Parameters’ domain in three flavour neutrino oscillations

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Abstract

We consider analytically the domain of the three mixing angles $\Theta_{ij}$ and the CP phase $\delta$ for three flavour neutrino oscillations both in vacuum and matter. Similarly to the quark sector, it is necessary and sufficient to let all the mixing angles $\Theta_{12}, \Theta_{13}, \Theta_{23}$ and $\delta$ be in the range $(0, \frac{\pi}{2})$ and $0 \leq \delta < 2\pi$, respectively. To exploit the full range of $\delta$ will be important in future when more precise fits are possible, even without CP violation measurements. With the above assumption on the angles we can restrict ourselves to the natural order of masses $m_1 < m_2 < m_3$. Considerations of the mass schemes with some negative $\delta m^2$’s, though for some reasons useful, are not necessary from the point of view of neutrino oscillation parametrization and cause double counting only. These conclusions are independent of matter effects.

1 Introduction

Three flavour neutrino oscillations are considered as a reliable mechanism to explain atmospheric and solar neutrino anomalies. The neutrino flavour eigenstates $\nu_\alpha = (\nu_e, \nu_\mu, \nu_\tau)$ are assumed to be combinations of mass eigenstates $\nu_i = (\nu_1, \nu_2, \nu_3)$

$$\nu_\alpha = \sum_{i=1}^{3} U_{\alpha i} \nu_i.$$  \hspace{1cm} (1)

Various parametrizations of the mixing matrix $U$ are possible for Dirac and Majorana neutrinos. All of them use three mixing angles $\Theta_{ij}$ (ij=12,13,23) and one (Dirac) or three (Majorana) CP phases. As the neutrino oscillation experiments are not sensitive to Majorana CP phases the same mixing matrix...
U as in the quark sector [1] is adopted (see e.g. [2] for discussion of various parametrizations)

\[
U = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}
\]

(2)

where \( c(s)_{ij} \equiv \cos(\sin)\Theta_{ij} \).

The mixing angles \( \Theta_{ij} \) can be defined to lie in the first quadrant by appropriately adjusting the neutrino and charged lepton phases, analogously to the quark sector [3]. To exhaust the full parameter space the CP phase \( \delta \) must be taken in the range \( 0 \leq \delta \leq 2\pi \). There is only one important difference between quark and neutrino sectors: alignment of absolute neutrino masses is unknown and, among others, normal and inverse neutrino mass hierarchy schemes are considered. It is also true that neutrino oscillations in vacuum will never be able to distinguish these two schemes. The argument is that in vacuum, without CP violation measurements, the oscillation probability depends only on \( \sin^2 \left( \delta m_{ij}^2 \frac{L}{4E} \right) \), and the sign of \( \delta m_{ij}^2 \), which decide about the mass scheme, is unmeasurable. Neutrino oscillations in matter would only give the chance to measure the sign of \( \delta m_{ij}^2 \). We would like to clarify the notion of using \( \delta m_{ij}^2 \) signs for neutrinos mixing parametrization [4]. We find the full domain of the three mixing angles \( \Theta_{ij} \) and \( \delta \) phase in the mixing matrix \( U \) in matter. It appears that parameter space in the matter case is exactly the same as in vacuum

\[
0 \leq \Theta_{ij} \leq \frac{\pi}{2}, \\
0 \leq \delta < 2\pi.
\]

(3)

In addition, in matter, it is not necessary to consider mass schemes with different mass arrangements \( \delta m_{ij}^2 = \pm|\delta m_{ij}^2| \). With the full range of parameters, it is enough to include only the “canonical” order of masses \( (m_1 < m_2 < m_3) \). All other mass schemes with negative \( \delta m_{ij}^2 \) are equivalent to that with \( \delta m_{21}^2 > 0 \) and \( \delta m_{32}^2 > 0 \) and a different region in the parameter space of Eq. 3. Finally we argue that exploring \( \delta \) in its full range will be important in future experiments and not necessarily in connection with the explicit measurements of CP violation effects.

In the next Section, in a simple analysis we find the domain of parameters for neutrino oscillation in vacuum. Even if the range Eq. 3 is well known from the quark sector, we discuss it as it is a suitable introduction to understand the more complicated case of neutrino oscillations in matter. This is presented in Section 3. Finally, the conclusions are gathered in Section 4.
2 Parameter space for neutrino oscillations in vacuum

The vacuum neutrino flavour oscillation probability for an initially produced $\nu_\alpha$ with an energy $E$ converted into detected $\nu_\beta$ after traveling a distance $L$ is given by

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{a>b} R_{\alpha\beta}^{ab} \sin^2 \Delta_{ab} - Y \sum_{\gamma} \epsilon_{\alpha\beta\gamma},$$

(4)

where

$$R_{\alpha\beta}^{ab} = \text{Re} \left[ W_{\alpha\beta}^{ab} \right],$$

(5)

$$Y = 8 \text{Im} \left[ W_{\epsilon\mu}^{12} \right] \sin \Delta_{21} \sin \Delta_{31} \sin \Delta_{32},$$

(6)

$$\Delta_{ab} = 1.27 \delta m_{ab}^2 [eV] \frac{L[km]}{E[GeV]},$$

(7)

and

$$\delta m_{ab}^2 = m_a^2 - m_b^2, \quad W_{\alpha\beta}^{ab} = U_{\alpha a} U_{\beta b} U_{\gamma a}^* U_{\gamma b}^*.$$

We can see that the mixing matrix elements $U_{\alpha a}$ enter the oscillation probability by $W_{\alpha\beta}^{ab}$ tensors which are invariant under the phase transformation

$$U_{\gamma c} \rightarrow e^{-i\delta}, U_{\gamma c} e^{i\epsilon_c}.$$

(8)

Freedom of this transformation can be used to show that all mixing angles $\Theta_{ij}$ originally belonging to the interval $\langle 0, 2\pi \rangle$ can be mapped onto the first quadrant $\Theta_{ij} \in \langle 0, \frac{\pi}{2} \rangle$. As the real and the imaginary parts of the phase factor $e^{i\delta}$ are allowed to change sign, the appropriate interval for $\delta$ is $\langle 0, 2\pi \rangle$.

Now we show in a way which will be useful in the more complicated case of neutrino oscillations in matter, that in fact the domain from Eq. 3 covers the full parameter space of possible neutrino transitions. First of all, from unitarity of the $U$ matrix follows that all $R_{\alpha\beta}^{ab}$ tensors can be expressed by squares of moduli of the $U$ matrix elements

$$R_{\alpha\beta}^{ab} = \frac{1}{2} \left( |U_{\gamma a}|^2 |U_{\gamma b}|^2 - |U_{\alpha a}|^2 |U_{\alpha b}|^2 - |U_{\beta a}|^2 |U_{\beta b}|^2 \right),$$

(9)

for $\alpha \neq \beta \neq \gamma, \ a \neq b$, and

$$R_{\alpha\alpha}^{ab} = |U_{\alpha a}|^2 |U_{\alpha b}|^2, \quad R_{\alpha\beta}^{aa} = |U_{\alpha a}|^2 |U_{\beta a}|^2,$$

(10)
otherwise.

The $|U_{e\alpha}|^2$ for $a=1,2,3$ and $|U_{\alpha\beta}|^2$ for $\alpha = \mu, \tau$ depend only on sine and cosine squares of $\Theta_{ij}$ and do not feel the transformations among the four quadrants. Only $|U_{aa}|^2$'s for $\alpha = \mu, \tau$ and $a=1,2$ depend linearly on sines and cosines of $\Theta_{ij}$ angles, namely

$$|U_{aa}|^2 = K_{aa} \pm S,$$

where $K_{aa}$ are still functions of $\cos^2 \Theta_{ij}$ and $\sin^2 \Theta_{ij}$, but

$$S = \frac{1}{2} F \cos \delta,$$

$$F = \sin 2\Theta_{12} \sin 2\Theta_{23} \sin \Theta_{13}.$$  

Exactly the same factor $F$ appears in the Jarlskog invariant (Eq. 6) [5]

$$J \equiv Im [W_{e\mu}^{12}] = \frac{1}{4} F \cos^2 \Theta_{13} \sin \delta.$$  

Only the $F$ factor is sensitive to the change of sign when the angles $\Theta_{ij}$ are mapped from the full domain $\langle 0, 2\pi \rangle$ to the final range $\langle 0, \frac{\pi}{2} \rangle$ ($n_{ij} = 0, \ldots, 3$)

$$F \left( \Theta_{12} + n_{12} \frac{\pi}{2}; \Theta_{13} + n_{13} \frac{\pi}{2}; \Theta_{23} + n_{23} \frac{\pi}{2} \right) \bigg|_{0 \leq \Theta_{ij} < \frac{\pi}{4}} = (-)^{n_{12} + n_{23}} f(n_{13}) F \left( \Theta_{12}; \Theta_{13}; \Theta_{23} \right)$$

where

$$f(n_{13}) = \begin{cases} 
+1 & \text{for } n_{13} = 0, 1 \\
-1 & \text{for } n_{13} = 2, 3
\end{cases}.$$  

In order to compensate the possible change of signs in Eq. 15 other factors in Eqs. 6, 14 and Eq. 12 must have the freedom to change sign. The only possible choices are the CP phase $\delta$ and the combination $\Delta$ in Eq. 6 defined as

$$\Delta = \sin \Delta_{21} \sin \Delta_{31} \sin \Delta_{32}.$$  

There are two possibilities.

- If $\delta \in \langle 0, 2\pi \rangle$ then a change of sign by $\sin \delta$ in Eq. 14 and $\cos \delta$ in Eq. 12 compensates the sign in Eq. 15. In this case the order of masses can be kept canonical, $m_1 < m_2 < m_3$. 

If $\delta \in (0, \pi)$ then $\cos \delta$ is able to compensate the sign in Eq. 12, but $\sin \delta > 0$, so $\Delta$ must be used in the CP violating $Y$ quantity (Eq. 6). In such a case, schemes with $\Delta > 0$ ([123],[231],[312]) are distinguishable from schemes with $\Delta < 0$ ([132],[321],[213]) in oscillation appearance experiments (see Fig. 1 for notation).

We see that the chosen $\Theta_{ij}$ and $\delta$ angles given in Eq. 3 exhaust the full parameter space. We can bind the CP violating phase to the smaller range

$$0 \leq \delta \leq \pi$$

and in the same time distinguish the neutrino mass schemes with $\Delta > 0$ (cyclic mass permutations from the canonical case) from $\Delta < 0$ cases (non-cyclic mass permutations of the canonical scheme). It is impossible to disentangle schemes inside these two groups. Therefore an approach with the canonical order of masses is clearer from the point of view of neutrino oscillation parametrization: a point (region) in the parameter space of Eq. 3 determines the scheme of masses and the mixture of the weak states in an unambiguously way. We will turn back to the interpretation of $\delta m_{ij}^2$ signs in the next section.

Presently, as statistical errors are large any subleading effects in neutrino oscillations are neglected and experimental data for neutrino (disappearance) oscillations are fitted by the formula where only sine and/or cosine squares of the $\Theta_{ij}$ mixing angles appear. Therefore we do not have to explore the full parameter space in Eq. 3. However, if the future precision improves and subleading effects are measured then it may be necessary to do it. Now we show an example where taking into account the $\delta$ phase is important even if CP violation is not measured. Let us consider atmospheric $\nu_{\mu} \rightarrow \nu_{\mu}$ disappearance probability in vacuum

$$P_{\mu\mu} = 1 - 4 \left[ (K_{\mu2}K_{\mu1} - S^2) \sin^2 \Delta_{21} + |U_{\mu3}|^2 (K_{\mu1} \sin^2 \Delta_{31} - K_{\mu2} \sin^2 \Delta_{32}) \right]$$

$$- 4 S \left[ (K_{\mu2} - K_{\mu1}) \sin^2 \Delta_{21} + |U_{\mu3}|^2 \left( \sin^2 \Delta_{31} - \sin^2 \Delta_{32} \right) \right]$$

(18)

where $K_{\mu i}$ (i=1,2) are defined in Eq. 11. We can see that there is a part proportional to $S$ which exists only if $\Delta_{21} \neq 0 \iff \Delta_{31} \neq \Delta_{32}$. In Fig. 2 $P_{\mu\mu}$ as function of $L/E$ is given for $\delta m_{21}^2 \equiv \delta m_{sol}^2 = 2.5 \cdot 10^{-4} eV^2$, $\delta m_{31}^2 \equiv \delta m_{atm}^2 = 2.5 \cdot 10^{-3} eV^2$, $\Theta_{23} = \Theta_{12} = \frac{\pi}{2}$ and $\Theta_{13} = 0.2$. $\delta$ is taken to be $0$ and $\pi$. The difference between $\delta = 0$ and $\delta = \pi$ cases can be easily seen. This difference diminishes with decreasing $\delta m_{sol}^2$. Taking into account some new results where exploration of large values of $\delta m_{sol}^2$ (even up to a scale of $\delta m_{atm}^2$) is discussed seriously [6],[7],[8],[9] this effect should be keep in mind when a precise, global analysis of oscillation data is undertaken, especially with incoming neutrino factory physics. Let us note, that only $\delta = 0$ is used presently. Usually it
is assumed, that in the CP conservation case it is allowed to take $\delta = 0$ and $0 \leq \Theta_{ij} \leq \frac{\pi}{2}$. Surprisingly it is not true. If CP is conserved, then the discussion given above implies that mapping the full range $0 \leq \Theta_{ij} < 2\pi$ onto $0 \leq \Theta_{ij} \leq \frac{\pi}{2}$ requires the term $\cos \delta$ in Eq. 12 to have two discrete values $\pm 1$.

3 Parameter space for neutrino oscillations in matter

The probability $P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^m$ of neutrino oscillations in matter of density $N_e$ is given by the vacuum formula (Eq. 4) with modified $U_{aa}, \Delta_{ab}$ and $J$ [10]

$$P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^m = \delta_{\alpha\beta} - 4 \sum_{a>b} Re[W_{\alpha\beta}^{(m)ab}] \sin^2 \Delta_{ab}^m - Y_m \sum_{\gamma} \epsilon_{\alpha\beta\gamma},$$

where

$$W_{\alpha\beta}^{(m)ab} = U_{\alpha a}^m U_{\beta b}^m U_{\alpha b}^m U_{\beta a}^m,$$

$$U_{aa}^m = \frac{N_a}{D_a} U_{aa} + \frac{A}{D_a} U_{ea} \left[ \left( \lambda_a^2 - m_b^2 \right) U_{ec}^* U_{ac} + \left( \lambda_a^2 - m_c^2 \right) U_{eb}^* U_{ab} \right],$$

$$\Delta_{ab}^m = 1.27 \frac{(\lambda_a^2 - \lambda_b^2) [eV]^2 L[km]}{E[GeV]},$$

$$Y_m = 8J^m \sin \Delta_{21}^m \sin \Delta_{31}^m \sin \Delta_{32}^m,$$

$$J^m = J \frac{(\lambda_1^2 - m_2^2) (\lambda_1^2 - m_3^2) (\lambda_2^2 - m_1^2) (\lambda_2^2 - m_3^2)}{D_1^2 D_2^2} \times \left\{ N_1 N_2 - N_2 A \left( \lambda_1^2 - m_b^2 \right) |U_{e1}|^2 - N_1 A \left( \lambda_2^2 - m_1^2 \right) |U_{e2}|^2 + A^2 |U_{e1}|^2 |U_{e2}|^2 \left[ \left( \lambda_2^2 - m_3^2 \right) \left( \lambda_1^2 - m_3^2 \right) + \left( \lambda_2^2 - m_3^2 \right) \left( \lambda_1^2 - m_2^2 \right) \right] - \left( \lambda_2^2 - m_3^2 \right) \left( \lambda_1^2 - m_2^2 \right) \right\},$$

$$N_a = \left( \lambda_a^2 - m_b^2 \right) \left( \lambda_a^2 - m_c^2 \right) - A \left[ \left( \lambda_a^2 - m_b^2 \right) |U_{ec}|^2 + \left( \lambda_a^2 - m_c^2 \right) |U_{eb}|^2 \right],$$

$$D_a^2 = N_a^2 + A^2 |U_{ea}|^2 \left[ \left( \lambda_a^2 - m_b^2 \right)^2 |U_{ec}|^2 + \left( \lambda_a^2 - m_c^2 \right)^2 |U_{eb}|^2 \right].$$
$\lambda_a^2$ denote the effective mass squares of neutrinos in matter and follow from diagonalization of an effective Hamiltonian

$$H_\nu = \frac{1}{2E} \left[ \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} + U^T \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U \right].$$

$m_a$ $(a=1,2,3)$ are neutrino masses and $A = 2\sqrt{2}E G_F N_e$. Using the Cardano formula we get

$$\lambda_1^2 = -\frac{a_2}{3} - \frac{1}{3} p \left( \cos \phi + \sqrt{3} \sin \phi \right),$$

$$\lambda_2^2 = -\frac{a_2}{3} - \frac{1}{3} p \left( \cos \phi - \sqrt{3} \sin \phi \right),$$

$$\lambda_3^2 = -\frac{a_2}{3} + \frac{2}{3} p \cos \phi,$$

where

$$p = \sqrt{a_2^2 - 3a_1},$$

$$\phi = \frac{1}{3} \arccos \left[ -\frac{1}{p^3} \left( a_2^3 - \frac{9}{2} a_1 a_2 + \frac{27}{2} a_0 \right) \right],$$

and

$$a_0 = -m_1^2 m_2^2 m_3 - A \left[ m_1^2 m_3^2 |Ue_2|^2 + m_2^2 m_3^2 |Ue_3|^2 + m_2^2 m_3 |Ue_1|^2 \right],$$

$$a_1 = m_2^2 m_3^2 + m_1^2 m_2^2 + m_1^2 m_3^2$$

$$+ A \left[ m_1^2 (1 - |Ue_1|^2) + m_2^2 (1 - |Ue_2|^2) + m_3^2 (1 - |Ue_3|^2) \right],$$

$$a_2 = -(m_1^2 + m_2^2 + m_3^2 + A).$$

Now we can proceed as in the vacuum case and consider $P_{\nu_a \nu_a}^m (\Theta_{ij}, \delta, \delta m_{ij}^2)$ in the full range of parameters. Similarly to the vacuum case the real parts of $W_{\alpha\beta}^{(m)ab}$ depend on $|U_{\alpha\beta}|^2$. These subsequently depend on vacuum mixing matrix elements

$$|U_{aa}|^2 = \frac{N_a^2}{D_a^2} |U_{aa}|^2 + 2 \frac{A N_a}{D_a^2} \left\{ (\lambda_a^2 - m_b^2) R_{ac} + (\lambda_c^2 - m_b^2) R_{co} \right\}$$

$$+ \frac{A^2}{D_a^2} |U_{ea}|^2 \left\{ (\lambda_a^2 - m_b^2)^2 |U_{ec}|^2 |U_{ac}|^2 + (\lambda_c^2 - m_b^2)^2 |U_{eb}|^2 |U_{ab}|^2 \right\}.$$
\[ + 2 \left( \lambda_a^2 - m_b^2 \right) \left( \lambda_a^2 - m_b^2 \right) R_{\alpha\alpha} \].

We can see that the mixing angles appear in the squared moduli \(|U_{\gamma c}|^2\) and inside the R tensors (Eqs. 9,10) which also depend on \(|U_{\gamma c}|^2\). So, as in the vacuum case, when the full domain \([0, 2\pi]\) is mapped onto \([0, \pi/2]\), non trivial signs appear only in the F factor (Eq. 13). Thus again, the change of signs can be compensated by \(|U_{\gamma c}|^2\) in Eq. 12.

In the CP violating part (Eq. 23) the vacuum mixing angles are found in the squared moduli of \(|U_{ei}|^2\) and J (see Eq. 24). Again, only the F factor in J (Eqs. 13,14) changes sign if the \(\Theta_{ij}\) angles are reduced to the first quadrant. If \(\delta \in (0, 2\pi)\) then \(|U_{ei}|^2\) and J (see Eq. 24) changes sign if the \(\Theta_{ij}\) angles are reduced to the first quadrant. If \(\delta \in (0, 2\pi)\) then \(|U_{ei}|^2\) and J (see Eq. 24) changes sign if the \(\Theta_{ij}\) angles are reduced to the first quadrant. If \(\delta \in (0, 2\pi)\) then \(|U_{ei}|^2\) and J (see Eq. 24) changes sign if the \(\Theta_{ij}\) angles are reduced to the first quadrant. If \(\delta \in (0, 2\pi)\) then \(|U_{ei}|^2\) and J (see Eq. 24) changes sign if the \(\Theta_{ij}\) angles are reduced to the first quadrant. If \(\delta \in (0, 2\pi)\) then \(|U_{ei}|^2\) and J (see Eq. 24) changes sign if the \(\Theta_{ij}\) angles are reduced to the first quadrant. If \(\delta \in (0, 2\pi)\) then \(|U_{ei}|^2\) and J (see Eq. 24) changes sign if the \(\Theta_{ij}\) angles are reduced to the first quadrant. If \(\delta \in (0, 2\pi)\) then \(|U_{ei}|^2\) and J (see Eq. 24) changes sign if the \(\Theta_{ij}\) angles are reduced to the first quadrant.

Let us start our considerations from the \(\Theta_{ij}\) angles. In the case of two flavour neutrino oscillations in vacuum the transition probability depends only on \(\sin^2 2\Theta \sin^2 \left[ \frac{\delta m^2}{16} \right] \). Then it is possible to limit the range of mixing angles to the first octant. The transition probability in matter depends on the combination [12]

\[ \left( \frac{A}{\delta m^2} - \cos 2\Theta \right)^2 \]  \hspace{1cm} (32)

The relative sign between \(\delta m^2\) and \(\cos 2\Theta\) is important, so two possibilities are considered

\[ \delta m^2 > 0 \quad \text{and} \quad 0 < \Theta < \frac{\pi}{2} \]

or

\[ \delta m^2 = \pm |\delta m^2| \quad \text{and} \quad 0 < \Theta < \frac{\pi}{4} \]

In the case of three flavour neutrino oscillations it is impossible to limit the range of \(\Theta_{ij}\) angles in this way, even in vacuum. Transition probabilities (Eq. 4)
depend not only on the product $\sin^2 \Theta_{ij} \cos^2 \Theta_{ij}$ but also on $\sin^2 \Theta_{ij}$ and $\cos^2 \Theta_{ij}$ separately. Since it is impossible to shrink the range of the mixing angles $\Theta_{ij}$ in the case of vacuum oscillations, the same will hold true for the matter case. In spite of that, various schemes (Fig. 1) are considered. Let us show that in this way the same angles are used twice when $0 < \Theta_{ij} < \frac{\pi}{2}$.

Neutrino oscillation formulae (Eqs. 28-30) are symmetric under permutation of neutrinos. Traditionally some scalar matrix $(1 \cdot \text{const})$ is removed from the effective hamiltonian (Eq. 27) giving the same physical predictions. For example, if $H_\nu$ is written in the form

$$M^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_1^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_3^2 \end{pmatrix},$$  \hfill (33)$$

then the matrix $1 \cdot m_1^2$ can be absorbed giving a common phase factor for all three neutrino flavours. In such a case we diagonalize the hamiltonian $H_\nu$ where

$$m_1^2 \rightarrow 0, \; m_2^2 \rightarrow \delta m_{21}^2, \; m_3^2 \rightarrow \delta m_{31}^2.$$  \hfill (34)$$

The new $a_i$ parameters derived from Eq. 30 are not symmetric under permutations of the masses anymore, they depend on $\delta m_{ij}^2$’s, namely

$$a_0 = -A\delta m_{21}^2\delta m_{31}^3|U_{e1}|^2,$$

$$a_1 = \delta m_{21}^2\delta m_{31}^3 + A \left[ \delta m_{21}^2 (1 - |U_{e2}|^2) + \delta m_{31}^3 (1 - |U_{e3}|^2) \right],$$

$$a_2 = -\delta m_{21}^2 - \delta m_{31}^3 - A.$$  \hfill (35)$$

Let us now calculate the eigenvalues for the case of negative $\delta m_{3i}^2$, i=1,2, i.e. $\delta m_{3i}^2 = -|\delta m_{3i}^2|$. We have

$$a_0(-|\delta m_{3i}^2|) = A|\delta m_{21}^2||\delta m_{31}^3||U_{e1}|^2,$$

$$a_1(-|\delta m_{3i}^2|) = -|\delta m_{21}^2||\delta m_{31}^3| + A \left[ |\delta m_{21}^2|(1 - |U_{e2}|^2) - |\delta m_{31}^3|(1 - |U_{e3}|^2) \right].$$

1 This statement is general. In the approximation with one dominating $\delta m^2$ scale some transition probabilities depend only on $\sin^2 2\Theta_{ij}$. For example, the short-baseline reactor disappearance probability $P_{\nu_e \rightarrow \nu_e} = 1 - \sin^2 \Theta_{13} \sin^2 \Delta_{23}$. However, this approximation seems to be questionable, even for present neutrino data [6] and quite probably a full theoretical framework without neglecting some $\delta m^2$ should be used in future.
Using these new parameters \( a_i(-|\delta m_{3i}^2|) \) different \( \lambda_i^2 \) eigenvalues are obtained. Are these new \( \lambda_i(-|\delta m_{3i}^2|) \) eigenvalues equal to the “canonical” \( \lambda_i \) calculated at some other point of the parameter space of Eq. 3? To show that they are, let us take the scheme \([312]\). This scheme (as any other in Fig. 1) is completely equivalent to the canonical one \([123]\). We have only to change the names of particles \( 2 \rightarrow 3, \ 1 \rightarrow 2, \ 3 \rightarrow 1 \) or more precisely replace \( U_{e1} \rightarrow U_{e2}, \ U_{e2} \rightarrow U_{e3}, \ U_{e3} \rightarrow U_{e1} \), \( \delta m_{23}^2 \rightarrow \delta m_{31}^2, \ \delta m_{21}^2 \rightarrow \delta m_{32}^2, \ \delta m_{13}^2 \rightarrow \delta m_{21}^2 \).

In the scheme \([312]\), as previously we subtract mass from the \( M^2 \) matrix. As now \( m_3 \) is the lightest mass, we have to diagonalized \( H_\nu \) with the following replacements

\[
m_1^2 \rightarrow 0, \quad m_2^2 \rightarrow |\delta m_{21}^2|, \quad m_3^2 \rightarrow -|\delta m_{31}^2|.
\]

The parameters \( a_i \) which we get are exactly the same as given by Eq. 36. Similarly we can check that any replacement \( \delta m_{ij}^2 \rightarrow -\delta m_{ij}^2 \) in the canonical parameters \( a_i([123]) \) is equivalent to the others given by one of the six schemes in Fig. 1. In this way we have proved that changing the signs of \( \delta m_{ij}^2 \) in the canonical \([123]\) eigenvalues is equivalent to evaluating \( \lambda_i^2 \)'s at some other point of the parameter space Eq. 3, schematically

\[
\lambda_i^2(-|\delta m_{ij}^2|) \sim \lambda_i^2([ijk]).
\]

We can see that using schemes with various permutations of masses does not confine the domain of the parameter space \( \Theta_{ij} \) and causes double counting only. However, we can find a practical reason for introducing \( \pm \delta m^2 \)'s. We have just shown that using various schemes is equivalent to using the \([123]\) scheme with different values of \( \Theta_{ij} \) angles in the parameter space. That is why we can reverse the situation by fixing angles to the same physical situation, i.e. \( \Theta_{12} \) can be connected with \( \pm \delta m_{12}^2 \) (oscillation of solar neutrinos), \( \Theta_{23} \) with \( \pm \delta m_{23}^2 \) (oscillation of atmospheric neutrinos) and \( \Theta_{13} \) with reactor neutrino oscillations.

Finally, let us consider the \( \delta \) phase in the case of matter neutrino oscillations. Can we bound it to the smaller range (Eq. 17) as in the vacuum case? There is a very elegant relationship between the universal CP-violating parameters \( J^m \) and \( J \) in matter and in vacuum \([13]\)

\[
J^m(\lambda_2^2 - \lambda_1^2)(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2) = J(m_2^2 - m_1^2)(m_3^2 - m_1^2)(m_3^2 - m_2^2).
\]

From this relation follows that the signs of \( \delta m_{ij}^2 \) and \( \delta \lambda_{ij}^2 \) are correlated. If \( \delta m_{ij}^2 \) changes sign, the same happens to \( \delta \lambda_{ij}^2 \). We conclude that for neutrino
oscillations in matter we have exactly the same situation as in the vacuum case. The basic domain of $\delta$ is $\langle 0, 2\pi \rangle$ and it can be restricted to $\langle 0, \pi \rangle$ and then the schemes with $\Delta > 0$ and $\Delta < 0$ are distinguishable.

4 Conclusions

We have proved in an analytical way that the ranges of the mixing angles $\Theta_{ij}$ and the CP violating phase $\delta$ are the same for three flavour neutrino oscillations in vacuum and in matter: $\Theta_{ij} \in \langle 0, \frac{\pi}{2} \rangle$, $\delta \in \langle 0, 2\pi \rangle$. It means that probabilities for three flavour neutrino oscillations can be described by points (more reliably by regions) in this parameters’ domain without using the signs of $\delta m^2_{ij}$ ($\delta m^2_{ij} > 0$, $i > j$). Contrary to the case of two neutrino oscillations in matter, the possibility of two signs for each $\delta m^2_{ij}$ does not restrict further the domain of the $\Theta_{ij}$ angles. Even though the signs of $\delta m^2_{ij}$'s are not needed, they are useful. Taking into account the signs of $\delta m^2_{ij}$ we can fix angles $\Theta_{ij}$ to a given scale of $\delta m^2_{ij}$. The range of the $\delta$ CP phase can be confined to $\delta \in \langle 0, \pi \rangle$ but then, only sets of schemes with cyclic ($\Delta > 0$) and odd ($\Delta > 0$) neutrino mass permutations are distinguishable to each other. A simple example has been given that $\delta$ can be important even for disappearance neutrino oscillation experiments.

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| Configuration | [123] | $\delta m_{21}^2 > 0$ | $\delta m_{32}^2 > 0$ | $\delta m_{31}^2 > 0$ | $\Delta > 0$ |
|--------------|--------|---------------------|---------------------|---------------------|----------------|
| 3 2 1        |        |                     |                     |                     |                |

| Configuration | [312] | $\delta m_{21}^2 > 0$ | $\delta m_{32}^2 < 0$ | $\delta m_{31}^2 < 0$ | $\Delta > 0$ |
|--------------|--------|---------------------|---------------------|---------------------|----------------|
| 2 1 3        |        |                     |                     |                     |                |

| Configuration | [231] | $\delta m_{21}^2 < 0$ | $\delta m_{32}^2 > 0$ | $\delta m_{31}^2 < 0$ | $\Delta > 0$ |
|--------------|--------|---------------------|---------------------|---------------------|----------------|
| 1 3 2        |        |                     |                     |                     |                |

| Configuration | [321] | $\delta m_{21}^2 < 0$ | $\delta m_{32}^2 < 0$ | $\delta m_{31}^2 < 0$ | $\Delta < 0$ |
|--------------|--------|---------------------|---------------------|---------------------|----------------|
| 2 3 1        |        |                     |                     |                     |                |

| Configuration | [213] | $\delta m_{21}^2 < 0$ | $\delta m_{32}^2 > 0$ | $\delta m_{31}^2 > 0$ | $\Delta < 0$ |
|--------------|--------|---------------------|---------------------|---------------------|----------------|
| 2 1 3        |        |                     |                     |                     |                |

| Configuration | [132] | $\delta m_{21}^2 > 0$ | $\delta m_{32}^2 < 0$ | $\delta m_{31}^2 > 0$ | $\Delta < 0$ |
|--------------|--------|---------------------|---------------------|---------------------|----------------|
| 3 1 2        |        |                     |                     |                     |                |

Fig. 1. Possible configurations of neutrino masses. The first one is called canonical. The first two ([123],[312]) are usually discussed in literature. All schemes are completely equivalent since marking neutrinos with numbers has no physical meaning. It is not important how a neutrino with number “i” couples to the $\alpha$ flavour. It is only meaningful how neutrinos of different masses couple to the $\alpha$ flavour.
Fig. 2. The effect of the CP-violating phase in the disappearance $\nu_\mu \rightarrow \nu_\mu$ transition (Eq. 18).