Schrödinger-Poisson system with steep potential well

Yongsheng Jiang and Huan-Song Zhou†
Wuhan Institute of Physics and Mathematics
Chinese Academy of Sciences, P.O.Box 71010, Wuhan 430071, China

Abstract: We study the following Schrödinger-Poisson system

\[
\begin{aligned}
-\Delta u + (1 + \mu g(x))u + \lambda \phi(x)u &= |u|^{p-1}u, \quad x \in \mathbb{R}^3, \\
-\Delta \phi &= u^2, \quad \lim_{|x| \to +\infty} \phi(x) = 0,
\end{aligned}
\]

where \(\lambda, \mu\) are positive parameters, \(p \in (1, 5)\), \(g(x) \in L^\infty(\mathbb{R}^3)\) is nonnegative and \(g(x) \equiv 0\) on a bounded domain in \(\mathbb{R}^3\), \(g(x) \neq g(|x|)\). For \(\mu = 0\), \((P_\lambda)\) was studied by Ruiz [J. Funct. Anal., 237(2006), 655-674]. However, if \(\mu \neq 0\) and \(g(x)\) is not radially symmetric, it is unknown whether \((P_\lambda)\) has a nontrivial solution for \(p \in (1, 2)\). In this paper, combining domain approximation and priori estimate we establish the boundedness and compactness of a (PS) sequence, then we prove for \(\mu > 0\) large that \((P_\lambda)\) with \(p \in (1, 2)\) has a ground state if \(\lambda > 0\) small and that \((P_\lambda)\) for \(p \in [3, 5)\) has a nontrivial solution for all \(\lambda > 0\). Moreover, some behaviors of the solutions of \((P_\lambda)\) as \(\lambda \to 0\), \(\mu \to +\infty\) and \(|x| \to +\infty\) are also discussed.

1 Introduction

In this paper, we are concerned with the existence of nontrivial solutions of the following Schrödinger-Poisson system

\[
\begin{aligned}
-\Delta u + V_\mu(x)u + \lambda \phi(x)u &= |u|^{p-1}u, \quad x \in \mathbb{R}^3, \\
-\Delta \phi &= u^2, \quad \lim_{|x| \to +\infty} \phi(x) = 0, \quad x \in \mathbb{R}^3,
\end{aligned}
\]

where \(V_\mu(x) = 1 + \mu g(x)\), \(\lambda\) and \(\mu\) are positive parameters, \(p \in (1, 5)\) and \(g(x)\) satisfies

\footnote{This work was supported by NSFC and CAS-KJCX3-SYW-S03.}


\footnote{Corresponding author: hszhou@wipm.ac.cn.}

\footnote{2000 Mathematical Subject Classification: 35J60. Key words: Schrödinger-Poisson, priori estimate, potential well, ground state, exponential decay.}
(G_1) \( g(x) \geq 0, \ g(x) \in L^\infty(\mathbb{R}^3) \).

(G_2) \( \Omega_0 = \{ x \in \mathbb{R}^3 : g(x) = 0 \} \) is bounded and has nonempty interior.

(G_3) \( \lim \inf_{|x| \to \infty} g(x) = 1 \). (Here 1 can be replaced by any positive number)

Problem (1.1) arisen in quantum mechanics, which is related to the study of nonlinear Schrödinger equation for a particle in an electromagnetic field or the Hatree-Fock equation, etc., see [6, 7, 10, 12, 17] and the references therein. If \( \mu = 0 \), the existence of solutions of problem (1.1) has been discussed for different ranges of \( p \), for examples, \([9, 10, 12]\) for \( p \in (3, 5) \), \([4]\) for \( p \in (2, 5) \), \([16]\) for \( p \in (2, 3) \) and \([2, 17, 3]\) for \( p \in (1, 5) \) or general nonlinearity, etc. For some \( \lambda_0 > 0 \), it was proved in \([16]\) that (1.1) with \( \mu = 0 \) and \( p \in (1, 2] \) has no nontrivial solution if \( \lambda \geq \lambda_0 \). Ruiz in \([17]\) proved that \( \lambda_0 = \frac{1}{4} \) and that (1.1) has solution if \( \lambda > 0 \) small enough. Some recent results in this direction was summarized in \([1]\). However, if \( \mu \neq 0 \), there are only a few results on (1.1) for \( p \in (2, 5) \) and some special potential functions, such as \([4, 22, 20, 21]\). In \([4]\), a ground state of (1.1) with \( p \in (3, 5) \) was obtained if \( V_\mu(x) \) satisfies

\[(V_1) \quad V_\infty = \lim_{|y| \to +\infty} V_\mu(y) \geq V_\mu(x) \text{ a.e. in } x \in \mathbb{R}^3, \text{ and the strict inequality holds on a positive measure set.}\]

It was also proved in \([22]\) that (1.1) still has a ground state for \( p \in (2, 3] \) if (V_1) holds and \( V_\mu(x) \) is weakly differentiable function such that

\[(V_2) \quad (\nabla V_\mu(x), x) \in L^\infty(\mathbb{R}^3) \cup L^{3/2}(\mathbb{R}^3) \text{ and } 2V_\mu(x) + (\nabla V_\mu(x), x) \geq 0 \text{ a.e. } x \in \mathbb{R}^3, \text{ where } (\cdot, \cdot) \text{ is the usual inner product in } \mathbb{R}^3.\]

If \( -\Delta u \) is replaced by \( -\varepsilon \Delta u \) in (1.1), the semiclassical states for this kind of problem was discussed very recently in \([18, 21]\) for special \( V_\mu(x) \), and in \([11]\) for \( \mu = 0 \), etc. Problem (1.1) with asymptotically linear nonlinearity was considered in \([20]\). To the authors’ knowledge, it seems still open if there exists a solution to problem (1.1) with \( \mu \neq 0 \) and \( p \in (1, 2) \). It is known that (V_1) is crucial in using concentration compactness principle if \( V_\mu(x) \) is not radially symmetric, (V_2) is important in getting the boundedness of a (PS) sequence. In this paper, we assume neither the conditions of (V_1) and (V_2) nor that the potential \( V_\mu(x) \) is radially symmetric. These lead to several difficulties in using variational method to get solutions of (1.1). The first is to prove that a (PS) sequence is bounded in \( H^1(\mathbb{R}^3) \), which is known to be hard for \( p \in (1, 2) \) because of the presence of the so called
nonlocal term \( \phi(x)u \) in (1.1). If \( \mu = 0 \), we can overcome this difficulty by using the decay property of functions in \( H^1_0(\mathbb{R}^3) \) [17], but this trick does not work when \( V_\mu(x) \) is not radially symmetric. Once we get a bounded (PS) sequence, then we have to find a new way to prove that the (PS) sequence converges to a solution because the concentration compactness principle cannot be used in our case due to the lack of conditions \( (V_1) \) etc. In this paper, we first consider the problem (1.1) on \( B_k = \{ x \in \mathbb{R}^3 : |x| < k \} \) for each \( k \in \mathbb{N} \), and we can easily get a solution \( u_k \) of (1.1) in \( H^1_0(B_k) \) since the Sobolev embedding \( H^1_0(B_k) \hookrightarrow L^q(B_k) \) for \( q \in (1, 6) \) is compact. Then, we establish a uniform priori estimate of \( L^\infty \) norm of \( u_k \). Finally, by adapting some techniques used in [20], we prove that \( \{ u_k \} \) converges to a nontrivial solution of (1.1). Moreover, a ground state of (1.1) for \( p \in (1, 2) \) is also proved in Section 6. However, our methods seem not useful for \( p \in [2, 3) \).

Notations: Throughout this paper, for \( k \in \mathbb{N} \), we denote 
\[
B_k = \{ x \in \mathbb{R}^3 : |x| < k \}.
\]
and define 
\[
C^1_B(\mathbb{R}^3) = \{ u \in C^1(\mathbb{R}^3) : u \in L^\infty(\mathbb{R}^3) \text{ and } |\nabla u| \in L^\infty(\mathbb{R}^3) \}, \tag{1.2}
\]
\[
\mathcal{D}_V = \{ u \in C^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V_\mu(x)u^2 \, dx < +\infty \}, \tag{1.3}
\]
\[
\mathcal{D}_k = \{ u \in C^{1,2}(B_k) : \int_{B_k} V_\mu(x)u^2 \, dx < +\infty \}. \tag{1.4}
\]
Clearly, \( \mathcal{D}_V \) and \( \mathcal{D}_k \) are Hilbert spaces, their scalar products are given by 
\[
\langle u, v \rangle_{\mathcal{D}_V} = \int_{\mathbb{R}^3} \nabla u \nabla v + V_\mu(x)uv \, dx, \quad \text{for any } u, v \in \mathcal{D}_V, \tag{1.5}
\]
\[
\langle u, v \rangle_k = \int_{B_k} \nabla u \nabla v + V_\mu(x)uv \, dx, \quad \text{for any } u, v \in \mathcal{D}_k. \tag{1.6}
\]
The norms of \( \mathcal{D}_V \) and \( \mathcal{D}_k \) are introduced by 
\[
\| u \|^2_{\mathcal{D}_V} = \langle u, u \rangle_{\mathcal{D}_V}, \quad \| u \|^2_k = \langle u, u \rangle_k. \tag{1.7}
\]
Clearly, \( \| \cdot \|_{\mathcal{D}_V} \) is an equivalent norm of \( H^1(\mathbb{R}^3) \). For \( p \in [1, +\infty] \), we denote the usual norm of \( L^p(\Omega) \) by \( | \cdot |_{L^p(\Omega)} \), and simply by \( | \cdot |_p \), if \( \Omega = \mathbb{R}^3 \).

Moreover, for any \( u \in \mathcal{D}_k \), the extension of \( u \) on \( \mathbb{R}^3 \) is defined by 
\[
\tilde{u}(x) = u(x) \text{ if } x \in B_k, \quad \tilde{u}(x) = 0 \text{ if } x \in B_k^c. \tag{1.8}
\]
We end this section by giving our main results. For the sake of simplicity, in many cases we just say $u$.

Definition 1.1 A pairs $(u, \phi) \in \mathcal{D}_V \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is said to be a (weak) solution of (1.1) if

$$I'(u)\varphi = \int_{\mathbb{R}^3} \nabla u \nabla \varphi + V(x)u \varphi dx + \lambda \int_{\mathbb{R}^3} \phi u \varphi dx - \int_{\mathbb{R}^3} |u|^{p+1} \varphi dx. \quad (1.10)$$

Moreover, if $u \in \mathcal{D}_V$ and $I'(u)\varphi = 0$ for all $\varphi \in \mathcal{D}_V$, Lemma 2.4 of [10] showed that $u$ and $\phi_u$ (simply by $\phi$, sometimes) satisfy (1.1) in the weak sense.

**Theorem 1.1** Let $p \in (1, 2)$ and (G1) to (G3) hold. Then there exists $\lambda_* \in (0, +\infty)$ and $C_{p, \lambda} = 2^{\frac{2}{p-2}} (2 - p)^{\frac{p-1}{2}} [p(p-1)\lambda^{-1}]^\frac{1}{p}$ such that problem (1.11) has one nontrivial solution $(u, \phi) \in \mathcal{D}_V \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ for any $\lambda \in (0, \lambda_*)$ and $\mu > \mu_1 = C_{p, \lambda}^{-1} - 1$. Moreover,

$$I(u) \in [\alpha, c_\lambda], \quad \|u\|_{\mathcal{D}_V} + |\nabla \phi|_2 \leq M_\mu := M(p, \lambda, \mu), \quad \text{and}$$

$$|u(x)| \leq C_{p, \lambda}, \quad 0 < \phi(x) \leq C_\mu := C(p, \lambda, \mu) \quad \text{a.e. in} \ x \in \mathbb{R}^3,$$

where $\alpha, c_\lambda > 0$ are independent of $\mu$, and $M_\mu$ is decreasing in $\mu$.

**Remark 1.1** We claim that $\lambda_* \leq c(p) = \frac{1}{4}(p - 1)^2(2 - p)^{\frac{2p-2}{p-1}}$. In fact, similar to the proof of [17, Theorem 4.1], we see that (1.11) has no any nontrivial solution in $\mathcal{D}_V \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ if $\lambda \geq c(p)$. Moreover, it is not difficult to check that $C_{p, \lambda} > 1$ if $\lambda \in (0, c(p))$. Hence $\mu_1 > 0$ in Theorem 1.1.

By Theorem 1.1 and Remark 1.1 we know that the existence of solutions of (1.11) for $p \in (1, 2)$ depends heavily on the parameter $\lambda$. However, if $p \in [3, 5)$, the situation is quite different and we have the following theorem.
Theorem 1.2 Let \( p \in [3, 5) \) and \((G_1)\) to \((G_3)\) hold. Then, for any \( \lambda > 0 \), there exist positive constants \( M_0 := M_0(\lambda), \) \( M_1 := M_1(\lambda) \) and \( M := M(\lambda) \), which are independent of \( \mu \) and nondecreasing in \( \lambda > 0 \), such that problem (1.1) has one nontrivial solution \((u, \phi) \in \mathcal{D}_V \times \mathcal{D}^{1,2}(\mathbb{R}^3)\) if \( \mu > \mu_2 = \max\{0, M_0(\lambda)^{p-1} - 1\} \), and \((u, \phi)\) satisfies
\[
|u(x)| \leq M_0 \text{ and } 0 < \phi(x) \leq M_1 \text{ a.e. in } x \in \mathbb{R}^3, \ ||u||_{\mathcal{D}_V} + |\nabla \phi|_2 \leq M.
\]
Moreover, there exist \( \alpha > 0 \) and \( c_\lambda > 0 \), independent of \( \mu \), such that \( I_\lambda(u) \in [\alpha, c_\lambda] \) where \( \alpha \) is also independent of \( \lambda > 0 \).

The following three theorems describe how the solutions of (1.1) behave when \( \lambda \to 0, \mu \to +\infty \) and \( |x| \to +\infty \), respectively.

**Theorem 1.3** Under the assumptions of Theorem 1.2. For each \( \lambda > 0 \) and \( \mu > \mu_2 \), let \( u_\lambda \) denote the solution of (1.1) obtained by Theorem 1.2. Then there exists a solution \( u_0 \in \mathcal{D}_V \setminus \{0\} \) for (1.1) with \( \lambda = 0 \) such that, along a subsequence,
\[ u_\lambda \to u_0 \text{ strongly in } \mathcal{D}_V \text{ as } \lambda \to 0. \]

**Theorem 1.4** For each \( \mu > 0 \) large, let \( u_\mu \) be a solution of (1.1) obtained by Theorem 1.1 or 1.2. Then, there is \( \tilde{u} \in H^1(\mathbb{R}^3) \) with \( \tilde{u}(x) = 0 \) a.e. in \( x \in \mathbb{R}^3 \setminus \Omega_0 \) and \( \tilde{u}(x) \not\equiv 0 \) in \( \Omega_0 \) such that, passing to a subsequence,
\[ u_\mu \to \tilde{u} \text{ in } H^1(\mathbb{R}^3) \text{ as } \mu \to +\infty. \]
Moreover, if \( \partial \Omega_0 \) is Lipschitz continuous, then \( \tilde{u} \in H^1_0(\Omega_0) \) and it is a weak solution of the following problem
\[
-\Delta u + u + \frac{\lambda}{|x|}(u^2 + \frac{1}{|x|})u = |u|^{p-1}u, \quad x \in \Omega_0, \quad u(x) = 0, \quad x \in \partial \Omega_0. \tag{1.12}
\]

**Theorem 1.5** For each \( \lambda \in (0, \lambda_*) \), let \( u_\mu \) be the solutions of (1.1) obtained by Theorem 1.1 or 1.2. If \( \mu > \mu_0 = \max\{3C_{p,\lambda}^{p-1} - 3, 3M_0^{p-1} - 3\} \), then there exist \( A > 0 \) and \( R_0 > 0 \) which are independent of \( \mu \) such that
\[ u_\mu(x) \leq A|x|^{-\frac{1}{2}}e^{-\frac{c}{|x|}(|x| - R_0)} \text{ for } |x| > R_0 \text{ and } \mu > \mu_0. \]

**Theorem 1.6** For \( \lambda_* \) and \( \mu_1 \) given by Theorem 1.1, let \( p \in (1, 2) \) and \((G_1)\) to \((G_3)\) hold. Then problem (1.1) has a ground state \((u, \phi) \in \mathcal{D}_V \times \mathcal{D}^{1,2}(\mathbb{R}^3)\) for each \( \lambda \in (0, \lambda_*) \) and \( \mu > \mu_1 \).
2  Nontrivial solution for (1.1) on $B_k$

For any $k \in \mathbb{N}$, we consider the following problem

$$\begin{cases}
-\Delta u + V(x)u + \lambda \phi(x)u = |u|^{p-1}u, & x \in B_k, \\
-\Delta \phi = u^2, & x \in B_k, \\
u(x) = \phi(x) = 0, & x \in \partial B_k.
\end{cases} \tag{2.1}$$

For $u \in \mathscr{D}_k$ and $\phi := \phi_u \in \mathscr{D}_{0,1}^1(B_k)$, define

$$I_k(u) := I_{\lambda,k}(u) \triangleq \frac{1}{2} \int_{B_k} |\nabla u|^2 + V(x)u^2 \, dx + \frac{\lambda}{4} \int_{B_k} \phi_u(x)u^2 \, dx - \frac{1}{p+1} \int_{B_k} |u|^{p+1} \, dx,$$\tag{2.2}

and $I_k \in C^1(\mathscr{D}_k, \mathbb{R})$, $(u, \phi_u) \in \mathscr{D}_k \times \mathscr{D}_{0,1}^1(B_k)$ is a weak solution of (2.1) if and only if $u$ is a nonzero critical point of $I_k(u)$. The main aim of this section is to prove that

**Theorem 2.1** Let $p \in (1,2) \cup [3,5)$ and $(G_1)$ to $(G_3)$ hold. Then there exist $\lambda_* > 0$ and $k_0 \in \mathbb{N}$ such that, for $k > k_0$, problem (2.1) has at least one nontrivial solution $(u_k, \phi_k) \in \mathscr{D}_k \times \mathscr{D}_{0,1}^1(B_k)$ with

$$I_k(u_k) = c_{\lambda,k} \in [\alpha, c_{\lambda}] \text{ for each } \lambda \in (0, \lambda_*),$$

where $\lambda_* = +\infty$ if $p \in [3,5)$ and $\lambda_* \in (0, +\infty)$ if $p \in (1,2)$, $\alpha$ and $c_{\lambda}$ are positive constants independent of $k$ and $\mu$ ( $\alpha$ is also independent of $\lambda$). Moreover, for $k > k_0$, $c_{\lambda_1,k} \leq c_{\lambda_2,k}$ if $\lambda_1 \leq \lambda_2$.

Before proving Theorem 2.1, we recall some results for the following Poisson equation on any smooth domain $\Omega \subset \mathbb{R}^3$, $\Omega$ may be unbounded,

$$\begin{cases}
-\Delta \phi(x) = u^2, & x \in \Omega, \\
\phi \in \mathscr{D}_{0,1}^1(\Omega).
\end{cases} \tag{2.3}$$

The following lemma is a result of Lax-Milgram theorem.

**Lemma 2.1** Let $u \in L^{12/5}(\Omega)$. Then (2.3) has a unique solution $\phi_u \in \mathscr{D}_{0,1}^1(\Omega)$ such that

$$|\nabla \phi_u|_{L^2(\Omega)} \leq S_0^{-1/2} |u|_{L^{12/5}(\Omega)}^{2/3}, \quad \int_{\Omega} \phi_u u^2(\omega) \, dx \leq S_0^{-1} |u|_{L^{12/5}(\Omega)}^{4}, \tag{2.4}$$

where $S_0 = \inf \{ \int_{\mathbb{R}^3} |\nabla u|^2 \, dx : u \in \mathscr{D}_{1,2}(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} |u|^6 \, dx = 1 \}$ is the Sobolev constant, which is independent of $\Omega$ and $u$. 

6
Lemma 2.2 ([17] Lemma 2.1) For \( \{ u_n \} \subset L^{12/5}(\Omega) \) with \( u_n \rightharpoonup u \) strongly in \( L^{12/5}(\Omega) \), let \( \phi_n \) and \( \phi \) be the unique solutions of (2.3) corresponding to \( u_n \) and \( u \), respectively. Then,

\[
\phi_n \rightharpoonup \phi \text{ strongly in } D^{1,2}_0(\Omega) \quad \text{and} \quad \int_{\Omega} \phi_n u_n^2 dx \rightharpoonup \int_{\Omega} \phi u^2 dx.
\]

Lemma 2.3 Let \( \{ u_k \} \subset L^{12/5}(\mathbb{R}^3) \) be such that \( u_k \rightharpoonup u \) strongly in \( L^{12/5}(\mathbb{R}^3) \) for some \( u \in L^{12/5}(\mathbb{R}^3) \), and \( \phi_k \in D^{1,2}_0(B_k) \) be the unique solution of (2.3) with \( \Omega = B_k \) and \( u = u_k \). If \( \tilde{\phi}_k \) is the extension of \( \phi_k \) on \( \mathbb{R}^3 \) defined as (1.8), then

\[
\tilde{\phi}_k(x) \xrightarrow{k} \phi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \text{ strongly in } D^{1,2}(\mathbb{R}^3),
\]

where \( \phi \) is essentially the unique solution of (2.3) with \( \Omega = \mathbb{R}^3 \).

Proof. For \( u \in L^{12/5}(\mathbb{R}^3) \), by Lemma 2.1 there is a unique \( \phi \in D^{1,2}(\mathbb{R}^3) \) such that

\[
\int_{\mathbb{R}^3} \nabla \phi \nabla \varphi dx = \int_{\mathbb{R}^3} u^2 \varphi dx, \quad \text{for any } \varphi \in D^{1,2}(\mathbb{R}^3). \tag{2.5}
\]

Moreover, \( \phi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \) by Theorem 9.9 of [14]. By Lemma 2.1 there is \( C > 0 \) independent of \( B_k \) and \( u_k \) such that

\[
|\nabla \tilde{\phi}_k|_2 \leq \left\{ \int_{B_k} |\nabla \phi_k|^2 dx \right\}^{1/2} \leq C |u_k|_{L^{12/5}(B_k)} \leq C |u_k|_{L^{12/5}(\mathbb{R}^3)}^{2/5},
\]

this implies that \( \{ \tilde{\phi}_k \} \) is bounded in \( D^{1,2}(\mathbb{R}^3) \), since \( u_k \rightharpoonup u \) strongly in \( L^{12/5}(\mathbb{R}^3) \) and \( \{ u_k \} \) is bounded in \( L^{12/5}(\mathbb{R}^3) \). Hence, there exists \( \tilde{\phi} \in D^{1,2}(\mathbb{R}^3) \) such that \( \tilde{\phi}_k \rightharpoonup \tilde{\phi} \) weakly in \( D^{1,2}(\mathbb{R}^3) \), that is,

\[
\int_{\mathbb{R}^3} \nabla \tilde{\phi}_k \nabla \varphi dx \rightharpoonup \int_{\mathbb{R}^3} \nabla \tilde{\phi} \nabla \varphi dx, \quad \text{for any } \varphi \in D^{1,2}(\mathbb{R}^3). \tag{2.6}
\]

By the definition of \( \tilde{\phi}_k \), (2.6) yields

\[
\int_{B_k} \nabla \phi_k \nabla \varphi dx = \int_{\mathbb{R}^3} \nabla \tilde{\phi}_k \nabla \varphi dx \rightharpoonup \int_{\mathbb{R}^3} \nabla \tilde{\phi} \nabla \varphi dx, \quad \text{for any } \varphi \in C^\infty_0(\mathbb{R}^3). \tag{2.7}
\]
For any \( \phi \in C_0^\infty(\mathbb{R}^3) \), since \( B_k \xrightarrow[k \to \infty]{} \mathbb{R}^3 \) we see that \( \text{supp} \phi \subset B_k \) for all \( k \in \mathbb{N} \) large. Noting that \( \phi_k \) satisfies (2.3) with \( u = u_k \) and \( \Omega = B_k \), it follows from \( u_k \xrightarrow[k \to \infty]{} u \) that
\[
\int_{B_k} \nabla \phi_k \nabla \phi dx = \int_{B_k} u_k^2 \phi dx + o(1) = \int_{\mathbb{R}^3} u_k^2 \phi dx + o(1) = \int_{\mathbb{R}^3} u^2 \phi dx + o(1),
\]
where and in what follows \( o(1) \) denotes the quantity which goes to 0 as \( k \to +\infty \). Therefore, combining (2.7) (2.8) and (2.5), we see that
\[
\int_{\mathbb{R}^3} \nabla \tilde{\phi}_k \nabla \phi dx = \int_{\mathbb{R}^3} u_k^2 \phi dx = \int_{\mathbb{R}^3} | \nabla \phi |^2 dx.
\]
On the other hand,
\[
\int_{\mathbb{R}^3} | \nabla \tilde{\phi}_k |^2 dx = \int_{B_k} | \nabla \phi_k |^2 dx = \int_{B_k} u_k^2 \phi_k dx = \int_{\mathbb{R}^3} u_k^2 \phi_k dx.
\]
By \( u_k \xrightarrow[k \to \infty]{} u \) strongly in \( L^{12/5}(\mathbb{R}^3) \), it follows from (2.9) that
\[
\int_{\mathbb{R}^3} | \nabla \tilde{\phi}_k |^2 dx \xrightarrow[k \to \infty]{} \int_{\mathbb{R}^3} | \nabla \phi |^2 dx,
\]
this implies that \( \tilde{\phi}_k \xrightarrow[k \to \infty]{} \phi \) in \( \mathcal{D}^{1,2}(\mathbb{R}^3) \). \( \square \)

Now, we are ready to prove Theorem 2.1. For using Mountain Pass Theorem, the following properties for \( I_k \) defined by (2.2) are required.

\textbf{Lemma 2.4} Let \( p \in (1, 2) \cup [3, 5) \) and (G1) to (G2) hold. We have that
\begin{itemize}
  \item[(i)] There are \( \rho > 0 \) and \( \alpha > 0 \) (independent of \( k \), \( \lambda \) and \( \mu \)) such that
    \[
    \inf_{u \in \mathcal{D}_k: \|u\|_k = \rho} I_{\lambda,k}(u) \geq \alpha > 0, \text{ for all } \lambda > 0, k \in \mathbb{N}.
    \]
  \item[(ii)] There exist \( \lambda_* \in (0, +\infty) \) if \( p \in (1, 2) \) or \( \lambda_* = +\infty \) if \( p \in [3, 5) \) and \( e \in \mathcal{D}^{1,2}(\mathbb{R}^3) \) such that \( \text{supp} e \subset B_k \), \( \|e\|_k > \rho \) and
    \[
    I_{\lambda,k}(e) < 0, \text{ for } \lambda \in (0, \lambda_*) \text{ and } k \in \mathbb{N} \text{ large},
    \]
    where \( e \) is independent of \( \lambda \) only if \( p \in (1, 2) \).
\end{itemize}
(iii) There is a constant $c_\lambda > 0$, independent of $k$ and $\mu$, such that

$$c_{\lambda,k} := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_{\lambda,k}(\gamma(t)) \leq c_\lambda < +\infty,$$

where $\Gamma_\lambda = \{ \gamma \in C([0,1], \mathcal{D}_k) : \gamma(0) = 0, \|\gamma(1)\| > \rho, I_{\lambda,k}(\gamma(1)) < 0 \}$. Moreover, for any fixed $k \in \mathbb{N}$, $0 < \alpha \leq c_{\lambda_1,k} \leq c_{\lambda_2,k}$ if $\lambda_1 \leq \lambda_2$.

Proof. (i) For $q \in (2,6)$, let

$$S = \inf \{ \int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx : u \in H^1(\mathbb{R}^3) \text{ and } |u|_q = 1 \},$$

which is independent of $k$ and $\mu$. For any $u \in \mathcal{D}_k \subset H^1_0(B_k)$, define $\tilde{u}$ as (1.8) and $\tilde{\mu} \in \mathcal{D}_V \subset H^1(\mathbb{R}^3)$. Then, Sobolev embedding implies that

$$\int_{B_k} |u|^{p+1} dx = \int_{\mathbb{R}^3} |\tilde{u}|^{p+1} dx \leq S^{-\frac{p+1}{2}} \|\tilde{u}\|_{H^1(\mathbb{R}^3)}^{p+1} \leq S^{-\frac{p+1}{2}} \|\tilde{u}\|_{H^1_0(\mathbb{R}^3)}^{p+1} = S^{-\frac{p+1}{2}} \|u\|_k^{p+1}.$$

Since $\lambda > 0$ and $\phi_u > 0$, it follows from (2.2) that

$$I_{\lambda,k}(u) \geq \frac{1}{2} \|u\|_k^2 - \frac{1}{p+1} S^{-\frac{p+1}{2}} \|u\|_k^{p+1}.$$

By $p > 1$, it is not difficult to see that there exist positive constants $\rho$, $\alpha$ (independent of $k$, $\lambda$ and $\mu$) such that (i) holds.

(ii) We separate the proof into two cases: $p \in (1,2)$ and $p \in [3,5)$.

If $p \in (1,2)$, let $w \in C_0^\infty(\mathbb{R}^3) \setminus \{0\}$ with supp\(w\) $\subset \Omega_0$. Since $\Omega_0 \subset B_k$ for $k \in \mathbb{N}$ large, there is $k_0 \in \mathbb{N}$ such that, for all $k > k_0$ and $t > 0$,

$$I_{0,k}(tw) \equiv I_{\lambda,k}(tw)|_{\lambda = 0} = \frac{1}{2} \int_{\Omega_0} |\nabla w|^2 + w^2 dx - \frac{1}{p+1} \int_{\Omega_0} |w|^{p+1} dx. \quad (2.10)$$

Clearly, there is $t_0 > 0$ large enough such that $e = t_0 w$ satisfying $I_{0,k}(e) < 0$ and $\|e\|_k = t_0 \|w\|_{H^1_0(\Omega_0)} > \rho$, here $e$ is independent of $k$, $\lambda$ and $\mu$. Since $I_{\lambda,k}(e) = I_{0,k}(e) + \frac{1}{4} \int_{\Omega_0} \phi ke^2 dx$ is continuous in $\lambda \geq 0$, combining $I_{0,k}(e) < 0$ and Lemma 2.1 we see that there exists $\lambda_* > 0$ small (independent of $k$ and $\mu$) such that $I_{\lambda,k}(e) < 0$ for all $\lambda \in (0, \lambda_*)$.

If $p \in [3,5)$, the above proof for $p \in (1,2)$ still works but by that way $\lambda$ has to be small. However, for $p \in [3,5)$, we can prove that (ii) holds for all $\lambda > 0$, that is $\lambda_* = +\infty$. In fact, by int$\Omega_0 \neq \emptyset$ we may assume that $B_{\varepsilon_0}(x_0) = \{ x \in \mathbb{R}^3 : |x - x_0| < \varepsilon_0 \} \subset \Omega_0$ for some $x_0 \in \Omega_0$ and $\varepsilon_0 > 0$. Taking $w \in C_0^\infty(\mathbb{R}^3)$ with supp\(w\) $\subset B_{\varepsilon_0}(0)$ and letting $w_t(x) = w(x - t)$. 

9
\[\hat{\Omega} = \{ x \mid \lambda \in \Omega \}\] since \( \lambda, k \in \mathbb{N} \) large enough. Since \( g(x) \equiv 0 \) on \( \Omega \), for \( t > 1 \), we have

\[
I_{\lambda,k}(w_t) = \frac{1}{2} \int_{\Omega_0} |\nabla w_t|^2 + w_t^2 \, dx + \frac{\lambda}{4} \int_{\Omega_0} \phi_n w_t^2 \, dx - \frac{1}{p+1} \int_{\Omega_0} |w_t|^{p+1} \, dx
\]

\[
= \frac{t^2}{2} \int_{\Omega_0} |\nabla w|^2 \, dx + \frac{t}{2} \int_{\Omega_0} w^2 \, dx + \frac{\lambda t^3}{4} \int_{\Omega_0} \phi_n w^2 \, dx - \frac{t^{2p-1}}{p+1} \int_{\Omega_0} |w|^{p+1} \, dx.
\]

Since \( p > 2 \), \( 2p - 1 > 3 \), we see that, for each \( \lambda > 0 \), there exists \( t_0 := t_0(\lambda) > 1 \) large (independent of \( k \) and \( \mu \)) such that

\[
I_{\lambda,k}(w_{t_0}) < 0 \quad \text{and} \quad \|w_{t_0}\|^2 = \frac{\lambda t_0^3}{2} \int_{\Omega_0} |\nabla w|^2 \, dx + \frac{t_0}{2} \int_{\Omega_0} w^2 \, dx > \rho^2.
\]

Hence (ii) is proved by taking \( e := e_\lambda \triangleq w_{t_0} \) (independent of \( k \) and \( \mu \)).

(iii) By the result of part (i), it is obvious that \( c_{\lambda,k} \geq \alpha > 0 \).

For \( p \in (1,2) \cup [3,5) \), by part (ii), there always exists \( e \in \mathcal{D}_k \) such that \( I_{\lambda,k}(e) < 0 \). Let \( \gamma_0(t) = te \) for \( t \in [0,1] \). Then \( \gamma_0(t) \in \Gamma_{\lambda} \) and

\[
c_{\lambda,k} \leq \max_{t \in [0,1]} I_{\lambda,k}(\gamma_0(t)) = \max_{t \in [0,1]} I_{\lambda,k}(te) \triangleq c_\lambda < +\infty.
\]

Note that \( \text{supp} \, e \subset \Omega_0 \) and \( e \) is independent of \( k \) and \( \mu \), so \( c_\lambda \) is also independent of \( k \) and \( \mu \). For each \( k > k_0 \) and for any \( u \in \mathcal{D}_k \), (2.2) implies that \( I_{\lambda_1,k}(u) \leq I_{\lambda_2,k}(u) \) if \( \lambda_1 \leq \lambda_2 \), then the definition of \( \Gamma_\lambda \) yields that \( \Gamma_{\lambda_2} \subseteq \Gamma_{\lambda_1} \). Therefore

\[
c_{\lambda_1,k} = \inf_{\gamma \in \Gamma_{\lambda_1}} \max_{t \in [0,1]} I_{\lambda_1,k}(\gamma(t)) \leq \inf_{\gamma \in \Gamma_{\lambda_2}} \max_{t \in [0,1]} I_{\lambda_2,k}(\gamma(t)) = c_{\lambda_2,k}.
\]

Proof of Theorem 2.1 Since \( I_k(0) = 0 \), it follows from Lemma 2.4 and the mountain pass theorem that, for each \( k \in \mathbb{N} \) large and \( \lambda \in (0,\lambda_* \) such that

\[
I_k(u_n) \xrightarrow{n \to +\infty} c_{\lambda,k} \quad \text{and} \quad I_k'(u_n) \xrightarrow{n \to +\infty} 0 \quad \text{in} \quad \mathcal{D}_k^*,
\]

where \( \mathcal{D}_k^* \) is the dual space of \( \mathcal{D}_k \). Hence, as \( n \to +\infty \),

\[
\frac{1}{2} \int_{B_k} |\nabla u_n|^2 + V_\mu(x) u_n^2 \, dx + \frac{\lambda}{4} \int_{B_k} \phi_n u_n^2 \, dx - \frac{1}{p+1} \int_{B_k} |u_n|^{p+1} \, dx = c_{\lambda,k} + o(1),
\]

\[
I'_k(u_n) = \int_{B_k} |\nabla u_n|^2 + V_\mu(x) u_n^2 \, dx + \lambda \int_{B_k} \phi_n u_n^2 \, dx - \int_{B_k} |u_n|^{p+1} \, dx, \tag{2.11}
\]

10
\[
\int_{B_k} \nabla u_n \nabla \varphi + V_\mu(x) u_n \varphi dx + \lambda \int_{B_k} \phi_n u_n \varphi dx - \int_{B_k} |u_n|^{p-1} u_n \varphi dx = o(1),
\]
for any \( \varphi \in \mathcal{D}_k \), where \( \phi_n \) denotes the unique solution of Poisson equation 

\(-\Delta \phi_n = u_n^2 \) in \( \mathcal{D}_k \). For each \( k \in \mathbb{N} \) large, if \( \{u_n\} = \{u_n^{(k)}\} \) is bounded in \( \mathcal{D}_k \), then there exists \( u_k \in \mathcal{D}_k \) such that, passing to a subsequence,

\[ u_n \rightharpoonup u_k \text{ weakly in } \mathcal{D}_k \text{ and strongly in } L^q(B_k) \] for \( q \in (1, 6) \),

since the embedding \( \mathcal{D}_k \subset H^{1,1}_0(B_k) \hookrightarrow L^q(B_k) \) is compact. Hence, Lemma 2.2 follows that \[ u \rightarrow 0 \text{ in } \mathcal{D}_k \text{ strongly and } \int_{B_k} \phi_n u_n^2 dx \rightarrow \int_{B_k} \phi u_k^2 dx \text{,} \]

Based on these facts, it is not difficult to see that \( u_n \rightharpoonup u_k \) strongly in \( \mathcal{D}_k \) by (2.12) and (2.13). Hence, \( I_k(u_k) = c_{\lambda,k} \in [\alpha, c_\lambda] \) by Lemma 2.4 and \( u_k \neq 0 \)

is a weak solution of (2.1). Therefore, to prove Theorem 2.1 we need only to show that \( \{u_n\} \) is bounded in \( \mathcal{D}_k \).

If \( p \in [3, 5) \), then \( \frac{1}{4} - \frac{1}{p+1} \geq 0 \). Since \( I_k'(u_n) \rightharpoonup 0 \) in \( \mathcal{D}_k^* \), subtracted \( \frac{1}{4} \times (2.12) \) from (2.11) which gives

\[ ||u_n||_k^2 \leq 8c_k + 2, \text{ for } n \text{ large enough,} \]

hence \( \{u_n\} \) is bounded in \( \mathcal{D}_k \). However, if \( p \in (1, 2) \), the properties of \( I_k(u) \) change greatly. In this case, we have to use other trick to get the boundedness of \( \{u_n\} \) in \( \mathcal{D}_k \). By \(-\Delta \phi_n = u_n^2 \) in \( \mathcal{D}_k^{1,2} \), we have that

\[
\sqrt{\lambda} \int_{B_k} |u_n|^3 dx = \sqrt{\lambda} \int_{B_k} \nabla \varphi_n \nabla |u_n| dx \leq \frac{1}{2} \int_{B_k} |\nabla u_n|^2 dx + \frac{\lambda}{2} \int_{B_k} |\nabla \phi_n|^2 dx
= \frac{1}{2} \int_{B_k} |\nabla u_n|^2 dx + \frac{\lambda}{2} \int_{B_k} \phi_n |u_n|^2 dx.
\]

Since \( I_k'(u_n) \rightharpoonup 0 \) in \( \mathcal{D}_k^* \), we may assume that \( ||I_k'(u_n)||_{\mathcal{D}_k^*} \leq 1 \) for \( n \) large.

Then, by (2.12) and (2.15) we see that, for \( n \in \mathbb{N} \) large,

\[
\frac{1}{2} \int_{B_k} |\nabla u_n|^2 + V_\mu u_n^2 dx + \sqrt{\lambda} \int_{B_k} |u_n|^3 dx \leq \int_{B_k} |u_n|^{p+1} dx + ||u_n||_k.
\]

If \( p \in (1, 2) \) and the Young’s inequality yields that

\[
\int_{B_k} |u_n|^{p+1} dx \leq \int_{B_k} \left[ \frac{(p+1)\varepsilon}{3} |u_n|^3 + \frac{2 - p}{3} \varepsilon^{-\frac{p+1}{p}} \right] dx
= \sqrt{\lambda} \int_{B_k} |u_n|^3 dx + \frac{2 - p}{3} \left( \frac{3\sqrt{\lambda}}{p+1} \right)^{\frac{p+1}{2-p}} |B_k|,
\]

by taking \( \varepsilon = \frac{3\sqrt{\lambda}}{p+1} \).
This and (2.16) imply that

\[
\frac{1}{2} \|u_n\|_k^2 \leq \|u_n\|_k + \frac{2 - p}{3} \left( \frac{3\sqrt{\lambda}}{p + 1} \right)^{-\frac{p+1}{2-p}} |B_k|, \text{ for } n \text{ large.}
\]

So, \( \{u_n\} = \{u_n^{(k)}\} \) is bounded in \( \mathcal{D}_k \) for each \( k \in \mathbb{N} \) large. □

3 \( L^\infty \)-norm priori estimates for solutions of problem (2.1)

The aim of this section is to establish priori estimates of \( L^\infty \)-norm for solutions of (1.1) and (2.1). If \( p \in (1, 2) \), Lemma 3.1 shows that the \( L^\infty \)-norm of solutions of (1.1) or (2.1) is bounded above by a constant depending only on \( \lambda \) and \( p \). If \( p \in [3, 5) \), similar to (2.14), we know that the solution \( u_k \) of (2.1) is uniformly bounded in \( \mathcal{D}_k \) for all \( k \in \mathbb{N} \) and so does in \( L^6(B_k) \), then in Lemma 3.3 we establish an estimate of \( |u_k|_{L^\infty(B_k)} \) by using \( |u_k|_{L^6(B_k)} \). These priori estimates are important in showing that the solution \( u_k \) of (2.1) is uniformly bounded with respect to the domain \( B_k \) and in showing that \( \tilde{u}_k \), the extension of \( u_k \) on \( \mathbb{R}^3 \), converges to a nontrivial solution of (1.1).

Lemma 3.1 For \( p \in (1, 2) \) and \( \Omega = B_k \) or \( \Omega = \mathbb{R}^3 \), let \( (u, \phi) \in H^1_0(\Omega) \times \mathcal{D}^{1,2}_0(\Omega) \) be a weak solution of the following problem

\[
\begin{cases}
-\Delta u + V(x)u + \lambda \phi(x)u = |u|^{p-1}u, & x \in \Omega, \\
-\Delta \phi = u^2, & x \in \Omega,
\end{cases}
\]

(3.1)

where \( \lambda > 0 \), \( V(x) \in L^\infty(\Omega) \) and \( V(x) \geq 1 \). Then

\[
|u(x)| \leq c_p \phi(x) \quad \text{and} \quad |u(x)| \leq C_{p,\lambda}, \quad \text{a.e. in } x \in \Omega,
\]

where \( c_p = (p - 1)(2 - p)^{\frac{2-p}{2-p}} \) and \( C_{p,\lambda} = \frac{1}{2} (2 - p) \left( \frac{pr}{2} \right)^{\frac{p}{2-p}} \).

Proof: By assumption, \( (u, \phi) \in H^1_0(\Omega) \times \mathcal{D}^{1,2}_0(\Omega) \) is a weak solution of (3.1), then, for any \( v \in H^1_0(\Omega) \), we have

\[
\int_\Omega \nabla u \nabla vdx + \int_\Omega V(x)uvdx + \lambda \int_\Omega \phi(x)uvdx - \int_\Omega |u|^{p-1}uvdx = 0, \quad (3.2)
\]

\[
\int_\Omega \nabla \phi \nabla vdx = \int_\Omega u^2vdx. \quad (3.3)
\]
Adding $c_p \int_\Omega u^2 v dx$ on both sides of (3.2), and using (3.3) we get that
\[
\int_\Omega \nabla u \nabla v dx + \int_\Omega [V(x)u + c_p u^2 - |u|^{p-1}u]v dx + \lambda \int_\Omega \phi(x)uv dx = c_p \int_\Omega \nabla \phi \nabla v dx,
\] for any $v \in H^1_0(\Omega)$. (3.4)

Based on this observation, we prove our lemma by the following two cases.

**Cases 1: $\Omega = B_k$.** We let
\[
w_1(x) = (u(x) - c_p \phi(x))^+ \quad \text{and} \quad \Omega_1 = \{x \in \Omega : w_1(x) > 0\}. \quad (3.5)
\]
then $w_1 \in H^1_0(\Omega)$ and $u(x)|_{\Omega_1} \geq c_p \phi(x) > 0$. Taking $v(x) = w_1(x)$ in (3.4), and using $V(x) \geq 1$, we see that
\[
\int_{\Omega_1} \nabla u \nabla w_1 dx + \int_{\Omega_1} [u + c_p u^2 - |u|^{p-1}u]w_1 dx \leq c_p \int_{\Omega_1} \nabla \phi \nabla w_1 dx. \quad (3.6)
\]
However, for all $t \geq 0$ we have $t + c_p t^2 - t^p \geq 0$ if $p \in (1,2)$ and $c_p = (p - 1)(2-p)\frac{2}{p-1}$. Then, (3.6) implies that $\int_{\Omega_1} \nabla u \nabla w_1 dx - c_p \int_{\Omega_1} \nabla \phi \nabla w_1 dx \leq 0$, that is,
\[
\int_{\Omega_1} \nabla (u - c_p \phi) \nabla w_1 dx = \int_{\Omega_1} |\nabla w_1|^2 dx = 0. \quad (3.7)
\]
Hence, $|\Omega_1| = 0$ or $w_1|_{\Omega_1} \equiv$ constant. By the definition of $\Omega_1$ and $w_1 \equiv 0$ in $\Omega \setminus \Omega_1$, then $u(x) \leq c_p \phi(x)$ a.e. in $\Omega$. On the other hand, replacing $u$ by $-u$ and repeating the above procedure, we see that $-u(x) \leq c_p \phi(x)$. Therefore
\[
|u(x)| \leq c_p \phi(x) \quad \text{a.e. in } x \in \Omega. \quad (3.8)
\]
To prove that $|u(x)| \leq C_{p,\lambda}$ a.e. in $x \in \Omega$, we let
\[
w_2(x) = (u(x) - C_{p,\lambda})^+ \quad \text{and} \quad \Omega_2 = \{x \in \Omega : w_2(x) > 0\}, \quad (3.9)
\]
then $w_2 \in H^1_0(\Omega)$ and $u(x)|_{\Omega_2} \geq C_{p,\lambda} > 0$. Taking $v(x) = w_2(x)$ in (3.2) and using (3.8) and $V(x) \geq 1$, it follows that
\[
\int_{\Omega_2} \nabla u \nabla w_2 + uw_2 dx \leq \int_{\Omega_2} |u|^{p-1}uw_2 - \frac{\lambda}{c_p} u^2 w_2 dx \leq \int_{\Omega_2} C_{p,\lambda} w_2 dx,
\]
here we used the fact that $\max_{t \geq 0} \{t^p - \frac{\lambda}{c_p} t^2\} = C_{p,\lambda}$ if $p \in (1,2)$. This yields
\[
\int_{\Omega_2} \nabla (u - C_{p,\lambda})^+ |^2 + |(u - C_{p,\lambda})^+|^2 dx = \int_{\Omega_2} \nabla [u - C_{p,\lambda}] \nabla w_2 + |u - C_{p,\lambda}| w_2 dx \leq 0,
\]
it follows that $|\Omega_2| = 0$, then $u(x) \leq C_{p,\lambda}$ a.e in $x \in \Omega$. Similarly, $-u(x) \leq C_{p,\lambda}$. Therefore

$$|u(x)| \leq C_{p,\lambda}, \text{ a.e. in } x \in \Omega. \quad (3.10)$$

**Case 2:** $\Omega = \mathbb{R}^3$. In this case, it is not sure if $w_1(x)$ given by (3.5) is in $H^1(\mathbb{R}^3)$, so we need to replace $w_1(x)$ in (3.5) and the correspondence by

$$w_\epsilon(x) = (u(x) - c_p \phi(x) - \epsilon)^+ \text{ for any } \epsilon > 0.$$

We claim that $w_\epsilon(x) \in H^1(\mathbb{R}^3)$ for any $\epsilon > 0$. In fact, by the second equation of (3.1) and Theorem 8.17 in [14], we know that

$$|\phi|_{L^\infty(B_1(y))} \leq C \left( |\phi|_{L^6(B_2(y))} + |u|^2_{L^p(B_3(y))} \right), \text{ for each } y \in \mathbb{R}^3,$$

where $C$ is a constant independent of $y \in \mathbb{R}^3$. This and $\phi \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ imply that $\phi(x) \xrightarrow{|x| \to +\infty} 0$. For any $y \in \mathbb{R}^3$, taking $\varphi \in C_0^\infty(B_3(y))$, it follows from (3.1) that

$$\int_{B_3(y)} \nabla u \nabla \varphi + b(x)u \varphi dx = \int_{B_3(y)} c(x) \varphi dx,$$

where $b(x) = V(x) + \lambda \phi(x)$ and $c(x) = |u|^{p-1}u(x)$. Clearly, $c(x) \in L^\infty_5(\mathbb{R}^3)$ and $b(x) \in L^\infty(B_{R_1}^c)$ for $R_1 > 0$ large enough since $\phi(x) \xrightarrow{|x| \to +\infty} 0$. For each $|y| > R_1 + 3$, since $p < 2$ and $\frac{6}{p} > \frac{3}{2}$, Theorem 8.17 in [14] implies that

$$|u|_{L^\infty(B_1(y))} \leq C \left( |u|_{L^2(B_2(y))} + |c(x)|_{L^\infty_5(B_3(y))} \right),$$

where $C > 0$ depends only on $p$ and $|b(x)|_{L^\infty(B_{R_1}^c)}$. This implies that

$$u(x) \to 0 \text{ as } |x| \to \infty. \quad (3.11)$$

Hence, there is $R_\epsilon > 0$ such that $\text{supp} w_\epsilon \subset B_{R_\epsilon}$, $w_\epsilon(x) \in H^1_0(B_{R_\epsilon})$ and $w_\epsilon(x) \in H^1(\mathbb{R}^3)$. Noting that $\nabla(u - c_p \phi) = \nabla(u - c_p \phi - \epsilon)$ for any $\epsilon > 0$, we see that (3.7) holds for $w_\epsilon$, then Poincare inequality implies that

$$\int_{\Omega_1} |w_\epsilon|^2 dx \leq C \int_{\Omega_1} |\nabla w_\epsilon|^2 dx = 0 \text{ since } w_\epsilon(x) \in H^1_0(B_{R_\epsilon}) \text{ and } w_\epsilon(x) \equiv 0 \text{ on } B_{R_\epsilon} \setminus \Omega_1. \text{ So, } u(x) \leq c_p \phi(x) + \epsilon \text{ a.e. in } \mathbb{R}^3 \text{ and (3.8) is obtained by letting } \epsilon \to 0.$$

By (3.11), there is $R_0 > 0$ such that $\text{supp}(u(x) - C_{p,\lambda})^+ \subset B_{R_0}$ and $w_2(x)$ given by (3.9) is in $H^1(\mathbb{R}^3)$, then exactly the same as in Case 1 we get (3.10). \(\square\)

To get the $L^\infty$-norm estimate of solutions to problem (2.1) for $p \in [3,5)$, we need the following general $L^\infty$-norm estimate for functions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ by using its $L^2^*$ norm, where $2^* = \frac{2N}{N-2}$.  

14
Lemma 3.2 Let $N \geq 3$, $p \in (1, \frac{N+2}{N-2})$ and let $u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \setminus \{0\}$ be a nonnegative function such that

$$
\int_{\mathbb{R}^N} \nabla u \nabla (h(u)\varphi) \, dx \leq \int_{\mathbb{R}^N} |u|^{p-1}u h(u)\varphi \, dx, \tag{3.12}
$$

holds for any nonnegative $\varphi \in C_0^\infty(\mathbb{R}^N)$ and any nonnegative piecewise smooth function $h$ on $[0, +\infty)$ with $h(0) = 0$ and $h' \in L^\infty(\mathbb{R}^3)$. Then, $u \in L^\infty(\mathbb{R}^N)$ and there exist $C_1 > 0$ and $C_2 > 0$, which depend only on $N$ and $p$, such that

$$
|u|_{\infty} \leq C_1 \left(1 + |u|_{2^*}^{C_2}\right) |u|_{2^*}. 
$$

Proof: The main idea of the proof is Moser iterations, which is somehow standard. For the sake of completeness, we give a proof in the appendix based on [19] and [13]. □

Lemma 3.3 For each $k \in \mathbb{N}$ large, let $p \in [3, 5)$ and $u_k$ be a solution of problem (2.1) given by Theorem 2.1. Then there exist constants $M_0(\lambda) > 0$ and $M(\lambda) > 0$, independent of $k$ and $\mu$, such that

$$
|u_k(x)| \leq M_0(\lambda) \text{ a.e. in } x \in B_k \text{ and } \|u_k\|_k + |\nabla \phi_k|_{L^2(B_k)} \leq M(\lambda). 
$$

Moreover, $M_0(\lambda)$ and $M(\lambda)$ are non-decreasing in $\lambda > 0$.

Proof. By Theorem 2.1 for any $k \in \mathbb{N}$ large, saying $k > k_0 \in \mathbb{N}$, there exist $u_k$ and $c_\lambda > 0$ such that

$$
I_k(u_k) = c_{\lambda,k} \leq c_\lambda \text{ and } I_k'(u_k) = 0 \text{ in } \mathcal{D}_k^*. \tag{3.13}
$$

Similar to the derivation of (2.14), we have

$$
|\nabla u_k|_{L^2(B_k)} \leq \|u_k\|_k \leq 2\sqrt{c_{\lambda,k}}, \tag{3.14}
$$

this and (2.4) yield

$$
|\nabla \phi_k|_{L^2(B_k)}^2 \leq C|u_k|_{L^4(B_k)}^4 \leq C\|u_k\|_k^4 \leq 16C\lambda^2, \tag{3.15}
$$

where $C > 0$ is independent of $k$, $\lambda$ and $\mu$. Let $M(\lambda) = \sup_{k > k_0, \mu > 0} \{2\sqrt{c_{\lambda,k}} + 4\sqrt{C}c_{\lambda,k}\}$, then $M(\lambda)$ is finite for each $\lambda > 0$ since $c_{k,\lambda} \leq c_\lambda$. Moreover, $M(\lambda)$ is non-decreasing in $\lambda$ since $c_{\lambda,k}$ is non-decreasing in $\lambda$ by Theorem...
Hence (3.14) and (3.15) imply that \( \|u_k\|_k + |\nabla \varphi_k|_{L^2(B_k)} \leq M(\lambda) \).
Define \( \tilde{u}_k \) as in (1.8), then \( \tilde{u}_k \in \mathcal{D}_V \) and supp\( \tilde{u}_k \subseteq B_k \), hence (3.14) yields
\[
\|\tilde{u}_k\|_{\mathcal{D}_V} \leq 2\sqrt{c_{\lambda,k}}.
\] (3.16)
For any \( \varphi \in C^\infty_0(\mathbb{R}^3) \) and any nonnegative piecewise smooth function \( h \) with \( h' \in L^\infty(\mathbb{R}^3) \) and \( h(0) = 0 \), let \( v = h(\tilde{u}_k^+)\varphi \), then \( v \in \mathcal{D}_k \) and \( I_k^\prime(u_k)v = 0 \), this yields
\[
\int_{\mathbb{R}^3} \nabla \tilde{u}_k^+ \nabla v + V_\mu(x)\tilde{u}_k^+ vdx + \lambda \int_{\mathbb{R}^3} \tilde{\varphi}_k \tilde{u}_k^+ vdx = \int_{\mathbb{R}^3} |\tilde{u}_k^+|^{p-1} \tilde{u}_k^+ vdx.
\]
Since \( \tilde{\varphi}_k \), \( V_\mu \) and \( \tilde{u}_k^+ \) are nonnegative, this implies that (3.12) holds for \( u = \tilde{u}_k^+ \) and \( N = 3 \). Then, by Lemma 3.2 and (3.16) there exist \( C_1 > 0 \) and \( C_2 > 0 \), depend only on \( p \), such that
\[
|\tilde{u}_k^+|_\infty \leq C_1(1 + |\tilde{u}_k^+|_{L^2}^2)|\tilde{u}_k^+|_{2^*} \leq \sup_{k > k_0, \mu > 0} \{ C_1(1 + e_{\lambda,k}^2) \} := M_0(\lambda).
\]
Similarly, we know that \( |\tilde{u}_k^-|_\infty \leq M_0(\lambda) \). So, \( |u_k|_{L^\infty(B_k)} \leq |\tilde{u}_k|_\infty \leq M_0(\lambda) \).
Moreover, \( M_0(\lambda) \) is nondecreasing in \( \lambda > 0 \) since \( c_{\lambda,k} \) is nondecreasing in \( \lambda \).
\( \square \)

4 Proofs of Theorems 1.1 and 1.2

For \( r > 0 \), let \( \xi_r \in C^\infty(\mathbb{R}^3) \) such that
\[
\xi_r(x) = \begin{cases} 1, & |x| > \frac{r}{4}, \\ 0, & |x| < \frac{r}{4}, \end{cases} \quad \text{with} \quad |\nabla \xi_r| \leq \frac{8}{r}. \quad (4.1)
\]
To prove our Theorems, we need the following two lemmas.

Lemma 4.1 Assume (G_1) (G_3) hold. Let \( p \in (1, 2) \), \( u \in \mathcal{D}_V \) and \( \phi \in L^6(\mathbb{R}^3) \) be such that \( |u|_\infty \leq L \) and \( |u(x)| \leq K\phi(x) \) a.e. in \( x \in \mathbb{R}^3 \), for some \( L > 0 \) and \( K > 0 \). Moreover, for all \( \eta \in C_B^1(\mathbb{R}^3) \) with \( \eta \geq 0 \), there holds
\[
\int_{\mathbb{R}^3} \nabla u \nabla (u\eta) + V_\mu(x)u^2\eta dx + \lambda \int_{\mathbb{R}^3} \phi u^2\eta dx = \int_{\mathbb{R}^3} |u|^{p+1}\eta dx, \quad (4.2)
\]
where \( \lambda > 0 \). Then, if \( \mu > \mu = \max\{0, L^{p-1} - 1\} \), there exists \( M_\mu > 0 \) (depends on \( L, K, p, \lambda, \mu \)) which is decreasing in \( \mu > \mu \) such that
\[
\|u\|_{\mathcal{D}_V} + \left\{ \int_{\mathbb{R}^3} \phi u^2 dx \right\}^{\frac{1}{2}} \leq M_\mu.
\]
Proof: For $\xi_\tau$, given by (4.1), taking $\eta = \xi_\tau$ in (4.2), it gives that

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + V_\mu(x)u^2) \xi_\tau dx + \lambda \int_{\mathbb{R}^3} \phi u^2 \xi_\tau dx = \int_{\mathbb{R}^3} |u|^{p+1} \xi_\tau dx - \int_{\mathbb{R}^3} u \nabla u \nabla \xi_\tau dx,$$

then, by $|u|_\infty \leq L$ and (4.1) we have

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + (1 + \mu g(x))u^2) \xi_\tau dx + \lambda \int_{\mathbb{R}^3} \phi u^2 \xi_\tau dx \leq L^{p-1} \int_{\mathbb{R}^3} u^2 \xi_\tau dx + \frac{8}{r} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx,$$

that is,

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + [1 + \mu g(x) - L^{p-1}]u^2) \xi_\tau dx \leq \frac{8}{r} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx. \quad (4.3)$$

Since $\mu > \bar{\mu} \geq L^{p-1} - 1$, for each $\tau \in (\bar{\mu}, \mu)$, there exists $\epsilon_\tau > 0$ such that $\tau(1 - \epsilon_\tau) > L^{p-1} - 1$. By (G3) we can find $R_\tau > 0$ such that $g(x) \geq 1 - \epsilon_\tau$ for all $|x| \geq R_\tau$, then $1 + \mu g(x) > 1 + \tau(1 - \epsilon_\tau) > L^{p-1}$ for all $|x| \geq R_\tau$ and there is $\delta_\tau > 0$ such that $1 + \mu g(x) > L^{p-1} + \delta_\tau$ for all $|x| \geq R_\tau$. So, if $\mu > \bar{\mu}$ and $r/4 > R_\tau$, it follows from (4.3) that, for each $\tau \in (\bar{\mu}, \mu)$,

$$\int_{|x| > \frac{r}{2}} (|\nabla u|^2 + \delta_\tau u^2) dx \leq \frac{8}{r} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx. \quad (4.4)$$

On the other hand, taking $\eta \equiv 1$ in (4.2), it gives

$$\int_{\mathbb{R}^3} |\nabla u|^2 + (1 + \mu g(x))u^2 dx \leq \int_{\mathbb{R}^3} L^{p-1} u^2 - \frac{\lambda}{K} u^3 dx. \quad (4.5)$$

Let $\beta(t) = L^{p-1} t^2 - \frac{\lambda}{K} t^3 (t \geq 0)$, then there exists $C^* := C(L, K, p, \lambda) \in (0, +\infty)$ such that $\beta(t) \leq C^* < +\infty$ for all $t \geq 0$. Therefore, if $r > 4R_\tau$ and $\mu > \bar{\mu}$, it follows from (4.4) and (4.5) that

$$\int_{\mathbb{R}^3} |\nabla u|^2 + (1 + \mu g(x))u^2 dx \leq \int_{|x| < \frac{r}{2}} \beta(u) dx + \int_{|x| \geq \frac{r}{2}} \beta(u) dx \leq \int_{|x| < \frac{r}{2}} C^* dx + L^{p-1} \int_{|x| \geq \frac{r}{2}} u^2 dx \leq C^* |B_{\frac{r}{2}}(0)| + L^{p-1} \frac{8}{r \delta_\tau} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx.$$ 

Take $r_\tau > 4R_\tau$ large enough such that $L^{p-1} \frac{8}{r \delta_\tau} < \frac{1}{7}$, then

$$\int_{\mathbb{R}^3} |\nabla u|^2 + (1 + \mu g(x))u^2 dx \leq 2C^* |B_{\frac{r}{2}}|.$$
By Sobolev embedding, $|u|_{p+1}$ is also bounded above by a constant depending on $C^*$ and $\tau$. This and (4.2) imply that there exists $M_\tau := M(L, K, p, \lambda, \tau)$ such that

$$\|u\|_{D^V} + \left\{ \int_{\mathbb{R}^3} \phi u^2 dx \right\}^{\frac{1}{2}} \leq M_\tau,$$

for each $\tau \in (\bar{\mu}, \mu)$. Let $M_\mu = \inf_{\tau \in (\bar{\mu}, \mu)} M_\tau$, then $M_\mu$ is decreasing in $\mu > \bar{\mu}$ and depends only on $L, K, p, \lambda, \mu$ such that

$$\|u\|_{D^V} + \left\{ \int_{\mathbb{R}^3} \phi u^2 dx \right\}^{\frac{1}{2}} \leq M_\mu. \quad \square$$

**Lemma 4.2** If $p > 1$, and (G1) (G3) hold, let $\{u_k\}$ be bounded in $D^V$ such that $|u_k|_{\infty} \leq M$ (for some $M > 0$) and

$$\int_{\mathbb{R}^3} \nabla u_k \nabla (u_k \eta) + V_\mu(x) u_k^2 \eta dx \leq \int_{\mathbb{R}^3} \|u_k\|_{p+1} \eta dx, \quad (4.6)$$

for all $\eta \in C^1_0(\mathbb{R}^3)$ with $\eta \geq 0$. Then, for each $\mu > \max\{0, \bar{M}p^{-1} - 1\}$, there exists $u \in D^V$ such that, passing to a subsequence,

$$|u_k - u|_q \xrightarrow{k} 0 \quad \text{for } q \in [2, 6). \quad (4.7)$$

**Proof:** Since $\{u_k\}$ is bounded in $D^V$, there exists $u \in D^V$ such that, passing to a subsequence,

$$u_k \xrightarrow{k} u \quad \text{weakly in } D^V, \quad u_k(x) \xrightarrow{k} u(x) \quad \text{a.e. in } x \in \mathbb{R}^3. \quad (4.8)$$

For $\xi_r$, given by (4.1), taking $\eta = \xi_r$ in (4.6),

$$\int_{\mathbb{R}^3} \nabla u_k \nabla (u_k \xi_r) + V_\mu(x) u_k^2 \xi_r dx \leq \int_{\mathbb{R}^3} \|u_k\|_{p+1} \xi_r dx.$$

Since $\mu > \max\{0, \bar{M}p^{-1} - 1\}$, $|u_k|_{\infty} \leq \bar{M}$ and (G3) holds, similar to the discussion of (4.4), there exists $R_\mu > 0$ and $\delta_\mu > 0$ such that, for all $R > 4R_\mu$, we have

$$\int_{|x| > R} (\nabla u_k)^2 + \delta_\mu u_k^2 dx < \frac{T}{R} \quad \text{uniformly for } k \in \mathbb{N} \text{ large}, \quad (4.9)$$
where \( T = \sup_{k \in \mathbb{N}} \|u_k\|_{\mathcal{D}_V} \). Since \( H^1(B_R) \hookrightarrow L^q(B_R) \) is compact for \( 1 \leq q < 6 \), passing to a subsequence, (4.8) implies that
\[
\phi_D \to u(x) \quad \text{in} \quad L^q(B_R) \quad \text{for} \quad 1 \leq q < 6.
\] (4.10)

For \( k \in \mathbb{N} \) large and any \( R > 4R_{\mu} \) large enough, we have
\[
\begin{align*}
|u_k - u|_q &\leq |u_k - u|_{L^q(B_R)} + |u_k - u|_{L^q(B_R^c)} \\
&\leq |u_k - u|_{L^q(B_R)} + |u_k - u|_{L^{\frac{6-q}{2}}(B_R^c)} \\
&\leq |u_k - u|_{L^q(B_R)} + C\|u_k - u\|_{H^1(R^3)} \left( |u|_{L^6(B_R)}^2 + |u_k|_{L^6(B_R^c)}^{\frac{6-q}{2}} \right) \\
&\leq |u_k - u|_{L^q(B_R)} + C \left( |u|_{L^6(B_R^c)}^2 + (T/R)^{\frac{6-q}{2}} \right) \quad \text{by (4.9)}.
\end{align*}
\]
By letting \( k \to +\infty \), then \( R \to +\infty \), we get (4.7). \( \square \)

**Lemma 4.3** Assume \( p \in (1,5) \). If \((G_1)\) \((G_3)\) hold and \( \{u_k\} \) is bounded in \( \mathcal{D}_V \) satisfying
\[
\|u_k\|_{\mathcal{D}_V} \leq \bar{M}, \quad \text{for some} \quad \bar{M} > 0.
\]
Let \( \phi_k \) be the solution of \(-\Delta \phi = u_k^2 \) in \( \mathcal{D}_V^{1,2}(B_k) \) and \( \tilde{\phi}_k \) be the extension of \( \phi_k \) on \( \mathbb{R}^3 \) defined as (1.8) such that, for all \( \eta \in C^1_B(\mathbb{R}^3) \) with \( \eta \geq 0 \),
\[
\int_{\mathbb{R}^3} \nabla u_k \nabla (u_k \eta) + V_\mu(x)u_k^2 \eta dx + \lambda \int_{\mathbb{R}^3} \tilde{\phi}_k u_k^2 \eta dx = \int_{\mathbb{R}^3} |u_k|^{p+1} \eta dx, \tag{4.11}
\]
where \( \lambda > 0 \). Then, for each \( \mu > \max\{0, M^{-1} - 1\} \), there exists \( u \in \mathcal{D}_V \) such that, passing to a subsequence,
\[
\|u_k - u\|_{\mathcal{D}_V} \xrightarrow{k} 0 \quad \text{and} \quad |\nabla (\tilde{\phi}_k - \phi)|_2 \xrightarrow{k} 0 \quad \text{with} \quad \phi = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|} dy. \tag{4.12}
\]

**Proof:** Since \( \lambda > 0 \) and \( \tilde{\phi}_k \geq 0 \), (4.11) implies (4.6), for each \( \mu > \max\{0, M^{-1} - 1\} \) applying Lemma 4.2 we have \( u \in \mathcal{D}_V \) such that,
\[
|u_k - u|_q \xrightarrow{k} 0 \quad \text{for} \quad q \in [2,6), \tag{4.13}
\]
This and Lemma 2.3 imply that, passing to a subsequence,
\[
\tilde{\phi}_k \xrightarrow{k} \phi = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|} dy \quad \text{strongly in} \quad \mathcal{D}_V^{1,2}(\mathbb{R}^3).
\]
Hence,
\[ \int_{\mathbb{R}^3} \tilde{\phi}_k u_k^2 dx \xrightarrow[k \to \infty]{} \int_{\mathbb{R}^3} \phi u^2 dx. \quad (4.14) \]
Then (4.12) follows from (4.13), (4.14) and (4.11) with $\eta \equiv 1$. □

Now, we are ready to prove Theorems 1.1 and 1.2. For $u_k \in \mathcal{D}_k$, $\phi_k \in \mathcal{D}_0^{1,2}(B_k)$ and $\alpha > 0$, $c_{\alpha} > 0$ given by Theorem 2.1, let $\tilde{u}_k \in \mathcal{D}_V$ and $\tilde{\phi}_k \in \mathcal{D}_1^{1,2}(\mathbb{R}^3)$ be the extensions of $u_k$ and $\phi_k$ on $\mathbb{R}^3$ defined as (1.3) respectively. Then,
\[ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \tilde{u}_k|^2 + V_\mu(x) \tilde{u}_k^2 dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} \tilde{\phi}_k(x) \tilde{u}_k^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |\tilde{u}_k|^p+1 dx = c_{\lambda,k}, \quad (4.15) \]
and $c_{\lambda,k} \in [\alpha, c_\alpha]$. Moreover, for any $\varphi \in \mathcal{D}_V$ with supp$\varphi \subseteq \bar{B}_k$, by $I_k^p(u_k) = 0$ we see that
\[ \int_{\mathbb{R}^3} \nabla \tilde{u}_k \nabla \varphi + V_\mu(x) \tilde{u}_k \varphi dx + \lambda \int_{\mathbb{R}^3} \tilde{\phi}_k \tilde{u}_k \varphi dx = \int_{\mathbb{R}^3} |\tilde{u}_k|^{p-1} \tilde{u}_k \varphi dx. \quad (4.16) \]
Note that for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, supp$\varphi \subseteq \bar{B}_k$ as $k \to +\infty$, therefore,
\[ \int_{\mathbb{R}^3} \nabla \tilde{u}_k \nabla \varphi + V_\mu(x) \tilde{u}_k \varphi dx + \lambda \int_{\mathbb{R}^3} \tilde{\phi}_k \tilde{u}_k \varphi dx = \int_{\mathbb{R}^3} |\tilde{u}_k|^{p-1} \tilde{u}_k \varphi dx + o(1). \quad (4.17) \]

**Proof of Theorem 1.1.** Let $\lambda_* \in (0, \lambda)$ be given by Theorem 2.1 and $\lambda_* \to +\infty$ if $p \in (1, 2)$. For $\lambda \in (0, \lambda_*)$, let $(u_k, \phi_k) \in \mathcal{D}_k \times \mathcal{D}_0^{1,2}(B_k)$ be a solution of (2.1) for each $k \in \mathbb{N}$ large. Applying Lemma 3.1 with $\Omega = B_k$, there exist $c_p > 0$ and $C_{p,\lambda} > 0$ such that
\[ |u_k(x)| \leq c_p \phi_k(x) \quad \text{and} \quad |u_k(x)| \leq C_{p,\lambda}, \; \text{a.e. in } x \in B_k. \]
Then,
\[ |\tilde{u}_k(x)| \leq c_p \tilde{\phi}_k(x) \quad \text{and} \quad |\tilde{u}_k(x)| \leq C_{p,\lambda}, \; \text{a.e. in } x \in \mathbb{R}^3. \]
Since $\tilde{u}_k \eta \in \mathcal{D}_k$ for any $\eta \in C_0^1(\mathbb{R}^3)$, it follows from $I_k'(u_k) = 0$ that (4.2) holds for $u = \tilde{u}_k$ and $\phi = \tilde{\phi}_k$, applying Lemma 4.1 with $L = C_{p,\lambda}$, $K = c_p$ and $u = \tilde{u}_k$, there is $M_{\mu} := M(p, \lambda, \mu) > 0$, which is decreasing in $\mu > \mu_1$, such that
\[ \|\tilde{u}_k\|_{\mathcal{D}_V} + |\nabla \tilde{\phi}_k|_2 = \|\tilde{u}_k\|_{\mathcal{D}_V} + \left\{ \int_{\mathbb{R}^3} \tilde{\phi}_k \tilde{u}_k^2 dx \right\}^{\frac{1}{2}} \leq M_{\mu}. \]

20
Similarly, \( \{\tilde{u}_k\} \) and \( \{\tilde{\phi}_k\} \) satisfy (4.11) by \( I'_k(u_k) = 0 \). Hence, Lemma 4.3 shows that there exists \( u \in \mathcal{D}_V \) such that, passing to a subsequence

\[
\tilde{u}_k \xrightarrow{k} u \quad \text{strongly in} \quad \mathcal{D}_V \quad \text{and} \quad \tilde{\phi}_k \xrightarrow{k} \phi_u \quad \text{strongly in} \quad \mathcal{D}^{1,2}(\mathbb{R}^3).
\]

(4.18)

Furthermore,

\[
\|u\|_{\mathcal{D}_V}^{50} + |\nabla \phi_u|_2 \leq M_\mu \quad \text{and} \quad |u|_\infty \leq C_{p, \lambda}.
\]

(4.19)

Combining (4.15) and (4.18), we have \( I(u) \in [\alpha, c_\lambda] \). On the other hand, by (4.17), for any \( \varphi \in C_0^\infty(\mathbb{R}^3) \), as \( k \to +\infty \), \( I'(u_k)\varphi = o(1) \), then, using (4.18) we see that

\[
I'(u)\varphi = 0.
\]

So, \( u \) is a nontrivial solution of (1.1) in \( \mathcal{D}_V \). By the second equation of (1.1) and Theorem 8.17 in [14], for any \( y \in \mathbb{R}^3 \), we have

\[
\sup_{B_1(y)} \phi_u \leq C(\|\phi_u\|_{L^6(B_2(y))} + |u|^2_{L^6(B_4(y))}),
\]

where \( C > 0 \) is a constant independent of \( u \) and \( y \in \mathbb{R}^3 \). This and (4.19) show that

\[
\sup_{x \in \mathbb{R}^3} \phi_u(x) \leq CM_\mu(1 + M_\mu) \triangleq C(p, \lambda, \mu) =: C_\mu,
\]

(4.20)

where \( C_\mu > 0 \) is a constant dependent on \( p, \lambda, \mu \). □

Proof of Theorem 1.2: For \( p \in [3, 5] \) and \( \lambda > 0 \), let \((u_k, \phi_k) \in \mathcal{D}_k \times \mathcal{D}^{1,2}(B_k)\) be a solution given by Theorem 2.1. By Lemma 3.3, we know that

\[
|u_k|_\infty \leq M_0(\lambda), \quad \|u_k\|_k + |\nabla \phi_k|_{L^2(B_k)} \leq M(\lambda),
\]

where \( M_0(\lambda) \) and \( M(\lambda) \) are given by Lemma 3.3. Hence,

\[
|\tilde{u}_k|_\infty \leq M_0(\lambda), \quad \|\tilde{u}_k\|_{\mathcal{D}_V} + |\nabla \tilde{\phi}_k|_2 \leq M(\lambda).
\]

Applying Lemma 4.3 with \( u_k = \tilde{u}_k \), there exists \( u \in \mathcal{D}_V \) such that

\[
\tilde{u}_k \xrightarrow{k} u \quad \text{in} \quad \mathcal{D}_V.
\]

(4.21)

Moreover,

\[
\|u\|_{\mathcal{D}_V}^{50} + |\nabla \phi_u|_2 \leq M(\lambda) \quad \text{and} \quad |u|_\infty \leq M_0(\lambda).
\]

(4.22)

Hence, (4.21) and (4.15) show that \( I(u) \in [\alpha, c_\lambda] \), where \( \alpha \) and \( c_\lambda \) are constants given by Theorem 2.1. On the other hand, by (4.17), for any \( \varphi \in C_0^\infty(\mathbb{R}^3) \), as \( k \to +\infty \), \( I'(u_k)\varphi = o(1) \), and it follows from (4.21) that
\( I'(u)\varphi = 0 \). So, \( u \) is a nontrivial solution of (1.1) in \( \mathcal{D}_V \). Noting (4.22), similar to the discussion of (4.20), we have

\[
\sup_{x \in \mathbb{R}^3} \phi_u(x) \leq CM(\lambda)(1 + M(\lambda)) =: M_1(\lambda),
\]

where \( M_1(\lambda) \) is a constant and independent of \( \mu \). By Lemma 3.3, \( M(\lambda) \) is non-decreasing in \( \lambda > 0 \), hence \( M_1(\lambda) \) is also non-decreasing in \( \lambda \). □

5 Proofs of Theorems 1.3 to 1.5

Proof of Theorem 1.3. By Theorem 1.2, if \( \mu > \mu_2(\lambda) \), problem (1.1) has always a solution \( u_\lambda \in \mathcal{D}_V \) for any \( \lambda \in (0, 1) \) such that

\[
\| u_\lambda \|_{\mathcal{D}_V} + |\nabla u_\lambda|_2 \leq M(\lambda) \leq M(1) \quad \text{and} \quad |u_\lambda|_\infty \leq M_0(\lambda) \leq M_0(1),
\]

since \( M_0(\lambda) \) and \( M(\lambda) \) are nondecreasing in \( \lambda > 0 \). Then, there exists \( u_0 \in \mathcal{D}_V \), passing to a subsequence, such that

\[
u_\lambda \rightharpoonup u_0 \quad \text{weakly in} \quad \mathcal{D}_V \quad \text{and} \quad u_\lambda \rightarrow u_0 \quad \text{a.e. in} \quad x \in \mathbb{R}^3.
\]

(5.1)

For any \( \varphi \in \mathcal{D}_V \), \( \int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda \varphi dx \leq |\phi_{u_\lambda}|_{C_0} |u_\lambda|_{L^2} |\varphi|_{L^2} \leq C(M(1)) \) and

\[
\int_{\mathbb{R}^3} \nabla u_\lambda \nabla \varphi + V(\mu(x))u_\lambda \varphi dx + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda \varphi dx = \int_{\mathbb{R}^3} |u_\lambda|^{p-1} u_\lambda \varphi dx,
\]

(5.2)

let \( \lambda \rightarrow 0 \), we have

\[
\int_{\mathbb{R}^3} \nabla u_0 \nabla \varphi + V(\mu(x))u_0 \varphi dx = \int_{\mathbb{R}^3} |u_0|^{p-1} u_0 \varphi dx,
\]

(5.3)

that is, \( u_0 \) is a weak solution of (1.1) with \( \lambda = 0 \). Since \( u_\lambda \eta \in \mathcal{D}_V \) for any \( \eta \in C^1_0(\mathbb{R}^3) \) with \( \eta \geq 0 \), by (5.2) and \( \lambda > 0 \), we see that

\[
\int_{\mathbb{R}^3} \nabla u_\lambda \nabla (u_\lambda \eta) + V(\mu(x))u_\lambda^2 \eta dx \leq \int_{\mathbb{R}^3} |u_\lambda|^{p+1} \eta dx.
\]

By Lemma 4.2, there is \( u^* \in \mathcal{D}_V \) such that \( u_\lambda \rightharpoonup u^* \) in \( L^q(\mathbb{R}^3) \) for \( q \in [2, 6) \). Hence \( u^* = u_0 \) by (5.1). Let \( \varphi = u_\lambda \) in (5.2) and \( \varphi = u_0 \) in (5.3), since \( \int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 dx = |\nabla u_\lambda|_2^2 \leq M^2(1) \), as \( \lambda \rightarrow 0 \), we have \( u_\lambda \rightharpoonup u_0 \) in \( \mathcal{D}_V \). This and \( I(u_\lambda) \in [\alpha, c_\lambda] \) for \( \lambda > 0 \) imply that \( I_0(u_0) \geq \alpha > 0 \), hence \( u_0 \neq 0 \). □

Proof of Theorem 1.4. To make it clear that \( \mathcal{D}_V \) depends on \( \mu \), here
we denote \( \mathcal{D}_V \) by \( \mathcal{D}_\mu \). Let \( \mu_0 > \max\{C_{p,\lambda}^{p-1} - 1, M_0^{p-1} - 1\} \) and \( \mu_0 > 0 \) by Remark 1.1. For \( \mu \geq \mu_0 \), let \( u_\mu \) be the solution given by Theorem 1.1 or Theorem 1.2 then \( \{u_\mu\} \) is bounded in \( D_{\mu_0} \) since \( M_\mu \leq M_{\mu_0} \). For some \( \bar{u} \in \mathcal{D}_{\mu_0} \subset H^1(\mathbb{R}^3) \) we may assume that

\[
u_\mu \to \bar{u} \text{ weakly in } \mathcal{D}_{\mu_0} \text{ and } u_\mu \to \bar{u} \text{ a.e. in } x \in \mathbb{R}^3, \text{ as } \mu \to +\infty. \tag{5.4}
\]

Then, Fatou’s lemma shows that

\[
\int_{\mathbb{R}^3} g(x)\bar{u}^2 \, dx \leq \lim_{\mu \to +\infty} \int_{\mathbb{R}^3} g(x)u_\mu^2 \, dx \leq \lim_{\mu \to +\infty} \frac{1}{\mu} \|u_\mu\|^2_{\mathcal{D}_\mu} \leq \lim_{\mu \to +\infty} \frac{M_{\mu_0}^2}{\mu} = 0,
\]

so (G_1) and (G_2) means that \( \bar{u}(x) = 0 \) a.e. in \( x \in \mathbb{R}^3 \setminus \Omega_0 \). On the other hand, for any \( \varphi \in \mathcal{D}_\mu \), by \( I'(u_\mu) = 0 \) we have

\[
\int_{\mathbb{R}^3} \nabla u_\mu \nabla \varphi + (1 + \mu g(x))u_\mu \varphi \, dx + \lambda \int_{\mathbb{R}^3} \varphi u_\mu u_\mu \varphi \, dx = \int_{\mathbb{R}^3} |u_\mu|^{p-1}u_\mu \varphi \, dx. \tag{5.5}
\]

For each \( \mu \geq \mu_0 \) and \( \eta \in C^1_B(\mathbb{R}^3) \) with \( \eta \geq 0 \), it follows from (5.5) and (G_1) that

\[
\int_{\mathbb{R}^3} \nabla u_\mu \nabla (u_\mu \eta) + (1 + \mu_0 g(x))u_\mu^2 \eta \, dx \leq \int_{\mathbb{R}^3} |u_\mu|^{p+1} \eta \, dx.
\]

Then, it follows from (5.4), Theorem 1.1 or 1.2 and Lemma 4.2 with \( \mathcal{D}_V = \mathcal{D}_{\mu_0} \) and \( \{u_k\} = \{u_\mu\} \) that

\[
|u_\mu - \bar{u}|_q \xrightarrow{\mu \to +\infty} 0 \text{ for } q \in [2,6), \tag{5.6}
\]

hence, Lemma 2.2 yields that

\[
\int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu^2 (x) \, dx \xrightarrow{\mu \to +\infty} \int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u}^2 (x) \, dx. \tag{5.7}
\]

Let \( \varphi \in C_0^\infty(\mathbb{R}^3) \), by (5.4) and (5.7) (see (3.18) in [15] for the details), we have

\[
\int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu (x) \varphi (x) \, dx \xrightarrow{\mu \to +\infty} \int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u} (x) \varphi (x) \, dx, \tag{5.8}
\]

and (5.8) holds also for \( \varphi \in H^1(\mathbb{R}^3) \) by the density of \( C_0^\infty(\mathbb{R}^3) \) in \( H^1(\mathbb{R}^3) \). Let \( \varphi = \bar{u} \) in (5.5) and in (5.8), then, by (5.4) and (5.6) we have

\[
\int_{\mathbb{R}^3} |\nabla \bar{u}|^2 + \bar{u}^2 \, dx + \lambda \int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u}^2 \, dx = \int_{\mathbb{R}^3} |\bar{u}|^{p+1} \, dx. \tag{5.9}
\]
By (5.6) and (5.5) with $\varphi = u_\mu$, we see that

$$
\int_{\mathbb{R}^3} |\nabla u_\mu|^2 + u_\mu^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu^2(x) dx \leq \int_{\mathbb{R}^3} |u_\mu|^{p+1} dx \xrightarrow{\mu \to +\infty} \int_{\mathbb{R}^3} |\bar{u}|^{p+1} dx.
$$

(5.10)

It follows from (5.9) and (5.10) that

$$
\lim_{\mu \to +\infty} \int_{\mathbb{R}^3} |\nabla u_\mu|^2 + u_\mu^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{u_\mu} u_\mu^2 dx \leq \int_{\mathbb{R}^3} |\nabla \bar{u}|^2 + \bar{u}^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u}^2 dx,
$$

this and (5.7) show that

$$
\lim_{\mu \to +\infty} \int_{\mathbb{R}^3} |\nabla u_\mu|^2 + u_\mu^2 dx \leq \int_{\mathbb{R}^3} |\nabla \bar{u}|^2 + \bar{u}^2 dx,
$$

and the lower semi-continuity of norm implies that

$$
u_\mu \to \bar{u}, \ \text{in } H^1(\mathbb{R}^3) \text{ as } \mu \to +\infty.
$$

(5.11)

Now, we claim that $\bar{u}(x) \not\equiv 0$ on $\Omega_0$. Otherwise, $\bar{u} = 0$ a.e. in $x \in \mathbb{R}^3$, then $|u_\mu|^{p+1} \xrightarrow{\mu \to +\infty} 0$ and $\int \phi_{u_\mu} u_\mu^2 dx \xrightarrow{\mu \to +\infty} 0$ by (5.7), hence $I(u_\mu) \to 0$ as $\mu \to +\infty$, which contradicts the fact that $I(u_\mu) \geq \alpha > 0$. Since $\partial \Omega_0$ is Lipschitz continuous, we have $\bar{u} \in H^1_0(\Omega_0)$ by $\bar{u}(x) = 0$, a.e. in $\mathbb{R}^3 \setminus \Omega_0$. It follows from (5.11), (5.8) and (5.5) that, for any $\varphi \in H^1_0(\Omega_0)$,

$$
\int_{\Omega_0} \nabla \bar{u} \nabla \varphi + \bar{u} \varphi dx + \frac{\lambda}{4\pi} \int_{\Omega \times \Omega} \frac{\bar{u}^2(y) \bar{u}(x) \varphi(x)}{|x - y|} dy dx = \int_{\Omega_0} |\bar{u}|^{p-1} \bar{u} \varphi dx,
$$

that is $\bar{u}$ is a weak solution of (1.12). $\square$

**Proof of Theorem 1.5**: This proof is motivated by that of Theorem 1.3 in [5]. Let $\mu_0 = \max\{3(M_0^{p-1} - 1), 3C_{p,\lambda}^{p-1} - 1\}$. By Theorem 1.1 or 1.2, the $L^\infty$-norms of $u_\mu$ and $\phi_{u_\mu}$ are bounded uniformly in $\mu \geq \mu_0$. By (G3), there exists $R_0 > 0$ such that $g(x) > \frac{\mu}{2}$ for $|x| \geq R_0$. Then, we have

$$
(-\Delta + \mu)u_\mu^2 = -2(W_\mu - \frac{H}{2})u_\mu^2 - 2|\nabla u_\mu|^2 \leq 0 \text{ for } |x| > R_0 \text{ and } \mu > \mu_0,
$$

(5.12)

where $W_\mu(x) = 1 + \mu g(x) + \lambda \phi_{u_\mu}(x) - |u_\mu(x)|^{p-1}$. On the other hand, let $w_\mu$ be the fundamental solution of $-\Delta + \mu$, then, it follows from Proposition 3.1 and Theorem 4.2 in [8] that $w_\mu$ is $C^\infty$ outside the origin and it is positive such that

$$
(-\Delta + \mu)w_\mu = 0 \text{ for } |x| > R_0 \text{ and }
$$

$$
w_\mu = |x|^{-\frac{(N-1)}{2}} e^{-\sqrt{\mu} |x|} (1 + o(1)) \text{ as } |x| \to +\infty, \text{ uniformly in } \mu \geq \mu_0.
$$

(5.13)
Let \( C_0 = \max\{M_0, C_{p,\lambda}\} \), \( \tilde{A} \geq \frac{C_0^2}{w_\mu(R_0)} \) and \( v(x) = u_\mu^2(x) - \tilde{A}w_\mu \). For \( \mu > \mu_0 \), it follows from (5.12) and (5.13) that
\[
\begin{cases}
(-\Delta + \mu)v \leq 0, & |x| > R_0, \\
v \leq 0, & |x| = R_0.
\end{cases}
\]
The maximum principle (Theorem 8.1 in \cite{14}) implies that \( u_\mu^2(x) \leq \tilde{A}w_\mu(x) \) for \( |x| > R_0 \).
By (5.14), there exists \( A > 0 \) independent of \( \mu > \mu_0 \) such that
\[
|u_\mu(x)| \leq A|x|^{-\frac{1}{2}}e^{-\frac{\sqrt{\mu}}{2}|x|-R_0} \quad \text{for } |x| > R_0 \text{ and } \mu > \mu_0.
\]

6 Ground state of (1.1) for \( p \in (1, 2) \)

**Proposition 6.1** Let \( \mathcal{N} = \{u \in \mathcal{D}_V \setminus \{0\} : I'(u) = 0 \text{ in } \mathcal{D}_V^*\} \) and \( \lambda_\ast \) be given by Theorem 1.1, \( C_{p,\lambda} \) be given by Lemma 3.1. Then, for each \( \lambda \in (0, \lambda_\ast) \) and \( \mu > C_{p,\lambda} - 1 \),

(i) \( \mathcal{N} \neq \emptyset \), and there exists \( C_p > 0 \) such that \( \|u\|_{\mathcal{D}_V} \geq C_p \) for all \( u \in \mathcal{N} \).

(ii) There exists \( C_{\lambda,\mu} > 0 \) such that \( \sup_{u \in \mathcal{N}} (\|u\|_{\mathcal{D}_V} + |\nabla \phi_u|_2) \leq C_{\lambda,\mu} \).

(iii) \( \mathcal{N} \subset \mathcal{D}_V \) is a compact set.

**Proof.** (i). Since \( \lambda \in (0, \lambda_\ast) \) and \( \mu > C_{p,\lambda} - 1 \), Theorem 1.1 implies that \( \mathcal{N} \neq \emptyset \). If \( u \in \mathcal{N} \), then \( I'(u) = 0 \) and \( u \neq 0 \), that is,
\[
\int_{\mathbb{R}^3} |\nabla u|^2 + V_\mu u^2 \, dx + \lambda \int_{\mathbb{R}^3} \phi_u u^2 \, dx = \int_{\mathbb{R}^3} |u|^{p+1} \, dx.
\]
Hence, \( \|u\|_{\mathcal{D}_V}^2 \leq \int_{\mathbb{R}^3} |u|^{p+1} \, dx \leq S^{p+1}\|u\|_{\mathcal{D}_V}^{p+1} \) and \( \|u\|_{\mathcal{D}_V} \geq S^{-\frac{p+1}{p-1}} := C_p \).

(ii). Since \( \|\cdot\|_{\mathcal{D}_V} \) and \( \|\cdot\|_{H^1(\mathbb{R}^3)} \) are equivalent norms in \( \mathcal{D}_V \), by Lemma 3.1 with \( \Omega = \mathbb{R}^3 \), we have that \( \sup_{u \in \mathcal{N}} |u|_\infty \leq C_{p,\lambda} \) and \( |u(x)| \leq c_p \phi_u(x) \) a.e.
in \( x \in \mathbb{R}^3 \). For \( u \in \mathcal{N} \), \( I'(u) = 0 \) and (1.2) holds. Then Lemma 4.1 implies that
\[
\|u\|_{\mathcal{D}_V} + |\nabla \phi_u|_2 = \|u\|_{\mathcal{D}_V} + \left\{ \int_{\mathbb{R}^3} \phi_u u^2 \, dx \right\}^{\frac{1}{2}} < C_{\lambda,\mu}, \quad \text{for some } C_{\lambda,\mu} > 0.
\]
(iii). Let \( \{ u_n \} \subset \mathcal{N} \), then \( I'(u_n) = 0 \) and (4.16) holds for \( u_n \). By part (ii) and Lemma 4.2, there exists \( u \in \mathcal{D}_V \) such that \( u_n \rightarrow u \) strongly in \( L^q(\mathbb{R}^3) \). This and Lemma 2.2 show that \( \phi_{u_n} \rightarrow \phi_u \) strongly in \( \mathcal{D}^{1,2}(\mathbb{R}^3) \). Thus, for any \( \varphi \in \mathcal{D}_V \),

\[
\int_{\mathbb{R}^3} \nabla u_n \nabla \varphi + V_\mu u_n \varphi dx \rightarrow \int_{\mathbb{R}^3} \nabla u \nabla \varphi + V_\mu u \varphi dx,
\]

\[
\int_{\mathbb{R}^3} \phi_{u_n} u_n \varphi dx \rightarrow \int_{\mathbb{R}^3} \phi_u u \varphi dx \text{ and } \int_{\mathbb{R}^3} |u_n|^{p-1} u_n \varphi dx \rightarrow \int_{\mathbb{R}^3} |u|^{p-1} u \varphi dx.
\]

It follows from \( I'(u_n) = 0 \) that \( I'(u) = 0 \) in \( \mathcal{D}_V^* \). Then, using \( I'(u_n) u_n = I'(u) u \) we know that \( u_n \rightarrow u \) strongly in \( \mathcal{D}_V \), this and (i) show that \( \| u \|_{\mathcal{D}_V} \geq C_\rho \), hence \( u \in \mathcal{N} \) and \( \mathcal{N} \) is a compact set in \( \mathcal{D}_V \). 

**Proof of Theorem 1.6.** By Proposition 6.1 (iii), we know that \( I \) is bounded below in \( \mathcal{N} \) and \( c_0 \triangleq \inf_{u \in \mathcal{N}} I(u) \rightarrow -\infty \), then there exists \( \{ u_n \} \subset \mathcal{N} \) such that \( I(u_n) \rightarrow c_0 \). So, Proposition 6.1 (ii), (iii) imply that, there exists \( u_0 \in \mathcal{N} \) such that \( I(u_0) = c_0 \) and \( I'(u_0) = 0 \), that is, \( u_0 \) is a ground state of (1.1). \( \Box \)

### 7 Appendix

**Proof of Lemma 3.2.** For \( \beta > 1 \), \( M > 1 \), the same as [19], we define a function \( H \in C^1(0, +\infty) \) by

\[
H(s) = \begin{cases} 
\beta^s, & s \in [0, M], \\
\beta M^{\beta-1} s - (\beta - 1) M^\beta, & s \in (M, \infty).
\end{cases}
\]

Let

\[
G(s) \triangleq \int_0^s |H'(t)|^2 dt = \begin{cases} 
\frac{\beta^2}{2^{\beta-1}} s^{2\beta-1}, & s \in [0, M], \\
\beta^2 M^{2(\beta-1)} s - 2\beta(\beta-1) M^{2\beta-1}, & s \in (M, \infty).
\end{cases}
\]

Then \( G(s) \) and \( H(s) \) are Lipschitz in \([0, +\infty)\), therefore, \( G(u) \) and \( H(u) \) are in \( \mathcal{D}^{1,2}(\mathbb{R}^N) \) if \( u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \). Moreover,

\[
s G(s) \leq s^2 H^2(s) \leq \beta^2 H^2(s).
\]

(7.1)

Let \( \overline{\eta}(x) \in C^\infty_c(\mathbb{R}^N) \) with \( \overline{\eta}(x) \geq 0 \) such that, for \( r_1 > r_2 > 0 \) (\( r_1 \) will be determined later)

\[
\overline{\eta}(x) \equiv 1 \text{ for } x \in B_{r_2}, \quad \overline{\eta}(x) \equiv 0 \text{ for } x \in B_{r_1}^c, \text{ and } |\nabla \overline{\eta}| \leq \frac{2}{r_1 - r_2}
\]

for each \( y \in \mathbb{R}^N \), setting \( \eta(x) = \overline{\eta}(x-y) \geq 0 \) and \( 0 \leq \eta^2 G(u) \in \mathcal{D}^{1,2}(\mathbb{R}^N) \) with compact support in \( B_{r_1}(y) \cap \{ x \in \mathbb{R}^3 : u(x) \neq 0 \} \). By (4.1) and (3.12) with \( \varphi = \eta^2 \) and \( h = G \),

\[
\int_{\mathbb{R}^N} \nabla u \nabla (\eta^2 G(u)) dx \leq \int_{\mathbb{R}^N} u^{p-1} u \eta^2 G(u) dx \leq \beta^2 \int_{\mathbb{R}^N} |u|^{p-1} \eta^2 H^2(u) dx.
\]

(7.2)
Noting that
\[ \nabla u \nabla (\eta^2 G(u)) = |\nabla u|^2 H^2(u) \eta^2 + 2 \nabla u \nabla G(u) \eta. \]
Also, by Young inequality and (7.1),
\[ |\nabla u \nabla G(u) \eta| = |\eta u^{-1/2} G^{1/2}(u) \nabla u| |u^{1/2} G^{1/2}(u) \nabla \eta| \]
\[ \leq \frac{1}{4} |\nabla u|^2 H^2(u) \eta^2 + 4 \beta^2 |\nabla \eta|^2 H^2(u). \]

Then, it follows from (7.2) that
\[ \int_{\mathbb{R}^N} |\nabla u|^2 |H'(u)|^2 \eta^2 \, dx \leq 16 \beta^2 \left( \int_{\mathbb{R}^N} |\nabla \eta|^2 H^2(u) \, dx + \int_{\mathbb{R}^N} |u|^\gamma - 1 |\nabla \eta|^2 H^2(u) \, dx \right). \]

Hence, by \( \beta > 1 \) and Hölder inequality we see that
\[ \int_{\mathbb{R}^N} |\nabla (H(u) \eta)|^2 \, dx \leq 2 \int_{\mathbb{R}^N} |\nabla \eta|^2 H^2(u) \, dx + 2 \int_{\mathbb{R}^N} |\nabla u|^2 |H'(u)|^2 \eta^2 \, dx \]
\[ \leq 34 \beta^2 \left( \int_{\mathbb{R}^N} |\nabla \eta|^2 H^2(u) \, dx + \int_{\mathbb{R}^N} |u|^\gamma - 1 |\nabla \eta|^2 H^2(u) \, dx \right) \]
\[ \leq 34 \beta^2 \left( \int_{\mathbb{R}^N} |\nabla \eta|^2 H^2(u) \, dx + |\eta H(u)|_{1,2}^2 |u|^\gamma - 1 |\nabla \eta|^2 H^2(u) \, dx \right). \] (7.3)

where \( \gamma = \frac{2^* - p - 1}{2} \) and \( |B_{r_1}| \) denotes the volume of \( B_{r_1} \). Taking \( \beta = \frac{2^* - p - 1}{\gamma} = \beta_0 \) and
\[ r_1 = |B_1|^{1/\gamma} \left( 68 \beta_0^2 |u|_{2,\gamma - 1}^\gamma + 1 \right)^{-\frac{\gamma}{2}} \leq \min \left\{|B_1|, |B_1|^{1/\gamma} \left( 68 \beta_0^2 |u|_{2,\gamma - 1}^\gamma \right)^{-\frac{\gamma}{2}} \right\}. \] (7.4)

Obviously, \( 34 \beta_0^2 |u|_{2,\gamma - 1}^\gamma |B_{r_1}| \leq 1/2 \) and \( |B_{r_1}| \leq 1 \). Then, (7.3) with \( \beta = \beta_0 \) gives that
\[ \int_{\mathbb{R}^N} |\nabla (H(u) \eta)|^2 \, dx \leq 68 \beta_0^2 \int_{\mathbb{R}^N} |\nabla \eta|^2 H^2(u) \, dx. \]

Hence, by Sobolev inequality and \( \eta H(u) \in \mathcal{S}^{1,2}_0(B_{r_1}(y)) \) as well as the definition of \( \eta \), we see that
\[ |H(u)|_{L^{2^*}(B_{r_2}(y))} \leq \left( \int_{\mathbb{R}^N} |H(u) \eta|^2 \, dx \right)^{1/2^*} \leq \mathcal{S}_0^{-1} \int_{\mathbb{R}^N} |\nabla (H(u) \eta)|^2 \, dx \]
\[ \leq 68 \mathcal{S}_0^{-1} \beta_0^2 \int_{\mathbb{R}^N} |\nabla \eta|^2 H^2(u) \, dx \]
\[ \leq \left( \frac{C \beta_0}{r_1 - r_2} \right)^2 |H(u)|_{L^2(B_{r_1}(y))}^2. \] (7.5)

where \( C = \sqrt{68} \mathcal{S}_0^{-1} \) and \( \mathcal{S}_0 \) is the Sobolev constant, which depends only on \( N \). By definition of \( H(s) \equiv s^{\beta} \) if \( M \to +\infty \). Noting that \( \beta = \beta_0 \) and \( 2 \beta_0 = 2^* \), it follows from (7.5) that
\[ |u|_{L^{2\beta_0^2}}(B_{r_2}(y)) \leq \left( \frac{C \beta_0}{r_1 - r_2} \right)^2 |u|_{L^{2^*}}^2(B_{r_1}(y)). \] (7.6)

For each \( i \geq 2 \), let \( r_i = \frac{2^{i+1} - 1}{4} \) \( (r_1 \) is given by (7.4). Take \( \tilde{\eta} \in C_0^\infty(\mathbb{R}^3) \) such that
\[ \tilde{\eta}(x) \equiv 1 \text{ for } x \in B_{r_{i+1}}, \quad \tilde{\eta}(x) \equiv 0 \text{ for } x \in \mathbb{R}^N \setminus B_{r_i} \text{ and } |\nabla \tilde{\eta}| \leq \frac{2}{r_i - r_{i+1}}. \]

27
Let $\delta = \frac{2\beta_0}{2\beta_0 + 1 - p}$ and $\delta \in (0, 1)$ by $p \in (1, \frac{N+2}{N-2})$. For $i \geq 2$, applying (7.3) with $\eta = \eta_i(x-y)$ and $\beta = \beta_i \geq \delta^{-1} > 1$, then noting that $|B_{r_i}| < |B_{r_{i+1}}| \leq 1$ and using Hölder inequality and the definition of $\eta_i$, we get

$$
\int_{\mathbb{R}^N} |\nabla (H(u)\eta)|^2 \, dx \leq 34\beta^2 \left( \int_{\mathbb{R}^N} |\nabla \eta|^2 H^2(u) \, dx + \int_{\mathbb{R}^N} |u|^{p-1} \eta^2 H^2(u) \, dx \right)
$$

$$
\leq 34\beta^2 \left[ \left( \frac{2}{r_i - r_{i+1}} \right)^2 |H(u)|^2_{L^{2^{*}(p-1)/2}(B_{r_{i+1}}(y))} + |H(u)|^2_{L^{2^{*}(p-1)/2}(B_{r_i}(y))} \right] |u|^{p-1} \eta^2_{r_i}(B_{r_{i+1}}(y))
$$

$$
\leq 34\beta^2 \left[ \frac{2}{r_i - r_{i+1}} \right]^2 + \left( \frac{2}{r_1 - r_2} \right)^2 \left( \frac{2}{r_1 - r_2} \right)^{(p-1)/\beta_0} |u|_{L^{2^{*}(p-1)/2}(B_{r_{i+1}}(y))} |H(u)|_{L^{2^{*}(p-1)/2}(B_{r_i}(y))}
$$

\begin{equation}
\leq \left( \frac{C\beta (1 + |u|_{L^{2^{*}(p-1)/2}(B_{r_{i+1}}(y))}}{r_i - r_{i+1}} \right)^2 |H(u)|_{L^{2^{*}(p-1)/2}(B_{r_{i+1}}(y))}^2.
\end{equation}

where $\bar{C}$ is a constant depending only on $N$ and $p$. Since $\eta_i H(u) \in \mathcal{S}_0^{1/2}(B_{r_i}(y))$, it follows from the Sobolev inequality that

$$
|H(u)|_{L^{2^{*}(p-1)/2}(B_{r_{i+1}}(y))} \leq S_0^{-1} \int_{\mathbb{R}^N} |\nabla (H(u)\eta)|^2 \, dx \leq \left( \frac{\bar{C} \beta (1 + |u|_{L^{2^{*}(p-1)/2}(B_{r_{i+1}}(y))})}{r_i - r_{i+1}} \right)^2 |H(u)|_{L^{2^{*}(p-1)/2}(B_{r_i}(y))}^2
$$

where $\bar{C} = \bar{C} \sqrt{S_0^{-1}}$ depends only on $N$ and $p$. Let $M \to +\infty$, and $H(u) = u^{\beta_i}$, then

$$
|u|_{L^{2^{*}\beta_i}(B_{r_{i+1}}(y))} \leq \left( \frac{\bar{C} \beta (1 + |u|_{L^{2^{*}(p-1)/2}(B_{r_{i+1}}(y))})}{r_i - r_{i+1}} \right)^{1/\beta_i} |u|_{L^{2^{*}\beta_{i-1}(B_{r_i}(y))}}.
$$

We can now perform the Moser iterations in a standard way and get that

$$
|u|_{L^{2^{*}\beta_i}(B_{r_{i+1}}(y))} \leq \prod_{l=2}^{i} \left( \frac{\bar{C} \beta (1 + |u|_{L^{2^{*}(p-1)/2}(B_{r_{i+1}}(y))})}{r_l - r_{l+1}} \right)^{1/\beta_i} |u|_{L^{2^{*}\beta_1}(B_{r_2}(y))}. \tag{7.7}
$$

$$
\leq \left( \frac{2}{\delta} f(i) \left( \frac{8 \bar{C} \beta (1 + |u|_{L^{2^{*}(p-1)/2}(B_{r_{i+1}}(y))})}{r_1} \right)^{g(i)} |u|_{L^{2^{*}\beta_1}(B_{r_2}(y))}.
$$

where $f(i) = \frac{2\beta^2}{\delta} + \frac{\delta^2 (1-\delta^{-2})^2}{(1-\delta)^2} + \frac{\delta^2 (1-\delta^{-2})}{1-\delta} \frac{2 \beta^2}{(1-\delta)^2}$ and $g(j) = \frac{\delta^2 (1-\delta^{-1})}{1-\delta} \frac{2 \beta^2}{1-\delta}$. Since $2^{*} \beta_i = 2\beta_0 + 1 - p < 2\beta_0$, by Hölder inequality, (7.3) and (7.6), we have

$$
|u|_{L^{2^{*}\beta_1}(B_{r_2}(y))} \leq |B_{r_2}|^{2\beta_0} \frac{2^{*} \beta_0}{r_1} |u|_{L^{2^{*}\beta_0}(B_{r_2}(y))} \leq |u|_{L^{2^{*}\beta_0}(B_{r_2}(y))} \leq \left( \frac{C \beta_0}{r_1 - r_2} \right)^{1/\beta_0} |u|_{2^{*}}.
$$

This and (7.7) show that

$$
|u|_{L^{2^{*}\beta_1}(B_{r_{1+2}}(y))} \leq |u|_{L^{2^{*}\beta_1}(B_{r_{1+1}}(y))} \leq \left( \frac{2}{\delta} f(i) \left( \frac{8 \bar{C} \beta (1 + |u|_{L^{2^{*}(p-1)/2}(B_{r_{1+1}}(y))})}{r_1} \right)^{g(i)} \left( \frac{C \beta_0}{r_1} \right)^{1/\beta_0} |u|_{2^{*}}.
$$

28
Let \( i \to +\infty \), and noting (7.4), there exist positive constant \( C_1(p, N), C_2(p, N) \) such that

\[
|u|_{L^\infty(B_{r_1/2}(y))} \leq \left( \frac{2 \delta}{\beta} \left( \frac{8 \bar{C}_1 (1 + |u|^{p-1/2})}{r_1} \right)^{\frac{p^2}{p-1}} \left( \frac{16 C \beta}{r_1} \right)^{1/\beta_0} |u|_{2^*}.
\]

\[
\leq \left( \frac{2 \delta}{\beta} \right)^{\frac{p^2}{p-1}} \left( \frac{8 \bar{C}_1 + 16 C \beta}{r_1} (1 + |u|^{p-1/2}) \right)^{\frac{p^2}{p-1}} + \frac{1}{\beta_0} |u|_{2^*}.
\]

\[
\leq C_1(p, N) (1 + |u|^{C_2(p, N)}) |u|_{2^*}.
\]

Since \( y \in \mathbb{R}^N \) and \( r_1 \) is fixed by (7.4), we have

\[
|u|_{\infty} \leq C_1(p, N) \left( 1 + |u|^{C_2(p, N)} \right) |u|_{2^*}. \quad \square
\]

References

[1] A. Ambrosetti. On Schrodinger-Poisson Systems. *Milan J. Math.*, 76(1):257–274, DEC 2008.

[2] A. Ambrosetti and D. Ruiz. Multiple bound states for the Schr"{o}dinger-Poisson problem. *Commun. Contemp. Math.*, 10(3):391–404, 2008.

[3] A. Azzollini, P. d’Avenia, and A. Pomponio. On the schr"{o}dinger-maxwell equations under the effect of a general nonlinear term. *preprint*.

[4] A. Azzollini and A. Pomponio. Ground state solutions for the nonlinear Schr"{o}dinger-Maxwell equations. *J. Math. Anal. Appl.*, 345(1):90–108, 2008.

[5] T. Bartsch, A. Pankov, and Z. Q. Wang. Nonlinear Schr"{o}dinger equations with steep potential well. *Commun. Contemp. Math.*, 3(4):549–569, 2001.

[6] V. Benci and D. Fortunato. An eigenvalue problem for the Schr"{o}dinger-Maxwell equations. *Topol. Methods Nonlinear Anal.*, 11(2):283–293, 1998.

[7] V. Benci and D. Fortunato. Solitary waves of the nonlinear klein-gordon equation coupled with maxwell equations. *Rev. Math. Phys.*, 14:409–420, 2002.
[8] F. A. Berezin and M. A. Shubin. The Schrödinger equation. Kluwer Academic Publishers Group, Dordrecht, 1991.

[9] G.M. Coclite. A multiplicity result for the nonlinear Schrödinger-Maxwell equations. Commun. Appl. Anal., 7(2-3):417–423, 2003.

[10] T. D’Aprile and D. Mugnai. Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations. Proc. Roy. Soc. Edinburgh Sect. A, 134(5):893–906, 2004.

[11] T. D’Aprile and J. Wei. On bound states concentrating on spheres for the maxwell-schrödinger equation. SIAM J. Math. Anal., 37:321–342, 2005.

[12] P. d’Avenia. Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations. Adv. Nonlinear Stud., 2(2):177–192, 2002.

[13] H. Egnell. Asymptotic results for finite energy solutions of semilinear elliptic equations. J. Differential Equations, 98(1):34–56, 1992.

[14] D. Gilbarg and N.S. Trudinger. Elliptic partial differential equations of second order. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[15] Y.S. Jiang and H.S. Zhou. Bound states for a stationary nonlinear Schrödinger-Poisson systems with sing-changing potential in $\mathbb{R}^3$. Acta. Math. Sci., 29B(4):1095–1104, 2009.

[16] H. Kikuchi. On the existence of a solution for elliptic system related to the Maxwell-Schrödinger equations. Nonlinear Anal., 67(5):1445–1456, 2007.

[17] D. Ruiz. The Schrödinger-Poisson equation under the effect of a nonlinear local term. J. Funct. Anal., 237(2):655–674, 2006.

[18] D. Ruiz and G. Vaira. Cluster solutions for the Schrödinger-Poisson-Slater problem around a local minimum of the potential. preprint.

[19] Neil S. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Ann. Scuola Norm. Sup. Pisa (3), 22:265–274, 1968.
[20] Z.P. Wang and H.S. Zhou. Positive solution for a nonlinear stationary Schrödinger-Poisson system in $\mathbb{R}^3$. *Discrete Contin. Dyn. Syst.*, 18(4):809–816, 2007.

[21] M.B. Yang, Z.F. Shen, and Y.H. Ding. Multiple semiclassical solutions for the nonlinear maxwell-schrödinger system. *Nonlinear Anal.*, in press.

[22] L.G. Zhao and F.K. Zhao. On the existence of solutions for the Schrödinger-Poisson equations. *J. Math. Anal. Appl.*, 346(1):155–169, 2008.