A CLASSICAL FUNCTIONAL GENERALIZATION OF THE FIRST BARNES
LEMMA

RAFFAELE MARCOVECCHIO

Abstract. We give a brief account and a simpler proof of a contour integral formula for the
Gauss hypergeometric function. Such formula is alternative to Barnes’s integral formula and
generalizes the first Barnes Lemma.

1. Introduction

The Gauss hypergeometric function (denoted by $F(a, b; c)$ throughout the present paper) has
been deeply studied, and several integral representations can be found in books dealing with
special functions (see e.g. [7 Sections 8.3, 8.8]). An important integral was discovered by Barnes
(see formula (3) below), who build an alternative theory of the function $F(a, b; c)$ based on such
integral formula. One useful feature of formulas of the type (3) relies in the possibility of applying
the saddle point method to obtain a precise asymptotic estimate of the function involved (see
the monography [5]). Another interesting property of (3) is that it possesses a wide range of
extensions to generalized hypergeometric series (see [6 Sections 4.6, 4.7]).

The contour integral formula proved in the present paper is not new (see [8 Section 14.53]
and [4 formula (15.6.7)]). However, we believe it is worth the present short note, because our
proof appears to be simpler than that in [8], and is independent of Barnes’s integral formula (3).
We remark that formula (6) encompasses (and, in the present note, relies on) the first Barnes
lemma (see (4) below), whose proof in [2] is very similar to the proof of Barnes’s integral formula
(3). Therefore our contribution allows one to use the residue theorem in the proof of the first
Barnes Lemma only. After that, one can prove the contour integral formula (6) as in the present
paper, and finally combine the two results to prove (3), with an argument similar to [8 Section
14.53], without applying the residue theorem a second time, as in [8]. Also, our argument is
very simple but apparently has been generally overlooked in this context, and may have further
applications.

The first Barnes lemma is often considered as an integral analogue of the Gauss summation
formula

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$  (1)

In addition, formula (6) can be seen as an integral analogue of the formula connecting the values
of hypergeometric functions of $z$ and $1 - z$ (see (5) below), and this is precisely the context where
(6) is used in [8 Section 14.53]. Let us also point out two formulas close to (6): the first one,
obtained in 1939 by S.O. Rice for his function $H_n(\xi, p; v) = {}_3F_2(-n, n + 1, \xi; 1, p; v)$ (see [3, Vol I, p.193]), and the second one, usually used in the proof of the second Barnes lemma (see e.g. [2, p.43]. We mention these formulas at the end of the present paper.

2. The main result and a few similar formulas

We denote by $(\xi)_n$ the product $\xi(\xi + 1) \cdots (\xi + n - 1)$ for any complex number $\xi$ and for any $n = 1, 2, \ldots$, and we put $(\xi)_0 = 1$. We say that $\xi$ is admissible if $\xi$ is not a negative integer nor 0.

The Gauss hypergeometric function $F(a, b; c; z)$ is defined over the unit disc $|z| < 1$ in the complex plane by the series

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} z^k,$$

where $a$, $b$ and $c$ are complex numbers and $c$ is admissible. Note that the series $F(a, b; c; z)$ may terminate: this happens when $a$ or $b$ are not admissible. In this case the function (2) is a polynomial in $z$, and could be defined even if $c$ is not admissible, provided that $\min\{a, b\} \leq c$.

Let $\Gamma(z)$ be the Euler gamma function, defined in the complex half-plane $\Re z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt,$$

and extended to a meromorphic function in the complex plane, with simple poles at $z = -n$ with residue $\frac{(-1)^n}{n} (n = 0, 1, 2, \ldots)$, for example by splitting the integration path $(0, \infty)$ in the union of $(0, 1)$ and $(1, \infty)$. Two main properties of the function $\Gamma(z)$ are important in the following: the Stirling formula

$$\log \Gamma(z) = (s - \frac{1}{2}) \log s - s + \frac{1}{2} \log(2\pi) + o(1),$$

valid for $|\arg z| < \pi - \delta$ for any $\delta > 0$, and the functional equations

$$\Gamma(z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z}, \quad \Gamma(z+1) = z\Gamma(z).$$

The Barnes integral representation (see e.g. [1, Theorem 2.4.1]) of the function (2) is given by

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b; c; z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s ds,$$

valid under the conditions that $|z| < 1$, $z \neq 0$ and $|\arg(-z)| < \pi$, and that $a$, $b$ and $c$ are admissible. The path $L$ of integration is curved, if necessary, in such a way that separates the poles $s = -a-n$ and $s = -b-n$ ($n = 0, 1, 2, \ldots$) at the left of $L$ from the poles $s = 0, 1, 2, \ldots$ at the right of $L$. In the sequel, we denote by $F(a, b; c; z)$ the analytic function defined for $z \notin [1, \infty)$ either by the series (2), if $|z| < 1$, or by the integral (3), if $z \notin [0, \infty)$.  

The first Barnes lemma (see e.g. [1] Theorem 2.4.2]) states that

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(a + s)\Gamma(b + s)\Gamma(c - s)\Gamma(d - s) = \frac{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)}{\Gamma(a + b + c + d)},
\]

provided that \(a + c, a + d, b + c\) and \(b + d\) are admissible. Using (3) and (4) one can prove (see [8, Sect. 14.53]) that

\[
F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b; 1 + a + b - c; 1 - z) + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - z)^{e-a-b} F(c - a, c - b; 1 + c - a - b; 1 - z)
\]

Using (4) we can prove an integral formula that encompasses (5), which is a generalization of (1). For this reason we named formula (6) below a functional generalization of the first Barnes lemma.

**Theorem 2.1.** [8 Section 14.53] Let \(a, b, c\) and \(z\) be complex numbers such that \(z \notin (-\infty, 0]\), and that \(a, c - a, b, c - b\) and \(c\) are admissible. Then

\[
\frac{\Gamma(a)\Gamma(c - a)\Gamma(b)\Gamma(c - b)}{\Gamma(c)} F(a, b; c; 1 - z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(a + s)\Gamma(b + s)\Gamma(c - a - b - s)\Gamma(-s) z^s ds,
\]

where the integration path \(L\) separates the poles \(s = -a - n\) and \(s = -b - n\) \((n = 0, 1, 2, \ldots)\) on the left of \(L\) from the poles \(s = n\) and \(s = a + b - c + n\) \((n = 0, 1, 2, \ldots)\) on the right of \(L\).

**Proof.** Suppose that \(|1 - z| < 1\). For any \(n = 0, 1, 2, \ldots\) we have

\[
(-1)^n \frac{d^n}{dz^n} F(a, b; c; 1 - z) \bigg|_{z=1} = \frac{(a)_{n}(b)_{n}}{(c)_{n}} = \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c + n)}.
\]

The integral at the right-hand side of (6) is an analytic function in the domain \(|\arg z| < 2\pi\) (see [5, Lemma 2.4]), which plainly contains the disc \(|1 - z| < 1\). This implies that the derivative of the integral in (6) with respect to \(z\) equals

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(a + s)\Gamma(b + s)\Gamma(c - a - b - s)\Gamma(-s) sz^{s-1} ds,
\]

this being an integral of the same type as in (6), once it is noticed that \(-s\Gamma(-s) = \Gamma(1 - s)\), and after substituting the variable \(s\) with \(t\) by putting \(s = 1 + t\), and then renaming \(t\) with \(s\). We thus have

\[
(-1)^n \frac{d^n}{dz^n} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(a + s)\Gamma(b + s)\Gamma(c - a - b - s)\Gamma(-s) z^s ds \bigg|_{z=1}
\]
\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(a + s)\Gamma(b + s)\Gamma(c - a - b - s)\Gamma(n - s)ds \quad (n = 0, 1, 2, \ldots).
\]

By (4) the last integral equals
\[
\frac{\Gamma(c - a)\Gamma(c - b)\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)},
\]
therefore (6) is proved for \(|1 - z| < 1\), because all the derivatives of both sides of (6) coincide at \(z = 1\). By analytic continuation (6) holds for \(z \not\in (-\infty, 0]\).

From (6), using Stirling’s formula, the residue theorem, and changing \(z\) into \(1 - z\), after a few simplifications one easily gets (5), very much as in the standard proofs of (3) and (4). Of course, it is possible to go the other way, which is the usual proof of (6).

Let us finish this short paper with two formulas formally close to (6): the first one (see [2, p.43]) is
\[
\sum_{n=0}^{\infty} \frac{\alpha_1^1 \alpha_2^1 \alpha_3^1}{n!\beta_1^1 \beta_2^1} = \frac{\Gamma(\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1 - \alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1 - \alpha_2)}
\]
\[
\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\alpha_1 + s)\Gamma(\alpha_2 + s)\Gamma(\beta_1 - \alpha_1 - \alpha_2 - s)\Gamma(-s)F(\alpha_3, -s; \beta_2; 1)ds,
\]
and is used in the standard proof of the second Barnes lemma. As to the second one, let us consider (see [3, Vol I, p.193]) the sequence of polynomials
\[
H_n(\xi, p; v) = \sum_{j=0}^{n} \frac{(-n)_j(n+1)_j(\xi)_j}{j!^2(p)_j} v^j \quad (n = 0, 1, 2, \ldots).
\]
Here \(\xi, p\) and \(v\) are complex numbers and \(p + n + 1\) is admissible. Then
\[
\Gamma(p - q)\Gamma(q)\Gamma(p - \xi)\Gamma(\xi)H_n(\xi, p; v)
\]
\[
= \frac{\Gamma(p)}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s)\Gamma(q - s)\Gamma(\xi - s)\Gamma(p - q - \xi + s)H_n(s, q; v)ds,
\]
where \(0 < \text{Re} \sigma < \text{Re} q\) and \(0 < \text{Re}(\xi - \sigma) < \text{Re}(q - p)\). It is worth noticing that the generating function of the sequence \(H_n\) is
\[
\sum_{n=0}^{\infty} t^n H_n(\xi, p; v) = \frac{1}{1 - t} F(\xi, 1/2; p; -4vt(1 - t)^{-2}).
\]
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References

[1] G.E. Andrews, R.A. Askey and R. Roy, *Special Functions*, The Encyclopedia of Mathematics and its applications 71 (G.-C.- Rota eds.), Cambridge University Press, Cambridge, 1999.
[2] W.N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935.
[3] A.Erdélyi et al. (Bateman manuscript project), *Higher transcendental functions*, McGrow-Hill, New York, 1953.
[4] NIST *Handbook of Mathematical Functions*, F.W.J. Oliver, D.W. Lozier, R.F. Boisvert and G.W. Clark (eds.), U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC and Cambridge University Press, Cambridge, 2010.
[5] R.B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*, The Encyclopedia of Mathematics and its applications 71 (G.-C.- Rota eds.), Cambridge University Press, Cambridge, 2001.
[6] L.J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.
[7] C.Viola, An introduction to special functions. Unitext 102, La Matematica per il 3+2, Springer, 2016.
[8] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Reprint of the fourth (1927) edition, Cambridge University Press, Cambridge, 1996.

Dipartimento di Ingegneria e Geologia, Università di Chieti-Pescara, Viale Pindaro 42, 65127 Pescara, Italy
E-mail address: raffaele.marcovecchio@unich.it