Spherical four-bar motion generation and axode generation in cam mechanism design

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Received 27 January 2016

Abstract
This work is an extension of the authors’ work where RRSS motion generation and RRSS axode generation were used to produce a spatial cam mechanism (D’Alessio et. al., 2013). Here, a cam system design method is presented where spherical four-bar motion generation and axode generation methods are applied. The primary advantage of the spherical four-bar linkage, compared to the RRSS linkage, is that the former can be scaled without adding error to the positions achieved by its coupler link. As an example, a concept prosthetic knee is developed to precisely achieve a group of prescribed femur positions.

Key words: Linkage, Spherical linkage, Motion generation, Axode, Cam mechanism

1. Introduction
1.1 The Spherical Four-Bar Linkage

The spherical four-bar linkage or 4R Spherical linkage (Figure 1a) is a spatial four-link mechanism where the crank link and follower links are bounded by revolute joints. While this characteristic is also true for the planar four-bar linkage, the joint axes in the planar linkage are all parallel while in the spherical four-bar linkage, the joint axes all intersect at a common point (the center of the sphere).

Because the dimensions of the spherical four-bar linkage are restricted to spherical space (as the name implies), the linkage can be scaled without changing the motion achieved by its links. Figure 1b includes two spherical four-bar linkages. The larger linkage was produced by scaling the smaller linkage about the center of the sphere (which is also the global coordinate frame origin). Both the original and scaled linkages in Figure 1b will achieve the same displaced values of coupler points \( p_1 \), \( q_1 \), and \( r_1 \) (and at the same driving link displacement angles). This characteristic holds true at any spherical four-bar linkage scale factor.

The scalability of the spherical four-bar linkage can be advantageous in engineering design, particularly when the designer has restrictions on the linkage workspace allowable. Linkage scaling was not an option in the author’s prior work on RRSS axode generation (D’Alessio et. al., 2013) since the displaced values of RRSS coupler points change as the linkage is scaled.
1.2 Published Works on Spherical Four-Bar Linkage Analysis and Synthesis

Noted contributions in the area of 4R spherical motion generation analysis and synthesis include McCarthy and Bodduluri (McCarthy and Bodduluri, 2000) and Gupta and Beloiu (Gupta and Beloiu, 1998) who considered the branching elimination in 4R spherical motion generation for three prescribed coupler positions. Hong and Erdman considered the synthesis of single phase and adjustable 4R spherical linkages for finite position synthesis (Hong and Erdman, 2005). Lee et. al. considered the synthesis of adjustable 4R spherical motion generators for expanded groups prescribed coupler positions (Lee et. al., 2009). Chu and Sun produced a numerical coupler curve atlas database of over six million 4R spherical linkage dimension types (Chu and Sun, 2010). Tse and Larochelle presented a method for approximating a finite set of n spatial locations with n orientations for 4R spherical linkages (Tse and Larochelle, 2000). Shen et. al. presented an expanded 4R spherical motion generation model that includes prescribed coupler loads (Shen et. al., 2008).

1.3 Scope of Current Work

D’Alessio et. al., presented a cam system design method where models for RRSS motion generation and RRSS axode generation are applied (D’Alessio et. al., 2013). This current work is an extension of the authors’ prior work. Here, a cam system design method is presented where spherical four-bar motion generation and axode generation methods are applied. The primary advantage of the spherical four-bar linkage in comparison to the RRSS linkage is that the former can be scaled without adding error to the positions achieved by its coupler link.

2. Spherical Four-Bar Motion Generation with Branch and Order Defect Rectification

Russell and Shen presented a nonlinear optimization model to synthesize branch and order defect-free spherical four-bar linkages to approximate N prescribed coupler positions (Russell and Shen, 2013). This optimization model includes the objective function

$$f(X) = \frac{1}{2} \sum_{j=2}^{N} \left( ||p_j - p||^2 + ||q_j - q||^2 + ||r_j - r||^2 \right)$$  \hspace{1cm} (1)

In Equation (1), \( X = (a_0, a_1, b_0, b_1, \theta_2, \ldots, \theta_N, \alpha_2, \ldots, \alpha_N) \) which are the design variables for the spherical four-bar linkage (Figure 1a). Variables \( p_j, q_j \) and \( r_j \) in Equation (1) represent the prescribed coupler positions and variables \( p, q \) and \( r \) represent the coupler positions achieved by the synthesized spherical four-bar motion generator using the spherical four-bar displacement model by Suh and Radcliffe (Suh and Radcliffe, 1978).

Along with the objective function, the optimization model includes two kinds of equality constraints. Equations (2) and (3) ensure the constant chord length of the crank and follower links respectively. Because the synthesized linkage is to lie on a sphere having a radius of 1, Equations (4) through (7) are included. These equations ensure that
the fixed and moving pivots of the synthesized spherical four-bar linkage lie on a sphere of unit radius. Variables \( a \) and \( b \) in Equations (2) and (3) are achieved by the synthesized spherical four-bar motion generator using the spherical four-bar displacement model by Suh and Radcliffe (Suh and Radcliffe, 1978).

\[
(a - a_0) \cdot (a - a_0) - (a_0 - a_0) = 0 \tag{2}
\]

\[
(b - b_0) \cdot (b - b_0) - (b_0 - b_0) = 0 \tag{3}
\]

\[
(a_0) \cdot (a_0) - 1 = 0 \tag{4}
\]

\[
(a_i) \cdot (a_i) - 1 = 0 \tag{5}
\]

\[
(b_0) \cdot (b_0) - 1 = 0 \tag{6}
\]

\[
(b_i) \cdot (b_i) - 1 = 0 \tag{7}
\]

The optimization model also includes two kinds of inequality constraints. Inequality (8) eliminates order defects (Balli and Chand, 2002, Mallik et. al., 1994) because it ensures constant counter-clockwise crank rotation (or clockwise rotation if \( \theta_j < \theta_{j-1} \)) is used. Inequality (9) eliminates branch defects (Balli and Chand, 2002, Mallik et. al., 1994) because it ensures a constant cross product of link \( b_0 - b \) and distance \( b_0 - a \). A change in branch would result in a change in sign in the cross-products \( (b - b_0) \times (a - b_0) \) and \( (b_i - b_0) \times (a_i - b_0) \).

\[
\begin{cases}
\theta_j > \theta_{j-1} \\
\theta_n < 2\pi
\end{cases}
\quad j = 2,3\ldots N \tag{8}
\]

\[
[(b - b_0) \times (a - b_0)] \cdot [(b_i - b_0) \times (a_i - b_0)] > 0 \tag{9}
\]

This combination of objective functions and constraints allow for the direct minimization of the error between the prescribed and achieved coupler positions while synthesizing the spherical four-bar crank and follower links. The order and branching constraints ensure the calculation of spherical four-bar motion generator solutions that are free of order and branch defects. For this work, the spherical four-bar motion generation model was implemented in the mathematical analysis software Matlab.

3. Axode Generation for the Spherical Four-Bar Linkage

An Instantaneous Screw Axis or ISA for the spherical four-bar linkage can be geometrically described as the line of intersection of the plane that includes \( a_0, a \) and the sphere center and the plane that includes \( b_0, b \) and the sphere center. Equation (10) is used to calculate a single point on the ISA (point \( X \) in Figure 2a). This equation calculates the point of intersection of the plane that includes \( a_0, a \) and the sphere center, and the line that passes through \( b_0 \) and \( b \). Equation (10) incorporates the spherical four-bar displacement model by Suh and Radcliffe to calculate the displaced moving pivots \( a \) and \( b \) (Suh and Radcliffe, 1978). Because the ISA will always include point \( X \) and the sphere center, the ISA unit vector (which we will call \( u \)) can be defined as \( u = \frac{X}{\|X\|} \). The ISA unit vectors for the initial spherical four-bar linkage form the fixed axode and the ISA unit vectors for the inverted spherical four-bar linkage (where the initial coupler link becomes the new ground and the initial ground becomes the new coupler link) form the moving axode (Figure 2b).

\[
X = b_0 + \frac{[(a - a_0) \times a_0] \cdot (b_0 - a_0)}{[(a - a_0) \times a_0] \cdot (b_0 - b)} (b - b_0) \tag{10}
\]
4. Cam Mechanism Design

Figure 3 illustrates a process diagram leading ultimately to cam mechanism design. As this figure illustrates, given a group of prescribed coupler positions, the spherical four-bar motion generation model is used to calculate the corresponding linkage dimensions. These dimensions are then incorporated in the spherical four-bar ISA model to generate the fixed and moving axodes for the linkage. The axodes will then be incorporated into the cam mechanism.

To produce the cam mechanism itself, the fixed and moving axodes are incorporated into component geometry and assembled. Because the rolling motion of the moving axode over the fixed axode will precisely replicate spherical four-bar coupler motion, the resulting cam mechanism will precisely replicate the coupler positions achieved by the synthesized spherical four-bar motion generator (from the 4R Spherical motion generation model in Figure 3).

5. Example
5.2 Spherical Four-Bar Motion Generation

Natural knee motion is itself cam-like (exhibiting rolling-sliding motion). Because cam systems are the standard for surgically-implanted artificial knees, they will also be applied for transfemoral prosthetic knee design here.

Figure 4 illustrates five femur bone positions with respect to the tibia bone (of the human leg). These positions represent discrete time points (0, 25, 50, 75 and 100%) during the stance phase of the gait cycle (Benoit et al., 2007). During the normal gait cycle, the femur and tibia move spatially with respect to each other (flexing and extending or moving back and forth). As a result, positions 2 and 3 nearly overlap (Figure 4 and Table 1). The remaining positions however are more distinct. Table 1 includes the spatial Cartesian coordinates (defined by points p, q and r) measured in each femur position. Using the femur positions as prescribed coupler motion input, the first step (as illustrated in Figure 3) is to synthesize a spherical four-bar linkage to approximate the femur positions.
Table 2 includes the initial values and calculated values for the spherical four-bar linkage dimensions. These dimensions were calculated using the spherical four-bar motion generation model in Section 2. The synthesized spherical four-bar linkage is included in Figures 6 and 7. Because this linkage, having a radius of unity (or 1mm), is difficult to see at the scale of the tibial and femur bones (Figure 5a), the solution is scaled by 45 (Figure 5a). This makes the linkage more visible within the bones (Figure 5b) and subsequently facilitates axode construction in CAD software. Tables 3 and 4 include the femur position coordinates achieved by the synthesized four-bar linkage as well as the difference magnitudes between the prescribed and achieved femur position coordinates respectively.

![Fig. 4 Prescribed femur positions](image)

![Fig. 5 (a) Original and scaled linkages and (b) spherical linkage location within femur and tibia bones](image)
Table 3 Femur position coordinates achieved by the synthesized spherical four-bar linkage

| Pos. # | p [m] | q [m] | r [m] |
|-------|-------|-------|-------|
| 1     | 37.50, 55, 185.81 | 82.50, 55, 185.81 | 82.50, 55, 132.19 |
| 2     | 37.50, 38.75, 189.87 | 82.46, 37.09, 190.21 | 83.03, 41.32, 136.76 |
| 3     | 37.48, 35.75, 190.46 | 82.44, 33.90, 190.82 | 83.07, 38.93, 137.43 |
| 4     | 37.78, 7.83, 193.57 | 82.70, 5.35, 193.62 | 83.45, 17.90, 141.49 |
| 5     | 42.73, –59.97, 183.12 | 87.70, –60.21, 181.77 | 86.53, –29.87, 137.57 |

Table 4 Difference magnitudes between the prescribed and achieved femur position coordinates

| Pos. # | p [m] | q [m] | r [m] |
|-------|-------|-------|-------|
| 1     | 0.00 | 0.00 | 0.00 |
| 2     | 0.05, 0.36, 1.25 | 0.03, 1.07, 0.97 | 0.19, 0.43, 1.02 |
| 3     | 0.52, 2.70, 0.36 | 0.29, 0.37, 0.14 | 0.36, 0.55, 0.13 |
| 4     | 1.62, 0.40, 3.67 | 1.64, 0.27, 2.64 | 0.30, 0.98, 2.82 |
| 5     | 5.77, 3.79, 13.45 | 5.69, 4.14, 11.81 | 4.35, 3.40, 12.38 |

5.2 Spherical Four-Bar Axode Generation

Using the dimensions of the synthesized spherical four-bar linkage as input, the second step (as illustrated in Figure 3) is to calculate its fixed and moving axodes over the crank motion range. The equations for the ISA point \( X \) and the ISA unit vector \( u \) were presented in Section 3. Table 5 includes the ISA points and unit vectors for the spherical four-bar linkage and its inversion over the crank rotation range (at \(-3.5^\circ\) increments). Figure 6 also illustrates the overlapping fixed and moving axode sections of the synthesized spherical four-bar linkage.

Fig. 6 Synthesized spherical four-bar linkage with fixed and moving axode sections

Table 5 Calculated ISA points \( X \) and unit vectors \( u \) for spherical four-bar linkage

| \( \theta \) | \( X \) [m] fixed axode | \( u \) fixed axode | \( X \) [m] moving axode | \( u \) moving axode |
|---------|-------------------|-------------------|----------------------|---------------------|
| 0°      | 0.82, –0.15, –0.58 | 0.81, –0.15, –0.57 | 0.82, –0.15, –0.58 | 0.81, –0.15, –0.57 |
| –3.5°   | 0.86, –0.10, –0.40 | 0.90, –0.11, –0.42 | 0.84, –0.13, –0.49 | 0.86, –0.13, –0.50 |
| –7°     | 0.89, –0.05, –0.25 | 0.96, –0.06, –0.26 | 0.85, –0.11, –0.41 | 0.89, –0.11, –0.43 |
| –10.5°  | 0.92, –0.01, –0.11 | 0.99, –0.01, –0.12 | 0.86, –0.08, –0.34 | 0.92, –0.09, –0.37 |
| –14°    | 0.95, 0.02, –0.00 | 0.99, 0.02, –0.00 | 0.87, –0.06, –0.28 | 0.95, –0.07, –0.31 |
| –17.5°  | 0.96, 0.05, 0.09  | 0.99, 0.05, 0.09  | 0.88, –0.04, –0.23 | 0.97, –0.04, –0.25 |
| –21°    | 0.98, 0.05, 0.15  | 0.99, 0.05, 0.15  | 0.88, –0.02, –0.18 | 0.98, –0.02, –0.20 |
| –24.5°  | 0.99, 0.04, 0.20  | 0.98, 0.04, 0.20  | 0.88, –0.01, –0.14 | 0.99, –0.01, –0.16 |
5.3 Cam Mechanism Design

Lastly, the fixed and moving axodes were incorporated as cam surfaces in a concept prosthetic knee design. Figure 7 illustrates exploded and assembly views of the prosthetic knee design along with the fixed and moving axode sections incorporated into the design. As illustrated in this figure, the fixed axode is included in the upper member of the prosthetic knee (the member that holds the amputee’s femur) and the moving axode is included in the lower member of the prosthetic knee (the member that represents the artificial tibia and foot).

Figure 8 illustrates the prosthetic knee in each of its five achieved femur positions. The pure rotation of the moving cam over the fixed cam precisely replicates the femur position coordinates achieved by the synthesized spherical four-bar linkage (Table 2).

In Figure 7, the moving cam component includes a pin the interfaces with a slot in the fixed cam component. The purpose of the pin and slot design is to make the cam mechanism a closed system and subsequently operable in any spatial orientation. By simulating the motion of the moving cam component over the fixed cam component, the precise path of the pin in the moving cam component can be traced. This path was then modeled into the fixed cam geometry as a pin slot.

No slip should occur between the cam surfaces of the concept prosthetic knee for proper operation. Incorporating a non-circular external gear section where the moving axode exists and a complementing non-circular internal gear section where the fixed axode exists (where the gear pitch surfaces are the axodes) is one possible design option to ensure that no cam surface slip occurs throughout the entire knee motion range.

Fig. 7 (a) Exploded and (b) assembly views of concept prosthetic knee design (patent pending)

Fig. 8 Concept prosthetic knee (patent pending) replicating femur positions
6. Discussion

As noted in Section 1.1, the primary advantage of the spherical four-bar linkage is that it can be scaled without compromising the positions achieved by its coupler link. This characteristic enables the user to freely determine the size of the linkage (rather than attempt to accommodate the size of the linkage as calculated).

A secondary advantage of the spherical four-bar linkage is that the specific location of its center can also be established prior to synthesis. Because the spherical four-bar linkage is centered at the origin of the coordinate frame used for its rigid-body points, the user can freely determine the spherical center of this linkage by measuring points \( p \), \( q \) and \( r \) with respect to the desired coordinate frame.

Therefore, the user determination of both scale and location are two advantages associated with the spherical four-bar linkage (not the RRSS linkage). Determining both the precise location and scale of the linkage (and subsequently its axodes) in an error-free manner enables designers to produce more practical prosthetic knee designs.

7. Conclusion

The design of a cam mechanism to replicate the coupler motion of a spherical four-bar linkage has been demonstrated in this work. Given a group of coupler positions (a series of femur positions in the example), the dimensions for a branch defect-free and order defect-free spherical four-bar linkage to approximate the coupler positions were calculated. Given the calculated dimensions of the spherical four-bar motion generator, fixed and moving axode sections for the given motion generator were calculated. Because the rolling motion of the moving axode over the fixed axode can replicate spherical four-bar coupler motion, these axode sections were incorporated into a cam mechanism (a cam-based prosthetic knee in the example) to precisely achieve the femur positions achieved by the synthesized spherical four-bar linkage. The primary advantages of a cam system designed using spherical four-bar linkage axodes (versus a cam system designed using RRSS linkage axodes), is that the spherical four-bar linkage can be scaled and/or relocated without adding error to the positions achieved by its coupler link (unlike the RRSS linkage).

References

Balli S. S. and Chand S., “Defects in link mechanisms and solution rectification,” *Mechanism and Machine Theory*, Vol. 37, No. 9, pp. 851–876 (2002).

Benoit D. L., Ramsey D. K., Lamontagne M., Xu L., Wretenberg P. and Renström P., “In vivo knee kinematics during gait reveals new rotation profiles and smaller translations,” *Clinical Orthopaedics and Related Research*, Vol. 454, No. 1, pp. 81-88 (2007).

Chu J. and Sun J., “Numerical atlas method for path generation of spherical four-bar mechanism,” *Mechanism and Machine Theory*, Vol. 45, No. 6, pp. 867–879 (2010).

D’Alessio J., Shen Q., Russell K. and Sodhi R. S., “A concept cam mechanism to replicate RRSS coupler motion,” *Journal of Advanced Mechanical Design, Systems, and Manufacturing*, Vol. 7, No. 2, pp. 187–194 (2013).

Gupta K. C. and Beloiu A. S., “Branch and circuit defect elimination in spherical four-bar linkages,” *Mechanism and Machine Theory*, Vol. 33, No. 5, pp. 491–504 (1998).

Hong B. and Erdman A. G., “A method for adjustable planar and spherical four-bar linkage synthesis,” *Journal of Mechanical Design*, Vol. 127, No. 3, pp. 456–463 (2005).

Lee W., Russell K., Shen Q., and Sodhi R. S., “On adjustable spherical four-bar motion generation for expanded prescribed positions,” *Mechanism and Machine Theory*, Vol. 44, No. 1, pp. 247–254 (2009).

Mallick A. K., Ghosh A., and Dittrich D., “Kinematic Analysis and Synthesis of Mechanisms,” CRC Press, Boca Raton (1994).

McCarthy M. J. and Bodduluri R. M., “Avoiding singular configurations in finite position synthesis of spherical 4R mechanisms,” *Mechanism and Machine Theory*, Vol. 35, No. 3, pp. 451–462 (2000).

Russell K. and Shen Q., “Expanded spatial four-link motion and path generation with order and branch defect elimination,” *Inverse Problems in Science and Engineering*, Vol. 21, No. 1, pp. 129–140 (2013).

Shen Q., Russell K., Sodhi R. S., and He Y., “Spherical four-bar motion generation with a prescribed rigid-body load,” *Transactions of the Canadian Society for Mechanical Engineering*, Vol. 32, No. 3–4, pp. 401–410 (2008).

Suh C. H. and Radcliffe C. W., “Kinematics and Mechanism Design,” John Wiley and Sons Inc., New York (1978).

Tse D. M. and Larochelle P. M., “Approximating spatial locations with spherical orientations for spherical mechanism design,” *Journal of Mechanical Design*, Vol. 122, No. 4, pp. 457–463 (2000).