\textbf{ABSTRACT:} We investigate the complete universe evolution in the framework of \( f(T) \) cosmology. We first study the requirements at the kinematic level and we introduce a simple scale factor with the necessary features. Performing a detailed analysis of the phase portrait we show that the universe begins in the infinite past from a phase where the scale factor goes to zero but the Hubble parameter goes to a constant, and its derivative to zero. Since these features resemble those of the Pseudo-Rip fate but in a reverted way, we call this initial phase as Pseudo-Bang. Then the universe evolves in a first inflationary phase, a cosmological turnaround and a bounce, after which we have a second inflationary regime with a successful exit. Subsequently we obtain the standard thermal history and the sequence of radiation, matter and late-time acceleration epochs, showing that the universe will result in an everlasting Pseudo-Rip phase. Finally, taking advantage of the fact that the field equations of \( f(T) \) gravity are of second order, and therefore the corresponding autonomous dynamical system is one dimensional, we incorporate the aforementioned kinematic features and we reconstruct the specific \( f(T) \) form that can dynamically generate the Pseudo-Bang cosmological scenario. Lastly, we examine the evolution of the primordial fluctuations showing that they are initially sub-horizon, and we show that the total fluid does not exhibit any singular behaviour at the phantom crossing points, while the torsional fluid experiences them as Type II singular phases.

\textbf{KEYWORDS:} \( f(T) \) gravity, Dark energy, Bounce cosmology, Pseudo-Bang, Pseudo-Rip
1 Introduction

According to accumulating observational evidence from different and various probes, the universe passed through two phases of accelerated expansion, one at very early and one at late cosmological times. In order to explain these phases one needs to proceed to a modification of the standard lore of cosmology. A first direction he could follow is to introduce new particles/fields while still remaining in the framework of general relativity, such as the inflaton field [1, 2] and the dark energy sector [3, 4].

A second direction is to construct gravitational modifications, which deviate from general relativity at particular scales, thus offering the extra degrees of freedom needed to describe the universe evolution [5, 6].

In order to build a modified gravity theory one modifies a specific feature of general relativity. Altering the Einstein-Hilbert Lagrangian leads to \( f(R) \) gravity [7, 8], Modified Gauss-Bonnet theory [9], Lovelock gravity [10], etc, while allowing for a scalar field coupled with curvature invariants gives rise to Horndeski [11] and generalized Galileon theories [12, 13]. Additionally, changing the spacetime dimensionality leads to the braneworld theories [14]. An interesting alternative is to start from the equivalent, teleparallel formulation of gravity [15–20] and construct modifications using combinations of torsional invariants, such as in \( f(T) \) gravity [21–23], in \( f(T, T_G) \) gravity [24, 25], in scalar-torsion theories [26, 27], etc. \( f(T) \) gravity proves to have interesting cosmological applications, being efficient in describing both the late-time acceleration and the inflationary phase, while its confrontation with observations leads to very satisfactory results [28–60]. Notably, it has been shown...
that reconciling Planck with the local value of $H_0$ in a six-parameter space is achievable within the exponential infrared $f(T)$ gravity [61].

On the other hand, since the standard inflationary Big Bang scenario faces the crucial problem of the initial singularity (unavoidable in the case where inflation is realized using a scalar field in the framework of general relativity [62]), a potential solution in terms of bouncing cosmologies has been introduced by considering Friedmann-Lemaître-Robertson-Walker (FLRW) models with a positive spatial curvature, where the matter sector is dominated by a massive scalar field [63] (see also [64, 65]). Since the bounce realization requires the violation of the null energy condition, it can be easily obtained in various modified gravity theories, such as the Pre-Big-Bang [66] and the Ekpyrotic [67, 68] models, higher-order corrected gravity [69–71], even with the earlier work of $f(R)$ gravity [72] as well as other $f(R)$ models [73, 74], braneworld scenarios [75, 76], non-relativistic gravity [77], massive gravity [78], loop quantum cosmology [80–82], Finsler gravity [83] etc, while it can be easily obtained within $f(T)$ gravity too [84]. Although trans-Planckian problems, which may arise in inflationary models due to dispersion law modifications for frequencies beyond the Planck scale, have been discussed in Ref. [85], it has been shown that these problems do not exist within inflationary scenarios as far as local Lorentz invariance is not broken even for those ultrahigh frequencies [86, 87]. Bounce cosmology proves efficient too to avoid these problems and bypasses the initial singularity [85, 88], while at the perturbation level it leads to scale-invariant power spectrum [89–92].

Nevertheless, the interesting question that appears is whether one can obtain a unified description of the whole universe evolution through modified gravity. In the classes where the initial singularity is bypassed through the bounce realization, this question includes the investigation of time intervals up to infinitely early times. On the other hand, in every cosmological scenario it is always interesting and necessary to study the fate of the universe in the asymptotically far future.

In the present work we are interested in investigating the complete universe evolution in the framework of $f(T)$ cosmology. In particular, we desire to construct a scenario that includes the standard observed thermal history of the universe, namely the sequence of radiation, matter and late-time acceleration epochs, and moreover bypasses the initial singularity. In order to achieve this we take advantage of the fact that the field equations of $f(T)$ gravity are second-ordered, and thus the corresponding autonomous dynamical system is one dimensional [93–96]. Hence, the resulting phase space can be systematically explored, while still being much more complex than the corresponding one of general relativity, thus allowing for significantly richer cosmological behavior.

We organize this manuscript as follows. In Section 2, we examine the necessary requirements on the kinematic level and we introduce a non-singular scale factor that can produce the unified universe evolution. In Section 3 we analyze in detail the resulting cosmology, which we name “Pseudo-Bang Scenario”, since its first phase presents the features of a Pseudo-Rip but in a reversed way. As we show, the universe passes through a phantom crossing, turnaround, bounce, inflation, radiation, matter and late-time acceleration eras, and asymptotically it results to a Pseudo-Rip phase. In Section 4 we reconstruct the $f(T)$ theory which can dynamically generate the above phase-portrait behavior, namely the unified Pseudo-Bang cosmological scenario. Additionally, we apply the energy conditions and the inertial-force approach, which determine the ripping behaviour, to verify that these phases fall within this classification. Finally, in Section 5 we summarize the obtained results.
2 Unifying bounce and late-time accelerated cosmologies

In this section we investigate the kinematics of a cosmological scenario that unifies the bouncing behavior with the standard thermal history of the universe and in particular with the late-time accelerated era. In particular, we desire to examine the necessary form of the Hubble parameter evolution $H(t)$ that is required to obtain the aforementioned unified evolution. In order to achieve this we apply the dynamical system approach and we focus our discussion to one-dimensional autonomous systems, i.e. where the Hubble derivative satisfies $\dot{H} = F(H)$ (as we will later see this is the case of $f(T)$ gravity).

In this case, the differential equation represents a vector field on a line ($H$-axis). We just need to draw the graph of $F(H)$ and then use it to sketch the vector field on the real line (the $H$-axis). Therefore, it is still convenient to depict $\dot{H}$ versus $H$, and then insert arrows on the $H$-axis to indicate the corresponding Hubble flow vector at each $H$ in a simple way. The arrows point to the right when $\dot{H} > 0$, to the left when $\dot{H} < 0$, while for $\dot{H} = 0$ there is no flow. Additionally, concerning the continuity and differentiability of $F(H)$, given an initial condition $H(t_i) = H_i$, the continuity of $F(H)$ guarantees the existence of a solution, while its differentiability guarantees the uniqueness of the solution (for more details see [97]). This approach allows for the visualization of all possible cosmological solutions as a graphical representation of the phase portrait, independently of the initial conditions, where every phase point can serve as an initial condition.

![Figure 1. Schematic representation of the structure of the phase space diagram in the $(H, \dot{H})$ plane, and the classification of the corresponding kinematic regions.](image)

In Fig. 1 we schematically present the structure of the phase space diagram in the $(H, \dot{H})$ plane. In general, the phase space has a Minkowskian origin at $(0, 0)$. We define the zero acceleration curve (dotted curve), which corresponds to $q \equiv -1 - \dot{H}/H^2 = 0$, and acts as a boundary between accelerated and decelerated regions. We split the phase space into four kinematic regions according to the values
of $H$ and $q$ in each region: The non-shaded region (I) represents an accelerated contraction, since $H < 0$ and $q < 0$. The shaded region (II) represents a decelerated contraction, since $H < 0$ and $q > 0$. The shaded region (III) represents a decelerated expansion, since $H > 0$ and $q > 0$. The non-shaded region (IV) represents an accelerated expansion, since $H > 0$ and $q < 0$. We mention that the positive (negative) $\dot{H}$ leads to phantom (non-phantom) cosmology, respectively. The transition from phantom to non-phantom or vice versa is allowed only through particular type of fixed points, characterized by infinite slope $d\dot{H}/dH \to \pm\infty$, which can be reached in finite time.

Having the above discussion in mind we deduce that it is easy to recognize complicated cosmological scenarios by following their phase trajectories and studying their qualitative behaviours (see [96] for more details). In the following subsections we present some specific evolution behaviors.

### 2.1 Standard bounce

Let us first investigate the conditions for the standard non-singular bounce realization. As it is known, this can be generated by employing a scale factor of the form [65]

$$a(t) = a_B \left[ \frac{3}{2} \gamma \alpha (t - t_B)^2 + 1 \right]^{\frac{1}{\gamma}}, \quad (2.1)$$

where the constant $a_B \equiv a(t_B)$ is the minimal scale factor at the bounce point $t_B$ and $\alpha$ is a positive parameter with dimensions $[T]^{-2}$. Moreover, $\gamma$ is the barotropic index related to the equation-of-state parameter of the cosmic fluid as

$$\gamma - 1 = w = p/\rho, \quad (2.2)$$

where $p$ and $\rho$ are the pressure and energy density respectively. Hence, for positive $\gamma, \alpha$ the above scale factor is indeed non-singular for finite times. As one can see, the scale factor (2.1) generates a symmetric phase portrait about $H = 0$ axis [95, 96]

$$\dot{H}\pm = \frac{3 \gamma H^2}{\pm \alpha - \sqrt{\alpha^2 - 6 \alpha \gamma H^2}}, \quad (2.3)$$

where $\dot{H}_+ (\dot{H}_-)$ denotes the branch $\dot{H} > 0 (\dot{H} < 0)$.

The non-singular bouncing cosmology phase portrait (2.3) is characterized by a double valued function, as presented in Fig. 2(a). Since the universe cannot reach the Minkowskian origin at a finite time, the universe is non-singular. We mention that the bounce occurs at $H = 0$ where $\dot{H}$ is positive. In fact, the $\dot{H} > 0$ regions require an effectively phantom cosmology. However, the crossing between phantom and non-phantom phase is possible only wherever the phase portrait is double-valued and has vertical slope at the crossing points (fixed points)\(^1\). Furthermore, it is obvious that the universe cannot result to a late-time accelerated expansion phase. This can be clearly seen in Fig. 2(a), since the last phase of the portrait on the $\dot{H}_-$ branch remains in the shaded region III eternally [94, 96].

\(^1\)The conditions to reach a fixed point in a finite time has been discussed in [98] (also see [96]).
2.2 Merging bounce with late-time acceleration

Let us try to modify the scale factor (2.1) to additionally obtain late-time accelerated expansion. One could think to impose a positive cosmological constant $\Lambda$ similarly to $\Lambda$CDM cosmology. This would impose a vertical shift of the phase portrait of Fig. 2(a) slightly upwards. Thus, the universe begins at a de Sitter phase with negative $H = -H_{de}$, evolving towards another de Sitter phase with positive $H = +H_{de}$ instead of the Minkowski phase at $H = 0$. However, the new feature is that the fixed points at the left and right boundaries of the portrait, i.e. at $H = \pm H_{\text{max}}$ where $H_{\text{max}}$ is the maximum value of the Hubble parameter, will not have infinite slopes anymore, and therefore the transition between phantom and non-phantom regimes, namely between $\dot{H}_+$ and $\dot{H}_-$ branches, cannot occur in a finite time. Thus, we need to find an alternative way to unify bounce and late-time acceleration. This is done in the following.

Since adding by hand a positive constant is not efficient, we proceed modifying the standard bounce scale factor (2.1) by introducing a correction exponential function as

$$a(t) = a_k e^{\beta(t-t_i)} \left[ \frac{3}{2} \gamma \alpha (t - t_i)^2 + 1 \right]^{\frac{1}{3\gamma}}, \quad (2.4)$$

where $a_k$ and $t_i$ are constants and $\beta$ is a positive dimension-full parameter with dimensions [T]$^{-1}$, whereas the modified scale factor (2.4) reduces to the usual bouncing model by setting $\beta = 0$. Unlike the bounce scale factor which has one local minimum at bounce $t = t_B$, the modified scale factor has a local minimum and local maximum at $t = t_i - \frac{1}{3\gamma \beta} \pm \sqrt{\frac{\alpha (\alpha - 6\beta^2 \gamma)}{a \beta \gamma}}$, which implies that $\alpha > 6\beta^2 \gamma$. In fact, these local critical points are associated with turnaround and bounce phases. The Hubble parameter
corresponding to (2.4) is

\[ H(t) = \beta + \frac{2\alpha(t - t_i)}{2 + 3\gamma\alpha(t - t_i)^2}, \]  

(2.5)

and its first derivative reads

\[ \dot{H}(t) = \frac{2\alpha \left[ 2 - 3\gamma\alpha(t - t_i)^2 \right]}{3 \left[ 2 + 3\gamma\alpha(t - t_i)^2 \right]^2}. \]  

(2.6)

Note that these expressions remain finite for positive \( \gamma, \alpha \), i.e. in the case where the bounce is non-singular. Inverting the above expression we acquire

\[ t_{\pm}(H) = t_i + \frac{\alpha \pm \sqrt{\alpha^2 - 6\gamma\alpha(H - \beta)^2}}{3\gamma\alpha(H - \beta)}, \]  

(2.7)

and thus inserting into (2.6) we obtain the phase portrait equation

\[ \dot{H}_{\pm}(H) = \frac{3\gamma(H - \beta)^2 \sqrt{\alpha^2 - 6\gamma\alpha(H - \beta)^2}}{\pm\alpha - \sqrt{\alpha^2 - 6\gamma\alpha(H - \beta)^2}}. \]  

(2.8)

Note that \( H = \beta \) is a fixed point and the time required to reach it is infinite.

The phase portrait graph of (2.8) is given in Fig. 2(b), which indeed shows that \( H \) and \( \dot{H} \) values are always finite. Moreover, we obtain fixed points, namely having \( \dot{H} = 0 \), at \( H = \beta \) (de Sitter origin) and at maximum positive and minimum negative values of the Hubble parameter \( H = \pm \sqrt{\frac{\alpha}{6\gamma}} + \beta \), respectively. We note that in a viable scenario, one needs \( |H|_{\text{max}} \gg \beta \) and subsequently \( \alpha \gg 6\beta^2\gamma \).

As it is clear from the graph, the modified scenario (2.4) shifts the phase portrait symmetry from \( H = 0 \) to \( H = \beta \). As a result, the Minkowskian origin of the conventional bounce is shifted to a de Sitter one, however this has crucial consequences on the cosmic evolution as we will discuss in Sec. 3. We mention that in this schematic graphical representation we have exaggerated the value of the parameter \( \beta \) in Fig. 2(b), in order to show clearly the breaking of the symmetry around \( H = 0 \). However, as we will see later on, when confronting with observational data the value of \( \beta \) will be a small positive number, and the deformation seen in 2(b) will be small, but still effective.

In the final phase of the portrait of Fig. 2(b), the universe evolves towards a de Sitter fixed point at \( H = \beta \), providing a late accelerated expansion phase. In addition, the fixed points at the phase portrait boundaries, \( H = \pm H_{\text{max}} \), are still having infinite slopes which is a necessary condition to allow for the crossing of the phantom divide in a smooth way (a detailed analysis of the phase portrait is given in Section 3).

In summary, up to now we have achieved our target to unify bounce and late acceleration in a single scenario. The last step is to determine the model parameters in order to obtain a viable scenario. The above scenario contains the four parameters \( a_k, t_i, \alpha \) and \( \beta \), in addition to the barotropic index (equation of state) parameter \( \gamma \), which takes the values \( \gamma = 1 \) for dust matter and \( \gamma = 4/3 \) for radiation. In order determine their values, one should thus choose four conditions from the observed universe history.

In standard Big Bang cosmology there is the initial singularity at cosmic time \( t = 0 \). However, since in our case \(-\infty < t < \infty \) it proves convenient to set \( t = 0 \) at the fixed point \( \dot{H} = 0 \) with
$H_{\text{max}} = \sqrt{\frac{\alpha}{\beta}} + \beta$, which is identified with point $G$ on the phase portrait of Fig. 2(b). From now on all time identifications are calculated from the fixed point $G$. This choice enables us to confront the model with the standard observational requirements of the cosmic thermal history. Imposing the fixed point phase condition $\dot{H} = 0$ in Eq. (2.6) and solving for the cosmic time $t$ we acquire

$$t^\pm = t_i \pm \sqrt{\frac{2}{3\gamma\alpha}},$$

(2.9)

where $t^-$ ($t^+$) identifies the fixed point phase at $H < 0$ ($H > 0$) regime at point $E$ ($G$) of the phase portrait Fig. 2(b). Setting $t^+ = 0$ as illustrated above, we determine that

$$t_i = -\sqrt{\frac{2}{3\gamma\alpha}}.$$

(2.10)

Therefore, we find that $t^- = 2t_i = -2\sqrt{\frac{2}{3\gamma\alpha}}$ at point $G$. This is the first out of the four conditions. For the other three we choose:

(i) For the present time we choose $t_0 \approx 4.3 \times 10^{17}$ s as it arises from standard cosmology, and we moreover normalize the present scale factor to $a_0 = a(t_0) = 1$.

(ii) At the end of inflation (point $K$ on Fig. 2(b)), which corresponds to zero acceleration $\ddot{a} = 0$ we need to have $t \sim 10^{-32}$ s as expected from standard cosmology.

(iii) For the late-time transition from deceleration to acceleration (point $L$), where the acceleration is again zero, namely $\ddot{a} = 0$, we impose a time $t \sim 2.4 \times 10^{17}$ s, which is consistent with the observed transition redshift $z_{\text{tr}} \sim 0.6 - 0.8$ [99]. Finally, alongside conditions (i) and (iii) we consider $\gamma = 1$ as cold dark matter is expected to dominate the evolution, while for condition (ii) we impose $\gamma = 4/3$ as radiation is expected to be dominant at the reheating phase by the end of inflation. Hence, we conclude that

$$a_K \sim 1.2 \times 10^{-33}, \quad \alpha \sim 8.6 \times 10^{62} \text{ s}^{-2}, \quad \beta \sim 6.2 \times 10^{-19} \text{ s}^{-1},$$

(2.11)

while inserting into (2.10) we find

$$t_i \sim -2.4 \times 10^{-32} \text{ s}.$$  

(2.12)

The numerical results show that $\alpha \gg 6\beta^2\gamma$, confirming the viability condition in order to have $|H|_{\text{max}} \gg \beta$, and also verifying that the modified scale factor (2.4) should have not only a local minimum as in the bounce scenario but also a local maximum which is associated with a turnaround phase. In the following we revisit these phases among other interesting features in more detail by investigating the corresponding phase portrait.

### 3 The Pseudo-Bang Scenario

In this section we utilize the phase portrait analysis in order to study the entire cosmic evolution and the stability of the scenario. As we described above, introducing the parameter $\beta$ in the scale factor results to a symmetry shift in the phase portrait, and the line of symmetry moves from Minkowski origin ($H = 0, \dot{H} = 0$) to de Sitter origin ($H = \beta, \dot{H} = 0$). This non-trivial shift seen in Fig. 2(b),
According to the numerical values of the model parameters exceptionally reached in finite time. As mentioned above, this configuration allows the universe to cross the phantom divide line smoothly.

Let us present briefly the key points of the phase portrait of Fig. 2(b). Point $A$ represents the de Sitter phase, that is the eternal phase of the universe as $t \to \pm \infty$. Points $B$, $D$, $K$ and $L$ represent transitions between acceleration and deceleration, which are characterized by $\ddot{a} = 0$. Points $C$ and $F$ represent the turnaround and bouncing points, which are characterized by $H = 0$ with $\dot{H} < 0$ and $\dot{H} > 0$, respectively. Points $E$ and $G$ represent a particular type of fixed points ($\dot{H} = 0$) that can be exceptionally reached in finite time. As mentioned above, this configuration allows the universe to cross the phantom divide line smoothly. According to the numerical values of the model parameters (2.11), we summarize the results in Table 1, estimating the representative values of the scale factor, Hubble parameter, and the energy scale $E \sim \sqrt{M_p H}$ at each point, mentioning the corresponding cosmological features.

In the following subsections we discuss the features of each point and its corresponding phase in more details.

### 3.1 From Pseudo-Bang origin to Pseudo-Rip fate

#### Pseudo-Bang origin

According to the phase portrait the cosmic time flows clockwise, where the origin has been shifted from Minkowski to a semi-stable de Sitter fixed point $A$, at which $H = \beta \sim 4.1 \times 10^{-43}$ GeV and

| Point | $t$(sec.) | $a$ | $H$(GeV) | $E$(GeV) | cosmological phase |
|-------|-----------|-----|-----------|-----------|-------------------|
| $A$   | $-\infty$ | 0   | $4.1 \times 10^{-43}$ | $9.9 \times 10^{-13}$ | Pseudo-Bang       |
| $B$   | $-1.9 \times 10^{18}$ | $3.3 \times 10^{-9}$ | $2.4 \times 10^{-43}$ | $7.6 \times 10^{-13}$ | transition I      |
| $C$   | $-8.0 \times 10^{17}$ | $4.3 \times 10^{-9}$ | $\approx 0$ | $\approx 0$ | turnaround         |
| $D$   | $-5.8 \times 10^{-32}$ | $1.6 \times 10^{-33}$ | $-6.4 \times 10^6$ | $3.9 \times 10^{12}$ | transition II     |
| $E$   | $-4.8 \times 10^{-32}$ | $1.5 \times 10^{-33}$ | $-6.8 \times 10^6$ | $4.0 \times 10^{12}$ | phantom crossing I|
| $F$   | $-2.4 \times 10^{-32}$ | $1.2 \times 10^{-33}$ | $\approx 0$ | $\approx 0$ | bounce            |
| $G$   | 0          | $1.5 \times 10^{-33}$ | $6.8 \times 10^6$ | $4.0 \times 10^{12}$ | phantom crossing II|
| $K$   | $1.0 \times 10^{-32}$ | $1.6 \times 10^{-33}$ | $6.4 \times 10^6$ | $3.9 \times 10^{12}$ | transition III    |
| $L$   | $2.4 \times 10^{17}$ | 0.6 | $2.2 \times 10^{-42}$ | $2.3 \times 10^{-12}$ | transition IV     |
| $A$   | $+\infty$ | $+\infty$ | $4.1 \times 10^{-43}$ | $9.9 \times 10^{-13}$ | Pseudo-Rip        |

**Table 1.** Approximate estimations of the Hubble and the Energy scale at the key points of the phase portrait of Fig. 2(b), using the parameter values (2.11). We refer to the cosmological phase transition I (at $B$) as a transition from accelerated to decelerated expansion, transition II (at $C$) as a transition from decelerated contraction to accelerated contraction, transition III (at $K$) as a transition from accelerated expansion to decelerated expansion and transition IV (at $L$) as a late-time transition from decelerated to accelerated expansion. All quantities at different phases are calculated for radiation except at the last two phases $L$ and $A$ we take $\gamma = 1$ as matter becomes dominating.
\[ \dot{H} = 0. \] Using the phase portrait equation (2.8), the flow time from any phase point \( H_0 \leq \beta \) on the \( H_- \) branch can be calculated by
\[
t = \int_{H_0}^{\beta} \frac{dH}{\dot{H}^{-}} \to -\infty,
\]
since \( 1 \leq \gamma \leq 2 \) to maintain the stability and the causality conditions. Therefore, the universe is eternal and has no initial finite-time singularity. It is straightforward to show that the time asymptotic of the scale factor (2.4) and the Hubble parameter (2.5) are respectively
\[
\lim_{t \to -\infty} a(t) = 0, \quad \lim_{t \to -\infty} H(t) = \beta^{-},
\]
given that \( \beta > 0 \) and \( \alpha > 0 \).

We mention that in the standard bounce the scale factor diverges as \( t \to -\infty \). On the other hand, in standard Big Bang cosmology the initial scale factor \( a(t) \to 0 \) and \( H(t) \to \infty \) as \( t \to 0 \). On the contrary, in the pure cosmological constant universe, the Hubble parameter has a finite constant value, but the universe cannot exhibit a decelerated expansion phase. Hence, one can realize that the present scenario is a novel one, in which the universe initial state is intermediate between the Big Bang and the de Sitter universe. Inspired by the Pseudo-Rip terminology (see below) we call this eternal phase as Pseudo-Bang, since it is characterized by \( a(t) \to 0 \) and \( H(t) \to \text{const.} > 0 \) as \( t \to -\infty \) (note that this is different from the emergent universe scenario which is characterized by \( a(t) \to \text{const.} \) at \( t \to -\infty \)). This is the first phase of the scenario at hand.

**Inflation**

Following the phase portrait of Fig. 2(b) clockwise, we identify the interval (i), which ends at point \( B \), in which the portrait cuts the zero acceleration curve \( \dot{H}_- = -H^2 \). Since the Hubble values are positive (as given in Table 1, \( 2.4 \times 10^{-43} \leq H(t) \leq 4.1 \times 10^{-43} \) GeV) in this interval, and \( \dot{H}_- > -H^2 \), the universe expands with acceleration. Such an unconventional initial phase represents a non-singular inflationary phase, unlike standard bounce cosmology which begins with a decelerated contraction phase. This is the second phase of the scenario at hand.

Remarkably this quasi de Sitter inflationary phase runs at low energy level which in effect might contribute to solve the theoretical problem of the cosmological constant. Additionally, since this phase is eternal, it has enough time to solve the usual problems of the Standard Model of cosmology. Furthermore, there is no need to compute the minimum e-fold number, \( N = -\ln(a_i/a_f) \), since \( a_i \) can always be chosen small enough to obtain a suitable \( N \). On the other hand, in the scenario at hand the comoving Hubble radius \( R_H = \frac{1}{a(t)H(t)} \), which we refer to as horizon, is infinite at the Pseudo-Bang origin as \( a(t) \to 0 \). Therefore, all modes are at sub-horizon scale \( k \gg R_H^{-1} \), or equivalently the physical wavelengths \( \lambda \ll \lambda_H \), where \( \lambda(t) = a(t)/k \) is the physical mode wavelength, \( \lambda_H = |H^{-1}| \) is the Hubble radius (we refer to the Hubble radius at Pseudo-Bang as \( \lambda_\beta \to 1/\beta \)). This is a necessary condition to have an initial causal universe. Consequently, we can assume that the primordial fluctuations are coherent, as indicated by the observations of acoustic peaks in the power spectrum of Cosmic Microwave Background (CMB) anisotropies. In this case, it is natural to assume that the quantum fluctuations around the initial vacuum state form the Bunch-Davies vacuum [102]. The evolution of the physical modes at different scales verses Hubble horizon is given in Fig. 3.
Figure 3. The time evolution of the wavelengths $\lambda = a(t)/k$, where $a(t)$ is given by Eq. (2.4), as well as the evolution of the Hubble horizon $\lambda_H = |H^{-1}|$, where $H$ is given by Eq. (2.5). The horizontal line identifies the minimal value of the Hubble radius at Pseudo-Bang origin $\lambda_{\beta} \to 1/\beta$, as well as its maximal value at Pseudo-Rip fate. The sub-graph in the upper left corner zooms at the wavelength and Hubble radius evolution during the bounce phase and the end of inflation II, followed by a decelerated expansion phase consistent with standard cosmology. The model parameters have been set as in (2.11). It is clear that at the Pseudo-Bang all modes are sub-horizon, namely $\lambda \ll \lambda_{\beta}$. The long wavelength modes ($k \lesssim 10^{-8}$) exit the Hubble horizon by the end of inflationary epoch I, while shorter wavelength modes ($k \gtrsim 10^{-3}$) still remain sub-horizon. At the turnaround and bounce points $\lambda_H$ becomes infinite, since $H = 0$, and all modes re-enter the horizon and become sub-horizon. Subsequently, long wavelength modes exit the horizon by the end of inflationary epoch II, while shorter wavelength modes are sub-horizon and they exit the horizon at the later accelerated expansion phase.

Turnaround

At point C on Fig. 2(b) the universe goes from dynamical region III (decelerated expansion) to another dynamical region II (decelerated contraction). Therefore, point C represents a turnaround point, $H = 0$ and $\dot{H} < 0$, at which the universe reaches a maximum size with a finite deceleration unlike Big Brake models [103]. This is the third phase of the scenario at hand.

At the turnaround point C, the Hubble parameter goes to zero. Using (2.5), (2.11) and (2.12), we determine the time at the turnaround as

$$t_C = -\frac{1}{3\gamma\beta} - \sqrt{\frac{2}{3\gamma\alpha}} - \frac{\sqrt{\alpha^2 - 6\gamma\alpha\beta^2}}{3\gamma\alpha\beta} \approx -8 \times 10^{17} \text{ s}. \quad (3.3)$$

Notably, in general relativity, at turnaround phase one should introduce unconventional matter species violating the null energy condition. This situation can be avoided in modified gravity making the gravitational sector to compensate the matter component at the turnaround point [104]. Finally, we
mention that Fig. 3 shows that the corresponding Hubble radius $\lambda_H$ becomes infinite, allowing the modes to re-enter the horizon and become sub-horizon again.

**Phantom crossing I**

At point $D$, the phase portrait intersects the zero acceleration curve $\dot{H} = -H^2$ for the second time, but at a negative value of $H \sim -6.4 \times 10^6$ GeV, as given in Table 1. The, the universe enters in a new phase of accelerated contraction $\dot{H} > -H^2$, namely interval (iv). Notably, the point $E$ is a fixed point which exceptionally can be reached in finite time. This can be proven, since

$$\lim_{H \to H_E} \dot{H}_\pm = 0, \quad \lim_{H \to H_E} \frac{d\dot{H}_\pm}{dH} = +\infty, \quad \text{and} \quad t \to \int_{H < H_E}^{H_E} H^{-1}_- dH = \text{finite}.$$ 

Additionally the phase portrait is double valued about $E$ which is a necessary condition to cross the phantom divide line through point $E$, for more detail see [96, 98]. At this point $E$, we calculate $H_E = \beta - \sqrt{\frac{2}{3\gamma}} \sim -6.8 \times 10^6$ GeV, then the time to reach is $t_E = 2t_i = -2 \sqrt{\frac{2}{3\gamma}} \sim -4.8 \times 10^{-32}$ s. This Phantom crossing I phase is the fourth epoch of the scenario at hand.

**Bounce**

At point $F$ the universe bounces from an accelerated contraction to accelerated expansion in an effective phantom regime, where $H_+ > 0$, as it is shown in Fig. 2(b). Using (2.5) and (2.10), we determine the time at the bounce point as

$$t_F = -\frac{1}{3\gamma\beta} - \sqrt{\frac{2}{3\gamma\alpha} + \frac{\sqrt{\alpha^2 - 6\gamma\alpha\beta^2}}{3\gamma\beta}} \approx -2.4 \times 10^{-32} \text{ s.} \quad (3.4)$$

At the bounce point $F$ one should investigate two major problems that are usually facing such models in GR: The first is known as ghost instability due to violation of the null energy condition, which has been shown to be avoided within modified gravity framework whereas the matter sector remains causal and stable [84, 93]. The second is more severe and called the anisotropy problem since any small anisotropy evolves as $\propto a(t)^{-6}$ and becomes dominant during the contraction phase at suitably small scale factors, thus destroying FLRW geometry [65]. Nevertheless, the usual way for its avoidance is the introduction of a super-stiff matter component $\gamma \gg 2$ which scales as $a(t)^{-3\gamma}$ and thus it grows faster and dominates over the anisotropy. This being the case as we will discuss this later on in Subsection 4.2. This bouncing phase is the fifth phase in the scenario at hand.

**Phantom crossing II**

Similar to the phase at the fixed point $E$, it can be shown that

$$\lim_{H \to H_G} \dot{H}_\pm = 0, \quad \lim_{H \to H_G} \frac{d\dot{H}_\pm}{dH} = -\infty, \quad \text{and} \quad t \to \int_{H > H_G}^{H_G} H^{-1}_- dH = \text{finite}.$$ 

Then, the fixed point $G$ can be reached in a finite time, whereas the portrait is double valued about it. Therefore, at the phase point $G$, the universe crosses the phantom divide line for the second time,
similarly to point $E$, but this time from phantom to non-phantom phase. At that point the Hubble parameter $H_G = \beta + \frac{\sqrt{\gamma}}{3} \sim 6.8 \times 10^6 \text{ GeV}$. We stress that we have set the time at point $G$ as $t_G = 0$ as discussed in detail in Subsection 2.2. The Phantom crossing II is the sixth phase of the evolution.

**Inflation II and graceful exit**

During the interval (vii) the universe evolves into quintessence regime just as in standard inflationary models. In the latter, the universe needs to enlarge itself around $10^{28}$ times ($\sim 60$ e-folds) to solve the horizon and flatness problems of standard cosmology [2]. However, in our scenario there is no need for this restriction, since the preceding phases are sufficient to solve these problems. Indeed, the model predicts just few e-folds during this accelerated expansion period as indicated in Table 1. Additionally, as the Table shows, the energy scales are $H \sim 10^6 \text{ GeV}$ and $E \sim 10^{12} \text{ GeV}$ at the graceful exit point, that is below the energy scale of the grand unified theory (GUT), and hence monopoles are not going to be produced, thus bypassing the monopole problem of standard cosmology. Moreover, as it has been shown, the scale invariant power spectrum can be produced in standard bouncing cosmology in the contraction phase, which occurs in the scenario at hand too. At the end of this interval, at point $K$, the universe gracefully exits into a decelerated expansion phase, which characterizes standard cosmology. This smooth transition period is essential to prepare the universe to begin the hot Big Bang nucleosynthesis process. The graceful exit from inflation II is the seventh phase of the present scenario.

**Radiation and matter epochs**

The eighth phase of the universe evolution is the standard cosmology phase of reheating, radiation and matter epochs. In this case, it is convenient to examine the behaviour of the phase portrait (2.8) at relevant epochs. In particular, for the $\dot{H}_-$ branch at $H \gg \beta$, we obtain the leading term $\dot{H}_- = -\frac{3}{2} \gamma H^2$ which reproduces the power-law standard cosmology phase portrait.

In order to obtain a fuller view and to examine the capability of the present scenario to predict a successful thermal evolution, we define the entropy $S$ of all particles in thermal equilibrium at temperature $\Theta$ in volume $V$. According to the first law of thermodynamics, in the expanding universe we have

$$\Theta dS = d(\rho V) + pdV,$$

with the integrability condition $\frac{\partial^2 S}{\partial \rho \partial \Theta} = \frac{\partial^2 S}{\partial V \partial \Theta} \quad [105]$, where the energy density and pressure satisfy

$$\frac{dp}{d\Theta} = \frac{\rho + p}{\Theta} \quad \Rightarrow \quad \frac{d\Theta}{\Theta} = \frac{\gamma - 1}{\gamma} \frac{d\rho}{\rho}.$$  \hspace{1cm} (3.6)

From the matter conservation equation $\dot{\rho} + 3 \gamma H \rho = 0$ it is implied that $\rho \propto a^{-3\gamma}$. Solving (3.6) we evaluate the temperature as $\Theta(t) = \Theta_0 a(t)^{-3(\gamma-1)}$, with $\Theta_0 \equiv \Theta(t_0)$ an arbitrary constant. Hence, this finally leads to

$$\Theta(t) = \Theta_0 a_k^{-3(\gamma-1)} e^{-3(\gamma-1)\beta(t-t_i)} \left[ \frac{3}{2} \gamma \alpha (t-t_i)^2 + 1 \right]^{1-\gamma}.$$  \hspace{1cm} (3.7)
We choose a boundary condition such that the temperature $\Theta \sim 2.73 \text{ K}$ at the present time $t_0 \sim 10^{17} \text{ s} > t_{eq}$, with a dust equation-of-state parameter $\gamma = 1$. This determines the value $\Theta_0 = 2.73 \text{ K}$.

In summary, at the contracting Pseudo-Bang phase the temperature increases and peaks up at the bounce point with $\Theta \sim 10^{27} \text{ K}$ (assuming relativistic species with barotropic index $\gamma \approx 1.28$), and then it decreases following the standard thermal history. In Fig. 4 we depict the temperature evolution of the Pseudo-Bang scenario on top of the corresponding one for the standard Big-Bang cosmology (we restrict the graph to the interval from Planck era to the radiation-matter equality, which is suitable for comparison since at later times both scenarios exhibit the same behaviour, namely the same matter epoch.

**Late-time acceleration**

The ninth phase of the scenario at hand is the late-time acceleration. In particular, on the phase portrait of Fig. 2(b) one can see that the universe exhibits a late-time transition from deceleration to acceleration, as indicated by point $L$. At this point, the phase portrait crosses the zero acceleration
curve for the last time, from $\dot{H}_+ < -H^2$ decelerated expansion region into $\dot{H}_+ > -H^2$ accelerated expansion, where $H > 0$ and $\dot{H}_+ < 0$. More precisely, we determine the current value of the Hubble parameter as expected, by taking the present time $t_0 = 4.3 \times 10^{17}$ s (which is consistent with the assumption $a(t_0) = 1$ or equivalently of redshift $z(t_0) = \frac{1}{a(t_0)} - 1 = 0$) and according to the estimated values of the model parameters (2.11), (2.12), and thus the Hubble function (2.5) gives the present Hubble value $H_0 \approx 66.6$ km/s/Mpc. Moreover, it is straightforward to show that at the deceleration-to-acceleration transition time $t_L = 2.4 \times 10^{17}$ s the corresponding redshift $z_{tr} = \frac{1}{a(t_L)} - 1 \approx 0.7$ lies within 1γ agreement with its measured value $z_{tr} = 0.72 \pm 0.05$ for $H_0 = 68 \pm 2.8$ km/s/Mpc [106].

The late acceleration is labeled on the phase portrait as interval (ix), and is confined between points $L$ and $A$. Notably, this last phase cannot be exhibited in the standard bounce of Fig. 2(a), where the universe continues the decelerated expansion eternally.

**Pseudo-Rip fate**

As we observe in Fig. 2(b), the phase portrait evolves towards the fixed point $A$ as a final fate. Similarly to (3.1), the time needed to reach that point in infinite. Given that $\beta > 0$ and $\alpha > 0$, the time asymptotic of the scale factor (2.4) and Hubble (2.5) have the limits $a(t) \to \infty$ and $H(t) \to \beta^+$ as $t \to \infty$. Among four possible final phases of expanding universe classified in Ref. [107], we deduce that in the scenario at hand the universe evolves towards a Pseudo-Rip which is an intermediate between the no Rip and the Little Rip. This phase is characterized by a monotonically increasing Hubble to a constant value, producing an inertial force which does not increase monotonically but peaks up at a particular future time and then decreases. In this case the bound structures dissociate if they are at or below a particular threshold that depends on the inertial force, for more details see Subsection 4.2. This is the tenth and final phase.

### 3.2 Poincaré patches

Recalling the Pseudo-Bang scale factor (2.4), one may realize that the hypersurface at the Pseudo-Bang $t = -\infty$ is not space-like but null. In this sense, the Pseudo-Bang resembles the de Sitter universe up to a sub-leading term, as it is clear from (2.5). Similarly to de Sitter, there can be particles coming to Pseudo-Bang from the other part of spacetime. To reveal the similarity between the Pseudo-Bang phase and de Sitter universe, we impose the transformation

$$\tau = \tilde{t} + \ln \left[ a_k^{-1/\beta} \left( \frac{3}{2} \gamma \alpha \tilde{t}^2 + 1 \right) \right],$$

where $\tau \to \pm \infty$ as $\tilde{t} \equiv t - t_i \to \pm \infty$, which leads to the Pseudo-Bang scale factor as $a(\tau) = e^{\beta \tau}$. Moreover, we write

$$\frac{d\tau}{d\tilde{t}} = 1 + \frac{3 \gamma \alpha \tilde{t}}{\frac{3}{2} \gamma \alpha \tilde{t}^2 + 1},$$

which implies that $d\tau$ differs from $d\tilde{t}$ by a sub-leading term only, and thus $d\tau/d\tilde{t} \to 1$ as $\tilde{t} \to \pm \infty$ ($\tau \to \pm \infty$). In this case, we write the FLRW metric at Pseudo-Bang/Rip as

$$ds^2 = d\tau^2 - e^{2\beta \tau} d\tilde{x}^2.$$  

(3.10)
The entire Pseudo-Bang space where the expanding patch (3.14) is constrained by $-\infty < \tau_+ < \infty$ and the contracting patch (3.15) is constrained by $-\infty < \tau_- < \infty$. The boundary $\tau_+ = \mp \infty$ is light-like. The phase portrait is invariant under $\tilde{t}^+ \to -\tilde{t}^-$ and $\beta \to -\beta$, and the matching between the two de Sitter semi-stable fixed points $H = \pm \beta$ is viable.

The symmetry of the obtained spacetime can be revealed by making use of a flat embedding of (3.10) into a 5-dimensional Minkowski spacetime,

$$d s_5^2 = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 - dX_4^2,$$

which can be realized as the hyperboloid

$$X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = -\beta^{-2}.$$  

For the coordinate transformation\footnote{We use $-\infty < \tau_+ < \infty$ to identify the expanding patch which represents only one-half of the entire de Sitter space, while $-\infty < \tau_- < \infty$ will be used to describe the other contracting patch.}

$$X_0 = \frac{1}{\beta} \sinh (\beta \tau_+) + \frac{\beta}{2} e^{\beta \tau_+} |\vec{x}|^2, \quad -\infty < \tau_+ < \infty$$

$$X_i = x_i e^{\beta \tau_+}, \quad -\infty < x_i < \infty (i = 1, 2, 3)$$

$$X_4 = \frac{1}{\beta} \cosh (\beta \tau_+) - \frac{\beta}{2} e^{\beta \tau_+} |\vec{x}|^2,$$

the induced metric is

$$d s_+^2 = d\tau_+^2 - e^{2\beta \tau_+} d\vec{x}_+^2.$$  

This shows that the above metric covers only half of the de Sitter spacetime, where Eqs. (3.13) restrict the spacetime configuration to the region $X_0 + X_4 = \beta^{-1} e^{\beta \tau_+} \geq 0$, that is the expanding Poincaré patch (EPP). The other half, $X_0 + X_4 \leq 0$, is covered by the metric

$$d s_-^2 = d\tau_-^2 - e^{2\beta \tau_-} d\vec{x}_-^2.$$
which represents the contracting Poincaré patch (CPP).

The entire manifold at Pseudo-Bang and Pseudo-Rip can be obtained via a Penrose–Carter diagram, as seen in Fig. 5(a). This diagram reveals that the boundary between the EPP and the CPP at $\tau_{\pm} = \mp \infty$ is light-like. Note that the solution is invariant under the transformations $t_+ \rightarrow -t_-$ and $\beta \rightarrow -\beta$. In terms of the phase-portrait terminology, one obtains that the phase portrait equation (2.8) may have another fixed point of de Sitter type at $H = -\beta$. In Subsection 3.1, we discussed the evolution corresponding to the expanding de Sitter origin, i.e. $H = +\beta$, as realized by the phase portrait of Fig. 2(b). Since the Pseudo-Bang resembles the de Sitter case, and $t \rightarrow -\infty$ is a null hypersurface, it may not represent a genuine beginning. For that reason we label the time of the portrait with the origin at $H = \beta$ as $-\infty < t_+ < \infty$. On the other hand, we label the time of the portrait with origin at $H = -\beta$ as $-\infty < t_- < \infty$, as seen in Fig. 5(b). One realizes that there is no need for junction conditions to match the two phases at $H = \pm \beta$. Indeed, the Pseudo-Bang in the expanding de Sitter (i.e. EPP), $t_+ \rightarrow -\infty$, $a(t_+) \rightarrow 0$ and $H(t_+) \rightarrow -\beta^+$, is naturally matched with the contracting de Sitter (i.e. CPP), $t_- \rightarrow +\infty$, $a(t_-) \rightarrow 0$ and $H(t_-) \rightarrow -\beta^-$. On the other hand, the Pseudo-Rip in the expanding de Sitter (i.e. EPP), $t_+ \rightarrow +\infty$, $a(t_+) \rightarrow \infty$ and $H(t_+) \rightarrow \beta^+$, is naturally matched with the contracting de Sitter (i.e. CPP), $t_- \rightarrow -\infty$, $a(t_-) \rightarrow \infty$ and $H(t_-) \rightarrow -\beta^-$. For the moment the whole analysis remains at the kinematic level, namely at the investigation of the required Hubble function evolution. Nevertheless, one should consider the phase portrait (2.8) as a field equation governing the dynamics of FLRW cosmology. Therefore, one should provide the theory, namely a modified gravity, that could dynamically produce such a Pseudo-Bang scenario. Amongst the different classes of gravitational modification there is one, i.e. $f(T)$ gravity, which has second-order fields equations and thus it allows for the construction of the phase portrait in a simple way [96] (in the particular case of Pseudo-Bang scenario the phase portrait should be (2.8)). Hence, in the following section we investigate the realization of the Pseudo-Bang scenario in the framework of $f(T)$ gravity. Notably, the junction conditions in $f(T)$ gravity has been investigated in [108], and used to match two branches of a phase portrait, see [109]. However, in our case, the matching between the EPP and CPP is naturally fulfilled with no need of junction conditions.

4 Pseudo-Bang scenario in $f(T)$ gravity

In this section we investigate the dynamical realization of the Pseudo-Bang cosmological scenario in the framework of $f(T)$ gravity. We first present a brief review of the latter and then we proceed to the analysis of its cosmological implications.

4.1 $f(T)$ gravity and cosmology

In the torsional formulation of gravity it proves convenient to use as dynamical variables the vierbeins fields $e_A(x^\mu)$, which at each manifold point $x^\mu$ form an orthonormal basis. In a coordinate basis they can be expressed as $e_A \equiv e^\mu_A \partial_\mu$, related to the metric through

$$g_{\mu\nu}(x) = \eta_{AB} e^A_\mu(x) e^B_\nu(x), \quad (4.1)$$
with Greek and Latin indices respectively running over coordinate and tangent space. One introduces the Weitzenböck connection \( \Gamma^w_{\mu\nu} \equiv e^A_{\mu} \partial_{\nu} e^A_{\alpha} \) [110], and therefore the resulting torsion tensor reads

\[
T^3_{\mu\nu} \equiv \Gamma^w_{\mu\nu} - \Gamma^w_{\nu\mu} = e^A_{\mu} (\partial_{\nu} e^A_{\alpha} - \partial_{\alpha} e^A_{\nu}).
\]

This torsion tensor contains all the geometrical information, and thus it describes the gravitational field. Through its contraction one acquires the torsion scalar

\[
T \equiv \frac{1}{4} T^\mu_{\rho\nu} T_{\rho\mu\nu} + \frac{1}{2} T^\mu_{\rho\nu} T_{\nu\rho\mu} - T_{\mu\rho} T^\rho_{\nu\nu},
\]

which is then used as the Lagrangian of the theory. Variation of the action in terms of the vierbeins gives identical equations with general relativity, and that is why this theory was named teleparallel equivalent of general relativity [19].

Inspired by the curvature modifications of gravity, in which one generalizes the Einstein-Hilbert action, one can start from teleparallel equivalent of general relativity and extend the Lagrangian to an arbitrary function of the torsion scalar \( T \). The resulting \( f(T) \) gravity is characterized by the action [23]

\[
S = \int d^4x \left[ \frac{1}{2\kappa^2} f(T) + L_m \right],
\]

with \( e = \det(e^\mu_\mu) = \sqrt{-g} \), \( \kappa^2 = 8\pi G \) the gravitational constant, and where for completeness we have added the matter Lagrangian \( L_m \). Variation of the action (4.4) with respect to the vierbein gives [23]

\[
\frac{1}{e} \partial_{\mu} \left( e_{\nu}^{\alpha} T_{\alpha}^{\mu\nu} \right) f_T - e_{\nu}^{\alpha} T_{\alpha}^{\mu\nu} \partial_{\mu} e_{\nu}^{\alpha} f_T + e_{\nu}^{\alpha} \partial_{\mu} T f_{TT} + \frac{1}{4} e_{\alpha}^{\nu} f(T) = \frac{k^2}{2} e^{\mu}_{\alpha} T_{\mu}^{\alpha},
\]

with \( f_T := \frac{df}{dT} \) and \( f_{TT} := \frac{d^2f}{dT^2} \). Additionally, we have defined the superpotential

\[
\gamma_{\alpha}^{\mu\nu} = \frac{1}{4} \left( T_{\alpha}^{\mu\nu} + T_{\alpha}^{\nu\mu} - T_{\mu}^{\nu\alpha} - T_{\nu}^{\mu\alpha} \right) + \frac{1}{2} \left( \delta_{\alpha}^{\mu} T_{\nu}^{\nu} - \delta_{\nu}^{\nu} T_{\alpha}^{\mu} \right),
\]

which is skew symmetric in the last pair of indices, as well as the energy-momentum tensor of the total matter fields (baryonic and dark matter and radiation) \( \mathcal{T}_{\mu}^{\nu} = e_{\alpha}^{\nu} \left( \begin{array}{c} \frac{k^2}{2} e_{\alpha}^{\mu} \end{array} \right) \), assumed it to be of a perfect fluid form

\[
\mathcal{T}_{\mu\nu} = \rho u_{\mu} u_{\nu} + p (u_{\mu} u_{\nu} - g_{\mu\nu}),
\]

with \( u_{\mu} = \delta_{\mu}^{\nu} \) the fluid 4-velocity, and \( \rho \) and \( p \) the energy density and pressure in its rest frame.

In order to apply \( f(T) \) gravity to a cosmological framework we consider the flat Friedmann-Lemaître-Robertson-Walker (FLRW) geometry

\[
ds^2 = dt^2 - a(t)^2 \delta_{ij} dx^i dx^j,
\]

which corresponds to the vierbein choice \( e^A_{\mu} = \text{diag}(1, a, a, a) \), with \( a(t) \) the scale factor (we impose the natural units \( c = \hbar = k_B = 1 \), while \( \kappa = 1/M_p \) with \( M_p = 2.4 \times 10^{18} \) GeV the reduced Planck mass). Inserting into (4.5) we extract the Friedmann equations as

\[
H^2 = \frac{k^2}{3} \rho - \frac{f}{6} + \frac{T f_T}{3},
\]

\[
H = -\frac{k^2 (\rho + p)}{2(1 + f_T + 2T f_{TT})},
\]

\[
L_{\text{warp}} = \int \sqrt{\gamma} \left[ \frac{1}{2\kappa^2} f(T) + L_m \right] d^4x,
\]

\[
L = \int \sqrt{\gamma} \left[ \frac{1}{2\kappa^2} f(T) + L_m \right] d^4x,
\]

\[
L_{\text{matter}} = \int \sqrt{\gamma} \left[ \frac{1}{2\kappa^2} f(T) + L_m \right] d^4x.
\]
where we have also made use of the useful relation that provides the torsion scalar in FLRW geometry, namely
\[
T(t) = -6 \left( \frac{\dot{a}}{a} \right)^2 = -6H^2. \quad (4.11)
\]
Additionally, assuming a barotropic equation of state for the matter fields of the form (2.2), namely \( p = (\gamma - 1)\rho \), the matter conservation equation reads
\[
\dot{\rho} + 3\gamma H \rho = 0. \quad (4.12)
\]
Note that the above Friedmann equations for \( f(T) = T - 2\Lambda \) (with \( \Lambda > 0 \)) coincide with \( \Lambda \)CDM cosmology.

As it was discussed in the Introduction, one of the advantages of \( f(T) \) gravity is that the Friedmann equation (4.9) gives \( \rho \equiv \rho(H) \). Consequently, the differential equation (4.10) represents a one-dimensional autonomous system of the form \( \dot{H} \equiv \mathcal{F}(H) \) as long as the equation of state \( p \equiv p(\rho) \) is assumed. Hence, we can always interpret it as a vector field on a line, applying one of the basic techniques of dynamics by drawing \( \dot{H} \) versus \( H \), which helps to analyze the cosmic model in a clear and transparent way even without solving the system. In order to fix our notation we follow [97] calling the above equation the phase portrait, while its solution \( H(t) \) is the phase trajectory. Thus, the phase portrait corresponds to any theory which can be drawn in an \( (\dot{H} - H) \) phase-space. In this space each point is a phase point and could serve as an initial condition.

Let us now extract the phase portrait of \( f(T) \) gravity. Inserting (4.11) into (4.9) and (4.10) we can express the total energy density and pressure for the matter fields as
\[
\rho = \frac{1}{2\kappa^2} \left[ f(H) - H f_H \right], \quad \text{(4.13)}
\]
\[
p = -\frac{1}{2\kappa^2} \left[ f(H) - H f_H - \frac{1}{3} \dot{H} f_{HH} \right] = \frac{1}{6\kappa^2} \dot{H} f_{HH} - \rho, \quad \text{(4.14)}
\]
where \( f_H := \frac{df}{dH} \) and \( f_{HH} := \frac{d^2f}{dH^2} \). Thus, using additionally the above linear equation of state we obtain the phase portrait equation for any \( f(T) \) theory as
\[
\dot{H} = 3\gamma \left[ \frac{f(H) - H f_H}{f_{HH}} \right] \equiv \mathcal{F}(H). \quad (4.15)
\]
This shows that the modified Friedmann equations of \( f(T) \) gravity represent a one-dimensional autonomous system [94, 96], and hence the theory in suitable for a phase portrait analysis. It is obvious that the phase portrait (4.15) reduces to the \( \Lambda \)CDM portrait by setting \( f(H) = -6H^2 - 2\Lambda \).

### 4.2 Pseudo-Bang scenario realization

In Section 2 and Section 3 we presented the Pseudo-Bang cosmological scenario, offering a kinematical picture based on the corresponding phase space structure. In this subsection we desire to reconstruct the \( f(H) \) (or equivalently \( f(T) \)) form which generates this Pseudo-Bang scenario phase portrait.
One could extract the corresponding phase portrait by inserting (2.8) into (4.15). However, in the Pseudo-Bang scenario the phase portrait is a double valued function, thence we expect two behaviors of $f(H)\pm$ for each Hubble value, one for the $H > 0$ branch which we denote by plus sign, and one for $H < 0$ branch which is labeled by negative sign. We mention that at a particular Hubble value $H$ the corresponding two values $f(H)_+$ and $f(H)_-$ characterize two different instants. Thus, in order to avoid the complexity of the double valued behaviour of the $f(H)$ function at this moment, we first evaluate it as a function of cosmic time, i.e. $f(t)$, which in this case is monotonic. Inserting the chain rule $f' = \dot{f}/\dot{T}$, $f'' = (\dot{T} \ddot{f} - \ddot{T} \dot{f})/\dot{T}^3$ in Eqs. (4.13) and (4.14), we rewrite the matter density and pressure as

$$\rho = \frac{1}{2\kappa^2} \left[ f(t) - \frac{H}{\dot{H}} f(t) \right],$$  
(4.16)

$$p = -\frac{1}{2\kappa^2} \left[ f(t) - \left( \frac{H}{\dot{H}} - \frac{H}{3H^2} \right) f(t) - \frac{1}{3H} \dot{f}(t) \right].$$  
(4.17)

Additionally, the conservation equation (4.12) can be integrated to give $\rho = \rho_0 e^{-3Ht}/\dot{T}$, where $\rho_0 \equiv \rho(t_0)$ is a constant. Recalling Eqs. (2.5) and (2.8) and combining with (4.16), the $f(t)$ function is finally obtained as

$$f(t) = \left( \beta + \frac{2a\tilde{t}}{2 + 3\alpha \gamma \tilde{t}^2} \right) \left\{ f_0 + 8\alpha \kappa^2 \rho_0 \int \frac{a_k^{-3\gamma} (\alpha \gamma \tilde{t}^2 - 2)e^{-3\beta \tilde{t}}}{(\alpha \gamma \tilde{t}^2 + 2)(\beta \gamma \tilde{t} + 2a\tilde{t} + 2\beta \tilde{t}^2)} d\tilde{t} \right\},$$  
(4.18)

where $f_0$ is an integration constant and $\tilde{t} = t - t_i$. One observes that the term associated with the constant $f_0$ is $H \propto \sqrt{-T}$, which acts as a divergence term and hence it has no contribution to the field equations. Therefore, we omit the $f_0$-term without losing the generality of the solution.

In the simple $\beta = 0$ case, which corresponds to the standard bounce cosmology of (2.1), the above integral can be easily solved and it produces the $f(T)$ function obtained in [84]. In the general case of $\beta \neq 0$, namely in the Pseudo-Bang cosmology of (2.4), the above integral is found to be

$$f(t) = \frac{18\kappa^2 \rho_0 \left[ \mathcal{A}e^{\theta} \text{Ei}(1, \mathcal{B}^+) - \mathcal{A}e^{-\theta} \text{Ei}(1, \mathcal{B}^-) + \frac{9}{2} \alpha \theta \psi e^{-3\beta \tilde{t}} \right]}{\alpha \theta \psi_k^{-\frac{3\gamma}{2}} (2 + 3\alpha \gamma \tilde{t}^2)},$$  
(4.19)

where $\theta = \beta \sqrt{-\gamma/\alpha}$. In the above expression $\text{Ei}(1, x)$ is the exponential integral, related to the Gamma function through $\text{Ei}(1, x) = \Gamma(0, x)$, and

$$\mathcal{A} \equiv \mathcal{A}(\tilde{t}) = \gamma \beta \left( \frac{2}{3} \beta + \frac{2}{3} a \tilde{t} + \alpha \gamma \beta \tilde{t}^2 \right),$$

$$\mathcal{B}^\pm \equiv \mathcal{B}^\pm(\tilde{t}) = 3\gamma \beta \tilde{t} \pm \theta.$$  

We mention that the above $f(t)$ function does not exhibit divergences since $\alpha$ and $\gamma$ are positive, and hence it is real and finite at all times $-\infty < t < +\infty$.

Using relation (2.7) which provides the cosmic time as a function of the Hubble parameter, as well as (4.11), one can transform $f(t)$ to $f(H)$ and then to $f(T)$. In particular, we first find

$$\tilde{t}_\pm(T) = \frac{\alpha \pm \sqrt{\alpha^2 - 6\alpha \gamma \beta^2 + 2\alpha \beta \gamma \sqrt{-6T} + \alpha \gamma T}}{3\gamma \alpha \left( \frac{1}{6} \sqrt{-6T} - \beta \right)},$$

$$
- 19 -$$
The reconstructed $f(T)$ gravity form that leads to the realization of the Pseudo-Bang cosmological scenario. It is a double-valued function, where $f(T)_+$ gives the evolution in the positive $\dot{H}$ branch, while $f(T)_-$ gives the evolution in the negative $\dot{H}$ branch. Note that the $f(T)_-$ branch practically coincides with $\Lambda$CDM paradigm (green dashed line), except that the latter has no $|T_{\text{max}}|$. Additionally, $f(T)_\pm$ function is finite at all times and at late times ($T \sim -6H_0^2 \sim 1.7 \times 10^{-35} \text{ s}^2$) $f(T)_-$ tends to a constant. The model parameters have been set as in (2.11).

Figure 6. The reconstructed $f(T)$ gravity form that leads to the realization of the Pseudo-Bang cosmological scenario. It is a double-valued function, where $f(T)_+$ gives the evolution in the positive $\dot{H}$ branch, while $f(T)_-$ gives the evolution in the negative $\dot{H}$ branch. Note that the $f(T)_-$ branch practically coincides with $\Lambda$CDM paradigm (green dashed line), except that the latter has no $|T_{\text{max}}|$. Additionally, $f(T)_\pm$ function is finite at all times and at late times ($T \sim -6H_0^2 \sim 1.7 \times 10^{-35} \text{ s}^2$) $f(T)_-$ tends to a constant. The model parameters have been set as in (2.11).

and then

$$f(T)_\pm = \frac{18\kappa^2 \rho_0 \left( A(i_\pm) e^{\theta_l} \text{Ei}(1, B^+(i_\pm)) - A(i_\pm) e^{-\theta_l} \text{Ei}(1, B^-(i_\pm)) + \frac{3}{2} \alpha \theta e^{-3\gamma \beta i_\pm} \right)}{\alpha \theta e^{3\gamma j i_\pm} \left( 2 + 3\gamma \alpha i_\pm^2 \right)},$$

which, as mentioned above, is a double-valued function. This is one of the main results of the present work, namely the $f(T)$ form that generates the scale factor (2.4), which in turn corresponds to the Pseudo-Bang cosmological scenario discussed in Section 3. We mention here that the above form contains the barotropic index of the total cosmic fluid $\gamma$ as a parameter. This is determined by the dominant component at each energy scale ($\gamma = 4/3$ for radiation epochs while $\gamma = 1$ for matter epoch). For a more realistic and complete expression one could impose one of the usual unified parameterizations for the total matter barotropic index that evolves smoothly from $\gamma = 4/3$ to $\gamma = 1$ during the cosmological evolution [111]. For completeness, in Fig. 6 we depict this $f(T)$ function versus $T$. As we can see it remains finite, in consistency with the non-singular universe evolution provided by the Pseudo-Bang scenario, while at late times it practically becomes constant and the scenario resembles general relativity with a cosmological constant. Finally, let us point out that our calculations exhibit a continuous behavior, hence “small” deviations from the above $f(T)$ form will only lead to “small” deviations from the resulting scale-factor evolution.
Before closing this Section we make some comments based on the equation-of-state behavior. This can be used to investigate the various phantom crossings and the quintom realization in the present scenario, as well as to verify its final phase as a Pseudo-Rip one.

Observing the Friedmann equations (4.9), (4.10) we can define an effective sector of torsional gravitational origin, characterized by an energy density and pressure of the form

$$\rho_T = \frac{1}{2\kappa^2} \left[ H f_H - f(H) + 6H^2 \right],$$

$$p_T = \frac{1}{6\kappa^2} \dot{H} (12 + f_{HH}) - \rho_T = \frac{1}{6\kappa^2} \left( 12\dot{H} - \frac{\ddot{H}}{H^2} f + \frac{\dddot{f}}{H^2} \right) - \rho_T.$$  (4.21)

The matter conservation implies conservation of the above torsional sector too, namely

$$\dot{\rho}_T + 3H (1 + w_T) \rho_T = 0,$$  (4.23)

where we have defined the corresponding equation-of-state parameter as

$$w_T \equiv \frac{p_T}{\rho_T} = -1 + \frac{1}{3} \frac{H (\dot{f} + 12\dot{H}^2) - H \dddot{f}}{H (f(t) - 6H^2) - H \dddot{f}}.$$  (4.24)

Finally, it proves convenient to introduce the total, or effective, energy density and pressure through

$$\rho_{eff} \equiv \rho + \rho_T$$  (4.25)

$$p_{eff} \equiv p + p_T,$$  (4.26)

and thus the total equation-of-state parameter of the universe becomes

$$w_{eff} \equiv \frac{p + p_T}{\rho + \rho_T} = -1 - \frac{2}{3} \frac{H}{H^2}.$$  (4.27)

A first observation is that at the initial Pseudo-Bang phase we have \( \lim_{t \to -\infty} w_T = \gamma - 1 \), while at the final phase we obtain \( \lim_{t \to \infty} w_T = -1 \). Proceeding forward we will examine the points at which the torsional equation of state diverges (despite the fact that for the scenario at hand, \( H, \dot{H} \) and \( \ddot{H} \) are always finite). This type of singularity is similar to Type II (sudden) singularity [112]. From (4.24) we deduce that \( w_T \to \pm \infty \) at the points at which \( \dot{H} = 0 \) or \( \frac{\dddot{f}}{H^2} = \frac{\dddot{f}}{H^2} - 6H \). According to the first condition, the torsional fluid is singular at the fixed points \( E \) and \( G \). We exclude the special case of the fixed point \( A \), since it is not reachable at a finite time. Consequently, \( w_T \) diverges at \( t_E = 2t_i = -2\sqrt{2/(3\gamma \alpha)} \) and at \( t_G = 0 \). Using (2.5), (2.6) and (4.18), the second condition above can be solved numerically to identify one further singular point of \( w_T \) at \( t_s \approx 1.2 \times 10^{17} \) s, having assumed that matter is dominated by dust \( \gamma = 1 \) at this epoch (other choices will not qualitatively alter the behaviour but just delay the singularity to later times). Hence, at \( t_i \) the torsional effective fluid transits from the quintessence \((-1 < w_T)\) to the phantom regime \((w_T < -1)\). At a later stage, namely around \( t \approx 2.2 \times 10^{17} \) s, it evolves from phantom into quintessence regime, and then evolves asymptotically towards the cosmological constant value, namely \( w_T \to -1 \) as \( t \to \infty \), which is the Pseudo-Rip fate. Hence, in the scenario at hand we have the quintom realization. Note that if \( w_T \) had remained in the
phantom regime and was approaching from there the cosmological constant boundary asymptotically, then the universe would result to a Little-Rip instead of the Pseudo-Rip phase.

Concerning the total (effective) equation of state we can easily see from (4.27) that \( \lim_{t \to \pm \infty} w_{\text{eff}} = -1 \). Additionally, since \( \dot{H} \) is always finite it becomes singular at the points where \( H = 0 \), namely at the turnaround point \( C \) and the bounce point \( F \). On the other hand, at the de Sitter points \( E \) and \( G \), \( w_{\text{eff}} \) crosses the phantom divide smoothly.

In order to provide a more transparent picture of the behavior of \( w_T \) and \( w_{\text{eff}} \), we depict them in Fig. 7 for the whole universe evolution, focusing additionally on the interval around 0, and keeping the notation of Fig. 2(b). At the Pseudo-Bang limit, the effective EoS becomes asymptotically a cosmological constant, i.e. \( w_{\text{eff}} \to -1 \). In the interval (i) we have \(-1 < w_{\text{eff}} < -1/3\), which represents an inflationary era. In interval (ii) we have \( w_{\text{eff}} > -1/3 \), and the universe expands with deceleration. However, the effective fluid \( w_{\text{eff}} \to +\infty \) at the turnaround point \( C \). In the interval (iii), \( w_{\text{eff}} \to -1/3 \), and the universe contracts with deceleration. In the interval (iv), \(-1 < w_{\text{eff}} < -1/3\), which represents an accelerated contraction era. Moreover, at point \( E \), the fixed point, we acquire the realization of the phantom-divide crossing from non-phantom to phantom regime. In the interval (v), at point \( F \), the universe experiences the bounce, where \( w_{\text{eff}} \to -\infty \). After the bounce, in the interval (vi), the effective fluid evolves as \( w_{\text{eff}} \to -1 \) in order to match the observable universe, where the phantom-divide crossing occurs at point \( G \). In the interval (vii) the effective fluid turns to quintessence regime \(-1 < w_{\text{eff}} < -1/3\), while in the interval (viii) the universe gracefully exits into the decelerated

**Figure 7.** The evolution of the torsional equation-of-state parameter \( w_T \) given in (4.24), and of the total, or effective, equation-of-state parameter \( w_{\text{eff}} \) given in (4.27), following the key points of the Pseudo-Bang scenario. Additionally, we draw the quintessence boundary (at \( w = -1/3 \)) and the phantom divide (at \( w = -1 \)) in order to facilitate an easy comparison with the phase portrait 2(b). The model parameters have been set as in (2.11).
era, where $-1/3 < w_{\text{eff}} < w$. In the interval (ix) the effective fluid returns back to the quintessence regime, where $-1 < w_{\text{eff}} < -1/3$. Finally, at the Pseudo-Rip limit $w_{\text{eff}}$ acts asymptotically as a cosmological constant, similarly to its initial phase.

Finally, let us make some comments on the identification of the last phase in the universe evolution as Pseudo-Rip, using the energy conditions and the inertial force interpretation, namely applying the same analysis that was used in [107] where the term Pseudo-Rip was first introduced. At the Pseudo-Bang limit, $t \to -\infty$, we have

\begin{equation}
\rho \to \infty, \quad p \to \infty, \quad 1 \leq \gamma \leq 2.
\end{equation}
\begin{equation}
\rho_T \to -\infty, \quad p_T \to -\infty, \quad w_T \to \gamma - 1.
\end{equation}
\begin{equation}
\rho_{\text{eff}} \to 3\beta^2/\kappa^2, \quad p_{\text{eff}} \to -3\beta^2/\kappa^2, \quad w_{\text{eff}} \to -1.
\end{equation}

Similarly, at the Pseudo-Rip limit, $t \to \infty$, we find

\begin{equation}
\rho \to 0, \quad p \to 0, \quad 1 \leq \gamma \leq 2.
\end{equation}
\begin{equation}
\rho_T \to 3\beta^2/\kappa^2, \quad p_T \to -3\beta^2/\kappa^2, \quad w_T \to -1.
\end{equation}
\begin{equation}
\rho_{\text{eff}} \to 3\beta^2/\kappa^2, \quad p_{\text{eff}} \to -3\beta^2/\kappa^2, \quad w_{\text{eff}} \to -1.
\end{equation}

Hence, the total energy density at asymptotically early and late times is non-zero, which is consistent with the fact that $H$ is constant at these phases. Although the energy scales at Pseudo-Bang and at Pseudo-Rip are almost equivalent, the scale factor and matter sector take two different extremes. As $H \to \beta^-$, at Pseudo-Bang, the scale factor $a \to 0$, the matter density $\rho \propto a^{-3}$, i.e. $\rho \to \infty$, the matter pressure $p \to \infty$ and the temperature $\Theta \to \infty$ for $\gamma > 1$. As $H \to \beta^+$, at Pseudo-Rip, the scale factor $a \to \infty$, the matter density $\rho \to 0$, the matter pressure $p \to 0$ and the temperature $\Theta \to 0$. Hence, these two phases are not similar, as one might think by comparing only their energy scales.

Concerning the inertial (ripping) force, for any two points separated by a comoving distance $l$, the relative acceleration between them is $l\ddot{a}/a$. Thus, the inertial force on a mass $m$ as seen by an observer at a comoving distance $l$ is given by [107, 113]

\begin{equation}
F_{\text{inert}} = ml\ddot{a}/a = ml(H + H^2).
\end{equation}

Using the phase portrait (2.8), in Fig. 8(a) we draw the inertial force as a function of the Hubble parameter. As we observe, the inertial force is very sensitive to the Hubble parameter changes at the phantom crossing points, since $dF_{\text{inert}}/dH \to \infty$ at these fixed points. Furthermore, in Fig. 8(b) we depict the inertial force time evolution, from which we can see that it will peak at a future time $\sim 5.3 \times 10^{17}$ s and then it will decrease asymptotically to a fixed limit $F_{\text{inert}} \to ml\beta^2$. Such a behavior, is the definition of a Pseudo-Rip fate [107], which distinguishes it from other Rips (in fact it is an intermediate between the no Rip and the Little Rip) and shows that the bound structures dissociate if they are at or below a particular threshold. Note that this behavior of $F_{\text{inert}}$ is closely related to the behavior of $w_T$, that is why we have added it in the same Figure for completeness.
5 Summary and Remarks

In the present work we studied the complete universe evolution in the framework of $f(T)$ cosmology. As a first step we investigated the necessary requirements at the kinematic level in order to describe the standard observed thermal history of the universe, namely the sequence of radiation, matter and late-time acceleration epochs, as well as being able to bypass the initial singularity. In particular, we introduced an exponential correction to the standard bouncing scale factor.

In order to investigate the cosmological behavior of the introduced scale factor, we performed a detailed analysis of the $(H, \dot{H})$ phase portrait. Firstly, we showed that the Minkowskian origin of the standard bounce universe is instead shifted to a de Setterian origin in the present scenario. This allows the universe to begin with an accelerated expansion phase, since in the infinite past $t \to -\infty$ the scale factor goes to 0, the Hubble parameter goes to a constant, and its derivative to 0. Since these features resemble those of the Pseudo-Rip fate [107] but in a revered way, we called the initial phase as Pseudo-Bang and the whole cosmological scenario as Pseudo-Bang scenario.

After the Pseudo-Bang, the universe evolves in a first inflationary phase, a cosmological turnaround and then experiences a bounce, after which we have a second inflationary regime with a successful exit. Then the universe follows the standard thermal history of the sequence of radiation, matter and late-time acceleration epochs. Finally, the last phase as $t \to \infty$ is an everlasting Pseudo-Rip one, where the scale factor goes to infinity, the Hubble parameter goes to a constant, and its derivative to 0. Interestingly enough, the origin and the fate of the universe are characterized by the same energy
scale, $E \sim \sqrt{M_p H} \sim 10^{-12}$ GeV. Additionally, after the turnaround and after the bounce the universe exhibits the crossing of the phantom divide twice through a particular type of fixed points at which the slope $dH/dH$ becomes infinite making them reachable in a finite time. These have almost the same energy scale, $E \sim 4 \times 10^{12}$ GeV, with a time separation $\sim 10^{-32}$ s, in a phantom regime. It is this second phantom crossing that provides the second inflationary realization. We examined the evolution of the primordial fluctuations versus the Hubble radius, and we showed that all fluctuations are initially sub-horizon, which is consistent with the observations of acoustic peaks in the power spectrum of CMB anisotropies. This allows to naturally assume Bunch-Davies vacuum conditions of the quantum fluctuations around the initial vacuum state. Interestingly, the Pseudo-Bang phase resembles de Sitter case up to a sub-leading term, whereas the hypersurface is not space-like but null. In this sense, the Pseudo-Bang phase is not a genuine beginning and it can be preceded by a contraction phase at $H = -\beta$.

Having investigated the kinematic requirements for the Pseudo-Bang cosmological scenario we examined its dynamical realization in the framework of $f(T)$ gravity. We took advantage of the fact that the field equations of $f(T)$ gravity are of second order, and therefore the corresponding autonomous dynamical system is one dimensional. This results to a phase space in the $(H, \dot{H})$ plane that can incorporate the aforementioned kinematic features of the Pseudo-Bang scale factor. Hence, we reconstructed the specific $f(T)$ form that can lead to the Pseudo-Bang cosmological scenario. Lastly, by studying the torsional and the total equation-of-state parameters we saw that the total, effective fluid does not exhibit any singular behaviour at the phantom crossing points, while the torsional fluid feels them as Type II singular phases.

In conclusion, we provided an $f(T)$ form that can generate the complete universe evolution from Pseudo-Bang to Pseudo-Rip, including the standard thermal history. It would be interesting to perform a full observational confrontation using data from Supernovae type Ia (SNIa), Baryonic Acoustic Oscillations (BAO), Cosmic Microwave Background (CMB) shift parameter, and Hubble parameter measurements, in order to examine whether the Pseudo-Bang scenario at late times is in agreement with observations. Moreover, we should perform a detailed perturbation analysis (allowing also for possible “soft” features [114]) in order to confront it with CMB temperature and polarization data, as well as with $f\sigma_8$ and growth data. Such investigations, although necessary, lie beyond the scope of the present work and are left for future projects.

Acknowledgments

We would like to thank the referees for valuable suggestions which improved the paper. W.E. gratefully acknowledges the technical assistance of M. Hashim to create the graphs of Fig. 7, and wishes to thank Adel Awad for helpful discussions about Penrose-Carter diagram. E.N.S work is supported in part by the USTC Fellowship for international professors.

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