ESTIMATES FOR OSCILLATORY INTEGRAL OPERATORS

Vyacheslav Rychkov

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Abstract

This thesis is devoted to asymptotic norm estimates for oscillatory integral operators acting on the $L^2$ space of functions of one real variable. The operators in question have compact support and an oscillatory kernel of the form $\exp(i\lambda S(x, y))$, where $S(x, y)$ is a smooth real phase function, and $\lambda$ is a large real number.

I study how the norm of the operator decays as $\lambda$ goes to infinity, and how the rate of this decay can be determined from the properties of the phase function $S(x, y)$.

For $C^\infty$ phase functions I prove results formulated in terms of the Newton polygon of $S(x, y)$, improving previously known estimates. My estimates are best possible or differ from the best possible ones by at most a power of $\log \lambda$.

I use two different methods. The first method is based on the geometric analysis of the zero set of the Hessian $S''_{xy}$ using the Puiseux decompositions. The second method is based on a stopping time argument.
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To the memory of my grandfathers,

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Chapter 1

Introduction

1.1 Formulation of the problem

My thesis studies asymptotic norm estimates for oscillatory integral operators acting on the $L^2$ space of functions of one real variable.

More precisely, I fix a real $C^\infty$ function $S(x, y)$ (called phase function) and consider a one-parameter family of operators of the form

$$T_\lambda f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x, y)} \chi(x, y) f(y) \, dy \quad (\lambda \in \mathbb{R}).$$

(1.1)

Here $\chi(x, y)$ is an unimportant $C^\infty$ cut-off function compactly supported in a small neighborhood of the origin in $\mathbb{R}^2$.

The operators $T_\lambda$ act on $L^2(\mathbb{R})$, and it is generally to be expected that for $\lambda \to \infty$ the norm $\|T_\lambda\|$ will decay. Typically, we will have

$$\|T_\lambda\| \leq C\lambda^{-\delta} \quad (\lambda \to \infty)$$

(1.2)

My thesis studies how the decay rate $\delta$ depends on the properties of the phase function $S(x, y)$. 

1
1.2 History and motivation

Hörmander [8] proved that if $S''_{xy} \neq 0$ on the support of $\chi$, then (1.2) is true with $\delta = 1/2$, and this is best possible.

Typical example falling under the scope of this result is $S(x, y) = xy$. For this choice of the phase functions $T_\lambda$ is a rescaled and cut-off version of the Fourier transform.

Some problems of harmonic analysis naturally lead to more general phase functions which do not necessarily satisfy Hörmander’s condition $S''_{xy} \neq 0$.

For instance, such phase functions arise in studying smoothing properties of generalized Radon transform associated with families of curves having various geometric degeneracies. The most direct connection exists between decay norm estimates for $T_\lambda$ and smoothing properties of the generalized Radon transform in the plane defined by

$$\mathcal{R}g(t, x) = \int_{-\infty}^{\infty} g(t + S(x, y), y)\chi(x, y) dy.$$  

Namely the decay estimate (1.2) implies that $\mathcal{R}$ is a smoothing operator of order $\delta$, that is, it acts from the Sobolev space $H^s(\mathbb{R}^2)$ into $H^{s+\delta}(\mathbb{R}^2)$ for any $s$ (see Phong and Stein [11]).

1.3 Known results for degenerate phase function

As I mentioned above, it is desirable to be able to determine the optimal exponent $\delta$ in (1.2) for phase functions with vanishing $S''_{xy}$, which are called degenerate.

My thesis addresses this problem in its local aspect. That is, I concentrate on the properties of $S(x, y)$ near the origin, and use the freedom to choose the support of the cut-off function $\chi(x, y)$ as small as I want. The typical form of results that I will state is going to be “There exists a small neighborhood of the origin $U$ such that if
As it was realized by Phong and Stein [9], the optimal exponent $\delta$ depends on the local properties of the phase function $S(x, y)$ at the origin via the Newton polygon of $S''_{xy}$.

The Newton polygon of $S''_{xy}$ is defined as follows. In the positive quadrant of the plane mark all the points with integer coordinates $(p, q)$ such that the partial derivative 

$$
\left( \frac{\partial}{\partial x} \right)^p \left( \frac{\partial}{\partial y} \right)^q S''_{xy}(0, 0) \neq 0.
$$

After that, take the marked point for which $p$ is minimal and add the vertical ray emanating from this point upward. Also, take the marked point for which $q$ is minimal and add the horizontal ray emanating from this point leftward. The Newton polygon is the convex hull of the set consisting of all marked points and two added rays (see Fig. 1).

Assume that the Newton polygon of $S''_{xy}$ is not empty, which means that not all partial derivatives of $S''_{xy}$ vanish at the origin. Denote by $t_0$ the parameter of intersection of the line $n_1 = n_2 = t$ with the boundary of the Newton polygon. The number

$$
\Delta = \frac{1}{2t_0 + 2}
$$

is called the Newton decay rate of $S(x, y)$ (this definition differs by a factor of $1/2$ from [9] and [12]).
Known norm estimates for $T_\lambda$ relevant for my work are the following:

- (Phong and Stein [9]) Lower estimate $\|T_\lambda\| \geq C\lambda^{-\Delta}$.

- (implicitly in Seeger [13],[14]) Almost sharp upper estimate $\|T_\lambda\| \leq C \varepsilon \lambda^{-\Delta+\varepsilon}$ for any $\varepsilon > 0$.

- (Phong and Stein [9]) Sharp upper estimate $\|T_\lambda\| \leq C\lambda^{-\Delta}$ under the additional assumption that $S(x, y)$ is real analytic.

The upper estimates are true provided the support of $\chi(x, y)$ is small enough. The lower estimate is true for $\chi(0,0) \neq 0$. In all three estimates $\lambda \to \infty$.

### 1.4 Main result of the thesis

The purpose of my thesis is to show that the sharp estimate proven by Phong and Stein in the real analytic case continues to hold in the $C^\infty$ case without loss of $\varepsilon$. There will be one possible exception, when one loses at most a power of log.

Consider the formal Taylor series of $S''_{xy}$ at the origin

$$S''_{xy}(x, y) \sim \sum_{p,q} c_{pq} x^p y^q, \quad c_{pq} = \frac{1}{p!q!} \partial_x^p \partial_y^q S''_{xy}(0,0).$$

I say that $S''_{xy}$ is *exceptionally degenerate*, if this series can be factored in the ring of formal power series $\mathbb{R}[[x,y]]$ into the product

$$U(x, y)(y - f(x))^N,$$

where

- $N \geq 2$,

- the series $f(x) \in \mathbb{R}[[x]]$ is of the form $f(x) = cx + \ldots$ with $c \neq 0$, and
• the series $U(x, y) \in \mathbb{R}[x, y]$ is invertible, that is its zeroth order term is nonzero.

Note that $\Delta = \frac{1}{N+2}$ for such a phase function.

The main result of the thesis is the following

**Theorem 1.1.** There exists a small neighborhood of the origin $V$ such that

(a) If $S''_{xy}$ is not exceptionally degenerate, and $\text{supp } \chi \subset V$, then

$$\|T_\lambda\| \leq C\lambda^{-\Delta} \quad (\lambda \to \infty).$$

(b) If $S''_{xy}$ is exceptionally degenerate, and $\text{supp } \chi \subset V$, then

$$\|T_\lambda\| \leq C\lambda^{-\frac{1}{N+2}}(\log \lambda)^{\frac{2N-1}{N+2}} \quad (\lambda \to \infty). \quad (1.3)$$

This theorem was proved in my paper [12], modulo an inessential improvement of the power of $\log \lambda$ in (1.3). In Chapter 5 I will show that in fact for $N = 2$ estimate (1.3) can be improved to the sharp one $\|T_\lambda\| \leq C\lambda^{-1/4}$. I do not know if a similar improvement is possible for $N \geq 3$. 

5
Chapter 2

Real analytic case

In this chapter I give an argument for the upper estimate of Phong and Stein in the real analytic case. This argument is somewhat simpler than the original proof. The main purpose is to familiarize the reader with the ideas and the technology, which will later be partially recycled in the proof of Theorem 1.1. I will explain what the main difficulty is going to be in generalizing to the $C^\infty$ case. To begin with, I review the lower bound. In this chapter no new results are proved.

2.1 Lower bound

The proof of the lower bound $\|T_\lambda\| \gtrsim \lambda^{-\Delta}$ is obtained by looking at the regions of the $(x, y)$ plane where $S(x, y) \approx \text{const} \sim 1/\lambda$ and restricting the operator to those regions.

2.1.1 Typical example

I consider the example of the polynomial phase function whose Newton polygon has only two corner points:

$$S(x, y) = c_1 x^{A_1} y^{B_1} + c_2 x^{A_2} y^{B_2} \quad (B_2 > B_1).$$  \hfill (2.1)
I also assume in this example that the segment joining points \((A_i, B_i)\) intersects the bisectrix of the \((A, B)\) plane (Fig. 2). This example captures the main idea of the proof in the general case.

We can expand \(S(x, y)\) as follows:

\[
S(x, y) = C x^A y^B \prod_{i=1}^{B_2-B_1} (y - \theta_i x^\gamma), \quad \gamma = \frac{A_1 - A_2}{B_2 - B_1}, \quad \theta_i \in \mathbb{C}.
\]

Notice that the exponent \(\gamma = \tan \theta\) in Fig. 2, while the number of branches \(B_2 - B_1\) is equal to the height of the triangle. The reader will see later that these features naturally extend to general phase functions.

Now notice that for \(y = C x^\gamma\) for generic \(C\) different from all \(\theta_i\) I have

\[
S(x, y) = \text{const.} x^{A_2 + \gamma B_1 + \gamma (B_2 - B_1)} \sim x^{A_2 + \gamma B_2}.
\] (2.2)

From the equation of the straight line passing through \((A_i, B_i)\),

\[
\frac{x - A_2}{y - B_2} = \frac{A_1 - A_2}{B_1 - B_2} = -\gamma,
\]

putting \(x = y = t_0\) I find

\[
t_0 = \frac{A_2 + \gamma B_2}{1 + \gamma}.
\]
I am looking for a region of the \((x, y)\)-plane where \(y \sim x^\gamma\) and \(S(x, y) \sim 1/\lambda\).

According to (2.2), this happens for

\[
x \sim \lambda^{-\frac{1}{\nu(1+\gamma)}}, \quad y \sim \lambda^{-\frac{\gamma}{\nu(1+\gamma)}}.
\]

It follows that I can find a rectangle \(R\) of size \(\delta_x \times \delta_y\) with sides parallel to the coordinate axes with

\[
\delta_x \sim \lambda^{-\frac{1}{\nu(1+\gamma)}}, \quad \delta_y \sim \lambda^{-\frac{\gamma}{\nu(1+\gamma)}},
\]

and such that \(\text{Re} \ e^{iAS(x, y)} > c > 0\) on \(R\).

It remains to invoke the following straightforward

Lemma 2.1. Suppose the kernel \(K(x, y)\) of an integral operator

\[
Tf(x) = \int K(x, y) f(y) dy \tag{2.3}
\]

is such that \(\text{Re} \ K(x, y) > c > 0\) on a rectangle of size \(\delta_x \times \delta_y\). Then the norm of the operator on \(L^2(\mathbb{R})\) satisfies \(\|T\| \geq c(\delta_x \delta_y)^{1/2}\).

Using the lemma, I get

\[
\|T_\lambda\| \gtrsim (\delta_x \delta_y)^{1/2} \sim \lambda^{-1/(2 \nu_0)} \tag{2.4}
\]

This is the correct answer, since I was looking at the Newton polygon of \(S\), which differs from the Newton polygon of \(S''_{xy}\) by a shift by vector \((1, 1)\).

2.1.2 General case

The general case turns out to be very similar to the example I have just considered. The argument does not use real analyticity and works generally in the \(C^\infty\) case.
I have an asymptotic expansion

\[ S(x, y) \sim \sum C_{AB} x^A y^B, \quad (2.5) \]

where \((A, B)\) runs through points inside the Newton polygon of \(S(x, y)\). I consider the edge of the Newton polygon intersecting the bisectrix (the main edge) and look at the region \(y \sim x^\gamma\), where \(\gamma\) is the exponent associated with this edge.

A simple check shows that in this region the terms in (2.5) coming from all the edges but the main one and from the inside of the Newton polygon are subleading. As a result, (2.2) is true as before, and (2.4) follows. ■

2.2 Upper bound

As I have just shown, the lower bound follows by restricting the operator to a rectangle where the phase function is effectively constant, so that the oscillatory behavior is suppressed. The upper bound, to which we proceed, is much trickier. It requires a decomposition of the \((x, y)\) plane into much bigger rectangles, on which the phase function does oscillate, in a controlled fashion.

2.2.1 Elementary tools

The following 3 elementary results are needed in the proof of the upper bound. In a sense, in most cases you just need to find the right combination of the tools which works, and do the algebra correctly.

All rectangles below are assumed to have sides parallel to the coordinate axes.

Lemma 2.2 (Size estimate). Let \(T\) be an integral operator of the form (2.3). Assume that the kernel \(K(x, y)\) is supported in a rectangle of size \(\delta_x \times \delta_y\) and bounded: \(|K| \leq 1\). Then \(T\) is bounded on \(L^2(\mathbb{R})\) with the norm \(\|T\| \leq (\delta_x \delta_y)^{1/2}\).
Lemma 2.3 (Oscillatory estimate). Let $T_\lambda$ be an oscillatory integral operator of the form (1.1). Assume that
(1) $\chi(x, y)$ is supported in a rectangle $R$ of size $\delta_x \times \delta_y$,
(2) $|\partial^n_y \chi| \leq C/\delta_y^n$ in $R$ for $n = 0, 1, 2$,
(3) $|S_{xy}''| \geq \mu > 0$ in $R$,
(4) $|\partial^n_y S_{xy}''| \leq C\mu/\delta_y^n$ in $R$ for $n = 0, 1, 2$.
Then $\|T_\lambda\| \leq \text{const}(\lambda \mu)^{-1/2}$ with const depending only on $C$.

Lemma 2.4 (Almost orthogonality). Let $\{R_j\}$ be a family of rectangles and $\{T_j\}$ be a family of integral operators of the form (2.3) such that
(1) the kernel of $T_j$ is supported in $R_j$,
(2) the family $\{R_j\}$ is almost orthogonal in the sense that for every rectangle $R_j$ the number of rectangles whose horisontal or vertical projections intersect those of $R_j$ is bounded by a constant $C$.
(3) $T_j$ are bounded on $L^2(\mathbb{R})$ with $\|T_j\| \leq A$ independent of $j$.
Then $T = \sum_j T_j$ is bounded on $L^2(\mathbb{R})$ with norm $\|T\| \leq \text{const.}A$, where const depends on $C$ only.

The proof of the size estimate is straightforward (consider $(Tf, g)$). The oscillatory estimate is a variant of the Operator van der Corput lemma of Phong and Stein [9]. The lemma is proved by a standard $TT^*$ argument. The assumptions made are enough to show, integrating by parts twice, that the kernel of $TT^*$ has the bound

$$K(x_1, x_2) \leq C \frac{\delta_y}{1 + \lambda^2 \mu^2 \delta_y^2 |x_1 - x_2|^2},$$

which implies the necessary norm estimate. We omit the details. The almost orthogonality is a trivial consequence of the Cotlar-Stein lemma.
2.2.2 Example

Once again, I start with an example. This time I take the $S''_{xy}$ rather than $S$ in the form of (2.1)

$$S''_{xy}(x, y) = c_1 x^{A_1} y^{B_1} + c_2 x^{A_2} y^{B_2} \quad (B_2 > B_1).$$

While I looked at the phase $S(x, y)$ when proving the lower bound, it is its second mixed derivative $S''_{xy}$ which is important for the upper bound.

I also assume again that the segment joining points $(A_i, B_i)$ is the main edge of the Newton polygon of $S''_{xy}$, that is it intersects the bisectrix of the $(A, B)$ plane (Fig. 2).

The proof starts by taking the dyadic decomposition of the $(x, y)$ plane into rectangles $R_{jk}$ of size $2^{-j} \times 2^{-k}$. I take a suitable smooth partition of unity fitted to this family of rectangles, and use it to localize the operator $T_{\lambda}$ to $R_{jk}$, that is, I consider the operators

$$T_{jk}f(x) = \int e^{i\lambda S(x, y)} \chi_{jk}(x, y) \chi(x, y)f(y)dy, \quad \sum T_{jk} = T_{\lambda},$$

where supp $\chi_{jk}$ is contained in the doubled rectangle $R_{jk}^*$ (Fig. 3).

I again look at the expansion

$$S''_{xy}(x, y) = Cx^{A_2} y^{B_1} \prod_{i=1}^{B_2-B_1} (y - \theta_i x^\gamma). \quad (2.6)$$
I notice that there are 3 important regions of parameters \((j, k)\). Case I: \(k \leq \gamma j - K\), where \(K\) is a large constant. Case II: \(k \geq \gamma j + K\). Case III: \(|k - \gamma j| \leq K\). These regions correspond to rectangles \(R_{jk}\) lying respectively well above, well below, and around the curve \(y = x^\gamma\) (Fig. 4).

Case I. By the size estimate (Lemma 2.2) I know the individual bounds

\[
\|T_{jk}\| \lesssim 2^{-(j+k)/2}.
\]  

(2.7)

To apply the oscillatory estimate, I need to know how large \(S_{xy}''\) is on \(R_{jk}\). It is easy to see from (2.6) that provided \(K\) is chosen large enough, I have the following estimates on \(R_{jk}^*\)

\[
|S_{xy}''| \sim 2^{-jA_2 - kB_2}.
\]  

(2.8)

In this particular situation it is easy to check that the remaining conditions of Lemma 2.3 are satisfied, so that I conclude

\[
\|T_{jk}\| \lesssim \lambda^{-1/2}2^{(jA_2 + kB_2)/2} \quad (k \leq \gamma j - K).
\]  

(2.9)

The most natural thing to do is to take a geometric mean \(a^{1-\theta}b^\theta\) of estimates (2.7) and (2.9), choosing \(\theta = 2\Delta\), so that the resulting estimate will have the desired \(\lambda\)
behavior $\lambda^{-\Delta}$. Remember that

$$\Delta = \frac{1}{2t_0 + 2} = \frac{1 + \gamma}{2(A_2 + 1) + 2\gamma(B_2 + 1)}.$$ 

Doing the algebra, I obtain the following estimate

$$\|T_{jk}\| \lesssim \lambda^{-\Delta 2^{D(k-\gamma j)}}, \quad D = \frac{B_2 - A_2}{2(A_2 + 1) + 2\gamma(B_2 + 1)}. \tag{2.10}$$

Now it’s time to invoke almost orthogonality. I split all the Case I rectangles into families indexed by a natural number $r$, putting into the $r$-th family all $R_{jk}$ such that

$$k - [\gamma j] = -r.$$

For each $r$, such a family is almost orthogonal, and so it follows from (2.10) by Lemma 2.4 that

$$\left\| \sum_{k - [\gamma j] = -r} T_{jk} \right\| \lesssim \lambda^{-\Delta 2^{-Dr}} \quad (r \geq K). \tag{2.11}$$

Notice that in general I have $B_2 \geq A_2$ (Fig. 2). Assume for the moment that $B_2 > A_2$. In this case $D > 0$, and I can sum estimate (2.11) over $r$ from $K$ to infinity, to get

$$\left\| \sum_{k - [\gamma j] \leq -K} T_{jk} \right\| \lesssim \lambda^{-\Delta}, \tag{2.12}$$

which is the required estimate.

If $B_2 = A_2$, I am going to start again from estimates (2.7) and (2.9) and use a completely different splitting into almost orthogonal families. In this case, I will put into the $r$-th family all $R_{jk}$ such that

$$j + k = r.$$
By almost orthogonality, it follows from (2.7) and (2.9) that

$$\left\| \sum_{j+k=r, \, k-\gamma j \leq -K} T_{jk} \right\| \lesssim \min(2^{-r/2}, \lambda^{-1/2} 2^r A_2/2).$$

(2.13)

The decreasing and increasing progressions under the minimum sign balance for

$$r = r_* = \frac{\log_2 \lambda}{A_2 + 1} \pm \text{const.}$$

Summing (2.13) in $r$, I get

$$\left\| \sum_{k-\gamma j \leq -K} T_{jk} \right\| \lesssim 2^{-r_*/2} \lesssim \lambda^{-1/(2A_2+2)},$$

(2.14)

which is exactly what is required.

**Case II.** It comes as no surprise that this case is going to be absolutely similar to Case I. The size estimate (2.7) stays the same, and the appropriate oscillatory estimate obtained analogously to (2.9) comes out to be

$$\|T_{jk}\| \lesssim \lambda^{-1/2} 2^{(jA_1+kB_1)/2} \quad (k \geq \gamma j + K).$$

If $B_1 = A_1$, I split into almost orthogonal families $j + k = r$ and arrive at the analogue of (2.14).

If $B_1 < A_1$ (notice that always $B_1 \leq A_1$), I do the same manipulation which led to (2.10), and get

$$\|T_{jk}\| \lesssim \lambda^{-1/2} 2^{D'(k-\gamma j)} \quad D' = \frac{B_1 - A_1}{2(A_1 + 1) + 2\gamma(B_1 + 1)}.$$

Notice that now $D' < 0$, which is exactly compatible with having to sum over $k-\gamma j \geq K$. I split into the almost orthogonal families $k - \gamma j = r \geq K$, and get the analogues of (2.11) and (2.12). Case closed.
Case III. Here I cannot get any reliable estimates on $S''_{xy}$ on the whole rectangle $R_{jk}$. Because of this, a further decomposition is required. However, the present example is a bit too special to demonstrate the method. I will deal with this situation in a more general setting in the next section.

2.2.3 General case

Now I am going to consider the general case of real analytic $S''_{xy}$. The basis of my consideration is going to be the following far-reaching generalization of (2.6) known as Puiseux theorem. This result is basically well known (see [4]).

First I look at the Newton polygon of $S''_{xy}$. In general, the polygon is going to have some number of finite edges and two infinite edges (Fig. 5). With each finite edge $\alpha$ joining points $(A_\alpha, B_\alpha)$ and $(A_{\alpha+1}, B_{\alpha+1})$, $B_{\alpha+1} > B_\alpha$, I associate numbers $\gamma_\alpha > 0$ and $n_\alpha \in \mathbb{N}$, where

$$\gamma_\alpha = \frac{A_\alpha - A_{\alpha+1}}{B_{\alpha+1} - B_\alpha}, \quad n_\alpha = B_{\alpha+1} - B_\alpha.$$

Let also $A$ and $B$ be the $x$ and $y$ coordinates of the infinite edges. Then the claim of the Puiseux theorem is that in a neighborhood of the origin there exists a factorization

$$S''_{xy} = U(x, y)x^A y^B \prod_{\alpha} \prod_{i=1}^{n_\alpha} (y - Y_{\alpha i}(x)), \quad (2.15)$$
where $Y_{\alpha i}(x)$ are convergent fractional power series with fractionality at most $x^{1/n!}$, $n = B + \sum n_{\alpha}$, whose expansion starts with

$$Y_{\alpha i}(x) = c_{\alpha i} x^{\gamma_{\alpha}} + \ldots \quad (c_{\alpha i} \in \mathbb{C} \text{ nonzero}),$$

and where $U(x, y)$ is a real analytic function with $U(0, 0) \neq 0$.

I see that looking at the Newton polygon alone gives me quite detailed information on the structure of the zero set $S''_{xy}$, as well as of its level sets. Now I am going to proceed along the lines of the example considered in the previous section.

I think of the $(x, y)$ plane, or rather of its positive quadrant, as split into pieces by curves $y = x^{\gamma_{\alpha}}$ (Fig. 6). Note that the way I number finite edges from right to left, numbers $\gamma_{\alpha}$ decrease with $\alpha$.

![Fig. 6](image)

Now I consider the dyadic partition of the positive quadrant into the rectangles $R_{jk}$, and the corresponding smooth partition of the $T_{\lambda}$ into the operators $T_{jk}$. The rectangles $R_{jk}$ fall into two categories, the ones which lie far away from any of the curves $y = x^{\gamma_{\alpha}}$, and the ones which lie close to one of these curves.

**Far away rectangles.**

Consider the rectangles lying between $y = x^{\gamma_{\alpha}}$ and $y = x^{\gamma_{\alpha+1}}$. To simplify the notation, I put $\alpha = 1$, but I do not assume that I am dealing with the rightmost finite
edge. The corresponding pairs of \((j, k)\) are singled out by the condition

\[
k - \gamma_1 j \leq -K, \quad k - \gamma_2 j \geq K
\]

\[(2.16)\]

\((K\) a large constant). It follows from (2.15) by straightforward algebra that on such a rectangle

\[
|S_{xy}''| \sim 2^{-jA_2-kB_2},
\]

which is a complete analogue of estimate (2.8).

Now if \(B_2 > A_2\), then I am again in the situation of Case I of the example from the previous section. I will split the rectangles into almost orthogonal families \(k - [\gamma_1 j] = -r\), resum, and get the estimate

\[
\left\| \sum T_{jk} \right\| \lesssim \lambda^{-\Delta_1},
\]

\[(2.17)\]

where the sum is taken over \((j, k)\) satisfying (2.16), and

\[
\Delta_1 = \frac{1}{2t_1 + 2},
\]

\[(2.18)\]

where \((t_1, t_1)\) is the point of intersection of the straight line passing through the edge \(\alpha = 1\) with the bisectrix (Fig. 7). Notice that I do not assume that \(\alpha = 1\) is the main edge of the Newton polygon.
Analogously if \( B_2 < A_2 \), I find myself in the Case II situation. So I will split the rectangles into the families \( k - [\gamma_2 j] = r \), resum, and get the estimate (2.17) with \( \Delta_2 \) instead of \( \Delta_1 \).

If \( A_2 = B_2 \), I as before split into families \( j + k = r \), and get the same estimates.

Fig. 8 shows the direction of resummation for all regions of the quadrant. Here \( \gamma_* \) denotes the exponent, corresponding to the main edge (the one intersecting the bisectrix).

Since obviously \( \Delta_\alpha \geq \Delta \) for all edges, the above discussion results in the needed estimate

\[
\left\| \sum_{\text{far away } R_{jk}} T_{jk} \right\| \lesssim \lambda^{-\Delta}.
\]

Rectangles which are close.
I look at the rectangles close to the curve $y = x^\gamma$, where $\gamma$ is one of exponents $\gamma_\alpha$. These rectangles satisfy the condition $|k - \gamma j| \leq K$ and form an almost orthogonal family. So it is sufficient to prove the bound

$$\|T_{jk}\| \lesssim \lambda^{-\Delta}$$

for each of these rectangles individually. The argument I give below is different from and shorter than the original proof of this estimate given by Phong and Stein (see [9], pp. 126–148).

It is easy to see from (2.15) that on $R_{jk}$ the $S''_{xy}$ has the following behavior

$$S''_{xy} \sim const.2^{-jA_2-j\gamma B_1}\prod_{i=1}^{n}(y - Y_i(x)),$$

where to simplify the notation I put $\alpha = 1$, so that the edge in question joins $(A_1, B_1)$ with $(A_2, B_2)$, but I am not going to assume that this is the rightmost edge. I also dropped the index $\alpha$ from $n_\alpha, \gamma_\alpha$, and $Y_\alpha(x)$.

The branches $y = Y_i(x)$ are in general complex-valued. I introduce

$$Z_i(x) = \text{Re}Y_i(x).$$

The $Z_i(x)$ are smooth analytic functions, and for $x \sim 2^{-j}$ I have

$$\frac{d}{dx}Z_i(x) \sim x^{\gamma-1} \sim 2^{-j(\gamma-1)} \sim const =: L.$$ (2.21)

Some of the curves $y = Z_i(x)$ may intersect the rectangle $R = R^*_{jk}$ (Fig. 9). The geometry of the problem suggests to take a Whitney decomposition of the set $R\setminus Z$, where

$$Z = \bigcup_i \{(x, y) : y = Z_i(x)\},$$

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into rectangles $R_l$ of size $2^{-m_l} \times L2^{-m_l}$ such that the distance from $R_l$ to $Z$ is $\sim 2^{-m_l}$ in anisotropic norm $|x| + L^{-1}|y|$.

The easiest way to arrange such a decomposition is to dilate the picture vertically by the factor of $L^{-1}$, do the usual Whitney decomposition, and contract back.

A quite obvious but important point follows from (2.21): For each $m$, the subfamily of the rectangles $R_l$ having $m_l = m$ is almost orthogonal.

Now I am going to smoothly localize $T_{jk}$ to $R_l^*$'s, denoting the corresponding partial operators $T_l$. This does not break almost orthogonality, and I have

$$\|T_{jk}\| \leq \sum_{m=j}^{\infty} \| \sum_{l \in Z} T_l \| \lesssim \sum_{m=j}^{\infty} \sup_{l : m_l = m} \| T_l \|. \quad (2.22)$$

I now turn to estimating the norm of $T_l$. Note that on $R_l^*$ I have

$$|y - Y_i(x)| \sim |y - Z_i(x)| + 2^{-j\beta_i} \sim 2^{-j(\gamma-1)-m_l} + 2^{-j\beta_i},$$

where $x^{\beta_i}$ is the first nonzero term in the expansion of $\text{Im} Z_i(x)$ ($\beta_i := \infty$ if this expansion is identically zero).

It follows that on $R_l^*$

$$|S_{xy}''| \sim 2^{-jA_2-j\gamma B_1} \prod_{i=1}^{n} (2^{-j(\gamma-1)-m_l} + 2^{-j\beta_i}) =: \mu.$$
It is also not difficult to see that supplementary conditions of Lemma 2.3

$$|\partial_y^{N} S'_{xy}| \lesssim \mu 2^{j(\gamma-1)N} \quad (N = 1, 2)$$

are satisfied on $R_l^*.$

Now I am prepared to apply the oscillatory estimate to $T_l.$ Namely, Lemma 2.3 gives

$$\|T_l\| \lesssim (\lambda \mu)^{-1/2} \leq \lambda^{-1/2} 2^{j(A_2 + B_3)}/2^{2^j(m_l - j)n}/2. \quad (2.23)$$

(The last inequality follows by taking the lower bound for $\mu$ ignoring contributions of $\beta_i,$ and also by using $n = B_2 - B_1.$)

As always, I also have the following size estimate:

$$\|T_l\| \lesssim 2^{-m_l - j(\gamma-1)/2}. \quad (2.24)$$

The natural decay in $\lambda$ that I expect is $\lambda^{-\Delta_1},$ where $\Delta_1$ has the same meaning as in (2.17), in particular,

$$\Delta_1 = \frac{1 + \gamma}{2(A_2 + 1) + 2(B_2 + 1)\gamma}.$$

So I take the geometric mean of (2.23) and (2.24) with the corresponding exponents $\theta = 2\Delta_1$ and $1 - \theta.$ As the reader may check, I get (see (2.18) for the definition of $t_1$)

$$\|T_l\| \lesssim \lambda^{-\Delta_1} 2^{-\tilde{m}(2t_1 - n)\Delta_1}, \quad \tilde{m} = m_l - j \geq 0. \quad (2.25)$$

Now please note that I may assume $\gamma \geq 1.$ Indeed, if $\gamma < 1,$ then I just switch to the adjoint of $T_{jk},$ which amounts to interchanging roles of $x$ and $y$ and transforms $\gamma \to 1/\gamma.$
Further, if \( \gamma > 1 \) strictly, then it is easy to see geometrically that necessarily \( t_1 > n/2 \) no matter how the edge of the Newton polygon lies. In this situation I can substitute (2.23) into (2.22), sum in \( \tilde{m} \), and get the required estimate (2.19) (note that \( \Delta_1 \geq \Delta \)).

The special case \( t_1 = n/2 \) can happen only if \( \gamma = 1 \) and the Newton polygon has only one finite edge joining points \((n,0)\) and \((0,n)\). In this case I avoid taking the geometric mean and substitute (2.23) and (2.24) directly into (2.22):

\[
\|T_{jk}\| \lesssim \sum_{\tilde{m}=0}^{\infty} \min(\lambda^{-1/2}2^{j\bar{n}/2+\tilde{m}\bar{n}/2}, 2^{-\tilde{m}-j}) \lesssim \lambda^{-1/(n+2)},
\]

as a simple analysis shows. (Find \( \tilde{m}_* = \tilde{m} \) for which the progressions balance. Consider the cases \( \tilde{m}_* < 0 \) and \( \tilde{m}_* \geq 0 \).) This is the right estimate in this particular case.

\[\blacksquare\]

### 2.2.4 Discussion and outlook

This finishes the proof of the upper bound in the real analytic case.

The main components of the proof, such as

- the use of the Puiseux expansion,
- the resummation procedures used to estimate far away from the branches,
- the Whitney decomposition method used close to the branches

are going to carry over to the \( C^\infty \) case either verbatim or with small modifications, as the reader will see in the coming chapters.

Jumping slightly ahead of time, I am going to say that the only crucial difference, actually the one responsible for the presence of \( \log \lambda \)'s in Theorem 1.1, is going to come from the possible occurrence of multiple real nondifferentiable branches. Namely, in the \( C^\infty \) case I may have a situation like the one shown in Fig. 10, when the zero set of
$S''_{xy}$ has several branches, which, although being close to each other to infinitely high order, are nevertheless non-coinciding and in fact nondifferentiable. This of course would be impossible in the real analytic case.

The problem with such a situation is that I lose condition (2.21), which was the basis of almost orthogonality, and the whole Whitney decomposition procedure is going to become useless near these multiple branches.

The way I am going to fight this difficulty will be to localize away from the branches by a very narrow cutoff. The Whitney decomposition will still work away from the branches, and near the branches I will have to use a completely different argument, based on a method due to Seeger [13].
Chapter 3

Smooth Puiseux theorem

The next 2 chapters are devoted to the proof of Theorem 1.1. While in the previous chapter, which was supposed to be expository, I was allowing myself to be informal at times, from now on I will strive to provide full details.

In this chapter I explain how the Puiseux expansion (2.13) generalizes from the real analytic to the $C^\infty$ case.

3.1 Algebraic notation

Proving theorems about $C^\infty$ functions often involves an intermediate step, when the analysis is done purely algebraically within the category of formal power series. I am going to employ this very strategy. Here, I will set up some algebraic notation.

First recall that for any ring $R$, the symbols $R[t]$ and $R[[t]]$ denote the rings of polynomials and, respectively, formal power series in indeterminate $t$ with coefficients from $R$. This notation can be iterated, e.g. $R[[x]][y]$ is the ring of polynomials in $y$ with coefficients which are elements of $R[[x]]$, and $R[[x, y]]$ is the ring of double formal power series.

Factorization formulas for $C^\infty$ function, which I am going to prove in this chapter, are going to be valid in a small neighborhood of the origin. Since I do not care how
small this neighborhood is, it will be convenient to formulate the results for function-germs rather than functions.

An identity involving several function-germs is defined to be true if there exist functions from the equivalence classes of these germs such that in the intersection of their domains of definition the identity is true in the usual sense.

Basically, this convention will spare me the necessity to repeat the phrase “There exists a small neighborhood of the origin $U$ such that in $U \ldots$” every time.

I will make use of the following rings of germs of $\mathbb{C}$-valued functions:

- $C((x))$ — continuous functions at the origin of $\mathbb{R}$;
- $C^\infty((x))$ and $C^\infty((x, y))$ — $C^\infty$ functions at the origin of $\mathbb{R}$ and $\mathbb{R}^2$, respectively;
- $C_+((x))$ and $C_+^\infty((x))$ — rings of one-sided germs; consist of (the equivalence classes of) functions $f(x)$ defined in a left half-neighborhood of zero of the form $[0, \varepsilon)$, where $\varepsilon > 0$ can depend on $f(x)$, which are continuous, respectively $C^\infty$, up to zero;
- $A_+((x^\gamma))$, $\gamma > 0$, — the subring of $C_+((x))$ consisting of germs $f(x)$, for which there exists a series $\overline{f}(x) \in \mathbb{C}[[x^\gamma]], \overline{f}(x) = \sum_{n=0}^{\infty} c_n x^{n\gamma}$, such that $f(x) \sim \overline{f}(x)$ in the sense that for any $N$

$$f(x) - \sum_{n=0}^{N} c_n x^{n\gamma} = O(x^{(N+1)\gamma}), \quad x \to 0.$$ 

Such an $\overline{f}(x)$ is uniquely determined and is called the asymptotic expansion of $f(x)$.

Notice that for the elements of $C^\infty((x, y))$, $C^\infty((x))$, and $C_+^\infty((x))$, I can talk about their Taylor series at the origin. A germ whose Taylor series is zero is called flat.

The rings of germs of $\mathbb{R}$-valued functions will be denoted by adding an $\mathbb{R}$ to the above notation, e.g. $\mathbb{R}C^\infty((x, y))$. 

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3.2 Puiseux decomposition of $C^\infty$ functions

Now I am going to state the main result of this chapter. The proof will be given in the following sections.

Let $F(x, y) \in \mathbb{R}C^\infty((x, y))$. Denote by $\Gamma = \Gamma(F)$ the Newton polygon of $F(x, y)$, and assume that $\Gamma \neq \emptyset$, so that $F$ is not flat.

Let $\alpha$ run through all compact edges of the boundary of $\Gamma$. For each edge $\alpha$ joining integer points $(A_\alpha, B_\alpha)$ and $(A_{\alpha}', B_{\alpha}')$, where $B_{\alpha}' > B_\alpha$, put

$$n_\alpha = B_{\alpha}' - B_\alpha, \quad \gamma_\alpha = \frac{A_\alpha - A_{\alpha}'}{B_{\alpha}' - B_\alpha}.$$

Let also $A$ be the $x$-coordinate of the vertical infinite edge, and $B$ be the $y$-coordinate of the horizontal infinite edge of $\Gamma$.

**Proposition 3.1.** In the above conditions, the germ $F(x, y)$ admits in the region $x, y > 0$ a factorization of the form

$$F(x, y) = U(x, y) \prod_{i=1}^{A} (x - X_i(y)) \prod_{i=1}^{B} (y - Y_i(x)) \prod_{\alpha} \prod_{i=1}^{n_\alpha} (y - Y_{\alpha i}(x)), \quad (3.1)$$

where

1. $U(x, y) \in \mathbb{R}C^\infty((x, y))$, $U(0, 0) \neq 0$,
2. all $X_i(x), Y_i(x) \in C_+(x)$, and $X_i(x), Y_i(x) = O(x^N)$ as $x \to 0$ for any $N > 0$,
3. all $Y_{\alpha i}(x) \in A_+(x^{1/n})$ for $n = B + \sum_{\alpha} n_\alpha$ with asymptotic expansions of the form $Y_{\alpha i}(x) = c_{\alpha i} x^{\gamma_\alpha} + \ldots$ as $x \to 0$, where $c_{\alpha i} \neq 0$,
4. if $Y(x)$ is any of the functions $Y_{\alpha i}(x)$, and if $f(x, y) = \prod (y - Y_{\alpha i}(x))$ is the product over all $i$ such that $Y_{\alpha i}(x)$ has exactly the same asymptotic expansion as $Y(x)$, then $f(x^{n_\alpha}, y) \in C^\infty_+(x)[y]$, if in (4) I additionally assume that the asymptotic expansion of $Y(x)$ is real, then $f(x, y)$ is also real.
This result copies (2.13) in the part that concerns the number of branches and the leading terms in their asymptotic expansion. However, there are also substantial differences, such as:

- in general, branches $X_i(y)$ and $Y_i(x)$ infinitely tangent to coordinate axes but not coinciding with them are present;
- I cannot claim that branches $Y_{\alpha i}(x)$ are differentiable; (4) is the best that is true in general.

These differences are for real, as very simple example show. For instance, one can take $S''_{xy} = y^2 + a(x)y + b(x)$ with a choice of coefficients so that the determinant oscillates around zero as $x \to 0$.

To the best of my knowledge, Puiseux decompositions of $C^\infty$ functions in the form of Proposition 3.1 or of a similar kind have not appeared in the literature before. However, granted Lemma 3.2 below, the proof of Proposition 3.1 follows a rather standard path, well known say in the singularity theory of $C^\infty$ and analytic functions, see e.g. Arnold et.al. [1], or Artin [2].

### 3.3 Preparation to the proof

The proof relies on the following result, which is well known in the theory of plane algebraic curves under the same generic name of the Puiseux theorem. A proof can be found in [17], p. 98ff, or [3], A.V.150.

**Lemma 3.2.** Let $\overline{F}(x, y) \in \mathbb{C}[[x]][y]$ be of the form

$$
\overline{F}(x, y) = y^n + \overline{c}_{n-1}(x)y^{n-1} + \ldots + \overline{c}_0(x), \quad \overline{c}_i(x) \in \mathbb{C}[[x]],
$$

where the zeroth order terms of all $c_i(x)$ vanish. Let $\alpha, n_{\alpha}, \gamma_\alpha, B$ be defined via the Newton polygon $\Gamma = \Gamma(\overline{F})$ in the same way as in the Proposition. Then there exists
a factorization

$$F(x, y) = y^B \prod_{\alpha} \prod_{i=1}^{n_{\alpha}} (y - \tilde{Y}_{\alpha i}(x)),$$

where the series $\tilde{Y}_{\alpha i}(x) \in \mathbb{C}[[x^{1/n}]]$ are of the form $\tilde{Y}_{\alpha i}(x) = c_{\alpha i}x^{r_{\alpha i}} + \ldots$ with $c_{\alpha i} \neq 0$.

The following lemma will be used to pass from factorizations of formal power series (obtained via Lemma 3.2) to factorizations of function-germs in the $C^\infty$ category. The proof uses standard technology usually applied in such situations.

**Lemma 3.3.** Let $P(x, y) \in C^\infty((x))[y]$ be of the form

$$P(x, y) = y^n + c_{n-1}(x)y^{n-1} + \ldots + c_0(x), \quad c_i(x) \in C^\infty((x)),$$

where $c_i(0) = 0$ for all $i$. Let $\bar{P}(x, y) \in \mathbb{C}[[x]][y]$ be the formal Taylor series of $P(x, y)$ at the origin. Let $\bar{Y}(x) \in \mathbb{C}[[x]]$ be a root of multiplicity $m$, $1 \leq m \leq n$, of $\bar{P}(x, y)$ considered as a polynomial in $y$, which means that

$$\bar{P}(x, \bar{Y}(x)) = \ldots = \bar{P}_y^{(m-1)}(x, \bar{Y}(x)) = 0, \quad \bar{P}_y^{(m)}(x, \bar{Y}(x)) \neq 0$$

as elements of $\mathbb{C}[[x]]$. Then there exist $m$ function-germs $Y_1(x), \ldots, Y_m(x) \in C((x))$ such that

1. all $Y_i(x) \sim \bar{Y}(x)$ as $x \to 0$,
2. all $P(x, Y_i(x)) = 0$ for $x > 0$,
3. $\prod_{i=1}^{m}(y - Y_i(x)) \in C^\infty((x))[y]$,
4. if we additionally assume that $P(x, y)$ and $\bar{Y}(x)$ are real, then $\prod_{i=1}^{m}(y - Y_i(x))$ is also real.

**Proof.** Let $\bar{Y}(x)$ be a $C^\infty$ function with the formal Taylor series $\bar{Y}(x)$, supplied by E. Borel’s theorem. Denote

$$\delta_i(x) = \frac{1}{i!} P_y^{(i)}(x, \bar{Y}(x)), \quad i = 0, \ldots, n.$$
Let the (nonzero by assumption) series $P_y^{(m)}(x, Y(x))$ starts with a term $cx^s$, $c \neq 0$, $s \in \mathbb{Z}_+$. Then I have $\delta_m(x) = cx^s + o(x^s)$ as $x \to 0$. On the other hand, the functions $\delta_0(x), \ldots, \delta_{m-1}(x)$ are flat.

I will be looking for $Y_i(x)$ of the form

$$Y(x) = \tilde{Y}(x) + \delta_m(x)\alpha(x),$$

where $\alpha(x)$ is an unknown continuous $\mathbb{C}$-valued function-germ such that $\alpha(x) = O(x^N)$ as $x \to 0$ for any $N > 0$.

By Taylor’s formula, the equation $P(x, Y(x)) = 0$ can be written as

$$\sum_{i=0}^{n} \delta_i(x)[\delta_m(x)\alpha(x)]^i = 0. \tag{3.3}$$

For small $x$, this is equivalent to the equation $w(x, \alpha(x)) = 0$ for the function $w(x, z)$ given by

$$w(x, z) = \sum_{i=0}^{n} \delta_i(x)[\delta_m(x)]^{i-m-1}z^i.$$  

Note that if $f, g \in C^\infty$, and $f$ is flat at the origin, while $g$ is not flat, then $f/g$ is $C^\infty$ near the origin and is flat. So the performed division by $[\delta_m(x)]^{m+1}$ is legitimate, and $w(x, z) \in C^\infty((x))[z]$.

On the complex circle $|z| = x^N$, $N > 0$, the term $z^m$ will dominate the other terms in $w(x, z)$ if $x$ is sufficiently small. By Rouche’s theorem it follows that the equation $w(x, z) = 0$ has for small fixed $x$ exactly $m$ roots in the disc $|z| < x^N$, which I denote $\alpha_i(x)$, $i = 1, \ldots, m$. I can arrange so that $\alpha_i(x)$ are continuous in $x$, and the previous argument shows that $\alpha_i(x) = O(x^N)$ for any $N > 0$.

We now prove (3). Since the functions $\alpha_i(x)$ enter the product in (3) in a symmetric way, it is sufficient to prove that the elementary symmetric polynomials $s_1, \ldots, s_m$ in $\alpha_i(x)$ are in $C^\infty((x))$. By the Newton relations (see [3], A.IV.70), it is sufficient to
prove the same for the functions

\[ p_k(x) = \sum_{i=1}^{m} [\alpha_i(x)]^k, \quad k = 1, \ldots, m. \]

However, by Cauchy’s formula I have that for small \( x \)

\[ p_k(x) = \frac{1}{2\pi i} \oint_{|z| = \varepsilon} \frac{z^k w'(x, z)}{w(x, z)} \, dz, \]

from where it is clear that \( p_k(x) \in C^\infty(\mathbb{R}) \), since nothing dramatic happens to \( w(x, z) \) on the circle \(|z| = \varepsilon\).

To prove (4), I notice that under the additional assumption made I can take \( \tilde{Y}(x) \) to be real. Then \( w(x, z) \in \mathbb{R}C^\infty(\mathbb{R})[z] \), and therefore non-real roots \( \alpha_i(x) \) will appear in conjugate pairs. Then all \( p_k(x) \) will be real, which implies (4).

Now I am going to combine two previous lemmas to prove

**Lemma 3.4.** Proposition 3.1 is true if \( F(x, y) \in \mathbb{R}C^\infty(\mathbb{R})[y] \).

**Proof.** By Lemma 3.2, the Taylor series \( \overline{F}(x, y) \in \mathbb{R}[x][y] \) of \( F(x, y) \) has a factorization (3.2). Consider the function \( P(x, y) = F(x^n, y) \). Its Taylor series has the form \( \overline{P}(x, y) = \overline{F}(x^n, y) \), and so factorizes as

\[ \overline{P}(x, y) = y^B \prod_{\alpha} \prod_{i=1}^{n_{\alpha}} (y - Y_{\alpha i}(x^n)). \]

Let \( \overline{Y}(x) \) be one of the series \( \overline{Y}_{\alpha i}(x^n) \in \mathbb{C}[x] \), and assume that among all the \( \overline{Y}_{\alpha i}(x^n) \) there are exactly \( m \) series coinciding with \( \overline{Y}(x) \). Then \( y = \overline{Y}(x) \) is a root of multiplicity \( m \) of the polynomial \( \overline{P}(x, y) \in \mathbb{R}[x][y] \), and by Lemma 3.3 I conclude that there exist \( m \) functions \( Y_i(x) \in C(\mathbb{R}) \), \( i = 1, \ldots, m \), such that (1)–(3) from the formulation of the lemma are true.

In view of (3), we can divide \( P(x, y) \) by \( \prod(y - Y_i(x)) \), and the result is again a
polynomial \( \tilde{P}(x, y) \) from \( C^\infty((x))[y] \). The Taylor polynomial of \( \tilde{P}(x, y) \) will be \( \overline{P}(x, y) \) divided by \( (y - \overline{Y}(x))^m \). Now we can apply Lemma 3.3 to \( \tilde{P}(x, y) \) choosing a different \( \overline{Y}(x) \) etc.

By repeating this operation several times, I get a complete factorization of \( P(x, y) \). The required factorization of \( F(x, y) \) is then obtained by the inverse substitution \( x \mapsto x^{1/n!} \). The property (5) is ensured by splitting off all real series \( \overline{Y}(x) \) before non-real ones in the above argument.

Proposition 3.1 will be reduced to Lemma 3.4 by means of the following Malgrange preparation theorem (see [6], p. 95).

Lemma 3.5. Let \( F(x, y) \in \mathbb{R}C^\infty((x, y)) \), and assume that \( F(0, y) \) is not flat, so that \( F(0, y) = cy^n + o(y^n) \), \( y \to 0 \), for some \( n \in \mathbb{Z}_+ \) and \( c \neq 0 \). Then there is a factorization

\[
F(x, y) = U(x, y)P(x, y),
\]

where

1. \( U(x, y) \in \mathbb{R}C^\infty((x, y)), U(0, 0) \neq 0 \),
2. \( P(x, y) \in \mathbb{R}C^\infty((x))[y] \) is of the form

\[
P(x, y) = y^n + c_{n-1}(x)y^{n-1} + \ldots + c_0(x),
\]

where all \( c_i(x) \in \mathbb{R}C^\infty((x)), c_i(0) = 0 \).

### 3.4 Proof of the Proposition

Notice that the Newton polygon is invariant with respect to multiplication by a nonzero \( C^\infty \) function (see Phong and Stein [4], p. 112). Therefore, for the functions \( F(x, y) \) such that \( F(0, y) \) is not flat (which is equivalent to having \( A = 0 \)) the proposition follows immediately from Lemmas 3.5 and 3.4.
Assume now that $A > 0$. In this case we must somehow separate the roots infinitely tangent to the $y$-axis. This can be done as follows. Since $F(x, y)$ is not flat at the origin, there exists a rotated orthogonal system of coordinates $(x', y')$ such that the restriction of $F$ to the $y'$-axis is not flat. So we can apply Lemma 3.5 to $F$ written in coordinates $(x', y')$. Let $P(x', y')$ be the arising polynomial.

If $y' = ax'$ is the equation of the old $y$-axis in the new coordinates, then $y' = ax'$ will be a root of multiplicity $A$ of $P(x', y') \in \mathbb{R}[[x']][y']$. So we can apply Lemma 3.3 and obtain $A$ roots $y' = Y_i(x')$, $i = 1, \ldots, A$, of $P(x', y') = 0$, such that $Y_i(x') \sim ax'$.

Moreover, by Lemma 3.3 (3),(4) we will have that $Q(x', y') = \prod (y' - Y_i(x'))$ is in $\mathbb{R}C^\infty((x'))[y']$. So we can divide $P(x', y')$ by $Q(x', y')$, and the quotient will be a $C^\infty$ function, which is no longer flat on the old $y$-axis.

Let $\tilde{F}(x, y)$ be this last quotient written in the old system of coordinates. Then the Newton polygon of $\tilde{F}$ is just $\Gamma(F)$ shifted $A$ units to the left. So we can factorize $\tilde{F}(x, y)$ as in the case $A = 0$ described above.

It remains to get a factorization of $Q(x', y')$ in the old coordinates. It is clear that the Taylor series of $Q$ written in the coordinates $(x, y)$ consists of one term $cx^A$. Interchanging the roles of $x$ and $y$ brings us back to the case $A = 0$, and the required factorization of the form $\prod (x - X_i(y))$ can be obtained as described above.
Chapter 4

Upper bound for smooth case

In this chapter I am going to prove Theorem [1.1]

4.1 Beginning of the proof

The proof starts just like in the real analytic case.

I decompose the operator $T_\lambda$ as

$$T_\lambda = \sum_{\pm} \sum_{j,k} T^{\pm \pm}_{jk},$$

where $T^{++}_{jk}$ is defined as

$$T^{++}_{jk}f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \chi_j(x) \chi_k(y) \chi(x,y) f(y) dy.$$ 

Here $\sum_j \chi_j(x) = 1$ is a smooth dyadic partition of unity on $\mathbb{R}^+$, so that the kernel of $T^{++}_{jk}$ is supported on the rectangle $R_{jk} = [2^{-j-1}, 2^{-j+1}] \times [2^{-k-1}, 2^{-k+1}]$. Three other $\pm$ combinations refer to the quadrants defined by specific signs of $x$ and $y$. We restrict ourselves with the positive quadrant, the other ones being exactly similar, and denote $T^{++}_{jk}$ by simply $T_{jk}$. 
By Proposition 3.1 applied to $F(x, y) = S''_{xy}(x, y)$, there is a neighborhood of the origin $V$ such that in $V \cap \mathbb{R}^2_+$ there exists a factorization of the form (3.1). I assume that $\text{supp} \chi \subset V$. The singular variety

$$Z = \{(x, y) \in V : F(x, y) = 0\}$$

now splits into branches corresponding to the factors in the RHS of (3.1). Note, however, that some of these branches may contain an imaginary component.

Let $R_{jk}^*$ denote the double of $R_{jk}$. I fix a large constant $D$ such that if the pair $(j, k)$ satisfies the condition $\min_{\alpha} |k - j\gamma_{\alpha}| \geq D$, then $y - c_{\alpha i}x^{\gamma_{\alpha}} \neq 0$ on $R_{jk}^*$ for all $c_{\alpha i}x^{\gamma_{\alpha}}$ occurring as the lowest order terms of the asymptotic expansions of $Y_{\alpha i}(x)$ in Proposition 3.1.

Let me number the compact edges $\alpha$ of the boundary of the Newton polygon $\Gamma(F)$ from right to left, so that

$$\gamma_1 > \gamma_2 > \cdots > \gamma_{\alpha_0},$$

where $\alpha_0$ is the total number of compact edges. Also put $\gamma' = \gamma_{\alpha_0}/2$ if $A > 0$, $\gamma' = 0$ otherwise; $\gamma'' = 2\gamma_1$ if $B > 0$, $\gamma'' = \infty$ otherwise.

Consider the following splitting of $T_\lambda$:

$$T_\lambda = \left(T' + T'' + \sum_{\nu = 1}^{\alpha_0 - 1} T_{\nu}\right) + (T_x + T_y) + \sum_{\nu = 1}^{\alpha_0} T'_{\nu}. \quad (4.1)$$

Here

$$T_{\nu} = \sum_{j\gamma_{\nu+1} \ll k \ll j\gamma_{\nu}} T_{jk}, \quad 1 \leq \nu \leq \alpha_0 - 1, \quad (4.2)$$

$$T' = \sum_{j\gamma' \ll k \ll j\gamma_0} T_{jk}, \quad T'' = \sum_{j\gamma_1 \ll k \ll j\gamma''} T_{jk},$$

($a \ll b$ stands for $a \leq b - D$) constitute the part of $T_\lambda$ supported relatively far away
from $\mathcal{Z}$. Further,

$$T_y = \sum_{k \ll \gamma_j} T_{jk}, \quad T_x = \sum_{\gamma''_j \ll k} T_{jk},$$

constitute the part of $T_\lambda$ supported near the branches of $\mathcal{Z}$ which are infinitely tangent to the coordinate axes. Finally

$$T^\nu = \sum_{\gamma_j - D < k < \gamma_j + D} T_{jk}, \quad 1 \leq \nu \leq \alpha_0,$$

are the part of $T_\lambda$ supported near all other branches of $\mathcal{Z}$.

In the following sections I will prove the upper bound claimed in Theorem 1.1 for the norms of all operators in the RHS of (4.1): $T'$, $T''$ and $T_\nu$ (Section 4.2), $T_x$ and $T_y$ (Section 4.3), and finally $T^\nu$ (Sections 4.4 and 4.5). This will prove Theorem 1.1.

### 4.2 Estimates far away from $\mathcal{Z}$

In this section, I prove that $\|T_\nu\| \leq C\lambda^{-\Delta}$ for each $\nu$. The reader will believe me that with minor modifications the argument given below will also produce the same estimate for $T'$, $T''$.

The proof is very similar to the argument given in Section 2.2.3 for far away rectangles. I will just provide some extra details about estimating the size of $F = S''_{xy}$ on the support of $T_{jk}$ and about checking conditions of Lemma 2.3. In what concerns subsequent resummation of the individual $\|T_{jk}\|$ estimates, the argument goes through verbatim.

Take an operator $T_{jk}$ entering the RHS of (4.2). I may reduce $V$ if necessary so that on the part of $\mathcal{Z}$ inside $V$ the functions $Y_{\alpha_i}$ do not differ much from the first terms of their asymptotic expansions. Assume that $T_{jk}$ is nonzero, which means that $R_{jk} \cap V \neq \emptyset$. Then it is clear from the definition of the constant $D$ that the factors
in the RHS of (3.1) can be estimated as follows for \((x, y) \in R_{jk}\) (see Fig. 10a):

\[
|x - X_i(y)| \approx 2^{-j},
\]

\[
|y - Y_i(x)| \approx 2^{-k},
\]

\[
|y - Y_{\alpha i}(x)| \approx \begin{cases} 2^{-k}, & \alpha \leq \nu, \\ 2^{-j\gamma_\alpha}, & \alpha > \nu. \end{cases}
\]  

(4.4)

(a \approx b \text{ means } C^{-1}b \leq a \leq Cb, \text{ where } C > 0 \text{ is an unimportant constant independent of } j, k, \lambda).\

Therefore it follows from (3.1) that on \(R_{jk}\)

\[
|F| \approx 2^{-jA_0 - kB_0} \prod_{\alpha \leq \nu} 2^{-kn_\alpha} \prod_{\alpha > \nu} 2^{-j\gamma_\alpha n_\alpha} =: \mu. \tag{4.5}
\]

The numbers \(\gamma_\alpha, n_\alpha\) can be found from the Newton polygon \(\Gamma(F)\) as described in Proposition 3.1. Using this information, I find that

\[
\mu = 2^{-jA_0' - kB_0'}.\]
I further claim that on $R_{jk}$

$$|\partial_y^n F| \lesssim \mu 2^{kn}, \quad n = 1, 2. \quad (4.6)$$

Indeed, when differentiating the RHS of (3.1) in $y$, the derivative can fall one either $U(x, y)$, or $\prod (x - X_i(y))$, or one of the remaining terms. In the first case, I simply get a bounded factor. In the second case, I get an even better factor of $O(2^{-kN})$ for any $N > 0$, since the product in question is a $C^\infty$ function whose Taylor series at the origin is $x^A$. Finally, in the third case I get a factor of the form $(y - Y_i(x))^{-1}$ or $(y - Y_{\alpha}(x))^{-1}$, which is $O(2^k)$ in view of (4.4). This argument works equally well for the second derivative, giving (4.6).

The rectangle $R_{jk}$ is of size $\delta_x \times \delta_y$ with $\delta_x \approx 2^{-j}$, $\delta_y \approx 2^{-k}$. So the conditions of Lemma 2.3 are satisfied, and I obtain the oscillatory estimate

$$\|T_{jk}\| \lesssim \lambda^{-1/2} 2^{(jA'_\nu + kB'_\nu)/2}. \quad (4.7)$$

On the other hand, the size estimate following from Lemma 2.2 is

$$\|T_{jk}\| \lesssim 2^{(j+k)/2}. \quad (4.8)$$

The rest of the proof goes through exactly as described in Section 2.2.3. Namely, I am going to split the operators $T_{jk}$ constituting $T_\nu$ into almost orthogonal families $k - [\gamma_\nu j] = -r$, or $k - [\gamma_{\nu+1} j] = r$, or $k + j = r$, depending if $B'_\nu$ is larger, smaller, or equal to $A'_\nu$. Then I am going to resum and get the $\lambda^{-\Delta}$ estimate for $\|T_\nu\|$. I will not repeat the details.
4.3 Estimates near the coordinate axes

In this section, I will prove the estimate $\|T_x\| \lesssim \lambda^{-\Delta}$. The same estimate will be true for $T_y$, since taking the adjoint of $T$ brings $T_y$ to the form of $T_x$. I may of course assume $B \geq 1$, since otherwise $\gamma'' = \infty$ and $T_x = 0$.

Notice that in the real analytic case there was no need to introduce this special localization along the coordinate axis. In the notation of Section 4.1, operator $T_y$ could be included into the $T'$ part and treated along the same lines as the $T_\nu$. Analogously $T_x$ could be united with $T''$. However, in the $C^\infty$ case the possible presence of the branches infinitely tangent to coordinate axes asks for this additional localization.

I represent $T_x$ as (see Fig. 10b)

$$T_x = \sum_j T_j, \quad T_j = \sum_{k: \gamma'' j \ll k} T_{jk},$$

and claim that

1. $\|T_j\| \lesssim \lambda^{-\Delta}$,
2. $\|T_j^* T_{j'}\| = 0$ for $|j - j'| \geq 2$,
3. $\|T_j T_{j'}^*\| \lesssim \lambda^{-2\Delta} 2^{-\varepsilon |j - j'|}$ for some $\varepsilon > 0$.

If I prove all these, the estimate $\|T_x\| \lesssim \lambda^{-\Delta}$ will follow from the Cotlar–Stein lemma.
I have
\[ T_j f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x, y)} \chi_j(x) \tilde{\chi}_j(y) \chi(x, y) f(y) \, dy, \]
where \( \tilde{\chi}_j = \sum_{\gamma'' \ll k} \chi_k \), so that the support of \( \tilde{\chi} \) is contained in \([0, C2^{-\gamma''}]\). The property (2) is obvious. Further, the operator \( T_j T_j^* \) has the kernel
\[ K(x_1, x_2) = \chi_j(x_1) \chi_{j'}(x_2) \int e^{i\lambda [S(x_1, y) - S(x_2, y)]} \tilde{\chi}_j(y) \tilde{\chi}_{j'}(y) \chi(x_1, y) \chi(x_2, y) \, dy. \]

I want to estimate this by the following variant of the standard van der Corput lemma (see [15], Corollary on p. 334).

**Lemma 4.1.** Let \( k \) be a positive integer, \( \Phi \in C^k[a, b], \Psi \in C^1[a, b] \), and assume that \( \Phi^{(k)} \geq \mu > 0 \) on \([a, b]\). If \( k = 1 \), assume additionally that \( \Phi' \) is monotonic on \([a, b]\).

Then
\[ \left| \int_{a}^{b} e^{i\lambda \Phi(y)} \Psi(y) \, dy \right| \lesssim (\lambda \mu)^{-1/k} \left( |\Psi(b)| + \int_{a}^{b} |\Psi'| \right). \]

Assume that \( j' \geq j \). I apply this lemma with \([a, b] = [0, C2^{-\gamma''}]\), \( k = B + 1 \geq 2\),
\[ \Phi(y) = S(x_1, y) - S(x_2, y), \]
\[ \Psi(y) = \tilde{\chi}_j(y) \tilde{\chi}_{j'}(y) \chi(x_1, y) \chi(x_2, y). \]

It is clear that \( |\Psi(b)| + \int_{a}^{b} |\Psi'| \lesssim 1 \). Further (recall that I denoted \( S''_{xy} = F \)),
\[ \Phi^{(B+1)}(y) = \partial_y^{B+1} S(x_1, y) - \partial_y^{B+1} S(x_2, y) = \int_{x_2}^{x_1} \partial_y^B F(x, y) \, dx. \]

Of all the terms arising when I differentiate (3.1) \( B \) times in \( y \), the term in which all derivatives fall on \( \prod(y - Y_j(x)) \) will dominate on the support of \( T_x \) after a possible reduction of \( V \). It follows that on the support of \( T_x \)
\[ |\partial_y^B F| \approx x^{A + \sum_n n_\alpha \gamma_\alpha} = x^{A_1}, \]

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where $(A_1, B)$ is the common vertex of the horizontal infinite edge of $\Gamma(F)$ and its first compact edge $\alpha = 1$.

By the previous remarks,

$$|\Phi^{(B+1)}(y)| \gtrsim |x_1^{A_1+1} - x_2^{A_1+1}|.$$  

In the case $j' = j$ I have $|x_1^{A_1+1} - x_2^{A_1+1}| \approx 2^{-jA_1}|x_1 - x_2|$ on the support of $K(x_1, x_2)$, so Lemma 4.1 gives

$$|K(x_1, x_2)| \leq 2^{jA_1/(B+1)}(\lambda|x_1 - x_2|)^{-1/(B+1)}.$$  

I apply the following variant of the Schur test (see e.g. [7], Theorem 5.2).

**Lemma 4.2.** Let $T$ be an integral operator on $L^2(\mathbb{R})$ with kernel $K(x, y)$,

$$Tf(x) = \int_{-\infty}^{\infty} K(x, y)f(y) \, dy.$$  

Assume that the quantities

$$M_1 = \sup_y \int |K(x, y)| \, dx, \quad M_2 = \sup_x \int |K(x, y)| \, dy$$  

are finite. Then $T$ is bounded with $\|T\| \leq (M_1M_2)^{1/2}.$

By this lemma and the estimate of $K(x_1, x_2)$ I have just obtained,

$$\|T_j T_j^*\| \lesssim 2^{jA_1/(B+1)} \int_0^{2^{-j}} (\lambda t)^{-1/(B+1)} \, dt \lesssim \lambda^{1/(B+1)} 2^{j(A_1-B)/(B+1)}.$$  

This of course implies the estimate

$$\|T_j\| \lesssim \lambda^{-1/(2B+2)} 2^{j(A_1-B)/(2B+2)}. \quad (4.9)$$
As usual, by Lemma 2.2 I also have a size estimate:

$$\|T_j\| \lesssim 2^{-j(1+\gamma'')/2} \leq 2^{-j(1+\gamma_1)/2}. \quad (4.10)$$

As the reader may check, taking the geometric mean of these two bounds which kills the $j$-factor gives exactly $\|T_j\| \lesssim \lambda^{-\Delta_1}$, with $\Delta_\nu$ defined as

$$\Delta_\nu = \frac{1 + \gamma_\nu}{2(1 + A_\nu) + 2(1 + B_\nu)\gamma_\nu}.$$

This implies (1) since all $\Delta_\nu \geq \Delta$.

In proving (3), I may assume $j' \geq j + 2$. Then $|x_1^{A_1+1} - x_2^{A_1+1}| \approx 2^{-j(A_1+1)}$ on the support of $K(x_1, x_2)$, whence by Lemma 4.1

$$|K(x_1, x_2)| \lesssim \lambda^{-1/(B+1)} 2^{j(A_1+1)/(B+1)} =: M.$$

The support of $K(x_1, x_2)$ is contained in the rectangle of size $\approx 2^{-j} \times 2^{-j'}$. Now Lemma 2.2 gives a bound improved by a factor of $M$:

$$\|T_j^* T_{j'}^*\| \lesssim M 2^{-(j+j')/2} = \lambda^{-1/(B+1)} 2^{j(A_1-B)/(B+1)} 2^{-J/2},$$

where I denoted $J = j' - j$. By multiplying the estimates (4.10) for $T_j$ and $T_{j'}$, I get another bound:

$$\|T_j^* T_{j'}^*\| \lesssim 2^{-j(1+\gamma_1)} 2^{-J(1+\gamma_1)/2}.$$

These two bounds have the form of (4.9) and (4.10) squared, but with an additional factor exponentially decreasing in $J$. Therefore it is clear that this time taking the geometric mean killing the $j$-factor will give

$$\|T_j^* T_{j'}^*\| \lesssim \lambda^{-2\Delta_1} 2^{-\varepsilon J}$$

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for some $\varepsilon > 0$. This implies (3) and concludes the treatment of $T_x$.

### 4.4 Estimates near $Z$

This next two sections are devoted to proving upper bounds for $T^\nu$.

Notice that the sum in (4.3) is almost orthogonal, since the $x$- and $y$-supports of $T_{jk}$ and $T_{{j'}k'}$ are disjoint for $|j - {j'}|$ larger than a fixed constant. Therefore it suffices to estimate each $T_{jk}$ from the RHS of (4.3) individually.

Fix such a $T_{jk}$. For quite a while the proof is going to proceed exactly like the argument in the part of Section 2.2.3 dealing with close rectangles. Analogously to (4.5), on $R_{jk}$

$$|F| \approx 2^{-jA} 2^{-j\gamma_\nu B} \prod_{\alpha < \nu} 2^{-j\gamma_\nu n_\alpha} \prod_{\alpha > \nu} 2^{-j\gamma_\alpha n_\alpha} \prod_{i=1}^{n_\nu} |y - Y_{\nu i}(x)| \quad (4.11)$$

where I ordered $Y_{\nu i}$ so that for $n'_{\nu} < i \leq n_\nu$ we have $\text{Re } c_{\nu i} = 0$ in $Y_{\nu i} = c_{\nu i} x^{\gamma_\nu} + \ldots$.

Let me quickly dispose of the case $n'_{\nu} = 0$, in which I can apply Lemma 2.3 (the condition (4) is easily checked) and Lemma 2.2 to get the oscillatory and size estimates

$$\|T_{jk}\| \lesssim \lambda^{-1/2} 2^{j(\gamma_\nu B_\nu + A_\nu)/2},$$

$$\|T_{jk}\| \lesssim 2^{-j(1+\gamma_\nu)/2}.$$

Now by taking the geometric mean killing the $j$-factor, I obtain the required estimate

$$\|T_{jk}\| \lesssim \lambda^{-\Delta_\nu} \leq \lambda^{-\Delta}.$$

Now assume that $n'_{\nu} > 0$. Denote $r_i(x) = \text{Re } Y_{\nu i}(x)$, and let $\tau_i(x) \in \mathbb{R}[[x^{1/n}]]$
be the asymptotic expansion of \( r_i(x) \) at zero. By E. Borel’s theorem, I can find real functions \( f_i(x) \) such that \( f_i(x^n) \in C^\infty \) and \( f_i(x) \sim \mathcal{T}_i(x) \) as \( x \to 0 \). Moreover, there is one case when I may and will take simply \( f_i(x) = Y_{\nu_i}(x) \). Namely, by Proposition 3.1, parts (4),(5), this is possible if the series \( Y_{\nu_i}(x) \) is real and different from any other \( Y_{\nu'}(x) \).

Let \( W \) be the union of the graphs of \( f_i(x) \) inside \( R_{jk} \):

\[
W = \bigcup_{i = 1}^{n'_\nu} \{(x, y) \in R_{jk}| y = f_i(x)\}.
\]

It is not difficult to see that on \( R_{jk} \)

\[
f'_i(x) \approx x^{\gamma_{\nu_i}-1} \approx 2^{-j(\gamma_{\nu_i}-1)} =: L.
\]

This suggests to consider a Whitney-type decomposition of \( R_{jk} \setminus W \) away from \( W \) into rectangles of size \( 2^{-m} \times L2^{-m} \). The easiest way to do this is to dilate the set \( R_{jk} \setminus W \) along the \( y \)-axis \( L^{-1} \) times, take the standard Whitney decomposition into the dyadic squares away from (the dilation of) \( W \), and contract everything to the original scale. As a result, I get a covering

\[
R_{jk} \setminus W \subset \bigcup R_l, \quad R_l \cap R_{jk} \neq \emptyset,
\]

where \( R_l \) are rectangles of size \( 2^{-m_l} \times L2^{-m_l} \), \( m_l \in \mathbb{Z}_+ \), such that the distance from \( R_l \) to \( W \) in the anisotropic norm \( |x| + L^{-1}|y| \) is of the order \( 2^{-m_l} \).

I claim that the rectangles \( R_l \) of fixed size form an almost orthogonal family, i.e. that for each \( R_l \) the number of rectangles \( R_{\nu'} \) with \( m_{\nu'} = m_l \) such that either the \( x \)- or the \( y \)-projections of \( R_l \) and \( R_{\nu'} \) intersect is bounded by a fixed constant independent of \( l \).
Consider the case of intersecting $y$-projections (the other case is similar). Then $R_{l'}$ is contained in in the horizontal strip passing through $R_l$ (see Fig. 11). By dilating along the $y$-axis, I may assume that $L = 1$. Since $\text{dist}(R_{l'}, W) \approx 2^{-m_l}$, there exists a point $A$ on the graph of one of the functions $f_i(x)$ such that $\text{dist}(R_{l'}, A) \approx 2^{-m_l}$. Let $B$ denote the point where the graph of $f_i(x)$ intersects the bottom of the strip. Since $f_i'(x) \approx L = 1$, I have $\text{dist}(A, B) \lesssim 2^{-m_l}$, and therefore $\text{dist}(R_{l'}, B) \lesssim 2^{-m_l}$. Thus all possible rectangles $R_{l'}$ are situated at a distance $\lesssim 2^{-m_l}$ from no more than $n'_{\nu}$ points where the bottom of the horizontal strip intersects $W$. This implies that the number of $R_{l'}$ in the horizontal strip is bounded by a fixed constant, and the almost orthogonality is verified.

Now let $R^*_{l} = (1 + \varepsilon)R_l$, where an $\varepsilon > 0$ is chosen so small that $\text{dist}(R^*_{l}, W) \approx 2^{-m_l}$ (in the anisotropic norm). Consider a smooth partition of unity $\sum_l \phi_l = 1$ on $\bigcup R_l$ with $\text{supp } \phi_l \subset R^*_{l}$, satisfying the natural differential inequalities. I am going to decompose $T_{jk}$ using this partition of unity. However, this decomposition will not be useful near the real multiple branches of $\mathcal{Z}$, since I will not have good control on the size of $F$ there. For now I am just going to localize away from those branches in the following way.

Let $\beta_i$ denote the power exponent of the first nonzero term $C x^{\beta_i}$ in the asymptotic expansion of $\text{Im} Y_{\nu_l}(x)$; $\beta_i := \infty$ if this expansion is identically zero. For a large fixed
number $Q$ I introduce the set

$$W_Q = \bigcup_{i=1}^{n'_i} \{(x, y) \in R_{j_k} ||y - f_i(x)|| \leq 2^{-j Q}\},$$

where * indicates that the union is taken over all $i$ such that $\beta_i = \infty$ and $f_i(\cdot) \neq Y_{\nu_i}(\cdot)$. By the choice of $f_i(x)$, this may happen only if the series $Y_{\nu_i}(x)$ is real and there are several $Y_{\nu_i'}(x)$ having $Y_{\nu_i}(x)$ as their asymptotic expansion. One can say that $W_Q$ is a tubular neighborhood of width $2^{-j Q}$ of the real multiple branches of $Z$ (see Fig. 12).

![Fig. 12](image)

The purpose of introducing $W_Q$ is that on $R_{j_k} \backslash W_Q$ I have (if $j$ is large enough, which can be achieved by a further contraction of $V$)

$$|y - Y_{\nu_i}(x)| \approx |y - f_i(x)| + 2^{-j \beta_i}. \quad (4.12)$$

Now let $\chi_Q$ be a smooth cut-off supported in the double of $W_Q$, $\chi_Q \equiv 1$ on $W_Q$. I consider the decomposition

$$T_{j_k} = T_Q + T^Q, \quad (4.13)$$

$$T_Q f(x) = \int e^{i \lambda S(x, y)} \chi_Q(x, y) \chi_j(x) \chi_k(y) \chi(x, y) f(y) dy,$$

$$T^Q = \sum_l T_l^Q.$$
\[ T^Q_i f(x) = \int e^{i \lambda S(x,y)} \phi_i(x,y) (1 - \chi_Q(x,y)) \chi_j(x) \chi_k(y) \chi(x,y) f(y) \ dy, \]

In the rest of this section I prove that \( \| T^Q \| \lesssim \lambda^{-\Delta} \). The operator \( T_Q \) will be dealt with in the next section.

Let \( T^Q_i \) be one of the operators from the decomposition of \( T^Q \), and assume that \( T^Q_i \neq 0 \), i.e. that \( R_j^* \cap (R_j \setminus W_Q) \neq \emptyset \). Fix a point \((x_l,y_l)\) in this last intersection. I claim that

\[ |y - f_i(x)| \approx |y_l - f_i(x_l)| \quad (4.14) \]

for \((x,y) \in R^*_j\) and \( i = 1, \ldots, n'_\nu \). Indeed, let \((x',y')\) and \((x'',y'')\) be points of \( R^*_j\) for which the value of \( |y - f_i(x)| \) is respectively minimal and maximal. Then

\[ |y'' - f_i(x'')| \leq |y' - f_i(x')| + |y'' - y'| + |f_i(x'') - f_i(x')| \]

\[ \lesssim |y' - f_i(x')| + L 2^{-m_l} \lesssim |y' - f_i(x')|, \]

since \( L^{-1} |y' - f_i(x')| \geq \text{dist}(R^*_j, W) \gtrsim 2^{-m_l} \). From this (4.14) follows.

Now from (4.11) and (4.12) we see that on \( R^*_j \)

\[ |F| \approx 2^{-j(\gamma_\nu B_\nu + A_\nu - \gamma_\nu n'_\nu)} \prod_{i=1}^{n'_\nu} (|y_l - f_i(x_l)| + 2^{-j \beta_i}) =: \mu_l. \]

It follows by Lemma 2.3 (the condition (4) needs to be checked, but this is easy) that \( \| T^Q_i \| \lesssim (\mu_l \lambda)^{-1/2} \).

I can get a lower bound on \( \mu_l \) by noting that \( |y_l - f_i(x_l)| \gtrsim L 2^{-m_l} = 2^{-j(\gamma_\nu - 1) - m_l} \).

This gives

\[ \mu_l \gtrsim 2^{-j(\gamma_\nu B_\nu + A_\nu)} 2^{-(m_l - j)n'_\nu}, \]

and therefore

\[ \| T^Q_i \| \lesssim \lambda^{-1/2} 2^{j(\gamma_\nu B_\nu + A_\nu)/2} 2^{(m_l - j)n'_\nu/2}. \quad (4.15) \]
On the other hand, by Lemma 2.2,

\[ \|T_Q^l\| \lesssim 2^{-m_l-j(\gamma_\nu-1)/2}. \tag{4.16} \]

Now it remains to resum the last two estimates by splitting the family of operators \(T_Q^l\) into almost orthogonal families \(l = \text{const}\). This is done exactly how I did it in Section 2.2.3 after Eq. (2.24)\(^\ast\).

This ends the proof of \(\|T^Q\| \lesssim \lambda^{-\Delta}\).

### 4.5 Estimates near multiple real branches

To finish the proof of the theorem, I must estimate the operator \(T_Q\) appearing in the decomposition (4.13) of \(T_{jk}\).

In the estimates below I can assume that \(\gamma_\nu \geq 1\), since this can be achieved by passing to the adjoint operator if necessary.

Further, I can assume that \(Q\) is chosen so large that the branches of \(Z\) having different asymptotic expansions become completely separated in the definition of \(W_Q\). Since such branches can be treated separately, I am reduced to the case when \(T_Q\) has the form

\[ T_Q f(x) = \int e^{i\lambda S(x,y)} \chi_{jkQ}(x,y)f(y) \, dy, \]

\[ \chi_{jkQ}(x,y) = \chi_j(x)\chi_k(y)\varphi(2^{jQ}(y-g(x))). \]

Here \(\varphi(t)\) is a \(C^\infty\) cut-off supported in \([-1,1]\), \(g(x^{n_l}) \in \mathbb{R}C^\infty\), \(g(x) = cx^{\gamma_\nu} + \ldots\), \(c \neq 0\), and in factorization (3.1) exactly \(N \geq 2\) functions \(Y_{\nu i}(x)\) have asymptotic expansion coinciding with that of \(g(x)\). I will assume that this happens for \(i = 1, \ldots, N\). I also re-denote \(W_Q = \{ (x,y) \in R_{jk} \|y-g(x)\| \leq 2^{-jQ}\}.\)

\(^\ast\)This resummation was unfortunately done in a wrong way in my paper [12]. The part of that paper from Eq. (5.6) and until the end of Section 5 has to be thrown out and substituted by the more careful argument I give in Section 2.2.3 of this thesis.
I write \( F(x, y) \) as
\[
F(x, y) = \tilde{U}(x, y) P(x, y),
\]
where \( P(x, y) = \prod_{i=1}^{N} (y - Y_{\nu i}(x)) \), and \( \tilde{U}(x, y) \) is the product of the rest of the terms in (3.1).

Since all the branches of \( \mathcal{Z} \) appearing in \( \tilde{U}(x, y) \) are well separated from \( W_Q \), there exists a constant \( M_1 \geq 0 \) such that
\[
|\tilde{U}| \approx 2^{-jM_1} \quad \text{on} \quad W_Q.
\]

Moreover, it can be seen directly that if \( S''_{xy} \) is exceptionally degenerate, we have \( M_1 = 0 \).

Further, by Proposition 3.1, parts (4), (5), I know that \( P(x, y) \in \mathbb{R}C^\infty_+((x^{1/n}))[y] \), so that \( P(x, y) \) is \( C^\infty \) in both variables on \( W_Q \). It is clear that
\[
\partial^N_y P(x, y) = \text{const} \neq 0. \quad (4.17)
\]

I claim that, more generally,
\[
\partial^k_x \partial^{N-k}_y P(x, y) \neq 0 \quad \text{on} \quad W_Q, \quad k = 0, \ldots, N. \quad (4.18)
\]

Denote \( Q(x, y) = P(x^{n_1}, y) \in C^\infty_+((x))[y] \). The Taylor series of \( Q(x, y) \) is
\[
\overline{Q}(x, y) = \prod_{i=1}^{N} (y - \overline{G}(x)), \quad \overline{G}(x) = \overline{g}(x^{n_1}).
\]

It is clear that
\[
[\partial^l_x \partial^{N-k}_y \overline{Q}](x, \overline{G}(x)) = 0, \quad 0 \leq l < k,
\]
\[
[\partial^k_x \partial^{N-k}_y \overline{Q}](x, \overline{G}(x)) \neq 0.
\]
Therefore the factorizations of $\partial_x^l \partial_y^{N-k} Q(x,y)$, $l < k$, which can be obtained as described in the proof of Lemma 3.4, will contain branches with the asymptotic expansion $G(x)$, while the factorization of $\partial_x^k \partial_y^{N-k} Q(x,y)$ will not contain such branches. This implies (1.18), provided that $Q$ is large enough, since $\partial_x^k \partial_y^{N-k} P(x,y)$ can be expressed as

$$(\partial_x^k \partial_y^{N-k} Q)(x^{1/n!}, y) + \sum_{l < k} c_l(x)(\partial_x^l \partial_y^{N-k} Q)(x^{1/n!}, y)$$

with coefficients $c_l(x)$ growing power-like as $x \to 0$.

In addition, the above argument gives an estimate

$$\partial_x^N P(x, y) \geq 2^{-jM_2} \quad \text{on} \quad W_Q,$$

for some constant $M_2 \geq 0$; $M_2 = 0$ if $S''_{xy}$ is exceptionally degenerate.

Denote $\sigma_j(x, y) = \frac{1}{j!} \partial_y^j P(x, y)$. Consider the decomposition

$$T_Q = \sum_{l=-C}^\infty T_l,$$

$$T_l f(x) = \int e^{i\lambda S(x,y)} \chi_{jkQ}(x, y) \chi_l(\sigma_0(x, y)) f(y) \, dy,$$

where $\chi_l(t)$ is the characteristic function of the set $2^{-l} \leq |t| \leq 2^{-l+1}$, $C$ is a constant.

I am going to prove the estimates:

$$\|T_l\| \lesssim 2^{-l/N + jM_2/2N}, \quad (4.20)$$

$$\|T_l\| \lesssim \lambda^{-1/2} (\log \lambda)^{1/2} 2^{l/2} l^{N-1/2} 2^j M_1/2. \quad (4.21)$$

The required bound for $T_Q$ can then be derived as follows.
Consider first the exceptionally degenerate case, when \( M_1 = M_2 = 0 \). I have

\[
\|T_Q\| \lesssim \sum_{l=0}^{\infty} \min(2^{-l/N}, \lambda^{-1/2} 2^{l/2} (\log \lambda)^{1/2} l^{N-1/2}).
\]

If it were not for the factor of \((\log \lambda)^{1/2} l^{N-1/2}\), the terms in parentheses would become equal for \( l = \frac{N}{N+2} \log_2 \lambda \), and I would have the best possible estimate \( \|T_Q\| \lesssim \lambda^{-1/(N+2)} \). In the present situation I am going to lose something, and to optimize the loss, I put \( t_0 = \frac{N}{N+2} \log_2 \lambda - k \log_2 \log_2 \lambda \) with indeterminate \( k \) and have the estimate

\[
\|T_Q\| \lesssim 2^{-l_0/N} + \lambda^{-1/2} 2^{t_0/2} (\log \lambda)^{1/2} t_0^{N-1/2}
\]

\[
\lesssim \lambda^{-1/(N+2)} \left[ (\log \lambda)^{k/N} + (\log \lambda)^{N-k/2-1/2} \right].
\]

The optimal value of \( k \) is \( k = \frac{2N^2-N}{N+2} \), which gives

\[
\|T_Q\| \lesssim \lambda^{-\frac{1}{N+2}} (\log \lambda)^{\frac{2N-1}{N+2}}
\]

in complete accordance with what is claimed in the theorem.

Assume now that \( S''_{xy} \) is not exceptionally degenerate. In this case the above argument gives in any case the estimate

\[
\|T_Q\| \leq C \varepsilon 2^{jM} \lambda^{-\frac{1}{N+2}+\varepsilon}
\]

for any \( \varepsilon > 0 \), with some constant \( M \). (I do not pursue the possibility of obtaining a log factor here, since as I will see in a moment, what I have is already good enough.)

I will need the following more general version of Lemma 2.2, which can be obtained immediately from Lemma 4.2.

\[\text{[1] Here I am being slightly more careful than in [12] and earn a marginal improvement in the power of } \log \lambda.\]
Lemma 4.3. (Phong and Stein [10], Lemma 1.6) Let $T$ be an integral operator with kernel $K(x,y)$, and assume that

1. $|K(x,y)| \leq 1$,
2. for each $y$, $K(x,y)$ is supported in an $x$-set of measure $\leq \delta_x$,
3. for each $x$, $K(x,y)$ is supported in a $y$-set of measure $\leq \delta_y$.

Then $\|T\| \leq (\delta_x \delta_y)^{1/2}$.

By this lemma, I certainly have the estimate

$$\|T_Q\| \lesssim 2^{-jQ/2}.$$ 

The idea is that now I can take the geometric mean of the last two estimates killing the $j$-factor and, if $Q$ is very large, this will introduce only a very small increase in the exponent of $\lambda$, actually tending to zero as $Q \to \infty$. Thus I have

$$\|T_Q\| \leq C_\varepsilon \lambda^{-\frac{1}{N+2} + \varepsilon}.$$ 

I am going to show that in the case under consideration $1/(N+2) > \Delta$. This allows me to choose and fix $Q$ from the very beginning so large that $\|T_Q\| \lesssim \lambda^{-\Delta}$, thus proving the theorem.

I show that in fact $1/(N+2) > \Delta$. Indeed, since I already have $N$ branches whose expansion starts with $cx^{\gamma}$, I know that $n_\nu \geq N$. Therefore $A_\nu = n_\nu \gamma_\nu + A'_\nu \geq N \gamma_\nu$, and

$$2 \Delta_\nu \leq \frac{1 + \gamma_\nu}{1 + N \gamma_\nu + \gamma_\nu} \leq \frac{2}{N + 2},$$

since $\gamma_\nu \geq 1$. Besides that, the equality holds if and only if $\gamma_\nu = 1, A_\nu = N, B_\nu = 0$. But this corresponds exactly to the exceptionally degenerate case, which is excluded.

I now turn to the proof of the claimed bounds for $T_i$. The proof of (1.20) is easy and is based on the following well-known
Lemma 4.4. (Christ [11], Lemma 3.3) Let \( f \in C^N[a,b] \) be such that \( f^{(N)} \geq \mu > 0 \) on \([a,b] \). Then for any \( \gamma > 0 \)

\[
|\{x \in [a,b] : |f(x)| \leq \gamma\}| \leq A_N(\gamma/\mu)^{1/N},
\]

where the constant \( A_N \) depends only on \( N \).

By this lemma, in view of (4.17) and (4.19), the kernel of \( T_l \) is supported in a \( y \)-set of measure \( \lesssim 2^{-l/N} \) for each \( x \), and in an \( x \)-set of measure \( \lesssim 2^{-l/N+jM_2/N} \) for each \( y \). Now (4.20) follows by Lemma 2.7.

Seeger’s method.

The proof of (4.21) constitutes the most intricate part of the whole argument. It is carried out by a variation of a method developed in Seeger [13], Section 3. The key idea is to take an additional dyadic localization in \( \sigma_j \), \( 1 \leq j \leq N-1 \). Let \( l \) be fixed; all constants below will however be independent of \( l \). Let \( \gamma = (\gamma_1, \ldots, \gamma_{N-1}) \) be a vector with integer components \(-C \leq \gamma_i \leq l \), \( C \) some constant. Denote

\[
\chi_{\gamma}(x,y) = \chi_{jkQ}(x,y)\chi_{l}(\sigma_0(x,y)\prod_{i=1}^{N-1} \chi_{\gamma_i}(\sigma_i(x,y)),
\]

where \( \chi_{\gamma_i}(t) \) is the characteristic function of the set \( 2^{-\gamma_i} \leq |t| \leq 2^{-\gamma_i+1} \) for \( \gamma_i < l \), and of the set \( |t| \leq 2^{-l+1} \) for \( \gamma_i = l \).

For an appropriate fixed \( C \) I have a decomposition

\[
T_l = \sum_{\gamma} T_{\gamma},
\]

\[
T_{\gamma}f(x) = \int e^{i\lambda S(x,y)}\chi_{\gamma}(x,y)f(y)\,dy.
\]
I am going to prove that for each \( \gamma \)

\[
\| T_\gamma \| \lesssim \lambda^{-1/2} (\log \lambda)^{1/2} 2^{jM_1/2}.
\] (4.22)

This will imply (4.21), since the number of \( T_\gamma \) in the decomposition of \( T_l \) is \( \lesssim l^{N-1} \).

The kernel of the operator \( T_\gamma^* T_\gamma \) has the form

\[
K(y_2, y_1) = \int e^{i\lambda [S(x, y_2) - S(x, y_1)]} \chi_\gamma(x) \chi_\gamma(x, y_1) dx.
\]

Assuming that \( y_2 > y_1 \), and using Taylor’s formula in \( y \) for \( P(x, y) \), I have

\[
[S(x, y_2) - S(x, y_1)]' = \int_{y_1}^{y_2} \widetilde{U}(x, y) P(x, y) dy
\] (4.23)

\[
= \int_{y_1}^{y_2} \widetilde{U}(x, y) \left[ \sum_{j=0}^{N} \sigma_j(x, y_1)(y - y_1)^j \right] dy
\]

\[
= \sum_{j=0}^{N} \sigma_j(x, y_1) \int_{y_1}^{y_2} \widetilde{U}(x, y)(y - y_1)^j dy.
\]

Notice that \( \int_{y_1}^{y_2} \widetilde{U}(x, y)(y - y_1)^j dy \approx 2^{-jM_1}(y_2 - y_1)^{j+1} \). So the RHS of (4.23) looks like a polynomial in \( y_2 - y_1 \) with dyadically restricted coefficients. To handle such polynomials, I need the following variant of Lemma 3.2 from [13]. I chose to give a proof, since I have found one simpler than in [13].

**Lemma 4.5.** For an integer \( N \geq 1 \), an integer vector \( r = (r_1, \ldots, r_N) \), \( r_i \geq 0 \), and a constant \( C > 0 \) consider the set \( \mathcal{P} = \mathcal{P}(r, C, N) \) of all polynomials of the form

\[
P(h) = 1 + \sum_{i=1}^{N} a_i h^i
\]

with real coefficients \( a_i \) satisfying

\[
|a_i| \in [C^{-1}2^{r_i}, C2^{r_i}] \quad \text{if} \quad r_i > 0,
\]

\[
|a_i| \leq C \quad \text{if} \quad r_i = 0.
\]
Then there exists a constant \( B = B(C,N) \), independent of \( r \), and a set \( E \in [0, 1] \) of the form

\[
E = [0, 2^{\beta_1}] \cup [2^{\alpha_2}, 2^{\beta_2}] \cup \ldots \cup [2^{\alpha_s}, 2^{\beta_s}],
\]

(4.24)
such that

1. \( \alpha_i, \beta_i \) are negative integers, \( \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_s < \beta_s \leq 0 \),
2. \( s \leq B; \beta_1 \geq -B \max(r_i); [(1 - \beta_s) + \sum_{j=1}^{s-1}(\alpha_{j+1} - \beta_j)] \leq B \),
3. \( |P(h)| \geq B^{-1} \) for \( h \in E \) for any \( P \in \mathcal{P} \).

**Proof.** Put \( r_0 = 0 \). Consider the convex set \( \Sigma \) given as the intersection of the half-planes lying above the lines \( y = r_i + ix, i = 0, \ldots, N \). The boundary of \( \Sigma \) consists of two infinite rays contained in straight lines \( y = 0 \) and \( y = r_N + Nx \), and of some (possibly zero) number of compact segments.

Let \( A_i, i = 1, \ldots, n \) be all the corner points of the boundary of \( \Sigma \) with the \( x \)-coordinates \( x_1 < x_2 < \ldots < x_n \). It is clear that \( n \leq N \). (In Fig. 12a \( N = 3, n = 3 \); in Fig. 12b \( N = 3, n = 2 \).)

An observation which will turn out to be important later: if \( r_k = 0 \), then the line \( y = r_k + kx \) cannot contain a compact segment of the boundary of \( \Sigma \). To see this, it
is sufficient to consider how the lines \( y = kx, \ 1 \leq k \leq N - 1 \), pass with respect to the lines \( y = 0 \) and \( y = r_N + Nx \).

I claim that for any \( P \in \mathcal{P} \) and for large enough \( B \)

\[
|P(h)| \geq B^{-1} \quad \text{if} \quad h \in [0, 1], \quad \log_2 h \notin \bigcup_{j=1}^{n} (x_j - B, x_j + B). \quad (4.25)
\]

First consider the case

\[
\log_2 h \in [x_j + B, x_{j+1} - B]. \quad (4.26)
\]

Let \( k \) be such that the boundary points \( A_j \) and \( A_{j+1} \) belong to the line \( y = r_k + kx \) (Fig. 12c).

By the above observation, \( r_k > 0 \). Since \( A_j \) and \( A_{j+1} \) lie above all the other lines \( y = r_i + ix \), I have for all \( i \)

\[
r_i + ix_j \leq r_k + kx_j, \quad r_i + ix_{j+1} \leq r_k + kx_{j+1}.
\]
From these two estimates it follows that
\[ |a_i h^i| \lesssim |a_k h^k| 2^{(i-k)(\log_2 h - x_j)} \quad (i < k), \]
\[ |a_i h^i| \lesssim |a_k h^k| 2^{(k-i)(x_{j+1} - 1 - \log_2 h)} \quad (i > k). \]

Using (4.26), I conclude
\[ |a_i h^i| \lesssim |a_k h^k| 2^{-|k-i|B}. \]

This estimate clearly implies \(|P(h)| \gtrsim |a_k h^k| \gtrsim 2^{r_k + k \log_2 h} \geq 1\), provided that \(B\) is large enough.

Second, if \(\log_2 h \leq x_1 - B\), then the same argument as above shows \(|P(h)| \gtrsim 1\).

Third, if \(\log_2 h \geq x_n + B\), then if \(r_N > 0\), I can show in the same way as above that \(|P(h)| \gtrsim |a_N h^N| \gtrsim 1\). If \(r_N = 0\), then \(x_n \geq 1\), and this region of \(h\)'s is irrelevant.

So (4.25) is verified. Finally, it is not difficult to see that the exceptional set in (4.23) satisfies (1)-(3).

Now if I take out the factor of \(2^{-l-j M_1(y_2 - y_1)}\), the expression in the RHS of (4.23) has the form of polynomial in \(h = y_2 - y_1\) falling under the scope of the lemma with \(r_i = l - \gamma_i\). So I have a set \(E\) of the form (4.24) such that
\[ |[S(x, y_2) - S(x, y_1)]'_{x}| \gtrsim 2^{-l-j M_1(y_2 - y_1)} \quad i f \quad y_2 - y_1 \in E. \]

I claim that this implies
\[ |K(y_2, y_1)| \lesssim 2^{l+j M_1} \lambda^{-1}(y_2 - y_1)^{-1} \quad (y_2 - y_1 \in E). \quad (4.27) \]

Indeed, this will follow from Lemma 4.1 with \(k = 1\), if I prove that there exists a constant \(C\) independent of \(y_1\) and \(y_2\) such that for fixed \(y_1\) and \(y_2\)

(1) the number of intervals of monotonicity of \([S(x, y_2) - S(x, y_1)]'_{x}\) considered as a
function of $x$ is less than $C$.

(2) the number of intervals comprising the $x$-set where $\chi_\gamma(x, y_1)\chi_\gamma(x, y_2)$ is non-zero is less than $C$.

To show (1), note that $\partial^N_x F(x, y) \neq 0$ on $W_Q$. It follows that

$$\partial^{N+1}_x [S(x, y_2) - S(x, y_1)] = \int_{y_1}^{y_2} \partial^N_x F(x, y) \, dy \neq 0.$$ 

Therefore, $[S(x, y_2) - S(x, y_1)]''_{xx}$ vanishes at most $N - 1$ times, which implies (1).

To show (2), it suffices to check that the number of intervals in the set $\{x|(x, y) \in W_Q, a \leq \sigma_j(x, y) \leq b\}$ is bounded by a constant independent of $a$ and $b$ for each $0 \leq j \leq N - 1$. However, this last statement follows from (1.18).

Unfortunately, to prove the claimed norm estimate for $T_\gamma$, I will need still another decomposition taking into account the form of the set $E$. Namely, for $1 \leq k \leq s$ and an integer $n$ I put

$$\chi_{kn}(y) = \psi(2^\beta_k y - n),$$

where $\psi(t)$ is the characteristic function of the interval $[0, 1]$, and consider the operators

$$T_{kn} f(x) = \int e^{i\lambda S(x, y)} \chi_{\gamma} (x, y) \chi_{kn}(y) f(y) \, dy.$$ 

I am going to prove by induction in $k$ that for each $n$

$$\|T_{kn}\| \lesssim \lambda^{-1/2}(\log \lambda) 2^{l/2} l^{1/2} 2^j M_1/2.$$ 

The statement for $k = s$ implies the required estimate (1.22), since $T_\gamma = \sum_n T_{sn}$, and the sum contains no more than $2^{-\beta_s} \leq C$ terms.

For $k = 1$, I use the kernel of the operator $T^*_{1n} T_{1n}$, which has the form

$$\chi_{1n}(y_1)\chi_{1n}(y_2)K(y_2, y_1),$$
where $K(y_2, y_1)$ is the kernel of $T_γ^*T_γ$. If this expression is not zero, then $|y_2 - y_1| \leq 2^β_1$. In view of (4.27), and also because $|K| \lesssim 1$, Lemma 4.2 gives

$$\|T_{1n}^*T_{1n}\| \lesssim \int_0^{2^β_1} \min(1, 2^l+jM_1^\lambda^{-1}t^{-1}) \, dt$$

$$\lesssim 2^l+jM_1^\lambda^{-1}\int_0^{\lambda^22^{-l-jM_1}} \min(1, t^{-1}) \, dt \lesssim 2^l+jM_1^\lambda^{-1}\log \lambda,$$

which is even better by a factor of $l$ than what I need.

The induction step is performed by using the decomposition

$$T_{k+1, n} = \sum_{n'} T_{kn'}.$$  

I will need the following variant of the Cotlar–Stein lemma, which can be proved by an easy adaptation of the standard proof given in [13], see e.g. Comech [8], Appendix.

**Lemma 4.6.** Let $T_i$ be a family of operators on a Hilbert space $H$ such that

1. $T_iT_{i'}^* = 0$ for $i \neq i'$,
2. $\sum_{i'} \|T_i^*T_{i'}\| \leq C$ with a constant $C$ independent of $i$.

Then $\|\sum T_i\| \leq C^{1/2}$.

I have $T_{kn}T_{kn'}^* = 0$ for $n' \neq n''$. Let us estimate the sum

$$\sum_{n''} \|T_{kn'}^*T_{kn''}\| \quad (4.28)$$

for a fixed $n'$. Since both $T_{kn'}$ and $T_{kn''}$ appear in the decomposition of $T_{k+1, n}$, I have $|n' - n''| \leq 2^{β_{k+1} - β_k}$. Further, the kernel of $T_{kn'}^*T_{kn''}$ has the form

$$\chi_{kn'}(y_1)\chi_{kn''}(y_2)K(y_2, y_1).$$
If this expression is different from zero, then
\[ 2^{\beta_k}|y_2 - y_1| \in \left[ |n' - n''| - 1, |n' - n''| + 1 \right]. \tag{4.29} \]

Assume first that
\[ 2^{\alpha_{k+1} - \beta_k} + 1 \leq |n' - n''| \leq 2^{\beta_{k+1} - \beta_k} - 1. \tag{4.30} \]

Then (4.29) implies \(|y_2 - y_1| \in E\), and I can use the estimate (4.27). By Lemma 4.2,
\[ \|T_{k'n'} T_{kn''}\| \lesssim \int_{2^{-\beta_k (|n' - n''| + 1)}}^{2^{-\beta_k (|n' - n''| + 1)}} 2^{l+jM_1} \lambda^{-1} t^{-1} dt \lesssim 2^{l+jM_1} \lambda^{-1} |n' - n''|^{-1}. \]

Therefore the part of the sum (4.28) over \(n''\) satisfying (4.30) is bounded by
\[ 2^{l+jM_1} \lambda^{-1} \sum_{m=2^{\alpha_{k+1} - \beta_k}}^{2^{\beta_{k+1} - \beta_k}} \frac{1}{m} \lesssim 2^{l+jM_1} \lambda^{-1} (\beta_{k+1} - \alpha_{k+1}) \lesssim 2^{l+jM_1} \lambda^{-1} l, \]
where I used the fact that by Lemma 4.5 (2) \(\beta_1 \geq -Bl\).

However, the number of \(n''\) which do not satisfy (4.30) is bounded by a constant in view of Lemma 4.5 (2), so the corresponding part of (4.28) is bounded by
\[ C \sup \|T_{kn''}\|^2 \lesssim l2^{l+jM_1} (\log \lambda) \lambda^{-1} \]
by the induction hypothesis.

By applying Lemma 4.6, I complete the induction step. Theorem 1.1 is now proven.
Chapter 5

Stopping time

This chapter stands somewhat separately from the rest of the thesis. Here I am developing a quite different method of proving upper norm bounds. This method is incomplete as it stands, and it is unclear if it is possible to make it complete. In its present form it is much less powerful compared to methods based on the geometric analysis of the zero set of \( S''_{xy} \) which I used above. However, I can use this method to prove that the estimate (1.3) from Theorem 1.1 can be improved to the optimal \( \lambda^{-1/4} \) in the case \( N = 2 \).

5.1 General idea

The main idea would be to try to organize an inductive process which would “resolve the singularity” of \( S''_{xy} \) by gradually decreasing the space under its Newton polygon, eventually reducing me to the non-degenerate case (Fig. 13).
This idea was first applied to oscillatory integral operators by Phong and Stein in [11]. Although the proof of that paper is incomplete as it stands (almost orthogonality claim on p. 114 of [11] is unjustified; see also Remark (c) on p. 150 of [9]), the argument can be saved at least in some partial cases [16]. Below I use a variation of the method of [11] and [16] to get a somewhat sharper result.

Still a full realization of the above idea remains elusive. The inductive process I can actually organize works well only for the simplest Newton polygons consisting of just one edge joining 2 points on the coordinate axes.

Unfortunately, this property may get destroyed already on the first step of the inductive process (Fig. 14).

However, it will not get destroyed provided that there are no integer points lying strictly inside the triangle OAB. The last condition is satisfied in the following two cases:

• $A = 1$ or $B = 1$ (Fig. 15)

• $A = B = 2$ (Fig. 16)
These are exactly the cases when I am able to produce final results by this method. In particular, the case \( A = B = 2 \) settles \( N = 2 \) in Theorem 1.1.

## 5.2 Results

I am going to prove the following

**Theorem 5.1.**

I. Assume that \( S(x, y) \) is a smooth phase function in the square \( Q = [0, 1]^2 \) and that \( F = S''_{xy} \) satisfies

\[
|F^{(1,0)}| \neq 0, \quad |F^{(0,B)}| \neq 0
\]

(5.1)

in \( Q \). Assume also that \( \chi \) is a smooth cutoff supported in \( Q \). Then the operator given by (1.1) is bounded on \( L^2 \) with

\[
\|T_\lambda\| \lesssim \lambda^{-\Delta}, \quad \Delta = \frac{B + 1}{2(2B + 1)}.
\]

II. If instead of (5.1) I assume that

\[
|F^{(2,0)}| \neq 0, \quad |F^{(0,2)}| \neq 0
\]

(5.2)

in \( Q \), then

\[
\|T_\lambda\| \lesssim \lambda^{-1/4}.
\]

I will need the following somewhat more quantitative auxiliary result, which implies Part I immediately, and to which Part II will also be later reduced.

**Theorem 5.2.** Let \( S(x, y) \) be a smooth phase function in the square \( Q = [0, 1]^2 \), \( \mu \) a real number. Assume that \( F = S''_{xy} \) satisfies in \( Q \)

\[
|F^{(1,0)}| \approx \mu, \quad |F^{(0,B)}| \approx \mu,
\]
and

$$|F^{(\alpha, \beta)}| \lesssim \mu$$

for \((\alpha, \beta) \in \{(0, B + 1), (0, B + 2), (1, 1), (1, 2), \ldots, (1, B - 1)\}\). Assume also that \(\chi\) is a smooth cutoff supported in \(Q\). Then the operator given by (1.1) is bounded on \(L^2\) with

$$\|T_\lambda\| \lesssim (\lambda \mu)^{-\Delta}, \quad \Delta = \frac{B + 1}{2(2B + 1)}.$$

Notice that the number \(\Delta\) in these results is the Newton decay rate corresponding to the Newton polygon with two vertices \((1, 0)\) and \((0, B)\). Analogously \(1/4\) is the right Newton decay rate for the \((2, 0)-(0, 2)\) Newton polygon. Notice that the \(N = 2\) exceptionally degenerate phase functions of Theorem 1.1 satisfy conditions of Theorem 5.2, Part II.

### 5.3 Proofs

**Proof of Theorem 5.2.** Induction on \(B\). For \(B = 0\) the result follows from Lemma 2.3.

Assume that \(B > 0\). I divide \(Q\) into equal rectangles of size \(\delta^B \times \delta, \; \delta = 1/2\). If for some of these rectangles the condition (S) below is satisfied, I put it into a numbered collection of rectangles \(\{R_k\}\). Otherwise I divide it further into equal rectangles of size \(\delta^B \times \delta, \; \text{now} \; \delta = 1/4, \text{etc.}\) The stopping condition for a rectangle \(R_k\) of size \(\delta^B_k \times \delta_k\) is

$$\min_{R_k^{**}} |F^{(0,n_k)}| \geq \mu \delta_k^{B-n_k} \quad \text{for SOME} \quad n_k \in \{0, \ldots, B - 1\}$$

(S)

(star means the doubled rectangle, double star means the quadrupled rectangle).

Eventually all \(Q\) up to a set of measure zero becomes decomposed into rectangles \(R_k\). The exceptional set is the intersection of the zero sets of \(F^{(0,n)}, n = 0, \ldots, B - 1\). This set is of measure zero, since \(F^{(0,B)} \neq 0\).
I claim that the covering \{R_k^*\} has finite multiplicity, that is for every \(k\) there are only finitely many \(l\)'s such that

\[ R_k^* \cap R_l^* \neq 0. \]  \hspace{1cm} (5.3)

It is sufficient to prove that (5.3) implies \(\delta_k \approx \delta_l\). Now if \(\delta_l \geq C\delta_k\), then it follows from (5.3) that

\[ (R_k^*)^{**} \subset R_l^* , \]  \hspace{1cm} (5.4)

where \(R^\#\) denotes the “parent” of \(R\), that is the rectangle out of which \(R\) was obtained in the \((2B, 2)\)-dyadic division process described above.

But it follows from (5.4) that already \(R_k^\#\) had to be retained and not divided further. This contradiction shows that necessarily \(\delta_k \approx \delta_l\), from which finite multiplicity follows.

Because of finite multiplicity, I can localize the operator \(T_\lambda\) to \(R_k^*\) by a smooth partition of unity satisfying the “right” differential bounds. Denote the part supported on \(R_k^*\) by \(T_k\).

I claim that as well as the lower bound \((S)\), the upper bound

\[ |F^{(0,n)}| \lesssim \mu \delta_k^{B-n} \]  \hspace{1cm} (U_n)

for EACH \(n \in \{0, \ldots, B-1\}\) is true on \(R_k^*\), and in fact on \(\overline{R} = (R_k^*)^{**}\).

The proof goes like this. Since \(R_k^\#\) was not retained, for each \(n = 0, \ldots, B-1\) there is a point \((x_n, y_n)\) in \(\overline{R}\) such that

\[ |F^{(0,n)}(x_n, y_n)| \leq \mu \delta_k^{B-n}. \]  \hspace{1cm} (5.5)

Now by assumption

\[ |F^{(1,n)}| \lesssim \mu \]  \hspace{1cm} (5.6)
in the whole \( Q \). Since the \( x \)-size of \( \overline{R} \) is \( \lesssim \delta_k^B \), it follows from (5.5) and (5.6) by Newton-Leibnitz applied in the \( x \)-direction that

\[
|F^{(0,n)}(x,y_n)| \leq \mu \delta_k^{B-n} \quad \forall n = 0, \ldots, B - 1, \tag{5.7}
\]

provided that \((x, y_n) \in \overline{R}\). (Fig. 17)

Now notice that for \( n = B \) \((U_n)\) is true by assumption in the whole \( Q \), and that \((U_{n-1})\) follows from \((U_n)\) and (5.7) by Newton-Leibnitz applied in the \( y \)-direction. So \((U_n)\) follows by induction for all \( n \) from \( B - 1 \) to 0.

The main reason I need \((U_n)\) is to show that for each \( \delta \), the subfamily of rectangles \( R_k^* \) with \( \delta_k = \delta \) is almost orthogonal.

Indeed, since \(|F^{(0,B)}| \geq \mu\) on \( Q \) and \(|F^{(0,B-1)}| \lesssim \mu \delta_k^B\) on \( R_k^* \), by Lemma 4.4 there are no more than \( const \) rectangles of the same \( y \)-size \( \sim \delta_k \) with intersecting \( x \)-projections.

Analogously, since \(|F^{(1,0)}| \geq \mu\) on \( Q \) and \(|F^{(0,0)}| \lesssim \mu \delta_k^B\) on \( R_k^* \), by Lemma 4.3 there are no more than \( const \) rectangles of the same \( x \)-size \( \sim \delta_k^B \) with intersecting \( y \)-projections.

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By almost orthogonality, I get

\[ \|T_\lambda\| \lesssim \sum_n \sup_{\delta_k = 2^{-n}} \|T_k\|. \quad (5.8) \]

Now the idea is to rescale \( T_k \) to a square of size \( \sim 1 \) by putting (Fig. 18)

\[ \tilde{S}(x, y) = S(\delta_k^B x, \delta_k y). \]

The norms on \( L^2 \) are related by

\[ \|T_k\| = \delta_k^{(B+1)/2} \|\tilde{T}_k\|. \quad (5.9) \]

After the rescaling, I get that the phase \( \tilde{S}(x, y) \) satisfies the conditions of Theorem 5.2 with \( n_k \) instead of \( B \) and with \( \mu \delta_k^{2B+1} \) instead of \( \mu \). Indeed, the main conditions

\[ |\widetilde{F}^{(0, n_k)}| = \delta_k^{B+n_k+1} |F^{(0, n_k)}| \approx \mu \delta_k^{2B+1}, \]

\[ |\widetilde{F}^{(1, 0)}| = \delta_k^{2B+1} |F^{(1, 0)}| \approx \mu \delta_k^{2B+1} \]

are satisfied. The auxiliary conditions are checked as follows:

1) \[ |\widetilde{F}^{(1, \beta)}| = \delta_k^{2B+\beta+1} |F^{(1, \beta)}| \lesssim \mu \delta_k^{2B+1} \text{ for } \beta = 1, \ldots, B - 1, \text{ where I used that by} \]
assumption $|F^{(1,\beta)}| \lesssim \mu$ on the whole $Q$.

2) $|\tilde{F}^{(0,n)}| = \delta_k^{B+n+1} |F^{(0,n)}| \lesssim \mu \delta_k^{2B+1}$ for $n = 0, \ldots, B$ by $(U_n)$, and for $n = B + 1$ by knowing that $|F^{(0,B+1)}| \lesssim \mu$ in the whole $Q$.

Thus it follows by the induction hypothesis that

$$\|\tilde{T}_k\| \lesssim \min(1, (\lambda \mu \delta_k^{2B+1})^{\frac{n_k+1}{2(2n_k+1)}}$$

It follows from (5.8), (5.9), and the fact that $n_k \leq B - 1$ that

$$\|T_\lambda\| \lesssim \sum_n 2^{-n(B+1)/2} \min(1, (\lambda \mu)^{-\frac{B}{2(2B-1)}}) \frac{B(2B+1)}{2(2B-1)} 2^{\frac{B(2B+1)}{2(2B-1)} n}).$$

Two progressions balance for

$$2^{n^*} \approx (\lambda \mu)^{\frac{B}{2(2B-1)}}.$$

Notice that the second progression is indeed increasing:

$$\frac{B(2B+1)}{2(2B-1)} - \frac{B+1}{2} = \frac{1}{2(2B-1)} > 0.$$

So it follows that

$$\|T_\lambda\| \lesssim 2^{-n^*(B+1)/2} \lesssim (\lambda \mu)^{-\Delta}.$$

This completes the induction step and the proof of the theorem. 

**Proof of Theorem 5.1, Part II.** I am going to reduce this result to the $B = 2$ case of Theorem 5.2. This reduction is in fact very similar to the proof of Theorem 5.2 itself.

I organize a dyadic decomposition of $Q$, this time into dyadic squares $R_k$ of size $\delta \times \delta$, $\delta \sim 2^{-n}$, with stopping condition

$$\min_{R_k^*} |F^{(0,1)}| \geq \delta_k \quad \text{OR} \quad \min_{R_k^*} |F^{(1,0)}| \geq \delta_k. \quad (S')$$
That is, if \((S')\) is satisfied, the \(R_k\) is retained, otherwise it is further subdivided into 4 squares of equal size, etc.

As before,

\[ Q = \bigcup R_k \]

up to a set of measure zero. I show that \(R^*_k\) form a covering of finite multiplicity in the same way as before, and split

\[ T_\lambda = \sum T_k, \quad \text{supp } T_k \subset R^*_k. \]

Then I prove that on \(R^*_k\)

\[ |F^{(0,1)}|, |F^{(1,0)}| \lesssim \delta_k. \quad (5.10) \]

The proof of these bounds is even simpler that that of \((U_n)\). They follow immediately by Newton-Leibnitz from the fact that \(R^*_k\) was not retained.

By Lemma 4.4 I conclude from \((5.10)\) and \((5.2)\) that for each \(n\) the \(R^*_k\) with \(\delta_k = 2^{-n}\) form an almost orthogonal family. This implies \((5.8)\).

To estimate \(\|T_k\|\), I rescale the operator to a square of size \(\sim 1\):

\[ \tilde{S}(x, y) = S(\delta_k x, \delta_k y), \]

\[ \|T_k\| = \delta_k \|\tilde{T}_k\|. \quad (5.11) \]

Assume that the stopping condition that was actually satisfied for \(R_k\) was \(|F^{(1,0)}| \geq \delta_k\) (the case of \(F^{(0,1)}\) being completely analogous because of the \(x-y\) symmetry). Then after rescaling

\[ |\tilde{F}^{(1,0)}| = \delta_k^3 |F^{(1,0)}| \approx \delta_k^4, \]

\[ |\tilde{F}^{(0,2)}| = \delta_k^4 |F^{(0,2)}| \approx \delta_k^4. \]
Conditions

\[ |\tilde{F}^{(\alpha,\beta)}| \lesssim \delta_k^4, \quad (\alpha, \beta) \in \{(0, 3), (0, 4), (1, 1), (1, 2)\}, \]

are also easily checked. So we see that the \( \tilde{T}_k \) satisfies the assumptions of Theorem 5.2 for \( B = 2 \) and \( \mu = \delta_k^4 \).

It follows that

\[ \|\tilde{T}_k\| \lesssim (\lambda \delta_k^4)^{-3/10}. \]

Going back to (5.8) and (5.11),

\[ \|T_\lambda\| \leq \sum_n 2^{-n} \min(1, \lambda^{-3/10} 2^{6n/5}). \]

The progressions are balanced for \( 2^{n*} = \lambda^{1/4} \), and thus

\[ \|T_\lambda\| \lesssim 2^{-n*} = \lambda^{-1/4}. \]

The theorem is proved. \( \blacksquare \)
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