Research Article

Application of Volterra Integral Equations in Dynamics of Multispan Uniform Continuous Beams Subjected to a Moving Load

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The dynamic behavior of multispan uniform continuous beam arbitrarily supported on its edges subjected to various types of moving noninertial loads is studied. Problem is solved by replacing a multispan structure with a single-span beam loaded with a given moving load and redundant forces situated in the positions of the intermediate supports. Redundant forces are obtained by solving Volterra integral equations of the first or the second order (depending on the stiffness of the intermediate supports) which are consistent deformation equations corresponding to each redundant. Solutions for the beam arbitrarily supported on its edges (pinned or fixed) due to a moving concentrated force and moving distributed load are given. The difficulty of solving Volterra integral equations analytically is bypassed by proposing a simple numerical procedure. Numerical examples of two- and three-span beam have been included in order to show the efficiency of the presented method.

1. Introduction

Many authors have considered the problem of vibrations in structural and mechanical engineering resulting from the moving load, because of both being interesting from the theoretical point of view and having a significant importance for the practice. This problem occurs in dynamics of bridges, roadways, railways, and runways as well as missiles, aircrafts, and other structures. Various types of structures and girders like beams, plates, shells, and frames have been considered. Also various models of moving loads have been assumed [1]. Both deterministic and stochastic approaches have been presented [2–4].

In most studies a single-span girder like a string, a beam, a plate, or a shell has been considered. The solution of the response of a finite, single-span beam subjected to a force moving with a constant velocity has a form of an infinite series and has been presented in many papers. Original solutions in a closed form for the aperiodic vibration of the finite, simply supported Euler-Bernoulli beam, Timoshenko beam, and a sandwich beam are given in the papers [5–7]. Also more complex systems like a double-string, a double-beam, or a suspension bridge have been considered as single-span girders [8–12]. An important and interesting problem is the vibrations of a multispan beam caused by a moving load. There are many structures, for example, bridges, which are multispan. There are not so many papers focused on the dynamic response problem of a multispan beam due to a moving load [13–31]. The vibrations of a multispan Bernoulli-Euler beam with an arbitrary geometry in each span subjected to moving forces [13–19], or moving masses [24, 25], or moving oscillators [26] have been considered. Also the vibrations of a multispan Timoshenko beam due to moving load have been considered [27–29]. Vibrations of multispan sandwich or composite beams are considered in the papers [30, 31]. The solutions for the vibration of a frame caused by a moving force are given in the paper [32].

In this paper the dynamic behavior of Euler-Bernoulli multispan uniform continuous beam system traversed by a moving load is analyzed. We combine analytical and
numerical procedures to present a solution for the case of a beam traversed by a constant force moving with the constant velocity. It is assumed that the stiffness and the mass of the beam in every span are the same but the lengths of the spans can be different. The problem is solved similar to the static force method but instead of a set of algebraic equations we have to solve a set of Volterra integral equations (first order when the beam rests on supports of infinite stiffness or the second order when the beam rests on elastic supports). It is difficult to solve these Volterra integral equations in analytical way; for this reason they should be solved numerically. The primary structure (primary beam) is an arbitrarily supported single-span beam. For this reason, in order to find the solution for multispan continuous beams using a set of the Volterra integral equations, in the first step the dynamic response of a finite, single-span beam subjected to a moving load and stationary point forces is considered. The presented algorithm is used to determine the vibrations of two- and three-span beams. The correctness of the algorithm has been tested using Finite Difference Method.

2. Vibrations of an Arbitrarily Supported Single-Span Beam under a Moving Load

Let us consider Euler-Bernoulli beam element of constant flexural rigidity $EI$ and constant mass $m$ per unit length subjected to a dynamic load $p(x,t)$. Equation of motion describing undamped vibrations $w(x,t)$ has the form

$$EI \frac{\partial^4 w(x,t)}{\partial x^4} + m \frac{\partial^2 w(x,t)}{\partial t^2} = p(x,t).$$

(1)

Let us assume that the beam is of finite length $L$ and has pinned or fixed supports on both ends. The beam rests also on $k$ arbitrarily located intermediate point supports and is subjected to a load moving with constant velocity $v$ (see Figure 1).

In the presented method we shall replace this structure with a single-span beam of the same length, mass, flexural rigidity, and boundary conditions at both ends, subjected to the same external moving load as the multispan beam (see Figure 2). Load function $p(x,t)$ from (1) depends on the type of moving load. In Sections 2.1 and 2.2 we shall consider two cases, namely, the cases of moving concentrated force and moving distributed load.

2.1. Vibrations of a Beam Subjected to a Moving Concentrated Force. In the following section of this paper we shall focus on the simplest but a very significant case of moving load, namely, the case of a single-span beam loaded with a constant force of magnitude $P$ moving with a constant velocity $v$. This case is shown on Figure 3.

Vibrations of a beam loaded with a moving constant force are described by

$$EI \frac{\partial^4 w^P(x,t)}{\partial x^4} + \sigma^2 \frac{\partial^2 w^P(x,t)}{\partial t^2} = P \delta(x - vt),$$

(2)

where the symbol $\delta(\cdot)$ denotes the Dirac delta. Superscript $P$ in the expression $w^P(x,t)$ denotes the factor initiating beam vibrations which in this case is moving force $P$. After introducing the dimensionless variables

$$\xi = \frac{x}{L},$$

$$T = \frac{vt}{L},$$

$$\xi \in [0,1], \ T \in [0,1].$$

(3)

Equation (2) takes the form

$$\left[ w^P(\xi,T) \right]^{IV} + \sigma^2 \ddot{w}^P(\xi,T) = P_0 \delta(\xi - T),$$

(4)

where $\sigma^2 = \frac{mv^2L^2}{EI}$ and $P_0 = \frac{PL^3}{EI}$. The roman numerals denote differentiation with respect to the spatial coordinate $\xi$ and the dots denote differentiation with respect to time $T$.

The solution of (4) is assumed to be in the form of the sine series:

$$w^P(\xi,T) = \sum_{n=1}^{\infty} Y_n^P(T) W_n(\xi),$$

(5)
Table 1: Constants $G_{1n}$, $G_{2n}$, $G_{3n}$, $G_{4n}$, $\gamma_n^2$ and eigenvalues $\lambda_n$ for different types of beam.

| Value | Pinned-pinned | Pinned-fixed | Beam type | Fixed-pinned | Fixed-fixed |
|-------|---------------|--------------|-----------|-------------|-------------|
| $\lambda_n$ | $n\pi$ | 3.927 for $n = 1$ | 3.927 for $n = 1$ | 4.730 for $n = 1$ | 4.730 for $n = 1$ |
| | | 7.069 for $n = 2$ | 7.069 for $n = 2$ | 7.853 for $n = 2$ | 7.853 for $n = 2$ |
| $\gamma_n^2$ | $0.5$ | 1 for $n > 1$ | 1 for $n > 1$ | 1.00001 for $n = 1$ | 1.00001 for $n = 1$ |
| $G_{1n}$ | 1 | $\frac{\sin \lambda_n}{\sinh \lambda_n}$ | $\frac{\sin \lambda_n + \cosh \lambda_n}{\sinh \lambda_n}$ | $\frac{-1}{\cosh \lambda_n - \sin \lambda_n}$ | $\frac{-1}{\cosh \lambda_n - \sin \lambda_n}$ |
| $G_{2n}$ | 0 | $\frac{1}{\cosh \lambda_n}$ | $\frac{\cosh \lambda_n + \sin \lambda_n}{\sinh \lambda_n}$ | $\frac{-1}{\cosh \lambda_n - \sin \lambda_n}$ | $\frac{-1}{\cosh \lambda_n - \sin \lambda_n}$ |
| $G_{3n}$ | 0 | $\frac{1}{\sin \lambda_n}$ | $\frac{\sin \lambda_n + \cosh \lambda_n}{\sinh \lambda_n}$ | $\frac{-1}{\cosh \lambda_n - \sin \lambda_n}$ | $\frac{-1}{\cosh \lambda_n - \sin \lambda_n}$ |
| $G_{4n}$ | 0 | $0$ | $1$ | $1$ | $1$ |

where $W_n(\xi)$ are the eigenfunctions of the beam and have the form

$$W_n(\xi) = G_{1n} \sin \lambda_n \xi + G_{2n} \cos \lambda_n \xi + G_{3n} \sinh \lambda_n \xi + G_{4n} \cosh \lambda_n \xi,$$

(Equation 6)

Eigenfunctions of the beam satisfy the homogeneous differential equation for the values $\lambda_n$:

$$\frac{d^4 W_n(\xi)}{d\xi^4} - \lambda_n^4 W_n(\xi) = 0.\quad (7)$$

Constants $G_{1n}$, $G_{2n}$, $G_{3n}$, $G_{4n}$ as well as eigenvalues $\lambda_n$ result from the boundary conditions for the single-span beam (see Table 1).

After substituting the expression (5) into (4) and using the orthogonality method one obtains the following set of ordinary differential equations:

$$\ddot{Y}_n^P(T) + \omega_n^2 Y_n^P = \frac{P_0}{\gamma_n^2 \sigma^2} W_n(T),\quad (8)$$

where

$$\gamma_n^2 = \int_0^1 [W_n(\xi)]^2 d\xi, \quad \omega_n^2 = \frac{\lambda_n^4}{\sigma^2}.\quad (9)$$

Assuming zero initial conditions,

$$Y_n^P(0) = 0, \quad \dot{Y}_n^P(0) = 0,\quad (10)$$

function $Y_n^P(T)$ can be presented as the sum of the particular solution $Y_{nA}^P(T)$ and the homogeneous solution $Y_{nS}^P(T)$ of (8):

$$Y_n^P(T) = Y_{nA}^P(T) + Y_{nS}^P(T),\quad (11)$$

where

$$Y_{nA}^P(T) = A_n \sin \lambda_n T + B_n \cos \lambda_n T + C_n \sin \lambda_n T + D_n \cosh \lambda_n T,\quad (12)$$

$$Y_{nS}^P(T) = E_n \sin \omega_n T + F_n \cos \omega_n T.$$

Function $Y_{nA}^P(T)$ is used to describe aperiodic vibrations of the beam and does not satisfy the initial conditions (10). Function $Y_{nS}^P(T)$ is used to describe free vibrations and results from the homogeneous equation $Y_{nS}^P(T) + \omega_n^2 Y_{nS}^P = 0$ and together with function $Y_{nA}^P(T)$ satisfies zero initial conditions of motion.

Constants $A_n$, $B_n$, $C_n$, and $D_n$ result from

$$(1) - \lambda_n^2 A_n + \omega_n^2 A_n = \frac{P_0}{\gamma_n^2 \sigma^2} G_{1n} \longrightarrow$$

$$A_n = \frac{P_n G_{1n}}{\gamma_n^2 \sigma^2 (\omega_n^2 - \lambda_n^2)},$$

$$(2) - \lambda_n^2 B_n + \omega_n^2 B_n = \frac{P_0}{\gamma_n^2 \sigma^2} G_{2n} \longrightarrow$$

$$B_n = \frac{P_n G_{2n}}{\gamma_n^2 \sigma^2 (\omega_n^2 - \lambda_n^2)},$$

$$(3) \lambda_n^2 C_n + \omega_n^2 C_n = \frac{P_0}{\gamma_n^2 \sigma^2} G_{3n} \longrightarrow$$

$$C_n = \frac{P_n G_{3n}}{\gamma_n^2 \sigma^2 (\omega_n^2 + \lambda_n^2)},$$

$$(4) \lambda_n^2 D_n + \omega_n^2 D_n = \frac{P_0}{\gamma_n^2 \sigma^2} G_{4n} \longrightarrow$$

$$D_n = \frac{P_n G_{4n}}{\gamma_n^2 \sigma^2 (\omega_n^2 + \lambda_n^2)}.\quad (14)$$
while constants $E_n$ and $F_n$ result from the zero initial conditions (10) and are equal to

$$E_n = -\frac{\lambda_n}{\omega_n} (A_n + C_n);$$

$$F_n = -B_n - D_n.$$  \hspace{1cm} (15)

After solving (8) taking into account the initial conditions (10) the response of the beam has the form

$$w^P(\xi, T) = w^P_A(\xi, T) + w^P_S(\xi, T),$$  \hspace{1cm} (16)

where

$$w^P_A(\xi, T) = \sum_{n=1}^{\infty} Y^P_n T W_n(\xi),$$

$$w^P_S(\xi, T) = \sum_{n=1}^{\infty} Y^P_n S_n T W_n(\xi).$$

If $\sigma = \lambda_n$ is fulfilled then the series in the solution (17) tends to infinity. Thus the resonance velocity $v_{\sigma,n}$ is equal to

$$v_{\sigma,n} = \frac{\lambda_n}{L} \sqrt{\frac{E I}{m}}.$$ \hspace{1cm} (18)

The function $w^P_S(\xi, T)$ in expression (16) describes the aperiodic vibrations and satisfies the nonhomogeneous differential equation (4). This function does not satisfy the initial conditions of motion (10). The function $w^P_S(\xi, T)$ is the free vibrations of the beam which satisfy the homogeneous differential equation (4) ($P = 0$) and together with the aperiodic function satisfy the initial conditions of motion (10).

Vibrations of the simply supported beam after taking into account that $W_n(\xi) = \sin n\pi\xi$ and $Y^P_n = 0.5$ have the form

$$w^P(\xi, T) = w^P_A(\xi, T) + w^P_S(\xi, T)$$

$$= 2P \sum_{n=1}^{\infty} \frac{\sin n\sigma T}{n^2 - \sigma^2} \left[ \frac{\sin n\pi\xi}{n^2} - \frac{\sin n\xi}{n^2 - \sigma^2} \right].$$

The first series in (19) which represents the aperiodic vibration is slower convergent than the second series which represents the free vibration. The function $w^P_A(\xi, T)$ can be presented in the closed form (see [5, 6]):

$$w^P_A(\xi, T) = P_0 \left[ \frac{\sin (1 - \xi)}{\pi^2 \sin \sigma} - \frac{1 - \xi}{\pi^2} \right]$$

for $\xi < T,$

$$w^P_A(\xi, T) = P_0 \left[ \frac{\sin T \sin \sigma (1 - \xi)}{\pi^2 \sin \sigma} - \frac{T (1 - \xi)}{\pi^2} \right]$$

for $\xi \geq T.$

The closed form solution is particularly important in the determination of bending moment and shear force which are equal to

$$M(\xi, T) = M_A(\xi, T) - 2PL \sum_{n=1}^{\infty} \frac{\sin \omega_n T \sin m\pi\xi}{(m\pi)^2 - \sigma^2},$$

where

$$M_A(\xi, T) = PL \frac{\sin (1 - T) \sin \sigma \xi}{\sigma \sin \sigma}$$

for $\xi < T,$

$$M_A(\xi, T) = PL \frac{\sin \sigma T \sin (1 - \xi)}{\sigma \sin \sigma}$$

for $\xi \geq T,$

$$Q(\xi, T) = Q_A(\xi, T) - 2P \sum_{n=1}^{\infty} \frac{\sin \omega_n T \cos m\pi\xi}{(m\pi)^2 - \sigma^2},$$

where

$$Q_A(\xi, T) = P \frac{\sin (1 - T) \cos \sigma \xi}{\sin \sigma}$$

for $\xi < T,$

$$Q_A(\xi, T) = -P \frac{\sin T \cos \sigma (1 - \xi)}{\sin \sigma}$$

for $\xi \geq T.$

2.2 Vibrations of a Beam Subjected to a Moving Distributed Load. In this section we shall analyze an arbitrarily supported single-span uniform beam subjected to a distributed uniform load $q$ moving with a constant velocity $v$ (see Figure 4). Equation of motion has the form

$$EI \frac{\partial^4 w^q(x, t)}{\partial x^4} + m \frac{\partial^2 \omega^q(x, t)}{\partial t^2} = q \left[ 1 - H(x - vt) \right],$$

where superscript $q$ in expression $\omega^q(x, t)$ denotes the factor initiating beam vibrations which in this case is moving distributed load $q$. After introducing dimensionless variables (3),

$$\left[ \omega^q(\xi, T) \right]^{IV} + \sigma^2 \omega^q(\xi, T) = q_0 \left[ 1 - H(\xi - T) \right].$$

Figure 4: A single-span beam subjected to a moving distributed load.
Expression $H(\cdot)$ denotes Heaviside step function:

$$H(\xi - T) = \begin{cases} 0 & \text{for } \xi \leq T \\ 1 & \text{for } \xi > T \end{cases},$$  \hfill (26)

and $q_0$ is equal to

$$q_0 = \frac{qL^4}{EI}. \hfill (27)$$

Solution of (25) can be found in the same way as for the case of moving concentrated force by using formulas (6) and (9) and expressions from Table 1. After substituting the expression

$$\int_0^T W_n(\xi) \, d\xi = \frac{G_1n - G_{3n} + G_{2n} \sin \lambda_n T - G_{1n} \cos \lambda_n T + G_{4n} \sinh \lambda_n T + G_{3n} \cosh \lambda_n T}{\lambda_n}$$

Integral from the formula above is equal to

$$Y^q_n(T) + \omega_n^2 Y'^q_n(T) = \frac{q_0}{\gamma_n^2 \sigma^2} \int_0^T W_n(\xi) \, d\xi. \hfill (29)$$

Function $Y^q_n(T)$ can be presented analogically to (10) and has the form

$$Y^q_n(T) = Y'^q_n(T) + \gamma_n \xi,$$  \hfill (31)

where

$$Y'^q_n(T) = H_n + A_n \sin \lambda_n T + B_n \cos \lambda_n T + C_n \sinh \lambda_n T + D_n \cosh \lambda_n T,$$  \hfill (32)

$$Y'^q_n(T) = E_n \sin \omega_n T + F_n \cos \omega_n T.$$  \hfill (33)

Function $Y'_{nA}(T)$ is used to describe aperiodic vibrations of the beam and does not satisfy the initial conditions (10). Function $Y'_{nA}(T)$ is used to describe free vibrations and results from the homogenous equation $Y''_{nA}(T) + \omega_n^2 Y'_{nA}(T) = 0$ and together with function $Y'^q_n(T)$ satisfies zero initial conditions. Constant $H_n$ is equal to

$$H_n = \frac{q_0 (G_{1n} - G_{3n})}{\gamma_n^2 \lambda_n^2}. \hfill (34)$$

Constants $A_n, B_n, C_n,$ and $D_n$ result from

$$A_n = \frac{q_0 G_{2n}}{\gamma_n^2 \sigma^2 \lambda_n (\omega_n^2 - \lambda_n^2)};$$

$$B_n = \frac{q_0 G_{3n}}{\gamma_n^2 \sigma^2 \lambda_n (\omega_n^2 - \lambda_n^2)};$$

$$C_n = \frac{q_0 G_{4n}}{\gamma_n^2 \sigma^2 \lambda_n (\omega_n^2 - \lambda_n^2)};$$

$$D_n = \frac{q_0 G_{5n}}{\gamma_n^2 \sigma^2 \lambda_n (\omega_n^2 - \lambda_n^2)}.$$  \hfill (35)

\section*{3. Vibrations of the Single-Span Beam under Concentrated Force}

In the next step let us consider vibrations of the beam under time-varying concentrated force $X(\xi,T)$ concentrated in the point $\xi$ (Figure 5).

In this case vibrations of the beam for the dimensionless variables (3) have the form

$$[\sigma^2 \omega^2 (\xi, T)]^{\frac{3}{I}} + \sigma^2 \omega^2 (\xi, T) = X_0 (T) \delta (\xi - \xi), \hfill (36)$$

where $\sigma^2 = m \nu (2L EI/3L)$. Superscript $X$ in expression $\sigma^2 \omega^2 (x, t)$ denotes the factor initiating beam vibrations which in this case is concentrated time-varying force $X(T)$. Following similarly to the case of the moving force, one obtains solution in the convolution form:

$$\sigma^2 \omega^2 (\xi, T) = \frac{L}{v} \int_0^T h_1(\xi, T - \tau) \, X(\tau) \, d\tau,$$  \hfill (37)
where the impulse response function $h_i(\xi, T)$ is equal to
\[
h_i(\xi, T) = \frac{1}{m v} \sum_{n=1}^{\infty} \sin \omega_n T W_n(\xi) W_n(\xi).
\]
(38)

The impulse response function $h_i(\xi, T)$ has been obtained from
\[
h_i^{IV}(\xi, T) + \sigma^2 h_i(\xi, T) = \frac{v L^2}{EI} \delta(T) \delta(\xi - \xi_i)
\]
and describes the vibrations of the beam caused by unit Dirac impulse at time $T$ acting at the point $\xi$. The solutions for a single-span beam presented above for a moving force and a concentrated force will be used to solve the problem of vibration of multispan beam.

The multispan beam can be treated as a single-span beam subjected to a given moving load and the $k$ redundant forces $X_i (i = 1, \ldots, k)$ in the mid-span supports (Figure 6). The deflections of the multispan beam under above load processes in the point of the mid-span supports are equal to zero. For this reason one obtains a set of $k$ Volterra integral equations of the first order:
\[
\sum_{j=1}^{k} d_{ij}(T) X_j(T) + \Delta_{iP/q}(T) = 0, \quad i = 1, 2, \ldots, k,
\]
(40)

where $d_{ij}(T)$ and $\Delta_{iP/q}(T)$ are the vertical displacement of a single-span beam in the point $\xi_i$ caused by a point force $X_j(T)$ and a given moving load (concentrated force or distributed load), respectively.

The functions $d_{ij}(T)$ and $\Delta_{iP/q}(T)$ are equal to
\[
d_{ij}(T) = h_j(\xi_i, T),
\]
\[
\Delta_{iP/q}(T) = w^{P/q}(\xi_i, T),
\]
(41)

where the impulse response function $h_j(\xi_i, T)$ is given by (38) and the function $\Delta_{iP/q}(T)$ is given by (5) or (28).

The coordinates $\xi_i (i = 1, \ldots, k)$ determine the points of the intermediate supports of the beam.

For comparison the static solutions have the form of
\[
d_{ij,\text{stat}}(T) = \frac{L^3}{EI} \sum_{n=1}^{\infty} W_n(\xi_i) W_n(\xi) \frac{\gamma^2 \lambda_n^4}{\gamma^2 \lambda_n^4},
\]
\[
\Delta_{iP,\text{stat}}(T) = P_0 \sum_{n=1}^{\infty} W_n(\xi_i) W_n(T) \frac{\gamma^2 \lambda_n^4}{\gamma^2 \lambda_n^4},
\]
\[
\Delta_{iq,\text{stat}}(T) = q_0 \sum_{n=1}^{\infty} W_n(\xi) \frac{\gamma^2 \lambda_n^4}{\gamma^2 \lambda_n^4} \left[ \int_0^T W_n(\xi) d\xi \right],
\]
(42)

and the set of integral equations (40) is replaced by the set algebraic equations
\[
\sum_{j=1}^{k} d_{ij,\text{stat}} X_j(T) + \Delta_{iP/q,\text{stat}}(T) = 0, \quad i = 1, 2, \ldots, k.
\]
(43)

After solving the Volterra integral equations the response of the multispan beam under moving load is given by solution for a single-span beam and has the form
\[
w_d(\xi, T) = w^{P/q}(\xi, T)
\]
\[
+ \sum_{j=1}^{k} \int_0^T d_{ij}(T - \tau) X_j(\tau) d\tau.
\]
(44)

The static displacement of the multispan beam under force $P$ concentrated in the point $T$ is equal to
\[
w_{\text{stat}}(\xi, T) = \frac{L^3}{EI} \sum_{n=1}^{\infty} \left[ \int_0^T P W_n(T) + \sum_{i=1}^{k} X_i W_n(\xi_i) \right] W_n(\xi) \frac{\gamma^2 \lambda_n^4}{\gamma^2 \lambda_n^4},
\]
(45)

while the static displacement of the multispan beam subjected to a load $q$ distributed on the length $T$ from the left end of the beam is equal to
The dynamic factor can be defined as the ratio of the dynamic to static displacement:

\[ \Psi(\xi, T) = \frac{w_d(\xi, T)}{w_{stat}(\xi, T)}. \]  

Let us assume that the intermediate supports are springs of the stiffness \( s_i \) (Figure 7). In this case, instead of the system of (40) we have a set \( k \) of the Volterra integral equations of the second order:

\[ \frac{L^3}{EI} \cdot \sum_{n=1}^{\infty} \left[ qL \left( \int_{0}^{T} W_n(\xi) \, d\xi \right) + \sum_{i=1}^{k} X_j W_n(\xi_i) \right] W_n(\xi) \frac{\lambda_n^2}{\gamma_n^2} \] \[ (46) \]

The dynamic factor can be defined as the ratio of the dynamic to static displacement:

\[ \Psi(\xi, T) = \frac{w_d(\xi, T)}{w_{stat}(\xi, T)}. \]  

3.1. Numerical Procedure. The set of the Volterra integral equations is difficult to solve analytically. For this reason a simple numerical procedure shall be applied. In the first step the time interval \([0,1]\) is divided into \( N \) equal time segments \( \Delta t = 1/N \) as it is shown on Figure 8. Collocation points \( \tau_R \) are placed in the middle of each segment. This allows us to replace the direct integration from formula (40) with the numeric integration by using the midpoint method [32]:

\[ \int_{0}^{T_R} d_{ij}(T_R - \tau) X_j(\tau) \, d\tau = \sum_{r=1}^{R} d_{ij}(T_R - \tau_r) X_j(\tau_r) \Delta \tau, \]  

where \( T_R = R\Delta t, \tau_r = (r - 0.5)\Delta t, r = 1, 2, \ldots, R, R = 1, 2, \ldots, N. \)

The main purpose of this method is to find values of the redundant forces \( X_j(T) \) for the collocation points \( \tau_R \). This way the \( k \) sets of the Volterra integral equations can be replaced with the \( k \) sets of \( N \) recurrent algebraic equations:

\[ \frac{L\Delta t}{V} \sum_{j=1}^{R} \sum_{r=1}^{R} d_{ij}(T_R - \tau_r) X_j(\tau_r) \Delta \tau = 0, \]  

\[ i = 1, 2, \ldots, k. \]  

\[ (50) \]
The number \( k \) is equal to the number of the redundant mid-span supports.

Let us take a look on an example of a two-span beam \( (k = 1) \). The first three and the last algebraic equations of the numerical procedure have the form

\[
R = 1 \quad \frac{L\Delta \tau}{v} [d_{11} (T_1 - \tau_1) X_1 (\tau_1) + \Delta_{1P/q} (T_1) ] = 0;
\]

\[
R = 2 \quad \frac{L\Delta \tau}{v} [d_{11} (T_2 - \tau_1) X_1 (\tau_1) + d_{11} (T_2 - \tau_2) X_1 (\tau_2) ] + \Delta_{1P/q} (T_2) = 0;
\]

\[
R = 3 \quad \frac{L\Delta \tau}{v} [d_{11} (T_3 - \tau_1) X_1 (\tau_1) + d_{11} (T_3 - \tau_2) X_1 (\tau_2) ] + d_{11} (T_3 - \tau_3) X_1 (\tau_3) + \Delta_{1P/q} (T_3) = 0;
\]

\[
R = N \quad \frac{L\Delta \tau}{v} \sum_{r=1}^{N} d_{11} (T_N - \tau_r) X_1 (\tau_r) + \Delta_{1P/q} (T_N) = 0,
\]

where \( T_R - \tau_R = 0.5\Delta \tau, T_R - \tau_{R-r} = \tau_1 + r\Delta \tau \). After finding values of the force \( X_1 \) in the mid-span redundant support at the time points \( \tau \), we are able to write the equation describing vibrations of the two-span beam:

\[
\begin{align*}
\omega \xi (T, T_R) &= \frac{L \Delta \tau}{v} \sum_{r=1}^{R} h_1 (\xi, T_R - \tau_r) X_1 (\tau_r) + \omega^{P/q} (\xi, T_R). \\
&= \frac{L \Delta \tau}{v} \sum_{r=1}^{R} b_{ij} \sum_{j=R}^{N} \frac{L \Delta \tau}{v} d_{ij} (T_j - \tau_r) X_1 (\tau_r) + \Delta_{1P/q} (T_j) = 0,
\end{align*}
\]

The size of the time step \( \Delta \tau \) selected for the numerical calculations has to be small enough in order to get an acceptable response. This size depends on the highest value \( \omega_{n} = \frac{\lambda}{2\pi} \) used in the series (38) and should be defined as

\[
\Delta \tau \leq \frac{2\pi}{\omega_{n,\text{max}}},
\]

which means that the number of time steps \( N \) has to be equal to

\[
N \geq \frac{\omega_{n,\text{max}}}{2\pi}.
\]

Equation (50) can be presented in the matrix form:

\[
\sum_{j=1}^{k} B_{ij} \cdot \overline{X}_{j} + \overline{\tau}_{i} = 0, \quad i = 1, 2, \ldots, k,
\]

where

\[
B_{ij} = \begin{bmatrix}
\cdot & 0 & 0 \\
\cdots & b_{ij, R_r} & 0 \\
\vdots & \ddots & \vdots \\
& & & \vdots \\
& & & \cdot & \vdots \\
& & & & \cdot & \vdots \\
& & & & & \cdot & \vdots \\
& & & & & & \cdot & \vdots \\
& & & & & & & \cdot \\
\end{bmatrix}_{N \times N},
\]

\[
b_{ij, R_r} = 0 \quad \text{for } r > R,
\]

\[
b_{ij, R_r} = \frac{L}{v} d_{ij} (T_r - \tau_r) \Delta \tau \quad \text{for } r = R,
\]

\[
b_{ij, R_r} = \frac{L}{v} d_{ij} (T_r - \tau_r) \Delta \tau \quad \text{for } r < R,
\]

\[
\overline{X}_{j} = \begin{bmatrix}
X_1 (T_j) \\
X_2 (T_j) \\
\vdots \\
X_N (T_j)
\end{bmatrix}, \quad j = 1, 2, \ldots, k,
\]

\[
\overline{\tau}_{i} = \begin{bmatrix}
\Delta_{1P/q} (T_1) \\
\Delta_{1P/q} (T_2) \\
\vdots \\
\Delta_{1P/q} (T_N)
\end{bmatrix}, \quad i = 1, 2, \ldots, k.
\]

As an example the matrix equation and its solution for the two-span beam have the form

\[
B_{11} \cdot \overline{X}_{1} + \overline{\tau}_{1} = \overline{0} \quad \rightarrow \quad \overline{X}_{1} = -B_{11}^{-1} \cdot \overline{\tau}_{1},
\]

and, for the three-span beam, they have the form

\[
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix} \cdot \begin{bmatrix}
\overline{X}_{1} \\
\overline{X}_{2}
\end{bmatrix} + \begin{bmatrix}
\overline{\tau}_{1} \\
\overline{\tau}_{2}
\end{bmatrix} = \begin{bmatrix}
\overline{0} \\
\overline{0}
\end{bmatrix} \quad \rightarrow \quad \begin{bmatrix}
\overline{X}_{1} \\
\overline{X}_{2}
\end{bmatrix} = - \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}^{-1} \cdot \begin{bmatrix}
\overline{\tau}_{1} \\
\overline{\tau}_{2}
\end{bmatrix}.
\]

4. Numerical Examples

Example 1 (two-span uniform beam). Figures 9 and 10 show a two-span uniform continuous beam considered as an example. The length of each span is equal to \( L/2 = 20 \) m. Flexural stiffness of the beam is equal to \( EI = 2.0 \text{ GNm}^2 \) and the mass per unit length is equal to \( m = 1000 \text{ kg/m} \). Beam is loaded in the first case with a point force of constant magnitude \( P = 10 \text{ kN} \) moving with a constant velocity \( v = 50 \text{ m/s} \) (Figure 9) and in the second case with distributed load \( q = 2 \text{ kN/m} \) moving with the same speed (Figure 10). In the numerical calculations undamped vibrations of points “a” and “b” placed in the center of each span were analyzed. Results shown on Figures 11–14 were obtained by dividing time \( T \) into \( N = 200 \) time steps. Results were also compared
5. Vibrations of the Multispan Beam: Volterra Integral Equations

Example 2 (three-span uniform beam). In the second example a three-span continuous beam (Figures 15 and 16) is...
considered. The length of each span is equal to $\frac{L}{3} = 20$ m. Flexural stiffness of the beam is equal to $EI = 2.0$ GNm$^2$ and the mass per unit length is equal to $m = 1000$ kg/m. Beam is loaded with a point force of constant magnitude $P = 10$ kN in the first case and with the uniform distributed load $q = 2$ kN/m in the second case. Both loads are moving with constant velocity $v = 100$ m/s. In the numerical calculations undamped vibrations of points “a”, “b”, and “c” placed in the center of each span were analyzed. Results shown on Figures 17–22 were obtained by dividing time $T$ into $N = 200$ time steps. Results were also compared with those obtained numerically by using Finite Difference Method and a very good agreement was observed. The dashed line on Figures 17–22 marks the influence line of the static deflection (case of $v = 0$).

Example 3 (three-span uniform beam with elastic supports). The third example is of a three-span uniform beam similar to the beam from Example 2 but elastically supported in points “1” and “2” (Figure 23). Stiffness of the mid-span supports is equal to $s = 50$ MN/m. Values of $L$, $EI$, $m$, $P$, and $v$ are the same as in the previous example. Undamped vibrations of points “a”, “b”, and “c” as well as dynamic deflections of supports “1” and “2” were determined by using Volterra integral equations of the second order. Time $T$ was divided into $N = 200$ time steps. Results shown on Figures 24–28 were also compared with those obtained for the beam with supports of infinite stiffness.

6. Conclusion

The method of determining the transverse vibration of multispans continuous beams, based on an application of Volterra integral equations, presented in this paper can be...
Figure 21: Dynamic deflection of point "b" due to a moving distributed load.

Figure 22: Dynamic deflection of point "c" due to a moving distributed load.

Figure 23: Three-span continuous uniform beam with elastic mid-span supports loaded with moving force.

Figure 24: Dynamic deflection of point "a" (middle of the first span).

Figure 25: Dynamic deflection of point "b" (middle of the second span).

Figure 26: Dynamic deflection of point "c" (middle of the third span).

Figure 27: Dynamic deflection of support "1".

Figure 28: Dynamic deflection of support "2".
successfully applied for arbitrarily supported uniform beams with constant flexural stiffness and constant mass per length. Formulas for undamped vibrations were given. After using appropriate transformations, this method can be applied also for other types of moving load such as a moving nonuniform distributed load, a moving moment, or moving series of point forces. To simplify calculations a numerical procedure that replaces the set of Volterra integral equations with the set of algebraic recurrent equations was presented. The main disadvantage of this method is that it can be applied only for uniform beams.

**Competing Interests**

The authors declare that there are no competing interests regarding the publication of this paper.

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