ON DESCARTES’ RULE OF SIGNS

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ABSTRACT. A sequence of \(d+1\) signs + and − beginning with a + is called a sign pattern (SP). We say that the real polynomial \(P := x^d + \sum_{j=0}^{d-1} a_j x^j, \quad a_j \neq 0\), defines the SP \(\sigma := (+, \text{sgn}(a_{d-1}), \ldots, \text{sgn}(a_0))\). By Descartes’ rule of signs, for the quantity \(\text{pos}\) of positive (resp. \(\text{neg}\) of negative) roots of \(P\), one has \(\text{pos} \leq c\) (resp. \(\text{neg} \leq p = d - c\)), where \(c\) and \(p\) are the numbers of sign changes and sign preservations in \(\sigma\); the numbers \(c - \text{pos}\) and \(p - \text{neg}\) are even. We say that \(P\) realizes the SP \(\sigma\) with the pair \((\text{pos, neg})\).

Key words: real polynomial in one variable; Descartes’ rule of signs; sign pattern

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1. Introduction

In the present paper we consider a problem which is a natural continuation of Descartes’ rule of signs. The latter states that the number of positive roots of a real univariate polynomial (counted with multiplicity) is majorized by the number of sign changes in the sequence of its coefficients. We focus on polynomials without zero coefficients. Such a polynomial (say, of degree \(d\)) is representable in the form \(P := x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0, \quad a_j \in \mathbb{R}^*\). Denoting by \(c\) and \(p\) the numbers of sign changes and sign preservations in the sequence \(1, a_{d-1}, \ldots, a_1, a_0\) and by \(\text{pos}\) and \(\text{neg}\) the number of positive and negative roots of \(P\) (hence \(c + p = \text{pos} + \text{neg} = d\)) one obtains the conditions

\[
\begin{align*}
\text{pos} &\leq c, \quad \text{neg} \leq p, \quad c + p = d, \\
c - \text{pos} &\in 2\mathbb{N} \cup 0, \quad p - \text{neg} \in 2\mathbb{N} \cup 0, \quad (-1)^{\text{pos}} = \text{sgn}(a_0)
\end{align*}
\]

(the condition \(\text{neg} \leq p\) results from Descartes’ rule applied to the polynomial \(P(-x)\)).

We call sign pattern (SP) a sequence of + or − signs of length \(d+1\) beginning with a +. We say that the polynomial \(P\) defines the SP \((+, \text{sgn}(a_{d-1}), \ldots, \text{sgn}(a_1), \text{sgn}(a_0))\). A pair \((\text{pos, neg})\) satisfying conditions (1.1) is called admissible. An admissible pair (AP) is called realizable if there exists a polynomial \(P\) with exactly \(\text{pos}\) positive distinct and exactly \(\text{neg}\) negative distinct roots.

Example 1. For \(c = 0\), the all-pluses SP is realizable with any AP (which is of the form \((0, d - 2k), k = 0, 1, \ldots, \lfloor d/2 \rfloor\), where \([\alpha]\) denotes the integer part of \(\alpha \in \mathbb{R}\)). Indeed, one can construct a polynomial \(P + t, t > 0\), one
encounters polynomials with exactly \(d - 2, d - 4, \ldots, d - 2[d/2]\) negative distinct roots and with no positive roots (as \(t\) increases, the polynomial \(P + t\) loses two-by-two its real roots; each time two coalescing real roots give birth to a complex conjugate pair).

**Proposition 1.** (1) Any SP with \(c = 1\) is realizable with any AP of the form 
\[(1, d - 1 - 2k), k \leq \lfloor (d - 1)/2 \rfloor.\]

(2) Any SP is realizable with the AP \((c, p)\).

Proposition 1 is proved in Section 2. Part (1) of it shows that in terms of the value of \(c\), the first truly nontrivial case is \(c = 2\). Its study is the object of the present paper. We should point out that due to the possibility to consider instead of the polynomial \(P(x)\) the polynomial \(P(-x)\) (this change exchanges the quantities \(c\) and \(p\) and the quantities \(pos\) and \(neg\)), it suffices to consider (for a given degree \(d\)) the cases with \(c \leq [d/2]\).

**Notation 1.** We denote by \(\Sigma_{m,n,q}\) the SP consisting of \(m \geq 1\) pluses followed by \(n \geq 1\) minuses followed by \(q \geq 1\) pluses, where \(m + n + q = d + 1\). For a given polynomial \(P\), we denote by \(PR\) the corresponding reverted polynomial, i.e. \(PR := x^dP(1/x)\). If the polynomial \(P\) defines the SP \(\Sigma_{m,n,q}\), then \(PR\) defines the SP \(\Sigma_{q,n,m}\). The roots of \(PR\) are the reciprocals of the roots of \(P\).

For small values of \(m\) or \(n\), we have the following result:

**Theorem 1.** (1) For \(n = 1, d \geq 2\), and for \(n = 2, d \geq 3\), any SP \(\Sigma_{m,n,q}\) is realizable with the AP \((0, d - 2)\).

(2) For \(n = 3\) and \(d \geq 5\), any SP \(\Sigma_{m,3,q}\) is realizable with the AP \((0, d - 2)\). For \(d = 4\), the SP \(\Sigma_{1,3,1}\) is not realizable with the AP \((0, 2)\).

(3) For \(n = 4\), the SP \(\Sigma_{m,4,q}\) is realizable with the AP \((0, d - 2)\) if \(q \geq 3, m \geq 3\) and \(d \geq 10\), or if \(m = 2\) and \(q \geq 6\) (hence \(d \geq 11\)).

(4) For \(m = 1\) and \(n \geq 4\), the SP \(\Sigma_{1,n,q}\) is not realizable with the AP \((0, d - 2)\).

Theorem 1 is proved in Section 3.

**Remarks 1.** (1) If a SP \(\Sigma_{m,n,q}\) is realizable with the AP \((0, d - 2)\), then it is realizable with any AP of the form \((0, d - 2k), k = 1, \ldots, \lfloor (d - 2)/2 \rfloor\). Indeed, if a polynomial \(P\) with distinct nonzero roots realizes the SP \(\Sigma_{m,n,q}\), then one can perturb \(P\) to make all its critical levels distinct. In the family \(P + t\) one encounters (for suitable positive values of \(t\)) polynomials with exactly \(d - 2k\) distinct negative roots and no positive ones, for \(k = 1, \ldots, \lfloor (d - 2)/2 \rfloor\). As \(t \geq 0\), the constant term of the polynomial \(P\) is positive hence \(P + t\) defines the SP \(\Sigma_{m,n,q}\).

(2) The exhaustive answer to the question which couples (SP, AP) are realizable for \(d \leq 8\) is given in [3, 11, 24] and [15]. From the results in these papers one deduces that for \(5 \leq d \leq 8\), the SP \(\Sigma_{m,4,q}\) is not realizable with the AP \((0, d - 2)\). For \(d \geq 9, n \leq 4\) and \(c = 2\), the only cases when the AP is \((0, d - 2)\) and which are not covered by Theorem 1 are the ones of \(\Sigma_{3,4,3}\) and \(\Sigma_{2,4,4}\) for \(d = 9\) and of \(\Sigma_{2,4,5}\) for \(d = 10\). These cases are settled by Proposition 2.

(3) The following result is proved in [2] (see Proposition 6 therein): If \(\kappa := (d - m - 1)/m((d - q - 1)/q) \geq 4\), then the SP \(\Sigma_{m,n,q}\) is not realizable with the AP \((0, d - 2)\). This seems to be the only result concerning nonrealizability of the couple \((\Sigma_{m,n,q}, (0, d - 2))\) known up to now. Part (4) of Theorem 1 implies nonrealizability of cases which are not covered by the cited result. These are \(\Sigma_{1,4,d-4}\) for \(d \geq 11\) (with \(\kappa = 3(d - 2)/(d - 4)\) which is \(\leq 27/7 < 4\) for \(d \geq 11\)).
Proposition 2. (1) For $d = 9$, the SPs $\Sigma_{3,4,3}$ and $\Sigma_{2,4,4}$ are not realizable with the AP $(0,7)$.

(2) For $d = 10$, the SP $\Sigma_{2,4,5}$ is not realizable with the AP $(0,8)$.

Proposition 2 is proved in Section 4. Our next result contains sufficient conditions for realizability of a SP $\Sigma_{m,n,q}$ with the AP $(0,d-2)$:

**Theorem 2.** The SP $\Sigma_{m,n,q}$ is realizable with the AP $(0,d-2)$ if

\[
L(d,m,n) := -dn^2 + 4dm + 4dn - 4m^2 - 4mn - 4d + 4m > 0.
\]

Theorem 2 is proved in Section 5.

**Remarks 2.** (1) Condition (1.2) is sharp in the following sense: in the two nonrealizable cases $(\Sigma_{3,4,3}, (0,7))$ and $(\Sigma_{2,4,5}, (0,8))$ (see Proposition 2) one has $L(d,m,n) = 0$.

(2) The condition of realizability (1.2) can be compared with the condition of nonrealizability $\kappa \geq 4$ (see part (3) of Remarks 1). To this end the latter can be given the following equivalent form:

\[3dm - dn - 3m^2 - 3mn + 3d + 3m + n - 2 \leq 0.
\]

(3) As the SPs $\Sigma_{m,n,q}$ and $\Sigma_{q,n,m}$ are simultaneously (non)realizable with the AP $(0,d-2)$ (see the definition of $D^R$ in Notation [I]), one can assume that $m \leq q$ hence $m \leq \lfloor d/2 \rfloor$. Condition (1.2) can be presented in the form

\[Ma - Na > 0 \quad \text{with} \quad M := 4m + 4n - n^2 - 4 \quad \text{and} \quad N := 4m + 4n - 4
\]

which allows, for given $n = n_0$, to find $m_0$ such that for $m \geq m_0$, one has $N/M < 2$. Then for $m \geq m_0$ and $d \geq 2m + n_0$, condition (1.2) is fulfilled and the corresponding SP is realizable with the AP $(0,d-2)$.

2. Proof of Proposition 1

Part (1). We remind the formulation of a **concatenation lemma** (see [2]):

**Lemma 1.** Suppose that the monic polynomials $P_1$ and $P_2$ of degrees $d_1$ and $d_2$ with SPs $(+,\sigma_1)$ and $(+,\sigma_2)$ respectively realize the pairs $(\text{pos}_1,\text{neg}_1)$ and $(\text{pos}_2,\text{neg}_2)$. Here $\sigma_j$ denote what remains of the SPs when the initial sign $+$ is deleted. Then

(1) if the last position of $\sigma_1$ is $+$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)$ realizes the SP $(1,\sigma_1,\sigma_2)$ and the pair $(\text{pos}_1 + \text{pos}_2,\text{neg}_1 + \text{neg}_2)$;  

(2) if the last position of $\sigma_1$ is $-$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)$ realizes the SP $(1,\sigma_1,-\sigma_2)$ and the pair $(\text{pos}_1 + \text{pos}_2,\text{neg}_1 + \text{neg}_2)$. Here $-\sigma_2$ is obtained from $\sigma_2$ by changing each $+$ by $-$ and vice versa.

For $d = 1$, the SP $(+,+)$ (resp. $(+,-)$) is realizable with the AP $(0,1)$ (resp. $(1,0)$) by the polynomial $x+1$ (resp. $x-1$). Applying Lemma 1 with $P_1$ and $P_2$ of the form $x \pm 1$ one realizes for $d = 2$ all the three SPs with $c = 0$ or $c = 1$ with the APs of the form $(0,2)$ or $(1,1)$.

Suppose that for $d = d_0 \geq 2$ all SPs with $c = 0$ or $c = 1$ are realizable by monic polynomials (denoted by $P$). Then to realize for $d = d_0 + 1$ a SP with $c = 0$ or $c = 1$ with the pair $(0,d_0 + 1)$ or $(1,d_0)$ it suffices to apply Lemma 1 with $P_1 = P$.
and with \( P_2 = x - 1 \) (resp. \( P_2 = x + 1 \)) if \( c = 0 \) and the last two signs of the SP defined by \( P \) are \((+,-)\) (resp. if \( c = 1 \) and these last two signs are \((-,-))\).

To realize for \( c = 1 \) a SP with any AP \( (1, d - 1 - 2k), k \leq [(d - 1)/2] \), it suffices to perturb a polynomial \( P \) realizing this SP with the pair \( (1, d - 1) \) so that all critical levels become distinct and then choose suitable values of \( t > 0 \) in the family of polynomials \( P - t \).

To prove part (2) one has to apply \( d - 1 \) times Lemma 1. When it is applied for the first time one sets \( P_1 := x - 1 \) (resp. \( P_1 := x + 1 \)) if the second entry of the SP is \(-\) (resp. \(+\)). Each time the polynomial \( P_2 \) equals \( x - 1 \) or \( x + 1 \).

3. Proof of Theorem

Part (1). For \( d = 2 \) and \( d = 3 \), the polynomials

\[
(x - 1)^2 + 1 = x^2 - 2x + 2, \quad (x + 2)((x - 2)^2 + 2) = x^3 - 2x^2 - 2x + 12
\]

realize the APs \((0,0)\) and \((0,1)\) with the SPs \( \Sigma_{1,1,1} \) and \( \Sigma_{1,2,1} \) respectively. If a polynomial \( P \) realizes a SP \( \Sigma_{m,n,q} \) (with \( n = 1 \) or \( 2 \)) with the AP \((0, d - 2)\), then the concatenation \( Q \) of \( P \) with \( x + 1 \) realizes the SP \( \Sigma_{m,n,q+1} \) with the AP \((0, d - 1)\), and the polynomial \( Q^R := x^d Q(1/x) \) (the reverted of \( Q \)) realizes the SP \( \Sigma_{q+1,n,m} \) with the AP \((0, d - 1)\). Thus by means of concatenation and reversion one can realize all SPs \( \Sigma_{m,1,q} \) and \( \Sigma_{m,2,q} \) with the AP \((0, d - 2)\).

Part (2). For \( d = 4 \), the nonrealizability of the SP \( \Sigma_{1,3,1} \) with the AP \((0, 2)\) is proved in [3]. For \( d = 5 \), the SP \( \Sigma_{1,3,2} \) is realizable with the AP \((0, 3)\), see [1]. To prove the first claim of part (2) one has to combine concatenation and reversion as in the proof of part (1) (applied to \( \Sigma_{1,3,2} \)).

Part (3). For \( d = 10 \), the polynomial

\[
(x + 1)^8(x^2 - 2.49x + 1.56) = x^{10} + 5.51x^9 + 9.64x^8 - 1.24x^7 - 25.76x^6 - 30.94x^5 - 2.24x^4 + 25.64x^3 + 24.76x^2 + 9.99x + 1.56
\]

defines the SP \( \Sigma_{3,4,4} \). (The quadratic factor is without real roots.) One can perturb its 8-fold root at \(-1\) so that the latter splits into 8 negative simple roots. Thus the perturbation realizes this SP with the AP \((0, 8)\). Similarly, for \( d = 11 \), a suitable perturbation of the polynomial

\[
(x + 1)^9(x^2 - 4.69x + 5.5) = x^{11} + 4.31x^{10} - 0.71x^9 - 35.34x^8 - 69.96x^7 - 2.94x^6 + 186.06x^5 + 335.04x^4 + 302.16x^3 + 156.79x^2 + 44.81x + 5.5
\]

realizes the SP \( \Sigma_{2,4,6} \) with the AP \((0, 9)\). As in the proof of parts (1) and (2), one deduces the realizability of all SPs as claimed by part (3) by applying concatenation and reversion.

Part (4). Suppose that the SP \( \Sigma_{1,n,q} \) with \( n \geq 4 \) is realizable by the polynomial

\[
P(x) = (x^{d-2} + c_1x^{d-3} + \cdots + c_{d-2})(x^2 - zx + y)
\]
where \( d \geq 5 \), \( z^2 < 4y \) and \( e_j > 0 \) is the \( j \)th elementary symmetric function of the moduli \( a_j \) of the negative roots of \( P \). (As \( y > 0 \), the coefficient of \( x \) of the quadratic factor must be negative, otherwise all coefficients of \( P \) will be positive, so \( z > 0 \)). Thus one obtains the conditions

\[
\begin{align*}
&z^2 < 4y, \\
e_1 - z < 0, \text{ i.e. } z > e_1 > 0, \\
e_2 - e_1z + y < 0, \text{ i.e. } e_1z > y + e_2, \\
e_3 - e_2z + e_1y < 0, \text{ i.e. } e_2z > e_3 + e_1y \quad \text{and} \\
e_4 - e_3z + e_2y < 0, \text{ i.e. } e_3z > e_4 + e_2y > 0.
\end{align*}
\]

Keeping in mind that \( e_j > 0 \), \( y > 0 \) and \( z > 0 \), one gets

\[
z > (e_4 + e_2y)/e_3 \quad \text{and} \quad z < 2\sqrt{y}, \text{ i.e.} \]

\[
2e_3\sqrt{y} > e_4 + e_2y \quad \text{and} \quad T(\sqrt{y}) := e_2y - 2e_3\sqrt{y} + e_4 < 0.
\]

The quadratic polynomial \( T \) has a positive discriminant \( e_4^2 - e_2e_4 \) (this follows from Newton’s inequalities). Hence it has two real roots, so the last inequality implies

\[
\sqrt{y} < \frac{e_3 + \sqrt{e_4^2 - e_2e_4}}{e_2}
\]

and as \( \sqrt{y} > \frac{z}{2} > \frac{e_1}{2} \), one deduces the condition

\[
e_1e_2 - 2e_3 < \sqrt{4e_3^2 - 4e_2e_4}.
\]

One has \( e_1e_2 - 2e_3 > 0 \). Indeed, every product \( a_i a_j a_k \) is encountered exactly two times in \( 2e_3 \) and three times in \( e_1e_2 \) (and there are also the products \( a_i^2a_j \) in \( e_1e_2 \)). Hence one can take squares of both hand-sides of inequality (3.3) and then divide by \( e_2 \) to obtain the condition

\[
e_1^2e_2 + 4e_4 < 4e_1e_3.
\]

We are going to show that for \( d \geq 3 \),

\[
e_1^2e_2 + 4e_4 > 4e_1e_3
\]

which contradiction proves part (4). For \( d = 3 \), one has \( e_1e_2 \geq 9e_3 \) (see Proposition 2 on page 2 of [6]). Suppose that (3.3) holds true up to degree \( d \geq 3 \). We proceed by induction on \( d \). Recall that we denote by \(-a_j\) the negative roots of \( P \). For degree \( d + 1 \), we have to show that

\[
(a_{d-1} + e_1)^2(a_{d-1}e_1 + e_2) + 4(a_{d-1}e_3 + e_4) > 4(a_{d-1} + e_1)(a_{d-1}e_2 + e_3),
\]

where \( e_j \) are the elementary symmetric functions of the quantities \( a_1, \ldots, a_{d-2} \), which is simplified to

\[
a_{d-1}^3 + 2a_{d-1}^2e_1^2 + a_{d-1}e_1^3 > 3a_{d-1}^2e_2 + 2a_{d-1}e_1e_2.
\]
Newton’s inequality $e_1^2 \geq \frac{2d}{d-1}e_2$ implies the following ones:

\[
2a_{d-1}^2 e_1^2 \geq 4a_{d-1}^2 \frac{d}{d-1}e_2 \quad \text{and}
\]

\[
a_{d-1} e_1^3 \geq \frac{2d}{d-1}a_{d-1} e_1 e_2
\]

(3.7)

From inequalities (3.7) we conclude that

\[
a_{d-1} e_1^3 + 2a_{d-1}^2 e_1^2 + a_{d-1} e_1^3 > 4a_{d-1}^2 \frac{d}{d-1}e_2 + \frac{2d}{d-1}a_{d-1} e_1 e_2
\]

> 3a_{d-1} e_1 e_2 + 2a_{d-1} e_1 e_2

which proves (3.6) and hence (3.5) as well.

4. Proof of Proposition 2

We give in detail the proof of part (1). For part (2), we point out only the differences w.r.t. the proof of part (1). These differences are only technical in character. In order to give easily references to the different parts of the proof, the latter are marked by $1^0, 2^0, \ldots, 6^0$.

Proof of part (1) of Proposition 2 $1^0$. Suppose that there exists a polynomial $P := RQ$, where

\[
R := (x + u_1) \cdots (x + u_7), \quad u_j > 0, \quad \text{and} \quad Q := x^2 + rx + s,
\]

which realizes one of the two SPs $\Sigma_{3,4,3}$ or $\Sigma_{2,4,4}$ with the AP $(0,7)$. We set $P := \sum_{j=0}^9 p_j x^j$ and $Q := (x-a)^2 + b, a \in \mathbb{R}, b \geq 0$. We show that for $b = 0$, there exists no polynomial satisfying the conditions

\[
p_3 < 0, \quad p_6 < 0, \quad \text{resp.} \quad p_4 < 0, \quad p_7 < 0.
\]

Hence this holds true also for $b > 0$ because $P = R \cdot Q|_{b=0} + bR$, and the polynomial $R$ has all coefficients positive. This in turn implies that for $b \geq 0$, there exists no polynomial $P$ realizing the SP $\Sigma_{3,4,3}$ or $\Sigma_{2,4,4}$. So from now on we concentrate on the case $b = 0$.

$2^0$. Suppose that a polynomial $P$ with $b = 0$ and $u_1 \geq u_2 \geq \cdots \geq u_7 \geq 0$ satisfying the left or right couple of inequalities (4.8) exists. We make the change of variables $x \mapsto u_1 x$ and after this we multiply $P$ by $(1/u_1)^9$ (these changes preserve the signs of the coefficients), so now we are in the case $u_1 = 1$. Denote by $\Delta \subset \mathbb{R}_7^+ = \{(u_2, u_3, \ldots, u_7, a)\}$ the set on which one has conditions (4.8). The closure $\overline{\Delta}$ of this set is compact. Indeed, one has $p_1 \geq 0$ hence

\[
1 + u_2 + \cdots + u_7 - 2a \geq 0 \quad \text{and} \quad u_j \leq 1 \quad \text{hence} \quad a \in [0,7/2].
\]

The set $\overline{\Delta}$ can be stratified according to the multiplicity vector of the variables $(u_2, \ldots, u_7)$ and the possible equalities $u_j = 0, u_i = 1$ and/or $a = 0$. Suppose that the set $\overline{\Delta}$ contains a polynomial satisfying the inequalities (4.8).

Remarks 3. (1) For this polynomial one has $a > 0$, otherwise all its coefficients are nonnegative. One has also $u_j > 0, j = 2, \ldots, 7$. Indeed, in the case of $\Sigma_{3,4,3}$ (resp. $\Sigma_{2,4,4}$), if three or more (resp. if four or more) of the variables $u_j$ are 0, then the polynomial $P$ has less than two sign changes in the sequence of its coefficients.
and by the Descartes rule of signs $P$ cannot have two positive roots counted with multiplicity. For $\Sigma_{3,4,3}$, if exactly one or two of the variables $u_j$ equal 0, then the polynomial $P$ is the product of $x$ with a polynomial defining the SP $\Sigma_{3,4,2}$ or of $x^2$ with a polynomial defining the SP $\Sigma_{3,4,1}$. However these SPs are not realizable with the APs $(0, 6)$ or $(0, 5)$ respectively, see [4] and [2]. For $\Sigma_{2,4,4}$, if exactly one, two or three of the variables $u_j$ equal 0, then $P$ is the product of $x$, $x^2$ or $x^3$ with a polynomial defining respectively the SP $\Sigma_{2,4,3}$, $\Sigma_{2,4,2}$ or $\Sigma_{2,4,1}$ which is not realizable with the AP $(0, 6)$, $(0, 5)$ or $(0, 4)$, see [4], [2] and [1].

(2) The set $\Delta$ being compact the quantity $p_3 + p_6$, resp. $p_4 + p_7$, attains its minimum $-\delta$ on it ($\delta > 0$). Consider the set $\Delta^* \subset \Delta$ on which one has $p_3 + p_6 \leq -\delta/2$, resp. $p_4 + p_7 \leq -\delta/2$. On this set one has $a \geq 2^{-9}\delta$. Indeed, $P = x^2R - 2axR + a^2R$, so any coefficient of $P$ is not less than $-2a\sigma$, where $\sigma$ is the sum of all coefficients of $R$ (they are all nonnegative); clearly $\sigma \leq 2^7$ (follows from $u_j \in [0,1]$).

(3) There exists $\delta_3 > 0$ such that on the set $\Delta^*$, one has also $u_j \geq \delta_3$. This follows from part (1) of the present remarks.

3$^0$. We need some technical lemmas:

**Lemma 2.** The minimum of the quantity $p_3 + p_6$, resp. $p_4 + p_7$, is not attained at a point of the set $\Delta^*$ with three or more distinct and distinct from 1 among the quantities $u_j$, $2 \leq j \leq 7$.

The lemmas used in the proof of part (1) of Proposition [2] are proved after the proof of part (1).

**Lemma 3.** Conditions (4,8) fail for $u_1 = u_2 = \cdots = u_7 = 1$ and any $a > 0$.

Thus to prove Proposition[2] we have to consider only the case when exactly one or two of the quantities $u_j$ are distinct from 1. We use the following result:

**Lemma 4.** For $d \geq 4$, set $P := RQ$, where $R := \prod_{i=1}^{d-2}(x + u_i)$, $u_i > 0$, $Q := (x - a)^2$. Then the coefficients $p_j$ of $P$, $j = 2, \ldots, d - 2$, are quadratic polynomials in $a$ with positive leading coefficients and with two distinct positive roots.

4$^0$. Further we consider several different cases according to the multiplicity of $u_{j_0}$, the smallest of the variables $u_j$. In the proofs we use linear changes $x \mapsto \chi x$, $\chi > 0$, followed by $P \mapsto \chi^{-9}P$. These changes preserve the signs of the coefficients; the condition $u_1 = 1$ is lost and the condition $u_{j_0} = 1$, $j_0 \neq 1$, is obtained. The aim of this is to have more explicit computations. In all the cases the polynomial $R$ is of the form $R = (x + 1)^{s_1}(x + v)^{s_2}(x + w)^{s_3}$, $s_1 + s_2 + s_3 = 7$, and one has $v > 1$, $w > 1$, but $v$ and $w$ are not necessarily distinct and we do not suppose that $v > w$ or $v < w$ (which permits us to assume that $s_2 \geq s_3$). Allowing the equality $v = w$ means treating together cases of exactly two or exactly three distinct quantities $u_j$ (counting also $u_1 = 1$). We list the triples $(s_1, s_2, s_3)$ defining the cases:

$$
(5, 1, 1) , \quad (4, 2, 1) , \quad (3, 3, 1) , \quad (3, 2, 2) , \quad (2, 4, 1) , \\
(2, 3, 2) , \quad (1, 5, 1) , \quad (1, 4, 2) \quad \text{and} \quad (1, 3, 3) .
$$

The cases when there are exactly two different quantities $u_j$ one of which is $u_1 = 1$ can be coded in a similar way. E.g. $(5, 2)$ means that $R = (x + 1)^5(x + u)^2$, $u > 1$.

The nonrealizability of these cases follows automatically from the one of the above
9 ones (when \( v \) and \( w \) coalesce), with the only exception of \( R = (x + 1)^6(x + w) \) (the case \((6, 1)\)).

**Lemma 5.** Conditions (4.8) fail in case \((6, 1)\).

5

We consider the SP \( \Sigma_{2,4,4} \) first. We compute using MAPLE the resultant \( \text{Res}(p_4, p_7, a) \) as a function of \( v \) and \( w \). Then we set \( v := 1 + V, \ w := 1 + W, \ V > 0, \ W > 0 \). In all 9 cases this resultant is a polynomial in \( V \) and \( W \) with all coefficients positive. Hence for no value of \( V > 0 \) and \( W > 0 \) do the coefficients \( p_4 \) and \( p_7 \) vanish together.

In all 9 cases, the leading coefficients of \( p_4 \) and \( p_7 \) considered as quadratic polynomials in \( a \) are positive. In fact, they are polynomials in \( v \) and \( w \) with all coefficients positive. For \( v = w = 2 \), we compute the two roots \( y_1 < y_2 \) of \( p_4 \) and the two roots \( y_3 < y_4 \) of \( p_7 \). In all 9 cases, one has \( y_1 < y_2 < y_3 < y_4 \). By continuity, these inequalities hold true for all values of \( v > 1 \) and \( w > 1 \). Hence the intervals \((y_1, y_2)\) and \((y_3, y_4)\) on which \( p_4 \) and \( p_7 \) are negative do not intersect for any \( v > 1, \ w > 1 \). This proves the proposition in the case of \( \Sigma_{2,4,4} \).

6

Consider now the SP \( \Sigma_{3,4,3} \). Recall that the polynomials \( P(x) \) and \( x^9 P(1/x) \) have one and the same numbers of positive and negative roots. Their roots are mutually reciprocal and they define the same SP. Hence the non-realizability of the case \((5, 1, 1)\) (resp. \((4, 2, 1)\), \((3, 3, 1)\), or \((3, 2, 2)\)) implies the one of \((1, 5, 1)\) (resp. \((2, 4, 1)\) and \((1, 4, 2)\), or \((1, 3, 3)\), or \((2, 3, 2)\)).

As in the case of \( \Sigma_{2,4,4} \), we express \( \text{Res}(p_4, p_6, a) \) as a polynomial of \( v \) and \( w \), and then of \( V \) and \( W \). In cases \((5, 1, 1)\), \((4, 2, 1)\) and \((3, 2, 2)\), this resultant has a single monomial with negative coefficient, this is \( UV \). We give the monomials \( VW \), \( V^2 \) and \( W^2 \) for these three cases:

\[
\begin{align*}
(5, 1, 1) & \quad -9408VW + 28224V^2 + 28224W^2, \\
(4, 2, 1) & \quad -18816VW + 47040V^2 + 28224W^2, \\
(3, 2, 2) & \quad -37632VW + 47040V^2 + 47040W^2.
\end{align*}
\]

The discriminants of these quadratic homogeneous polynomials are negative hence they are nonnegative (and positive for \( V > 0, \ W > 0 \)). In the case of \((3, 3, 1)\), there are exactly two monomials with negative coefficients, namely \( VW \) and \( V^2W \). The resultant equals

\[
(-28224VW + 56448V^2 + 28224W^2) + V(-42336VW + 127008V^2 + 282240W^2) + \cdots
\]

(we skip all other monomials; their coefficients are positive). The two quadratic homogeneous polynomials have negative discriminants, so they are positive for \( V > 0, \ W > 0 \).

The rest of the reasoning goes by exact analogy with the case of \( \Sigma_{2,4,4} \).

\( \square \)

**Proof of Lemma** Denote by \( v_1, \ v_2 \) and \( v_3 \) three distinct and distinct from 1 of the variables \( u_j \). We prove that one can choose \( v_1^*, \ v_2^*, \ v_3^*, \ a^* \in \mathbb{R} \) such that the infinitesimal change \( v_j \mapsto v_j + \varepsilon v_j^*, \ j = 1, 2, 3, \ a \mapsto a + \varepsilon a^* \), \( \varepsilon > 0 \), results in \( p_\mu \mapsto p_\mu + \varepsilon p_\mu^* + o(\varepsilon), \ p_\nu \mapsto p_\nu + \varepsilon p_\nu^* + o(\varepsilon) \), where \((\mu, \nu) = (3, 6) \) or \((4, 7)\) and \( p_\mu^* < 0, \ p_\nu^* < 0 \). Hence locally the quantity \( p_\mu + p_\nu \) is not minimal.
Set $P := (x + v_1)^{α_1}(x + v_2)^{α_2}(x + v_3)^{α_3}(x - a)^2P^†$, where $a$, $-v_1$, $-v_2$ and $-v_3$ are not roots of $P^†$ and $α_j$ are the multiplicities of roots $-v_j$ of $P$. Set $P_{v_i} := P/(x + v_i)$, $P_a := P/(x - a)$, $P_{v_i,v_j} := P/((x + v_i)(x + v_j))$, $P_{a,v_i,v_j} := P/((x + a)(x + v_i)(x + v_j))$ etc. Then the above infinitesimal change transforms $P$ into

$$P + ε\tilde{P} + o(ε) , \text{ where } \tilde{P} := \sum_{j=1}^{3} \alpha_j v_j^*P_{v_j} - 2a^* P_a .$$

We show that one can choose $v_j^*$ and $a^*$ such that the coefficients of $x^μ$ and $x^ν$ of the polynomial $\tilde{P}$ (where $(μ, ν) = (3, 6)$ or $(4, 7)$) are both negative from which the lemma follows. To this end we observe that each of the polynomials $P_{v_i}$ and $(x^B + P)$ of the polynomials $P_{v_i,v_j}$ has all roots real and by Rolle’s theorem this is the case of $(x^B + P)v_i$ for $B \neq 0$, the first two of them are not collinear. As $E \neq 0$ (see parts (2) and (3) of Remarks 3), then for $B = 0$, the second and fourth of these vectors are not collinear and the choice of $v_j^*$ and $a^*$ is possible.

If $(μ, ν) = (4, 7)$, then the 2-vectors of coefficients of $x^4$ and $x^7$ equal $(A, 0)$, $(B, 0)$, $(C, 1)$ and $(D, A)$. One has either $A \neq 0$ or $B \neq 0$. Indeed, the polynomial $P^o$ has all roots real and by Rolle’s theorem this is the case of $(P^o)^′$ and $(P^o)^″$ as well. If $A = B = C = D = 0 \neq E$ (resp. $A = B = C = 0 \neq D$ or $A = B = 0 \neq C$), then $P^o$ (resp. $(P^o)^′$ or $(P^o)^″$) has not all roots real. Thus either $(A, 0)$, $(C, 1)$ or $(B, 0)$, $(C, 1)$ are not collinear and the choice of $v_j^*$ and $a^*$ is possible.

**Proof of Lemma 3** For the polynomial $(x + 1)^7(x - a)^2$, we list its coefficients $p_3$, $p_4$, $p_6$ and $p_7$ and their roots:

| $p_3$ | $= 7 - 42a + 35a^2$ | $p_4$ | $= 21 - 70a + 35a^2$ |
|-------|-----------------|-------|-----------------|
| 0.2   | 1               | 0.36… | 1.63…          |
| $p_6$ | $= 35 - 42a + 7a^2$ | $p_7$ | $= 21 - 14a + a^2$ |
| 1     | 5               | 1.70… | 12.2…          |

Hence for no value of $a \geq 0$ does one have the left or the right two of conditions (4.8) together.

**Proof of Lemma 4** Set $R := r_{d-2}x^{d-2} + r_{d-3}x^{d-3} + \ldots + r_0$, $r_j > 0$, $r_{d-2} = 1$. The polynomial $R$ has $d - 2$ negative roots. Hence Newton’s inequalities hold true:

$$(4.9) \left( \frac{r_k}{\binom{d - 2}{k}} \right)^2 \geq \left( \frac{r_{k-1}}{\binom{d - 2}{k-1}} \right) \left( \frac{r_{k+1}}{\binom{d - 2}{k+1}} \right) , k = 1, \ldots, d - 3 .$$

The coefficient $p_{k+1}$ equals $a^2r_{k+1} - 2ar_k + r_{k-1}$, $k = 1, \ldots, d - 3$, $r_{k+1} > 0$. This quadratic polynomial has two distinct positive roots if and only if $r_k^2 > r_{k-1}r_{k+1}$. 

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These inequalities result from (4.10) because \(\binom{d-2}{k} > \binom{d-2}{k+1} (d-2-k)\) (the latter inequality is equivalent to \((k+1)/(d-1-k) > 1\), which is true). □

**Proof of Lemma 5.** In case (6,1), with \(P = (x + 1)^0(x + w)(x - a)^2\), one has

\[
\begin{align*}
p_3 &= 1 + 6w - 12a - 30wa + 15a^2 + 20wa^2, \\
p_4 &= 6 + 15w - 30a - 40wa + 20a^2 + 15wa^2, \\
p_5 &= 20 + 15w - 30a - 12wa + 6a^2 + wa^2 \quad \text{and} \\
p_7 &= 15 + 6w - 12a - 2wa + a^2.
\end{align*}
\]

For \(w = 1\), the roots of \(p_4\) (resp. of \(p_7\)) equal 0.36 . . . and 1.63 . . . (resp. 1.70 . . . and 12.29 . . .). As \(\text{Res}(p_4, p_7, a) = 7056 + 2520w + 540w^2 + 3960w^3 + 1800w^4\) has no positive roots, for any \(w > 0\) fixed, the two intervals of values of \(a\) for which \(p_4 < 0\) or \(p_7 < 0\), do not intersect. Hence the couple of conditions \(p_4 < 0, p_7 < 0\) fails.

One has \(\text{Res}(p_3, p_6, a) = 7056(w - 1)^2(w + 1)^2\), so only for \(w = 1\) do the polynomials \(p_3\) and \(p_6\) have a root in common. For \(w = 1/2, w = 1\) and \(w = 2\), the roots of \(p_3\) and \(p_6\) equal respectively

\[
\begin{align*}
w &= 1/2 \ 0.17\ldots, \ 0.90\ldots \ \text{and} \ 0.91\ldots, \ 4.62\ldots; \\
w &= 1 \quad 0.2, \ 1 \ \text{and} \ 1, \ 5; \\
w &= 2 \quad 0.21\ldots, \ 1.09\ldots \ \text{and} \ 1.10\ldots, \ 5.64\ldots.
\end{align*}
\]

Hence again the intervals of values of \(a\) for which \(p_3 < 0\) or \(p_6 < 0\) do not intersect and the couple of conditions \(p_3 < 0, p_6 < 0\) fails. □

**Proof of part (2) of Proposition 3.** In the analog of part 1° of the proof of part (1), we set \(R := (x + u_1) \cdots (x + u_8), u_j > 0\), and the analog of inequalities (4.8) reads \(p_5 < 0, p_8 < 0\).

2°. In the analog of part 2° we make the change of variables \(x \mapsto u_1x\) and then we multiply \(P\) by \((1/u_1)^{10}\). We denote by \(\Delta \subset \mathbb{R}_+^8 = \{(u_2, u_3, \ldots, u_8, a)\}\) the set on which one has the conditions \(p_5 < 0, p_8 < 0\). On the closure \(\overline{\Delta}\) of this set one has \(p_1 \geq 0\) hence

\[1 + u_2 + \cdots + u_8 - 2a \geq 0 \text{ and } u_j \leq 1 \text{ hence } a \in [0, 4].\]

The analog of Remarks 3 reads:

**Remarks 4.** (1) One has \(u_j > 0, j = 2, \ldots, 8\). Indeed, if exactly one of the quantities \(u_j\) is 0, then \(P = XY\), where the polynomial \(Y\) defines the SP \(\Sigma_{2,4,4}\) which by part (1) of Proposition 3 is impossible. If more than one of the quantities \(u_j\) is 0, then see part (1) of Remarks 3 about \(\Sigma_{2,4,4}\).

(2) In the proof of part (2) of Proposition 3 we define the set \(\Delta^* \subset \overline{\Delta}\) as the one on which one has \(p_5 + p_8 < \delta/2\). On this set one has \(a \geq 2^{10} \delta\). Indeed, as \(P = x^2R - 2axR + a^2R\), any coefficient of \(P\) is not less than \(-2a\sigma\), where \(\sigma\) is the sum of all coefficients of \(R\) (they are all nonnegative); clearly \(\sigma \leq 2^8\) (follows from \(u_j \in [0, 1]\)).

3°. The analog of Lemma 2 reads: *The minimum of the quantity \(p_5 + p_8\) is not attained at a point of the set \(\Delta^*\) with three or more distinct and distinct from 1 among the quantities \(u_j, 2 \leq j \leq 8\).*
The proof is much the same as the one of Lemma 2. One sets \((\mu, \nu) = (5, 8)\). Each of the polynomials \(P_v\) and \(P_a\) is a linear combination of \(P^0 := P_{v_1, v_2, v_3, a} := x^6 + Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F, xP^0, x^2P^0\) and \(x^3P_{v, w, a}\). The 2-vectors of coefficients of \(x^6\) and \(x^5\) of the polynomials \(P^0, xP^0, x^2P^0\) and \(x^3P^0\) equal \((A, 0), (B, 0), (C, 1)\) and \((D, A)\) respectively. If \(A \neq 0\) or \(B \neq 0\), there are two noncollinear among the first three of these vectors and the choice of \(v^*_s\) and \(a^*\) is possible. If \(A = B = 0\), then, as \(F \neq 0\), either the polynomial \(P^0\) or one of its derivatives is not with all roots real which is a contradiction.

The analog of Lemma 3 reads: Conditions \(p_5 < 0, p_8 < 0\) fail for \(u_1 = \cdots = u_8 = 1\) and any \(a > 0\).

Here’s the proof of this. For the polynomial \((x+1)^8(x-a)^2\), we list its coefficients \(p_5, p_8\) and their roots:

\[
p_5 = 28(2 - 5a + 2a^2) , \quad p_8 = 28 - 16a + a^2 ,
\]

\[
0.5 \quad 2 \quad 2 \quad 14
\]

Hence for no value of \(a \geq 0\) does one have \(p_5 < 0, p_8 < 0\).

We remind that Lemma 3 is formulated for any \(d \geq 4\).

4\textsuperscript{o}. In the analog of part 4\textsuperscript{o} of the proof, one has \(R = (x + 1)^s_1(x + v)^s_2(x + w)^s_3, s_1 + s_2 + s_3 = 8\), and one has to consider the following cases of exactly three different quantities \(u_j\):

\[
(6, 1, 1) , \quad (5, 2, 1) , \quad (4, 3, 1) , \quad (4, 2, 2) , \quad (3, 4, 1) , \quad (3, 3, 2) \\
(2, 5, 1) , \quad (2, 4, 2) , \quad (2, 3, 3) , \quad (1, 6, 1) , \quad (1, 5, 2) \quad \text{and} \quad (1, 4, 3)
\]

The cases with exactly two different quantities \(u_j\) are treated in the same way. The exceptional case is the one with \(R = (x + 1)^7(x + w)\) (the case \((7, 1)\)).

**Lemma 6.** The conditions \(p_5 < 0, p_8 < 0\) fail in case \((7, 1)\).

**Proof.** Set \(P := (x + 1)^7(x + w)(x - a)^2\). Then

\[
p_5 = 21 + 35w - 70a - 70wa + 35a^2 + 21wa^2 \quad \text{and} \quad p_8 = 21 + 7w - 14a - 2wa + a^2 .
\]

One has \(\text{Res } (p_5, p_8, a) = 3969(w + 2)^2(w - 1)^2\). We list the roots of \(p_5\) and \(p_8\) for \(w = 1/2, w = 1\) and \(w = 2\):

\[
\begin{align*}
&w = 1/2 \quad 0.45 \ldots , \quad 1.850 \ldots \text{ and } \quad 1.865 \ldots , \quad 13.13 \ldots ; \\
&w = 1 \quad 0.5 \quad 2 \quad \text{and} \quad 2 \quad 14 ; \\
&w = 2 \quad 0.54 \ldots , \quad 2.18 \ldots \text{ and } \quad 2.21 \ldots , \quad 15.78 \ldots .
\end{align*}
\]

As in the proof of Lemma 5, we conclude that the conditions \(p_5 < 0, p_8 < 0\) fail for \(w > 0\). \(\square\)

5\textsuperscript{o}. We compute \(\text{Res } (p_5, p_8, a)\) as a function of \(v\) and \(w\) and then set \(v := 1 + V, w := 1 + W\). Our aim is to show that in all 12 cases, the leading coefficients of \(p_5\) and \(p_8\) considered as quadratic polynomials in \(a\) are positive. The rest of the reasoning is done by analogy with part 5\textsuperscript{o} of the proof of part (1) of Proposition 2.

6\textsuperscript{o}. It is in the analog of 6\textsuperscript{o} that there is much more technical work to be done. Of the twelve cases listed in 4\textsuperscript{o}, in three there is a single monomial with a negative
coefficient, and this is $UV$. We list the coefficients of the monomials $UV$, $U^2$ and $V^2$ of the cases $(6, 1, 1)$, $(5, 2, 1)$ and $(4, 2, 2)$ respectively:

$$(-10206, 35721, 35721), \ (-20412, 61236, 35721), \ (-40824, 61236, 61236).$$

Everywhere in $6^0$ quadratic and biquadratic polynomials have negative discriminants. There are four cases in which exactly two monomials have negative signs, namely $(4, 3, 1)$, $(3, 4, 1)$, $(3, 3, 2)$ and $(2, 4, 2)$ in which we give only the monomials forming quadratic homogeneous polynomials with negative discriminants (multiplied by 1 or $U$); we skip all other monomials (their coefficients are positive):

$$(-30618UV + 76545U^2 + 35721V^2) + U(-10206UV + 221130U^2 + 91854V^2)$$

$$(-40824UV + 81648U^2 + 35721V^2) + U(-81648UV + 326592U^2 + 122472V^2)$$

$$(-61236UV + 76545U^2 + 61236V^2) + U(-20412UV + 221130U^2 + 81648V^2)$$

$$(-81648UV + 81648U^2 + 61236V^2) + U(-163296UV + 326592U^2 + 108864V^2)$$

In the cases $(2, 5, 1)$ and $(1, 6, 1)$ there are four and five negative monomials respectively. These cases are treated in a similar way:

$$(-51030UV + 76545U^2 + 35721V^2) + U(-187110UV + 391230U^2 + 153090V^2) + U^2(-245430UV + 868725U^2 + 297270V^2) + U^3(-86670UV + 1094472U^2 + 352350V^2)$$

and

$$(-6804UV + 6804U^2 + 3969V^2) + U(-22680UV + 28728U^2 + 12474V^2) + U^2(-29052UV + 50436U^2 + 15849V^2) + U^3(-3252UV + 24628U^2 + 3672V^2) + U^4(-16848UV + 47088U^2 + 10368V^2).$$

In the case $(2, 3, 3)$, there are four negative monomials which we include in polynomials as follows:

$$(-91854UV + 76545U^2 + 76545V^2) + (-59778U^2V^2 + 273375U^4 + 273375V^4)$$

$$(-30618U^2V + 221130U^3 + 221130V^3 - 30618UV^2).$$

For the third polynomial in brackets its corresponding inhomogeneous polynomial

$$-30618x^2 + 221130x^3 + 221130 - 30618x$$

has one negative and two complex conjugate roots. For a univariate real polynomial with positive leading coefficient and having only negative and complex conjugate roots we say that it is of type $P$. It is clear that the homogeneous polynomial corresponding to a type $P$ univariate polynomial (we say that it is also of type $P$) is nonnegative.

In the case $(1, 5, 2)$, there are seven negative monomials:
\((-102060UV + 76545U^2 + 61236V^2) + U(-374220UV + 391230U^2 + 136080V^2)\)
\[\] + \((-490860U^3V + 868725U^4 + 10530U^2V^2 + 369360UV^3 + 79704V^4)\]
\[\] + \(U^2(513540U^3 - 210600UV^2 - 173340U^2V + 109472U^3)\)
\[\] + \(U^2(-215190U^2V^2 + 85545U^4 + 379215V^4)\)
\[\] + \(U^4(-64116UV + 300060U^2 + 176760V^2)\).

The third and fourth of the polynomials in brackets are of type P hence nonnegative.

Finally, in the case \((1, 4, 3)\) we have also seven negative monomials:
\((-122472UV + 81648U^2 + 76545V^2) + \(-383940U^2V^2 + 565056U^4 + 273375V^4)\)
\[\] + \((326592U^3V + 244944U^2V - 40824UV^2 + 221130V^3)\)
\[\] + \(U(-359649U^2V^2 + 552096U^4 + 557928V^4)\)
\[\] + \((75816U^3V^3 + 332928U^6 + 79065V^6)\)
\[\] + \(U(-15066U^3V^3 + 126720U^6 + 138096V^6)\).

The third and the last two polynomials in brackets are of type P. \(\square\)

5. Proof of Theorem \[\text{[2]}\]

Consider the polynomial \(P = (x + 1)^{d-2}(x^2 - zx + y)\), where the quadratic factor
has no real roots, i.e. \(z^2 < 4y\). Hence \(y > 0\) and \(z > 0\) (otherwise all coefficients of \(P\)
must be positive). If the polynomial \(P\) defines the SP \(\Sigma_{m,n,q}\), then any perturbation
of \(P\) with \(d - 2\) distinct negative roots close to \(-1\) defines also the SP \(\Sigma_{m,n,q}\). We
expand \(P\) in powers of \(x\):
\[P := x^d + \sum_{j=1}^{d} p_j x^{d-j},\]
where \(p_j = C^j_{d-2} - C^j_{d-2}z + C^j_{d-2}y\) with \(C^\mu = 0\) if \(\mu < \nu\). The coefficients of \(P\)
define the SP \(\Sigma_{m,n,q}\), so
\[p_j > 0 \text{ for } j = 1, \ldots, m - 1 \text{ and for } j = m + n, \ldots, d, \text{ and} \]
\[p_j < 0 \text{ for } j = m, m + 1, \ldots, m + n - 1.\]
The latter inequalities (combined with \(z < 2\sqrt{y}\)) yield:
\[C^j_{d-2} + C^j_{d-2} < C^j_{d-2} < 2C^j_{d-2} \sqrt{y}, \quad j = m, m + 1, \ldots, m + n - 1.\]
This means that:
\[C^j_{d-2} \frac{C^j_{d-2} - \sqrt{j-1}}{C^j_{d-2}} < \sqrt{y} < C^j_{d-2} \frac{C^j_{d-2} + \sqrt{j-1}}{C^j_{d-2}}, \quad j = m, m + 1, \ldots, m + n - 1. \tag{5.10}\]
where $\delta_{j-1} := (C_{d-2}^{j-1})^2 - C_{d-2}^{j-2}C_{d-2}^{j} > 0$. Indeed, the polynomial $C_{d-2}^j - 2C_{d-2}^{j-1}\sqrt{y} + C_{d-2}^{j-2}y$ (quadratic in $\sqrt{y}$) has a positive discriminant $\delta_{j-1}$ hence its value is negative precisely when $\sqrt{y}$ is between its roots. Set

$$Q^\pm(k) := (C_{d-2}^k \pm \sqrt{\delta_k})/C_{d-2}^{k-1}.$$ 

**Lemma 7.** One has $Q^\pm(k) = \frac{d-k-1}{k}(1 \pm A(k))$, where

$$A(k) = \sqrt{1 - \frac{k(d-k-2)}{(k+1)(d-k-1)}} = \sqrt{\frac{(d-1)}{(k+1)(d-k-1)}}.$$ 

**Lemma 8.** The quantities $Q^\pm(k)$ are decreasing functions in $k$ (for $k=1, 2, \ldots, \lfloor \frac{d}{2}\rfloor$).

Lemmas 7 and 8 are proved after the proof of Theorem 2. It follows from Lemma 8 that one can find a value of $y$ satisfying conditions (5.10) if

$$(5.12)\quad Q^-(m-1) < Q^+(m+n-2)$$

or equivalently

$$(5.13)\quad a-f < aB + fG,$$

where

$$a = \frac{d-m}{m-1} > 0 \quad B = \sqrt{1 - \frac{(m-1)(d-m-1)}{m(d-m)}}$$

$$f = \frac{d-m-n+1}{m+n-2} > 0 \quad G = \sqrt{1 - \frac{(m+n-2)(d-m-n)}{(m+n-1)(d-m-n+1)}}$$

One has

$$a-f = \frac{(n-1)(d-1)}{(m+n-2)(m-1)} > 0$$

which permits to take squares in (5.13) to obtain the condition :

$$(5.14)\quad H := \frac{(a-f)^2 - (aB)^2 - (fG)^2}{2af} < GB$$

which is equivalent to (5.12). If $H < 0$, then (5.14) is trivially true. If $H \geq 0$, then (5.14) is equivalent to $H^2 < (GB)^2$, i.e. to (1.2) (the latter equivalence can be proved using MAPLE). Theorem 2 is proved.
Proof of Lemma 7

\[ \delta_k = (C_{d-2}^k)^2 - C_{d-2}^{k-1}C_{d-2}^{k+1} \]

\[ = \left( \frac{(d-2) \ldots (d-k-1)}{k!} \right)^2 - \left( \frac{(d-2) \ldots (d-k)}{(k-1)!} \right) \cdot \left( \frac{(d-2) \ldots (d-k-2)}{(k+1)!} \right) \]

\[ = \left( \frac{(d-2) \ldots (d-k)}{(k+1)!} \right)^2 \cdot \left( \frac{(d-2) \ldots (d-k)}{(k+1)!} \right) \cdot \left( \frac{(d-2) \ldots (d-k-2)}{(k+1)!} \right) \]

\[ = \left( \frac{(d-2)!}{(d-k)!} \right)^2 \cdot (k+1)(d-1)(d-k-1). \]

We substitute this expression of \( \delta_k \) in (5.11) to obtain

\[ Q^\pm(k) = \frac{\frac{(d-k-1)!}{(d-2)!}}{\frac{(d-2)!}{(d-k-1)!} \pm \sqrt{\delta_k}} \]

\[ = \frac{d-k-1}{k} \pm \frac{1}{k(k+1)} \sqrt{(k+1)(d-1)(d-k-1)} \]

\[ = \frac{d-k-1}{k} \pm \sqrt{\left( \frac{(d-k-1)^2}{k^2} - \frac{(d-k-1)(d-k-2)}{k(k+1)} \right)} \]

\[ = \frac{d-k-1}{k} \left( 1 \pm \sqrt{1 - \frac{k(d-k-2)}{(k+1)(d-k-1)}} \right). \]

\[ \square \]

Proof of Lemma 8 Both factors \( \frac{d-k-1}{k} \) and \( 1 + A(k) \) of \( Q^+(k) \) are decreasing in \( k \) (for \( k = 1, 2, \ldots, \left[ \frac{d}{2} \right] \)) hence \( Q^+(k) \) is also decreasing. We represent the quantity \( Q^-(k) \) in the form

\[ Q^-(k) = \left( \frac{d-k-2}{k+1} \right) / \left( 1 + \sqrt{\frac{d-1}{(k+1)(d-k-1)}} \right) \]

The inequality \( Q^-(k) > Q^-(k+1) \) is equivalent to

\[ \frac{d-k-2}{k+1} + \frac{d-k-2}{k+1} \sqrt{\frac{d-1}{(k+2)(d-k-2)}} > \frac{d-k-3}{k+2} + \frac{d-k-3}{k+2} \sqrt{\frac{d-1}{(k+1)(d-k-1)}} \]

This follows from \( \frac{d-k-2}{k+1} > \frac{d-k-3}{k+2} \), \( \frac{1}{(k+1)\sqrt{k+2}} > \frac{1}{(k+2)\sqrt{k+1}} \) and \( (d-k-2)(d-k-1) > (d-k-3)^2 \). \[ \square \]
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