A Trustful Monad for Axiomatic Reasoning with Probability and Nondeterminism

Reynald Affeldt\textsuperscript{1}, Jacques Garrigue\textsuperscript{2}, David Nowak\textsuperscript{3}, and Takafumi Saikawa\textsuperscript{4}

\textsuperscript{1}National Institute of Advanced Industrial Science and Technology, Cyber Physical Security Center, Japan
\textsuperscript{2}Nagoya University, Graduate School of Mathematics, Japan
\textsuperscript{3}CNRS & Lille University, CRIStAL, France
\textsuperscript{4}Nagoya University, Graduate School of Mathematics, Japan

Abstract

The algebraic properties of the combination of probabilistic choice and nondeterministic choice have long been a research topic in program semantics. This paper explains a formalization in the Coq proof assistant of a monad equipped with both choices: the geometrically convex monad. This formalization has an immediate application: it provides a model for a monad that implements a non-trivial interface which allows for proofs by equational reasoning using probabilistic and nondeterministic effects. We explain the technical choices we made to go from the literature to a complete Coq formalization, from which we identify reusable theories about mathematical structures such as convex spaces and concrete categories, and that we integrate in a framework for monadic equational reasoning.

1 Introduction

In their ICFP paper “Just do It: Simple Monadic Equational Reasoning” \cite{Gibbons2011}, the authors present an axiomatic approach to reason about programs with effects using equational reasoning, thus recovering one of the appeals of pure functional programming. This approach uses monads to encapsulate the effects, hence the name \textit{monadic equational reasoning}. In particular, to handle the effects of probability and nondeterminism, Gibbons and Hinze propose a combination of two interfaces: one for monads equipped with an operator for probabilistic choice and one for monads equipped with an operator for nondeterministic choice. It was later observed that in the proposed combination the authors “got [the algebraic properties that characterise their interaction] wrong” \cite{Abou-Saleh2016}. The problem was that right-distributivity of bind over probabilistic choice combined with distributivity of probabilistic choice over nondeterministic choice resulted in an inconsistent theory. Fortunately, the previous work in question \cite{Gibbons2011} was not relying on this mistake.

The example above shows that there is a need for a formal account of the consistency of such a theory. One way to achieve it is to construct a monad realizing the theory, which is, in our case, the combination of algebraic theories of probabilistic and nondeterministic choices. Monadic equational reasoning is not the only motivation to provide a formalized monad. Indeed, such a monad could be used to give semantics to programs mixing probabilities and nondeterminism (e.g., \cite{Kaminski2016}). The infrastructure needed to formalize such a monad could be used to formalize further foundational results in an area which is blooming (e.g., \cite{Bouchi2020a, Mio2020, Goy2020}).

In this paper, we provide a framework with which we formalize a monad with an interface representing the combined algebraic theory of probabilistic and nondeterministic choices;
we moreover verify the axiomatization of this theory and illustrate it with an example. While many sets of axioms have been suggested as axiomatizations of the combination of probabilistic and nondeterministic choice, only few give rise to interesting models [Mislove et al., 2004, Keimel and Plotkin, 2017]. We will stick here to Gibbons and Hinze’s axiomatization, removing just the incriminated right-distributivity. This gives us a trustful monad to reproduce Gibbons and Hinze’s examples of monadic equational reasoning.

We can rely on a large body of work to model formally the combination of probabilistic and nondeterministic choice (e.g., Mislove, 2000, Varacca and Winskel, 2006, Beaulieu, 2008, Tix et al., 2009, Gibbons, 2012, Keimel and Plotkin, 2017, Cheung, 2017, and much more if we consider concurrency). So what should be a monad modeling this axiomatization? Since we already have the finite powerset monad and the finitely-supported distributions monad for these two choices, one could think of composing them. At first sight, monadic distributive laws look like a candidate approach but unfortunately it has been proved that distributivity between these two monads is impossible [Varacca and Winskel, 2006, Proposition 3.2]. A very recent result [Goy and Petrisan, 2020] indicates that weak distributive laws provide a solution to this composition problem. A more direct approach is to rethink the construction of a model of the intended monad by looking into what it should be more precisely. The presence of probabilistic choice suggests that sets of distributions might be a model, like it is the case with the probability monad. Yet, the semantics must also be convex-closed because if two distributions \( d_1 \) and \( d_2 \) are possible outcomes, so is any convex combination \( p d_1 + (1 - p) d_2 \) \((0 \leq p \leq 1)\) of them [Gibbons, 2012, Sect. 5.2]. Convexity is in particular necessary to allow for idempotence of probabilistic choice. Unfortunately these observations do not readily lead to a formalization, as they leave many technical details unsettled. In his PhD thesis, Cheung derives a monad (called the geometrically convex monad) for the theory resulting from the combination of the effects of probability and nondeterminism [Cheung, 2017, Chapter 6]. It highlights in particular the central role of convex spaces [Stone, 1949, Jacobs, 2010, Fritz, 2015] to formalize convexity without resorting to vector spaces.

**Contributions** In this paper, we provide a construction of the geometrically convex monad that can be formalized by integrating reusable components (some obtained by adapting existing work and some created for this occasion). To the best of our knowledge, this is the first formalization of the monad that combines probabilistic and nondeterminism choices while retaining idempotence of probabilistic choice. It has been carried out in the Coq proof assistant [The Coq Development Team, 2019a]. This construction is original; in particular, we adapt the pencil-and-paper construction of Cheung [Cheung, 2017] to an infinitary setting using Beaulieu’s operator for infinite nondeterministic choice [Beaulieu, 2008, Def. 3.2.3]. We partly build on previous formalization work: theories of convex spaces [Affeldt et al., 2020a], interfaces for monadic equational reasoning, and finitely-supported probability distributions [Affeldt et al., 2019].

The new components that complete the construction of the geometrically convex monad are: a formalization of the (non-empty) convex powerset functor [Bonchi et al., 2017, Sect. 5.1] and affine functions (based on convex spaces), a formalization of semicomplete semilattice structures (related to Beaulieu’s work), and an original formalization of concrete categories. They are built in a reusable way following in particular the methodology of packed classes [Garillot et al., 2009].

We will discuss how our choices allow these distinct formalizations to fit together. All these formal libraries are now available to tackle similar formalizations that are already numerous as explained above. Our formalization of the geometrically convex monad already has a direct application: it is used to complete an existing formalization of monadic equational reasoning called Monae [Affeldt et al., 2019]. The latter comes with concrete monads modeling several interfaces except the one that combines probabilistic and nondeterministic choices, because it is arguably more difficult than the others. Our work improves the trusted base of this practical tool by filling this hole.

**Paper Outline** In Sect. 2, we clarify our formalization target by reviewing the formalization of monadic equational reasoning we aim at extending. We explain the operators of interest and
their properties, and we give an overview of the construction of the geometrically convex monad. In Sect. 3, we give an overview of a formalization of convex spaces, an important ingredient of our construction to represent probabilistic choice, convex sets, hulls, and affine functions. In Sect. 4, we explain the formalization of semicomplete semilattice structures, which provide an operator to represent a nondeterministic choice compatible with the probabilistic choice. In Sect. 5, we define several adjunctions, from which we derive the geometrically convex monad through composition. In Sect. 6, we verify that the geometrically convex monad can be equipped with the combined choice and that the latter enjoys the expected properties. Finally, we show that the monad we have formalized can be used to support monadic equational reasoning; we provide in Sect. 8 a complete mechanization of the Monty Hall problem as presented by Gibbons. We further comment on related work in Sect. 9 and conclude in Sect. 10.

About Notations For the sake of clarity, we try to display the Coq source code as it is. However, to limit the amount of code, we often indicate the surrounding namespace using a comment instead of displaying the precise Coq constructs (most of the time, this means that the name of the surrounding Module appears as a comment for the reader to figure out the fully qualified names). To further ease reading, we perform some beautification using \LaTeX symbols instead of ASCII art. When there are too many details, we omit parts of the source code (and mark them as “...”) and instead provide a paraphrase and indicate to the reader where to look in the formalization. In the prose, we use as much as possible standard mathematical notations, sometimes augmented to avoid too much overloading (for example, we note \( f@X \) the direct image of the set \( X \) by \( f \) but \( F#(g) \) the application of the functor \( F \) to the morphism \( g \)).

About the Formalization This paper comes with a Coq formalization which is available online as open source software [Infotheo, 2020] [Monae, 2020].

2 Formalization Target and Approach

Our goal is to construct a monad that combines probabilistic and nondeterministic choices, as intended by Gibbons et al. [Gibbons and Hinze, 2011]. Here, we review an existing formalization in Coq of Gibbons et al.’s monads and their interfaces [Affeldt et al., 2019]: our formalization target is the model of the monad of type \texttt{altProbMonad}.

2.1 An Existing Hierarchy of Probability-related Monads

Figure 1 provides an excerpt of an existing hierarchy of effects formalized [Affeldt et al., 2019] in Coq that includes the ones by Gibbons et al. [Gibbons and Hinze, 2011] [Gibbons, 2012] (amended as suggested by Abou-Saleh et al. [Abou-Saleh et al., 2016]). The complete hierarchy can be found in the online development [Monae, 2020] file hierarchy.v.

We assume given two types functor and monad for endofunctors and monads on Coq’s Type universe. The type monad is equipped with a join operator Join and a unit operator Ret. In this section we rather use the bind operator, defined as \( m \gg f \equiv \text{Join}(M#(f) m) \) for the monad \( M \). The precise definitions of functor and monad are not relevant at this stage but can be found in related work [Affeldt et al., 2019 Sect. 2.1].

Note that these so-called “types” are actually data-structures that provide the same functionality as type classes in Agda [The Agda Team, 2020] or Idris [Brady, 2013], i.e., providing an implementation for such a type amounts to defining an instance of the corresponding type class. Moreover, thanks to implicit coercions, this implementation itself can be used as a type, so that assuming \( M : \text{monad} \) allows one to write the type \( M \circ \tau \) of computations resulting in a value of type \( \tau \) inside the monad \( M \). The other nodes represent various monad types, that extend monad through the incremental additions of mixins, using the methodology of packed classes [Garillot et al., 2009].
We first extend the type monad into the type of the probability monad probMonad. The interface of probMonad takes the form of a mixin that introduces an operator for probabilistic choice $a \triangleleft p \triangleright b$, where $a$ and $b$ are computations and $p$ is a probability, i.e., a real number $p$ such that $0 \leq p \leq 1$. The intuition is that the computation $a \triangleleft p \triangleright b$ represents the computation $a$ with probability $p$ or the computation $b$ with probability $1 - p$. The properties, or axioms, of the interface are identity axioms (lines 5 and 6), skewed commutativity (line 7), idempotence (line 8), quasi-associativity (line 9), and the fact that bind left-distributes over probabilistic choice.

In Coq, the type prob is for probabilities. The notation $\%:pr$ turns a real number into a probability when possible. The notation $p.~$ is for $1 - p$ (often written $\overline{p}$ on paper). Skewed commutativity allows to derive one of the identity axioms from the other; here we are just preserving the original interface from Gibbons and Hinze [Gibbons and Hinze, 2011].

The monad type probDrMonad extends probMonad with right-distributivity of bind over probabilistic choice. We do not display its implementation because we do not model this monad in this paper; we mention it for the sake of completeness.

The monad type altMonad introduces an operator $\square$ for nondeterministic choice. Besides associativity of nondeterministic choice (line 17 below), it also states that bind left-distributes over nondeterministic choice (line 18), as specified by the following mixin:

---

\[\text{Gibbons and Hinze actually call “choice” and use the identifier alt for what we call nondeterministic choice; they call nondeterministic choice a combination of choice and failure [Gibbons and Hinze, 2011 Sect. 4.3].}\]
Gibbons and Hinze do not require right-distributivity (i.e., \( m \gg (\lambda x.k_1 x \triangle k_2 x) = (m \gg k_1) \triangle (m \gg k_2) \)) by default, due in particular to undesirable interactions with non-idempotent effects [Gibbons and Hinze, 2011, Sect. 4.2].

The monad type \texttt{altCIMonad} extends \texttt{altMonad} with commutativity and idempotence of non-deterministic choice, as expressed by the following mixin, where \texttt{op} \( x y \) stands for \( x \triangle y \):

\begin{verbatim}
(* Module MonadAltCI. *)
Record mixin_of (M : Type → Type) (op : ∀ {T}, M T → M T → M T) : UU1 :=
  Mixin { _ : ∀ T (x : M T), op x x = x ;
  _ : ∀ T (x y : M T), op x y = op y x }.
\end{verbatim}

Finally, in the monad type \texttt{altProbMonad}, probabilistic choice distributes over nondeterministic choice is expressed by another mixin, where \texttt{op} \( p x y \) is intended to denote \( x \triangleright p \triangleright y \):

\begin{verbatim}
(* Module MonadAltProb. *)
Record mixin_of (M : altCIMonad) (op : prob → ∀ {T}, M T → M T → M T) :=
  Mixin { _ : ∀ T p (x y z : M T), op p x (y \triangle z) = op p x y \triangle op p x z }.
\end{verbatim}

**Implementation of Inheritance Relations with Packed Classes**

Up to now, we have only shown the mixin part of the inheritance hierarchy. The packed class methodology [Garillot et al., 2009] actually contains three ingredients: mixins, classes, and structures. For example, here are the class and structure definitions for \texttt{altProbMonad}.

\begin{verbatim}
27 (* Module MonadAltProb. *)
28 Record class_of (m : Type → Type) := Class {
29  base : MonadAltCI.class_of m ;
30  mixin_prob : MonadProb.mixin_of
31 (Monad.Pack (MonadAlt.base (MonadAltCI.base base))) ;
32  mixin_altProb : @mixin_of
33 (MonadAltCI.Pack base) (@MonadProb.choice _ mixin_prob) }.
34 Structure altProbMonad : Type := Pack {
35  m :> Type → Type ; class : class_of m }.
\end{verbatim}

(In the code above, the modifier @ disables implicit arguments and the type declaration :> turns the corresponding structure field into a coercion.) The class definition inherits from \texttt{altCIMonad} through its class (line 24), and extends it with two mixins: the one we have seen for \texttt{probMonad} (line 30) and the additional distributivity axiom we have just defined (line 32). The structure (line 34) then packages together the type constructor \( m \) with the class defined above. Finally, the triple mixin-class-structure is completed with additional coercions and unification hints (provided by the \texttt{Canonical} command [Mahboubi and Tassi, 2013] of COQ) to achieve the inheritance relations depicted in Fig. 1.

**Sample Programs and Proof**

Last, for the sake of illustration, we reproduce sample programs by Gibbons [Gibbons, 2012, Sect. 5.1] and a simple proof by monadic equational reasoning using the operators we have introduced so far in the syntax of \texttt{MONAE}. Here is a biased coin, with probability \( p \) of returning \texttt{true} and probability \( \overline{p} \) of returning \texttt{false}:

\begin{verbatim}
Definition bcoin {M : probMonad} (p : prob) : M bool :=
  Ret true ◄ p ▷ Ret false.
\end{verbatim}

Here is an arbitrary nondeterministic choice between Booleans:

\begin{verbatim}
Definition arb {M : altMonad} : M bool :=
  Ret true □ Ret false.
\end{verbatim}

Using the \texttt{do} notation instead of the bind operator, these two programs can be used to make a probabilistic choice followed by a an arbitrary choice:

\begin{verbatim}
Definition coinarb p : M bool :=
  (do c ← bcoin p ; (do a ← arb; Ret (a == c) : M _) )%Do.
\end{verbatim}
Using Monae, one can prove that coinarb and arb are actually the same program by means of mere rewritings:

**Lemma** coinarb_spec p : coinarb p = arb.

**Proof.**

rewrite /coinarb /bcoin prob_bindDl !bindretf.
by rewrite /arb !alt_bindDl !bindretf eqxx eq_sym altC choicemm.
Qed.

Each lemma corresponds to an axiom from an interface. In order, prod_bindDl corresponds to left-distributivity of bind over probabilistic choice, bindretf corresponds to the fact that Ret is the left neutral of bind, alt_bindDl corresponds to left-distributivity of bind over nondeterministic choice, altC corresponds to the commutativity of nondeterministic choice, and choicemm to the idempotence of probabilistic choice. Other rewrite invocations are to unfold definitions or applying properties of equality. See Sect. 8 for a larger example, and related work for more proofs by monadic equational reasoning [Gibbons and Hinze, 2011, Gibbons, 2012, Mu, 2019a, Mu, 2019b] and their mechanization [Affeldt et al., 2019].

### 2.2 Alternative Axiomatizations

As we mentioned in the introduction, the axiomatization of combined choice we have followed is not the only possible one. We will consider shortly two other possible axiomatizations for which non-trival models are known.

The first one is obtained by replacing the distributivity axiom added in altProbMonad by the distributivity of nondeterministic choice over probabilistic choice:

\[ \square (x \triangleleft p \triangleright z) = (x \triangleleft p \triangleright y) \square (x \triangleleft p \triangleright z). \]

Keimel et al. [Keimel and Plotkin, 2017] have shown that in this case probabilities different from 0 and 1 become indistinguishable (i.e., \( x \triangleleft p \triangleright y = x \triangleleft q \triangleright y \) for any \( 0 < p, q < 1 \)). The algebraic theory of combined choice then boils down to a bisemilattice (two semilattices with their operators mutually distributing over each other). This is equivalent to having both distributivity laws. While it can be modeled by a powerset monad, the structure is poor, as probability information is lost, so we did not try to formalize this axiomatization. Another way to reach the same axiomatization is to inherit from probDrMonad rather than probMonad [Abou-Saleh et al., 2016, Sect. 3]. It appears that, while left-distributivity of bind over probabilistic choice is fine alone, right-distributivity can be used to deduce the distributivity of nondeterministic choice over probabilistic choice from its dual, which leads to the same collapse of probability information as above.

The second one is obtained by keeping the same distributivity axiom as in altProbMonad, but removing the idempotence of probabilistic choice from probMonad, i.e., we lose the equality \( x \triangleleft p \triangleright x = x \). Varacca [Varacca and Winskel, 2006] has shown that this relaxed probMonad can be modeled by a monad of real quasi-cones, which distributes over the finite powerset monad modeling altCIMonad. As a result, one can use Beck’s construction [Beck, 1969] to create a monad combining both. While this is a clever approach, the loss of the idempotence axiom can be problematic depending on the application, and Varacca presents in the same paper another construction using a convex powerset functor to obtain a model including the idempotence axiom, in a way similar to the geometrically convex powerdomain [Mislove, 2000, Tix et al., 2009].

Ultimately, the only way to be sure that our choice of axioms follows our expectations, is to provide a model where we can check that different computations can be properly distinguished (which we will do with the geometrically convex monad in Sect. 7.2).

### 2.3 Formalization of the Geometrically Convex Monad: Overview

As already hinted at in the introduction (Sect. 1), a computation using the monadic operations defined in the type altProbMonad can be modeled by a non-empty convex set of finitely-supported probability distributions. Cheung provides a construction for such a monad and calls the resulting
monad the geometrically convex monad [Cheung, 2017, Chapter 6]. It is built by composition of adjunctions, as depicted in Fig. [2]. The latter depicts three categories related by two adjunctions. The category \( \text{Mod}^{(\text{PROB})} \) corresponds to spaces with a convexity operator (for probabilistic choice) and the category \( \text{Mod}^{(\text{PROB} \triangleright \text{NDET})} \) corresponds to spaces with a convexity operator and a binary operator (for nondeterministic choice). The geometrically convex monad results from the composed adjunction \( F_1 \circ F_0 \dashv U_0 \circ U_1 \). We can derive the monad \( U_0 \circ U_1 \circ F_1 \circ F_0 \) directly from this adjunction [Mac Lane, 1998].

![Figure 2: Cheung’s original diagram of adjunctions [Cheung, 2017 Fig. 6.1]](image)

Now that we have given an overview of the construction of the geometrically convex monad, let us take a step back to think ahead what we need to achieve its formalization. First, we need a formalization of convex spaces. This work has actually started independently [Affeldt et al., 2020a, Affeldt et al., 2020b] and provides a formalization of convex spaces that can be easily reused (among others, it develops a theory of convex functions). Second, we need a formalization of probability distributions that can be used as an instance of convex spaces and that can be used to form the probability monad. Such a formalization happens to be available in the form of a theory of finitely-supported probability distributions [Affeldt et al., 2019], which comes as an enhancement of a theory of finite probability distributions [Affeldt et al., 2014] which could not be used to build a genuine monad because their type is not an endofunction. Third, we can draw inspiration from our previous work on formalizing monadic equational reasoning [Monae, 2020]. This work contains in particular a formalization of the basic elements of Cheung’s construction (functors, adjunctions, monads, etc.) in the specialized setting of the category \( \text{Set} \). Our experience with this work led us more precisely to the following technical insights: (1) packed classes are a satisfactory approach to formalize the needed mathematical structures, (2) affine functions can be accommodated to act as morphisms provided one uses concrete categories to generalize from the specialized setting using the category \( \text{Set} \). The very last bit of the story was to understand precisely the proofs by Cheung’s to realize that an infinitary operator for the representation of the nondeterministic choice was called for (namely, Beaulieu’s operator already mentioned in Sect. 1).

We are now ready to recast Cheung’s definition into the Coq formalization we will explain in this paper. Figure 3 depicts four concrete categories related by three adjunctions. Each category is named after a Coq type to which it corresponds. The category \( \mathcal{C}_T \) corresponds to Coq’s type \( \text{Type} \). The latter actually represents a countably infinite hierarchy of types \( \text{Type}_0, \text{Type}_1, \text{etc.} \) such that \( \text{Type}_i \) is a subtype of \( \text{Type}_{i+1} \). By default, Coq hides the indices to the user. We can regard \( \text{Type} \) as

![Figure 3: Adjunctions between the categories involved in the construction of the geometrically convex monad](image)
a category by seeing each Type as a Grothendieck universe \citep{Timany:2016}. The category \( \mathcal{C} \) corresponds to types satisfying the axiom of choice (i.e., equipped with a choice function). The type \texttt{choiceType} \citep[Sec. 3.1]{Garillot:2009} comes from the Mathematical Components library (hereafter, MATHCOMP \citep{MathComp:2007}). The category \( \mathcal{C}_V \) corresponds to \texttt{Mod}(\texttt{PROB}) and the category \( \mathcal{C}_S \) corresponds to a subcategory of \texttt{Mod}(\texttt{PROB}+\texttt{NDET}) with an infinitary operator for nondeterministic choice instead of a binary one; the details of these two categories are one of the purposes of this paper. The three adjunctions are composed of six functors. The unit and counit of \( F_C \circ U_C \) are \( \eta_C \) and \( \varepsilon_C \) respectively (resp. \( \eta_0, \varepsilon_0 \) for \( F_0 \circ U_0 \) and \( \eta_1, \varepsilon_1 \) for \( F_1 \circ U_1 \)). In particular, \( U_C, U_0, \) and \( U_1 \) are forgetful functors, which makes \( F_C, F_0, \) and \( F_1 \) free functors. The desired monad \( P_\Delta = P_\Delta^\text{right} \circ P_\Delta^\text{left} \) is obtained by composing adjunctions:

\[
P_\Delta^\text{left} = F_1 \circ F_0 \circ F_C \circ U_C \circ U_0 \circ U_1 = P_\Delta^\text{right}.
\]

Our setting features three adjunctions while Cheung’s has only two. The additional adjunction is the one between \texttt{Type} and \texttt{choiceType}. It comes from the fact that the formalization of monadic equational reasoning we build upon \citep{Affeldt:2019} represents monads as endofunctors over \texttt{Type}, whereas our construction requires types to be equipped with a choice function\footnote{Actually, these choice functions are not used in our development itself, but \texttt{choiceType}s are required due to our use of the \texttt{FINMAP} library \citep{Cohen:2010} (which builds upon MATHCOMP). See Sect. \ref{sec:convexity} for more details.}. In practice, the functor \( F_C \) only amounts to adding a choice function to the type, without changing the values.

Note that, since we assume the existence of such a choice function for all types, we are actually adding the axiom of choice to the ambient logic, which is known to be sound in Coq \citep{TheCoqDevelopmentTeam:2019}. It is simpler to assume a well known axiom than to try to define all our monads on \texttt{choiceType}, and prove that all the types we use can actually be equipped with a concrete choice function.

## 3 Convexity Toolbox

The formalization of the geometrically convex monad naturally calls for a formal theory of convexity. As alluded to in Sect. \ref{sec:convexity} it can be used to represent the probabilistic choice, convex spaces (needed for the categories \( \mathcal{C}_V \) and \( \mathcal{C}_S \)), non-empty convex sets (to represent computations in a monad modeling \texttt{altProbMonad}), convex hulls (to represent nondeterminism), and also to represent the morphisms of the categories \( \mathcal{C}_V \) and \( \mathcal{C}_S \) (these morphisms are affine functions) and the (non-empty) convex powerset functor \( F_1 \). For that purpose, we extend an existing formalization of convex spaces \citep{Affeldt:2020}.

We recall our formalization of convex spaces in Sect. \ref{sec:convexity} its axiom system leads to a formalization of convex sets and convex hulls, as explained in Sect. \ref{sec:convexhull}. We extend this formalization with affine functions and their properties in Sect. \ref{sec:convexity}. The most relevant file of the online development for this section is \citep{Infotheo:2020} file \texttt{convex_choice.v}.

### 3.1 Formalization of Convex Spaces

A convex space (a.k.a. barycentric calculus \citep{Stone:1949}) is an algebraic structure allowing convex combinations of its elements by an operator satisfying several equational axioms. The interface is in fact similar to the interface of the \texttt{probMonad} we saw in Sect. \ref{sec:probMonad}. It provides an operator \( a \triangleright p \triangleright b \) where \( a \) and \( b \) are elements of the convex space and \( p \) is a probability. The axioms about the operator are similar to the ones already explained in Sect. \ref{sec:probMonad} (the reader can observe a difference of presentation for the axiom of quasi-associativity but it is not relevant). Of course, contrary to \texttt{probMonad}, convex spaces have no axiom about a bind operator.

\[
(* \ Module \ ConvexSpace. \ *)
\]

\[
\text{Record \ \\[mixin\ of\ (T : \texttt{choiceType}) : \texttt{Type} := \text{Class} \{}
\quad \text{conv : \texttt{prob} \to T \to T \to T \text{ where } "a \triangleright p \triangleright b" := \text{(conv p a b)};}
\}
\]
∀ a b, a ▷ 1 %: pr ▷ b = a ;
_ : ∀ p a b, a ▷ p ▷ b = b ▷ p .¬ %: pr ▷ a ;
_ : ∀ (p q : prob) (a b c : T), a ▷ p ▷ (b ▷ q ▷ c) = (a ▷ [r_of p, q] ▷ b) ▷ [s_of p, q] ▷ c }.

The notation [s_of p, q] stands for ¬ p ¬ q; the notation [r_of p, q] stands for p /slash.left ¬ p ¬ q. Here we assume the carrier type of convex spaces to be a choiceType.

The above mixin is used to define the type convType using the packed classes methodology (that we briefly overviewed in Sect. 2.1).

We can show for example that the real numbers form a convex space by taking the averaging function λp x y. px + ¬ py to be the operator. Similarly, finitely-supported probability distributions form a convex space with the operator λp d_1 d_2. pd_1 + ¬ pd_2 where d_1 and d_2 are distributions.

We will later need a generalization of the binary operator a ▷ p ▷ b to n points, namely ◄d f, where f consists of n points and d is a distribution of n probabilities.

### 3.2 Convex Sets and Convex Hulls

We use convex spaces to define convex sets and convex hulls. As already said in Sect. 2.3, we put ourselves in a classical setting that extends the logic of COQ with a number of axioms known to be compatible with it. Concretely, we use the axioms provided by MathCOMP-Analysys, an extension of MathCOMP for classical analysis [Affeldt et al., 2018]. In this setting, Prop and bool are equivalent (strong excluded middle), and we can freely embed Prop-valued formulas such as ∀ x, P x into bool using a notation: ‘[< ∀ x, P x>] : bool. From MathCOMP-Analysys, we also reuse a library of “sets”. Here “sets” means “sets of elements of a specific type”. They are represented by Prop-valued characteristic functions, and thus not necessarily finite. The type set A stands for sets over the type A.

A set D is convex when any convex combination of any two points is still inside D:

Variable A : convType.

Definition is_convex_set (D : set A) : bool :=
‘ [< ∀ x y t, D x → D y → D (x ▷ t ▷ y)>].

The hull of a set X is the set of points p such that p is the convex combination of points belonging to X. The notation [set p : T | P p] is for sets defined by comprehension.

Definition hull (T : convType) (X : set T) : set T :=
[set p : T | ∃ n (g : 'I_n → T) d, g 0' setT ⊆ X ∧ p = ◄d g].

We represent the n points to be combined as g_0, g_1, ..., hence the function g : 'I_n → T from 'I_n, the MathCOMP type of natural numbers smaller than n. The notation g 0' setT is for the direct image g@ (setT) where setT is the full set (these are part of the library of sets that comes with the MathCOMP-Analysys library).

### 3.3 Affine Functions

We are interested in affine functions because they are used for the morphisms of the categories C_V and C_S (Sect. 2.3). For example, in real analysis, affine functions correspond to the functions of the form x → ax + b. But the real line is just one example of convex space. In fact, the generic operator of convex spaces provides an easy, generic definition. First, we introduce a predicate that applies to a function f, a pair of points x and y, and a probability t:

Variables (T U : convType).

Definition affine_function_at (f : T → U) x y t :=
f (x ◄ t ▷ y) = f x ◄ t ▷ f y.

We use this predicate to characterize affine functions by a type for COQ functions packaged with the following axiom:
(* Module AffineFunction. *)

Variables (U V : convType).

Definition axiom (f : U → V) := ∀ x y t, affine_function_at f x y t.

This packaging is done in such a way that affine functions can be used as ordinary functions; see the online development [Infotheo, 2020] for details. Hereafter, the type of affine functions from the convex space U to the convex space V is denoted by \{affine U → V\}.

As a sample proposition, we can observe that convex hulls are preserved by affine functions:

Proposition image_preserves_convex_hull (f : \{affine T → U\}) (Z : set T) :
  f ∩ (hull Z) = hull (f ∩ Z).

This property will be used to define the functor \(F_1\), whose action on morphisms defined by the direct image needs to preserve convex hulls.

4 Semicomplete Semilattice Structures

In this section, we define generic structures that provide an operator to represent nondeterministic choice in a way that is compatible with probabilistic choice. The most relevant file in the online development is [Infotheo, 2020, file neset.v].

As a prerequisite, we introduce the type of non-empty sets. The type neset T is the type of sets over T that have at least one element. As a convenience, this type comes with a postfix notation \%:ne such that \%:ne s is the non-empty set corresponding to the set s. This notation infers the proof of non-emptiness in several situations such as when s is a singleton set, the image of a non-empty set, the union of non-empty sets, etc. using Coq’s canonical structures [Mahboubi and Tassi, 2013].

4.1 Semicomplete Semilattice

The first structure we introduce provides a unary operator \(\op\) that turns a non-empty set of elements into a single element (line 66). The first axiom of this structure says that this operator applied to a singleton set returns the sole element of the set (line 67). The second axiom starting at line 68 collapses a non-empty collection \(\big\{\op\\) of non-empty sets into one element:

\[
\big\{\op\ (\bigcup_i f_i) \%:ne = \op (\bigcup_i (\op @' (f_i @' s))) \%:ne \big\}.
\]

The theory defined by this mixin is similar to Beaulieu’s theory for infinite nondeterministic choice [Beaulieu, 2008, Def. 3.2.3]. The difference is that the right-hand side of the second axiom in Beaulieu’s work is expressed by means of a partition of the indexing set. We prefer to avoid partitions because in our experience they cause technical difficulties in formal proofs.

Hereafter we denote by \(\big\) the operator introduced by the above mixin and use the mixin to define the type semiCompSemiLattType of semicomplete semilattices [Bergman, 2015, p. 185].

4.2 Combining Semicomplete Semilattice with Convex Space

We now extend the structure of semicomplete semilattices from the previous section (Sect. 4.1) with an axiom that captures the interaction between the operator \(\big\) and probabilistic choice. This interaction is akin to a distribution law that can be stated informally as follows:

\[
x ∩ p \big I = \big((λ y. x ∩ p \big y) @ (I))
\]
Formally, this axiom is provided as a mixin parameterized by a semicomplete semilattice and a ternary operator $\text{op}$ indexed by a probability:

```coq
(* Module SemiCompSemiLattConvType. *)
Record mixin_of (L : semiCompSemiLattType) (op : prob → L → L → L) :=
  Mixin { _ : ∀ (p : prob) (x : L) (I : neset L),
    op p x (⊔ I) = ⊔ ((op p x) ⊎ I) }.
```

We use this mixin to extend the type of semicomplete semilattices to the type of semicomplete semilattice convex spaces (semiCompSemiLattConvType in Coq scripts) that inherits both the properties of semicomplete semilattices (Sect. 4.1) and the properties of convex spaces (Sect. 3.1). The methodology to achieve this multiple inheritance is again the one of packed classes.

We conclude this section with a sample property of the operator $\bigcup$ that is both important and non-trivial:

```coq
Variable L : semiCompSemiLattConvType.
Lemma lub_op_hull (X : neset L) :
  $\bigcup$ (hull X)%:ne = $\bigcup$ X.
```

The proof is as follows. First, we lift the operator of convex spaces ($\langle p \triangleright \rangle$) from points to sets of points; we denote this lifted operator by ($\langle p \triangleright \rangle$). We use this lifted operator to define a new binary operator $X :\triangleright: Y := \bigsqcup_{p \in [0,1]} X :\triangleright: Y$. Second, we show that $\text{hull } X = \bigsqcup X :\triangleright: X :\triangleright: \cdots :\triangleright: X$.

Then, we show that $\bigcup(X) = \bigcup(X :\triangleright: X :\triangleright: \cdots :\triangleright: X)$, using the property introduced by semicomplete semilattice convex spaces. Finally, we conclude the proof by appealing to the properties of semicomplete semilattices.

We will later provide a concrete example of use of the lemma `lub_op_hull`. It can also be used to establish technical results from Beaulieu’s work (e.g., [Beaulieu, 2008, p. 56, l. 3]) or similar ones as in Varacca and Winskel’s work (e.g., [Varacca and Winskel, 2006, Lemma 5.6]).

### 4.3 Instances with Non-empty Convex Sets

The definitions of semicomplete semilattices and of semicomplete semilattice convex spaces that we have provided in the previous sections are just interfaces. To instantiate them, it turns out that it suffices to use non-empty convex sets instead of mere non-empty sets. This is this instance that we will use in particular to produce the adjunction $F_1 \dashv U_1$ (Fig. 3).

Thus we start by extending the type `neset` of non-empty sets into the type `necset` of non-empty convex sets, using the definition from the Sect. 3.2 (and again the methodology of packed classes).

We then instantiate the semicomplete semilattice operator on non-empty convex sets using union and hull operators ($A$ below is a convex space):

$$
\bigcup : \text{neset} (\text{necset } A) \rightarrow \text{necset } A
$$

$$
X \mapsto \text{hull} (\bigcup_{x \in X} x)
$$

This gives us in particular the type `necset_semiCompSemiLattConvType A` a generic instance of `semiCompSemiLattConvType` where the carrier consists of non-empty convex sets over a `convType A`. We will use this type as the object part of $F_1 : C_V \rightarrow C_S$.

The structures and instances explained in this section can be summarized as the hierarchy pictured in Fig. 4.

```
semiCompSemiLattType

semiCompSemiLattConvType

necset_semiCompSemiLattConvType

convType

Figure 4: Hierarchy of semicomplete semilattices structures (dashed lines are for instances).
```
5 Formalization of Category Theory based on Concrete Categories

The purpose of this section is to provide a formalization of enough category theory to construct the geometrically convex monad. This formalization is interesting in itself because it features an original use of concrete categories through their shallow embedding. It also fits our application because it comes as a conservative extension of Monae [Affeldt et al., 2019]. The most relevant file in the accompanying development is [Monae, 2020, file category.v].

5.1 Formalization based on Concrete Categories

5.1.1 Shallow Embedding of Concrete Categories

As we saw in Sect. 2.3, we need to formalize several categories to formalize the geometrically convex monad; this is in contrast with Monae, which could get along on the sole category of sets Set. Among the various possibilities, we chose to favor a definition akin to a shallow embedding: it lets us use the typing relation of Coq to declare elements of an object and apply morphisms to them as if morphisms were ordinary Coq functions. The starting idea is to represent categories with a universe à la Tarski, i.e., a type with an interpretation operation, or realizer, allowing us to regard terms of this type as Types (the function el below at line 78). In this setting, we can then look at the morphisms of a category through the realizer and identify the set of morphisms between two objects as a subset of the function space between two realized objects (via the predicate defining the hom-set at line 79).

76 (* Module Category. *)
77 Record mixin_of (obj : Type) : Type := Mixin {
78   el : obj → Type ; (* interpretation operation, "realizer" *)
79   inhom : ∀ A B, (el A → el B) → Prop ; (* subset of morphisms * )
80   _ : ∀ A, @inhom A A idfun ; (* idfun is in inhom * )
81   _ : ∀ A B C (f : el A → el B) (g : el B → el C),
82   inhom f → inhom g → inhom (g ° f) (* inhom is closed by composition * )}.
84 Structure type : Type := Pack {
85   carrier : Type ; class : mixin_of carrier }.

This definition has two salient features. First, the parameter obj lets us choose how we index our objects and use those indices to declare morphisms (e.g., A and B in f : el A → el B). Second, we can use morphisms as functions and apply them to elements, as illustrated by the following script:

Variable C : category.
Variable A B : C.
Variable x : el A.
Variable f : {hom A, B}.
Check f x : el B.

Here, {hom A, B} is essentially the type of functions el A → el B equipped with a proof that they are morphisms (i.e., f such that inhom A B f holds); there is a coercion from {hom A, B} to el A → el B. Last, observe that, thanks to the shallow embedding, the laws of units and composition are unnecessary because they are valid definitionally.

The resulting encoding is by no way ad hoc: it actually corresponds to a shallow embedding of concrete categories. A category C is said to be concrete if it comes with a faithful functor from C to Set, that is, a functor whose action on each hom-set is injective. The indexing type obj and the realizer el together form the object part. The function inhom represents the hom-sets of C by their images. For the category-savy, the following diagram explains how the morphism part $F_{Mor}$
of the faithful functor $F$ is represented through its image in the hom-sets of $\textbf{Set}$.

\[
\begin{array}{ccc}
\text{Im}(F_{\text{Mor}} |_{C(A,B)}) & \xrightarrow{!} & * \\
C(A,B) & \xrightarrow{F_{\text{Mor}} |_{C(A,B)}} & \textbf{Set}(F(A), F(B)) \\
\downarrow_{\text{inhom } A\ B} & & \downarrow_{\text{Prop}}
\end{array}
\]

Let $C(A,B)$ be a hom-set of $C$, which is mapped by $F_{\text{Mor}}$ (restricted to $C(A,B)$) injectively into the corresponding hom-set $\textbf{Set}(F(A), F(B))$ of $\textbf{Set}$. Note that $\textbf{Set}(F(A), F(B))$ appears in the Coq code as the type $\text{el } A \rightarrow \text{el } B$. The triangle on the left is the image decomposition of $F_{\text{Mor}} |_{C(A,B)}$. The square on the right is a pullback diagram, with $\text{inhom } a\ b$ being the characteristic morphism of the image $\text{Im}(F_{\text{Mor}} |_{C(A,B)})$.

Except some hard examples (including homotopy categories), many abstract categories can be concretized, i.e., we can find some faithful functor from the category of $\textbf{Set}$ and rephrase it in our framework. The categories in this paper are concretized just by injections, this is also the case for slice categories. Other examples require some encoding of objects and morphisms (e.g., product categories).

### 5.1.2 Categories to Build the Geometrically Convex Monad

In this section, we instantiate our definition of concrete categories with the categories that were described in Sect. 2.3.

#### The Categories $C_T$ and $C_C$

We want to define the category $C_T$, i.e., the situation in which $C$ is $\textbf{Set}$, a.k.a. $\textbf{Type}$. We just need to instantiate $F_{\text{Mor}} |_{C}$ to the identity function and keep all the morphisms. Technically, this amounts to instantiate the mixin of the previous section with the identity function $\text{fun } x : \textbf{Type} \Rightarrow x$ as the realizer and the third argument of $\text{Category.Mixin}$ to be the true predicate $\text{fun } _\_ \_ \Rightarrow \text{True}$, so that the faithful functor for the concrete category is full (i.e., surjective on hom-sets):

\[
\text{Definition } \text{Type_category_mixin} : \text{Category.mixin_of } \textbf{Type} :\!
\begin{array}{l}
\text{@Category.Mixin } \textbf{Type} (\text{fun } x : \textbf{Type} \Rightarrow x) (\text{fun } _\_ \_ \Rightarrow \text{True}) \\
\text{(fun} \Rightarrow 1) (\text{fun } _\_ \_ \Rightarrow 1).
\end{array}
\]

\[
\text{Definition } \text{Type_category} := \text{Category.Pack Type_category_mixin}.
\]

(The identifier $1$ is a proof of $\text{True}$ in the standard library of Coq.)

Using this setting, we can now use the type $\textbf{Type}$ of Coq as if it were actually the category $C_T$. The very last ingredient is the declaration of $\text{Type_category}$ as a canonical instance of categories:

\[
\text{Variable } A : \textbf{Type}.
\]

\[
\text{Fail Variable } x : \text{el } A.
\]

\[
\text{Canonical Type_category}.
\]

\[
\text{Variable } x : \text{el } A.
\]

The command $\text{Canonical}$ (that we already mentioned for its use in the packed classes methodology in Sect. 2) provides a unification hint to Coq’s type-checker to automatically endow $\textbf{Type}$ with a structure of category when needed. The other instances of categories in this section are also made canonical but we only display the mixins which hold the relevant information.

Similarly to the category $C_T$, to define the category $C_C$, we take the function $\text{fun } x : \textbf{choiceType} \Rightarrow \textbf{Choice.sort} x$, that returns the carrier type (in $\textbf{Type}$) of its argument (we make $\textbf{Choice.sort}$ appear explicitly here but it is actually an implicit coercion in Coq). Again the faithful functor is full:

\[
\text{Definition } \text{choiceType_category_mixin} : \text{Category.mixin_of } \textbf{choiceType} := \\
\text{@Category.Mixin } \textbf{choiceType} (\text{fun } x : \textbf{choiceType} \Rightarrow \textbf{Choice.sort} x) \\
(\text{fun } _\_ \_ \Rightarrow \text{True}) (\text{fun} \Rightarrow 1) (\text{fun } _\_ \_ \Rightarrow 1).
\]
The Category of Convex Spaces $C_V$

The objects are convex spaces (Sect. 3.1) and the morphisms are affine functions (between convex spaces), which can be enforced by using the axiom from Sect. 3.3. In our formalization, the objects are indexed by the type of convex spaces $\text{convType}$, and realized by its coercion into $\text{Type}$. Contrary to the previous two examples, being affine is not just a true predicate and requires us to prove that the identity function over a convex space is affine (proof $\text{affine_function_id_proof}$) and that the composition of affine functions is affine (proof $\text{affine_function_comp_proof}$):

```coq
Definition convType_category_mixin : Category.mixin_of convType :=
  @Category.Mixin convType (fun A => A) AffineFunction.axiom (* Sect. 3.3 *)
  affine_function_id_proof affine_function_comp_proof'.
```

The Category of Semicomplete Semilattice Convex Spaces $C_S$

The objects are semicomplete semilattice convex spaces (Sect. 4.2) and the morphisms are affine functions $f$ such that $f@\big(\bigcup X\big) = \bigcup(f@X)$ for any non-empty convex set $X$. We can show that identity functions are such functions (proof $\text{lub_op_affine_id_proof}$) and that composition preserves these properties (proof $\text{lub_op_affine_comp_proof}$), leading to the following definition of $C_S$:

```coq
Definition semiCompSemiLattConvType_category_mixin :
  Category.mixin_of semiCompSemiLattConvType :=
  @Category.Mixin semiCompSemiLattConvType (fun U => U) LubOpAffine.class_of
  lub_op_affine_id_proof lub_op_affine_comp_proof.
```

5.2 Formalization of Functors, Natural Transformations, and Monads

We now formalize functors, natural transformations, and monads using the concrete categories formalized in the previous section. In the following, $C$ and $D$ are two categories.

We encode a functor from $C$ to $D$ as an action on objects represented by a function $m : C \to D$ (line 112 below) and an action on morphisms represented by a function $f : \forall A B, \{\text{hom } A, B\} \to \{\text{hom } m A, m B\}$ (line 113) equipped with proofs that $f$ preserves the identity (line 114) and composition (line 115):

```coq
Record mixin_of (C D : category) (m : C \to D) : Type :=
  Mixin { f : \forall (A B : Type), \{\text{hom } A, B\} \to \{\text{hom } m A, m B\} ;
  _ : FunctorLaws.id f ;
  _ : FunctorLaws.comp f }.
```

By way of comparison, functors in MONAE [Affeldt et al., 2019] were specialized to the category $\text{Set}$ of sets and functions (the type $\text{Type}$ of Coq being interpreted as the category $\text{Set}$):

```coq
Record mixin_of (m : Type \to Type) : Type :=
  Class { f : \forall (A B : Type), (A \to B) \to m A \to m B ;
  _ : FunctorLaws.id f ;
  _ : FunctorLaws.comp f }.
```

It is clear that the new, more general setting introduced above improves on this specialized setting because it makes it possible to talk about morphisms that are, e.g., affine functions. Hereafter, we denote by $F \circ g$ the application of a functor $F$ to a morphism $g$.

Let $F$ and $G$ be two functors from $C$ to $D$. We encode a natural transformation from $F$ to $G$ as a family of maps $f : \forall A, \{\text{hom } F A,G A\}$ (hereafter, denoted by $F \to G$) such the naturality predicate holds:

```coq
Definition naturality (f : F \to G) := \forall A B (h : \{\text{hom } A, B\}),
  (G \circ h) \circ f A = (f B) \circ (F \circ h).
```
When \( F \rightsquigarrow G \) is packaged together with a proof of naturality, we have a genuine natural transformation that we denote by \( F \rightsquigarrow G \) (mind the shorter arrow).

Finally, we define a monad as an endofunctor \( M \) equipped with two natural transformations: \( \text{ret} \) from the identify functor (denoted by \( F\text{Id} \)) to \( M \), and \( \text{join} \) from the composition of \( M \) with itself (denoted by \( M \circ M \)) to \( M \). The proofs of naturality appear at lines [127] and [128]. These two natural transformations furthermore satisfy three coherence conditions (lines [129], [130], and [131]):

```
(* Module Monad. *)
Record mixin_of (C : category) (M : functor C C) : Type := Mixin {
  ret : \forall A, \{\text{hom} A, M A\} ;
  join : \forall A, \{\text{hom} M (M A), M A\} ;
  _ : naturality F\text{Id} M ret ;
  _ : naturality (M \circ M) M join ;
  _ : \forall A, join A \circ \text{ret} (M A) = \text{id} ;
  _ : \forall A, join A \circ M # \text{ret} A = \text{id} ;
  _ : \forall A, join A \circ M # join A = join A \circ join (M A) }.
```

We already said above that our formalization of functors generalizes the one of MONAE, the formal framework for monadic equational reasoning on which our work is based. Our formalization of monads also generalizes the one of MONAE in a conservative way. Concretely, we provide a function \( \text{Monad\_of\_category\_monad} \) that given a monad (as defined just above) over the category \( C_T \), returns a monad as defined in MONAE (over \( \text{Type} \), regarded as the category \( \text{Set} \)). This way, it will be possible to (1) prove that our formalization of the geometrically convex monad satisfies the expected axioms and (2) retrofit it back into MONAE.

### 5.3 Formalization of Adjoint Functors

We use adjoint functors to build the geometrically convex monad. In this section, we recall the lemmas used for this construction and give a brief overview of their formalization. We do not provide all the technical details because these lemmas are well-known lemmas and their formalization follows naturally from the definitions we saw so far.

#### 5.3.1 Definition of Adjunction

Two functors \( F \) and \( G \) are adjoint (denoted by \( F \dashv G \)) when there are two natural transformations \( \eta : 1 \rightsquigarrow G \circ F \) and \( \varepsilon : F \circ G \rightsquigarrow 1 \) such that \( \eta \) and \( \varepsilon \) satisfy the triangular laws \( \forall c. \varepsilon(F c) \circ F # (\eta c) = \text{id} \) (triangular left) and \( \forall d. G # (\varepsilon d) \circ \eta(G d) = \text{id} \) (triangular right).

In Coq, we provide the notation \( F \dashv G \) for the following type (where the categories \( C \) and \( D \) are implicit arguments):

```
AdjointFunctors.t : \forall C D : category, functor C D \rightarrow functor D C \rightarrow Type
```

To build an adjunction, one needs to provide two natural transformations \( \text{eta} \) and \( \text{eps} \) together with the proofs that they satisfy the triangular laws. The corresponding constructor has the following type (where all arguments except the proofs of the triangular laws are implicit):

```
AdjointFunctors.mk : \forall (C D : category) (F : functor C D) (G : functor D C) (eta : F\text{Id} \rightarrow G \circ F) (eps : F \circ G \rightarrow F\text{Id}),
  TriangularLaws.left eta eps \rightarrow TriangularLaws.right eta eps \rightarrow F \dashv G
```

#### 5.3.2 Composition of Adjunction

It is well-known that two adjunctions \( F \dashv G \) (with unit/counit \( \eta/\varepsilon \)) and \( F' \dashv G' \) (with unit/counit \( \eta'/\varepsilon' \)) can be composed to form another adjunction \( F' \circ F \dashv G \circ G' \) by taking the unit to be \( \lambda A. G # (\eta'(F_A)) \circ \eta_A \) and the counit to be \( \lambda A. \varepsilon'_A \circ F' # (\varepsilon(G_A')) \). Using the constructs we have defined so far, we provide a Coq function that performs this composition:
adj_comp : \forall (C0 C1 C2 : category) (F : functor C0 C1) (G : functor C1 C0), F \dashv G \to
\forall (F' : functor C1 C2) (G' : functor C2 C1), F' \dashv G' \to
F' \circ F \dashv G \circ G'

5.3.3 Monad Defined by Adjointness

It is well-known that an adjunction \( F \dashv G \) gives rise to a monad \( G \circ F \) by taking \( \eta \) to be the unit and \( \lambda A. G \#(\varepsilon(F_A)) \) to be the join operator. In our formalization, this construction takes the form of the following function:

\[
\text{Monad_of_adjoint} : \forall (C D : \text{category}) (F : \text{functor } C D) (G : \text{functor } D C),
F \dashv G \to \text{monad } C
\]

Observe that contrary to \text{Monae} where all monads are over the category \text{Set}, here our monad is over some category \( C \) which appears explicitly in the type.

6 Adjoint Functors for the Geometrically Convex Monad

At this point, we have explained the formalization of all the elements necessary to construct the geometrically convex monad: convex spaces and affine functions in Sect. 3, semicomplete semilattice structures in Sect. 4, and category theory in Sect. 5. In this section, we explain the formalization of the adjunctions explained in Sect. 2.3. The most relevant file from the accompanying development is [Monae, 2020, file \text{gcm_model.v}].

6.1 The Adjunction \( F_C \dashv U_C \)

The raison d’être of the adjunction \( F_C \dashv U_C \) in our formalization is essentially technical: it comes from the use of the Coq type \text{Type} in \text{Monae} and the need to use a \text{choiceType} in the definition of finitely-supported distributions.

Let us first define the functor \( F_C \) from \( C_T \) to \( C_C \). The action on objects consists in turning a type in \text{Type} into a \text{choiceType}. This is performed by the function \text{choice_of_Type} which relies on an axiom inherited from a MATHCOMP library and whose validity is explained elsewhere [Affeldt et al., 2018, Sect. 5.2]. The action on morphisms turns a morphism \( f : T \to U \) into the same morphism but with type \text{choice_of_Type} T \to \text{choice_of_Type} U:

\[
\text{Definition } \text{hom_choiceType} (A B : \text{choiceType}) (f : A \to B) : \{\text{hom } A, B\} :=
\text{HomPack (I : InHom (f : el A \to el B))}.
\]

\text{Local Notation} \( C_T \) := \text{Type_category}.

\[
\text{Definition } \text{free_choiceType_mor} (T U : C_T) (f : \{\text{hom } T, U\}) :\{\text{hom } m T, m U\} := \text{hom_choiceType} (f : m T \to m U).
\]

The purpose of the function \text{hom_choiceType} is to turn a Coq function between two \text{choiceTypes} into a morphism of the category \( C_C \). Here, \text{I} (that we already saw in Sect. 5.1.2) acts as a trivial proof that \( f \) is indeed a morphism; it is sufficient because in this category all functions are morphisms (the notation \text{HomPack} is just a smart constructor [Monae, 2020]). The functor laws are trivially proved and together with the definitions above, this leads to the definition of the functor \text{free_choiceType} of type \text{functor } C_T \to C_C.

The definition of the corresponding forgetful functor \( U_C \) is similar. The main difference is that instead of using the function \text{choice_of_Type} to augment a type in \text{Type}, we use the coercion \text{Choice.sort} that retrieves the carrier type of a \text{choiceType} (see \text{forget_choiceType} in [Monae, 2020, file \text{gcm_model.v}]).

The unit \( \eta_C : 1 \to U_C \circ F_C \) and the counit \( \varepsilon \) : \( F_C \circ U_C \to 1 \) are also essentially identity functions and the proofs of the triangular laws are therefore trivial.
6.2 The Adjunction $F_0 \dashv U_0$

The second adjunction $F_0 \dashv U_0$ corresponds to the probability monad [Giry, 1982]. It relies on an existing formalization of finitely-supported distributions [Affeldt et al., 2019, Sect. 6.2] that we recall briefly. In the definition of $\textsf{FSDist.t}$ below, the first field (line 149) is a finitely-supported function $f$ from the $\textsf{choiceType}$ $A$ to the type of real numbers from the standard Coq library; this function evaluates to 0 outside its support $\textsf{finsupp f}$. The second (anonymous) field (line 150) contains proofs that (1) the probability function outputs positive reals and that (2) its outputs sum to 1.

It is important to observe that $\textsf{FSDist.t}$ has type $\textsf{choiceType} \to \textsf{choiceType}$ and can therefore be used to build an endofunctor and a monad on top of it. Hereafter, $\{$dist $A$\} \to$ $\textsf{FSDist.t A}$.

6.2.1 Functors

The action on morphisms of $F_0$ is the map of the probability monad associated with finitely-supported distributions. Indeed, let $\cdot \et \cdot \enskip \cdot$ be the operation of the convex space of finitely-supported distributions (see Sect. 3.1) and let $\gg = \gg$ be the bind operator of the probability monad. We have $(d_1 \et p \gg d_2) \gg f = (d_1 \gg f) \et p \gg (d_2 \gg f)$, which is equivalent to the map of the probability monad being affine.

In Coq, we define the action on morphisms of $F_0$ as follows, where $\textsf{FSDistfmap}$ is the map operation of the probability monad:

\[
\text{Definition free_convType_mor (A B : choiceType) (f : \{hom A, B\}) :}
\{\text{hom FSDist_convType A, FSDist_convType B}\} :=
\text{\@Hom.Pack C V _ _ _ (FSDistfmap f) (FSDistfmap_affine f)}.
\]

The type $\textsf{FSDist.convType A}$ is the type of convex spaces of finitely-supported distributions over $A$ and $\textsf{FSDistfmap_affine}$ is the proof that $\textsf{FSDistfmap f}$ is affine.

We can show that $\textsf{free_convType_mor}$ satisfies the functor laws (proofs $\textsf{free_convType_mor_id}$ and $\textsf{free_convType_mor_comp}$), leading to the definition of the functor $F_0$ (recall the definitions of Sect. 5.2):

\[
\text{Definition free_convType : functor C V C C} :=
\text{Functor.Pack (Functor.Mixin free_convType_mor_id free_convType_mor_comp)}.
\]

The constructors $\textsf{Functor.Mixin}$ and $\textsf{Functor.Pack}$ are respectively for the mixin and the type of functors explained in Sect. 5.2.

The forgetful functor $U_0$ of type $\textsf{functor C V C C}$ is just formalized by substituting the category $\textsf{C V}$ by the category $\textsf{C C}$ in morphisms (see $\textsf{forget_convType}$ in [Monae, 2020, file gcm_model.v]).

6.2.2 Counit / unit

The counit is the natural transformation $\varepsilon_0 : F_0 \circ U_0 \Rightarrow 1_{\textsf{C V}}$ essentially defined by the following function:

\[
\varepsilon_0 : \{\text{dist C}\} \to C
\]

\[
d \mapsto \langle \bullet \rangle_d \textsf{finsupp} (d).
\]

In this definition, $C$ is a $\textsf{convType}$; the operation “$\langle \bullet \rangle_d$” has been explained in Sect. 3.1. Intuitively, $\varepsilon_0$ corresponds to the computation of a barycenter.

The unit is the natural transformation $\eta_0 : 1_{\textsf{C C}} \Rightarrow U_0 \circ F_0$ defined by the point-supported distribution $\textsf{FSDist1.d}$:

\[
\eta_0 : C \Rightarrow \{\text{dist C}\}
\]

\[
x \mapsto \textsf{FSDist1.d} x.
\]
The proofs of the triangular laws required us to substantially enrich the theory of finitely-supported distributions used in Monae. The reason can be understood by looking at the proof of the left triangular law $\text{triL}0$. The latter essentially amounts to prove that we have for any probability distribution $d$: 

$$\left(\text{FSDistfmap} \text{FSDist1.} \cdot \text{d} \text{finsupp}(\text{FSDistfmap} \text{FSDist1.} \cdot \text{d}) \right) = \text{d}.$$ 

One can observe that this statement involves distributions of distributions 

\text{Check} \text{FSDistfmap (@FSDist1. C) \cdot d : \{dist \{dist C\}}). 

whose properties called for new lemmas. Comparatively, the proof of the right triangular law $\text{triR}0$ is simpler.

### 6.3 The Adjunction $\mathcal{F}_1 \dashv \mathcal{U}_1$

The third adjunction $\mathcal{F}_1 \dashv \mathcal{U}_1$ corresponds to the nondeterminism part of the geometrically convex monad, giving a nondeterminism monad over the category $\mathcal{C}_V$ of convex spaces. It consists of the (non-empty) convex powerset functor $\mathcal{F}_1$ and a corresponding forgetful functor $\mathcal{U}_1$.

#### 6.3.1 Functors

The action on objects of $\mathcal{F}_1$ is $\text{neset\_semiComp\_SemiLatt\_ConvType}$, explained in Sect. 4.3. The action on morphisms of $\mathcal{F}_1$ is defined by the direct image $f \mapsto \lambda X. f \theta(X)$ (where $X$ is a non-empty convex set):

- **Variables** $(A B : \text{convType})$ (f : {hom A, B}).
- **Definition** \text{free\_semiComp\_SemiLatt\_ConvType\_mor’} 
  - $(X : \text{neset\_convType A}) : \text{neset\_convType B} := \text{NECSet.Pack}$ (* definition using the direct image omitted *).

We can show that the image of a morphism is still a morphism: it is affine and preserves $\sqcup$ (because convex hulls are preserved by taking the direct image along affine functions—Sect. 3.3):

- **Definition** \text{free\_semiComp\_SemiLatt\_ConvType\_mor} :
  - $\{\text{hom \text{neset\_semiComp\_SemiLatt\_ConvType} A,} \text{neset\_semiComp\_SemiLatt\_ConvType B}\} := \text{@Hom.Pack C_S \_ \_ \_ free\_semiComp\_SemiLatt\_ConvType\_mor’}$
    - $\text{LubUpAffine.Class free\_semiComp\_SemiLatt\_ConvType\_mor’\_affine}$
    - $\text{free\_semiComp\_SemiLatt\_ConvType\_mor’\_lub\_op\_morph}$.

To be more precise, this is the lemma $\text{free\_semiComp\_SemiLatt\_ConvType\_mor’\_lub\_op\_morph}$ that uses the lemma $\text{image\_preserves\_convex\_hull}$ explained in Sect. 5.5.

Finally, we show that the action on morphisms satisfies the functor laws, leading to the following definition of $\mathcal{F}_1$:

- **Definition** \text{free\_semiComp\_SemiLatt\_ConvType} :
  - functor $\mathcal{C}_V \mathcal{C}_S := \text{Functor.Pack (Functor.Mixin free\_semiComp\_SemiLatt\_ConvType\_mor\_id}$
    - $\text{free\_semiComp\_SemiLatt\_ConvType\_mor\_comp)}$.

  Like for the adjunction $\mathcal{F}_0 \dashv \mathcal{U}_0$, the forgetful functor $\mathcal{U}_1$ of type functor $\mathcal{C}_S \mathcal{C}_V$ is just formalized by substituting the category $\mathcal{C}_S$ by the category $\mathcal{C}_V$ in morphisms (see $\text{forget\_semiComp\_SemiLatt\_ConvType}$ in [Monae, 2020 file gcm_model.v]).

#### 6.3.2 Counit / unit

Let us explain how we implement the counit $\varepsilon_1 : \mathcal{F}_1 \circ \mathcal{U}_1 \Rightarrow 1_{\mathcal{C}_S}$.

- It is exactly the $\sqcup$ operator seen in Sect. 4.3.

$$\varepsilon_1 : \text{neset}(\text{neset} T) \rightarrow \text{neset} T$$

$$X \mapsto \sqcup X.$$
We need to show that it is natural, that it preserves the operator $\bigsqcup$, i.e., $\varepsilon_1(\bigsqcup (X)) = \bigsqcup (\varepsilon_1 @ (X))$ (for that purpose we use the lemma lub_op_hull from Sect. 4.2), and that it is affine, i.e., $\varepsilon_1 (X \triangleleft p \triangleright Y) = \varepsilon_1 X \triangleleft p \triangleright \varepsilon_1 Y$.

Let us comment on the proof that $\varepsilon_1$ preserves the nondeterministic choice to highlight a key difference with Cheung’s work [Cheung, 2017]. From the proof that $\varepsilon_1$ preserves the infinitary nondeterministic choice [Monae, 2020, lemma eps1''_lub_op_morph, file gcm_model.v], we can derive the proof that it preserves the binary nondeterministic choice [Infotheo, 2020, lemma lub_op_lub_binary_morph, file necset.v]. In contrast, Cheung proves the binary version directly. Cheung’s setting is finitary but his proofs rely on an implicit connection between finitary and infinitary uses of convex hulls which make them incomplete (at best). This manifests concretely by the use of an undefined infinitary operator [Cheung, 2017, p. 160]. We think that there is a way to make sense of his proof, seeing it as using finitary operators on finite sets whose convex hulls correspond to the infinite sets appearing in his proof, but the theory underlying that reading is completely omitted. Anyway, we have experienced that an infinitary setting is more comfortable for formal proofs. Those are the main reasons why we think that formalization is best performed in an infinitary setting.

The unit $\eta_1 : 1_C \sim U_1 \circ F_1$ is the singleton map, which is easily shown to be natural and affine.

\[ \eta_1 : \text{necset } T \rightarrow \text{nset} (\text{necset } T) \]
\[ X \mapsto \{ T \} \]

We call the corresponding triangular laws triL1 and triR1.

6.4 Putting it All Together

6.4.1 Formalization of the Geometrically Convex Monad

We use the proofs of the triangular laws of Sections 6.1, 6.2.2, and 6.3.2 to create the three adjunctions $\mathcal{F}_C \dashv \mathcal{U}_C$, $\mathcal{F}_0 \dashv \mathcal{U}_0$, and $\mathcal{F}_1 \dashv \mathcal{U}_1$:

Definition AC := AdjointFunctors.mk triLC triRC.
Definition A0 := AdjointFunctors.mk triL0 triR0.
Definition A1 := AdjointFunctors.mk triL1 triR1.

The definition of these adjunctions has been given in Sect. 5.3.1.

We then build the adjunction resulting from the composition of the three adjunctions we have just defined, using the function of Sect. 5.3.2:

Definition Agcm := adj_comp AC (adj_comp A0 A1).

Finally, we obtain the geometrically convex monad from the resulting adjunction using the generic lemma explained at the end of Sect. 5.3.3:

Definition Mgcm := Monad_of_adjoint Agcm.

The very last step is to use the function Monad_of_category_monad of Sect. 5.2 to recover a monad compatible with the MONAE formal framework of monadic equational reasoning.

Definition gcm := Monad_of_category_monad Mgcm.

6.4.2 Informal Description of the Join of the Monad

At this stage, it is worth taking a step back to check that the join of the monad we have built indeed corresponds to the intuition one can have of the execution of a program mixing probabilistic choice and nondeterministic choice. Provided we ignore the function $\varepsilon_C$ (the counit of the adjunction

\[^3\text{We can also recover the probability monad of Affeldt et al., 2019 which is definitionally equal to Monad_of_category_monad (Monad_of_adjoint (adj_comp AC A0)).}\]
$F_C \rightarrow U_C$, which, as we already explained in Sect. 6.1, is here essentially for technical reasons), the join operator can informally be explained as the following function:

$$\varepsilon_1 \circ (\lambda X. \varepsilon_0(\varepsilon_0(X))).$$

The input of this function is indeed $\text{necest} \{\text{dist} \{\text{necest} \{\text{dist} T}\}\}$, i.e., it takes non-empty sets of distributions. The function $\varepsilon_0$ (Sect. 6.2.2) computes barycenters, so that when applied the right-hand side of the function composition returns an object of type $\text{necest} \{\text{necest} \{\text{dist} T\}\}$. The function $\varepsilon_1$ (Sect. 6.3.2) computes the hull of the union of its input, which results in an object of type $\text{necest} \{\text{dist} T\}$, as expected.

7 The Properties of Combined Choice of the Geometrically Convex Monad

The very last step of our construction is to show that the geometrically convex monad (that we obtained as a result of the previous section—Sect. 6) satisfies the expected distributivity axioms that we discussed in Sect. 2.1 and to check that it is meaningful, i.e., that it really distinguishes the different choice operators. This corresponds to [Monae, 2020 file altprob_model.v] in the accompanying development.

7.1 The Geometrically Convex Monad has the Properties of Combined Choice

First, we start by defining nondeterministic choice for the geometrically convex monad using a binary version of the operator $\sqcup$ of Sect. 4.1:

Definition alt A (x y : gcm A) : gcm A := x $\sqcup$ y.

We construct a monad $\text{gcmA}$ implementing $\text{altMonad}$ by proving the following properties, which are essentially consequences of the properties of the operator $\sqcup$:

Lemma altA A : associative (@alt A).
Lemma bindaltDl : BindLaws.left_distributive (@monad.Bind gcm) alt.
Definition gcmA : altMonad := MonadAlt.Pack ...

We extend the monad $\text{gcmA}$ to the monad $\text{gcmACI}$ that implements $\text{altCIMonad}$:

Lemma altxx A : idempotent (@Alt gcmA A).
Lemma altC A : commutative (@Alt gcmA A).
Definition gcmACI : altCIMonad := MonadAltCI.Pack ...

Second, we go on defining probabilistic choice for the geometrically convex monad using the operator of convex spaces:

Definition choice p A (x y : gcm A) : gcm A := x $\prec$ p $\triangleright$ y.

Most properties are direct consequences of the properties of convex spaces, and they lead to the definition of the monad $\text{gcmp}$ that implements $\text{probMonad}$:

Lemma choice0 A (x y : gcm A) : x $\prec$ 0%:pr $\triangleright$ y = y.
Lemma choice1 A (x y : gcm A) : x $\prec$ 1%:pr $\triangleright$ y = x.
Lemma choiceC A p (x y : gcm A) : x $\prec$p $\triangleright$ y = y $\prec$ p.~%:pr $\triangleright$ x.
Lemma choiceemm A p : idempotent (@choice p A).
Lemma choiceA A (p q r s : prob) (x y z : gcm A) :
  p = (r * s) $:$ R $\land$ s.~ = (p.~ * q.~)%R $\rightarrow$
  x $\ll$ p $\triangleright$ (y $\ll$ q $\triangleright$ z) = (x $\ll$ r $\triangleright$ y) $\ll$ s $\triangleright$ z.
Definition gcmp : probMonad := MonadProb.Pack ...

Finally, we prove left-distributivity of bind over the probabilistic choice and right-distributivity of the probabilistic choice over the nondeterministic choice.
Lemma bindchoiceD1 p : BindLaws.left_distributive (@monad.Bind gcm) (@choice p)
Lemma choicealtDr A (p : prob) :
  right_distributive (fun x y : gcmACI A ⇒ x \ p ▷ y) Alt.
and use these lemmas to instantiate altProbMonad into the monad gcmAP:
Definition gcmAP : altProbMonad := MonadAltProb.Pack ...
This completes the construction of the monad proposed by Gibbons et al. [Gibbons and Hinze, 2011, Abou-Saleh et al., 2016].

7.2 The Combined Choice is not a Trivial Theory
We conclude this section with a formal check that probabilistic choice in our axiom system of combined choice is not trivial, meaning that it indeed distinguishes different probabilities. It is sufficient to check that there exists a model which is not trivial in this sense, and our construction of geometrically convex monad serves this purpose nicely:

Example gcmAP_choice_nontrivial (p q : prob) :
  p \ q \→
  Ret true \ p ▷ Ret false \ Ret true \ q ▷ Ret false :> gcmAP bool.
Proof.
...
Qed.

Here :> gcmAP bool indicates the type of this inequality, which forces the resolution of monadic operations inside our instance of altProbMonad. The proof just requires to unfold definitions and provides further evidence that the geometrically convex monad is not a trivial model.

8 Application: Mechanization of the Monty Hall Problem
As an application of altProbMonad, we provide a mechanization of the Monty Hall problem using probability and nondeterminism as described by Gibbons [Gibbons, 2012, Sect. 6.1] (we have also mechanized a purely probabilistic variant [Gibbons, 2012, Sect. 6] as well as a forgetful variant [Gibbons, 2012, Sect. 7.2]).

Let us recall the Monty Hall problem. The player is given a choice of three doors: there is a car behind one door and there are goats behind the other doors. First, the player picks one door and the host opens one of the other doors behind which there is a goat. The player is then asked whether he/she wants to stick to his/her first choice or switch to the other door. It turns out that the best strategy is to switch, even though this appears to be counterintuitive for many, as shown by the controversy the problem sparked when first exposed in the specialized press.

8.1 Problem Setting
Let us consider the datatype door consisting of three different doors A, B, and C (doors is a list consisting of these three doors). The host hides the car behind one of the three doors chosen nondeterministically (hence altMonad) (below, def has type door unless quantified):
Definition hide_n {M : altMonad} : M door := arbitrary def doors.
The function arbitrary takes a default element and a list and returns an element of the list chosen nondeterministically (or the default element if the list is empty). It is defined using standard functions as follows:
Definition arbitrary {M : altMonad} {A : Type} (def : A) : seq A → M A :=
  foldr1 (Ret def) (fun x y ⇒ x ▷ y) \ o map Ret.

The player picks one of the doors uniformly at random (using probMonad):
Definition pick {M : probMonad} : M door := uniform def doors.
The function `uniform` is defined using the binary probabilistic choice as follows:

```ocaml
Fixpoint uniform {M : probMonad} {A : Type} (def : A) (s : seq A) : M A :=
  match s with
  | [] => Ret def
  | [x] => Ret x
  | x :: xs =>
    Ret x \ (IZR (Z_of_nat (size (x :: xs))))%:pr \ uniform def xs
end.
```

The host teases the player by opening a door, which is nor the one hiding the car neither the one picked by the player, chosen nondeterministically:

```ocaml
Definition tease_n {M : altMonad} (h p : door) : M door :=
  arbitrary def (doors \ [:: h; p]).
```

We can now arrange above elements chronologically to represent a game, the latter being parameterized by the strategy of the player:

```ocaml
(* generic game *)
Definition monty {M : monad} hide pick tease (strategy : door \ door \ M door) :=
  do h ← hide ;
  do p ← pick ;
  do t ← tease h p ;
  do s ← strategy p t ;
  Ret (s == h).

(* nondeterministic variant *)
Variable M : altProbMonad.
Definition play_n (strategy : door \ door \ M door) : M bool :=
  monty hide_n (pick def) tease_n strategy.
```

We finally provide the two possible strategies. The “stick” strategy is defined by returning the already-chosen door:

```ocaml
Definition stick {M : monad} (p t : door) : M door :=
  Ret p.
```

The “switch” strategy is defined by returning the other door (the one that was nor picked neither used for teasing):

```ocaml
Definition switch {M : monad} (p t : door) : M door :=
  Ret (head def (doors \ [:: p; t])).
```

### 8.2 Switch is Better than Stick

One can prove that the “switch” strategy is better than the “stick” strategy by comparison with a biased coin, defined as follows (definition from Sect. 2.1 reproduced here for the convenience of the reader):

```ocaml
Definition bcoin {M : probMonad} (p : prob) : M bool :=
  Ret true \ p \ Ret false.
```

More precisely, one can show that the “switch” strategy is as good as a 2/3-biased coin (recall from Sect. 2.1 that \( (1/3) \%:pr \) is the probability \( 1 - 1/3 = 2/3 \)):

```ocaml
Lemma monty_switch : play_n (switch def) = bcoin (1/3).%:pr.
```

The proof goes as follows.

1. The left-hand side `play_n (switch def)` can be rewritten as:

```ocaml
hide_n ≫= (fun h => pick def ≫= (fun p => tease_n h p ≫= (fun t => Ret (h == head def (doors \ [:: p; t])))))
```
This step essentially amounts to use the property that the unit is the left neutral of bind.

2. The rightmost continuation can furthermore be rewritten to lead to:

```plaintext
hide_n ≫ (fun h ⇒ pick def ≫ (fun p ⇒ tease_n h p ≫
  (if h == p then Ret false else Ret true)))
```

This step is essentially by case analysis on h == p and observation of the expression

```plaintext
head def (doors \ \ [:: p; t]).
```

When h == p, this expression cannot be h. When h ≠ p, it is h.

3. Since teasing does not influence the outcome anymore, the left-hand side can furthermore be simplified into:

```plaintext
hide_n ≫ (fun h ⇒ pick def ≫ (fun p ⇒ Ret (h ≠ p)))
```

The main lemma needed for this step can be stated in a generic way as follows:

**Lemma arbitrary_inde** (M : altCIMonad) T (a : T) s U (m : M U) :

```plaintext
0 < size s → arbitrary a s >> m = m.
```

4. The last step produces the expected biased coin bcoin (/ 3). This is captured by the following lemma:

**Lemma bcoin23E** :

```plaintext
arbitrary def doors ≫
  (fun h ⇒ uniform def doors ≫ (fun p ⇒ Ret (h ≠ p))) =
  bcoin (/ 3).%
```

Its proof essentially appeals to the properties of probabilistic choice as specified by the interface of probMonad seen in Sect. 2.1 and to the fact that bind left-distributes over nondeterministic choice, a property of altMonad.

On the other hand, the “stick” strategy is as good as a 1/3-biased coin:

**Lemma monty_stick** : play_n stick = bcoin (/ 3)%.

The proof is a bit simpler. It suffices to observe that the teasing does not influence the outcome and use the lemma arbitrary_inde. It is completed by computations similar to the last step of the proof for the “switch” strategy and uses the fact that bind left-distributes over nondeterministic choice and probabilistic choice.

As the reader has observed in this section, the example of the Monty Hall problem uses only the interfaces of the involved monads, including the altProbMonad, and we know for sure that its interface is correct since we have a formal model since Sect. 6.4.1.

9 Related Work

We have already commented on several related work throughout this paper. We add in this section further comments that are better explained now that we have completed the technical presentation of our contributions.

The formalization of convex spaces comes from [Affeldt et al., 2020a]. This work develops applications of convex spaces such as convex and concave functions and formalizes equivalences between various axiomatizations of binary and multiary convex operators. Here, we use the multiary convex combination operator in Sect. 6.2, we further develop the theory of affine functions, and we extend convex spaces to build the convex powerset functor.

In our formalization of semicomplete semilattices (in Sect. 1), the nondeterministic choice is modeled as an infinitary operator. This is similar to Beaulieu’s “infinite nondeterministic
choice” [Beaulieu, 2008, Def. 3.2.3] and, at first sight, looks different from Cheung’s approach, who models nondeterministic choice as a binary operator [Cheung, 2017, Sec. 6.3.1]. In Sect. 6.3.2, we explained that Cheung also implicitly uses an infinitary version of his operator and that we find an infinitary operator to be more comfortable and clearer from the viewpoint of formalization.

The monad for probability and nondeterminism can also be presented using finitely-generated convex sets of distributions [Bonchi et al., 2019, Sect. 3.1]. Here, we did not insist on having finitely-generated convex sets because our first attempt at doing so led to technically involved formal proofs. Now that we have completed our formalization, it should be easier to extend it with finitely-generated convex sets. Indeed, looking at [Bonchi et al., 2020b], we recognize several technical results that we have already formalized (e.g., parts of Lemma 4.4). Concretely, the approach would start by defining the data structure for the non-empty finitely-generated convex sets by adding an axiom for the existence of a finite generator to the type necset, and then by replaying and fixing the proofs (the category part of our framework should stay unchanged). This could open the door to the construction of an executable model.

The geometrically convex monad is not the first example of a formalized monad that combines probabilistic and nondeterministic choices: Tassarotti and Harper [Tassarotti and Harper, 2019] already formalized in Coq the indexed valuation monad by Varacca and Winskel that we already mentioned in Sect. 2.2. In this monad, probabilistic is not idempotent and therefore it is not suitable for our purpose. Our formalization looks arguably more modular than the one by Tassarotti and Harper who build their monad in a direct manner.

We have been formalizing one model that combines probabilistic and nondeterministic choices: the one advocated by Gibbons et al. because it fits well with functional programming. pGCL [McIver and Morgan, 2005] is another such model that has been formalized in the Isabelle/HOL proof assistant [Cock, 2014] (as such it qualifies as the first formalized model that provides both probabilistic and nondeterministic choices). However, its default semantics is given in different terms (using predicate transformers, no category theory involved, refinement instead of equations) so that the formalizations of the geometrically convex monad and of pGCL turn out to be different tasks. McIver and Morgan’s book [McIver and Morgan, 2005] contains also another semantics, the relational demonic semantics whose mathematical construction (Definition 5.4.4) is similar to Cheung’s. Yet, it is not presented as a monad with algebraic laws, which is a crucial aspect of our framework, and it has not been formalized.

There is a number of formalizations of category theory in proof assistants (many of which being listed by Gross et al. [Gross et al., 2014]). However, we could not find a readily usable formalization of concrete categories in Coq. For example, UniMath is a large Coq library that aims at formalizing mathematics using the univalent point of view [Voevodsky et al., 2014]. It contains a substantial formalization of abstract categories but does not seem to feature a formalization of concrete categories. Since we needed only a handful of theorems about category theory, we formalized concrete categories from scratch and developed their theories as a generalization of Monae (in Sect. 5).

The idea of using categories as a package to handle functions with proofs was already presented by McBride [McBride, 1999, Chapter 7, Section 3.1]. He also proposed the use of concrete categories for such a lightweight use of category theory, noting that the convertibility of terms is an easier way than propositional equality to handle the equational laws for morphisms, such as unit and associativity laws. His formal definition of categories differs from ours in that it is also indexing on hom-sets, while in our definition, hom-sets are embedded as predicates. This difference further affects later definitions such as functors. Our definition makes it clearer that concrete categories are shallow embeddings of categories.

10 Conclusion and Future Work

In this paper, we proposed a formalization in the Coq proof assistant of an infinitary version of the geometrically convex monad, a monad that combines probabilistic and nondeterministic choice with idempotence of probabilistic choice. To the best of our knowledge, this is the first formaliza-
tion of such a monad. Our development led us to develop several formal mathematical theories of broader interest such as a formalization of the convex powerset functor and a formalization of concrete categories. A direct application was to complete an existing formalization of monadic equational reasoning which was lacking the model of the combined interface of probabilistic and nondeterministic choices and which we illustrated with an extended example. We could also use our model to check that the probabilistic operator does not collapse with Gibbons et al.’s choice of axioms.

We formalized an infinitary nondeterministic choice operator. As we discussed in Sect. 9 it would be interesting to formalize a finitary one with the insights from recent work on finitely-generated convex sets [Bonchi et al., 2020a]. Our experiment is an example of combination of two monads that requires a substantial amount of work. There also exist a number of generic results about the combination of monads such as distributive laws [Zwart and Marsden, 2018] or weak ones [Goy and Petrisan, 2020] that would deserve formalization. By introducing a formalization of concrete categories to support the construction of the geometrically convex monad, our work also raises the question of the generalization of MONAE [Affeldt et al., 2019] from its specialization to the Set category.

Acknowledgments We acknowledge the support of the JSPS KAKENHI Grant Number 18H03204 and the JSPS-CNRS bilateral program “FoRmal tools for IoT seCurity” (PRC2199), and thank all the participants of these projects for fruitful discussions. We also thank Cyril Cohen and Shinya Katsumata for guidance about the formalization of monads, Kazunari Tanaka who contributed to the formalization of categories, Jeremy Gibbons and Joseph Tassarotti for their comments.

References

[Abou-Saleh et al., 2016] Abou-Saleh, F., Cheung, K.-H., and Gibbons, J. (2016). Reasoning about probability and nondeterminism. In POPL workshop on Probabilistic Programming Semantics.

[Affeldt et al., 2018] Affeldt, R., Cohen, C., and Rouhling, D. (2018). Formalization techniques for asymptotic reasoning in classical analysis. J. Formalized Reasoning, 11(1):43–76.

[Affeldt et al., 2020a] Affeldt, R., Garrigue, J., and Saikawa, T. (2020a). Formal adventures in convex and conical spaces. In Benzmüller, C. and Miller, B. R., editors, 13th International Conference on Intelligent Computer Mathematics (CICM 2020), Bertinoro, Italy, July 26–31, 2020, volume 12236 of Lecture Notes in Computer Science, pages 23–38. Springer.

[Affeldt et al., 2020b] Affeldt, R., Garrigue, J., and Saikawa, T. (2020b). Reasoning with conditional probabilities and joint distributions in Coq. Computer Software, 37(3):79–95.

[Affeldt et al., 2014] Affeldt, R., Hagiwara, M., and Sénizergues, J. (2014). Formalization of Shannon’s theorems. Journal of Automated Reasoning, 53(1):63–103.

[Affeldt et al., 2019] Affeldt, R., Nowak, D., and Saikawa, T. (2019). A hierarchy of monadic effects for program verification using equational reasoning. In Hutton, G., editor, 13th International Conference on Mathematics of Program Construction (MPC 2019), Porto, Portugal, October 7–9, 2019, volume 11825 of Lecture Notes in Computer Science, pages 226–254. Springer.

[Beaulieu, 2008] Beaulieu, G. (2008). Probabilistic Completion of Nondeterministic Models. PhD thesis, Faculty of Graduate and Postdoctoral Studies, University of Ottawa.

[Beck, 1969] Beck, J. (1969). Distributive laws. In B., E., editor, Seminar on Triples and Categorical Homology Theory, number 80 in Lecture Notes in Mathematics, pages 119–140. Springer.
Bergman, G. (2015). *An Invitation to General Algebra and Universal Constructions*. Universitext. Springer International Publishing.

Bonchi, F., Silva, A., and Sokolova, A. (2017). The power of convex algebras. In Meyer, R. and Nestmann, U., editors, *28th International Conference on Concurrency Theory (CONCUR 2017), September 5–8, 2017, Berlin, Germany*, volume 85 of LIPIcs, pages 23:1–23:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik.

Bonchi, F., Sokolova, A., and Vignudelli, V. (2019). The theory of traces for systems with nondeterminism and probability. In *34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2019), Vancouver, BC, Canada, June 24–27, 2019*, pages 1–14. IEEE.

Bonchi, F., Sokolova, A., and Vignudelli, V. (2020a). Presenting convex sets of probability distributions by convex semilattices and unique bases. [https://arxiv.org/abs/2005.01670](https://arxiv.org/abs/2005.01670) arXiv cs.LO.

Bonchi, F., Sokolova, A., and Vignudelli, V. (2020b). The theory of traces for systems with nondeterminism, probability, and termination. [https://arxiv.org/abs/1808.00923v4](https://arxiv.org/abs/1808.00923v4) arXiv cs.LO.

Brady, E. (2013). *Idris, a general-purpose dependently typed programming language: Design and implementation*. *J. Funct. Program.*, 23(5):552–593.

Cheung, K.-H. (2017). *Distributive Interaction of Algebraic Effects*. PhD thesis, Merton College, University of Oxford.

Cock, D. (2014). *Leakage in Trustworthy Systems*. PhD thesis, School of Computer Science and Engineering, The University of New South Wales, Sydney, Australia.

Cohen and Sakaguchi, K. (2015). A finset and finmap library: Finite sets, finite maps, multisets and order. Available at [https://github.com/math-comp/finmap](https://github.com/math-comp/finmap). Last stable release: 1.5.0 (2020).

Fritz, T. (2015). Convex spaces I: Definition and examples. [https://arxiv.org/abs/0903.5522](https://arxiv.org/abs/0903.5522) arXiv math.MG. First version: 2009.

Garillot et al., (2009). Garillot, F., Gonthier, G., Mahboubi, A., and Rideau, L. (2009). Packaging mathematical structures. In Berghofer, S., Nipkow, T., Urban, C., and Wenzel, M., editors, *22nd International Conference on Theorem Proving in Higher Order Logics (TPHOLs 2009), Munich, Germany, August 17–20, 2009*, volume 5674 of *Lecture Notes in Computer Science*, pages 327–342. Springer.

Gibbons, J. (2012). Unifying theories of programming with monads. In Wolff, B., Gaudel, M., and Feliachi, A., editors, *Revised Selected Papers of the 4th International Symposium on Unifying Theories of Programming (UTP 2012), Paris, France, August 27–28, 2012*, volume 7681 of *Lecture Notes in Computer Science*, pages 23–67. Springer.

Gibbons and Hinze, R. (2011). Just do it: simple monadic equational reasoning. In Chakravarty, M. M. T., Hu, Z., and Danvy, O., editors, *Proceeding of the 16th ACM SIGPLAN international conference on Functional Programming, ICFP 2011, Tokyo, Japan, September 19–21, 2011*, pages 2–14. ACM.

Giry, M. (1982). A categorical approach to probability theory. In Banaschewski, B., editor, *Categorical aspects of topology and analysis*, volume 915 of *Lecture Notes in Mathematics*, pages 68–85. Springer.
[Goy and Petrisan, 2020] Goy, A. and Petrisan, D. (2020). Combining probabilistic and non-deterministic choice via weak distributive laws. In 35th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2020), Saarbrücken, Germany, July 8–11, 2020, pages 454–464. ACM.

[Gross et al., 2014] Gross, J., Chlipala, A., and Spivak, D. I. (2014). Experience implementing a performant category-theory library in Coq. In Klein, G. and Gamboa, R., editors, Interactive Theorem Proving, pages 275–291, Cham. Springer International Publishing.

[Infotheo, 2020] Infotheo (2020). A Coq formalization of information theory and linear error-correcting codes. https://github.com/affeldt-aist/infotheo. Open source software. Since 2009.

[Jacobs, 2010] Jacobs, B. (2010). Convexity, duality and effects. In IFIP TCS, volume 323 of IFIP Advances in Information and Communication Technology, pages 1–19. Springer.

[Kaminski et al., 2016] Kaminski, B. L., Katoen, J., Matheja, C., and Olmedo, F. (2016). Weakest precondition reasoning for expected run-times of probabilistic programs. In Thieman, P., editor, 25th European Symposium on Programming Languages and Systems (ESOP 2016), Eindhoven, The Netherlands, April 2–8, 2016, volume 9632 of Lecture Notes in Computer Science, pages 364–389. Springer.

[Keimel and Plotkin, 2017] Keimel, K. and Plotkin, G. D. (2017). Mixed powerdomains for probability and nondeterminism. Logical Methods in Computer Science, 13(1:2):1–84.

[Mac Lane, 1998] Mac Lane, S. (1998). Categories for the Working Mathematician, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, Berlin and New York, second edition.

[Mahboubi and Tassi, 2013] Mahboubi, A. and Tassi, E. (2013). Canonical structures for the working coq user. In Blazy, S., Paulin-Mohring, C., and Pichardie, D., editors, 4th International Conference on Interactive Theorem Proving (ITP 2013), Rennes, France, July 22–26, 2013, volume 7998 of Lecture Notes in Computer Science, pages 19–34. Springer.

[Mathematical Components Team, 2007] Mathematical Components Team (2007). Mathematical Components library. https://github.com/math-comp/math-comp. Development version. Last stable version 1.10 (2019) available on the same website.

[McBride, 1999] McBride, C. (1999). Dependently Typed Functional Programs and their Proofs. PhD thesis, University of Edinburgh.

[McIver and Morgan, 2005] McIver, A. and Morgan, C. (2005). Abstraction, Refinement and Proof for Probabilistic Systems. Monographs in Computer Science. Springer.

[Mio and Vignudelli, 2020] Mio, M. and Vignudelli, V. (2020). Monads and quantitative equational theories for nondeterminism and probability. https://arxiv.org/abs/2005.07509. arXiv cs.LO.

[Mislove et al., 2004] Mislove, M., Ouaknine, J., and Worrell, J. (2004). Axioms for probability and nondeterminism. Electronic Notes in Theoretical Computer Science, 96:7 – 28. Proceedings of the 10th International Workshop on Expressiveness in Concurrency.

[Mislove, 2000] Mislove, M. W. (2000). Nondeterminism and probabilistic choice: Obeying the laws. In Palamidessi, C., editor, 11th International Conference on Concurrency Theory (CONCUR 2000), University Park, PA, USA, August 22–25, 2000, volume 1877 of Lecture Notes in Computer Science, pages 350–364. Springer.

[Monae, 2020] Monae (2020). Monadic effects and equational reasoning in Coq. https://github.com/affeldt-aist/monae. Open source software. Since 2019.
[Mu, 2019a] Mu, S.-C. (2019a). Calculating a backtracking algorithm: An exercise in monadic program derivation. Technical Report TR-IIS-19-003, Institute of Information Science, Academia Sinica.

[Mu, 2019b] Mu, S.-C. (2019b). Equational reasoning for non-deterministic monad: A case study of Spark aggregation. Technical Report TR-IIS-19-002, Institute of Information Science, Academia Sinica.

[Stone, 1949] Stone, M. H. (1949). Postulates for the barycentric calculus. *Annali di Matematica Pura ed Applicata*, 29:25–30.

[Tassarotti and Harper, 2019] Tassarotti, J. and Harper, R. (2019). A separation logic for concurrent randomized programs. *Proc. ACM Program. Lang.*, 3(POPL):64:1–64:30.

[The Agda Team, 2020] The Agda Team (2020). *The Agda User Manual*. Available at https://agda.readthedocs.io/en/v2.6.0.1 Version 2.6.0.1.

[The Coq Development Team, 2019a] The Coq Development Team (2019a). *The Coq Proof Assistant Reference Manual*. Inria. Available at https://coq.inria.fr Version 8.10.2.

[The Coq Development Team, 2019b] The Coq Development Team (2019b). The logic of coq. Available at https://github.com/coq/coq/wiki/The-Logic-of-Coq Part of the Coq FAQ.

[Timany and Jacobs, 2016] Timany, A. and Jacobs, B. (2016). Category theory in Coq 8.5. In Kesner, D. and Pientka, B., editors, 1st International Conference on Formal Structures for Computation and Deduction (FSCD 2016), June 22–26, 2016, Porto, Portugal, volume 52 of LIPIcs, pages 30:1–30:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik.

[Tix et al., 2009] Tix, R., Keimel, K., and Plotkin, G. D. (2009). Semantic domains for combining probability and non-determinism. *Electron. Notes Theor. Comput. Sci.*, 222:3–99.

[Varacca and Winskel, 2006] Varacca, D. and Winskel, G. (2006). Distributing probability over nondeterminism. *Mathematical Structures in Computer Science*, 16(1):87–113.

[Voevodsky et al., 2014] Voevodsky, V., Ahrens, B., Grayson, D., et al. (2014). UniMath — a computer-checked library of univalent mathematics. Available at https://github.com/UniMath/UniMath Last stable release 0.1 (2016).

[Zwart and Marsden, 2018] Zwart, M. and Marsden, D. (2018). Don’t try this at home: No-go theorems for distributive laws. https://arxiv.org/abs/1811.06460 arXiv math.CT.