ON A KIND OF SELF-SIMILAR SETS WITH COMPLETE OVERLAPS

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Abstract. Let $E$ be the self-similar set generated by the iterated function system

$$f_0(x) = \frac{x}{\beta}, \quad f_1(x) = \frac{x + 1}{\beta}, \quad f_{\beta+1} = \frac{x + \beta + 1}{\beta}$$

with $\beta \geq 3$. Then $E$ is a self-similar set with complete overlaps, i.e., $f_0 \circ f_{\beta+1} = f_1 \circ f_1$, but $E$ is not totally self-similar. We investigate all its generating iterated function systems, give the spectrum of $E$, and determine the Hausdorff dimension and Hausdorff measure of $E$ and of the sets which contain all points in $E$ having finite or infinite different triadic codings.

1. Introduction

Let $\beta \geq 3$, and let $E_\beta$ be the self-similar set generated by the iterated function system (IFS)

$$f_d(x) = \frac{x + d}{\beta}, \quad d \in \{0, 1, \beta + 1\}.$$  

Then $E_\beta$ is the unique non-empty compact set in the real line satisfying $E_\beta = \bigcup_{d \in \{0, 1, \beta + 1\}} f_d(E_\beta)$ (cf. [12]). It is easy to check $f_0 \circ f_{\beta+1} = f_1 \circ f_1$. Then the self-similar set $E_\beta$ has complete overlaps.

Our interest in $E_\beta$ comes from expansions in non-integer bases (for the surveys see [13, 18]). One example is expansions with digit set $\{0, 1, \beta\}$. For $\beta > 1$, let $F_\beta$ be the attractor of the IFS

$$\phi_d(x) = \frac{x + d}{\beta}, \quad d \in \{0, 1, \beta\}.$$  

Then $F_\beta$ is a self-similar set with overlaps since $\phi_0(F_\beta) \cap \phi_1(F_\beta) \neq \emptyset$. There has been considerable interest in $F_\beta$. For example, Ngai and Wang [16] investigated the Hausdorff dimension of $F_\beta$. Zou et al. [21] considered the set of points in $F_\beta$ having a unique $\beta$-expansion. Yao and Li [19] gave all the generating IFSs of $F_\beta$. Dajani et al. [3] described the size of the set of bases $\beta$ for which there exists $x \in F_\beta$ having finite or countably many different $\beta$-expansions and the set of $x \in F_\beta$ which have exactly finite or countable $\beta$-expansions.

There are two striking differences between $E_\beta$ and $F_\beta$, one is that the total self-similarity (see [2] for its first appearance) fails in $E_\beta$, as we will explain later. Another is that by the obvious fact that $\phi_0 \circ \phi_\beta = \phi_1 \circ \phi_0$ we have $\phi_{1k0} = \phi_{0\beta k}$ for any positive integer $k$, which is an important property in discussing $F_\beta$. However, we do not see this property in $E_\beta$.

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We will abbreviate $E_\beta$ to $E$ if no confusion arises. Observe that for each point $x \in E$ there exists an infinite sequence $(d_i) \in \{0, 1, \beta + 1\}^\mathbb{N}$ such that

\begin{equation}
    x = \lim_{n \to \infty} f_{d_1 \ldots d_n}(0) := \lim_{n \to \infty} f_{d_1} \circ \cdots \circ f_{d_n}(0) = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}.
\end{equation}

The infinite sequence $(d_i)$ is called a *coding* of $x$. Since $f_0(\beta + 1) = f_{11}$, a point $x \in E$ may have multiple codings. By (1.1) it follows that

$$E = \left\{ \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} : d_i \in \Omega \right\},$$

where $\Omega := \{0, 1, \beta + 1\}$ is the *alphabet* which will be fixed throughout the paper.

\[ 0 \quad \gamma_\beta := \frac{\beta + 1}{\beta - 1} \quad \Delta \]

\[ f_0 \quad f_1 \quad f_{\beta + 1} \quad \Delta_1 \]

\[ f_{00} \quad f_{0(\beta + 1)} = f_{11} \quad f_{(\beta + 1)0} \quad f_{(\beta + 1)(\beta + 1)} \quad \Delta_2 \]

**Figure 1.** The first three levels $\Delta, \Delta_1, \Delta_2$ of basic intervals of $E_\beta$ with $\beta = 5$.

Let $\Delta = [0, \gamma_\beta]$ be the convex hull of $E$, where $\gamma_\beta := (\beta + 1)/(\beta - 1)$. Then for $n \in \mathbb{N}$ let

$$\Delta_n := \bigcup_{i \in \Omega^n} f_i(\Delta)$$

be the union of all level-$n$ basic intervals of $E$ (see Figure 1 for the first three levels of basic intervals), where $\Omega^n := \{d_1 \ldots d_n : d_i \in \Omega\}$ is the set of all blocks of length $n$ with respect to the alphabet $\Omega$. Since $\beta \geq 3$, the basic interval $f_{10}(\Delta)$ is on the right-hand side of $f_{01}(\Delta)$, and their intersection is nonempty (in fact is a singleton only if $\beta = 3$).

We write

$$H := \Delta \setminus \Delta_1 = (f_1(\gamma_\beta), f_{\beta + 1}(0)), \quad \text{and} \quad H_1 = f_1(H).$$

Then $H$ and $H_1$ are open intervals for all $i \in \Omega^* := \bigcup_{n=1}^{\infty} \Omega^n$. By a *hole* of $E$ we mean an open interval $(a, b)$ with $a, b \in E$ and $(a, b) \cap E = \emptyset$. Then $H$ is the biggest hole of $E$. Unlike the situation in $F_\beta$, we emphasize that $H_1$ is not necessarily a hole of $E$. For example, $H_0 = f_0(H)$ is not a hole since $H_0 \cap f_{10}(E) \neq \emptyset$. By [5, Proposition 2.1] it follows that $E$ is not *totally self-similar*.

As a new class of self-similar sets with complete overlaps we first investigate all the generating iterated function systems (IFSs) of $E$. We call a nonempty compact set $F \subseteq \mathbb{R}^d$ a *self-similar set* if it is a finite union of its self-similar copies; this is, there exists a family of contractive similitudes $F = \{\psi_i(x) = \rho_i O_i x + b_i\}_{i=1}^N (N \geq 2$ is an integer) such that
\( F = \bigcup_{i=1}^{N} \psi_i(F) \), where \( \rho_i \in (0, 1) \), \( O_i \) is a \( d \times d \) orthogonal matrix and \( b_i \in \mathbb{R}^d \) is a translation. The family \( F \) is called a generating IFS for \( F \). It is well known that \( F \) determines \( F \) uniquely, but not vice versa.

The question of determining all the generating IFSs for a given self-similar set was initiated by Feng and Wang [10] and was later studied by Deng and Lau [6, 7], and Yao [20]. The requirement of some separation condition (open set condition or strong separation condition, etc.) is crucial in their proof. If the assumption of separation condition is dropped, the problem will be more complicated. Dajani et al. [5] first gave an answer to this problem for a kind of self-similar sets which are totally self-similar.

Our first main result is on the generating IFSs of \( E \), which satisfies neither of the conditions stated in the above paragraph.

**Theorem 1.1.** If \( g \) is an affine map such that \( g(E) \subseteq E \), then \( g = f_i \) for some \( i \in \Omega^* \).

During the past fifty years the question on the spectrum of overlapping fractals have been extensively studied because of its close connections with infinite Bernoulli convolutions and expansions in noninteger bases. Many works have been devoted to this topic, see [1, 8, 9, 11, 17] and the references therein.

Motivated by the work of [5] we define the spectrum of \( E \) by

\[
\ell_\beta := \inf \left\{ \left| \sum_{i=0}^{n-1} d_i \beta^i \right| : d_i \in \Omega - \Omega = \{0, \pm 1, \pm \beta, \pm (\beta + 1)\}, n \in \mathbb{N} \right\}.
\]

In our second result we show that the spectrum \( \ell_\beta \) of \( E \) is constant for all \( \beta \geq 3 \).

**Proposition 1.2.** For any \( \beta \geq 3 \), we have \( \ell_\beta = 1 \).

Note that \( E \) is a self-similar set having overlaps. Then a point \( x \in E \) may have multiple codings. Motivated by the work of [3] we introduce the following subsets of \( E \). For \( k \in \mathbb{N} \cup \{\aleph_0, 2\aleph_0\} \) let

\[
E^{(k)} := \{x \in E : x \text{ has precisely } k \text{ different codings with alphabet } \Omega\}.
\]

Then \( E = E^{(2\aleph_0)} \cup E^{(\aleph_0)} \cup \bigcup_{k=1}^{\infty} E^{(k)} \). Our final result is on the Hausdorff dimension and Hausdorff measures of \( E \) and \( E^{(k)} \).

**Proposition 1.3.**

(i) The Hausdorff dimensions of \( E \) and \( E^{(2\aleph_0)} \) are given by

\[
\dim_H E = \dim_H E^{(2\aleph_0)} = \frac{\log(3 + \sqrt{5}) - \log 2}{\log \beta} =: s.
\]

Furthermore, \( \mathcal{H}^s(E) = \mathcal{H}^s(E^{(2\aleph_0)}) \in (0, \infty) \).

(ii) If \( k = \aleph_0 \) or \( k \in \mathbb{N} \) is not of the form \( 2^m \), then \( E^{(k)} = \emptyset \). Otherwise, the set \( E^{(2^m)} \) with \( m \in \{0, 1, 2, \ldots\} \) has the same Hausdorff dimension given by

\[
\dim_H E^{(2^m)} = \frac{\log r}{\log \beta} =: t,
\]

where \( r \approx 2.24698 \) is the root of \( x^3 - 2x^2 - x + 1 = 0 \). Furthermore, \( \mathcal{H}^t(E^{(1)}) \in (0, \infty) \), and \( \mathcal{H}^t(E^{(2^m)}) = \infty \) for any \( m \in \mathbb{N} \).
The rest of the paper is arranged as follows. In the next section we discuss all the generating IFSs of $E$ and establish Theorem 1.1. In the last Section we investigate the spectrum, the Hausdorff dimension and Hausdorff measure of $E$ and of the sets of points in $E$ having finite or infinite different $\beta$-expansions, and prove both Proposition 1.2 and Proposition 1.3.

2. Generating iterated function systems of $E$

In this section we will investigate all generating iterated function systems of $E$, and prove Theorem 1.1. First we prove the asymmetry of $E$.

**Lemma 2.1.** $-E + c \notin E$ for any $c \in \mathbb{R}$.

**Proof.** Suppose on the contrary that $-E + c \subseteq E$ for some $c \in \mathbb{R}$. Recall that $0$ is the minimum of $E$, and $\gamma_\beta$ is the maximum of $E$. Then

\begin{equation}
- E + c \subseteq [c - \gamma_\beta, c] \quad \text{with} \quad c - \gamma_\beta \text{ and } c \in E.
\end{equation}

If $c \neq \gamma_\beta$, then one of $c$ and $\gamma_\beta - c$ is not in $[0, \gamma_\beta]$, therefore is not in $E$, leading to a contradiction with (2.1). So we must have $c = \gamma_\beta$. Then

\begin{equation}
- E + \gamma_\beta \subseteq E.
\end{equation}

However, by using $\beta \geq 3$ it is easy to check that

\[ -f_{1(\beta+1)}(0) + \gamma_\beta = -\frac{2\beta + 1}{\beta^2} + \frac{\beta + 1}{\beta - 1} \in \left(0, \frac{\beta + 1}{\beta - 1}\right) = H, \]

where $H$ is the largest hole of $E$. This leads to a contradiction with (2.2). Therefore, $-E + c \notin E$ for any $c \in \mathbb{R}$. \quad \square

In the next lemma we show that for an affine map $g(x) = mx + b$, if $g(E) = \mu E + b \subseteq E$, then $g(E)$ can not intersect both $f_1(E)$ and $f_{\beta+1}(E)$.

**Lemma 2.2.** If $g(E) \subseteq E$, then either $g(E) \subseteq f_0(E) \cup f_1(E)$ or $g(E) \subseteq f_{\beta+1}(E)$.

**Proof.** It suffices to prove that $g(E)$ can not intersect both $f_0(E) \cup f_1(E)$ and $f_{\beta+1}(E)$. Conversely, assume that

\[ g(E) \cap (f_0(E) \cup f_1(E)) \neq \emptyset \quad \text{and} \quad g(E) \cap f_{\beta+1}(E) \neq \emptyset. \]

Then both $E$ and $\Delta = [0, \gamma_\beta]$ are the attractors of the IFS $\{f_0, f_1, f_{\beta+1}, g\}$. So they should be the same, leading to a contradiction. \quad \square

The following lemma plays an important role in the proof of Theorem 1.1. For a compact set $A \subseteq \mathbb{R}$ denote by $A_{\max}$ and $A_{\min}$ the largest and smallest elements of $A$, respectively. Then $E_{\min} = 0$ and $E_{\max} = \gamma_\beta$.

**Lemma 2.3.** Assume that $\mu E + b \subseteq E$.

(i) If $(\mu E + b)_{\max} < f_{01}(0)$, then $\mu E + b \subseteq f_0(E)$;
(ii) If $f_{01}(\gamma_\beta) < (\mu E + b)_{\min} < f_{\beta+1}(0)$, then $\mu E + b \subseteq f_1(E)$;
(iii) If $(\mu E + b)_{\min} > f_1(\gamma_\beta)$, then $\mu E + b \subseteq f_{\beta+1}(E)$. 

Proof. In view of Figure 1 (i) and (iii) are obvious. We only need to prove (ii).

Suppose $\mu E + b \subseteq E$ and $f_{01}(\gamma \beta) < (\mu E + b)_{\min} < f_{\beta + 1}(0)$. By Lemma 2.2 and $f_0(\beta + 1) = f_{11}$ it follows that

$$
\mu E + b \subseteq (f_0(E) \cup f_1(E)) \cap f_1(\Delta)
= (f_0(\beta + 1)(E) \cap f_1(E)) \cup f_1(E) = (f_{11}(E) \cap f_1(\Delta)) \cup f_1(E) \subseteq f_1(E).
$$

This completes our proof. \qed

2.1. Determination of $\mu$ for $\mu E + b \subseteq E$. Assume $\mu E + b \subseteq E$. Since $E$ is a compact set, it follows that $|\mu| \leq 1$. In view of Lemma 2.1 it is only necessary to consider $0 < |\mu| < 1$ and $b \in \mathbb{R}$. We will show that the only possibility is $\mu = \beta^{-n}$ for some $n \in \mathbb{N}$.

**Proposition 2.4.** If $g(E) = \mu E + b \subseteq E$ with $0 < |\mu| < 1$ and $b \in \mathbb{R}$, then $\mu = \beta^{-n}$ for some $n \in \mathbb{N}$.

The proof of Proposition 2.4 will be split into several lemmas.

**Lemma 2.5.** Let $0 < |\mu| < 1$. If $\mu E + b \subseteq E$, then $0 < |\mu| < 1/\beta$ or $\mu = 1/\beta$.

**Proof.** Suppose $\mu E + b \subseteq E$. Then by Lemma 2.2 we have

$$
\text{either } \mu E + b \subseteq \frac{E + \beta + 1}{\beta} \quad \text{or} \quad \mu E + b \subseteq \frac{E}{\beta} \cup \frac{E + 1}{\beta}.
$$

In the first case, by observing the lengths of the convex hulls on both sides of the inclusion we have $0 < |\mu| \leq 1/\beta$. In the following we will discuss the latter case

$$
(2.3) \quad \mu E + b \subseteq \frac{E}{\beta} \cup \frac{E + 1}{\beta} = f_0(E) \cup f_1(E).
$$

Note that each hole of $\mu E + b$ is mapped by a hole of $E$, and the largest hole $H$ of $E$ has length $A := f_{\beta + 1}(0) - f_1(\gamma \beta)$.

(1). We first consider the case that $\mu E + b$ contains a hole including the interval $H_1 = f_1(H)$. Say $(c, d)$ is the gap of $E$ such that

$$
(2.4) \quad \mu(c, d) + b \supseteq H_1.
$$

Then we claim that

$$
(2.5) \quad (c, d) = H.
$$

If $(c, d) \neq H$, then by noting that the second largest hole of $E$ has length $A/\beta$, it follows that

$$
|\mu|(d - c) \leq |\mu|A < A/\beta,
$$

leading to contradiction with (2.4). This proves (2.5).

- If $\mu > 0$, then by (2.3) and (2.5) it follows that $\mu f_{\beta + 1}(E) + b \subseteq f_{1(\beta + 1)}(E)$. This implies $\mu \in (0, 1/\beta]$.
- If $\mu < 0$, then again by (2.3) and (2.5) we have $\mu f_0(E) + b \subseteq f_{1(\beta + 1)}(E)$. This gives $\mu \in [-1/\beta, 0)$. Suppose $\mu = -1/\beta$. Then

$$
-f_0(E) + b = -\frac{1}{\beta} f_0(E) + b \subseteq f_{1(\beta + 1)}(E).
$$

This will lead to a contradiction with Lemma 2.1. So, $\mu \in (-1/\beta, 0)$. 
(II). Next we consider the case that $\mu E + b$ does not contain a hole including $H_1$. Then by (2.3) we either have $\mu E + b \subseteq f_{1(\beta + 1)}(E)$ which implies $0 < |\mu| \leq \beta^{-2} < 1/\beta$, or

\begin{equation}
\mu E + b \subseteq f_{00}(E) \cup f_{01}(E) \cup f_{10}(E) \cup f_{11}(E) = f_0(\Delta) \cap E
\end{equation}

which also gives $0 < |\mu| \leq 1/\beta$. In the following it suffices to prove $\mu \neq -1/\beta$.

Suppose on the contrary that $\mu = -1/\beta$. Then by (2.6) we have $-\beta^{-1} E + b \subseteq f_0(\Delta) \cap E$. By the same argument as in the proof of Lemma 2.1 it follows that $b = \gamma_\beta/\beta$. Therefore,

\begin{equation}
\begin{aligned}
-\frac{E}{\beta^2} + \gamma_\beta \beta &\subseteq -\frac{E}{\beta} + \gamma_\beta \beta \subseteq E.
\end{aligned}
\end{equation}

Note by $\beta \geq 3$ that

\begin{equation}
\left( -\frac{E}{\beta^2} + \frac{\gamma_\beta}{\beta} \right)_{\text{min}} = \frac{\gamma_\beta(\beta - 1)}{\beta^2} = \frac{\beta + 1}{\beta^2} > \frac{2}{\beta(\beta - 1)} = f_{01}(\gamma_\beta).
\end{equation}

Then by (2.6), (2.7) and Lemma 2.3 (ii) it follows that $-\beta^{-2} E + \beta^{-1} \gamma_\beta \subseteq f_1(E) = \beta^{-1}(E + 1)$, which is equivalent to

\begin{equation}
-\frac{E}{\beta} + \frac{2}{\beta - 1} \subseteq E.
\end{equation}

So,

\begin{equation}
-\frac{E}{\beta^2} + \frac{2}{\beta - 1} \subseteq -\frac{E}{\beta} + \frac{2}{\beta - 1} \subseteq E.
\end{equation}

Note that

\begin{equation}
\left( -\frac{E}{\beta^2} + \frac{2}{\beta - 1} \right)_{\text{min}} = -\frac{\gamma_\beta}{\beta^2} + \frac{2}{\beta - 1} \in (f_{01}(\gamma_\beta), f_{\beta+1}(0)).
\end{equation}

By Lemma 2.3 (ii) it follows that

\begin{equation}
-\frac{E}{\beta^2} + \frac{2}{\beta - 1} \subseteq f_1(E) = \frac{E + 1}{\beta}.
\end{equation}

This implies $-\beta^{-1} E + \gamma_\beta \subseteq E$. One can check that $(-\beta^{-1} E + \gamma_\beta)_{\text{min}} > f_1(\gamma_\beta)$. By Lemma 2.3 (iii) we can deduce

\begin{equation}
-\frac{E}{\beta} + \gamma_\beta \subseteq f_{\beta+1}(E) = \frac{E + \beta + 1}{\beta}.
\end{equation}

This contradicts to Lemma 2.1. So $\mu \neq -1/\beta$, and we complete the proof. \hfill \Box

In the following we will show that if $\mu E + b \subseteq E$ with $0 < |\mu| < 1/\beta$, then $\beta \mu E + c \subseteq E$ for some $c \in \mathbb{R}$ (see Lemma 2.9). To prove this we need the following two lemmas.

**Lemma 2.6.** Let $\mu E + b \subseteq E$ with $0 < \mu < 1/\beta$. If

\begin{equation}
b \geq \frac{1}{\beta} - \frac{2\mu}{\beta - 1} \quad \text{and} \quad b > \frac{1 - \mu - \mu \beta}{\beta - 1},
\end{equation}

then $\beta \mu E + c \subseteq E$ for some $c \in \mathbb{R}$.

**Proof.** Suppose $\mu E + b \subseteq E$. By Lemma 2.2 it follows that

either $\mu E + b \subseteq f_{\beta+1}(E) = \frac{E + \beta + 1}{\beta}$ or $\mu E + b \subseteq f_0(E) \cup f_1(E)$.

In the first case we have $\beta \mu E + c := \beta \mu E + b \beta - \beta - 1 \subseteq E$. So it suffices to consider the latter case.
Suppose $\mu E + b \subseteq f_0(E) \cup f_1(E)$. Note that $\mu \in (0, 1/\beta)$ and $A = f_{\beta+1}(0) - f_1(\gamma\beta)$. Then the largest hole of $\mu E + b$ has length $\mu A$, which is strictly less than $A/\beta$. This implies that either

$$\mu E + b \subseteq f_{1(\beta+1)}(E) \subseteq f_1(E) = \frac{E + 1}{\beta}$$

or

$$(2.9) \quad \mu E + b \subseteq f_{00}(E) \cup f_{01}(E) \cup f_{10}(E) \cup f_{11}(E) = f_0(\Delta) \cap E.$$ 

In the first case, we have $\beta \mu E + c \subseteq E$ by letting $c = b\beta - 1$. For the second case as in (2.9) we need some effort.

Since $\mu \in (0, 1/\beta)$ and $\beta \geq 3$, by the first inequality in (2.8) it follows that

$$\left( \mu \cdot \frac{E + \beta + 1}{\beta} + b \right)_{\text{min}} = \mu \left( 1 + \frac{1}{\beta} \right) + b > \frac{1}{\beta} \geq \frac{2}{\beta(\beta - 1)} = f_0(\gamma\beta).$$

Clearly, by (2.9) we have

$$\left( \mu \cdot \frac{E + \beta + 1}{\beta} + b \right)_{\text{min}} \leq (\mu E + b)_{\text{max}} < f_{\beta+1}(0).$$

So by Lemma 2.3 (ii) it follows that

$$\mu \cdot \frac{E + \beta + 1}{\beta} + b \subseteq f_1(E) = \frac{E + 1}{\beta},$$

or, equivalently

$$\mu E + T(b) := \mu E + b\beta + \mu\beta + \mu - 1 \subseteq E.$$ 

Furthermore, by the second inequality in (2.8) we have $T(b) > b$.

Repeating the above process we have either $\beta \mu E + c \subseteq E$ for some $c \in \mathbb{R}$ or

$$(2.10) \quad \mu E + T^n(b) \subseteq E \quad \text{for all} \quad n \in \mathbb{N}.$$ 

We will finish the proof by showing that the case in (2.10) is impossible. It is easy to check that

$$(2.11) \quad T^n(x) = \beta^n \left( x - \frac{1 - \mu - \mu\beta}{\beta - 1} \right) + \frac{1 - \mu - \mu\beta}{\beta - 1} \quad \forall n \in \mathbb{N}.$$ 

Therefore, by (2.8) and (2.11) it follows that $T^n(b) \nearrow +\infty$ as $n \to \infty$. This leads to a contradiction with (2.10). \hfill \Box

Remark 2.7. The proof of Lemma 2.6 implies the following fact: Let $\mu E + b \subseteq E$ with $0 < \mu < 1/\beta$. If

$$b \geq \frac{1}{\beta} - \frac{2\mu}{\beta - 1} \quad \text{and} \quad \mu E + b \subseteq f_0(\Delta) \cap E,$$

then $\mu E + b\beta + \mu\beta + \mu - 1 \subseteq E$. This fact will be used later.

Lemma 2.8. Let $\mu E + b \subseteq E$ with $0 < \mu < 1/\beta$. If

$$(2.12) \quad b \geq \frac{1}{\beta} - \frac{2\mu}{\beta - 1} \quad \text{and} \quad b = \frac{1 - \mu - \mu\beta}{\beta - 1},$$

then $\mu E + b^* \subseteq E$ for some $b^* \neq b$. Furthermore, we have $b^* > b$ unless $\mu \leq 1/(\beta^2 + 1)$. 

Proof. Suppose \( b = (1 - \mu - \mu \beta) / (\beta - 1) \). Then

\[
(\mu E + b)_{\text{max}} = \frac{1}{\beta - 1} < \frac{\beta + 1}{\beta} = f_{\beta + 1}(0).
\]

So \( \mu E + b \subseteq f_0(E) \cup f_1(E) \). By (2.12) it follows that

\[
0 \leq \mu \leq \frac{1}{\beta(\beta - 1)}.
\]

We consider the following two cases.

**Case 1.** If \( 1 / (\beta^2 + 1) < \mu \leq 1 / (\beta(\beta - 1)) \), then

\[
\left( \mu \cdot \frac{E}{\beta^2} + b \right)_{\text{max}} = \frac{\mu \gamma \beta}{\beta^2} + \frac{1}{\beta - 1} - \mu \gamma \beta = \frac{1}{\beta - 1} - \mu \frac{(\beta + 1)^2}{\beta^2}
\]

\[
< \frac{1}{\beta - 1} - \frac{1}{\beta^2 + 1} \cdot \frac{(\beta + 1)^2}{\beta^2}
\]

\[
\leq \frac{1}{\beta} = f_{10}(0),
\]

where in the last inequality we have used \( \beta \geq 3 \). So, by Lemma 2.3 (i) it follows that

\( \mu E + b \subseteq \beta^{-1} E \); which is equivalent to

\[
\mu \cdot \frac{E}{\beta} + \beta b \subseteq E.
\]

Since \( 1 / (\beta^2 + 1) < \mu \leq 1 / (\beta(\beta - 1)) \), it follows that

\[
\left( \mu \cdot \frac{E}{\beta} + \beta b \right)_{\text{min}} = \frac{\beta}{\beta - 1} - \frac{\beta(\beta + 1)}{\beta - 1} \mu \in \left( \frac{2}{\beta(\beta - 1)}, \frac{\beta + 1}{\beta} \right) = (f_{01}(\gamma \beta), f_{\beta + 1}(0)).
\]

By Lemma 2.3 (ii) it follows that

\[
\mu \cdot \frac{E}{\beta} + \beta b \subseteq \frac{E + 1}{\beta}.
\]

Thus \( \mu E + b^* \subseteq E \) with \( b^* := \beta^2 b - 1 \). Since \( \beta \geq 3 \) and \( \mu \leq 1 / (\beta(\beta - 1)) \), by an easy computation we obtain \( b^* > b \).

**Case 2.** If \( \mu \leq 1 / (\beta^2 + 1) \), then by using \( \beta \geq 3 \) and (2.12) we have

\[
\left( \mu \cdot \frac{E + 1}{\beta} + b \right)_{\text{min}} = \frac{\mu}{\beta} + \frac{1 - \mu - \mu \beta}{\beta - 1} \geq \frac{1}{\beta} \geq \frac{2}{\beta(\beta - 1)} = f_{01}(\gamma \beta).
\]

Therefore, by Lemma 2.3 (ii) we get

\[
\mu \cdot \frac{E + 1}{\beta} + b \subseteq \frac{E + 1}{\beta}.
\]

This is equivalent to \( \mu E + b^* \subseteq E \) with \( b^* = \mu - 1 + \beta b \). Obviously, by (2.12) we have \( b^* < b \). □

Now we are ready to prove the following lemma.

**Lemma 2.9.** Let \( \mu E + b \subseteq E \) with \( 0 < |\mu| < 1 / \beta \). Then there exists \( c \in \mathbb{R} \) such that

\[
\beta \mu E + c \subseteq E.
\]
Proof. Since the case for $\mu < 0$ can be proved similarly, we only consider the case for $\mu > 0$.

By repeating the same process as in the proof of Lemma 2.6, we have either $\beta \mu E + c \subseteq E$ for some $c \in \mathbb{R}$, or

\[(2.13) \quad \mu E + b \subseteq f_0(\Delta) \cap E.\]

In the following it suffices to consider the case in (2.13). Clearly, by (2.13) it follows that

\[0 \leq b = (\mu E + b)_{\min} \leq f_0(\gamma \beta) = \gamma \beta.\]

If $b \geq f_0(\gamma \beta) = 2/(\beta(\beta - 1))$, then by Lemma 2.3 (ii) it follows that $\mu E + b \subseteq f_1(E) = \beta^{-1}(E + 1)$, which implies $\mu \beta E + c \subseteq E$ with $c = b\beta - 1$. So we only need to consider $b \in [0, 2/(\beta(\beta - 1))]$. It is convenient to divide the proof into the following two cases.

**Case 1.** $0 \leq b < 1/\beta - 2\mu/(\beta - 1)$. Then

\[
\left(\frac{\mu E + 1}{\beta} + b\right)_{\text{max}} = \mu \frac{\gamma \beta + 1}{\beta} + b < \frac{2\mu}{\beta - 1} + \frac{1}{\beta} - \frac{2\mu}{\beta - 1} = \frac{1}{\beta} = f_0(0).
\]

Therefore, we have

\[
\mu \cdot \frac{E + 1}{\beta} + b \subseteq \frac{E}{\beta}
\]

by Lemma 2.3 (i), which is equivalent to $\mu E + b \beta + \mu \subseteq E$. Define $S(x) := \beta x + \mu$. Then

\[(2.14) \quad S^n(x) = \beta^n \left( x + \frac{\mu}{\beta - 1} \right) - \frac{\mu}{\beta - 1}.
\]

Note that $S(b) \neq b$. So there exists a unique positive integer $m_0$ such that

\[S^{m_0}(b) \geq \frac{1}{\beta} - \frac{2\mu}{\beta - 1} \quad \text{and} \quad S^{m_0-1}(b) < \frac{1}{\beta} - \frac{2\mu}{\beta - 1}.
\]

We continue the above process by replacing $\mu E + b \subseteq E$ with $\mu E + S(b) \subseteq E$ to draw

\[\mu E + S^{m_0}(b) \subseteq E,
\]

which is reduced to the below case.

**Case 2.** $1/\beta - 2\mu/(\beta - 1) \leq b < 2/(\beta(\beta - 1))$. There are three cases to consider.

(2A). $b > (1 - \mu - \mu \beta)/(\beta - 1)$. Then by Lemma 2.6 there exists $c \in \mathbb{R}$ such that $\beta \mu E + c \subseteq E$.

(2B). $b < (1 - \mu - \mu \beta)/(\beta - 1)$. Then by Remark 2.7 we have

\[\mu E + T(b) = \mu E + b\beta + \mu\beta + \mu - 1 \subseteq E.
\]

Since $T(b) < b$ and

\[T^n(b) = \beta^n \left( b - \frac{1 - \mu - \mu \beta}{\beta - 1} \right) + \frac{1 - \mu - \mu \beta}{\beta - 1},
\]

so there exists a unique positive integer $n_0$ such that

\[T^{n_0-1}(b) \geq \frac{1}{\beta} - \frac{2\mu}{\beta - 1} \quad \text{and} \quad T^{n_0}(b) < \frac{1}{\beta} - \frac{2\mu}{\beta - 1}.
\]

Then we have $\mu E + T^{n_0}(b) \subseteq E$ by using Remark 2.7 for $n_0 - 1$ times.

Suppose $\mu E + T^{n_0}(b) \subseteq E$. If $0 \leq b' := T^{n_0}(b) < 1/\beta - \mu \gamma \beta$, then $(\mu E + b')_{\max} < 1/\beta$. By Lemma 2.3 (i) we have $\mu E + b' \subseteq f_0(E)$, which implies $\beta \mu E + c \subseteq E$ with $c = \beta b'$.
If \( 1/\beta - \mu \gamma \beta \leq b' < 1/\beta - 2\mu/(\beta - 1) \), then by Case 1 it follows that \( \mu E + \beta b' + \mu \subseteq E \).

We claim
\[
 f_{01}(\gamma \beta) < \left( \frac{\mu E + \beta + 1}{\beta} + \beta b' + \mu \right)_{\min} < f_{\beta + 1}(0),
\]
or equivalently,
\[
(2.15) \quad \frac{2}{\beta(\beta - 1)} < 2\mu + \beta b' + \frac{\mu}{\beta} < \frac{\beta + 1}{\beta}.
\]

Since \( b' < 1/\beta - 2\mu/(\beta - 1) \), one can verify the right inequality of (2.15) directly. For the left inequality in (2.15) we note from the hypothesis in Case (2B) that
\[
\frac{1}{\beta} - \frac{2\mu}{\beta - 1} < b < \frac{1}{\beta} - \frac{\mu - \mu \beta}{\beta - 1}.
\]

This implies \( \mu < 1/(\beta(\beta - 1)) \). Using this and the fact that \( \beta \geq 3 \) we can prove the left inequality of (2.15). So by (2.15) and Lemma 2.3 (ii) it follows that
\[
\mu E + \beta + 1 \leq \beta b' + \mu \subseteq E,
\]
which implies
\[
\mu E + \beta^2 b' + 2\beta \mu + \mu - 1 \subseteq E.
\]

Using \( \mu < 1/(\beta(\beta - 1)) \) and the fact \( \beta \geq 3 \) one can verify that
\[
b'' := \beta^2 b' + 2\beta \mu + \mu - 1 > \frac{1 - \mu - \mu \beta}{\beta - 1} > \frac{1}{\beta} - \frac{2\mu}{\beta - 1}.
\]

Then by Lemma 2.6 we conclude that \( \beta \mu E + c \subseteq E \) for some \( c \in \mathbb{R} \).

(2C). \( b = (1 - \mu - \mu \beta)/(\beta - 1) \). Then by Lemma 2.8 we have \( \mu E + b^* \subseteq E \) with \( b^* \neq b \). Furthermore, by Lemma 2.8 it follows that if \( \mu > 1/(\beta^2 + 1) \), then we have \( b^* > b \). In this case we can conclude by Lemma 2.6 that \( \beta \mu E + c \subseteq E \) for some \( c \in \mathbb{R} \).

In the following we assume \( \mu \leq 1/(\beta^2 + 1) \). Then by the proof of Lemma 2.8 it follows that \( \mu E + b^* \subseteq E \) with
\[
(2.16) \quad b^* = \mu - 1 + \beta b = \mu - 1 + \beta \frac{1 - \mu - \mu \beta}{\beta - 1} < b.
\]

So, either we can get \( \beta \mu E + c \subseteq E \) for some \( c \in \mathbb{R} \), or we can reduce to Case 1 that
\[
b^* < \frac{1}{\beta} - \frac{2\mu}{\beta - 1} \quad \text{and} \quad S^n(b^*) = b \quad \text{for some } n \in \mathbb{N},
\]
where \( S(x) = \beta x + \mu \).

Suppose \( S^n(b^*) = b = (1 - \mu - \mu \beta)/(\beta - 1) \). Then by (2.14) it follows that
\[
b^* = \frac{1 - \mu \beta - \mu \beta^n}{\beta^n(\beta - 1)}.
\]

Combined with (2.16) we obtain
\[
\mu = \frac{\beta^n - 1}{\beta^{n+2} - \beta}.
\]

Using \( \mu \leq 1/(\beta^2 + 1) \) and \( \beta \geq 3 \) this implies \( n = 1 \). So
\[
\mu = \frac{1}{\beta(\beta + 1)} \quad \text{and} \quad b^* = \frac{1 - 2\mu \beta}{\beta(\beta - 1)} = \frac{1}{\beta(\beta + 1)}.
\]
We will finish the proof in Case (2C) by proving
\[
\mu E + b^* = \frac{E + 1}{\beta(\beta + 1)} \not\subseteq E.
\]

Suppose \((E + 1)/(\beta(\beta + 1)) \subseteq E\). Since
\[
\frac{E + 1}{\beta(\beta + 1)} = \frac{\gamma\beta + 1}{\beta(\beta + 1)} = \frac{2}{\beta^2 - 1} < \frac{2}{\beta(\beta - 1)} = f_{01}(\gamma\beta),
\]
by Lemma 2.3(i) it follows that
\[
\frac{E + 1}{\beta(\beta + 1)} \subseteq \frac{E}{\beta} \implies \frac{E + 1}{\beta + 1} \subseteq E.
\]
One can check that \(\mu E + \hat{b} := (E + 1)/(\beta + 1) \subseteq E\) satisfies the conditions in Lemma 2.6. Then by Lemma 2.6 we get
\[
\frac{\beta}{\beta + 1} E + c \subseteq E \quad \text{for some } c \in \mathbb{R}.
\]
This leads to a contradiction with Lemma 2.5 since \(\beta/(\beta + 1) > 1/\beta\).

Therefore, for \(\mu E + b \subseteq E\) with \(\mu \in (0, 1/\beta)\) we must have \(\beta \mu E + c \subseteq E\) for some \(c \in \mathbb{R}\). This completes the proof. \(\square\)

**Proof of Proposition 2.4.** By Lemma 2.5, we have either \(0 < |\mu| < \frac{1}{\beta}\) or \(\mu = \frac{1}{\beta}\). Suppose \(\mu \neq \beta^{-n}\) for any positive integer \(n\). Then there exists a positive integer \(k\) such that \(\beta^{-(k+1)} < \mu < \beta^{-k}\) or \(\beta^{-(k+1)} \leq -\mu < \beta^{-k}\). Using Lemma 2.9 for \(k\) times yields
\[
\beta^k \mu E + c_k \subseteq E \quad \text{for some } c_k \in \mathbb{R},
\]
where \(\beta^k \mu \in (1/\beta, 1)\) or \(\beta^k \mu \in (-1, -1/\beta)\). This leads to a contradiction with Lemma 2.5 \(\square\)

2.2. Determination of \(b\) for \(\mu E + b \subseteq E\). First consider the case for \(\mu = \beta^{-1}\).

**Lemma 2.10.** If \(g(E) = \beta^{-1} E + b \subseteq E\), then \(b \in \{f_0(0), f_1(0), f_{\beta+1}(0)\}\)

**Proof.** Suppose \(\beta^{-1} E + b \subseteq E\). Then by Lemma 2.2 we have
\[
either \frac{E}{\beta} + b \subseteq f_{\beta+1}(E) = \frac{E + \beta + 1}{\beta} \quad \text{or} \quad \frac{E}{\beta} + b \subseteq f_0(E) \cup f_1(E).
\]
In the first case, we clearly have \(b = f_{\beta+1}(0)\), and we pay attention to the second case.

Suppose \(\beta^{-1} E + b \subseteq f_0(E) \cup f_1(E)\). Then either \(\beta^{-1} E + b \subseteq f_0(\Delta) \cap E\) or \(\mu E + b\) contains the hole \(H_1 = (f_{11}(\gamma\beta), f_{1(\beta+1)}(0))\) of length \(A/\beta\), where \(A\) is the length of the largest hole \(H = (f_1(\gamma\beta), f_{\beta+1}(0))\) of \(E\). If \(\beta^{-1} E + b \subseteq f_0(\Delta) \cap E\), then \(b = 0 = f_1(0)\).

If \(\beta^{-1} E + b\) contains the hole \(H_1\), note that every hole of \(\beta^{-1} E + b\) is mapped by a hole of \(E\) with scaling \(1/\beta\), then this hole must be mapped by exactly the largest hole \(H\) of \(E\). Thus,
\[
\frac{1}{\beta} \cdot f_1(\gamma\beta) + b = f_{11}(\gamma\beta) \quad \text{and} \quad \frac{1}{\beta} \cdot f_{\beta+1}(0) + b = f_{1(\beta+1)}(0).
\]
This yields \(b = 1/\beta = f_1(0)\), completing the proof. \(\square\)
Proof of Theorem 1.1. Suppose $g(E) = \mu E + b \subseteq E$ with $0 < |\mu| < 1$ and $b \in \mathbb{R}$. By Proposition 2.4 it follows that $\mu = \beta^{-n}$ for some $n \in \mathbb{N}$. In the following we will prove by induction on $n$ that if $\mu = \beta^{-n}$, then $b = f_i(0)$ for some $i \in \Omega^n$.

When $n = 1$, this has been proved in Lemma 2.10. Assume $n \geq 1$ and suppose $b \in \{f_i(0) : i \in \Omega^n\}$ for $\mu = \beta^{-n}$. We will prove that $b \in \{f_i(0) : i \in \Omega^{n+1}\}$ for $\mu = \beta^{-(n+1)}$.

**Case A.** $b \in f_{\beta+1}(E)$. Then by Lemma 2.3 (iii) we have $\beta^{-(n+1)}E + b \subseteq f_{\beta+1}(E)$. Thus

$$\beta^{-n}E + \beta b - 1 \subseteq E.$$ 

By induction we have $\beta b - 1 = f_j(0)$ for some $j \in \Omega^n$. It follows that $b = f_{(\beta+1)j}(0)$ with $(\beta + 1)j \in \Omega^{n+1}$.

**Case B.** $b \in f_0(E) \cup f_1(E)$. If $b \geq \beta^{-1}$, then by Lemma 2.3 (ii) we have $\beta^{-(n+1)}E + b \subseteq f_1(E)$. In this case, $b = f_j(0)$ for some $j \in \Omega^n$. Now we suppose $b < \beta^{-1}$. If $b < \beta^{-1} - \beta^{-(n+1)}\gamma_{\beta}$, then $(\beta^{-(n+1)}E + b)_{\max} < 1/\beta$. By Lemma 2.3 (i) we have

$$\beta^{-(n+1)}E + b \subseteq \frac{1}{\beta},$$

and therefore $b = f_0(0) + f_1(0)$ for some $j \in \Omega^n$. In the following we assume $\beta^{-1} - \beta^{-(n+1)}\gamma_{\beta} \leq b < \beta^{-1}$. Then there exists a unique $m \geq n + 1 \geq 2$ such that

$$\frac{1}{\beta} - \frac{\gamma_{\beta}}{\beta^{m+1}} > b \geq \frac{1}{\beta} - \frac{\gamma_{\beta}}{\beta^{m}}. \tag{2.18}$$

We claim that

$$b \in \{f_{01(\beta+1)^{m-2}0}, f_{01(\beta+1)^{m-2}1}, f_{01(\beta+1)^{m-1}0}\}. \tag{2.19}$$

By (2.18) we have $(\beta^{-(n+1)}E + b)_{\max} < 1/\beta$. So by Lemma 2.3 (i) it follows that $\beta^{-(m+1)}E + b \subseteq \beta^{-1}E$, which is equivalent to

$$\beta^{-m}E + b \beta \subseteq E.$$ 

Then by (2.18) and using $\beta \geq 3$ it follows that

$$\beta \geq 1 - \frac{\gamma_{\beta}}{\beta^{m-1}} \geq 1 - \frac{\gamma_{\beta}}{\beta} \geq \frac{1}{\beta} \quad \text{and} \quad \beta^{-m}\gamma_{\beta} + b \beta < 1 < \frac{\beta + 1}{\beta}.$$ 

So, by Lemma 2.3 (ii) we have $\beta^{-m}E + b \beta \subseteq f_1(E)$, which implies

$$\beta^{-(m-1)}E + b \beta^2 - 1 \subseteq E.$$ 

If $m = 2$, then by Lemma 2.10 we prove (2.19). Otherwise, for $m \geq 3$ it follows from (2.18) that

$$\beta \beta^2 - 1 \geq \beta - 1 - \frac{\gamma_{\beta}}{\beta^{m-2}} \geq \gamma_{\beta} - \frac{\gamma_{\beta}}{\beta^{m-2}} \geq \frac{\beta + 1}{\beta}. \tag{2.20}$$

Therefore, by Lemma 2.3 (iii) we have $\beta^{-(m-1)}E + \beta \beta^2 - 1 \subseteq \beta^{-1}(E + \beta + 1)$ which implies

$$\beta^{-(m-2)}E + \beta(\beta \beta^2 - 1) - (\beta + 1) \subseteq E.$$ 

If $m = 3$, then by Lemma 2.10 we get (2.19). Otherwise, for $m \geq 4$ it follows by (2.20) that

$$\beta(\beta \beta^2 - 1) - (\beta + 1) \geq \beta \left(\gamma_{\beta} - \frac{\gamma_{\beta}}{\beta^{m-2}}\right) - (\beta + 1) = \gamma_{\beta} - \frac{\gamma_{\beta}}{\beta^{m-3}} \geq \frac{\beta + 1}{\beta}.$$
We continue the above process for \( m - 3 \) times to get
\[
\frac{E}{\beta} + \beta^{m-2}(b\beta^2 - 1) - (1 + \beta)(1 + \beta + \cdots + \beta^{m-3}) \subseteq E.
\]
Therefore by Lemma 2.10 we have
\[
\beta^{m-2}(b\beta^2 - 1) - (1 + \beta)(1 + \beta + \cdots + \beta^{m-3}) \in \{ f_0(0), f_1(0), f_{\beta+1}(0) \}.
\]
This proves (2.21), establishing the claim.

Case I. \( b = f_{01(\beta+1)^m-2}(0) = f_{01(\beta+1)^m-2}(0) \). If \( m = n + 1 \), then \( b = f_{01(\beta+1)^{n-1}}(0) \), and we are done. Now we suppose \( m \geq n + 2 \). Then we claim that \( \beta^{-n+1}E + b \subseteq E \) is impossible.

Note that \( y = \pi(0^{m-n-2}10^k(\beta + 1)^\infty) \in E \) for any \( k \geq 2 \), where \( \pi \) is the natural projection map from the symbolic space \( \Omega^N \) to \( E \). Then one can check for \( k \) sufficiently large that
\[
\beta^{-(n+1)}y + f_{01(\beta+1)^m-2}(0) \in H_{01(\beta+1)^m-2} = \left( f_{01(\beta+1)^m-2}(1), f_{01(\beta+1)^{m-1}}(0) \right),
\]
where the open interval \( H_{01(\beta+1)^m-2} \) is a hole of \( E \). So \( \beta^{-(n+1)}E + b \not\subseteq E \).

Case II. \( b = f_{01(\beta+1)^m-1}(0) \). Here we also show that \( \beta^{-(n+1)}y + b \not\subseteq E \) for all \( m \geq n + 1 \).

Note that \( z = \pi(0^{m-n-1}(\beta + 1)0^k(\beta + 1)^\infty) \in E \) for all \( k \in \mathbb{N} \). Observe by using \( \beta \geq 3 \) that
\[
f_{01(\beta+1)^m-1}(1) \leq f_1(0) + f_{\beta+1}(0).
\]
Then by (2.21) one can prove for \( k \) sufficiently large that
\[
\beta^{-(n+1)}z + f_{01(\beta+1)^m-1}(0) \in H_{01(\beta+1)^m-1} = \left( f_{01(\beta+1)^m-1}(1), f_{01(\beta+1)^{m-1}}(0) \right)
\]
with \( H_{01(\beta+1)^m-1} \) being a hole of \( E \). So \( \beta^{-(n+1)}y + b \not\subseteq E \).

Case III. \( b = f_{01(\beta+1)^m-1}(0) \). By the same argument as in (B1) it follows that \( \beta^{-(n+1)}E + b \not\subseteq E \) for all \( m \geq n + 1 \).

Hence, for \( \beta^{-(n+1)}E + b \subseteq E \) we have \( b \in \{ f_j(0) : j \in \Omega^{n+1} \} \). This completes the proof by induction.

\section{Spectrum and unique/multiple expansions}

In this section we first prove Proposition 1.2 which shows that the spectrum of \( E \) is constant if \( \beta \geq 3 \).

\textbf{Proof of Proposition 1.2} First, let \( n = 0 \) and \( d_0 = 1 \), then we have \( \ell_\beta \leq 1 \). Now we prove the other direction. It suffices to prove the following claim: \textit{for any} \( n \geq 0 \) \textit{and any} \( \sum_{i=0}^n d_i \beta^i \neq 0 \) \textit{with} \( d_n \neq 0 \) \textit{we have} \( \sum_{i=0}^n d_i |\beta^i| \geq 1 \). We will prove this by induction on \( n \).

Clearly, the claim holds true for \( n = 0 \). Suppose it holds for all \( n < k \) for some positive integer \( k \). Now we consider \( n = k \). Let \( \sum_{i=0}^k d_i \beta^i \neq 0 \) with \( d_i \in \{ 0, \pm 1, \pm \beta, \pm (\beta + 1) \} \) and \( d_k \neq 0 \). Then \( |d_k| \in \{ 1, \beta, \beta + 1 \} \). We consider the following three cases.

Case I. \( |d_k| = \beta \) or \( \beta + 1 \). Then by using \( |d_i| \leq \beta + 1 \) and \( \beta \geq 3 \) it follows that
\[
\left| \sum_{i=0}^k d_i \beta^i \right| \geq \beta \cdot \beta^k - (\beta + 1) \sum_{i=0}^{k-1} \beta^i = \beta^{k+1} \left( 1 - \frac{\beta^1}{\beta^2} \right) + \frac{\beta + 1}{\beta - 1} > 1
\]
as desired.
Case II. $|d_k| = 1$. Without loss of generality we assume $d_k = 1$. If $d_{k-1} = -\beta$ or $-(\beta+1)$, then we can rewrite

$$\sum_{i=0}^{k} d_i \beta^i = \sum_{i=0}^{k-1} d'_i \beta^i$$

with $d'_{k-1} = d_{k-1} + \beta$ and $d'_i = d_i$ for all $0 \leq i < k-1$. By the induction hypothesis it follows that $|\sum_{i=0}^{k} d_i \beta^i| \geq 1$. If $d_{k-1} \notin \{-\beta, -(\beta+1)\}$, then by using $|d_i| \leq \beta + 1$ and $\beta \geq 3$ it follows that

$$\left| \sum_{i=0}^{k} d_i \beta^i \right| \geq \beta^k - \beta^{k-1} - (\beta + 1) \sum_{i=0}^{k-2} \beta^i = \beta^{k-1} \frac{\beta^2 - 3\beta}{\beta - 1} + \frac{\beta + 1}{\beta - 1} > 1.$$

By induction this proves the claim, and then completes the proof. \qed

In the following we will investigate the Hausdorff dimensions and Hausdorff measures of $E$ and $E^{(k)}$. Note that $E^{(k)}$ is the set of points in $E$ having precisely $k$ different codings with respect to the alphabet $\Omega$. For this we need the following property of $E$.

**Lemma 3.1.** $f_0(E) \cap f_1(E) = f_{0(\beta+1)}(E) = f_{11}(E)$.

**Proof.** By $f_{0(\beta+1)} = f_{11}$ we have

$$f_0(E) \cap f_1(E) \subseteq f_0(E) \cap f_1(\Delta) = f_{0(\beta+1)}(E) \cap f_1(\Delta) = f_{11}(E) \cap f_1(\Delta) = f_{11}(E) = f_{0(\beta+1)}(E).$$

On the other hand, since $f_{0(\beta+1)}(E) \subseteq f_0(E)$ and $f_{11}(E) \subseteq f_1(E)$, using $f_{0(\beta+1)} = f_{11}$ again it follows that

$$f_0(E) \cap f_1(E) \supseteq f_{0(\beta+1)}(E) \cap f_{11}(E) = f_{11}(E) = f_{0(\beta+1)}(E).$$

This completes the proof. \qed

Now we are ready to prove Proposition 1.3.

**Proof of Proposition 1.3.** The proof is similar to that of [4, Theorem 2]. For completeness we sketch the main idea.

Note by Lemma 3.1 that for any point $x \in E$, if $x$ has a coding containing the block 11 or $0(\beta+1)$, then $x$ has at least two different codings by observing the substitution $11 \sim 0(\beta+1)$. On the other hand, if $x$ has two different codings, say $(c_i)$ and $(d_i)$ (without loss of generality we assume $c_1 < d_1$), then we must have $c_1 = 0$ and $d_1 = 1$. So, $x \in f_0(E) \cap f_1(E)$. By Lemma 3.1 it follows that $x \in f_{0(\beta+1)}(E) = f_{11}(E)$. This implies that $c_1 c_2 = 0(1 + \beta)$ and $d_1 d_2 = 11$. Therefore, $x \in E$ has multiple codings if and only if its codings contain the block 11 or $0(\beta+1)$.

Note that $f_{11} = f_{0(\beta+1)}$. One can verify that the set $E$ is a graph-directed set satisfying the open set condition. More precisely, let $X_A$ be the subshift of finite type with the forbidden block $0(\beta+1)$. Then

$$X_A = \left\{ (d_i) \in \Omega^\mathbb{N} : A_{d_i, d_{i+1}} = 1 \right\},$$
where $A$ is the transition matrix with states 0, 1 and $\beta + 1$ given by

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$ 

Then $E = \pi_\beta(X_A)$ is a graph-directed set. By a well-known result of [15] it follows that

$$\dim_H E = \log r_A \log \beta = \log(3 + \sqrt{5}) - \log 2 \log \beta,$$

where $r_A = (3 + \sqrt{5})/2$ is the spectral radius of $A$. Since the matrix $A$ is irreducible, or equivalently, the subshift of finite type $X_A$ is transitive with respect to the left shift, we conclude that $H^\dim_H E(E) \in (0, \infty)$. Then (i) follows from the second part that

$$\dim_H E^{(k)} < \dim_H E \quad \text{for any } k \neq 2^{\aleph_0} \quad \text{and} \quad E = E^{(2^{\aleph_0})} \cup E^{(\aleph_0)} \cup \bigcup_{k=1}^{\infty} E^{(k)},$$

which we will prove below.

Now we consider the subset $E^{(k)}$. Let $U := E^{(1)}$ be the set of $x \in E$ having a unique coding. Observe that a point in $E$ has multiple codings if and only if its codings contain the block 11 or 0$(\beta + 1)$. In other words, the set $U$ consists of all $x \in E$ with its unique coding containing neither 11 nor 0$(\beta + 1)$. Therefore, $U = \pi_\beta(X_B)$. So, the Hausdorff dimension of $U$ is given by (cf. [15])

$$\dim_H U = \frac{\log r_B}{\log \beta},$$

where $r_B \approx 2.24698$ is the spectral radius of $B$, which satisfies the equation $r^3 - 2r^2 - r + 1 = 0$. Furthermore, since the matrix $B$ is also irreducible, the corresponding Hausdorff measure $H^\dim_H U(U) \in (0, \infty)$.

Let $U$ be the set of all unique codings of points in $U$. Note that for any $k \in \mathbb{N}$, any $x \in E^{(k)}$ has precisely $k$ different codings, and all of these codings must be end in $U$. Thus,

$$E^{(k)} \subseteq \bigcup_{n=0}^{\infty} \bigcup_{1 \in \Omega^n} f_1(U),$$

which implies $\dim_H E^{(k)} \leq \dim_H U$.

If $k = 2^m$ for some $m \in \mathbb{N}$, then one can verify that

$$(3.1) \quad \Lambda_{n,m} := \pi_\beta(\{d_1 \ldots d_n(0(\beta + 1))^m c_1 c_2 \ldots : d_n = c_1 = \beta + 1, \ d_1 \ldots d_n \in B_n(U), (c_i) \in U\})$$
is a subset of \(E^{(2^m)}\) for any \(n \in \mathbb{N}\), where \(B_n(U)\) is the set of admissible blocks of length \(n\) in \(U\). This implies that
\[
\dim_H E^{(k)} \geq \dim_H \Lambda_{n,m} = \dim_H U.
\]
Furthermore, note that for different \(n, n'\) the sets \(\Lambda_{n,m}\) and \(\Lambda_{n',m}\) are disjoint. Write \(t := \dim_H U\) and denote by \(U(\beta + 1)\) the set of \(x \in U\) with its unique coding beginning with \(\beta + 1\). Since \(X_B\) is transitive, we have
\[
(3.2) \quad \mathcal{H}^t(U(\beta + 1)) > 0.
\]
So, by (3.1) it follows that
\[
\mathcal{H}^t(E^{(2^m)}) \geq \mathcal{H}^t(\bigcup_{n=1}^{\infty} \Lambda_{n,m}) = \sum_{n=1}^{\infty} \mathcal{H}^t(\Lambda_{n,m})
\]
\[
= \sum_{n=1}^{\infty} \sum_{d_1...d_n \in B_n(U), d_n = \beta+1} \mathcal{H}^t(f_{d_1...d_n(0(\beta+1))^m}(U(\beta + 1)))
\]
\[
= \beta^{-2mt} \mathcal{H}^t(U(\beta + 1)) \sum_{n=1}^{\infty} \left( \sum_{d_1...d_n \in B_n(U), d_n = \beta+1} \beta^{-nt} \right)
\]
\[
\geq \beta^{-2mt} \mathcal{H}^t(U(\beta + 1)) \sum_{n=1}^{\infty} C = \infty,
\]
where the last inequality follows by using the Perron-Frobenius Theorem (cf. [14]) that
\[
\sum_{d_1...d_n \in B_n(U), d_n = \beta+1} \beta^{-nt} \geq C > 0
\]
for any \(n \geq 1\).

Finally, we prove that \(E^{(k)} = \emptyset\) for any other \(k \neq 2^m\). Let \(x \in E^{(k)}\). Then \(x\) has multiple codings. So there exists a smallest integer \(n_1 \geq 0\) such that
\[
T^{n_1}(x) \in f_0(E) \cap f_1(E) = f_{0(\beta+1)}(E) = f_{11}(E),
\]
where \(T : E \to E\) is the inverse map of \(f_0, f_1\) and \(f_{\beta+1}\). This implies that all codings of \(x_1 := T^{n_1}(x)\) begin with either \(0(\beta + 1)\) or \(11\). Note that there exists a unique word \(d_1...d_{n_1} \in B_{n_1}(U)\) such that \(x = f_{d_1...d_{n_1}}(x_1)\). So, any coding of \(x\) either begins with \(d_1...d_{n_1}0(\beta + 1)\) or begins with \(d_1...d_{n_1}11\). Since \(f_{d_1...d_{n_1}0(\beta+1)} = f_{d_1...d_{n_1}11}\), there exists a unique \(y_1 \in E\) such that
\[
x = f_{d_1...d_{n_1}0(\beta+1)}(y_1) = f_{d_1...d_{n_1}11}(y_1).
\]
Now we proceed with the same argument on \(y_1\) instead of \(x\). Then there exist a smallest integer \(n_2 \geq 0\), a unique word \(d_{n_1+1}...d_{n_1+n_2}\) and a unique \(y_2 \in E\) such that
\[
y_1 = f_{d_{n_1+1}...d_{n_1+n_2}0(\beta+1)}(y_1) = f_{d_{n_1+1}...d_{n_1+n_2}11}(y_2).
\]
If there exists \(m \in \mathbb{N}\) such that the above procedure stops after \(m\) steps, then \(x\) has precisely \(2^m\) different codings. If the above procedure never stops, then \(x\) has a continuum of different codings. So, \(E^{(k)} = \emptyset\) for all \(k \neq 2^m\). This completes the proof.
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