Height-unmixed of Tensor Product of Lattices

Ali Molkhasi

Department of Mathematics, Faculty of Mathematical Sciences, University of Farhangian, Tabriz, Iran

Copyright ©2016 by authors, all rights reserved. Authors agree that this article remains permanently open access under the terms of the Creative Commons Attribution License 4.0 International License

Abstract  We investigate WB-height-unmixed of tensor product of distributive lattices: Cohen-Macaulay rings related to tensor products of distributive lattices are constructed using the method of Stanly and Reisner.

Keywords  Distributive Lattices, Tensor Product, Cohen-Macaulay Rings, Stanley-Reisner Ring, Unmixedness

2010 AMS(MOS) Subject Classification: 13H10, 08B10, 08B25

1 Introduction

All rings and algebras considered in this paper are commutative with identity elements and $k$ stands for a perfect field. Suitable background on depth of algebraic geometry and Cohen-Macaulay rings is [3], [8], and [11].

We recall that tensor products were introduced in J. Anderson and N. Kimura [1] and G. A. Fraser [5]. Our aim in this paper is to prove that the Cohen-Macaulay property is inherited by tensor products of $k$-algebras and tensor products of distributive lattices. Tensor products of semilattices and related structures have already been the source of many publications ([5] and [7]). We denote by $A \otimes B$ the tensor product of $A$ and $B$, which $A$ and $B$ are $\{\lor, 0\}$-semilattices.

In this paper, we suppose $k$ is a perfect field and $F$ and $K$ be two extension fields of $k$ such that $F \otimes K$ is Noetherian. Then we prove that if $L_1$ and $L_2$ are distributive lattices, then the ring $(F \otimes K)[L_1 \otimes L_2][X_1, X_2, \ldots]$ is WB-height-unmixed. Finally, WB-height-unmixed is related to universally catenary and approximately Cohen-Macaulay ring.

2 Tensor Product and Cohen-Macaulay Rings

Let $A$ be a lattice. If $a, b \in A$, the join and meet of $a$ and $b$ are written as $a \lor b$ and $a \land b$, respectively. If $a_i \in A$ for $i \in I$, where $I$ is any non-empty set, then the join and meet of $\{a_i : i \in I\}$, if they exist, are denoted by $\lor_{i \in I} a_i$ and $\land_{i \in I} a_i$, respectively. The smallest and largest elements of $A$, if they exist, are denoted by 0 and 1, respectively. Tensor products of distributive lattices are well-known, but it exists in many other categories equipped with a forgetful functor to $\text{Set}$ [14]. Examples of such categories that matters to us are boolean algebra, distributive lattices, semilattices with zero, etc. Given $A$, $B$, and $X$, three object of the same category, a bimorphism from $A \times B$ to $X$ is a set theoretic map $f : A \times B \rightarrow X$ such that for all $a \in A$ and for all $b \in B$, the mappings $f(a, \_): B \rightarrow X$ and $f(\_, b): A \rightarrow X$ are morphisms. Being given an object $X$ of the category and a bimorphism $i : A \times B \rightarrow X$, we say that $X$ is a tensor product of $A \times B$ if for every object $C$ and every bimorphism $f : A \times B \rightarrow C$, there exist a unique morphism $h : X \rightarrow C$, such that $f = h \circ i$. Tensor products are unique up to isomorphisms and they are denoted by $A \otimes B$.

The bimorphism $i$ is not surjective but its image generates $A \otimes B$, thus we call generating elements (of $A \otimes B$) those coming from $A \otimes B$ and we will write $i(a, b) = a \otimes b$. Now, we have the following definitions from [5].

Definition 2.1. Let $A$, $B$, and $C$ be distributive lattices. A function

$$f : A \times B \rightarrow C$$

is a bihomomorphism if the functions $g_a : B \rightarrow C$ defined by $g_a(b) = f(a, b)$ and $h_b : A \rightarrow C$ defined by $h_b(a) = f(a, b)$ are homomorphisms for all $a \in A$ and $b \in B$.

Definition 2.2. Let $A$ and $B$ be distributive lattices. A distributive lattice $C$ is a tensor product of $A$ and $B$ (in the category $\mathcal{D}$), if there exists a bihomomorphism $f : A \times B \rightarrow C$, such that $C$ is generated by $f(A \times B)$ and for any distributive lattice $D$ and any bihomomorphism $g : A \times B \rightarrow D$, there is a homomorphism $h : C \rightarrow D$ satisfying $g = hf$. 


Note that since \( f(A \times B) \) generates \( C \), the homomorphism \( h \) is necessarily unique. Let \( A \) and \( B \) be distributive lattices. Then a tensor product of \( A \) and \( B \) in the category \( \mathcal{D} \) exists and is unique up to isomorphism (see [5]).

The tensor product of \( A \) and \( B \) is denoted by \( A \otimes B \) and the image of \((a, b)\) under the canonical bihomomorphism \( f : A \times B \to A \otimes B \) is written as \( a \otimes b \). Now, \( A \otimes B \) is the distributive lattice generated by the elements \( a \otimes b \) \((a \in A, b \in B)\), subject to the bihomomorphic conditions:

\[
(a_1 \lor a_2) \otimes b = (a_1 \otimes b) \lor (a_2 \otimes b),
\]

\[
(a_1 \land a_2) \otimes b = (a_1 \otimes b) \land (a_2 \otimes b),
\]

\[
a \otimes (b_1 \land b_2) = (a \otimes b_1) \land (a \otimes b_2),
\]

and

\[
a \otimes (b_1 \lor b_2) = (a \otimes b_1) \lor (a \otimes b_2),
\]

for all \( a, a_1, a_2 \in A \) and \( b, b_1, b_2 \in B \). Every element of \( A \otimes B \) can be written in the form \( \bigvee_{i=1}^{n} (a_i \otimes b_i) \) for some \( a_i \in I \) and \( b_i \in B, i = 1, 2, \ldots, n \).

For terminology and basic results of lattice theory and universal algebra, consult Birkhoff [4] and Gratzer [10]. In this paper, relationship among tensor product of the Cohen-Macaulay rings, tensor product of the distributive lattices, WB-height-unmixed and Stanley-Reisner ring are considered.

We first recall the definition of an algebra with straightening lows in [6], then recall WB-height-unmixed in ideal of a commutative ring in [11].

**Definition 2.3.** Given a commutative ring \( R \) with identity element. If \( P \) is a finite partially ordered set, then we say that \( \mathfrak{A} \) is a \( \text{ASL} \) (algebra with straightening lows) on \( P \) over \( R \) if the followings hold:

\( \text{ASL-0.} \) An injective map \( P \to \mathfrak{A} \) is given, \( \mathfrak{A} \) is a graded \( R \)-algebra generated by \( P \), and each element of \( P \) is a homogeneous of positive degree. We call a product of elements of \( P \) a monomial in \( P \). In general, a monomial \( M \) is a map \( P \to \mathbb{N}_0 \) and we denote \( M = \prod_{x \in P} a_x M(x) \) such that it also stands for an element of \( \mathfrak{A} \). A monomial in \( P \) of the form

\[
x_{i_1} \cdots x_{i_l}
\]

with \( x_{i_1} \leq \cdots \leq x_{i_l} \) is called standard.

\( \text{ASL-1.} \) The set of standard monomials in \( P \) is an \( R \)-free basis of \( \mathfrak{A} \).

\( \text{ASL-2.} \) For \( x, y \in P \) that \( x \nleq y \) and \( y \nleq x \), there is an expression of the form

\[
xy = \sum_M c^x_M M \quad (c^x_M \in R)
\]

where the sum is taken over all standard monomials

\[
M = x_1 \cdots x_{r_M} \quad (x_1 \leq \cdots \leq x_{r_M})
\]

with \( x_1 < x, y \) and \( M = \deg(xy) \).

The most simple example of an \( \text{ASL} \) on \( P \) over \( R \) is the Stanley-Reisner ring \( R[P] = R[x \mid x \in P]/(xy \mid x \nleq y, y \nleq x) \). The Stanley-Reisner rings play central role in theory of \( \text{ASL} \). For the proof of the following theorem, see [6].

**Theorem 2.4.** If \( R \) is Cohen-Macaulay ring, and if \( P \) is a distributive lattice, then \( R[P] \) is Cohen-Macaulay.

Let \( I \) is ideal of the ring \( R \). Then we say that an ideal \( I \) is unmixed if \( I \) has no embedded prime divisors or, in modern language, if the associated prime ideals of \( R/I \) are the minimal prime ideals of \( I \). We know a prime ideal \( p \) is a weak Bourbaki associated prime of the ideal \( I \) of the ring \( R \) if for some \( a \in R \) it is a minimal ideal of the form \( I : a \).

We recall that an ideal is WB-height-unmixed, if it is height-unmixed with respect to the set of weak Bourbaki associated primes, which an ideal is height-unmixed if all the associated primes of \( I \) have equal height.

Before proving main theorem let us note that a ring is regular if for all prime ideal \( p \) of \( R \), \( R_p \) is a regular local ring. Recently, in [2], Bouchiba and Kabbaj showed that if \( R \) and \( S \) are \( k \)-algebras such that \( R \otimes_k S \) is Noetherian then \( R \otimes_k S \) is a Cohen-Macaulay ring if and only if \( R \) and \( S \) are Cohen-Macaulay rings. Now we are ready to present our main theorem.

**Theorem 2.5.** Let \( k \) be a perfect field, \( F \) and \( K \) be two extension fields of \( k \) such that \( F \otimes K \) is Noetherian. If \( L_1 \) and \( L_2 \) are distributive lattices, then the ring \( (F \otimes K)[L_1 \otimes L_2] \) is WB-height-unmixed.

**Proof.** By ([4], Theorem 2.6), \( L_1 \otimes L_2 \) is distributive lattice if \( L_1 \) and \( L_2 \) are distributive lattices. On the other hand, in [15] and [2], it is proved that the tensor product of two extension fields of \( k \) is not necessarily Noetherian. Here, for all two \( k \)-algebras \( F \) and \( K \) we have \( F \otimes K \) is Noetherian and \( k \) is perfect field. From the note on page 49 of [13] we will have that \( F \otimes K \) is regular ring. By using the following well-known chain we conclude \( F \otimes K \) is Cohen-Macaulay ring:

\begin{align*}
\text{Regular} & \implies \text{Complete intersection} \implies \text{Gorenstein} \\
& \implies \text{Cohen Macaulay}
\end{align*}

On the other hand, by applying that Theorem 2.4, if \( R \) is Cohen-Macaulay and \( P \) is distributive lattice, then \( R[P] \) is Cohen-Macaulay ring. Here, we will have \( (F \otimes K)[L_1 \otimes L_2] \) is Cohen-Macaulay ring. By ([12], Theorem 2.3), \( (F \otimes K)[L_1 \otimes L_2][X_1, X_2, \ldots] \) is WB-height-unmixed.

\[\square\]
We will say that a Noetherian ring $R$ is catenary if every saturated chain joining prime ideals $p$ and $q$ has (maximal) length height $q/p$ such that $p \subseteq q$. Also, we say that $R$ is universally catenary if all the polynomial rings $R[X_1, X_2, \ldots]$ are catenary.

**Theorem 2.6.** Let $k$ be a perfect field, $F$ and $K$ be two extension fields of $k$ such that $F \otimes K$ is Noetherian. If $L_1$ and $L_2$ are distributive lattices, then the ring $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary.

**Proof.** In Theorem 2.5, we showed that $(F \otimes K)[L_1 \otimes L_2]$ is a Cohen-Macaulay ring. By applying that ([3], Theorem 2.1.12), we see that every Cohen-Macaulay ring is universally catenary. Therefore, $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary.

**Corollary 2.7.** Let $k$ be a perfect field, $F$ and $K$ be two extension fields of $k$ such that $F \otimes K$ is Noetherian. If $L_1$ and $L_2$ are distributive lattices, then any polynomial algebra over $(F \otimes K)[L_1 \otimes L_2]$ is Cohen-Macaulay ring.

**Theorem 2.8.** Let $k$ be a perfect field, $F$ and $K$ be two extension fields of $k$ such that $F \otimes K$ is Noetherian. If $L_1$ and $L_2$ are distributive lattices, then any polynomial algebra over $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary.

**Corollary 2.9.** Let $k$ be a perfect field, $F$ and $K$ be two extension fields of $k$ such that $F \otimes K$ is Noetherian. If $L_1$ and $L_2$ are distributive lattices, then any quotient of $(F \otimes K)[L_1 \otimes L_2]$ is universally catenary.

Now, assume that $(R, m)$ is a local ring with $\dim(R) = d$. We say that $R$ is an approximately Cohen-Macaulay ring if either $\dim(R) = 0$ or there exists an element $r$ of $m$ such that $R/a^n R$ is a Cohen-Macaulay ring of dimension $d - 1$ for every integer $n > 0$ ([9]). A ring $R$ is called an approximately Cohen-Macaulay ring if the ring $R_0$ is an approximately Cohen-Macaulay ring, for all prime ideals $p$ of $R$.

**Corollary 2.10.** Let $k$ be a perfect field and $F$ and $K$ be nonzero $k$-algebras such that $F \otimes K$ is Noetherian. Assume that $F$ is not a Cohen-Macaulay ring. If $F \otimes K$ is an approximately Cohen-Macaulay ring, then $K[X_1, X_2, \ldots]$ is universally catenary.

**Acknowledgements**

The author is deeply grateful of Professor I. Zamani for his careful reading of the paper and valuable suggestions.

**References**

[1] J. A. Anderson and N. Kimura, The tensor product of semilattices, Semigroup Forum, 16 (1978), 83-88.

[2] S. Bouchiba and S. Kabbaj, Tensor products of Cohen-Macaulay rings Solution to a problem of Grothendieck, Journal of Algebra, 252 (2002), 65–73.

[3] W. Bruns and J. Herzog, Cohen-Macaulay rings, Cambridge University Press, Cambridge, 1997.

[4] G. Birkhoff, Lattice theory, 3rd ed. (Colloq. Publ., Vol. 25, Amer. Math. Soc, Providence, R.I., 1967).

[5] G. A. Fraser, The semilattice tensor product of distributive semilattices, Trans. Amer. Math. Soc., 217 (1976), 183-194.

[6] C. De Concini and D. Eisenbud, Hodge algebras, asterisque, societe mathematique de france, 91 (1982).

[7] G. Gratzer and F. Wehrung, Tensor products of semilattices with zero, Journal of Pure and Applied Algebra, 147 (2000), 273–301.

[8] R. Gilmer, Multiplicative ideal theory, Marcel Dekker, New York, 1972.

[9] S. Goto, Approximately Cohen-Macaulay Rings, J. Algebra 76(1) (1982), 214–225.

[10] G. Gratzer, Universal algebra, Van Nostrand, Princeton, N.J., 1968.

[11] I. Kaplansky, Commutative ring theory, University of Chicago Press, Chicago, 1974.

[12] A. Molkhasi, Polynomials, $\alpha$-ideals and the principal lattice, Journal of Siberian Federal University, 4(3) (2011), 292–297.

[13] R. Y. Sharp, Simplifications in the theory of tensor products of field extensions, J. London Math. Soc., 15 (1977), 48–50.

[14] Z. Shmuely, The tensor product of distributive lattices, Algebra Universalis, 9 (1979), 281–296.

[15] P. Vamos, On the minimal prime ideals of a tensor product of two fields, Math. Proc. Camb. Phil. Soc. 84 (1978), 25–35.