear, and can be extended to a $\mathbb{k}[\widetilde{\mathbb{M}}_\sigma]$-linear automorphism $\bar{\xi}_*$ of $D^*_\sigma$ uniquely (of course, the same is true for $\psi'_\sigma$). Since

$$\text{Hom}^{D^0(\text{Mod} \mathbb{k}[\mathbb{M}_\sigma])}(D^*_\sigma, D^*_\sigma) = \mathbb{k}[\widetilde{\mathbb{M}}_\sigma]$$

and $D^*_\sigma$ is a cochain complex of injective modules, the automorphism $\bar{\xi}_*$ is homotopic to the multiplication by $c$ for some $0 \neq c \in \mathbb{k}$. Moreover, since $D^*_\sigma$ is of the form (1.1), $\bar{\xi}_*$ is equal to the multiplication by $c$. Since the same is true for $\psi'_\sigma$, we have $\psi'_\sigma = c' \xi_*$ for some $0 \neq c' \in \mathbb{k}$. Hence $\phi'_\sigma := c' \xi_*$ satisfies the desired condition.  

For each $i \in \mathbb{Z}$, $+I^*_R$ is an $R$-submodule of $I^*_R$. However $+I^*_R$ is not a subcomplex of $I^*_R$. This problem occurs even in the semigroup ring case. See Remark 3.2.
Let $\kappa : {}^+I^*_R \longrightarrow I^*_R$ be the collection of the natural injections $^+I^i_R \rightarrow I^i_R$ (since this is not a chain map, we use the symbol “$\hookrightarrow$”). The similar map $\kappa_\sigma : {}^+I^*_\sigma \rightarrow I^*_\sigma$ is not a chain map in general again. For each $i$, $^+I^i_\sigma$ is a direct summand of $I^i_\sigma$ as an $k[M_\sigma]$-module, the $i$-th component $\mu^i_\sigma : I^i_\sigma \longrightarrow {}^+I^i_\sigma$ of the chain map $\mu_\sigma : I^*_\sigma \longrightarrow {}^+I^*_\sigma$ satisfies $\mu^i_\sigma \circ \kappa^i_\sigma = \text{Id}$.

**Lemma 5.7.** The composition $^+I^*_R \hookrightarrow I^*_R \xrightarrow{\psi} D^*_R \xrightarrow{\lambda} D^*_R$ is a chain map.

*Proof.* It suffice to check that
\[
\partial^{i+1}_R \circ (\lambda \circ \psi \circ \kappa)(y) = (\lambda^{i+1} \circ \psi^{i+1} \circ \kappa^{i+1}) \circ \partial^i_R(y)
\]
for all “homogeneous” element $y$ (i.e., $y \in (^+I^i_R)_a$ for some $a \in |\mathcal{M}|$), since any element of $^+I^i_R$ is a sum of these elements. Then we can regard $y \in ^+I^i_\sigma$ for some $\sigma \in \mathcal{X}$. We have the following commutative diagram.

\[
\begin{array}{ccc}
{^+I^i_R} & \kappa^i & I^i_R \\
\downarrow & \downarrow & \downarrow \\
{^+I^i_R} & \kappa^i & I^i_R
\end{array}
\begin{array}{ccc}
\psi^i & \phi^i & D^i_R \\
\downarrow & \downarrow & \downarrow \\
\psi^i & \phi^i & D^i_R
\end{array}
\begin{array}{ccc}
\lambda^i & \lambda^i & D^i_R \\
\downarrow & \downarrow & \downarrow \\
\lambda^i & \lambda^i & D^i_R
\end{array}
\]

The commutativity of the left square is clear, that of the middle one is Lemma 5.5, and that of the right one follows from the fact that the composition $R \hookrightarrow \tilde{R} \rightarrow k[M_\sigma]$ coincides with the composition $R \rightarrow k[M_\sigma] \hookrightarrow k[\tilde{M}_\sigma]$.

By Lemma 5.6, we have $\lambda^i_\sigma \circ \psi^i_\sigma \circ \kappa^i_\sigma = \phi^i_\sigma \circ \mu^i_\sigma \circ \kappa^i_\sigma = \phi^i_\sigma$. Since $\phi_\sigma$ is a chain map, we are done. \hfill \square

Let $\phi$ denote the chain map $J^*_R \rightarrow D^*_R$ constructed in Lemma 5.7. To prove Theorem 5.2, we will show that $\phi$ is a quasi-isomorphism by a slightly indirect way.

**Definition 5.8.** Let $R = k[\mathcal{M}]$ be a toric face ring. We say an $R$-module $M$ is $|\tilde{\mathcal{M}}|$-graded if the following are satisfied:

(i) $M = \bigoplus_{a \in |\tilde{\mathcal{M}}|} M_a$ as $k$-vector spaces;

(ii) Let $a \in |\mathcal{M}|$ and $b \in |\tilde{\mathcal{M}}|$. If $a + b$ exists (equivalently, $a, b \in \tilde{M}_\sigma$ for some $\sigma \in \mathcal{X}$), then $x^a M_b \subseteq M_{a+b}$. Otherwise, $x^a M_b = 0$.

Let $\text{Mod}_{\tilde{\mathcal{M}}} R$ denote the subcategory of $\text{Mod} R$ whose objects are $|\tilde{\mathcal{M}}|$-graded and homomorphisms are $f : M \rightarrow N$ with $f(M_a) \subseteq N_a$ for all $a \in |\tilde{\mathcal{M}}|$.

We say $M \in \text{Mod}_{\tilde{\mathcal{M}}} R$ is $|\mathcal{M}|$-graded, if $M = \bigoplus_{a \in |\mathcal{M}|} M_a$. Let $\text{Mod}_{\mathcal{M}} R$ denote the subcategory of $\text{Mod}_{\tilde{\mathcal{M}}} R$ consisting of $|\mathcal{M}|$-graded modules.

Clearly, $\text{Mod}_{\tilde{\mathcal{M}}} R$ and $\text{Mod}_{\mathcal{M}} R$ are abelian categories. It is easy to see that $R \in \text{Mod}_{\mathcal{M}} R$ and $\tilde{R} \in \text{Mod}_{\tilde{\mathcal{M}}} R$. Moreover, $I^*_R$ (resp. $^+I^*_R$) is a cochain complex in $\text{Mod}_{\mathcal{M}} R$ (resp. $\text{Mod}_{\tilde{\mathcal{M}}} R$).

**Definition 5.9.** For each $a \in |\tilde{\mathcal{M}}|$, there is a unique cell $\sigma \in \mathcal{X}$ with $a \in \text{int}(C_\sigma)$ (equivalently, $a \in \tilde{M}_\sigma$ and $\sigma$ is the minimal one with this property). This cell $\sigma$ is denoted by $\text{supp}(a)$. 
An \( R \)-module \( M \in \text{Mod} \, R \) is said to be squarefree if it is \(|\mathcal{M}|\)-graded (not \(|\tilde{\mathcal{M}}|\)-graded), finitely generated, and the multiplication map \( M_a \ni x \mapsto ax \in M_{a+b} \) is bijective for all \( a, b \in |\mathcal{M}| \) with \( \text{supp}(a) \supset \text{supp}(b) \).

For example, \( \mathbb{k}[M_\sigma] \) and \( R \) itself are squarefree \( R \)-modules. In \cite{12}, squarefree modules over a cone-wise normal toric face ring play a key role. Many properties are lost in the non-normal case. For example, \( +I_R^* \) is no longer a complex of squarefree modules. In fact, \( +I_R^* \) is \(|\tilde{\mathcal{M}}|\)-graded, not \(|\mathcal{M}|\)-graded. However, the next result still holds.

**Lemma 5.10** (c.f. \cite{12} Lemma 4.2). Let \( \text{Sq} \, R \) be the full subcategory of \( \text{Mod}_{\mathcal{M}} \, R \) consisting of squarefree modules. Then \( \text{Sq} \, R \) is an abelian category with enough injectives, and indecomposable injectives are objects isomorphic to \( \mathbb{k}[M_\sigma] \) for some \( \sigma \in \mathcal{X} \). The injective dimension of any object is at most \( d \).

The proof is similar to the cone-wise normal case (\cite{12}), and we omit it here. We just remark that \( \text{Sq} \, R \) is equivalent to the category of finitely generated left \( \Lambda \)-modules, where \( \Lambda \) is the incidence algebra of \( \mathcal{X} \) (as a poset) over \( \mathbb{k} \).

Let \( \text{Inj-Sq} \) be the full subcategory of \( \text{Sq} \, R \) consisting of all injective objects, that is, finite direct sums of copies of \( \mathbb{k}[M_\sigma] \) for various \( \sigma \in \mathcal{X} \). Then the bounded homotopy category \( K^b(\text{Inj-Sq}) \) is equivalent to \( D^b(\text{Sq} \, R) \). We have an exact functor
\[
\text{Hom}_R^*(–, +I_R^*) : K^b(\text{Inj-Sq}) \to D^b(\text{Mod} \, R)^{\text{op}}.
\]

Similarly, we have an exact functor
\[
\text{Hom}_R^*(–, D_R^*) : K^b(\text{Inj-Sq}) \to D^b(\text{Mod} \, R)^{\text{op}}.
\]

The chain map \( \phi : +I_R^* \to D_R^* \) gives a natural transformation \( \Phi : \text{Hom}_R^*(–, +I_R^*) \to \text{Hom}_R^*(–, D_R^*) \).

**Theorem 5.11.** If \( R \) is seminormal, \( \Phi \) is a natural isomorphism.

**Proof.** By virtue of \cite{7} Proposition 7.1, it suffices to show that
\[
\Phi(\mathbb{k}[M_\sigma]) : I_\sigma^* = \text{Hom}_R^*(\mathbb{k}[M_\sigma], I_R^*) \to \text{Hom}_R^*(\mathbb{k}[M_\sigma], D_R^*) = D_\sigma^*
\]
is a quasi-isomorphism for all \( \sigma \in \mathcal{X} \). Since \( \Phi(\mathbb{k}[M_\sigma]) = \text{Hom}_R^*(\mathbb{k}[M_\sigma], \phi) \), it is factored as \( I_\sigma^* \xrightarrow{\psi} I_\sigma^* \xrightarrow{\phi} D_\sigma^* \xrightarrow{\lambda} D_\sigma^* \). As shown in the proof of Lemma 5.6, this coincides with the quasi-isomorphism \( \phi_\sigma \) of Lemma 5.6.

The proof of Theorem 5.12. The theorem follows from Theorem 5.11. In fact, \( R \in \text{Sq} \, R \), and \( \phi : J_R^* \to D_R^* \) coincides with the isomorphism \( \Phi(R) : \text{Hom}_R^*(R, J_R^*) \to \text{Hom}_R^*(R, D_R^*) \).

The converse of Theorem 5.12 also holds.

**Proposition 5.12.** Let \( R = \mathbb{k}[\mathcal{M}] \) be a toric face ring. If \( +I_R^* \) is quasi-isomorphic to the dualizing complex \( D_R^* \), then \( R \) is seminormal.

**Proof.** Recall that \( +\mathcal{M} := \{ +M_\sigma \}_{\sigma \in \mathcal{X}} \) forms a monoidal complex, and the toric face ring \( +R = \mathbb{k}[+\mathcal{M}] \) is the seminormalization of \( R \). Since \( +I_R^* = +I_R^* \), the proof of the latter half of Theorem 5.12 also works here.
6. Local cohomologies

Recall that a monoidal complex \( \mathcal{M} = \{ \mathbf{M}_\sigma \}_{\sigma \in \mathcal{X}} \) is a collection of additive submonoids \( \mathbf{M}_\sigma \) of lattices \( \mathbf{L}_\sigma \cong \mathbb{Z}^{\dim \sigma + 1} \) for each \( \sigma \in \mathcal{X} \), and we have an injective homomorphisms \( \iota_{\sigma, \tau} : \mathbf{L}_\tau \to \mathbf{L}_\sigma \) for all \( \sigma, \tau \in \mathcal{X} \) with \( \sigma \geq \tau \). Set

\[
\mathcal{L} := \lim_{\sigma \in \mathcal{X}} \mathbf{L}_\sigma.
\]

Note that \( \mathcal{L} \) is no longer a group in general. Since all \( \iota_{\sigma, \tau} \) is injective, we can regard \( \mathbf{L}_\sigma \) as a subset of \( \mathcal{L} \). Let \( a, b \in \mathcal{L} \). If there is some \( \sigma \in \mathcal{X} \) with \( a, b \in \mathbf{L}_\sigma \), we have \( a + b \in \mathbf{L}_\sigma \subseteq \mathcal{L} \). If there is no \( \sigma \in \mathcal{X} \) with \( a, b \in \mathbf{L}_\sigma \), then \( a + b \) does not exist.

However, any \( a \in \mathcal{L} \) has \( -a \in \mathcal{L} \). We can regard that \( |\mathcal{M}| \subseteq \mathcal{L} \), and the structure of \( \mathcal{L} \) defined above and that of \( |\mathcal{M}| \) are compatible with this injection.

**Definition 6.1.** Let \( R := \mathbb{k}[\mathcal{M}] \) be a toric face ring. Then \( M \in \text{Mod } R \) is said to be \( \mathcal{L} \)-graded if the following conditions are satisfied;

1. \( M = \bigoplus_{a \in \mathcal{L}} M_a \) as \( \mathbb{k} \)-vector spaces;
2. \( x^a M_b \subseteq M_{a+b} \) if \( a \in \mathbf{M}_\sigma \) and \( b \in \mathbf{L}_\sigma \) for some \( \sigma \in \mathcal{X} \), and \( x^a M_b = 0 \) otherwise.

Let \( \text{Mod}_\mathcal{L} R \) be the category of \( \mathcal{L} \)-graded \( R \)-modules and \( R \)-homomorphisms \( f : M \to N \) with \( f(M_a) \subseteq N_a \) for all \( a \in \mathcal{L} \).

Clearly, \( \text{Mod}_\mathcal{M} R \) and \( \text{Mod}_\mathcal{M}^{-} R \) are full subcategories of \( \text{Mod}_\mathcal{L} R \). Note that \( T_\sigma := \{ x^a \mid a \in \mathbf{M}_\sigma \} \subseteq R \) is a multiplicatively closed subset. As shown in [12, Lemma 2.1], the localization \( T_{\sigma}^{-1} R \) is \( \mathcal{L} \)-graded.

Well, set

\[
\mathcal{C}_{\mathcal{L}}^i := \bigoplus_{\dim \sigma = i-1} T_{\sigma}^{-1} R
\]

and define \( \partial : \mathcal{C}_{\mathcal{L}}^i \to \mathcal{C}_{\mathcal{L}}^{i+1} \) by

\[
\partial(x) = \sum_{\tau \geq \sigma \divides \tau} \varepsilon(\tau, \sigma) \cdot \iota_{\tau, \sigma}(x)
\]

for \( x \in T_{\sigma}^{-1} R \subseteq \mathcal{C}_{\mathcal{L}}^i \), where \( \varepsilon \) is an incidence function on \( \mathcal{X} \) and \( \iota_{\tau, \sigma} \) is a natural map \( T_{\sigma}^{-1} R \to T_{\tau}^{-1} R \) for \( \sigma \leq \tau \). Then \( (\mathcal{C}_{\mathcal{L}}^i, \partial) \) forms a cochain complex in \( \text{Mod}_\mathcal{L} R \):

\[
0 \to \mathcal{C}_R^0 \to \mathcal{C}_R^1 \to \cdots \to \mathcal{C}_R^d \to 0.
\]

We set \( m := (x^a \mid 0 \neq a \in |\mathcal{M}|) \). This is a maximal ideal of \( R \). The following result has been proved by Ichim and Römer [8] in the case \( \mathcal{M} \) comes from a fan in \( \mathbb{R}^d \), and Okazaki and the present author in the general case. (The proofs are essentially the same.)

**Proposition 6.2** ([8, Theorem 4.2], [12, Proposition 3.2]). For any \( R \)-module \( M \), we have

\[
H^i_m(M) \cong H^i(\mathcal{C}_R^\bullet \otimes_R M),
\]

for all \( i \). In particular, \( H^i_m(R) \) is \( \mathcal{L} \)-graded.

**Corollary 6.3.** Let \( \mathcal{X} \) be a CW complex supporting \( R = \mathbb{k}[\mathcal{M}] \), and \( X \) the underlying topology of the underlying space of \( \mathcal{X} \). Then we have \( [H^i_m(R)]_0 \cong \widetilde{H}^{i-1}(X; \mathbb{k}) \), where \( 0 \) is the zero element of \( \mathcal{L} \) and \( \widetilde{H}^{i-1}(X; \mathbb{k}) \) is the \( i \)th reduced cohomology of \( X \) with the coefficients in \( \mathbb{k} \).
Proposition 6.5. Let \( R = \mathbb{k}[\mathcal{M}] \) be a toric face ring, and \( \ast R \) its seminormalization. Then we have
\[
H^i_m(\ast R) \cong [H^i_m(R)]_{-|\mathcal{M}|}
\]
as \( \mathcal{L} \)-graded \( \mathbb{k} \)-vector spaces for all \( i \).

Proof. It is easy to see that
\[
\{ a \in |\mathcal{M}| \mid |T^\sigma_{-1} R|_{-a} \neq 0 \} = \mathbb{Z} M_{\sigma} \cap C_{\sigma} = \{ a \in |\mathcal{M}| \mid |T^\sigma_{-1} (\ast R)|_{-a} \neq 0 \}
\]
for all \( \sigma \in \mathcal{X} \). Hence we have \((\mathcal{C}_R^\ast)^{-a} = (\mathcal{C}_R)^{-a}\) for all \( a \in |\mathcal{M}| \). Now the assertion follows from the following computation;
\[
[H^i_m(R)]_{-|\mathcal{M}|} \cong [H^i(\mathcal{C}_R^\ast)]_{-|\mathcal{M}|} \cong [H^i(\mathcal{C}_R)]_{-|\mathcal{M}|} \cong [H^i_m((\ast R)) ]_{-|\mathcal{M}|} \cong H^i_m(\ast R).
\]
Here the second "\( \cong \)" follows from the fact stated above, and the last one is Lemma 6.4. \( \square \)

Remark 6.6. In some sense, Proposition 6.5 generalizes and refines the results and the problem in §4 of Nguyen [11] (especially, [11, Theorem 4.3]). However, the toric face rings in [11] are assumed to have nice multigradings, while the "\( \mathcal{L} \)-grading" of our \( \mathbb{k}[\mathcal{M}] \) is not the grading in the usual sense.

Corollary 6.7. Let \( R = \mathbb{k}[\mathcal{M}] \) be a toric face ring, and \( \ast R \) its seminormalization. If \( R \) is Cohen-Macaulay, then so is \( \ast R \).
Proof. We prove the contrapositive: if \( ^+R \) is not Cohen-Macaulay, then \( R \) is also. Assume that \( ^+R \) is not Cohen-Macaulay. Then there is some \( 0 \leq i < \dim R \) with \( H^{-i}(^+I_R^i) \neq 0 \). For \( a \in |\widetilde{\mathcal{M}}| \), the cochain complex \( [^+I_R^i]_a \) of \( k \)-vector spaces is isomorphic to the \( k \)-dual of \( [C^*_a]_{-a} \). Hence it follows that \( H^{-i}_m(\mathcal{R}) \neq 0 \). By Proposition 6.3, we have \( H^{-i}_m(R) \neq 0 \), and hence the localization \( R_m \) is not Cohen-Macaulay. \( \Box \)

**Proposition 6.8.** For a monoidal complex \( \mathcal{M} = \{ M_\sigma \}_{\sigma \in \mathcal{X}} \), set \( \widetilde{\mathcal{M}} := \{ L_\sigma \cap C_\sigma \}_{\sigma \in \mathcal{X}} \) as before. Let \( R := k[\mathcal{M}] \) and \( \widetilde{R} := k[\widetilde{\mathcal{M}}] \) be their toric face rings. If \( R \) is Cohen-Macaulay, then so is \( \widetilde{R} \). Moreover, \( H^{-i}_m(\mathcal{R}) \neq 0 \) implies \( H^{-i}_m(\widetilde{\mathcal{R}}) \neq 0 \).

**Lemma 6.9.** With the same notation as in Proposition 6.8, \( H^i(D_R^*) \neq 0 \) implies \( H^i(\widetilde{D}_{\widetilde{R}}^*) \neq 0 \).

Proof. Recall that \( D_R^* \cong I_R^* \). If \( H^i(D_R^*)(\cong H^i(I_R^*)) \neq 0 \), then there is \( a \in |\widetilde{\mathcal{M}}| \) with \( [H^i(I_R^*)]_a \neq 0 \). Set \( \sigma := \text{supp}(a) \) (i.e., \( a \in \widetilde{\mathcal{M}}_\sigma \cap \text{int}(C_\sigma) \)). Since \( H^i(I_R^*) \) is a squarefree \( \widetilde{R} \)-module, we have \( [H^i(I_R^*)]_a \cong [H^i(I_{\widetilde{R}}^*)]_b \) for all \( b \in |\mathcal{M}| \) with \( \text{supp}(b) = \sigma \).

For \( b \in \mathcal{M}_\sigma \) with \( \text{supp}(b) = \sigma \), we have \( b \in \mathcal{M}_\tau \) for all \( \tau \in \mathcal{X} \) with \( \tau \geq \sigma \). In this case, regarding \( b \in |\mathcal{M}| \subset |\widetilde{\mathcal{M}}| \), we have \( [\widetilde{D}_{\widetilde{R}}^*]_b \) as cochain complexes of \( k \)-vector spaces, and hence \( [H^i(I_{\widetilde{R}}^*)]_b \cong [H^i(I_{\widetilde{R}}^*)]_b \neq 0 \). \( \Box \)

The proof of Proposition 6.8. By Proposition 6.3 and Corollary 6.7, we may assume that \( R \) is seminormal. Then \( ^+R \cong D_R^* \) by Theorem 5.2, and the assertion easily follows from Lemma 6.9.

Let \( R = k[\mathcal{M}] \) be a general toric face ring, \( ^+R = k[\mathcal{M}]^\ast \) its seminormalization, and \( \widetilde{R} = k[\widetilde{\mathcal{M}}] \). Proposition 6.8 and Corollary 6.7 state that \( R \) is Cohen-Macaulay \( \implies \) \( ^+R \) is Cohen-Macaulay \( \implies \) \( \widetilde{R} \) is Cohen-Macaulay.

By a result of Caijun [1] (see also [12]), the Cohen-Macaulay property of \( \widetilde{R} \) is a topological property of the underlying space \( X \) of \( \mathcal{X} \), while it may depend on \( \text{char}(k) \).

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