BRST Quantization of a Particle in AdS₅

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Abstract

We perform the quantization of a massive particle propagating on $AdS_5$. We use the twistor formulation in which the action can be brought into a quadratic form. We construct the BRST operator which commutes with $AdS_5$ isometries forming $SU(2,2)$. The condition of a consistent BRST quantization requires that the AdS energy $E$ is quantized in units of the $AdS_5$ radius $R$, $E = \frac{1}{2\pi}(N_a + N_b + 4)$, with $N_a, N_b$ being some non-negative integers. We also argue that the mass operator will be identified with the moduli of the $U(1)$ central extension $Z$ of the $SU(2,2)\mid 4$ algebra in the supersymmetric case. The spectrum of physical states with vanishing ghost number contains a particular subset of ‘massless’ $SU(2,2)$ multiplets (including the bosonic part of the ‘novel short’ supermultiplets). We hope that our results will help to quantize also the string on $AdS_5$.

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The quantization of string theory on $AdS_5 \times S^5$ is a complicated problem. One of the main difficulties is due to the non-linearities, present even in the bosonic part. The action is not quadratic and hence difficult to quantize. It has been first suggested [1] to use the supertwistors forming the fundamental representation of $SU(2, 2|4)$ as the fundamental variables of the theory, as they should linearize the symmetries of the string action. A toy model world-sheet action using supertwistor variables was suggested and its quantization resulted in the spectrum of states forming unitary irreducible representations of $SU(2, 2|4)$. The connection of this supertwistor action with linearly realized $SU(2, 2|4)$ to the gauge-fixed string action with non-linearly realized $SU(2, 2|4)$ was not clear however. Progress was made in [2] where a twistor parametrization of $AdS_5$ was found. The results were applied to a massive particle propagating on $AdS_5$, and it was shown that the resulting action is a simple quantum mechanical system with an $U(1) \times SU(2)$ gauge symmetry and a linear $SU(2, 2)$ global symmetry. Of course, the goal is eventually to use the results for a quantization of strings on $AdS_5 \times S^5$, thus solving a long-standing problem. In this paper we consider again the massive particle on $AdS_5$. Already in the particle case the problems analogous to those in string theory exist. We consider the action of [2] and argue by counting of the number of degrees of freedom that quantum-mechanically the twistor version is equivalent to the space-time version of the particle theory propagating in $AdS_5$ space. We then find a gauge leading to the action in a quadratic form and perform its BRST quantization, thus giving hope and further support for the program of the quantization of the string.

The action of a massive particle propagating in $AdS_5$ space is proportional to the invariant length of the world line

$$S = -m \int ds$$

where

$$-ds^2 = d\tilde{x}^\tilde{m} \tilde{g}_{\tilde{m} \tilde{n}} d\tilde{x}^\tilde{n} = \rho^2 dx^2 + R^2 \left( \frac{d\rho}{\rho} \right)^2 .$$

This action can also be presented in an equivalent form

$$S = \int d\tau \left[ \tilde{P}_\tilde{m} \partial_\tau \tilde{x}^\tilde{m} - \frac{e}{2} \left( \tilde{P}_\tilde{m} \tilde{g}^{\tilde{m} \tilde{n}} \tilde{P}_{\tilde{n}} + m^2 \right) \right]$$

and also in a clearly reparametrization invariant form:

$$S = \int d\tau \left[ \sqrt{\tilde{g}} g^{\tau \nu} \partial_\tau \tilde{x}^{\tilde{m}} \partial_\tau \tilde{x}^\tilde{n} \tilde{g}_{\tilde{m} \tilde{n}} - \frac{\sqrt{\tilde{g}}}{2} m^2 \right] .$$

One can gauge-fix the reparametrization symmetry but to the best of our understanding, there is no clear way to make the action quadratic using any of these forms of the classical action. The only exception is the massless case. Note that the actions
and (4) consistently contain the limit $m \to 0$, as opposed to (3). For $m = 0$, by going to the conformally flat metric of $AdS_5$, the factor $g_{\bar{m}\bar{n}}$ in (3) can be completely absorbed into the worldline metric, and the action becomes quadratic upon gauge fixing the reparametrization symmetry. However, since the mass parameter prevents us from doing the same successfully in the massive case, our strategy is to first write the classical Lagrangian of a massive particle in $AdS_5$ using different field variables.

A good candidate for a set of coordinates are twistors. These variables are well defined in the context of the 4-dimensional space $\mathbb{R}^4$ and realize the conformal symmetry $SO(2,4) \sim SU(2,2)$ linearly. Since this is also the isometry group of $AdS_5$ (with 4d Minkowski space as its boundary) it seemed natural to extend the twistor construction to include the radial coordinate $\rho$. This was achieved in [2]. For this purpose two twistors

$$Z^I = \left( \frac{\lambda^I_{\bar{a} \dot{a}}}{\bar{\mu}^{\bar{a} \dot{a}}} \right).$$

with $I = 1, 2$ needed to be introduced instead of one. The space-time momenta $P$ are then replaced by the bilinears of the upper components of the twistors, $\lambda$

$$P_{\alpha \dot{\alpha}} = 2\lambda^I_{\alpha \dot{\alpha}} \lambda^I_{\dot{\alpha} \alpha},$$

$$P_{\rho} = -\frac{i}{2\rho^2} \left( \varepsilon^{\alpha \beta} \varepsilon^{IJ} \lambda^I_{\alpha} \lambda^J_{\beta} - c.c. \right).$$

The 5d space-time coordinates $x^m, \rho$ are encoded in the twistor construction through the relation between the upper and lower twistor components:

$$\bar{\mu}^{\bar{a} \dot{a}} = -i \varepsilon^{\alpha \beta} \lambda^I_{\alpha} + \frac{\epsilon^{IJ} \lambda^I_{\dot{\alpha} J}}{\rho} \bar{\mu}^{\bar{a} \dot{a}}\bar{\mu}^{\bar{a} \dot{a}},$$

with $\epsilon^{12} = 1$. (7) automatically implies 3 real constraints the twistors have to satisfy:

$$\bar{\mu}^{\bar{a} \dot{a}} \sigma^{\dot{a} \bar{a}} Z^J = 0,$$

where $\sigma^{\dot{a} \bar{a}}$ are the Pauli matrices, and where

$$\bar{Z}^I = (Z^I)^\dagger H, \quad \text{with} \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the conjugate twistor.

This new twistor construction for $AdS_5$ was then applied [4] to the dynamics of a massive particle propagating on this space. The action (3) in standard two-component notation (see [4] for detailed conventions) is given by

$$S = -\int d\tau \left[ \frac{1}{2} P_{\alpha \dot{\alpha}} \dot{x}^\alpha - P_{\rho} \dot{\rho} - \frac{e}{2R^2} \left( \frac{1}{2\rho^2} P_{\alpha \dot{\alpha}} P^\alpha_{\dot{\alpha}} - \rho^2 P^2_{\rho} - m^2 R^2 \right) \right].$$
Using relations (6) and (7) one can rewrite the action (10) as follows

\[ S = -i \int d\tau \bar{Z}^I \partial_\tau Z^I - \frac{e}{2R^2} \left[ \frac{1}{4} (\bar{Z}^I \delta^{IJ} Z^J)^2 - (mR)^2 \right], \]  

(11)

where the twistor pair is subject to constraints (8). The reparametrization constraint, the variation of the action over the worldline metric, implies a fourth constraint

\[ \frac{1}{2} |\bar{Z}^I \delta^{IJ} Z^J| = mR. \]  

(12)

All four constraints are quadratic in the twistors, and the classical action can be brought into the form [2]

\[ S_{cl} = -i \int d\tau \left( \bar{Z}^I \partial_\tau Z^I - i u^a \phi_a \right), \]  

(13)

where \( u^a(\tau) \) are 4 Lagrange multipliers to the 4 constraints

\[ \phi_a = \bar{Z}^I t^I_a Z^J - 2 \delta_a^0 sR, \quad t^I_a = \{ \delta^{IJ}, (\hat{\sigma}_i)^{IJ} \} \]  

(14)

and \( s = \pm m \). Note that there is an ambiguity in imposing the constraint \( \phi^0 \). The original particle action induces (12), hence \( \phi^0 \) can be stated with \( \mp 2mR \). Therefore, we have the situation that two different twistor actions are equivalent to one particle action. Fortunately, we will also see that both twistor theories are related by a global symmetry, and therefore equivalent.

This classical twistor Lagrangian (13) is equivalent to the original classical particle action in AdS\(_5\) space since both theories result in the same classical equations of motion. The twistor Lagrangian has an \( U(1) \times SU(2) \) gauge symmetry which acts on the fields as

\[ \delta \bar{Z}^I = i \xi^a(\tau) t^I_a Z^J, \quad \delta u^0 = \partial_\tau \xi^0(\tau), \quad \delta u^i = \partial_\tau \xi^i(\tau) + 2 \epsilon^{ijk} u^j \xi^k(\tau), \]  

(15)

with local parameters \( \xi^a(\tau) \). Also, the action has a global \( SU(2, 2) \) symmetry since any bilinear combination of twistors (or their space-time derivatives) of the form

\[ \bar{Z}^I(\tau_1) Z^J(\tau_2) = \mu^a(\tau_1) \lambda^J_\alpha(\tau_2) + \bar{\lambda}_\alpha^I(\tau_1) \bar{\mu}^{\alpha J}(\tau_2) \]  

(16)

is invariant under the global \( SU(2, 2) \)

\[ \delta \bar{Z}^I = -g \bar{Z}^I = -(e^A T_A) \bar{Z}^I, \]  

(17)

where the transformations are those of the fundamental representation.
The 4 gauge symmetries of the action are consequences of the fact that the twistor pair is subject to the four real first class constraints. The canonical Poisson brackets $[q, p]_{P.B.} = 1$ following from the twistor action (11) are

\[
[Z^I_A, \bar{Z}^I_B]_{P.B.} = i\delta^I_J \delta_{AB}, \quad [Z^I_A, \bar{Z}^J_B]_{P.B.} = [\bar{Z}^I_A, \bar{Z}^J_B]_{P.B.} = 0, \quad (18)
\]

where $A, B$ label the four twistor components $\lambda_\alpha, \bar{\mu}^\dot{\alpha}$. With (18) it is easy to verify that the constraints form an $U(1) \times SU(2)$ algebra

\[
[\phi_0, \phi_i]_{P.B.} = 0, \quad [\phi_i, \phi_j]_{P.B.} = -2\varepsilon_{ijk}\phi_k. \quad (19)
\]

At this point it is instructive to compare the degrees of freedom of (3) and (13). The original space-time action had 5 degrees of freedom $(\tilde{x}_\mu, \tilde{P}_\mu)$ and one gauge symmetry (or one first class constraint), leading to one pair of Faddeev-Popov ghosts. The net number of physical degrees of freedom is $5 - 1 = 4$. The twistor action has 8 degrees of freedom $Z^I, \bar{Z}^I$ and 4 gauge symmetries (or 4 first class constraints). This leads to four pairs of Faddeev-Popov ghosts, and the net number of physical degrees of freedom is $8 - 4 = 4$. Thus the total number of physical degrees of freedom in both action coincides which provides an argument that they are equivalent quantum-mechanically.

The quantization can be performed both in the Lagrangian form in the field space, as well as in canonical form. The first one allows to find a gauge in which the gauge-fixed action is quadratic, showing that the theory is free. The BRST symmetry of this action proves that the path integral is gauge-independent.

The operator form of the quantization in the canonical space has the advantage that one can construct the BRST operator so that the Hamiltonian is given by the commutator of the BRST operator with the gauge fermion $\Psi$,

\[
H = [Q_{BRST}, \Psi] \quad \Rightarrow \quad [H, Q_{BRST}] = 0 \quad (20)
\]

and therefore the Hamiltonian commutes with the BRST operator by construction. This leads to a simple definition of the physical states in terms of the oscillators of the quantized theory. It is particularly important in our case to find both, the generators of the global $SU(2, 2)$ symmetry, as well as the BRST operator which is the global quantum counterpart of the $U(1) \times SU(2)$ local gauge symmetry in terms of the free oscillators of our theory. Since the action (13) realizes the symmetry $SU(2, 2)$ linearly we would expect that the states of the theory fall into $SU(2, 2)$ representations. In [3, 10] these representations (more precisely, those of $SU(2, 2|4)$) were constructed by an oscillator method. We will later see that a subset of their representations built the Fock space of our theory.

In order for their construction to be valid in our quantum theory one has to verify that all $SU(2, 2)$ generators commute with the BRST operator $Q_{BRST}$. We will find that this is indeed the case.
As our theory has constraints, we will use Dirac’s methods of quantization of constrained systems \([5]\). These methods in the modern form in application to the path integral were developed by Fradkin and his collaborators \([6]\) and by Becchi, Rouet, Stora and Tyutin \([7]\).

To proceed with the Lagrangian BRST quantization we add gauge fixing conditions through gauge functions \(\chi^a\). For simplicity we take the \(\chi^a\) to be functions only of the classical fields (Lagrange multipliers and twistors), but not of their derivatives. The gauge fixed action is

\[
\mathcal{L}_{g.f.}(\Phi_{cl}, b_a, c^a, \pi_a) = \mathcal{L}_{cl}(\Phi_{cl}) + Q\Psi . \quad \Phi_{cl} = \{u, Z, \bar{Z}\}
\]

where \(\Psi\) is the gauge fermion of the form

\[
\Psi = b_a \chi^a(\Phi_{cl})
\]

and the \(b_a\) are anti-ghosts. The BRST action on the fields of the classical action is defined as follows. On the fields of the classical action it is given by eqs. (15) with the parameter of the gauge transformation \(\xi^a(\tau)\) replaced by a product of the ghost field and a global anticommuting parameter \(\Lambda\). In addition we have to specify the standard BRST transformation on ghosts, anti-ghosts and multipliers to the gauge fixing condition

\[
\begin{align*}
\delta_{BRST}\Phi_{cl} &= \delta_{cl}\Phi_{cl}(\xi^a(\tau) \to c^a(\tau)\Lambda), \\
\delta_{BRST}b_a &= \pi_a \Lambda, \\
\delta_{BRST}c^a &= -\frac{1}{2} f^a_{bc} c^b c^c \Lambda \\
\delta_{BRST}\pi_a &= 0
\end{align*}
\]

Using the notation \(\delta_{BRST}X = QX\Lambda\) we can easily verify that \(Q^2 = 0\). Therefore, the action above is BRST invariant since the classical action is invariant under BRST transformations and the gauge fixing term is invariant due to the nilpotency of \(Q\).

\[
QL_{g.f.} = Q\mathcal{L}_{cl} + Q^2\Psi = 0.
\]

The BRST invariant action in this class of gauges can be rewritten as

\[
\mathcal{L}_{g.f.} = \mathcal{L}_{cl}(\Phi_{cl}) + \pi_a \chi^a(\Phi_{cl}) + b_a \frac{\partial \chi^a(\Phi_{cl})}{\Phi_{cl}} Q\Phi_{cl}.
\]

The path integral is independent on the choice of the gauge-fixing function \(\chi^a(\Phi_{cl})\). With the simple choice

\[
\chi^0 = u^0 - 1 \quad \chi^i = u^i
\]

the action becomes

\[
\mathcal{L}_{g.f.} = -i \bar{Z}^I \partial_\tau Z^I - u^0(\bar{Z}^I \delta^I J Z^J - sR) + \bar{\pi}_i u^i + \pi_0 (u^0 - 1) + b_a \partial_\tau c^a.
\]
The cubic terms in the classical action and in the ghost action proportional to $u^i$ have been absorbed by a redefinition of $\pi_i$ into $\tilde{\pi}_i$. The dependence of the classical action on $u^0(\tau)$ is not absorbed into the redefinition of $\pi_0$. Since this is a reparametrization related symmetry, and $u^0$ is related to $e$, it is not attractive to gauge away the worldline metric, but rather to choose a gauge which sets $u^0(\tau)$ to a constant. The last step is simply to integrate over $\tilde{\pi}_i$ and $\pi_0$, or equivalently to solve the equations for these variables. The final form of the gauge-fixed action in this gauge is quadratic in all fields

$$S_{g.f.} = -i \int d\tau \tilde{Z}^I \partial_\tau Z^I - i(\tilde{Z}^I \delta^{IJ} Z^J - 2sR) + i b_\alpha \partial_\tau c^\alpha . \quad (28)$$

This shows that the resulting theory is free and we may use oscillators to construct the spectrum. The BRST symmetry is inherited from the general case described above since we have just taken a particular gauge and excluded the auxiliary fields. We may also perform the quantization in the elegant and powerful formalism of Batalin and Vilkovisky [8]. This can be found in the appendix. From (28) one again derives the Poisson brackets

$$[Z^I_A, \tilde{Z}^J_B]_{P.B.} = i \delta_{AB} \delta^{IJ} . \quad (29)$$

One might be tempted to identify $Z^I$ ($\tilde{Z}^I$) as creation (annihilation) operators, but we will wait with the construction of states (i.e. the definition of a vacuum and creation and annihilation operators) until later. In fact, we will find that there is a more suitable choice.

Let us first construct the $SU(2, 2)$ generators explicitly in terms of the quantized twistors. We start from the classical action (11) and will obtain the $SU(2, 2)$ generators by a Noether procedure. The twistors transform in the fundamental representation of $SU(2, 2)$ with $\delta Z^I = -(\epsilon^A T_A) Z^I$. The fundamental of $SU(2, 2)$ is the (chiral) spinor representation of $SO(4, 2)$ and the generators are the $SO(4, 2)$ gamma-matrices, $M_{\hat{m}\hat{n}} = \frac{1}{4} \hat{\gamma}_{\hat{m}\hat{n}}$. We choose $\hat{\gamma}_{mn} = \gamma_{mn}, \hat{\gamma}_{mS} = \gamma_m \gamma_5, \hat{\gamma}_{mT} = -\gamma_m, \gamma_{TS} = \gamma_5,$ where the $\gamma_m, \gamma_5$ are 5-dimensional gamma-matrices. The translation between the spinor representation of $SO(2, 4)$ and the fundamental of $SU(2, 2)$ goes through the unitary similarity matrix which relates the hermitian conjugate $\gamma$-matrices to the original ones

$$\left(\hat{\gamma}_{\hat{m}\hat{n}}\right)^\dagger = -A \hat{\gamma}_{\hat{m}\hat{n}} A^{-1} , \quad A^2 = 1 . \quad (30)$$

The matrix $A$ is identified with the $SU(2, 2)$ metric $H$ and is given by

$$A = -i \gamma_0 . \quad (31)$$

It can be checked that in the twistor basis for the gamma-matrices [3] it gives the metric as in (9). Therefore the variation of the twistor, which is in the first place an $SO(2, 4)$-spinor is

$$\delta Z^I = -\frac{1}{4} \hat{\Lambda}^{\hat{m}\hat{n}} \hat{\gamma}_{\hat{m}\hat{n}} Z^I . \quad (32)$$
In general, the variation of the action is
\[ \delta S = \int d\tau (\partial_\tau \epsilon^\Lambda) J_\Lambda \] (33)
for local parameters \( \epsilon^\Lambda \). The \( J_\Lambda \) are the conserved currents (in this case, as we deal with a quantum mechanical system, they are right away the charges). Applying this to the case at hand
\[ \delta S = \frac{i}{4} \int d\tau (\partial_\tau \hat{\Lambda}^\hat{\mu}\hat{\eta}^\hat{\nu}) \hat{Z}_I^I \hat{Z}^I. \] (34)
We derive the \( SU(2, 2) \) charges in terms of twistors
\[ \hat{M}_{\hat{\mu}\hat{\nu}} = \frac{i}{4} \hat{Z}_I^I \hat{Z}^I. \] (35)
These are the 15 hermitian generators of \( SU(2, 2) \sim SO(2, 4) \). The Noether procedure guarantees that they satisfy the algebra of Poisson brackets. To elevate this to the quantum theory, we promote the Poisson brackets defined above to the commutators
\[ [\hat{Z}_A^I, \hat{Z}_B^J]_{P.B.} = i\delta_{AB}\delta^{IJ} \Rightarrow [\hat{Z}_A^I, \hat{Z}_B^J] = \delta_{IJ}. \] (36)
In terms of components the commutation relations are
\[ [\mu^\alpha_I, \lambda^J_\beta] = \delta^\alpha_\beta\delta^{IJ}, \quad [\bar{\lambda}_\bar{\alpha}^I, \bar{\mu}^\bar{\beta}J] = \delta_{\bar{\alpha}\bar{\beta}}\delta^{IJ}. \] (37)
We will also have to deal with normal ordering ambiguities, but first we would like to give the explicit correspondence between this twistor construction and the oscillator construction of [10].

To construct the Fock space we have to identify the annihilation and creation operators. We have found in (36) and (37) that one can view \( \lambda \) and \( \bar{\mu} \) as creation operators, or any linear combination of the two (the same holds, of course, for the annihilation operators). The commutation relations of our dynamical system are closely related to those of the oscillator method used in [10] to construct the UIR’s of \( SU(2, 2) \). Therefore, we would like to make use of the results obtained there.

The oscillator construction is built upon a set of creation (annihilation) operators \( a^r, b^s (a_r, b_r) \) satisfying \( a_r = (a^r)^\dagger, b_r = (b^r)^\dagger \). As the twistors as such do not satisfy this condition a change of basis is required. Thus, we obtain the following picture:

\[ \text{AdS-spacetime} \implies \text{twistors} \implies \text{oscillator construction} \]

\[ (x^m, \rho) \implies \hat{Z}_I^I \left( \begin{array}{c} \lambda^I_\alpha \\ \bar{\mu}^{I\dot{\alpha}} \end{array} \right) \implies \Psi_I^I \left( \begin{array}{c} a^I_r \\ -b^I_s \end{array} \right) . \]

To explain the second part of this connection we make the change of basis explicit. The change of basis between \( \gamma \)-matrices in the oscillator basis as in [10], denoted by
γ′, and the twistor basis, denoted by γ, is given by the following unitary similarity transformation
\[ \dot{\gamma}_{\dot{m}\dot{n}} = S \gamma_{\dot{m}\dot{n}} S^\dagger. \]  
(38)

where
\[ S = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1_2 & 1_2 \\ -1_2 & 1_2 \end{array} \right). \]  
(39)

This induces the following rotation on the twistors
\[ \Psi^I = S \bar{Z}^I, \quad \bar{\Psi}^I = \bar{Z}^I S^\dagger. \]  
(40)

Now we identify the oscillators in the spinors Ψ 
\[ \Psi^I = \left( \begin{array}{c} a^r I \\ -b^r I \end{array} \right), \quad \bar{\Psi}^I \equiv -i \Psi^I \gamma_0 = \left( \begin{array}{c} a^r I \\ b^r I \end{array} \right). \]  
(41)

The commutation relations for th Ψ follow from the ones of \[ Z^I \] 
\[ [\bar{\Psi}_A^I, \Psi_B^J] = \mathbb{S}_A^C [\bar{Z}_C^I, Z_D^J] \mathbb{S}^\dagger_B = \delta_{AB} \delta^{IJ}, \]  
(42)

which in more detail read
\[ [a^r_I, a^{sJ}] = \delta_r^s \delta^{IJ}, \quad [b^r_I, b^{sJ}] = \delta_r^s \delta^{IJ}, \]  
(43)

as desired. In this basis the SU(2,2) metric \( H = \mathbb{A} \) is given by \( \text{diag}(1,1,-1,-1) \). The global SU(2,2) generators are
\[ \hat{M}_{\dot{m}\dot{n}} = \frac{i}{4} \bar{\Psi}^I \gamma_{\dot{m}\dot{n}}^I \Psi^I \]  
(44)

in the oscillator basis, and the gauge-fixed action (28) reads
\[ S_{g.f.} = -i \int d\tau \bar{\Psi}^I \partial_\tau \Psi^I - i \left( \bar{\Psi}^I \delta^{IJ} \Psi^J - 2sR \right) + ib_a \partial_\tau c^a. \]  
(45)

Now, that we have related the quantized twistors \( Z^I \) to the creation \( (a^r, b^r) \) and annihilation \( (a^r, b^r) \) operators of [10] we have proven that the oscillator construction of UIR of SU(2,2) in [10] follows from the quantization of our action. Before we revisit this construction let us first reexpress the SU(2,2) generators explicitly in the \( a, b \) basis. Since we want to define them as quantum operators we have to tackle the problem of normal ordering. In the following the index \( \dot{m} \) takes values \{0,\ l,\ 5,\ 6\}, where \( l = 1, 2, 3 \). We find that the following generators build the SU(2,2) algebra on the quantum level:
\[ :\hat{M}_{lm}: = -\frac{1}{4} \varepsilon_{lmn} \left[ a^{rI} a^s_r (\hat{\sigma}_m)^s - b^{rI} b^s_r (\hat{\sigma}_m)^s \right], \]  
(5.1)
The generators $\hat{M}_{l5} = -\frac{1}{4} \left[ a^{rI} a^I_r (\hat{\sigma}_I)^s_r + b^{rI} b^I_r (\hat{\sigma}_I)^s_r \right]$, $\hat{M}_{06} = \frac{1}{4} \left[ a^{rI} a^I_r + b^{rI} b^I_r + 4 \right]$, $\hat{M}_{0l} = \frac{i}{4} \left[ a^{rI} b^I_r (\hat{\sigma}_I)^s_r - a^I_r b^I_s (\hat{\sigma}_I)^s_r \right]$, $\hat{M}_{6l} = \frac{1}{4} \left[ a^{rI} b^I_r (\hat{\sigma}_I)^s_r + a^I_r b^I_s (\hat{\sigma}_I)^s_r \right]$, $\hat{M}_{05} = \frac{1}{4} \left[ a^{rI} b^I_s (\hat{\sigma}_I)^s_r + a^I_r b^I_r (\hat{\sigma}_I)^s_r \right]$, $\hat{M}_{65} = \frac{i}{4} \left[ a^{rI} b^I_r (\hat{\sigma}_I)^s_r - a^I_r b^I_r (\hat{\sigma}_I)^s_r \right]$. (46)

The generators $\hat{M}_{l5} + \frac{1}{2} \epsilon_{lmn} M_{mn} :$, $\hat{M}_{05} - \frac{1}{2} \epsilon_{lmn} \hat{M}_{mn} :$, $\hat{M}_{06} :$ generate the maximal compact subgroup $SU(2)_L \times SU(2)_R \times U(1)_E$, which provides the quantum numbers of the states. The operator $\hat{M}_{06} :$ corresponds to the AdS energy and generates translations along the global timelike vector field of the $AdS_5$ space. In the conformal interpretation it is the conformal Hamiltonian, $\frac{1}{2} (P_0 + R^2 K_0)$. This generator is the only global $SU(2, 2)$ generator which suffers from a normal ordering ambiguity, because of the tracelessness of the Pauli matrices. This ambiguity can be resolved by demanding that the $SU(2, 2)$ is a global symmetry of the quantum theory and since we are dealing with a simple algebra (therefore every operator appears on the r.h.s. of the algebra) the quantum operators are given necessarily by (46). These satisfy the algebra of commutation relations

$$[ : \hat{M}_{\hat{m} \hat{n}} : , : \hat{M}_{\hat{p} \hat{q}} : ] = -i (\hat{\eta}_{\hat{m} \hat{p}} : \hat{M}_{\hat{q} \hat{n}} : - \hat{\eta}_{\hat{n} \hat{p}} : \hat{M}_{\hat{q} \hat{m}} : )$$. (47)

For convenience, we introduce the number operators

$$N_a = a^{rI} a^I_r , \quad N_b = b^{rI} b^I_r$$. (48)

The AdS energy

$$E = \frac{2}{R} : \hat{M}_{06} : = \frac{1}{2R} (N_a + N_b + 4)$$ (49)

is quantized in units of $R^{-1}$.

The Fock space of (zero ghost number) states is constructed as in [10] with just two generations of oscillators. The vacuum is defined by

$$a_r |0\rangle = b_r |0\rangle = 0$$. (50)

The lowest weight type representations (lwtr) are then constructed by selecting a lowest weight vector (lwv) $| \Omega \rangle$ annihilated by all $SU(2, 2)$ generators of the form

$$L_{rs} \equiv (a_r)^I (b_s)^I$$. (51)
The lwv transforms irreducibly under the maximal compact subgroup \( S(U(2) \times U(2)) \), 
\[ L^0 \equiv (a^r)^I(a_s)^I \oplus (b^r)^I(b_s)^I, \]
and the representations are then constructed by the repeated action of
\[ L^{+rs} \equiv (a^r)^I(b^s)^I \] on \( |\Omega\rangle \), i.e. the representations are of the form
\[ |\Omega\rangle, \; L^+|\Omega\rangle, \; L^+L^+|\Omega\rangle, \; L^+L^+L^+|\Omega\rangle, \ldots \] (53)

Finally, let us proceed with the Hamiltonian quantization. The physical states of the theory can now be constructed using the BRST operator. As usual, for the first class constraints \([\phi_a, \phi_b] = f_{ab}^c \phi_c\) one defines
\[ Q_{\text{BRST}} = c^a \phi_a - \frac{1}{2} f_{ab}^c c^a c^b b_c, \]
which in our case becomes
\[ Q_{\text{BRST}} = c^a \phi_a + i \varepsilon_{ijk} c^i d^j b_k. \] (54)

It is easily verified that one has indeed \( Q_{\text{BRST}}^2 = 0 \). There are no normal ordering issues, since the only expression which could lead to such ambiguities, \( \phi_0 \), drops out of the calculation of \( Q_{\text{BRST}}^2 \).

The physical states \(|\Omega\rangle\) of the theory must also be constructed in a way that respects the global symmetry of our action. Fortunately, by construction, our BRST operator commutes with the generators of the global \( SU(2, 2) \), because the constraints in \( Q_{\text{BRST}} \) contain the twistors only in the \( SU(2, 2) \) invariant bilinear form of (16). Therefore, the construction of representations above is consistent with the BRST condition on physical states \(|\Omega\rangle\) given by
\[ Q_{\text{BRST}}|\Omega\rangle = 0 \quad |\Omega\rangle \neq Q_{\text{BRST}}|\tilde{\Omega}\rangle \] (55)

As usual, we will focus here on the states with vanishing ghost number
\[ Q_{\text{BRST}}|\Omega\rangle_{N_{gh}=0} = c^a \phi_a |\Omega\rangle_{N_{gh}=0} = 0, \] (56)
which are by construction annihilated by the (normal ordered) BRST constraints
\[ :\phi_0: = :\hat{Z}^I \hat{Z}^I - 2s R: = :\hat{\Psi}^I \hat{\Psi}^I - 2s R: = (N_a - N_b + 4 - 2(s + s_0)R), \]
\[ :\phi_i: = :\hat{Z}^I (\hat{\sigma}_i)^{IJ} \hat{Z}^J : = :\hat{\Psi}^I (\hat{\sigma}_i)^{IJ} \hat{\Psi}^J : = a^r I a^r I (\hat{\sigma}_i)^{IJ} - b^r J b^r J (\hat{\sigma}_i)^{IJ} \] (57)

In the \( SU(2) \) constraints :\( \phi_i : \) there is no normal ordering constant, because the Pauli matrices are traceless. The nonvanishing commutators of above constraints are
\[ [:\phi_i :, :\phi_j :] = -2i \varepsilon_{ijk} :\phi_k :. \] (58)

However, :\( \phi_0 : \) has a normal ordering ambiguity expressed in the new parameter \( s_0 \). This ambiguity can neither be resolved by requiring nilpotency of the BRST operator, nor closure of the algebra. Hence, the theory seems to be consistent for all choices of \( s_0 \).
However the classical twistor action (11) and the gauge-fixed actions (28) and (45) are “CPT invariant” for the transformations

\[ \Psi^I \xrightarrow{\text{CPT}} -\gamma_5^I (\Psi^I)^*, \quad \Rightarrow \quad a \xrightarrow{\text{CPT}} b, \quad b \xrightarrow{\text{CPT}} a, \]  

(59)
since \( \Psi^I \) is a Dirac spinor, and

\[ \tau \xrightarrow{\text{CPT}} -\tau, \quad c^a \xrightarrow{\text{CPT}} c^a, \quad b_a \xrightarrow{\text{CPT}} b_a \quad \text{and} \quad s \xrightarrow{\text{CPT}} -s. \]  

(60)

We see that under a “CPT” transformation the parameter \( s \) necessarily changes sign. If we want to extend this symmetry to a quantum symmetry and construct a “CPT invariant” spectrum we need that the constraints transform into themselves under this CPT. This leads us to \( s_0 = 2 \) as then under CPT the constraint \( :\phi_0: \xrightarrow{\text{CPT}} -:\phi_0: \). This constraint then gives that on physical states

\[ Z \equiv sR = \frac{1}{2}(N_a - N_b). \]  

(61)

Since this generator commutes with all \( SU(2, 2) \) generators all states of a given representation carry the same eigenvalue of \( Z \). In the full supersymmetric case a \( U(1) \) operator (central charge) does appear in the r.h.s. of the superconformal algebra [10], which takes in the bosonic case precisely the form of \( Z \). This suggests that in fact the mass \( mR \) is to be identified with the modulus of the central charge in the superconformal theory. However, a full clarification of this point requires a supersymmetric generalization of this work.

Now, let us discuss the representations of the theory which form the physical states. We will not attempt a full classification in this publication, rather have a first glance at the problem. We analyze the representations of [10] with two generations of oscillators and check whether they are admitted by the BRST constraints (i.e. \( SU(2) \) invariance).

The first state allowed (with the choice \( s_0 = 2 \)) is the vacuum state

\[ |\Omega\rangle = |0\rangle. \]  

(62)

The corresponding multiplet arises from the “massless (super)particle” and gives rise (in the supersymmetric case) to the supergravity multiplet of \( \mathcal{N} = 8 \) supergravity.

Generically, \( SU(2, 2) \) representations with hvv

\[ |\Omega\rangle = a^{r_1 l_1} a^{r_2 l_2} \ldots a^{r_n l_n} |0\rangle, \]  

(63)

are not annihilated by the BRST operator, and therefore not physical states of our theory.

\footnote{We mean CPT invariance in the 4-dimensional sense as in [10].}
To construct admissible representations is rather simple, as we only need to use $SU(2)$ representation theory (where the $SU(2)$ is part of the BRST symmetry and not be confused with any $SU(2,2)$ subgroup). The oscillators transform in the fundamental representation of $SU(2)$. For example the Young tableaux of an $SU(2)$ invariant state is given by

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\end{array}
, 1
\]

representing the lwv

\[
|\Omega\rangle = \epsilon_{r_1r_2}\epsilon_{I_1I_2}...\epsilon_{r_{2j-1}r_{2j}}a^{r_1I_1}a^{r_2I_2}a^{r_3I_3}a^{r_4I_4}...a^{r_{2j-1}I_{2j-1}}a^{r_{2j}I_{2j}}|0\rangle.
\]

The lwv of representations of this type have $AdS$ energies

\[
ER = 2 + \frac{1}{2}N_a = 2 + j,
\]

where $N_a$ is a non-negative even integer, and the entire multiplet has

\[
Z = \frac{1}{2}N_a = j.
\]

These are the multiplets which in the supersymmetric case were called in [10] novel short multiplets. The lwv of the “CPT conjugate” multiplets [10], obtained by replacing $a$ with $b$ oscillators, have the same $AdS_5$ energy, but $Z = -j$.

Above we gave examples of $SU(2,2)$ UIR’s which are physical states of the theory, and of UIR’s which are not. A complete study of the admitted states remains to be done.

In conclusion, we have quantized the dynamics of a massive particle propagating on $AdS_5$. We applied the BRST quantization formalism to the action following from the twistor parametrization of $AdS_5$. The action turned out to be free and we specified the Fock space of the theory. We found that the states fall into $SU(2,2)$ representations with integral value of the “central charge” $Z$, corresponding to the mass of the particle. They correspond to the bosonic part of the novel short supermultiplets [10] of states on $AdS_5$.

Given the success of twistor variables in the particle case we hope that the procedure can be generalized and may help also in the quantization of the string on $AdS_5$.

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**BV quantization of the particle on AdS$_5$ in a nutshell**

In the BV method we introduce for every field (including the ghosts) an antifield with opposite statistics. The space of fields and antifields are called the Fields. We denote them generically with $\Phi = \{\phi^{A},\phi^{A*}\}$. The antifields are assigned a ghostnumber $gh(\Phi)$ such that

$$gh(\phi^{A}) + gh(\phi^{A*}) = -1.$$  

The various fields of our model are given in table

| fields | antifields |
|--------|------------|
| name   | stat | gh | name   | stat | gh |
| $\bar{Z}^{I}$ | +   | 0  | $\bar{Z}^{*I}$ | -   | -1 |
| $\bar{Z}^{I}$ | +   | 0  | $Z^{*I}$ | -   | -1 |
| $u^{a}$ | +   | 0  | $u^{*a}$ | -   | -1 |
| $c^{a}$ | -   | 1  | $c^{*a}$ | +   | -2 |

Table 1: The BV Fields of the model

One introduces the antibracket of two functions of the Fields $F, G$

$$(F,G) = \frac{\partial F}{\partial \phi^{A}} \frac{\partial G}{\partial \phi^{*A}} - \frac{\partial F}{\partial \phi^{*A}} \frac{\partial G}{\partial \phi^{A}}.$$  

In the summation over $A$ also an integration over $\tau$ is understood. The extended action for the model is

$$S_{ext} = -i\bar{Z}^{I}\partial_{\tau}Z^{I} - u^{a}(\bar{Z}^{I}t_{a}^{IJ}Z^{J} - 2\delta_{a}^{0}mR) + iZ^{*I}t_{a}^{IJ}Z^{J}c^{a} - i\bar{Z}^{I}t_{a}^{IJ}Z^{*J}c^{*a} + u^{*a}(\partial_{\tau}c^{a} + 2\epsilon^{ijk}u^{i}c^{k}\delta_{a}^{j}) + c^{*}_{i}\epsilon^{ijk}c^{j}c^{k}$$

and is obtained by requiring the classical master equation

$$(S_{ext},S_{ext}) = 0.$$  

13
The space of physical observables is defined through the antibracket cohomology. We have the nilpotent (through Jacobi-identities and (71)) operator

$$SF = (F, S).$$  \(72\)

The physical observables are functions of the fields of ghostnumber 0 and two functions describe the same physical observable if they are in the same cohomology class. We define canonicle transformations as transformations which preserve the antibracket. It is then clear that this does not affect the physical observables. It turns out that gauge-fixing can be performed by a canonical transformation. In our case we take the transformation

$$u^*_a = b_a, \quad u^a = \delta^a_0 - b^* a,$$  \(73\)

provides an appropriate gauge-fixing. What it means is that we choose a different subset in the space of Fields of what we call fields. Indeed, we consider $u^*_a$ as a field and give it the name $b_a$. It has ghostnumber $-1$ hence its usual name antighost. The gauge-fixed action gets the form

$$S_{g.f.} = -i\bar{Z}^I \partial_\tau Z^I - (\bar{Z}^I \dot{Z}^I - 2mR) + b_a \partial_\tau c^a + i \bar{Z}^*_{Ia} \dot{Z}^J \dot{c}^a - i \bar{Z}^*_{Ia} \dot{Z}^{*J} \dot{c}^a + 2b^i \epsilon^{ijk} b^j \dot{c}^k + c^*_i \epsilon^{ijk} c^j \dot{c}^k. \quad \text{(74)}$$

The antifield-independent part of this action coincides with action (28). The BRST operator $\Omega$ is identified through

$$\Omega f(\phi) = (f(\phi), S)|_{\phi^* = 0}. \quad \text{(75)}$$

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