Bounding toric singularities with normalized volume

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Abstract

We study the normalized volume of toric singularities. As it turns out, there is a close relation to the notion of (nonsymmetric) Mahler volume from convex geometry. This observation allows us to use standard tools from convex geometry, such as the Blaschke–Santaló inequality and Radon’s theorem to prove nontrivial facts about the normalized volume in the toric setting. For example, we prove that for every \( \varepsilon > 0 \) there are only finitely many \( \mathbb{Q} \)-Gorenstein toric singularities with normalized volume at least \( \varepsilon \). From this result it directly follows that there are also only finitely many toric Sasaki–Einstein manifolds of volume at least \( \varepsilon \) in each dimension. Additionally, we show that the normalized volume of every toric singularity is bounded from above by that of the rational double point of the same dimension. Finally, we discuss certain bounds of the normalized volume in terms of topological invariants of resolutions of the singularity. We establish two upper bounds in terms of the Euler characteristic and of the first Chern class, respectively. We show that a lower bound, which was conjectured earlier by He, Seong, and Yau, is closely related to the nonsymmetric Mahler conjecture in convex geometry.
1 | INTRODUCTION

The study of log terminal singularities has played a central role in the development of birational geometry, in particular, within the minimal model program [33] and for the algebraic theory of K-stability [34]. Among log terminal singularities, toric singularities are the most well-known, as they are closely related to orbifold singularities [6, 10]. The combinatorial nature of toric singularities allows us to use methods from convex geometry to study them [6]. There are two important invariants for log terminal singularities, the normalized volume [22] and the minimal log discrepancy [28, 29]. The former is related to K-stability [20], while the latter is related to termination of flips [31]. Minimal log discrepancies are well-understood for toric singularities [3]. In [14], the authors conjecture that log terminal singularities with normalized volume bounded away from zero, are bounded up to deformations. This conjecture is closely related to work on deformations for exceptional singularities [13, 27]. Our first result is a positive answer for this conjecture in the setting of \(\mathbb{Q}\)-Gorenstein toric singularities (see Corollary 3.5).

**Theorem 1.** Let \(d\) be a positive integer and \(\epsilon > 0\) be a positive real number. There are only finitely many \(d\)-dimensional toric \(\mathbb{Q}\)-Gorenstein singularities with \(\hat{\text{vol}}(X) > \epsilon\).

In particular, it is expected that the normalized volume of \(d\)-dimensional log terminal singularities can only accumulate to zero. As a consequence of Theorem 1, we prove this conjecture in the case of \(\mathbb{Q}\)-Gorenstein toric singularities.

**Corollary 1.** Let \(d\) be a positive integer. Then, the set

\[ V_{T,d} := \{ \hat{\text{vol}}(X) \mid X \text{ is a } \mathbb{Q}\text{-Gorenstein-dimensional toric singularity} \}, \]

only accumulates to zero.

As a corollary, we also obtain that there are only finitely many toric Sasaki–Einstein manifolds of volume at least \(\epsilon\) in each dimension.

**Corollary 2.** Let \(n\) be a positive integer and \(\epsilon > 0\) be a positive real number. There are only finitely many toric Sasaki–Einstein manifolds \(X\) of dimension \(n\) with \(\text{vol}(X) > \epsilon\).

The previous corollary is tightly related to [17, Theorem 1.1]. See also [34, Theorem 6.5 and Remark 6.7]). It is expected that the rational double point of dimension \(d\) is the mildest among all \(d\)-dimensional singularities. This principle holds for almost all known invariants of singularities. In particular, it is expected that rational double points of dimension \(d\) compute the highest normalized volume among all log-terminal singularities. This is conjectured by Spotti and Sun [32]. The previous conjecture is known up to dimension three due to the work of Li, Liu, and Xu [21, 24]. In this direction, we prove the following theorem.
Theorem 2. Let $X$ be a $d$-dimensional $\mathbb{Q}$-Gorenstein toric singularity.

(1) If $d = 2$, then $\hat{\text{vol}}(X) \leq 2$ and the equality holds precisely when $X$ is the two-dimensional $A_1$ singularity.

(2) If $d \geq 3$, then $\hat{\text{vol}}(X) \leq \frac{16}{27}d^d$ and the equality holds precisely when $X$ is $\mathbb{A}^{d-3} \times C$ where $C$ is the three-dimensional $A_1$ singularity.

In the proof of the previous theorems, we will compute the normalized volume of the toric singularity by using techniques of convex geometry. Particularly, the nonsymmetric Mahler volume \cite{30}. Well-established tools from convex geometry, such as the Blaschke–Santaló inequality \cite{4} and Radon’s theorem will be used to prove the boundedness theorem for toric singularities using normalized volume and the upper bound for the normalized volume, respectively.

The paper is organized as follows. In Section 2, we introduce some preliminaries related to toric singularities and normalized volume. In Section 3, we prove the boundedness result for toric singularities using normalized volume. In Section 4, we show that the normalized volume of a toric singularity is at most the normalized volume of a rational double point. Finally, in Section 5, we will establish some relations between topological invariants, such as the Euler characteristic and the first Chern class, and the normalized volume of toric singularities.

2 | PRELIMINARIES

In this section, we recall some preliminaries regarding toric singularities, normalized volume, and the Blaschke–Santaló inequality. For the basics of toric geometry we refer the reader to \cite{6,10}.

Definition 2.1. Let $N$ be a free finitely generated abelian group of rank $d$. Let $M := \text{Hom}(N, \mathbb{Z})$ be its dual and $\langle \cdot, \cdot \rangle$ the natural pairing. We denote by $N_\mathbb{Q}$ (resp., $M_\mathbb{Q}$) the associated $\mathbb{Q}$-vector spaces. Analogously, we denote by $N_\mathbb{R}$ and $M_\mathbb{R}$ the corresponding $\mathbb{R}$-vector spaces. Let $\sigma \subset N_\mathbb{Q}$ be a strictly convex rational polyhedral cone. We denote by $\sigma^\vee \subset M_\mathbb{Q}$ its dual cone, that is, the cone spanned in $M_\mathbb{Q}$ by elements that are nonnegative on $\sigma$. Then, the affine variety

$$X(\sigma) := \text{Spec}(\mathbb{k}[M \cap \sigma^\vee])$$

is an affine toric variety. The affine variety $X(\sigma)$ has dimension $d$. It admits the action of the $d$-dimensional algebraic torus

$$\mathbb{T} := \text{Spec}(\mathbb{k}[M]).$$

The affine variety $X(\sigma)$ admits a unique closed point that is torus invariant. We say that a singularity of a normal variety $(X; x)$ is toric if it is isomorphic to $(X(\sigma); x_0)$, where $X(\sigma)$ is an affine toric variety and $x_0$ is the unique torus invariant closed point.

In what follows, we are mostly concerned with toric singularities. Then, when writing $X(\sigma)$, we mean the toric singularity associated to the affine toric variety, that is, the statements
that we write for singularities hold after possibly shrinking around the closed torus invariant point.

**Definition 2.2.** Let $X(\sigma)$ be a $d$-dimensional toric singularity. The set of the extremal rays of $\sigma$ is denoted by $\sigma(1)$. For each extremal ray $\rho$ of $\sigma \subset N_\mathbb{Q}$, we denote by $v_\rho$ its primitive lattice generator (in the lattice $N$). The singularity $X(\sigma)$ is Gorenstein (resp., $\mathbb{Q}$-Gorenstein) if and only if there exists $u \in M$ (resp., $u \in M_\mathbb{Q}$) for which $\langle u, v_\rho \rangle = 1$ for each $\rho$. If $X(\sigma)$ is a $\mathbb{Q}$-Gorenstein toric singularity, then the minimal positive integer $\ell$ for which $\ell K_{X(\sigma)} \sim 0$ is called the Gorenstein index of the singularity. In the notation from above this is exactly the minimal positive integer $\ell$ such that $\ell u \in M$.

The following remark allows us to write $\mathbb{Q}$-Gorenstein toric singularities as cones over lattice polytopes.

**Remark 2.3.** Let $X(\sigma)$ be a $d$-dimensional $\mathbb{Q}$-Gorenstein toric singularity. Let $\ell$ be the Gorenstein index of the singularity and $u \in \frac{1}{\ell} M$ as above. Then the convex hull $P' := \text{conv}\{v_\rho \mid \rho \in \sigma(1)\}$ lies in the hyperplane $\{\langle \cdot, u \rangle = 1\} := \{v \in N_\mathbb{R} \mid \langle u, v \rangle = 1\}$ and we have $\sigma = \mathbb{R}_{\geq 0} \cdot P'$. We have the following exact sequence of lattices

$$0 \rightarrow N_u \rightarrow N \rightarrow \mathbb{Z} \rightarrow 0,$$

where $N_u = \{\langle \cdot, u \rangle = 0\} \cap N$. After choosing a splitting of the above sequence we obtain isomorphisms $N \cong N_u \times \mathbb{Z}$ and $N_\mathbb{R} \cong (N_u \otimes \mathbb{R}) \times \mathbb{R}$, respectively. With these isomorphisms we have $P' = P \times \{\ell\}$ for some lattice polytope $P \subset N_u \otimes \mathbb{R}$ and we obtain

$$\sigma = \text{cone}(P, \ell) := \mathbb{R}_{\geq 0} \cdot (P \times \{\ell\}).$$

Different choices for the splitting of our sequence lead to unimodularly equivalent polytopes. Hence, our choice of $P$ was unique up to unimodular equivalence. Let us emphasize that the reverse statement is not true, that is, unimodular equivalent polytopes $P$ and $Q$ do not necessarily lead to unimodular equivalent polytopes $P \times \{1\}$ and $Q \times \{1\}$.

Observe that $P$ has dimension $n = d - 1$ and the singularity has dimension $d$. We will fix this notation for the rest of the article, that is, $d$ will stand for the dimension of the toric singularity (or the cone), while $n$ will stand for the dimension of the polytope (or the corresponding projective toric variety).

Now, we turn to define $\mathbb{Q}$-Gorenstein log terminal singularities and their normalized volume.

**Definition 2.4.** Let $x \in X$ be a $\mathbb{Q}$-Gorenstein singularity. Let $\pi : Y \to X$ be a projective birational morphism from a normal variety $Y$. Let $E \subset Y$ be a prime divisor. We define the log discrepancy of $X$ with respect to $E$, to be

$$a_E(X) := 1 - \text{coeff}(K_Y - \pi^*(K_X)).$$

We say that $X$ is log terminal if $a_E(X) > 0$ for every prime divisor $E$ over $X$. We say that $x \in X$ is a log terminal singularity if some neighborhood of $x$ in $X$ is log terminal. For a log terminal $\mathbb{Q}$-Gorenstein singularity the log discrepancy can be regarded as a function $a_X : \text{DVal}(X) \to (0, \infty)$.
with

\[ a_X(v_E) := a_E(X), \]

from the space of divisorial valuations \( \text{DVal}(X) \) over \( X \). In [18], the authors prove that this function can be extended to a function \( a_X : \text{Val}(X) \to (0, \infty) \) from the space of valuations over \( X \). We denote by \( \text{Val}(X, x) \) the space of valuations over \( X \) with center \( x \).

The volume of a valuation over a singularity is defined in [8].

**Definition 2.5.** Let \( x \in X \) be a normal algebraic singularity of dimension \( n \). The volume of a valuation \( v \in \text{Val}(X, x) \) is defined to be

\[ \text{vol}_X(v) := \frac{\ell(\mathcal{O}_{X,x}/a_m(v))}{m^n/n!}. \]

Here, \( a_m(v) := \{ f \in \mathcal{O}_{X,x} | v(f) \geq m \} \) and \( \ell \) denotes the Artinian length of the module.

Using the two previous definitions, we can define the normalized volume. The following is a special case of a definition due to Li [20].

**Definition 2.6.** Let \( X \) be a \( d \)-dimensional \( \mathbb{Q} \)-Gorenstein log terminal singularity. The normalized volume function is a function

\[ \widehat{\text{vol}} : \text{Val}(X, x) \to (0, \infty) \]

defined by

\[ \widehat{\text{vol}}(v) = a_X(v)^d \text{vol}_X(v) \]

if \( a_X(v) < \infty \) and as \( \infty \) otherwise. The normalized volume of the \( \mathbb{Q} \)-Gorenstein log terminal singularity \( x \in X \) is defined to be

\[ \widehat{\text{vol}}(X, x) := \inf \left\{ \widehat{\text{vol}}(v) \mid v \in \text{Val}(X, x) \right\}. \]

By the work of Blum, we know that the infimum in the definition of normalized volume of a singularity is indeed a minimum [5]. The following proposition allows to compute the normalized volume of \( d \)-dimensional \( \mathbb{Q} \)-Gorenstein toric singularities.

**Proposition 2.7** [23, section 2.2.2.1; 25]. Let \( X = X(\sigma) \) a \( d \)-dimensional \( \mathbb{Q} \)-Gorenstein toric singularity. Then normalized volume \( \widehat{\text{vol}}(X) \) of \( X \) coincides with the minimum of the volume of a truncated cone

\[ \sigma^\vee(\xi) := \{ \langle \xi, \cdot \rangle \leq 1 \} \cap \sigma^\vee \]

with \( \xi \in \text{conv}\{v_\rho \mid \rho \in \sigma(1)\} \), that is,

\[ \widehat{\text{vol}}(X) = \min\{d! \text{vol}\sigma^\vee(\xi) \mid \xi \in \text{conv}\{v_\rho \mid \rho \in \sigma(1)\}\}. \]
**Definition 2.8.** Let \( P \subset \mathbb{R}^n \) be a convex body. It is **polar dual** is defined to be

\[
P^* := \{ y \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1 \text{ for all } x \in P \}.
\]

We finish this section by recalling the Blaschke–Santaló inequality for the product of the volume of a convex body and its dual.

**Theorem 2.9 (Blaschke–Santaló Inequality).** Given a convex body \( K \subset \mathbb{R}^n \), such that the barycentre of \( K \) coincides with the origin. Then the following inequality holds

\[
\text{vol}(K)\text{vol}(K^*) \leq \omega_n^2,
\]

where \( K^* \) denotes the polar dual of \( K \) and \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \).

# 3 PROOF OF BOUNDEDNESS IN THE TORIC CASE

In this section, we prove the boundedness of toric singularities using normalized volume. We start this section by proving some lemmata regarding the dual polytope. Throughout this section, we work with a polytope \( P \subset \mathbb{R}^n \) that is not necessarily a lattice polytope.

In addition to the cone \( \text{cone}(P, \ell) \subset \mathbb{R}^{n+1} \) over a polytope \( P \subset \mathbb{R}^n \) we will denote the corresponding **truncated cone** as follows.

\[
\text{tr. cone}(P, \ell) := \text{conv}(P \times \{\ell\} \cup \{0\})
\]

**Lemma 3.1.** For a polytope \( P \subset \mathbb{R}^n \) containing the origin, one has

\[
\text{cone}(P, \ell)^\vee = \text{cone}(P^*, 1/\ell).
\]

**Proof.** This follows easily from the equation

\[
\langle (v, \ell), (u, 1/\ell) \rangle = \langle v, u \rangle + 1.
\]

\( \square \)

**Lemma 3.2.** For \( P \subset \mathbb{R}^n \) containing the origin, \( \sigma = \text{cone}(P, \ell) \) and \( \xi_0 = (0, \ell) \in P \times \{\ell\} \) we have

\[
\text{vol} \sigma^\vee(\xi_0) = \frac{1}{(n+1) \cdot \ell} \text{vol} P^*.
\]

**Proof.** It follows from Lemma 3.1 that \( \sigma^\vee(0, \ell) = \text{tr. cone}(P^*, 1/\ell) \). This is a pyramid of height \( 1/\ell \) over the polytope \( P^* \). Hence, we obtain the result from the volume formula for a pyramid. \( \square \)

**Theorem 3.3** [11, Theorem 3.3.]. Given an interior point \( u \in \sigma^\vee \). Then the volume of a truncated cone

\[
[\langle \xi, \cdot \rangle \leq 1] \cap \sigma^\vee = \sigma^\vee(\xi)
\]

is minimized among all choices \( \xi \) such that \( \langle \xi, u \rangle = 1 \) if and only if \( u \) coincides with the barycenter of \( [\langle \xi, \cdot \rangle = 1] \cap \sigma^\vee \).
Theorem 3.4. Assume $X = X(\sigma)$ is a $d$-dimensional toric singularity of Gorenstein index $\ell$. Then the following inequalities hold

$$\frac{1}{(d-1)!} \leq \text{vol}(\text{conv}\{v_\rho \mid \rho \in \sigma(1)\}) < \frac{\omega_{d-1}^2}{\ell \cdot d \cdot \text{vol}(X)}.$$  \hspace{1cm} (1)

In particular, we have

$$\ell < \frac{(d-1)! \cdot \omega_{d-1}^2}{d \cdot \text{vol}(X)}.$$  \hspace{1cm} (2)

Proof. Having Gorenstein index $\ell$ implies that for an appropriate choice of coordinates we have $\sigma = \text{cone}(P, \ell)$ and $\text{conv}\{v_\rho \mid \rho \in \sigma(1)\} = P \times \{\ell\}$.

Now, assume $\xi = (\chi, \ell)$ minimizes $\text{vol}^{\vee}(\cdot, -)$. We consider

$$T : \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^{d-1} \times \mathbb{R}; (v, h) \mapsto (v - h \cdot \chi / \ell, h).$$

Then we set

$$\sigma_0 = T \sigma = \text{cone}(P_0, \ell)$$

with $P_0 = P - \chi$. As, $T \in \text{SL}(n, \mathbb{R})$ preserves volumes, we see that $\xi_0 = T \xi = (0, \ell)$ minimizes $\text{vol}^{\vee}_0(\cdot, -)$ among all elements of $P_0 \times \{\ell\}$ and

$$\text{vol}(\sigma_0^{\vee}(\xi_0)) = \text{vol}((T \sigma)^{\vee}(T \xi)) = \text{vol}(\sigma^{\vee}(\xi)).$$

Now, by Lemma 3.1 we have $\sigma_0^{\vee} = \text{cone}(P_0^n, 1 / \ell)$ and obviously for all $v \in \mathbb{R}^{n-1}$ we have $(\langle v, \ell \rangle, (0, 1 / \ell)) = 1$. As $\xi_0$ was a minimizer for $\text{vol}^{\vee}_0(\cdot, -)$, Theorem 3.3 implies that $u = 0$ is the barycentre of $P_0$. Now, we can apply the Blaschke–Santaló inequality and obtain

$$\text{vol} P_0 \cdot \text{vol} P_0^n < \omega_{d-1}^2.$$

By Proposition 2.7, we have $\text{vol}(X) = \text{vol}(\sigma^{\vee}(\xi)) = \text{vol}(\sigma_0^{\vee}(\xi_0))$. Then the inequality

$$\text{vol} P = \text{vol} P_0 < \frac{\omega_{d-1}^2}{\ell \cdot d \cdot \text{vol}(X)}$$

follows from Lemma 3.2. As $P$ is a lattice polytope it has volume at least $1/(d - 1)!$ giving rise to the inequality on the left-hand side of (1).

Corollary 3.5. In each dimension and for any $\varepsilon > 0$ there are only finitely many toric $\mathbb{Q}$-Gorenstein singularities with $\text{vol}(X) > \varepsilon$.

Proof. By claim (2) of Theorem 3.4, it follows that the Gorenstein index of $X$ is bounded from above by

$$\frac{(d - 1)! \cdot \omega_{d-1}^2}{d \cdot \varepsilon}.$$
For a fixed Gorenstein index $\ell$ every toric singularity of that index is uniquely (up to unimodular equivalence) determined by a lattice polytopes $P$ via $X = X(\text{cone}(P, \ell))$. Then by (1) of Theorem 3.4, we have

$$\text{vol} P < \frac{\omega_{d-1}^2 \cdot \epsilon^{-1}}{\ell \cdot d}.$$ 

Now, by [19, Theorem 2] it follows that there are only finitely many equivalence classes of polytopes with such a bounded volume.

The following result is merely a reformulation of Corollary 3.5 in terms of Sasaki–Einstein geometry. For the details on the connections to Sasaki–Einstein geometry, we refer the reader to [22, section 3.3].

**Corollary 3.6.** In each dimension and for any $\epsilon > 0$, there are only finitely many toric Sasaki–Einstein manifolds of volume at least $\epsilon$.

## 4 SINGULARITIES OF MAXIMAL VOLUME

The aim of this section is to find, in each dimension, the $\mathbb{Q}$-Gorenstein toric singularity with the largest normalized volume.

### 4.1 Toric singularities and Santaló points

In this subsection, we draw a connection between the normalized volume of a toric singularity and the Santaló points of the corresponding polytope.

As explained in the preliminaries any $\mathbb{Q}$-Gorenstein toric singularity of dimension $d$ is isomorphic to $\text{cone}(P, \ell)$ where $P$ is a $(d-1)$-dimensional lattice polytope and $\ell$ is a positive integer. First, we relate the normalized volume of $X$ with the volume of the polar dual of certain translation of $P$. To define this translation of $P$ we use the concept of Santaló point.

**Definition 4.1.** Let $P$ be a convex polytope, we define the Santaló point $\chi$ of $P$ to be the point in $\text{int}(P)$ that minimizes the volume of $(P - \chi)^*$. In what follows, we write $P^\chi := (P - \chi)^*$.

The normalized volume of a $\mathbb{Q}$-Gorenstein singularity $X = \text{cone}(P, \ell)$ and the volume of $P^\chi$ are related by the following lemma. The proof of the following lemma is essentially contained in the proof of Theorem 3.4. We give a complete proof for the sake of clarity.

**Lemma 4.2.** Let $X = \text{cone}(P, \ell)$ be a $\mathbb{Q}$-Gorenstein toric singularity of dimension $d$. Then we can write

$$\tilde{\text{vol}}(X) = \frac{(d - 1)!}{\ell} \text{vol}(P^\chi).$$
Proof. Let \( \sigma = \text{cone}(P, \ell') \) be the cone associated to \( X \). By Proposition 2.7, we have

\[
\frac{\hat{\text{vol}}(X)}{d!} = \min\{\text{vol}\sigma^Y(\xi) \mid \xi = (\chi, \ell), \chi \in \text{int}(P)\}.
\]

Let \( \xi_0 = (\chi_0, \ell') \) be the minimizer of \( \text{vol}\sigma^Y(\xi) \) with \( \chi_0 \in \text{int}(P) \). So, we have

\[
\frac{\hat{\text{vol}}(X)}{d!} = \text{vol}\sigma^Y(\chi_0).
\]

Let \( T : \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^{d-1} \times \mathbb{R}; (v, h) \mapsto (v - h \cdot \chi / \ell', h) \). Note that \( T\sigma = \text{cone}(P - \chi_0, \ell') \) and \( T\xi = (0, \ell') \). As \( T \in \text{SL}(n, \mathbb{R}) \) it preserves volume, so

\[
\frac{\hat{\text{vol}}(X)}{d!} = \text{vol}\sigma^Y(\xi_0) = \text{vol}((T\sigma)^Y(T\xi_0)) = \frac{1}{d!} \text{vol}(P - \chi_0)^x,
\]

where the last equality follows from Lemma 3.2. The right-hand side of equality (3) is minimized precisely when \( \chi_0 \) is the Santaló point of \( P \). We conclude that \( \hat{\text{vol}}(X) = (d-1)! \text{vol}(P \chi) \ell^{-1} \) as claimed.

In what follows, we set \( n := (d - 1) \) to be the dimension of the lattice polytope \( P \). The previous lemma reduces the problem of finding toric singularities with maximal normalized volume to the following proposition.

**Proposition 4.3.** Let \( P \) be a lattice polytope of dimension \( n \) and \( \chi \in P \) be its Santaló point. Assume that \( P \) is not the standard simplex. Then the inequality

\[
\text{vol}(P^x) \leq \frac{16 (n + 1)^{(n+1)}}{27 n!}
\]

holds, with equality only when \( P \) is the iterated pyramid over the unit square.

To prove the previous proposition, we will introduce several lemmata in the following four subsections. Then Proposition 4.3 and Theorem 2 will be proved in Subsection 4.6

### 4.2 The Mahler volume

Given a convex body \( P \) with Santaló point \( \chi \). The product \( \text{vol}(P)\text{vol}(P^x) \) is known as the (nonsymmetric) Mahler volume of \( P \). In the case of a simplex the Santaló point coincides with the barycentre and we obtain the following (well-known) statement about the nonsymmetric Mahler volume of a simplex.

**Lemma 4.4.** Let \( \Delta \) be a simplex of dimension \( n \) and \( \chi \in \Delta \) the Santaló point, then we have

\[
\text{vol}(\Delta)\text{vol}(\Delta^x) = \frac{(n + 1)^{(n+1)}}{(n!)^2}.
\]
**Proof.** This can be easily checked for the standard simplex. Then, the statement follows from the fact that the Mahler volume is invariant under affine transformations.

The well-known nonsymmetric Mahler conjecture states that the $n$-dimensional simplex is the minimizer of the Mahler volume among all convex bodies of dimension $n$.

### 4.3 Convex hull of $n+2$ points

In this subsection, we recall some lemmas regarding the volume of $n$-dimensional lattice polytopes that are the convex hull of $n+2$ points. In the following lemmas, two affine subspaces of $\mathbb{R}^n$ are called *complementary* if they only intersect at one point and their dimensions add up to $n$.

**Lemma 4.5.** Let $P \subset \mathbb{R}^n$ be a polytope of dimension $n$ given as convex hull of $n+2$ points. Then $P = \text{conv}(\Delta \cup \Delta')$, such that $\Delta$ and $\Delta'$ are simplices of dimensions $p$ and $q$, respectively, spanning complementary affine subspaces and $\Delta \cap \Delta'$ consists of exactly one point.

**Proof.** This is a consequence of Radon’s theorem, see also the proof of [1, Theorem 5.3].

In the setting of Lemma 4.5, we say that $P$ has a $(p, q)$-partition. Note, that this notion is symmetric in $p$ and $q$. The unique intersection point of Lemma 4.5 will be called the Radon point of the partition. The following example shows that $(p, q)$ is in general not unique.

**Example 4.6.** Consider a pyramid $P_n$ over a lattice simplex $S_n$ in $\mathbb{R}^n$ with respect to the vertex $v_0$. Let $F_1$ and $F_2$ be two disjoint faces of $S_n$ of dimensions $p$ and $q$ with $p+q = n-1$. Let $\Delta_i := \text{conv}(F_i, v_0)$. Then $P_n = \text{conv}(\Delta_1 \cup \Delta_2)$ and $\Delta_1 \cap \Delta_2 = \{v_0\}$. The simplices $\Delta_1$ and $\Delta_2$ span complementary affine subspaces. Note that the previous construction gives a $(p, q)$ partition of $P_n$ for every $1 \leq p \leq n-1$.

**Lemma 4.7.** Let $P$ be a lattice polytope as in Lemma 4.5. Then

$$\text{vol}(P) \geq \frac{\min\{p \mid \text{Phas a}(p, q)\text{-partition}\} + 1}{n!}.$$  

**Proof.** Assume that $p \in \mathbb{N}$ is minimal such that $P$ admits a $(p, q)$-partition. Assume that such a partition is given by the two complementary simplices $\Delta$ and $\Delta'$, which intersect in the Radon point $o$. Then we obtain a triangulation of $P$ as follows. For every facet $F < \Delta$ we consider $Q_F = \text{conv}(F \cup \{o\})$ and $P_F = \text{conv}(F \cup \Delta') = \text{conv}(Q_F \cup \Delta')$.

Then $\{Q_F\}_F$ and $\{P_F\}_F$ induce triangulations of $\Delta$ and $P$, respectively. We claim that all the $P_F$ are of full dimension. Indeed, assume this is not the case. Then, on the one hand, one of the $P_F$ spans a proper affine subspace of $\mathbb{R}^n$ and, on the other hand, $F \subset P_F \cap \Delta$. Hence, we actually have $F = P_F \cap \Delta$. We conclude for the Radon point that

$$o \in \Delta \cap P_F = F.$$  

But then, we also have a $(p-1, q+1)$-partition of $P$, given by $F$ and $\text{conv}(\Delta' \cup \{v_F\})$, where $v_F$ is the unique vertex of $\Delta$, which is not contained in $F$. Hence, $p$ was not minimal. This leads to a contradiction.
Now, we are given a triangulation of $P$ into $(p + 1)$ full-dimensional lattice simplices. This implies that $\text{vol}(P) \geq \frac{(p + 1)}{n!}$.

We recall the following special version of [2, Theorem 5.1] for convex polytopes.

**Theorem 4.8.** Let $P$ be a lattice polytope in $\mathbb{R}^n$ that admits a $(p, q)$-partition. Let $\chi$ be the Santaló point of $P$. Then we have

$$\text{vol}(P^\chi) \text{vol}(P) = \frac{(p + 1)^{p+1}(q + 1)^{q+1}}{p!q!(p + q)!}.$$

**Proposition 4.9.** Given a polytope $P$ as the convex hull of $n + 2$ lattice points and $p \geq 1$ is minimal such that a $(p, q)$-partition exists. Let $\chi \in P$ be the Santaló point. Then, we have

$$\text{vol}(P^\chi) \leq \frac{(p + 1)^p(q + 1)^{(q+1)}}{p!q!}.$$

**Proof.** The inequality follows from Lemma 4.5, Lemma 4.7, and the equality in Theorem 4.8.

### 4.4 An arithmetic lemma

In this subsection, we prove the following arithmetic lemma.

**Lemma 4.10.** Let $p_0 < n$ be positive integers and $k > 0$ be a real number. Assume that the inequality

$$\frac{(p + 1)^p(q + 1)^{q+1}}{p!q!} < \frac{kn^{(n+1)}}{n!}$$

holds for $p = p_0$ and $q_0 = n - p_0$. Then, the inequality (4) holds for any $p \geq p_0$ with $p < n$ and $q = n - p$.

**Proof.** The claim is equivalent to

$$\binom{n}{p} = \frac{n!}{p!q!} \leq \frac{kn^{(n+1)}}{(p + 1)^p(q + 1)^{(q+1)}} = : f_k(n, p).$$

We now argue by induction on $p$. We have $\binom{n}{p+1} = \binom{n}{p} \cdot \frac{q}{p+1}$ and

$$f_k(n, p + 1) = f_k(n, p) \cdot \frac{q}{(p + 2)\left(\frac{p+2}{p+1}\right)^p \left(\frac{q}{q+1}\right)^{q+1}}.$$

By induction hypothesis, it is enough to show that

$$\frac{q}{p + 1} < \frac{q}{(p + 2)\left(\frac{p+2}{p+1}\right)^p \left(\frac{q}{q+1}\right)^{q+1}}.$$
for $p_0 \leq p < n$ or equivalently

$$\left(\frac{p + 2}{p + 1}\right)^{p+1} \left(\frac{q}{q + 1}\right)^{q+1} \leq 1.$$  (5)

To obtain the latter, observe that

$$\sqrt[n+2]{\frac{p + 2}{p + 1}}^{p+1} \left(\frac{q}{q + 1}\right)^{q+1} < \frac{(p + 1) \cdot \frac{p+2}{p+1} + (q + 1) \cdot \frac{q}{q+1}}{n + 2} = \frac{n + 2}{n + 2} = 1$$

holds by the inequality between arithmetic and geometric mean. □

### 4.5 Mahler volume of pyramids

In this subsection, we recall how to compute the Mahler volume of a pyramid. The following lemma is taken from [26, Lemma 11].

**Lemma 4.11.** Let $P = \text{conv}(F, x_0)$ be a pyramid in $\mathbb{R}^n$ where $F$ is a $(n - 1)$-dimensional lattice polytope and $x_0$ is not in the hyperplane containing $F$. Let $\chi$ and $s$ be the Santaló points of $P$ and $F$, respectively. Then

$$\frac{\text{vol}(P)\text{vol}(P\chi)}{(n+1)^{n+1}n^{n+2}} = \frac{\text{vol}(F)\text{vol}(F_s)}{(n+1)^{n+1}n!}.$$  

### 4.6 Toric singularities of maximal volume

In this section, we prove Theorem 2. First we prove Proposition 4.3.

**Proof of Proposition 4.3.** If $P$ is a simplex with $\text{vol}(P) > 2/n!$, then the claim follows from Lemma 4.4. If it is not a simplex, then $P$ contains a lattice polytope $Q$ spanned by $n + 2$ lattice points. If $Q \subset P$ and $z \in Q$ is the Santaló point of $Q$, then it follows that $\text{vol}P^z \leq \text{vol}P^z < \text{vol}Q^z$. Hence, it is sufficient to prove the statement for lattice polytopes $P$, which are spanned by $n + 2$ lattice points. Let now be $P$ such a polytope and $\chi$ its Santaló point.

If $\text{vol}(P) \geq 4/n!$, then we set $p = 1$ and $p = 2$ in Theorem 4.8 and obtain

$$\text{vol}P^x \leq \frac{n^n}{(n - 1)!} < \frac{16}{27} \cdot \frac{(n + 1)^{(n+1)}}{n!}$$  (6)

and

$$\text{vol}P^x \leq \frac{27(n - 1)^{(n-1)}}{8(n - 2)!} < \frac{16}{27} \cdot \frac{(n + 1)^{(n+1)}}{n!},$$  (7)

respectively. Here, the leftmost inequalities of (6) and (7) follow from Theorem 4.8 by taking into account that $\text{vol}(P) > 4/n!$ holds. The rightmost inequalities are equivalent to $(1 - 1/(n +
$1)^{n+1} < 16/27$ and $n/(n-1) \cdot (n-2/(n+1))^{n+1} < 2^7/3^6$, respectively. By taking logarithms and differentiating in $n$ we can see that both sequences are strictly increasing and approach $e^{-1} < 16/27$ and $e^{-2} < 2^7/3^6$, respectively. Hence, the claim follows for $p = 1$ and $p = 2$. Now, we consider the case $p = 3$. Here, Proposition 4.9 gives the left inequality of

$$\text{vol}(P^X) \leq \frac{4^3(n-2)^{(n-2)}}{6(n-3)!} < \frac{16}{27} \cdot \frac{(n+1)^{(n+1)}}{n!}.$$ \hspace{1cm} (8)

To see that the right inequality holds, we may consider the sequence

$$\frac{n(n-1)}{(n-2)^2} \cdot \left(1 - \frac{3}{n+1}\right)^{n+2},$$ \hspace{1cm} (9)

which is strictly increasing and has the limit $e^{-3} < 1/18$. Hence, every element of that sequence is strictly less than $1/18$, which is equivalent to the rightmost inequality in (8).

For $p \geq 3$, we have

$$\text{vol}(P^X) \leq \frac{(p+1)^p(q+1)^{q+1}}{p!q!} < \frac{16}{27} \cdot \frac{(n+1)^{(n+1)}}{n!},$$

where the first inequality is provided by Proposition 4.9 while the second inequality is given by Lemma 4.10 with $k = 16/27$ and $p_0 = 3$.

If $\text{vol}(P) = 3/n!$, then by [9] we have that $P$ is an iterated pyramid over one of the following polytopes:\footnote{Using the notation of [16, Theorem 1.3].}

$$P^{(3)}_1 := \text{conv}\{0, 2e_1, e_2, e_1 + e_2\}$$

$$Q^{(3)}_1 := \text{conv}\{0, e_1, e_2, e_3, e_1 + e_2 - 2e_3\}$$

$$Q^{(3)}_2 := \text{conv}\{0, e_1, e_2, e_3, e_4, -e_1 - e_2 + e_3 + e_4\}.$$ We address one after the other. By Theorem 4.8, the polytope $P^{(3)}_1$ has a Mahler volume strictly less than 8. Hence, for an iterated pyramid over that polytope by applying Lemma 4.11, we obtain

$$\text{vol}(P)\text{vol}(P^X) < \frac{8 \cdot (2)!^2}{3^3} \cdot \frac{(n+1)^{(n+1)}}{n!} < \frac{32}{27} \cdot \frac{(n+1)^{(n+1)}}{n!}.$$ As $\text{vol}(P) = 3/n!$, we have $\text{vol}(P^X) < 16/27 \cdot (n+1)^{n+1}/n!$.

Similarly, for $Q^{(3)}_1$ by Theorem 4.8 we get a value less than 9. Now, Lemma 4.11 gives

$$\text{vol}(P)\text{vol}(P^X) < \frac{9 \cdot (3)!^2}{4^4} \cdot \frac{(n+1)^{(n+1)}}{n!} < \frac{81}{64} \cdot \frac{(n+1)^{(n+1)}}{n!}.$$ For the corresponding iterated pyramid. As $\text{vol}(P) = 3/n!$ we obtain $\text{vol}(P^X) < 16/27 \cdot (n+1)^{n+1}/n!$. 
Finally, one calculates 243/32 for the Mahler volume of $Q_2^{(3)}$ by using Theorem 4.8. Applying Lemma 4.11, we obtain

$$\frac{\text{vol}(P)\text{vol}(P^\chi)}{\text{vol}(P^\chi)} = \frac{243 \cdot (4!)^2}{32 \cdot 5^5} \cdot \frac{(n + 1)^{(n+1)}}{n!} = \frac{4374}{3125} \cdot \frac{(n + 1)^{(n+1)}}{n!}.$$  

In particular, as $\text{vol}(P) = 3/n!$ we see that

$$\text{vol}(P^\chi) = \frac{1458}{3125} \cdot \frac{(n + 1)^{(n+1)}}{n!} < \frac{16}{27} \cdot \frac{(n + 1)^{(n+1)}}{n!}.$$  

If $\text{vol}(P) = 2/n!$, then by [9] $P$ is an iterated pyramid over the unit square and we indeed get equality by Lemma 4.11.

Now we turn to prove Theorem 2.

**Proof of Theorem 2.** The statement is clear for two-dimensional singularities (see, e.g., [35, Corollary 1.4]).

Let $X = X(\sigma)$ where $\sigma = \text{cone}(P, \ell)$ for a lattice polytope $P$ of dimension at least 2. By Lemma 4.2, we have

$$\widetilde{\text{vol}}(X) = \frac{(d - 1)!}{\ell} \text{vol}(P^\chi),$$

where $\chi$ is the Santaló point of $P$. By Proposition 4.3, we conclude that

$$\widetilde{\text{vol}}(X) \leq (d - 1)! \cdot \left( \frac{16}{27} \frac{d^d}{(d - 1)!} \right) = \frac{16}{27} d^d.$$  

If the equality holds in this inequality, then also by Proposition 4.3 the polytope $P$ is an iterated pyramid over the unit square, so $X \simeq \mathbb{A}^{d-3} \times C$ where $C$ is the three-dimensional $A_1$ singularity.

## 5 | TOPOLOGICAL INVARIANTS AND THE NORMALIZED VOLUME

In [15], the authors try to find bounds on the normalized volume of a toric Gorenstein singularity in terms of topological invariants of certain (crepant) resolutions. In the toric setting, we suggest the following conjecture.

**Conjecture 5.1.** Let $X$ be a toric Gorenstein singularity and $\bar{X} \to X$ be a toric resolution of singularities. Then

$$\widetilde{\text{vol}}(X) \geq \frac{d^d}{\chi(X)},$$

where $\chi$ denotes the Euler characteristic.
The previous conjecture is inspired by and closely related to [15, Conjecture 5.3]. We prove that the nonsymmetric Mahler conjecture implies Conjecture 5.1.

**Proposition 5.2.** Assume that the nonsymmetric Mahler conjecture holds in dimension $n$. Then Conjecture 5.1 holds for toric Gorenstein singularities of dimension $d = n + 1$.

**Proof.** Let $X$ be a toric Gorenstein singularity of dimension $d$. Then $X$ is defined by the cone $\sigma = \text{cone}(P, 1)$ with $\dim(P) = n$. Now, a toric resolution corresponds to a regular triangulation of that cone, that is, a subdivision into simplicial cones each of them spanned by elements of a lattice basis of $\mathbb{Z}^{n+1}$ (see, e.g., [10, section 2.6]). By the crepant hypothesis, such a triangulation of $\sigma$ induces a triangulation of the polytope $P$. The triangulation of the polytope $P$ has facets of volume at most $1/n!$. Hence, the value $n!\text{vol}(P)$ is a lower bound for the number of maximal cones in the triangulation of $\sigma$. On the other hand, the number of maximal cones (and therefore of torus fixed points in the resolution) coincides with the Euler characteristic $\chi(\hat{X})$ (see, e.g., [7, Corollary 11.6]). We obtain an inequality

$$\hat{\text{vol}}(X)\chi(\hat{X}) \geq n!\text{vol}(P)\chi(\hat{X}). \quad (10)$$

The Mahler conjecture implies that

$$\text{vol}(P)\text{vol}(P^\vee) \geq \frac{(n+1)(n+1)}{(n!)^2}, \quad (11)$$

where $\chi$ is the Santaló point of $P$. Putting Equations (10) and (11) together, we obtain

$$\hat{\text{vol}}(X)\chi(\hat{X}) \geq (n+1)(n+1) = d^d$$

finishing the proof.

In a similar direction, we prove the following proposition connecting the Blaschke–Santaló inequality to an upper bound for the normalized volume of toric singularities.

**Proposition 5.3.** Let $X$ be a $\mathbb{Q}$-Gorenstein toric singularity of dimension $d$ and $\hat{X} \to X$ a toric crepant partial resolution. Then

$$\hat{\text{vol}}(X) < \frac{(d-1)!\omega_{d-1}^2}{\chi(\hat{X})}.$$ 

**Proof.** Write $X = \text{cone}(P, \ell)$. A toric crepant partial resolution $\hat{X} \to X$ induces a subdivision of $P$ into lattice polytopes of volume at least $1/n!$. Hence, $\chi(\hat{X}) \leq n!\text{vol}(P)$ in this case. Now, from the inequality (1) in Theorem 3.4, we obtain the desired inequality.

Note, that for the case $d = 2$ both Conjecture 5.1 and Proposition 5.3 have been shown earlier in [12].

In [15, Conjecture 5.5], the authors propose the following conjecture.
**Conjecture 5.4.** Let $P$ be a reflexive polytope of dimension $n$. Let $Y$ be the Gorenstein toric Fano variety associated to the face fan of $P$ and let $\tilde{Y}$ be a crepant resolution of $Y$. Let $X$ be the cone over $Y$ with respect to an anti-canonical embedding. Then

$$\widehat{\text{vol}}(X) \leq \int_{\tilde{Y}} c_1(\tilde{Y})^{n-1}.$$ 

We finish this section by proving a more general version of the previous conjecture.

**Proposition 5.5.** Let $Y$ be a Gorenstein Fano variety of dimension $n$. Let $X$ be the cone over an anti-canonical embedding of $Y$. Assume that $\tilde{Y} \to Y$ is a crepant resolution. Then

$$\widehat{\text{vol}}(X) \leq \int_{\tilde{Y}} c_1(\tilde{Y})^{n-1}.$$ 

Furthermore, the equality holds if and only if $Y$ is $K$-semistable.

**Proof.** We have

$$\int_{\tilde{Y}} c_1(\tilde{Y})^{d-1} = (-K_{\tilde{Y}})^{d-1} = (-K_Y)^{d-1} = \text{vol}_{\text{wt}}(X).$$

The equality in the middle follows from the fact that $\tilde{Y} \to Y$ is crepant. The valuation $\text{wt}$ is the one given by the weight with respect to the $\mathbb{C}^*$-action coming with the cone structure of $X$. Moreover, for the anti-canonical embedding the log discrepancy $a_X(\text{wt})$ is 1. This follows, for example, from Step 3 of the proof of Theorem 1 in [27, section 3] (in the notation of this paper our assumption assures $D_i = -K_Y$). Hence, we obtain

$$\int_{\tilde{Y}} c_1(\tilde{Y})^{d-1} = \text{vol}_{\text{wt}}(X) \geq \widehat{\text{vol}}(X) := \inf_v \text{vol}_v(X).$$

Here, $v$ runs over all valuation on $X$ with center at the vertex and log discrepancy $a_X(v) = 1$. $\square$

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