A STUDY OF COMPARISON, EXISTENCE AND REGULARITY OF VISCOSITY AND WEAK SOLUTIONS FOR QUASILINEAR EQUATIONS IN THE HEISENBERG GROUP

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Abstract. In this manuscript, we are interested in the study of existence, uniqueness and comparison of viscosity and weak solutions for quasilinear equations in the Heisenberg group. In particular, we highlight the limitation of applying the Euclidean theory of viscosity solutions to get comparison of solutions of sub-elliptic equations in the Heisenberg group. Moreover, we will be concerned with the equivalence of different notions of weak solutions under appropriate assumptions for the operators under analysis. We end the paper with an application to a Radó property.

1. Introduction and main results. The main concern of this paper is to analyse weak and viscosity solutions of quasilinear equations in divergence form in the Heisenberg group:

\[- \text{div}_0 (A(p, \nabla_0 u)) = f(p, u),\]

where \( A = A(p, \xi) : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2 \), \( \text{div}_0 \) and \( \nabla_0 \) are the intrinsic divergence and gradient operators in the Heisenberg group. We refer the reader to Section 2 for definitions and notation. Observe that under the symmetric condition:

\[ \frac{\partial A_1}{\partial \xi_2} = \frac{\partial A_2}{\partial \xi_1}, \]

to study viscosity solutions of (1) is equivalent to consider viscosity solutions of:

\[- \text{tr} \begin{pmatrix} \frac{\partial A_1}{\partial \xi_1} & \frac{\partial A_1}{\partial \xi_2} \\ \frac{\partial A_2}{\partial \xi_1} & \frac{\partial A_2}{\partial \xi_2} \end{pmatrix} \nabla_0^{2*} u - \text{div}_0 (A(\cdot, \nabla_0 u)) = f(p, u),\]

where \( \nabla_0^{2*} u \) is the symmetrized Hessian of \( u \) in the Heisenberg group and where \( \text{div}_0 (A(\cdot, \nabla_0 u)) \) is the horizontal divergence of \( p \to A(p, \xi) \) with respect to \( p \) at

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\[ \xi = \nabla_0 u. \] Therefore, we shall also treat the theory of viscosity solutions to second-order equations in non-divergence form:

\[ - \operatorname{tr} \left[ M(p, \nabla_1 u(p)) \nabla^2_0 u \right] = F(p, u, \nabla_1 u), \quad (2) \]

where \( M \) is a matrix field.

As a motivation for considering differential equations with intrinsic structures in the Heisenberg group, let us consider the sub-elliptic Laplacian operator:

\[ \Delta_0 := X_1^2 + X_2^2 \]

where \( X_1 \) and \( X_2 \) are given as in \((3)\). In terms of Euclidean derivatives, the sub-elliptic Laplace equation may be written as:

\[ -\operatorname{tr} \left( \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \\ 2y & -2x & 4(x^2 + y^2) \end{pmatrix} \nabla^2 u \right) = 0. \]

Observe that the matrix:

\[ \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \\ 2y & -2x & 4(x^2 + y^2) \end{pmatrix} \]

is not uniformly elliptic for all \((x, y, z)\). Hence the results from the Euclidean theory of viscosity solutions of uniformly elliptic operators are not applied directly (see the fundamental works \([11, 5]\) and the references therein).

With respect to distributional or weak solutions, we may write the sub-Laplace equation in terms of the Euclidean divergence operator as:

\[ -\operatorname{div} (A(p, \nabla u)) = 0 \]

where:

\[ A(p, \xi) := \begin{pmatrix} \xi_1 + 2y \xi_3 \\ \xi_2 - 2x \xi_3 \\ 2y \xi_1 - 2x \xi_2 + 4(x^2 + y^2) \xi_3 \end{pmatrix}, \quad p = (x, y, z), \quad \xi = (\xi_1, \xi_2, \xi_3). \]

The matrix \( \partial_\xi A \) is not uniformly elliptic for all \( p \), and hence the comparison and uniqueness results for weak solutions in the Euclidean framework for uniformly elliptic operators are not applied (see the seminal work \([36]\) and the reference therein).

Therefore, in order to obtain satisfactory existence, uniqueness and regularity results, we apply an intrinsic theory derived from the differential and metric structures of the Heisenberg group as a Carnot group.

The Heisenberg group is a sub-Riemannian group. Recently, different applications of the geometry of sub-Riemannian manifolds have been considered. As an example, the roto-translation group serves as a model for the structure of the visual cortex. Indeed, in \([8]\), the authors consider the sub-Riemannian cortical model of image completion which gives rise to a diffusion driven motion by mean curvature. The roto-translation group appears as a natural model for the process of lifting of a plane image to a regular surface. We refer the reader to \([9, 10, 8, 6]\) and the references therein for details and more applications of sub-Riemannian structures.

The theory of viscosity solutions has been applied to differential equations in the Heisenberg group since the 80’s. However, one of the main limitations to apply directly the Euclidean theory is the loss of uniform ellipticity. Moreover, the lack of a Crandall-Ishii Lemma intrinsic to sub-Riemannian structures is also a limitation.
To the best of our knowledge, the available Crandall-Ishii Lemma in sub-Riemannian groups and, in particular, Carnot groups, depends on the corresponding Euclidean result via a translation of sub-elliptic jets into Euclidean jets. We refer the reader to [4] and [2] for a derivation of the Crandall-Ishii Lemma in the Heisenberg group and Carnot groups. A related issue is the development of comparison results for viscosity solutions. We refer the reader to [28] for a survey of comparison results.

Towards getting comparison principles, it is usual to assume that a given second order operator $F = F(p, u, \nabla_1 u, \nabla_2 u, *_{0}^u)$ does not depend on the spatial variable $p$, or the first-derivative variable $\nabla_1 u$, and also assume that $F$ has bounded away from zero derivatives $\partial F/\partial u$ (strict monotonicity in $u$). For examples of these results we quote [28], [31, Proposition 4.1], [30, Theorem 2.1] (here the authors remove the strictly increasing assumptions but they assume a sign condition). At this point we would like to quote the works [1] and [29] where the authors propose various form of partial nondegeneracy to weaken the uniform ellipticity assumption and apply their results to some sub-elliptic second order equations.

In our work, any viscosity solution will be an Euclidean solution (see Definition 2.3). This allows to apply comparison principles from the Euclidean theory to the sub-elliptic equations. However, as we will see in Section 3 and specifically in Remark 1, standard assumptions in the Euclidean context used to get comparison, such as (3.14) in [11], (6) and (10) in [1], or the Lipschitz hypothesis on $G$ in [1, Theorem 3.2], are not satisfied in general by the operators treated here. To overcome this scenario, we employ intrinsic techniques from the sub-Riemannian framework.

On the other hand, the theory of weak solutions has also been developed in sub-Riemannian groups. For completeness, we provide, in the appendix, a full proof of existence of weak solutions for the equations treated here and in John domains. Most of the literature on existence is developed in non-characteristic domains [46, 44, 43, 17, 45, 47, 42, 39, 41, 40]. However, following [34, Theorem 3.2.1], the class of non-characteristic domains is a subclass of John domains, and hence our results are slightly more general.

We make now some comments on the relation between weak and viscosity solutions. In the Euclidean setting, the relation is well understood from the works [27, 22, 25] and [23]. In the Heisenberg scenario, the basic reference is [3].

Having taking into account the above comments, we now summarize our main contributions and the organization of the manuscript.

- **Comparison result for semicontinuous viscosity solutions of (2).** In order to obtain this result, we do not assume that the differential operator has partial derivatives with respect to $u$ bounded away from zero. Also, we will not assume any sign condition. We refer the reader to Section 3.
- **Equivalence of weak and viscosity solutions.** Based on the above results and regularity theory, we derive the equivalence of the notions of solutions considered in this work. We refer to Section 4.1.
- **Radó type property.** As an application of the above results, we provide a Radó type theorem for horizontal critical points. See Section 5.
- **Existence of weak (or distributional) solutions to (1).** This will achieve by standard methods by introducing obstacle problems. We prove existence in John domains. See the Appendix.

2. Preliminaries. Basic notation. We shall use the following standard notation in the work. The Euclidean interior product is denoted by $\langle \cdot, \cdot \rangle$. If $\gamma, \beta \in \mathbb{R}^n$, the
vector $\gamma \oplus \beta$ is defined as the vector in $\mathbb{R}^{2n}$ whose first $n$ entries are those of $\gamma$, followed by the components of $\beta$. The Euclidean norm of a vector in $\mathbb{R}^{2n}$ will be denoted by $\| \cdot \|$. For matrices, by $A < B$ and $A \leq B$ we mean that $B - A$ is positive definite and positive semi-definite, respectively. Also, we denote by $S^n(\mathbb{R})$ the set of $n \times n$ symmetric and positive semi-definite matrices with real coefficients. The trace of a matrix $A$ is denoted by $\text{tr}(A)$. By $\Omega$ we shall always denote an open and bounded domain in $\mathbb{R}^3$ where $\partial \Omega$ has Lebesgue measure zero. We also introduce the following functional spaces:

$$LSC(\Omega) = \{ u : \Omega \to \mathbb{R} : u \text{ is lower semicontinuous in } \Omega \}.$$  

$$USC(\Omega) = \{ u : \Omega \to \mathbb{R} : u \text{ is upper semicontinuous in } \Omega \}.$$  

2.1. **Heisenberg group.** We denote by $\mathcal{H}$ the first-order Heisenberg group whose underlying manifold is $\mathbb{R}^3$, and whose group operation is given by:

$$p \cdot q = (x_0 + x_1, y_0 + y_1, z_0 + z_1 + 2(x_1y_0 - x_0y_1)),$$

for all $p = (x_0, y_0, z_0)$, $q = (x_1, y_1, z_1) \in \mathbb{R}^3$. The group $\mathcal{H}$ is a Lie group with Lie algebra $\mathfrak{h}$ generated by the left-invariant basis:

$$X_1 = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z} = -\frac{1}{4}[X_1, X_2],$$  

where $p = (x, y, z) \in \mathbb{R}^3$. We equip $\mathfrak{h}$ with an interior product (a Riemann structure) so that the frame (3) is orthonormal. We recall that the exponential mapping is a global diffeomorphism that takes the vector $xX_1 + yX_2 + zX_3$ in the Lie algebra $\mathfrak{h}$ to the point $(x, y, z)$ in the Lie group $\mathcal{H}$. This allows us to identify vectors in $\mathfrak{h}$ with points in $\mathcal{H}$. For $t > 0$, we denote by $\delta_t$ the dilation of the group given as:

$$\delta_t(p) = (tx, ty, t^2z)$$

The two dimensional linear space generated by the vectors $X_1(p)$ and $X_2(p)$ is denoted by $\mathcal{H}_0$. The distribution $\mathcal{H}_0$ is called the horizontal distribution.

2.1.1. **Metric structure and Calculus in $\mathcal{H}$.** The metric structure on $\mathcal{H}$ is given by the Carnot-Carathéodory distance (CC distance in brief) which is defined as follows: an absolutely continuous curve $\gamma \in W^{1,2}((0,1), \mathbb{R}^3)$ is said to be horizontal if there is a control $v \in L^2((0,1), \mathbb{R}^3)$ such that:

$$\gamma'(t) = \sum_{i=1}^2 v_i(t)X_i(\gamma(t))$$

for a.e. $t$ in $(0,1)$. For any $p, q \in \mathcal{H}$, the CC distance between $p$ and $q$ is defined as:

$$d_{CC}(p, q) := \inf \left\{ \left( \int_0^1 |v(t)|^2 dt \right)^{1/2} \right\}$$

where the infimum is taken over all horizontal curves $\gamma$, with associated control $v$, so that $\gamma(0) = p$ and $\gamma(1) = q$.

For computational purposes, we shall also use a smooth gauge out of the diagonal defined as follows:

$$|p|_{\mathcal{H}} := \left( (x^2 + y^2)^2 + z^2 \right)^{1/4}, \text{ for } p = (x, y, z) \in \mathcal{H}.$$  

This gauge is comparable to the CC distance. The corresponding distance is:

$$d_{\mathcal{H}}(p, q) := |q^{-1} \cdot p|_{\mathcal{H}}, \text{ for all } p, q \in \mathcal{H}.$$
Also, for any \( p \in H \) and \( \delta > 0 \), we write:

\[
B_H(p, \delta) := \{ q \in H : |q^{-1} \cdot p|_H < \delta \},
\]
to denote the ball in the Heisenberg group with center at \( p \) and radius \( \delta \).

Given a smooth function \( u \) in \( \mathbb{R}^3 \), and a multi-index \( I = (i_1, i_2, i_3) \), the derivative \( X^I u \) is defined by:

\[
X^I u = X_1^{i_1} X_2^{i_2} X_3^{i_3} u.
\]
The function \( u \) belongs to \( C^k_H(\Omega) \) if \( X^I u \) is continuous in \( \Omega \) for all multi-indices \( I \) such that:

\[
d(I) := i_1 + i_2 + 2i_3 \leq k.
\]

In general, the class of \( C^k_H \) functions is larger than the class of Euclidean \( C^k(\Omega) \) (see [16, Remark 5.9] for examples).

For a Euclidean smooth function \( u : \mathbb{R}^3 \to \mathbb{R} \), Taylor expansion around \( 0 \) implies:

\[
u(p) = u(0) + \langle \nabla u(0), p \rangle + \frac{1}{2} \langle \nabla^2 u(0) p, p \rangle + o(|p|^2),
\]
for \( p \to 0 \), and where \( \nabla u \) and \( \nabla^2 u \) stand for the Euclidean gradient and Hessian of \( u \), respectively. The horizontal Taylor expansion of \( u \) at \( 0 \) is:

\[
u(p) = u(0) + \langle \nabla_1 u(0), p \rangle + \frac{1}{2} \langle \nabla^2_0 u(0) p, p \rangle + o(|p|^2),
\]
where the gradient of \( u \) with respect to the frame (3) is:

\[
\nabla_1 u := (X_1 u) X_1 + (X_2 u) X_2 + (X_3 u) X_3,
\]
and the symmetrized horizontal second derivative matrix, denoted by \( \nabla^2_0 u \), is given by:

\[
\nabla^2_0 u := \begin{bmatrix} X_1^2 u & \frac{1}{2}(X_1 X_2 u + X_2 X_1 u) \\ \frac{1}{2}(X_1 X_2 u + X_2 X_1 u) & X_2^2 u \end{bmatrix}.
\]

In addition, the horizontal gradient of \( u \) is:

\[
\nabla_0 u := (X_1 u) X_1 + (X_2 u) X_2.
\]
Finally, for a vector field \( V = (V_1, V_2) \), its horizontal divergence operator is defined by:

\[
div_0 V := X_1 V_1 + X_2 V_2.
\]

2.1.2. Horizontal Sobolev spaces. In order to treat variational problem in \( H \), we consider horizontal Sobolev spaces:

\[
S^{1,t}_H(\Omega) := \{ u \in L^r(\Omega) : X_1 u, X_2 u \in L^r(\Omega) \}
\]
equipped with the norm:

\[
\| u \|_{S^{1,t}_H(\Omega)} := \| u \|_{L^r(\Omega)} + \| \nabla_0 u \|_{L^r(\Omega)}.
\]

Also, we define the Sobolev space \( S^{0,t}_H(\Omega) \) as the closure of \( C_0^\infty(\Omega) \) with respect to the above norm in \( S^{1,t}_H(\Omega) \). Usually, we shall write \( \| \cdot \|_t \) for \( \| \cdot \|_{L^r(\Omega)} \).

The following is the Sobolev-Poincaré inequality in the Heisenberg group in John domains (see [18, Corollary 1.6, Theorem 1.30] and also [12] for global results). We recall that a bounded open set \( \Omega \) in a metric space \((M,d)\) is a John domain if there exist a point \( p_0 \in \Omega \) and \( C > 0 \) such that for every \( p \in \Omega \) there exists a
continuous rectifiable curve parametrized by arclength $\gamma : [0, T] \to \Omega$, $T \geq 0$, such that $\gamma(0) = p$, $\gamma(T) = p_0$ and:

$$\text{dist}(\gamma(t), \partial \Omega) \geq C t,$$

for all $t \in [0, T]$. We refer the reader to [6] and the references therein for further properties of John domains. As an illustration, we point out that in view of [33, Theorem 1.3], any $C^{1,1}$ domain in the Heisenberg group is a John domain.

**Theorem 2.1.** Let $\Omega \subset \mathcal{H}$ be a John domain. If $1 \leq t < Q = 4$, then there exists a constant $C = C(t, Q) > 0$ such that:

$$\left( \int_\Omega |u|^t \right)^{1/t} \leq \left( \int_\Omega |\nabla_0 u|^t \right)^{1/t}$$

for all $u \in S_{0,t}^{1,4}(\Omega)$ and where:

$$\frac{1}{t^*} := \frac{1}{t} - \frac{1}{Q}.$$ 

In particular:

$$\left( \int_\Omega |u|^t \right)^{1/t} \leq \left( \int_\Omega |\nabla_0 u|^t \right)^{1/t}$$

for all $u \in S_{0,t}^{1,4}(\Omega)$ and all $t \in [1, +\infty)$.

**2.1.3. Convolution and approximation to identity in $\mathcal{H}$.** Let $\rho \in C_0^\infty(\Omega)$ such that:

$$0 \leq \rho \leq 1, \quad \int_\Omega \rho = 1, \quad \rho(p) = \rho(p^{-1}), \quad \text{and supp } \rho \subset B_{\mathcal{H}}(0, 1).$$

For any $h > 0$, we let:

$$\rho_h(p) := h^{-4} \rho(\delta_{1/h}(p)).$$

For $f \in L^1(\Omega)$, we let:

$$f_h(p) = (\rho_h \ast f)(p) := \int_\Omega \rho_h(p \cdot q^{-1}) f(q).$$

Then the following happens (see for instance [15, 16] and [34]):

**Theorem 2.2.** Let $t \in [1, +\infty)$. We have:

- If $f \in S_{1,t}^1(\Omega)$, then there exists a sequence $\{u_n\} \subset S_{1,t}^1(\Omega) \cap C^\infty(\Omega)$ such that $u_n \to f$ in $S_{1,t}^1(\Omega)$.
- $\text{supp } f_h \subset B_{\mathcal{H}}(0, h) \cdot \text{ supp } f$.
- If $f \in S_{1,t}^1(\Omega)$ and has compact support, then $f_h \in C_0^\infty(\Omega)$ and $f_h \to f$ in $S_{1,t}^1(\Omega)$.
- If $f \in S_{1,1}^1(\Omega)$, then in any subdomain $\Omega'$, we have $\nabla_0 f_h = (\nabla_0 f)_h$ for all $h$ sufficiently small.

For a more complete discussion of the Heisenberg group and more general Carnot groups, we refer the reader to [6, 15] and [28].
2.2. Viscosity solutions. The basic reference in what follows is [11] in the Euclidean setting and [4] in the Heisenberg group scenario. Let \( u \in USC(\Omega) \). The second-order superjet of \( u \) at \( p \) is defined as follows:

\[
\mathcal{J}^{2,+}(u, p) := \left\{ (\eta, \mathcal{X}) \in \mathbb{R}^3 \times S^2(\mathbb{R}) \mid u(q) \leq u(p) + \langle \eta, p^{-1} \cdot q \rangle + \frac{1}{2} \langle \mathcal{X}(p^{-1} \cdot q)u, (p^{-1} \cdot q)_{0} \rangle + o(d_{\mathcal{H}}(p, q)^{2}) \right\},
\]

as \( q \to p \). Here, \((p^{-1} \cdot q)_{0}\) is the projection of \( p^{-1} \cdot q \) onto the horizontal distribution \( \mathcal{H}_{0} \). Similarly, for \( v \in LSC(\Omega) \), the second-order subjet of \( v \) at the point \( p \) namely \( \mathcal{J}^{2,-}(v, p) \) is defined.

It is well-known (see [4] and [2, Lemma 2.2]) that subelliptic jets may be seen as appropriate derivatives of test functions touching the given function by above or below. More precisely, if \( u \) is upper semicontinuous, let us consider:

\[
\mathcal{K}^{+2}(u, p) := \left\{ (\nabla_{1}\phi(p), \nabla_{0}^{2,*}\phi(p)) \mid \text{ so that } \phi \in C^{2}_{\mathcal{H}} \right\}
\]

and similarly define \( \mathcal{K}^{-2}(v, p) \) for test functions touching the lower semicontinuous function \( v \) from below around \( p \). Hence, by the results in [4] and [2], it follows that:

\[
\mathcal{J}^{2,+}(u, p) = \mathcal{K}^{+2}(u, p)
\]

and:

\[
\mathcal{J}^{2,-}(v, p) = \mathcal{K}^{-2}(v, p).
\]

Finally, we shall also consider the theorectic closure of the sets defined above. We define \( \mathcal{J}^{2,+}(u, p) \) as the set of \((\eta, \mathcal{X})\) in \( \mathbb{R}^3 \times S^2(\mathbb{R}) \) so that there exists a sequence \((p_{n}, u(p_{n}), \eta_{n}, \mathcal{X}_{n})\) converging to \((p, u(p), \eta, \mathcal{X})\) satisfying \((\eta_{n}, \mathcal{X}_{n}) \in \mathcal{J}^{2,+}(u, p_{n})\) for all \( n \). In a similar way, we define \( \mathcal{J}^{2,-}(v, p) \).

In the next definition, we introduce the notion of viscosity solutions of general second-order equations driven by an operator \( \mathcal{F} \).

**Definition 2.3.** Let \( \mathcal{F} \) be a continuous function in \( \Omega \times \mathbb{R} \times \mathbb{R}^3 \times S^2(\mathbb{R}) \). Also, assume that \( \mathcal{F} \) is degenerate elliptic, that is:

\[
\mathcal{F}(p, r, \eta, \mathcal{X}) \leq \mathcal{F}(p, r, \eta, \mathcal{Y}) \quad \text{for all } \mathcal{X}, \mathcal{Y} \in S^2(\mathbb{R}), \ \mathcal{Y} \leq \mathcal{X}.
\]

An upper (resp. lower) semicontinuous function \( u : \Omega \to \mathbb{R} \cup \{ -\infty \} \) (resp. \( \mathbb{R} \cup \{ +\infty \} \)) is a subsolution (resp. supersolution) of:

\[
\mathcal{F}(p, u, \nabla_{1}u, \nabla_{0}^{2,*}u) = 0,
\]

in \( \Omega \) if for any \( p \in \Omega \):

1. \( u(p) < +\infty \) (resp. \( u(p) > -\infty \));
2. for every \((\eta, \mathcal{X}) \in \mathcal{J}^{2,+}(u, p)\) (resp. \((\eta, \mathcal{Y}) \in \mathcal{J}^{2,-}(v, p)\)) there holds:

\[
\mathcal{F}(p, u, \eta, \mathcal{X}) \leq 0,
\]

(resp. \( \mathcal{F}(p, u, \eta, \mathcal{Y}) \geq 0 \)).

Finally, a viscosity solution is both a viscosity subsolution and a viscosity supersolution.
2.3. A sub-elliptic Maximum Principle. The following result is a standard tool towards proving comparison principle for viscosity solutions in sub-Riemannian structures. For a complete discussion see [2].

Theorem 2.4. Let \( u \in USC(\Omega) \) and \( v \in LSC(\Omega) \). Let:
\[
\phi(p \cdot q^{-1}) := (p_1 - q_1)^m + (p_2 - q_2)^m + (p_3 - q_3 + 2(p_1 q_2 - q_1 p_2))^m,
\]
with \( m \geq 4 \) (so that \( \phi \in C^2_\bar{\Omega} \)) and assume that for each \( \tau > 0 \), \( p_\tau \) and \( q_\tau \) are points in \( \Omega \) at which \( u(p) - v(q) - \tau \phi(p \cdot q^{-1}) \) has a local maximum. Moreover, suppose that:
\[
\sup_{\Omega}(u - v) > 0.
\]
Then, there exist vectors \( \eta_\tau = \nabla^2 \phi(p_\tau \cdot q_\tau^{-1}) \) and matrices \( X_\tau, Y_\tau \) in \( S^2(\mathbb{R}) \) so that:
\[
(\tau \eta_\tau, X_\tau) \in J^{2,1}(u, p_\tau), \quad (\tau \eta_\tau, Y_\tau) \in J^{2,1}(v, q_\tau)
\]
and for any vectors \( \xi, \zeta \in \mathbb{R}^2 \), we have:
\[
\langle X_\tau \xi, \xi \rangle - \langle Y_\tau \zeta, \zeta \rangle \leq \tau (\nabla^2 \phi(p_\tau \cdot q_\tau^{-1})(\xi - \zeta), \xi - \zeta) + \tau \langle \mathcal{M}_\tau(\xi \otimes \zeta), \xi \otimes \zeta \rangle + \tau \|M_\tau\|^2 \|M^t(p_\tau) \xi \otimes M^t(q_\tau) \zeta\|^2,
\]
where \( \mathcal{M}_\tau \) is the matrix of Euclidean second-order derivatives:
\[
\mathcal{M}_\tau := \begin{bmatrix}
\nabla^2 \phi(p_\tau \cdot q_\tau^{-1}) & \nabla^2 \phi(p_\tau \cdot q_\tau^{-1}) \\
\nabla^2 \phi(p_\tau \cdot q_\tau^{-1}) & \nabla^2 \phi(p_\tau \cdot q_\tau^{-1})
\end{bmatrix}
\]
the matrix \( M \) denotes:
\[
M(p) = \begin{pmatrix}
1 & 0 & 2y \\
0 & 1 & -2x
\end{pmatrix},
\]
with \( p = (x, y, z) \in \Omega \), and \( \mathcal{M}_\tau \) is given by:
\[
\mathcal{M}_\tau = \begin{pmatrix}
0 & \frac{1}{2}(W_\tau - W^*_\tau) \\
\frac{1}{2}(W_\tau^* - W_\tau) & 0
\end{pmatrix},
\]
with:
\[
W_\tau^* = X_\tau^* Y_\tau \phi(p_\tau, p_\tau),
\]
for all \( i, j = 1, 2 \). In particular, up to a constant, we have:
\[
X_\tau - Y_\tau \leq \tau \|M_\tau\|^2 I.
\]

3. Uniqueness of viscosity solutions for quasilinear equations. In this section we consider quasilinear equations of the form:
\[
- \text{tr} \left[ M(p, \nabla u) \nabla^2 u(p) \right] = F(p, u, \nabla u),
\]
where the matrix field \( M \) splits as follows:
\[
M(p, \eta) = M_1(p) + M_2(\eta), \quad p \in \overline{\Omega}, \eta \in \mathbb{R}^3,
\]
where \( M_1 \) and \( M_2 \) are matrix fields. Writing:
\[
M_1(p) = z_1(p) \otimes z_1(p) + z_2(p) \otimes z_2(p) + \cdots + z_k(p) \otimes z_k(p)
\]
and:
\[
M_2(\xi) = w_1(\xi) \otimes w_1(\xi) + w_2(\xi) \otimes w_2(\xi) + \cdots + w_n(\xi) \otimes w_n(\xi),
\]
we assume that:
(H1) There are constants $C > 0$ and $\alpha > 0$ so that:
\[ \| w_i(\xi) \| \leq C \| \xi \|^\alpha, \quad \text{for all } \xi \in \mathbb{R}^3, \text{ and } i = 1, \ldots, k, \]

(H2) For all $p_0 \in \Omega$, there exist a constant $C_{p_0} > 0$ and $r_0 > 0$ so that:
\[ \| z_i(p) - z_i(q) \| \leq C_{p_0} \phi(p \cdot q^{-1})^{1/m}, \quad \text{for all } p, q \in B_\mathcal{H}(p_0, r_0), i = 1, \ldots, n, \]

\[ \text{where } \phi \text{ is as in (7) with } m = 4 + 2\alpha. \]

(H3) For all $p_0 \in \Omega$ there exist $r_0 > 0$, $C_{p_0} > 0$ and an increasing function $h : (0, +\infty) \rightarrow (0, +\infty)$ such that:
\[ F(p, r, \xi) - F(q, s, \xi) \leq C_{p_0} [ \| p \cdot q^{-1} \| \mathcal{H} + \phi(p \cdot q^{-1})^{1/m} \| \xi \| - h(r - s)], \]

for all $p, q \in B_\mathcal{H}(p_0, r_0), s, r \in \mathbb{R}$ so that $s < r$ and $\xi \in \mathbb{R}^3$.

The following result is the comparison principle for solutions of (11).

**Theorem 3.1.** Assume hypotheses (H1)-(H3). Let $u \in USC(\bar{\Omega})$ be a subsolution of (11) and $v \in LSC(\bar{\Omega})$ be a supersolution of (11) so that $u \leq v$ on $\partial \Omega$. Then:
\[ u \leq v \quad \text{in } \Omega. \]

**Proof.** Reasoning by contradiction, suppose that:
\[ \sup_{\Omega} (u - v) \geq 0. \tag{12} \]

Now we apply the subelliptic maximum principle to $u - v$. In order to do so, we shall check that if $p_0, q_0 \in \bar{\Omega}$ are so that:
\[ M_\tau := \sup_{\bar{\Omega} \times \bar{\Omega}} \{ u(p) - v(q) - \tau \phi(p \cdot q^{-1}) \} \equiv u(p_0) - v(q_0) - \tau \phi(p_0 \cdot q_0^{-1}), \tag{13} \]

then we indeed have that $p_\tau$ and $q_\tau$ belong to $\Omega$. To prove the statement, observe that in view of the compactness of $\bar{\Omega}$, there exist $p_0, q_0 \in \bar{\Omega}$ so that, up to a subsequence that we do not re-label:
\[ p_\tau \rightarrow p_0 \text{ and } q_\tau \rightarrow q_0 \text{ as } \tau \rightarrow \infty. \tag{14} \]

To reach a contradiction, assume that $p_0 \in \partial \Omega$. Observe that by (13), we have for any $p \in \bar{\Omega}$:
\[ u(p_\tau) - v(q_\tau) - \tau \phi(p_\tau \cdot q_\tau^{-1}) \geq u(p) - v(p), \tag{15} \]

which, together with the boundedness of $u$ and $-v$ from above in $\bar{\Omega}$, imply that there is a constant $C > 0$ independent of $\tau$ so that:
\[ \tau \phi(p_\tau \cdot q_\tau^{-1}) \leq C, \quad \text{for all } \tau. \]

Hence, we must have:
\[ \lim_{\tau \rightarrow \infty} \phi(p_\tau \cdot q_\tau^{-1}) = 0. \]

Thus $p_0 = q_0$ in (14). Moreover, by (15), it follows:
\[ \tau \phi(p_\tau \cdot q_\tau^{-1}) \leq u(p_\tau) - v(q_\tau) - u(p_0) + v(p_0). \]

Taking limsup and recalling (14), we certainly derive:
\[ \lim_{\tau \rightarrow \infty} \tau \phi(p_\tau \cdot q_\tau^{-1}) = 0. \tag{16} \]

By (12), there is $\hat{p} \in \Omega$ so that $(u - v)(\hat{p}) > 0$. Another application of (13), together with (16), give:
\[ u(p_0) - v(p_0) \geq \limsup_{\tau \rightarrow \infty} \{ u(p_\tau) - v(q_\tau) - \tau \phi(p_\tau \cdot q_\tau^{-1}) \} \geq u(\hat{p}) - v(\hat{p}) > 0, \]
where the latter inequality follows by the choice of \( \hat{p} \). Hence we arrive at \( u(p_0) - v(p_0) > 0 \), which contradicts the boundary assumption. Then \( p_0 \in \Omega \) and we derive \( p_r, q_r \in \Omega \) for all sufficiently large \( \tau \).

By Theorem 2.4, for each \( \tau > 0 \), there exist \( \eta \in \mathbb{R}^3 \) and symmetric matrices \( \mathcal{X}_\tau, \mathcal{Y}_\tau \in S^2(\mathbb{R}) \) such that:

\[
(\tau \eta, \mathcal{X}_\tau) \in \mathcal{J}^{2,+}(u, p_r), \quad (\tau \eta, \mathcal{Y}_\tau) \in \mathcal{J}^{2,-}(v, q_r).
\]

Then

\[
0 \leq \text{tr}(M(p_r, \tau \eta), \mathcal{X}_\tau - M(q_r, \tau \eta), \mathcal{Y}_\tau) + F(p_r, u(p_r), \tau \eta) - F(q_r, v(q_r), \tau \eta).
\]  

(17)

Note that

\[
\text{tr}(M(p_r, \tau \eta), \mathcal{X}_\tau - M(q_r, \tau \eta), \mathcal{Y}_\tau) = \text{tr}
\left(\sum_{i=1}^{k} z_i(p_r) \otimes z_i(p_r) + \sum_{j=1}^{n} w_j(\tau \eta) \otimes w_j(\tau \eta) \right) \mathcal{X}_\tau
\]

\[
- \text{tr}
\left(\sum_{i=1}^{k} z_i(q_r) \otimes z_i(q_r) + \sum_{j=1}^{n} w_j(\tau \eta) \otimes w_j(\tau \eta) \right) \mathcal{Y}_\tau
\]

\[
= \sum_{j=1}^{n} \langle \mathcal{X}_\tau w_j(\tau \eta), w_j(\tau \eta) \rangle + \sum_{i=1}^{k} \langle \mathcal{X}_\tau z_i(p_r), z_i(p_r) \rangle
\]

\[
- \sum_{j=1}^{n} \langle \mathcal{Y}_\tau w_j(\tau \eta), w_j(\tau \eta) \rangle - \sum_{i=1}^{k} \langle \mathcal{Y}_\tau z_i(q_r), z_i(q_r) \rangle
\]

The sub-elliptic Maximum Principle implies by (8) the estimates:

\[
\langle \mathcal{X}_\tau w_j(\tau \eta), w_j(\tau \eta) \rangle - \langle \mathcal{Y}_\tau w_j(\tau \eta), w_j(\tau \eta) \rangle
\]

\[
\leq \tau \left[ \langle \mathcal{M}_\tau (w_j(\tau \eta) \oplus w_j(\tau \eta)), w_j(\tau \eta) \oplus w_j(\tau \eta) \rangle + \| \mathcal{M}_\tau \|^2 \| w_j(\tau \eta) \oplus w_j(\tau \eta) \|^2 \right]
\]

\[
= \tau \| \mathcal{M}_\tau \|^2 \| w_j(\tau \eta) \oplus w_j(\tau \eta) \|^2
\]

and:

\[
\langle \mathcal{X}_\tau z_i(p_r), z_i(p_r) \rangle - \langle \mathcal{Y}_\tau z_i(q_r), z_i(q_r) \rangle
\]

\[
\leq \tau \left[ \langle \nabla^2 \phi(p_r \cdot q_r^{-1}) (z_i(p_r) - z_i(q_r), z_i(p_r) - z_i(q_r)) \rangle
\]

\[
+ \langle \mathcal{M}_\tau (z_i(p_r) \oplus z_i(q_r)), z_i(p_r) \oplus z_i(q_r) \rangle + \| \mathcal{M}_\tau \|^2 \| z_i(p_r) \oplus z_i(q_r) \|^2 \right]
\]

\[
= \tau \left[ \langle \nabla^2 \phi(p_r \cdot q_r^{-1}) (z_i(p_r) - z_i(q_r), z_i(p_r) - z_i(q_r)) \rangle
\]

\[
+ \langle (1/2)(W_r - W_r^t) z_i(q_r), z_i(p_r) \rangle - \langle (1/2)(W_r - W_r^t) z_i(p_r), z_i(q_r) \rangle
\]

\[
+ \| \mathcal{M}_\tau \|^2 \| z_i(p_r) \oplus z_i(q_r) \|^2 \right]
\]

for \( i = 1, \ldots, k \) and \( j = 1, \ldots, n \). Observe that the terms:

\[
\langle (1/2)(W_r - W_r^t) z_i(q_r), z_i(p_r) \rangle - \langle (1/2)(W_r - W_r^t) z_i(p_r), z_i(q_r) \rangle
\]
may be rewritten as:
\[
\frac{1}{2} \langle (W_\tau - W_\tau^t)(z_i(q_\tau) - z_i(p_\tau)), z_i(q_\tau) \rangle + \frac{1}{2} \langle (W_\tau - W_\tau^t)z_i(q_\tau), z_i(p_\tau) - z_i(q_\tau) \rangle
\]
\[
= \langle (W_\tau - W_\tau^t)(z_i(q_\tau) - z_i(p_\tau)), z_i(q_\tau) \rangle.
\]
Hence:
\[
\langle \mathcal{X}_\tau z_i(p_\tau), z_i(p_\tau) \rangle - \langle \mathcal{Y}_\tau z_i(q_\tau), z_i(q_\tau) \rangle
\]
\[
\leq \tau \left[ \| \nabla_0^2 \phi(p_\tau, q_\tau^{-1}) \| \| z_i(p_\tau) - z_i(q_\tau) \|^2 + \| W_\tau - W_\tau^t \| \| z_i(p_\tau) - z_i(q_\tau) \| \right]
\]
\[
+ \| \mathcal{M}_\tau \| \| z_i(p_\tau) \ominus z_i(q_\tau) \|^2
\]
for \( i = 1, ..., k \). Since \( \Omega \) is bounded, we derive the bounds:
\[
\left| \frac{\partial \phi}{\partial x_i}(p_\tau, q_\tau) \right| \leq C \phi(p_\tau, q_\tau)^{(m-1)/m}, \text{ for } i = 1, 2, 3.
\]
and:
\[
\left| \frac{\partial^2 \phi}{\partial x_i \partial x_j}(p_\tau, q_\tau) \right| \leq C \phi(p_\tau, q_\tau)^{(m-2)/m}, \text{ for } i, j = 1, 2, 3.
\]
Hence:
\[
\| \eta_\tau \| \leq C \phi(p_\tau, q_\tau)^{(m-1)/m}, \| \nabla_0^2 \phi(p_\tau, q_\tau^{-1}) \|, \| \mathcal{M}_\tau \| \leq C \phi(p_\tau, q_\tau)^{(m-2)/m},
\]
and:
\[
\| W_\tau - W_\tau^t \|
\]
\[
= 4m \left| \left( (p_{r,3}-q_{r,3}+2(p_{r,2}q_{r,2}-p_{r,2}q_{r,1}))^{m-1} \right) \right|
\]
\[
\leq C \phi(p_\tau, q_\tau)^{(m-1)/m}.
\]
Therefore:
\[
\langle \mathcal{X}_\tau z_i(p_\tau), z_i(p_\tau) \rangle - \langle \mathcal{Y}_\tau z_i(q_\tau), z_i(q_\tau) \rangle \leq \tau C \left( \phi(p_\tau, q_\tau^{-1}) + \phi(p_\tau, q_\tau^{-1})^{2(m-2)/m} \right)
\]
and:
\[
\langle \mathcal{X}_\tau w_j(\tau \eta_\tau), w_j(\tau \eta_\tau) \rangle - \langle \mathcal{Y}_\tau w_j(\tau \eta_\tau), w_j(\tau \eta_\tau) \rangle \leq \tau^{1+2\alpha} \phi(p_\tau, q_\tau^{-1})^{2m+2m\alpha-4-2\alpha/m}
\]
where we have used (H1) and (H2).
Hence, we derive:
\[
\text{tr} \left( M(p_\tau, \tau \eta_\tau) \mathcal{X}_\tau - M(q_\tau, \tau \eta_\tau) \mathcal{Y}_\tau \right)
\]
\[
\leq \tau \left( \phi(p_\tau, q_\tau^{-1}) + \phi(p_\tau, q_\tau^{-1})^{2(m-2)/m} \right) + \tau^{1+2\alpha} \phi(p_\tau, q_\tau^{-1})^{2m+2m\alpha-4-2\alpha/m}
\]
converges to 0 as \( \tau \to \infty \) if \( m = 4 + 2\alpha \).

Let us now estimate the second term. By (15) with \( p = \hat{p} \) we have \( u(p_\tau) > v(q_\tau) \), hence by (H3) we derive:
\[
F(p_\tau, u(p_\tau), \tau \eta_\tau) - F(q_\tau, v(q_\tau), \tau \eta_\tau)
\]
\[
\leq C \nu_0 \left[ \| p_\tau \cdot q_\tau^{-1} \| + \tau \phi(p_\tau, q_\tau^{-1})^{1/m} \| \eta_\tau \| - h(u(p_\tau) - v(q_\tau)) \right]
\]
\[
\leq C \| p_\tau \cdot q_\tau^{-1} \| \nu_0 + \tau \phi(p_\tau, q_\tau^{-1}) - h(u(\hat{p}) - v(\hat{p}))].
\]
The last term converges to \( -h(u(\hat{p}) - v(\hat{p})) \) < 0 as \( \tau \to \infty \) which contradicts (17).
This ends the proof of the theorem. \( \square \)
Examples.

- **Hamilton-Jacobi equations.** The comparison result may be applied to Hamilton-Jacobi equations perturbed by a viscosity term:
  \[-\nu \Delta_0 u = f(p, u, \nabla u)\]
  where \(\nu\) is a positive constant and \(\Delta_0\) denotes the sub-elliptic Laplacian operator in \(H\).

- **Degenerate p-Laplacian type equations.** Consider the p-Laplacian operator for \(p > 2\):
  \[F(\eta, X) = - (||\eta||^{p-2} \text{tr}(X) + (p-2)||\eta||^{p-4} \text{tr}(X \eta \otimes \eta)) = -\text{tr} (M(\eta)X),\]
  where:
  \[M(\eta) := ||\eta||^{p-2} I + (p-2)||\eta||^{p-4} \eta \otimes \eta\]
  for \(\eta \neq 0\) and we let \(M(0) = 0\). Observe that we may write:
  \[M(\eta) = ||\eta||^{p-2} e_1 \otimes ||\eta||^{p-2} e_1 + ||\eta||^{p-2} e_2 \otimes \eta + (p-2)||\eta||^{p-4} \eta \otimes ||\eta||^{p-4} \eta.\]

  Hence, in view of Theorem 3.1, the comparison principle holds.

**Remark 1.** The following assumption, together with strict monotonicity in \(u\), is used in [11, Theorem 3.3] to derive a comparison principle for fully non-linear operators \(F(p, u, \nabla u, \nabla^2 u) = 0\) in the Euclidean framework: there is a modulus of continuity \(\omega_0 : [0, \infty) \to [0, \infty)\) so that
  \[F(q, r, \tau(p-q), Y) - F(p, r, \tau(p-q), X) \leq \omega_0 (\tau ||p-q||^2 + ||p-q||)\]
  for all \(p, q \in \Omega, r \in \mathbb{R}\) and \(X, Y\) in the space of (3 x 3)-symmetric matrices so that the next structural condition holds:
  \[-3\tau \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\tau \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},\]
  where \(I\) is the (3 x 3)-identity matrix. We claim that, as a function depending on Euclidean quantities, the operator (11) does not meet, in general, the estimate (18).

  Indeed, denote by \(\sigma(p)\) and \(\sigma^*(p)\) the following matrices:
  \[\sigma^T(p) := \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix}, \quad \sigma^*(p) := \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix}, \quad p = (x, y, z),\]
  and let:
  \[F(p, r, \xi, X) = -\text{tr} \left[ M(\sigma^*(p)\xi) \sigma(p)^T X \sigma(p) \right] + r\]
  where:
  \[M(\eta) = \omega(\eta) \otimes \omega(\eta), \quad \eta \in \mathbb{R}^3\]
  and:
  \[\omega(\eta) = ||\eta||^\alpha (1, 1).\]
  We assume that \(\alpha > 2\). Observe \(\|\omega(\eta)\| \leq \sqrt{2} ||\eta||^\alpha\) and so \(\omega\) verifies (H1). Also, (H2) and (H3) from page 9 are also verified.

  Consider the matrices:
  \[X_r = \tau^{-1/2} A^2 = \tau^{-1/2} I, \quad Y_r = \frac{3\tau^{1/2}}{3\tau - \tau^{-1/2}} A^2 = \frac{3\tau^{1/2}}{3\tau - \tau^{-1/2}} I,\]
  (21)
where:

\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

By [21, Theorem 7.7.6], the matrices \( X_\tau \) and \( Y_\tau \) satisfy (19).

Next, we define:

\[
(p_\tau)_1 = 2(p_\tau)_2, \quad (q_\tau)_1 = 2(q_\tau)_2, \quad (p_\tau)_3 = 2(q_\tau)_3, \quad (p_\tau)_2 = \sqrt{2}(q_\tau)_2, \quad (22)
\]

and

\[
[(q_\tau)_2]^2 = \frac{2\tau^{-1/2}}{3\tau - 2\tau^{-1/2}}, \quad (q_\tau)_3 = \frac{1}{\sqrt{\tau \ln(\tau)}}. \quad (23)
\]

Observe that:

\[
\tau \| p_\tau - q_\tau \| ^2 \to 0, \quad \text{but} \quad \tau^{1+\gamma}\| p_\tau - q_\tau \| ^2 \text{ does not tend to } 0, \quad (24)
\]
as \( \tau \to +\infty \), for all \( \gamma > 0 \).

The left hand side of (18) is given by:

\[
\mathcal{F}(q_\tau, r, \tau(p_\tau - q_\tau), Y_\tau) - \mathcal{F}(p_\tau, r, \tau(p_\tau - q_\tau), X_\tau)
\]

\[
\begin{equation}
= \tau^{-1/2}\| A\sigma (p_\tau) \omega (\tau \sigma^*(p_\tau)(p_\tau - q_\tau)) \|^2 - \frac{3\tau^{1/2}}{3\tau - \tau^{-1/2}} \| A\sigma (q_\tau) \omega (\tau \sigma^*(q_\tau)(p_\tau - q_\tau)) \|^2. \quad (25)
\end{equation}
\]

Note that

\[
A\sigma (p_\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2(p_\tau)_2 & 2(p_\tau)_1 \end{pmatrix}, \quad A\sigma (q_\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2(q_\tau)_2 & 2(q_\tau)_1 \end{pmatrix}.
\]

Then

\[
\| A\sigma (p_\tau) \omega (\tau \sigma^*(p_\tau)(p_\tau - q_\tau)) \|^2 = 2\tau^{2\alpha} ||(\sigma(p_\tau)(p_\tau - q_\tau)) ||^{2\alpha} \{1 + 2[(p_\tau)_1 - (p_\tau)_2]^2\},
\]

\[
\| A\sigma (q_\tau) \omega (\tau \sigma^*(q_\tau)(p_\tau - q_\tau)) \|^2 = 2\tau^{2\alpha} ||(\sigma(q_\tau)(p_\tau - q_\tau)) ||^{2\alpha} \{1 + 2[(q_\tau)_1 - (q_\tau)_2]^2\}.
\]

From (25) and the definitions of \( p_\tau \) and \( q_\tau \), we obtain:

\[
\begin{equation}
\mathcal{F}(q_\tau, r, \tau(p_\tau - q_\tau), Y_\tau) - \mathcal{F}(p_\tau, r, \tau(p_\tau - q_\tau), X_\tau)
\]

\[
\begin{equation}
= 2\tau^{2\alpha - 1/2} \left\{ 1 + 2[(p_\tau)_2]^2 \right\} \left( ||\sigma^*(p_\tau)(p_\tau - q_\tau)||^{2\alpha} - ||\sigma^*(q_\tau)(p_\tau - q_\tau)||^{2\alpha} \right) \]

\[
+ ||\sigma^*(q_\tau)(p_\tau - q_\tau)||^{2\alpha} \left( 1 + 2[(p_\tau)_2]^2 \right) - \frac{3\tau}{3\tau - \tau^{-1/2}} \left\{ 1 + 2[(q_\tau)_2]^2 \right\} \right\}. \quad (26)
\end{equation}
\]

Note that:

\[
\sigma^*(p_\tau)(p_\tau - q_\tau) = \begin{pmatrix} 1 & 0 & 2(p_\tau)_2 \\ 0 & 1 & -2(p_\tau)_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (p_\tau)_1 - (q_\tau)_1 \\ (p_\tau)_2 - (q_\tau)_2 \\ (p_\tau)_3 - (q_\tau)_3 \end{pmatrix}
\]

\[
= [(p_\tau) - (q_\tau)] + 2[(p_\tau)_3 - (q_\tau)_3]l_{p_\tau},
\]

\[
\sigma^*(q_\tau)(p_\tau - q_\tau) = [(p_\tau) - (q_\tau)] + 2[(p_\tau)_3 - (q_\tau)_3]l_{q_\tau},
\]

where:

\[
l_{p_\tau} = \begin{pmatrix} (p_\tau)_2 \\ -(p_\tau)_1 \\ 0 \end{pmatrix}, \quad l_{q_\tau} = \begin{pmatrix} (q_\tau)_2 \\ -(q_\tau)_1 \\ 0 \end{pmatrix}.
\]
By (22) and (23), it follows that:
\[ \|\sigma^*(p_r)(p_r - q_r)\|^2 - \|\sigma^*(q_r)(p_r - q_r)\|^2 \geq 0. \] (27)
and hence \( \|\sigma^*(p_r)(p_r - q_r)\|^{2\alpha} - \|\sigma^*(q_r)(p_r - q_r)\|^{2\alpha} \geq 0. \) Then by appealing to (26) and (27) we obtain
\[ F(q_r, r, \tau(p_r - q_r), Y_r) - F(p_r, r, \tau(p_r - q_r), X_r) \]
\[ \geq \frac{2\tau^{\beta-1/2}}{3\tau - \tau^{-1/2}} \left[ \tau^{1+\gamma}\|p_r - q_r\|^2 \right]^{\alpha} \|\sigma^*(q_r)(p_r - q_r)\|^{2\alpha} \frac{\tau^{\beta-2}}{\|p_r - q_r\|^{2\alpha}} \right]^{\alpha} \]
\[ \left[ (3\tau - \tau^{-1/2}) \{1 + 2([p_r]_2)^2\} - 3\tau \{1 + 2([q_r]_2)^2\} \right] \]
\[ = \frac{4\tau}{3\tau - \tau^{-1/2}} \|\sigma^*(q_r)(p_r - q_r)\|^{2\alpha} \tau^{\beta-2} \left[ \tau^{1+\gamma}\|p_r - q_r\|^2 \right]^{\alpha}, \] (28)
where \( \beta := \alpha(1 - \gamma) \) and we have used the definitions of \( p_r \) and \( q_r \). Finally, applying the inequality:
\[ \frac{\|\sigma^*(q_r)(p_r - q_r)\|^{2\alpha}}{\|p_r - q_r\|^{2\alpha}} \geq \|\sigma^*(q_r)^{-1}\|^{-2\alpha}, \]
and the choice \( \gamma = (\alpha - 2)/\alpha \) in (28), it follows that:
\[ F(q_r, r, \tau(p_r - q_r), Y_r) - F(p_r, r, \tau(p_r - q_r), X_r) \]
\[ \geq \frac{4\tau}{3\tau - \tau^{-1/2}} \|\sigma^*(q_r)^{-1}\|^{-2\alpha} \left[ \tau^{1+\gamma}\|p_r - q_r\|^2 \right]^{\alpha} \rightarrow +\infty \text{ as } \tau \rightarrow +\infty. \] (29)
In view of (24), the above expression (29) contradicts hypothesis (18).

4. Weak solutions for equations in divergence form. Throughout this section, we shall study second-order equations in divergence form as follows:
\[ -\text{div}_0(A(p, \nabla_0 u)) = f(p, u), \quad p \in \Omega, \] (30)
where \( \Omega \) is a John domain, \( A : \overline{\Omega} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and \( f : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R} \).

We refer the reader to the Appendix for the proofs of existence and uniqueness of weak solutions of equations like (30). We highlight the parallelism between the assumptions on \( A \) and \( f \) in (30) to obtain the mentioned results to the corresponding well-known conditions in the Euclidean scenario. Moreover, even do one can not apply directly the Euclidean results in the Heisenberg framework, the use of slightly modifications of Euclidean techniques works in the sub-Riemannian setting.

The notion of weak solutions is provided in the next definition.

**Definition 4.1.** A function \( u \in S^{1,1}_0(\Omega) \) is a weak solution to equation (30) if:
\[ \int_{\Omega} \langle A(p, \nabla_0 u), \nabla_0 v \rangle = \int_{\Omega} f(p, u)v \]
for all \( v \in S^{1,1}_0(\Omega) \).

4.1. **Equivalence of viscosity and weak solutions.** In this section, we shall prove the equivalence of the notions of viscosity and weak solutions to equations in divergence form. We first state the following regularity result for viscosity solutions.

**Theorem 4.2.** Let \( u \in C(\overline{\Omega}) \) be a viscosity supersolution to
\[ -\text{div}_0(A(p, \nabla_0 u)) = f(p, u), \quad p \in \Omega, \]
where $A$ is smooth and satisfies the polynomial growth rate ($Hw1$). Moreover, we assume that the matrix:

$$
\begin{pmatrix}
\frac{\partial A_1}{\partial \xi_1} & \frac{\partial A_1}{\partial \xi_2} \\
\frac{\partial A_2}{\partial \xi_1} & \frac{\partial A_2}{\partial \xi_2}
\end{pmatrix}
$$

is positive semi-definite, that the following symmetric property holds:

$$
\frac{\partial A_1}{\partial \xi_2} = \frac{\partial A_2}{\partial \xi_1},
$$

and finally that $f$ satisfies ($Hw2$). Then $u \in S_{1,\ell}^1(\Omega)$.

**Proof.** For each $\epsilon > 0$, consider the inf-convolution in the Heisenberg group:

$$
u^\epsilon(x) := \inf_{y \in \Omega} \left\{ u(y) + \frac{1}{2\epsilon} |x \cdot y - 1|_H^4 \right\}.
$$

Hence, there exists $r(\epsilon) \to 0$ such that:

$$
u^\epsilon(x) := \inf_{y \in B_H(x, r(\epsilon)) \cap \Omega} \left\{ u(y) + \frac{1}{2\epsilon} |x \cdot y - 1|_H^4 \right\}.
$$

By the results in [37], the sequence $\{\nu^\epsilon\}$ is increasing, (Euclidean) semi-concave and converges uniformly to $u$ in $\Omega$. Hence:

$$
\psi(p) = \nu^\epsilon + C\|p\|^2
$$

is concave. As in [32, Lemma 2.1], we derive that $\nu^\epsilon$ is a viscosity supersolution of:

$$
-\text{div}_0(A(p, \nabla_0 u^\epsilon)) = f^\epsilon(p, u),
$$
in $\Omega_\epsilon := \{p \in \Omega : \text{dist}_H(p, \partial \Omega) > r(\epsilon)\}$, and where:

$$
f^\epsilon(p, r) := \inf_{q \in B_H(p, r(\epsilon))} f(q, r).
$$

Moreover since $A$ is smooth and $\nu^\epsilon$ is almost everywhere twice differentiable, we have:

$$
-\text{div}_0(A(p, \nabla_0 u^\epsilon)) \geq f^\epsilon(p, u) \quad \text{ (32)}
$$
a.e. in $\Omega_\epsilon$ (indeed, by Aleksandrov’s Theorem, $(\nabla_0 u^\epsilon(p), \nabla_0^{2,*} u^\epsilon(p)) \in \mathcal{J}^{2,-}(u^\epsilon, p)$ for a.e. $p$ in $\Omega_\epsilon$). Let $\varphi \in C_0^\infty(\Omega)$ and take $\epsilon$ small enough so that $K := \text{supp} \varphi \subset \Omega_\epsilon$. Hence, (32) implies:

$$
-\int_{\Omega} \text{div}_0(A(p, \nabla_0 u^\epsilon)) \varphi \geq \int_{\Omega} f^\epsilon(x, u^\epsilon) \varphi.
$$

Choose $\{\psi_j\}$ as a sequence of mollifications of $\psi$. Then, $\psi_j$ is smooth and concave in $\Omega$, $\psi_j \to \psi$ in $S_{1,\ell}^q(\Omega)$ for all $q \in [1, \infty)$, $\|\nabla_0 \psi_j\|_{L^\infty(K)}$ is uniformly bounded (since the Euclidean gradients are locally uniformly bounded [20]), and by [13, pag. 242]:

$$
\nabla_0^{2,*} \psi_j \to \nabla_0^{2,*} \psi \quad \text{a.e as } j \to \infty.
$$

We let:

$$
u^\epsilon_j(p) := \psi_j(p) + \frac{C}{2\epsilon}\|p\|^2, \quad p \in \Omega.
$$

Integration by parts and Dominated Convergence Theorem give:

$$
\lim_{j \to \infty} \int_{\Omega} -\text{div}_0(A(p, \nabla_0 u^\epsilon_j)) \varphi = \int_{\Omega} \langle A(p, \nabla_0 u^\epsilon), \nabla \varphi \rangle.
$$

(35)
In the sequel we shall prove that:
\[
\int_{\Omega} \langle A(p, \nabla_0 u^\epsilon), \nabla_0 \varphi \rangle \geq -\int_{\Omega} \text{div}_0 (A(p, \nabla_0 u^\epsilon)) \varphi. \tag{36}
\]

From $\nabla^2 u^{\epsilon,j} \leq \frac{1}{2} I$, the decomposition (see [2, Lemma 3.1]):
\[
\nabla^2_{0^*} u^{\epsilon,j} = \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix} \nabla^2 u^{\epsilon,j} \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix}^T + \mathcal{M}(p, \nabla_0 u^{\epsilon,j})
\]
where $\mathcal{M} = 0$ in the Heisenberg scenario, and by the uniform (in $j$) local boundedness of $\nabla_0 u^{\epsilon,j}$, we obtain up to a multiplicative constant:
\[
\langle \nabla^2_{0^*} u^{\epsilon,j} \eta, \eta \rangle = \left\langle \nabla^2 u^{\epsilon,j} \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix}^T \eta, \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix}^T \eta \right\rangle 
\leq \frac{1}{\epsilon} \left\| \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix}^T \eta \right\|^2 \leq \left( \frac{1}{\epsilon} + C_\epsilon \right) \| \eta \|^2.
\]

By assumption, the matrix:
\[
\begin{pmatrix}
\partial A_1 & \partial A_1 \\
\partial A_2 & \partial A_2 \\
\partial \xi_1 & \partial \xi_2 \\
\partial \xi_1 & \partial \xi_2
\end{pmatrix}
\]
is positive semidefinite. Hence by the local boundedness of $\nabla_0 u^{\epsilon,j}$ (with a bound independent of $j$) and recalling (31):

\[
-\text{div}_0 \left( A(p, \nabla_0 u^{\epsilon,j}) \right) = -\text{tr} \left( \begin{pmatrix} \partial A_1 & \partial A_1 \\
\partial A_2 & \partial A_2 \\
\partial \xi_1 & \partial \xi_2 \\
\partial \xi_1 & \partial \xi_2
\end{pmatrix} \nabla^2_{0^*} u^{\epsilon,j} \right) -\text{div}_0 \left( A(\cdot, \nabla_0 u^{\epsilon,j}) \right) \geq c_\epsilon \text{ in } K,
\]
for some $c_\epsilon \in \mathbb{R}$ and where $\text{div}_0 (A(\cdot, \nabla_0 u^{\epsilon,j}))$ is the horizontal divergence of $p \rightarrow A(p, \xi)$ with respect to $p$ at $\xi = \nabla_0 u^{\epsilon,j}$. By Fatou’s Lemma:

\[
\liminf_{j \to \infty} \int_{\Omega} -\text{div}_0 \left( A(p, \nabla_0 u^{\epsilon,j}) \right) \varphi \geq \int_{\Omega} \liminf_{j \to \infty} \left( -\text{div}_0 \left( A(p, \nabla_0 u^{\epsilon,j}) \right) \right) \varphi.
\]

Hence, combining (35) and the pointwise convergence of $-\text{div}_0 \left( A(p, \nabla_0 u^{\epsilon,j}) \right)$ to $-\text{div}_0 \left( A(p, \nabla_0 u^\epsilon) \right)$ (which follows from the pointwise convergence of $\nabla^2_{0^*} \psi_j$ to $\nabla_{0^*} \psi$, (34) and the smoothness of $A$) we derive (36). So far, we have obtained:

\[
\int_{\Omega} \langle A(p, \nabla_0 u^\epsilon), \nabla_0 \varphi \rangle \geq \int_{\Omega} f_e(x, u^\epsilon) \varphi
\]
for any $\varphi \in C_0^\infty(\Omega)$. As in [32, Lemma 2.3], we have the following Caccioppoli’s estimate:

\[
\int_{\Omega} |\nabla_0 u^\epsilon|^4 \zeta^t \leq C(\Omega, \alpha_2, \gamma_2) \left( \text{osc}_K u^\epsilon \right)^t \int_K \|\nabla_0 \zeta\|^t + \text{osc}_K u^\epsilon \tag{37}
\]
for all $\zeta \in C_0^\infty(\Omega)$ and where $K := \text{supp} \zeta$. Since $u^\epsilon$ is increasing in $\epsilon$ and converges uniformly to $u$ in $\Omega$, we have:

\[
\text{osc}_K u^\epsilon \leq \sup_K u - \inf_K u^\epsilon \leq \epsilon_0
\]
for all $\epsilon < \epsilon_0$. Hence, by (37), the sequence $\{\nabla_0 u^\epsilon\}$ is uniformly bounded in $L^t$ on compact sets. The previous fact together with the uniform convergence of $u^\epsilon$ to $u$ in $\Omega$ and the boundedness of $u$ in $\Omega$ imply that $\{u^\epsilon\}$ is a bounded sequence.
in \( S^{1,t}(K) \) for each compact \( K \subset \Omega \). Hence, up to a subsequence, \( \{u^\varepsilon\} \) converges weakly to \( v \in S^{1,t}_{loc}(\Omega) \) in \( S^{1,t}(\Omega) \) on compact set. Since the embedding \( S^{1,t} \) in \( L^t \) is compact for all \( t \) (see [34, Corollary 4.1.13]), we derive that \( u = v \) a.e. and hence \( u \in S^{1,t}_{loc}(\Omega) \). \(\square\)

In the remaining of this section, we shall prove the equivalence of the notions of viscosity and weak solutions. To do so, we shall assume the following:

\textbf{(H)} The solution from Theorem 6.1 is continuous up to the boundary.

Moreover, we assume that \( A \) and \( f \) satisfy the structure assumptions (Hw1)-(Hw5), where in (Hw4) we require that \( A \) is strictly monotone, that \( A \) has continuous partial derivatives satisfying (31), that:

\[
M = \begin{pmatrix}
\frac{\partial A_1}{\partial \xi_1} & \frac{\partial A_1}{\partial \xi_2} \\
\frac{\partial A_2}{\partial \xi_1} & \frac{\partial A_2}{\partial \xi_2}
\end{pmatrix}, \quad F = \text{div}_0 (A(\cdot, \nabla_0 u)) + f(p, u)
\]

verify (H1)-(H3) from Section 3 and that \( M \geq 0 \).

\textbf{Remark 2.} Assumption \textbf{(H)} may be satisfied if \( t > 4 \). Moreover, under continuous Dirichlet condition \( \vartheta \in C(\partial \Omega) \), the continuity up to the boundary of the solution from Theorem 6.1 (which is unique by Theorem 6.2) may be obtained for some classes of divergence form equations as in [38, Theorem 3.13].

\textbf{Theorem 4.3.} Consider the following equation in divergence form:

\[
- \text{div}_0 (A(p, \nabla_0 u)) = f(p, u) \text{ in } \Omega,
\]

(38)

If \( u \in C(\Omega) \) is a weak solution to (38), then \( u \) is a viscosity solution as well. Conversely, if \( u \in C(\overline{\Omega}) \) is a viscosity solution to (38), then \( u \) is a weak solution.

\textbf{Proof.} Let \( u \in S^{1,t}(\Omega) \cap C(\Omega) \) be a weak solution to (38). Suppose that \( u \) is not a viscosity solution. Without loss of generality, assume that \( u \) is not a viscosity supersolution. Then there exist \( p_0 \in \Omega \) and \( \varphi \in C^2_H(\Omega) \) such that:

\[
u(p_0) = \varphi(p_0), \quad u(p) > \varphi(p) \quad \text{for all } p \neq p_0,
\]

(39)

but:

\[- \text{div}_0 (A(p_0, \nabla_0 \varphi(p_0))) < f(p_0, u(p_0)).\]

By continuity, there is \( \delta > 0 \) such that:

\[- \text{div}_0 (A(p, \nabla_0 \varphi(p))) < f(p, u(p)). \]

(40)

for \( p \in B_H(p_0, \delta) \). Let:

\[
m =: \inf_{\partial B_H(p_0, \delta)} (u - \varphi).
\]

By (39), \( m > 0 \). Let \( \tilde{\varphi} := \varphi + m/2 \). By (40), \( \tilde{\varphi} \) is a weak subsolution of (38) in \( B_H(p_0, \delta) \). Moreover, \( u \) is a weak supersolution of (38) in \( B_H(p_0, \delta) \) and \( u > \tilde{\varphi} \) on \( \partial B_H(p_0, \delta) \). By Theorem 6.2 and the continuity of \( u \) and \( \varphi \) in \( B_H(p_0, \delta) \) we arrive at the contradiction \( u \geq \tilde{\varphi} \) in \( B_H(p_0, \delta) \). Hence, \( u \) is a viscosity solution to (38).

To prove the converse, assume that \( u \in C(\overline{\Omega}) \) is a viscosity solution to (38). From Theorem 6.1 and assumption \textbf{(H)}, there is a weak solution \( v \in S^{1,t}(\Omega) \cap C(\Omega) \) to (38) satisfying:

\[v - u \in S^{1,t}_{0}(\Omega) \text{ and } v = u \text{ on } \partial \Omega.\]
Here, recall that $u \in S_{loc}^{1,1}(\Omega)$ by Theorem 4.2. Since we already proved that weak solutions are viscosity solutions, we derive by the comparison result Theorem 3.1, that $u = v$ in $\Omega$ and thus $u$ is a weak solution.

**Remark 3.** Observe that we do not need continuity up to the boundary for the implication weak $\Rightarrow$ viscosity. To derive that $u \in C(\Omega)$ (and further regularity such as $C^1(\Omega)$ or $C^{1,\alpha}_{loc}(\Omega)$) for weak solutions to some classes of quasilinear equations, we refer the reader to [7, Theorem 1.1], [38, Theorem 3.13] and [39, Theorem 1] and the references therein.

With respect to the imposed regularity $C(\Omega)$ on viscosity solutions in the implication viscosity $\Rightarrow$ weak, we point out that under continuous Dirichlet conditions, the continuity up to the boundary has been derived for linear equations in divergence form in [35]. Moreover, Lipschitz viscosity solutions (and hence $S_{loc}^{1,1}(\Omega)$ solutions) have been obtained in [31] for some classes of second order equations in non-divergence form.

5. **A Radó type result for homogeneous equations.** The classical theorem of T. Radó says that, if the continuous complex function $f(z)$ is analytic when $f(z) \neq 0$, then it is analytic in its whole domain of definition. This result has been generalized in many ways. In [24] a Radó type theorem for the solutions of the so-called $p$-harmonic equation was proved, this is

$$\text{div}(|\nabla u|^{p-2}\nabla u) = 0, \text{ in } \mathcal{O},$$

where $\mathcal{O}$ is open bounded set of $\mathbb{R}^n$. They show that a function $u \in C^1(\mathcal{O})$ which is $p$-harmonic in $\mathcal{O}\setminus\{x \in \mathcal{O} : u(x) = 0\}$ is indeed $p$-harmonic in the whole domain $\mathcal{O}$.

The objective of this section is to prove of following theorem

**Theorem 5.1.** Let $\Omega$ be a John domain. Let $u \in C^1(\Omega) \cap C(\bar{\Omega})$. Assume that all of the hypothesis of Theorem 4.3 are satisfied and that $A(\cdot, 0)$ is divergence free:

$$-\text{div}_0(A(\cdot, 0)) = 0.$$

If $u$ is a weak solution of:

$$-\text{div}_0 (A(p, \nabla_0 u)) = 0,$$

in the set:

$$\Omega\setminus\{p \in \Omega : \nabla_0 u(p) = 0\},$$

then $u$ is weak solution of equation (41) in the whole $\Omega$.

To prove the previous theorem we need a different definition for viscosity solutions.

**Definition 5.2.** Let $\mathcal{F}$ be a continuous function in $\Omega \times \mathbb{R} \times \mathbb{R}^2 \times S^2(\mathbb{R})$. An upper (resp. lower) semicontinuous function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ (resp. $\mathbb{R} \cup \{+\infty\}$) is a subsolution (resp. supersolution) of:

$$\mathcal{F}(p, u, \nabla_0 u, \nabla_0^2 u) = 0,$$

in $\Omega$ if for any $p_0 \in \Omega$:

(i) $u(p) < +\infty$ (resp. $u(p) > -\infty$);  
(ii) $u(p_0) = \varphi(p_0)$;  
(iii) $u(p) < \varphi(p)$ (resp. $u(p) > \varphi(p)$), when $p \neq p_0$;

in $\Omega$ if for any $p_0 \in \Omega$:
Finally, a viscosity solution is both a viscosity subsolution and a viscosity supersolution.

**Proposition 1** (Equivalence of definitions). Assume that \( \mathcal{F} \) satisfy

1. \( \mathcal{F}(p, r, 0, 0) = 0 \);
2. \( \mathcal{F} \) is elliptic, this is:

\[
\mathcal{F}(p, r, 0, \mathcal{Y}) \geq \mathcal{F}(p, r, 0, \mathcal{X}),
\]

for all \( p, q \in \Omega, r \in \mathbb{R} \) and \( \mathcal{X}, \mathcal{Y} \in S^2(\mathbb{R}) \) such that \( \mathcal{Y} \leq \mathcal{X} \).

An upper (resp., lower) semicontinuous function \( u \) is a subsolution (resp., supersolution) defined as in Definition 2.3 (in \( \Omega \)) if and only if it is a subsolution (resp., supersolution) in \( \Omega \) in the sense of Definition 5.2.

**Proof.** The proof follows the line of [14, Proposition 3.1] and we just give some details for convenience of the reader. First, Definition 2.3 implies Definition 5.2.

To prove the converse, assume that \( u \) is a subsolution in the sense of Definition 5.2. Suppose there are a smooth function \( \varphi \) and \( \hat{p} \in \Omega \) such that:

\[
\max_{\Omega} (u - \varphi)(p) = (u - \varphi)(\hat{p}) = 0.
\]

Define:

\[
\Psi_{\epsilon}(p, q) := u(p) - \frac{1}{\epsilon}|q^{-1} \cdot p|_{\mathcal{H}}^{4} - \varphi(q),
\]

and hence the envelope:

\[
\Psi^{*}(p, q) := \lim_{\epsilon \to 0} \sup_{\epsilon} \Psi_{\epsilon}(p, q) = \begin{cases} u(p) - \varphi(p) & \text{if } p = q \\ -\infty & \text{if } p \neq q \end{cases}
\]

attains a maximum at \((p^{*}, q^{*})\). By the convergence of maximizers ([19, Lemma 2.2.5]), we may take \( p^{*}, q^{*} \) converging to \( \hat{p}, \hat{p} \) respectively as \( \epsilon \to 0 \) such that \( \Psi_{\epsilon} \) attains a maximum at \((p^{*}, q^{*})\).

It follows that \( q \mapsto -(1/\epsilon)|q^{-1} \cdot p^{*}|_{\mathcal{H}}^{4} - \varphi(q) \) has a maximum at \( q^{*} \), which, by [14, Proposition 2.2], implies that:

\[
-\frac{1}{\epsilon} \nabla_{0}^{q} f(p^{*}, q^{*}) = \nabla_{0} \varphi(q^{*});
\]

\[
-\frac{1}{\epsilon} \nabla_{0}^{2+q} f(p^{*}, q^{*}) \leq \nabla_{0}^{2+q} \varphi(q^{*}),
\]

where \( f(p, q) = |q^{-1} \cdot p|_{\mathcal{H}}^{4} \). If \( \nabla_{0} \varphi(q^{*}) \neq 0 \) for a subsequence of \( \epsilon \to 0 \), then the maximality of \( \Psi \) at \((p^{*}, q^{*})\) implies that:

\[
p \mapsto u(p) - \frac{1}{\epsilon} f(p^{*}, q^{*}) - \varphi(p \cdot (p^{*})^{-1} \cdot q^{*}),
\]

attains a maximum at \( p^{*} \in \Omega \). Denote \( \varphi^{*}(p) = \varphi(p \cdot (p^{*})^{-1} \cdot q^{*}) \). We apply Definition 5.2 to get

\[
\mathcal{F}(p^{*}, \varphi^{*}(p^{*}), \nabla_{0} \varphi^{*}(p^{*}), \nabla_{0}^{2+q} \varphi^{*}(p^{*})) \leq 0.
\]

Since the derivative of the right multiplication tends to 0 as \( \epsilon \to 0 \) and its second derivatives are 0, we have:

\[
\nabla_{0} \varphi^{*}(p^{*}) \to \nabla_{0} \varphi(\hat{p}) \quad \text{and} \quad \nabla_{0}^{2+q} \varphi^{*}(p^{*}) \to \nabla_{0}^{2+q} \varphi(\hat{p}) \quad \text{as} \quad \epsilon \to 0.
\]
Hence:
\[ F(\hat{p}, \varphi(\hat{p}), \nabla_0 \varphi(\hat{p}), \nabla_0^{2, \ast} \varphi(\hat{p})) \leq 0. \]
On the other hand, if \( \nabla_0 \varphi(q^\epsilon) = 0 \) for all sufficiently small \( \epsilon > 0 \), then from (42) we have that \( \nabla_0 f(p^\epsilon, q^\epsilon) = 0 \), which by [14, Proposition 2.1] yields that:
\[ p_i^\epsilon = q_i^\epsilon \text{ for } i = 1, 2. \]
Hence:
\[ \nabla_0 \varphi(q^\epsilon) = 0 \text{ and } \nabla_0^{2, \ast} \varphi(q^\epsilon) = 0. \]
On the other hand, by passing to the limit in (42) and (43), we have
\[ \nabla_0 \varphi(\hat{p}) = 0 \text{ and } \nabla_0^{2, \ast} \varphi(\hat{p}) \geq 0 \] (44)
Hence by the first assumption on \( F \):
\[ F(\hat{p}, \varphi(\hat{p}), \nabla_0 \varphi(\hat{p}), \nabla_0^{2, \ast} \varphi(\hat{p})) \leq 0. \]

Now we are ready to prove the Proposition 5.1

**Proof of Theorem 5.1.** Suppose that \( u \in C^1_H(\Omega) \cap C(\Omega) \). We first prove that Definition 5.2 is satisfied. Let \( p_0 \in \Omega \) and suppose that \( \varphi \) is a test function, touching \( u \) from below at the point \( p_0 \). By [14, Proposition 2.2]:
\[ \nabla_0 \varphi(p_0) = \nabla_0 u(p_0). \]
Hence, if \( \nabla_0 u(p_0) = 0 \), there is nothing to prove. On the other hand, if \( \nabla_0 u(p_0) \neq 0 \), then it is clear, according to Definition 5.2, that \( -\text{div}_0(A(p, \nabla_0 \varphi(p_0))) \leq f(p_0, u(p_0)) \).
Thus \( u \) is a viscosity supersolution at \( p_0 \) in the sense of the Definition 5.2. Since \( p_0 \) is arbitrary in \( \Omega \), we obtain that \( u \) is a viscosity supersolution in the whole \( \Omega \). Similarly, \( u \) is seen to be a viscosity subsolution in the sense of the Definition 5.2. Now Proposition 1 says that \( u \) is a viscosity solution in the sense of Definition 2.3. Finally, an application of Theorem 4.3 yields the desired result.

6. **Appendix.**

6.1. **Existence of weak solutions.** Throughout this section, the set \( \Omega \) is a John domain in the Heisenberg group. We make the following assumptions for all \( p \in \Omega \) and \( \xi, \xi_1, \xi_2 \in \mathbb{R}^3 \):

(Hw1) There exist \( \alpha_1 \geq 0 \) and \( \gamma_1 \in L^{t/(t-1)}_{\text{loc}}(\Omega, \mathbb{R}) \), \( t > 1 \) such that:
\[ \|A(p, \xi)\| \leq \alpha_1 \|\xi\|^{t-1} + \gamma_1(p). \]

(Hw2) There are \( \alpha_2 \geq 0 \) and \( \gamma_2 \in L^{t/(t-1)}_{\text{loc}}(\Omega, \mathbb{R}) \), \( t > 1 \) such that:
\[ |f(p, r)| \leq \alpha_2 |r|^{t-1} + \gamma_2(p). \]

(Hw3) There exists \( \beta > 0 \) so that:
\[ \langle A(p, \xi), \xi \rangle \geq \beta \|\xi\|^t. \]

(Hw4) The operator \( A \) is continuous in \( \xi \) and monotone, that is:
\[ \langle A(p, \xi_1) - A(p, \xi_2), \xi_1 - \xi_2 \rangle \geq 0. \]
\( f = f(p, r) \) is non-increasing and continuous in \( r \).

**Remark 4.** Observe that the assumptions (Hw1) and (Hw2) imply that the integrals in Definition 4.1 are well-defined.

In the following theorem, we prove existence of weak solutions.

**Theorem 6.1.** There exists \( u \in \mathcal{S}^{1,\eta}(\Omega) \) solving equation (30).

**Proof.** Firstly, we shall prove that there is a function \( u \in \mathcal{K}_{\psi, \vartheta}(\Omega) := \{ v : v \geq \psi \text{ a.e. in } \Omega, v - \vartheta \in \mathcal{S}_{\text{loc}}^{1,\eta}(\Omega), v = \vartheta \text{ on } \partial \Omega \} \), where \( \psi : \Omega \to [-\infty, +\infty], \vartheta : \Omega \to \mathbb{R} \) and \( \vartheta \in \mathcal{S}_{\text{loc}}^{1,\eta}(\Omega) \), such that

\[
\int_{\Omega} \langle A(p, \nabla_0 u), \nabla_0 v - \nabla_0 u \rangle \geq \int_{\Omega} f(p, u) \cdot (v - u),
\]

whenever \( v \in \mathcal{K}_{\psi, \vartheta}(\Omega) \). Let \( X := \mathcal{S}^{1,\eta}(\Omega), X' \) its dual, and:

\[
\mathbb{K} := \{ u \in X : u \in \mathcal{K}_{\psi, \vartheta}(\Omega) \}.
\]

It is clear that \( \mathbb{K} \) is convex and closed. Next, define the operator \( \mathcal{A} : \mathbb{K} \to X' \) by:

\[
\langle \mathcal{A}(v), u \rangle := \int_{\Omega} \langle A(p, \nabla_0 v), \nabla_0 u \rangle - \int_{\Omega} f(p, v) \cdot u
\]

for all \( u \in X \) and \( v \in \mathbb{K} \). Observe that \( \mathcal{A}(v) \in X' \) by assumptions (Hw1), (Hw2) and Hölder’s inequality. Also, \( \mathcal{A} \) is monotone, since \( f \) is decreasing in the second variable and \( \mathcal{A} \) is monotone.

To show that \( \mathcal{A} \) is coercive, let \( v, \phi \in \mathbb{K} \). Then:

\[
\langle \mathcal{A}(v) - \mathcal{A}(\phi), v - \phi \rangle \geq \int_{\Omega} \langle A(p, \nabla_0 v), \nabla_0 v - \nabla_0 \phi \rangle - \int_{\Omega} \langle A(p, \nabla_0 \phi), \nabla_0 v - \nabla_0 \phi \rangle
\]

By Hypothesis (Hw1), (Hw3) and by Hölder’s inequality, we deduce:

\[
\int_{\Omega} \langle A(p, \nabla_0 v), \nabla_0 v - \nabla_0 \phi \rangle - \int_{\Omega} \langle A(p, \nabla_0 \phi), \nabla_0 v - \nabla_0 \phi \rangle \\
\geq \beta \left( \| \nabla_0 v \|^2_t \right) - \alpha_1 \int_{\Omega} \| \nabla_0 v \|^2_{t-1} \| \nabla_0 \phi \| - \alpha_1 \int_{\Omega} \| \nabla_0 \phi \|^2_{t-1} \| \nabla_0 v \| \\
- \| \nabla_0 \phi \|^2_t \left( \int_{\Omega} \gamma_1^{t/(t-1)} \right)^{\frac{t-1}{t}} - \| \nabla_0 v \|^2_t \left( \int_{\Omega} \gamma_1^{t/(t-1)} \right)^{\frac{t-1}{t}} \\
\geq \beta 2^{t-1} \| \nabla_0 v - \nabla_0 \phi \|^2_t - \alpha_1 \| \nabla_0 \phi \|^2_t \left( \int_{\Omega} \gamma_1^{t/(t-1)} \right)^{\frac{t-1}{t}} \\
- \alpha_1 \| \nabla_0 \phi \|^2_{t-1} \left( \int_{\Omega} \gamma_1^{t/(t-1)} \right)^{\frac{t-1}{t}} + \| \nabla_0 \phi - \nabla_0 v \|^2_t \left( \int_{\Omega} \gamma_1^{t/(t-1)} \right)^{\frac{t-1}{t}} \\
- 2 \| \nabla_0 \phi \| \left( \int_{\Omega} \gamma_1^{t/(t-1)} \right)^{\frac{t-1}{t}},
\]

where we used that for all \( s > 0 \) and all \( a, b \geq 0 \), it holds:

\[
(a + b)^s \leq \max\{1, 2^{s-1}\}(a^s + b^s).
\]

On the other hand, since \( v \) and \( \phi \) belong to \( \mathbb{K} \), it follows that:

\[
\| v - \phi \|_t = \| v - \vartheta + \vartheta - \phi \|_t \leq C \| \nabla_0 v - \nabla_0 \phi \|_t,
\]
by Sobolev-Poincaré inequality (Theorem 2.1). Hence, if \( \|v - \phi\|_{S^1, r}(\Omega) \to \infty \), then the same happens to \( \|\nabla_0 \phi - \nabla_0 v\|_{l} \). In order to finish the proof that \( \mathcal{A} \) is coercive, we proceed to estimate the right hand side in (47) when divided by \( \|v - \phi\|_{S^1, r}(\Omega) \).

- Firstly, we have by Poincaré inequality:
  \[
  \frac{\beta^{2t-1} \|\nabla_0 v - \nabla_0 \phi\|^t_t}{\|v - \phi\|_{S^1, r}(\Omega)} \geq C \beta^{2t-1} \|\nabla_0 v - \nabla_0 \phi\|^{t-1}_t. \tag{48}
  \]

  Next, observe that:
  \[
  - \alpha_1 \|\nabla_0 \phi\|_t \|\nabla_0 \phi - \nabla_0 v\|^{t-1}_t \geq -\alpha_1 \|\nabla_0 \phi\|_t \|\nabla_0 \phi - \nabla_0 v\|^{t-2}_t. \tag{49}
  \]

  If \( t < 2 \), then:
  \[
  -\alpha_1 \|\nabla_0 \phi\|_t \|\nabla_0 \phi - \nabla_0 v\|^{t-2}_t \to 0 \text{ as } \|v - \phi\|_{S^1, r}(\Omega) \to \infty.
  \]

  In contrast, when \( t \geq 2 \), we use (48) to dominate (49). Hence, in any case:
  \[
  \frac{\beta^{2t-1} \|\nabla_0 v - \nabla_0 \phi\|^t_t}{\|v - \phi\|_{S^1, r}(\Omega)} - \frac{\alpha_1 \|\nabla_0 \phi\|_t \|\nabla_0 \phi - \nabla_0 v\|^{t-1}_t}{\|v - \phi\|_{S^1, r}(\Omega)} \to \infty \text{ as } \|v - \phi\|_{S^1, r}(\Omega) \to \infty.
  \]

- For the term:
  \[
  -\alpha_1 \|\nabla_0 \phi\|^{t-1}_t (\|\nabla_0 \phi\|_t + \|\nabla_0 \phi - \nabla_0 v\|_t) - \|\nabla_0 \phi - \nabla_0 v\|_t \left( \int_{\Omega} \gamma_1^{t/(t-1)} \right)^{\frac{t-1}{t}} \tag{50}
  \]

  we obtain the lower bound:
  \[
  -\alpha_1 \|\nabla_0 \phi\|^{t-1}_t (\|\nabla_0 \phi\|_t + \|\nabla_0 \phi - \nabla_0 v\|_t) - \|\nabla_0 \phi - \nabla_0 v\|_t \left( \int_{\Omega} \gamma_1^{t/(t-1)} \right)^{\frac{t-1}{t}} \]

  \[
  \geq -\alpha_1 \frac{\|\nabla_0 \phi\|_t}{\|v - \phi\|_{S^1, r}(\Omega)} - \alpha_1 \|\nabla_0 \phi\|^{t-1}_t - \left( \int_{\Omega} \gamma_1^{t/(t-1)} \right)^{\frac{t-1}{t}}.
  \]

  Therefore, the term (50) remains bounded and hence (48) dominates. Finally, the last term in (47) divided by \( \|v - \phi\|_{S^1, r}(\Omega) \) converges to zero.

Combining (46), (47) and the above estimates, we derive:
\[
\frac{\langle \mathcal{A}(v) - \mathcal{A}(\phi), v - \phi \rangle}{\|v - \phi\|_{S^1, r}(\Omega)} \to \infty
\]

as \( \|v - \phi\|_{S^1, r}(\Omega) \to \infty \). This finishes the proof that \( \mathcal{A} \) is coercive.

So far, we have that \( \mathcal{A} \) is monotone, coercive and moreover demicontinuous (continuous from \( \mathbb{K} \) with the norm topology of \( S^1, r(\Omega) \) to \( X' \) with the weak topology). By [26, Corollary III 1.8] there exists \( u \in \mathbb{K} \) such that:
\[
\langle \mathcal{A}(u), v - u \rangle \geq 0,
\]

for all \( v \in \mathbb{K} \). This means that there is a function \( u \in \mathcal{K}_{\psi, \varphi} \) such that
\[
\int_{\Omega} \langle \mathcal{A}(p, \nabla_0 u), \nabla_0 v - \nabla_0 u \rangle \geq \int_{\Omega} f(p, u) \cdot (v - u),
\]

for all \( v \in \mathcal{K}_{\psi, \varphi} \). Choosing \( \psi \equiv -\infty \) and being \( u \in \mathcal{K}_{\psi, \varphi} \) a solution of the corresponding obstacle problem, we have \( u + \varphi \) and \( u - \varphi \) are in \( \mathcal{K}_{\psi, \varphi} \) for all \( \varphi \in \mathcal{C}_0^\infty(\Omega) \). Then:
\[
\int_{\Omega} \langle \mathcal{A}(p, \nabla_0 u), \nabla_0 \varphi \rangle \geq \int_{\Omega} f(p, u) \cdot \varphi
\]
and:
\[ \int_{\Omega} \langle A(p, \nabla_0 u), \nabla_0 \varphi \rangle \leq \int_{\Omega} f(p, u) \cdot \varphi. \]
That is:
\[ \int_{\Omega} \langle A(p, \nabla_0 u), \nabla_0 \varphi \rangle = \int_{\Omega} f(p, u) \cdot \varphi \]
as desired. We conclude by density. □

**Remark 5.** Observe that the weak solution \( u \) founded above satisfies the weak boundary condition \( u - \vartheta \in S^1_0(\Omega) \) and the Dirichlet boundary value \( u = \vartheta \) on \( \partial \Omega \). Moreover, when \( t > 4 \), we have by horizontal Sobolev imbedding that \( u \in C(\Omega) \).

### 6.2. Comparison principle for weak solutions.

In what follows, for any \( u, v \in S^{1,t}(\Omega) \), we mean by \( u \leq v \) on \( \partial \Omega \) that \( u \leq v \) almost everywhere on a neighbourhood of \( \partial \Omega \). Hence, we assume that \( u \) and \( v \) are defined in \( \Omega \).

**Theorem 6.2.** Let \( \Omega \) be a John domain. Let us assume that \( A \) is strictly monotone in \( \xi \) and \( f \) is non-increasing in \( r \). Let \( u \in S^{1,t}(\Omega) \) be a subsolution of (30) and \( v \in S^{1,t}(\Omega) \) be a supersolution of (30). If \( u \leq v \) in \( \partial \Omega \), then \( u \leq v \) in \( \Omega \).

The proof is a straightforward adaptation of the results in [36, Section 3].

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