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Singular Cauchy problem for the general Euler-Poisson-Darboux equation

General Euler-Poisson-Darboux equation

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Abstract: In this paper we obtain the solution of the singular Cauchy problem for the Euler-Poisson-Darboux equation when differential Bessel operator acts by each variable.

Keywords: Bessel operator, Euler-Poisson-Darboux equation, Singular Cauchy problem

MSC: 26A33, 44A15

1 Introduction

The classical Euler-Poisson-Darboux equation has the form

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + k \frac{\partial u}{\partial t}, \quad u = u(x,t), \quad x \in \mathbb{R}^n, \quad t > 0, \quad -\infty < k < \infty. \quad (1)$$

The operator acting by $t$ in (1) is called the Bessel operator. For the Bessel operator we use the notation (see. [1], p. 3)

$$(B_k)_t = \frac{\partial^2}{\partial t^2} + \frac{k}{t} \frac{\partial}{\partial t}.$$ 

The Euler-Poisson-Darboux equation for $n = 1$ appears in Euler’s work (see [2], p. 227). Further Euler’s case of (1) was studied by Poisson in [3], Riemann in [4] and Darboux in [5] (for the history of this issue see also in [6], p. 532 and [7], p. 527). The generalization of it was studied in [8]. When $n \geq 1$ the equation (1) was considered, for example, in [9, 10]. The Euler-Poisson-Darboux equation appears in different physics and mechanics problems (see [11–15]). In [16] (see also [17], p. 243) and in [18] there were different approaches to the solution of the Cauchy problem for the general Euler-Poisson-Darboux equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \gamma_i \frac{\partial u}{\partial x_i} + \frac{\partial^2 u}{\partial t^2} + k \frac{\partial u}{\partial t}, \quad 0 < \gamma_i, \quad i = 1, \ldots, n, \quad k > 0 \quad (2)$$

with the initials conditions

$$u(x, 0) = f(x), \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0. \quad (3)$$

The Cauchy problem with the nonequal to zero first derivative by $t$ of $u$ for the (2) (and for (1)) is incorrect. However, if we use the special type of the initial conditions containing the nonequal to zero first derivative by $t$ of $u$ then such Cauchy problem for the (2) will be solvable. Following [17] and [19] we will use the term

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singular Cauchy problem in this case. The abstract Euler-Poisson-Darboux equation (when in the left hand of (2) an arbitrary closed linear operator is presented) was studied in [20–22].

In this article we consider the solution of the problem (2)-(3) when \(-\infty < k < +\infty\) and its properties. Besides this, we get the formula for the connection of solution of the problem (2)-(3) and solution of a simpler problem. Also using the solution of the problem (2)-(3) we obtain solution of the singular Cauchy problem for the equation (2) when \(k < 1\) with the conditions

\[
u(x, 0) = 0, \quad \lim_{t \to 0} t^k \frac{\partial u}{\partial t} = \varphi(x).\]

(4)

### 2 Property of general Euler-Poisson-Darboux equations’ solutions

In this section we give some necessary definitions and obtain two fundamental recursion formulas for solution of (2).

Let

\[
\mathbb{R}^n_\Omega = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ x_1 > 0, \ldots, x_n > 0\}
\]

and \(\Omega\) is open set in \(\mathbb{R}^n\), which is symmetric correspondingly to each hyperplane \(x_i = 0\), \(i = 1, \ldots, n\). \(\Omega_+ = \Omega \cap \mathbb{R}^n_+\) and \(\overline{\Omega}_+ = \Omega \cap \mathbb{R}^n\)

We have \(\Omega_+ \subseteq \mathbb{R}^n_+\) and \(\overline{\Omega}_+ \subseteq \mathbb{R}^n\). Consider the set \(C^m(\Omega_+), m \geq 1\), consisting of differentiable functions on \(\Omega_+\) by order \(m\). Let \(C^m(\overline{\Omega}_+)\) be the set of functions from \(C^m(\Omega_+)\) such that all their derivatives by \(x_i\) for all \(i = 1, \ldots, n\) are continuous up to the \(x_i = 0\). Class \(C^m_n(\overline{\Omega}_+)\) consists of functions from \(C^m(\overline{\Omega}_+)\) such that

\[
\frac{\partial^{\alpha + 1}}{\partial x_1^{\alpha + 1}}|_{x=0} = 0
\]

for all non-negative integers \(k \leq \frac{m-1}{2}\) and all \(x_i, i = 1, \ldots, n\) (see [1], p. 21). A multi-index \(\gamma = (\gamma_1, \ldots, \gamma_n)\) consists of fixed positive numbers \(\gamma_i > 0, i = 1, \ldots, n\) and \(|\gamma| = \gamma_1 + \ldots + \gamma_n\).

We consider the multidimensional Euler-Poisson-Darboux equation wherein the Bessel operator acts in each of the variables:

\[
(\triangle_\gamma)_{\mu} u = (B_k)_{\mu} u, \quad -\infty < k < \infty, \quad u = u^k(x, t), \quad x \in \mathbb{R}^n, \quad t > 0,
\]

(5)

where

\[
(\triangle_\gamma)_{\mu} = \triangle_\gamma = \sum_{i=1}^{n} (B_{\gamma_i})_{x_i} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \gamma_i \frac{\partial}{\partial x_i},
\]

\[
(B_k)_\mu = \left[ \frac{\partial^2}{\partial t^2} + \frac{k}{t} \frac{\partial}{\partial t} \right], \quad k \in \mathbb{R}.
\]

Equation (5) we will call the general Euler-Poisson-Darboux equation.

**Statement 2.1.** Let \(u^k = u^k(x, t)\) denote the solution of (5) when the next two fundamental recursion formulas hold

\[
u^k = t^{1-k} u^{2-k},
\]

(7)

\[
u^k_t = t u^{k+2}.
\]

(8)

**Proof.** Following [23] we prove (7). Putting \(w = t^{k-1} v, v = u^k\) we have

\[
w_t = (k - 1) t^{k-2} v + t^{k-1} v_t = \frac{k - 1}{t} w + t^{k-1} v_t,
\]

\[
w_{tt} = (k - 1)(k - 2) t^{k-3} v + (k - 1) t^{k-2} v_t + (k - 1) t^{k-2} v_t + t^{k-1} v_{tt} = \frac{(k - 1)(k - 2)}{t^2} w + 2(k - 1) t^{k-2} v_t + t^{k-1} v_{tt},
\]
Here we present the solutions of the problem (2)-(3) for different values of \( k \) for which we obtain solution of (2)-(4) in the next section, and get formula for the connection of solution of problem (2)-(3) and solution of simpler problem when \( k = 0 \) in (2).

In \( \mathbb{R}^n \) we will use multidimensional generalized translation corresponding to multi-index \( \gamma \):

\[
\gamma T^t = \gamma_1 T^t_{x_1} \ldots \gamma_n T^t_{x_n},
\]

where \( \gamma = (\gamma_1, \ldots, \gamma_n) \) is a multi-index.

If \( w = t^{k-1}v \) satisfies the equation

\[
\Delta \gamma w = w_{tt} + \frac{2 - k}{t}w_t,
\]

then using (9) we get

\[
t^{k-1} \Delta \gamma v = t^{k-1} \left( v_{tt} + \frac{k}{t} v_t \right)
\]

which means that \( v \) satisfies the equation

\[
\Delta \gamma v = v_{tt} + \frac{k}{t} v_t.
\]

Denoting \( v = u^{2-k} \) we obtain (7).

Now we prove the (8). Let \( tw = v_t, v = u^k \). We obtain

\[
w_t = -\frac{1}{t^2}v_t + \frac{1}{t} v_{tt},
\]

\[
w_{tt} = \frac{2}{t^3} v_t - \frac{2}{t^2} v_{tt} + \frac{1}{t} v_{ttt}.
\]

We find now \( \frac{k+2}{t} w_t \):

\[
\frac{k+2}{t} w_t = -\frac{k+2}{t^2} v_t + \frac{k+2}{t} v_{tt}.
\]

Then we get

\[
w_{tt} + \frac{k+2}{t} w_t = \frac{2}{t^3} v_t - \frac{2}{t^2} v_{tt} + \frac{1}{t} v_{ttt} - \frac{k+2}{t^3} v_t + \frac{k+2}{t^2} v_{tt} =
\]

\[
= \frac{1}{t} v_{ttt} - \frac{k}{t^3} v_t + \frac{k}{t^2} v_{tt} + \frac{1}{t} \left( v_{ttt} - \frac{k}{t^2} v_t + \frac{k}{t} v_{tt} \right) = \frac{1}{t} \frac{\partial}{\partial t} \left( v_{tt} + \frac{k}{t} v_t \right)
\]

or

\[
w_{tt} + \frac{k+2}{t} w_t = \frac{1}{t} \frac{\partial}{\partial t} \left( v_{tt} + \frac{k}{t} v_t \right).
\]

Recursion formulas (7) and (8) allow us to obtain, from a solution \( u_k \) of equation (5), the solutions of the same equation with the parameter \( k + 2 \) and \( 2 - k \), respectively. Both formulas are proved for Euler-Poisson-Darboux equation

\[
\frac{\partial^i u}{\partial t^i} + \frac{k}{T} \frac{\partial u}{\partial t} - \Delta u = 0.
\]

### 3 Weighted spherical mean and the first Cauchy problem for the general Euler-Poisson-Darboux equation

Here we present the solutions of the problem (2)-(3) for different values of \( k \) for which we obtain solution of (2)-(4) in the next section, and get formula for the connection of solution of problem (2)-(3) and solution of simpler problem when \( k = 0 \) in (2).

In \( \mathbb{R}^n \) we will use multidimensional generalized translation corresponding to multi-index \( \gamma \):

\[
\gamma T^t = \gamma_1 T^t_{x_1} \ldots \gamma_n T^t_{x_n},
\]
where each $\gamma T^\gamma_x f(x)$ is defined by the formula (see [24])

$$\gamma T^\gamma_x f(x) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^\infty f(x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + \tau_i^2 - 2x_i\tau_i \cos \alpha_i}, x_{i+1}, \ldots, x_n) \sin^{\gamma-1} \alpha_i \, d\alpha_i. $$

The below-considered weighted spherical mean generated by a multidimensional generalized translation $\gamma T^\gamma_x$ has the form (see [25])

$$M^\gamma_f(x; r) = \frac{1}{|S_1^\gamma(n)|} \int_{S_1^\gamma(n)} \gamma T^\gamma_x f(x) \theta^\gamma \, dS,$$

where $\ theta^\gamma = \prod_{i=1}^n \theta_i^{\gamma_i}, \ S_1^\gamma(n) = \{\theta; |\theta| = 1, \theta \in \mathbb{R}^n\}$ and the coefficient $|S_1^\gamma(n)|_\gamma$ is computed by the formula

$$|S_1^\gamma(n)|_\gamma = \int_{S_1^\gamma(n)} \prod_{i=1}^n |x_i|^{\gamma_i} \, dS = \frac{n!}{\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)} \frac{1}{2^n \Gamma\left(\frac{n+|\gamma|}{2}\right)}$$

(see [26], p. 20, formula (1.2.5) in which we should put $N=n$). Construction of a multidimensional generalized translation and the weighted spherical mean are transmutation operators (see [27]).

Theorems 3.1-3.4 have been proved in [28]. We give formulations of these theorems here because they will be needed in the next section.

**Theorem 3.1.** The weighted spherical mean of $f \in C_{2v}$ satisfies the general equation Euler–Poisson–Darboux equation

$$(\Delta_\gamma)_x M^\gamma_f(x; t) = (B_k)_x M^\gamma_f(x; t), \quad k = n + |\gamma| - 1$$

and the conditions

$$M^\gamma_f(x; 0) = f, \quad (M^\gamma_f)_t(x; 0) = 0.$$

This theorem has been proved in [25]).

We give theorems on the solution of the Cauchy problem for the general Euler–Poisson–Darboux equation for the remaining values of $\gamma$.

$$(\Delta_\gamma)_x u = (B_k)_x u, \quad u = u^k(x, t), \quad x \in \mathbb{R}^n, \quad t > 0, \quad u^k(x, 0) = f(x), \quad u^k_t(x, 0) = 0.$$

**Theorem 3.2.** Let $f \in C_{2v}$. Then for the case $k > n + |\gamma| - 1$ the solution of (15)–(16) is unique and given by

$$u^k(x, t) = \frac{2^n \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n+|\gamma|+1}{2}\right) \prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)} \int \left[ \gamma T^\gamma y f(x) \right] (1-|y|^2)^{\frac{k-n+|\gamma|-1}{2}} y^\gamma \, dy. \quad (17)$$

Using weighted spherical mean we can write

$$u^k(x, t) = \frac{2^{1-k} \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n+|\gamma|+1}{2}\right) \Gamma\left(\frac{\gamma+i+1}{2}\right)} \int_0^t \left(t^2 - r^2\right)^{\frac{k-n+|\gamma|-1}{2}} \rho_{n+|\gamma|-1} M^\gamma_f(x; r) \, dr. \quad (18)$$

**Theorem 3.3.** If $f \in C_{2v}^{k+1}$ then the solution of (15)–(16) for $k < n + |\gamma| - 1, k \neq -1, -3, -5, \ldots$

$$u^k(x, t) = t^{1-k} \left(\frac{\partial}{\partial t}\right)^m (t^{k+2m-1} u^{k+2m}(x, t), \quad m = \min\{k+2, 0\} \quad m \geq \frac{n+|\gamma|+1}{2} \quad (19)$$

where $m$ is a minimum integer such that $m \geq \frac{n+|\gamma|+1}{2}$ and $u^{k+2m}(x, t)$ is the solution of the Cauchy problem

$$(B_{k+2m})_x u^{k+2m}(x, t) = (\Delta_\gamma)_x u^{k+2m}(x, t), \quad (20)$$
\[ u^{k+2m}(x,0) = \frac{f(x)}{(k+1)(k+3)\ldots(k+2m-1)}, \quad u_t^{k+2m}(x,0) = 0. \tag{21} \]

The solution of (15)–(16) is unique for \( k \geq 0 \) and not unique for negative \( k \).

**Theorem 3.4.** If \( f \in C_\text{ev}^{1+k} \) is \( B \)–polyharmonic of order \( \frac{1+k}{2} \) then one of the solutions of the Cauchy problem (20)–(21) for the \( k=-1, -3, -5, \ldots \) is given by
\[ u^{-1}(x,t) = f(x), \tag{22} \]
\[ u^k(x,t) = f(x) + \sum_{h=1}^{\frac{k+1}{2}} \frac{\Delta_h^k f}{(k+1)\ldots(k+2h-1)} \frac{t^{2h}}{2 \cdot 4 \cdot \ldots \cdot 2h}, \quad k = -3, -5, \ldots \tag{23} \]

The solution of (15)–(16) is not unique for negative \( k \).

The theorem 3.5 contains the explicit form of the transmutation operator for the solution. Definition, methods of construction and applications of the transmutation operators can be found in [27, 29, 30].

**Theorem 3.5.** Let \( k > 0 \). The twice continuously differentiable on \( \mathbb{R}^{n+1}_+ \) solution \( u = u^k(x,t) \) of the Cauchy problem
\[ (\Delta_\gamma)_x u = (B_k)_x u, \quad u = u^k(x,t), \quad x \in \mathbb{R}^n_x, \quad t > 0, \tag{24} \]
\[ u^k(x,0) = f(x), \quad u_t^k(x,0) = 0 \tag{25} \]

such that \( u^k_i(x,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n,t) = 0 \), \( i = 1, \ldots, n \) is connected with the twice continuously differentiable on \( \mathbb{R}^n_x \times \mathbb{R} \) solution \( w = w(x,t) \) of the Cauchy problem
\[ (\Delta_\gamma)_x w = w_{tt}, \quad w = w(x,t), \quad x \in \mathbb{R}^n_x, \quad t \in \mathbb{R}, \tag{26} \]
\[ w(x,0) = f(x), \quad w_t(x,0) = 0 \tag{27} \]

such that \( w_i(x,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n,t) = 0 \), \( i = 1, \ldots, n \) by formula
\[ u^k(x,t) = (\mathcal{P}_1^\lambda)_\alpha w(x,\alpha t), \tag{28} \]
where \( (\mathcal{P}_1^\lambda)_\alpha \) is transmutation Poisson operator (see [24]) acting by \( \alpha \)
\[ (\mathcal{P}_1^\lambda)_\alpha g(\alpha) = \frac{2\Gamma(\lambda+1)}{\sqrt{\pi}\Gamma(\frac{1}{2})} \frac{1}{\tau^{\frac{1}{2}\lambda}} \int_0^\tau g(\alpha)[\tau^2 - \alpha^2]^{\frac{1}{2}\lambda-\frac{1}{2}} d\alpha. \]

**Proof.** The fact that the function \( u^k \) defined by the equality (28) satisfies the conditions (31) is obvious. Let us show that \( u^k \) defined by (28) satisfies (24)
\[ (\Delta_\gamma)_x u = (\mathcal{P}_1^\lambda)_\alpha (\Delta_\gamma)_x w(x,\alpha t) = (\mathcal{P}_1^\lambda)_\alpha w_{\xi \xi}(x,\alpha t) = \frac{2\Gamma(k+1)}{\sqrt{\pi}\Gamma(\frac{k+1}{2})} \int_0^1 (\Delta_\gamma)_x w(x,\alpha t)[1 - \alpha^2]^{\frac{1}{2}-1} d\alpha, \]
where \( \xi = \alpha t \). Further integrating by parts we obtain
\[ \frac{du^k}{dt} = \frac{2\Gamma(k+1)}{\sqrt{\pi}\Gamma(\frac{k+1}{2})} \int_0^1 \alpha w_{\xi}(x,\alpha t)[1 - \alpha^2]^{\frac{1}{2}-1} d\alpha = \]
\[ = \left\{ u = w_{\xi}(x,\alpha t), dv = \alpha[1 - \alpha^2]^{\frac{1}{2}-1} d\alpha, du = tw_{\xi}(x,\alpha t) d\alpha, v = \frac{1}{k}[1 - \alpha^2]^{\frac{1}{2}} \right\} = \]
Thus we have unique representation of

\[ w(x, t) = \frac{2 \Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi \Gamma\left(\frac{k}{2}\right)}} \int_0^t w_{\xi \xi}(x, \alpha t)[1 - \alpha^2]^{\frac{k}{2}} d\alpha = \frac{2 \Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi \Gamma\left(\frac{k}{2}\right)}} \int_0^t w_{\xi \xi}(x, \alpha t)[1 - \alpha^2]^\frac{k}{2} d\alpha. \]

For \( \frac{\partial^2 u^k}{\partial t^2} \) we have

\[ \frac{\partial^2 u^k}{\partial t^2} = \frac{2 \Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi \Gamma\left(\frac{k}{2}\right)}} \int_0^t w_{\xi \xi}(x, \alpha t)[1 - \alpha^2]^{\frac{k}{2} - 1} d\alpha = \frac{2 \Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi \Gamma\left(\frac{k}{2}\right)}} \int_0^t (\Delta_\gamma)_{x} w(x, \alpha t) \alpha^2 [1 - \alpha^2]^{\frac{k}{2} - 1} d\alpha. \]

Finally,

\[ \frac{\partial^2 u^k}{\partial t^2} + \frac{k}{t} \frac{\partial u^k}{\partial t} = \frac{2 \Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi \Gamma\left(\frac{k}{2}\right)}} \left[ \int_0^t (\Delta_\gamma)_{x} w(x, \alpha t) \alpha^2 [1 - \alpha^2]^{\frac{k}{2} - 1} d\alpha + \int_0^t (\Delta_\gamma)_{x} w(x, \alpha t) [1 - \alpha^2]^\frac{k}{2} d\alpha \right] = \frac{2 \Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi \Gamma\left(\frac{k}{2}\right)}} \int_0^t (\Delta_\gamma)_{x} w(x, \alpha t) [1 - \alpha^2]^{\frac{k}{2} - 1} d\alpha = (\Delta_\gamma)_{x} u^k. \]

Thus the function \( u^k \) defined by equality (28) satisfies the problem (24)–(31).

Let us prove that from the relation (28) we can uniquely obtain a solution of the problem (26)–(27). By introducing new variables \( \alpha t = \sqrt{\gamma}, t = \sqrt{\gamma} \), we get

\[ y^{\frac{k+1}{2}} u^k(x, \sqrt{\gamma}) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi \Gamma\left(\frac{k}{2}\right)}} \int_0^\sqrt{\gamma} w(x, \sqrt{\gamma}) (y - \tau)^{\frac{k}{2} - 1} d\tau. \]

Let \( k > 0 \) then \( y^{\frac{k+1}{2}} u^k(x, \sqrt{\gamma}) \) is the Riemann-Liouville left-sided fractional integral of the order \( \frac{k}{2} \) (see [31], p. 33):

\[ y^{\frac{k+1}{2}} u^k(x, \sqrt{\gamma}) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi \Gamma\left(\frac{k}{2}\right)}} \left( I_{\frac{k}{2}, 1} \frac{w(x, \sqrt{\gamma})}{\sqrt{\gamma}} \right) (\gamma). \]

Thus we have unique representation of \( w(x, \sqrt{\gamma}) \) (see [31], p. 44, theorem 24)

\[ w(x, \sqrt{\gamma}) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{k+1}{2}\right)} \left( D_{\frac{k}{2}, \gamma} y^{\frac{k+1}{2}} u^k(x, \sqrt{\gamma}) \right) (\tau) \]

or

\[ w(x, t) = \frac{2}{\Gamma\left(n - \frac{k}{2}\right)} \left( \frac{d}{2t} \right)^n \int_0^t \frac{u^k(x, z) z^k}{(t^2 - z^2)^{\frac{n}{2} - 1}} dz. \]

\[ \Box \]

4 The second Cauchy problem for the general Euler-Poisson-Darboux equation

In this section we obtain solution of (2)-(4).

**Theorem 4.1.** If \( \varphi \in C_{2\nu}^{\left[\frac{n+1+k}{2}\right]} \) then the solution \( v = v^k(x, t) \) of

\[ (\Delta_\gamma)_{x} v = (B_k)v, \quad 0 < \gamma_i, \quad i = 1, \ldots, n, \quad k < 1, \quad x \in \mathbb{R}^n, \quad t > 0, \]

(29)
Proof. Taking into account formula (33) we obtain

\[ v^k(x, 0) = 0, \quad \lim_{t \to 0} t^k \frac{\partial v}{\partial t} = \varphi(x) \]  

\[ (\triangle) v = t^{q-2} \varphi \]  

is given by

\[ v^k(x, t) = \frac{2^{-q} \Gamma \left( \frac{3-k}{2} \right)}{(1-k) \Gamma \left( \frac{3-k+2q}{2} \right)} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^q \left( t^{1-k+2q} v^{2-k+2q} \right) \]

\[ \times \left( t^{1-k+2q} \int_{B^2_t(n)} \left[ \gamma T^\gamma \varphi(x) \right] (1-|y|^2)^{2-k+2q-\gamma+1} y^\gamma dy \right) \]

if \( n + |\gamma| + k \) is not an odd integer and

\[ v^k(x, t) = \frac{2^{-q} \Gamma \left( \frac{3-k}{2} \right)}{(1-k) \Gamma \left( \frac{3-k+2q}{2} \right)} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^q \left( t^{n+|\gamma|-2} M_\gamma\omega^r(x; t) \right) \]

if \( n + |\gamma| + k \) is an odd integer, where \( q \geq 0 \) is the smallest positive integer number such that \( 2-k+2q \geq n+|\gamma|-1 \).

Let \( q \geq 0 \) be the smallest positive integer number such that \( 2-k+2q \geq n+|\gamma|-1 \) i.e. \( q = \left\lceil \frac{n+|\gamma|+k-1}{2} \right\rceil \)

and let \( v^{2-k+2q}(x, t) \) be a solution of (29) when we take \( 2-k+2q \) instead of \( k \) such that

\[ v^{2-k+2q}(x, 0) = \varphi(x), \quad v^{2-k+2q}_t(x, 0) = 0. \]  

Then by property (7) we obtain that

\[ v^{2-k+2q} = t^{1-k+2q} v^{2-k+2q} \]

is a solution of the equation

\[ (\triangle) v = t^{q-2} \varphi \]

Further, applying \( q \)-times the formula (8) we obtain that

\[ \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^q v^{2-k+2q} = \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^q \left( t^{1-k+2q} v^{2-k+2q} \right) \]

is a solution of the (29).

Let’s consider

\[ v^k(x, t) = \frac{2^{-q} \Gamma \left( \frac{3-k}{2} \right)}{(1-k) \Gamma \left( \frac{3-k+2q}{2} \right)} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^q \left( t^{1-k+2q} v^{2-k+2q} \right) \]

(32)

We have shown that (32) satisfies the equation (29).

Now we will prove that \( v^k \) satisfies the conditions (31). For \( v^k \in C^q_\omega(\Omega+, \gamma) \) we have the formula (see [19], p.9)

\[ \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^q \left( t^{1-k+2q} v^{2-k+2q} \right) = \sum_{s=0}^{q} \frac{2^{q-s} C^q_s \Gamma \left( \frac{1-k}{2} + q + 1 \right)}{\Gamma \left( \frac{1-k}{2} + s + 1 \right)} t^{1-k+2s} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^s v^{2-k+2q} \]

(33)

Taking into account formula (33) we obtain \( v^k(x, 0) = 0 \) and

\[ \lim_{t \to 0} t^k v^k(x, t) = \frac{2^{-q} \Gamma \left( \frac{3-k}{2} \right)}{(1-k) \Gamma \left( \frac{3-k+2q}{2} \right)} \lim_{t \to 0} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^q \left( t^{1-k+2q} v^{2-k+2q} \right) = \]

\[ = \frac{2^{-q} \Gamma \left( \frac{3-k}{2} \right)}{(1-k) \Gamma \left( \frac{3-k+2q}{2} \right)} \lim_{t \to 0} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^q \sum_{s=0}^{q} \frac{2^{q-s} C^q_s \Gamma \left( \frac{1-k}{2} + q + 1 \right)}{\Gamma \left( \frac{1-k}{2} + s + 1 \right)} t^{1-k+2s} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^s v^{2-k+2q} = \]

\[ = \frac{1}{1-k} \lim_{t \to 0} t^k \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^q \left( t^{1-k} v^{2-k+2q} \right) = \frac{1}{1-k} \lim_{t \to 0} t^k \left( t^{1-k} v^{2-k+2q} + t^{1-k} v^{2-k+2q}_t \right) = \]
Now we obtain the representation of $v^k$ through the integral. Using formula (18) we get

$$v^{2-k+2q} = \frac{2^q \Gamma \left( \frac{3-k+2q}{2} \right)}{\Gamma \left( \frac{3-k+2q-n+1}{2} \right) \Gamma \left( n+|\gamma|-1 \right)} \int_0^1 (1-r^2)^{\frac{1-k+2n-|\gamma|}{2}} r^{n+|\gamma|-1} M^j_{\psi}(x; rt) \, dr.$$

If $2-k+2q > n + |\gamma| - 1$ then by applying (32) and (33) we write

$$v^k = \frac{2^{-q} \Gamma \left( \frac{3-k}{2} \right) \sum_{s=0}^{q} 2^{-s} C_q^s \Gamma \left( \frac{1-k+q}{2} + 1 \right) t^{1-k+2s} \left( \frac{1}{t} \partial_t \right)^s v^{2-k+2q} =$$

$$= \Gamma \left( \frac{3-k}{2} \right) \sum_{s=0}^{q} \frac{C_q^s t^{1-k+2s}}{2! \Gamma \left( \frac{3-k+2q}{2} + s \right)} \left( \frac{1}{t} \partial_t \right)^s v^{2-k+2q} =$$

$$= \Gamma \left( \frac{3-k+2q}{2} \right) \Gamma \left( \frac{1-k}{2} \right) \Gamma \left( n+|\gamma|-1 \right) \sum_{s=0}^{q} \frac{C_q^s t^{1-k+2s}}{2^s \Gamma \left( \frac{3-k+2q-n+1}{2} \right) \Gamma \left( n+|\gamma|-1 \right)} \int_0^1 (1-r^2)^{-\frac{1-k+2n-|\gamma|}{2}} r^{n+|\gamma|-1} \left( \frac{1}{t} \partial_t \right)^s M^j_{\psi}(x; rt) \, dr.$$

If $2-k+2q = n + |\gamma| - 1$ then $v^{2-k+2q} = M^j_{\psi}(x; t)$ and

$$v^k = \frac{2^{-q} \Gamma \left( \frac{1-k}{2} \right)}{\Gamma \left( \frac{3-k+2q}{2} \right)} \left( \frac{1}{t} \partial_t \right)^q \left( t^{n+|\gamma|-2} M^j_{\psi}(x; t) \right) =$$

$$= \frac{2^{-1-q} \Gamma \left( \frac{1-k}{2} \right) \sum_{s=0}^{q} 2^{-s} C_q^s \Gamma \left( \frac{3-k+2q-n+1}{2} \right) \Gamma \left( n+|\gamma|-1 \right) t^{1-k+2s} \left( \frac{1}{t} \partial_t \right)^s M^j_{\psi}(x; t) =$$

$$= \sum_{s=0}^{q} \frac{C_q^s \Gamma \left( \frac{1-k}{2} \right) \Gamma \left( \frac{3-k+2q-n+1}{2} \right) \Gamma \left( n+|\gamma|-1 \right)}{2^{s+1} \Gamma \left( \frac{3-k+2q}{2} + s \right)} t^{1-k+2s} \left( \frac{1}{t} \partial_t \right)^s M^j_{\psi}(x; t).$$
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