THE REFLECTION PRINCIPLE AND CALDERÓN PROBLEMS WITH PARTIAL DATA

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1. Introduction

Let $M_0$ be a smooth Riemann surface with boundary, equipped with a metric $g$. A complex line bundle $E$ on $M_0$ has a trivialization $E \simeq M_0 \times \mathbb{C}$, thus there is a non-vanishing smooth section $s: M_0 \to E$, and a connection $\nabla$ on $E$ induces a complex valued 1-form $iX$ on $M_0$ (where $i = \sqrt{-1} \in \mathbb{C}$) defined by $\nabla s = s \otimes iX$, which means that $\nabla(fs) = s \otimes (d + iX)f$ if $d$ is the exterior derivative. The associated connection Laplacian ($*$ is the Hodge operator with respect to $g$) is the operator

$$\Delta^X := \nabla^X \ast \nabla^X = - \ast (d \ast + iX \Wedge \ast)(d + iX)$$

acting on complex valued functions (sections of $E$). When $X$ is real valued, this operator is often called the magnetic Laplacian associated to the magnetic field $dX$, and the connection 1-form $X$ can be seen as a connection 1-form on the principal bundle $M_0 \times S^1$ by identifying $i\mathbb{R} \subset \mathbb{C}$ with the Lie algebra of $S^1$. This also corresponds to a Hermitian connection, in the sense that it preserves the natural Hermitian product on $E$. Let $V$ be a complex valued function on $M_0$ and assume that the 1-form $X$ is real valued, and consider the magnetic Schrödinger Laplacian associated to the couple $(X, V)$

$$L_{X,V} := \Delta^X + V = - \ast (d \ast + iX \Wedge \ast)(d + iX) + V.$$

If $H^s(M_0)$ denotes the Sobolev space with $s$ derivatives in $L^2$ and $\Gamma \subset \partial M_0$ is an open subset such that $\partial M_0 \setminus \Gamma$ contains an open segment, we define the partial Cauchy data space of $L_{X,V}$ to be

$$\mathcal{C}_{X,V,\partial M_0 \setminus \Gamma} := \{ (u, \nabla^X_{\nu} u |_{\partial M_0 \setminus \Gamma}) \mid u \in H^1(M_0), \text{supp}(u |_{\partial M_0}) \subset \partial M_0 \setminus \Gamma, L_{X,V}u = 0 \}$$

where $\nu$ is the outward pointing unit normal vector field to $\partial M_0$ and $\nabla^X_{\nu} u := (\nabla^X u)(\nu)$.

The first natural inverse problem is to see if the Cauchy data space determines the connection form $X$ and the potential $V$ uniquely, and one easily sees that it is not the case since there are gauge invariances in the problem: for instance, conjugating $L_{X,V}$ by $e^f$ with $f = 0$ on $\partial M_0 \setminus \Gamma$, one obtains the same partial Cauchy data space but with a Laplacian associated to the connection $\nabla^{X+df}$, therefore it is not possible to identify $X$ but rather one should expect to recover the connection $\nabla^X$ modulo isomorphism.

It was shown in [13] and [1] that, in the special case when $\Gamma = \emptyset$, the Cauchy data uniquely determines the connection $\nabla^X$ up to unitary bundle isomorphisms which are identity on the boundary and the potential $V$. This was done in [13] through showing that the Cauchy data determines the integrals of $X$ along closed loops modulo integer multiples of $2\pi$. For planar domains, this result was first proved by Imanuvilov-Yamamoto-Uhlmann in [18] assuming only partial data measurement.

For these types of results in Euclidean domains of dimensions three and higher, we refer the readers to the works of Henkin-Novikov [25], Sun [28] [29], Nakamura-Sun-Uhlmann in [24].
Kang-Uhlmann in [19], and for partial data Dos Santos Ferreira-Kenig-Sjöstrand-Uhlmann in [6]. For simply connected planar domains, Imanuvilov-Yamamoto-Uhlmann in [18] deal with the case of general second order elliptic operators for partial data measurement, and Lai [21] deals with the special case of magnetic Schrödinger operator for full data measurement.

For $s \in \mathbb{N}, p \in [1, \infty]$, let us denote by $W^{s,p}(M_0)$ and $W^{s,p}(M_0; T^* M_0)$ the Sobolev spaces consisting of functions and 1-forms respectively with $s$ derivatives in $L^p$. If $X_1, X_2 \in W^{3,p}(M_0; T^* M_0)$ and $V_1, V_2 \in W^{2,p}(M_0)$ for $p$ large, we assume that the partial Cauchy data spaces for $L_{X_1,V_1}$ and $L_{X_2,V_2}$ agree
\begin{equation}
\mathcal{E}_{X_1, V_1} \partial M_0 \Gamma = \mathcal{E}_{X_2, V_2} \partial M_0 \Gamma.
\end{equation}

As the Cauchy data is invariant under the gauge transformation $X \mapsto X + d\zeta$ for $\zeta \in W^{4,p}(M_0) \cap H^1_0(M_0)$, we may assume without loss of generality that
\begin{equation}
\iota_\nu(X_1 - X_2) = 0.
\end{equation}

The main result of this paper is the following generalization of the results of [13]:

**Theorem 1.1.** Let $X_1, X_2 \in W^{3,p}(M_0; T^* M_0)$ be real-valued 1-forms and $V_1, V_2 \in W^{2,p}(M_0)$ be functions such that they satisfy [3] and [4]. Then there exists a non-vanishing function $\Theta$ with $\Theta |_{\partial M_0 \Gamma} = 1$ such that $iX_1 = iX_2 + \Theta^{-1} d\Theta$ and $V_1 = V_2$.

To simplify the geometry it is sometimes convenient to consider larger $\Gamma$. As such we will prove the following auxiliary theorem.

**Theorem 1.2.** Let $X_1, X_2 \in W^{3,p}(M_0; T^* M_0)$ be real-valued 1-forms and $V_1, V_2 \in W^{2,p}(M_0)$ be functions such that they satisfy [3] and [4]. Then there exists a subset $\Gamma_0 \subset \partial M_0$ containing $\Gamma$ with $\partial M_0 \Gamma_0$ a connected open segment $\partial M_0$, and a non-vanishing function $\Theta$ with $\Theta |_{\partial M_0 \Gamma_0} = 1$ such that $iX_1 = iX_2 + \Theta^{-1} d\Theta$ and $V_1 = V_2$.

Note that, unlike Theorem [13] we may assume without loss of generality in Theorem [12] that $\partial M_0 \Gamma$ consists of a small line segment along the boundary. The fact that Theorem [12] follows from Theorem [11] is a simple exercise in unique continuation and gauge transformation.

An approach to treat this problem in the case when $X_1 = X_2 = 0$ was developed in [12]. The technique was based on ideas of [17] and [3] of constructing CGO vanishing on $\Gamma$ whose phase is stationary at a prescribed point. One then applies stationary phase expansion at the critical points to extract point-wise information on the coefficients.

There are two difficulties when applying this technique to prove Theorem [12]. First, the presence of first order terms in the boundary integral identity causes derivatives of the phase function to appear in the integrand and thus prevent one from obtaining the desired information at the critical points of the phase function. Second, one needs to construction CGO with higher regularity via a "shifted" Carleman estimate. The standard methods of shifting loses track of the boundary structure (see e.g. [6]) and therefore it is not clear how one can construct CGO with $H^1_{\text{reg}}$ estimates and at the same time vanish on $\Gamma$. Chung in [4] resolved the "shifting" issue in $\mathbb{R}^n$ for $n \geq 3$ and our approach is partially inspired by his ideas. In the planar case, Imanuvilov-Uhlmann-Yamamoto in [18] overcame these difficulties by direct computation and our method, based more on geometry, differs significantly from their approach.

The first difficulty is resolved through the use of a new boundary integral identity:
Proposition 1.3. Under the assumptions of (3) and (4), if one sets \( A_j := \pi_{0,1} X_j \), then there exists an open boundary component \( \Gamma_0 \) containing \( \bar{\Gamma} \) with \( \partial M_0 \setminus \bar{\Gamma} \) an open segment of \( \partial M_0 \), such that one can find non-vanishing functions \( F_{A_j} \in W^{2,p}(M_0) \cap W^{4,p}_{\text{loc}}(M_0) \) solving
\[
F_{A_j}^{-1} \bar{\partial} F_{A_j} = iA_j, \quad |F_{A_j}|_{\Gamma_0} = 1 \quad j = 1, 2 \quad \text{with} \quad F_{A_1} |_{\partial M_0 \setminus \bar{\Gamma}} = F_{A_2} |_{\partial M_0 \setminus \bar{\Gamma}}.
\]
Furthermore, for any pair of \( \{F_{A_1}, F_{A_2}\} \) satisfying (5) and solutions \( u_j \) to
\[
L_{X_j, V_j} u_j = 0 \quad u_j |_{\bar{\Gamma}} = 0
\]
one has
\[
0 = \int_{M_0} \left\{ \langle |F_{A_1}|^{-2} - |F_{A_2}|^{-2} \rangle \bar{\partial} \bar{u}_1, \bar{\partial} \bar{u}_2 \rangle + \frac{1}{2} \langle (Q_2 |F_{A_2}|^2 - Q_1 |F_{A_1}|^2) \bar{u}_1, \bar{u}_2 \rangle \right\}
\]
where \( \bar{u}_j = F_{A_j} u_j \) and \( Q_j = *dX_j + V_j \).

Note that as both solutions are differentiated only by \( \bar{\partial} \) we can then construct CGO (in Section 5) which are compatible with this differential operator so that the difficulty of the phase function appearing in the integrand would not occur. Arriving at (6) requires one to see how assumption (3) leads to the existence of a holomorphic extension of the function \( F_{A_j} F_{A_2}^{-1} |_{\partial M_0 \setminus \bar{\Gamma}} \) for any non-vanishing solutions of \( F_{A_j}^{-1} \bar{\partial} F_{A_j} = iA_j \). This is achieved by considering the double of Riemann surfaces and exploit the symmetry of the holomorphic extension problem under reflection.

The second difficulty, the one of ”shifting” the Carleman estimate, will be treated again by using the reflection principle. In this case we double the bordered Riemann surface and extend the harmonic Carleman weight with reflection principle. On the doubled surface we ”shift” the Carleman estimate with the semiclassical pseudodifferential operator \( \langle hD \rangle^{-1} \) as in [9]. We then use symmetry to see that this shift operation on the doubled surface actually leaves a large portion of the original boundary intact.

In addition to highlighting the geometric nature of this problem, the approach outlines here allows one to extending the setting of [18] to general surfaces. Furthermore, the program described here can be applied to study a wide range of inverse problems involving the connection Laplacian. In a series of forthcoming articles we will use the approach outlined here to treat:

1. The partial Cauchy data problem for the Hodge Laplacian on surfaces (see [5] for the higher dimensional case),
2. The partial Cauchy data problem for Dirac systems (the full data case was considered in [11]),
3. Inverse scattering on surfaces in the presence of magnetic potentials (the special case when \( X_1 = X_2 = 0 \) was considered in [10]).

The systematic approach developed here will facilitate future discussions which naturally follow the identifiability result we prove - that of stability, analytic reconstruction, and numerical reconstruction.

2. Harmonic and Holomorphic Morse Functions on a Riemann Surface

2.1. Riemann surfaces. We start by recalling few elementary definitions and results about Riemann surfaces, see for instance [9] for more details. Let \((M, g)\) be a compact connected smooth Riemannian surface with boundary \( \partial M \). The surface \( M \) can be considered as a subset of a compact Riemannian surface, for instance by taking the double of \( M \).
The conformal class of \( g \) on the closed surface \( M \) induces a structure of closed Riemann surface, i.e. a closed surface equipped with a complex structure via holomorphic charts \( z_α : U_α \to \mathbb{C} \). The Hodge star operator \(*\) acts on the cotangent bundle \( T^*M \), its eigenvalues are \( \pm i \) and the respective eigenspace \( T^*_1,0M := \ker(\star + i\text{Id}) \) and \( T^*_{0,1}M := \ker(\star - i\text{Id}) \) are sub-bundles of the complexified cotangent bundle \( \mathbb{C}T^*M \) and the splitting \( \mathbb{C}T^*M = T^*_1,0M \oplus T^*_{0,1}M \) holds as complex vector spaces. Since \(*\) is conformally invariant on 1-forms on \( M \), the complex structure depends only on the conformal class of \( g \). In holomorphic coordinates \( z = x + iy \) in a chart \( U_α \), one has 
\[ * (udx + vdy) = -vdx +udy \]
and 
\[ T^*_1,0M|_{U_α} \cong \mathbb{C}dz, \quad T^*_{0,1}M|_{U_α} \cong \mathbb{C}d\bar{z} \]
where \( dz = dx + idy \) and \( d\bar{z} = dx - idy \). We define the natural projections induced by the splitting of \( \mathbb{C}T^*M \)
\[ \pi_{1,0} : \mathbb{C}T^*M \to T^*_1,0M, \quad \pi_{0,1} : \mathbb{C}T^*M \to T^*_{0,1}M. \]

The exterior derivative \( d \) defines the De Rham complex \( 0 \to \Lambda^0 \to \Lambda^1 \to \Lambda^2 \to \cdots \) where \( \Lambda^k := \Lambda^kT^*M \) denotes the real bundle of \( k \)-forms on \( M \). Let us denote \( \mathcal{C}Λ^k \) the complexification of \( \Lambda^k \), then the \( \partial \) and \( \bar{\partial} \) operators can be defined as differential operators \( \partial : \mathcal{C}Λ^0 \to T^*_1,0M \) and \( \bar{\partial} : \mathcal{C}Λ^0 \to T^*_{0,1}M \) by
\begin{align}
\partial f &= \pi_{1,0}df, \\
\bar{\partial} f &= \pi_{0,1}df,
\end{align}
they satisfy \( d = \partial + \bar{\partial} \) and are expressed in holomorphic coordinates by
\[ \partial f = \partial z f dz, \quad \bar{\partial} f = \partial \bar{z} f d\bar{z}. \]
with \( \partial z := \frac{1}{2}(\partial x - i\partial y) \) and \( \partial \bar{z} := \frac{1}{2}(\partial x + i\partial y) \). Similarly, one can define the \( \partial \) and \( \bar{\partial} \) operators from \( \mathcal{C}Λ^1 \) to \( \mathcal{C}Λ^2 \) by setting
\[ \partial(\omega_{1,0} + \omega_{0,1}) := d\omega_{0,1}, \quad \bar{\partial}(\omega_{1,0} + \omega_{0,1}) := d\omega_{1,0} \]
if \( \omega_{0,1} \in T^*_{0,1}M \) and \( \omega_{1,0} \in T^*_1,0M \). In coordinates this is simply
\[ \partial(udz + vdy) = \partial v \wedge d\bar{z}, \quad \bar{\partial}(udz + vdy) = \partial u \wedge dz. \]
There is a natural operator, the Laplacian acting on functions and defined by
\[ \Delta f := -2i \star \partial \bar{\partial} f = d^*d \]
where \( d^* \) is the adjoint of \( d \) through the metric \( g \) and \(*\) is the Hodge star operator mapping \( \Lambda^2 \to \Lambda^0 \) and induced by \( g \) as well.

2.2. Maslov Index and Boundary value problem for the \( \overline{\partial} \) Operator. In this subsection we consider the setting where \( M \) is an oriented Riemann surface with boundary \( \partial M \) and \( M_0' \) is a submanifold of \( M \) such that \( \partial M \cap \partial M_0' \neq \emptyset \). Denote by \( \Gamma_0' \subseteq \partial M \) an open subset of \( \partial M \) which compactly contains \( \partial M \cap \partial M_0' \). We assume in addition that \( \partial M \setminus \Gamma_0' \) contains an open set.

Following [23] (see also [12]), we adopt the following notations: let \( E \to M \) be a complex line bundle with complex structure \( J : E \to E \) and let \( D : C^∞(M, E) \to C^∞(M, T^*_1,0M_0' \otimes E) \) be a Cauchy-Riemann operator with smooth coefficients on \( M \), acting on sections of the bundle \( E \). Observe that in the case when \( E = M \times \mathbb{C} \) is the trivial line bundle with the natural complex structure on \( M \), then \( D \) can be taken to be the operator \( \overline{\partial} \) introduced in (7). For \( q > 1 \), we define
\[ D_F : W^{t,q}_F(M, E) \to W^{t-1,q}_F(M, T^*_1,0M \otimes E) \]
where $F \subset E$ |$\partial M$ is a totally real subbundle (i.e. a subbundle such that $JF \cap F$ is the zero section) and $D_F$ is the restriction of $D$ to the $L^q$-based Sobolev space with $\ell$ derivatives and boundary condition $F$

$$W^\ell q_\delta(M, E) := \{ \xi \in W^\ell q(M, E) \mid \xi(\partial M) \subset F \}.$$ \n
The boundary Maslov index for a totally real subbundle $F \subset E|_{\partial M}$ of a complex vector bundle is defined in generality in Appendix C.3 of [23], we only recall the definition in our setting

**Definition 2.1.** Let $E = M \times \mathbb{C}$ and $\partial M = \cup_{j=1}^m \partial_j M$ be a disjoint union of $m$ circles. The boundary Maslov index $\mu(E, F)$ is the degree of the map $\rho \circ \Lambda : \partial M \to \partial M$ where 

$$\Lambda|_{\partial M} : S^1 \cong \partial M \to GL(1, \mathbb{C})/GL(1, \mathbb{R})$$

is the natural map assigning to $z \in S^1$ the totally real subspace $F_z \subset \mathbb{C}$, where $GL(1, \mathbb{C})/GL(1, \mathbb{R})$ is the space of totally real subbundles of $\mathbb{C}$, and $\rho : GL(1, \mathbb{C})/GL(1, \mathbb{R}) \to S^1$ is defined by $\rho(A, GL(1, \mathbb{R})) := A^2/|A|^2$.

In this setting, we have the following boundary value Riemann-Roch theorem stated in [23]:

**Theorem 2.2.** Let $E \to M$ be a complex line bundle over an oriented compact Riemann surface with boundary and $F \subset E|_{\partial M}$ be a totally real subbundle. Let $D$ be a smooth Cauchy-Riemann operator on $E$ acting on $W^\ell q(M, E)$ for some $q > 1$ and $\ell \in \mathbb{N}$. Then

1) The following operators are Fredholm

$$D_F : W^\ell q(M, E) \to W^{\ell-1,q}(M, T_{0,1}^*M \otimes E)$$

$$D_F^* : W^\ell q(M, T_{0,1}^*M \otimes E) \to W^{\ell-1,q}(M, E).$$

2) The real Fredholm index of $D_F$ is given by

$$\text{Ind}(D_F) = \chi(M) + \mu(E, F)$$

where $\chi(M)$ is the Euler characteristic of $M$ and $\mu(E, F)$ is the boundary Maslov index of the subbundle $F$.

3) If $\mu(E, F) < 0$, then $D_F$ is injective, while if $\mu(E, F) + 2\chi(M) > 0$ the operator $D_F$ is surjective.

As an application, we obtain the following (here and in what follows, $H^m(M) := W^{m,2}(M)$):

**Proposition 2.3.** (i) For $q > 1$ and $k \in \mathbb{N}_0$, there exists a bounded operator

$$\bar{\partial}^{-1} : W^{k,q}(M, T_{0,1}^*M) \to \{ u \in W^{k+1,q}(M) \mid u |_{\Gamma_0} \in \mathbb{R} \}$$

satisfying $\bar{\partial}\bar{\partial}^{-1} = 1d$. 

(ii) If $\chi \in C_0^\infty(M)$ is supported in a complex charts $U$ bi-holomorphic to a bounded open set $\Omega \subset \mathbb{C}$ with complex coordinate $z$, then as operators

$$\bar{\partial}^{-1} \chi = \chi' \bar{T} \chi + K$$

where $\chi' \in C_0^\infty(U)$ are such that $\chi' \chi = \chi$, $K$ has a smooth kernel on $M \times M$ and $\bar{T}$ is given in the complex coordinate $z \in U \cong \Omega$ by

$$\bar{T} (fd\bar{z}) = \frac{1}{\pi} \int_U \frac{f(z')}{z - z'} dz'd\bar{z}'$$
where \( dv(z) = \alpha^2(z)dz_1dz_2 \) is the volume form of \( g \) in the chart.

(iii) For \( m > 1/2 \), let \( f \in H^m(\partial M) \) be a real valued function, then there exists a holomorphic function \( v \in H^{m+\frac{1}{2}}(M) \) such that \( \text{Re}(v)|_{\Gamma_0} = f \). Furthermore, \( v \) can be chosen so that \( ||v||_{H^{m+\frac{1}{2}}(M)} \leq C_m ||f||_{H^m(M)} \).

(iv) For \( k \in \mathbb{N} \) and \( q > 1 \), the space of \( W^{k,q}(M) \) holomorphic functions on \( M \) which are real valued on \( \Gamma_0 \) is infinite dimensional.

**Proof.** (i) Let \( L \in \mathbb{N} \) be arbitrary large and let us identify the boundary as a disjoint union of circles \( \partial M = \bigcup_{i=1}^n \partial_i M \) where each \( \partial_i M \simeq S^1 \). Since \( \Gamma_0 \) can be chosen so that \( \partial M \setminus \Gamma_0' \) is as small as we like, it is sufficient to assume that \( \partial M \setminus \Gamma_0' \) is a connected non-empty open segment of \( \partial_i M = S^1 \), and which can thus be defined in a coordinate \( \theta \) (respecting the orientation of the boundary) by \( \partial M \setminus \Gamma_0' = \{ \theta \in S^1 \mid 0 < \theta < 2\pi/k \} \) for some integer \( k \). Define the totally real subbundle of \( F \subset E|_{\partial M} = \bigcup_{j=1}^m (\partial_j M \times \mathbb{C}) \) by the following: on \( \partial_i M \simeq S^1 \) parametrized by \( \theta \in [0, 2\pi) \), define \( F_\theta = e^{ia(\theta)}\mathbb{R} \subset \mathbb{C} \), where \( a : [0, 2\pi] \to \mathbb{R} \) is a smooth nondecreasing function such that \( a(\theta) = 0 \) in a neighbourhood \( [0, \epsilon] \) of 0, \( a(2\pi/k) = 2L\pi \) for some \( L \in \mathbb{N} \), and \( a(\theta) = 2L\pi \) for all \( \theta > 2\pi/k \). In particular \( F_\theta = \mathbb{R} \) is constant for \( \theta \notin \partial M \setminus \Gamma_0' \). For the rest of \( \partial_2 M, \ldots, \partial_n M \), we just let \( F|_{\partial_i M} = S^1 \times \mathbb{R} \). The map \( \Lambda \) in Definition 2.1 is then given on \( \partial_i M \) by \( \Lambda(e^{i\theta}) = e^{ia(\theta)}\text{GL}(1, \mathbb{R}) \) and on \( \partial_2 M, \ldots, \partial_n M \) by \( \Lambda(e^{i\theta}) = e^{i\theta}\text{GL}(1, \mathbb{R}) \), therefore the Maslov index \( \mu(E, F) \) is given by the degree of the map \( e^{i\theta} \to e^{2ia(\theta)} \) on \( S^1 \), and this is given by \( (a(2\pi) - a(0))/2\pi = 2L \). By theorem 2.2 \( D_F \) is surjective if \( 2\chi(M) + 2L > 0 \). Since \( L \) can be taken as large as we want this establishes the solvability assertion of (i).

To obtain the estimate, we fix \( L \) large enough so that \( 2\chi(M) + 2L > 0 \) and consider the splitting given by \( W^{k+1,q}(M) = \ker D_F + (\ker D_F)\perp \). By taking a projection one sees that for all \( \omega \in W^{k,q}(M) \) there exists a unique element \( u \in (\ker D_F)\perp \) such that \( \partial u = \omega \). Therefore we conclude that \( D_F : (\ker D_F)\perp \to W^{k,q}(M, T^{*}_{\Gamma_0'}M) \) is a linear bijection and the uniform boundedness principle gives the desired estimate.

(ii) Observe that \( \tilde{\partial}^{-1}\tilde{\partial} - 1 \) maps \( W^{k,p}_F(M) \) into \( \ker \tilde{\partial} \cap W^{k,p}_F(M) \) which is a finite dimensional space spanned by some smooth functions \( \psi_1, \ldots, \psi_n \) (by elliptic regularity) on \( M \). Assuming that \( (\psi_j)_j \) is an orthonormal basis in \( L^2 \), this implies that, on \( W^{1,2}_F(M) \)

\[
\tilde{\partial}^{-1}\tilde{\partial} = 1 - \Pi \quad \text{where} \quad \Pi = \sum_{k=1}^n \psi_k(\cdot, \psi_k)_{L^2(M)}.
\]

Now we also have

\[
\partial\chi'\tilde{T}\chi = \chi + [\tilde{\partial}, \chi']\tilde{T}\chi
\]

and the last operator on the right has a smooth kernel in view of \( \chi \nabla \chi' = 0 \) and the fact that \( T \) has a smooth kernel outside the diagonal \( z = z' \). Now since \( \chi' \in C_\infty(M) \subset W^{1,2}_F(M) \), we can multiply by \( \tilde{\partial}^{-1} \) on the left of the last identity and obtain

\[
\tilde{\partial}^{-1}\chi = \chi'\tilde{T}\chi - \Pi\chi'\tilde{T}\chi - \tilde{\partial}^{-1}[\partial, \chi']\tilde{T}\chi.
\]

The last two operator on the right have a smooth kernel on \( M \times M \), in view of the smoothness of \( \psi_k \) and the kernel of \( [\tilde{\partial}, \chi']\tilde{T}\chi \), and since \( \tilde{\partial}^{-1} \) maps \( C_\infty(M, T^{*}_{\Gamma_0'}M) \) to \( C_\infty(M) \).

(iii) Let \( w \in H^{m+\frac{1}{2}}(M) \) be a real function with boundary value \( f \) on \( \partial M \), then by (i) there exists \( R \in H^{m+1/2}(M) \) with \( ||R||_{H^{m+\frac{1}{2}}(M)} \leq C ||w||_{H^{m+\frac{1}{2}}(M)} \leq C ||f||_{H^m(\partial M)} \) such that
\[ \frac{\partial r}{\partial R} = -\partial w \] and \( R \) purely real on \( \Gamma'_0 \), thus \( v := iR + w \) is holomorphic such that \( \text{Re}(v) = f \) on \( \Gamma'_0 \).

(iv) Taking the subbundle \( F \) as in the proof of (i), we have that \( \dim \ker D_F = \chi(M) + 2L \) if \( L \) satisfies \( 2\chi(M) + 2L > 0 \), and since \( L \) can be taken as large as we like, this concludes the proof.

\[ \square \]

**Lemma 2.4.** Let \( \{p_0, p_1, \ldots, p_n\} \subset M \) be a set of \( n + 1 \) disjoint points. Let \( c_1, \ldots, c_K \in \mathbb{C}, N \in \mathbb{N}, \) and let \( z \) be a complex coordinate near \( p_0 \) such that \( p_0 = \{z = 0\} \). Then if \( p_0 \in \text{int}(M) \), there exists a holomorphic function \( f \) on \( M \) with zeros of order at least \( N \) at each \( p_j \), such that \( f \) is real on \( \Gamma'_0 \) and \( f(z) = c_0 + c_1z + \ldots + c_Kz^K + O(|z|^{K+1}) \) in the coordinate \( z \). If \( p_0 \in \partial M \), the same is true except that \( f \) is not necessarily real on \( \Gamma'_0 \).

**Proof.** First, using linear combinations and induction on \( K \), it suffices to prove the Lemma for any \( K \) and \( c_0 = \cdots = c_{K-1} = 0 \), which we now show. Consider the subbundle \( F \) as in the proof of (i) in Proposition 2.3. The Maslov index \( \mu(E, F) \) is given by \( 2L \) and so for each \( N \in \mathbb{N} \), one can take \( L \) large enough to have \( \mu(E, F) + 2\chi(M) \geq 2N(1 + n) \). Therefore by Theorem 2.2 the dimension of the kernel of \( \bar{\partial}F \) will be greater than \( 2(n+1)N \). Now, since for each \( p_j \) and complex coordinate \( z_j \) near \( p_j \), the map \( u \to (u(p_j), \partial_{z_j}u(p_j), \ldots, \partial_{z_j}^{N-1}u(p_j)) \in \mathbb{C}^N \) is linear, this implies that there exists a non-zero element \( u \in \ker D_F \) which has zeros of order at least \( N \) at all \( p_j \).

First, assume that \( p_0 \in \text{int}(M) \) and we want the desired Taylor expansion at \( p_0 \) in the coordinate \( z \). In the coordinate \( z \), one has \( u(z) = \alpha z^M + O(|z|^{M+1}) \) for some \( \alpha \neq 0 \) and \( M \geq N \). Define the function \( r_K(z) = \chi(z)z^{K-1}z^{-M+K} \) where \( \chi(z) \) is a smooth cut-off function supported near \( p_0 \) and which is 1 near \( p_0 = \{z = 0\} \). Since \( M \geq N > 1 \), this function has a pole at \( p_0 \) and trivially extends smoothly to \( M \setminus \{p_0\} \), which we still call \( r_K \). Observe that the function is holomorphic in a neighbourhood of \( p_0 \) but not at \( p_0 \) where it is only meromorphic, so that in \( M \setminus \{p_0\} \), \( \bar{\partial}r_K \) is a smooth and compactly supported section of \( T^*_0M \) and therefore trivially extends smoothly to \( M \) (by setting its value to be 0 at \( p_0 \)) to a one form denoted \( \omega_K \). By the surjectivity assertion in Corollary 2.3 there exists a smooth function \( R_K \) satisfying \( \bar{\partial}R_K = -\omega_K \) and that \( R_K |_{\Gamma'_0} \in \mathbb{R} \). We now have that \( R_K + r_K \) is a holomorphic function on \( M \setminus \{p_0\} \) meromorphic with a pole of order \( M - K \) at \( p_0 \), and in coordinate \( z \) one has \( z^{M-K}(R_K(z) + r_K(z)) = c_K + O(|z|) \). Setting \( f_K = u(R_K + r_K) \), we have the desired holomorphic function. Note that \( f \) also vanish to order \( N \) at all \( p_1, \ldots, p_n \). This achieves the proof.

Now, if \( p_0 \in \partial M \) we can consider a slightly larger manifold \( M' \) containing \( M \) and we apply the the result above.

We conclude this subsection with the following estimate for the operator \( \bar{\partial}^{-1}e^{2i\psi/h} \).

**Lemma 2.5.** Let \( U \) be an open subset compactly contained in \( M \) and for \( q, p \in [1, \infty] \). Let \( \psi \) be a real valued smooth Morse function on \( M \) and let \( \bar{\partial}^{-1} := \bar{\partial}^{-1}e^{2i\psi/h} \) where \( \bar{\partial}^{-1} \) is the right inverse of \( \bar{\partial} : W^{1,p}(M) \to L^p(T^*_0M) \) constructed in Proposition 2.3. Let \( q \in (1, \infty) \) and \( p > 2 \), then there exists \( C > 0 \) independent of \( h \) such that for all \( \omega \in W^{1,p}_0(U, T^*_0M) \)

\[ ||\bar{\partial}^{-1}\omega||_{L^q(M)} \leq C h^{2/3} ||\omega||_{W^{1,p}(M, T^*_0M)} \text{ if } 1 \leq q < 2 \]

\[ ||\bar{\partial}^{-1}\omega||_{L^q(M)} \leq C h^{1/q} ||\omega||_{W^{1,p}(M, T^*_0M)} \text{ if } 2 \leq q \leq p. \]
There exists $\epsilon > 0$ and $C > 0$ such that for all $\omega \in W^{1,p}_c(M, T_{0,1}^* M)$
\begin{equation}
||\partial_\psi^{-1}\omega||_{L^2(M)} \leq C h^{1+\epsilon}||\omega||_{W^{1,p}(M, T_{0,1}^* M)}.
\end{equation}

**Proof.** Observe that the estimate (10) is a direct corollary of (9) and (8) by using interpolation. We recall the Sobolev embedding $W^{1,p}(M) \subset C^\alpha(M)$ for $\alpha \leq 1 - 2/p$ if $p > 2$, and we shall denote by $\bar{T}$ the Cauchy-Riemann inverse of $\partial_\xi$ in $\mathbb{C}$:
\[ \bar{T}(f dz) := \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\xi)}{z - \xi} d\xi_1 d\xi_2 \]
where $\xi = \xi_1 + i\xi_2$. If $\Omega, \Omega' \subset \mathbb{C}$ are bounded open sets, then the operator $\mathbb{I}_{\Omega'} T$ maps $L^p(\Omega)$ to $L^p(\Omega')$. Since $\omega$ is compactly supported in a chart $U$ biholomorphic to a bounded domain $\Omega \subset \mathbb{C}$, and since the estimates will be localized, we can assume with no loss of generality that $\psi$ has only one critical point, say $z_0 \in \Omega$ (in the chart). The expression of $\partial_\psi^{-1}(f dz)$ in complex local coordinates in the chart $\Omega$ satisfies
\[ \partial_\psi^{-1}(f(z) dz) = \chi(z) T(e^{-2i\psi/h} f) + K(e^{-2i\psi/h} f dz) \]
where $K$ is an operator with smooth kernel and $\chi \in C_0^\infty(\mathbb{C})$ is identically 1 on $U$.

Let us first prove (8). Let $\chi_\delta \in C_0^\infty(\mathbb{C})$ be a function which is equal to 1 for $|z - z_0| > 2\delta$ and to 0 in $|z - z_0| \leq \delta$, where $\delta > 0$ is a parameter that will be chosen later (it will depend on $h$). Using Minkowski inequality, one can write when $q < 2$
\begin{equation}
||\chi T((1 - \chi_\delta)e^{-2i\psi/h} f)||_{L^q(\mathbb{C})} \leq \int_{\Omega} ||\chi(\cdot - \xi)||_{L^q(\mathbb{C})} |(1 - \chi_\delta(\xi))f(\xi)| d\xi_1 d\xi_2
\end{equation}
\begin{equation}
\leq C||f||_{L^\infty} \int_{\Omega} |(1 - \chi_\delta(\xi))| d\xi_1 d\xi_2 \leq C\delta^2||f||_{L^\infty}.
\end{equation}

On the support of $\chi_\delta$, we observe that since $\chi_\delta = 0$ near $z_0$, we can use
\[ T(e^{-2i\psi/h} \chi_\delta f) = \frac{1}{2} ih[e^{-2i\psi/h} \chi_\delta f - T(e^{-2i\psi/h} \partial(\chi_\delta f))] \]
and the boundedness of $T$ on $L^q$ to deduce that for any $q < 2$
\begin{equation}
||\chi T(\chi_\delta e^{-2i\psi/h} f)||_{L^q(\mathbb{C})} \leq C h \left(||\frac{\chi_\delta f}{\partial \psi}||_{L^q} + ||\frac{f}{\partial \psi} \chi_\delta \partial \chi_\delta}{||\chi_\delta f||_{L^q}}\right).
\end{equation}
The first term is clearly bounded by $\delta^{-1}||f||_{L^\infty}$ due to the fact that $\psi$ is Morse. For the last term, observe that since $\psi$ is Morse, \[ \frac{1}{|\partial \psi|} \leq \frac{c}{|z - z_0|} \text{ near } z_0, \]
therefore
\[ ||\frac{f \chi_\delta}{\partial \psi}||_{L^q} \leq C ||f||_{L^\infty} \left(\int_\delta^1 r^{1-2\delta}\, dr\right)^{1/q} \leq C \delta^{2/2-q}||f||_{L^\infty}. \]
The second term can be bounded by $||\frac{\partial \chi_\delta}{\partial \psi}||_{L^q} \leq ||f||_{L^\infty}||\frac{\partial \chi_\delta}{\partial \psi}||_{L^q}$, with that the $||\frac{\partial \chi_\delta}{\partial \psi}||_{L^q}$ grows like $\delta^{-2}$, $\partial \chi_\delta$ is only supported in a neighbourhood of radius $2\delta$. Therefore, we obtain
\[ ||\frac{f \partial \chi_\delta}{\partial \psi}||_{L^q} \leq \delta^{2/q-2}||f||_{L^\infty}. \]
The third term can be estimated by
\[ ||\chi_\delta \frac{\partial f}{\partial \psi}||_{L^q} \leq C ||\partial f||_{L^p}||\chi_\delta||_{L^q} \leq C \delta^{-1}||\partial f||_{L^p}. \]
Combining these four estimates with (12) we obtain
\[ \| \chi T(\chi \delta e^{-2i\psi/h} f) \|_{L^q(\mathbb{C})} \leq h^2\| f \|_{W^{1,p}(\delta^{-1} + 2/q - 2)} \].

Combining this and (11) and optimizing by taking \( \delta = h^{1/3} \), we deduce that
\[ \| \chi T(\chi \delta e^{-2i\psi/h} f) \|_{L^q(\mathbb{C})} \leq h^{2/3}\| f \|_{W^{1,p}} \]
if \( q < 2 \). We now move on to the smoothing part given by \( K(e^{-2i\psi/h} f) \). Take \( \chi \) to be a compactly supported function in \( \Omega \) such that it is equal to 1 on the support of \( f \), we see that \( K(e^{-2i\psi/h} f) = K(e^{-2i\psi/h}(f - \chi f(z_0))) + f(z_0)K(e^{-2i\psi/h} \chi) \). By applying stationary phase, we easily see that \( \| f(z_0)K(e^{-2i\psi/h} \chi) \|_{L^q} \leq Ch\| f \|_{C^q} \) for any \( q \in [1, \infty] \). For the first term, we write \( \tilde{f} := f - \chi f(z_0) \) and we integrate by parts to get, for some smoothing operator \( K' \)
\[ K(e^{-2i\psi/h} \tilde{f}) = hK'(e^{-2i\psi/h} \tilde{f}) + \frac{h}{2i}K(e^{-2i\psi/h} \partial_z \left( \frac{\tilde{f}}{\partial z} \right)) \).

By the fact that \( K \) and \( K' \) are smoothing, we see that for all \( k \in \mathbb{N} \)
\[ \| K(e^{2i\psi/h} \tilde{f}) \|_{C^k} \leq hC \left( \| f \|_{L^\infty} + \| \partial_z \left( \frac{\tilde{f}}{\partial z} \right) \|_{L^1} \right) \]
Using the fact that \( \psi \) is Morse, the Sobolev embedding \( W^{1,p} \subset C^\alpha \) for \( \alpha = 1 - 2/p \) and \( \tilde{f}(z_0) = 0 \), we can estimate the last term by \( C\| f \|_{W^{1,p}} \) if \( p > 2 \). Therefore,
\[ \| K(e^{2i\psi/h} f) \|_{L^q} \leq Ch\| f \|_{W^{1,p}} \]
for any \( q \in [1, \infty] \) and \( p > 2 \). Combining (13) and (14) we see that (8) is established.

Let us now turn our attention to the case when \( \alpha > q \geq 2 \), one can use the boundedness of \( T \) on \( L^q \) and thus
\[ \| \chi T((1 - \chi \delta)e^{-2i\psi/h} f) \|_{L^q(\mathbb{C})} \leq \| (1 - \chi \delta)e^{-2i\psi/h} f \|_{L^q(\Omega)} \leq C\delta^\frac{2}{q} \| f \|_{L^\infty}. \]

Now since \( \chi_\delta = 0 \) near \( z_0 \), we can use
\[ T(e^{-2i\psi/h} \chi \delta f) = \frac{1}{2i\hbar} \left[ e^{-2i\psi/h} \chi \delta f + T(e^{-2i\psi/h} \partial_z \left( \chi \delta f \right)) \right] \]
and the boundedness of \( T \) on \( L^q \) to deduce that for any \( q \leq p \), (12) holds again with all the terms satisfying the same estimates as before so that
\[ \| T(e^{-2i\psi/h} \chi \delta f) \|_{L^q} \leq Ch\| f \|_{W^{1,p}} (\delta^{2/q - 2} + \delta^{-1}) \leq Ch\delta^{2/q - 2} \| f \|_{W^{1,p}} \]
since now \( q \geq 2 \). Now combine the above estimate with (15) and take \( \delta = h^{1/2} \) we get
\[ \| T(e^{-2i\psi/h} f) \|_{L^q} \leq h^{1/q} \| f \|_{W^{1,p}} \]
for \( 2 \leq q \leq p \). The smoothing operator \( K \) is controlled by (14) for all \( q \in [1, \infty] \) and therefore we obtain (9). \( \square \)
2.3. Morse holomorphic functions with prescribed critical points. The main result of this section is the following

**Proposition 2.1.** Let \( \hat{p} \) be an interior point of \( M \) and \( \epsilon > 0 \) small. Then there exists a holomorphic function \( \Phi \) on \( M \) which is Morse on \( M \) (up to the boundary) and real valued on \( \Gamma'_0 \), which has a critical point \( \hat{p}' \) at distance less than \( \epsilon \) from \( \hat{p} \) and such that \( \text{Im}(\Phi(\hat{p}')) \neq 0 \).

Let \( \Omega \) be a connected open set of \( M^D \) such that \( \partial \) is a smooth surface with boundary, with \( \bar{M} \subset \partial \subset M^D \) and \( \Gamma'_0 \subset \partial \Omega \). Fix \( k > 2 \) a large integer, we denote by \( C^k(\bar{\Omega}) \) the Banach space of \( \mathcal{C}^k \) real valued functions on \( \bar{\Omega} \). Then the set of harmonic functions on \( \bar{\Omega} \) which are in the Banach space \( C^k(\bar{\Omega}) \) (and smooth in \( \bar{\Omega} \) by elliptic regularity) is the kernel of the continuous map \( \Delta : C^k(\bar{\Omega}) \to C^{k-2}(\bar{\Omega}) \), and so it is a Banach subspace of \( C^k(\bar{\Omega}) \). The set \( \mathcal{H} \subset C^k(\bar{\Omega})(\mathcal{K}) \) of harmonic functions \( u \) in \( C^k(\bar{\Omega}) \) such there exists \( v \in C^k(\bar{\Omega}) \) harmonic with \( u + iv \) holomorphic on \( \bar{\Omega} \) is a Banach subspace of \( C^k(\bar{\Omega}) \) of finite codimension. Indeed, let \( \{\gamma_1, ..., \gamma_N\} \) be a homology basis for \( \bar{\Omega} \), then

\[
\mathcal{H} = \ker L, \quad \text{with} \quad L : \ker \Delta \cap C^k(\bar{\Omega}) \to \mathbb{C}^N \quad \text{defined by} \quad L(u) := \left( \frac{1}{\pi i} \int_{\gamma_j} \partial u \right)_{j=1,...,N}.
\]

For all \( \tilde{\Gamma}_0 \subset \partial \Omega \) such that the complement of \( \tilde{\Gamma}_0 \) contains an open subset, we define

\[
\mathcal{H}_{\tilde{\Gamma}_0} := \{ u \in \mathcal{H}; u|_{\tilde{\Gamma}_0} = 0 \}.
\]

We now show

**Lemma 2.6.** The set of functions \( u \in \mathcal{H}_{\tilde{\Gamma}_0} \) which are Morse in \( \Omega \) is residual (i.e. a countable intersection of open dense sets) in \( \mathcal{H}_{\tilde{\Gamma}_0} \) with respect to the \( C^k(\bar{\Omega}) \) topology.

**Proof.** We use an argument very similar to those used by Uhlenbeck [30]. We start by defining \( m : \Omega \times \mathcal{H}_{\tilde{\Gamma}_0} \to T^*\Omega \) by \( (p, u) \mapsto (p, du) \in T^*_p \Omega \). This is clearly a smooth map, linear in the second variable, moreover \( m_u := m(\cdot, u) = (\cdot, du(\cdot)) \) is Fredholm since \( \Omega \) is finite dimensional. The map \( u \) is a Morse function if and only if \( m_u \) is transverse to the zero section, denoted \( T^*_p \Omega \), of \( T^*\Omega \), ie. if

\[
\text{Image}(D_p m_u + T_{m_u(p)}(T^*_p \Omega)) = T_{m_u(p)}(T^*_p \Omega), \quad \forall p \in \Omega \text{ such that } m_u(p) = (p, 0),
\]

which is equivalent to the fact that the Hessian of \( u \) at critical points is non-degenerate (see for instance Lemma 2.8 of [30]). We recall the following transversality theorem ([30, Th.2]):

**Theorem 2.7.** Let \( m : X \times \mathcal{H}_{\tilde{\Gamma}_0} \to W \) be a \( C^k \) map, where \( X, \mathcal{H}_{\tilde{\Gamma}_0}, \) and \( W \) are separable Banach manifolds with \( W \) and \( X \) of finite dimension. Let \( W' \subset W \) be a submanifold such that \( k > \max(1, \dim X - \dim W + \dim W') \). If \( m \) is transverse to \( W' \) then the set \( \{ u \in \mathcal{H}_{\tilde{\Gamma}_0}; m_u \text{ is transverse to } W' \} \) is dense in \( \mathcal{H}_{\tilde{\Gamma}_0} \), more precisely it is a residual set.

We want to apply it with \( X := \Omega, W := T^*\Omega \) and \( W' := T^*_p \Omega \), and the map \( m \) is defined above. We have thus proved Lemma 2.6 if one can show that \( m \) is transverse to \( W' \). Let \( (p, u) \) such that \( m(p, u) = (p, 0) \in W' \). Then identifying \( T_{(p,0)}(T^*_p \Omega) \) with \( T^*_p \Omega \oplus T^*_p \Omega \), one has

\[
D_{(p,u)} m(z, v) = (z, dv(p) + \text{Hess}_p(u)z)
\]

where \( \text{Hess}_p u \) is the Hessian of \( u \) at the point \( p \), viewed as a linear map from \( T_p \Omega \) to \( T^*_p \Omega \). To prove that \( m \) is transverse to \( W' \) we need to show that \( (z, v) \mapsto (z, dv(p) + \text{Hess}_p(u)z) \) is onto from \( T^*_p \Omega \oplus \mathcal{H}_{\tilde{\Gamma}_0} \) to \( T^*_p \Omega \oplus T^*_p \Omega \), which is realized for instance if the map \( v \to dv(p) \) from \( \mathcal{H}_{\tilde{\Gamma}_0} \) to \( T^*_p \Omega \) is onto. But from Lemma 2.4 we know that there exist holomorphic functions
Lemma 2.9. Let \( \tilde{\Gamma}_0 \) be an open set of the boundary. Let \( \phi : \partial \tilde{\Gamma}_0 \to \mathbb{R} \) be a harmonic function with \( \phi|_{\tilde{\Gamma}_0} = 0 \). Let \( p \in \tilde{\Gamma}_0 \) be a critical point of \( \phi \), then it is nondegenerate if and only if \( \partial_v \partial_u \phi \neq 0 \) where \( \partial_v \) and \( \partial_u \) denote respectively the tangential and normal derivatives along the boundary.

Proof. By Riemann mapping theorem, there is a conformal transformation mapping a neighbourhood of \( p \) in \( \partial \tilde{\Gamma}_0 \) to a half-disc \( D := \{ |z| < \epsilon, \text{Im}(z) \geq 0 \} \) and \( \partial \tilde{\Gamma}_0 = \{ \text{Im}(z) = 0 \} \) near \( p \). Denoting \( z = x + iy \), one has \( (\partial_x^2 + \partial_y^2)\phi = 0 \) in \( D \) and \( \partial_x^2 \phi|_{y=0} = 0 \), which implies \( \partial_y^2 \phi(p) = 0 \). Since \( \partial_v = e^f \partial_y \) and \( \partial_u = e^f \partial_x \) for some smooth function \( f \), and since \( d\phi(p) = 0 \), the conclusion is then straightforward.

Let \( N^* \partial \tilde{\Gamma}_0 \) be the conormal-bundle of \( \partial \tilde{\Gamma}_0 \) and \( N^* \tilde{\Gamma}_0 \) be the restriction of this bundle to \( \tilde{\Gamma}_0 \). Denote the zero sections of these bundles respectively by \( N^*_0 \partial \tilde{\Gamma}_0 \) and \( N^*_0 \tilde{\Gamma}_0 \). We now define the map

\[
b : \tilde{\Gamma}_0 \times \mathcal{H}_{\tilde{\Gamma}_0} \to N^* \tilde{\Gamma}_0, \quad b(p, u) := (p, \partial_v u).
\]

For a fixed \( u \in \mathcal{H}_{\tilde{\Gamma}_0} \), we also define \( b_u(\cdot) := b(\cdot, u) \). Simple computations yield the

Lemma 2.10. Suppose that \( p \in \tilde{\Gamma}_0 \) is such that \( \partial_v u(p) = 0 \), then \( \partial_v \partial_u u(p) \neq 0 \) if and only if

\[
\text{Image}(D_p b_u) + T_{(p, 0)}(N^*_0 \tilde{\Gamma}_0) = T_{(p, 0)}(N^* \tilde{\Gamma}_0).
\]
Proof. This can be seen by the fact that for all \( p \in \tilde{\Gamma}_0 \) such that \( b_u(p) = (p,0) \),

\[
D_p b_u : T_p \tilde{\Gamma}_0 \to T_{(p,0)}(\tilde{\Gamma}_0) \simeq T_p \tilde{\Gamma}_0 \oplus N_0^* \tilde{\Gamma}_0
\]

is given by \( w \mapsto (w, \partial_r \partial_u u(p)w) \).

At a point \((p,u)\) such that \( b(p,u) = 0\), a simple computation yields that the differential

\[
D_{(p,u)} b : T_p \tilde{\Gamma}_0 \times \mathcal{H}_{\tilde{\Gamma}_0} \to T_{(p,0)}(\tilde{\Gamma}_0) \text{ is given by } \dfrac{1}{(w',u')} \mapsto (w, \partial_r \partial_u u(p)w + \partial_r u(p)') .
\]

This observation combined with Lemma 2.8 shows that for all \((p,u)\),

\[
0 \\in \tilde{\Gamma}_0 \text{ is transverse to } N_0^* \tilde{\Gamma}_0 \text{ at } (p,0) .
\]

Now we can apply Theorem 2.7 with \( X = \tilde{\Gamma}_0, W = N^* \tilde{\Gamma}_0 \) and \( W' = N_0^* \tilde{\Gamma}_0 \) we see that the set \( \{u \in \mathcal{H}_{\tilde{\Gamma}_0} : b_u \text{ is transverse to } N_0^* \tilde{\Gamma}_0\} \) is residual in \( \mathcal{H}_{\tilde{\Gamma}_0} \). In view of Lemmas 2.11 and 2.6 we deduce the

**Lemma 2.11.** The set of functions \( u \in \mathcal{H}_{\tilde{\Gamma}_0} \) such that \( u \) has no degenerate critical point on \( \tilde{\Gamma}_0 \) is residual in \( \mathcal{H}_{\tilde{\Gamma}_0} \).

Observing the general fact that finite intersection of residual sets remains residual, the combination of Lemma 2.11 and Lemma 2.6 yields

**Corollary 2.12.** The set of functions \( u \in \mathcal{H}_{\tilde{\Gamma}_0} \) which are Morse in \( \mathcal{O} \) and have no degenerate critical points on \( \tilde{\Gamma}_0 \) is residual in \( \mathcal{H}_{\tilde{\Gamma}_0} \) with respect to the \( C^k(\overline{\mathcal{O}}) \) topology. In particular, it is dense.

We are now in a position to give a proof of the main proposition of this section.

**Proof of Proposition 2.1** As explained above, choose \( \mathcal{O} \) in such a way that \( \overline{\mathcal{O}} \) is a smooth surface with boundary, containing \( \Gamma_0' \), that \( \Gamma_0' \subset \partial \mathcal{O} \) and \( \mathcal{O} \) contains \( \partial M \setminus \overline{\Gamma_0'} \). Let \( \tilde{\Gamma}_0 \) be an open subset of the boundary of \( \overline{\mathcal{O}} \) such that the closure of \( \Gamma_0' \) is contained in \( \tilde{\Gamma}_0 \) and \( \partial \mathcal{O} \setminus \overline{\tilde{\Gamma}_0} \neq \emptyset \). Let \( \tilde{p} \) be an interior point of \( \Gamma_0' \). By Lemma 2.4 there exists a holomorphic function \( f = u + iv \) on \( \tilde{\mathcal{O}} \) such that \( f \) is purely real on \( \tilde{\Gamma}_0 \), \( v(\tilde{p}) = 1 \), and \( df(\tilde{p}) = 0 \) (thus \( v \in \mathcal{H}_{\tilde{\Gamma}_0} \)).

By Corollary 2.12 there exist a sequence \( (v_j)_j \) of Morse functions \( v_j \in \mathcal{H}_{\tilde{\Gamma}_0} \) such that \( v_j \to v \) in \( C^k(M) \) for any fixed \( k \) large. By Cauchy integral formula, there exist harmonic conjugates \( u_j \) of \( v_j \) such that \( f_j := u_j + iv_j \to f \) in \( C^k(M) \). Let \( \epsilon > 0 \) be small and let \( U \subset \mathcal{O} \) be a neighbourhood containing \( \tilde{p} \) and no other critical points of \( f \), and with boundary a smooth circle of radius \( \epsilon \). In complex local coordinates near \( \tilde{p} \), we can identify \( \partial f \) and \( \partial f_j \) to holomorphic functions on an open set of \( \mathcal{C} \). Then by Rouche’s theorem, it is clear that \( \partial f_j \) has precisely one zero in \( U \) and \( v_j \) never vanishes in \( U \) if \( j \) is large enough.

Fix \( \Phi \) to be one of the \( f_j \) for \( j \) large enough. By construction, \( \Phi \) is Morse in \( \mathcal{O} \) and has no degenerate critical points on \( \overline{\Gamma_0'} \subset \tilde{\Gamma}_0 \). We notice that, since the imaginary part of \( \Phi \) vanishes on all of \( \tilde{\Gamma}_0 \), it is clear from the reflection principle applied after using the Riemann mapping theorem (as in the proof of Lemma 2.9) that no point on \( \overline{\mathcal{O}} \) can be an accumulation point for critical points. Now \( \partial M \setminus \overline{\Gamma_0'} \) is contained in the interior of \( \mathcal{O} \) and therefore no points on \( \partial M \setminus \overline{\Gamma_0'} \) can be an accumulation point of critical points. Since \( \Phi \) is Morse in the interior of \( \mathcal{O} \), there are no degenerate critical points on \( \partial M \setminus \overline{\Gamma_0'} \). This ends the proof. \( \square \)
2.4. Doubling of Riemann Surfaces. We describe the construction of a double of a bordered Riemann surface outlined in [9]. Let $M$ and $M'$ be two copies of a bordered Riemann surface. We construct the closed surface $M^D := \bar{M} \cup M'$ by identifying points $p \in \partial M$ with its copy $p' \in \partial M'$. We take in the interior of $M$ the existing holomorphic coordinates while on $M'$ the holomorphic coordinates are precisely the complex conjugate of those on $M$. To construct coordinate charts along the boundary $\partial M$, if $U$ is a small neighbourhood in $M^D$ containing $p \in \partial M$ such that $U \cap \partial M$ is an open segment we take a holomorphic chart which maps $U \cap M$ conformally to the upper half plane such that $U \cap \partial M$ is mapped to a segment of the real axis. We can then apply the reflection principle to obtain a holomorphic coordinate chart around $p \in \partial M$.

Let $M$ be a bordered Riemann surface which is isometric to the flat cylinder $([0, \epsilon] \times S^1, dt^2 + d\theta^2)$ near each of its boundary components. If $q_0 \in \partial M$, define $M_0' \subset M$ by removing a small interior closed half-disks around $q_0$ of radius $\delta > 0$ and let $\Gamma'$ be defined by $\Gamma' := \partial M_0' \cap \partial M$. If one denote by $M^D := M^D \setminus \bar{B}_\delta(q_0)$ with $B_\delta(q_0) := \{p \in M^D \mid d(p, q_0) < \delta\}$, then one has that $M_0' = M^D \cap M$. That is, $M_0'$ is half of the surface obtained by removing a whole disk from $M^D$.

On every doubled Riemann surface $M^D$ there exists an anti-conformal involution $R$ satisfying $R(M) = M'$ and is the identity on the boundary $\partial M$. Since the metric $g$ on $M$ is assumed to be of the form $dt^2 + d\theta^2$ near $\partial M$, it extends smoothly to a metric on $M^D$ by the relation $R^* g = g$. It is easily checked that if $\Phi$ is a holomorphic function on $M_0'$ satisfying the boundary condition $\Phi |_{\Gamma'} = \Phi$, then $\Phi$ extends to be a holomorphic function on $M^D$ by the relation $(R^* \Phi) = \Phi$. Similarly, if $\eta$ is a holomorphic 1-form with boundary condition $\partial_{\partial M_0'} \eta |_{\Gamma'} = \eta$, then $\eta$ extends to be a holomorphically to $M^D$ by the relation $R^* \eta = \eta$.

Conversely, if $\Phi$ is a holomorphic function on $M^D$, we say it is conjugate even/odd if $(R^* \Phi) = \pm \Phi$ and we adopt the same terminology for holomorphic forms. It is easily seen that the set of even holomorphic functions/1-forms are precisely the reflected ones described above.

2.5. Boundary Values of Meromorphic Functions. In this section we characterize the boundary value of holomorphic/meromorphic functions on the surface $M^D$. These characterizations will be useful in boundary identification and in proving Proposition [13]. We begin by stating a well-understood orthogonality condition for boundary values of holomorphic functions (see eg. [13]).

**Proposition 2.2.** Let $f \in W^{2- \frac{1}{p}, p}(\partial M^D)$ be a complex valued function. Then $f$ is the restriction of a holomorphic function which is differentiable up to the boundary if and only if

$$\int_{\partial M^D} f i^*_\partial M^D \eta = 0$$

for all 1-forms $\eta \in C^\infty(M^D; T^*_1 \partial M^D)$ satisfying $\partial \eta = 0$.

We would like to generalize this statement to that of meromorphic functions with prescribed poles of certain order. As such we consider the following

**Lemma 2.13.** Let $\{p_0, \ldots, p_N\} \subset \partial M^D \cup \partial \hat{M}^D$ be a discrete set of points. If $f \in W^{2- \frac{1}{p}, p}(\partial M^D)$ is a complex valued function satisfying

$$\int_{\partial M^D} f i^*_\partial M^D \eta = 0$$

for all 1-forms $\eta \in C^\infty(M^D; T^*_1 \partial M^D)$ satisfying $\partial \eta = 0$.
for all holomorphic 1-forms $\eta \in C^\infty(\tilde{M}^D; T^*_0\tilde{M}^D)$ with the property $\eta(p_j) = 0$ to $k$-th order, then $f$ is the restriction of a meromorphic function which is smooth up to the boundary and whose only poles lie in the interior points $\{p_0, ... , p_N\} \cap M^D$. Furthermore the poles are of order at most $k$.

**Proof.** Let $a$ be a holomorphic function which is smooth up to the boundary with isolated zeros on $M^D \cup \partial M^D$ such that $a$ vanishes to exactly $k$-th order at $\{p_0, ... , p_N\}$. Such functions can be constructed by compactly embedding $M^D$ into a slightly larger surface with boundaries and apply Lemma 2.3. If $f \in W^{2, \frac{1}{2}^p}(\partial M^D)$ is a complex function satisfying the hypothesis then one has

$$\int_{\partial M^D} (af)^i_{\partial M^D} \eta = 0$$

for all holomorphic 1-forms $\eta$. By Proposition 2.2 we have that $af \in W^{2, \frac{1}{2}^p}(\partial M^D)$ extends to a holomorphic function which we denote by $G_a$. Clearly,

$$f = \frac{G_a}{a} \big|_{\partial M^D} \in W^{2, \frac{1}{2}^p}(\partial M^D)$$

is the restriction of the meromorphic function $\frac{G_a}{a}$ and since the zeros of $a$ are isolated, this meromorphic function is continuous up to the boundary. As such, the singularities of $\frac{G_a}{a}$ are precisely the interior zeros of $a$.

Let us now consider another holomorphic function $a'$ with isolated zeroes vanishing exactly to $k$-th order at $\{p_0, ... , p_N\}$. By using Lemma 2.4 we may construct $a'$ in such a way that $a$ and $a'$ do not have common zeroes in the interior other than $\{p_0, ... , p_N\} \cap M^D$. We repeat the above argument for $a'$ to show that $f = \frac{G_{a'}}{a'} \big|_{\partial M^D} \in W^{2, \frac{1}{2}^p}(\partial M^D)$ for some holomorphic function $G_{a'}$.

Unique continuation for meromorphic functions forces the identity $\frac{G_{a'}}{a} = \frac{G_a}{a}$. The fact that the only common interior zeroes for $a$ and $a'$ are $\{p_0, ... , p_N\} \cap M'_i$ ensures that they are the only poles and that they are of order at most $k$. Thus we conclude that $f$ extends to a meromorphic function differentiable up to the boundary whose only poles are $\{p_0, ... , p_N\} \cap M^D$ of degree at most $k$.

Observe that if $R$ is the involution defined in Section 2.4, then every holomorphic function $F$ and 1-form $\eta$ can be decomposed into their conjugate even and odd part by writing

$$\Phi = \Phi + (\Phi^{*}) \frac{1}{2} + \Phi - (\Phi^{*}) \frac{1}{2} \quad \text{and} \quad \eta = \eta + (\eta^{*}) \frac{1}{2} + \eta - (\eta^{*}) \frac{1}{2}.$$ 

As one can transform between conjugate even and odd functions via multiplication with $i \in \mathbb{C}$, one has that a smooth function $f \in W^{2, \frac{1}{2}^p}(\partial M^D)$ satisfies $\int_{\partial M^D} f_{\partial M^D} \eta = 0$ for all conjugate even holomorphic 1-forms vanishing to $k$-th order at $\{p_1, ... , p_N, R(p_1), ... , R(p_N)\} \subset M^D \cup \partial M^D$ iff $\int_{\partial M^D} f_{\partial M^D} \eta = 0$ for all holomorphic 1-forms vanishing to $k$-th order at the same points.

This discussion combined with Lemma 2.13 gives the following condition for being the boundary value of a meromorphic function on $M^D$.

**Lemma 2.14.** Let $f \in W^{2, \frac{1}{2}^p}(\partial M^D)$ and $\{p_1, ... , p_N, R(p_1), ... , R(p_N)\}$ be a discrete set of points in $M^D \cup \partial M^D$. The function $f$ is the boundary value of a meromorphic function in $M^D$ with poles at $\{p_1, ... , p_N, R(p_1), ... , R(p_N)\} \cap M^D$ of at most order $k$ if $\int_{\partial M^D} f_{\partial M^D} \eta = 0$ for all conjugate even holomorphic 1-forms $\eta$ vanishing to order $k$ at $\{p_1, ... , p_N, R(p_1), ... , R(p_N)\}$. 


3. Shifted Carleman Estimates and $H^1$ Solvability

In this section, we prove a shifted Carleman estimate on a Riemann surface using harmonic Morse weights. The estimate will have boundary conditions similar to the ones established in [4]. We show the following estimate for $M_0'$, $M$, and $\Gamma'$ described in Section 2.4.

**Proposition 3.1.** Let $\varphi : M \to \mathbb{R}$ be a $C^k(M)$ harmonic Morse function for $k$ large such that $\partial \varphi \mid \Gamma_0' = 0$ for some open subset $\Gamma_0' \subset \partial M$ compactly containing $\Gamma'$. For all $X \in W^{1, \infty}(M)$, $V \in L^\infty(M)$ there exists $h_0 > 0$ such that for all $u \in C_0^\infty(M_0')$ and $h \leq h_0$ we have

$$
\|e^{-\varphi/h} L_{X,V} e^{\varphi/h} u \|_{H^{-1}_s(M)} \geq C \sqrt{h} \|u\| + \|d \varphi u\|
$$

Note that since $\Delta_{s,K}$ is equivalent to $\Delta_\varphi$ it suffices to prove Proposition 3.1 for a conformal representative of $g$ which is isometric to the flat cylinder near $\partial M$. The important feature in Proposition 3.1 is that $\Gamma'$ is the common boundary component of $M_0'$ and $M$. This allows us to deduce the following semiclassical solvability while controlling the solution on a part of the boundary.

**Corollary 3.1.** Let $\varphi$ be as in Proposition 3.1. Then for all $f \in L^2(M_0')$ there exists a solution $u \in H_0^1(M)$ of the boundary value problem

$$
e^{-\varphi/h} L_{X,V} e^{\varphi/h} u = f \text{ in } M_0', \quad u \mid_{\Gamma'} = 0,
$$

satisfying the estimate $\|u\| + \|hdu\| \leq \sqrt{h} \|f\|$. We start the proof by modifying the weight as follows: Let $\Gamma_0' \subset \partial M$ be an open subset compactly containing $\Gamma'$ so that $\partial M \\setminus \Gamma_0'$ contains on open subset. If $\varphi_0 : M \to \mathbb{R}$ is a real valued harmonic Morse function with critical points $\{p_1, \ldots, p_N\}$ in $M \cup \partial M$ and $\partial \varphi \varphi_0 \mid \Gamma_0' = 0$, we let $\varphi_j : M \to \mathbb{R}$ be harmonic functions with boundary condition $\partial \varphi \varphi_j \mid \Gamma_0' = 0$ such that $p_j$ is not a critical point of $\varphi_j$ for $j = 1, \ldots, N$. Their existence is ensured by Lemma 2.4. For all $\epsilon > 0$, we define the convexified weight $\varphi_\epsilon := \varphi - \frac{h}{2\epsilon} (\sum_{j=0}^N |\varphi_j|^2)$. By Lemma 2.8 we can choose $\varphi_j$ such that $\partial \varphi \varphi_j = 0$ on $\Gamma_0'$.

As the normal derivatives of $\varphi_j$ along $\Gamma_0'$ all vanish, the even extensions of $\varphi_j$ to the double $M^D$ (which we denote again by $\varphi_j$) are harmonic on some connected bordered surface $M_0^D \subset M^D$ which compactly contains $M^D$. We note that if $\varphi_0$ is Morse on $M \cup \partial M$, then its extension is Morse on $M^D$.

3.1. Shifted Estimate on $M^D$. In this section let $M$, $M^D$, and the metric $g$ be as described in the construction given in Section 2.4. We prove in the setting the following estimate:

**Proposition 3.2.** There exists an $h_0 > 0$ such that for all $h \in (0, h_0)$ and $u \in C_0^\infty(M^D)$ we have

$$
\|e^{-\varphi_\epsilon/h} L_{X,V} e^{\varphi_\epsilon/h} u \|_{H^{-1}_s(M^D)} \geq \frac{C h}{\epsilon} \sqrt{h} \|u\| + \|d \varphi u\| + \|d \varphi_\epsilon u\| + \|hdu\|_{H^{-1}_s(M^D))}
$$

**Proof.** By Lemma 3.2 of [11] one has the $L^2$ Carleman estimate

$$
\|e^{-\varphi_\epsilon/h} L_{X,V} e^{\varphi_\epsilon/h} u \| \geq \frac{C h}{\epsilon} \sqrt{h} \|u\| + \|d \varphi u\| + \|d \varphi_\epsilon u\| + \|hdu\|
$$

for all $u \in C_0^\infty(M_0^D)$. Now let $\chi \in C_0^\infty(M^D)$ be a cutoff so that $\chi = 1$ on $M^D$ and apply the above inequality to $\chi (hD)^{-1} u$ for $u \in C_0^\infty(M^D)$ where $(hD)^{-1}$ is the elliptic semiclassical
pseudodifferential operator obtained by quantizing the symbol \( \langle \xi \rangle^{-1} := (1 + |\xi|^2)^{-1/2} \in S^{-1}(T^*M^D) \). Standard commutator calculus yields that
\[
\|e^{-\varphi_\epsilon/h^2} \Delta_g e^{\varphi_\epsilon/h} u\|_{H^{-1}_sc(M)} + \|e^{-\varphi_\epsilon/h^2} \Delta_g \chi e^{\varphi_\epsilon/h} \langle hD \rangle^{-1} u\| + \|\chi e^{-\varphi_\epsilon/h^2} \Delta_g e^{\varphi_\epsilon/h}, (hD)^{-1} u\|
\geq C \frac{\hbar}{\epsilon} (\sqrt{\hbar} \|u\| + \|ud\phi\| + \|ud\phi\| + \|hdu\|_{H^{-1}_sc(M^D)})
\tag{17}
\]
We compute the second term directly to obtain
\[
\|e^{-\varphi_\epsilon/h^2} \Delta_g \chi e^{\varphi_\epsilon/h} \langle hD \rangle^{-1} u\| \leq \hbar^2 \|u\| + \hbar \|ud\phi\| + \hbar \|hdu\|_{H^{-1}_sc(M^D)}
\]
and see that it can therefore be absorbed into the right side of inequality (17). Similarly if we write \( e^{-\varphi_\epsilon/h^2} \Delta_g e^{\varphi_\epsilon/h} = A + iB \) where
\[
Au = h^2 \Delta_g u - |d\phi|^2 u, \quad Bu = \text{div}_g(ud\phi) + \langle d\phi, du \rangle,
\]
we see that the third term on the left side of (17) can be written as
\[
[e^{-\varphi_\epsilon/h^2} \Delta_g e^{\varphi_\epsilon/h}, \langle hD \rangle^{-1}] = \text{Op}_h(\{a + ib, \langle \xi \rangle^{-1}\}) + h^2 \text{Op}_h(S^{-1}(T^*M^D))
\]
which leads to the estimate
\[
\|\chi[A + iB, \langle hD \rangle^{-1}] u\| \leq h \|hdu\|_{H^{-1}_sc(M^D)} + \hbar^2 \|u\|
\]
and therefore can again be absorbed into the right side of inequality (17).

\[\square\]

3.2. Reflection Argument. In this section we apply a reflection argument to prove Proposition 3.1. We first prove the estimate for the special case when \( X = V = 0 \).

**Lemma 3.2.** For all \( u \in C_0^\infty(M_0^\epsilon) \) we have that
\[
\|e^{-\varphi_\epsilon/h^2} \Delta_g e^{\varphi_\epsilon/h} u\|_{H^{-1}_sc(M)} \geq \frac{\hbar}{\epsilon} (\sqrt{\hbar} \|u\| + \|d\phi u\| + \|d\phi u\| + \|hdu\|_{H^{-1}_sc(M^D)}).
\]

**Proof.**
If \( u \) is an element of \( C_0^\infty(M_0^\epsilon) \), let \( \tilde{u} \) denote its odd reflection which is an element of \( C_0^\infty(M^D) \) which extends trivially to a smooth odd function on \( M^D \). We can now apply Lemma 3.2 to the compactly supported function \( \tilde{u} \in C_0^\infty(M^D) \) to obtain
\[
\|e^{-\varphi_\epsilon/h^2} \Delta_g e^{\varphi_\epsilon/h} \tilde{u}\|_{H^{-1}_sc(M^D)} \geq \frac{\hbar}{\epsilon} (\sqrt{\hbar} \|\tilde{u}\| + \|d\phi \tilde{u}\| + \|d\phi \tilde{u}\| + \|h\tilde{du}\|_{H^{-1}_sc(M^D)}).
\]
We now would like to use the symmetry of \( \tilde{u} \) with respect to the pull-back by \( R \) to argue that this estimate is comparable to the analogous one on \( M \). This can be done with the help of the following

**Lemma 3.3.** Let \( \tilde{u} \in C^\infty(M^D) \) be an odd function with respect to the involution \( R \), that is, \( R^* \tilde{u} = -\tilde{u} \), then
\[
\|\tilde{u}\|_{H^{-1}_sc(M^D)} = \sqrt{2} \|\tilde{u}\|_{H^{-1}_sc(M)}.
\]
Indeed, since \( \tilde{u} \) is odd and \( \varphi_r \) is even we have that \( e^{-\varphi_r/h^2}\Delta_g e^{\varphi_r/h^2} \tilde{u} \) is also a smooth odd function on \( M^D \). Thus we can apply Lemma 3.3 to \( e^{-\varphi_r/h^2}\Delta_g e^{\varphi_r/h^2} \tilde{u} \) to obtain
\[
\|e^{-\varphi_r/h^2}\Delta_g e^{\varphi_r/h^2} \tilde{u}\|_{H^{-1}_{scl}(M)} \geq C \frac{h}{\epsilon} \sqrt{\text{h}} \|u\| + \|d\varphi u\| + \|d\varphi_r u\| + \|h d\tilde{u}\|_{H^{-1}_{scl}(M^D)} \\
\geq C \frac{h}{\epsilon} \sqrt{\text{h}} \|u\| + \|d\varphi u\| + \|d\varphi_r u\| + \|h du\|_{H^{-1}_{scl}(M)}
\]

We complete this subsection we must provide

**Proof of Lemma 3.3.** We compute directly the \( H^{-1}_{scl}(M^D) \) norm of \( u \in C^\infty(M^D) \).
\[
\|u\|_{H^{-1}_{scl}(M^D)} := \sup_{v \in H^{1}(M^D), \|v\|_{H^{1}_{scl}(M^D)} \leq 1} \langle u, v \rangle = \int_{M^D} u \hat{v}
\]
where \( \hat{v} \) is the unique maximizer in \( \hat{v} \in H^1(M^D) \) with \( \|\hat{v}\|_{H^1_{scl}(M^D)} = 1 \). We decompose \( \hat{v} \) into its odd and even parts by writing
\[
\hat{v}(x, y) = \frac{\hat{v} + R^* \hat{v}}{2} + \frac{\hat{v} - R^* \hat{v}}{2} := \hat{v}^+ + \hat{v}^-.
\]
Observe that since \( u \) is odd by assumption we have
\[
\int_{M^D} u \hat{v}^+ = \int_{M^D} R^* u R^* \hat{v}^+ = - \int_{M^D} u \hat{v}^-, \int_{M^D} u \hat{v}^- = 2 \int_{M} u \hat{v}^-
\]
and thus we can write
\[
(18) \quad \|u\|_{H^{-1}_{scl}(M^D)} = \int_{M^D} u \hat{v} = \int_{M^D} u \hat{v}^- = 2 \int_{M} u \hat{v}^-.
\]
Note that since \( \int_{M^D} \hat{v}^+ \hat{v}^- = 0 \) and \( \int_{M^D} \langle d\hat{v}^+, d\hat{v}^- \rangle = \int_{M^D} \Delta \hat{v}^+ \hat{v}^- = 0 \), we can write the \( H^1_{scl}(M^D) \) norm of \( \hat{v} \) as
\[
1 = \|\hat{v}\|_{H^1_{scl}(M^D)}^2 = \|\hat{v}^+\|_{H^1_{scl}(M^D)}^2 + \|\hat{v}^-\|_{H^1_{scl}(M^D)}^2.
\]
From this we can conclude that \( \hat{v}^- \) is in the unit ball of \( H^1_{scl}(M^D) \) and by the uniqueness of maximizer we have that \( \hat{v}^- = \hat{v} \). Furthermore, since \( \hat{v}^- \) is odd, it vanishes along the fixed points of the involution \( R \). As the involution \( R \) fixes the boundary \( \partial M \), this means that \( v^- |_{\partial M} = 0 \) and therefore \( \hat{v}^- |_{M} \in H^1_0(M) \) with semiclassical norm \( \|\hat{v}^-\|_{H^1_{scl}(M)} = \frac{1}{\sqrt{2}} \). So by \( [18] \) we have that
\[
\|u\|_{H^{-1}_{scl}(M^D)} = 2 \int_{M} u \hat{v}^- \leq 2 \sup_{v \in H^1_0(M), \|v\|_{H^1_{scl}(M)} \leq \frac{1}{\sqrt{2}}} \int_{M} uv = \sqrt{2}\|u\|_{H^{-1}_{scl}(M)}
\]
This inequality goes the other direction by observing that for odd functions \( u \in C^\infty(M^D) \) we have
\[
\|u\|_{H^{-1}_{scl}(M^D)} \geq \sup_{v \in H^1_0(M), \|v\|_{H^1_{scl}(M)} \leq \frac{1}{\sqrt{2}}} \int_{M} uv + \sup_{v \in H^1_0(M), \|v\|_{H^1_{scl}(M)} \leq \frac{1}{\sqrt{2}}} \int_{R(M)} u R^* v = \sqrt{2}\|u\|_{H^{-1}_{scl}(M)}.
\]
3.3. Proof of Proposition 3.1. By Lemma 3.2 we have for $u \in C^\infty_0(M'_0)$ the estimate for the Laplacian with convexified weights:

$$
\|e^{-\varphi/h}h^2 \Delta_g e^{\varphi/h}u\|_{H^{-1}_{rad}(M)} \geq C \frac{h}{\epsilon} \sqrt{h} \|u\| + \|d\varphi u\| + \|h d\varphi u\|_{H^{-1}_{rad}(M)}.
$$

If we replace $\Delta_g$ by the operator $L_{X,V} := (d + iX)^*(d + iX) + V$ we will obtain errors on the left side:

$$
\|e^{-\varphi/h}h^2 L_{X,V} e^{\varphi/h}u\|_{H^{-1}_{rad}(M)} \geq C \frac{h}{\epsilon} \sqrt{h} \|u\| + \|d\varphi u\| + \|h \langle X, du \rangle\|_{H^{-1}_{rad}(M)} \geq C \frac{h}{\epsilon} \sqrt{h} \|u\| + \|d\varphi u\| + \|h d\varphi u\|_{H^{-1}_{rad}(M)}
$$

for some $Q \in L^\infty$. Since $X \in W^{1,\infty}(M)$ and $Q \in L^\infty(M)$ all the errors on the left side can be absorbed into the right side of the inequality. We now replace $u$ in the above estimate by $e^{-\varphi/h}e^{\frac{1}{2} \sum_{j=1}^N \psi_j^2} u$ so that $e^{-\varphi/h}e^{\frac{1}{2} \sum_{j=1}^N \psi_j^2} u = e^{\varphi/h}u$ and the estimate follows.

\[\square\]

4. Boundary Determination

We begin the section by stating the local boundary determination result. The statement was proven in the Euclidean case by [2] and [26]. A slight generalization to the case of Riemann surfaces was done in [13]. The results are statement for the global Dirichlet-Neumann map but as the methods are local they can be handled without generalization to show

**Proposition 4.1.** Let $X_1, X_2 \in W^{3,p}(M; T^* M_0)$ be real-valued 1-forms and $V_1, V_2 \in W^{2,p}(M_0)$ be functions. If assumptions (3) and (4) are satisfied then $i^*_0 \partial M_0 X_1 \|_{\partial M_0 \Gamma} = i^*_0 \partial M_0 X_2 \|_{\partial M_0 \Gamma}$.

An immediate consequence of this is the following. If $M$ is a surface containing $M_0$ such that $\Gamma \subset \partial M_0 \cap \partial M$ and $M \setminus M_0$ is simply connected, then there exists $W^{1,\infty}(M)$ and $L^\infty(M)$ extensions of $X_j$ and $V_j$ respectively such that $X_1 |_{M \setminus M_0} = X_2 |_{M \setminus M_0}$, $\langle \nu, X_1 - X_2 \rangle |_{\partial M} = 0$, and $V_1 |_{M \setminus M_0} = V_2 |_{M \setminus M_0}$. Furthermore, on the surface $M$ the Cauchy data holds for $V_j$ replaced by $-\bar{K} V_j$. As such we may assume without loss of generality that for each connected component of $\partial M$ there exists an interior neighbourhood which is isometric to the flat cylinder $[0, \epsilon] \times S^1$ with metric $dt^2 + d\theta^2$ ([22]). Furthermore, if $q_0 \in \partial M_0 \setminus \Gamma$ and $\delta > 0$ are chosen so that $M_0$ is contained in $M'_0 := M \setminus B_\delta(q_0)$ with $B_\delta(q_0) := \{ p \in M_D \mid d(p, q_0) < \delta \}$, then on the surface $M_0$ one again has $\xi_{X_1, V_1, \partial M_0 \setminus \Gamma} = \xi_{X_2, V_2, \partial M_0 \setminus \Gamma}$ and $\langle \nu, X_1 - X_2 \rangle |_{\partial M_0} = 0$. Here $\Gamma' := \partial M'_0 \cap \partial M$ contains $\Gamma$. We summarize this discussion in the following

**Corollary 4.1.** Let $M$ and $M_0$ be the surfaces defined above. There exists $W^{1,\infty}(M)$ and $L^\infty(M)$ extensions to the coefficients $X_j$ and $V_j$ respectively such that on $M$ one has $X_1 |_{M \setminus M_0} = X_2 |_{M \setminus M_0}$, $V_1 |_{M \setminus M_0} = V_2 |_{M \setminus M_0}$. On the surface $M'_0$ one has $\langle \nu, X_1 - X_2 \rangle |_{\partial M'_0} = 0$ and the Cauchy data satisfies $\xi_{X_1, V_1, \partial M'_0 \setminus \Gamma} = \xi_{X_2, V_2, \partial M'_0 \setminus \Gamma}$.

The advantage in working with $M'_0 \subset M$ with flat cylindrical metric near $\partial M$ is that its double as a subset of $M_D$ with metric given by $R * g = g$ is a manifold with both smooth metric and boundary.
4. Boundary Values of $F_{A_j}$. Let $M$ and $M'_0$ be the surface constructed in the previous section. Let $X \in W^{1,\infty}(M,\bar{T}'M)$ be a real-valued 1-form on $M$ which can be decomposed into its $T^\alpha_{01}M$ and $T^\alpha_{10}M$ component which we denote by $A$ and $\bar{A}$ respectively. If $\Gamma' := \partial M'_0 \cap \partial M$, Proposition 2.3 asserts that for all $p \in (1, \infty)$ one can find $\alpha \in W^{2,p}(M)$ which is real-valued along $\Gamma'$ solving $\bar{\partial}\alpha = A$ so that

$$\bar{\partial}e^{i\alpha} = ie^{i\alpha}A \text{ in } M'_0, \quad |e^{i\alpha}| |\Gamma' = 1.$$

Of course, $e^{i\alpha}$ is not the non-vanishing solution to this boundary value problem. Indeed, one can multiply $e^{i\alpha}$ by any non-vanishing holomorphic function which is unitary along $\Gamma'$ to obtain another solution. It turns out the solutions of these boundary value problems are closely related to the Cauchy data of $L_{X,V}$.

**Proposition 4.2.** If $X_j \in W^{1,\infty}(M,\bar{T}'M)$ are real valued 1-forms and $V_j \in L^{\infty}(M)$ for $j = 1, 2$ satisfy $\langle \nu, X_1 - X_2 \rangle |_{\partial M} = 0$ and for $p \in (1, \infty)$ large, let $\alpha_j \in W^{2,p}(M)$ be a solution of

$$\bar{\partial}\alpha_j = A_j, \quad \alpha_j |_{\Gamma' = \R}.$$

Suppose $\mathcal{E}_{X_1, V_1, \partial M'_0 \setminus \Gamma'} = \mathcal{E}_{X_2, V_2, \partial M'_0 \setminus \Gamma'}$, then

i) $e^{i(\alpha_1 - \alpha_2)} |_{\partial M'_0 \setminus \Gamma'}$ extends to a non-vanishing holomorphic function $\Psi$ on $M'_0$ which is unitary along $\Gamma'$. Furthermore, $\Psi |_{M_0} \in C^{\infty}(M_0)$ up to the boundary.

ii) $e^{i(\bar{\alpha}_1 - \bar{\alpha}_2)} |_{\partial M'_0 \setminus \Gamma'}$ extends to a non-vanishing antiholomorphic function $\Psi$ on $M'_0$ which is unitary along $\Gamma'$. Furthermore, $\Psi |_{M_0} \in C^{\infty}(M_0)$ up to the boundary.

**Proof.** Since (ii) and (i) are equivalent we will only prove (ii).

Since $\alpha_j |_{\Gamma' = \R}$, one can define a Lipschitz piece-wise smooth function $F_{1,2} \in W^{1,\infty}(\bar{M}^D)$ on $\bar{M}^D \cup \partial \bar{M}^D$ by

$$F_{1,2} = \begin{cases} e^{i(\bar{\alpha}_2 - \bar{\alpha}_1)} & \text{on } M'_0 \\ R^* e^{i(\alpha_2 - \alpha_1)} & \text{on } R(M'_0). \end{cases}$$

In fact one can show that $F_{1,2} \in W^{2,p}(\bar{M}^D)$. Indeed, since $\text{Im}(\alpha_1 - \alpha_2)$ vanishes along $\Gamma'$ by assumption, its odd extension across $\Gamma'$ is an element of $W^{2,p}(\bar{M}^D)$. To show that $F_{1,2} \in W^{2,p}(\bar{M}^D)$ we need to check that the even extension across $\Gamma'$ of $\text{Re}(\alpha_1 - \alpha_2)$ has two derivatives as well. This is equivalent to showing that $\partial_{\nu} \text{Re}(\alpha_1 - \alpha_2)$ vanishes along $\Gamma'$. This can be done by using $\langle \nu, X_1 - X_2 \rangle = 0$ along $\partial M$ (Corollary 4.1) and the fact that

$$0 = \langle \nu, X_1 - X_2 \rangle = \langle \nu, (\bar{A}_1 - A_2) + (A_1 - \bar{A}_2) \rangle = \langle \nu, \partial(\bar{\alpha}_1 - \alpha_2) + \partial(\bar{\alpha}_1 - \bar{\alpha}_2) \rangle.$$

Writing this out in boundary normal coordinates yields that $\partial_{\nu} \text{Re}(\alpha_1 - \alpha_2) = 0$ along $\Gamma'$ and thus $F_{1,2} \in W^{2,p}(\bar{M}^D)$.

We have the following Lemma for the boundary value of $F_{1,2} |_{\partial \bar{M}^D} \in W^{2-\frac{1}{p},p}(\partial \bar{M}^D)$ defined by (20):

**Lemma 4.2.** The function $F_{1,2} |_{\partial \bar{M}^D}$ has an antiholomorphic extension $\Psi$ into the surface $\bar{M}^D$.

Assuming Lemma 4.2, we need to show that $\Psi$ is non-vanishing. To this end we switch the indices 1 and 2 in (20) to show that $F_{2,1} |_{\partial \bar{M}^D} = F_{1,2}^{-1} |_{\partial \bar{M}^D}$ is the boundary value of an
antiholomorphic function on $\tilde{M}^D$. By uniqueness, this antiholomorphic function must be $\Psi^{-1}$ and we have that $\Psi$ is non-vanishing.

We now show that $\Psi |_{\Gamma'}$ is unitary. To this end, observe that $F_{1,2} |_{\partial \tilde{M}^D}$ satisfies the symmetry condition $(R^*F_{1,2})^{-1} |_{\partial \tilde{M}^D} = F_{1,2} |_{\partial \tilde{M}^D}$. By uniqueness this implies that the antiholomorphic function $(R^*\Psi)^{-1}$ is identical to $\Psi$. As such, since $R$ is the identity on $\Gamma'$, we have that $(\Psi^{-1})'(p) = \Psi(p)$ for all $p \in \Gamma'$; that is, $\Psi |_{\Gamma'}$ is unitary. Restricting the function $\Psi$ to $M'_0$ we have the desired antiholomorphic extension to $e^{i(\bar{a}_1 - \bar{a}_2)} |_{\partial M'_0 \setminus \Gamma'}$. The smoothness of $\Psi$ on the closure of $M_0$ follows from the fact that $M_0$ is compactly contained in $\tilde{M}^D$. □

An immediate consequence of Proposition 4.2 is the following

**Corollary 4.3.** There exists an open subset $\Gamma_0 \subset \partial M_0$ containing $\Gamma$ whose complement $\partial M_0 \setminus \Gamma_0$ contains an open subset such that for all $p \in (1, \infty)$ one can choose solutions $F_{A_j}, F_{\bar{A}_j} \in W^{2,p}(M_0) \cap W^{4,p}_{loc}(M_0)$ solving

\begin{equation}
\bar{\partial} F_{A_j} = i A_j F_{A_j} \quad \text{in} \quad M_0, \quad |F_{A_j}| |_{\Gamma_0} = 1
\end{equation}

and

\begin{equation}
\partial F_{\bar{A}_j} = i \bar{A}_j F_{\bar{A}_j} \quad \text{in} \quad M_0, \quad |F_{\bar{A}_j}| |_{\Gamma_0} = 1
\end{equation}

such that $F_{A_1} |_{\partial M_0 \setminus \Gamma_0} = F_{A_2} |_{\partial M_0 \setminus \Gamma_0}$ and $F_{A_1} |_{\partial M_0 \setminus \Gamma_0} = F_{A_2} |_{\partial M_0 \setminus \Gamma_0}$.

**Proof.** We will only prove the statement for $F_{A_j}$ as the one for $F_{\bar{A}_j}$ can be achieved by the same argument. Let $M$ be a surface with boundary containing $M_0$ such that $\Gamma \subset \partial M_0 \cap \partial M$. Define $M'_0$ by removing a small half-disk around boundary point $q_0 \in \partial M \setminus \Gamma$ such that $M_0 \subset M'_0$ and $\Gamma' := \partial M'_0 \cap \partial M$ compactly contains $\Gamma$.

By Corollary 4.1 there exists $W^{1,\infty}(M)$ extensions of $X_j$ and $V_j$ respectively such that $e_{X_1, V_1, \partial M'_0 \setminus \Gamma'} = e_{X_2, V_2, \partial M'_0 \setminus \Gamma'}$, $X_1 = X_2$ on $M \setminus M_0$, and $\langle \nu, X_1 - X_2 \rangle |_{\partial M_0} = 0$. Lemma 2.3 shows that for all $p \in (1, \infty)$ if denotes $A_j := \pi_{0,1} X_j$ then there exists $\alpha_j \in W^{2,p}(M)$ solving

\[ \bar{\partial} \alpha_j = A_j, \quad \alpha_j |_{\Gamma'} \in \mathbb{R}. \]

Observe that since $X_j |_{\partial M} \in W^{3,\infty}(M_0)$ elliptic regularity stipulates that $\alpha_j \in W^{4,p}(M_0)$ for all $p \in (1, \infty)$. Proposition 4.2 asserts that the boundary value $e^{i(\alpha_1 - \alpha_2)} |_{\partial M'_0 \setminus \Gamma'}$ extends to a non-vanishing holomorphic function $\Psi$ on $M'_0$ which is unitary along $\Gamma'$ and smooth on the closure of $M_0$.

Setting $F_{A_1} := e^{i a_1}$ and $F_{A_2} := \Psi e^{i a_2}$ one has that

\[ \bar{\partial} F_{A_j} = F_{A_j} A_j \quad \text{in} \quad M'_0, \quad F_{A_1} = F_{A_2} \quad \text{on} \quad \partial M'_0 \setminus \Gamma', \quad |F_{A_j}| |_{\Gamma'} = 1 \quad \text{on} \quad \Gamma'. \]

Furthermore, using the fact that $X_1 = X_2$ in $M'_0 \setminus M_0$ one sees that $F_{A_1} F_{A_2}^{-1}$ is holomorphic in $M'_0 \setminus M_0$. The boundary condition $F_{A_1} = F_{A_2}$ on $\partial M'_0 \setminus \Gamma'$ forces $F_{A_1} = F_{A_2}$ in $M'_0 \setminus M_0$ and therefore if one defines $\Gamma_0 := \partial M_0 \cap \partial M'_0 \subset \Gamma'$ one has $F_{A_1} = F_{A_2} |_{\partial M_0 \setminus \Gamma_0}$ and $|F_{A_j}| |_{\Gamma_0} = 1$ on $\Gamma_0$. □

It remains to prove Lemma 4.2 and it is the goal of the next subsection.
4.2. **Proof of Lemma 4.2** The strategy which we will follow is to use the equivalence of the Cauchy data $C_{X_1,v_1,\partial M_0^\Gamma} = C_{X_2,v_2,\partial M_0^\Gamma}$ on $M_0'$ to derive an orthogonality condition similar to the one in Lemma 2.14 on the double $\mathcal{M}$. This will be done through the standard boundary integral identity, assuming that $C_{X_1,v_1,\partial M_0^\Gamma} = C_{X_2,v_2,\partial M_0^\Gamma}$ on $M_0'$,

\begin{equation}
0 = \int_{M_0'} \bar{u}_2(L_{X_2,v_2} - L_{X_1,v_1})u_1 \, d\text{vol}_g
\end{equation}

\[= \int_{M_0'} \bar{u}_2(A_1 - A_2) \wedge \partial u_1 - \bar{u}_2(\bar{A}_1 - \bar{A}_2) \wedge \partial u_1 + \bar{u}_2(V_1 - V_2)u_1 \, d\text{vol}_g\]

for all solutions $u_j$ of $L_{X_j,V_j}u_j = 0$ on $M_0'$ and vanishing on $\Gamma' \subset M_0'$.

Let $\Phi$ be the Morse holomorphic function on $M$ given by Proposition 2.1 which is real valued along $\Gamma'$. If $\{p_1,\ldots,p_N\}$ are critical points of $\Phi$ in $M_0' \cup \partial M_0'$, we consider the set of anti- holomorphic 1-forms $b \in W^{2,\infty}(M_0, T^*M_0)$ satisfying

\begin{equation}
t_{0,b}\,b\,|_{\Gamma'} = 0 \quad \text{for } k\text{-th order for } j = 1,\ldots,N\n\end{equation}

For all such $b$, $\Phi$ and $\alpha_j$ satisfying (19) the ansatz given by

\begin{equation}
\bar{u}_0 := e^{\frac{\Phi}{h} + i\alpha_1 \nu_1} b - e^{\frac{\Phi}{h} - i\alpha_1} \bar{b}
\end{equation}

vanishes along $\Gamma'$. We denote by $\frac{\partial}{\partial \Phi}$ the unique function satisfying $\frac{\partial}{\partial \Phi} \bar{\Phi} = b$. Since $b$ vanishes to $k$-th order at all critical points of $\Phi$, this function is an element of $W^{2,\infty}(M_0')$.

Writing $L_{X,V} = (d + iX)^*(d + iX) + V$ as

\begin{equation}
L_{X,V} = -2i \ast e^{-i\alpha} \bar{\partial} |e^{i\alpha}|^{-2} \bar{\partial} e^{i\alpha} + Q = -2i \ast e^{-i\alpha} \bar{\partial} |e^{i\alpha}|^2 \partial e^{i\alpha} + \bar{Q}
\end{equation}

for some $Q, \bar{Q} \in L^\infty(M_0')$, one sees that the ansatz $u_0$ satisfies

\[e^{-\Phi/h}L_{X_1,V_1}u_0 = O_{L^\infty}(h), \quad u_0 \mid_{\Gamma'} = 0.\]

To obtain a solution one then applies Corollary 3.1 to obtain $u_1$ solving $L_1u_1 = 0$ of the form

\begin{equation}
u_1 = u_0 + e^{\frac{\Phi}{h}} b, \quad u_1 \mid_{\Gamma'} = 0, \quad \|r_1\| + \|h r_1\| \leq Ch\sqrt{h}.
\end{equation}

Using (20) again we can also directly show that

\[e^{\frac{\Phi}{h}}L_{X,V}(e^{-\Phi/h} e^{-i\alpha} - e^{-\Phi/h} e^{-i\alpha}) = O_{L^\infty}(1), \quad \left(e^{-\Phi/h} e^{-i\alpha} - e^{-\Phi/h} e^{-i\alpha}\right) \mid_{\Gamma'} = 0.\]

Therefore, by applying Corollary 3.1 again we obtain solutions $u_2$ to $L_{X_2,V_2}u_2 = 0$ of the form

\begin{equation}
u_2 = (e^{-\Phi/h} e^{-i\alpha} - e^{-\Phi/h} e^{-i\alpha}) + e^{\frac{\Phi}{h}} b, \quad u_2 \mid_{\Gamma'} = 0, \quad \|r_2\| \leq C\sqrt{h}.
\end{equation}

Simple computation from expression (27) yields that

\[\partial u_1 = -e^{\frac{\Phi}{h}} e^{-i\alpha} b + e^{\frac{\Phi}{h}} O_{L^2}(\sqrt{h}), \quad \bar{\partial} u_1 = e^{\frac{\Phi}{h}} e^{-i\alpha} \bar{b} + e^{\frac{\Phi}{h}} O_{L^2}(\sqrt{h}).\]

Combining this with the expression (28) and plug them into (23) we obtain

\[0 = \int_{M_0} e^{i(\alpha_2 - \alpha_1)} (A_1 - A_2) \wedge b - e^{i(\alpha_2 - \alpha_1)} (\bar{A}_1 - \bar{A}_2) \wedge \bar{b} + o(1).\]

Using $\partial e^{i\alpha_j} = ie^{i\alpha_j} A_j, \partial e^{-i\alpha_j} = -ie^{-i\alpha_j} \bar{A}_j$, and $\partial b = 0$ we obtain in the limit $h \to 0$,

\[0 = \int_{M_0} \partial(e^{i(\alpha_2 - \alpha_1)} b) - \partial(e^{i(\alpha_2 - \alpha_1)} \bar{b}) = \int_{\partial M_0} e^{i(\alpha_2 - \alpha_1)} i\partial u_1 \wedge b - e^{i(\alpha_2 - \alpha_1)} i\partial u_1 \wedge \bar{b}.
\]
The antiholomorphic 1-form $b$ satisfies the boundary condition given in (24) so that
\[ \iota_{\partial M_0^*}^* b - \iota_{\partial M_0^*}^* \tilde{b} = 0 \quad \text{on } \Gamma' \]
and $\alpha_j |\nu| \in \mathbb{R}$ on $\Gamma$ by (19). Therefore the integrand in above boundary integral identity vanishes on $\Gamma$ to give
\[ (29) \quad 0 = \int_{\partial M_0^* \Gamma'} (\tilde{F}_A^*)^{-1} \tilde{F}_{A_2} \iota_{\partial M_0^*}^* b - F_{A_2} F_{A_1}^{-1} \iota_{\partial M_0^*}^* \tilde{b}. \]
for all antiholomorphic 1-form $b$ satisfying (24).

Note that since $\iota_{\partial M_0^*}^* b |\nu| \in \mathbb{R}$, the antiholomorphic 1-form $b$ on $M_0^*$ extends to a conjugate even antiholomorphic 1-form $\eta$ on $\tilde{M}^D$. Expressed in the antiholomorphic 1-form $\eta$ and the function $F_{1,2}$ defined in (20), the integral in (29) can be written as an integral along $S^1 = \partial \tilde{M}^D$ to give
\[ 0 = \int_{\partial M_0^* \Gamma'} F_{1,2} \iota_{\partial M_0^*}^* \eta + \int_{R(\partial M_0^* \Gamma')} F_{1,2} \iota_{\partial M_0^*}^* \eta = \int_{\partial \tilde{M}^D} F_{1,2} \iota_{\partial \tilde{M}^D}^* \eta. \]
As $b$ vary over the space of antiholomorphic 1-forms on $M_0^*$ satisfying (24), its conjugate even extension $\eta$ vary over the space of all conjugate even antiholomorphic 1-forms on $\tilde{M}^D$ vanishing at $\{p_1, ..., p_N, R(p_1), R(p_N)\}$. Therefore, by Lemma 2.14, the function $F_{1,2} |_{\partial \tilde{M}^D}$ is the boundary value of an antimeromorphic function $\Psi$ on $\tilde{M}^D$ with poles at
\[ \{p_1, ..., p_N, R(p_1), R(p_N)\} \cap \tilde{M}^D. \]

We would like to show that the antimeromorphic extension $\Psi$ is actually antiholomorphic by showing that all poles are removable. To this end construct by Lemma 2.4 a holomorphic function $\tilde{\varphi}$ on $M$ which is real valued along $\Gamma$ such that $p_1$ is not a critical point of $\tilde{\varphi}$. We can then use the perturbation argument of Lemma 2.6 to ensure that it is Morse. By applying the same argument with $\tilde{\varphi}$ in place of $\varphi$ we can assert that $F_{1,2} |_{\partial \tilde{M}^D}$ extends to a antimeromorphic function $\tilde{\Psi}$ for which $p_1$ and $R(p_1)$ are not poles. By uniqueness $\Psi$ and $\tilde{\Psi}$ are identical since they have the same boundary value. Therefore we can conclude that $\Psi$ has a removable singularity at $p_1$ and $R(p_1)$. Applying the same argument for the other points we have that $\Psi$ is antiholomorphic.

4.3. **Proof of Proposition 1.3.** An immediate consequence of Proposition 1.2 is the new boundary integral identity of Proposition 1.3 which is more convenient for recovering information about first-order coefficients. Let $F_{A_1}$ and $F_{A_2}$ be non-vanishing functions solving (21) and by Corollary 4.3 we can choose them to satisfy $F_{A_1} = F_{A_2}$ on the line segment $\partial M_0 \Gamma_0$ for some $\Gamma_0 \subset \partial M_0$ containing $\Gamma$. Similarly, Corollary 4.3 allows one to make the analogues choice for $F_{A_1}$ solving $\partial F_{A_1} = \bar{A}_j$ on $\Gamma_0$ for systems
\[ (30) \quad \begin{pmatrix} 0 & 0 \\ \tilde{\varphi} & 0 \end{pmatrix} \begin{pmatrix} \tilde{u}_j \omega_j \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \tilde{v}_j & \tilde{\omega}_j \end{pmatrix} \begin{pmatrix} \tilde{u}_j \omega_j \\ 0 \end{pmatrix} = 0, \quad \tilde{u}_j |_{\Gamma_0} = 0 \]
where \( v_j = \frac{1}{2} |F_{A_j}|^2 Q_j, \quad v'_j = -|F_{A_j}|^2, \) and \( Q_j = *dX_j + V_j. \) Setting \( u_j = F_{A_j}^{-1} \tilde{u}_j \) and \( \omega_j = F_{A_j}^{-1} \tilde{\omega}_j \), system (30) is equivalent to \((u_j, \omega_j)\) solving the system

\[
\begin{pmatrix}
0 & (\partial + iA_j)^* \\
\partial + iA_j & 0
\end{pmatrix}
\begin{pmatrix}
u_j \\
\omega_j
\end{pmatrix}
+ \begin{pmatrix}
Q_j \\
0
\end{pmatrix}
\begin{pmatrix}
u_j \\
\omega_j
\end{pmatrix} = 0, \quad u_j|_{\Gamma_0} = 0
\]

and this holds if and only if \( L_{X_j} v_j u_j = 0. \) Consequently, if a pair \((\tilde{u}_1, \tilde{\omega}_1)\) solves (30) with \( \tilde{u}_1|_{\Gamma_0} = 0 \), then by the fact that \( C_{C2} = C_{C2}' \), there exists a \( u_0 \) solving \( L_{X_2, \gamma} u_0 \) such that \( (u_1|_{\partial M}, (d + iX_1)u_1|_{\partial M \setminus \Gamma}) = (u_2|_{\partial M}, (d + iX_2)u_2|_{\partial M \setminus \Gamma}) \). By equation (31) this means that \( (u_1|_{\partial M}, \omega_1|_{\partial M \setminus \Gamma}) = (u_2|_{\partial M}, \omega_2|_{\partial M \setminus \Gamma}) \). As we have chosen \( F_{A_1} \) and \( F_{A_2} \) so that \( F_{A_1} = F_{A_2} \) on \( \partial M \setminus \Gamma_0 \), we conclude that \((\tilde{u}_1|_{\partial M}, \tilde{\omega}_1|_{\partial M \setminus \Gamma_0}) = (\tilde{u}_2|_{\partial M}, \tilde{\omega}_2|_{\partial M \setminus \Gamma_0})\). We therefore conclude that the systems (30) for \( j = 1, 2 \) has the same partial Cauchy data

\[
\{(\tilde{u}_j, \tilde{\omega}_j|_{\partial M \setminus \Gamma_0}) \mid \text{supp}(\tilde{u}_j) \subset \partial M \setminus \Gamma_0, (0 \quad \tilde{\partial}^* \tilde{\partial} \quad 0) \begin{pmatrix}
u_j \\
\omega_j
\end{pmatrix} + (v_j \quad 0 \quad v'_j) \begin{pmatrix}
u_j \\
\omega_j
\end{pmatrix} = 0\}
\]

Standard boundary integral identity for first order systems then yields that for any two sets of solutions \((\tilde{u}_j, \tilde{\omega}_j)\),

\[
\int_{M_0} \begin{pmatrix}
\tilde{u}_2 \\
\tilde{\omega}_2
\end{pmatrix} : \begin{pmatrix}
v_1 - v_2 \quad 0 \quad 0 \\
0 \quad v'_1 - v'_2 \quad 0
\end{pmatrix} \begin{pmatrix}
\tilde{u}_1 \\
\tilde{\omega}_1
\end{pmatrix} = 0
\]

provided that \( \tilde{u}_1 \) and \( \tilde{u}_2 \) vanishes on \( \Gamma_0 \). The boundary integral identity (6) follows by definition of \( v_j \) and \( v'_j \).

\[\square\]

5. Construction of CGO - Part I

In this section we construct complex geometries solving \( L_{X,V} u = 0 \) which vanish on \( \Gamma_0 \subset \partial M_0 \). The solutions we construct here will be inserted into boundary integral identity (6) to show that \( |F_{A_1}| = |F_{A_2}| \).

Let \( \Phi \) be a holomorphic Morse function on \( M_0 \) which is real valued on \( \Gamma_0 \). Suppose \( \{p_0, \ldots, p_N\} \) are the critical points of \( \Phi \) in \( M_0 \) with \( p_0 \) in the interior. We apply Lemma 2.4 to construct antiholomorphic 1-form \( b \) on \( M \) smooth up to the boundary such that \( b(p_j) = 0 \) to \( k \)-th order at \( p_1, \ldots, p_N \) and \( b(p_0) \neq 0 \). Let \( F_A \in W^{2,p}(M_0) \cap W^{4,2}_0(M_0) \) be a non-vanishing function for large \( p \in (1, \infty) \) satisfying \( \tilde{\partial} F_A = iAF_A \) and \( |F_A| = 1 \) on \( \Gamma_0 \). We choose a smooth cut-off \( \chi \in C_0^\infty(M_0) \) supported in a small neighbourhood of \( p_0 \) and define

\[
u'_0 := F_A^{-1} e^{\Phi/h} \tilde{\partial}^{-1} e^{-2i\psi/h} \chi |F_A|^2 b + h(1 - \chi) |F_A|^{-1} \frac{\partial}{\partial \Phi} \]

where \( \tilde{\partial}^{-1} : C_0^\infty(\text{supp}(\chi)) \rightarrow C^\infty(M) \) is the operator constructed in Proposition 2.3. Using Lemma 2.5 and direct computation gives

\[
\|e^{\phi/h} u'_0\| \leq C h^{\frac{1}{2} + \epsilon}, \quad e^{-\phi/h} L_{X,V}(u'_0) = O_{L^2}(h^{\frac{1}{2} + \epsilon}).
\]

We now compute the boundary value of \( u'_0 \) along \( \Gamma_0 \).

**Lemma 5.1.** The boundary value for the ansatz \( u_0 \) in (32) has the boundary condition

\[
u'_0|_{\Gamma_0} = F_A^{-1} e^{\Phi/h}(h f_0 + h e^{-2i\psi(p_0)/h} f_1 + h^2 f_h)
\]
for some \( f_0 \) and \( f_1 \) in \( C^\infty(\partial M_0) \) independent of \( h \) and \( f_h \) satisfies \( \| f_h \|_{C^\infty(\partial M_0)} \leq C \).

**Proof.** Along the subset \( \Gamma_0 \subset \partial M_0 \) we have that \( F_A^{-1} |\Gamma_0 = \overline{F_A} |\Gamma_0 \) and \( \Phi |\Gamma_0 \in \mathbb{R} \). Therefore, along \( \Gamma_0 \) the ansatz \( u'_0 \) has the expression

\[
u'_0 |\Gamma_0 = F_A^{-1} e^{\Phi/h} (\overline{\partial} e^{-2i\psi/h} |F_A|^2 b + h b \frac{\partial b}{\partial \Phi}) |\Gamma_0 .
\]

The boundary value along \( \Gamma_0 \) of the second term of \( u'_0 \) can be written down directly. For the first term in (32), let \( \chi' \) be a smooth function on \( M_0 \) whose support is disjoint from that of \( \chi \) and \( \chi' = 1 \) in a neighbourhood of \( \partial M_0 \). By Proposition 2.3 we have that \( \chi' \overline{\partial}^{-1} \chi \) is an operator with smooth kernel. Therefore in a coordinate system which identifies \( \chi \) and \( \chi' \), we can compute explicitly both the principal and the remainder term in the stationary phase expansion. That is,

\[
(\chi' \overline{\partial}^{-1} \chi |F_A|^2 b)(\tilde{z}) = \int \frac{e^{-2i\psi(z)/h} \chi(z)}{2} K(z, \tilde{z}) |F_A(z)|^2 dz \wedge d\tilde{z}.
\]

We may assume that the support of \( \chi \) is chosen to be so small such that we can apply Morse Lemma we obtain a change of variable \( w = \gamma(z) \) with \( \gamma(0) = 0 \) such that

\[
(\chi' \overline{\partial}^{-1} \chi |F_A|^2 b)(\tilde{z}) = e^{-2i\psi(p_0)/h} \int_D e^{2i w Q w}/h \chi(w) K(w, \tilde{z}) |F_A(w)|^2 dw \wedge d\tilde{w}
\]

for some diagonal matrix \( Q \) with entries \( \pm 1 \) on the diagonal.

With this quadratic phase we can compute explicitly both the principal and the remainder term in the stationary phase expansion. That is,

\[
(\chi' \overline{\partial}^{-1} \chi |F_A|^2 b)(\tilde{z}) = h e^{-2i\psi(p_0)/h} K(0, \tilde{z}) + h^2 \int_0^1 (1 - t) J(th, \hat{K}(\cdot, \tilde{z})) dt
\]

where

\[
J(h, \hat{K}(\cdot, \tilde{z})) = \int e^{i h (\xi, Q^{-1} \xi)} (\xi, Q^{-1} \xi) \mathcal{T}_w (\hat{K}(w, \tilde{z}) \chi(w) |F_A(w)|^2) (\xi) d\xi \wedge d\tilde{\xi}
\]

with \( \mathcal{T}_w \) denoting the classical Fourier transform with respect to the variable \( w \). We claim that \( J(h, \hat{K}(\cdot, \tilde{z})) \), is a smooth function in \( \tilde{z} \) whose \( C^0(M_0) \) norm is bounded independently of \( h > 0 \). Indeed, for any multi-index \( \beta \) standard oscillatory integral arguments give

\[
D_{\tilde{z}}^\beta J(h, \hat{K}(\cdot, \tilde{z})) = \int e^{i h (\xi, Q^{-1} \xi)} (\xi, Q^{-1} \xi) \mathcal{T}_w (D_{\tilde{z}}^\beta \hat{K}(w, \tilde{z}) \chi(w) |F_A(w)|^2) (\xi) d\xi \wedge d\tilde{\xi}
\]

for some constant coefficient third order pseudodifferential operator \( a_3 \) in the variable \( w \). Using the fact that \( D_{\tilde{z}}^\beta \hat{K}(w, \tilde{z}) \chi(w) |F_A(w)|^2 \) is a compactly supported \( W^{3,p} \) function in \( w \) for all \( p \in [1, \infty) \) we can estimate the right side by using Holder’s inequality

\[
|D_{\tilde{z}}^\beta J(h, \hat{K}(\cdot, \tilde{z}))| \leq \| (\xi, Q^{-1} \xi) \|_{L^p} \| a_3(D_{\tilde{z}}) D_{\tilde{z}}^\beta \hat{K}(w, \tilde{z}) \chi(w) |F_A(w)|^2 \|_{L^p}
\]

The fact that \( \hat{K}(w, \tilde{z}) \) is smooth and compactly supported in both variables gives the desired uniform estimate in \( \tilde{z} \).

Plugging this estimate into (34) we conclude that

\[
F_A^{-1} \overline{\partial}^{-1} \chi |F_A|^2 b |_{\partial M_0} = F_A^{-1}(he^{-2i\psi(p_0)/h} f_1 + h^2 f_h)
\]

where \( f_1 \in C^\infty(\partial M_0) \) and \( \| f_h \|_{C^\infty(\partial M_0)} \leq C \) for all \( h > 0 \). This completes the proof. \( \square \)
Note that since $F_A^{-1} = \tilde{F}_A$ on $\Gamma_0$, we can apply Corollary 2.3 and construct holomorphic functions $a_0$, $a_1$, $a_h$, and antiholomorphic functions $\tilde{a}_0$, $\tilde{a}_1$, $\tilde{a}_h$ such that

$$F_A^{-1}a_j + \tilde{F}_A\tilde{a}_j = F_A^{-1}f_j \text{ on } \Gamma_0, \quad j = 1, 2, h.$$  

Furthermore, as all the $C^k$ norm of $f_j$ are bounded, we apply the estimates in Corollary 2.3 to get that $\|a_j\|_{C^k(M_0)} + \|\tilde{a}_j\|_{C^k(M_0)} \leq C$ independent of $h > 0$. Therefore by (32) and (33) we have that the ansatz

$$(35) \quad u_j'' := u_j' - h(e^{\Phi/h}F_A^{-1}(a_0 + e^{-2i\phi(p)/h}a_1 + ha_h) + e^{\Phi/h}\tilde{F}_A(\tilde{a}_0 + e^{-2i\phi(p)/h}\tilde{a}_1 + h\tilde{a}_h))$$

with $u_j'$ given by (32) satisfies

$$(36) \quad \|e^{-\phi/h}u_j''\| \leq C h^{1+\epsilon}, \quad e^{-\phi/h}L_{X,V}u_j'' = O(h^{\frac{1}{2}+\epsilon}) \text{ in } M_0, \quad u_j'' \mid_{\Gamma_0} = 0.$$  

Extend the $O_L(h^{\frac{1}{2}+\epsilon})$ remainder on the right side trivially to $M_0'$ and applying Corollary 3.1 with we arrive at the following

**Proposition 5.2.** There exists solutions to $L_{X,V}u = 0$ in $M_0$ of the form

$$u = u_j'' + e^{\phi/h}r, \quad u \mid_{\Gamma_0} = 0, \quad \|r\| + \|hdr\| \leq Ch^{1+\epsilon}, \quad \|e^{-\phi/h}u\| \leq C h^{\frac{1}{2}+\epsilon}$$

where $u_j''$ be the ansatz given by (35).

Direct computation gives the following Lemma

**Lemma 5.3.** Let $u$ be the solution to $L_{X,V}u = 0$ constructed in Proposition 5.2. We then have that

$$\partial F_Au = e^{\Phi/h}|F_A|^2b - h\partial(e^{\Phi/h}|F_A|^2\tilde{a}_0 + e^{-2i\phi(p)/h}\tilde{a}_1 + h\tilde{a}_h)) + he^{\Phi/h}R_0 + \partial(e^{\phi/h}F_Ar)$$

for some $R_0 \in L^\infty(M_0)$ and $r$ satisfying the estimate $\|r\| + \|hdr\| \leq Ch^{1+\epsilon}$.  

6. **Construction of CGO - Part II**

In this section we construct complex geometric optics to recover the zeroth order term of the operator $L_{X,V}$. The presentation here is essentially a repeat of [12] and we only include it here for completeness and convenience of the reader. Let $p_0 \in \text{int}(M_0)$ be the critical point of a Morse holomorphic function $\Phi = \phi + i\psi$ on $M$ which is purely real on $\Gamma_0$. By Proposition 2.1 such points form a dense subset of $M$. Given such a holomorphic function, the purpose of this section is to construct, for $X \in W^{1,p}(M_0)$ and $V \in W^{2,p}(M_0)$, solutions $u$ on $M_0$ of $((d + iX)^*d + iX) + V)u = 0$ of the form

$$\begin{align*}
(37) \quad u &= (e^{\Phi/h}(F_A^{-1}a + \tilde{F}_Aa_1) + e^{\Phi/h}(\tilde{F}_A\tilde{a} + F_A^{-1}\tilde{r}_1) + he^{\Phi/h}F_A^{-1}a_0 + he^{\Phi/h}\tilde{F}_A\tilde{a}_0) + e^{\phi/h}r_2
\end{align*}$$

with $u \mid_{\Gamma_0} = 0$ for $h > 0$ small, where $a$ is holomorphic and $F_A \in W^{1,p}(M_0)$ is a non-vanishing function solving $\partial F_A = iAF_A$, $\tilde{a}_0, \tilde{a}_0 \in H^2(M_0)$ are antiholomorphic, moreover $a(p_0) \neq 0$ and $a$ vanishes to high order at all other critical points $p' \in M_0$ of $\Phi$. Furthermore, we ask that the holomorphic function $a$ is purely imaginary on $\Gamma_0$. The existence of such a holomorphic function is a consequence of Lemma 2.4.

The remainder terms $r_1, \tilde{r}_1, r_2$ will be controlled as $h \to 0$ and have particular properties near the critical points of $\Phi$. More precisely, $r_2$ will be a $O_{L^2}(h^{3/2}|\log h|)$ and $r_1, \tilde{r}_1$ will be of the form $h\tilde{r}_{12} + o_L(h)$ and $\tilde{r}_{12} + o_L(h)$ respectively where $\tilde{r}_{12}, \tilde{r}_{12}$ are independent of $h$, which can be used to obtain sufficient informations from the stationary phase method in the identification process.
6.1. Construction of \( r_1 \). We shall construct \( r_1 \) to satisfy
\[
e^{-\Phi/h}((d+iX)(d+iX)+V)e^{\Phi/h}(F_A^{-1}a+\tilde{F}Ar_1) = O_{L^2}(h|\log h|)
\]
and \( r_1 = r_{11} + hr_{12} \). Using (20) we can write, for some \( Q, \tilde{Q} \in W^{2,p}(M_0) \)
\[
L_{X,V} = -2i* F_A \partial |F_A|^{-2} \partial F_A + Q = -2i* F_A^{-1} \partial |F_A|^2 \partial \tilde{F}_A^{-1} + \tilde{Q}
\]
where \( A = \pi_{0,1}X \) and \( F_A \in W^{4,p}(M_0) \) is a non-vanishing function solving \( \partial F_A = iAF_A \) and unitary along \( \Gamma_0 \). Such functions are given by Proposition 1.3.

We let \( G \) be the Green operator of the Laplacian on the smooth surface with boundary \( M_0 \) with Dirichlet condition, so that \( \Delta_{\theta}G = \text{Id} \) on \( L^2(M_0) \). In particular this implies that \( \partial \bar{\partial}G = \frac{i}{2} \ast^{-1} \) where \( \ast^{-1} \) is the inverse of \( \ast \) mapping functions to 2-forms. We will search for \( r_1 \in H^2(M_0) \) satisfying \( ||r_1||_{L^2} = O(h) \) and
\[
e^{-2i\psi/h}|F_A|^2 \partial e^{2i\psi/h}r_1 = -\partial G(aQ) + \omega + O_{L^2}(h|\log h|)
\]
where \( \omega \) is a smooth holomorphic 1-form on \( M_0 \). Indeed, using the fact that \( \Phi \) is holomorphic we have
\[
e^{-\Phi/h}L_{X,V}e^{\Phi/h} = -2i*F_A^{-1} \partial e^{-\Phi/h}|F_A|^2 \partial F_A^{-1} + \omega + O_{L^2}(h|\log h|)
\]
for some \( Q, \tilde{Q} \in W^{2,p}(M_0) \). Applying \(-2i*\partial \) to (39), we obtain (note that \( \partial G(aQ) \in C^{2,\alpha}(M_0) \) by elliptic regularity)
\[
e^{-\Phi/h}L_{X,V}e^{\Phi/h}F_Ar_1 = -aQ + O_{L^2}(h|\log h|).
\]

We will choose \( \omega \) to be a smooth holomorphic 1-form on \( M_0 \) such that at all critical point \( p' \) of \( \Phi \) in \( M_0 \), the form \( \beta := \partial G(aQ) - \omega \) with value in \( T_{\Gamma_0}^*M_0 \) vanish to the highest possible order. Writing \( \beta = \beta(z)dz \) in local complex coordinates, \( \beta(z) \) is \( C^{2+\alpha} \) by elliptic regularity and we have \(-2i\partial \bar{\partial} \beta(p') = \partial^2 \beta(p') = 0 \) at each critical point \( p' \neq p_0 \) by construction of the function \( a \). Therefore, we deduce that at each critical point \( p' \neq p_0 \), \( \partial G(aQ) \) has Taylor series expansion \( \sum_{j=0}^2 c_j z^j + O(|z|^{2+\alpha}) \). That is, all the lower order terms of the Taylor expansion of \( \partial G(aQ) \) around \( p' \) are polynomials of \( z \) only.

**Lemma 6.1.** Let \( \{p_0, \ldots, p_N\} \) be finitely many points on \( M_0 \) and let \( \theta \) be a \( C^{2,\alpha} \) section of \( T_{\Gamma_0}^*M_0 \). Then there exists a \( C^k \) holomorphic function \( f \) on \( M_0 \) with \( k \in \mathbb{N} \), such that \( f \) vanishes to high order at the points \( \{p_1, \ldots, p_N\} \) and \( \omega = \partial f \) satisfies the following: in complex local coordinates \( z \) near \( p_0 \), one has \( \partial^2 \omega(p_0) = \partial^2 \omega(p_0) \) for \( \ell = 0, 1, 2 \), where \( \theta = \theta(z)dz \) and \( \omega = \omega(z)dz \).

**Proof.** This is a direct consequence of Lemma 2.4.

Applying this to the form \( \partial G(aQ) \) and using the observation we made above, we can construct a \( C^k \) holomorphic form \( \omega \) such that in local coordinates \( z \) centered at a critical point \( p' \) of \( \Phi \) (i.e. \( p' = \{z = 0\} \) in this coordinate), we have for \( \beta = -\partial G(aQ) + \omega = \beta(z)dz \)
\[
|\partial^m \beta(z)| = O(|z|^{2+\alpha-\ell-m}), \text{ for } \ell + m \leq 2, \text{ if } p' \neq p_0
\]
\[
|\beta(z)| = O(|z|), \text{ if } p' = p_0.
\]

Now, we let \( \chi_1 \in C_0^\infty(M_0) \) be a cutoff function supported in a small neighbourhood \( U_{p_0} \) of the critical point \( p_0 \) and identically 1 near \( p_0 \), and \( \chi \in C_0^\infty(M_0) \) is defined similarly with \( \chi = 1 \) on the support of \( \chi_1 \). We will construct \( r_{11} = r_{11} + hr_{12} \) in two steps: first, we will construct \( r_{11} \) to solve equation (39) locally near the critical point \( p_0 \) of \( \Phi \) and then we will
construct the global correction term \( r_{12} \) away from \( p_0 \) by using the extra vanishing of \( \beta \) in (40) at the other critical points.

We define locally in complex coordinates centered at \( p_0 \) and containing the support of \( \chi \)

\[
r_{11} := \chi e^{-2i\psi/h} R(e^{2i\psi/h} \chi_1 |F_A|^{-2}) \beta
\]

where \( Rf(z) := -(2\pi i)^{-1} \int_{\mathbb{C}} \frac{1}{z-z'} f(z') \, dz' \) for \( f \in L^\infty \) compactly supported is the classical Cauchy-Riemann operator inverting locally \( \partial z \) \((r_{11} \) is extended by 0 outside the neighbourhood of \( p \)). The function \( r_{11} \) is in \( C^{\frac{3}{2}}(M_0) \) and we have

\[
e^{-2i\psi/h} \partial(e^{2i\psi/h} r_{11}) = \chi_1 (-\partial G(aQ) + \omega + \eta)
\]

with \( \eta := e^{-2i\psi/h} R(e^{2i\psi/h} \chi_1 |F_A|^{-2}) \partial \chi \).

We then construct \( r_{12} \) by observing that \( \partial \psi \) vanishes to order \( 2 + \alpha \) at critical points of \( \Phi \) other than \( p \) (from (40)), and \( \partial \chi = 0 \) in a neighbourhood of any critical point of \( \psi \), so we can find \( r_{12} \) satisfying

\[
2ir_{12} \partial \psi = (1 - \chi_1) \beta |F_A|^{-2}.
\]

This is possible since both \( \partial \psi \) and the right hand side are valued in \( T^*_{p_0} M_0 \), \( \partial \psi \) has finitely many isolated zeroes on \( M_0 \): \( r_{12} \) is then a function which is in \( C^{\frac{3}{2}}(M_0 \setminus P) \) where \( P := \{ p_1, \ldots, p_N \} \) is the set of critical points other than \( p_0 \), it extends to a \( C^{1,\alpha}(M_0) \) and it satisfies in local complex coordinates \( \zeta \) near each \( p_j \)

\[
|\partial^m \partial_{\zeta} r_{12}(\zeta)| \leq C|\zeta - p_j|^{1+\alpha-m-l}, \quad m + l \leq 2.
\]

by using also the fact that \( \partial \psi \) can be locally be considered as holomorphic function with a zero of order 1 at each \( p_j \). This implies that \( r_1 \in H^2(M_0) \) and we have

\[
e^{-2i\psi/h} |F_A|^2 \partial(e^{2i\psi/h} r_1) = \beta + h \partial r_{12} + \eta = -\partial G(aQ) - \omega + h \partial r_{12} + \eta.
\]

Now the first error term \( ||\partial r_{12}||_{H^1(M_0)} \) is bounded by

\[
||\partial r_{12}||_{H^1(M_0)} \leq C \left( \left\| \frac{(1 - \chi_1)b(z)}{\partial z \psi(z)} \right\|_{H^2(U_{p_0})} \right) \leq C
\]

for some constant \( C \), where we used the fact that \( \frac{(1 - \chi_1)b(z)}{\partial z \psi(z)} \) is in \( H^2(U_{p_0}) \) and independent of \( h \). To deal with the \( \eta \) term, we need the following

**Lemma 6.2.** The following estimates hold true

\[
||\eta||_{H^2} = O(\|\log h\|), \quad ||\eta||_{H^1} \leq O(h \log h), \quad ||r_1||_{L^2} = O(h), \quad ||r_1 - h\bar{r}_{12}||_{L^2} = o(h)
\]

where \( \bar{r}_{12} \) solves \( 2i\bar{r}_{12} \partial \psi = \beta |F_A|^{-2} \) is independent of \( h \) and \( H^2 \) near the boundary \( \partial M_0 \).

**Proof.** We start by observing that since \( \beta \) vanishes to high order at all critical points of \( \Phi \) except for the interior point \( p_0 \in M \), one has that \( \bar{r}_{12} \) is in \( H^2 \) in a neighbourhood of the boundary \( \partial M \). Furthermore,

\[
||r_1 - h\bar{r}_{12}||_{L^2} = \left\| \chi e^{-2i\psi/h} R(e^{2i\psi/h} \chi_1 \beta |F_A|^{-2}) - h \frac{\chi_1 \beta |F_A|^{-2}}{2i \partial z \psi} \right\|_{L^2(U_{p_0})},
\]

\[
||r_1||_{L^2} \leq \left\| \chi e^{-2i\psi/h} R(e^{2i\psi/h} \chi_1 \beta |F_A|^{-2}) - h \frac{\chi_1 \beta |F_A|^{-2}}{2i \partial z \psi} \right\|_{L^2(U_{p_0})} + h||\bar{r}_{12}||_{L^2(M_0)}
\]
The first term is estimated in Proposition 2.7 of [17], it is a \( o(h) \), while the \( \|\tilde{r}_{12}\|_{L^2} \) is independent of \( h \). Now are going to estimate the \( H^2 \) norms of \( \eta \). Locally in complex coordinates \( z \) centered at \( p_0 \) (i.e. \( p_0 = \{z = 0\} \)), we have

\[
(43) \quad \eta(z) = -\partial_z \chi(z)e^{-2ivz/h} \int_C e^{2ivz/h} \frac{1}{\bar{z} - \xi} \chi_1(\xi)\beta(\xi)|F_A(\xi)|^{-2}d\xi_1d\xi_2, \quad \xi = \xi_1 + i\xi_2.
\]

Since \( \beta \) is \( C^{2,\alpha} \) in \( U \), we decompose \( \beta(\xi) = \langle \nabla \beta(0), \xi \rangle + \tilde{\beta}(\xi) \) using Taylor formula, so we have \( \tilde{\beta}(0) = \partial_\xi \tilde{\beta}(0) = 0 \) and we split the integral \([13]\) with \( \langle \nabla \beta(0), \xi \rangle \) and \( \tilde{\beta}(\xi) \). Since the integrand with the \( \langle \nabla \beta(0), \xi \rangle \) is smooth and compactly supported in \( \xi \) (recall that \( \chi_1 = 0 \) on the support of \( \partial_z \chi_1 \)), we can apply stationary phase to get that

\[
\left| \partial_\xi \chi(z)e^{-2ivz/h} \int_C e^{2ivz/h} \frac{1}{\bar{z} - \xi} \chi_1(\xi)\beta(\xi)|F_A(\xi)|^{-2}d\xi_1d\xi_2 \right| \leq C h^2
\]

uniformly in \( z \). Now set \( \tilde{\beta}_z(\xi) = \partial_\xi \chi(z)\chi_1(\xi)\tilde{\beta}(\xi)/(\bar{z} - \xi) \) which is \( C^{2,\alpha} \) in \( \xi \) and smooth in \( z \).

Let \( \theta \in C_0^\infty([0,1]) \) be a cutoff function which is equal to 1 near 0 and set \( \theta_h(\xi) := \theta(|\xi|/h) \), then we have by integrating by parts

\[
(44) \quad \int_C e^{2ivz/h}|F_A(\xi)|^{-2}\tilde{\beta}_z(\xi)d\xi_1d\xi_2 = h^2 \int_{\text{supp}(\chi_1)} e^{2ivz/h} \theta_h \left( \frac{1 - \theta_h(\xi)}{2i\partial_\xi}\partial_\xi \left( \frac{|F_A(\xi)|^{-2}\tilde{\beta}_z(\xi)}{2i\partial_\xi} \right) \right) d\xi_1d\xi_2

- h \int_{\text{supp}(\chi_1)} e^{2ivz/h} \theta_h(\xi) \partial_\xi \left( \frac{|F_A(\xi)|^{-2}\tilde{\beta}_z(\xi)}{2i\partial_\xi} \right) d\xi_1d\xi_2.
\]

Using polar coordinates with the fact that \( \tilde{\beta}_z(0) = 0 \), it is easy to check that the second term in \([44]\) is bounded uniformly in \( z \) by \( Ch^2 \). To deal with the first term, we use \( \tilde{\beta}_z(0) = \partial_\xi \tilde{\beta}_z(0) = \partial_\xi \tilde{\beta}_z(0) = 0 \) and a straightforward computation in polar coordinates shows that the first term of \([44]\) is bounded uniformly in \( z \) by \( Ch^2|\log(h)| \). We conclude that

\[
||\eta||_{L^2} \leq C||\eta||_{L^\infty} \leq Ch^2|\log h|,
\]

It is also direct to see that the same estimates holds with a loss of \( h^{-2} \) for any derivatives in \( z, \bar{z} \) of order less or equal to 2, since they only hit the \( \chi(z) \) factor, the \( (\bar{z} - \xi)^{-1} \) factor or the oscillating term \( e^{-2ivz/h} \). So we deduce that

\[
||\eta||_{H^2} = O(|\log h|).
\]

and this ends the proof. \( \square \)

We summarize the result of this section with the following

**Lemma 6.3.** Let \( k \in \mathbb{N} \) be large and \( \Phi \in C^k(M_0) \) be a holomorphic function on \( M_0 \) which is Morse in \( M_0 \) with a critical point at \( p_0 \in \text{int}(M_0) \). Let \( a \in C^k(M_0) \) be a holomorphic function on \( M_0 \) purely imaginary on \( \Gamma_0 \) and vanishing to high order at every critical point of \( \Phi \) other than \( p \). Then there exists \( r_1 \in H^2(M_0) \) such that \( r_1 = h\tilde{r}_{12} + o_L(h) \) with \( \tilde{r}_{12} \in L^2 \) independent of \( h \) and

\[
e^{-\Phi/h}L_{X,V}e^{\Phi/h}(F_A^{-1}a + F_A r_1) = O_{L^2}(h|\log h|).
\]

One can follow the same construction for the antiholomorphic phase \( \tilde{\Phi} \) in place of \( \Phi \). Indeed, repeating the above argument in this case yields

\[
ne^{-\tilde{\Phi}/h}L_{X,V}e^{\tilde{\Phi}/h}(\tilde{F}_A^{-1}a + \tilde{F}_A r_1) = O_{L^2}(h|\log h|).
\]
Lemma 6.4. Let \( k \in \mathbb{N} \) be large and \( \Phi \in C^k(M_0) \) be a holomorphic function on \( M_0 \) which is Morse in \( M_0 \) with a critical point at \( p_0 \in \text{int}(M_0) \). Let \( a \in C^k(M_0) \) be a holomorphic function on \( M \) purely imaginary on \( \Gamma_0 \) and vanishing to high order at every critical point of \( \Phi \) other than \( p \). Then there exists \( r_1' \in H^2(M_0) \) such that \( r_1' = h \tilde{r}_{12} + o_{L^2}(h) \) with \( \tilde{r}_{12} \in L^2 \) independent of \( h \) and
\[
e^{-\Phi/h} L_{X,V} e^{\Phi/h} (F_A \tilde{u} + F_A^{-1} r_1') = O_{L^2}(h |\log h|).
\]

6.2. Construction of \( a_0 \). We have constructed the correction terms \( r_1 \) which solves the Schrödinger equation to order \( h \) as stated in Lemma 6.3. In this subsection, we will construct a holomorphic function \( a_0 \) which annihilates the boundary value of the solution on \( \Gamma_0 \). In particular, we have the following

Lemma 6.5. There exists a holomorphic function \( a_0 \in H^2(M_0) \) and an antiholomorphic function \( \tilde{a}_0 \in H^2(M_0) \) independent of \( h \) such that
\[
e^{-\Phi/h} L_{X,V} (e^{\Phi/h} (F_A^{-1} a + \bar{F}_A r_1') + e^{\Phi/h} (\tilde{F}_A \tilde{a} + F_A^{-1} \bar{r}_1')) + h e^{\Phi/h} F_A^{-1} a_0 + h e^{\Phi/h} \bar{F}_A \tilde{a}_0 = O_{L^2}(h |\log h|)
\]
and
\[
e^{\Phi/h} (F_A^{-1} a + \bar{F}_A r_1') + e^{\Phi/h} (\tilde{F}_A \tilde{a} + F_A^{-1} \bar{r}_1') + h e^{\Phi/h} F_A^{-1} a_0 + h e^{\Phi/h} \bar{F}_A \tilde{a}_0) |_{\Gamma_0} = 0.
\]

Proof. First, notice that \( h^{-1} r_1 |_{\partial M_0} = \tilde{r}_{12} |_{\partial M_0} \in H^{3/2}(\partial M_0) \) and \( h^{-1} \tilde{r}_2 |_{\partial M_0} = \tilde{r}_{12} |_{\partial M_0} \in H^{3/2}(\partial M_0) \) are independent of \( h \). Using part (iii) of Proposition 2.3, one can construct \( a_0, \tilde{a}_0 \in H^2(M_0) \) holomorphic and antiholomorphic respectively such that \( [a_0 + \tilde{a}_0] |_{\Gamma_0} = -(\tilde{r}_{12} + \bar{\tilde{r}}_{12}) |_{\Gamma_0} \). Since \( \Phi \) is purely real on \( \Gamma_0 \) and \( F_A \) is unitary on \( \Gamma_0 \), we see that
\[
e^{\Phi/h} (F_A^{-1} a + \bar{F}_A r_1') + e^{\Phi/h} (\tilde{F}_A \tilde{a} + F_A^{-1} \bar{r}_1') + h e^{\Phi/h} F_A^{-1} a_0 + h e^{\Phi/h} \bar{F}_A \tilde{a}_0) |_{\Gamma_0} = 0.
\]
This combined with the asymptotic given by Lemma 6.3 and Lemma 6.4 completes the proof.

We can extend the \( O_{L^2}(|h \log h|) \) remainder in Lemma 6.5 trivially to all of \( M_0' \) and apply Corollary 6.1 to obtain the following CGO:

Proposition 6.1. There exist solutions to \( L_{X,V} u = 0 \) with boundary condition \( u |_{\Gamma_0} = 0 \) of the form \( 6.8 \) with \( r_1, r_1', \ a_0, \tilde{a}_0 \) constructed in the previous sections and \( r_2 \) satisfying \( \| r_2 \|_{H^1_{\text{rad}}} = O(h^{3/2} |\log h|) \).

7. Recovery of Coefficients

7.1. Recovering the Modulus of \( F_{A_j} \). We assume that \( \mathcal{E}_{X_1, V_1, \partial M_0 \setminus \Gamma} = \mathcal{E}_{X_2, V_2, \partial M_0 \setminus \Gamma} \). By Proposition 7.3 we have that there exists a portion of the boundary \( \Gamma_0 \) containing \( \Gamma \) whose complement contains an open set and non-vanishing solutions \( F_{A_j} \in W^{2,p}(M_0) \cap W^{3,p}_{\text{loc}}(M_0) \) to \( \partial F_{A_j} = A_j F_{A_j} \) with \( |F_{A_j}| |_{\Gamma_0} = 1 \) such that \( F_{A_1} |_{\partial M_0 \setminus \Gamma_0} = F_{A_2} |_{\partial M_0 \setminus \Gamma_0} \). Furthermore, if \( L_{X_j, V_j} u_j = 0 \) with \( u_j |_{\Gamma_0} = 0 \) then the boundary integral identity
\[
0 = \int_{M_0} \langle (|F_{A_1}|^{-2} - |F_{A_2}|^{-2}) \bar{\delta} u_1, \bar{\delta} u_2 \rangle + \frac{1}{2} \langle (Q_2 |F_{A_2}|^2 - Q_1 |F_{A_1}|^2) \bar{u}_1, \bar{u}_2 \rangle
\]
holds for \( \bar{u}_j := F_{A_j} u_j \).

The main result of this subsection is to show that the \( F_{A_1} \) and \( F_{A_2} \) chosen above have the same modulus. More precisely,
Proof of Lemma 7.1. If \( c_{X_1,V_1,\partial M_0} = c_{X_2,V_2,\partial M_0} \) and \( F_A \) are chosen as above then \( |F_A| = |F_{A_2}| \).

**Proof.** If \( \tilde{p} \) is any interior point of \( M_0 \) and \( B_\epsilon(\tilde{p}) \) is a neighbourhood of the point, then by Proposition 7.1 there exists a Morse holomorphic function \( \Phi = \phi + i\psi \) on \( M \) which is real valued along \( \Gamma_0 \) with a critical point \( p_0 \) in \( B_\epsilon(\tilde{p}) \). If \( \{p_0,\ldots,p_N\} \) are the critical points of \( \Phi \), we can construct by Lemma 2.4 an antiholomorphic 1-form \( b \) which vanishes to order \( k \) at \( \{p_1,\ldots,p_N\} \) and \( b(p_0) \neq 0 \). We have the following Lemma which we will prove at the end of the subsection:

**Lemma 7.1.** For all such \( \Phi \) and \( b \) we have the following asymptotic as \( h \to 0 \):

\[
0 = \int_{M_0} (|F_{A_0}|^2 - |F_{A_1}|^2) b^2 e^{-2i\psi/h} + o(h)
\]

Since \( b \) vanishes at all critical points of \( \Phi \) except for \( p_0 \), (45) has stationary phase expansion

\[
0 = h e^{2i\psi(p_0)/h} (|F_{A_2}(p_0)|^2 - |F_{A_1}(p_0)|^2) + o(h)
\]

which implies that \( |F_{A_2}(p_0)|^2 - |F_{A_1}(p_0)|^2 = 0 \). Since \( \epsilon > 0 \) can be chosen arbitrarily small, the continuity of \( F_{A_2} \) then gives that \( |F_{A_2}(\tilde{p})|^2 = |F_{A_1}(\tilde{p})|^2 \) for any \( \tilde{p} \in M_0 \).

It remains to prove Lemma 7.1.

**Proof of Lemma 7.1.** By Proposition 1.3 we have that if \( L_{X_1,V_1} u_j = 0 \) and \( u_j |_{\Gamma_0} = 0 \) then

\[
0 = \int_{M_0} \langle (|F_{A_2}|^2 - |F_{A_1}|^2) \partial \bar{u}_1, \partial \bar{u}_2 \rangle + \frac{1}{2} \langle (Q_2 |F_{A_2}|^2 - Q_1 |F_{A_1}|^2) \bar{u}_1, \bar{u}_2 \rangle
\]

where \( \bar{u}_j = F_{A_0} u_j \) and \( Q_j = *dX_j + V_j \).

If \( \Phi \) and \( b \) are as given in the statement of the Lemma, let \( u_1 \) be the solution to \( L_{X_1,V_1} u_1 = 0 \) given by Proposition 5.2 for the phase \( \Phi \) and let \( u_2 \) be the solution to \( L_{X_2,V_2} u_2 = 0 \) given by Proposition 5.2 for the phase \( -\Phi \). That is,

\[
u_1 = u_0^\prime + e^{\phi/h} r_1, \quad u_2 = u_0'' - e^{-\phi/h} r_2
\]

where \( u_0^\prime, u_0'' \) are the ansatz given by (35) for \( \pm \Phi \) respectively. Plugging these solutions into this identity and using the estimate on \( u_j \) in Proposition 5.2 in conjunction with the identity in Lemma 7.3, the boundary integral identity becomes

\[
0 = \int_{M_0} (|F_{A_2}|^2 - |F_{A_1}|^2) b^2 e^{-2i\psi/h}
\]

\[
- \int_{M_0} \langle (1 - |F_{A_1}|^2 |F_{A_2}|^2) e^{\phi/h} b, \bar{\partial}(e^{-\phi/h} |F_{A_2}|^2 A_h + e^{\phi/h} F_{A_2} r) \rangle
\]

\[
- \int_{M_0} \langle (|F_{A_1}|^2 |F_{A_2}|^2 - 1) \bar{\partial}(e^{\phi/h} |F_{A_2}|^2 A_h + e^{\phi/h} F_{A_2} r), e^{-\phi/h} b \rangle
\]

\[
+ h^2 \int_{M_0} \langle (|F_{A_1}|^2 |F_{A_2}|^2 - 1) \bar{\partial}(e^{\phi/h} |F_{A_2}|^2 A_h + e^{\phi/h} F_{A_2} r), \bar{\partial}(e^{-\phi/h} |F_{A_2}|^2 A_h + e^{-\phi/h} F_{A_2} r) \rangle
\]

\[
+ o(h)
\]

where \( A_h := \bar{a}_0 + e^{-2i\psi(p_0)/h} a_1 + h a_h \) and \( A_h' := \bar{a}_0 + e^{-2i\psi(p_0)/h} a_1' + h a_h' \) are antiholomorphic functions depending on the parameter \( h > 0 \).

The second term can be estimated by taking the adjoint of \( \bar{\partial} \) and using that

\[
|F_{A_1}||_{\partial M_0} = |F_{A_2}||_{\partial M_0}
\]
to obtain
\[
(47) \quad h \int_{M_0} \langle (1 - |F_{A_2}|^{-2})|F_{A_1}|^2 e^{\Phi/h} \partial (e^{-\Phi/h} |F_{A_2}|^2 A_h + e^{-\phi/h} F_{A_2} r) \rangle = -h \int_{M_0} e^{-2i\psi/h} \partial (|F_{A_2}|^{-2} |F_{A_1}|^2 \wedge b (|F_{A_2}|^2 A_h + e^{-\phi/h} F_{A_2} r).
\]

By Proposition [5.2] the remainder \( r \) satisfies the estimate \( \| r \| \leq C h^{1+\varepsilon} \). This combined with the fact that \( \int e^{2i\psi/h} f = o(1) \) for all \( f \in L^1 \) independent of \( h \) gives that (47) can be estimated by
\[
(48) \quad h \int_{M_0} \langle (1 - |F_{A_2}|^{-2})|F_{A_1}|^2 e^{\Phi/h} \partial (e^{-\Phi/h} |F_{A_2}|^2 A_h + e^{-\phi/h} F_{A_2} r) \rangle = o(h).
\]

We have then that the second term of (46) can be estimated by \( o(h) \). The third term of (46) can be treated the same way to obtain
\[
(49) \quad h \int_{M_0} \langle (|F_{A_2}|^{-2} - |F_{A_2}|^{-2}) \partial (e^{\Phi/h} |F_{A_1}|^2 A_h' + e^{\phi/h} F_{A_2} r'), e^{-\phi/h} b \rangle = o(h).
\]

Therefore, plugging the estimates of (48) and (49) into (46) we have
\[
(50) \quad 0 = \int_{M_0} \langle (|F_{A_2}|^{-2} - |F_{A_2}|^{-2}) \partial (e^{\Phi/h} |F_{A_1}|^2 A_h' + e^{\phi/h} F_{A_2} r') \rangle + o(h).
\]

For the remaining integral we integrate by parts again to obtain
\[
0 = \int_{M_0} \langle (|F_{A_2}|^{-2} - |F_{A_2}|^{-2}) |F_{A_1}|^2 b \rangle e^{-2i\psi/h} + h^2 \int_{M_0} \langle (e^{\Phi/h} |F_{A_1}|^2 A_h' + e^{\phi/h} F_{A_1} r') \rangle \partial (|F_{A_2}|^{-2} - |F_{A_2}|^{-2}), \partial (e^{-\Phi/h} |F_{A_2}|^2 A_h + e^{-\phi/h} F_{A_2} r) \rangle + h^2 \int_{M_0} \langle (e^{\Phi/h} |F_{A_1}|^2 A_h' + e^{\phi/h} F_{A_1} r') \rangle (|F_{A_1}|^{-2} - |F_{A_2}|^{-2}) \Delta_g (e^{-\Phi/h} |F_{A_2}|^2 A_h + e^{-\phi/h} F_{A_2} r) + o(h).
\]

Using the fact that \( A_h = \tilde{a}_0 + e^{-2i\psi/p_0} \tilde{a}_1 + h \tilde{a}_h \) with \( \| a_h \|_{C^\alpha} \) independent of \( h \) and
\[
e^{\phi/h} \Delta_g e^{-\phi/h} r = 2 e^{\phi/h} (X_2, d e^{-\phi/h} r) + (V_2 + |X_2|^2) r + O L^2 (h^{1+\varepsilon}), \quad |r|_{H^1_{L^2}} \leq C h^{1+\varepsilon},
\]
we have that the above expression becomes
\[
0 = \int_{M_0} \langle (|F_{A_2}|^{-2} - |F_{A_2}|^{-2}) |F_{A_1}|^2 b \rangle e^{-2i\psi/h} + o(h)
\]
and the proof is complete.

\[\square\]

7.2. Gauge Equivalence of \( X_1 \) and \( X_2 \). The purpose of this subsection is to prove the first assertion of Theorem 1.2. More precisely,

**Proposition 7.2.** There exists an open subset of the boundary \( \Gamma_0 \subset \partial M_0 \) compactly containing \( \Gamma \) with \( \partial M_0 \setminus \Gamma_0 \) an open segment and a non-vanishing function \( \Theta \) such that
\[
X_1 - X_2 = d\Theta/\Theta, \quad \Theta |_{\partial M_0 \setminus \Gamma_0} = 1.
\]
Proof. By Lemma 7.1 we can choose non-vanishing functions \( F_{A_j} \) satisfying \( \partial F_{A_j} = iA_j F_{A_j} \) with boundary condition \( |F_{A_j}| \mid_{\Gamma_0} = 1 \) such that
\[
F_{A_1} \mid_{\partial M_0 \setminus \Gamma_0} = F_{A_2} \mid_{\partial M_0 \setminus \Gamma_0} \quad \text{and} \quad |F_{A_1}| = |F_{A_2}| \mid_{M_0}.
\]
Observe that if we define \( F_{\bar{A}_j} := F_{A_j}^{-1} \), it is a solution to \( \partial F_{\bar{A}_j} = i\bar{A}_j F_{\bar{A}_j} \) with boundary condition \( |F_{\bar{A}_j}| \mid_{\Gamma_0} = 1 \) such that
\[
F_{\bar{A}_1} \mid_{\partial M_0 \setminus \Gamma_0} = F_{\bar{A}_2} \mid_{\partial M_0 \setminus \Gamma_0} \quad \text{and} \quad |F_{\bar{A}_1}| = |F_{\bar{A}_2}| \mid_{M_0}.
\]
Therefore, \( \Theta := \frac{F_{\bar{A}_1}}{F_{\bar{A}_2}} = \frac{F_{A_1}}{F_{A_2}} \) is a function mapping \( M_0 \) to the unit circle \( S^1 \subset \mathbb{C} \) solving the differential equation
\[
\partial \Theta / \Theta = i(A_1 - A_2), \quad \partial \Theta / \Theta = i(\bar{A}_1 - \bar{A}_2)
\]
and thus \( d\Theta / \Theta = i(X_1 - X_2) \) with \( \Theta \mid_{\partial M_0 \setminus \Gamma_0} = 1 \) and the proof is complete. \( \square \)

7.3. Identifying Zeroth Order Term. The purpose of this section is to prove that under the assumptions of Theorem 1.2 \( V_1 = V_2 \). In conjunction with Proposition 7.2 this completes the proof of Theorem 1.2. The argument presented here is almost identical to that of of [12] which we repeat here for the convenience of the reader.

We begin by observing that due to Proposition 7.2 the operators \( d + iX_1 \) and \( d + iX_2 \) are gauge equivalent. Therefore, we can assume, by taking a gauge transformation, that \( X := X_1 = X_2 \in W^{3,p}(M_0) \) and that
\[
\mathcal{L}_{X,V_1,\partial M_0} = \mathcal{L}_{X,V_2,\partial M_0}.
\]
So by repeating the same boundary determination argument in the appendix of [12] we can conclude that \( V_1 \mid_{\partial M_0 \setminus \Gamma_0} = V_2 \mid_{\partial M_0 \setminus \Gamma_0} \).

If we let \( \alpha \in W^{1,p}(M_0) \) be a solution of
\[
\partial \alpha = A := \pi_{0,1}X, \quad i\alpha \mid_{\Gamma_0} \in \mathbb{R}
\]
given by Proposition 2.3 and set \( F_A = e^{i\alpha} \) we have by Proposition 6.3
\[
0 = \int_{M_0} (V_2 - V_1) |F_A|^2 u_1 \bar{u}_2
\]
for all \( u_j \) solving
\[
L_{X,V_j} u_j = 0 \quad u_j \mid_{\Gamma_0} = 0 \quad \text{for} \quad j = 1, 2.
\]
Let \( p_0 \in M_0 \) be an interior point such that there exits a holomorphic Morse function \( \Phi \) on \( M \) with \( \Phi \mid_{\Gamma_0} \in \mathbb{R} \). We also require that \( \operatorname{Im}(\Phi(p_0)) \neq 0 \). Such points are dense on \( M_0 \) by Proposition 2.4. Let \( a \) be a holomorphic function which is purely imaginary on \( \Gamma_0 \) such that \( a(p_0) \neq 0 \) and \( a \) vanishes to high order at all other critical points of \( \Phi \). One can construct such a holomorphic function by Lemma 2.4. Applying Proposition 6.1 to both \( \Phi \) and \( -\Phi \) yields solutions to \( L_{X,V,j} u_j = 0 \) which are of the form:
\[
u_1 = (e^{\Phi/h}(F_A^{-1}a + F_A \tilde{r}_1) + e^{\Phi/h}(F_A \tilde{a} + F_A^{-1}r_1) + he^{-\Phi/h}F_A^{-1}a_0 + he^{-\Phi/h}F_A \tilde{a}_0) + e^{\phi/h}r_2
\]
\[
u_2 = (e^{-\Phi/h}(F_A^{-1}a + F_A \tilde{s}_1) + e^{-\Phi/h}(F_A \tilde{a} + F_A^{-1}s_1) + he^{-\Phi/h}F_A^{-1}a_0 + he^{-\Phi/h}F_A \tilde{a}_0) + e^{-\phi/h}s_2
\]
where
\[
\tilde{r}_1 = h\tilde{r}_1 + o_{L^2}(h), \quad r'_1 = h\tilde{r}'_1 + o_{L^2}(h), \quad s_1 = h\tilde{s}_1 + o_{L^2}(h), \quad s'_1 = h\tilde{s}'_1 + o_{L^2}(h)
\]
with \( \tilde{r}_1, \tilde{r}'_1, \tilde{s}_1, \tilde{s}'_1 \in L^2(M_0) \) independent of \( h \) and \( \|r_2\|_{L^2} + \|s_2\|_{L^2} = o(h) \).
Plug these solutions into the integral identity we have that
\[
0 = \int_{M_0} (V_2 - V_1)|F_A|^4(e^{2i\psi/h}|F_A|^{-2}|\alpha|^2 + e^{-2i\psi/h}|F_A|^2|\alpha|^2 + g_0 + hg_1) + o(h)
\]
for some \(g_0, g_1 \in L^2(M_0)\) independent of \(h\).

**Lemma 7.2.** In the limit as \(h \to 0\) the following asymptotic holds:
\[
\int_{M_0} (V_2 - V_1)|F_A|^2e^{2i\psi/h}|\alpha|^2 + (V_2 - V_1)|F_A|^6e^{-2i\psi/h}|\alpha|^2 = hC_+ e^{2i\psi(p_0)/h}(V_2 - V_1)(p_0) + hC_- e^{-2i\psi(p_0)/h}(V_2 - V_1)(p_0) + o(h).
\]
Here \(C_+\) and \(C_-\) are non-zero constants independent of \(h\).

Using Lemma 7.2 we have that
\[
0 = \int_{M_0} (V_1 - V_2)g_0 + O(h)
\]
and therefore
\[
0 = \int_{M_0} (V_2 - V_1)|F_A|^4(e^{2i\psi/h}|F_A|^{-2}|\alpha|^2 + e^{-2i\psi/h}|F_A|^2|\alpha|^2 + h g_1) + o(h).
\]
Using Lemma 7.2 again we get that
\[
0 = C_+ e^{2i\psi(p_0)/h}(V_2 - V_1)(p_0) + C_- e^{-2i\psi(p_0)/h}(V_1 - V_2)(p_0) + \int_{M_0} (V_1 - V_2)|F_A|^4g_1 + o(1)
\]
for constants \(C_+\) independent of \(h\). Since \(\psi(p_0) \neq 0\) we can choose a sequence of \(h \to 0\) such that \(e^{2i\psi(p_0)/h} = e^{-2i\psi(p_0)/h} = 1\) and another sequence \(h \to 0\) such that \(e^{2i\psi(p_0)/h} = e^{-2i\psi(p_0)/h} = -1\) to obtain
\[
\int_{M_0} (V_1 - V_2)|F_A|^4g_1 = 0.
\]
Therefore, we have that
\[
0 = C_+ e^{2i\psi(p_0)/h}(V_2 - V_1)(p_0) + C_- e^{-2i\psi(p_0)/h}(V_1 - V_2)(p_0) + o(1).
\]
Again we choose a sequence \(h \to 0\) such that \(e^{2i\psi(p_0)/h} = i\) and another sequence \(h \to 0\) such that \(e^{-2i\psi(p_0)/h} = e^{-2i\psi(p_0)/h} = 1\) we can obtain \((V_1 - V_2)(p_0) = 0\).

In order to complete the proof we must provide the

**Proof of Lemma 7.2.** Let \(\chi\) be a smooth cutoff function on \(M_0\) which is identically 1 everywhere except inside a small ball containing \(p_0\) and no other critical point of \(\Phi\), and \(\chi = 0\) near \(p_0\). Setting \(V := V_2 - V_1\) we split the oscillatory integral in two parts:
\[
\int_{M_0} (e^{2i\psi/h}|F_A|^2 + e^{-2i\psi/h}|F_A|^6)V|\alpha|^2 = \int_{M_0} \chi(e^{2i\psi/h}|F_A|^2 + e^{-2i\psi/h}|F_A|^6)V|\alpha|^2
\]
\[
+ \int_{M_0} (1 - \chi)(e^{2i\psi/h}|F_A|^2 + e^{-2i\psi/h}|F_A|^6)V|\alpha|^2
\]
The phase \(\psi\) has nondegenerate critical points, therefore, a standard application of the stationary phase at \(p_0\) gives
\[
\int_{M_0} (1 - \chi)(e^{2i\psi/h}|F_A|^2 + e^{-2i\psi/h}|F_A|^6)V|\alpha|^2 = hC_+ e^{2i\psi(p_0)/h}V(p_0) + hC_- e^{-2i\psi(p_0)/h}V(p_0) + o(h).
\]
Define the potential \( \tilde{V}(\cdot) := V(\cdot) - V(p_0) \in C^{1,\alpha}(M_0) \), then we show that
\[
(51) \quad \int_{M_0} (1 - \chi)(e^{2i\psi/h}|F_A|^2 + e^{-2i\psi/h}|F_A|^6)\tilde{V}|a|^2 = o(h).
\]
Indeed, first by integration by parts and using \( \Delta_g \psi = 0 \) one has
\[
\int_{M_0} (1 - \chi)e^{2i\psi/h}|F_A|^2\tilde{V}|a|^2 = \frac{h}{2i} \int_{M_0} \langle de^{2i\psi/h}, d\psi \rangle |F_A|^2\tilde{V}(1 - \chi)|a|^2 d\nu_g
\]
and
\[
\int_{M_0} (1 - \chi)e^{-2i\psi/h}|F_A|^6\tilde{V}|a|^2 = -\frac{h}{2i} \int_{M_0} \langle de^{-2i\psi/h}, d\psi \rangle |F_A|^6\tilde{V}(1 - \chi)|a|^2 d\nu_g
\]
but we can see that \( \langle d((1 - \chi)|F_A|^6|a|^2\tilde{V})/d|\psi|^2, d\psi \rangle \in L^1(M_0) \): this follows directly from the fact that \( \tilde{V} \) is in the Hölder space \( C^{1,\alpha}(M_0) \) and \( \tilde{V}(p_0) = 0 \), and from the non-degeneracy of Hess(\( \psi \)). It then suffice to observe that \( \int e^{\pm 2i\psi/h} f = o(1) \) for all \( f \in L^1(M_0) \) to conclude that (51) holds. Using similar argument, we now show that
\[
\int_{M_0} \chi(e^{2i\psi/h}|F_A|^2 + e^{-2i\psi/h}|F_A|^6)V|a|^2 = o(h).
\]
Indeed, since \( a \) vanishes to large order at all boundary critical points of \( \psi \), we may write
\[
\int_{M_0} \chi(e^{2i\psi/h}|F_A|^2 + e^{-2i\psi/h}|F_A|^6)V|a|^2 d\nu_g = \frac{h}{2i} \int_{M_0} \langle (de^{2i\psi/h}, d\psi) |F_A|^2 - (de^{-2i\psi/h}, d\psi) |F_A|^6 \rangle \frac{\chi V|a|^2}{|d\psi|^2} d\nu_g
\]
\[
= \frac{h}{2i} \int_{M_0} e^{2i\psi/h} \nabla_g (V \chi |F_A|^2 |a|^2 \nabla \psi) - e^{-2i\psi/h} \nabla_g (V \chi |F_A|^6 |a|^2 \nabla \psi) + \frac{h}{2i} \int_{M_0} (e^{2i\psi/h} - e^{-2i\psi/h}) V|a|^2 d\nu_g \cdot \nabla \psi.
\]
Here the expression for the boundary integral is obtained by using the fact that \( V_1 = V_2 \) on \( \partial M_0 \setminus \Gamma_0 \) from boundary determination and \( |F_A| = 1 \) on \( \Gamma_0 \) by construction.

For the interior integral we use the fact that \( \int e^{\pm 2i\psi/h} f = o(1) \) for all \( f \in L^1(M_0) \) to conclude that
\[
-\frac{h}{2i} \int_{M_0} (e^{2i\psi/h} \nabla_g (V \chi |F_A|^2 |a|^2 \nabla \psi) - e^{-2i\psi/h} \nabla_g (V \chi |F_A|^6 |a|^2 \nabla \psi)) = o(h)
\]
and for the boundary integral, we observe that on \( \Gamma_0 \), \( \psi = 0 \) by construction so \( (e^{2i\psi/h} - e^{-2i\psi/h}) = 0 \). Therefore
\[
\int_{M_0} \chi(e^{2i\psi/h} + e^{-2i\psi/h})V|a|^2 d\nu_g = o(h)
\]
and the proof is complete. \( \square \)
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