ASYMPTOTIC ANALYSIS AND ENERGY QUANTIZATION FOR THE LANE-EMDEN PROBLEM IN DIMENSION TWO

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Abstract. We complete the study of the asymptotic behavior, as $p \to +\infty$, of the positive solutions to

$$
\begin{align*}
-\Delta u &= u^p \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega \\
u &> 0 \quad \text{in } \Omega
\end{align*}
$$

when $\Omega$ is any smooth bounded domain in $\mathbb{R}^2$, started in [4]. In particular we show quantization of the energy to multiples of $8\pi e$ and prove convergence to $\sqrt{e}$ of the $L^\infty$-norm, thus confirming the conjecture made in [4].

1. Introduction

This paper focuses on the asymptotic analysis, as $p \to +\infty$, of families of solutions to the Lane-Emden problem

$$
\begin{align*}
-\Delta u &= u^p \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega \\
u &> 0 \quad \text{in } \Omega
\end{align*}
$$

where $\Omega$ is any smooth bounded planar domain.

This line of investigation started in [10, 11] for families $u_p$ of least energy solutions, for which a one-point concentration behavior in the interior of $\Omega$ is proved, as well as the $L^\infty$-bounds

$$\sqrt{e} \leq \lim_{p \to +\infty} \|u_p\|_{L^\infty} \leq C \quad \text{(1.1)}$$

and the following estimate

$$\lim_{p \to +\infty} p\|\nabla u_p\|_2^2 = 8\pi e. \quad \text{(1.2)}$$

The bound in (1.1) was later improved in [1], where it was shown that for families of least energy solutions the following limit holds true:

$$\lim_{p \to +\infty} \|u_p\|_{L^\infty} = \sqrt{e}. \quad \text{(1.2)}$$

Moreover in [1] and [7] the Liouville equation in the whole plane

$$
\begin{align*}
-\Delta U &= e^U \quad \text{in } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} e^U \, dx &= 8\pi
\end{align*}
$$

was identified to be a limit problem for the Lane-Emden equation. Indeed in [1] it was proved that suitable rescalings around the maximum point of any least energy solution to

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The solutions converge, in $C^2_{\text{loc}}(\mathbb{R}^2)$, to the regular solution
\[ U(x) = \log \left( \frac{1}{1 + \frac{1}{8}|x|^2} \right)^2 \]
(1.4)
of (1.3). Hence least energy solutions exhibit only one concentration point and the local limit profile is given by (1.4). More general solutions having only one peak have been recently studied in [6], where their Morse index is computed and connections with the question of the uniqueness of positive solutions in convex domains are shown.

Observe that when $\Omega$ is a ball any solution to $(P_p)$ is radial by Gidas, Ni and Nirenberg result ([9]) and so the least energy is the unique solution for any $p > 1$.

In general in non-convex domains there may be families of solutions to $(P_p)$ other than the least energy ones. This is the case, for instance, of those found in [8] when the domain $\Omega$ is not simply connected, which have higher energy, precisely
\[ \lim_{p \to +\infty} p\|\nabla u_p\|_2^2 = 8\pi e \cdot k, \]
for any fixed integer $k \geq 1$. These solutions exhibit a concentration phenomenon at $k$ distinct points in $\Omega$ as $p \to +\infty$ and their $L^\infty$-norm satisfies the same limit as in (1.2).

The question of characterizing the behavior of any family $u_p$ of solutions to $(P_p)$ naturally arises. An almost complete answer has been recently given in [4] in any general smooth bounded domain $\Omega$, under the uniform energy bound assumption
\[ p\|\nabla u_p\|_2^2 \leq C \]
(1.5)
(see also [2], where this general asymptotic analysis was started and the related papers [3, 5]). The results in [4] show that under the assumption (1.5) the solutions to $(P_p)$ are necessarily spike-like and that the energy is quantized. More precisely in [4, Theorem 1.1] it is proved that, up to a subsequence, there exists an integer $k \geq 1$ and $k$ distinct points $x_i \in \Omega$, $i = 1, \ldots, k$, such that, setting
\[ S = \{x_1, \ldots, x_k\}, \]
one has
\[ \lim_{p \to +\infty} \sqrt{p} u_p = 0 \quad \text{in} \quad C^2_{\text{loc}}(\overline{\Omega} \setminus S) \]
(1.6)
and the energy satisfies
\[ \lim_{p \to +\infty} p\|\nabla u_p\|_2^2 = 8\pi \sum_{i=1}^k m^2_i, \]
(1.7)
where $m_i$’s are positive constants given by
\[ m_i = \lim_{\delta \to 0} \lim_{p \to +\infty} \max_{B_\delta(x_i)} u_p \]
(1.8)
which satisfy
\[ m_i \geq \sqrt{e}. \]
(1.9)
Furthermore the location of the concentration points is shown to depend on the Dirichlet Green function $G$ of $-\Delta$ in $\Omega$ and on its regular part $H$
\[ H(x, y) = G(x, y) + \frac{\log(|x - y|)}{2\pi} \]
(1.10)
according to the following system
\[ m_i \nabla_x H(x_i, x_i) + \sum_{\ell \neq i} m_\ell \nabla_x G(x_i, x_\ell) = 0, \]
and moreover
\[ \lim_{p \to +\infty} p u_p = 8\pi \sum_{i=1}^k m_i G(\cdot, x_i) \text{ in } C^2_{\text{loc}}(\bar{\Omega} \setminus S). \]

In [4, Lemma 4.1] it is also proved that a suitable rescaling of \( u_p \) around each concentration point, in the spirit of the one done in [1] for the least energy solutions, converges to the regular solution \( U \) in (1.4).

Observe that (1.6) and (1.9) immediately imply the following bound on the \( L^\infty \)-norm:
\[ \sqrt{e} \leq \lim_{p \to +\infty} \| u_p \|_\infty \leq C. \] (1.11)

In [4] it was conjectured that for all solutions to \( (P_p) \), under the assumption (1.5), one should have the equality in (1.9).

Here we complete the analysis in [4] proving this conjecture, namely we show the following:

**Theorem 1.1.**
\[ m_i = \sqrt{e}, \quad \forall i = 1, \ldots, k. \]

This result implies, by (1.6) and (1.9), a sharp improvement of (1.11):

**Theorem 1.2 (\( L^\infty \)-norm limit).** Let \( u_p \) be a family of solutions to \( (P_p) \) and assume that (1.5) holds. Then
\[ \lim_{p \to +\infty} \| u_p \|_\infty = \sqrt{e}. \]

On the other side, by (1.7), Theorem 1.1 implies a quantization of the energy to integer multiples of \( 8\pi e \) as \( p \) goes to infinity. Our final asymptotic results can be summarized as follows:

**Theorem 1.3** (Complete asymptotic behavior & quantization). Let \( u_p \) be a family of solutions to \( (P_p) \) and assume that (1.5) holds. Then there exist a finite number \( k \) of distinct points \( x_i \in \Omega, \ i = 1, \ldots, k \) and a sequence \( p_n \to +\infty \) as \( n \to +\infty \) such that setting
\[ S := \{ x_1, \ldots, x_k \} \]
once has
\[ \lim_{n \to \infty} \sqrt{p_n} u_{p_n} = 0 \text{ in } C^2_{\text{loc}}(\bar{\Omega} \setminus S). \] (1.12)
The concentration points \( x_i, \ i = 1, \ldots, k \) satisfy the system
\[ \nabla_x H(x_i, x_i) + \sum_{i \neq \ell} \nabla_x G(x_i, x_\ell) = 0. \] (1.13)
Moreover
\[ \lim_{n \to \infty} p_n u_{p_n}(x) = 8\pi \sqrt{e} \sum_{i=1}^k G(x, x_i) \text{ in } C^2_{\text{loc}}(\bar{\Omega} \setminus S) \] (1.14)
and the energy satisfies
\[ \lim_{n \to \infty} p_n \int_{\Omega} |\nabla u_{p_n}(x)|^2 \, dx = 8\pi e \cdot k. \] (1.15)
2. Proof of Theorem 1.1

Let \( k \geq 1 \) and \( x_i \in \Omega, i = 1, \ldots, k \) be as in the introduction and let us keep the notation \( u_p \) to denote the corresponding subsequence of the family \( u_p \) for which the results in [4] hold true.

In particular (see [4, Theorem 1.1 & Lemma 4.1]) for \( r > 0 \) such that \( B_{3r}(x_j) \subset \Omega, \forall j = 1, \ldots, k \) and \( B_{3r}(x_j) \cap B_{3r}(x_i) = \emptyset, \forall j = 1, \ldots, k, j \neq i \), letting \( y_{i,p} \in \Omega \) be the sequence defined as

\[
u_p(y_{i,p}) := \max_{B_{2r}(x_i)} u_p
\] (2.16)

it follows that

\[
\lim_{p \to +\infty} y_{i,p} = x_i,
\] (2.17)

\[
\lim_{p \to +\infty} u_p(y_{i,p}) = m_i,
\] (2.18)

\[
\lim_{p \to +\infty} \varepsilon_{i,p} \left( := \left[ p u_p(y_{i,p})^{p-1} \right]^{-1/2} \right) = 0
\] (2.19)

and setting

\[
w_{i,p}(y) := \frac{p}{u_p(y_{i,p})} (u_p(y_{i,p} + \varepsilon_{i,p} y) - u_p(y_{i,p})), \quad y \in \Omega_{i,p} := \frac{\Omega - y_{i,p}}{\varepsilon_{i,p}},
\] (2.20)

then

\[
\lim_{p \to +\infty} w_{i,p} = U \text{ in } C^2_{\text{loc}}(\mathbb{R}^2),
\] (2.21)

where \( U \) is as in (1.4).

Furthermore by the result in [4, Proposition 4.3 & Lemma 4.4] we have that for any \( \gamma \in (0, 4) \) there exists \( R_{\gamma} > 1 \) such that

\[
w_{i,p}(z) \leq (4 - \gamma) \log \frac{1}{|z|} + \tilde{C}_{\gamma}, \quad \forall i = 1, \ldots, k
\] (2.22)

for some \( \tilde{C}_{\gamma} > 0 \), provided \( R_{\gamma} \leq |z| \leq \frac{R}{\varepsilon_{i,p}} \) and \( p \) is sufficiently large.

The pointwise estimate (2.22) implies the following uniform bound, which will be the key to use the dominated convergence theorem in the proof of Theorem 1.1:

**Lemma 2.1.**

\[
0 \leq \left( 1 + \frac{w_{i,p}(z)}{p} \right)^p \leq \begin{cases} 1 & \text{for } |z| \leq R_{\gamma} \\ \frac{1}{C_{\gamma} |z|^{4-\gamma}} & \text{for } R_{\gamma} \leq |z| \leq \frac{R_{\gamma}^2}{\varepsilon_{i,p}} \end{cases}
\] (2.23)

**Proof.** Observe that by (2.17)

\[
B_r(y_{i,p}) \subset B_{2r}(x_i), \text{ for } p \text{ sufficiently large,}
\]
as a consequence

\[
w_{i,p} \leq 0, \text{ in } B_{\frac{r}{\varepsilon_{i,p}}}(0) (\subset \Omega_{i,p}), \text{ for } p \text{ large,}
\] (2.24)
which implies the first bound in (2.23).
For $p$ sufficiently large, by (2.24) and (2.22), we also get the second bound in (2.23):

$$0 \leq \left(1 + \frac{w_{j,p}(z)}{p}\right)^p = e^{p\log\left(1 + \frac{w_{j,p}(z)}{p}\right)} \leq e^{w_{j,p}(z)} \leq C\gamma \frac{1}{|z|^{1-\gamma}}$$

for $R_\gamma \leq |z| \leq \frac{r}{\varepsilon_{j,p}}$.

$\square$

**Proof of Theorem 1.1.** Observe that by the assumption (1.5) and Hölder inequality

$$(0 \leq) \int_\Omega u_p^p(x)dx \leq p^{1+1/p} \Omega_{1/p+1/p} \left[p \int_\Omega |\nabla u_p|^2 dx\right]^{p/2}$$

$$= p \int_\Omega |\nabla u_p|^2 dx + o_p(1) \leq C + o_p(1),$$

so that, by the properties of the Green function $G$,

$$\int_{\Omega \setminus B_2(x_j)} G(y_{j,p}, x) u_p^p(x)dx \leq C \int_{\Omega \setminus B_2(x_j)} u_p^p(x)dx$$

$$\leq C \int_{\Omega} u_p^p(x)dx = O\left(\frac{1}{p}\right) \quad (2.25)$$

and similarly, observing that for $p$ large enough the points $y_{j,p} \in B_r(x_j)$ by (2.17) and $B_r(x_j) \subset B_{r_j}(y_{j,p}) \subset B_{2r}(x_j)$, also

$$\int_{B_{2r}(x_j) \setminus B_r(y_{j,p})} G(y_{j,p}, x) u_p^p(x)dx \leq \int_{\{z < |x - x_j| < 2r\}} G(y_{j,p}, x) u_p^p(x)dx$$

$$\leq C \int_{\Omega} u_p^p(x)dx = O\left(\frac{1}{p}\right). \quad (2.26)$$

By the Green representation formula, using the previous estimates, we then get

$$u_p(y_{j,p}) = \int_\Omega G(y_{j,p}, x) u_p^p(x)dx$$

$$= \int_{B_{2r}(x_j)} G(y_{j,p}, x) u_p^p(x)dx + \int_{\Omega \setminus B_{2r}(x_j)} G(y_{j,p}, x) u_p^p(x)dx \quad (2.25)$$

$$= \int_{B_r(y_{j,p})} G(y_{j,p}, x) u_p^p(x)dx + o_p(1) \quad (2.26)$$

$$= \int_{B_{r_j}(y_{j,p})} G(y_{j,p}, x) u_p^p(x)dx + o_p(1) \quad (2.20)$$

$$= \int_{B_{r_j}(0)} G(y_{j,p} + \varepsilon_{j,p}z) \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz + o_p(1) \quad (1.10)$$

$$= \frac{u_p(y_{j,p})}{p} \int_{B_{r_j}(0)} H(y_{j,p} + \varepsilon_{j,p}z) \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz$$

$$- \frac{u_p(y_{j,p})}{2\pi p} \int_{B_{r_j}(0)} \log |z| \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz$$

$$- \frac{u_p(y_{j,p})}{2\pi p} \int_{B_{r_j}(0)} \log |z| \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz$$
where \( z \in B_{\frac{r}{2}}(0) \) and this implies that again by the dominated convergence theorem, using (2.21) and the uniform bounds in (2.23) we can apply the dominated convergence theorem, and since the function \( z \mapsto \frac{1}{|z|^{1+\gamma}} \) is integrable in \( \{|z| > R_\gamma\} \) choosing \( \gamma \in (0, 2) \) we deduce

\[
\lim_{p \to +\infty} u_p(y_{j,p}) \int_{B_{\frac{r}{2}}(0)} H(y_{j,p}, y_{j,p} + \varepsilon_{j,p} z) \left( 1 + \frac{w_{j,p}(z)}{p} \right)^p \, dz = m_j \int_{\mathbb{R}^2} e^{U(z)} = 8\pi m_j H(x_j, x_j),
\]

From which

\[
A_p := \frac{u_p(y_{j,p})}{p} \int_{B_{\frac{r}{2}}(0)} H(y_{j,p}, y_{j,p} + \varepsilon_{j,p} z) \left( 1 + \frac{w_{j,p}(z)}{p} \right)^p \, dz = o_p(1). \tag{2.28}
\]

For the second term in (2.27) we apply again the dominated convergence theorem, using (2.23) and observing now that the function \( z \mapsto \frac{\log |z|}{|z|^{\gamma}} \) is integrable in \( \{|z| > R_\gamma\} \) and that \( z \mapsto \log |z| \) is integrable in \( \{|z| \leq R_\gamma\} \). Hence we get

\[
\lim_{p \to +\infty} u_p(y_{j,p}) \int_{B_{\frac{r}{2}}(0)} \log |z| \left( 1 + \frac{w_{j,p}(z)}{p} \right)^p \, dz = m_j \int_{\mathbb{R}^2} \log |z| e^{U(z)} \, dz < +\infty
\]

and this implies that

\[
B_p := -\frac{u_p(y_{j,p})}{2\pi p} \int_{B_{\frac{r}{2}}(0)} \log |z| \left( 1 + \frac{w_{j,p}(z)}{p} \right)^p \, dz = o_p(1). \tag{2.29}
\]

Finally for the last term in (2.27) let us observe that by the definition of \( \varepsilon_{j,p} \) in (2.19)

\[
\log \varepsilon_{j,p} = -\frac{(p - 1)}{2} \log u_p(y_{j,p}) - \frac{1}{2} \log p, \tag{2.30}
\]

again by the dominated convergence theorem

\[
\lim_{p \to +\infty} \int_{B_{\frac{r}{2}}(0)} \left( 1 + \frac{w_{j,p}(z)}{p} \right)^p \, dz = \int_{\mathbb{R}^2} e^{U(z)} = 8\pi \tag{1.3},
\]

and it follows

\[
C_p := -\frac{u_p(y_{j,p})}{2\pi p} \int_{B_{\frac{r}{2}}(0)} \left( 1 + \frac{w_{j,p}(z)}{p} \right)^p \, dz \tag{2.31}
\]

\[
= -\frac{u_p(y_{j,p})}{2\pi p} \varepsilon_{j,p} (8\pi + o_p(1)) \tag{2.32}
\]

\[
= u_p(y_{j,p}) \left[ \frac{(p - 1)}{p} \log u_p(y_{j,p}) + \frac{\log p}{p} \right] (2 + o_p(1)).
\]
Substituting (2.28), (2.29) and (2.32) into (2.27) we get
\[ u_p(y_{j,p}) = u(y_{j,p}) \left[ \frac{(p-1)}{p} \log u_p(y_{j,p}) + \log \frac{p}{p} \right] (2 + o_p(1)) + o_p(1), \]
passing to the limit as \( p \to +\infty \) and using (2.18) conclude that
\[ \log m_j = \frac{1}{2}. \]
\[ \square \]

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