Scattering of Solitary Waves in Granular Media

Lautaro Vergara

\textit{Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile}

(Dated: today)

A detailed numerical study of the scattering of solitary waves by a barrier, in a granular media with Hertzian contact, shows the existence of secondary multipulse structures generated at the interface of two ”sonic vacua”, which have a similar structure as the one previously found by Nesterenko and coworkers.

PACS numbers: 46.40.Cd; 45.70.-n; 47.20. Ky

INTRODUCTION

Nesterenko \textsuperscript{1} noticed that the propagation of a perturbation in a chain of beads in Hertzian contact possesses soliton-like features. Several studies, theoretical as well as experimental, \textsuperscript{2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15} have confirmed the existence of such soliton-like pulses. Despite the great deal of recent work on the subject \textsuperscript{3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}, the physics of granular media remains a challenge and new effects are there to be discovered and studied.

The simplest granular systems are one-dimensional chain of elastic spheres. It is known \textsuperscript{1} that in the regime where Hertz’s static solution for the contact of elastic spheres applies, the spheres may be considered as point masses interacting through massless nonlinear springs with elastic force \( F = k \delta^{3/2} \), where \( \delta \) is the overlap of contacts and \( k \) is the spring constant (a function of the material properties). Let \( v_i(t) \) represents the displacement of the center of the \( i \)-th sphere from its initial equilibrium position, and assume that the \( i \)-th sphere, of mass \( m \), has neighbor of different radius (and/or mechanical properties). Then, in absence of load and in a frictionless medium, the equation of motion for the \( i \)-th sphere reads

\[
m \frac{d^2 v_i}{dt^2} = k_1 (v_{i-1} - v_i)^{3/2} - k_2 (v_i - v_{i+1})^{3/2},
\]

(1)

where it is understood that the brackets take the argument value if they are positive and zero otherwise, ensuring that the spheres interact only when in contact.

Nesterenko, Lazaridi and Sibiryakov \textsuperscript{16} studied experimentally as well numerically the interaction of a solitary wave with the boundary of two ”sonic vacua” using a setup similar as the one shown in Figure 1, with a wall at the right hand side. They also analyzed the case where all spheres are made of the same material and that both ends of the chain are free to move.

In this work we make a detailed numerical study of the propagation of solitary waves in a linear chain of beads composed of two types of bead sizes, as shown in Figure 1, repeating the analysis done in \textsuperscript{16}. It will be assumed that all spheres are made of the same material and that both ends of the chain are free to move.

Consider a set of spheres with two different radius \( a \) and \( b \). It is known that adjacent spheres of radius \( a \) and \( b \) will interact with a force \( F = k \delta^{3/2} \), where

\[
k = \frac{\sqrt{ab/(a+b)}}{2 \theta},
\]

(2)

with

\[
\theta = \frac{3(1-\nu^2)}{4E}
\]

(3)

and \( E \) is the Young modulus and \( \nu \) the Poisson ratio of the bead material.

We will consider the scattering of solitary waves in a setup as that shown in Fig. 1, similar to the one used in \textsuperscript{16, 20}, where there are \( M \) beads, of which \( N \) has radius \( a \) and mass \( m_1 \) and \( L \) radius \( b \) and mass \( m_2 \); \( a < b \). The displacements of the beads are governed by a set of eqs. of motion that can be readily obtained from the successive application of eq. (1), having in mind that the eq. of motion for the first (resp. the last) sphere only includes the second (resp. the first) term, in case there is no wall (as we here assume).

For numerical convenience, we rescale the variables, defining

\[
u_i = \left( \frac{k_1}{m_1} \right)^{-2} v_i.
\]

(4)

In the first part of our analysis of the effect of the interface on the dynamics of solitary waves, the parameters of the system are chosen as \( N = 60, \ L = 60, \ a = 1, \ b = 2, \ m_1/m_2 = 1/8 \). We assume that initially all beads are at rest, except for the first bead at the left side of the chain. This bead is supposed to have a nonzero value of velocity in order to generate the soliton-like perturbation in the
chain. That is, we assume that the initial conditions are

\[
\begin{align*}
  u_i(0) &= 0, \ i = 1, \ldots, M, \\
  \dot{u}_1(0) &= 1, \\
  \dot{u}_i(0) &= 0, \ i = 2, \ldots, M,
\end{align*}
\]

In this case, the scattering process is as follows: the solitary wave arrives at the interface between the two "sonic vacua" and it splits into a transmitted and a reflected solitary wave as already observed in [16]. In addition to what reported there, we have observed the presence of extra multipulse structures.

In Figure 2 we have plotted the velocity of the pulses as a function of the bead number for different times \( t = 140, 165, 190, 220 \) and plots where shifted by 0.025 for clarity. At \( t = 140 \) it is observed (part of) the reflected solitary wave and behind it two pulses appear. In addition, an unresolved perturbation around the interface is present. As shown in the figure, at later times multipulse structures appears.

The perturbations in the multipulse structure possess much less energy than the solitary waves; comparing the velocity of the reflected solitary wave with the velocity of the first pulse in the multipulse structure shows that in the case at hand, the perturbation has around 3\% of the velocity of the solitary wave.

Figure 3 shows the velocity of bead 54 as a function of time. The large peak corresponds to the reflected solitary wave, while the velocity of the incident solitary wave is not shown. The two multipulse structures observed here have their counterparts in the structures observed at \( t = 240 \) in Figure 2.

It is also interesting to study the dependence of the effects that we have observed on the strength of the impact by the first bead. To that end we use as initial conditions

\[
\begin{align*}
  u_i(0) &= 0, \ i = 1, \ldots, M, \\
  \dot{u}_1(0) &= 3, \\
  \dot{u}_i(0) &= 0, \ i = 2, \ldots, M.
\end{align*}
\]

Figure 4 shows the dependence of velocity of beads 54 as a function of time, for these initial conditions. As expected, all perturbations are enhanced and because of the larger magnitude of the initial velocity, perturbations are shifted to earlier times as compared with Figure 3. No big differences are found between both cases, indicating that (at least to these orders of magnitude) the impact velocity has no incidence in the the appearance of the multipulse structures.

Before continuing with our analysis we remark that since the effects that we observe are small it is important to be sure of the accuracy of the numerics. We have numerically studied the system of equations by using an explicit Runge-Kutta method of 5th order based on the Dormand-Prince coefficients, with local extrapolation. As step size controller we have used the proportional-integral step control, which gives a smooth step size sequence. We have tested our numerics by comparing with results given in the literature as the one shown in [12] and [17], obtaining a good agreement with them. In addition, as a crude check of numerical integration accuracy we have found that the final kinetic energy of the transmitted solitary wave in the heavy system differs from its initial kinetic energy in that system by \( 10^{-7} \).

When studying the dependence of the multipulse structure as a function of the ratio between the beads, one observes that the number of pulses in the multipulse structure increases when the mass ratio \( m_1/m_2 \) decreases and viceversa. Also notice that for lower mass ratios \( m_1/m_2 \)
the multipulse structure take more time to appear. This is shown in Figures 5 where the velocity of bead 54 as a function of time is plotted. Mass ratios are chosen as $m_1/m_2 = 1/1.6^3$, $m_1/m_2 = 1/2^3$ and $m_1/m_2 = 1/2.5^3$. The curves corresponding to the last two mass ratios have been shifted by 0.05 units for clarity.

![Pulse velocity vs Time for different mass ratios](image)

**FIG. 5:** Velocity of bead 54 as a function of time for three different mass ratio.

One could ask whether similar structures also appear in the case when the solitary wave moves from the "heavy" system to the "light" system (from right to left in Figure 1). This system was first analyzed in [16] and here we reanalyze their numerical calculation. To this end we choose the system’s parameters as $a = 2, b = 1, m_1/m_2 = 8$, with 30 big beads and 170 small ones and the initial conditions.

![Snapshots at times ranging from t = 34.4 to t = 40 showing the way the pulse velocity deforms to produce the first multipulse structure. Plots were shifted by 10 units for display reasons.](image)

**FIG. 6:** Snapshots at times ranging from $t = 34.4$ to $t = 40$ showing the way the pulse velocity deforms to produce the first multipulse structure. Plots were shifted by 10 units for display reasons.

The scattering process is as follows: the solitary wave comes to the interface and no backscattered solitary wave is observed but a multipulse structure is generated as shown in Figure 6. This is a remarkable phenomenon whose origin is not completely clear to us (apart from the fact that it originates in the discreteness and nonlinearity of the system in question). It was first observed by Nesterenko and coworkers in [16].

The new effect that we report here is that a time after it appears a second multipulse structure, with similar characteristics than the first one but with less energy. This is shown in Figure 7.

![Grain displacements with respect to their original equilibrium position and pulse velocity as a function of bead number at $t = 115$. The pulse velocity has been scaled by a factor 6.](image)

**FIG. 7:** Grain displacements with respect to their original equilibrium position and pulse velocity as a function of bead number at $t = 115$. The pulse velocity has been scaled by a factor 6.

Notice that there are some similarities between this effect and the one observed before. Indeed, when the perturbation travels from the light to the heavy system, the multipulse structures are generated at the interface, leaving some energy behind the interface. If we zoom-in Figure 7 we notice that something similar happens here. Indeed, in Figure 8 and 9 we show how the secondary multipulse structure is generated by a "seed" of energy that remains at the interface after the primary multipulse structure has been generated.

![Zoom-in of Fig. 7, at time $t = 60$, showing the "seed" of energy behind the interface and part of the primary multipulse structure.](image)

**FIG. 8:** Zoom-in of Fig. 7, at time $t = 60$, showing the "seed" of energy behind the interface and part of the primary multipulse structure.

Using a detailed numerical approach, we have studied the scattering of solitary waves in a system with "sonic vacua" in a granular media with Hertzian contact. We have studied the same system as the one analyzed in [16] (except for the existence of a wall, that makes no difference in the final results) and found that, in addition to what observed there, multipulse structures with smaller energies emerge. As far as we know, this kind of secondary multipulse structures have not been observed, yet, in experiments. We are aware of a new experimental
method suitable for the investigation of solitary waves at walls (we thank Prof. Francisco Melo for giving us a copy of his paper [18] after the completion of this work). We expect that this method could be used for studying the effects shown here.

Although we have given some information about the formation of the secondary multipulse structures shown here, their origin is not completely clear to us but it is interesting to notice that similar multipulse structures appear in the scattering of two solitary waves [19] (although in that case only primary multipulse structures appear). We agree with them that part of the explanation lies in the discreteness of the medium through which the solitary waves propagate.

Finally, it is worth to mention that although the effects shown here are small, in our opinion they are important in the sense that they form part of the dynamics of the scattering of solitary waves by interfaces and then deserve to be studied and their origin further clarified.

Acknowledgements

I thank Dr. Stéphane Job for a talk that originated my attention on this interesting subject and his interest in preliminary results that encouraged me to go forward. I want to acknowledge fruitful discussions with Dr. Raúl Labbé and Dr. Stéphane Job. I would specially like to thank useful and encouraging correspondence with Prof. Vitali F. Nesterenko.

* Electronic address: lvergara@laucia.usach.cl
[1] V.F. Nesterenko, Zh. Prik. Mekh. Tekh. Fiz. 5 (1983) 733
[2] G. Friesecke and J.A.D. Wattis, Commun. Math. Phys. 161 (1994) 391
[3] S. Sen and R.S. Sinkovits, Phys Rev E 54 (1996) 6857
[4] C. Coste, E. Falcon, and S. Fauve, Phys Rev E 56 (1997) 6104
[5] S. Sen, M. Manciu and J.D Wright, Phys Rev E 57 (1998) 2386
[6] E.J. Hinch and S. Saint-Jean, Proc. R. Soc. London, Ser. A 455 (1999) 3201
[7] J. Hong and A. Xu, Phys. Rev. E 63 (2001) 061310
[8] M. Manciu, S. Sen and A.J. Hurd, Physica D 157 (2001) 226
[9] S. Sen and M. Manciu, Phys. Rev. E 64 (2001) 056605
[10] J. Lee, S. Park and Y. Yu, Phys. Rev. E 67 (2003) 066607
[11] S. Sen et al., in Modern Challenges in Statistical Mechanics: Patterns, Noise and the Interplay of Nonlinearity and Complexity, edited by V. M. Kenkre and K. Lindenberg, AIP Conference Proceedings 658 357 (2003) 357
[12] M. Nakagawa, J. H. Agui, D. T. Wu, and D. V. Extramiana, Granular Matter 4 (2003) 167
[13] A. Rosas and K. Lindenberg, Phys. Rev. E 69 (2004) 037601
[14] The Granular State, S. Sen and M.L. Hunt (Eds.), Mater. Res. Soc. Symp. Proc. No. 627, Material Research Society, Pittsburg, 2001
[15] C. Daraio, V. F. Nesterenko, E. Herbold, and S. Jin, "Strongly Nonlinear Waves in a Chain of Teflon Beads", cond-mat/0503299
[16] Dynamics of Heterogeneous Materials V.F. Nesterenko, Springer-Verlag New York, 2001; V.F. Nesterenko, A.N. Lazaridi and E.B. Sibiryakov, Jour. Aplied Mech. Tech. Phys., 36 (1995) 166
[17] A. Chatterjee, Phys. Rev. E 59 (1999) 5912
[18] How Hertzian solitary waves interact with boundaries in a 1-D granular medium, S. Job, F. Melo, A. Sokolow and S. Sen, submitted for publication to Physical Review Letters.
[19] S. Sen, M. Manciu and A.J. Hurd, Phys. Rev. E 63 (2001) 016614
[20] A series of experiments with similar setups have been done by F. Melo and coworkers at the Nonlinear Physics Laboratory of the Physics Department of Universidad de Santiago de Chile