Confinement in Polyakov Gauge

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We approach the non-perturbative regime in finite temperature QCD within a formulation in Polyakov gauge. The construction is based on a complete gauge fixing. Correlation functions are then computed from Wilsonian renormalisation group flows. First results for the confinement-deconfinement phase transition for SU(2) are presented. Within a simple approximation we obtain a second order phase transition within the Ising universality class. The critical temperature is computed as $T_c \approx 305$ MeV.

PACS numbers: 05.10.Cc, 12.38.Aw, 11.10.Wx

\section{I. INTRODUCTION}

One of the remaining problems in low energy QCD is the quantitative field theoretical description of the confinement-deconfinement phase transition. Apart from its genuine importance for a first principle understanding of the confining physics in QCD it also is a key input for the evaluation of the QCD phase diagram.

In the past decade much progress has been achieved both in continuum studies as well as with lattice computations for our understanding of the low energy sector of QCD, for reviews see e.g. \cite{1,2,3,4,5}. For an analytical description of the low energy sector, topological degrees of freedom are likely to play an important role for the confinement-deconfinement phase transition as well as for chiral symmetry breaking, see e.g. \cite{6}. The latter has been very successfully described within instanton models, whereas the confining properties of the theory are harder to incorporate within semi-classical descriptions. Indeed, tracking down those topological degrees of freedoms relevant for confinement in the physical vacuum has its intricacies as the physical vacuum is more likely to contain a rather dense packing of topological configurations, making their detection difficult. Moreover, models of confinement are rather based on topological defects instead of stable topological objects, the construction of which out of these defects is plagued by non-localities.

Still, these defects are manifest in the Polyakov loop, the order parameter in pure Yang-Mills theory \cite{7}, and can be extracted by an appropriate gauge fixing, see e.g. \cite{8,9}. Gauge fixing is also mandatory in most continuum formulations of QCD for removing the redundant gauge degrees of freedom. This is mostly seen as a liability of such an approach, as a formulation of QCD in gauge-variant variables complicates the access to gauge invariant observables. However, gauge fixing is nothing but a reparameterisation of the path integral and can be used for even facilitating the computation of at least a subset of observables. Indeed, this point of view has been exploited much in the discussion of confinement mechanisms based on topological defects. More recently it also has become clear that these are not competing physics mechanisms but rather different facets of the same global physics picture which still awaits a fully gauge invariant description, see e.g. \cite{10}. Despite this final step we have learned much from the combined investigations which together built a nearly complete mosaic.

The effective potential of the Polyakov loop has also been used as an input for effective field theories that give some access to the QCD phase diagram \cite{11}. At finite temperature and vanishing density, these models have led to impressive results in particular for thermodynamical quantities. At finite chemical potential, the back-reaction of the matter sector to the gauge sector is difficult to quantify in these models, and the chiral and confinement-deconfinement phase transitions are sensitive to the details of this back-reaction. This also holds true for the question of a quarkyonic phase with confinement and chiral symmetry at finite density \cite{12}. For an extension of these models one has to resort to a field-theoretical description of the gauge sector which allows to systematically study the impact of a finite chemical potential on the dynamics of the gauge field, \cite{13}.

In summary, the evaluation of Green functions of the Polyakov loop allows for a direct access to the physics in the strongly coupled sector of QCD, and in particular the confinement-deconfinement phase transition. In the present work we initiate a non-perturbative study of QCD in Polyakov gauge. In this gauge the Polyakov loop takes a particularly simple form and is directly related to the temporal component of the gauge field. After integrating-out the spatial components of the gauge field, and formulated with Polyakov loop variables, the gauge field sector of QCD resembles a scalar model. The dynamics of low energy Yang-Mills theory is then incorporated by evaluating Wilsonian flows for the effective action \cite{14,15,16,17}. We derive the flow equation for QCD in Polyakov gauge, and solve it for the full effective potential of the Polyakov loop. Due to the formulation in Polyakov gauge a simple truncation already suffices to encode the physics of the confinement-deconfinement phase transition. The results include the temperature dependence of the Polyakov loop, and the critical temperature. We also compare the present approach to lattice studies \cite{18}, and to a recent continuum computation in Landau gauge \cite{19}. 
II. QCD IN POLYAKOV GAUGE

In QCD with static quarks the expectation value of a static quark \( \langle q(\vec{x}) \rangle \) serves as an order parameter for confinement. It is proportional to the free energy \( F_0 \) of such a state, \( \langle q(\vec{x}) \rangle \sim \exp(-\beta F_q) \), where \( \beta = 1/T \) is the inverse temperature. Hence in the confining phase at low temperature, the expectation value vanishes, whereas at high temperatures in the deconfined phase, it is non-zero. The Polyakov loop variable, \( \mathcal{P} \), is the creation operator for a static quark, \[
L(\vec{x}) = \frac{1}{N_c} \text{tr} \mathcal{P}(\vec{x}) ,
\]
where the trace in \([1]\) is done in the fundamental representation, and the Polyakov loop operator is a Wegner-Wilson loop in time direction, \[
\mathcal{P}(\vec{x}) = \mathcal{P} \exp \left( 1g \int_0^\beta dx_0 A_0(x_0, \vec{x}) \right) .
\] (2)

Here \( \mathcal{P} \) stands for path ordering. We conclude that \( \langle q(\vec{x}) \rangle \sim \langle L(\vec{x}) \rangle \), and thus the negative logarithm of the Polyakov loop expectation value relates to the free energy of a static fundamental color source. Moreover, \( \langle L \rangle \) measures whether center symmetry is realised by the ensemble under consideration, see e.g. \([\text{I, II, III, IV, V, VI}])\.

More specifically we consider gauge transformations \( U(x_0, x) \) with \( U(0, \vec{x}) U^{-1}(\beta, \vec{x}) = Z \), where \( Z \) is a center element. In \( SU(2) \) the center is \( \mathbb{Z}_2 \), whereas in physical QCD with \( SU(3) \) it is \( \mathbb{Z}_3 \). Under such center transformations the Polyakov loop operator \( \mathcal{P}(\vec{x}) \) in \([2]\) is multiplied with a center element \( Z \), \[
\mathcal{P}(\vec{x}) \rightarrow Z \mathcal{P}(\vec{x}) ,
\] (3)
and so does the Polyakov loop, \( L(\vec{x}) \rightarrow Z L(\vec{x}) \). Hence, a center-symmetric confining (disordered) ground state ensures \( \langle L \rangle = 0 \), whereas deconfinement \( \langle L \rangle \neq 0 \) signals the ordered phase and center-symmetry breaking, \[
T < T_c : \quad \langle L(\vec{x}) \rangle = 0 , \quad F_q = \infty ,
\]
\[
T > T_c : \quad \langle L(\vec{x}) \rangle \neq 0 , \quad F_q < \infty .
\] (4)

The expectation value of the Polyakov loop can be deduced from the equations of motion of its effective potential \( \mathcal{V}_\text{eff}(\langle L \rangle) \). We shall argue, that the computation of the latter greatly simplifies within an appropriate choice of gauge. Indeed, gauge fixing is nothing but the choice of a specific parameterisation of the path integral, and a conveniently chosen parameterisation can simplify the task of computing physical observables.

In the present case our choice of gauge is guided by the demand of a particularly simple representation of the Polyakov loop variable \([1]\). A gauge ensuring time-independent \( A_0 \) leads to both, a trivial integration in \([2]\) and renders the path ordering irrelevant. Having done this one can still rotate the Polyakov loop operator \( \mathcal{P}(\vec{x}) \), \([2]\), into the Cartan subgroup. The above properties are achieved for time-independent gauge field configurations in the Cartan subalgebra, i.e. \( A_0(t_0, \vec{x}) = A_0^0(\vec{x}) \). For \( SU(2) \) this reads \[
A_0(t_0, \vec{x}) = A_0(\vec{x}) e^{i3 \over 2} ,
\] (5)
and entails a particularly simple relation between \( A_0 \) and \( L \), \[
L(\vec{x}) = \cos \frac{1}{2} g \beta A_0(\vec{x}) ,
\] (6)
Note that this simple relation is not valid on the level of expectation values of \( L \) and \( A_0 \), in \( SU(2) \) we have \( \langle L \rangle \neq \cos \frac{1}{2} g \beta \langle A_0 \rangle \). However, in the present work we consider an approach that gives direct access to the effective potential \( \mathcal{V}_\text{eff}(\langle A_0 \rangle) \) for the gauge field, as distinguished to those for the Polyakov loop, \( \mathcal{U}_\text{eff}(\langle L \rangle) \) \(^1\).

Here, we argue that \( L(\langle A_0 \rangle) \) also serves as an order parameter: to that end we show that the order parameter \( \langle L[A_0] \rangle \) is bounded from above by \( L[\langle A_0 \rangle] \). It follows that \( L[\langle A_0 \rangle] \) is non-vanishing in the center-broken phase. Furthermore we show that in the center-symmetric phase with vanishing order parameter, \( \langle L[A_0] \rangle = 0 \), also the observable \( L[\langle A_0 \rangle] \) vanishes. For the sake of simplicity we restrict the explicit argument to \( SU(2) \), but it straightforwardly extends to general \( SU(N) \). First we note that we can use \([3]\) for expressing the expectation value of \( A_0 \) in terms of \( L \), \[
\frac{1}{2} g \beta \langle A_0 \rangle = \langle \arccos L \rangle .
\] (7)

We emphasise that the rhs of \([7]\) defines an observable as it is the expectation value of an gauge invariant object. This observable happens to agree with \( \langle A_0 \rangle \) in Polyakov gauge. It follows from the Jensen inequality that the expectation value of the Polyakov loop, the order parameter for confinement, is bounded from above by \( L[\langle A_0 \rangle] \), see \([9]\), \[
L[\langle A_0 \rangle] \geq \langle L[\langle A_0 \rangle] \rangle .
\] (8)
for gauge fields \( g \beta \langle A_0 \rangle/2 \in [0, \pi/2] \). Note that it is sufficient to consider the above interval due to periodicity and center symmetry of the potential. This means we restrict the Polyakov loop expectation value to \( \langle L \rangle \geq 0 \). Negative values of \( \langle L \rangle \) are then obtained by center transformations, \( L \rightarrow \pm L \). Eq. \([8]\) is easily proven for \( SU(2) \) from \([9]\) as \( \cos(x) \) is concave for \( x \in [0, \pi/2] \). Thus, for \( \langle L \rangle > 0 \) it necessarily also follows that \( g \beta \langle A_0 \rangle/2 < \pi/2 \).

In turn we can show that \( g \beta \langle A_0 \rangle/2 < \pi/2 \), if the Polyakov loop variable \( \langle L[A_0] \rangle \) vanishes. This then entails that \( L[\langle A_0 \rangle] = 0 \). To that end we expand \( L \) about

\(^1\) A reformulation in terms of the Polyakov loop variable only along the lines outlined in \([17]\) is in progress.
its mean value $\langle L \rangle$, that is $L = \langle L \rangle + \delta L$. Inserting this expansion into (7) we arrive at

$$\frac{1}{2} g \beta(A_0) = \arccos(\langle L \rangle) - \frac{1}{\sqrt{1 - (\langle L \rangle)^2}} \delta L + O \left( (\delta L^2) \right).$$

In the center-symmetric phase $\langle L \rangle = 0$, c.f. [4]. Under center transformations $L$ transforms according to (3) $L \rightarrow Z L$ with $Z = \pm 1$ and hence $\delta L \rightarrow Z \delta L$. It follows that $(\delta L^{2n+1}) = Z(\delta L^{2n+1}) = 0$, and all odd powers in (9) vanish. The even powers vanish since arccos is an odd function and hence has vanishing even Taylor coefficients arccos$(2n)(0)$. Thus, in the center-symmetric phase we have

$$\frac{1}{2} g \beta(A_0) = \arccos(\langle L \rangle) = \frac{\pi}{2}.$$

In summary we have shown

$$T < T_c : \quad L[\langle A_0 \rangle] = 0 \quad \iff \quad \frac{1}{2} g \beta(A_0(\vec{x})) = \frac{\pi}{2},$$

$$T > T_c : \quad L[\langle A_0 \rangle] \neq 0 \quad \iff \quad \frac{1}{2} g \beta(A_0(\vec{x})) < \frac{\pi}{2}.$$ (11)

We conclude that $\langle A_0 \rangle$ in Polyakov gauge serves as an order parameter for the confinement-deconfinement (order-disorder) phase transition, as does $L[\langle A_0 \rangle]$. Thus, we only have to compute the effective potential $V_{\text{eff}}[\langle A_0 \rangle]$ in order to extract the critical temperature, and e.g. critical exponents. This potential is more easily accessed than that for the Polyakov loop. It is here were the specific gauge comes to our aid as it allows the direct physical interpretation of a component of the gauge field. This property has been already exploited in the literature, where it has been shown that $\langle A_0 \rangle$ in Polyakov gauge is sensitive to topological defects related to the confinement mechanism [8, 9].

### III. Quantisation

We proceed by discussing the generating functional of Polyakov gauge Yang-Mills theory. For its derivation we use the Faddeev-Popov method. Specifying to $SU(2)$, the Polyakov gauge [15] is implemented by the gauge fixing conditions

$$\delta_0 \text{tr} \sigma_3 A_0 = 0, \quad \text{tr} (\sigma_1 \pm i \sigma_2) A_0 = 0,$$ (12)

where the $\sigma_i$ are the Pauli matrices. However, the gauge fixing (12) is not complete. It is unchanged under time-independent gauge transformations in the Cartan subgroup. These remaining gauge degrees of freedom are completely fixed by the following conditions,

$$\delta_1 \int d x_0 \text{tr} \sigma_3 A_1 = 0, \quad \delta_2 \int d x_0 dx_1 \text{tr} \sigma_3 A_2 = 0, \quad \delta_3 \int d x_0 dx_1 dx_2 \text{tr} \sigma_3 A_3 = 0.$$ (13)

The gauge fixings (13) are integral conditions and are the weaker the more integrals are involved. Basically they eliminate the corresponding zero modes. This can be seen directly upon putting the theory into a box with periodic boundary conditions, $T^4$, see e.g. [3].

The gauge fixing conditions (12), (13) lead to the Faddeev-Popov determinant

$$\Delta_F[A] = (2T)^2 \left[ \prod_x \sin^2 \left( \frac{g A_{0}(\vec{x})}{2T} \right) \right],$$ (14)

the computation of which is detailed in appendix A. The integration over the longitudinal gauge fields precisely cancels the Faddeev-Popov determinant in the static approximation $\partial_i A_{0}^i = 0$, see Appendix A. Finally we are left with the action

$$S_{\text{eff}}[A] \simeq \frac{1}{2} \beta \int d^4 x A_0 \delta^2 A_0$$

$$- \frac{1}{2} \int d^4 x A_{+}^{a,i} \left[ (D_0)^2_{ab} + \partial^2 \gamma^{ab} \right] A_{+}^{a,i} + O(A_{+}^{3})$$

with $D_0^{ab} = \partial_0 \delta^{ab} + A_0^{ab} \delta^{ab}$ and transversal spatial gauge fields, $\partial_i A_{\perp}^i = 0$. The generating functional of Yang-Mills theory in Polyakov gauge then reads

$$Z[J] = \int d A_0 d A_{\perp} \exp \left\{ -S_{\text{eff}}[A] \right\}$$

$$+ \int d^4 x J_0 A_0 + \int d^4 x J_{\perp} A_{\perp} \right\}.$$ (16)

In (16) we have normalised the temporal component $J_0$ of the current with a factor $\beta$. The classical action $S_{\text{eff}}$ is inherently non-local as it contains one-loop terms, the Faddeev-Popov determinant as well as the integration over the longitudinal gauge fields. Instead of computing $Z[J]$ in (16) we shall compute the effective action $\Gamma$ within a functional renormalisation group approach [3, 4, 14, 15, 16, 17]. To that end we introduce an infrared cut-off for the transversal spatial gauge fields and in the temporal gauge fields by modifying the action, $S \rightarrow S_{\text{eff}} + \Delta S_k[A_0] + \Delta S_{\perp,k}[A_{\perp}]$, with infrared scale $k$, and cut-off terms

$$\Delta S_k[A_0] = \frac{1}{2} \beta \int d^4 x A_0 R_{0;k} A_0$$

$$\Delta S_{k,\perp}[A_{\perp}] = \int d^4 x A_{\perp} R_{\perp;k} A_{\perp}.$$ (17)

The regulators $R_k$ in (17) are chosen to be momentum-dependent and required to provide masses at low momenta and to vanish at large momenta. For $k \rightarrow 0$ they vanish identically.

They can be written as one single regulator $R_{A,\mu\nu}$, which is a block-diagonal matrix in field space with entries $R_{A,00} = R_{0;k}$ and $R_{A,i,j} = R_{\perp;k} \Pi_{\perp,i,j}$, where the transversal projector is defined by

$$\Pi_{\perp,i,j} = \delta_{ij} - \frac{p_i p_j}{p^2}.$$ (18)
The above structure is induced by the fact the \( A_{\perp,i} \) are transversal, and hence \( R_{\perp,k} \) only couples to the transversal degrees of freedom.

The flow of the cut-off dependent effective action \( \Gamma_k \) is then given by Wetterich’s equation \([14, 15, 16]\) for Yang-Mills theory \([3, 17]\) in Polyakov gauge,

\[
\partial_t \Gamma_k = \beta \int \frac{d^4p}{(2\pi)^3} \left( \frac{1}{\Gamma_k^{(2)} + R_A} \right) \partial_t R_{0,k} \\
+ \frac{T}{2} \sum_{n \in \mathbb{Z}} \int \frac{d^4p}{(2\pi)^3} \left( \frac{1}{\Gamma_k^{(2)} + R_A} \right) \partial_t R_{\perp,k} ,
\]

where \( t \) is the RG time \( t = \ln(k/\Lambda) \), and \( \Lambda \) is some reference scale.

**IV. APPROXIMATION SCHEME**

Eq. \((19)\) together with an initial effective action at some initial ultraviolet scale \( k = \Lambda_{UV} \) provides a definition of the full effective action at vanishing cut-off scale \( k = 0 \) via the integrated flow. For the solution of \((19)\) we have to resort to approximations to the full effective action. In gauge theories such an approximation also requires the control of gauge invariance, see e.g. \([17]\).

Here we shall argue that in Polyakov gauge a rather simple approximation to the full effective action already suffices to describe the confinement-deconfinement phase transition, and, in particular, to estimate the critical temperature. We compute the flow of the effective action \( \Gamma[A_0, \tilde{A}_{\perp}] \) in the following truncation

\[
\Gamma_k[A_0, \tilde{A}_{\perp}] = \beta \int d^4x \left( -\frac{Z_0}{2} A_0 \tilde{\phi}^2 A_0 + V_k[A_0] \right) \\
- \frac{1}{2} \int d^4x \tilde{A}_{\perp}^a \left[ (D_0^a)_{ab} + \tilde{\phi}^2 \delta^{ab} \right] \tilde{A}_{\perp}^b ,
\]

with \( k \)-dependent wave function renormalisations \( Z_0, Z_i \). The effective action \((20)\) relates to the order parameter \( \langle L(\vec{x}) \rangle \) as well as its two point correlation \( \langle L(\vec{x}) L(\vec{y}) \rangle \) via the effective potential \( V_{\text{eff}}[A_0] = V_k[A_0] \) as explained in section \([11]\). The expectation value \( \langle L(\vec{x}) \rangle \), or \( \langle \langle A_0 \rangle \rangle \), is used to determine the phase transition temperature \( T_c \) as well as critical exponents. The temperature-dependence of the Polyakov loop two-point function relates to the string tension. In the confining phase, for \( T < T_c \), and large separations \( |\vec{x} - \vec{y}| \to \infty \), the two-point function falls off like

\[
\lim_{|\vec{x} - \vec{y}| \to \infty} \langle L(\vec{x}) L(\vec{y}) \rangle_c \simeq \exp \{-\beta \sigma |\vec{x} - \vec{y}|\} .
\]

Here, \( \langle \cdots \rangle_c \) stands for the connected part of the related correlation function, i.e. \( \langle L(\vec{x}) L(\vec{y}) \rangle_c = \langle L(\vec{x}) L(\vec{y}) \rangle - \langle L(\vec{x}) \rangle \langle L(\vec{y}) \rangle \). In turn, its Fourier transform shows the

\[
\text{momentum dependence}
\]

\[
\lim_{|p| \to 0} \langle L(0) L(p) \rangle_c \simeq \lim_{|p| \to 0} \frac{1}{\pi^2} \frac{\beta \sigma}{(\beta \sigma)^2 + p^2} = \frac{1}{\pi^2} \frac{1}{(\beta \sigma)^2} .
\]

We conclude that the Polyakov loop variable has a massive propagator. This directly relates to a massive propagator of \( A_0 \) in Polyakov gauge.

The approximation scheme is fully set by specifying the regulators \( R_{0,k} \) and \( R_{\perp,k} \). Naively one would identify the cut-off parameter \( k \) in the regulators with the physical cut-off scale \( k_{\text{phys}} \). For general regulators this is not possible and one deals with two distinct physical cut-off scales, \( k_{0,\text{phys}} \) and \( k_{\perp,\text{phys}} \) related to \( R_{0,k} \) and \( R_{\perp,k} \) respectively, for a detailed discussion see \([14]\). However, within the approximation \((20)\) it is crucial to have a unique effective cut-off scale \( k_{\text{phys}} = k_{0,\text{phys}} = k_{\perp,\text{phys}} \), as different physical cut-off scales \( k_{0,\text{phys}} \neq k_{\perp,\text{phys}} \) necessarily introduce a momentum transfer into the flow which carries part of the physics. This momentum transfer is only fully captured with a non-local approximation to the effective action. In turn, a local approximation such as \((20)\) requires \( k_{0,\text{phys}} = k_{\perp,\text{phys}} \). In other word, a local approximation works best if the momentum transfer in the flow is minimised. More details about such a scale matching and its connection to optimisation \([17, 20]\) can be found in \([17]\). Note in this context that in the present case we also have to deal with the subtlety that \( A_0 \) only depends on spatial coordinates whereas \( \tilde{A}_{\perp} \) is space-time dependent. However, the requirement of minimal momentum transfer in the flow is a simple criterion which is technically accessible.

More specifically we restrict ourselves to regulators \([21]\)

\[
R_{A,00} = Z_0 R_{\text{opt},k}(\vec{p}^2) , \quad R_{A,ij} = Z_i \Pi_{\perp,ij}(\vec{p}) R_{\text{opt},k}(\vec{p}^2) ,
\]

where \( R_{\text{opt},k}(\vec{p}^2) = (k^2 - \vec{p}^2) \delta(k^2 - \vec{p}^2) \).

The detailed scale-matching argument is deferred to Appendix \([C]\) and results in a relation \( k_\perp = k_\perp(k) \) depicted in Fig. 17 in the appendix. It is left to determine the effective cut-off scale \( k_{\text{phys}} \). This cut-off scale can be determined from the numerical comparison of the flows of appropriate observables: one computes the flow with the three-dimensional regulator \( R_{\text{opt},k}(\vec{p}^2) \) in \( [23] \), as well as with the four-dimensional regulator \( R_{\text{opt},k_{\text{phys}}}(\vec{p}^2) \). Then the respective physical scales are identified, i.e. \( k_{\perp,\text{phys}}(k_\perp) = k_{\text{phys}} \). The results for this matching procedure are depicted in Fig. 18 in Appendix \([C]\). Another estimate comes from the flow related to the three-dimensional \( A_0 \)-fluctuations, where we can directly identity \( k_{\text{phys}} = k \). We use the above choices as limiting cases for an estimate of the systematic error in our computation. Our explicit results are obtained for the best choice that works in all physics constraints.
V. FLOW

We are now in the position to integrate the flow equation (19). To begin with, we can immediately integrate out the spatial gauge fields $\vec{A}_\perp$ for $Z_i = 1$, that is the second line in (19). This part of the flow only carries an explicit dependence on the cut-off $k$, details of the calculation can be found in Appendix B. It results in a non-trivial effective potential $V_{\perp,k}[A_0]$ that approaches the Weiss potential [22] in the limit $k/T \to 0$, and falls off like $\exp(-\beta k_\perp(k)) \cos(g \beta A_0)$ in the UV limit $k/T \to \infty$, see Fig. 1. In terms of the effective action, after the integration over $\vec{A}_\perp$, we are led to an effective action of $A_0$,

$$\Gamma_k[A_0] = \beta \int d^4x \left( \frac{Z_0}{2} (\partial_\perp A_0)^2 + \Delta V_k[A_0] + V_{\perp,k}[A_0] \right).$$

Eq. (25) follows from (29) with $\Gamma_k[A_0] = \Gamma_k[A_0, \vec{A}_\perp = 0]$, and

$$V_k[A_0] = \Delta V_k[A_0] + V_{\perp,k}[A_0].$$

The full effective potential is given by $V_{\textrm{eff}}[A_0] = \Delta V_{k=0}[A_0] + V_{\perp,k=0}[A_0]$. We are left with the task to determine $\Delta V_k$, which is the part of the effective potential induced by $A_0$-fluctuations. In Polyakov gauge, these fluctuations carry the confining properties of the Polyakov loop variable, whereas the spatial fluctuations generate a deconfining effective potential for $A_0$, see Appendix B. We emphasise that this structure is not present for spatial confinement which is necessarily also driven by the spatial fluctuations, and solely depends on these fluctuations in the high temperature limit. We hope to report on this matter in the near future.

Here we proceed with the integration of the flow for the potential $\Delta V_k$. To that end we reformulate the flow (19) as a flow for $\Delta V_k$ with the external input $V_{\perp,k}$, see (28). The flow equation for $\Delta V_k$ reads

$$\beta \frac{\partial}{\partial t} \Delta V_k = \frac{1}{2} \int d^3p \frac{2 \pi^3}{3} \frac{Z_0 \vec{p}^2 + \partial_\perp^2 (\Delta V_k + V_{\perp,k}) + R_{0,k}}{\partial_\perp^2 (\Delta V_k + V_{\perp,k}) + R_{0,k}}.$$ (27)

With the specific regulator $R_k$ in (23) we can perform the momentum integration analytically. We also introduce the scalar field $\varphi = g \beta A_0$, and arrive at

$$\beta \frac{\partial}{\partial \varphi} \Delta V_k = \frac{1}{3} \frac{2(1 + \eta_0/5)k^2}{1 + \frac{2}{k^2} \partial_\varphi^2 (\Delta V_k + V_{\perp,k})},$$ (28)

where the coupling $g_k^2$ has to run with the effective cut-off scale $k_{\textrm{phys}}$, and is estimated by an appropriate choice of the running coupling $\alpha_s$,

$$g_k^2 = \frac{g^2}{Z_0}, \quad \text{with} \quad g_k^2 = 4 \pi \alpha_s(k_{\textrm{phys}}^2),$$ (29)

see also Appendix B. This also entails that the anomalous dimension $\eta_0$ is linked to the running coupling by

$$\eta_0 = -\partial_\varphi \log \alpha_s(k_{\textrm{phys}}^2).$$ (30)

At its root (28) is an equation for the dimensionless effective potential $\tilde{V} = \beta^4 V_k$ in terms of $\tilde{V}_\perp = \beta^4 V_{\perp,k}$ and $\tilde{\Delta} V = \beta^4 \Delta V_k$. The infrared RG-scale $k$ naturally turns into the modified RG-scale $k = k/\beta$, that is all scales are measured in temperature units. Then the flow equation is of the form

$$\partial_k \Delta \tilde{V} = \frac{2}{3(2\pi)^2} \frac{(1 + \eta_0/5)k^2}{1 + \frac{2}{k^2} \partial_\varphi^2 (\tilde{V}_\perp + \Delta \tilde{V})}.$$ (31)

The potential $\tilde{V}$ and hence $\Delta \tilde{V}$ has a field-independent contribution which is related to the pressure. For the present purpose it is irrelevant and we can conveniently normalise the flow (31) such that it vanishes at fields where $\partial_\varphi^2 (\tilde{V}_\perp + \Delta \tilde{V}) = 0$. This is achieved by subtracting $2(1 + \eta_0/5)k^2/(3(2\pi)^2)$ in (31) and we are left with

$$\partial_k \Delta \tilde{V} = \frac{1}{6\pi^2} \left( 1 + \frac{\eta_0}{5} \right) \frac{g_k^2 \partial_\varphi^2 (\tilde{V}_\perp + \Delta \tilde{V})}{1 + \frac{2}{k^2} \partial_\varphi^2 (\tilde{V}_\perp + \Delta \tilde{V})}.$$ (32)

where we have kept the notation $\partial_k \Delta \tilde{V}$ for $\partial_k \Delta \tilde{V} - 2(1 + \eta_0/5)k^2/(3(2\pi)^2)$. In this form it is evident, that the flow vanishes for fields where $\partial_\varphi^2 (\tilde{V}_\perp + \Delta \tilde{V}) = 0$, i.e. once a region of the potential becomes convex, this part is frozen, unless the external input $V_{\perp}$ triggers the flow again.

We close this section with a discussion of the qualitative features of (22). It resembles the flow equation of a real scalar field theory, and due to $V_{\perp}$, the flow is initialised in the broken phase. It relies on two external inputs, $V_{\perp}$ and $\alpha_s$.

The first input, $V_{\perp}$, is computed in a perturbative approximation to the spatial gluon sector, and its computation is deferred to Appendix B. It is shown in Fig. 1 for various values of the RG time $k$, and approaches the perturbative Weiss potential [22] for vanishing cutoff $k = 0$. We have argued that within Polyakov gauge this approx-

![FIG. 1: $\tilde{V}_\perp$ for different values of $k$](image)
again that this is not so for the question of spatial confinement, and the related potential of the spatial Wilson loops.

The second input is \( \alpha_s = g^2_s/(4\pi) \), the running gauge coupling. It runs with the physical cut-off scale \( k_{\text{phys}} \) derived in Appendix C, \( \alpha_s = \alpha_s(k^2_{\text{phys}}) \). In the present work we model \( \alpha_s \) with a temperature-dependent coupling that runs into a three-dimensional fixed point \( \alpha_{s,3d}k_{\text{phys}}/T \) for low cut-off scales \( k_{\text{phys}}/T \ll 1 \). This choice carries some uncertainty as the running coupling in Yang-Mills theory is not universal beyond two loop order. Here we have chosen the Landau gauge couplings \( \alpha_{\text{Landau},4d}(k^2_{\text{phys}}) \) at cut-off scales \( k_{\text{phys}}/T \gg 1 \), see \[2, 1, 23, 24, 27, 28\]. The corresponding three-dimensional fixed point \( \alpha_{s,3d} = 1.12 \) is obtained from \[23\]. A specific choice for such a running coupling is given in Fig. 2.

This error includes that related to our specific choice of the running coupling. For example, a viable alternative choice to Fig. 2 is provided by the background field coupling derived in \[23\] which is covered by the above error estimate.

VI. INTEGRATION

The numerical solution of \[32\] is done on a suitably chosen grid or parameterisation of \( \Delta \dot{V} \) and its derivatives. As \( \dot{V}, \dot{V}_1 \) and \( \Delta \dot{V} \) are periodic, one is tempted to solve \[32\] in a Fourier decomposition, see e.g. \[27\]. However, as can be seen already at the example of the perturbative Weiss potential \( V_W = V_{\perp,0} \), \[19\], this periodicity is deceiving. The Weiss potential is polynomial of order four in \( \tilde{\phi} = \varphi \mod 2\pi \), its periodicity comes from the periodic \( \varphi \), \[22\]. Consequently the third derivative \( \partial_\varphi^2 V_W \) jumps at \( \varphi = 2\pi n \) with \( n \in \mathbb{Z} \). Moreover, \( \partial_\varphi^2 V_W(\varphi \to 0_+) = \partial_\varphi^2 V_W(\varphi \to 0_-) \neq 0 \). A periodic expansion of \( V_W \), e.g. in trigonometric functions cannot capture this property at finite order. This does not only destabilise the parameterisation, but also fails to capture important physics: the flow of the position of the minima is proportional to \( \partial_\varphi \dot{V} \). This follows from \( \partial_\varphi \dot{V}(\varphi_{\text{min}}) = 0 \). Expanding this identity leads to

\[
\partial_\varphi \dot{V}(\varphi_{\text{min}}) = \frac{\partial_\varphi \dot{V}(\varphi)}{V''(\varphi)} \bigg|_{\varphi = \varphi_{\text{min}}},
\]

where \( \dot{V}' = \partial_\varphi \dot{V} \) and \( \dot{V}'' = \partial_\varphi^2 \dot{V} \). The flow \( \partial_\varphi \dot{V}(\varphi) \) is proportional to \( \partial_\varphi \dot{V} \), which e.g. can be seen by taking the \( \varphi \)-derivative of \[19\]. Hence, as a Fourier-decomposition enforces \( \partial_\varphi \dot{V} = 0 \) at any finite order, the minimum does not flow in such an approximation, and the theory always remains in the deconfined phase. Note also that the resulting effective potential at \( \dot{k} = 0 \) for smooth periodic potentials and flows vanishes identically as it has to be convex. In the present case this is not so, as the potential is rather polynomial (in \( \tilde{\phi} \)) and convexity does not enforce a vanishing effective potential.

In turn, a standard polynomial expansion about the minimum \( \rho_{\text{min}} \) already captures the flow towards the confining phase. Here, however, we use a grid evaluation of the flow of \( \Delta V \) with \( \varphi \in [0, 2\pi] \) while taking special care of the boundary conditions at \( \varphi = 0, 2\pi \): we have extrapolated the second derivative to \( \varphi = 0 \) and \( \varphi = 2\pi \). It suffices to use a first order extrapolation, and we have explicitly checked that the resulting flow is insensitive to the precision of the extrapolation.

An alternative procedure is an expansion in terms of Chebyshev polynomials that also works quite well and is also a very fast and efficient way of integrating the flow. A comparison between the results obtained on a grid and with Chebyshev polynomials shows that both parameterisations agree nicely and the corresponding flows deviate from each other only for small values of \( k \). This is due to

FIG. 2: \( \alpha_s \) for temperatures \( T = 0, 150, 300, 600 \) MeV

The normalisation of the momentum scale has been done by the comparison of continuum Landau gauge propagators to their lattice analogues. Fixing the lattice string tension to \( \sqrt{\sigma} = 440 \) MeV, we are led to the above momentum scales. For a comparison with the Landau gauge results obtained in \[19\] we have also computed the temperature-dependence of the Polyakov loop by using \( \alpha_{\text{Landau},4d} \) for all cut-off scales. Indeed, this overestimates the strength of \( \alpha_s \), as can be seen from Fig. 2. However, qualitatively this does not make a difference: for infrared scales far below the temperature scale, \( \dot{k} \to 0 \), the flow switches off for fields \( \varphi \) with \( \partial_\varphi^2(V_{\perp} + \Delta V) \geq 0 \), that is for the convex part of the potential. This happens both for \( g^2_{\perp} \to \text{const} \) and for \( g^2_{\perp}(k^2 \to 0) \sim \dot{k} \). In other words, the minimum of the potential freezes out in this regime. For the non-convex part of the potential, \( \partial_\varphi^2(V_{\perp} + \Delta V) < 0 \), the flow does not tend to zero but simply flattens the potential, thus arranging for convexity of the effective potential \( V_{\text{eff}} = V_{\text{con}} \). The uncertainty in \( g^2_{\perp} \) is taken into account by evaluating the limiting cases. Together with the error estimate on the physical cut-off scale \( k_{\text{phys}} \) in Appendix C this leads to an estimate for the systematic error of the results presented below.
an expected failure of the standard Chebyshev-expansion for those small $k$ where the position of the minimum is almost settled and the potential flattens out in the regions that are not convex. This is better resolved with a grid than with polynomials. On a grid implementation we see the potential becoming convex as $k \to 0$.

VII. RESULTS

In Fig. 3 we show the full effective potential for temperatures ranging from $T = 500$ MeV in the deconfined phase to $T = 250$ MeV in the confined phase. The expectation value $\langle \phi \rangle$ in the center-broken deconfined phase is given by the transition point between decreasing part of the potential for small $\varphi$ and the flat region in the middle of the plot. It can also be explicitly computed from (33). In the center-symmetric confined phase it is just given by the minimum at $\varphi = \pi$.

![FIG. 3: Full effective potential $V_{eff}$, normalised to 0 at $\varphi = 0$](image)

The temperature-dependence of the order parameter $L[\langle A_0 \rangle] = \cos(\langle \varphi/2 \rangle)$ is shown in Fig. 4 and we observe a second order phase transition from the confined to the deconfined phase at a critical temperature

$$T_c = 305^{+40}_{-55} \text{ MeV}, \quad T_c/\sqrt{\sigma} = 0.69^{+0.04}_{-0.12}, \quad (34)$$

with the string tension $\sqrt{\sigma} = 440$ MeV. The corresponding value on the lattice is $T_c/\sqrt{\sigma} = 0.709$, which agrees within the errors with our result. The estimate of the systematic error in (34) is dominated by that of the uncertainty of the identification of $k_{phys}$, see Appendix C.

We would also like to comment on the difference of the temperature-dependence of $L[\langle A_0 \rangle]$ depicted in Fig. 3 and that of the Polyakov loop $\langle L[A_0] \rangle$. It has been shown in section 1I that in the confined phase they both vanish and both are non-zero in the deconfined phase. However, the Jensen inequality entails that the present observable $L[\langle A_0 \rangle]$ takes bigger values than the Polyakov loop $\langle L[A_0] \rangle$, which is in agreement with lattice results.

![FIG. 4: Temperature dependence of the Polyakov loop $L[\langle A_0 \rangle] = \cos(\langle \varphi/2 \rangle)$ in $SU(2)$](image)

The critical physics should not depend on this issue. Here we compute the critical exponent $\nu$, a quantity well-studied in the $O(1)$ model which is in the same universality class as $SU(2)$ Yang-Mills theory. Moreover, in Polyakov gauge the effective action $\Gamma[A_0]$ after integrating-out the spatial gauge field is close to that of an $O(1)$-model. Studies using the FRG in local potential approximation with an optimised cut-off for the $O(1)$ model yield $\nu = 0.65$, see [28]. The critical exponent is related to the screening mass of temporal gauge field by

$$m^2(T) \propto |T - T_c|^{2\nu}, \quad (35)$$

where $m^2 = V''(\varphi_{min,0})/2$. We have computed the temperature-dependence of the screening mass in the confined phase near the phase transition, and extracted the critical exponent $\nu$ from a linear fit to the data. This is shown in Fig. 5. The fit yields the anticipated value of

$$\nu = 0.65^{+0.02}_{-0.01}, \quad (36)$$

for the critical exponent $\nu$. The critical exponent $\beta$ agrees within the errors with the Ising exponent $\beta = 0.33$.

Finally we would like to compare the results obtained here with the results of [19]. There, the effective potential $V_{eff}[A_0]$ was computed from the flow of Landau gauge propagators within a background field approach in Landau-DeWitt gauge. In this gauge the confining properties of the theory are encoded in the non-trivial momentum dependence of the gluon and ghost propagators. Indeed, in [19] the effective potential $V_K$ was computed solely from this momentum dependence and was not fed back into the flow. In $SU(2)$ Landau gauge Yang-Mills this is expected to be a good approximation with the exception of temperatures close to the phase transition, see [19]. The back-reaction of the effective potential is particularly important for the critical physics, and the value of the critical temperature.

For the comparison we have computed the present flow with the zero-temperature running coupling in Fig. 6 for
in Fig. 6. The coincidence between the two gauges is

all temperatures. This mimics the approximation used in [19], which implicitly relies on the zero-temperature running coupling $\alpha_s$. We also remark that the quantity $L(\langle A_0 \rangle)$ in general is gauge-dependent, and only the critical temperature derived from it is not. However, in Landau-DeWitt gauge with backgrounds $A_0$ in Polyakov gauge temporal fluctuations about this background include those in Polyakov gauge. For this reason we might expect a rather quantitative agreement for the quantity $L(\langle A_0 \rangle)$ in both approaches. The results for the temperature dependence of the Polyakov loop are depicted in Fig. 6. The coincidence between the two gauges is

very remarkable, particularly since the mechanisms driving confinement are quite different in the different approaches, as are the approximations used in both cases. This provides further support for the respective results. It also sustains the argument concerning the lack of gauge dependence made above. The quantitative deviations in the vicinity of the phase transition are due to the truncation used in [19], that cannot encode the correct critical physics yet, as has been already discussed there.

VIII. SUMMARY AND OUTLOOK

In the present work we have put forward a formulation of QCD in Polyakov gauge. We have argued that this gauge is specifically well-adapted for the investigation of the confinement-deconfinement phase transition as the order parameter, the Polyakov loop expectation value $\langle L(A_0) \rangle$, has a simple representation in terms of the temporal gauge field. Moreover, we have shown that $L(\langle A_0 \rangle)$ also serves as an order parameter. In summary this allows us to access the phase transition within a simple approximation to the full effective action.

The computation was done for the gauge group $SU(2)$, where we observe a second order phase transition at a critical temperature of $T_c = 305_{-52}^{+40}$ MeV, as well as the Ising critical exponents $\nu$ and $\beta$ to the precision achieved within our approximation. The temperature-dependence of the order parameter $L(\langle A_0 \rangle)$ agrees well with a recent computation in Landau gauge [19]. This is very remarkable: firstly the latter computation is technically different as in Landau gauge the full momentum-dependence of the propagators is needed to cover confinement. Second, the order parameter $L(\langle A_0 \rangle)$ is gauge dependent, only the critical temperature is not.

In the present analysis we used several external inputs which we plan to remove in future work. First of all we proceed with computing the running coupling within Polyakov gauge, that is the momentum-dependence of the temporal gauge field. As it is one of the advantages of the computation in Polyakov gauge that the momentum dependence of Green functions is rather mild we expect only minor deviations from the computations shown here. As argued in the present work, the momentum-dependence of the $A_0$-propagator also gives access to the string tension. For a description of spatial confinement one has to treat the spatial components of the gauge field beyond the present perturbative approximation. Moreover, the present analysis is extended to $SU(3)$, which is conceptually straightforward but technically more challenging. For the matter sector one can revert to the plethora of results with the present renormalisation group methods, ranging from results in effective theories to that in QCD-based approaches, see e.g. [4, 13, 15, 29, 30].

Acknowledgements – We thank J. Braun, H. Gies, F. Lamprecht, D. F. Litim, A. Maas and B.-J. Schaefer for discussions. We thank O. Jahn for discussions and collaboration at an early state of this project. FM acknowledges financial support from the state of Baden-Württemberg and the Heidelberg Graduate School of Fundamental Physics.
APPENDIX A: FADDEEV-POPOV DETERMINANT

From the gauge fixing functionals (12) and (13) we can compute the Faddeev-Popov determinant given by

$$\Delta_{FP}[A] = \det \left[ \frac{\delta F^0(A^\omega)}{\delta \omega^\lambda} \right] , \quad (A1)$$

where $A^\omega$ is the gauge transformed gauge field $A$. For infinitesimal gauge transformations it is given by

$$A^\omega_\mu = A_\mu - (\partial_\mu \sigma^a + igA^b_\mu [\sigma^a, \sigma^b]) \omega^a . \quad (A2)$$

In the following we use the representation $\omega^a \sigma^a = \omega^+ \sigma^- + \omega^- \sigma^+ + \omega^3 \sigma^3$, and the related derivatives w.r.t. $\omega^+, \omega^3$. The matrix elements related to $\omega$-derivatives of $F^+$ read

$$\frac{\delta F^+(A^\omega)}{\delta \omega^+} = -\text{Tr} \sigma^+ \left( \partial_0 \sigma^- + iA^3_0 [\sigma^-, \sigma^+] \right) ,$$

$$\frac{\delta F^+(A^\omega)}{\delta \omega^-} = 0 ,$$

$$\frac{\delta F^+(A^\omega)}{\delta \omega^3} = -\text{Tr} \sigma^+ \left( \partial_0 \sigma^3 + iA^3_0 [\sigma^3, \sigma^-] \right) . \quad (A3)$$

Evaluating the traces (A3), (A4), (A5) we can compute the Faddeev-Popov determinant. Again we only concentrate on the terms dependent on $A_0$, and use the gauge fixing condition $A^+_0 = A^-_0 = 0$ for eliminating some of the off-diagonal elements,

$$\Delta_{FP}[A] = -\det \left[ \begin{array}{ccc} 0 & 0 & 0 \\ -4ig \int dx_0 \partial_1 A^+ - \cdots & 4ig \int dx_0 \partial_1 A^- - \cdots & 1/2(\partial_0^2 + \int dx_0 \partial_1^2 + \cdots) \\ 0 & 0 & 0 \end{array} \right]$$

$$= -\det[\partial_0 + igA^3_0] = \frac{1}{2} \left( \partial_0^2 + \int dx_0 \partial_1^2 + \int dx_0 dx_1 \partial_2^2 + \int dx_0 dx_1 dx_2 \partial_3^2 \right) . \quad (A6)$$

Using the third gauge fixing condition, $\partial_0 A^3_0 = 0$, we can write the Faddeev-Popov determinant as

$$\Delta_{FP}[A] = \frac{1}{2} \det \left[ \left( \partial_0^2 + (gA^3_0)^2 \right) \right] \det[\partial_0^2 + \cdots] .$$

We note that the second determinant in (A7) is independent of the gauge fields and hence can be absorbed in the normalisation of the path integral. The first determinant is evaluated in frequency space, we get

$$\prod_{\vec{x}} \left( (gA^3_0(\vec{x})) \prod_{n=1}^{n=\infty} \left( (2\pi T n)^2 - (gA^3_0(\vec{x}))^2 \right) \right)^2 . \quad (A7)$$

Multiplying the determinant (A7) with a further constant normalisation

$$\mathcal{N} = \left( \prod_{n=1}^{n=\infty} (2\pi T n)^2 \right)^{-2} , \quad (A8)$$

we arrive at

$$\mathcal{N} \det [G_{A_0}] = \prod_x (gA^3_0(x))^2 \prod_{n=1}^{n=\infty} \left( 1 - \left( \frac{gA^3_0(x)}{2\pi n T} \right)^2 \right) . \quad (A9)$$

Eq. (A9) is just a product representation of the sine-function, $\sin(x) = x \prod_{n=1}^{n=\infty} \left( 1 - \frac{x^2}{(\pi n)^2} \right)$, and the final result for the Faddeev-Popov determinant is

$$\Delta_{FP}[A] = \mathcal{N}'(2T)^2 \left[ \prod_x \sin^2 \left( \frac{gA^3_0(x)}{2T} \right) \right] , \quad (A10)$$

where $\mathcal{N}'$ is a further normalisation constant.
APPENDIX B: INTEGRATING OUT SPATIAL GLUONS

After integrating out the longitudinal gauge fields the action \( S_{\text{eff}} = \frac{1}{4} \int d^4 x F^a_{\mu \nu} F^a_{\mu \nu} \) reads

\[
S_{\text{eff}} = -\frac{1}{2} \beta \int d^3 x Z_0 A_0 \partial^2 A_0 - \frac{1}{2} \int d^4 x A^a_i \left[ (\partial_0^2 + \partial_i^2) \delta_{ij} - \partial_i \partial_j \right] + 2g f^{ab3} (A_0 \partial_0 + g^2 A_0^2 (\delta^{ab} - \delta^i \delta^j) \delta_{ij}) A_j^a + O(A^3) \tag{B1}
\]

Writing \( A_0^3 = \varphi/(g \beta) + a_0 \), where \( \varphi \) is a constant and \( a_0 \) the fluctuating field, this expression is given to second order in the fluctuating fields by

\[
S_{YM} \approx \frac{1}{2} \int d^3 x \left\{ Z_0 (\partial a_0)^2 - 2 \varphi f^{ab3} (\partial a_0 A_i^a) A_i^b + \varphi^2 (\delta^{ab} - \delta^i \delta^j) A_i^a A_j^b - A_i^a \left( (\partial_0^2 + \partial_i^2) \delta_{ij} - \partial_i \partial_j \right) A_j^a \right\} = \frac{1}{2} \int d^3 x \left\{ (\partial a_0)^2 - A_0^2 (\partial_0^2 - \partial_i \partial_j) A_j^a - A_0^a D_0^{ac} D_0^{cb} A_b^c \right\}, \tag{B2}
\]

where we have defined

\[
D_0^{ab} = \partial_0 \delta^{ab} + A_0^3 g f^{ab3}. \tag{B3}
\]

In the present work we neglect back-reactions of the \( A_0 \) potential on the transverse gauge fields. Assuming an expansion around \( A_0^a = 0 \), \( \Gamma^{(2)} \) is block-diagonal, like the regulators, cf. eq. \((28)\), and we can decompose the flow equation \((19)\) into a sum of two contributions, schematically written as

\[
\partial_t \Gamma_k = \frac{1}{2} \begin{array}{c} \text{Tr} \left( \frac{1}{\Gamma^{(2)}_k + R_A} \right) \partial_t R_k + \text{Tr} \left( \ln(S_{YM}^{(2)} + R_A) \right) \end{array} \tag{B4}
\]

The first term on the rhs encodes the quantum fluctuations of \( A_0 \), the second line encodes those of the transversal spatial components of the gauge field. In the present truncation the second line is a total derivative w.r.t. \( t \), and does not receive contributions from the first term. Therefore we can evaluate the flow of the second contribution, and use its output \( V_{\perp,k}(A_0) \) as an input for the remaining flow.

The computation is done for the regulators \((23)\). As explained below \((22)\) in section \(V\) the cut-off parameters \( k_\perp \) and \( k_\perp \) in \( R_k \) for the fluctuations of \( A_0 \) and \( R_{k,\perp} \) for the fluctuations of \( \vec{A}_\perp \) respectively satisfy a non-trivial relation \( k_0 \perp = k_\perp(k) \) for coinciding physical infrared cut-offs \( k_0 \) for \( A_0 \) and \( k_\perp \) for \( \vec{A}_\perp \). The computation is similar to those done in one loop perturbation theory in \( SU(2) \) by Weiss \((22)\), the only difference being the infrared cut-off. We infer from the second line in \((B3)\) that

\[
V_{\perp,k} = V_{\perp,k}^{UV} + \frac{1}{2} \text{Tr} \left[ \ln(S_{YM}^{(2)} + R_A) \right]_{\text{eff}}^{k} \tag{B5}
\]

\[
= V_W + T \sum_n \int \frac{dT}{(2\pi)^2} \delta(k_\perp^2 - p^2) \ln(k_\perp^2 + D_0^2). \tag{B6}
\]

In \((15)\) we have used that \( V_{\perp,k}^{UV} \to 0 \) up to a constant term, and have added and subtracted the Weiss potential \( V_W \) \((22)\),

\[
V_W(\varphi) = -(\varphi - \pi)^2/(6\beta^4) + (\varphi - \pi)^4/(12\beta^2 \beta^4), \tag{B7}
\]

with the dimensionless \( \varphi = g \beta A_0 \), and \( \varphi = \varphi \mod 2\pi \). Alternatively one can simply put \( \Lambda_{UV} = 0 \), even though this seems to be counter-intuitive. We also have used that with \((15)\) it follows \( \text{tr} \Pi_\perp = 2 \). Performing the Matsubara sum and neglecting terms independent of the temporal gauge fields, the resulting effective potential is given by

\[
V_{\perp,k} = \frac{4T}{(2\pi)^2} \int_0^k \frac{dT}{(2\pi)^2} \left\{ \ln \left( 1 - 2 \cos(\varphi)e^{-\beta k_\perp} \right) + e^{-2\beta k_\perp} \right\} + V_W. \tag{B7}
\]

From \((B7)\) we deduce that the potential \( V_{\perp,k} \) approaches \( V_W \) in the limit \( k \to 0 \) and vanishes like \( e^{-2\beta k_\perp} \cos(\varphi) \) for \( k \to \infty \). From eq. \((B7)\) we can now extract the flow of the effective potential, by setting \( V_{\text{eff},k} = \Delta V_k + V_{\perp,k} \).

Then we get

\[
\partial_t \Delta V_k = \frac{1}{2} \int \frac{dT}{(2\pi)^2} \frac{\delta \varphi k_\perp^2 \theta(k_\perp^2 - p^2) + 2k_\perp k_\perp \partial_k^2 \Delta(V_k^\perp + V_{\perp,k})}{k_\perp^2 + g_k^2 \beta^2 \Delta(V_k^\perp + V_{\perp,k})}, \tag{B8}
\]

with the input \( V_{\perp,k} \) given in \((B7)\) and \( \delta \varphi = \partial_k \ln Z_0 \). The factor \( g_k^2 \beta^2 \) arises from the fact that we parametrise the potential in terms of \( \varphi \) rather than in \( A_0 \), and \( g_k^2 = g^2/Z_0 \) is nothing but the running coupling at momentum \( p^2 \sim k^2_{\perp,\text{phys}} \). Thus we estimate \( g_k^2 = 4\pi \alpha_s(p^2) \approx k^2_{\perp,\text{phys}} \). Note that \( g_k \) is an RG-invariant. The momentum integration can be performed analytically, and we are led to

\[
\beta \partial_k \Delta V_k = \frac{2}{3(2\pi)^2} \frac{(1 + \eta_0/5)k^2}{1 + g_k^2 \beta^2 \delta \varphi (V_{\perp,k} + \Delta V_k)}, \tag{B9}
\]

where \( \eta_0 \) is given by

\[
\eta_0 = -\partial_k \log \alpha_s, \tag{B10}
\]

as the consistent choice in the given truncation.
APPENDIX C: MATCHING SCALES

The flow of the temporal component of the gauge field, \( A_0(\vec{x}) \), is computed with a three-dimensional regulator, see \( ^{[23]} \). In Polyakov gauge \( A_0(\vec{x}) \) only depends on the spatial coordinates, whereas the spatial components \( A_{\perp}(x) \) are four-dimensional fields. For cut-off scales far lower than the temperature, \( k/T \ll 1 \), also the spatial gauge fields are effectively three-dimensional fields as only the Matsubara zero mode propagates. Hence in this regime we can identify \( k = k_{\perp} \). For large cut-off scales, \( k/T \gg 1 \), the \( A_0 \)-flow decouples from the theory. A comparison between the two flows can only be done after the summation of the spatial flow over the Matsubara frequencies. In the asymptotic regime \( k/T \gg 1 \) this leads to the relation

\[
\frac{1}{k} \simeq \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + k_{\perp}^2} \rightarrow \frac{1}{2k_{\perp}}, \quad (C1)
\]

The crossover between these asymptotic regimes happens at about \( k/T = 1 \). This crossover is implemented with the help of an appropriately chosen interpolating function \( f \).

\[
\frac{T}{k^2} f(k/T) = T \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + k_{\perp}^2}, \quad (C2)
\]

A natural choice for \( f(k/T) \) is depicted in Fig. 7 and has been used in the computation. A more sophisticated adjustment of the relative scales can be performed within a comparison of the flow of momentum-dependent observables such as the wave function renormalisation \( Z_0 \). The peak of these flows in momentum space is directly related to the cut-off scale. Indeed, the function \( f \) carries the physical information of the peak of the flow at some momentum scale. Scanning the set of \( f \) gives some further access to the uncertainty in such a procedure. The effective cut-off scales \( k_{\text{phys}}(k_0) \) and \( k_{\perp, \text{phys}}(k_\perp) \) in the flows of the temporal gluons and of spatial gluons respectively do not match in general. If solving the flow within a local truncation as chosen in the present work we have to identify the two effective cut-off scales, \( k_{\text{phys}}(k_0) = k_{\perp, \text{phys}}(k_\perp) = k_{\text{phys}} \), leading to a non-trivial relation \( k_0 = k_0(k_\perp) \). Moreover, the effective cut-off scale has to be used in the running coupling \( \alpha_s = \alpha_s(\hat{p}^2 = k_{\text{phys}}^2) \).

\[
\text{FIG. 7: } \hat{k}_{\perp}/\hat{k} \text{ as function of } \hat{k}.
\]

It is left to determine the physical cut-off scale \( k_{\text{phys}} \) from either the flow of the spatial gauge fields as \( k_{\perp, \text{phys}}(k_\perp) \) or from the temporal flow \( k_{0, \text{phys}}(k_0) \). We first discuss the spatial flow. For an optimised regulator depending on all momentum directions, \( p^2 \), we have the relation \( k_{\text{phys}} = k_\perp \). Hence the relation \( k_{\perp, \text{phys}}(k_\perp) \) can be computed if comparing the flows for a specific observable with three-dimensional regulator \( R_{\text{opt}, k_\perp}(p^2) \). \( ^{[23]} \), with flows with four-dimensional regulator \( R_{\text{opt}, k_{\text{phys}}}(p^2) \). Here, as a model example, we choose the effective potential of a \( \phi^4 \)-theory. This leads to the relation \( k_{\text{phys}}(k_\perp) \) displayed in Fig. 8. We remark that the relation in Fig. 8 depends on the dimension \( d \) of the theory, and flatten to \( k_{\text{phys}}(k) = k \) for \( d \rightarrow \infty \). In other words, \( \lim_{k \rightarrow \infty} k_{\text{phys}}(k)/k \) is proportional to \( d/(d-1) \). Moreover, for momentum-dependent observables the crossover rather resembles the relation \( k_\perp(k_0) \) as it is more sensitive to the propagator than to the momentum integral of the propagator. Indeed, for the three-dimensional field \( A_0(\vec{x}) \) the cut-off scale \( k_0 \) is another natural choice for the physical cut-off scale, \( k_{0, \text{phys}}(k_0) = k_0 \), even though it underestimates the importance of the spatial flow for the correlators of the temporal gauge field. In summary, we take the above two extremal choices \( k_{\text{phys}} = k_{\perp, \text{phys}} \) depicted in Fig. 8 and \( k_{\text{phys}} = k_{0, \text{phys}}(k_0) = k_0 \) as a broad estimate of the systematic error in the present computation.

\[
\text{FIG. 8: } \hat{k}_{\text{phys}}(\hat{k}) \text{ from the comparison of flows with three-dimensional regulators and four-dimensional regulators.}
\]
