In this work quantum metrology techniques are applied to the imaging of objects with a non-uniform refractive spatial profile. A sensible improvement on the classical accuracy is shown to be found when the "Twin Beam State" (TWB) is used. In particular exploiting the multimode spatial correlation, naturally produced in the Parametric Down Conversion (PDC) process, allows a 2D reconstruction of complex spatial profiles, thus enabling an enhanced imaging. The idea is to use one of the spatially multimode beam to probe the sample and the other as a reference to reduce the noise. A similar model can be also used to describe wave front distortion measurements. The model is meant to be followed by a first experimental demonstration of such enhanced measurement scheme.

**Keywords**: Quantum; Imaging; Enhanced; Refractive; Gradient-Index; Schlieren.

### 1. Introduction

In recent years quantum states of light have been proven successful in the enhancement of a variety of measurement schemes [1], such as undetected photon imaging [2], quantum illumination [3-5], super resolution [6, 7], ghost imaging [8-13], interferometry [14, 17] and absorption imaging [18-21]. In particular a fundamental limit in the accuracy of classical schemes is the Shot Noise Limit (SNL) [22, 23], that bounds the uncertainty in the estimation of a parameter to scale as the inverse square root of the photons involved. Schemes that enable to surpass the SNL are of paramount importance in settings where the energy that can be used is limited, as it is the case, for example, when dealing with biological samples [24] that

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Fig. 1. *Wide field imaging of a refractive profile*. A sample is illuminated by a spatial multimode beam. At each position the beam is deflected at a different angle, altering the intensity distribution. In the dashed shape the configuration in which the beam is correlated to another, used as reference, is pictured.

could be damaged by the radiation. Sub SNL measurements have been realized, using squeezed states of light, for interferometry [23, 26], beam displacement measurements [27–29], and recently sub SNL wide field absorption imaging has been achieved [20, 30, 31], using quantum correlated states. A state often used in such schemes is the Twin Beam (TWB) state produced by the process of Parametric Down Conversion (PDC) [32, 33] or four wave mixing [34, 35]. In PDC a laser pump interacts with a non-linear crystal creating, as a result, a pair of photons correlated both in position and momentum. This state is particularly interesting not only because the use of quantum correlations allows a reduction of the uncertainty of an estimation below the SNL, but also because of the spatial multimode nature of the PDC process, that automatically enables wide field imaging, meaning that a 2D spatial profile can be imaged with a single exposure. It can be expected that the TWB state, similarly as it is in the case of absorption imaging, can be used to achieve sub SNL measurements of non uniform refractive profiles and aim of this work is, in fact, to investigate the improvements that the use of quantum correlations would bring to such measurements. Classically different techniques are used to image the refractive profile of an object. Between those the Schlieren scheme [36] focuses on the imaging of the gradient of the refractive profile $\nabla n$. Considering a beam interacting with the object, using ray optics [37], can be seen that a deflection of a certain angle is produced, proportional to $\nabla n$ and in the Schlieren configuration this is in turn proportional to the difference in detected intensity, with and
without the object. Thus, for each point of the object the angle of deflection is retrieved measuring the change in the intensity distribution at the detection plane, by means of a multipixel detector.

In this paper, similarly, we analyze the possible quantum advantage achievable in a scheme where the deflection is estimated by a measurement of the intensity distribution, so that the uncertainty of the estimation depends on the statistics of the detected photons. This problem is similar to the beam displacement problem analyzed in Ref. [28], where the entire beam is deflected of a certain angle and it is detected by a quadrant detector. The difference is that the structure causing the deflection in our case is more complex, in the sense that at each position of the sample incoming light is deflected at a different angle, or no angle at all, as pictured in figure 1. The object is considered to be illuminated by a spatially incoherent source with a certain pattern, e.g., the TWB state. The results, after the interaction, is a measured intensity distribution where deflected and non-deflected parts of the probe pattern sum up in intensity at each pixels. Interference effects are not considered here given the incoherent properties of the multimode source.

2. The Model

The analysis of the interaction of the beam with the object can be carried out from a phenomenological point of view as depicted in figure 2A.

In the simplified scheme pictured a single mode, labeled $\hat{a}$ goes through a region with non-uniform refractive index, called an impurity, and, as a result, is deflected downwards of an angle $\alpha$. In turn at the detection plane, close to the object, photons will be detected in a shifted position. The detector are positioned such that the one labeled "1" intercepting the first mode, when unperturbed, while an adja-
cent detector of the same size, labeled "2", receives photons only when the photon is deflected. The deflection is assumed small enough that the beam never exceeds the position of detector 2 at detection. In figure 2.B a second mode, labeled \( \hat{b} \) and considered independent from the first mode, is added, so that detector 2 in this case collects photons from \( \hat{b} \) but also part of the photons from \( \hat{a} \) due to the deflection. This last configuration mimics the situation one have in wide field imaging where the object can be illuminated simultaneously by different modes at different positions. The following analysis refers to this elementary scheme, but the situation can be generalized to the situation in which a gradient is present all across the object, producing local deflection.

We develop a quantum statistical model in which the deflection in figure 2.B is represented as the result of a beam splitter (BS) acting on the mode \( \hat{a} \) as showed in figure 3.A. The BS is characterized by its transmission coefficient \( \tau \), the fraction of transmitted photons. The angle of deflection is then proportional to the reflectance \( 1 - \tau \) where the constant of proportionality depends on the particular spatial distribution of the mode. Estimating the angle of deflection of figure 2.B is then equivalent to the estimation of the coefficient \( \tau \) in the scheme 3.A.

2.1. Direct scheme and SNL

Referring to the configuration of figure 3.A the estimation of \( \tau \) can be carried out using the estimator \( \hat{E} \):

\[
\hat{E} = \frac{\hat{n}_1 - \hat{n}_2}{\hat{n}_1 + \hat{n}_2}
\]

where \( \hat{n}_1 \) and \( \hat{n}_2 \) are the photon number operators detected from detectors 1 and 2 respectively. The choice of this estimator, where the role of the denominator is to attenuate the fluctuations, follows from the fact that it allows to reach the Ultimate Quantum Limit in the estimation of a BS parameter when the second mode \( \hat{b} \) is not considered \[38\]. The estimator \( \hat{E} \) is defined using a ratio of operator and is mean value can be found expanding equation (1) for small fluctuations around the operator mean value, that at the zero-th order is just:

\[
\langle \hat{E} \rangle = \frac{\langle \hat{n}_1 - \hat{n}_2 \rangle}{\langle \hat{n}_1 + \hat{n}_2 \rangle} \approx \frac{\langle \hat{n}_1 \rangle - \langle \hat{n}_2 \rangle}{\langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle} = \frac{(2\tau - 1)N_a - N_b}{N_a + N_b}
\]

where \( N_a = \langle \hat{n}_a \rangle = \langle \hat{a}^\dagger \hat{a} \rangle \) and \( N_b = \langle \hat{n}_b \rangle = \langle \hat{b}^\dagger \hat{b} \rangle \) are the mean number of photons in modes \( \hat{a} \) and \( \hat{b} \). All the mean values \( \langle \cdot \rangle \) are taken on the initial state of the field, \( \rho_a \otimes \rho_b \). Those states will be specified by means of their photons statistics, and given the physical configuration under analysis, from now on we will consider \( \rho_a \) and \( \rho_b \) equal, as the two modes are produced by the same source. An estimation of \( \tau \) can be found solving equation (2) using the fact that \( N_a \) and \( N_b \) can be considered parameters as they can be determined with arbitrary accuracy in a preliminary characterization of the experimental apparatus, in absence of the sample under test.
The variance $\langle \Delta^2 \hat{E} \rangle$ can be obtained with the propagation of the uncertainty on $\hat{n}_1$ and $\hat{n}_2$ and expressed in terms of the statistic of the input modes $\hat{a}$ and $\hat{b}$.

From the well known BS relations $\hat{a}_t = \sqrt{\tau} \hat{a} + i \sqrt{1 - \tau} \hat{v}$ and $\hat{a}_r = i \sqrt{1 - \tau} \hat{a} + \sqrt{\tau} \hat{v}$, the statistic of the transmitted and reflected modes $\hat{a}_t$ and $\hat{a}_r$ is easily found to be:

$$\langle \hat{n}_t \rangle = \langle \hat{a}_t^\dagger \hat{a}_t \rangle = \tau N_a$$
$$\langle \hat{n}_r \rangle = \langle \hat{a}_r^\dagger \hat{a}_r \rangle = (1 - \tau) N_a$$
$$\langle \Delta^2 \hat{n}_t \rangle = \tau N_a (\tau F + 1 - \tau)$$
$$\langle \Delta^2 \hat{n}_r \rangle = N_a (1 - \tau) (F (1 - \tau) + \tau)$$
$$\langle \Delta \hat{n}_t \Delta \hat{n}_r \rangle = \tau (1 - \tau) N_0 (F - 1)$$

The Fano factor $F = \langle \Delta^2 \hat{n}_a \rangle / \langle \hat{n}_a \rangle$ was introduced to characterize the statistic of the input state. States with $F < 1$, i.e. characterized by sub-Poissonian fluctuation, are considered non-classical states of light. The statistic of $\hat{n}_1$ follows directly from relations (3) since from scheme A it coincides with $\hat{n}_1$. To determine the statistic of $\hat{n}_2$ we use the fact that $\hat{a}$ and $\hat{b}$ are independent so that we have:

$$\langle \hat{n}_2 \rangle = \langle \hat{n}_b \rangle + \langle \hat{n}_r \rangle$$
$$\langle \Delta \hat{n}_1 \Delta \hat{n}_2 \rangle = \langle \Delta \hat{n}_1 \Delta \hat{n}_r \rangle$$
$$\langle \Delta^2 \hat{n}_2 \rangle = \langle \Delta^2 \hat{n}_b \rangle + \langle \Delta^2 \hat{n}_r \rangle$$

So that for $\hat{n}_2$ we get:

$$\langle \hat{n}_2 \rangle = N_b + (1 - \tau) N_a$$
$$\langle \Delta^2 \hat{n}_2 \rangle = F N_b + (1 - \tau)^2 F N_a + \tau (1 - \tau) N_a$$
$$\langle \Delta \hat{n}_1 \Delta \hat{n}_2 \rangle = \tau (1 - \tau) (F N_a - N_a)$$

Fig. 3. A. Model of beam deflection. Schematic representation of the situation of fig. B. The deflection of the beam is modeled with a BS of transmission $0 \leq \tau \leq 1$, where $1 - \tau$ is proportional to the angle of deflection $\alpha$. B. Correlated scheme. The scheme pictures a deflection measurement. A correlate source is used to produce pairs of correlated modes, $\hat{a}$ correlated to $\hat{a}_c$ and $\hat{b}$ to $\hat{b}_c$. $\hat{a}$ and $\hat{b}$ probe the object, while their respective correlated modes are used as reference.
Using equations 3 and 5 we can propagate the uncertainty from equation 1. Assuming \( N_a = N_b = N \) we have:

\[
\langle \Delta^2 \hat{E} \rangle \approx \frac{F \tau^2}{2N} + \frac{\tau(1-\tau)}{N} \tag{6}
\]

This uncertainty can be propagated to the parameter \( \tau \) as:

\[
\Delta \tau = \sqrt{\frac{\langle \Delta^2 \hat{E} \rangle}{\partial \langle \hat{E} \rangle / \partial \tau}} \tag{7}
\]

So that:

\[
\Delta \tau = \sqrt{\frac{F \tau^2}{2N} + \frac{\tau(1-\tau)}{N}} \tag{8}
\]

The minimum fluctuation that can be achieved with "classical" states is the one obtained with coherent states, with \( F = 1 \), setting the SNL for this scheme:

\[
\Delta \tau_{SNL} = \sqrt{\frac{\tau^2}{2N} + \frac{\tau(1-\tau)}{N}} \tag{9}
\]

### 2.2. Correlated scheme

In order to take advantage of quantum correlations, we propose another scheme, depicted in figure 3.B. A source is used to produce spatially separated pairs of correlated modes. In picture 3.B the modes testing the object, \( \hat{a} \) and \( \hat{b} \), are correlated to the modes \( \hat{a}_c \) and \( \hat{b}_c \) respectively, that act as a reference. The aim of this scheme is to exploit correlations in photon numbers to improve the accuracy over the direct scheme. The degree of correlation, for a pair of generic modes \( \hat{i} \) and \( \hat{j} \), is expressed by the noise reduction factor \[30\] \( \sigma \) defined as:

\[
\sigma = \frac{\langle \Delta^2 (\hat{n}_i - \hat{n}_j) \rangle}{\langle \hat{n}_i + \hat{n}_j \rangle} \tag{10}
\]

With this configuration the parameter \( \tau \) can be computed using the estimator \( \hat{E}_C \):

\[
\hat{E}_C = \frac{\hat{n}_1 - (\hat{n}_2 - \hat{n}_2^c)}{\hat{n}_1^c} \tag{11}
\]

The choice of this estimator is arbitrary but motivated by the fact that the correlation of \( \hat{n}_2 \) and \( \hat{n}_2^c \) should allow to reduce the fluctuation of the bracket term at the numerator, meanwhile normalizing by \( \hat{n}_1^c \) compensates for the fluctuation of \( \hat{n}_1 \). For small fluctuation in photon numbers, the mean value can be approximated, as done before as:

\[
\langle \hat{E}_C \rangle \approx \frac{\langle \hat{n}_1 - (\hat{n}_2 - \hat{n}_2^c) \rangle}{\langle \hat{n}_1^c \rangle} = 2\tau - 1 \tag{12}
\]

The calculation of the uncertainty is similar to the one showed in the previous section and will not be reported. The result is:

\[
\Delta \tau_C = \sqrt{\frac{\tau(1-\tau)}{N} + \frac{(2\tau - 1)^2 \sigma}{4N} + \frac{\sigma}{2N}} \tag{13}
\]
that depends only on the measured mean number of photons $N$ in the reference beam and on the measured noise reduction factor in absence of the sample’s perturbation.

3. Results and Discussion

From equations 9 and 13 a comparison of the performance in the estimation with different input states can be made. In particular for the direct scheme, of section 2.1 we consider each mode of the multimode beam to be, alternatively, in one of the following states:

- The Fock state, eigenstate of the photon number operator of the field so that $F_{\text{Fock}} = 0$
- The coherent state, eigenstate of the annihilation operator with a Poissonian photon number distribution, hence $F_{\text{coh}} = 1$
- The thermal state, a mixed state characterized by the Bose-Einstein distribution at thermal equilibrium, $P(n) \equiv \frac{N^n}{(1+N)^n}$, having then $F_{\text{th}} = 1 + N$, $N$ being the main number of photons.

The differential scheme will analyzed in the case of the TWB state:

$$|\psi\rangle_{\text{TWB}} = \sum_n c(n)|n\rangle_{\tilde{k}_t,\omega}|n\rangle_{-\tilde{k}_t,-\omega}$$

where $\tilde{k}_t$ and $\omega$ are the transverse momentum and frequency of the mode and $|c(n)|^2$ is a thermal like distribution with parameter $N$. From 13 it is clear that the quantum nature of the state resides in its entanglement, as tracing out either one of the modes would give a thermal statistic for the other. Moreover it is easy to see that for this state, due to the perfect photon number correlation, the noise reduction factor is $\sigma_{\text{TWB}} = 0$.

In figure 4.A the uncertainty $\Delta\tau$ on the estimation is plotted against the parameter $\tau$ in the case of each of the states discussed. The curves are obtained by simply substituting the Fano factor of the different states considered in equation 9 for the direct scheme, and $\sigma = 0$ in 13 for the correlated case. The minimum uncertainty attainable in the estimation of a BS parameter is 38 reached, for every value of $\tau$ by both the TWB and the Fock state. It is not surprising that the Fock state reaches the lower bound to the uncertainty, since the estimation is based on photon number measurement, for which this state has no noise. When lossless channels are considered, the use of quantum correlations allows to erase the quantum noise present in the probe beam, by exploiting the the information on the photon number fluctuation measured in the reference beam, reproducing the situation in which the field is prepared in a Fock state. The coherent state, plotted in green, is a useful reference for the performance of the TWB state, since as mentioned before, the former represent the SNL and so the limit achievable with classical states. The advantage of TWB over the SNL gets more evident in the region of high $\tau$, corresponding to
Fig. 4. A. Uncertainty on the estimation of the beam splitter parameter $0 < \tau < 1$, modelling a beam deflection. Referring to scheme \(3\)A the input states are Fock(blue), coherent(green) and thermal state(orange). The TWB state result, the dashed line in red coinciding with the Fock state, refers to scheme \(3\)B. B. Uncertainty on the estimation of the beam splitter parameter $0 < \tau < 1$, modelling a beam deflection, with efficiency $\eta = 0.9$. The uncertainty of the measurement scheme \(3\)A is plotted in the case of optical efficiency $\eta = 0.9$, meaning that a fraction $1 - \eta$ of the initial number of photons are lost. The input states considered are Fock(blue), coherent(green) and thermal(orange) state. The TWB state result, plotted in red, refers to scheme \(3\)B, where the efficiency is considered $\eta = 0.9$ in both the probe and reference channel.

low deflections. The thermal state is, as expected, the worst one and is reported to show the disadvantage in the use of light modes in noisier states unless quantum correlation are used.

Up until now, possible photon losses have not been considered, although they are unavoidable in any real optical scheme. Since optical losses are random processes, that add a certain amount of noise, sub-Poissonian behavior and quantum correlations are strongly affected by them. The Fano factor and the NRF measured in case of a fraction $0 \leq 1 - \eta \leq 1$ of photons lost in the channel are:

\[
F_\eta = \eta F + 1 - \eta \\
\sigma_\eta = \eta \sigma + 1 - \eta
\]

where for $\sigma_\eta$ equal losses on the correlated channels have been assumed.

In figure 4B the uncertainty is reported in the case of an high, but not perfect, efficiency, $\eta = 0.9$, evaluated by substituting expressions 15 into equations 9 and 13. In this scenario the performance of the TWB state does not coincide anymore with the one of the Fock state but it becomes slightly worse. An interesting feature is that the uncertainty of the TWB estimation does not approach zero as $\tau \to 0$ and as a consequence the TWB performs worst than any other configuration in the high deflections region. This is a consequence of the choice of 11 as an estimator, and can be eliminated with a different one. The advantage, however, of 11 over other tested estimators, and the reason why it has been chosen here, is that it allows to improve the sensitivity for small deflections, the one we are more interested in. In this region the TWB state approaches the result of the Fock state, even in presence of losses, and gives a sensible improvement over the SNL.

Finally in figure 5 we report the maximum value of the detected noise reduction...
Quantum enhanced imaging of non uniform refractive profiles

4. Conclusions

In this work, a simple quantum model describing the measurement of a refractive profile, based on the change of the intensity distribution of a beam after the interaction with a sample, has been elaborated to investigate a possible quantum enhancement in the sensitivity. The deflections caused on the spatially multimode beam interacting with the test object, were modeled using a beam splitter transformation with transmission coefficient $\tau$, where the angle of deflection $\alpha$ is proportional to $1 - \tau$. A direct measurement scheme was compared to a correlated one, where quantum correlations are used to improve the accuracy. In particular we found that the TWB state, a state characterized by entanglement in photon number between pairs of spatio-temporal modes, overcomes the Shot noise limit (SNL) both in the ideal lossless case, reported in figure 4A and in presence of losses shown in figure 4B. Moreover, we have shown that only a correlation level well above the classical bound (noise reduction factor $\sigma_{\text{max}} < 0.7$) allows to overcome the SNL, as reported in figure 5.

This results show the possibility to reach a quantum enhancement for wide field imaging of refractive profiles inducing an intensity perturbation in the near field, using a TWB configuration. The analysis performed in this work is meant to be followed by a wide field experimental realization of the differential scheme with the
TWB state. Twin beams are, in fact, currently routinely generated in quantum optics laboratories, and they have already been used for sub shot noise imaging of absorption profiles. Thus the scheme suggested in this work for refractive profile measurements is feasible with the current technology.

Realizing sub SNL wide field imaging is especially important when there is a limit on the energy that can be used to probe samples. For this reason sub SNL imaging of refractive profiles would have useful application, for example, in the analysis of quasi transparent biological sample, giving complementary information to the one obtained using other measurements.

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