Reflections at infinity of time changed RBMs on a domain with Liouville branches

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Abstract

Let $Z$ be the transient reflecting Brownian motion on the closure of an unbounded domain $D \subset \mathbb{R}^d$ with $N$ number of Liouville branches. We consider a diffusion $X$ on $\overline{D}$ having finite lifetime obtained from $Z$ by a time change. We show that $X$ admits only a finite number of possible symmetric conservative diffusion extensions $Y$ beyond its lifetime characterized by possible partitions of the collection of $N$ ends and we identify the family of the extended Dirichlet spaces of all $Y$ (which are independent of time change used) as subspaces of the space $BL(D)$ spanned by the extended Sobolev space $H^1_e(D)$ and the approaching probabilities of $Z$ to the ends of Liouville branches.

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1 Introduction

The boundary problem of a Markov process $X$ concerns all possible Markovian prolongations $Y$ of $X$ beyond its life time $\zeta$ whenever $\zeta$ is finite. For a conservative but transient Markov process, we can still consider its extension, after a time change to speed up the original process. Let $Z = (Z_t, Q_z)$ be a conservative right process on a locally compact separable metric space $E$ and $\partial$ be the point at infinity of $E$. Suppose $Z$ is transient relative to an excessive measure $m$: for the 0-order resolvent $R$ of $Z$, $Rf(z) < \infty, m$-a.e. for some strictly positive function (or equivalently, for any non-negative function) $f \in L^1(E; m)$. Then

$$Q_z \left( \lim_{t \to \infty} Z_t = \partial \right) = 1 \quad \text{for q.e. } x \in E,$$

if $Rf$ is lower semicontinuous for any non-negative Borel function $f$ (\cite{[FTa]}). The last condition is not needed when $X$ is $m$-symmetric (\cite{[CF2]}). Here, 'q.e.' means 'except for an $m$-polar set.

Take any strictly positive bounded function $f \in L^1(E; m)$. Then $A_t = \int_0^t f(Z_s)ds, \ t \geq 0$ is a strictly increasing PCAF of $Z$ with $E^Q_z[A_{\infty}] = Rf(x) < \infty$ for q.e. $x \in E$. 

1
The time changed process $X = (X_t, \zeta, P_x)$ of $Z$ by means of $A$ is defined by

$$X_t = Z_{\tau t}, \quad t \geq 0, \quad \tau = A^{-1}, \quad \zeta = A_\infty, \quad P_x = Q_x, \quad x \in E.$$  

Since $P_x(\zeta < \infty, \lim_{t \to \zeta} X_t = \partial) = P_x(\zeta < \infty) = 1$ for q.e. $x \in E$, the boundary problem for $X$ at $\partial$ makes perfect sense. We denote $X$ also by $X^f$ to indicate its dependence on the function $f$. For different choices of $f$, $X^f$ have a common geometric structure related each other only by time changes. Thus a study of the boundary problem for $X = X^f$ is a good way to have a closer look at the geometric behaviors of a conservative transient process $Z$ around $\partial$. A strong Markov process $\tilde{X}$ on a topological space $\tilde{E}$ is said to be an extension of $X$ on $E$ if (i) $E$ can be embedded homomorphically as a dense open subset of $\tilde{E}$, (ii) the part process of $\tilde{X}$ killed upon leaving $E$ has the same distribution as $X$, and (iii) $\tilde{X}$ has no sojourn on $\tilde{E} \setminus E$; that is, $\tilde{X}$ spends zero Lebesgue amount of time on $\tilde{E} \setminus E$.

In this paper, we consider as $Z$ the transient reflecting Brownian motion on the closure of an unbounded domain $D \subset \mathbb{R}^d$ with $N$ number of Liouville branches. Our main aim is to prove in Section 5 that a time changed process $X^f$ of $Z$ admits essentially only a finite number of possible symmetric conservative diffusion extensions $Y$ beyond its lifetime. They are characterized by the partition of the collection of $N$ ends. Moreover, all the corresponding extended Dirichlet spaces $(\mathcal{E}^Y, \mathcal{F}^Y_e)$ are identified in terms of the extended Dirichlet space of $Z$ and the approaching probabilities of $Z$ to the ends of Liouville branches in an extremely simple manner. These extended Dirichlet spaces are independent of the choice of $f$. The $L^2$-generator of each extension $Y$ is also characterized in Section 6 by means of zero flux conditions at the ends of branches. Each extension $Y$ may be called a many point reflection at infinity of $X^f$ generalizing the notion of the one point reflection in \[CF3\] in the present specific context. The characterization of possible extensions also uses quasi-homeomorphism and equivalence between Dirichlet forms. See the Appendix, Section 8 of this paper for details.

In fact, our results are valid for a time changed process $X^\mu$ of $Z$ by means of a more general finite smooth measure $\mu$ on $D$ than $f(x)dx$. This is demonstrated in Section 7.

Although we formulate our results for the reflecting Brownian motion on an unbounded domain in $\mathbb{R}^d$ with several Liouville branches, all of them except for Theorem 6.1 remain valid without any essential change for the reflecting diffusion process associated with the uniformly elliptic second order self-adjoint partial differential operator with measurable coefficients that was constructed in [G] and [PT16]. Since we need strong Feller property of the reflecting diffusion process, we assume the underlying unbounded domain is Lipschitz in the sense of [PT16]; see Remark 5.3. Thus we are effectively investigating common path behaviours at infinity holding for such a general family of diffusion processes.

Acknowledgement This paper is a direct outgrowth of our paper \[CF1\] and Chapter 7 of our book \[CF2\]. In relation to them, we had very valuable discussions with Krzysztof Burdzy on boundaries of transient reflecting Brownian motions. We would like to express our sincere thanks to him.
2 Preliminaries

For a domain $D \subset \mathbb{R}^d$, let us consider the spaces

$$BL(D) = \{ u \in L^2_{\text{loc}}(D) : |\nabla u| \in L^2(D) \}, \quad H^1(D) = BL(D) \cap L^2(D), \quad (2.1)$$

The space $BL(D)$ called the Beppo Levi space was introduced by J. Deny and J. L. Lions [DL] as the space of Schwartz distributions whose first order derivatives are in $L^2(D)$, which can be identified with the function space described above. The quotient space $\dot{BL}(D)$ of $BL(D)$ by the space of all constant functions on $D$ is a real Hilbert space with inner product

$$D(u,v) = \int_D \nabla u(x) \cdot \nabla v(x) dx.$$  

See §1.1 of V.G, Maz’ja [M] for proofs of the above stated facts, where the space $BL(D)$ is denoted by $L^2_{\text{loc}}(D)$ and studied in a more general context of the spaces $L^p(D)$, $\ell \geq 1$, $p \geq 1$.

Define

$$\mathcal{E}, \mathcal{F} = (\frac{1}{2}D, H^1(D)), \quad (2.2)$$

which is a Dirichlet form on $L^2(D)$. The collection of those domains $D \subset \mathbb{R}^d$ for which (2.2) is regular on $L^2(D)$ will be denoted by $\mathcal{D}$. It is known that $D \in \mathcal{D}$ if $D$ is either a domain of continuous boundary or an extendable domain relative to $H^1(D)$ (cf. [CF1, p 866]). For $D \in \mathcal{D}$, the diffusion process $Z$ on $\overline{D}$ associated with (2.2) is by definition the reflecting Brownian motion (RBM in abbreviation) which is known to be conservative. Furthermore, the space $BL(D)$ is nothing but the reflected Dirichlet space of the form (2.2) ([CF2, §6.5]). The Dirichlet form (2.2) is either recurrent or transient and the latter case occurs only when $d \geq 3$ and $D$ is unbounded. For $D_1, D_2 \in \mathcal{D}$ with $D_1 \subset D_2$, (2.2) is transient for $D_2$ whenever so it is for the smaller domain $D_1$. If (2.2) is recurrent, then we have the identity

$$BL(D) = H^1_e(D)$$

where $H^1_e(D)$ denotes the extended Dirichlet space of the form (2.2) or of the RBM $Z$ ([CF2]) that may be called the extended Sobolev space of order 1.

Suppose $D \in \mathcal{D}$ and (2.2) is transient. Then $H^1_e(D)$ is a Hilbert space with inner product $\frac{1}{2}D$ possessing the space $C_c^\infty(\overline{D})$ as its core. $H^1_e(D)$ can be regarded as a proper closed subspace of the quotient space $BL(D)$. Define

$$\mathcal{H}^*(D) = \{ u \in BL(D) : D(u,v) = 0 \text{ for every } v \in H^1_e(D) \}. \quad (2.3)$$

Any function $u \in BL(D)$ admits a unique decomposition

$$u = u_0 + h, \quad u_0 \in H^1_e(D), \quad h \in \mathcal{H}^*(D). \quad (2.4)$$

Any function $h \in \mathcal{H}^*(D)$ is of finite Dirichlet integral and harmonic on $D$. Furthermore, the quasi-continuous version of $h$ is harmonic on $\overline{D}$ with respect to the RBM $Z$. In what follows, we restrict our attention to the case where the form (2.2) is transient and so we assume that $d \geq 3$ and $D \in \mathcal{D}$ is unbounded.
Definition 2.1 A domain $D \in \mathcal{D}$ is called a Liouville domain if the form $(2.2)$ is transient and $\dim \mathbb{H}^{\ast}(D) = 1$.

A domain $D \in \mathcal{D}$ is a Liouville domain if and only if the form $(2.2)$ is transient and any function $u \in \text{BL}(D)$ admits a unique decomposition
\[ u = u_0 + c, \quad \text{where } u_0 \in H^1_c(D) \text{ and } c \in \mathbb{R}. \quad (2.5) \]
We shall denote by $c(u)$ the constant $c$ in (2.5) uniquely associated with $u \in \text{BL}(D)$ for a Liouville domain $D$.

A trivial but important example of a Liouville domain is $\mathbb{R}^d$ with $d \geq 3$, see M. Brelot [3]. Another important example of a Liouville domain is provided by an unbounded uniform domain that has been shown by P. Jones [1] (see also [HK]) to be an extendable domain relative to the space $\text{BL}(D)$.

A domain $D \subset \mathbb{R}^d$ is called a uniform domain if there exists $C > 0$ such that for every $x, y \in D$, there is a rectifiable curve $\gamma$ in $D$ connecting $x$ and $y$ with $\text{length}(\gamma) \leq C|x - y|$, and moreover
\[ \min\{|x - z|, |z - y|\} \leq C \text{dist}(z, D^c) \quad \text{for every } z \in \gamma. \]
It was proved in Theorem 3.5 of [CF1] that any unbounded uniform domain is a Liouville domain in the sense of Definition 2.1. An unbounded uniform domain is such a domain that is broadened toward the infinity. The truncated infinite cone $C_{A, a} = \{(r, \omega) : r > a, \omega \in A\} \subset \mathbb{R}^d$ for any connected open set $A \subset S^{d-1}$ with Lipschitz boundary is an unbounded uniform domain. To the contrary, $(2.2)$ is recurrent for the cylinder $D = \{(x, x') \in \mathbb{R}^d : x \in \mathbb{R}, |x'| < 1\}$. See R. G. Pinsky [P] for transience criteria for other types of domains. On the other hand, it has been shown in [CF2, Proposition 7.8.5] that $(2.2)$ is transient but $\dim(\mathbb{H}(D)) = 2$ for a special domain
\[ D = B_1(0) \cup \left\{ (x, x') \in \mathbb{R}^d : x \in \mathbb{R}, |x| > |x'| \right\}, \quad d \geq 3. \quad (2.6) \]
with two symmetric cone branches. Here $B_r(\phi)$, $r > 0$, denotes an open ball with radius $r$ centered at the origin. This domain is not uniform because of a presence of a bottleneck. We shall consider much more general domains than this. But before proceeding to the main setting of the present paper, we state a simple property of Liouville domains:

Proposition 2.2 For $D_1, D_2 \in \mathcal{D}$ with $D_1 \subset D_2$, Suppose $D_1$ is a Liouville domain and $D_2 \setminus D_1$ is bounded. Then $D_2$ is a Liouville domain. Furthermore, for any $u \in \text{BL}(D_2)$, it holds that $c(u) = c(u|_{D_1})$.

Proof. The proof is similar to that of [CF1, Proposition 3.6]. Note that $(2.2)$ is transient for $D_2$. We show that any $u \in \text{BL}(D_2)$ admits a decomposition $(2.5)$ with $u_0 \in H^1_c(D_2)$ and $c = c(u|_{D_1})$. Due to the normal contraction property of $\text{BL}(D_2)$ and the transience of $(\mathbb{R}^d, H^1_c(D_2))$, we may assume that $u$ is bounded on $D_2$. By noting that $u|_{D_1} \in \text{BL}(D_1)$ and $D_1$ is a Liouville domain, we let $c = c(u|_{D_1})$ and $u_0(x) = u(x) - c$, $x \in D_2$. Then $u_0|_{D_1} \in H^1_c(D_1)$. To prove that $u_0 \in H^1_c(D_2)$, choose an open ball $B_r(\phi) \supset D_2 \setminus D_1$ and a function $w \in C_{\infty}^c(\mathbb{R}^d)$ with $w(x) = 1$, $x \in B_r(\phi)$. Clearly $wu_0 \in H^1_c(D_2)$.
It remains to show \((1 - w)u_0 \in H^1_e(D_2)\). Take \(g_n \in H^1(D_1)\) converging to \(u_0\) a.e. on \(D_1\) and in the Dirichlet norm on \(D_1\). By truncation, we may assume that \(g_n\) is uniformly bounded on \(D_1\). Then
\[
\int_{D_2} |\nabla [(1 - w(x))g_n(x)]|^2 dx \\
\leq 2 \sup_{x \in \mathbb{R}^d} (1 - w(x))^2 \int_{D_1} |\nabla g_n(x)|^2 dx + 2 \sup_{x \in D_1} |g_n(x)|^2 \int_{\mathbb{R}^d} |\nabla w(x)|^2 dx,
\]
which is uniformly bounded in \(n\), yielding by the Banach-Saks theorem that \((1 - w)u_0 \in H^1_e(D_2)\). 

We shall work under the regularity condition
\((A.1)\) \(D\) is of a Lipschitz boundary \(\partial D\), which means the following: there are constants \(M > 0\), \(\delta > 0\) and a locally finite covering \(\{U_j\}_{j \in J}\) of \(\partial D\) such that, for each \(j \in J\), \(D \cap U_j\) is a upper part of a graph of a Lipschitz continuous function under an appropriate coordinate system with the Lipschitz constant bounded by \(M\) and \(\partial D \subset \bigcup_{j \in J} \{x \in U_j : \text{dist}(x, \partial U_j) > \delta\}\). According to [FTo], there exists then a conservative diffusion process \(Z = (Z_t, Q_x)\) on \(D\) associated with the regular Dirichlet form \((2.2)\) on \(L^2(D)\) whose resolvent \(\{G^Z_\alpha; \alpha > 0\}\) has the strong Feller property in the sense that
\[
G^Z_\alpha(bL^1(D)) \subset bC(\overline{D}). \tag{2.7}
\]
\(Z\) is a precise version of the RBM on \(\overline{D}\). In particular, the transition probability of \(Z\) is absolutely continuous with respect to the Lebesgue measure.

Under the condition \((A.1)\) and the transience assumption on \((2.2)\), the RBM \(Z = (Z_t, Q_x)\) on \(\overline{D}\) enjoys the properties that
\[
Q_x \left( \lim_{t \to \infty} Z_t = \partial \right) = 1 \quad \text{for every } x \in \overline{D}, \tag{2.8}
\]
where \(\partial\) denotes the point at infinity of \(\mathbb{R}^d\), and
\[
Q_x \left( \lim_{t \to \infty} u(Z_t) = 0 \right) = 1 \quad \text{for every } x \in \overline{D}, \tag{2.9}
\]
for any \(u \in H^1_e(D)\), \(u\) being taken to be quasi-continuous. See [CP2 §7.8, (4o)].

In the rest of this paper, we fix a domain \(D\) of \(\mathbb{R}^d\), \(d \geq 3\), satisfying \((A.1)\) and
\((A.2)\) \(D \setminus \overline{B_r(\emptyset)} = \bigcup_{j=1}^N C_j\)
for some \(r > 0\) and an integer \(N\), where \(C_1, \ldots, C_N\) are Liouville domains with Lipschitz boundaries such that \(\overline{C_1}, \ldots, \overline{C_N}\) are mutually disjoint. \(D\) may be called a Lipschitz domain with \(N\) number of Liouville branches.

Let \(\partial_j\) be the point at infinity of the unbounded closed set \(\overline{C_j}\) for each \(1 \leq j \leq N\). Denote the \(N\)-points set \(\{\partial_1, \ldots, \partial_N\}\) by \(F\) and put \(\overline{D'} = \overline{D} \cup F\). \(\overline{D'}\) can be made to be a
compact Hausdorff space if we employ as a local base of neighborhoods of each point $\partial_j \in F$ the neighborhoods of $\partial_j$ in $\overline{C_j} \cup \{\partial_j\}$. $D$ may be called the $N$-points compactification of $D$.

Obviously the Dirichlet form $\langle 2.2 \rangle$ is transient for $D$. We shall verify in Section 4 that $\dim(\mathbb{H}^n(D)) = N$. Here we note the following implication of Proposition $\langle 2.2 \rangle$ if a domain $D$ is of the type $\langle A.2 \rangle$ for different $0 < r_1 < r_2$, and if $D$ is a domain with $N$ number of Liouville branches relative to $r_2$, then so it is relative to $r_1$.

3 Approaching probabilities of RBM $Z$ and limits of BL-functions along $Z_t$

For each $1 \leq j \leq N$, define the approaching probability of the RBM $Z = (Z_t, Q_x)$ to $\partial_j$ by

$$\varphi_j(x) = Q_x \left( \lim_{t \to \infty} Z_t = \partial_j \right), \quad x \in \overline{D}. \quad (3.1)$$

Proposition 3.1 It holds that

$$\sum_{j=1}^{N} \varphi_j(x) = 1 \quad \text{for every} \quad x \in \overline{D}, \quad (3.2)$$

and, for each $1 \leq j \leq N$,

$$\varphi_j(x) > 0 \quad \text{for every} \quad x \in \overline{D}. \quad (3.3)$$

Proof. $\langle 3.2 \rangle$ is a consequence of $\langle 2.8 \rangle$. As $\varphi_j$ is a non-negative harmonic function on the domain $D$, it is either identically zero on $D$ or strictly positive on $D$. Since $\varphi_j(x) = Q_t \varphi_j(x), \ x \in \overline{D}$, where $Q_t$ is the transition semigroup of the RBM $Z$, which has a strictly positive transition density kernel, the above dichotomy extends from $D$ to $\overline{D}$.

Suppose $\varphi_j(x) \equiv 0$ on $\overline{D}$. Then by $\langle 2.8 \rangle$

$$Q_x \left( \sigma_{\partial B_r(\emptyset)} < \infty \right) = 1, \quad \text{for any} \ x \in \overline{C_j} \setminus B_{r+1}(\emptyset). \quad (3.4)$$

Let $Z^j = (Z^j_t, Q_x^j), \ x \in \overline{C_j}$, be the RBM on $\overline{C_j}$, which is transient as $C_j$ is a Liouville domain. Since $Z$ and $Z^j$ share the common part process on $\overline{C_j} \setminus \partial B_r(\emptyset), \langle 3.4 \rangle$ remains valid if $Q_x$ is replaced by $Q_x^j$. By the Markov property of $Z^j$ and the conservativeness of $Z^j$, we have

$$Q_x^j \left( \sigma_{\partial B_r(\emptyset)} \circ \theta_t < \infty \text{ for every integer } t \right) = 1,$$

for any $x \in \overline{C_j} \setminus B_{r+1}(\emptyset)$. This however contradicts to the transience property $\langle 2.8 \rangle$ of $Z^j$.

$\square$

Proposition 3.2 For any $u \in BL(D)$, let $c_j(u) = c(u|_{C_j})$ for $1 \leq j \leq N$. Then

$$Q_x \left( Z_{\infty} = \partial_j, \ \lim_{t \to \infty} u(Z_t) = c_j(u) \right) = Q_x \left( Z_{\infty} = \partial_j \right), \ x \in \overline{D}, \ 1 \leq j \leq N. \quad (3.5)$$

If $c_j(u) = 0$ for every $1 \leq j \leq N$, then $u \in H^1_e(D)$.
Proof. We prove (3.5) for \( j = 1 \). Let \( r > 0 \) be the radius in (A.2) and \( Z^1 = (Z^1_t, Q^1_x) \) be the RBM on \( \overline{C_1} \). The hitting times of \( B_r(o) \) and \( B_{R}(o) \) for \( R > r \) will be denoted by \( \sigma_r \) and \( \sigma_{R} \), respectively. Observe that \( Z \) and \( Z^1 \) share in common the part process on \( \overline{C_1} \cap \partial B_{r} \). Since \( C_1 \) is a Liouville domain, we see from (2.5) and (2.9) that

\[
Q^1_x \left( \lim_{t \to \infty} u(Z^1_t) = c_1(u) \right) = 1 \quad \text{for every} \ x \in \overline{C_1}.
\]

For \( R > r \), we consider the event \( \Gamma_R = \{ Z_{\sigma_R} \in \overline{C_1}, \ \sigma_r \circ \theta_{\sigma_R} = \infty \} \).

Then \( \Gamma_R \cap \{ Z_{\infty} = \partial \} \) increases as \( R \) increases and \( \{ Z_{\infty} = \partial_1 \} = \bigcup_{R > r} [\Gamma_R \cap \{ Z_{\infty} = \partial \}] \).

In view of (2.8), we have for \( x \in \overline{D} \),

\[
Q_x(Z_{\infty} = \partial_1) = \lim_{R \to \infty} Q_x(\Gamma_R \cap \{ Z_{\infty} = \partial \}) = \lim_{R \to \infty} Q_x(\Gamma_R)
\]

\[
= \lim_{R \to \infty} \mathbb{E}^{Q_x} \left[ Q_{Z_{\sigma_R}}(\sigma_r = \infty); Z_{\sigma_R} \in \overline{C_1} \right]
\]

\[
= \lim_{R \to \infty} \mathbb{E}^{Q_x} \left[ Q_{Z_{\sigma_R}}(\sigma_r = \infty); Z_{\sigma_R} \in \overline{C_1} \right]
\]

\[
= \lim_{R \to \infty} \mathbb{E}^{Q_x} \left[ Q_{Z_{\sigma_R}}(\sigma_r = \infty, \lim_{t \to \infty} u(Z^1_t) = c_1(u)); Z_{\sigma_R} \in \overline{C_1} \right].
\]

In exactly the same way, we can see that \( Q_x(Z_{\infty} = \partial_1, \lim_{t \to \infty} u(Z_t) = c_1(u)) \) equals the last expression in the above display, proving (3.5) for \( j = 1 \).

Suppose \( u \in \text{BL}(D) \) satisfies \( c_j(u) = 0 \) for every \( 1 \leq j \leq N \). Then \( u|_{C_j} \in H^1(D) \) for every \( 1 \leq j \leq N \) and we can conclude as the proof of Proposition 2.2 that \( u \in H^1(D) \). \( \square \)

We remark that, in view of Proposition 2.2, the constants \( c_j(u), 1 \leq j \leq N, \) in the above proposition are independent of the choice of the radius \( r \) in (A.2).

4 Reflecting extension \( X^* \) of a time changed RBM \( X \) and dimension of \( \mathbb{H}^*(D) \)

Fix a strictly positive bounded integrable function \( f \) on \( \overline{D} \) and define

\[
A_t = \int_0^t f(Z_s) ds, \quad t \geq 0.
\]

(4.1)

\( A_t \) is a positive continuous additive functional (PCAF) of the RBM \( Z = (Z_t, Q_x) \) on \( \overline{D} \) in the strict sense with full support. Notice that

\[
Q_x (A_\infty < \infty) = 1 \quad \text{for every} \ x \in \overline{D},
\]

(4.2)

because \( \mathbb{E}^{Q_x} [A_\infty] = G_{0+}^Z f(x) < \infty \) for a.e. \( x \in \overline{D} \) due to the transience of \( Z \) ([CF2, Proposition 2.1.3]) and hence

\[
Q_x (A_\infty = \infty) = Q_x (A_\infty \circ \theta_t = \infty) = \mathbb{E}^{Q_x} [Q_{Z_t}(A_\infty = \infty)] = 0 \quad \text{for every} \ x \in \overline{D},
\]

(4.3)
on account of the stated absolute continuity of the transition function of $Z$.

Let $X = (X_t, \zeta, P_x)$ be the time changed process of $Z$ by means of $A$:

$$X_t = Z_{\tau t}, \quad \tau = A^{-1}, \quad \zeta = A_{\infty}, \quad P_x = Q_x \text{ for } x \in \mathcal{D}.$$  

The Markov process $X = X^f$ is a diffusion process on $\mathcal{D}$ symmetric with respect to the measure $m(dx) = f(x)dx$ and the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of $X$ on $L^2(\mathcal{D}; m)$ is given by

$$\mathcal{E} = \frac{1}{2} \mathcal{D}, \quad \mathcal{F} = H^1_D(D) \cap L^2(\mathcal{D}; m). \quad (4.4)$$

Since the extended Dirichlet space and the reflected Dirichlet space are invariant under a time change by a fully supported PCAF ($\text{[CF2, Cor.5.2.12, Prop.6.4.6]}$), these spaces for $\mathcal{E}$ are still given by $H^1_D(D)$ and $\mathcal{BL}(D)$, respectively. But the life time $\zeta$ of $X$ is finite $P_x$-a.s. for every $x \in \mathcal{D}$ in view of (4.2) so that we may consider the problem of extending $X$ after $\zeta$, particularly, from $\mathcal{D}$ to its N-points compactification $\mathcal{D}^* = \mathcal{D} \cup F$ with $F = \{\partial_1, \cdots, \partial_N\}$.

We can rewrite the approaching probability $\varphi_j$ of $Z$ to $\partial_j$ defined by (3.1) as

$$\varphi_j(x) = P_x(\zeta < \infty, \, \mathcal{X}_{\zeta-} = \partial_j), \quad x \in \mathcal{D}, \quad 1 \leq j \leq N, \quad (4.5)$$
in terms of the time changed process $X$. The measure $m(dx) = f(x)dx$ is extended from $\mathcal{D}$ to $\mathcal{D}^*$ by setting $m(F) = 0$. An $m$-symmetric conservative diffusion process $X^*$ on $\mathcal{D}$ will be called a symmetric conservative diffusion extension of $X$ if its part process on $\mathcal{D}$ being killed upon hitting $F$ is equivalent in law with $X$. The resolvent of $X$ is denoted by $\{G^X_\alpha, \alpha > 0\}$.

**Proposition 4.1** There exists a unique symmetric conservative diffusion extension $X^*$ of $X$ from $\mathcal{D}$ to $\mathcal{D}^* = \mathcal{D} \cup F$. The process $X^*$ is recurrent. Let $(\mathcal{E}, \mathcal{F})$ and $\mathcal{F}_e$ be the Dirichlet form of $X^*$ on $L^2(\mathcal{D}, m) (= L^2(\mathcal{D}; m))$ and its extended Dirichlet space, respectively. Then

$$\mathcal{F}_e = H^1_D(D) \oplus \left\{ \sum_{j=1}^{N} c_j \varphi_j : \forall c_j \in \mathbb{R} \right\} \subset \mathcal{BL}(D), \quad (4.6)$$

$$\mathcal{E}^*(u, v) = \frac{1}{2} \mathcal{D}(u, v), \quad u, v \in \mathcal{F}_e^*. \quad (4.7)$$

**Proof.** We apply a general existence theorem of a many-point extension formulated in $\text{[CF2, Theorem 7.7.4]}$ to the $m$-symmetric diffusion $X$ on $\mathcal{D}$ and the $N$-points compactification $\mathcal{D}^* = \mathcal{D} \cup F$ of $\mathcal{D}$. We verify conditions (M.1), (M.2), (M.3) for $X$ required in this theorem. $\psi_j(x) := P_x(\zeta < \infty, X_{\zeta-} = \partial_j)$ is positive for every $x \in \mathcal{D}, 1 \leq j \leq N$, by (3.3) and (4.5), and so (M.1) is satisfied. Since $m(\mathcal{D}) = \int_{\mathcal{D}} f dx < \infty$, the $m$-integrability (M.2) of the function $u_\alpha^{(j)}(x) = E_x[e^{-\alpha \zeta}; X_{\zeta-} = \partial_j], \quad x \in \mathcal{D}$, is trivially fulfilled, $1 \leq j \leq N$. For any $1 \leq j \leq N$ and any compact set $V \subset \mathcal{D}$, $\inf_{x \in V} G^X_\alpha \psi_j(x)$ is positive because $G^X_\alpha \psi_j = G^X_{\alpha+} u_\alpha^{(j)} = G^Z_{\alpha+} (u_\alpha^{(j)})$ is lower semi-continuous on account of (2.7) and $u_\alpha^{(j)}$ is positive on $\mathcal{D}$. Accordingly, condition (M.3) is also satisfied.

Therefore there exists an $m$-symmetric diffusion extension $X^*$ of $X$ from $\mathcal{D}$ to $\mathcal{D}^*$ admitting no killing on $F$. We can then use a general characterization theorem $\text{[CF2, Theorem}$ 4.4.1] to the...
Let $\tau$ be the exit time of $Z$ from the set $D \cap B_n(\phi)$, $n \geq 1$. Then $\{\varphi_j(Z_{\tau_n})\}_{n \geq 1}$ is a bounded $Q_x$-martingale and possesses an a.s. limit $\Phi$ with $\varphi_j(x) = E^{Q_x} [\Phi]$. By (3.5),

$$\Phi = \sum_{k=1}^{N} c^{(j)}_k 1_{\{Z_{\infty}=\partial_k\}}. \tag{4.10}$$

For $k \neq j$, put $F_{k,n} = C_k \cap \{|x|=n\}$. Then by (3.5) again

$$c^{(j)}_k \varphi_k(x) = \lim_{n \to \infty} E^{Q_x} [\varphi_j(Z_{\tau_n}) 1_{\{Z_{\infty}=\partial_k\}}] \leq \limsup_{n \to \infty} E^{Q_x} [\varphi_j(Z_{\tau_n}) 1_{\{Z_{\tau_n} \in C_k\}}]$$

$$= \limsup_{n \to \infty} E^{Q_x} [Q_x (Z_{\infty} \circ \theta_{\tau_n} = \partial_j, Z_{\tau_n} \in C_k \mid F_{\tau_n})]$$

$$\leq \lim_{n \to \infty} Q_x (Z_{\infty} = \partial_j, \sigma_{F_{k,n}} < \infty) = 0,$$

yielding $c^{(j)}_k = 0, \; k \neq j$. Taking $Q_x$-expectation in (4.10) proves the claim (4.9).

Next for any $u \in BL(D)$, let $u_0 = u - \sum_{j=1}^{N} c_j(u) \varphi_j$. Then $u_0 \in BL(D)$ with $c_j^{(a_0)} = 0$ for every $1 \leq j \leq N$. So by Proposition 3.2, $u_0 \in H^{1}_D$. This establishes (4.8). The linear independence of $\{\varphi_j; 1 \leq j \leq N\}$ follows from (4.9), while (4.6) and (4.8) yield the last assertion of the theorem.

**Theorem 4.2** \(\dim(\mathbb{H}^*(D)) = N\) and

$$\mathbb{H}^*(D) = \left\{ \sum_{j=1}^{N} c_j \varphi_j : c_j \in \mathbb{R} \right\}. \tag{4.8}$$

The \(m\)-symmetric conservative diffusion extension \(X^*\) of the time changed RBM \(X\) constructed in Proposition 4.1 is a reflecting extension of \(X\) in the sense that the extended Dirichlet space \((\mathcal{F}^*_x, \mathcal{E}^*)\) of \(X^*\) equals \((BL(D), \frac{1}{2}D)\) the reflected Dirichlet space of \(X\).

**Proof.** By Proposition 4.1 \(\{\varphi_j; 1 \leq j \leq N\} \subset \mathbb{H}^*(D) \subset BL(D)\). For \(1 \leq j, k \leq N\), let \(c^{(j)}_k = c_k(\varphi_j)\). We claim that

$$c^{(j)}_k = \delta_{jk}, \quad 1 \leq k \leq N. \tag{4.9}$$

Let $\tau_n$ be the exit time of $Z$ from the set $\bar{D} \cap B_n(\phi), \; n \geq 1$. Then $\{\varphi_j(Z_{\tau_n})\}_{n \geq 1}$ is a bounded $Q_x$-martingale and possesses an a.s. limit $\Phi$ with $\varphi_j(x) = E^{Q_x} [\Phi]$. By (3.5),

$$\Phi = \sum_{k=1}^{N} c^{(j)}_k 1_{\{Z_{\infty}=\partial_k\}}. \tag{4.10}$$

For $k \neq j$, put $F_{k,n} = C_k \cap \{|x|=n\}$. Then by (3.5) again

$$c^{(j)}_k \varphi_k(x) = \lim_{n \to \infty} E^{Q_x} [\varphi_j(Z_{\tau_n}) 1_{\{Z_{\infty}=\partial_k\}}] \leq \limsup_{n \to \infty} E^{Q_x} [\varphi_j(Z_{\tau_n}) 1_{\{Z_{\tau_n} \in C_k\}}]$$

$$= \limsup_{n \to \infty} E^{Q_x} [Q_x (Z_{\infty} \circ \theta_{\tau_n} = \partial_j, Z_{\tau_n} \in C_k \mid F_{\tau_n})]$$

$$\leq \lim_{n \to \infty} Q_x (Z_{\infty} = \partial_j, \sigma_{F_{k,n}} < \infty) = 0,$$

yielding $c^{(j)}_k = 0, \; k \neq j$. Taking $Q_x$-expectation in (4.10) proves the claim (4.9).

Next for any $u \in BL(D)$, let $u_0 = u - \sum_{j=1}^{N} c_j(u) \varphi_j$. Then $u_0 \in BL(D)$ with $c_j^{(a_0)} = 0$ for every $1 \leq j \leq N$. So by Proposition 3.2, $u_0 \in H^{1}_D$. This establishes (4.8). The linear independence of $\{\varphi_j; 1 \leq j \leq N\}$ follows from (4.9), while (4.6) and (4.8) yield the last assertion of the theorem.

**Remark 4.3** This theorem for the special domain (2.6) was stated in [CF2, Proposition 7.8.5]). We take this opportunity to mention that the proof of the latter given in the book [CF2] contained a flaw (on the third line of page 386), that should be corrected in the above way.
5  Partitions Π of F and all possible symmetric diffusion extensions Y of a time changed RBM X

We continue to consider the N-points compactification $\overline{D}^* = \overline{D} \cup F$ of $\overline{D}$ introduced at the end of Section 1. A map Π from the boundary set $F = \{\partial_1, \ldots, \partial_N\}$ onto a finite set $\widehat{F} = \{\widehat{\partial}_1, \ldots, \widehat{\partial}_\ell\}$ with $\ell \leq N$ is called a partition of F. We let $\overline{D}^\Pi, \ast = \overline{D} \cup \widehat{F}$. We extend the map Π from $F$ to $\overline{D}^\ast$ by setting $\Pi x = x$, $x \in \overline{D}$, and introduce the quotient topology on $\overline{D}^\Pi, \ast$ by Π. In other words, we employ $\mathcal{U}_\Pi = \{ U \subset \overline{D}^\Pi, \ast : \Pi^{-1}(U) \text{ is an open subset of } \overline{D} \}$ as the family of open subsets of $\overline{D}^\Pi, \ast$. Then $\overline{D}^\Pi, \ast$ is a compact Hausdorff space and may be called an $\ell$-points compactification of $\overline{D}$ obtained from $\overline{D}$ by identifying the points in the set $\Pi^{-1}\widehat{\partial}_i \subset F$ as a single point $\widehat{\partial}_i$ for each $1 \leq i \leq \ell$.

Given a partition Π of $F$, the approaching probabilities $\widehat{\varphi}_i$ of the RBM $Z = (Z_t, Q_x)$ to $\widehat{\partial}_i \in \widehat{F}$ are defined by

$$\varphi_i(x) = \sum_{j \in \Pi^{-1}\widehat{\partial}_i} \varphi_j(x), \quad x \in \overline{D}, \quad 1 \leq i \leq \ell. \quad (5.1)$$

As in the preceding section, we define the time changed process $X = (X_t, \zeta, P_x)$ on $\overline{D}$ of $Z$ by means of a strictly positive bounded integrable function $f$ on $\overline{D}$. The measure $m(dx) = f(x)dx$ is extended from $\overline{D}$ to $\overline{D}^\Pi, \ast$ by setting $m(\widehat{F}) = 0$. Just as in Proposition 4.1 there exists then a unique $m$-symmetric conservative diffusion extension $X^\Pi, \ast$ of $X$ from $\overline{D}$ to $\overline{D}^\Pi, \ast$ and the Dirichlet form $(\mathcal{E}^\Pi, \ast, \mathcal{F}^\Pi, \ast)$ of $X^\Pi, \ast$ on $L^2(\overline{D}^\Pi, \ast ; m) (= L^2(D; m))$ admits the extended Dirichlet space $(\mathcal{F}^\Pi, \ast, \mathcal{E}^\Pi, \ast)$ expressed as

$$\mathcal{F}^\Pi, \ast = H^1_e(D) \oplus \left\{ \sum_{i=1}^\ell c_i \varphi_i : c_i \in \mathbb{R} \right\} \subset BL(D), \quad (5.2)$$

$$\mathcal{E}^\Pi, \ast(u, v) = \frac{1}{2} D(u, v), \quad u, v \in \mathcal{F}^\Pi, \ast. \quad (5.3)$$

$X^\Pi, \ast$ is recurrent. $\mathcal{E}^\Pi, \ast$ is a quasi-regular Dirichlet form on $L^2(\overline{D}^\Pi, \ast ; m)$.

We now prove that the family $\{X^\Pi, \ast : \Pi \text{ is a partition of } F \}$ exhausts all possible $m$-symmetric conservative diffusion extensions of the time changed RBM $X$ on $\overline{D}$.

Let $E$ be a Lusin space into which $\overline{D}$ is homeomorphically embedded as an open subset. The measure $m(dx) = f(x)dx$ on $\overline{D}$ is extended to $E$ by setting $m(E \setminus \overline{D}) = 0$. Let $Y = (Y_t, P^Y_x)$ be an $m$-symmetric conservative diffusion process on $E$ whose part process on $\overline{D}$ is identical in law with $X$. We denote by $(\mathcal{E}^Y, \mathcal{F}^Y)$ and $\mathcal{F}^Y_e$ the Dirichlet form of $Y$ on $L^2(E; m)$ and its extended Dirichlet space. We call $Y$ an $m$-symmetric conservative diffusion extension of $X$. The following theorem extends [CF1, Theorem 3.4].

**Theorem 5.1** There exists a partition Π of $F$ such that, as Dirichlet forms on $L^2(\overline{D}; m)$,

$$\langle \mathcal{E}^Y, \mathcal{F}^Y \rangle = \langle \mathcal{E}^\Pi, \ast, \mathcal{F}^\Pi, \ast \rangle. \quad (5.4)$$
To this end, we consider a finite measure \( \nu \) and Theorem 7.1.6 in it applies to \( \nu \) quasi support of \( E \) respectively. Thus we are in the same setting as in Section 7.1.6 in it applies to \( Y \) and \( \tilde{F} \).

\[ \mathcal{F}^Y_e = H^1_e(D), \quad \mathcal{E}^Y = \frac{1}{2}D, \tag{5.5} \]

\[ (\mathcal{F}^Y)^{\text{ref}} = \text{BL}(D) = H^1_e(D) \oplus \mathbb{H}^*(D), \quad (\mathcal{E}^Y)^{\text{ref}} = \frac{1}{2}D, \tag{5.6} \]

respectively.

\( \mathcal{E}^Y \) is a quasi-regular Dirichlet form on \( L^2(E; m) \) and \( Y \) is properly associated with it by virtue of Z.-M. Ma and M. Röckner [MR]. By Chen-Ma-Röckner [CMR], \( \mathcal{E}^Y \) is therefore quasi homeomorphic with a regular Dirichlet form. In particular, via a quasi homeomorphism \( j \) in [CF2 Theorems 3.1.13], we can assume that \( E \) is a locally compact separable metric space, \( \mathcal{E}^Y \) is a regular Dirichlet form on \( L^2(E; m) \), \( Y \) is an associated Hunt process on \( E \), and \( \tilde{F} := E \setminus \overline{D} \) is quasi-closed. Since \( Y \) is a conservative extension of the non-conservative process \( X \), \( \tilde{F} \) must be non \( \mathcal{E}^Y \)-polar. \( Y \) can be also shown to be irreducible as in the proof of [CF2, Lemma 7.2.7 (ii)]. Thus we are in the same setting as in §7.1 of [CF2] and Theorem 7.1.6 in it applies to \( Y \) and \( \tilde{F} \).

Every function in \( \mathcal{F}^Y_e \) will be taken to be \( \mathcal{E}^Y \)-quasi continuous. As \( Y \) is a diffusion with no killing inside, the jumping measure \( J \) and the killing measure \( K \) in the Beurling-Deny decomposition of \( \mathcal{E}^Y \) vanish so that we have by [CF2, Theorem 7.1.6]

\[ H^1_e(D) \subset \mathcal{F}^Y_e \subset \text{BL}(D), \quad \mathcal{H}^Y := \{ \mathbf{H}u : u \in \mathcal{F}^Y_e \} \subset \mathbb{H}^*(D), \tag{5.7} \]

\[ \mathcal{E}^Y(u, u) = \frac{1}{2}D(u, u) + \frac{1}{2}\mu^e_{\mathbf{H}u}(\tilde{F}), \quad u \in \mathcal{F}^Y_e, \tag{5.8} \]

where \( \mathbf{H}u(x) = \mathbf{E}^Y_x [u(Y_{\sigma_{\tilde{F}}})], \ x \in E \).

Let us prove that

\[ \mu^e_{\mathbf{H}u}(\tilde{F}) = 0 \quad u \in \mathcal{H}^Y. \tag{5.9} \]

To this end, we consider a finite measure \( \nu \) on \( E \) defined by

\[ \nu(B) = \int_D \mathbf{P}^Y_x \left( Y_{\sigma_{\tilde{F}}} \in B, \ \sigma_{\tilde{F}} < \infty \right) m(dx), \quad B \in \mathcal{B}(E). \]

\( \nu \) vanishes off \( \tilde{F} \) and charges no \( \mathcal{E}^Y \)-polar set. In view of [CF2, Lemma 5.2.9 (i)], \( \tilde{F} \) is a quasi support of \( \nu \) in the following sense: \( \nu(E \setminus \tilde{F}) = 0 \) and \( \tilde{F} \subset \tilde{F} \text{ q.e. for any quasi closed set } \tilde{F} \text{ with } \nu(E \setminus \tilde{F}) = 0 \).

Now, for \( u \in \mathcal{H}^Y, \) \( (5.6) \) and \( (5.7) \) imply that \( u = \sum_{j=1}^N c_j \varphi_j \) for some constants \( c_j \).

Take \( \tilde{F} = \{ x \in E : u(x) \in \{ c_1, \ldots, c_N \} \} \). Since \( u \) is quasi continuous, \( \tilde{F} \) is a quasi closed set. As \( u \) is continuous along the sample path of \( Y \) (cf. [CF2, Theorem 3.1.7]), we have \( \nu(E \setminus \tilde{F}) = \mathbb{P}_m(u(Y_{\sigma_{\tilde{F}}}) \notin \{ c_1, \ldots, c_N \}) = 0 \) on account of Proposition 3.2 and (4.9).
Accordingly, \( \tilde{F} \subset \hat{F} \) q.e., namely, \( u \) takes only finite values \( \{c_1, \ldots, c_N\} \) q.e. on \( \tilde{F} \). By the \textit{energy image density property} of \( \mu(u) \) due to N. Bouleau and F. Hirsch [BH] (cf. [CF2, Theorem 4.3.8]), we thus get (5.9).

Relation (5.7) and Proposition 3.2(ii) imply that every function \( u \in \mathcal{H}^Y(\subset BL(D)) \) admits a limit \( u(\partial_j) \) at each boundary point \( \partial_j \in F \) along the path of \( Z \). Define an \textit{equivalence relation} \( \sim \) on \( F \) by \( \partial_j \sim \partial_k \) if and only if \( u(\partial_j) = u(\partial_k) \). Then \( \mathcal{H}^Y = \{ \sum_{i=1}^{t} c_i\hat{\varphi}_i : c_i \in \mathbb{R} \} \) for \( \hat{\varphi}_i \) define by (5.1). Hence (5.2), (5.3), (5.7), (5.8) and (5.9) lead us to the desired identity (5.4).

Remark 5.2 (i) For different choices of \( f \), the family of all symmetric conservative extensions \( Y \) of \( X^f \) is invariant up to time changes because it shares a common family of extended Dirichlet spaces (5.2)-(5.3). The same can be said for more general time changed RBM \( X^\mu \), which will be formulated in Section 7.

(ii) We can replace the conservativness assumption on \( Y \) by a weaker one that \( Y \) is a proper extension of \( X \) with no killing on \( E \setminus \tilde{D} \). Then the above theorem remains valid if \( X^\Pi, \ast \) is allowed to be replaced by its subprocess being killed upon hitting some (but not all) \( \hat{\partial}_i \).

Remark 5.3 (Symmetric diffusion for a uniformly elliptic differential operator)

Given measurable functions \( a_{ij}(x), 1 \leq i, j \leq d \), on \( D \) such that

\[
a_{ij}(x) = a_{ji}(x), \quad \Lambda^{-1}|\xi|^2 \leq \sum_{1 \leq i, j \leq d} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad x \in D, \; \xi \in \mathbb{R}^d,
\]

for some constant \( \Lambda \geq 1 \), we consider a Dirichlet form

\[
(E, F) = (a, H^1(D))
\]

on \( L^2(D) \) where

\[
a(u, v) = \int_D \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) dx, \quad u, v \in H^1(D).
\]
If we replace the Dirichlet form (2.2) on $L^2(D)$ and the associated RBM $Z$ on $\overline{D}$, respectively, by the Dirichlet form (5.11) on $L^2(D)$ and the associated reflecting diffusion process on $\overline{D}$ constructed in [FTo], all results from Section 3 to Section 5 still hold without any change as we shall see now.

By this replacement, the extended Dirichlet space and the reflected Dirichlet space are still $H^1_e(D)$ and $BL(D)$, respectively, although the inner product $\frac{1}{2}D$ is replaced by $a$. The transience of (5.11) is equivalent to that of (2.2). The space $H^*\ast(D)$ is now defined by (2.3) with $a$ in place of $\frac{1}{2}D$. But, by noting that $a(c,c) = 0$ for any constant $c$ and by taking the characterization of a Liouville domain stated below Definition 2.1 into account, we readily see that $D \in \mathcal{D}$ is a Liouville domain relative to (5.11) if and only if so it is relative to (2.2).

\[\blacksquare\]

**Remark 5.4 (All possible symmetric conservative diffusion extensions of a one-dimensional minimal diffusion)** Consider a minimal diffusion $X$ on a one-dimensional open interval $I = (r_1, r_2)$ with no killing inside for which both boundaries $r_1, r_2$ are regular. Let $E$ be a Lusin space into which $I$ is homeomorphically embedded as an open subset. The speed measure $m$ of $X$ is extended to $E$ by setting $m(E \setminus I) = 0$. Let $Y$ be an $m$-symmetric conservative diffusion extension of $X$ from $I$ to $E$. Then, by removing some $m$-polar open set for $Y$ from $\tilde{F} = E \setminus I$, a homeomorphic image of $Y$ is identical with either the two point extension of $X$ to $[r_1, r_2]$ or its one-point extension to the one-point compactification of $I$. This fact was implicitly indicated in [F2, §5] and [F3, §5] without proof. This can be shown in a similar manner to the proof of Theorem 5.1 by establishing the counterpart of the identity (5.9) and by noting that, for the one-point and two-point extensions of $X$, every non-empty subset of the state space has a positive 1-capacity uniformly bounded away from zero due to the bound [CF2, (2.2.31)] and so a quasi-homeomorphism is reduced to a homeomorphism.

To put it another way, Theorem 5.1 reveals that the time changed RBM $X$ on an unbounded domain with $N$-Liouville branches has a very similar structure to the one-dimensional diffusion only by changing two boundary points to $N$ boundary points.

\[\blacksquare\]

We note that the connected sum of non-parabolic manifolds being studied by Y. Kuz’menko and S. Molchanov [KM], A. Grigor’yan and L. Salloff-Coste [GS] bears a strong similarity to the present paper in the setting although the main concern in these papers was the heat kernel estimates.

### 6 Characterization of $L^2$-generator of extension $Y$ by zero flux condition at infinity

For a strictly positive bounded integrable function $f$ on $D$, we put $m(dx) = f(x)dx$ and denote by $(\cdot, \cdot)$ the inner product for $L^2(D; m)$. Let $Y$ be any $m$–symmetric conservative diffusion extension of the time changed process $X = X^f = (X_t, \zeta, P_x)$ of the RBM $Z$
on $\overline{D}$. Let $\Pi : F \mapsto \{\hat{\partial}_1, \ldots, \hat{\partial}_\ell\}$, $\ell \leq N$, be the corresponding partition of the boundary $F = \{\partial_1, \ldots, \partial_N\}$ appearing in Theorem 5.1. The Dirichlet form $(\mathcal{E}^Y, \mathcal{F}^Y)$ of $Y$ on $L^2(D; m)$ is then described as
\[
\mathcal{F}^Y = \left\{ u = u_0 + \sum_{i=1}^{\ell} c_i \hat{\varphi}_i : u_0 \in H^1_0(D) \cap L^2(D; m), \ c_i \in \mathbb{R} \right\},
\]
and
\[
\mathcal{E}^Y(u, v) = \frac{1}{2} \mathcal{D}(u, v), \quad u, v \in \mathcal{F}^Y,
\]
where $\hat{\varphi}_i$, $1 \leq i \leq \ell$, are defined by (5.1).

Let $\mathcal{A}$ be the $L^2$-generator of $Y$, that is, $\mathcal{A}$ is a self-adjoint operator on $L^2(D; m)$ such that $u \in \mathcal{D}(\mathcal{A})$, $\mathcal{A} u = v \in L^2(D; m)$ if and only if $u \in \mathcal{F}^Y$ with $\mathcal{E}^Y(u, w) = -(v, w)$ for every $w \in \mathcal{F}^Y$. In view of Proposition 3.2 the condition (7.3.4) of [CF2] is fulfilled by $Y$. Therefore Theorem 7.7.3 (vii) of [CF2] is well applicable in getting the following characterization of $\mathcal{A}$:
\[
u \in \mathcal{D}(\mathcal{A}) \quad \text{if and only if} \quad u \in \mathcal{D}(\mathcal{L}) \text{ and } \mathcal{N}(u)(\hat{\partial}_i) = 0, \ 1 \leq i \leq \ell.
\]
In this case, $\mathcal{A} u = \mathcal{L} u$.

Here $\mathcal{L}$ is a linear operator defined as follows: $u \in \mathcal{D}(\mathcal{L})$, $\mathcal{L} u = v \in L^2(D; m)$ if and only if $u \in \text{BL}(D) \cap L^2(D; m)$ and $\frac{1}{2} \mathcal{D}(u, w) = -(v, w)$ for every $w \in H^1_0(D) \cap L^2(D; m)$, or equivalently, for every $w \in C^1_c(D)$. $\mathcal{N}(u)(\hat{\partial}_i)$ is the flux of $u$ at $\hat{\partial}_i$ defined by
\[
\mathcal{N}(u)(\hat{\partial}_i) = \frac{1}{2} \mathcal{D}(u, \hat{\varphi}_i) + (\mathcal{L} u, \hat{\varphi}_i), \quad 1 \leq i \leq \ell.
\]

It can be readily verified that $u \in \mathcal{D}(\mathcal{L})$ if and only if $u \in \text{BL}(D) \cap L^2(D; m)$, $\Delta u$ in the Schwartz distribution sense is in $L^2(D)$ and
\[
\mathcal{D}(u, w) + \int_D \Delta u(x) \cdot w(x) dx = 0 \quad \text{for every } w \in C^1_c(\overline{D}). \tag{6.1}
\]
In this case, $\mathcal{L} u(x) = \frac{1}{2f(x)} \Delta u(x)$, $x \in D$. The equation (6.1) can be interpreted as the requirement that the generalized normal derivative of $u$ vanishes on $\partial D$. Thus we have

**Theorem 6.1** $u \in \mathcal{D}(\mathcal{A})$ if and only if $u \in \text{BL}(D) \cap L^2(D; m)$, $\Delta u$ in the Schwartz distribution sense belongs to $L^2(D)$, the equation (6.1) is satisfied and
\[
\left( \mathcal{N}(u)(\hat{\partial}_i) \right) = \frac{1}{2} \mathcal{D}(u, \hat{\varphi}_i) + \frac{1}{2} \int_D \Delta u(x) \hat{\varphi}_i(x) dx = 0, \quad 1 \leq i \leq \ell. \tag{6.2}
\]
In this case,
\[
\mathcal{A} u(x) = \frac{1}{2f(x)} \Delta u(x), \quad \text{a.e. on } D. \tag{6.3}
\]

Suppose $u \in \mathcal{D}(\mathcal{A})$ is smooth on $\overline{D}$. Then $\frac{\partial u}{\partial n} = 0$ on $\partial D$ due to the condition (6.1) so that the zero flux condition (6.2) at $\hat{\partial}_i$ can be expressed as
\[
\lim_{r \to \infty} \int_{D \cap \partial B_r(o)} u_r(x) \hat{\varphi}_i(x) d\sigma_r(dx) = 0, \quad 1 \leq i \leq \ell, \tag{6.4}
\]
where $\sigma_r$ is the surface measure on $\partial B_r(o)$. 14
The last part of Section 7.6 (4°) of [CP2] has treated a very special case of the above where \( D = \mathbb{R}^d \), \( d \geq 3 \), and \( Y \) is the one-point reflection at the infinity of \( \mathbb{R}^d \) of a time changed Brownian motion on \( \mathbb{R}^d \).

In [F3], the \( L^2 \)-generator of any symmetric diffusion extension \( Y \) of a one-dimensional minimal diffusion \( X \) is identified. In this case, the Dirichlet form of \( Y \) admits its reproducing kernel which enables us to identify also the \( C_b \)-generator of \( Y \), recovering the general boundary condition due to W. Feller and K. Itô-H. P. McKean.

### 7 Extensions of more general time changed RBMs

All the results in Sections 4-6 except for (6.3) hold for more general time changed RBMs than \( X^f \). Let \( Z = (Z_t, Q_x) \), \( f \), \( X = X^f = (X_t, \zeta, P_x) \), \( X^* = (X^*_t, P^*_x) \) be as in Section 4.

We consider a finite smooth measure \( \mu \) on \( \mathcal{D} \) with full quasi-support \( \mathcal{D} \) relative to the Dirichlet form \((\mathcal{E}, \mathcal{F})\) of (2.2). Let \( A^\mu \) be the PCAF of \( Z \) with Revuz measure \( \mu \) and \( X^\mu = (X^\mu_t, \zeta^\mu, P^\mu_x) \) be the time changed process of \( Z \) by \( A^\mu \). The Markov process \( X^\mu \) is \( \mu \)-symmetric and its Dirichlet form \((\mathcal{E}^{X^\mu}, \mathcal{F}^{X^\mu})\) on \( L^2(\mathcal{D}; \mu) \) is given by

\[
\mathcal{E}^{X^\mu} = \frac{1}{2} D, \quad \mathcal{F}^{X^\mu} = H^1(D) \cap L^2(\mathcal{D}; \mu).
\]

#### Proposition 7.1
It holds that

\[
Q_x(A^\mu_\infty < \infty) = 1 \quad \text{for q.e. } x \in \mathcal{D}, \tag{7.2}
\]

\[
P^\mu_x(\zeta^\mu < \infty, X^\mu_{\zeta^\mu-} = \partial_i) = \varphi_i(x) > 0, \quad \text{for q.e. } x \in \overline{\mathcal{D}} \text{ and } 1 \leq i \leq N. \tag{7.3}
\]

**Proof.** Fix a strictly positive bounded integrable function \( h_0 \). By the transience of \( Z \) and [CP2] Theorem A.2.13 (v)), \( G^{\mathcal{D}}_{0+, h_0}(x) < \infty \) for q.e. \( x \in \overline{\mathcal{D}} \). For integer \( k \geq 1 \), let

\[
\Lambda_k := \left\{ x \in \overline{\mathcal{D}} : G^{\mathcal{D}}_{0+, h_0}(x) \leq 2^k \right\} \quad \text{and} \quad h(x) = \sum_{k=1}^{\infty} 2^{-2k} 1_{\Lambda_k}(x) h_0(x).
\]

Then \( h \) is a strictly positive bounded integrable function on \( \overline{\mathcal{D}} \) with \( G^{\mathcal{D}}_{0+, h}(x) \leq 1 \) q.e. on \( \overline{\mathcal{D}} \). From [CP2] (4.1.3), we have

\[
\int_{\overline{\mathcal{D}}} \mathbb{E}^{Q_x}[A^\mu_\infty] h(x) dx = \langle G^{\mathcal{D}}_{0+, h}, \mu \rangle \leq \mu(\overline{\mathcal{D}}) < \infty. \tag{7.4}
\]

It follows that \( \mathbb{E}^{Q_x}[A^\mu_\infty] < \infty \) a.e \( x \in \overline{\mathcal{D}} \) and hence q.e. \( x \in \mathcal{D} \) by [CP2] Theorem A.2.13 (v)), yielding (7.2). (7.3) follows from (7.2) and Proposition 3.1 \( \square \)

Since \( \mu(dx) = f(x) dx \) has its quasi-support \( \overline{\mathcal{D}} \) relative to \((\mathcal{E}, \mathcal{F})\), the Dirichlet form \((\mathcal{E}^X, \mathcal{F}^X)\) of (4.1) shares the common quasi-notion with \((\mathcal{E}, \mathcal{F})\) ([CP2] Theorem 5.2.11]). Hence the quasi-support of \( \mu \) relative to \((\mathcal{E}^X, \mathcal{F}^X)\) is still \( \overline{\mathcal{D}} \).

The Dirichlet form \((\mathcal{E}^*, \mathcal{F}^*)\) on \( L^2(\mathcal{D}^*, m) \) of \( X^* \) is quasi-regular. According to the quasi-homeomorphism method already used in Section 4, we may assume it to be regular. The
measure \( \mu \) on \( \overline{D} \) is extended to \( \overline{D}^* \) by setting \( \mu(F) = 0 \). We claim that the quasi-support of \( \mu \) relative to this Dirichlet form equals \( \overline{D}^* \) by using a criteria \([CF2\) Theorem 3.3.5].

Assume that \( u \in F^* \) is \( E^* \)-quasi-continuous and that \( u = 0 \) \( \mu \)-a.e. Then \( u|_{\overline{D}} \) is \( E^ \)-quasi-continuous \([CF2\) Theorem 3.3.8]) so that \( u = 0 \) q.e. on \( \overline{D} \). According to the same reference, there exists a Borel \( m \)-polar set \( C \subset \overline{D} \) relative to \( X^* \) such that \( u(x) = 0 \) for every \( x \in \overline{D} \setminus C \). Since \( u \) is continuous along the path of \( X^* \) \([CF2\) Theorem 3.1.7]), we have for each \( 1 \leq i \leq N \)

\[
P^*_m\left(u(\partial_i) = \lim_{t \uparrow \sigma_F} u(X^*_i), \sigma_C = \infty, \sigma_F < \infty, X^*_{\sigma_F} = \partial_i \right) = P_m(\zeta < \infty, X_{\zeta_\partial} = \partial_i) > 0,
\]

and so \( u \) vanishes on \( F \) and hence q.e. on \( \overline{D}^* \), as was to be proved.

**Theorem 7.2** There exists a unique \( \mu \)-symmetric conservative diffusion \( \tilde{X}^{*,\mu} \) on \( \overline{D}^* \) which is a q.e. extension of \( X^* \) in the sense that the part of the former on \( \overline{D} \) coincides in law with the latter for q.e. starting points \( x \in \overline{D} \). The extended Dirichlet space of \( \tilde{X}^{*,\mu} \) equals \((BL(D), \frac{1}{2}D)\) the reflected Dirichlet space of \( X^\mu \).

**Proof.** Let \( B^0_t \) and \( B_t \) be the PCAFs of \( X \) and \( X^* \), respectively, with Revuz measure \( \mu \). According to \([CF2\) Proposition 4.1.10]

\[
B^0_t = B_{t \wedge \sigma_F}.
\] (7.5)

Let \( \tilde{X}^\mu \) and \( \tilde{X}^{*,\mu} \) be the time changed processes of \( X \) and \( X^* \) by means of \( B^0_t \) and \( B_t \), respectively. The Markov process \( \tilde{X}^\mu \) is then the part of \( \tilde{X}^{*,\mu} \) on \( \overline{D} \) by (7.5). Since \( X^* \) is recurrent, so is \( \tilde{X}^{*,\mu} \) in view of \([CF2\) Theorem 5.2.5]. Therefore \( \tilde{X}^{*,\mu} \) is a \( \mu \)-symmetric conservative diffusion extension of \( X^\mu \).

On the other hand, the Dirichlet form of \( \tilde{X}^\mu \) on \( L^2(\overline{D}; \mu) \) is identical with \((\mathcal{E}, F^\mu)\) the Dirichlet form of \( X^\mu \) on \( L^2(\overline{D}; \mu) \), and consequently \( \tilde{X}^{*,\mu} \) is a q.e. extension of \( X^\mu \). The last statement follows from the invariance of extended and reflected Dirichlet spaces under time changes by fully supported PCAFs.

The uniqueness of such a \( \mu \)-symmetric conservative Markovian extension of \( X^\mu \) to \( \overline{D}^* \) follows from \([CF2\) Theorem 7.7.3].

Similarly, all results in Section 4 and 5 with \( \mu \) in place of \( dm = fdx \) remain valid except for (6.3).

**Remark 7.3** One can give an alternative proof of Theorem 7.2 without invoking the time change of \( X^* \) but still using the quasi-regularity of \((\mathcal{E}^*, \mathcal{F}^*)\). Indeed, the following proposition combined with (7.3) and \([CF2\) Theorem 7.7.3] readily yields Theorem 7.2.

Each function in \( \mathcal{F}^*_\epsilon \) is taken to be \( \mathcal{E}^* \)-quasi continuous. Define

\[
\hat{\mathcal{F}} = \mathcal{F}^*_\epsilon \cap L^2(\overline{D}; \mu) \quad \text{and} \quad \hat{\mathcal{E}}(u, v) = \mathcal{E}^*(u, v) = \frac{1}{2}D(u, v) \text{ for } u, v \in \hat{\mathcal{F}}.
\] (7.6)

**Proposition 7.4** (i) \((\hat{\mathcal{E}}, \hat{\mathcal{F}})\) is a quasi-regular Dirichlet form on \( L^2(\overline{D}^*; \mu) \).
(ii) Its associated strong Markov process \( \hat{X} \) on \( \overline{D}^* \) is a \( \mu \)-symmetric conservative diffusion which is a q.e. extension of \( X^\mu \).

(iii) Each \( \partial_j \) is non-\( \mathcal{E} \)-polar.

Proof. (i) As \( \overline{D} \) is a quasi-support of \( \mu \), \( u = 0 \) \( \mu \)-a.e. for \( u \in \hat{\mathcal{E}} \) implies \( u = 0 \) a.e. on \( \overline{D} \) and \( D(u, u) = 0 \). This together with the transience of \( (\mathcal{F}_e^*, \mathcal{E}_e^*) \) implies that \( (\mathcal{E}, \hat{\mathcal{F}}) \) is a well defined Dirichlet form on \( L^2(\overline{D}; \mu) \).

Since \( (\mathcal{E}_e^*, \mathcal{E}_e^*) \) is a quasi-regular Dirichlet form on \( L^2(\overline{D}; m) \), by [CF2, Remark 1.3.9], there is an increasing sequence of compact subsets \( \{F_k\} \) of \( \overline{D} \) so that

(a) there is an increasing sequence of compact subsets \( \{F_k\} \) of \( \overline{D} \) so that \( \cup_{k \geq 1} F_k^* \) is \( \mathcal{E}_1^* \)-dense in \( \mathcal{F}_e^* \).

(b) there is an \( \mathcal{E}_1^* \)-dense of countable set \( \Lambda_0 := \{f_j; j \geq 1\} \) of bounded functions of \( \mathcal{F}_e^* \) so that \( \{f_j; j \geq 1\} \subset \mathcal{C}\{\mathcal{F}_k\} \) and they separate points of \( \cup_{k \geq 1} F_k \).

By the contraction of the Dirichlet form, we may and do assume without loss of generality that for every integer \( n \geq 1 \) and \( f \in \Lambda_0 \), \( ((-n) \vee f) \land n \in \Lambda_0 \). We claim that \( \cup_{k \geq 1} F_k^* \subset \mathcal{E}_1^* \)-dense in \( \hat{\mathcal{F}}_b \). Let \( u \in \hat{\mathcal{F}}_b \). Since \( \hat{\mathcal{F}}_b = \mathcal{F}_b^* \), there are \( u_k \in \mathcal{F}_b^* \) so that \( u_k \to u \) in \( \mathcal{E}_1^* \)-norm. Using truncation if needed, we may and do assume \( \|u_k\|_\infty \leq \|u\|_\infty + 1 \). Taking a subsequence if needed, we may also assume that \( u_k \) converges to \( u \) \( \mathcal{E}_e^* \)-q.e. on \( \overline{D} \). Since \( \mu \) is a finite smooth measure, we conclude that \( u_k \) is \( \hat{\mathcal{E}}_1 \)-convergent to \( u \). This proves the claim. As \( \hat{\mathcal{F}}_b \) is \( \mathcal{E}_1 \)-dense in \( \hat{\mathcal{F}} \), it follows that \( \{F_k\} \) is an \( \mathcal{E} \)-nest on \( \overline{D}^* \).

A similar argument shows that \( \Lambda_0 \subset \hat{\mathcal{F}}_b = \mathcal{F}_b^* \) is \( \mathcal{E}_1 \)-dense in \( \hat{\mathcal{F}}_b \) and hence in \( \hat{\mathcal{F}} \). This proves the assertion (i).

(ii) Since \( 1 \in \hat{\mathcal{F}} \) and \( D(1, 1) = 0 \), the associated \( \mu \)-symmetric diffusion \( \hat{X} \) on \( \overline{D}^* \) is recurrent and conservative. For \( R > r \), take \( \psi \in C_c^\infty(D) \) with \( \psi = 1 \) on \( B_{R+1}(0) \). Then, for any bounded \( \psi \in \hat{\mathcal{F}} \), \( \psi u \in H^1_e(D) \) and so

\[
\{v \in \hat{\mathcal{F}} : v = 0 \text{ q.e. on } \overline{D} \setminus B_R(0)\} = \{v \in H^1_e(D) \cap L^2(\overline{D}; \mu) : v = 0 \text{ q.e. on } \overline{D} \setminus B_R(0)\},
\]

namely, the part of \( \hat{\mathcal{E}} \) on \( D \cap B_R(0) \) coincides with the part of \( \mathcal{E}X^\mu \) on \( D \cap B_R(0) \). By letting \( R \to \infty \), we see that the part of \( \hat{\mathcal{E}} \) on \( \overline{D} \) coincides with \( \mathcal{E}X^\mu \), proving (ii).

(iii) The non-\( \mathcal{E} \)-polarity of \( \partial_j \) follows from (ii) and (7.3). \( \square \)

8 Appendix: equivalence and quasi-homeomorphism

In dealing with boundary problems for symmetric Markov processes, it is convenient to introduce an equivalence of Dirichlet spaces following [FOT1, A.4] as will be stated below.

We say that a quadruplet \( (E, m, \mathcal{E}, \mathcal{F}) \) is a Dirichlet space if \( E \) is a Hausdorff topological space with a countable base, \( m \) is a \( \sigma \)-finite positive Borel measure on \( E \) and \( \mathcal{E} \) with domain \( \mathcal{F} \) is a Dirichlet form on \( L^2(E; m) \). The inner product in \( L^2(E; m) \) is denoted by \( (\cdot, \cdot)_E \).

For a given Dirichlet space \( (E, m, \mathcal{E}, \mathcal{F}) \), the notions of an \( \mathcal{E} \)-nest, an \( \mathcal{E} \)-polar set, an \( \mathcal{E} \)-quasi-continuous numerical function and ‘\( \mathcal{E} \)-quasi-everywhere’ (‘\( \mathcal{E} \)-q.e.’ in abbreviation) are
defined as in [CF2, Definition 1.2.12]. The quasi-regularity of the Dirichlet space is defined just as in [CF2, Definition 1.3.8]. We note that the space $F_b = \mathcal{F} \cap L^\infty(E; m)$ is an algebra.

**Remark 8.1** In Section 1.2 and the first half of Section 1.3 of [CF2], it is assumed that

$$\text{supp}[m] = E.$$  \hfill (8.1)

We need not assume it. Generally, if we let $E' = \text{supp}[m]$, then $E \setminus E'$ is $\mathcal{E}$-polar according to the definition of the $\mathcal{E}$-polarity. If $(E, m, \mathcal{E}, \mathcal{F})$ is quasi-regular, so is $(E', m|_{E'}, \mathcal{E}, \mathcal{F})$ accordingly. Therefore we may assume (8.1) if we like by replacing $E$ with $E'$.

Given two Dirichlet spaces

$$(E, m, \mathcal{E}, \mathcal{F}), \quad (\tilde{E}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}),$$

we call them equivalent if there is an algebraic isomorphism $\Phi$ from $\mathcal{F}_b$ onto $\tilde{\mathcal{F}}_b$ preserving three kinds of metrics: for $u \in \mathcal{F}_b$

$$\|u\|_\infty = \|\Phi u\|_\infty, \quad (u, u)_E = (\Phi u, \Phi u)_{\tilde{E}}, \quad \mathcal{E}(u, u) = \tilde{\mathcal{E}}(\Phi u, \Phi u).$$

One of the two equivalent Dirichlet spaces is called a representation of the other.

The underlying spaces $E, \tilde{E}$ of two Dirichlet spaces (8.2) are said to be quasi-homeomorphic if there exist $\mathcal{E}$-nest $\{F_n\}$, $\tilde{\mathcal{E}}$-nest $\{\tilde{F}_n\}$ and a one to one mapping $q$ from $E_0 = \cup_{n=1}^\infty F_n$ onto $\tilde{E}_0 = \cup_{n=1}^\infty \tilde{F}_n$ such that the restriction of $q$ to each $F_n$ is a homeomorphism onto $\tilde{F}_n$. $\{F_n\}$, $\{\tilde{F}_n\}$ are called the nests attached to the quasi-homeomorphism $q$. Any quasi-homeomorphism is quasi-notion-preserving.

We say that the equivalence $\Phi$ of two Dirichlet spaces (8.2) is induced by a quasi-homeomorphism $q$ of the underlying spaces if

$$\Phi u(\tilde{x}) = u(q^{-1}(\tilde{x})), \quad u \in \mathcal{F}_b, \quad \tilde{m} - \text{a.e. } \tilde{x}.$$  \hfill (8.3)

Then $\tilde{m}$ is the image measure of $m$ and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is the image Dirichlet form of $(\mathcal{E}, \mathcal{F})$.

**Theorem 8.2** Assume that two Dirichlet spaces (8.2) are quasi-regular and that they are equivalent. Let $X = (X_t, \mathbb{P}_x)$ (resp. $\tilde{X} = (\tilde{X}_t, \tilde{\mathbb{P}}_x)$) be an $m$-symmetric right process on $E$ (resp. an $\tilde{m}$-symmetric right process on $\tilde{E}$) properly associated with $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ (resp. $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(\tilde{E}; \tilde{m})$). Then the equivalence is induced by a quasi-homeomorphism $q$ with attached nests $\{F_n\}$, $\{\tilde{F}_n\}$ such that $\tilde{X}$ is the image of $X$ by $q$ in the following sense: there exist an $m$-inessential Borel subset $N$ of $E$ containing $\cap_{n=1}^\infty F_n$ and an $\tilde{m}$-inessential Borel subset $\tilde{N}$ of $\tilde{E}$ containing $\cap_{n=1}^\infty \tilde{F}_n$ so that $q$ is one to one from $E \setminus N$ onto $\tilde{E} \setminus \tilde{N}$ and

$$\tilde{X}_t = q(X_t), \quad \tilde{\mathbb{P}}_x = \mathbb{P}_{q^{-1}x}, \quad \tilde{x} \in \tilde{E} \setminus \tilde{N}.$$  \hfill (8.3)

**Proof.** Since both Dirichlet spaces in (8.2) are assumed to be quasi-regular, they are equivalent to some regular Dirichlet spaces and the equivalences are induced by some quasi-homeomorphisms $q_1, q_2$ in view of [CF2, Theorem 1.4.3]. Since two Dirichlet spaces in (8.2)
are also assumed to be equivalent, so are the corresponding two regular Dirichlet spaces, the equivalence being induced by a quasi-homeomorphism $g_3$ on account of [FOT1 Theorem A.4.2] combined with [CF2 Theorem 1.2.14]. Hence the equivalence of the quasi-regular Dirichlet spaces in (8.2) is induced by the quasi-homeomorphism $q = q_1 \circ q_3 \circ q_2^{-1}$ between $E$ and $\tilde{E}$. Let $\{F_n\}, \{\tilde{F}_n\}$ be the nests attached to $q$.

According to [CF2 Theorem 3.1.13], we may assume without loss of generality that both $X$ and $\tilde{X}$ are Borel right processes. Further the $E$-polarity is equivalent to the $m$-polar for $X$. By virtue of [CF2 Theorem A.2.15], we can therefore find an $m$-inessential Borel set $N_1 \subset E$ containing $\cap_{n=1}^{\infty} F_n^c$. Consider the set $\tilde{N}_1 \subset \tilde{E}$ defined by $q(E \setminus N_1) = \tilde{E} \setminus \tilde{N}_1$. $\tilde{N}_1$ is an $\tilde{E}$-polar Borel set and $q$ is one to one from $E \setminus N_1$ onto $\tilde{E} \setminus \tilde{N}_1$.

Define the process $\tilde{X} = (\tilde{X}_t, \tilde{P}_x)$ by

$$\tilde{X}_t = q(X_t), \quad \tilde{P}_x = P_{q^{-1}\tilde{x}}, \quad \tilde{x} \in \tilde{E} \setminus \tilde{N}_1.$$ 

On account of [FFY Lemma 3.1], we can then see that $\tilde{X}$ is an $\tilde{m}$-symmetric Markov process on $\tilde{E} \setminus \tilde{N}_1$ properly associated with the Dirichlet form $(\tilde{E}, \tilde{F})$ on $L^2(\tilde{E}; \tilde{m})$. Since the $\tilde{m}$-symmetric Borel right process $\tilde{X}$ is also properly associated with the Dirichlet form $(\tilde{E}, \tilde{F})$ on $L^2(\tilde{E}; \tilde{m})$, the same method as in the proof of [CF2, Theorem 3.1.12] combined with [CF2 Theorem A.2.15] leads us to finding an $\tilde{m}$-inessential Borel set $\tilde{N}$ containing $\tilde{N}_1$ for $\tilde{X}$ such that the Markov processes $\tilde{X}\big|_{\tilde{E}\setminus \tilde{N}}$ and $\tilde{X}\big|_{\tilde{E}\setminus \tilde{N}}$ are identical in law. It now suffices to define the set $N$ by $E \setminus N = q^{-1}(\tilde{E} \setminus \tilde{N})$. 

\[\square\]

**Remark 8.3** Owing to the works of S. Albeverio, Z.-M. Ma, M. R"ockner and P. J. Fitzsimmons, the quasi-regularity of a Dirichlet form has been known to be not only a sufficient condition but also a necessary one for the existence of a properly associated right process. It is further shown in [CMR] that a Dirichlet form is quasi-regular if and only if it is quasi-homeomorphic to a regular Dirichlet form on a locally compact separable metric space. These facts are formulated by Theorem 1.5.3 and Theorem 1.4.3, respectively, of [CF2] under the assumption (8.1) which is not needed actually. But we may assume it without loss of generality as will be seen below.

Indeed, let $E$ be a Lusin space, $m$ be a $\sigma$-finite measure on $E$ and $X$ be an $m$-symmetric Borel right process on $E$. Then, for $E_0 = \text{supp}[m]$, $E \setminus E_0$ is an $m$-negligible open set so that it is $m$-polar for $X$ by [CF2 Theorem A.2.13 (iii)]. Hence, by [CF2 Theorem A.2.15], there exists a Borel set $E_1 \subset E_0$ such that $E \setminus E_1$ is $m$-inessential for $X$. $E_1$ is the support of $m|_{E_1}$ because, for any $x \in E_1$ an any neighborhood $O(x)$ of $x$, $m(O(x) \cap E_1) = m(O(x)) - m(O(x) \cap (E \setminus E_1)) > 0$. Hence it suffices to replace $E$ by $E_1$.

In Theorem 5.1 the extension process $Y$ is assumed to live on a Lusin space $E$ into which $\overline{D}$ is homeomorphically embedded as an open subset. In this particular case, the above set $E_1$ can be chosen to contain $\overline{D}$ on account of the proof of [CF2, Theorem A.2.15]. Therefore, in Theorem 5.1 (resp. Remark 5.4), we can assume more strongly that $\overline{D}$ (resp. $I$) is homeomorphically embedded into the state space $E$ of $Y$ as a dense open subset. 

\[\square\]
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