ON THE GARDNER-ZVAVITCH CONJECTURE: SYMMETRY IN THE INEQUALITIES OF BRUNN-MINKOWSKI TYPE

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ABSTRACT. In this paper, we study the conjecture of Gardner and Zvavitch from [23], which suggests that the standard Gaussian measure $\gamma$ enjoys $\frac{1}{n}$-concavity with respect to the Minkowski addition of symmetric convex sets. We prove this fact up to a factor of 2: that is, we show that for symmetric convex $K$ and $L$,

$$\gamma(\lambda K + (1 - \lambda)L) \geq \lambda \gamma(K)^{\frac{1}{n}} + (1 - \lambda)\gamma(L)^{\frac{1}{n}}.$$ 

Further, we show that under suitable dimension-free uniform bounds on the Hessian of the potential, the log-concavity of even measures can be strengthened to $p$-concavity, with $p > 0$, with respect to the addition of symmetric convex sets.

1. INTRODUCTION

We shall work in the Euclidean $n$-dimensional space $\mathbb{R}^n$. The unit ball shall be denoted by $B_2^n$ and the unit sphere by $S^{n-1}$. The Lebesgue volume of a measurable set $A \subset \mathbb{R}^n$ is denoted by $|A|$.

Recall that a measure $\mu$ on $\mathbb{R}^n$ is called log-concave if for every pair of Borel measurable sets $K$ and $L$,

$$\mu(\lambda K + (1 - \lambda)L) \geq \mu(K)^{\lambda} \mu(L)^{1 - \lambda}. \quad (1)$$

More generally, $\mu$ is called $p$-concave for $p \geq 0$, if

$$\mu(\lambda K + (1 - \lambda)L)^p \geq \lambda \mu(K)^p + (1 - \lambda)\mu(L)^p. \quad (2)$$

Log-concavity corresponds to the limiting case $p = 0$. By Hölder’s inequality, if $p > q$, and a measure is $p$-concave, it is also $q$-concave.

Borell’s theorem ensures, roughly, that a measure with log-concave density is log-concave [5]. Further, the celebrated Brunn-Minkowski inequality states that for all Borel sets $K$ and $L$, and for every $\lambda \in [0, 1]$,

$$|\lambda K + (1 - \lambda)L|^{\frac{1}{n}} \geq \lambda |K|^{\frac{1}{n}} + (1 - \lambda)|L|^{\frac{1}{n}}. \quad (3)$$

See more on the subject in Gardner’s survey [22], and some classical textbooks in Convex Geometry, e.g. Bonnesen, Fenchel [4], Schneider [40]; see also Figalli, Maggi, Pratelli [20], Figalli, Jerison [21] for a sharpening of (3), effective in the case when $A$ and $B$ have very different shape. In view of Hölder’s inequality, (3) implies the log-concavity of the Lebesgue measure:

$$|\lambda K + (1 - \lambda)L| \geq |K|^{\lambda} |L|^{1 - \lambda}. \quad (4)$$

The homogeneity of the Lebesgue measure ensures that, in fact, (4) is equivalent to (3). However, this is not the case for general (non-homogenous) measures $\mu$ on $\mathbb{R}^n$: the log-concavity property (1) does not entail the stronger inequality

$$\mu(\lambda K + (1 - \lambda)L)^{\frac{1}{n}} \geq \lambda \mu(K)^{\frac{1}{n}} + (1 - \lambda)\mu(L)^{\frac{1}{n}}. \quad (5)$$

Date: August 10, 2018.

2010 Mathematics Subject Classification. Primary: 52.

Key words and phrases. Convex bodies, log-concave, Brunn-Minkowski.
In fact, (5) cannot hold in general for a probability measure: if \( K \) is fixed, and \( L \) is shifted far away from the origin, then the left hand side of (5) is close to zero (thanks to the decay of the measure at infinity), while the right hand side is bounded from below by a fixed number.

Gardner and Zvavitch conjectured [23] that for the standard Gaussian measure \( \gamma \), for any pair of symmetric convex sets \( K \) and \( L \), and for any \( \lambda \in [0, 1] \), one has

\[
\gamma(\lambda K + (1 - \lambda)L)^{\frac{1}{n}} \geq \lambda \gamma(K)^{\frac{1}{n}} + (1 - \lambda)\gamma(L)^{\frac{1}{n}}.
\]

In fact, initially they considered the possibility that (6) may hold for sets \( K \) and \( L \) containing the origin, but a counterexample to that was constructed by Nayar and Tkocz [35]. Symmetry, on the other hand, seems to play crucial role in the improvement of isoperimetric type inequalities. One simple example when such phenomenon occurs is Poincaré inequality: for any smooth \( 2\pi \)-periodic function \( \psi \) on \( \mathbb{R} \),

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi^2 \, dx - \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi \, dx \right)^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\dot{\psi}|^2 \, dx,
\]

and in the case when \( \psi \) is also even, one has a stronger inequality:

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi^2 \, dx - \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi \, dx \right)^2 \leq \frac{1}{8\pi} \int_{-\pi}^{\pi} \dot{\psi}^2 \, dx.
\]

In general, given a log-concave probability measure \( \mu \) with density \( e^{-V} \) such that \( \nabla^2 V \geq k_1 I_d, \quad k_1 > 0 \), one has

\[
\int \psi^2 \, d\mu - \left( \int \psi \, d\mu \right)^2 \leq \frac{1}{k_1} \int |\nabla \psi|^2 \, d\mu;
\]

this follows from Brascamp-Lieb inequality [9]. Cordero-Erasquin, Fradelizi and Maurey [16] proved a strengthening of (7), which implies, in particular, that for even functions and uniformly log-concave measures with even densities the following inequality holds:

\[
\int \psi^2 \, d\mu - \left( \int \psi \, d\mu \right)^2 \leq \frac{1}{2k_1} \int |\nabla \psi|^2 \, d\mu.
\]

In the recent years, a number of conjectures has appeared, concerning the improvement of the inequalities of Brunn-Minkowski type under additional symmetry assumptions. For instance, in the case of the Gaussian measure, Schechtman, Schlumprecht and Zinn [41] obtained an exciting inequality in the style of the conjecture of Dar [17]; Tehranchi [42] has recently found an extension of their results, which is also a strengthening of the celebrated Gaussian correlation conjecture, recently proved by Royen [37] (see also Latala, Matlak [30]).

One of the most famous such conjectures is the Log-Brunn-Minkowski conjecture of Böröczky, Lutwak, Yang and Zhang (see [6], [7], [8]). It states that for all symmetric convex bodies \( K \) and \( L \) with support functions \( h_K \) and \( h_L \),

\[
|\lambda K +_0 (1 - \lambda)L| \geq |K|^\lambda |L|^{1-\lambda},
\]

where \( +_0 \) stands for the geometric mean of convex sets \( K \) and \( L \):

\[
\lambda K +_0 (1 - \lambda)L := \{ x \in \mathbb{R}^n : (x, u) \leq h_K^\lambda(u)h_L^{1-\lambda}(u) \forall u \in S^{n-1} \}.
\]

Böröczky, Lutwak, Yang and Zhang [6] showed that the Log-Brunn-Minkowski conjecture holds for \( n = 2 \). Saroglou [33] and Cordero-Erasquin, Fradelizi, Maurey [16] proved that (9) is true when the convex sets \( K \) and \( L \) are unconditional (that is, they are symmetric with
respect to every coordinate hyperplane). The conjecture was verified in the neighborhood of the Euclidean ball by Colesanti, the second named author and Marsiglietti [14], [15]. In [28], the first named author and E. Milman found a relation between the Log-Brunn-Minkowski conjecture and the second eigenvalue problem for certain elliptic operators. In addition, the local Log-Brunn-Minkowski conjecture was verified in [28] for the cube and for $l_q$-balls, $q \geq 2$, when the dimension is sufficiently large. Saroglou [39] showed that the validity of (9) for all convex bodies is equivalent to the validity of the analogous statement for an arbitrary log-concave measure.

In [32], the second named author, Marsiglietti, Nayar and Zvavitch proved that the Log-Brunn-Minkowski conjecture is stronger than the conjecture of Gardner and Zvavitch. In fact, if (9) was proved to be true, then (5) would hold for any even log-concave measure $\mu$ and for all symmetric convex $K$ and $L$. Therefore, (5) holds for all unconditional log-concave measures and unconditional convex sets, as well as for all even log-concave measures and symmetric convex sets in $\mathbb{R}^2$.

The main result of this paper is the following.

Theorem 1.1. Let $\mu$ be a symmetric log-concave measure on $\mathbb{R}^n$ with density $e^{-V(x)}$, for some convex function $V : \mathbb{R}^n \to \mathbb{R}$.

We shall assume that $k_1, k_2 > 0$ are such constants that

\begin{align}
\nabla^2 V & \geq k_1 I, \\
\Delta V & \leq k_2 n.
\end{align}

Consider their ratio $R = \frac{k_2}{k_1} \geq 1$.

Then, for any pair of symmetric convex sets $K$ and $L$, and for any $\lambda \in [0, 1]$, one has

\begin{align}
\mu(\lambda K + (1 - \lambda)L)^{\frac{1}{n}} & \geq \lambda \mu(K)^{\frac{1}{n}} + (1 - \lambda)\mu(L)^{\frac{1}{n}},
\end{align}

where

\[ c = c(R) = \frac{2}{(\sqrt{R} + 1)^2} > 0. \]

Further, we outline a more general observation.

Proposition 1.2. In the notation of Theorem 1.1 under the assumptions (11) and (12), for any pair of convex sets $K$ and $L$ which satisfy

\[ \int_K \nabla V d\mu = \int_L \nabla V d\mu = 0, \]

and for any $\lambda \in [0, 1]$, one has

\begin{align}
\mu(\lambda K + (1 - \lambda)L)^{\frac{1}{n}} & \geq \lambda \mu(K)^{\frac{c'}{n}} + (1 - \lambda)\mu(L)^{\frac{c'}{n}},
\end{align}

where

\[ c' = c'(R) = \frac{1}{R + 1} > 0. \]

Recall that the standard Gaussian measure $\gamma$ is the measure with the density $\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{|x|^2}{2}}$. In this case, $\nabla V = x$, $\nabla^2 V = I$, and hence $k_1 = k_2 = R = 1$. Therefore, the Proposition 1.2 implies:
Corollary 1. Let $\gamma$ be the standard Gaussian measure. For any pair of convex sets $K$ and $L$ with barycenters at the origin, i.e.

$$\int_K x d\gamma(x) = \int_L x d\gamma(x) = 0,$$

and for any $\lambda \in [0, 1]$, one has

$$\gamma(\lambda K + (1 - \lambda) L) \geq \lambda \gamma(K) + (1 - \lambda) \gamma(L). \tag{15}$$

We reduce the problem to its infinitesimal version following the approach of [11], [14], [15], [25], [26], [27], [28]. In particular, we use a Bochner-type identity obtained in [25]. The arguments are based on the application of the elliptic boundary value problem $Lu = F$ with Neumann boundary condition $u_\nu = f$. Our main result corresponds to the simplest choice of $F$, namely $F = 1$. However, we demonstrate that a choice of non-constant $F$ can lead to sharp estimates (see Section 6). This is an important observation which we believe can be useful for further developments. In Section 6 we also prove that constant $c$ in (13) can be estimated by the parameter

$$\inf_K \left[ 1 - \frac{1}{n \mu(K)} \int_K \langle (\nabla^2 V + \frac{1}{n} \nabla V \otimes \nabla V)^{-1} \nabla V, \nabla V \rangle d\mu \right],$$

where infimum is taken over all symmetric convex sets.

This paper is organized as follows. In Section 2, we outline the high-level structure of the proof of Theorem 1.1 with the goal of indicating the main steps in the estimate. In Sections 3, 4 and 5 we proceed with the said steps, one at a time. In the end of Section 5 we include the sketch of the proof of Proposition 1.2. In Section 6 we discuss some concluding remarks: namely, in subsection 6.1 we formulate a more general version of Theorem 1.1 in subsection 6.2 we discuss a more general approach to the proof which recovers the result of Gardner and Zvavitch about dilates of convex bodies.

Acknowledgment. The first named author was supported by RFBR project 17-01-00662 and DFG project RO 1195/12-1; the first named author has been funded by the Russian Academic Excellence Project “5-100” and supported in part by the Simons Foundation. The second named author is supported by the NSF CAREER DMS-1753260. The work was partially supported by the National Science Foundation under Grant No. DMS-1440140 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2017 semester. The authors are grateful to Emanuel Milman for fruitful discussions.

2. High-level structure of the proof

We shall work in $\mathbb{R}^n$. $K$ below stands for a symmetric convex body and $\mu$ for a log-concave measure with even density $e^{-V}$, where $V$ is twice continuously differentiable. The norm sign $|| \cdot ||$ with respect to a matrix shall stand for Hilbert-Schmidt norm. We shall assume without loss of generality that the boundary of $K$ is $C^2$-smooth and $K$ is strictly convex; the general bound follows by approximation.

In this section, we outline the steps of the proof by gradually introducing several definitions and lemmas.

Definition 2.1. We shall define the Gardner-Zvavitch constant $C_0 = C_0(\mu)$ to be the largest number so that for all symmetric convex sets $K$, $L$, and for any $\lambda \in [0, 1]$

$$\mu(\lambda K + (1 - \lambda) L) \geq \lambda \mu(K) + (1 - \lambda) \mu(L). \tag{16}$$
It can be verified, by considering small balls centered at the origin, that $C_0$ cannot be larger than 1. Note also by Hölder’s inequality that (16) implies (15) for all $c \in [0, C_0]$. Therefore, we shall be concerned with estimating $C_0$ from below.

Define the weighted Laplace operator $L$ associated with the measure $\mu$:

\[ \int_{\mathbb{R}^n} v \cdot Lu \, d\mu = - \int_{\mathbb{R}^n} \langle \nabla v, \nabla u \rangle d\mu. \]

**Definition 2.2.** Define $C_1 = C_1(\mu)$ to be the largest number, such that for every $u \in C^2(K)$ with $Lu = 1_K$,\[ \frac{1}{\mu(K)} \int_K ||\nabla^2 u||^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle d\mu \geq \frac{C_1(\mu)}{n}. \]

The first key step in our proof is outlined in the following Lemma:

**Lemma 2.3.** $C_0(\mu) \geq C_1(\mu)$.

Next, we conclude with two more lemmas.

**Lemma 2.4.** Assume that $\nabla^2 V \geq k_1 I_d$. Then for every $\epsilon \in [0, 1]$\[ C_1(\mu) \geq \frac{1}{\mu(K)} \int_K \frac{1}{(\nabla^2 |V|) + \frac{1}{1-\epsilon}} d\mu. \]

**Lemma 2.5.** Assume that $\nabla^2 V \geq k_1 I_d$ and $\Delta V \leq k_2 n$. Then

1. There exists an $0 < \epsilon < 1$ such that\[ \frac{1}{\mu(K)} \int_K \frac{1}{(\nabla^2 |V|) + \frac{1}{1-\epsilon}} d\mu \geq \frac{2}{(\sqrt{R} + 1)^2}, \]
   where $R = \frac{k_2}{k_1}$.
2. In the case $\epsilon = 0$,
   \[ \frac{1}{\mu(K)} \int_K \frac{1}{n k_1} + 1 \, d\mu \geq \frac{1}{1 + R}. \]

**Proof of the Theorem 1.1:** the Theorem follows immediately from Lemma 2.3, Lemma 2.4 and the first part of Lemma 2.5.

In the following sections we shall prove each of the Lemmas separately.

### 3. Proof of Lemma 2.3

The proof of Lemma 2.3 is a combination of a variational argument, integration by parts, and an application of Cauchy inequality. We start by introducing the variational argument.

#### 3.1. Variational argument

The infinitesimal versions of Brunn-Minkowski type inequalities have been considered and extensively studied in Bakry, Ledoux [1], Bobkov [2], [3], Colesanti [11], [12], Hug, Saorín-Gómez [13], the second named author, Marsiglietti [14], [15], the first named author, Milman [27], [24], [28], and many others.

Let $h$ be the support function of a $C^{2, +}$ convex body $K$, and let $\psi \in C^2(\mathbb{S}^{n-1})$; then

\[ h_s := h + s\psi \in C^{2, +}(\mathbb{S}^{n-1}), \]

(18)
if $s$ is sufficiently small (say $|s| \leq a$ for some appropriate $a > 0$). Hence for every $s$ in this range there exists a unique $C^{2, +}$ convex body $K_s$, with the support function $h_s$. For an interval $I$, we define the one-parameter family of convex bodies: 

$$K(h, \psi, I) := \{K_s : h_{K_s} = h + s\psi, s \in I\}.$$ 

**Lemma 3.1.** Assume that $\mu$ is a symmetric log-concave measure with continuously differentiable density. Inequality 

$$\mu(\lambda K + (1 - \lambda)L)^\# \geq \lambda \mu(K)^\# + (1 - \lambda)\mu(L)^\#.$$ 

holds for $\mu$ if and only if for every one-parameter family $K(h, \psi, I)$, with even $h$ and $\psi$, 

$$\frac{d^2}{ds^2} [\mu(K_s)] \bigg|_{s=0} \cdot \mu(K_0) \leq \frac{n - e}{n} \left( \frac{d}{ds} [\mu(K_s)] \right)_{s=0}^2.$$ 

**Proof.** Assume first that $\mu$ satisfies (19) on the system $K(h, \psi, I)$. Then the equality $h_{K_s} = h + s\psi, s \in I$, and the linearity of support function with respect to Minkowski addition, imply that for every $s, t \in I$ and for every $\lambda \in [0, 1]$ 

$$K_{\lambda s + (1 - \lambda)t} = \lambda K_s + (1 - \lambda)K_t.$$ 

Inequality (19) implies 

$$\mu(K_{\lambda s + (1 - \lambda)t})^\# = \mu(\lambda K_s + (1 - \lambda)K_t)^\# \geq \lambda \mu(K_s)^\# + (1 - \lambda)\mu(K_t)^\#,$$ 

which means that the function $\mu(K_s)^\#$ is concave on $I$.

Conversely, suppose that for every system $K(h, \psi, I)$ the function $\mu(K_s)^\#$ has non-negative second derivative at 0, i.e. (20) holds. We observe that this implies concavity of $\mu(K_s)^\#$ on the entire interval $I$. Indeed, given $s_0$ in the interior of $I$, consider $h = h + s_0\psi$, and define a new system $\tilde{K}(\tilde{h}, \psi, J)$, where $J$ is a new interval such that $\tilde{h} + s\psi = h + (s + s_0)\psi \in C^{2, +}$ for every $s \in J$. Then the second derivative of $\mu(K_s)^\#$ at $s = s_0$ is negative, as it is equal to the second derivative of $\mu(K_s)^\#$ at $s = 0$. On the other hand, the concavity $\mu(K_s)^\#$ on the family $K(h, \psi, I)$ is equivalent to the validity of (19) on this family. It remains to observe that arbitrary $K$ and $L$ may be “embedded” into such a family with $h = h_K$ and $\psi = h_L - h_K$. 

The normal vector to the boundary of $K$ at the point $x$ shall be denoted by $n_x$. We remark that without loss of generality we may assume that $K$ is strictly convex and $C^{2, -}$-smooth, and hence the normal is unique; the general case may be derived by approximation. We shall denote $\mu_{\partial K}(x) = e^{-V(x)} \cdot H_{n-1}|_{\partial K}$, where $H_{n-1}$ stands for the $(n - 1)$-dimensional Hausdorff measure; the notation $\nabla_{\partial K}$ means the boundary gradient (i.e., the projection of the gradient onto the support hyperplane). The second quadratic form of $\partial K$ shall be denoted by $II$, and the weighted mean curvature at a point $x$ is given by 

$$H_x = tr(II) - \langle \nabla V, n_x \rangle.$$ 

The following proposition was shown by the first named author and Milman [27], [24]:

**Proposition 3.2.** Let $f(x) = \psi(n_x)$. Then 

$$\mu(K_s)'|_{s=0} = \int_{\partial K} f(x) d\mu_{\partial K}(x);$$ 

$$\mu(K_s)''|_{s=0} = \int_{\partial K} (H_x f^2 - \langle II^{-1}\nabla_{\partial K} f, \nabla_{\partial K} f \rangle) d\mu_{\partial K}(x).$$ 

Lemma 3.1 and Proposition 3.2 imply:
Corollary 2. Suppose for $C \in \mathbb{R}$, for any convex body $K$ with $\mu(K) = 1$ and for any function $f(x) \in C^2(\partial K)$ one has

\begin{equation}
\int_{\partial K} \left( H_x f^2 - \langle II^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle \right) \, d\mu_{\partial K}(x) - \frac{n - C}{n \mu(K)} \left( \int_{\partial K} f(x) \, d\mu_{\partial K}(x) \right)^2 \leq 0.
\end{equation}

Then $C_0 \geq C$.

3.2. Integration by parts. The following Bochner-type identity was obtained by the first named author and Milman [25]. This is a generalization of a classical result of R.C. Reilly.

Proposition 3.3. Let $u \in C^2(K)$ and $f = \langle \nabla u, n_x \rangle \in C^1(\partial K)$. Then

\begin{equation}
\int_K (Lu)^2 \, d\mu = \int_K \left[ ||\nabla^2 u||^2_{HS} + \langle \nabla^2 V \nabla u, \nabla u \rangle \right] \, d\mu + \int_{\partial K} (H_x f^2 - 2 \langle \nabla_{\partial K} u, \nabla_{\partial K} f \rangle + \langle II \nabla_{\partial K} u, \nabla_{\partial K} f \rangle) \, d\mu_{\partial K}(x).
\end{equation}

3.3. Proof of Lemma 2.3. In view of Corollary 2 it is sufficient to verify (21) with $C = C_1(\mu)$. Fix a $C^1$ function $f : \partial K \to \mathbb{R}$. In the case when $\int_{\partial K} f \, d\mu_{\partial K} = 0$, we automatically get (21) with an arbitrary constant $C$. Therefore, it remains to consider the case $\int_{\partial K} f \, d\mu_{\partial K} \neq 0$. If $\int_{\partial K} f \, d\mu_{\partial K} \neq 0$, after a suitable renormalization one can assume that $\int_{\partial K} f \, d\mu_{\partial K} = \mu(K)$. Solve the Poisson equation

$$Lu = 1$$

with the Neumann boundary condition

$$\langle \nabla u(x), n_x \rangle = f(x).$$

Note that the necessary and sufficient consistency condition

$$\mu(K) = \int_K Lu \, d\mu = \int_{\partial K} f \, d\mu_{\partial K}$$

is fulfilled.

Applying (22) and definition of $C_1(\mu)$ one obtains

$$\mu(K) \geq \frac{C_1(\mu)}{n} \mu(K) + \int_{\partial K} \left( H_x f^2 - 2 \langle \nabla_{\partial K} u, \nabla_{\partial K} f \rangle + \langle II \nabla_{\partial K} u, \nabla_{\partial K} f \rangle \right) \, d\mu_{\partial K}(x).$$

Recall that for a positive-definite matrix $A$,

\begin{equation}
\langle Ax, x \rangle + \langle A^{-1} y, y \rangle \geq 2 \langle x, y \rangle.
\end{equation}

Hence

$$\int_{\partial K} \left( H_x f^2 - \langle II^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle \right) \, d\mu_{\partial K}(x) - \frac{n - C_1(\mu)}{n} \mu(K) \leq 0.$$

This finishes the proof of Lemma 2.3 \(\square\)

4. Proof of Lemma 2.4

Suppose $u$ is an even $C^2$-smooth function on $K$ with $Lu = 1_K$. To prove the lemma, it suffices to show that

\begin{equation}
\int_K \left[ ||\nabla^2 u||^2_{HS} + \langle \nabla^2 V \nabla u, \nabla u \rangle \right] \, d\mu \geq \int_K \frac{1}{\|\nabla u\|^2 \{1+\varepsilon\} n_k} \frac{1}{1-\varepsilon} \, d\mu.
\end{equation}

By Cauchy inequality,

\begin{equation}
\int_K ||\nabla^2 u||^2_{HS} \, d\mu \geq \frac{1}{n} \int_K |\nabla u|^2 \, d\mu.
\end{equation}
Note that the symmetry assumption implies
\begin{equation}
\int_K u_{x_i} d\mu = 0.
\end{equation}

By the Brascamp–Lieb inequality, we have
\begin{equation}
\int_K u_{x_i}^2 d\mu \leq \int_K (\nabla^2 V)^{-1} \nabla u_{x_i}, \nabla u_{x_i}) d\mu.
\end{equation}

Applying the lower bound for $\nabla^2 V$ and summing up in $i = 1, \ldots, n$, we get
\begin{equation}
\int_K ||\nabla^2 u||_{HS}^2 d\mu \geq k_1 \int_K |\nabla u|^2 d\mu.
\end{equation}

In addition, we observe that $\nabla^2 V \geq k_1 I d$ implies that
\begin{equation}
\int_K (\nabla^2 V \nabla u, \nabla u) d\mu \geq k_1 \int_K |\nabla u|^2 d\mu.
\end{equation}

Let $\epsilon > 0$; note that (25), (27) and (28) imply:
\begin{equation}
\int_K (||\nabla^2 u||_{HS}^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle) d\mu \geq \int_K \left( \frac{1 - \epsilon}{n} |\Delta u|^2 + k_1 (1 + \epsilon) |\nabla u|^2 \right) d\mu.
\end{equation}

Writing $\Delta u = Lu + \langle \nabla V, \nabla u \rangle = 1 + \langle \nabla V, \nabla u \rangle$, we get that the left hand side of (29) equals to
\begin{equation}
\int_K \left[ \frac{1 - \epsilon}{n} + 2 \langle \nabla u, \frac{1 - \epsilon}{n} \nabla V \rangle + \langle A_\epsilon \nabla u, \nabla u \rangle \right] d\mu,
\end{equation}

where
\begin{equation*}
A_\epsilon = \frac{1 - \epsilon}{n} \nabla V \otimes \nabla V + k_1 (1 + \epsilon) I d.
\end{equation*}

Using (23) once again, this time with $A = A_\epsilon, x = -\nabla u$ and $y = -\frac{1 - \epsilon}{n} \nabla V$, we see that (30) is greater than or equal to
\begin{equation}
\int_K \frac{1 - \epsilon}{n} \left( 1 - \frac{1 - \epsilon}{n} \langle A_\epsilon^{-1} \nabla V, \nabla V \rangle \right) d\mu.
\end{equation}

We observe that for any vector $z \in \mathbb{R}^n$, for all $a, b \in \mathbb{R}$,
\begin{equation}
(a Id + bz \otimes z)^{-1} z = \frac{z}{a + b |z|^2}.
\end{equation}

Applying (32) with $z = \nabla V$, we rewrite (31) as
\begin{equation}
k_1 (1 + \epsilon) \int_K \frac{1}{|\nabla V|^2 + k_1 n \frac{1 + \epsilon}{1 - \epsilon}} d\mu = \int_K \frac{d\mu}{\frac{|\nabla V|^2}{k_1 (1 + \epsilon)} + \frac{n}{1 - \epsilon}}.
\end{equation}

The proof is complete. □
5. PROOF OF LEMMA 2.5

We shall need the following Lemma, where symmetry is used in the crucial way: namely, we use the simple fact that log-concave even functions on the real line are concave at zero.

**Lemma 5.1.** Under the assumptions of Theorem 1.1,
\[ \int_K |\nabla V(x)|^2 d\mu \leq \int_K \Delta V d\mu. \]

**Proof.** Pick an arbitrary \( \theta \in \mathbb{S}^{n-1} \). By Prekopa-Leindler inequality, the function
\[ g(t) = \int_K e^{-V(x+\theta t)} dx \]
is log-concave in \( t \). In particular,
\[ g(0)g''(0) - g'(0)^2 \leq 0. \]

Note that
\[ g'(0) = -\int_K \frac{\partial V}{\partial \theta} e^{-V(x)} dx = 0, \]
as we assumed that \( V \) is even and \( K \) is symmetric. Therefore, by (34),
\[ g''(0) = \int_K \left( \frac{\partial^2 V}{\partial \theta^2} + \left( \frac{\partial V}{\partial \theta} \right)^2 \right) e^{-V(x)} dx \leq 0. \]

Applying (36) with \( \theta = e_i \) and summing up in \( i = 1, ..., n \), we obtain the conclusion of the lemma. \( \square \)

5.1. **Proof of Lemma 2.5.** Using Jensen’s inequality and concavity of the function \( \frac{1}{1+x} \), one obtains
\[ \frac{1}{\mu(K)} \int_K \frac{1}{\|\nabla V\|^2 + 1} d\mu \geq \frac{1}{\mu(K)} \int_K \frac{1}{\|\nabla V\|^2 + 1} d\mu \leq \frac{R}{1+\epsilon} + \frac{1}{1-\epsilon}. \]

Next, we apply Lemma 5.1 along with the assumption \( \Delta V \leq nk_2 \), to get
\[ \frac{1}{\mu(K)} \int_K \frac{1}{\|\nabla V\|^2 + 1} d\mu \geq \frac{1}{R + 1 - 2\sqrt{R}}. \]

where, as before, \( R = \frac{k_2}{k_1} \).

Plugging the optimal
\[ \epsilon = \frac{R + 1 - 2\sqrt{R}}{R - 1}, \]
we get the conclusion of the first part of Lemma 2.5.

Plugging \( \epsilon = 0 \), we get
\[ \frac{1}{\mu(K)} \int_K \frac{1}{\|\nabla V\|^2 + 1} d\mu \geq \frac{1}{R + 1}, \]
which implies the second part of Lemma 2.5. \( \square \)

**Remark 5.2.** We note that Lemma 2.4 is essentially optimal in the case of the standard Gaussian measure, i.e. when \( V = \frac{|x|^2}{2} \). Indeed, e.g. by Laplace method,
\[ \delta(n) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2n}} \frac{1 + o(1)}{2n}. \]
Further, the sequence $\delta(n)$ is decreasing. Indeed, by Fubini theorem,
\[
\delta(n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{-\frac{|x|^2}{2}} dx dt \\
\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{-\frac{|x|^2}{2}} dx dt = \delta(n-1).
\]

Lastly, we point out that
\[
\delta(1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} ds = \sqrt{e} \int_{1}^{\infty} e^{-\frac{s^2}{2}} ds \approx 0.66,
\]
and hence for all $n$,
\[
\delta(n) \in [0.5, 0.67].
\]

Sketch of the proof of Proposition 1.2. We outline, that the symmetry assumption is used in the proof twice: in (26) and (35). Observe that the condition which is actually used for (35) is
\[
(38) \int_K \frac{\partial V}{\partial x_i} d\mu = 0,
\]
for all $i = 1, \ldots, n$. We may completely avoid the use of (26) and consider only the case $\epsilon = 0$, in which case, as shown above,
\[
C_0 \geq \frac{1}{\mu(K)} \int_K \frac{1}{|\nabla V|^2 + 1} d\mu \geq \frac{1}{1 + R}.
\]

We conclude that for any pair of convex sets $K$ and $L$, which satisfy
\[
(39) \int_K \frac{\partial V}{\partial x_i} d\mu = \int_L \frac{\partial V}{\partial x_i} d\mu = 0,
\]
and for any $\lambda \in [0, 1]$, one has (6) with constant $\frac{1}{1+R}$. \hfill \Box

Note, for that matter, that in the case of the standard Gaussian measure the optimal choice is $\epsilon = 0$.

6. CONCLUDING REMARKS

6.1. An improved estimate. We outline a sharper, more general estimate for the Gardner-Zvavitch constant in the following.

Theorem 6.1. Let $C$ be a collection of convex bodies in $\mathbb{R}^n$ closed under scalar multiplication and Minkowski addition. Let
\[
C := C(\mu, C) = \sup_{\epsilon \in [0,1]} \inf_{K \in C} \left[ 1 - \frac{1}{n \mu(K)} \int_K \langle A^{-1} \nabla V, \nabla V \rangle d\mu \right],
\]
where
\[
A = \nabla^2 V + \frac{1}{n} \nabla V \otimes \nabla V + \frac{\epsilon}{(1-\epsilon) C(K, \mu)} Id
\]
and $C(\mu, K)$ is the Poincaré constant of $\mu|_K$. Then, for all $K, L \in C$, and for every $\lambda \in [0, 1]$
\[
\mu(\lambda K + (1-\lambda) L) \overset{G}{\geq} \lambda \mu(K) \overset{G}{\geq} (1-\lambda) \mu(L) \overset{G}{\geq}.
\]

In particular,
\[
C \geq \inf_{K \in C} \left[ 1 - \frac{1}{n \mu(K)} \int_K \langle (\nabla^2 V + \frac{1}{n} \nabla V \otimes \nabla V)^{-1} \nabla V, \nabla V \rangle d\mu \right].
\]
Proof. Consider an arbitrary even \( u : K \to \mathbb{R} \) such that \( Lu = 1_K \). Then, by (25), along with the fact that \( \Delta u = 1 + \langle \nabla V, \nabla u \rangle \),

\[
\int_K \|\nabla^2 u\|^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle d\mu \geq \int_K \frac{1}{n} [1 + \langle \nabla V, \nabla u \rangle]^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle d\mu
\]

\[
= \int_K \frac{1}{n} + 2 \langle \nabla V, \nabla u \rangle + \langle \langle \nabla^2 V + \frac{1}{n} \nabla V \otimes \nabla V \rangle \nabla u, \nabla u \rangle d\mu.
\]

Next we apply the Poincaré inequality to every \( u_{x_i} \):

\[
\int_K u_{x_i}^2 d\mu \leq C(K, \mu) \int_K |\nabla u_{x_i}|^2 d\mu.
\]

Thus

\[
\int_K |\nabla u|^2 d\mu \leq C(K, \mu) \int_K \|\nabla^2 u\|^2 d\mu,
\]

and for every \( \varepsilon \in [0, 1] \) one has

\[
\int_K \|\nabla^2 u\|^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle d\mu \geq \frac{\varepsilon}{C(K, \mu)} \int_K |\nabla u|^2 d\mu
\]

\[
+ (1 - \varepsilon) \int_K \frac{1}{n} + 2 \langle \nabla V, \nabla u \rangle + \langle \langle \nabla^2 V + \frac{1}{n} \nabla V \otimes \nabla V \rangle \nabla u, \nabla u \rangle d\mu
\]

\[
= (1 - \varepsilon) \left( \int_K \frac{1}{n} + 2 \langle \nabla V, \nabla u \rangle + \langle A \nabla u, \nabla u \rangle d\mu \right).
\]

Applying (23) with the positive-definite matrix \( A \) and Lemma 2.3 we complete the proof.

The Theorem 1.1 follows directly from Theorem 6.1. It is possible that \( C(\mu, C) \) can be estimated for the class of symmetric convex sets under less restrictive assumptions than \( \nabla^2 V \geq k_1 I_d \) and \( \Delta V \leq n \), however it is not clear to us at the moment.

6.2. The case of non-constant \( F \), and the Gardner-Zvavitch conjecture for dilates.

Definition 6.2. For a \( C^2 \)-smooth even function \( F : K \to \mathbb{R} \), with \( \int_K F d\mu \neq 0 \), let \( C_F \) be the largest number, such that for every \( u \in C^2(K) \) with \( Lu = F \),

\[
\int_K \|\nabla^2 u\|^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle d\mu \geq \int_K F^2 d\mu - \frac{n - C_F}{n \mu(K)} \left( \int_K F d\mu \right)^2.
\]

We define

\[
C_2(\mu) := \sup_F C_F,
\]

where the supremum runs over all \( C^2 \)-smooth even functions \( F : K \to \mathbb{R} \), with \( \int_K F d\mu \neq 0 \).

We observe the following straightforward

Claim 3. \( C_2(\mu) \geq C_1(\mu) \).

Note, that the proof of Lemma 2.3 implies, in fact, a stronger statement:

Lemma 6.3. \( C_0(\mu) \geq C_2(\mu) \).

It is possible that in the case of the standard Gaussian measure, the only sub-optimal place in our argument is the application of Lemma 2.3 in place of the stronger statement of Lemma 6.3; indeed, solving the Neumann system with \( F \neq 1_K \) could lead to a better bound, however our current proof of Lemma 2.4 does not allow us to use this freedom.

Finally, we outline the following
**Lemma 6.4.** Let $K$ be a barycentered convex body, let $\gamma$ be the Gaussian measure and let $u(x) = \frac{x^2}{2}$ on $K$. Let $F = Lu = n - |x|^2$ on $K$. Then

$$\int_K \left( ||\nabla^2 u||^2 + \langle \nabla^2 V u, \nabla u \rangle d\gamma \right) \geq \int_K F^2 d\gamma - \frac{n-1}{n\gamma(K)} \left( \int_K F d\gamma \right)^2.$$  

**Proof.** For all $x \in K$, $L\frac{x^2}{2} = n - x^2$; $||\nabla^2 \frac{x^2}{2}||^2 = n$; $|\nabla \frac{x^2}{2}|^2 = x^2$. Hence, (41) rewrites:

$$n\gamma(K) + \int_K x^2 d\gamma \geq n^2\gamma(K) - 2n \int_K x^2 d\gamma + \int_K x^4 d\gamma - \left( n^2\gamma(K) - 2n \int_K x^2 d\gamma + \frac{1}{\gamma(K)} \left( \int_K x^2 d\gamma \right)^2 \right) + \frac{1}{n} \left( n^2\gamma(K) - 2n \int_K x^2 d\gamma + \frac{1}{\gamma(K)} \left( \int_K x^2 d\gamma \right)^2 \right),$$

and rearranging, we get

$$\left[ \int_K x^4 d\gamma - \frac{1}{\gamma(K)} \left( \int_K x^2 d\gamma \right)^2 - 2 \int_K x^2 d\gamma \right] + \left[ - \int_K x^2 d\gamma + \frac{1}{n\gamma(K)} \left( \int_K x^2 d\gamma \right)^2 \right] \leq 0.$$  

Recall the B-Theorem from [16]:

$$\int_K x^4 d\gamma - \frac{1}{\gamma(K)} \left( \int_K x^2 d\gamma \right)^2 - 2 \int_K x^2 d\gamma \leq 0;$$

recall also the corollary of Lemma 5.1

$$- \gamma(K) + \frac{1}{n} \int_K x^2 d\gamma \leq 0.$$  

Applying (44) and (45) we obtain the validity of (43), which in turn implies the validity of (41). □

As a consequence of Lemma 6.3 and Lemma 6.4, we observe the conjecture of Gardner and Zvavitch in the case when $K$ and $L$ are dilates. This result was previously obtained by Gardner and Zvavitch [23], where the authors also used (44). We include the following proposition merely for completeness.

**Proposition 6.5.** Let $K = aL$ for some $a > 0$ be barycentered convex bodies. Then

$$\gamma(\lambda K + (1 - \lambda)L)^\frac{1}{2} \geq \lambda \gamma(K)^\frac{1}{2} + (1 - \lambda)\gamma(L)^\frac{1}{2}.$$  

**Sketch of the proof:** Recall, from the proof of Lemma 3.1, that arbitrary $K$ and $L$ may be “embedded” into a family $K(h, \psi, I)$ with $h = h_K$ and $\psi = h_L - h_K$. Recall as well that the boundary condition in the Neumann problem we considered is given by $f(x) = \psi(n_x) = h_L(n_x) - h_K(n_x)$. In the case when $L = aK$, we are dealing with

$$f(x) = (a - 1)h_K(n_x) = (a - 1)\langle x, n_x \rangle.$$
Repeating the proof of the Theorem 1.1 in this specific case, we see that to verify the Proposition, it suffices to show that for some $u : K \to \mathbb{R}$ with
\begin{equation}
\langle \nabla u, n_x \rangle = f(x) = (a - 1) \langle x, n_x \rangle,
\end{equation}
one has
\begin{equation}
\int_K \|\nabla^2 u\|^2 + \langle \nabla^2 \nabla u, \nabla u \rangle d\gamma \geq \int_K (Lu)^2 d\gamma - \frac{n-1}{n\gamma(K)} \left( \int_K L u d\gamma \right)^2.
\end{equation}

It remains to note that $u = (a - 1) R \frac{|x|^2}{2}$ satisfies (46), and that Lemma 6.4 along with the homogeneity of (47), implies the validity of (47) for $u = (a - 1) R \frac{|x|^2}{2}$. □

**Remark 6.6.** Note that Proposition 6.3 implies the validity of the conjecture of Gardner and Zvavitch in dimension 1, since every pair of symmetric intervals are dilates of each other. Further, directly verifying (47) in the case $n = 1$ boils down to the elementary inequality
\[ \alpha(R) := \int_0^R (t^4 - 3t^2) e^{-\frac{t^2}{2}} dt \leq 0, \]
which follows from the fact that $\alpha(0) = \alpha(\pm \infty) = 0$, $\alpha(R)$ decreases on $[0, \sqrt{3}]$ and increases on $[\sqrt{3}, +\infty]$. It of course also follows from (44) and (45), but that would be an overkill.

It is curious to note that Lemma 2.3 is also sharp when $n = 1$: for every $u : [-R, R] \to \mathbb{R}$ with $Lu = 1$ and with the boundary condition $u'(R) = -u'(-R)$, one has
\[ \beta(R) := \frac{\int_{-R}^R ((u'')^2 + (u')^2) e^{-\frac{t^2}{2}} dt}{\int_{-R}^R e^{-\frac{t^2}{2}} dt} \geq 1. \]
In fact, the equality is never attained unless $R = 0$, and $\beta(R) \to 1$ as $R \to 0$. A routine computation shows that $\beta(R)$ is strictly increasing in $R$, and $\beta(R) \to 1$ as $R \to \infty$. Furthermore, $\beta(R)$ grows very fast.

This indicates that our proof of Lemma 2.3 is sub-optimal, at least in the case $n = 1$: we replace the term which includes $|\nabla u|^2$ with the much smaller term, while $|\nabla u|^2$ has large growth. Constant $\frac{1}{2}$ which we get after such replacement is attained when $R = \infty$, and in fact the estimate decreases as $R$ increases, contrary to the actual behavior of $\beta(R)$.

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