Calculation of temperature oscillations in thermal layer of regular solids at unsteady coefficient of heat transfer

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Abstract. The temperature oscillations in thermal layer of regular solids, caused by periodic oscillations of the ambient temperature and heat transfer coefficient, are analysed. A half-space, a plate, a space with cylindrical channel, a cylinder, a space with spherical cavity are considered as solids. Exact solutions of the corresponding cyclic problems of heat conduction in the complex and real forms of trigonometric Fourier series are obtained. Methods of simplified solution of heat conduction problem based on boundary condition modification are considered. Comparison of exact solutions with approximate ones is carried out and the admissibility of the proposed approximate solutions is shown.

1. Problem definition
In a number of technical devices and structures cyclic heat transfer between their elements and external fluid medium with periodically changing temperature takes place, therefore in solids temperature oscillations occur; this affects both operating efficiency of devices and their durability as temperature oscillations cause thermocyclic stresses. Processes of this kind take place in rotor blades of gas turbines, in internal combustion engines parts, heat exchangers of regenerative type, in elements of buildings and Earth crust which are periodically heated by the Sun.

Solutions of cyclic classical problems of heat conduction based on Fourier phenomenology are given in [1,2]. Modern researches [3–5] of cyclic processes of heat conduction are based mostly on more difficult models of heat transfer in a solid. At the same time for practical problems Fourier classical model of heat conduction usually provides the acceptable level of results accuracy what is possible to appreciate by its use in applied researches [6–9]. One of such problems is considered in this work.

Special feature of cyclic processes of heat conduction is that in the absence of any internal periodically operating heat sources the temperature oscillations decay with removal from a surface of a solid. High-frequency temperature oscillations, characterized by small oscillation range and decay in the surface layer of a solid called thermal layer, are of particular interest. To calculate the temperature oscillations and thermocyclic stresses in thermal layer, we can solve the problems of heat conduction and thermoelasticity in one-dimensional approximation. At the same time, depending on a shape of a solid in the neighborhood of the considered surface point, it is necessary to use one of the following models: half-space, cylinder, space with the cylindrical channel, sphere, and space with a spherical cavity. Heat transfer between a solid and fluid can be described by means of Newton-Richmann law. It includes heat-transfer coefficient which in real processes depends on time that creates difficulties of fundamental nature at the analytical solution of the heat conduction problem [10]. For cyclic processes
of heat conduction it is possible to define the stable field of temperature by means of R.S. Minasyan’s method [11]. The received analytical solution at the same time is rather cumbersome and unhandy for use therefore it is advisable to receive an approximate solution of the heat transfer problem. In the presented work several approaches connected with modification of a boundary condition are considered. The first approach is connected with replacement of unsteady coefficient of heat transfer with its period average value. The second approach which was offered by the author earlier in [12] assumes replacement of unsteady coefficient of heat transfer with equivalent constant one. The third approach which was offered by the author earlier in [13] assumes replacement of the third type boundary condition with the second type boundary condition. All approaches are used for calculation of temperature oscillations in six canonical solids – a half-space, a plate, a space with cylindrical channel, a cylinder, a space with a spherical cavity, a sphere.

2. Solution of the problem

For convenience we will set correspondence for a half-space $j = 0$, for a plate $j = 1$, for a space with cylindrical channel $j = 2$, for a cylinder $j = 3$, for a space with a spherical cavity $j = 4$, for a sphere $j = 5$. The steady-state cyclic temperature in all bodies satisfies the generalized heat conduction equation

$$\frac{\partial T}{\partial t} = a_1 \frac{1}{\eta^m} \frac{\partial}{\partial \eta} \left( \eta^m \frac{\partial T}{\partial \eta} \right), \eta \in \mathcal{D}_\eta, t > -\infty, j = 0,5, \quad (1)$$

periodicity condition

$$T(\eta, t + \mathcal{T}) = T(\eta, t), \eta \in \mathcal{D}_\eta, t > -\infty, j = 0,5, \quad (2)$$

the third type boundary condition on the solid surface

$$-\kappa \lambda \frac{\partial T(\eta_w, t)}{\partial \eta} = \alpha [T(\eta_w, t) - T_0], t > -\infty, j = 0,5, \quad (3)$$

one of the conditions

$$\frac{\partial T(0, t)}{\partial \eta} = 0, t > -\infty, j = 1,3,5, \quad (4)$$

$$\lim_{\eta \to +\infty} \left( \eta^m \frac{\partial T}{\partial \eta} \right) = 0, t > -\infty, j = 0,2,4. \quad (5)$$

Here $T = T(\eta, t)$ is a temperature field of a solid, $K; \eta$ is generalized space variable, $m$; $t$ is time, $s$; $m$ is coordinate system number; $D_\eta$ is combination of inner points of region; $\mathcal{T}$ is the cycle period, $s$; $\eta_w = R \text{ sign } j$ is value of $\eta$ on region boundary; $R$ is characteristic linear dimension of region, $m$; $\kappa = (-1)^{j+1}$; $a$ is thermal diffusivity, $m^2/s$; $\lambda$ is heat conduction coefficient, $W/(m^2 \cdot K)$; $T_f = T_f(t)$ is fluid temperature, $K$; $\alpha = \alpha(t) > 0$ is coefficient of heat transfer, $W/(m^2 \cdot K)$. By statement of the problem the boundary functions are confined and periodic:

$$T_f(t + \mathcal{T}) = T_f(t), \alpha(t + \mathcal{T}) = \alpha(t), t > -\infty.$$  

The condition (4) at $j = 1$ expresses symmetry of the temperature field in the case of $x = 0$, and at $j = 3,5$ expresses boundedness of temperature. The condition (5) expresses equality to zero of the heat flow through a coordinate surface which is infinitely far from the origin of coordinates. Key parameters of the problem for regions of computation are specified in table 1, where $x$ is the Cartesian coordinate, $m$; $\rho$ is polar radius, $m$; $r$ is radial coordinate, $m$.

Table 1. Key parameters of the problem (1) – (5).

| $j$  | 0   | 1   | 2   | 3   | 4   | 5   |
|------|-----|-----|-----|-----|-----|-----|
| $m$  | 0   | 0   | 1   | 1   | 2   | 2   |
| $\eta$ | $x$ | $x$ | $\rho$ | $\rho$ | $r$ | $r$ |
| $\eta_w$ | 0   | $R$ | $R$ | $R$ | $R$ | $R$ |
| $\mathcal{D}_\eta$ | $(0, +\infty)$ | $[0, R)$ | $(R, +\infty)$ | $(0, R)$ | $(R, +\infty)$ | $[0, R)$ |
| $\kappa$ | 1  | -1  | -1  | 1   | -1  | 1   |
For convenience we will proceed to nondimensional variables

\[ \hat{\eta} = \frac{\eta}{\chi}, \hat{\eta}_w = \frac{\eta_w}{\chi}, \hat{\chi} = \omega t, \hat{R} = \frac{R}{\chi}, \hat{\Theta} = \frac{T - \langle T_f \rangle}{\Delta T_f}, \hat{\Theta}_f = \frac{T_f - \langle T_f \rangle}{\Delta T_f}, \text{Bi} = \frac{\alpha \chi}{\lambda}. \]

Taking them into account the boundary value problem (1) – (5) will become:

\[
\frac{\partial \hat{\Theta}}{\partial \hat{t}} = \frac{1}{\hat{\eta}^m} \frac{\partial}{\partial \hat{\eta}} \left( \hat{\eta}^m \frac{\partial \hat{\Theta}}{\partial \hat{\eta}} \right), \hat{\eta} \in D_\eta, \hat{t} > -\infty, j = 0, 5; \tag{6}
\]

\[
\hat{\Theta}(\hat{\eta}, \hat{t} + 2\pi) = \hat{\Theta}(\hat{\eta}, \hat{t}), \hat{\eta} \in D_\eta, \hat{t} > -\infty, j = 0, 5; \tag{7}
\]

\[
-\kappa \frac{\partial \hat{\Theta}(\hat{\eta}_w, \hat{t})}{\partial \hat{\eta}} = \text{Bi} [\hat{\Theta}(\hat{\eta}_w, \hat{t}) - \hat{\Theta}_f], \hat{t} > -\infty, j = 0, 5; \tag{8}
\]

\[
\frac{\partial \hat{\Theta}(0, \hat{t})}{\partial \hat{\eta}} = 0, \hat{t} > -\infty, j = 1, 3, 5; \tag{9}
\]

\[
\lim_{\hat{\eta} \to +\infty} \left( \hat{\eta}^m \frac{\partial \hat{\Theta}}{\partial \hat{\eta}} \right) = 0, \hat{t} > -\infty, j = 0, 2, 4. \tag{10}
\]

Here \( \chi = \sqrt{a / \omega} \) is the characteristic linear scale of process, \( m; \omega = 2\pi / T \) is the angular frequency of process, \( s^{-1}; \Delta T_f \) is range of fluid temperature, \( K; \langle T_f \rangle \) is period average fluid temperature, \( K \).

As at solving of the problem the theory of Fourier series will be used, we will give the main designations. Periodic function with the \( 2\pi \) period \( f = f(\hat{t}) \) meeting Dirichlet conditions can be expanded into a trigonometric Fourier series. Complex form of Fourier series is

\[
f = \sum_{n=-\infty}^{+\infty} c^n f e^{i n \hat{t}},
\]

where \( i = \sqrt{-1} \) is imaginary unit;

\[
c^n f = \frac{1}{2\pi} \int_{0}^{2\pi} f e^{-i n \hat{t}} d\hat{t}, n \in \mathbb{Z};
\]

\( \mathbb{Z} \) is set of integers. Real form of Fourier series

\[
f = \sum_{n=0}^{+\infty} \kappa_n \left[ A^n f \cos(n\hat{t}) + B^n f \sin(n\hat{t}) \right],
\]

where \( \kappa_n = (1 + \delta_{0,n})^{-1}; \delta_{i,j} \) is Kronecker delta;

\[
A^n f = \frac{1}{\pi} \int_{0}^{2\pi} f \cos(n\hat{t}) d\hat{t}, B^n f = \frac{1}{\pi} \int_{0}^{2\pi} f \sin(n\hat{t}) d\hat{t}, n \in \mathbb{Z}.
\]

It is obvious that

\[
c^n f = A^n f - i B^n f, A^n f = A^n f, B^n f = -A^n f, B^n f = -i A^n f, n \in \mathbb{Z}.
\]

We will designate a period average value of function

\[
\langle f \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} f d\hat{t},
\]

function range

\[
\Delta f = \max_{0 \leq \hat{t} < 2\pi} f - \min_{0 \leq \hat{t} < 2\pi} f.
\]

It is convenient to seek the solution of the boundary problem (6) – (10) in the complex form of trigonometric Fourier series.
It is easy to be convinced by direct substitution that function of the (11) form meets the condition (7). Having used series (11) in (6), (8) – (10), we will obtain a boundary problem

\[
\frac{1}{\hat{\eta}^m} \frac{d}{d\hat{\eta}} \left( \hat{\eta}^m \frac{dC_n^\Theta}{d\hat{\eta}} \right) - inC_n^\Theta = 0, \hat{\eta} \in D_{\hat{\eta}}, n \in \mathbb{Z}, j = 0,5; 
\]

\[
-\kappa \frac{dC_n^\Theta(\hat{\eta}_w)}{d\hat{\eta}} = \sum_{k=-\infty}^{k=+\infty} C_{n-k}^\Theta(\hat{\eta}_w) - C_{n}^\Theta, n \in \mathbb{Z}, j = 0,5; 
\]

\[
\frac{dC_n^\Theta(0)}{d\hat{\eta}} = 0, n \in \mathbb{Z}, j = 1,3,5; 
\]

\[
\lim_{\eta \to \pm \infty} \left( \hat{\eta}^m \frac{dC_n^\Theta}{d\hat{\eta}} \right) = 0, n \in \mathbb{Z}, j = 0,2,4. 
\]

It is not difficult to obtain the solution (12) – (15) in a form

\[
C_n^\Theta = C_n^{\bar{\Theta}}(\phi_n + i \text{sign} n \psi_n), 
\]

where \( \phi_n = \phi_n(\eta), \psi_n = \psi_n(\eta) \), and constants of integration are found from the infinite system of the linear algebraic equations with an infinite number of variables

\[
\sum_{k=-\infty}^{k=+\infty} [\kappa (\Lambda_k + i \text{sign} k \Omega_k) \delta_{n,k} + C_{n-k}^{\text{Bi}}(\Phi_k + i \text{sign} k \Psi_k)] C_n^{\bar{\Theta}} = C_n^{\text{Bi}\bar{\Theta}}, n \in \mathbb{Z}, 
\]

which can be solved approximately by a reduction method. Hence at \( \text{Bi} = \text{const} \) we find explicit dependences

\[
\tilde{C}_n^{\Theta} = C_n^{\bar{\Theta}} \left[ (\Phi_n + \kappa \Lambda_n / \text{Bi}) + i \text{sign} n (\Psi_n + \kappa \Omega_n / \text{Bi}) \right]^{-1}, n \in \mathbb{Z}. 
\]

Here

\[
\Phi_n = \phi_n(\hat{\eta}_w), \Psi_n = \psi_n(\hat{\eta}_w), \Lambda_n = \frac{d\phi_n(\hat{\eta}_w)}{d\hat{\eta}}, \Omega_n = \frac{d\psi_n(\hat{\eta}_w)}{d\hat{\eta}}, n \in \mathbb{Z}; 
\]

\[
\phi_0 = 1, \psi_0 = 0, j = 0,5; 
\]

\[
\phi_n = e^{-\hat{\eta} \sqrt{|n|/2}} \cos \left( \hat{\eta} \sqrt{|n|/2} \right), \psi_n = -e^{-\hat{\eta} \sqrt{|n|/2}} \sin \left( \hat{\eta} \sqrt{|n|/2} \right), |n| \in \mathbb{N}, j = 0; 
\]

\[
\phi_n = \text{ch} \left( \hat{\eta} \sqrt{|n|/2} \right) \cos \left( \hat{\eta} \sqrt{|n|/2} \right), \psi_n = \text{sh} \left( \hat{\eta} \sqrt{|n|/2} \right) \sin \left( \hat{\eta} \sqrt{|n|/2} \right), |n| \in \mathbb{N}, j = 1; 
\]

\[
\phi_n = \text{ker} \left( \hat{\eta} \sqrt{|n|} \right), \psi_n = \text{kel} \left( \hat{\eta} \sqrt{|n|} \right), |n| \in \mathbb{N}, j = 2; 
\]

\[
\phi_n = \text{ber} \left( \hat{\eta} \sqrt{|n|} \right), \psi_n = \text{bei} \left( \hat{\eta} \sqrt{|n|} \right), |n| \in \mathbb{N}, j = 3; 
\]

\[
\phi_n = \frac{e^{-\hat{\eta} \sqrt{|n|/2}} \cos \left( \hat{\eta} \sqrt{|n|/2} \right)}{\hat{\eta} \sqrt{|n|}}, \psi_n = -\frac{e^{-\hat{\eta} \sqrt{|n|/2}} \sin \left( \hat{\eta} \sqrt{|n|/2} \right)}{\hat{\eta} \sqrt{|n|}}, |n| \in \mathbb{N}, j = 4; 
\]

\[
\phi_n = \frac{\text{sh} \left( \hat{\eta} \sqrt{|n|/2} \right) \cos \left( \hat{\eta} \sqrt{|n|/2} \right)}{\hat{\eta} \sqrt{|n|}}, \psi_n = \frac{\text{ch} \left( \hat{\eta} \sqrt{|n|/2} \right) \sin \left( \hat{\eta} \sqrt{|n|/2} \right)}{\hat{\eta} \sqrt{|n|}}, |n| \in \mathbb{N}, j = 5; 
\]

\( \mathbb{N} \) is set of positive integers; \( \text{ber} \hat{\eta}, \text{bei} \hat{\eta}, \text{ker} \hat{\eta}, \text{kei} \hat{\eta} \) are Kelvin functions.

From a complex form of Fourier series (11) it is possible to proceed to a real one

\[
\tilde{\Theta} = \sum_{n=0}^{n=\infty} \kappa_n [A_n^\Theta \cos(n\tilde{t}) + B_n^\Theta \sin(n\tilde{t})], 
\]

where

\[
A_n^\Theta = \text{Re} A_n \phi_n + \text{Im} B_n \psi_n, B_n^\Theta = \text{Re} B_n \phi_n - \text{Im} A_n \psi_n, n \in \mathbb{Z}, 
\]
and constants of integration are found from the infinite system of the linear algebraic equations with an infinite number of variables

\[
\sum_{k=0}^{+\infty} \left\{ \left[ \kappa \Delta_k \delta_{n,k} + \kappa'_k \left( A_{k,n}^{Bi} + A_{k,-n}^{Bi} \right) \Phi_k - \kappa'_k \left( B_{k,n}^{Bi} + B_{k,-n}^{Bi} \right) \Psi_k \right] \bar{A}_k^\mu \right. \\
\left. + \left[ \kappa \Omega_k \delta_{n,k} + \kappa'_k \left( B_{k,n}^{Bi} + B_{k,-n}^{Bi} \right) \Phi_k + \kappa'_k \left( A_{k,n}^{Bi} + A_{k,-n}^{Bi} \right) \Psi_k \right] \bar{B}_k^\mu \right) = A_n^{Bi \bar{\theta}_f}, \quad n \in \mathbb{N}_0;
\]

\[
\sum_{k=0}^{+\infty} \left\{ \left[ -\kappa \Lambda_k \delta_{n,k} + \kappa'_k \left( B_{k,n}^{Bi} - B_{k,-n}^{Bi} \right) \Phi_k + \kappa'_k \left( A_{k,n}^{Bi} - A_{k,-n}^{Bi} \right) \Psi_k \right] \bar{A}_k^\mu \right. \\
\left. + \left[ \kappa \Lambda_k \delta_{n,k} - \kappa'_k \left( A_{k,n}^{Bi} - A_{k,-n}^{Bi} \right) \Phi_k + \kappa'_k \left( B_{k,n}^{Bi} - B_{k,-n}^{Bi} \right) \Psi_k \right] \bar{B}_k^\mu \right) = B_n^{Bi \bar{\theta}_f}, \quad n \in \mathbb{N}.
\]

Here \( \kappa'_n = \kappa_n / 2 \); \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \). At \( Bi = \text{const} \) we have

\[
\bar{A}_n = A_0^{\bar{\theta}_f}, \quad \bar{B}_n = 0, \quad A_n^{\bar{\theta}} = \frac{(\Phi_n + \kappa \Lambda_n / Bi) A_n^{\bar{\theta}_f} - (\Psi_n + \kappa \Omega_n / Bi) B_n^{\bar{\theta}_f}}{(\Phi_n + \kappa \Lambda_n / Bi)^2 + (\Psi_n + \kappa \Omega_n / Bi)^2},
\]

\[
B_n^{\bar{\theta}} = \frac{(\Phi_n + \kappa \Lambda_n / Bi) B_n^{\bar{\theta}_f} + (\Psi_n + \kappa \Omega_n / Bi) A_n^{\bar{\theta}_f}}{(\Phi_n + \kappa \Lambda_n / Bi)^2 + (\Psi_n + \kappa \Omega_n / Bi)^2}, \quad n \in \mathbb{N}_0.
\]

Hence it follows that \( \langle \bar{\theta} \rangle = 0 \) at \( Bi = \text{const} \).

For approximate solution of the problem the temperature field is divided into a constant component \( \langle \Theta \rangle \) and an oscillating component \( \vartheta = \Theta - \langle \Theta \rangle \) \([12,13]\). At first the constant component of the temperature field is defined in consequence of the boundary problem solution

\[
\frac{1}{\eta^n} \frac{d}{d\eta} \left( \eta^n \frac{d\langle \Theta \rangle}{d\eta} \right) = 0, \quad \eta \in \mathcal{D}_\eta, \quad j = 0,5; \quad (16)
\]

\[
-\kappa \frac{d\langle \Theta \rangle(\eta_w)}{d\eta} = \langle Bi \rangle \left[ \langle \Theta \rangle(\eta_w) - \langle \Theta_f \rangle \right], \quad j = 0,5; \quad (17)
\]

\[
\frac{d\langle \Theta \rangle(0)}{d\eta} = 0, \quad j = 1,3,5; \quad (18)
\]

\[
\lim_{\eta \to +\infty} \left( \eta^n \frac{d\langle \Theta \rangle}{d\eta} \right) = 0, \quad j = 0,2,4. \quad (19)
\]

Here

\[
\langle \Theta_f \rangle = \int_0^{2\pi} Bi \bar{\Theta}_f d\xi / \int_0^{2\pi} Bi d\xi.
\]

From (16) – (19) it follows that \( \langle \bar{\theta} \rangle = \langle \bar{\theta}_f \rangle \). Then the oscillating component of the temperature field is defined in consequence of the boundary problem solution

\[
\frac{\partial \vartheta}{\partial \xi} = \frac{1}{\eta^n} \frac{\partial}{\partial \eta} \left( \eta^n \frac{\partial \vartheta}{\partial \eta} \right), \quad \eta \in \mathcal{D}_\eta, \xi > -\infty, \quad j = 0,5; \quad (20)
\]

\[
\vartheta(\eta, \xi + 2\pi) = \vartheta(\eta, \xi), \quad \eta \in \mathcal{D}_\eta, \xi > -\infty, \quad j = 0,5; \quad (21)
\]

\[
\frac{\partial \vartheta(0, \xi)}{\partial \eta} = 0, \quad \xi > -\infty, \quad j = 1,3,5; \quad (22)
\]

\[
\lim_{\eta \to +\infty} \left( \eta^n \frac{\partial \vartheta}{\partial \eta} \right) = 0, \quad \xi > -\infty, \quad j = 0,2,4. \quad (23)
\]

in which one of the boundary conditions on the solid surface is established as

\[
-\kappa \frac{\partial \vartheta(\eta_w, \xi)}{\partial \eta} = \langle Bi \rangle [\vartheta(\eta_w, \xi) - \vartheta_f], \quad \xi > -\infty, \quad j = 0,5; \quad (24)
\]
\[ -\kappa \frac{\partial \hat{\theta} (\hat{\eta}, \hat{t})}{\partial \hat{\eta}} = B_i \left[ \hat{\theta} (\hat{\eta}, \hat{t}) - \hat{\theta}_f \right], \quad \hat{t} > -\infty, \quad j = 0,5; \quad (25) \]

\[ -\kappa \frac{\partial \hat{\theta} (\hat{\eta}, \hat{t})}{\partial \hat{\eta}} = B_i (\hat{\theta}_f)_{B_i} - \hat{\theta}_f), \quad \hat{t} > -\infty, \quad j = 0,5. \quad (26) \]

Here

\[ B_i e = \int_0^{2\pi} B_i (\hat{\theta}_f - (\hat{\theta}_f)_{B_i}) \hat{\theta}_f \, d\hat{t} \int_0^{2\pi} \hat{\theta}_f^2 \, d\hat{t}. \]

Solution (20) – (26) has an expanded form

\[ \hat{\theta} = \sum_{n=-\infty}^{+\infty} \hat{C}_n^\theta (\phi_n + i \text{ sign } n \psi_n) = \sum_{n=0}^{+\infty} \kappa_n \left[ (\hat{A}_n^\theta \phi_n + \hat{B}_n^\theta \psi_n) \cos(n\hat{t}) + (\hat{B}_n^\theta \phi_n - \hat{A}_n^\theta \psi_n) \sin(n\hat{t}) \right], \]

where \( \hat{A}_0^\theta = \hat{B}_0^\theta = \hat{C}_0^\theta = 0; \) at boundary condition (24)

\[ \hat{C}_n^\theta = C_n^\theta / \left[ ((\phi_n + \kappa \Lambda_n / (\langle Bi \rangle)) + i \text{ sign } n (\Psi_n + \kappa \Omega_n / (\langle Bi \rangle)) \right]^{-1}, \quad |n| \in \mathbb{N}, \]

\[ \hat{A}_n^\theta = (\phi_n + \kappa \Lambda_n / (\langle Bi \rangle)) A_{n}^f = (\Psi_n + \kappa \Omega_n / (\langle Bi \rangle) B_{n}^f + (\Psi_n + \kappa \Omega_n / (\langle Bi \rangle))^2, \]

\[ B_n^\theta = (\phi_n + \kappa \Lambda_n / (\langle Bi \rangle)) B_{n}^f + (\Psi_n + \kappa \Omega_n / (\langle Bi \rangle)) A_{n}^f / (\phi_n + \kappa \Lambda_n / (\langle Bi \rangle))^2 + (\Psi_n + \kappa \Omega_n / (\langle Bi \rangle))^2, \quad n \in \mathbb{N}; \]

at boundary condition (25)

\[ \hat{C}_n^\theta = C_n^\theta / \left[ ((\phi_n + \kappa \Lambda_n / (\langle Bi \rangle)) + i \text{ sign } n (\Psi_n + \kappa \Omega_n / (\langle Bi \rangle)) \right]^{-1}, \quad |n| \in \mathbb{N}, \]

\[ \hat{A}_n^\theta = (\phi_n + \kappa \Lambda_n / (\langle Bi \rangle)) A_{n}^f + (\Psi_n + \kappa \Omega_n / (\langle Bi \rangle) B_{n}^f / (\phi_n + \kappa \Lambda_n / (\langle Bi \rangle))^2 + (\Psi_n + \kappa \Omega_n / (\langle Bi \rangle))^2, \]

\[ B_n^\theta = (\phi_n + \kappa \Lambda_n / (\langle Bi \rangle)) B_{n}^f + (\Psi_n + \kappa \Omega_n / (\langle Bi \rangle)) A_{n}^f / (\phi_n + \kappa \Lambda_n / (\langle Bi \rangle))^2 + (\Psi_n + \kappa \Omega_n / (\langle Bi \rangle))^2, \quad n \in \mathbb{N}; \]

at boundary condition (26)

\[ \hat{C}_n^\theta = -\kappa \left( C_n^\theta (\hat{\theta}_f)_{Bi} - C_n^\theta (\hat{\theta}_f)_{Bi} \right) / \left( \Lambda_n + i \text{ sign } n \Omega_n \right)^{-1}, \quad |n| \in \mathbb{N}, \]

\[ \hat{A}_n^\theta = -\kappa / \left( \Lambda_n (A_n^\theta (\hat{\theta}_f)_{Bi} - A_n^\theta (\hat{\theta}_f)_{Bi}) - \Omega_n (B_n^\theta (\hat{\theta}_f)_{Bi} - B_n^\theta (\hat{\theta}_f)_{Bi}) \right), \]

\[ B_n^\theta = -\kappa / \left( \Lambda_n (B_n^\theta (\hat{\theta}_f)_{Bi} - B_n^\theta (\hat{\theta}_f)_{Bi}) + \Omega_n (A_n^\theta (\hat{\theta}_f)_{Bi} - A_n^\theta (\hat{\theta}_f)_{Bi}) \right), \quad n \in \mathbb{N}. \]

The solution of the problem under the boundary condition (BC) (24) is the simplest one, however, as it will be shown further, its accuracy is not always acceptable.

### 3. Examples

To illustrate the use of the submitted solutions, calculations for two important cases of cyclic heat transfer were carried out. In the first case harmonic temperature oscillations of fluid and heat transfer take place:

\[ \hat{\theta}_f = \frac{1}{2} \cos \hat{t}, \quad Bi = \langle Bi \rangle (1 + \zeta \cos \hat{t}). \]

These conditions of heat transfer are generalization of the simplest cyclic conditions of heat transfer

\[ \hat{\theta}_f = \frac{1}{2} \cos \hat{t}, \quad Bi = \text{ const} \]

for a case of an unsteady heat transfer. In the second case the body consistently interacts with two fluid mediums with various heat transfer intensity when
\[ \hat{\theta}_f = \frac{1}{2} \text{sign}(\sin \hat{t}), \quad \text{Bi} = \langle \text{Bi} \rangle [1 - \zeta \text{sign}(\sin \hat{t})]. \]

Here \( \zeta \in [0, 1] \) is the parameter, which determines degree of heat transfer unsteadiness. Calculations were carried out at parameters \( \langle \text{Bi} \rangle = 0.1 \) and \( \hat{R} = 4 \). The choice of the specified parameters is caused by the fact that by their applying the temperature oscillations take place, which can be considered as small from the point of view of given approximate methods of cyclic problem of heat conduction solution, but are rather big to be of practical interest. Besides, the sizes of solids at \( \hat{R} = 4 \) are rather big for thermal layer formation in them, but the curvature of boundary surfaces is not small enough to have no effect on temperature oscillations in a thermal layer.

Temperature oscillations on the solid surface are of the greatest interest in practice. We will consider further an oscillating component of the temperature field on the solid surface \( \theta_w = \theta(\hat{t}_w, \hat{t}) \). In figure 1 temperature oscillations on the half-space surfaces, calculated by means of exact and approximate solutions of heat conduction problem, are shown. It follows that at harmonic oscillations of fluid temperature and heat transfer the approximate solutions to the problem with BC (24) and (25) describe oscillations of body temperature badly. This is explained by the fact that, at constant heat transfer coefficient, the solution of the heat conduction problem can reproduce only the lowest harmonic, although the harmonic with a multiplicity of \( n = 2 \) in this case is also essential. The approximate solution to the problem with BC (26) is deprived of this shortcoming and reproduces two lowest harmonics; therefore, it is well agreed with exact solution. At piecewise constant boundary functions, all approximate solutions describe temperature oscillations qualitatively correctly, at the same time the largest accuracy is provided when using BC (25), and the smallest accuracy is achieved, when using BC (24).

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure1.png}
\caption{Temperature oscillations on half-space surface at boundary functions \( \hat{\theta}_f = 0.5 \cos \hat{t}, \text{Bi} = 0.1(1 + \cos \hat{t}) \) (a) and \( \hat{\theta}_f = 0.5 \text{sign}(\sin \hat{t}), \text{Bi} = 0.1[1 - 0.5 \text{sign}(\sin \hat{t})] \) (b): 1 – BC (8); 2 – BC (24); 3 – BC (25); 4 – BC (26).}
\end{figure}

The important integral characteristic of temperature oscillations is temperature range. Dependences \( \Delta \theta_w \) on \( \zeta \) for two conditions of heat transfer are given in figures 2 and 3. According to them, it follows that use of BC (24) is admissible only at small oscillations of heat transfer as it does not account for dependence \( \Delta \theta_w \) on \( \zeta \). Use of BC (25) allows us to account for dependence \( \Delta \theta_w \) on \( \zeta \) qualitatively, however the error of the approximate solution is small either at piecewise constant boundary functions, or at small \( \zeta \). Use of BC (26) in all cases ensures determination accuracy of \( \Delta \theta_w \), acceptable
for engineering calculations, which raises with reduction of $\Delta \theta_w$. and at $\Delta \theta_w \approx 0.1$ the approximate solution overestimates $\Delta \theta_w$ by 7% approximately.

![Figure 2. Dependence $\Delta \theta_w$ on $\zeta$ at $\bar{\theta}_f = 0.5 \cos \bar{\xi}$, $Bi = 0.1(1 + \zeta \cos \bar{\xi})$

for $j = 0$ (a), $j = 1$ (b), $j = 2$ (c), $j = 3$ (d), $j = 4$ (e), $j = 5$ (f):

1 – BC (8); 2 – BC (24); 3 – BC (25); 4 – BC (26).]
Figure 3. Dependence $\Delta \theta_w$ on $\zeta$ at $\hat{\Theta}_f = 0.5 \text{sign}(\sin \hat{t})$, $\text{Bi} = 0.1 [1 - \zeta \text{sign}(\sin \hat{t})]$ for $j = 0$ (a), $j = 1$ (b), $j = 2$ (c), $j = 3$ (d), $j = 4$ (e), $j = 5$ (f):
1 – BC (8); 2 – BC (24); 3 – BC (25); 4 – BC (26).

Conclusion
At the example of regular solids approximate calculation methods of small temperature oscillations in a thermal layer of a solid at unsteady coefficient of heat transfer are considered. It is established that use in calculations of a period average heat transfer coefficient is justified only at small heat transfer oscillations. At piecewise constant boundary functions high accuracy is provided by use of an equivalent heat transfer coefficient. The most universal method is use of an equivalent second type boundary condition which ensures the accuracy of temperature oscillations determination, acceptable for engineering calculations, under any conditions of heat transfer.
Acknowledgments
The reported study was funded by RFBR according to the research project № 18-31-00090.

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