On the crossing relation in the presence of defects

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The OPE of local operators in the presence of defect lines is considered both in the rational CFT and the $c > 25$ Virasoro (Liouville) theory. The duality transformation of the 4-point function with inserted defect operators is explicitly computed. The two channels of the correlator reproduce the expectation values of the Wilson and ’t Hooft operators, recently discussed in Liouville theory in relation to the AGT conjecture.

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1. Introduction

1.1. The problem

Topological defects are defined algebraically as operators commuting with the left and right copies of the chiral algebra \([1]\), and in particular are invariant under diffeomorphisms,

\[
\left[ L_n, X \right] = \left[ \bar{L}_n, X \right] = 0. \tag{1.1}
\]

We are concerned in this note with the 4-point crossing relation in the presence of defect operators: for trivial defects it reduces to the standard Belavin-Polyakov-Zamolodchikov duality relation for the correlators of local 2d fields. The presence of a defect line inserted between two local operators modifies their operator product expansion (OPE), resulting in creation of defect fields, and one is interested in the computation of the corresponding OPE coefficients. A special case of this extended BPZ relation has been exploited in \([2]\) to derive a general formula for the relative coefficients of the OPE of local fields of integer spin in the rational non-diagonal theories. Here we restrict to the diagonal theories where the computation of the duality transformation in the presence of defects is a straightforward consequence of two basic identities in CFT: the pentagon identity for the quantum 6j symbols (the fusing matrices) and the Moore-Seiberg \([3]\), \([4]\) torus identity, an equation for the 1-point modular matrix.

As a side result of this computation, extended to the non-rational \(c > 25\) Virasoro theory (Liouville CFT), one obtains an explicit general expression for the expectation value of the ’t Hooft loop operator in Liouville theory, defined as the dual to that of the Wilson loop operator. The formula essentially reproduces the recently proposed ad hoc expression \([5]\), \([6]\) and thus confirms the assumed duality of the two operators.

The effect of some defect lines as creating "disorder" fields when attached to the local fields was pointed out already in \([2]\). It was later thoroughly analysed in \([7]\) and in particular the precise conditions on the type of the defect fusion algebra leading to Kramers-Wanniers type duality in the rational case were described; for more on the defect fields from a TFT point of view see \([8]\), \([9]\). A topological defect interpretation of the Wilson loop operator in the 2d rational theories has been discussed e.g. in \([10]\), \([11]\). The construction of the loop operators in \([5]\), \([6]\) does not refer to defects.\(^1\)

\(^1\) The present work was in an advanced stage when it was announced \([12]\) that a parallel work on the defect interpretation of the construction in \([3]\), \([4]\) is under way.
1.2. Preliminaries on the topological defects

Let us summarise some of the consequences of the definition (1.1) studied in [1], [2]. In a rational CFT with a set \( \{I \ni j\} \) of representations the solutions of (1.1) read

\[
X_x = \sum_{j,\bar{j}} \frac{\mathcal{Z}_{j,\bar{j}}}{\sqrt{S_{1j}S_{1\bar{j}}}} \Psi^{(j,\bar{j};\alpha,\alpha')} P^{(j,\bar{j};\alpha,\alpha')},
\]

(1.2)

where \( P^{(j,\bar{j};\alpha,\alpha')} \) are projectors in the representation spaces \( (\mathcal{V}_j \otimes \bar{\mathcal{V}}_{\bar{j}})^{\beta} \), and the sum is restricted to pairs \((j,\bar{j})\), allowed by the nonzero values of the given modular invariant matrix \( \sum_{j,\bar{j}} Z^2_{j,\bar{j}} \). As in the computation of the cylinder partition function, which leads to non-negative integer matrix representations (nim-reps) \( n_{ja}^b \) of the Verlinde algebra [13], [14]

\[
n_i n_j = \sum_s N_{ij}^s n_s, \quad T n_j = n_j^*, \quad j \in I
\]

\[
\Leftrightarrow n_{ja}^b = \sum_{l,\bar{l},\alpha,\alpha'} \frac{S_{il}}{S_{1l}} \psi^{(l,\alpha)} \psi_{l,\alpha'}^{(l,\alpha)^*}, \quad \psi \psi^\dagger = Id = \psi^\dagger \psi, \quad \text{(1.3)}
\]

one constructs partition functions on the torus \( Z_{x_1 x_2 \ldots x_{n-1}} \) (or, on the cylinder \( Z_{b|x_1 x_2 \ldots x_n,a} \)), inserting arbitrary number of defect operators. They are sesquilinear (respectively linear) combinations of the characters of the chiral algebra representations with non-negative integer coefficients \( \mathcal{\tilde{V}}_{ij;x_1 \ldots x_{n-1}}^{x^n}, i,\bar{i} \in I \) (resp \( n_{ja}^b \)). The case of two defects on the torus leads to an equation for the multiplicities analogous to (1.3)

\[
\mathcal{\tilde{V}}_{i_1 j_1} \mathcal{\tilde{V}}_{i_2 j_2} = \sum_{i_3,j_3} N_{i_1 i_3} N_{j_1 j_3} \mathcal{\tilde{V}}_{i_3 j_3}, \quad \mathcal{\tilde{V}}_{ij;x_1 \ldots x_{n-1}}^{x^n} = Z_{ij}, \quad T \mathcal{\tilde{V}}_{ij} = \mathcal{\tilde{V}}_{ij^*},
\]

\[
\Leftrightarrow \mathcal{\tilde{V}}_{ij;x} = \sum_{l,\bar{l},\alpha,\alpha'} \frac{S_{il}S_{\bar{l}l}}{S_{1l}S_{1\bar{l}}} \psi^{(l,\alpha,\alpha')} \psi_{y}^{(l,\alpha;\alpha')^*}.
\]

(1.4)

The classification of the topological defects amounts in the classification of the NIM-reps (1.4). In the sl(2) related cases it confirms the results of Ocneanu, visualised by his generalised ADE diagrams, with vertices associated with the set of defects [15]. Another distinguished set of non-negative integers \( \mathcal{\tilde{N}}_{ij;x} \) is provided by the identity contribution of left and right characters in the torus partition function with three inserted defects, \( Z_{x_1 x_2 x_3} \). This set serves as structure constants of an associative, in general non-commutative, algebra, the fusion algebra of defects (=Ocneanu graph algebra), as it allows to compute the fusion of two defects

\[
X_{x} X_y = \sum_{z} \mathcal{\tilde{N}}_{yz} X_z.
\]

(1.5)
The action of the defects on the boundary states

\[ X_x |a\rangle = \sum_c \tilde{n}_{ax}^c |c\rangle , \]

\[ \tilde{n}_{ax}^c = \sum_{j,\alpha,\beta} \psi^{(j,\alpha)}_{a} \Psi^{(j,\alpha,\beta)} \psi^{(j,\beta)*} \sqrt{S_{1j} S_{1j}} , \alpha, \beta = 1, \ldots Z_{j,j} \]

introduces another set of non-negative integers \( \{ \tilde{n}_{ax}^c \} \), interpreted as the multiplicities \( n_{1;x;a}^c = \tilde{n}_{ax}^c \) of the identity character in the cylinder partition functions in the presence of one defect, \( Z_{b;x,a} = \sum_j n_{j;x;a}^b \chi_j(\tau) = \sum_j (\tilde{n}_x n_j)_a^b \chi_j(\tau) \). Combining relations (1.5) and (1.6) implies that this set of matrices provides NIM-reps of the Ocneanu algebra

\[ \tilde{n}_x \tilde{n}_y = \sum_z \tilde{N}_{xy}^z \tilde{n}_z . \]

Thus the computation of the partition functions on the torus and the cylinder with an arbitrary number of defects inserted is reduced to the knowledge of several basic structure constants, or, equivalently, the knowledge of the sets of unitary matrices \( \{ \psi, \Psi \} \) in (1.3), (1.4), e.g.,

\[ \tilde{V}_{ij^*;x}^z = \sum_y \tilde{N}_{xy}^z \tilde{V}_{ij^*;1}^y , \tilde{V}_{ij^*;x_1x_2}^z = \sum_u \tilde{N}_{x_1x_2}^u \tilde{V}_{ij^*;1}^u \tilde{V}_{ij^*;u}^z . \]

In the field interpretation, these multiplicities encode the possible holomorphic - antiholomorphic content \((i, j)\) of defect fields; in general they correspond to non-local 2d fields.

In the case of a ”diagonal” theory, i.e., described by a modular matrix with \( Z_{j,j} = \delta_{j,j} \), the sets of defects and boundaries can be identified with the set \( \mathcal{I} \) of representations of the chiral algebra and \( n, \tilde{n}, \tilde{N} \) coincide with the Verlinde multiplicities \( \mathcal{N} \), while

\[ \tilde{V}_{ij} = \mathcal{N}_i \mathcal{N}_j , \tilde{V}_{ij;1}^y = \mathcal{N}_i \mathcal{N}_j^y , \]

etc. Accordingly the matrices diagonalising these multiplicities reduce to the modular matrix, \( \Psi^{(j,j);\alpha,\beta} = \delta_{j,j} \delta_{\alpha,1} \delta_{\beta,j} S_{xj} \), and the eigenvalue in the r.h.s. of (1.2) is expressed by the ratio of modular matrix elements \( S_{xj}/S_{1j} \). In the diagonal case the action of the defect operator \( X_x \) (on \( (\mathcal{V}_j \otimes \bar{\mathcal{V}}_j) \)) coincides with the action of its chiral analog acting on \( \mathcal{V}_j \) (one to one with the Ishibashi states)

\[ X_x^I = \sum_j \frac{S_{xj}}{S_{1j}} \sum_k |j,k\rangle, \langle j,k| \]

\[ \text{etc.} \]
In the WZW case this operator can be identified \[11\] with a generalised Casimir operator \[16\] (see also \[17\] for the Virasoro minimal models) giving precise meaning of the Wilson loop operator \[10\]. The twisted partition function in the diagonal case can be interpreted alternatively in terms of related operators, associated with the two cycles of the torus

\[
\hat{X}_x(a)\chi_p(\tilde{\tau}) = \text{Tr}_p(e^{2\pi i \tilde{\tau}(L_0-c/24)}X^I_x) = \frac{S_{xp}}{S_{1p}}\chi_p(\tilde{\tau})
\]

\[
\hat{X}_x(b)\chi_p(\tau) = \sum_i S_{pi}^x S_{xi}^p S_{1i} \chi_i(\tilde{\tau}) = \sum_s N_{xp}^s \chi_s(\tau)
\]

This reproduces the monodromy operators in the derivation of the Verlinde formula \[18\].

In the Liouville case the sum in (1.2) is replaced by an integral over a continuous series of representations. The modular matrices were computed in \[19\] in the two basic cases - with the second representation \(x\) also belonging to the continuous principal series, or, with \(x\) belonging to the \(c > 25\) infinite discrete degenerate series. The latter is parametrised by a pair \((m,n)\) of positive integers, i.e., \(x\) labels a factor representation of a degenerate Verma module of scaling dimension \(\triangle(x) = x(Q-x)\), with \(x = x_{mn} = -\frac{m-1}{2}b - \frac{n-1}{2b}\), \(Q = b + \frac{1}{b}\). Thus, there are two types of defects in the Liouville theory, with two different ratios of modular matrix elements in (1.2), corresponding to the two types FZZ \[20\] or ZZ \[19\], respectively, of boundary states; this has been discussed in detail recently in \[21\].

For the purpose of the comparison with recent work \[5\], \[6\] on the relation of Liouville theory to the 4d supersymmetric gauge theories \[22\], we perform in more detail also the explicit computation in the Liouville case. We will restrict mostly to the ZZ case, a quasi-rational theory, similar in many respects to the rational \(c < 1\) theory. The main section 2 deals with the diagonal rational case. The technical difference with the Liouville theory is in the use of different normalisation of the chiral vertex operators - traditionally in the minimal models the \(\mathbb{Z}_2\) symmetry is effectively fixed, the reflected operators are identified, and in particular one can use bases of conformal blocks for which the fusing matrices are unitary. The three parts of section 3 deal with the Liouville case and contain: a collection of basic formulae (sect. 3.1); more details on the Liouville torus identity and the crossing relation in the Liouville case (sect. 3.2); an explicit example compared with the proposed expression for the expectation value of the ’t Hooft operator in \[5\], \[6\] (sect. 3.3). The Appendix recalls the OPE formula of \[2\].
2. The duality relation in the rational case

For simplicity of notation we shall restrict to the \( \text{sl}(2) \) case, but will keep the conjugation so that the higher rank generalisation is straightforward. In the rational case one can choose a unitary fusing matrix \( F \).

As in [2] we consider a 4-point function of four local fields with insertion of two defects; for the purpose here we shall restrict to the diagonal case with scalar fields; the second label will be suppressed,

\[
G = \langle 0 | \Phi_{a_4}(\bar{z}_4, \bar{z}_4) \Phi_{a_2}(\bar{z}_2, \bar{z}_2) X_x \Phi_{a_1}(\bar{z}_1, \bar{z}_1) X_x | 0 \rangle
\]

(2.1)

In the second line we have used that in this channel the defects are diagonalised, applying the OPE expansion of the local fields and inserting the diagonal version of the general formula for the 2-point function

\[
\langle 0 | \Phi_{(\star, \alpha)} X_x \Phi_{(\star, \beta)} | 0 \rangle = \delta_{\gamma, \gamma'} \delta_{\delta, \delta'} \frac{\Psi_{x}^{(J, \alpha)}(z_{a_3}, \bar{z}_{a_3})}{\Psi_{1}^{(J, \beta)}(z_{a_4}, \bar{z}_{a_4})} \langle 0 | \Phi_{(\star, \alpha)} | 0 \rangle.
\]

(2.2)

In the third line of (2.1) we have performed the braiding of the chiral blocks. In the new channel the defect modifies the OPE expansions of two local fields

\[
G = \langle 0 | \Phi_{a_3}(z_3, \bar{z}_3) X_x \Phi_{a_2}(z_2, \bar{z}_2) \Phi_{a_1}(z_1, \bar{z}_1) X_x \Phi_{a_4}(z_4, \bar{z}_4) | 0 \rangle.
\]

(2.3)

It remains to perform the initial summation over \( j \) in (2.1), i.e.,

\[
A^{(x)}_{\gamma, \delta} = \sum_{j} \frac{S_{xj}}{S_{1j}} F_{j\gamma} \left[ a_4 a_1 \atop a_3 a_2 \right] F_{j\delta}^* \left[ a_4 a_1 \atop a_3 a_2 \right], \quad \text{with } A^{(1)}_{\gamma, \delta} = \delta_{\gamma, \delta}.
\]

(2.4)

The first step is, using a standard relation derived from the pentagon identity, to rewrite the product of F matrices (2.4) as

\[
F_{j\gamma} \left[ a_4 a_1 \atop a_3 a_2 \right] F_{j\delta}^* \left[ a_4 a_1 \atop a_3 a_2 \right] = d_j \sqrt{d_{\gamma} d_{\delta}} F_{a_1 a_3} \left[ a_2 \gamma \atop j a_4 \right] F_{a_4 a_2} \left[ a_3^* \delta^* \atop j a_1 \right].
\]

(2.5)
We have also used the relation derived from the pentagon identity, taking into account the symmetries of the 6j symbols and using that \( F_{11} \begin{bmatrix} a & a^* \\ a & a \end{bmatrix} = 1/d_a \):

\[
F_{1m} \begin{bmatrix} j & i \\ j & i^* \end{bmatrix} F_{m1} \begin{bmatrix} i^* & i \\ j & j \end{bmatrix} = \frac{d_m}{d_j d_j} N_{ji}^m. \tag{2.6}
\]

It implies that for a unitary \( F \) the values of \( F_{1c} \) and \( F_{y1} \) are determined up to a sign in terms of a square root of a ratio of q-dimensions; the positive sign is chosen. We next apply the pentagon identity itself, representing the product in (2.5) as a sum

\[
F_{a_1^* a_2} \begin{bmatrix} a_3^* & \delta^* \\ j & a_1 \end{bmatrix} F_{a_1 a_3^*} \begin{bmatrix} a_2 & \gamma \cr j & a_4^* \end{bmatrix} = \sum_y F_{ya_3^*} \begin{bmatrix} a_2 & \gamma \cr a_2 & \delta^* \end{bmatrix} F_{a_1^* y} \begin{bmatrix} \gamma & \delta^* \cr a_1 & a_1 \end{bmatrix} F_{a_1 a_2} \begin{bmatrix} a_2 & y \cr j & a_1 \end{bmatrix}
\]

\[= \sum_y F_{a_3^* y} \begin{bmatrix} \gamma^* & \delta \cr a_2 & a_2 \end{bmatrix} F_{a_4^* y} \begin{bmatrix} \gamma & \delta^* \cr a_1 & a_1 \end{bmatrix} F_{y^* j} \begin{bmatrix} a_2 & a_1 \cr a_2 & a_1^* \end{bmatrix} F_{y^* 1} \begin{bmatrix} j & j^* \cr a_2 & a_2 \end{bmatrix}. \tag{2.7}\]

Inserting this into (2.4) we can now perform the summation over \( j \), using the MS torus identity [3, 4]. It can be written e.g. as

\[
\sum_s S_{ri}(s) F_{qs} \begin{bmatrix} j_1 & j_2 \\ r & r \end{bmatrix} e^{2\pi i (\Delta_r - \Delta_y + \Delta_{j_1} + \Delta_{j_2} - \Delta_m)} F_{sm} \begin{bmatrix} i & j_1 \\ i & j \end{bmatrix} F_{mp} \begin{bmatrix} i & j \\ i & i \end{bmatrix} = S_{qi}(p) F_{rp} \begin{bmatrix} j_2 & j_1 \\ q & q \end{bmatrix}. \tag{2.8}\]

Taking \( r = 1 \), hence \( s = 1 \), gives an expression for the modular matrix \( S(p) \) of one point functions on the torus

\[
S_{ji}(p) = \frac{S_{1i}}{F_{1p} \begin{bmatrix} j & j^* \\ j & j \end{bmatrix}} \sum_m e^{i\pi (2(\Delta_r + \Delta_y - \Delta_m) - \Delta_p)} F_{1m} \begin{bmatrix} j & i \\ j & j^* \end{bmatrix} F_{mp} \begin{bmatrix} i & i \end{bmatrix}, \tag{2.9}\]

and \( S_{ji}^*(p) = e^{i\pi \Delta_p} S_{ji}(p^*) \). Inserting the transposed version of (2.9) in (2.8) - taken for \( p = 1 \), and inverting with \( F_{qs}^{-1} \) gives a ”fusion” like representation for the product of two S-matrices,

\[
\frac{d_x}{F_{y^* 1} \begin{bmatrix} a_2 & a_2^* \\ a_2 & a_2 \end{bmatrix}} \sum_j S_{xj} F_{y^* j} \begin{bmatrix} a_2 & a_1 \cr a_2^* & a_1^* \end{bmatrix} F_{a_1^* 1} \begin{bmatrix} j & j^* \cr a_2 & a_2 \end{bmatrix} = \frac{S_{a_2 x}(y^*)}{S_{11}} \frac{S_{a_1 x}(y)}{S_{11}} e^{i\pi \Delta_y} \frac{S_{a_2 x}(y)}{S_{11}} \frac{S_{a_1 x}(y)}{S_{11}}. \tag{2.10}\]

In particular, for \( y = 1 \) (2.10) reproduces, using (2.6), the Verlinde formula for the fusion multiplicity \( N \). Combining (2.5), (2.7), (2.10) we get finally for (2.4),

\[
d_x A_{\gamma, \delta}^{(x)} = \sqrt{\frac{d_x d_\delta}{d_{a_1} d_{a_2} d_{a_3} d_{a_4}} \sum_y F_{a_3 y} \begin{bmatrix} \gamma & \delta^* \\ a_2^* & a_2 \end{bmatrix} \frac{S_{a_2 x}(y)}{S_{11}} \frac{S_{a_1 x}(y)}{S_{11}} F_{a_4 y} \begin{bmatrix} \gamma & \delta^* \\ a_1 & a_1 \end{bmatrix}}. \tag{2.11}\]
The range of $y$ in (2.11) is determined by the multiplicities $N_{xy}^x, N_{ay}^a$ of the 1-point blocks on the torus, transformed by the 1-point modular matrix $S_{ax}(y), a = a_1, a_2$. We see that we can interpret $y$ as a defect label, appearing in the defect fusion multiplicity $N_{xy}^x$. In turn the values of $y$ restrict the possible pairs $(\gamma, \delta)$ of representations labelling the left and right conformal blocks in (2.1) as dictated by the multiplicity $N_{\gamma\delta}^y$, implicit in the $F$ matrices. Altogether this corresponds to the first of the relations in (1.8) for $z = x$,

$$\tilde{V}_{\gamma^*;x}^y = \sum_{y} \tilde{N}_{xy}^x \tilde{V}_{\gamma^*;1}^y = \sum_{y} N_{xy}^x N_{\gamma^*}^y. \quad (2.12)$$

In other words, $\tilde{V}_{\gamma^*;1}^y$ is the multiplicity of a defect field $y \Phi_{(\gamma,\delta)}(z,\bar{z})$ that is created in the OPE of two local operators, modified by the inserted defect line operator $X_x$ as in (2.3),

$$\Phi_{a_1}(z_1, \bar{z}_1) X_x \Phi_{a_4}(z_4, \bar{z}_4) = z_{14}^{-\Delta_{\gamma_1}^i} \bar{z}_{14}^{-\Delta_{\delta_1}^i} \sum_{\gamma,\delta,y} d_{a_1,a_4;x}^{(\gamma,\delta);y} \Phi_{(\gamma,\delta)}(z_2, \bar{z}_2) + \ldots \quad (2.13)$$

and the OPE coefficients in (2.13) are determined from (2.11) up to a normalisation of the defect field 2-point function. Alternatively, inserting

$$\delta_{y,y'} = \sum_q F_{yq} \left[ x \delta^* \right]_{\gamma} F_{y'q} \left[ x \delta^* \right]_{\gamma}$$

we can rewrite (2.11) in terms of the modular matrix transforming a 2-point chiral block on the torus

$$S_{i,a;x,q}(\gamma_1, \gamma_2) = F^{-1}(S \otimes 1)F = \sum_{y} F_{yq} \left[ x \gamma_2 \right]_{\gamma_1} S_{ix}(y) F_{aq} \left[ \gamma_1 \gamma_2 \right]_{i \ i} \quad (2.14)$$

In the higher rank cases it depends on four more indices, their range being determined by the fusion multiplicities $N_{\gamma_1 a}^i, N_{\gamma_2 i}^a$ and $N_{\gamma_1 q}^x, N_{\gamma_2 x}^q$ of the chiral vertex operators involved. For (2.11) we have

$$d_x A_{\gamma,\delta}^{(x)} = \sqrt{d_{\gamma} d_{\delta}} \sum_q \frac{S_{a_1,a_4;x,q}^{(\gamma,\delta)}}{\sqrt{d_{a_2} d_{a_3}}} \frac{S_{a_2,a_3;x,q}^{(\gamma,\delta)}}{\sqrt{d_{a_1} d_{a_4}}}. \quad (2.15)$$

The summation here corresponds to (2.12) rewritten as

$$\tilde{V}_{\gamma^*;x}^y = \sum_{y} \tilde{N}_{xy}^x \tilde{V}_{\gamma^*;1}^y = \sum_{q} N_{\gamma q}^x N_{\delta q}^y \quad (2.16)$$
Taking $\gamma = 1 = \delta$ in (2.11) represents the leading contribution of the identity block in (2.1), with the restrictions $a_3 = a_2^*, a_4 = a_1^*$. Using (2.2) this reproduces indeed the r.h.s. of (2.11) for these values.

In the analog of the chiral interpretation (1.11) here, the modular transformation of the characters is replaced by braiding (fusing) of the conformal blocks on the sphere

$$\hat{X}_x(a)G_p(\hat{z}) = \langle 0 | \phi^1_{a_4,a_1}(\hat{z}_4) \phi^4_{a_3,p}(\hat{z}_3) X^I_{a_2,a_1} (\hat{z}_2) \phi^a_{a_1,0}(\hat{z}_1) | 0 \rangle = \frac{S_{xp}}{S_{1p}} G_p(\hat{z})$$

and formula (2.11) gives alternative expression for $A_p^{(x)}$.

The MS torus identity has been encountered in the boundary CFT in [23], [14]; it has been observed that in the diagonal case the two Cardy-Lewellen bulk-boundary eqs [24], [25] both originate in this identity. The first equation corresponds to (2.8) (with boundary labels $q, r$ and $F_{rp}, F_{qs}$ substituted by the $3j$ symbols - the boundary field OPE coefficients), while the second equation is identified with the (transposed version of the) relation (2.10). Accordingly the bulk-boundary structure constant in the diagonal theory is proportional to the 1-point modular matrix.

The duality transformation relating the correlators (2.1) and (2.3) has been recently discussed in [21], following [2] and comparing with the permutation brane approach; this consideration does not yield, however, explicit formulae like (2.11), (2.15).

3. The Liouville case

3.1. Collection of Liouville formulae

The quantum $6j$ symbols $F$ for the Liouville theory have been computed in [26], [27]. It is convenient to change the normalisation of the chiral blocks, $(F)G_\beta \rightarrow G_\beta$

$$N(\beta_3, \beta_2, \beta_1) = \frac{\Gamma_b(Q)\Gamma_b(2\beta_1)\Gamma_b(2\beta_2)\Gamma(2Q-2\beta_3)}{\Gamma_b(2Q-\beta_123)\Gamma_b(\beta_123)\Gamma_b(\beta_23)\Gamma_b(\beta_123)}$$

($\beta_{123} = \sum_i \beta_i$, $\beta_{12}^3 = \beta_1 + \beta_2 - \beta_3$, etc.) so that $G_\beta$ transform with the matrices

$$G_{\beta_5, \beta_6} \begin{bmatrix} \beta_3 & \beta_2 \\ \beta_4 & \beta_1 \end{bmatrix} = \frac{N(\beta_6, \beta_3, \beta_2)N(\beta_4, \beta_6, \beta_1)}{N(\beta_4, \beta_3, \beta_5)N(\sigma_4, \beta_5, \beta_1)} F_{\beta_5, \beta_6} \begin{bmatrix} \beta_3 & \beta_2 \\ \beta_4 & \beta_1 \end{bmatrix}.$$
The Liouville bulk 3-point constant is then given by

\[ N(\beta_3, \beta_2, \beta_1)N(Q - \beta_3, Q - \beta_2, Q - \beta_1) = 2\pi \lambda \frac{\Gamma(2b)}{2\pi} \prod_i W(Q - \beta_i) C(\beta_3, \beta_2, \beta_1)^{-1}. \] (3.3)

Here \( \lambda := \frac{\pi \mu \Gamma(b^2)}{\Gamma(1 - b^2)} \) and we have used the ZZ variable \[ W(\alpha) = \frac{\Gamma_b(2\alpha)}{\Gamma_b(2\alpha - Q)} \lambda^{\frac{Q - 2\alpha}{b^2}} (= -2W(iP)^{ZZ}). \] (3.4)

Recall that the product of \( W(\alpha) \) in (3.4) and its reflected counterpart \( W(Q - \alpha) \) is proportional to a modular matrix element, while the ratio gives the bulk reflection amplitude,

\[ \frac{W(\alpha)W(Q - \alpha)}{W(\alpha)} = \frac{S_b(2\alpha)}{S_b(2\alpha - Q)} = -4 \sin \pi b(2\alpha - Q) \sin \frac{\pi Q}{b}(2\alpha - Q) =: S_{0\alpha} \]

\[ \frac{W(Q - \alpha)}{W(\alpha)} = \frac{\Upsilon_b(2\alpha)}{\Upsilon_b(2\alpha - Q)} \lambda^{\frac{Q - 2\alpha}{b^2}} = S(\alpha). \] (3.5)

More generally, for the case of a degenerate representation \( x \) and generic charge \( \alpha \) the modular matrix reads \[ S_{x, m, n} = -4 \sin \pi b m(2\alpha - Q) \sin \frac{\pi m}{b}(2\alpha - Q) = \hat{S}_{x, m, n} - \hat{S}_{x - m, n} \] (3.6)

where

\[ \hat{S}_{\beta \alpha} = 2\cos \pi (2\alpha - Q)(2\beta - Q) \] (3.7)

is the FZZ type modular matrix, computed for two generic representations.

The Weyl reflected charge in the second line of (3.6) corresponds to the only singular vector at generic \( b^2 \) of the reducible Virasoro module of highest weight \( \Delta(x_{m, n}) \). This relation, coming from the character formula for the degenerate representations, extends to other quantities of the theory, e.g., the corresponding fusion multiplicities, \( \hat{N}_{\alpha \beta \gamma} \) and \( N_{\alpha x_{m, n} \gamma} \)

\[ N_{\alpha x_{m, n} \gamma} = \hat{N}_{\alpha x_{m, n} \gamma} - \hat{N}_{\alpha x_{-m, n} \gamma}. \] (3.8)

Here the l.h.s. is a finite sum of delta functions, while \( \hat{N}_{\alpha \beta \gamma} \) is given \[ \text{by an integral formula of Verlinde type, i.e., it is diagonalised by } \sqrt{2} \hat{S}_{\alpha \delta} \text{ in } (3.7) \] and its eigenvalues (1-dimensional representations) are given by the ratios \( \hat{S}_{\alpha \delta} / S_{0\delta} \).

The \( F \) matrix is invariant under reflection \( \beta_i \rightarrow Q - \beta_i \) of any of the indices \[ \text{equivalent to a complex conjugation for pure imaginary } iP = Q - 2\beta. \] Extended to arbitrary
values of the charges, the gauged $G$ matrix (3.2) satisfies the standard symmetry relations with the star operation understood as a reflection, $\beta^* = Q - \beta$

$$G_{\beta_5, \beta_6} \left[ \begin{array}{c} \beta_3 \\ \beta_4 \\ \beta_2 \\ \beta_1 \end{array} \right] = G_{\beta_5, \beta_6} \left[ \begin{array}{c} \beta_4^* \\ \beta_3^* \\ \beta_1^* \\ \beta_2^* \end{array} \right] = G_{\beta_5, \beta_6} \left[ \begin{array}{c} \beta_2 \\ \beta_3 \\ \beta_1 \\ \beta_2 \end{array} \right] .$$  

(3.9)

The locality of the scalar 4-point function is rewritten in terms of the $G$ matrices as

$$\int d\gamma \frac{S_{\beta_0}}{S_{\gamma_0}} G_{\beta \gamma} \left[ \begin{array}{c} \alpha_4 \\ \alpha_1 \\ \alpha_3 \\ \alpha_2 \end{array} \right] G_{\beta^* \gamma'} \left[ \begin{array}{c} \alpha_4^* \\ \alpha_1^* \\ \alpha_3^* \\ \alpha_2^* \end{array} \right] = \delta(\gamma - \gamma') .$$  

(3.10)

The integrals here and below run along $\frac{Q}{2} + i i^{\mathbb{R}}$. We shall exploit the relation of the fusion matrices to the Liouville boundary field OPE coefficients $C$ [28],

$$G_{\sigma_2, Q - \beta_3} \left[ \begin{array}{c} \beta_2 \\ \sigma_3 \\ \beta_1 \end{array} \right] = \frac{N(Q - \beta_3, \beta_2, \beta_1) R(\sigma_3, Q - \beta_3, \sigma_1)}{R(\sigma_3, \beta_2, \sigma_1) R(\sigma_2, \beta_1, \sigma_1)} C_{\sigma_2, Q - \beta_3} \left[ \begin{array}{c} \beta_2 \\ \sigma_3 \\ \beta_1 \end{array} \right] ,$$  

(3.11)

where $R$ is the ratio of the two gauge factors;

$$R^{-1}(\sigma_2, \gamma, \sigma_3) = \frac{g(\sigma_2, \gamma, \sigma_3)}{N(\sigma_2, \gamma, \sigma_3)} = \frac{\pi^{\alpha_3 - \alpha_2} S_b(\gamma + \sigma_2 - \sigma_3) S_b(\gamma + \sigma_3 - \sigma_2)}{S_b(\gamma)} .$$  

(3.12)

The OPE coefficients are related to the coefficients of the boundary field 3-point functions

$$C_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1} = C_{\sigma_2, Q - \beta_3} \left[ \begin{array}{c} \beta_2 \\ \sigma_3 \\ \beta_1 \end{array} \right] = S(\sigma_3, \beta_3, \sigma_1) C_{\sigma_2, \beta_3} \left[ \begin{array}{c} \beta_2 \\ \sigma_3 \\ \beta_1 \end{array} \right],$$  

$$S(\sigma_3, \beta, \sigma_1) = \frac{g(\sigma_3, Q - \beta, \sigma_1)}{g(\sigma_3, \beta, \sigma_1)}$$

(3.13)

with the boundary reflection amplitude [28] defined in the second line. In the case when the three charges $\beta_i$ in (3.11) are constrained by a charge conservation condition, the 3-point function $C_{\beta_3, \beta_2, \beta_1}^{\sigma_3, \sigma_2, \sigma_1}$ develops poles. The residue corresponds to the correlator, which can be computed in the half-plane Coulomb gas formulation of [28]. We shall denote it and the residues of the corresponding $G$ in (3.11) by the same letters. For $\sum_i \beta_i = Q = 0$ (absence of screening charges) the residue is 1, so in these cases $G$ reduces to the gauge factor in (3.11). Furthermore any $C$ related by a reflection to a trivial one is also simple, being obtained by applying the boundary reflection matrix as in (3.13). This modifies one (or two, or three) of the ratios (3.12) in (3.11), replacing $g(\sigma_4, \gamma, \sigma_3)$ with $g(\sigma_4, Q - \gamma, \sigma_3)$. Examples of $G$ matrix elements obtained this way will be used below:

$$G_{\beta Q} \left[ \begin{array}{c} \alpha^* \\ \gamma \end{array} \right] = \frac{1}{d_\alpha} \frac{\sin \pi b Q \sin \frac{\pi}{Q} Q}{\sin \pi b (2Q - \alpha) \sin \frac{\pi}{Q} (2Q - \alpha)} = G_{\beta 0} \left[ \begin{array}{c} \gamma^* \\ \alpha \end{array} \right] ,$$

$$G_{\alpha^* \gamma} \left[ \begin{array}{c} \alpha^* \\ Q \end{array} \right] = G_{\alpha \gamma} \left[ \begin{array}{c} 0 \\ \gamma \end{array} \right] = 1 ,$$

(3.14)
where the quantum dimension $d_\alpha = \frac{\sin \frac{\alpha}{2}}{\sin \frac{\alpha}{2^2}}$ appears; furthermore

$$G_{Q+b, \alpha \pm b/2} \begin{bmatrix} \alpha - \frac{b}{2} \\ \alpha Q + \frac{b}{2} \end{bmatrix} = \mp \frac{\sin \pi b^2}{\sin \pi b(2\alpha - Q)},$$

$$G_{\alpha \pm b/2, Q+b} \begin{bmatrix} Q + \frac{b}{2} \\ \alpha \end{bmatrix} = \mp \frac{\sin \pi b(2\alpha \mp Q - Q)}{\sin \pi 2b^2}. \tag{3.15}$$

More generally, denoting

$$G_2(\sigma_3, \beta, \sigma_1) := \frac{S(\sigma_3, \beta, \sigma_1)}{W(Q - \beta)} = S_b(2\beta - Q) \frac{S_b(\sigma_2 + \sigma_1 - \beta)S_b(Q - \beta + \sigma_2 - \sigma_1)}{S_b(\beta + \sigma_2 + \sigma_1 - Q)S_b(\beta + \sigma_2 - \sigma_1)} \tag{3.16}$$

we can write a compact formula for the general Coulomb gas boundary coefficients $C$, obtained as a residue from the Ponsot-Teschner (PT) formula [28],

$$C_{\sigma_2, Q-\beta_3} \begin{bmatrix} \beta_2 \\ \beta_3 \end{bmatrix} = 2\pi \text{Res}_{\beta_{123}=Q+mb+n/b=0} C_{\sigma_2, Q-\beta_3} \begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_3 \end{bmatrix} = g(Q - \beta_3, \beta_2, \beta_1)^{-1} \times \frac{S_b(2\beta_2 + mb + \frac{n}{b})S_b(2\beta_1)}{S_b(2\beta_2)S_b(2\beta_1 + mb + \frac{n}{b})} \sum_{k=0}^{m} \sum_{p=0}^{n} \frac{G_2(\sigma_3 - \frac{(k-m)b}{2} - \frac{p-n}{2b}, Q - \beta_3 + \frac{(k-m)b}{2} + \frac{p-n}{2b}, \sigma_1)}{G_2(\sigma_3, Q - \beta_3, \sigma_1)} \times \frac{G_2(\sigma_3 - \frac{kb}{2} - \frac{p}{2b}, Q - \beta_2 - \frac{kb}{2}, \sigma_2)}{G_2(\sigma_3, Q - \beta_2, \sigma_2)} (-1)^{m(p+1)+n(k+1)+mn} \frac{1}{S_b((k+1)b)S_b((m-k+1)b)S_b(\frac{p+1}{b})S_b(\frac{n-p+1}{b})} \tag{3.17}$$

Further some of the $\beta_i$ charges in (3.17) can be set to degenerate values; it is a polynomial in the boundary parameters $2\cos \pi b(2\sigma_i - Q), 2\cos \frac{\pi}{6}(2\sigma_i - Q)$. Rewritten in terms of finite products of sine-functions (3.17) admits analytic continuation to the region $c < 1$ [29] and in this sense the integral formulœ of [26], [28] are universal.

The following relations follow from the pentagon identity,

$$G_{ci} \begin{bmatrix} j & k \\ b & a \end{bmatrix} = G_{bk} \begin{bmatrix} i^* & j \\ a & c \end{bmatrix} \frac{G_{cQ} \begin{bmatrix} k^* & k \\ a & a \end{bmatrix}}{G_{bQ} \begin{bmatrix} i^* & i \\ a & a \end{bmatrix}} = G_{b^{*k}} \begin{bmatrix} j & i^* \\ c^{*} & a^{*} \end{bmatrix} \frac{d_i}{d_k} \tag{3.18}$$

where in the second equality we have used the (Coulomb gas) values (3.14), particular for the chosen gauge, and the third equality is obtained repeating the first one. This relation is derived alternatively by using (3.11) and the cyclic symmetry of the boundary 3-point coefficients in the l.h.s. of (3.13). In particular (3.18) implies

$$G_{0i} \begin{bmatrix} j & k \\ j & k^* \end{bmatrix} = G_{0Q} \begin{bmatrix} k^* & k \\ k^* & k^* \end{bmatrix} = G_{Qi} \begin{bmatrix} k & j \\ k & j^* \end{bmatrix} = \frac{d_i}{d_k}. \tag{3.19}$$
Here we have replaced delta function singularities on both sides with the residue values; for the precise details of treatment of these singularities see Appendix B of [28].

In all these relations it is assumed that the triples of representations are consistent with the corresponding fusion multiplicities. With this data the analogs of the first two steps (2.4), (2.7) in the rational case now read

\[
G_{\beta \gamma}^{\left[ \begin{array}{c} \alpha_4 \alpha_1 \\ \alpha_3 \alpha_2 \end{array} \right]} G_{\delta \epsilon}^{\left[ \begin{array}{c} \alpha_4 \alpha_1 \\ \alpha_3 \alpha_2 \end{array} \right]} = G_{\alpha_1 \alpha_2}^{\left[ \begin{array}{c} \alpha_2 \gamma \\ \beta \alpha_4 \end{array} \right]} G_{\alpha_3 \alpha_2}^{\left[ \begin{array}{c} \alpha_3 \delta^* \\ \beta \alpha_1 \end{array} \right]} \frac{d_\gamma d_\delta}{d_\alpha d_\beta} \tag{3.20}
\]

\[
S_{rx}(s) \int dm e^{2\pi i(\Delta(x) - \Delta(m))} G_{sm}^{x j_2 \alpha j_1} G_{mp}^{j_1 j_2 x x} = e^{i\pi(\Delta(p) - \Delta(j_1) - \Delta(j_2))} \int dq S_{q x}(p) G_{sq}^{r j_2 \alpha j_1} G_{rp}^{j_2 j_1 q q} \tag{3.21}
\]

The identity is gauge invariant and can be rewritten in terms of the F matrix with

\[
(F)S_{ax}(s) = S_{ax}(s) \frac{N(\alpha, s, \alpha)}{N(x, s, x)} = (F)S_{ax}(Q - s). \tag{3.22}
\]

It depends on the range of the representations, here symbolically written in general form; in the FZZ case the notation \( \hat{S}_{\alpha \beta}(p) \) will be used. Setting \( r = 0 = s \), hence \( q = j_2 = j_1^* = \alpha \), one obtains expressions for the 1-point modular matrices, integral in the generic case. In the case of our main interest, when \( x = x_{m,n} \) is degenerate, the first integral in (3.21) is replaced by a finite sum with the Coulomb gas expressions of the \( G \) matrices appearing,

\[
S_{ax}(p) = \frac{S_{0x}}{G_{0p}^{x x}} \sum_u e^{i\pi(2(\Delta(x) + \Delta(\alpha) + \Delta(u)) - \Delta(p))} G_{qu}^{x x} G_{up}^{x x} \tag{3.23}
\]

\[
= \frac{S_{j0}}{G_{pQ}^{x x}} \sum_u e^{i\pi(2(\Delta(x) + \Delta(\alpha) + \Delta(u)) - \Delta(p))} G_{pu}^{x x} G_{u0}^{x x} \tag{3.23}
\]

(where \( u = \alpha + kb + l/b \) with the range of the integers \( k, l \) restricted by the fusion rule of the degenerate \( x_{m,n} \)). The second equality follows from the pentagon identities. Note that \( S_{ax}(Q) = \frac{d_\alpha}{d_\alpha} S_{ax} \) from (3.22). As in the rational case, (3.23) provides for \( p = 0 \) an
alternative formula for the modular matrix (3.6), which can be also written in integral form

\[ S_{\alpha x, m, n} = S_{00} \int d\gamma e^{2\pi i(\Delta(\alpha) + \Delta(x, m, n) - \Delta(\gamma))} d\gamma N_{\alpha x, m, n} \gamma \]  

(3.24)

This and the analogous formula for \( \hat{S}_{\alpha \beta} / S_{00} \) with \( N_{\alpha x, m, n} \gamma \) replaced by \( \hat{N}_{\alpha \beta} \gamma \) can be checked as in the rational case. In the generic integral analog of (3.23) \( \hat{S}_{\alpha \beta}(p) \) stands in the l.h.s. of (3.23) and an alternative representation can be obtained similarly to the computation in [30] of the FZZ bulk-boundary constant, by solving the set of finite difference equations obtained when setting \( j_2 = -b/2 \) in the integral analog of (3.23), equivalent to the basic identity (3.21); see also below.

The expression (3.23) simplifies if \( 2\alpha = p^* \), or \( p \) (or if \( 2x = p \), or \( p^* \)) since, as discussed above, the \( G \) matrix in the r.h.s is simple, does not involve a summation and reduces to a product of gauge factors. The simplest examples are provided taking (3.23) for \( x = x_{2,1} = -b/2 \) and \( p = -b \), or \( p = Q + b \). For these values the sum contains two terms, \( u = \alpha \pm b/2 \), \( e^{-i\pi \Delta(-b)} = -e^{i\pi 2b} \) and one computes the 1-point modular matrices using the fundamental \( G \) matrices (3.15), (3.14), (3.19)

\[ S_{\alpha, -b/2}(-b) \]
\[ = \frac{d_\alpha}{d_{-b}} 2i e^{i\pi Q b} \sin \pi 2b \alpha \sin \pi b(2\alpha - 2Q) \]
\[ \sin \pi b^2 \]
\[ \]
\[ S_{\alpha, -b/2}^\beta(Q + b) \]
\[ = d_{-b} \frac{\pi Q b}{2} e^{i\pi Q b} 2i \sin \pi b^2. \]

(3.25)

Next we set \( p = 0 \) in (3.21) and apply (3.23) for the sum in the l.h.s. of (3.21), using also (3.14), or, changing notation, \( r = x_{1} \), \( \alpha = x_{2} \), \( q = \beta \), we get

\[ \int d\beta \frac{S_{x\bar{\beta}} S_{00}}{G_{y^* \beta}} \left[ \frac{\alpha_2 \alpha_1}{\alpha_2 \alpha_1^*} \right] G_{\alpha_4 Q} \left[ \frac{\beta \beta^*}{\alpha_2 \alpha_2} \right] = \frac{S_{\alpha_2 x}(y^*)}{S_{00}} \frac{e^{i\pi \Delta(y)}}{d_2^2} \frac{S_{\alpha_1 x}(y)}{S_{00}} \]

(3.26)

This relation is what we need when evaluating the analog of (2.4). Indeed we have an additional factor \( S_{\beta_0} / S_{00} \), coming from the measure in (3.14), which cancels the denominator of the defect eigenvalue \( S_{x\bar{\beta}} / S_{00} \). Combining with (3.20) we finally obtain

\[ A_{\gamma, \delta}^{(x)} = \int d\beta \frac{S_{x\bar{\beta}} S_{00}}{G_{\gamma \gamma}} \left[ \frac{\alpha_4 \alpha_1}{\alpha_4 \alpha_1^*} \right] G_{\beta^* \delta} \left[ \frac{\alpha_4 \alpha_1}{\alpha_2 \alpha_2} \right] \]
\[ = d_\gamma d_\delta \sum_y G_{\alpha_4 y^*} \left[ \frac{\gamma^* \delta}{\alpha_2 \alpha_2} \right] \frac{S_{\alpha_2 x}(y^*)}{S_{0x}} \frac{e^{i\pi \Delta(y)}}{d_y} \frac{S_{\alpha_1 x}(y)}{S_{0x}} G_{\alpha_4 y} \left[ \frac{\gamma \delta^*}{\alpha_1 \alpha_1} \right] = d_\gamma \sum_y B_{\gamma, \delta}(y) \]

(3.27)
We have replaced the integral in (3.20) by a sum once again using the fact that for degenerate representations the fusion matrices are represented by residues of the initial singular expressions. Using that \( S_{\alpha x}(Q) = \frac{d_x}{d_\alpha} S_{\alpha x} \) and (3.14) one checks that for trivial defect \( x = 0 \) (hence \( y = 0 \)) (3.27) reduces to \( A_{\gamma, \delta}^{(0)} = d_\gamma \delta (\gamma - \delta) \) in agreement with (3.10).

For \( x \) in the continuous series, the integral analog of (3.27) holds, with the ratio \( \frac{\tilde{S}_{x\beta}}{S_{00}} \) in the l.h.s. (confirmed [21] as a defect eigenvalue), while \( \hat{S}_{\alpha x}(y^*)/S_{0x}, \hat{S}_{\alpha x}(y)/S_{0x} \) will appear in the r.h.s.; the remaining q-dimension factors are unchanged.

The generic analog of (3.26) considered as an expression for \( \hat{S}_{\alpha x}(y^*) \) simplifies for the choice \( \alpha_1 = y/2 \) of the other charge by the mechanism discussed above: this is more transparent in the transposed version of (3.26) obtained using the identities (3.18) and \( \hat{S}_{\alpha x}(y) = \frac{d_\alpha}{d_\alpha} \tilde{S}_{x\alpha}(y^*) \). On the r.h.s. appears the constant \( R e_{\alpha_1=x_0} \hat{S}_{x\alpha_1}(y^*) \), while with this choice the \( G \) matrix in the l.h.s. of (3.26) is replaced by a Coulomb gas correlator from (3.11):

\[
G_{y^*, \beta}
\left[
\begin{array}{c}
\alpha_2 \\
\alpha_2 \beta
\end{array}
\right] =
\left[
\begin{array}{cc}
\frac{y}{2} & \frac{y}{2}
\end{array}
\right]
G_{Q, \frac{y}{2}, \frac{y}{2}}
\left[
\begin{array}{cc}
\beta & \beta^*
\end{array}
\right]
\left[
\begin{array}{cc}
\alpha_2 & \alpha_2
\end{array}
\right] =
\frac{1}{d_y} G_{\beta, y}
\left[
\begin{array}{cc}
Q & \frac{y}{2} \\
\alpha_2 & \alpha_2
\end{array}
\right] =
\frac{S_{00}}{S_b(\beta - \frac{y}{2} + \alpha_2)}
S_b(\beta - \frac{y}{2} + \alpha_2) + Q \right).
\]

The transposed version of (3.26) (taken with either of the two modular matrices \( \hat{S}_{x\beta} \) or \( S_{x\beta} \)) is to be compared, with the corresponding Cardy-Lewellen type equation for the bulk-boundary reflection coefficient. Special cases of this equation have been used in [20], [19] for the determination of the half-plane bulk 1-point functions in the two Liouville cases. The general equation has been exploited in [31] to give an alternative derivation of the FZZ bulk-boundary constant \( R_{\alpha}(\alpha, y) \), computed in [30]. In obtaining (3.28) here we have followed a similar argument to that given in [31] - indeed one recognises in the expression (3.28) the Fourier transform \( \tilde{R}(\alpha_2, y; \beta - Q/2) \) up to \( \beta \)-independent factors. The resulting expression for \( \hat{S}_{x\alpha}(y) \) can be written in terms of \( b \)-deformed hypergeometric functions (see [27], [32])

\[
\hat{S}_{\alpha x}(y)e^{i\pi \Delta(y)/2} = \frac{d_\alpha}{d_x} \hat{S}_{\alpha x}(y^*)e^{i\pi \Delta(y)/2}
= \sum_{\pm} \frac{S_b(\pm(2\alpha - Q) + y)}{S_b(\pm(2\alpha - Q))} e^{i\pi(\pm(2\alpha - Q) + y)(2\alpha - Q)} F_b(y, \pm(2\alpha - Q) + y; \pm(2\alpha - Q) + Q; 2\alpha - Q)
= \frac{S_b(2\alpha + y - Q)}{S_b(2\alpha - Q)} \sum_{\pm} e^{i\pi(2\alpha + y - Q)(2\alpha - Q)} F_b(y, \pm(2\alpha + y - Q; 2\alpha; \pm(2\alpha - Q))
\]

(3.29)
For $y = 0$ it reproduces $\hat{S}_{x\alpha}$ in (3.24) and the residue of $\hat{S}_{x\alpha}(y^*)$ at $\alpha = y/2$ is consistent with the FZZ analog of eqn (3.26) at this value. The required symmetries like the one in the first line of (3.29) are checked exploiting relations for the $b$-deformed hypergeometric functions. Note that the ratio
\[
\frac{\hat{S}_{x\alpha}(y)e^{i\pi \Delta(y)/2}}{W(\alpha)g(\alpha, y, \alpha)} = \frac{F_{\hat{S}_{x\alpha}(y)}e^{i\pi \Delta(y)/2}}{W(\alpha)g(x, y, x)}
\]
has the correct properties under reflections with respect to the bulk $\alpha$ and the boundary $y$ field charges as required for the FZZ bulk-boundary constant $R_x(\alpha, y)$.

For (odd) degenerate $y = y_{2m+1,2n+1} = -mb - n/b$, one furthermore obtains, accounting for the residues of the poles of the integrand defining the hypergeometric functions a formula for degenerate $x = x_{r,s}$: the computation is analogous to the derivation of the ZZ bulk-boundary coefficient in [31]. Namely, denoting $\gamma_{k,l} = kb + l/b$,
\[
S_{x_{r,s},\alpha}(y) = S_{\alpha x_{r,s}}(y^*) \frac{d_{\alpha}}{d_{x_{r,s}}} = \hat{S}_{x_{r,s},\alpha}(y) - \hat{S}_{x_{r,s},\alpha}(y) = e^{-i\pi \Delta(y)} \sum_{k=0}^{m} \sum_{l=0}^{n} S_{x_{r,s}, \gamma_{k,l}, y \frac{y}{2} \alpha} \times
\]
\[
\frac{(-1)^{kn+ml} S_b((m + 1)b) S_b\left(\frac{n+1}{b}\right)}{S_b((m + 1 - k)b) S_b\left(\frac{n+1-l}{b}\right) S_b((k + 1)b) S_b\left(\frac{n+1}{b}\right) S_b(2\alpha + \gamma_{k,l}) S_b(2Q - y - \gamma_{k,l} - 2\alpha)}.
\]
(3.30)
The values of $y$ are restricted by the degenerate fusion multiplicity, $0 \leq m \leq r-1$, $0 \leq n \leq s-1$. The formula (3.30) provides alternative to (3.23) representation for $S_{\alpha x_{r,s}}(Q - y_{2m+1,2n+1})$. It is checked to reproduce the particular example in (3.25).

3.3. The 't Hooft operator: example

Here we compute (3.27) for the simplest example considered in [3], [6], namely $x = -b/2$, so that $y$ takes the values $y = 0, -b$.

The $G$ matrices for $\delta = \gamma \pm b$ are all straightforward to compute, being related as explained above, to trivial boundary OPE coefficients,
\[
G_{\sigma_4, Q^{+b}} \left[ \begin{array}{cc} Q - \gamma & \gamma + b \\ \sigma_3 & \sigma_3 \end{array} \right] = -\frac{d_{b/2} d_{\sigma_3}}{d_{\gamma + b}} \sin \pi 2\sigma_3 b \sin \pi b(2\sigma_3 - 2Q) G_{\sigma_4, Q^{-b}} \left[ \begin{array}{cc} Q - \gamma & \gamma + b \\ \sigma_3 & \sigma_3 \end{array} \right],
\]
\[
G_{\sigma_4, Q^{+b}} \left[ \begin{array}{cc} Q - \gamma & \gamma - b \\ \sigma_3 & \sigma_3 \end{array} \right] = -\frac{d_{-b/2} d_{\sigma_3}}{d_{\gamma + b}} \sin \pi 2\sigma_3 b \sin \pi b(2\gamma - 2Q) G_{\sigma_4, Q^{-b}} \left[ \begin{array}{cc} Q - \gamma & \gamma - b \\ \sigma_3 & \sigma_3 \end{array} \right],
\]
(3.31)
For the only matrix that involves two terms (the case \(m = 1, n = 0\) in \((3.17)\)) one gets
\[
G_{\sigma_4,-b} \begin{bmatrix} Q-\gamma & \gamma \\ \sigma_3 & \sigma_3 \end{bmatrix} = -\frac{d_{-b}}{d_{\sigma_3}} \frac{\cos \pi b(2\gamma-Q) \cos \pi b(2\sigma_3-Q) + \cos \pi b(2\sigma_4-Q) \cos \pi b^2}{\sin \pi 2\sigma_3 b \sin \pi b(2\sigma_3 - 2Q)} \\
= G_{\sigma_4,Q+b} \begin{bmatrix} Q-\sigma_3 & \sigma_3 \\ \gamma & \gamma \end{bmatrix}.
\]
(3.32)

Besides \((3.25)\) we also need
\[
S_{\alpha_1,-\frac{b}{2}} = 2 \cos \pi b(2\alpha_1 - Q), \quad S_{\alpha_2,-\frac{b}{2}}(Q) = \frac{d_{b}}{d_{\alpha_2}} S_{\alpha_2,-\frac{b}{2}} = d_{-\frac{b}{2}} 2 \cos \pi b(2\alpha_2 - Q). \quad (3.33)
\]
Let us change the notation \((\alpha_3, \alpha_2, \alpha_1, \alpha_4) \rightarrow (\sigma_4, \sigma_3, \sigma_2, \sigma_1)\), so that \(B^{(x)}_{\gamma,\delta}(y)\), as defined in \((3.27)\), reads
\[
B^{(x)}_{\gamma,\delta}(y) = \frac{d_{\delta}}{d_{\gamma}} G^*_{\gamma,\alpha} \left[ \frac{\gamma^*}{\sigma_3} \frac{\delta^*}{\sigma_3} \right] S_{\sigma_4 x}(y^*) S_{\sigma_3 y}(y) G^*_{\gamma y} \left[ \frac{\gamma}{\sigma_2} \frac{\delta}{\sigma_2} \right].
\]
(3.34)
Collecting all formulae obtained from \((3.31), (3.32)\) with the proper conjugations and change of variables, and using also \((3.14), (3.23), (3.33)\), we obtain for \((3.34)\):
\[
B^{(-b/2)}_{\gamma,\gamma-b}(-b) = -4 \sin \pi b(Q+\sigma_4^3 - \gamma) \sin \pi b(Q+\sigma_3^3 - \gamma) \sin \pi b(2Q - \gamma - \sigma_{12}) \sin \pi b(\sigma_{12} - \gamma) \\
\quad \sin \pi b(2\gamma - Q) \sin \pi b(2\gamma - 2Q)
\]
(3.35)
\[
B^{(-b/2)}_{\gamma,\gamma+b}(-b) = -4 \sin \pi b(Q+\gamma - \sigma_{34}) \sin \pi b(\gamma + \sigma_{34} - Q) \sin \pi b(\gamma + \sigma_1^3) \sin \pi b(\gamma + \sigma_1^2) \\
\quad \sin \pi b(2\gamma - Q) \sin \pi b(2\gamma - 2Q)
\]
(3.36)
\[
B^{(-b/2)}_{\gamma,\gamma}(0) = \frac{\cos \pi b(2\gamma - Q) \cos \pi b(2\sigma_3 - Q) - \cos \pi b(2\sigma_4 - Q) \cos \pi b^2}{d_{-\frac{b}{2}} \sin \pi b2\gamma \sin \pi b(2\gamma - 2Q)} \\
\quad 4(\cos \pi b(2\gamma - Q) \cos \pi b(2\sigma_2 - Q) - \cos \pi b(2\sigma_1 - Q) \cos \pi b^2)
\]
(3.37)
where \(\sigma_{12} = \sigma_1 + \sigma_2, \sigma_1^2 = \sigma_1 - \sigma_2\), etc. The above, normalised by \(d_{-b/2}\), expressions for \(B_{\gamma,\delta}(-p)/d_{-b/2}\) should be compared with formulae \((5.32-34)\) of [8]. Apart from, presumably, a sign typo in \((5.34)\):
\[
\sin \pi b(\alpha - m_{12} - b) \sin \pi b(\alpha - m_{34} - b) \rightarrow \sin \pi b(\alpha - m_{12} + b) \sin \pi b(\alpha - m_{34} + b)
\]
the formulae coincide for \((\gamma; \sigma_4, \sigma_3, \sigma_2, \sigma_1) \rightarrow (\alpha; m_4, m_3, m_2, Q - m_1)\), i.e., up to a reflection of one of the charges. Here \(\sigma_1 = \alpha_4\) is the charge of the first vertex operator in the Wilson loop channel, cf. [3.1].
The sum of the two terms in (3.36) can be cast into a form which makes explicit the symmetry $(\sigma_2, \sigma_3) \rightarrow (\sigma_4, \sigma_1)$. It coincides (up to a relative sign) with the term $H_0$ in (5.39) of [5] upon identification of the charges: $2\sigma_j - Q = 2im_j$, $2\gamma - Q = 2iP$. In the initial basis of conformal blocks the duality relation reads

$$
\int d\beta C(\alpha_4, \alpha_3, \beta) C(\beta^*, \alpha_2, \alpha_1) \frac{S_{\alpha\beta}}{S_{0\beta}} |(F)G_\beta(\alpha_4, \alpha_3, \alpha_2, \alpha_1; \bar{z})|^2
$$

$$
= \int d\gamma \int d\delta \sum_y C(\alpha_3, \alpha_2, (\gamma, \delta)) C((\gamma^*, \delta^*), \alpha_1, \alpha_4) \frac{W(\gamma)}{W(\delta)} B_{\gamma, \delta}(y) \times
$$

$$(F)G_\gamma(\alpha_3, \alpha_2, \alpha_1, \alpha_4; z)(F)G^*_\delta(\alpha_3, \alpha_2, \alpha_1, \alpha_4; \bar{z})$$

where we have denoted (consistent with (3.3) for $\gamma = \delta$)

$$C(\alpha_3, \alpha_2, (\gamma, \delta)) C((\gamma^*, \delta^*), \alpha_1, \alpha_4) \frac{W(\delta)}{W(\gamma)} :=$$

$$\frac{(2\pi)^2 \lambda^{\gamma\delta} \prod_{i=1}^4 W(Q - \alpha_i)}{N(\alpha_3^2, \gamma, \alpha_2)N(\gamma, \alpha_4, \alpha_1) N^*(\alpha_3^2, \delta, \alpha_2)N^*(\delta, \alpha_4, \alpha_1)}$$

$$= C(\alpha_3, \alpha_2, \gamma) C(Q - \gamma, \alpha_1, \alpha_4) \frac{N(\alpha_3, Q - \gamma, Q - \alpha_2) N(\alpha_1, Q - \alpha_4, \gamma)}{N(\alpha_3, Q - \delta, Q - \alpha_2) N(\alpha_1, Q - \alpha_4, \delta)}.$$ 

Multiplying $B_{\gamma, \gamma \pm b}^{(-b/2)}$ in (3.35) with the ratio of $N$-factors, relative to the diagonal constant in the last line in (3.38), we get expressions invariant under any reflection $\alpha_i \rightarrow Q - \alpha_i$ of the four charges. These normalised expressions coincide with $H_\pm$ in (5.38) of [5] (up to an overall factor $2\pi$) under the above identification of the charges.

**Appendix A. The defects and the OPE coefficients of local fields**

In the non-diagonal rational cases the identity contribution in the duality relation for the correlators (2.1), (2.3) is nontrivial and implies an explicit formula [2] for the relative OPE coefficients of local fields $\Phi_{I;\alpha}(z, \bar{z}), I = (i, \bar{i}), \alpha = 1, ..., Z_{\bar{i}i}$ of arbitrary integer spin

$$\Phi_{I;\alpha}(z, \bar{z}) = \sum_{j, \bar{j}, k, \bar{k}, \gamma, t, \bar{t}} d_{I;\alpha}(J;\beta) \left( \phi_k^{t}(z) \otimes \phi_{\bar{k}}^{\bar{t}}(\bar{z}) \right)_{\alpha, \beta}.$$  

Namely (restricting to the $sl(2)$ case) one obtains

$$\sum_{k, \bar{k}, \gamma, \gamma'} d_{I;\alpha}(J;\beta) d_{I;\alpha'}(J;\beta') \frac{\Psi_{x}^{(K;\gamma, \gamma')}}{\Psi_{x}^{(1)}} = \frac{\Psi_{x}^{(I;\alpha, \alpha')}}{\Psi_{x}^{(1)}} \hspace{1cm} (A.2)$$
Using the unitarity of $\Psi$ one gets an expression for the product of OPE coefficients which involves a summation over the complete set of defects for the given modular invariant.\footnote{2}

The $\Psi$-ratios in (A.2) serve as 1-dimensional representations of an associative, commutative algebra, dual to the fusion algebra of defects. This universal algebra generalises the Pasquier algebra \cite{33} associated with each of the ADE $\text{nim}$-reps in (1.3), which determines the subset of OPE coefficients with scalar labels only \cite{34}.

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\footnote{2 We use the opportunity to correct some inaccuracies in \cite{3}: formula (A.2) is slightly more general than what stated in \cite{2} and reproduces also some of the signs in the case with non-commutative $\tilde{N}$, the $D_{\text{even}}$ series: note that there are different bases for the pair of doubled fields in this case and accordingly different bases for the $\Psi$ matrices.}
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