A quantum complexity lower bound from differential geometry

Adam R. Brown

Differential geometry has long found applications in physics in general relativity and related areas. More recently, it was proposed by Nielsen that the tools of differential geometry, when applied to the unitary group, might be used to bound the complexity of quantum operations. The Bishop–Gromov bound—a cousin of the focusing lemmas used to prove the Penrose–Hawking black hole singularity theorems—is a differential geometry result that gives an upper bound on the rate of growth of the volume of geodesic balls in terms of the Ricci curvature. Here I apply the Bishop–Gromov bound to Nielsen’s complexity geometry to prove lower bounds on the quantum complexity of a typical unitary. For a broad class of models, the typical complexity is shown to be exponentially large in the number of qubits. This technique gives results that are tighter than all known lower bounds in the literature, as well as establishing lower bounds for a much broader class of complexity geometry metrics than has hitherto been bounded. This method thus realizes the original vision of Nielsen, which was to apply the tools of differential geometry to study quantum complexity.

Received: 15 January 2022
Accepted: 16 November 2022
Published online: 9 January 2023

Geometric ideas were introduced into the study of quantum complexity by Nielsen and collaborators1–5. Their vision was that by considering a definition of complexity that replaces quantum gates with a smooth path through unitary space, the tools of differential geometry might be brought to bear on proving complexity lower bounds. So far this vision has not been realized. Here I will use a theorem from differential geometry—the Bishop–Gromov bound6, (https://en.wikipedia.org/wiki/Bishop-Gromov_inequality)—to prove a complexity lower bound for the complexity geometry. I address the following basic question: for the complexity geometry on N qubits, does the complexity of a typical operator grow exponentially with N? For one particular complexity geometry—what below I will call the ‘cliff’ metric—this question has already been answered in the affirmative by Nielsen et al.2, using a non-geometric technique that leverages previous results from gate complexity. Here I will use a tool from differential geometry to rederive this result in a way that makes no mention of gates, ε balls or error budgets, and improves the exponent. I will argue that my improved lower bound is tight. Next, I will apply this technique to many different complexity geometries, and show that for a broad class of metrics, many much less highly curved than the cliff metric, the complexity of a typical unitary is still exponential in N.

Review of gate complexity and complexity geometry

The complexity of a transformation quantifies how hard it is to implement. The group of transformations we will be interested in implementing is U(2^N), the purity-preserving linear functions on N qubits. Let us consider two different sets of primitive operations we might use to build elements of this group, which will yield two different definitions of complexity.

Gate complexity

For gate complexity, we compile complex unitaries by arranging gates in a circuit. Usually the primitive gates may only act on a small number (often two) of qubits at a time. There may either be a discrete set of primitive gates (for example, CNOT plus Hadamard plus a random phase are known to be universal7) or a continuous set (for example, we may permit any transformation U(2^2) on the two qubits). The value of
the complexity is then the number of primitive gates required to build the target unitary.

We are interested in the number of gates required to synthesize typical elements of $U(2^N)$. For a discrete gate set, we must introduce a tolerance $\epsilon$ if we aspire to reach every element, whereas for a continuous gate set we can hit every element exactly. Either way, we can derive an exponential lower bound just by counting. For discrete gates, we count the number of balls in $U(2^N)$, namely $e^{O(\log (\epsilon^{-1} N^4 ))}$, and then count the number of different circuits with $c$ gates, $e^{O(c \log (N^4))}$; for continuous gate sets we count the number of degrees of freedom in $U(2^N)$, namely $4^N$, and then count the dimensionality of the space of gates, $O(1)$; in both cases, therefore, we cannot hope to have fully explored the majority of the unitary group until

$$e_{\text{gates}}[U(2^N)] \geq 4^N \tag{1}$$

For many sets of primitive gates (including the continuous two-qubit gate set $U(2^2)$) we can, with a little more work, prove that this complexity lower bound is approximately tight. For gate complexity the question of the complexity of a typical unitary thus has a simple and settled answer. Not so for complexity geometry.

**Complexity geometry**

Complexity geometry introduces a new metric on the unitary group that stretches directions that are hard to move in, so that complex unitaries are pushed farther away. The complexity of a unitary may then be defined as the length of the shortest path connecting $U$ to the identity. We thus compile a complex unitary by gliding through $U(2^N)$ on a continuous path, guided by a (possibly time-dependent) Hamiltonian.

A complete basis for the Hamiltonians in the tangent space of $U(2^N)$ is given by the generalized-Pauli operators. A $k$-local generalized-Pauli operator $\sigma_k$ is the product of $k$ single-qubit Pauli operators and $N-k$ identity operators that act on the other qubits. For example, $(\sigma_k)_0 \equiv \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes (\sigma_1)_3 \otimes \mathbb{1}_4 \otimes ... \otimes \mathbb{1}_N$ is a 3-local generalized-Pauli operator, where the numerical subscript indicates which qubit the operator acts on. There are

$$\mathcal{N}_k \equiv \binom{N}{k} 3^k \tag{2}$$

different $k$-local (weight $k$) generalized-Pauli operator's, for a total of

$$\sum_{k=0}^{N} \binom{N}{k} 3^k = 4^N \text{ } \mathcal{N}_k \text{ is peaked at } k = \frac{1}{4} N.$$ In this basis, the complexity distance between $U$ and $U + dU$ is then defined by

$$ds^2 = \sum_{\mathcal{Q}} \text{Tr} [i dU U^\dagger \sigma_k ] \text{Tr} [i dU U^\dagger \sigma_k ] \tag{3}$$

(See also Section 1 of ref. 8 for a pedagogical introduction and refs. 9–20 for other recent work.) Here we have normalized the trace so that $\text{Tr} [1] = 1$ and $\mathcal{Q}$ is a positive-definite matrix. Were $\mathcal{Q}$ the identity matrix, $\mathcal{Q} = \mathbb{1}_d$, this would recover the standard 'bi-invariant' Riemannian metric—sometimes called the 'Killing metric'—on $U(2^n)$, which has a maximum separation (distance) of $N$. In the Killing metric, it is equally 'easy' to move in any direction. In the complexity geometry, we change $\mathcal{Q}$ so that while it may still be inexpensive to move in some directions (for example, in the directions of Hamiltonians that act non-trivially on only two qubits), other directions may be assigned a large 'penalty factor' $\gamma_k$ to reflect that moving in highly non-local directions is hard. The $\gamma_k$ we will consider here will be diagonal in the generalized-Pauli basis, and will have a diagonal value that depends only on the $k$ locality, so that the metric is fully specified by the penalty schedule $\mathcal{Q}_k$ that assigns a penalty factor to each value of $k \leq N$. The infinitesimal distance in off-diagonal directions is then determined by Pythagoras' theorem, since equation (3) has an L2 norm. Finally, notice that the metric equation (3) is 'right-invariant', which means the distance from $\gamma$ to $U$ is the same as the distance from $\gamma U^\dagger$ to $U U^\dagger$, as a consequence of which the complexity geometry is homogeneous.

We want to know the complexity of a typical operator as a function $N$ and the penalty schedule $\mathcal{Q}_k$. Unlike for gate complexity, we cannot just count. We cannot count points because the number of points reachable by even an arbitrarily short path is uncountable. Moreover, we cannot even count dimensions, since there is no restriction on how often the path may change direction, so the dimensionality of the space of even arbitrarily short paths is infinity. To find a lower bound to the complexity of a typical unitary, we cannot use counting. Instead we must measure volume. But first let us note some useful lemmas.

The first useful lemma is that the cliff-metric complexity distance is upper bounded by the gate complexity. This is because any at-most-2-qubit gate (that is, any element of $U(2^2)$) can be synthesized by evolving with an at-most-2-local Hamiltonian for a Killing distance of at most $\pi$. Thus, even for the 'infinite-cliff' metric $\mathcal{Q}_{k \geq 2} = 1$, $\mathcal{Q}_{k \geq 3} = \infty$, the distance to a given unitary is never more than $\pi$ times the number of 1- and 2-local gates in the smallest circuit that reaches the same unitary,

$$e_{\text{geom, cliff}}[U] \leq \pi \epsilon e_{\text{gates}}[U] \tag{4}$$

The second useful lemma relates distances in two different complexity geometries. The more expensive penalty schedule must have concomitantly longer distances, so

$$\forall k, \mathcal{Q}_k \geq \gamma_k \rightarrow e_{\text{geom, } \mathcal{Q}_k}[U] \geq e_{\text{geom, } \gamma_k}[U] \tag{5}$$

As a special case, increasing every $\gamma_k$ in a penalty schedule to be equal to the maximum penalty, $\gamma_k = \max_k \gamma_k$, gives a rescaled version of the Killing metric, and so upper bounds

$$e_{\text{geom, } \gamma_k}[U] \leq \pi \max_k \sqrt{\gamma_k} \tag{6}$$

**The Bishop–Gromov bound**

The volume of a geodesic ball of radius $c$ in $d$-dimensional hyperbolic space is given by $v_{\text{vol}}(c)$ where, for unit vectors $X$,

$$v_{\text{vol}}(c) \equiv \Omega_d-1 \int_0^c e^d \left( \sqrt{1- \frac{d-1}{\Omega_d} \sinh \left( \frac{\sqrt{-\Omega_d} [x^\mu [X^\mu [X^\tau]]X^\tau]]_c}{\sqrt{2c^2}} \right)} \right)^{d-1} \tag{7}$$

Since hyperbolic space is isotropic, the ‘min’ here is redundant as the Ricci curvature is the same in all directions (and so $\mathcal{R}_{\mu \nu [X^\mu [X^\tau]]X^\tau]}$ is independent of $X$). However, the Bishop–Gromov theorem14, (https://en.wikipedia.org/wiki/Bishop-Gromov_inequality) says that this same expression upper bounds the volume of geodesic balls in any $d$-dimensional homogeneous space, even when it is not isotropic

**Bishop–Gromov theorem:** $v_{\text{vol}}(c) \leq v_{\text{vol}}(c)$ (8)

Although calculating the exact volume of the geodesic ball is hard, the Bishop–Gromov theorem gives a simple upper bound in terms of a single local geometric quantity. (For mathematicians, the Bishop–Gromov bound is a venerable ‘comparison theorem’ of differential geometry that needs no introduction, but physicists may find it helpful to think of it as arising from the $(d+0)$-dimensional Raychaudhuri equation15, as discussed in the Supplementary Information.)

**Proof strategy overview**

Our proof strategy is, schematically,
bound curvature below $\rightarrow$ bound volume$_{\text{ball}}(C)$
above $\rightarrow$ bound complexity below

as I will now explain. The complexity metric has an infinite number of
points but only a finite (although generally double-exponentially large) volume,

$$\text{volume} \triangleq (U(2^N)) = \omega_d \prod_{k \leq N} (\binom{N}{k})^k$$

Here $\omega_d$ is the volume of the unit $d = 4^d$ dimensional
unitary group $U(2^N)$ with the bi-invariant metric. $\mathbb{S}^1 = 1$, and the quantity in the square-
root is the determinant of the metric. Since the complexity of an operator is defined as its geometric distance
from the identity, the volume of unitaries with complexity less than $\mathcal{C}$ is simply the volume of
the geodesic ball with radius $\mathcal{C}$. We cannot hope to be able to synthesize
the median unitary until this geodesic ball has engulfed (half of)
the space

$$\text{volume}_{\mathcal{C}}(U(2^N)) = \frac{1}{2} \text{volume}_{\mathcal{C}}(U(2^N))$$

Combining equations (8), (10) and (11), we will lower bound $\mathcal{C}_{\text{typical}}$ using

$$\text{volume}_{\text{typical}} \geq \text{volume}_{\mathcal{C}}(U(2^N)) = \frac{1}{2} \omega_d \prod_{k \leq N} (\binom{N}{k})^k$$

In summary, our proof strategy for bounding below the complexity
distance one must travel to reach a typical unitary will be by observing
in equation (10) that the volume of the whole complexity geometry is huge,
and then observing in equation (8) that the volume of the ball made by shooting geodesics out from a point in every direction
does not grow very fast as a function of its radius. Combining these
observations tells us that a geodesic ball must have a ginormous radius
before it can hope to have encompassed an appreciable fraction of the
complexity geometry.

The Ricci curvature
To evaluate the Bishop–Gromov bound, we must calculate the most neg-
ative component of the Ricci curvature. The curvature of right-invariant
metrics was investigated by Milnor\(^\text{21}\) (note mathematicians use the
mirror convention and call them 'left-invariant'). As recounted in the
Supplementary Information, the Ricci curvature is diagonal in the
generalized-Pauli basis and given by

$$\mathcal{R}_{\sigma_0} = \sum_{\mathcal{C}} \left( \frac{\text{Tr}([\sigma_0, C])^2}{4} \right)^2 \mathcal{C} \left( \mathcal{C} \right) \left( \mathcal{C} \right) \mathcal{C} \left( \mathcal{C} \right) \mathcal{C} \left( \mathcal{C} \right)$$

Define $\#\text{overlap}_{\text{same}}$ as the number of qubits to which $\sigma_0$ and $\sigma_1$
assign the same SU(2) Pauli operators ($\sigma_0$ or $\sigma_1$) and $\#\text{overlap}_{\text{diff}}$ as the number
of qubits to which they both assign a Pauli operator (that is, not $\neq$)
but the Pauli operators are different. Then $\frac{1}{2} \text{Tr}([\sigma_0, C])^2$ will be 1 if $\#\text{overlap}_{\text{diff}}$
is odd and 0 otherwise. When the commutator is non-zero it is
another generalized-Pauli operator with weight

$$\text{weight}([\sigma_0, C]) = \text{weight}(\sigma_0) + \text{weight}(\sigma_1)$$

These formulae, plus some combinatorics, are sufficient to cal-
culate the Ricci curvature for every direction and every penalty schedule.

Approximations
The Bishop–Gromov technique can give a precise lower bound, but
here we will extract only the exponential behaviour in $N$. This will allow us
to make a number of simplifications.

First, since we are interested in a lower bound, we will drop the
positive term in equation (13). Since $\frac{1}{2} \left\{ \sigma_0, \left( \frac{1}{2} [\sigma_0, C_0] \right) \right\} = \sigma_1$ (or zero
when $\sigma_0$ and $\sigma_1$ commute) we can reorder the sum to write

$$\mathcal{R}_{\sigma_0}^\mathcal{C} \geq \frac{-2}{\mathcal{C}} \sum_{\sigma_1} \left( \text{Tr}([\sigma_0, C_0])^2 \right)^{\frac{1}{2}} \left( \frac{\mathcal{C} \left( [\sigma_0, C_0] \right) \mathcal{C} \left( [\sigma_0, C_0] \right) \mathcal{C} \left( [\sigma_0, C_0] \right) \mathcal{C} \left( [\sigma_0, C_0] \right)}{4} \right)$$

Second, to make the double-exponentially large terms more man-
able, we can take the $4^{\text{th}}$ root of equation (12), which to good
approximation becomes

$$(\Omega_4)^{\frac{1}{2}} \text{L sinh} \left( \frac{\mathcal{C}_{\text{typical}}}{\mathcal{L}} \right) \geq \left( \frac{1}{2} \right)^{\frac{1}{2}} \left( \mathcal{L}_\mathcal{C} \right)^{\frac{1}{2}} \mathcal{C}_{\text{typical}}$$

where $L^2 \equiv 4^{\mathcal{C}} \left( -\min_{[\mathcal{C}_\mathcal{C}]} \mathcal{X}, \mathcal{X} \right)^{\mathcal{C}}$ and $\mathcal{C}_{\text{av}}$ is the geometric mean
of the penalty factors. The $\left( \frac{1}{2} \right)^{\frac{1}{2}}$ term is very close to one and can be dropped: in this
high-dimensional negatively curved space, geodesic balls grow so
rapidly that the bounds we can place on the typical complexity are very
close to the bounds we can place on the worst-case complexity (the
‘diameter’), and henceforth we will elide the two. We can also neglect
the difference between the volume of the unitary $\omega_d$ and the
volume of the unit sphere $\mathcal{C}_{\text{av}}$, since the ratio is the trifling
$\left( \frac{\mathcal{L}_\mathcal{C}}{\omega_d} \right)^\mathcal{C} \approx \frac{3}{4}$

Equation (16) then implies

$$\mathcal{C}_{\text{typical}} \geq \min \left( \sqrt{\mathcal{C}_{\text{av}}}, L \right) \equiv \min \left( \sqrt{\mathcal{C}_{\text{av}}}, \sqrt[4]{\frac{4^{\mathcal{C}}}{-\min_{[\mathcal{C}_\mathcal{C}]} \mathcal{X}, \mathcal{X} \mathcal{X} \mathcal{X}}} \right)$$

Complexity lower bounds for specific metrics
In the Supplementary Information, I apply the main result of this
article, equation (17), to a number of specific metrics. The results are
summarized in Table. 1. For the particular example of the cliff metric,
$\mathcal{J}_1 = 2 = 1$, $\mathcal{J}_{k>3} = \mathcal{J}_{\text{max}}$, the bound as a function of $\mathcal{J}_{\text{max}}$ is plotted
in Fig. 1.

Discussion
Here I used the Bishop–Gromov theorem to find a lower bound to the
complexity of a typical unitary in the complexity geometry, for a range
of penalty schedules. We saw that the Bishop–Gromov technique was
able to prove a lower bound of $4^{\mathcal{C}_\mathcal{C}}$ for the cliff schedule, and also lower
bounds for the other schedules that were also exponential in $N$. For the
special case of the cliff metric (and unlike for the other metrics,
tm there was already an exponential lower bound known due to the work
of Nielsen et al.\(^\text{22}\), namely $4^{\mathcal{C}_\mathcal{C}}$. Both this article and ref. 2 use
methods that are non-constructive, in the sense that even though both assure
you that almost all unitaries are highly complex, neither presents you with
a certified hard unitary. However my method is non-constructive
even given an oracle for gate complexity, whereas ref. 2 proved
that if a unitary is exponentially hard to approximate in gate complexity, it
is also exponentially hard to reach in the cliff metric.
In the literature, there have been two major motivations for study-
ing complexity geometry. The original motivation\(^\text{23}\) saw complexity
Table 1 | Summary of results. Summary of results for some example penalty schedules, omitting subexponential factors. On the left is the best known bound before this article, from ref. 2. On the right are the results derived in the Supplementary Information, using equation (17)

| Metric Type                      | Previous best lower bound | Lower bound in this article |
|----------------------------------|---------------------------|-----------------------------|
| Cliff metric                     |                           |                             |
| $J_{k=2} = 1$                    | $\min(\sqrt{q}, 4^{N^2})$ | $\min(\sqrt{q}, 4^{N^2/2})$ |
| $J_{k=3} = q$                    |                           |                             |
| Binomial metric                  |                           |                             |
| $J_q = (N_q q)^{1/2} \equiv \left(\begin{pmatrix} N \cr q \end{pmatrix} q\right)^{1/2}$ for some $q > 0$ | $2^{n/2}$ | |
| Exponential metric               |                           |                             |
| $J_k = x^k$ for some $x > 1$     |                           |                             |
| Any $N$ of the basis directions have penalty $q$ | (none) | $\min(\sqrt{q}, (4^k x^k N^{1/4})^{1/4})$ |

This inequality means that if the growth of the geodesic ball ever falls behind the Bishop–Gromov pace, then it can never catch up. Since we know that the geodesic ball must have engulfed the whole space by the time the diameter is reached, and since we can find an upper bound for the diameter (see, for example, equation (6)), this allows us to find a lower bound to the volume growth.

The technique developed here should also be able to find a lower bound for the diameters of right-invariant metrics on other compact Lie groups, in addition to the unitary group. I have focused on the unitary group because it has been the subject of the most previous study, and because it is the group most applicable to quantum complexity. Let us make a mathematical conjecture. In Fig. 1, we saw that for the cliff metric with $q < 4^k$ the Bishop–Gromov bound was able to determine (the exponential part of) the diameter exactly, whereas for $q > 4^k$ there was a gap between the upper bound and the Bishop–Gromov lower bound. My conjecture is that the Bishop–Gromov lower bound is tight:

\[
\frac{d}{d\varepsilon} \left. \text{volume}_{\text{ball}}(\varepsilon) \right|_{\varepsilon = 0} \leq 0. 
\]  (18)

Let us describe two pieces of circumstantial evidence in favour of this conjecture.

The first piece of evidence is that the Bishop–Gromov bound cannot be tightened further. In the Supplementary Information, I show that by using a ‘staircase’ technique, we can enhance the Bishop–Gromov lower bound on the complexity of the cliff metric from the naive $2^{n/2}$ to the improved $2^n$. This is as good as it gets: there is no better shape for the staircase that gives an even stronger lower bound. Equation (17) tells us that to get a lower bound above $2^n$ we need both $\min_k [\mathcal{J}_k \mathcal{S}_k]$ to be exponentially small (or positive) and $\mathcal{J}_{\max} > 4^k$. However, if $\mathcal{J}_2 = 1$ and $\mathcal{J}_{\max} > 4^k$, then there must exist $k$ such that $\mathcal{J}_k > 4\mathcal{J}_{k-1}$. Equation (15) tells us that this value of $k$ will make at least an $O(1)$ negative contribution to $\mathcal{J}_k$. To turn this into a rigorous proof, we need to confirm that the positive terms in equation (13) cannot fully cancel these negative contributions, and thereby give an upper bound on the most negative component of the Ricci curvature when there is a sufficiently large ratio between $\mathcal{J}_2$ and $\mathcal{J}_{\max}$.
For the second piece of circumstantial evidence, consider a new definition of complexity I will call zig-zag complexity. In zig-zag complexity you move in piecewise-linear segments. Within each segment, you move with a fixed $\mathcal{H}^F \leq 1$ time-independent 2-local Hamiltonian, for a Killing distance $\delta$. Then for the next segment you pick some new fixed 2-local Hamiltonian. The complexity of a path is defined as $\mathcal{C} = \delta \mathcal{S}$, where $\mathcal{S}$ is the number of segments. This definition interpolates between the infinite-cliff complexity geometry when $\delta = 0$ and a model that is only somewhat more powerful than the gate definition when $\delta = 1$. To reach every simple dimension-counting tells us we need $\mathcal{S} \geq 2\mathcal{N}$, so that we have enough ‘fine motor skills’ to fill out the dimensionality of the neighbouring of a point. However, even for $\mathcal{S} = 4^2$, if $\delta$ is too small we do not have enough ‘gross motor skills’ to reach every $\epsilon$ ball. Consider starting with a complexity geometry path and ‘coarse-graining’ into a zig-zag path by dividing up the path into segments of length $\delta$, and within each segment applying the time-average Hamiltonian. This introduces a per-segment killing error of about $\delta^2$, for a total killing error of $\mathcal{S} \delta^2$. This suggests that the critical values are $\mathcal{S} = 4^2$ and $\delta = 2^{-\mathcal{N}}$, and that the zig-zag diameter of the unitary group is $4^2 \delta$ for $\delta > 2^{-\mathcal{N}}$ and $4^2 \delta$ for $\delta < 2^{-\mathcal{N}}$. Taking $\delta = 0$ recovers equation (19).

Nielsen’s original vision was to use the tools of differential geometry to find a lower bound to the complexity of the complexity geometry, and then use that to prove new lower bounds on gate complexity. This article has realized the first half of that vision. When I proved complexity lower bounds for the complexity geometry, I also via equation (4) implicitly proved lower bounds for gate complexity. But those lower bounds were too weak to be new. The problem is that the gate definition of complexity is so much less permissive than the complexity geometry definition that in this case it is easier to directly prove limitations on gate complexity rather than to indirectly prove limitations via the complexity geometry. The Bishop–Gromov bound has proven itself to be a powerful tool for establishing complexity lower bounds for the complexity geometry—and provided an important existence proof that it is possible to prove complexity lower bounds using differential geometry—but more progress will be required to realize the totality of Nielsen’s vision.

Online content

Any methods, additional references, Nature Portfolio reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at https://doi.org/10.1038/s41567-022-01884-6.
20. Hackl, L. & Myers, R. C. Circuit complexity for free fermions. J. High Energy Phys. 2018, 139 (2018).

21. Raychaudhuri, A. Relativistic cosmology I. Phys. Rev. 98, 1123–1126 (1955).

22. Milnor, J. Curvatures of left invariant metrics on lie groups. Adv. Math. 21, 93–329 (1976).

23. Susskind, L. Computational complexity and black hole horizons. Fortsch. Phys. 64, 24–43 (2016).

24. Stanford, D. & Susskind, L. Complexity and shock wave geometries. Phys. Rev. D 90, 126007 (2014).

25. Brown, A. R., Roberts, D. A., Susskind, L., Swingle, B. & Zhao, Y. Holographic complexity equals bulk action? Phys. Rev. Lett. 116, 191301 (2016).

26. Brown, A. R., Roberts, D. A., Susskind, L., Swingle, B. & Zhao, Y. Complexity, action, and black holes. Phys. Rev. D 93, 086006 (2016).

27. Brown, A. R. & Susskind, L. Second law of quantum complexity. Phys. Rev. D 97, 086015 (2018).

28. Brown, A. R., Freedman, M. H., Lin, H. W. & Susskind, L. Effective geometry, complexity, and universality. Preprint at arXiv:2111.12700 [hep-th].

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

© The Author(s), under exclusive licence to Springer Nature Limited 2023
Data availability
Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Acknowledgements
I thank H. Lin, L. Susskind and, in particular, M. Freedman.

Competing interests
The author declares no competing interests.

Additional information
Supplementary information The online version contains supplementary material available at https://doi.org/10.1038/s41567-022-01884-6.

Correspondence and requests for materials should be addressed to Adam R. Brown.

Peer review information Nature Physics thanks Michal Heller, Lucas Hackl and the other, anonymous, reviewer(s) for their contribution to the peer review of this work.

Reprints and permissions information is available at www.nature.com/reprints.