Real intersection theory (II)

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Abstract
Continuing from part (I), we develop properties of real intersection theory that turns out to be an extension of the well-established theory in algebraic geometry.

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1 Introduction

Intersection in mathematics has a long history. But the systemically developed theories only started appearing in the 20th century. They are centered around the quotient rings that are derived from freely generated Abelian groups of non-quotient objects. While the quotient rings fit into the known axiomatic system well, the non-quotient groups do not.

For instance in classical approach the topological intersection for a real compact manifold $X$ in homology or/and cohomology (quotient groups) are obtained from the non quotient objects – singular chains. Once the theory is set up, quotient groups are well-adapted to the axiomatic environment, but the groups of singular chains are not. Using cohomology we can let $H^i(X;\mathbb{Z})$ be the cohomology of degree $i$ with integer coefficients. The product is defined in an elaborated method through the intersection of chains as the cup product,

$$\cup : H^p(X;\mathbb{Q}) \times H^q(X;\mathbb{Z}) \rightarrow H^{p+q}(X;\mathbb{Z}). \quad (1.1)$$

The product gives a ring structure to the cohomology. Then the focus is shifted to the framework of rings which provides rich mathematical structures. However such a ring structure does not exist on the groups of chains. So the groups of singular chains plays a supporting role behind the cohomology groups.

Another example is the intersection in algebraic geometry defined by William Fulton in [3]. Its non-quotient objects are algebraic cycles, which form an Abelian group $Z$. Let $X$ be a smooth projective variety over a field. Let $CH(X)$ be the Chow groups (quotient groups). Then there is an elaborated product map through the intersection of algebraic cycles,

$$\bullet : CH(X) \times CH(X) \rightarrow CH(X) \quad (1.2)$$

such that $CH(X)$ is a ring. The study of algebraic cycles has been motivated by the structure of the Chow rings. So the non-quotient $Z$ group plays a supporting role since it does not have the similar ring structure.
Both ring structures $\cup, \bullet$ are classically known to be compatible with each other and functorial between spaces. These quotiented intersections play important roles in the more general cohomological theory that later generalizes both quotient rings in the derive category. They created rich algebraic structures that flourish beyond the original transcendental geometry. However the flourishing has been one-sided focusing on quotient groups mainly through homological algebra. In this paper we try to look into the other direction which studies the non-quotient objects. Our direction shows that the global invariant – cohomology ring may contain non-trivial geometry without quotient in the local charts. More specifically, the intersection in cohomology which is considered to be a quotient object, is related to the geometric measure in transcendental geometry, which is a non-quotient object. The study of non-quotient objects for quotient objects is the real intersection theory.

On the technical side, we let $\mathcal{X}$ be a connected, oriented manifold of dimension $m$. In [9], using local charts, we defined the subspace $\mathcal{C}(\mathcal{X})$ called Lebesgue currents in the space of currents such that there is a bilinear homomorphism – the intersection of currents, $[\cdot \wedge \cdot] : \mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{X}) \rightarrow \mathcal{C}(\mathcal{X})$.

\[
(T_1, T_2) \rightarrow [T_1 \wedge T_2].
\]

The intersection is extrinsic, for it depends on De Rham data consisting of a De Rham covering of $\mathcal{X}$. But it satisfies an important intrinsic relation,

\[
supp([T_1 \wedge T_2]) \subset supp(T_1) \cap supp(T_2).
\]

where the support is the focus of the theory. In this paper which is not self-contained, we continue [9] to establish sufficient properties for the applications. It is a detailed demonstration that by adjusting the extrinsic data, the well-known operations on quotient objects can be carried over to non-quotient objects. As a conclusion we give a short but immediate application that proves the generalized Hodge conjecture on 3-folds.

2 Basic properties

Let $\mathcal{X}$ be a connected oriented manifold. Let $T_1, T_2$ be two arbitrary singular chains. Then $T_1 \cap T_2$ may not be a singular chain. To catch the essence of

\[\footnote{Intrinsically it is impossible.}\]
such an intersection we are going to use a tool — current, the notion created by G. de Rham in 1950’s, [2].

We denote the space of $C^\infty$ forms with a compact support by $\mathcal{D}(\mathcal{X})$, and its topological dual — the space of currents by $\mathcal{D}'(\mathcal{X})$. If necessary we add a superscript (a subscript) to denote the codimension (the dimension) of currents. The functional of a current $T$ is denoted by

$$\int_T (\bullet).$$  \hfill (2.1)

\textbf{Lemma 2.1.} Let $\mathcal{Z} \subset \mathcal{X}$ be a submanifold. Let

$$\mathcal{Z} \xhookrightarrow{i} \mathcal{X}$$

be the inclusion map. Let

$$\mathcal{D}(\mathcal{X}, \mathcal{Z}) = \{ \phi \in \mathcal{D}(\mathcal{X}) : \phi|_{\mathcal{Z}} = 0 \},$$  \hfill (2.2)

where $\phi|_{\mathcal{Z}}$ is the pullback of the $C^\infty$-differential form by the inclusion map. So

$$\mathcal{D}(\mathcal{X}, \mathcal{Z}) \subset \mathcal{D}(\mathcal{X}).$$  \hfill (2.3)

Then the sequence of topological dual

$$\mathcal{D}'(\mathcal{Z}) \xrightarrow{i^*} \mathcal{D}'(\mathcal{X}) \xrightarrow{R} \mathcal{D}'(\mathcal{X}, \mathcal{Z})$$  \hfill (2.4)

is exact, where $'$ stands for the topological dual and $R$ is the restriction map.

\textit{Proof.} We may assume $\mathcal{Z}$ is compact. It is trivial that $R \circ i_* = 0$. Let’s show

$$\ker(R) \subset \text{Im}(i_*).$$

Let $U$ be a tubular neighborhood of $\mathcal{Z}$ and $j : U \to \mathcal{Z}$ be a projection induced from the normal bundle structure of $U$. Let $h$ be a $C^\infty$ function on $\mathcal{X}$ such that it has a compact support in $U$ and it is 1 on $\mathcal{Z}$. For any $T \in \mathcal{D}'(\mathcal{X})$, we define a current $T'$ on $\mathcal{Z}$.
Let $T \in \ker(R)$. We would like to show
\[ i_*(T') = T. \]

It suffices to show that for any testing form of $\phi$ on $\mathcal{X}$
\[ \int_T h^*(\phi|_Z) = \int_T \phi, \]
or
\[ \int_T \left( h^*(\phi|_Z) - \phi \right) = 0. \] (2.6)

Since $h^*(\phi|_Z) - \phi$ vanishes on $Z$, the formula (2.6) holds. If $Z$ is non-compact, we can use a partition of unity to have the same proof. We complete the proof.

In [9], we have developed the local and global calculations of intersection of Lebesgue currents. We refer the readers to [2] and [9] for the necessary background material. Let’s give a summary. It started with the De Rham’s regularization.

**Theorem 2.2.** (G. de Rham)

Let $\mathcal{X}$ be a connected, oriented manifold. Let $\epsilon$ be a small positive number. There are linear operators $R_\epsilon$ and $A_\epsilon$ on $\mathcal{D}'(\mathcal{X})$ satisfying

1. a homotopy formula
\[ R_\epsilon T - T = bA_\epsilon T + A_\epsilon bT. \] (2.7)

where $b$ is the boundary operator.

2. $\text{supp}(R_\epsilon T), \text{supp}(A_\epsilon T)$ are contained in any given neighborhood of $\text{supp}(T)$ provided $\epsilon$ is sufficiently small.

3. $R_\epsilon T$ is $C^\infty$;

4. If $T$ is $C^r$, $A_\epsilon T$ is $C^r$.

5. If a smooth differential form $\phi$ varies in a bounded set and $\epsilon$ is bounded above, then $R_\epsilon \phi, A_\epsilon \phi$ are bounded.
(6) As \( \epsilon \to 0 \),
\[
R_\epsilon T(\phi) \to T(\phi), A_\epsilon T(\phi) \to 0
\]
uniformly on each bounded set \( \phi \).

In this paper we use the notations \( R^X_\epsilon \), \( A^X_\epsilon \) to replace De Rham’s notations \( R_\epsilon \), \( A_\epsilon \).

**Definition 2.3.** (for De Rham’s regularization)
(a) We call \( R^X_\epsilon \) from Theorem 2.2 the De Rham’s smoothing operator, 
\( A^X_\epsilon \) from Theorem 2.2 the De Rham’s homotopy operator, and the 
associated regularization the De Rham’s regularization.
(b) We define De Rham data to be all items in the construction

We continue to have

**Theorem 2.4.** (B. Wang 2019)
Let \( X \) be a connected, oriented manifold of dimension \( m \), equipped with a 
De Rham data. Let \( T_1, T_2 \in \mathcal{C}(X) \) such that \( \dim(T_1) + \dim(T_2) \geq m \).

(1) Then there is a family of \( C^\infty \) forms, the kernel \( g_\epsilon(x,y) \) on \( X \times X \) 
where \( \epsilon > 0 \) such that
\[
R^X_\epsilon(T_2) = \int_{y \in T_2} g_\epsilon(x,y)
\]
where the right hand side is defined by the fibre integral similar to the fibre 
integral of the projection
\[
X \times X \to X(1st \ copy) \\
(x, y) \to x.
\]
Furthermore \( g_\epsilon(x,y) \) is a closed form, which in case with a compact \( X \) is 
Poincaré dual to the diagonal of \( X \times X \) in the cohomology group \( \mathbb{R} \)

(2) There exists a subspace \( \mathcal{C}(X) \) of \( \mathcal{D}'(X) \), defined in geometric measure 
theory, such that for \( T_1, T_2 \in \mathcal{C}(X) \) continuous functional
\[
\phi \to \lim_{\epsilon \to 0} \int_{T_1} R^X_\epsilon(T_2) \wedge \phi
\]
\[
\mathcal{D}(X) \to \mathbb{R}
\]
\[2\text{In the language of [2], } g_\epsilon(x,y) \text{ is homologous to the diagonal for each non-zero } \epsilon.\]
exists and lies in $C(\mathcal{X})$, denoted by
\[ [T_1 \wedge T_2]. \quad (2.10) \]

**Proposition 2.5.** Let $\mathcal{X}$ be a manifold equipped with a De Rham data. Let $i : Z \hookrightarrow \mathcal{X}$ be a submanifold. Let $T \in C(\mathcal{X})$. Then there is a current denoted by $[Z \wedge T]_Z$ in $Z$ such that
\[ i_*([Z \wedge T]_Z) = [Z \wedge T]. \quad (2.11) \]

**Proof.** For any $\phi \in \mathcal{D}(\mathcal{X}, Z)$,
\[ \int_{[Z \wedge T]} \phi = \lim_{\epsilon \to 0} \int_Z R_{\epsilon}^X(T) \wedge \phi = 0. \quad (2.12) \]

Then by Lemma 2.1, there is a current in $Z$ satisfying (2.11).

\[ \square \]

### 2.1 Basic properties

**Property 2.6.**

Let $\mathcal{X}$ a connected, oriented $C^\infty$ manifold of dimension $m$. Assume it is equipped with a De Rham data. In [9], we defined the intersection of currents,
\[ C(\mathcal{X}) \times C(\mathcal{X}) \to C(\mathcal{X}) \]
\[ (T_1, T_2) \to [T_1 \wedge T_2], \quad (2.13) \]
on the subspace $\mathcal{C}(\mathcal{X}) \subset \mathcal{D}'(\mathcal{X})$ consisting of Lebesgue currents. For all Lebesgue currents $T_1, T_2$, the intersection $[T_1 \wedge T_2]$ has properties:

1. (Supportivity) $\text{supp}([T_1 \wedge T_2]) \subset \text{supp}(T_1) \cap \text{supp}(T_2)$. \quad (2.14)
2. (Closedness) The intersection current $[T_1 \wedge T_2]$ is closed if $T_1, T_2$ are.
(3) (Graded commutativity) There is a graded-commutativity. If
\[ \deg(T_1) = p, \deg(T_2) = q, \]
then
\[ [T_1 \wedge T_2] = (-1)^{pq}[T_2 \wedge T_1] \] (2.15)
or equivalently
\[
\lim_{\epsilon \rightarrow 0} \lim_{\epsilon' \rightarrow 0} \int_{\mathcal{X}} R_{\epsilon}(T_1) \wedge R_{\epsilon'} T_2 \wedge \phi = \lim_{\epsilon \rightarrow 0} \lim_{\epsilon' \rightarrow 0} \int_{\mathcal{X}} R_{\epsilon}(T_1) \wedge R_{\epsilon'} T_2 \wedge \phi.
\] (2.16)
for the test from \( \phi \in \mathcal{D}(\mathcal{X}) \).

(4) (Cohomologicity) Let \( \mathcal{X} \) be compact. We use \( \langle T \rangle \) to denote the cohomology class represented by a closed current \( T \). If \( T_1, T_2 \) are closed, in \( H(\mathcal{X}; \mathbb{R}) \), we have
\[
\langle T_1 \rangle \cup \langle T_2 \rangle = \langle [T_1 \wedge T_2] \rangle.
\] (2.17)
Hence if the cohomology \( \langle T_1 \rangle, \langle T_2 \rangle \) are integral, so is \( \langle [T_1 \wedge T_2] \rangle \).

(5) (Associativity) There is an associativity
\[
\left[[T_1 \wedge T_2] \wedge T_3\right] = [T_1 \wedge [T_2 \wedge T_3]].
\] (2.18)

(6) (Leibniz rule) If \( bT_1, bT_2 \) are Lebesgue and \( \deg(T_1) = p \), then
\[
d[T_1 \wedge T_2] = [dT_1 \wedge T_2] + (-1)^p [T_1 \wedge dT_2].
\] (2.19)

Proof.

(1) The proof is in [9].

(2) Let \( \phi \) be a test form. By the definition
\[
\int_{b[T_1 \wedge T_2]} \phi = \lim_{\epsilon \rightarrow 0} \int_{T_1} R_{\epsilon} T_2 \wedge d\phi = \pm \int_{T_1} dR_{\epsilon} T_2 \wedge \phi
\] (2.20)
According to the homotopy (2.7),
\[ bR_\varepsilon T_2 - bT_2 = bbA_\varepsilon T_2 - bA_\varepsilon bT_2 \] (2.21)
Because \( T_2 \) is closed,
\[ bR_\varepsilon T_2 = 0. \]
So \([T_1 \wedge T_2]\) is closed.

(3) (Graded commutativity). The proof is in [9].

(4) Let \( \phi \) be a closed \( C^\infty \) form of degree \( deg(T_1) + deg(T_2) \). Denote the intersection number in cohomology by \( \text{int}(\cdot, \cdot) \). Hence the intersection number,
\[ \text{int}(\langle [T_1 \wedge T_2] \rangle, \langle \phi \rangle) \] (2.22)
is a well-defined real number that equals to
\[ \lim_{\varepsilon \to 0} \int_{T_1} R_\varepsilon (T_2) \wedge \phi. \] (2.23)
(by De Rham’s theorem) which is
\[ (-1)^{deg^2(T_1)} \lim_{\varepsilon \to 0} \int_{T_2 \wedge \phi} R_\varepsilon (T_1) \] (2.24)
(by the community (3)). Then we use (2.24) as De Rham’s Kronecker index
\[ T_1 \wedge (T_2 \wedge \phi)[1] \]
which is well-defined between the currents \( T_1 \) and \( T_2 \wedge \phi \). We obtain that
\[ \text{int}(\langle [T_1 \wedge T_2] \rangle, \langle \phi \rangle) = T_1 \wedge (T_2 \wedge \phi)[1] \] (2.25)

On the compact manifold, the Kronecker index only depends the cohomology classes (see section 20, chapter IV [2]). Hence (2.25) is the intersection number
\[ \text{int}(\langle T_1 \rangle, \langle T_2 \wedge \phi \rangle) \] (2.26)
which by the associativity of the intersection in cohomology is the same as
\[ \text{int}(\langle T_1 \rangle \cup \langle T_2 \rangle, \langle \phi \rangle) \] (2.27)
Since $\phi$ is any closed form,
\[ \langle T_1 \cup T_2 \rangle = \langle [T_1 \wedge T_2] \rangle. \quad (2.28) \]

(5) Let degrees of $T_i$ be $p_i$. The intersection of Lebesgue currents is still Lebesgue. Therefore we have a triple intersection expressed as a functional on the test forms $\bullet$.

Then
\[
\int T_1 \wedge T_2 \wedge T_3 (\bullet) = \lim_{\epsilon_3 \to 0} \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \int_X R_{\epsilon_1}(T_1) \wedge R_{\epsilon_2}(T_2) \wedge R_{\epsilon_3}(T_3) \wedge (\bullet). \quad (2.29)
\]

By the commutitativity in Proposition 4.4 of [9],
\[
\int T_1 \wedge [T_2 \wedge T_3] (\bullet) = \int (-1)^{p_1(p_2+p_3)} [T_2 \wedge T_3 \wedge T_1] (\bullet)
\]
\[
= (-1)^{p_1(p_2+p_3)} \lim_{\epsilon_1 \to 0} \lim_{\epsilon_2 \to 0} \lim_{\epsilon_3 \to 0} \int_X R_{\epsilon_2}(T_2) \wedge R_{\epsilon_3}(T_3) \wedge R_{\epsilon_1}(T_1) \wedge (\bullet)
\]
\[
= \lim_{\epsilon_3 \to 0} \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \int_X R_{\epsilon_1}(T_1) \wedge R_{\epsilon_2}(T_2) \wedge R_{\epsilon_3}(T_3) \wedge (\bullet)
\]
\[
= \int [T_1 \wedge T_2 \wedge T_3] (\bullet). \quad (2.30)
\]

(6) (Leibniz Rule) Let $\phi \in \mathcal{D}(\mathcal{X})$ be a test form. Let
\[ \deg(T_1) = p, \deg(T_2) = q. \]
Then
\[ b[T_1 \wedge T_2](\phi) = \lim_{\epsilon \to 0} \int_{T_1} R_\epsilon T_2 \wedge d\phi \]
\[ = \lim_{\epsilon \to 0} \int_{T_1} \left( (-1)^q (R_\epsilon T_2 \wedge \phi) + (-1)^{q+1} dR_\epsilon T_2 \wedge \phi \right) \]
\[ = \lim_{\epsilon \to 0} \int_{(-1)^{q}bT_1} R_\epsilon T_2 \wedge \phi + \lim_{\epsilon \to 0} \int_{(-1)^{q+1}T_1} dR_\epsilon T_2 \wedge \phi \]
(\(bT_1, bT_2\) are Lebesgue)
\[ = \int (-1)^q [bT_1 \wedge T_2] \phi + \int (-1)^{q+1} [T_1 \wedge dT_2] \phi \]

Hence
\[ b[T_1 \wedge T_2] = (-1)^q [bT_1 \wedge T_2] + (-1)^{q+1} [T_1 \wedge dT_2]. \quad (2.31) \]

After change the sign, we found (2.31) is the same as (2.19).

\[ \square \]

### 2.2 Product and Inclusion

Intersection theory in algebraic geometry has functoriality. We study its extension to the real intersection theory.

**Definition 2.7.**

(1) Let \( \mathcal{U}_1, \mathcal{U}_2 \) be the De Rham data for the manifolds \( \mathcal{X}_1, \mathcal{X}_2 \) respectively. Let \( B^1_i \subset U_i, B^2_j \subset U_j \) be the De Rham covering from \( \mathcal{U}_1, \mathcal{U}_2 \) and \( f^1_i, f^2_j \) are the associated convolution functions. Then given an order of the De Rham’s covering \( B^1_i \times B^2_j \subset U^1_i \times U^2_j \) of \( \mathcal{X}_1 \times \mathcal{X}_2 \), we define the De Rham data on the product \( \mathcal{X}_1 \times \mathcal{X}_2 \) by taking the direct product of given data in each manifold. For instance, we define the convolution functions \( (f^1_i, f^2_j) \) on each open set \( B^1_i \times B^2_j \). We call it the product De Rham data.

(2) Given \( (B^1_i, B^2_j) \subset (U^1_i, U^2_j) \) the product De Rham covering of \( \mathcal{X}_1 \times \mathcal{X}_2 \) that has the given order denoted by
\[ (U^{i_1}_1, U^{j_1}_2), (U^{i_2}_1, U^{j_2}_2), \cdots, \]
we construct new De Rham data on $X_1$ by setting ordered De Rham covering as

$$U_i^1, U_i^2, U_i^3, \ldots$$

and on $X_2$ by setting the De Rham covering as

$$U_j^1, U_j^2, U_j^3, \ldots$$

These new De Rham data are called projection De Rham data with respect to the product De Rham data.

The important implication of these new De Rham data is the following projection formula.

**Proposition 2.8.** (Projection formula) Let $X_1 \times X_2$ be two manifolds equipped with a product De Rham data, $X_2$ equipped with a projection De Rham data. Let $P_2 : X_1 \times X_2 \rightarrow X_2$ be the projection, and $\sigma \in C(X_2)$, and $T \in C(X_1 \times X_2)$. Then

1. $R^{X_1 \times X_2}_\epsilon(\mathcal{X}_1 \otimes \sigma) = (P_2)^*(R^{X_2}_\epsilon(\sigma))$. (2.32)

2. Let $X_1$ be compact. Then

$$[(P_2)_*(T) \wedge \sigma] = (P_2)_*[T \wedge (X_1 \otimes \sigma)].$$

**Proof.** (1). Assume $X_1, X_2$ are equipped with De Rham data, $U_1, U_2$ respectively. Let’s give a product De Rham data to $X_1 \times X_2$ and projection De Rham data to $X_1$ and $X_2$. Each chart in the product De Rham covering $(U_i^1, U_j^2)$ gives a De Rham’s smoothing operators on $X_1 \times X_2$, denoted by $R^{X_1 \times X_2}_\epsilon(i,j)$. Each chart in the projection De Rham covering $U_j^2$ gives a De Rham’s smoothing operator on $X_2$, denoted by $R^{X_2}_\epsilon(i,j)$. We claim for any $\sigma \in \mathcal{D}'(X)$,

**Claim 2.9.**

$$R^{X_1 \times X_2}_\epsilon(i,j)(1 \otimes \sigma) = 1 \otimes R^{X_2}_\epsilon(i,j)(\sigma)$$

(2.34)

on $X_1 \times X_2 - \partial(B_i \times B_j)$.

where 1 is the current $X_1$. Let’s consider both sides of (2.34) restricted to a neighborhood of a point $(q_i, q_j)$ (both sides globally are not continuous, but in a neighborhood they are). If $(q_i, q_j) \notin B_i \times B_j$. Then both sides
are restricted to the \( 1 \otimes \sigma \) in the neighborhood. If \((q_i, q_j) \in B_i^1 \times B_j^2\), let \( \phi \in \mathcal{D}^{\dim(X_1)+\dim(\sigma)}(X_1 \times X_2) \) be supported in the neighborhood.

\[
\int_{R^X_{1 \times X_2,j}(1 \otimes \sigma)} \phi \\
= \int_{x_1 \in B_i} \int_{x_2 \in B_j} R^X_{\epsilon,j}(\sigma) \wedge \phi(x_1, x_2) \\
= \int_{1 \otimes R^X_{\epsilon,j}(\sigma)} \phi
\]

(we should note that the partial degree of \( \phi(x_1, x_2) \) in \( x_1 \) is maximal). Then we obtain that (2.34) holds on the \( X_1 \times X_2 - \partial(B_i \times B_j) \).

Continuing from Claim 2.9, we glue each piece \((U^i_1, U^j_2)\) by taking De Rham’s compositions of smoothing operators on both sides of (2.34) to obtain that

\[
R^X_{\epsilon \times X_2}(1 \otimes \sigma) = 1 \otimes R^X_{\epsilon}(\sigma)
\]

(2.35) on \( X_1 \times X_2 - \cup_{i,j} \partial(B_i \times B_j) \),

where \( R^X_{\epsilon} \) is obtained from the projection De Rham data with respect to the product De Rham data. Now since both sides of (2.35) are \( C^\infty \), by the continuity, (2.35) holds on the closure \( X_1 \times X_2 \). This completes the proof of part (1).

(2). Since \( X_1 \) is compact, \( P_2 \) is proper. Then the pushforward \((P_2)_*\) of currents is well-defined. Let \( \phi \) be a test form on \( X_2 \). We use projection De Rham data on \( X_2 \) and product De Rham data on \( X_1 \times X_2 \) to find

\[
\int_{[(P_2)_*(T)\wedge \sigma]} \phi = \lim_{\epsilon \to 0} \int_{(P_2)_* T} R^X_{\epsilon}(\sigma) \wedge \phi \\
= \lim_{\epsilon \to 0} \int_{T} P^*_2(R^X_{\epsilon}(\sigma) \wedge \phi) \\
= \lim_{\epsilon \to 0} \int_{T} (1 \otimes R^X_{\epsilon}(\sigma)) \wedge P^*_2(\phi)
\]
(Use part (1))
\[
\lim_{\epsilon \to 0} \int_T R^X_{\epsilon} (1 \otimes \sigma) \wedge P^*_2 (\phi)
\]
\[
= \int_{[T \wedge (X_1 \times X)]} P^*_2 (\phi)
\]
\[
= \int_{(P_2)_*[T \wedge (X_1 \times X)]} \phi
\]

This completes the proof. \qed

Proposition 2.10. (reduction to the diagonal) Let $X$ be a compact manifold. Let $T_1, T_2 \in \mathcal{C}(X)$. Let $X$ be equipped with a De Rham data $\mathcal{U}$. We give the product De Rham data to $X \times X$ and the associated projection De Rham data to each copy $X$. With these De Rham data we have the reduction to diagonal,
\[
[T_1 \wedge T_2] = (P_2)_*[T_1 \otimes T_2] \wedge \Delta
\] (2.36)
where $P_2 : X \times X \to X$ (2nd copy) is the projection, the left hand side of intersection occurs in the second copy of $X$, and $\Delta$ is the diagonal.

Proof. With induced De Rham data there is the projection formula (Proposition 2.8),
\[
[T_1 \wedge T_2] = (P_2)_*[T_1' \wedge (X \otimes T_2)]
\] (2.37)
where $T_1'$ is a current in $X \times X$ such that
\[
(P_2)_* T_1' = T_1.
\]

Now we claim

Claim 2.11. with the projection De Rham data also on the first copy $X$,
\[
(P_2)_*[T_1 \otimes X] \wedge \Delta = T_1
\] (2.38)
\[
(T_1 \otimes X) \wedge (X \otimes T_2) = T_1 \otimes T_2.
\] (2.39)

\footnote{Each copy $X$ will have different De Rham data depending on the order of the product De Rham data.}
For (2.38), we let $\phi \in \mathcal{D}(X)$. Then

$$\int_{(P_2)_*([T_1 \otimes X] \wedge \Delta)} \phi$$

$$= \int_{[(T_1 \otimes X) \wedge \Delta]} (P_2)^*(\phi)$$

$$= \lim_{\epsilon \to 0} \int_\Delta R^X_{\epsilon X}(T_1 \otimes X) \wedge (P_2)^*(\phi)$$

(Use projection formula, Proposition 2.8)

$$= \lim_{\epsilon \to 0} \int_\Delta (P_1)^*(R^X_{\epsilon X}(T_1)) \wedge (P_2)^*(\phi)$$

(where $P_1 : X \times X \to X(1st \ copy)$ is the projection then identify $X \simeq \Delta$)

$$= \lim_{\epsilon \to 0} \int_X R^X_{\epsilon X}(T_1) \wedge \phi$$

$$= \int_{T_1} \phi.$$  

Hence (2.38) is true. For (2.39), we let $\phi \in \mathcal{D}(X)$. Then by the projection formula

$$R^X_{\epsilon X \times \epsilon X}(X \times T_2) = R_X(T_2)$$  

(2.40)

Then

$$\int_{(T_1 \otimes X) \wedge (X \otimes T_2)} \phi$$

$$= \lim_{\epsilon \to 0} \int_{T_1 \otimes X} R^X_{\epsilon X}(1 \otimes T_2) \wedge \phi$$

$$= \lim_{\epsilon \to 0} \int_{T_1 \otimes X} 1 \otimes R^X_{\epsilon X}(T_2) \wedge \phi$$

$$= \lim_{\epsilon \to 0} \int_X R^X_{\epsilon X}(T_2) \wedge (\int_{x_1 \in T_1} \phi(x_1, x_2))$$

$$= \int_{T_2} \int_{x_1 \in T_1} \phi(x_1, x_2)$$

$$= \int_{T_1 \otimes T_2} \phi.$$  

(2.41)
Hence with projection De Rham data on both $\mathcal{X}$ and product De Rham data on $\mathcal{X} \times \mathcal{X}$, Claim 2.11 is true. Then in (2.37), we replace $T'$ by $(T_1 \otimes \mathcal{X}) \wedge \Delta$ and apply the associativity and commutativity to have

$$[T_1 \wedge T_2] = (P_2)_* \left( [(T_1 \otimes \mathcal{X}) \wedge \Delta \wedge (\mathcal{X} \otimes T_2)] \right)$$

$$= (P_2)_* \left( [(T_1 \otimes \mathcal{X}) \wedge (\mathcal{X} \otimes T_2) \wedge \Delta] \right)$$

(by (2.39))

$$= (P_2)_* \left( [(T_1 \otimes T_2) \wedge \Delta] \right).$$

This completes the proof. 

Let $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ be the inclusion of a submanifold of dimension $n$.

Proposition 2.12. (associativity)

There exist De Rham data $\mathcal{U}_\mathcal{Z}, \mathcal{U}_\mathcal{X}$ on $\mathcal{Z}, \mathcal{X}$ respectively such that for any cellular chain $W \subset \mathcal{Z}$ and $\sigma \in \mathcal{C}(\mathcal{X})$,

$$i_*([W \wedge [\sigma \wedge \mathcal{Z}]_\mathcal{Z}]) = [i_* W \wedge \sigma] \quad (2.42)$$

where the notation for $[\sigma \wedge \mathcal{Z}]_\mathcal{Z}$ is defined in Proposition 2.5, and $W$, to abuse the notation, denotes the current $\in \mathcal{D}(\mathcal{Z})$.

Proof. Let $j : E \to \mathcal{Z}$ be a tubular neighborhood of $\mathcal{Z}$ in $\mathcal{X}$. Thus $E$ is diffeomorphic to a vector bundle of rank $r$. We denoted the bundle also by $E$. Let $\mathcal{U}$ be a De Rham data for $\mathcal{Z}$ such that each De Rham chart $U_i$ lies in the trivialization. So

$$j^{-1}(U_i) =: U_i \times \mathbb{R}^r \quad (2.43)$$

where $=: $ denotes a fixed diffeomorphism. Let $B \subset \mathbb{R}^r$ be the De Rham chart in the step 3 of the construction of De Rham's regularization in [9]. Then $E$ is equipped with a De Rham data $\mathcal{U}_E$ whose De Rham covering is
\[ j^{-1}(U_i), \text{all } i. \text{ Then we extend } \mathcal{U}_E \text{ (arbitrarily) to } \mathcal{X} \text{ to have a De Rham data } \mathcal{U}_X. \text{ We denote the smoothing operator associated to the chart } j^{-1}(U_i) \text{ by } R^{X,i}_e. \text{ Let } \xi \in \mathscr{D}(\mathcal{X}) \text{ be a function that is 1 on } \mathcal{W} \text{ and has support in a neighborhood of } Z \text{ inside of } E. \text{ Let } \mathcal{W}' = \xi j^{-1}(\mathcal{W}) \text{ be the current } \in \mathscr{D}'(\mathcal{X}). \text{ By the associativity,}
\[ [\mathcal{W}' \land [Z \land \sigma]] = [[\mathcal{W}' \land Z] \land \sigma]. \tag{2.44} \]

Notice that with vector bundle structure, \( \mathcal{W}' \) in a neighborhood of \( \mathcal{W} \) meets \( Z \) transversely at the open cells of \( \mathcal{W} \). By Proposition 3.1 below,
\[ [\mathcal{W}' \land Z] = i_* \mathcal{W}. \]

so
\[ [\mathcal{W}' \land [Z \land \sigma]] = [i_* \mathcal{W} \land \sigma]. \tag{2.45} \]

Let \( \phi \in \mathscr{D}(Z) \). By using the partition of unity, we may assume \( \mathcal{W} \) lies in a single trivialization
\[ j^{-1}(U_i) =: U_i \times \mathbb{R}^r \]

of \( E \). Then we calculate
\[
\int_{[[Z \land \sigma] \land \mathcal{W}']} j^*(\phi)
\]
\[ = \lim_{(e,e') \to 0} \int_Z R^X_e(\sigma) \land (R^{X}_e(\mathcal{W}')) \land j^*(\phi)
\]
\[ = \lim_{(e,e') \to 0} \int_Z R^X_e(\sigma) \land (R^{X,i_1}_e \circ \cdots \circ R^{X,i_h}_e(\mathcal{W}')) \land j^*(\phi)
\]
(\text{where } i_1 < \cdots < i_h, \text{ and apply Claim 2.9 to the trivialization (2.43) })
\[
= \lim_{(e,e') \to 0} \int_Z R^X_e(\sigma) \land (1 \otimes R^Z_{e,i_1} \circ \cdots \circ R^Z_{e,i_h} (\mathcal{W}')) \land j^*(\phi)
\]
\[ = \lim_{e' \to 0} \int_{[Z \land \sigma]} (1 \otimes R^Z_{e'}(\mathcal{W})) \land j^*(\phi) \tag{2.46}
\]
\[ = \int_{[[Z \land \sigma] \land i_* \mathcal{W}]} j^*(\phi)
\]
\[ = \int_{[[Z \land \sigma] \land i_* \mathcal{W}]} \phi
\]


3 Dependence of the local data

The intersection of currents is extrinsic because De Rham data is extrinsic. The real intersection theory stresses inseparable nature of “intrinsic” and “extrinsic”. Let’s look into the dependence of the De Rham data to see how much is intrinsic and how much is extrinsic. We begin with the real case.

3.1 Real case

It is well-known that on a manifold, if two sub manifolds meet transversally at another submanifold, then the intersection should be defined as the intersectional manifold. The more useful version is its extension in algebraic geometry. The following proposition says that the transversal intersection is a particular case of the intersection of currents, where the dependence of De Rham data disappears.

Proposition 3.1. Let $\mathcal{X}$ be a manifold of dimension $m$. If $T_1, T_2$ are cells of real dimension $p, q$ with $p+q \geq m$, and the intersection $T_1 \cap T_2$ is transversal at a connected, manifold $V$ of dimension $p + q - m$. We assume $V$ at each point can be oriented concordant with $T_1, T_2$. Then $[T_1 \wedge T_2]$ is independent of $\mathcal{U}$. Furthermore it is the current of integration over $V$.
Proof. Let’s set up the coordinates for the cells. Let $\mathcal{X} = \mathbb{R}^m$ have linear basis $e_1, \ldots, e_m$ and coordinates $x_1, \ldots, x_m$. Set up the subspaces,

\[
\mathbb{R}^p = \text{span}(e_1, \ldots, e_p)
\]
\[
\mathbb{R}^q = \text{span}(e_{m-q+1}, \ldots, e_m)
\]
\[
\mathbb{R}^{p+q-m} = \text{span}(e_{m-q+1}, \ldots, e_p)
\]

Let $T_1 = \Delta^p \subset \mathbb{R}^p$ be the polyhedron defined by

\[
\{ \sum_{i=1}^{p} |x_i| < 1 \}
\] (3.1)

Similarly $T_2 = \Delta^q$ is defined by

\[
\{ \sum_{i=m-q+1}^{m} |x_i| < 1 \},
\] (3.2)

$V = \Delta^p \cap \Delta^q$ is defined by

\[
\{ \sum_{i=m-q+1}^{p} |x_i| < 1 \}.
\] (3.3)

Let $\pi_{p+q-m} : \mathbb{R}^m \to V$ be the projection. The proof has two steps.

1st step: Notice $[T_1 \wedge T_2]$ has a compact support, hence $[T_1 \wedge T_2]$ is also evaluated at the forms in $C^\infty(\mathbb{R}^m)$. Let $\phi \in C^\infty(\mathbb{R}^m)$ have degree $p + q - m$. Let $\xi \in \mathcal{D}(\mathbb{R}^m)$ that is equal to 1 on a bounded neighborhood $K$ of $\Delta^p \cap \Delta^q$. We consider a $C^\infty$ form

\[
\xi(\phi - \pi^*_p \pi_{p+q-m}(\phi|_V))
\] (3.4)

which has a compact support in $K$ and is a limit of $C^\infty$ forms $\psi_n \in \mathcal{D}(K)$ compactly supported in $K - \overline{V}$ as $n \to \infty$. Because $[T_1 \wedge T_2]$ is supported on $V$, but $\psi_n$ is supported outside of $V$,

\[
\int_{[T_1 \wedge T_2]} \psi_n = 0
\]

for all $n$. By the continuity of the functional $[T_1 \wedge T_2]$,

\[
\int_{[T_1 \wedge T_2]} \phi = \int_{[T_1 \wedge T_2]} \pi^*_p \pi_{p+q-m}(\phi|_V).
\] (3.5)
Recall that the $C^\infty$ form $\pi^*_{p+q-m}(\phi|_V)$ is called a local constant slicing in \[9\]. Therefore it is a closed form.

Now we apply the homotopy formula (2.7). Let $\phi \in \mathcal{D}(\mathbb{R}^m)$ such that

$$supp(\phi) \cap (\partial(\Delta^p) \cup \partial(\Delta^q)) = \emptyset.$$

For arbitrary De Rham’s regularization $R'_\epsilon, A'_\epsilon$ with fixed sufficiently small real numbers $\epsilon_1, \epsilon_2$, we apply the homotopy formula (2.7) to have

\[
\int_{[T_1 \land T_2]} \pi^*_{p+q-m}(\phi|_V) = \int_{[(bA'_1 + A'_1, bT_1) \land (bA'_2 + A'_2, bT_2)]} \pi^*_{p+q-m}(\phi|_V) + \int_{[R'_1 \land R'_2]} \pi^*_{p+q-m}(\phi|_V) 
\]

Now we calculate the first integral (3.7)

\[
\int_{[(bA'_1 + A'_1, bT_1) \land (bA'_2 + A'_2, bT_2)]} \pi^*_{p+q-m}(\phi|_V) = \lim_{\epsilon \to 0} \int_{[R, (bA'_1 + A'_1, bT_1) \land R, (bA'_2 + A'_2, bT_2)]} \pi^*_{p+q-m}(\phi|_V) 
\]

(because $supp(bT_i) \cap supp(\phi) = \emptyset$.)

\[
= \lim_{\epsilon \to 0} \int_{[bR, A'_1, T_1 \land bR, A'_2, T_2]} \pi^*_{p+q-m}(\phi|_V) = \pm \lim_{\epsilon \to 0} \int_{[R, A'_1, T_1 \land R, A'_2, T_2]} d(\pi^*_{p+q-m}(\phi|_V)) = 0
\]

This shows that

\[
\int_{[T_1 \land T_2]} \pi^*_{p+q-m}(\phi|_V) = \int_{[R'_1 \land R'_2]} \pi^*_{p+q-m}(\phi|_V). 
\]

We observe that the right hand side of (3.9) does not involve the De Rham’s smoothing operator $R'_\epsilon$, thus the current $[T_1 \land T_2]$ is independent of the choice of De Rham data $U$. \[4\]

\[4\]The technique of using the arbitrary $R'_\epsilon$ is due to De Rham §20, [2].
2nd step: To calculate the intersection \([T_1 \land T_2]\). By the 1st step, we can choose a particular De Rham’s data \(U\) that has one chart \(x_1, \cdots, x_m\) for \(\mathbb{R}^m\). Also we choose a \(C^\infty\) convolution function \(f(x)\) supported in a neighborhood of a unit ball \(B\) satisfying

\[
\int_{\mathbb{R}^m} f(x) d\mu = 1. \tag{3.10}
\]

where \(d\mu = dx_1 \land \cdots \land dx_m\). Let \(\vartheta_\epsilon(x) = f(\frac{x}{\epsilon}) d\mu_\epsilon\).

Let

\[
\kappa : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \quad (x, y) \mapsto x - y, \tag{3.11}
\]

Denote the coordinates \((x_1, \cdots, x_{m-q})\) by \(x_1, (x_{m-q+1}, \cdots, x_p)\) by \(x_2\) and \(x_{i+1}, \cdots, x_m\) by \(x_3\). Similarly for the second copy of \(\mathbb{R}^m\) in (3.11), the corresponding coordinates are denoted by \(y_1, y_2, y_3\) respectively.

Let

\[
g(x_1, x_2, x_3, y_1, y_2, y_3) = \kappa^*(\vartheta_\epsilon). \tag{3.12}
\]

Let \(\phi \in \mathcal{D}(\mathbb{R}^m)\) be a test form. Then we calculate the current

\[
\int_{[T_1 \land T_2]} \phi = \lim_{\epsilon \to 0} \int_{T_1} R_\epsilon(T_2) \land \phi
\]

\[
= \lim_{\epsilon \to 0} \int_{T_1} \int_{(y_2, y_3) \in \mathbb{R}_j} g(x_1, \frac{x_2}{\epsilon}, \frac{x_3}{\epsilon}, 0, 0, \frac{y_2}{\epsilon}, \frac{y_3}{\epsilon}) \land \phi(\frac{x_1}{\epsilon}, x_2, 0) \tag{3.13}
\]

where \(\phi(\frac{x_1}{\epsilon}, x_2, 0)\) is a test form, i.e. \(C^\infty\) form on \(T_1\) with a compact support. Now applying the fibre integral to that over \(T_1\), we obtain

\[
\int_{[T_1 \land T_2]} \phi
\]

\[
= \lim_{\epsilon \to 0} \int_{x_2 \in \mathbb{R}^{i+j-m}} \int_{x_1 \in \mathbb{R}^{m-j}} \int_{(y_2, y_3) \in \mathbb{R}_j} g(x_1, \frac{x_2}{\epsilon}, 0, 0, \frac{y_2}{\epsilon}, \frac{y_3}{\epsilon}) \land \phi(\frac{x_1}{\epsilon}, x_2, 0) \tag{3.14}
\]

Then we make a change of variables,

\[
\frac{x_1}{\epsilon} \to x_1, \quad \frac{x_2}{\epsilon} \to y_2, \quad \frac{x_3}{\epsilon} \to y_3. \tag{3.15}
\]
Then
\[
\int_{[T_1 \wedge T_2]} \phi = \pm \lim_{\epsilon \to 0} \int_{x_2 \in \mathbb{R}^{i+j-m}} \int_{x_1 \in \mathbb{R}^{m-j}} \int_{(y_2, y_3) \in \mathbb{R}^j} g(x_1, \frac{x_2}{\epsilon}, 0, 0, y_2, y_3) \wedge \phi(\epsilon x_1, x_2, 0)
\]  
(3.16)

Then we notice for each fixed \(x_2\), the fibre integral
\[
\int_{y_2, y_3 \in \mathbb{R}^j, x_1 \in \mathbb{R}^{m-j}} g(x_1, \frac{x_2}{\epsilon}, 0, 0, y_2, y_3)
\]  
(3.17)
by formula (3.10), is 1. Therefore we obtain that
\[
[T_1 \wedge T_2](\phi) = \int_{\mathbb{R}^{i+j-m}} \phi(0, x_2, 0)
\]  
(3.18)

Thus
\[
[T_1 \wedge T_2](\phi) = \int_{V} \phi|_{V}.
\]  
(3.19)
where \(\phi|_{V}\) is the restriction \(\phi(0, x_2, 0)\) of \(\phi\) to \(V\). We complete the proof.

For non transversal intersection, there is no classification in general sit-  
uation because the intersection may not have the notion of dimension. But  
for some specific examples, the notion of the dimension exists.

**Proposition 3.2.** (real excess intersection) The intersection of currents  
\([T_1 \wedge T_2]\) depends on \(U\).

*Proof.* We prove it by using an example. Let \(\mathcal{X} = \mathbb{R}^2\), and be equipped with the De Rham data consisting of single chart \(\mathbb{R}^2\) with the convolution function \(f\) satisfying
\[
\int_{\mathbb{R}^2} f(x_1, x_2) dx_1 \wedge dx_2 = 1
\]  
(3.20)

where \(x_1, x_2\) are Euclidean coordinates of \(\mathbb{R}^2\). Let \(T_1 = T_2\) be the current of integration over the finite piece of the parabola
\[
x_1 = x_2^2
\]  
(3.21)
containing the origin $0$. Since $T_1, T_2$ are singular chains, $[T_1 \wedge T_2]$ exists. Let $\phi(x)$ be a test function with a compact support. Denote the second copy of $\mathbb{R}^2$ for the De Rham’s regularization by $y_1, y_2$. Then we calculate

$$
\int_{[T_1 \wedge T_2]} \phi
$$

$$
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_{x \in T_1} \int_{y \in T_2} f\left(\frac{x_1 - y_1}{\epsilon}, \frac{x_2 - y_2}{\epsilon}\right) \phi(x_1, x_2) (dx_1 - dy_1) \wedge (dx_2 - dy_2)
$$

(3.22)

Substitute $x_1 = x_2, y_1 = y_2$ for $T_1, T_2$, we obtain that

$$
\int_{[T_1 \wedge T_2]} \phi
$$

$$
\lim_{\epsilon \to 0} \frac{2}{\epsilon^2} \int_{x_2 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} f\left(\frac{x_2 - y_2}{\epsilon}, \frac{x_2 - y_2}{\epsilon}\right) \phi(x_1, x_2) (x_2 - y_2) dy_2 \wedge dx_2.
$$

(3.23)

Next we make a change of the variables

$$
\begin{cases}
  u = \frac{(x_2 - y_2)}{\epsilon} \\
  v = x_2 + y_2.
\end{cases}
$$

(3.24)

Then

$$
[T_1 \wedge T_2](\phi)
$$

$$
\lim_{\epsilon \to 0} \int_{u \in \mathbb{R}} \int_{v \in \mathbb{R}} uf(uv, u) \phi\left(\frac{\epsilon u + v}{2}, \frac{\epsilon u + v}{2}\right) dv \wedge du
$$

(3.25)

$$
\int_{(u,v) \in \mathbb{R}^2} uf(uv, u) \phi\left(\frac{v}{2}, \frac{v}{2}\right) dv \wedge du.
$$

Then the functional

$$
\phi \to \int_{(u,v) \in \mathbb{R}^2} uf(uv, u) \phi\left(\frac{v}{2}, \frac{v}{2}\right) dv \wedge du
$$

(3.26)

defines a current supported on $T_1$. So the intersection current

$$
[T_1 \wedge T_2]
$$

(which is (3.26)) is supported on $T_1$, depending on the convolution function $f$. 

\qed
Example 3.3. (real proper intersection)

Let $\mathcal{X} = \mathbb{R}^2$ be equipped with the De Rham data consisting of single chart $\mathbb{R}^2$ with the convolution function $f$ satisfying

$$\int_{\mathbb{R}^2} f(x_1, x_2) dx_1 \wedge dx_2 = 1 \quad (3.27)$$

where $x_1, x_2$ are Euclidean coordinates of $\mathbb{R}^2$.

Case 1: Let $T_1$ be a line through the origin $0$ and $T_2$ is another line segment through the origin. Then it is known that

$$[T_1 \wedge T_2] = \delta_0$$

if the order matches with the orientation of $\mathbb{R}^2$. (for instance see Proposition 3.1).

Case 2: Continuing from the setting in case 1, let $T_2$ be the line $x_1 = 0$. Let $T_1$ be a pieces of parabola

$$x_1 = x_2^2, x_2 \in (-1, 1). \quad (3.28)$$

Let’s calculate $[T_1 \wedge T_2]$. Let $\phi(x)$ be a test function supported in a neighborhood of the origin. We denote the second copy of $\mathbb{R}^2$ for De Rhams’ regularization by $(y_1, y_2)$. Then

$$\int_{[T_1 \wedge T_2]} \phi = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_{x_1 \in T_1} \int_{y_2 \in \mathbb{R}} f\left(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon} - \frac{y_2}{\epsilon}\right) \phi(x_1, x_2) dy_2 \wedge dx_1. \quad (3.29)$$

Let

$$f_1(x_1) = \int_{y_2 \in \mathbb{R}} f(x_1, -y_2) dy_2.$$

Now we continue (3.29) to have

$$\int_{[T_1 \wedge T_2]} \phi \quad \parallel \quad \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{(x_1, x_2) \in T_1} f_1\left(\frac{x_1}{\epsilon}\right) \phi(x_1, x_2) dx_1 \quad (3.30)$$

$$\phi(0) \left( \int_0^\infty f_1(x_1) dx + \int_0^\infty f_1(x_1) dx_1 \right) = 0,$$
So

\[[T_1 \wedge T_2] = 0\]

for all convolution function \(f\) in the De Rham data. This example shows the formula

\[\supp([T_1 \wedge T_2]) = \supp(T_1) \cap \supp(T_2)\]

does not hold for singular chains.

Case 3: Continuing from the setting in case 2, let \(T_2\) be the line \(x_1 = 0\).
Let \(T_1\) be a piece of the cubic curve

\[x_1 = x_2^3, x_2 \in (-1, 1)\].

(3.31)

The same calculation in case 2 shows if order of \(T_1, T_2\) is concordant with orientation of \(\mathbb{R}^2\), then

\[\int_{[T_1 \wedge T_2]} \phi = \phi(0)\].

(3.32)

Hence

\[[T_1 \wedge T_2] = \delta_0\]

(3.33)

where \(\delta_0\) is the \(\delta\)-function at the origin. So the intersection is independent choice of convolution function \(f\) in De Rham data.

Remark All three cases in Example 3.3 belong to the case of “real proper intersection” (not fully defined) which by the step 1 of Proposition 3.1 is independent of De Rham data. They also coincide with Kronecker index \(T_1 \wedge T_2[1]\) defined by De Rham. However the multiplicity associated to the “proper” components is different from that of the complex cases. Thus such an intrinsic problem of determination of the multiplicity has no satisfactory answer.

3.2 Complex case

Proposition 3.4. Let \(f : X \to Y\) be a regular map between two smooth projective varieties. Let \(W\) be a \(p\) dimensional algebraic cycle of \(X\). Then the current \(f_*[W]\) is the current of integration over the cycle

\[f_*W\].
where \([W]\) stands for the current of integration over the algebraic set.

**Proof.** Let \(W = \sum_i a_i W_i\) where \(W_i\) are irreducible and \(a_i\) are non-zero integers. Let \(f_* W = \sum_i b_i S_i\) where \(b_i\) are non-zero integers divisible by non-zero \(a_i\) and \(S_i\) are irreducible subvarieties. Let \(|W_0|\) be the open sets of the support \(|W|\) such that \(f\) is smooth. Then correspondingly \(f(|W_0|) = \bigcup_i S_i^0\), where \(S_i^0\) are open sets of \(S_i\). By the definition of the push-forward of algebraic cycles, the map

\[
f : W_i \rightarrow S_i
\]

is a finite to one morphism, and \(f\) is restricted to an etal morphism on \(f^{-1}(S_i^0)\). Then using currents, we have

\[
f_*[f^{-1}(S_i^0)] = \frac{b_i}{a_i} S_i^0.
\]

(3.35)

Taking the closure and the sum over \(i\), we obtain that

\[
f_*(\sum i a_i [W_i]) = \sum_i b_i [S_i].
\]

(3.36)

Since for algebraic cycles, we have

\[
f_*(\sum_i a_i W_i) = \sum_i b_i S_i,
\]

(3.37)

we complete the proof.

\(\Box\)

**Theorem 3.5.** Let \(X\) be a smooth projective variety of dimension \(n\) over \(\mathbb{C}\). Let \(T_1, T_2\) be irreducible subvarieties of \(X\) of dimension \(p, q\). To abuse the notations, the currents of integration over them are also denoted by \(T_1, T_2\) respectively. Assume \(T_1 \cap T_2\) is proper. Then with an arbitrary De Rham data \(\mathcal{U}\) on \(X\), the current \([T_1 \wedge T_2]\) is independent of \(\mathcal{U}\), and equals to the current of integration over the algebraic cycle

\[
T_1 \cdot T_2,
\]

where \(T_1 \cdot T_2\) is the Fulton’s intersection ([3]) defined as the linear combination of all irreducible components of the scheme

\[
T_1 \cap T_2.
\]

Furthermore the assertion extends to algebraic cycles linearly.
Remark Theorem shows that $[T_1 \wedge T_2]$ in this case is intrinsic. This is the principle for the real intersection theory: every intrinsically defined intersection in the past can be interpreted as an intersection of currents that are independent of De Rham data.

Proof. Let’s fix the cycle $T_2$. By example 11.4.2, [3], there is an algebraic cycle $E_1$ rationally equivalent to $T_1$ such that $A$ meets $T_2$ transversely (at an open set of each irreducible support). Without losing the generality, let’s have a simplified setting as follows. Let $V \subset X \times \mathbb{P}^1$ be an irreducible subvariety, and $P_1 : V \to X, P_2 : V \to \mathbb{P}^1$ are the projections. Let $T_2 \subset X$ be an irreducible subvariety. Assume the cycle of the scheme $P_2^{-1}(1)$ is $E_1$ and the cycle of the scheme $P_2^{-1}(0)$ is $T_1$, where 0, 1 are two points of $\mathbb{P}^1$. Let $I$ be a real curve in $\mathbb{P}^1$ connecting 0, 1. Next we consider two objects: currents and algebraic cycles. Using the currents, according to Proposition 4.12 below in section 4, we have the formula

$$[T_1 \wedge T_2] - [E_1 \wedge T_2] = d \left( (P_1)_*(V_I(T_2)) \right) \quad (3.38)$$

where $V_I(T_2)$ is the current

$$[V \wedge (T_2 \otimes I)]. \quad (3.39)$$

Next we consider the algebraic cycles in intersection theory where two rationally equivalent algebraic cycles are homotopic. More precisely we have the equation in singular cycles

$$T_1 \cdot T_2 - E_1 \cdot T_2 = d \left( (P_1)_*(V'_I(T_2)) \right), \quad (3.40)$$

where $V'_I(T_2)$ is the singular cycle defined by the semi-algebraic set in the complex manifold $X \times \mathbb{P}^1$, $T_1 \cdot T_2, E_1 \cdot T_2$ are singular cycles obtained from the triangulation of the intersectional algebraic cycles, and $d$ is the differential operator on the singular chains (the boundary operator with a sign). We should note that the current of integration over $V'_I(T_2)$ is the current $V_I(T_2)$. Next we convert the equation (3.40) to that in currents to have

$$[T_1 \cdot T_2] - [E_1 \cdot T_2] = d \left( (P_1)_*(V_I(T_2)) \right). \quad (3.41)$$
Hence
\[ [T_1 \land T_2] - [E_1 \land T_2] = [T_1 \cdot T_2] - [E_1 \cdot T_2] \tag{3.42} \]
(where the difference of two sides should be noticed). By Proposition 3.1, since \(E_1\) meets \(T_2\) transversely,
\[ [E_1 \land T_2] = [E_1 \cdot T_2]. \tag{3.43} \]
Thus
\[ [T_1 \land T_2] = [T_1 \cdot T_2]. \]
We complete the proof.

**Theorem 3.6.** Let \(X\) be a smooth projective variety of dimension \(n\) over \(\mathbb{C}\). Let \(T_1, T_2\) be subvarieties of \(X\) of codimension \(p, q\). The currents of integration over them are also denoted by \(T_1, T_2\) respectively. Assume \(T_1 \cap T_2\) is an excess intersection. Then
\[ [T_1 \land T_2] \tag{3.44} \]
in general depends on the De Rham data \(\mathcal{U}\).

**Proof.** Let’s give an example where \([T_1 \land T_2]\) is dependent of De Rham data. Let \(\mathbb{P}^2\) be a projective space over \(\mathbb{C}\) with affine coordinates \((z_1, z_2)\). Let \(T_1\) be the hyperplane \(z_2 = 0\), and \(T_2 = T_1\). First it is not zero because its reduction to cohomology group is non-zero. Choose two open sets as De Rham’s covering: \(U_1\), the finite affine plane, and a small neighborhood \(U_2\) of the infinity \(\mathbb{P}^1 \subset \mathbb{P}^2\). Choose real Euclidean coordinates \(x_1, y_1, x_2, y_2\) for \(U_1\) such that
\[ z_1 = x_1 + iy_2, \quad z_2 = x_2 + iy_2. \]
Use these open covering and Euclidean coordinates to have a De Rham data for \(\mathbb{P}^2\) with a convolution function \(h(x_1, x_2, y_1, y_2)\) of the unit ball \(B\) in \(U_1\). Then we see in \(U_1\),
\[ R^1_\epsilon(T_2) = -\frac{1}{\epsilon^4} \int \int_{(x', y') \in \mathbb{R}^2} h\left(\frac{x_1 - x'}{\epsilon}, \frac{x_2}{\epsilon}, \frac{y_1 - y'}{\epsilon}, \frac{y_2}{\epsilon}\right) dx' \land dy' \land dx \land dy, \tag{3.45} \]
where \( x_i', y_i' \) are the Euclidean coordinates for the second factor in the smoothing operator. The composing with another local smoothing operator from \( U_2 \) will not change the smooth current \( R^1(T_2) \) in \( B \). Thus for a test form \( \phi \) supported in \( B \), the integral

\[
\int_{T_1} R^B(B) \wedge \phi = \int_{x_2 = y_2 = 0} (\cdots) dx_2 \wedge dy_2 = 0. \tag{3.46}
\]

This shows with this type of De Rham data,

\[
[T_1 \wedge T_2] \tag{3.47}
\]

is zero on \( U \cap T_1 \). Hence \([T_1 \cap T_2] \) is a 0-dimensional current supported at the infinity point of \( T_1 \). Since the \( \infty = \mathbb{P}^1 \) is arbitrary, \([T_1 \wedge T_2] \) is supported on an arbitrary set determined by the De Rham data.

\[\square\]

**Example 3.7.**

The intersection of currents of integration over algebraic cycles always exists because algebraic cycles are Lebesgue. But its intersection depends on De Rham data. Theorem 3.5 asserts that in case of a proper intersection, it is actually independent of De Rham data. But for excess intersection, the situation still goes back to the dependence of De Rham data, and the interpretation is different from Fulton’s.

| Intersection     | Cycle                      | Cycle class                        | Support             |
|------------------|----------------------------|------------------------------------|---------------------|
| algebraic \( T_1 \cdot T_2 \) | not well-defined         | well-defined in the Chow ring, and cohomology ring | \( T_1 \cap T_2 \) |
| current \( [T_1 \wedge T_2] \) | well-defined, but \( U \)-dependent | not well-defined in the Chow ring, well-defined in cohomology ring | \( T_1 \cap T_2 \) |
4 Tools for application

4.1 Correspondence of a current

Lemma 4.1. Let $\mathcal{X}, \mathcal{Y}$ be two compact manifolds, and $P_\mathcal{X}$ be the projection $\mathcal{X} \times \mathcal{Y} \to \mathcal{X}$.

Then the image of the projection
\[(P_\mathcal{X})_* : \mathcal{C}(\mathcal{X} \times \mathcal{Y}) \to \mathcal{D}'(\mathcal{X})\]
lies in $\mathcal{C}(\mathcal{X})$.

Proof. Notice there is a coordinates chart of $\mathcal{X} \times \mathcal{Y}$ satisfying that the coordinates planes of $\mathcal{X}$ are also the coordinates planes for $\mathcal{X} \times \mathcal{Y}$. Thus the two conditions of Lebesgue currents for $\mathcal{X}$ are implied by that for $\mathcal{X} \times \mathcal{Y}$. 

Definition 4.2. Let $\mathcal{X}, \mathcal{Y}$ be two compact manifolds.

Let
\[F \in \mathcal{C}(\mathcal{X} \times \mathcal{Y}) \quad (4.1)\]
be a Lebesgue current. Let $\mathcal{U}$ be a product De Rham data on $\mathcal{X} \times \mathcal{Y}$. Let $P_\mathcal{X}, P_\mathcal{Y}$ be the projections $\mathcal{X} \times \mathcal{Y} \to \mathcal{X}, \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$.

Define pull-back of currents
\[F^*(T)\]
by
\[F^* : \mathcal{C}(\mathcal{Y}) \to \mathcal{C}(\mathcal{X})
T \mapsto (P_\mathcal{X})_*[F \wedge ([\mathcal{X} \otimes T])]. \quad (4.2)\]

Define the push-forward $F_*(T)$ of currents by
\[F_* : \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{Y})
T \mapsto (P_\mathcal{Y})_*[F \wedge (T \otimes [\mathcal{Y}])] \quad (4.3)\]
Proposition 4.3. Let $X, Y, Z$ be three smooth projective varieties over $\mathbb{C}$. Let $F_{XY}, F_{YZ}$ be algebraic cycles on $X \times Y$ and $Y \times Z$ respectively, and represent finite correspondences ([6]). Then

$$ (F_{YZ})_* \circ (F_{XY})_* = (F_{YZ} \circ F_{XY})_* , $$

(4.4)

where $F_{YZ} \circ F_{XY}$ denotes the multiplication of algebraic correspondences. Similarly

$$ (F_{XY})^* \circ (F_{YZ})^* = (F_{XY}^t \circ F_{YZ}^t)^* , $$

(4.5)

where $F_{XY}^t, F_{YZ}^t$ are the transposes of the correspondences.

Proof. Because they are finite correspondences, hence

$$ (F_{XY} \times Z) \cdot (X \times F_{YZ}) $$

(4.6)

is a well-defined algebraic cycle $\Gamma$ in $X \times Y \times Z$. Let $\sigma \in C(X)$. By Theorem 3.5, evaluations of both sides of (4.4) is equal to

$$ (P_Z)_*[(\sigma \otimes Y \otimes Z) \wedge \Gamma] , $$

(4.7)

where $P_Z : X \times Y \times Z \to Z$ is the projection. So (4.4) is proved. The (4.5) is the transpose of (4.4).

Proposition 4.4. Let $X, Y$ be compact complex manifolds. The pull-back and push-forward of currents extend Gillet and Soulé’s push-forward of currents and smooth pull-back of currents.

Proof. In [4], Gillet and Soulé defined operations on the currents on compact complex manifolds. They include push-forward for proper maps and pull-back for smooth maps. We verify that these operations coincide with ours.

Let

$$ f : X \to Y $$

(4.8)
be a regular map. Let $F$ be its graph. Let $T$ be a Lebesgue current on $X$. Let $\phi$ be a $C^\infty$ form on $Y$. We use product De Rham data on $X \times Y$ and projection De Rham data on $X$ and $Y$. Then

$$
\int_{F^\ast(T)} \phi \\
= \lim_{\epsilon \to 0} \int_F R^X_{\epsilon \times Y} (T \times Y) \wedge (P_Y)^\ast(\phi)
$$

(by Proposition 2.8, the projection formula)

$$
= \lim_{\epsilon \to 0} \int_F (P_X)^\ast R^X_\epsilon (T) \wedge (P_Y)^\ast(\phi)
$$

(4.9)

$$
= \lim_{\epsilon \to 0} \int_X R^X_\epsilon (T) \wedge f^\ast(\phi)
$$

$$
= \int_T f^\ast(\phi).
$$

This shows

$$
F^\ast(T) = f^\ast(T)
$$

where $f^\ast$ is defined as the dual of the pullback on forms in 1.1.4, ([4]).

Now let

$$
\begin{align*}
f : X & \to Y \\
F & \subset X \times Y
\end{align*}
$$

(4.10)

be a smooth map. Let $F$ be its graph. Let $\phi$ be a test form on $X$.

$$
\int_{F^\ast(T)} \phi = \int_{(P_X)^\ast[F \wedge (X \times T)]} \phi \\
= \lim_{\epsilon \to 0} \int_F R^X_{\epsilon \times Y} (X \times T) \wedge (P_X)^\ast(\phi)
$$

(by Proposition 2.8, the projection formula)

(4.11)

$$
= \lim_{\epsilon \to 0} \int_F (P_Y)^\ast(R^Y_\epsilon(T)) \wedge (P_X)^\ast(\phi).
$$

Notice

$$
P_Y : F \to Y
$$

(4.12)
is isomorphic to $f$ which is also smooth. Then we apply the fibre integral to have

$$
\lim_{\epsilon \to 0} \int_F (P_Y)^* (R^Y_\epsilon (T)) \wedge (P_X)^*(\phi) = \int_T f_*(\phi).
$$  \hspace{1cm} (4.13)

Thus

$$
F^*(T) = f^*(T).
$$  \hspace{1cm} (4.14)

We complete the proof.

\textbf{Proposition 4.5.} Let $\mathcal{X}, \mathcal{Y}$ be two compact manifolds. Let

$$
F \in C(\mathcal{X} \times \mathcal{Y})
$$  \hspace{1cm} (4.15)

be a homogeneous closed, Lebesgue current.

(a) Let $T$ be a Lebesgue current of $\mathcal{X}$ or $\mathcal{Y}$. Then $\text{supp}(F_*(T))$ is contained in the set

$$
P_Y \left( \text{supp}(F) \cap (\text{supp}(T) \times \mathcal{Y}) \right);$$

$\text{supp}(F^*(T))$ is contained in the set

$$
P_X \left( \text{supp}(F) \cap (\mathcal{X} \times \text{supp}(T)) \right).$$

(b) If $T_1, T_2$ are Lebesgue and closed (rspt. homologous to zero) in $\mathcal{X}$ and $\mathcal{Y}$ respectively, then $F_*(T_1), F_*(T_2)$ are also closed (rspt. homologous to zero).

\textit{Proof.} (a) Let $S$ be a Lebesgue current on $\mathcal{X} \times \mathcal{Y}$. Let $a \notin P_Y(\text{supp}(S))$. Then there is a neighborhood $B_a \subset \mathcal{Y}$ of $a$, such that

$$
(\mathcal{X} \times B_a) \cap \text{supp}(S) = \emptyset.
$$

Then for any $\phi \in \mathcal{D}(\mathcal{Y})$ supported in $B_a$,

$$
\int_S (P_Y)^*(\phi) = 0.
$$  \hspace{1cm} (4.16)

Then (4.16) says

$$
a \notin \text{supp}((P_Y)_*(S)).$$
Hence So
\[ \text{supp}((P_Y)_*(S)) \subset P_Y(\text{supp}(S)). \] (4.17)

Similarly
\[ \text{supp}((P_X)_*(S)) \subset P_X(\text{supp}(S)). \] (4.18)

Now we consider our case. Applying the assertion (4.17) for
\[ S = F \wedge (T \otimes Y), \]
together with part (1), property 2.6,
\[ \text{supp}((P_Y)_*[F \wedge (T \times Y)]) \]
\[ \cap \]
\[ \text{supp} \left( P_Y(\text{supp}([F \wedge (T \times Y)])) \right) \]
\[ \cap \]
\[ \text{supp} \left( P_Y(\text{supp}(F) \cap (\text{supp}(T) \times Y)) \right). \] (4.19)

The proof of
\[ \text{supp}(F^*(T)) \subset P_X \left( \text{supp}(F) \cap (X \times \text{supp}(T)) \right). \] (4.20)
is similar.

(b) By property 2.6, the currents
\[ [F \wedge (T_1 \times Y)], [F \wedge (X \times T_2)] \]
are closed. Therefore \( F^*T_2, F_4T_1 \) are closed. If they are homologous to zero, then by the property 2.6,
\[ [F \wedge (T_1 \times Y)], [F \wedge (X \times T_2)] \]
are homologous to zero in \( \mathcal{X}, \mathcal{Y} \). Thus \( F^*T_2, F_4T_1 \) are homologous to zero.

We complete the proof \( \square \)
Example 4.6. Let $X, Y$ be two smooth projective varieties over $\mathbb{C}$, 

$$f : X \dashrightarrow Y$$

be a rational map. Then there is graph 

$$F \subset X \times Y. \quad (4.21)$$

Once $X \times Y$ is equipped with De Rham data (which does not have any requirements for $X, Y$), there are homomorphisms $F_*, F^*$

$$F_* : \mathcal{C}(X) \to \mathcal{C}(Y)$$

$$F^* : \mathcal{C}(Y) \to \mathcal{C}(X). \quad (4.22)$$

When $\mathcal{C}(X), \mathcal{C}(Y)$ are reduced to cohomology, $F_*, F^*$ are reduced to the usual cohomological correspondences.

### 4.2 Functoriality

The real intersection theory is not functorial in the usual category of algebraic varieties. However if we attach De Rham data, then it has a functoriality.

**Proposition and Definition 4.7.** Let $\text{Cord}(\mathbb{C})$ be the category whose objects are the pairs of a smooth projective variety over $\mathbb{C}$, and a De Rham data on it, denoted by $(X, U)$. The morphisms are finite correspondences of $X \times Y$.

**Proof.** The verification of the category is done in [6].

**Definition 4.8.** Let $k$ be a whole number. Let $(X, U) \in \text{Cord}(\mathbb{C})$. Define $N_k \mathcal{C}(X)$ to be the linear span of Lebesgue currents

$$T \in \mathcal{C}(X)$$
satisfying
(1) \( \text{dim}(T) \leq k \), OR \( \text{dim}(T) \geq 2n - k \), OR,
(II) \( \text{supp}(T) \) lies in an algebraic set \( A \) of dimension
\[ \leq \frac{\text{dim}(T) + k}{2}. \]

A current in \( N_k \mathcal{C}(X) \) will be called \( N_k \) leveled. The objects \( N_k \mathcal{C}(X) \) with usual group homomorphisms form a subcategory of the Abelian category, denoted by \( N_k \mathcal{C} \).

**Remark** De Rham data \( \mathcal{U} \) plays no role in both categories \( \text{Cord}(\mathbb{C}) \) and \( N_k \mathcal{C} \). However the functoriality needs De Rham data.

**Proposition and Definition 4.9.** The maps
\[ (X, \mathcal{U}) \to N_k \mathcal{C}(X) \] (4.23)
and
\[ \text{Hom in } \text{Cord}(\mathbb{C}) \to \text{Hom in } N_k \mathcal{C} \]
\[ \Gamma \to \Gamma^*, \quad (\text{covariant}) \]
\[ \Gamma \to \Gamma^*, \quad (\text{contravariant}) \] (4.24)
define a covariant functor and a contravariant functor.

**Proof.** Let \((X, \mathcal{U}_X), (Y, \mathcal{U}_Y)\) be two objects in \( \text{Cord}(\mathbb{C}) \). Let \( F \in \mathcal{Z}(X \times Y) \) be a homomorphism in \( \text{Cord}(\mathbb{C}) \). The corresponding homomorphism in category \( N_k \mathcal{C} \) is defined is defined to be \( F_* \)
\[ \sigma \to (P_Y)_* [F \wedge (\sigma \otimes Y)] \]
where \( \sigma \in N_k \mathcal{C}(X) \). It suffices to show \( F_* \) satisfies two conditions:
(a) \( F_* \) maps a \( N_k \) leveled current to a \( N_k \) leveled current.
(b) The map satisfies the composition criterion, i.e. if \( X, Y, W \) are smooth projective varieties over \( \mathbb{C} \), and \( Z_1, Z_2 \) algebraic cycles are finite correspondences between \( X, Y \) and \( Y, W \), then
\[ (Z_2 \circ Z_1)_* = (Z_2)_* \circ (Z_1)_*, \] (4.25)
where \( Z_2 \circ Z_1 \) is the composition of finite correspondences.

Proof of (a): By the definition of the intersection of currents, we obtain
\[
\deg(F_*(\sigma)) = \deg(\sigma).
\]

Let \( A \) be an algebraic set containing \( \sigma \) such that the level of \( \sigma \) is
\[
k = \deg(\sigma) - 2\deg_C(A).
\]

For any algebraic set \( A \), since \( F \) is a finite correspondence,
\[
\deg(F_*(A)) = \deg(A). \quad (4.26)
\]

The level of \( F_*(\sigma) \) is
\[
\deg(F_*(\sigma)) - 2\deg(F_*(A))
\]
which is equal to
\[
k = \deg(\sigma) - 2\deg_C(A).
\]

The proof for contravariant \( F^* \) is similar.

Proof of (b): This is Proposition 4.3.

\[\square\]

Let
\[
\begin{align*}
\kappa : & C(X) \to D'(X) \\
\quad T & \to dT.
\end{align*}
\]

Let \( B(X) = \kappa^{-1}(C(X)) \) be the sublinear space that at least contain singular chains and \( C^\infty \) forms. By Leibniz rule, part (6) of Property 2.6, the intersection \([ \cdot \wedge \cdot ]\) send \( B(X) \times B(X) \) to \( B(X) \). So we let
\[
N_k B(X) = N_k C(X) \cap B(X).
\]

**Proposition 4.10.** Then
(a) \( N_\bullet B(X) \) forms a decreasing filtration of complex of \( B(X) \)
(b) its spectral sequence \( E_\bullet \) converges to the \( \mathbb{R} \) coefficiented, algebraically leveled filtration defined as
\[
N_k(X) = \sum_{r=\dim(X)-k}^{2\dim(X)-k} N^r H^{2r+k}(X), k = 0, \ldots
\]

on the total real cohomology \( H(X; \mathbb{R}) \) (See [8] for details on leveled filtration).
4 TOOLS FOR APPLICATION

Proof. (a). Let $T \in \mathcal{N}_k \mathcal{B}(X)$. If $\text{dim}(T) \in [0, k + 1] \cup [2n - k - 1, 2n]$, then by the definition $T$ is $\mathcal{N}_{k+1}$ leveled. If $\text{dim}(T) \in (k + 1, 2n - k - 1)$, then $\text{dim}(T) \in (k, 2n - k)$. If $\text{dim}(T)$ is not in above cases, there is an algebraic set $A$ such that

$$\text{supp}(T) \subset A, \quad \text{and dim}(A) \leq \frac{\text{dim}(T) + k}{2}.$$ 

This implies that

$$\text{supp}(T) \subset A, \quad \text{and dim}(A) \leq \frac{\text{dim}(T) + k + 1}{2}.$$ 

So

$$T \in \mathcal{N}_{k+1} \mathcal{B}(X).$$

This shows that

$$\mathcal{N}_0 \mathcal{B}(X) \subset \cdots \mathcal{N}_k \mathcal{B}(X) \subset \mathcal{N}_{k+1} \mathcal{B}(X) \subset \cdots \subset \mathcal{B}(X)$$

form a filtration. Since the differential $d$ on the differential form preserves the support, $d$ maps $\mathcal{N}_k \mathcal{B}(X)$ to $\mathcal{N}_k \mathcal{B}(X)$. Thus $(\mathcal{N}_\bullet \mathcal{B}, d)$ is a decreasing filtration of the complex.

(b) Because $\mathcal{B}(X)$ includes singular chains, the limit of the spectral sequence,

$$\sum_{p,q} Gr^p(H^{p+q}(\mathcal{N}_\bullet \mathcal{B}(X)))$$

is a filtration on the cohomology

$$H(X; \mathbb{R}).$$

Notice that by the definition, this filtration is the algebraically leveled filtration.

\[ \square \]

4.3 Family of currents
Definition 4.11. Let $S$ and $\mathcal{X}$ be manifolds equipped with De Rham data. Let $S \times \mathcal{X}$ be equipped with the product De Rham data. Let $I \in C(S \times X)$ be a homogeneous Lebesgue current. Let $P_{\mathcal{X}}$ be the projection $S \times \mathcal{X} \to \mathcal{X}$.

We denote

$$((P_{\mathcal{X}})_*[(\{s\} \otimes \mathcal{X}) \wedge I])$$

by $I_s$. The set $\{I_s\}$ for all such $s$ in $S$ will be called a family of of currents parametrized by $S$.

Remark We should note that we have abused the notation to use $P_{\mathcal{X}}$ for its restriction $P_{\mathcal{X}}|_{\cdot}$ to a subset. The family $I_s$ depends on extrinsic De Rham data.

Proposition 4.12. Let $\mathcal{X}$ be a manifold. Let $S$ be real one dimensional. Let $I_\epsilon \subset S$ be diffeomorphic to a finite closed interval of $\mathbb{R}$ with two end points $0$ and $\epsilon > 0$. Let $S$ be equipped with De Rham data, $S \times \mathcal{X}$ be equipped with a product De Rham data and $\mathcal{X}$ be equipped with corresponding the projection De Rham data. Let $J$ be a Lebesgue current $S \times \mathcal{X}$.

Then for a closed current $T \in C(\mathcal{X})$, there is a well-defined current $J_{I_\epsilon}(T)$ on $\mathbb{R} \times \mathcal{X}$ such that

$$(1) \quad [J_{I_\epsilon} \wedge T] - [J_0 \wedge T] = (-1)^k \epsilon d((P_{\mathcal{X}})_* J_{I_\epsilon}(T)) + (-1)^p [I_\epsilon \otimes T] \wedge dJ$$

where $k = \deg(J), p = \deg(T), P_{\mathcal{X}} : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ is the projection.

$$\quad \quad \quad \quad \quad \quad \text{(4.31)}$$

$$(2) \quad (P_{\mathcal{X}})_* J_{I_\epsilon}(T) = \epsilon (P_{\mathcal{X}})_* J_{I_1}(T)$$

$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad [[I_\epsilon \otimes T] \wedge dJ] = \epsilon [[I_1 \otimes T] \wedge dJ],$$

and $[J_{I_\epsilon} \wedge T]$ is continuous in $\epsilon$ ( in the topology of currents).\textsuperscript{5}

\textsuperscript{5} The continuity for higher dimensional parameter space $S$ does not hold
Proof. Let \( \mathbb{R} \times \mathcal{X} \) be equipped with a product De Rham data. We defined
\[
\mathcal{J}_\mathcal{I}(T) = [\mathcal{J} \wedge (I_\varepsilon \otimes T)].
\]
(4.33)
Then we calculate
\[
d\mathcal{J}_\mathcal{I}(T)
= [d(I_\varepsilon \otimes T) \wedge \mathcal{J}] + (-1)^{p+1}[(I_\varepsilon \otimes T) \wedge d\mathcal{J}]
\]
(by associativity and commutativity, Property 2.6.)
\[
= (-1)^{kp}[((\{\varepsilon\} - \{0\}) \otimes \mathcal{X}) \wedge \mathcal{J} \wedge (\mathbb{R} \otimes T)] + (-1)^{p+1}[(I_\varepsilon \otimes T) \wedge d\mathcal{J}]
\]
(4.34)
By Definition 4.11,
\[
(P_X)_*[((\{\varepsilon\} - \{0\}) \otimes \mathcal{X}) \wedge \mathcal{J} - [\mathcal{J} - [\mathcal{J}_0]] = [\mathcal{J}_\varepsilon - [\mathcal{J}_0]].
\]
Thus applying the projection formula, Proposition 2.8, we obtain
\[
(P_X)_*[((\{\varepsilon\} - \{0\}) \otimes \mathcal{X}) \wedge \mathcal{J} \wedge (\mathbb{R} \otimes T)] = [\mathcal{J}_\varepsilon \wedge T] - [\mathcal{J}_0 \wedge T].
\]
(4.35)
Combination of (4.34), (4.35) is the assertion (4.31). We complete the proof.
(2) We apply the construction (4.33). Then (4.32) follows from the equality of currents,
\[
I_\varepsilon = \varepsilon I_1.
\]
Finally by part (1) and formula (4.32) the family \([\mathcal{J}_\varepsilon \wedge T]\) is continuous in \(\varepsilon\).
\[\square\]

Example 4.13. Let \( X \) be a smooth projective variety of dimension \( n \) over \( \mathbb{C} \). Let \( T \) be a closed Lebesgue current representing a non-zero primitive cohomology class in \( H^n(X; \mathbb{Q}) \). Let
\[
V \subset \mathbb{P}^1 \times X.
\]
(4.36)
be a Lefschetz pencil in \( X \). Assume \( \mathbb{P}^1 \times X \) is equipped with a product De Rham data. Let
\[
\mathcal{I} = [V \wedge (\mathbb{P}^1 \times T)].
\]
(4.37)
be the intersection current. Then its fibre currents \( \mathcal{I}_t \) are exact for all \( t \). But \( \mathcal{I} \) is not because \( T \) is not.
Example 4.14. Let $\mathcal{X}$ be a manifold and $T$ a non-zero homogeneous Lebesgue current in $\mathcal{X}$. Let $S^1 \times \mathcal{X}$ be equipped with a product De Rham data. Let $t_0$ be a point of $S^1$. Then $\mathcal{I} = \{t_0\} \otimes T$ gives a family of currents by Definition 4.11. Notice $\mathcal{I}_t = 0$ for all $t$ including $t_0$, but $\mathcal{I}$ is non-zero, and not closed provided $T$ is not.

5 Generalized Hodge conjecture on 3-folds

We give an application in complex geometry.

Theorem 5.1. Generalized Hodge conjecture is correct on a 3-fold $X$.

Proof. Let $X$ be a smooth projective variety over $\mathbb{C}$. In the cohomology vector space with rational coefficients, we denote the coniveau filtration of coniveau $i$ and degree $2i + k$ by

$$N^i H^{2i+k}(X)$$

(defined in [5] where the coniveau filtration is called the arithmetic filtration) and the linear span of sub-Hodge structures of Hodge coniveau $i$ and degree $2i + k$ by

$$M^i H^{2i+k}(X).$$

The generalized Hodge conjecture asserts that

$$M^i H^{2i+k}(X) = N^i H^{2i+k}(X). \quad (5.1)$$

for all existing $i, k$. For a 3-fold $X$, the only non-trivial cases are

$$M^1 H^2(X) = N^1 H^2(X)? \quad (5.2)$$
$$M^1 H^3(X) = N^1 H^3(X)? \quad (5.3)$$

The formula (5.2) is well-known as Lefschetz (1, 1) theorem. For (5.3), Delign’s lemma 8.2.8, [1] already implies that

$$N^1 H^3(X) \subset M^1 H^3(X).$$
Thus it is sufficient to prove
\[ M^1 H^3(X) \subset N^1 H^3(X). \] (5.4)

Let \( L \subset H^3(X; \mathbb{Q}) \) be a sub-Hodge structure of coniveau 1. So it is polarized. In [7], Voisin used the intermediate Jacobian to prove a cohomological result that says there is a smooth projective curve \( C \), and a Hodge cycle
\[ \Psi \in \text{Hdg}^4(C \times X) \] (5.5)
such that
\[ \Psi^* (H^1(C; \mathbb{Q})) = L. \] (5.6)
where \( \Psi^* \) is defined as the image
\[ \langle P \rangle^* \left( \Psi \cup ((\bullet) \otimes 1)) \right), \] (5.7)
of the cohomological map \( \langle P \rangle \) with the projection \( P : C \times X \to X \). Next we convert the cohomological expression (5.7) to current’s expression as
\[ P^* \left[ T \wedge ((\bullet) \otimes 1)) \right], \] (5.8)
where \( T \) is any current representing \( \Psi \). (The formula (5.8) is the key turning point of the proof that has no difference from (5.7) in cohomology. However it carries the information of the support which is absent in (5.7)). Next we analyze the current \( T \). Notice \( \langle P \rangle^* \) is a Hodge morphism, so \( \langle P \rangle^* (\Psi) \) is a Hodge cycle in \( X \). By the Lefschetz (1, 1) theorem \( \langle P \rangle^* (\Psi) \) is algebraic on \( X \). So there is a singular cycle \( T_\Psi \) on \( C \times X \) representing the class \( \Psi \) such that the projection in currents satisfies
\[ P^* (T_\Psi) = S + bW \] (5.9)
where \( S \) is a current of integration over the algebraic cycle \( S \), and \( bW \) is an exact Lebesgue current of dimension 4 in \( X \). (adjust \( \Psi \) so \( S \) is non-zero). Consider another current in \( C \times X \)
\[ T := T_\Psi - [\epsilon] \otimes bW \] (5.10)
denoted by $T$, where $[e]$ is a current of evaluation at a point $e \in C$. Note $T$ is Lebesgue. By adjusting the singular chain $W$ continuously, we can assume the projection of the support of $T$ satisfies

$$P(supp(T)) = supp(P_*(T)),$$

(5.11)
i.e. the projection of the support is the support of the projection. (See appendix for the proof). Thus we have the projection of currents

$$P_*(T) = S.$$ 

(5.12)

Let $\Theta$ be the collection of closed Lebesgue currents on $C$ representing the classes in $H^1(C; \mathbb{Q})$. Now we apply the real intersection theory to establish the correspondence of currents (in section 4.1),

$$T_*(\Theta)$$

(5.13)
defined as (5.8). It gives a family of currents (parametrized by $\Theta$) supported on the support of the current

$$P_*(T) = S,$$

which is the integration over an algebraic cycle $S$, i.e. the family of currents are all supported on the algebraic set $|S|$. This is a criterion for coniveau filtration in terms of currents, i.e. for $\beta \in T_*(\Theta)$, the cohomology class $\langle \beta \rangle$ of $\beta$ satisfies

$$\langle \beta \rangle \in ker \left( H^3(X; \mathbb{Q}) \to H^3(X - |S|; \mathbb{Q}) \right)$$

(5.14)

By Proposition 4.5 and Voisin’s assertion (5.6), the collection of cosmological classes of the currents in $T_*(\Theta)$ consists of all classes in $L$. This shows $L \subset N^1H^3(X)$. We complete the proof.

\[ \square \]

**Appendices**

**A  Support of the projection**

In this Appendix, we study the supports of cellular cycles in a Cartesian product.
Let $\mathcal{X}$ be a compact manifold of dimension $n$. We use the following setting in algebraic topology. A $p$-singular simplex $S$ consists of three elements: a $p$-dimensional polyhedron $\Delta^p$ in $\mathbb{R}^v$, an orientation of $\mathbb{R}^v$, and a $C^\infty$ map $f$ of $\mathbb{R}^v$ to $\mathcal{X}$. A chain is a linear combination of singular simplexes. The support $|S|$ of $S$ is the image of $S$ in $\mathcal{X}$. A point in $S$ is a point in $|S|$.

Let $\mathcal{Y}$ be another compact manifold of dimension $m$. Let $P : \mathcal{Y} \times \mathcal{X} \rightarrow \mathcal{X}$ (A.1) be the projection.

**Definition A.1.** Let $\sigma$ be a $C^\infty$ $p$-singular simplex of $\mathcal{Y} \times \mathcal{X}$. Let $a$ be an interior point of $\sigma$. If

$$P^{-1} \circ P(a) \cap \sigma$$

is a finite set, we say $\sigma$ is finite at $a$. If $\sigma$ is finite at all interior points of $\sigma$, we say $\sigma$ is finite to $\mathcal{X}$. The chain is finite if each simplex in the chain is finite.

**Proposition A.2.** For any $C^\infty$ $p$-singular simplex $\sigma$ in the coordinates chart of $\mathcal{Y} \times \mathcal{X}$ with $p \leq \dim(\mathcal{X})$, there is barycentric subdivision (multiple times) of $\sigma$

$$Sd(\sigma) = \sum_{\text{finite } i} C_i,$$

(A.2)

such that each simplex $C_i$ is homotopic to another simplex finite to $\mathcal{X}$ and the homotopy is a constant on the $\partial Sd(\sigma)$

**Proof.** Let

$$\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n$$

be the coordinate's charts for $\mathcal{Y}, \mathcal{X}, \mathcal{Y} \times \mathcal{X}$ respectively such that $p \leq n$. We would like to show that there is a multi barycentric subdivision to divide $\sigma$ to a chain $\sum_{i=0}^{N} \sigma_i$ (a sum of smaller regular cells $\sigma_i$) such that there are homotopy $\sigma_i'$ for each $\sigma_i$ that is finite to $\mathbb{R}^n$, and boundary of $\sigma_i'$ is the same as that of $\sigma_i$. We use a claim to construct such small simplex $\sigma_i$. 

Claim A.3. Let \( g : \mathbb{R}^k \rightarrow \mathbb{R}^l \) be a \( C^\infty \) map with \( l \geq k \). Let \( q \in \mathbb{R}^k \) be a point. Then there is an open ball \( B \) of \( q \) and continuous map \( g' : \mathbb{R}^k \rightarrow \mathbb{R}^l \) such that

1) \( g \) is homotopically deformed to \( g' \) such that at all points on \( \partial B \) and the boundary \( D \) of the unit ball \( g \) is fixed under the homotopy,
2) \( g' \) in \( B \setminus D \) is \( C^\infty \) and finite to one to its image in \( \mathbb{R}^l \).

Proof. of Claim A.3: Let \( \theta_1, \theta_2 \) be two analytic functions on \( \mathbb{R}^k \) such that \( \theta_1 = \epsilon, \theta_2 = 0 \) define \( \partial B \) and \( D \) where \( \epsilon \) is the radius of \( B \). We consider the homotopy

\[
(1 - t)g + t(g + \theta_1 \theta_2 h), \quad t \in [0, 1].
\]

where \( h \) is some \( C^\infty \) function. Thus \( g \) is homotopic to

\[
\left( g + \theta_1 \theta_2 h \right)
\]

The determinant of a maximal minor of the differential \( J \) of

\[
\left( g + \theta_1 \theta_2 h \right)
\]

is a polynomial in \( \theta_1, \theta_2 \) whose coefficients are \( C^\infty \) functions of \( h \). Thus for a small \( \epsilon \), by choosing a suitable \( h \), the determinant is non-zero for all points in \( B \) with \( \theta_2 \neq 1 \) and \( \theta_1 \neq \epsilon \), i.e. the differential \( J \) has full rank. By mean value theorem

\[
\left( g + \theta_1 \theta_2 h \right)
\]

is 1-to-1 to its image when restricted to \( B \setminus D \).

It satisfies required conditions in Claim A.3.

Now let \( f \) be the composition of

\[
(\Delta^p)' \rightarrow \mathcal{Y} \times \mathcal{X} \rightarrow \mathcal{X}
\]

where \( (\Delta^p)' \) is a neighborhood of \( \Delta^p \). Next we cover \( \Delta^p \) with finitely many balls \( B_i, i = 1, \cdots, l \) and the homotopy in Claim A.3 for each \( B_i \). Consider the first open set \( B_1 \). Applying Claim A.3, \( f \) is homotopic to \( f_1 : (\Delta^p)' \rightarrow \mathcal{X} \) such that the homotopy fix the map \( f \) on \( \partial B_1 \) and \( D \) and \( f \) is homotopic
to \( f_1 \) which is finite-to-one on \( B_1 \). Then we repeat the homotopy from \( f_1 \) to 
\( f_2 \), from \( f_2 \) to \( f_3 \), \( \cdots \), from \( f_{l-1} \) to \( f_l \). Finally, we obtain a continuous map 
\( f_l \) which is finite-to-one in each \( B_i \) and is equal to \( f \) on \( D \). Let \( C_i \) be the 
barycentric subdivisions obtained from the covering \( B_i, i = 1, \cdots, l \). Then \( f_l \) 
is homotopy to \( f \). We complete the proof. 

\[ \square \]

**Proposition A.4.** \textit{For any cellular cycle} \( S \) \textit{in} \( Y \times X \), \textit{of dimension} \( p < \dim(X) \), \( S \) \textit{is homopotic to a cycle finite to} \( X \).

**Proof.** Let 
\[
S = \sum_i C_i \quad (A.6)
\]
and each cell \( C_i \) satisfies Proposition A.2 with a homotopy \( h_i \). Then there 
is synchronized homotopy with the same parameter \( t \in [0, 1] \) such that \( C_i \) is 
homotopic to another cell \( C'_i \) 1-to-1 to \( X \), but the boundary is fixed. Since the 
boundaries are not changed, this synchronized homotopy are glued together 
to yield a homotopy of the cycle \( S \),
\[
S' = \sum_i C'_i. \quad (A.7)
\]

\[ \square \]

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