Semiclassical matrix elements for a chaotic propagator in the scar function basis

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Abstract

A semiclassical approximation for the matrix elements of a quantum chaotic propagator in the scar function basis has been derived. The obtained expression is solely expressed in terms of canonical invariant objects. For our purpose, we have used the recently developed, semiclassical matrix elements of the propagator in coherent states, together with the linearization of the flux in the neighborhood of a classically unstable periodic orbit of chaotic two-dimensional systems. The expression derived here is successfully verified to be exact for a (linear) cat map, after the theory is adapted to a discrete phase space appropriate to a quantized torus.

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1. Introduction

The Gutzwiller trace formula provides a tool for the semiclassical evaluation of the energy spectrum of a classically chaotic Hamiltonian system in terms of canonical invariants of periodic orbits (POs). However, the number of long POs required to resolve the spectrum increases exponentially with the Heisenberg time $T_H$ [1]. For this reason, the approach is limited to eigenenergies close to the ground state [2]. Of course, Gutzwiller’s formula is very attractive because it is given in terms of canonical invariants, and for this reason, a lot of work has been dedicated to improve this theory [3]. In particular, the paper by Mehlig and Wilkinson [4] formulates the Gutzwiller trace formula using coherent states. This manifestly canonically invariant formulation allows us to separate the contribution of each PO to the quantum evolution operator.

The semiclassical theory of short POs developed by Vergini and co-workers [5–10] is a formalism where the number of used POs increases only linearly with the mean energy density,
allowing us to obtain all the quantum information of a chaotic Hamiltonian system in terms of a very small number of short POs. The key elements in this theory are wavefunctions related to short unstable POs and then the evaluation of matrix elements between these wavefunctions is crucial.

In this context, these wavefunctions, named scar functions because of their strong connection with the scarring phenomenon [11], are not restricted to a PO; it additionally includes dynamical information up to the Ehrenfest time and, as a result, is influenced by pieces of the stable and unstable manifolds near the PO. These wavefunctions define an optimal basis in chaotic systems [5, 12] and they have attracted increasing interest in closed systems [13]. Moreover, they have been shown to be crucial for the understanding of long lived resonances in open systems [14–16].

In recent work, estimates for the asymptotic behavior of off-diagonal matrix elements [9] and asymptotic expansions for matrix elements in quantum cat maps [10] were derived. Also, recently a general semiclassical expression in phase space for the scar functions was obtained explicitly in terms of the classical invariants that generate the dynamics of the system [17].

In order to perform further developments to this theory of short POs, the semiclassical evaluation of matrix elements in a scar function basis set is an important objective. With these matrix elements at hand, the energy spectrum can be obtained without requiring an explicit computation of scar functions. This is the purpose of this paper. For our objective, we use the recently developed semiclassical matrix elements of the quantum propagator between coherent states [18]. After that, we perform the needed time integrals to obtain the matrix elements of the propagator between scar functions.

Also, for two degrees of freedom Hamiltonian systems, the dynamics is studied entirely within a surface of section transversal to the PO. That is, the full dynamics is studied through a two-dimensional section map. With the purpose of showing the validity of our approximation, we have compared the general expression found here with numerical calculations in a ‘realistic’ system, the cat map, i.e. the quantization of linear symplectic maps on the torus. As shown by Keating [19] in this case, the semiclassical theory is exact, making these maps an ideal probe for our expression. After the formalism is adapted for a torus phase space, we see that the semiclassical expression deduced here coincides exactly with the numerically computed matrix elements for the cat maps.

Of course, in order to include nonlinear contributions, a deeper understanding of the dynamics up to the Ehrenfest time is required; in this respect, enormous efforts were recently carried out in such a direction for the diagonal matrix elements [20].

This paper is organized as follows. In section 2, we introduce the definition of scar functions in terms of coherent states. Hence, we review the construction of the semiclassical matrix elements of the propagator in the coherent state basis [18]. We then show the utility of introducing the Weyl representation, and that our approach avoids any complex trajectory.

Section 3 is devoted to obtaining a semiclassical expression for the matrix elements of the propagator in the scar function basis. For that purpose, we need to perform a linearization of the flux close to POs and express the classical evolutions in the stable and unstable directions. Also, we obtain the expressions for a system with a discrete time evolution.

In section 4, we study the particular case of the cat map where not only the semiclassical theory is exact but also the linear approximation is valid throughout the torus. After the semiclassical expressions deduced here are adapted for a torus phase space, we see that they coincide exactly with the numerically computed matrix elements for the cat maps.
2. Coherent state matrix elements

Scar function states are the main objects of study of the current work. According to [21–25], the scar function $|\psi_{X}^\phi\rangle$ of parameter $\phi$ constructed on a single periodic point $X = (P, Q)$ is defined as

$$
|\psi_{X}^\phi\rangle = \int_{-\infty}^{\infty} dr \ e^{i\phi r} f_T (t) \hat{U}^t |X\rangle ,
$$

where $T = \ln \hbar$ is the Ehrenfest time, $|X\rangle$ is a coherent state centered at the point $X$ on the PO and $\hat{U}^t$ is the unitary propagator that governs the quantum evolution of the system, while $f_T (t)$ is a decaying function that takes negligible values for $|t| > T/2$. Without loss of generality, it will be convenient for our purpose to choose $f_T (t) = e^{- (T/2)^2}$, that is, the scar function is

$$
|\psi_{X}^\phi\rangle = \int_{-\infty}^{\infty} dr \ e^{i\phi r} \hat{U}^t |X\rangle .
$$

These wavefunctions have been shown, in the Husimi representation, to live in the immediate vicinity [25], while the Wigner function also shows hyperbolic fringes asymptotic to the stable and unstable manifolds [17]. Wigner functions with a hyperbolic structure have been spotted in previous works. For instance, the pioneer work of Berry [26] shows this phenomenon for the spectral Wigner function in continuous systems, while for maps, this has been shown in [27]. In the paper of Nicacio et al [28], the hyperbolic fringes are observed for a superposition of two squeezed states with orthogonal squeezing directions. As mentioned by Nicacio et al [28], the scar functions are superpositions of Gaussian states with different degrees (and directions) of squeezing, i.e. they are generalized Gaussian cat states. The purpose of this work is to study semiclassically the matrix elements of the unitary propagator in the scar function basis:

$$
\langle \psi_{X_1}^\phi | \hat{U}^t | \psi_{X_2}^\phi \rangle.
$$

From the definition of scar functions (1), we obtain

$$
\langle \psi_{X_1}^\phi | \hat{U}^t | \psi_{X_2}^\phi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 \ e^{i(\phi t_2 - \phi t_1)} f_T (t_1) f_T (t_2) \langle X_1 | \hat{U}^{t_1 + t_2} | X_2 \rangle. (3)
$$

Hence, we need to calculate $\langle X_1 | \hat{U}^t | X_2 \rangle$, i.e. the matrix elements of the propagator in the coherent state basis. The semiclassical matrix elements of the propagator in the coherent state basis have been obtained in [18]; we will here reproduce the main steps of the procedure.

Let us write the propagator $\hat{U}^t$ in terms of its, symplectically invariant, center or Weyl–Wigner symbol $U^\alpha (x)$ [29],

$$
\hat{U}^t = \frac{1}{(\pi \hbar)^{3/2}} \int dx U^\alpha (x) \hat{R}_{x} \quad \text{and} \quad U^\alpha (x) = \text{tr}[\hat{R}_{x} \hat{U}^t].
$$

(4)

where $\int dx$ is an integral over the whole phase space of $L$ degrees of freedom, while $\hat{R}_{x}$ denotes the set of reflection operators through points $x = (p, q)$ in phase space [29, 30] (see the appendix). Hence, the coherent state matrix elements can be written in terms of reflections as

$$
\langle X_1 | \hat{U}^t | X_2 \rangle = \left( \frac{1}{\pi \hbar} \right)^L \int \langle X_1 | U^\alpha (x) \hat{R}_x | X_2 \rangle \ dx. (5)
$$

The coherent states at points $X = (P, Q)$ in phase space are obtained by translating to $X$ the ground state of the harmonic oscillator; its position representation is

$$
\langle q | X \rangle = \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \exp \left[ - \frac{\omega}{2 \hbar} (q - Q)^2 + \frac{i}{\hbar} \left( q - Q \frac{P}{2} \right) \right].
$$

(6)
For simplicity, unit frequency ($\omega = 1$) and mass ($m = 1$) are chosen for the harmonic oscillator without loss of generality. The overlap of two coherent states is then

$$\langle X|x' \rangle = \exp \left[ -\frac{(X - x')^2}{4\hbar} - \frac{i}{2\hbar}X \wedge X' \right].$$

with the wedge product

$$X \wedge X' = PQ - QP = (JX) \cdot X'.$$

The second equation also defines the symplectic matrix $J$, i.e.

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ (8)

As shown in the appendix, the action of the reflection operator $R_\xi$ on a coherent state $|X\rangle$ is the $x$-reflected coherent state

$$R_\xi |X\rangle = e^{\frac{i\pi X \cdot \xi}{\hbar}} |2x - X\rangle.$$ (9)

Inserting (9) and (7) into (5), the propagator in coherent states is obtained from the Weyl propagator

$$\langle X_1|\hat{U}|X_2 \rangle = \frac{1}{(\pi\hbar)^2} \int \frac{d^2x}{2\pi^2} \exp \left[ \frac{i}{\hbar} S_j(x) + i\frac{\xi}{2} \alpha'_j \right] \exp \left[ \frac{i}{\hbar} S\xi(x) - \xi \cdot x \right] dx,$$ (10)

with $\xi_0 \equiv (X_1 - X_2)$ being the chord joining the points $X_1$ and $X_2$, while $\bar{X} = \frac{1}{2} (X_1 + X_2)$ denotes their mid point.

Also, the semiclassical approximation for the propagator in the Weyl representation was performed in [29] so that

$$U_j(x)_{\text{sc}} = \sum_\gamma 2^{\frac{1}{2}} \exp \left[ \frac{i\hbar}{4} S\xi(x) + i\frac{\xi}{2} \alpha'_j \right] \frac{\det (M_j + 1)}{\det (M_j + 1)^{\frac{1}{2}}}.$$ (11)

where the summation is over all the classical orbits $\gamma$ whose center lies at the point $x$ after having evolved a time $t$ [29]. Then, $S\xi_j(x)$ is the classical center generating function of the orbit from which the chord $\xi$ joining the initial and final points of the orbit is obtained:

$$\xi = -J\frac{\partial S\xi_j(x)}{\partial x}.$$ (12)

$M_j$ stand for the monodromy matrix and $\alpha'_j$ its Morse index.

The metaplectic operators form a ‘double covering’ of the symplectic matrices since this property gives contributions to the Morse index [4]. If we follow the evolution of the symplectic matrix as the trajectory evolves, each time $M_j$ crosses a manifold where det($M_j + 1$) = 0 (caustic), the path contribution undergoes a divergence changing the sign from $-\infty$ to $\infty$. This change of the sign lets the quantum phase proceed by $\pi$. The Morse index $\alpha'_j$ therefore changes by $\pm 1$ when crossing caustics [26, 31].

For sufficiently short times such that the variational problem has a unique solution, there will be a single chord. Although for longer times, there will be bifurcations producing more chords. In the case of a single orbit, the corresponding Morse index $\alpha'_j = 0$.

The semiclassical approximation for the propagator (11) is inserted into (10) so that

$$\langle X_1|\hat{U}_{\text{sc}}|X_2 \rangle = \left( \frac{2}{\pi\hbar} \right)^{\frac{1}{2}} \sum_\gamma \exp \frac{1}{\hbar} S\xi_j(x) \int \exp \frac{1}{\hbar} \left( \bar{X} - x \right)^2 \exp \frac{i}{\hbar} \left[ S\xi_j(x) - \xi_0 \cdot x \right] dx.$$ (13)

In order to perform the phase space integral in (13), it must be noted that classical orbits that start near $X_2$ and end near $X_1$ will have an important contribution in (13). These orbits have
their center points close to $\bar{X}$. Hence, let us expand the center action up to quadratic terms near the midpoint $\bar{X}$, so that,

$$S'_{\gamma}(x) = S'_{\gamma}(\bar{X}) + \bar{x} \cdot x' + x'^{\dagger} B_t x' + O(x'^{3})$$  \hspace{1cm} (14)

with $x = \bar{X} + x'$ and $S'_{\gamma}(\bar{X})$ is the action of the orbit through the point $\bar{X}$ for which the chord $\bar{x}$ is

$$\bar{x} = -\mathcal{J} \frac{\partial S'_{\gamma}(x)}{\partial x} \bigg|_{\bar{X}},$$

while the symmetric matrix $B_t$ is the Cayley representation of $\mathcal{M}'_{\gamma}$:

$$\mathcal{J}B_t = \frac{1 - \mathcal{M}'_{\gamma}}{1 + \mathcal{M}'_{\gamma}} = \frac{1}{2} \frac{\partial^2 S'_{\gamma}(x)}{\partial x^2}.$$ \hspace{1cm} (15)

Let us define the action $\tilde{S}'_{\gamma}(\bar{X}) = S'_{\gamma}(\bar{X}) + \hbar \pi \alpha$, in order to include the Morse index in the action. After the linearization of the flux around the middle point, expression (14) is inserted into (13); hence, we obtain

$$\langle \chi | \hat{U}_{\text{sc}} | \chi_2 \rangle = \left( \frac{2}{\pi \hbar} \right)^L \sum_{\gamma} \exp \frac{\hbar}{\pi} \left[ \tilde{S}'_{\gamma}(\bar{X}) - \xi_0 \wedge \bar{X} + \frac{1}{2} x_1 \wedge x_2 \right] \times I,$$

with

$$I = \int \exp \left[ -\frac{1}{\hbar} x'^{\dagger} C x' + i x'^{\dagger} B_t x' + (\bar{x} - \xi_0) \wedge x' \right] dx',$$

being a quadratic integral. The matrix $C$ is the quadratic form that denotes the scalar product,

$$x'^{\dagger} = x', x' = x'^{\dagger} C x',$$

where $x'^{\dagger}$ denotes the transposed vector. We now perform exactly the quadratic integral, using

$$I = \int \exp \left[ -\frac{1}{\hbar} x'^{\dagger} \mathcal{V}_l x' + \frac{1}{\hbar} (\bar{X} \wedge x') \right] dx' = \frac{(\pi \hbar)^L}{\sqrt{\det \mathcal{V}_l}} \exp \left[ -\frac{1}{4\hbar} y'^{\dagger} \mathcal{V}_l^{-1} y' \right].$$ \hspace{1cm} (18)

From equation (16)

$$\mathcal{V}_l = C - iB_t$$ \hspace{1cm} (19)

and

$$y = i\mathcal{J}(\bar{x} - \xi_0) = 2i\mathcal{J}\delta_l,$$

where $\bar{x} = x_f - x_i$ is the chord that joins $x_f$ and $x_i$, respectively, the final and initial points of the orbit of center $\bar{X}$. The last expression defines the point shift $\delta_l$, so that

$$\delta_l = \frac{1}{2}(\bar{x} - \xi_0) = x_f - x_i = x_2 - x_i.$$ \hspace{1cm} (20)

Note that the point shift $\delta_l$ is zero if there is a classical orbit starting from the point $X_2$ and ending at $X_1$. Inserting (18) into (16), we obtain for the propagator in coherent states,

$$\langle \chi_1 | \hat{U}_{\text{sc}} | \chi_2 \rangle = 2 \exp \frac{\hbar}{2} \left[ \tilde{S}'(\bar{X}) - \frac{1}{2} x_1 \wedge x_2 \right] \times \exp \left[ -\frac{1}{2\hbar} n_i \mathcal{V}_l \delta_l \right],$$ \hspace{1cm} (22)

with the complex matrix $\mathcal{V}_l$ and the point shift $\delta_l$ defined, respectively, in (19) and (21), while

$$\mathcal{V} = \mathcal{J}^\dagger \mathcal{V}_l^{-1} \mathcal{J}.$$

In order to separate amplitude and phase terms in (22), it is useful to write

$$\mathcal{V} = \mathcal{J} \mathcal{V}_l \mathcal{J}^{-1} = \mathcal{J} \frac{1}{C - iB_t} \mathcal{J} = \mathcal{C}_l - i\mathcal{B}_l,$$

\hspace{1cm} (23)
with the real matrices

\[ C_t = \Re(\tilde{V}) \quad \text{and} \quad B_t = -\Im(\tilde{V}). \]

Also,

\[ \det V_t = | \det V_t | e^{i\varepsilon}, \quad \text{(24)} \]

with \(| \det V_t |\) denoting the modulus and \(\varepsilon\) the argument.

Hence, inserting (23) and (24) into the matrix elements of the coherent state propagator (22), we obtain

\[
\langle X_1 | \hat{U}_{t\text{SC}} | X_2 \rangle = 2^{L} \sum_{\gamma} \frac{1}{| \det V_1 | | \det (M'_\gamma + 1) |} \exp \left[ -\frac{\delta_t \bar{C}_t \delta_t}{\hbar} \right] \times \exp \left[ \frac{\bar{S}'_\gamma (\bar{X}) - \frac{1}{2} X_1 \wedge X_2 + \delta_t \bar{B}_t \delta_t + \frac{\hbar}{2} \frac{\varepsilon}{2} \right]. \quad \text{(25)}
\]

The last expression of the semiclassical matrix elements between two coherent states of the quantum propagator is entirely expressed in terms of real classical objects, namely the action \(S'_\gamma (\bar{X})\) of the classical real orbit whose midpoint is \(\bar{X}\), the point shift \(\delta_t\), the monodromy matrix \(M'_\gamma\) and its Cayley representation \(B_t\) and \(C_t\), the scalar product form. We must note that the phase term in the second line of the expression is clearly separated from the amplitude ones in the first line. In this way, it is important to remark the Gaussian term that dampens the amplitude for large values of the point shift \(\delta_t\), that is, for orbits centered on \(\bar{X}\) that start far from the point \(X_2\) (then end far from \(X_1\)). So, the main contribution to the sum over classical orbits in (25) will come from the particular orbit \(\gamma\), centered in \(\bar{X}\), whose initial point \(x_i\) lies closest to \(X_2\). Other orbit contributions will be highly damped by the exponential term involving the point shift \(\delta_t\). Then, only this particular orbit will be taken into account in the next sections in order to calculate the matrix elements for scar functions.

It must be mentioned here that extensive work has been previously performed for the coherent state matrix elements of the propagator. In particular, a complete semiclassical derivation was performed by Baranger et al. [32], while dos Santos and de Aguiar performed a Weyl ordering treatment in [33]. Although mathematically correct, both constructions involve an analytic continuation to complex trajectories; while expression (25), derived originally in [18], has the peculiarity to avoid complex trajectories, only the real canonical variables of the classical system are needed.

Also, note that, if \(t = 0\), the quantum propagator is just the identity operator in Hilbert space, the classical symplectic matrix is the identity, the center action is null and so are the symmetric matrix \((B_{t=0} = 0)\) and the chord \(\bar{\xi} = 2\delta_t - X_2 + X_1 = 0\). Hence,

\[
\langle X_1 | \hat{U}_{0\text{SC}} | X_2 \rangle = \langle X_1 | X_2 \rangle = 2^{L} \frac{\exp \left[ \frac{1}{4\hbar} \frac{1}{| \det (2) |} \left[ -\frac{1}{2} X_1 \wedge X_2 \right] \right]}{| \det (2) |} \times \exp \left[ -\frac{1}{4\hbar} (X_2 - X_1)^2 \right] \times \exp \left[ -\frac{(X_2 - X_1)^2}{2\hbar} - \frac{i}{2\hbar} X_1 \wedge X_2 \right],
\]

and we recover the result (7) for the overlap of coherent states. As we have seen in [18], expression (25) is exact for the case of linear systems.

3. Matrix elements for scar functions

For the study of the matrix elements of the quantum propagator in the scar function basis, we must insert the expression for the matrix elements of the propagator in the coherent state
basis (25) in the scar function matrix elements (expression (3)). In that way, we obtain

\[
\langle \psi_i^\phi | \hat{U}_{\text{SC}} | \psi_i^\phi \rangle = 2^L \left( - \frac{1}{2} X_1 \wedge X_2 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}r_1 \mathrm{d}r_2 \exp \left( i \theta \phi \sigma - \theta \phi \delta \right) f_{\tau} (t_1) f_{\tau} (t_2) \\
\times \exp \left[ \frac{i}{\hbar} \left[ \hat{S}_{\phi} (X) + \delta^\phi_{Xr} \hat{C}_{\phi} \delta_{Xr} + \hbar \frac{\delta^\phi_{Xr}}{2} \right] \right] \times \exp \left[ - \frac{\delta^\phi_{Xr} \hat{C}_{\phi} \delta_{Xr}}{\hbar} \right].
\]

(26)

where \( t_R = t + t_2 - t_1 \). Note that, in (26), only the contribution of a unique orbit is taken into account. As we have already mentioned, the contributions of orbits with longer point shifts were neglected. With the choice for \( f_{\tau} (t) \) made in (2) and performing the change of variables \( t_s = t_2 + t_1 \), and \( t_t = t_1 - t_2 \), it is possible to separate part of the time integrals so that

\[
\langle \psi_i^\phi | \hat{U}_{\text{SC}} | \psi_i^\phi \rangle = A \exp \left( - \frac{1}{2} \langle X_1 \wedge X_2 \rangle \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}r_1 \mathrm{d}r_2 \exp \left( -2 i \frac{\delta^\phi_{Xr}}{\hbar} \right) e^{\frac{1}{4} (\theta \phi \sigma - \theta \phi \delta)} \\
\times \exp \left[ \frac{i}{\hbar} \left[ \hat{S}_{\phi} (X) + \delta^\phi_{Xr} \hat{C}_{\phi} \delta_{Xr} + \hbar \frac{\delta^\phi_{Xr}}{2} \right] \right] \times \exp \left[ - \frac{\delta^\phi_{Xr} \hat{C}_{\phi} \delta_{Xr}}{\hbar} \right].
\]

(27)

Performing the first time integral, defining \( A = T \sqrt{\frac{\pi}{2}} e^{-2 t_1^2} \) and changing the variables to \( t' = t - t_s \), we obtain

\[
\langle \psi_i^\phi | \hat{U}_{\text{SC}} | \psi_i^\phi \rangle = A \exp \left( - \frac{1}{2} \langle X_1 \wedge X_2 \rangle \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}r_1 \mathrm{d}r_2 \exp \left( -2 i \frac{\delta^\phi_{Xr}}{\hbar} \right) e^{\frac{1}{4} (\theta \phi \sigma - \theta \phi \delta)} \\
\times \exp \left[ \frac{i}{\hbar} \left[ \hat{S}_{\phi} (X) + \delta^\phi_{Xr} \hat{C}_{\phi} \delta_{Xr} + \hbar \frac{\delta^\phi_{Xr}}{2} \right] \right] \times \exp \left[ - \frac{\delta^\phi_{Xr} \hat{C}_{\phi} \delta_{Xr}}{\hbar} \right].
\]

(28)

Equation (28) expresses the semiclassical approximation of the matrix elements of the quantum propagator in the scar function basis uniquely in terms of classical objects. However, at this point, expression (28) has the inconvenience that we need the evaluation of each one of these classical objects, for all times of integration involved. In what follows, we will obtain an explicit expression for the classical objects involved in (28).

For that purpose, we will first perform the study on a surface of section that is transversal to the flux and passing through \( \bar{X} \), in analogy with classical Poincaré surfaces of section. The flux restricted to this section is now a map on the section; for this map, the time is discrete and time integrals must be replaced by summations.

The study of autonomous fluxes through a map on the surface of section is a standard procedure; in the case of billiards, this is done through the well-known Birkhoff coordinates. Also, quantum surface of section methods are shown to be exact [34] for general Hamiltonian systems.

For this procedure, we can choose coordinates near the PO of period \( \tau \) where \( X_2 \) belongs, such that one coordinate is the energy \( E \) and the conjugate coordinate is the time along the orbit. With this choice of coordinates, a point \( x = (\bar{x}, t, E) \) with now \( \bar{x} \) being a \((2L - 2)\) vector on the so-called central surface of section [29]. In order to perform our study on this surface of section near the fixed point \( X_2 \), we have linearized the flux in the neighborhood of the orbit through \( X_2 \). That is, for the orbit \( \gamma \) that starts from \( x_i \) and ends at \( x_f \), \( x_f = \mathcal{L}_\gamma (x_i) \approx \mathcal{M}_\gamma x_i \), where \( \mathcal{M}_\gamma \) is the symplectic matrix denoting this linearized time evolution. As shown in [29], in the transformation \( x_f = \mathcal{M}_\gamma x_i \) for times \( t \) that are integer multiples of \( \tau \), \( t = n \tau \), the points \( x_f = (\bar{x}_f, t_f, E_f) \) and \( x_i = (\bar{x}_i, t_i, E_i) \) on the surface of section have the same energy \( (E_f = E_i) \) and time along the orbit \( (t_f = t_i) \), so we can write

\[
\mathcal{M}_\gamma = \begin{pmatrix} m_{\gamma} & 0 \\ 0 & 1 \end{pmatrix}
\]

(29)
\[
\det \left[ 1 + \mathcal{M}' \right] = 4 \det \left[ 1 + m' \right], \tag{30}
\]
where \( m' \) is now the \((2L - 2) \times (2L - 2)\) symplectic matrix for the center map determined by the orbit \( \gamma \) on the surface section, i.e.
\[
\tilde{x}_f = m'_c \tilde{x}_i.
\]

From now on, the \(2L\)-dimensional autonomous flux is studied through the \(2L - 2\) map on the mentioned surface of section. Also, the point \( X_2 \) on the PO of the flux is a fixed point for the map on the section.

For values of the time that are integer multiples of \( \tau \) and for points \( \tilde{x} \) on the surface of section, expression (28) for the matrix elements of the propagator takes now the following form (replacing time integrations by summations):
\[
\langle \psi_{X_1} | \hat{U}^t | \psi_{X_2} \rangle = \exp \left( \frac{i}{\hbar} \left[ -\frac{1}{2} \mathcal{X}_1 \wedge \mathcal{X}_2 \right] A \sum_{n = -\infty}^{\infty} e^{-2(t - t')^2/T^2} e^{-\frac{i}{\hbar} (\phi_2 + \phi_1)(t - t')} \right) \times \exp \left[ \frac{1}{\hbar} \left[ \bar{\mathcal{X}}_2 (\mathcal{X}) + \bar{\mathcal{Y}}_f \delta_f + \hbar \hat{\mathcal{F}} \right] \right] \times \exp \left[ -\frac{1}{\hbar^2} \bar{G} \bar{F} \delta_c \right], \tag{31}
\]
with \( t = n \tau \) and \( t' = n' \tau \) with \( n \) and \( n' \) being integer numbers. The last expression represents the semiclassical propagator matrix elements in the scar function basis on the surface of section that cuts transversally the PO at the point \( X_2 \). In order to deal with the infinite time summation in (31), we perform a cutoff for values of \(|t - t'|\) greater than \( T \), the Ehrenfest time, beyond which the time-dependent Gaussian became negligible. Remember also the discussion according to the choice of the function \( J_f(t) \) in (1).

As we have already mentioned, we need to evaluate the classical objects involved in (31) to perform the time summation. Let us first obtain expressions for the point shifts \( \delta \) and the center action \( \tilde{S}_f(\mathcal{X}) \) in terms of the monodromy matrix. For that purpose, we linearize the flux in the neighborhood of the fixed point \( X_2 \). That is,
\[
x_f = \mathcal{X}_1 + \delta_1 = \mathcal{L}'_f(x_i) = \mathcal{L}'_f(X_2 - \delta_f) \approx \mathcal{M}'_f(X_2) - \mathcal{M}'_f \delta_f = X_2 - \mathcal{M}'_f \delta_f,
\]
where \( \mathcal{M}'_f \) is the symplectic matrix denoting this linearized time evolution in the neighborhood of \( X_2 \); the last equality holds because \( X_2 \) is a fixed point. Resolving for \( \delta_f \), we obtain
\[
\delta_f = -\frac{1}{\mathcal{M}'_f + 1} (X_1 - X_2). \tag{32}
\]

Equivalently, we can perform the linearization using the center generating function near the point \( X_2 \), so that
\[
S'_f(x) = S'_f(X_2) + (x - X_2)^T \mathcal{B}_f (x - X_2) + O(x^3). \tag{32}
\]

Note that the linear term \( \xi_2 \wedge (x - X_2) \) is not present here since \( X_2 \) is a fixed point; hence, the chord \( \xi_2 \) passing through it is null, \( \xi_2 = -\mathcal{J} \frac{\partial S'_f(x)}{\partial x} \big|_{X_2} \) = 0. Also, for \( X_2 \) being a fixed point
\[
S'_f(X_2) = i S_k,
\]
where \( S_k \) is the action of the PO in \( X_2 \) in which the Morse index \( \alpha'_f = \alpha_f + \hbar \alpha_f \). Let us define the action \( S_{X_2} = S_{X_2} + \hbar^{\frac{1}{2}} \alpha_f \) in order to include the Morse index in the action.

The chord \( \xi \) of the orbit \( \gamma \) centered at \( X \) is obtained by performing the derivative of the center generating function (32):
\[
\xi = -\mathcal{J} \frac{\partial S'_f(x)}{\partial x} \big|_X = -2 \mathcal{J} \mathcal{B}_f (X - X_2) = -\mathcal{J} \mathcal{B}_f \xi_0.
\]
so that (as already seen)
\[
\delta_t = \frac{1}{2}(\xi - \xi_0) = -\frac{1}{2}(J B_t + 1) \xi_0 = -\frac{1}{\mathcal{M}'_t + 1} \xi_0.
\]  

(33)

Hence, the center generating function in the middle point \( \tilde{X} \) is
\[
\tilde{S}'_t(\tilde{X}) = i \tilde{S}_X + (\tilde{X} - X_2)^\dagger B_t (\tilde{X} - X_2) = i \tilde{S}_X + \frac{1}{2} \xi_0^\dagger B_t \xi_0.
\]  

(34)

It is important to mention that the summation to be performed in (31) is a summation on the orbits \( \gamma \) that starts near \( X_2 \) and after a time \( t \) ends up near \( X_1 \), having \( \tilde{X} \) as the center point; this is a sum on the family of heteroclinic orbits as already seen in [9]. Inserting expressions (33) and (34), respectively, for the point shifts \( \delta_t \) and the center action \( \tilde{S}'_t(\tilde{X}) \) in the scar function expressions (28), we obtain
\[
\langle \phi_{X_1}^{(s)} | \hat{U} | \phi_{X_1}^{(s)} \rangle = A \exp \left( \frac{i}{\hbar} \left[ -\frac{1}{2} X_1 \wedge X_2 \right] \sum_{n'=\infty}^\infty e^{-2i(t-t')^2/\hbar^2} \exp \left[ -\frac{\xi_0^\dagger E_t \xi_0}{\hbar} \right] \right),
\]  

(35)

with the matrices \( E_t \) and \( D_t \) so that
\[
E_t = \left( \frac{1}{\mathcal{M}'_t + 1} \right)^\dagger \tilde{C}_t \left( \frac{1}{\mathcal{M}'_t + 1} \right) \quad \text{and} \quad D_t = \left( \frac{1}{\mathcal{M}'_t + 1} \right)^\dagger \tilde{B}_t \left( \frac{1}{\mathcal{M}'_t + 1} \right).
\]  

(36)

Equation (35) gives a general expression only in terms of classical objects; its difference from (31) is that we have used the linearization around the PO in order to express both the point shift and the center generating function only in terms of the monodromy matrix of the linearized transformation. In this way, the semiclassical approximation of the scar function’s matrix elements involves uniquely the action of the classical orbit \( \tilde{S}_X \), the scalar product \( C \) for the symplectic basis of vectors and the monodromy matrices \( \mathcal{M}'_t \). From the former, we obtain its Cayley representation \( B_t \) through equation (15), after which the complex matrix \( V_t \) is obtained with (19) and (24) expressing its exponential form, while the real matrices \( \tilde{C}_t \) and \( \tilde{B}_t \) defined in (23) allow us to obtain \( D_t \) and \( E_t \) though (36).

Although in order to perform the time summation, we need the classical objects for all the different times involved. As we will show, for that purpose, it will be convenient to express them in the basis of eigenvectors of the symplectic matrix. For the case of a map with one degree of freedom (corresponding to a two degrees of freedom flux), these are the stable and unstable vector bases \( (\zeta_u, \zeta_s) \) where the eigenvalues of the symplectic matrix \( \mathcal{M}'_t \) are \( \exp(-\lambda t) \) and \( \exp(\lambda t) \) (\( \lambda \) is the stability or the Lyapunov exponent of the orbit).

Let us then define \( x_u \) and \( x_s \) as canonical coordinates along the stable and unstable directions, respectively, such that \( x = (x_u, x_s) = x_u \zeta_u + x_s \zeta_s \) with \( \zeta_u \wedge \zeta_s = 1 \). As the basis formed by \( (\zeta_u, \zeta_s) \) is non-orthonormal, the scalar product of two vectors takes the form
\[
x_1 \cdot x_2 = x_1^\dagger C x_2 = \left[ \zeta_u^2 x_{1u} x_{2u} + \zeta_s^2 x_{1s} x_{2s} + \zeta_u \zeta_s (x_{1u} x_{2s} + x_{1s} x_{2u}) \right].
\]

That is, the scalar product matrix is
\[
C = \begin{bmatrix}
\zeta_u^2 & \zeta_u \zeta_s \\
\zeta_u \zeta_s & \zeta_s^2
\end{bmatrix}
\]

(37)

with \( \zeta_u^2 = \tilde{C}_u \zeta_u \zeta_s \) and \( \zeta_s^2 = \tilde{C}_s \zeta_u \zeta_s \). Since the transformation from the orthonormal basis \( (i, j) \) to the basis \( (\tilde{C}_u, \tilde{C}_s) \) is symplectic
\[
\det C = \zeta_u^2 \zeta_s^2 - (\tilde{C}_u \zeta_s)^2 = 1.
\]
Also, in the \((\tilde{\zeta}_u, \tilde{\zeta}_s)\) basis,
\[
\mathcal{M}'_t + 1 = 2 \cosh \left( \frac{\lambda t}{2} \right) \begin{bmatrix} e^{i\lambda/2} & 0 \\ 0 & e^{-i\lambda/2} \end{bmatrix};
\]
(38)

hence,
\[
| \det (\mathcal{M}'_t + 1) | = 4 \cosh^2 \left( \frac{\lambda t}{2} \right)
\]
(39)
is easily obtained only in terms of \(\lambda\) and \(t\). Analogously,
\[
\mathcal{M}'_t - 1 = 2 \sinh \left( \frac{\lambda t}{2} \right) \begin{bmatrix} e^{i\lambda/2} & 0 \\ 0 & -e^{-i\lambda/2} \end{bmatrix},
\]
while \(B_t\), the Cayley parameterization of \(\mathcal{M}'_t\), is in this basis
\[
B_t = \begin{bmatrix} 0 & \tanh (t\lambda/2) \\ \tanh (t\lambda/2) & 0 \end{bmatrix}.
\]
(40)

Hence, using the expression of the symmetric matrix \(B_t\) (40) and the scalar product (37), we obtain the complex matrix
\[
\mathcal{V}_t = C - iB_t = \begin{bmatrix} \tilde{\zeta}_u^2 - \tilde{\zeta}_s \tilde{\zeta}_s & -i \tanh (t\lambda/2) \\ \tilde{\zeta}_u \tilde{\zeta}_s & \tilde{\zeta}_u^2 \end{bmatrix},
\]
for which the complex determinant
\[
\det \mathcal{V}_t = [1 + \tanh^2 (t\lambda/2) + 2i\tilde{\zeta}_u \tilde{\zeta}_s \tanh (t\lambda/2)],
\]
with modulus
\[
|\det \mathcal{V}_t| = \sqrt{[1 + \tanh^2 (t\lambda/2)]^2 + (2\tilde{\zeta}_u \tilde{\zeta}_s \tanh (t\lambda/2))^2}
\]
(43)
and argument
\[
\epsilon = \arctan \frac{2\tilde{\zeta}_u \tilde{\zeta}_s \tanh (t\lambda/2)}{1 + \tanh^2 (t\lambda/2)}
\]
(44)
can be explicitly written in terms of the time and the Lyapunov exponent. Now, inverting the matrix \(\mathcal{V}_t\) (41), we obtain
\[
\mathcal{V}_t^{-1} = \frac{1}{\det \mathcal{V}_t} \begin{bmatrix} \tilde{\zeta}_u^2 & -\tilde{\zeta}_u \tilde{\zeta}_s + i \tanh (t\lambda/2) \\ -\tilde{\zeta}_u \tilde{\zeta}_s & \tilde{\zeta}_u^2 + i \tanh (t\lambda/2) \end{bmatrix} = \frac{1}{\det \mathcal{V}_t} (C^{-1} + iB_t).
\]

Also, we must note that since the matrix \(\mathcal{V}_t\) is symmetric, we obtain that
\[
\tilde{\mathcal{V}} = J^\dagger \mathcal{V}_t^{-1} J = \frac{\tilde{\mathcal{V}}_t}{\det \mathcal{V}_t} \left( \Re (\det \mathcal{V}_t) - i\Im (\det \mathcal{V}_t) \right) (C - iB_t) = \tilde{\mathcal{C}}_t - i\tilde{\mathcal{B}}_t.
\]
(45)

Hence, in the stable and unstable vector bases \((\tilde{\zeta}_u, \tilde{\zeta}_s)\), the real matrices \(\tilde{\mathcal{C}}_t\) and \(\tilde{\mathcal{B}}_t\) take the form
\[
\tilde{\mathcal{C}}_t = \Re (\tilde{\mathcal{V}}) = \frac{1}{|\det \mathcal{V}_t|^2} [C(1 + \tanh^2(t\lambda/2)) - 2B_t \tilde{\zeta}_u \tilde{\zeta}_s \tanh(t\lambda/2)],
\]
(46)
and
\[
\tilde{\mathcal{B}}_t = -\Im (\tilde{\mathcal{V}}) = \frac{1}{|\det \mathcal{V}_t|^2} [B_t(1 + \tanh^2(t\lambda/2)) + 2C \tilde{\zeta}_u \tilde{\zeta}_s \tanh(t\lambda/2)]
\]
(47)
with the symmetric matrix \(B_t\), the scalar product matrix \(C\) and the determinant \(\det \mathcal{V}_t\), respectively, given by expressions (40), (37) and (43). Inserting expressions (38), (46) and (47) into the definition of the symmetric matrices \(D_t\) and \(E_t\) (36), we obtain
\[
D_t = -2 \frac{\tanh (t\lambda/2)}{\det_1} \begin{bmatrix} -\zeta_u^2 (\tilde{\zeta}_u \tilde{\zeta}_s) e^{-i\lambda} & 1 + \tanh^2 (t\lambda/2) + 2(\tilde{\zeta}_u \tilde{\zeta}_s)^2 \\ 1 + \tanh^2 (t\lambda/2) + 2(\tilde{\zeta}_u \tilde{\zeta}_s)^2 & -\zeta_u^2 (\tilde{\zeta}_u \tilde{\zeta}_s) e^{i\lambda} \end{bmatrix}
\]
(48)
and

\[ E_t = -\frac{1 + \tanh^2 (\lambda t/2)}{\det_1} \begin{bmatrix} -\xi_1^2 e^{-\lambda t/2} & \xi_0 \xi_1 e^\lambda t/2 \\ \xi_0 \xi_1 e^{-\lambda t/2} & 2 \sinh^2 (\lambda t/2) + 1 & -\xi_1^2 e^{\lambda t/2} \end{bmatrix} \]  

(49)

where we have defined

\[ \det_1 = 4 \cosh^2 (\lambda t/2) |\det \mathcal{V}|^2. \]

It is important to note that (48), (49), (40), (42) and (39) are, respectively, explicit expressions of the symmetric matrices \( \mathcal{D}_t, \mathcal{E}_t \) and \( \gamma_t \) and the determinants \( |\det \mathcal{V}| \) and \( |\det (\mathcal{M}_t') + 1| \) for any value of the time \( t \). Inserting these expressions into (35), the time summation can be numerically performed. In this way, we obtain a semiclassical expression for the matrix elements of the propagator in the scar function basis entirely in terms of classical features such as the chord \( \xi_0 \) that joins the points \( X_2 \) and \( X_1 \), the action of the PO \( \bar{S}_{X}, \) the stable and unstable vectors \( \zeta_u, \zeta_s \) and the Lyapunov exponent \( \lambda \).

4. Scar function matrix elements for the cat map

Now, the present theory is applied to the cat map, i.e. the linear automorphism on the 2-torus generated by the 2 \times 2 symplectic matrix \( \mathcal{M} \) that takes a point \( x_\cdot \) to a point \( x_+: \) \( x_+ = \mathcal{M} x_\cdot \mod(1) \). In other words, there exists an integer two-dimensional vector \( \mathbf{m} \) such that \( x_+ = \mathcal{M} x_\cdot - \mathbf{m} \). Equivalently, the map can also be studied in terms of the center generating function [36]. This is defined in terms of center points

\[ x \equiv \frac{x_+ + x_-}{2} \]  

(50)

and chords

\[ \xi \equiv x_+ - x_- = -\mathcal{J} \frac{\partial S(x, \mathbf{m})}{\partial x} \]  

(51)

where

\[ S(x, \mathbf{m}) = xBx + x (\mathcal{J} - \mathcal{M}) \mathbf{m} + \frac{1}{2} \mathbf{m}(\mathcal{B} + \mathcal{J})\mathbf{m} \]  

(52)

is the center generating function. Here, \( \mathcal{B} \) is a symmetric matrix (the Cayley parameterization of \( \mathcal{M} \), as in (40), while

\[ \mathcal{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]  

(53)

We will study here the cat map with the symplectic matrix

\[ \mathcal{M} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \text{ and symmetric matrix } \mathcal{B} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}. \]  

(54)

This map is known to be chaotic (ergodic and mixing) as all its POs are hyperbolic. The periodic points \( x_l \) of the integer period \( l \) are labeled by the winding numbers \( \mathbf{m} \), so that

\[ x_l = \left( \frac{p_l}{q_l} \right) = (\mathcal{M}^l - 1)^{-1} \mathbf{m}. \]  

(55)

The first periodic points of the map are the fixed points at \( (0, 0) \) and \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and the POs of period 2 are \( \left\{ (0, \frac{1}{2}), (\frac{1}{2}, 0) \right\}, \left\{ (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}) \right\}, \left\{ (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}) \right\}, \left\{ (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}) \right\} \) and \( \left\{ (0, \frac{1}{2}), (0, \frac{1}{2}) \right\} \). The eigenvalues of \( \mathcal{M} \) are \( e^{\lambda x} \) and \( e^{\lambda u} \) with \( \lambda = \ln(2 + \sqrt{3}) \approx 1.317 \). This is then the stability exponent for the fixed points, whereas the exponents must be doubled.
for orbits of period 2. All the eigenvectors have directions $\vec{\xi}_v = \left(-\frac{\sqrt{3}}{3}, \frac{1}{2}\right)$ and $\vec{\xi}_u = \left(1, \frac{1}{\sqrt{3}}\right)$ corresponding to the stable and unstable directions, respectively.

Quantum mechanics on the torus implies a finite Hilbert space of dimension $N = \frac{1}{2\pi \eta}$, and that positions and momenta are defined to have discrete values in a lattice of separation $\frac{1}{N}$ [35, 30]. The cat map was originally quantized by Hannay and Berry [35] in the coordinate representation and the propagator is

$$U_{\mathcal{M}}(x) = \frac{2}{|\text{det}(\mathcal{M} + 1)|^2} \sum_m e^{i2\pi N[S(x, m)]},$$

(57)

where the center points are represented by $x = (\frac{a}{N}, \frac{b}{N})$ with $a$ and $b$ being integer numbers in $[0, N - 1]$ for odd values of $N$ [30]. There exists an alternative definition of the torus Wigner function which also holds for even $N$.

The fact that the symplectic matrix $\mathcal{M}$ has equal diagonal elements implies time-reversal symmetry and then the symmetric matrix $B$ has no off-diagonal elements. This property will be valid for all the powers of the map and, using (58), we can see that it implies quantum symmetry

$$U_{\mathcal{M}}^l(p, q) = (U_{\mathcal{M}}^l(-p, q))^* = (U_{\mathcal{M}}^l(p, -q))^*,$$

(59)

for any integer value of $l$.

It has been shown [35] that the unitary propagator is periodic (nilpotent) in the sense that for any value of $N$, there is an integer $k(N)$ such that

$$\hat{U}_{\mathcal{M}}^{k(N)} = e^{i\phi}.$$

Hence, the eigenvalues of the map lie on the $k(N)$ possible sites

$$\left\{\exp\left[i(2m\pi + \phi)/k(N)\right]\right\}, \quad 1 \leq m \leq k(N).$$

(60)

For the cases where $k(N) < N$, there are degeneracies and the spectrum does not behave as expected for chaotic quantum systems. In spite of the peculiarities in this map, a very weak nonlinear perturbation of cat maps restores the universal behavior of non-degenerate chaotic quantum system spectra [37]. Eckhardt [38] has argued that typically, the eigenfunctions of cat maps are random.

The scar Wigner function on the torus depends on the definition of the periodic coherent state [39], with $(\rho) = P$ and $(\eta) = Q$. In accordance with (6)

$$\langle X | q_k \rangle = \sum_{j=-\infty}^{\infty} \exp\left\{-\frac{1}{\hbar} \left[iP \left(j + \frac{Q}{2} - k/N\right) + \frac{1}{2} (j + Q - k/N)^2\right]\right\}.$$

(61)

The scar function is then defined on the torus as

$$|\varphi_{X, \phi}\rangle = \sum_{l=-\infty}^{\infty} e^{i\phi} e^{-\frac{1}{2\eta^2}} U_{\mathcal{M}}^{l} |X\rangle.$$

(62)
Indeed, the construction on the torus from the plane is obtained in terms of averages (real and imaginary parts) the exact matrix elements between the two scar functions constructed on the fixed points of the cat map. The third column shows the respective semiclassical approximation, using expression (35), of the matrix elements shown in the second column. The fourth column displays (real and imaginary parts) the exact matrix elements between the two scar functions constructed on the fixed points for one iteration of the quantum propagator for the cat map. The fifth column shows the respective semiclassical approximation, using expression (35), of the matrix elements shown in the fourth column.

| $N$ | $\langle \psi^I_{X_1} | \psi^I_{X_2} \rangle$ | $\langle \psi^I_{X_1} | \psi^I_{X_2} | \text{SC} \rangle$ | $\langle \psi^I_{X_1} | \hat{U} | \psi^I_{X_2} \rangle$ | $\langle \psi^I_{X_1} | \hat{U} | \psi^I_{X_2} | \text{SC} \rangle$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| 100 | 0.321 701 30    | 0.321 701 28    | 0.364 484 90 + i0.419 063 829 | 0.364 484 89 + i0.419 063 830 |
| 101 | 0, 0            | 0, 0            | 0, 0            | 0, 0            |
| 104 | 0.330 824 19    | 0.330 824 18    | 0.375 201 32 + i0.397 240 638 | 0.375 201 33 + i0.397 240 637 |
| 200 | 0.364 685 29    | 0.364 685 30    | 0.453 266 51 + i0.358 242 569 | 0.453 266 50 + i0.358 242 570 |

Remember that for maps, time has discrete values; then, the time integral in (2) has in this case been replaced by a summation. Also, as we have already discussed, for our numerical computations, we truncate the sum for times $|t| > T/2$, where the Gaussian damping term became negligible.

In order to construct operators or functions on the torus, we have to periodize the operation in the Hilbert space of the torus is $[30]$. In this way, the coherent state matrix elements for any operator on the torus are obtained taking in both cases the torus periodization (63). As we can observe, the semiclassical expression (35) is exact in this case. This fact is not surprising since the cat map is equivalent to a quadratic Hamiltonian system. Also, we have verified that, as previously seen in [10], these matrix elements are not null only for values of $N$ that are multiples of 4 and in these cases, the matrix elements $\langle \psi^I_{X_1} | \psi^I_{X_2} \rangle$ are real numbers.
5. Discussion

The semiclassical theory of short POs developed by Vergini and co-workers [5–10] is a formalism where the number of used POs needed to obtain the spectrum of a classically chaotic system increases only linearly with the mean energy density, allowing us to obtain all the quantum information of a chaotic Hamiltonian system in terms of a very small number of short POs. The key elements in this theory are wavefunctions related to short unstable POs and then the evaluation of matrix elements between these wavefunctions is crucial.

In this work, by means of the Weyl representation, we have obtained a semiclassical expression for these matrix elements of the propagator in the scar function basis entirely in terms of the classical canonical invariants such as the chord that joins the points $X_2$ and $X_1$, the action of the PO, the stable and unstable vectors $\vec{\zeta}_u$, $\vec{\zeta}_s$ and the Lyapunov exponent $\lambda$. Also, the comparison with a system whose semiclassical limit is exact has allowed us to correctly check the exactness of the obtained expression up to quadratic Hamiltonian systems.

As already seen [10, 20], with these matrix elements at hand, the spectrum of the propagator can be obtained without requiring an explicit computation of scar functions. Of course, in order to include nonlinear contributions, a deeper understanding of the dynamics up to the Ehrenfest time is required; in this respect, enormous efforts were recently carried out in such a direction for the diagonal matrix elements [20].

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Appendix. Reflection operators in phase space

Among the several representations of quantum mechanics, the Weyl–Wigner representation is the one that performs a decomposition of the operators that act on the Hilbert space, on the basis formed by the set of unitary reflection operators. In this appendix, we review the definition and some properties of these reflection operators.

First of all, we construct the family of unitary operators

$$\hat{T}_q = \exp(-i\hbar^{-1} q_{\hat{p}}), \quad \hat{T}_p = \exp(i\hbar^{-1} p_{\hat{q}}),$$

and following [29], we define the operator corresponding to a general translation in phase space by $\xi = (p, q)$ as

$$\hat{T}_\xi \equiv \exp\left(\frac{i}{\hbar} \xi \wedge \hat{x} \right) = \exp\left[\frac{i}{\hbar} (p_{\hat{q}} - q_{\hat{p}})\right] \quad (A.2)$$

$$= \hat{T}_q \hat{T}_p \exp\left[\frac{-i}{2\hbar} p_{\hat{q}}\right] = \hat{T}_p \hat{T}_q \exp\left[\frac{i}{2\hbar} p_{\hat{q}}\right],$$

where naturally $\hat{x} = (\hat{p}, \hat{q})$. In other words, the order of $\hat{T}_p$ and $\hat{T}_q$ affects only the overall phase of the product, allowing us to define the translation as above. $\hat{T}_\xi$ is also known as a Heisenberg operator. Acting on the Hilbert space, we have

$$\hat{T}_\xi |q_a\rangle = e^{\frac{i}{\hbar} (q_{\hat{a}} + \frac{1}{2}\xi)} |q_a + q\rangle$$

and

$$\hat{T}_\xi |p_a\rangle = e^{-\frac{i}{\hbar} (p_{\hat{a}} + \frac{1}{2}\xi)} |p_a + p\rangle.$$
We, hence, verify their interpretation as translation operators in phase space. The group property is maintained within a phase factor:

$$\hat{T}_{\xi_1} \hat{T}_{\xi_2} = \hat{T}_{\xi_1 + \xi_2} \exp\left[-\frac{i}{\hbar} (\xi_1 \wedge \xi_2)\right]. \quad (A.6)$$

where $D_3$ is the symplectic area of the triangle determined by two of its sides. Evidently, the inverse of the unitary operator $\hat{T}_{\xi}^{-1} = \hat{T}_{-\xi}^\dagger = \hat{T}_{-\xi}$.

The set of operators corresponding to phase space reflections $\hat{R}_x$ about points $x = (p, q)$ in phase space is formally defined in [29] as the Fourier transform of the translation (or Heisenberg) operators

$$\hat{R}_x \equiv \left(4\pi \hbar\right)^{-\frac{1}{2}} \int dx \hat{T}_x.$$ \quad (A.7)

Their actions on the coordinate and momentum bases are

$$\hat{R}_x |q_a\rangle = e^{2i(q-q_a)p/\hbar} |2q - q_a\rangle \quad (A.8)$$

$$\hat{R}_x |p_a\rangle = e^{2i(p-p_a)q/\hbar} |2p - p_a\rangle , \quad (A.9)$$

displaying the interpretation of these operators as reflections in phase space. Also, using the coordinate representation of the coherent state (6) and the action of reflection on the coordinate basis (A.8), we can see that the action of the reflection operator $\hat{R}_x$ on a coherent state $|X\rangle$ is

$$\hat{R}_x |X\rangle = \exp\left(i \hbar X \wedge x\right) |2x - X\rangle . \quad (A.10)$$

This family of operators has the property that they are a decomposition of the unity (completeness relation)

$$\hat{1} = \frac{1}{2\pi \hbar} \int dx \hat{R}_x,$$ \quad (A.11)

and also they are orthogonal in the sense that

$$\text{Tr}[\hat{R}_{x_1} \hat{R}_{x_2}] = 2\pi \hbar \delta(x_2 - x_1). \quad (A.12)$$

Hence, an operator $\hat{A}$ can be decomposed in terms of reflection operators as follows:

$$\hat{A} = \frac{1}{2\pi \hbar} \int dx A_W(x) \hat{R}_x . \quad (A.13)$$

With this decomposition, the operator $\hat{A}$ is mapped on a function $A_W(x)$ living in phase space, the so-called Weyl–Wigner symbol of the operator. Using (A.12), it is easy to show that $A_W(x)$ can be obtained by performing the following trace operation:

$$A_W(x) = \text{Tr}[\hat{R}_x \hat{A}].$$

Of course, as shown in [29], the Weyl symbol also takes the usual expression in terms of matrix elements of $\hat{A}$ in the coordinate representation

$$A_W(x) = \int \langle q - \frac{Q}{2} | \hat{A} | q + \frac{Q}{2} \rangle \exp\left(-i \hbar pQ\right) dQ.$$

It was also shown in [29] that reflection and translation operators have the following composition properties:

$$\hat{R}_{x} \hat{T}_{\xi} = \hat{T}_{x - \xi/2} e^{-i x \wedge \xi}, \quad (A.14)$$

$$\hat{T}_{\xi} \hat{R}_{x} = \hat{T}_{x + \xi/2} e^{i x \wedge \xi}, \quad (A.15)$$

$$\hat{R}_{x_1} \hat{R}_{x_2} = \hat{T}_{2(x_2-x_1)} e^{2i x_2 \wedge x_1}, \quad (A.16)$$
so that
\[ \hat{R}_x \hat{R}_y = \hat{1}. \] (A.17)

Now using (A.16) and (A.15), we can compose three reflections so that
\[ \hat{R}_{x_2} \hat{R}_{x_1} \hat{R}_{x_3} = e^{i \Delta_2 (x_2-x_1) \eta_{x_2-x_3}}, \] (A.18)
where \( \Delta_1 (x_2, x_1, x) = 2(x_2 - x) \wedge (x_1 - x) \) is the area of the oriented triangle whose sides are centered at the points \( x_2, x_1 \) and \( x \), respectively.

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