THE COMPLETE LIST OF GENERA OF QUOTIENTS OF THE $\mathbb{F}_{q^2}$-MAXIMAL HERMITIAN CURVE FOR $q \equiv 1 \pmod{4}$

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Abstract. Let $\mathbb{F}_{q^2}$ be the finite field with $q^2$ elements. Most of the known $\mathbb{F}_{q^2}$-maximal curves arise as quotient curves of the $\mathbb{F}_{q^2}$-maximal Hermitian curve $\mathcal{H}_q$. After a seminal paper by Garcia, Stichtenoth and Xing [15], many papers have provided genera of quotients of $\mathcal{H}_q$, but their complete determination is a challenging open problem. In this paper we determine completely the spectrum of genera of quotients of $\mathcal{H}_q$ for any $q \equiv 1 \pmod{4}$.

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1. Introduction

Let $q = p^n$ be a prime power, $\mathbb{F}_{q^2}$ the finite field with $q^2$ elements, and $\mathcal{X}$ be a projective, absolutely irreducible, non-singular algebraic curve of genus $g$ defined over $\mathbb{F}_{q^2}$. The curve $\mathcal{X}$ is called $\mathbb{F}_{q^2}$-maximal if the number $|\mathcal{X}(\mathbb{F}_{q^2})|$ of its $\mathbb{F}_{q^2}$-rational points attains the Hasse-Weil upper bound $q^2 + 1 + 2gq$. Maximal curves have been investigated for their applications in Coding Theory. Surveys on maximal curves are found in [12, 13, 14, 16, 37, 38] and [24, Chapter 10].

A well-known and important example of an $\mathbb{F}_{q^2}$-maximal curve is the Hermitian curve $\mathcal{H}_q$. It is defined as any $\mathbb{F}_{q^2}$-rational curve which is projectively equivalent to the plane curve $Y^{q+1} - X^{q+1} + Z^{q+1} = 0$. For fixed $q$, the curve $\mathcal{H}_q$ has the largest genus $g(\mathcal{H}_q) = q(q - 1)/2$ that an $\mathbb{F}_{q^2}$-maximal curve can have. The full automorphism group $\text{Aut}(\mathcal{H}_q)$ is isomorphic to $PGU(3,q)$, the group of projectivities of $PG(2,q^2)$ commuting with the unitary polarity associated with $\mathcal{H}_q$. The automorphism group $\text{Aut}(\mathcal{H}_q)$ is extremely large with respect to the value $g(\mathcal{H}_q)$. Indeed it is known that the Hermitian curve is the unique curve of genus $g \geq 2$ up to isomorphisms admitting an automorphism group of order at least equal to $16g^4$.

By a result commonly referred to as the Kleiman-Serre covering result, see [28] and [29, Proposition 6], a curve $\mathcal{X}$ defined over $\mathbb{F}_{q^2}$ which is $\mathbb{F}_{q^2}$-covered by an $\mathbb{F}_{q^2}$-maximal curve is $\mathbb{F}_{q^2}$-maximal as well. In particular, $\mathbb{F}_{q^2}$-maximal curves can be obtained as Galois $\mathbb{F}_{q^2}$-subcovers of an $\mathbb{F}_{q^2}$-maximal curve $\mathcal{X}$, that is, as quotient curves $\mathcal{X}/G$ where $G$ is a finite automorphism group of $\mathcal{X}$. Most of the known maximal curves are Galois covered by the Hermitian curve; see for instance [15, 17, 18, 19, 22, 30, 32, 33] and the references therein.

The first example of a maximal curve not Galois covered by the Hermitian curve is due to Garcia and Stichtenoth [17]. This curve is $\mathbb{F}_{q^2}$-maximal and it is not Galois covered by $\mathcal{H}_{17}$. It is a special case of the $\mathbb{F}_{q^2}$-maximal GS curve, which was later shown not to be Galois covered by $\mathcal{H}_{q^2}$ for any $q > 3$, [22, 30]. Giulietti and Korchmáros [21] provided an $\mathbb{F}_{q^2}$-maximal curve, nowadays referred to as the GK curve, which is not covered by the Hermitian curve $\mathcal{H}_{q^2}$ for any $q > 2$. Two generalizations of the GK curve were introduced by Garcia, Güneri and Stichtenoth [17] and by Beelen and Montanucci [3]. Both these generalizations are $\mathbb{F}_{q^n}$-maximal curves, for any $q$ and odd $n \geq 3$. Also, they are not Galois covered by the Hermitian curve $\mathcal{H}_{q^2}$.
for \( q > 2 \) and \( n \geq 5 \), see [11] [3]; the Garcia-Güneri-Stichtenoth’s generalization is also not Galois covered by \( \mathcal{H}_2 \) for \( q = 2 \), see [22].

A challenging open problem is the determination of the spectrum \( \Gamma(q^2) \) of genera of \( \mathbb{F}_{q^2} \)-maximal curves, for given \( q \). Apart from the examples listed above, most of the known values in \( \Gamma(q^2) \) have been obtained from quotient curves \( \mathcal{H}_q / G \) of the Hermitian curve, which have been investigated in many papers. The most significant cases are the following:

- \( G \) fixes an \( \mathbb{F}_{q^2} \)-rational point of \( \mathcal{H}_q \); see [2] [18] [1].
- \( G \) fixes a self-polar triangle in \( \text{PG}(2, q^2) \ \backslash \ \mathcal{H}_q \); see [8].
- \( G \) normalizes a Singer subgroup of \( \mathcal{H}_q \) acting on three \( \mathbb{F}_{q^6} \)-rational points of \( \mathcal{H}_q \); see [18] [7].
- \( G \) has prime order; see [6].
- \( G \) fixes neither points nor triangles in \( \text{PG}(2, q^6) \); see [33].

From the results already obtained in the literature (see [8] [33] and the references therein), in order to obtain the complete list of genera of quotients \( \mathcal{H}_q / G, G \leq \text{Aut}(\mathcal{H}_q) \), only the following cases still have to be analyzed:

1. \( G \) fixes an \( \mathbb{F}_{q^2} \)-rational point \( P \notin \mathcal{H}_q, p > 2 \).
2. \( G \) fixes a point \( P \in \mathcal{H}_q(\mathbb{F}_{q^2}), p = 2 \) and \( |G| = p^d \) where \( p^d \leq q \) and \( d \mid (q - 1) \).

In this paper a complete analysis of Case 1 is given provided that \( q \) is congruent to 1 modulo 4. This provides the complete list of genera of quotient of the Hermitian curve under this assumption.

More precisely, this paper is organized as follows. Section 2 provides a collection of necessary preliminary results on the Hermitian curve and its automorphism group. In Section 3 a complete analysis of Case 1 is given for \( q \equiv 1 \) (mod 4). Section 4 contains the complete list of genera of quotients of the Hermitian curve for \( q \equiv 1 \) (mod 4) joining our results with the ones already obtained in the literature.

## 2. Preliminary results

Throughout this paper \( q = p^n \), where \( p \) is a prime number and \( n \) is a positive integer. The Deligne-Lusztig curves defined over a finite field \( \mathbb{F}_q \) were originally introduced in [9]. Other than the projective line, there are three families of Deligne-Lusztig curves, named Hermitian curves, Suzuki curves and Ree curves. The Hermitian curve \( \mathcal{H}_q \) arises from the algebraic group \( ^2A_2(q) = \text{PGU}(3, q) \) of order \( q^3 + 1 \). It has genus \( (q-1)/2 \) and is \( \mathbb{F}_{q^2} \)-maximal. This curve is \( \mathbb{F}_{q^2} \)-isomorphic to the following curves:

\[
X^{q+1} - Y^{q+1} - Z^{q+1} = 0;
\]

\[(1)\]

\[
X^qZ + XZ^q - Y^{q+1} = 0.
\]

(2)

The automorphism group \( \text{Aut}(\mathcal{H}_q) \) is isomorphic to the projective unitary group \( \text{PGU}(3, q) \), and it acts on the set \( \mathcal{H}_q(\mathbb{F}_{q^2}) \) of all \( \mathbb{F}_{q^2} \)-rational points of \( \mathcal{H}_q \) as \( \text{PGU}(3, q) \) in its usual 2-transitive permutation representation. The combinatorial properties of \( \mathcal{H}_q(\mathbb{F}_{q^2}) \) can be found in [22]. The size of \( \mathcal{H}_q(\mathbb{F}_{q^2}) \) is equal to \( q^3 + 1 \), and a line of \( \text{PG}(2, q^2) \) has either 1 or \( q + 1 \) common points with \( \mathcal{H}_q(\mathbb{F}_{q^2}) \), that is, it is either a tangent line or a chord of \( \mathcal{H}_q(\mathbb{F}_{q^2}) \). Furthermore, a unitary polarity is associated with \( \mathcal{H}_q(\mathbb{F}_{q^2}) \) whose isotropic points are those of \( \mathcal{H}_q(\mathbb{F}_{q^2}) \) and isotropic lines are the tangent lines to \( \mathcal{H}_q(\mathbb{F}_{q^2}) \), that is, the tangents to \( \mathcal{H}_q \) at the points of \( \mathcal{H}_q(\mathbb{F}_{q^2}) \).

The following classification of subgroups of \( \text{PGU}(3, q) \) goes back to Mitchell [31] and Hartley [23]; see also [33].

**Theorem 2.1.** Let \( G \) be a nontrivial subgroup of \( \text{PGU}(3, q) \). Then one of the following cases holds.

(i) \( G \) is contained in the maximal subgroup of \( \text{PGU}(3, q) \) of order \( q^3(q^2 - 1) \) which stabilizes an \( \mathbb{F}_{q^2} \)-rational point of \( \mathcal{H}_q \).

(ii) \( G \) is contained in the maximal subgroup \( M_q \) of \( \text{PGU}(3, q) \) of order \( q(q - 1)(q + 1)^2 \) which stabilizes an \( \mathbb{F}_{q^2} \)-rational point of \( \text{PG}(2, q^2) \ \backslash \ \mathcal{H}_q \); equivalently, \( M_q \) stabilizes a chord of \( \mathcal{H}_q(\mathbb{F}_{q^2}) \).
Theorem 2.2. \( \bullet \) Following. In particular, a linear collineation \( \Phi_{\mathbb{F}_q} : (X,Y,Z) \mapsto (X^q,Y^q,Z^q) \) of \( \mathrm{PG}(2,\mathbb{F}_q) \).

Theorem 2.3. (see [10, Chapt. XII, Par. 260], [26, Kap. II, Hauptsatz 8.27], [24, Thm. A.8]) Non-tame subgroups:

(i) The cyclic group \( C_d \) of order \( d \), where \( d \mid (q \pm 1) \).

(2) The dihedral group \( D_{2m} = \langle \delta, \epsilon \mid \delta^{2m} = 1, \epsilon^2 = \delta^m, \epsilon^{-1} \delta \epsilon = \delta^{-1} \rangle \cong C_{2m} \circ C_4 \) of order \( 4m \), where \( 1 < m \mid \frac{q-1}{2} \).

Non-tame subgroups:

(1) \( E_{p^k} \rtimes C_d \), where \( d \mid \gcd(p^k - 1, q - 1) \), \( k \leq n \), and \( E_{p^k} \) is elementary abelian of order \( p^k \).

(2) \( \mathrm{SL}(2,5) \), when \( p = 3 \) and \( q^2 \equiv 1 \pmod{5} \);

(3) \( \mathrm{SL}(2,3) \), when \( p \geq 5 \);

(4) \( \mathrm{TL}(2,p^k) \cong \langle \mathrm{SL}(2,p^k), d_\pi \rangle \), where \( k \mid n \), \( n/k \) is even,

\[
d_\pi = \begin{pmatrix} w & 0 & 0 \\ 0 & w^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\( w = \xi^{\frac{k+1}{2}} \), and \( \mathbb{F}_q^{*} = \langle \xi \rangle \).

Theorem 2.3. (see [10] Chapt. XII, Par. 260], [26] Kap. II, Hauptsatz 8.27], [24] Thm. A.8]) The following is the complete list of subgroups of \( \mathrm{PGL}(2,q) \) up to conjugacy:

(i) the cyclic group of order \( h \) with \( h \mid (q \pm 1) \);

(ii) the elementary abelian \( p \)-group of order \( p^f \) with \( f \leq n \);

(iii) the dihedral group of order \( 2h \) with \( h \mid (q \pm 1) \);

(iv) the alternating group \( A_4 \) for \( p > 2 \), or \( p = 2 \) and \( n \) even;

(v) the symmetric group \( S_4 \) for \( 16 \mid (q^2 - 1) \);

(vi) the alternating group \( A_5 \) for \( p = 5 \) or \( 5 \mid (q^2 - 1) \);

(vii) the semidirect product of an elementary abelian \( p \)-group of order \( p^k \) by a cyclic group of order \( h \), with \( h \leq n \) and \( h \mid \gcd(p^f - 1, q - 1) \);

(viii) \( \mathrm{PSL}(2,p^f) \) for \( f \mid n \);

(ix) \( \mathrm{PGL}(2,p^f) \) for \( f \mid n \).

In our investigation it is useful to know how an element of \( \mathrm{PGU}(3,q) \) of a given order acts on \( \mathrm{PG}(2,\mathbb{F}_q) \), and in particular on \( \mathcal{H}_q(\mathbb{F}_q) \). This can be obtained as a corollary of Theorem 2.1 and is stated in Lemma 2.2 with the usual terminology of collineations of projective planes; see [25]. In particular, a linear collineation \( \sigma \) of \( \mathrm{PG}(2,\mathbb{F}_q) \) is a \((P,\ell)\)-perspectivity, if \( \sigma \) preserves each line through the point \( P \) (the center of \( \sigma \)), and fixes each point on the line \( \ell \) (the axis of \( \sigma \)). A \((P,\ell)\)-perspectivity is either an elation or a homology according to \( P \in \ell \) or \( P \notin \ell \). This classification result was obtained in [34].
Lemma 2.4. For a nontrivial element \( \sigma \in \text{PGU}(3,q) \), one of the following cases holds.

(A) \( \text{ord}(\sigma) \mid (q+1) \) and \( \sigma \) is a homology, with center \( P \in \text{PG}(2,q^2) \setminus \mathcal{H}_q \) and axis \( \ell \) which is a chord of \( \mathcal{H}_q(\mathbb{F}_{q^2}) \); \( (P,\ell) \) is a pole-polar pair with respect to the unitary polarity associated to \( \mathcal{H}_q(\mathbb{F}_{q^2}) \).

(B) \( p \not\mid \text{ord}(\sigma) \) and \( \sigma \) fixes the vertices \( P_1, P_2, P_3 \) of a non-degenerate triangle \( T \subset \text{PG}(2,q^2) \).

(B1) \( \text{ord}(\sigma) \mid (q+1) \); \( P_1, P_2, P_3 \in \text{PG}(2,q^2) \setminus \mathcal{H}_q \), and the triangle \( T \) is self-polar with respect to the unitary polarity associated to \( \mathcal{H}_q(\mathbb{F}_{q^2}) \).

(B2) \( \text{ord}(\sigma) \mid (q^2-1) \) and \( \text{ord}(\sigma) \mid (q+1) \); \( P_1 \in \text{PG}(2,q^2) \setminus \mathcal{H}_q \) and \( P_2, P_3 \in \mathcal{H}_q(\mathbb{F}_{q^2}) \).

(C) \( \text{ord}(\sigma) = p \) and \( \sigma \) is an elastion with center \( P \in \mathcal{H}_q(\mathbb{F}_{q^2}) \) and axis \( \ell \) which is tangent to \( \mathcal{H}_q \) at \( P \), such that \( (P,\ell) \) is a pole-polar pair with respect to the unitary polarity associated to \( \mathcal{H}_q(\mathbb{F}_{q^2}) \).

(D) \( \text{ord}(\sigma) = p \) with \( p \not= 2 \), or \( \text{ord}(\sigma) = 4 \) and \( p = 2 \); \( \sigma \) fixes a point \( P \in \mathcal{H}_q(\mathbb{F}_{q^2}) \) and a line \( \ell \) which is tangent to \( \mathcal{H}_q \) at \( P \), such that \( (P,\ell) \) is a pole-polar pair with respect to the unitary polarity associated to \( \mathcal{H}_q(\mathbb{F}_{q^2}) \).

(E) \( \text{ord}(\sigma) = p \cdot d \) where \( d \) is a nontrivial divisor of \( q+1 \); \( \sigma \) fixes two points \( P \in \mathcal{H}_q(\mathbb{F}_{q^2}) \) and \( Q \in \text{PG}(2,q^2) \setminus \mathcal{H}_q \), the polar line \( PQ \) of \( P \), and the polar line of \( Q \) which is another line through \( P \).

Throughout the paper, a nontrivial element of \( \text{PGU}(3,q) \) is said to be of type (A), (B), (B1), (B2), (B3), (C), (D), or (E), as given in Lemma 2.4.

To compute the genus of a quotient curve we make use of the Riemann-Hurwitz formula; see [35, Theorem 3.4.13]. For any subgroup \( G \) of \( \text{PGU}(3,q) \), the cover \( \mathcal{H}_q \to \mathcal{H}_q/G \) is a Galois cover defined over \( \mathbb{F}_{q^2} \) and the degree \( \Delta \) of the different divisor is given by the Riemann-Hurwitz formula, namely \( \Delta = (2g(\mathcal{H}_q) - 2) - |G(2\mathcal{H}_q/G) - 2) \). On the other hand, \( \Delta = \sum_{\sigma \in G \setminus \{ id \}} i(\sigma) \), where \( i(\sigma) \geq 0 \) is given by the Hilbert’s different formula [35, Thm. 3.8.7], namely \( i(\sigma) = \sum_{P \in \mathcal{H}_q \setminus \{P\}} \psi_P(\sigma(t) - t) \), where \( t \) is a local parameter at \( P \).

By analyzing the geometric properties of the elements \( \sigma \in \text{PGU}(3,q) \), it turns out that: there are only a few possibilities for \( i(\sigma) \). This is obtained as a corollary of Lemma 2.4 and stated in the following theorem, which is proved in 334.

Theorem 2.5. For a nontrivial element \( \sigma \in \text{PGU}(3,q) \) one of the following cases occurs.

1. If \( \text{ord}(\sigma) = 2 \) and \( 2 \mid (q+1) \), then \( \sigma \) is of type (A) and \( i(\sigma) = q+1 \).
2. If \( \text{ord}(\sigma) = 3 \) and \( 3 \mid (q+1) \) and \( \sigma \) is of type (B3), then \( i(\sigma) = 3 \).
3. If \( \text{ord}(\sigma) = 2 \) and \( \sigma \) is of type (B1), then \( i(\sigma) = 0 \).
4. If \( \text{ord}(\sigma) = 2 \) and \( \sigma \) is of type (B1), then \( i(\sigma) = 0 \).
5. If \( \text{ord}(\sigma) = 2 \) and \( \sigma \) is of type (B2), then \( i(\sigma) = 2 \).
6. If \( \text{ord}(\sigma) = 2 \) and \( \sigma \) is of type (B3), then \( i(\sigma) = 3 \).
7. If \( \text{ord}(\sigma) = 2 \) and \( \sigma \) is of type (B3), then \( i(\sigma) = 3 \).
8. If \( \text{ord}(\sigma) = 2 \) and \( \sigma \) is of type (B3), then \( i(\sigma) = 3 \).
9. If \( \text{ord}(\sigma) = 2 \) and \( \sigma \) is of type (B3), then \( i(\sigma) = 3 \).
10. If \( \text{ord}(\sigma) = 2 \) and \( \sigma \) is of type (B3), then \( i(\sigma) = 3 \).

In order to characterize the genera of \( \mathcal{H}_q/G \) for any subgroup \( G \) of \( \text{PGU}(3,q) \) under the assumption \( q \equiv 1 \) (mod 4), it is sufficient to consider the case \( G \leq \mathcal{M}_q \), where \( \mathcal{M}_q \) is the maximal subgroup (ii) in Theorem 2.4. In fact, if \( G \not\leq \mathcal{M}_q \):

- the genera \( g(\mathcal{H}_q/G) \) for the subgroups \( G \leq \text{PGU}(3,q) \) stabilizing an \( \mathbb{F}_{q^2} \)-rational point of \( \mathcal{H}_q \) are characterized in [4, Theorem 1.1];
- the genera \( g(\mathcal{H}_q/G) \) for the subgroups \( G \leq \text{PGU}(3,q) \) stabilizing a self-polar triangle of \( \text{PG}(2,q^2) \) are characterized in [8, Section 3];
- the genera \( g(\mathcal{H}_q/G) \) for the subgroups \( G \leq \text{PGU}(3,q) \) stabilizing a Frobenius-invariant triangle in \( \mathcal{H}_q(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2}) \) are characterized in [4, Proposition 4.2];
- the genera \( g(\mathcal{H}_q/G) \) for the subgroups \( G \leq \text{PGU}(3,q) \) which do not stabilize any point or triangle are characterized in [33].
3. The maximal subgroup $\mathcal{M}_q$ for $q \equiv 1 \pmod{4}$

Let $\mathcal{M}_q$ be the maximal subgroup of $\text{PGU}(3, q)$ stabilizing a point $P \in \text{PG}(2, q^2) \setminus \mathcal{H}_r$ and its polar line $\ell$, which is a chord of $\mathcal{H}_r(F_{q^2})$. For any odd $q$, the structure of $\mathcal{M}_q$ was already given in [5, Section 3] and [32, Section 3] as a semidirect product $\mathcal{M}_q = \Gamma \rtimes C$, where $\Gamma$ is the commutator subgroup of $\mathcal{M}_q$ and is isomorphic to $\text{SL}(2, q)$, while $C$ is a cyclic group of order $q+1$; this description was used to compute the genera $q(\mathcal{H}_r/G)$ for some $G \leq \mathcal{M}_q$, namely when $G$ is a tame subgroup of $\Gamma$ in [5], and when $G = (G \cap \Gamma) \rtimes (G \cap C)$ in [32].

Henceforth, we assume $q \equiv 1 \pmod{4}$ and use a different description of $\mathcal{M}_q$. Let $\mathcal{H}_r$ be given by the model $[1]$. Up to conjugation in $\text{PGU}(3, q)$, we can assume that $P = (0, 0, 1)$ and $\ell$ has equation $Z = 0$. As pointed out in the proof of Theorem 2.5 in [3],

$$\mathcal{M}_q = \left\{ \begin{pmatrix} a & \zeta c^q & 0 \\ c & \zeta c^q & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, c \in F_{q^2}, a^{q+1} - c^{q+1} = 1, \zeta^{q+1} = 1 \right\}$$

and the commutator subgroup of $\mathcal{M}_q$ is

$$H = \left\{ \begin{pmatrix} a & c^q & 0 \\ c & a^q & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, c \in F_{q^2}, a^{q+1} - c^{q+1} = 1 \right\} \cong \text{SL}(2, q).$$

The center $Z$ of $\mathcal{M}_q$ is cyclic of order $q+1$ and is made by the elements of type (A) with center $P$; see [32, Section 3]. Hence,

$$Z = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$  

The intersection of $I = H \cap Z$ has order 2 and is generated by the unique involution $\iota = \text{diag}(-1, -1, 1)$ of $H$. Since $\frac{q+1}{2}$ is odd, the group generated by $H$ and $Z$ is a direct product $HZ = H \times \Omega$, where $\Omega \cong C_{\frac{q+1}{2}}$ is the subgroup of $Z$ of order $\frac{q+1}{2}$.

The group $HZ$ has index 2 in $\mathcal{M}_q$, and contains exactly one involution $\iota$. Let $\beta$ be any involution of $\mathcal{M}_q$ different from $\iota$, for instance $\beta = \text{diag}(-1, 1, 1)$; obviously, $\beta$ normalizes both $H = \mathcal{M}_q$ and $Z = Z(\mathcal{M}_q)$. Then

$$\mathcal{M}_q = (H \rtimes \langle \beta \rangle) \rtimes \Omega \cong (\text{SL}(2, q) \rtimes C_2) \rtimes C_{\frac{q+1}{2}}.$$  

With the notation of [24, Section 9] we also denote $H \rtimes \langle \beta \rangle$ by $\text{SU}^\pm(2, q)$, meaning that $H \rtimes \langle \beta \rangle$ consists of the elements of $\mathcal{M}_q$ with determinant 1 or $-1$; here, the determinant of an element $\alpha \in \mathcal{M}_q$ is the determinant of the representative matrix of $\alpha$ having entry 1 in the third row and column. Note that $H \rtimes \langle \beta \rangle$ is conjugated to $H \rtimes \langle \gamma \rangle$ in $\mathcal{M}_q$ for any involution $\gamma \in \mathcal{M}_q \setminus \{\iota\}$, because $\gamma$ and $\beta$ are conjugated in $\mathcal{M}_q$ (see [27, Lemma 2.2]) and $H$ is normal in $\mathcal{M}_q$. Note also that $\mathcal{M}_q$ contains no elements of type $(B3)$ or $(D)$.

For any $G \leq \mathcal{M}_q$, we will use the following notation:

$$G_\pm = G \cap \text{SU}^\pm(2, q), \quad G_H = G \cap H, \quad G_\Omega = G \cap \Omega, \quad \omega = |G_\Omega|.$$  

We determine in the following proposition the subgroups of $\text{SU}^\pm(2, q)$.

**Proposition 3.1.** The following is the complete list of subgroups of $\text{SU}^\pm(2, q) \leq \mathcal{M}_q$ for $q \equiv 1 \pmod{4}$.

- The subgroups of $H$, listed in Theorem 2.2.
- Cyclic groups of order 2.
- Groups $\text{SL}(2, 3) \rtimes C_2 \cong \text{SmallGroup}(48, 29)$ when $p \geq 5$ and $8 \nmid (q - 1)$.
- Cyclic groups of order $2d > 2$, where either $d \mid (q - 1)$ and $d \nmid \frac{q-1}{2}$, or $d \mid (q + 1)$.
- Abelian groups $C_d \rtimes C_2$, where $d \mid (q + 1)$ and $d$ is even.
• Dihedral groups of order $2d$, where $d \mid (q \pm 1)$.
• Groups $\text{Dic}_m = \langle \alpha, \epsilon \mid \alpha^{4m} = 1, \epsilon^2 = \alpha^{2m}, \epsilon^{-1}\alpha = \alpha^{2m-1}\rangle$ of order $8m$ extending a subgroup $\text{Dic}_m$ of $H$, when $m \mid \frac{q-1}{2}$ and $m \nmid \frac{q+1}{2}$.
• Groups $\text{Dic}_m \rtimes C_2$ of order $8m$ extending a subgroup $\text{Dic}_m$ of $H$, when $1 < m \mid \frac{q+1}{2}$.
• Groups $E_{p^k} \rtimes C_{2d}$, where $k \leq n$, $d \mid \gcd(p^k-1,q-1)$, $d$ is even, $E_{p^k}$ is elementary abelian of order $p^k$, and $C_{2d}$ is cyclic of order $2d$.
• Groups $\text{SU}^\pm(2,p^k) \cong SL(2,p^k) \rtimes C_2$, where $k \mid n$ and $n/k$ is odd.

**Proof.** Let $G \leq H \rtimes \langle \beta \rangle$ and assume $G \nleq H$, so that $G_H = G \cap H$ has index $2$ in $G$. We may assume that $G$ has order greater than 2, that is $G_H$ is a nontrivial subgroup of $H$. If $\iota$ is the unique involution of $H$, we denote by $\bar{G}$ and $\bar{G}_H$ the images of $G$ and $G_H$ under the canonical epimorphism $\text{SU}^\pm(2,q) \to \text{SU}^\pm(2,q)/\langle \iota \rangle$. Since $\iota$ is the kernel of the action of $\text{SU}^\pm(2,q)$ on the line $\ell$, the action of $\text{SU}^\pm(2,q)/\langle \iota \rangle$ on the $q+1$ points of $\mathcal{H}_q \cap \ell$ is equivalent to the action of a subgroup of $\text{PGL}(2,q)$, as they have the same order, $\text{SU}^\pm(2,q)/\langle \iota \rangle = \text{PGL}(2,q)$. Note that $|\bar{G}| = |G_H|$ or $|\bar{G}| = 2|\bar{G}|$ according to $|G_H|$ being even or odd, respectively. We classify $G_H$ according to Theorem 2.2.

• Suppose $G_H = \text{SL}(2,5)$. Then $\text{PGL}(2,q)$ has a subgroup $\bar{G}$ of order $120$ and $\bar{G}$ contains $\bar{G}_H \cong A_5$, a contradiction to Theorem 2.4.

• Suppose $G_H = \Sigma_4$. As in the previous point, $\text{PGL}(2,q)$ has a subgroup of order $48$ containing a subgroup isomorphic to $S_4$, a contradiction to Theorem 2.3.

• Suppose $G_H = E_{p^k} \rtimes C_{2d}$ with $k \leq n$ and $d \mid \gcd(p^k-1,q-1)$. The unique Sylow $p$-subgroup $E_{p^k}$ of $G$ is normal in $G$, and hence $G$ fixes the unique fixed point $Q \in \ell$ of $E_{p^k}$ on $\mathcal{H}_q$; see [24] Lemma 11.129). If $d$ is odd, then $|G| = 2|G_H|$ and $G \setminus G_H$ contains an involution $\omega$, which is of type (A) and has center on $\ell$: this contradicts, since $\omega$ cannot fix any point on $\ell \cap \mathcal{H}_q$ by Lemma 2.3. Then $d$ is even. By [24] Lemma 11.44), $G = E_{p^k} \rtimes C_{2d}$. Such a group $G = E_{p^k} \rtimes C_{2d}$ actually exists in $\mathcal{M}_q$; for instance, as in [18], use the model (2) for $\mathcal{H}_q$, assume up to conjugacy that $P = (0,1,0)$, and define

$$G = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid c^{q^k} + c = 0 \right\} \rtimes \left\{ \begin{pmatrix} 0 & p^k+1 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a^{2d} = 1 \right\} \cong E_{p^k} \rtimes C_{2d}. $$

Then $G \geq M_q$. Since $M_q = \text{SU}^\pm(2,q) \rtimes C_{2q^2}$ and $\gcd(|G|, \frac{q+1}{2}) = 1$, this implies $G \leq \text{SU}^\pm(2,q)$.

• Suppose $G_H = \text{SL}(2,p^k)$ with $k \mid n$. Then $|\bar{G}| = |\text{PGL}(2,p^k)|$ and $\bar{G}$ contains $G_H \cong \text{PSL}(2,p^k)$, so that $\bar{G} = \text{PGL}(2,p^k)$ by Theorem 2.4. Let $G_1$ be a subgroup of $\text{SU}^\pm(2,q)$ with $G_H \leq G_1$ and $[G_1 : G_H] = 2$. Clearly, $\bar{G}_1 = \bar{G} = \text{PGL}(2,p^k)$: we show that $G_1 = G$. Choose $\delta \in \bar{G}$ of order $o(\delta) = p^k - 1$ if $4 \mid (p^k - 1)$, or $o(\delta) = p^k + 1$ if $4 \nmid (p^k + 1)$. Let $\alpha \in G$ and $\bar{G}_1 \in G_1$ be preimages of $\delta$, that is $\bar{\alpha} = \bar{\alpha}_1 = \delta$; their order is $o(\alpha) = o(\alpha_1) = 2 \cdot o(\delta)$ and divides $2(q - 1)$. This implies in particular $\alpha, \alpha_1 \notin G_H$, so that $\bar{G} = \langle \bar{G}_H, \alpha \rangle$ and $\bar{G}_1 = \langle \bar{G}_1, \alpha_1 \rangle$. The group $\text{PGU}(3,q)$ contains a unique cyclic subgroup $C$ such that $\delta \in C$ and $[C : \langle \delta \rangle] = 2$; see Lemma 2.3; hence, $\langle \alpha \rangle = \langle \alpha_1 \rangle$. Thus, $G_1 = G$: there is at most one subgroup of $\text{SU}^\pm(2,q)$ containing $G_H$ with index 2.

If $n/k$ is even, then $G = \text{TL}(2,p^k)$ by Theorem 2.4. Assume that $n/k$ is odd. Then the group

$$G = \left\{ \begin{pmatrix} a & c^q & 0 \\ c & a^q & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, c \in \mathbb{F}_{p^{2k}}, a^{p^k+1} - c^{p^k+1} = 1 \right\} \rtimes \left\{ \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \zeta = \pm 1 \right\} $$

is a subgroup of $\text{SU}^\pm(2,q)$; this can be proved in analogy to the proof of [33] Lemma 3.10], i.e. showing by induction on $n/k$ that the condition $a^{p^k+1} - c^{p^k+1} = 1$ is equivalent to $a^{q^1} - c^{q^1} = 1$ for any $a, c \in \mathbb{F}_{p^{2k}}$. Clearly, $G \cong \text{SU}^\pm(2,p^k) = \text{SL}(2,p^k) \rtimes C_2$. 


• Suppose $G_H = TL(2, p^k)$. Then $\text{PGL}(2, q)$ has a subgroup $\hat{G}$ such that $|\hat{G}| = p^k(p^{2k} - 1)$ and $\text{PGL}(2, p^k) = \hat{G} \leq G$, a contradiction to Theorem 2.3.

• Suppose $G_H = C_d = \langle \alpha \rangle$ cyclic of order $d \mid (q \pm 1)$, so that $|G| = 2d$. Firstly, we prove that the conditions in the statement when $G$ is abelian or dihedral are necessary for the existence of $G$; secondly, we show that such conditions are also sufficient.

Note that, if $\alpha \in SU^\pm(2, q) \setminus H$ and $o(\alpha) > 2$, then $o(\alpha) \mid (q - 1)$. In fact, if $2 < o(\alpha) \mid (q - 1)$, then $\alpha$ is of type (B2) and fixes exactly two points other than $P$, say $Q, R \in \ell \cap \mathcal{H}_q$; but the pointwise stabilizer $S$ of $\{Q, R\}$ in $\text{PGU}(3, q)$ is cyclic of order $q^2 - 1$ (see Lemma 2.4), and $|S \cap H| = q - 1$, which implies $\alpha \in H$. Hence, if $d \mid (q - 1)$ and $G$ is cyclic, then $d = \frac{q - 1}{2}$.

We can assume that $d > 2$. A generator $\delta$ of $G_H$ is either of type (B1) or of type (B2) and has three fixed points $P, Q, R$, where $Q, R \in \ell$; since $G_H$ is normal in $G$, $G$ acts on $\{Q, R\}$. Let $\gamma \in G \setminus G_H$. If $\gamma(Q) = Q, \gamma(R) = R$, and $d \mid (q - 1)$, then $G$ is cyclic because the pointwise stabilizer of $\{P, Q, R\}$ in $\text{PGU}(3, q)$ is cyclic. If $\gamma(Q) = Q, \gamma(R) = R$, and $d \mid (q + 1)$, then $G$ is contained in the pointwise stabilizer $C_{q+1} \times C_{q+1}$ of $\{P, Q, R\} \subset \text{PG}(2, q^2) \setminus H_q$, see Section 3; hence, $G = C_d \times C_2$ (which is $C_{2d}$ when $d$ is odd). Assume that $\gamma$ interchanges $Q$ and $R$. Then $\gamma^2$ fixes $\ell$ pointwise, so that either $\gamma$ is an involution or $\gamma^2 = \iota$, the unique element of type (A) in $H$. If $\gamma^2 = \iota$, then $\gamma$ has order $4 \mid (q - 1)$, and hence $\gamma \in G_H$, a contradiction. Then $\gamma$ is an involution, and $G$ is a semidirect (eventually direct) product $G_H \ltimes \langle \gamma \rangle$; do so by $Q, R \in \text{PG}(2, q^2) \setminus H_q$ the two points of $\ell$ fixed by $\gamma$ (which are the center of $\gamma$ and the intersection between $\ell$ and the axis of $\gamma$; see Lemma 2.4). Since $d > 2$ and $\delta$ acts with orbits of length $d$ on $\ell \setminus \{Q, R\}$, $\delta$ cannot commute with $\gamma$ unless $\{Q, R\} = \{\tilde{Q}, \tilde{R}\}$, which implies $\delta$ being of type (B1) and hence $d \mid (q + 1)$. If $\delta$ and $\gamma$ do not commute, then $G$ induces a dihedral group $G \leq \text{PGL}(2, q)$ of order $\frac{d \cdot q}{\gcd(2d, i)} \cdot 2$, and $G$ is dihedral itself.

Conversely, we show that abelian and dihedral groups $G$ as in the statement actually exist in $SU^\pm(2, q) \setminus H$. To this aim, we make use of other models of $\mathcal{H}_q$ and provide groups $G$ which fix $P$ whose order is coprime to $\frac{q - 1}{2}$; this assures that $G \leq SU^\pm(2, q)$.

A cyclic group $G$ of order $2d$ with $d \mid (q - 1)$ and $d \nmid \frac{q - 1}{2}$ is provided as follows: $\mathcal{H}_q$ has equation (2), $P = (0, 1, 0)$, and $G$ is generated by $\text{diag}(a^{\sigma+1}, a, 1)$, where $0(a) = 2d$; if $G \leq H$, multiply its generator with $\text{diag}(-1, 1, 1) \in SU^\pm(2, q) \setminus H$.

A dihedral group $G$ of order $2d$ with $d \mid (q - 1)$ is provided as follows: $\mathcal{H}_q$ has equation (2), $P = (0, 1, 0)$, and $G$ is generated by $\text{diag}(a^{\sigma+1}, a, 1)$ and

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where $o(a) = d$.

A cyclic group $G$ of order $2d$ with $2 < d \mid (q + 1)$ is provided as follows: $\mathcal{H}_q$ has equation (11), $P = (0, 0, 1)$, and $G$ is generated by $\text{diag}(\lambda, \lambda^i, 1)$, where $o(\lambda) = 2d$ and $\gcd(2d, i) = 1$. After $\lambda$ is chosen, there exists only one value $i \in \{2, \ldots, 2d - 1\}$ such that $\text{diag}(\lambda, \lambda^i, 1) \in H$; hence we can choose $i$ such that $G \not\leq H$.

An abelian group $G = C_d \times C_2$ with $d \mid (q + 1)$ is provided as follows: $\mathcal{H}_q$ has equation (11), $P = (0, 0, 1)$, $G_H$ is generated by $\text{diag}(\lambda, \lambda^i, 1)$ as in the previous point, and $G$ is generated by $G_H$ together with $\text{diag}(-1, 1, 1) \in SU^\pm(2, q) \setminus H$.

A dihedral group of order $2d$ with $d \mid (q + 1)$ is provided as follows: $\mathcal{H}_q$ has equation (11), $P = (0, 0, 1)$, $G_H$ is generated by $\text{diag}(\lambda, \lambda^i, 1)$ as in the previous point, and $G$ is generated by $G_H$ together
with
\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

• Suppose \( G_H = \text{Dic}_m = \langle \delta, \epsilon \rangle \) with \( m \mid \frac{2m}{q-1} \), \( o(\delta) = 2m \), and \( o(\epsilon) = 4 \). Note that both \( \delta \) and \( \epsilon \) are of type (B2); let \( Q \) and \( R \) be the points on \( \ell \cap \mathcal{H}_q \) fixed by \( \delta \). Let \( \alpha \in G \setminus G_H \). Since \( \langle \delta \rangle \) is normal in \( G \), \( \alpha \) acts on \( \{Q, R\} \). Up to replacing \( \alpha \) with \( \alpha \epsilon \), we can assume that \( \alpha(Q) = Q \) and \( \alpha(R) = R \). Since the pointwise stabilizer \( S \) of \( \{Q, R\} \) in \( \text{PGU}(3, q) \) is cyclic, \( \langle \delta, \alpha \rangle \) is cyclic of order \( 4m \); up to replacing \( \alpha \) with a generator of \( \langle \delta, \alpha \rangle \), we can assume that \( \alpha \) is of type (B2) and has order \( 4m \). Therefore \( G = \langle \alpha, \epsilon \rangle \) with \( \alpha^{2m} = \epsilon^2 \). Since \( G \not\subseteq H \) and \( |S \cap H| = q - 1 \), we have \( o(\alpha) = 4m \mid (q - 1) \).

Such a subgroup \( G \) actually exists in \( \text{SU}^\pm(2, q) \) and is determined uniquely up to conjugation, as follows. Let \( \mathcal{H}_q \) have equation \( x^2 + y^2 + z^2 = 0 \), up to conjugation we have \( P = (0, 1, 0), Q = (1, 0, 0), R = (0, 0, 1) \). The only element of order \( 4m \) in \( \text{PGU}(3, q) \) fixing \( \{P, Q, R\} \) pointwise is \( \alpha = \text{diag}(\alpha^{q+1}, a, 1) \), where \( a \) is a primitive \( 4m \)-th root of unity. Any element of order 4 in \( \text{PGU}(3, q) \) fixing \( P \) and interchanging \( Q \) and \( R \) has the form
\[
\epsilon = \begin{pmatrix}
0 & 0 & \gamma \\
0 & 1 & 0 \\
-\gamma^{-1} & 0 & 0
\end{pmatrix},
\]
where \( \gamma^{q+1} = 1 \). By direct checking, \( \epsilon^{-1} \alpha \epsilon = \alpha^{2m-1} \) and \( G = \langle \alpha, \epsilon \rangle \) has order \( 8m \). Since \( G \leq \mathcal{M}_q \) and \( \gcd(|G|, \frac{2m}{q-1}) = 1 \) we have \( G \leq \text{SU}^\pm(2, q) \). Also, the assumptions \( m \mid \frac{2m}{q-1} \) and \( m \mid \frac{2m}{q-1} \) imply \( \alpha \notin H \) and \( \alpha^2 \in H \), so that \( G_H = \langle \alpha^2, \epsilon \rangle \cong \text{Dic}_m \). The elements of \( G \) are \( \alpha^i \) and \( \alpha^i \epsilon \), with \( i = 0, \ldots, 4m - 1 \). By direct checking, \( \alpha^{2m} \) and \( \alpha^i \epsilon \) with odd \( i \) are involutions and hence of type (A), while \( \alpha^i \) with \( j \neq 0, 2m \) and \( \alpha^i \epsilon \) with even \( k \) are of type (B2).

• Suppose \( G_H = \text{Dic}_m = \langle \delta, \epsilon \rangle \) with \( m \mid \frac{2m}{q-1} \), \( o(\delta) = 2m \), and \( o(\epsilon) = 4 \). Denote by \( Q \) and \( R \) the points other than \( P \) fixed by \( \delta \); we have \( Q, R \in \ell \setminus \mathcal{H}_q \). Let \( \alpha \in G \setminus G_H \). If \( o(\alpha) = 4m \), then \( o(\alpha) \mid (q^2 - 1) \) and \( o(\alpha) \mid (q + 1) \), so that \( \alpha \) is of type (B2) and \( \alpha^2 \) is of type (A), a contradiction to the fact that \( \ell \) is the only element of type (A) in \( H \); hence, \( o(\alpha) \neq 4m \). Since \( \langle \delta \rangle \) is normal in \( G \), the subgroup \( K = \langle \delta, \alpha \rangle \) has order \( 4m \). As \( K \) is not cyclic, we have shown above that either \( K \) is a direct product \( C_{2m} \times C_2 \), or \( K \) is a dihedral group \( C_{2m} \rtimes C_2 \). We can then assume that \( \alpha \) is an involution.

We show that we can also assume \( K = \langle \delta \rangle \times \langle \alpha \rangle \cong C_{2m} \times C_2 \). The number of subgroups of order 4 generated elements of \( G_H \setminus \langle \delta \rangle \) is equal to \( m \) and hence is odd. This implies that \( \alpha \) normalizes \( \langle \zeta \rangle \) for some \( \zeta \in G_H \setminus \langle \delta \rangle \) with \( o(\zeta) = 4 \); up to conjugation, \( \zeta = \epsilon \). Let \( \mathcal{H}_q \) have equation \( (x, y, z) \) and assume up to conjugation that \( P = (0, 0, 1), Q = (1, 0, 0), \) and \( R = (0, 0, 1) \). If \( o(\alpha(Q) = Q \) and \( o(R) = R \), then \( \alpha \) is represented by a diagonal matrix and commutes with \( \delta \). Suppose that \( \alpha \) and \( \delta \) do not commute, so that \( \alpha \) interchanges \( Q \) and \( R \). Then
\[
\delta = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda^{-1} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \alpha = \begin{pmatrix}
0 & a & 0 \\
a^{-1} & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \epsilon = \begin{pmatrix}
1 & 0 & 0 \\
0 & -\epsilon^{-1} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
for some \((q + 1)\)-th roots of unity \( \lambda, a, \epsilon \) with \( o(\lambda) = 2m \). Since \( \alpha \) normalizes \( \epsilon \), either \( a^2 = -\epsilon^2 \) if \( \alpha \epsilon = \epsilon \), or \( a^2 = e^2 \) if \( o(\alpha \epsilon = \epsilon^{-1} \). In the former case, \( \alpha \epsilon \) is a diagonal matrix, so that \( \alpha \epsilon \) fixes \( \{Q, R\} \) pointwise; but \( o(\alpha \epsilon) = 4 \), so that \( \alpha \) is of type (B2): a contradiction. Hence, \( o(\alpha \epsilon = \epsilon^{-1} \); this implies \( o(\alpha \epsilon = 2 \); and either \( \alpha \epsilon = \text{diag}(1, -1, 1) \) or \( \alpha \epsilon = \text{diag}(-1, 1, 1) \). We can then replace \( \alpha \) with \( \alpha \epsilon \), so that \( K = \langle \delta \rangle \times \langle \alpha \rangle \) and \( G \) is a product \( K \langle \epsilon \rangle \cong (C_{2m} \times C_2)C_4 \), with \( |K \cap \langle \epsilon \rangle| = 2 \).

Such a group \( G = \langle \langle \delta \rangle \times \langle \alpha \rangle \rangle \langle \epsilon \rangle \cong (C_{2m} \times C_2)C_4 \) actually exists in \( \text{SU}^\pm(2, q) \), as the choice in \( (4) \) shows. The elements in \( \langle \delta \rangle \times \langle \alpha \rangle \rangle of order greater than 2 are of type (B1); the three involutions in \( \langle \delta \rangle \times \langle \alpha \rangle \rangle are of type (A); the elements \( \delta \epsilon \) of \( G_H \setminus \langle \delta \rangle \) have order 4 and are of type (B2); the remaining elements have the form \( \delta \epsilon \alpha \epsilon \), are involutions, and are of type (A).
Suppose $G_H = \text{SL}(2,3)$ with $p \geq 5$. Then $G$ is a subgroup of order 24 with a subgroup $\bar{G}_H \cong A_4$; from Theorem 2.3. $\bar{G}$ isomorphic to $S_4$. By direct checking with MAGMA [1], the unique groups $L$ admitting a normal subgroup isomorphic to $\text{SL}(2,3)$ and such that the factor group of $L$ over the unique involution of $\text{SL}(2,3)$ is isomorphic to $\text{SmallGroup}(48,28) \cong \bar{S}_4$ and $\text{SmallGroup}(48,29)$.

We show that, if $L_1$ and $L_2$ are subgroups of $\text{SU}^\pm(2,q)$ containing $G_H$ with index $[L_1 : G_H] = [L_2 : G_H] = 2$, then $L_1 = L_2$. By direct inspection on $\text{SmallGroup}(48,28)$ and $\text{SmallGroup}(48,29)$, both $L_1$ and $L_2$ are generated by $G_H$ together with any element of order 8, whose square lies in $G_H$. Also, any cyclic subgroup $C_4$ of order 4 of $G_H$ is contained is contained both in a cyclic subgroup $C_8$ of order 4 of $L_1$ and in a cyclic subgroup $C_8$ of order 8 of $L_2$. The group $C_4$ is generated by an element of type (B2) with two fixed points $Q, R \in \ell \cap H_q$; thus, $C_8^q$ and $C_8^q$ act on $\{Q, R\}$. If a generator $\alpha_i$ of $C_8^q (i \in \{1, 2\})$ interchanges $Q$ and $R$, then $\alpha_i^2$ is of type (A), a contradiction to Lemma 2.3. Thus, both $C_8^q$ and $C_8^q$ fix $\{Q, R\}$ pointwise. Since the pointwise stabilizer of $\{Q, R\}$ in $\text{PGU}(3, q)$ is cyclic, this implies $C_8^q = C_8^q$ and hence $L_1 = L_2$.

If $8 \mid (q - 1)$, then by Theorem 2.3 there $H$ already contains a subgroup $\bar{S}_4 \cong \text{SmallGroup}(48,28)$ having a subgroup isomorphic to $\text{SL}(2,3)$. Hence, there exists no subgroup $G$ of $\text{SU}^\pm(2,q)$ with $G_H \cong \text{SL}(2,3)$.

If $8 \nmid (q - 1)$, then $G \leq \text{SU}^\pm(2,q)$ with $G_H \cong \text{SL}(2,3)$ does exist, and can be constructed as follows. Let $H_q$ be given by Equation (2); up to conjugation, $P = (0,1,0)$ and $\ell : Y = 0$. Choose $\lambda, \mu \in F_q$ and $c, e \in F_q^2$ such that $\lambda^2 = -1$, $\mu^2 = \lambda, c^2 = \frac{\lambda + 1}{2}$, and $e = \mu c$. Define
\[
\begin{align*}
\alpha_1 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\alpha_2 &= \begin{pmatrix} 0 & 0 & -\lambda c \\ 0 & 1 & 0 \\ -\frac{\lambda}{c} & 0 & 0 \end{pmatrix}, \\
\alpha_3 &= \begin{pmatrix} 0 & 0 & c \\ 0 & 1 & 0 \\ -\frac{1}{c} & 0 & 0 \end{pmatrix}, \\
\xi &= \begin{pmatrix} \frac{\lambda + 1}{2} & 0 & \frac{\lambda - 1}{2}c \\ 0 & 1 & 0 \\ c & 0 & \frac{1 - \lambda}{2} \end{pmatrix}, \\
\gamma &= \begin{pmatrix} 0 & 0 & c \\ 0 & 1 & 0 \\ e^{-1} & 0 & 0 \end{pmatrix}.
\end{align*}
\]

By direct checking, the following holds.
- $\alpha_1, \alpha_2, \alpha_3$ have order 4; $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \iota = \text{diag}(1, -1, 1)$; $\alpha_1 \alpha_2 \alpha_3 = \text{id}$. Hence, $Q_8 = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is a quaternion group.
- $\alpha_1, \alpha_2, \alpha_3 \in M_q$ since they preserve the equation (2) of $H_q$ and fix $P$; $\alpha_1, \alpha_2, \alpha_3$ are of type (B2). The fixed points other than $P$ are $P_1 = (1,0,0)$ and $P_3 = (0,0,1)$ for $\alpha_1$; $Q_1 = (-c, 0, 1)$ and $Q_3 = (c, 0, 1)$ for $\alpha_2$; $R_1 = (-\lambda c, 0, 1)$ and $R_3 = (\lambda c, 0, 1)$ for $\alpha_3$.
- $\xi$ has order 3 and $\xi \in M_q$. For $i = \{1, 3\}$, we have $\xi(Q_i) = Q_i, \xi(Q_i) = R_i$; and $\xi(Q_i) = R_i$; since the pointwise stabilizer of two points in $\ell \cap H_q$ is cyclic, this implies that $\xi$ normalizes $Q_8$.
- $\xi$ does not commute with $\alpha_j$. Then $\langle \alpha_1, \alpha_2, \alpha_3, \xi \rangle = Q_8 \rtimes C_3$ is isomorphic to $\text{SL}(2,3)$.
- $\xi$ has no fixed points in $\text{PG}(2, F_q^2) \setminus \ell$; this implies by Lemma 2.3 that $\xi$ is of type (B1).
- $\gamma$ is an involution of $M_q$ satisfying $\gamma(P_1) = P_3, \gamma(Q_1) = R_1$, and $\gamma(Q_3) = R_3$: this implies that $\gamma$ normalizes $Q_8$. Also, $\gamma \xi \gamma = \alpha_2 \xi \in Q_8 \rtimes C_3$: this implies that $\gamma$ normalizes $Q_8 \rtimes C_3 \cong \text{SL}(2,3)$.
- Therefore, $G = \langle \alpha_1, \alpha_2, \alpha_3, \xi, \gamma \rangle$ is a subgroup of $M_q$ of order 48, with a subgroup of index 2 isomorphic to $\text{SL}(2,3)$. Since $G$ contains a Klein four-group $\langle i, \gamma \rangle$, we have $G \cong \text{SmallGroup}(48,29)$. We have $\gcd(|G|, \frac{24}{2}) \in \{1, 3\}$; recall that $M_q \rtimes C_3 \rtimes \mathbb{Z}_2$. If $\gcd(|G|, \frac{24}{2}) = 1$, then $G \leq \text{SU}^\pm(2,q)$.

If $\gcd(|G|, \frac{24}{2}) = 3$, then all elements of order 3 in $G$ are of type (B1) as they are conjugated to $\xi$; this implies again $G \leq \text{SU}^\pm(2,q)$, because any element $\psi$ of order 3 in $M_q \setminus \text{SU}^\pm(2,q)$ is of type (A).

In fact, let $H_q$ have equation (1); up to conjugation, $P = (0,0,1)$ and $\psi$ fixes $Q = (1,0,0)$ and $R = (0,1,0)$. This implies $\psi = \text{diag}(u, v, 1)$ with $u^3 = v^3 = 1$. If $\psi$ is not of type (A), then $u \neq 1, v \neq 1, u \neq v$; hence, $v = u^{-1}$ and $\psi \in \text{SU}^\pm(2,q)$.
We will make use of the following remark, which can be easily proved in analogy to [3] Remark 4.1.

**Remark 3.2.** Let \( G \leq M_q \) be such that \( G/G_\Omega \) is generated by elements whose order is coprime to \( |G_\Omega| \). Then \( G = G_k \times G_\Omega \). If in addition \( G/G_\Omega \) is generated by elements of odd order, then \( G = G_H \times G_\Omega \).

Now we compute the genera of quotient curves \( H_q/G \) for all subgroups \( G \) of \( M_q = SU^\pm(2,q) \times \Omega \). The factor group \( G/G_\Omega \) is isomorphic to a subgroup of \( SU^\pm(2,q) \). Hence, we will consider the different possibilities for \( G/G_\Omega \) given by Proposition 3.1 and Theorem 2.2.

**Lemma 3.3.** Let \( G \leq M_q \) be such that one of the following cases holds:

1. \( G/G_\Omega \) is cyclic of order a divisor of \( q + 1 \) different from 2.
2. \( G/G_\Omega \) is dihedral of order 4m with \( 1 < m \leq \frac{q-1}{2} \).
3. \( G/G_\Omega \cong C_d \times C_2 \) where \( d \mid (q+1) \) and \( d \) is even.
4. \( G/G_\Omega \) is dihedral of order 2d with \( 1 < d \mid (q+1) \).
5. \( G/G_\Omega \cong Dic_m \times C_2 \) with \( 1 < m \leq \frac{q-1}{2} \).

Then \( G \) is contained in the maximal subgroup of \( PGU(3,q) \) stabilizing a self-polar triangle \( T \subset PG(2,q^2) \setminus H_q \).

If \( G \leq M_q \) is such that \( G/G_\Omega \cong E_{p^k} \rtimes C \), where \( E_{p^k} \) is elementary abelian of order \( p^k \) and \( C \) is cyclic, then \( G \) is contained in the maximal subgroup of \( PGU(3,q) \) stabilizing a point of \( H_q(F_{q^2}) \).

**Proof.** If \( \langle \alpha_1 G_\Omega, \ldots, \alpha_r G_\Omega \rangle \) is a normal subgroup of \( G/G_\Omega \), then \( \langle \alpha_1, \ldots, \alpha_r \rangle \) is a normal subgroup of \( G \), because \( G_\Omega \) is central in \( G \).

- Let \( G/G_\Omega \cong E_{p^k} \rtimes C \). Then \( E_{p^k} \) has a unique fixed point on \( H_q(F_{q^2}) \); see [24] Lemma 11.129. As \( E_{p^k} \) is normal in \( G \), \( G \) fixes this point.

- Let \( G/G_\Omega = \langle \alpha G_\Omega \rangle \times \langle \gamma G_\Omega \rangle \) satisfy assumption (3), with \( \alpha^d, \gamma^2 \in G_\Omega \). We show that \( o(\alpha), o(\gamma) \mid (q+1) \); this implies that \( \alpha \) and \( \gamma \) are of type (A) or (B1), \( \langle \alpha, \gamma \rangle \) fixes pointwise a triangle \( T = \{ P, Q, R \} \) with \( Q, R \in \ell(F_{q^2}) \setminus H_q \), and hence \( G \) fixes \( T \) pointwise, the claim.

Suppose by contradiction that \( o(\alpha) 
\mid (q+1) \). By Lemma 2.4 \( o(\alpha) 
2(q+1) \), \( \alpha \) is of type (B2), and \( \alpha^2 \) is of type (A) with center \( P \). Let \( Q, R \in \ell \cap H_q \) be the fixed points of \( \alpha \) other than \( P \). Since the pointwise stabilizer of \( \{ Q, R \} \) is cyclic unlike \( G/G_\Omega \), \( \gamma \) interchanges \( Q \) and \( R \). Let \( H_q \) have equation \( \overline{2} \); up to conjugation, \( P = (0,1,0), Q = (1,0,0), R = (0,0,1) \); hence,

\[
\alpha = \begin{pmatrix}
-1 & 0 & 0 \\
0 & a & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \gamma = \begin{pmatrix}
0 & 0 & c \\
0 & 1 & 0 \\
d & 0 & 0
\end{pmatrix},
\]

where \( c^{q+1} = d^{q+1} = a^{2(q+1)} = 1 \neq q^{q+1} \). Then \( \alpha \gamma \neq \gamma \alpha \), a contradiction. Suppose that \( o(\gamma) 
(q+1) \). Then swap the roles of \( \alpha \) and \( \gamma \) in the argument above to obtain a contradiction.

- Let \( G/G_\Omega = \langle \alpha G_\Omega \rangle \) satisfy assumption (1), with \( |G/G_\Omega| = d \). By Lemma 2.4 \( o(\alpha) \mid (q^2 - 1) \). Together with \( o(\alpha) \mid d(q+1) \), this yields \( o(\alpha) \mid 2(q+1) \). If \( o(\alpha) 
(q+1) \), then \( \alpha \) is of type (B2) and \( \alpha^2 \in G_\Omega \), a contradiction. Hence, \( o(\alpha) \mid (q+1) \), and \( \alpha \) is of type (A) or (B1). Since \( \langle \alpha \rangle \) is normal in \( G \), \( G \) acts on the points fixed by \( \alpha \); as \( G \) fixes \( P \), this implies that \( G \) stabilizes a self-polar triangle \( \{ P, Q, R \} \subset PG(2,q^2) \setminus H_q \).

- Let \( G/G_\Omega = \langle \alpha G_\Omega, \xi G_\Omega \rangle \cong Dic_n \) or \( G/G_\Omega = \langle \alpha G_\Omega, \xi G_\Omega, \varepsilon G_\Omega \rangle \cong Dic_n \rtimes C_2 \) satisfy assumption (2) or (5), respectively, with \( 6 \leq o(\alpha G_\Omega) = 2n \mid (q+1) \). Using the normality of \( \langle \alpha \rangle \) in \( G \) and arguing as in the previous point, we have that \( \alpha \) is of type (A) or (B1), and \( G \) stabilizes a self-polar triangle \( T \subset PG(2,q^2) \setminus H_q \).

- Let \( G/G_\Omega = \langle \alpha \rangle \times \langle \gamma \rangle \) satisfy assumption (4), with \( \alpha^d, \gamma^2 \in G_\Omega \). If \( d = 2 \), then \( G \) satisfies also assumption (3) and the claim was already proved. If \( d > 2 \), then \( \langle \alpha \rangle \) is normal in \( G \); arguing as in the previous point, \( \alpha \) is of type (A) or (B1), and \( G \) stabilizes a self-polar triangle \( T \subset PG(2,q^2) \setminus H_q \). □
Lemma 3.3 provides cases for $G/G_\Omega$ which do not need to be considered in the following, since $G$ is contained in a maximal subgroup of $\text{PGU}(3,q)$ for which $g(H_q/G)$ has already been computed in the literature. Namely, if $G/G_\Omega \cong E_p^s \times C$, then $g(H_q/G)$ is computed in \cite[Theorem 1.1]{2}; if $G/G_\Omega$ satisfies one of the assumptions (1) to (5), then $g(H_q/G)$ is computed in \cite[Proposition 3.4]{3}.

**Proposition 3.4.** Let $G \leq \mathcal{M}_q$ be such that $G/G_\Omega \cong \text{SL}(2,5)$, with $q^2 \equiv 1 \pmod{5}$. Then

$$g(H_q/G) = \frac{(q+1)(q-1-2\omega) + 180\omega - 20r - 48s}{240\omega},$$

where

$$r = \begin{cases} 4\omega & \text{if } 3 \mid (q-1), \\ q + 1 + 2\omega & \text{if } 3 \mid q, \\ 0 & \text{if } 3 \mid (q+1), \ 3 \nmid \omega, \end{cases} \quad s = \begin{cases} 2\omega & \text{if } 5 \mid (q-1), \\ 0 & \text{if } 5 \mid (q+1), \ 5 \nmid \omega, \\ q + 1 & \text{if } 5 \mid \omega. \end{cases}$$

**Proof.** By Remark 3.2, $G = G_H \times G_\Omega$ with $G_H \cong \text{SL}(2,5)$. The nontrivial elements of $G$ are as follows.

- 20 elements of order 3 in $G_H$: they are of type (B2), (C), or (B1), according to $3 \mid (q-1)$, $3 \mid q$, or $3 \mid (q+1)$, respectively.
- $2\omega - 1$ elements in $\langle \iota \rangle \times G_\Omega$: they are of type (A).
- $30\omega$ elements obtained as the product of a nontrivial element of order 4 in $G_H$ by an element of $G_\Omega$: they are of type (B2).
- $20\omega$ elements obtained as the product of an element of order 6 in $G_H$ by an element of $G_\Omega$: they are of type (B2), (E), or (B1), according to $3 \mid (q-1)$, $3 \mid q$, or $3 \mid (q+1)$, respectively.
- $48\omega$ elements obtained as the product of an element $\eta \in G_H$ of order 5 or 10 by an element $\theta \in G_\Omega$. If $5 \mid (q-1)$, they are of type (B2). If $5 \mid (q+1)$ and $5 \nmid \omega$, they are of type (B1).

If $5 \mid \omega$, 48 of them are of type (A), the other ones are of type (B1). Namely, if $o(\eta) = 10$ then $\eta\theta$ is of type (B1); if $o(\eta) = 5$ and $\{P,Q,R\}$ are the fixed points of $\eta$, then there are exactly 2 choices for $\theta \in G_\Omega$ such that $\eta\theta$ is of type (A), one with center $Q$, the other with center $R$.

- $20(\omega - 1)$ elements obtained as the product of an element of order 3 in $G_H$ by a nontrivial element of $G_\Omega$: they are of type (B2) or (E) if $3 \mid (q-1)$ or $3 \mid q$, respectively. If $3 \not\mid (q+1)$, either they are all of type (B1) or there are 48 of them of type (A), according to $5 \nmid \omega$ or $5 \mid \omega$, arguing as in the previous case.

The claim follows by direct computation using the Riemann-Hurwitz formula and Theorem 2.5.

**Proposition 3.5.** Let $G \leq \mathcal{M}_q$ be such that $G/G_\Omega \cong \hat{\Sigma}_4 \cong \text{SmallGroup}(48,28)$, with $p \geq 5$ and $8 \mid (q-1)$. Then

$$g(H_q/G) = \frac{(q+1)(q-1-2\omega) + 36\omega - 16r}{96\omega},$$

where

$$r = \begin{cases} 2\omega & \text{if } 3 \mid (q-1), \\ 0 & \text{if } 3 \mid (q+1), \ 3 \nmid \omega, \\ q + 1 & \text{if } 3 \mid \omega. \end{cases}$$

**Proof.** By Remark 3.2 and Proposition 3.1, $G = G_H \times G_\Omega$ with $G_H \cong \text{SmallGroup}(48,28)$. The nontrivial elements of $G$ are as follows. Since $SU^\pm(2,q)$ has no elements of type (A) with odd order, the type of any element in $G_H$ is uniquely determined by its order. The product of $\iota$ by an element of $G_\Omega$ has type (A), and the product of an element of type (B2) in $G_H$ by an element of $G_\Omega$ has type (B2). The product of an element $\eta$ of type (B1) in $G_H$ by an element of $G_\Omega$ has type (B1), unless $o(\eta) \mid \omega$: if $o(\eta) \mid \omega$, then $G_\Omega$ contains exactly 2 elements $\theta_1, \theta_2$ such that $\eta\theta_1$ and $\eta\theta_2$ are of type (A); here, this argument applies to the elements of order 3. Now the claim follows by direct computation with the Riemann-Hurwitz formula and Theorem 2.5.
Proposition 3.6. Let $G \leq \mathcal{M}_q$ be such that $G/G_{22} \cong SL(2,3)$ with $p \geq 5$. If $3 | (q - 1)$, then

$$g(H_q/G) = (q + 1)(q - 1 - 2\omega) + 4\omega.$$  

If $3 | (q + 1)$, then one of the following cases holds:

$$g(H_q/G) = (q + 1)(q - 1 - 2\omega) + 36\omega - 8(q + 1)(\gcd(3, \omega) - 1).$$

$$g(H_q/G) = (q + 1)(q - 9 - 2\omega) + 36\omega,$$

with $3 \nmid \omega$;

$$g(H_q/G) = (q + 1)(q - 1 - 2\omega) + 36\omega,$$

with $3 \mid \omega$, $3 \nmid (q + 1)$. All cases (5) to (8) actually occur, for some $G$ as in the assumptions.

Proof. Assume $3 | (q - 1)$. By Remark 3.2, $G = G_H \times G_{22}$ with $G_H \cong SL(2,3)$. The nontrivial elements of $G$ are as follows: $2\omega - 1$ elements of type $(A)$ in $\langle i \rangle \times G_{22}$; $22\omega$ elements of order divisible by 3 or 4, which are of type (B2). Equation (5) follows by the Riemann-Hurwitz formula and Theorem 2.5.

For the rest of the proof, assume $3 | (q + 1)$. Let $G/G_{22} = Q_8 \times \langle \xi G_{22} \rangle$, where $Q_8$ is a quaternion group and $\xi \notin G_{22}$ satisfies $\xi^3 \in G_{22}$. Since $Q_8$ and $G_{22}$ have coprime orders, $Q_8$ is induced by a subgroup $\langle \alpha, \omega \rangle$ of $G$ isomorphic to $Q_8$.

Suppose that there exists $\xi \in G_H$ inducing $\xi G_{22}$; this can be assumed when $3 \nmid \omega$. Then $G = G_H \times G_{22}$, and the nontrivial elements of $G$ are as follows: $2\omega - 1$ elements of type $(A)$ in $\langle i \rangle \times G_{22}$; $6\omega$ elements of order divisible by 4, which are of type (B2). If $3 \mid \omega$, any other element is of type (B1). If $3 \mid \omega$, then there are 8·2 elements of order 3 and type (A); namely, for any element $\eta \in G_H$ of order 3 there are exactly 2 elements $\theta_1, \theta_2 \in G_{22}$ such that $\eta^\theta_1, \eta^\theta_2$ are of type (A); any other element is of type (B1). Equation (6) follows by the Riemann-Hurwitz formula and Theorem 2.5.

For the rest of the proof, we can assume that $\xi \notin G_H$ for any $\xi \in G$ inducing $\xi G_{22}$. As the subgroups of $G$ of order 3 are conjugated by elements of $Q_8$, no element of $G_H$ induces an element of order 3 in $G/G_{22}$.

- Suppose that $o(\xi) = 3$. Since $\xi \notin G_H$ and $\xi \notin G_{22}$, this implies that $\xi$ is of type $(A)$ with center on $\ell$ and axis passing through $P$. Note that $G_H = Q_8$. Note also that $3 \nmid \omega$; otherwise, $G_H$ has an element $\rho$ of order $\rho$, and either $\rho$ or $\rho^2$ is an element of type $(B1)$ and order 3 lying in $G_H$, a contradiction. The nontrivial elements of $G$ are as follows: $2\omega - 1$ elements of type $(A)$ in $\langle i \rangle \times G_{22}$; $8\omega$ elements of order 3 and type (A); $6\omega$ elements of order a multiple of 4 and type (B2); any other element is of type (B1). Equation (7) follows by the Riemann-Hurwitz formula and Theorem 2.5.

Such a group $G$ actually exists in $\mathcal{M}_q$. In fact, let $3 \mid \omega$ and $(Q_8 \times C_3) \times C_\omega$ be the subgroup isomorphic to $SL(2,3) \times G_{22}$ constructed above, with $Q_8 \times C_3 \leq H$. Let $\eta$ be a generator of $C_3 \leq H$ and $\rho$ be an element of order 3 in $\Omega$. Then $\eta^\rho$ is an element of order 3 and type $(A)$ not in $\Omega$, such that $G := (Q_8 \times \langle \eta^\rho \rangle) \times C_\omega$ is the desired group.

- Now suppose that $o(\xi) > 3$. Up to composing with an element of $G_{22}$, we can assume that $\xi$ is a 3-element, of order $3^k$, $k \geq 2$. As $\xi^3$ is a nontrivial element of $G_{22}$, $\xi$ is not of type $(A)$, and hence is of type (B1). As $G_{22}$ is cyclic, $\langle \xi^3 \rangle$ is the Sylow 3-subgroup of $G_{22}$. Then $G \cong (Q_8 \times C_{3^k}) \times C_{\omega/3^k-1}$, where $C_{3^k} = \langle \xi \rangle$ and $3^k \nmid \omega$. The nontrivial elements of $G$ are as follows:

  - Elements of $Q_8 \times C_{3^k}$. There are $2 \cdot 3^k - 1$ elements of type $(A)$ in $\langle i \rangle \times C_{3^k-1}$; $6 \cdot 3^{k-1}$ elements of type (B2), as the product of an element of order 4 in $Q_8$ by an element of $C_{3^k-1}$; $2(3^k - 3^{k-1})$ elements of type (B1) in $\langle i \rangle \times C_{3^k} \setminus (\langle i \rangle \times C_{3^k-1})$.

The $6(3^k - 3^{k-1})$ elements $\sigma$ obtained as the product of an element $\alpha$ of order 4 in $Q_8$ by an element $\gamma$ of order $3^k$ in $C_{3^k}$ are of type (B1). In fact, $o(\sigma) \in \{3^k, 2 \cdot 3^k, 4 \cdot 3^k\}$. If $o(\sigma) = 4 \cdot 3^k$, then $\sigma$ is of type (B2) and $o(\sigma^4) \mid (q + 1)$, so that $\sigma^4 \in G_H$ with $o(\sigma^4) = 3^k$, a contradiction. If
Proposition 3.7. Let $G$ be cyclic of order $d$ and $G$ stabilizes pointwise a self-polar triangle $\{P, Q, R\} \subset PG(2, q^2) \setminus H_q$, or

$$g(H_q/G) = \frac{(q-1)(q+1-\omega \cdot \gcd(d, 2))}{2d\omega}.$$ 

Both cases occur.

Proof. Let $\alpha G_\Omega$ be a generator of $G/G_\Omega$. If $d = 1$, the claim is trivial. Suppose $d = 2$. Since $|G_\Omega|$ is odd, $G$ is cyclic of order $2d | (q+1)$ and fixes pointwise a self-polar triangle $\{P, Q, R\} \subset PG(2, q^2) \setminus H_q$. Suppose $d > 2$. Then $\alpha$ is of type (B2) and $G$ is cyclic. The number of elements of type (A) is either $\omega - 1$ or $2\omega - 1$, according to $d$ odd or $d$ even, respectively; any other nontrivial element is of type (B2).

The claim follows by direct computation with Theorem 2.5. □

Proposition 3.8. Let $G \leq M_q$ be such that $G/G_\Omega \cong Dic_n$, with $1 < n | \frac{q^3-1}{2}$. Then

$$g(H_q/G) = \frac{(q-1)(q+1-2\omega)}{8\omega^2}.$$ 

Proof. By Remark 3.2 and Proposition 3.1, $G = G_H \times G_\Omega$ with $G_H \cong Dic_n$. Any nontrivial element $\sigma \in G$ is of type (A) if $\sigma \in \langle \iota \rangle \times G_\Omega$, and of type (B2) otherwise. The claim follows by Theorem 2.5. □

Proposition 3.9. Let $G \leq M_q$ be such that $G/G_\Omega \cong SL(2, p^k)$ with $k | n, r = n/k$. Then

$$g(H_q/G) = 1 + \frac{q^2 - q - 2 - \Delta}{2p^k(p^k - 1)\omega},$$

where

$$\Delta = (p^{2k} - 1)(q + 2) + p^{2k} - 1 + q + 1 + p^k(p^k + 1)(p^k - 3)\omega + p^k(p^k - 1)^2(\gcd(r, 2) - 1) + 2(p^{2k} - 1)(\omega - 1) + 2(\omega - 1)(q + 1) + p^{2k}(p^k - 1)^2(\omega - 1)(\gcd(r, 2) - 1) + (\gcd(\omega, p^k + 1) - 1)p^k(p^k - 1)(q + 1)(2 - \gcd(r, 2)).$$

Proof. By Remark 3.2, $G = G_H \times G_\Omega$ with $G_H \cong SL(2, p^k)$. Clearly $(p^k - 1) | (q - 1)$, while $(p^k + 1) | (q - 1)$ or $(p^k + 1) | (q + 1)$ according to $r$ even or $r$ odd, respectively. The nontrivial elements in $G_H$ are classified in the proof of [32] Proposition 4.3] as follows:

- $p^{2k} - 1$ elements of order $p$ and type (C);
- $p^{2k} - 1$ elements of order $p$ times a nontrivial divisor of $q + 1$, which are of type (E);
- $1$ involution $\iota$, of type (A);
- $p^k(p^k + 1)(p^k - 1)$ elements of order a divisor of $p^k - 1$ different from 2, which are of type (B2);
• $p^k(p^k-1)/2$ elements of order a divisor of $p^k + 1$ different from 2, which are of type (B1) or (B2) according to $r$ odd or $r$ even, respectively.

The product of an element $\sigma \in G_H$ by a nontrivial element $\tau \in G_\Omega$ is as follows. If $\sigma$ is of type (C) or (E), then $\sigma \tau$ is of type (E). If $\sigma \in \langle \iota \rangle$, then $\sigma \tau$ is of type (A). If $\sigma$ is of type (B2), then $\sigma \tau$ is of type (B2).

If $\sigma$ is of type (B1) and $o(\sigma) \mid |G_\Omega|$, then $G_\Omega$ contains exactly 2 elements $\tau_1, \tau_2$ such that $\sigma \tau_1, \sigma \tau_2$ are of type (A); otherwise, $\sigma \tau$ is of type (B1). If $r$ is even, there are no elements of type (B1). Assume $r$ odd. The elements of type (B1) together with $\langle \iota \rangle$ form $\mathbb{Z}/(p^k-1)$ cyclic groups of order $p^k + 1$ which intersect pairwise in $\langle \iota \rangle$. Then, the number of elements $\sigma \tau$ of type (A) with $\sigma$ of type (B1) is $\frac{p^k(p^k-1)}{2} \cdot (\gcd(\omega, p^k + 1) - 1) \cdot 2$.

Now the claim follows by direct computation with Theorem 2.5.

**Proposition 3.10.** Let $G \leq M_q$ be such that $G/G_\Omega \cong TL(2, p^k)$, where $k \mid n$ and $n/k$ is even. Then

$$g(H_q/G) = 1 + \frac{q^2 - q - 2 - \Delta}{4p^k(p^2k - 1)\omega},$$

where

$$\Delta = (p^2k - 1)(q + 2) + p^2k - 1 + q + 1 + p^k(p^k - 1)(p^k - 3)\omega + p^k(p^k - 1)^2 + 2(p^{2k} - 1)(\omega - 1)
+ 2(\omega - 1)(q + 1) + p^k(p^k - 1)^2(\omega - 1) + 2p^k(p^{2k} - 1)\omega.$$

**Proof.** By Remark 3.2 and Proposition 3.1, $G = G_H \times G_\Omega$ with $G_H = \langle L, \delta \rangle \cong TL(2, p^k), L \cong SL(2, p^k)$, $o(\delta) = 2(p^k - 1)$. The nontrivial elements in $L \times G_\Omega$ are already classified according to their type in the proof of Proposition 3.9. Every element in $G_H \setminus L$ is of type (B2); see the proof of [32, Proposition 4.4]. Hence, for every $\sigma \in G_H \setminus L$ and $\tau \in G_\Omega$, $\sigma \tau$ is of type (B2). The claim now follows by direct computation with Theorem 2.5.

The case $G/G_\Omega$ isomorphic to a cyclic subgroup of $SU^\pm(2, q)$ of order 2 not in $H$ has already been considered in Proposition 3.7.

**Proposition 3.11.** Let $G \leq M_q$ be such that $G/G_\Omega \cong SL(2, 3) \rtimes C_2 \cong SmallGroup(48, 29)$, with $p \geq 5$ and $8 \mid (q - 1)$. Then

$$g(H_q/G) = \frac{(q + 1)(q - 2\omega - 13) + 60\omega - 16r}{96\omega},$$

where $r = \begin{cases} 
2\omega & \text{if } 3 \mid (q - 1), \\
0 & \text{if } 3 \mid (q + 1), 3 \nmid \omega, \\
q + 1 & \text{if } 3 \nmid \omega.
\end{cases}$

**Proof.** By Remark 3.2 and Proposition 3.1, $G = G_\pm \times G_\Omega$ with $G_\pm \cong SmallGroup(48, 29)$ and $G_H \cong SL(2, 3)$. By Lemma 2.4, $G_\pm$ contains 13 involutions, of type (A); 18 elements of order 8 or 4, of type (B2); 16 elements of order 6 or 3, which are of type (B2) or (B1) according to $3 \mid (q - 1)$ or $3 \mid (q + 1)$. The element $\sigma \tau$, where $\sigma \in G_\pm$ and $\tau \in G_\Omega \setminus \{id\}$, is as follows. If $\sigma$ is the unique involution $\iota$ of $G_H$, then $\sigma \tau$ is of type (A). If $\sigma$ is an involution different from $\iota$, then $\sigma \tau$ is of type (B1). If $\sigma$ is of type (B2), then $\sigma \tau$ is of type (B2). If $\sigma$ has order 6 and is of type (B1), then $\sigma \tau$ is of type (B1). If $\sigma$ has order 3 and is of type (B1), then there are $\gcd(|G_\Omega|, 3) - 1$ elements of $G_\Omega$ such that $\sigma \tau$ is of type (A), while for any other $\tau \in G_\Omega \sigma \tau$ is of type (B1); in fact, $\sigma \tau$ is of type (A) if and only if $o(\tau) = 3$.

The claim follows by the Riemann-Hurwitz formula and Theorem 2.5.

**Proposition 3.12.** Let $G \leq M_q$ be such that $G/G_\Omega$ is cyclic of order $2d > 2$, where either $d \mid (q - 1)$ and $d \nmid \frac{q + 1}{2}$, or $d \mid (q + 1)$. Assume also that $G$ does not stabilize any self-polar triangle $T \subset PG(2, q^2) \setminus H_q$. Then either $d \mid (q - 1)$ and $d \nmid \frac{q + 1}{2}$, or $d = 2$; in both cases,

$$g(H_q/G) = \frac{(q - 1)(q + 1 - 2\omega)}{4d\omega}.$$
Whenever \( d \) satisfies the numerical assumptions, a subgroup \( G \leq \mathcal{M}_q \) with \( g(\mathcal{H}_q/G) \) as in Equation (9) exists.

**Proof.** Let \( \alpha G_\Omega \) be a generator of \( G/G_\Omega \).

Suppose that \( d \mid (q-1) \) and \( d \nmid \frac{q-1}{2} \). Then \( \alpha(\alpha) \mid (q^2 - 1) \) and \( \alpha \) is of type (B2). The group \( G \) fixes pointwise the 2 fixed points \( Q, R \in \ell \cap \mathcal{H}_q \) other than \( P \). Hence, \( G \) is cyclic; we can assume that \( G = \langle \alpha \rangle \). Since \( \gcd(\alpha(\alpha), q + 1) = 2 \), \( G \) contains \( 2\omega - 1 \) elements of type (A) and \( 2d\omega - 2 \omega \) elements of type (B2); Equation (9) follows. Such a group \( G \) does exist in \( \mathcal{M}_q \), being generated by any element of type (B2) and order \( 2d\omega \).

Suppose that \( d \mid (q + 1) \). If \( 2d \mid (q + 1) \), then Lemma [3.3] implies that \( G \) stabilizes a self-polar triangle \( T \subset PG(2, q^2) \setminus \mathcal{H}_q \); hence, we can assume that \( 2d \nmid (q + 1) \), that is, \( d \) is even and \( d/2 \) is odd. Then \( 4 \mid \alpha(\alpha) \) and \( \alpha \) is of type (B2). Also, \( \alpha(\alpha^2) \mid (q + 1) \) implies that \( \alpha^4 \in G_\Omega \), so that \( d = 2 \). As above, \( G \) is cyclic, and we can assume that \( \alpha \) is a generator of \( G \). The group \( G \) has \( 2\omega - 1 \) elements of type (A) and \( 4\omega - 2 \omega \) elements of type (B2); Equation (9) follows. Such a group \( G \) does exist in \( \mathcal{M}_q \), being generated by any element of type (B2) and order \( 4\omega \).

**Proposition 3.13.** Let \( G \leq \mathcal{M}_q \) be such that \( G/G_\Omega \) is dihedral of order \( 2d \), with \( 2 < d \mid (q - 1) \). Then

\[
g(\mathcal{H}_q/G) = \frac{(q + 1)(q - 1 - \gcd(d, 2) \cdot \omega - d) + 2\omega(d + \gcd(d, 2))}{4d\omega}.
\]

**Proof.** By Remark [3.2] \( G = G_\pm \times G_\Omega \), where \( G_\pm = \langle \alpha G_\Omega, \gamma G_\Omega \rangle \) is dihedral; we can assume that \( \alpha(\alpha) = d \) and \( \alpha(\gamma) = 2 \). Such a group \( G \) actually exists in \( \mathcal{M}_q \), as shown in the proof of Proposition [3.1] using the model \( [2] \) of \( \mathcal{H}_q \). The nontrivial elements of \( G \) are as follows: \( \gcd(d, 2) \cdot \omega - 1 \) elements of type (A) and center \( P \) in \( G_\Omega \) or \( \langle \iota \rangle \times G_\Omega \), according to \( d \) odd or \( d \) even; \( d \) elements of type (A) with center on \( \ell \), in \( G_\pm \); \( (d - \gcd(d, 2))\omega \) elements of type (B2) in \( \langle \alpha \rangle \times G_\Omega ; d\omega - 1 \) elements of type (B1) as the product of an involution in \( G_\pm \setminus \langle \iota \rangle \) by an element of \( G_\Omega \). The claim follows by direct computation with Theorem [2.5].

**Proposition 3.14.** Let \( G \leq \mathcal{M}_q \) be such that \( G/G_\Omega \cong \text{Dic}_m \), where \( m \mid \frac{q-1}{2} \) and \( m \nmid \frac{q-1}{4} \). Then

\[
g(\mathcal{H}_q/G) = 1 + \frac{q^2 - q - 2 - [(2m + 2\omega - 1)(q + 1) + 4\omega(3m - 1)]}{16m\omega}.
\]

**Proof.** By Remark [3.2] \( G = G_\pm \times G_\Omega \) with \( G_\pm \cong \text{Dic}_m \). From the proof of Proposition [3.1] the nontrivial elements \( G_\pm \) are exactly: 1 involution \( \iota \); \( 2m \) other involutions of type (A) with center on \( \ell \); \( 6m - 2 \) elements of type (B2). The nontrivial elements of \( G_\Omega \) are of type (A). The product \( \sigma \tau \) with \( \sigma \in G_\pm \) and \( \tau \in G_\Omega \setminus \{id\} \) is as follows: of type (A), if \( \sigma = \iota \); of type (B1), if \( \sigma \) is an involution different from \( \sigma \); of type (B2), if \( \sigma \) is of type (B2). The claim follows by direct computation with Theorem [2.5].

**Proposition 3.15.** Let \( G \leq \mathcal{M}_q \) be such that \( G/G_\Omega \cong \text{SU}^\pm(2, p^k) \cong \text{SL}(2, p^k) \rtimes C_2 \), where \( k \mid n \) and \( n/k \) is odd. Then

\[
g(\mathcal{H}_q/G) = 1 + \frac{q^2 - q - 2 - \Delta}{4p^k(p^{2k} - 1)\omega},
\]

where

\[
\Delta = (q + 1) + p^{2k}(p^k + 1)(p^k - 3) + (p^{2k} - 1)(q + 3) + p^{2k}(p^k - 1)(q + 1) + p^{2k}(p^{2k} - 1) + (2\omega - 2)(q + 1)
+ 2(p^{2k} - 1)(\omega - 1) + 2p^{2k}(p^k - 1)(p^{2k} - 2)(\omega - 1) + 2p^{2k}(p^k)(q + 1)(\gcd(p^k + 1, \omega - 1)).
\]

**Proof.** By Remark [3.2] and Proposition [3.1] \( G = G_\pm \times G_\Omega \) with \( G_\pm \cong \text{SU}^\pm(2, p^k) \) and \( G_\Omega \cong \text{SL}(2, p^k) \).

The nontrivial elements of \( G_\Omega \) are classified according to their type in the proof of [3.2] Proposition [4.3]. Namely, \( G_\Omega \) contains exactly: 1 element \( \iota \) of type (A); \( \frac{p^{2k}(p^k - 1)^2}{2} \) elements of type (B1), forming \( \frac{p^{2k}(p^k - 1)}{2} \) cyclic groups of order \( p^k + 1 \) with pairwise intersection \( \langle \iota \rangle \); \( \frac{p^{2k} + 1}{2} \) elements of type (B2), forming \( \frac{(p^k + 1)}{2} \) cyclic groups of order \( p^k - 1 \) with pairwise intersection \( \langle \iota \rangle ; p^{2k} - 1 \) elements of type (C), forming
$p^k + 1$ elementary abelian groups of order $p^k$ with trivial pairwise intersection; $p^{2k} - 1$ elements of type (E), contained in $p^k + 1$ cyclic groups of order $2p^k$ with pairwise intersection $(i)$. The elements of $G_{\pm} \setminus G_H$ are classified as follows.

- $G_{\pm} \setminus G_H$ contains exactly $p^k(p^k - 1)$ elements of type (A) and $\frac{p^k(p^k - 1)^2}{2}$ elements of type (B1).

In fact, let $\alpha \in M_q \setminus (\langle i \rangle \times \Omega)$ be of type (A) or (B1), and $\{Q, R\} \in \PGU(3, p^{2k}) \setminus H_q$ be the fixed points of $\alpha$ on $\ell$. Let $H_q$ have equation (11); up to conjugation, $G_{\pm}$ is made by $\mathbb{F}_{p^{2k}}$-rational elements, as pointed out in Equation (13). Also, up to conjugation, $P = (0, 0, 1)$, $Q = (1, 0, 0)$, and $R = (0, 1, 0)$, so that $\alpha = \text{diag}(\lambda, \mu, 1)$ with $\lambda^{o(\alpha)} = \mu^{o(\alpha)} = 1$. For any $(p^k + 1)$-th root of unity $\lambda$, $\alpha \in G_{\pm} \setminus G_H$ if and only if $\det(\alpha) = -1$, i.e., $\mu = -\lambda^{-1}$. Note that $-\lambda^{-1} \neq \lambda$ since $4 \nmid (p^k + 1)$. Hence, after the choice of $\{Q, R\}$, there are exactly 2 elements of type (A) (namely, $\alpha = \text{diag}(1, -1, 1)$ and $\alpha = \text{diag}(-1, 1, 1)$) and $p^k - 1$ elements of type (B1) (namely, $\alpha = \text{diag}(\lambda, -\lambda^{-1}, 1)$ with $\lambda \neq \pm 1$).

Let $\mathcal{F}$ be the free abelian group on $r$ elements, as pointed out in Equation (3). Also, up to conjugation, for any element of type (E) in $H_q$, $\alpha = \text{diag}(\lambda, -\lambda^{-1}, 1)$ with $\lambda \neq \pm 1$.

There are exactly $\frac{p^{2k} - 1}{2}(p^k - 1)$ choices for $\{Q, R\}$. In fact, $Q$ can be chosen as anyone of the $\mathbb{F}_{p^{2k}}$-rational points of $\ell$ which are not on $H_q$; then $R$ is uniquely determined as the pole of the line $PQ$ with respect to the unitary polarity associated to $H_q(\mathbb{F}_{q^2})$.

- As $p \nmid |G : H_q|$, there are no $p$-elements in $G \setminus G_H$. Also, any element of type (E) in $M_q$ is the product $\sigma \tau$ of a $p$-element of type (C) by an element $\tau$ of type (A) in $\langle i \rangle \times \Omega$ (i.e., $\tau$ has center $P$). Since $G_{\pm} \cap (\langle i \rangle \times \Omega) = \langle i \rangle$, there are no elements of type (E) in $G_{\pm} \setminus G_H$.

- Any other element of $G_{\pm} \setminus G_H$ is of type (B2); their number is

$$p^k(p^{2k} - 1) \cdot \frac{(p^k - 1)^2}{2} = \frac{p^k(p^k - 1)(2p^k + 2 - 2p^k + 1)}{2} = \frac{p^k(p^{2k} - 1)}{2}.$$ The nontrivial elements of $G_\Omega$ are of type (A). The elements $\sigma \tau$ with $\sigma \in G_{\pm} \setminus \{id\}$ and $\tau \in G_\Omega \setminus \{id\}$ are classified as follows.

- If $\sigma$ is of type (C) or (E), then $\sigma \tau$ is of type (E).
- If $\sigma = \iota$, then $\sigma \tau$ is of type (A).
- If $\sigma$ is an involution different from $\iota$, then $\sigma \tau$ is of type (B1).
- If $\sigma$ is of type (B2), then $\sigma \tau$ is of type (B2).
- Let $\sigma$ be of type (B1) and $\{Q, R\}$ be the points fixed by $\sigma$ on $\ell$; we have $\frac{p^k(p^k - 1)}{2}$ choices for $\{Q, R\}$. Arguing as above, we can use the model $(11)$ for $H_q$, assume that $\{P, Q, R\}$ is the fundamental triangle, and that $\sigma = \text{diag}(\lambda, -\lambda^{-1}, 1)$ or $\sigma = \text{diag}(\lambda, -\lambda^{-1}, 1)$, with $\lambda^{p^{2k} + 1} = 1$, $\lambda \neq \pm 1$. If $o(\lambda) \mid \omega$, then there exists exactly one $\tau \in G_{\Omega} \setminus \{id\}$ such that $\sigma \tau$ is of type (A); otherwise, $\sigma \tau$ is of type (B1). Altogether, when $\sigma$ ranges over the elements of type (B1), there are exactly $\frac{p^k(p^k - 1)}{2} \cdot (\gcd(\omega, p^k + 1) - 1) \cdot 4$ elements $\sigma \tau$ of type (A); the other ones are of type (B1).

Now the claim follows by direct computation with Theorem 2.5. □

4. The complete list of genera of quotients of $H_q$ for $q \equiv 1 \pmod{4}$

This section provides the explicit complete list of genera of quotients $H_q[G, G \leq \PGU(3, q)]$ for $q = p^n \equiv 1 \pmod{4}$, and hence it gives a collection of the results obtained in this paper together with the results already obtained in the literature. It will be divided into subsections corresponding to the maximal subgroup of $\PGU(3, q)$ containing $G$. The case in which $G \leq \PGU(3, q)Q, Q \in \PGU(2, q^2) \setminus H_q$ will not be repeated here as it was already described in the previous sections.

4.1. $G \leq \PGU(3, q)P, P \in H_q(\mathbb{F}_{q^2})$.

Let $P \in H_q(\mathbb{F}_{q^2})$ and suppose that $G \leq \PGU(3, q)P$, the stabilizer of $P$ in $\text{Aut}(H_q) = \PGU(3, q)$. Since $\PGU(3, q)$ acts transitively on $H_q(\mathbb{F}_{q^2})$, we can assume that $P = P_{\infty}$, that is, the unique point at infinity of
the model $y^{q+1} = x^q + x$ of $\mathcal{H}_q$. The complete list for odd values of $q$ of genera of quotients $\mathcal{H}_q / G$ where $G \leq \text{PGU}(3, q)_{\infty}$ was determined in [2] and partially in [13].

In these papers, an element $\sigma \in \text{PGU}(3, q)_{\infty}$ is uniquely described as a triple of elements in $F_{q^2}$, $\sigma = [a, b, c]$. The authors associate to $G$ three sets, namely

$$G_1 = \{a \mid [a, b, c] \in G\}, \quad G_2 = \{b \mid [1, b, c] \in G\}, \quad G_3 = \{c \mid [1, 0, c] \in G\}$$

with $|G_1| = g_1$, $|G_2| = p^{g_2}$, and $|G_3| = p^{g_3}$.

In [15] it is proved that the genus of the quotient $\mathcal{H}_q / G$ depends just on the values $g_i$. Namely, the following theorem is proved.

**Theorem 4.1.** (see [15] and [2] Lemma 4.1) Let $G \leq \text{PGU}(3, q)_{\infty}$, $G_i$, and $g_i$ be defined as above, and let $d = \gcd(g_1, q + 1)$. Then

$$g(\mathcal{H}_q / G) = \frac{q - p^{g_3}}{2|G|} (q - (d - 1)p^{g_2}).$$

Defining $r$ and $u$ to be the smallest integers such that $p^r \equiv 1 \pmod{g_1}$ and $p^u \equiv 1 \pmod{g_1/d}$ respectively, then $|G| = h_1p^{g_2r + g_3u}$.

At this point the authors provide necessary and sufficient conditions to the triple $(g_1, g_2, g_3)$ to be associated with an existing subgroup $G$ of $\text{PGU}(3, q)_{\infty}$, so that Theorem [11] gives the complete list of genera when $G$ is in $\mathcal{H}_q(\mathbb{F}_{q^2})$. The authors observed that when $g_1 | (q^2 - 1)$ is fixed it is sufficient to consider the cases $0 \leq g_2 \leq 2n/r$ and $0 \leq g_3 < n/u$ as otherwise $\mathcal{H}_q / G$ would be rational.

**Theorem 4.2.** ([2] Theorems 4.3 and 5.8) Define $r$ and $u$ as above. Let $M_G(p, n) = \{(g_1, g_2, g_3) : g_1 | (q^2 - 1), 0 \leq g_2 \leq 2n/r, \text{ and } 0 \leq g_3 < n/u \text{ for some } G \leq \text{PGU}(3, q)_{\infty}\}$. Then

- If $g_1 | (q - 1)$ then for every $0 \leq g_2 \leq 2n/r$ and $0 \leq g_3 < n/u$ $(g_1, g_2, g_3) \in M_G(p, n)$.
- If $g_1 | (q^2 - 1)$ but $g_1 \nmid (q - 1)$ then for every $0 \leq g_2 \leq 2n/r$ and $0 \leq g_3 < n/u$ $(g_1, g_2, g_3) \in M_G(p, n)$.

Moreover, this is the complete list of elements in $M_G(p, n)$.

4.2. $G \leq \text{PGU}(3, q)_{T}$, $T$ a self-polar triangle in $\text{PG}(2, q^2) \setminus \mathcal{H}_q$.

Let $T = \{P_1, P_2, P_3\}$ be a self-polar triangle in $\text{PG}(2, q^2) \setminus \mathcal{H}_q$ and $\text{PGU}(3, q)_{T}$ be the maximal subgroup of $\text{PGU}(3, q)$ stabilizing $T$. We have $\text{PGU}(3, q)_{T} = (C_{q+1} \times C_{q+1}) \rtimes S_3$, where $C_{q+1} \times C_{q+1}$ stabilizes $T$ pointwise and $S_3$ acts faithfully on $T$.

The genera of quotients $\mathcal{H}_q / G$ where $G \leq \text{PGU}(3, q)_{T}$ are completely classified in [8] as follows according to the action of $G$ in $T$. We define $G_T = G \cap (C_{q+1} \times C_{q+1})$.

**Theorem 4.3.** ([8] Theorem 3.1) Let $q + 1 = \prod_{i=1}^{\ell} p_i^{r_i}$ be the prime factorization of $q + 1$.

(i) For any divisors $a = \prod_{i=1}^{\ell} p_i^{u_i}$ and $b = \prod_{i=1}^{\ell} p_i^{v_i}$ of $q + 1$ $(0 \leq s_i, t_i \leq r_i)$, let $c = \prod_{i=1}^{\ell} p_i^{u_i}$ be such that, for all $i = 1, \ldots, \ell$, we have $u_i = \min\{s_i, t_i\}$ if $s_i \neq t_i$, and $s_i \leq u_i \leq r_i$ if $s_i = t_i$. Define $d = a + b + c - 3$. Let $e = \frac{abc}{\gcd(a, b)} \prod_{i=1}^{\ell} p_i^{r_i}$, where for all $i$’s $v_i$ satisfies $0 \leq v_i \leq r_i - \max\{s_i, t_i, u_i\}$. We also require that, if $p_i = 2$ and either $2 \mid abc$ or $2 \mid \gcd(a, b, c)$, then $v_i = 0$. Then there exists a subgroup $G$ of $C_{q+1} \times C_{q+1}$ such that

$$g(\mathcal{H}_q / G) = \frac{(q + 1)(q - 2 - d) + 2e}{2e}.\]
Proposition 4.4. (Proposition 3.4) Let $q$ be odd.

(i) Let $\ell, a, c,$ and $e$ be positive integers satisfying $e \mid (q + 1)^2, c \mid (q + 1), \ell \mid c, d \mid a, ac \mid e, \frac{a}{c} \mid (q + 1),$ and $\gcd\left(\frac{a}{ac}, \frac{a}{c}\right) = 1$. If $2 \mid a$ or $2 \nmid c,$ we also require that $2 \nmid \frac{a}{c}$. Then there exists a subgroup $G \leq (C_{q+1} \times C_{q+1}) \rtimes S_3$ of order $2e$ such that $|G \cap (C_{q+1} \times C_{q+1})| = e$ and

\[
g(H_q/G) = \frac{(q + 1)(q - 2a - c + 1 - h) - 2k + 4e}{4e},
\]

where

\[
(h,k) = \begin{cases} 
(\frac{5}{2}, \frac{3}{2}) & \text{if } 2a \mid (q + 1); \\
(\frac{1}{2}, 0) & \text{if } 2a \mid (q + 1), 2a \nmid c; \\
(0, e) & \text{if } 2a \mid c, 2\ell \mid (q + 1); \\
(0, 0) & \text{if } 2a \mid c, 2\ell \mid (q + 1), 2\ell \nmid c; \\
(\frac{1}{2}, 0) & \text{if } 2a \mid c, 2\ell \mid c.
\end{cases}
\]

(ii) Conversely, if $G \leq (C_{q+1} \times C_{q+1}) \rtimes S_3$ and $G \cap (C_{q+1} \times C_{q+1})$ has index 2 in $G$, then the genus of $H_q/G$ is given by Equation (12), where $e = |G|/2$; without loss of generality, $a - 1$ is the number of homologies in $G$ with center $P_1$ which is equal to the number of homologies in $G$ with center $P_2$, and $c - 1$ is the number of homologies in $G$ with center $P_3$; $\ell = \frac{a(3)}{2}$ for some $\beta \in G \setminus G_T$; $\ell, a, c, e$ satisfy the numerical assumptions in point (i).

Proposition 4.5. (Proposition 3.5) Let $q$ be such that $3 \nmid (q + 1)$.

(i) Let $a$ and $e$ be positive integers satisfying $e \mid (q + 1)^2, a^2 \mid e, \frac{a}{a} \mid (q + 1), 2 \nmid \frac{a}{a}$, and $\gcd\left(\frac{a}{a}, a\right) = 1$. We also require that there exists a positive integer $m \leq \frac{a}{a}$ such that $\frac{a}{a} \mid (m^2 - m + 1)$. Then there exists a subgroup $G \leq (C_{q+1} \times C_{q+1}) \rtimes S_3$ of order $3e$ such that $|G \cap (C_{q+1} \times C_{q+1})| = e$ and

\[
g(H_q/G) = \frac{(q + 1)(q - 3a + 1) + 2e}{6e}.
\]

(ii) Conversely, if $G \leq (C_{q+1} \times C_{q+1}) \rtimes S_3$ and $G \cap (C_{q+1} \times C_{q+1})$ has index 3 in $G$, then the genus of $H_q/G$ is given by Equation (13), where $e = |G|/3$; the number of homologies in $G$ with center $P_i$ is $a - 1$ for $i = 1, 2, 3$; there exist $\ell$ and $m$ such that $a, c, \ell, m$ satisfy the numerical assumptions in point (i).

Proposition 4.6. (Proposition 3.6) Let $q$ be such that $3 \mid (q + 1)$.

(i) Let $a, e,$ and $\ell$ be positive integers satisfying $e \mid (q + 1)^2, a^2 \mid e, \frac{a}{a} \mid (q + 1), 2 \nmid \frac{a}{a}$, $\gcd\left(\frac{a}{a}, a\right) = 1$, and $\ell \mid (q + 1)$. We also require that there exists a positive integer $m \leq \frac{a}{a}$ such that $\frac{a}{a} \mid (m^2 - m + 1)$. Then there exists a subgroup $G \leq (C_{q+1} \times C_{q+1}) \rtimes S_3$ of order $3e$ such that $|G \cap (C_{q+1} \times C_{q+1})| = e$ and

\[
g(H_q/G) = \frac{(q + 1)(q - 3a + 1) + h \cdot e}{6e},
\]

with

\[
h = \begin{cases} 
2 & \text{if } a \nmid \frac{a}{a} + 1; \\
0 & \text{if } a \nmid \frac{a}{a} + 1, \ell \mid \frac{a}{a} + 1; \\
6 & \text{if } a \mid \frac{a}{a} + 1, \ell \mid \frac{a}{a} + 1.
\end{cases}
\]

(ii) Conversely, if $G \leq (C_{q+1} \times C_{q+1}) \rtimes S_3$ and $G \cap (C_{q+1} \times C_{q+1})$ has index 3 in $G$, then the genus of $H_q/G$ is given by Equation (14), where $e = |G|/3$; the number of homologies in $G$ with center $P_i$ is $a - 1$ for $i = 1, 2, 3$; there exist $\ell$ and $m$ such that $a, c, \ell, m$ satisfy the numerical assumptions in point (i).

Proposition 4.7. (Proposition 3.7)
(i) Let $a$ be a divisor of $q+1$. We choose $e = a^2$ if $3 \mid (q+1)$ or $3 \mid a$; else $e \in \{a^2, 3a^2\}$. Then there exists a subgroup $G \leq (C_{q+1} \times C_{q+1}) \rtimes S_3$ of order $6e$ such that $|G \cap (C_{q+1} \times C_{q+1})| = e$ and

$$g(H_q/G) = \frac{(q+1)(q - 3a + 1 - \frac{3e}{2}) - 2r - 3s + 12e}{12e},$$

where

$$r = \begin{cases} \frac{2e}{3} & \text{if } q \equiv 0 \text{ or } 1 \text{ (mod 3) and } a \nmid \frac{q+1}{2}, \\ 2e & \text{if } q \equiv 2 \text{ (mod 3) and } a \nmid \frac{q+1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

and $s = \begin{cases} \frac{4e}{3} & \text{if } q \equiv 2 \text{ (mod 3) and } a \nmid \frac{q+1}{2}, \\ 0 & \text{otherwise.} \end{cases}$

(ii) Conversely, if $G \leq (C_{q+1} \times C_{q+1}) \rtimes S_3$, then one of the following holds:

$$g(H_q/G) = \begin{cases} \frac{1}{2} \left( \frac{q^2-q+1}{\nu} - 1 \right); & q \equiv 2 \text{ (mod 3) and } 3 \nmid \nu; \\ \frac{q^2-q+1}{6\nu}, & q \equiv 2 \text{ (mod 3) and } 3 \mid \nu; \\ \frac{q^2-q+1-3\nu}{6\nu}, & q \equiv 2 \text{ (mod 3) and } 3 \nmid \nu. \end{cases}$$

When $q \equiv 2$ (mod 3) and $3 \nmid \nu$, then both the third and the fourth line in Equation (16) are obtained for some $G \leq \PGU(3,q)$.

4.3. $G \leq \PGU(3,q)$, $T$ a triangle in $H_q(F_{q^2}) \setminus H_q(F_{q^2})$.

Let $T = \{P_i, P_2, P_3\}$ be a triangle in $H_q(F_{q^2}) \setminus H_q(F_{q^2})$ which is invariant under the Frobenius collineation $\Phi_{q^2}$ of $\PG(2,q^6)$, and $\PGU(3,q)_T$ be the maximal subgroup of $\PGU(3,q)$ stabilizing $T$. We have $\PGU(3,q)_T = S \times C_3$, where $S \cong C_{q^2-q+1}$ is a Singer subgroup stabilizing $T$ pointwise and acting semiregularly on $\PG(2,q^2)$; $C_3$ acts faithfully on $T$.

The genera of quotients $H_q/G$ where $G \leq \PGU(3,q)_T$ are completely classified in [6] whenever $p \nmid |G|$. The case $p \mid |G|$ happens only if $p = 3$. This case was not considered in [6], and can be dealt with using Theorem 2.5 and the fact that each element in $G \setminus G_T$ has order 3.

Theorem 4.8. (6 Proposition 4.2)

(i) Let $\nu \mid (q^2 - q + 1)$. Then there exists $G \leq \PGU(3,q)_T$ such that $|G_T| = \nu$ and one of the following holds:

$$g(H_q/G) = \begin{cases} \frac{1}{2} \left( \frac{q^2-q+1}{\nu} - 1 \right); & q \equiv 2 \text{ (mod 3) and } 3 \nmid \nu; \\ \frac{q^2-q+1}{6\nu}, & q \equiv 2 \text{ (mod 3) and } 3 \mid \nu; \\ \frac{q^2-q+1-3\nu}{6\nu}, & q \equiv 2 \text{ (mod 3) and } 3 \nmid \nu. \end{cases}$$

When $q \equiv 2$ (mod 3) and $3 \mid \nu$, then both the third and the fourth line in Equation (16) are obtained for some $G \leq \PGU(3,q)$.

(ii) Conversely, let $G \leq \PGU(3,q)_T$. If $G = G_T$, then $g(H_q/G)$ is given by the first line in Equation (16) with $\nu = |G|$. If $G \neq G_T$, then $g(H_q/G)$ is given by the second or third or fourth line in Equation (16) with $\nu = |G|/3$.

4.4. $G \leq \PGU(3,q)$ has no fixed points or triangles.

The genus of quotient $H_q/G$ with $G \leq \PGU(3,q)$ was computed in [33] whenever $G$ has no fixed points or triangles. The equations of this section describe the genera of such quotients $H_q/G$.

Theorem 4.9. (33) For any integer $\bar{q}$ provided by one of the Equations (17) to (27), there exists $G \leq \PGU(3,q)$ such that $g(H_q/G) = \bar{q}$ and $G$ has no fixed points or triangle.

Conversely, if $G \leq \PGU(3,q)$ has no fixed points or triangles, then $g(H_q/G)$ is given by one of the Equations (17) to (27).
(17) \[ \frac{q^2 - 34q + 289}{432}, \quad \frac{q^2 - 10q + 25}{144}, \quad \frac{q^2 - 10q + 25}{72}, \]

where \( G \cong \text{PGU}(3, 2), G \cong \text{PSU}(3, 2), G \cong \text{SmallGroup}(36, 9), \) respectively.

(18) \[ \frac{q^2 - 16q + 103 - 24\gamma - 20\delta}{120}, \quad \text{when} \quad p = 5 \quad \text{or} \quad 5 \mid (q^2 - 1), \quad G \cong A_5, \]

\[ \delta = \begin{cases} 2, & \text{if either } p = 3 \text{ or } 3 \mid (q - 1), \\ 0, & \text{if } 3 \mid (q + 1), \end{cases} \quad \text{and} \quad \gamma = \begin{cases} 0, & \text{if } 5 \mid (q + 1), \\ 2, & \text{if } p = 5 \text{ or } 5 \mid (q - 1). \end{cases} \]

(19) \[ \frac{q^2 - q - 2 - \Delta}{q(q + 1)(q - 1)} + 1, \]

where \( q = q^h, \bar{q} \neq 3, G \cong \text{PSL}(2, q), \) and

- \[ \Delta = 2(q - 2)(\bar{q} + 1) + 2\bar{q}(\bar{q} + 1) \left( \frac{\bar{q} - 1}{2} \right)^2 + \bar{q}(\bar{q} + 1)(q + 1) + \delta \bar{q}(\bar{q} + 1) \left( \frac{q + 1}{2} - 1 \right), \quad \text{if } \bar{q} \equiv 1 \pmod{4}, \]

- \[ \Delta = 2(\bar{q} - 1)(q + 1) + 2\bar{q}(\bar{q} + 1) \left( \frac{\bar{q} - 1}{2} \right)^2 + \bar{q}(q + 1) + \delta \bar{q}(\bar{q} + 1) \left( \frac{\bar{q} + 1}{2} - 2 \right), \quad \text{if } \bar{q} \equiv 3 \pmod{4}, \]

with \( \delta = \begin{cases} 2, & \text{if } h \text{ is even}, \\ 0, & \text{otherwise}; \end{cases} \)

(20) \[ \frac{q^2 - q - 2 - \Delta}{2q(q + 1)(q - 1)} + 1, \]

where \( q = q^h, \bar{q} \neq 3, G \cong \text{PGL}(2, q), \) and

\[ \Delta = 2(\bar{q} - 1)(\bar{q} + 1) + \frac{\bar{q}(\bar{q} + 1)}{2} (q + 1) + \frac{\bar{q}(\bar{q} - 1)}{2} (q + 1) + 2\frac{\bar{q}(\bar{q} + 1)}{2} (\bar{q} - 1 - 2) + \frac{\bar{q}(\bar{q} - 1)}{2} (\bar{q} + 1 - 2) \]

and

\[ \delta = \begin{cases} 2, & \text{if } h \text{ is even}, \\ 0, & \text{otherwise}. \end{cases} \]

(21) \[ \frac{q^2 - 22q + 229 - 56\alpha - 48\beta}{336}, \quad \text{when} \quad p = 7 \quad \text{or} \quad \sqrt{-7} \notin \mathbb{F}_q \]

where

\[ \alpha = \begin{cases} 0, & \text{if } 3 \mid (q + 1), \\ 2, & \text{otherwise}; \end{cases} \quad \beta = \begin{cases} 0, & \text{if } 7 \mid (q + 1), \\ 3, & \text{if } 7 \mid (q^2 - q + 1), \\ 2, & \text{otherwise}. \end{cases} \]

(22) \[ \frac{q^2 - 10q + 25}{72}, \quad \frac{q^2 - 16q + 55}{120}, \quad \frac{q^2 - 10q + 25}{144}, \]

\[ \frac{q^2 - 46q + 205}{720}, \quad \frac{q^2 - 46q + 205}{1440}, \quad \text{when} \quad q = 5^n, \quad n \text{ is odd}, \]

where \( G \cong \text{SmallGroup}(36, 9), G \cong A_5, G \cong \text{PSU}(3, 2), G \cong A_6, G \cong \text{SmallGroup}(720, 765), \) respectively.

(23) \[ \frac{q^2 - 46q + 493 - 80\alpha - 144\gamma}{720}, \quad \frac{q^2 - 16q + 103 - 20\alpha - 24\gamma}{120}, \quad \frac{q^2 - 10q + 25}{72}, \]
THE COMPLETE LIST OF GENERA OF QUOTIENTS OF THE $\mathbb{F}_{q^2}$-MAXIMAL HERMITIAN CURVE FOR $q \equiv 1 \pmod{4}$

when either $p = 3$ and $n$ is even, or $\sqrt{5} \in \mathbb{F}_q$ and $\mathbb{F}_q$ contains no primitive cube roots of unity, with

$$\alpha = \begin{cases} 2, & \text{if } p = 3, \\ 0, & \text{otherwise} \end{cases}$$

and $G \cong A_6$, $G \cong A_5$, $G \cong \text{SmallGroup}(36,9)$, respectively.

\begin{equation}
\frac{q^2 - 106q + 2665 - 720\beta}{5040}, \quad \frac{q^2 - 46q + 205}{720}, \quad \frac{q^2 - 22q + 229 - 48\beta}{336},
\end{equation}

\begin{equation}
\frac{q^2 - 26q + 105}{240}, \quad \frac{q^2 - 16q + 55}{120}, \quad \frac{q^2 - 10q + 25}{72},
\end{equation}

where

$q = 5^n$, $n$ is odd,

$\beta = \begin{cases} 0, & \text{if } 7 \mid (q + 1), \\ 3, & \text{otherwise}, \end{cases}$

and $G \cong A_7$, $G \cong A_6$, $G \cong \text{PSL}(2,7)$, $G \cong A_5 \rtimes C_2$, $G \cong A_5$, $G \cong \text{SmallGroup}(36,9)$, respectively.

\begin{equation}
1 + \frac{q^2 - q - 2 - \Delta}{2q^3(q^3 + 1)(q^2 - 1)}, \quad \text{when } \bar{q} = p^k, \ k \mid n, \ n/k \text{ is odd}, \ G \cong \text{PGU}(3,p^k),
\end{equation}

where

$$\Delta = (\bar{q} - 1)(q^3 + 1) \cdot (q + 2) + (q^3 - \bar{q})(q^3 + 1) \cdot 2 + \bar{q}(q^4 - \bar{q}^3 + q^2) \cdot (q + 1)$$

$$+ (q^2 - \bar{q} - 2) \frac{(q^3 + 1)q^2}{2} \cdot 2 + (\bar{q} - 1)\bar{q}(q^3 + 1)q^2 \cdot 1 + (q^2 - \bar{q})\frac{\bar{q}^6 + \bar{q}^5 - \bar{q}^4 - \bar{q}^3}{3} \cdot \gamma,$$

with

$\gamma = \begin{cases} 3, & \text{if } (\bar{q}^2 - \bar{q} + 1) \mid (q^2 - q + 1), \\ 0, & \text{if } (\bar{q}^2 - \bar{q} + 1) \mid (q + 1). \end{cases}$

\begin{equation}
\frac{3(q^2 - q - 2 - \Delta)}{2q^3(q^2 - 1)(q^3 + 1)} + 1, \quad \text{when } \bar{q} = p^k, \ k \mid n, \ n/k \text{ is odd}, \ 3 \mid (q + 1), \ G \cong \text{PSU}(3,p^k),
\end{equation}

where

$$\Delta = (\bar{q} - 1)(q^3 + 1) \cdot (q + 2) + (q^3 - \bar{q})(q^3 + 1) \cdot 2 + ((\bar{q} + 1)/3 - 1)(q^4 - q^3 + q^2) \cdot (q + 1)$$

$$+ ((q^2 - 1)/3 - (\bar{q} + 1)/3) \frac{(q^3 + 1)q^2}{2} \cdot 2 + (\bar{q} - 1)((\bar{q} + 1)/3 - 1)(q^3 + 1)q^2 \cdot 1 + ((\bar{q}^2 - \bar{q} + 1)/3 - 1)\frac{\bar{q}^6 + \bar{q}^5 - \bar{q}^4 - \bar{q}^3}{3} \cdot \delta,$$

with

$\delta = \begin{cases} 3, & \text{if } (\bar{q}^2 - \bar{q} + 1)/3 \mid (q^2 - q + 1), \\ 0, & \text{if } (\bar{q}^2 - \bar{q} + 1)/3 \mid (q + 1). \end{cases}$

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