Trigonometric integrable tops from solutions of associative Yang-Baxter equation

T. Krasnov ♦ ♯ A. Zotov ♦ ♯ § ♮

♦ – Steklov Mathematical Institute of Russian Academy of Sciences, Gubkina str. 8, Moscow, 119991, Russia
♯ – ITEP, B. Cheremushkinskaya str. 25, Moscow, 117218, Russia
§ – National Research University Higher School of Economics, Usacheva str. 6, Moscow, 119048, Russia
♮ – Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudny, Moscow region, 141700, Russia

E-mails: timofei.krasnov@phystech.edu, zotov@mi-ras.ru

Abstract

We consider a special class of quantum non-dynamical $R$-matrices in the fundamental representation of $GL_N$ with spectral parameter given by trigonometric solutions of the associative Yang-Baxter equation. In the simplest case $N = 2$ these are the well-known 6-vertex $R$-matrix and its 7-vertex deformation. The $R$-matrices are used for construction of the classical relativistic integrable tops of the Euler-Arnold type. Namely, we describe the Lax pairs with spectral parameter, the inertia tensors and the Poisson structures. The latter are given by the linear Poisson-Lie brackets for the non-relativistic models, and by the classical Sklyanin type algebras in the relativistic cases. In some particular cases the tops are gauge equivalent to the Calogero-Moser-Sutherland or trigonometric Ruijsenaars-Schneider models.
1 Introduction

In this paper we discuss GL$_N$ integrable Euler-Arnold type tops [3] defined by the equations of motion

$$\dot{S} = [S, J(S)], \quad S = \sum_{i,j=1}^{N} E_{ij} S_{ij} \in \text{Mat}(N, \mathbb{C}),$$

(1.1)

where $\{S_{ij}, i, j = 1, ..., N\}$ is the set of dynamical variables, $\{E_{ij}\}$ is the standard basis in Mat($N, \mathbb{C}$), and $J(S)$ is a linear map on $S$

$$J(S) = \sum_{i,j,k,l=1}^{N} J_{ijkl} E_{ij} S_{lk} \in \text{Mat}(N, \mathbb{C})$$

(1.2)

with components $J_{ijkl}$ independent of dynamical variables. The model is not integrable in the general case but for special choices of $J(S)$ only. The construction of integrable tops under consideration goes back to E. Sklyanin’s paper [29] (see also [12]). The idea was to formulate the classical analogue of the models described by the inverse scattering method. In this way the classical spin chains were described and the quadratic Poisson structures were obtained via the classical limit of the exchange (RLL) relations.

The GL$_N$ top can be viewed as the model obtained through the classical limit from the 1-site spin chain. The rational models of this type were described in [1, 18]. Here we use a specification of the above mentioned results based on trigonometric $R$-matrices satisfying the associative Yang-Baxter equation (AYBE) [13, 23]:

$$R^{h}_{12}(z_{12}) R^{\eta}_{23}(z_{23}) = R^{\eta}_{13}(z_{13}) R^{h}_{12}(z_{12}) + R^{\eta-h}_{23}(z_{23}) R^{h}_{13}(z_{13}), \quad z_{ab} = z_{a} - z_{b}.$$  

(1.3)

It was shown in [20] that solution of (1.3) satisfying also additional properties of skew-symmetry:

$$R^{h}_{12}(z) = - R^{h}_{21}(-z) = - P_{12} R^{h}_{12}(-z) P_{12}, \quad P_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji},$$

(1.4)

Equations (1.4) describe rotation of a rigid body in $N$-dimensional (complex) space. In this respect $J(S)$ is the inverse inertia tensor.

$P_{12}$ in (1.4) and below is the permutation operator. In particular, for any pair of matrices $A, B \in \text{Mat}(N, \mathbb{C})$ with $\mathbb{C}$-valued matrix elements: $(A \otimes B) P_{12} = P_{12} (B \otimes A)$. 

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1
unitarity
\[ R^h_{12}(z) R^h_{21}(-z) = f^h(z) \ 1_N \otimes 1_N \] (1.5)
and the local expansions³
\[ \text{Res}_{\hbar=0} R^h_{12}(z) = 1_N \otimes 1_N = 1_{N^2}, \quad \text{Res}_{z=0} R^h_{12}(z) = P_{12} \] (1.6)
(1_N – is N × N identity matrix) leads to explicit constructions of the Lax pair L(z), M(z) ∈ Mat(N, ℂ). That is the Lax equations
\[ \dot{L}(z) = [L(z), M(z)] \] (1.7)
are equivalent to the equations of motion (1.1) identically in spectral parameter z. All the data of the models including their Hamiltonians, the Lax pairs, the Poisson structures and the inertia tensors (i.e., J(S)) are given in terms of coefficients of expansion of the R-matrices near \( \hbar = 0 \) and \( z = 0 \). For example, in the relativistic case the Lax pair is as follows:
\[ L^n(z) = \text{tr}_2 (R^h_{12}(z) S_2), \quad M^n(z) = -\text{tr}_2 (r_{12}(z) S_2), \] (1.8)
where \( S_2 = 1_N \otimes S \), and \( r_{12}(z) \) – is the classical r-matrix. See Section 3 for details. The Planck constant plays the role of the relativistic deformation parameter \( \eta \). In some special case it is identified with the corresponding parameter in the Ruijsenaars-Schneider model.

Notice that together with the properties (1.4) and (1.5) a solution of (1.3) satisfies also the custom Yang-Baxter equation
\[ R^h_{12}(z_1 - z_2) R^h_{13}(z_1 - z_3) R^h_{23}(z_2 - z_3) = R^h_{23}(z_2 - z_3) R^h_{13}(z_1 - z_3) R^h_{12}(z_1 - z_2), \] (1.9)
so that such solution of (1.3) is then a true quantum R-matrix by convention. Sometimes the following property holds true as well⁴:
\[ R^h_{12}(z) P_{12} = R^z_{12}(\hbar). \] (1.10)
This allows to relate the coefficients of expansion (of R-matrices) near \( \hbar = 0 \) and \( z = 0 \) with each other.

The paper is organized as follows. In Section 2 we describe the set of well-known trigonometric R-matrices satisfying conditions (1.3)–(1.6), and briefly describe the general classification of such solutions of (1.3) suggested by T. Schedler and A. Polishchuk [30, 24]. We will show that a representative example of the classification is given by the so-called non-standard trigonometric R-matrix [2], which generalizes the GL₂ 7-vertex R-matrix [9] for \( N > 2 \). In Section 3 we review the construction of integrable tops and evaluate all the data for the general case and the non-standard R-matrix. Using (1.3) we also prove that the classical quadratic r-matrix structure provides the classical Sklyanin type Poisson structure. This results in getting the classification of the trigonometric Sklyanin Poisson structures, and it is parallel to the classification of solutions of the associative Yang-Baxter equation. In Section 4 we consider a special top corresponding to rank one matrix S and related to the non-standard R-matrix. It turns out that this model is gauge equivalent to the Ruijsenaars-Schneider [26] or the Calogero-Moser-Sutherland [8] models. Explicit changes of variables are described.

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³Here we imply that R-matrices have simple poles at \( z = 0 \) and \( \hbar = 0 \) only, and no higher order poles. The classical limit is given by the expansion near \( \hbar = 0 \). It is the first one condition in (1.10). See [30, 24].

⁴The condition (1.10) is related to the finite Fourier transformations. See [30] for details.
2 Trigonometric $R$-matrices and AYBE

We begin with the properties of well-known $R$-matrices and then proceed to the general case.

2.1 Standard and non-standard $R$-matrices

Consider the following examples of $R$-matrices

\[ R^n_{12}(z) = \sum_{i,j,k,l=1}^{N} R^n_{ijkl}(z) E_{ij} \otimes E_{kl} \]  

(2.1)

- The $\mathbb{Z}_N$-invariant $A_{N-1}$ trigonometric $R$-matrix [9, 22, 16]:

\[ (R_1)^n_{ijkl}(z) = \delta_{ij} \delta_{kl} \delta_{ik} \frac{N}{2} \left( \coth(Nz/2) + \coth(N\eta/2) \right) + \]

\[ + \delta_{ij} \delta_{kl} \varepsilon(i \neq k) \frac{N e^{(i-k)\eta - \text{sign}(i-k)N\eta/2}}{2 \sinh(N\eta/2)} + \delta_{il} \delta_{kj} \varepsilon(i \neq k) \frac{N e^{(i-k)z - \text{sign}(i-k)Nz/2}}{2 \sinh(Nz/2)}, \]

where hereinafter we use

\[ \varepsilon(A) = \begin{cases} 
1, & \text{if } A \text{ is true}, \\
0, & \text{if } A \text{ is false}.
\end{cases} \]  

(2.2)

- Baxterization of the (trigonometric) Cremmer-Gervais $R$-matrix [4, 2]:

\[ (R_2)^n_{ijkl}(z) = \delta_{ij} \delta_{kl} \delta_{ik} \frac{N}{2} \left( \coth(Nz/2) + \coth(N\eta/2) \right) + \]

\[ + \delta_{ij} \delta_{kl} \varepsilon(i \neq k) \frac{N e^{(i-k)\eta - \text{sign}(i-k)N\eta/2}}{2 \sinh(N\eta/2)} + \delta_{il} \delta_{kj} \varepsilon(i \neq k) \frac{N e^{(i-k)z - \text{sign}(i-k)Nz/2}}{2 \sinh(Nz/2)} + \]

\[ + N \delta_{i+k,j+l} \left( \varepsilon(i<j<k) e^{(i-j)z+(j-k)\eta} - \varepsilon(k<j<i) e^{(i-j)z+(j-k)\eta} \right). \]

(2.4)

It differs from the previous one [2,2] by the last line. Let us comment on how it is related to the Cremmer-Gervais $R$-matrix. First, one should perform the gauge transformation

\[ R^n_{12}(z - w) \rightarrow \tilde{R}^n_{12}(z, w) = D_1(z) D_2(w) R^n_{12}(z) D_1^{-1}(z) D_2^{-1}(w) \]  

(2.5)

with the diagonal matrix $D_{ij}(z) = \delta_{ij} e^{-jz}$. For (2.1) $\tilde{R}^n_{12}(z, w) = \tilde{R}^n_{12}(z - w)$. The result is

\[ (\tilde{R}_2)^n_{ijkl}(z) = \delta_{ij} \delta_{kl} \delta_{ik} \frac{N}{2} \left( \coth(Nz/2) + \coth(N\eta/2) \right) + \]

\[ + \delta_{ij} \delta_{kl} \varepsilon(i \neq k) \frac{N e^{(i-k)\eta - \text{sign}(i-k)N\eta/2}}{2 \sinh(N\eta/2)} + \delta_{il} \delta_{kj} \varepsilon(i \neq k) \frac{N e^{\text{sign}(i-k)Nz/2}}{2 \sinh(Nz/2)} + \]

\[ + N \delta_{i+k,j+l} \left( \varepsilon(i<j<k) e^{(j-k)\eta} - \varepsilon(k<j<i) e^{(j-k)\eta} \right). \]

(2.6)
Consider the Cremmer-Gervais $R$-matrix $[10]$. It is free of spectral parameter:

$$R_{12}^{CG,a} = q^{-1/N} \left( q \sum_{i=1}^{N} E_{ii} \otimes E_{ii} + q \sum_{i>j}^{N} q^{-2(i-j)/N} E_{ii} \otimes E_{jj} + q^{-1} \sum_{i<j}^{N} q^{-2(i-j)/N} E_{ii} \otimes E_{jj} - (q - q^{-1}) \sum_{i<j}^{N} \sum_{k=1}^{j-i-1} q^{2k/N} E_{ij-k,i} \otimes E_{i+k,j} + (q - q^{-1}) \sum_{i>j}^{N} \sum_{k=0}^{i-j-1} q^{-2k/N} E_{j+k,i} \otimes E_{i-k,j} \right).$$

(2.7)

Next, introduce

$$R_{12}^{CG,a}(x) = x R_{12}^{CG,a} - x^{-1} \left( R_{12}^{CG,a} \right)^{-1}. \tag{2.8}$$

Finally,

$$(\tilde{R}_2)^{\eta}_{12}(z) = \frac{N}{4 \sinh(Nz/2) \sinh(N\eta/2)} R_{12}^{CG,a}(x)^T, \tag{2.9}$$

where ”$T$” means the transpose of matrix $(R_{ij,kl} \rightarrow R_{ji,ik})$ and $x = e^{-\eta/2-Nz/2}$, $q = e^{-N\eta/2}$.

- **Non-standard trigonometric $R$-matrix $[2]$:**

$$R_{ij,kl}^{\eta}(z) = \delta_{ij} \delta_{kl} \delta_{ik} \frac{N}{2} \left( \coth(Nz/2) + \coth(N\eta/2) \right) +$$

$$+ \delta_{ij} \delta_{kl} \delta_{ik} N e^{(i-k)\eta} \frac{\text{sign}(i-k)N\eta/2}{2 \sinh(N\eta/2)} + \delta_{ij} \delta_{kl} \delta_{ik} N e^{(i-k)z} \frac{\text{sign}(i-k)Nz/2}{2 \sinh(Nz/2)} +$$

$$+ N \delta_{i+k,j+l} \left( \varepsilon(i < j < k) e^{(i-j)z + (j-k)\eta} - \varepsilon(k < j < i) e^{(i-j)z + (j-k)\eta} \right) +$$

$$+ N \delta_{i+k,j+l+N} \left( \delta_{i,N} e^{-jz-l\eta} - \delta_{k,N} e^{iz+j\eta} \right). \tag{2.10}$$

It differs from the previous one $[2,4]$ by the last line, which provides in $N = 2$ case the 7-vertex deformation $[9]$ of the 6-vertex $R$-matrix.

**Properties of $R$-matrices.**

Briefly, all the $R$-matrices $[2,2]$, $[2,4]$ and $[2,10]$ satisfy the associative Yang-Baxter equation $[13]$, the skew-symmetry property $[1,4]$, the unitarity property $[1,5]$, and therefore, the Yang-Baxter equation $[1,9]$. Moreover, all of them satisfy the Fourier symmetry $[1,10]$. The gauge transformed $R$-matrix $[2,6]$ does not satisfy $[1,10]$ while the rest of the properties hold true.

In order to summarize the properties of the above $R$-matrices introduce notations for the last lines of $[2,4]$ and $[2,10]:$ $\Delta_1 R_{ij,k,l}^{\eta}(z) = (R_{ij,k,l}^{R}(z) - (R_{ij,k,l}^{1}(z)$ and $\Delta_2 R_{ij,k,l}^{\eta}(z) = (R_{ij,k,l}^{R}(z) - (R_{ij,k,l}^{2}(z), i.e.

$$\Delta_1 R_{ij,k,l}^{\eta}(z) = N \delta_{i+k,j+l} \left( \varepsilon(i < j < k) e^{(i-j)z + (j-k)\eta} - \varepsilon(k < j < i) e^{(i-j)z + (j-k)\eta} \right), \tag{2.11}$$

$$\Delta_2 R_{ij,k,l}^{\eta}(z) = N \delta_{i+k,j+l+N} \left( \delta_{i,N} e^{-jz-l\eta} - \delta_{k,N} e^{iz+j\eta} \right) \tag{2.12}$$

and consider the following linear combination:

$$R_{ij,kl}^{\eta}(z) = A_0 (R_{ij,kl}^{1}(z) + A_1 \Delta_1 R_{ij,kl}^{\eta}(z) + A_2 \Delta_2 R_{ij,kl}^{\eta}(z)), \tag{2.13}$$

where $A_0$, $A_1$ and $A_2$ are some constants. For example, for $A_0 = A_1 = A_2 = 1$ $[2,13]$ yields $[2,10]$. To summarize:
Proposition 2.1 For any \( A_0, A_1 \) and \( A_2 \) (2.13) satisfies the properties (1.4), (1.10) and (1.5) with
\[
f^\eta(z) = A_0^2 N^2 \left( \frac{1}{\sinh^2(N\eta/2)} - \frac{1}{\sinh^2(Nz/2)} \right),
\]
that is (2.13) is non-degenerated iff \( A_0 \neq 0 \).

The associative Yang-Baxter equation (1.3) holds true for all \( R \)-matrices (2.2), (2.4) and (2.10).

The linear combination (2.13) satisfies (1.3) in the following cases:
1. \( A_0 = A_1 \neq 0, A_2 \) - any,
2. \( A_0 \neq 0, A_1 = A_2 = 0 \)
3. \( A_0 = A_1 = 0, A_2 \) - any.

The latter means that the \( R \)-matrix (2.12) satisfies (1.3).

Let us also mention two special cases:
a.) In the case \( 5 \) \( N = 2, 3 \) the combination (2.13) satisfies (1.3) for \( A_0, A_1 \) - any, and \( A_2 = 0 \).

b.) for \( N = 4 \) and \( A_0 = A_2 = 0 \) (2.13) does not satisfy (1.3) while the Yang-Baxter equation (1.9) holds true.

The case 2 from the Proposition can be verified directly. Instead of a direct proof of cases 1. and 3. we will show (in the next paragraph) that the non-standard \( R \)-matrix (2.10) is contained in the general classification. Next, we can apply the gauge transformation (2.5) with
\[
D_{ij} = \delta_{ij} e^{-j\Lambda}
\]
(2.15) to (2.10). In terms of components it leads to \( R_{ij,kl}^\eta(z) \rightarrow e^{(j+l-i-k)\Lambda} R_{ij,kl}^\eta(z) \). Therefore, the last line of (2.10) is multiplied by \( e^{-N\Lambda} \):
\[
R_{ij,kl}^\eta(z) = \delta_{ij}\delta_{kl}\frac{N}{2} \left( \coth(Nz/2) + \coth(N\eta/2) + \sum_{i \neq k} N e^{(i-k)\eta - \text{sign}(i-k)N\eta/2} \frac{\delta_{il}\delta_{jk}(i \neq k)}{2\sinh(N\eta/2)} + \sum_{i < j} N e^{(i-j)z + (j-k)\eta} \delta_{ij}\delta_{kl} \right) + N e^{-N\Lambda} \frac{\delta_{i+k,j+l+N}}{\sinh(N\eta/2)} \left( \delta_{iN} e^{-jz-l\eta} - \delta_{kN} e^{i z + j \eta} \right).
\]
(2.16)

By taking the limit \( \Lambda \rightarrow \pm \infty \) we come to the cases 1. with \( A_2 = 0 \) or to the case 3.

At last, consider
- \( R \)-matrix for the affine quantized algebra \( \hat{U}_q(gl_N) \) [14, 25]:
\[
R_{12}^{xy,\eta}(z) = \frac{N}{2} \left( \coth(Nz/2) + \coth(N\eta/2) \right) \sum_{i=1}^{N} E_{ii} \otimes E_{ii} + \frac{(N/2)}{\sinh(N\eta/2)} \sum_{i \neq j} E_{ii} \otimes E_{jj} + \frac{(N/2)}{\sinh(Nz/2)} \sum_{i < j} \left( E_{ij} \otimes E_{ji} e^{Nz/2} + E_{ji} \otimes E_{ij} e^{-Nz/2} \right).
\]
(2.17)

\( ^{5} \)In fact, for \( N = 2 \) case \( A_1 \) is not necessary since \( \Delta_1 R^\eta(z) = 0 \) in this case.
It is used for construction of GL\(_N\) XXZ spin chains and is usually written in different normalization:

\[
R_{12}^{xxz,q}(x) = \frac{4}{N} \sinh(Nz/2) \sinh(N\eta/2) R_{12}^{xxz,\eta}(z) = \\
= \left( xq - \frac{1}{xq} \right) \sum_{i=1}^{N} E_{ii} \otimes E_{ii} + \left( x - \frac{1}{x} \right) \sum_{i \neq j}^{N} E_{ii} \otimes E_{jj} + \left( q - \frac{1}{q} \right) \sum_{i \neq j}^{N} x^{\text{sign}(j-i)} E_{ij} \otimes E_{ji},
\]  

(2.18)

where \( x = e^{Nz/2}, \ q = e^{N\eta/2} \). The XXZ \( R \)-matrix is the Baxterization of the Drinfeld’s one [11]:

\[
\left( R_{12}^{\text{Dr},q} \right)^\pm = q^{\pm 1} \sum_{i=1}^{N} E_{ii} \otimes E_{ii} + \sum_{i \neq j}^{N} E_{ii} \otimes E_{jj} \pm (q - q^{-1}) \sum_{i > j}^{N} E_{ij} \otimes E_{ji}.
\]  

(2.19)

Namely, 

\[
\tilde{R}_{12}^{xxz,q}(x) = x R_{12}^{\text{Dr},q} - x^{-1} \left( R_{12}^{\text{Dr},q} \right)^{-1}.
\]  

(2.20)

The \( R \)-matrix (2.17) satisfies Yang-Baxter equation (1.9). It is skew-symmetric and unitary (1.5) with

\[
f^n(z) = \frac{N^2}{4} \left( \frac{1}{\sinh^2(N\eta/2)} - \frac{1}{\sinh^2(Nz/2)} \right).
\]  

(2.21)

The associative Yang-Baxter equation (1.3) for (2.17) holds true in the \( N = 2 \) case. For \( N > 2 \) the difference of the l.h.s. and the r.h.s. from (1.3) is not zero though it is independent of spectral parameters:

\[
R_{12}^h(z_{12}) R_{23}^\eta(z_{23}) - R_{13}^\eta(z_{13}) R_{12}^h(z_{12}) - R_{23}^{-\eta}(z_{23}) R_{13}^\eta(z_{13}) = \\
= - \frac{N^2}{8 \cosh(N\hbar/4) \cosh(N\eta/4) \cosh(N(h - \eta)/4)} \sum_{i \neq j \neq k \neq i}^{N} E_{ii} \otimes E_{jj} \otimes E_{kk}.
\]  

(2.22)

The latter statement is verified by direct computation. We do not consider the XXZ \( R \)-matrix for construction of integrable tops in this paper. It is of course possible, but our method requires (1.3) to be valid.

2.2 General classification

Here we briefly describe the classification [30, 24] of trigonometric solutions to associative Yang-Baxter equation (1.3) with the properties of skew-symmetry (1.4) and unitarity (1.5). As noted previously, this is sufficient condition for satisfying the Yang-Baxter equation (1.9) as well. So that we deal with the quantum non-dynamical \( R \)-matrices. Another goal of the Section is to show how the non-standard \( R \)-matrix (2.10) arises from the classification.

General solution of (1.3) is given in terms of combinatorial construction called the associative Belavin-Drinfeld structure. Consider \( S = \{1, ..., N\} \) – a finite set of \( N \) elements. Say, \( S \) is the set of \( N \) vertices on a circle numerated from 1 to \( N \) (the extended Dynkin diagram of \( A_{N-1} \) type). Let \( C_0 \) be a transitive cyclic permutation acting on \( S \), and \( \Gamma_{C_0} \) be its graph, i.e. the set of ordered pairs \( \Gamma_{C_0} = \{(s, C_0(s)) \mid s \in S\} \).

Define another one transitive cyclic permutation \( C \) and a pair of proper subsets \( \Gamma_1, \Gamma_2 \subset \Gamma_{C_0} \) related by \( C \): \( (C \times C) \Gamma_1 = \Gamma_2 \), where the action means \( (C \times C)(i,j) = (C(i), C(j)) \). So that \( C \times C \) provides the induced bijective map \( \tau \): \( \Gamma_1 \overset{C \times C}{\rightarrow} \Gamma_2 \). The set \((\Gamma_1, \Gamma_2, \tau)\) is an example of the Belavin-Drinfeld triple [6].
Here the action of $\tau$ is extended to larger sets. Namely, it is extended to $\tau : P_1 \xrightarrow{C \times C} P_2$, where $P_{1,2}$ are the following sets:

$$P_i = \{(s, C_0^k(s)) : (s, C_0(s)) \in \Gamma_i, \ldots, (C_0^{k-1}(s), C_0^k(s)) \in \Gamma_i, (C_0^k(s), C_0^{k+1}(s)) \notin \Gamma_i\}. \quad (2.23)$$

From the transitivity of $C$ and the choice of $\Gamma_{1,2}$ to be proper subsets of $\Gamma_{C_0}$ it follows that there exists a number $k$ such that $(C \times C)^k \Gamma_1 \notin \Gamma_1$. Similarly, there exist $k_1, k_2$ with the property $(C_0 \times C_0)^{k_i+1} \Gamma_i \notin \Gamma_i$, $i = 1, 2$. Therefore, $P_i$ are well-defined finite sets, and $\tau$ is the bijective map between them.

Then the general answer for trigonometric $R$-matrix based on $(C_0, C, \Gamma_1, \Gamma_2)$ is as follows:

$$R_{12}^B(z) = \frac{N}{2} \left( \coth(Nz/2) + \coth(N\eta/2) \right) \sum_i E_{ii} \otimes E_{ii} +$$

$$+ \sum_{0 < n < N, i = C^n(k)} \frac{e^{m_n} E_{ii} \otimes E_{kk}}{e^{Nz} - 1} + \sum_{0 < m < N, k = C^n(i)} \frac{e^{m_z} E_{ik} \otimes E_{kl}}{e^{Nz} - 1} +$$

$$+ \sum_{0 \leq m < N, \eta > 0, i = C^n(j), \tau^n(j, i) = (k, l)} \left( e^{-m_n - m_z} E_{ij} \otimes E_{kl} - e^{m_n + m_z} E_{kl} \otimes E_{ij} \right), \quad (2.24)$$

where the sums are over all possible values of indices – elements of $S$. In particular, the last sum is over all $i,j,k,l \in \{1, \ldots, N\}$ and positive $m,n$ for which the $\tau^n(j, i)$ is defined, i.e. $(j, i) \in P_1$ and $\tau^n(j, i) = (k, l) \in P_2$ with $i = C^n(j)$. The $R$-matrix is skew-symmetric and unitary \[^{[1,5]}\] with $f^n(z) \ [221]$. The answer \[^{(2.24)}\] is given up to some gauge transformations. See the details in \[^{[30, 24]}\].

**Example.** Consider example with the cyclic permutations

$$C_0 : \begin{array}{c|c} s & C_0(s) \\ \hline 1 & N \\ 2 & 1 \\ 3 & 2 \\ \vdots & \vdots \\ N & N-1 \\ N-1 & 1 \\ \end{array}$$

and the proper subsets $\Gamma_{1,2} \subset \Gamma_{C_0} = \{(s, C_0(s))\}$ given by

$$\Gamma_1 = \left\{(1, N), (2, 1), (3, 2), \ldots, (N - 1, N - 2)\right\}, \quad (2.26)$$

$$\Gamma_2 = (C \times C) \Gamma_1 = \left\{(2, 1), (3, 2), \ldots, (N - 1, N - 2), (N, N - 1)\right\}. \quad (2.27)$$

To construct $P_1$ consider the action of $C_0 \times C_0$ on the elements of $\Gamma_1 \ [2.26]$

$$\begin{align*}
(1, N) &\xrightarrow{C_0 \times C_0} (N, N - 1) \notin \Gamma_1, \\
(2, 1) &\xrightarrow{C_0 \times C_0} (1, N) \xrightarrow{C_0 \times C_0} (N, N - 1) \notin \Gamma_1, \\
(3, 2) &\xrightarrow{C_0 \times C_0} (2, 1) \xrightarrow{C_0 \times C_0} (1, N) \xrightarrow{C_0 \times C_0} (N, N - 1) \notin \Gamma_1, \\
&\vdots \\
(N - 1, N - 2) &\xrightarrow{C_0 \times C_0} \ldots \xrightarrow{C_0 \times C_0} (2, 1) \xrightarrow{C_0 \times C_0} (1, N) \xrightarrow{C_0 \times C_0} (N, N - 1) \notin \Gamma_1.
\end{align*} \quad (2.28)$$
According to the definition (2.25) we get the following set for \( P_1 \):

\[
P_1 = \left\{ (1, N), \ (2, 1), \ (2, N), \ (3, 2), \ (3, 1), \ (3, N), \ldots \ (N-1, N-2), (N-1, N-3), \ldots, (N-1, 1), (N-1, N) \right\} \tag{2.29}
\]

In a similar way from (2.27) we obtain the set of \( P_2 \):

\[
P_2 = (C \times C)P_1 = \left\{ (2, 1), \ (3, 2), \ (3, 1), \ (4, 3), \ (4, 2), \ (4, 1), \ldots \ (N, N-1), (N, N-2), \ldots, (N, 2), (N, 1) \right\} \tag{2.30}
\]

The bijection between \( P_1 \) and \( P_2 \) induced by \( C \times C \) is the map \( \tau \).

**Proposition 2.2** The \( R \)-matrix (2.24) reproduces the non-standard one (2.10) for the case of the associative Belavin-Drinfeld structure (2.25)-(2.27).

**Proof:** The first lines of (2.24) and (2.10) coincide. Consider the first term from the second line of (2.24):

\[
\frac{N}{e^{N\eta} - 1} \sum_{0 < n < N, i = C^n(k)} N e^{-\frac{N\eta}{2}} \sum_{0 < n < N, i = C^n(k)} e^{mn} E_{ii} \otimes E_{kk} \tag{2.31}
\]

Due to the definition of \( C \) (2.25) for the summation index \( n \) we have: \( n = i-k \) if \( i > k \) and \( n = N-k+i \) for \( i < k \). In this way we reproduce the first term in the second line of (2.10). Similar consideration for the second term in the second line of (2.24) yields that the total second line of (2.24) coincides with the second line of (2.10).

Next, consider the first sum in the last line of (2.24) and subdivide it into two parts:

\[
\sum_{0 < m < N, n > 0, \ i = C^m(j), \ e^\eta(j, i) = (k, l)} Ne^{-\frac{m\eta}{2}} E_{ij} \otimes E_{kl} = \left( \sum' + \sum'' \right) Ne^{-\frac{m\eta}{2}} E_{ij} \otimes E_{kl} \tag{2.32}
\]

where the sums \( \sum' \) and \( \sum'' \) are defined as follows. The total sum is over \( i, j, k, l \) such that \( (j, i) \in P_1 \) and \( (k, l) \in P_2 \). Then the sum \( \sum'' \) is over the digonal elements \( (1, N), \ldots, (N-1, N) \) among \( (j, i) \in P_1 \) (2.29), and the sum \( \sum' \) is over the rest of the elements among \( (j, i) \in P_1 \) (it is the lower triangular part of (2.29)).

From (2.29) and (2.30) it follows that \( j > i \) and \( k > l \) for the elements in the \( \sum' \). Moreover, \( i + k = j + l \) for these elements, and \( k > j \) since the map \( P_1 \rightarrow P_2 \) is generated by \( C \times C \). Therefore, \( i < j < k \) holds true. Also, from \( i = C^m(j) \) we have \( m = j - i \). And finally, \( C^m(j) = k \), so that \( n = k - j \). In this way we showed that the sum \( \sum' \) provides the first term in the third line of (2.10).

For the elements of the sum \( \sum'' \) we have \( i = N > j \) and \( i + k = j + l + N \). Since \( N = C^m(j) \) we have \( j = m \). On the other hand \( C^m(N) = k \), so that \( k = n \). In this way the sum \( \sum'' \) is shown to be the first term in the last line of (2.10).

\footnote{Condition \( i+k = j+l \) is verified directly for \( n = 1 \) by comparing (2.29) and (2.30). To make next application of \( \tau \) one should determine Image(\( \tau \)) \( \cap P_1 \subset P_1 \), i.e. each time we return back to a subset in \( P_1 \). This is why condition \( i+k = j+l \) is independent of \( n \).}
In the same way (by subdividing into two parts) the second term in the last line of (2.31) is shown to be equal to the sum of the second terms in the third and the fourth lines of (2.10).

Let us comment on the origin of the general classification. It comes from non-trivial limiting procedures (trigonometric limits) \[2, 7, 28\] starting from the elliptic case, where the classification is rather simple. It is based on the M. Atiyah’s classification of bundles over elliptic curves. The elliptic \(R\)-matrix is fixed by its poles structure (1.6) and quasiperiodic boundary conditions on a torus given by powers of \(N \times N\) matrices \(I_1^k, I_2^l \ (k, l = 1, ..., N - 1)\), where \(I_1 = \text{diag}(\exp(4\pi i/N), \exp(2\pi i/N), ..., 1)\) and \((I_2)_{ij} = \varepsilon(i = j + 1 \mod N)\). The non-dynamical \(R\)-matrix corresponds to \(\gcd(k, N) = 1\) and \(\gcd(l, N) = 1\). Otherwise, elliptic moduli appear, which play the role of dynamical variables.

### 3 Integrable tops

Below we describe the relativistic and the non-relativistic tops constructed by means of \(R\)-matrices satisfying (1.3)-(1.6). Our consideration uses results of \[18, 20\]. For the relativistic models the classical \(r\)-matrix structure is quadratic, while in the non-relativistic case it is linear. In its turn the relativistic models admit two natural (and equivalent) Lax representations: the first one includes explicit dependence on the relativistic parameter \(\eta\). It is based on the quantum \(R\)-matrix. And the second one is based on the classical \(r\)-matrix. The Lax pair in this description is independent of \(\eta\).

Consider a solution of the associative Yang-Baxter equation (1.3) with the properties\(^7\) (1.4) and (1.5) and the following expansions near \(\hbar = 0\) (the classical limit):

\[
R_{12}^h(z) = \frac{1}{h} 1_N \otimes 1_N + r_{12}(z) + h m_{12}(z) + O(h^2)
\]

and near \(z = 0\)

\[
R_{12}^h(z) = \frac{1}{z} P_{12} + R_{12}^{h,(0)} + z R_{12}^{h,(1)} + O(z^2),
\]

\[
R_{12}^{h,(0)} = \frac{1}{h} 1_N \otimes 1_N + r_{12}^{(0)} + O(h), \quad r_{12}(z) = \frac{1}{z} P_{12} + r_{12}^{(0)} + O(z).
\]

From the skew-symmetry (1.4) we have

\[
\begin{align*}
  r_{12}(z) &= -r_{21}(-z), & m_{12}(z) &= m_{21}(-z), \\
  R_{12}^{h,(0)} &= -R_{21}^{h,(0)}, & r_{12}^{(0)} &= -r_{21}^{(0)}.
\end{align*}
\]

If the Fourier symmetry (1.10) holds true\(^8\) as well then

\[
\begin{align*}
  R_{12}^{z,(0)} &= r_{12}(z) P_{12}, \\
  R_{12}^{z,(1)} &= m_{12}(z) P_{12}, \\
  r_{12}^{(0)} &= r_{12}^{(0)} P_{12}.
\end{align*}
\]

Let us summarize the results from \[20\]. Consider \(R\)-matrix, which obeys equations (1.3)-(1.6) and has the expansions (3.1)-(3.3). Then the Lax equations

\[
\dot{L}(z, S) = [L(z, S), M(z, S)]
\]

\(^7\)In fact, it is enough \[20\] to have any one of (1.4) or (1.5) conditions. In any case, we deal with \(R\)-matrices satisfying both properties except the case \(A_0 = A_1 = 0\) in (2.13), where the unitarity is degenerated.

\(^8\)The right multiplication of \(R\)-matrix (2.1) by \(P_{12}\) provides \(R_{ijkl} \rightarrow R_{iklj}\).
are equivalent to equations

\[ \dot{S} = [S, J(S)] \]  

(3.7)

in the following cases

- **Relativistic top:**

\[ L^\eta(z, S) = \text{tr}_2(R_1^\eta(z)S_2), \quad M^\eta(z, S) = -\text{tr}_2(r_{12}(z)S_2) \]  

(3.8)

and

\[ J^\eta(S) = \text{tr}_2\left((R_1^\eta(0) - r_{12}^{(0)})S_2\right), \]  

(3.9)

- **Non-relativistic top:**

\[ L(z, S) = \text{tr}_2(r_{12}(z)S_2), \quad M(z, S) = \text{tr}_2(m_{12}(z)S_2) \]  

(3.10)

and

\[ J(S) = \text{tr}_2(m_{12}(0)S_2). \]  

(3.11)

These formulae can be easily written through \( R \)-matrix components (2.1). For example, the Lax matrix (3.8) is of the form

\[ L^\eta(z, S) = \sum_{i,j,k,l=1}^{N} R_{ijkl}^\eta(z)S_{lk}E_{ij}, \]  

(3.12)

due to \( \text{tr}(E_{kl}S) = S_{lk} \). Equivalently,

\[ L^\eta(z, S) = \sum_{i,j=1}^{N} L_{ij}^\eta(z, S)E_{ij}, \quad L_{ij}^\eta(z, S) = \sum_{k,l=1}^{N} R_{ijkl}^\eta(z)S_{lk}, \]  

(3.13)

and for (3.9), (3.11)

\[ J^\eta(S) = \sum_{i,j=1}^{N} E_{ij}J_{ij}^\eta(S), \quad J_{ij}^\eta(S) = \sum_{k,l=1}^{N} (R_{ijkl}^\eta(0) - r_{ijkl}^{(0)})S_{lk}, \]  

(3.14)
\[ J(S) = \sum_{i,j=1}^{N} E_{ij}J_{ij}(S), \quad J_{ij}(S) = \sum_{k,l=1}^{N} m_{ijkl}(0)S_{lk}. \]

**Classical Sklyanin algebras and \( r \)-matrix structures.** In this paragraph we show that any solution of the associative Yang-Baxter equation (1.3) with the properties provides (1.4)–(1.6) and the local expansions (3.1)–(3.4) provides the quadratic Poisson structures of Sklyanin type. The quadratic \( r \)-matrix structure [29]

\[ c_2\{L_1^\eta(z, S), L_2^\eta(w, S)\} = [L_1^\eta(z, S)L_2^\eta(w, S), r_{12}(z - w)], \]  

(3.15)

where \( c_2 \neq 0 \) is arbitrary constant, leads to the following Poisson brackets

\[ c_2\{S_1, S_2\} = [S_1 S_2, r_{12}^{(0)}] + [L_1^\eta(0)(S)S_2, P_{12}], \quad L_1^\eta(0)(S) = \text{tr}_3(R_{13}^\eta(0)S_3). \]  

(3.16)

for the defined above Lax matrices. These brackets are easily obtained (see [18]) by taking residues at \( z = 0 \) and \( w = 0 \) of both sides of (3.15). Being written in components (3.16) takes the form:

\[ c_2\{S_{ij}, S_{kl}\} = (L_{il}^\eta(0)S_{kj} - L_{kj}^\eta(0)S_{il}) + \sum_{a,b=1}^{N} (S_{ia}S_{kb}r_{aj,bl}^{(0)} - r_{ia,kb}S_{aj}S_{bd}), \]  

(3.17)
where
\[ L_{ij}^{\eta,(0)} = \sum_{k,l=1}^{N} R_{ij,kl}^{\eta,(0)} S_{lk}. \] (3.18)

The proof of equivalence of (3.16) and (3.17) is based on the degeneration of (1.3)
\[ R_{12}^h(x) R_{23}^h(y) = R_{13}^h(x + y) r_{12}(x) + r_{23}(y) R_{13}^h(x + y) - \partial_h R_{13}^h(x + y), \] (3.19)
oc弟子the limit \( \eta \to h \) in (1.3).

**Proposition 3.1** For the Lax matrix (3.8) defined by R-matrix satisfying the associative Yang-Baxter equation (1.3) together with the properties (3.1)-(3.5) the Poisson brackets (3.16) are equivalently written in the r-matrix form (3.15).

**Proof:** Plugging the Lax matrix (3.8) into (3.15) we get the following expression for the l.h.s. of (3.15):
\[ \text{tr}_{3,4} \left( R_{13}^n(z) R_{24}^n(w) \{ S_3, S_4 \} \right) \]
\[ = \text{tr}_{3,4} \left( R_{13}^n(z) R_{24}^n(w) \left( [S_3 S_4, r_{14}^{(0)}] + [L_3^{\eta,(0)} S_3 S_4, P_{34}] \right) \right), \] (3.20)
and we are going to prove that it is equal to the r.h.s. of (3.15):
\[ \text{r.h.s.} = \text{tr}_{3,4} \left( \left( R_{13}^n(z) R_{24}^n(w) r_{12}(z - w) - r_{12}(z - w) R_{13}^n(z) R_{24}^n(w) \right) S_3 S_4 \right). \] (3.21)
Let us rewrite the expression in the brackets of (3.21) using (3.19), which we represent in the form (the skew-symmetry (1.14) is also used)
\[ R_{24}^n(w) r_{12}(z - w) = -R_{21}^n(w - z) R_{14}^n(z) + R_{14}^n(z) R_{24}^n(w) - \partial_n R_{24}^n(w) \] (3.22)
for the first term in (3.21), and
\[ r_{12}(z - w) R_{24}^n(w) = -R_{14}^n(z) R_{24}^n(w - z) + R_{24}^n(w) r_{14}(z) - \partial_n R_{24}^n(w) \] (3.23)
for the second one. Due to \( [R_{13}^n(z), \partial_n R_{24}^n(w)] = 0 \) we have
\[ R_{13}^n(z) R_{24}^n(w) r_{12}(z - w) - r_{12}(z - w) R_{13}^n(z) R_{24}^n(w) = \]
\[ = R_{14}^n(z) R_{21}^n(w - z) R_{13}^n(z) - R_{13}^n(z) R_{21}^n(w - z) R_{14}^n(z) + \]
\[ + R_{13}^n(z) r_{14}(z) R_{24}^n(w) - R_{24}^n(w) r_{14}(z) R_{13}^n(z). \] (3.24)
The second line of (3.24) is cancelled out after substitution into (3.21) since it is skew-symmetric under renaming the numbers of the tensor components 3 ↔ 4. Therefore, the expression (3.21) is simplified to
\[ \text{r.h.s.} = \text{tr}_{3,4} \left( \left( R_{13}^n(z) r_{14}(z) R_{24}^n(w) - R_{24}^n(w) r_{14}(z) R_{13}^n(z) \right) S_3 S_4 \right). \] (3.25)
Next, transform the latter expression using further degeneration of (1.3), corresponding to \( z \to 0 \) in (3.22) and (3.23):
\[ R_{13}^n(z) r_{14}(z) = r_{34}^{(0)} R_{13}^n(z) + R_{14}^n(z) R_{43}^{(0)} - \partial_2 R_{14}^n(z) P_{34} + \partial_n R_{13}^n(z), \] (3.26)
\[ r_{14}(z) R_{13}^n(z) = R_{13}^n(z) r_{34}^{(0)} + R_{43}^{(0)} R_{14}^n(z) - \partial_2 R_{13}^n(z) P_{34} + \partial_n R_{13}^n(z). \] (3.27)

\(^9\)The R-matrices \( R_{13}^n(z) \) and \( R_{24}^n(w) \) commute since they are defined in different tensor components.
Finally, the third line of (3.28) after substitution into (3.21) results in the second term of the r.h.s. of component 5 of (3.32) are equal to each other (the equality of the second terms is verified similarly):

The latter equality is verified as follows. Let us show that the first terms in the upper and lower lines of (3.29) vanishes being substituted into (3.21). Indeed, on one hand

\[ \text{tr}_{3,4} \left( \partial_z R_{13}^{(0)}(z)R_{13}^{(0)}(z)S_3S_4 \right) = \text{tr}_{3,4} \left( P_{34}\partial_z R_{13}^{(0)}(z)R_{24}^{(0)}(w)S_3S_4 \right), \tag{3.29} \]

and, on the other hand,

\[ \text{tr}_{3,4} \left( R_{13}^{(0)}(z)R_{13}^{(0)}(z)S_3S_4 \right) = \text{tr}_{3,4} \left( \partial_z R_{13}^{(0)}(z)R_{13}^{(0)}(z)S_3S_4 \right) = \text{tr}_{3,4} \left( P_{34}\partial_z R_{13}^{(0)}(z)R_{24}^{(0)}(w)S_3S_4 \right). \tag{3.30} \]

The second line of (3.29) after substitution into (3.21) results in the first term of the r.h.s. of (3.30):

\[ \text{tr}_{3,4} \left( r_{34}^{(0)} R_{13}^{(0)}(z)R_{24}^{(0)}(w) - R_{24}^{(0)}(w)R_{13}^{(0)}(z)S_3S_4 \right) = \text{tr}_{3,4} \left( R_{13}^{(0)}(z)S_3S_4 \left( [S_3S_4, r_{34}^{(0)}] \right) \right). \tag{3.31} \]

Finally, the third line of (3.28) after substitution into (3.21) results in the second term of the r.h.s. of (3.30):

\[ \text{tr}_{3,4} \left( \left( R_{14}^{(0)}(z)R_{43}^{(0)}(w) - R_{24}^{(0)}(w)R_{43}^{(0)}(w) \right)S_3S_4 \right) = \text{tr}_{3,4} \left( R_{13}^{(0)}(z)S_3S_4 \left( L_3^{(0)}(S)S_3S_4 - P_{34}L_3^{(0)}(S)S_3S_4 \right) \right). \tag{3.32} \]

The latter equality is verified as follows. Let us show that the first terms in the upper and lower lines of (3.29) are equal to each other (the equality of the second terms is verified similarly):

\[ \text{tr}_{3,4} \left( R_{13}^{(0)}(z)R_{24}^{(0)}(w) - R_{13}^{(0)}(z)S_3S_4 \right) = \text{tr}_{3,4} \left( R_{13}^{(0)}(z)L_3^{(0)}(S)S_3S_4 \right) = \text{tr}_{3,4} \left( \partial_z R_{13}^{(0)}(z)R_{24}^{(0)}(w) \right) = \text{tr}_{3,4} \left( P_{34}\partial_z R_{13}^{(0)}(z)S_3S_4 \right) = \text{tr}_{3,4} \left( R_{13}^{(0)}(z)S_3S_4 \right). \tag{3.33} \]

The last step is to take the trace over the third tensor component (then $P_{34}$ vanishes), and rename the component 5 $\leftrightarrow$ 3.

To summarize: we deduce the $r$-matrix structure (3.15) from the brackets (3.10). The converse statement (when the brackets are derived from the $r$-matrix structure) requires the property that $A = 0$ should follow from $\text{tr}_2(R_{12}^{(0)}(z)A_2) = 0$ for a generic matrix $A \in \text{Mat}(N, \mathbb{C})$. It is true for the matrices under consideration due to the local behavior (3.2).

In the non-relativistic limit we are left with the linear $r$-matrix structure

\[ c_1 \{ L_1(z, S), L_2(w, S) \} = [L_1(z, S) + L_2(w, S), r_{12}(z - w)], \tag{3.34} \]
which provides the Poisson-Lie brackets on $\mathfrak{gl}_N^*$ Lie coalgebra ($c_1 \neq 0$ is an arbitrary constant):

$$c_1 \{S_1, S_2\} = [S_2, P_{12}]$$  \hspace{1cm} (3.35)

or

$$c_1 \{S_{ij}, S_{kl}\} = S_{kj} \delta_{il} - S_{il} \delta_{kj}.$$  \hspace{1cm} (3.36)

The Poisson structures (3.15)-(3.16) and (3.34)-(3.35) provide the Hamiltonians generating the Euler-Arnold equations (3.7). In the relativistic case the Hamiltonian is given by

$$H^{\text{rel}} = \frac{1}{c_2} \text{tr}(S),$$  \hspace{1cm} (3.37)

and for the non-relativistic case we have

$$H^{\text{non-rel}} = \frac{1}{2c_1} \text{tr}(SJ(S)).$$  \hspace{1cm} (3.38)

In the relativistic case the Hamiltonian is linear, while the Poisson structure is quadratic (in variables $S$), and vice versa for non-relativistic models.

### 3.1 The case of non-standard $R$-matrix

In order to describe the tops explicitly it is enough to write down all $R$-matrices and related coefficients of expansions entering (3.8)-(3.14). Below is the summary based on the $R$-matrix (2.16):

$$R^\eta_{ij,kl}(z) = \delta_{ij} \delta_{kl} \frac{N}{2} \left( \coth(Nz/2) + \coth(N\eta/2) \right) +$$

$$\begin{align*}
\delta_{ij} \delta_{kl} \varepsilon(i \neq k) & \left( \frac{N e^{(i-k)\eta} \cdot \text{sign}(i-k) N\eta/2}{2 \sinh(N\eta/2)} \right) + \\
\delta_{il} \delta_{kj} \varepsilon(i \neq k) & \left( \frac{N e^{(i-k)z} \cdot \text{sign}(i-k) Nz/2}{2 \sinh(Nz/2)} \right) + \\
N \delta_{i+k,j+l} & \left( \varepsilon(i<j<k) - \varepsilon(k<j<i) \right) + \\
N e^{-NA} & \left( \delta_{i+k,j+l+N} \left( \delta_{iN} e^{-jz-l\eta} - \delta_{kN} e^{jz+l\eta} \right) \right).
\end{align*}$$  \hspace{1cm} (3.39)

The classical $r$-matrix:

$$r_{ij,kl}(z) = \delta_{ij} \delta_{kl} \frac{N}{2} \coth(Nz/2) +$$

$$\begin{align*}
\delta_{ij} \delta_{kl} \varepsilon(i \neq k) & \left( \frac{N \text{sign}(i-k)}{2} \right) + \\
\delta_{il} \delta_{kj} \varepsilon(i \neq k) & \left( \frac{N e^{(i-k)z} \cdot \text{sign}(i-k) Nz/2}{2 \sinh(Nz/2)} \right) + \\
N e^{(i-j)z} & \left( \varepsilon(i<j<k) - \varepsilon(k<j<i) \right) + \\
N e^{-NA} & \left( \delta_{i+k,j+l+N} \left( e^{-jz} \delta_{iN} - e^{jz} \delta_{kN} \right) \right).
\end{align*}$$  \hspace{1cm} (3.40)

The next coefficient in the expansion (3.11):

$$m_{ij,kl}(z) = \delta_{ij} \delta_{kl} \frac{N^2}{12} + \delta_{ij} \delta_{kl} \varepsilon(i \neq k) \left( \frac{(i-k)^2}{2} - \frac{N^2}{12} - \frac{N}{2} |i-k| \right) +$$

$$\begin{align*}
N(j-k) e^{(i-j)z} & \delta_{i+k,j+l} \left( \varepsilon(i<j<k) - \varepsilon(k<j<i) \right) - N e^{-NA} & \left( \delta_{i+k,j+l+N} \left( e^{-jz} \delta_{iN} + e^{jz} \delta_{kN} \right) \right).
\end{align*}$$  \hspace{1cm} (3.41)
Its value at $z = 0$ entering the inverse inertia tensor in the non-relativistic case (3.11) or (3.14):

$$m_{ij,kl}(0) = \delta_{ij} \delta_{kl} \frac{N^2}{12} + \delta_{ij} \delta_{kl} \varepsilon(i \neq k) \left( \frac{(i-k)^2}{2} - \frac{N^2}{12} - \frac{N}{2} |i-k| \right) +$$

$$+ N(j-k) \delta_{i+k,j+l} \left( \varepsilon(i<j<k) - \varepsilon(k<j<i) \right) - N e^{-NA} \delta_{i+k,j+l+N} \left( l \delta_{ij} + j \delta_{kN} \right).$$

The coefficient from the expansions (3.22) and (3.33) entering the relativistic inverse inertia tensor (3.9) or (3.14):

$$R_{ij,kl}^{(0)} = r_{iklj}^{(0)} = \delta_{ij} \delta_{kl} \frac{N}{2} \coth(N\eta/2) +$$

$$+ \delta_{ij} \delta_{kl} \varepsilon(i \neq k) \frac{Ne^{(i-k)\eta - \text{sign}(i-k)N\eta/2}}{2 \sinh(N\eta/2)} + \delta_{il} \delta_{kj} \varepsilon(i \neq k) \left( i - k \right) - \frac{N \text{sign}(i-k)}{2} +$$

$$+ N e^{(j-k)\eta} \delta_{i+k,j+l} \left( \varepsilon(i<j<k) - \varepsilon(k<j<i) \right) + N e^{-NA} \delta_{i+k,j+l+N} \left( e^{-\eta} \delta_{iN} - e^{\eta} \delta_{kN} \right)$$

and

$$r^{(0)}_{ij,kl} = \left( \delta_{ij} \delta_{kl} \varepsilon(i \neq k) + \delta_{il} \delta_{kj} \varepsilon(i \neq k) \right) \left( i - k \right) - \frac{N \text{sign}(i-k)}{2} +$$

$$+ N \delta_{i+k,j+l} \left( \varepsilon(i<j<k) - \varepsilon(k<j<i) \right) + N e^{-NA} \delta_{i+k,j+l+N} \left( \delta_{iN} - \delta_{kN} \right).$$

**Lax pairs.** The Lax matrix of the relativistic top constructed by means of (3.39) is of the following form. For $i = j$:

$$L_{ii}(z) = \frac{N}{2} \left( \coth(Nz/2) + \coth(N\eta/2) \right) S_{ii} +$$

$$+ \frac{N}{2 \sinh(N\eta/2)} \left( e^{-N\eta/2} \sum_{k=1}^{i-1} e^{(i-k)\eta} S_{kk} + e^{N\eta/2} \sum_{k=i+1}^{N} e^{(i-k)\eta} S_{kk} \right),$$

for $i < j$:

$$L_{ij}(z) = \frac{N \exp(Nz/2 + (i-j)z)}{2 \sinh(Nz/2)} S_{ij} + \sum_{k=j+1}^{N} e^{(i-j)z + (j-k)\eta} S_{i-j+k,k},$$

and for $i > j$:

$$L_{ij}(z) = \frac{N \exp(-Nz/2 + (i-j)z)}{2 \sinh(Nz/2)} S_{ij} - \sum_{k=1}^{j-1} e^{(i-j)z + (j-k)\eta} S_{i-j+k,k} -$$

$$- N e^{-NA} e^{(i-j)z + \eta} S_{i-j,N} + \delta_{iN} N e^{-NA} \sum_{k=j+1}^{N} e^{-jz + (j-k)\eta} S_{k-j,k}.$$

From the definitions (3.8), (3.10) and the expansion (3.11) it follows that

$$-M^{(0)}(z) = L(z) = \text{Res}_{\eta=0} \left( \eta^{-1} L^{(0)}(z) \right), \quad M(z) = \text{Res}_{\eta=0} \left( \eta^{-2} L^{(0)}(z) \right).$$

Similarly, the expansion (3.2) near $z = 0$ yields

$$L^{(0)}(z) = \frac{1}{z} S + L^{(0)}(S) + O(z), \quad L^{(0)}(S) = \text{tr}_2 \left( R_{12}^{(0)} S_2 \right) = \text{Res}_{z=0} \left( z^{-1} L^{(0)}(z) \right).$$
Example: GL₂ top. In this case we deal with the following quantum

\[
R^h(z) = \begin{pmatrix}
\coth(z) + \coth(h) & 0 & 0 & 0 \\
0 & \sinh^{-1}(h) & \sinh^{-1}(z) & 0 \\
0 & \sinh^{-1}(z) & \sinh^{-1}(h) & 0 \\
-4 e^{-2\Lambda} \sinh(z + h) & 0 & 0 & \coth(z) + \coth(h)
\end{pmatrix}
\]  
(3.50)

and classical

\[
r(z) = \begin{pmatrix}
\coth(z) & 0 & 0 & 0 \\
0 & 0 & \sinh^{-1}(z) & 0 \\
0 & \sinh^{-1}(z) & 0 & 0 \\
-4 e^{-2\Lambda} \sinh(z) & 0 & 0 & \coth(z)
\end{pmatrix}
\]  
(3.51)

R-matrices. In the relativistic case this provides the Lax pair

\[
L^\eta(z, S) = \begin{pmatrix}
S_{11} \left( \coth(z) + \coth(\eta) \right) + \frac{S_{22}}{\sinh(\eta)} & S_{12} & \frac{S_{22}}{\sinh(z)} \\
\frac{S_{21}}{\sinh(z)} - 4 e^{-2\Lambda} S_{12} \sinh(z + \eta) & S_{22} \left( \coth(z) + \coth(\eta) \right) + \frac{S_{11}}{\sinh(\eta)} \\
\frac{S_{21}}{\sinh(z)} - 4 e^{-2\Lambda} \sinh(z) S_{12} & \coth(z) S_{22}
\end{pmatrix}
\]  
(3.52)

\[
M^\eta(z, S) = - \begin{pmatrix}
\coth(z) S_{11} & S_{12} & \frac{S_{22}}{\sinh(z)} \\
\frac{S_{21}}{\sinh(z)} - 4 e^{-2\Lambda} \sinh(z) S_{12} & \coth(z) S_{22}
\end{pmatrix}
\]  
(3.53)

and the inverse inertia tensor

\[
J^\eta(S) = \begin{pmatrix}
\coth(\eta) S_{11} + \frac{S_{22}}{\sinh(\eta)} & 0 \\
-4 e^{-2\Lambda} \sinh(\eta) S_{12} & \frac{S_{11}}{\sinh(\eta)} + \coth(\eta) S_{22}
\end{pmatrix}
\]  
(3.54)

In the non-relativistic case the Lax matrix is defined by (3.53): \(L(z, S) = -M^\eta(z, S)\). The accompany matrix is as follows

\[
M(z, S) = \frac{1}{6} \begin{pmatrix}
2 S_{11} - S_{22} & 0 \\
-24 e^{-2\Lambda} \cosh(z) S_{12} & -S_{11} + 2 S_{22}
\end{pmatrix}
\]  
(3.55)

The inverse inertia tensor acquires the form:

\[
J(S) = \frac{1}{6} \begin{pmatrix}
2 S_{11} - S_{22} & 0 \\
-24 e^{-2\Lambda} S_{12} & -S_{11} + 2 S_{22}
\end{pmatrix}
\]  
(3.56)

• Relativistic top (\(\eta\)-independent description):

Another one description for the relativistic top is available, which is similar to original construction 29. Instead of usage of the quantum \(R\)-matrix (3.8) consider the traceless part of the non-relativistic Lax matrix and supplement it by the scalar term \(s_01_N\):

\[
\tilde{L}(z, S) = s_01_N + L(z, S) - \frac{1_N}{N} \text{tr}L(z, S), \quad s_0 = \frac{\text{tr}S}{N},
\]  
(3.57)
where $s_0$ is a dynamical variable. In fact, it is the Hamiltonian since $\text{tr}\tilde{L} = Ns_0$. The Lax equations do not change because $L(z, S)$ and $\tilde{L}(z, S)$ differ from each other by only a scalar matrix. So that the $M$-matrix for (3.57) is the same as in (3.8). However, the Poisson structures are different (see below). It happens because of the bihamiltonian structure in this kind of models [15, 18].

As was mention in [18] (see also [32]) there is a relation between the Lax matrices (3.8) and (3.57). Similarly, to the rational case we have

$$L^n(z - \frac{\eta}{N}, \tilde{L}(\frac{\eta}{N}, S)) = \frac{\text{tr}(L^n(z - \frac{\eta}{N}, S))}{\text{tr}(S)} \tilde{L}(z, S).$$

These relation can be verified directly using explicit formulae (3.45)-(3.47).

The quadratic Poisson structure takes the form

$$\{\tilde{L}_1(z, S), \tilde{L}_2(w, S)\} = \frac{1}{e_2} [\tilde{L}_1(z, S)\tilde{L}_2(w, S), r_{12}(z - w)],$$

and provides the following Poisson brackets:

$$c_2\{S_1, S_2\} = s_0[S_2, P_{12}] + [S_1S_2, r^{(0)}_{12}] + [\text{tr}_3(r^{(0)}_{13}S_3)S_2, P_{12}].$$

The latter is verified similarly to the $\eta$-dependent case (3.15)-(3.16).

### 3.2 The case of general $R$-matrix

The summary of the integrable tops data in the general case is based on the expansions of the $R$-matrix (2.24):

$$R_{12}^0(z) = \frac{N}{2} \left( \coth(Nz/2) + \coth(N\eta/2) \right) \sum_i E_{ii} \otimes E_{ii} + \frac{N}{e^{N\eta} - 1} \sum_{0<n<N, i=C^n(k)} e^{n\eta}E_{ii} \otimes E_{kk} + \frac{N}{e^{Nz} - 1} \sum_{0<m<N, k=C^m(i)} e^{mz}E_{ik} \otimes E_{kl} +$$

$$+ \sum_{0<m<N, n>0, i = C^n(j), \tau^n(j,i) = (k,l)} N \left( e^{-n\eta-mz}E_{ij} \otimes E_{kl} - e^{n\eta+mz}E_{kl} \otimes E_{ij} \right),$$

The classical $r$-matrix and the next coefficient of the classical limit (3.1) are as follows:

$$r_{12}(z) = \frac{N}{2} \coth(Nz/2) \sum_i E_{ii} \otimes E_{ii} +$$

$$+ \sum_{0<n<N, i=C^n(k)} \left( n - \frac{N}{2} \right) E_{ii} \otimes E_{kk} + \frac{N}{e^{Nz} - 1} \sum_{0<m<N, k=C^m(i)} e^{mz}E_{ik} \otimes E_{kl} +$$

$$+ \sum_{0<m<N, n>0, i = C^n(j), \tau^n(j,i) = (k,l)} N \left( e^{-mz}E_{ij} \otimes E_{kl} - e^{mz}E_{kl} \otimes E_{ij} \right)$$

and

$$m_{12}(z) = \frac{N^2}{12} \sum_i E_{ii} \otimes E_{ii} + \frac{1}{12} \sum_{0<n<N, i=C^n(k)} \left( 6n^2 - 6nN + N^2 \right) E_{ii} \otimes E_{kk} -$$

$$- \sum_{0<m<N, n>0, i = C^n(j), \tau^n(j,i) = (k,l)} Nn \left( e^{-mz}E_{ij} \otimes E_{kl} + e^{mz}E_{kl} \otimes E_{ij} \right).$$
The first non-trivial coefficients from the expansions (3.2), (3.3) are of the form:

\[ R_{12}^{η(0)} = \frac{N}{2} \coth(Nη/2) \sum_i E_{ii} \otimes E_{ii} + \]

\[ + \frac{N}{e^{Nη} - 1} \sum_{0<n<N, i=C^n(k)} e^{mn} E_{ii} \otimes E_{kk} + \sum_{0<m<N, k=C^m(i)} \left( m - \frac{N}{2} \right) E_{ik} \otimes E_{ki} + \]

\[ + \sum_{0 < m < N, n > 0, \ i = C^n(j), r^n(j, i) = (k, l)} N \left( e^{-mn} E_{ij} \otimes E_{kl} - e^{mn} E_{kl} \otimes E_{ij} \right) \]

and

\[ r_{12}^{(0)} = \sum_{0<n<N, i=C^n(k)} \left( n - \frac{N}{2} \right) E_{ii} \otimes E_{kk} + \sum_{0<m<N, k=C^m(i)} \left( m - \frac{N}{2} \right) E_{ik} \otimes E_{ki} + \]

\[ + \sum_{0 < m < N, n > 0, \ i = C^n(j), r^n(j, i) = (k, l)} N \left( E_{ij} \otimes E_{kl} - E_{kl} \otimes E_{ij} \right) . \]

\[ (3.64) \]

\[ (3.65) \]

4 Relation to Ruijsenaars-Schneider model

Introduce the matrix \( [2]^{10} \)

\[ g(z, q) = \Xi(z, q) D^{-1}(q) \in \text{Mat}(N, \mathbb{C}), \]

where

\[ \Xi_{ij}(z, q) = e^{(i-1)(z-\bar{q}_j)} + (-1)^N e^{-(z-\bar{q}_j)} \delta_{1N} \]

and

\[ D_{ij}(q) = \delta_{ij} \prod_{k \neq i} \left( e^{-\bar{q}_i} - e^{-\bar{q}_k} \right). \]

The matrices depend on \( z \) and the set of variables \( q_1, ..., q_N \). The variables \( \bar{q}_1, ..., \bar{q}_N \) are obtained by transition to the center of mass frame:

\[ \bar{q}_i = q_i - \frac{1}{N} \sum_{k=1}^{N} q_k . \]

The determinant of the matrix \( \Xi \) is as follows:

\[ \det \Xi(z, q) = e^{zN(N-1)/2} (1 - e^{-Nz}) \prod_{i > j} (e^{-\bar{q}_i} - e^{-\bar{q}_j}) . \]

That is \( \Xi(z, q) \) is degenerated at \( z = 0 \).

Our statement is that the following matrix

\[ L^{RS}(z) = g^{-1}(z) g(z + \eta) e^{P/c}, \quad P = \text{diag}(p_1, p_2, ..., p_N) \]

\[ (4.6) \]

\[ ^{10} \text{It is the intertwining matrix relating the non-standard } R\text{-matrix and the trigonometric Felder’s dynamical } R\text{-matrix through the quantum IRF-Vertex correspondence.} \]
The Lax matrix of the trigonometric Ruijsenaars-Schneider model. More precisely,

\[
L_{ij}^{RS}(z) = e^{\frac{N-2}{4} \eta} \sinh(\eta/2) \left( \coth \left( \frac{Nz}{2} \right) + \coth \left( \frac{q_i - q_j + \eta}{2} \right) \right) e^{p_i/c} \prod_{k \neq j} \frac{\sinh \left( \frac{q_i - q_k + \eta}{2} \right)}{\sinh \left( \frac{q_i - q_k}{2} \right)}.
\]  

(4.7)

The proof is obtained by direct verification, which is similar to calculations performed in [18] in the rational case. One should introduce the set of elementary symmetric polynomials \( \sigma_k(q) \) of \( N \) variables \( \{e^{-q_1}, ..., e^{-q_N}\} \)

\[
\prod_{k=1}^{N} (\zeta - e^{-q_k}) = \sum_{k=0}^{N} (-1)^k \zeta^k \sigma_k(q)
\]  

(4.8)

and \( N \) sets of similar functions \( \{\bar{\sigma}_{k,i}(q), i = 1, ..., N\} \) defined for the sets \( \{e^{-q_1}, ..., e^{-q_N}\} \setminus \{e^{-q_i}\} \) of \( N - 1 \) variables each:

\[
\prod_{k \neq i}^{N} (\zeta - e^{-\bar{q}_k}) = \sum_{k=0}^{N} (-1)^k \zeta^k \bar{\sigma}_{k,i}(q).
\]  

(4.9)

The inverse of \( \Xi \) is then written as follows:

\[
(\Xi^{-1})_{ij}(z, q) = \frac{(-1)^{j-1} e^{(N-j+1)z}}{e^{Nz} - 1} \frac{\left( \bar{\sigma}_{j-1,i}(q) + e^{-\bar{q}_i} \bar{\sigma}_{j,i}(q) e^{-Nz} \right)}{\prod_{k \neq i} (e^{-\bar{q}_i} - e^{-\bar{q}_k})}.
\]  

(4.10)

Consider the gauge transformed Lax matrix

\[
L^n(z) = g(z) \tilde{L}^{RS}(z) g^{-1}(z) = g(z + \eta) e^{p_i/c} g^{-1}(z)
\]

(4.11)

Then \( L^n(z) = \text{tr} \left( R^n_{ij}(z) S_2(p, q) \right) \)

(4.12)

with the non-standard \( R \)-matrix \( \mathbf{2} \), where \( \Lambda = \sqrt{-1} \pi \). Put it differently, the matrix \( (4.12) \) coincides with \( (4.15) \) when \( \Lambda = \sqrt{-1} \pi \). The change of variables is as follows:

\[
S_{ij}(p, q) = \frac{(-1)^j \sigma_j(q) e^{(i-1)\eta}}{N} \sum_{n=1}^{N} \prod_{k: k \neq n} \frac{e^{p_n/c}}{e^{-q_n} - e^{-\bar{q}_k}} \left( e^{-(i-1)\eta} + \frac{(-1)^N \delta_{iN}}{e^{N(\eta-q_n)}} \right).
\]  

(4.13)

The Poisson structure for \( (p, q) \) variables is canonical, i.e.

\[
\{p_i, q_j\} = \delta_{ij} \quad \text{or} \quad \{p_i, \bar{q}_j\} = \delta_{ij} - \frac{1}{N}.
\]  

(4.14)

After some tedious calculations it can be verified that the Poisson brackets \( \{S_{ij}(p, q), S_{kl}(p, q)\} \) evaluated through \( (4.14) \) coincide with \( (3.17) \) with \( c_2 = Nc \) and \( r_{12}^{(0)} \) from \( (3.44) \). In particular, it is useful to notice for the proof that the matrix \( (4.13) \) is of rank 1, i.e.

\[
S_{ij}(p, q) = a_i(p, q) b_j(q),
\]

\[
a_i(p, q) = \frac{e^{(i-1)\eta}}{N} \sum_{n=1}^{N} \prod_{k: k \neq n} \frac{e^{p_n/c}}{e^{-q_n} - e^{-\bar{q}_k}} \left( e^{-(i-1)\eta} + \frac{(-1)^N \delta_{iN}}{e^{N(\eta-q_n)}} \right), \quad b_j(q) = (-1)^j \sigma_j(q).
\]  

(4.15)

\[\text{[11] The origins of factorization of the Lax pairs (4.6) and (4.11), (4.12) are discussed in [31].}\]
In this case $S_{ij}S_{kl} = S_{il}S_{kj}$, and the Poisson structure \((3.17)\) takes the (relatively simple) form:

$$\{S_{ij}, S_{kl}\} = \frac{1}{Nc}(L_{ij}^{\eta(0)} S_{kj} - L_{kj}^{\eta(0)} S_{il}) + \frac{2}{Nc}(k - i)S_{ij}S_{kl} +$$

$$+\frac{\varepsilon(i > k)}{c} \sum_{p=0}^{i-k-1} S_{i-p,j}S_{k+p,l} + \frac{\varepsilon(i < k)}{c} \sum_{p=0}^{k-i-1} S_{i+p,j}S_{k-p,l} +$$

$$+\frac{(-1)^N \delta_{kN}}{c} \sum_{p=1}^{i-1} S_{i-p,j}S_{p,l} - \frac{(-1)^N \delta_{kN}}{c} \sum_{p=1}^{k-1} S_{pj}S_{k-p,l}. \quad (4.16)$$

**Non-relativistic limit.** The Calogero-Moser-Sutherland models appears from the above results by taking the non-relativistic limit, when $\eta = \nu/c$ and $c \to \infty$. The Lax matrix arising from (4.17) is of the form\(^{12}\):

$$L_{ij}^{CM}(z) = \delta_{ij} (\dot{q}_i + \nu \coth(Nz)) + \nu(1 - \delta_{ij}) \left( \coth \left( \frac{q_i - q_j}{2} \right) + \coth(Nz) \right), \quad (4.17)$$

$$\dot{q}_i = p_i + \nu(N - 2) - \nu \sum_{k \neq i}^{N} \coth \left( \frac{q_i - q_j}{2} \right).$$

Similarly, the non-relativistic top \((3.10)\) comes from \((3.9)\). The gauge transformation \((4.11)\) holds on at the level of non-relativistic models as well \([17, 31]\). That is

$$L(z) = \text{tr} \left( r_{12} S_2(p,q) \right) = g(z)L^{CM}(z)g^{-1}(z). \quad (4.18)$$

The residue of both parts of the latter relation provides explicit change of variables, or the non-relativistic limit of \((4.15)\):

$$S_{ij}(p,q) = a_i(p,q)b_j(q), \quad b_j(q) = (-1)^j \sigma_j(q),$$

$$a_i(p,q) = \frac{1}{N} \sum_{n=1}^{N} \frac{(p_n + (i-1)\nu)}{\prod_{k \neq n} (e^{-q_n} - e^{-q_k})} \left( e^{-(i-1)\bar{q}_n} + (-1)^N \delta_{iN} e^{\bar{q}_n} \right) - N\nu(-1)^N \delta_{iN} e^{\bar{q}_n}. \quad (4.19)$$

The Poisson brackets $\{S_{ij}(p,q), S_{kl}(p,q)\}$ computed via the canonical structure \((4.14)\) reproduce \((3.36)\) with $c_1 = N$, and the value of the Casimir functions are given by the powers of the Calogero-Moser-Sutherland coupling constant

$$\text{tr}(S^k) = \nu^k. \quad (4.20)$$

Thus, the Calogero-Moser-Sutherland model is gauge equivalent to the non-relativistic top with special values of the Casimir functions corresponding to the coadjoint orbit (of $GL_N$ group) of minimal dimension. Apart from the gauge transformation we obtain explicit change of variables $(p_i, q_j) \to (a_i(p,q), b_j(q))$, where $b_i$ are elementary symmetric functions. These variables are known in the quantum Calogero-Moser-Sutherland model \([21]\).\(^{13}\)

\(^{12}\)It is easy to verify that $p_i \to \dot{q}_i(p,q)$ with $\dot{q}_i(p,q)$ from \((4.14)\) is a canonical map, i.e. $\{\dot{q}_i(p,q), q_j\} = \delta_{ij}$.\(^{13}\)Let us mention that there is another one application of the associative Yang-Baxter equation to the models of the Calogero-Moser-Sutherland type and related long-range spin chains \([19, 27]\).
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