Representations of the exceptional and other Lie algebras with integral eigenvalues of the Casimir operator

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Abstract

The uniformity, for the family of exceptional Lie algebras \( \mathfrak{g} \), of the decompositions of the powers of their adjoint representations is well-known now for powers up to the fourth. The paper describes an extension of this uniformity for the totally antisymmetrised \( n \)-th powers up to \( n = 9 \), identifying (see Tables 3 and 6) families of representations with integer eigenvalues \( 5, \ldots, 9 \) for the quadratic Casimir operator, in each case providing a formula (see eq. (11) to (15)) for the dimensions of the representations in the family as a function of \( D = \text{dim} \ \mathfrak{g} \). This generalises previous results for powers \( j \) and Casimir eigenvalues \( j, j \leq 4 \). Many intriguing, perhaps puzzling, features of the dimension formulas are discussed and the possibility that they may be valid for a wider class of not necessarily simple Lie algebras is considered.

1 Introduction

After noting some conventions in Sec. 1.1, we describe quite carefully the context of this paper, in Sec. 1.2. This enables us to indicate briefly in Sec. 1.3 the scope of this paper, and highlight the new results it obtains.

1.1 Notation and conventions

We are concerned with simple complex Lie algebras \( \mathfrak{g} \) and with their irreducible representations. Irreducibility is understood over the field of complex numbers. We note that we use the informal abbreviation irrep for irreducible representation.

We focus in particular on the family of

\[ \mathcal{F} = \{ \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8 \} \]  \hspace{1cm} (1)
of simple Lie algebras. They feature in the extension of the last line of the Freudenthal magic square [1] that is given in [2]. These algebras are well-known to form a family in some profound sense whose ramifications probably have not yet been fully exhausted.

Our work depends heavily on access to a large body of data for the Lie algebras \( g \), especially lists for the exceptional Lie algebras of irreducible representations \( R \) classified by highest weights which give the corresponding dimension and eigenvalue \( c^{(2)}(R) \) of the quadratic Casimir operator \( C^{(2)}(R) \). We have created a C++ program to provide this and related information given the Cartan matrix as the only input. We note also that valuable general sources of data regarding Lie algebras are available, e.g. [3] and [4].

We use a normalisation in which \( c^{(2)}(g) = 1 \) for the adjoint representation and therefore \( c^{(2)}(R) = \langle \Lambda_R, \Lambda_R + 2\delta \rangle \) where \( \Lambda_R \) denotes the highest weight of \( R \), \( \delta \) is the half-sum of positive roots of \( g \) and \( \langle \cdot, \cdot \rangle \) denotes the Cartan–Killing form on the space of weights.

We refer to irreps here often by their dimension because our studies are concerned with dimension formulas for families of representations of Lie algebras \( g \). When we need to refer to irreps by their highest weight or Dynkin coordinate specification, we adopt the conventions that follow from the Cartan matrices of \( g \) used by [3,4,5]. We also often omit commas between the coordinates, here always integers less than ten.

The diagram automorphisms of the algebras \( g \in F \) are \( S_2 \) for \( \mathfrak{a}_2 \) and \( \mathfrak{e}_6 \), \( S_3 \) for \( \mathfrak{d}_4 \), and the trivial group for all the others. As the adjoint irrep \( \text{ad} \) is always mapped to itself under diagram automorphisms, the constituents of the complete decomposition of its tensor powers \( \text{ad} \otimes j \) are either self-conjugate or pairs of complex conjugate irreps for \( \mathfrak{a}_2 \) and \( \mathfrak{e}_6 \). For \( \mathfrak{d}_4 \), the constituents are either irreps that are stable under triality or triples and sextuples of irreps that are related by triality.

### 1.2 Background material

The first property of the family \( F \) to be noted concerns the structure of \( \text{ad} \otimes \text{ad} \). We write this as

\[
\text{ad} \otimes \text{ad} = (\text{ad} \otimes \text{ad})_A \oplus (\text{ad} \otimes \text{ad})_S. \tag{2}
\]

For the antisymmetric piece we have a universal result, i.e. one that is valid for each simple compact \( g \)

\[
(\text{ad} \otimes \text{ad})_A = \text{ad} \oplus X_2, \tag{3}
\]

where \( X_2 \) denotes a representation of \( g \) of dimension

\[
\dim X_2 = \frac{1}{2} D(D - 3), \tag{4}
\]

where \( D = \dim g \). For \( \mathfrak{a}_2 \), \( X_2 \) is the representation \( 20 = 10 + \overline{10} = (3,0) \oplus (0,3) \), a pair of conjugate irreps. For the exceptional Lie algebras, see Table 1.

A universal and important property of the family of irreps \( X_2 \) is the result

\[
c^{(2)}(X_2) = 2. \tag{5}
\]

It is of special relevance to the work described here, because families \( X_j \) with

\[
c^{(2)}(X_j) = j, \tag{6}
\]

for \( j \leq 4 \) are known to appear in the \( j \)-th antisymmetric tensor power of \( \text{ad} \), and we extend this knowledge beyond \( j = 4 \) here.
The result corresponding to (3) for the symmetric piece of $ad \otimes ad$ is not universal, but for each $g \in \mathcal{F}$ we have a result of the form $$(ad \otimes ad)_{S} = R_{1} \oplus R_{2} \oplus R_{3},$$ (7)

defining three families of irreps as given in Table 1.

|   | $g_{2}$ | $f_{4}$ | $e_{6}$ | $e_{7}$ | $e_{8}$ |
|---|---------|---------|---------|---------|---------|
| $ad$ | 14      | 52      | 78      | 133     | 248     |
|      | (10)    | (1000)  | (000001)| (1000000)| (00000010)|
| $X_{2}$ | 77'     | 1274    | 2925    | 8645    | 30380   |
|      | (03)    | (0100)  | (001000)| (0100000)| (00000100)|
| $R_{1}$ | 1       | 1       | 1       | 1       | 1       |
| $R_{2}$ | 27      | 324     | 650     | 1539    | 3875    |
|      | (02)    | (0002)  | (100010)| (0000100)| (10000000)|
| $R_{3}$ | 77      | 1053    | 2430    | 7371    | 27000   |
|      | (20)    | (2000)  | (000002)| (2000000)| (000000020)|

Table 1: Irreps of $g$ for $ad \otimes ad$.

The analysis just discussed for $ad \otimes ad$ gives rise to a natural conjecture — the Deligne conjecture [7] — that the $j$-th tensor powers of $ad$ for the exceptionals possess uniform decompositions into irreps. That this does indeed happen, defining further families of irreps, has been comprehensively confirmed by algebraic computation in [8], using [9], for $j = 3, 4$ and established independently of computational procedures in [10].

We do not need these details here. As far as we can determine, the first full explicit analysis of $ad \otimes ad$ appears in [6].

The important fact here is that (8) defines only one new family beyond those already understood from the study of $ad \otimes ad$ which we denote by $X_{3}$ (Table 2). The irreps involved here possess two notable properties, natural analogues of (4) and (5). Their dimension is given by a polynomial in $D$,

$$d_{3}(D) = \frac{1}{24}D(D - 1)(D - 8),$$ (9)

valid for all $g \in \mathcal{F}$. The factor $D - 8$ in (9) reflects that fact that $a_{2}$ has no irrep with $c^{(2)} = 3$ and hence no member in the $X_{3}$ family. For $a_{1}$, the the structure of [8] also collapses, and there is no representation with $c^{(2)} = 3$ in $ad^{\wedge 3}$ either. However, for $D = 3$, the dimension
formula (9) gives the answer \(-5\), and \(a_1\) does indeed have a representation of dimension 5 with \(c^{(2)} = 3\) which is, however, not contained in \(ad^{\otimes 3}\). No systematic understanding has been obtained of why negative values of the dimension formula are often seen to happen and seem to make some sort of sense. More examples follow in our work for \(j = 5, \ldots, 9\).

Finally, we remark that the three irreps that occur for \(d_4\) (Table 2), are related by diagram automorphisms.

Next we note that analysis of \(ad^{\wedge 4}\) brings into the discussion only one further family of representations beyond those that were mastered within the discussion (not reviewed here, but see [8]) of \(ad^{\otimes 3}\). We denote this family as \(X_4\) (Table 2).

The two irreps, \((0022)\) and \((020010)\), that are listed for \(e_6\) are related by diagram automorphisms as are \((0022)\), \((000100)\), and \((0001000)\), for \(d_4\). The dimension formula

\[ d_4(D) = \frac{1}{4} D(D - 1)(D - 3)(D - 14), \]  

already indicates that \(a_1\) and \(g_2\) have no member in the \(X_4\) family. Indeed, there do not exist any irreps of \(a_1\) and \(g_2\) with \(c^{(2)} = 4\). For \(a_2\) we have again the phenomenon that (10) gives a negative result, here \(-70\). Indeed, \(a_2\) has got exactly one pair of conjugate irreps with \(c^{(2)} = 4\) which have dimension 35 + 35. For all irreps listed in Table 2, (6) is satisfied.

The dimension formulas given here in (4), (9) and (10) are equivalent to results given in [7] and [8], where other parametrisations of family properties are used: see Sec. 1.4 below. The \(c^{(2)}\) eigenvalues can also be found in these sources.

### 1.3 Summary of new results

We now turn to the problem of extending uniformity properties for \(g \in F\) into the case of \(ad^{\otimes 5}, \ldots, ad^{\otimes 9}\). A systematic extension would seem to entail massive computational effort, but confirmation that the nice picture known for \(j \leq 4\) does not stop at \(j = 4\) can be provided.

Looking at \(ad^{\wedge j}\) for \(j = 2, 3, 4\) motivates easy but compelling conjectures. It is natural to expect that there exist, for higher \(j\) values, identifiable families \(X_j\) of representations of \(g \in F\) occurring in the decomposition of \(ad^{\wedge j}\), that they satisfy (5) and that nice dimension formulas exist.
The purpose of this paper then is to attain such knowledge by confrontation of the cases of \( j = 5, \ldots, 9 \). In fact we are able to provide an identification of the members of families \( X_5, \ldots, X_9 \) of representations of \( g \in F \) that satisfy (33) and establish the dimension formulas

\[
\begin{align*}
\text{d}_5(D) &= \frac{1}{5!} D(D-3)(D-6)(D^2 - 21D + 8), \\
\text{d}_6(D) &= \frac{1}{6!} D(D-1)(D-10)(D^3 - 34D^2 + 181D - 144), \\
\text{d}_7(D) &= \frac{1}{7!} D(D-2)(D-3)(D-8)(D^3 - 50D^2 + 529D - 120), \\
\text{d}_8(D) &= \frac{1}{8!} D(D-1)(D-3)(D-6)(D^4 - 74D^3 + 1571D^2 - 9994D + 4200), \\
\text{d}_9(D) &= \frac{1}{9!} D(D-1)(D-3)(D-4)(D-14)(D-26)(D^3 - 60D^2 + 491D - 120).
\end{align*}
\]

We display information in Tables 3 and 6 that describe in full the assignments of representations for the members of the families \( X_5, \ldots, X_9 \) for all Lie algebras \( g \in F \). There are various features of these results that need, and will receive, consideration.

1. The occurrence of the quadratic, cubic and quartic polynomials in (11)–(15) which do not have rational factors.

2. The status of the table entries for \( d_j \) for each \( g \) when \( j \) exceeds the first \( j_0 \)-value for which \( d_j \) is not positive. For \( a_2, g_2, d_4, f_4 \) we have\(^1\) \( j_0 = 3, 4, 7, 10 \).

3. The appearance of direct sums of several irreducible representations that are not related by diagram automorphisms. This feature is new compared with the results of [8].

4. The occurrence of negative values of the dimension formulas (11)–(15).

5. The fact that the dimension formulas (11)–(15) give integer results for any integral \( D \).

6. The question of whether these patterns extend beyond the members of the family \( F \).

The ensuing material is organised as follows. For comparison with the work of others in Sec. 1.4, we mention parametrisations alternative to \( D = \dim g \). In Sec. 2 we explain our construction of the dimension formulas (11)–(15). Sec. 3 then poses the obvious question: are the results discussed here for \( g \in F \) universal? The results for \( j = 2 \) are well-known to be

\(^1\)These numbers are related to the highest integer \( j \) for which \( \text{ad}^{\otimes j} \) contains a Casimir eigenspace of eigenvalue \( j \).
universal in that they apply, not only to $g \in F$, but to all simple $g$. To what extent if any does a similar statement hold for higher $j$? We are unable to give a systematic algebraic analysis of the situation, but can easily gain some insight into it, by giving an empirical analysis of the cases of the simple Lie algebras $b_2, b_3, c_3, a_3, \ldots, a_5$. Further insight, partially motivated by the appearance of $D - 6$ factors in $d_5(D)$ and $d_6(D)$, comes from study in Sec. 4 of the cases of $a_1 \oplus a_1$, and the corresponding three-fold and four-fold direct sum. Sec. 5 contains a conclusion and a list of the most obvious open questions.

### 1.4 Alternative parametrisations

In general in our work, we prefer to give formulas for the dimensions of and the quadratic Casimir eigenvalues of members of a family of irreps of the Lie algebras in $F$ as a function of the single parameter $D = \dim g$, but other parameters are used in the literature. To aid comparison of our discussion with related work in other sources, we have drawn up Table 4.

The parameter $\alpha$ is used in [7] and [8], while $m$ is used in [10] and [12]. The connection between the different parameters can be obtained from

$$D = \frac{2(3m+7)(5m+8)}{m+4},$$

$$\frac{1}{\alpha} = 3(m+2) = h^\vee,$$

(16)

where $h^\vee$ is the dual Coxeter number, (p. 37 of [3]). Also $\alpha$ is related to the eigenvalue of the Casimir operator of the members of the $R_3$ family

$$c^{(2)}(R_3) = 2(1 + \alpha).$$

(17)

For the exceptional algebras in the last line of the Freudenthal magic square, the $m$ values in Table 4 have this interpretation: the division algebra used in their Freudenthal construction has dimension $m$.

### 2 The dimension formulas

In the study of $ad^{\wedge j}$ up to $j = 4$ [8] it was sufficient to identify the irreducible component of the highest weight in the antisymmetric power $ad^{\wedge j}$ and then to determine the direct sum $X_j$ of all irreps that can be obtained from the former by the application of diagram automorphisms. The dimension formulas (9) and (10) then agree with the interpolation polynomial for which $d_j(\dim g) = \dim X_j$ for all algebras $g \in F$ for which the corresponding $X_j$ satisfies (6). In
Table 5: The highest positive roots $x_j$ of $\mathfrak{f}_4$ in terms of the simple roots $\alpha_j$ and their height.

| Root | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | Height |
|------|------------|------------|------------|------------|--------|
| $x_{24}$ | 2          | 3          | 4          | 2          | 11     |
| $x_{23}$ | 1          | 3          | 4          | 2          | 10     |
| $x_{22}$ | 1          | 2          | 4          | 2          | 9      |
| $x_{21}$ | 1          | 2          | 3          | 2          | 8      |
| $x_{20}$ | 1          | 2          | 3          | 1          | 7      |
| $x_{19}$ | 1          | 2          | 2          | 2          | 7      |
| $x_{18}$ | 1          | 2          | 2          | 1          | 6      |
| $x_{17}$ | 1          | 1          | 2          | 2          | 6      |
| $x_{16}$ | 0          | 1          | 2          | 2          | 5      |
| $x_{15}$ | 1          | 1          | 2          | 1          | 5      |
| $x_{14}$ | 1          | 2          | 2          | 0          | 5      |
| :      | :          | :          | :          | :          | :      |

our notation, it is already a non-trivial fact that a polynomial in $D = \dim \mathfrak{g}$ is sufficient to parametrise the relevant dimensions for all algebras $\mathfrak{g} \in \mathcal{F}$.

At $j = 5$, however, the same strategy does not result in any simple dimension formula at all. The solution is to modify the strategy and to choose $X_j$ to be the entire Casimir eigenspace of $ad^{\wedge j}$, i.e. the maximal $X_j$ that satisfies (6), or if there is no non-trivial subspace with this property when the algebra $\mathfrak{g}$ has dropped out of the full picture, to choose a suitable direct sum of (other) irreps of $\mathfrak{g}$ that satisfy (6). We have arrived at this result purely empirically, searching for simple and in particular polynomial dimension formulas, and we have found the following rule$^2$ which characterizes the direct summands of $X_j$.

The rule specifies how to select $j$ distinct roots of $\mathfrak{g}$ whose sum is the highest weight of an irrep that is contained in $ad^{\wedge j}$. Whenever an algebra $\mathfrak{g}$ has not yet dropped out of the full picture (as explained above), the rule finds all irreps that both occur in $ad^{\wedge j}$ and have $c^{(2)} = j$. If the algebra has dropped out, the rule finds some other irreps in $ad^{\wedge j}$.

We explain the procedure for $\mathfrak{f}_4$ whose roots are given in Table 5. Consider the root lattice of $\mathfrak{f}_4$, drawn as a directed graph in Figure 1. The vertices correspond to the roots $x_k$ and are numbered as in the table. There is a directed arrow from $x_k$ to $x_\ell$, denoted by a pair $(x_k, x_\ell)$, if and only if $x_\ell = x_k + \alpha$ for some simple root $\alpha$.

Let $X = \{x_1, x_2, \ldots\}$ denote the set of all roots. For $j = 1, 2, \ldots$ consider subsets $S \subset X$ of cardinality $|S| = j$. A subset $S$ is called admissible if for each $x \in S$ and each arrow $(x, y)$ in the graph, we have also $y \in S$.

Observe that in the $\mathfrak{f}_4$ example, the highest root $x_{24}$ is contained in any non-empty ad-

$^2$We thank J. Landsberg for bringing the article [13] to our attention in which the Casimir $j$ eigenspace of $ad^{\wedge j}$ is characterized by an algebraic condition equivalent to the rule we state here.
Figure 1: A part of the root lattice of $f_4$ as a directed graph. The vertices correspond to the roots $x_k$ and are numbered as in Table 5.

missible subset. Similarly, the second highest root $x_{23}$ is contained in any admissible set of cardinality at least 2, and so on.

**Observation:** Given an admissible subset $S$, $|S| = j$, the weight

$$w := \sum_{x \in S} x$$

is the highest weight of an irrep of $\mathfrak{g}$ which is contained in $ad^{\wedge}j$. Any irrep of $c^{(2)} = j$ that occurs in $ad^{\wedge}j$ can be found from this rule.

Given the root lattices of the algebras $\mathfrak{g} \in \mathcal{F}$, we can use this rule in order to obtain a list of all irreps that are both contained in $ad^{\wedge}j$ and also have $c^{(2)} = j$. This information forms the basis for the higher dimension formulas.

A point for $d_4$ regarding admissible sets is worth noting. For $d_4$, $d_4(28) = 3 \cdot 3675$ involving a triple of irreps with $c^{(2)} = 4$ which occur in $ad^{\wedge}4$. But $d_4$ also has $1925 = (0300)$ with $c^{(2)} = 4$ which is not part of $ad^{\wedge}4$. If we examine the admissible sets for $d_4$ at $j = 4$, one finds sets for the 3675 dimensional irreps, but not for 1925.

We obtain the dimension formula (11),

$$d_5(D) = \frac{1}{30}D(D - 3)(D - 6)(D^2 - 21D + 8),$$

as the interpolation polynomial using the data for six algebras of the family $\mathcal{F}$ from Table 8.

If we hoped that the right side of (19) would be the product of factors linear in $D$, like the formula for $d_j(D)$ for lower $j$ then we have been disappointed. However the expectation was based on viewing these formulas, as in [10], in relation to the Weyl formula for the dimensions of irreps of Lie algebras, and that view is valid as long as the families $X_j$ involve only irreps (up to diagram automorphisms), i.e. for $j \leq 4$. But it is not valid for $j = 5$ and the assignments for $X_5$ already made, and so the basis for the hope has gone.

In case it might be thought that the use of the parameter $m$ of Table 4 might improve the status of (19), we note that (16) implies

$$D^2 - 21D + 8 = \frac{6(15m^2 + 67m + 68)(10m^2 + 27m + 28)}{(m + 4)^2},$$

in which each one of the quadratic expressions in view has discriminant 409 and does not have rational factors.
The dimension formula \textbf{(19)} gives negative values for \(a_2\) and \(g_2\) (Table 3), in each case referring to the unique irrep of the Lie algebra in question with \(c^{(2)} = 5\).

Although it would not be correct to assign 924 of \(g_2\) to \(X_5\), since it does not occur in the decomposition for \(g_2\) of \(ad^5\), the situation is similar to the one found for \(a_2\) at the previous stage: we found there not a proper member of the \(X_4\) family for \(a_2\), but one with the correct value of \(c^{(2)}\) and the negative of the correct dimension. Such things are prevalent also in the work of \([7]\) and \([8]\), but not explained.

As one goes to higher \(j\) in an effort to push the search for families and for dimension formulas for them higher, one expects more and more algebras to drop out of the full picture much in the way that \(a_2\) did beyond \(j = 2\) and \(g_2\) did beyond \(j = 3\). This will happen just because \(\dim ad^5\) eventually becomes too small to contain any irrep of \(c^{(2)} = j\). We also expect that for Lie algebras that have dropped out of the full picture, \(i.e.\) out of correctly assigning members to families, use of dimension formulas will continue to yield in modulus representations carrying the correct Casimir eigenvalue for the family in question.

As we are looking for a dimension formula \(d_j(D)\) which is a polynomial of degree \(j\) in \(D\), we need \(j + 1\) Lie algebras to fix its coefficients and then another Lie algebra in order to confirm that the dimension formula contains non-trivial information.

Table 6 shows the values \(d_j(D)\) of the dimension formulas \textbf{(12)}–\textbf{(15)}, \(j = 6,\ldots, 9\) and the assignment of representations \(X_j\). For \(a_1\), \(d_6 = -7\) and the seven dimensional irrep of \(a_1\) has \(c^{(2)} = 6\), but \(d_7 = d_8 = d_9 = 0\) as expected.

For \(j = 6, 7\), we have obtained the dimension formulas \textbf{(11)} and \textbf{(12)} as the interpolation polynomial for the eight values \(d_j(D)\) where \(D = \dim g\) for the eight Lie algebras \(g \in \mathcal{F}\). Whenever \(ad^{c^{(j)}}\) contains irreps of \(c^{(2)} = j\), then we choose \(X_j\) to be their direct sum and \(d_j(\dim g) = \dim X_j\). Whenever \(ad^{c^{(j)}}\) does not contain any irreps of \(c^{(2)} = j\), so that \(g\) has dropped out of the full picture, then \(d_j(\dim g)\) is the sum or difference of the dimensions of all irreps of \(g\) with \(c^{(2)} = j\). In this case, the signs of the summands are chosen by trial and error so that we obtain a ‘simple’ dimension formula, \(i.e.\) one which has as many linear factors as possible, which has only ‘small’ coefficients and for which \(d_j(D)\) is an integer for any integral \(D\). By experimentation with these interpolations, we always find a unique choice of signs which dramatically, absolutely dramatically, simplifies the interpolation polynomial.

For \(j = 7\), it is of course a trivial fact that we can use the data of 8 algebras in order to uniquely fix a polynomial \(d_j(D)\) of degree 7. The simplicity of the resulting formula \textbf{(13)} is, however, a highly non-trivial property.

Comparing the dimension formulas \textbf{(10)}–\textbf{(13)}, the following general pattern emerges: We have \(d_j(0) = 0\), and the leading term is \(\frac{1}{j}D^j\). For \(j = 8\), we now assume these two conditions and can therefore determine a polynomial of degree eight from only seven additional data points. We employ all algebras \(g \in \mathcal{F}\) except for \(\mathfrak{d}_4\) and obtain \textbf{(14)}.

For \(\mathfrak{d}_4\), we discover the following exception from the rules stated so far. The dimension formula \textbf{(14)} yields \(d_8(28) = -554400\). There exist indeed representations of \(\mathfrak{d}_4\) with \(c^{(2)} = 8\), namely \((0106) \oplus (0160) \oplus (6100)\) of dimension \(3 \cdot 15015\) and \((1213) \oplus (1231) \oplus (3211)\) of dimension \(3 \cdot 169785\), and indeed \(d_8(28)\) is the negative sum of their dimensions. However, \(\mathfrak{d}_4\) has got further representations with \(c^{(2)} = 8\) that do not play any role in the dimension formula \(d_8(D)\), namely \((0044) \oplus (4004) \oplus (4040)\) with dimension \(3 \cdot 35035\).

For \(j = 9\), we again assume the two conditions and determine a polynomial of degree 9 from eight data points, making use of all eight algebras \(g \in \mathcal{F}\).
| $d_0$ | $\mathfrak{g}_2$ | $\mathfrak{g}_4$ | $\mathfrak{h}_4$ | $\ell_4$ | $\mathfrak{h}_6$ | $\ell_6$ | $\ell_8$ |
|---|---|---|---|---|---|---|---|
| 28 | $\oplus(06)$ | (0006) | (0112) | (120100) | (0001101) | (01000001) | 267413986840 |
| 28 | $\oplus(60)$ | (0060) | (01005) | (120100) | (0001101) | (01000001) | 267413986840 |
| $d_7$ | 0 | 1254 | $-\mathfrak{g}_2$ | $-\mathfrak{g}_4$ | $-\mathfrak{h}_4$ | $-\ell_4$ | $-\ell_6$ | $-\ell_8$ |
| $d_8$ | $-125$ | 3003 | $-\mathfrak{g}_2$ | $-\mathfrak{g}_4$ | $-\mathfrak{h}_4$ | $-\ell_4$ | $-\ell_6$ | $-\ell_8$ |
| $d_9$ | 80 | 0 | $-\mathfrak{g}_2$ | $-\mathfrak{g}_4$ | $-\mathfrak{h}_4$ | $-\ell_4$ | $-\ell_6$ | $-\ell_8$ |

Table 6: Irreps related to (12)–(15).
3 Dimension formulas for simple $\mathfrak{g}$ not in $\mathcal{F}$

The formulas (4), (9), (10), (11)–(15) have been derived and discussed in the context of the extended family $\mathcal{F}$ of simple Lie algebras $\mathfrak{g}$ that include the exceptionals.

It is known however that (4) is universal: a well-defined representation $X_2$ of each simple $\mathfrak{g}$ has its dimension given by (4) and the eigenvalue $c^{(2)}(X_2) = 2$ of its quadratic Casimir operator. It is natural to ask if the other dimension formulas (9), (10), (11)–(15) are likewise universal, and, if not, what, if anything, they can tell us for $\mathfrak{g} \not\in \mathcal{F}$.

No systematic algebraic approach is available, but a large body of data can readily be assembled, e.g. using our programs, MAPLE, and references such as [5, 4]. Some indication of the limitations, if any, of the applicability of (9)–(15) to $\mathfrak{g} \not\in \mathcal{F}$ can certainly be gained by reference to the cases of $\mathfrak{b}_2\sim \mathfrak{c}_2$, $\mathfrak{b}_3$, $\mathfrak{c}_3$, $\mathfrak{a}_3$, $\mathfrak{a}_4$, $\mathfrak{a}_5$.

The entries of Table 7 for each $\mathfrak{g}$ and each $X_j$ show representations of $\mathfrak{g}$—often direct sums of irreps—all with the $c^{(2)}(X_j) = j$. The dimension formulas $d_j(D) = \text{dim} \mathfrak{g}$, yield sums or differences of the dimensions of the irreducible components of $X_j$. Note that the feature that $d_j(D)$ gives a difference of the dimensions of irreps was for $\mathfrak{g}_2$ first encountered at $j = 7$.

If we extend Table 7 to $\mathfrak{b}_4$ and $\mathfrak{c}_4$, we encounter at $j = 5$ the same exception from the rules that we have already seen for $\mathfrak{b}_4$ at $j = 8$, namely that the algebra has already dropped out of the full picture (as explained above), and there exist many irreps with $c^{(2)} = j$ only some of which are relevant for the dimension formula.

It is a striking observation that the simple Lie algebras of Table 7 all fit into the general pattern. In particular, the fact that $\mathfrak{b}_2$ does not have any irrep of $c^{(2)} = 6$ can be seen as an ‘explanation’ of the linear factor $D - 10$ in (12). It is then a natural question to ask which are the Lie algebras of dimension 1, 2, 4, 6 and 26 that cause the other integer roots of the dimension formulas (11)–(15).

4 Some further studies

4.1 The factor $(D - 6)$ in $d_5(D)$ and $d_8(D)$

To account for the presence of the factors $(D - 6)$, consider the case of $\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_1$, employing the Cartan matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

and a Cartan-Killing form with no relative scaling of the two $\mathfrak{a}_1$ summands, so that the algebra has an $S_2$ group of diagram automorphisms.

Let $(j, k)$ denote the irrep of dimension $(j + 1)(k + 1)$, so that $ad = (2, 0) \oplus (0, 2)$. We list in Table 8 irreps of $\mathfrak{a}_1 \oplus \mathfrak{a}_1$ with integer eigenvalues $n$ of the quadratic Casimir operator, with $d_n(6)$ alongside for comparisons of the type made systematically made in previous cases.

The entries for $c^{(2)} = 5$ and $c^{(2)} = 8$ explain the $(D - 6)$ factors in $d_5(D)$ and $d_8(D)$, and all the other entries follow precisely a now familiar pattern. Only one entry needs any comment:

$$d_6(6) = 11 = 25 - 2 \cdot 7,$$

where $25 = \text{dim}(4, 4)$, $7 = \text{dim}(6,0) = \text{dim}(0,6)$.
Table 7: The dimension formulas $d_j(D)$ for some algebras $g$ and the corresponding representations $X_j$ in highest weight notation.

|   | $b_2$ | $b_3$ | $c_3$ | $a_3$          | $a_4$          | $a_5$          |
|---|-------|-------|-------|----------------|----------------|----------------|
| $d_2$ | 35   | 189   | 189   | $45 + \overline{45}$ | $126 + \overline{126}$ | $280 + \overline{280}$ |
| $X_2$ | (12) | (101) | (210) | $\oplus (012)$ | $\oplus (0005)$ | $\oplus (01002)$ |
| $d_3$ | 30   | 294   | 385   | $35 + \overline{35}$ | 1024 | $3675$ |
|      |      | +616  | +525  |                    | $+224 + \overline{224}$ | $+840 + \overline{840}$ |
| $X_3$ | (30) | (004) | (030) | $\oplus (004)$ | (1111) | (11011) |
|      |      | $\oplus (202)$ | $\oplus (301)$ | $\oplus (121)$ | $\oplus (0103)$ | $\oplus (00014)$ |
| $d_4$ | $-105$ | 1386  | 2205  | 105             | $1701 + \overline{1701}$ | $12250 + \overline{12250}$ |
|      |      | +819  |       |                 | $+1176 + \overline{126}$ | $+6720 + \overline{1050}$ |
| $X_4$ | (14) | (104) | (121) | (040)           | $\oplus (0220)$ | $(21101) \oplus (10112)$ |
|      |      | $\oplus (310)$ | $\oplus (021)$ | $\oplus (5000)$ | $\oplus (0005)$ | $\oplus (00014)$ |
| $d_5$ | $-84$ | 378   | 2457  | $-189$          | $-189 + \overline{729}$ | $36750 + \overline{36750}$ |
|      |      |       | -2079 |                | $-729 + \overline{729}$ | $+34496$ |
|      | $-154$ |       |       |                 |                 | $+12936 + \overline{12936}$ |
|      |      |       |       |                 |                 | $+462 + \overline{462}$ |
| $X_5$ | (06) | (500) | (022), | (501)          | (1310) | $(01121) \oplus (12110)$ |
|      |      |       | (501) | $\oplus (0105)$ | $\oplus (0131)$ | $\oplus (20202)$ |
|      |      |       |       | $\oplus (222)$ |                 | $\oplus (32001) \oplus (10023)$ |
|      |      |       |       |                 |                 | $\oplus (60000) \oplus (00006)$ |
| $d_6$ | $-9009$ | -11319 | -735 | $-8624 + \overline{8624}$ | $169785$ |
|      |      | -3003 | -875 | $-924 + \overline{924}$ |                 | $+43120 + \overline{43120}$ |
|      |      | +1001 | -875 | $+1176 + \overline{1176}$ |                 | $+25200 + \overline{25200}$ |
| $X_6$ | $-114$ | (321), | (141) | (3202) $\oplus (2023)$ | $(11211)$ |
|      |      | (610), |       | $\oplus (6001) \oplus (1006)$ |                 | $\oplus (23010) \oplus (01032)$ |
|      |      | (004) | $\oplus (214)$ | $\oplus (0500) \oplus (0050)$ |                 | $\oplus (03200) \oplus (00230)$ |
Representations of the exceptional and other Lie algebras...

\[
c^{(2)} = n \quad \text{irreps} \quad d_n(6)
\]

|   |   |   |
|---|---|---|
| 1 | \(2, 0\) \(\oplus\) \(0, 2\) | 6 |
| 2 | \(2, 2\) | 9 |
| 3 | \(4, 0\) \(\oplus\) \(0, 4\) | \(-10\) |
| 4 | \(4, 2\) \(\oplus\) \(2, 4\) | \(-30\) |
| 5 | none | 0 |
| 6 | \((6, 0) \oplus (0, 6), (4, 4)\) | 11 |
| 7 | \((6, 2) \oplus (2, 6)\) | \(-42\) |
| 8 | none | 0 |
| 9 | \((6, 4) \oplus (4, 6)\) | 70 |

Table 8: Data for \(a_1 \oplus a_1\).

We note also that all the irreps that feature here are either self-conjugate or else occur as conjugate pairs, as the \(S_2\) invariance of \(ad\) requires.

4.2 \(a_1 \oplus a_1 \oplus a_1\)

In this case we use as Cartan matrix twice the unit matrix, again with no relative scales, so that the algebra has a group \(S_3\) of diagram automorphisms. Table 9 displays data about all the irreps with integral values of the Casimir operator. The notation \((j, k, l)\) denotes the irrep with dimension \((j + 1)(k + 1)(l + 1)\), so that \(ad = (2, 0, 0) \oplus (0, 2, 0) \oplus (0, 0, 2)\). To keep the displays as brief as is reasonable, the notation \(r \cdot (a, b, c)\) denotes the direct sum of all \(r\) distinct permutations of \((a, b, c)\). In view of the automorphism group \(S_3\), we may have \(r = 3\) and \(r = 6\). Again we can check that all the data conforms to the expected pattern.

4.3 \(a_1 \oplus a_1 \oplus a_1 \oplus a_1\)

This example was treated to see one further automorphism group at work. But few surprises were expected or found. Everything is in full accord with expectation. We do not display the data that would make a table like Tables 9 and 10 but note only the situation for \(d_9(12)\). The set of irreps, in notation similar to that used in previous subsections, that have \(c^{(2)} = 9\) is

\[
12 \cdot (6, 4, 0, 0), \quad 4 \cdot (4, 4, 4, 0), \quad 4 \cdot (7, 1, 1, 1), \quad 4 \cdot (6, 2, 2, 2),
\]

with dimensions \(12 \cdot 35, 4 \cdot 125, 4 \cdot 64, 4 \cdot 189\), and we have

\[
d_9(12) = -836 = 12 \cdot 35 - 4 \cdot (189 + 125).
\]

We note that the resolution (24) does not employ the irreps \(4 \cdot (7, 1, 1, 1)\), but such an omission is also familiar in previous cases. There is some curious numerology in the \(n = 9\) case: \(64 = (189 - 125)\), and similar things are seen for lower \(n\) cases.
Representations of the exceptional and other Lie algebras

\[ c^{(2)} = n \]

| irreps                  | \( d_n(9) \)   |
|-------------------------|---------------|
| \((2, 2, 2), 3 \cdot (4, 0, 0)\) | 12 = 27 - 3 \cdot 5 |
| \(6 \cdot (4, 2, 0)\)   | -90 = -6 \cdot 15 |
| \(3 \cdot (4, 2, 2)\)   | -135 = -3 \cdot 45 |
| \(3 \cdot (4, 4, 0), 3 \cdot (6, 0, 0)\) | 54 = 3 \cdot 25 - 3 \cdot 7 |
| \(6 \cdot (6, 2, 0), 3 \cdot (4, 4, 2)\) | 3 \cdot 75 - 6 \cdot 21 |
| \(3 \cdot (6, 2, 2)\)   | -189 = -3 \cdot 63 |
| \(6 \cdot (6, 4, 0), (4, 4, 4)\) | 85 = 6 \cdot 35 - 125 |

Table 9: Data for \(a_1 \oplus a_1 \oplus a_1\).

But we leave the analysis here, without including much further data with features that are of a qualitatively similar nature to what has been presented. The fact however is that everything follows a coherent if far from understood pattern.

5 Conclusion

We have extended the dimension formulas of [8] for a particular family up to the ninth power of the adjoint representation. The formulas (11)–(15) describe a further striking uniformity of the Lie algebras of the exceptional series and our results of Sec. 3 and 4 indicate even a uniformity beyond that. The formulas were obtained by inspection of a large amount of data from tables and from computer calculations and finally by a considerable amount of trial and error until we had found the appropriate rules that give rise to ‘simple’ formulas. The fact that ‘easy’ formulas such as (11)–(15) with coefficients smaller than a few thousand give rise to integers of up to sixteen digits which precisely correspond to dimensions of representations of the exceptional Lie algebras, deserves to be seen as of real significance.

We conclude by listing some particular observations and first ideas that come to mind.

- The formulas (11)–(15) for \(d_j(D)\) are polynomials in \(D = \dim \mathfrak{g}\). From [8], we might have expected only rational functions in \(\alpha, m\) or \(D\). In particular, \(\dim Y_j\) in the notation of [8] (\(Y_j\) is the highest weight component of the \(j\)-th totally symmetric tensor power of \(ad\)) is not a polynomial in \(D\).

- With the rational functions of [8], one can search for those values of the parameter \(\alpha\) for which the result is an integer and thus obtain a list of all algebras that conform to the family pattern. In our case, however, the formulas for \(d_j(D)\) give integer results for any integral \(D\).

- Whenever the dimension formula ceases to give a strictly positive answer and \(ad^{\wedge j}\) does not contain any representation of the desired \(c^{(2)}\), we can successfully describe the phenomenology of the situation, but do not have a satisfactory explanation of why it occurs.
• The leading coefficient of the formula for $d_j(D)$ is $1/j!$. This looks like a growth rate of the dimension $d_j(D)$ of the family member $X_j$ if the dimension $D$ of the underlying Lie algebra $\mathfrak{g}$ tends to infinity.

• Our data confirm that the dimension formulas extend to other simple Lie algebras not in the family $F$ and beyond that to some non-simple Lie algebras. A particularly strong indication for this is the fact that the algebras $\mathfrak{b}_2$ and $\mathfrak{a}_1 \oplus \mathfrak{a}_1$ ‘explain’ some of the integer roots of the polynomials (11), (12) and (14).

It is an interesting question of whether one can identify for each linear factor $(D - m)$ of the dimension formulas $d_j(D)$ a Lie algebra of dimension $m$ for which there exists no irrep with $c^{(2)} = j$. For large $j$, however, there is hardly any simple Lie algebra other than $\mathfrak{a}_1$. It is therefore crucial to go beyond simple Lie algebras and to include more examples in order to prove or disprove the conjecture. In this context, it is a striking observation that $d_9(D)$ of (15) has so many linear factors.

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