A spectral method for half-integer spin fields based on spin-weighted spherical harmonics

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Abstract
We present a new spectral scheme for analysing functions of half-integer spin-weight on the two-sphere and demonstrate the stability and convergence properties of our implementation. The dynamical evolution of the Dirac equation on a manifold with spatial topology of $S^2$ with a pseudo-spectral method is also demonstrated.

Keywords: numerical relativity, spectral method, Dirac equation, spin-weighted spherical harmonics

(Some figures may appear in colour only in the online journal)

1. Introduction
In a recent paper [3] we presented a spectral method for tensorial partial differential equations on geometries with a spherical component. We showed how to implement the ‘eth’-formalism based on spin-weighted spherical harmonics following the work by Newman, Penrose and others [9, 23]. The restriction to tensorial equations implied the use of spin-weighted functions with integer spin-weights. In the present work we extend our method to also include functions with half-integer spin-weight on spherical geometries. In particular, we show how the evolution of dynamical half-integer spin fields can be accomplished. Our motivation for doing so stems mainly from general relativity: spinorial fields describing elementary particles such as electrons or neutrinos are fundamental sources for the Einstein equations and are studied for instance as test fields on Kerr–Newman backgrounds, see [10] and references
therein. However, there are also interesting developments in condensed matter physics in relation to the description of graphene \[5, 39\] for which our method might be relevant. The numerical treatment of the Dirac field is particularly interesting as its evolution can be rather peculiar and sometimes even counter-intuitive as visualized by Thaller \[34, 35\].

It is well known that when working with \(S^2\) (and other compact geometries) coordinate singularities can lead to instabilities that spoil the accuracy of numerical schemes. These issues can be avoided by working with spectral methods where coordinate-independent manipulation of expressions in terms of a well-defined basis of functions upon which the action of differential operators reduces to algebraic manipulation is possible \[4, 11\]. Thus, issues of coordinate singularities are dealt with automatically by the method.

Our spectral method makes use of the spin-weighted spherical harmonics (SWSH) \[9, 23, 25, 26\] which can be shown to be equivalent to the well-known Wigner \(D\)-functions \[6, 29, 33\] and thus provide a complete, orthonormal basis for \(L^2(SU(2))\). In the description of half-integer spin fields, we have found it convenient to work directly with the half-integer SWSH due to the particularly simple action of the associated \(\delta\) and \(\delta'\) differential operators which may be thought of as constituting covariant derivative operators on the sphere \[9\]. As the action of the aforementioned operators on the SWSH reduces to simple algebraic manipulation we may translate PDE systems into coupled (infinite dimensional) ODE systems and thus construct a spectral evolution scheme for an initial value problem (IVP) of interest.

Our work extends that of \[3, 15\] into a spectral algorithm for SWSH with half-integer spin-weight \(s\). The method we present inherits several desirable properties from \[3, 15\]. In particular it is theoretically exact if a minimum number of grid points are used at a given band-limit \(L\). Furthermore the same algorithmic complexity of \(\mathcal{O}(L^3)\) is achievable for spectral transformations. In brief, \[3, 15\] seek to calculate values for the integer SWSH over \(S^2\) by mapping the sphere into the 2-torus (thus allowing for fast Fourier transforms; FFT to be used) and relate the SWSH to the reduced Wigner \(d\)-functions evaluated at \(2\pi\). For a short summary of past work related to the integer case see \[3\]. The approach to the half-integer case is in principle similar, however suitable modification of the 2-torus map must be made in order to account for the periodicity of spinor fields. In addition, the recursion relation \[36\] that allows for evaluation of the Wigner \(d\)-functions must be modified.

As a toy model we numerically explore the dynamics associated with a Dirac equation on a two-dimensional manifold with spatial topology of \(S^2\) as an IVP. This will allow for a test of the spin-weighted spectral transformations we present in the context of evolution equations. Common techniques for treating the numerical problem of the Dirac equation consist of: FD schemes formulated on a flat-lattice in configuration space \[12, 13\], on a grid within a finite-volume in momentum-space \[22\] and using methods based on the split-step operator technique \[7, 8, 20, 21\]. Particular to the FD approach special care must be taken so as to avoid the Fermion-doubling problem \[24\]. Elimination of spurious modes introduced to the solution may be accomplished by means of nonlocal approximation for the spatial derivative operator \[32, 38\] or by staggered-grid schemes \[12, 13\]. As remarked upon in \[13\] the issue of spurious modes is not particular to the Dirac equation but can occur whenever a symmetric FD approximant is used for a first derivative on a uniform grid. We seek to avoid the above issue entirely by making use of the global approximation to functions and derivative operators that spectral methods provide \[14\].

The equations of motion (EOM) that we derive for the Dirac equation in the aforementioned geometric setting result in a coupled system of variable-coefficient linear hyperbolic PDEs. When performing a SWSH decomposition this results in product terms that must be further simplified. From past experience \[3\] we have found that while it is possible to make use of Clebsch–Gordan expansions (see section 2), i.e., working entirely in the space of
coefficients, it is far more convenient to instead work with a pseudo-spectral (PS) method (see section 5.1), and thus we follow the latter approach in this work.

This paper is structured as follows: in section 2 we recall basic properties of the half-integer SWSH together with the δ-formalism. In sections 2.1 and 2.2 respectively we discuss the forward and backward spectral transformations of spin-weighted functions on the sphere. In section 3 we present the required modification to the recursion relation for computing the reduced Wigner functions at $\pi/2$. Subsequently, in section 4 we perform consistency checks on our implementation of the algorithm. Error pairs and the property of exponential convergence are analysed. In section 5 we proceed by numerically solving the 2+1 dimensional Dirac equation as an IVP on a curved geometry, specifying to cases with spatial topology of $S^2$. We find EOM adapted to the spin-weighted formalism. In section 5.1 we briefly recall the PS method. In section 5.2 we construct numerical solutions to the EOM and inspect convergence properties of the numerical solutions obtained together with conserved currents. Section 6 concludes.

2. SWSH summary

In this section we briefly summarize key properties of the SWSH that we will make use of in the presentation of our half-integer spectral algorithm and in the PS method for solution of dynamical systems. For further details we refer the reader to [3, 6, 9, 23, 25, 26].

Geometrically, the spin-weighted functions are sections of certain line bundles over the two-sphere $S^2$. They have a representation in terms of ordinary functions over patches of $S^2$ which depend on the choice of coordinates and the choice of an orthonormal frame at points of the patch. The behaviour of this representation under changes of the frame is captured by the spin-weight assigned to the function. Here, we will deal only with representations of the spin-weighted functions given in terms of the standard polar coordinates $(\theta, \varphi)$ and the orthonormal frame defined in terms of the coordinate derivative vectors

$$e_{\theta} = \partial_{\theta}, \quad e_{\varphi} = \frac{1}{\sin \theta} \partial_{\varphi}. \quad (2.1)$$

Equivalently, these vectors can be obtained as real and imaginary parts of the complex linear combination

$$m = \frac{1}{\sqrt{2}} (e_{\theta} - ie_{\varphi}) = \frac{1}{\sqrt{2}} \left( \partial_{\theta} - \frac{i}{\sin \theta} \partial_{\varphi} \right)$$

which, together with its complex conjugate, satisfies the orthonormality relations

$$m \cdot m = 0, \quad m \cdot m = 0, \quad m \cdot m = 1$$

at all points of $S^2$ covered by the polar coordinates with respect to the standard metric on $S^2$.

Every smooth function $f_s$ on $S^2$ with spin-weight $s$ can be expressed as:

$$f_s(\theta, \varphi) = \lim_{L \to \infty} \sum_{l=|s|}^{L} \sum_{m=-l}^{l} \alpha_{l m} Y_{l m}(\theta, \varphi), \quad (2.2)$$

where the $Y_{l m}$ are the SWSH, which form a complete orthonormal basis for the complex vector space of spin-weight $s$ functions.

Recall that the action of the $\delta$ and $\delta^\prime$ differential operators on $f_s$ serves to raise and lower the spin-weight respectively; when viewed as maps with respect to polar coordinates we have:

$$\delta: f_s \to s + f_s, \quad \delta[ f_s ] = \partial_{\theta} \partial_{\varphi} f_s = -i \csc \theta \partial_{\varphi} f_s = 1 \cot \theta f_s, \quad (2.3)$$
The SWSH may be written explicitly as:

\[ Y_{lm}(\theta, \phi) = \frac{\sqrt{2l + 1}}{4\pi} \exp(i\vartheta\phi) d_{lm}^\vartheta(\theta), \]  

(2.5)

where \( d_{lm}^\vartheta(\theta) \) is the reduced Wigner \( d \)-function:

\[
 d_{lm}^\vartheta(\theta) = \sum_{r = \max(0, m-n)}^{\min(l+n, l-n)} (-1)^{n-m+r} \frac{\sqrt{(l+m)!(l-m)!(l+n)!(l-n)!}}{r!(l+m-r)!(l-r-n)!(r-m+n)!} 
 \times \cos^{2l-2r+n-m} \left( \frac{\vartheta}{2} \right) \sin^{2r-m+n} \left( \frac{\vartheta}{2} \right).
\]

(2.6)

Under complex conjugation the SWSH satisfy:

\[ \bar{Y}_{lm} = (-1)^{l+m-n} Y_{-l,-m} \implies Y_{lm} = (-1)^{l-m} Y_{l,-m}. \]

(2.7)

The SWSH satisfy the orthonormality relation:

\[
 \left\langle Y_{s_{l_{1}m_{1}}, Y_{s_{l_{2}m_{2}}} \right\rangle = \int_0^{2\pi} \int_0^{\pi} Y_{s_{l_{1}m_{1}}}^{\ast} Y_{s_{l_{2}m_{2}}} \sin \theta \, d\theta \, d\varphi = \delta_{s_{l_{1}}} \delta_{m_{1}m_{2}}.
\]

(2.8)

A particularly useful property the SWSH possess is that under the action of \( \delta \) and \( \delta' \) the action of equations (2.3) and (2.4) reduces to a 'ladder algebra':

\[
 \delta [ Y_{lm}(\theta, \phi) ] = -\sqrt{(l-s)(l+s+1)} Y_{l+1m}(\theta, \phi),
\]

(2.9)

\[
 \delta' [ Y_{lm}(\theta, \phi) ] = \sqrt{(l+s)(l-s+1)} Y_{l-1m}(\theta, \phi),
\]

(2.10)

which we will exploit when performing decompositions of dynamical equations, allowing for a map from a PDE system to an (infinite dimensional) ODE system (see section 5). In addition to this we have the commutator expression:

\[
 [\delta, \delta'] Y_{lm}(\theta, \phi) = -2\vartheta \, Y_{lm}(\theta, \phi).
\]

(2.11)

A product of two SWSH with spin-weights \( s_{1} \) and \( s_{2} \) is a function of spin-weight \( s_{1} + s_{2} \) and, therefore, can be written as a finite linear combination of SWSH

\[
 s_{1}Y_{l_{1}m_{1}}(\theta, \phi) s_{2}Y_{l_{2}m_{2}}(\theta, \phi) = \sum_{i \in \Lambda} C_i (s_{1}, l_{1}, m_{1}; s_{2}, l_{2}, m_{2}) Y_{l_{1}+l_{2}, m_{1}+m_{2}}(\theta, \phi),
\]

(2.12)

where \( \Lambda' := \{ \max(|l_{1} - l_{2}|, |s_{1} + s_{2}|, |m_{1} + m_{2}|), ..., l_{1} + l_{2} \} \) and the coefficients are related to the usual Clebsch–Gordan coefficients, see e.g. [3, 29].

2.1. Forward transformation

We now describe our numerical algorithm for evaluation of the forward transform \( \mathcal{T} \): \( f \mapsto (\alpha_{m}) \). As a first step we introduce the notation \( \Delta_{mn}^{\vartheta} := d_{mn}^{\vartheta}(\pi/2) \) which allows for the rewriting of equation (2.6) as [28]:

\[
 d_{mn}^{\vartheta}(\theta) = \sum_{q=-m}^{m} \Delta_{qn}^{\vartheta} e^{-iq\vartheta}\Delta_{mn}^{\vartheta},
\]

(2.13)

following from a factoring of rotations [36]. In particular, note that \( d_{mn}^{\vartheta}(\theta) = \bar{d}_{mn}(\theta) \) (see equation (2.6)). We defer the details of how the \( \Delta \) elements are calculated together with their symmetry properties to section 3.
Define the functional:

\[ I_{mn} \left[ f(\theta, \varphi) \right] := \int_0^{2\pi} \int_0^\pi e^{-im\theta} e^{-in\varphi} f(\theta, \varphi) \sin \theta \ d\theta d\varphi. \]  

(2.14)

This integral can be evaluated exactly—we now describe our method which is based on [15]. Combining equations (2.2), (2.5), (2.13) and (2.14) results in:

\[
\sum_{l=1}^{\infty} \left[ \Delta_q^l \Delta_q^l \right] = \pi \Delta_q^l J_{qm} \Delta_q^l J_{qm},
\]

(2.15)

\[
= i^{-m} \sqrt{\frac{2l + 1}{4\pi}} \sum_{q=m}^{l} \Delta_q^{l,l} J_{qm} \Delta_q^{l,l},
\]

(2.16)

where

\[ J_{qm} := I_{qm} + (-1)^{2l+m+n} I_{-q,m}. \]  

(2.17)

In order to make use of existing FFT algorithms in the evaluation of the quadrature in equation (2.14), we must extend a spin-weighted function \( f \) with data sampled on the domain \( D := \{(\theta, \varphi) | \theta \in [0, \pi], \varphi \in [0, 2\pi] \} \) to a \( 4\pi \) periodic function on \( \tilde{D} := \{(\theta, \varphi) | \theta \in [0, 4\pi], \varphi \in [0, 4\pi] \} \). This is accomplished using the intrinsic symmetries of the \( Y_m(\theta, \varphi) \) together with the expansion ansatz of equation (2.2). For convenience we define the domains \( D_1, \ldots, D_{IV} \) (see Figure 1). We have \( D = D_1 \) upon which \( f(\theta, \varphi) = f_1(\theta, \varphi) \). For \( D_II \) put \( f_{II}(\theta, \varphi) = -f_1(\theta, \varphi - 2\pi) \). Next, define \( s_g: D_1 \cup D_{II} \to \mathbb{C} \) by

\[
s_g(\theta, \varphi) := \begin{cases} 
  f_1(\theta, \varphi) & (\theta, \varphi) \in D_1, \\
  f_{II}(\theta, \varphi) & (\theta, \varphi) \in D_{II}.
\end{cases}
\]

(2.18)
Now, define \( h : D_1 \cup D_{II} \cup D_{III} \cup D_{IV} \to \mathbb{C} \) by
\[
h(\theta, \varphi) = \begin{cases} \sin(\theta, \varphi) & (\theta, \varphi) \in D_1 \cup D_{II} \\ (1)^{i+1} \sin(2\pi - \theta, (\varphi - \pi) \mod 4\pi) & (\theta, \varphi) \in D_{III} \cup D_{IV} \end{cases}
\] (2.19)

Finally, we arrive at the extended function \( F : \bar{D} \to \mathbb{C} \) defined by
\[
F(\theta, \varphi) := \begin{cases} h(\theta, \varphi) & (\theta, \varphi) \in D_1 \cup D_{II} \cup D_{III} \cup D_{IV} \\ -h(\theta - 2\pi, \varphi) & (\theta, \varphi) \in D_{V} \cup D_{VI} \cup D_{VII} \cup D_{VIII} \end{cases}
\] (2.20)

The smooth function \( F(\theta, \varphi) \) is 4\( \pi \) periodic in \( \theta \) and \( \varphi \), thus we may write:
\[
F(\theta, \varphi) = \sum_{\kappa=0}^{K} \sum_{m=0}^{M} F_{km} e^{i k \theta} e^{i m \varphi},
\] (2.21)

In order to evaluate (2.14) we now identify
\[
I_{mn} = I_{\mu} \equiv I_{\mu} = \int_{0}^{2\pi} \int_{0}^{\pi} e^{-i m \theta/2} e^{-i n \varphi/2} F(\theta, \varphi) \sin \theta \, d\theta \, d\varphi,
\] (2.22)

where we can extend the indices to all integers, i.e., \( m, n \in \mathbb{Z} \). This is permissible as both functions are equal in the region of integration. Upon substitution with equation (2.21) we find:
\[
I_{mn} = \sum_{i,j=0}^{K} F_{ij} W_{\theta}(i-m) W_{\varphi}(j-n),
\] (2.23)

with
\[
W_{\theta}(\tau) := \int_{0}^{\pi} e^{i \tau \theta/2} \sin \theta \, d\theta = \begin{cases} \pm \frac{\pi}{2} & \tau = \pm 2 \\ \frac{1+\tau}{4-\tau^2} & \tau \in \mathbb{Z} \setminus \{\pm 2\} \end{cases},
\] (2.24)
\[
W_{\varphi}(\rho) := \int_{0}^{2\pi} e^{i \rho \varphi/2} \, d\varphi = \begin{cases} 2\pi & \rho = 0 \\ \frac{2\pi}{\rho} (1 - (-1)^{\rho}) & \rho \in \mathbb{Z} \setminus \{0\} \end{cases}.
\] (2.25)

Equation (2.23) amounts to a two-fold discrete convolution in spectral space. By the convolution theorem, this implies that we may equivalently consider pointwise multiplication of the inverse transforms of \( W_{\theta} \) and \( W_{\varphi} \) with \( F \). Let \( N_{\theta} \) and \( N_{\varphi} \) denote the number of grid points over \( \theta \) and \( \varphi \) on the containerization of \( S^2 \) that the function \( f(\theta, \varphi) \) is to be sampled at. Upon extension to the doubly 4\( \pi \) periodic domain (see figure 1) we work instead with the extended function \( F(\theta, \varphi) \). Furthermore, in order to simplify numerical construction of the extension, the number of samples on the extended domain is taken to be \( N_{\theta} = 4N_{\theta} - 3 \) and \( N_{\varphi} = 2N_{\varphi} - 1 \). Correspondingly, the spatial sampling intervals are now given by \( \Delta \theta = \frac{4\pi}{N_{\theta} - 1} \) and \( \Delta \varphi = \frac{4\pi}{N_{\varphi} - 1} \). In order to satisfy the Nyquist condition, such that spurious aliasing does not occur, we impose \( N_{\theta} = N_{\varphi} = 2(L + 2 - 1/2) + 1 \), where \( L \) is the harmonic that the function \( f(\theta, \varphi) \) is band-limited to.

Under these choices a convenient form of the quantities discussed above that is directly amenable to numerical work is given by:

---

\(^2\) More efficient samplings on the sphere may be possible [19] however the overall asymptotic complexity of our proposed algorithm will not be affected and thus we do not explore this issue here.
\[
\hat{W}_\theta(\gamma) := \sum_{\delta=0}^{N_\theta-2} e^{i\gamma\Delta\theta/2}W_\theta\left(\frac{N_\theta - 1}{2} - \delta\right) \quad \gamma = 0, \ldots, N_\theta - 2
\]

\[
\hat{W}_\varphi(\epsilon) := \sum_{\zeta=0}^{N_\varphi-2} e^{i\epsilon\Delta\varphi/2}W_\varphi\left(\frac{N_\varphi - 1}{2} - \zeta\right) \quad \epsilon = 0, \ldots, N_\varphi - 2
\]

together with:

\[
I_{\mu\nu}\left[F(\theta, \varphi)\right] = \frac{1}{(N_\theta - 1)(N_\varphi - 1)} \sum_{\gamma=0}^{N_\theta-2} \sum_{\epsilon=0}^{N_\varphi-2} e^{-i\mu\gamma\Delta\theta/2}e^{-i\nu\epsilon\Delta\varphi/2} \\
\times \hat{W}_\theta(\gamma)\hat{W}_\varphi(\epsilon) F(\gamma, \epsilon).
\]

Where \(\mu \in \{0, \ldots, N_\theta - 2\}\) and \(\nu \in \{0, \ldots, N_\varphi - 2\}\). Note that in this approach, we must discard all values of \(I_{\mu\nu}\) for which \(\mu\) and \(\nu\) are even integers. In order to recover \(I_{\mu\nu}\) from equation (2.28) we take \(\mu = \frac{1}{2}(N_\theta - 4m - 1)\) and \(\nu = \frac{1}{2}(N_\varphi - 4n - 1)\). Overall we find an algorithmic complexity of \(O(L^3)\) as the integrals \(I_{\mu\nu}\) may be evaluated exactly in \(L \log L\) operations by performing a two-dimensional FFT and each component of \(\Delta_{\mu\nu}\) (as required by equation (2.16)) can be computed using \(O(1)\) floating point evaluations—see section 3 and also [15, 36]. We remark that if the analysis of strictly real data is desired then it is possible to attain a linear increase in the execution speed of transformations (specifically the FFT component by a factor of approximately 2). However, as we are primarily interested in applying transformations for the solution of EOMs that govern the dynamics of complex fields we do not explore this further.

2.2. Backward transformation

We now describe the algorithm for evaluation of the backward (inverse) transform \(F^{-1}\). \((\alpha_{ln}) \mapsto f\). The backward spherical harmonic transform maps the expansion coefficients \(\alpha_{ln}\) for \(|l| \leq L\), to a function on (a dense subset of) \(S^2\). Because we are working with band-limited functions we can, at the analytical level, perfectly reconstruct the original function. To this end, equation (2.2) must be evaluated. As the inverse transform does not contain integrals, issues of quadrature accuracy do not arise. Define:

\[
K_{mn}[\alpha_{ln}] := i^{-n} \sum_{l=|s|}^{L} \sqrt{\frac{2l+1}{4\pi}} \Delta_{m,s}^{l} \alpha_{ln} \Delta_{m,n}^{l}
\]

which allows for:

\[
f(\theta, \varphi) = \sum_{m=-\frac{1}{2}(N_\theta-1)+\frac{1}{2}}^{\frac{1}{2}(N_\theta-1)+\frac{1}{2}} \sum_{n=-\frac{1}{2}(N_\varphi-1)+\frac{1}{2}}^{\frac{1}{2}(N_\varphi-1)+\frac{1}{2}} e^{im\phi}e^{in\theta}K_{mn}.
\]

Use of equation (3.1) permits a rewriting of equation (2.30) as:

\[
K_{mn}[\alpha_{ln}] = \begin{cases} 
    i^{s-n} \sum_{l=|s|}^{L} \sqrt{\frac{2l+1}{4\pi}} \Delta_{m,s}^{l} \alpha_{ln} \Delta_{m,n}^{l} & m \geq \frac{1}{2} \\
    i^{s-n} \sum_{l=|s|}^{L} \sqrt{\frac{2l+1}{4\pi}} (-1)^{2l+s+n} \Delta_{m,s}^{l} \alpha_{ln} \Delta_{m,n}^{l} & m \leq \frac{1}{2}
\end{cases},
\]

7
which allows for a reduction in computation time. In equation (2.30) we may use an FFT directly, discarding values of \( f(\vartheta, \varphi) \) for which \( \vartheta > \pi \) and \( \varphi > 2\pi \). If the input data to the FFT library is Hermitean then another linear increase \((\sim 2)\) in the execution speed of a transform is possible however, this again does not change the overall algorithmic complexity \( \mathcal{O}(L^3) \).

3. Wigner \( \Delta \) and recursion

Here we answer the question of how to compute the \( \Delta_{mn}^l \) required in the forward and backward transforms. We will base our numerical algorithm for computation of an arbitrary \( \Delta_{mn}^l \) on recursion and exploitation of symmetries. 

First we note the symmetry properties (inherited from \( d_{mn}^l(\theta) \)):

\[
\Delta_{-m,n}^l = (-1)^{l+m} \Delta_{mn}^l, \quad (3.1)
\]

\[
\Delta_{m,-n}^l = (-1)^{l-m} \Delta_{mn}^l, \quad (3.2)
\]

\[
\Delta_{mn}^l = (-1)^{l-m} \Delta_{mn}^l, \quad (3.3)
\]

Using an approach similar to [36] one can derive an analogous Trapani–Navaza style recursion from equation (2.6) for half-integer \( l, m, n \) values:

\[
\Delta_{ll}^l = \frac{1}{2} \Delta_{l-1,l-1}^{l-1}, \quad (3.4)
\]

\[
\Delta_{ml}^l = \sqrt{\frac{l(l+1)(l+m)(l+m+1)}{2(l+m)(m+1)}} \Delta_{m-1,l-1}^{l-1}, \quad (3.5)
\]

\[
\Delta_{mn}^l = m \sqrt{2} \Delta_{mn,n+1}^l \quad (n = l - 1). \quad (3.6)
\]

\[
\Delta_{mn}^l = \frac{2m}{\sqrt{(l-n)(l+n+1)}} \Delta_{m,n+1}^l - \sqrt{\frac{(l-n-1)(l+n+2)}{(l-n)(l+n+1)}} \Delta_{m,n+2}^l \quad (|n| \leq l - 2). \quad (3.7)
\]

\[
\Delta_{l,l}^l = \sqrt{\frac{2l}{2l+1}} \Delta_{l-1,l-1}^{l-1}. \quad (3.8)
\]

Note that by making use of the symmetries provided by equations (3.1)–(3.3) similar recursion relations may be constructed connecting different combinations of subscript indices.

We visualize the possible values of \( \Delta_{mn}^l \) up to some maximal band-limit \( L \) as being arranged in a square pyramidal lattice with \( \Delta_{mn}^l \) values corresponding to \( l = 1/2 \) and \(-1/2 \leq m, n \leq 1/2\) occupying the top-most plane; \( l = 3/2 \) and \(-3/2 \leq m, n \leq 3/2\) the next plane down and so forth. In a recursive approach one can thus initialize with a single value of \( \Delta_{mn}^l \) and with equations (3.4), (3.5) and (3.8) (together with symmetries) compute those values of \( \Delta_{mn}^l \) constrained to the surface of the pyramidal lattice. For each fixed \( l \) value those \( \Delta_{mn}^l \) that occupy the interior of the pyramidal structure can be computed using equations (3.6) and (3.7) (together with symmetries).
We note that symmetries allow for a reduction in the total number of elements that must be calculated explicitly via recursion to \( (2^l - 1)/2 \) entries for a fixed plane and \( L^2 \) total entries for a choice of maximal band-limit \( L \).

In our implementation we initialize with \( \Delta_{\text{init}} = -\Delta \) and iterate such that for a given \( l \) the \( m, n \) indices obey the condition \( n < l \) and \( |m| > l \) and \( |n| > l \).

Working with double precision arithmetic, we have found that the above scheme remains stable up to a band-limit of \( L \approx 5173/2 \). Due to exponential convergence [4] this \( L \) will in practical situations be far above the resolution required to accurately sample fields for evolution equations and thus is not a concern for this work. We note however, that by instead working with ratios such as \( \Delta_{lm} = \Delta_{lm}/\Delta_{l-1,m} \) we have verified that it is possible to construct a stable type of hybrid recursion in analogy to [3], for \( L > 5173/2 \).

4. SWSH consistency checks

As a first check on the consistency of the half-integer SWSH algorithm presented above we construct error pairs. Here one populates coefficients with random data whereupon a transformation is applied so as to construct the spatial representation of the corresponding function; this spatial function is transformed back to coefficients and the associated error may be examined. This procedure may be summarized as:

\[
\mathcal{E}(n) = \frac{\| \mathcal{A}(n) - \mathcal{A}(n) \|}{\| \mathcal{A}(n) \|}
\]

We note that symmetries allow for a reduction in the total number of elements that must be calculated explicitly via recursion to \( ((2l + 1)/2)^2 \) entries for a fixed plane and \( (2L + 3)(2L + 1)(L + 1) \) total entries for a choice of maximal band-limit \( L \).

In our implementation we initialize with \( \Delta_{l/2,-1/2} = 1/\sqrt{2} \) and iterate such that for a given \( l \) the \( m, n \) indices obey the condition \( n < l \) and \( |m| > l \) and \( |n| > l \).

Working with double precision arithmetic, we have found that the above scheme remains stable up to a band-limit of \( L \approx 5173/2 \). Due to exponential convergence [4] this \( L \) will in practical situations be far above the resolution required to accurately sample fields for evolution equations and thus is not a concern for this work. We note however, that by instead working with ratios such as \( \Delta_{lm} = \Delta_{lm}/\Delta_{l-1,m} \) we have verified that it is possible to construct a stable type of hybrid recursion in analogy to [3], for \( L > 5173/2 \).
The real and imaginary parts of \( \tilde{a}_{slm} \) we generate by sampling from the continuous uniform random distribution on the interval \([-1, 1)\). The numerical error associated with the specification of equation (4.1) is shown in figures 2(a) and (b). We see that our implementation is indeed consistent, accurate and stable. Furthermore the scaling \( L(n) \) of \( \epsilon_{\text{rel rms}} \) (figure 2(b)) for the half-integer spin-weight uniform random data matches the scaling observed for integer spin-weight Gaussian random data observed in [15].

The product of two spin-weighted functions \( f \) and \( g \) on \( S^2 \) is a function with spin-weight \( s_1 + s_2 \). If \( f \) and \( g \) are comprised of a finite number of SWSH then their product is also and thus by selecting a sufficiently large \( L \) we exactly sample the resulting product function. We can also check the property of exponential convergence; to this end define \( A_l = \langle \tilde{a}_{slm} \rangle_m = \sum_m |\tilde{a}_{slm}|/(2I + 1) \), a measure of the average magnitude of coefficients at a fixed \( l \) value. We expect that given smooth test functions \( A_l \) should behave as \( A_l \sim \alpha \exp(-\kappa l) \) \((\alpha, \kappa \in \mathbb{R})\) for large \( l \) [4, 16]. Smooth half-integer spin-weighted functions may be constructed by taking a finite number of SWSH and modulating by the exponential of a smooth spin-weight 0 function. Introduce:

\[
g_Y(\theta, \phi) = i_0 Y_{3/2,1/2}(\theta, \phi) + 1.1 Y_{1,1}(\theta, \phi),
\]
\[
h_Y(\theta, \phi) = Y_{1/2}(-\theta, \phi).
\]
\[
0g_0^2 f(\theta, \phi) = 1.3 Y_{3/2,1/2}(\theta, \phi) + i Y_{1/2}(-\theta, \phi),
\]
\[
0g_0^2 k(\theta, \phi) = 0.71 Y_{1/2}(-\theta, \phi),
\]
\[
1/2 i^2 h(\theta, \phi) = [1/2 Y_{1/2,3/2} \times Y_{1/2,3/2}(\theta, \phi)]^3,
\]
\[
-1/2 i^2 k(\theta, \phi) = 0.9 Y_{3/2,1/2}(\theta, \phi) + 0.9 Y_{3/2,1/2}(\theta, \phi),
\]
\[
-1/2 i^2 k(\theta, \phi) = 0.9 Y_{3/2,1/2}(\theta, \phi).
\]

In figure 3 we show \( A_l \) for various combinations that test the various properties outlined. In particular, we observe that functions comprised of a finite number of SWSH are completely captured if \( L \) is greater than the highest harmonic of the linear combination. Products of spin-weighted functions (each individual function comprised of a finite linear combination of SWSH) also are exactly resolved to within numerical error for appropriate \( L \). Furthermore, we observe the crucial property of exponential convergence for smooth functions.

### 5. 2+1 Dirac equation

In order to test the SWSH half-integer algorithm we now numerically solve the Dirac equation on a three-dimensional Lorentz manifold \((\mathcal{M}, g)\) with topology \( \mathbb{R} \times S^2 \). Since the Dirac equation on a three-dimensional manifold is less familiar than its four-dimensional counterpart we present a detailed derivation in the appendix and simply state the result here.

We consider the manifold \( \mathcal{M} \sim \mathbb{R} \times S^2 \) with the metric

\[
g = dt \otimes dt - F^{-2}(t, \theta, \phi)(d\theta \otimes d\theta + \sin^2 \theta \, d\phi \otimes d\phi),
\]

where \((\theta, \phi)\) are standard polar coordinates for the two-sphere. The function \( F(t, \theta, \phi) \) is a conformal factor relating the induced metric at every instance of time \( t \) on \( S^2 \) to the standard metric of the unit two-sphere.
According to the appendix the Dirac equation on $\mathcal{M}$ is given by the following two equations for a spinor $\psi$:

$$\partial^T \psi + \psi \partial_T \psi = -i \frac{1}{2} \gamma^\alpha \partial_\alpha \psi,$$

(5.2)

$$\partial^T \psi + \psi \partial_T \psi = -i \frac{1}{2} \gamma^\alpha \partial_\alpha \psi.$$

(5.3)

The spinor components $\psi_\pm$ have spin-weight $\pm \frac{1}{2}$, while $\psi$ has vanishing spin-weight.

5.1. PS method

While performing a spectral decomposition of variable coefficient PDEs such as equations (5.2) and (5.3) by assuming functions may be expanded as in equation (2.2) and then products decomposed as in equation (2.12) is possible (see [3] for examples of such expansions in the integer SWSH case) we have found that instead it is far simpler to use a PS approach.

We outline this as follows: Given the coefficients $\tilde{f}_{l,m}$ and $\tilde{g}_{l,m}$ which represent the functions $f$ and $g$ sampled at a band-limit $L$ then the coefficients $\tilde{a}_{l,m}$ corresponding to

3 Indeed the pseudo-spectral method generalizes more readily to nonlinear problems.
the associated point-wise product $s^f \cdot s^g \tilde{\cdot} s^s$ can be calculated by performing the transformations:

$$F^{-1}: (s_i f_{l,m}) \mapsto s_i f, \quad F^{-1}: (s_i \tilde{g}_{l,m}) \mapsto s_i \tilde{g},$$

subsequently taking the pointwise product and transforming:

$$F: s^f \cdot s^g \rightarrow (s_i + s_i \partial_{l,m}),$$

we find an approximation to an expansion utilizing equation (2.12) directly. We emphasize that this method also easily allows one to take into account the action of the $\delta$, $\delta'$ operators on spin-weighted functions by embedding their action as multiplication (see equations (2.9) and (2.10)) in coefficient space, together with taking account of their spin raising and lowering properties when transforms are performed.

We now recast the PDE system of equations (5.2) and (5.3) as an (infinite dimensional) ODE system using the PS method. It is convenient to define the auxiliary fields:

$$\Xi := \tilde{F}^\prime \psi, \quad \Phi := \frac{1}{2} \psi \delta F, \quad \Psi := \psi \frac{F'}{F},$$

(5.4)

where $\pm$ corresponds to a spin-weight of $\pm 1/2$. Using these fields we find that equations (5.2) and (5.3) can be written as:

$$\psi_{-l,m}(t) = -i \mu \psi_{-l,m}(t) - \Phi_{-l,m}(t) - \Xi_{-l,m}(t) + \Psi_{-l,m}(t)$$

$$\psi_{+l,m}(t) = i \mu \psi_{+l,m}(t) - \Phi_{+l,m}(t) - \Xi_{+l,m}(t) + \Psi_{+l,m}(t)$$

(5.5)

We note that in the case of $F = 1$, equations (5.2) and (5.3) yield a constant-coefficient PDE system which may be directly decomposed as:

$$\psi_{-l,m}(t) = -i \mu \psi_{-l,m}(t) - \left( l + \frac{1}{2} \right) \psi_{+l,m}(t)$$

$$\psi_{+l,m}(t) = i \mu \psi_{+l,m}(t) + \left( l + \frac{1}{2} \right) \psi_{-l,m}(t)$$

(5.6)

In this case, a solution is readily arrived at:

$$\psi_{\pm,lm}(t) = \frac{1}{\alpha t} \left[ \alpha t \cos(\alpha t) \pm i \mu \sin(\alpha t) \right] \psi_{\pm,lm}(0) \pm \left( l + \frac{1}{2} \right) \sin(\alpha t) \psi_{+l,m}(0)$$

(5.7)

where $\alpha t := \sqrt{(l + \frac{1}{2})^2 + \mu^2}$.

When $F \neq 1$ we numerically construct the spatial representation of the function using equation (2.5) with spatial sampling to coincide with the half-integer SWSH transformation. We will also find it useful to introduce the rescaled current component:

$$\tilde{j}^0(t, \theta, \varphi) := \frac{1}{F^2(t, \theta, \varphi)} j^0(t, \theta, \varphi),$$

(5.8)

which with the PS method can be computed using the rescaled fields $\tilde{\psi}_\pm := \psi_\pm / F^2$. From equations (A.11) and (5.8) the probability $Q(\Sigma)$ can be computed via:

$$Q(\Sigma) = \sum_{l,m} (\tilde{\psi}_{-l,m}\tilde{\psi}_{+l,m} + \tilde{\psi}_{-l,m}\tilde{\psi}_{+l,m})$$

(5.9)
5.2. Numerical solutions and convergence tests

In this section we examine numerical solutions to the EOM equations (5.2) and (5.3) under three conditions: the case \( F = 1 \) (corresponding to a \( S^2 \) spatial geometry), where we numerically solve the system of equations given by the decomposition of equation (5.6); the case of a static deformation (time-independent) of \( F \); a time-dependent \( F \). In the latter two cases we work with the decomposition of equation (5.5). In each case we are free to select arbitrary initial data for \( \psi_{\pm} \). As convergence tests must be performed for any numerical calculation we begin by briefly summarizing the procedure we follow (based on [2]) for doing so, subsequently presenting our results. Aside from presenting convergence for the fields \( \psi_{\pm} \) we also verify that the continuity equation for the current is obeyed and probability is conserved in each case examined.

In using the PS approach of section 5.1 to compute the time-evolution of \( \psi_{\pm} \) we solve a truncated \( l \leq L \) ODE system for \( \psi_{\pm,lm}(t) \). Suppose that we use an explicit, temporal integrator with a fixed time-step \( \delta t \). The numerically calculated solution for this choice of \( \delta t \) we denote as \( \psi_{\pm,lm}(t; \delta t) \). Assume that there exists a Taylor expansion of \( \psi_{\pm,lm}(t; \delta t) \) about the exact solution \( \psi_{\pm,lm}(t) \) with error constant terms \( E_n \):

\[
\psi_{\pm,lm}(t; \delta t) = \psi_{\pm,lm}(t) + \sum_{n=1}^{\infty} \delta t^n E_n.
\]  

(5.10)

Suppose now that the integrator is of order \( p \) so that \( E_p < 0 \). Thus:

\[
\psi_{\pm,lm}(t; \delta t) - \psi_{\pm,lm}(t) = \delta t^p E_p + \mathcal{O}(\delta t^{p+1}).
\]  

(5.11)

By successively rescaling \( \delta t \) with a constant (here 2) and comparing \( \psi_{\pm,lm}(t; \delta t/2^k) \) with \( \psi_{\pm,lm}(t) \) we find:

\[
2^k \left\{ \psi_{\pm,lm} \left( t; \frac{1}{2^k} \delta t \right) - \psi_{\pm,lm}(t) \right\} \to \delta t^p E_p.
\]  

(5.12)

Equation (5.12) may be used in the case when an analytical solution is known. If this is not the case we may instead successively compare \( \psi_{\pm,lm}(t; \delta t/2^k) \) and \( \psi_{\pm,lm}(t; \delta t/2^{k+1}) \) which yields:

\[
2^k \left\{ \psi_{\pm,lm} \left( t; \frac{1}{2^k} \delta t \right) - \psi_{\pm,lm} \left( t; \frac{1}{2^{k+1}} \delta t \right) \right\} \to \left( 1 - \frac{1}{2^k} \right) \delta t^p E_p = \delta t^p \tilde{E}_p,
\]  

(5.13)

this is known as a self-consistent convergence test (SCCT) [2]. Note that in both cases (equations (5.12) and (5.13)) we may also make comparisons of functions in the spatial representation by transforming from coefficient-space. In particular, we will consider the maximum, absolute, relative error metric:

\[
\epsilon_i \left( \phi_{\pm}(t, \theta, \varphi), \psi_{\pm}(t, \theta, \varphi) \right) = \max_{m,n} \left| 1 - \frac{\phi_{\pm}(t, m \Delta \theta, n \Delta \varphi)}{\psi_{\pm}(t, m \Delta \theta, n \Delta \varphi)} \right|,
\]  

(5.14)

which allows for the comparison of the two solutions \( \phi_{\pm}, \psi_{\pm} \) at sampling nodes \( m \Delta \theta, n \Delta \varphi \). We measure error in this manner such that convergence properties of solutions and any potential instability that may result from the SWSH transformation algorithm may be simultaneously inspected.

Having described the methods we use to test convergence of solutions we now examine some example cases. We fix the mass-parameter to be \( \mu = 1.2 \) at the outset and choose a
temporal range of $t \in [0, 5]$ throughout. These choices have been made so as to allow for observation of multiple oscillations of $t_l m, \psi_{\pm}$ modes (see equation (5.7)) while simultaneously avoiding exact integer multiple of the frequencies $(2) l \omega \pi$. Note that all specified initial conditions $t_0 \psi = \psi^{\pm}$ are constructed with data as indicated below and then the corresponding coefficients $\psi_{l m}^{\pm}$ rescaled by an overall factor so that $Q$ is normalized to 1 at $t = 0$. For each numerical evolution the temporal integrator we choose is the standard, explicit Runge–Kutta 4th order method (RK4); thus $p = 4$ in equations (5.12) and (5.13).

For $F = 1$ we select the smooth initial data:

$$\psi(t, \theta, \varphi) \big|_{t=0} = -1/2 \hat{k}(\theta, \varphi), \psi_{e}(t, \theta, \varphi) \big|_{t=0} = 1/2 \hat{j}(\theta, \varphi)$$

(5.15)

with $-1/2 \hat{k}(\theta, \varphi)$ given by equation (4.8) and $1/2 \hat{j}(\theta, \varphi)$ given by equation (4.5). The system to solve is thus specified by equations (5.6) and (5.15) upon mapping the latter $\psi_{l m}^{\pm}$ to coefficient space using the SWSH transformation algorithm. The results of convergence tests (based on equation (5.12) with the analytical solution of equation (5.7)) of our numerical solution are shown in figure 4. We find excellent agreement with the expected 4th order convergence in time for initial data and parameters specified. In particular, we see that the numerical error of the solution is dominated by the temporal discretization. In
figures 5(a) and (b) we provide representative examples of the maximum absolute value of the three-divergence $|\nabla A|$ (expected to be 0 by equation (A.2)) and probability conservation $Q$ (again expected to be 0) at a fixed number of time-steps (temporal discretization). This allows for the examination of the effect of varying the band-limit $L$ on the evolution of the smooth initial data of equation (5.15). In figure 5(a) we observe that the numerical error is greatest for $L = 129/2$ and minimizes for $L = 17/2$—the reason for this can be explained as follows: from the evolution equation (equation (5.6)) we see that coupling does not occur between distinct $l, m$ modes, only between fixed $l, m$ of $\psi_{l,m}$. Hence we see that if for given $l, m$ it is the case that $\psi_{l,m} |_{b=0} = \psi_{l,m} |_{b=0} = 0$ then to within numerical tolerance these coefficients should remain 0 over the course of the evolution. Thus for a finite $L$ we effectively provide a choice of initial data which can be viewed as exact at the specified $L$. The error arising from the spectral components of $\psi_L$ can hence be entirely attributed to that of the SWSH transformation itself, which scales with increasing $L$ as shown in figure 2. This behaviour can also be observed in figure 4 by comparing the numerical error associated with components $\psi_L$ at a fixed $k$ for differing $L$. In figure 5 we examine the conservation of probability with the quantity $|1 - Q|$, where $Q$ is calculated according to equation (5.9) and as $F = 1$ we may avoid rescaling to $\psi_L$ as is required in more general cases. As no SWSH transformations are required to compute this quantity once $\psi_{l,m}(t)$ is known we find that upon changing $L$ the previous associated numerical error is not accumulated and the results for $|1 - Q|$ at differing $L$ coincide.
For a time independent $F$ we select the (finite $L$) initial data
\[ t = 0 \]
\[ \psi \big|_{t = 0} = \pm 2, 1, i, 2i, 1, \ (5.16) \]
and $F$ we choose as:
\[ F_{0,0} = 4, \quad F_{1,-1} = -F_{1,1} = \frac{1}{2}, \quad F_{4,2} = F_{4,-2} = \frac{1}{10}. \ (5.18) \]

Together with the above specifications we make use of auxiliary fields as defined in equation (5.4) with $\Psi(t) = 0 \Rightarrow \Psi_{\text{lin}}(t) = 0$ and the EOM decomposition of equation (5.5).

In this case instead of comparing with an analytical solution we perform a self-consistent convergence based on equation (5.13). The result of this is presented in figure 6. We observe the expected 4th order convergence in time for the numerical solution $\psi_{\text{lin}}(t, \theta, \varphi)$ at the band limits tested. In figures 7(a) and (b) we show representative examples of how the numerical

Figure 6. SCCT of numerical solution $\psi_{\text{lin}}$ (spatial representation) for the case of static $F$ (equation (5.18)). $\varepsilon_{t} (\psi_{\text{lin}}) = \varepsilon_{t} (\psi_{\text{lin}}(t; \hat{\Delta}^{k}, \theta, \varphi), \psi_{\text{lin}}(t; \hat{\Delta}^{k+1}, \theta, \varphi))$ is displayed at $t = 5$ for a number of steps $N = 2^{k} \times 100$. Blue (dashed), 'd': $L = 17/2$, $\varepsilon_{t} (\psi_{\text{lin}})$; green (dashed), 'b': $L = 17/2$, $\varepsilon_{t} (\psi_{\text{lin}})$; red (dashed), 'o': $L = 33/2$, $\varepsilon_{t} (\psi_{\text{lin}})$; cyan (dashed), 'x': $L = 33/2$, $\varepsilon_{t} (\psi_{\text{lin}})$; black (dashed), '□': $L = 65/2$, $\varepsilon_{t} (\psi_{\text{lin}})$; magenta (dashed), 'o': $L = 65/2$, $\varepsilon_{t} (\psi_{\text{lin}})$; blue (solid), '+'$: 129/2, \varepsilon_{t} (\psi_{\text{lin}})$; green (solid), '-'$: 129/2, \varepsilon_{t} (\psi_{\text{lin}})$. The thick black line corresponds to an expected convergence of 4th order in time. (See the text for discussion.)

For a time independent $F$ we select the (finite $L$) initial data $\psi_{\text{lin}}(t)_{t = 0}:$
\[ \psi_{\text{lin}} = 2, \quad \psi_{\text{lin}} = \psi_{\text{lin}} = 1, \]
\[ \psi_{\text{lin}} = 2, \quad \psi_{\text{lin}} = \psi_{\text{lin}} = 1, \quad \psi_{\text{lin}} = 5, \quad \psi_{\text{lin}} = 3, \]
\[ \psi_{\text{lin}} = 2, \quad \psi_{\text{lin}} = 1, \quad \psi_{\text{lin}} = 2, \quad \psi_{\text{lin}} = 1, \] (5.17)
The solution obeys the continuity equation and probability conservation respectively. Note that in order to calculate $Q$ for this case we rescale fields as in equation (5.9) and use the full prescription of the PS method (see section 5.1). In contrast to the $F = 1$ case we now have a variable coefficient EOM and coupling between distinct $l, m$ modes of $\psi_{\pm}$. Thus, sampling at a sufficiently high band-limit $L$ in addition to choosing a sufficiently fine temporal grid is required in order to resolve conservation properties accurately. Finally we consider a time-dependent $F$. The selection we make is linear interpolation in time between initial $g(\theta, \varphi)$ and final $h(\theta, \varphi)$ static deformations of $\Sigma$. That is, $F(t, \theta, \varphi) := (1 - t/t_f) g(\theta, \varphi) + (t/t_f) h(\theta, \varphi)$ and $t_f = 5$. The (real) functions $g$ and $h$ we choose to have non-zero $g_{l,m}$ and $h_{l,m}$ coefficients:

$$
\begin{align*}
  g_{0,0} &= 8, & g_{n,-1} &= -\frac{5}{2}, & g_{4,-2} &= \frac{1}{10} \\
  h_{0,0} &= 8, & h_{n,-1} &= -\frac{5}{2}, & h_{4,-2} &= \frac{1}{10}
\end{align*}
$$

(5.19)

Initial data $\Psi_{\pm}(t)_{t=0}$ is selected as in equations (5.16) and (5.17). Together with the above specifications we again make use of auxiliary fields as defined in equation (5.4) (note however that $\Psi_{\pm}(t) \neq 0$) and the EOM decomposition of equation (5.5). We again perform a self-consistent convergence based on equation (5.13). The result of this is presented in figure 8. Once again, excellent agreement with the expected 4th order convergence is observed with respect to the temporal scheme. We are again in a situation with a variable coefficient EOM.

---

4 The amount of raw data that can be generated at higher $L$ can grow dramatically—during the calculation we thus only retain the coefficients $\Psi_{\pm}(t)$ at evenly interspersed points on the temporal grid as presented in figures.
however now it also becomes non-autonomous. The continuity equation, together with probability conservation must be obeyed in this case also, and indeed we find that provided a sufficiently high band-limit \( L \) is chosen we may resolve these stated properties in our numerical solution (see figures 9(a) and (b)).

5.3. Dirac equation: collapsing background geometry

Having analysed the numerical properties of our implementation of a PS method for the Dirac equation we now turn to potential applications. We consider a (2+1) dimensional analogue of the usual Friedmann–Robertson–Walker (FRW) space–time in comoving coordinates. The particular physical scenario we wish to model is an imploding Universe with pressure \( P \) and density \( \rho \) equal to zero. We take the scale factor to be

\[
a(t) = \frac{1}{1 + t} = F(t)^{-1}.
\]

Thus, we see that at \( t = 0 \) the background spatial geometry coincides with that of \( S^3 \) upon which initial data for the Dirac equation must be provided.

For many practical purposes it is sufficient to consider an initial Gaussian state when discussing the dynamics of the Dirac equation. For a \((n - 1)\)-sphere embedded in \( \mathbb{R}^n \) the von Mises–Fisher distribution serves as an analogue to the planar Gaussian distribution [18]. The probability density function is given by:
\[
\begin{align*}
&\text{where } \kappa > 0, \|X_0\| = 1 \text{ and } L_m(z) \text{ is the modified Bessel function of the first kind. The parameter } \kappa \text{ may be thought of as analogous to the reciprocal of the variance of the Gaussian distribution and } X_0 \text{ as the mean direction about which the points } X \text{ cluster. We choose initial data with average momentum 0 where } \\
&j_1^2(\mu, \phi) = j_1(\mu, \phi) = 0. \\
&\text{For the mass parameter we take } \mu = 1.2 \text{ and evolve equations (5.2) and (5.3) on the temporal range } t \in [0, 0.99]. \\
&\text{As before we use explicit RK4 as the temporal integrator and upon performing SCCT (figure 10) find 4th order convergence in time as expected. Once again the continuity equation must hold and probability must be conserved. These properties are checked for the present numerical calculation in figures 11(a) and (b).} \\
&\text{For each chosen band-limit we observe growth in numerical error of several orders of magnitude as } t \to 0.99. \text{ This is reasonable when we take into account that the spatial geometry is shrinking to a point at } a(t) \to 0 \text{ as } t \to 0.99. \\
&\text{We now fix the band-limit as } L = 129/2 \text{ together with temporal grid } N_t = 6.4 \times 10^4. \text{ For the aforementioned resolutions snapshots of the dynamics are displayed on an Aitoff–Hammer projection [30] in figure 12. In figure 12(a) we see that the probability density } f_1^0 \text{ is initially Gaussian in character and subsequently due to the rotational symmetry of the initial state about the mean direction } X_0 \text{ we observe evolution towards a ring-like structure indicating dynamics governed by dispersion (figure 12(b)). In figure 12(c) we see evolution towards the antipode of } X_0, \text{ however we find that this does not coincide with a reconstruction of the initial condition in the limit } t \to 1 \text{ (figure 12(d)). Observe that the average amplitude of } \\
&\lim_{t \to 0} f_1^0(t) \text{ increases dramatically; this is to be expected due to conservation of probability } Q.
\end{align*}
\]
6. Conclusion

We have presented a new spectral algorithm for half-integer spin-weighted functions on $S^2$ based on SWSH by extending the method for the integer case presented in [3, 15]. Our implementation of this spectral algorithm shows excellent agreement with the theory we have presented. Indeed we find that the numerical error scaling comparable to the integer SWSH algorithm [15] and that exponential convergence properties are displayed as anticipated. Furthermore, the expected algorithmic complexity of $O(L^3)$ is retained. In addition, we have outlined how one may construct the $2+1$ Dirac equation on a curved space–time, adapting the result to the $\delta$-formalism and for a geometry with spatial topology of $S^2$. Viewing the Dirac equation as an IVP we demonstrated how a PS approach can be combined with the half-integer SWSH algorithm and analysed several different situations with distinct spatial geometries, examples including linear interpolation between distinct spatial configurations with time and the case of an imploding $2+1$ FRW-like model. The numerical solutions that we

and the form of the volume element for this particular geometry used in its calculation (equation (A.11)).
Test of the continuity equation and probability conservation for the case of $F(t)^{-1} = a(t) = 1 - t$. We fix the time-step $\delta t$ to be $0.9964 \times 10^4$. $N_t$ indicates the number of time-steps taken from $t = 0$ (to the maximum $t = 0.99$). (a) Maximum over all sampled nodes in the spatial representation of $\mathcal{J}$. (b) Conservation of probability $Q$ with time. In both subfigures: blue ‘◦’: $L = 17/2$; red ‘*’: $L = 33/2$; green ‘+’: $L = 65/2$; black ‘□’: $L = 129/2$. (See the text for discussion.)

Figure 12. Aitoff–Hammer projection of the probability density $j^0$ associated with the $(2 + 1)$ dimensional Dirac equation on FRW background. Note that during the course of the time evolution the spatial radius $a(t)$ of the spherical projection in each sub-figure is distinct, furthermore the colour scaling also varies. See the text for discussion.
constructed obey anticipated convergence rates for the temporal solver used (explicit RK4) and physical invariants (current continuity and probability) are shown to be preserved during the time-evolution to an excellent degree.

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Appendix. The Dirac equation on a three-dimensional Lorentz manifold

Let $\mathcal{V}$ be a $n$-dimensional real vector space equipped with a quadratic form $\eta$ with signature $n - 2$, i.e., $\eta$ can be represented in diagonal form as $\eta = \text{diag}(1, -1, \ldots, -1)$. Let $\mathcal{c}(\mathcal{V}, \eta)$ be the Clifford algebra associated to $(\mathcal{V}, \eta)$. This is the unique algebra with unit $1$ and so called structure map $\gamma : \mathcal{V} \to \mathcal{c}(\mathcal{V}, \eta)$ such that for every $v \in \mathcal{V}$ the relation $\gamma(v)\gamma(v) = \eta(v, v)1$ holds. Let $(e_1, \ldots, e_n)$ be a basis of $\mathcal{V}$ and define $\eta_{ab} = \eta(e_a, e_b)$ and $\gamma_a = \gamma(e_a)$. Then, by polarization of the defining relation, one finds the well-known Clifford-Dirac anti-commutator relations

$$\{\gamma_a, \gamma_b\} = \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}1.$$

Irreducible representations $\mathcal{S}$ of the Clifford algebras have dimensions $N = 2^{n/2}$ for $n$ even and $N = 2^{(n-1)/2}$ for $n$ odd. It is well known (see [26]) that these representations can be constructed recursively from the lower dimensional ones. The representation space $\mathcal{S}$ of the Clifford algebra $\mathcal{c}(\mathcal{V}, \eta)$ is called the spin space. Depending on the particular case, the spin space is equipped with certain invariant structures, see [26, 27, 31] for detailed discussions.

The commutators of the generators $\gamma_{ab} := \eta_{ab}1$ generate a Lie algebra with commutation relations

$$[\gamma_{ab}, \gamma_{cd}] = \eta_{cb}\gamma_{ad} - \eta_{ca}\gamma_{bd} - \eta_{da}\gamma_{bc} + \eta_{db}\gamma_{ac}.$$  

These are the commutation relations for the Lie algebra $\mathfrak{o}(\eta)$ of the (pseudo-) orthogonal group $O(\eta)$ associated with the bilinear form $\eta$. This Lie algebra acts on the subspace $\gamma(\mathcal{V}) \subset \mathcal{c}(\mathcal{V}, \eta)$ as follows: for any skew bi-vector $w_{ab}$ and every vector $v^a$ we define $\gamma(w) : w_{ab}\gamma_{ab}$ and $\gamma(v) = v^a\gamma_a$, then the action of $\mathfrak{o}(\eta)$ is via commutation in $\mathcal{c}(\mathcal{V}, \eta)$

$$\mathfrak{o}(\eta) \times \gamma(\mathcal{V}) \to \gamma(\mathcal{V}), \quad (\gamma(w), \gamma(v)) \mapsto \gamma(w)\gamma(v) - \gamma(v)\gamma(w).$$

We can see the corresponding action on $\mathcal{V}$ from the explicit calculation

$$\gamma(w)\gamma(v) - \gamma(v)\gamma(w) = w^{ab}v^c(\gamma_{ab}\gamma_c - \gamma_c\gamma_{ab}) = w^{ab}v^c(\eta_{bc}\gamma_a - \eta_{ac}\gamma_b) = 2w^{ab}v^b\gamma_a.$$  

Thus, the action on $\gamma(\mathcal{V})$ corresponds to the linear mapping defined by $2w^a\gamma_a$ on $\mathcal{V}$, which is clearly anti-symmetric with respect to $\eta$. The same Lie algebra also acts on the spin-space $\mathcal{S}$ via the representation of the Clifford algebra

$$\mathfrak{o}(\eta) \times \mathcal{S} \to \mathcal{S}, \quad (\gamma(w), \psi) \mapsto \gamma(w)\psi.$$  

In order to make use of spinors on the manifold $\mathcal{M}$ we associate at every point $x \in \mathcal{M}$ the Clifford algebra $\mathcal{c}(T_x\mathcal{M}, g_x)$ and its representation space which we denote by $\mathcal{S}_x$. The
rigorous construction is invariantly described in terms of (associated) vector bundles but we will not go into the details here. Instead we refer to [17] for a complete description. The collection of all spin spaces \( S_x \) forms the spin bundle \( S(\mathcal{M}) \) over \( \mathcal{M} \) and sections of this bundle are referred to as spinor fields or simply spinors. Similarly, we obtain the Clifford bundle as the collection of all Clifford algebras \( \mathcal{C}(T_x\mathcal{M}, g_x) \). The structure maps defined at every point \( x \) yield a bundle map \( \gamma \) from the tangent bundle \( T_x\mathcal{M} \) to the Clifford bundle, assigning to every tangent vector \( v \in T_x\mathcal{M} \) an element \( \gamma_x(v) \) in the Clifford algebra at \( x \).

The connection \( \nabla \) on \( \mathcal{M} \) can be lifted to a connection also denoted by \( \nabla \) on \( S(\mathcal{M}) \) (see [37]). It can be uniquely characterized by the fact that the structure map \( \gamma \) and the invariant structures on the spin spaces are covariantly constant. With this connection one can define the Dirac operator on \( \mathcal{M} \) as follows: we pick a basis \( (e_1, \ldots, e_n) \) of the tangent space \( T_x\mathcal{M} \) at each \( x \in \mathcal{M} \). Now the anti-commutator relations are

\[
\{ \gamma_a, \gamma_b \} = 2g_{ab} \mathbf{1}
\]

which hold at every \( x \in \mathcal{M} \) with the appropriate definition of \( g_{ab} \) and \( \gamma_a \). The Dirac operator \( \mathcal{D} \) acting on spinor fields \( \psi \) is defined by

\[
\mathcal{D}\psi := g^{ab} \gamma( e_a ) \gamma_x \psi = \gamma^a \gamma_x \psi.
\]

(A.1)

In this paper we assume the vector space \( \mathcal{V} \) to have dimension 3 and to be equipped with a metric \( \eta \) with signature \((+, -,-)\). The corresponding Clifford algebra \( \mathcal{C}(\mathcal{V}, \eta) \) has a four-dimensional real representation [27, 31], which may conveniently be described within \( M(2, \mathbb{C}) \) as the real algebra generated by the ‘Dirac matrices’

\[
\gamma_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.
\]

The spin space is \( \mathbb{C}^2 \) (regarded as a four-dimensional real vector space) and carries a sesquilinear form (Hermitean inner product) defined by

\[
\langle \psi, \phi \rangle := \psi^* \gamma_0 \phi,
\]

where \( \psi^* \) is the Hermitean conjugate of \( \psi \). For given spinors \( \psi = [\psi_1, \psi_2]^T \) and \( \phi = [\phi_1, \phi_2]^T \) this product is

\[
\langle \psi, \phi \rangle = -\bar{\psi}_1 \phi_1 + \bar{\psi}_2 \phi_2.
\]

The generators are symmetric with respect to this product, i.e., we have

\[
\langle \psi, \gamma_a \phi \rangle = \langle \gamma_a \psi, \phi \rangle.
\]

Given two spinors \( \psi \) and \( \phi \) there is a naturally defined covector \( \alpha = \alpha(\phi, \psi) \) with components \( \alpha_a = \langle \phi, \gamma_a \psi \rangle \). When \( \psi = \phi \), then this covector is

\[
\alpha = \begin{bmatrix} |\psi_1|^2 + |\psi_2|^2, -\bar{\psi}_1 \psi_2 - \bar{\psi}_2 \psi_1, i(\bar{\psi}_2 \psi_1 - \bar{\psi}_1 \psi_2) \end{bmatrix}.
\]

Hence, it is real and null whenever \( \langle \psi, \psi \rangle = 0 \).

The commutators of the Dirac matrices \( \gamma_a \) are infinitesimal generators of the spin group \( \text{spin}(1, 2) \). They can be written here in form

\[
\gamma_a \gamma_b - \gamma_b \gamma_a = 2i\epsilon_{abc} \gamma_c.
\]

\[5\text{ Note, that in the general case there are global topological obstructions to the existence of these bundles, see [17, 26]. However, since we are interested mainly in the three-dimensional case where these obstructions do not exist we will not discuss them here.} \]
Choosing an orthonormal frame \((e_0, e_1, e_2)\) on \(\mathcal{M}\) we can express the Dirac operator on \(\mathcal{M}\) with respect to the chosen frame explicitly. To this end we need the Ricci rotation coefficients for the frame defined by

\[ V_{e_0} e_0 = \Gamma_{ab}^c e_c. \]

The condition that the structure map be covariantly constant translates into the equation

\[ \Gamma_{ab}^c \Gamma_{c}^d = \gamma_b \gamma_a - \gamma_a \gamma_b \]

where the \(\gamma_a\), which depend on the point \(x \in \mathcal{M}\), are endomorphisms of the spin space \(S_x\). More precisely, they are linear combinations of the infinitesimal generators of \(\text{Spin}(1, 2)\). This equation can be solved uniquely for the \(\gamma_a\) and yields

\[ \gamma_a = -\frac{1}{2} \Gamma_{bd}^c e_c b^d. \]

The connection defined in this way also leaves the complex structure and the sesquilinear product invariant. With these ‘spin coefficient’ matrices we can express the Dirac operator acting on an arbitrary spinor \(\psi\) as

\[ D \psi = \eta^{ab} \gamma_a (e_b (\psi) + \gamma_b \psi). \]

The Dirac equation on \(\mathcal{M}\) can be obtained from a variational principle. For any given spinor field \(\psi\) with compact support on \(\mathcal{M}\) we can write down the action functional

\[ \mathcal{A}[\psi] = \int_{\mathcal{M}} i \langle \psi, D \psi \rangle + \mu \langle \psi, \psi \rangle \, d\mathcal{M}, \]

where here and later on \(d\mathcal{M}\) denotes the invariant volume form on a manifold \(\mathcal{M}\). This defines a real number since the imaginary part can be rewritten as

\[ i \int_{\mathcal{M}} \langle \psi, D \psi \rangle + \langle \psi, D \psi \rangle \, d\mathcal{M} = i \int_{\mathcal{M}} V_a \alpha^a \, d\mathcal{M} = 0. \]

Variation with respect to \(\psi\) yields the Dirac equation

\[ i \partial_t \psi + \mu \psi = 0. \]

To every solution \(\psi\) of the Dirac equation the covector \(\alpha_a\) defines a conserved current \(j^a\) denoted by

\[ j^a = \langle \psi, \gamma^a \psi \rangle. \]

Taking the divergence of this equation we find

\[ V_{a} j^{a} = \langle V_a \psi, \gamma^a \psi \rangle + \langle \psi, \gamma^a V_a \psi \rangle = \langle D \psi, \psi \rangle + \langle \psi, D \psi \rangle = (i \omega_{\psi} \psi) + \langle \psi, i \mu \psi \rangle = 0. \quad (A.2) \]

With this current we can define a conserved quantity in the usual way. Let \(t\) be a time coordinate for \(\mathcal{M}\) defined on an interval \(I = [0, T]\) for some arbitrary \(T\) and let \(\Sigma\) be a two-dimensional manifold. We assume that \(I \times \Sigma\) is embedded into \(\mathcal{M}\) as a three-dimensional submanifold \(\mathcal{V}\) via the embedding \(\iota: I \times \Sigma \hookrightarrow \mathcal{M}\). Then \(\iota_t: \Sigma \to \mathcal{M}, x \mapsto (t, x)\) embeds \(\Sigma\) as a space-like hyper-surface \(\Sigma_t\) into \(\mathcal{M}\). Let \(S = d\Sigma\) be the boundary of \(\Sigma\), which \(\iota_t\) embeds as a 1-dimensional submanifold \(S_t\) into \(\mathcal{M}\). Now the boundary of \(\mathcal{V}\) is \(\partial\Sigma = \Sigma_0 \cup \mathcal{T} \cup \Sigma_T\) where \(\mathcal{T} = \bigcup_{t \in I} S_t\).

Integrating the divergence equation over \(\mathcal{V}\) and using the generalized Stokes’ theorem we obtain
0 = \int_{\mathcal{V}} V_{\alpha} j^{\alpha} \, d\mathcal{V} = \int_{\partial \mathcal{V}} j^{\alpha} \, d\mathcal{Y}_{\alpha},

where we denote the hyper-surface forms on the boundary \( \partial \mathcal{V} \) by \( d \mathcal{Y}_{\alpha} \). Introducing the future-pointing time-like normal \( t_{\alpha} \) to the hyper-surfaces \( \Sigma_{t} \) and the outward normal \( n_{\alpha} \) to \( \mathcal{T} \) we can write

\[
\int_{\Sigma_{t}} j^{\alpha} t_{\alpha} \, d\Sigma = \int_{\Sigma_{r}} j^{\alpha} t_{\alpha} \, d\Sigma + \int_{\mathcal{F}} j^{\alpha} n_{\alpha} \, d\mathcal{T}.
\]

Defining the scalar ‘charge’ (also interpreted as probability) \( Q \) and its flux \( J \) by

\[
Q(t) = \int_{\Sigma_{t}} j^{\alpha} t_{\alpha} \, d\Sigma, \quad J(t) = \int_{\Sigma_{t}} j^{\alpha} n_{\alpha} \, dS
\]

we can write the balance law

\[
Q(T) = Q(0) + \int_{0}^{T} J(t) \, dt
\]

which holds for arbitrary values of \( T \). With appropriate boundary conditions (for instance, when \( \Sigma \) has no boundary as we assume below) for the Dirac field one can make the flux vanish so that the balance law turns into the conservation equation for \( Q \):

\[
Q(T) = Q(0).
\]

We now specialize to the case \( \mathcal{M} \sim \mathbb{R} \times S^2 \) with the metric

\[
g = dt \otimes dt - F^{-2}(t, \theta, \varphi) \left( d\theta \otimes d\theta + \sin^2 \theta \, d\varphi \otimes d\varphi \right),
\]

where \((\theta, \varphi)\) are standard polar coordinates for the two-sphere. The function \( F(t, \theta, \varphi) \) is a conformal factor relating the induced metric at every instant of time \( t \) on \( S^2 \) to the standard metric of the unit two-sphere. We choose the orthonormal frame as

\[
e_0 = \partial_t, \quad e_1 = F \partial_\theta, \quad e_2 = F \text{csc} \theta \, \partial_\varphi.
\]

The non-vanishing Ricci rotation coefficients are

\[
\Gamma^1_{10} = - \frac{T_1}{F}, \quad \Gamma^2_{20} = - \frac{T_2}{F}, \quad \Gamma^1_{12} = - \frac{T_\theta}{\sin \theta}, \quad \Gamma^2_{22} = - (F_\theta - F \cot \theta).
\]

From these we obtain the non-vanishing ‘spin coefficient’ matrices

\[
Y_1 = \frac{i}{2} \begin{bmatrix} T_\theta / \sin \theta & \frac{F_\theta}{F} \end{bmatrix}, \quad Y_2 = \frac{i}{2} \begin{bmatrix} F_\theta - F \cot \theta & - \frac{F_\theta}{F} \end{bmatrix}.
\]

In order to make use of the spin-weighted formalism we need to identify the spin-weights of the quantities involved. To this end we consider the infinitesimal rotations of tangent vectors to the sphere as induced by the \((1, 2)\)-Lorentz group \( O(\eta) \) at every point on \( \mathcal{M} \). Consider the infinitesimal rotation given by \( \frac{1}{2} \gamma_{12} = \frac{i}{2} \gamma_0 \) acting as follows on the two frame vectors on the sphere

\[
\gamma(e_1) = \gamma_1 = \frac{1}{2} [\gamma_{12}, \gamma_1] = \gamma_2, \quad \gamma(e_2) = \gamma_2 = \frac{1}{2} [\gamma_{12}, \gamma_2] = - \gamma_1.
\]

In terms of the complex null-vector \( m = \frac{1}{\sqrt{2}} (e_1 - ie_2) \) this is

\[
\gamma(m) \mapsto i \gamma(m),
\]

\[
(25)
\]}
so \( \frac{1}{2} \gamma_{12} \) generates the frame rotations \( m \mapsto e^{im} m \). But it also acts on the spin-space at each point: for every spinor \( \psi = [\psi_1, \psi_2]^T \) we have

\[
\frac{1}{2} \gamma_{12} \psi = \frac{i}{2} \left[ \psi_1 \right] \frac{i}{2} \left[ \psi_2 \right] - \frac{i}{2} \left[ \psi_1 \right] \psi_2.
\]

Therefore, assigning spin-weight \( \pm \frac{1}{2} \) to \( m \) results in the spin-weights \( \pm \frac{1}{2} \) and \( \mp \frac{1}{2} \) for \( \psi_1 \) and \( \psi_2 \), respectively. Henceforth, we denote the spinor \( \psi \) therefore by \( \psi = [\psi_-, \psi_+]^T \). Since \( \mathcal{F} \) is a scalar function it has spin-weight 0.

Now we are in a position to write down the Dirac equation. Using (A.1), (A.6), (A.7) and (A.8) we find the two equations

\[
\partial_t \psi_- = -i \mu \psi_- - \frac{1}{2} \partial^\mathcal{F} \psi_+ - \mathcal{F} \partial^\mathcal{F} \psi_- + \frac{\mathcal{F}}{F} \psi_- ,
\]

\[
\partial_t \psi_+ = i \mu \psi_+ - \frac{1}{2} \partial^\mathcal{F} \psi_- - \mathcal{F} \partial^\mathcal{F} \psi_+ + \frac{\mathcal{F}}{F} \psi_+ .
\]

The probability or charge density is

\[
j^0 = |\psi_+|^2 + |\psi_-|^2 .
\]

We take \( \mathcal{V} \) in the form \((0, t) \times S^2\) with hyper-surfaces of constant time \( \Sigma_t \sim S^2 \) then the ‘charge’ integral as defined in (A.3) yields the expression

\[
Q(\tau) = \int_{S^2} \left| \psi_+(\tau, \theta, \varphi) \right|^2 + \left| \psi_-(\tau, \theta, \varphi) \right|^2 \sin \theta \, d\theta \, d\varphi \, F^2(\tau, \theta, \varphi).
\]

Since \( \Sigma_t \sim S^2 \) has no boundary, (A.4) implies that for every \( \tau \in (0, T) \)

\[
Q(\tau) = Q(0) = \text{const}.
\]

References

[1] Ashcroft N W and Mermin N D 1976 Solid State Physics (Philadelphia, PA: Saunders)
[2] Baumgarte T and Shapiro S 2010 Numerical Relativity: Solving Einstein’s Equations on the Computer (Cambridge: Cambridge University Press)
[3] Beyer F, Daszuta B, Frauendiener J and Whale B 2014 Numerical evolutions of fields on the two-sphere using a spectral method based on spin-weighted spherical harmonics Class. Quantum Grav. 31 075019
[4] Boyd J P 2001 Chebyshev and Fourier Spectral Methods 2nd edn (New York: Dover)
[5] Castro Neto A H, Guinea F, Peres N M R, Novoselov K S and Geim A K 2009 The electronic properties of graphene Rev. Mod. Phys. 81 109–62
[6] Dray T 1986 A unified treatment of Wigner D-functions, spin-weighted spherical harmonics, and monopole harmonics J. Math. Phys. 27 781–92
[7] Fillion-Gourdeau F, Lorin E and Bandrauk A D 2012 Numerical solution of the time-dependent Dirac equation in coordinate space without fermion-doubling Comput. Phys. Commun. 183 1403–15
[8] Fillion-Gourdeau F, Lorin E and Bandrauk A D 2014 A split-step numerical method for the time-dependent Dirac equation in 3D axisymmetric geometry J. Comput. Phys. 272 559–87
[9] Goldberg J N, Macfarlane A J, Newman E T, Rohrlich F and Sudarshan E C G 1967 Spin-spherical harmonics and \( \delta \) J. Math. Phys. 8 2155
[10] Finster F, Kamran N, Smoller J and Yau S-T 2006 Decay of solutions of the wave equation in the Kerr geometry Commun. Math. Phys. 264 465503
[11] Grandclément P and Novak J 2009 Spectral methods for numerical relativity Living Rev. Relativ. 12 1
[12] Hammer R and Pötz W 2014 Staggered grid leap-frog scheme for the Dirac equation Comput. Phys. Commun. 185 40–52
[13] Hammer R, Pötz W and Arnold A 2014 Single-cone real-space finite difference scheme for the time-dependent Dirac equation J. Comput. Phys. 265 50–70
[14] Hesthaven J, Gottlieb S and Gottlieb D 2007 Spectral methods for time-dependent problems Cambridge Monographs on Applied and Computational Mathematics (Cambridge: Cambridge University Press)
[15] Huffenberger K M and Wandelt B D 2010 Fast and exact spin-s spherical harmonic transforms Astrophys. J. Suppl. Ser. 189 255–60
[16] Katznelson Y 2004 An Introduction to Harmonic Analysis 3rd edn (Cambridge: Cambridge University Press)
[17] Lawson H B and Michelsohn M-L 1990 Spin Geometry (Princeton, NJ: Princeton University Press)
[18] Mardia K and Jupp P 2009 Directional Statistics (Wiley Series in Probability and Statistics) (New York: Wiley)
[19] McEwen J D and Wiaux Y 2011 A novel sampling theorem on the sphere IEEE Trans. Signal Process. 59 5876–87
[20] Mocken G R and Keitel C H 2004 Quantum dynamics of relativistic electrons J. Comput. Phys. 199 558–88
[21] Mocken G R and Keitel C H 2008 FFT-split-operator code for solving the Dirac equation in 2 + 1 dimensions Comput. Phys. Commun. 178 868–82
[22] Momberger K, Belkacem A and Srensen A H 1996 Numerical treatment of the time-dependent Dirac equation in momentum space for atomic processes in relativistic heavy-ion collisions Phys. Rev. A 53 1605–22
[23] Newman E T and Penrose R 1966 Note on the Bondi–Metzner–Sachs group J. Math. Phys. 7 863–70
[24] Nielsen H and Ninomiya M 1981 A no-go theorem for regularizing chiral fermions Phys. Lett. B 105 219–23
[25] Penrose R and Rindler W 1984 Spinors and Space–Time (Two-Spinor Calculus and Relativistic Fields vol 1) (Cambridge: Cambridge University Press)
[26] Porteous I 1995 Clifford Algebras and the Classical Groups (Cambridge: Cambridge University Press)
[27] Risbo T 1996 Fourier transform summation of Legendre series and D-functions J. Geod. 70 383–96
[28] Sakurai J and Tuan S 1994 Modern Quantum Mechanics (Reading, MA: Addison-Wesley)
[29] Snyder J P 1987 Map Projections: A Working Manual (Professional Paper 1395) (Geological Survey (US))
[30] Sparling G A J The magic of twistor theory University of Pittsburgh (unpublished)
[31] Stacey R 1982 Eliminating lattice fermion doubling Phys. Rev. D 26 468–72
[32] Sugiura M 1990 Unitary Representations and Harmonic Analysis: An Introduction (Amsterdam: North-Holland)
[33] Thaller B 2004 Visualizing the kinematics of relativistic wave packets arXiv:quant-ph/0409079
[34] Trapani S and Navaza J 2006 Calculation of spherical harmonics and Wigner d functions by FFT. Applications to fast rotational matching in molecular replacement and implementation into AMoRe Acta Cryst. A 62 262–9
[35] Trautman A 2008 Connections and the Dirac operator on spinor bundles J. Geom. Phys. 58 238–52
[36] Twozydlo J, Groth C W and Beenakker C W J 2008 Finite difference method for transport properties of mass-less Dirac fermions Phys. Rev. B 78 235438
[37] Vozmediano M, Katsnelson M and Guinea F 2010 Gauge fields in graphene Phys. Rep. 496 109–48