Braided quantum groups related to the quantum "\(ax + b\)" group.

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Abstract

We consider quantum group theory on the Hilbert space level. We find all unitary representations of three braided quantum groups related to the quantum "\(ax + b\)" group. First we introduce an auxiliary braided quantum group, which is apparently not related to the quantum "\(ax + b\)" group, but easy to work with. We find all unitary representations of this quantum group. Then we use this result to find all unitary representations of another braided quantum group, whose \(C^*\)-algebra is generated by two (out of three) generators generating (in the sense of Woronowicz) \(C^*\)-algebra of the quantum "\(ax + b\)"-group. We find all unitary representations of this other braided quantum group. This is the most difficult result needed to classify all unitary representations of the quantum "\(ax + b\)" group.

key words: unbounded operators – Hilbert space – braided quantum group
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1 Introduction

There are three levels on which one can consider quantum group theory, namely the Hopf \(\ast\)-algebra level, the \(C^*\)-algebra level and the Hilbert space level. In this paper we restrict ourselves to this last level, i.e. we consider unbounded operators acting on Hilbert space and encounter various problems related to their domains and selfadjoint extensions of sum of such operators.

Let \(G\) be a set of closed operators acting on a Hilbert space, invariant under direct sum and unitary tranformations, i.e. an operator domain. The group structure may be then introduced by an associative map from \(G \times G\) into \(G\). Then, rouhgly speaking, \(G\) is quantum group. If the Cartesian product \(G \times G\) is non-trivial, i.e. operators from the first copy of \(G\) do not commute with operators from the second copy of it, then \(G\) is a braided quantum group.

Braided quantum groups considered in this paper are related to the quantum deformation of the "\(ax + b\)" group (i.e. group of affine mappings of the real line). They are called \(A\), \(N\) and \(M\). Operator domain \(A\) is commutative, which means that \(A\) may be identified with a locally compact space. This space is a sum of three half-lines with common origin. Similarly, \(N\) can be identified with a locally compact classical space consisting of four

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half-lines with common origin. The only quantum, or non-classical, property of \( A \) and \( N \) is non-triviality of the Cartesian products \( A \times A \) and \( N \times N \). The operator domain \( M \) is purely quantum - \( M \) is not commutative and the Cartesian product \( M \times M \) is non-trivial, i.e. there is non-trivial braiding.

In this paper we find all unitary representations of the quantum groups \( A, N \) and \( M \). It turns out that every unitary representation of \( A \) is a direct integral of one-dimensional representations. Since these representations are already found \[22\], we are able to give a formula for all unitary representation of the braided quantum group \( A \). Having this, it is not difficult to find all unitary representations of the braided quantum group \( N \). It is done in Section \[3\].

The most interesting result of this paper is finding all unitary representations of the braided quantum group \( M \). However, the straightforward way to do it requires coping with operator functions of noncommuting unbounded operators. Functions of unbounded, but commuting, operators are much easier to handle, and that is why we start with the case of commutative operator domain \( N \). So it is a great advantage that we invented a simple trick that allows us to “translate” our previous results concerning \( N \) to the case of \( M \). This way the problem of finding all unitary representations of \( M \) is solved in Section \[3\]. This result is essential for classification of all unitary representations of the quantum group ”ax+b”, which is achieved in our forthcoming paper \[16\].

In the remaining part of this section we introduce some non-standard notation and notions used in this paper.

### 1.1 Notation

We denote Hilbert spaces by \( \mathcal{H} \) and \( \mathcal{K} \), the set of all closed operators acting on \( \mathcal{H} \) by \( C(\mathcal{H}) \), the set of bounded operators by \( B(\mathcal{H}) \) and the sets of compact and unitary ones by \( CB(\mathcal{H}) \) and \( \text{Unit}(\mathcal{H}) \), respectively. The set of all continuous vanishing at infinity functions on a space \( X \) will be denoted by \( C_\infty(X) \). We consider only separable Hilbert spaces, usually infinite-dimensional. We denote scalar product by \( (\cdot|\cdot) \) and it is antilinear in the first variable. We consider mainly unbounded linear operators. All operators considered are densely defined. We use functional calculus of selfadjoint operators \[13, 14, 18\]. We also use the symbol sign \( T \) for partial isometry obtained from polar decomposition of an selfadjoint operator \( T \). The end of a proof will be marked by \( \square \).

We use a non-standard, but very useful notation for orthogonal projections and their images \[22\], as explained below. Let \( a \) and \( b \) be strongly commuting selfadjoint operators acting on a Hilbert space \( \mathcal{H} \). Then by spectral theorem there exists a common spectral measure \( dE(\lambda) \) such that

\[
a = \int_{\mathbb{R}^2} \lambda dE(\lambda, \mu), \\
b = \int_{\mathbb{R}^2} \mu dE(\lambda, \mu).
\]

For every complex measurable function \( f \) of two variables

\[
f(a, b) = \int_{\mathbb{R}^2} f(\lambda, \lambda') dE(\lambda, \lambda').
\]

Let \( f \) be a logical sentence and let \( \chi(f) \) be 0 if is false, and 1 otherwise. If \( \mathcal{R} \) is a binary relation on \( \mathbb{R} \) then \( f(\lambda, \lambda') = \chi(\mathcal{R}(\lambda, \lambda')) \) is a characteristic function of a set

\[
\Delta = \{(\lambda, \lambda') \in \mathbb{R}^2 : \mathcal{R}(\lambda, \lambda')\}
\]
and assuming that $\Delta$ is measurable $f(a,b) = E(\Delta)$. From now on we will write $\chi(\mathcal{R}(a,b))$ instead of $f(a,b)$:

$$\chi(\mathcal{R}(a,b)) = \int_{\mathbb{R}^2} \chi(\mathcal{R}(\lambda,\lambda')) dE(\lambda,\lambda') = E(\Delta).$$

Image of this projector will be denoted by $\mathcal{H}(\mathcal{R}(a,b))$, where ‘$\mathcal{H}$’ is a Hilbert space, on which operators $a,b$ act.

Thus we defined symbols $\chi(a > b)$, $\chi(a^2 + b^2 = 1)$, $\chi(a = 1)$, $\chi(b < 0)$, $\chi(a \neq 0)$ etc. They are orthonormal projections on appropriate spectral subspaces. For example $\mathcal{H}(a = 1)$ is an eigenspace of operator $a$ for eigenvalue 1 and $\chi(a = 1)$ is orthonormal projector on this eigenspace.

Generally, whenever $\Delta$ is a measurable subset of $\mathbb{R}$, then $\mathcal{H}(a \in \Delta)$ is spectral subspace of an operator $a$ corresponding to $\Delta$ and $\chi(a \in \Delta)$ is its spectral projection.

### 1.2 Zakrzewski relation

Let $-\pi < h < \pi$. The Zakrzewski relation was introduced in [22] by

**Definition 1.1** Let $R$ and $S$ be self-adjoint operators acting on Hilbert space $\mathcal{H}$. Operators $R$ and $S$ are in Zakrzewski relation $R \sim S$ if

1. sign $R$ commutes with $S$ and sign $S$ commutes with $R$
2. On subspace $(\ker R) \perp \cap (\ker S) \perp$ we have

$$|R|^l |S|^k = e^{ihlk} |S|^k |R|^l$$

for any $l, k \in \mathbb{R}$.

Observe, that whenever we say that certain $R$ and $S$ satisfy Zakrzewski relation we also have to specify parameter $h$.

**Example 1.2** Let $\hat{q}$ and $\hat{p}$ denote the position and momentum operators in Schrödinger representation, i.e. we set $\mathcal{H} = L^2(\mathbb{R})$. Then the domain of $\hat{q}$

$$D(\hat{q}) = \{ \psi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} x^2 |\psi(x)|^2 dx < \infty \}$$

and $\hat{q}$ is multiplication by coordinate operator on that domain

$$(\hat{q}\psi)(x) = x\psi(x).$$

The domain of $\hat{p}$ consists of all distributions from $L^2(\mathbb{R})$ such that

$$D(\hat{p}) = \{ \psi \in L^2(\mathbb{R}) : \psi' \in L^2(\mathbb{R}) \}$$

and for any $\psi \in D(\hat{p})$

$$(\hat{p}\psi)(x) = \frac{h}{i} \frac{df(x)}{dx},$$

where $-\pi < h < \pi$.

Operators $e^{\hat{p}}$ and $e^{\hat{q}}$ acting on $L^2(\mathbb{R})$ satisfy Zakrzewski relation.
Remark 1.3 Two strictly positive, i.e. positive and invertible, selfadjoint operators satisfy Zakrzewski relation iff, they satisfy Weyl commutation relations $[14]$.

Example 1.4 Let $R$ and $S$ act on Hilbert space $L^2(\mathbb{R})^\oplus 4$.

Let

$$ R = \begin{bmatrix}
    0 & e^\hat{p} & 0 & 0 \\
    0 & 0 & 0 & e^\hat{p} \\
    0 & e^\hat{p} & 0 & 0 \\
    0 & 0 & e^\hat{p} & 0 \\
\end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix}
    0 & e^\hat{q} & 0 & 0 \\
    0 & 0 & e^\hat{q} & 0 \\
    0 & 0 & 0 & e^\hat{q} \\
    0 & 0 & 0 & 0 \\
\end{bmatrix}, $$

Hence

$$ \text{sign } R = \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & -1 & 0 \\
    0 & 0 & 0 & -1 \\
\end{bmatrix} \quad \text{and} \quad \text{sign } S = \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & -1 \\
\end{bmatrix}. $$

Moreover $|R|$ and $|S|$ satisfy Weyl commutation relations, where

$$ |R| = \begin{bmatrix}
    e^\hat{q} & 0 & 0 & 0 \\
    0 & e^\hat{q} & 0 & 0 \\
    0 & 0 & e^\hat{q} & 0 \\
    0 & 0 & 0 & e^\hat{q} \\
\end{bmatrix} \quad \text{and} \quad |S| = \begin{bmatrix}
    e^\hat{p} & 0 & 0 & 0 \\
    0 & e^\hat{p} & 0 & 0 \\
    0 & 0 & e^\hat{p} & 0 \\
    0 & 0 & 0 & e^\hat{p} \\
\end{bmatrix}. $$

All pairs of operators satisfying Zakrzewski relations are in some sense builded from operators $e^\hat{q}$ and $e^\hat{p}$. Precisely, S. L. Woronowicz [22] proved

Proposition 1.5 Let $R$ and $S$ be operators acting on Hilbert space $\mathcal{H}$, let $\ker R = \ker S = \{0\}$ and let $R \circ S$. Then every pair $(R, S)$ is unitarily equivalent to the pair $(u \otimes e^\hat{p}, v \otimes e^\hat{q})$ acting on Hilbert space $\mathcal{K} \otimes L^2(\mathbb{R})$, where $u, v$ are unitary, selfadjoint and mutually commuting operators acting on Hilbert space $\mathcal{K}$.

$$ u = u^* = u^{-1}, \quad v = v^* = v^{-1} \quad uv = vu $$

Selfadjoint extensions of sum $R + S$, where $R \circ S$, proved to be very important in constructing quantum deformation of the “ax + b” group and were studied by Woronowicz in [22]. Operator $R+S$ is symmetric, but in general not selfadjoint. To make the thing worse, sometimes there are even no selfadjoint extensions of such a sum. However, a selfadjoint extension of the sum $R + S$ exists, if there exists a selfadjoint operator $\tau$, such that $\tau$ anticommutes with $R$ and $S$ and

$$ \tau^2 = \chi(e^{\frac{\phi}{2}}RS < 0). $$

Any selfadjoint extension is described uniquely by this operator $\tau$, so we denote it by $[R+S]_\tau$. It is given by

$$ [R+S]_\tau = (R+S)^*|_{D(R+S)+D((R+S)^*) \cap \mathcal{H}(\tau=1)}. $$
### 1.3 Operator domains and operator functions

We introduce now two notions very important for understanding this paper, operator domains and operator functions. They are generalization of notion of function and its domain to the case of function of “non-commuting operator variables”.

The definitions are chosen in such a way that operator functions and domains respect symmetries of Hilbert space. We are more precise below.

**Definition 1.6** Let for any Hilbert space $\mathcal{H}$ be given a subset $\mathcal{D}_\mathcal{H} \subset C(\mathcal{H})^N$. We say that $\mathcal{D}$ is $N$-dimensional operator domain if:

1. For any Hilbert space $\mathcal{H}$ and $\mathcal{K}$ and for any unitary operator $V : \mathcal{H} \to \mathcal{K}$ and for any element
   
   $$x = (x_1, x_2, ..., x_N) \in \mathcal{D}_\mathcal{H} \subset C(\mathcal{H})^N$$
   
   we have
   
   $$VxV^* = (Vx_1V^*, Vx_2V^*, ..., Vx_NV^*) \in \mathcal{D}_\mathcal{K}$$

2. For any space with measure $(\Lambda, \mu)$ and for any measurable field $\mathcal{F}$ of Hilbert spaces $\{\mathcal{H}(\lambda)\}_{\lambda \in \Lambda}$ and for any measurable field of closed operators $\{a(\lambda)\}_{\lambda \in \Lambda}$ we have
   
   $$\int_{\Lambda} \oplus a(\lambda)d\mu(\lambda) \in D_{\mathcal{F}}^{\oplus} \mathcal{H}(\lambda)d\mu(\lambda)$$

   iff, $a(\lambda) \in D_{\mathcal{H}(\lambda)}$ for $\mu-$ almost all $\lambda \in \Lambda$.

The notion of a measurable field of closed operators is not widely known, but it can be easily reduced to the more popular notion of a measurable field of bounded operators. For any $T \in C(\mathcal{H})$ its $z$-transform is defined by

$$z_T = T(I + T^*T)^{-\frac{1}{2}}.$$ 

Observe that $z_T$ is a bounded operator and $T$ is uniquely determined by $z_T$. For more details see [26].

We say that a field of closed operators

$$\Lambda \ni \lambda \to a(\lambda) \in C(\mathcal{H}(\lambda))$$

is measurable if

$$\Lambda \ni \lambda \to z_{a(\lambda)} \in B(\mathcal{H}(\lambda))$$

is a measurable field of bounded operators. Then there exists a unique operator

$$a \in C(\int_{\Lambda}^{\oplus} \mathcal{H}(\lambda)d\mu(\lambda))$$

such that

$$z_a = \int_{\Lambda}^{\oplus} z_{a(\lambda)}d\mu(\lambda).$$

\footnote{Its definition, as well as the definition of a measurable field of closed operators, can be found in [21]. See also [4].}
We call operator $a$ a direct integral of the field (1.3) and denote it by $\int_{\Lambda} a(\lambda) d\mu(\lambda)$. For bounded operators a notion of direct integral introduced above coincides with that used in [3].

In particular, when $\Lambda$ is a countable set and $\mu$ is a counting measure $H = \bigoplus_{\lambda \in \Lambda} H(\lambda)$ and condition 2. takes form

$$\bigoplus_{\lambda \in \Lambda} a(\lambda) \in D_{H(\lambda)}$$

iff

$$a(\lambda) \in D_{H(\lambda)} \quad \text{for any} \quad \lambda \in \Lambda .$$

Observe that Hilbert space $H$ plays a role of a variable in this scheme, i.e. difference between an operator domain $D$ and a set $D_H$ is such as between a function $f$ and its value at a point $x, f(x)$. All operator domains considered are not closed in the norm topology.

Observe that an operator domain is a category; the bounded intertwining operators are its morphisms. For more details see [21] and [26].

Example 1.7 The set

$$A_H = \{(R, \rho) \in C(H)^2 \mid R = R^*, \rho = \rho^* \quad \text{and} \quad R\rho = \rho R \quad \text{and} \quad \rho^2 = \chi(R < 0)\}$$

is an operator domain. This operator domain was considered in [22].

Example 1.8 The set

$$N_H = \{(K, \kappa) \in C(H)^2 \mid K = K^*, \kappa = \kappa^* \quad \text{and} \quad K\kappa = \kappa K \quad \text{and} \quad \kappa^2 = \chi(K \neq 0)\}$$

is also an operator domain.

This operator domain will be discussed in Section 2.

Example 1.9 The set

$$M_H = \{(b, \beta) \in C(H)^2 \mid b = b^*, \beta = \beta^* \quad \text{and} \quad b\beta = -\beta b \quad \text{and} \quad \beta^2 = \chi(b \neq 0)\}$$

is also an operator domain. This operator domain will be discussed in Section 3.

These examples show that the description of operator domains is similar to a description of manifold given by a set of equations. For example the sphere

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

is described by giving coordinates $(x_1, x_2, x_3)$ and relations between them $(x_1^2 + x_2^2 + x_3^2 = 1)$. Therefore unbounded operators entering descriptions of operator domains can be thought of as “coordinates on a quantum space”.

The operator functions can be thought of as a recipee what to do with a $N$-tuple of closed operators $(a_1, a_2, ..., a_N)$ to obtain another closed operator $F(a_1, a_2, ..., a_N)$.

Definition 1.10 Let $D$ be an operator domain and let for any Hilbert space $\mathcal{H}$ be given a map $F_\mathcal{H} : D_\mathcal{H} \to C(\mathcal{H})$. We say that $F$ is a measurable operator function if
1. For any Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and for any unitary operator $V : \mathcal{H} \to \mathcal{K}$ and for any element $x \in \mathcal{D}_\mathcal{H}$

we have

$$F_\mathcal{K}(VxV^*) = VF_\mathcal{H}(x)V^*$$

2. For any space with a measure $(\Lambda, \mu)$ and for any measurable field of Hilbert spaces $\{\mathcal{H}(\lambda)\}_{\lambda \in \Lambda}$ and for any $a \in \mathcal{D}_\mathcal{H}$ having decomposition

$$a = \int_\Lambda a(\lambda) d\mu(\lambda) \in D^{\oplus} f_{\lambda}^{\oplus} \mathcal{H}(\lambda)d\mu(\lambda)$$

the field of operators $\{F_{\mathcal{H}(\lambda)}(a(\lambda))\}_{\lambda \in \Lambda}$ is measurable and

$$F^{\oplus}_{\lambda} f_{\lambda}^{\oplus} \mathcal{H}(\lambda)d\mu(\lambda)(a) = \int_\Lambda F_{\mathcal{H}(\lambda)}(a(\lambda))d\mu(\lambda).$$

For example, if $(a_1, a_2, ..., a_N) \in D$ and operators $a_i$ are normal, mutually strongly commuting and for any Hilbert space $\mathcal{H}$ their joint spectrum is contained in a set $\Lambda \subset \mathbb{C}$, then measurable operator functions on $D$ are simply measurable functions on $\Lambda$. It shows that the above definition is a generalisation of the functional calculus of measurable functions of strongly commuting normal operators to the case of non-commuting closed, but not necessary normal, ones. However in this paper we consider only operator functions of selfadjoint, but often non-commuting, operators.

We also need a notion of an operator maps between operator domains.

**Definition 1.11** Let $M$ be an operator domain and let $N$ be a $k$-dimensional operator domain. Moreover, let $F = (F^1, F^2, ..., F^k)$,

where $F^i$ for $i = 1, 2, ..., k$ are operator functions on $M$. If for any Hilbert space $\mathcal{H}$ and for any $m \in \mathcal{M}_\mathcal{H}$ we have

$$F_{\mathcal{H}}(m) = (F^1_{\mathcal{H}}(m), F^2_{\mathcal{H}}(m), ..., F^k_{\mathcal{H}}(m)) \in \mathcal{N}_\mathcal{H}$$

then we call $F$ an operator map from an operator domain $M$ into an operator domain $N$.

1.4 Quantum groups and braided quantum groups on the Hilbert space level

The definition of quantum group is still under construction, there are however many examples and one knows approximately what a quantum group should be.

Let $G$ be an operator domain and let $G \times G$ denote an operator domain

$$G \times G := \{(x, y) \mid x, y \in G \text{ and } xy = yx\}$$

Let $\cdot$ be an operator map

$$\cdot : G \times G \ni (x, y) \to xy \in G$$

Roughly speaking, a quantum group $G$ is such an operator domain $G$ equipped with an associative operator map $\cdot$. 


Example 1.12 (Quantum $SU_q(2)$ group) Let us define an operator domain

$$SU_q(2) = \left\{ (\alpha, \gamma) \in B(H) : \begin{array}{l} \alpha \alpha^* + \gamma \gamma^* = I; \\ \alpha \gamma = q \gamma \alpha; \\ \gamma \gamma^* = \gamma^* \gamma; \\ \alpha^* \gamma = q^* \gamma^* \alpha; \\ \alpha^* \alpha + q^2 \gamma \gamma^* = I \end{array} \right\}.$$ 

Moreover, let us define an operator map $\cdot$ by

$$\cdot : SU_q(2) \times SU_q(2) \ni ((\alpha_1, \gamma_1), (\alpha_2, \gamma_2)) \to (\alpha, \gamma) \in SU_q(2)$$

where

$$\alpha = \alpha_1 \alpha_2 - q \gamma_1^* \gamma_2 \quad \text{and} \quad \gamma = \gamma_1 \alpha_2 + \alpha_1^* \gamma_2$$

This operation is associative, and $SU_q(2)$ together with this operation forms a quantum group. Since in the definition of the operation domain $SU_q(2)$ we restricted ourselves to the bounded operators, $SU_q(2)$ is a compact quantum group.

Example 1.13 (Quantum “$ax+b$” group) Let us define an operator domain

$$G_H = \left\{ (a, b, \beta) \in C(H)^3 : \begin{array}{l} a > 0 \\ a \rightarrow b \\ (b, \beta) \in M_H \\ a \beta = \beta a \end{array} \right\},$$

where $M_H$ is the operator domain defined in Example 1.12.

Group operation $\oplus$ on $G_H$ is given by

$$\oplus : G_H \times G_H \ni ((a_1, b_1, \beta_1), (a_2, b_2, \beta_2)) \to (a, b, \beta) \in G_H$$

where

$$a = a_1 \otimes a_2 \quad \text{and} \quad b = [a_1 \otimes b_2 + b_1 \otimes I]_{(1)k(\beta_1 \otimes \beta_2) \chi(b_1 \otimes b_2 < 0)}$$

Formula for $\beta$ is much more complicated. This additional “generator” $\beta$ is needed to ensure the existence of a selfadjoint extension of $a_1 \otimes b_2 + b_1 \otimes I$.

To make group operation $\oplus$ associative one has assume that

$$h = \pm \frac{\pi}{2k + 3}, \quad \text{where} \quad k = 0, 1, 2, \ldots$$

and $k$ is the same as chosen in formula for selfadjoint extension of $a_1 \otimes b_2 + b_1 \otimes I$.

An operator domain $G$ with operation $\oplus$ defined as above forms the quantum “$ax+b$” group $[22]$. Observe that from $a \rightarrow b$ follows that operators $a$ and $b$ are not bounded. Therefore the quantum “$ax+b$” group is non-compact.

The theory of non-compact quantum groups is more difficult, more interesting and less developed than that of compact ones. The most important examples of non-compact quantum groups are the quantum $E(2)$ group, quantum Lorentz group and quantum groups ”$ax+b$" and ”$az+b$" $[24], [23], [11], [12], [23], [27]."
What we study in this paper are braided quantum groups. The main difference between a braided quantum group and a quantum group is that a group operation on a braided quantum group $G$ is defined on an operator domain

$$G^2 := \{ (x, y) \mid x, y \in G \text{ and } x, y \text{ satisfy certain relations} \}.$$ 

Usually we do not assume that operators from both copies of $G$ commute, so in general a braided quantum group is not a quantum group. A group operation on a braided quantum group $G$ should be the operator map

$$\triangledown : G^2 \ni (x, y) \to x \triangledown y \in G$$

which is associative.

Roughly speaking, a braided quantum group $G$ is an operator domain $G$ equipped with such an operator map $\triangledown$.

The objects $N$, $A$ and $M$ investigated in this paper are examples of braided quantum groups.

## 2 The braided quantum group $N$

Let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space. Consider operators $R$ and $\rho$ acting on $\mathcal{H}$ and such that

$$R = R^* \quad \text{and} \quad \rho = \rho^* \quad \text{and} \quad \rho R = R \rho \quad \text{and} \quad \rho^2 = \chi(R \neq 0).$$

Let

$$N_\mathcal{H} = \{ (R, \rho) \in C(\mathcal{H}) : R = R^* \quad \rho = \rho^* \quad \rho R = R \rho \quad \rho^2 = \chi(R \neq 0) \}. \quad (1)$$

It is easily seen that $N$ is an operator domain. To ensure existence of selfadjoint extensions of a sum $R + S$, where $(R, \rho), (S, \sigma) \in N_\mathcal{H}$, we introduce an additional condition for pairs $(R, \rho), (S, \sigma) \in N_\mathcal{H}$ to fulfill. This condition is

$$R - \circ S \quad \text{and} \quad S \rho = - \rho S \quad \text{and} \quad R \sigma = - \sigma R \quad \text{and} \quad \rho \sigma = \sigma \rho. \quad (2)$$

If pairs $(R, \rho), (S, \sigma) \in N_\mathcal{H}$ satisfy this condition we write

$$(R, \rho), (S, \sigma) \in N^2_\mathcal{H},$$

i.e.

$$N^2_\mathcal{H} = \{ (R, \rho), (S, \sigma) \in N_\mathcal{H} : R - \circ S \quad \text{and} \quad S \rho = - \rho S \quad \text{and} \quad R \sigma = - \sigma R \quad \text{and} \quad \rho \sigma = \sigma \rho \}.$$

Observe, that $N^2$ is also an operator domain.

To give a formula for selfadjoint extensions of a sum $R + S$ we need the quantum exponential function.

### 2.1 The special function $V_\theta$ and the quantum exponential function $F_\hbar$

The special function $V_\theta$ is defined by

$$V_\theta(x) = \exp \left\{ \frac{1}{2\pi i} \int_0^\infty \log(1 + a^{-\theta}) \frac{da}{a + e^{-x}} \right\}$$
for any $x \in \mathbb{C}$ such that $|\Im x| < \pi$. $V_\theta$ can be extended to a function meromorphic on $\mathbb{C}$. Let

$$
\Omega_h^+ = \{ r \in \mathbb{C} \setminus \{0\} : \arg r \in [0, h] \} \\
\Omega_h^- = \{ r \in \mathbb{C} \setminus \{0\} : \arg r \in [-\pi, h - \pi] \} \\
\Delta = \Omega_h^+ \times \{0\} \cup \Omega_h^- \times \{-1, 1\}
$$

The quantum exponential function $F_h$ is a $q$-analogue of the exponential function fit for operators satisfying commutation rules of the type $[2]$. What we mean exactly by this is explained in Proposition 2.2. The function $F_h$ is defined for any $(r, \rho) \in \Delta$ by

$$
F_h(r, \rho) = [1 + i\rho(-r)^{\frac{\pi}{h}}] V_\theta(\log r)
$$

In particular, for $(r, \rho) \in \Delta_{\text{real}} := \{(r, \rho) \in \Delta : r \in \mathbb{R}\}$

$$
F_h(r, \rho) = \begin{cases} 
V_\theta(\log r) & \text{dla } r > 0 \text{ and } \rho = 0 \\
\{1 + i\rho|r|^{\frac{\pi}{h}}\} V_\theta(\log |r| - \pi i) & \text{dla } r < 0 \text{ and } \rho = \pm 1
\end{cases}
$$

Axiomatic introduction of $F_h$ and some other properties of $F_h$ and $V_\theta$ can be found in Section 2 of [22]. The function $F_h$ is extended to the closure of $\Delta_{\text{real}}$ by setting

$$
F_h(0, \rho) = 1.
$$

2.2 The group operation on $N$

Observe that if $(R, \rho) \in N_\mathcal{H}$ then $R$ commutes with $\rho \chi(R < 0)$ and the joint spectrum of these operators is the closure of $\Delta_{\text{real}}$. Therefore $F_h(R, \rho \chi(R < 0))$ is well defined. Moreover, since $(R, \rho \chi(R < 0)) \in A_\mathcal{H}$ and

$$
F_h : \Delta_{\text{real}} \rightarrow \mathbb{C}
$$

is a measurable function, we know from Introduction that $F_h$ is an operator function defined on $A$.

Let $((R, \rho), (S, \sigma)) \in N_\mathcal{H}^2$. Assume that $\ker S = \{0\}$. This assumption is not very restrictive since every $S$ is a direct sum of invertible $S_1$ and $S_2 = 0$, and the case $S_2 = 0$ is trivial. Define

$$
T = e^{i\frac{h}{\pi} S^{-1} R} \quad \tau = (-1)^k \rho \sigma,
$$

where $k \in \mathbb{N}$ and $k$ are related to $h$ by

$$
h = \pm \frac{\pi}{2k + 3}.
$$

Define also

$$
[R + S]_{\tau \chi(T < 0)} = F_h(T, \tau \chi(T < 0)) S F_h(T, \tau \chi(T < 0))
$$

and

$$
\tilde{\sigma} = F_h(T, \tau \chi(T < 0)) S F_h(T, \tau \chi(T < 0)).
$$

Since $F_h(T, \tau \chi(T < 0))$ is an unitary operator, we see that $([R + S]_{\tau \chi(T < 0)}, \tilde{\sigma}) \in N_\mathcal{H}$. Therefore an operation

$$
\odot_N : N_\mathcal{H}^2 \rightarrow N_\mathcal{H} \\
(R, \rho) \odot_N (S, \sigma) = ([R + S]_{\tau \chi(T < 0)}, \tilde{\sigma})
$$
is well defined. Moreover, it is not difficult to prove that this operation is associative. The operator domain $N$ together with the operation $\boxplus_A$ is a braided quantum group.

Let $A$ be the operator domain as in Example 1.7. Define an operator map

$$\varphi : N_H \to A_H$$

for any $(R, \rho) \in N_H$ by

$$\varphi(R, \rho) = (R, \rho \chi(R < 0)). \quad (6)$$

The operator domain $A^2$ and the group operation $\boxplus_A$ on $A$ were described in [22]. We prove now a colorally, which enables us to apply results obtained for $A$ in that paper to (1).

We prove (see Corollary 2.1 below) that for any $((R, \rho), (S, \sigma)) \in \mathcal{N}^H_2$ we have

$$\varphi((R, \rho) \boxplus_A (S, \sigma)) = \varphi(R, \rho) \boxplus_A \varphi(S, \sigma).$$

Moreover, next Colorally 2.2 states that

$$F_\hbar(\varphi((R, \rho) \boxplus_A (S, \sigma))) = F_\hbar(\varphi(R, \rho)) F_\hbar(\varphi(S, \sigma)).$$

**Corollary 2.1** Let $((R, \rho), (S, \sigma)) \in \mathcal{N}^H_2$. Define

$$\hat{\tau} = (-1)^k \rho \chi(R < 0) \sigma \chi(S < 0) + (-1)^k \sigma \chi(S < 0) \rho \chi(R < 0)$$

and

$$\hat{\sigma} = \sigma \chi([R + S]_{\tau \chi(T < 0)} < 0).$$

Then

$$\tau \chi(T < 0) = \hat{\tau} \quad \text{and} \quad \hat{\sigma} \chi(S < 0) = \hat{\sigma}.$$

**Proof:** Since $R$ and $S$ satisfy Zakrzewski relation, it follows that $R$ commutes with sign $S$ and $S$ commutes with sign $R$. Hence

$$R \sigma \chi(S < 0) = \sigma \chi(S < 0) R \quad \text{and} \quad S \rho \chi(R < 0) = \rho \chi(R < 0) S.$$ 

This means that if $((R, \rho), (S, \sigma)) \in \mathcal{N}^H_2$ then $(R, \rho \chi(R < 0), S, \sigma \chi(S < 0))$ satisfies assumptions of Theorem 6.1 [22]. By this theorem the sum $R + S$ has a selfadjoint extension, determined uniquely by a reflection operator $\hat{\tau}$ such that

$$\hat{\tau} = (-1)^k \rho \sigma \chi([R < 0]) \sigma \chi(S < 0) + (-1)^k \sigma \chi(S < 0) \rho \chi(R < 0).$$

Since $\sigma$ anticommutes with $R$, it follows that $\chi(R < 0) \sigma = \sigma \chi(R > 0)$ and analogously for $\rho$ and $S$. Hence

$$\hat{\tau} = (-1)^k \rho \sigma \{\chi(R > 0) \chi(S < 0) + \chi(S > 0) \chi(R < 0)\} = (-1)^k \rho \sigma \chi(e^{i\pi/2} S^{-1} R < 0).$$

Comparing this result with formula (6) we see that $\hat{\tau} = \tau$. It remains to prove that

$$\hat{\sigma} = \sigma \chi([R + S]_{\tau \chi(T < 0)} < 0).$$

Compute

$$\hat{\sigma} = \hat{\sigma} F_\hbar(T, \tau \chi(T < 0))^* \sigma \chi(S < 0) F_\hbar(T, \tau \chi(T < 0)) =
\sigma F_\hbar(T, \tau \chi(T < 0))^* \chi(S < 0) F_\hbar(T, \tau \chi(T < 0)) = \sigma \chi([R + S]_{\tau \chi(T < 0)} < 0). \quad \square$$

By Theorem 6.1 [22] we obtain
Proposition 2.2 Let \((R, \rho), (S, \sigma) \in N_2^2 \) and let \( \ker S = \{0\} \). Then
\[
F_\hbar([R + S]_{T < 0}, \tilde{\sigma} \chi([R + S]_{T < 0} < 0)) = F_\hbar(R, \rho \chi(R < 0))F_\hbar(S, \sigma \chi(S < 0)). \tag{7}
\]
The above Proposition explains why \( F_\hbar \) is called the quantum exponential function. Moreover, the quantum exponential function \( F_\hbar \), as the classical exponential one, is the only function (up to a parameter) satisfying the exponential equation (7) (see theorem below).

Theorem 2.3 Let \((R, \rho), (S, \sigma) \in N_2^2 \) and let \( f : \Delta_{\text{real}} \to S^1 \) be a measurable function. Then the following conditions are equivalent

a). \( f(\varphi(R, \rho))f(\varphi(S, \sigma)) = f(\varphi((R, \rho) \oplus_N (S, \sigma))) \)

b). There exists \( M \geq 0 \) and \( \mu = \pm 1 \), such that
\[
f(\varphi(r, \rho)) = F_\hbar(\varphi(Mr, \mu \rho)) \text{ for a.a.} (r, \rho) \in \mathbb{R} \times \{-1, 1\}.
\]

Proof: b). \( \Rightarrow \) a). We first consider the case \( M = 0 \). Then
\[
F_\hbar(Mr, \mu \rho) = F_\hbar(0, \mu \rho) = 1,
\]
because by Theorem 1.1 \[22\]
\[
\lim_{r \to 0} F_\hbar(r, \rho) = 1.
\]
It is easily seen that if \( M > 0 \) and \( \mu = \pm 1 \) then
\[
((MR, \mu \rho), (MS, \mu \sigma)) \in N_2^2 \]
and \( \ker MS = \{0\} \). Thus assumptions of Corollary \[22\] are satisfied and therefore function
\[
f(r, \rho \chi(r < 0)) = F_\hbar(Mr, \mu \rho \chi(Mr < 0))
\]
satisfies \[23\].

b). \( \Leftarrow \) a). By Corollary \[21\] we may apply Theorem 7.1 \[22\], and b) follows. \(\square\)

2.3 The matrix representation of \( N \)
Consider \((R, \rho), (S, \sigma) \in N_2^2 \). Since \( R \rightarrow S \), operators \( R \) and \( S \) commute with sign \( R \) and sign \( S \). Therefore we may introduce notation
\[
\mathcal{H}_{++} = \mathcal{H}(R > 0) \cap \mathcal{H}(S > 0)
\]
\[
\mathcal{H}_{+-} = \mathcal{H}(R > 0) \cap \mathcal{H}(S < 0)
\]
\[
\mathcal{H}_{-+} = \mathcal{H}(R < 0) \cap \mathcal{H}(S > 0)
\]
\[
\mathcal{H}_{--} = \mathcal{H}(R < 0) \cap \mathcal{H}(S < 0)
\]
Then \( \mathcal{H} = \mathcal{H}_{++} \oplus \mathcal{H}_{+-} \oplus \mathcal{H}_{-+} \oplus \mathcal{H}_{--} \).

Any vector \( \psi \) from the space \( \mathcal{H} \) is represented by
\[
\psi = \begin{bmatrix}
\psi_{++} \\
\psi_{+-} \\
\psi_{-+} \\
\psi_{--}
\end{bmatrix}
\]
where \( \psi_{++} \in \mathcal{H}_{++}, \psi_{+-} \in \mathcal{H}_{+-}, \psi_{-+} \in \mathcal{H}_{-+} \) and \( \psi_{--} \in \mathcal{H}_{--} \). Therefore operators acting on \( \mathcal{H} \) are represented by \( 4 \times 4 \) matrices. Moreover, since \( \rho \) is selfadjoint and \( \rho^2 = \chi(R \neq 0) \), we see that maps \( \rho : \mathcal{H}_{--} \to \mathcal{H}_{--} \) and \( \rho : \mathcal{H}_{+-} \to \mathcal{H}_{+-} \) are mutually inverse. Similarly maps \( \rho : \mathcal{H}_{-+} \to \mathcal{H}_{-+} \) and \( \rho : \mathcal{H}_{+-} \to \mathcal{H}_{-+} \) are mutually inverse. Since \( \sigma \) is selfadjoint and \( \sigma^2 = \chi(S \neq 0) \), we see that maps \( \sigma : \mathcal{H}_{--} \to \mathcal{H}_{--} \) and \( \sigma : \mathcal{H}_{+-} \to \mathcal{H}_{+-} \) are mutually inverse. Also maps \( \sigma : \mathcal{H}_{-+} \to \mathcal{H}_{-+} \) and \( \sigma : \mathcal{H}_{++} \to \mathcal{H}_{++} \) are mutually inverse. Therefore Hilbert spaces \( \mathcal{H}_{--}, \mathcal{H}_{-+}, \mathcal{H}_{+-}, \mathcal{H}_{++} \) and \( \mathcal{H}_{++} \) are unitarily equivalent. In what follows we simply assume that \( \mathcal{H}_{++} = \mathcal{H}_{-+} = \mathcal{H}_{+-} = \mathcal{H}_{--} \) and denote this Hilbert space by \( \mathcal{H}_+ \).

Then

\[
\rho = \begin{bmatrix}
0 & I & 0 & 0 \\
I & 0 & 0 & 0 \\
0 & 0 & 0 & I \\
0 & 0 & I & 0
\end{bmatrix}, \quad \text{and} \quad \sigma = \begin{bmatrix}
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
I & 0 & 0 & 0 \\
0 & I & 0 & 0
\end{bmatrix}.
\]

Hence the matrix representation of operator \( \tau := (-1)^k \rho \sigma \) is

\[
\tau = \begin{bmatrix}
0 & 0 & 0 & I \\
0 & 0 & I & 0 \\
0 & I & 0 & 0 \\
I & 0 & 0 & 0
\end{bmatrix}.
\]

Since operators \( R \) and \( S \) commute with sign \( R \) and with sign \( S \), they are represented by diagonal matrices

\[
R = \begin{bmatrix}
R_+ & 0 & 0 & 0 \\
0 & R_+ & 0 & 0 \\
0 & 0 & -R_+ & 0 \\
0 & 0 & 0 & -R_+
\end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix}
S_+ & 0 & 0 & 0 \\
0 & -S_+ & 0 & 0 \\
0 & 0 & S_+ & 0 \\
0 & 0 & 0 & -S_+
\end{bmatrix},
\]

where \( R_+ \) and \( S_+ \) are restrictions to \( \mathcal{H}_+ \) of \( R \) and \( S \), respectively. Clearly \( R_+ \) and \( S_+ \) are selfadjoint and strictly positive and \( R_+ \approx S_+ \). Moreover

\[
T = e^{i\frac{\tau}{2}} S^{-1} R = \begin{bmatrix}
T_+ & 0 & 0 & 0 \\
0 & -T_+ & 0 & 0 \\
0 & 0 & -T_+ & 0 \\
0 & 0 & 0 & T_+
\end{bmatrix},
\]

where \( T_+ = e^{i\frac{\tau}{2}} S_+^{-1} R_+ \). \( T_+ \) is selfadjoint and strictly positive.

### 2.4 Matrix elements

Consider strictly positive operators \( R \) and \( S \), such that \( R \approx S \). For example, \( R_+ \) and \( S_+ \) are such operators. Since \( R \approx S \), by Corollary 1.5 the pair \((R,S)\) is unitarily equivalent to \((u \otimes e^\delta, v \otimes e^\delta)\), where \( u, v \) are unitary, selfadjoint and commuting operators, i.e. \( \text{Sp } u, \text{Sp } v \subset \{-1,1\} \). Assume that

\[
R_o = e^{i\delta} \quad \text{and} \quad S_o = e^{i\delta}.
\]

Denote the complex conjugation operator by \( J_o \). Then for any \( w \in L^2(\mathbb{R}) \)

\[
(J_o w)(t) = \overline{w(t)},
\]

where \( \overline{w(t)} \) denotes the complex conjugate of \( w(t) \).
where $t \in \mathbb{R}$. Note that $J_o$ is an antilinear operator such that

$$(J_o)^2 = I \quad \text{and} \quad (J_o)^* = J_o \quad \text{and} \quad J_o R_o J_o = J_o e^{\hat{p}} J_o = e^{-\hat{p}} = R_o^{-1}$$

and

$$J_o S_o J_o = J_o e^{\hat{q}} J_o = e^{\hat{q}} = S_o.$$ 

Therefore by Corollary 1.5 for any $R - \circ S$ there exists an antilinear operator $J$, such that

$$J^2 = I \quad \text{and} \quad J^* = J$$

and

$$J R J = R^{-1} \quad \text{(8)}$$

and

$$J S J = S \quad \text{(9)}.$$ 

Since $J$ is antilinear and $J^* = J$ then for any $w, v \in \mathcal{H}$

$$\langle w | J v \rangle = \langle v | J^* w \rangle = \langle v | J w \rangle \quad \text{(10)}.$$ 

Moreover, define an operator $F$ by

$$F = e^{i \frac{\pi}{4} e^{-i \log_2 S \frac{\pi}{2}} e^{-i \log_2 T \frac{\pi}{2}}}.$$ \hspace{1cm} (11)

Note that

$$F^* = F^{-1} \quad \text{(12)}.$$ 

It is not very difficult to see that if $R = R_o$ and $S = S_o$ then $F$ is the Fourier transform.

By (8), (9) and (11)

$$F R F^{-1} = S \quad \text{(13)}$$

and

$$F S F^{-1} = R^{-1} \quad \text{(14)}.$$ 

Moreover

$$F J = J F^{-1} \quad \text{and} \quad F^{-1} J = J F \quad \text{(14)}$$

and

$$J T J = F T^{-1} F^{-1} \quad \text{(15)}.$$ 

We use the notion of generalized eigenvectors. It is well known that a selfadjoint operator with continuous spectrum acting on $\mathcal{H}$ does not have eigenvectors. Still one can show that in the general case the generalized eigenvectors are continuous linear functionals on a certain dense locally convex subspace $\Phi \subset \mathcal{H}$, provided with a much stronger topology than $\mathcal{H}$. Then we get the same formulae as for discreet spectrum provided we replace scalar product by the duality relation between $\Phi$ and $\Phi'$. This will be explained by the example below, for general considerations see [10].

**Example 2.4** Let $\mathcal{H} = L^2(\mathbb{R})$ and

$$R = e^{\hat{p}} \quad \text{and} \quad S = e^{\hat{q}} \quad \text{and} \quad T = e^{i \frac{\pi}{2}} S^{-1} R = e^{\hat{p} - \hat{q}}.$$ \hspace{1cm} (16)
These operators have continuous spectra, so they do not have eigenvectors. There are however tempered distributions on \( \mathbb{R} \) such that for every function \( f \) from the Schwartz space of smooth functions on \( \mathbb{R} \) decreasing rapidly at infinity \( \mathcal{S}(\mathbb{R}) \) we have

\[
\langle f | R | \Omega_r \rangle = r \langle f | \Omega_r \rangle \quad \text{and} \quad \langle f | S | \Phi_s \rangle = s \langle f | \Phi_s \rangle \quad \text{and} \quad \langle f | T | \Psi_t \rangle = t \langle f | \Psi_t \rangle.
\]

(17)

Such \( | \Omega_r \rangle \), \( | \Phi_s \rangle \) and \( | \Psi_t \rangle \) are called generalized eigenvectors of operators \( R, S \) and \( T \) with eigenvalues respectively \( r, s \) and \( t \).

An example of generalized eigenvectors of operators (16) is

\[
| \Omega_r \rangle = \frac{1}{\sqrt{2\pi \hbar}} e^{\frac{i}{\hbar} x \log r} \quad \text{and} \quad | \Phi_s \rangle = \delta(\log s - x) \quad \text{and} \quad | \Psi_t \rangle = \frac{1}{\sqrt{2\pi \hbar}} e^{\frac{i}{\hbar} x \log t}.
\]

Moreover, we will use notation of a type \( \langle \Omega_r | \Phi_s \rangle \).

It should be understood in the following way: for any \( f \in \mathcal{S}(\mathbb{R}) \) we have

\[
\langle \Omega_r | f \rangle = \int_{\mathbb{R}} \langle \Omega_r | \Phi_s \rangle \langle \Phi_s | f \rangle ds.
\]

(18)

To shorten notation from now on we skip the integration symbol, i.e. we write

\[
\langle \Omega_r | f \rangle = \langle \Omega_r | \Phi_s \rangle \langle \Phi_s | f \rangle.
\]

instead of (18) The generalized eigenvectors \( \Omega \) are said to have the Dirac \( \delta \) normalization if

\[
\langle \Omega_r | \Omega_s \rangle = \delta(r - s),
\]

where \( \delta \) is the Dirac \( \delta \) distribution. Note that generalized eigenvectors \( \Omega \), \( \Phi \) and \( \Psi \) given above have the Dirac \( \delta \) normalization.

Let \( | \Omega_r \rangle \) be a generalized eigenvector of \( R \) with real eigenvalue \( r \) and with Dirac delta normalization. Analogously, let \( | \Phi_s \rangle \) and \( | \Psi_t \rangle \) denote generalized eigenvectors of operators \( S \) and \( T \) with real eigenvalues respectively \( s \) and \( t \) and with Dirac delta normalization. Note that by (13) for any \( w \in \mathcal{H} \)

\[
\langle w | S \Phi_s \rangle = \langle w | J S J \Phi_s \rangle.
\]

Since \( J \) is an antilinear operator and \( S \) commutes with \( J \), it follows

\[
\langle w | J S J \Phi_s \rangle = \langle S J \Phi_s | J w \rangle = \langle J S \Phi_s | J w \rangle = \langle w | J^2 S \Phi_s \rangle = \langle w | S \Phi_s \rangle = s \langle w | \Phi_s \rangle.
\]

Comparing above formulae we obtain

\[
\langle S J \Phi_s | J w \rangle = s \langle w | \Phi_s \rangle = s \langle w | J^2 \Phi_s \rangle = s \langle J \Phi_s | J w \rangle,
\]

so

\[
S | J \Phi_s \rangle = s | J \Phi_s \rangle.
\]

Hence

\[
| J \Phi_s \rangle = | \Phi_s \rangle.
\]

(19)

Moreover observe, that by (13)

\[
FRF^{-1} | \Phi_r \rangle = r | \Phi_r \rangle.
\]
Hence
\[ R|F^{-1}\Phi_r\rangle = r|F^{-1}\Phi_r\rangle , \]
so
\[ |F^{-1}\Phi_r\rangle = |\Omega_r\rangle . \]  \hspace{1cm} (20)

Applying (19) and (21), and then (12) and (14), and again (20) and (13) and (15), we obtain
\[ \langle \Omega_r|\Phi_s\rangle = \langle F^{-1}\Phi_r|J\Phi_s\rangle = \langle \Phi_r|FJ\Phi_s\rangle = \]
\[ = \langle \Phi_r|JF^{-1}\Phi_s\rangle = \langle \Phi_r|J\Omega_s\rangle = \langle \Omega_s|J\Phi_r\rangle = \langle \Omega_s|\Phi_r\rangle , \]
so
\[ \langle \Omega_r|\Phi_s\rangle = \langle \Omega_s|\Phi_r \rangle . \]  \hspace{1cm} (21)

We proceed to deriving our next formula. Note that
\[ e^{i\frac{\log^2 t}{2h}}\langle \Omega_r|\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle = \langle \Omega_r|e^{i\frac{\log^2 T}{2h}}\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle = \langle F^{-1}\Phi_r|e^{i\frac{\log^2 T}{2h}}\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle = \]
\[ = \langle \Phi_r|e^{i\frac{\log^2 T}{h}}e^{-i\frac{\log^2 T}{2h}}e^{i\frac{\log^2 T}{2h}}\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle = \langle \Phi_r|e^{i\frac{\log^2 T}{h}}e^{-i\frac{\log^2 T}{2h}}\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle = \]
\[ = e^{i\frac{\log T}{2h}}\langle e^{i\frac{\log^2 T}{h}}\Phi_r|\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle = e^{i\frac{\log^2 T}{h}}\langle \Phi_r|\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle . \]

It means, we have the formula
\[ e^{-i\frac{\log T}{2h}}\langle \Omega_r|\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle = e^{-i\frac{\log^2 T}{h}}\langle \Phi_r|\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle . \]  \hspace{1cm} (22)

We prove now that
\[ \langle \Omega_r|V_\theta(\log T)^*|\Phi_s\rangle = c_h e^{-i\frac{\log^2 T}{h}}\langle \Phi_s|V_\theta(\log T)|\Phi_r\rangle . \]  \hspace{1cm} (23)

Compute the left hand side of this formula
\[ LHS = \langle \Omega_r|V_\theta(\log T)^*|\Phi_s\rangle = \langle \Omega_r|V_\theta(\log T)^*|\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle = \]
\[ = \overline{V_\theta(\log t)}\langle \Omega_r|\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle . \]

Moreover by formula 1.36 [22], for any \( t \in \mathbb{R} \)
\[ \overline{V_\theta(\log t)} = e^{-i\frac{\log^2 T}{2h}}V_\theta(-\log t) , \]  \hspace{1cm} (24)
where \( c'_h = e^{i\left(\frac{\theta_T}{T} + \frac{\theta_T^2}{2T} + \frac{\theta_T^3}{6T}\right)} . \) Hence
\[ L = e^{-i\frac{\log^2 T}{h}}c'_h e^{i\frac{\log^2 T}{2h}}V_\theta(-\log t)\langle \Omega_r|\Psi_t\rangle\langle \Psi_t|\Phi_s\rangle . \]

Compute now the right hand side of (23)
\[ RHS = c'_h e^{-i\frac{\log^2 T}{h}}\langle \Phi_s|V_\theta(\log t)|\Psi_t\rangle\langle \Psi_t|\Phi_r\rangle = c'_h V_\theta(\log t)e^{-i\frac{\log^2 T}{h}}\langle \Phi_s|\Psi_t\rangle\langle \Psi_t|\Phi_r\rangle . \]  \hspace{1cm} (25)

Note that by (11) and (12)
\[ T|e^{i\frac{\log^2 T}{h}}J\Psi_t\rangle = e^{-i\frac{\log^2 T}{2h}}e^{i\frac{\log T}{2h}}F^{-1}|J\Psi_t\rangle . \]
Moreover by (13),
\[
e^{-i \frac{\log^2 T}{2 \hbar} e^{i \frac{\theta}{\hbar}} F^{-1} J} \Psi_t = e^{-i \frac{\log^2 T}{2 \hbar} e^{i \frac{\theta}{\hbar}} F^{-1} J T^{-1} J} \Psi_t =
\]
\[= t^{-1} e^{-i \frac{\log^2 T}{2 \hbar} e^{i \frac{\theta}{\hbar}} F^{-1} J} \Psi_t = t^{-1} |e^{i \frac{\log^2 T}{2 \hbar} J} \Psi_t|,
\]
so
\[|e^{i \frac{\log^2 T}{2 \hbar} J} \Psi_t| = |\Psi_{t^{-1}}|.
\]
Hence
\[V_\theta(\log t) \langle \Phi_s | \Psi_t \rangle \langle \Psi_t | \Phi_r \rangle = V_\theta(- \log t) \langle \Phi_s | \Psi_{t^{-1}} \rangle \langle \Psi_{t^{-1}} | \Phi_r \rangle =
\]
\[= V_\theta(- \log t) e^{i \frac{\log^2 T}{\hbar}} |J \Psi_t \rangle \langle e^{-i \frac{\log^2 T}{\hbar} J} | \Phi_r \rangle =
\]
\[= V_\theta(- \log t) e^{-i \frac{\log^2 T}{\hbar} e^{i \frac{\theta}{\hbar}} |J \Psi_t \rangle \langle J \Psi_t | \Phi_r \rangle =
\]
\[= V_\theta(- \log t) e^{-i \frac{\log^2 T}{\hbar} e^{i \frac{\theta}{\hbar}} \langle \Psi_t | J \Phi_s \rangle \langle J \Phi_s | \Phi_r \rangle =
\]
\[= V_\theta(- \log t) e^{-i \frac{\log^2 T}{\hbar} e^{i \frac{\theta}{\hbar}} \langle \Psi_t | \Phi_s \rangle \langle \Phi_s | \Phi_r \rangle =
\]
Therefore
\[V_\theta(\log t) \langle \Phi_s | \Psi_t \rangle \langle \Psi_t | \Phi_r \rangle = V_\theta(- \log t) e^{-i \frac{\log^2 T}{\hbar} e^{i \frac{\theta}{\hbar}} \langle \Psi_t | \Phi_s \rangle \langle \Phi_s | \Phi_r \rangle =
\]
(26)
In fact, we have even proved a more general formula, namely for any measurable function \(f\) we have
\[f(\log t) \langle \Phi_s | \Psi_t \rangle \langle \Psi_t | \Phi_r \rangle = f(- \log t) e^{-i \frac{\log^2 T}{\hbar} e^{i \frac{\theta}{\hbar}} \langle \Psi_t | \Phi_s \rangle \langle \Phi_s | \Phi_r \rangle =
\]
(27)
We use this formula in our forthcoming paper [17] on the quantum ‘az+b’ group at roots of unity.

Comparing (27) and (24) and using (22) and (26) we get (23). Using again formulæ (1.36) (22) with \(z = - \log t - i \pi\), where \(t \in \mathbb{R}\), and (2), we obtain
\[\overline{V_\theta(\log t - i \pi)} = c_\theta(-1)^k e^{-i \frac{\log^2 T}{2 \hbar} e^{i \frac{\theta}{\hbar}} V_\theta(- \log t - i \pi) =
\]
Using the above formula and the same method as for derivation of (23), we obtain formulæ for some matrix elements we will soon find very useful
\[\langle \Omega_r | V_\theta(\log T - i \pi I)^* \Phi_s \rangle = i(-1)^k c_\theta e^{-i \frac{\log^2 T}{2 \hbar} e^{i \frac{\theta}{\hbar}} \langle \Phi_s | T^* V_\theta(\log T - i \pi I) | \Phi_r \rangle =
\]
(28)
and
\[\langle \Omega_r | T^* V_\theta(\log T - i \pi I)^* | \Phi_s \rangle = i(-1)^k c_\theta e^{-i \frac{\log^2 T}{2 \hbar} e^{i \frac{\theta}{\hbar}} \langle \Phi_s | V_\theta(\log T - i \pi I) | \Phi_r \rangle =
\]
(29)

2.5 Unitary representations of \(N\)

We find now a formula for all unitary representations of \(N\) acting on a Hilbert space \(\mathcal{K}\).

**Definition 2.5** Let for any Hilbert space \(\mathcal{H}\) there exists a map
\[V_\mathcal{H} : N_\mathcal{H} \to \text{Unit}(\mathcal{K} \otimes \mathcal{H})
\]
such that
1. For any \((R, \rho) \in \mathcal{N}_H\) and for any operators \(v \in \text{Unit}(\mathcal{H}, \mathcal{H}')\) we have
\[
(id_K \otimes v^*)V_H(R, \rho)(id_K \otimes v) = V_{H'}(v^* Rx, v^* \rho y)
\].

2. For any space with measure \((\Lambda, \mu)\) and for any measurable field of Hilbert spaces \(\{H(\lambda)\}_{\lambda \in \Lambda}\) and for any measurable fields of closed operators \(\{R(\lambda)\}_{\lambda \in \Lambda}\) and \(\{\rho(\lambda)\}_{\lambda \in \Lambda}\), such that \((R(\lambda), \rho(\lambda)) \in \mathcal{N}_H(\lambda)\) we have
\[
\int_{\Lambda} V_{H(\lambda)}(R(\lambda), \rho(\lambda))d\mu(\lambda) = V_{\int_{\Lambda} H(\lambda)d\mu(\lambda)}\left(\int_{\Lambda} R(\lambda)d\mu(\lambda), \int_{\Lambda} \rho(\lambda)d\mu(\lambda)\right).
\]

3. For any \(((R, \rho), (S, \sigma)) \in \mathcal{N}_H^2\) we have
\[
V_H(R, \rho)V_H(S, \sigma) = V_H(R, \rho) \prod_{\mathcal{N}} (S, \sigma).
\]

Then we call \(V\) a **unitary representation** of \(N\) on Hilbert space \(K\).

In what follows we omit the subscript \(H\) in \(V_H\).

We prove now a formula for all unitary representations of \(N\) on a Hilbert space \(K\).

**Theorem 2.6** A map
\[
V_K : \mathcal{N}_H \rightarrow \text{Unit}(K \otimes \mathcal{H})
\]
is a unitary representation of \(N\) iff, there exists \((M, \mu) \in \mathcal{N}_K\), such that for any \((R, \rho) \in \mathcal{N}_H\) we have
\[
V(R, \rho) = F_h(M \otimes R, \mu \otimes \rho)(M \otimes R < 0)).
\]

By \(F_h(M \otimes R, \mu \otimes \rho)(M \otimes R < 0))\) we mean
\[
F_h(M \otimes R, \mu \otimes \rho)(M \otimes R < 0)) = \int_{R \times (-1,1)} F_h(R, \rho)(R < 0) \otimes dE_{M, \mu}(z),
\]
where \(dE_{M, \mu}\) is the joint spectral measure of strongly commuting operators \(M\) and \(\mu\), acting on Hilbert space \(\mathcal{K}\).

We proceed to prove Theorem 2.6.

**Proof:** \(\Leftarrow\) Observe that if \((R, \rho) \in \mathcal{N}_H\) and \((M, \mu) \in \mathcal{N}_K\) then also
\[
(M \otimes R, \mu \otimes \rho) \in \mathcal{N}_{K \otimes H}.
\]

Therefore we may apply Theorem 2.3, which is our claim.

\(\Rightarrow\) We follow the proof of Theorem 4.2. We first outline the proof. We show that if \(V\) is a unitary representation of \(N\), then
\[
V(r, g)V(s, \sigma) = V(s, \sigma)V(r, g)
\]
for any \(r, s \in \mathbb{R} \setminus \{0\}\) and \(g, \sigma \in \{-1, 1\}\). Then we find formula for all unitary representations of \(N\) acting on \(\mathbb{C}\). Using spectral decomposition theorem completes the proof.

Our proof starts with the observation that since \(R\) and \(\rho\) commute and \(\text{Sp} \rho = \{-1, 1\}\), the function \(V\) may be written as
\[
V(R, \rho) = V_1(R) + (I_K \otimes \rho)V_2(R),
\]
(31)
where

\[ V_1(R) = \frac{1}{2}(V(R, 1) + V(R, -1)) \quad \text{and} \quad V_2(R) = \frac{1}{2}(V(R, 1) - V(R, -1)). \]

Then

\[
V(R, \rho) = \begin{bmatrix}
V_1(R_+) & V_2(R_+) & 0 & 0 \\
V_2(R_+) & V_1(R_+) & 0 & 0 \\
0 & 0 & V_1(-R_+) & V_2(-R_+) \\
0 & 0 & V_2(-R_+) & V_1(-R_+)
\end{bmatrix}
\]

and

\[
V(S, \sigma) = \begin{bmatrix}
V_1(S_+) & 0 & V_2(S_+) & 0 \\
0 & V_1(-S_+) & 0 & V_2(-S_+) \\
V_2(S_+) & 0 & V_1(S_+) & 0 \\
0 & V_2(-S_+) & 0 & V_1(-S_+)
\end{bmatrix}.
\]

Hence

\[
V(R, \rho)V(S, \sigma) = \begin{bmatrix}
V_1(R_+)V_1(S_+) & V_2(R_+)V_1(-S_+) & V_1(R_+)V_2(S_+) & V_2(R_+)V_2(-S_+) \\
V_2(R_+)V_1(S_+) & V_1(R_+)V_1(-S_+) & V_2(R_+)V_2(S_+) & V_1(R_+)V_2(-S_+) \\
V_1(-R_+)V_2(S_+) & V_2(-R_+)V_2(-S_+) & V_1(-R_+)V_1(S_+) & V_2(-R_+)V_1(-S_+) \\
V_2(-R_+)V_2(S_+) & V_1(-R_+)V_2(-S_+) & V_2(-R_+)V_1(S_+) & V_1(-R_+)V_1(-S_+)
\end{bmatrix}.
\]

Moreover

\[
F_h(T, \tau\chi(T < 0)) = \begin{bmatrix}
V_\theta(\log T_+) & 0 & 0 & 0 \\
0 & V_\theta(\log T_+ - i\pi I) & i(-1)^k T_+^k V_\theta(\log T_+ - i\pi I) & 0 \\
0 & i(-1)^k T_+^k V_\theta(\log T_+ - i\pi I) & V_\theta(\log T_+ - i\pi I) & 0 \\
0 & 0 & 0 & V_\theta(\log T_+)
\end{bmatrix}.
\]

Hence

\[
X := V([R + S]_{\tau\chi(T < 0)}, \tilde{\sigma}) = \begin{bmatrix}
X(1, 1) & X(1, 2) & X(1, 3) & 0 \\
X(2, 1) & X(2, 2) & X(2, 3) & X(2, 4) \\
X(3, 1) & X(3, 2) & X(3, 3) & X(3, 4) \\
0 & X(4, 2) & X(4, 3) & X(4, 4)
\end{bmatrix},
\]
Moreover where

\[
\begin{align*}
X(1, 1) &= V_\theta(\log T_+) V_1(S_+) V_\theta(\log T_+) \\
X(1, 2) &= i(-1)^k V_\theta(\log T_+) V_2(S_+) (T_+)^\pi V_\theta(\log T_+ - i\pi I) \\
X(1, 3) &= V_\theta(\log T_+) V_2(S_+) V_\theta(\log T_+ - i\pi I) \\
X(2, 1) &= -i(-1)^k (T_+)^\pi V_\theta(\log T_+ - i\pi I) V_2(S_+) V_\theta(\log T_+) \\
X(2, 2) &= V_\theta(\log T_+ - i\pi I) V_1(-S_+) V_\theta(\log T_+ - i\pi I) + T_+^\pi V_\theta(\log T_+ - i\pi I) V_1(S_+) T_+^\pi V_\theta(\log T_+ - i\pi I) \\
X(2, 3) &= i(-1)^k V_\theta(\log T_+ - i\pi I) V_1(-S_+) T_+^\pi V_\theta(\log T_+ - i\pi I) - i(-1)^k T_+^\pi V_\theta(\log T_+ - i\pi I) V_1(S_+) V_\theta(\log T_+ - i\pi I) \\
X(2, 4) &= V_\theta(\log T_+ - i\pi I) V_2(-S_+) V_\theta(\log T_+) \\
X(3, 1) &= V_\theta(\log T_+ - i\pi I) V_2(S_+) V_\theta(\log T_+) \\
X(3, 2) &= -i(-1)^k T_+^\pi V_\theta(\log T_+ - i\pi I) V_1(-S_+) V_\theta(\log T_+ - i\pi I) + i(-1)^k V_\theta(\log T_+ - i\pi I) V_1(S_+) T_+^\pi V_\theta(\log T_+ - i\pi I) \\
X(3, 3) &= T_+^\pi V_\theta(\log T_+ - i\pi I) V_1(-S_+) T_+^\pi V_\theta(\log T_+ - i\pi I) + V_\theta(\log T_+ - i\pi I) V_1(S_+) V_\theta(\log T_+ - i\pi I) \\
X(3, 4) &= -i(-1)^k (T_+)^\pi V_\theta(\log T_+ - i\pi I) V_2(-S_+) V_\theta(\log T_+) \\
X(4, 2) &= V_\theta(\log T_+) V_2(-S_+) V_\theta(\log T_+ - i\pi I) \\
X(4, 3) &= i(-1)^k V_\theta(\log T_+) V_2(-S) (T_+)^\pi V_\theta(\log T_+ - i\pi I) \\
X(4, 4) &= V_\theta(\log T_+) V_1(-S_+) V_\theta(\log T_+) \\
\end{align*}
\]

We prove that

**Proposition 2.7** For any \( r, s \in \mathbb{R} \) and \( \varrho, \sigma \in \{-1, 1\} \) we have

\[
V(r, \varrho)V(s, \sigma) = V(s, \sigma)V(r, \varrho). \tag{32}
\]

Moreover

\[
V_2(r)V_2(-s) = 0 = V_2(-s)V_2(r). \tag{33}
\]

**Proof:** We prove that

\[
\begin{align*}
V_1(r)V_1(s) &= V_1(s)V_1(r) \tag{33} \\
V_1(-r)V_1(-s) &= V_1(-s)V_1(-r) \tag{34} \\
V_1(r)V_1(-s) &= V_1(-s)V_1(r) \tag{35} \\
V_2(r)V_2(s) &= V_2(s)V_2(r) \tag{36} \\
V_2(-r)V_2(-s) &= V_2(-s)V_2(-r) \tag{37} \\
V_2(r)V_2(-s) &= 0 = V_2(-s)V_2(r) \tag{38} \\
V_1(r)V_2(s) &= V_2(s)V_1(r) \tag{39} \\
V_1(-r)V_2(-s) &= V_2(-s)V_1(-r) \tag{40} \\
V_1(r)V_2(-s) &= V_2(-s)V_1(r) \tag{41} \\
V_1(-r)V_2(s) &= V_2(s)V_1(-r) \tag{42}
\end{align*}
\]

The formulae (21), (23) and (28), and (29) will be of great use throughout the proof. First we prove the formula (33).
Compute
\[ \langle \Omega_r | V_1(R_+) V_1(S_+) | \Phi_s \rangle = \langle \Omega_r | X(1,1) | \Phi_s \rangle = \langle \Omega_r | V_\theta(\log T_+)^* V_1(S_+) V_\theta(\log T_+) | \Phi_s \rangle . \]

Hence
\[ V_1(r) V_1(s) = \langle \Omega_r | \Phi_s \rangle^{-1} \langle \Omega_r | V_\theta(\log t)^* | \Psi_s \rangle \langle \Psi_s | \Phi_z \rangle V_1(\tilde{s}) | \langle \Phi_z | \Psi_z \rangle | \Psi_z | V_\theta(\log t) | \Phi_s \rangle . \]

Therefore applying (23) we get
\[ V_1(r) V_1(s) = \langle \Omega_r | \Phi_s \rangle^{-1} \langle \Omega_r | V_\theta(\log T_+)^* | \Phi_s \rangle \langle \Phi_z | V_\theta(\log T_+) | \Phi_s \rangle = \frac{c'_i e^{-i \log \phi_s}}{\langle \Omega_r | \Phi_s \rangle^{-1} \langle \Phi_z | V_\theta(\log T_+) | \Phi_s \rangle} \langle \Phi_z | V_\theta(\log T_+) | \Phi_s \rangle . \]

Note that \( \langle \Phi_z | V_\theta(\log T_+) | \Phi_s \rangle \langle \Phi_z | V_\theta(\log T_+) | \Phi_s \rangle \) is symmetric with respect to swapping \( r \) and \( s \). By (21), the same holds for \( \langle \Omega_r | \Phi_s \rangle \) and the remaining terms of the above formula depend on neither \( r \) nor \( s \). Therefore
\[ V_1(r) V_1(s) = V_1(s) V_1(r) , \]
i.e. (33) holds.

In the same manner one can prove formulae (34) and (36) and (37).

The proof of the formula (33) is slightly different. Compute
\[ \langle \Omega_r | V_1(R_+) V_1(-S_+) | \Phi_s \rangle = \langle \Omega_r | X(2,2) | \Phi_s \rangle = \langle \Omega_r | V_\theta(\log T_+ - i \pi I)^* V_1(-S_+) V_\theta(\log T_+ - i \pi I) | \Phi_s \rangle + \langle \Omega_r | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(S_+) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_s \rangle . \]

Hence
\[ V_1(r) V_1(-s) = \langle \Omega_r | \Phi_s \rangle^{-1} \times \langle \Omega_r | V_\theta(\log t - i \pi I)^* \Psi_s \rangle V_1(-\tilde{s}) | \langle \Phi_z | \Psi_z \rangle \rangle V_\theta(\log \tilde{t} - i \pi I) \Phi_s \rangle + \langle \Omega_r | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(S_+) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_s \rangle \} = \langle \Omega_r | \Phi_s \rangle^{-1} \times \{ V_1(-\tilde{s}) \langle \Omega_r | V_\theta(\log T_+ - i \pi I)^* \Phi_z \rangle | \langle \Phi_z | \Psi_z \rangle \rangle V_\theta(\log \tilde{t} - i \pi I) \Phi_s \rangle + V_1(\tilde{s}) \langle \Omega_r | T_{\mp} V_\theta(\log T_+ - i \pi I)^* \Phi_z \rangle | \langle \Phi_z | \Psi_z \rangle \rangle V_\theta(\log T_+ - i \pi I) \Phi_s \rangle \} . \]

Therefore by (28) and (29) we obtain
\[ V_1(r) V_1(-s) = i(-1)^r c'_i e^{-i \log \phi_s} \langle \Omega_r | \Phi_s \rangle^{-1} \times \{ V_1(-\tilde{s}) \langle \Phi_z | T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle \langle \Phi_z | V_\theta(\log T_+ - i \pi I) | \Phi_s \rangle + V_1(\tilde{s}) \langle \Phi_z | V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle \langle \Phi_z | T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_s \rangle \} . \]

On the other hand
\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]

\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]

\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]

\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]

\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]

\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]

\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]

\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]

\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]

\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]

\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]

\[ \langle \Omega_s | V_1(-R_+) V_1(S_+) | \Phi_r \rangle = \langle \Omega_s | X(3,3) | \Phi_r \rangle = \langle \Omega_s | T_{\mp} V_\theta(\log T_+ - i \pi I)^* V_1(-s) T_{\mp} V_\theta(\log T_+ - i \pi I) | \Phi_r \rangle + \]
\[ + \langle \Omega_s | V_\theta(\log T_+ - i\pi I)^* V_1(s) \times V_\theta(\log T_+ - i\pi I) | \Phi_r \rangle. \]

Hence
\[ V_1(-s)V_1(r) = \langle \Omega_s | \Phi_r \rangle^{-1} \times \]
\[ \times \left\{ \langle \Omega_s | t_\pi V_\theta(\log t - i\pi)^* | \Psi_\ell \rangle \langle \Psi_\ell | \Phi_s \rangle \langle \Phi_s | V_1(-s) | \Psi_\ell \rangle \langle \Psi_\ell | t_\pi V_\theta(\log t - i\pi) | \Phi_r \rangle + \langle \Omega_s | V_\theta(\log t - i\pi I)^* | \Psi_\ell \rangle \langle \Psi_\ell | \Phi_s \rangle \langle \Phi_s | V_1(s) | \Psi_\ell \rangle \langle \Psi_\ell | V_\theta(\log t - i\pi I) | \Phi_r \rangle \right\} = \]
\[ = \langle \Omega_s | \Phi_r \rangle^{-1} \times \left\{ V_1(-\tilde{s}) \langle \Omega_s | t_\pi T^\pi_+ V_\theta(\log T_+ - i\pi I)^* | \Phi_s \rangle \langle \Phi_s | T^\pi_+ V_\theta(\log T_+ - i\pi I) | \Phi_r \rangle + V_1(\tilde{s}) \langle \Omega_s | V_\theta(\log T_+ - i\pi I)^* | \Phi_s \rangle \langle \Phi_s | V_\theta(\log T_+ - i\pi I) | \Phi_r \rangle \right\}. \]

Therefore by (28) and (29)
\[ V_1(-s)V_1(r) = i(-1)^k c^k \theta e^{-1/k} \langle \Omega_s | \Phi_r \rangle^{-1} \times \]
\[ \times \left\{ V_1(\tilde{s}) \langle \Phi_s | T^\pi_+ V_\theta(\log T_+ - i\pi I) | \Phi_r \rangle + V_1(\tilde{s}) \langle \Phi_s | T^\pi_+ V_\theta(\log T_+ - i\pi I) | \Phi_r \rangle \right\}. \]

Applying (21) we see that (43) and (44) are the same. Thus
\[ V_1(r)V_1(-s) = V_1(-s)V_1(r) \]
so (33) holds.

The proofs of formulae (41), (38) and (40) and (42) are exactly the same so we omit them.

In order to prove (38) one has to observe additionally that
\[ X(1, 4) = X(4, 1) = 0. \]

Thus we proved formulae (38) \div (42).

Adding (33) and (38) and substracting (33) from (38) yields
\[ V_1(r)\{V_1(s) + V_2(s)\} = V_1(r)V(s, 1) = \{V_1(s) + V_2(s)\}V_1(r) = V(s, 1)V_1(r). \]

Hence
\[ V_1(r)V(s, 1) = V(s, 1)V_1(r) \quad \text{and} \quad V_1(r)V(s, -1) = V(s, -1)V_1(r) \quad (45) \]

In the same manner we can see that
\[ V(r, g)V(s, \sigma) = V(s, \sigma)V(r, g) \]
for any \( r, s \in \mathbb{R} \setminus \{0\} \) and \( g, \sigma \in \{0, 1\} \). Moreover (38) holds
\[ V_2(r)V_2(-s) = 0 = V_2(-s)V_2(r). \]

We stress that satisfying (22) and (38) is necessary, but not sufficient condition for \( V \) to be a representation of \( N \).

We find now a formula for all unitary representations of \( N \) acting on \( \mathbb{C} \).
Proposition 2.8  All unitary representations of $N$ acting on $\mathbb{C}$ are of the form

$$V(R, \rho) = F_h(MR, \mu \rho \chi(MR < 0))$$

where $M \in \mathbb{R}$ and $\mu = \pm 1$.

Proof: We first find solutions of equation (38). There are 3 cases

1. for any $r \in \mathbb{R}_+$ we have $V_2(r) = 0$ and $V_2(-r) \neq 0$.

Then:

$$V(r, \rho) = \begin{cases} V_1(r) & \text{for } r > 0 \\ V_1(r) + \rho V_2(r) & \text{for } r < 0 \end{cases},$$

so

$$V(R, \rho) = V_1(R) + \rho \chi(R < 0) V_2(R) = V(R, \rho \chi(R < 0)) = V(\phi(R, \rho)).$$

Similarly

$$V(S, \sigma) = V_1(S) + \sigma \chi(S < 0) V_2(S) = V(S, \sigma \chi(S < 0)) = V(\phi(S, \sigma)).$$

Moreover, since $V$ is representation of $N$

$$V(\phi(R, \rho)) V(\phi(S, \sigma)) = V(\phi((R, \rho) \mathbb{D}_N(S, \sigma))).$$

Hence by Theorem 2.3

$$V(R, \rho) = F_h(MR, \mu \rho \chi(R < 0)),$$

where $M \geq 0$ and $\mu = \pm 1$.

2. For any $r \in \mathbb{R}_+$ we have $V_2(-r) = 0$ and $V_2(r) \neq 0$.

In the same manner as in the previous case we conclude that

$$V(R, \rho) = F_h(-MR, \mu \rho \chi(R > 0)) = F_h(MR, \mu \rho \chi(MR < 0)),$$

where $\tilde{M} = -M \leq 0$ and $\mu = \pm 1$.

3. For any $r \in \mathbb{R}_+$ we have $V_2(r) = 0$ and $V_2(-r) = 0$.

Then $V(R, \rho) = V_1(R)$ and $V(S, \sigma) = V_1(S)$. Note that

$$V_1(R) V_1(S) \neq V_1([R + S]_{\mu \chi(e^{\frac{\pi}{2} S^{-1} R} < 0)}),$$

since $[R + S]_{\mu \chi(e^{\frac{\pi}{2} S^{-1} R} < 0)}$ depends on $\rho$ and on $\sigma$, while the left hand side does not.

Therefore we conclude that $V = V_1$ is not a representation of $N$.

Thus we proved that all unitary representations of $N$ acting on Hilbert space $\mathbb{C}$ are

$$V(R, \rho) = F_h(MR, \mu \rho \chi(MR < 0)),$$

where $M \in \mathbb{R}$ and $\mu = \pm 1$.

Now we turn to the case of representations of $N$ acting on arbitrary Hilbert space $\mathcal{K}$. Note that if dim $\mathcal{K} = k < \infty$, then from commutation of unitary operators $V(r, \rho)$ and $V(s, \sigma)$ follows the existence of an orthonormal basis diagonalizing matrices $V(r, \rho)$ and
$V(s, \sigma)$ for any $r, s \in \mathbb{R}$ and $\rho, \sigma \in \{-1, 1\}$. Thus the problem reduces to finding solutions of $k$ scalar equations

$$V_o(R, \rho)V_o(S, \sigma) = V_o((R, \rho) \circlearrowleft_N (S, \sigma)),$$

where complex-valued function $V_o$ is defined on $\mathbb{R} \times \{-1, 1\}$. The same conclusion can be drawn for an arbitrary separable Hilbert space $K$. The reason is that operators $V(r, \rho)$ and $V(s, \sigma)$ belong to commutative $*$-subalgebra of $B(K)$. Therefore, by spectral theorem and its consequences [4, Chapter X] operators $V(r, \rho)$ and $V(s, \sigma)$ have the same spectral measure

$$V(r, \rho) = \int_{\mathbb{R} \times \{-1, 1\}} V_o(r, \rho, t) dE_K(t) \quad \text{and} \quad V(s, \sigma) = \int_{\mathbb{R} \times \{-1, 1\}} V_o(s, \sigma, t) dE_K(t),$$

where $dE_{R, \rho}$ is the joint spectral measure of strongly commuting operators $R$ and $\rho$.

Thus theorem 2.6 reduces to the already proved Proposition 2.8.

\[\square\]

3 The braided quantum group $M$

The main goal of this paper is finding all unitary representations of the operator domain $M$, which will be introduced below. This operator domain is very close to the operator domain corresponding to the quantum 'ax+b' group. To emphasize that we use the same letters $b$ and $\beta$, which denoted operators generating quantum 'ax+b' group in [23]. Theorem 3.3 we prove in this section will be crucial in our next paper [16].

Consider operators $b$ and $\beta$ such that

$$b = b^* \quad \text{and} \quad \beta = \beta^* \quad \text{and} \quad \beta b = -b \beta \quad \text{and} \quad \beta^2 = \chi(b \neq 0). \quad (46)$$

Define an operator domain $M$ by

$$M = \{(b, \beta) \in \mathcal{C}(H)^2 | b = b^*, \beta = \beta^*, \beta b = b \beta, \beta^2 = \chi(b \neq 0)\}$$

and $M^2$ by

$$M^2 = \{(b, \beta) \in \mathcal{C}(H)^2 | b = b^*, \beta = \beta^*, \beta b = b \beta, \beta^2 = \chi(b \neq 0)\}.$$

The colorally below allows us to define group operation on $M$.

**Corollary 3.1** Let $((b, \beta), (d, \delta)) \in M^2$ and let $\ker d = \{0\}$. Let

$$f = e^{\frac{\Delta}{2}}d^{-1}b \quad \text{and} \quad \phi = \pm \beta \delta \chi(e^{\frac{\Delta}{2}}d^{-1}b < 0)$$

and

$$[b + d]_\phi = F_h(f, \phi)^*dF_h(f, \phi) \quad \text{and} \quad \delta = F_h(f, \phi)^*\delta F_h(f, \phi).$$

Then

$$([b + d]_\phi, \delta) \in M.$$

---

\[2\] In fact, generators are $b$ and $ib\beta$
Proof: We first show that \([b + d]_\phi\) is a selfadjoint operator. Operator \(\phi\) is a reflection operator corresponding to \(e^{\frac{ib}{\hbar}}d^{-1}b\), since \(\phi\) is selfadjoint, \(\phi^2 = \chi(e^{\frac{ib}{\hbar}}d^{-1}b < 0)\) and \(\phi\) anticommutes with \(b + d\). Operator \(b + d\) is selfadjoint, if \(e^{\frac{ib}{\hbar}}bd \geq 0\) (see Theorem 5.4 [22]). Hence \([b + d]_\phi = F_h(f, \phi)^*dF_h(f, \phi)\) is an selfadjoint extension of sum of selfadjoint operators \(e^{\frac{ib}{\hbar}}fd\) and \(d\) (see Theorem 6.1 and 4.1 [22]), corresponding to the reflection operator \(\phi\).

We prove now that \(\tilde{\delta}\) is a selfadjoint operator. From

\[
\tilde{\delta} = F_h(f, \phi)^*\delta F_h(f, \phi)
\]

follows that \(\tilde{\delta}\) is selfadjoint, since it is unitarily equivalent to selfadjoint operator \(\delta\). Moreover, note that \([b + d]_\phi\) commutes with \(\tilde{\delta}\), because \(d\) commutes with \(\delta\).

In order to prove that \(([b + d]_\phi, \tilde{\delta}) \in M_\mathcal{H}\), we have to check that

\[
\tilde{\delta}^2 = \chi([b + d]_\phi \neq 0)
\]

to this end compute

\[
\tilde{\delta}^2 = F_h(f, \phi)^*\delta F_h(f, \phi)F_h(f, \phi)^*\delta F_h(f, \phi) = F_h(f, \phi)^*\delta F_h(f, \phi) = F_h(f, \phi)^*\chi(d \neq 0)F_h(f, \phi) = \chi(F_h(f, \phi)^*dF_h(f, \phi) \neq 0) = \chi([b + d]_\phi \neq 0)
\]

We are now in a position to define operation on \(M\) by

\[
\oplus_M: M^2_\mathcal{H} \rightarrow M_\mathcal{H}
\]

\((b, \beta) \oplus_M (d, \delta) = ([b + d]_\phi, \tilde{\delta})\).

Operation \(\oplus_M\) is associative. Moreover, \(M\) with this operation forms a braided quantum group.

3.1 The matrix representation of \(M\)

Let \(\ker b = \ker d = \{0\}\). Since \(b \rightarrow d\) it follows that \(b\) and \(d\) commute with \(\sign b\) and \(\sign d\). Therefore we may introduce notation

\[
\mathcal{H}_{++} = \mathcal{H}(b > 0) \cap \mathcal{H}(d > 0)
\]
\[
\mathcal{H}_{+-} = \mathcal{H}(b > 0) \cap \mathcal{H}(d < 0)
\]
\[
\mathcal{H}_{-+} = \mathcal{H}(b < 0) \cap \mathcal{H}(d > 0)
\]
\[
\mathcal{H}_{--} = \mathcal{H}(b < 0) \cap \mathcal{H}(d < 0)
\]

Then \(\mathcal{H} = \mathcal{H}_{++} \oplus \mathcal{H}_{+-} \oplus \mathcal{H}_{-+} \oplus \mathcal{H}_{--}\). Every vector \(\psi \in \mathcal{H}\) we represent by

\[
\psi = \begin{bmatrix}
\psi_{++} \\
\psi_{+-} \\
\psi_{-+} \\
\psi_{--}
\end{bmatrix},
\]

where \(\psi_{++} \in \mathcal{H}_{++}, \psi_{+-} \in \mathcal{H}_{+-}, \psi_{-+} \in \mathcal{H}_{-+}\) and \(\psi_{--} \in \mathcal{H}_{--}\). Therefore operators acting on \(\mathcal{H}\) will be represented by \(4 \times 4\) matrices. From (10) follows that maps \(\beta : \mathcal{H}_{++} \rightarrow \mathcal{H}_{--}\) and \(\beta : \mathcal{H}_{-+} \rightarrow \mathcal{H}_{++}\) and \(\beta : \mathcal{H}_{--} \rightarrow \mathcal{H}_{+-}\) and \(\beta : \mathcal{H}_{+-} \rightarrow \mathcal{H}_{-+}\) are mutually inverse. Similarly by (11) we obtain that maps \(\delta : \mathcal{H}_{++} \rightarrow \mathcal{H}_{+-}\) and \(\delta : \mathcal{H}_{--} \rightarrow \mathcal{H}_{++}\) and \(\delta : \mathcal{H}_{-+} \rightarrow \mathcal{H}_{--}\) and \(\delta : \mathcal{H}_{+-} \rightarrow \mathcal{H}_{-+}\) are mutually inverse.
\( \mathcal{H}_++ \rightarrow \mathcal{H}_+ \) and \( \delta : \mathcal{H}_- \rightarrow \mathcal{H}_- \) and \( \delta : \mathcal{H}_- \rightarrow \mathcal{H}_- \) are mutually inverse. Hence Hilbert spaces \( \mathcal{H}_++, \mathcal{H}_-, \mathcal{H}_+ \) and \( \mathcal{H}_- \) are unitarily equivalent. In what follows for simplicity we assume that \( \mathcal{H}_++ = \mathcal{H}_- = \mathcal{H}_+ = \mathcal{H}_- \) and denote this Hilbert space by \( \mathcal{H}_+ \). With this notation we have the following representations of \( \beta \) and \( \delta \):

\[
\beta = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \delta = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & I & 0 \end{bmatrix}.
\]

Since \( b \) anticommutes with \( \beta \) and commutes with \( \delta \) and \( d \) anticommutes with \( \delta \) and commutes with \( \beta \), they will be represented as follows:

\[
b = \begin{bmatrix} b_+ & 0 & 0 & 0 \\ 0 & b_+ & 0 & 0 \\ 0 & 0 & -b_+ & 0 \\ 0 & 0 & 0 & -b_+ \end{bmatrix}, \quad \text{and} \quad d = \begin{bmatrix} d_+ & 0 & 0 & 0 \\ 0 & -d_+ & 0 & 0 \\ 0 & 0 & d_+ & 0 \\ 0 & 0 & 0 & -d_+ \end{bmatrix},
\]

where \( b_+ \) and \( d_+ \) are restrictions of \( b \) and \( d \) to \( \mathcal{H}_+ \). It is easily seen that \( b_+ \) and \( d_+ \) are selfadjoint and strictly positive and \( b_+ \prec d_+ \). Moreover

\[
f = e^{ih}d^{-1}b = \begin{bmatrix} f_+ & 0 & 0 & 0 \\ 0 & -f_+ & 0 & 0 \\ 0 & 0 & -f_+ & 0 \\ 0 & 0 & 0 & f_+ \end{bmatrix},
\]

where \( f_+ = e^{ih}d^{-1}b_+ \) is selfadjoint and strictly positive. Hence

\[
\phi = (-1)^k \beta \delta \chi (e^{ih}d^{-1}b < 0) = (-1)^k \beta \delta \chi (f < 0) = \begin{bmatrix} 0 & 0 & 0 & (-1)^k I \\ 0 & 0 & (1)^k I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

where \( k \in \mathbb{N} \).

### 3.2 From \( N \) to \( M \)

As we said in Introduction, the operator domain \( N \) is auxiliary, what we are really interested in are unitary representations of \( M \). However, \( N \) was easier to work with, because it was commutative, so we found formula for all unitary representations of it. It would be very nice now to have an operator map from \( N \) into \( M \), which would allow us to “transfer” results obtained for \( N \) to \( M \). Such an operator map is constructed below

**Proposition 3.2** Let \( (\mathcal{R}, \rho), (\mathcal{S}, \sigma) \in N^2_\mathcal{H} \). Define a map

\[
\phi : N_\mathcal{H} \rightarrow M_{C^2 \otimes \mathcal{H}}
\]

for any \( (\mathcal{R}, \rho) \in N_\mathcal{H} \) by

\[
\varphi(\mathcal{R}, \rho) = \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}, \quad \begin{bmatrix} 0 & \rho \\ \rho & 0 \end{bmatrix}.
\]

Then \( \phi \) is an operator map and for any \( (\mathcal{R}, \rho), (\mathcal{S}, \sigma) \in N^2_\mathcal{H} \) we have

\[
\varphi ((\mathcal{R}, \rho) \oplus_N (\mathcal{S}, \sigma)) = \varphi(\mathcal{R}, \rho) \oplus_M C_{2 \otimes \mathcal{H}} \varphi(\mathcal{S}, \sigma).
\]
Proof: We first prove that

\[
\begin{bmatrix}
R & 0 \\
0 & -R
\end{bmatrix} + \begin{bmatrix}
S & 0 \\
0 & -S
\end{bmatrix} = \begin{bmatrix}
[R + S]_\tau & 0 \\
0 & -[R + S]_\tau
\end{bmatrix}.
\]

Since selfadjoint extensions are determined uniquely by reflection operators, it is enough to check that \( I_{C^2} \otimes \tau = \phi \). We already know that \( \phi = (-1)^k \beta \delta \). Hence

\[
\phi = (-1)^k \beta \delta = (-1)^k \begin{bmatrix}
0 & \rho \\
\rho & 0
\end{bmatrix} \begin{bmatrix}
0 & \sigma \\
\sigma & 0
\end{bmatrix} = (-1)^k \begin{bmatrix}
\rho \sigma & 0 \\
0 & \rho \sigma
\end{bmatrix}
\]

and

\[
\tau = (-1)^k \rho \sigma.
\]

Therefore selfadjoint extensions given by this reflection operators are the same.

It remains to prove that

\[
\varphi(\tilde{\sigma}) = \tilde{\delta}.
\]

To this end, note that

\[
\varphi(\tilde{\sigma}) = \begin{bmatrix}
0 & \tilde{\sigma} \\
\tilde{\sigma} & 0
\end{bmatrix} =
\begin{bmatrix}
0 & F_h(T, \tau \chi(T < 0))^* \sigma F_h(T, \tau \chi(T < 0)) \\
F_h(T, \tau \chi(T < 0))^* \sigma F_h(T, \tau \chi(T < 0)) & 0
\end{bmatrix} = F_h(f, \phi)^* \delta F_h(f, \phi) = \tilde{\delta}.
\]

\[
\square
\]

3.3 Unitary representations of \( M \)

We can now formulate our main result.

**Theorem 3.3** \( U \) is an unitary representation of \( M \) acting on Hilbert space \( K \), if there exists such \((g, \gamma) \in M_K\) that for any \((b, \beta) \in M_H\)

\[
U(b, \beta) = F_h(g \otimes b, (\gamma \otimes \beta) \chi(g \otimes b < 0)). \quad (47)
\]

**Proof:** We first show that if \( U \) is a representation of \( M \), it has form \([47]\). Let \( \varphi : N_H \to M_{C^2 \otimes H} \) be the operator map considered before. Let \((b, \beta)\) denote the following element from \( M_{C^2 \otimes H} \)

\[
(b, \beta) := \varphi(R, \rho) = \left( \begin{bmatrix}
R & 0 \\
0 & -R
\end{bmatrix}, \begin{bmatrix}
0 & \rho \\
\rho & 0
\end{bmatrix}\right).
\]

Let

\[
V(R, \rho) := U(b, \beta) = U(\varphi(R, \rho)) \, .
\]

Since \( U \) is a unitary representation of \( M \), it follows by Proposition 3.2 that \( V \) is a unitary representation of \( N \) on \( K \otimes C^2 \). Next by Theorem 2.6 we get

\[
V(R, \rho) = U \left( \begin{bmatrix}
R & 0 \\
0 & -R
\end{bmatrix}, \begin{bmatrix}
0 & \rho \\
\rho & 0
\end{bmatrix}\right) = F_h(M \otimes R, (\mu \otimes \rho) \chi(M \otimes R < 0)) \quad (48)
\]
where \((M, \mu) \in N_{K \otimes \mathbb{C}_2}\). We find the conditions on \((M, \mu) \in N_{C_2 \otimes K}\), under which (48) holds. We know that \(U, \phi\) and \(V\) are operator maps. Let a unitary operator \(\hat{U}\) be given by

\[
\hat{U} := I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I_{\mathcal{H}}.
\]

Apply \(\text{ad}_{\hat{U}}\) to the left hand side of (48). We obtain

\[
\left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I_{\mathcal{H}} \right)^* U \left( \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}, \begin{bmatrix} 0 & \rho \\ \rho & 0 \end{bmatrix} \right) \left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I_{\mathcal{H}} \right) = U \left( \begin{bmatrix} 0 & \rho \\ \rho & 0 \end{bmatrix} \right).
\]

Hence

\[
\hat{U}^* F_h(M \otimes R, \mu \otimes \rho) \hat{U} = F_h(-M \otimes R, \mu \otimes \rho).
\] (49)

Observe that for any \(t \in \mathbb{R} \setminus \{0\}\) the pair \((tR, \rho)\) belongs to \(N_{\mathcal{H}}\) if only \((R, \rho)\) does. Therefore we may put \(tR\) instead of \(R\) in (49). The function \(F_{h}\) is not injective, however the family \(F_{h}(t \cdot)\) separates points of \(\Delta_{\text{real}}\). Therefore from (49) follows

\[
\left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) M \left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = -M
\]

and

\[
\left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \mu \left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = \mu.
\]

Let us introduce another unitary operator

\[
\hat{V} := I_K \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes I_{\mathcal{H}}.
\]

Applying \(\text{ad}_{\hat{V}}\) to the left hand side of (48) we obtain

\[
\left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I_{\mathcal{H}} \right)^* U \left( \begin{bmatrix} r & 0 \\ 0 & -r \end{bmatrix}, \begin{bmatrix} 0 & \rho \\ \rho & 0 \end{bmatrix} \right) \left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I_{\mathcal{H}} \right) = U \left( \begin{bmatrix} 0 & \rho \\ \rho & 0 \end{bmatrix} \right).
\]

Hence

\[
\hat{V}^* F_h(M \otimes R, \mu \otimes \rho) \hat{V} = F_h(M \otimes R, -\mu \otimes \rho),
\]

so

\[
\left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) M \left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = M
\]

and

\[
\left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \mu \left( I_K \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = -\mu.
\]

Therefore \(M\) and \(\mu\) have form

\[
M = g \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \mu = \gamma \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]
where \( g \) and \( \gamma \) are operators acting on Hilbert space \( K \). Note that since \((M, \mu) \in N_{K \otimes \mathbb{C}^2}\), we see that \( g, \gamma \) are selfadjoint, \( g \) anticommutes with \( \gamma \) and \( \gamma^2 = \chi(g \neq 0) \). It means that \((g, \gamma) \in M_K\). Hence

\[
U \left( \left[ \begin{array}{cc} R & 0 \\ 0 & -R \end{array} \right], \left[ \begin{array}{cc} 0 & \rho \\ \rho & 0 \end{array} \right] \right) = F_h\left( g \otimes \left[ \begin{array}{cc} R & 0 \\ 0 & -R \end{array} \right], (\gamma \otimes \left[ \begin{array}{cc} 0 & \rho \\ \rho & 0 \end{array} \right]) \chi(g \otimes \left[ \begin{array}{cc} R & 0 \\ 0 & -R \end{array} \right] < 0) \right),
\]
so

\[
U(b, \beta) = F_h\left( g \otimes b, (\gamma \otimes \beta) \chi(g \otimes b < 0) \right),
\]
where \((g, \gamma) \in M_K\).

We proved that every unitary representation of \( M \) has form (47). What is left is to show that every operator function given by (47) is a unitary representation of \( M \). To this end it is sufficient to show

**Proposition 3.4** Let \((g, \gamma) \in M_K\) and \(((b, \beta), (d, \delta)) \in M^2_{\mathcal{H}}\). For any \((b, \beta) \in M_{\mathcal{H}}\) define a map by

\[
\varphi_{(g, \gamma)} : M_{\mathcal{H}} \ni (b, \beta) \mapsto (g \otimes b, \gamma \otimes \beta) \in N_{K \otimes \mathcal{H}}
\]
Then

\[
\varphi_{(g, \gamma)} ((b, \beta) \otimes M(d, \delta)) = \varphi_{(g, \gamma)} (b, \beta) \otimes N_{\varphi(g, \gamma)}(d, \delta).
\]

**Proof:** We first have to check that selfadjoint extensions on both sides of the above formula are the same, i.e. that \( g \otimes [b + d] \) equals \( [g \otimes b + g \otimes d] \). Since \([g \otimes b + g \otimes d]_{\tau} = g \otimes [b + d]_{\tau}\) and selfadjoint extensions are given uniquely by reflection operators, it is enough to check that \( \phi = \tau|_{\mathcal{H}} \). We know that

\[
\phi = (-1)^k (\beta \delta \chi(e^{i\beta} d^{-1} b < 0)
\]
and

\[
\tau = (-1)^k (\gamma \otimes \beta) (\gamma \otimes \delta) \chi(e^{i\beta} (g \otimes b)^{-1} (g \otimes d) < 0) =
\]
\[
= (-1)^k (\gamma^2 \otimes \beta \delta) \chi(I \otimes e^{i\beta} b^{-1} d < 0).
\]
Since \( \gamma^2 = \chi(g \neq 0) \), it follows that

\[
\tau = (-1)^k (I \otimes \beta \delta) \chi(I \otimes e^{i\beta} b^{-1} d < 0).
\]
Obviously

\[
\tau|_{\mathcal{H}} = (-1)^k \beta \delta \chi(e^{i\beta} db < 0) = \phi,
\]
so selfadjoint operators determined by \( \tau \) and \( \phi \) are also the same.

Secondly, we have to prove that

\[
\widetilde{\gamma \otimes \delta} = (\gamma \otimes \delta).
\]
Left hand side is by definition

\[
LHS = F_h\left( I_K \otimes f, I_K \otimes \phi \right)^* (\gamma \otimes \delta) F_h\left( I_K \otimes f, I_K \otimes \phi \right) =
\]
\[
= (I_K \otimes F_h(f, \phi))^* (\gamma \otimes \delta) (I_K \otimes F_h(f, \phi)) = RHS,
\]
which completes the proof of Proposition 3.4. \( \square \)

This finishes also the proof of Theorem 3.3. \( \square \)

This result will prove extremely useful in our next paper \([13]\), where we find all unitary representations of the quantum 'ax+b' group. We also use formula \([17]\) derived in this paper in \([17]\).
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