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ABSTRACT

Isostable reduction is a powerful technique that can be used to characterize behaviors of nonlinear dynamical systems using a basis of slowly decaying eigenfunctions of the Koopman operator. When the underlying dynamical equations are known, previously developed numerical techniques allow for high-order accuracy computation of isostable reduced models. However, in situations where the dynamical equations are unknown, few general techniques are available that provide reliable estimates of the isostable reduced equations, especially in applications where large magnitude inputs are considered. In this work, a purely data-driven inference strategy yielding high-accuracy isostable reduced models is developed for dynamical systems with a fixed point attractor. By analyzing steady-state outputs of nonlinear systems in response to sinusoidal forcing, both isostable response functions and isostable-to-output relationships can be estimated to arbitrary accuracy in an expansion performed in the isostable coordinates. Detailed examples are considered for a population of synaptically coupled neurons and for the one-dimensional Burgers’ equation. While linear estimates of the isostable response functions are sufficient to characterize the dynamical behavior when small magnitude inputs are considered, the high-accuracy reduced order model inference strategy proposed here is essential when considering large magnitude inputs.

I. INTRODUCTION

With an ever-growing abundance of data, there is an increasing demand for data-driven strategies that can be used to both predict and control the behaviors of dynamical systems. While a variety of projection based techniques are well-suited for dynamical systems that can be accurately approximated by local linearization, Koopman analysis has emerged as an essential technique in the study of dynamical systems with fundamentally nonlinear behaviors. The key feature of Koopman analysis is that it allows the observables of a nonlinear dynamical system to be studied in terms of the evolution of eigenfunctions that propagate in accordance with their associated eigenvalues. As such, complicated nonlinear behaviors can be understood in terms of the evolution of eigenfunctions that propagate in accordance with their associated eigenvalues. Using the Koopman operator framework, it is generally possible to make predictions about the evolution of observables by considering their behavior on a low-dimensional manifold comprised of a finite number of Koopman eigenmodes.
Koopman eigenmodes can often be computed numerically if the underlying dynamical equations are known. However, when the underlying dynamical equations are unknown, the Koopman operator must be approximated using the temporal evolution of system observables (i.e., data). Dynamical mode decomposition (DMD) is one such data-driven method\cite{Willis2015} that can be used to identify eigenmodes from a snapshot sequence of data. Extended DMD,\cite{Lu2017} deep learning approaches,\cite{Wang2018} and delay-embedding techniques have also been useful in certain applications. A commonality among these data-driven Koopman analysis strategies is that they attempt to identify a low-dimensional representation of the infinite-dimensional Koopman operator.

Because the Koopman framework often results in a low-dimensional dynamical representation for the evolution of system observables, there has been a growing interest in applying this strategy to control problems. Time-varying inputs make the system behavior non-autonomous, which adds an additional layer of complexity to applications involving control. For such systems, the associated Koopman eigenmodes and eigenvalues are time-dependent. With this in mind, a number of Koopman-based control strategies have been developed that augment the system measurements with knowledge of the applied control in order to characterize the dynamical behavior in response to inputs.\cite{Liu2015,Recht2015} Related strategies have been successful that consider a finite number of constant controls, compute the associated Koopman eigenfunctions, and switch between control states as necessary.\cite{Korda2016} An alternative strategy is to identify Koopman eigenfunctions of the autonomous, unforced system and work directly in the resulting eigenfunction basis.\cite{McInnes2018} This is the approach taken by the isostable reduction framework\cite{McInnes2018,Korda2018} that uses a basis of the most slowly decaying eigenfunctions associated with the Koopman operator. In many applications, an accurate reduced order model can be obtained using a small number of isostable coordinates,\cite{McInnes2018,Korda2018} making control and analysis possible in a substantially reduced order setting.

The characterization of isostable coordinate dynamics in response to an applied input is a fundamental challenge when working in an isostable reduced coordinate basis. This can be done, for instance, by approximating the gradient of the isostable coordinate field with respect to the applied inputs. Previous work has focused on identifying linear approximations of the isostable response functions with respect to a fixed point attractor\cite{Seidman1980} or along specific trajectories.\cite{Korda2018} These strategies work in the limit that the applied inputs are small in magnitude but suffer when larger magnitude inputs are considered. This limitation can be addressed by considering high-order expansions of the isostable response functions in a basis of isostable coordinates. Recent work\cite{Korda2018} details such a method for approximating these isostable response functions in oscillatory dynamical systems to arbitrary orders of accuracy; however, this strategy requires knowledge of the underlying model equations. Currently, no existing strategies have been developed for identifying high-accuracy approximations of the isostable response functions using data-driven techniques.

In this work, a purely data-driven strategy is developed for identifying reduced order isostable-based models in dynamical systems with stable fixed point attractors. By analyzing steady-state model outputs in response to sinusoidal forcing, both the isostable response functions and the isostable-to-output relationships can be estimated to arbitrary orders of accuracy in an expansion performed in the basis of isostable coordinates. This proposed method can be considered an extension of the technique suggested in Ref.\cite{Seidman1980}, which considered only the first order accurate dynamics of the isostable reduced equations. The current work allows for estimates of higher order accuracy terms, which are critically important when considering large magnitude inputs. The organization of this paper is as follows: Section II gives the necessary background on the isostable reduced coordinate framework and also reframes the numerical isostable reduction techniques from Ref.\cite{Korda2018} for use in dynamical systems with stable fixed points. Section III details a data-driven strategy for inferring isostable-based reduced order models using information about steady-state model outputs in response to sinusoidal forcing. Section IV provides examples that include systems with relevance to both neuroscientific and fluid flow applications. Section V gives concluding remarks.

II. BACKGROUND

A. The isostable coordinate system

Consider an ordinary differential equation

\[ \dot{x} = F(x) + U(t), \]

where \( x \in \mathbb{R}^N \) denotes the state, \( F \) represents the nominal dynamics, and \( U(t) \) is an external input. Let \( \phi(t, x) \) denote the unperturbed flow of \( (1) \), and suppose that \( (1) \) has a stable fixed point denoted by \( x_0 \). Near the fixed point, local linearization yields

\[ \Delta x = J\Delta x + O((\Delta x)^2), \]

where \( \Delta x \equiv x - x_0 \) and \( J \) is the Jacobian of \( F \) evaluated at \( x_0 \). While \( (2) \) is only accurate in a close vicinity of the fixed point, it nonetheless can be used in conjunction with the isostable coordinate framework to characterize the infinite-time decay of solutions in the fully nonlinear basin of attraction of \( x_0 \). To do so, let \( w_k \), \( \nu_k \), and \( \lambda_k \) be left eigenvectors, right eigenvectors, and eigenvalues of \( J \), respectively, ordered so that \( |\text{Re}(\lambda_k)| \leq |\text{Re}(\lambda_{k+1})| \). Isostable coordinates associated with a subset of eigenvalues with the smallest magnitude real components can be defined explicitly according to

\[ \psi_j(x) = \lim_{t \to \infty} \left( w_j^T (\phi(t, x) - x_0) \exp(-\lambda_j t) \right), \]

where \( ^T \) indicates the matrix transpose. Intuitively, in Eq. \( (3) \), the term \( w_j^T (\phi(t, x) - x_0) \) selects for the component of the decay in the \( y_j \) direction—multiplying by the exponentially growing term and taking the limit as time approaches infinity yields the isostable coordinate. The definition \( (3) \) is closely aligned with the one given in Ref.\cite{Seidman1980}; however, other definitions that compute Fourier averages of observables\cite{Korda2018} can also be used. As a point of emphasis, the constructive definition of isostable coordinates \( (3) \) is only possible for a subset of eigenvalues with the smallest magnitude real components.\cite{Korda2018} Isostable coordinates associated with larger magnitude eigenvalues must be defined implicitly as level sets of Koopman eigenfunctions with decay rates that are governed by their associated \( \lambda_j \).

The utility of the transformation to isostable coordinates relies on the fact that many dynamical systems have behavior that is governed by their slowest decaying modes of the Koopman operator.
By truncating the rapidly decaying components and retaining only a small number of the slowly decaying isostables, a reduced order model can be obtained. Such isostable coordinate systems have been successfully applied in a variety of control applications.\textsuperscript{33,30–34}

B. Reduced order modeling using isostable coordinates

When \( U(t) = 0 \), under the evolution of \( \dot{x} = F(x) \), all isostable coordinates decay exponentially according to \( \dot{\psi}_k = \lambda_k \psi_j \) in the entire nonlinear basin of attraction of the fixed point. In the analysis that follows, the dynamics of the \( M \) slowest decaying isostable coordinates will be explicitly considered, and the rest will be truncated (i.e., by taking all truncated isostable coordinates to be equal to zero at all times). The reduced order dynamics of Eq. (1) on a hypersurface determined by the non-truncated isostable coordinates then follows

\[
\dot{\psi}_k = \lambda_k \psi_k + I_k(\psi_1, \ldots, \psi_M) \cdot U(t), \quad k = 1, \ldots, M,
\]

\[
x = x_0 + G(\psi_1, \ldots, \psi_M),
\]

where \( I_k \) denotes the gradient of \( \dot{\psi}_k \), \( G \) maps the isostable coordinates to the state, and the dot denotes the dot product. For low-dimensional systems (with \( N \leq 3 \), it is sometimes possible to compute the isostable coordinates directly for all states \( x \) and infer the functions \( I_k \) and \( G \) numerically.\textsuperscript{\textsuperscript{35}} This approach, however, does not scale well for higher-dimensional systems. Alternatively, it is usually more feasible to Taylor expand both \( I_k \) and \( G \) in powers of the isostable coordinates as

\[
G(\psi_1, \ldots, \psi_M) \approx \sum_{k=1}^{M} \left[ \psi_k g_1^k \right] + \sum_{j=1}^{M} \sum_{k=1}^{M} \left[ \psi_j \psi_k g_{jk}^k \right]
\]

\[
+ \sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{k=1}^{M} \left[ \psi_j \psi_k g_{ijk}^k \right] + \ldots.
\]

(5)

\[
I_k(\psi_1, \ldots, \psi_M) \approx \sum_{k=1}^{M} \left[ \psi_k I_1^k \right] + \sum_{j=1}^{M} \sum_{k=1}^{M} \left[ \psi_j \psi_k I_{jk}^k \right]
\]

\[
+ \sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{k=1}^{M} \left[ \psi_j \psi_k I_{ijk}^k \right] + \ldots.
\]

(6)

For the purposes of this work, Eq. (4) will be considered a \( j \)th order accurate model if the expansion of \( G \) from (5) and each \( I_k \) from (6) are computed to \( j \)th and \( j - 1 \)th order accuracy in the isostable coordinates, respectively. This convention generally assumes that \( U(t) \) and each \( \psi_k \) are \( O(\epsilon) \) terms where \( 0 < \epsilon \ll 1 \) so that the resulting reduction (4) retains terms up to and including \( O(\epsilon) \) accuracy. Note that these models often perform well even when the magnitude of input is larger than \( O(\epsilon) \) (see, for example, Ref. 28).

Using methods adapted from Ref. 28 that are given in Appendix A, it is possible to directly solve for each \( g_{jk}^k \) and \( I_{ijk}^k \) when the underlying model equations are known. As a concrete example, a simple dynamical system is considered,

\[
\begin{align*}
\dot{x}_1 &= \mu x_1 + u_{10}(t), \\
\dot{x}_2 &= \lambda (-x_1 + x_2 + x_1^2 + x_2^2),
\end{align*}
\]

with \( \lambda = -1 \) and \( \mu = -0.05 \). The model (7) has a finite-dimensional Koopman invariant subspace\textsuperscript{\textsuperscript{\textsuperscript{36}}} making it an ideal testbed for developing nonlinear model identification strategies. Equation (7) has a stable fixed point at \( x_1 = x_2 = 0 \) with associated eigenvalues \( \lambda_1 = -0.05 \) and \( \lambda_2 = -1 \). Trajectories tend to approach along the \( x_2 \) nullcline as shown in Fig. 1. Due to the separation between the eigenvalues, a single isostable reduced model can be determined by identifying the unknown \( g_{jk}^k \) and \( I_{ijk}^k \) terms from Eqs. (5) and (6), respectively, using methods detailed in Appendix A.

Level sets of the isostable coordinates and the gradient of the isostable coordinates are shown in panels B and C of Fig. 1. The resulting reduced dimension model of the form (4) is simulated in response to the input shown in panel E. Corresponding outputs are given in panel F. For the model (7), third order accuracy is sufficient to match the full model behavior.

The procedure described in Fig. 1 explicitly requires knowledge of the underlying model equations. In practical applications, underlying model equations are generally not known \textit{a priori}. Alternatively, the temporal evolution of model observables is typically all that can be measured. With this in mind, Secs. III and IV will illustrate a purely data-driven nonlinear model identification strategy.

III. NONLINEAR MODEL IDENTIFICATION OF REDUCED ORDER MODELS FROM INPUT-OUTPUT RELATIONSHIPS

A. Overview of the proposed approach

In the analysis to follow, input–output relationships will be used to identify the necessary terms of an isostable reduced model based on (4). To proceed, it will be assumed that \( U(t) = Au(t) \), with \( A = [A_1, \ldots, A_N]^T \in \mathbb{R}^N \) and \( u(t) \in \mathbb{R}^n \); in other words, \( U(t) \) is explicitly assumed to be a rank-1 input. Using this assumed structure on \( U(t) \), the isostable coordinate reduction from (4) will be written as

\[
\dot{\psi}_k = \lambda_k \psi_j + I_A(\psi_1, \ldots, \psi_M)u(t),
\]

\[
k = 1, \ldots, M,
\]

\[
y = y_0 + G_f(\psi_1, \ldots, \psi_M).
\]

(9)

Above, \( y \in \mathbb{R}^N \) is a collection of observables and \( G_f \) maps the isostable coordinates to the system observables. As with (4), \( G_f \) will be Taylor expanded in the basis of isostable coordinates,

\[
G_f(\psi_1, \ldots, \psi_M) \approx \sum_{k=1}^{M} \left[ \psi_k g_1^k \right] + \sum_{j=1}^{M} \sum_{k=1}^{M} \left[ \psi_j \psi_k g_{jk}^k \right]
\]

\[
+ \sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{k=1}^{M} \left[ \psi_j \psi_k g_{ijk}^k \right] + \ldots.
\]

(10)
FIG. 1. The isostable-based model order reduction strategy is applied to the simple model (7) with nominal dynamics represented in panel A. If the full model equations are known, it is straightforward to compute the necessary terms of the reduced order models from Eqs. (5) and (6) that only include the slowest decaying isostable dynamics; these terms are shown in panels B–D for a single isostable coordinate. Once the terms of the reduced order model are identified, it can be used to predict the response to inputs, provided the terms of $G$ and $I_1$ are computed to sufficiently high-order accuracy in the isostable coordinate. In response to the example input in panel E, resulting outputs are shown in panel F for reduced order models computed to various orders of accuracy.
Similarly, the isostable response to inputs from (8) will be written in terms of the Taylor expansion,

\[
I_{nA}(\psi_1, \ldots, \psi_M) \approx P_{nA}^0 + \sum_{k=1}^{M} \left[ \psi_k P_{nA}^k \right] + \frac{\varepsilon}{1!} \left[ \psi_0 P_{nA}^0 \right] + \sum_{j=1}^{M} \left[ \psi_j \psi_0 P_{nA}^j \right] + \cdots, \tag{11}
\]

where, for instance, \( P_{nA}^0 = I_{nA} \cdot A \). In order to aid in the following analysis, each \( \psi_n \) from Eq. (8) will be asymptotically expanded in orders of \( \varepsilon \) as

\[
\psi_n(t) = \psi_n^{(0)}(t) + \varepsilon \psi_n^{(1)}(t) + \varepsilon^2 \psi_n^{(2)}(t) + \cdots, \tag{12}
\]

where \( 0 < \varepsilon \ll 1 \). In the proposed model identification strategy, applied inputs \( u(t) \) are assumed to be order \( \varepsilon \) terms. With this in mind, substituting (12) into (8) and collecting the \( O(1) \) terms yields \( \psi_n^{(0)} = \lambda_n \psi_n^{(0)} \); consequently, in the limit that time approaches infinity, \( \psi_n^{(0)} \) tends to zero. As such, for simplicity of exposition, it will be assumed that each \( \psi_n^{(0)} \) is zero for all the time.

Figure 2 illustrates the fundamental principles behind the proposed data-driven model inference strategy. In response to a sinusoidal input \( u(t) = \varepsilon \sin(\omega t) \), Eq. (8) can be used to analytically solve for the steady-state response \( y_{1,ss}(t), \ldots, y_{M,ss}(t) \) to arbitrary orders of accuracy in the expansion (12). These steady-state solutions can be used in conjunction with Eq. (9) to represent the steady-state output, \( y_{ss}(t) \), to arbitrary orders of accuracy in \( \varepsilon \). As shown in the analysis to follow, \( y_{ss}(t) \) can be decomposed into Fourier modes, with modes of frequency \( k\omega \) being \( O(\varepsilon^k) \) terms. Analytical relationships between the coefficients \( \tau_k(\omega) \) and \( \sigma_k(\omega) \) of each Fourier mode can be obtained in terms of the asymptotic expansions from Eqs. (10) and (11). This subsequently allows for the implementation of the model fitting procedure to infer the individual terms of these expansions to arbitrary orders of accuracy in the isostable coordinates. The resulting reduced order model is fully nonlinear and can be used to accurately characterize the system output in response to arbitrary inputs.

**B. Inferring first order accurate models for single output systems**

For the moment, it will be assumed that \( \zeta = 1 \), i.e., that the observable \( y \) is of dimension 1. Higher-dimensional outputs will be considered in Sec. III G. A similar model inference strategy valid for only first order accurate models was considered in Ref. 26. To begin, \( u(t) = \varepsilon \sin(\omega t) \) will be applied, and only the steady-state dynamics will be considered. Collecting the \( O(\varepsilon) \) terms after substituting (12) into (8) yields

\[
\psi_n^{(1)} = \lambda_n \psi_n^{(1)} + \varepsilon P_{nA}^0 \sin(\omega t). \tag{13}
\]

The steady state, the solution to (8), is

\[
\psi_n^{(1)} = \psi_n^{(1)} + \frac{P_{nA}^0}{\omega^2 + \lambda_n^2} (\lambda_n \sin(\omega t) + \omega \cos(\omega t))
= \lambda_n \psi_n^{(1)} + P_{nA}^0 \sin(\omega t) + \varepsilon c_{nA}^0(\omega) P_{nA}^0 \cos(\omega t), \tag{14}
\]

where \( \lambda_n \psi_n^{(1)} \in \mathbb{C} \) and \( c_{nA}^0(\omega) \in \mathbb{C} \) are defined appropriately. Considering (9), to leading order in \( \varepsilon \), the output is

\[
y_{ss}(w, t) = y_0 + \varepsilon \sum_{k=1}^{M} \left[ \lambda_k(\omega) g_k^0 \psi_k + \sigma_k(\omega) g_k^0 \psi_k \right] \sin(\omega t) + O(\varepsilon^2). \tag{15}
\]

Multiplying both sides of (15) by either \( \sin(\omega t) \) or \( \cos(\omega t) \), integrating over one period, and considering the steady-state outputs that result when using \( q \) different frequencies \( \omega_1, \ldots, \omega_q \), one can write the matrix equation

\[
\frac{1}{\varepsilon} \Gamma_1 = \Xi \Psi \Gamma_1 + O(\varepsilon), \tag{16}
\]

where \( \Xi \) contains the frequencies \( \omega_1, \ldots, \omega_q \), and \( \Psi \) contains the \( \lambda_n \psi_n^{(1)} \) terms. In the next section, this will be applied to nonlinear, isostable-based models of a chaotic system.
where

\[
\Gamma_t = \begin{bmatrix}
\omega_1 \int_{0}^{2\pi/\omega_1} y_u(w_1, t) \sin(\omega_1 t) dt \\
\omega_1 \int_{0}^{2\pi/\omega_1} y_u(w_1, t) \cos(\omega_1 t) dt \\
\vdots \\
\omega_q \int_{0}^{2\pi/\omega_q} y_u(w_q, t) \sin(\omega_q t) dt \\
\omega_q \int_{0}^{2\pi/\omega_q} y_u(w_q, t) \cos(\omega_q t) dt
\end{bmatrix} \in \mathbb{C}^{2q \times 1},
\]

\[
\Xi_t = \begin{bmatrix}
c_1^0(\omega_1) & \cdots & c_M^0(\omega_1) \\
c_1^1(\omega_1) & \cdots & c_M^1(\omega_1) \\
\vdots & & \vdots \\
c_1^0(\omega_q) & \cdots & c_M^0(\omega_q)
\end{bmatrix} \in \mathbb{C}^{q \times M},
\]

and \(Y_t = [g_t^1, \ldots, g_t^M]^T \in \mathbb{C}^{M \times 1}\). The individual terms of \(Y_t\) can then be estimated, for instance, by taking

\[
Y_t = \frac{1}{\epsilon^2} \Xi_t^\dagger \Gamma_t,
\]

where \(\dagger\) is the Moore–Penrose pseudoinverse. As explained in Ref. 36, isostable reduced models of the form (8) and (9) can always be scaled so that \(g_{t,0}^1\) can be chosen arbitrarily. For this reason, it will be assumed that \(g_{t,0}^1 = 1\) for all \(T\) so that the relationship (16) can be used to identify the unknown \(g_{t,1}^1, \ldots, g_{t,M}^1\) terms of a discussion of how the transition frequencies \(\omega_1, \ldots, \omega_q\) can be chosen is provided in Sec. III F.

C. Estimation of linearized eigenvalues

Knowledge of the eigenvalues of the Jacobian from (2) (i.e., the decay rates of isostable coordinates) is necessary to identify the terms of reduced order models. These can be inferred using a two step procedure that involves first obtaining a coarse estimate of the eigenvalues from noisy data and subsequently refining these estimates using the relationship (16). To obtain a coarse estimation of the linearized eigenvalues as described below, it is assumed that low intensity background noise prevents the system from reaching the stable fixed point in the absence of stimulation. If no background noise is present, one could alternatively apply a small magnitude, random input that simulates a white noise process.

1. Coarse estimation of linearized eigenvalues

A coarse approximation of the linearized eigenvalues can be obtained using noisy, unperturbed model output. This strategy is based on similar strategies implemented in Refs. 13 and 27 and represents a variation of the delay-embedding approach proposed in Ref. 6. To begin, consider an isostable reduced model (8) with output (9). It will be assumed that the output \(y(t) = O(\epsilon)\) and that each eigenvalue is not repeated. Consider a time series of output data taken every \(\Delta t\) seconds. This time series can be stored as a matrix \(Y \in \mathbb{R}^{K \times L}\), where

\[
Y = \begin{bmatrix} y_{00} & y_{01} & \cdots & y_{0,L-1} \\ y_{10} & y_{11} & \cdots & y_{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{K-1,0} & y_{K-1,1} & \cdots & y_{K-1,L-1} \end{bmatrix},
\]

with \(y_t \in \mathbb{R}^K\) defined according to

\[
y_t = \begin{bmatrix} y(\Delta t t) - y_0 \\ y(\Delta t t + \Delta t) - y_0 \\ \vdots \\ y(\Delta t t + \Delta t (K - 1)) - y_0 \end{bmatrix}.
\]

As a preliminary step in the analysis, proper orthogonal decomposition (POD) [37,38] can be employed to decompose the time series contained in \(Y\) into a set of representative modes. Assuming that \(L > K\), the POD modes \(\phi_i\) can be computed by finding the eigenvectors of the covariance matrix \(YY^T\). The corresponding eigenvalues \(\lambda_i^{\text{POD}}\) give a sense of the importance of the temporal fluctuations captured by the associated POD mode. In order to accurately represent the output data, a total of \(N_{\text{POD}}\) modes can be chosen so that \(\sum_{i=1}^{N_{\text{POD}}} \lambda_i^{\text{POD}} / \sum_{i=1}^{K} \lambda_i^{\text{POD}} \approx 1\). Defining \(\Phi \in \mathbb{R}^{K \times N_{\text{POD}}}, \) such that

\[
\Phi = [\phi_1, \ldots, \phi_{N_{\text{POD}}}],
\]

any column of \(Y\) can be projected onto the new POD basis as

\[
y_t - y_0 = \sum_{j=1}^{N_{\text{POD}}} \phi_j \hat{\psi}_{j,0} + \overline{\text{res}}^{\text{ISO}} = \Phi \mu_t + \overline{\text{res}}^{\text{ISO}},
\]

where \(\mu_t = [\mu_1^T \ldots \mu_{N_{\text{POD}}}^T]^T\) is a vector of POD coefficients associated with \(y_0\) and \(\overline{\text{res}}^{\text{ISO}}\) is a small residual term that stems from the fact that some of the POD modes are truncated. Simultaneously, considering (10), to leading order \(\epsilon\), the output is given by

\[
y(t) - y_0 = \sum_{k=1}^{M} \psi_{k}^T g_t^y.
\]

Recalling that \(\psi_k = \lambda_k \psi_k\) so that \(\hat{\psi}_k(t) = \exp(\lambda_k t) \psi_k(0)\), one can write

\[
y_t - y_0 = P^T \Psi_t + \overline{\text{res}}^{\text{ISO}} ,
\]

where \(\Psi_t = [\hat{\psi}_1(\Delta t t) \ldots \hat{\psi}_M(\Delta t t)]^T\), \(P \in \mathbb{C}^{K \times M}\), where the kth row and jth column are equal to \(g_t^y \exp(\lambda_j (t - 1))\), and \(\overline{\text{res}}^{\text{ISO}}\) is a small residual term that results from truncating both the \(O(\epsilon^3)\) terms and the most rapidly decaying isostable coordinates. Assuming both residuals negligible in (20) and (22) and noticing that the left hand sides are identical, one can write

\[
\mu_t = \Phi^T P^T \Psi_t ,
\]

\[
\Psi_t = P^T \Phi^T \Psi_t.
\]

Above, Eq. (23) uses the fact that the POD modes are orthogonal, and Eq. (24) assumes that the columns of \(P\) are linearly independent so that \(P^T P\) gives the identity matrix. Letting \(\Lambda = \text{diag}(\lambda_1, \Delta t), \ldots, (\lambda_M, \Delta t))\), note that \(\psi_t = \Lambda \Psi_{t+1}\) Using this information, along with (23) and (24), one finds that

\[
\mu_{t+1} = \Phi^T P \Lambda^T \Phi^T \Psi_t = \Lambda \mu_t.
\]

Additionally, as discussed in Ref. 13, when residuals \(\overline{\text{res}}^{\text{ISO}}\) and \(\overline{\text{res}}^{\text{POD}}\) are small enough that they are negligible, \(\Lambda_t\) shares eigenvalues with \(\Lambda\). This allows each \(\lambda_m\) term to be estimated from data.
by estimating the matrix $A_\mu$ according to

$$A_\mu = X^+ X^-,$$  \(26\)

where $X^+ = [\mu_1, \ldots, \mu_{l-1}]$ and $X^- = [\mu_0, \ldots, \mu_{l-2}]$. Subsequently, each eigenvalue $\lambda_\mu$ of $A_\mu$ can be related to the decay rates of the isostable coordinates from (8) according to

$$\lambda_j = \frac{\lambda_{1,j}}{K\Delta t}.$$  \(27\)

Note that the accuracy of (27) will be influenced by the magnitude of the residuals $r^{\text{SO}}_k$ and $r^{\text{POD}}_k$, which depend on factors such as noise, truncation of the POD modes, and truncation of the $O(\epsilon^2)$ terms from (21). In practice, the above approach gives a coarse approximation of the eigenvalues that comprise the reduced order model (8).

2. Refinement of eigenvalue estimates

In some cases, it can be useful to refine these estimates in conjunction with relationships derived in Sec. III B. To do so, recalling that each term $p_{\lambda}^{k,A}$ is assumed to be equal to 1, consider the relationship (16) that can be used to estimate the first order accurate terms of the reduction

$$h(g_1^{(t)}, \ldots, g_L^{(t)}, \lambda_1, \ldots, \lambda_M) = \pi \sum_{I} - \frac{1}{\epsilon} \Gamma_1.$$  \(28\)

When $\lambda_1, \ldots, \lambda_M$ are known exactly, it is possible to find values of each $g_1^{(t)}, \ldots, g_L^{(t)}$ so that $h(g_1^{(t)}, \ldots, g_L^{(t)}, \lambda_1, \ldots, \lambda_M) \approx 0$, for instance, by using Eq. (17). Supposing that only an estimate of $\lambda_1, \ldots, \lambda_M$ is available (perhaps using the method from Sec. III C) with a corresponding estimate of $g_1^{(t)}, \ldots, g_L^{(t)}$, denote $Z = [g_1^{(t)}, \ldots, g_L^{(t)}; \lambda_1, \ldots, \lambda_M]^T$, and let $Z_0$ be the initial estimate. One can update this estimate of $Z_0$ according to the Newton iteration $Z_1 = Z_0 + \Delta Z$ where

$$\Delta Z = - \left( \frac{\partial h}{\partial Z} \right)^+ h(Z_0).$$  \(29\)

Above, $\Delta Z$ represents the least squares solution to

$$\Delta Z = -(h(Z_0)) \Delta Z = -h(Z_0)$$

and attempts to update the estimated parameters so that $||h(Z_0) + h(Z_0 + \Delta Z)|| \approx ||h(Z_0)|| \approx 0$ where $|| \cdot ||$ denotes the Euclidean norm. The update (29) can be repeated until convergence is achieved. Note that convergence is not guaranteed; like any Newton iteration, this strategy works best when starting with a good initial estimation of $Z$. Additionally, if convergence is not achieved, it may be necessary to change the number of isostable coordinates considered in the reduction.

D. Inferring second order accurate models for single output systems

Using the strategies detailed in Secs. III B and III C, a first order accurate model of the form (8) can be fit to the input–output data. Such a model may be sufficient to characterize the dynamical behavior when small magnitude inputs are considered. However, higher accuracy models must be obtained when considering inputs with larger magnitudes. Strategies for identifying these higher order terms are discussed below and in Sec. III E.

Once the $O(\epsilon)$ dynamics have been determined, the steady-state behavior of the $O(\epsilon^2)$ terms of (12) can be used to determine the second order accurate terms of (10) and (11). To this end, recalling that $u(t) = \epsilon \sin(\omega t)$, one can substitute the asymptotic expansions (11) and (12) into (8) and collect $O(\epsilon^2)$ terms to yield

$$\dot{\psi}_n^{(2)} = \lambda_n \psi_n^{(2)} + \sum_{k=1}^M \psi_k^{(1)} p_{\lambda}^{k,A} \sin(\omega t).$$  \(30\)

The steady-state dynamics of each $\psi_k^{(1)}$ are given by (14) so that in a steady state, (30) can be written as

$$\dot{\psi}_n^{(2)} = \lambda_n \psi_n^{(2)} + \sum_{k=1}^M \psi_k^{(1)} p_{\lambda}^{k,A} \sin(\omega t).$$  \(31\)

Using the trigonometric product-to-sum identities from (B3) in Appendix B, one can rewrite (31) as

$$\dot{\psi}_n^{(2)} = \lambda_n \psi_n^{(2)} + \frac{1}{2} \sum_{k=1}^M \left[ \psi_k^{(1)} p_{\lambda}^{k,A} \left( \sin(2\omega t) c_{\lambda,k}(\omega) - \cos(2\omega t) s_{\lambda,k}(\omega) + s_{\lambda,k}(\omega) \right) \right].$$  \(32\)

Because (32) is a periodically forced linear equation, one can verify that the steady-state solution is

$$\dot{\psi}_n^{(2)} = \frac{-1}{2(\lambda_n^2 + 4\omega^2)} \sum_{k=1}^M \left[ \sin(2\omega t) p_{\lambda}^{k,A} \left( \lambda_n c_{\lambda,k}(\omega) + 2\omega s_{\lambda,k}(\omega) \right) + \cos(2\omega t) p_{\lambda}^{k,A} \left( -\lambda_n s_{\lambda,k}(\omega) + 2\omega c_{\lambda,k}(\omega) \right) \right] - \sum_{k=1}^M \frac{\psi_k^{(1)} p_{\lambda}^{k,A}}{2\lambda_n} \left[ c_{\lambda,k}(\omega) + i s_{\lambda,k}(\omega) \right].$$  \(33\)
where $s_{i,j}^j(\omega) \in \mathbb{C}$ and $s_{i,j}^k(\omega) \in \mathbb{C}$ are defined appropriately. Once again, considering the output given by (9), to leading order, the $O(\epsilon^3)$ terms of the output equations are

$$y_u(\omega, t) - y_0 = O(\epsilon) + \sum_{j=1}^{M} \left[ \psi_j^{(1)} g_j^0 \right] + \sum_{j=1}^{M} \sum_{k=1}^{j} \left[ \psi_j^{(1)} \psi_k^{(1)} g_k^0 \right] + O(\epsilon^3)$$

$$= O(\epsilon) + \epsilon^3 \sum_{j=1}^{M} \sum_{k=1}^{j} \left[ s_{i,j}^k(\omega) \sin(2\omega t) + s_{i,j}^k(\omega) \cos(2\omega t) - \frac{s_{i,j}^{0}(\omega) \beta_{j,k}^0}{2\lambda_j} \right]$$

$$+ \sum_{j=1}^{M} \sum_{k=1}^{j} \left[ \sum_{i=1}^{M} g_i^0 \left( s_{i,j}^{k}(\omega) \nu_{i,k}^0 + c_{i,j}^{0}(\omega) \nu_{i,k}^0 \cos(\omega t) \right) \left( s_{i,j}^{k}(\omega) \nu_{i,k}^0 \sin(\omega t) + c_{i,j}^{0}(\omega) \nu_{i,k}^0 \cos(\omega t) \right) \right] + O(\epsilon^3),$$

(34)

where the $O(\epsilon)$ terms are given by (15). Once again, trigonometric product-to-sum identities from (B3) can be used to rewrite (34) according to

$$y_u(\omega, t) - y_0 = O(\epsilon) + \epsilon^3 \sum_{j=1}^{M} \sum_{k=1}^{j} \left[ s_{i,j}^k(\omega) \sin(2\omega t) + s_{i,j}^k(\omega) \cos(2\omega t) - \frac{s_{i,j}^{0}(\omega) \beta_{j,k}^0}{2\lambda_j} \right]$$

$$+ \frac{1}{2} \sum_{j=1}^{M} \sum_{k=1}^{j} \left[ -s_{i,j}^{0}(\omega) \nu_{i,k}^0 \beta_{j,k}^0 \nu_{i,k}^0 \cos(2\omega t) + c_{i,j}^{0}(\omega) \nu_{i,k}^0 \beta_{j,k}^0 \nu_{i,k}^0 \cos(2\omega t) + s_{i,j}^{0}(\omega) \nu_{i,k}^0 \beta_{j,k}^0 \nu_{i,k}^0 \sin(2\omega t) \right.$$

$$+ c_{i,j}^{0}(\omega) \nu_{i,k}^0 \beta_{j,k}^0 \nu_{i,k}^0 \cos(2\omega t) + s_{i,j}^{0}(\omega) \nu_{i,k}^0 \beta_{j,k}^0 \nu_{i,k}^0 + c_{i,j}^{0}(\omega) \nu_{i,k}^0 \beta_{j,k}^0 \nu_{i,k}^0 \cos(2\omega t)] + O(\epsilon^3).$$

(35)

Notice that the $O(\epsilon^3)$ terms are a linear combination of sines, cosines, and constants. Also, recall that terms of the form $g_j^0$ and $\beta_{j,k}^0$ can be estimated independently for all $j$ using (16). Thus, the relationships (35) can be rewritten as

$$y_u(\omega, t) - y_0 = O(\epsilon) + \epsilon^3 \sum_{j=1}^{M} \sum_{k=1}^{j} \left[ s_{i,j}^k(\omega) \sin(2\omega t) + s_{i,j}^k(\omega) \cos(2\omega t) - \frac{s_{i,j}^{0}(\omega) \beta_{j,k}^0}{2\lambda_j} \right]$$

$$+ \left( \sum_{j=1}^{M} \sum_{k=1}^{j} \left[ \sum_{i=1}^{M} g_i^0 \left( s_{i,j}^{k}(\omega) \nu_{i,k}^0 + c_{i,j}^{0}(\omega) \nu_{i,k}^0 \cos(\omega t) \right) \left( s_{i,j}^{k}(\omega) \nu_{i,k}^0 \sin(\omega t) + c_{i,j}^{0}(\omega) \nu_{i,k}^0 \cos(\omega t) \right) \right] \right) + O(\epsilon^3),$$

(36)

where $\beta_{j,k}^1(\omega), \ldots, \beta_{j,k}^M(\omega) \in \mathbb{C}$ are defined so that (35) and (36) are identical. Finally, recalling that the $O(\epsilon)$ terms from (36) are a linear combination of sines and cosines with frequency $\omega_k$ employing a similar approach that was used to identify (16), one can multiply both sides of (36) by either $\sin(2\omega t)$, $\cos(2\omega t)$, or 1 and integrate over one period using $q$ different frequencies, $\omega_1, \ldots, \omega_q$, to yield an equation of the form

$$\frac{1}{\epsilon^2} \Gamma_2 = \pi \mathbb{Z}_2 \mathbb{Y}_2 + O(\epsilon),$$

(37)

where $\mathbb{Y}_2 \in \mathbb{C}^{(\frac{3q^2+4q}{2}) \times 1}$ is a vector containing all of the terms in the second order accurate expansion, i.e., terms of the form $\beta_{j,k}^0$ and $g_j^0$, and $\mathbb{Z}_2 \in \mathbb{C}^{(\frac{3q^2+4q}{2}) \times 1}$. Much like for (16), the second order accurate terms of the isostable reduced model (8) that comprise $\mathbb{Y}_2$ can be estimated according to

$$\mathbb{Y}_2 = \frac{1}{\epsilon^2} \mathbb{Z}_2 \Gamma_2.$$  

(38)

E. Inferring reduced models to arbitrary orders of accuracy for single output systems

Sections III B and III D detail a strategy for inferring first and second order accurate reduced order isostable models from output data. As illustrated in Appendix C, it is possible to identify the terms of the expansions (10) and (11) to arbitrary orders of accuracy using linear relationships of the form (C6). Intuitively, this is made possible by Lemma C.1, which states that under the application of an
input \( u(t) = \epsilon \sin(\omega t) \), the steady-state output can be written as

\[
y_\alpha(\omega, t) - y_0 = \sum_{k=0}^{j-1} \left[ \tau_k(\omega) \sin(k\omega t) + \sigma_k(\omega) \cos(k\omega t) \right] + \epsilon^j \left[ \tau_j(\omega) \sin(j\omega t) + \sigma_j(\omega) \cos(j\omega t) \right] + O(\epsilon^{j+1}),
\]

where \( \tau_k(\omega) \in \mathbb{C} \) and \( \sigma_k(\omega) \in \mathbb{C} \). In other words, the oscillatory components of the response with frequency \( j\omega \) first appear in the \( O(\epsilon^j) \) terms. These specific terms can be isolated by multiplying \( j \in \mathbb{Z} \) by either \( \sin(j\omega t) \) or \( \cos(j\omega t) \) and taking the integral over a single period. Subsequently, relationships between the terms that comprise each \( \tau_k(\omega) \) and \( \sigma_k(\omega) \) can be derived to infer the unknown coefficients required for a \( j \)th order reduction as long as the coefficients of the first, second, …, \( j - 2 \)th and \( j - 1 \)th order reductions have been identified. Explicit details of this fitting strategy are given in Appendix C.

As a final note, as the order of accuracy required increases, the complexity of the necessary terms grows rapidly. For instance, the \( O(\epsilon) \) terms of the steady-state response are written succinctly in (19); the \( O(\epsilon^2) \) terms of the steady-state output (36) are much more complicated. For reductions that are valid to higher orders of accuracy, it is advisable to compute the terms of each matrix \( \Sigma_j \) and \( R_j \) from (C6) using a symbolic computational package.

F. List of steps necessary for obtaining reduced order models

The following summarizes a list of steps required to estimate the terms of (8) and (9) with the expansions (10) and (11) used for \( G_j \) and each \( I_{j,j} \), respectively, computed to arbitrary order accuracy in the isostable coordinates.

Step 1. Using the strategy given in Sec. III C 1, decide on the number of isostable coordinates to use in the reduction and obtain a coarse estimate for their unperturbed decay rates. It is advisable to choose a number of POD modes, \( N_{\text{POD}} \), so that \( \sum_{j=1}^{N_{\text{POD}}} s_j^{\text{POD}} / \sum_{j=1}^{N_{\text{POD}}} \approx 1 \). Choosing \( N_{\text{POD}} \) so that this ratio is closer to 1 will allow more of the dynamical behavior to be captured but will also increase the risk of overfitting. Step 1 can be skipped if the eigenvalues are known \textit{a priori}. Step 1 can also be skipped by making an initial guess for the number of isostable coordinates and their associated decay rates and updating these guesses with the Newton iteration from Step 4.

Step 2. Once the number of isostable coordinates has been determined, use a symbolic computational package to compute the terms of \( \Sigma_j \) for each \( j \) up to the desired order of accuracy. These matrices are defined as part of (16) and (37) for first order and second order accuracy relationships, respectively. Third order accuracy terms and higher are defined in (C14). For third order accurate reductions and higher, associated terms of the vectors \( R_j \), as defined in (C14) will also need to be computed symbolically.

Step 3. For multiple frequencies \( \omega \), apply an input \( u(t) = \epsilon \sin(\omega t) \). Use the resulting steady-state responses to determine the terms of each vector \( \Gamma_j \), defined according to Eqs. (16), (37), and (C7). As a general heuristic, the frequencies of the sinusoids used during the model identification procedure should be chosen to match the expected frequency content of the inputs that will be applied after the model has been inferred. The number of frequencies required for the fitting grows with the number of unknown coefficients in the reduced order model described by Eqs. (8)–(11); the number of unknown coefficients grows, in turn, as the number of isostable coordinates and the order of accuracy of the Taylor expansions used in (10) and (11) increases. Undersampling with too few distinct frequencies will result in models that are not fully resolved. In the examples considered in this work, once enough samples are obtained, additional samples neither improve nor degrade the accuracy of the resulting reduced order model. Also, as discussed in the provided examples, it can be helpful to use larger values of \( \epsilon \) when fitting higher order accuracy terms, although this is not absolutely necessary.

Step 4. Apply the strategy from Sec. III C 2 to refine the eigenvalue estimates obtained in Step 1. If the Newton iteration does not converge, one can repeat Steps 1 and 2 using a smaller number of isostable coordinates (or by using a different guess for the initial isostable coordinates).

Step 5. Once a refined estimate for the eigenvalues associated with the decay rates of the isostable coordinates is obtained, this information can be used to evaluate \( \Sigma_1 \) and obtain an estimate for the first order accurate terms using (17). This information can be used to compute a numerical approximation for \( \Sigma_2 \) and subsequently estimate the second order accurate terms of the expansion according to (38). In general, once the \( j \)th order accuracy terms are computed, as discussed in Appendix C, numerical approximations of \( \Sigma_{j+1} \) and \( R_{j+1} \) can be computed, and the relationship (C6) can be used to estimate the \( j + 1 \)th order terms. This process can be repeated until the terms of (10) and (11) are computed to the desired orders of accuracy.

G. Inferring reduced output models for multiple outputs

Systems with multiple outputs can be considered with straightforward modifications to the model identification algorithms. To do so, suppose that the state dynamics of a general differential equation of the form (1) can be characterized by an isostable coordinate framework according to (4). As in (8), it will be assumed that a rank-1 input can be applied of the form \( U(t) = Au(t), A = [A_1 \ldots A_N]^T \in \mathbb{R}^N \), and \( u(t) \in \mathbb{R} \) yielding isostable dynamics of the form (8). Additionally, suppose that there are \( N_y \) outputs of the form

\[
y_m = y_{m0} + G_{m\alpha}(\psi_1, \ldots, \psi_M),
\]

\[
m = 1, \ldots, N_y.
\]

Analogous to (10), each \( G_{m\alpha} \) can be Taylor expanded in terms of the isostable coordinates.
\[ G_{m,n}(\psi_1, \ldots, \psi_M) \approx \sum_{k=1}^{M} \left[ \psi_k \psi_{m,k}^R \right] + \sum_{j=1}^{M} \sum_{k=1}^{J} \left[ \psi_j \psi_{m,j} \right] \]
\[ + \sum_{i=1}^{M} \sum_{j=1}^{J} \left[ \psi_i \psi_j \psi_{m,j} \right] + \cdots \]  \hspace{1cm} (41)

for \( m = 1, \ldots, N_p \). The isostatic dynamics in multiple output scenarios can still be represented according to (11); adding extra outputs does not change the isostatic coordinate dynamics. Alternatively, each additional coordinate yields an additional \( G_{m,n}(\psi_1, \ldots, \psi_M) \) for which the terms of the expansion from (41) must be fit.

To identify a first order accurate model with multiple outputs, for example, each \( g_{m,n} \) can be fit to the output data. To do so, the techniques from Secs. III B–III E can be used to identify equations of the form (16)
\[ \frac{1}{\epsilon} \Gamma_{m,1} = \pi \sum_{i=1}^{M} \psi_i \psi_{m,i}, \]
\[ m = 1, \ldots, N_p, \]  \hspace{1cm} (42)

where similar to the definition given as part of (16), \( \Gamma_{m,1} \) is a vector calculated considering the steady-state behavior of \( y_{m} \) when applying input at various frequencies and \( \psi_1 \) is a vector of terms characterizing the first order accurate isostable dynamics from (11) and the first order accurate terms of the expansion of \( G_{m,n} \) from (41). Note that the terms of \( \psi_1 \) are identical for all output relationships and were defined as part of (16). As done in Sec. III B, one can assume that \( \bar{g}_{m,n}^{(P)} = 1 \) for all \( n \) so that the relationship (42) can be used to identify \( g_{m,n}^{(P)} \) for all \( m \).

Once the first order terms are identified, the second order accurate terms can be inferred by identifying equations of the form (37) for each output
\[ \frac{1}{\epsilon} \Gamma_{m,2} = \pi \Gamma_{m,2}, \]
\[ m = 1, \ldots, N_p, \]  \hspace{1cm} (43)

where \( \Gamma_{m,2} \) and \( \Gamma_{m,2} \) are defined analogously to \( \Gamma_{m,1} \) and \( \Gamma_{m,2} \) for each output \( y_{m} \). Note that \( \sum_{i=1}^{M} \psi_i \) was defined as part of (37). All relationships of the form (43) for \( m = 1, \ldots, N_p \) can be used to infer the second order terms of each \( \Gamma_{m,1} \) from (11) and of each \( G_{m,n} \) from (41). In general, once the \( j \)th order accurate terms have been estimated, a similar strategy can be used to infer the \( j + 1 \)th order terms of the expansion by identifying equations of the form (C14) for each output
\[ \Gamma_{m,j+1} = \pi \sum_{i=1}^{M} \left[ \sum_{j=1}^{J} \psi_{m,j} \right] + O(\epsilon^{j+2}), \]  \hspace{1cm} (44)

where \( \Gamma_{m,j+1} \), \( \sum_{j=1}^{J} \psi_{m,j} \), and \( R_{m,j+1} \) are defined analogously to \( \Gamma_{m,j}, \sum_{j=1}^{J} \psi_{m,j} \), and \( R_{m,j} \) from (C14) for each input \( y_{m} \) and \( \sum_{j=1}^{J} \psi_{m,j} \) is defined as part of Eq. (C14).

As a final consideration, note that the number of terms to fit grows in proportion to the number of outputs. In situations where there are a large number of outputs, it can be useful to first apply a reduction strategy such as POD to the outputs and consider a subset of the most important POD coefficients to be the system output rather than fitting coefficients for the full set of outputs. This is intuitively similar to Galerkin projection methods for obtaining numerical solutions of partial differential equations, for example, as performed in Ref. 39. This suggested approach will be illustrated in the example from Sec. IV C.

### IV. EXAMPLES

#### A. Reduced order modeling of a simple dynamical system with additive noise

As a preliminary illustration of the proposed reduced order model identification strategy, consider the simple dynamical system from (7) with white noise added to the \( x_1 \) variable,
\[ \dot{x}_1 = \mu x_1 + u_1(t) + \sqrt{2D}\eta(t), \]
\[ \dot{x}_2 = \lambda(-x_1 + x_2 + x_1^2 + x_2^2), \]  \hspace{1cm} (45)

where \( \lambda = -1, \mu = -0.05, \) and \( \eta(t) \) is an independent and identically distributed zero-mean white noise process with intensity \( D = 0.0005 \). The model output is taken to be \( y(t) = x_1(t) \). Note that the only difference between (7) and the above equation is the addition of noise. In Sec. II B, a reduced order model was determined with knowledge of the underlying model equation (7). Here, a reduced order model of the form (8) with output of the form (9) will be inferred strictly from measurements of model output without any assumed knowledge of the underlying system dynamics. To do so, (45) is simulated for 20000 time units with a time step of 0.1 using the stochastic simulation algorithm presented in Ref. 40. The resulting data are stored in a matrix of the form (18) using \( K = 100 \), and POD is performed on the resulting data. Using a single POD mode, \( \lambda_{1,POD}^2 / \sum_{j=1}^{K} \lambda_{1,POD}^2 = 0.87 \). The coarse eigenvalue estimation strategy detailed in Sec. III C.1 results in an initial eigenvalue estimate of \( \lambda_1 = -0.0322 \).

Next, the stimulus \( u_1(t) = \epsilon \sin(\omega t) \) is applied taking \( \omega \in [0.02, 0.025, 0.03, 0.035, 0.04] \). Using \( \epsilon = 0.01 \), the steady-state output is averaged over 100 cycles in order to compute the terms of \( \Gamma_1 \) according to (16). An example input–output relationship in a steady state is shown in Panels A and B of Fig. 3. The terms of \( \Gamma_1 \) from (37) and \( \Gamma_1 \) from (C7) are computed taking \( \epsilon = 0.01 \) and \( \sqrt{0.01} \), respectively. Choosing \( \epsilon \) in this manner helps to emphasize the contributions of the higher order terms when estimating the associated coefficients of the expansions (10) and (11). Auxiliary matrices \( \Sigma_1, \Sigma_2, \Sigma_3, \) and \( R_i \) used to fit the first, second, and third order coefficients of the reduction are computed symbolically. After obtaining a preliminary estimate of \( \epsilon_j^R \), the Newton iteration suggested by (29) is employed to refine the estimate of the principal eigenvalue of \( \lambda_1 = -0.0462 \) which is relatively close to the true value of \( \lambda_1 = -0.05 \) for the noiseless system (7). Second and third order terms are computed in succession using (38) and (C6), respectively. Panels C and D show estimates of \( \psi_{m,j} \) for \( x_1 \) and \( x_2 \) obtained from first, second, and third order accurate estimates of the reduced isostable dynamics for models of the form (8) with output (9). These estimates are similar to results from panels C and D from Fig. 1 that were obtained with knowledge of the full model equations. The resulting estimates of the reduced order model parameters can be used to predict the response to other inputs. Starting from an
FIG. 3. Results from applying the model inference strategy described in Sec. III to the simple model (45). Panel A shows an example of a sinusoidal input with the black line in panel B giving the corresponding model output after allowing the transient dynamics to decay. Fourier coefficients are computed for the first, second, and third harmonics by averaging the output over 100 cycles, and the associated Fourier series representation is shown as a red line. Using inputs of various frequencies and magnitudes, the corresponding outputs are measured, and this information is used to identify reduced order models valid to various orders of accuracy as described in Sec. III. Curves from panels C and D characterize the response of the isostable coordinate to inputs and the output as a function of the isostable coordinate, respectively. Notice that these curves are similar to those shown in panels C and D of Fig. 1, which were obtained with full knowledge of the underlying dynamical equations. The input shown in panel E is applied to the model equation (45), and the output in panel F is compared to the output predicted from the reduced order models valid to different orders of accuracy.
initial condition at $x_1 = x_2 = 0$, the model (45) is simulated taking $u_x(t) = 0.08(\sin(0.17t) + \sin(0.175t) + \sin(0.175t))$ as shown in panel E. The full model output is compared to the outputs from reduced models of various orders of accuracy. For results shown here, the reduced models do not contain any noise terms. Much like the results presented in Fig. 1, the third order accurate reduced model fully replicates the output from the full simulations. The first and second order accurate models perform decently, especially in moments when the magnitude of the input is small, but are not sufficient to replicate the full system behavior. Note here that $u_x(t)$ is comprised of higher frequency sinusoids than those used to infer the reduced order models. Additionally, the applied input is periodic but has a period that is significantly larger than the sinusoids used to infer the reduced order model.

B. Spike rates of neural populations

Next, the proposed reduced model inference strategy is applied to a more complicated system describing the spike rates of a population of synaptically coupled neurons from Ref. 41. The model equations are

$$
CV_i = -I_i(V_i) - I_{ax}(V_i, h_i) - I_k(V_i, h_i) - I_T(V_i, r_i) \\
+ I_i^0 - \frac{g_{syn}}{N} \sum_{j=1}^{N} s_j(V_i - E_{syn}) + \sqrt{2D} \eta_i(t) + u(t),
$$

(46)

$\dot{h}_i = (h_\infty(V_i) - h_i) / \tau_h(V_i),$

$\dot{r}_i = (r_\infty(V_i) - r_i) / \tau_r(V_i),$

$\dot{s}_i = \frac{a(1 - s)}{1 + \exp(-(V_i - V_T)/\sigma_T)} - b_i.$

Here, $N = 1000$ gives the total number of neurons in the population; $V_i, s_i, h_i$, and $r_i$ represent the transmembrane voltage, a synaptic variable, and two gating variables assigned to neuron $i$, respectively; the conductance $g_{syn} = 0.4 \text{mS/cm}^2$ sets the coupling strength; $E_{syn} = -100 \text{mV}$ is the reversal potential of the neurotransmitter resulting in inhibitory coupling; $u(t)$ is an injected current common to all neurons; $C = 1 \mu \text{F/cm}^2$ is the membrane capacitance; $\sqrt{2D} \eta_i(t)$ is an independent and identically distributed zero-mean white noise process with intensity $D = 1$; and $I_i^0$ is the baseline current of neuron $i$ and is drawn from a normal distribution with a mean of 5 and a variance of $1 \mu \text{A/cm}^2$. The characteristics of the synaptic current are determined by the parameters $a = 3, V_\infty = -20 \text{mV}, \sigma_T = 0.8 \text{mV}$, and $\beta = 1$. The reader is referred to Ref. 41 for a full explanation of the remaining functions that determine the ionic currents $I_L, I_{Na}, I_K, I_T$ and the behavior of the gating variables. The observable for (46) is taken to be the firing rate $R(t)$, where neuron $i$ is defined to fire at the moment $V_i$ crosses $-25 \text{mV}$ with a positive slope. The firing rate defined over a sliding window

$$
FR(t) = \frac{\text{Total number of neurons that fired in the interval } [t-W, t]}{W},
$$

(47)

where $W = 1.5 \text{ ms}$ is the width of the window.

Figure 4 shows an example of the typical behavior of the model (46) with output (47) in response to input. Panel A (respectively, D) shows an example of a weak (respectively strong) sinusoidal input. Voltage traces from ten representative neurons within the population are shown in panel B (respectively E) with the corresponding firing rate shown in panel C (respectively F). Toward obtaining a reduced order model to characterize the input–output relationships of this model, (46) is simulated taking $\Delta t = 0.015 \text{ ms}$ for a total of 5000 ms. The resulting data are stored in a matrix of the form (18).

![FIG. 4](image_url) Panel A shows sinusoidal inputs to (46) with the resulting voltage traces from a subset of neurons and firing rates shown in panels B and C, respectively. The same information is shown in panels D–F using a larger magnitude input with a different frequency. For small inputs, $FR(t)$ appears somewhat sinusoidal. When larger magnitude inputs are used, the resulting firing rates are non-sinusoidal, indicating the importance of nonlinear models to characterize the output in response to inputs.
taking $K = 50$, and POD is performed on the resulting data. Using two POD modes yields $\sum_{j=1}^{2}(\lambda_j^{\text{POD}}) / \sum_{j=1}^{K}(\lambda_j^{\text{POD}}) = 0.93$. Using the coarse eigenvalue estimation strategy from Sec. III C 1 in conjunction with these two POD modes yields two complex conjugate eigenvalues $\lambda_{1,2} = -65 \pm 1.65i$.

A reduced order model is obtained by considering the firing rate in response to sinusoidal input $u(t) = \epsilon \sin(\omega t)$ $\mu_A$/cm$^2$ with $\omega \in \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$. Taking $\epsilon = 0.1$, initial transients are allowed to decay, and the steady-state output is averaged over 700 ms to compute the terms of $\Gamma_j$ from (16). In a similar manner, $\epsilon = 1$ is used to compute the terms of $\Gamma_2$ from (37). Terms of the auxiliary matrices $\Sigma_1$ and $\Sigma_2$ from (16) and (37), respectively, are computed symbolically. After obtaining preliminary estimates of the first order terms, the Newton iteration from (29) is employed to obtain a refined estimate of $\lambda_{1,2} = -0.19 \pm 0.68i$ for the principal eigenvalues. Second order terms are subsequently computed using (38). Third order accurate terms were also considered; however, the resulting reduced order models did not perform better than the second order accurate models, and the results are not shown.

The predictive capability of the resulting reduced order model is considered with two different inputs shown in Fig. 5. In both the full (46) and reduced model equations, no input is applied for the first 200 ms of simulations to allow for any transient behavior to die out. Subsequently, a small magnitude input $u(t) = \sin(0.1t) + \cos(0.23t) + \sin(-0.49t)$ is applied for 100 ms and is shown in panel A. Resulting outputs from the full and reduced models are shown in panel B with the absolute error between the reduced and full models shown in panel C. Analogous results are shown in panels D–F for a larger magnitude input $u(t) = 2.5(\sin(0.2t) + \cos(0.35t) + \sin(0.63t))$. Note that each neuron from the full model equation (46) has a white noise term added to the voltage equations, while the reduced order models are deterministic—this contributes to a jagged appearance of the full model outputs. First and second order reduced models have comparable performance when a weaker input is applied. As the inputs become stronger, the performance of the first order accurate model begins to degrade, while the second order accurate model still performs well. Note that while the terms of the reduced order models were determined by considering the steady-state behavior in response to purely sinusoidal forcing, they accurately replicate the transient response to the inputs considered in panels A and D (which are not periodic on the time scales considered).

### C. One-dimensional Burgers’ equation

As a final example, consider the 1D Burgers’ equation

$$\frac{\partial w}{\partial t} = \frac{1}{\text{Re}} \frac{\partial^2 w}{\partial x^2} - w \frac{\partial w}{\partial x}. \quad (48)$$

Here, $w$ is the state on the spatial domain $x \in [0, 1]$, and Re is a viscosity term that is analogous to the Reynolds number from the Navier–Stokes equation. The boundary condition at $x = 0$ is defined to be $w(0,t) = 0.3 + u(t)$, where 0.3 is the nominal value and $u(t)$ serves as a time-varying input. The boundary condition at $x = 1$ is held fixed at a value of 0.3. Because of its similar structure to the Navier–Stokes equation, the Burgers’ equation is often used as a test bed for model reduction techniques in fluid flow applications.

In this example, $\text{Re} = 10$ is used, and the model output is taken to be the state on the entire spatial domain. Toward determining appropriate POD modes to represent the model outputs, an input $w(t) = 0.3 + 0.7 \sin(t^2/20)$ is applied for $t = 100$ time units in order to excite system modes over a wide range of frequencies. The model response to the time-varying output is shown in panels (a) and (b) of Fig. 6, respectively. POD is performed on the resulting snapshots taken at $\Delta t = 0.01$ increments with the spatial domain discretized into 152 equally spaced elements. Taking five POD modes yields $\sum_{j=1}^{5}(\lambda_j^{\text{POD}}) / \sum_{j=1}^{152}(\lambda_j^{\text{POD}}) = 0.99994$ and captures the majority of the deviations from the steady-state solutions. These resulting POD modes are shown in panel (c) and are ranked in descending order of their importance.

A reduced order model using three isostable coordinates will be used with an initial guess of $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = -3$. As discussed in Sec. III G, the POD coefficients associated with the five
FIG. 6. Panel (a) shows the response of the 1D Burgers’ equation to the time-varying left boundary condition shown in panel (b). POD modes are extracted from the model output and plotted in panel (c) as colored lines. The steady-state solution when \( u(t) = 0 \) is shown for reference as a black line.

FIG. 7. The input \( w_{L}(t) \) shown in panel (a) is applied to the full model (48) as well as the first and second order reduced models. Panel (b) shows the resulting \( L^2 \) error between full and reduced order models. Panel (c) gives a snapshot of the solution profiles for each model at \( t = 50 \). Analogous results are shown in panels (d)–(f) when the stronger input from panel (d) is applied. The second order model output has about two orders of magnitude less error than the first order model. These differences are more pronounced when larger magnitude inputs are used as evidenced by panel (f).
POD modes from Fig. 6 are taken as the system output. A reduced order model is obtained by considering the steady-state model output (i.e., the time course of the POD coefficients in a steady state) in response to input $u(t) = \epsilon \sin(\omega t)$ with $\omega = \{0.1, 0.2, \ldots, 2.0\}$. Taking $\epsilon = 0.05$ (respectively 0.5), the terms of $\Gamma_1$ (respectively $\Gamma_2$) from (16) [respectively (37)] are computed. Terms of auxiliary matrices $\Xi_1$ and $\Xi_2$ from (16) and (37), respectively, are computed using a symbolic computational package.

The Newton iteration from (29) is applied to obtain a refined estimate of the principal eigenvalues. This Newton iteration is performed as described in Sec. III C 2 using the first POD mode as the output. This iteration converges to $\lambda_1 = -1.22, \lambda_2 = -4.62$, and $\lambda_3 = -32.06$. The eigenvalues that result from the Newton iteration are robust to changes in the initial guess for the eigenvalues. First and second order terms of the reduced functions (i.e., the functions that characterize the isostable dynamics and the isostable-to-output relationships for each POD coefficient) are then computed using relationships of the form (42) and (43). Reduced order models beyond second order accuracy are not considered for this example.

Figure 7 compares the outputs predicted by the resulting reduced order models with isostable dynamics of the form (8) and model outputs (41) to the full model dynamics from (48). Here, $w_{red}(x,t) = \sum_{j=1}^{3} \phi_j(x)p_j(t)$, where $\phi_j(x)$ and $p_j(t)$ are the $j$th POD modes and POD coefficients, respectively. Starting from a steady state, the left boundary condition is taken to be $w_0(t) = 0.3 + 0.05(\sin(0.2t) + \sin(0.35t) + \sin(0.63t))$. The logarithm of the $L^2$ error defined as $E^2(t) = \int_0^L (w_{red}(x,t) - w_{full}(x,t))^2 dx$ is shown in panel (b), where $w_{red}$ and $w_{full}$ come from the reduced and full model simulations, respectively. The solution profiles at $t = 50$ are shown in panel (c). Panels (d)–(f) give analogous results using a larger magnitude input $w_0(t) = 0.3 + 0.5(\sin(0.2t) + \sin(0.35t) + \sin(0.63t))$. In both cases, the second order accurate models result in $L^2$ errors that are approximately two orders of magnitude smaller than the $L^2$ errors from the first order accurate models. The differences in errors between these reduced order models are particularly pronounced when using stronger inputs.

V. CONCLUSION

In applications where the underlying model equations are known, it is relatively straightforward to numerically compute isostable-based reduced order models directly from the dynamical equations. However, in most experimental applications, model equations are not known precisely, and data-driven techniques must be used instead. In this work, a general, data-driven strategy is developed for inferring nonlinear isostable reduced models of the form (8) and (9).

The strategy proposed in this work leverages an isostable-based coordinate system that characterizes the behavior of the level sets of the slowest decaying Koopman eigenfunctions. Using this isostable coordinate basis as a foundation, one can show that in response to inputs of the form $u(t) = \epsilon \sin(\omega t)$, steady-state model outputs can be written in terms of a Fourier series expansion with a fundamental frequency $\omega$. Furthermore, components of the output with frequency $\omega$ are shown to be $O(\epsilon^2)$ terms. These key insights are used to develop the proposed model inference strategy developed in Secs. III B–III E. Using this strategy, it is possible to consider the steady-state output in response to a sinusoidal input at various frequencies to infer high-accuracy reduced order models of the form (8) and (9). The proposed model inference strategy is illustrated for three example systems. In each example, a linear model that only considers the first order terms in the expansions of (10) and (11) performs reasonably well when small magnitude inputs are applied. However, when considering larger magnitude inputs, nonlinear reduced order models are necessary to capture the dynamical behavior.

While it is technically possible to infer the terms of the expansions (10) and (11) to arbitrary orders of accuracy in the isostable coordinates using the proposed method, the robustness of the model fitting procedure suffers as the desired order of accuracy increases. Indeed, considering the structure of (C6), the output measurements that comprise $\Gamma_j$ are multiplied by $1/\epsilon^j$; this amplifies the influence of any errors or uncertainties associated with the measurements of the steady-state output. This effect can be partially mitigated by increasing the amplitude of the sinusoidal inputs as higher order accuracy terms are considered (this strategy was used in the examples considered in this work). As an additional consideration, the $j$th order accuracy terms depend on the $j - 1$th order accuracy terms; therefore, any errors in the computation of lower order accuracy terms will be compounded when computing higher order accuracy terms.

For the simple model (45) from Sec. IV A, the third order terms of (10) and (11) were inferred reliably. In the more complicated neural spike rate example from Sec. IV B, the performance of the reduced models was significantly improved by the addition of the second order terms, but the third order terms did not improve the performance—in this example, it is possible that the contribution from the third order terms is simply negligible. It would be worthwhile to investigate other statistical techniques to infer the reduced order terms from the relationships (16), (37), and (C14) for the first, second, and arbitrary order accuracy terms, respectively, perhaps involving either approximate Bayesian computation (ABC) or deep learning approaches.

Future work will investigate the possibility of extending the proposed methods for use in oscillatory dynamical systems. The “direct method” is a well-established technique that can be used to determine reduced order equations of the form $\dot{\theta} = \omega + Z(\theta)u(t)$ from experimental data, where $\theta \in \mathbb{D}$ is the phase, $Z(\theta)$ captures the effect of an input $u(t)$ on the phase, and $\omega$ is the unperturbed frequency of oscillations. The reduced order dynamical models considered in this work from Eq. (4) are similar in structure to the models that were investigated in Ref. 28 for oscillatory dynamical systems. It would be of interest to investigate whether the data-driven model inference techniques proposed here can be adapted for use in applications that involve limit cycle oscillators.

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APPENDIX A: ASYMPTOTIC EXPANSIONS OF ISOSTABLE REDUCED MODELS

In Ref. 28, a strategy was devised for computing asymptotic expansions of the state and the gradient of the isostable coordinates...
for periodic orbit attractors in terms of the isostable coordinates themselves. A nearly identical approach can be used to compute the individual terms of (5) and (6) for stable fixed point attractors. The following results can be obtained by following the same arguments as those presented in Ref. 28 but taking the attractor to be a fixed point instead of a periodic orbit.

First, assuming \( U(t) = 0 \), the individual terms of \( G(\psi_1, \ldots, \psi_M) \) from (5) can be found by taking the time derivatives of \( \Delta x = G(\psi_1, \ldots, \psi_M) \) yielding

\[
\frac{d\Delta x}{dt} = \sum_{k=1}^{M} [g^k \dot{\psi}_k] + \sum_{j=1}^{M} \sum_{k=1}^{j} [g^k (\lambda_j + k_\lambda) \psi_j \dot{\psi}_k] + \cdots. \tag{A1}
\]

Above, the relationship \( \dot{\psi}_k = \lambda_k \psi_k \) is used when taking time derivatives using the product rule. Letting \( f(x) = [f_1, \ldots, f_n]^T \), a direct expansion of (1) in terms of \( \Delta x \) yields

\[
\frac{d\Delta x}{dt} = J(...) \rightleftharpoons \sum_{i=1}^{n} \frac{\partial \Delta x}{\partial x_i} \vec{v}(f_i^{(0)}), \tag{A2}
\]

where \( \otimes \) is the Kronecker product, \( \vec{v}(\cdot) \) is an operator that stacks each column of a matrix into a single column vector, and higher order partial derivatives are defined recursively as

\[
f_{j,k} = \frac{\partial \vec{v}(f_{j,k-1})}{\partial x^k} \in \mathbb{R}^{n \times (k-1) \times N}. \tag{A3}
\]

Above, all partial derivatives are evaluated at \( x_0 \), and the notation \( \otimes \) is shorthand for \( \Delta x \otimes \Delta x \otimes \Delta x \in \mathbb{R}^{k \times N} \). By equating (A1) and (A2), substituting \( \Delta x \) from (5) into (A2), and matching the resulting powers in the isostable coefficients, one finds the resulting relationship

\[
0 = (J - (\lambda_i + \lambda_j + \lambda_k + \cdots) I_d) g^{rk} + q^{rk}, \tag{A4}
\]

where \( I_d \) is an appropriately sized identity matrix and \( q^{rk} \in \mathbb{R}^N \) is a function of the lower order terms of the expansion. For example, \( q^{321} \) (a third order term) might depend on \( g^2 \) or \( g^1 \) (a first or second order term) but will not depend on \( q^{321} \) (a third order term). Equation (A4) is nearly identical to Eq. (18) from Ref. 28 except for the fact that the time derivative of \( g^{rk} \) is zero since the attractor is a fixed point. For this reason, \( g^{rk} \) can be computed as the solution to a linear matrix equation

\[
\text{The individual terms of } I_n \text{ from (6) can be computed in a similar fashion. Starting with Eq. (39) from Ref. 28, for any isostable-based coordinate system, the following holds for any state in the basin of attraction of the associated attractor:}
\]

\[
\frac{dI_n}{dt} = - (\frac{\partial F_T}{\partial x} - \lambda_n I_d) I_n. \tag{A5}
\]

Direct differentiation of (6) provides the time derivative of the gradient of the isostable coordinate,

\[
\frac{dI_n}{dt} = \sum_{k=1}^{M} [\psi_k \lambda_k I_n^k] + \sum_{j=1}^{M} \sum_{k=1}^{j} \psi_j \dot{\psi}_k (\lambda_j + \lambda_k) I_n^k + \cdots. \tag{A6}
\]

Above, the relationship \( \dot{\psi}_k = \lambda_k \psi_k \) is used when taking time derivatives using the product rule. Additionally, letting \( f' \equiv \frac{\partial F_T}{\partial x} \), where all partial derivatives are evaluated at \( x_0 \), near the fixed point, one can write the asymptotic expansion

\[
\frac{\partial F_T}{\partial x} \bigg|_{x_0 = \Delta x} = f' + \lambda_n I_d, \tag{A7}
\]

where \( \lambda_n \) is a column vector and \( f'^{n+1} \) was defined in (A3). Following the strategy from Ref. 28, one can substitute (A7), (A6), and (A2) into (A5), replace any remaining \( \Delta x \) terms with \( G(\psi_1, \ldots, \psi_M) \) from (5), and match the resulting terms of the isostable coefficients to find the relationships

\[
0 = -(f' + (-\lambda_i + \lambda_i + \lambda_j + \lambda_k + \ldots) I_d) I_n^{rk} + q^{rk}, \tag{A8}
\]

where \( q^{rk} \) is comprised of only lower order terms of the expansion of \( I_n(\psi_1, \ldots, \psi_M) \); for example, \( q^{321} \) (a third order term) might depend on \( I_1 \) or \( I_2 \) (a first or second order term) but will not depend on \( I_3 \) (a third order term). Equation (A8) is nearly identical to Eq. (44) from Ref. 28 except for the fact that the time derivative of \( I_n^{rk} \) is zero since the attractor is a fixed point. Once again, each \( I_n^{rk} \) can be found as the solution to a linear matrix equation.

A detailed description of strategies for computation of the terms \( q^{rk} \) and \( q^{rk} \) from Eqs. (A4) and (A8), respectively, is discussed in Ref. 28 and is not repeated here.

APPENDIX B: STEADY-STATE BEHAVIOR OF THE ISOSTABLE COORDINATE EXPANSION IN RESPONSE TO A SINUSOIDAL INPUT

The relationships (14) and (33) illustrate how the steady-state behavior of the first and second order accurate terms of the isostable coordinate expansions in response to sinusoidal input \( u(t) = \epsilon \sin(\omega t) \) can be written as a Fourier series with a fundamental frequency \( \omega \) using a finite number of terms. Theorem B.1 shows that this pattern holds for all orders of accuracy in the expansion.

Theorem B.1: Under the application of \( u(t) = \epsilon \sin(\omega t) \), in a steady state, any \( \psi_{n,k}^{(j)} \) term from (12) can be written in the following form:

\[
\psi_{n,k}^{(j)} = \sum_{k=0}^{M} \left[ a_{n,k}^{(j)}(\omega) \sin(k\omega t) + b_{n,k}^{(j)}(\omega) \cos(k\omega t) \right], \tag{B1}
\]

where \( a_{n,k}^{(j)}(\omega) \in \mathbb{C} \) and \( b_{n,k}^{(j)}(\omega) \in \mathbb{C} \).
Proof. Theorem B.1 can be proven by induction. From Eqs. (14) and (33), this relationship holds for each $\psi^{(1)}_{n,a}$ and $\psi^{(2)}_{n,a}$, respectively. Suppose that (B1) holds for each $\psi^{(1)}_{n,a}, \psi^{(2)}_{n,a}, \psi^{(3)}_{n,a}, \ldots, \psi^{(j)}_{n,a}$ and for all $n$. Then, considering the dynamics specified by (8) and using the asymptotic expansions (11) and (12), for any $\psi_{a}$, one can collect all $O(\varepsilon^{j+1})$ terms giving the steady-state dynamics

$$\psi^{(j+1)}_{n,a} = \lambda_{a} \psi^{(j+1)}_{n,a} + \sum_{b_{1}=1}^{M} \left( \sin(oCt) \psi^{(j)}_{1,b_{1}a} \right) + \sum_{a_{1}+a_{2}+\cdots+a_{j}=j} \left( \sin(oCt) \left[ \sum_{b_{1}=1}^{M} \sum_{b_{2}=1}^{M} \left( \psi^{(a_{1})}_{b_{1},1} \psi^{(a_{2})}_{b_{2},1} \lambda_{b_{1}b_{2}} \right) \right] \right) + \cdots$$

$$+ \sum_{a_{1}+a_{2}+\cdots+a_{j}=j} \left( \sin(oCt) \left[ \sum_{b_{1}=1}^{M} \sum_{b_{1}}^{M} \sum_{b_{1}}^{M} \left( \psi^{(a_{1})}_{b_{1},1} \psi^{(a_{2})}_{b_{1},1} \lambda_{b_{1}b_{2}} \right) \right] \right).$$

(B2)

Consider any term of the form $\sin(oCt)\psi^{(a_{1})}_{b_{1},1} \psi^{(a_{2})}_{b_{2},1} \ldots \psi^{(a_{j})}_{b_{j},1}$ for which $a_{1} + a_{2} + \cdots + a_{m} = j$ from (B2). From (B1), terms of this form can be written as the product of sines and cosines with frequencies that are multiples of $\omega$. Consider the trigonometric product-to-sum identities,

$$\sin(u) \sin(v) = \frac{1}{2} (\cos(u-v) - \cos(u+v)),$$

$$\cos(u) \cos(v) = \frac{1}{2} (\cos(u-v) + \cos(u+v)),$$

$$\sin(u) \cos(v) = \frac{1}{2} (\sin(u+v) + \sin(u-v)).$$

(B3)

By converting all products to sums and noting, for instance, that $\psi^{(a_{1})}_{b_{1},1} \psi^{(a_{2})}_{b_{2},1}$ terms have a maximum frequency of $a_{1} \omega$, one can write

$$\sin(oCt)\psi^{(a_{1})}_{b_{1},1} \psi^{(a_{2})}_{b_{2},1} \ldots \psi^{(a_{j})}_{b_{j},1} = \sum_{k=0}^{j+1} \left[ \alpha_{k}^{(a_{1})}(\omega) \sin(koCt) + \beta_{k}^{(a_{1})}(\omega) \cos(koCt) \right].$$

(B4)

where $\alpha_{k}^{(a_{1})}(\omega) \in \mathbb{C}$ and $\beta_{k}^{(a_{1})}(\omega) \in \mathbb{C}$. Using the product-to-sum formulas, all terms containing sines and cosines in (B2) can be decomposed into the form (B4). Therefore, (B2) can be rewritten in the form

$$\psi^{(j+1)}_{n,a} = \lambda_{a} \psi^{(j+1)}_{n,a} + \sum_{k=1}^{j+1} \left[ \gamma^{(k)}_{a,j+1}(\omega)(\sin(koCt) + \lambda_{a} \sin(koCt)) = \delta^{(k)}_{a,j+1}(\omega)(\lambda_{a} \cos(koCt) - \cos(koCt)) \right.$$

(B5)

where each $\gamma^{(k)}_{a,j+1}(\omega) \in \mathbb{C}$ and $\delta^{(k)}_{a,j+1}(\omega) \in \mathbb{C}$. Finally, noticing that (B5) is simply a periodically forced linear ordinary differential equation, one finds that the steady-state dynamics are

$$\psi^{(j+1)}(t) = \sum_{k=1}^{j+1} -\gamma^{(k)}_{a,j+1}(\omega)(koCt \cos(koCt) + \lambda_{a} \sin(koCt)) = \delta^{(k)}_{a,j+1}(\omega)(\lambda_{a} \cos(koCt) - \cos(koCt)),$$

(B6)

which is of the form (B1), thereby completing the proof. \qed

Remark: Theorem B.1 states that in response to an input $u(t) = \varepsilon \sin(oCt)$, the $O(\varepsilon)$ terms can be represented exactly as a Fourier series with a fundamental frequency $\omega$ and a maximum frequency $\omega$. This can be exploited to identify the terms of the expansions from (10) and (11) as explained in Appendix C.

APPENDIX C: FITTING ISOSTABLE REDUCED MODELS TO ARBITRARY ORDER ACCURACY USING INPUT-OUTPUT RELATIONSHIPS

Linear relationships (16) and (37) can be used to fit the first and second order terms of the expansion (10) and (11), respectively, when considering a system with a single output. Similar relationships can be obtained for any desired orders of accuracy. To show how this can be done, first consider the following lemma:

Lemma C.1: Under the application of $u(t) = \varepsilon \sin(oCt)$, for a single output system, in a steady state, the output relationship from (9) can be written in the following form:

$$y_{a}(\omega, t) - y = \sum_{k=0}^{j} \left[ t_{k}(\omega) \sin(koCt) + \sigma_{k}(\omega) \cos(koCt) \right] + O(\varepsilon^{j+1}),$$

(C1)

where each $t_{k}(\omega) \in \mathbb{C}$ and $\sigma_{k}(\omega) \in \mathbb{C}$ is an $O(\varepsilon^{m})$ term with $m \geq j$.

Remark: Lemma C.1 implies that under the application of $u(t) = \varepsilon \sin(oCt)$, oscillatory components of frequency $\omega$ of the steady-state output are $O(\varepsilon^{j})$ terms.

Proof. Consider any term from the Taylor expansion of $G_{j}(\psi_{1}, \ldots, \psi_{a})$ from (10) of the form $\sum_{b_{1}=1}^{M} \sum_{b_{2}=1}^{M} \cdots \sum_{b_{j-1}=1}^{M}$
\[ \sum_{\sum_{b_1=1}^{b_p} \cdots \sum_{b_{p-1}=1}^{b_1}} \left[ \psi_{b_1, \ldots, b_p}^{h_1, \ldots, h_p} \right] \cdot \text{where } p \in \mathbb{N}. \] Considering the asymptotic expansions of the isostable coordinates from Eq. (12), in a steady state, one can rewrite these sums as
\[ \sum_{\sum_{b_1=1}^{b_p} \cdots \sum_{b_{p-1}=1}^{b_1}} \left[ \psi_{b_1, \ldots, b_p}^{h_1, \ldots, h_p} \right] = \sum_{k=1}^{j-1} \left( \sum_{a_1+a_2+\cdots+a_p=k} \left[ e^\epsilon \sum_{b_1=1}^{b_p} \cdots \sum_{b_{p-1}=1}^{b_1} \left[ \psi_{b_1, \ldots, b_p}^{a_1, \ldots, a_p} \right] \right] \right) + O(\epsilon^{j+1}). \tag{C2} \]

Above, all terms have been rewritten to emphasize the orders of \( \epsilon \) in the Taylor expansion. Using (B1), in a steady state, terms of the form \( \psi_{b_1}^{n_1} \) can be written as a Fourier series expansion with fundamental frequency \( \omega \) and a maximum frequency of \( j \omega \). Considering the product-to-sum identities from (B3), this implies that Eq. (C2) can be rewritten as
\[ \sum_{\sum_{b_1=1}^{b_p} \cdots \sum_{b_{p-1}=1}^{b_1}} \left[ \psi_{b_1, \ldots, b_p}^{h_1, \ldots, h_p} \right] = \sum_{k=0}^{j-1} \left[ t_{k,1}(\omega) \sin(k\omega t) + \sigma_{k,1}(\omega) \cos(k\omega t) \right] + \sum_{a_1+a_2+\cdots+a_p=k} \left[ e^\epsilon \sum_{b_1=1}^{b_p} \cdots \sum_{b_{p-1}=1}^{b_1} \left[ \psi_{b_1, \ldots, b_p}^{a_1, \ldots, a_p} \right] \right] + O(\epsilon^{j+1}), \tag{C3} \]

where each \( t_{k,1}(\omega) \) and \( \sigma_{k,1}(\omega) \) contain the contributions from the \( O(\epsilon), \ldots, O(\epsilon^{j-1}) \) terms of the expansion (C2). To arrive at the simplification (C3), one can note that when using the product-to-sum identities (B3) when simplifying each \( O(\epsilon^j) \) term for \( k < j \), \( \omega \) is the maximum possible frequency of the resulting sinuoids. Considering the \( O(\epsilon^j) \) terms in (C3), continuing the simplification in this manner yields
\[ \sum_{\sum_{b_1=1}^{b_p} \cdots \sum_{b_{p-1}=1}^{b_1}} \left[ \psi_{b_1, \ldots, b_p}^{h_1, \ldots, h_p} \right] = \sum_{k=0}^{j-1} \left[ t_{k,2}(\omega) \sin(k\omega t) + \sigma_{k,2}(\omega) \cos(k\omega t) \right] + \left[ t_{j,2}(\omega) \sin(j\omega t) + \sigma_{j,2}(\omega) \cos(j\omega t) \right] + O(\epsilon^{j+1}), \tag{C4} \]

where terms that are proportional to \( \sin(k\omega t) \) [respectively \( \cos(k\omega t) \)] are absorbed into the new constants \( t_{k,2}(\omega) \) [respectively \( \sigma_{k,2}(\omega) \) \( \in \mathbb{C} \). Above, note that \( t_{j,2}(\omega) \) and \( \sigma_{j,2}(\omega) \) are \( O(\epsilon^j) \) terms. Because any of the terms of the expansion (10) can be written in the form (C4), it is possible to write \( y_n(t) = y_0(t) \) in the form of (C1) where the terms proportional to \( \sin(j\omega t) \) and \( \cos(j\omega t) \) are \( O(\epsilon^j) \) terms, which completes the proof. \( \square \)

Lemma C.1 will be used to show how higher order terms of the expansions (10) and (11) can be determined from the steady-state behavior. Toward this goal, (B1) will be rewritten as
\[ \psi_{b_1}^{(j)}(t) = \alpha_{b_1}^{(j)}(\omega) \sin(j\omega t) + \beta_{b_1}^{(j)}(\omega) \cos(j\omega t) + \sum_{k=0}^{j-1} \left[ \alpha_{b_1}^{(k)}(\omega) \sin(k\omega t) + \beta_{b_1}^{(k)}(\omega) \cos(k\omega t) \right], \tag{C5} \]

in order to emphasize the highest frequency terms in each \( \psi_{b_1}^{(j)} \). The following theorem can now be stated:

**Theorem C.1.** Consider an isostable coordinate reduction of the form (8) with output equation (9) with a single output and \( M \) isostable coordinates where all \( \lambda_n \) for \( n = 1, \ldots, M \) are known. Suppose for all \( k = 1, \ldots, j \) that the coefficients \( \alpha_{b_1}^{(k)}(\omega) \) and \( \beta_{b_1}^{(k)}(\omega) \) from (C5) have been computed; these represent the coefficients of (C5) that are proportional to the highest frequency terms of each \( \psi_{b_1}^{(j)}(t) \). Suppose also that all terms \( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{2^j-1} \) have been estimated for all \( n \); i.e., all terms associated with the \( j \)-th and \( j-1 \)-th order accurate expansions in the isostable coordinates from (10) and (11), respectively, have been estimated. Then,

Statement C1: By applying a series of \( q \) inputs of the form \( u = \epsilon \sin(\omega t) \) with \( \omega = \omega_1, \omega_2, \ldots, \omega_q \) and measuring the resulting steady-state behavior \( y_n(t, \omega) \), all terms of the form \( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{2^j-1} \) for all \( n \) can be estimated using the linear relationship
\[ \tau_{j+1}^{(n)} = \Xi_{j+1}^{(n)} \left( \frac{1}{\epsilon^{j+1} \pi} \Gamma_{j+1} - \mathbf{R}_{j+1} \right) + O(\epsilon), \tag{C6} \]

where
\[ \Gamma_{j+1} = \left[ \begin{array}{c} \alpha_0 \int_0^{\frac{\pi}{\omega_1}} y_n(w_1, t) \sin((j+1)\omega t) dt \\ \alpha_0 \int_0^{\frac{\pi}{\omega_1}} y_n(w_1, t) \cos((j+1)\omega t) dt \\ \vdots \\ \alpha_{q} \int_0^{\frac{\pi}{\omega_q}} y_n(w_q, t) \sin((j+1)\omega t) dt \\ \alpha_{q} \int_0^{\frac{\pi}{\omega_q}} y_n(w_q, t) \cos((j+1)\omega t) dt \end{array} \right] \in \mathbb{C}^q, \tag{C7} \]

where \( \tau_{j+1}^{(n)} \in \mathbb{C}^{q+1} \) is a vector containing \( q+1 \) total unknown elements and is comprised solely of terms of the form \( \tilde{y}^{(2^j-1)} \).
\( \forall \) in \( \mathbb{C}^{n} \) are appropriately defined matrices comprised solely of previously estimated terms, and \( \dagger \) denotes the pseudoinverse.

Statement C2: The terms \( \alpha_{\nu,k}^{j+1}(\omega) \) and \( \beta_{\nu,k}^{j+1}(\omega) \) of \( \psi_{\nu,k}^{(j+1)} \) from the expansion (C5) can be estimated for all \( n \) once \( \gamma_{j+1} \) has been estimated according to (C6).

**Proof of Statement C1.** Consider the steady-state dynamics of \( \psi_{\nu,k}^{(j+1)} \) governed by (B2). These dynamics can be rewritten as a sum of sines and cosines according to (B5),

\[
\psi_{\nu,k}^{(j+1)} = \lambda_{\nu} \psi_{\nu,k}^{(j+1)} + \gamma_{\nu,k}^{(j+1)}(\omega) \sin((j+1)\omega t) + \delta_{\nu,k}^{(j+1)}(\omega) \cos((j+1)\omega t)
\]

in the above equation, the main focus will be on the terms proportional to \( \sin((j+1)\omega t) \) and \( \cos((j+1)\omega t) \) in the above equation. When substituting (C5) into (B2) and subsequently applying the product-to-sum identities from (B3), the only way to yield terms that are proportional to \( \sin((j+1)\omega t) \) and \( \cos((j+1)\omega t) \) is to consider the highest frequency terms of each \( \psi_{\nu,k}^{(j)} \) [i.e., the highest frequency terms only depend on \( \alpha_{\nu,k}^{j} \) and \( \beta_{\nu,k}^{j} \) for \( k = 1, \ldots, j \)]. With this in mind, considering the structure of (B2) and comparing to the structure of (C8), one finds that the terms \( \gamma_{\nu,k}^{(j+1)} \) and \( \delta_{\nu,k}^{(j+1)} \) can be written as

\[
\gamma_{\nu,k}^{(j+1)}(\omega) = \sum_{b_{1}=1}^{M} X_{\nu,b_{1}}(\omega) \alpha_{\nu,b_{1}}^{j} + \sum_{b_{1}=1}^{M} \sum_{b_{2}=1}^{b_{1}} b_{2} \beta_{\nu,b_{2}}^{j} \alpha_{\nu,b_{1}}^{j} + \cdots + \sum_{b_{1}=1}^{M} b_{1} b_{2} \cdots \sum_{b_{j-1}=1}^{b_{j-1}} b_{j} \beta_{\nu,b_{j}}^{j} \alpha_{\nu,b_{j-1}}^{j}.
\]

\[
\delta_{\nu,k}^{(j+1)}(\omega) = \sum_{b_{1}=1}^{M} X_{\nu,b_{1}}(\omega) \beta_{\nu,b_{1}}^{j} + \sum_{b_{1}=1}^{M} \sum_{b_{2}=1}^{b_{1}} b_{2} \beta_{\nu,b_{2}}^{j} \beta_{\nu,b_{1}}^{j} + \cdots + \sum_{b_{1}=1}^{M} b_{1} b_{2} \cdots \sum_{b_{j-1}=1}^{b_{j-1}} b_{j} \beta_{\nu,b_{j}}^{j} \beta_{\nu,b_{j-1}}^{j}.
\]

where each term of the form \( X_{\nu,b_{1}}^{b_{2}} \) is \( \in \mathbb{C} \) and \( X_{\nu,b_{1}}^{b_{2}} \) is \( \in \mathbb{C} \) only depends on the highest frequency terms of each \( \psi_{\nu,k}^{(j)} \) [i.e., only depend on \( \alpha_{\nu,k}^{j} \) and \( \beta_{\nu,k}^{j} \) for \( k = 1, \ldots, j \), which are known by assumption]. Because (C8) is of the form (B5) (i.e., a linear, periodically forced ordinary differential equation), the solution \( \psi_{\nu,k}^{(j+1)}(t) \) is given by (B6). Note that from \( \gamma_{\nu,k}^{(j+1)}(\omega) \) and \( \delta_{\nu,k}^{(j+1)}(\omega) \) in (C9), the only unknown terms are of the form \( b_{1} b_{2} \cdots b_{j} \). Substituting (C9) into (B6) and calling \( \alpha_{\nu,k}^{(j+1)}(\omega) \) and \( \beta_{\nu,k}^{(j+1)}(\omega) \) the coefficients proportional to the highest frequency terms of \( \psi_{\nu,k}^{(j+1)} \), one can write

\[
\alpha_{\nu,k}^{(j+1)}(\omega) = \sum_{b_{1}=1}^{M} \sum_{b_{2}=1}^{b_{1}} \cdots \sum_{b_{j-1}=1}^{b_{j-1}} \sum_{b_{j}=1}^{b_{j}} [X_{\nu,b_{1}}^{b_{2}}(\omega) \alpha_{\nu,b_{2}}^{j} \beta_{\nu,b_{1}}^{j} + \cdots + X_{\nu,b_{1}}^{b_{2}}(\omega) \beta_{\nu,b_{2}}^{j} \alpha_{\nu,b_{1}}^{j} + \cdots + X_{\nu,b_{1}}^{b_{2}}(\omega) \beta_{\nu,b_{2}}^{j} \beta_{\nu,b_{1}}^{j} + \cdots + X_{\nu,b_{1}}^{b_{2}}(\omega) \beta_{\nu,b_{2}}^{j} \beta_{\nu,b_{1}}^{j}]
\]

\[
\beta_{\nu,k}^{(j+1)}(\omega) = \sum_{b_{1}=1}^{M} \sum_{b_{2}=1}^{b_{1}} \cdots \sum_{b_{j-1}=1}^{b_{j-1}} \sum_{b_{j}=1}^{b_{j}} \beta_{\nu,b_{1}}^{j} \beta_{\nu,b_{1}}^{j} + \cdots + \beta_{\nu,b_{1}}^{j} \beta_{\nu,b_{1}}^{j} \beta_{\nu,b_{1}}^{j} + \cdots + \beta_{\nu,b_{1}}^{j} \beta_{\nu,b_{1}}^{j} \beta_{\nu,b_{1}}^{j}.
\]
identities to (C12) only result when considering the highest frequency terms of each \( \psi_{n,b}^j \). With this in mind, terms that are proportional to \( \sin((j+1)t) \) and \( \cos((j+1)t) \) can be computed solely with knowledge of each \( \alpha_{n,b}^{j+1}(\omega) \) and \( \beta_{n,b}^{j+1}(\omega) \) for \( k \leq j + 1 \). By assumption, each \( \alpha_{n,b}^{j+1}(\omega) \) and \( \beta_{n,b}^{j+1}(\omega) \) is already known for \( k \leq j \) for all \( n \). Considering the relationship (C10), the terms \( \alpha_{n,j+1}^{j+1}(\omega) \) and \( \beta_{n,j+1}^{j+1}(\omega) \) are known for all \( n \) except for the contributions from terms of the form \( \psi_{n,b}^{j+1} \).

By substituting (C5) into (C12), expanding using product-to-sum identities (B3), and keeping in mind the structure of the unknown terms of \( \alpha_{n,b}^{j+1}(\omega) \) and \( \beta_{n,b}^{j+1}(\omega) \) from (C10), one can collect terms appropriately as in (C13) to find that \( \tau_{j+1}(\omega) \) and \( \sigma_{j+1}(\omega) \) can be written as

\[
\tau_{j+1}(\omega) = \sum_{n=1}^{M} \left[ \sum_{b_1=1}^{b_j} \sum_{b_2=1}^{b_j} \cdots \sum_{b_{j+1}=1}^{b_j} \sum_{b_{j+2}=1}^{b_j} \left( \sum_{n_{j+2} b_{j+2}} \psi_{n,b}^j \right) \right] + \sum_{n=1}^{M} \left[ \sum_{b_1=1}^{b_j} \sum_{b_2=1}^{b_j} \cdots \sum_{b_{j+1}=1}^{b_j} \sum_{b_{j+2}=1}^{b_j} \left( \sum_{n_{j+2} b_{j+2}} \psi_{n,b}^j \right) \right] + \sum_{n=1}^{M} \left[ \sum_{b_1=1}^{b_j} \sum_{b_2=1}^{b_j} \cdots \sum_{b_{j+1}=1}^{b_j} \sum_{b_{j+2}=1}^{b_j} \left( \sum_{n_{j+2} b_{j+2}} \psi_{n,b}^j \right) \right] + r_{j+1}(\omega),
\]

\[
\sigma_{j+1}(\omega) = \sum_{n=1}^{M} \left[ \sum_{b_1=1}^{b_j} \sum_{b_2=1}^{b_j} \cdots \sum_{b_{j+1}=1}^{b_j} \sum_{b_{j+2}=1}^{b_j} \left( \sum_{n_{j+2} b_{j+2}} \psi_{n,b}^j \right) \right] + \sum_{n=1}^{M} \left[ \sum_{b_1=1}^{b_j} \sum_{b_2=1}^{b_j} \cdots \sum_{b_{j+1}=1}^{b_j} \sum_{b_{j+2}=1}^{b_j} \left( \sum_{n_{j+2} b_{j+2}} \psi_{n,b}^j \right) \right] + \sum_{n=1}^{M} \left[ \sum_{b_1=1}^{b_j} \sum_{b_2=1}^{b_j} \cdots \sum_{b_{j+1}=1}^{b_j} \sum_{b_{j+2}=1}^{b_j} \left( \sum_{n_{j+2} b_{j+2}} \psi_{n,b}^j \right) \right] + r_{j+1}(\omega),
\]

where each \( \sum_{n_{j+2} b_{j+2}} \psi_{n,b}^j \) is a vector containing \( \eta_{j+1} \) total elements of the unknown terms \( \psi_{n,b}^j \) and \( \sum_{b_{j+2}} \). The vectors \( \ni_{j+1} \) and \( \RA_{j+1} \) are known vectors that is comprised of all remaining terms. The relationship (C6) follows immediately.

Proof of Statement C2. Estimation of \( \eta_{j+1} \) according to (C6) provides an estimation of all terms of the form \( I_{n,b}^j \). Consequently, all elements of the right hand side of (C10) are known providing an estimate for the terms \( \alpha_{n,b}^{j+1}(\omega) \) and \( \beta_{n,b}^{j+1}(\omega) \) for all \( n \).

Remark: Theorem C1 provides a strategy to iteratively estimate the terms of the expansions (10) and (11) that are part of the reduced order model (8) with output (9). For example, Eq. (17) provides a relatively straightforward expression of this form that can be used to infer the first order accurate terms. Equations for identifying second order and higher accuracy terms are more complicated, and it is advisable to compute the terms of matrices \( \Xi_b \) and \( I \) using a symbolic computational package. All equations of the form (C6) require knowledge of all lower order terms of the expansions (10) and (11).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

REFERENCES

1. P. Benner, S. Gugercin, and K. Willcox, “A survey of projection-based model reduction methods for parametric dynamical systems,” SIAM Rev. 57(4), 483–531 (2015).
2. M. Budišić, R. Mohr, and I. Mezić, “Applied Koopmanism,” Chaos 22(4), 047510 (2012).
3. I. Mezić, “Analysis of fluid flows via spectral properties of the Koopman operator,” Annu. Rev. Fluid Mech. 45, 357–378 (2013).
4. A. Mauroy and I. Mezić, “Global stability analysis using the eigenfunctions of the Koopman operator,” IEEE Trans. Automat. Contr. 61(11), 3356–3369 (2016).
5. S. Bagheri, “Koopman–mode decomposition of the cylinder wake,” J. Fluid Mech. 726, 596–623 (2013).
6. H. Arbabi and I. Mezić, “Ergodic theory, dynamic mode decomposition, and computation of spectral properties of the Koopman operator,” SIAM J. Appl. Dyn. Syst. 16(4), 2096–2126 (2017).
7. B. Lusch, J. N. Kutz, and S. L. Brunton, “Deep learning for universal linear embeddings of nonlinear dynamics,” Nat. Commun. 9(1), 4950 (2018).
8. S. Bagheri, “Koopman-mode decomposition of the cylinder wake,” J. Fluid Mech. 641(1), 115–127 (2009).
9. C. W. Rowley, I. Mezić, S. Bagheri, P. Schlatter, and D. S. Henningson, “Spectral analysis of nonlinear flows,” J. Fluid Mech. 641(1), 115–127 (2009).
10. N. Kutz, S. L. Brunton, B. W. Brunton, and J. L. Proctor, Dynamic Mode Decomposition: Data-Driven Modeling of Complex Systems (Society for Industrial and Applied Mathematics, Philadelphia, PA, 2016).
11. M. O. Williams, I. G. Kevrekidis, and C. W. Rowley, “A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition,” J. Nonlinear Sci. 25(6), 1307–1346 (2015).
12. E. Yeung, S. Kundu, and N. Hoda, “Learning deep neural network representations for Koopman operators of nonlinear dynamical systems,” in 2019 American Control Conference (IEEE, 2019), pp. 4832–4839.
13. D. Wilson, “A data-driven phase and isostable reduced modeling framework for oscillatory dynamical systems,” Chaos 30(1), 013121 (2020).
14. J. L. Proctor, S. L. Brunton, and J. N. Kutz, “Dynamic mode decomposition with control,” SIAM J. Appl. Dyn. Syst. 15(1), 142–161 (2016).
15. J. L. Proctor, S. L. Brunton, and J. N. Kutz, “Generalizing Koopman theory to allow for inputs and control,” SIAM J. Appl. Dyn. Syst. 17(1), 909–930 (2018).
16. M. O. Williams, S. M. Hemati, S. T. M. Dawson, I. G. Kevrekidis, and C. W. Rowley, “Extending data-driven Koopman analysis to actuated systems,” IFAC-PapersOnLine 49(18), 704–709 (2016).
17. M. Korda and I. Mezić, “Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control,” Automatica 93, 149–160 (2018).
18. S. Peitz and S. Klus, “Koopman operator-based model reduction for switched-system control of PDEs,” Automatica 106, 184–191 (2019).
19. E. Kaiser, J. N. Kutz, and S. L. Brunton, “Data-driven discovery of Koopman eigenfunctions for control,” arXiv:1707.01146 (2017).
20. A. Mauroy, I. Mezić, and J. Moehlis, “Isostable, isochrons, and Koopman spectrum for the action–angle representation of stable fixed point dynamics,” Phys. D 261, 19–30 (2013).
21. D. Wilson and J. Moehlis, “Extending phase reduction to excitable media: Theory and applications,” SIAM Rev. 57(2), 201–222 (2015).
22. D. Wilson and J. Moehlis, “Isostable reduction with applications to time-dependent partial differential equations,” Phys. Rev. E 94(1), 012211 (2016).
A. Sootla and A. Mauroy, “Geometric properties of isostables and basins of attraction of monotone systems,” IEEE Trans. Automat. Contr. 62(12), 6183–6194 (2017).

D. Wilson and B. Ermentrout, “Greater accuracy and broadened applicability of phase reduction using isostable coordinates,” J. Math. Biol. 76(1–2), 37–66 (2018).

D. Wilson and B. Ermentrout, “Phase models beyond weak coupling,” Phys. Rev. Lett. 123(16), 164101 (2019).

D. Wilson and S. Djouadi, “Isostable reduction and boundary feedback control for nonlinear convective flow,” in Proceedings of the 58th IEEE Conference on Decision and Control (IEEE, 2019).

D. Wilson and B. Ermentrout, “Augmented phase reduction of (not so) weakly perturbed coupled oscillators,” SIAM Rev. 61(2), 277–315 (2019).

S. Shirasaka, W. Kurebayashi, and H. Nakao, “Phase-amplitude reduction of transient dynamics far from attractors for limit-cycling systems,” Chaos 27(2), 023119 (2017).

J. N. Kutz, J. L. Proctor, and S. L. Brunton, “Applied Koopman theory for partial differential equations and data-driven modeling of spatio-temporal systems,” Complexity 2018, 6010634.

E. M. Izhikevich, Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting (MIT Press, London, 2007).