Multifractal theory within quantum calculus

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Abstract

Within framework of the quantum calculus, we represent the partition function and the mass exponent of a multifractal, as well as the average of random variables distributed over self-similar set, on the basis of the deformed expansion in powers of the difference $q - 1$. For the partition function, such expansion is shown to be determined by binomial-type combinations of the Tsallis entropies related to manifold deformations, while the mass exponent expansion generalizes known relation $\tau_q = D_q(q - 1)$. We find the physical average related to the escort probability in terms of the deformed expansion as well. It is demonstrated the mass exponent can acquire a singularity that relates to a phase transition of the multifractal set in the course of its deformation.

Key words: Multifractal set; Deformation; Power series
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1 Introduction

Fractal conception [1] has become widespread idea of contemporary science (see Refs. [2] – [5], for review). Characteristic feature of fractal sets is known to be the self-similarity: if one takes a part of the whole set, it looks like the original set after a deformation $\lambda \neq 1$. Formal basis of the self-similarity is the power-law function $F \sim \lambda^h$ with a Hurst exponent $h$ (for time series, value $F$ is reduced to the fluctuation amplitude and $\lambda$ is the interval size within which this amplitude is determined). While the simple case of monofractal is characterized by a single exponent $h$, a multifractal system is described by a continuous spectrum of exponents (singularity spectrum) $h(q)$ [6]. Along
this way, a self-similarity degree \( q \) represents the exponent of a homogeneous function being characteristic function of self-similar systems \([7]\) (so, within nonextensive thermostatistics, this exponent expresses the escort probability \( P_i \propto p_i^q \) throughout the original one \( p_i \) \([8,9]\)). In physical applications, a central role is played by the partition function \( Z_q \propto l^{\tau(q)} \) \((l \) is a characteristic size of boxes covering multifractal) to be determined by the mass exponent \( \tau(q) \) connected with the generalized Hurst exponent \( h(q) \) by the relation \( \tau(q) = qh(q) - 1 \).

Since fractals are scale invariant sets, it is natural to apply the quantum calculus to description of multifractals because quantum analysis is based on the Jackson derivative which yields variation of a function with respect to the deformation of its argument \([10,11]\). This Letter is devoted to such a description with using deformed series of expansions of the partition function, the mass exponent and the averages of random variables over deformed powers of the difference \( q - 1 \). As shows the consideration in Section 2, the coefficients of the partition function expansion are reduced to binomial-type combinations of the Tsallis entropies related to manifold deformations \( \lambda^m \) with growing powers \( m = 0,1,\ldots \). Respectively, Section 3 shows that escort probability is generated by the mass exponent whose expansion into deformed power series represents the generalization of well-known relation \( \tau_q = D_q(q-1) \) with fractal dimension spectra \( D_q \) corresponding to manifold deformations. According to Section 4, above expansion permits also to expresses physical averages over the escort probability \( P_i \) in terms of related averages based on the ordinary one \( p_i \). Section 5 contains discussion of results obtained, and in Appendix we adduce necessary information from the quantum calculus.

2 Partition function

Following the standard scheme \([6,2]\), we consider a multifractal set covered by boxes \( i = 1,2,\ldots,W \) with number \( W \to \infty \). The peculiarity of self-similar sets is that the probability to occupy a box \( i \) is determined by its size \( l_i \to 0 \) according to the power-law relation \( p_i = l_i^\alpha \) characterized by a Hölder exponent \( \alpha > 0 \). All properties of a multifractal is known to be determined by the partition function

\[
Z_q = Z_q\{p_i\} := \sum_{i=1}^{W} p_i^q
\]  

which according to the normalization condition takes the value \( Z_q = 1 \) at \( q = 1 \). Because \( p_i \leq 1 \) for all boxes \( i \), the function (1) decreases monotonically from maximum magnitude \( Z_q = W \) related to \( q = 0 \). In the simplest case of the flat distribution \( p_i = 1/W \) fixed by the statistical weight \( W \gg 1 \), one has the exponential decay \( Z_q = W^{1-q} \).
Our approach is based on the expansion of the partition function (1) into the deformed series (23) in powers of the difference $q - 1$ [10]

$$Z_q^\lambda = -\sum_{n=0}^{\infty} \frac{S^{(n)}_\lambda}{[n]_\lambda!} (q - 1)_\lambda^{(n)}. \quad (2)$$

Here, the deformed powers $(q - 1)_\lambda^{(n)}$ are determined by the binomial (24), while the kernels $S^{(n)}_\lambda = S^{(n)}_\lambda \{ p_i \}$ are defined by the $n$-fold action of the Jackson derivative:

$$S^{(n)}_\lambda = - (qD_q^\lambda)^n Z_q \big|_{q=1}. \quad (3)$$

By using the method of induction with accounting the definitions (22) and (1), one obtains the explicit expression

$$S^{(n)}_\lambda = -(\lambda - 1)^{-n} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} Z_{\lambda^m} \quad (4)$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ are binomial coefficients. In absence of deformation ($\lambda = 1$), all coefficients (4) equal zero, apart from the single term $S^{(0)}_1 = -1$. Inserting these coefficients into the series (2) gives $Z_1 = 1$ as desired.

It is easily to see that the set of kernels (4) with $n > 0$ is expressed in terms of the Tsallis entropy [8]

$$S_\lambda \{ p_i \} := -\frac{\sum_i p_i^\lambda - 1}{\lambda - 1} = -\frac{Z_\lambda - 1}{\lambda - 1}. \quad (5)$$

With growth of the deformation parameter $\lambda$, this entropy decreases monotonically with inflection in the point $\lambda = 1$ where the expression (5) takes the Boltzmann-Gibbs form $S_1 = -\sum_i p_i \ln(p_i)$. In the case $n = 0$, the coefficient (4) does not depend on the deformation parameter $\lambda$ to give the value $S^{(0)}_\lambda = -1$. At $n = 1$, one obtains [12]

$$S^{(1)}_\lambda \{ p_i \} := -\frac{Z_{\lambda q} - Z_q}{\lambda - 1} \big|_{q=1} = -\frac{\sum_i p_i^\lambda - 1}{\lambda - 1} = S_\lambda \{ p_i \}. \quad (6)$$

Respectively, in the second order $n = 2$, Eq.(4) yields

$$S^{(2)}_\lambda \{ p_i \} = -\frac{1 - 2 \sum_i p_i^\lambda + \sum_i p_i^{2\lambda}}{(\lambda - 1)^2}. \quad (7)$$

With accounting the definition (5) one has

$$S^{(2)}_\lambda = -\frac{2}{\lambda - 1} S_\lambda + \frac{\lambda + 1}{\lambda - 1} S_\lambda^2. \quad (8)$$
In arbitrary order $n$, combination of equations (4) and (5) arrives at the sum

$$S^{(n)}_\lambda = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \frac{\lambda^m - 1}{(\lambda - 1)^n} S^m_\lambda, \quad (9)$$

which expresses in explicit form a generalized entropy of a multifractal by means of contributions of the Tsallis entropies related to different powers of the set deformation. In the limit $\lambda \to 0$, when $S^0_\lambda \sim W$, accounting of all terms in the sum over $m = 0, 1, \ldots, n$ arrives at cancellation of the denominator $(\lambda - 1)^n$ so that $S^{(n)}_0 \to W$; similarly, at $\lambda \to 1$, one obtains the Boltzmann-Gibbs limit $S^{(n)}_1 \to S^1 = -\sum_i p_i \ln(p_i)$ as desired. Finally, in the case $\lambda \to \infty$, where $S^m_\lambda \sim \lambda^{-m}$, the term related to $m = 0$ is reduced to zero so that main contribution gives the term with $m = 1$ to arrive at the sign-changing asymptotics $S^{(n)}_\lambda \sim (-1)^{n+1} \lambda^{-n}$.

For the flat distribution, when $Z^0_\lambda = W^{1-\lambda}$, the dependence (5) is characterized by the asymptotics $S^0_\lambda \sim W [1 - \ln(W)\lambda]$ in the limit $\lambda \ll 1$ and $S^1_\lambda \sim 1/\lambda$ at $\lambda \gg 1$. As a result, with the $\lambda$ growth the coefficients (4) increase from the value $S^{(n)}_0 = W$ to the Boltzmann magnitude $S^{(n)}_1 = W \ln(W)$ and then decay to the asymptotics $S^{(n)}_\lambda \sim (-1)^{n+1} \lambda^{-n}$.

### 3 Characteristic function of escort probability

Following pseudo-thermodynamic picture of multifractal sets [9], let us define effective values of the free energy $\tau_q$, the internal energy $\alpha$ and the entropy $f$:

$$\tau_q := \frac{\ln(Z_q)}{\ln(l)}, \quad \alpha := -\sum_i p_i \ln p_i, \quad f := -\sum_i p_i \ln(1/l). \quad (10)$$

Here, $l \ll 1$ is the size of characteristic box in phase space, $p_i$ and $P_i$ are original and escort probabilities to be connected with the definition

$$P_i := \frac{p^q_i}{\sum_i p^q_i} = \frac{p^q_i}{Z_q}. \quad (11)$$

Inserting this equation into the second expression (10), one obtains the well-known Legendre transformation [6,2]

$$\tau_q = q\alpha_q - f(\alpha_q) \quad (12)$$

where $q$ plays the role of the inverse temperature and the internal energy is specified with the state equation

$$\alpha_q = \frac{d\tau_q}{dq}. \quad (13)$$
It is easily to convince the escort probability (11) is generated by the characteristic function being the mass exponent
\[ \tau_q := \frac{\ln(Z_q)}{\ln(l)} = \frac{\ln(\sum_i p_i^q)}{\ln(l)}. \] (14)

Really, one has
\[ P_i = \frac{\ln(l)}{q} p_i \frac{\partial \tau_q}{\partial p_i} = q^{-1} \frac{\partial \ln(Z_q)}{\partial \ln(p_i)}. \] (15)

By analogy with Eq.(2), one can expand the function (14) into the deformed series
\[ \tau_q^\lambda = \sum_{n=1}^{\infty} D^{(n)}_\lambda \frac{(q-1)^{(n)}}{n!} \] (16)
being the generalization of the known relation \( \tau_q = D_q(q-1) \) connecting the mass exponent \( \tau_q \) with the multifractal dimension spectrum \( D_q \) [2]. Similarly to the equations (3) and (4), the kernel \( D^{(n)}_\lambda \) is determined by the \( n \)-fold action of the Jackson derivative:
\[ D^{(n)}_\lambda := (qD_q^\lambda)^n \bigg|_{q=1} = (\lambda - 1)^{-n} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \tau_{\lambda^m}. \] (17)

Noteworthy, the term with \( n = 0 \) is absent in the series (16). At \( n = 1 \), one reproduces the ordinary relation \( D^{(1)}_\lambda = \tau_\lambda/(\lambda - 1) \), while the kernels \( D^{(n)}_\lambda \) with \( n \geq 2 \) include terms being proportional to \( \tau_{\lambda^m}/(\lambda - 1)^n \) to correspond to manifold deformations \( \lambda^m, 1 < m \leq n \). Generally, the definition (17) yields a hierarchy of the multifractal dimension spectra related to different multiplication factors \( n \) of the set deformation.

In trivial case of the flat distribution\(^1\), when the partition function is \( Z_\lambda = W^{1-\lambda} \), the definition (14) yields the mass exponent \( \tau_\lambda = D(1 - \lambda) \) with the fractal dimension \( D = \frac{\ln(W)}{\ln(1/l)} \) which tends to \( D = 1 \) when the size of covering boxes \( l \) goes to the inverse statistical weight \( 1/W \). Thus, we can conclude the flat distribution relates to a monofractal with dimension \( D_\lambda = \frac{\ln(1-\lambda)}{\ln(1/l)} \) tending to a smooth one-dimensional set at deformation \( \lambda \to 1/l \).

\(^1\) It is worthwhile to stress the expression (15) is not applicable in this case because fixation of the statistical weight \( W \) makes impossible variation of the probability \( p_i = 1/W \).
4 Random variable distributed over multifractal

Let us consider an observable $\phi_i$ distributed over a multifractal set. According to Ref. [8], its mean value

$$\langle \phi \rangle := \sum_i \phi_i P_i \quad (18)$$

is determined by the escort probability (11). With accounting Eqs. (15) – (17), the average (18) can be expressed in terms of the sum

$$\langle \phi \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(q-1)^n}{(\lambda - 1)^n} \sum_{m=0}^n (-1)^{n-m} \binom{n}{m} \lambda^m \langle \phi \rangle^m \quad (19)$$

where the specific average

$$\langle \phi \rangle^m = \sum_i \phi_i P_i (\lambda^m) \quad (20)$$

is introduced. At $m = 0$, the escort distribution $P_i(\lambda^m)$ is reduced to the original one $p_i$ so that related terms in the sum (19) are proportional to the ordinary average $\langle \phi \rangle_1 = \sum_i \phi_i p_i$. On the other hand, in absence of deformation ($\lambda = 1$), the only term with $n = 0$ determines the mean value (18): $\langle \phi \rangle = \langle \phi \rangle_1$. In the limit $\lambda \to 0$, the escort probability reduced to flat distribution $P_i(0) = 1/W$ for arbitrary form of the ordinary distribution so that average (18) is determined by the simple expression $\langle \phi \rangle = W^{-1} \sum_i \phi_i$. Finally, in the case $\lambda \to \infty$, main contribution is given by the terms with $m = 0$ and $m = 1$ so that the average (19) takes the form

$$\langle \phi \rangle \simeq \langle \phi \rangle_1 + (q-1)\left( \langle \phi \rangle^1 - \lambda^{-1} \langle \phi \rangle \right) \quad (21)$$

5 Conclusions

A principle peculiarity of above consideration is that the expansions (2), (16) and (19) depend of both deformation parameter $\lambda$ and self-similarity degree $q$. As shows the example of using the flat distribution in the end of Section 3, the former determines the fractal dimension $D_\lambda = \frac{\ln(\lambda)}{\ln(1/\lambda)}$ in the simple case of the monofractal generated by deformation $\lambda \gg 1$ of initial cell of size $l \ll 1$. With passage to general case of multifractal, we obtain the fractal dimension spectrum defined by the series (17). Along this line, the self-similarity degree $q$ plays the role of a free parameter whose variation describes the multifractal spectrum, while value of the deformation parameter $\lambda$ fixes the scale of the multifractal resolution to be determined by external conditions in the case of natural objects.
As pointed out in Introduction, our study is based on the Jackson derivative (22) which gives variation of a function with respect to scale choice of its argument. This derivative expresses the coefficients of the deformed Taylor expansion (23) in powers of the deformed difference \( q - 1 \) defined by the binomial (24). In this way, the expansion (2) of the partition function (1) is determined by the coefficients (9) being binomial-type combinations of the Tsallis entropies (5) related to manifold deformations \( \lambda^m \) with growing powers \( m = 0, 1, \ldots \). According to definitions (10) and (11) the mass exponent (14) is expressed by the Legendre transformation (12) to generate the escort probability (15). On the other hand, the expansion of the mass exponent into deformed power series (16) represents generalization of known relation \( \tau_q = D_q(q - 1) \) with fractal dimension spectra (17) corresponding to manifold deformations. Moreover, making use of the deformed expansion (23) arrives at the expression (19) of the physical averages (18) corresponding to the escort probability \( P_i \) in terms of the averages (20) related to multiple deformations.

Under deformation \( \lambda \), above multifractal characteristics vary in the following manner. The generalized entropies (9) tend to the statistical weight \( W \) in the limit \( \lambda \to 0 \), take the Boltzmann-Gibbs magnitude \( S^{(n)}_1 = -\sum_i p_i \ln(p_i) \) at \( \lambda = 1 \) and arrive at the sign-changing asymptotics \( S^{(n)}_\lambda \sim (-1)^{n+1}\lambda^{-n} \) when \( \lambda \to \infty \). What about the relation (19) between the physical average \( \langle \phi \rangle = \sum_i \phi_i P_i \) and the ordinary one \( \langle \phi \rangle_1 = \sum_i \phi_i p_i \), one has the simple expression \( \langle \phi \rangle = W^{-1} \sum_i \phi_i \) in the limit \( \lambda \to 0 \), the trivial relation \( \langle \phi \rangle = \langle \phi \rangle_1 \) in absence of deformation \( (\lambda = 1) \), and the difference \( \langle \phi \rangle - \langle \phi \rangle_1 \) being proportional to the factor \( q - 1 \) at \( \lambda \to \infty \).

From physical point of view, the most tractable is the expansion of the mass exponent (16) whose coefficients represent fractal dimension spectra. As shows the equality (17), the fractal dimensions \( D^{(n)}_\lambda \) related to different powers \( n \) of the series (16) can change their signs with variation of the deformation \( \lambda \). Because, within the pseudo-thermodynamic formalism, the mass exponent \( \tau \) and the self-similarity degree \( q \) relate to the free energy and the inverse temperature respectively, such a change means that thermodynamic potential can obtain a singularity at some deformations \( \lambda \). Within framework of the physical presentation [9], this singularity means the phase transition in the course of the deformation of the multifractal set.

Appendix

Quantum analysis is known to be based on the Jackson derivative [10,11]

\[
D^\lambda_x := \frac{\lambda x \partial_x - 1}{(\lambda - 1)x}, \quad \partial_x \equiv \frac{\partial}{\partial x}
\]  

(22)
where $\lambda$ is a deformation parameter. The deformed Taylor expansion reads:

$$f(x) = f(a) + \frac{(x-a)^{(1)}\lambda}{[1]!} D_x^1 f(x) \bigg|_{x=a} + \frac{(x-a)^{(2)}\lambda}{[2]!} \left(D_x^2 f(x)\right) \bigg|_{x=a} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(x-a)^{(n)}\lambda}{[n]!} \left(D_x^n f(x)\right) \bigg|_{x=a}. \tag{23}$$

Here, the deformed binomial

$$(x+y)^{(n)}_\lambda = (x+y)(x+\lambda y)(x+\lambda^2 y)\ldots(x+\lambda^{n-1} y)$$

is determined by the coefficients

$$\left[\begin{array}{c} n \\ m \end{array}\right]_\lambda = \frac{[n]!}{[m]! [n-m]!} \tag{25}$$

where generalized factorials $[n]! = [1]! [2]! \ldots [n]!$ are given by the basic deformed numbers

$$[n]_\lambda = \frac{\lambda^n - 1}{\lambda - 1}. \tag{26}$$

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