EXTENDED COUNTERPOINT SYMMETRIES AND CONTINUOUS COUNTERPOINT

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Abstract. A counterpoint theory for the whole continuum of the octave is obtained from Mazzola’s model via extended counterpoint symmetries, and some of its properties are discussed.

1. Introduction

Mazzola’s model for first species counterpoint is interesting because it predicts the rules of Fux’s theory (in particular, the forbidden parallel fifths) reasonably well. It is also generalizable to microtonal equally tempered scales of even cardinality, and offers alternative understandings of consonance and dissonance distinct from the one explored extensively in Europe. In this paper we take some steps towards an effective extension of the whole model from a microtonal equally tempered scale into another, and not just of the mere consonances and dissonances, as it was done by the author in his doctoral dissertation [1].

First, we provide a definition of an extended counterpoint symmetry that preserves the characteristics of the counterpoint of one scale in the refined one. Then, we see that the progressive granulation of a specific example suggest an infinite counterpoint with a continuous polarity, different from the one that Mazzola himself proposed; a comparison of both alternatives calls for a deeper examination of the meaning of counterpoint extended to the full continuum of frequencies within the octave.

We must warn the reader that just a minimum exposition of Mazzola’s counterpoint model is done, and hence we refer to his treatise The Topos of Music [3] (whose notation we use here) and an upcoming comprehensive reference [5] for further details.

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2. Some Definitions and Notations

Let $R$ be a finite ring of cardinality $2k$. A subset $S$ of $R$ of such that $|S| = k$ is a dichotomy. It is often denoted by $(S/\overline{S})$ to make the complement explicit. The group
\[ \overline{GL}(R) = R \ltimes R^\times = \{ e^u v : u \in R, v \in R^\times \} \]
is called the affine group of $R$, its members are the affine symmetries. It acts on $R$ by
\[ e^u v(x) = vx + u; \]
this action is extended to subsets in a pointwise manner. A dichotomy $S$ is called self-complementary if there exists an affine symmetry $p$ (its quasipolarity) such that $p(S) = \overline{S}$. A self-complementary dichotomy is strong if its quasipolarity $p$ is unique, in which case $p$ is called its polarity.

Of particular interest are the strong dichotomies of $\mathbb{Z}_{2k}$, since this ring models very well the equitempered $2k$-tone scales modulo octave and Mazzola discovered that the set of classical consonances is a strong dichotomy. For counterpoint, the self-complementary dichotomies of the dual numbers
\[ \mathbb{Z}_{2k}[\epsilon] = \{ a + \epsilon b : a, b \in \mathbb{Z}_{2k}, \epsilon^2 = 0 \} \]
are even more interesting, since they are used in Mazzola’s counterpoint model as counterpoint intervals. More specifically, given a counterpoint interval $a + \epsilon b$, $a$ represents the cantus firmus, and $b$ the interval between $a$ and the discantus, and from every strong dichotomy $(K/D)$ with polarity $p = e^u.v$ in $\mathbb{Z}_{2k}$ we can obtain the induced interval dichotomy
\[ (K[\epsilon]/D[\epsilon]) = \{ x + \epsilon k : x \in \mathbb{Z}_{2k}, k \in K \} \]
in $\mathbb{Z}_{2k}[\epsilon]$. It is easily proved that, for every cantus firmus, there exists a quasipolarity $q_x[\epsilon]$ that leaves its tangent space $x + \epsilon K$ invariant.

A symmetry $g \in \overline{GL}(\mathbb{Z}_{2k}[\epsilon])$ is a counterpoint symmetry of the consonant interval $\xi = x + \epsilon k \in K[\epsilon]$ if
\begin{enumerate}
  \item the interval $\xi$ belongs to $g(D[\epsilon])$,  
  \item it commutes with the quasipolarity $q_x[\epsilon]$,  
  \item the set $g(K[\epsilon]) \cap K[\epsilon]$ is of maximal cardinality among those obtained with symmetries that satisfy the previous two conditions.
\end{enumerate}

Given a counterpoint symmetry $g$ for a consonant interval $\xi$, the members of the set $g(K[\epsilon]) \cap K[\epsilon]$ are its admitted successors; they represent the rules of counterpoint in Mazzola’s model. It must also be noted that it can be proved that the admitted successors only need
to be calculated for intervals of the form $0 + \epsilon.k$, and then suitably transposed for the remaining intervals.

3. Extending counterpoint symmetries

Let $(X_n/Y_n)$ be a strong dichotomy in $\mathbb{Z}_n$ where

$$g_1 = e^{\epsilon.t_1}(u_1 + \epsilon.u_1v_1) : \mathbb{Z}_n[\epsilon] \to \mathbb{Z}_n[\epsilon]$$

is a contrapuntal symmetry for the consonant interval $\epsilon.y \in X_n[\epsilon]$, with

$$p_n = e^{\epsilon.r_1}w_1$$

the polarity of $(X_n/Y_n)$. This means that if $s \in X_n$ and $p_n[\epsilon] = e^{\epsilon.r_1}w_1$ is the induced quasipolarity then

$$t_1 = y - u_1p_n(s) \quad \text{and} \quad p_n[\epsilon](\epsilon.t_1) = g_1(\epsilon.r_1),$$

as it is proved in [3, p. 652]. If $a : X_n \leftrightarrow X_{an} : x \mapsto ax$ is an embedding of dichotomies, then

$$p_{an} \circ a = a \circ p_n$$

(where $p_{an} = e^{\epsilon.t_2}w_2$ is the polarity of $(X_{an}/Y_{an})$) and, evidently,

$$p_{an}[\epsilon] \circ a = a \circ p_n[\epsilon].$$

In particular, $ar_1 = r_2$.

Suppose there is a symmetry

$$g_2 = e^{\epsilon.t_2}(u_2 + \epsilon.u_2v_2) : \mathbb{Z}_{an}[\epsilon] \to \mathbb{Z}_{an}[\epsilon]$$

such that $a \circ g_1 = g_2 \circ a$, then

$$t_2 = at_1 \quad \text{and} \quad au_2 = au_1.$$

From this we deduce

$$t_2 = at_1 = ay - au_1p_1(s)$$

$$= ay - u_2ap_n(s)$$

$$= ay - u_2p_{an}(as)$$

where $as \in X_{an}$, and

$$p_{an}[\epsilon](\epsilon.t_2) = p_{an}[\epsilon](\epsilon.at_1)$$

$$= ap_n[\epsilon](\epsilon.t_1) = ag_1(\epsilon.r_1)$$

$$= g_2(\epsilon.ar_1) = g_2(\epsilon.r_2).$$

This means that $g_2$ is almost a contrapuntal symmetry for $\epsilon.ay$, except for the maximization of the intersection $g_2X_{an}[\epsilon] \cap X_{an}[\epsilon]$. Now we can define a extended counterpoint symmetry with respect the embedding $a$ as a symmetry $g_2 \in \overline{GL}(\mathbb{Z}_{an}[\epsilon])$ that satisfy

1. $a \circ g_1 = g_2 \circ a$ with $g_1$ a (extended or not) contrapuntal symmetry for $\epsilon.y$, and
(2) $g_2X_{an}[\varepsilon] \cap X_{an}[\varepsilon]$ has the maximum cardinality among the symmetries with the above property.

Note that extended counterpoint symmetries preserve the admitted successors of $\varepsilon.y \in Z_n[\varepsilon]$, since otherwise the restriction $g_2|Z_n[\varepsilon]$ of a extended counterpoint symmetry would be a symmetry such that the intersection $g_2|Z_n[\varepsilon]X_n[\varepsilon] \cap X_n[\varepsilon]$ is bigger than the corresponding intersection for any counterpoint symmetry. This is a contradiction.

**Remark 3.1.** In particular, extended counterpoint symmetries always exist in the case of the embedding $2 : Z_n \to Z_{2n}$, because all the elements of $GL(Z_n)$ are coprime with 2. Thus, for any $\varepsilon.y \in \lim_{k \to \infty} X_{2k.n}[\varepsilon]$, there exist a extended contrapuntal symmetry in the limit $\lim_{k \to \infty} Z_{2k.n}[\varepsilon]$ which is the limit of extended counterpoint symmetries.

**Example 3.2.** Let $X_6 = \{0, 2, 3\} \subseteq Z_6$. The consonant interval $\varepsilon.2 \in Z_6[\varepsilon]$ has $e^{\varepsilon.3}(1+\varepsilon.3)$ as its only counterpoint symmetry and 15 admitted successors. The extended counterpoint symmetries of $\varepsilon.4 \in X_{12} = \{0, 1, 4, 5, 6, 9\} \subseteq Z_{12}$ with respect to the embedding 2 are $e^{\varepsilon.6}.(1 + \varepsilon.6)$ and $e^{\varepsilon.6}.(7 + \varepsilon.6)$. The number of extended admitted successors is 48.

4. A More Detailed Example

In Example 4.11 of [1], it is shown that there exists a strong dichotomy in $Z_{24}$ that can be extended progressively (via the embedding Lemma 4.5 of [1]) towards a dense dichotomy in $S^1$ with polarity $x \mapsto xe^{i\pi}$, which is the antipodal map. Analogously, the dichotomy

$$U_0 = \{0, 1, 3, \ldots, 7, 10\}$$

in $Z_{16}$ can be completed in each step using the dichotomy

$$V_i = \{0, \ldots, |U_i| - 1\},$$

so we have the inductive definition

$$U_{i+1} = 2U_i \cup (2V_i + 1), \quad i \geq 1,$$

which is a strong dichotomy of $Z_{2^{i+1}}$, in each case with polarity $e^{2\pi i}$. Note that the injective limit of the $U_i$ in $S^1$ is dense in one hemisphere.

The standard counterpoint symmetries for $U_0$ and successively extended counterpoint symmetries for $Z_{512}$ are listed in Table [1]. With “successively extended” we mean that they are those who commute with the extended counterpoint symmetries of $Z_{256}$, which in turn commute with those of $Z_{128}$, and so on down to $Z_{16}$. In most cases the linear part is $-1$, and in fact it is remarkable that all of them have no dual component.
### Table 1

| Interval | Symmetries for $\mathbb{Z}_{16}$ | $|gX[\varepsilon] \cap X[\varepsilon]|$ | Extended symmetries for $\mathbb{Z}_{512}$ | $|gX[\varepsilon] \cap X[\varepsilon]|$ |
|----------|----------------------------------|---------------------------------|----------------------------------------|-----------------------------------|
| 0        | $e^{3\pi i/3}$, $e^{3\pi i/6}$, $e^{3\pi i/11}$ | 96                              | $e^{3\pi i/2}$, $e^{3\pi i/11}$ | 82432                             |
| 1        | $e^{3\pi i/10}$                  | 112                             | $e^{3\pi i/12}$                  | 98816                             |
| 3        | $e^{3\pi i/2}$, $e^{3\pi i/9}$, $e^{3\pi i/11}$ | 96                              | $e^{3\pi i/2}$, $e^{3\pi i/11}$ | 82432                             |
| 4        | 7                                | 112                             | 7                                    | 75264                             |
| 5        | $e^{3\pi i/4}$, $e^{3\pi i/6}$, $e^{3\pi i/7}$ | 112                             | $e^{3\pi i/20}$                  | 76800                             |
| 6        | $e^{3\pi i/6}$                  | 112                             | $e^{3\pi i/16}$                  | 76800                             |
| 7        | $e^{3\pi i/5}$, $e^{3\pi i/11}$, $e^{3\pi i/15}$ | 96                              | $e^{3\pi i/20}$                  | 76800                             |

Table 1. A set of consonances in $\mathbb{Z}_{16}$, their respective counterpoint symmetries and number of admitted successors, and their extended counterpoint symmetries when embedded in $\mathbb{Z}_{512}$, with the corresponding number of extended admitted successors.

### 5. A Possible Continuous Counterpoint

The previous calculations suggest the following constructions that enable a continuous and compositionally useful counterpoint. First, we consider the space $S^1 \subseteq \mathbb{C}$ (which represents the continuum of intervals modulo octave), with the action of the group $G = \mathbb{R}/\mathbb{Z} \ltimes \mathbb{Z}_2$ given by

$$e^t \psi(x) = \begin{cases} x \exp(2\pi it), & v = 1, \\ \overline{x} \exp(2\pi it), & v = -1. \end{cases}$$

We define the set of consonances $(K/D)$ as the image of $[0, \frac{1}{2})$ under the map $\phi : [0, 1] \mapsto S^1 : t \mapsto e^{2\pi it}$, which musically means that we consider as consonant any interval greater or equal than the unison but smaller than the tritone (within an octave). Apart from the identity, no
element of $G$ leaves $(K/D)$ invariant, thus it is strong and its polarity is $e^{1/2}$.

Now, for counterpoint, we consider the torus $T = S^1 \times S^1$, with the first component for the cantus firmus and the second for the discantus interval. Let $G$ act on $T$ in the following manner:

$$e^t \nu(x, y) = (v^t x, e^t y);$$

this action is suggested by the fact that all the linear parts of the affine symmetries of counterpoint intervals have no dual component.

Thus the set of consonant intervals is $(K[e]/D[e]) = (S^1 \times K/S^1 \times D)$, the self-complementary function for any $\xi \in T$ which fixes its tangent space is $e^{1/2}1$, and it commutes with any element of $G'$. Also $\xi = (0, k) \in g(D[e])$ for a $g \in G'$ if and only if

$$g = e^t 1, \quad t \in (k, k + 1/2) \quad \text{or} \quad g = e^t (-1), \quad t \in [k - 1/2, k).$$

And here comes a delicate point. If we wish to preserve the idea of cardinality maximization, it would be reasonable to ask the set of infinite admitted successors to attain certain maximum. A possibility is to gauge these sets in terms of the standard measure in $T$ since, for instance, the affine morphisms

$$g = \begin{cases} 
    e^{k-1/2}(-1), & k \in \phi([0, 1/4]), \\
    e^{k-1/2}1, & k \in \phi([1/4, 1/2]), 
\end{cases}$$

maximize the measure of the intersection $(gX[e]) \cap X[e]$. The musical meaning of this alternative is that the admitted successors of consonant intervals below the minor third are all the consonant intervals above it, and vice versa. The minor third is special, because it has any consonant interval as an admitted successor.

But, in terms of the new perspective of homology introduced by Mazza-\[4,\] we observe first that $T$ is homeomorphic to $T$ itself with respect to the Kuratowski closure operator induced by the quasipolarity $e^{1/2}1$. This is so because, for in each section $x \times S^1$, the self-complementary function is the antipodal morphism, thus each $x \times S^1$ is homeomorphic to the projective line, which in turn is homeomorphic to $x \times S^1$ itself [2, p. 58]. Furthermore, any $g \in G'$ which leaves $\xi$ out of $g(X[e])$ is such that $(g(X[e])) \cap X[e]$ is homotopically equivalent to $S^1$, except when such intersection is empty. Therefore, $H_1((g(X[e])) \cap X[e]) = \mathbb{Z}$ is always the group of maximum rank when it satisfies the rest of the conditions of counterpoint symmetries. This

\[\text{By the way, this happens only with the self-complementary function itself.}\]
implies that, except for itself, any counterpoint interval can be an admitted counterpoint successor, which is clearly an undesirable outcome.

6. SOME FINAL REMARKS

In the version of infinite counterpoint that maximizes measure, we arrive to some peculiar features:

(1) Certainly there are no culs-de-sac.
(2) The only consonance that has all the other consonances as admitted successors is the minor third.
(3) All the intervals smaller than the minor third admit only larger intervals as successors, while all those greater admit only smaller ones.
(4) Although it is continuous regarding its induced quasipolarity and the cantus firmus can be chosen to be a continuous function of time, the discantus cannot be continuous in the standard topology.

All of these seem to be very close to the general principles of counterpoint. Unfortunately, this specific instance is not a natural extension of the discrete version; their relation is mainly axiomatic. On the other hand, the restriction of the linear part of the morphisms to \( \mathbb{Z}_2 \), although not entirely artificial, feels too limited with respect to the original finite model.

In fact, the selected dichotomy for the continuous example is a particularly nice one that permits a simple analysis, but by no means it is the only possible one. Considering that the general linear parts for counterpoint symmetries can be recovered as “windings” of \( S^1 \), carefully constructed infinite dichotomies could yield more complicated homology groups that make the algebro-topological approach far more interesting.

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