L²-theory for non-symmetric Ornstein–Uhlenbeck semigroups on domains

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Abstract. We prove that the mild solution of the stochastic evolution equation \( dX(t) = AX(t) \, dt + dW(t) \) on a Banach space \( E \) has a continuous modification if the associated Ornstein–Uhlenbeck semigroup is analytic on \( L^2 \) with respect to the invariant measure. This result is used to extend recent work of Da Prato and Lunardi for Ornstein–Uhlenbeck semigroups on domains \( \mathcal{O} \subseteq E \) to the non-symmetric case. Denoting the generator of the Ornstein–Uhlenbeck semigroup by \( \mathcal{L}_\mathcal{O} \), we obtain sufficient conditions in order that the domain of \( \sqrt{-\mathcal{L}_\mathcal{O}} \) be a first-order Sobolev space.

1. Introduction

In this paper, we present new results on analytic Ornstein–Uhlenbeck semigroups associated with the linear Cauchy problem

\[
dX(t) = AX(t) \, dt + dW(t),
\]

where \( A \) is the generator of a \( C_0 \)-semigroup on a Banach space \( E \) and \( W \) is a cylindrical Brownian motion, and use them to extend recent work of Da Prato and Lunardi [11] for Ornstein–Uhlenbeck semigroups on domains (see also [9]) to the non-symmetric case. The approach in [11] is based on the Feynman–Kac formula and uses the pathwise continuity of the Ornstein–Uhlenbeck process in a crucial way. Our first main result (Theorem 2.6) asserts that in the non-symmetric case, pathwise continuity still holds provided the Ornstein–Uhlenbeck semigroup is analytic on \( L^2(E, \mu_\infty) \). Here, \( \mu_\infty \) denotes an invariant measure whose existence we assume throughout. Further, new results concern the \( \mu_\infty \)-almost sure pointwise convergence of analytic Ornstein–Uhlenbeck semigroups to the projection onto the constant functions in \( L^p(E, \mu_\infty) \) (Theorem 2.10) and a Poincaré inequality for analytic Ornstein–Uhlenbeck semigroups (Theorem 2.11).

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The construction and discussion of the main properties of the Ornstein–Uhlenbeck semigroup \( P_\Omega(t) \) \( t \geq 0 \) and its generator \( L_\Omega \) on an open domain \( \Omega \) in \( E \), presented in Sections 3 and 4, are extensions of their symmetric counterparts in \([11]\); see also \([9,35]\) for earlier work. In contrast, the domain identification of \( D(\sqrt{-L_\Omega}) \) in \( L^2(E, \mu_\infty) \) as a first-order Gaussian Sobolev space is essentially trivial in the symmetric case but requires substantial effort in the non-symmetric case. In order to establish this identification, in Sect. 5 we adapt arguments from recent work by Maas and the second named author \([27,28]\). As an application, we prove a Poincaré inequality for the gradient on \( \Omega \) in the direction of \( H \).

All spaces are real. The domain and range of a (possibly unbounded) linear operator \( A \) are denoted by \( D(A) \) and \( R(A) \), respectively. Our terminology, in so far unexplained, follows \([21,26–28,31]\).

2. Analytic Ornstein–Uhlenbeck semigroups

Let \( E \) be a real Banach space and \( H \) a real Hilbert space, continuously embedded into \( E \) by means of a bounded injective linear operator \( i : H \to E \), and with inner product \([\cdot, \cdot]\). We fix a probability space \((\Omega, \mathbb{P})\) and let \( W^H \) be an \( H \)-cylindrical Brownian motion, that is, a linear mapping \( W^H : L^2(\mathbb{R}^+; H) \to L^2(\Omega, \mathbb{P}) \) satisfying

1. for all \( f \in L^2(\mathbb{R}^+; H) \) the random variable \( W^H f \) is centred Gaussian distributed;
2. for all \( f, g \in L^2(\mathbb{R}^+; H) \), we have

\[
\mathbb{E}(W^H f W^H g) = \int_0^\infty [f(t), g(t)] \, dt.
\]

For \( t \geq 0 \) and \( h \in H \), we put

\[W^H(t)h := W^H(\mathbb{1}_{[0,t]} \otimes h)\]

and note that \((W^H(t)h)_{t \geq 0}\) is a Brownian motion, which is standard if and only if \( \|h\| = 1 \). Moreover, two such Brownian motions \((W^H(t)h)_{t \geq 0}\) and \((W^H(t)h')_{t \geq 0}\) are independent if and only if \([h, h'] = 0\).

Let \( S = (S(t))_{t \geq 0} \) be a \( C_0 \)-semigroup of bounded linear operators, with generator \( A \), on \( E \). Throughout this paper, we shall make the following standing assumption.

ASSUMPTION 2.1. The linear stochastic Cauchy problem

\[dX(t) = AX(t) \, dt + dW^H(t), \quad t \geq 0,\]  

(2.1)

admits an invariant measure.

Note that in (2.1), we suppress the inclusion mapping \( i \) and identify \( H \) with a linear subspace of \( E \). Recall that a Radon measure \( \mu \) on \( E \) is said to be invariant for the
problem (2.1) if the following holds. Whenever \( X_0 \) is an \( E \)-valued random variable, independent of \( W^H \) and with distribution \( \mu \), the initial value problem
\[
\begin{align*}
\mathrm{d}X(t) &= AX(t)\,\mathrm{d}t + \mathrm{d}W^H(t), \quad t \geq 0, \\
X(0) &= X_0,
\end{align*}
\]
is well posed and its unique mild solution is stationary (with distribution \( \mu \)). Necessary and sufficient conditions for well-posedness can be found in \([31]\).

In order to arrive at a useful equivalent formulation of Assumption 2.1, we need the following terminology. The reproducing kernel Hilbert space \( H_\mu \) associated with a centred Gaussian Radon measure \( \mu \) on \( E \) is the closure in \( L^2(E,\mu) \) of the dual space \( E^* \) (identifying functionals \( x^* \in E^* \) with the functions \( x \mapsto \langle x, x^* \rangle \) in \( L^2(E,\mu) \)). The mapping
\[
i_\mu : H_\mu \to E, \quad i_\mu x^* = \int_E \langle x, x^* \rangle x \,\mathrm{d}\mu(x)
\]
is continuous and injective, and its adjoint is given by
\[
i^*_\mu : E^* \to H_\mu, \quad i^*_\mu x^* = \langle \cdot, x^* \rangle.
\]
Here and in what follows, we identify \( H_\mu \) with its dual using the Riesz representation theorem.

Using this terminology, Assumption 2.1 is satisfied if and only if there exists a centred Gaussian Radon measure \( \mu_\infty \) on \( E \) whose reproducing kernel Hilbert space \( i_\infty : H_\infty \to E \) satisfies
\[
\|i^*_\infty x^*\|^2_{H_\infty} = \int_0^\infty \|i^*_\infty S^*(t)x^*\|^2_{H} \,\mathrm{d}t, \quad x^* \in E^*.
\]
(see \([13,21]\)). The measure \( \mu_\infty \) is then invariant.

On the space \( B_b(E) \) of bounded Borel functions \( f : E \to \mathbb{R} \), we define the operators \( P(t), t \geq 0, \) by
\[
P(t)f(x) := \mathbb{E} f(X^x(t)), \quad t \geq 0, \quad x \in E,
\]
where
\[
X^x(t) := S(t)x + \int_0^t S(t-s) \,\mathrm{d}W^H(s)
\]
is the mild solution of the problem (2.1) with initial value \( x \); the existence and uniqueness of this solution are implicit in the Assumption 2.1. These operators satisfy \( P(0) = I \) and \( P(t+s) = P(t)P(s) \) for all \( t \) and \( s \geq 0 \). For all \( 1 \leq p < \infty \), the family \( P = (P(t))_{t \geq 0} \) extends to a \( C_0 \)-contraction semigroup on \( L^p(E,\mu_\infty) \) satisfying
\[
\int_E P(t)f \,\mathrm{d}\mu_\infty = \int_E f \,\mathrm{d}\mu_\infty, \quad f \in L^p(E,\mu_\infty), \quad t \geq 0.
\]
(2.4)
Throughout this paper, we make the following assumption.
ASSUMPTION 2.2. The semigroup $P$ is analytic on $L^2(E, \mu_{\infty})$.

In statements like these, we always tacitly pass to the complexifications of the operators and the spaces involved. Thus, what we are assuming is that $P_{\mathbb{C}}$ is analytic on $L^2(E, \mu_{\infty}; \mathbb{C})$. This assumption implies (see [26]) that $P_{\mathbb{C}}$ is in fact an analytic contraction semigroup on $L^p(E, \mu_{\infty}; \mathbb{C})$ for all $1 < p < \infty$.

Necessary and sufficient conditions for this assumption to be satisfied are presented in [21, Theorem 8.3]; this result extends previous results by Fuhrman [18] and Gołdys [19]. As a corollary to this result (see [21, Theorem 9.2]), Assumption 2.2 holds if the semigroup $S$ restricts to an analytic semigroup on $H$ which is contractive with respect to some Hilbertian norm. This sufficient condition is close to being necessary: if Assumption 2.2 holds, then $S$ restricts to a bounded analytic semigroup on $H$ [28].

REMARK 2.3. In applications to parabolic SPDEs, the above sufficient condition is usually satisfied, the typical situation being that $A$ is a second-order elliptic operator on some domain $D \subseteq \mathbb{R}^d$ and $H = L^2(D)$.

We proceed with a discussion of some consequences of Assumptions 2.1 and 2.2 that will be needed later on.

Let $U : H_{\infty} \to H$ be the linear operator with initial domain $i_{\infty}^*(E^*)$, defined by

$$Ui_{\infty}^*x^* := i_{\infty}^*x^*, \quad x^* \in E^*.$$  

This operator is densely defined, and, by [20, Theorem 3.5], the analyticity of $P$ on $L^2(E, \mu_{\infty})$ implies that $U$ is closable. From now on, we denote by $U$ its closure and by $D(U)$ the domain of this closure.

Let $\phi : H_{\infty} \to L^2(E, \mu_{\infty})$ be the isometric embedding given by

$$(\phi(i_{\infty}^*x^*))(\cdot) := \langle \cdot, x^* \rangle, \quad x^* \in E^*.$$  

In order to simplify notations a bit, we shall write

$$\phi_h(x) := (\phi(h))(x).$$  

When $H_0$ is a linear subspace of $H_{\infty}$ and $k \geq 0$ is an integer, we denote by $\mathcal{F}C^k_b(E, H_0)$ the vector space of all $\mu_{\infty}$-almost everywhere defined functions $f : E \to \mathbb{R}$ of the form

$$f := \varphi(\phi_{h_1}, \ldots, \phi_{h_n}), \quad x \in E,$$

with $n \geq 1$, $\varphi \in C^k_b(\mathbb{R}^n)$, and $h_1, \ldots, h_n \in H_0$. Here, $C^k_b(\mathbb{R}^n)$ is the space of all bounded continuous functions with bounded continuous derivatives up to order $k$.

For $f \in \mathcal{F}C^1_b(E, D(U))$, we define the Fréchet derivative $D^H f : E \to H$ of $f$ in the direction of $H$ by

$$D^H f := \sum_{j=1}^n \partial_j \varphi(\phi_{h_1}, \ldots, \phi_{h_n}) \otimes U h_j.$$
The closability of $U$ implies that $D^H$ is closable from $L^p(E, \mu_\infty)$ to $L^p(E, \mu_\infty; H)$ for all $p \in [1, \infty)$ (see [20, Theorem 3.5] and [21, Proposition 8.7]). Henceforth, by slight abuse of notation, we denote by $D^H$ its closure in $L^p(E, \mu_\infty)$ and write

$$W_1^H(E, \mu_\infty) := D_p(D^H)$$

for the domain of this closure in $L^p(E, \mu_\infty)$. Furthermore, we write $D(D^H) := D_2(D^H)$.

Under Assumption 2.1, $S$ maps $H_\infty$ into itself, and the restriction $S_\infty = S|_{H_\infty}$ is a $C_0$-contraction semigroup on $H_\infty$. We shall denote its generator by $A_\infty$. The next result is taken from [21] and [26].

**PROPOSITION 2.4.** Let Assumptions 2.1 and 2.2 be satisfied. There exists a unique bounded operator $B \in L(H)$ such that

$$i B i\ast x^* = -i_\infty A_\infty^\ast i_\infty\ast x^*, \quad x^* \in D(A^\ast).$$

This operator satisfies $B + B^\ast = I$ and $[Bh, h] = \frac{1}{2} \|h\|^2_H$ for all $h \in H$.

Let $l$ be the sesquilinear form defined by

$$l(f, g) := [BD^H f, D^H g] \quad f, g \in D(l) := D(D^H),$$

where $B \in \mathcal{L}(H)$ is the bounded operator described in Proposition 2.4. This form is closed, densely defined, accretive, and sectorial on $L^2(E, \mu_\infty)$. Let us denote by $D^{H^\ast} BD^H$ the associated sectorial operator with domain consisting of all functions $f \in D(D^H)$ such that $BD^H f \in D(D^{H^\ast})$. This domain is a core for $D(D^H)$ (see [32, Lemma 1.25]) and we have (see [26]):

**PROPOSITION 2.5.** Let Assumptions 2.1 and 2.2 be satisfied. The generator $L$ of the semigroup $P$ on $L^2(E, \mu_\infty)$ equals

$$L = -D^{H^\ast} BD^H.$$
For $t \geq s \geq 0$, we have
\[
\mathbb{E}\langle Z(t), x^* \rangle\langle Z(s), x^* \rangle = \mathbb{E}\langle Z(0), x^* \rangle\langle Z(t-s), x^* \rangle \\
= \mathbb{E}\langle Y_\infty, x^* \rangle\langle S(t-s)Y_\infty + X^0(t-s), x^* \rangle \\
= \mathbb{E}\langle Y_\infty, x^* \rangle\langle S(t-s)Y_\infty, x^* \rangle \\
= \langle Q_\infty S^*(t-s)x^*, x^* \rangle,
\]
where we used that $Y_\infty$ and $X^0(t)$ are independent for every $t \geq 0$.

For $t \geq s \geq 0$, by the above we have
\[
0 \leq \mathbb{E}\langle Z(t) - Z(s), x^* \rangle^2 = \mathbb{E}\langle Z(t), x^* \rangle^2 + \mathbb{E}\langle Z(s), x^* \rangle^2 - 2\mathbb{E}\langle Z(t), x^* \rangle\langle Z(s), x^* \rangle \\
= 2\langle Q_\infty x^*, x^* \rangle - 2\langle Q_\infty x^*, S^*(t-s)x^* \rangle \\
= 2\langle Q_\infty (I - S^*(t-s))x^*, x^* \rangle.
\]
The authors would like to thank Ben Goldys for showing this argument.

Since $\mu_\infty$ is a Radon measure, the closure $E_0$ of its reproducing kernel Hilbert space $H_\infty$ in $E$ is separable, and we have $\mu_\infty(E_0) = 1$. The invariance of $H_\infty$ under $S$ (see [6,30]) implies that also $E_0$ is invariant under $S$. The covariance operator $Q_t$ of $\mu_t$ satisfies
\[
\langle Q_t x^*, x^* \rangle = \int_0^t \|i^* S^*(s) x^* \|_H^2 \, ds \leq \langle Q_\infty x^*, x^* \rangle, \quad x^* \in E^*,
\]
and therefore Anderson’s inequality implies that $\mu_t(E_0) = 1$ for all $t \geq 0$. It follows that for all $t \geq 0$, $X_t \in E_0$ almost surely. Since $H$ is contained in $E_0$ (by [21, Proposition 2.6]), this argument shows that without loss of generality, we may assume that $E$ is separable.

The analyticity of $P$ on $L^2(E, \mu_\infty)$ implies that the operator $Q_\infty A^* x^*$, which is well defined on the domain $D(A^*)$, extends to a bounded operator from $E^*$ to $E$. In fact, we have
\[
\| Q_\infty A^* x^* \| \leq \| i \| \| Q_\infty A^* x^* \|_H \leq \| i \| \| B \| \| i^* x^* \|_H \leq \| i \|^2 \| B \| \| x^* \|.
\]
By a standard argument, this implies that for all $x^* \in E^*$,
\[
\mathbb{E}\langle Z(t) - Z(s), x^* \rangle^2 = 2\langle Q_\infty (I - S^*(t-s))x^*, x^* \rangle \leq M_T |t - s| \| i \|^2 \| B \| \| x^* \|^2, \tag{2.6}
\]
where $M_T = \sup_{0 \leq t \leq T} \| S(t) \|$. The process $(Z, x^*)$ being Gaussian, the Kolmogorov continuity criterion then implies that the process $(Z, x^*)$ has a continuous modification. By (2.6) and the stationarity of $Z$, the conditions of [5, Proposition 1] are satisfied, and we conclude that the Gaussian process $((Z(t), x^*))_{(t,x^*) \in [0,T] \times B_{E^*}}$ has a continuous modification. The existence of a continuous modification of $(Z(t))_{t \in [0,T]}$ then follows from [17, Theorem 1.2].

\[\square\]
REMARK 2.7. The problem of existence of a continuous version for the mild solution of (2.1) has been discussed by many authors. If the inclusion mapping $i : H \to E$ is $\gamma$-radonifying (if $E$ is a Hilbert space, this is equivalent to $I$ being Hilbert–Schmidt), a continuous version exists if $E$ has type 2; this follows by the factorization method of Da Prato, Kwapień, and Zabczyk [10]. For Hilbert spaces $E$, the result is due to Smoleński [34]; the type 2 case follows from Millet and Smoleński [29] combined with a result of Rosiński and Suchanecki [33] (see also [31]). The special case with $E$, a Hilbert space, had been treated before by [34]. In the general case considered here ($E$ an arbitrary Banach space, $i : H \to E$ bounded and injective), a continuous version exists if $S$ is analytic on $E$ [4]. Analyticity of $P$ does not in general imply analyticity of $S$ (a counterexample can be found in [28]), so our Theorem 2.6 is not contained as a special case in the result in [4]. In the converse direction, we mention that neither does the analyticity of $S$ imply that of $P$; a counterexample is due to Fuhrman [18].

REMARK 2.8. Examples of ‘Ornstein–Uhlenbeck-like’ processes without continuous version are presented in [10,23]; in both references, these processes arise as mild solutions of an equation of the form

$$dU = AU(t) \, dt + B \, dW^H$$

with an unbounded densely defined closed linear operator $B : D(B) \subseteq H \to E$.

In the rest of this paper, we will always work with a continuous version of $X$ whose existence is guaranteed by Theorem 2.6.

We continue with two almost everywhere convergence results. The first concerns the behaviour of $P(t)$ as $t \downarrow 0$. It follows from the $L^p$-boundedness of the maximal function

$$Mf(x) := \sup_{t > 0} |P(t)f(x)|;$$

see [8] and [27, Proposition 8.5] (where the present setting is considered).

THEOREM 2.9. Let Assumptions 2.1 and 2.2 be satisfied and let $1 < p < \infty$. For all $f \in L^p(E, \mu_\infty)$ we have

$$\lim_{t \downarrow 0} P(t)f(x) = f(x) \quad \text{for } \mu_\infty\text{-almost all } x \in E.$$

The second result concerns the behaviour of $P(t)$ as $t \to \infty$. Below, we shall only need the part (1) (with $p = 2$) (see also [14, Proposition 10.1.1] for a partial result in this direction).

THEOREM 2.10. Fix $1 \leq p < \infty$.

1. If Assumption 2.1 is satisfied, then for all $f \in L^p(E, \mu_\infty)$ we have

$$\lim_{t \to \infty} P(t)f = \int_E f \, d\mu_\infty \quad \text{in } L^p(E, \mu_\infty).$$
(2) If Assumptions 2.1 and 2.2 are satisfied, then for all \( f \in L^p(E, \mu_\infty) \) we have
\[
\lim_{t \to \infty} P(t) f(x) = \int_E f \, d\mu_\infty \quad \text{for } \mu_\infty\text{-almost all } x \in E.
\]

Proof. The proof of the first statement follows by second quantisation and using the fact [21, Proposition 2.4] that \( S^*_\infty \) is strongly stable. The details are as follows.

First, we consider the case \( p = 2 \). For all \( h_1, \ldots, h_n \in H_\infty \), we have
\[
\lim_{t \to \infty} (S^*_\infty(t))^{\otimes n} (h_1 \otimes \cdots \otimes h_n) = \lim_{t \to \infty} S^*_\infty(t) h_1 \otimes \cdots \otimes S^*_\infty(t) h_n = 0,
\]
from which it follows that \( (S^*_\infty)^{\otimes n} := S^*_\infty \otimes \cdots \otimes S^*_\infty \) (\( n \) times) is strongly stable on \( H^{\otimes n}_\infty := H_\infty \otimes \cdots \otimes H_\infty \). By restricting to the symmetric tensor products \( H^{\otimes n}_\infty \) and taking direct sums, it follows that the second quantised semigroup
\[
\Gamma(S^*_\infty) := \bigoplus_{n=0}^{\infty} (S^{\otimes n}_\infty)^* \]
is strongly stable on the closed subspace \( \bigoplus_{n=1}^{\infty} H^{\otimes n}_\infty \) of \( \bigoplus_{n=0}^{\infty} H^{\otimes n}_\infty \). Under the Wiener-Itô isometry, the latter space is mapped isometrically onto \( L^2(E, \mu_\infty) \), and the first summand \( H^{\otimes 0}_\infty \) is mapped onto the one-dimensional subspace spanned by the constant function \( 1 \). Moreover, under this isometry, the semigroup \( \bigoplus_{n=0}^{\infty} (S^{\otimes n}_\infty)^* \) corresponds to \( P \) in the sense that the following diagram commutes:
\[
\begin{array}{ccc}
\bigoplus_{n=0}^{\infty} H^{\otimes n}_\infty & \xrightarrow{G} & \bigoplus_{n=0}^{\infty} (S^{\otimes n}_\infty)^* \\
\downarrow & & \downarrow \\
L^2(E, \mu_\infty) & \xrightarrow{P(t)} & L^2(E, \mu_\infty)
\end{array}
\]
the isomorphism on the vertical arrows being the Wiener-Itô isomorphism (see [6,30]).

As a result, we find that the semigroup \( P \) is strongly stable on \( L^2(E, \mu_\infty) \otimes \mathbb{R} 1 \). Since \( P(t) 1 = 1 \) and \( (\int_E f \, d\mu_\infty) 1 \) equals the orthogonal projection of \( f \) onto \( \mathbb{R} 1 \), this gives the first assertion for \( p = 2 \).

Next, let \( 2 < p < \infty \) be arbitrary, and choose \( p < q < \infty \) arbitrarily. Since \( P \) is contractive on \( L^q(E, \mu_\infty) \), for all \( f \in L^q(E, \mu_\infty) \) we have, by convexity,
\[
\begin{align*}
\| P(t) f - \int_E f \, d\mu_\infty \|_p & \leq \| P(t) f - \int_E f \, d\mu_\infty \|_2^{1-\theta} \| P(t) f - \int_E f \, d\mu_\infty \|_q^\theta \\
& \leq \| P(t) f - \int_E f \, d\mu_\infty \|_2^{1-\theta} (2 \| f \|_q)^\theta,
\end{align*}
\]
where \( 0 < \theta < 1 \) satisfies \( \frac{1-\theta}{2} + \frac{\theta}{q} = \frac{1}{p} \). The right-hand side tends to 0 as \( t \to \infty \). Since \( L^q(E, \mu_\infty) \) is dense in \( L^p(E, \mu_\infty) \) and \( P \) is contractive on \( L^p(E, \mu_\infty) \), this implies the first assertion for \( 2 < p < \infty \).
Next, let $1 \leq p < 2$. For $f \in L^2(E, \mu_\infty)$, the $L^2$-convergence implies the $L^p$-convergence by Hölder’s inequality. Since $L^2(E, \mu_\infty)$ is dense in $L^p(E, \mu_\infty)$ and $P$ is contractive on $L^p(E, \mu_\infty)$, this gives the first assertion for $1 \leq p < 2$.

For the proof of (2), we fix $1 < p < \infty$. We shall identify a dense subspace of functions for which the asserted $\mu_\infty$-almost everywhere convergence does hold. By the $L^p$-boundedness of the maximal function, which follows from the analyticity of $P$ by [27, Proposition 8.5], the set of all functions for which we have $\mu_\infty$-almost everywhere convergence is norm-closed in $L^p(E, \mu_\infty)$ and the proof is complete.

For $h \in H_\infty$ define

$$K_h := \exp \left( \phi_h - \frac{1}{2} \|h\|_{H_\infty}^2 \right).$$

As is well known, these functions belong to $L^p(E, \mu_\infty)$ and their linear span is dense in $L^p(E, \mu_\infty)$. Moreover, from the identity

$$K_h = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\phi_h^n),$$

with $I_n$ the orthogonal projection in $L^2(E, \mu_\infty)$ onto the $n$th Wiener-Itô chaos, it follows that

$$\int_E K_h \, d\mu_\infty = 1.$$

By second quantisation,

$$P(t)K_h = KS^*_\infty(t)h.$$

The proof will be finished by observing that for $\mu_\infty$-almost all $x \in E$ we have

$$\lim_{t \to \infty} P(t)K_h(x) = \lim_{t \to \infty} \exp \left( \phi_{S^*_\infty(t)h}(x) - \frac{1}{2} \|S^*_\infty(t)h\|_{H_\infty}^2 \right) = \int_E K_h \, d\mu_\infty.$$

In this computation, we used that $\lim_{t \to \infty} S^*_\infty(t)h = 0$ in $H_\infty$, from which we shall deduce next that $\lim_{t \to \infty} \phi_{S^*_\infty(t)h} = 0$ $\mu_\infty$-almost surely. Once this has been shown the proof is complete.

We start by noting that

$$P(t)\phi_h = \phi_{S^*_\infty(t)h}.$$

Hence by the $L^2$-boundedness of the maximal function,

$$\left\| \sup_{t > 0} |\phi_{S^*_\infty(t)h}| \right\|_{L^2(E, \mu_\infty)} \lesssim \|h\|_{H_\infty}.$$

By the semigroup property, this implies that

$$\left\| \sup_{t > T} |\phi_{S^*_\infty(t)h}| \right\|_{L^2(E, \mu_\infty)} = \left\| \sup_{t > 0} |\phi_{S^*_\infty(t+T)h}| \right\|_{L^2(E, \mu_\infty)} \lesssim \|S^*_\infty(T)h\|_{H_\infty}.$$
The right-hand side of this expression tends to 0 as $T \to \infty$. Having observed this, the proof can be finished with a standard Borel–Cantelli argument. With Chebyshev’s inequality, we find times $T_n \to \infty$ such that

$$\mu_\infty \left( x \in E : \sup_{t > T_n} |\phi S_\infty^{(t)} h(x)| > \frac{1}{2^n} \right) < \frac{1}{2^n}.$$ 

By the Borel–Cantelli lemma, it follows that

$$\mu_\infty \left( x \in E : \sup_{t > T_n} |\phi S_\infty^{(t)} h(x)| > \frac{1}{2^n} \text{ for infinitely many } n \right) = 0.$$

Hence, for $\mu_\infty$-almost all $x \in E$, we can find $n_0$ (depending on $x$) such that

$$\sup_{t > T_n} |\phi S_\infty^{(t)} h(x)| \leq \frac{1}{2^n} \text{ for all } n \geq n_0.$$

Clearly, that implies that $\lim_{t \to \infty} \phi S_\infty^{(t)} h(x) = 0$ for $\mu_\infty$-almost all $x \in E$. □

The next result is an extension of [7, Theorem 3.3, Corollary 3.4], where the stronger assumption was made that $\|S_\infty^{(t)}\| \leq e^{-wt}$ for some $w > 0$ and all $t \geq 0$.

THEOREM 2.11. (Poincaré inequality) Let Assumptions 2.1 and 2.2 be satisfied. If the semigroup $S_\infty^{(t)}$ is uniformly exponentially stable, then there is a constant $C$ such that for all $\phi \in W^{1,2}_H(E, \mu_\infty)$, we have

$$\int_E (\phi - \bar{\phi})^2 \, d\mu_\infty \leq C \int_E \|D^H \phi\|^2 \, d\mu_\infty.$$

Proof. By [28], $S$ restricts to a bounded analytic $C_0$-semigroup $S_H := S|_H$ on $H$, and by [21, Theorem 5.4], this semigroup is uniformly exponentially stable, say $\|S_H^{(t)}\| \leq Me^{-wt}$ with $M \geq 1$ and $w > 0$.

Next, we note (see [27, Theorem 5.6]) that $P(t) f \in W^{1,2}_H(E, \mu_\infty)$ and $D^H P(t) f = (P(t) \otimes S_H^{(t)})) D^H f$. Hence, $\mu_\infty$-almost everywhere we have

$$\|D^H P(t) f\|_{H^2}^2 = \|P(t) \otimes S_H^{(t)} D^H f\|_{H^2}^2 \leq M^2 e^{-2wt} P(t)(\|D^H f\|_{H^2}^2). \quad (2.7)$$

Combining (2.7) with Proposition 2.5 and Theorem 2.10, as in [11, Proposition 2.2(a)], the desired result follows a method of Deuschel–Stroock [15] (following the lines of the proof of [14, Proposition 10.5.2], using the expression for $L$ as given in [26]; this produces the constant $M^2/2w$).

3. The Feynman–Kac semigroup on $L^2(E, \mu_\infty)$

In this section and the next, we extend the results of [11, Section 3] to the non-symmetric setting. Our proofs follow those of [11] closely, with some modifications necessitated by the non-selfadjointness of $L$. Another subtle difference concerns the
assumptions on the domain $\mathcal{O}$, which we take to be open as in [9,35]; in [11], closed domains are considered (in this connection see also Remark 4.2). For the convenience of the reader (and for the sake of mathematical rigour), we have therefore decided to write out all proofs in detail.

We shall always assume that Assumptions 2.1 and 2.2 are satisfied without repeating this at every instance. We fix an non-empty open subset $\mathcal{O}$ in $E$ satisfying

$$\mu_\infty(\mathcal{O}) > 0$$

and a bounded continuous function $V : E \to [0, 1]$ which satisfies

$$V(x) = 0, \quad x \in \overline{\mathcal{O}},
V(x) > 0, \quad x \in \partial \overline{\mathcal{O}}. \quad (3.1)$$

For $f \in B_b(E), x \in E$, and $\varepsilon > 0$ set

$$P_\varepsilon(t)f(x) := \mathbb{E}\left[ f(X^x(t))e^{-\frac{1}{\varepsilon} \int_0^t V(X^x(r))dr} \right]. \quad (3.2)$$

By standard arguments, $P_\varepsilon = (P_\varepsilon(t))_{t \geq 0}$ is a semigroup of linear contractions on $B_b(E)$, the so-called Feynman–Kac semigroup associated with $-L + \frac{1}{\varepsilon} V$.

**Proposition 3.1.** (cf. [11, Proposition 3.1]) For all $f \in B_b(E)$ and $\varepsilon > 0$,

$$\int_E (P_\varepsilon(t)f)^2 d\mu_\infty \leq \int_E f^2 d\mu_\infty.$$

As a consequence, $P_\varepsilon$ is uniquely extendable to a $C_0$-semigroup of contractions on $L^2(E, \mu_\infty)$.

**Proof.** Using the Cauchy–Schwarz inequality, for $f \in B_b(E)$, we have

$$\begin{align*}
(P_\varepsilon(t)f(x))^2 &= (\mathbb{E}\left[ f(X^x(t))e^{-\frac{1}{\varepsilon} \int_0^t V(X^x(r))dr} \right])^2 \\
&\leq \mathbb{E}[f^2(X^x(t))e^{-\frac{2}{\varepsilon} \int_0^t V(X^x(r))dr}] \\
&\leq \mathbb{E}[f^2(X^x(t))] = P(t)f^2(x).
\end{align*}$$

Integrating with respect to $\mu_\infty$ and using (2.4), we obtain

$$\int_E (P_\varepsilon(t)f)^2 d\mu_\infty \leq \int_E P(t)f^2 d\mu_\infty = \int_E f^2 d\mu_\infty.$$

This shows that the operators $P_\varepsilon(t)$ are contractive on $L^2(E, \mu_\infty)$. To see that the resulting semigroup $P_\varepsilon$ is strongly continuous, note that for all $f \in C_b(E)$ the mapping $t \mapsto P_\varepsilon(t)f(x)$ is continuous for each $x \in E$ by the path continuity of $t \mapsto X^x(t)$. Hence, by dominated convergence, $\lim_{t \downarrow 0} P_\varepsilon(t)f = f$ in $L^2(E, \mu_\infty)$ for all $f \in C_b(E)$. By density and uniform boundedness, the strong continuity of $P_\varepsilon$ follows from this. \qed
From now on, unless stated otherwise, we shall denote by $P_\varepsilon$ the $C_0$-semigroup of contractions on $L^2(E, \mu_\infty)$ whose existence is assured by the above proposition. Our next aim is to identify $L - \frac{1}{\varepsilon} V$ as its generator.

For fixed $\lambda > 0$ and $f \in L^2(E, \mu_\infty)$, let us consider the resolvent equation
\[ \lambda \phi_\varepsilon - L \phi_\varepsilon + \frac{1}{\varepsilon} V \phi_\varepsilon = f. \] (3.3)

PROPOSITION 3.2. (cf. [11, Proposition 3.2]). Equation (3.3) has a unique solution $\phi_\varepsilon \in D(L)$, and the following estimates hold:
\[ \int_E \phi_\varepsilon^2 \, d\mu_\infty \leq \frac{1}{\lambda^2} \int_E f^2 \, d\mu_\infty, \] (3.4)
\[ \int_E \|D^H \phi_\varepsilon\|_H^2 \, d\mu_\infty \leq \frac{2}{\lambda} \int_E f^2 \, d\mu_\infty, \] (3.5)
\[ \int_E \phi_\varepsilon^2 V \, d\mu_\infty \leq \frac{\varepsilon}{\lambda} \int_E f^2 \, d\mu_\infty, \] (3.6)
\[ \int_E \|D^H \phi_\varepsilon\|_H^2 V \, d\mu_\infty \leq \frac{\varepsilon^{1/2}}{\lambda^{1/2}} \int_E f^2 \, d\mu_\infty. \] (3.7)

Proof. We know that the form $l$ defined in (2.5) is closed, densely defined, sectorial, and accretive. Since
\[ \left|\left[1 \varepsilon V f, f\right]_{L^2(E, \mu_\infty)}\right| \leq \frac{1}{\varepsilon} \|V\|_\infty \|f\|_{L^2(E, \mu_\infty)}^2, \]
the KLMN theorem (see [24, Theorem VI.1.33]) shows that the form associated with $-L + \frac{1}{\varepsilon} V$ is closed, densely defined, and sectorial. It is also accretive since $-L + \frac{1}{\varepsilon} V \geq -L \geq 0$. Therefore, $-L + \frac{1}{\varepsilon} V$ is maximal accretive, and (3.3) has a unique solution $\phi_\varepsilon \in D(L)$. Thus,
\[ \int_E \phi_\varepsilon^2 \, d\mu_\infty = \left\|\left(\lambda - L + \frac{1}{\varepsilon} V\right)^{-1} f\right\|_{L^2(E, \mu_\infty)}^2 \leq \frac{1}{\lambda^2} \|f\|_{L^2(E, \mu_\infty)}^2 = \frac{1}{\lambda^2} \int_E f^2 \, d\mu_\infty. \]

Let us now multiply both sides of (3.3) by $\phi_\varepsilon$ and integrate over $E$:
\[ \lambda \int_E \phi_\varepsilon^2 \, d\mu_\infty - \int_E L \phi_\varepsilon \cdot \phi_\varepsilon \, d\mu_\infty + \frac{1}{\varepsilon} \int_E V \phi_\varepsilon^2 \, d\mu_\infty = \int_E f \phi_\varepsilon \, d\mu_\infty. \] (3.8)

Since $[Bu, u] = \frac{1}{2} \|u\|_H^2$,
\[ - \int_E L \phi_\varepsilon \cdot \phi_\varepsilon \, d\mu_\infty = \int_E \left[BD^H \phi_\varepsilon, D^H \phi_\varepsilon \right] \, d\mu_\infty = \frac{1}{2} \int_E \|D^H \phi_\varepsilon\|_H^2 \, d\mu_\infty. \]

Substituting this identity in (3.8) yields
\[ \frac{1}{2} \int_E \|D^H \phi_\varepsilon\|_H^2 \, d\mu_\infty \leq \int_E f \phi_\varepsilon \, d\mu_\infty \leq \left(\int_E f^2 \, d\mu_\infty\right)^{1/2} \left(\int_E \phi_\varepsilon^2 \, d\mu_\infty\right)^{1/2} \leq \frac{1}{\lambda} \int_E f^2 \, d\mu_\infty, \]
where we used the Cauchy–Schwarz inequality and (3.4). We also notice from (3.8) that
\[ \frac{1}{\varepsilon} \int_E V \phi_t^2 \, d\mu_\infty \leq \int_E f \phi_t \, d\mu_\infty \leq \frac{1}{\lambda} \int_E f^2 \, d\mu_\infty. \]
Next, multiplying both sides of (3.3) by \( \phi_t V \) and integrating give
\[ \lambda \int_E \phi_t^2 V \, d\mu_\infty - \int_E L \phi_t \cdot \phi_t V \, d\mu_\infty + \frac{1}{\varepsilon} \int_E V \phi_t^2 \, d\mu_\infty = \int_E f \phi_t V \, d\mu_\infty. \quad (3.9) \]
Repeating the reasoning following (3.8), from (3.9) we infer
\[ \frac{1}{2} \int_E \|D^H \phi_t\|_H^2 V \, d\mu_\infty \leq \int_E f \phi_t V \, d\mu_\infty \leq \left( \int_E f^2 \, d\mu_\infty \right)^{1/2} \left( \int_E \phi_t^2 V \, d\mu_\infty \right)^{1/2} \]
\[ \leq \left( \int_E f^2 \, d\mu_\infty \right)^{1/2} \left( \int_E \phi_t^2 V \, d\mu_\infty \right)^{1/2} \leq \frac{\varepsilon^{1/2}}{\lambda^{1/2}} \int_E f^2 \, d\mu_\infty. \]
\[ \square \]
To prove that \( L - \frac{1}{\varepsilon} V \) is the generator of the semigroup \( (P_\varepsilon(t))_{t \geq 0} \), we need the following result. Let
\[ \mathcal{C} := \text{span}\{P(t)f : t > 0, \ f \in C_b(E)\}. \]

**Lemma 3.3.** \( \mathcal{C} \) is a core for \( D(L) \), and we have \( \mathcal{C} \subseteq D(L) \cap C_b(E) \).

**Proof.** Since \( C_b(E) \) is dense in \( L^2(E, \mu_\infty) \) and contained in \( \overline{\mathcal{C}} \), \( \mathcal{C} \) is dense in \( L^2(E, \mu_\infty) \). Since \( P \) is analytic on \( L^2(E, \mu_\infty) \), \( \mathcal{C} \) is contained in \( D(L) \). Moreover, \( \mathcal{C} \) is \( P \)-invariant, and therefore, \( \mathcal{C} \) is a core for \( D(L) \). Finally, it is immediate from (2.2) that \( P(t) f \in C_b(E) \) for all \( t > 0 \) and \( f \in C_b(E) \), so \( \mathcal{C} \subseteq D(L) \cap C_b(E) \). \( \square \)

Let \( M_\varepsilon \) be the infinitesimal generator of \( P_\varepsilon \) on \( L^2(E, \mu_\infty) \).

**Proposition 3.4.** (cf. [11, Proposition 3.3]) We have \( D(M_\varepsilon) = D(L) \) and
\[ M_\varepsilon = L - \frac{1}{\varepsilon} V. \quad (3.10) \]

**Proof.** Let us show that \( D(L) \subseteq D(M_\varepsilon) \) and that the identity (3.10) holds on \( D(L) \). Then, since both \( M_\varepsilon \) and \( L - \frac{1}{\varepsilon} V \) are semigroup generators, the identity \( D(M_\varepsilon) = D(L) \) follows.

Fix \( f \in D(L) \cap C_b(E) \) and \( x \in E \). For all \( t > 0 \),
\[ P_\varepsilon(t)f(x) - f(x) = \mathbb{E}[f(X^x(t))e^{-\frac{1}{\varepsilon} \int_0^t V(X^x(s)) \, ds}] - f(x) \]
\[ = \mathbb{E}[f(X^x(t))] - f(x) + \mathbb{E}[f(X^x(t))(e^{-\frac{1}{\varepsilon} \int_0^t V(X^x(s)) \, ds} - 1)]. \]
Dividing both sides by $t$ and letting $t \downarrow 0$, by pathwise continuity and dominated convergence, we obtain

$$\frac{1}{t} (P_\varepsilon(t)f - f) \to Lf - \frac{1}{\varepsilon} Vf$$

in $L^2(E, \mu_\infty)$. It follows that $f \in \mathcal{D}(M_\varepsilon)$ and $M_\varepsilon f = Lf - \frac{1}{\varepsilon} Vf$.

Let now $f \in \mathcal{D}(L)$ be arbitrary. Let $f_n \to f$ in $\mathcal{D}(L)$ with $f_n \in \mathcal{C}$, where $\mathcal{C}$ is as in Lemma 3.3. Then, $f_n \to f$ in $L^2(E, \mu_\infty)$, $L f_n \to L f$ in $L^2(E, \mu_\infty)$, and $\frac{1}{\varepsilon} V f_n \to \frac{1}{\varepsilon} V f$ in $L^2(E, \mu_\infty)$. Therefore, $M_\varepsilon f_n = L f_n - \frac{1}{\varepsilon} V f_n \to L f - \frac{1}{\varepsilon} V f$ in $L^2(E, \mu_\infty)$. Since $M_\varepsilon$ is closed, this implies that $f \in \mathcal{D}(M_\varepsilon)$ and $M_\varepsilon f = Lf - \frac{1}{\varepsilon} Vf$. \hfill \square

4. The semigroup $P_{\mathcal{O}}(t)$

On $B_b(\mathcal{O})$, following [9,35], we define the operators $P_{\mathcal{O}}(t)$ for $t \geq 0$ by

$$P_{\mathcal{O}}(t)f(x) := \mathbb{E}[f(X^x(t))1_{\{\tau_\mathcal{O}^x > t\}}], \quad x \in \mathcal{O}.$$ 

Here,

$$\tau_\mathcal{O}^x := \inf\{t > 0 : X^x(t) \in \overline{\mathcal{O}}\}$$

is the entrance time of $\overline{\mathcal{O}}$ corresponding to the initial value $x$. As $\tau_\mathcal{O}^x > 0$ for all $x \in \mathcal{O}$, it is clear that $P_{\mathcal{O}}(0)f = f$, and an easy calculation based on (2.3) shows that $P_{\mathcal{O}}(t)P_{\mathcal{O}}(s)f = P_{\mathcal{O}}(t+s)f$ for all $t, s \geq 0$.

For $\varepsilon > 0$ let

$$\mathcal{O}_\varepsilon := \{x \in \mathcal{O} : d(x, \overline{\mathcal{O}}) > \varepsilon\}.$$ 

Let $V_\varepsilon : \mathbb{E} \to [0, 1]$ be the potential defined by

$$V_\varepsilon(x) = \frac{1}{\varepsilon} d(x, \mathcal{O}_\varepsilon) \wedge 1.$$ 

Note that $V_\varepsilon \equiv 0$ on $\overline{\mathcal{O}_\varepsilon}$ and $V_\varepsilon \equiv 1$ on $\overline{\mathcal{O}}$. In the results below, we denote by $P_\varepsilon$ the strongly continous semigroup of contractions on $B_b(E)$ generated by $L - \frac{1}{\varepsilon} V_\varepsilon$.

For functions $f \in B_b(\mathcal{O})$, we define

$$\tilde{f}(x) := \begin{cases} f(x), & x \in \mathcal{O}, \\ 0, & x \in \overline{\mathcal{O}}. \end{cases}$$

**Proposition 4.1.** (cf. [11, Proposition 3.5]) For all $f \in B_b(\mathcal{O}), x \in \mathcal{O},$ and $t \geq 0$,

$$\lim_{\varepsilon \downarrow 0} P_\varepsilon(t)\tilde{f}(x) = P_{\mathcal{O}}(t)f(x).$$
Proof. For $t = 0$, the result is trivial, so we may assume that $t > 0$. Fix $x \in \mathcal{O}$.

On the set $\{ \tau^\mathcal{O}_x > t \}$, we have $X^x(s) \in \mathcal{O}$ for all $s \in [0, t]$, and therefore, $V_\varepsilon(X^x(s)) = 0$ for all $s \in [0, t]$ provided $\varepsilon > 0$ is sufficiently small. If, on the other hand, $\omega \in \{ \tau^\mathcal{O}_x \leq t \}$, then by path continuity we have $X^x(t_0(\omega), \omega) \in \partial \mathcal{O}$ for some $t_0(\omega) \in (0, t]$, and therefore, $V_\varepsilon(X^x(t_0(\omega)), \omega) = 1$ for all $\varepsilon > 0$. Hence, for some small enough $\delta(\omega) > 0$, we have $V_\varepsilon(X^x(s, \omega)) \geq \frac{1}{2}$ for all $s \in [t_0(\omega) - \delta(\omega), t_0(\omega)]$.

Then,

$$
\int_0^t V_\varepsilon(X^x(s, \omega)) \, ds \geq \int_{t_0(\omega) - \delta(\omega)}^{t_0(\omega)} V_\varepsilon(X^x(s, \omega)) \, ds \geq \frac{1}{2} \delta(\omega) > 0,
$$

and therefore, $\limsup_{\varepsilon \downarrow 0} e^{-\frac{1}{2} \int_0^t V_\varepsilon(X^x(s, \omega)) \, ds} \leq \lim_{\varepsilon \downarrow 0} e^{-\frac{\delta(\omega)}{2}} = 0$.

Using these facts, by dominated convergence we obtain

$$
\lim_{\varepsilon \downarrow 0} P_\varepsilon(t) \tilde{f}(x) = \mathbb{E}[f(X^x(t)) \mathbf{1}_{\{\tau^\mathcal{O}_x > t\}}] + \lim_{\varepsilon \downarrow 0} \int_{\{\tau^\mathcal{O}_x \leq t\}} \tilde{f}(X^x(t)) e^{-\frac{1}{\varepsilon}} \int_0^{t_0(\omega)} V_\varepsilon(X^x(s)) \, ds \, d\mathbb{P}
$$

$$
= P_\mathcal{O}(t) f(x) + \lim_{\varepsilon \downarrow 0} \int_{\{\tau^\mathcal{O}_x \leq t\}} \tilde{f}(X^x(t)) e^{-\frac{1}{\varepsilon}} \int_0^{t_0(\omega)} V_\varepsilon(X^x(s)) \, ds \, d\mathbb{P}
$$

$$
= P_\mathcal{O}(t) f(x).
$$

\[\Box\]

REMARK 4.2. The papers [11] considers closed domains $K$ are used instead of open sets $\mathcal{O}$. This has the advantage that one can work with one potential $V$ which vanishes on $K$ and is strictly positive outside $K$. In this setting, however, we do not see how to prove the analogue Proposition 4.1 without any assumptions on the boundary of $K$ (the problem being the identity $P_K(0) f(x) = f(x)$ for points $x \in \partial K$, which in general need not hold).

PROPOSITION 4.3. The semigroup $P_\mathcal{O}$ has a unique extension to a $C_0$-semigroup of contractions on $L^2(\mathcal{O}, \mu_\infty)$.

Proof. First, we prove that each of the operators $P_\mathcal{O}(t)$ extends uniquely to a contraction on $L^2(\mathcal{O}, \mu_\infty)$. By the Cauchy–Schwarz inequality, for all $f \in B_b(\mathcal{O})$ and $x \in \mathcal{O}$, we have

$$
(P_\mathcal{O}(t) f(x))^2 = (\mathbb{E}[\tilde{f}(X^x(t)) \mathbf{1}_{\{\tau^\mathcal{O}_x > t\}}])^2
$$

$$
\leq \mathbb{E}[\tilde{f}^2(X^x(t)) \mathbf{1}_{\{\tau^\mathcal{O}_x > t\}}] \leq P(t) \tilde{f}^2(x).
$$

Hence,

$$
\int_{\mathcal{O}} (P_\mathcal{O}(t) f)^2 \, d\mu_\infty \leq \int_{E} P(t) \tilde{f}^2 \, d\mu_\infty = \int_{E} \tilde{f}^2 \, d\mu_\infty = \int_{\mathcal{O}} f^2 \, d\mu_\infty.
$$

This proves the asserted contractivity.
To prove strong continuity on $L^2(\mathcal{O}, \mu_\infty)$, first let $f \in B_b(\mathcal{O})$. Then, by the path continuity of $X^x$, for all $x \in \mathcal{O}$, we have $\lim_{t \downarrow 0} X^x(t) = x$ and $\tau^x_\mathcal{O} > 0$, and therefore,

$$\lim_{t \downarrow 0} P_{\mathcal{O}}(t) f(x) = \lim_{t \downarrow 0} \mathbb{E}[f(X^x(t))1_{\{\tau^x_\mathcal{O} > t\}}] = f(x)$$

by dominated convergence. Again by dominated convergence, this implies that $\lim_{t \downarrow 0} P_{\mathcal{O}}(t) f = f$ in $L^2(\mathcal{O}, \mu_\infty)$. For general $f \in L^2(\mathcal{O}, \mu_\infty)$, strong continuity in $L^2(\mathcal{O}, \mu_\infty)$ follows by density. \hfill \Box

From now on, $P_{\mathcal{O}}$ always denotes the $C_0$-semigroup of contractions on $L^2(E, \mu_\infty)$ whose existence is assured by the proposition. We denote by $L_\mathcal{O}$ its generator.

**PROPOSITION 4.4.** (cf. [11, Proposition 3.7]) For all $f \in L^2(\mathcal{O}, \mu_\infty)$ and $t > 0$,

$$\lim_{\varepsilon \downarrow 0} (P_\varepsilon(t)f) |_{\mathcal{O}} = P_{\mathcal{O}}(t)f \text{ in } L^2(\mathcal{O}, \mu_\infty). \quad (4.1)$$

Moreover, for all $\lambda > 0$ with $\lambda \in \varrho(L - \frac{1}{\varepsilon} V_\varepsilon)$ we have $\lambda \in \varrho(L_{\mathcal{O}})$ and

$$\lim_{\varepsilon \downarrow 0} \left( R \left( \lambda, L - \frac{1}{\varepsilon} V_\varepsilon \right) \tilde{f} \right) |_{\mathcal{O}} = R(\lambda, L_{\mathcal{O}}) f \text{ in } L^2(\mathcal{O}, \mu_\infty). \quad (4.2)$$

Here, for an operator $A$ and $\lambda \in \varrho(A)$, $R(\lambda, A) := (\lambda - A)^{-1}$ denotes the associated resolvent operator.

**Proof.** First let $f \in C_b(\mathcal{O})$. Then for all $x \in \mathcal{O}$, we have the pointwise bounds

$$|P_\varepsilon(t)\tilde{f}(x)| = |\mathbb{E}[\tilde{f}(X^x(t))e^{-\frac{t}{\varepsilon} \int_0^t V_\varepsilon(X^x(s))ds}]| \leq \|\tilde{f}\|_\infty$$

and

$$|P_{\mathcal{O}}(t) f(x)| = |\mathbb{E}[f(X^x(t))1_{\{\tau^x_\mathcal{O} > t\}}]| \leq \|f\|_\infty.$$ 

Hence by Proposition 4.1 and dominated convergence theorem, we obtain

$$\lim_{\varepsilon \downarrow 0} \|P_\varepsilon(t)\tilde{f})|_{\mathcal{O}} - P_{\mathcal{O}}(t)f\|_{L^2(\mathcal{O}, \mu_\infty)} = 0$$

for all $f \in C_b(E)$. Since $P_\varepsilon$ and $P_{\mathcal{O}}$ are contractive in $L^2(E, \mu_\infty)$ and $L^2(\mathcal{O}, \mu_\infty)$, respectively, this convergence extends to arbitrary $f \in L^2(\mathcal{O}, \mu_\infty)$.

Finally, (4.2) follows from (4.1) by taking Laplace transforms. \hfill \Box

Recalling the definition $W_{H}^{1, 2}(\mathcal{O}, \mu_\infty) := D(D^H)$, we now define

$$\hat{W}_{H}^{1, 2}(\mathcal{O}, \mu_\infty) := \{ f \in L^2(\mathcal{O}, \mu_\infty) : \tilde{f} \in D(D^H), \ D^H \tilde{f} = 0 \text{ } \mu_\infty\text{-a.e. on } \overline{\mathcal{O}} \}.$$ 

Thus, $\hat{W}_{H}^{1, 2}(\mathcal{O}, \mu_\infty)$ is the natural domain of the part of $D^H$ in $L^2(\mathcal{O}, \mu_\infty)$. We shall study this operator in more detail in the next section.
THEOREM 4.5. (cf. [11, Theorem 3.8]) For all \( \lambda > 0 \) and \( f \in L^2(\mathcal{O}, \mu_\infty) \), we have \( \phi := R(\lambda, L_\mathcal{O}) f \in \dot{W}^{1,2}_H(\mathcal{O}, \mu_\infty) \) and

\[
\lambda \int_\mathcal{O} \phi v \, d\mu_\infty + \int_\mathcal{O} [B D^H \phi, D^H v] \, d\mu_\infty = \int_\mathcal{O} f v \, d\mu_\infty \quad \forall v \in \dot{W}^{1,2}_H(\mathcal{O}, \mu_\infty).
\]

(4.3)

Proof. Fix \( \lambda > 0 \) and \( f \in L^2(\mathcal{O}, \mu_\infty) \). For \( \varepsilon > 0 \) set

\[
\phi_\varepsilon := R(\lambda, M_\varepsilon) \tilde{f} = R(\lambda, L - \frac{1}{\varepsilon} V_\varepsilon) \tilde{f}.
\]

Then, \( \phi_\varepsilon \in D(M_\varepsilon) = D(L) \), so \( \phi_\varepsilon \in D(D^H) = W^{1,2}_H(E, \mu_\infty) \), and by (3.4) and (3.5) (applied to the potentials \( V_\varepsilon \)), we obtain

\[
\| \phi_\varepsilon \|^2_{W^{1,2}_H(E, \mu_\infty)} = \| \phi_\varepsilon \|^2_{L^2(E, \mu_\infty)} + \| D^H \phi_\varepsilon \|^2_{L^2(E, \mu_\infty; H)} \leq \left( \frac{1}{\lambda^2} + \frac{2}{\lambda} \right) \| f \|^2_{L^2(\mathcal{O}, \mu_\infty)}.
\]

Therefore, there exists a sequence \( \varepsilon_j \to 0 \) and a function \( \psi \in W^{1,2}_H(E, \mu_\infty) \) such that \( \phi_{\varepsilon_j} \to \psi \) weakly in \( W^{1,2}_H(E, \mu_\infty) \) as \( j \to \infty \). Let us prove that \( \psi = \tilde{\phi} \).

For every \( g \in L^2(\mathcal{O}, \mu_\infty) \), by (4.2) we have

\[
\int_\mathcal{O} \psi g \, d\mu_\infty = \lim_{j \to \infty} \int_\mathcal{O} \phi_{\varepsilon_j} g \, d\mu_\infty = \int_\mathcal{O} \phi g \, d\mu_\infty.
\]

Thus, \( \psi |_{\mathcal{O}} = \phi \). Next, we want to prove that \( \psi |_{\mathcal{C} \mathcal{O}} = 0 \). The weak convergence \( \phi_{\varepsilon_j} \to \psi \) in \( W^{1,2}_H(E, \mu_\infty) \) implies weak convergence in \( L^2(E, \mu_\infty) \) and hence in \( L^2(\mathcal{C} \mathcal{O}, \mu_\infty) \). Recalling that \( V_\varepsilon \equiv 1 \) on \( \mathcal{C} \mathcal{O} \), we obtain

\[
\int_{\mathcal{C} \mathcal{O}} \psi^2 \, d\mu_\infty = \lim_{j \to \infty} \int_{\mathcal{C} \mathcal{O}} \phi_{\varepsilon_j} \psi \, d\mu_\infty = \lim_{j \to \infty} \int_{\mathcal{C} \mathcal{O}} \phi_{\varepsilon_j} \psi_{\varepsilon_j} \, d\mu_\infty.
\]

Using (3.6),

\[
\left| \int_{\mathcal{C} \mathcal{O}} \phi_{\varepsilon_j} \psi \psi_{\varepsilon_j} \, d\mu_\infty \right| \leq \left( \int_{\mathcal{C} \mathcal{O}} |\phi_{\varepsilon_j}|^2 \psi_{\varepsilon_j} \, d\mu_\infty \right)^{1/2} \left( \int_{\mathcal{C} \mathcal{O}} |\psi|^2 \psi_{\varepsilon_j} \, d\mu_\infty \right)^{1/2} \leq \left( \frac{\varepsilon_j}{\lambda} \int_E |\tilde{f}|^2 \, d\mu_\infty \right)^{1/2} \left( \int_{\mathcal{C} \mathcal{O}} |\psi|^2 \psi_{\varepsilon_j} \, d\mu_\infty \right)^{1/2}.
\]

Upon letting \( j \to \infty \), we obtain that \( \psi |_{\mathcal{C} \mathcal{O}} = 0 \) \( \mu_\infty \)-almost everywhere.

By what has been proved so far, \( \phi_{\varepsilon_j} \to \tilde{\phi} \) weakly in \( W^{1,2}_H(E, \mu_\infty) \).

Next, we will prove that \( (D^H \tilde{\phi}) |_{\mathcal{C} \mathcal{O}} = 0 \) \( \mu_\infty \)-almost everywhere. By (3.5), the functions \( D^H \phi_\varepsilon \) are uniformly bounded in \( L^2(E, \mu_\infty) \), and therefore there exists a (possibly different) sequence \( \varepsilon_j \to 0 \) and a function \( \xi \in W^{1,2}_H(E, \mu_\infty) \) such that \( D^H \phi_{\varepsilon_j} \to \xi \) weakly in \( L^2(E, \mu_\infty) \) as \( j \to \infty \). Then, arguing as before,

\[
\int_{\mathcal{C} \mathcal{O}} \xi^2 \, d\mu_\infty = \lim_{j \to \infty} \int_{\mathcal{C} \mathcal{O}} D^H \phi_{\varepsilon_j} \xi \psi_{\varepsilon_j} \, d\mu_\infty.
\]
Using (3.7),
\[ \left| \int_{\mathcal{O}} D^H \phi_{\varepsilon_j} \xi V_{\varepsilon_j} \, d\mu_\infty \right| \leq \left( \int_{\mathcal{O}} \| D^H \phi_{\varepsilon_j} \|^2_H V_{\varepsilon_j} \, d\mu_\infty \right)^{1/2} \left( \int_{\mathcal{O}} |\xi|^2 V_{\varepsilon_j} \, d\mu_\infty \right)^{1/2} \]
\[ \leq \left( \frac{\varepsilon_j^{1/2}}{\lambda^{1/2}} \int_E |\tilde{f}|^2 \, d\mu_\infty \right)^{1/2} \left( \int_{\mathcal{O}} |\xi|^2 \, d\mu_\infty \right)^{1/2}. \]

Upon letting \( j \to \infty \), we obtain that \( \xi|_{\mathcal{O}} = 0 \) \( \mu_\infty \)-almost everywhere. Moreover, the closedness (and hence, by the Hahn–Banach theorem, weak closedness) of \( D^H \) gives \( D^H \tilde{\phi} = \xi \). This proves that \( (D^H \tilde{\phi})|_{\mathcal{O}} = 0 \) \( \mu_\infty \)-almost everywhere.

Combining what we have proved so far, we see that \( \phi \in \dot{W}^{1,2}_H(\mathcal{O}, \mu_\infty) \). Next, we multiply the identity \( \lambda \phi_{\varepsilon_j} - L \phi_{\varepsilon_j} + \frac{1}{\varepsilon_j} V_{\varepsilon_j} \phi_{\varepsilon_j} = \tilde{f} \) with \( v \), where \( v \in \dot{W}^{1,2}_H(\mathcal{O}, \mu_\infty) \).

Upon integrating over \( E \setminus (\mathcal{O} \setminus \mathcal{O}_\varepsilon) \) and noting that \( V_{\varepsilon_j} \tilde{v} \equiv 0 \) on this set, we obtain
\[ \int_{E \setminus (\mathcal{O} \setminus \mathcal{O}_\varepsilon)} (\lambda - \tilde{L}) \phi_{\varepsilon_j} \tilde{v} \, d\mu_\infty = \int_{E \setminus (\mathcal{O} \setminus \mathcal{O}_\varepsilon)} (\lambda - \tilde{L}) \phi_{\varepsilon_j} \tilde{v} \, d\mu_\infty \]
\[ + \frac{1}{\varepsilon_j} \int_{E \setminus (\mathcal{O} \setminus \mathcal{O}_\varepsilon)} V_{\varepsilon_j} \phi_{\varepsilon_j} \tilde{v} \, d\mu_\infty = \int_{E \setminus (\mathcal{O} \setminus \mathcal{O}_\varepsilon)} \tilde{f} \tilde{v} \, d\mu_\infty. \]

Passing to the limit for \( j \to \infty \) and using Proposition 2.5, we obtain
\[ \lambda \int_{E} \phi_{\varepsilon_j} \tilde{v} \, d\mu_\infty + \int_{E} [BD^H \phi_{\varepsilon_j}, D^H \tilde{v}] \, d\mu_\infty = \int_{E} (\lambda - \tilde{L}) \phi_{\varepsilon_j} \tilde{v} \, d\mu_\infty = \int_{E} \tilde{f} \tilde{v} \, d\mu_\infty. \]
This proves (4.3). \( \square \)

It follows from this theorem that \( D(L_\mathcal{O}) \subseteq \dot{W}^{1,2}_H(\mathcal{O}, \mu_\infty) \). In particular, the space \( \dot{W}^{1,2}_H(\mathcal{O}, \mu_\infty) \) is dense in \( L^2(\mathcal{O}, \mu_\infty) \).

Consider the bilinear form (recall that we work over the real scalars)
\[ (f, g) \mapsto \int_\mathcal{O} [BD^H f, D^H g] \, d\mu_\infty. \quad f, g \in \dot{W}^{1,2}_H(\mathcal{O}, \mu_\infty). \]

It is an easy consequence of the identity \([Bh, h] = \frac{1}{2} \| h \|^2_H\) (see Proposition 2.4) that this form is densely defined, continuous, accretive, and closed. Arguing as in [27, Proposition 4.3], we see that it is in fact sectorial, and therefore, we can define a closed densely defined operator \(-M_\mathcal{O}\), which we will call the Dirichlet Ornstein–Uhlenbeck operator, on \( L^2(\mathcal{O}, \mu_\infty) \) with this form in the usual way (see [32, Section 1.2.3]), and \( M_\mathcal{O} \) generates a strongly continuous analytic contraction semigroup on \( L^2(\mathcal{O}, \mu_\infty) \).

**THEOREM 4.6.** We have \( L_\mathcal{O} = M_\mathcal{O} \). As a consequence, the semigroup \( P_\mathcal{O} \) is a strongly continuous analytic contraction semigroup on \( L^2(\mathcal{O}, \mu_\infty) \).

**Proof.** Using the notation of the previous proposition, from (4.3) it follows that if \( f \in L^2(\mathcal{O}, \mu_\infty) \) and \( \lambda > 0 \), then \( \phi = R(\lambda, L_\mathcal{O}) f \in D(M_\mathcal{O}) \) and
\[ \lambda \phi - M_\mathcal{O} \phi = f = \lambda \phi - L_\mathcal{O} \phi. \]
It follows that $\mathcal{D}(L_\mathcal{O}) \subseteq \mathcal{D}(M_\mathcal{O})$ and that $L_\mathcal{O} = M_\mathcal{O}$ on $\mathcal{D}(L_\mathcal{O})$. Since both operators are semigroup generators, this implies that $\mathcal{D}(L_\mathcal{O}) = \mathcal{D}(M_\mathcal{O})$ and $L_\mathcal{O} = M_\mathcal{O}$. □

We conclude this section with a gradient estimate for non-symmetric Ornstein–Uhlenbeck semigroups. Da Prato and Lunardi studied the symmetric case (see [11, Section 3.3, consequence (iii), and Proposition 3.9]).

**Theorem 4.7.** (Gradient estimates) For all $f \in L^2(E, \mu_\infty)$

$$\|D^H p_\varepsilon(t) f\|_{L^2(E, \mu_\infty; H)} \leq \frac{C}{\sqrt{t}} \|f\|_{L^2(E, \mu_\infty)},$$

and for all $f \in L^2(\mathcal{O}, \mu_\infty)$

$$\|D^H p_\varepsilon(t) f\|_{L^2(\mathcal{O}, \mu_\infty; H)} \leq \frac{C}{\sqrt{t}} \|f\|_{L^2(\mathcal{O}, \mu_\infty)}.$$

**Proof.** Using (3.5) and setting $t = \frac{1}{\lambda}$, we observe that, for all $g \in L^2(E, \mu_\infty)$,

$$\left\| D^H \left( I - tL + \frac{t}{\varepsilon} V_\varepsilon \right)^{-1} g \right\|_{L^2(E, \mu_\infty; H)} \leq \sqrt{\frac{2}{t}} \|g\|_{L^2(E, \mu_\infty)}.$$  

Then, using this estimate with the $L^2$-contractivity of $p_\varepsilon(t)$ and its $L^2$-analyticity, we obtain

$$\|D^H p_\varepsilon(t) f\|_{L^2(E, \mu_\infty; H)} \leq \sqrt{\frac{2}{t}} \| (I - tL + \frac{t}{\varepsilon} V_\varepsilon) p_\varepsilon(t) f\|_{L^2(E, \mu_\infty)} \leq \frac{C_\varepsilon}{\sqrt{t}} \|f\|_{L^2(E, \mu_\infty)},$$

with a constant $C_\varepsilon$ which, as an inspection of the proof shows, can be uniformly bounded from above independently of $\varepsilon > 0$. Applying the method of proof of the inequality (3.5) on the identity (4.3) yields

$$\|D^H R(\lambda, L_\mathcal{O}) f\|_{L^2(\mathcal{O}, \mu_\infty; H)} \leq \sqrt{\frac{2}{\lambda}} \|f\|_{L^2(\mathcal{O}, \mu_\infty)}.$$ (4.4)

Then, arguing as above we obtain

$$\|D^H p_\varepsilon(t) f\|_{L^2(\mathcal{O}, \mu_\infty; H)} \leq \frac{C}{\sqrt{t}} \|f\|_{L^2(\mathcal{O}, \mu_\infty)}.$$ □

### 5. Boundedness of the Riesz transform for $L_\mathcal{O}$

In this section, we obtain sufficient conditions for the boundedness on $L^2(\mathcal{O}, \mu_\infty)$ of the Riesz transform associated with $L_\mathcal{O}$. Observe that when $L_\mathcal{O}$ is selfadjoint (i.e.
when \( B = \frac{1}{2} I \), this follows from the identities
\[
\| (-L_O)^{1/2} f \|_{L^2(O, \mu_\infty)}^2 = -\int_O L_O f \cdot f \, d\mu_\infty
\]
\[
= \frac{1}{2} \int_O [D^H_O f, D^H_O f] \, d\mu_\infty = \frac{1}{2} \| D^H_O f \|_{L^2(O, \mu_\infty; H)}^2.
\]

In order to discuss the non-selfadjoint case, we need to introduce some auxiliary operators.

We begin by defining the operator \( D^H_O \) with domain \( D(D^H_O) := \hat{W}^{1,2}_H(O, \mu_\infty) \) by
\[
D^H_O f := D^H \tilde{f}, \quad f \in D(D^H_O).
\]

By the definition of \( \hat{W}^{1,2}_H(O, \mu_\infty) \), \( D^H \tilde{f} \) vanishes \( \mu_\infty \)-almost everywhere on \( \partial O \), so that it can indeed be identified with an element of \( L^2(O, \mu_\infty; H) \).

**Lemma 5.1.** The operator \( D^H_O \) is closed and densely defined in \( L^2(O, \mu_\infty) \).

**Proof.** We have already seen that \( D(D^H_O) = \hat{W}^{1,2}_H(O, \mu_\infty) \) is dense in \( L^2(O, \mu_\infty) \). To see that \( D^H_O \) is closed, let \( f_n \in D(D^H_O) \) be such that \( f_n \to f \) in \( L^2(O, \mu_\infty) \) and \( D^H_O f_n \to g \) in \( L^2(O, \mu_\infty; H) \). Then, \( \tilde{f}_n \to \tilde{f} \) in \( L^2(E, \mu_\infty) \) and \( D^H \tilde{f}_n \to \tilde{g} \) in \( L^2(E, \mu_\infty; H) \), so \( \tilde{f} \in D(D^H_H) \) and \( D^H \tilde{f} = \tilde{g} \). But this is the same as saying that \( f \in D(D^H_H) \) and \( D^H_O f = g \).

Thanks to this lemma, the adjoint operator \( D^H_O^* = (D^H_O)^* \) is well defined as a closed densely defined operator on \( L^2(O, \mu_\infty; H) \).

The next lemma is a straightforward consequence of the definition of \( L_O \) in terms of the bilinear form \( l_O \).

**Lemma 5.2.** We have
\[
D(L_O) = \{ f \in D(D^H_O) : BD^H_O f \in D(D^H_O^*) \} = \{ f \in D(D^H_O) : D^H_O f \in D(D^H_O^*B) \},
\]
and for all \( f \in D(L_O) \) we have
\[
L_O f = -D^H_O^* (BD^H_O) f = -(D^H_O^* B) D^H_O f.
\]

Consider the form
\[
l_O^I(F, G) := \int_O [D^H_O^* F, D^H_O^* G]_H \, d\mu_\infty
\]
for \( F, G \in D(l_O^I) := D(D^H_O^*) \). This form is accretive, densely defined, and closed, and since it is symmetric, it is sectorial. Therefore the associated operator, which we denote by \( D^H_O D^H_O^* \), is densely defined, closed, and selfadjoint, with domain
\[
D(D^H_O D^H_O^*) = \{ F \in D(D^H_O^*) : D^H_O^* F \in D(D^H_O) \}.
\]
Since $B$ is bounded and coercive we have equivalences of norms

$$\|Bu\| \approx \|u\| \approx \|B^*u\|.$$ 

As a consequence, $B$ is boundedly invertible. By the argument of [27, Proposition 5.1]), it follows from [2, Proposition 7.1] that the operator

$$L_\mathcal{O} := -D_\mathcal{O}^HD_\mathcal{O}^{H*}B$$

with domain

$$\mathcal{D}(D_\mathcal{O}^HD_\mathcal{O}^{H*}B) = \{F \in L^2(\mathcal{O}; H) : BF \in \mathcal{D}(D_\mathcal{O}^HD_\mathcal{O}^{H*})\}$$

is closed, densely defined, and sectorial. In particular, $L_\mathcal{O}$ generates a bounded analytic semigroup, denoted by $P_\mathcal{O}(t)$, on $L^2(\mathcal{O}, \mu_{\infty}; H)$ (see [16, Theorem 4.6]).

**Lemma 5.3.** For all $g \in \mathcal{D}(L_\mathcal{O})$ and $t > 0$, we have $(I - tL_\mathcal{O})^{-1}g \in \mathcal{D}(D_\mathcal{O}^H)$ and

$$D_\mathcal{O}^H(I - tL_\mathcal{O})^{-1}g = (I - tL_\mathcal{O})^{-1}D_\mathcal{O}^Hg \quad (5.1)$$

and

$$(I - tL_\mathcal{O})^{-1}D_\mathcal{O}^{H*}BD_\mathcal{O}^Hg = D_\mathcal{O}^{H*}B(I - tL_\mathcal{O})^{-1}D_\mathcal{O}^Hg. \quad (5.2)$$

**Proof.** The set $\mathcal{S} := \{f \in \mathcal{D}(L_\mathcal{O}) : L_\mathcal{O}f \in \mathcal{D}(D_\mathcal{O}^H)\}$ is dense (it contains the dense set $\mathcal{R} = \{R(\lambda, L_\mathcal{O})g : \lambda > 0, g \in \mathcal{D}(L_\mathcal{O})\}$ and invariant under $P_\mathcal{O}(t)$, and therefore, it is a core for $\mathcal{D}(L_\mathcal{O})$.

For all $f \in \mathcal{S}$, we have, using Lemma 5.2 to justify the formal computation,

$$D_\mathcal{O}^HL_\mathcal{O}f = -D_\mathcal{O}^HD_\mathcal{O}^{H*}BD_\mathcal{O}^Hf = L_\mathcal{O}D_\mathcal{O}^Hf.$$ 

Multiplying the resulting identity

$$D_\mathcal{O}^H(I - tL_\mathcal{O})f = (I - tL_\mathcal{O})D_\mathcal{O}^Hf$$

on the left by $(I - tL_\mathcal{O})^{-1}$ and taking $f = (I - tL_\mathcal{O})^{-1}g$ with $g \in \mathcal{D}(L_\mathcal{O})$ (in which case we have $f \in \mathcal{S}$), the identity in (5.1) is obtained for functions $g \in \mathcal{D}(L_\mathcal{O})$.

Next, $\mathcal{D}(L_\mathcal{O})$ is a core for $\mathcal{D}(D_\mathcal{O}^H)$ and by (5.1), for all $f \in \mathcal{D}(D_\mathcal{O}^H)$ we have

$$[D_\mathcal{O}^Hf, B(I - tL_\mathcal{O})^{-1}D_\mathcal{O}^Hg] = [D_\mathcal{O}^Hf, BD_\mathcal{O}^H(I - tL_\mathcal{O})^{-1}g]$$

$$= -[f, L_\mathcal{O}(I - tL_\mathcal{O})^{-1}g] = -[f, (I - tL_\mathcal{O})^{-1}L_\mathcal{O}g]$$

$$= [f, (I - tL_\mathcal{O})^{-1}D_\mathcal{O}^{H*}BD_\mathcal{O}^Hg].$$

This show that $B(I - tL_\mathcal{O})^{-1}D_\mathcal{O}^Hg$ is in $\mathcal{D}(D_\mathcal{O}^{H*})$ and (5.2) holds. \qed
By standard semigroup theory, the above lemma implies the identity
\[ P_{\mathcal{O}}(t) D_{\mathcal{O}}^H f = D_{\mathcal{O}}^H P_{\mathcal{O}}(t) f, \]
first for \( f \in D(L_{\mathcal{O}}) \) and then for \( f \in D(D_{\mathcal{O}}^H) \), using that \( D(L_{\mathcal{O}}) \) is a core for \( D(D_{\mathcal{O}}^H) \). In particular, we see that the semigroup \( P_{\mathcal{O}} \) maps \( R(D_{\mathcal{O}}^H) \) into itself. From now on, we shall always consider \( P_{\mathcal{O}} \) as a semigroup on this space. By a slight abuse of notation, its generator, which is the part of \( L_{\mathcal{O}} \) in \( R(D_{\mathcal{O}}^H) \), will be denoted again by \( L_{\mathcal{O}} \).

On the product space \( L^2(E, \mu_{\infty}) \oplus R(D_{\mathcal{O}}^H) \), we now consider the operator
\[
\Pi_{\mathcal{O}} := \begin{pmatrix} 0 & D_{\mathcal{O}}^H B^* \\ D_{\mathcal{O}}^H & 0 \end{pmatrix}
\]
with domain \( D(\Pi_{\mathcal{O}}) = D(D_{\mathcal{O}}^H) \oplus D(D_{\mathcal{O}}^H B^* B) \), where, by the same abuse of notation, we denote by \( D_{\mathcal{O}}^H B^* \) the domain of the part of \( D_{\mathcal{O}}^H B \) in \( R(D_{\mathcal{O}}^H) \).

A densely defined closed linear operator \( A \) is called \textit{bisectorial} if \( i\mathbb{R}\backslash\{0\} \subseteq \varrho(A) \) and
\[
\sup_{t \in \mathbb{R}\backslash\{0\}} \| (I - itA)^{-1} \| < \infty.
\]

A standard Taylor expansion argument implies that there exists an \( \theta \in (0, \frac{1}{2}\pi) \) such that the open bisector of angle \( \theta \) around the imaginary axis belongs to \( \varrho(A) \) and the above uniform boundedness estimate extends to this bisector.

Let us look at following result (see [1, Section (H)]) which uses McIntosh’s notion of a bounded functional calculus. Let \( \theta \in (0, \frac{1}{2}\pi) \) be given. A sectorial operator \( T \) on a Banach space \( F \) admits a \textit{bounded functional calculus of angle} \( \theta \) if the Dunford functional calculus of \( T \) extends to a bounded homomorphism
\[
H^\infty(\Sigma_\theta) \to \mathcal{L}(F), \quad f \mapsto f(T).
\]
Here, \( \Sigma_\theta = \{ z \in \mathbb{C}\backslash\{0\} : |\arg(z)| < \theta \} \) is the open sector in the complex right half-plane with aperture \( \theta \). For a detailed treatment, we refer the reader to [1,22,25]. The bounded functional calculus
\[
H^\infty(\Sigma_\theta \cup -\Sigma_\theta) \to \mathcal{L}(F), \quad f \mapsto f(T)
\]
for bisectorial operators \( T \) on \( E \) is defined similarly.

\textbf{Proposition 5.4.} If \( \Pi \) is a bisectorial operator on a Hilbert space \( \mathcal{H} \), then \( \Pi^2 \) is sectorial on \( \mathcal{H} \) and for each \( \theta \in (0, \frac{\pi}{4}) \), the following assertions are equivalent:

1. \( \Pi \) admits a bounded functional calculus on a bisector of angle \( \theta \);
2. \( \Pi^2 \) admits a bounded functional calculus on a sector of angle \( 2\theta \).
Now, we are ready to state and prove the first main result of this section. Examples where the conditions of the theorem are fulfilled are given subsequently.

**THEOREM 5.5.** Suppose that $-L_O$ admits a bounded holomorphic functional calculus on $\overline{R(D_H^O)}$. Then,

$$D(D_H^O) = D((-L_O)^{1/2}),$$
$$D(D_H^O B) = D((-L_O)^{1/2}),$$

(5.3)

with equivalence of the homogeneous seminorms

$$\| D_H^O f \|_{L^2(O, \mu_\infty; H)} \equiv \| (-L_O)^{1/2} f \|_{L^2(O, \mu_\infty)},$$
$$\| D_H^O B g \|_{L^2(O, \mu_\infty)} \equiv \| (-L_O)^{1/2} g \|_{L^2(O, \mu_\infty; H)}.$$  

(5.4)

**Proof.** We shall prove that $\Pi_O$ is bisectorial on $L^2(O, \mu_\infty) \oplus \overline{R(D_H^O)}$. Assuming this for the moment, we first show how the result follows from this.

Since $-L_O$ and $-L_O^T$ have bounded functional calculi on suitable sectors of angle $< \pi/2$ (for $-L_O$, this follows from the fact that $L_O$ generates an analytic contraction semigroup), the same is true for $\Pi_O^2$, and hence, by Proposition 5.4, $\Pi_O$ has a bounded functional calculus on a bisector of angle $< \pi/2$. This implies the boundedness of the operators $\Pi_O/\sqrt{\Pi_O^2}$ and of $\sqrt{\Pi_O^2}/\Pi_O$ (apply the functional calculus of $\Pi_O$ to the bounded holomorphic functions $z/\sqrt{z^2}$ and $\sqrt{z^2}/z$). By a standard argument, this implies (5.3) and (5.4); we refer to [3,27] for the details.

The bisectoriality of $\Pi_O$ remains to be proved. Fix $t \in \mathbb{R} \setminus \{0\}$ and consider the operator matrix

$$R_t := \begin{pmatrix} (I - t^2 L_O)^{-1} & it(I - t^2 L_O)^{-1} D_H^O B \\ itD_H^O (I - t^2 L_O)^{-1} & (I - t^2 L_O)^{-1} \end{pmatrix}.$$ 

By Lemma 5.3, the identity $R_t (I - it \Pi_O) = I$ holds on the linear subspace of all $(g, G) \in L^2(O, \mu_\infty) \oplus \overline{R(D_H^O)}$ with $g \in D(L_O)$ and $G = D_H^O g'$ with $g' \in D(L_O)$. Since $D(L_O)$ is a core for $D(D_H^O)$, this linear subspace is dense and the identity extends to all pairs $(g, G) \in L^2(O, \mu_\infty) \oplus \overline{R(D_H^O)}$.

This shows that $R_t$ equals the resolvent $(I - it \Pi_O)^{-1}$ defined on $L^2(O, \mu_\infty) \oplus \overline{R(D_H^O)}$. Let us now study the boundedness of each of the entries of the matrix $R_t$.

We have already seen that

$$\|(I - t^2 L_O)^{-1}\|_{L^2(O, \mu_\infty)} \leq 1$$

and

$$\|(I - t^2 L_O)^{-1}\|_{L^2(O, \mu_\infty; H)} \leq C,$$

with a constant $C$ independent of $t \in \mathbb{R} \setminus \{0\}$.

Taking $\lambda = \frac{1}{t^2}$ in (4.4), we obtain

$$\|t D_H^O (I - t^2 L_O)^{-1}\|_{L^2(O, \mu_\infty; H)} \leq 2.$$  

(5.5)
admits a bounded holomorphic calculus on $L^2(\mathcal{O}, \mu_\infty; H)$. Then, using Lemma 5.2 to see that $L^*_\mathcal{O} = (D^*_\mathcal{O}B)D^*_\mathcal{O}$ implies $L^*_\mathcal{O} = D^*_\mathcal{O}(D^*_\mathcal{O}B)^* = D^*_\mathcal{O}(B^*D^*_\mathcal{O}) = (D^*_\mathcal{O}B^*)D^*_\mathcal{O}$, and using (5.5) with $L^*_\mathcal{O}$ instead of $L^*_\mathcal{O}$, we obtain

$$
\|tB^*D^*_\mathcal{O}(I - t^2L^*_\mathcal{O})^{-1}\|_{L^2(\mathcal{O}, \mu_\infty; H)} \leq \|B\|\|tD^*_\mathcal{O}(I - t^2L^*_\mathcal{O})^{-1}\|_{L^2(\mathcal{O}, \mu_\infty; H)} \leq 2\|B\| \leq 1,
$$

and by duality we obtain

$$
\|t(I - t^2L^*_\mathcal{O})^{-1}D^*_\mathcal{O}B\|_{L^2(\mathcal{O}, \mu_\infty)} \leq 1.
$$

As a consequence, the operators $(I - it\Pi_\mathcal{O})^{-1}$ are uniformly bounded on the space $L^2(\mathcal{O}, \mu_\infty) \oplus \overline{R(D^*_\mathcal{O})}$ for all $t \in \mathbb{R}\setminus\{0\}$. By standard arguments, this implies that $\Pi_\mathcal{O}$ is bisectorial on $L^2(\mathcal{O}, \mu_\infty) \oplus \overline{R(D^*_\mathcal{O})}$. □

The condition that $-L^*_\mathcal{O}$ has a functional calculus is satisfied when $L^*_\mathcal{O}$ is selfadjoint. Indeed, then $L^*_\mathcal{O}$ is selfadjoint as well, and $-L^*_\mathcal{O}$, being non-negative and selfadjoint, admits a bounded holomorphic calculus.

Open problem. If $-L$ admits a bounded holomorphic calculus on $\overline{R(D^*_\mathcal{O})}$, does $-L^*_\mathcal{O}$ admit a bounded holomorphic calculus on $\overline{R(D^*_\mathcal{O})}$?

An affirmative answer would imply that the condition of Theorem 5.5 is always satisfied in case $H = E = \mathbb{R}^n$ (as is explained in the discussion below [27, Theorem 2.2]).

In order to state a second open problem, we need to introduce some notations. We begin with a lemma which asserts that we can define $D^*_\mathcal{O} = \hat{D}^*_\mathcal{O} \otimes I$ as a closed and densely defined operator from $L^2(\mathcal{O}, \mu_\infty)$ to $L^2(\mathcal{O}, \mu_\infty; H \hat{\otimes} H)$. Here, and in what follows, we denote by $\otimes$ and $\hat{\otimes}$ the algebraic tensor product and the completed Hilbert space tensor product respectively.

**Lemma 5.6.** The mapping $D^*_\mathcal{O} : \hat{W}^{1,2}_{1H}(\mathcal{O}, \mu_\infty) \otimes H \to L^2(\mathcal{O}, \mu_\infty; H \hat{\otimes} H)$ defined by

$$
D^*_\mathcal{O}(f \otimes h) := D^*_{\mathcal{O}}f \otimes h
$$

is closable as an operator from $L^2(\mathcal{O}, \mu_\infty; H)$ to $L^2(\mathcal{O}, \mu_\infty; H \hat{\otimes} H)$.

**Proof.** Suppose $F_n \to 0$ in $L^2(\mathcal{O}, \mu_\infty; H)$, with each $F_n \in \hat{W}^{1,2}_{1H}(\mathcal{O}, \mu_\infty) \otimes H$, and $D^*_\mathcal{O}F_n \to G$ in $L^2(\mathcal{O}, \mu_\infty; H \hat{\otimes} H)$. Then, for each $h \in H$, $[F_n, h] \to 0$ in $L^2(\mathcal{O}, \mu_\infty)$, $[F_n, h] \in \hat{W}^{1,2}_{1H}(\mathcal{O}, \mu_\infty)$, and

$$
[D^*_\mathcal{O}F_n, h] = D^*_\mathcal{O}[F_n, h] \to [G, h]
$$

in $L^2(\mathcal{O}, \mu_\infty; H)$. The closedness of $D^*_\mathcal{O}$ implies $[G, h] = 0$ in $L^2(\mathcal{O}, \mu_\infty; H)$ for all $h \in H$, and therefore, $G = 0$ in $L^2(\mathcal{O}, \mu_\infty; H \hat{\otimes} H)$. □
By a slight abuse of notation, from now one we shall denote by \( D^H_\mathcal{O} \) the closure of this operator and by \( D(D^H_\mathcal{O}) \) its domain. The closed operator \( D^H \) with domain \( D(D^H) \)
is defined similarly (see, e.g., [27, Section 11]).

Let
\[
\hat{W}^{2,2}_H(\mathcal{O}, \mu_\infty) = \{ f \in D(D^H_\mathcal{O}) : D^H_\mathcal{O} f \in D(D^H) \} = \{ f \in L^2(\mathcal{O}, \mu_\infty) : \tilde{f} \in D(D^H), D^H f \in D(D^H), \ D^H \tilde{f} = 0 \ \mu_\infty\text{-a.e. on } \bar{\mathcal{O}}, \ D^H(D^H f) = 0 \ \mu_\infty\text{-a.e. on } \bar{\mathcal{O}} \}.
\]

With respect to the norm \( \| f \|^2_{\hat{W}^{2,2}_H(\mathcal{O}, \mu_\infty)} = \| f \|^2 + \| D^H_\mathcal{O} f \|^2 + \| D^H(D^H f) \|^2 \), this space is a Hilbert space.

**Open problem.** Under what conditions on \( A \) and \( \mathcal{O} \) do we have a continuous inclusion
\[
D(L_\mathcal{O}) \subseteq \hat{W}^{2,2}_H(\mathcal{O}, \mu_\infty)?
\]

For \( \mathcal{O} = \mathcal{E} \), this inclusion is obtained in [27, Proposition 11.1(iii)]. In the case of non-trivial open domains \( \mathcal{O} \subset \mathcal{E} \), Da Prato and Lunardi [12] obtained the inclusion in the case \( A = I \) under suitable regularity conditions on \( \partial \mathcal{O} \) and showed that the inclusion may fail if no such conditions are imposed. The methods of [27] seem not to adapt very well to the domain setting.

We conclude with a Poincaré inequality for \( L_\mathcal{O} \).

**THEOREM 5.7.** (Poincaré inequality for \( L_\mathcal{O} \)) Suppose that \( \mu_\infty(\bar{\mathcal{O}}) > 0 \) and that \( S_\infty \) is uniformly exponentially stable. If \( -L_\mathcal{O} \) admits a bounded holomorphic functional calculus on \( \Re(D^H_\mathcal{O}) \), there is a constant \( C \) such that for all \( u \in \hat{W}^{1,2}(\mathcal{O}, \mu_\infty) \), we have
\[
\| u \|^2_{L^2(\mathcal{O}, \mu_\infty)} \leq C \| D^H_\mathcal{O} u \|^2_{L^2(\mathcal{O}, \mu_\infty; \mathcal{H})}.
\]

**Proof.** The proof is a modification of [11, Proposition 3.9].

**Step 1**—In this step, we prove that \( 0 \in \varrho(L_\mathcal{O}) \).

Since \( P_\mathcal{O} \) is a contraction semigroup on \( L^2(\mathcal{O}, \mu_\infty) \) (see Proposition 4.3), the spectrum of \( L_\mathcal{O} \) is contained in the closed left-half plane. Therefore, if \( 0 \in \sigma(L_\mathcal{O}) \), it belongs to the approximate point spectrum of \( L_\mathcal{O} \). This means that there is a sequence \( (u_n)_{n \geq 1} \) in \( D(L_\mathcal{O}) \) such that \( \| u_n \|^2_{L^2(\mathcal{O}, \mu_\infty)} = 1 \) for all \( n \geq 1 \) and \( \lim_{n \to \infty} L_\mathcal{O} u_n = 0 \) in \( L^2(\mathcal{E}, \mu_\infty) \). Then,
\[
\int_{\mathcal{O}} \| D^H_\mathcal{O} u_n \|^2_{\mathcal{H}} d\mu_\infty = 2 \int_{\mathcal{O}} [B^H_\mathcal{O} u_n, D^H_\mathcal{O} u_n] d\mu_\infty = -2[ L_\mathcal{O} u_n, u_n ] \to 0.
\]

Hence \( \lim_{n \to \infty} D^H_\mathcal{O} u_n = 0 \) in \( L^2(\mathcal{E}, \mu_\infty) \). Therefore, by Theorem 2.11, \( \lim_{n \to \infty} (\tilde{u}_n - \bar{u}_n) = 0 \) in \( L^2(\mathcal{E}, \mu_\infty) \). But then \( \lim_{n \to \infty} \| \tilde{u}_n \|^2_{L^2(\mathcal{E}, \mu_\infty)} = 1 \), which means that
\[ \tilde{u}_n \to 1 \text{ in } L^2(E, \mu_\infty). \] Passing to a subsequence, we may also assume that the convergence holds \( \mu_\infty \)-almost everywhere. But this contradicts the fact that \( \tilde{u}_n \) vanishes on the set \( \mathcal{C} \mathcal{O} \) which has positive \( \mu_\infty \)-measure by assumption.

Step 2—By Step 1, \( L_\mathcal{O} \) is boundedly invertible, and then \( (-L_\mathcal{O})^{1/2} \) is boundedly invertible as well. Consequently, for \( u \in W^{1,2}(\mathcal{O}, \mu_\infty) = \mathcal{D}((-L_\mathcal{O})^{1/2}) = \mathcal{D}(D^H_\mathcal{O}) \), we have, by the equivalence of seminorms of Theorem 5.5,

\[
\| u \|_{L^2(\mathcal{O}, \mu_\infty)} \leq \| (-L_\mathcal{O})^{-1/2} \| \| (-L_\mathcal{O})^{1/2} u \|_{L^2(\mathcal{O}, \mu_\infty)} \\
\approx \| (-L_\mathcal{O})^{-1/2} \| \| D^H_\mathcal{O} u \|_{L^2(\mathcal{O}, \mu_\infty; H)}.
\]

\[ \square \]

Note that if \( \mu_\infty(\mathcal{C} \mathcal{O}) = 0 \), then we have a canonical identification \( L^2(\mathcal{O}, \mu_\infty) = L^2(E, \mu_\infty) \), and under this identification, we have \( D^H_\mathcal{O} = D^H \) and \( L_\mathcal{O} = L \). Then, it follows from Theorem 2.11 that

\[
\| u - \bar{u} \|_{L^2(\mathcal{O}, \mu_\infty)} \leq C \| D^H_\mathcal{O} u \|_{L^2(\mathcal{O}, \mu_\infty; H)}.
\]

REMARK 5.8. Step 1 in the above proof could be simplified (along the lines of [11]) if we knew that \( L \) has compact resolvent.

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