Abstract

We define the notion of a Catalan pair (which is a pair of binary relations \((S, R)\) satisfying certain axioms) with the aim of giving a common language to most of the combinatorial interpretations of Catalan numbers. We show, in particular, that the second component \(R\) uniquely determines the pair, and we give a characterization of \(R\) in terms of forbidden configurations. We also propose some generalizations of Catalan pairs arising from some slight modifications of (some of the) axioms.

1 Introduction

A famous exercise of [St1] proposes to the reader to show that every item of a long list of combinatorial structures provides a possible interpretation of the well-known sequence of Catalan numbers. In addition, since its appearance, many new combinatorial instances of Catalan numbers (in part due to Stanley as well [St2]) have been presented by several authors ([BEM, Cl, MM, MaSh, MaSe], to cite only a few). What makes Stanley’s exercise even more scary is the request for an explicit bijection for each couple of structures: even the more skillful and bold student will eventually give up, frightened by such a long effort.

The motivation of the present work lies in the attempt of making the above job as easier as possible. We propose yet another instance of Catalan numbers, by showing that they count pairs of binary relations satisfying certain axioms. Of course this is not the first interpretation of Catalan numbers in terms of binary relations. For instance, a well-known appearance of Catalan numbers comes from considering the so-called similarity relations;
these have been introduced by Fine [F] and further studied by several authors [GP, M, Sh]. However, what we claim to be interesting in our setting is that fairly every known Catalan structure (or, at least, most of the main ones) can be obtained by suitably interpreting our relations in the considered framework. From the point of view of our student, this approach should result in a quicker way to find bijections: indeed, it will be enough to guess the correct translation of any two Catalan structures in terms of our binary relations to get, as a bonus, the desired bijection. We hope to make this statement much clearer in section 3 where, after the definition of a Catalan pair and the proofs of some of its properties (pursued on sections 2), we explicitly describe some representations of Catalan pairs in terms of well-known combinatorial objects.

The rest of the paper is devoted to show that Catalan pairs are indeed a concept that deserves to be better investigated. In section 3 we show that any Catalan pair is uniquely determined by its second component, and we also provide a characterization of such a component in terms of forbidden configurations (which, in our case, are forbidden posets). In addition, we look at what happens when the second component of a Catalan pair has some specific properties, namely when it determines a connected posets or a (possibly distributive) lattice. We also observe that the first component of a Catalan pair does not uniquely determine the pair itself, and we give a description of Catalan pairs having the same first component. Finally, we propose some generalizations of Catalan pairs: in section 6 we see how to modify the axioms in order to obtain pairs of relations associated with other important integer sequences, such as Schröder numbers and central binomial coefficients; moreover we propose a slight, and very natural, modification of the crucial axiom in the definition of a Catalan pair and give an account on what this fact leads to.

Throughout the paper, the reader will find a (not at all exhaustive) series of open problems. We hope they can serve to stimulate future research on these topics.

2 Catalan pairs

In what follows, given any set $X$, we denote $D = D(X)$ the diagonal of $X$, that is the relation $D = \{(x, x) \mid x \in X\}$. Moreover, if $\theta$ is any binary relation on $X$, we denote by $\overline{\theta}$ the symmetrization of $\theta$, i.e. the relation $\overline{\theta} = \theta \cup \theta^{-1}$.

2.1 Basic definitions

Given a set $X$ of cardinality $n$, let $\mathcal{O}(X)$ be the set of strict order relations on $X$. By definition, this means that $\theta \in \mathcal{O}(X)$ when $\theta$ is an
irreflexive and transitive binary relation on $X$. In symbols, this means that $	heta \cap D = \emptyset$ and $\theta \circ \theta \subseteq \theta$.

Now let $(S, R)$ be an ordered pair of binary relations on $X$. We say that $(S, R)$ is a Catalan pair on $X$ when the following axioms are satisfied:

(i) $S \in \mathcal{O}(X)$; \hspace{1cm} (ord $S$)

(ii) $R \in \mathcal{O}(X)$; \hspace{1cm} (ord $R$)

(iii) $\overline{R} \cup \overline{S} = X^2 \setminus D$; \hspace{1cm} (tot)

(iv) $\overline{R} \cap \overline{S} = \emptyset$; \hspace{1cm} (inters)

(v) $S \circ R \subseteq R$; \hspace{1cm} (comp)

Remarks.

1. Observe that, since $S$ and $R$ are both strict order relations, the two axioms (tot) and (inters) can be explicitly described by saying that, given $x, y \in X$, with $x \neq y$, exactly one of the following holds: $xSy$, $xRy$, $ySx$, $yRx$.

2. Axiom (comp) could be reformulated by using strict containment, i.e. $S \circ R \subset R$. In fact, it is not difficult to realize that equality cannot hold since $X$ is finite. However we prefer to keep our notation, thus allowing to extend the definition of a Catalan pair to the infinite case.

3. From the above axioms it easily follows that $S \cap S^{-1} = \emptyset$.

In a Catalan pair $(S, R)$, $S$ (resp. $R$) will be referred to as the first (resp. second) component. Two Catalan pairs $(S_1, R_1)$ and $(S_2, R_2)$ on the (not necessarily distinct) sets $X_1$ and $X_2$, respectively, are said to be isomorphic when there exists a bijection $\xi$ from $X_1$ to $X_2$ such that $xS_1 y$ if and only if $\xi(x)S_2\xi(y)$ and $xR_1 y$ if and only if $\xi(x)R_2\xi(y)$. As a consequence of this definition, we say that a Catalan pair has size $n$ when it is defined on a set $X$ of cardinality $n$. The set of isomorphism classes of Catalan pairs of size $n$ will be denoted $\mathcal{C}(n)$. We will be mainly interested in the set $\mathcal{C}(n)$, even if, in several specific cases, we will deal with “concrete” Catalan pairs. However, in order not to make our paper dull reading, we will use the term “Catalan pair” when referring both to a specific Catalan pair and to an element of $\mathcal{C}(n)$. In the same spirit, to mean that a Catalan pair has size $n$, we will frequently write “$(S, R) \in \mathcal{C}(n)$”, even if $\mathcal{C}(n)$ is a set of isomorphism classes. In each situation, the context will clarify which is the exact meaning of what we have written down.

As an immediate consequence of the definition of a Catalan pair (specifically, from the fact that all the axioms are universal propositions), the following property holds.
Proposition 2.1 Let \((S, R)\) be a Catalan pair on \(X\). For any \(\bar{X} \subseteq X\), denote by \(\bar{S}\) and \(\bar{R}\) the restrictions of \(S\) and \(R\) to \(\bar{X}\), respectively. Then \((\bar{S}, \bar{R})\) is a Catalan pair on \(\bar{X}\).

2.2 First properties of Catalan pairs

In order to get trained with the above definition, we start by giving some elementary properties of Catalan pairs. All the properties we will prove will be useful in the rest of the paper.

Proposition 2.2 Given a Catalan pair \((S, R)\), the following properties hold:

1. \(S \circ R^{-1} \subseteq R^{-1}\);
2. \(R \circ S \subseteq R \cup S\);

Proof.

1. If \(xSyR^{-1}z\), then \(xSy\) and \(zRy\). Since \(x\) and \(z\) are necessarily distinct (this follows from axiom (inters)), it must be either \(zRx, xRz, zSx\) or \(xSz\). It is then easy to check that the three cases \(xRz, zSx, xSz\) cannot hold. For instance, if \(xRz\), then \(xRxRy\), whence \(xRy\), against (inters) (since, by hypothesis, \(xSy\)). Similarly, the reader can prove that both \(zSx\) and \(xSz\) lead to a contradiction. Thus \(zRx\), i.e. \(xR^{-1}z\).

2. Suppose that \(xRySz\). Once again, observe that the elements \(x\) and \(z\) are necessarily distinct, thus it must be either \(xRz, xSz, zRx\) or \(zSx\). Similarly as above, it can be shown that neither \(zRx\) nor \(zSx\) can hold. For instance, in the first case, from \(zRxRy\) we deduce \(zRy\), but we have \(ySz\) by hypothesis. The case \(zSx\) can be similarly dealt with.

\[\blacksquare\]

Remark. As a consequence of this proposition, we have that, in the definition of a Catalan pair, axiom (comp) can be replaced by:

\[S \circ \overline{R} \subseteq \overline{R}.\]

The above property will be useful in the sequel, when we will investigate the properties of the relation \(R\).

Proposition 2.3 Let \((S, R)\) be a pair of binary relations on \(X\) satisfying axioms (ord \(S\)), (ord \(R\)), (tot) and (inters). Then axiom (comp) is equivalent to:

\[\overline{S} \circ R \subseteq R \cup S^{-1}.\]

(comp*)
Proof. Assume that axiom (comp) holds and let $xSyRz$. Since $xSy$, we have two possibilities: if $xSy$, then $xSyRz$ and $xRz$. Instead, if $ySx$, then, being also $yRz$, we get that both the cases $xSx$ and $zRx$ cannot occur. Therefore it must be either $zSx$ or $xRz$, which means that $(x, z) \in R \cup S^{-1}$.

Conversely, assume that condition (comp*) holds, and suppose that $xSyRz$. We obviously deduce $xSyRz$, and so we have either $xRz$ or $zSx$. If $zSx$, then $zSxSy$, whence $zSy$, against the hypothesis $yRz$. Therefore it must be $xRz$. ■

2.3 Catalan pairs are enumerated by Catalan numbers

To show that the cardinality of $\mathcal{C}(n)$ is given by the $n$-th Catalan number $C_n$ we will provide a recursive decomposition for the structure $s$ of $\mathcal{C}(n)$. We recall that the sequence $C_n$ of Catalan numbers starts 1, 1, 2, 5, 14, 42, . . . (sequence A000108 in [Sl]) and has generating function $\frac{1-\sqrt{1-4x}}{2x}$.

Given two Catalan pairs, say $(S, R) \in \mathcal{C}(n)$ and $(S', R') \in \mathcal{C}(m)$, suppose that $S$ and $R$ are defined on $X = \{x_1, \ldots, x_n\}$, whereas $S'$ and $R'$ are defined on $Y = \{y_1, \ldots, y_m\}$, with $X \cap Y = \emptyset$. We define the composition of $(S, R)$ with $(S', R')$ to be the pair of relations $(S'', R'')$ on the set $\{z\} \cup X \cup Y$ of cardinality $n + m + 1$, defined by the following properties:

(i) $S''$ and $R''$, when restricted to $X$, coincide with $S$ and $R$, respectively;
(ii) $S''$ and $R''$, when restricted to $Y$, coincides with $S'$ and $R'$, respectively;
(iii) for every $x \in X$ and $y \in Y$, it is $xR''y$;
(iv) for every $x \in X$, it is $xS''z$;
(v) for every $y \in Y$, it is $zR''y$;
(vi) no further relation exists among the elements of $\{z\} \cup X \cup Y$.

For the composition we will use the standard notation, so that $(S'', R'') = (S, R) \circ (S', R')$.

Remarks.

1. The above definition of composition can be clearly given in a more compact form by setting $S'' = S \cup S' \cup (X \times \{z\})$ and $R'' = R \cup R' \cup ((X \cup \{z\}) \times Y)$.

2. From the above definition it follows that $S''$ is a strict order relation on $\{z\} \cup X \cup Y$ and $z$ is a maximal element of $S''$. Indeed, if $zS't$, for some $t$, then necessarily $t \in Y$ (from (iv)), but from (v) we would also have $zR''t$, against (vi). Similarly, it can be proved that $R''$ is a strict order relation on $\{z\} \cup X \cup Y$ and $z$ is a minimal element of $R''$. 

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**Proposition 2.4** Let $\alpha = (S, R) \in \mathcal{C}(n)$ and $\beta = (S', R') \in \mathcal{C}(m)$ be two Catalan pairs as above. Then $\alpha \circ \beta = (S'', R'') \in \mathcal{C}(n + m + 1)$.

*Proof.* The fact that $S'', R'' \in \mathcal{O}(\{z\} \cup X \cup Y)$ is stated in remark 2 above. Moreover, if $t, w \in \mathcal{O}(\{z\} \cup X \cup Y)$, with $t \neq w$, then the following cases are possible:

- both $t$ and $w$ belong to $X$ or $Y$: in this case $(t, w)$ belongs to exactly one among the relations $S, S^{-1}, R, R^{-1}, (S')^{-1}, (R')^{-1}$.
- $t$ belongs to $X$ and $w$ belongs to $Y$: then $tR''w$, and no further relation exists between $t$ and $w$; the case $t \in Y$ and $w \in X$ can be treated analogously.
- $t = z$ and $w \in X$: then the only relation between $t$ and $w$ is $t(S'')^{-1}w$; and similarly, if $w \in Y$, we have only $tR''w$.

As a consequence, we can conclude that $R'' \cup S'' = (\{z\} \cup X \cup Y)^2 \setminus \mathcal{D}$ and $R'' \cap S'' = \emptyset$.

Finally, suppose that $t(S'' \circ R'')w$. If $t, w$ both belong to $X$ or else to $Y$, then it is immediate to see that $tR''w$. Otherwise, suppose that $t$ and $w$ are both different from $z$: then necessarily $t \in X$ and $w \in Y$, and so $tR''w$. Finally, the cases $t = z$ and $w = z$ cannot occur, as a consequence of remark 2 above. Thus, we can conclude that, in every case, $tR''w$, whence $S'' \circ R'' \subseteq R''$. ■

**Lemma 2.1** Given a Catalan pair $(S, R)$ on $X$, let $x, y$ be two distinct (if any) maximal elements of $S$. Then there exists no element $t \in X$ such that $tSx$ and $tSy$.

*Proof.* If not, since $x$ and $y$ are maximal for $S$, then necessarily $x \bar{R}y$. If there were an element $t \in X$ such that $tSx$ and $tSy$, then, from $tSx \bar{R}y$, we would get $tRy$, against the fact that $tSy$. ■

Lemma 2.1 essentially states that the principal ideals generated by the maximal elements of $X$ (with respect to $S$) are mutually disjoint.

**Proposition 2.5** Let $\gamma = (S'', R'')$ be a Catalan pair of size $l \geq 1$. Then there exist unique Catalan pairs $\alpha = (S, R)$ and $\beta = (S', R')$ such that $\gamma = \alpha \circ \beta$.

*Proof.* Suppose that $\gamma$ is defined on $X_l$ of cardinality $l$ and let $M(S'')$ be the set of the maximal elements of $S''$. It is clear that $M(S'') \neq \emptyset$, since $X_l$ is finite. Define the set $\Phi$ to be the set of all elements of $M(S'')$ which are minimal with respect to $R''$. We claim that $|\Phi| = 1$. Indeed, since the
elements of $M(S'')$ are an antichain of $S''$, then necessarily they constitute a chain of $R''$, and so the minimum of such a chain is the only element of $\Phi$. Set $\Phi = \{x_0\}$, we can split $X_l$ into three subsets, $\{x_0\}$, $X$ and $Y$, where $X = \{x \in X_l \mid xS''x_0\}$ and $Y = \{x \in X_l \mid x_0 R''x\}$. The reader can easily check that the above three sets are indeed mutually disjoint. To prove that their union is the whole $X_l$, let $x \in X_l$ and suppose that $xS''x_0$ does not hold. Since $x_0$ is maximal for $S''$, then necessarily $x_0 R''x$. Suppose, ab absurdo, that $xR''x_0$. Denoting by $y$ the unique (by the above lemma) element of $M(S'')$ for which $xS''y$, we would have $xR''x_0 R''y$, and so $xR''y$, a contradiction. Thus we can conclude that $x_0 R''x$, as desired. Finally, define $\alpha = (S, R)$ and $\beta = (S', R')$ as the restrictions of $(S'', R'')$ to the sets $X$ and $Y$, respectively. The fact that $\alpha$ and $\beta$ are Catalan pairs follows from proposition 2.1 whereas the proof that $\alpha \circ \beta = \gamma$ is left to the reader. The uniqueness of the above described decomposition follows from the fact that $|\Phi| = 1$, i.e. there is only one possibility of choosing $x_0$ so that it satisfies the definition of composition of Catalan pairs.

Proposition 2.6 For any $n \in \mathbb{N}$, we have:

$$|C(n + 1)| = \sum_{k=0}^{n} |C(k)| \cdot |C(n - k)|. \quad (2)$$

Since $|C(0)| = 1$, we therefore have that $|C(n)| = C_n$, the $n$-th Catalan number.

Proof. By proposition 2.5 giving a Catalan pair of size $n + 1$ is the same as giving two Catalan pairs of sizes $k$ and $n - k$, for a suitable $k$. On the other hand, by proposition 2.4 any two Catalan pairs of sizes $k$ and $n - k$ can be merged into a Catalan pair of size $n + 1$. These arguments immediately imply formula (2).

3 Combinatorial interpretations of Catalan pairs

In this section we wish to convince the reader that fairly every combinatorial structure counted by Catalan numbers can be interpreted in terms of Catalan pairs. More precisely, we deem that any Catalan structure can be described using a suitable Catalan pair $(S, R)$, where $S$ and $R$ are somehow naturally defined on the objects of the class. To support this statement, we will take into consideration here five examples, involving rather different combinatorial objects, such as matchings, paths, permutations, trees and partitions. For each of them, we will provide a combinatorial interpretation in terms of Catalan pairs.
3.1 Perfect noncrossing matchings and Dyck paths

Our first example will be frequently used throughout all the paper. Given a set $A$ of even cardinality, a perfect noncrossing matching of $A$ is a noncrossing partition of $A$ having all the blocks of cardinality 2. There is an obvious bijection between perfect noncrossing matchings and well formed strings of parentheses.

A graphical device to represent a perfect noncrossing matching of $A$ consists of drawing the elements of $A$ as points on a straight line and join with an arch each couple of corresponding points in the matching. Using this representation, we can define the following relations on the set $X$ of arches of a given perfect noncrossing matching:

- for any $x, y \in X$, we say that $xSy$ when $x$ is included in $y$;
- for any $x, y \in X$, we say that $xRy$ when $x$ is on the left of $y$.

The reader is invited to check that the above definition yields a Catalan pair $(S, R)$ on the set $X$.

**Example.** Let $X = \{a, b, c, d, e, f, g\}$, and let $S$ and $R$ be defined as follows:

$$S = \{(b, a), (f, e), (f, d), (e, d), (g, d)\}$$

$$R = \{(a, c), (a, d), (a, e), (a, f), (a, g), (b, c), (b, d), (b, e), (b, f), (b, g),
(c, d), (c, e), (c, f), (c, g), (e, g), (f, g)\}.$$ 

It is easy to check that $(S, R)$ is indeed a Catalan pair on $X$ of size 7, which can be represented as in figure 1(a).

![Figure 1](image.png)

**Figure 1:** The graphical representation of a Catalan pair in terms of a noncrossing matching, and the associated Dyck path.

An equivalent way to represent perfect noncrossing matchings is to use Dyck paths: just interpret the leftmost element of an arch as an up step and
the rightmost one as a down step. For instance, the matching represented in figure 1(a) corresponds to the Dyck path depicted in figure 1(b). Coming back to Catalan pairs, the relations $S$ and $R$ are suitably interpreted using the notion of tunnel. A tunnel in a Dyck path is a horizontal segment joining the midpoints of an up step and a down step, remaining below the path and not intersecting the path anywhere else. Now define $S$ and $R$ on the set $X$ of the tunnels of a Dyck paths by declaring, for any $x, y \in X$:

- $xSy$ when $x$ lies above $y$;
- $xRy$ when $x$ is completely on the left of $y$.

See again figure 1 for an example illustrating the above definition.

3.2 Pattern avoiding permutations

Let $n, m$ be two positive integers with $m \leq n$, and let $\pi = \pi(1) \cdots \pi(n) \in S_n$ and $\nu = \nu(1) \cdots \nu(m) \in S_m$. We say that $\pi$ contains the pattern $\nu$ if there exist indices $i_1 < i_2 < \ldots < i_m$ such that $(\pi(i_1), \pi(i_2), \ldots, \pi(i_m))$ is in the same relative order as $(\nu(1), \ldots, \nu(m))$. If $\pi$ does not contain $\nu$, we say that $\pi$ is $\nu$-avoiding. See [B] for plenty of information on pattern avoiding permutations. For instance, if $\nu = 123$, then $\pi = 524316$ contains $\nu$, while $\pi = 632541$ is $\nu$-avoiding.

We denote by $S_n(\nu)$ the set of $\nu$-avoiding permutations of $S_n$. It is known that, for each pattern $\nu \in S_3$, $|S_n(\nu)| = C_n$ (see, for instance, [B]).

It is possible to give a description of the class of 312-avoiding permutations by means of a very natural set of Catalan pairs. More precisely, let $[n] = \{1, 2, \ldots, n\}$; for every permutation $\pi \in S_n$, define the following relations $S$ and $R$ on $[n]$:

- $iSj$ when $i < j$ and $(j, i)$ is an inversion in $\pi$ (see, for instance, [B] for the definition of inversion);
- $iRj$ when $i < j$ and $(i, j)$ is a noninversion in $\pi$.

**Proposition 3.1** The permutation $\pi \in S_n$ is 312-avoiding if and only if $(S, R)$ is a Catalan pair of size $n$.

**Proof.** The axioms (i) to (iv) in the definition of a Catalan pair are satisfied by $(S, R)$ for any permutation $\pi$, as the reader can easily check. Moreover, $\pi$ is 312-avoiding if and only if, given any three positive integers $i < j < k$, it can never happen that both $(j, i)$ and $(k, i)$ are inversions and $(j, k)$ is a noninversion. This happens if and only if $S \circ R$ and $S$ are disjoint. But, from the above definitions of $S$ and $R$, it must be $S \circ R \subseteq R \cup S$, whence $S \circ R \subseteq R$. ■
The present interpretation in terms of 312-avoiding permutations can be connected with the previous ones using Dyck paths and perfect noncrossing matchings, giving rise to a very well-known bijection, whose origin is very hard to be traced back (see, for instance, [P]). We leave all the details to the interested reader.

3.3 Plane trees

By means of the well-known bijection between perfect noncrossing matchings and plane trees [St1], the previous example allows us to give an interpretation of Catalan pairs in terms of plane trees. The details are left to the reader.

3.4 Noncrossing partitions

Let \( \mathcal{P}_n \) be the set of noncrossing partitions on the linearly ordered set \( X_n = \{x_1, x_2, \ldots, x_n\} \). Each \( p \in \mathcal{P}_n \) determines an equivalence relation \( \sim_p \) on \( X_n \). Given a generic element \( x \in X_n \), we will denote its equivalence class with \( [x]_{\sim_p} \).

Given \( x \in X_n \), we set \( u(x) = \max_{y < [x]_{\sim_p}} y \). Thus \( u(x) \) is given by the greatest lower bound of the elements in \( [x]_{\sim_p} \) minus 1. Observe that \( u(x) \) need not be defined for all \( x \).

Given \( p \in \mathcal{P}_n \), define two relations \( S \) and \( R \) as follows:

- \( S \) is the transitive closure of the relation \( \{(x, u(x)) \mid x \in X_n\} \);
- \( xRy \) when \( x < y \) and \( (x, y) \) is not in \( S \).

Then the pair \( (S, R) \) is indeed a Catalan pair on \( X_n \), and it induces an obvious bijection between noncrossing partitions and plane trees. Figure 2 depicts the noncrossing partition corresponding to the Catalan pair \( (S, R) \) represented in figure 1.

![Figure 2: The noncrossing partition corresponding to the Catalan pair represented in figure 1.](image-url)
4 Properties of the posets defined by $S$ and $R$

In the present section we investigate some features of the posets associated with the (strict) order relations $R$ and $S$. In the sequel, a poset will be denoted using square brackets, e.g. $[X, R]$ and $[X, S]$. An immediate observation which follows directly from the definition of a Catalan pair is the following, which we state without proof.

**Proposition 4.1** Given a finite set $X$, consider the graphs $X_1$ and $X_2$ determined by the Hasse diagrams of the posets $[X, R]$ and $[X, S]$. Then $X_1$ and $X_2$ are edge-disjoint subgraphs of the complete graph $K(X)$ on $X$ whose union gives the whole $K(X)$.

4.1 The poset defined by $R$

From the point of view of Catalan pars, it turns out that the strict order relation $R$ completely defines a Catalan pair. To prove this, we first need a technical definition which will be useful again later.

Given a strict order relation $R$ on $X$, define the relation $\sim_R$ on the set $X$ by declaring $x \sim_R y$ when, for all $z$, it is $zRx$ if and only if $zRy$. It is trivial to show that $\sim_R$ is an equivalence relation. In what follows, the equivalence classes of $\sim_R$ will be denoted using square brackets.

**Lemma 4.1**

(i) If $x \sim_R y$, then $xRy$.

(ii) It is $x \sim_R y$ if and only if, for all $z$, $zRx$ iff $zRy$ and $xRz$ iff $yRz$.

(iii) If $(S, R)$ is a Catalan pair, then, for all $x, y \in [z]_{\sim_R}$, it is $xSy$ or $ySx$, i.e. $S$ is a total order on each equivalence class of $\sim_R$.

(iv) Suppose $(S, R)$ is a Catalan pair. If $xSy$ and $x \not\sim_R y$, then there exists $a \in X$ such that $aRx$ and $aSy$.

(v) For all $x, y \in X$, it is $xRy$ iff $[x]_{\sim_R} [y]_{\sim_R}$ (where the extension of $\sim_R$ to sets has an obvious meaning).

**Proof.**

(i) Just observe that, if $x \sim_R y$, then $\overline{xRy}$ would imply $\overline{xRx}$, which is false.

(ii) Notice that, given that $x \sim_R y$, if $zRx$, then obviously $\overline{zRx}$, whence $\overline{zRy}$. If we had $yRz$, then, since $zRx$, it would also be $yRx$, which is impossible thanks to the preceding statement (i). The fact that $xRz$ implies $yRz$ can be dealt with analogously.

(iii) Obvious after (i).
(iv) From \( x \not\sim_R y \) it follows, by definition, that either there exists \( a \in X \) such that \( a \bar{R} x \) and \( a \not\bar{R} y \), or there exists \( b \in X \) such that \( b \bar{R} x \) and \( b \not\bar{R} y \). The second possibility cannot occur since, if such an element \( b \) existed, then, from the hypothesis \( xS_y \) and from (I), we would have \( x \not\bar{R} b \), a contradiction. Thus an element \( a \in X \) with the above listed properties exists. In particular, since \( a \not\bar{R} y \), it must be \( a \bar{S} y \). If we had \( y \bar{S} a \), then, from \( x \bar{S} y \), it would follow \( x \bar{S} a \), a contradiction. Therefore it must be \( a \bar{S} y \), as desired.

(v) Suppose that \( x \bar{R} y \). If \( a \not\sim_R x \), applying (ii) it follows that \( a \bar{R} y \). Now, if it is also \( b \not\sim_R y \), applying (ii) once more yields \( a \bar{R} b \), which implies the thesis. ■

Theorem 4.1 If \((S_1, R), (S_2, R)\) are two Catalan pairs on \(X\), then they are isomorphic.

Proof. From lemma 4.1(iii), each equivalence class of the relation \( \sim_R \) is linearly ordered by the order relations \( S_1 \) and \( S_2 \).

Define a function \( F \) mapping \( X \) into itself such that, if \( x \in X \) and there are exactly \( k \geq 0 \) elements in \([x]_{\sim_R}\) less than \( x \) with respect to the total order \( S_1 \), then \( F(x) \) is that element in \([x]_{\sim_R}\) having exactly \( k \) elements before it in the total order given by \( S_2 \).

It is trivial to see that \( F \) is a bijection. Since \( x \sim_R F(x) \), using lemma 4.1(v), we get that \( x \bar{R} y \) iff \( F(x) \bar{R} F(y) \).

To prove that \( xS_1 y \) implies \( F(x)S_2 F(y) \) it is convenient to consider two different cases. First suppose that \( x \sim_R y \); in this case our thesis directly follows from the definition of \( F \). On the other hand, if \( x \not\sim_R y \), using lemma 4.1(iv), there exists an element \( a \in X \) such that \( a \bar{R} x \) and \( a \bar{S}_1 y \). Thus, considering the Catalan pair \((S_2, R)\), it cannot be \( F(x) \bar{R} F(y) \), since this would imply (by lemma 4.1(v)) that \( x \bar{R} y \), against \( x \bar{S}_1 y \). Therefore it must be \( F(x) \bar{S}_2 F(y) \). More precisely, we get \( F(x)S_2 F(y) \), since, from \( F(y) \bar{S}_2 F(x) \bar{R} a \), we would derive \( F(y) \bar{R} a \) and so \( y \bar{R} a \), which is impossible. With an analogous argument, we can also prove that \( F(x)S_2 F(y) \) implies \( xS_1 y \), which concludes the proof that \( F \) is an isomorphism between \((S_1, R)\) and \((S_2, R)\). ■

For the rest of the paper, we set \( R(n) = \{ [X, R] \mid (\exists S)(S, R) \in C(n) \} \).

The posets \( [X, R] \in R(4) \) are those depicted in figure 3.

Among the possible 16 nonisomorphic posets on 4 elements, the two missing posets are those shown in figure 3. They are respectively the poset \( 2 + 2 \) (i.e. the direct sum of two copies of the 2-element chain) and the poset \( Z_4 \), called fence of order 4 (see, for instance, [C, MZ, S1]).

The rest of this section is devoted to proving that the absence of the two posets \( 2 + 2 \) and \( Z_4 \) is not an accident.
Proposition 4.2 If \([X, R] \in R(n)\), then \([X, R]\) does not contain any subposet isomorphic to \(2 + 2\) or \(Z_4\).

Proof. Let \((S, R) \in C(n)\) and suppose, ab absurdo, that \(2 + 2\) is a subposet of \([X, R]\). Then, denoting with \(x, z\) and \(y, t\) the minimal and maximal elements of an occurrence of \(2 + 2\) in \([X, R]\), respectively, and supposing that \(xRy\) and \(zRt\), we would have, for instance, \(tSxRy\). By proposition 2.3, since \(tRy\), it is \(ySt\). However, we also have \(ySzRt\) and \(yRt\), whence \(tSy\), which yields a contradiction with the previous derivation.

Similarly, suppose that \(Z_4\) is a subposet of \([X, R]\). Then, supposing that \(xRy, xRt\) and \(zRt\), we have \(zSxRy\), whence, by proposition 2.3, \(ySz\). However, it is also \(ySzRt\), which implies \(yRt\), and this is false. \(\blacksquare\)

We will now prove that the converse of the above proposition is also true, thus providing an order-theoretic necessary and sufficient condition for a strict order relation \(R\) to be the second component of a Catalan pair.

Proposition 4.3 Let \(R \in \mathcal{O}(X)\) such that \([X, R]\) does not contain subposets isomorphic to \(2 + 2\) or \(Z_4\). Then \([X, R] \in R(n)\).

Proof. Given \(X = \{x_1, \ldots, x_n\}\), we define a binary relation \(S = S(R)\) on \(X\) by making use of the equivalence relation \(\sim_R\) defined at the beginning of this section. More precisely:

- if \(x_i \sim_R x_j\) and \(i < j\), set \(x_iSx_j\);
- if \( x \sim_R y \) and \( x \not\sim_R y \), set:
  
  i) \( xSy \), when there exists \( z \in X \) such that \( z \not\sim_R x \) and \( z \not\sim_R y \);  
  
  ii) \( ySx \), when there exists \( z \in X \) such that \( z \not\sim_R x \) and \( z \not\sim_R y \).

We claim that \((S, R) \in \mathcal{O}(n)\).

It is trivial to show that axioms \((\text{tot})\) and \((\text{inters})\) in the definition of a Catalan pair are satisfied.

Next we show that axiom \((\text{comp})\) holds. Indeed, suppose that \( xSyRq \) and \( xRq \). From lemma \text{4.1} \((ii)\), it would follow that \( x \not\sim_R y \). Thus, from \( xSy \) and the definition of \( S \), we deduce that there is an element \( z \) such that \( z \not\sim_R x \) and \( z \not\sim_R y \). The reader can now check that the four elements \( x, y, q, z \) determine a subposet of \([X, R]\) isomorphic either to \( 2 + 2 \) or \( Z_4 \), which is not allowed.

Using an analogous argument, it can be shown that \( S \circ R^{-1} \subseteq R^{-1} \), fact that will be useful below.

Finally, it remains to prove axiom \((\text{ord} S)\), i.e. that \( S \in \mathcal{O}(X) \). The fact that \( S \) is irreflexive is evident from its definition. To prove the transitivity of \( S \), we first need to prove that, given \( x, y \in X \), the two relations \( xSy \) and \( ySx \) cannot hold simultaneously. Indeed, if \( x, y \in X \) were such that \( xSy \) and \( ySx \), then it could not be \( x \sim_R y \) and so, by definition, there would exist two elements \( z, q \in X \) such that \( z \not\sim_R x \), \( z \not\sim_R y \), \( q \not\sim_R x \) and \( q \not\sim_R y \). It is not difficult to prove that the four elements \( x, y, z, q \) have to be all distinct (using the irreflexivity of \( R \) and \( S \)). Now, if we consider the poset determined by these four elements, in all possible cases a forbidden poset comes out, and we have reached a contradiction. Now suppose to have \( xSySt \); we want to prove that necessarily \( xSt \). The cases in which we have \( x \sim_R y \) and/or \( y \sim_R t \) can be dealt with using the definition of \( S \). Moreover, if \( x \sim_R y \) and \( y \sim_R t \), let \( z, q \) such that \( z \not\sim_R x \), \( z \not\sim_R y \), \( q \not\sim_R x \) and \( q \not\sim_R y \). Thanks to the first part of this proof (namely axiom \((\text{comp})\) and the fact that \( S \circ R^{-1} \subseteq R^{-1} \)), from \( xSyRq \) it follows that \( q \not\sim_R x \). On the other hand, if we had \( x \not\sim_R t \), since it is \( x \not\sim_R y \) and \( t \not\sim_R y \), it would be \( tSy \) (by the definition of \( S \)), which is impossible since, by hypothesis, \( ySt \), and we have just shown that the last two relations lead to a contradiction. Therefore we must have \( x \not\sim_R t \), which, together with \( q \not\sim_R t \) and \( q \not\sim_R x \), implies that \( xSt \), as desired. \[ \]

In order to clarify the construction of \( S(R) \) given in the proof of proposition 4.3, consider the poset \( R \in \mathcal{R}(9) \) shown in figure 5(a). It is \( x_1 \sim_R x_2 \), hence \( x_1 S x_2 \). Similarly we get \( x_3 S x_6 \). Moreover, for any fixed \( i = 1, \ldots, 8 \), we have \( x_9 \sim_R x_i \), and there exists \( x_j, j \neq i \), such that \( x_i \not\sim_R x_j \), so we have \( x_i S x_9 \). Similarly we have \( x_2 S x_4, x_3 S x_4, x_7 S x_5, x_7 S x_6, x_8 S x_5, x_8 S x_6 \), and we finally obtain the Catalan pair \((S, R)\) represented by the matching depicted in figure 5(b).
Remark. Observe that, as a byproduct of the last proposition, we have found a presumably new combinatorial interpretation of Catalan numbers: $C_n$ counts nonisomorphic posets which are simultaneously $(2 + 2)$-free and $Z_4$-free.

**Open problem 1.** We have shown that $(S, R)$ is a Catalan pair if and only if $[X, R]$ does not contain neither $2 + 2$ nor $Z_4$. The class of $2 + 2$-free posets have been deeply studied, see for example [Fis] or the more recent paper [BMCDK]. What about $Z_4$-free posets?

**Open problem 2.** Can we define some interesting (and natural) partial order relation on the set $R(n)$? Maybe some of the combinatorial interpretations of Catalan pairs can help in this task.

### 4.2 Imposing some combinatorial conditions on the posets in $R(n)$

In this section we impose some conditions on the relation $R$ and provide the corresponding combinatorial descriptions in terms of noncrossing matchings and/or Dyck paths and/or 312-avoiding permutations.

a) *Connected posets.* First of all notice that a generic $[X, R] \in R(n)$ necessarily has at most one connected component of cardinality greater than...
than one (this follows at once from the poset avoidance conditions found in the previous section). It is not difficult to see that, in the interpretation by means of noncrossing matchings, the fact that \([X, R]\) is connected means that 1 and \(2n\) are not matched. From this observation it easily follows that \([X, R]\) corresponds to a non elevated Dyck path and to a 312-avoiding permutation not ending with 1. This also gives immediately the enumerations of the Catalan pairs \((S, R) \in \mathcal{C}(n)\) such that \([X, R]\) is connected. Indeed, elevated Dyck paths of semilength \(n\) are known to be enumerated by the sequence \(C_{n-1}\) of shifted Catalan numbers, whence we get immediately that the number of connected posets belonging to \(R(n)\) is given by \(c_n = C_n - C_{n-1}\) when \(n \geq 2\), whereas \(c_0 = c_1 = 1\). The resulting generating function is therefore

\[
\frac{1 - x + 2x^2 - (1 - x)\sqrt{1 - 4x}}{2x}.
\]

Figure 6: A connected poset \([X, R]\), and the corresponding perfect matching.

b) Lattices. In order to enumerate those posets of \(R(n)\) which are also lattices, it is convenient to interpret Catalan pairs as Dyck paths. The following proposition then holds (where \(U\) and \(D\) denote up and down steps of Dyck paths, respectively).

**Proposition 4.4** Let \([X, R] \in R(n)\) and \(P\) be its associated Dyck path. Then \([X, R]\) is a lattice if and only if \(P\) starts and ends with a peak and does not contain the pattern \(DDUU\).

**Proof.** The fact that \(P\) must have a peak both at the beginning and at the end stems from the fact that a finite lattice must have a minimum and a maximum. If \(P\) contains the pattern \(DDUU\), then denote by \(x, y, z, t\) the four tunnels associated with the four steps of the pattern. It is immediate to see that \(z\) and \(t\) are both sups of \(x\) and \(y\) in \([X, R]\), which implies that such a poset is not a lattice. Now suppose that \(P\) does not contain the pattern \(DDUU\). Given \(x, y \in X\) incomparable with respect to \(R\), then, in the associated path \(P\), \(x\) and \(y\) are represented by two tunnels lying one above the other (say, \(x\) above \(y\)). Consider the down step \(D_y\) belonging to \(y\). It is obvious that \(D_y\) is
not isolated, i.e. it is either followed or preceded by at least another
down step. Now take the first up step coming after $D_y$. Since $P$
avoidsthe pattern $DDUU$, such an up step must be followed by a down step,
thus originating a tunnel $z$. It is not difficult to show that $z$ is the least
upper bound of $x$ and $y$. Thus, since any two elements of $X$ have a
least upper bound, we can conclude that $[X, R]$ is a lattice, as desired.

\[\square\]

![Figure 7: A lattice $[X, R]$, and the corresponding perfect matching.](image)

As a consequence of the last proposition, we are now able to enumerate
Catalan pairs $(S, R)$ such that $[X, R]$ is a lattice. Indeed, the sequence
counting Dyck paths avoiding the pattern $DDUU$ is A025242 in [Sl]
(see also [STT]).

**Open problem 3.** It seems to be a quite difficult task to provide a
purely order-theoretic characterization of the lattices $[X, R]$ arising in
this way.

c) **Distributive lattices.** To understand when $R$ gives rise to a distribu-
tive lattice is undoubtedly a much easier task. Indeed, in order $[X, R]$
to be a distributive lattice, it is necessary that it does not contain the
two sublattices $M_3$ and $N_5$ [DP], shown in figure [8] This means that,
in the associated matching, at most two arches can be nested and no
consecutive sets of nested arches can occur. Equivalently, the associ-
ated Dyck path has height\(^1\) at most 2, and no consecutive factors of
height 2 can occur. Therefore, an obvious argument shows that the
sequence $d_n$ counting distributive lattices in $R(n)$ satisfies the recur-
rence $d_n = d_{n-1} + d_{n-3}$, with $d_0 = d_1 = d_2 = 1$, having generating
function $\frac{1}{1-x-x^2}$, whence $d_n = \sum_i \binom{n-2i}{i}$ (sequence A000930 in [Sl]). In

\(^1\)The *height* of a Dyck path is the maximum among the ordinates of its points.
this case, we can also give a structural characterization of distributive lattices in $R(n)$: they are all those expressible as

$$\bigoplus_{i=1}^{r} n_i \oplus 2^2 \oplus n_{r+1},$$

where $\oplus$ denotes the linear (or ordinal) sum of posets, $n_i$ is the $n$-element chain and $2^2$ is the Boolean algebra having 4 elements (see [DP] for basic notions and notations on posets).

Figure 8: The lattices $M_3$ and $N_5$, and the corresponding perfect matchings.

4.3 The poset defined by $S$

Similarly to what has been done for $R$, we can define the set $S(n) = \{[X, S] \mid (\exists R)(S, R) \in C(n)\}$. The posets in $S(n)$ have an interesting combinatorial characterization, which is described in the next proposition.

**Proposition 4.5** If $[X, S] \in S(n)$, then the Hasse diagram of $[X, S]$ is a forest of rooted trees, where the roots of the trees are the maximal elements of $S$ and $xSy$ if and only if $y$ is a descendant of $x$ in one of the tree of the forest.

**Proof.** First observe that, thanks to lemma 2.1, the poset $[X, S]$ has $k$ connected components, where $k$ is the number of its maximal elements. Now take $x, y$ belonging to the same connected component and suppose that $xSy$. We claim that the set of all lower bounds of $\{x, y\}$ is empty. Indeed, if we had $z$ such that $zSx$ and $zSy$, then, supposing (without lose of generality) that $xRy$, it would be $zRy$, a contradiction. Thus, the Hasse diagram of each connected component of $[X, S]$ is a direct acyclic graph, that is a tree, rooted at its maximum element, and this concludes our proof. ■

As a consequence of the previous proposition, we have the following result.

**Corollary 4.1** There is a bijection between $S(n)$ and the set of rooted trees with $n + 1$ nodes.
Proof. Just add to the Hasse diagram of each element \([X, S]\) of \(S(n)\) a new root, linking such a root with an edge to the maximum of each connected component.

Below the rooted tree on 6 nodes associated with \((X, S) \in S(5)\) is shown, where \(S = \{(x_2, x_1), (x_4, x_3), (x_5, x_3)\}\).

The above corollary implies that \(|S(n)|\) is given by the number of rooted trees having \(n + 1\) nodes, which is sequence A000081 in [Sl].

Unlike it happens with \(R\), the order relation \(S\) does not uniquely determine a Catalan pair. This should be clear by examining the following two perfect noncrossing matchings, which are associated with the same \(S\), but determine a different \(R\).

This fact is of course an obvious consequence of our last result, since Catalan pairs are enumerated by Catalan numbers. Recall that a rooted tree can be seen as a graph-isomorphism class of plane rooted trees. Since we have shown in section 3.3 that Catalan pairs can be interpreted by using plane rooted trees, it easily follows that, given \(S \in S(n)\), the set of Catalan pairs \((S, R)\) can be interpreted as the set of all plane rooted trees which are isomorphic (as graphs) to the Hasse diagram of \([X, S]\). Figure 9 gives an illustration of this situation, by showing the rooted tree \(T\) associated with a given \(S\) and all the plane rooted trees representing the associated Catalan pairs, together with the alternative representation as perfect noncrossing matchings.

5 Generalizations of Catalan pairs

In this section we see how a slight modification of the axioms defining Catalan pairs determines some combinatorial structures and number
sequences, mostly related with permutations. In particular, we focus our attention on axiom (comp).

We notice that axiom (comp) is the reason since Catalan pairs can be represented using perfect noncrossing matchings. If we relax such a condition, we are able to represent some classes of permutations which, in general, include 312-avoiding ones.

Consider all pairs of relations $(S, R)$ on a set $X$ satisfying axioms (ord $S$), (ord $R$), (tot) and (inters). In this situation, we call $(S, R)$ a factorial pair on $X$. The set of all factorial pairs on $X$ will be denoted $\mathcal{F}(X)$. As we did for Catalan pairs, we work up to isomorphism, and $\mathcal{F}(n)$ will denote the isomorphism class of factorial relations on a set $X$ of $n$ elements.

Each pair $(S, R) \in \mathcal{F}(X)$ can be graphically represented using perfect matchings, extending the encoding given in section 3.1. In the matching determined by a factorial pair, however, two distinct arches can cross, as shown in figure 10.

The interpretation of the first component of a factorial pair, $S$, is the same as for Catalan pairs, and corresponds to inclusion of arches. The second component $R$ still describes the reciprocal position of two arches but, more generally, we have to consider the reciprocal positions $l(x)$ (left) and $r(x)$ (right) of the two vertices of an arch $x$. Specifically, we have $xRy$ if and only if $l(x)$ lies on the left of $l(y)$ and $r(x)$ lies on the left of $r(y)$.

Example. Let $(S, R) \in \mathcal{F}(4)$ represented in figure 10. Using the notations of figure 10 on the set of arches $\{x, y, z, t, w\}$ we have $S = \{(z, x), (z, y), (z, t), (z, w), (t, x), (t, y)\}$ and $R =$.
\{(x, y), (x, w), (y, w), (t, w)\}.

It is clear that, for any set \(X\), \(\mathcal{C}(X) \subseteq \mathcal{F}(X)\). Moreover, using an obvious extension of the bijection given in section 3.1, it turns out that \(|\mathcal{F}(n)| = n!|\). More precisely, we have the following proposition.

**Proposition 5.1** Every factorial pair \((S, R)\) of size \(n\) can be uniquely represented as a permutation \(\pi \in S_n\).

**Proof.** Given \(\pi \in S_n\), just define \(S\) and \(R\) as in section 3.1. ■

Given a factorial pair \((S, R)\), we call the permutation \(\pi\) found in the above proposition its permutation representation. See again figure 10 for an example.

Now we come to the main point of the present section, and show how relaxing axiom (\text{comp}) naturally leads to a family of interesting combinatorial structures which, in some sense, interpolates between the analogous combinatorial interpretations of Catalan pairs and factorial pairs.

Denote by \(\mathcal{F}_{h,k}(X)\) the class of all pairs of relations \((S, R)\) on the set \(X\) satisfying axioms (\text{ord} \(S\)), (\text{ord} \(R\)), (\text{tot}), (\text{inters})\), and such that (\text{comp}) is replaced by the weaker axiom:

\[S^h \circ R^k \subseteq R \quad (\text{comp} (h, k)).\]

The next proposition (whose easy proof is left to the reader) illustrates how the sets \(\mathcal{F}_{h,k}(X)\) are related to Catalan and factorial pairs.

**Proposition 5.2**

\(i\) \(\mathcal{C}(X) = \mathcal{F}_{1,1}(X)\).

\(ii\) For all \(h\) and \(k\) we have that \(\mathcal{F}_{h,k}(X) \subseteq \mathcal{F}(X)\).

\(iii\) If \(a \leq b\), then \(\mathcal{F}_{a,k}(X) \subseteq \mathcal{F}_{b,k}(X)\) and \(\mathcal{F}_{h,a}(X) \subseteq \mathcal{F}_{h,b}(X)\).

Each element of the family \(\{\mathcal{F}_{h,k}(X) : h, k \geq 1\}\), where \(X\) is finite, can be characterized in terms of permutations avoiding a set of patterns. For example, consider the two families \(\mathcal{F}_{h,1}(X)\) and \(\mathcal{F}_{1,k}(X)\). The following two
propositions completely characterize them in terms of pattern avoiding permutations. The proofs of both propositions easily follow from the bijection given in proposition 5.1. In both propositions (as well as in the subsequent corollary) \( X \) denotes a set having \( n \) elements.

**Proposition 5.3** The permutation representation of \( F_{1,k}(X) \) is given by \( S_n((k + 2)12 \cdots k(k + 1)) \).

**Proposition 5.4** The permutation representation of \( F_{h,1}(X) \) is given by \( S_n(\pi_2, \pi_3, \ldots, \pi_{h+1}) \), where \( \pi_i \in S_{h+2} \), for every \( 2 \leq i \leq h + 1 \), and \( \pi_i \) is obtained from \( (h + 2)(h + 1) \cdots 21 \) by moving \( i \) to the rightmost position.

**Corollary 5.1** The cardinality of \( F_{2,1}(X) \) is given by the \( n \)-th Schröder number.

*Proof.* From the previous proposition we get that the permutation representation of \( F_{2,1}(X) \) is given by \( S_n(4312, 4213) \). In [K] it is shown that the above set of pattern avoiding permutations (or, more precisely, the one obtained by reversing both patterns) is counted by Schröder numbers. \( \blacksquare \)

**Open problem 4.** The enumeration of the sets \( F_{h,k}(X) \) has to be almost completely carried out, except for some specific cases. For instance, concerning \( F_{3,1}(X) \), proposition 5.4 states that its permutation representation is given by \( S_n(53214, 54213, 54312) \). The first terms of its counting sequence are 1, 2, 6, 24, 117, 652, 3988, . . . , which are not in [Sl].

6 Other kinds of generalizations

Among the possible combinatorial interpretations of Catalan pairs we have mentioned Dyck paths. In this section we show how some slight modifications of the axioms for Catalan pairs allow us to define different pairs of binary relations, which are naturally interpreted as some well-known families of lattice paths and then determine well known number sequences. We assume that the reader is familiar with the most common families of lattice paths, such as Schröder and Grand-Dyck paths.

As usual, we deal with pairs of binary relations \((S, R)\), both defined on a set \( X \) of cardinality \( n \) (this will still be expressed by saying that \((S, R)\) is a pair of size \( n \)). The axioms \( S \) and \( R \) are required to satisfy are the same as the axioms for Catalan pairs, except for the fact that we do not impose irreflexivity for \( S \). It is immediate to see that all the remaining axioms are coherent with our new assumption.
6.1 Unrestricted reflexivity

Let $\mathcal{U}(n)$ be the set of pairs of binary relations $(S, R)$ of size $n$, satisfying axioms (ord $R$), (tot), (inters), (comp), such that $S$ is a transitive relation and, as it was in the case of Catalan pairs, $S \cap S^{-1} = \emptyset$. Of course, since we are not imposing irreflexivity on $S$, given $x \in X$, we may have either $xSx$ or $x \not\in Sx$.

A possible combinatorial interpretation of the elements of $\mathcal{U}(n)$ can be obtained by means of a slight modification of the notion of a perfect non-crossing matching. Loosely speaking, we can introduce two different kinds of arches, namely solid and dotted arches, imposing that, when $xSx$, the arch corresponding to $x$ is dotted. These objects will be called two-coloured perfect noncrossing matchings (briefly, two-coloured matchings).

It is evident that, for any Catalan pair $(S, R) \in \mathcal{C}(n)$, we can define exactly $2^n$ different elements $(S', R') \in \mathcal{U}(n)$ with the property that

$$R = R', \quad \text{and} \quad S = S' \setminus \mathcal{D}.$$ 

Hence the number of elements of $\mathcal{U}(n)$ is $2^n C_n$, (sequence A052701 in [Sl]).

We obtain some more interesting combinatorial situations by giving specific axioms for the behavior of $S$ with respect to the diagonal $\mathcal{D}(X)$.

6.2 Grand-Dyck paths and central binomial coefficients

Recall that a Grand-Dyck path of semi-length $n$ is a lattice path from $(0, 0)$ to $(2n, 0)$ using up $(1, 1)$ and down $(1, -1)$ steps. The number of Grand-Dyck paths of semi-length $n$ is given by the central binomial coefficient $\binom{2n}{n}$ [St1]. We can represent a Grand-Dyck path by using a two-coloured matching, with the convention that for the parts of the path lying above the $x$-axis we use solid arches, whereas for the parts of the paths lying below the $x$-axis we use dotted arches (see figure 11).

![Figure 11: A Grand-Dyck path and its representation as a two-coloured matching.](image)
Of course, not every two-coloured matching represents a Grand-Dyck path. Indeed, we must add the following constraint: if an arch $x$ is contained into an arch $y$, then $x$ and $y$ are either both solid or both dotted.

In order to give a correct axiomatization of what can be called a Grand-Dyck pair, just add to the axioms for $\mathcal{U}(n)$ the following one:

- if $xSy$, then $xSx$ if and only if $ySy$. (choose)

Denote by $G(n)$ the resulting set of pairs of binary relations, called Grand-Dyck pairs of size $n$. It is evident that, interpreting the relations $S$ and $R$ as in the case of (one-coloured) matchings, and adding the convention that, if $xSx$, then $x$ is a dotted arch, we get precisely the set of two-coloured matchings.

For instance, referring to the example in figure 11, $R$ and $S$ are as follows:

\begin{align*}
R &= \{(x, y), (x, u), (x, v), (x, z), (y, z), (y, w), (u, v), (u, z), (v, z), (v, w)\}, \\
S &= \{(u, y), (v, y), (w, z), (u, u), (v, v), (y, y)\}.
\end{align*}

Axiom (choose) can be reformulated in a more elegant way.

**Proposition 6.1** Let $D(S) = \{(x, x) \in X^2 \mid xSx\}$. Then axiom (choose) is equivalent to

$$D(S) \circ S = S \circ D(S).$$

**Proof.** Using (choose), it is easy to see that $x(D(S) \circ S)y$ if and only if $xD(S)xSy$ if and only if $xSyD(S)y$ if and only if $x(S \circ D(S))y$. Conversely, suppose that $xSy$. If $xSx$, then $x(D(S) \circ S)y$. However, by hypothesis, this is equivalent to $x(S \circ D(S))y$, whence $ySy$. ■

### 6.3 Schröder paths and Schröder numbers

Recall that a Schröder path of semi-length $n$ is a path from $(0, 0)$ to $(2n, 0)$ using up steps $(1, 1)$, down steps $(1, -1)$, and horizontal steps of length two $(2, 0)$, and remaining weakly above the $x$-axis.

We can represent Schröder paths by using two-coloured matchings as well. We can essentially adopt the same representation as for Dyck paths, just using dotted arches to represent horizontal steps. According to such a representation, dotted arches can be contained into other arches, but they cannot contain any arch (see figure 12). This condition precisely identifies those two-coloured matchings representing Schröder paths among all two-coloured matchings.

Let $S(n) \subseteq \mathcal{U}(n)$ denote the set of pairs of relations $(S, R)$ on $X$ of cardinality $n$ satisfying the following axiom:
• if $x S x$, then $x$ is a minimal element for $S$. (min)

Since each combinatorial interpretations of this kind of pairs of relations
is counted by Schröder numbers, we call them Schröder pairs of size $n$.

Also for axiom (min) an equivalent formulation can be given which is
analogous to that of proposition 6.1 and whose proof is left to the reader:

$$S \circ D(S) = D(S).$$

Notice that, in this case, $S$ does not commute with $D(S)$; more precisely,

$$S \circ D(S) \subseteq D(S) \circ S.$$

For example, referring to the matching representation of the Schröder
path given in figure 12 we have $y(D(S) \circ S)x$, but $(y, x) \notin S \circ D(S)$.

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