Error estimates of semi-discrete and fully discrete finite element methods for the Cahn-Hilliard-Cook equation *

Ruisheng Qi\textsuperscript{a}, Xiaojie Wang\textsuperscript{b}

\textsuperscript{a} School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Qinhuangdao, China
qirsh@neuq.edu.cn
\textsuperscript{b} School of Mathematics and Statistics, Central South University, Changsha, China
x.j.wang7@csu.edu.cn and x.j.wang7@gmail.com

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Abstract

In two recent publications [Kovács, Larsson, and Mesforush, SIAM J. Numer. Anal. 49(6), 2407-2429, 2011] and [Furihata, et al., SIAM J. Numer. Anal. 56(2), 708-731, 2018], strong convergence of the semi-discrete and fully discrete finite element methods are, respectively, proved for the Cahn-Hilliard-Cook (CHC) equation, but without convergence rates revealed. The present work aims to fill the left gap, by recovering strong convergence rates of (fully discrete) finite element methods for the CHC equation. More accurately, strong convergence rates of the full discretization are analyzed, realized by using Galerkin finite element methods based on piecewise continuous polynomials of degree at most \(r-1\), \(r \geq 2\) for the spatial discretization and the backward Euler method for the temporal discretization. Different from the stochastic Allen-Cahn equation, the presence of the unbounded elliptic operator in the front of the cubic nonlinearity in the underlying model makes the error analysis much more challenging and demanding. To address such difficulties, several new estimates and techniques are introduced. It is shown that the fully discrete scheme possesses convergence rates of order \(O(h^{\min\{\gamma, r\}} |\ln h|)\) in space and order \(O(k^{\min\{\gamma, r\}} |\ln k|)\) in time, where \(\gamma \in [3, 4]\) from the assumption \(\|A^{\frac{\gamma}{2}} Q^{\frac{1}{2}}\|_{\mathcal{L}_2} < \infty\) is used to characterize the spatial correlation of the noise process. In particular, a classical convergence rate of order almost \(O(h^4 + k)\) can be reached, even in multiple spatial dimension, when \(r = 4\) and the aforementioned assumption is fulfilled with \(\gamma = 4\).

Key words. Cahn-Hilliard-Cook equation, finite element method, strong convergence rates.

1 Introduction

Over the last twenty years, numerical approximations of stochastic partial differential equations (SPDEs) with globally Lipschitz coefficients have been extensively and well studied, see the monographs \cite{22, 27, 33} and references therein. By contrast, numerical analysis of SPDEs with non-globally Lipschitz coefficients is, in our opinion, at an early stage and far from being well-understood. A typical SPDE model with non-globally Lipschitz coefficients is the stochastic Allen-Cahn equation, which has been numerically studied by many researchers

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Recently, see e.g., [1, 5, 9, 12, 15, 16, 18, 20, 21, 23, 24, 31, 32, 34]. As another prominent SPDE model with non-globally Lipschitz coefficients, the Cahn-Hilliard-Cook (CHC) equations, also named stochastic Cahn-Hilliard equation in the literature, are, however, much less investigated. As far as we know, only a few publications are devoted to the numerical research of the CHC equation [10, 13, 17, 19, 25, 30]. Particularly, strong convergence of the semi-discrete and fully discrete finite element methods are, respectively, proved in [25] and [17] for the CHC equation, but without convergence rates recovered. The present article attempts to fill the left gap, by recovering strong convergence rates of the (fully discrete) finite element methods for the CHC equation.

Let $D \subset \mathbb{R}^d, d \in \{1, 2, 3\}$ be a bounded open spatial domain with smooth boundary and let $H := L_2(D; \mathbb{R})$ be the real separable Hilbert space endowed with the usual inner product and norm. Throughout the paper we are interested in the following Cahn-Hilliard-Cook equation perturbed by noise in $\dot{H} := \{v \in H : \int_D v \, dx = 0\},$

$$\begin{cases}
    \frac{du}{dt} - \Delta u + f(u), & \text{in } \Omega, \forall t \in [0, T], \\
    u(0, x) = u_0, & \text{in } D,
\end{cases}$$

where $f(s) = s^3 - s, s \in \mathbb{R}$.

Following the framework of [13] we rewrite (1.1) as an abstract evolution equation of the form,

$$\begin{cases}
    \frac{dX}{dt} + A(AX(t) + F(X(t))) = 0, & t \in (0, T], \\
    X(0) = X_0,
\end{cases}$$

where $-A$ is the Laplacian with homogeneous Neumann boundary conditions, generating an analytic semigroup $E(t)$ in $\dot{H}$. Similarly as in [17, 25], $\{W(t)\}_{t \geq 0}$ is assumed to be an $\dot{H}$-valued $Q$-Wiener process on a filtered probability $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$. The nonlinear mapping $F$ is supposed to be a Nemytskij operator, given by $F(u)(x) = f(u(x)), x \in D$. Moreover, $X(t)$ and $X(t) + AF(X(t))$ represent a scaled concentration and the chemical potential, respectively.

The deterministic version of such equation is used to describe the complicated phase separation and coarsening phenomena in a melted alloy [6, 8] that is quenched to a temperature at which only two different concentration phases can exist stably. The existence and uniqueness of the solution to (1.2) have been already proved by Da Prato and Debussche [13]. The space-time regularities of the weak solution of (1.2) have been examined in [17, 28]. The first goal of this paper is to provide further results on the regularities of the mild solution to (1.2) based on existing results from [13, 17, 28]. Under further assumptions specified later, particularly including

$$\|A^{\frac{\gamma}{4}}Q^\frac{1}{2}\|_{L_2} < \infty, \text{ for some } \gamma \in [3, 4],$$

Theorems [4.3, 2.6] tell us that, the underlying problem (1.2) admits a unique mild solution $X(t)$, given by

$$X(t) = E(t)X_0 - \int_0^t E(t-s)AF(X(s)) \, ds + \int_0^t E(t-s)\, dW(s),$$

which enjoys the following spatial-temporal regularity properties,

$$X \in L_\infty([0, T]; L^p(\Omega; \dot{H}^\gamma)), \forall p \geq 1,$$

and for $\forall p \geq 1$ and $0 \leq s < t \leq T$,

$$\|X(t) - X(s)\|_{L^p(\Omega; \dot{H}^\beta)} \leq C(t-s)^{\min\{\frac{\gamma}{4}, \frac{3}{2}\}}, \beta \in [0, \gamma].$$

Here $\dot{H}^\alpha := D(A^\frac{\alpha}{2}), \alpha \in \mathbb{R}$ and the parameter $\gamma \in [3, 4]$ coming from [15, 30] quantifies the spatial regularity of the covariance operator $Q$ of the driving noise process.

The second aim of this article is devoted to error estimates of the finite element approximation of (1.2). Let $\mathcal{V}_h \subset H^1(D) \cap H$ be the space of continuous functions that are piecewise polynomials of degree at most $r-1$
for some integer \( r \geq 2 \) and \( X_h(t) \in \tilde{V}_h \) be the finite element spatial approximation of the mild solution \( X(t) \), which can be represented by

\[
X_h(t) = E_h(t)P_hX_0 - \int_0^t E_h(t-s)P_hPF(X_h(s))\,ds + \int_0^t E_h(t-s)P_h\,dW(s), \quad t \in [0, T].
\]  

(1.7)

Here \( E_h(t) := e^{-tA_h^2} \) is the strongly continuous semigroup generated by the discrete Laplace operator \( A_h \). The resulting spatial approximation error, as implied by Theorem 4.1, is measured as follows,

\[
\|X(t) - X_h(t)\|_{L^p(\Omega; \mathcal{H})} \leq C h^r |\ln h|, \quad \kappa := \min\{\gamma, r\}.
\]  

(1.8)

To arrive at the above error estimate, we introduce an auxiliary approximation process \( \tilde{X}_h \), defined by

\[
\tilde{X}_h(t) = E_h(t)P_hX_0 - \int_0^t E_h(t-s)P_hPF(X_h(s))\,ds + \int_0^t E_h(t-s)P_h\,dW(s), \quad t \in [0, T],
\]  

(1.9)

and split the considered error \( \|X(t) - X_h(t)\|_{L^p(\Omega; \mathcal{H})} \) into two parts:

\[
\|X(t) - X_h(t)\|_{L^p(\Omega; \mathcal{H})} \leq \|X(t) - \tilde{X}_h(t)\|_{L^p(\Omega; \mathcal{H})} + \|\tilde{X}_h(t) - X_h(t)\|_{L^p(\Omega; \mathcal{H})}. \tag{1.10}
\]

In a semigroup framework, one can straightforwardly treat the first error term and show \( \|X(t) - \tilde{X}_h(t)\|_{L^p(\Omega; \mathcal{H})} = O(h^\kappa |\ln h|) \), with the aid of the well-known estimates for the error operators \( \Psi_h(t) := E(t) - E_h(t)P_h \) and \( \Phi_h(t) := E(t)A - E_h(t)A_hP_h \) and uniform moment bounds of \( \tilde{X}_h(t) \) and \( X(t) \). Further, we subtract (1.9) from (1.7) to eliminate the stochastic convolution and the remaining term \( \tilde{e}(t) := \tilde{X}_h(t) - X_h(t) \) is thus differentiable and satisfies

\[
\frac{d}{dt}\tilde{e}_h(t) + A_h^2\tilde{e}_h(t) = A_hP_h(F(X_h(t)) - F(X(t))), \quad \tilde{e}_h(0) = 0,
\]  

(1.11)

whose solution is given by

\[
\tilde{e}_h(t) = \int_0^t E_h(t-s)A_hP_h\left(F(X_h(s)) - F(X(s))\right)\,ds.
\]  

(1.12)

Note that the tough term \( \|\tilde{e}(t)\|_{L^p(\Omega; \mathcal{H})} \) can not be handled directly. However, we turn things around and derive

\[
\left\| \int_0^t |\tilde{e}_h(s)|^2\,ds \right\|_{L^p(\Omega; \mathcal{H})} = O(h^{2\kappa} |\ln h|^2) \tag{1.11}
\]

instead, after fully exploiting (1.11), the monotonicity of the nonlinearity, regularity properties of \( X_h(t), \tilde{X}_h(t) \) and \( X(t) \), and the previous error estimate of \( \|X(t) - \tilde{X}_h(t)\|_{L^p(\Omega; \mathcal{H})} \). Equipped with the key error estimate of \( \left\| \int_0^t |\tilde{e}_h(s)|^2\,ds \right\|_{L^p(\Omega; \mathcal{H})} \) and (1.12), we can smoothly show \( \|\tilde{X}_h(t) - X_h(t)\|_{L^p(\Omega; \mathcal{H})} = O(h^\kappa |\ln h|) \) (see (4.23)-(4.29) and accordingly (1.8) holds.

Discretizing the semi-discrete problem by an implicit backward Euler time-stepping scheme, we also investigate a fully discrete discretization, given by

\[
X^n_h = E_{k,h}X^{n-1}_h - kE_{k,h}A_hP_hPF(X^n_h) + E_{k,h}P_h\Delta W_n,
\]  

(1.13)

where \( E_{k,h} := (I + kA_h^2)^{-1} \) and \( X_h^n \) is regarded as the fully discrete approximation of \( X(t_n) \). By essentially exploiting discrete versions of arguments as used in the semi-discrete case, one can obtain the following strong approximation error bound,

\[
\|X(t_n) - X^n_h\|_{L^p(\Omega; \mathcal{H})} \leq C(h^\kappa |\ln h| + k^\gamma |\ln k|), \quad \kappa := \min\{\gamma, r\}.
\]  

(1.14)

It is important to mention that, the presence of the operator \( A \) in the front of the non-globally Lipschitz (cubic) nonlinearity in the underlying model causes essential difficulties in the error analysis for the approximations and the error analysis becomes much more challenging than that of the stochastic Allen-Cahn equation (see [35] and relevant comments in [17, 25]). More specifically, our error analysis heavily relies on the new arguments.
mentioned above, a priori strong moment bounds of the numerical approximations, and a rich variety of error estimates for finite element approximation of the corresponding deterministic linear problem. Some estimates can be derived from existing ones in [17,26,28]. Nevertheless, estimates available in [17,26,28] are far from being enough for the purpose of the error analysis. For example, the strong moment bounds [3.11] and [5.7] and the error estimates of integral form such as (1.6), (1.7), (1.8), and (1.9) are completely new.

Finally, we add some comments on a few closely relevant works. A finite difference scheme is examined in [10] for the problem (1.2) and convergence (with rates) in probability is established. The authors of [19] used a general perturbation results and exponential integrability properties of the exact and numerical solutions to prove strong convergence rates for the spatial spectral Galerkin approximation (no time discretization) in one spatial dimension. In [17,26,28], strong convergence of finite element methods for (1.2) was proved, but with no rate obtained. The analysis in [17,26,28] is based on proving a priori moment bounds with large exponents and in higher order norms using energy arguments and bootstraping followed by a pathwise Gronwall argument in the mild solution setting. When we are almost ready to submit the present manuscript, we are aware of an interesting preprint [11] submitted to arXiv in mid-December. There strong convergence rates of a fully discrete scheme are obtained, done by a spatial spectral Galerkin method and a temporal accelerated implicit Euler method for the stochastic Cahn-Hilliard equation. To the best of our knowledge, strong convergence rates of finite element methods for the CHC equation are missing in the existing literature.

The outline of this paper is as follows. In the next section, some preliminaries are collected and well-posedness of the considered problem is elaborated. Section 3 is devoted to the uniform moment bounds of the semi-discrete finite element approximation. Based on the uniform moment bounds obtained in section 3, we derive the error estimates for the semi-discrete problem in section 4. Section 5 focuses on the uniform moment bounds of the solution to the fully discrete problem and section 6 provides error estimates of the backward Euler-finite element full discretization.

2 The Cahn-Hilliard-Cook equation

Throughout this paper, we use $\mathbb{N}$ to denote the set of all positive integers and denote $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Given a separable $\mathbb{R}$-Hilbert space $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$, by $\mathcal{L}(H)$ we denote the Banach space of all linear bounded operators from $H$ to $H$. Also, we denote by $\mathcal{L}_2(H)$ the Hilbert space consisting of all Hilbert-Schmidt operators from $H$ into $H$, equipped with the inner product and the norm,

$$\langle \Gamma_1, \Gamma_2 \rangle_{\mathcal{L}_2(H)} = \sum_{j=1}^{\infty} \langle \Gamma_1 \phi_j, \Gamma_2 \phi_j \rangle, \quad \|\Gamma\|_{\mathcal{L}_2(H)} = \sum_{j=1}^{\infty} \|\Gamma \phi_j\|^2,$$

(2.1)

independent of the choice of orthonormal basis $\{\phi_j\}$ of $H$. If $\Gamma \in \mathcal{L}_2(H)$ and $L \in \mathcal{L}(H)$, then $\Gamma L, L \Gamma \in \mathcal{L}_2(H)$ and

$$\|\Gamma L\|_{\mathcal{L}_2(H)} \leq \|L\|_{\mathcal{L}(H)} \|\Gamma\|_{\mathcal{L}_2(H)}, \quad \|L \Gamma\|_{\mathcal{L}_2(H)} \leq \|L\|_{\mathcal{L}(H)} \|\Gamma\|_{\mathcal{L}_2(H)}.$$

(2.2)

2.1 Abstract framework and main assumptions

In this subsection, we formulate main assumptions concerning the operator $A$, the nonlinear term $F$, the noise process $W(t)$ and the initial data $X_0$.

**Assumption 2.1 (Linear operator $A$)** Let $D$ be a bounded convex domain in $\mathbb{R}^d$ for $d \in \{1, 2, 3\}$ with sufficiently smooth boundary and let $H = L_2(D, \mathbb{R})$ be the real separable Hilbert space endowed with the usual inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\| = (\cdot, \cdot)^{\frac{1}{2}}$, and $\hat{H} = \{v \in H : \int_D v \, dx = 0\}$. Let $-A : D(A) \subset \hat{H} \to \hat{H}$ be the Laplacian with homogeneous Neumann boundary conditions, defined by $-Au = \Delta u$ with $u \in \text{dom}(A) := \{v \in H^2(D) : \frac{\partial v}{\partial \nu} = 0, \text{ on } \partial D\}$.

Such assumptions guarantee that the operator $A$ is positive definite, selfadjoint, bounded, linear on $\hat{H}$ with compact inverse. Let $P : H \to \hat{H}$ denote the orthogonal projection. Then $(I - P)v = |D|^{-1} \int_D v \, dx$ is the
average of \(v\). When extended to \(H\) as \(Av := APv\), for \(v \in H\), the linear operator \(A\) has an orthonormal basis \(\{e_j\}_{j \in \mathbb{N}_0}\) of \(H\) with corresponding eigenvalues \(\{\lambda_j\}_{j \in \mathbb{N}_0}\) such that
\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \rightarrow \infty.
\] (2.3)
Note that the first eigenfunction is a constant, i.e., \(e_0 = |D|^{-\frac{1}{2}}\) and \(\{e_j\}_{j \in \mathbb{N}}\) forms an orthonormal basis of \(\dot{H}\). By the spectral theory, we can define the fractional powers of \(A\) on \(\dot{H}\) in a simply way, e.g., \(A^{\alpha}v = \sum_{j=1}^{\infty} \lambda_j^\alpha (v,e_j)e_j\), \(\alpha \in \mathbb{R}\). Then define the Hilbert space \(\dot{H}^\alpha := D(A^{\frac{\alpha}{2}})\) with the inner product \((A^{\frac{\alpha}{2}} - A^{\frac{\alpha}{2}}\cdot, \cdot)\) and the associated norm \(|\cdot|_{\alpha} := \|A^{\frac{\alpha}{2}}\cdot\|.\) It is known that for integer \(\alpha \geq 0\), \(\dot{H}^\alpha\) is a subspace of \(H^\alpha(D) \cap \dot{H}\) characterized by certain boundary conditions and the norm \(|\cdot|_{\alpha}\) is equivalent on \(\dot{H}^\alpha\) to the standard Sobolev norm \(\|\cdot\|_{H^\alpha(D)}\).

Thanks to (2.3), the operator \(-A^2\) can generate an analytic semigroup \(E(t) = e^{-tA^2}\) on \(H\) and
\[
E(t)v = e^{-tA^2}v = \sum_{j=0}^{\infty} e^{-t\lambda_j^2} (v,e_j)e_j = \sum_{j=1}^{\infty} e^{-t\lambda_j^2} (v,e_j)e_j + (v,e_0) = Pe^{-tA^2}v + (I - P)v, \quad v \in H.
\] (2.4)
By expansion in terms of the eigenbasis of \(A\) and using Parseva’s identity, one can easily obtain
\[
\|A^\mu E(t)\|_{\mathcal{L}(\dot{H})} \leq Ct^{-\frac{\mu}{2}}, \quad t > 0, \quad \mu \geq 0,
\] (2.5)
\[
\|A^{-\nu}(I - E(t))\|_{\mathcal{L}(\dot{H})} \leq Ct^{\frac{\nu}{2}}, \quad t \geq 0, \quad \nu \in [0, 2],
\] (2.6)
\[
\int_{\tau_1}^{\tau_2} \|A^\nu E(s)v\|^2 ds \leq C|\tau_2 - \tau_1|^{1-\nu}\|v\|^2, \quad \forall v \in \dot{H}, \quad \nu \in [0, 1],
\] (2.7)
\[
\|A^{2\rho} \int_{\tau_1}^{\tau_2} E(\tau - \sigma)v d\sigma\| \leq C|\tau_2 - \tau_1|^{1-\rho}\|v\|, \quad \forall v \in \dot{H}, \quad \rho \in [0, 1].
\] (2.8)
The next assumption specifies the nonlinearity of the CHC equation.

**Assumption 2.2** (Nonlinearity) Let \(F : L_6(D; \mathbb{R}) \rightarrow H\) be a deterministic mapping given by
\[
F(v)(x) = f(v(x)) = v^3(x) - v(x), \quad x \in D, \quad v \in L_6(D; \mathbb{R}).
\] (2.9)
Here and below, by \(L_p(D; \mathbb{R}), p \geq 1\) \((L_1(D)\) or \(L_6\) for short) we denote a Banach space consisting of \(r\)-times integrable functions. By \(V := C(D, \mathbb{R})\) we denote a Banach space of continuous functions with a usual norm. It is easy to check that, for any \(v, \psi, \psi_1, \psi_2 \in L_6(D),\)
\[
(F'(v)(\psi))(x) = f'(v(x))\psi(x) = (3v^2(x) - 1)\psi(x), \quad x \in D,
\]
\[
(F''(v)(\psi_1, \psi_2))(x) = f''(v(x))\psi_1(x)\psi_2(x) = 6v(x)\psi_1(x)\psi_2(x), \quad x \in D.
\] (2.10)
Moreover, there exists a constant \(C\) such that
\[
-(F(u) - F(v), u - v) \leq \|u - v\|^2, \quad u, v \in L_6(D),
\] (2.11)
\[
\|F(u) - F(v)\| \leq C\|u - v\|(1 + \|u\|_V + \|v\|_V), \quad u, v \in V.
\] (2.12)

**Assumption 2.3** (Noise process) Let \(\{W(t)\}_{t \in [0,T]}\) be a standard \(\dot{H}\)-valued \(Q\)-Wiener process on the stochastic basis \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,T]})\), where the covariance operator \(Q \in \mathcal{L}(\dot{H})\) is bounded, self-adjoint and positive semi-definite, satisfying
\[
\|A^{1-2\gamma}Q^\frac{\gamma}{2}\|_{L_2} < \infty, \quad \text{for some} \quad \gamma \in [3, 4].
\] (2.13)

**Assumption 2.4** (Initial data) Let \(X_0 : \Omega \rightarrow \dot{H}\) be \(\mathcal{F}_0/\mathcal{B}(\dot{H})\)-measurable and satisfy, for a sufficiently large number \(p_0 \in \mathbb{N},\)
\[
\mathbb{E}[\|X_0\|_{\gamma}^{p_0}] < \infty,
\] (2.14)
where \(\gamma \in [3, 4]\) is the parameter from (2.13).
We remark that the assumption on the initial value can be relaxed, but at the expense of having the constant $C$ later depending on $T^{-1}$, by exploring the smooth effect of the semigroup $E(t), t \in [0,T]$ and standard non-smooth data error estimate \cite{36}.

2.2 Regularity results of the model

This part is devoted to the well-posedness of the underlying problem (1.2) and the space-time regularity properties of the mild solution. Existence, uniqueness, and regularity of weak and mild solutions to (1.2) have been studied in \cite{13,25}. The relevant result is stated as follows.

**Theorem 2.5** Under Assumptions 2.1-2.3, the problem (1.2) admits a weak solution $X(t)$, which is continuous almost surely and satisfies the equation

$$\langle X(t), v \rangle - \langle X_0, v \rangle + \int_0^t \langle X(s), A^2 v \rangle + \langle F(X(s)), Av \rangle \, ds = \langle W(t), v \rangle, \text{ a.s., } \forall v \in H^4 = D(A^2), t \in [0,T]. \quad (2.15)$$

In addition, $X(t)$ is also a mild solution, given by \cite{14}, satisfying

$$\sup_{s \in [0,T]} \|X(s)\|_{L^p(\Omega;H^1)} < \infty, \quad \forall p \geq 1. \quad (2.16)$$

To validate (2.16), one can simply adapt the proof of \cite[Theorem 3.1]{25}, in which it was shown that $E[J(X(t))] + \mathbb{E}[\int_0^t J'(X(s)) \, ds] \leq C(t)$ by introducing the following Lyapunov functional

$$J(u) = \frac{1}{2} \|\nabla u\|^2 + \int_D \Phi(u) \, dx, \quad u \in H^1, \quad (2.17)$$

where $\Phi(s) := \frac{1}{4}(s^2 - 1)^2$ is a primitive of $f(s) = s^3 - s$. Evidently, the above estimate (2.16) together with the fact $H^1 \subset L_6(D)$ suffices to ensure

$$\sup_{s \in [0,T]} \|F(X(s))\|_{L^p(\Omega;H)} \leq C(1 + \sup_{s \in [0,T]} \|X(s)\|_{L^p(\Omega;H^1)})^3 < \infty, \quad (2.18)$$

and similarly

$$\sup_{s \in [0,T]} \|f'(X(s))\|_{L^p(\Omega;L_3)} + \sup_{s \in [0,T]} \|f''(X(s))\|_{L^p(\Omega;L_3)} < \infty. \quad (2.19)$$

Accordingly, we have further properties of the mild solution as follows.

**Theorem 2.6** Under Assumptions 2.1-2.3, the mild solution (1.4) enjoys the following regularity

$$\sup_{s \in [0,T]} \|X(t)\|_{L^p(\Omega;H^\gamma)} < \infty, \quad \forall p \geq 1, \quad (2.20)$$

and, for $\forall \beta \in [0, \gamma]$

$$\|X(t) - X(s)\|_{L^p(\Omega;H^\beta)} \leq C|t - s|^{\min\left\{\frac{\beta}{2}, \frac{\gamma - \beta}{2}\right\}}, \quad 0 \leq s < t \leq T. \quad (2.21)$$

To prove Theorem 2.6, we introduce some basic inequalities. Recall first the following embedding inequalities,

$$H^1 \subset L_6(D) \quad \text{and} \quad H^3 \subset C(D;\mathbb{R}), \quad \text{for} \quad \delta > \frac{d}{2}, \quad d \in \{1,2,3\}. \quad (2.22)$$

With (2.22) at hand, one can further show, for any $\delta > \frac{d}{2}$ and any $x \in L_1(D)$,

$$\|A^{-\frac{d}{2}}Px\| = \sup_{v \in H} \frac{|\langle Px, A^{-\frac{d}{2}}v \rangle|}{\|v\|} \leq \sup_{v \in H} \frac{|\|P_x\|_{L_1} A^{-\frac{d}{2}}v\|}{\|v\|} \leq C \sup_{v \in H} \frac{|\|Px\|_{L_1} \|v\|}{\|v\|} \leq C \|x\|_{L_1}. \quad (2.23)$$
Similarly, one can see that, for any $x \in L^p(D)$,
\begin{equation}
\|A^{-\frac{1}{2}}Px\| = \sup_{v \in H} \frac{\|P_x, A^{-\frac{1}{2}}v\|}{\|v\|} \leq \sup_{v \in H} \frac{\|Px\|_{L^p}}{\|v\|} \|A^{-\frac{1}{2}}v\|_{L^p} \leq C \sup_{v \in H} \frac{\|x\|_{L^p}}{\|v\|} \|v\| \leq C \|x\|_{L^p}.
\end{equation}
(2.24)

Since for integer order $l \in \mathbb{N}_0$, the norm $\|\cdot\|_{l}$ is equivalent on $\hat{H}^l$ to the standard Sobolev norm $\|\cdot\|_{H^l(D)}$ and $H^l(D)$, $l \in \mathbb{N}, l \geq 2$ is an algebra, one can find a constant $C = C(l)$ such that, for any $f, g \in \hat{H}^l$,
\begin{equation}
\|fg\|_{H^l(D)} \leq C\|f\|_{H^l(D)}\|g\|_{H^l(D)} \leq C\|A^{\frac{1}{2}}f\|\|A^{\frac{1}{2}}g\|.
\end{equation}
(2.25)

**Proof of Theorem 2.7.** We start by proving a preliminary spatial-temporal regularity of the mild solution, which will be used later in a bootstrapping argument. By using (2.5) with $\mu = 0, \frac{\delta_0 + 2}{2}$, (2.7) with $p = 1$ and (2.18), one can observe that, for any fixed $\delta_0 < 2$,
\begin{equation}
\|X(t)\|_{L^p(\Omega; \hat{H}^{\delta_0})} \leq \|E(t)X_0\|_{L^p(\Omega; \hat{H}^{\delta_0})} + \int_0^t \|E(t - s)APF(X(s))\|_{L^p(\Omega; \hat{H}^{\delta_0})} ds + \left( \int_0^t \|\frac{A}{\lambda}E(t - r)Q^\frac{1}{2}\|_{\mathcal{L}_2}^2 dr \right)^{\frac{1}{2}} \leq C\left( \|X_0\|_{L^p(\Omega; \hat{H}^{\delta_0})} + \int_0^t \|E(t - s)\|_{L^p(\Omega; \hat{H}^{\delta_0})} ds + \|A^{\frac{1}{2}}Q^\frac{1}{2}\|_{\mathcal{L}_2} \right) \leq C\|X_0\|_{L^p(\Omega; \hat{H}^{\delta_0})} + C \sup_{s \in [0, T]} \|F(X(s))\|_{L^p(\Omega; H)} + C\|Q^\frac{1}{2}\|_{\mathcal{L}_2} < \infty,
\end{equation}
(2.26)

where we also used the Burkholder-Davis-Gundy-type inequality and the fact $Av = APv$, for any $v \in H$. Concerning the temporal regularity of the mild solution, we utilize (2.5)-(2.7), (2.18) and the Burkholder-Davis-Gundy-type inequality to get, for $\beta \in [0, \delta_0]$,
\begin{equation}
\|X(t) - X(s)\|_{L^p(\Omega; \hat{H}^\beta)} \leq \|E(t - s) - I\|_{L^p(\Omega; \hat{H}^\beta)} \|X(s)\|_{L^p(\Omega; \hat{H}^\beta)} + \int_s^t \|E(t - r)APF(X(r))\|_{L^p(\Omega; \hat{H}^\beta)} dr + C \left( \int_s^t \|\frac{A}{\lambda}E(t - r)Q^\frac{1}{2}\|_{\mathcal{L}_2}^2 dr \right)^{\frac{1}{2}} \leq C(t - s)^{\frac{\delta_0 - \alpha}{4}} \left( \|X(s)\|_{L^p(\Omega; \hat{H}^{\delta_0})} + \|A^{\frac{\delta_0 - \alpha}{4}}Q^\frac{1}{2}\|_{\mathcal{L}_2} \right) + C \int_s^t (t - r)^{-\frac{\delta_0 - \alpha}{4}} \|F(X(r))\|_{L^p(\Omega; H)} dr \leq C(t - s)^{\frac{\delta_0 - \alpha}{4}} \left( \sup_{s \in [0, T]} \|X(s)\|_{L^p(\Omega; \hat{H}^{\delta_0})} + \sup_{s \in [0, T]} \|F(X(s))\|_{L^p(\Omega; H)} + \|Q^\frac{1}{2}\|_{\mathcal{L}_2} \right) \leq C(t - s)^{\frac{\delta_0 - \alpha}{4}}.
\end{equation}
(2.27)

In the sequel, we aim to show a stronger spatial regularity of the mild solution. First, the above two estimates in a combination with (2.24) imply, for $\delta_0 \in \left(\frac{4}{7}, 2\right)$,
\begin{equation}
\|A^{-\frac{1}{2}}P(F(X(s)) - F(X(t)))\|_{L^p(\Omega; H)} \leq \|F(X(s)) - F(X(t))\|_{L^p(\Omega; \hat{H}^\delta)} \|X(s) - X(t)\|_{L^2(\Omega; H)} \left(1 + \left( \sup_{s \in [0, T]} \|X(s)\|_{L^p(\Omega; \hat{H}^{\delta_0})} \right)^2 \right) \leq C(t - s)^{\frac{\delta_0}{4}}.
\end{equation}
(2.28)
This together with $\text{(2.13), (2.14)-(2.15)}$ and the Burkholder-Davis-Gundy-type inequality shows, for $\delta_0 \in (\frac{3}{2}, 2)$,

$$\|X(t)\|_{L^p(\Omega; H^2)} \leq \|E(t)X_0\|_{L^p(\Omega; H^2)} + \int_0^t \|E(t-s)A^2 P\left(F(X(s)) - F(X(t))\right)\|_{L^p(\Omega; H)} ds$$

$$+ \left\| \int_0^t E(t-s)A^2 PF(X(t)) ds \right\|_{L^p(\Omega; H)} + \left\| \int_0^t AE(t-s)dW(s) \right\|_{L^p(\Omega; H)}$$

$$\leq C\|X_0\|_{L^p(\Omega; H^2)} + C\int_0^t (t-s)^{-\frac{3}{4}} \|A^{-\frac{1}{2}} P(F(X(s)) - F(X(t)))\|_{L^p(\Omega; H)} ds$$

$$+ C\|F(X(t))\|_{L^p(\Omega; H)} + C \left( \int_0^t \|AE(t-s)Q^\frac{1}{2}\|_{L^2} ds \right)^{\frac{1}{2}}$$

$$\leq C\left(\|X_0\|_{L^p(\Omega; H^2)} + \int_0^t (t-s)^{\frac{8\delta_0 - 5}{3}} ds + \sup_{s \in [0, T]} \|F(X(s))\|_{L^p(\Omega; H)} + \|Q^\frac{1}{2}\|_{L^2}\right) < \infty.$$

Taking the above estimate and (2.25) into account implies

$$\sup_{s \in [0, T]} \|PF(X(s))\|_{L^p(\Omega; H^2)} \leq C \sup_{s \in [0, T]} \|F(X(s))\|_{L^p(\Omega; H^2(D))} \leq C\left(1 + \left( \sup_{s \in [0, T]} \|X(s)\|_{L^{2p}(\Omega; H^2)}\right)^3\right) < \infty.$$

Bearing this in mind and following the proof of (2.26), we can prove

$$\|X(t)\|_{L^p(\Omega; H^3)} < \infty,$$

which combined with (2.25) yields

$$\sup_{s \in [0, T]} \|PF(X(s))\|_{L^p(\Omega; H^3)} \leq C\sup_{s \in [0, T]} \|F(X(s))\|_{L^p(\Omega; H^3(D))} \leq C\left(1 + \left( \sup_{s \in [0, T]} \|X(s)\|_{L^{2p}(\Omega; H^2)}\right)^3\right) < \infty.$$

Finally, by repeating the same lines in the proof of (2.26) and (2.27), one can show (2.20) and (2.21). Hence this finishes the proof of this theorem. □

As a direct consequence of Theorem 2.6, the following lemma holds.

**Lemma 2.7** Under Assumptions (2.4), (2.2) the following results hold

$$\sup_{s \in [0, T]} \|A^{\frac{1}{2}} f'(X(s))\|_{L^p(\Omega; L^3)} < \infty, \quad \forall p \geq 1,$$

and, for $\beta \in \{0, 1, 2\}$

$$\|P\left(F(X(t)) - F(X(s))\right)\|_{L^p(\Omega; H^\beta)} \leq C|t - s|^\min\left(\frac{\beta}{2}, \frac{3}{2} - \frac{\beta}{2}\right), \quad 0 \leq s < t \leq T.$$

**Proof of lemma 2.7** Note first that $f'(X(s)) = 6X^2(s) - 1$. Thus, from (2.22), (2.20) and Hölder’s inequality, it follows that

$$\sup_{s \in [0, T]} \|A^{\frac{1}{2}} f'(X(s))\|_{L^p(\Omega; L^3)} \leq \sup_{s \in [0, T]} \|\nabla(6X^2(s) - 1)\|_{L^p(\Omega; L^3)}$$

$$\leq C \sup_{s \in [0, T]} \left( \|\nabla X(s)\|_{L^{2p}(\Omega; L^3)} \|X(s)\|_{L^{2p}(\Omega; L^3)} \right)$$

$$\leq C\left(1 + \left( \sup_{s \in [0, T]} \|X(s)\|_{L^{2p}(\Omega; H^2)}\right)^2\right) < \infty.$$
To show (3.3), we first apply (2.24), Sobolev’s inequality \( \| u \|_{L^2} \leq \| u \|_{H^1} \leq |u|_{1} \), and Hölder’s inequality to get

\[
\| A^{\frac{1}{2}} P(F(X(t)) - X(s)) \| \leq \| \nabla (F(X(t), \cdot) - F(X(s), \cdot)) \| \leq \| (X(t, \cdot) - X(s, \cdot)) \cdot \nabla (X^2(t, \cdot) + X(t)X(s, \cdot) + X^2(s, \cdot)) \|
\]

\[
+ \| \nabla (X(t, \cdot) - X(s, \cdot)) \cdot (X^2(t, \cdot) + X(t, \cdot)X(s, \cdot) + X^2(s, \cdot)) \| + |X(t) - X(s)|_1
\]

\[
\leq C \| X(t) - X(s) \|_{L^2}(\| \nabla X(t) \|_{L^2} + \| \nabla X(s) \|_{L^2})(\| X(t) \|_V + \| X(s) \|_V)
\]

\[
+ C \| X(t) - X(s) \|_1(1 + \| X(t) \|_V^2 + \| X(s) \|_V^2)
\]

\[
\leq C \| X(t) - X(s) \|_1(1 + |X(t)|^2 + |X(s)|^2).
\]

(2.36)

This in conjunction with (2.24), (2.25), (2.12) and (2.22) enables us to obtain

\[
\| P(F(X(t)) - F(X(s))) \|_{L^p(\Omega; \dot{H}^p)}
\]

\[
\leq C \| X(t) - X(s) \|_{L^p(\Omega; \dot{H}^p)}(1 + \| X(t) \|_{L^2(\Omega; \dot{H}^2)}^2 + \| X(s) \|_{L^2(\Omega; \dot{H}^2)}^2)
\]

(2.37)

\[
\leq C |t - s|^{\min(\frac{1}{2}, \frac{\alpha}{\alpha - 1})}.
\]

Hence the proof of this lemma is complete. \( \square \)

3 The finite element spatial semi-discretization

In this section, we consider the finite element spatial semi-discretization of the CHC equation and show uniform-in-time moment bounds of the solution to semi-discrete finite element problem, which will be used later on the convergence analysis. Throughout the proofs, \( C \) denotes a generic nonnegative constant that is independent of the discretization parameters \( h \) and \( k \) and may change from line to line.

3.1 Basic elements of FEM

Before coming to semi-discrete finite element methods for the stochastic problem (1.2), we introduce some notation and operators on the finite element space. Let \( V_h \subset H^1(D), h \in (0, 1] \) be the space of continuous functions that are piecewise polynomials of degree at most \( r - 1 \) for some integer \( r \geq 2 \) over the quasiuniform triangulation \( T_h \) of \( D \). Also, we define \( \dot{V}_h = PV_h \) by

\[
\dot{V}_h = \{ v_h \in V_h : \int_D v_h \, dx = 0 \}.
\]

Then we introduce a discrete Laplace operator \( A_h : V_h \rightarrow V_h \) defined by

\[
(A_h v_h, \chi_h) = a(v_h, \chi_h) := (\nabla v_h, \nabla \chi_h), \quad \forall v_h \in V_h, \, \chi_h \in V_h.
\]

(3.1)

The operator \( A_h \) is selfadjoint, positive semidefinite on \( V_h \), but positive definite on \( \dot{V}_h \). Therefore, \( A_h \) has an orthonormal eigenbasis \( \{ e_{j, h} \}_{j=1}^{N_h} \) in \( \dot{V}_h \) with corresponding eigenvalues \( \{ \lambda_{j, h} \}_{j=1}^{N_h} \), satisfying

\[
0 \leq \lambda_{1, h} \leq \cdots \leq \lambda_{j, h} \leq \cdots \leq \lambda_{N_h}.
\]

(3.2)

Let \( \lambda_{0, h} = 0 \) and \( e_{0, h} = |D|^{-\frac{1}{2}} \). Then \( \{ e_{j, h} \}_{j=0}^{N_h} \) forms an orthonormal basis of \( V_h \). Moreover, we introduce a discrete norm on \( \dot{V}_h \), defined by

\[
|v_h|_{\alpha, h} = \| A_h^{\alpha} v_h \| = \left( \sum_{j=1}^{N_h} \lambda_{j, h}^\alpha |(v, e_{j, h})|^2 \right)^{\frac{1}{2}}, \quad v_h \in \dot{V}_h, \, \alpha \in \mathbb{R},
\]

(3.3)
In addition, we introduce a Riesz representation operator \( R_h : \hat{H}^1 \to \hat{V}_h \) defined by
\[
 a(R_h v, \chi_h) = a(v, \chi_h), \quad \forall v \in \hat{H}^1, \chi_h \in \hat{V}_h,
\]
and a generalized projection operator \( P_h : H \to V_h \) given by
\[
 (P_h v, \chi_h) = (v, \chi_h), \quad \forall v \in H, \chi_h \in V_h.
\]
It is clear that \( P_h \) is also a projection operator from \( \hat{H} \) to \( \hat{V}_h \) and
\[
P_h A = A_h R_h.
\]
In what follows, we give an extra assumption on the operators \( R_h \) and \( P_h \), which is crucial in the error analysis of finite element approximation of \( (1.2) \).

**Assumption 3.1** For the operators \( R_h \) and \( P_h \) and some integer \( r \geq 2 \), we assume that there exists a constant \( C \) independent of \( h \) such that
\[
 |(I - R_h)v|_1 + |(I - P_h)v|_1 \leq C h^{\beta - i}|v|_\beta, \quad \forall v \in \hat{H}^\beta, \quad i = 0, 1, \beta \in [1, r].
\]
This holds with \( r = 2 \) in the case of a bounded convex polygonal domain \( D \). As commented in [28], for higher-order elements the situation is more complicated and we refer to standard textbooks on the finite element method. Furthermore, we assume that \( P_h \) is bounded with respect to the \( \hat{H}^1 \) and \( L_4 \) norms and that the operator \( A_h P_h A^{-1} \) is bounded, that is,
\[
 \| P_h v \|_{L_4} \leq C \| v \|_{L_4}, \quad \forall v \in L_4,
\]
\[
 |P_h v|_1 \leq C |v|_1, \quad \forall v \in \hat{H}^1,
\]
\[
 \| A_h P_h v \| \leq C |v|_2, \quad \forall v \in \hat{H}^2.
\]
This holds, for example, if the mesh \( T_h \) is quasi-uniform. The inverse inequality \( \| A_h \|_{\mathcal{L}(\hat{H})} \leq C h^{-2} \) combined with Assumption 3.1 helps us to obtain (3.11) as follows
\[
 \| A_h P_h v \| \leq \| A_h P_h (I - R_h)v \| + \| P_h Av \| \leq C h^{-2} \| (I - R_h)v \| + C |v|_2 \leq C |v|_2.
\]
Moreover, the operators \( A \) and \( A_h \) obey
\[
 C_1 \| A_h^\alpha P_h v \| \leq \| A_h^\alpha v \| \leq C_2 \| A_h^\alpha P_h v \|, \quad v \in \hat{H}^\alpha, \quad \alpha \in [-1, 1],
\]
and
\[
 \| v_h \|_V \leq C \| A_h v_h \|, \quad \forall v_h \in \hat{V}_h.
\]
Throughout the following error analysis, we always assume the above assumptions are fulfilled.

### 3.2 Moment bounds of the approximation

In this subsection, we come to the semi-discrete finite element approximation of the stochastic problem and provide some useful moment bounds. We mention that moment bounds for the semi-discrete solution will be derived based on some properties of the discrete analytic semigroup below and some known results in [25].
Theorem 3.2
Let $E_h$ as in (2.4), the analytic semigroup $E_h(t)$ generated by the discrete operators $-A_h^2$ can be given as follows:

$$E_h(t)P_h v = e^{-tA_h^2}P_h v = \sum_{j=0}^{N_h} e^{-t\lambda_{j,h}}(P_h v, e_{j,h})e_{j,h} = PE_h(t)P_h v + (I - P)v. \quad (3.16)$$

Since $\hat{V}_h$ is finite-dimensional and $F$ is a polynomial of particular structure, one can easily check that the problem (3.15) admits a unique solution $X_h(t) \in \hat{V}_h$, adapted, continuous almost surely, satisfying both

$$X_h(t) = E_h(t)P_h X_0 - \int_0^t E_h(t-s)A_h P_h F(X_h(s)) ds + \int_0^t E_h(t-s)P_h dW(s). \quad (3.18)$$

We recall our assumption that $X_0 \in \hat{H}$, so that $X_h(0) = P_h X_0 \in \hat{V}_h$ and hence $X_h(t) \in \hat{V}_h$, for $t > 0$.

**Theorem 3.2** Let $X_h(t)$ be the solution of (3.15). If Assumptions 2.1-2.3 are valid, then $\forall p \geq 1$

$$\sup_{s \in [0,T]} \|A_h X_h(t)\|_{L_p(\Omega; H)} + \left\| \int_0^T |A_h X_h(s) + P_h F(X_h(s))|^2 ds \right\|_{L_p(\Omega; \mathbb{R})} < \infty. \quad (3.19)$$

Before showing this theorem, we introduce a spatially discrete version of (2.5)-(2.8), which plays an important role in deriving the moment bounds of $X_h(t)$.

**Lemma 3.3** Under Assumption 2.7, the following estimates for $E_h(t)$ hold.

(i) Let $\mu \geq 0$. There exists a constant $C$ such that

$$\|A_h E_h(t)P_h v\| \leq C t^{-\frac{\mu}{2}} \|v\|, \quad \forall v \in \hat{H}. \quad (3.20)$$

(ii) Let $\nu \in [0,2]$. There exists a constant $C$ such that

$$\|A_h^{-\nu}(I - E_h(t))P_h v\| \leq C t^{\nu} \|v\|, \quad \forall v \in \hat{H}. \quad (3.21)$$

(iii) There exists a constant $C$ such that

$$\left\| \int_0^t A_h^2 E_h(t)P_h v ds \right\| \leq C \|v\|, \quad \forall v \in \hat{H}. \quad (3.22)$$

(iv) There exists a constant $C$ such that

$$\left( \int_0^t \|A_h E_h(s)P_h v\|^2 ds \right)^{\frac{1}{2}} \leq C \|v\|, \quad \forall v \in \hat{H}. \quad (3.23)$$

**Proof of Theorem 3.2** By a slightly modification of the proof of [25, Theorem 3.1], we obtain

$$\left\| J(X_h(s)) \right\|^p_{L_p(\Omega; \mathbb{R})} + \left\| \int_0^T |A_h X_h(s) + P_h F(X_h(s))|^2 ds \right\|^p_{L_p(\Omega; \mathbb{R})}$$

$$\leq C \left( 1 + \|J(P_h X_0)\|^p_{L_p(\Omega; \mathbb{R})} + \sup_{t \in [0,T]} \left\| \int_0^t (A_h X_h(s) + P_h F(X_h(s)), P_h dW(s)) \right\|^p_{L^1(\Omega; \mathbb{R})} \right).$$

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By assumptions (3.9) and (3.10), it follows \( \|J(P_h X_0)\|_{L^p(\Omega; \mathbb{R})} < \infty \). Then, by applying the Burkholder-Davis-Gundy-type inequality, one can find that

\[
\begin{align*}
\left\| \sup_{s \in [0,T]} J(X_h(s)) \right\|_{L^p(\Omega; \mathbb{R})}^p + \left\| \int_0^T |A_h X_h(s) + P_h F(X_h(s))|^2 \, ds \right\|_{L^p(\Omega; \mathbb{R})}^p & \\
\leq C \left( 1 + \left\| \int_0^T \|Q^\frac{1}{2}(A_h X_h(s) + P_h F(X_h(s)))\|^2 \, ds \right\|_{L^2(\Omega; \mathbb{R})}^{\frac{p}{2}} \right) \\
\leq C \left( 1 + \left\|Q^\frac{1}{2}\right\|_{L^2(\hat{H})}^{\frac{p}{2}} \right) \int_0^T \|A_h X_h(s) + P_h F(X_h(s))\|^2 \, ds \, ds \right\|_{L^p(\Omega; \mathbb{R})}^p \\
\leq C \left( 1 + \frac{\|Q^\frac{1}{2}\|_{L^2(\hat{H})}^{\frac{p}{2}}}{2\varepsilon} + \varepsilon \left\| \int_0^T |A_h X_h(s) + P_h F(X_h(s))|^2 \, ds \right\|_{L^p(\Omega; \mathbb{R})}^p \right),
\end{align*}
\]

(3.25)

where we also used the fact \( \|Q^\frac{1}{2}\|_{L^2(\hat{H})} \leq \left\|Q^\frac{1}{2}\right\|_{L^2(\mathbb{R})} < \infty \). Taking \( \varepsilon > 0 \) small enough in (3.25), we conclude that

\[
\left\| \sup_{s \in [0,T]} J(X_h(s)) \right\|_{L^p(\Omega; H^1)}^p + \left\| \int_0^T |A_h X_h(s) + P_h F(X_h(s))|^2 \, ds \right\|_{L^p(\Omega; \mathbb{R})}^p < \infty.
\]

(3.26)

It remains to bound \( \|A_h X_h(t)\|_{L^p(\Omega; \mathbb{H})} \). From the definition of the Lyapunov functional \( J(\cdot) \) and noting \( \Phi(s) = \frac{1}{t}(s^2 - 1)^2 \), it follows that

\[
|v|_1^2 \leq 2J(v), \quad \forall v \in \hat{H},
\]

(3.27)

which implies

\[
\sup_{s \in [0,T]} \|X_h(s)\|_{L^p(\Omega; \mathbb{H})}^p \leq C \sup_{s \in [0,T]} \left[ \|J(X_h(s))\|_{L^p(\Omega; \mathbb{R})}^{\frac{p}{2}} \right] \leq C.
\]

(3.28)

Using (2.22) shows

\[
\sup_{s \in [0,T]} \|F(X_h(s))\|_{L^p(\Omega; \mathbb{H})} \leq C \left( \sup_{s \in [0,T]} \|X_h(s)\|_{L^p(\Omega; \mathbb{H})} + \left( \sup_{s \in [0,T]} \|X_h(s)\|_{L^p(\Omega; \mathbb{H})}^{\frac{3}{2}} \right) \right).
\]

(3.29)

With the above estimate, one can follow the same lines of the proof of (2.23) to show, for \( \delta_0 \in (\frac{\pi}{2}, 2) \),

\[
\sup_{s \in [0,T]} \|A_h^{-\frac{s}{2}} X_h(s)\|_{L^p(\Omega; \mathbb{H})} \leq C \left( \sup_{s \in [0,T]} \|A_h^{-\frac{s}{2}} P_h X_0\|_{L^p(\Omega; \mathbb{H})} + \sup_{s \in [0,T]} \|F(X_h(s))\|_{L^p(\Omega; \mathbb{H})} + \|Q^\frac{1}{2}\|_{L^2(\mathbb{R})} \right) \leq C \left( \|X_0\|_{L^p(\Omega; \mathbb{H})}^{\frac{1}{2}} + 1 \right) \leq \infty,
\]

(3.30)

where in the second inequality we also used (3.11). Similarly to (2.27) in the previous proof, we obtain

\[
\|X_h(t) - X(s)\|_{L^p(\Omega; \mathbb{H})} \leq C|t - s|^{\frac{\delta_0}{2}},
\]

(3.31)

which combined with (2.22) and (3.11) yields,

\[
\|A_h^{-\frac{s}{2}} P_h (F(X_h(s)) - F(X_h(t)))\|_{L^p(\Omega; \mathbb{H})} \leq C \left( \|A^{-\frac{s}{2}} P_h (F(X_h(s)) - F(X_h(t)))\|_{L^p(\Omega; \mathbb{H})} \right) \leq C \left( \|F(X_h(s)) - F(X_h(t))\|_{L^p(\Omega; \mathbb{H})} \right) \leq C \left( \|X_h(s) - X_h(t)\|_{L^p(\Omega; \mathbb{H})} + \left( \sup_{s \in [0,T]} \|X_h(s)\|_{L^p(\Omega; \mathbb{H})} \right)^{\frac{3}{2}} \right) \leq C|t - s|^{\frac{\delta_0}{2}}.
\]

(3.32)
Combining this with (3.11), (3.29), (3.33), (3.28) and (5.20) with \( \mu = 0, \frac{5}{4} \) gives, for \( \delta_0 \in (\frac{5}{4}, 2) \),

\[
\| A_hX_h(t) \|_{L^p(\Omega; H^1)} \leq \| A_hE_h(t)P_hX_0 \|_{L^p(\Omega; H^1)} + \int_0^t \| E_h(t-s)A_h^2P_hP(\Gamma(X_h(s)) - F(X_h(t))) \|_{L^p(\Omega; H^1)} \, ds
\]

\[
+ \int_0^t \| E_h(t-s)A_h^2P_hP(X_h(s)) - F(X_h(t)) \|_{L^p(\Omega; H^1)} \, ds + C\left( \int_0^t \| A_hE_h(t-s)P_hQ^{\frac{2}{3}} \|_{L^2}^2 \, ds \right) \left( \int_0^t \| A_hE_h(t-s)P_hQ^{\frac{2}{3}} \|_{L^2}^2 \, ds \right)
\]

\[
\leq C\| X_0 \|_{L^p(\Omega; H^1)} + \int_0^t \| \delta \|^{\frac{\kappa-5}{2}} \, ds + \sup_{s \in [0,T]} \| F(X(s)) \|_{L^p(\Omega; H^1)} + \| Q^{\frac{2}{3}} \|_{L^2} < \infty,
\]

where the Burkholder-Davis-Gundy inequality was also used. Hence this finishes the proof of this theorem. \( \Box \)

4 Strong convergence rates of the FEM semi-discretization

In this part, our target is to derive the error estimates of the semi-discrete finite element approximation of the stochastic problem (1.2). The convergence analysis heavily relies on the moment bounds obtained in subsection 3.2 and the corresponding deterministic error estimates. The spatial approximation error is measured as follows

**Theorem 4.1** Let \( X(t) \) be the weak solution of (1.2) and \( X_h(t) \) be the solution of (3.1) (for \( t > 0 \) and \( \forall p \in [1, \infty) \), if Assumptions 2.1, 2.3 are valid, then for \( t > 0 \) and \( \forall p \in [1, \infty) \),

\[
\| X(t) - X_h(t) \|_{L^p(\Omega; H^1)} \leq C h^\kappa | \ln h |, \text{ with } \kappa = \min \{ \gamma, r \}. \quad (4.1)
\]

Moreover, for the "chemical potential" \( Y(t) := AX(t) + PF(X(t)) \) and its approximation \( Y_h(t) := A_hX_h(t) + P_hPF(X_h(t)) \), we have, for \( t > 0 \)

\[
\| Y(t) - Y_h(t) \|_{L^p(\Omega; H^1)} \leq C(1 + t^{-1}) h^\kappa | \ln h |, \text{ with } \kappa = \min \{ \gamma - 2, r - 1 \}. \quad (4.2)
\]

Its proof is postponed after we have been well-prepared with some deterministic semi-discrete error estimates in the following lemma. Define the semi-discrete approximation operators \( \Psi_h(t) \) and \( \Phi_h(t), t \in [0, T] \) as follows

\[
\Psi_h(t) := E(t) - E_h(t)P_h \quad \text{and} \quad \Phi_h(t) := AE(t) - A_hE_h(t)P_h, \quad t \in [0, T]. \quad (4.3)
\]

**Lemma 4.2** Under Assumption 2.4, the following estimates for the error operators \( \Psi_h(t) \) and \( \Phi_h(t) \) hold.

(i) Let \( \beta \in [1, r] \). There exists a constant \( C \) such that

\[
\| \Psi_h(t)v \| \leq C h^\beta | v |_{\beta}, \quad \forall v \in \tilde{H}^\beta. \quad (4.4)
\]

(ii) Let \( \alpha \in [1, r] \). There exists a constant \( C \) such that

\[
\| \Phi_h(t)v \| \leq C h^\alpha t^{-1} | v |_{\alpha-2}, \quad \forall v \in \tilde{H}^{\alpha-2}. \quad (4.5)
\]

(iii) Let \( \nu \in [1, r] \). There exists a constant \( C \) such that

\[
\left( \int_0^t \| \Psi_h(s)v \|^2 \, ds \right)^{\frac{1}{2}} \leq C h^\nu | \ln h | | v |_{\nu-2}, \quad \forall v \in \tilde{H}^{\nu-2}. \quad (4.6)
\]
(iv) Let \( \mu \in [0, r] \). There exists a constant \( C \) such that
\[
\left( \int_0^t \| \Phi_h(s)v \|^2 \, ds \right)^{\frac{1}{2}} \leq Ch^\mu |\ln h||v|_{\mu}, \quad \forall v \in H^\mu.
\] (4.7)

(v) Let \( \theta \in [1, r] \). There exists a constant \( C \) such that
\[
\left\| \int_0^t \Phi_h(s)v \, ds \right\| \leq Ch^{\theta}|v|_{\theta-2}, \quad \forall v \in H^{\theta-2}.
\] (4.8)

Proof of Lemma 4.2. The estimates (4.1) and (4.6) are shown in [28, Theorem 2.1]. Taking (3.8) into account, we can make a slight modification of the proof of \([17, (5.6)]\) in the case \( \delta = 0 \) to prove (4.6). For (4.7), the error bounds follow by a simple interpolation between the cases \( \mu = 0 \) and \( \mu = r \). The case \( \mu = 0 \) immediately follows from (2.7) with \( \theta = 0 \) and (3.23). For \( \mu = r \), we use (3.23), (3.7), (3.8), (3.8) with \( i = 0 \) and \( \beta = r \) and (4.6) with \( \nu = r \) to get
\[
\left( \int_0^t \| \Phi_h(s)v \|^2 \, ds \right)^{\frac{1}{2}} \leq \left( \int_0^t \| (E(s) - E_h(s)P_h)Av \|^2 \, ds \right)^{\frac{1}{2}} + \left( \int_0^t \| A_hE_h(s)P_h(R_h - I)v \|^2 \, ds \right)^{\frac{1}{2}}
\]
\[
\leq Ch^\mu |\ln h||v|_r + C||(R_h - I)v|| \leq Ch^\mu |\ln h||v|_r.
\] (4.9)

Finally, the interpolation argument finally concludes the proof of (4.7). Similarly as before, we use (3.7) to split the term \( \| \int_0^t \Phi_h(s)v \, ds \| \) into two parts:
\[
\left\| \int_0^t \Phi_h(s)v \, ds \right\| = \left\| \int_0^t (A^2E(s)A^{-1} - A_h^2E_h(s)R_hA^{-1})v \, ds \right\|
\]
\[
\leq \left\| \int_0^t (A^2E(s) - A_h^2E_h(s)P_h)A^{-1}v \, ds \right\| + \left\| \int_0^t A_h^2E_h(s)P_h(R_h - I)A^{-1}v \, ds \right\|
\]
\[
\leq \left\| \int_0^t \Psi_h(s)A^{-1}v \, ds \right\| + \left\| \int_0^t E_h(s)P_h(R_h - I)A^{-1}v \, ds \right\|.
\] (4.10)

For the first term, we use the fundamental theorem of calculus, (3.8) with \( i = 0 \) and \( \beta = \theta \) and (4.4) with \( \beta = \theta \) to show, for \( \theta \in [1, r] \),
\[
\left\| \int_0^t \Psi_h(s)A^{-1}v \, ds \right\| = \| (\Psi_h(t) - \Psi_h(0))A^{-1}v \| \leq \| \Psi_h(t)A^{-1}v \| + \| (I - P_h)A^{-1}v \| \leq Ch^{\theta}|v|_{\theta-2}.
\] (4.11)

Similarly, we combine the boundness of \( \| E_h(s)P_h \| \) with (3.8) to yield
\[
\left\| \int_0^t E_h(s)P_h(R_h - I)A^{-1}v \, ds \right\| = \| (E_h(t) - I)P_h(R_h - I)A^{-1}v \| \leq Ch^{\theta}|v|_{\theta-2},
\] (4.12)

which together with (4.11) and (4.10) implies (4.8). This ends the proof of this lemma. □

At the moment, we are well-prepared to prove Theorem 4.1.

Proof of Theorem 4.1. Since \( A \) does not commute with \( P_h \), the usual arguments splitting the error \( X(t) - X_h(t) \) into \( (I - P_h)X(t) \) and \( P_hX(t) - X_h(t) \) dose not work here. To prove this theorem, we propose a different approach and introduce a new auxiliary problem:
\[
d\tilde{X}_h(t) + A_h(\tilde{A}_h \tilde{X}_h(t) + P_hF(X(t))) \, dt = P_h \, dW(t), \quad X_h(0) = P_hX_0,
\] (4.13)

whose unique solution can be written as, in the mild form,
\[
\tilde{X}_h(t) = E_h(t)P_hX_0 - \int_0^t E_h(t - s)A_hP_hP_hF(X(s)) \, ds + \int_0^t E_h(t - s)P_h \, dW(s).
\] (4.14)
In view of (2.30), (3.11), (3.20) and (3.23) and the Burkholder-Davis-Gundy inequality, we acquire that

\[
\|A_h \tilde{X}_h(t)\|_{L^p(\Omega; H)} \leq \|A_h E_h(t) P_h X_0\|_{L^p(\Omega; H)} + \int_0^t \|E_h(t-s) A_h^2 P_h P F(X(s))\|_{L^p(\Omega; H)} \, ds \\
+ C \left( \int_0^t \|A_h E_h(t-s) P_h Q^h\|_{L^2}^2 \, ds \right)^{\frac{1}{2}} \\
\leq \|A_h P_h X_0\|_{L^p(\Omega; H)} + C \int_0^t (t-s)^{-\frac{1}{2}} \|A_h P_h P F(X(s))\|_{L^p(\Omega; H)} \, ds + C \|Q^h\|_{L^2} \\
\leq C \left( \|X_0\|_{L^p(\Omega; H)} + \int_0^t (t-s)^{-\frac{1}{2}} \|PF(X(s))\|_{L^p(\Omega; H)} \, ds \right) + C \|Q^h\|_{L^2} < \infty.
\]

(4.15)

Now, we separate the considered error term \(\|X(t) - X_h(t)\|_{L^p(\Omega; H)}\) as

\[
\|X(t) - X_h(t)\|_{L^p(\Omega; H)} \leq \|X(t) - \tilde{X}_h(t)\|_{L^p(\Omega; H)} + \|\tilde{X}_h(t) - X_h(t)\|_{L^p(\Omega; H)}.
\]

(4.16)

The first error term \(\|X(t) - \tilde{X}_h(t)\|_{L^p(\Omega; H)}\) is treated in a standard way. Subtracting (4.14) from (1.4) yields

\[
\|X(t) - \tilde{X}_h(t)\|_{L^p(\Omega; H)} \leq \|\Psi_h(t) X_0\|_{L^p(\Omega; H)} + \left\| \int_0^t \Phi_h(t-s) P F(X(s)) \, ds \right\|_{L^p(\Omega; H)} \\
+ \left\| \int_0^t \Psi_h(t-s) \, dW(s) \right\|_{L^p(\Omega; H)} \\
:= I_1 + I_2 + I_3.
\]

(4.17)

Below we treat \(I_1, I_2\) and \(I_3\), separately. For \(I_1\), we utilize (4.4) with \(\beta = \kappa\) to derive

\[
I_1 \leq C h^\kappa \|X_0\|_{L^p(\Omega; H^r)}, \quad \kappa = \min\{\gamma, r\}.
\]

(4.18)

Similarly, employing (2.30), (2.31), (4.5) with \(\alpha = r\) and (4.8) with \(\theta = r\) enables us to obtain, for \(r \in \{2, 3, 4\},

\[
I_2 \leq \left\| \int_0^t \Phi_h(t-s) P F(X(t)) \, ds \right\|_{L^p(\Omega; H)} + \left\| \int_0^t \Phi_h(t-s) P (F(X(t)) - F(X(s))) \right\|_{L^p(\Omega; H)} \, ds \\
\leq C h^r \ln h \|PF(X(t))\|_{L^p(\Omega; H^{r-2})} + C h^r \ln h \int_0^t (t-s)^{-\frac{1}{2}} \|PF(X(t)) - F(X(s))\|_{L^p(\Omega; H^{r-2})} \, ds \\
\leq C h^r \ln h \sup_{s \in [0,T]} \|PF(X(t))\|_{L^p(\Omega; H^{r-2})} + C h^r \ln h \int_0^t (t-s)^{-\frac{1}{2}} \, ds \\
\leq C h^r \ln h.
\]

(4.19)

To bound \(I_3\), we combine the Burkholder-Davis-Gundy-type inequality and (4.10) with \(\nu = \kappa\) to arrive at

\[
I_3 \leq C_{\nu} \left( \int_0^t \|\Psi_h(t-s) Q^h\|_{L^2}^2 \, ds \right)^{\frac{1}{2}} \leq C h^\kappa \ln h \|A^\frac{\nu}{2} Q^h\|_{L^2} \leq C h^\kappa \ln h.
\]

(4.20)

Putting the above estimates together yields

\[
\|X(t) - \tilde{X}_h(t)\|_{L^p(\Omega; H)} \leq C h^\kappa \ln h.
\]

(4.21)

Next we turn our attention to the error \(\bar{e}_h(t) := \tilde{X}_h(t) - X_h(t)\), which is differentiable and satisfies

\[
\frac{d}{dt} \bar{e}_h(t) + A_0^2 \bar{e}_h(t) = A_h P_h (F(X(s)) - F(X_h(s))), \quad \bar{e}_h(0) = 0.
\]

(4.22)
Multiplying both sides of (1.22) by $A_h^{-1} \tilde{e}_h$ and using (2.11), (3.14), (2.22) and the fact $\|\tilde{e}_h\|^2 \leq |\tilde{e}_h|_1 |\tilde{e}_h|_{-1,h}$ result in

$$\frac{1}{2} \frac{d}{dt} |\tilde{e}_h(s)|^2_{-1,h} + |\tilde{e}_h(s)|^2 = \left( F(\tilde{X}_h(s)) - F(X_h(s)), \tilde{e}_h(s) \right) + \left( F(X(s)) - F(\tilde{X}_h(s)), \tilde{e}_h(s) \right)$$

$$\leq \frac{1}{2} \|\tilde{e}_h(s)\|^2 + \frac{1}{2} \|F(X(s)) - F(\tilde{X}_h(s))\|^2$$

$$\leq \frac{3}{2} |\tilde{e}_h(s)|_1 |\tilde{e}_h(s)|_{-1,h} + C \|X(s) - \tilde{X}_h(s)\|^2 \left( 1 + \|\tilde{X}_h(s)\|_{L^2} + \|X(s)\|_{L^2} \right)$$

(4.23)

Integrating over $[0, t]$ and then using Gronwall’s inequality suggest that

$$|\tilde{e}_h(t)|^2_{-1,h} + \int_0^t |\tilde{e}_h(s)|^2 ds \leq C \int_0^t \|X(s) - \tilde{X}_h(s)\|^2 (1 + \|A_h \tilde{X}_h(s)\| + |X(s)|^2_H) ds. \quad (4.24)$$

By employing (4.21), (4.15) and (2.20), one can arrive at

$$\left\| |\tilde{e}_h(s)|^2_{1, h} \right\|_{L^p(\Omega; H)} \leq C \left( \int_0^t \|X(s) - \tilde{X}_h(s)\|^2 (1 + \|A_h \tilde{X}_h(s)\|^4 + |X(s)|^2_H) \right) ds$$

$$\leq C \left( \int_0^t \|X(s) - \tilde{X}_h(s)\|^4_{L^p(\Omega; H)} ds \right)^{\frac{3}{2}} \left( \int_0^t (1 + \|A_h \tilde{X}_h(s)\|^8_{L^p(\Omega; H^2)} + \|X(s)\|^8_{L^p(\Omega; H^2)}) ds \right)^{\frac{1}{2}}.$$ (4.25)

At the moment we employ the above estimate to bound $\|\tilde{e}_h(t)\|_{L^p(\Omega; H)}$, which can be splitted into two terms:

$$\|\tilde{e}_h(t)\|_{L^p(\Omega; H)} = \left\| \int_0^t E_h(t-s)A_h P_h P \left( F(X(s)) - F(X_h(s)) \right) ds \right\|_{L^p(\Omega; H)}$$

$$\leq \left\| \int_0^t E_h(t-s)A_h P_h P \left( F(X(s)) - F(\tilde{X}_h(s)) \right) ds \right\|_{L^p(\Omega; H)}$$

$$+ \left\| \int_0^t E_h(t-s)A_h P_h P \left( F(\tilde{X}_h(s)) - F(X_h(s)) \right) ds \right\|_{L^p(\Omega; H)}$$

$$:= J_1 + J_2.$$ (4.26)

With the same arguments of the proof of (4.25), we obtain

$$J_1 \leq C \int_0^t (t-s)^{-\frac{1}{2}} \|F(\tilde{X}_h(s)) - F(X(s))\|_{L^p(\Omega; H)} ds$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|X(s) - \tilde{X}_h(s)\|_{L^2(\Omega; H)} (1 + \|A_h \tilde{X}_h(s)\|^2_{L^2(\Omega; H)} + \|X(s)\|^2_{L^2(\Omega; H^2)}) ds$$

$$\leq C h^2 \ln h.$$ (4.27)

To bound the term $J_2$, we first adapt the similar arguments used in the proof of (2.36) and also use (3.14) to get

$$\left\| A_h^{\frac{1}{2}} P_h P \left( F(X_h(s)) - F(\tilde{X}_h(s)) \right) \right\| \leq \left\| A_h^{\frac{1}{2}} P \left( F(X_h(s)) - F(\tilde{X}_h(s)) \right) \right\|$$

$$\leq C |\tilde{e}_h(s)|_1 (1 + \|A_h X_h(s)\|^2 + \|A_h \tilde{X}_h(s)\|^2). \quad (4.28)$$
This combined with (4.25), (4.15) and (3.19) yields
\[
J_2 \leq \int_0^t (t-s)^{-\frac{1}{2}} \| A_h \frac{\partial}{\partial t} P_h (F(X_h(s) - \ddot{X}_h(s))) \|_{L^p(\Omega; \mathbb{R})} \, ds
\]
\[
\leq C \left( \int_0^t (t-s)^{-\frac{1}{2}} \| \ddot{X}_h(s) \|_1 (1 + \| A_h \ddot{X}_h(s) \|^2 + \| A_h X_h(s) \|^2) \, ds \right)^{\frac{1}{2}}_{L^p(\Omega; \mathbb{R})}
\]
\[
\leq C \left( \int_0^t \| \ddot{X}_h(s) \|^2_{L^{(p)}(\Omega; \mathbb{R})} \right)^{\frac{1}{2}} \left( \int_0^t (t-s)^{-\frac{1}{2}} (1 + \| A_h \ddot{X}_h(s) \|^2 + \| A_h X_h(s) \|^2) \, ds \right)^{\frac{1}{2}}_{L^p(\Omega; \mathbb{R})}
\]
\[
\leq C \| \ddot{X}_h(t) \|^2_{L^{(p)}(\Omega; \mathbb{R})} \int_0^t (t-s)^{-\frac{1}{2}} (1 + \| A_h \ddot{X}_h(s) \|^2 + \| A_h X_h(s) \|^2) \, ds \right)^{\frac{1}{2}}_{L^p(\Omega; \mathbb{R})}
\]
\[
\leq C h^\alpha \| \ln h \|.
\]

Finally, gathering the estimates of $J_1$ and $J_2$ together gives
\[
\| \ddot{X}_h(t) - X_h(t) \|_{L^p(\Omega; \mathbb{H})} \leq C h^\alpha \| \ln h \|,
\]
which combined with (4.21) shows (4.1).

We are now in a position to verify (4.22). Similarly as before, we need to treat the two terms $\| Y(t) - \ddot{Y}_h(t) \|_{L^p(\Omega; \mathbb{H})}$ and $\| \ddot{Y}_h(t) - Y_h(t) \|_{L^p(\Omega; \mathbb{H})}$, where $\ddot{Y}_h(t) := A_h \ddot{X}_h(t) + P_h P F(X(t))$. By (1.4) and (4.14), the error $\| Y(t) - \ddot{Y}_h(t) \|_{L^p(\Omega; \mathbb{H})}$ can be decomposed as follows:
\[
\| Y(t) - \ddot{Y}_h(t) \|_{L^p(\Omega; \mathbb{H})} \leq \|(I - P_h) PF(X(t)) \|_{L^p(\Omega; \mathbb{H})} + \|(A E(t) - A_h E_h(t)) P_h\|_{L^p(\Omega; \mathbb{H})}
\]
\[
+ \left( \int_0^t (A^2 E(t-s) - A_h^2 E_h(t-s)) P_h P F(X(s)) \, ds \right)^{\frac{1}{2}}_{L^p(\Omega; \mathbb{H})}
\]
\[
+ \left( \int_0^t (A E(t-s) - A_h E_h(t-s) P_h) \, dW(s) \right)^{\frac{1}{2}}_{L^p(\Omega; \mathbb{H})}
\]
\[
:= L_1 + L_2 + L_3.
\]
Using (3.8), (2.30) and (4.5) with $\alpha = 2$ gives
\[
L_1 \leq \epsilon \| \Phi_h(t) \|_{L^p(\Omega; \mathbb{H})} \leq \epsilon \| \ddot{X}_h(t) \|_{L^p(\Omega; \mathbb{H})} \leq \epsilon \| \ddot{X}_h(t) \|_{L^p(\Omega; \mathbb{H})} + C h^2 t^{-1} \| X_0 \|_{L^p(\Omega; \mathbb{H})} \leq C h^2 (1 + t^{-1}).
\]

To deal with the term $L_2$, we use (3.7) and the definition of the operator $\Phi_h(t)$ in (4.3) to get
\[
L_2 \leq \left( \int_0^t \| \phi_h(t-s) A P F(X(s)) \|_{L^p(\Omega; \mathbb{H})} \right)^{\frac{1}{2}}_{L^p(\Omega; \mathbb{H})}
\]
\[
+ \left( \int_0^t A_h^2 E_h(t-s) P_h (R_h - I) P F(X(s)) \, ds \right)^{\frac{1}{2}}_{L^p(\Omega; \mathbb{H})}
\]
\[
:= L_{21} + L_{22}.
\]
Owing to (2.30), (2.34) with $\beta = 2$, (3.8) with $p = 2$ and (4.5) with $\alpha = 2$, we infer
\[
L_{21} \leq \left( \int_0^t \| \phi_h(t-s) A P F(X(t)) \|_{L^p(\Omega; \mathbb{H})} + \int_0^t \| \phi_h(t-s) A P (F(X(s)) - F(X(t))) \|_{L^p(\Omega; \mathbb{H})} \, ds \right)^{\frac{1}{2}}_{L^p(\Omega; \mathbb{H})}
\]
\[
\leq C h^2 \| P F(X(t)) \|_{L^p(\Omega; \mathbb{H})} + C h^2 \int_0^t (t-s)^{-1} \| F(X(s)) - F(X(t)) \|_{L^p(\Omega; \mathbb{H})} \, ds
\]
\[
\leq C h^2 \| P F(X(s)) \|_{L^p(\Omega; \mathbb{H})} + C h^2 \int_0^t (t-s)^{-1} (t-s)^{\frac{1}{2}} \, ds
\]
\[
\leq C h^2.
\]

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Likewise, we rely on (3.30), (3.32), (3.34) with $\beta = 2$, (3.20) with $\mu = 2$ to derive

$$L_{22} \leq \int_0^t \| A_n^2 E_h(t-s) P_h(R_h - I) P( F(X(s)) - F(X(t)) \) \|_{L^p(\Omega; H)} \, ds$$

$$+ \int_0^t A_n^2 E_h(t-s) P_h(R_h - I) P F(X(t)) \|_{L^p(\Omega; H)} \, ds$$

$$\leq C \int_0^t (t-s)^{-1} \| (I - R_h) A^{-1}_{L(\Omega)} \| P( F(X(s)) - F(X(t)) \) \|_{L^p(\Omega; H)} \, ds$$

$$+ C \|(R_h - I) F(X(t))\|_{L^p(\Omega; H)}$$

$$\leq C h^2 \int_0^t (t-s)^{-1} \| (I - R_h) A^{-1}_{L(\Omega)} \| \| P( F(X(s)) - F(X(t)) \) \|_{L^p(\Omega; H)} \, ds$$

$$\leq C h^2,$$

which together with (4.34) implies

$$L_2 \leq C h^2. \quad (4.35)$$

Similarly as in (4.20), utilizing (4.17) with $\mu = \gamma - 2$ and the Burkholder-Davis-Gundy inequality one can infer

$$L_3 \leq C_p \left( \int_0^t \| \Phi_h(t-s) Q^\frac{1}{2} \|_{L^2}^2 \, ds \right)^\frac{1}{2} \leq C h^{-2} |ln h| \| \Phi_h(t) \|_{L^2} \leq C h^{-2} |ln h|. \quad (4.37)$$

Gathering (4.32), (4.36) and (4.37) together implies

$$\| Y(t) - Y_h(t) \|_{L^p(\Omega; H)} \leq C h^{-2} |ln h|(1 + t^{-1}). \quad (4.38)$$

To handle the error $\| Y(t_n) - Y_h(t_n) \|_{L^p(\Omega; H)}$, we first apply (2.12), (3.14), (2.22), (2.20) and (3.19) to achieve

$$\| P( F(X(t)) - F(X_h(t)) \) \|_{L^p(\Omega; H)} \leq (1 + \| X(s) \|_{L^p(\Omega; V)}^2 + \| X_h(s) \|_{L^p(\Omega; V)}^2) \cdot \| X(t) - X_h(t) \|_{L^p(\Omega; H)}$$

$$\leq (1 + \| X(s) \|_{L^p(\Omega; H)}^2 + \| X_h(s) \|_{L^p(\Omega; H)}^2) \cdot \| X(t) - X_h(t) \|_{L^p(\Omega; H)}$$

$$\leq C h^\kappa |ln h|. \quad (4.39)$$

This combined with (3.20) with $\mu = \frac{3}{2}$ and the inverse inequality $\| A_n^\frac{3}{2} \|_{L(\Omega)} < C h^{-1}$ enables us to obtain

$$\| Y_h(t_n) - Y_h(t_n) \|_{L^p(\Omega; H)} \leq \int_0^t \| A_n^2 E_h(t-s) P_h( F(X(s)) - F(X_h(s)) \) \|_{L^p(\Omega; H)} \, ds$$

$$+ \| P( F(X(t)) - F(X_h(t)) \) \|_{L^p(\Omega; H)} \leq C h^{-1} \int_0^t (t-s)^{-\frac{3}{2}} \| F(X(s)) - F(X_h(s)) \|_{L^p(\Omega; H)} \, ds + C h^\kappa |ln h|$$

$$\leq C h^{\kappa-1} |ln h|, \quad (4.40)$$

which in conjunction with (4.38) implies (4.2) and hence this finishes the proof. □

5 The finite element full-discretization and its moment bounds

In this section, we proceed to study a full discretization of (1.2) and provide useful moment bounds in the convergence analysis of the full discretization. Let $k = T/N$, $N \in \mathbb{N}$ be a time step-size and $t_n = kn$, 

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\( n = 1, 2, \cdots, N \). We discretize (3.10) in time with an implicit backward Euler scheme and the resulting fully discrete problem is to find \( \mathcal{F}_n \)-adapted \( \dot{V}_h \)-valued random variable \( X^n_h \) such that
\[
X^n_h - X^{n-1}_h - kA^n_hX^n_h + kA^n_hP_hF(X^n_h) = P_h\Delta W_n, \quad X^n_0 = P_hX_0, \quad \text{with} \quad \Delta W_n = W(t_n) - W(t_{n-1}). \tag{5.1}
\]
Similarly to the semi-discrete case, by introducing a family of operators \( \{E^n_{k,h}\}_{n=1}^N \):
\[
E^n_{k,h}v := (1 + kA_h)^{-n}v_h = \sum_{j=0}^{N_h} (1 + k\lambda_{j,h}^2)^{-n}(v_h, e_{j,h})e_{j,h}, \quad \forall v_h \in V_h, \tag{5.2}
\]
the solution of (5.1) can be expressed by the following form
\[
X^n_h = E^n_{k,h}P_hX_0 - k\sum_{j=1}^{n} A_hE^{n-j+1}_{k,h}P_hF(X^n_h) + \sum_{j=1}^{n} E^{n-j+1}_{k,h}P_h\Delta W_j. \tag{5.3}
\]
The next theorem offers a priori strong moment bounds for the fully-discrete approximations.

**Theorem 5.1** Let \( X^n_h \) be the solution of (5.1). If Assumptions 2.1-2.3 are valid, there exist two constants \( C \) and \( k_0 \) such that for all \( k \leq k_0, h > 0 \) and \( \forall p \geq 1, \)
\[
\sup_{1 \leq n \leq N} \|A_hX^n_h\|_{L^p(\Omega; H)} + \left\| \sum_{j=1}^{n} k|A_hX^n_h + P_hF(X^n_h)|^2 \right\|_{L^p(\Omega; H)} < \infty. \tag{5.4}
\]
To show it, we need to introduce some smooth properties of the operator \( E^n_{k,h} \). With the notation \( r(z) = (1 + z)^{-1} \), one can write \( E^n_{k,h} = r(kA^n_h)^n \). As shown in the proof of [36, Theorem 7.1], there exist two constants \( C \) and \( c \) such that
\[
|r(z) - e^{-z}| \leq Cz^2, \quad z \in [0, 1], \tag{5.5}
\]
and
\[
|r(z)| \leq e^{-cz}, \quad \forall z \in [0, 1]. \tag{5.6}
\]
These two inequalities suffice to ensure that, for \( n = 1, 2, 3, \cdots, \)
\[
|r(z)^n - e^{-zn}| \leq \left| (r(z) - e^{-z}) \sum_{l=1}^{n-1} r(z)^{n-1-l}e^{-zl} \right| \leq Cn z^2 e^{-c(n-1)z}, \quad \text{for} \quad z \in [0, 1]. \tag{5.7}
\]
Additionally, we need a temporal discrete version of Lemma 3.3 which will play an important role in deriving the moment bounds of \( X^n_h \) and will be validated in Appendix.

**Lemma 5.2** Under Assumption 2.1, the following estimates for \( E^n_{k,h} \) hold.

(i) Let \( \mu \in [0, 2] \). There exists a constant \( C \) such that
\[
\|A^n_hE^n_{k,h}P_hv\| \leq Ct_n^{-\frac{\mu}{2}}\|v\|, \quad \forall v \in H, \quad n = 1, 2, 3, \cdots, \tag{5.8}
\]

(ii) Let \( \nu \in [0, 2] \). There exists a constant \( C \) such that
\[
\|A^n_h^{-\nu}(I - E^n_{k,h})P_hv\| \leq Ct_n^{-\frac{\nu}{2}}\|v\|, \quad \forall v \in \dot{H}. \tag{5.9}
\]

(iii) There exists a constant \( C \) such that
\[
\left\| k \sum_{j=1}^{n} A^n_h E^n_{k,h} P_h v \right\| \leq C\|v\|, \quad \forall v \in \dot{H}. \tag{5.10}
\]

(iv) There exists a constant \( C \) such that
\[
\left( k \sum_{j=1}^{n} \|A_h E^n_{k,h} P_h v\|^2 \right)^{\frac{1}{2}} \leq C\|v\|, \quad \forall v \in \dot{H}. \tag{5.11}
\]
Equipped with the above preparations, we are ready to show Theorem 5.1.

Proof of Theorem 5.1: Following almost the same step of the proof of \cite{17} Throem 4.3, one observes

\[
\left\| \sup_{1 \leq j \leq N} J(X_h^n) \right\|_{L^p(\Omega; \mathbb{R})}^p + \left\| \sum_{j=1}^N k_j |A_h X_h^n + P_h F(X_h^n)| \right\|_{L^p(\Omega; \mathbb{R})}^p \\
\leq (1 + \|J(P_h X_0)\|_{L^p(\Omega; \mathbb{R})}^p + \|P_h X_0\|_{L^p(\Omega; \mathbb{R})}^p + \|Q_h^\frac{1}{2} (A_h P_h X_0 + P_h F(P_h X_0))\|_{L^p(\Omega; \mathbb{R})}^p)^{\frac{2p}{2p+1}}
\]

\[
\leq C(1 + \|J(P_h X_0)\|_{L^p(\Omega; \mathbb{R})}^p + \|P_h X_0\|_{L^p(\Omega; \mathbb{R})}^p + \|Q_h^\frac{1}{2} (A_h P_h X_0 + P_h F(P_h X_0))\|_{L^p(\Omega; \mathbb{R})}^p)^{\frac{2p}{2p+1}}
\]

Hence this bounds \( \sum_{j=1}^N k_j |A_h X_h^n + P_h F(X_h^n)|^2 \) for all \( \gamma \geq 0 \) and it remains to bound the term \( \|A_h X_h^n\|_{L^p(\Omega; \mathbb{R})} \). To this end, we need more space-time regularity properties of \( X_h^n \). As in (5.25), using (3.9)-(3.11) and applying the fact \( \|Q_h^\frac{1}{2}\|_{C^0} < \infty \) guarantee

\[
\sup_{1 \leq j \leq N} \|X_h^n\|_{L^p(\Omega; \mathbb{R})} \leq C \sup_{1 \leq j \leq N} \left\| J(X_h^n) \right\|_{L^1(\Omega; \mathbb{R})}^{\frac{1}{2}} < \infty.
\]

Evidently, the above estimate suffices to ensure

\[
\sup_{1 \leq j \leq N} \|F(X_h^n)\|_{L^p(\Omega; \mathbb{R})} \leq \left( \sup_{1 \leq j \leq N} \|X_h^n\|_{L^p(\Omega; \mathbb{R})} + \sup_{1 \leq j \leq N} \|X_h^n\|_{L^p(\Omega; \mathbb{R})} \right) < \infty.
\]

Finally, using the above estimate and following the same lines of the proof of (3.33), we can show

\[
\sup_{1 \leq n \leq N} \|A_h X_h^n\|_{L^p(\Omega; \mathbb{R})} < \infty.
\]

The proof of this theorem is complete. \( \Box \)

6 Strong convergence rates of the FEM full-discretization

The main goal of this part is to analyze mean-square convergence rates of the fully discrete finite element method (5.1). Similarly to the semi-discrete case, the deterministic error estimates and the moment bounds of the fully discrete finite element solution together play a key role in our convergence analysis.

The next theorem states the main result of this section, concerning strong convergence rates of the FEM full-discretization.

Theorem 6.1: Let \( X(t) \) be the weak solution of (1.2) and \( X_h^n \) be the solution of (5.1). If Assumptions 2.1-2.3 are valid, there exist two positive constants \( C \) and \( k_0 \) such that for all \( k \leq k_0, h > 0 \) and all \( \gamma \geq 1 \)

\[
\|X(t_n) - X_h^n\|_{L^p(\Omega; \mathbb{R})} \leq C(h^\kappa |\ln h| + k^\frac{2}{3} |\ln k|) \quad \text{with} \quad \kappa = \min\{\gamma, r\}.
\]

Moreover, for the "chemical potential" \( Y(t) := AX(t) + PF(X(t)) \) and its approximation \( Y_h^n := A_h X_h^n + P_h PF(X_h^n) \), we have

\[
\|Y(t_n) - Y_h^n\|_{L^p(\Omega; \mathbb{R})} \leq C(1 + t_n^{-1}) \left( h^\iota |\ln h| + k^\frac{2}{3} |\ln k| \right) \quad \text{with} \quad \iota = \min\{\gamma - 2, r - 1\}.
\]
Let $\beta \in [1, r]$. There exists a constant $C$ such that, for $t > 0$

$$\|\Psi_{k,h}(t)v\| \leq C(h^\beta + k^{\frac{\beta}{2}})|v|_{\beta}, \quad \forall x \in \dot{H}^\beta.$$  \hfill (6.4)

(ii) Let $\alpha \in [1, r]$. There exists a constant $C$ such that, for $t > 0$

$$\|\Phi_{k,h}(t)v\| \leq C(h^\alpha + k^{\frac{\alpha}{2}})t^{-1}|v|_{\alpha-2}, \quad \forall v \in \dot{H}^{\alpha-2}. \hfill (6.5)$$

(iii) Let $\nu \in [1, r]$. There exists a constant $C$ such that, for $t > 0$

$$\left(\int_0^t \|\Psi_{k,h}(s)v\|^2 ds\right)^{\frac{1}{2}} \leq C(h^{\nu}|\ln h| + |\ln k|k^{\frac{\nu}{2}})|v|_{\nu-2}, \quad v \in \dot{H}^{\nu-2}. \hfill (6.6)$$

(iv) Let $\mu \in [0, r]$. There exists a constant $C$ such that, for $t > 0$

$$\left(\int_0^t \|\Phi_{k,h}(s)v\|^2 ds\right)^{\frac{1}{2}} \leq C(h^{\mu}|\ln h| + k^{\frac{\mu}{2}}|\ln k|)|v|_{\mu}, \quad v \in \dot{H}^{\mu}. \hfill (6.7)$$

(v) Let $\varrho \in [1, r]$. There exists a constant $C$ such that, for $t > 0$

$$\left\|\int_0^t \Phi_{k,h}(s)v \, ds\right\| \leq C(h^\varrho + k^{\frac{\varrho}{2}})|v|_{\varrho-2}, \quad v \in \dot{H}^{\varrho-2}. \hfill (6.8)$$

Proof of Lemma 6.2. The estimates (6.4) and (6.6) can be handled by a simple modification of the proof of [28, Theorem 2.2]. For (6.4), it follows from (4.15), (2.6) with $\mu = 2$ and (2.6) with $\nu = \frac{\varrho}{2}$, for $\forall t \in [t_{n-1}, t_n)$

$$\|\Phi_{k,h}(t)v\| \leq \|A(E(t) - E(t_n))v\| + \|(AE(t_n) - A_hE_h(t_n)P_h)v\| + \|A_h(E_h(t_n) - E^n_k(t_n))P_hv\|$$

$$\leq \|A^2E(t_n)A^{-\frac{\varrho}{2}}(I - E(t - t_n))A^{\frac{\varrho}{2}}v\| + Ct_n^{-1}h^{\alpha}|v|_{\alpha-2} + \|A_h(E_h(t_n) - E^n_k(t_n))P_hv\|$$

$$\leq Ct_n^{-1}(h^\alpha + k^{\frac{\alpha}{2}})|v|_{\alpha-2} + \|A_h(E_h(t_n) - E^n_k(t_n))P_hv\|. \hfill (6.9)$$

Hence it remains to bound $\|A_h(E_h(t_n) - E^n_k(t_n))P_hv\|$. Due to (3.20) with $\mu = 2$ and (5.8) with $\mu = 2$, we have

$$\|A_h(E_h(t_n) - E^n_k(t_n))P_hv\| \leq C_t_n^{-1}\|A_h^{-1}P_hv\|. \hfill (6.10)$$

On the other hand, [28, Theorem 4.4] shows

$$\|A_h(E_h(t_n) - E^n_k(t_n))P_hv\| \leq Ck_t_n^{-1}\|A_hP_hv\|. \hfill (6.11)$$

An interpolation between this estimate and (6.10) shows, for $\beta \in [0, 4]$ and $t \in [t_{n-1}, t_n)$,

$$\|A_h(E_h(t_n) - E^n_k(t_n))P_hv\| \leq Ct_n^{-1}k^{\frac{\beta}{2}}\|A_h^{\frac{\beta}{4}}P_hv\| \leq Ct^{-1}k^{\frac{\beta}{2}}\|A_h^{\frac{\beta}{4}}P_hv\|, \hfill (6.12)$$

which, assigning $\beta = \alpha \in [1, r]$ and together with (6.31) and (3.13) implies (6.5). Repeating the same argument of the proof of (4.7), we can show (6.7). For (6.8), it holds, for $\varrho \in [1, r]$ and $t \in [t_n, t_{n+1})$, $n \geq 0$,

$$\left\|\int_0^t \Phi_{k,h}(s)v \, ds\right\| \leq \left\|\int_{t_n}^t \Phi_{k,h}(s)v \, ds\right\| + \left\|\int_0^{t_n} \Phi_{k,h}(s)v \, ds\right\|. \hfill (6.13)$$
Further, by virtue of \(3.13\), \(3.16\) with \(\nu = \frac{9}{2}\) and \(3.8\) with \(\mu = \frac{4e-\theta}{2}\),
\[
\left\| \int_{t_n}^{t} \Phi_{k,h}(s) v \, ds \right\| = \left\| A^{-1} (E(t) - E(t_n)) v \right\| + (t - t_n) \| A_h E_{k,h}^{n+1} P_h v \|
\leq \left\| E(t_n) \|_{\left[ H^1 \right]} \right\| A^{-1} (I - E(t - t_n)) A_h E_{k,h}^{n+1} \|_{\left[ H^1 \right]} \| A_h E_{k,h}^{n+1} \|_{\left[ H^1 \right]} \| A_h E_{k,h}^{n+1} P_h v \|
\leq C (k^\theta + (t - t_n) t_{n+1}^{-\frac{3e-\theta}{2}}) \| v \|_{\theta - 2}
\leq C k^\theta \| v \|_{\theta - 2}.
\]
Next, we have the following bound by \(3.18\)
\[
\left\| \int_{0}^{t_n} \Phi_{k,h}(s) v \, ds \right\| \leq \left\| \int_{0}^{t_n} \Phi_{h}(s) v \, ds \right\| + \left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (E_h(s) - E_h(t_j)) A_h P_h v \, ds \right\|
+ \left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (E_h(s) - E_h(t_j)) A_h P_h v \, ds \right\|
\leq C h^\theta \| v \|_{\theta - 2} + \left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (E_h(s) - E_h(t_j)) A_h P_h v \, ds \right\| + \left\| \sum_{j=1}^{n} (E_h(t_j) - E_h^{j}) A_h P_h v \right\|
\]
where we employ Parseval’s identity and the fact \(\lambda_{i,h}^{-\frac{e-\theta}{2}} (1 - e^{-((t-s)\lambda_{i,h}^2)}) \leq C k^\theta\), \(0 \leq \theta \leq 4\) to derive
\[
\left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (E_h(s) - E_h(t_j)) A_h P_h v \, ds \right\|^2 = \left\| \sum_{j=1}^{n} \sum_{i=1}^{N_h} \int_{t_{j-1}}^{t_j} (e^{-\lambda_{i,h}^2} - e^{-t_j \lambda_{i,h}^2}) \lambda_{i,h} (P_h v, e_{i,h}) e_{i,h} \, ds \right\|^2
\leq C k^\theta \sum_{i=1}^{N_h} \int_{0}^{t_n} \lambda_{i,h}^2 h e^{-\lambda_{i,h}^2 t} \, ds \right\|^2 \lambda_{i,h}^{e-2} (P_h v, e_{i,h})^2
\leq C k^\theta \left\| A_h^{e-2} P_h v \right\|^2
\leq C k^\theta \| v \|_{\theta - 2}^2.
\]
and similarly,
\[
\left\| \sum_{j=1}^{n} k (E_h(t_j) - E_h^{j}) A_h P_h v \right\|^2 = \left\| \sum_{j=1}^{n} k \sum_{i=1}^{N_h} (e^{-t_j \lambda_{i,h}^2} - r (k \lambda_{i,h}^2) \lambda_{i,h} (P_h v, e_{i,h}) e_{i,h} \right\|^2
\leq \sum_{i=1}^{N_h} \left\| k (e^{-t_j \lambda_{i,h}^2} - r (k \lambda_{i,h}^2) \lambda_{i,h} (P_h v, e_{i,h}) e_{i,h} \right\|^2
\]
and similarly,
\[
\left\| \sum_{j=1}^{n} k (E_h(t_j) - E_h^{j}) A_h P_h v \right\|^2 = \left\| \sum_{j=1}^{n} k \sum_{i=1}^{N_h} (e^{-t_j \lambda_{i,h}^2} - r (k \lambda_{i,h}^2) \lambda_{i,h} (P_h v, e_{i,h}) e_{i,h} \right\|^2
\leq \sum_{i=1}^{N_h} \left\| k (e^{-t_j \lambda_{i,h}^2} - r (k \lambda_{i,h}^2) \lambda_{i,h} (P_h v, e_{i,h}) e_{i,h} \right\|^2.
\]
Next we consider two cases: \(k \lambda_{i,h}^2 \leq 1\) and \(k \lambda_{i,h}^2 > 1\). For all summands with \(k \lambda_{i,h}^2 \leq 1\), we get by \(3.20\)
\[
\left| \sum_{j=1}^{n} (e^{-jk \lambda_{i,h}^2} - r (k \lambda_{i,h}^2) \lambda_{i,h} \right| \leq C \lambda_{i,h}^5 k^2 \sum_{j=1}^{n} j e^{-c(j-1)k \lambda_{i,h}^2} \leq C \lambda_{i,h}^5 k \int_{0}^{\infty} (r + k) e^{-c r \lambda_{i,h}^2} \, dr \leq C \lambda_{i,h}^5 k \left( \frac{1}{c \lambda_{i,h}^2} + \frac{k}{c \lambda_{i,h}^2} \right) \leq C \lambda_{i,h}^3 k^2.
\]
For all summands with $k\lambda_{l,h}^2 > 1$, utilizing the fact $\sup_{s\in[0,\infty)} se^{-s} < \infty$ yields

$$
\left| \sum_{j=1}^{n} \left( e^{-jk\lambda_{l,h}^2} - r(-k\lambda_{l,h}^2)^j \right) \lambda_{l,h} \right| \leq C \left( k\lambda_{l,h} e^{-k\lambda_{l,h}^2} \sum_{j=1}^{n} e^{-(j-1)} + \frac{k\lambda_{l,h}}{1 + k\lambda_{l,h}^2} \sum_{j=1}^{n} 2^{-j+1} \right)
\leq C k^{\frac{2}{p}} \lambda_{l,h}^{\frac{2}{p}} \left( k\lambda_{l,h}^2 \right)^{1 - \frac{1}{p}} \left( e^{-k\lambda_{l,h}^2} + \frac{1}{k\lambda_{l,h}^2} \right)
\leq C k^{\frac{2}{p}} \lambda_{l,h}^{\frac{2}{p}} .
$$

(6.19)

Altogether, this together with (3.11) and (3.13) proves

$$
\left\| \sum_{j=1}^{n} \left( E_h(t_j) - E_h^l \right) A_h P_h v \right\|^2 \leq C k^{\frac{2}{p}} \sum_{i=1}^{N_h} A_{l,h}^{q_{i,h} - 2} \left( P_h v, e_{i,h} \right) \leq C k^{\frac{2}{p}} \| A_h^{q_{i,h} - 2} P_h v \|^2 \leq C k^{\frac{2}{p}} \| v \|^2_{q_{i,h} - 2} .
$$

(6.20)

Finally, plunging (6.14)-(6.16) and (6.20) into (6.13) shows (6.8). □

Subsequently, we are well-prepared to show Theorem 6.1.

**Proof of Theorem 6.1.** Similarly to the semi-discrete case, by introducing the auxiliary problem,

$$
\tilde{X}^n_h - \tilde{X}_h^n = k A_h \left( A_h \tilde{X}_h^n + P_h F(X(t_n)) \right) = P_h \Delta W^n, \quad \tilde{X}_h^0 = P_h X_0,
$$

we decompose the considered error $\| X(t_n) - X_h^n \|_{L^p(\Omega; H)}$ into two parts:

$$
\| X(t_n) - X_h^n \|_{L^p(\Omega; H)} \leq \| X(t_n) - \tilde{X}_h^n \|_{L^p(\Omega; H)} + \| \tilde{X}_h^n - X_h^n \|_{L^p(\Omega; H)} .
$$

(6.23)

Owing to (2.30), (5.8), (5.11), (3.11) and the Burkholder-Davis-Gundy-type inequality, one can derive that, for any $n \in \{1, 2, \cdots, N\}$,

$$
\| A_h \tilde{X}_h^n \|_{L^p(\Omega; H)} \leq \| A_h E_h^n P_h X_0 \|_{L^p(\Omega; H)} + k \sum_{j=1}^{n} \| A_h^{2} E_h^{n-j+1} P_h P F(X(t_j)) \|_{L^p(\Omega; H)}
+ \| \sum_{j=1}^{n} A_h E_h^{n-j+1} P_h \Delta W^j \|_{L^p(\Omega; H)}
\leq C \| A_h P_h X_0 \|_{L^p(\Omega; H)} + C k \sum_{j=1}^{n} t_{n-j+1}^{-\frac{1}{2}} \| A_h P_h P F(X(t_j)) \|_{L^p(\Omega; H)}
+ C \left( k \sum_{j=1}^{n} \| A_h E_h^{n-j+1} P_h Q^2 \|_{L^2}^{-2} \right)^{\frac{1}{2}}
\leq C \left( 1 + \| X_0 \|_{L^p(\Omega; H^2)} + k \sum_{j=1}^{n} t_{n-j+1}^{-\frac{1}{2}} \sup_{s \in [0, T]} \| P F(X(s)) \|_{L^p(\Omega; H^2)} + \| Q^2 \|_{L^2} \right) < \infty .
$$

(6.24)

At first we handle the estimate of error $\| X(t_n) - \tilde{X}_h^n \|_{L^p(\Omega; H)}$. Subtracting (6.22) from (1.4), the error $X(t_n) - \tilde{X}_h^n$
can be splitted into the following three parts

\[
\|X(t_n) - \tilde{X}^n_k\|_{L^p(\Omega; \dot{H})} = \|(E(t_n) - E^n_{k,h} P_h)X_0\|_{L^p(\Omega; \dot{H})}
+ \left\| \int_0^{t_n} E(t_n - s) APF(X(s)) \, ds - k \sum_{j=1}^n A_h E^{n-j+1}_{k,h} P_h PF((X(t_j))) \right\|_{L^p(\Omega; \dot{H})}
+ \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left( E(t_n - s) - E^{n-j+1}_{k,h} P_h \right) dW(s) \right\|_{L^p(\Omega; \dot{H})}
:= \mathbb{I} + \mathbb{J} + \mathbb{K}.
\]

In what follows, we will treat the tree terms \(I, J, K\) separately. By using \((6.3)\) with \(\beta = \kappa = \min\{\gamma, r\}\), the first term \(I\) can be estimated as follow,

\[
I \leq C(h^n + k^\frac{r}{2}) \|X_0\|_{L^p(\Omega; \dot{H}^n)}.
\]

To bound the term \(J\), we decompose it into three further terms:

\[
\mathbb{J} \leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left\| E(t_n - s) AP(F(X(s)) - F(X(t_j))) \right\|_{L^p(\Omega; \dot{H})} \, ds
+ \left\| \int_0^{t_n} \Phi_{k,h}(t_n - s) PF(X(t_n)) \, ds \right\|_{L^p(\Omega; \dot{H})}
+ \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \Phi_{k,h}(t_n - s) P(F(X(t_j)) - F(X(t_n))) \right\|_{L^p(\Omega; \dot{H})} \, ds
= : \mathbb{J}_1 + \mathbb{J}_2 + \mathbb{J}_3.
\]

Then we treat the above three terms separately. Noting that, for \(s \in [t_{j-1}, t_j]\)

\[
X(t_j) = E(t_j - s)X(s) - \int_s^{t_j} E(t_j - \sigma)APF(X(\sigma)) \, d\sigma + \int_s^{t_j} E(t_j - \sigma) \, dW(\sigma),
\]

and using Taylor’s formula helps us to split \(\mathbb{J}_1\) into four terms:

\[
\mathbb{J}_1 \leq \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} E(t_n - s) APF'(X(s))(E(t_j - s) - I)X(s) \, ds \right\|_{L^p(\Omega; \dot{H})}
+ \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} E(t_n - s) APF'(X(s)) \int_s^{t_j} E(t_j - \sigma) AF(X(\sigma)) \, d\sigma \, ds \right\|_{L^p(\Omega; \dot{H})}
+ \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} E(t_n - s) APF'(X(s)) \int_s^{t_j} E(t_j - \sigma) \, dW(\sigma) \, ds \right\|_{L^p(\Omega; \dot{H})}
+ \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} E(t_n - s) APR_F(X(s), X(t_j)) \, ds \right\|_{L^p(\Omega; \dot{H})}
:= \mathbb{J}_{11} + \mathbb{J}_{12} + \mathbb{J}_{13} + \mathbb{J}_{14}.
\]

Here the remainder term \(R_F\) reads,

\[
R_F(X(s), X(t_j)) = \int_0^1 P''(X(s) + \lambda(X(t_j) - X(s)))(X(t_j) - X(s))(X(t_j) - X(s))(1 - \lambda) \, d\lambda.
\]
In view of (6.30) with \( \nu = \frac{\gamma}{2}, (2.19), (2.23) \) and (2.24), we derive, for any fixed \( \delta_0 \in (\frac{1}{2}, 2) \) and \( \gamma \in [3, 4], \)

\[
\mathbb{J}_{11} \leq C \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (t_n - s)^{-\frac{2+\delta_0}{4}} \| A^{-\frac{\delta_0}{4}} PF'(X(s))(E(t_j - s) - I)X(s) \|_{L^p(\Omega; \dot{H})} \, ds \\
\leq C \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (t_n - s)^{-\frac{2+\delta_0}{4}} \| F'(X(s))(E(t_j - s) - I)X(s) \|_{L^p(\Omega; L^1)} \, ds \\
\leq C \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (t_n - s)^{-\frac{2+\delta_0}{4}} \| f'(X(s)) \|_{L^{2p}(\Omega; H)} \| (E(t_j - s) - I)X(s) \|_{L^{2p}(\Omega; H')} \, ds \\
\leq C k^{\frac{\gamma}{4}} \int_0^t (t_n - s)^{-\frac{2+\delta_0}{4}} \, ds \sup_{s \in [0, T]} \| f'(X(s)) \|_{L^{2p}(\Omega; H')} \sup_{s \in [0, T]} \| X(s) \|_{L^{2p}(\Omega; H')} \\
\leq C k^{\frac{\gamma}{4}}.
\]

For the term \( \mathbb{J}_{12} \), using similar arguments as in the proof of (6.30), with (2.30) used instead, implies

\[
\mathbb{J}_{12} \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_s^{t_j} (t_n - s)^{-\frac{2+\delta_0}{4}} \| F'(X(s))E(t_{j+1} - \sigma)AF(X(\sigma)) \|_{L^p(\Omega; L^1)} \, d\sigma \, ds \\
\leq C \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_s^{t_j} (t_n - s)^{-\frac{2+\delta_0}{4}} \| f'(X(s)) \|_{L^{2p}(\Omega; H)} \| PF(X(\sigma)) \|_{L^{2p}(\Omega; H')} \, d\sigma \, ds \\
\leq C k \int_0^t (t_n - s)^{-\frac{2+\delta_0}{4}} \, ds \sup_{s \in [0, T]} \| f'(X(s)) \|_{L^{2p}(\Omega; H')} \sup_{s \in [0, T]} \| PF(X(s)) \|_{L^{2p}(\Omega; H')} \\
\leq C k.
\]

To estimate \( \mathbb{J}_{13} \), we recall the stochastic Fubini theorem (see [14, Theorem 4.18]) and the Burkholder-Davis-Gundy-type inequality to obtain

\[
\mathbb{J}_{13} = \left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \chi_{[s, t_j]}(\sigma)E(t_n - s)APF'(X(s))E(t_j - \sigma) \, dW(\sigma) \, ds \right\|_{L^p(\Omega; \dot{H})} \\
= \left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \chi_{[s, t_j]}(\sigma)E(t_n - s)APF'(X(s))E(t_j - \sigma) \, d sdW(\sigma) \right\|_{L^p(\Omega; \dot{H})} \\
\leq \left( \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left\| \int_{t_{j-1}}^{t_j} \chi_{[s, t_j]}(\sigma)E(t_n - s)APF'(X(s))E(t_j - \sigma) \, dW(\sigma) \right\|^2_{L^p(\Omega; L^2)} \, ds \right)^{\frac{1}{2}} \\
\leq \left( \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left\| \int_{t_{j-1}}^{t_j} \chi_{[s, t_j]}(\sigma)E(t_n - s)APF'(X(s))E(t_j - \sigma) \, Q^{\frac{1}{2}} \, ds \right\|^2_{L^p(\Omega; L^2)} \, d\sigma \right)^{\frac{1}{2}},
\]

where \( \chi_{[s, t_j]}(\cdot) \) stands for the indicator function defined by \( \chi_{[s, t_j]}(t) = 1 \) for \( t \in [s, t_j] \) and \( \chi_{[s, t_j]}(t) = 0 \) for \( t \notin [s, t_j] \).
where in the first inequality we used similar arguments as in (6.30). Thus, this together with (6.30), (6.31) and

\[ J \]

To bound the term \( J_{14} \), we use (2.24) and (2.48) to infer

\[
 J_{14} \leq C \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} (t_n - s)^{-\frac{2+\delta}{4}} \| R_P(X(s), X(t_j)) \|_{L^p(\Omega; L_1)} \, ds
\]

\[
 \leq C \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} (t_n - s)^{-\frac{2+\delta}{4}} \| X(t_j) - X(s) \| \| f''((1 - \lambda)X(s) + \lambda X(t_j)) \|_{L_4} \| X(t_j) - X(s) \|_{L_4} \| R_P(\Omega; L_1) \| \, ds
\]

\[
 \leq C \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} (t_n - s)^{-\frac{2+\delta}{4}} \| X(t_j) - X(s) \|_{L^p(\Omega; H)} \| X(t_j) - X(s) \|_{L^p(\Omega; H)} \| R_P(\Omega; L_1) \|_{L^p(\Omega; L_4)} \| f''(X(s)) \|_{L^p(\Omega; L_4)} \sup_{s \in [0, T]} \| f''(X(s)) \|_{L^p(\Omega; L_4)}
\]

\[
 \leq C \kappa \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} (t_n - s)^{-\frac{2+\delta}{4}} \| X(t_j) - X(s) \|_{L^p(\Omega; H)} \| R_P(\Omega; L_1) \|_{L^p(\Omega; L_4)} \| f''(X(s)) \|_{L^p(\Omega; L_4)} \sup_{s \in [0, T]} \| f''(X(s)) \|_{L^p(\Omega; L_4)} \leq C k,
\]

where in the first inequality we used similar arguments as in (6.30). Thus, this together with (6.30), (6.31) and (6.32) leads to, for \( \gamma \in [3, 4] \),

\[
 J_1 \leq C k^{\gamma/2}.
\]

Concerning the term \( J_2 \), we apply (2.8) and (2.8) with \( q = \kappa = \min\{\gamma, r\} \) to get

\[
 J_2 \leq C (h^\kappa \ln h + k^{\gamma/2} \ln k) \| P_F(X(t_n)) \|_{L^p(\Omega; H^{s-2})}
\]

\[
 \leq C (h^\kappa \ln h + k^{\gamma/2} \ln k) \| P_F(X(t_n)) \|_{L^p(\Omega; H^2)}
\]

\[
 \leq C (h^\kappa \ln h + k^{\gamma/2} \ln k).
\]

With regard to \( J_3 \), by employing (6.5) with \( \alpha = \kappa \) and (2.3) with \( \beta = 2 \), one can observe that

\[
 J_3 \leq C (h^\kappa + k^{\gamma/2}) \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} (t_n - s)^{-1} A^{\gamma/2} (F(X(t_j)) - F(X(t_n))) \|_{L^p(\Omega; H)} \, ds
\]

\[
 \leq C (h^\kappa + k^{\gamma/2}) \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_{j}} (t_n - s)^{-1} t_n^{\gamma-2} \, ds
\]

\[
 \leq C (h^\kappa + k^{\gamma/2}),
\]
where we used the fact $\kappa - 2 = \min\{r - 2, \gamma - 2\} \leq 2$. Gathering the above three estimates together leads to

$$\mathcal{J} \leq C(h^\kappa |\ln h| + k^{\tilde{\kappa}} |\ln k|).$$

(6.38)

For the term $\mathcal{K}$, we utilize the Burkholder-Davis-Gundy-type inequality and employ (6.4) with $\nu = \kappa$ to obtain

$$\mathcal{K} \leq \left( \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|\Psi_{k,h}(t_n-s)\|_{L^2_h}^2 \right)^{\frac{1}{2}} \leq C(h^\kappa |\ln h| + k^{\tilde{\kappa}} |\ln k|) \|A^{\frac{2}{\nu}} Q^\frac{1}{\nu}\|_{L^2}.$$

(6.39)

At last, putting the above estimates together implies

$$\|X(t_n) - \bar{X}_h^n\|_{L^p(\Omega;H)} \leq C(h^\kappa |\ln h| + k^{\tilde{\kappa}} |\ln k|).$$

(6.40)

Next we turn our attention to the error $\bar{e}_h^n := X_h^n - \bar{X}_h^n$, which is the solution of the following error problem

$$\bar{e}_h^n - \bar{e}_h^{n-1} + kA_h^2\bar{e}_h^n = -kA_hP_h F(X^n_h) + kA_h P_h F(X(t_n)), \quad \bar{e}_h^0 = 0,$$

(6.41)

which can be reformulated as

$$\bar{e}_h^n = k \sum_{j=1}^n A_h E_{k,h}^{n-j+1} P_h (F(X(t_j)) - F(X_h^n)).$$

(6.42)

Multiplying both sides of (6.41) by $A_h^{-1} \bar{e}_h^n$ yields

$$(\bar{e}_h^n - \bar{e}_h^{n-1}, A_h^{-1} \bar{e}_h^n) + k(\nabla \bar{e}_h^n, \nabla \bar{e}_h^n) = k(-F(X^n_h) + F(\bar{X}_h^n), \bar{e}_h^n) + k(-F(\bar{X}_h^n) + F(X(t_n)), \bar{e}_h^n).$$

(6.43)

Furthermore, note that $\bar{e}_h^0 = 0$ and $\frac{1}{2}(|e_h^n|^2 - |e_h^{n-1}|^2) \leq (\bar{e}_h^n, A_h^{-1} \bar{e}_h^n)$. Taking these facts into account, one can follow a similar way as in the proof of (4.23) to arrive at

$$\frac{1}{2}(|e_h^n|^2 - |e_h^{n-1}|^2) \leq \frac{k}{2} |e_h^n|^2 + \frac{9k}{8} |e_h^{n-1}|^2 + Ck \|X^n_h - X(t_n)\|^2 (1 + |X_h^n|_{H^1}^4 + |X(t_j)|_{H^1}^4).$$

(6.44)

Summation on $n$ and applying the Gronwall inequality yield

$$\sum_{j=1}^n |\bar{e}_h^n|^2 \leq C k \sum_{j=1}^n \|X^n_h - X(t_j)\|^2 (1 + |X_h^n|_{H^1}^4 + |X(t_j)|_{H^1}^4),$$

(6.45)

which combined with (6.20), (6.24) and (6.40) leads to

$$\left\|k \sum_{j=1}^n |\bar{e}_h^n|^2 \right\|_{L^p(\Omega;\mathbb{R})} \leq C k \sum_{j=1}^n \|X^n_h - X(t_j)\|^2 (1 + |X_h^n|_{H^1}^4 + |X(t_j)|_{H^1}^4)$$

$$\leq C k \sum_{j=1}^n \|X^n_h - X(t_j)\|^2_{L^{2p}(\Omega;H)} (1 + \|A_h \bar{X}_h^n\|^4_{L^{2p}(\Omega;H)} + \|X(t_j)\|^4_{L^{2p}(\Omega;H^2)}),$$

(6.46)

$$\leq C(h^\kappa |\ln h| + k^{\tilde{\kappa}} |\ln k|)^2.$$
As in (4.27), using (6.40), (5.8), (6.24) and (2.20) leads to

\[ A \leq C k \sum_{j=1}^{n} t_{n-j+1}^{-\frac{1}{2}} \| F(X(t_j)) - F(\tilde{X}_k^j) \|_{L^p_\omega(H)} \]
\[ \leq C k \sum_{j=1}^{n} t_{n-j+1}^{-\frac{1}{2}} \| X(t_j) - \tilde{X}_k^j \|_{L^p_\omega(\Omega; H)} (1 + \| X(t_j) \|_{L^p_\omega(\Omega; H^2)} + \| A_h \tilde{X}_k^j \|_{L^p_\omega(\Omega; H^2)}) \]
\[ \leq C (h^n \ln h + k^{\frac{1}{2}} \ln k) k \sum_{j=1}^{n} t_{n-j+1}^{-\frac{1}{2}} (1 + \sup_{s \in [0,T]} \| X(s) \|_{L^p_\omega(\Omega; H^2)} + \sup_{1 \leq j \leq N} \| A_h \tilde{X}_k^j \|_{L^p_\omega(\Omega; H^2)}) \]
\[ \leq C (h^n \ln h + k^{\frac{1}{2}} \ln k). \] (6.48)

For the term \( B \), using similar calculations performed in (4.28) we obtain

\[ \| A_{\frac{k}{2}}^T P_h P(\tilde{X}_k^j) - F(X_k^j) \| \leq C \| \tilde{X}_k^j - X_k^j \|_1 (1 + \| A_h \tilde{X}_k^j \| + \| A_h X_k^j \|^2). \] (6.49)

Combining this with (6.46), (6.24) and (5.4) helps us to derive

\[ \| X(t) - X_h(t) \|_{L^p_\omega(\Omega; H)} \leq C (h^n \ln h + k^{\frac{1}{2}} \ln k), \]
\[ \| X(t) - X_h(t) \|_{L^p_\omega(\Omega; H)} \leq C (h^n \ln h + k^{\frac{1}{2}} \ln k), \] (6.50)

which together with (6.48), (6.47) and (6.40) shows (6.1).

In the sequel, we focus on the error \( \| Y(t) - \tilde{Y}_h^n \|_{L^p_\omega(\Omega; H)} \). Similarly to the semi-discrete case, we first consider the error \( \| Y(t_n) - \tilde{Y}_h^n \|_{L^p_\omega(\Omega; H)}, \) where \( \tilde{Y}_h^n = A_h \tilde{X}_h^n - P_h PF(X(t_n)) \). By (6.22) and (1.4), we have

\[ \| Y(t_n) - \tilde{Y}_h^n \|_{L^p_\omega(\Omega; H)} \leq \| (AE(t_n) - A_h E_{k,h} P_h) X_0 \|_{L^p_\omega(\Omega; H)} + \| (I - P_h) PF(X(t_n)) \|_{L^p_\omega(\Omega; H)} \]
\[ + \left( \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} A^2 P(t_s) P(t_s) - A^2_{k,h} E_{k,h}^{n-j+1} P_h PF(X(t_j)) ds \right) \]
\[ \leq \| A E(t_n) - A_h E_{k,h} P_h X_0 \|_{L^p_\omega(\Omega; H)} + \| (I - P_h) PF(X(t_n)) \|_{L^p_\omega(\Omega; H)} \]
\[ + \left( \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| A E(t_s) - A_h E_{k,h}^{n-j+1} P_h \|^2 \| \circ \|_{L^p_\omega(\Omega; H)}\right)^{\frac{1}{2}} \]
\[ := L_1 + L_2 + L_3. \] (6.51)

In the same spirit as in (4.32) but employing (6.5) with \( \alpha = 2 \) instead we obtain

\[ L_1 \leq C h^2 \sup_{s \in [0,T]} \| PF(X(s)) \|_{L^p_\omega(\Omega; H^2)} + C (h^2 + k^{\frac{1}{2}}) t_n^{-1} \| X_0 \|_{L^p_\omega(\Omega; H)} \]
\[ \leq C (h^2 + k^{\frac{1}{2}})(1 + t_n^{-1}). \] (6.52)
Before treating the term $L_2$, we need its further decomposition as follows

\[
L_2 \leq \left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} A^2 E(t_n - s)P(F(X(s)) - F(X(t_j))) \, ds \right\|_{L^p(\Omega; H)} + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left\| (A^2 E(t_n - s) - A^2 E_{n-j+1}^h P_h) P(F(X(t_j)) - F(X(t_n))) \right\|_{L^p(\Omega; H)} \, ds
\]

\[
+ \left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (A^2 E(t_n - s) - A^2 E_{n-j+1}^h P_h) PF(X(t_n)) \, ds \right\|_{L^p(\Omega; H)} (6.53)
\]

Owing to (2.31) with $\beta = 1$ and (2.5) with $\mu = \frac{3}{2}$, we derive

\[
L_{21} \leq C \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (t_n - s)^{-\frac{3}{4}} \left\| A^\frac{3}{2} P(F(X(s)) - F(X(t_j))) \right\|_{L^p(\Omega; H)} \, ds \leq C k^{\frac{3}{2}} \int_{0}^{t_n} (t_n - s)^{-\frac{3}{4}} \, ds \leq C k^{\frac{3}{2}}. \quad (6.54)
\]

As above, using (5.8), (6.53) with $\alpha = 2$, (6.8) with $\mu = 2$ and (2.5) with $\beta = 2$ implies

\[
L_{22} \leq \left\| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \Phi_{k,h}(t_n - s) AP(F(X(t_j)) - F(X(t_n))) \right\|_{L^p(\Omega; H)} \, ds
\]

\[
+ \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} A^2 E_{n-j+1}^h P_h(I - R_h) P(F(X(t_j)) - F(X(t_n))) \left\|_{L^p(\Omega; H)} \, ds \leq C(h^2 + k^{\frac{1}{2}}) \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} ((t_n - s)^{-1} + t_{n-j+1}^{-1}) \left\| AP(F(X(t_j)) - F(X(t_n))) \right\|_{L^p(\Omega; H)} \, ds
\]

\[
\leq C(h^2 + k^{\frac{1}{2}}) \sum_{j=1}^{n-1} k(t_{n-j}^{-1} + t_{n-j+1}^{-1}) \leq C(h^2 + k^{\frac{1}{2}}). \quad (6.55)
\]

For the term $L_{23}$, after splitting it into two terms by (6.7), we utilize (5.10), (2.30), (6.8) and (5.8) with $\varrho = 2$ to show

\[
L_{23} \leq \left\| \int_{0}^{t_n} \Phi_{k,h}(t_n - s) APF(X(t_n)) \, ds \right\|_{L^p(\Omega; H)} + \left\| \sum_{j=1}^{n} k A^2 E_{n-j+1}^h P_h(R_h - I) PF(X(t_n)) \right\|_{L^p(\Omega; H)}
\]

\[
\leq C(h^2 |\ln h| + k^{\frac{1}{2}} |\ln k|) \left\| PF(X(t_n)) \right\|_{L^p(\Omega; H^2)} + C \left\| (R_h - I) PF(X(t_n)) \right\|_{L^p(\Omega; H)}
\]

\[
\leq C(h^2 |\ln h| + k^{\frac{1}{2}} |\ln k|) \sup_{s \in [0,T]} \left\| PF(X(s)) \right\|_{L^p(\Omega; H^2)}. \quad (6.56)
\]

Putting the above three estimates together ensures

\[
L_2 \leq C(h^2 |\ln h| + k^{\frac{1}{2}} |\ln k|). \quad (6.57)
\]

Now we are in a position to bound the term $L_3$. In the light of (6.7) with $\mu = \frac{2}{2}$, we derive

\[
L_3 \leq \left( \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left\| \Phi_{k,h}(t_n - s) Q^2 \right\|_{L^2}^2 \, ds \right)^{\frac{1}{2}} \leq C(h^{\gamma-2} |\ln h| + k^{\frac{2-2}{4}} |\ln k|) || A^{\frac{2-2}{4}} Q^2 ||_{L^2}, \quad (6.58)
\]
which together with \((6.57)\) and \((6.59)\) shows
\[
\|Y(t_n) - \tilde{Y}_h^n\|_{L^p(\Omega; H)} \leq C(h^{-2}\ln h + k^{3/4})(1 + t_n^{-1}). \tag{6.59}
\]
So we only need to bound the error \(\|\tilde{Y}_h^n - Y_h^n\|_{L^p(\Omega; H)}\). Using the same arguments as in the proof of \((4.39)\) yields
\[
\|P(F(X(t_j)) - F(X_h^n))\|_{L^p(\Omega; H)} \leq C(1 + \sup_{s \in [0, T]} \|X(s)\|^2_{L^p(\Omega; H^2)} + \sup_{1 \leq j \leq N} \|A_k X_h^j\|^2_{L^p(\Omega; H)})
\]
\[
\cdot \|X(t_j) - X_h^n\|_{L^p(\Omega; H)}
\]
\[
\leq C(h^\kappa |\ln h| + k^{3/4} |\ln k|). \tag{6.60}
\]
This combined with the inverse inequality \(\|A_h^{\frac{1}{2}}\|_{L(\mathcal{H})} \leq Ch^{-1}\) and \(t_n^{-1} \leq Ck^{-1}\) helps us to arrive at
\[
\|\tilde{Y}_h^n - Y_h^n\|_{L^p(\Omega; H)} \leq \sum_{j=1}^n k\|E_{k,n}^{n-j+1} A_h^{\frac{j}{4}} P_h P(F(X(t_j)) - F(X_h^n))\|_{L^p(\Omega; H)}
\]
\[
+ \|P(F(X(t_n)) - F(X_h^n))\|_{L^p(\Omega; H)}
\]
\[
\leq C \min\{h^{-1}, k^{-\frac{1}{2}}\} \sum_{j=1}^n k^{j} t_{n-j+1} \|P(F(X(t_j)) - F(X_h^n))\|_{L^p(\Omega; H)}
\]
\[
+ C(h^\kappa |\ln h| + k^{3/4} |\ln k|)
\]
\[
\leq C(h^\kappa - 1 |\ln h| + k^{\frac{n-1}{2}} |\ln k|). \tag{6.61}
\]
Finally, a combination of \((6.61)\) and \((6.59)\) gives \((6.2)\), as required. 

7 Appendix: Proof of Lemma 5.2

To prove \((5.8)\), we refer to [17] (2.10)]. For \((5.9)\), the case \(\nu = 0\) is already included in \((5.8)\) with \(\mu = 0\). Thus it remains to show \(\nu = 2\). Let \(\{\lambda_j, h\}_{j=1}^{N_h}\) be the positive eigenvalues of \(A_h\) with corresponding orthonormal eigenvectors \(\{e_{j,h}\}_{j=1}^{N_h} \subset \hat{V}_h\). By using the expansion of \(P_h v\) in terms of \(\{e_{j,h}\}_{j=1}^{N_h}\), we know
\[
\|A_h^{-2}(I - E_{k,h}^n) P_h v\|^2 = \left\| \sum_{j=1}^{N_h} \lambda_j^{-2}(1 - r(k\lambda_j^2)^n)(v, e_{j,h}) e_{j,h}\right\|^2 = \sum_{j=1}^{N_h} \lambda_j^{-4}(1 - r(k\lambda_j^2)^n)^2(v, e_{j,h})^2. \tag{7.1}
\]
Due to Taylor’s formula, there exists a constant \(C\) such that for all \(j = 1, 2, \cdots, N_h\)
\[
|1 - r(k\lambda_j^2)^n| = \left| 1 - \frac{1}{(1 + k\lambda_j^2)^n}\right| \leq t_n \lambda_j^2 h. \tag{7.2}
\]
This combined with \((7.1)\) shows the case \(\nu = 2\). The intermediate cases follow by interpolation. Finally, we will show \((5.10)\). As in \((7.1)\), Parseval’s identity yields
\[
\left\| \sum_{j=1}^{n} \sum_{i=1}^{N_i} \lambda_i^2 r(k\lambda_i^2)^j(v, e_{i,h}) e_{i,h}\right\|^2 = \sum_{i=1}^{N_i} \left(k \sum_{j=1}^{n} \lambda_i^2 r(k\lambda_i^2)^j\right)^2 (v, e_{i,h})^2. \tag{7.3}
\]
Thus, it suffices to show that there exists a constant \(C\) such that \(i = 1, 2, \cdots, N_h\).
\[
k \sum_{j=1}^{n} \lambda_i^2 r(k\lambda_i^2)^j \leq C. \tag{7.4}
\]
First, we consider all summands with $k\lambda_{i,h}^2 \leq 1$. In this case, using (5.6) implies

$$k \sum_{j=1}^{n} \lambda_{i,h}^2 r(k\lambda_{i,h}^2)^j \leq k \sum_{j=1}^{n} \lambda_{i,h}^2 e^{-ct\lambda_{i,h}^2} \leq \int_{0}^{tn} \lambda_{i,h}^2 e^{-cs\lambda_{i,h}^2} \, ds \leq \frac{1}{c}(1 - e^{-ct\lambda_{i,h}^2}) \leq \frac{1}{c},$$

(7.5)

For all summands with $k\lambda_{i,h}^2 \geq 1$, we have the following estimates by $r(k\lambda_{i,h}^2) \leq \frac{1}{2}$

$$k \sum_{j=1}^{n} \lambda_{i,h}^2 r(k\lambda_{i,h}^2)^j \leq n \sum_{j=1}^{n} r(k\lambda_{i,h}^2)^{j-1} \leq n \sum_{j=1}^{n} 2^{-(j-1)} \leq 2.$$

(7.6)

Altogether, this shows (7.4). The proof of (5.10) is similar to (5.11) and we omit it. □

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