COBORDISM CATEGORIES AND PARAMETRIZED MORSE THEORY

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Abstract. Fix a tangential structure $\theta : B \longrightarrow BO(d + 1)$ and an integer $k < d/2$. In this paper we determine the homotopy type of a cobordism category $\text{Cob}_{\theta}^{mf,k}$, where morphisms are given by $\theta$-cobordisms $W : P \rightsquigarrow Q$ equipped with a choice a Morse function $h_W : W \longrightarrow [0, 1]$, with the property that all critical points $c \in W$ of $h_W$ satisfy: $k < \text{index}(c) < d - k + 1$. In particular, we prove that there is a weak homotopy equivalence $B\text{Cob}_{\theta}^{mf,k} \simeq \Omega^\infty hW^k_\theta$, where $hW^k_\theta$ is a certain Thom spectrum associated to the space of Morse jets on $\mathbb{R}^{d+1}$.

In the special case that $k = -1$, the equivalence $B\text{Cob}_{\theta}^{mf,-1} \simeq \Omega^\infty hW^{-1}_\theta$ follows from the work of Madsen and Weiss in [12], used in their celebrated proof of the Mumford conjecture. Following the methods of Madsen and Weiss we use the weak equivalence $B\text{Cob}_{\theta}^{mf,k} \simeq \Omega^\infty hW^k_\theta$ to give an alternative proof the “high-dimensional Madsen-Weiss theorem” of Galatius and Randal-Williams from [8], which identifies the homology of the moduli spaces, $\text{BDiff}((S^n \times S^n)^{#g}, D^{2n})$, in the limit $g \rightarrow \infty$.

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1. Introduction

To state our results, we begin by defining a topological category whose objects are given by closed manifolds embedded in high-dimensional Euclidean space, and whose morphisms are given by embedded cobordisms equipped with a Morse function. Fix an integer $d \geq 0$ and a tangential structure $\theta : B \longrightarrow BO(d + 1)$.

Definition 1.1. Objects of the (non-unital) topological category $\text{Cob}_\theta^{mf}$ are given by closed $d$-dimensional submanifolds $M \subset \mathbb{R}^\infty$, equipped with a $\theta$-structure $\hat{\ell}_M : TM \oplus e^1 \longrightarrow \theta^* \gamma^{d+1}$. The morphisms $M_0 \rightsquigarrow M_1$ are given by pairs $(t, W)$ where $t \in (0, \infty)$ and $W \subset [0, t] \times \mathbb{R}^\infty$ is a compact submanifold equipped with a $\theta$-structure $\hat{\ell}_W : TW \longrightarrow \theta^* \gamma^{d+1}$, subject to the following conditions:
(i) the height function, $W \hookrightarrow [0, t] \times \mathbb{R}^\infty \xrightarrow{\text{proj}} [0, t]$, is a Morse function;

(ii) $W \cap \{(0, t) \times \mathbb{R}^\infty\} = (M_0 \times \{0\}) \cup (M_1 \times \{1\})$ as a $\theta$-manifold, and furthermore the intersection is orthogonal.

We will always denote the height function from (i) by $h_W : W \to [0, t]$. By condition (ii) $W$ is a cobordism between $M_0$ and $M_1$. Composition in this category is defined in the usual way by concatenation of cobordisms.

The category $\text{Cob}_\theta^{mf}$ is topologized so that for any two objects $M_0, M_1$, there is a homotopy equivalence

\begin{equation}
\text{Cob}_\theta^{mf}(M_0, M_1) \simeq \coprod_{[W]} \left( \text{Bun}(TW, \theta^* \gamma^{d+1}; \hat{\ell}_{M_0} \sqcup \hat{\ell}_{M_1}) \times \text{Morse}(W; M_0, M_1) \right) \sslash \text{Diff}(W, \partial W),
\end{equation}

where the disjoint union is taken over all diffeomorphism classes of compact manifolds $W$, whose boundary is equipped with an identification, $\partial W = M_0 \sqcup M_1$. The space $\text{Morse}(W; M_0, M_1)$ is the space of all Morse functions $f : W \to [0, 1]$ having 0 and 1 as regular values, with $f^{-1}(0) = M_0$ and $f^{-1}(1) = M_1$. The space $\text{Bun}(TW, \theta^* \gamma^{d+1}; \hat{\ell}_{M_0} \sqcup \hat{\ell}_{M_1})$ consists of all $\theta$-structures on $W$ that agree with $\hat{\ell}_{M_0} \sqcup \hat{\ell}_{M_1}$ when restricted to the boundary.

Using the work of Madsen and Weiss from [12], the homotopy type of the classifying space $B\text{Cob}_\theta^{mf}$ can be identified with the infinite loopspace of a familiar Thom spectrum. Let $G_\theta^{mf}(\mathbb{R}^\infty)$ denote the space of tuples $(V, l, l, \sigma)$ where:

- $V \subset \mathbb{R}^\infty$ is a $(d + 1)$-dimensional vector subspace,
- $\hat{l}$ is a $\theta$-orientation on $V$,
- $l : V \to \mathbb{R}$ is a linear functional,
- $\sigma : V \otimes V \to \mathbb{R}$ is a symmetric bilinear form,

with the property that $(l, \sigma)$ satisfies the Morse condition: if $l = 0$, then $\sigma$ is non-degenerate. Let $U_{d+1, \infty} \to G_\theta^{mf}(\mathbb{R}^\infty)$ denote the canonical $(d + 1)$-dimensional vector bundle and let $h_W^\theta$ denote the Thom spectrum associated to the $-(d + 1)$-dimensional virtual bundle, $-U_{d+1, \infty} \to G_\theta^{mf}(\mathbb{R}^\infty)$. The theorem stated below follows from the work of Madsen and Weiss in [12].

**Theorem A** (Madsen-Weiss, 2002). For any tangential structure $\theta : B \to BO(d)$, there is a weak homotopy equivalence, $B\text{Cob}_\theta^{mf} \simeq \Omega^{\infty-1}h_W^\theta$.

**Remark 1.2.** In [12], Madsen and Weiss prove a weak homotopy equivalence $|\mathcal{W}_\theta| \simeq \Omega^{\infty-1}h_W^\theta$, where $|\mathcal{W}_\theta|$ is the representing space of a sheaf $\mathcal{W}_\theta$. The cobordism category $\text{Cob}_\theta^{mf}$ is never officially uttered in their paper, but by importing an extra argument from [5] one can prove the weak equivalence $B\text{Cob}_\theta^{mf} \simeq |\mathcal{W}_\theta|$; this is explained in Sections 1.1 and 2.2.

In this paper we prove a generalization of Theorem A for subcategories of $\text{Cob}_\theta^{mf}$ consisting of cobordisms $(t, W) : M \leadsto N$ whose associated Morse function $h_W : W \to [0, t]$ is subject to certain constraints.

**Definition 1.3.** Let $k \geq -1$ be an integer. The objects of the subcategory $\text{Cob}_\theta^{mf, k} \subset \text{Cob}_\theta^{mf}$ are given by those $M$ for which the map $\ell_M : M \to B$ (which underlies the $\theta$-structure $\hat{\ell}_M$) is $k$-connected. The morphisms of $\text{Cob}_\theta^{mf, k}$ are given by those $(t, W) : P \leadsto Q$ such that all critical points $c \in W$ of the associated Morse function $h_W : W \to [0, t]$ satisfy: $k < \text{index}(c) < d - k + 1$. It is immediate from the definition that $\text{Cob}_\theta^{mf, -1} \simeq \text{Cob}_\theta^{mf}$.
Our main theorem identifies the homotopy type of the classifying space $BC\text{obe}^{mf,k}_\theta$ in the case that $k < d/2$. We need to construct a generalization of the spectrum $hW_\theta$. For $k \in \mathbb{Z}_{\geq -1}$, we define $G^\text{mf}_\theta(\mathbb{R}^\infty)^k \subset G^\text{mf}_\theta(\mathbb{R}^\infty)$ to be the subspace consisting of those $(V, \ell, l, \sigma)$ subject to the following condition: if $l = 0$, then $\sigma$ is non-degenerate with $k < \text{index}(\sigma) < d - k + 1$. As before, $hW^k_\theta$ is defined to be the Thom spectrum associated to the virtual bundle, $-U_{d+1,\infty} \rightarrow G^\text{mf}_\theta(\mathbb{R}^\infty)^k$. The main theorem of this paper is stated below.

**Theorem B.** Let $-1 \leq k < d/2$. Suppose that the tangential structure $\theta : B \rightarrow BO(d + 1)$ is chosen so that $B$ satisfies Wall’s finiteness condition $F(k)$ (see [18]). Then there is a weak homotopy equivalence, $BC\text{obe}^{mf,k}_\theta \simeq \Omega^{\infty-1}hW^k_\theta$.

We note that the case of Theorem B for $k = -1$ is covered by Theorem A.

**Remark 1.4.** In [12], the weak homotopy equivalence $BC\text{obe}^{mf}_\theta \simeq \Omega^{\infty-1}hW_\theta$ from Theorem A constitutes a major step in Madsen and Weiss’ proof of the Mumford Conjecture, which identifies the homology of the moduli space $\text{BDiff}((S^1 \times S^1)^{\#g}, D^2)$ in the limit as $g \rightarrow \infty$. Proceeding in a way similar to Madsen and Weiss, we will use the weak equivalences $BC\text{obe}^{mf,k}_\theta \simeq \Omega^{\infty-1}hW^k_\theta$ from Theorem B to recover the “High-dimensional Madsen-Weiss theorem” of Galatius and Randal-Williams from [3], which identifies the homology of $\text{BDiff}((S^n \times S^n)^{\#g}, D^{2n})$ in the limit as $g \rightarrow \infty$ for all $n \geq 2$. We give an outline of how this is done later in the introduction in Subsection 1.2 after discussing another application of Theorem B.

1.1. **Outline of the proof of Theorem B.** The main technical ingredient in Madsen and Weiss’ proof of Theorem C is Vassiliev’s $h$-principle [16] applied to certain spaces of proper Morse functions. We refer the reader to [12, Section 4] for the precise formulation of how the $h$-principle is used and for the precise definition of the function spaces that they apply it to. Now, there does not exist any version of Vassiliev’s $h$-principle for such spaces of proper Morse functions $f : W \rightarrow \mathbb{R}$ whose critical points $c \in W$ satisfy the condition $k < \text{index}(c) < d - k + 1$. For this reason, our proof of Theorem B must follow a different strategy than the proof of Theorem A in [12].

Our first step is to replace the classifying space $BC\text{obe}^{mf,k}_\theta$ with a weakly equivalent space of manifolds that is more flexible and easier to work with.

**Definition 1.5.** For $k \in \mathbb{Z}_{\geq -1}$, the space $D_{\theta}^{mf,k}$ consists of $(d + 1)$-dimensional submanifolds $W \subset \mathbb{R} \times \mathbb{R}^\infty$, equipped with a $\theta$-structure $\ell_W : TW \rightarrow \theta^*\gamma^{d+1}$, subject to the following conditions:

(i) the map $W \leftarrow \mathbb{R} \times \mathbb{R}^\infty \stackrel{\text{proj}}{\rightarrow} \mathbb{R}$ is a proper Morse function for which all critical points $c \in W$ satisfy: $k < \text{index}(c) < d - k + 1$;

(ii) the map $\ell_W : W \rightarrow B$ is $k$-connected.

The space of manifolds $D_{\theta}^{mf,k}$ is topologized by the same procedure used in [7, Section 2]. In the case that $k = -1$, the space $D_{\theta}^{mf,-1}$ is a model for the representing space of the sheaf $W_\theta$ considered by Madsen and Weiss in [12].

Using an argument from [5, Section 4], it follows that for all $k$ there is a weak homotopy equivalence, $BC\text{obe}^{mf,k}_\theta \simeq D_{\theta}^{mf,k}$. Using a version of the Pontryagin-Thom construction from [12] we construct a weak map

$$P_{\theta}^k : D_{\theta}^{mf,k} \rightarrow \Omega^{\infty-1}hW^k_\theta.$$ 

In the special case that $k = -1$, Madsen and Weiss prove that this map gives the weak homotopy equivalence $D_{\theta}^{mf,-1} \simeq \Omega^{\infty-1}hW^{-1}_\theta$. Our strategy is to prove Theorem B by induction on the integer
k, using Madsen and Weiss’ result for \( k = -1 \) as the base case of the induction. In order to get the induction argument running, we need to introduce one more tool.

We let \( G_\theta^{mf}(\mathbb{R}^\infty)_{loc} \subset G_\theta^{mf}(\mathbb{R}^\infty) \) denote the subspace consisting of those \( (V, \ell, l, \sigma) \) for which \( l = 0 \). For \( k \in \mathbb{Z}_{\geq -1} \), we define \( G_\theta^{mf}(\mathbb{R}^\infty)_{loc}^{[k,d-k+1]} \subset G_\theta^{mf}(\mathbb{R}^\infty)_{loc} \) to be the subspace consisting of those \( (V, \ell, l, \sigma) \) for which \( \text{index}(\sigma) \in \{ k, d - k + 1 \} \).

**Definition 1.6.** For \( k \in \mathbb{Z}_{\geq -1} \), the space \( D_{\theta,loc}^{mf,[k,d-k+1]} \) is defined to consist of pairs \( (\bar{x}, \phi) \) where:

(i) \( \bar{x} \subset \mathbb{R} \times \mathbb{R}^\infty \) is a 0-dimensional submanifold with the property that \( \bar{x} \cap ((-\delta, \delta) \times \mathbb{R}^\infty) \) is finite for all \( \delta \in (0, \infty) \);

(ii) \( \phi : \bar{x} \rightarrow G_\theta^{mf}(\mathbb{R}^\infty)_{loc}^{[k,d-k+1]} \) is a map.

There is a map

\[
L_k : D_{\theta}^{mf,[k-1]} \rightarrow D_{\theta,loc}^{mf,[k,d-k+1]}
\]

deﬁned by sending \( W \in D_{\theta}^{mf,[k]} \) to the pair \( (\bar{x}, \phi) \) where \( \bar{x} \) is the set of critical points of degrees \( k \) and \( d - k + 1 \) for the height function \( \hat{W} : \mathbb{R} \times \mathbb{R}^\infty \rightarrow \mathbb{R} \), and \( \phi \) is the map which sends each \( x \in \bar{x} \) to the tuple \( (T_x W, \ell_W | x, d^2 h_W | x) \), where \( d^2 h_W | x \) is the Hessian of \( h_W \) at the critical point \( x \).

Notice that for any \( k \), the subspace \( D_{\theta}^{mf,[k]} \subset D_{\theta}^{mf,[k-1]} \) is contained in the fibre \( L_k^{-1}(\emptyset) \subset D_{\theta,loc}^{mf,[k-1]} \) of the map \( L_k \) over the empty element \( \emptyset \in D_{\theta,loc}^{mf,[k,d-k+1]} \). As a result, the inclusion \( D_{\theta}^{mf,[k]} \hookrightarrow D_{\theta,loc}^{mf,[k-1]} \) induces a map

\[
D_{\theta}^{mf,[k]} \rightarrow \text{hofibre}(D_{\theta}^{mf,[k-1]} \xrightarrow{L_k} D_{\theta,loc}^{mf,[k,d-k+1]})
\]

The main technical theorem that we prove in this paper is stated below.

**Theorem 1.7.** Suppose that \( k < d/2 \). Suppose that \( B \) satisfies Wall’s ﬁniteness condition \( F(k) \). Then the map from (1.4) is a homotopy equivalence, and thus the following sequence of maps

\[
D_{\theta}^{mf,[k]} \xrightarrow{L_k} D_{\theta,loc}^{mf,[k,d-k+1]}
\]

is a homotopy fibre sequence.

Let us now sketch how to use the fibre sequence from Theorem 1.7 to prove Theorem B. Let \( hW_{\theta,loc}^{[k,d-k+1]} \) denote the suspension spectrum \( \Sigma^\infty \left( G_\theta^{mf}(\mathbb{R}^\infty)_{loc}^{[k,d-k+1]} \right) \). For all \( k \) there is a scanning map

\[
P_{\theta,loc}^{[k,d-k+1]} : D_{\theta,loc}^{mf,[k,d-k+1]} \rightarrow \Omega^\infty \Omega^{-1} hW_{\theta,loc}^{[k,d-k+1]},
\]

and it follows from the Barratt-Priddy-Quillen theorem (or [7, Theorem 3.12]) that this scanning map is a weak homotopy equivalence for all \( k \). Following [12, Section 3] there is a cofibre sequence of spectra, \( hW_{\theta}^{k} \rightarrow hW_{\theta}^{k-1} \rightarrow hW_{\theta,loc}^{[k,d-k+1]} \). Furthermore, the maps \( P_{\theta,loc}^{[k,d-k+1]} \) and \( P^k \) can be modeled in such a way so that the following diagram is homotopy commutative,

\[
\begin{array}{ccc}
D_{\theta}^{mf,[k]} & \xrightarrow{L_k} & D_{\theta,loc}^{mf,[k,d-k+1]} \\
\downarrow & & \downarrow \\
\Omega^\infty \Omega^{-1} hW_{\theta}^{k} & \rightarrow & \Omega^\infty \Omega^{-1} hW_{\theta,loc}^{[k,d-k+1]} \\
\end{array}
\]
With this homotopy commutative diagram established, we may use Theorem 1.7 to prove Theorem B by induction on $k$. Indeed, in the case that $k = 0$, the vertical map $\mathcal{D}^{\text{mf}, -1}_\theta \to \Omega^{\infty - 1} h W^{-1}_\theta$ is a weak homotopy equivalence as proven by Madsen and Weiss. Since the right-vertical map is always a weak homotopy equivalence for all $k$ and since both rows are fibre sequences, it follows that the left-vertical map is a weak homotopy equivalence as well. Continuing by induction for all $k < d/2$ yields Theorem B.

1.2. Homology fibrations and stabilization. We now describe how to use Theorem B to recover the high-dimensional analogue of the Madsen-Weiss theorem proven by Galatius and Randall-Williams in [8]. We turn our attention to the case where $d = 2n$ and $k = n$. In this case the height function $h_W : W \to \mathbb{R}$ associated to an element $W \in \mathcal{D}^{\text{mf}, n}_\theta$ is a submersion. With $d = 2n$ and $k = n$, the sequence from the statement of Theorem 1.7 fails to be a fibre sequence. However, it turns out that we can recover a homological fibre sequence upon applying a certain ‘stabilization process’ to the spaces $\mathcal{D}^{\text{mf}, k}_\theta$. In order to describe this stabilization process, we will need to specify a certain subspace of $\mathcal{D}^{\text{mf}, k}_\theta$.

For what follows, fix a $d$-dimensional compact submanifold $L \subset [0, \infty) \times \mathbb{R}^{\infty - 1}$ that agrees with $[0, \infty) \times \partial L$ near $\{0\} \times \mathbb{R}^{\infty - 1}$.

**Definition 1.8.** Fix a $\theta$-structure $\hat{\ell}_L : TL \oplus \varepsilon^1 \to \theta^* \gamma^{d+1}$. For all $k \in \mathbb{Z}_{\geq -1}$, the subspace $\mathcal{D}^{\text{mf}, k}_{\theta, L} \subset \mathcal{D}^{\text{mf}, k}_\theta$ consists of those $W$ for which,

$$W \cap (\mathbb{R} \times [0, \infty) \times \mathbb{R}^{\infty - 1}) = \mathbb{R} \times L,$$

as a $\theta$-manifold.

It can be shown whenever $k < d/2$, for all such manifolds $L$, the inclusion induces a weak homotopy equivalence, $\mathcal{D}^{\text{mf}, k}_{\theta, L} \simeq \mathcal{D}^{\text{mf}, k}_\theta$. Let $d = 2n$ and suppose that $L \neq \emptyset$. Let $P$ denote the boundary $\partial L$ and let $\hat{\ell}_P$ denote the restriction of $\hat{\ell}_L$ to $P = \partial L$.

**Definition 1.9.** Consider the manifold $S^n \times S^n$. Fix a $\theta$-structure $\hat{\ell}_0 : T(S^n \times S^n) \oplus \varepsilon^1 \to \theta^* \gamma^{d+1}$ with the property that the underlying map $\ell_0 : S^n \times S^n \to B$ is null-homotopic. Choose an embedding $\phi : S^n \times S^n \to (0, 1) \times \mathbb{R}^{\infty - 1}$ with image disjoint from the manifold $(0, 1) \times P$. Let

$$H(P) \subset [0, 1] \times \mathbb{R}^{\infty - 1}$$

be the submanifold obtained by forming the connected sum of $[0, 1] \times P$ with $\phi(S^n \times S^n)$ along an embedded arc connecting $[0, 1] \times P$ with $\phi(S^n \times S^n)$. Fix a $\theta$-structure $\hat{\ell}_{H(P)}$ on $H(P)$ that agrees with $\hat{\ell}_P$ on $\{0\} \times P$ and $\{1\} \times P$, and that agrees with $\hat{\ell}_0$ on the connected sum factor $S^n \times S^n$.

We consider $H(P)$ to be a self-cobordism of the $\theta$-manifold $P$.

Let $\hat{H}(P) \subset \mathbb{R} \times [0, \infty) \times \mathbb{R}^{\infty - 1}$ be the submanifold defined by the union

$$\hat{H}(P) = \mathbb{R} \times H(P) \bigcup_{\mathbb{R} \times \{1\} \times P} \mathbb{R} \times (L + e_1)$$

where $(L + e_1) \subset [1, \infty) \times \mathbb{R}^{\infty - 1}$ denotes the manifold obtained by translating $L \subset [0, \infty) \times \mathbb{R}^{\infty - 1}$ by one unit in the first coordinate. Using $\hat{H}(P)$, we define a map

$$\hat{\mathcal{D}}^{\text{mf}, k}_{\theta, P} : \mathcal{D}^{\text{mf}, k}_{\theta, L} \to \mathcal{D}^{\text{mf}, k}_{\theta, L}$$

by sending $W \in \mathcal{D}^{\text{mf}, k}_{\theta, L}$ to the union, $(W \setminus (\mathbb{R} \times L)) \cup \hat{H}(P)$, and then rescaling the second coordinate of the ambient space. Since $S^n \times S^n$ is $(n - 1)$-connected, it follows that 1.7 restricts to a map
\(D_{\theta,L}^{n,f,k} \longrightarrow D_{\theta,L}^{n,f,k}\) for all \(k \leq n = d/2\). The map \(- \cup \hat{H}(P)\) may be iterated, and for each \(k \leq d/2\) we define

\[
D_{\theta,L}^{n,f,k,\text{stab}} := \text{hocolim} \left( D_{\theta,L}^{n,f,k} - \cup \hat{H}(P) \longrightarrow D_{\theta,L}^{n,f,k} - \cup \hat{H}(P) \longrightarrow \cdots \right),
\]

Now, for \(k < d/2\) we prove that the map \(- \cup \hat{H}(P) : D_{\theta,L}^{n,f,k} \longrightarrow D_{\theta,L}^{n,f,k}\) is a weak homotopy equivalence, and thus we obtain the weak homotopy equivalence \(D_{\theta,L}^{n,f,k,\text{stab}} \simeq D_{\theta,L}^{n,f,k}\) when \(k < d/2\). For \(k = n = d/2\) however, this weak homotopy equivalence does not hold.

Our main theorem relates \(D_{\theta,L}^{n,f,n,\text{stab}}\) to the homotopy fibre of \(L_n : D_{\theta,L}^{n,f,n-1,\text{stab}} \longrightarrow D_{\theta,L}^{n,f,n+1}\).

As before, the inclusion \(D_{\theta,L}^{n,f,n,\text{stab}} \hookrightarrow D_{\theta,L}^{n,f,n-1,\text{stab}}\) factors through the fibre \(L_n^{-1}(\emptyset)\), and thus induces a map

\[
(1.8) \quad D_{\theta,L}^{n,f,n,\text{stab}} \longrightarrow \text{hofibre} \left( D_{\theta,L}^{n,f,n-1,\text{stab}} \xrightarrow{L_n} D_{\theta,L}^{n,f,n+1} \right).
\]

Using the homological stability theorem from [6] we prove the following theorem.

**Theorem 1.10.** Let \(d = 2n\) and suppose that \(B\) satisfies Wall’s finiteness condition \(F(n)\). Then the map \((1.8)\) is an acyclic map.

By invoking the same argument given Section 1.1 to derive Theorem B from Theorem 1.7 we obtain the following corollary which is the analogue of Theorem B in the case that \(d = 2n\) and \(k = n\).

**Corollary 1.11.** Let \(d = 2n\) and \(k = n\). Suppose that \(B\) satisfies Wall’s finiteness condition \(F(n)\). Then there is an acyclic map, \(D_{\theta,L}^{n,f,n,\text{stab}} \longrightarrow \Omega^{-1}hW^n_{\theta}\).

The high-dimensional version of the Madsen-Weiss theorem, proven by Galatius and Randal-Williams in [8], can be recovered from Corollary 1.11 by simply reinterpreting the spaces involved in the statement of the above corollary. In particular, the spectrum \(hW^n_{\theta}\) is homotopy equivalent to the Madsen-Tillmann spectrum \(\Sigma^{-1}MT_{\theta} \simeq hW^n_{\theta}\) and the space \(D_{\theta,L}^{n,f,n,\text{stab}}\) is homotopy equivalent to the Stable moduli space of \(\theta\)-manifolds studied in \([8]\) and \([6]\). We will show how to recover their result from the above corollary in Section 11. We remark that our method of proof specializes to Madsen and Weiss’ proof of the Mumford Conjecture \([12]\) in the \(n = 1\) case.

1.3. **Organization.** In Section 2 we cover the basic definitions and state the main theorems for us to prove. In Section 3 we define an alternative model for the spaces \(D_{\theta}^{n,f,k}\) which will facilitate the proofs of the main technical theorems of the paper. The proof that this newly defined space is weakly equivalent to \(D_{\theta}^{n,f,k}\) follows an argument from [12] almost verbatim, and thus we relegate its proof to the appendix. Sections 4 through 10 involve all of the technical work of the paper. The propositions proven in those sections are all stated in terms of the definitions given in Section 3 and thus the agenda for those sections is set and described in Section 3. In Section 11 we show how to use our results to recover the theorem of Galatius and Randal-Williams from [8]. Appendix A is devoted to giving the proof of a technical result stated in Section 8.

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2. Spaces of Manifolds and the Pontryagin-Thom Construction

2.1. Spaces of manifolds. In this section we define some spaces to be used throughout the paper. We first recall the definition of a tangential structure.

**Definition 2.1.** A tangential structure is a map \( \theta : B \to BO(m) \). A \( \theta \)-structure on a \( l \)-dimensional manifold \( M \) (with \( l \leq m \)) is defined to be a bundle map \( \hat{\ell}_M : TM \oplus \epsilon^{m-1} \to \theta^*\gamma^m \), i.e. a fibrewise linear isomorphism. The pair \((M, \hat{\ell}_M)\) is called an \( l \)-dimensional \( \theta \)-manifold.

For what follows, fix an integer \( d \in \mathbb{Z}_{\geq 0} \) and a tangential structure \( \theta : B \to BO(d+1) \). Below we define a space of \((d+1)\)-dimensional \( \theta \)-manifolds.

**Definition 2.2.** Let \( N \in \mathbb{Z}_{\geq 0} \). Fix a \((d-1)\)-dimensional, closed, submanifold \( P \subset \mathbb{R}^{\infty-1} \) and a \( \theta \)-structure \( \hat{\ell}_P \) on \( P \). The space \( D_{\theta,P} \) consists of pairs \((W, \hat{\ell}_W)\) where \( W \subset \mathbb{R} \times (-\infty, 0] \times \mathbb{R}^{\infty-1} \) is a \((d+1)\)-dimensional submanifold (not necessarily compact), and \( \hat{\ell}_W \) is a \( \theta \)-structure on \( W \), subject to the following conditions:

(i) \( \partial W = \mathbb{R} \times P = W \cap (\mathbb{R} \times \{0\} \times \mathbb{R}^{\infty-1}) \);

(ii) \( W \) agrees with \( \mathbb{R} \times (-\infty, 0] \times P \) near \( \mathbb{R} \times \{0\} \times \mathbb{R}^{\infty-1} \) as a \( \theta \)-manifold, where the space \( \mathbb{R} \times (-\infty, 0] \times P \) is equipped with the \( \theta \)-structure induced by \( \hat{\ell}_P \).

(iii) the map \( W \xrightarrow{\text{proj.}} \mathbb{R} \times (-\infty, 0] \times \mathbb{R}^{\infty-1} \xrightarrow{\text{proj.}} \mathbb{R} \), is a proper map. We note that this condition implies that for any \( \delta \in (0, \infty) \), the space \( W \cap \{[-\delta, \delta] \times (-\infty, 0] \times \mathbb{R}^{\infty-1}\} \) is compact.

The space \( D_{\theta,P} \) is topologized by the same process that was employed in [7, Section 2]; we do not repeat their construction here.

We will also have to work with spaces of compact \( d \)-dimensional manifolds with prescribed boundary.

**Definition 2.3.** Let \( N \in \mathbb{Z}_{\geq 0} \). Fix a \((d-1)\)-dimensional closed submanifold \( P \subset \mathbb{R}^{\infty-1} \), and a \( \theta \)-structure \( \hat{\ell}_P \) on \( P \). The space \( N_{\theta,P} \) consists of pairs \((M, \hat{\ell}_M)\) where \( M \subset (-\infty, 0] \times \mathbb{R}^{\infty-1} \) is a \( d \)-dimensional compact submanifold, and \( \hat{\ell} : TM \oplus \epsilon^1 \to \theta^*\gamma^{d+1} \) is a \( \theta \)-structure, subject to the following conditions:

(i) \( \partial M = M \cap \{0\} \times \mathbb{R}^{\infty-1} = P \),

(ii) \( M \) agrees with \((-\infty, 0] \times P \) near \( \{0\} \times \mathbb{R}^{\infty-1} \) as a \( \theta \)-manifold, where \((-\infty, 0] \times P \) is equipped with the \( \theta \)-structure induced by \( \hat{\ell}_P \).

The space \( N_{\theta,P} \) is topologized in the usual way by identifying,

\[
(2.1) \quad N_{\theta,P} \cong \coprod_{[M]} \left( \text{Emb}(M, (-\infty, 0] \times \mathbb{R}^{\infty-1}; i_P) \times \text{Bun}(TM \oplus \epsilon^1, \theta^*\gamma^{d+1}; \hat{\ell}_P) \right) / \text{Diff}(M, P),
\]

where the union ranges over all diffeomorphism classes of compact manifolds \( M \), equipped with an identification \( \partial M = P \). The space \( \text{Emb}(M, (-\infty, 0] \times \mathbb{R}^{\infty-1}; i_P) \) consists of all neat embeddings \( M \to (-\infty, 0] \times \mathbb{R}^{\infty-1} \) that agree with the inclusion \( i_P : P \hookrightarrow \mathbb{R}^{\infty-1} \) on \( \partial M = P \). The space \( \text{Bun}(TM \oplus \epsilon^1, \theta^*\gamma^{d+1}; \hat{\ell}_P) \) consists of all \( \theta \)-structures on \( M \) that agree with \( \hat{\ell}_P \) on \( \partial M \). The action of \( \text{Diff}(M, P) \) is given by the formula \( ((\phi, \hat{\ell}), f) \mapsto (\phi \circ f, f^*\hat{\ell}) \).

Both of the spaces \( D_{\theta,P} \) and \( N_{\theta,P} \) depend on a choice of closed \((d-1)\)-dimensional \( \theta \)-manifold \((P, \hat{\ell}_P)\). Later on it will be important to let these spaces vary with the manifold \((P, \hat{\ell}_P)\). It will be useful to specify the space of manifolds that \((P, \hat{\ell}_P)\) is drawn from.
**Definition 2.4.** We define $\mathcal{M}_\theta$ to be the space consisting of pairs $(P, \hat{\ell}_P)$ where $P \subset \mathbb{R}^{\infty-1}$ is a closed $(d - 1)$-dimensional submanifold, and $\hat{\ell}_P$ is a $\theta$-structure on $P$. The space $\mathcal{M}_\theta$ is topologized as a quotient space similar to (2.1).

**Notational Convention 2.5.** From here on out, for elements of the spaces $\mathcal{M}_\theta$, $N_{\theta,P}$, and $D_{\theta,P}$ we will use the notation $W := (W, \hat{\ell}_W)$. Since the $\theta$-structure has $W$ as a subscript there is no real loss of information with this notational convention. For such an element, we will denote by $\ell_W : W \to B$ the underlying map associated to the $\theta$-structure $\hat{\ell}_W$.

At the beginning of this section, we fixed an integer $d$ and tangential structure $\theta : B \to BO(d + 1)$. We will continue to use this convention throughout the whole paper. The space $D_{\theta,P}$ will always consist of $(d + 1)$-dimensional manifolds, $N_{\theta,P}$ will always consist of $d$-dimensional manifolds, and $\mathcal{M}_\theta$ will always consist of $(d - 1)$-dimensional manifolds.

**2.2. Spaces of manifolds equipped with morse functions.** In this section we define the main spaces of interest. Fix an integer $d \in \mathbb{Z}_{\geq 0}$ and a tangential structure $\theta : B \to BO(d + 1)$. We must fix some more notation first.

**Notational Convention 2.6.** Let $P \in \mathcal{M}_\theta$. For any element $W \in D_{\theta,P}$, we will let $h_W : W \to \mathbb{R}$ denote the function, $W \to \mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{\infty-1} \xrightarrow{\text{proj}} \mathbb{R}$. We refer to this function as the *height function*. For any subset $K \subseteq \mathbb{R}$, we write $W|_K = W \cap h_W^{-1}(K)$ if it is a manifold. If $\hat{\ell}$ is a $\theta$-structure on $W$ and $W|_K$ a submanifold of $W$, then we write $\hat{\ell}|_K$ for the restriction of $\hat{\ell}$ to $W|_K$.

**Definition 2.7.** Let $P \in \mathcal{M}_\theta$. We define $D_{\theta,P}^{mf} \subset D_{\theta,P}$ to be the subspace consisting of those $W$ for which the height function $h_W : W \to \mathbb{R}$ is a Morse function. For each integer $k$, the subspace $D_{\theta,P}^{mf,k} \subset D_{\theta,P}^{mf}$ consists of those $W$ subject to the following further conditions:

(i) For all critical points $c \in W$ of $h_W : W \to \mathbb{R}$, the index of $c$ satisfies the inequality $k < \text{index}(c) < d - k + 1$.

(ii) The map $\ell_W : W \to B$ is $k$-connected.

Using the above definition we define cobordism categories.

**Definition 2.8.** Let $P \in \mathcal{M}_\theta$. The (non-unital) topological category $\text{Cob}_{\theta,P}^{mf}$ has $N_{\theta,P}$ for its space of objects. The morphism space is the following subspace of $\mathbb{R} \times D_{\theta,P}^{mf}$: A pair $(t,W)$ is a morphism if there exists an $\varepsilon > 0$ such that

$$W|_{(-\infty, \varepsilon)} = (\infty, \varepsilon) \times W|_0 \quad \text{and} \quad W|_{(t-\varepsilon, \infty)} = (t-\varepsilon, \infty) \times W|_t$$

as $\theta$-manifolds. The source of such a morphism is $W|_0$ and the target is $W|_t$, equipped with their respective restrictions of the $\theta$-structure $\hat{\ell}_W$ on $W$. Composition is defined in the usual way by concatenation of cobordisms.

Let $k$ be an integer. The subcategory $\text{Cob}_{\theta,P}^{mf,k} \subset \text{Cob}_{\theta,P}^{mf}$ has for its objects those $M \in \text{Ob} \text{Cob}_{\theta,P}^{mf}$ for which the map $\ell_M : M \to B$ is $k$-connected. It has for its morphisms those $(t,W) \in \text{Mor} \text{Cob}_{\theta,P}^{mf}$ such that $W$ is contained in the subspace $D_{\theta,P}^{mf,k} \subset D_{\theta,P}^{mf}$.

This paper is concerned with the homotopy type of the classifying space $B\text{Cob}_{\theta,P}^{mf,k}$ for different values of $k$. The following proposition is proven in the same way as [7, Theorems 3.9 and 3.10]. We omit the proof.

**Proposition 2.9.** For all integers $k$ and $P \in \mathcal{M}_\theta$, there is a weak homotopy equivalence,

$$B\text{Cob}_{\theta,P}^{mf,k} \simeq D_{\theta,P}^{mf,k}.$$
2.3. The Pontryagin-Thom construction. Fix an integer \( d \in \mathbb{Z}_{\geq 0} \) and a tangential structure \( \theta : B \to BO(d+1) \). We will keep these choices fixed for the rest of the paper. Our main theorem concerns the homotopy type of the space \( D^m_{\theta,P} \) for \( k < d/2 \).

**Definition 2.10.** For integers \( k \) and \( m \), let \( G^m_\theta(\mathbb{R}^m)^k \) denote the space of tuples \((V,l,\sigma)\) where:

(i) \( V \subset \mathbb{R}^m \) is a \((d+1)\)-dimensional linear subspace equipped with a \( \theta \)-orientation \( l \);

(ii) \( l : V \to \mathbb{R} \) is a linear functional and \( \sigma : V \otimes V \to \mathbb{R} \) is a symmetric bilinear form subject to the following condition: if \( l = 0 \), then \( \sigma \) is non-degenerate with \( k < \text{index}(\sigma) < d - k + 1 \).

We use the spaces \( G^m_\theta(\mathbb{R}^m)^k \) to define a Thom spectrum. For each \( N \in \mathbb{N} \) we let

\[
U_{d+1,N} \to G^m_\theta(\mathbb{R}^{d+1+N})^k
\]

denote the canonical \((d+1)\)-dimensional vector bundle, and we let \( U_{d+1,N}^\perp \to G^m_\theta(\mathbb{R}^{d+1+N})^k \) denote the orthogonal complement bundle, which has \( N \)-dimensional fibres. For each \( N \), the bundle \( U_{d+1,N} \) pulls back to the bundle \( \epsilon^1 \oplus U_{d+1,N}^\perp \) by the inclusion \( G^m_\theta(\mathbb{R}^{d+1+N})^k \to G^m_\theta(\mathbb{R}^{d+1+N+1})^k \), and thus this induces a map of Thom spaces

\[
(2.2) \quad S^1 \wedge \text{Th}(U_{d+1,N}^\perp) \cong \text{Th}(\epsilon^1 \oplus U_{d+1,N}^\perp) \to \text{Th}(U_{d+1,N+1}^\perp).
\]

We define \( hW_\theta^k \) to be the spectrum whose \((d+1+N)\)th space is given by the Thom space \( \text{Th}(U_{d+1,N}^\perp) \), and with structure maps given by \((2.2)\).

We let \( G^m_\theta(\mathbb{R}^{d+1+\infty})^k \) denote the colimit of the spaces \( G^m_\theta(\mathbb{R}^{d+1+N})^k \) taken as \( N \to \infty \) and let \( U_{d+1,\infty} \to G^m_\theta(\mathbb{R}^{d+1+\infty})^k \) denote the canonical \((d+1)\)-dimensional vector bundle. It follows easily that \( hW_\theta^k \) is homotopy equivalent to the Thom spectrum associated to the virtual bundle \( -U_{d+1,\infty} \to G^m_\theta(\mathbb{R}^{d+1+\infty})^k \), and thus the above definition agrees with the definition of \( hW_\theta^k \) given in the introduction.

We now construct a zig-zag of maps between the spaces \( D^m_{\theta,P} \) and \( \Omega^{-1} hW_\theta^k \). We will need to use a slight modification of the space \( D^m_{\theta,P} \).

**Definition 2.11.** Fix an integer \( N \in \mathbb{N} \). We define \( \tilde{D}^m_{\theta,P} \subset D^m_{\theta,P} \) to be the subspace consisting of those \( W \) that satisfy the following conditions:

(i) \( W \) is contained in the subspace \( \mathbb{R} \times (-\infty,0] \times \mathbb{R}^{d+N-1} \);

(ii) the exponential map \( \exp : TW \to \mathbb{R} \times \mathbb{R}^{d+N} \) has injectivity radius greater than or equal to 1 with respect to the Euclidean metric on \( \mathbb{R} \times \mathbb{R}^{d+N} \).

It follows by a standard argument using the *tubular neighborhood theorem* that the natural map induced by the inclusions, \( \colim_{N \to \infty} \tilde{D}^m_{\theta,P} \to D^m_{\theta,P} \), is a weak homotopy equivalence.

**Construction 2.1.** For each \( N \in \mathbb{N} \) we construct a map

\[
(2.3) \quad \tilde{D}^m_{\theta,P} : \tilde{D}^m_{\theta,P} \to \Omega^{d+N} \text{Th}(U_{d+1,N}^\perp).
\]

Let \( W \in \tilde{D}^m_{\theta,P} \). The 2-jet of the morse function \( h_W : W \to \mathbb{R} \) equips the tangent space \( TW \) with the data \((l,\sigma)\), where \( l \) is a linear functional (the differential of \( h_W \)) and \( \sigma \) is a symmetric bilinear form (the quadratic differential of \( h_W \)). Since \( h_W \) is a Morse function it follows that \((l,\sigma)\) satisfies the Morse condition. Furthermore, we have \( k < \text{index}(\sigma) < d - k + 1 \). It follows that the Gauss map for the vertical tangent bundle \( TW \) yields a map \( \tau : W \to G^m_\theta(\mathbb{R}^{d+1+N})^k \). The inclusion \( W \subset \mathbb{R} \times (-\infty,0] \times \mathbb{R}^{d+N-1} \) induces the bundle trivialization

\[
TW \oplus \nu_W \cong \epsilon^{d+1+N}
\]
where $\nu_W$ is the normal bundle of $W$. From this bundle trivialization it follows that the map $\tau$ is covered by a bundle map $\hat{\tau}^\perp : \nu_W \to U^\perp_{d+1,N}$. Let $U \subset \mathbb{R} \times \mathbb{R}^{d+N}$ be the geodesic tubular neighborhood of $W$ of radius 1 (this exists by Definition 2.11). This tubular neighborhood $U$ determines a collapse map $\pi_U : \mathbb{R} \times S^{d+N} \to \Theta(\nu_W)$, and composing with $\Theta(\hat{\tau}^\perp)$ we obtain, $\mathbb{R} \times S^{d+N} \to \Theta(\hat{U}^\perp_{d+1,N})$. Precomposing this map with the inclusion $\{0\} \times S^{d+N} \hookrightarrow \mathbb{R} \times S^{d+N}$ and then forming the adjoint yields an element, $\hat{\Theta}_{\theta}^k(N)(W) \in \Omega^{d+N}\Theta(U^\perp_{d+1,N})$. This is our definition of the map $\hat{\Theta}_{\theta}^k$ from (2.3). We denote by

$$
\hat{\Theta}_{\theta}^k : \lim_{N \to \infty} \hat{D}_{\theta,P}^m \to \Omega^{-1}hW^k_	heta
$$

the map induced by the maps $\hat{\Theta}_{\theta}^k$ in the limit $N \to \infty$.

The following theorem was proven by Madsen and Weiss in [12].

**Theorem 2.12** (Madsen-Weiss 2007). Let $P \in \mathcal{M}_\theta$. Suppose that $P$ is null-bordant as a $\theta$-manifold. Then the map, $\hat{\Theta}^{-1}_{\theta} : \lim_{N \to \infty} \hat{D}_{\theta,P}^{m-1} \to \Omega^{-1}hW^{-1}_\theta$, is a weak homotopy equivalence.

We remark that Theorem A stated in the introduction is obtained by combining the above theorem with the weak homotopy equivalence $D_{\theta,P}^{m-1} \simeq \text{Cob}_{\theta,P}^{m-1}$ from Proposition 2.9. The next theorem below generalizes the above result to the case where $k < d/2$.

**Theorem 2.13.** Let $k < d/2$ and suppose that $\theta : B \to BO(d+1)$ is such that $B$ satisfies Wall’s finiteness condition $F(k)$ (see [13]). Let $P \in \mathcal{M}_\theta$. Suppose that $P$ is null-bordant as a $\theta$-manifold (this includes the case $P = \emptyset$). Then the map, $\hat{\Theta}_{\theta}^k : \lim_{N \to \infty} \hat{D}_{\theta,P}^{m,k,N} \to \Omega^{-1}hW^k_{\theta}$, is a weak homotopy equivalence.

Combining Theorem 2.13 with Proposition 2.9 implies Theorem B from the introduction. Theorem 2.13 is the main result that we are after and its proof is the subject of the rest of the paper.

### 2.4. The Localization Sequence

We proceed to construct the localization sequence that was discussed in Section 1.1. We begin with a preliminary definition.

**Definition 2.14.** For each integer $m$, we let $G^m_\theta(\mathbb{R}^m)_{\text{loc}} \subset G^m_\theta(\mathbb{R}^m)$ denote the subspace consisting of those tuples $(V,l,\sigma)$ with the property that $l = 0$. Such elements of $G^m_\theta(\mathbb{R}^m)_{\text{loc}}$ will be denoted by the pair $(V,\sigma)$ since the $l$ term is redundant. The space $G^m_\theta(\mathbb{R}^\infty)_{\text{loc}}$ is defined to be the direct limit of the spaces $G^m_\theta(\mathbb{R}^m)_{\text{loc}}$ taken as $m \to \infty$. Let $k$ be an integer. The subspace $G^m_\theta(\mathbb{R}^m)_{\text{loc}}^{k,d-k+1} \subset G^m_\theta(\mathbb{R}^m)_{\text{loc}}$ consists of those $(V,\sigma)$ such that $\text{index}(\sigma) \in \{k,d-k+1\}$. As before, $G^m_\theta(\mathbb{R}^\infty)_{\text{loc}}^{k,d-k+1} \subset G^m_\theta(\mathbb{R}^\infty)_{\text{loc}}$ is defined to be the limiting space.

**Definition 2.15.** We define $D^m_{\theta,\text{loc}}$ to be the space of tuples $(\bar{x};(V,\sigma))$ where $\bar{x} \subset \mathbb{R} \times \mathbb{R}^\infty$ is a zero-dimensional submanifold (not necessarily compact), and $(V,\sigma) : \bar{x} \to G^m_\theta(\mathbb{R} \times \mathbb{R}^\infty)_{\text{loc}}$ is a map, subject to the following condition: for all $\delta > 0$, the set $\bar{x} \cap ((-\delta,\delta) \times \mathbb{R}^\infty)$ is a finite set (or in other words is a compact 0-manifold). We will need to define certain subspaces of $D^m_{\theta,\text{loc}}$. For an integer $k$, we define $D^m_{\theta,\text{loc}}^{k,d-k+1}$ to be the space consisting of those $(\bar{x};(V,\sigma))$ for which,

$$
(V,\sigma)(x) = (V(x),\sigma(x)) \in G^m_\theta(\mathbb{R} \times \mathbb{R}^\infty)_{\text{loc}}^{k,d-k+1} \text{ for all } x \in \bar{x}.
$$

The spaces $D^m_{\theta,\text{loc}}^{k,d-k+1} \subset D^m_{\theta,\text{loc}}$ are topologized using the same process carried out in [7, Section 2.1].
We have a map $L : D_{\theta,P}^{\mf} \to D_{\theta,\loc}^{\mf}$ defined by sending $W \in D_{\theta,P}^{\mf}$ to the pair $(\bar{x}; (V, \sigma))$ where:

- $\bar{x} \subset W$ is the set of critical points of the height function $h_W : W \to \mathbb{R}$;
- for each $x \in \bar{x}$, $V(x)$ is the tangent space $T_x W$ and $\sigma(x) : T_x W \otimes T_x W \to \mathbb{R}$ is the Hessian associated to the height function $h_W$.

For each $k$ we have the map
\begin{equation}
L_k : D_{\theta,\loc}^{\mf,k-1} \to D_{\theta,\loc}^{\mf,\{k,d-k+1\}}
\end{equation}
defined by composing the map $L$ with the projection $D_{\theta,\loc}^{\mf,k-1} \to D_{\theta,\loc}^{\mf,\{k,d-k+1\}}$ that sends $(\bar{x}; (V, \sigma))$ to the element
\[
(\bar{x}\{k,d-k+1\}; (V, \sigma)|_{\bar{x}\{k,d-k+1\}}) \in D_{\theta,\loc}^{\mf,\{k,d-k+1\}},
\]
where $\bar{x}\{k,d-k+1\} \subset \bar{x}$ is the subset consisting of all $x \in \bar{x}$ with $\text{index}(\sigma(x)) \in \{k, d - k + 1\}$. The theorem below is a restatement of Theorem 1.7 from the introduction. The proof of this theorem is carried out over the course of the whole paper.

**Theorem 2.16.** Let $0 < k < d/2$ and let $P \in \mathcal{M}_{\theta,d-1}$ be null-bordant as a $\theta$-manifold (including the case that $P = \emptyset$). Suppose that $B$ satisfies Wall’s finiteness condition $F(k)$. Then the sequence
\begin{equation}
D_{\theta,P}^{\mf,k} \xrightarrow{L_k} D_{\theta,P}^{\mf,k-1} \xrightarrow{L_k} W_{\theta,\loc}^{\mf,\{k,d-k+1\}}
\end{equation}
is a homotopy fibre sequence.

We will refer to the homotopy fibre sequence in (2.6) as the localization sequence in degree $k$. Below we show how to derive Theorem 2.13 (and hence Theorem B) using the localization sequence.

### 2.5. Theorem B from Theorem 2.16

As discussed in Section 1.1, we will prove Theorem 2.13 (and Theorem B) using Theorem 2.16. An outline for the proof of Theorem B was provided in the introduction. In this section we fill in the details.

We need to construct an infinite loopspace to map $D_{\theta,\loc}^{\mf,\{k,d-k+1\}}$ to. For each $N \in \mathbb{N}$, let $U_{d+1,N}^{\loc} \to G_{\theta}^{\mf}(\mathbb{R}^{d+1+N})^{\{k,d-k+1\}}_{\loc}$ denote the restriction of the canonical bundle
\[U_{d+1,N} \to G_{\theta}^{\mf}(\mathbb{R}^{d+1+N})\]
to the subspace $G_{\theta}^{\mf}(\mathbb{R}^{d+1+N})^{\{k,d-k+1\}}_{\loc}$. We let $U_{d+1,N}^{\loc,\perp} \to G_{\theta}^{\mf}(\mathbb{R}^{d+1+N})^{\{k,d-k+1\}}_{\loc}$ denote the restriction of the orthogonal complement $U_{d+1,N}^{\perp}$. We define $hW_{\theta,\loc}^{\{k,d-k+1\}}$ to be the spectrum whose $(d + 1 + N)$th space is given by the Thom space, $\text{Th}(U_{d+1,N}^{\loc} \oplus U_{d+1,N}^{\loc,\perp})$. For each $N$, the $N$-th structure map is the map between Thom spaces induced by the bundle map
\[U_{d+1,N}^{\loc} \oplus U_{d+1,N}^{\loc,\perp} \oplus 1 \to U_{d+1,N+1}^{\loc} \oplus U_{d+1,N+1}^{\loc,\perp}\]
that covers the inclusion,
\[G_{\theta}^{\mf}(\mathbb{R}^{d+1+N})^{\{k,d-k+1\}}_{\loc} \to G_{\theta}^{\mf}(\mathbb{R}^{d+1+N})_{\loc}.
\]
It is easy to see that $hW_{\theta,\loc}^{\{k,d-k+1\}}$ is homotopy equivalent to the suspension spectrum,
\[\Sigma^\infty \left( G_{\theta}^{\mf}(\mathbb{R}^{d+1+N})^{\{k,d-k+1\}}_{\loc} \right),\]
and thus it coincides with the spectrum discussed in the introduction. For any \( k \), we have a sequence of spectrum maps

\[
\begin{align*}
\mathbf{hW}_\theta^k & \rightarrow \mathbf{hW}_\theta^{k-1} \rightarrow \mathbf{hW}_{\theta,\text{loc}}^{\{k,d-k+1\}},
\end{align*}
\]

where the first map is induced by the inclusion \( G^\text{mf}_\theta(\mathbb{R}^{d+1+N})^{k-1} \rightarrow G^\text{mf}_\theta(\mathbb{R}^{d+1+N})^k \) (or rather the bundle map that covers it) and the second map is induced by the inclusion of Thom spaces

\[\Theta \left( U^\text{loc,⊥}_{d+1,N} \right) \rightarrow \Theta \left( U^\text{loc}_{d+1,N} \oplus U^\text{loc,⊥}_{d+1,N} \right).\]

By replicating the same arguments from [12, Section 2] we obtain the following proposition:

**Proposition 2.17.** The sequence of maps, \( \mathbf{hW}_\theta^k \rightarrow \mathbf{hW}_\theta^{k-1} \rightarrow \mathbf{hW}_{\theta,\text{loc}}^{\{k,d-k+1\}} \), is a cofibre sequence of spectra.

**Proof sketch.** The complement of \( G^\text{mf}_\theta(\mathbb{R}^{d+1+N})_{\text{loc}}^{\{k,d-k+1\}} \) in \( G^\text{mf}_\theta(\mathbb{R}^{d+1+N})^{k-1} \) is given by the subspace \( G^\text{mf}_\theta(\mathbb{R}^{d+1+N})^k \). The normal bundle of \( G^\text{mf}_\theta(\mathbb{R}^{d+1+N})_{\text{loc}}^{\{k,d-k+1\}} \) in \( G^\text{mf}_\theta(\mathbb{R}^{d+1+N})^{k-1} \) is isomorphic to the dual of the canonical bundle, \( U^\text{loc}_{d+1,N} \rightarrow G^\text{mf}_\theta(\mathbb{R}^{d+1+N})_{\text{loc}}^{\{k,d-k+1\}} \). From this observation we obtain a cofibre sequence of spaces, \( G^\text{mf}_\theta(\mathbb{R}^{d+1+N})^{k-1} \rightarrow G^\text{mf}_\theta(\mathbb{R}^{d+1+N})^k \rightarrow \Theta(U^\text{loc}_{d+1,N}) \), which in turn leads to the cofibre sequence

\[
\Theta(U^\text{loc}_{d+1,N}) \rightarrow \Theta(U^\text{loc}_{d+1,N}) \rightarrow \Theta(U^\text{loc}_{d+1,N} \oplus U^\text{loc,⊥}_{d+1,N}).
\]

The proposition follows by observing that these spaces are the \((d+1+N)\)th spaces of the spectra \( \mathbf{hW}_\theta^k \), \( \mathbf{hW}_\theta^{k-1} \), and \( \mathbf{hW}_{\theta,\text{loc}}^{\{k,d-k+1\}} \) respectively. \( \square \)

We want to map the space \( D^\text{mf}_{\theta,\text{loc}}^{\{k,d-k+1\}} \) to the infinite loopspace \( \Omega^{-1}\mathbf{hW}_{\theta,\text{loc}}^{\{k,d-k+1\}} \). As in the previous section we won’t be able to define the map “on the nose”. We will need to work with a slightly different model of \( D^\text{mf}_{\theta,\text{loc}}^{\{k,d-k+1\}} \).

**Definition 2.18.** Fix \( N \in \mathbb{N} \). We define \( \hat{D}^\text{mf}_{\theta,\text{loc}}^{\{k,d-k+1\},N} \subset \hat{D}^\text{mf}_{\theta,\text{loc}}^{\{k,d-k+1\}} \) to be the subspace consisting of those \((\bar{x};(V,\sigma))\) for which:

(i) \( \bar{x} \) is contained in the subspace \( \mathbb{R} \times \mathbb{R}^{d+N}; \)

(ii) \( \text{dist}(x,y) > 2 \) for all pairs of distinct points \( x,y \in \bar{x} \).

It follows by a standard argument that the natural map induced by inclusion

\[
\colim_{N \to \infty} \hat{D}^\text{mf}_{\theta,\text{loc}}^{\{k,d-k+1\},N} \rightarrow \hat{D}^\text{mf}_{\theta,\text{loc}}^{\{k,d-k+1\}}
\]

is a weak homotopy equivalence. We proceed to define a map

\[
\hat{D}^\text{mf}_{\theta,\text{loc}}^{\{k,d-k+1\},N} \rightarrow \Omega^{-1}\mathbf{hW}_{\theta,\text{loc}}^{\{k,d-k+1\}}.
\]

We do this in the construction below.

**Construction 2.2.** To define (2.8) we will construct for each \( N \in \mathbb{N} \) a map

\[
\hat{D}^\text{mf}_{\theta,\text{loc}}^{\{k,d-k+1\},N} \rightarrow \Omega^{d+N}\Theta(U^\text{loc}_{d+1,N} \oplus U^\text{loc,⊥}_{d+1,N}).
\]

Let \((\bar{x};(V,\sigma)) \in \hat{D}^\text{mf}_{\theta,\text{loc}}^{\{k,d-k+1\},N}\). By condition (ii) of Definition 2.18 there is a geodesic tubular neighborhood \( U \subset \mathbb{R} \times \mathbb{R}^{d+N} \) of \( \bar{x} \) of radius 1. Since the normal bundle \( \nu_{\bar{x}} \) of \( \bar{x} \) is naturally isomorphic to the trivial bundle \( e^{d+N+1} \) (\( \bar{x} \) is a discrete set of points), the tubular neighborhood
$U$ determines a collapse map $c : \mathbb{R} \times S^{d+N} \to S^{d+N} \times \bar{x}$. Composing this collapse map with $\text{Id}_{S^{d+N}} \times (V, \sigma)$ we obtain

$$\mathbb{R} \times S^{d+N} \xrightarrow{c} S^{d+N} \times \bar{x} \xrightarrow{\text{Id}_{S^{d+N}} \times (V, \sigma)} S^{d+N} \times G_{\theta}^{\text{mf}}(\mathbb{R} \times \mathbb{R}^{d+N})^{k,d-k+1}_{\text{loc}+}.$$ Precomposing this map with the inclusion $\{0\} \times S^{d+N} \hookrightarrow \mathbb{R} \times S^{d+N}$ and forming the adjunction yields the desired element of $\Omega^{d+N} \text{h}(U_{d+1,N}^{\text{loc}+} \oplus U_{d+1,N}^{\text{loc}+})$. This construction determines the map (2.9). Letting $N \to \infty$, yields the map $\hat{P}_k^{\theta,\text{loc}}$ of (2.8).

The following proposition is a special case of the Barrat-Priddy-Quillen theorem for configurations spaces consisting of points with labels in the space $G_{\theta}^{\text{mf}}(\mathbb{R} \times \mathbb{R}^{\infty})^{k,d-k+1}_{\text{loc}+}$. This proposition can also be viewed as a special case of [7, Theorem 3.12] applied to spaces of zero-dimensional manifolds with maps to a background space.

**Proposition 2.19.** For all $k$, the map (2.8) induces the weak homotopy equivalence,

$$\colim_{N \to \infty} \widetilde{\mathcal{D}}_{\theta,\text{loc}}^{\text{mf},(k,d-k+1),N} \simeq \Omega^{\infty-1} \text{hW}_{\theta,\text{loc}}^{k,d-k+1}.$$  

We emphasize that unlike Theorem 2.13, the map (2.8) is a weak homotopy equivalence for all choices of $k$. With the above constructions in place, we are now in a position to put everything together and prove Theorem 2.13 using Theorem 2.16.

**Proof of Theorem 2.13, assuming Theorem 2.16.** Let $k < d/2$. The maps defined in Constructions 2.1 and 2.2 fit together into the commutative diagram

$$\begin{array}{ccc}
\colim_{N \to \infty} \widetilde{\mathcal{D}}_{\theta,\text{loc}}^{\text{mf},k,N} & \simeq & \Omega^{\infty-1} \text{hW}_{\theta}^{k} \\
\colim_{N \to \infty} \widetilde{\mathcal{D}}_{\theta,\text{loc}}^{\text{mf},k-1,N} & \simeq & \Omega^{\infty-1} \text{hW}_{\theta}^{k-1} \\
\colim_{N \to \infty} \widetilde{\mathcal{D}}_{\theta,\text{loc}}^{\text{mf},(k,d-k+1),N} & \simeq & \Omega^{\infty-1} \text{hW}_{\theta,\text{loc}}^{k,d-k+1}. 
\end{array}$$

The maps of the bottom row are induced by the sequence (2.17), the maps of the top row are induced by the localization sequence in the statement of Theorem 2.16, and the maps of the middle row are defined analogously to the maps on the top row. Commutativity is easily checked by tracing through the constructions. By Proposition 2.17 the bottom row is a homotopy fibre sequence. With this commutative diagram established, the proof of Theorem 2.13 follows by the same induction argument given in Section 1.1 (page 9) used in our outline of the proof of Theorem B, where the base of the induction is given by Theorem 2.12 (or Theorem A).

With the above argument out of the way, we can now devote all of our resources to proving Theorem 2.16.

### 3. Homotopy Colimit Decompositions

#### 3.1. A homotopy colimit decomposition.

We now proceed as in [12, Section 5] to express the homotopy type of $\mathcal{D}_{\theta,P}^{\text{mf},k}$ as a homotopy colimit of spaces of compact $d$-dimensional manifolds.
Definition 3.4. Fix \(\theta\) spaces defined below will not contain will keep this structure fixed for the rest of the paper. To prevent notational overcrowding, the bilinear form \(S_{V,\sigma}\) a typical element of this mapping space will be denoted \((V,\sigma)\) where \(j\) is an injective map over \(\{0, \ldots, d+1\}\) from \(s\) to \(t\), and \(\varepsilon\) is a function \(t \setminus j(s) \to \{-1, +1\}\). The composition of two morphisms \((j_1, \varepsilon_1) : s \to t\) and \((j_2, \varepsilon_2) : t \to u\) is \((j_2j_1, \varepsilon_3)\), where \(\varepsilon_3\) agrees with \(\varepsilon_2\) outside \(j_2(t)\) and \(\varepsilon_1 \circ j_2^{-1}\) on \(j_2(t \setminus j_1(s))\).

We will define a space-valued contravariant functor on the category \(\mathcal{K}\). Before doing this we will need to fix some notational conventions.

Notational Convention 3.2. Let \((V, \sigma) \in G^\text{mf}(\mathbb{R}^\infty)_{\text{loc}}\). The standard Euclidean inner product \(\langle - , - \rangle\) on the ambient space \(\mathbb{R}^\infty\) induces an inner product \(\langle - , - \rangle_V\) on the \((d+1)\)-dimensional subspace \(V\). Let \(T_V : V \to V^*\) denote the linear isomorphism, \(v \mapsto \langle v, - \rangle_V\). Using \(T_V\), the bilinear form \(\sigma\) determines a self-adjoint linear operator \(\tilde{\sigma} : V \to V\) defined by,

\[
\tilde{\sigma}(v) = T_V^{-1}(\sigma(v, -)).
\]

The form \(\sigma\) is non-degenerate if and only if \(\tilde{\sigma}\) is an isomorphism. Since \(\tilde{\sigma}\) is self-adjoint, the vector space \(V\) decomposes as \(V = V^+ \oplus V^-\) where \(V^\pm\) are the negative and positive eigenspaces of \(\tilde{\sigma}\). Notice that \(\sigma\) is negative definite on the subspace \(V^-\) and \(\text{index}(\sigma) = \dim(V^-)\).

Notational Convention 3.3. Let \(t \in \mathcal{K}\). We will need to consider the mapping space,

\[
(G^\text{mf}(\mathbb{R}^\infty)_{\text{loc}})^t = \text{Maps}(t, G^\text{mf}(\mathbb{R}^\infty)_{\text{loc}}).
\]

A typical element of this mapping space will be denoted \((V, \sigma)\) with

\[
(V, \sigma)(i) = (V(i), \sigma(i)) \in G^\text{mf}(\mathbb{R}^\infty)_{\text{loc}}
\]

for \(i \in t\). Let \((V, \sigma) \in (G^\text{mf}(\mathbb{R}^\infty)_{\text{loc}})^t\). For each \(i \in t\) we may form the unit spheres and disks \(S(V^\pm(i))\) and \(D(V^\pm(i))\) defined with respect to the inner product \(\langle - , - \rangle_V\) induced from the inner product in the ambient space. We will denote,

\[
D(V^+) \times_t S(V^-) := \bigsqcup_{i \in t} D(V^+(i)) \times S(V^-(i)).
\]

The \(\theta\)-orientations on the vector spaces \(V(i)\) induce a \(\theta\)-structure on \(D(V^+) \times_t S(V^-)\) which we will denote by \(\hat{\ell}_t : T(D(V^+) \times_t S(V^-)) \to \theta^\ast_d, d+1\). We may think of \(D(V^+) \times_t S(V^-)\) as being a fibre bundle over \(t\) equipped with a fibrewise \(\theta\)-structure.

Fix once and for all an integer \(d \in \mathbb{Z}_{\geq 0}\) and a tangential structure \(\theta : B \to BO(d+1)\). We will keep this structure fixed for the rest of the paper. To prevent notational overcrowding, the spaces defined below will not contain \(\theta\) in their notation.

Definition 3.4. Fix \(P \in \mathcal{M}_\theta\) and let \(t \in \mathcal{K}\). The set \(\mathcal{W}_{P,t}\) consists of tuples \((M, (V, \sigma), e)\) where:

(i) \(M\) is an element of \(\mathcal{N}_{\theta, P}\) (where \(\mathcal{N}_{\theta, P}\) was defined in Definition 2.3);

(ii) \((V, \sigma)\) is an element of \((G^\text{mf}(\mathbb{R}^\infty)_{\text{loc}})^t\) that satisfies, \(\delta(i) = \text{index}(\sigma(i))\) for all \(i \in t\), where recall that \(\delta : t \to \{0, \ldots, d+1\}\) is the labeling function.

(iii) \(e : D(V^+) \times_t D(V^-) \to (-\infty, 0) \times \mathbb{R}^{-1}\) is an embedding subject to the following further conditions:
(a) \( e^{-1}(M) = D(V^+) \times_t S(V^-) \);

(b) the induced \( \theta \)-structure \( \hat{\ell}_t \) on \( D(V^+) \times_t S(V^-) \) agrees with \( \hat{\ell}_M \).

We also have a local version of the above definition.

**Definition 3.5.** Let \( t \in \mathcal{K} \). The space \( \mathcal{W}_{\text{loc},t} \) consists of pairs \( ((V, \sigma), e) \) where:

(i) \( (V, \sigma) \) is an element of \( (G^\text{mf}_\theta(\mathbb{R}^\infty)_{\text{loc}})^t \) that satisfies \( \delta(i) = \text{index}(\sigma(i)) \) for all \( i \in t \).

(ii) \( e : D(V^+) \times_t D(V^-) \to (-\infty, 0) \times \mathbb{R}^{\infty-1} \) is a smooth embedding.

We need to describe how to make the correspondences \( t \mapsto \mathcal{W}_{P,t} \) and \( t \mapsto \mathcal{W}_{\text{loc},t} \) into contravariant functors on \( \mathcal{K} \). The case with \( \mathcal{W}_{\text{loc},(-)} \) is easy; if \( (j, \varepsilon) : s \to t \) is a morphism, the map \( (j, \varepsilon)^* : \mathcal{W}_{\text{loc},t} \to \mathcal{W}_{\text{loc},s} \) is given by sending an element \( (V, e) \in \mathcal{W}_{\text{loc},t} \) to the element in \( \mathcal{W}_{\text{loc},s} \) obtained by precomposing \( V \) with \( j : s \to t \), and then restricting the embedding \( e \). Notice that in this definition the function \( \varepsilon \) played no role.

Describing the functor \( t \mapsto \mathcal{W}_{P,t} \) will take more work. Let \( (j, \varepsilon) : s \to t \) be a morphism in \( \mathcal{K} \). If \( j \) is bijective, there is an obvious identification \( \mathcal{W}_{P,t} \cong \mathcal{W}_{P,s} \) and this is the induced map. We may assume that \( j \) is an inclusion \( s \hookrightarrow t \). We may then reduce to the case where \( t \setminus s \) has exactly one element, \( a \). This case has two subcases: \( \varepsilon(a) = +1 \) and \( \varepsilon(a) = -1 \).

**Definition 3.6.** Let \( (j, \varepsilon) : s \to t \) be a morphism in \( \mathcal{K} \) where \( j \) is an inclusion and \( t \setminus s = \{a\} \) with \( \varepsilon(a) = +1 \). We describe the induced map \( (j, \varepsilon)^* : \mathcal{W}_{P,t} \to \mathcal{W}_{P,s} \). Let \( (M, (V, \sigma), e) \) be an element of \( \mathcal{W}_{P,t} \). Map this to an element of \( \mathcal{W}_{P,s} \) by pulling \( M \) and \( V \) back along the inclusion \( s \times X \to t \times X \), and restricting \( e \) accordingly.

**Definition 3.7.** Let \( (j, \varepsilon) : s \to t \) be a morphism in \( \mathcal{K} \) where \( j \) is an inclusion and \( t \setminus s = \{a\} \) with \( \varepsilon(a) = -1 \). The induced map \( \mathcal{W}_{P,t} \to \mathcal{W}_{P,s} \) is defined as follows. Let \( (M, (V, \sigma), e) \) be an element of \( \mathcal{W}_{P,t} \). Map this to the element \((\tilde{M}, (\tilde{V}, \tilde{\sigma}), \tilde{e})\) in \( \mathcal{W}_{P,s} \) where:

(i) \( \tilde{M} \) is the element of \( N_{a,p} \) given by,

\[
\tilde{M} = (M \setminus e(S(V^-) \times_a D(V^+)) \cup e(D(V^-) \times_a S(V^+)),
\]

where \( S(V^-) \times_a D(V^+) \subset S(V^-) \times_t D(V^+) \) is the component of \( S(V^-) \times_t D(V^+) \) that corresponds to \( a \in t \).

(ii) \( (\tilde{V}, \tilde{\sigma}) \) is the restriction of \( (V, \sigma) \) to \( s \);

(iii) \( \tilde{e} \) is obtained from \( e \) by restriction.

The definitions above make the assignment \( t \mapsto \mathcal{W}_{P,t} \) into a contravariant functor on \( \mathcal{K} \).

We will need to work with a “restricted index” version of \( \mathcal{W}_{P,t} \). For each integer \( k \), let \( \mathcal{K}^k \subset \mathcal{K} \) denote the full subcategory consisting of those \( t \in \mathcal{K} \) whose reference map \( \delta : t \to \{0, \ldots, d+1\} \) has its image in the subset \( \{k+1, \ldots, d-k\} \subset \{0, \ldots, d+1\} \). Similarly, we let \( \mathcal{K}^{k,\{d-k+1\}} \subset \mathcal{K} \) be the full subcategory consisting of those objects \( t \) with \( \delta(t) \subset \{k, d-k+1\} \).

**Definition 3.8.** Let \( k \) be an integer. For \( t \in \mathcal{K}^k \) we define \( \mathcal{W}_{P,t}^{k} \subset \mathcal{W}_{P,t} \) to be the subset consisting of those \( (M, (V, \sigma), e) \) for which the map \( \ell_M : M \to B \) is \( k \)-connected. In this way \( t \mapsto \mathcal{W}_{P,t}^{k} \) defines a functor on \( \mathcal{K}^k \).

Let \( p_{\{k,d-k+1\}} : \mathcal{K}^{k-1} \to \mathcal{K}^{\{k,d-k+1\}} \) denote the projection functor, which is defined by sending an object \( t \in \mathcal{K}^{k-1} \) to the subset \( t_{\{k,d-k+1\}} \subset t \) consisting of all points with label contained
in \( \{ k, d - k + 1 \} \). For \( t \in \mathcal{K}^{k-1} \), we define
\[(3.1) \quad \mathcal{W}_{\text{loc},t}^{(k,d-k+1)} := \mathcal{W}_{\text{loc},p(k,d-k+1)}(t).\]
In this way we may view \( \mathcal{W}_{\text{loc},(-)}^{(k,d-k+1)} \) as a functor on \( \mathcal{K}^{k-1} \). For all \( t \in \mathcal{K}^{k-1} \) we have a map
\[(3.2) \quad \mathcal{W}_{P,t}^{k-1} \rightarrow \mathcal{W}_{\text{loc},t}^{(k,d-k+1)}\]
defined by sending \( (M, (V, \sigma), e) \) to \( ((V, \sigma)|_{(k,d-k+1)}, e|_{(k,d-k+1)}) \), which is the element obtained by restricting \( (V, \sigma) \) and \( e \) to the components of \( t \) with labels in \( \{ k, d - k + 1 \} \). This map yields a natural transformation of contravariant functors on \( \mathcal{K}^{k-1} \).

3.2. Homotopy colimits. We turn our attention to the homotopy colimit, \( \text{hocolim}_{t \in \mathcal{K}} \mathcal{W}_{P,t}^{k} \). Before stating the main theorem, let us recall a particular model for the homotopy colimit of a functor (or contravariant functor) from a small category to the category of topological spaces.

**Definition 3.9.** Let \( \mathcal{C} \) be a small category and let \( \mathcal{F} \) be a contravariant functor from \( \mathcal{C} \) to the category of topological spaces. The *transport category* \( \mathcal{C} \uparrow \mathcal{F} \) has as its objects, pairs \((C,x)\) where \( C \in \text{Ob} \mathcal{C} \) and \( x \in \mathcal{F}(C) \). A morphism \( (B,x) \rightarrow (C,y) \) in \( \mathcal{C} \uparrow \mathcal{F} \) is a morphism \( f : C \rightarrow B \) in \( \mathcal{C} \) such that \( F(f)(x) = y \). The *homotopy colimit* \( \text{hocolim}_{C \in \mathcal{C}} \mathcal{F}(C) \) is defined to be the classifying space \( B(\mathcal{C} \uparrow \mathcal{F}) \).

The following theorem is a generalization of the main result from [12, Section 5]. We give its proof in Appendix A. The theorem is proven by repeating the same steps from [12, Section 5] almost verbatim. In Appendix A we provide sketches of the main steps of the proof and refer the reader to the relevant construction in [12] for the details.

**Theorem 3.10.** For all \( k \) and \( P \in \mathcal{M}_\theta \), there exists spaces \( \mathcal{L}_\theta^{k-1} \) and \( \mathcal{L}_{\theta,\text{loc}}^{(k,d-k+1)} \) together with a commutative diagram
\[
\begin{array}{ccc}
\text{hocolim}_{t \in \mathcal{K}^{k-1}} \mathcal{W}_{P,t}^{k-1} & \overset{\simeq}{\longrightarrow} & \mathcal{L}_\theta^{k-1} \\
\downarrow & & \downarrow \simeq \\
\text{hocolim}_{t \in \mathcal{K}^{k-1}} \mathcal{W}_{\text{loc},t}^{(k,d-k+1)} & \overset{\simeq}{\longrightarrow} & \mathcal{L}_{\theta,\text{loc}}^{(k,d-k+1)} \\
& & \downarrow \simeq \\
& & \text{D}_{\theta,P}^{\text{mf},(k,d-k+1)}
\end{array}
\]
such that all horizontal maps are weak homotopy equivalences.

3.3. Increased Connectivity. In view of the above theorem, Theorem 2.16 translates to the statement that the sequence \( \text{hocolim}_{t \in \mathcal{K}^{k-1}} \mathcal{W}_{P,t}^{k-1} \rightarrow \text{hocolim}_{t \in \mathcal{K}^{k-1}} \mathcal{W}_{P,t}^{k} \rightarrow \text{hocolim}_{t \in \mathcal{K}^{k-1}} \mathcal{W}_{\text{loc},t}^{(k,d-k+1)} \), is a homotopy fibre sequence whenever \( k < d/2 \). We will break the proof of this result down into two intermediate theorems. To state them we will need to define an intermediate space sitting in between \( \mathcal{W}_{P,t}^{k} \) and \( \mathcal{W}_{P,t}^{k-1} \).

**Definition 3.11.** For each \( t \in \mathcal{K}^{k} \), we define \( \mathcal{W}_{P,t}^{k,c} \subset \mathcal{W}_{P,t}^{k} \) to be the subspace consisting of those \((M, (V, \sigma), e)\) such that the \( \theta \)-structure \( \ell_M \) satisfies the following connectivity condition: the restricted map, \( \ell|_{M \setminus \text{Im}(e)} : M \setminus \text{Im}(e) \rightarrow B \), is \((k+1)\)-connected.

Let \((j, \varepsilon) : t \rightarrow s\) be a morphism in \( \mathcal{K}^{k} \). If \((M, (V, \sigma), e) \in \mathcal{W}_{P,s}^{k,c}\), it can be verified that the image \((j, \varepsilon)^*(M, (V, \sigma), e) \in \mathcal{W}_{P,t}^{k,c}\) actually lies in the subset, \( \mathcal{W}_{P,t}^{k,c} \subset \mathcal{W}_{P,t}^{k} \). Indeed, the surgery
performed on $(M, (V, \sigma), e)$ as a result of this morphism does not destroy the connectivity condition from Definition 3.11. Thus, the correspondence $t \mapsto W_{t}^{k,c}$ defines a functor on $K^{k}$, and we can form the homotopy colimit, $\text{hocolim}_{t \in K^{k}} W_{t}^{k,c}$. With this space defined, Theorem 2.16 follows by combining the two theorems stated below.

**Theorem 3.12.** Let $k < d/2$. Suppose that $B$ satisfies Wall’s finiteness condition $F(k)$. Then the inclusion, $\text{hocolim}_{t \in K^{k}} W_{t}^{k,c} \hookrightarrow \text{hocolim}_{t \in K^{k}} W_{t}^{k}$, is a weak homotopy equivalence.

To state our next theorem we will need to fix a basepoint in the space $\text{hocolim}_{t \in K^{k}} W_{t}^{k,c}$. Notice that the space $W_{\emptyset}^{k,d-k-1}$ has exactly one element, namely the one determined by the empty set. This element determines an element in the homotopy colimit which we denote by, $\emptyset \in \text{hocolim}_{t \in K^{k}} W_{t}^{k,d-k-1}$.

**Theorem 3.13.** Let $k < d/2$. The map, $\text{hocolim}_{t \in K^{k}} W_{t}^{k-1,c} \to \text{hocolim}_{t \in K^{k}} W_{t}^{k,d-k+1}$, is a quasi-fibration. The fibre over the point $\emptyset$ is given by the space $\text{hocolim}_{t \in K^{k}} W_{t}^{k}$.

The proof of Theorem 3.12 is carried out in Section 4. Theorem 3.13 is proven over the course of Sections 5 through 10.

### 3.4. Fibre sequences of homotopy colimits.

In the following subsection we state a technique for establishing the fibre sequences in Theorem 3.13. The following proposition is well known. Its proof can be found in [12, Corollary B.3].

**Proposition 3.14.** Let $C$ be a small category and let $u : G_{1} \to G_{2}$ be a natural transformation between functors from $C$ to the category of topological spaces. Suppose that for each morphism $f : a \to b$ in $C$, the map

$$f_{*} : \text{hofibre} (u_{a} : G_{1}(a) \to G_{2}(a)) \to \text{hofibre} (u_{b} : G_{1}(b) \to G_{2}(b))$$

is a weak homotopy equivalence. Then for any object $c \in \text{Ob} C$ the inclusion

$$\text{hofibre} (u_{c} : G_{1}(c) \to G_{2}(c)) \hookrightarrow \text{hofibre} (\text{hocolim} G_{1} \to \text{hocolim} G_{2})$$

is a weak homotopy equivalence.

### 4. Parametrized Surgery

This section is devoted to proving Theorem 3.12 which states that the inclusion

$$\text{hocolim}_{t \in K^{k}} W_{t}^{k,c} \hookrightarrow \text{hocolim}_{t \in K^{k}} W_{t}^{k}$$

is a weak homotopy equivalence whenever $k < d/2$ and the space $B$ is $F(k)$.

#### 4.1. The category of multiple surgeries.

To prove the above weak homotopy equivalence we will have to work with a category that is a generalization of [12, Definition 6.6]. For what follows, let $\Omega$ be the infinite set used in the definition of $K$ (Definition 3.1). Let $M \subset \mathbb{R}^{\infty}$ be a $d$-dimensional submanifold equipped with a $\theta$-structure $\ell_{M} : TM \oplus e^{1} \to \theta^{*} \gamma^{d+1}$. Suppose that the map $\ell_{M} : M \to B$ is $(k-1)$-connected.
**Definition 4.1.** With $M$ and $\hat{\ell}_M$ as above, let $S^{k-1}_M$ be the category defined as follows. An object is a tuple $(t, \hat{\ell}, e)$ where $t \subset \Omega$ is a finite subset, $\hat{\ell}$ is a $\theta$-structure on $t \times D^k \times D^{d-k+1}$, and $e : t \times D^k \times D^{d-k+1} \to \mathbb{R}^\infty$ is a smooth embedding, subject to the following conditions:

(i) $e^{-1}(M) = t \times S^{k-1} \times D^{d-k+1}$ and the restriction of $\hat{\ell}_M$ to the image of $e$ agrees with the structure $\hat{\ell}$ on $t \times D^k \times D^{d-k+1}$.

(ii) Consider the surgered manifold, $\tilde{M} = M \setminus e(t \times S^{k-1} \times D^{d-k+1}) \sqcup e(t \times D^k \times S^{d-k})$. Let $\hat{\ell}_{\tilde{M}}$ be the $\theta$-structure on $\tilde{M}$ induced by $\hat{\ell}_M$ and $\hat{\ell}$. We require the map $\hat{\ell}_{\tilde{M}} : \tilde{M} \to B$ to be $k$-connected.

A morphism $(t, e, \hat{\ell}) \to (s, e', \hat{\ell}')$ is defined to be an injective map $j : t \hookrightarrow s$ such that $j^*e' = e$ and $j^*\hat{\ell}' = \hat{\ell}$. The object space of this category is topologized as a subspace of,

$$ \prod_{t \subset \Omega} \left[ \text{Emb}(t \times D^k \times D^{d-k+1}, \mathbb{R}^\infty) \times \text{Bun}(T(t \times D^k \times D^{d-k+1}), \theta^*\gamma^{d+1}) \right]. $$

The morphism space is topologized in a similar way.

The main technical ingredient used in the proof of Theorem 3.12 is the following proposition, which we view as a generalization of [12, Proposition 6.7 (Page 916)].

**Proposition 4.2.** Let $k \leq d/2$ and suppose that $\theta : B \to BO(d+1)$ is such that $B$ satisfies Wall’s finiteness condition $F(k)$. Let $M \subset \mathbb{R}^\infty$ be a $d$-dimensional $\theta$-manifold with the property that $\ell_M : M \to B$ is $(k-1)$-connected. Then the classifying space $BS^{k-1}_M$ is weakly contractible.

The proof of the above proposition will require the use of techniques from the following subsection. The proof of Proposition 4.2 will be completed in Section 4.3.

4.2. **Bases for fibrewise surgery.** We begin by making a definition.

**Definition 4.3.** Let $X$ be a smooth manifold. Let $M \subset X \times \mathbb{R}^\infty$ be a smooth $(\dim(X) + d)$-dimensional submanifold for which the projection $\pi : M \to X$ is a submersion (we do not assume that this submersion is proper). Let $\hat{\ell} : T^\pi M \oplus e^1 \to \theta^*\gamma^{d+1}$ be a fibrewise $\theta$-structure with the property that for all $x \in X$, the map $\ell_x : M_x \to B$ is $(k-1)$-connected, where $M_x = \pi^{-1}(x)$ denotes the fibre over $x$. We will denote by $\text{Sub}_{\theta,k-1}(X)$ the set of all such pairs $(M, \hat{\ell})$.

**Definition 4.4.** Fix once and for all an infinite set $\Omega$. A **fibrewise surgery basis** (of degree $k$) for an element $(M, \hat{\ell}) \in \text{Sub}_{\theta,k-1}(X)$ is given by the following data:

- a locally finite covering $\mathcal{U} = \{U_\alpha \mid \alpha \in \Gamma\}$ of $X$;
- a collection of finite subsets $\Lambda_\alpha \subset \Omega$ together with fibrewise embeddings

$$ \varphi_\alpha : U_\alpha \times \Lambda_\alpha \times D^k \times D^{d-k+1} \to X \times \mathbb{R}^\infty, \quad \alpha \in \Gamma, $$

for which $\varphi_\alpha^{-1}(M) = U_\alpha \times \Lambda_\alpha \times S^{k-1} \times D^{d-k+1}$;
- for each $\alpha \in \Gamma$, a fibrewise $\theta$-structure, $\phi_\alpha : U_\alpha \times \Lambda_\alpha \times T(D^k \times D^{d-k+1}) \to \theta^*\gamma^{d+1}$, that extends the fibrewise $\theta$-structure, $\hat{\ell} \circ D\varphi_\alpha$.

The data described above is required to satisfy the following conditions:

(i) For $\alpha \in \Gamma$ and $x \in U_\alpha$, let $(\tilde{M}_x(\alpha), \tilde{\ell}_x(\alpha))$ denote the $\theta$-manifold obtained from $(M_x, \ell_x)$ by performing $\theta$-surgery with respect to the data $(\varphi_\alpha, \phi_\alpha)$. We require that for each $\alpha \in \Gamma$ and $x \in U_\alpha$, the map $\tilde{\ell}_x(\alpha) : \tilde{M}_x(\alpha) \to B$ is $k$-connected.

(ii) For all $\alpha \neq \beta$, $\varphi_\alpha(U_\alpha \times \Lambda_\alpha \times D^k \times D^{d-k+1}) \cap \varphi_\beta(U_\beta \times \Lambda_\beta \times D^k \times D^{d-k+1}) = \emptyset$. 
A fibrewise surgery basis defined as above will be denoted by the tuple \((\mathcal{U}, \varphi, \phi, \Gamma)\).

The following theorem is the main technical ingredient used in the proof of Proposition 4.2.

This can be viewed as a generalization of \(12\) Lemma 6.8)

**Theorem 4.5.** Let \(k \leq d/2\) and let \((M, \ell) \in \text{Sub}_{\beta, k-1}(X)\). Suppose that \(B\) and \(M_x\) satisfy Wall's finiteness condition \(F(k)\) for all \(x \in X\). Then \((M, \ell)\) admits a fibrewise surgery basis.

For our next step in proving Theorem 4.5, we formulate and prove a slightly weaker version of Definition 4.4.

**Definition 4.6.** Let \((M, \ell) \in \text{Sub}_{\beta, k-1}(X)\). A fibrewise surgery semi-basis is a tuple \((\mathcal{U}, \varphi, \phi, \Gamma)\) exactly as in Definition 4.4 except now we drop condition (iii). In particular, the images of the embeddings \(\varphi_i\) are allowed to have non-empty intersections.

**Lemma 4.7.** Let \(k \leq d/2\) and let \((M, \ell) \in \text{Sub}_{\beta, k-1}(X)\). Suppose that \(B\) and \(M_x\) satisfy Wall's finiteness condition \(F(k)\) for all \(x \in X\). Then there exists a sequence of fibrewise surgery semi-bases \((\mathcal{U}^n, \varphi^n, \phi^n, \Gamma^n)\), \(n \in \mathbb{N}\), that satisfies the following condition: If \(i > j\), then

\[
\varphi_i^j(U^n_i \times \Lambda^n_i \times D^k \times D^{d-k+1} \setminus \Lambda^n_j \times D^k \times D^{d-k+1}) = \emptyset
\]

whenever \(\alpha \in \Gamma^i\) and \(\beta \in \Gamma^j\).

**Proof.** We will explicitly construct a family of fibrewise surgery semi-bases \((\mathcal{U}^n, \varphi^n, \phi^n, \Gamma^n)\) satisfying the slightly weaker condition,

\[
\varphi_i^j(U^n_i \times \Lambda^n_i \times D^k \times \{0\}) \setminus \varphi_j^i(U^n_j \times \Lambda^n_j \times D^k \times \{0\}) = \emptyset,
\]

whenever \(i > j\) and \(\alpha \in \Gamma^i\) and \(\beta \in \Gamma^j\). With such embeddings constructed satisfying this condition, the stronger condition from the statement of the lemma is achieved by precomposing the embeddings \(\varphi_i^j\) with a self-embedding \(D^k \times D^{d-k+1} \to D^k \times D^{d-k+1}\) with image sufficiently close to the core \(D^k \times \{0\}\). We construct the family \((\mathcal{U}^m, \varphi^m, \phi^m, \Gamma^m)\), \(m \in \mathbb{N}\) satisfying (4.2) by induction on \(m\). Fix an integer \(n\) and suppose that the fibrewise surgery semi-bases \((\mathcal{U}^i, \varphi^i, \phi^i, \Gamma^i)\) have been constructed for all \(i < n\). We show how to construct \((\mathcal{U}^n, \varphi^n, \phi^n, \Gamma^n)\). Let \(x \in X\) and consider the fibre \(M_x\). By assumption, the map \(\ell_x : M_x \to B\) is \((k-1)\)-connected. Since \(B\) and \(M_x\) satisfy Wall's finiteness condition \(F(k)\), it follows from [11] Proposition 4] that there exists the following data:

- a finite subset \(\Lambda_x \subset \Omega\);
- an embedding \(\varphi_x : \Lambda_x \times D^k \times D^{d-k+1} \to \{x\} \times \mathbb{R}^\infty\) with \(\varphi_x^{-1}(M) = \Lambda_x \times S^{k-1} \times D^{d-k+1}\);
- a \(\theta\)-structure \(\phi_x : T(\Lambda_x \times D^k \times D^{d-k+1}) \to \theta^* \gamma^{d+1}\) that extends

\[
T(\Lambda_x \times S^{k-1} \times D^{d-k+1}) \oplus \ell^1 \xrightarrow{D \varphi_x} TM_x \oplus \ell^1 \xrightarrow{\ell_x} \theta^* \gamma^{d+1},
\]

with the property that the map \(\ell_x : \tilde{M}_x \to B\) is \(k\)-connected, where \((\tilde{M}_x, \ell_x)\) is the \(\theta\)-manifold obtained from \((M_x, \ell_x)\) by performing surgery with respect to the data \((\varphi, \phi)\). Since the collection of embeddings \(\varphi^{i,j}_x\) with \((i, \alpha) \in \{0, \ldots, n-1\} \times (\prod_{i=1}^{n-1} \Gamma^i)\) is finite, we may perturb \(\varphi_x\) so as to make \(\varphi_x(\Lambda_x \times S^{k-1} \times \{0\}) \subset M_x\) transverse to the submanifolds \(\varphi^{i,j}_x(\Lambda^{i,j}_x \times S^{k-1} \times \{0\}) \cap M_x\) for all \((i, \alpha) \in \{0, \ldots, n-1\} \times (\prod_{i=1}^{n-1} \Gamma^i)\). Since \(k \leq d/2\) and \(\dim(M) = d\), this transversality condition implies that

\[
\varphi_x(\Lambda_x \times S^{k-1} \times \{0\}) \cap \varphi^{i,j}_x(\Lambda^{i,j}_x \times S^{k-1} \times \{0\}) = \emptyset,
\]
for all \((i, \alpha) \in \{0, \ldots, n - 1\} \times (\coprod_{i=1}^{n-1} \Gamma^i)\). With the disjointness condition achieved in \(M\), we may perturb \(\varphi_x(\Lambda_x \times D^k \times \{0\}) \subset \mathbb{R}^\infty\), keeping \(\varphi_x(\Lambda_x \times S^{k-1} \times \{0\}) \subset M\) fixed, so as to achieve
\[
\varphi_x(\Lambda_x \times D^k \times \{0\}) \bigcap \varphi^i_x(\Lambda^i_x \times D^k \times \{0\}) = \emptyset,
\]
for all \((i, \alpha) \in \{0, \ldots, n - 1\} \times (\coprod_{i=1}^{n-1} \Gamma^i)\). Since \(\pi : M \to X\) is a locally trivial fibre bundle there exists:

- a neighborhood \(U_x \subset X\) of \(x\),
- a fibrewise embedding \(\varphi_x : U_x \times \Lambda \times D^k \times D^{d-k+1} \to U_x \times \mathbb{R}^\infty\) that extends \(\varphi_x\), and
- a \(\theta\)-structure \(\phi_x : U_x \times T(\Lambda \times D^k \times D^{d-k+1}) \to \theta^* \gamma^{d+1}\) that extends both \(\phi_x\) and the fibrewise \(\theta\)-structure, \(U_x \times T(\Lambda \times S^{k-1} \times D^{d-k+1}) \to \theta^* \gamma^{d+1}\).

Furthermore, by choosing the neighborhood \(U_x\) to be sufficiently small (containing the point \(x \in X\)) we may achieve the condition,
\[
(4.3) \quad \varphi_x(U_x \times \Lambda \times D^k \times \{0\}) \bigcap \varphi^i_x(U^i_x \times \Lambda^i \times D^k \times \{0\}) = \emptyset,
\]
for all \((i, \alpha) \in \{0, \ldots, n - 1\} \times (\coprod_{i=1}^{n-1} \Gamma^i)\). By carrying out the same construction for every point \(x \in X\) we obtain:

- an open cover \(\{U_x \mid x \in X\}\);
- a collection of finite subsets \(\Lambda_x \subset \Omega, x \in X\);
- a collection of fibrewise embeddings \(\varphi_x : U_x \times \Lambda \times D^k \times D^{d-k+1} \to X \times \mathbb{R}^\infty, x \in X\);
- a collection of \(\theta\)-structures \(\phi_x : U_x \times T(\Lambda \times D^k \times D^{d-k+1}) \to \theta^* \gamma^{d+1}, x \in X\);

satisfying the same conditions stated above. Since \(X\) is paracompact, \(\{U_x \mid x \in X\}\) admits a locally-finite subcover. Let \(U^n = \{U^n_\alpha \mid \alpha \in \Gamma^n\}\) be such a locally-finite subcover of \(\{U_x \mid x \in X\}\) and let \(\varphi^n_\alpha\) and \(\phi^n_\alpha\) denote the embedding and \(\theta\)-structure corresponding to \(U^n_\alpha\) for each \(\alpha \in \Gamma^n\). The tuple \((U^n, \varphi^n, \phi^n, \Gamma^n)\) then is the \(n\)-th fibrewise surgery semi-basis satisfying the condition in the statement of the lemma. \(\square\)

4.3. **Proof of Theorem 4.5** We now show how to use Lemma 4.7 from the previous subsection to prove Theorem 4.5. Let \((M, \hat{\ell}) \in \text{Sub}_{\theta, \ell - 1}(X)\). By Lemma 4.7 there exists a sequence \((U^n, \varphi^n, \phi^n, \Gamma^n)\), \(n \in \mathbb{N}\), of fibrewise surgery semi-bases that satisfy the condition from the statement of Lemma 4.7. If \(i > j\), then
\[
\varphi^i_\alpha(U^i_\alpha \times \Lambda^i \times D^k \times D^{d-k+1}) \bigcap \varphi^j_\beta(U^j_\beta \times \Lambda^j \times D^k \times D^{d-k+1}) = \emptyset
\]
for all \(\alpha \in \Gamma^i\) and \(\beta \in \Gamma^j\). With this sequence of fibrewise surgery semi-bases chosen, we set
\[
\Gamma = \coprod_{n=1}^{\infty} \Gamma^n.
\]
Using the sequence \((U^n, \varphi^n, \phi^n, \Gamma^n)\), we define a space,
\[
(4.4) \quad P \subset \mathbb{N} \times \Gamma \times X,
\]
by setting,
\[
P = \{(n, \alpha, x) \in \mathbb{N} \times \Gamma \times X \mid \alpha \in \Gamma^n \text{ and } x \in U^n_\alpha \text{ for some } U^n_\alpha \in U^n\}.
\]
Since \(\coprod_{\alpha \in \Gamma^n} U^n_\alpha = X\) for all \(n \in \mathbb{N}\), it follows that the projection map \(P \to \mathbb{N} \times X\) is a surjective, local homeomorphism, i.e. it is an *etale map*. Now, the space \(P\) has the structure of a topological
Consider the geometric realization $|\mathcal{P}_\bullet|$ of the augmentation map $|\mathcal{P}_\bullet| \rightarrow X$, whose existence is guaranteed by Lemma 4.8. For each $m \in \mathbb{N}$ and $\alpha \in \Gamma^m$, we define

$$W_{m,\alpha} := \zeta^{-1}(V_{m,\alpha}) \subset X.$$  

The collection $\mathcal{W} := \{W_{m,\alpha} \mid m \in \mathbb{N}, \alpha \in \Gamma^m\}$ is an open cover of $X$. We define fibrewise embeddings

$$\Psi_{m,\alpha} : W_{m,\alpha} \times \Lambda^m_\alpha \times D^k \times D^{d-k+1} \rightarrow W_{m,\alpha} \times \mathbb{R}^\infty$$

by setting,

$$\Psi_{m,\alpha} = \varphi^m_\alpha \circ \pi_m \circ \zeta|_{W_{m,\alpha}}.$$  

Similarly define $\theta$-structures

$$\hat{\psi}_{m,\alpha} : W_{m,\alpha} \times T(\Lambda^m_\alpha \times D^k \times D^{d-k+1}) \rightarrow \theta^* \gamma^{d+1},$$

by the formula,

$$\hat{\psi}_{m,\alpha} = \theta^m_\alpha \circ D\pi_m \circ D\zeta|_{W_{m,\alpha}}.$$
It follows that the tuple \((W, \Psi, \psi, \Gamma)\) is a fibrewise surgery semi-basis, namely that it satisfies conditions (i) and (ii) of Definition 4.4. We now argue that it also satisfies condition (iii) as well, namely that

\[
\text{Im}(\Psi_{m,\alpha}) \cap \text{Im}(\Psi_{n,\beta}) = \emptyset
\]

whenever \((m,\alpha) \neq (n,\beta)\). Suppose first \(m \neq n\). By how the initial family \((U^n, \varphi^n, \phi^n, \Gamma^n)\) was chosen, it follows that \(\text{Im}(\varphi_{m,\alpha}^n) \cap \text{Im}(\varphi_{n,\beta}^n) = \emptyset\) for any \(\alpha\) or \(\beta\). By how \(\Psi\) was constructed, it follows then that \(\text{Im}(\Psi_{m,\alpha}) \cap \text{Im}(\Psi_{n,\beta}) = \emptyset\) whenever \(m \neq n\). For the other case \((n = m, \text{but } \alpha \neq \beta)\), observe that \(W_{m,\alpha} \cap W_{m,\beta} = \emptyset\) whenever \(\alpha \neq \beta\), which follows from the fact that

\[
V_{m,\alpha} \cap V_{m,\beta} = \emptyset
\]

for all \(m\), whenever \(\alpha \neq \beta\). Since the \(\Psi_{m,\alpha}\) are fibrewise embeddings, it is then automatic that

\[
\text{Im}(\Psi_{m,\alpha}) \cap \text{Im}(\Psi_{m,\beta}) = \emptyset
\]

for all \(m\), whenever \(\alpha \neq \beta\). We have proven that \(\text{Im}(\Psi_{m,\alpha}) \cap \text{Im}(\Psi_{m,\beta}) = \emptyset\) in all cases that \((m,\alpha) \neq (n,\beta)\). It follows that \((W, \Psi, \psi, \Gamma)\) is a fibrewise surgery basis for \((M, \hat{\ell})\). This completes the proof of Theorem 4.5.

With the proof of Theorem 4.5 complete, we are now in a position to prove Proposition 4.2, which we do below in the following subsection.

4.4. Proof of Proposition 4.2. For the purposes of proving Proposition 4.2 it will be convenient to work with a sheaf model for the topological category \(S_{M}^{k-1}\). For what follows let \(\text{Mfds}\) denote the category whose objects are smooth manifolds without boundary, and whose morphisms are given by smooth maps. We refer the reader to [12, Section 2.4] for background on the concordance theory of sheaves on \(\text{Mfds}\). The sheaf model for \(S_{M}^{k-1}\) is defined below:

**Definition 4.9.** Let \(M\) be as in Definition 4.1. For \(X \in \text{Mfds}\), the category \(S_{M}^{k-1}(X)\) is defined as follows. An object consists of a finite subset \(t \subset \Omega\), a \(\theta\)-structure \(\hat{\ell}\) on \(t \times D^k \times D^{d-k+1}\), and a fibrewise embedding \(e: X \times t \times S^{k-1} \times D^{d-k+1} \to X \times \mathbb{R}^\infty\) subject to the following conditions:

(i) \(e^{-1}(X \times M) = X \times t \times S^{k-1} \times D^{d-k+1}\).

(ii) On each fibre of the projection to \(X\), \(\hat{\ell}_M\) agrees with the restriction of \(\hat{\ell}\) to \(M\).

(iii) For each \(x \in X\), let \(\tilde{M}_x\) denote the \(\theta\)-manifold obtained by performing \(\theta\)-surgery on \(M_x\) via \(e|_{\{x\} \times S^{k-1} \times D^{d-k+1}}\), and the \(\theta\)-structure \(\hat{\ell}\). We require that \(\hat{\ell}_{\tilde{M}_x} : \tilde{M}_x \to B\) be \(k\)-connected for all \(x \in X\).

A morphism from \((t, e, \hat{\ell}) \to (s, e', \hat{\ell}')\) is an injective map \(j : t \to s\) such that \(j^*e' = e\) and \(j^*\hat{\ell}' = \hat{\ell}\).

From the above definition, the assignment \(X \to S_{M}^{k-1}(X)\) defines a *category-valued sheaf* on \(\text{Mfds}\). Clearly the topological category \(S_{M}^{k-1}\) is weak homotopy equivalent to the representing space \(|S_{M}^{k-1}|\) (see [12, Definition 2.6]). Proposition 4.2 asserts that the classifying space \(BS_{M}^{k-1}\) is weakly contractible. In order to prove this we need to work with a sheaf model for the classifying space \(BS_{M}^{k-1}\). We invoke a general construction from [12]. Choose once and for all an uncountably infinite set \(J\).

**Definition 4.10** (Cocycle sheaves). Let \(\mathcal{F}\) be a sheaf on \(\text{Mfds}\) taking values in the category of small categories. There is an associated set valued sheaf \(\beta\mathcal{F}\). For a manifold \(X\), an element \(\beta\mathcal{F}(X)\) is a pair \((\mathcal{U}, \Phi)\) where \(\mathcal{U} = \{U_j \mid j \in J\}\) is a locally finite open cover of \(X\) indexed by \(J\), and \(\Phi\) is a certain collection of morphisms defined as follows: Given a non-empty finite subset \(R \subset J\), let
$U_R$ denote the intersection \( \cap_{j \in R} U_j \). Then \( \Phi \) is a collection \( \varphi_{RS} \in \text{Mor} \mathcal{F}(U_S) \) indexed by the pairs \( R \subseteq S \) of non-empty finite subsets of \( J \), subject to the conditions:

(i) \( \varphi_{RR} = \text{Id}_{c_R} \) for an object \( c_R \in \text{Ob} \mathcal{F}(U_R) \);

(ii) For each non-empty finite subset \( R \subseteq S \), \( \varphi_{RS} \) is a morphism from \( c_S \) to \( c_R|_{U_S} \);

(iii) For all triples \( R \subseteq S \subseteq T \) of finite non-empty subsets of \( J \), we have \( \varphi_{RT} = (\varphi_{RS}|_{U_T}) \circ \varphi_{ST} \).

By [12, Theorem 4.1.2], for any category-valued sheaf \( \mathcal{F} \) there is a weak homotopy equivalence

\[
|\beta \mathcal{F}| \simeq B|\mathcal{F}|,
\]

where the right-hand side is the classifying space of the topological category \( | \mathcal{F} | \). By this theorem from [12], to prove Proposition 4.2 it will suffice to prove that the representing space \( |\beta(S^{k-1}_M)^{\text{op}}| \) is weakly contractible, where \( (S^{k-1}_M)^{\text{op}} \) is the sheaf that assigns to each manifold \( X \), the opposite category to \( S^{k-1}_M(X) \). To do this it will suffice to show that the sheaf \( \beta(S^{k-1}_M)^{\text{op}} \) is weakly equivalent to the constant sheaf sending any manifold \( X \) to a singleton. This reduces to the following proposition (compare with [12, Proof of Proposition 6.7 (Page 916)]):

**Proposition 4.11.** Let \( X \) be given with a closed subset \( A \) and a germ \( s \in \text{colim}_U \beta(S^{k-1}_M)^{\text{op}}(U) \), where \( U \) ranges over the neighborhoods of \( A \) in \( X \). Then \( s \) extends to an element of \( \beta(S^{k-1}_M)^{\text{op}}(X) \).

**Proof of Proposition 4.11.** We will show that this follows as a consequence of Theorem 4.5. Choose an open neighborhood \( U \) of \( A \) in \( X \) such that the germ \( s \in \text{colim}_U \beta(S^{k-1}_M)^{\text{op}}(U) \) can be represented by some \( s_0 \in \beta(S^{k-1}_M)^{\text{op}}(U) \). The information contained in \( s_0 \) includes a locally finite covering of \( U \) by open subsets \( U_j \) for \( j \in J \). It also includes a choice of object \( \psi_{RR} \in \text{Ob} S^{k-1}_M(U_R) \) for each finite nonempty subset \( R \) of \( J \).

Next, choose an open \( X_0 \subset X \) such that \( U \cup X_0 = X \) and the closure of \( X_0 \) in \( X \) avoids \( A \). Let \( N \) be the open subset of \( M \times X_0 \) obtained by removing from \( M \times X_0 \) the closures of the embedded sphere bundles determined by the various \( \varphi_{RR}|_{U_R \cap X_0} \). Consider the projection \( \pi : N \rightarrow X_0 \), which is a submersion. Let \( \hat{\ell} : T^n N \oplus \epsilon^1 \rightarrow \theta^* \gamma^{d+1} \) be the fibrewise \( \theta \)-structure obtained by restricting \( \hat{\ell}_M \) to each fibre of \( \pi \). By assumption, the map \( \ell_M : M \rightarrow B \) is \((k-1)\)-connected. Since each fibre \( N_x = \pi^{-1}(x) \) is obtained from \( M \) by deleting a finite number of disjoint embedded copies of \( S^{k-1} \times D^{d-k+1} \), it then follows by a general position argument that the map \( \ell_x : N_x \rightarrow B \) is \((k-1)\)-connected as well, for all \( x \in X_0 \). It follows that the pair \((N, \hat{\ell})\) determines an element of the set \( \text{Sub}_{d,k-1}(X_0) \) (see Section 4.2). Since \( \pi_k(M) \) is finitely generated by assumption it follows that \( \pi_k(N_x) \) is finitely generated for all \( x \in X \), since \( N_x \) is obtained from \( M \) by deleting a finite number of copies of \( S^{k-1} \times D^{d-k+1} \). We have shown that the element \((N, \hat{\ell}) \in \text{Sub}_{d,k-1}(X_0) \) satisfies the hypotheses of Theorem 4.5. We may then apply Theorem 4.5 to obtain a fibrewise surgery basis \((\mathcal{V}, \varphi, \phi, \Gamma)\) for \((N, \hat{\ell})\).

We now use the fibrewise surgery basis \((\mathcal{V}, \varphi, \phi, \Gamma)\) to construct an element of \( \beta(S^{k-1}_M)^{\text{op}}(X) \) that extends the element \( s \in \beta(S^{k-1}_M)^{\text{op}}(U) \). Let us first fix some notation. Let \( U = \{ U_i \mid i \in J \} \) be the covering of \( U \subset X \) associated to the element \( s \in \beta(S^{k-1}_M)^{\text{op}}(U) \) that was specified in first paragraph of the proof. Let \( V = \{ V_i \mid i \in \Gamma \} \) be the covering associated to the chosen fibrewise surgery basis, \((\mathcal{V}, \varphi, \phi, \Gamma)\). Since the indexing set \( J \) is uncountably infinite we may assume that \( \Gamma \) is a subset of \( J \) with the property that \( U_i = \emptyset \) for all \( i \in \Gamma \). We let \( \mathcal{V}' = \{ V_i' \mid i \in J \} \) denote the
covering of \( X_0 \) obtained by setting

\[
V'_i = \begin{cases} 
V_i & \text{if } i \in \Gamma, \\
\emptyset & \text{if } i \notin \Gamma.
\end{cases}
\]

Defined in this way, it follows that for any \( i \in J \) the set \( U_i \) is non-empty only if \( V'_i \) is empty. With this notation in place, we proceed to define a new covering of \( X \). For \( j \in J \), define

\[
Y_j := \begin{cases} 
U_j & \text{if } U_j \neq \emptyset, \\
V'_j & \text{if } V'_j \neq \emptyset, \\
\emptyset & \text{else}.
\end{cases}
\]

The set \( \{ Y_j \mid j \in J \} \) defines a locally finite covering for \( X \). For finite \( R \subset J \) with nonempty \( Y_R \), we can write \( Y_R = U_S \cap V'_T \) for disjoint subsets \( S, T \) of \( R \) with \( S \cup T = R \). Let \( \varphi_{RR} \in \text{Ob} S_{M}^{k-1}(Y_R) \) be the object given by the union of \( \psi_{SS}|_{Y_R} \) and the embeddings \( \varphi_j|_{Y_R} \) for all \( j \in T \) (where recall that \( \psi_{SS} \) was the object associated to the element \( s \) specified in the first paragraph of the proof). Since these embeddings \( \psi_{SS}|_{Y_R} \) and the \( \varphi_j|_{Y_R} \) are disjoint by construction, it follows that \( \varphi_{RR} \) determines a well defined element of \( \text{Ob} S_{M}^{k-1}(Y_R) \). The covering \( \{ Y_j \mid j \in J \} \) together with the collection of objects \( \varphi_{RR} \) for finite non-empty subsets \( R \subset J \) is an element of \( \beta(S_{M}^{k-1})^{\mathsf{op}}(X) \) that extends \( s \in \beta(S_{M}^{k-1})^{\mathsf{op}}(U) \). This proves Claim \ref{claim_4.11}.

With Claim \ref{claim_4.11} established, the proof of Proposition \ref{prop_4.2} is now complete.

### 4.5. Implementation of fibrewise surgery

In this section we show how to use the above results (namely Proposition \ref{prop_4.2} to prove Theorem \ref{thm_3.10}). We follow closely the methods of \cite[Section 6]{12} but with a few modifications.

First, let \( \mathcal{K}^{(k)} \subset \mathcal{K} \) be the full subcategory on those \( t \) where all elements in \( t \) are labelled by the integer \( k \). Let \( \mathcal{K}^{k-1,(k)^c} \subset \mathcal{K}^{k-1} \) denote the full subcategory consisting of those \( s \) such that all points in \( s \) have labels in the set \( \{ k+1, \ldots, d-k+1 \} \). The category \( \mathcal{K}^{k-1} \) factors as the product \( \mathcal{K}^{k-1} = \mathcal{K}^{k-1,(k)^c} \times \mathcal{K}^{(k)} \), and thus there is a homeomorphism,

\[
\text{hocolim}_{t \in \mathcal{K}^{k-1,(k)^c}} \text{hocolim}_{s \in \mathcal{K}^{(k)}} \mathcal{W}_{P,s,t}^{k-1} \cong \text{hocolim}_{u \in \mathcal{K}^{k-1}} \mathcal{W}_{P,u}^{k-1}.
\]

By the homotopy invariance of homotopy colimits, to prove Theorem \ref{thm_3.10} it will suffice to show that for any \( t \in \mathcal{K}^{k-1,(k)^c} \), the map

\[
\text{hocolim}_{s \in \mathcal{K}^{(k)}} \mathcal{W}_{P,s,t}^{k-1,c} \rightarrow \text{hocolim}_{s \in \mathcal{K}^{(k)}} \mathcal{W}_{P,s,t}^{k-1}
\]

is a weak homotopy equivalence. Taking the homotopy colimit of these maps with \( t \) ranging over \( \mathcal{K}^{k-1,(k)^c} \) will then imply Theorem \ref{thm_3.10}. We will need to introduce some new constructions.

**Definition 4.12.** We call a diagram, \( s \xrightarrow{(j_1, \varepsilon_1)} t \xrightarrow{(j_2, \varepsilon_2)} u \), in \( \mathcal{K}^{(k)} \) **special** if

1. \( j_2(u) \) contains \( j_1(s) \), and
2. \( \varepsilon_1 = +1, \varepsilon_2 = -1 \).
In this situation we define the space $W_{P,u \to t}^{k-1,c}$ by means of the pullback diagram

$$
\begin{array}{ccc}
W_{P,u \to t}^{k-1,c} & \longrightarrow & W_{P,u}^{k-1,c} \\
\downarrow & & \downarrow \\
W_{P,t}^{k-1} & \longrightarrow & W_{P,u}^{k-1}.
\end{array}
$$

The special diagrams $s \to t \leftarrow u$ with a fixed $s$ are objects of a category $D_s^{(k)}$ where the morphisms are commutative diagrams in $K^{(k)}$ of the form,

$$
\begin{array}{ccc}
s & \longrightarrow & t \\
\downarrow & & \downarrow \\
\longrightarrow & & \leftarrow
\end{array}
$$

with special rows.

The rule taking a special diagram $s \to t \leftarrow u$ to $W_{P,u \to t}^{k-1,c}$ defines a contravariant functor on $D_s^{(k)}$ (see the discussion in [12, Page 918]). There is also a natural transformation from that functor on $D_s^{(k)}$ to the constant functor with value $W_{P,s}^{k-1}$, determined by the composition

$$
W_{P,u \to t}^{k-1,c} \longrightarrow W_{P,t}^{k-1} \longrightarrow W_{P,s}^{k-1},
$$

for $s \to t \leftarrow u$ in $D_s^{(k)}$.

The following lemma is the same as [12, Lemma 6.10]. We provide a sketch of the proof.

**Lemma 4.13.** For any object $s \in K^{(k)}$, the natural transformation (4.10) induces a weak homotopy equivalence

$$
\begin{array}{ccc}
\text{hocolim}_{(s \to t \leftarrow u) \in D_s^{(k)}} W_{P,u \to t}^{k-1,c} & \cong & W_{P,s}^{k-1}.
\end{array}
$$

**Proof sketch.** Let $(M,e) := (M,(V,\sigma),e)$ be an element of $W_{P,s}^{k-1}$. It will suffice to prove that the homotopy fibre of the map

$$
\text{hocolim}_{(s \to t \leftarrow u) \in D_s^{(k)}} W_{P,u \to t}^{k-1,c} \longrightarrow W_{P,s}^{k-1}
$$

over $(M,e)$ is weakly contractible. By [12, Lemma 6.11], this homotopy fibre is weakly homotopy equivalent to the homotopy colimit

$$
\text{hocolim}_{(s \to t \leftarrow u) \in D_s^{(k)}} \left(\text{hofibre}_{(M,e)} \left[ W_{P,u \to t}^{k-1,c} \to W_{P,s}^{k-1} \right]\right).
$$

By the same argument given in the proof of [12, Lemma 6.10], this homotopy colimit can be identified with the classifying space $BS_{M\setminus \text{Im}(e)}^{k-1}$. More precisely, this homotopy colimit is actually identified with the classifying space of the edgewise subdivision of $S_{M\setminus \text{Im}(e)}^{k-1}$, but this classifying space is homotopy equivalent to $BS_{M\setminus \text{Im}(e)}^{k-1}$ none-the-less. The manifold $M \setminus \text{Im}(e)$ satisfies the hypotheses of Proposition 4.2 and thus $BS_{M\setminus \text{Im}(e)}^{k-1}$ is weakly contractible. This concludes the proof of the lemma. □
Definition 4.14. Let $\mathcal{D}^{(k)}$ be the category of all special diagrams $s \to t \leftarrow u$ in $\mathcal{K}^{(k)}$. A morphism in $\mathcal{D}^{(k)}$ is a commutative diagram

\[
\begin{array}{ccc}
\mathsf{s} & \rightarrow & \mathsf{t} \\
\downarrow & & \downarrow \\
\mathsf{s'} & \rightarrow & \mathsf{t'}
\end{array}
\]

in $\mathcal{K}^{(k)}$ with special rows. The rule taking an object $s \to t \leftarrow u$ of $\mathcal{D}^{(k)}$ to $W^{k-1,c}_{s\to t\leftarrow u}$ defines a contravariant functor. We may then form the homotopy colimit, $\underleftarrow{\text{hocolim}}\ W^{k-1,c}_{\mathcal{D}^{(k)}; s\to t\leftarrow u}$.

Each morphism $(j, \varepsilon) : s' \rightarrow s$ in $\mathcal{K}^{(k)}$ induces a functor $(j, \varepsilon)^* : \mathcal{D}^{(k)}_s \rightarrow \mathcal{D}^{(k)}_{s'}$, which in turn induces a map,

\[
\underleftarrow{\text{hocolim}}\ W^{k-1,c}_{\mathcal{D}^{(k)}_s; s\to t\leftarrow u} \rightarrow \underleftarrow{\text{hocolim}}\ W^{k-1,c}_{\mathcal{D}^{(k)}_{s'}; s\to t\leftarrow u}.
\]

Construction 4.2. We will construct a map

\[
(4.11) \quad \underleftarrow{\text{hocolim}}\ W^{k-1,c}_{\mathcal{D}^{(k)}; s\to t\leftarrow u} \rightarrow \underleftarrow{\text{hocolim}}\ W^{k-1,c}_{\mathcal{D}^{(k)}; s\to t\leftarrow u}.
\]

In order to define this map it will be useful to construct a particular model for the double homotopy colimit appearing on the left-hand side. We build this model as follows. Let $\mathcal{D}_{\bullet \bullet}$ be the bi-simplicial space where $D_{p,q}$ consists of tuples

\[
(s_0 \rightarrow \cdots \rightarrow s_p; (s_p \rightarrow t_0 \leftarrow u_0) \rightarrow \cdots \rightarrow (s_p \rightarrow t_q \leftarrow u_q); (M, (V, \sigma), e))
\]

where

- $s_0 \rightarrow \cdots \rightarrow s_p$ is a sequence of morphisms in $\mathcal{K}^{(k)}$;
- $(s_p \rightarrow t_0 \leftarrow u_0) \rightarrow \cdots \rightarrow (s_p \rightarrow t_q \leftarrow u_q)$ is a sequence of morphisms in $\mathcal{D}^{(k)}_{s_p}$;
- $(M, (V, \sigma), e)$ is an element of $W^{k-1,c}_{p, u \rightarrow t_q}$.

The face and degeneracy maps are defined in the obvious way. By unpacking the definition of homotopy colimit (Definition 4.9), it can be seen that the geometric realization $|\mathcal{D}_{\bullet \bullet}|$ agrees with the double homotopy colimit,

\[
\underleftarrow{\text{hocolim}}\left[\begin{array}{c}
\underleftarrow{\text{hocolim}}\ W^{k-1,c}_{\mathcal{D}^{(k)}; s\to t\leftarrow u}
\end{array}\right].
\]

Let $\tilde{D}_{\bullet}$ be the simplicial space with $p$-simplices given by tuples

\[
((s_0 \rightarrow t_0 \leftarrow u_0) \rightarrow \cdots \rightarrow (s_p \rightarrow t_p \leftarrow u_p); (M, (V, \sigma), e))
\]

where

- $(s_0 \rightarrow t_0 \leftarrow u_0) \rightarrow \cdots \rightarrow (s_p \rightarrow t_p \leftarrow u_p)$ is a sequence of morphisms in $\mathcal{D}^{(k)}$;
- $(M, (V, \sigma), e)$ is an element of $W^{k-1,c}_{p, u \rightarrow t_p}$.

It follows from the definitions that $\underleftarrow{\text{hocolim}}\ W^{k-1,c}_{\mathcal{D}^{(k)}; s\to t\leftarrow u}$ agrees with the geometric realization $|\tilde{D}_{\bullet}|$.

Now, any $(p, q)$-simplex

\[
(s_0 \rightarrow \cdots \rightarrow s_p; (s_p \rightarrow t_0 \leftarrow u_0) \rightarrow \cdots \rightarrow (s_p \rightarrow t_q \leftarrow u_q); (M, (V, \sigma), e)) \in \mathcal{D}_{p, q}
\]
determines a \((p + q + 1)\)-simplex in \(\overline{D}_{p+q+1}\), namely the element
\[
\left( (s_0 \rightarrow t_0 \leftarrow u_0) \rightarrow \cdots \rightarrow (s_p \rightarrow t_0 \leftarrow u_0) \xrightarrow{\text{id}} (s_p \rightarrow t_0 \leftarrow u_0) \rightarrow \cdots \rightarrow (s_p \rightarrow t_q \leftarrow u_q); \ (M, (V, \sigma), e) \right),
\]
where the maps \(s_i \rightarrow t_0\) for \(i \leq p\) are defined by precomposing \(s_p \rightarrow t_0\) with \(s_i \rightarrow s_p\). Notice that in the above formula the middle map \((s_p \rightarrow t_0 \leftarrow u_0) \xrightarrow{\text{id}} (s_p \rightarrow t_0 \leftarrow u_0)\) is the identity morphism. For each \((p, q)\) this association defines a map, \(F_{p,q} : D_{p,q} \rightarrow \overline{D}_{p+q+1}\). Using these maps we construct a homotopy
\[
[0, 1] \times D_{p,q} \times \Delta^p \times \Delta^q \rightarrow \overline{D}_{p+q+1} \times \Delta^{p+q+1}
\]
by the formula
\[
(r, (\bar{x}; t; s)) \mapsto (F_{p,q}(\bar{x}); (1-r)t, rs),
\]
where \(((1-r)t, rs)\) is considered to be an element of \(\Delta^{p+q+1}\). These maps for all \((p, q)\) geometrically realize to yield a homotopy
\[
(4.12) \quad j_t : |D_{\bullet, \bullet}| \rightarrow |\overline{D}_{\bullet}|, \quad t \in [0, 1].
\]
The promised map \((4.11)\) is defined to be \(j_0\).

The proof of the following lemma is similar to [12, Proof of Theorem 6.5 (page 920)].

**Lemma 4.15.** The inclusion, \(\operatorname{hocolim}_{s \in K^{(k)}} W_{P,s}^{k-1,c} \longrightarrow \operatorname{hocolim}_{s \in K^{(k)}} W_{P,s}^{k-1}\), is a weak homotopy equivalence.

**Proof.** For each \(s \in K^{(k)}\), we denote
\[
X_s := \operatorname{hocolim}_{(s \rightarrow t \leftarrow u) \in D_s} W_{P,u \rightarrow t}^{k-1,c} \quad \text{and} \quad X := \operatorname{hocolim}_{(s \rightarrow t \leftarrow u) \in D^{(k)}} W_{P,u \rightarrow t}^{k-1,c}.
\]
Construction \((4.2)\) yields the homotopy \(j_t : \operatorname{hocolim}_{s \in K^{(k)}} X_s \rightarrow X, \quad t \in [0, 1]\). Consider the diagram
\[
(4.13) \quad \begin{array}{ccc}
& X_s & \\
& j_t & \downarrow \pi_1 \\
\operatorname{hocolim}_{s \in K^{(k)}} W_{P,s}^{k-1} & s & \\
\downarrow \pi_2 & & \\
\operatorname{hocolim}_{s \in K^{(k)}} W_{P,u}^{k-1,c} & & \operatorname{hocolim}_{u \in K^{(k)}} W_{P,u}^{k-1,c},
\end{array}
\]
where \(\iota\) is the inclusion, and \(\pi_1\) and \(\pi_2\) are the maps induced by the functors
\[
p_1 : D^{(k)} \longrightarrow K^{(k)} \quad \text{and} \quad p_2 : D^{(k)} \longrightarrow K^{(k)}
\]
given by the projections
\[
(4.14) \quad (s \rightarrow t \leftarrow u) \mapsto s \quad \text{and} \quad (s \rightarrow t \leftarrow u) \mapsto u
\]
respectively. We will explain the map \(s\) (which will so happen to be a section of \(\pi_2 \circ j_1\)) shortly.
Now, the two functors \(p_1, p_2 : D^{(k)} \rightarrow K^{(k)}\) defined in \((4.14)\) are connected by a zig-zag of natural transformations. Namely, there is a natural transformation from the functor \(p_1\) to
\[
p : D^{(k)} \rightarrow K^{(k)}, \quad (s \rightarrow t \leftarrow u) \mapsto t,
\]
defined by associating to the object \((s \rightarrow t \leftarrow u)\) the arrow \(s \rightarrow t\). Similarly, there is a natural transformation from \(p\) to \(p_2\) defined in the same way. This zig-zag of natural transformations

\[ p_1 \Leftarrow p \Rightarrow p_2 \]
yields a homotopy between the maps \(\iota \circ \pi_2\) and \(\pi_1\) in the diagram (4.13), and thus the lower triangle is homotopy commutative.

Now, the composite \(\pi_1 \circ j_0\) is just the homotopy colimit (taken over \(s \in K^{(k)}\)) of the weak homotopy equivalences from Lemma 4.13, \(\text{hocolim} \ W_{P,u \rightarrow t}^{k-1,c} \cong W_{P,s}^{k-1}\). By the homotopy invariance of homotopy colimits it follows that \(\pi_1 \circ j_0\) is a weak homotopy equivalence.

By the homotopy commutativity of the lower triangle of (4.13), it follows that the composite

\[ \iota \circ \pi_2 \circ j_0 : \text{hocolim}\ X_s \rightarrow \text{hocolim} W_{P,s}^{k-1} \]
is a weak homotopy equivalence as well; this implies that \(\iota\) induces a surjection on all homotopy groups \(\pi_i(-)\). To finish the proof we will need to show that it induces an injection on homotopy groups; to do this we need to use the map \(s\), which still needs to be described. Indeed, for each \(s \in K^{(k)}\) there is a trivial special diagram \(s \rightarrow s \leftarrow s\) in \(D_s\). For each \(s\), the association

\[ s \mapsto (s \rightarrow s \leftarrow s) \]
determines the natural transformation, \(W_{P,s}^{k-1,c} \rightarrow W_{P,s}^{k-1,c}\). The map \(s\) in diagram (4.13) is defined to be the homotopy colimit over \(s \in K^{(k)}\) of these maps. By unpacking the definition of the homotopy \(j_t\) in Construction 1.2 it follows that,

\[ \pi_2 \circ j_1 \circ s = \text{Id}, \]
and thus we see that \(s\) is a section of \(\pi_2 \circ j_1\). This proves that \(\pi_2 \circ j_1\) induces a surjection on all homotopy groups (and thus \(\pi_2 \circ j_t\) induces a surjection on homotopy groups for all \(t \in [0,1]\)). From the weak homotopy equivalence (4.15) it follows that \(\pi_2 \circ j_t\) induces an injection (for all \(t \in [0,1]\)), and thus we have proven that \(\pi_2 \circ j_t\) is a weak homotopy equivalence. In view of (4.13) again, the two-out-of-three property then implies that \(\iota\) is a weak homotopy equivalence.

Below we show how to use Lemma 4.15 to prove Theorem 3.12.

**Proof of Theorem 3.12.** As discussed in the beginning of the subsection, to prove the theorem it will suffice to show that for any \(t \in K^{k-1,(k)^c}\), the map

\[ \text{hocolim} W_{P,s,t}^{k-1,c} \rightarrow \text{hocolim} W_{P,s,t}^{k-1} \]
is a weak homotopy equivalence. With this weak homotopy equivalence established, the desired weak homotopy equivalence is obtained by taking the homotopy colimit over \(t \in K^{k-1,(k)^c}\).

Fix \(t \in K^{k-1,(k)^c}\). For \(s \in K^{(k)}\), consider the map

\[ W_{P,s,t}^{k-1} \rightarrow W_{\text{loc},t}, \]
defined by sending \((M,(V,\sigma),e) \in W_{P,s,t}^{k-1}\) to \(((V,\sigma)|_t, e|_t)\), which is the component of \(((V,\sigma),e)\) that corresponds to the subset \(t\). It is easily verified that (4.16) is a Serre fibration. Let \(\tilde{W}_{P,s,t}^{k-1}\) denote the fibre of this map over some point

\[ ((V',\sigma'),e') \in W_{\text{loc},t}. \]
Keeping $t$ constant and taking the homotopy colimit over $s \in \mathcal{K}^{(k)}$, we obtain a fibre sequence

$$\text{hocolim}_{s \in \mathcal{K}^{(k)}} \mathcal{W}^{k-1}_{P,s:t} \longrightarrow \text{hocolim}_{s \in \mathcal{K}^{(k)}} \mathcal{W}^{k-1}_{P,s:t} \longrightarrow \mathcal{W}_{\text{loc},t}.$$

We obtain a similar fibre sequence involving $\text{hocolim}_{s \in \mathcal{K}^{(k)}} \mathcal{W}^{k-1}_{P,s:t}$ and a map of fibre sequences

$$(4.18)$$

$$\text{hocolim}_{s \in \mathcal{K}^{(k)}} \mathcal{W}^{k-1}_{P,s:t} \longrightarrow \text{hocolim}_{s \in \mathcal{K}^{(k)}} \mathcal{W}^{k-1}_{P,s:t} \longrightarrow \mathcal{W}_{\text{loc},t} \longrightarrow \text{Id} \longrightarrow \mathcal{W}_{\text{loc},t}.$$ 

We claim that the top-horizontal map is a weak homotopy equivalence. Indeed, $\mathcal{W}^{k-1}_{P,s:t}$ can be identified with the subspace of $\mathcal{W}^{k-1}_{P,s:t}$ consisting of those $(M,(V,\sigma),e) \in \mathcal{W}^{k-1}_{P,s:t}$ for which the manifold $M$ contains $\mathcal{E}'(S(V) \times_t D(V^+))$. Setting $P' = P \cup \mathcal{E}'(S(V) \times_t D(V^+))$, it follows that there are homeomorphisms

$$\mathcal{W}^{k-1}_{P,s:t} \cong \mathcal{W}^{k-1}_{P',s},$$

$$\mathcal{W}^{k-1}_{P,s:t} \cong \mathcal{W}^{k-1}_{P',s},$$

(see also Proposition 6.2). By these homeomorphisms the top-horizontal map of (4.18) can be identified with the inclusion

$$\text{hocolim}_{s \in \mathcal{K}^{(k)}} \mathcal{W}^{k-1}_{P,s:t} \subset \text{hocolim}_{s \in \mathcal{K}^{(k)}} \mathcal{W}^{k-1}_{P',s},$$

and thus by Lemma 4.15 it is a weak homotopy equivalence (Lemma 4.15 holds for all choice of $P' \in \mathcal{M}_\theta$). Since the top-horizontal map in (4.18) is a weak homotopy equivalence, it follows that the middle-horizontal map is a weak homotopy equivalence as well, since the vertical columns are fibre sequences. This concludes the proof of Theorem 3.12.

5. Decomposing the Localization Sequence

In this section we embark on the proof of Theorem 3.13. This theorem asserts that the map

$$\text{hocolim}_{t \in \mathcal{K}^{k-1}} \mathcal{W}^{k-1}_{P,t} \longrightarrow \text{hocolim}_{t \in \mathcal{K}^{k-1}} \mathcal{W}_{\text{loc},t}^{k,d-k-1}$$

is a quasi-fibration. Let $t \in \mathcal{K}^{k-1}$. We may uniquely write $t = t' \cup s$ with $t' \in \mathcal{K}^k$ and $s \in \mathcal{K}^{k,d-k+1}$. Consider the map

$$(5.1)$$

$$\mathcal{W}^{k-1}_{P,t' \cup s} \longrightarrow \mathcal{W}_{\text{loc},s}^{k,d-k-1}, \quad (M,(V,\sigma),e) \mapsto ((V,\sigma),e).$$

This map is easily seen to be a Serre-fibration. We will need to analyze its fibres.

**Definition 5.1.** For each $s \in \mathcal{K}^{k,d-k+1}$, let us fix once and for all an element

$$V_s := ((V_s,\sigma_s),e_s) \in \mathcal{W}_{\text{loc},s}^{k,d-k-1}.$$

In the case that $s = \emptyset$, we will still denote the single element in $\mathcal{W}_{\text{loc},\emptyset}^{k,d-k-1}$ by $\emptyset$. For each $t \in \mathcal{K}^k$, we let $\tilde{\mathcal{W}}^{k-1}_{P,t,s}$ denote the fibre of the map $\mathcal{W}^{k-1}_{P,t} \longrightarrow \mathcal{W}_{\text{loc},s}^{k,d-k-1}$ over the element $V_s \in \mathcal{W}_{\text{loc},s}^{k,d-k-1}$.
The lemma below follows from gluing together the Serre fibrations
\[ W_{k-1,c}^{k,t \in \mathcal{K}} \rightarrow W_{loc,s}^{\{k,d-k-1\}} \]
and letting \( t \) range over \( \mathcal{K}^k \), while keeping \( s \) constant.

**Lemma 5.2.** For all \( s \in \mathcal{K}^{\{k,d-k+1\}} \), the following sequence is a homotopy fibre sequence,
\[ \text{hocolim}_{t \in \mathcal{K}^k} W_{k-1,c}^{k,t \in \mathcal{K}} \rightarrow \text{hocolim}_{t \in \mathcal{K}^k} W_{loc,s}^{\{k,d-k-1\}}. \]

We now state the main theorem of this section. Its proof is carried out over the course of Sections 6 through 10.

**Theorem 5.3.** Let \( k < d/2 \). Then for any morphism \((j, \varepsilon) : s \rightarrow s'\) in \( \mathcal{K}^{\{k,d-k+1\}} \) the induced map
\[ (j, \varepsilon)^* : \text{hocolim}_{t \in \mathcal{K}^k} W_{k-1,c}^{k,t \in \mathcal{K}} \rightarrow \text{hocolim}_{t \in \mathcal{K}^k} W_{k-1,c}^{k,t \in \mathcal{K}} \]
is a weak homotopy equivalence.

By taking the homotopy colimit of the weak homotopy equivalences from Theorem 5.3 over \( s \in \mathcal{K}^{\{k,d-k+1\}} \) we obtain the following corollary.

**Corollary 5.4.** Let \( k < d/2 \). Then the map
\[ \text{hocolim}_{s \in \mathcal{K}^{\{k,d-k+1\}}} \text{hocolim}_{t \in \mathcal{K}^k} W_{k-1,c}^{k,t \in \mathcal{K}} \rightarrow \text{hocolim}_{s \in \mathcal{K}^{\{k,d-k+1\}}} W_{loc,s}^{\{k,d-k-1\}} \]
is a quasi-fibration. The fibre over \( \emptyset \) is given by \( \text{hocolim}_{t \in \mathcal{K}^k} W_{k-1,c}^{k,t \in \mathcal{K}} \).

**Proof.** Each morphism \((j, \varepsilon) : s \rightarrow s'\) in \( \mathcal{K}^{\{k,d-k+1\}} \) induces a map of fibre sequences
\[
\begin{array}{ccc}
\text{hocolim}_{t \in \mathcal{K}^k} W_{k-1,c}^{k,t \in \mathcal{K}} & \rightarrow & \text{hocolim}_{t \in \mathcal{K}^k} W_{k-1,c}^{k,t \in \mathcal{K}} \\
\downarrow & & \downarrow \ \\
\text{hocolim}_{t \in \mathcal{K}^k} W_{k-1,c}^{k,t \in \mathcal{K}} & \rightarrow & \text{hocolim}_{t \in \mathcal{K}^k} W_{k-1,c}^{k,t \in \mathcal{K}} \\
\downarrow & & \downarrow \ \\
W_{loc,s'}^{\{k,d-k-1\}} & \rightarrow & W_{loc,s}^{\{k,d-k-1\}}
\end{array}
\]
By Theorem 5.3, the top horizontal map is a weak homotopy equivalence. As a consequence of this, it follows from Proposition 3.14 that the map
\[ \text{hocolim}_{s \in \mathcal{K}^{\{k,d-k+1\}}} \text{hocolim}_{t \in \mathcal{K}^k} W_{k-1,c}^{k,t \in \mathcal{K}} \rightarrow \text{hocolim}_{s \in \mathcal{K}^{\{k,d-k+1\}}} W_{loc,s}^{\{k,d-k-1\}} \]
is a quasi-fibration, with fibre over \( \emptyset \) given by \( \text{hocolim}_{t \in \mathcal{K}^k} W_{k-1,c}^{k,t \in \mathcal{K}} \). This concludes the proof of the corollary.

Using the above result, Theorem 3.13 follows by making a few simple identifications.
Proof of Theorem 6.13 (assuming Proposition 5.3). We first observe that
\[ \text{hocolim}_{t \in K} W_{P,t,0}^{k-1,c} = \text{hocolim}_{t \in K} W_{P,t}^k. \]
This identification follows immediately from the definition of the spaces. From this it follows that the fibre of
\[ \text{hocolim}_{s \in K^{(k,d-k+1)}} \text{hocolim}_{t \in K^k} W_{P,t,0}^{k-1,c} \rightarrow \text{hocolim}_{s \in K^{(k,d-k+1)}} W_{\text{loc,s}}^{k \times d-k-1} \]
over the element 0 is given by the space \( \text{hocolim}_{t \in K^k} W_{P,t}^k \). Using the product factorization of the indexing categories \( K^k \cong K^{k-1} \times K^{(k,d-k+1)} \), we obtain the following homeomorphism,
\[ \text{hocolim}_{s \in K^{(k,d-k+1)}} \text{hocolim}_{t \in K^k} W_{P,t,0}^{k-1,c} \cong \text{hocolim}_{u \in K^{k-1}} W_{P,u}^{k-1,c}. \]

Theorem 3.13 then follows by plugging the above identifications in to the Serre fibration obtained as a result of Corollary 5.4.

### 6. Cobordism Categories and Stability

In this section we begin to tackle Theorem 5.3 which was the main technical device used to prove Theorem 6.13. Let \( k < d/2 \). Theorem 5.3 states that for any morphism \((j, \varepsilon) : s \rightarrow s'\) in \( K^{(k,d-k+1)} \), the induced map
\[ (j, \varepsilon)^* : \text{hocolim}_{t \in K^k} W_{P,t,s}^{k-1,c} \rightarrow \text{hocolim}_{t \in K^k} W_{P,t,s'}^{k-1,c}, \]
is a weak homotopy equivalence. Below, we define a generalization of these maps and will prove a generalization of Theorem 5.3. We must first identify the spaces \( W_{P,t,s}^{k-1,c} \) (with \( t \in K^k \) and \( s \in K^{(k,d-k+1)} \)) with something else a bit more familiar.

**Definition 6.1.** For each \( s \in K^{(k,d-k+1)} \), fix once and for all an element \( S_s \in M_\emptyset \) with the following properties: (i) \( S_s \cap P = \emptyset \), (ii) \( S_s \) is diffeomorphic to \( s \times S^{k-1} \times S^{d-k} \), and (iii) the \( \theta \)-structure \( \ell_s \) admits an extension to the \( \theta \)-structure on \( s \times D^k \times D^{d-k+1} \) (where we use a diffeomorphism \( s \times S^{k-1} \times S^{d-k} \cong S_s \) from (ii) to identify \( S_s \) with \( s \times S^{k-1} \times S^{d-k} \)). It follows that the \( (d-1) \)-dimensional manifold \( P \cup S_s \) equipped with the structure \( \ell_P \cup \ell_s \), defines an element of the space \( M_0 \), see Definition 2.3.

**Proposition 6.2.** For all \( t \in K^k \) and \( s \in K^{(k,d-k+1)} \) there is a homotopy equivalence,
\[ W_{P,t,s}^k \cong W_{P,t,s}^{k-1,c}. \]

**Proof.** Fix \((V, \sigma) \in (G^\text{mf}_{D}^{(k,d-k+1)})^s \). Choose an embedding
\[ e : D(V^-) \times_s D(V^+) \rightarrow (-\infty, 0] \times \mathbb{R}^{\infty-1} \]
with image disjoint from \( P \), and with \( e(S(V^-) \times_s S(V^+) = S_s \). We define a map
\[ W_{P,t,s}^k \rightarrow W_{P,t,s}^{k-1,c} \]
by sending \((M, (V', \sigma'), e')\) to the element,
\[ (M \cup e(S(V^-) \times_s D(V^+) \cup (V' \cup V, \sigma' \cup \sigma'), e' \cup e). \]
Since \( \ell_M : M \rightarrow B \) is \( k \)-connected, it follows that the element
\[ (M \cup e(S(V^-) \times_s D(V^+) \cup (V' \cup V, \sigma' \cup \sigma'), e' \cup e) \]
is contained in the space $\tilde{\mathcal{W}}_{P,t,s}^{k-1,c}$, and thus the map is well-defined. A homotopy inverse

\begin{equation}
\tilde{\mathcal{W}}_{P,t,s}^{k-1,c} \to \mathcal{W}_{P,t}^{k}
\end{equation}

to (6.2) by sending $(M, (V, \sigma), e)$ to $(M \setminus e(S(V^-) \times_b D(V^+)), (V, \sigma)|_{t}, e|_{D(V^-) \times_t D(V^+)})$. Checking that this map is indeed a homotopy inverse is straightforward and we leave it to the reader. \[ \square \]

We will now reinterpret the maps (6.1) using the homeomorphism (6.2) from the above proposition. We will actually consider a generalization of these maps and will prove a generalization of Theorem 5.3. We will re-interpret these maps as being induced by morphisms from a cobordism category, with objects given by $(d-1)$-dimensional closed manifolds and morphisms given by $d$-dimensional cobordisms defined below.

**Definition 6.3.** The (non-unital) topological category $\text{Cob}_{\theta,d}$ has $\mathcal{M}_{\theta}$ (see Definition 2.4) for its space of objects. A morphism $W : P \rightsquigarrow Q$ is a compact $d$-dimensional submanifold

\[ W \subset [0,1] \times \mathbb{R}^{\infty-1} \]

equipped with a $\theta$-structure $\hat{\ell}_W : TW \oplus e^1 \to \theta^* \gamma^{d+1}$, for which there exists $0 < \varepsilon < 1/2$ such that

\[ W \cap \{0,1\} \times \mathbb{R}^{\infty-1} = \partial W, \]

\[ W \cap ([0, \varepsilon) \times \mathbb{R}^{\infty-1}) = [0, \varepsilon) \times P, \]

\[ W \cap ((1-\varepsilon,1] \times \mathbb{R}^{\infty-1}) = (1-\varepsilon,1] \times Q. \]

Let $\Phi : [0,2] \times \mathbb{R}^{\infty-1} \to [0,1] \times \mathbb{R}^{\infty-1}$ be the diffeomorphism given by $\Phi(t, x) = (t/2, x)$, where $t \in [0,2]$ and $x \in \mathbb{R}^{\infty-1}$. For morphisms $W : P \rightsquigarrow Q$ and $W' : Q \rightsquigarrow R$, composition is defined by

\begin{equation}
W' \circ W = \Phi(W' \cup W).
\end{equation}

The category $\text{Cob}_{\theta,d}$ is topologized in the standard way following [5], [8].

**Remark 6.4.** With the composition rule in (6.2), the category is not strictly associative; it is associative up to homotopy. This lack of strict associativity won’t affect any of our latter constructions.

We will need to consider certain subcategories of $\text{Cob}_{\theta,d}$.

**Definition 6.5.** Let $l \in \mathbb{Z}_{\geq -1}$ be an integer. The topological subcategory $\text{Cob}_{\theta,d}^l \subset \text{Cob}_{\theta,d}$ has the same objects. Its morphisms consist of those cobordisms $W : P \rightsquigarrow Q$ for which the pair $(W, P)$ is $l$-connected.

Fix an integer $k$. We now proceed to make $P \mapsto \mathcal{W}_{P,t}^k$ into a functor on $\text{Cob}_{\theta,d}^{k-1}$. Let $\psi : (-\infty,1] \to (-\infty,0]$ be a diffeomorphism that is the identity on $(-\infty, -1)$ and let

\[ \Psi := \psi \times \text{Id}_{\mathbb{R}^{\infty-1}} : (-\infty,1] \times \mathbb{R}^{\infty-1} \to (-\infty,0] \times \mathbb{R}^{\infty-1}. \]

Let $W : P \rightsquigarrow Q$ be a morphism in $\text{Cob}_{\theta,d}^{k-1}$. For any $t \in K^k$, such a morphism determines a map

\begin{equation}
S_W : \mathcal{W}_{P,t}^k \to \mathcal{W}_{Q,t}^k
\end{equation}

by sending $(M, (V, \sigma), e) \in \mathcal{W}_{P,t}^k$ to the element $(\Psi(M \cup_P W), (V, \sigma), e) \in \mathcal{W}_{Q,t}^k$. Let us denote $M \cup_P W := \Psi(M \cup_P W)$. Notice that since $(W, P)$ is $(k-1)$-connected, the new $\theta$-structure $\ell_{M \cup_P W}$ on $M \cup_P W$ is such that $\ell_{M \cup_P W} : M \cup_P W \to B$ is $k$-connected, and thus determines
Then the induced map $\theta, d$ conditions:

Let $0$ suffice to prove that such a morphism induces a weak homotopy equivalence $\theta, d$ or $\ell$.

Proof of Theorem 5.3 assuming Theorem 6.7.

Theorem 6.7 implies Proposition 5.3, which is the whole point of these new definitions.

$P \mapsto \text{hocolim}_{t \in K^k} W^k_{P,t}$, $W^k_{P,t'}$ $\xrightarrow{S}$ $W^k_{Q,t'}$

$\xrightarrow{(j, \varepsilon)^*}$ $\xrightarrow{(j, \varepsilon)^*}$

commutes. For morphisms $W : P \rightsquigarrow Q$ in $\text{Cob}_{\theta,d}^{k-1}$ we will study the effect of the induced map

$S_W : \text{hocolim}_{t \in K^k} W^k_{P,t} \longrightarrow \text{hocolim}_{t \in K^k} W^k_{Q,t}$.

In this way we may view the correspondence $P \mapsto \text{hocolim}_{t \in K^k} W^k_{P,t}$ to be a functor on the category $\text{Cob}_{\theta,d}^{k-1}$. In order to formulate our main theorem we will need to make some preliminary definitions.

**Definition 6.6.** Let $W : P \rightsquigarrow Q$ be a morphism in $\text{Cob}_{\theta,d}$. Let $0 \leq l < d$, and suppose that $W$ is diffeomorphic relative $P$ to the trace of a surgery along an embedding $f : S^l \times D^{d-l-1} \longrightarrow P$.

(i) The cobordism $W : P \rightsquigarrow Q$ is said to be trivial of degree $l + 1$ if the embedding $f$ factors through an embedding $\mathbb{R}^{d-l-1} \hookrightarrow P$.

(ii) The cobordism $W : P \rightsquigarrow Q$ is said to be primitive of degree $l + 1$ if there exists another embedding $g : D^l \times S^{d-l-1} \longrightarrow P$ such that $f(S^l \times \{0\})$ and $g(\{0\} \times S^{d-l-1})$ intersect transversally in $P$ at exactly one point.

Notice that the above definition only makes sense in the case that $0 \leq l < d$. It will be useful to us latter on to make the following convention. A morphism $W : P \rightsquigarrow Q$ in $\text{Cob}_{\theta,d}$ is said to be primitive of degree 0 if $W$ is isomorphic (as a $\theta$-manifold) to the disjoint union $([0,1] \times P) \sqcup S^d$, where $([0,1] \times P)$ is equipped with the $\theta$-structure induced by $\hat{\ell}_P$, and $S^d$ is equipped with some $\theta$-structure $\hat{\ell}_S$ that admits an extension to $D^d$. Equivalently, we may also refer to the same cobordism $W$ as being trivial of degree $d$.

Our main result of this section is the following theorem:

**Theorem 6.7.** Let $0 \leq k < d/2$. Let $W : P \rightsquigarrow Q$ be a morphism in $\text{Cob}_{\theta,d}^{k-1}$. Suppose that $W$ is a composite of elementary cobordisms $V_m \circ \cdots \circ V_1$, such that each $V_i$ satisfies one of the following conditions:

(a) $V_i$ is primitive of degree $k$,

(b) $V_i$ is trivial of degree $d - k$, or

(c) the pair $(V_i, \partial_{in} V_i)$ is $(d - k)$-connected, where $\partial_{in} V_i$ is the source object of the morphism $V_i$.

Then the induced map $S_W : \text{hocolim}_{t \in K^k} W^k_{P,t} \longrightarrow \text{hocolim}_{t \in K^k} W^k_{Q,t}$ is a weak homotopy equivalence.

The proof of the above theorem will span the next three sections. Let us now show how Theorem 6.7 implies Proposition 5.3, which is the whole point of these new definitions.

Proof of Theorem 5.3 assuming Theorem 6.7. Let $s \in K^{k,d-k+1}$. Let $s' = s \sqcup \{x\}$, where $x$ has either label $k$ or $d - k + 1$. Let $(j, \varepsilon) : s \longrightarrow s'$ be a morphism with $j : s \hookrightarrow s'$ the inclusion. It will suffice to prove that such a morphism induces a weak homotopy equivalence

$$\text{hocolim}_{t \in K^k} \hat{W}_{P,t,s}^{k-1,c} \xrightarrow{\sim} \text{hocolim}_{t \in K^k} \hat{W}_{P,t,s'}^{k-1,c}.$$
We will show that the composite map

\[ \text{hocolim}_{t \in \mathcal{K}^k} W^k_{P \sqcup s', t} \xrightarrow{\cong} \text{hocolim}_{t \in \mathcal{K}^k} W^{k-1,c}_{P \sqcup s', t} \xrightarrow{\cong} \text{hocolim}_{t \in \mathcal{K}^k} W^{k-1,c}_{P \sqcup s, t} \]

agrees with the map induced by \( S_W \), for some cobordism \( W : P \sqcup s' \to P \sqcup s \) covered by Theorem 6.7. The vertical maps in the diagram are the homotopy equivalences \( 6.2 \) and \( 6.3 \) from the proof of Proposition 6.2. Let us describe that cobordism. Recall that \( S_s \) is the product \( s \times S^{k-1} \times S^{d-k} \) and \( S_{s'} = S_s \sqcup (\{x\} \times S^{k-1} \times S^{d-k}) \). Let \( W_1 \) be the manifold given by

\[ (S_{s'} \sqcup P) \times [0,1] \bigcup_{\{x\} \times S^{k-1} \times S^{d-k}} \{x\} \times D^k \times S^{d-k}. \]

Similarly, let \( W_2 \) be the manifold given by

\[ (S_{s'} \sqcup P) \times [0,1] \bigcup_{\{x\} \times S^{k-1} \times S^{d-k}} \{x\} \times S^{k-1} \times D^{d-k+1}. \]

Both of these define cobordisms from \( P_{s'} \) to \( P_s \). Tracing through the definition we see that \( 6.6 \) coincides with the map induced by \( S_{W_1} \) when the label on \( x \) is \( k \) and \( \varepsilon = +1 \), or in the case when the label on \( x \) is \( d-k+1 \) and \( \varepsilon = -1 \). Similarly, \( 6.6 \) coincides with the map induced by \( S_{W_2} \) when the label on \( x \) is \( d-k+1 \) and \( \varepsilon = -1 \), or when the label on \( x \) is \( k \) and \( \varepsilon = +1 \). Now, we observe that the cobordisms \( W_1 \) and \( W_2 \) are both covered by Theorem 6.7. Indeed, \( W_1 \) is constructed by first attaching a primitive \( k \)-handle and then attaching a \( d \)-handle; this is covered by conditions (a) and (c). Similarly, \( W_2 \) is constructed by attaching a \( (d-k+1) \)-handle and then a \( d \)-handle. Since \( P \) is assumed to be non-empty, it follows from Theorem 6.7 that in all cases, the map under consideration is a weak homotopy equivalence. This completes the proof of Theorem 5.3 assuming Theorem 6.7.

\[ \square \]

Remark 6.8. The argument in the above proof does also include the case where \( k = 0 \). In this case, it follows that \( S_{s'} = s \times S^{k-1} \times S^{d-k} = \emptyset \), and the cobordism \( W_1 \) is isomorphic (as a \( \theta \)-manifold) to \( ([0,1] \times P) \sqcup S^d \). By Definition 6.6, the cobordism \( W_1 \) is primitive of degree-0 (or equivalently trivial of degree-\( d \)) and thus it is covered by Theorem 6.7 parts (a) or (b). Part (c) of Theorem 6.7 is not needed in this case. The cobordism \( W_2 \) from the above proof is irrelevant in this \( k = 0 \) case and need not be considered.

Below we introduce some notation to keep us organized while proving Theorem 6.7.

Notational Convention 6.9. We let \( \mathcal{V}^{k-1} \subset \text{Coh}_{g,d}^{k-1} \) be the subcategory consisting of all morphisms \( W : P \to Q \) with the property that \( S_W : \text{hocolim}_{t \in \mathcal{K}^k} W^k_{P \sqcup t} \longrightarrow \text{hocolim}_{t \in \mathcal{K}^k} W^{k-1,c}_{Q \sqcup t} \) is a weak homotopy equivalence. To prove Theorem 6.7 we will need to show that \( \mathcal{V}^{k-1} \) contains all morphisms that satisfy one of the conditions (a), (b), or (c).

Remark 6.10. We remark that our proof of Theorem 6.7 is conceptually similar to the homological stability theorem proven by Galatius and Randal-Williams in [6]. Of course in our case, Theorem 6.7 asserts that the maps under consideration are weak homotopy equivalences rather than homology equivalences. In view of what is done in Section 11 (Theorem 11.6 in particular), one can actually
consider the homological stability theorem of [6] as the analogue of Theorem 6.7 for the case that \( d = 2n \) and \( k = n \).

7. The morphism \( H_{k,d-k}(P) \)

In this section we begin the process of proving Theorem 6.7 by verifying that

\[
S_{W} : \text{hocolim}_{t \in \mathcal{K}^{k}} W^{k}_{P,t} \longrightarrow \text{hocolim}_{t \in \mathcal{K}^{k}} W^{k}_{Q,t}
\]

is a weak homotopy equivalence for certain basic cobordisms \( W : P \rightsquigarrow Q \). In particular we will prove the special case of Theorem 6.7 for when \( k = 0 \), see Remark 7.2.

7.1. The morphism \( H_{k,d-k}(P) \). Fix an object \( P \in \text{Ob} \text{Cob}_{\theta,d}^{k-1} \). We construct a morphism

\[
(7.1) \quad H_{k,d-k}(P) : P \rightsquigarrow P
\]

of \( \text{Cob}_{\theta,d}^{k-1} \) as follows. Choose an embedding \( j : S^{k} \times S^{d-k} \rightarrow (0,1) \times \mathbb{R}^{\infty-1} \) with image disjoint from \([0,1] \times P\). Fix a \( \theta \)-structure \( \hat{\ell}_{k,d-k} \) on \( S^{k} \times S^{d-k} \) with the property that \( \ell_{k,d-k} : S^{k} \times S^{d-k} \rightarrow B \) is null-homotopic. We then let \( H_{k,d-k}(P) \subset [0,1] \times \mathbb{R}^{\infty-1} \) be the cobordism obtained by forming the connected sum of \([0,1] \times P\) with \( j(S^{k} \times S^{d-k}) \) along some embedded arc connecting the two submanifolds. We equip \( H_{k,d-k}(P) \) with a \( \theta \)-structure that agrees with \( \hat{\ell}_{k,d-k} \) on the connect-summand of \( S^{k} \times S^{d-k} \), and with \( \ell_{[0,1] \times P} \) away from the connect-summand. This self-cobordism \( H_{k,d-k}(P) : P \rightsquigarrow P \) equipped with its \( \theta \)-structure yields a morphism in \( \text{Cob}_{\theta,d}^{k-1} \). We have the following proposition.

**Proposition 7.1.** Let \( 0 \leq k < d/2 \). Then for any non-empty object \( P \in \text{Ob} \text{Cob}_{\theta,d}^{k-1} \), the morphism \( H_{k,d-k}(P) : P \rightsquigarrow P \) is contained in \( \mathcal{Y}^{k-1} \) (see Notational Convention 6.3).

**Remark 7.2.** Notice that in the case that \( k = 0 \), for any \( P \in \text{Ob} \text{Cob}_{\theta,d}^{k-1} \) the cobordism \( H_{0,d}(P) \) is isomorphic to a \( \theta \)-manifold to \( ([0,1] \times P) \sqcup S^{d} \), where the sphere \( S^{d} \) is equipped with a \( \theta \)-structure that extends to the disk. By Definition 6.5 the cobordism \( H_{0,d}(P) \) is then primitive of degree 0 (or equivalently trivial of degree \( d \)). In this case, Proposition 7.1 then implies Theorem 6.7 (parts (a) and (b)), and thus implies Theorem 5.3 (recall from Remark 6.8 that part (c) of Theorem 6.7 is not needed in this \( k = 0 \) case). We remark that what we do in the \( k = 0 \) case is similar to what was done in [12] Section 6.3 “Annihilation of \( \mathcal{D}-\)Spheres”.

We will show that \( S_{H_{k,d-k}(P)} : \text{hocolim}_{t \in \mathcal{K}^{k}} W^{k}_{P,t} \longrightarrow \text{hocolim}_{t \in \mathcal{K}^{k}} W^{k}_{Q,t} \) is homotopic to the identity map. The proof will be easiest if we reformulate the statement to one about the classifying space of the transport category, \( \mathcal{K}^{k} \mid W^{k}_{P,(-)} \). Recall that an object of \( \mathcal{K}^{k} \mid W^{k}_{P,(-)} \) is a pair \((t, x)\) where \( t \in \mathcal{K}^{k} \) and \( x \in W^{k}_{P,t} \). A morphism in \( \mathcal{K}^{k} \mid W^{k}_{P,(-)} \) from \((t, x)\) to \((s, y)\) is defined to be a morphism \((j, \varepsilon) : t \rightarrow s \) in \( \mathcal{K}^{k} \) such that the induced map \((j, \varepsilon)^{*} : W^{k}_{P,s} \longrightarrow W^{k}_{P,t}\) sends \( y \) to \( x \). The homotopy colimit, \( \text{hocolim}_{t \in \mathcal{K}^{k}} W^{k}_{P,t} \), is then by definition the classifying space of this category, i.e.

\[
\text{hocolim}_{t \in \mathcal{K}^{k}} W^{k}_{t} = B \left( \mathcal{K}^{k} \mid W^{k}_{P,(-)} \right).
\]

The map \( S_{H_{k,d-k}(P)} \) then induces a self-functor of \( \mathcal{K}^{k} \mid W^{k}_{P,(-)} \) which we denote by

\[
(7.2) \quad T^{k}(P) : \mathcal{K}^{k} \mid W^{k}_{P,(-)} \longrightarrow \mathcal{K}^{k} \mid W^{k}_{P,(-)}.
\]
To prove Proposition 7.1 it will suffice to show that this functor induces a weak equivalence on classifying spaces. To do this we will construct a zig-zag of natural transformations

\[ \mathbf{T}^k(P) \rightleftharpoons s_k \rightleftharpoons \text{cyl} \]

of self-functors of \( \mathcal{K}^k \mid \mathcal{W}^k_{P(-)} \), where \( \text{cyl} : \mathcal{K}^k \mid \mathcal{W}^k_{P(-)} \to \mathcal{K}^k \mid \mathcal{W}^k_{P(-)} \) is a homotopy equivalence of topological categories. Since natural transformations of functors induce homotopies between their induced maps on classifying spaces, it will follow that \( B|\mathbf{T}^k(P)| \) is homotopic to \( B|\text{cyl}| \) and thus is a weak homotopy equivalence. We proceed to define the functors \( \text{cyl} \) and \( s_k \).

**Definition 7.3.** The functor \( \text{cyl} : \mathcal{K}^k \mid \mathcal{W}^k_{P(-)} \to \mathcal{K}^k \mid \mathcal{W}^k_{P(-)} \) is defined as follows. Let

\[ (t, (M, (V, \sigma), e)) \in \text{Ob}(\mathcal{K}^k \mid \mathcal{W}^k_{P(-)}). \]

Let \( M' \subset (-\infty, 1] \times \mathbb{R}^{\infty-1} \) be the submanifold defined by the pushout, \( M' = M \cup_P ([0, 1] \times P) \). By keeping \( V \) and \( e \) the same, the tuple \( (t, (M', (V, \sigma), e)) \) determines a well defined element of \( \text{Ob}(\mathcal{K}^k \mid \mathcal{W}^k_{P(-)}). \). The functor \( \text{cyl} \) is defined by

\[ \text{cyl}(t, (M, (V, \sigma), e)) = (t, (M', (V, \sigma), e)), \]

with \( M' \) defined as above.

Next, we define the natural transformation \( s_k : \mathcal{K}^k \mid \mathcal{W}^k_{P(-)} \to \mathcal{K}^k \mid \mathcal{W}^k_{P(-)} \). To do this we need to specify some preliminary data. Fix once and for all a symmetric bilinear form

\[ \sigma_{\mathbb{R}^{d+1}} : \mathbb{R}^{d+1} \otimes \mathbb{R}^{d+1} \to \mathbb{R} \]

of index \( d-k \). Fix a \( \theta \)-structure \( \hat{\ell}_{\mathbb{R}^{d+1}} \) on \( \mathbb{R}^{d+1} \). For \( \mathbb{R}^{d+1} \) equipped with \( \hat{\ell}_{\mathbb{R}^{d+1}} \), the pair \( (\mathbb{R}^{d+1}, \sigma_{\mathbb{R}^{d+1}}^{d-k}) \) determines an element of the space \( G_{\theta}^\text{mf}(\mathbb{R}^\infty)_\text{loc} \).

**Definition 7.4.** The functor \( s_k : \mathcal{K}^k \mid \mathcal{W}^k_{P(-)} \to \mathcal{K}^k \mid \mathcal{W}^k_{P(-)} \) is defined as follows. Let \( (t, (M, (V, \sigma), e)) \in \text{Ob}(\mathcal{K}^k \mid \mathcal{W}^k_{P(-)}). \) Fix an embedding

\[ \phi : D^{d-k} \times D^{k+1} \to (0, 1) \times \mathbb{R}^{\infty-1}, \]

that satisfies the following conditions:

(i) \( \phi^{-1}([0, 1] \times P) = S^{d-k-1} \times D^{k+1} \);

(ii) the restriction of \( \hat{\ell}_{[0,1] \times P} \) to \( S^{d-k-1} \times D^{k+1} \) agrees with the \( \theta \)-structure \( \hat{\ell}_{\mathbb{R}^{d+1}}|_{S^{d-k-1} \times D^{k+1}} \).

(iii) the surgered manifold,

\[ \left( ([0, 1] \times P) \setminus \phi(S^{d-k-1} \times D^{k+1}) \right) \cup \phi(D^{d-k} \times S^k), \]

agrees with

\[ \mathcal{H}_{k,d-k}(P) \subset [0, 1] \times \mathbb{R}^{\infty-1} \]

as a \( \theta \)-manifold.

Since \( \mathcal{H}_{k,d-k}(P) \) is obtained by forming the connected sum of \( S^k \times S^{d-k} \) with \( [0, 1] \times P \), in order to satisfy condition (ii), it will suffice to choose \( \phi \) so that the embedding

\[ \phi|_{S^{d-k-1} \times D^{k+1}} : S^{d-k-1} \times D^{k+1} \to [0, 1] \times P \]

factors through some disk \( D^d \hookrightarrow [0, 1] \times P \). Indeed, performing surgery on an embedded \((d-k-1)\)-sphere that bounds a disk has the effect of creating a connect-summand of \( S^{d-k} \times S^k \).

We proceed to define \( s_k(t, (M, (V, \sigma), e)) \):
(i) Let \( M' = M \cup_P ([0,1] \times P) \) be as in Definition 7.3.

(ii) Let \( t' \in \mathcal{K}^k \) be the object obtained by adjoining one more point, \( x \), to \( t \), with label \( d - k \).

(iii) The element \((V', \sigma') \in \text{Maps}(t', G^\text{mf}_g(\mathbb{R}^\infty)_{\text{loc}})\) is defined by setting
\[
(V'(i), \sigma'(i)) = (V(i), \sigma(i))
\]
for \( i \in t \), and \((V'(x), \sigma'(x)) = (\mathbb{R}^{d+1}, \sigma_{d-k}^{d+1})\).

(iv) The embedding \( e' : D((V')^-) \times t' D((V')^+) \to (-1,0] \times (-1,1)^{\infty-1} \) is set equal to \( e \) over \( t' \), and to agree with \( \phi \) from (7.4) on the component corresponding to the element \( x \in t' \).

We define \( s_k(t, (M, (V, \sigma), e)) \) by setting
\[
(7.5) \quad s_k(t, (M, V, e)) := (t', (M', (V', \sigma'), e')).
\]
Since any morphism \( s : t \to t' \in \mathcal{K}^k \) induces a unique morphism \( s \sqcup \{x\} \to t \sqcup \{x\} \), it follows that the operation \( s_k \) is functorial.

We now have all of the ingredients ready to finish the proof of Proposition 7.1.

**Proof of Proposition 7.1.** The proof will be complete once we construct the natural transformations
\[
(7.6) \quad \mathcal{T}^k(P) \xrightarrow{s_k} \text{cyl}.
\]
First, let \( b_{d-k} : \mathcal{K}^k \to \mathcal{K}^k \) be the functor defined by the formula \( b_{d-k}(t) = t \sqcup \{x\} \), where \( x \) is a point with label \( d - k \). For \( t \in \mathcal{K}^k \), let
\[
(i_t, \varepsilon_\pm^t) : t \to b_{d-k}(t)
\]
be the morphisms (one for each case \(+1\) or \(-1\)) defined by letting \( i_t \) be the inclusion
\[
t \mapsto s_k(t) = t \sqcup \{x\},
\]
and \( \varepsilon_\pm^t(x) = \pm 1 \). The assignments
\[
t \mapsto (i_t, \varepsilon_+^t) \quad \text{and} \quad t \mapsto (i_t, \varepsilon_-^t)
\]
yield two different natural transformations between the identity functor and the functor \( b_{d-k} \). We will use these natural transformations to construct the natural transformations (7.6). Let
\[
(t, (M, (V, \sigma), e)) \in \mathcal{K}^k \uplus \mathcal{W}^k_P(\cdot).
\]
The morphism \((i_t, \varepsilon_+^t) : t \to b_{d-k}(t)\) induces a morphism
\[
(i_T, \varepsilon_+^T)^* : s_k(b_{d-k}(t), (M, (V, \sigma), e)) \to \text{cyl}(t, (M, (V, \sigma), e))
\]
in \( \mathcal{K}^k \uplus \mathcal{W}^k_P(\cdot) \). This has the effect of forgetting the extra surgery data \( \phi \) defined in (7.4) over the point \( x \in b_{d-k}(t) \). This assignment \((t, (M, (V, \sigma), e)) \mapsto (i_T, \varepsilon_+^T)^*\) did not depend on the data \((M, (V, \sigma), e)\), and thus it defines a natural transformation between \( s_k \) and \( \text{cyl} \).

For the other natural transformation \( s_k \Rightarrow \mathcal{T}^k(P) \), we observe that \((i_t, \varepsilon_-^t) : t \to b_{d-k}(t)\) induces a morphism
\[
(i_t, \varepsilon_-^t)^* : s_k(b_{d-k}(t), (M, (V, \sigma), e)) \to \mathcal{T}^k(P)(t, (M, (V, \sigma), e)).
\]
Indeed, \((i_t, \varepsilon_-^t)^*\) is defined by performing the fibrewise surgery associated to the point \( x = b_{d-k}(t) \setminus t \). Since this surgery is given by the embedding \( \phi \) from (7.4), it follows that this surgery creates a connected sum factor of \( S^k \times S^{d-k} \), which is precisely what the functor \( \mathcal{T}^k(P) \) does. This assignment
\[
(t, (M, (V, \sigma), e)) \mapsto (i_t, \varepsilon_-^t)^*
\]
yields the desired natural transformation between $s_k$ and $T^k(P)$. It follows that the map onclassifying spaces induced by $T^k(P)$ is homotopic to the map on classifying spaces induced by $\text{cyl}$.Since $\text{cyl}$ induces a weak homotopy equivalence on classifying spaces (which is actually homotopic to the identity map) it follows that $T^k(P)$ induces a weak homotopy equivalence as well. Thiscompletes the proof of the proposition. □

8. Surgeries and Traces

8.1. Surgeries and Traces. We now use Proposition 7.1 to show that many other cobordisms beyond $H_{k,d-k}(P)$ are contained in $\mathcal{V}^{k-1}$. We will need to work with morphisms in the category $\text{Cob}^{k-1}_{\theta,d}$ that arise as the traces of surgeries on objects in $\text{Cob}^{k-1}_{\theta,d}$. In order to do this we introduce a series of new general constructions.

**Construction 8.1** (Standard Trace). Fix a non-negative integer $l < d$. We fix once and for all a submanifold

$$T_l \subset [0,1] \times D^l \times D^{d-l} \subset \mathbb{R}^l \times \mathbb{R}^{d-l}$$

with the following properties:

(a) The height function, $T_l \hookrightarrow [0,1] \times D^l \times D^{d-l} \xrightarrow{\text{proj}} [0,1]$, is Morse with one critical point in the interior with index $l$.

(b) The intersections with $\{0\} \times S^{l-1} \times D^{d-l}$, $\{1\} \times D^l \times S^{d-l-1}$, and $[0,1] \times S^{l-1} \times S^{d-l-1}$ are given by:

$$T_l \cap (\{0\} \times S^{l-1} \times D^{d-l}) = \{0\} \times S^{l-1} \times D^{d-l},$$

$$T_l \cap (\{1\} \times D^l \times S^{d-l-1}) = \{1\} \times D^l \times S^{d-l-1},$$

$$T_l \cap ([0,1] \times S^{l-1} \times S^{d-l-1}) = [0,1] \times S^{l-1} \times S^{d-l-1}.\)$$

(c) The boundary $\partial T_l$ is given by the union

$$\partial T_l = (\{0\} \times S^{l-1} \times D^{d-l}) \cup (\{1\} \times D^l \times S^{d-l-1}) \cup ([0,1] \times S^{l-1} \times S^{d-l-1}).$$

We refer to $T_l$ as the trace of surgery along $S^{l-1} \times D^{d-l} \subset S^{l-1} \times \mathbb{R}^{d-l}$. The Morse trajectories associated to the height function $T_l \longrightarrow [0,1]$ determine embeddings:

$$\beta_0 : (D^l \times D^{d-l}, S^{l-1} \times D^{d-l}) \longrightarrow (T_l, \{0\} \times S^{l-1} \times D^{d-l}),$$

$$\beta_1 : (D^l \times D^{d-l}, D^l \times S^{d-l-1}) \longrightarrow (T_l, \{1\} \times D^l \times S^{d-l-1}).$$

We call these embeddings the standard core handles associated to $T_l$. We note any choice of $\theta$-structure $\phi : T(D^l \times D^{d-l}) \oplus \epsilon^1 \longrightarrow \theta^* \gamma^{d+1}$ that extends $\hat{\ell}$ determines a $\theta$-structure on $T_l$ uniquely up to homotopy.

We use the above trace construction to define morphisms in the category $\text{Cob}^{k-1}_{\theta,d}$.

**Definition 8.1** (Trace of a surgery). Let $P \in \text{Ob \ Cob}^{k-1}_{\theta,d}$. Let $\sigma : \mathbb{R}^l \times \mathbb{R}^{d-l} \longrightarrow \mathbb{R}^{\infty-1}$ be anembedding that satisfies, $\sigma^{-1}(P) = S^{l-1} \times \mathbb{R}^{d-l}$. Let $\phi : T(\mathbb{R}^l \times \mathbb{R}^{d-l}) \longrightarrow \theta^* \gamma^d$ be a $\theta$-structure that extends $\hat{\ell}_P \circ D\sigma : T(S^{l-1} \times D^{d-l}) \oplus \epsilon^1 \longrightarrow \theta^* \gamma^d$. The pair $(\sigma, \phi)$ will be called surgery data of degree $l-1$. We denote by $\text{Surg}_{l-1}(P)$ the set of all such surgery data $(\sigma, \phi)$ of degree $l-1$.

Using $\sigma := (\sigma, \phi) \in \text{Surg}_{l-1}(P)$ we define a morphism in $\text{Cob}^{k-1}_{\theta,d}$,

$$T(\sigma) : P \hookrightarrow P(\sigma),$$

(8.1)
by setting,

\[ T(\sigma) = \left[ (I \times P) \setminus (I \times \sigma(S^{l-1} \times D^{d-l})) \right] \cup (\text{Id}_I \times \sigma)(T_i). \]

The manifold \( P(\sigma) \) is defined to be \( T(\sigma) \cap (\{1\} \times \mathbb{R}^{\infty-1}) \). The \( \theta \)-structure \( \hat{\ell}_{T(\sigma)} \) on \( T(\sigma) \) is the one induced by the extended \( \theta \)-structure \( \tilde{\phi} \).

**Definition 8.2** (Dual surgery). Let \( \sigma \in \text{Surg}_{l-1}(P) \) and consider the trace \( T(\sigma) : P \rightsquigarrow P(\sigma) \).

The embedding \( \beta_1 : (D^l \times D^{d-l}, D^l \times S^{d-l-1}) \rightarrow (T_i, \{1\} \times D^l \times S^{d-l-1}) \) from Definition 8.1 determines an embedding \( \beta_1(\sigma) : (D^l \times D^{d-l}, D^l \times S^{d-l-1}) \rightarrow (T(\sigma), P(\sigma)) \).

Let

\[ \bar{\sigma} : \mathbb{R}^l \times \mathbb{R}^{d-l} \rightarrow \mathbb{R}^{\infty-1} \]

be an embedding with \( \bar{\sigma}^{-1}(P(\sigma)) = \mathbb{R}^l \times S^{d-l-1} \), that extends \( \beta_1(\sigma) \). Let

\[ \tilde{\phi} : T(\mathbb{R}^l \times \mathbb{R}^{d-l}) \oplus \epsilon^1 \rightarrow \theta^* \gamma^{d+1} \]

be a \( \theta \)-structure obtained by restricting \( \hat{\ell}_{T(\sigma)} \). The pair \( \bar{\sigma} = (\bar{\sigma}, \tilde{\phi}) \) defines surgery data in \( P(\sigma) \) of degree \( d-l-1 \). We call \( \bar{\sigma} \) the surgery dual to \( \sigma \).

**Remark 8.3.** In the above definition, the construction of the dual surgery \( \bar{\sigma} = (\bar{\sigma}, \tilde{\phi}) \) depends on choices. However, the \( \theta \)-diffeomorphism class of the trace \( T(\bar{\sigma}) \) does not depend on any of these choices made. In particular, the \( \theta \)-diffeomorphism class of the resulting surgered manifold \( P(\sigma)(\bar{\sigma}) \) does not depend on the choices, and it follows that \( P(\sigma)(\bar{\sigma}) \cong P \) for all such choices of \( \bar{\sigma} \). For this reason, this indeterminacy in the definition of the dual surgery \( \bar{\sigma} \) won’t affect any of our results.

We will also have to form the traces of multiple disjoint surgeries simultaneously.

**Definition 8.4** (Simultaneous surgeries). Let \( (\sigma_1, \phi_1), \ldots, (\sigma_p, \phi_p) \) be a collection of surgery data for \( P \in \text{Ob Cob}_{\theta, d}^{k-1} \) that satisfies, \( \sigma_i(\mathbb{R}^{l_i} \times \mathbb{R}^{d-l_i}) \cap \sigma_j(\mathbb{R}^{l_j} \times \mathbb{R}^{d-l_j}) = \emptyset \), for all \( i \neq j \). We may form their simultaneous trace

\[ T(\sigma_1, \ldots, \sigma_p) : P \rightsquigarrow P(\sigma_1, \ldots, \sigma_p) \]

by setting \( T(\sigma_1, \ldots, \sigma_p) \) equal to,

\[ \left[ (I \times P) \setminus \bigcup_{i=1}^{p} (I \times \sigma_i(S^{l_i-1} \times D^{d-l_i})) \right] \bigcup \bigcup_{i=1}^{p} (\text{Id}_I \times \sigma_i)(T_i). \]

The associated \( \theta \)-structure \( \ell_{T(\sigma_1, \ldots, \sigma_p)} \) on \( T(\sigma_1, \ldots, \sigma_p) \) is defined in the same way as above using \( \phi_1, \ldots, \phi_p \).

**Construction 8.2** (Interchange of Disjoint Surgeries). Let \( P \in \text{Ob Cob}_{\theta, d}^{k-1} \). Let \( \sigma_1 \) and \( \sigma_2 \) be disjoint surgery data for \( P \) of degrees \( l_1 \) and \( l_2 \) respectively (by disjoint we mean that \( \text{Im}(\sigma_1) \cap \text{Im}(\sigma_2) = \emptyset \)). Consider the traces:

\[ T(\sigma_1) : P \rightsquigarrow P(\sigma_1), \]
\[ T(\sigma_2) : P \rightsquigarrow P(\sigma_2). \]

Notice that the cobordism \( T(\sigma_2) \) agrees with the cylinder \([0, 1] \times P \subset [0, 1] \times \mathbb{R}^{\infty-1} \) outside of the submanifold \((\text{Id}_I \times \sigma_2)(T_1) \subset T(\sigma_2)\). Furthermore,

\[ P(\sigma_2) \setminus \text{Im}(\sigma_2) = P \setminus \text{Im}(\sigma_2). \]

Since the images of \( \sigma_1 \) and \( \sigma_2 \) are disjoint, it follows that

\[ \{1\} \times \sigma_1(S^{l_1} \times D^{d-l_1}) \subset P(\sigma_2) \setminus \text{Im}(\sigma_2). \]
Thus, \( \sigma_1 \) determines surgery data of degree \( l_1 \) in \( P(\sigma_2) \). We denote this new surgery data by \( \mathcal{R}_{\sigma_2}(\sigma_1) \). Its trace yields a cobordism

\[
(8.3) \quad T(\mathcal{R}_{\sigma_2}(\sigma_1)) : P(\sigma_2) \hookrightarrow P(\sigma_1, \sigma_2).
\]

The manifold on the right is precisely the result of the target of the simultaneous trace described in Definition 8.1. It follows directly from the construction that

\[
(8.4) \quad T(\mathcal{R}_{\sigma_2}(\sigma_1)) \circ T(\sigma_2) = T(\sigma_1, \sigma_2)
\]

as cobordisms from \( P \) to \( P(\sigma_1, \sigma_2) \).

**Definition 8.5.** Let \( P \in \text{ObCob}^{k-1}_{\theta,d} \) and let \( \sigma \in \text{Surg}_{l-1}(P) \). We denote by \( \mathcal{L}(P, \sigma) \) the set of all surgeries \( \alpha \in \text{Surg}_{d-1-1}(P) \) that satisfy:

(i) \( \alpha(D^{d-l} \times D^l) \cap \sigma(D^l \times D^{d-l}) = \emptyset \);

(ii) the embedding \( \alpha|_{S^{d-l-1} \times \{0\}} : S^{d-l-1} \times \{0\} \rightarrow P \setminus \sigma(S^{l-1} \times \{0\}) \) is isotopic to the embedding

\[
\sigma|_{\{x\} \times S^{d-l-1}} : \{x\} \times S^{d-l-1} \rightarrow P \setminus \sigma(S^{l-1} \times \{0\})
\]

for some \( x \in S^{l-1} \).

For any surgery datum \( \sigma \), the set \( \mathcal{L}(P, \sigma) \) is always non-empty. To find an element of \( \mathcal{L}(P, \sigma) \) one simply takes the thickening of a small displacement of \( \sigma(\{x\} \times S^{d-l-1}) \). The following proposition is proven directly by unpacking Construction 8.2.

**Proposition 8.6.** For \( P \in \text{ObCob}^{k-1}_{\theta,d} \), let \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) be mutually disjoint surgery data. Suppose that \( \sigma_3 \in \mathcal{L}(P, \sigma_2) \). Then \( \mathcal{R}_{\sigma_1}(\sigma_3) \in \mathcal{L}(P(\sigma_1), \mathcal{R}_{\sigma_1}(\sigma_2)) \).

8.2. **First results.** We now apply the constructions from the previous subsection. We need one more definition (which is just a rewording of Definition 8.6 (part (i)).

**Definition 8.7.** Let \( P \in \text{ObCob}^{k-1}_{\theta,d} \). Surgery data \( \sigma \in \text{Surg}_{l}(P) \) (of any degree \( d \)) is said to be trivial if the embedding \( \sigma|_{S^{l} \times D^{d-1-1}} : S^{l} \times D^{d-1-1} \rightarrow P \) factors through some embedding \( \mathbb{R}^{d-1} \hookrightarrow P \). We denote the set of all trivial surgery data of degree \( l \) by \( \text{Surg}^{\text{tr}}_{l}(P) \subset \text{Surg}_{l}(P) \).

**Proposition 8.8.** Let \( P \in \text{ObCob}^{k-1}_{\theta,d} \). Let \( \sigma \in \text{Surg}^{\text{tr}}_{l}(P) \), with \( i = k - 1 \) or \( d - k - 1 \). Let \( \alpha \in \mathcal{L}(P, \sigma) \). Then the cobordism \( T(\sigma, \alpha) : P \hookrightarrow P(\sigma, \alpha) \) lies in \( \mathcal{V}^{k-1} \).

**Proof.** We first prove the proposition in the special case that \( P \cong S^{d-1} \). With this assumption, since \( \alpha \in \mathcal{L}(P, \sigma) \) it follows directly from Definition 8.3 that the embedded spheres \( \sigma(S^{i}), \alpha(S^{d-i-2}) \subset P \) have linking number equal to 1 (this linking number is well defined because \( P \cong S^{d-1} \)). It follows that the manifold obtained by attaching handles to \( P \) along \( \sigma(S^{i} \times D^{d-i-1}) \) and \( \alpha(S^{d-i-2} \times D^{i+1}) \) is diffeomorphic to \(([0,1] \times P)\#(S^{k} \times S^{d-k})\). From this it follows that \( T(\sigma, \alpha) \) is equivalent to the cobordism \( H_{k,d-k}(P) \), and thus by Proposition 7.1 \( T(\sigma, \alpha) \) is contained in \( \mathcal{V}^{k-1} \). This proves the proposition in the special case that \( P \cong S^{d-1} \). For the general case, we observe that since \( \sigma \) is a trivial surgery by assumption, it follows that both \( \sigma(S^{i} \times D^{d-i-1}) \) and \( \alpha(S^{d-i-2} \times D^{i+1}) \) are contained in a disk in \( P \). Using this fact, the general case reduces to the case where \( P \cong S^{d-1} \) which was proven above.

\[\square\]

The next theorem provides a partial upgrade of the previous proposition. Its proof requires significantly more work and thus is postponed until the next section.
**Theorem 8.9.** Let $P \in \text{Ob \text{Cob}}^{k-1}_{\theta,d}$. Let $\sigma \in \text{Surg}_{k-1}(P)$ be an arbitrary surgery and let $\alpha \in \mathcal{L}(P,\sigma)$. Then $T(\sigma,\alpha)$ lies in $\mathcal{V}^{k-1}$.

Before embarking on the proof of the above theorem we present a corollary.

**Corollary 8.10.** Let $P \in \text{Ob \text{Cob}}^{k-1}_{\theta,d}$. Let $T(\sigma) : P \rightarrow P(\sigma)$ be the trace of a trivial surgery in degree $d-k-1$. Then $T(\sigma)$ lies in $\mathcal{V}^{k-1}$.

**Proof.** Denote $\sigma_0 := \sigma$. Let $\sigma_1 \in \mathcal{L}(P,\sigma_0)$, and then choose $\sigma_2 \in \mathcal{L}(P,\sigma_1)$ so that $\text{Im}(\sigma_2) \cap \text{Im}(\sigma_0) \cup \text{Im}(\sigma_1) = \emptyset$.

We consider the composite

$$P \xrightarrow{T(\sigma_0)} P(\sigma_0) \xrightarrow{T(\mathcal{R}_{\sigma_0}(\sigma_1))} P(\sigma_0,\sigma_1) \xrightarrow{T(\mathcal{R}_{\sigma_0,\sigma_1}(\sigma_2))} P(\sigma_0,\sigma_1,\sigma_2).$$

The first composite $T(\mathcal{R}_{\sigma_0}(\sigma_1)) \circ T(\sigma_0)$ is equal to $T(\sigma_0,\sigma_1)$ and thus is in $\mathcal{V}^{k-1}$ by Proposition 8.8 because $\sigma_0$ is a trivial surgery of degree $d-k-1$. The second composite, $T(\mathcal{R}_{\sigma_0,\sigma_1}(\sigma_2)) \circ T(\mathcal{R}_{\sigma_0}(\sigma_1))$, is equal to $T(\mathcal{R}_{\sigma_0}(\sigma_1),\mathcal{R}_{\sigma_0}(\sigma_1))$. Now, $\mathcal{R}_{\sigma_0}(\sigma_1)$ is a surgery of degree $k-1$, and $\mathcal{R}_{\sigma_0}(\sigma_1)$ is contained in $\mathcal{L}(P(\sigma_0),\mathcal{R}_{\sigma_0}(\sigma_1))$. It follows from Theorem 8.9 that the cobordism $T(\mathcal{R}_{\sigma_0,\sigma_1}(\sigma_2)) \circ T(\mathcal{R}_{\sigma_0}(\sigma_1))$ lies in $\mathcal{V}^{k-1}$. Combining both of our observations it follows that the map, $\underset{t \in K_k}{\text{hocolim}} W_{P(\sigma_0),t}^{k} \rightarrow \text{hocolim} W_{P(\sigma_0,\sigma_1),t}^{k}$, induced by the cobordism $T(\mathcal{R}_{\sigma_0}(\sigma_1))$ induces an injection and surjection on all homotopy groups, and thus is in $\mathcal{V}^{k-1}$. Finally, it follows that $T(\sigma_0) = T(\sigma)$ lies in $\mathcal{V}^{k-1}$ by the 2-out-of-3 property.

We will derive one more corollary. To state it we need one more definition and a lemma. The following definition is a rewording of Definition 6.6, part (ii).

**Definition 8.11.** Let $P \in \text{Ob \text{Cob}}^{k-1}_{\theta,d}$. A surgery $\sigma \in \text{Surg}_{d-j}(P)$ is said to be primitive if there exists another surgery $\alpha \in \text{Surg}_{d-j-1}(P)$ such that $\sigma(S^j \times \{0\}) \subset P$ and $\alpha(S^{d-j-1} \times \{0\}) \subset P$ intersect transversally in $P$ at exactly one point. We denote by $\text{Surg}_{d-j}(P)$ the set of all primitive surgeries in $P$ of degree $j$.

The following lemma says that the condition of being a trivial surgery is “dual” to the condition of being primitive.

**Lemma 8.12.** Let $P \in \text{Ob \text{Cob}}^{k-1}_{\theta,d}$ and $l$ be a non-negative integer. Let $\sigma \in \text{Surg}_{d-l}^{\text{tr}}(P)$ be a trivial surgery. Then its dual $\hat{\sigma}$ is a primitive surgery. Similarly, if $\sigma$ is a primitive surgery then its dual $\hat{\sigma}$ is trivial.

**Proof.** Let $\sigma$ be a primitive surgery in $P$ of degree $l$. We first observe that in the special case where

(i) $P \cong S^l \times S^{d-l-1}$, and

(ii) $\sigma$ corresponds to (a thickening of) the standard inclusion $S^l \times \{0\} \hookrightarrow S^l \times S^{d-l-1}$, it follows that $P(\sigma) \cong S^{d-1}$, and thus the dual $\hat{\sigma} \in \text{Surg}_{d-l-1}^{\text{tr}}(S^{d-1})$ has no choice but to be trivial. We claim that the general case of the lemma can always be reduced to this special case. Indeed, if $\sigma$ is primitive, then by definition there exists another embedding $\alpha : D^l \times S^{d-l-1} \rightarrow P$ such that $\sigma(S^l \times \{0\})$ and $\alpha(\{0\} \times S^{d-l-1})$ intersect transversally in exactly one point. By shrinking down in the normal direction if necessary, we may assume that the intersection $\sigma(S^l \times D^{d-l-1}) \cap \alpha(D^l \times S^{d-l-1})$ is diffeomorphic to a disk. It follows that there is a diffeomorphism

$$S^l \times S^{d-l-1} \setminus \text{Int}(D^{d-1}) \cong \sigma(S^l \times D^{d-l-1}) \cup \alpha(D^l \times S^{d-l-1}).$$
The surgery \( \sigma \) is contained in the image of \( S^l \times S^{d-l-1} \setminus \text{Int}(D^{d-1}) \) in \( P \) under this diffeomorphism, and it follows that \( \sigma \) corresponds to the standard embedding, \( S^l \times \{0\} \hookrightarrow S^l \times S^{d-l-1} \setminus \text{Int}(D^{d-1}) \).

The manifold obtained from \( S^l \times S^{d-l-1} \setminus \text{Int}(D^{d-1}) \) by performing surgery along this standard embedding is diffeomorphic to a disk, and thus it follows that the dual surgery \( \bar{\sigma} \) factors through a disk in \( P(\sigma) \). This proves that \( \bar{\sigma} \) is trivial whenever \( \sigma \) is primitive.

The other claim in the statement of the lemma, that \( \bar{\sigma} \) is primitive whenever \( \sigma \) is trivial, is proven by a similar argument and we leave that to the reader. \( \square \)

**Corollary 8.13.** Let \( k < d/2 \) and \( P \in \text{Cob}_{k,d}^{k-1} \). Then for any primitive surgery \( \sigma \in \text{Surg}_{k-1}(P) \), the trace \( T(\sigma) \) belongs to \( \mathcal{V}^{k-1} \).

**Proof.** Consider the surgered manifold \( P(\sigma) \) and the surgery dual to \( \sigma \), \( \bar{\sigma} \in \text{Surg}_{d-k-1}(P(\sigma)) \). Since \( \sigma \) was a primitive surgery, it follows by Lemma 8.12 that the dual surgery \( \bar{\sigma} \) is a trivial surgery. By Theorem 8.10 it then follows that \( T(\bar{\sigma}) \in \mathcal{V}^{k-1} \). We then consider the composite

\[
P(\sigma) \overset{T(\sigma)}{\longrightarrow} P \overset{T(\sigma)}{\longrightarrow} P(\sigma).
\]

It follows easily by inspection that the composite \( T(\sigma) \circ T(\bar{\sigma}) \) is equivalent to the cobordism \( H_{k,d-k}(P) \) and thus is contained in \( \mathcal{V}^{k-1} \) as well. The corollary then follows by the two-out-of-three property. \( \square \)

**Remark 8.14.** We note that by Corollaries 8.10 and 8.13, we have established parts (a) and (b) of Theorem 6.7. These corollaries of course depend on Theorem 8.9 which we prove in the next section. Once Theorem 8.9 is proven, the only thing left to do is prove part (c) of Theorem 6.7.

### 9. Resolving Nontrivial Handle Attachments

This section is geared toward proving Theorem 8.9 which asserts that if \( \sigma \) is an arbitrary surgery in degree \( k-1 \) and \( \alpha \in \mathcal{L}(P, \sigma) \), then the trace \( T(\sigma, \alpha) \) lies in \( \mathcal{V}^{k-1} \).

#### 9.1. A semi-simplicial resolution

We will need to work with a semi-simplicial space analogous to the one constructed in [6, Section 4.3].

**Definition 9.1.** Let \( l < d \). Fix an element \( z := (M, (V, \sigma), e) \in \mathcal{W}_P^k \) together with the following data:

- an embedding \( \chi : S^{l-1} \times (1, \infty) \times \mathbb{R}^{d-l-1} \longrightarrow P \);
- a 1-parameter family \( \hat{\ell}_t^{\text{std}}, t \in (2, \infty) \), of \( \theta \)-structures on \( D^l \times D^{d-l} \) such that
  \[
  \hat{\ell}_t^{\text{std}}|_{\partial D^l \times D^{d-l}} = \chi^* \hat{\ell}_P|_{\partial D^l \times (t \cdot e_1 + D^{d-l})},
  \]
  where \( e_1 \in (1, \infty) \times \mathbb{R}^{d-k-1} \) is the basis vector corresponding to the first coordinate.

We define \( X_0(z; \chi) \) to be the set of triples \((t, \phi, \hat{L})\) consisting of \( t \in (2, \infty) \), an embedding \( \phi : (D^l \times D^{d-l}, \partial D^l \times D^{d-l}) \longrightarrow (M, P) \), and a path of \( \theta \)-structures \( \hat{L} \) on \( D^l \times D^{d-l} \) such that:

1. the restriction of the map \( \phi \) to \( \partial D^l \times D^{d-l} \) satisfies the equation \( \phi(y, v) = (y, v + t \cdot e_1) \) for all \((y, v) \in \partial D^l \times D^{d-l}\).
2. the image \( \phi(D^l \times D^{d-l}) \) is disjoint from the image of the embedding \( e \), associated to the element \( z = (M, (V, \sigma), e) \).
(iii) the family of \( \theta \)-structures \( \hat{L} \) satisfies: \( \hat{L}(0) = \phi^* \hat{\ell}_M, \hat{L}(1) = \hat{\ell}_t^{\text{std}} \), and \( \hat{L}(s)|_{\partial D^l \times D^{d-t}} \) is independent of \( s \in [0,1] \).

For \( p \in \mathbb{Z}_{\geq 0} \), let \( X_p(z; \chi) \subset (X_0(z; \chi))^{p+1} \) be the subspace consisting of those
\[
\left( (t_0, \phi_0, \hat{L}_0), \ldots, (t_p, \phi_p, \hat{L}_p) \right)
\]
such that
(i) the images \( \phi_i(D^l \times D^{d-l}) \) are disjoint for \( i = 0, \ldots, p \),
(ii) \( t_0 < t_1 < \cdots < t_p \).

The collection \( X_\bullet(z; \chi) \) has the structure of a semi-simplicial space, where the \( i \)th face map forgets the entry \( (t_i, \phi_i) \). The definition of the semi-simplicial space \( X_\bullet(z; \chi) \) does definitely depend on the one-parameter family \( \hat{\ell}_t^{\text{std}} \). However, we drop it from the notation in order to save space.

We now combine all of the semi-simplicial spaces \( X_\bullet(z; \chi) \) together into one semi-simplicial space augmented over \( W^k_{P,t} \).

**Definition 9.2.** Choose an integer \( k \). Fix the following data:
- an object \( t \in K^k \);
- an object \( P \in \text{Ob Cob}^{k-1}_{\theta,d} \);
- an embedding \( \chi : \partial D^l \times (1, \infty) \times \mathbb{R}^{d-l-1} \to P \), and one parameter family of \( \theta \)-structures \( \hat{\ell}_t^{\text{std}} \) as in Definition 9.1.

For \( p \in \mathbb{Z}_{\geq 0} \), we define \( M^k_{P,t}(\chi)_p \) be the space of tuples \( (y; x) \) with \( y \in W^k_{P,t} \) and \( x \in X_p(y; \chi) \). The assignment \([p] \mapsto M^k_{P,t}(\chi)_p \) defines a semi-simplicial space \( M^k_{P,t}(\chi)_\bullet \). The forgetful maps
\[
M^k_{P,t}(\chi)_p \to W^k_{P,t}, \quad (y; x) \mapsto y
\]
yield an augmented semi-simplicial space, \( M^k_{P,t}(\chi)_\bullet \to M^k_{P,t}(\chi)_-1 = W^k_{P,t} \).

We have the following proposition.

**Proposition 9.3.** Let \( P \in \text{Ob Cob}^{k-1}_{\theta,d} \). Let \( \chi : \partial D^l \times (1, \infty) \times \mathbb{R}^{d-l-1} \to P \) and \( \hat{\ell}_t^{\text{std}} \) be as in Definition 9.1. If \( l \leq k < d/2 \), then the map induced by augmentation
\[
|M^k_{P,t}(\chi)_\bullet| \to M^k_{P,t}(\chi)_-1 = W^k_{P,t}
\]
is a weak homotopy equivalence.

**Proof.** To prove the theorem we will need to replace \( M^k_{P,t}(\chi)_\bullet \) with a weakly equivalent semi-simplicial space that is a bit more flexible to work with. For each \( p \in \mathbb{Z}_{\geq 0} \), let \( \widetilde{M}^k_{P,t}(\chi)_p \) consist of those tuples \( ((M_i, (V, \sigma), e); (t_0, \phi_0, \hat{L}_0), \ldots, (t_p, \phi_p, \hat{L}_p)) \) defined as in Definition 9.2 (and Definition 9.1), except now we relax condition (i) and only require, \( \phi_i(D^l \times \{0\}) \cap \phi_j(D^l \times \{0\}) = \emptyset \) whenever \( i \neq j \). It follows by an argument similar the one used in [8 Proposition 6.9] that the inclusion \( M^k_{P,t}(\chi)_\bullet \to \widetilde{M}^k_{P,t}(\chi)_\bullet \) is a level-wise weak homotopy equivalence. To prove the proposition it will suffice to prove that \( |\widetilde{M}^k_{P,t}(\chi)_\bullet| \to \widetilde{M}^k_{P,t}(\chi)_-1 \) is a weak homotopy equivalence. Now, the augmented semi-simplicial space \( \widetilde{M}^k_{P,t}(\chi)_\bullet \to \widetilde{M}^k_{P,t}(\chi)_-1 \) is an augmented topological flag complex in the sense of [8 Definition 6.1]. By [8 Theorem 6.2], to prove the proposition it will suffice to verify that it has the following three properties (compare with the proof of Proposition 2.9):

(i) The augmentation map \( \varepsilon_0 : \widetilde{M}^k_{P,t}(\chi)_0 \to \widetilde{M}^k_{P,t}(\chi)_-1 \) has local lifts of any map from a disk.
(ii) For any \( x \in \tilde{M}_{P,t}^k(\chi)_{-1} \) the fibre \( \epsilon_0^{-1}(x) \) is non-empty.

(iii) For any \( x \in \tilde{M}_{P,t}^k(\chi)_{-1} \), given any non-empty set of zero-simplices \( v_1, \ldots, v_m \in \epsilon_0^{-1}(x) \), there exists another zero simplex \( v \in \epsilon_0^{-1}(x) \) such that \( (v, v_i) \in \tilde{M}_{P,t}^k(\chi)_1 \) for all \( i = 1, \ldots, m \).

Property (i) follows from the fact that \( \epsilon_0 : \tilde{M}_{P,t}^k(\chi)_0 \to \tilde{M}_{P,t}^k(\chi)_{-1} \) is clearly a locally trivial fibre bundle. Property (iii) follows from a general position argument. Indeed, let \((t_1, \phi_1, \hat{L}_1), \ldots, (t_m, \phi_m, \hat{L}_m)\) be a collection of zero simplices in \( \epsilon_0^{-1}(M, (V, \sigma), e) \), for some element \( (M, (V, \sigma), e) \in \tilde{M}_{P,t}^k(\chi)_{-1} \).

Let \( \phi = \phi_1 \). Since \( l \leq k < d/2 \), we may perturb \( \phi \) to a new embedding \( \phi' \) with

\[
\phi'(D^l \times \{0\}) \cap \phi_i(D^l \times \{0\}) = \emptyset, \quad \text{for } i = 1, \ldots, m.
\]

By choosing \( t' \) with \( t' \neq t_i \) for all \( i \) and setting \( \hat{L}' = \hat{L}_1 \), it follows that \((t, \phi', \hat{L}')\) is a zero simplex with \((t', \phi', \hat{L}')\) and \((t_i, \phi_i, \hat{L}_i)\) \( \in \tilde{M}_{P,t}^k(\chi)_1 \) for all \( i \).

We now just have to verify property (ii). Let \((M, (V, \sigma), e) \in \tilde{M}_{P,t}^k(\chi)_{-1} \). Pick a number \( t \in (2, \infty) \) and consider the commutative square

\[
\begin{array}{ccc}
\partial D^l \times D^{d-l-1} & \xrightarrow{\chi(-, - + t e_i)} & P \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
D^l \times D^{d-l} & \xrightarrow{\ell_{\overline{M}} \ell_{t}^{\text{std}}} & M \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
B.
\end{array}
\]

By definition of \( \tilde{M}_{P,t}^k(\chi)_{-1} \) the map \( \ell_M : M \to B \) is \( k \)-connected. Since \( l \leq k \), it follows that the lift \( g \) in diagram (9.1) exists, making both the upper and lower triangles commute. Since both \( \ell_M \) and \( \ell_t^{\text{std}} \) are covered by bundle maps, this provides a bundle map \( \hat{g} : T(D^l \times D^{d-l}) \to M \) covering \( g \), such that \( \hat{g} \circ \hat{L}_t \) is homotopic to \( \hat{\ell}_t^{\text{std}} \) through bundle maps, via a homotopy that is constant over \( \partial D^l \times D^{d-l} \). By the Smale-Hirsch theorem the pair \((g, \hat{g})\) may be homotoped relative to \( \partial D^l \times D^{d-l} \) to a pair of the form \((\phi, D(\phi))\), for an immersion \( \phi \) and \( D\phi \) its differential. Since \( k < \dim(M)/2 = d/2 \), by putting \( \phi(D^l \times \{0\}) \) into a general position we may assume that the restriction \( \phi \) to \((D^l \times \{0\}) \cup (\partial D^l \times D^{d-l})\) is an embedding. By precomposing \( \phi \) with an isotopy of \( D^l \times D^{d-l} \) that compresses \( D^l \times D^{d-l} \) down to the core \((D^l \times \{0\}) \cup (\partial D^l \times D^{d-l})\), we may then make \( \phi \) into an embedding. The \( \theta \)-structure \( \phi^* \hat{L}_t = \hat{L}_t \circ D\phi \) is still homotopic to \( \hat{\ell}_t^{\text{std}} \), and if we choose such a homotopy, \( \hat{L}_t \), then we have constructed an element \((t, \phi, \hat{L}_t) \in \epsilon_0^{-1}(M, (V, \sigma), e) \). This establishes condition (ii) and concludes the proof of the proposition.

The correspondence \( t \mapsto (\mathcal{M}_{P,t}^k(\chi)_1) \to \tilde{M}_{P,t}^k(\chi)_{-1} \) defines a functor from \( \mathcal{K}^k \) to the category of augmented semi-simplicial spaces. Taking the levelwise homotopy colimit yields the augmented semi-simplicial space, \( \text{hocolim}_{t \in \mathcal{K}^k} \mathcal{M}_{P,t}^k(\chi)_1 \to \text{hocolim}_{t \in \mathcal{K}^k} \tilde{M}_{P,t}^k(\chi)_{-1} \). We have the following corollary.

**Corollary 9.4.** Let \( P \in \text{ObCob}_{0,d}^{k-1} \). Let \( \chi : \partial D^l \times (1, \infty) \times \mathbb{R}^{d-l-1} \to P \) and \( \hat{\ell}_t^{\text{std}} \) be as in Definition 9.3. If \( l \leq k < d/2 \), then the map induced by augmentation

\[
|\text{hocolim}_{t \in \mathcal{K}^k} \mathcal{M}_{P,t}^k(\chi)_1| \to \text{hocolim}_{t \in \mathcal{K}^k} \tilde{M}_{P,t}^k(\chi)_{-1}
\]

is a weak homotopy equivalence.

**Proof.** By Proposition 9.3 the map \( |\mathcal{M}_{P,t}^k(\chi)_1| \to \tilde{M}_{P,t}^k(\chi)_{-1} \) is a weak homotopy equivalence for all \( t \in \mathcal{K}^k \). By the homotopy invariance of homotopy colimits, the augmentation map induces
a weak homotopy equivalence, $\text{hocolim}_{t \in K^k} \mathcal{M}^k_{P,t}(\chi)_{\bullet} \xrightarrow{\simeq} \text{hocolim}_{t \in K^k} \mathcal{M}^k_{P,t}(\chi)_{-1}$. The corollary then follows by combining this with the weak homotopy equivalence,
\[
\text{hocolim}_{t \in K^k} |\mathcal{M}^k_{P,t}(\chi)_{\bullet}| \simeq |\text{hocolim}_{t \in K^k} \mathcal{M}^k_{P,t}(\chi)_{\bullet}|.
\]

\[\square\]

9.2. Resolving the map $S_{T(\sigma)}$. We now show how to use the above semi-simplicial resolution to prove Theorem 8.9. For the rest of this section, let us fix once and for all an object $P \in \text{Ob Cob}_{\theta,d}^{k-1}$. For each $l \in \mathbb{Z}_{\geq 0}$, fix once and for all a diffeomorphism
\[
\gamma_l : D^l \times D^{d-l} \xrightarrow{\simeq} D^l \times D^{d-l} \bigcup D^{l-1} \times [0, 1] \times D^{d-l}
\]
which is the identity on $D^l(1/2) \times D^{d-l}$ (where $D^l(1/2)$ is the disk centered at the origin with radius $1/2$).

Construction 9.1 (A semi-simplicial map). Let $P \in \text{Ob Cob}_{\theta,d}^{k-1}$. Let
\[
\chi : \partial D^l \times (1, \infty) \times \mathbb{R}^{d-l-1} \rightarrow P
\]
be as in Definition 9.1. Let $\sigma \in \text{Surg}_{j-1}(P)$ and suppose that
\[
\sigma(D^{j+1} \times D^{d-j-1}) \cap \chi(\partial D^l \times (1, \infty) \times \mathbb{R}^{d-l-1}) = \emptyset.
\]
From this disjointness condition, we can consider $\chi$ to be an embedding into $P(\sigma)$ as well and thus the semi-simplicial space $\mathcal{M}^k_{P(\sigma),t}(\chi)_{\bullet}$ is well defined. Let
\[
(W; (t_0, \phi_0, L_0), \ldots, (t_p, \phi_p, \tilde{L}_p)) \in \mathcal{M}^k_{P,t}(\chi)_{P}.
\]
To this $p$-simplex we associate a new $p$-simplex in $\mathcal{M}^k_{P(\sigma),t}(\chi)_{\bullet}$ as follows: First note that from the definition of $T(\sigma)$ (Definition 8.1) for each $i$ we have $[0, 1] \times \phi_i(S^{l-1} \times D^{d-l}) \subset T(\sigma)$. For each $i$, we may define a new embedding $\tilde{\phi}_i(\sigma) : D^l \times D^{d-l} \rightarrow W \cup_P T(\sigma)$, by precomposing
\[
(D^l \times D^{d-l}) \bigcup ([0, 1] \times (S^{l-1} \times D^{d-l})) \xrightarrow{\phi_i \cup [\text{Id}_{[0,1]} \times \partial \phi_i]} W \cup_P T(\sigma).
\]
with the diffeomorphism
\[
\gamma_l : (D^l \times D^{d-l}) \xrightarrow{\simeq} D^l \times D^{d-l} \bigcup [0, 1] \times (S^{l-1} \times D^{d-l}).
\]
Using this diffeomorphism $\gamma_l$, each $\tilde{L}_i$ can also be extended cylindrically up the trace $T(\sigma)$, to yield new one-parameter families of $\theta$-structures $\tilde{L}_0(\sigma), \ldots, \tilde{L}_p(\sigma)$, satisfying the conditions of Definition 9.1 with respect to $(t_0, \phi_0(\sigma)), \ldots, (t_p, \phi_p(\sigma))$. The correspondence
\[
\left( M; (t_0, \phi_0, L_0), \ldots, (t_p, \phi_p, \tilde{L}_p) \right) \mapsto \left( M \cup T(\sigma); (t_0, \phi_0(\sigma), L_0(\sigma)), \ldots, (t_p, \phi_p(\sigma), \tilde{L}_p(\sigma)) \right)
\]
defines a map
\[
T(\sigma)_p : \mathcal{M}^k_{P,t}(\chi)_p \rightarrow \mathcal{M}^k_{P(\sigma),t}(\chi)_p.
\]
It is clear that $d_i \circ T(\sigma)_p = T(\sigma)_p \circ d_i$ for all face maps $d_i$, and thus we obtain a semi-simplicial map
\[
T(\sigma)_{\bullet} : \mathcal{M}^k_{P,t}(\chi)_{\bullet} \rightarrow \mathcal{M}^k_{P(\sigma),t}(\chi)_{\bullet}.
\]
Notice that for $p = -1$, the map $T(\sigma)_{-1}$ is precisely the map from Section 6

$$S_{T(\sigma)} : W^k_{P, t} \rightarrow W^k_{P(\sigma), t},$$

induced by the cobordism $T(\sigma) : P \rightsquigarrow P(\sigma)$. Thus the semi-simplicial map $T(\sigma)_* \text{ covers } S_{T(\sigma)}$.

We will need to use one more construction.

**Construction 9.2.** Let $P \in \text{Cob}^{k-1}_{d,t}$ be the same object fixed from the beginning of this subsection. Let $\chi : \partial D^l \times (1, \infty) \times \mathbb{R}^{d-l-1} \rightarrow P$ and $\hat{\ell}_t$ be as in Definition 9.1 with respect to $P$. Choose an extension of the embedding $\chi$,

$$\bar{\chi} : D^l \times (1, \infty) \times \mathbb{R}^{d-l-1} \rightarrow \mathbb{R}^\infty$$

with the properties:

- $\bar{\chi}^{-1}(0) = \partial D^l \times (1, \infty) \times \mathbb{R}^{d-l-1}$,
- $\bar{\chi}|_{\partial D^l \times (1, \infty) \times \mathbb{R}^{d-l-1}} = \chi$.

For $i \in \mathbb{Z}_{\geq 0}$, we define an embedding

$$\chi_i : D^l \times D^{d-l} \rightarrow \mathbb{R}^\infty$$

by the formula

$$\chi_i(x, y) = \bar{\chi}(x, 3(i + 1)e_0 + y)$$

(in the formula $3(i + 1)e_0 + y$, the term $e_0$ denotes the basis vector corresponding to the first coordinate in $(1, \infty) \times \mathbb{R}^{d-k-1}$). For each $i$, we define a $\theta$-structure

$$\hat{L}_i(\chi) : TD^l \times D^{d-l} \rightarrow \theta^* \gamma^d$$

by setting $\hat{L}_i(\chi) = \chi_i \hat{\ell}_t^{\text{std}} |_{3i+1}$. Equipped with the $\theta$-structure $\hat{L}_i(\chi)$, the embedding $\chi_i$ is surgery data of degree $l - 1$ in the manifold $P$.

For each $i$, consider the surgery data given by $\chi_i$. For $p \in \mathbb{Z}_{\geq 0}$, let us denote by $\chi(p)$ the truncated sequence of surgery data $(\chi_0, \chi_1, \ldots, \chi_p)$. We let $\bar{\chi}(p)$ denote the sequence of surgery data $(\bar{\chi}_0, \bar{\chi}_1, \ldots, \bar{\chi}_p)$, where each $\bar{\chi}_i$ is the surgery in $P(\chi(p))$ dual to $\chi_i$ (see Definition 8.2). Since the embeddings $\bar{\chi}_i$ are all disjoint, we may form their (simultaneous) trace, $T(\bar{\chi}(p)) : P(\chi(p)) \rightsquigarrow P$, and consider its induced map $S_{T(\bar{\chi}(p))} : W^k_{P(\chi(p)), t} \rightarrow W^k_{P, t}$. By Construction 8.1, the trace $T(\bar{\chi}(p))$ is equipped with embeddings

$$\psi_i : (D^l \times D^{d-l}, S^{l-1} \times D^{d-l}) \rightarrow (T(\bar{\chi}(p)), P)$$

for $i = 0, \ldots, p$, where $\psi_i(D^l \times D^{d-l}) \cap \psi_j(D^l \times D^{d-l}) = \emptyset$ for $i \neq j$. These embeddings are the standard core handles as described in Construction 8.1. By how $\chi_i$ was defined, it follows that each restricted embedding $\partial \psi_i = \psi_i|_{S^{l-1} \times D^{d-l}}$ satisfies the equation

$$\partial \psi_i(x, y) = \chi_i(x, y + 3(i + 1) \cdot e_0)$$

for all $(x, y) \in D^l \times D^{d-l}$. Using this, we may define a map

$$\mathcal{H}(\chi)_p : W^k_{P(\chi(p)), t} \rightarrow \mathcal{M}^k_{P, t}(\chi)_p$$

by sending $x \in W^k_{P(\chi(p)), t}$ to the $p$-simplex given by

$$\left( S_{T(\bar{\chi}(p))}(x); \ (1, \psi_0, \hat{L}_0(\chi), \ldots, (3i + 1, \psi_i, \hat{L}_i(\chi)), \ldots, (3p + 1, \psi_p, \hat{L}_p(\chi)) \right),$$

where each $\hat{L}_i(\chi)$ is considered to be the constant one-parameter family of $\theta$-structures as defined above. By how each $\chi_i$ was defined, it follows that this formula does indeed give a well defined map.
The following lemma is similar to [6, Lemma 6.10].

**Lemma 9.5.** For all $p \in \mathbb{Z}_{\geq 0}$, the map induced by $H(\chi)_p,$
\[ \text{hocolim}_{t \in k^k} W_{P(\chi(p)), t}^k \longrightarrow \text{hocolim}_{t \in k^k} M_{P, t}(\chi)_p, \]
is a weak homotopy equivalence.

**Proof.** It will suffice to show that $W_{P(\chi(p)), t}^k \longrightarrow M_{P, t}(\chi)_p$ is a weak homotopy equivalence for all $t$. We will do this explicitly for the case that $p = 0$; the case for general $p$ being similar. Let $E_P(\chi)$ denote the set of tuples $(t, \phi, \hat{L})$ with

1. $t \in (1, \infty),$
2. $\phi : \left(D^k \times D^{d-l}, \partial D^k \times D^{d-l}\right) \longrightarrow \left((\infty, 0] \times \mathbb{R}^{\infty-1}, \{0\} \times P\right)$ is an embedding,
3. $\hat{L}$ is a one-parameter family of $\theta$-structures on $D^k \times D^{d-l},$

subject to conditions similar to those from Definition 9.1. Since the target space of the embedding $\phi$ in the above definition is infinite dimensional Euclidean space, it follows that $E_P(\chi)$ is weakly contractible. There is a natural transformation
\[ (9.8) \quad M_{P, t}(\chi)_0 \longrightarrow E_P(\chi) \]
defined by
\[ \left((M, (V, \sigma), e), (t, \phi, \hat{L})\right) \mapsto (t, i_M \circ \phi, \hat{L}), \]
where on the right $i_M : (M, P) \hookrightarrow ((\infty, 0] \times \mathbb{R}^{\infty-1}, \{0\} \times P)$ is the inclusion map. This natural transformation has the concordance lifting property. The proof then follows by identifying $W_{P(\chi(0)), t}^k$ with the fibre of (9.8). This is similar to what was done in [6, Lemma 6.10] and we leave this step to the reader. \hfill \Box

With the above constructions in place we can now prove Theorem 8.9.

**Proof of Theorem 8.9.** Let $P \in \text{Ob Cob}_{\partial, d}^{k-1}$ be as in the statement of Theorem 8.9. Let $\sigma \in \text{Surg}_{k-1}(P)$. Let $\alpha \in \mathcal{L}(P, \sigma)$. Choose an embedding $\chi : \partial D^k \times (1, \infty) \times \mathbb{R}^{d-k-1} \longrightarrow P$ as in Definition 9.1 but with the extra properties:

1. $\chi(\partial D^k \times (1, \infty) \times \mathbb{R}^{d-k-1}) \cap \sigma(\partial D^k \times D^{d-k}) = \emptyset,$
2. $\chi(\partial D^k \times (1, \infty) \times \mathbb{R}^{d-k-1}) \cap \sigma(D^{d-k} \times D^k) = \emptyset,$
3. $\chi|_{\partial D^k \times \{0\}} : S^{k-1} \longrightarrow P$ is isotopic to the embedding $\sigma|_{S^{k-1} \times \{0\}}.$

Such an embedding $\chi$ can always be found by picking a small displacement of $\sigma$. Chosen in this way, we consider the semi-simplicial space $M_{P, t}^k(\chi)_*$. We may utilize Constructions 9.1 and 9.2 for these particular choices of $\chi$ and $\sigma$.

Let $p \in \mathbb{Z}_{\geq 0}$. Consider the cobordism $T(\mathcal{R}_{\chi(p)}(\sigma, \alpha)) : P(\chi(p)) \rightsquigarrow P(\mathcal{R}_{\chi(p)}(\sigma, \alpha)),$ and its map induced by $S_T(\mathcal{R}_{\chi(p)}(\sigma, \alpha))$,
\[ (9.9) \quad \text{hocolim}_{t \in k^k} W_{P(\chi(p)), t}^k \longrightarrow \text{hocolim}_{t \in k^k} W_{P(\mathcal{R}_{\chi(p)}(\sigma, \alpha)), t}. \]

By condition (c), it follows that in the manifold $P(\chi(p))$ the embedding $\sigma|_{S^{k-1} \times \{0\}} : S^{k-1} \longrightarrow P(\chi(p))$ is null-homotopic, and since $k < d/2$ it follows further that this embedding factors through an embedding $\mathbb{R}^{d-1} \hookrightarrow P(\chi(p))$. It follows that $\mathcal{R}_{\chi(p)}(\sigma)$ is a trivial surgery, and thus by Proposition 8.8 the map (9.9) is a homotopy equivalence, and we conclude that the cobordism $T(\mathcal{R}_{\chi(p)}(\sigma, \alpha))$ lies in $\mathcal{V}^{k-1}$ and thus (9.9) is a weak homotopy equivalence. Let us denote the map (9.9) by $F_p$. 


Recall the map
\[ \mathcal{H}(\chi)_p : \hocolim_{t \in \mathcal{K}^k} \mathcal{W}_P^{k}(\chi(p)), t \longrightarrow \hocolim_{t \in \mathcal{K}^k} \mathcal{M}_{P,t}^{k}(\chi)_p, \]
which is a weak homotopy equivalence by Lemma 9.5. By the disjointness conditions (a) and (b), this map induces a similar map
\[ \mathcal{H}(\chi)'_p : \hocolim_{t \in \mathcal{K}^k} \mathcal{W}_P^{k}(\mathcal{R}_{X(p)}(\sigma, \alpha)), t \longrightarrow \hocolim_{t \in \mathcal{K}^k} \mathcal{M}_{P(t),t}^{k}(\chi)_p \]
defined in a similar way, and which is also a weak homotopy equivalence. We obtain the commutative diagram,
\[ \begin{array}{c}
\hocolim_{t \in \mathcal{K}^k} \mathcal{W}_P^{k}(\chi(p)), t & \xrightarrow{F_p} & \hocolim_{t \in \mathcal{K}^k} \mathcal{W}_P^{k}(\mathcal{R}_{X(p)}(\sigma, \alpha)), t \\
\cong & \hocolim_{t \in \mathcal{K}^k} \mathcal{M}_{P,t}^{k}(\chi)_p & \cong \hocolim_{t \in \mathcal{K}^k} \mathcal{M}_{P(t),t}^{k}(\chi)_p \end{array} \]
By commutativity, it follows that \( T(\sigma, \alpha)_p \) is a weak homotopy equivalence for each \( p \) and thus \( T(\sigma, \alpha)_p \) induces a weak homotopy equivalence on geometric realization,
\[ |\hocolim_{t \in \mathcal{K}^k} \mathcal{M}_{P,t}^{k}(\chi)_p| \longrightarrow |\hocolim_{t \in \mathcal{K}^k} \mathcal{M}_{P(\sigma, \alpha)_p}, t(\chi)_p|. \]
Theorem 8.9 then follows from the commutative diagram
\[ \begin{array}{c}
|\hocolim_{t \in \mathcal{K}^k} \mathcal{M}_{P,t}^{k}(\chi)_p| & \cong & |\hocolim_{t \in \mathcal{K}^k} \mathcal{M}_{P(\sigma, \alpha)_p}, t(\chi)_p| \\
\cong & \hocolim_{t \in \mathcal{K}^k} \mathcal{W}_P^{k}, t & \hocolim_{t \in \mathcal{K}^k} \mathcal{W}_P^{k}, t \\
\end{array} \]
whose vertical arrows are weak homotopy equivalences by Corollary 9.4.

10. Higher Index Handles

In this section we prove the following proposition.

**Proposition 10.1.** Let \( k < d/2 \) and \( P \in \text{Ob Cob}_{\theta,d}^{k-1} \). Let \( \sigma \in \text{Surg}_j(P) \) with \( d - k \leq j < d - 1 \). Then the trace \( T(\sigma) : P \rightsquigarrow P(\sigma) \) lies in \( \mathcal{V}^{k-1} \).

The proof of the above proposition will use Corollary 8.10 together with the results developed in the following subsection.

10.1. Primitive surgeries. We will need to work with primitive surgeries. Let \( P \in \text{Ob Cob}_{\theta,d}^{k-1} \). Recall from Definition 8.11 that a surgery \( \sigma \in \text{Surg}_j(P) \) is said to be primitive if there exists another surgery \( \alpha \in \text{Surg}_{d-j-1}(P) \) such that \( \sigma(S^j \times \{0\}) \subset P \) and \( \alpha(S^{d-j-1} \times \{0\}) \subset P \) intersect transversally in \( P \) at exactly one point. As before we denote by \( \text{Surg}_{j}^{pr}(P) \subset \text{Surg}_{j}(P) \) the set of all primitive surgeries of degree \( j \).

Now, let \( \sigma \in \text{Surg}_{j}^{pr}(P) \) and let \( \alpha \in \text{Surg}_{d-j-1}(P) \) be such that the spheres \( \sigma(S^j \times \{0\}) \) and \( \alpha(S^{d-j-1} \times \{0\}) \) intersect transversally at one point in \( P \). Consider the dual surgery data \( \bar{\alpha} \in P(\alpha) \) and its trace \( T(\bar{\alpha}) : P(\alpha) \rightsquigarrow P \). The lemma below follows from the handle cancelation technique used in the proof of the H-cobordism theorem [13, Theorem 5.4].
Lemma 10.2. The underlying manifold of the composite cobordism
\[ T(\sigma) \circ T(\alpha) : P(\alpha) \sim P(\sigma) \]
is diffeomorphic to the cylinder, \([0, 1] \times P(\alpha)\), and thus the morphism \(T(\sigma) \circ T(\alpha)\) lies in \(\mathcal{V}^{k-1}\).

Our first result is the following:

Proposition 10.3. Let \(k < d/2\) and \(P \in \text{Ob Cob}_{d,k}^{k-1}\). Let \(\sigma \in \text{Surg}_{d-k}^{\text{pr}}(P)\) be a primitive surgery. Then the trace \(T(\sigma) : P \sim P(\sigma)\) lies in \(\mathcal{V}^{k-1}\).

Proof. Let \(\alpha \in \text{Surg}_{d-1}(P)\) be such that \(\alpha(S^{k-1} \times \{0\})\) and \(\sigma(S^{d-k} \times \{0\})\) intersect transversally in \(P\) at exactly one point. Let \(\bar{\alpha} \in \text{Surg}_{d-1}(P(\alpha))\) be dual to \(\alpha\). By the primitivity of \(\alpha\) it follows that \(\bar{\alpha}\) is a trivial surgery. By Corollary 8.10 the trace \(T(\bar{\alpha}) : P(\alpha) \sim P\) lies in \(\mathcal{V}^{k-1}\).

We then consider the composite cobordism, \(P(\alpha) \xrightarrow{T(\bar{\alpha})} P \xrightarrow{T(\sigma)} P(\sigma)\). By Lemma 10.2 its underlying manifold is diffeomorphic to a cylinder, thus it lies in \(\mathcal{V}^{k-1}\). By the 2-out-of-3 property it follows that \(T(\sigma)\) lies in \(\mathcal{V}^{k-1}\) as well. This concludes the proof of the proposition. \(\square\)

The following proposition is proven using the exact same argument as the previous one.

Proposition 10.4. Let \(l < k < d/2\). Suppose that it has been proven that the trace of every element of \(\text{Surg}_{d-l-1}^{\text{pr}}(P)\) is contained in \(\mathcal{V}^{k-1}\). Then it follows that the trace of every element of \(\text{Surg}_{d-l-1}^{\text{pr}}(P)\) is contained in \(\mathcal{V}^{k-1}\).

10.2. Proof of Proposition 10.1. We will need to use the semi-simplicial space constructed in Definition 9.2. For what follows, let \(P \in \text{Ob Cob}_{d,l}^{k-1}\), and let \(\sigma \in \text{Surg}_{d-l}(P)\) be with \(d-l < d-1\). Choose an embedding
\[ \chi : \partial D^{l-1} \times (1, \infty) \times \mathbb{R}^{d-l} \rightarrow P \]
and a one parameter family of \(\theta\)-structures \(\theta_t^{\text{std}}\) as in Definition 9.1 (with respect to \(P\)), with the following additional properties:

(a) \(\chi(\partial D^{l-1} \times (1, \infty) \times \mathbb{R}^{d-l}) \cap \sigma(\partial D^{d-l+1} \times D^{l-1}) = \emptyset\);

(b) the embedding \(\chi|_{\partial D^{l-1}} : \partial D^{l-1} \rightarrow P \setminus \sigma(\partial D^{d-l+1} \times \{0\})\) is isotopic to
\[ \sigma|_{\{s\} \times \partial D^{l-1}} : \{s\} \times \partial D^{l-1} \rightarrow P \setminus \sigma(\partial D^{d-l+1} \times \{0\}) \]
for some \(s \in \partial D^{d-l+1}\).

For \(p \in \mathbb{Z}_{\geq 0}\), let \(\chi(p) = (\chi_0, \ldots, \chi_p)\) be the list of mutually disjoint surgery data defined exactly as in Construction 9.2 using the embedding \(\chi\). Notice that in order for such an embedding to exist, it is necessary and sufficient that \(d-l < d-1\). Consider the transported surgery data \(\mathcal{R}_{\chi(p)}(\sigma)\) in \(P(\chi(p))\). The main lemma that we will need is the following:

Lemma 10.5. Let \(\chi\) and \(\sigma\) be chosen so that they satisfy conditions (a) and (b) as above. Let \(p \in \mathbb{Z}_{\geq 0}\). Then the surgery \(\mathcal{R}_{\chi(p)}(\sigma) \in \text{Surg}_{d-l}(P(\chi(p)))\) is a primitive surgery.

Proof. Let us first assume that \(p = 0\). Pick a point \(s \in \partial D^{d-l+1}\). Let \(\hat{\chi}_0 : D^{l-1} \rightarrow P\) be an embedding with the following properties:

- \(\hat{\chi}_0(D^{l-1}) \subset P \setminus \text{Int}(\chi_0(S^{l-2} \times D^{d-l+1}))\),
- \(\hat{\chi}_0^{-1}(\chi_0(S^{l-2} \times \{s\})) = S^{l-2}\).

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\( \hat{\chi}_0(D^{l-1}) \) intersects \( \sigma(S^{d-l} \times \{0\}) \) transversally in \( P \setminus \text{Int}(\chi_0(S^{d-2} \times D^{d-l+1})) \) at exactly one point.

It follows from condition (b) that such an embedding \( \hat{\chi}_0 \) can always be found; we may choose \( \hat{\chi}_0 \) to be the inclusion of a fibre of the normal disk-bundle of \( \sigma(S^{d-l} \times \{0\}) \). Let \( D_-^{l-1} \subset P \) denote \( \hat{\chi}_0(D^{l-1}) \). Consider the surgered manifold
\[
P(\chi_0) = P \setminus \text{Int}(\chi_0(S^{d-2} \times D^{d-l+1})) \bigcup D_-^{l-1} \times S^{d-l}.
\]
Notice that by how \( D_-^{l-1} \) was chosen, \( D_-^{l-1} \) is contained in \( P(\chi_0) \). Let \( D_+^{l-1} \subset P(\chi_0) \) be the disk \( D_-^{l-1} \times \{s\} \subset D_-^{l-1} \times S^{d-l} \) coming from the right-hand side of the above union. The union
\[
D_-^{l-1} \cup D_+^{l-1} \subset P(\chi_0)
\]
is an embedded \((l-1)\)-sphere which is easily seen to have trivial normal bundle. Let use denote by \( \beta : S^{l-1} \to P(\chi_0) \) the inclusion of this sphere. By construction, \( \beta(S^{l-1}) \) intersects the sphere \( \sigma(S^{d-l} \times \{0\}) \subset P(\chi_0) \) transversally at a single point. This proves that \( \mathcal{R}_{\chi_0}(\sigma) \) is primitive. Now suppose that \( p > 0 \). If we perform surgery on the first embedding \( \chi_0 \), by the above argument \( \mathcal{R}_{\chi_0}(\sigma) \) is primitive in \( P(\chi_0) \). We then observe that \( \beta(S^{l-1}) \subset P(\chi_0) \) is disjoint from the embedding that corresponds to the surgery data \( \mathcal{R}_{\chi_0}(\chi_1) \), and thus \( \beta(S^{l-1}) \) determines an embedded sphere with trivial normal bundle (with the same name), \( \beta(S^{l-1}) \subset P(\chi_1, \chi_2) \). In \( P(\chi_1, \chi_2) \), the sphere \( \beta(S^{l-1}) \) still intersects \( \sigma(S^{d-l} \times \{0\}) \) transversally at a single point, and thus \( \mathcal{R}_{\chi_1, \chi_2}(\sigma) \) is primitive. The proof of the lemma then follows by proceeding inductively.

With the above lemma established we may now prove Proposition 10.1.

**Proof of Proposition 10.1.** We prove the theorem by induction on the value \( j = d - l \geq d - k \). The induction step is based on the following claim:

**Claim 10.6.** Suppose that it has been proven that the trace of every element of \( \text{Surg}_{d-l-1}^k(P) \) is contained in \( \mathcal{V}^{k-1} \). Then it follows that \( T(\sigma) \in \mathcal{V}^{k-1} \) for all \( \sigma \in \text{Surg}_{d-l}(P) \).

Let us explain how this claim implies Proposition 10.1. By Corollary 8.10 all trivial surgeries \( \alpha \in \text{Surg}_{d-k-1}^k(P) \) have their trace \( T(\alpha) \) contained in \( \mathcal{V}^{k-1} \). By the above claim, it follows that all surgeries in \( \text{Surg}_{d-k}(P) \) (trivial or not) have trace contained in \( \mathcal{V}^{k-1} \). This establishes the base case of the induction. We then may apply Claim 10.6 again to find that all surgeries of degree \( d - k + 1 \) have trace contained in \( \mathcal{V}^{k-1} \). Continuing by induction then establishes Proposition 10.1.

We now proceed to prove Claim 10.6. Suppose it has been proven that the trace of every element of \( \text{Surg}_{d-l-1}^k(P) \) is contained in \( \mathcal{V}^{k-1} \). Let \( \sigma \in \text{Surg}_{d-l}(P) \) and let the embedding
\[
\chi : \partial D^{l-1} \times (1, \infty) \times \mathbb{R}^{d-l} \to P
\]
be chosen exactly as in 10.1. For each \( p \in \mathbb{Z}_{\geq 0} \), consider the commutative diagram
\[
\begin{array}{ccc}
\text{hocolim} \mathcal{W}_{P(\chi(p))}^k & \xrightarrow{T(\sigma)p} & \text{hocolim} \mathcal{W}_{P(\chi(p)), t}^k \\
\text{hocolim} \mathcal{M}_{P, t}(\chi)_p & \xrightarrow{\mathcal{H}(\chi)_p} & \text{hocolim} \mathcal{M}_{P, t}(\chi)_p
\end{array}
\]
where the top-horizontal map is induced by concatenation with \( T(\mathcal{R}_{\chi(p)}(\sigma)) \), the bottom horizontal map is induced by concatenation with \( T(\sigma) \), and the vertical maps are the weak homotopy equivalences defined as in Construction 9.2. By Lemma 10.5 \( \mathcal{R}_{\chi(p)}(\sigma) \) is a primitive surgery of degree.
d − l. Since the trace of a trivial surgery of degree d − l − 1 is contained in \(\mathcal{V}^{k−1}\) by assumption, it follows by Proposition 10.4 that the top vertical map in the above diagram is a weak homotopy equivalence. By commutativity, it then follows that the bottom horizontal map is a weak homotopy equivalence, for all \(p \in \mathbb{Z}_{≥0}\). It then follows that \(T(σ)\) induces a homotopy equivalence on geometric realization. Claim 10.6 then follows by considering the commutative square

\[\begin{array}{ccc}
\text{hocolim}_{t ∈ K^k} \mathcal{M}_{P,t} (χ) & \xrightarrow{\sim} & \text{hocolim}_{t ∈ K^k} |\mathcal{M}_{P(σ),t} (χ)| \\
\downarrow \simeq & & \downarrow \simeq \\
\text{hocolim}_{t ∈ K^k} \mathcal{W}^k_{P,t} & \xrightarrow{S_{T(σ)}} & \text{hocolim}_{t ∈ K^k} \mathcal{W}^k_{P(σ),t},
\end{array}\]

whose vertical arrows are weak homotopy equivalences by Corollary 9.4.

10.3. Index d-handles. In Proposition 10.4, it was required that the degree of the surgery \(σ\) be strictly less than \(d − 1\). We now handle the final case where \(σ\) has deg(\(σ\)) = 0.

Proposition 10.7. Let \(P \in \text{ObCob}^{k−1}_{θ,d}\) be an object with strictly more than one path component. Let \(σ ∈ \text{Surg}_{d−1}(P)\). Then \(T(σ) \in \mathcal{V}^{k−1}\).

Proof. We will prove that \(σ\) is primitive. We first observe that for dimensional reasons, the sub-manifold \(σ(S^{d−1}) \subset P\) is a whole path component of \(P\). Let \(S^0 \hookrightarrow P\) be an embedding that sends one point to \(σ(S^{d−1}) \subset P\), and the other point of \(S^0\) to a different path component of \(P\). Since \(P\) has more than one path component, such an embedding exists. The existence of this embedding proves that \(σ\) is primitive. Since it has been proven that trace of every surgery of degree \(d − 2\) is contained in \(\mathcal{V}^{k−1}\), Proposition 10.3 then implies that \(T(σ) \in \mathcal{V}^{k−1}\). This concludes the proof of the proposition.

We now deal with the general case where the object \(P\) is path-connected, and hence be diffeomorphic to \(S^{d−1}\).

Proposition 10.8. Let \(P \in \text{ObCob}^{k−1}_{θ,d}\) be any object (that is possibly path-connected). Let \(σ ∈ \text{Surg}_{d−1}(P)\). Then \(T(σ) ∈ \mathcal{V}^{k−1}\).

The proof of the above proposition will require a new construction.

Definition 10.9. Let \(P \in \text{ObCob}^{k−1}_{θ,d}\). Choose once and for all a standard \(θ\)-structure \(\hat{L}_{D^d}\) on the disk \(D^d\). Fix an element \(z := (M, (V, σ), e) ∈ \mathcal{W}^k_{P,t}\). We define \(Y_0(z)\) to be the space of pairs \((φ, \hat{L})\) where:

- \(φ : D^d → W\) is an embedding with image disjoint from the image of \(e\);
- \(\hat{L}(t), t ∈ [0, 1]\), is a one-parameter family of \(θ\)-structures on \(D^d\) with \(\hat{L}(0) = φ^*\hat{L}_W\), and \(\hat{L}(1) = \hat{L}_{D^d}\).

For \(p ∈ \mathbb{Z}_{≥0}\), the space \(Y_p(z)\) consists of \((p + 1)\)-tuples \(((φ_0, \hat{L}_0), \ldots, (φ_p, \hat{L}_p))\), that satisfy:

\[φ_i(D^d) ∩ φ_j(D^d) = ∅ \quad \text{whenever} \quad i ≠ j.\]

The assignment \([p] → Y_p(z)\) defines a semi-simplicial space. For each \(p ∈ \mathbb{Z}_{≥0}\), the space \(D^k_{P,t,p}\) is defined to consist of tuples, \((z; ((φ_0, \hat{L}_0), \ldots, (φ_p, \hat{L}_p)))\), where \(z ∈ \mathcal{W}^k_{P,t}\) and \(((φ_0, \hat{L}_0), \ldots, (φ_p, \hat{L}_p)) ∈ Y_p(z)\). The assignment \([p] → D^k_{P,t,p}\) defines a semi-simplicial space, \(D^k_{P,t,•}\). The forgetful maps \(D^k_{P,t,p} → \mathcal{W}^k_{P,t}, \quad (z; ((φ_0, \hat{L}_0), \ldots, (φ_p, \hat{L}_p)) → z\).
yield an augmented semi-simplicial space, \( D^k_{P,t,\bullet} \rightarrow D^k_{P,t,-1} \) with \( D^k_{P,t,-1} = W^k_{P,t} \). It follows from the definition that \( t \mapsto D^k_{P,t,\bullet} \) defines a contravariant functor on \( K^k \) valued in augmented semi-simplicial spaces. By taking the level-wise homotopy colimits over \( K^k \), we obtain the augmented semi-simplicial space, \( \hocolim_{t \in K^k} D^k_{P,t,\bullet} \rightarrow \hocolim_{t \in K^k} D^k_{P,t,-1} \).

We have the following proposition.

**Proposition 10.10.** For all \( P \in \text{Ob} \mathbf{Cob}^k_{\theta,d} \), the map induced by augmentation,

\[
\left| \hocolim_{t \in K^k} D^k_{P,t,\bullet} \right| \rightarrow \hocolim_{t \in K^k} D^k_{P,t,-1},
\]

is a weak homotopy equivalence.

**Proof.** For each \( t \in K^k \), the augmentation map \( |D^k_{P,t,\bullet}| \rightarrow D^k_{P,t,-1} \) is a weak homotopy equivalence. This follows by the exact same argument employed in the proof of Proposition 9.3 (in this case it even easier because there is no need to use the Smale-Hirsch theorem). From the homotopy invariance of homotopy colimits, it then follows that the induced map

\[
\hocolim_{t \in K^k} |D^k_{P,t,\bullet}| \rightarrow \hocolim_{t \in K^k} D^k_{P,t,-1}
\]

is a weak homotopy equivalence. The proof then follows from the homotopy equivalence

\[
\hocolim_{t \in K^k} |D^k_{P,t,\bullet}| \simeq |\hocolim_{t \in K^k} D^k_{P,t,\bullet}|,
\]

see also the proof of Corollary 9.3. \( \square \)

We now prove Proposition 10.8.

**Proof of Proposition 10.8.** Fix \( P \in \text{Ob} \mathbf{Cob}^k_{\theta,d} \). Let \( \sigma \in \text{Surg}_{d-1}(P) \) and consider the trace \( T(\sigma) : P \rightarrow P(\sigma) \). If \( P \) has more than one path component, then \( T(\sigma) \) belongs to \( V^{k-1} \) by Proposition 10.7, and so there is nothing to show. So, assume that \( P \) is path connected. Consider the augmented semi-simplicial space \( D^k_{P,t,\bullet} \rightarrow D^k_{P,t,-1} \). For each integer \( p \in \mathbb{Z}_{\geq 0} \), fix an object \( S_p \in \text{Ob} \mathbf{Cob}^k_{\theta,d} \), with the properties:

(i) there is a diffeomorphism \( S_p \cong (S^{d-1})^{\cup(p+1)} \);

(ii) \( S_t \) is disjoint from \( P \) as a submanifold of \( \mathbb{R}^\infty \);

(iii) on each component of \( S_p \), the \( \theta \)-structure \( \hat{\ell}_p \) agrees with \( \hat{\ell}_{D_d|S_{d-1}} \), via the diffeomorphism from part (i).

By condition (i), it follows that the disjoint union \( P \sqcup S_p \) is an object of \( \mathbf{Cob}^k_{\theta,d} \). By the same argument used in the proof of Lemma 9.5 for each \( p \in \mathbb{Z}_{\geq 0} \) there is a weak homotopy equivalence,

\[
\mathcal{H}_{t,p} : W^k_{P \sqcup S_p,t} \xrightarrow{\simeq} D^k_{P,t,p}
\]

(see also the constructions in Section 9.2). For each \( p \in \mathbb{Z}_{\geq 0} \) we may consider the commutative diagram,

\[
\begin{array}{ccc}
\hocolim_{t \in K^k} W^k_{P \sqcup S_p,t} & \xrightarrow{-\cup T(\sigma)} & \hocolim_{t \in K^k} W^k_{P(\sigma) \sqcup S_p,t} \\
\mathcal{H}_{t,p} : |\cup T(\sigma)| \nleftarrow \hocolim_{t \in K^k} D^k_{P,t,p} & \xrightarrow{\simeq} & \hocolim_{t \in K^k} D^k_{P(\sigma),t,p} \\
\end{array}
\]
Since the object \( P \sqcup S_\sigma \) is not path-connected, it follows from Proposition 10.7 that the top vertical map in (10.3) is a weak homotopy equivalence. By commutativity of the diagram it follows that the bottom-horizontal map is a weak homotopy equivalence as well. By geometrically realizing it follows that \(- \cup T(\sigma)\) induces the weak homotopy equivalence,

\[
| \operatorname{hocolim}_{t \in K} D^k_{P,t,\bullet} | \xrightarrow{\sim} | \operatorname{hocolim}_{t \in K} D^k_{P(\sigma),t,\bullet} |.
\]

The proposition then follows by commutativity of the diagram,

\[
\begin{array}{c}
| \operatorname{hocolim}_{t \in K} D^k_{P,t,\bullet} | \\
\xrightarrow{\sim} \\
\downarrow \\
| \operatorname{hocolim}_{t \in K} W^k_{P,t} | \\
\xrightarrow{\sim} \\
\downarrow \\
| \operatorname{hocolim}_{t \in K} W^k_{P(\sigma),t} | \end{array}
\]

\[\xrightarrow{\sim} \]

\[
\begin{array}{c}
| \operatorname{hocolim}_{t \in K} D^k_{P,t,-1} | \\
\xrightarrow{=} \\
\downarrow \\
| \operatorname{hocolim}_{t \in K} W^k_{P,t} | \\
\xrightarrow{=} \\
\downarrow \\
| \operatorname{hocolim}_{t \in K} W^k_{P(\sigma),t} | \end{array}
\]

By collecting our results from the last three sections, we observe that we have proven Theorem 6.7. Indeed, the theorem follows by combining Corollary 8.10, Corollary 8.13, Proposition 10.1, and Proposition 10.8.

11. Stable Moduli Spaces of Even Dimensional Manifolds

Recall that Theorem 2.13 required the condition that \( k < d/2 \). In this section we show how to recover a modified version of Theorem 2.13 in the case the \( d = 2n \) and \( k = n \). As described in the introduction, this will let us recover the proof of the Madsen-Weiss theorem from [12] and the theorem of Galatius and Randal-Williams from [8]. To state this modified version of Theorem 2.13 we will need a new construction.

In this section we assume that \( d = 2n \) for a positive integer \( n \). Fix a tangential structure \( \theta : B \to B(d+1) \). Let \( P \in \text{Ob} \mathbf{Cob}_\theta^{d,d} \). We consider the functor \( t \mapsto W^{n-1,c}_{P,t} \) from Definition 3.11. Recall from Subsection 7.1 the self-cobordism \( S_{H_{n,n}(P)} : P \sim P \) and the map \( S_{H_{n,n}(P)} : W^{n-1,c}_{P,t} \to W^{n-1,c}_{P,t} \) that it induces. By the same argument employed in the proof of Proposition 7.1 it follows that upon taking homotopy colimits, the map \( S_{H_{n,n}(P)} \) induces a weak homotopy equivalence

\[
\operatorname{hocolim}_{t \in K^{n-1}} W^{n-1,c}_{P,t} \xrightarrow{\sim} \operatorname{hocolim}_{t \in K^{n-1}} W^{n-1,c}_{P,t} .
\]

**Definition 11.1.** Using the map \( S_{H_{n,n}(P)} \) above, for each \( k \leq n \) and \( t \in K^k \) we define

\[ W^{k,\text{stb}}_{P,t} := \operatorname{hocolim} \left[ W^k_{P,t} \to W^k_{P,t} \to \cdots \right] \]

where the direct system is given by iterating the map \( S_{H_{n,n}(P)} \). The space \( W^{k,\text{stb}}_{P,t} \) is defined similarly.

We will mainly be interested in the space \( W^{n-1,c,\text{stb}}_{P,t} \).
Proposition 11.2. The inclusion
\[ \hocolim_{t \in K^{n-1}} W_{P,t}^{n-1,c} \rightarrow \hocolim_{t \in K^{n-1}} W_{P,t}^{n-1,c,\text{stb}} \]
is a weak homotopy equivalence.

Proof. The proposition follows from (11.1) combined with the weak homotopy equivalence
\[ \hocolim_{t \in K^{n-1}} \left( \hocolim_{s} \left[ W_{P,s}^{n-1,c} \rightarrow W_{P,t}^{n-1,c} \rightarrow \cdots \right] \right) \simeq \hocolim_{t \in K^{n-1}} \left[ \hocolim_{s} W_{P,s}^{n-1,c} \rightarrow \hocolim_{s} W_{P,t}^{n-1,c} \rightarrow \cdots \right]. \]

We now analyze the fibres of the map, \( W_{P,t}^{n-1,c,\text{stb}} \rightarrow W_{\text{loc},t}^{[n,n+1]} \).

Definition 11.3. For each \( s \in K^{[n,n+1]} \), let us fix once and for all an element
\[ ((V_s, \sigma_s), e_s) \in W_{\text{loc},s}^{[n,n+1]} \].

As before, we denote the unique element in \( W_{\text{loc},\emptyset}^{[n,n+1]} \) by \( \emptyset \). We let \( W_{P,s}^{n-1,c,\text{stb}} \) denote the fibre of the map \( W_{P,s}^{n-1,c,\text{stb}} \rightarrow W_{\text{loc},s}^{[n,n+1]} \) over the element \( ((V_s, \sigma_s), e_s) \in W_{\text{loc},s}^{[n,n+1]} \).

Proceeding as in Section 6 for each \( t \in K^{[n,n+1]} \) we fix once and for all a closed submanifold
\( S_t \subset \mathbb{R}^\infty \)
disjoint from the manifold \( P \), and diffeomorphic to the product \( s \times S^{n-1} \times S^n \). Choose a \( \theta \)-structure \( \hat{\ell}_S \) on \( S_t \) that admits an extension to a \( \theta \)-structure on \( t \times D^n \times S^n \). The manifold \( P \sqcup S_t \) equipped with the structure \( \hat{\ell}_P \sqcup \hat{\ell}_S \) defines an object in the cobordism category \( \text{Cob}_{\theta,2n} \).

By the same augment used in Proposition 6.2 we obtain:

Proposition 11.4. For all \( P \in \text{Ob} \text{Cob}_{\theta,2n} \) and \( t \in K^{[n,n+1]} \) there is a homeomorphism
\[ W_{P,t}^{n,\text{stb}} \rightarrow \bigoplus_{[M]} \text{BDiff}_{\theta,n}(M, P) \]
where the union ranges over all diffeomorphism classes of compact manifolds \( M \) equipped with an identification \( \partial M \rightarrow P \). The space \( \text{BDiff}_{\theta,n}(M, P) \) is the homotopy quotient
\[ \text{Bun}_n(TM \oplus e^1, \theta^* \gamma^{d+1}; \hat{\ell}_P) \rightarrow \text{Diff}(M, P) \]
where \( \text{Bun}_n(TM \oplus e^1, \theta^* \gamma^{d+1}; \hat{\ell}_P) \) is the space of \( \theta \)-structures \( \hat{\ell}_M \) on \( M \), that agree with \( \hat{\ell}_P \) on the boundary, such that the underlying map \( \ell_M : M \rightarrow B \) is \( n \)-connected. It follows that the limiting space \( W_{P,\emptyset}^{n,\text{stb}} \) agrees with the stable moduli space of \( \theta \)-manifolds studied in [8] and in [6].

As was observed in Section 6, any morphism \( W : P \rightarrow Q \) in \( \text{Cob}_{\theta,2n}^{n-1} \) induces a map
\[ s_W : W_{P,\emptyset}^{n,\text{stb}} \rightarrow W_{Q,\emptyset}^{n,\text{stb}}, \ M \mapsto M \cup_P W. \]
The theorem stated below is a translation of the homological stability theorem of Galatius and Randal-Williams proven in [6].
Theorem 11.5. For any morphism $W : P \rightsquigarrow Q$ in $\text{Cob}^n_{\theta,d}$, the map $S_W : \mathcal{W}^n_{P,\emptyset} \rightarrow \mathcal{W}^n_{Q,\emptyset}$ is an Abelian homological equivalence.

Corollary 11.6. For any morphism $(j, \varepsilon) : \mathfrak{t} \rightarrow \mathfrak{s}$ in $\mathcal{K}^{n-1}$, the induced map

$$(j, \varepsilon)^* : \mathcal{W}^{n-1,\text{c, stb}}_P \rightarrow \mathcal{W}^{n-1,\text{c, stb}}_P$$

is an Abelian homological equivalence.

Proof. This is proven in the same way as Theorem 5.3 using Theorem 11.5 instead of Theorem 6.7. The map $(j, \varepsilon)^* : \mathcal{W}^{n-1,\text{c, stb}}_P \rightarrow \mathcal{W}^{n-1,\text{c, stb}}_P$ fits into a commutative diagram

$$
\begin{array}{ccc}
\mathcal{W}^{n-1,\text{c, stb}}_P & \xrightarrow{(j, \varepsilon)^*} & \mathcal{W}^{n-1,\text{c, stb}}_P \\
\downarrow & & \downarrow \\
\mathcal{W}^{n,\text{stb}}_P \cong S_W & \xrightarrow{S_W} & \mathcal{W}^{n,\text{stb}}_P
\end{array}
$$

where $W : P \sqcup S_b \rightsquigarrow P \sqcup S_b$ is some morphism in $\text{Cob}^{n-1}_{\theta,2n}$, and the vertical maps are the homotopy equivalences from Proposition 11.4. By Theorem 11.5, $S_W$ is an Abelian homological equivalence and so it follows by commutativity that $(j, \varepsilon)^*$ is as well.

Corollary 11.11 from the introduction concerned an acyclic map. We will need to use a homological version of Theorem 5.3 for the case $d = 2n$ and $k = n$. We need to introduce a new definition and a preliminary result.

Definition 11.7. A map $f : X \rightarrow Y$ between topological spaces is said to be an acyclic homology fibration if for all $y \in Y$, the inclusion, $f^{-1}(y) \hookrightarrow \text{hofibre}_y(f)$, is an acyclic map.

The proposition below is a generalization of Proposition 3.14. Its proof follows from [14, Proposition 4.4].

Proposition 11.8. Let $\mathcal{C}$ be a small category and let $u : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a natural transformation between functors from $\mathcal{C}$ to the category of topological spaces. Suppose that for each morphism $f : a \rightarrow b$ in $\mathcal{C}$, the map

$$f_* : \text{hofibre}(u_a : \mathcal{G}_1(a) \rightarrow \mathcal{G}_2(a)) \rightarrow \text{hofibre}(u_b : \mathcal{G}_1(b) \rightarrow \mathcal{G}_2(b))$$

is an Abelian homology equivalence. Then for any object $c \in \text{Ob}\, \mathcal{C}$, the inclusion

$$\text{hofibre}(u_c : \mathcal{G}_1(c) \rightarrow \mathcal{G}_2(c)) \hookrightarrow \text{hofibre}(\text{hocolim}\, \mathcal{G}_1 \rightarrow \text{hocolim}\, \mathcal{G}_2)$$

is an Abelian homology equivalence.

Proof. Let the simplicial space $X_\bullet$ denote the nerve of the transport category $\mathcal{G}_1|\mathcal{C}$, and let $Y_\bullet$ denote the nerve of $\mathcal{G}_2|\mathcal{C}$. The homotopy colimits of $\mathcal{G}_1$ and $\mathcal{G}_2$ are obtained by taking the geometric realizations of the these simplicial spaces $X_\bullet$ and $Y_\bullet$ respectively. The natural transformation induces a simplicial map $u_\bullet : X_\bullet \rightarrow Y_\bullet$. The condition in the statement of the proposition implies that for all $p \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq p$, the face map $d_i$ induces an Abelian homology equivalence,

$$\text{hofibre}(u_p : X_p \rightarrow Y_p) \xrightarrow{\simeq} \text{hofibre}(u_{p-1} : X_{p-1} \rightarrow Y_{p-1})$$

(a similar statement holds true for the degeneracy maps). It then follows from [14, Proposition 4.4] that the inclusion, $\text{hofibre}(u_0 : X_0 \rightarrow Y_0) \hookrightarrow \text{hofibre}(u_\bullet : |X_\bullet| \rightarrow |Y_\bullet|)$, is an Abelian homology equivalence. This completes the proof of the proposition.
The next corollary follows by combining Theorem 11.6 with Proposition 11.8. By applying Proposition 11.8 to the above corollary we obtain the following result.

**Corollary 11.9.** Let \( d = 2n \). The localization map
\[
\hocolim_{s \in K^{n-1}} W^{n-1,c,\text{stb}}_{P,s} \rightarrow \hocolim_{s \in K^{n-1}} W^{[n,n+1]}_{\text{loc},s}
\]
is an acyclic homology fibration with fibre over \( \emptyset \) given by the space, \( W^{n,\text{stb}}_{P,\emptyset} \).

**Proof.** By Proposition 11.8 it follows that the inclusion
\[
W^{n,\text{stb}}_{P,\emptyset} \hookrightarrow \text{hofibre} \left( \hocolim_{s \in K^{n-1}} W^{n-1,c,\text{stb}}_{P,s} \rightarrow \hocolim_{s \in K^{n-1}} W^{[n,n+1]}_{\text{loc},s} \right)
\]
is an abelian homology equivalence. Both of the spaces \( \hocolim_{s \in K^{n-1}} W^{n-1,c,\text{stb}}_{P,s} \) and \( \hocolim_{s \in K^{n-1}} W^{[n,n+1]}_{\text{loc},s} \) are infinite loopspaces and thus have Abelian fundamental group. As a result, the above homotopy fibre has Abelian fundamental group as well. It follows from this fact that the above Abelian homology equivalence is an Acyclic map. \( \square \)

Combining the above corollary with Theorem 3.10, we obtain Theorem 11.10 stated in the introduction.

Finally we show how to use the above results to recover the theorem of Galatius and Randal-Williams of [8]. Using the commutative diagram
\[
\begin{array}{ccc}
W^{n,\text{stb}}_{P,\emptyset} & \rightarrow & \text{hofibre} \left( \hocolim_{t \in K^{n-1}} W^{n-1,c,\text{stb}}_{P,t} \rightarrow \hocolim_{s \in K^{n-1}} W^{[n,n+1]}_{\text{loc},s} \right) \\
\downarrow & & \downarrow \\
\Omega^{\infty-1} hW^n & \rightarrow & \Omega^{\infty-1} hW_{\emptyset}^{n-1} \\
\downarrow & & \downarrow \\
\Omega^{\infty-1} hW_{\emptyset}^{n} & \rightarrow & \Omega^{\infty-1} hW^{[n,d-n+1]}_{\theta,\text{loc}} \\
\end{array}
\]

By Corollary 11.9 it follows that the right-vertical map
\[
W^{n,\text{stb}}_{P,\emptyset} \rightarrow \Omega^{\infty-1} hW^n_{\emptyset}
\]
is an acyclic map. Let \( \theta_{2n} \) denote the tangential structure obtained by restricting \( \theta \) to \( BO(2n) \). Let \( MT_{\theta_{2n}} \) denote the *Madsen-Tillmann* spectrum associated to \( \theta_{2n} \). The theorem of Galatius and Randal-Williams is then recovered by combining this with the homotopy equivalence of spectra
\[
\Sigma^{-1} MT_{\theta_{2n}} \simeq hW^n_{\emptyset}.
\]

Below, we sketch how this homotopy equivalence of spectra is obtained. This is basically the same as what was done in [12] Section 3.1.

**Construction 11.1.** Let \( G_{\theta_{2n}}(\mathbb{R}^\infty) \) be the space of pairs \((V, \hat{\ell}_V)\) where \( V \leq \mathbb{R}^\infty \) is a \( 2n \)-dimensional vector subspace and \( \hat{\ell}_V \) is a \( \theta_{2n} \)-orientation on \( V \). We let \( U_{2n,\infty} \rightarrow G_{\theta_{2n}}(\mathbb{R}^\infty) \) denote the canonical vector bundle. The spectrum \( MT_{\theta_{2n}} \) is defined to be the Thom spectrum associated to the virtual vector bundle \(-U_{2n,\infty} \rightarrow G_{\theta_{2n}}(\mathbb{R}^\infty)\). There is a map
\[
G_{\theta_{2n}}(\mathbb{R}^\infty) \rightarrow G_{\theta_{2n}}(\mathbb{R}^\infty)^n
\]
defined by sending \((V, \hat{\ell}_V)\) to the element \((\mathbb{R} \times V, \hat{\ell}_{\mathbb{R} \times V}, \ell_V, \sigma_0)\), where:

- \( \mathbb{R} \times V \leq \mathbb{R} \times \mathbb{R}^\infty \cong \mathbb{R}^\infty \) is the product vector space,
- \( \hat{\ell}_{\mathbb{R} \times V} \) is the product \( \theta \)-orientation induced by \( \hat{\ell}_V \).
• \(l : \mathbb{R} \times V \to \mathbb{R}\) is the product projection,
• \(\sigma_0 : (\mathbb{R} \times V) \otimes (\mathbb{R} \times V) \to \mathbb{R}\) is the trivial bilinear form.

By Definition 2.10, it follows that \(G^{mf}_\theta(\mathbb{R}^\infty)^n\) is precisely the subspace of \(G^{mf}_\theta(\mathbb{R}^\infty)\) consisting of those \((V, \ell, l, \sigma)\) such that \(l \neq 0\), and so the above map (11.4) is well defined. The virtual bundle \(-U_{2n+1, \infty} \to G^{mf}_\theta(\mathbb{R}^\infty)^n\) pulls back to \(-U_{2n, \infty} \to G_\theta(\mathbb{R}^\infty)\) under the map (11.4) and so it follows that this map induces a map of spectra \(\Sigma^{-1} \Theta_{2n} \to hW^n_\theta\). A homotopy inverse of (11.4) is defined by sending \((V, \ell, l, \sigma)\) to the element \((\ker(l), \hat{\ell}', \ker(\ell))\). It follows that (11.4) induces the homotopy equivalence of spectra \(\Sigma^{-1} \Theta_{2n} \simeq hW^n_\theta\).

**Appendix A. Proof of Theorem 3.10**

In this section we prove Theorem 3.10 which asserts that for all choices of \(k\) and \(P\), there is a commutative diagram

\[
\begin{array}{ccc}
\underset{t \in K^{k-1}}{\text{hocolim}} W_{k-1, P, t} & \cong & L^{k-1} \\
\downarrow & & \downarrow \\
\underset{t \in K^{k-1}}{\text{hocolim}} W_{\{k, d-k+1\}, \text{loc}, t} & \cong & D^{mf, \{k, d-k+1\}}_{\text{loc}, \text{loc}} \\
\end{array}
\]

such that the horizontal maps are weak homotopy equivalences. The proof uses several constructions, all of which are directly analogous to constructions from [12, Section 5], where Madsen and Weiss prove the special case of this theorem for \(k = -1\). We provide only a sketch of the proof, mainly for the purpose of showing that the constructions are well defined for all \(k\). We refer the reader to the relevant sections of [12] for technical details.

**A.1. The bottom row.** In this section we construct the bottom row of (A.1). Our first step is to construct an intermediate space sitting in-between \(\text{hocolim}_{t \in K^{k-1}} W_{\{k, d-k+1\}, \text{loc}, t}\) and \(D^{mf, \{k, d-k+1\}}_{\theta, \text{loc}}\).

**Definition A.1.** For \(t\), the space \(L_{\text{loc}, t}\) consists of tuples \(((\bar{x}; V, \sigma), \delta, \phi)\) where

\(\begin{align*}
& (i) \ (\bar{x}; V, \sigma) \in D^{mf}_{\theta, \text{loc}}; \\
& (ii) \ \delta : \bar{x} \to \{-1, 0, +1\} \text{ is a function}; \\
& (iii) \ \phi : t \xrightarrow{\cong} \delta^{-1}(0) \text{ is a bijection};
\end{align*}\)

subject to the following condition: the height function \(h_{\bar{x}} : \bar{x} \to \mathbb{R}\) is bounded below on \(\delta^{-1}(+1)\) and is bounded above on \(\delta^{-1}(-1)\).

We need to describe how the correspondence \(t \mapsto L_{\text{loc}, t}\) defines a contravariant functor on \(\mathcal{K}\). Let \((j, \varepsilon) : s \to t\) be a morphism in \(\mathcal{K}\). The induced map \((j, \varepsilon)^* : L_{\text{loc}, t} \to L_{\text{loc}, s}\) sends an element \(((\bar{x}; V, \sigma), \delta, \phi) \in L_{\text{loc}, t}\) to \(((\bar{x}; V, \sigma), \delta, \phi')\) where \(\phi' = \phi \circ j\) and

\[
\delta'(y) = \begin{cases} 
\varepsilon(s) & \text{if } y = \phi(s) \text{ where } s \in s \setminus j(t), \\
\delta(y) & \text{otherwise}.
\end{cases}
\]

For each integer \(k\) we define \(L_{\text{loc}, t}^{\{k, d-k+1\}} \subset L_{\text{loc}, t}\) in the usual way. For each \(t \in \mathcal{K}^{\{k, d-k+1\}}\) there is a forgetful map \(L_{\text{loc}, t}^{\{k, d-k+1\}} \to D^{mf, \{k, d-k+1\}}_{\text{loc}}\). Furthermore, for every morphism \((j, \varepsilon) : s \to t\)
these forgetful maps fit into a commutative diagram

\[
\begin{array}{ccc}
\mathcal{L}^{k,d-k+1}_{\text{loc,}t} & \rightarrow & \mathcal{L}^{k,d-k+1}_{\text{loc,}a} \\
\downarrow & & \downarrow \\
\mathcal{D}^{\text{inf}, [k,d-k+1]}_{\text{loc}} & \rightarrow & \mathcal{D}^{\text{inf}, [k,d-k+1]}_{\text{loc}}
\end{array}
\]

and thus the forgetful maps induce a map, \( \text{hocolim}_{t \in \mathcal{K}^{[k,d-k+1]}} \mathcal{L}^{k,d-k+1}_{\text{loc,}t} \rightarrow \mathcal{D}^{\text{inf}, [k,d-k+1]}_{\text{loc}} \). The following proposition is proven in the same way as \cite[Proposition 5.16]{[12]}

**Proposition A.2.** For all \( k \), the map \( \text{hocolim}_{t \in \mathcal{K}^{[k,d-k+1]}} \mathcal{L}^{k,d-k+1}_{\text{loc,}t} \rightarrow \mathcal{D}^{\text{inf}, [k,d-k+1]}_{\text{loc}} \) is a weak homotopy equivalence.

We now need to compare \( \text{hocolim}_{t \in \mathcal{K}^{[k,d-k+1]}} \mathcal{L}^{k,d-k+1}_{\text{loc,}t} \) to \( \text{hocolim}_{t \in \mathcal{K}^{[k,d-k+1]}} \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \).

**Definition A.3.** For \( t \), define \( \mathcal{W}^{\text{loc,}t}_{\text{loc}} \) to be the subspace of \( G_m^0(\mathbb{R}^\infty)_{\text{loc}}^t \), consisting of those \( (V,\sigma) \) such that \( d(i) = \text{index}(\sigma(i)) \) for all \( i \in t \). As with \( \mathcal{W}^{\text{loc,}t} \), the correspondence \( t \mapsto \mathcal{W}^{\text{loc,}t} \) defines a contravariant functor on \( \mathcal{K} \). The subspace \( \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \subset \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \) is defined in the same way as before.

**Proposition A.4.** For all \( t \in \mathcal{K}^{[k,d-k+1]} \), the forgetful map

\( \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \rightarrow \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \) is a weak homotopy equivalence.

**Proof.** We first observe that the map \( \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \rightarrow \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \) is a Serre fibration. The lemma then follows from the fact that the fibre over a point \( (V,\sigma) \) is given by the space of embeddings \( D(V^-) \times_t D(V^+) \rightarrow \mathbb{R}^\infty \), which is a weakly contractible space. \( \square \)

We now consider the map

(A.3) \( \mathcal{L}^{k,d-k+1}_{\text{loc,}t} \rightarrow \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \), \( ((\tilde{x};V,\sigma),\delta,\phi) \mapsto ((V,\sigma)|_{\delta^{-1}(0)}) \circ \phi \),

viewing \( ((V,\sigma)|_{\delta^{-1}(0)}) \circ \phi \) as a function from \( t \) to \( G_m^0(\mathbb{R}^\infty)_{\text{loc}} \). This clearly defines a natural transformation of functors on \( \mathcal{K}^{(k,d-k+1)} \) and thus induce a map

\[ \text{hocolim}_{t \in \mathcal{K}^{[k,d-k+1]}} \mathcal{L}^{k,d-k+1}_{\text{loc,}t} \rightarrow \text{hocolim}_{t \in \mathcal{K}^{[k,d-k+1]}} \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \].

The following proposition is proven in the same way as \cite[Proposition 5.20]{[12]}

**Proposition A.5.** For all \( t \in \mathcal{K}^{[k,d-k+1]} \), the map \( \mathcal{L}^{k,d-k+1}_{\text{loc,}t} \rightarrow \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \) from (A.3) is a weak homotopy equivalence.

It follows from the above proposition that the induced map

\( \text{hocolim}_{t \in \mathcal{K}^{[k,d-k+1]}} \mathcal{L}^{k,d-k+1}_{\text{loc,}t} \rightarrow \text{hocolim}_{t \in \mathcal{K}^{[k,d-k+1]}} \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \)

is a weak homotopy equivalence. Combining this with Propositions A.2 and A.4 yields the zig-zag of weak homotopy equivalences

\[ \mathcal{D}^{\text{inf}, [k,d-k+1]}_{\text{loc}} \overset{\sim}{\longrightarrow} \text{hocolim}_{t \in \mathcal{K}^{[k,d-k+1]}} \mathcal{L}^{k,d-k+1}_{\text{loc,}t} \overset{\sim}{\longrightarrow} \text{hocolim}_{t \in \mathcal{K}^{[k,d-k+1]}} \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \overset{\sim}{\longrightarrow} \text{hocolim}_{t \in \mathcal{K}^{[k,d-k+1]}} \mathcal{W}^{k,d-k+1}_{\text{loc,}t} \]

This establishes the bottom row of weak equivalences from the statement of Theorem 3.10.
A.2. The long trace. We now proceed to establish the top row of the diagram. This will require the use of some new constructions to be carried out over the course of the next three subsections. We begin by recalling a construction from [12 Section 5.2]. Let \((V, \sigma) \in G^\text{mf}_0(\mathbb{R}^\infty)_\text{loc}\). Observe that the function

\[ f_V : V \to \mathbb{R}, \quad f_V(v) := \sigma(v, v) \]

is a Morse function on \(V\) with exactly one critical point at the origin with index equal to \(\text{index}(\sigma)\).

**Definition A.6.** Let \((V, \sigma) \in G^\text{mf}_0(\mathbb{R}^\infty)_\text{loc}\). The subspace \(\text{sdl}(V, \sigma) \subset V\) is defined by

\[ \text{sdl}(V, \sigma) = \{ v \in V \mid ||v_+||^2 ||v_-||^2 \leq 1 \}, \]

where \(v = (v_-, v_+)\) is the coordinate representation of \(v\) using the splitting \(V = V^- \oplus V^+\) given by the negative and positive eigenspaces of \(\sigma\). The norm on \(V\) is the one induced by the Euclidean inner product on the ambient space in which the subspace \(V\) is contained.

Given \((V, \sigma) \in G^\text{mf}_0(\mathbb{R}^\infty)_\text{loc}\), the formula

\[ v \mapsto (f_V(v), ||v_-||v_+, ||v_-||^{-1}v_-) \]

defines a smooth embedding

\[ \phi : \text{sdl}(V, \sigma) \backslash V^+ \to \mathbb{R} \times D(V^+) \times S(V^-), \]

with complement \([0, \infty) \times \{0\} \times S(V^-)\). It respects boundaries and is a map over \(\mathbb{R}\), where we use the restriction of \(f_V\) on the source and the function \((t, x, y) \mapsto t\) on the target.

For \(t \in K\), let \((V, \sigma) \in \hat{W}_{\text{loc}, t}\). We may form the space

\[ \text{sdl}(V, \sigma; t) = \bigsqcup_{i \in t} \text{sdl}(V(i), \sigma(i)). \]

We let

\[ V^\pm(t) \subset \text{sdl}(V, \sigma; t) \]

denote the subspace given by the disjoint union \(\bigsqcup_{i \in t} V^\pm(i)\). By applying the embedding \(\phi\) with \(i \in t\) over each component

\[ \text{sdl}(V(i), \sigma(i)) \subset \text{sdl}(V, \sigma, t), \]

we obtain an embedding

\[ \phi_t : \text{sdl}(V, \sigma; t) \backslash V^+(t) \to \mathbb{R} \times D(V^+) \times_t S(V^-), \]

which has complement \([0, \infty) \times \{0\} \times_t S(V^-)\).

**Construction A.1.** Let \((M, (V, \sigma), e) \in W_{Pt}\). The submanifold

\[ \text{Trc}(e) \subset \mathbb{R} \times \mathbb{R}^\infty \]

is defined to be the pushout of the diagram

\[ \text{sdl}(V, \sigma; t) \backslash V^+(t) \quad \xrightarrow{\quad} \mathbb{R} \times M \setminus ([0, \infty) \times e(\{0\} \times_t S(V^-))) \]

\[ \text{sdl}(V, \sigma; t). \]

The \(\theta\)-orientation \(\hat{l}_V\) on \(V\) together with the \(\theta\)-structure \(\hat{l}_M\) on \(M\) determines a \(\theta\)-structure \(\hat{l}_{\text{Trc}(e)}\) on \(\text{Trc}(e)\). The height function

\[ \text{Trc}(e) \to \mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R} \]
is Morse. It has one critical point for each \( i \in t \), with index equal to \( \dim(V^-(i)) \), all of which have value 0. It follows that \( \text{Trc}(e) \) equipped with its \( \theta \)-structure \( \hat{\ell}_{\text{Trc}(e)} \) determines an element of the space \( W_{\theta,P} \). The correspondence \( (M,(V,\sigma),e) \mapsto \text{Trc}(e) \) defines a map

\[ \text{Trc} : W_{P,t} \to D_{\theta,P}^{mf} . \]

It is easily verified that for each integer \( k \), the map \( \text{Trc} \) restricts to a map, \( \text{Trc}_k : W^k_{P,t} \to D^k_{\theta,P} \).

A.3. Couplings. The following definition is analogous to [12] Definition 5.24. First we must fix some notation. If \((\bar{x};(V,\sigma)) \in D^\text{mf}_{\theta,\text{loc}} \), we may form the space

\[ \text{sdl}(V,\sigma;\bar{x}) = \bigcup_{x \in \bar{x}} \text{sdl}(V(x),\sigma(x)) . \]

Let \( h_\bar{x} : \bar{x} \to \mathbb{R} \) denote the height function, i.e. the projection \( \bar{x} \to \mathbb{R} \times \mathbb{R}^\infty \xrightarrow{\text{proj}} \mathbb{R} \). The function

\[ f_{V,\bar{x}} : \text{sdl}(V,\sigma;\bar{x}) \to \mathbb{R} \]

is defined by the formula

\[ f_{V,\bar{x}}(z) = f_{V(x)}(z) + h_\bar{x}(x), \]

if \( z \) is contained in the component \( \text{sdl}(V,\sigma;\bar{x})|_x \), where \( f_{V(x)} : \text{sdl}(V(x),\sigma(x)) \to \mathbb{R} \) is the function from \([A.4]\).

**Definition A.7.** Let \( W \in D^\text{mf}_{\theta,P} \) and \((\bar{x};(V,\sigma)) \in D^\text{mf}_{\theta,\text{loc}} \) be elements with \( L(W) = (\bar{x};V,\sigma) \), where recall that \( L \) is the localization map \([A.3]\). A **coupling** between \( W \) and \((\bar{x};V,\sigma)\) is an embedding

\[ \lambda : \text{sdl}(V,\sigma;\bar{x}) \to \mathbb{R} \times \mathbb{R}^\infty \]

that satisfies the following conditions:

(i) \( \lambda(\text{sdl}(V,\sigma;\bar{x})) \subseteq W; \)

(ii) \( f_{V,\bar{x}} = h_W \circ \lambda \), where recall that \( h_W : W \to \mathbb{R} \) is the height function on \( W; \)

(iii) \( \hat{\ell}_{\text{sdl}(V,\sigma)} = \lambda^* \hat{\ell}_W \).

Using the definition of a coupling we define a new space as follows:

**Definition A.8.** The space \( L_{\theta,P} \) consists of tuples \((W,(\bar{x};V,\sigma),\lambda) \) where \( W \in D^\text{mf}_{\theta,P} \) and \((\bar{x};V,\sigma) \in D^\text{mf}_{\theta,\text{loc}} \) are elements with \( L(W) = (\bar{x};V,\sigma) \), and \( \lambda \) is a coupling between these two elements. For each integer \( k \), we define \( L^k_{\theta,P} \subseteq L_{\theta,P} \) to be the subspace consisting of those \((W,(\bar{x};V,\sigma),\lambda) \) for which \( W \in D^k_{\theta,P} \) and \((\bar{x};V,\sigma) \in D^k_{\theta,\text{loc}} \).

The following proposition is proven in the same way as [12] Proposition 5.28.

**Proposition A.9.** For all \( k \) the forgetful map

\[ L^k_{\theta,P} \to D^k_{\theta,P} , (W,(\bar{x};V,\sigma),\lambda) \mapsto W \]

is a weak homotopy equivalence.

We need to define one more space sitting in between the spaces \( L^k_{\theta,P} \) and \( \text{hocolim}_{t \in K^k} W^k_{P,t} \). The following definition is essentially the same as [12] Definition 5.30.

**Definition A.10.** Fix \( t \in K \). The space \( L_{P,t} \) consists of tuples \((W,(\bar{x};V,\sigma),\lambda,\delta,\phi) \) where:

- \((W,(\bar{x};V,\sigma),\lambda) \in L_{\theta,P}; \)

\begin{itemize}
\item $\delta : \bar{x} \rightarrow \{-1, 0, +1\}$ is a function;
\item $\phi : t \xrightarrow{\delta^{-1}} \delta^{-1}(0)$ is a function over the set $\{0, 1, \ldots, d + 1\}$;
\end{itemize}
subject to the following conditions: the height function, $h_\bar{x} : \bar{x} \rightarrow \mathbb{R}$, admits a lower bound on $\delta^{-1}(+)$ and an upper bound on $\delta^{-1}(-1)$.

We need to describe how the correspondence $t \mapsto \mathcal{L}_{P, t}$ defines a contravariant functor on $\mathcal{K}$. Let $(j, \varepsilon) : s \mapsto t$ be a morphism in $\mathcal{K}$. The induced map $(j, \varepsilon)^* : \mathcal{L}_{P, t} \rightarrow \mathcal{L}_{P, s}$ sends an element $(W, (\bar{x}; V, \sigma), \lambda, \delta, \phi) \in \mathcal{L}_{P, t}$ to $(W, (\bar{x}; V, \sigma), \lambda, \delta', \phi')$ where $\delta' = \delta \circ j$ and

$$\delta'(y) = \begin{cases} 
\varepsilon(s) & \text{if } y = \phi(s) \text{ where } s \in s \setminus j(t), \\
\delta(y) & \text{otherwise.}
\end{cases}
$$

(A.8)

For each integer $k$ we define $\mathcal{L}^k_{P, t} \subset \mathcal{L}_{P, t}$ in the usual way. For each $t \in \mathcal{K}^k$ there is a forgetful map $\mathcal{L}_{P, t}^k \rightarrow \mathcal{L}_{P, t}^k$. Furthermore, for every the morphism $(j, \varepsilon) : s \mapsto t$ these forgetful maps fit into a commutative diagram

$$\begin{array}{ccc}
\mathcal{L}_{P, t}^k & \xrightarrow{s} & \mathcal{L}_{P, s}^k \\
\downarrow & & \downarrow \\
\mathcal{L}_{P, t}^k & \xrightarrow{j} & \mathcal{L}_{P, t}^k
\end{array}
$$

and thus the forgetful maps induce a map, $\text{hocolim}_{t \in \mathcal{K}^k} \mathcal{L}_{P, t}^k \rightarrow \mathcal{L}_{P, t}^k$. The following proposition is proven by running the same argument from the proof of [12 Proposition 5.16].

**Proposition A.11.** For all $k$, the map, $\text{hocolim}_{t \in \mathcal{K}^k} \mathcal{L}_{P, t}^k \rightarrow \mathcal{L}_{P, t}^k$, is a weak homotopy equivalence.

We have weak homotopy equivalences

$$\mathcal{D}_{P, t}^{mf, k} \xrightarrow{\simeq} \mathcal{L}_{P, t}^k \xrightarrow{s} \text{hocolim}_{t \in \mathcal{K}^k} \mathcal{L}_{P, t}^k.$$

It remains to construct a weak homotopy equivalence between $\text{hocolim}_{t \in \mathcal{K}^k} \mathcal{L}_{P, t}^k$ and $\text{hocolim}_{t \in \mathcal{K}^k} \mathcal{W}_{P, t}^k$. Defining the required map will require a new construction which we carry out in the following subsection. This construction is essentially the same as what was done in [12 Pages 893-894].

**A.4. Regularization.** Choose once and for all a diffeomorphism $\psi : \mathbb{R} \rightarrow (-\infty, 0)$ such that $\psi(t) = t$ for $t < -\frac{1}{2}$, and a smooth nondecreasing function $\varphi : [0, 1] \rightarrow [0, 1]$ such that $\varphi(x) = x$ for $x$ close to 0, and $\varphi(x) = 1$ for $x$ close to 1. Let

$$\psi_x(t) = \varphi(x) t + (1 - \varphi(x)) \psi(t)$$

for $x \in [0, 1]$. Then $\psi_0 = \psi$ embeds $\mathbb{R}$ in $\mathbb{R}$ with image $(-\infty, 0)$, whereas each $\psi_x$ for $x > 0$ is a diffeomorphism $\mathbb{R} \rightarrow \mathbb{R}$. Let $(V, \sigma) \in G^{mf}_{\theta}(\mathbb{R}^\infty)_{\text{loc}}$. Recall form (A.4) the Morse function $f_V : V \rightarrow \mathbb{R}$. We define functions

$$f^+_V : \text{sd}(V, \sigma) \setminus V^+ \rightarrow \mathbb{R},$$

$$f^-_V : \text{sd}(V, \sigma) \setminus V^- \rightarrow \mathbb{R},$$

by the formulae

(A.9)

$$f^+_V(v) = \psi_x^{-1}(t),$$

$$f^-_V(t) = (-\psi_x)^{-1}(-t),$$
where $t = f_V(v)$ and $x = \|v_\pm\|^2/v_\mp^2$. These functions agree with $f_V$ on open subsets that contain the entire boundary and the sets
\[
\{\omega \in \text{sdl}(V, g) | f_V(\omega) \leq -1\},
\{\omega \in \text{sdl}(V, g) | f_V(\omega) \geq +1\},
\]
respectively. The following proposition can be verified by hand.

**Proposition A.12.** The functions $f^\mp_V$ defined in (A.9) are proper submersions.

Let $(\bar{x}; (V, \sigma)) \in \mathcal{D}^{mf}_{\theta, \text{loc}}$. The functions (A.9) determine functions
\[
\begin{align*}
&f^+_V (V, \sigma; \bar{x}) \setminus V^+ \rightarrow \mathbb{R}, \\
&f^-_V (V, \sigma; \bar{x}) \setminus V^- \rightarrow \mathbb{R},
\end{align*}
\]
(A.10)
defined by
\[
f^\pm_V (z) = f^+_V (z) + h(x)
\]
for $z$ on $\text{sdl}(V, g)|_\bar{x}$ for each $x \in \bar{x}$. By Proposition A.12 it follows that the functions $f^\mp_V$ are proper submersions as well. In the construction below, we use these functions to define a map $\mathcal{L}_{P_t} \rightarrow W_{P_t}$.

**Construction A.2.** Let $(W, (\bar{x}; V, \sigma), \lambda, \delta, \phi)$ of $\mathcal{L}_{P_t}$. Let $(V_{+1}, \sigma_{+1}), (V_{-1}, \sigma_{-1})$, and $(V_0, \sigma_0)$ denote the restrictions of $(V, \sigma)$ to the subsets
\[
\delta^{-1}(+1), \delta^{-1}(-1), \delta^{-1}(0) \subset \bar{x}.
\]
Let us denote the submanifold
\[
W^{rg} = W \setminus \lambda(V_{+1}^+ \cup V_0^+ \cup V_{-1}^-).
\]
We define a function $f^{rg}: W^{rg} \rightarrow \mathbb{R}$ by
\[
f^{rg}(z) = \begin{cases} 
  f(z) & \text{if } z \in \text{Im}(\lambda), \\
  f^+_V (v) & \text{if } z = \lambda(v) \text{ and } v \in V_+ \cup V_0, \\
  f^-_V (v) & \text{if } z = \lambda(v) \text{ and } v \in V_-.
\end{cases}
\]
(A.11)
Let $j_{W^{rg}}: W \rightarrow (-\infty, 0] \times \mathbb{R}^{\infty-1}$ denote the composite
\[
W \rightarrow \mathbb{R} \times (-\infty, 0] \times \mathbb{R}^{\infty-1} \rightarrow (-\infty, 0] \times \mathbb{R}^{\infty-1}.
\]
We re-embed $W^{rg}$ into $\mathbb{R} \times (-1, 0] \times (-1, 1)^{\infty-1}$ using the product map
\[
f^{rg} \times j_{W^{rg}}: W^{rg} \rightarrow \mathbb{R} \times (-\infty, 0] \times \mathbb{R}^{\infty-1},
\]
and we let $\tilde{W} \subset \mathbb{R} \times (-\infty, 0] \times \mathbb{R}^{\infty-1}$ denote its image. By construction, the height function
\[
h_{\tilde{W}}: \tilde{W} \rightarrow \mathbb{R}
\]
is a proper submersion, and thus
\[
M := h_{\tilde{W}}^{-1}(0)
\]
is a compact $d$-dimensional submanifold of $\{0\} \times (-\infty, 0] \times \mathbb{R}^{\infty-1}$ with $\partial M = P$. The restriction of $\lambda$ gives an embedding
\[
c: D(V_0^+) \times_t S(V_0^-) \rightarrow M.
\]
This embedding together with $M$ and $(V_0, \sigma_0)$ determines an element
\[
(M, (V_0, \sigma_0), c) \in W_{P_t}.$
The correspondence, \((W, (\bar{x}; V, \sigma), \lambda, \delta, \phi) \mapsto (M, (V_0, \sigma_0), e)\), defines a map
\[(A.12) \quad L^k_{P,t} \longrightarrow W^k_{P,t}.\]

It is easily verified that for each \(k \in \mathbb{Z}_{\geq 0}\) this map restricts to a map, \(L^k_{P,t} \longrightarrow W^k_{P,t}\). By what was observed in [12], this map defines a natural transformation of functors on \(\mathcal{K}\).

The following proposition is the same as [12] Proposition 5.36.

**Proposition A.13.** For all \(k\) and \(t\), the map \(L^k_{P,t} \longrightarrow W^k_{P,t}\) is a weak homotopy equivalence.

**Proof.** We use Construction [A.1] to define a homotopy inverse. Indeed, we define a map
\[(A.13) \quad W^k_{P,t} \longrightarrow L^k_{P,t}\]
by sending \((M, (V, \sigma), e)\) to the tuple \((\text{Trc}(M, (V, \sigma), e), (\bar{x}; V', \sigma'), \lambda, \delta, \phi)\), where:

(i) \(\text{Trc}(M, (V, \sigma), e)\) is defined as in Construction [A.1]

(ii) \(\bar{x} \subset \text{Trc}(M, (V, \sigma), e)\) is the set of critical points of the height function
\[\text{Trc}(M, (V, \sigma), e) \longrightarrow \mathbb{R}.\]

Induced by the construction of \(\text{Trc}(M, (V, \sigma), e)\) there is natural bijection, \(\bar{x} \cong t\).

(iii) \((V', \sigma')\) is given by the composite \(\bar{x} \cong t \xrightarrow{(V, \sigma)} G^\text{mf}_\theta(\mathbb{R})\text{loc}\), where the first map is the bijection from (ii).

(iv) \(\lambda\) is the coupling coming from the embedding \(\text{sd}(V, \sigma; t) \hookrightarrow \text{Trc}(M, (V, \sigma), e)\) used in the construction of \(\text{Trc}(M, (V, \sigma), e)\);

(v) \(\delta : \bar{x} \longrightarrow \{0\}\) is the constant function.

(vi) \(\phi : \delta^{-1}(0) = \bar{x} \cong t\) is the bijection from (iii).

By what was observed above, it follows that the map \([A.13]\) defines a homotopy inverse to \([A.13]\) (see also [12] Proposition 5.36]). \(\square\)

The above proposition implies that the natural transformation of \([A.12]\) induces a weak homotopy equivalence
\[\text{hocolim}_{t \in \mathbb{K}_t} L^k_{P,t} \cong \text{hocolim}_{t \in \mathbb{K}_t} W^k_{P,t}.\]

Putting this together with the weak homotopy equivalences from Propositions [A.9] and [A.11] we obtain the zig-zag of weak homotopy equivalencies
\[\text{hocolim}_{t \in \mathbb{K}_t} L^k_{\theta,P} \cong \text{hocolim}_{t \in \mathbb{K}_t} W^k_{P,t}.\]

This establishes the top row of the diagram [A.1]. Proving that it is commutative is a straightforward verification done by tracing through the constructions. This completes the proof of Theorem [3.10] which was the objective of this appendix.
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