Discrete Modes in Gravitational Waves from the Big-Bang

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Abstract

We develop a new approach to gravitational waves in which the Einstein equations acquire the gauge $R + 2\kappa \mathcal{L}_m = 2\Lambda$, with $\delta(R + 2\kappa \mathcal{L}_m) = 0$, which resolve the problem of spurious back-reaction effects. We develop an example in which the matter Lagrangian is described by the scalar (inflaton) field. There are only three dynamical solutions. In one of them the universe is initially static but begins to increase until an inflationary stage. We calculate the dynamics of GW in this primordial stage of the universe. There should be an infinite number of discrete modes in order to the fields can be quantized. Finally, we calculate the energy density due to the gravitational waves.

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I. INTRODUCTION

When one tries to describe the early universe, GR is applied beyond its domain of validity. The quantum effects which dominate in this epoch are expected to resolve the singularity. In particular, the existence of the cosmological singularity in the framework of Loop Quantum Gravity (LQG)\cite{1} has been subject of study in the last years. However, at the present time, it is not possible to realize a consistent quantum gravity theory which leads to the unification of gravitation with the other forces. In particular, the theory of gravitational waves (GW) is a rich subject that brings together different domains such as general relativity, field theory, astrophysics and cosmology. At present various gravitational-wave detectors, after decades of developments, have reached a sensitivity where there are significant chances of detection, and future improvements are expected to lead, in a few years, to advanced detectors with even better sensitivities \cite{4}. There are good reasons to expect that the Universe is permeated also by a stochastic background of GW generated in the early universe. In particular, the fossil GW becoming from the big-bang should be a great success if it were detected. Recently has been reported GW produced during the inflationary epoch \cite{5} which are compatible with a energy scale of $10^{16}$ GeV. However, more data are required to confirm the above situation. During the inflationary expansion, the universe suffered an exponential accelerated expansion driven by a scalar (inflaton) field with an equation of state close to a vacuum dominated one\cite{6,7}. The most conservative assumption is that the energy density $\rho = P/\omega$ is due to a cosmological parameter which is constant and the equation of state is given by a constant $\omega = -1$, describing a vacuum dominated universe with pressure $P$ and energy density $\rho$. Inflationary cosmology can be recovered from a 5D vacuum\cite{8–10}, and is very consistent with current observations of the temperature anisotropy of the Cosmic Microwave Background (CMB)\cite{11}. The most popular model of supercooled inflation is chaotic inflation\cite{12}, but there are many models which are good candidates. In this model the expansion of the universe is driven by a single scalar field called inflaton. At some initial epoch, presumably the Planck scale, the scalar field is roughly homogeneous and dominates the energy density, which remains almost constant during all the inflationary epoch. It is well known that the inflationary cosmology also generates a background of gravitational waves \cite{14}. Dark energy cosmological scenarios have been intensively studied in the last years \cite{15}. The scenarios there described can explain the generation of gravitational waves
on cosmological, but not on astrophysical scales.

In this work we shall study a formalism to study gravitational waves with the aim to obtain a consistent formalism with a quantum field theory in a curved space-time. It is well known that when one tries to study a quantum field theory using GR appear some problems with back reaction effects. These effects become from spurious terms that one must include when the variation of the stress tensor is done. After it we shall study an application to the early big-bang universe to study the pre-inflationary dynamics of the universe. Finally, we shall study the emission of GW from the early pre-inflationary universe, in order to obtain its spectral density.

II. FORMALISM OF GRAVITATIONAL WAVES

In this section we shall revise the formalism of GW to obtain the wave equations in a consistent manner in which spurious back reaction effects can be avoided.

A. Variation of the matter action

We consider the general action \( I \) which describes gravitation and matter

\[
I = \int_V d^4x \sqrt{-g} \left[ \frac{R}{2\kappa} + \mathcal{L}_m \right],
\]

where \( g \) is the determinant of the covariant background tensor metric \( g_{\mu\nu} \), \( R = g^{\mu\nu} R_{\mu\nu} \) is the scalar curvature, \( R^\alpha_{\mu\nu\alpha} = R_{\mu\nu} \) is the covariant Ricci tensor and \( \mathcal{L}_m \) is an arbitrary Lagrangian density which describes matter\(^1\). If we consider an orthogonal base, the curvature tensor will be given written in terms of the Levi-Civita connections

\[
R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta\gamma} - \Gamma^\alpha_{\beta\gamma\delta} + \Gamma^\epsilon_{\beta\delta} \Gamma^\alpha_{\epsilon\gamma} - \Gamma^\epsilon_{\beta\gamma} \Gamma^\alpha_{\epsilon\delta}.
\]

The variation of the action matter will be

\[
\delta\left[ \sqrt{-g}\mathcal{L}_m (g^{\mu\nu}, g^{\mu\nu}) \right] = \sqrt{-g} \left\{ \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} - \frac{1}{2} g^{\mu\nu} \mathcal{L}_m \right\} \delta g^{\mu\nu} + \left[ \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}_{\lambda\lambda}} \right] \delta g^{\mu\nu}_{\lambda\lambda}.
\]

\(^1\) In this paper we shall consider some Lagrangian density related to a metric tensor which is symmetric and free of nonmetricity.
By using the fact that $\delta g^{\mu\nu,\lambda} = g^{\gamma\mu} g^{\nu\beta} g_{\gamma\lambda,\beta}$, we obtain that
\[
\delta \left[ \sqrt{-g} L_m \left( g^\mu\nu, g^\mu\nu, \lambda \right) \right] = \sqrt{-g} \delta g^{\mu\nu} \left\{ \frac{1}{2} T_{\mu\nu} + \left[ \frac{\delta L_m}{\delta g^\mu\nu, \lambda} \right] \delta^\beta_{\lambda,\beta} \right\},
\]
where $\delta^\beta_{\lambda,\beta} = 0$, so that
\[
\delta \left[ \sqrt{-g} L_m \left( g^\mu\nu, g^\mu\nu, \lambda \right) \right] = \frac{1}{2} \sqrt{-g} \delta g^{\mu\nu} T_{\mu\nu}.
\]
Here, we have used the generic definition for the Energy-Momentum (EM) tensor: $T_{\mu\nu} = 2 \frac{\delta L_m}{\delta g^{\mu\nu}} - g_{\mu\nu} L_m$.

\[\text{B. Variation of the gravitational action on a curved spacetime}\]

Now we consider the gravitational action. Its variation is
\[
\delta \left[ \sqrt{-g} R \right] = \delta \left[ \sqrt{-g} g^{\alpha\beta} R_{\alpha\beta} \right] = \sqrt{-g} \left[ \delta g^{\alpha\beta} G_{\alpha\beta} + g^{\alpha\beta} \delta R_{\alpha\beta} \right],
\]
where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ is the Einstein tensor. As can be demonstrated, the second term is zero and can be written as the tetra-divergence of some tetra-vector with components $W^\mu$:
\[
g^{\alpha\beta} \delta R_{\alpha\beta} = \nabla_\mu W^\mu = 0,
\]
with
\[
W^\mu = \frac{1}{2} \left( \delta g^{\lambda\nu} g^{\alpha\mu} \left( \delta g_{\alpha\nu,\lambda} + g_{\lambda\nu,\alpha} - g_{\alpha\lambda,\nu} \right) + g^{\alpha\mu} g^{\lambda\nu} \left( \delta g_{\alpha\nu,\lambda} + \delta g_{\lambda\nu,\alpha} - \delta g_{\alpha\lambda,\nu} \right) - \delta g^{\mu\nu} g^{\alpha\lambda} \left( g_{\alpha\nu,\lambda} + g_{\lambda\nu,\alpha} - g_{\alpha\lambda,\nu} \right) - g^{\alpha\lambda} g^{\mu\nu} \left( g_{\alpha\nu,\lambda} + \delta g_{\lambda\nu,\alpha} - \delta g_{\alpha\lambda,\nu} \right) \right),
\]
where we have made use of the fact that $\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma}$. Therefore, if we require that $\delta I = 0$, we obtain the Einstein equations which provide us the dynamics of the system and the constrictions for $W^\mu$, which are the standard result of general relativity
\[
\nabla_\mu W^\mu = 0.
\]

\[\text{C. Variation of the Einstein equations}\]

Now, in order to study the dynamics of the metric fluctuations we must variate the Einstein equations (9):
\[
\delta G_{\alpha\beta} = -\kappa \delta T_{\alpha\beta},
\]
where
\[
\nabla_\mu W^\mu = 0.
\]
with the constrictions \((10)\). Using the fact that \(R = g^{\alpha\beta}R_{\alpha\beta}\) and \(T = g^{\alpha\beta}T_{\alpha\beta}\), we obtain that \(\delta R = \kappa \delta T\), and we obtain that the equation \((11)\) can be re-written as

\[
\delta R_{\alpha\beta} = -\kappa \delta S_{\alpha\beta},
\]

where we have introduced the tensor \(S_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\), which takes into account matter as a source of the Ricci tensor \(R_{\alpha\beta}\). Another manner to write the equation \((12)\) is in terms of the tetra-vector \(W^\mu\), is

\[
\nabla_\beta W_\alpha = -\kappa \delta \left( T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right),
\]

which explicitly holds

\[
\delta g'^{\nu}_{\alpha,\beta\nu} - g'^{\nu\gamma}_{\alpha,\beta\nu} \delta g_{\alpha\gamma} - g'^{\nu\gamma}_{\alpha,\beta\nu} \delta g_{\beta\gamma} + g'^{\nu\gamma}_{\alpha,\beta\nu} \delta g_{\alpha\gamma,\beta} = \frac{\kappa}{2} \left[ \delta T_{\alpha\beta} - \frac{1}{2} \delta ( T g_{\alpha\beta} ) \right],
\]

where we have used \((10)\). The tetra-vector components \(W_\alpha\) can be written as

\[
W_\alpha = \frac{1}{2} \left\{ \delta g^{\gamma\theta}_{\alpha,\gamma} ( g_{\theta,\gamma} + g_{\gamma,\theta} - g_{\alpha\gamma,\theta} ) + g^{\gamma\theta} ( \delta g_{\alpha\theta,\gamma} + \delta g_{\gamma,\alpha} - \delta g_{\alpha\gamma,\theta} ) - g^{\gamma\theta} ( \delta g^{\beta}_{\alpha,\beta} ( g_{\theta,\beta} + g_{\beta,\theta} - g_{\gamma,\beta} ) - ( \delta g_{\gamma,\alpha} + \delta g_{\theta,\alpha,\gamma} - \delta g_{\theta,\alpha} ) ) \right\}.
\]

To finalize we can propose the existence of a tensor field \(\Psi_{\alpha\beta}\), such that \(\delta R_{\alpha\beta} \equiv \nabla_\beta W_\alpha \equiv \Box \Psi_{\alpha\beta} = -\kappa \delta S_{\alpha\beta}\), and hence

\[
W_\alpha = \nabla^\beta \Psi_{\alpha\beta}.
\]

This mean that the gravitational waves can be described in two manners, as the resulting of the wave equation of motion for the tensor field \(\Psi_{\alpha\beta}\):

\[
\Box \Psi_{\alpha\beta} = -\kappa \delta S_{\alpha\beta},
\]

or as the solution of a vectorial differential equation

\[
\nabla_\beta W_\alpha = -\kappa \delta S_{\alpha\beta}.
\]

In both cases the constriction equation is given by eq. \((10)\). The second expression has the adventage that the field \(W_\alpha\) is gauge-invariant under transformations \(W_\alpha \rightarrow W_\alpha + \nabla_\alpha \Omega\), when the scalar function \(\Lambda\) holds \(\Box \Omega = 0\). Hence, the tetra-vector field \(W^\mu\) will be more appropiated than the tensor field \(\Psi_{\alpha\beta}\) in order to describe the fluctuations of the metric because \(W^\mu\) is gauge-invariant and therefore it is conserved on the 4D-hypersurface: \(\nabla_\mu W^\mu = 0\). This last condition implies that the fluctuations of the physical fields must comply \(g^{\alpha\beta} \delta S_{\alpha\beta} = 0\).
D. The local vacuum and dynamics of gravitational waves

We shall introduce the following gauge equations

\[ R + 2\kappa \mathcal{L}_m = 2\Lambda, \]  
\[ \delta (R + 2\kappa \mathcal{L}_m) = 0, \]  

(19) \hspace{2cm} (20)

where \( \Lambda \) is the cosmological constant. Physically, the eq. (19) means that for each point of the space-time the action density is a constant related with the cosmological constant. The second eq. (20) means that its variation is also null and each alteration of the space-time is locally produced by a local variation of the physical fields of the system which we are considering. Another manner to see this gauge consists in adopt the philosophical point of view that matter (i.e. the physical fields) and space-time distortions appears jointly and both must be viewed as a unity. If we adopt this gauge the equations (17) take the final form

\[ \nabla_\beta W_\alpha - \Lambda \delta g_{\alpha\beta} = -2\kappa \delta \left( \frac{\delta \mathcal{L}_m}{\delta g^{\alpha\beta}} \right). \]  

(21)

If we define \( \nabla_\beta \bar{W}_\alpha = \nabla_\beta W_\alpha - \Lambda \delta g_{\alpha\beta} \), and we make \( \bar{W}_\alpha = \nabla_\beta \bar{\Psi}_{\alpha\beta} \), we obtain the following wave equation for \( \bar{\Psi}_{\alpha\beta} \):

\[ \Box \bar{\Psi}_{\alpha\beta} = -2\kappa \delta \left( \frac{\delta \mathcal{L}_m}{\delta g^{\alpha\beta}} \right). \]  

(22)

The equations (19) and (20) avoid some ambiguities of the Einstein equations when we study field theories in curved space-times. These problems are responsible for the gauge-invariance problems of general relativity. In our approach the gauge-invariance of the metric fluctuations is preserved because both, the action density, \( \frac{R}{2\kappa} + \mathcal{L}_m \), and its first variation, are also invariants. This is our main result, which is valid for an arbitrary physical system with an arbitrary Lagrangian density \( \mathcal{L}_m \). In the following section we shall study the dynamics of gravitational waves in the very early universe.

III. AN EXAMPLE: GW FROM THE PRIMORDIAL BIG-BANG OF THE UNIVERSE

As an example we can study the example which describes the gravitational waves in the primordial universe. If the expansion is driven by a scalar field \( \varphi(x^\alpha) \) which is minimally
coupled to gravity

\[ I = \int_V d^4x \sqrt{-g} \left[ \frac{R}{2\kappa} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \tag{23} \]

where \( \kappa = 8\pi G \), \( G \) is the gravitational constant, \( \sqrt{-g} = a^3(t) \) is the volume of the manifold \( \mathcal{M} \) and \( g_{\mu\nu} = \text{diag}[1, -a^2, -a^2, -a^2] \) are the components of the diagonal tensor metric. The dynamics of the scalar field being given by the equation

\[ \ddot{\phi} + 3H \dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + V'(\phi) = 0. \tag{24} \]

Given the quantum nature of the fields \( \phi \) and \( \Pi^0 = \frac{\delta L_m}{\delta \dot{\phi}} \) it seems convenient to use the quantization procedure. To do it, we impose the commutation relations

\[ \left[ \varphi(t, \vec{R}), \Pi^0(\varphi)(t, \vec{R}') \right] = i \delta^{(3)} (\vec{x} - \vec{x}'). \tag{25} \]

### A. The scalar field dynamics

We shall consider a semiclassical approach to the scalar field: \( \varphi(x^\alpha) = \phi(t) + \delta\phi(x^\alpha) \), such that \( \phi(t) = \langle E|\varphi|E \rangle \) is the background solution that describe the dynamics on the background metric. Here, \(|E\rangle\) is some quantum state such that \( \langle E|\varphi|E \rangle \) denotes the expectation value of the \( \varphi \) on the 3D Euclidean hypersurface and \( \phi(t) \) is the background solution of the equation

\[ \ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0. \tag{26} \]

Furthermore \( \delta\phi(x^\alpha) \) are the fluctuations with respect to the background, such that \( \langle E|\delta\varphi|E \rangle = 0 \). The dynamics of the fluctuations \( \delta\phi \) can be approximated to

\[ \ddot{\delta\phi} + 3H \dot{\delta\phi} - \frac{1}{a^2} \nabla^2 \delta\phi + V''(\phi) \delta\phi = 0. \tag{27} \]

Here, \( V'' \equiv \frac{\delta^2 V}{\delta \varphi^2} \mid_{\phi} \equiv m^2 \) gives the square mass of the inflaton field related to the density potential \( V(\varphi) \). From the Einstein equations, we obtain that \( R + 4\Lambda - \kappa T = 0 \), so that after make use of the fact that \( T = 4V(\phi) - \dot{\phi}^2 \) and the condition (19), we obtain

\[ V(\phi) = \frac{3\Lambda}{\kappa}, \tag{28} \]

\[ \dot{\phi}^2 + \frac{6}{\kappa} \left( 2H^2 + \dot{H} \right) = \frac{8\Lambda}{\kappa}. \tag{29} \]

From the Eq. \( 29 \) it is obvious that \( V(\dot{\phi}) \) is a constant, and the Eqs. \( 29 \) with \( 26 \) provide us with the dynamics of \( \phi(t) \) and \( H(t) \). Since \( V' = 0 \), hence the dynamics of the background
scalar field $\phi$ decouples with gravity. From the Eq. (26) we obtain general solution for the scalar field

$$\dot{\phi}(t) = \frac{C}{a^3(t)}.$$  

(30)

If we replace this solution in (29), we obtain that there are only three possible solutions.

1. The static universe

The more trivial case with $C \neq 0$ and $a(t) = \text{const.}$, for $\Lambda > 0$. This solution give us a null Hubble parameter and hence do not describes an expanding universe.

2. The bang of the primordial universe (or pre-inflation)

The more interesting case is with $C = 0$ (i.e., with $\dot{\phi} = 0$), for $\Lambda > 0$. The dynamics of the Hubble parameter begin given by the equation $12H^2 + 6\dot{H} = 8\Lambda$, and the solution is

$$H(t) = \sqrt{\frac{2\Lambda}{3}} \tanh \left[2 \sqrt{\frac{2\Lambda}{3}} t\right],$$  

(31)

which is related to a scale factor

$$a(t) = \frac{a_0}{\left[1 - \tanh \left(2 \sqrt{\frac{2\Lambda}{3}} t\right)\right]^{1/4} \left[\tanh \left(2 \sqrt{\frac{2\Lambda}{3}} t\right) + 1\right]^{1/4}},$$  

(32)

with $a_0 = \sqrt{\frac{3}{2\Lambda}}$. This interesting case presents a new paradigm in cosmology because describes an universe with a Hubble parameter that increases from a null value to an asymptotically constant value $H(t)_{t\gg G^{1/2}} \rightarrow \sqrt{\frac{2}{3}}\Lambda$, describing the creation of the universe and its transition from a static state to an accelerated de Sitter inflationary expansion. The Hubble parameter was plotted in the figure (11). A special case of this case with $\dot{H} = 0$ describes a de Sitter inflationary expansion governed by the cosmological constant $\Lambda > 0$ with a scale factor $a(t)/a_0 = e^{\sqrt{\frac{2\Lambda}{3}} t}$ and $C = 0$. This is the asymptotic solution of (32), for very large times.

3. The oscillating universe

To finalize there is a case with $C = 0$ and $\Lambda < 0$ in which the solution of $a(t)$ is oscillating [see figure (2)]

$$a(t) = \frac{a_0}{\left[\sec^2 \left(2 \sqrt{-\frac{2\Lambda}{3}} t\right)\right]^{1/4}}.$$  

(33)
The Hubble parameter for this case is

$$H(t) = -\sqrt{\frac{-2\Lambda}{3}} \tan \left[ 2\sqrt{\frac{-2\Lambda}{3}} t \right].$$  \hfill (34)

This interesting case describes an universe that fails to progress. It could be assimilated to a bouncing universe [16].

B. GW from the primordial bang of the universe

We shall study GW for the case $2$, which are the more interesting. Due to the fact that

$$\delta L_m \delta g^\alpha\beta = \varphi,_{\alpha} \varphi,_{\beta},$$

the linearized variation $\delta \left( \delta L_m \delta g^\alpha\beta \right) = \varphi,_{\alpha} \varphi,_{\beta} - \phi,_{\alpha} \phi,_{\beta}$, will be

$$\delta \left( \delta L_m \delta g^\alpha\beta \right) = \phi,_{\alpha} \delta \phi,_{\beta},$$  \hfill (35)

so that

$$\Box \bar{\Psi}_{\alpha\beta} = -2\kappa \phi,_{\alpha} \delta \phi,_{\beta}.$$  \hfill (36)

Using (30) we obtain that $\phi,_{\alpha} = 0$, $\forall\alpha$, so that finally we obtain the equation of motion for $\Psi_{\alpha\beta}$

$$\Box \bar{\Psi}_{\alpha\beta} = 0.$$  \hfill (37)

C. Transverse-Traceless (TT) Gauge

The expansion of the tensor field $\bar{\Psi}_{\alpha\beta}(t, \vec{x})$ can be expanded as

$$\bar{\Psi}_{\alpha\beta}(t, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \sum_{A=+,x} \int d^3k \ e^A_{ab}(\hat{z}) \left[ B_k e^{i\vec{k}.\vec{x}} \xi_k(t) + B_k e^{-i\vec{k}.\vec{x}} \xi^*_k(t) \right],$$  \hfill (38)

where $a, b = 1, 2$, denote the transverse polarizations $+, \times$, on the plane with normal co-linear with $\vec{k}$ and $e^A_{ab}$ are the components of the polarization tensor, such that $e^A_{ab} e^{A'}_{ab} = \delta^A_{A'}$.

In the frame where $\vec{k}$ is along the $\hat{z}$ direction these polarizations are

$$e^+_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ab}, \quad e^\times_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ab},$$  \hfill (39)

with $a, b$ spanning the $(x, y)$ plane.
In order to solve the equations for the gravitational waves we shall use the TT gauge, which is represented by the following conditions

$$\bar{\Psi}_{0\mu} = 0, \quad \bar{\Psi}^i_j = 0, \quad \nabla^j \bar{\Psi}_{ij} = 0. \quad (40)$$

The equation of motion for the modes $\xi_k(t)$ is

$$\ddot{\xi}_k(t) + 3 \frac{\dot{a}}{a} \dot{\xi}_k(t) + \frac{k^2}{a(t)^2} \xi_k(t) = 0, \quad (41)$$

which is the same that for the scalar field fluctuations $\delta \phi$, because the scalar field fluctuations are massless due to the fact the potential $V$ is a constant

$$\delta \phi(t, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left[ B_k \ e^{i\vec{k} \cdot \vec{x}} \xi_k(t) + B_k^\dagger \ e^{-i\vec{k} \cdot \vec{x}} \xi_k^*(t) \right]. \quad (42)$$

The annihilation and creation operators $B_k$ and $B_k^\dagger$ satisfy the usual commutation algebra

$$[B_k, B_k'] = \delta^{(3)}(\vec{k} - \vec{k}'), \quad [B_k, B_k'] = [B_k^\dagger, B_k^\dagger] = 0. \quad (43)$$

Using the commutation relation (25) and the Fourier expansions (38) and (42), we obtain the normalization condition for the modes. For convenience we shall re-define the dimensionless time: $\tau = b \ t$, where $b = \sqrt{\frac{2A}{3}} = \frac{1}{a_0}$, so that the normalization condition for $\xi_k(\tau)$ is

$$\xi_k(t) \frac{d\xi_k^*(\tau)}{d\tau} - \xi_k^*(t) \frac{d\xi_k(\tau)}{d\tau} = i \left( \frac{a_0}{a(\tau)} \right)^3, \quad (44)$$

where the asterisk denotes the complex conjugated. For the case $2$ in which the Hubble parameter and the scale factor are given respectively by (31) and (32), the general solution for $\xi_k(\tau)$ is

$$\xi_k(\tau) = C_1 \frac{\sinh(\tau)}{\sqrt{2 \cosh^2(\tau) - 1}} \mathrm{Hn} \left[ -1, \frac{k^2 - 1}{4}; 0, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}; -\tanh^2(\tau) \right] + C_2 \frac{\cosh(\tau)}{\sqrt{2 \cosh^2(\tau) - 1}} \mathrm{Hn} \left[ -1, \frac{k^2 + 1}{4}; -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}; -\tanh^2(\tau) \right], \quad (45)$$

where $\mathrm{Hn}[a; q; \alpha, \beta, \gamma, \delta, z] = \sum_{j=0}^{\infty} c_j \ z^j$ is the Heun function. Since the Heun functions are written as infinity series, we can make a series expansion in the both sides of (44), in order to obtain the restrictions for the coefficients $C_1$ and $C_2$, and the wavenumber values $k$. From
the zeroth order expansion (in $\tau$), we obtain that $C_2 = i C_1 / 2$. Hence, we shall choose $C_1 = 1$ results that $C_2 = i / 2^2$.

Of course the series is infinite so that there are infinite values of quasi-normal modes with finite wavenumbers $k_n^{(N)}$ which provide us the quantization of both, the scalar and tensor fields $\delta \phi$, $\bar{\Psi}_{\alpha \beta}$, respectively.

In general, the modes with real-$k_{n}^{(N)}$ corresponds to modes of GW that propagates from the white hole. Modes which are purely imaginary denotes unstable gravitons which is created from (corresponding to the signature $-$), or absorbed by (corresponding to the signature $+$) the horizon of the white hole which has an initial size $a_0 = 1 / b = \sqrt{3 / 2 \Lambda}$. They are discrete values that the modes with polarization $+$ can take. On the other hand, modes with complex-$k_{n}^{(N)}$ correspond to those with polarization $\times$ of GW that propagates with both, the real $\Re(k_{n}^{(N)})$ and the imaginary parts $\Im(k_{n}^{(N)})$. They are absorbed or created, depending on their signatures.

We can calculate the two-point expectation value for the fluctuations of the spacetime due to gravitational waves at the spatial points $\vec{x}$. If the spatial position in the interior of the exploiting source is denoted by $\vec{x}'$, we have

$$\langle E | \bar{\Psi}^2 | E \rangle (\tau, \vec{x}, \vec{x}) = 2 i \sum_{A=+,\times} \sum_{N=1}^{\infty} \sum_{n=1}^{2N} \sin \left[ \vec{k}_{n}^{(N)}, (\vec{x} - \vec{x}') \right] \xi_{k_{n}^{(N)}}(\tau) \xi^{*}_{k_{n}^{(N)}}(\tau).$$

Hence, integrating on all the points of the spherical source of ratio $b^{-1}$, we obtain

$$\langle E | \bar{\Psi}^2 | E \rangle (\tau, \vec{x}, \theta) = 8 \pi i \sum_{A=+,\times} \sum_{N=1}^{\infty} \sum_{n=1}^{2N} \xi_{k_{n}^{(N)}}(\tau) \xi^{*}_{k_{n}^{(N)}}(\tau) I^{(N,n)}(|\vec{x}|, \theta),$$

where $0 < \theta < \pi / 2$ is the angle between $\vec{x}$ and $\vec{x}'$, and the function $I^{(N,n)}(|\vec{x}|, \theta)$ for the $N$-th

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2 The polynomial expansion of $\xi_k(t) \frac{d\xi_k(t)}{dt} - \xi_k(t) \frac{d\xi_k(t)}{dt} = i \left( \frac{a_0}{a(\tau)} \right)^3$, is given by

$$\xi_k(t) \frac{d\xi_k(t)}{dt} - \xi_k(t) \frac{d\xi_k(t)}{dt} - i \left( \frac{a_0}{a(\tau)} \right)^3 = \sum_{N=1}^{\infty} f_N(k) \tau^N = 0,$$

where $f_N(k_{n}^{(N)}) = 0$, for each $N$. There are $2N$ modes for each $N$-th order of the expansion.
order in the expansion is given by
\[
I^{(N,n)}(|\vec{x}|, \theta) = \frac{\cos \left[ k_n^{(N)} \left( |\vec{x}| - \cos(\theta) \right) \right] \left( \frac{\cos(\theta)}{b} \right)^2 \left( \frac{k_n^{(N)}}{b} \right)^2 + 2 \cos \left( k_n^{(N)} |\vec{x}| \right)}{\left( \frac{k_n^{(N)}}{b} \right)^3 \cos^3(\theta)} + 2 \sin \left[ k_n^{(N)} \left( |\vec{x}| - \cos(\theta) \right) \right] \left( \frac{\cos(\theta)}{b} \right) \frac{\cos(\theta)}{b} \left( \frac{\cos(\theta)}{b} \right) \frac{2}{k_n^{(N)} |\vec{x}|} \left( \frac{\cos(\theta)}{b} \right)^3 \cos^3(\theta)
\]
for $|\vec{x}| > 1/b = \sqrt{\frac{3}{2\Lambda}}$. Finally, we can calculate the energy density due to gravitational waves, $\rho_{gw} = \langle E|\dot{\Psi}^2|E \rangle (t, \vec{x}, \theta)$
\[
\rho_{gw}(\tau, \vec{x}, \theta) = 8\pi i \sum_{A=+, \times} \sum_{N=1}^{\infty} \sum_{n=1}^{2N} \xi_{k_n^{(N)}}(\tau) \dot{\xi}_{k_n^{(N)}}(\tau) I^{(N,n)}(|\vec{x}|, \theta),
\]
which up to fifth order in the series expansion with respect to $\tau$, is
\[
\rho_{gw}(\tau, \vec{x}, \theta) = 8\pi i \sum_{A=+, \times} \sum_{N=1}^{\infty} \sum_{n=1}^{2N} \left[ 1 + \sum_{n=1}^{2N} F_{n}^{(N)} \right] I^{(N,n)}(|\vec{x}|, \theta),
\]
where the coefficients $F_{n}^{(N)}(k_n^{(N)})$, are
\[
F_{n}^{(1)}(k_n^{(1)}) = \frac{1}{3} \left[ 1 - \left( k_n^{(1)} \right)^2 \right], \quad \tag{51}
\]
\[
F_{n}^{(2)}(k_n^{(2)}) = \frac{1}{9} \left[ \frac{53}{20} \left( k_n^{(2)} \right)^4 + \frac{23}{5} \left( k_n^{(2)} \right)^2 - 50 \right], \quad \tag{52}
\]
\[
F_{n}^{(3)}(k_n^{(3)}) = -\left[ \frac{109}{1260} \left( k_n^{(3)} \right)^6 - \frac{37}{140} \left( k_n^{(3)} \right)^4 - \frac{1693}{630} \left( k_n^{(3)} \right)^2 + \frac{128}{63} \right], \quad \tag{53}
\]
\[
F_{n}^{(4)}(k_n^{(4)}) = \left[ \frac{319}{28350} \left( k_n^{(4)} \right)^8 - \frac{6281}{56700} \left( k_n^{(4)} \right)^6 - \frac{49877}{56700} \left( k_n^{(4)} \right)^4 - \frac{4807}{56700} \left( k_n^{(4)} \right)^2 + \frac{527}{28} \right], \quad \tag{54}
\]
\[
F_{n}^{(5)}(k_n^{(5)}) = -\left[ \frac{1493}{1871100} \left( k_n^{(5)} \right)^{10} - \frac{6943}{467775} \left( k_n^{(5)} \right)^8 - \frac{351}{1925} \left( k_n^{(5)} \right)^6 + \frac{1839031}{1871100} \left( k_n^{(5)} \right)^4 + \frac{463679}{53460} \left( k_n^{(5)} \right)^2 - \frac{63541}{8316} \right]. \quad \tag{55}
\]
In the table we have included the square wavenumbers for the first five orders in the expansion.
IV. FINAL COMMENTS

We have studied a new approach to gravitational waves in which the Einstein equations acquire a gauge governed by the cosmological constant. This gauge removes the inconsistencies on the right side of the Einstein equations and excludes spurious terms which are responsible for back reaction effects when we make the first variation of the Einstein equations. A simple example where the matter Lagrangian is described by a scalar field is studied and the resulting dynamics is surprising. There are only three dynamical solutions. i) The first one describes that a primordial universe remains static. ii) The second one describes a primordial universe which is initially static but begins to increase until an inflationary stage. iii) The third solution (governed by a negative cosmological solution) describes an eternally oscillating (bouncing) universe from the initial state.

We have explored the study of gravitational waves for the case ii), because describes a pre-inflationary universe that evolves towards an asymptotic inflationary phase. Finally, we have calculated the dynamics of the GW in this stage. The normalization conditions for the modes impose that the wavenumbers for both, the inflation fluctuations and $\Psi_{\alpha\beta}$ are an infinity number of discrete values (which can be real, imaginary or complex). This is an important result which gives us the exact values for the frequencies at the big-bang.

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FIG. 1: Evolution of $H(t)$ during the primordial bang, for $\Lambda = 10^{-8} G^{-1/2}$. The universe approaches to a asymptotic de Sitter expansion from a initial static universe.
FIG. 2: Evolution of the scale factor $a(t)$ for the universe with negative cosmological constant
$\Lambda = -10^{-8} G^{-1/2}$. Notice that this universe collapses and bounce cyclically, but fails to progress.