EFFECTS OF FEAR AND ANTI-PREDATOR RESPONSE IN A DISCRETE SYSTEM WITH DELAY

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ABSTRACT. In this paper a discrete-time two prey one predator model is considered with delay and Holling Type-III functional response. The cost of fear of predation and the effect of anti-predator behavior of the prey is incorporated in the model, coupled with inter-specific competition among the prey species and intra-specific competition within the predator. The conditions for existence of the equilibrium points are obtained. We further derive the sufficient conditions for permanence and global stability of the co-existence equilibrium point. It is observed that the effect of fear induces stability in the system by eliminating the periodic solutions. On the other hand the effect of anti-predator behavior plays a major role in de-stabilizing the system by giving rise to predator-prey oscillations. Finally, several numerical simulations are performed which support our analytical findings.

1. Introduction. The presence of predators often induces non-lethal effects on the prey which cause changes in behavior, physiology or even foraging pattern of the prey species [12, 7, 20]. A field study in [19] reported that the fear of predation of large carnivores caused significant cascading effects across multiple trophic levels. The experiment was conducted in several coastal Gulf Islands where playbacks of large carnivore vocalizations were used for one month which reduced the intertidal foraging time of the mesocarnivore raccoon (Procyon lotor) by 66% further leading to a dramatic increase of the mesocarnivore’s prey namely intertidal crab (by 97%), intertidal fish (by 81%), polychaete worms (by 59%) and subtidal red rock crabs (by 61%). The effects cascaded down to the level of primary producers, as the increased abundance of red rock crabs contributed to the reduced density of its prey periwinkle snail which are significant grazers [8]. Such trophic cascades triggered due to the fear of predation has caught the attention of several ecologists and are widely studied in [15, 5, 9, 21].

Anti-predator behavior is the evolutionary adapted response of a prey when subjected to predation. This tactical strategy adopted by the prey generally involves a counterattacking technique wherein the adult prey attacks the juvenile predator.
during a predator-prey interaction to lower the future predation pressure. The role reversal is often observed in classical predator-prey systems [4], for example, the authors in [14] showed that the adult prey spider mite (Schizotetranychus celarius) attack and kill their juvenile predator, phytoseiid mite (Typhlodromus bambusae). Often the adult prey kills their juvenile predator without consuming them, suggesting that the anti-predator behavior is potentially used by the prey to reduce future predation risk or competition [1, 6, 10].

After the reintroduction of gray wolves (Canis lupus) in Yellowstone National Park in 1995–1996, the population density of the elk has steadily declined from 17,000 in 1995 to 5,800 in 2019 [16]. Several studies have reported trophic cascade effects in YNP after the reintroduction of the wolves [12, 13, 3]. As the wolves in YNP mainly prey on elks, the reduced density of the elks have led to more availability of forage for the bison, which in turn has a positive effect on the bison population. This has triggered a secondary trophic cascade whereby the wolf predation has an indirect effect on the increase in the bison population which has further significant effects on the ecosystem down the food chain [11]. A recent study [18] of wolf-bison interaction in Yellowstone National Park (YNP) found that the bison stood their ground and faced the wolves in 75% of the recorded encounters. During bison-wolf encounters the bison often exhibits anti-predator behavior leading to unsuccessful hunts which in turn has negative correlation with the wolf density.

In this paper we have studied the effects of anti-predator behavior together with the effect of fear of predation on a two-prey one-predator system with delay. The model is analogous to the ecological scenario in YNP.

The paper is organized as follows. In Section (2), a discrete-time two prey one predator is formulated which is followed by the investigation of the existence and local stability of the equilibrium points in Section (3). The conditions of permanence and global asymptotic stability of the interior equilibrium point are obtained in Section (4) and (5) respectively. In Section (6), numerical simulations are obtained to illustrate the rich dynamics of the system and verify the analytical findings. Finally, in Section (7) brief discussion on the biological interpretations and the significance of our findings are put forward.

2. Model formulation. The conditions of local stability and permanence of the elk-bison-wolf population in YNP were discussed in our previous study [2]. In [2] inter-specific competition among the prey and intra-specific competition among the wolf population was considered and the following discrete-time model was proposed

\[
\begin{align*}
    x_1(k+1) &= x_1(k) \exp \left[ r_1(1 - \alpha x_1(k)) - \frac{b_1 x_1(k) z(k)}{1 + x_1^2(k)} - g_1 x_2(k) \right] \\
    x_2(k+1) &= x_2(k) \exp \left[ r_2(1 - \beta x_2(k)) - \frac{b_2 x_2(k) z(k)}{1 + x_2^2(k)} - g_2 x_1(k) \right] \\
    z(k+1) &= z(k) \exp \left[ \frac{c_1 x_1^2(k)}{1 + x_1^2(k)} + \frac{c_2 x_2^2(k)}{1 + x_2^2(k)} - dz(k) - d_1 \right]
\end{align*}
\]

where \( x_1(0) > 0, x_2(0) > 0, z(0) > 0 \) and all the parameters are positive. Here \( x_1, x_2 \) and \( z \) represents the population densities of elk, bison and wolf respectively. The parameters \( r_i, b_i, c_i \) and \( g_i \) \((i = 1, 2)\) represents the intrinsic growth rates, predation coefficients, food conversion coefficients and inter-specific competition coefficients respectively. The parameters \( d \) and \( d_1 \) are respectively the intra-specific competition and the mortality rate coefficients of the predator.

In this study we aim to study the effect of the fear of predation in the elk population coupled with the effect of anti-predator behavior exhibited by the bison...
population. We assume that the birth rate of the elk population is negatively affected due to the fear induced by the predators. This phenomenon is incorporated in the model by multiplying the birth rate \( r_1 \) of the prey (elk) population with a non-increasing function of the predator (wolf) population size, given by \( \frac{m_2}{1 + h z} \), where \( h \) represents the level of induced fear. We consider the functional form \( \frac{m_2}{1 + h z} \) which represents the effect of anti-predator behavior exhibited by the bison population \( x_2 \). Here, \( m \) represents the anti-predator rate and \( \frac{m_2}{h} \) is the maximum anti-predator efficiency. This choice of the functional response restricts the infinite increase of anti-predator efficiency with increase in predator density. We further consider discrete time delays \( \tau_1, \tau_2 > 0 \) in the predator response terms which represents the gestation time of the predators. Thus we obtain the system

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1(k) \exp \left[ \frac{r_1}{1 + h z} - px_1(k) - q - \frac{b_1 x_2(k) z(k)}{1 + h z} - g_1 x_2(k) \right] \\
\frac{dx_2}{dt} &= x_2(k) \exp \left[ r_2 (1 - \beta x_2(k)) - \frac{b_2 x_2(k) z(k)}{1 + h z} - g_2 x_1(k) \right] \\
\frac{dz}{dt} &= z(k) \exp \left[ \frac{c_1 x_2^2(k - [\tau_1])}{1 + h z} + \frac{c_2 x_2^2(k - [\tau_2])}{1 + h z} - dz(k) - d_1 - \frac{m x_2(k) z(k)}{1 + a z(k)} \right]
\end{align*}
\]

(2)

3. Existence of equilibria.

1. The system (2) always has the trivial equilibrium point \( E_0(0,0,0) \) and the axial equilibrium point \( E_1 \left( 0, \frac{1}{\beta}, 0 \right) \).

2. If \( r_1 > q \) then the equilibrium \( E_2 \left( \frac{r_1 - q}{p}, 0, 0 \right) \) exists.

3. If \( \frac{p}{q} < r_1 - q < \frac{p q}{2} \) the predator free equilibrium \( E_3 \left( \frac{c_2 x_2^2}{b \beta} \right), \frac{c_2 x_2^2}{b \beta}, 0 \) exists.

4. If \( c_1 > d_1, r_1 > q \) and \( \bar{x}_1 > \frac{d_1}{c_1 - d_1} \), the one prey free equilibrium \( E_4 \left( x_1, 0, \frac{c_2 x_2^2 - d_4 (x_2^2 + 1)}{a_7 x^2 + 1} \right) \) exists, where \( \bar{x}_1 \) is the positive solution of the equation \( a_7 x^7 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \) where the \( a_i \)'s are given by \( a_7 = dp \left[ d + h(1 - d_1) \right] \), \( a_6 = \left( dq \left(c_1 h + d\right) - d_2 r_1 - d_1 dhq \right) \), \( a_5 = b_1 c_1 \left(c_1 h + d\right) - b_1 d_1 h - b_1 d_1 \left(c_1 h + d\right) + b_1 d_1^2 h + 2 dp \left( c_1 h + d \right) + d^2 p - 3 d_1 dhp \), \( a_4 = 2 dq \left(c_1 h + d\right) + d^2 q - 3 d^2 r_1 - 3 d_1 dhq \), \( a_3 = -b_1 c_1 d_1 h - b_1 d_1 \left(c_1 h + d\right) + b_1 c_1 d_2 + 2 b_1 d_1^2 h - b_1 d_1 d + dp \left( c_1 h + d \right) + 2 d^2 p - 3 d_1 dhp \), \( a_2 = dq \left(c_1 h + d\right) + 2 d^2 q - 3 d^2 r_1 - 3 d_1 dhq \), \( a_1 = b_1 d_1^2 h - b_1 d_1 d + d^2 p - d_1 dhq \), \( a_0 = - d^2 \left(r_1 - q\right) + d_1 dhq \).

If \( c_1 > d_1 \) and \( r_1 > q \), we get \( a_0 a_7 < 0 \), thus the existence of at least one positive solution \( \bar{x}_1 \) is guaranteed.

**Theorem 1.** One prey free equilibrium \( E_5 \left( 0, x'_2, z' \right) \) of system (2) exists if \( c_2 > d_1, \)

\( \frac{1}{\beta} > \sqrt{\frac{d_1}{c_2 - d_1}} \), \( Q > 0 \) and \( S < 0 \) where,

\[
Q = (az + 1) \left( 2 c_2 x_2 (az + 1) - m z (x_2^2 + 1)^2 \right),
\]

\[
S = b_2 z \left( x_2^2 - 1 \right) - \beta r_2 \left( x_2^2 + 1 \right)^2.
\]
Proof. \( x' \) and \( z' \) are the positive solutions of \( f(x, y) = 0 \) and \( g(x, y) = 0 \) where

\[
\begin{align*}
\tag{3a}\quad f(x, y) & \equiv r_2(1 - \beta x^2) - \frac{b_2 x z}{1 + x^2} = 0, \\
\tag{3b}\quad g(x, y) & \equiv \frac{c_2 x^2}{1 + x^2} - dz - \frac{d_1}{1 + \alpha z} = 0.
\end{align*}
\]

From (3a) when \( z \to 0 \), let \( x^2 \to x^2 f \) where, \( x^2 f = \frac{1}{\beta} > 0 \). Again from (3b) when \( z \to 0 \), let \( x^2 \to x^2 g \) where, \( x^2 g = \frac{q d_1}{c^2} - \frac{d_1}{\alpha} > 0 \). Considering, \( \frac{1}{\beta} > \frac{q d_1}{c^2} - \frac{d_1}{\alpha} \), we get \( x^2 g < x^2 f \).

From (3a) we have \( \frac{dz}{dx} = -\left( \frac{\partial f}{\partial z} \right) / \left( \frac{\partial f}{\partial x} \right) = R/S \), where \( R = b_2 x z (x^2 + 1)^2 \) and \( S = b_2 z (x^2 + 1) - \beta r_2 (x^2 + 1)^2 \). Again it can be noted that \( R > 0 \) and if \( S < 0 \) then \( \frac{dz}{dx} < 0 \).

Hence if the sufficient conditions hold, \( x_{2g} < x_{2f} \) and \( \frac{dz}{dx} > 0 \) for \( g(x, y) \) while \( \frac{dz}{dx} < 0 \) for \( f(x, y) \). Thus the isoclines (3a) and (3b) intersect at the point \((x'_{2}, z')\) in the \( x_{2} - z \) plane (fig. 1) and the existence of the equilibrium point \( E_{5}(0, x'_{2}, z') \) is established.

**Fig. 1.** Existence of the equilibrium point \( E_{5}(0, x_{2}', z') \) of system (2) with the parameter values \( r_2 = 2, b_2 = 1, \beta = 1.5, g_2 = 0.03, d = 1, d_1 = 0.1, \alpha = 5, m = 0.025, c_2 = 1 \). For the chosen parameters the conditions of Theorem 1 are verified with \( Q = 8.73724 > 0 \), \( S = -2.89871 < 0 \).
4. **Permanence.** If there exists positive constants \( M_1^{x_1}, M_1^{x_2}, M_1^z, N_1^{x_1}, N_1^{x_2} \) and \( N_1^z \) such that

\[
\begin{align*}
N_1^{x_1} & \leq \liminf x_1(k) \leq \limsup x_1(k) \leq M_1^{x_1}, \\
N_1^{x_2} & \leq \liminf x_2(k) \leq \limsup x_2(k) \leq M_1^{x_2}, \\
N_1^z & \leq \liminf z(k) \leq \limsup z(k) \leq M_1^z,
\end{align*}
\]

holds, then the system (2) is said to be permanent.

**Proposition 1.1** If we assume that

\[
c_1 + c_2 - d_1 > 0. \tag{H1}
\]

Then for every solution \( x_1(k), x_2(k) \) and \( z(k) \) of system (2) we have:

\[
\begin{align*}
\limsup x_1(k) & \leq M_1^{x_1} = \frac{\exp(r_1 - 1)}{p}, \tag{4a} \\
\limsup x_2(k) & \leq M_1^{x_2} = \frac{\exp(r_2 - 1)}{r_2 \beta}, \tag{4b} \\
\limsup z(k) & \leq M_1^z = \frac{\exp(c_1 + c_2 - d_1 - 1)}{d}. \tag{4c}
\end{align*}
\]

**Proposition 1.2** If we assume that

\[
\begin{align*}
\frac{r_1}{1 + hM_1^z} - q - b_1 M_1^z M_1^{x_1} - g_1 M_1^{x_2} > 0, \tag{H2} \\
r_2 - b_2 M_1^z M_1^{x_2} - g_2 M_1^{x_1} > 0, \tag{H3} \\
\frac{c_1(N_1^{x_1})^2}{1 + (N_1^{x_1})^2} + \frac{c_2(N_1^{x_2})^2}{1 + (N_1^{x_2})^2} > m M_1^{x_2} M_1^z + d_1. \tag{H4}
\end{align*}
\]

where \( M_1^{x_1}, M_1^{x_2} \) and \( M_1^z \) are defined in (4a), (4b) and (4c) respectively. Then for every solution \( x_1(k), x_2(k) \) and \( z(k) \) of system (2) we have:

\[
\begin{align*}
\liminf x_1(k) & \leq N_1^{x_1} = \frac{1}{p} \left( \frac{r_1}{1 + hM_1^z} - q - b_1 M_1^z M_1^{x_1} - g_1 M_1^{x_2} \right) \exp \left[ \frac{r_1}{1 + hM_1^z} - p M_1^{x_1} - q - b_1 M_1^z M_1^{x_1} - g_1 M_1^{x_2} \right], \tag{5a} \\
\liminf x_2(k) & \leq N_1^{x_2} = \frac{1}{r_2 \beta} \left( r_2 - b_2 M_1^z M_1^{x_2} - g_2 M_1^{x_1} \right) \exp \left[ r_2 (1 - \beta M_1^{x_2}) - b_2 M_1^z M_1^{x_2} - g_2 M_1^{x_1} \right], \tag{5b} \\
\liminf z(k) & \leq N_1^z = \frac{1}{d} \left( \frac{c_1(N_1^{x_1})^2}{1 + (N_1^{x_1})^2} + \frac{c_2(N_1^{x_2})^2}{1 + (N_1^{x_2})^2} - d_1 - m M_1^{x_2} M_1^z \right) \exp \left[ \frac{c_1(N_1^{x_1})^2}{1 + (N_1^{x_1})^2} + \frac{c_2(N_1^{x_2})^2}{1 + (N_1^{x_2})^2} - d_1 - d M_1^z - m M_1^{x_2} M_1^z \right]. \tag{5c}
\end{align*}
\]

**Theorem 2.** Assuming that \( \text{(H1), (H2), (H3) and (H4)} \) holds, then system (2) is permanent.

**Proof.** It can be observed that the result follows directly from the proofs of Proposition 1.1 and 1.2. \qed
5. **Global asymptotic stability.** Using an iterative scheme, we obtain the sufficient conditions for global asymptotic stability (GAS) of the interior equilibrium point $E(x^*_1, x^*_2, z^*)$.

**Lemma 1.** Consider the function $f(x) = x \exp(a - bx)$, where a and b are positive real constants. Then, $f(x)$ is non-decreasing for $x \in (0, \frac{1}{b}]$.

**Lemma 2.** Assuming that sequence $\{x_n\}$ satisfies $x_{n+1} = x_n \exp(p - qx_n)$, $n \in \mathbb{N}$ where $p, q$ are positive constants and $x_0 > 0$, then

1. If $p < 2$, we get $\lim_{n \to \infty} x_n = \frac{p}{q}$.

2. If $p \leq 1$, then $x_n \leq \frac{1}{q}$, $n = 2, 3, \ldots$

**Lemma 3.** Assuming two functions $\phi, \theta : \mathbb{Z}_+ \times [0, +\infty) \to [0, +\infty)$ satisfy $\phi(n, x) \leq \theta(n, x)$ for $n \in \mathbb{Z}_+$ and $x \in [0, +\infty)$ and $\theta(n, x)$ is non-decreasing for $x > 0$. If the sequences $x_n$ and $y_n$ are the non-negative solutions of the recurrence relations given by $x_{n+1} = \phi(n, x_n)$ and $y_{n+1} = \theta(n, y_n)$ for $n = 0, 1, 2, \ldots$ respectively with $x_0 \leq y_0 \ (x_0 \geq y_0)$ then we must have $x_n \leq y_n \ (x_n \geq y_n)$ for all $n = 0, 1, 2, \ldots$

**Theorem 3.** Assuming the conditions established in (H1), (H2), (H3) and (H4) and considering

\[
\begin{align*}
\rho_1 &= q < 1 \\
\rho_2 &= 1 \\
c_1 + c_2 - d_1 &\leq 1
\end{align*}
\]

(H5) (H6) (H7)

the co-existence equilibrium point $E(x^*_1, x^*_2, z^*)$ of system (2) is globally asymptotically stable.

**Proof.** Let $(x_1(k), x_2(k), z(k))$ be any solution of system (2), with $x_1(0) > 0$, $x_2(0) > 0$ and $z(0) > 0$. Let,

\[
\begin{align*}
U_1 &= \limsup_{k \to +\infty} x_1(k), \quad L_1 = \liminf_{k \to +\infty} x_1(k) \\
U_2 &= \limsup_{k \to +\infty} x_2(k), \quad L_2 = \liminf_{k \to +\infty} x_2(k) \\
U_3 &= \limsup_{k \to +\infty} z(k), \quad L_3 = \liminf_{k \to +\infty} z(k)
\end{align*}
\]

We claim that $L_1 = U_1 = x^*_1$, $L_2 = U_2 = x^*_2$ and $L_3 = U_3 = z^*$. From proposition (1.1) and (1.2) we have $U_1 \leq M_1^{x_1}, \ U_2 \leq L_1^{x_2}, \ U_3 \leq M_1^{z}, \ L_1 \leq N_1^{x_1}, \ L_2 \geq N_1^{x_2}$ and $L_3 \geq N_1^z$.

**Part 1(a)** We aim to show $U_1 \leq M_2^{x_1} \leq M_1^{x_1}$, where $M_2^{x_1} = \frac{r_1}{1 + h(N_1^z - \epsilon)} - q - \frac{b_1 N_1^{x_1} N_1^z}{1 + (M_1^{x_1} + r_1)^2} - g_1(N_1^{x_1} - \epsilon)$. For any arbitrarily small $\epsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that for $k > k_1$ we obtain from first equation of (2)

\[
x_{1}(k+1) \leq x_{1}(k) \exp \left( \frac{r_1}{1 + h(N_1^z - \epsilon)} - q - \frac{b_1 (N_1^{x_1} - \epsilon)(N_1^z - \epsilon)}{1 + (M_1^{x_1} + r_1)^2} - g_1(N_1^{x_1} - \epsilon) - px_1(k) \right). \quad (6a)
\]

Let us consider the auxiliary equation $u(k+1) = u(k) \exp \left[ \frac{r_1}{1 + h(N_1^z - \epsilon)} - q - \frac{b_1(N_1^{x_1} - \epsilon)(N_1^z - \epsilon)}{1 + (M_1^{x_1} + r_1)^2} - g_1(N_1^{x_1} - \epsilon) - pu(k) \right]$. By result (ii) of Lemma 2, we have $u(k) \leq \frac{1}{p}$ for all $k > k_1$, since using (H5) we have $\frac{r_1}{1 + h(N_1^z - \epsilon)} - q - \frac{b_1(N_1^{x_1} - \epsilon)(N_1^z - \epsilon)}{1 + (M_1^{x_1} + r_1)^2} - g_1(N_1^{x_1} - \epsilon) \leq r_1 - q \leq 1$. From Lemma 1, we have
\[ f(u) = u \exp \left[ \frac{-r_1}{1 + M_1^\alpha} - q \right] - b_1(N_1^{x_1} - \epsilon)(N_1^\epsilon - \epsilon) - g_1(N_1^{x_2} - \epsilon) - pu \] is non-decreasing for any \( u \in \left( 0, \frac{1}{p} \right) \). Thus from Lemma 3, \( x_1(k) \leq u(k) \) for all \( k > k_1 \) which further gives \( \limsup_{k \to +\infty} x_1(k) \leq \limsup_{k \to +\infty} u(k) \). Consequently from result (i) of Lemma 2 and considering the arbitrariness of \( \epsilon > 0 \) we get, \( U_1 \leq M_1^{x_1} = \frac{1}{p} \left[ \frac{r_1}{1 + hN_1^\epsilon} - q - b_1(N_1^{x_1} - \epsilon) - g_1(N_1^{x_2} - \epsilon) \right] \). Hence, for arbitrarily small \( \epsilon > 0 \) there exists \( n_2 > n_1 \) such that \( x_1(k) \leq M_1^{x_1} + \epsilon \) for all \( n > n_2 \).

Also, we have \( M_1^{x_1} \leq \frac{1}{p} \left( \frac{r_1}{1 + hN_1^\epsilon} \right) \leq \frac{r_1}{p} \leq \frac{\exp(r_1-1)}{p} = M_1^{x_1} \), using the fact \( \frac{\exp(x-1)}{x} \geq 1 \) for \( x > 0 \). Hence, our claim is established.

**Part 1(b)** We aim to show that \( U_2 \leq M_2^{x_2} \leq M_1^{x_1} \), where \( M_2^{x_2} = \frac{1}{r_2} (r_2 - b_2 N_2^{x_2} - g_2 N_1^{x_1}) \). Proceeding in a similar manner as in part-1a, from the second equation of system (2) for some \( k > k_2 \), we obtain

\[
x_2(k+1) \leq x_2(k) \exp \left[ r_2 - \frac{b_2(N_2^{x_2} - \epsilon)(N_1^\epsilon - \epsilon)}{1 + (M_1^{x_1} + \epsilon)^2} - g_2(N_1^{x_1} - \epsilon) - r_2 \beta x_2(k) \right]
\] (6b)

with \( U_2 = \limsup_{k \to +\infty} x_2(k) \leq \frac{1}{r_2 \beta} \left( r_2 - \frac{b_2(N_2^{x_2} - \epsilon)(N_1^\epsilon - \epsilon)}{1 + (M_1^{x_1} + \epsilon)^2} - g_2(N_1^{x_1} - \epsilon) \right), \) by using the (H6). From the arbitrariness of \( \epsilon > 0 \) we get \( U_2 \leq M_2^{x_2} = \frac{1}{r_2} (r_2 - b_2 N_2^{x_2} - g_2 N_1^{x_1}) \). Thus for arbitrarily small \( \epsilon > 0 \) there exists \( k_3 > k_2 \) such that \( x_2(k) \leq M_2^{x_2} + \epsilon \) for all \( k > k_3 \). We further get \( M_2^{x_2} \leq \frac{1}{\beta} \leq \frac{\exp(r_2-1)}{r_2} = M_1^{x_1} \), using the fact \( \frac{\exp(x-1)}{x} \geq 1 \) for \( x > 0 \). Hence our claim is established.

**Part 1(c)** We aim to show that \( U_3 \leq M_2^{x_2} \leq M_1^{x_1} \), where \( M_2^{x_2} = \frac{1}{d} \left( c_1(M_2^{x_1})^2 + c_2(M_2^{x_2})^2 + c_1 M_2^{x_1} + c_2 M_2^{x_2} - d_1 \right) \). Proceeding in a similar manner as in part-1a, from the third equation of system (2) for some \( k > k_3 \), we obtain

\[
z(k+1) \leq z(k) \exp \left[ \frac{c_1(M_2^{x_1})^2 + c_2(M_2^{x_2})^2}{1 + (M_2^{x_1} + \epsilon)^2} + \frac{c_1(M_2^{x_1})^2}{1 + (M_2^{x_2} + \epsilon)^2} - d_1 \right. - \frac{d M_1^{x_1}}{1 + \alpha M_1^{x_1}} - dz(k) \] (6c)

with \( U_3 = \limsup_{k \to +\infty} z(k) \leq \frac{1}{d} \left( c_1(M_2^{x_1})^2 + c_2(M_2^{x_2})^2 - d_1 - \frac{m N_2^{x_2} N_1^{x_1}}{1 + \alpha M_1^{x_1}} \right), \) by using the (H7). From the arbitrariness of \( \epsilon > 0 \) we get \( U_3 \leq M_2^{x_2} = \frac{1}{d} \left( c_1(M_2^{x_1})^2 + c_2(M_2^{x_2})^2 - d_1 - \frac{m N_2^{x_2} N_1^{x_1}}{1 + \alpha M_1^{x_1}} \right). \) Thus for arbitrarily small \( \epsilon > 0 \) there exists \( k_4 \) such that \( z(k) \leq M_2^{x_2} + \epsilon \) for all \( k > k_4 \). We further get \( M_2^{x_2} \leq \frac{c_1 + c_2 - d_1}{c_1 + c_2 - d_1} \leq \frac{\exp(c_1 + c_2 - d_1) - 1}{c_1 + c_2 - d_1} = M_1^{x_1} \), using (H1) and the fact \( \frac{\exp(x-1)}{x} \geq 1 \) for \( x > 0 \). Hence our claim is established.

**Part 2(a)** We aim to show that \( L_1 \geq N_2^{x_1} \geq N_1^{x_1} \), where \( N_2^{x_1} = \frac{1}{p} \left( \frac{r_1}{1 + hM_2^{x_2}} - q - b_1 M_1^{x_1} M_2^{x_2} - g_1 M_2^{x_2} \right) \). Proceeding as in part-1a, from the first equation of system
(2) for $k > k_4$ we have

$$x_1(k + 1) \geq x_1(k) \exp \left[ \frac{r_1}{1 + h(M_2^z + \epsilon)} - q - \frac{b_1(M_2^z + \epsilon)(M_2^z + \epsilon)}{1 + (N_1^z - \epsilon)^2} \right] - px_1(k),$$

with $L_1 = \liminf_{k \to +\infty} x_1(k) \geq \frac{1}{p} \left( \frac{r_1}{1 + h(M_2^z + \epsilon)} - q - \frac{b_1(M_2^z + \epsilon)(M_2^z + \epsilon)}{1 + (N_1^z - \epsilon)^2} \right)$ and by using (H5). From the arbitrariness of $\epsilon > 0$ we have $L_1 \geq N_2^{z_1} = \frac{1}{p} \left( \frac{r_1}{1 + h(M_2^z)} - q - \frac{b_1M_1^z M_2^z}{1 + (N_1^z)^2} - g_1(M_2^z) \right)$. Thus for arbitrarily small $\epsilon > 0$ there exists $k_5 > k_4$ such that $x_1(k) \geq N_2^{z_1} - \epsilon$ for all $k > k_5$. Furthermore, using the fact $M_2^z \leq M_1^z$ and $M_2^{z_1} \leq M_1^{z_1}$, (i = 1, 2) we have,

$$N_2^{z_1} \geq \frac{1}{p} \left( \frac{r_1}{1 + h(M_2^z) - q - \frac{b_1M_1^z M_2^z}{1 + (N_1^z)^2}} \right) \geq \frac{1}{p} \left( \frac{r_1}{1 + h(M_2^z) - q - b_1M_1^z M_2^z - g_1(M_2^z)} \right) \geq \frac{1}{p} \left( \frac{r_1}{1 + h(M_2^z) - q - b_1M_1^z M_2^z - g_1(M_2^z)} \right) \geq \frac{1}{p} \left( \frac{r_1}{1 + h(M_2^z) - q - b_1M_1^z M_2^z - g_1(M_2^z)} \right) = N_1^{z_1},$$

since from (4a) we have $pM_1^{z_1} = \exp(r_1 - 1) \geq r_1$.

Hence our claim is established.

**Part 2(b)** We aim to show that, $L_2 \geq N_2^{z_2} \geq N_1^{z_2}$, where $N_2^{z_2} = \frac{1}{r_2^2} \left( r_2 - \frac{b_2M_2^z M_2^z}{1 + (N_2^z)^2} - g_2(M_2^z) \right)$. Proceeding as in part-1a, from the second equation of system (2) for $k > k_5$ we have

$$x_2(k + 1) \geq x_2(k) \exp \left[ r_2 - \frac{b_2(M_2^z + \epsilon)(M_2^z + \epsilon)}{1 + (N_1^z - \epsilon)^2} - g_2(M_2^z + \epsilon) - r_2\beta x_2(k) \right],$$

with $L_2 = \liminf_{k \to +\infty} x_2(k) \geq \frac{1}{r_2^2} \left( r_2 - \frac{b_2(M_2^z + \epsilon)(M_2^z + \epsilon)}{1 + (N_1^z - \epsilon)^2} - g_2(M_2^z + \epsilon) \right)$ by using (H6). From the arbitrariness of $\epsilon > 0$ we get $L_2 \geq N_2^{z_2} = \frac{1}{r_2^2} \left( r_2 - \frac{b_2M_2^z M_2^z}{1 + (N_2^z)^2} - g_2M_2^{z_2} \right)$. Hence, for an arbitrarily small $\epsilon > 0$ there exists $k_6 > k_5$ such that $x_2(k) \geq N_2^{z_2} - \epsilon$ for all $k > k_6$. Finally, using the fact $M_2^z \leq M_1^z$ and $M_2^{z_2} \leq M_1^{z_2}$, (i = 1, 2) we have,

$$N_2^{z_2} \geq \frac{1}{r_2^2} \left( r_2 - \frac{b_2M_2^z M_2^z}{1 + (N_2^z)^2} - g_2M_2^{z_2} \right) \geq \frac{1}{r_2^2} \left( r_2 - \frac{b_2M_2^z M_2^z}{1 + (N_2^z)^2} - g_2M_2^{z_2} \right) \geq \frac{1}{r_2^2} \left( r_2 - \frac{b_2M_2^z M_2^z}{1 + (N_2^z)^2} - g_2M_2^{z_2} \right) = N_1^{z_2},$$

since from (4b) we obtain $\beta M_1^{z_1} = \exp(\frac{r_2 - 1}{r_2}) \geq 1$.

Hence our claim is established.

**Part 2(c)** We aim to show that, $L_3 \geq N_2^{z_2} \geq N_1^{z_1}$, where $N_2^{z_2} = \frac{1}{r_2^2} \left( c_1(N_2^{z_2})^2 - d_1 - M_2^{z_2} M_2^{z_2} \right)$. Proceeding as in part-1a, from the third equation of system (2) for $k > k_6$ we have

$$z(k + 1) \geq z(k) \exp \left[ c_1(N_2^{z_2} - \epsilon)^2 - d_1 - M_2^{z_2} + c_2(N_2^{z_2} - \epsilon)^2 \right],$$

$$d_k(k).$$
with \( L_3 = \lim \inf_{k \to +\infty} z(k) \geq \frac{1}{\delta} \left( c_1(N_n^{z_1} - \epsilon)^2 + c_2(N_n^{z_2} - \epsilon)^2 - d_1 - m(M_{n}^{z_2} + \epsilon) \times \right) (M_n^{z} + \epsilon) \) by using (H7). From the arbitrariness of \( \epsilon > 0 \) we get \( L_3 \geq N_2^z = \frac{1}{\delta} \left( c_1(N_n^{z_1})^2 + c_2(N_n^{z_2})^2 - d_1 - mM_1^{z_2}M_1^z \right) \). Thus, for an arbitrarily small \( \epsilon > 0 \) there exists \( k_7 > k_6 \) such that \( z(k) \geq N_2^z - \epsilon \) for all \( k > k_7 \). Finally, using the fact \( M_n^{z_2} \leq M_1^{z_2}, M_n^{z} \leq M_1^{z} \) and \( N_n^{z} \geq N_1^{z} \), (i = 1, 2) we have,

\[
N_2^z \geq \frac{1}{\delta} \left( c_1(N_n^{z_1})^2 + c_2(N_n^{z_2})^2 - d_1 - mM_1^{z_2}M_1^z \right) \exp \left[ -\frac{c_1(N_n^{z_1})^2}{1+(N_n^{z_1})^2} - \frac{c_2(N_n^{z_2})^2}{1+(N_n^{z_2})^2} \right] \\
= \frac{1}{\delta} \left( c_1(N_n^{z_1})^2 + c_2(N_n^{z_2})^2 - d_1 - mM_1^{z_2}M_1^z \right) \exp \left[ -\frac{c_1(N_n^{z_1})^2}{1+(N_n^{z_1})^2} - \frac{c_2(N_n^{z_2})^2}{1+(N_n^{z_2})^2} \right] \\
\geq \frac{1}{\delta} \left( c_1(N_n^{z_1})^2 + c_2(N_n^{z_2})^2 - d_1 - mM_1^{z_2}M_1^z \right) \exp \left[ -\frac{c_1(N_n^{z_1})^2}{1+(N_n^{z_1})^2} - \frac{c_2(N_n^{z_2})^2}{1+(N_n^{z_2})^2} \right] \\
= N_1^z, \text{ since from (4c) we obtain } dM_1^z = \exp(c_1 + c_2 - d_1 - 1) \geq c_1 + c_2 - d_1.
\]

Hence our claim is established.

**Part 3.** Using the above iteration process repeatedly and following part-1a we can obtain the sequences \( \{M_n^{x_1}\}, \{M_n^{x_2}\}, \{M_n^{z}\} \) and again following part-2a we can obtain the sequences \( \{N_n^{x_1}\}, \{N_n^{x_2}\}, \{N_n^{z}\} \) such that

\[
M_n^{x_1} = \frac{1}{p} \left( \frac{r_1}{1 + hN_n^{x_1} - q - b_1N_n^{x_1} - N_n^{z_1} - g_1N_{n-1}^{x_1}} \right); \\
M_n^{x_2} = \frac{1}{r_2} \left( r_2 - b_2N_n^{x_1} - N_n^{z_1} - g_2N_{n-1}^{x_1} \right); \\
M_n^{z} = \frac{1}{d} \left( \frac{c_1(M_n^{x_1})^2}{1 + (M_n^{x_1})^2} + \frac{c_2(M_n^{x_2})^2}{1 + (M_n^{x_2})^2} - d_1 - mN_n^{x_1} - N_n^{z_1} \right); \\
N_n^{x_1} = \frac{1}{p} \left( r_1 \frac{M_n^{x_1} + M_n^{z}}{1 + (N_n^{x_1})^2} - q - b_1M_n^{x_1} - M_n^{z} \right); \\
N_n^{x_2} = \frac{1}{r_2} \left( r_2 - b_2M_n^{x_1} - M_n^{z} \right); \\
N_n^{z} = \frac{1}{d} \left( \frac{c_1(N_n^{x_1})^2}{1 + (N_n^{x_1})^2} + \frac{c_2(N_n^{x_2})^2}{1 + (N_n^{x_2})^2} - d_1 - mM_n^{x_1}M_n^{z} \right); \text{ for all } n \geq 2.
\]

We obviously have \( N_2^{z_1} \leq L_1 \leq U_1 \leq M_2^{z_1}, N_2^{x_2} \leq L_2 \leq U_2 \leq M_2^{x_2} \) and \( N_2^{z} \leq L_3 \leq U_3 \leq M_2^{z} \). We further aim to show that \( \{N_n^{x_1}\}, \{N_n^{x_2}\}, \{N_n^{z}\} \) are monotonically increasing and \( \{M_n^{x_1}\}, \{M_n^{x_2}\}, \{M_n^{z}\} \) are monotonically decreasing by the method of mathematical induction. It clearly holds for \( n = 2 \). Let us assume for \( n = r(r \geq 2) \), \( N_r^{x_1} \geq N_{r-1}^{x_1}, N_r^{x_2} \geq N_{r-1}^{x_2}, N_r^{z} \geq N_{r-1}^{z}, M_r^{x_1} \leq M_{r-1}^{x_1}, M_r^{x_2} \leq M_{r-1}^{x_2} \) and \( M_r^{z} \leq M_{r-1}^{z} \) holds. Then we have

\[
M_{r+1}^{x_1} = \frac{1}{p} \left( \frac{r_1}{1 + hN_n^{x_1} - q - b_1N_n^{x_1} - N_n^{z_1} - g_1N_{n-1}^{x_1}} \right) \leq \frac{1}{p} \left( \frac{r_1}{1 + hN_n^{x_1} - q - b_1N_n^{x_1} - N_n^{z_1} - g_1N_{n-1}^{x_1}} \right) \\
= M_{r+1}^{x_1}.
\]

Similarly, we can obtain \( M_{r+1}^{x_2} \leq M_{r+1}^{x_1} \) and \( M_{r+1}^{z} \leq M_{r+1}^{x_2} \). Therefore, by mathematical induction \( \{M_n^{x_1}\}, \{M_n^{x_2}\}, \{M_n^{z}\} \) are monotonically decreasing. Again we have,
\( N_{r+1}^{x_1} = \frac{1}{p} \left( \frac{r}{1+hn_3} - q - b_1 \frac{M_{r+1}^{x_1} M_{r+1}^{z_1}}{1+(N_{r+1}^{x_1})^2} - g_1 M_{r+1}^{z_2} \right) \geq \frac{1}{p} \left( \frac{r}{1+hn_3} - q - b_1 \frac{M_{r+1}^{x_1} M_{r+1}^{z_1}}{1+(N_{r+1}^{x_1})^2} \right) - g_1 M_{r+1}^{z_2} = N_{r+1}^{x_1} \).

Similarly, we can obtain \( N_{r+1}^{x_2} \geq N_{r+1}^{x_2} \) and \( N_{r+1}^{z_1} \geq N_{r+1}^{z_1} \). Hence, by mathematical induction \( \{N_{n+1}^{x_1}\}, \{N_{n+1}^{x_2}\}, \{N_{n+1}^{z_1}\} \) are monotonically increasing.

Let \( \lim_{n \to \infty} M_{n+1}^{x_1} = m_1 \), \( \lim_{n \to \infty} M_{n+1}^{x_2} = m_2 \), \( \lim_{n \to \infty} M_{n+1}^{z_1} = m_3 \), \( \lim_{n \to \infty} N_{n+1}^{z_1} = n_1 \), \( \lim_{n \to \infty} N_{n+1}^{x_2} = n_2 \) and \( \lim_{n \to \infty} N_{n+1}^{z_1} = n_3 \). Taking \( n \to \infty \) in (10) we get,

\[
\begin{align*}
    m_1 &= \frac{1}{p} \left( \frac{r}{1+hn_3} - q - \frac{b_1 n_1 n_3}{1+m_1^2} - g_1 n_2 \right), \\
    m_2 &= \frac{1}{r_2 \beta} \left( r_2 - \frac{b_2 n_2 n_3}{1+m_2^2} - g_2 n_1 \right), \\
    m_3 &= \frac{1}{d} \left( \frac{c_1 m_1^2}{1+m_1^2} + \frac{c_2 m_2^2}{1+m_2^2} - d_1 - \frac{mn_2 n_3}{1+am_3^2} \right), \\
    n_1 &= \frac{1}{p} \left( \frac{r}{1+hn_3} - q - \frac{b_1 m_1 m_3}{1+n_1^2} - g_1 m_2 \right), \\
    n_2 &= \frac{1}{r_2 \beta} \left( r_2 - \frac{b_2 m_2 m_3}{1+n_2^2} - g_2 m_1 \right), \\
    n_3 &= \frac{1}{d} \left( \frac{c_1 n_1^2}{1+n_1^2} + \frac{c_2 n_2^2}{1+n_2^2} - d_1 - \frac{mn_2 n_3}{1+am_3^2} \right).
\end{align*}
\]

Evidently, we have from (11), \( m_1 = n_1 = x_1^*, m_2 = n_2 = x_2^* \) and \( m_3 = n_3 = z^* \). Hence, the global asymptotic stability of \( (x_1^*, x_2^*, z^*) \) is obtained and subsequently the theorem is established. \( \square \)

6. Numerical simulations. Using Matlab14 we perform numerical simulations to study the effect of fear and anti-predator behavior on system (2).

Example 1. Considering the parameter values \( r_1 = 1, h = 1, p = 1, q = 0.01, b_1 = 0.6, b_2 = 0.6, g_1 = 0.02, g_2 = 0.03, d = 1, d_1 = 0.02, \alpha = 5, \beta = 1.5, m_1 = 0.01, c_1 = 0.3, c_2 = 0.3, \tau_1 = 2, \tau_2 = 2 \) we obtain the time series for \( r_2 = 0.9 \) and 3 respectively in fig. (2).

The change in dynamics of system (2) is illustrated in fig. (2a) and (2b) for \( r_2 \) = 0.9 and \( r_2 \) = 3, respectively. The chosen parameter values along with \( r_2 = 0.9 \) satisfy the sufficient conditions for GAS obtained in (H1) - (H7). Whereas for \( r_2 = 3 \) the condition obtained in (H6) is violated and \( \frac{c_1 n_1^2}{1+n_1^2} + \frac{c_2 n_2^2}{1+n_2^2} - m_1^2 M_1^2 - d_1 = -0.0292253 \), also violates the condition in (H4). Thus, fig. (2a) and (2b) validates the analytical results obtained in Theorem 3.

Example 2. Considering the parameter values \( r_1 = 2, h = 3.8, p = 0.3, q = 0.6, b_1 = 1.5, b_2 = 1, \beta = 1.5, g_1 = 0.08, g_2 = 0.03, d = 1, d_1 = 0.1, \alpha = 5, m = 0.025, c_2 = 1, c_1 = 1.4, \tau_1 = 2, \tau_2 = 2 \) we obtain the phase portraits and time series for \( r_2 = 2 \) and 2.6 respectively in fig. (3). It can be observed that the increase in \( r_2 \) leads to the existence of more than one fixed points. Beyond a threshold value of \( r_2 = 2.8 \) the system exhibits chaotic dynamics which is observed in the bifurcation diagram in fig. (4a). The existence of chaos is verified using the chaos 0-1 test as shown in (4b). The region of stability of system (2) is obtained in fig. (5) for the varying parameters \( m \) and \( r_2 \). It can be observed that at higher anti-predator rates the system becomes unstable even for lower growth rate \( (r_2) \) of the prey displaying the anti-predator behavior. The fact that increase in \( m \) leads to the loss of stability...
(a) For the chosen set of parameters and \( r_2 = 0.9 \), the conditions for GAS obtained in (H1) - (H7) are satisfied.

(b) For the chosen set of parameters and \( r_2 = 3 \), the conditions for GAS obtained in (H4) and (H6) are violated.

Fig. 2. Time series of (2) with the parameter values \( r_1 = 1, h = 1, p = 1, q = 0.01, b_1 = 0.6, b_2 = 0.6, g_1 = 0.02, g_2 = 0.03, d = 1, d_1 = 0.02, \alpha = 5, \beta = 1.5, m = 0.01, c_1 = 0.3, c_2 = 0.3, \tau_1 = 2, \tau_2 = 2 \).

is justified by the condition obtained in (H4), where increase in \( m \) beyond a critical threshold value tends to violate the criterion obtained in (H4). Below a threshold value of the anti-predator rate (\( m \)), increase in the growth rate of the prey leads to 2, 3 and 4 period oscillations which eventually leads to chaotic dynamics for higher values of the growth rate. (See fig. (5)).

Fig. 3. Phase portraits and time series of (2) with the parameter values \( r_1 = 2, h = 3.8, p = 0.3, q = 0.6, b_1 = 1.5, b_2 = 1, \beta = 1.5, g_1 = 0.08, g_2 = 0.03, d = 1, d_1 = 0.1, \alpha = 5, m = 0.025, c_2 = 1, c_1 = 1.4, \tau_1 = 2, \tau_2 = 2 \).

(a) \( r_2 = 2 \)  
(b) \( r_2 = 2.6 \)  
(c) \( r_2 = 2.7 \)

Example 3. Considering the parameter values \( r_1 = 2, p = 0.3, q = 0.6, b_2 = 1, \beta = 1.5, g_1 = 0.08, g_2 = 0.03, d = 1, d_1 = 0.1, \alpha = 5, c_2 = 1, c_1 = 1.4, b_1 = 1.5, \tau_1 = 2, \tau_2 = 2 \).
(a) Bifurcation plot of system (2) for the varying parameter $r_2$.

(b) Chaos 0-1 test of system (2) for the varying parameter $r_2$.

Fig. 4. Bifurcation diagram and chaos 0-1 test of system (2) with the parameter values $r_1 = 2, h = 3.8, p = 0.3, q = 0.6, b_1 = 1.5, b_2 = 1, \beta = 1.5, g_1 = 0.08, g_2 = 0.03, d = 1, d_1 = 0.1, \alpha = 5, m = 0.025, c_2 = 1, c_1 = 1.4, \tau_1 = 2, \tau_2 = 2$, showing the existence of chaos with increase in $r_2$.

Fig. 5. Stability region of system (2) for the varying parameters $r_2$ and $m$. The system shows stable dynamics in region A, period-2 oscillations in region B, period-4 oscillations in region C, period-3 oscillations in region D and chaotic dynamics in region E.

2, $\tau_2 = 2$ we obtain the bifurcation diagram and chaos 0-1 test of system (2) for the varying parameters $h$ and $m$ in fig. (6). It can be observed that the effect of fear removes the periodic oscillations and stabilizes the system. But on the other hand the anti-predator behavior induces periodic oscillations which further amplifies with increase in $m$, thus de-stabilizing the system.
EFFECTS OF FEAR AND ANTI-PREDATOR RESPONSE IN A DISCRETE SYSTEM

Example 4. Considering the parameter values $r_1 = 2, p = 0.3, q = 0.6, b_1 = 1.5, b_2 = 1, \beta = 1.5, g_1 = 0.08, g_2 = 0.03, d = 1, d_1 = 0.1, \alpha = 5, c_2 = 1, c_1 = 1.4, \tau_1 = 2, \tau_2 = 2$. The nature of dynamics of (2) is studied for varying values of $\tau_{1,2}$. In chaos 0-1 test, time $\{z(n)\}_{n=1}^{N}$ is translated to $p_c(n) = \sum_{j=1}^{n} z(j) \cos c j, q_c(n) = \sum_{j=1}^{n} z(j) \sin c j$, where $c \in \left(0, \frac{\pi}{2}\right)$ and $n = 1, 2, ..., N$ with $n \ll N$. For convenience, we refer to $(p_c, q_c)$ as $(p, q)$. The $(p, q)$ plots can quantify regular and chaotic dynamics, where regular geometrical shapes correspond to regular dynamics, whereas irregular shapes indicate chaotic dynamics. It can be observed in fig. (7) that increase in gestation delay has a positive correlation with the irregular dynamics of system (2). As the value of $\tau_{1,2}$ is increased from $\tau_{1,2} = 0$ to $\tau_{1,2} = 2$, it can be observed that the $(p, q)$ plot in fig. (7) transforms from a regular geometrical shape to an irregular shape which is indicative of the existence of chaotic dynamics.

Example 5. Considering the parameter values $r_1 = 2, r_2 = 3.7, p = 0.3, q = 0.6, b_2 = 1, \beta = 1.5, g_1 = 0.08, g_2 = 0.03, d = 1, d_1 = 0.1, \alpha = 1, c_2 = 1, c_1 = 1.4, \tau_1 = 2, \tau_2 = 2$ the change in dynamics of system (2) is studied for varying values of $r_2$. It can be observed in fig. (8) that increase in $r_2$ from $r_2 = 2$ to $r_2 = 4.5$ transforms the dynamics of the system from regular to chaotic. Thus increase in $r_2$ induces irregular dynamics in system (2) which correlates with the numerical simulations obtained in example 2.

7. Discussions. In this paper we studied the effects of fear and anti-predator behavior in a discrete-delayed model with two prey and one predator. The proposed
model (2) is analogous to the ecological relationship of elk-bison-wolf in YNP. The analytical conditions for permanence and global stability along with some numerical simulations are obtained to study the rich dynamics of the system.

It is observed in fig. (6) that the effect of fear has an overall stabilizing effect on the system, although it reduces the population density of the elk ($x_1$). It might be justified by the following set of events, that at higher levels of fear the elk population becomes more vigilant and often migrate to unfavorable regions of the national park at the cost of higher mortality rate but with lesser chance of interaction with the wolves. This in turn would increase the wolf-bison interactions and induce a natural stability in the system by neutralizing the secondary trophic cascades. Thus the effect of reduced predation on the elk population can increase their growth rate ($r_1$).
which counter balances the negative effect of fear \((h)\) on the elk population density. This phenomenon can be explained by the term \(r_1 \frac{r_1}{r_1 + M_1^z}\) in \((H2)\) where increase in \(r_1\) can balance the effect of increase in \(h\) without violating the sufficient condition for global stability in \((H2)\). Also, at high intra-specific competition rate \((d)\) among the wolves, the value of \(M_1^z\) decreases (refer to \((4c)\)), thus diminishing the effects of fear. It is also observed in fig. \((5)\) and fig. \((8)\) that higher density of bison population induces instability in the system and also leads to chaotic dynamics. As the bison are not allowed to move freely outside YNP due to fears that they might transmit brucellosis to cattle [17], hence it is very crucial that the bison population in the park is maintained below a certain threshold by utilizing optimal harvesting techniques. It is also observed that fear of predation can not cause extinction of the prey or predator species in system \((2)\). Moreover, the population density of the predator is more affected by the effect of fear than that of the prey population, since due to the increased vigilance and awareness in the prey about the presence of its predator, makes it difficult for the predator to kill the prey, thus affecting the growth rate of the predator indirectly. Furthermore, the presence of gestation delay plays a crucial role in influencing the nature of dynamics of system \((2)\). It is observed in fig. \((7)\) that the delay induces instability and gives rise to irregular dynamics.

Analogous to the ecological scenario in YNP, we introduced the effects of fear, anti-predator behavior and delay to the model studied in \([2]\). The modified model \((2)\) in this study captures a more realistic scenario of the elk-bison-wolf interaction in YNP and posits the complex dynamics of the system. It is observed that underlying effects of fear and anti-predator behavior significantly influences the dynamics of system and plays a major role in maintaining the predator-prey balance in YNP. In our future work, we aim to study the change in dynamics of this system by incorporating the effects of migration and optimal harvesting of the prey population.

8. **Conflict of interest.** The authors declare that there is no conflict of interests regarding the publication of this paper.

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Appendix A. Proofs

A.1. Proof of Proposition 1.1 Let us assume that there exists \( l \in \mathbb{N} \) such that \( x_1(l + 1) \geq x_1(l) \). Then,
\[
\frac{r_1}{1 + r_2 z(l)} - px_1(l) - q - \frac{b_{1x_1} z(l)}{1 + r_2 z(l)} - \gamma_1 x_2(l) \geq 0.
\]
Hence, \( x_1(l) \leq \frac{r_1}{p} \). \hspace{1cm} (12a)

Using the fact that \( \max_{x \in \mathbb{R}} \{x \exp(u - vx)\} = \frac{\exp(u-1)}{v} \), where \( u, v > 0 \), we have from (12a),
\[
x_1(l + 1) \leq x_1(l) \exp \left( r_1 - px_1(l) \right) \leq \frac{\exp(r_1 - 1)}{p} = M_1^{r_1} \hspace{1cm} (12b)
\]
We claim that, \( x_1(k) \leq M_1^{r_1} \) for \( k \geq l \). By a contradiction approach, let us assume that there exists a \( t_1 \in \mathbb{N} \), where \( t_1 > l \), such that \( x_1(t_1) > M_1^{r_1} \). Then, \( t_1 > l + 1 \). Let \( \tilde{t}_1 > l + 1 \) be the smallest possible integer such that \( x_1(\tilde{t}_1) > M_1^{r_1} \) and hence we get, \( x_1(\tilde{t}_1 - 1) \leq M_1^{r_1} \). Subsequently, we get \( x_1(\tilde{t}_1 - 1) < x_1(t_1) \), which further leads to \( x_1(\tilde{t}_1 - 1) < x_1(\tilde{t}_1) \leq M_1^{r_1} \) using the result from (12b). This being a a contradiction, the claim is established.

Now assuming \( x_1(k + 1) < x_1(k) \) for all \( k \in \mathbb{N} \), let \( \lim_{k \to +\infty} x_1(k) = \overline{x}_1 \). We assert that \( \overline{x}_1 \leq \frac{r_1}{p} \). Again by a contradiction approach, let us assume that \( \overline{x}_1 > \frac{r_1}{p} \). In
the first equation of (2) taking limit $k \to +\infty$ we get
$$
\lim_{k \to +\infty} \left[ \frac{r_1}{1 + b_1 z(k)} - px_1(k) - q - \frac{b_1 x_1(z(k))}{1 + x_1^2(k)} - g_1 x_2(k) \right] = 0.
$$
This is a contradiction since,\
$$
\left( \frac{r_1}{1 + b_1 z(k)} - px_1(k) - q - \frac{b_1 x_1(z(k))}{1 + x_1^2(k)} - g_1 x_2(k) \right) \leq r_1 - px_1(k) \leq r_1 - p_M < 0
$$
for large enough $k \in \mathbb{N}$. Thus the claim is established.

Using the fact $\exp(x-1) \leq 1$ for $x > 0$, we have $\frac{r_1}{p} \leq \frac{\exp(r_1-1)}{r_1} \left( \frac{r_1}{p} \right) = \frac{\exp(r_1-1)}{p} = M_1^+$. Hence it follows that (4a) holds.

Proceeding in a similar manner, the proof of (4b) can be obtained and it is omitted here.

Now we proceed to the proof of (4c). We assume that there exists a $l \in \mathbb{N}$ such that $z(l+1) \geq z(l)$. Then,\
$$
\left[ c_1 x_1^2(l-\tau_1) + \frac{c_2 x_2^2(l-\tau_2)}{1 + x_2^2(l-\tau_2)} - dz(l) - d_1 - \frac{mx_2(z(l))}{1 + \alpha z(l)} \right] \geq 0.
$$
Hence we get,

$$
z(l) \leq \frac{c_1 + c_2 - d_1}{d}. \tag{12c}
$$

Using the fact that $\max_{x \in \mathbb{R}} \{ x \exp(u - vx) \} = \frac{\exp(u-1)}{v}$, where $u, v > 0$, we have from (12c),

$$
z(l+1) \leq z(l) \exp(c_1 + c_2 - d_1 - dz(l)) \leq \frac{\exp(c_1 + c_2 - d_1 - 1)}{d} = M_1^+. \tag{12d}
$$

We assert that $z(k) \leq M_1^+$ for $k \geq l$. If possible, we assume that there exists $t_2 \in \mathbb{N}$ where $t_2 > l$ such that $z(t_2) > M_1^+$. Then we have $t_2 > l + 1$. Let $\tilde{t}_2 > l + 1$ be the smallest positive integer such that $z(\tilde{t}_2) > M_1^+$. Then we must have $z(\tilde{t}_2 - 1) \leq M_1^+$ which further gives $z(\tilde{t}_2 - 1) < z(\tilde{t}_2)$ by using (12d). This leads to the contradiction that $z(\tilde{t}_2) \leq M_1^+$. Hence, our assertion is established.

Now, we assume that $z(k+1) < z(k)$ for all $k \in \mathbb{N}$ and let $\lim_{k \to +\infty} z(k) = \tau$. We prove our claim $\tau \leq \frac{c_1 + c_2 - d_1}{d}$, by way of contraction. If possible, let us assume that $\tau > \frac{c_1 + c_2 - d_1}{d}$. In the third equation of (2) taking limit $k \to +\infty$ we get

$$
\lim_{k \to +\infty} \left[ c_1 x_1^2(k-\tau_1) + \frac{c_2 x_2^2(k-\tau_2)}{1 + x_2^2(k-\tau_2)} - dz(k) - d_1 - \frac{mx_2(z(k))}{1 + \alpha z(k)} \right] = 0,
$$
which is a contradiction as\
$$
\left[ c_1 x_1^2(k-\tau_1) + \frac{c_2 x_2^2(k-\tau_2)}{1 + x_2^2(k-\tau_2)} - dz(k) - d_1 - \frac{mx_2(z(k))}{1 + \alpha z(k)} \right] \leq c_1 + c_2 - d\tau - d_1 < 0
$$
for large enough $k \in \mathbb{N}$. Hence the claim is proved.

Using the fact $\exp(x-1) \geq 1$ for $x > 0$ and from (H1), we have $\frac{c_1 + c_2 - d_1}{d} \leq \exp(c_1 + c_2 - d_1 - 1) \left( \frac{c_1 + c_2 - d_1}{d} \right) = \exp(c_1 + c_2 - d_1 - 1) = M_1^+$. Thus, it follows that (4c) holds.

Hence, proposition 1.1 is proved.
A.2. Proof of Proposition 1.2 Let us assume that there exists \( l \in \mathbb{N} \) such that \( x_1(l+1) \leq x_1(l) \). Then,
\[
\frac{r_1}{1+hz(l)} - px_1(l) - q - \frac{b_1 x_1(l)z(l)}{1+x_1^2(l)} - g_1 x_2(l) \leq 0
\]
or, \( x_1(l) \geq \frac{1}{p} \left( \frac{r_1}{1+hM_1^2} - q - b_1 M_1^{x_1} - g_1 M_1^{x_2} \right) \) \hspace{1cm} (13a)

Hence we get, \( x_1(l+1) \geq \frac{1}{p} \left( \frac{r_1}{1+hM_1^2} - q - b_1 M_1^{x_1} - g_1 M_1^{x_2} \right) \exp \left[ \frac{r_1}{1+hM_1^2} - pM_1^{x_1} - q - b_1 M_1^{x_1} - g_1 M_1^{x_2} \right] = N_1^{x_1} \). \hspace{1cm} (13b)

We assert that \( x_1(k) \geq N_1^{x_1} \) for any \( k \geq l \). To establish our assertion, we consider that if possible there exists a \( t_3 \in \mathbb{N} \) where \( t_3 > l \) such that \( x_1(t_3) < N_1^{x_1} \). Then \( t_3 > l + 1 \) and let \( t_3 > l + 1 \) be the smallest integer satisfying \( x_1(t_3) < N_1^{x_1} \). Then \( x_1(t_3-1) \geq N_1^{x_1} \) which further gives \( x_1(t_3-1) > x_1(t_3) \geq N_1^{x_1} \), using (13b). Thus, we arrive at a contradiction, and hence our assertion is established.

Now, let us assume that \( x_1(k+1) > x_1(k) \) for all \( k \in \mathbb{N} \) and \( \lim_{k \to +\infty} x_1(k) = x_1 \). We claim that \( x_1 \geq \frac{1}{p} \left( \frac{r_1}{1+hM_1^2} - q - b_1 M_1^{x_1} - g_1 M_1^{x_2} \right) \). To establish our claim, by way of contradiction we consider that \( x_1 < \frac{1}{p} \left( \frac{r_1}{1+hM_1^2} - q - b_1 M_1^{x_1} - g_1 M_1^{x_2} \right) \).

Taking limit \( k \to +\infty \) in the first equation of (2) we have, \( \lim_{k \to +\infty} \left[ \frac{r_1}{1+hz(k)} - px_1(k) \right] = 0 \), which is a contradiction as, \( \liminf_{k \to +\infty} \left[ \frac{r_1}{1+hz(k)} - px_1(k) \right] \geq \frac{r_1}{1+hM_1^2} - px_1(k) - q - b_1 M_1^{x_1} - g_1 M_1^{x_2} > 0 \) for large enough \( k \in \mathbb{N} \). Hence, our claim is established. Using (H2) we further have, \( N_1^{x_1} \leq \frac{1}{p} \left( \frac{r_1}{1+hM_1^2} - q - b_1 M_1^{x_1} - g_1 M_1^{x_2} \right) \leq x_1 \).

Thus it follows that (5b) holds.

The proof of (5c) can be obtained by a similar analysis, and the details are omitted here.

Finally, we proceed to the proof of (5c). Let us assume that there exists some \( l \in \mathbb{N} \) such that \( z(l+1) \leq z(l) \). Then we have,
\[
\left[ \frac{c_1 x_1^2(l-z(l))}{1+x_1^2(l-z(l))} + c_2 x_2^2(l-z(l)) - dz(l) - d_1 - \frac{m x_2(l-z(l))}{1+\alpha z(l)} \right] \leq 0
\]
or, \( z(l) \geq \frac{1}{d} \left( \frac{c_1 (N_1^{x_1})^2}{1+(N_1^{x_1})^2} + \frac{c_2 (N_1^{x_2})^2}{1+(N_1^{x_2})^2} - d_1 - mM_1^{x_2} M_1^{x_1} \right) \) \hspace{1cm} (13c)

Using (13c) we have,
\[
z(l+1) \geq \frac{1}{d} \left( \frac{c_1 (N_1^{x_1})^2}{1+(N_1^{x_1})^2} + \frac{c_2 (N_1^{x_2})^2}{1+(N_1^{x_2})^2} - d_1 - mM_1^{x_2} M_1^{x_1} \right) \exp \left[ \frac{c_1 (N_1^{x_1})^2}{1+(N_1^{x_1})^2} + \frac{c_2 (N_1^{x_2})^2}{1+(N_1^{x_2})^2} - d_1 - dM_1^{x_1} - mM_1^{x_2} M_1^{x_1} \right] = N_1^{x_1} \). \hspace{1cm} (13d)

We claim that \( z(k) \geq N_1^{x_1} \) for all \( k \geq l \). If possible, let us assume that there exists a \( t_4 \in \mathbb{N} \) where \( t_4 > l \) such that \( z(t_4) < N_1^{x_1} \). Then \( t_4 > l + 1 \) and let \( t_4 > l + 1 \) be the smallest integer such that \( z(t_4) < N_1^{x_1} \). Then \( z(t_4-1) \geq N_1^{x_1} \), which further leads to \( z(t_4-1) > z(t_4) \geq N_1^{x_1} \) using (13d). Therefore, we arrive at a contradiction and hence our assertion is established.
Now, let us assume that $z(k+1) > z(k)$ for all $k \in \mathbb{N}$ and let $\lim_{k \to +\infty} z(k) = \hat{z}$. We claim that $\hat{z} \geq \frac{1}{d} \left( \frac{c_1(N^{e_1})^2}{1+(N^{e_1})^2} + \frac{c_2(N^{e_2})^2}{1+(N^{e_2})^2} - d_1 - mM_1^{x_2}M_1^{\hat{z}} \right)$. To establish our claim, by way of contradiction we consider that $\hat{z} < \frac{1}{d} \left( \frac{c_1(N^{e_1})^2}{1+(N^{e_1})^2} + \frac{c_2(N^{e_2})^2}{1+(N^{e_2})^2} - d_1 - mM_1^{x_2}M_1^{\hat{z}} \right)$. Taking limit $k \to +\infty$ in the last equation of (2) we get, $\lim_{k \to +\infty} \left[ \frac{c_1x_1^2(k-\tau_1)}{1+x_1^2(k-\tau_1)} + \frac{c_2x_2^2(k-\tau_2)}{1+x_2^2(k-\tau_2)} - dz(k) - d_1 - \frac{m_2x_2(k)z(k)}{1+\alpha(z(k))} \right] = 0$, which is a contradiction since $\lim_{k \to +\infty} \left[ \frac{c_1x_1^2(k-\tau_1)}{1+x_1^2(k-\tau_1)} + \frac{c_2x_2^2(k-\tau_2)}{1+x_2^2(k-\tau_2)} - dz(k) - d_1 - \frac{m_2x_2(k)z(k)}{1+\alpha(z(k))} \right] \geq \frac{c_1(N^e_1)^2}{1+(N^{e_1})^2} + \frac{c_2(N^e_2)^2}{1+(N^{e_2})^2} - d_1 - m_2M_1^{x_2}M_1^{\hat{z}} > 0$ for large enough $k \in \mathbb{N}$.

Hence, our claim is established.

Using the fact $\exp(x-1) \geq x$, for $x > 0$, we get that $M_1^{\hat{z}} = \frac{\exp(c_1+c_2-d_1-1)}{d} \geq \frac{c_1+c_2-d_1}{d}$, or, $d_1 + dM_1^{\hat{z}} \geq c_1 + c_2$ which further implies that $N_1^{\hat{z}} \leq \frac{1}{d} \left( \frac{c_1(N^e_1)^2}{1+(N^{e_1})^2} + \frac{c_2(N^e_2)^2}{1+(N^{e_2})^2} - (c_1+c_2) - mM_1^{x_2}M_1^{\hat{z}} \right)$ or,

$N_1^{\hat{z}} \leq \frac{1}{d} \left( \frac{c_1(N^e_1)^2}{1+(N^{e_1})^2} + \frac{c_2(N^e_2)^2}{1+(N^{e_2})^2} - d_1 - m_2M_1^{x_2}M_1^{\hat{z}} \right) \exp \left[ -\frac{c_1(N^e_1)^2}{1+(N^{e_1})^2} - \frac{c_2(N^e_2)^2}{1+(N^{e_2})^2} - (c_1+c_2) - mM_1^{x_2}M_1^{\hat{z}} \right]$

or, $N_1^{\hat{z}} \leq \frac{1}{d} \left( \frac{c_1(N^e_1)^2}{1+(N^{e_1})^2} + \frac{c_2(N^e_2)^2}{1+(N^{e_2})^2} - d_1 - m_2M_1^{x_2}M_1^{\hat{z}} \right) \leq \hat{z}$.

Thus, it follows that (5c) holds. Hence, proposition 1.2 is proved.

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