Large deviation principle for the intersection measure of Brownian motions on unbounded domains

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Abstract

Consider the intersection measure $\ell_{IS}^t$ of $p$ independent Brownian motions on $\mathbb{R}^d$. In this article, we prove the large deviation principle for the normalized intersection measure $t^{-p}\ell_{IS}^t$ as $t \to \infty$, before exiting a (possibly unbounded) domain $D \subset \mathbb{R}^d$ with smooth boundary. This is an extension of [W. König and C. Mukherjee: Communications on Pure and Applied Mathematics, 66(2):263–306, 2013] which deals with the case $D$ is bounded. The essential contribution of this paper is to prove the so-called super-exponential estimate for the intersection measure of killed Brownian motions on such $D$ by an application of the Chapman-Kolmogorov relation. As a consequence, the new argument in this paper gives not only an extension to unbounded domains but also a simpler proof even for bounded domains.

Keywords: Intersection measure; Large deviations

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1 Introduction

Analysis of the intersection of the Brownian paths begins with the series of studies by Dvoretzky, Erdős, Kakutani and Taylor [DEK50, DEK54, DEKT57], which give the following dichotomy: for $p$ independent Brownian motions $B^{(1)}, \ldots, B^{(p)}$ on $\mathbb{R}^d$, the paths intersect, i.e., $B^{(1)}(0, \infty) \cap \cdots \cap B^{(p)}(0, \infty) \neq \emptyset$ almost surely if $d - p(d - 2) > 0$, and do not intersect almost surely if $d - p(d - 2) \leq 0$. Motivated by physical problems such as the configurations of interacting polymers, two random measures that measure the intensity of the intersections of the paths have been introduced. One is called the intersection local time and the other is called the intersection measure. A brief overview is given in Section 1.2. Large deviation principles for this kind of measures have recently been applied by Mukherjee [Muk17] to study a model of mutually interacting polymers, and by Adams, Bru and König [ABK06] to prove the Gross-Pitaevskii formula for the model of particles with Dirac interaction potential.

In this paper, we consider the (mutual) intersection measure introduced by König and Mukherjee [KM13], which is formally written as

$$\ell_{IS}^t(A) = \int_A \left[ \int_{[0,t]^p} \prod_{i=1}^p \delta_x(B^{(i)}(s_i)) ds_1 \cdots ds_p \right] dx \quad \text{for } A \subset \mathbb{R}^d \text{ Borel}$$

under the regime $d - p(d - 2) > 0$, where $\delta_x$ is the Dirac measure at $x$ (see Section 1.2 for a precise definition). Here and in the following, the superscript “IS” means “InterSection”. A contribution of this paper is the Donsker-Varadhan type large deviation principle for the intersection measure $\ell_{IS}^t(dx)$ as $t \to \infty$, before exiting an unbounded domain $D \subset \mathbb{R}^d$ with smooth boundary. This is roughly written as

$$\mathbb{P} \left( (t^{-p}\ell_{IS}^1, t^{-1}\ell_{IS}^{(1)}, \ldots, t^{-1}\ell_{IS}^{(p)}) \approx \mu, t < \tau_D^{(1)} \wedge \cdots \wedge \tau_D^{(p)} \right) \approx \exp \left\{ -t \sum_{i=1}^p \frac{1}{2} \int_D |\nabla \psi^{(i)}|^2 dx \right\}$$

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as $t \to \infty$ (see Theorem 0 for a precise meaning), where $\ell_t^{(i)}$ and $\tau_D^{(i)}$ are the occupation measure and the exit time from $D$ of the process $B^{(i)}$, respectively, and $\mu = (\mu; \mu^{(1)}, \ldots, \mu^{(p)})$ is a tuple of a Radon measure and $p$ probability measures on $D$ of the form $\frac{d\mu}{dx} = \prod_{i=1}^{p} \frac{d\mu^{(i)}}{dx}$ and $\psi^{(i)} = \sqrt{\frac{d\mu^{(i)}}{dx}} \in W_{0}^{1,2}(D)$, the Sobolev space with zero boundary values. Previously, Köning and Mukherjee [KM13] showed such large deviation result when $D$ is bounded. As we will note at the end of Section 1.2 later, the boundedness of the domain is essential for their proof. In view of applications to models from physics such as the ones in [AmK17, ABK16], however, it is desirable to extend the result to unbounded domains, in particular to the whole space. It is this extension that the main result of this paper provides. In addition, our technical improvements give a simpler proof of [KM13] when the domain is bounded.

This paper is organized as follows. In Section 1.1, we recall the definition of the intersection measure and state our main result (Theorem 1.1), the large deviation principle of the intersection measure. Section 1.2 summarizes earlier works related to our main result. Section 2 is the outline of the proof of the main result. In this section, we state the super-exponential estimate (Theorem 2.1), a key theorem to prove the main result. We prove this in the following Section 3. In Section 4 and 5, we prove the large deviation lower and upper bound, respectively. Finally in Section 6, we discuss an extension of the main theorems from Brownian motions to other processes such as the stable processes.

1.1 Settings and main results

Suppose $D \subset \mathbb{R}^d$ be a (possibly unbounded) domain with smooth boundary. Let $\partial$ be a point added to $D$ so that $D_0 := D \cup \{\partial\}$ is the one-point compactification of $D$. A killed Brownian motion $X$ in $D$ is the process given by

$$X_t = \begin{cases} B_t, & t < \tau_D, \\ \partial, & t \geq \tau_D, \end{cases}$$

where $B$ is a Brownian motion on $\mathbb{R}^d$ and $\tau_D = \inf\{t > 0 : B_t \notin D\}$ is the exit time of $B$ from $D$. We write the continuous transition density function and the $\alpha$-order resolvent density function of $X$ by $p_t(x,y)$ and $r_\alpha(x,y)$, respectively.

Set the ball average kernel $q_\varepsilon(x,y)$ by

$$q_\varepsilon(x,y) := \frac{1}{|B(x,\varepsilon)|}1_{B(x,\varepsilon)}(y), \quad \varepsilon > 0, x,y \in \mathbb{R}^d$$

and the ball average operator $T_\varepsilon$ by

$$T_\varepsilon f(x) = \int_{\mathbb{R}^d} q_\varepsilon(x,y)f(y)dy, \quad \varepsilon > 0, x \in \mathbb{R}^d, f \in \mathcal{B}_0(\mathbb{R}^d),$$

where $B(x,\varepsilon)$ is the open ball $\{y \in \mathbb{R}^d; |x-y| < \varepsilon\}$ and $\mathcal{B}_0(\mathbb{R}^d)$ is the set of bounded Borel functions on $\mathbb{R}^d$. We note that $T_\varepsilon$ is $L^r(\mathbb{R}^d)$-contractive and strongly continuous as $\varepsilon \to 0$ for any $r \geq 1$.

Suppose that $p \geq 2$ is an integer with $d - p(d - 2) > 0$. Let $X^{(1)}, \ldots, X^{(p)}$ be independent killed Brownian motions in $D$. Throughout this article, we fix their initial points $x^{(1)}_0, \ldots, x^{(p)}_0 \in D$. We write $\tau^{(1)}_D, \ldots, \tau^{(p)}_p$ as their exit times from $D$, respectively. For each $\varepsilon > 0$, we define the approximated (mutual) intersection measure $\ell^{\text{IS}}_{t,\varepsilon}$ of $X^{(1)}, \ldots, X^{(p)}$ up to time $t$ by

$$\langle \ell^{\text{IS}}_{t,\varepsilon}, f \rangle = \int_D f(x) \prod_{i=1}^{p} \int_0^t q_\varepsilon(x, X^{(i)}_s)ds dx$$

for $f \in \mathcal{B}_0(D)$, with the convention $q_\varepsilon(x, X^{(i)}_s) = 0$ when $s \geq \tau^{(i)}_D$. It is well known that for each $t > 0$, there exists a random measure $\ell^{\text{IS}}_t$ such that $\ell^{\text{IS}}_{t,\varepsilon}$ converges vaguely to $\ell^{\text{IS}}_t$ in $\mathcal{M}(D)$ and that

$$\lim_{\varepsilon \to 0} \mathbb{E}[(\langle f, \ell^{\text{IS}}_{t,\varepsilon} \rangle - \langle f, \ell^{\text{IS}}_t \rangle)^k] = 0 \quad \text{for any integer } k \geq 1 \text{ and } f \in C_K(D),$$

where $\mathcal{M}(D)$ is the set of Radon measures on $D$ equipped with the vague topology $\tau_v$ and $C_K(D)$ is the set of continuous functions on $D$ with compact support. The limit $\ell^{\text{IS}}_t$ is called the (mutual)
intersection measure of $X^{(1)}, \ldots, X^{(p)}$ up to time $t$. For detail, see [KIM13] or the author’s previous paper [Mar21] for example.

Before stating our main results, we recall the definition of large deviation principle. Usually the large deviation principle is defined for families of probability measures, but in this paper we define for families of sub-probability measures. Note that most of the basic properties of large deviation principle (e.g., contraction principle) also hold in this case.

Let $\mathcal{X}$ be a topological space. A function $I : \mathcal{X} \to [0, +\infty]$ is called a rate function (resp. good rate function) if for any $\alpha \geq 0$ the level set $\{ x \in \mathcal{X} : I(x) \leq \alpha \}$ is closed (resp. compact) in $\mathcal{X}$. We say that the family of sub-probability measures $\{ \mathcal{P}_t \}_{t \geq 0}$ on $\mathcal{X}$ satisfies the (full) large deviation principle (LDP in abbreviation) as $t \to \infty$ with rate function $I$ if

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathcal{P}_t(F) \leq -\inf_{\mu \in F} I(\mu) \quad \text{for all closed set } F \subset \mathcal{X}$$

and

$$\liminf_{t \to \infty} \frac{1}{t} \log \mathcal{P}_t(G) \geq -\inf_{\mu \in G} I(\mu) \quad \text{for all open set } G \subset \mathcal{X}.$$  

(1.3) and (1.4)

The statements (1.3) and (1.4) are referred to as the LDP upper and lower bounds, respectively. We also define the weak LDP by replacing all closed sets with all compact sets in the definition of LDP upper bound (1.3).

Our main result is the weak LDP for the intersection measure. This is a natural formulation for unbounded domains since (full) LDP fails to hold even for a single empirical measure when $D = \mathbb{R}^d$. When the domain $D$ is unbounded, we need to consider the case that some mass of the (normalized) occupation measure of a Brownian motion escapes to infinity. Hence, for the occupation measure, it is natural to consider the LDP on the space $\mathcal{M}_{1}(D_0)$, the set of probability measures on $D_0$ equipped with the weak topology, even when $D$ is the whole space $\mathbb{R}^d$. We can see that this is equivalent to the set of sub-probability measures $\mathcal{M}_{\leq 1}(D)$ equipped with the vague topology.

Define the function $I : \mathcal{M}(D) \times (\mathcal{M}_{1}(D))^p \to [0, +\infty]$ by

$$I(\mu; \mu^{(1)}, \ldots, \mu^{(p)}) := \begin{cases} \frac{1}{2} \sum_{i=1}^p \int_D |\nabla \psi^{(i)}|^2 dx, & \text{if } \psi^{(i)} = \sqrt{\frac{d\mu^{(i)}}{dx}} \in W^{1,2}_0(D) \text{ and } \int_0^t d\mu^{(i)} = \frac{d\mu}{dx}, \\ \infty, & \text{otherwise} \end{cases}$$  

(1.5)

for $(\mu; \mu^{(1)}, \ldots, \mu^{(p)}) \in \mathcal{M}(D) \times (\mathcal{M}_{1}(D))^p$, where $\mathcal{M}_{1}(D)$ is the set of probability measures on $D$ equipped with the weak topology $\tau_\nu$. Write the occupation measure $\ell_t^{(i)}$ of $X^{(i)}$ up to $t$ by $(f, \ell_t^{(i)}) = f \int_0^t f(X_s^{(i)}) ds$ for bounded Borel functions $f$ on $D$. The following weak LDP is our main result.

**Theorem 1.1.** On the space $\mathcal{M}(D) \times (\mathcal{M}_{1}(D))^p$, the law of the tuple $(t^{-p(I_{t}^{\nu})}; t^{-1}\ell_t^{(1)}, \ldots, t^{-1}\ell_t^{(p)})$ satisfies the weak LDP as $t \to \infty$ under $\mathbb{P}(\cdot, t < \tau_D^{(1)} \wedge \cdots \wedge \tau_D^{(p)})$, with the rate function $I$.

In fact we will show a stronger result (Theorem 2.2), in which the weak LDP upper bound is replaced by the full LDP upper bound on the space $\mathcal{M}(D) \times (\mathcal{M}_{\leq 1}(D), \tau_\nu)^p$, where $\mathcal{M}_{\leq 1}(D)$ is the set of sub-probability measures on $D$ equipped with the vague topology $\tau_\nu$.

**Remark 1.2.** The above result holds not only for Brownian motion but for more general processes. In Remark 3.3 and at the end of Section 6, we list the conditions for the process we used in the proof.

When the domain $D$ is bounded, there is no difference between weak and full LDP’s. Thus we recover [KIM13, Theorem 1.1].

**Proposition 1.3.** Suppose $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary. Then, on the space $\mathcal{M}(D) \times (\mathcal{M}_{1}(D))^p$, the law of the tuple $(t^{-p(I_{t}^{\nu})}; t^{-1}\ell_t^{(1)}, \ldots, t^{-1}\ell_t^{(p)})$ satisfies the full LDP as $t \to \infty$ under $\mathbb{P}(\cdot, t < \tau_D^{(1)} \wedge \cdots \wedge \tau_D^{(p)})$, with the good rate function $I$.  

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By technical improvements of the proof of the so-called super-exponential estimate (Theorem 2.1), our proof is simpler than theirs.

1.2 Related works

In this section, we briefly review some works which are related to this paper. The interested reader may refer, for example, to [LG92, Che10] for more information on intersection properties of Brownian motions.

Let us first recall another random measure called the intersection local time which is mentioned at the beginning. This measure can be formally written as

\[ \alpha(A) = \int_{\mathbb{R}^d} \left[ \prod_{i=1}^{p} \delta_x(B(i)(s_i))ds_1 \ldots ds_p \right] dx \quad \text{for} \ A \subset [0, \infty)^p \text{Borel}. \]  

(1.6)

The precise meaning and a construction of this measure can be found in the work by Geman, Horowitz and Rosen [GHR84], and Le Gall [LG92], for example. It should be emphasized that this measure \( \alpha \) is different to our \( \ell^B \) since the former measures \( \text{when} \) the Brownian paths intersect, while the latter measures \( \text{where} \) the Brownian paths intersect. These two measures have the same total mass \( \alpha([0, t]^p) = \ell^B_t([0, \mathbb{R}^d]) \), but otherwise there seems to be no other direct relation. In fact, Geman, Horowitz and Rosen [GHR84] constructed a more general object formally written as

\[ \alpha(x, A) = \int_{\mathbb{R}^d} \prod_{i=1}^{p-1} \delta_{x_i}(B^{(i+1)}(s_{i+1}) - B^{(i)}(s_i))ds_1 \ldots ds_p \quad \text{for} \ A \subset [0, \infty)^p \text{Borel} \]  

(1.7)

for each \( x = (x_1, \ldots, x_{p-1}) \in ([0, \mathbb{R}^d])^{p-1} \), which is supported on the set \( \{(s_1, \ldots, s_p) \in [0, \infty)^p : B^{(i+1)}(s_{i+1}) - B^{(i)}(s_i) = x_i \text{ for all } i = 1, \ldots, p - 1 \} \). The two measures in (1.6) and (1.7) are related as \( \alpha(ds_1 \ldots ds_p) = \alpha(0, ds_1 \ldots ds_p) \). Note that (1.7) has the spatial variable \( x \) but its role is still different to that of \( x \) in (1.6).

Now let us turn to the earlier studies on large deviations of Brownian intersections. Most of the works are about the total mass of the intersection measure. König and Mörters [KM02] investigated upper tail asymptotics of the random variable \( \ell^B_\infty (D) \) (this is equal to \( \alpha([0, \tau_D^{(1)}] \times \cdots \times [0, \tau_D^{(p)}]) \) when \( D \) is bounded. Chen [Che04] studied the law of the iterated logarithm about \( \ell^B_t([0, \mathbb{R}^d]) \) (this equals \( \alpha([0, t]^p) \)) as \( t \to \infty \) and showed that the asymptotics of the logarithmic moment generating function \( \log \mathbb{E} \exp\{\theta \ell^B_t([0, \mathbb{R}^d])^{1/p}\} \) as \( t \to \infty \) can be represented as a variational formula. Chen and Rosen [CR05] proved that similar results also hold for \( p \) independent \( \alpha \)-stable processes on \( \mathbb{R}^d \) with \( 0 < \alpha < 2 \).

We also review recent progress of moving from weak to full LDP for the occupation and intersection measures of Brownian motions. We repeatedly noted that the full LDPs fail to hold even for the occupation measure of single Brownian motion when \( D = \mathbb{R}^d \). In [Ma10], a translation-invariant compactification was developed, and a full LDP for the (embedded) “orbits” of the occupation measure was proved in such a compactification and the theory was applied to treat shift-invariant functionals of occupation measures. The case of intersection measures is different, as these are not shift-invariant functionals of occupation measures (because of contributions coming from two different translations corresponding to each occupation measure). However, the refined construction [Muk13] of the aforementioned theory only needs a certain kind of “diagonal-shift invariance”, which the intersection measure satisfies. The point is, while the intersection measure \( \ell^B_t(dx) \) satisfies only a weak LDP (which is also the main result of the present article), its total mass \( \ell^B_t([0, \mathbb{R}^d]) \) can still satisfy full LDP estimates on the whole space as has been shown using the aforementioned methods.

It is the work by König and Mukherjee [KM13] that first considered (and established) the LDP for the intersection measure \( \ell^B_t \). In that paper, they used the eigenvalue expansion of the transition density, and hence their argument requires that the domain \( D \) is bounded.
2 Proof outline

Our proof follows the same three steps as in [KMIK]:

- LDP for the approximated intersection measure $\ell_{t,\varepsilon}^{\mathsf{IS}}$,
- Convergence of the rate function as $\varepsilon \to 0$,
- Super-exponential estimate for $\ell_{t,\varepsilon}^{\mathsf{IS}} - \ell_{t}^{\mathsf{IS}}$.

The first two steps do not require new ideas and will be done in Sections 2 and 3. The technical novelty of this paper lies in the third step. More precisely, we show the following extension of [KMIK, Proposition 2.3]. Let $D \subset \mathbb{R}^d$ be a (possibly unbounded) domain with smooth boundary.

**Theorem 2.1** (Super-exponential estimate). For each $f \in C_K(D)$ and $\varepsilon > 0$, there exists positive constant $C(\varepsilon)$, which depends on $p$ and $f$ and is independent of $t$ and $k$, such that $\lim_{\varepsilon \to 0} C(\varepsilon) = 0$ and

$$
\mathbb{E}[\langle \ell_{t,\varepsilon}^{\mathsf{IS}}, f \rangle - \langle \ell_{t}^{\mathsf{IS}}, f \rangle] \leq e^{pt}(k!)^p C(\varepsilon)^k \quad \text{for any } k \geq 1 \text{ and } t > 0.
$$

We prove this in the following Section 4.

We note that Theorem 2.1 and the Markov inequality imply that the random variables $\{t^{-p}\ell_{t,\varepsilon}^{\mathsf{IS}}\}_{t,\varepsilon}$ are exponentially good approximations of $\{t^{-p}\ell_{t}^{\mathsf{IS}}\}_t$, that is, it holds that

$$
\lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}(d(t^{-p}\ell_{t,\varepsilon}^{\mathsf{IS}}, t^{-p}\ell_{t}^{\mathsf{IS}}) > \delta) = -\infty
$$

for every $\delta > 0$, where $d$ is a metric on $\mathcal{M}(D)$ associated with $\tau_v$.

Once Theorem 2.1 (and hence the exponentially good approximation) is proved, we can deduce the following large deviation result by a similar argument as in the proof of [KMIK, Theorem 1.1]. Let $\ell_{t}^{(1)}, \ldots, \ell_{t}^{(p)}$ be the occupation measure of $X^{(1)}, \ldots, X^{(p)}$ up to $t$, respectively. We extend the definition of $\mathbf{I}$ from $\mathcal{M}(D) \times (\mathcal{M}(D))^p$ to $\mathcal{M}(D) \times (\mathcal{M}_{\leq 1}(D))^p$ canonically and write the extended function as $\bar{\mathbf{I}}$.

**Theorem 2.2.**

(i) On the space $(\mathcal{M}(D), \tau_v) \times (\mathcal{M}_{1}(D), \tau_v)^p$, the law of the tuple $(t^{-p}\ell_{t}^{\mathsf{IS}}; t^{-1}\ell_{t}^{(1)}, \ldots, t^{-1}\ell_{t}^{(p)})$ satisfies the LDP lower bound as $t \to \infty$ under $\mathbb{P}(\cdot, t < \tau_{D}^{(1)} \wedge \cdots \wedge \tau_{D}^{(p)})$, with the rate function $\bar{\mathbf{I}}$.

(ii) On the space $(\mathcal{M}(D), \tau_v) \times (\mathcal{M}_{\leq 1}(D), \tau_v)^p$, the law of the tuple $(t^{-p}\ell_{t}^{\mathsf{IS}}; t^{-1}\ell_{t}^{(1)}, \ldots, t^{-1}\ell_{t}^{(p)})$ satisfies the LDP upper bound as $t \to \infty$ under $\mathbb{P}(\cdot, t < \tau_{D}^{(1)} \wedge \cdots \wedge \tau_{D}^{(p)})$, with the good rate function $\mathbf{I}$.

We remark that Theorem 2.2 (i) is exactly the same as the LDP lower bound of Theorem 2.1. Theorem 2.2 (ii) implies the LDP upper bound of Theorem 2.1 because all compact sets of $(\mathcal{M}(D), \tau_v) \times (\mathcal{M}_{1}(D), \tau_v)^p$ are closed in $(\mathcal{M}(D), \tau_v) \times (\mathcal{M}_{\leq 1}(D), \tau_v)^p$. Therefore Theorem 2.1 is proved in this way.

As for (i), unlike the LDP upper bound (ii), the LDP lower bound on the space $(\mathcal{M}(D), \tau_v) \times (\mathcal{M}_{\leq 1}(D), \tau_v)^p$ does not hold in general. Indeed, for the open set $G = \mathcal{M}(D) \times (\mathcal{M}_{\leq 1}(D), \tau_v)^p$ of $(\mathcal{M}(D), \tau_v) \times (\mathcal{M}_{\leq 1}(D), \tau_v)^p$ we have $-\inf_G \mathbf{I} = 0$. On the other hand, we can find that the value

$$
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P}\left((t^{-p}\ell_{t}^{\mathsf{IS}}; t^{-1}\ell_{t}^{(1)}, \ldots, t^{-1}\ell_{t}^{(p)}) \in G, t < \tau_{D}^{(1)} \wedge \cdots \wedge \tau_{D}^{(p)}\right)
$$

may be negative.
3 Proof of Theorem 2.1: super-exponential estimate

First, we heuristically state our idea of the proof of the super-exponential estimate. For simplicity, we assume that the processes $X^{(1)}, \ldots, X^{(p)}$ have the same initial point $x_0 \in D$ and we only consider the case $k$ is an even integer and $f = 1_U$, the indicator function of a relatively compact open subset $U$ of $D$. Note that an analogy of the Le Gall’s moment formula

$$E[(\ell_{t}^{IS}, f)^k] = \int_{D^k} f(x_1) \cdots f(x_k) \prod_{i=1}^{k} \left( \sum_{\sigma} H_i^{(i)}(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \right) dx_1 \cdots dx_k$$

holds for the intersection measure $\ell_{t}^{IS}(dx)$, where

$$H_i(x_1, \ldots, x_k) := \int_{[0,\infty]^k} 1_{\{\sum_{j=1}^{k} s_j \leq t\}} h_s(x_1, \ldots, x_p) ds_1 \cdots ds_k$$

and

$$h_s(x_1, \ldots, x_p) := p_{s_1}(x_0, x_1)1_{U}(x_1) \cdots p_{s_k}(x_{k-1}, x_k)1_{U}(x_k).$$

This type of moment formulae are firstly obtained in [1,6] for the intersection local time (ILT), and the same method also works for the intersection measure (see [4, Lemma 5.1] for example). This formula and a straightforward calculation give that for sufficient small $\varepsilon > 0$

$$E[(\ell_{t}^{IS}, f) - (\ell_{t}^{IS}, f)]^k \leq (k!)^p \|T_\varepsilon - \text{id}\|^k_{L^p(D^k)}.$$

Then our goal is to estimate the function $(T_\varepsilon - \text{id})^k h_s$ with respect to the integral $\int_{[0,\infty]^k} 1_{\{\sum_{j=1}^{k} s_j \leq t\}} ds_1 \cdots ds_k$ and then the norm $\| \cdot \|_{L^p(D^k)}$.

Now, fix small $\delta > 0$ and focus on the regime $s_1, \ldots, s_p \geq \delta$ where we need a new idea. By setting $u_j = s_j - \delta$, the Chapman-Kolmogorov relations

$$p_{u_1+\delta}(x_0, x_1) = \int_D p_{\frac{u_1+\delta}{2}}(x_0, z_1)p_{\frac{u_1}{2}}(z_1, x_1)dz_1,$$

$$p_{u_j+\delta}(x_{j-1}, x_j) = \int_D \int_D p_{\frac{u_k}{2}}(x_{j-1}, y_1)p_{\frac{u_j}{2}}(y_j, z_j)p_{\frac{u_k}{2}}(z_j, x_j)dy_1dz_j$$

and then the norm $\| \cdot \|_{L^p(D^k)}$.

To summarize, the variables $x_1, \ldots, x_k$ appear in different functions. This allows us to apply $(T_\varepsilon - \text{id})^k$ separately. Hence all what we need are estimates on the functions $(x)

$$(T_\varepsilon - \text{id})[p_{\frac{u_k}{2}}(z_{j-1}, \cdot)1_{U}(\cdot)](x), \ 1 \leq j \leq k-1 \text{ and } p_{\frac{u_k}{2}+u_k}(y_k, x)$$
uniformly over $y_2, \ldots, y_{k-1}, y_k, z_1, \ldots, z_{k-1}$ and $u_k$, with respect to proper norms.

Remark 3.1. In the previous work [1,6], when the domain $D$ is bounded, König and Mukherjee proved the super-exponential estimated by using the eigenvalue expansion

$$p_t(x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \psi_n(x)\psi_n(y)$$
to separate the variables instead of the Chapman-Kolmogorov equation, where $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $\psi_n \in W_0^{1,2}(D)$ satisfies $-\frac{1}{2}\Delta \psi_n = \lambda_n \psi_n$.

We now prove Theorem 2.3.

Proof of Theorem 2.3. Fix $f \in C_K(D)$ and take a relatively compact open set $U$ with $\text{supp}[f] \subset U$. For $\varepsilon > 0$ and $\delta > 0$, set

$$C_1(\varepsilon, \delta) := \sup_{x \in D} \left\{ \int_D \left\| (T_\varepsilon - \text{id}) [p_{\frac{1}{2}}(z, \cdot)p_{\frac{1}{2}}(\cdot, y)1_{U}(\cdot)] \right\|_{L^p(D)} dy \right\}$$

and

$$C_2(\delta) := \sup_{x \in D} \left\{ \int_0^\delta \int_0^\delta p_s(x, y) ds dy \right\}^{1/p}.$$

We also write the constant

$$C_3 := \sup_{x \in D} \left\{ \int_D r_1(x, y)^p dy \right\}^{1/p}.$$

We easily find that $\lim_{\delta \to 0} C_2(\delta) = 0$ and $C_3 < \infty$ because of the bound $p_s(x, y) \leq (2\pi)^{-d/2} \exp\{-|x - y|^2/2\delta\}$ and the assumption $d - p(d - 2) > 0$ (see equation (2.10) and Theorem A.1 of [Che14] for example). In Lemma 2.4 later, we will see that $\lim_{\delta \to 0} C_1(\varepsilon, \delta) = 0$. Therefore, we can derive the conclusion (2.4) as soon as we show the following: for sufficiently small $\delta > 0$ and $\varepsilon > 0$ with $C_1(\varepsilon, \delta) + C_2(\delta) < 1$, it holds that

$$\mathbb{E}[\| (I_{t,\varepsilon}^S, f) - (I_t^S, f)^k \|] \leq e^{\varepsilon(k!)} \| f \|_\infty \left\{ 16(C_3 + 1)(C_2(\delta) + C_1(\varepsilon, \delta))^{\frac{k}{2}} \right\}^{p\delta},$$

for any $k \geq 1$ and $t > 0$.

Once (3.3) is obtained, we can choose decreasing sequences $\{\delta_n\}_{n=1}^\infty$ and $\{\varepsilon_n\}_{n=1}^\infty$ which converge to zero as $n \to \infty$ such that $\{C_2(\delta_n) + C_1(\varepsilon_n, \delta_n)\}_{n=1}^\infty$ also converges to zero as $n \to \infty$. To obtain the conclusion (2.4), set $C(\varepsilon) := \| f \|_\infty \left\{ 16(C_3 + 1)(C_2(\delta_n) + C_1(\varepsilon_n, \delta_n))^{\frac{k}{2}} \right\}^p$ for $\varepsilon_{n+1} < \varepsilon \leq \varepsilon_n$. For $\varepsilon > \varepsilon_1$, set $C(\varepsilon) := 2\| f \|_\infty C_3^p$ because of the estimate $\mathbb{E}[\| (I_{t,\varepsilon}^S, f) - (I_t^S, f)^k \|] \leq e^{\varepsilon(k!)} C(\varepsilon)^k$ obtained by the same computation as Mor21, (2.2).

From now on, we prove (3.3). For small $\varepsilon > 0$ such that $d(\text{supp}[f], D \setminus U) \geq \varepsilon$, we have

$$\langle I_{t,\varepsilon}^S, f \rangle = \int_D f(x) \left[ \prod_{i=1}^p \int_0^\varepsilon q_{\varepsilon}(x, X_s^{(i)}) ds \right] dx = \int_D f(x) \left[ \prod_{i=1}^p \int_0^\varepsilon q_{\varepsilon}(x, X_s^{(i)}) 1_{U}(X_s^{(i)}) ds \right] dx.$$

In the following, we fix an even integer $k \geq 2$. Set

$$H_k^{(i)}(x_1, \ldots, x_k) := \int_{(0,\infty)^k} \left\{ \sum_{j=1}^k s_j \leq t \right\} \left[ \int_D \prod_{j=1}^k p_{s_j}(x_{j-1}, x_j) 1_U(x_j) \nu^{(i)}(dz_0) \right] ds_1 \cdots ds_k,$$

where $\nu^{(i)} = \delta_{x(i)}$ (Dirac’s delta measure) is the initial distribution of $X^{(i)}$. By Le Gall’s moment formula (3.1), we have

$$\mathbb{E}[\| (I_{t,\varepsilon}^S, f) - (I_t^S, f)^k \|] = \int_{D^k} f(x_1) \cdots f(x_k) \left[ \prod_{i=1}^p \left( \sum_{\sigma \in S_k} (T_\varepsilon - \text{id})^{\otimes k} (H_k^{(i)})(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \right) dx_1 \cdots dx_k \right].$$

(3.6)

Fix $\delta > 0$. We decompose $H_k^{(i)}$ as

$$H_k^{(i)}(x_1, \ldots, x_k) = \sum_{A \subset \{1, \ldots, k\}} H_k^{(i)}(A; x_1, \ldots, x_k),$$
where for \( A \subset \{1, \ldots, k\} \) we set
\[
H_t^{(i)}(A; x_1, \ldots, x_k) := \int_D \prod_{j=1}^k 1_{\{s_j \leq t\}} \prod_{j \in A} 1_{[0, \delta]}(s_j)p_{s_j}(x_{j-1}, x_j) \int U(x_j) \prod_{j \in A'} 1_{[\delta, \infty]}(s_j)p_{s_j}(x_{j-1}, x_j) \int U(x_j) \, ds_1 \cdots ds_k \nu(t)(dx_0).
\]

(3.7)

When \#A is large, we obtain the contribution of \( C_2(\delta) \) from the indices in \( A \) in the above function \( H_t^{(i)}(A; \cdot) \). On the other hand, when \#A is small, we obtain the contribution of \( C_1(\varepsilon, \delta) \) from (some subset of) \( A^c \). More precisely, we have the following proposition:

**Proposition 3.2.** Let \( k \geq 3 \) be an integer, \( A \subset \{1, \ldots, k\} \) and \( \delta > 0 \).

(i) When \#A > \( \frac{k}{4} \), it holds that
\[
\| (T_{\varepsilon} - \text{id})^{\otimes k} H_t^{(i)}(A; \cdot) \|_{L^p(D^k)} \leq e^t 2^k (C_3 + 1)^k C_2(\delta)^{\frac{k}{4}}.
\]

(ii) When \#A \leq \( \frac{k}{4} \), it holds that
\[
\| (T_{\varepsilon} - \text{id})^{\otimes k} H_t^{(i)}(A; \cdot) \|_{L^p(D^k)} \leq e^t 2^k (C_3 + 1)^k C_1(\varepsilon, \delta)^{\frac{k}{4}}.
\]

We postpone the proof Proposition 3.2 to the next section and complete the proof of Theorem 2.1 first. We have for \( k \geq 3 \),
\[
\| (T_{\varepsilon} - \text{id})^{\otimes k} H_t^{(i)} \|_{L^p(D^k)} \leq \sum_{A \subset \{1, \ldots, k\}} \sum_{\#A > \frac{k}{4}} \| (T_{\varepsilon} - \text{id})^{\otimes k} H_t^{(i)}(A; \cdot) \|_{L^p(D^k)} + \sum_{A \subset \{1, \ldots, k\}} \sum_{\#A \leq \frac{k}{4}} \| (T_{\varepsilon} - \text{id})^{\otimes k} H_t^{(i)}(A; \cdot) \|_{L^p(D^k)}
\]
\[
\leq e^t 2^k (C_3 + 1)^k \{ C_2(\delta)^{\frac{k}{4}} + C_1(\varepsilon, \delta)^{\frac{k}{4}} \}
\]
\[
\leq e^t 2^k (C_3 + 1)^k 2(C_2(\delta) + C_1(\varepsilon, \delta))^{\frac{k}{4}}.
\]

Here we used \( C_2(\delta) + C_1(\varepsilon, \delta) < 1 \). By combining this with (3.1), we have for any even integer \( k \geq 4 \),
\[
\mathbb{E}[\| (T_{\varepsilon; t}, f) - \langle \ell_t^{(S)}, f \rangle \|^k] \leq \| f \|^k \sum_{i=1}^p \| (T_{\varepsilon} - \text{id})^{\otimes k} H_t^{(i)}(A; \cdot) \|^k_{L^p(D^k)}
\]
\[
\leq e^t (k!)^p \| f \|^k \mathbb{E}[\min \{ 8(C_3 + 1)(C_2(\delta) + C_1(\varepsilon, \delta))^{\frac{k}{4}} \}^{p(k+1)}].
\]

This proves (3.3) for even integer \( k \geq 4 \). By applying Jensen’s inequality to the above inequality, we have the desired bounds (3.3) for \( k = 1, 2 \). For any odd integer \( k \geq 3 \), by combining Jensen’s inequality with the above inequalities for \( k + 1 \), we have
\[
\mathbb{E}[\| (T_{\varepsilon; t}, f) - \langle \ell_t^{(S)}, f \rangle \|^k] \leq \mathbb{E}[\| (T_{\varepsilon; t}, f) - \langle \ell_t^{(S)}, f \rangle \|^k+1]^{\frac{k}{k+1}}
\]
\[
\leq \left( e^t (k+1)! \mathbb{E}[\| f \|^k+1 \min \left\{ 8(C_3 + 1)(C_2(\delta) + C_1(\varepsilon, \delta))^{\frac{k}{4}} \right\}]^{p(k+1)} \right)^{\frac{1}{k+1}}
\]
\[
\leq e^t (k!)^p \| f \|^k \mathbb{E}[\min \left\{ 16(C_3 + 1)(C_2(\delta) + C_1(\varepsilon, \delta))^{\frac{k}{4}} \right\}]^{p(k+1)}.
\]

This proves (3.3) and thus we complete the proof of Theorem 2.1. □

**Remark 3.3.** The above proof uses the following three conditions:
\[
\lim_{\varepsilon \to 0} C_1(\varepsilon, \delta) = 0, \quad \lim_{\delta \to 0} C_2(\delta) = 0, \quad C_3 < \infty.
\]

Theorem 2.1 holds not only for Brownian motion but for processes that satisfy these conditions. As a representative example, we discuss the stable process in Section 1.
3.1 Proof of Proposition \[\text{Proposition 3.1 (i)}\]

In this section, we prove Proposition \[\text{Proposition 3.1 (i)}\]. As we stated in the previous section, we will obtain the contribution $C_2(\delta)$ from the indices in $A$. Fix $A \subset \{1, \ldots, k\}$ with $\#A > \frac{k}{3}$. We have

$$H^{(i)}_{t}(A; x_1, \ldots, x_k)$$

$$= \int_{D} \int_{[0, \infty)^k} 1_{\{\sum_{j=1}^{k} s_j \leq t\}} \prod_{j \in A} 1_{[0, \delta]}(s_j) p_{s_j}(x_{j-1}, x_{j}) 1_U(x_j) \prod_{j \in A^c} 1_{[\delta, \infty]}(s_j) p_{s_j}(x_{j-1}, x_{j}) 1_U(x_j) \, ds_1 \cdots ds_k \nu^{(i)}(dx_0)$$

$$\leq e^t \int_{D} \prod_{j \in A} \int_{0}^{\delta} p_{s_j}(x_{j-1}, x_{j}) \, ds_j \prod_{j \in A^c} r_1(x_{j-1}, x_{j}) \nu^{(i)}(dx_0).$$

Recall the notations $C_2(\delta)$ defined in (3.3) and $C_3$ in (3.4). Then we have

$$\|H^{(i)}_{t}(A; \cdot)\|_{L^p(D^k)} \leq e^t C_2(\delta)^{\#A} C_3^{\#A'} \leq e^t C_2(\delta)^{\frac{k}{2}} (C_3 + 1)^k$$

and hence, by combining this with the $L^p$-contractivity of the operator $T_{\varepsilon}$, we have

$$\|(T_{\varepsilon} - \text{id})^{\otimes k}[H^{(i)}_{t}(A; \cdot)]\|_{L^p(D^k)} \leq 2^k e^t C_2(\delta)^{\frac{k}{2}} (C_3 + 1)^k,$$

which completes the proof.

3.2 Proof of Proposition \[\text{Proposition 3.2 (ii); in case of } k = 3, A = \emptyset\]

In proving Proposition \[\text{Proposition 3.2 (ii)}\], we first deal with a simple case: $k = 3$ and $A = \emptyset$. The argument in this case contains a key estimate that we will be used in the general case.

Until the end of the next section, we simply write multiple integral $\int_{[0, \infty)^k} ds_1 \cdots ds_k$ as $\int_{[0, \infty)^k} ds$, and $\int_{D^p} \prod_{i=1}^{p} d\mathbf{z}^{(i)}$ as $\int_{D^p} d\mathbf{z}$. Recall the definition of $H^{(i)}_{t}(\emptyset; x_1, x_2, x_3)$ in (3.4). By the change of variables, we have

$$H^{(i)}_{t}(\emptyset; x_1, x_2, x_3)$$

$$= \int_{D} \nu^{(i)}(dx_0) \int_{[0, \infty)^3} ds \ 1_{\{s_1 + s_2 + s_3 \leq t - 3\delta\}}$$

$$\left( p_{s_1 + \delta}(x_0, x_1) 1_U(x_1) \right) \left( p_{s_2 + \delta}(x_1, x_2) 1_U(x_2) \right) \left( p_{s_3 + \delta}(x_2, x_3) 1_U(x_3) \right)$$

and then, the Chapman-Kolmogorov equations

$$p_{s_1 + \delta}(x_0, x_1) = \int_{D} p_{\frac{1}{2} + s_1}(x_0, z_1) p_{\frac{1}{2}}(z_1, x_1) \, dz_1,$$

$$p_{s_2 + \delta}(x_1, x_2) = \int_{D} \int_{D} p_{\frac{1}{2}}(x_1, y_2) p_{\frac{1}{2}}(y_2, z_2) p_{\frac{1}{2}}(z_2, x_2) \, dy_2 \, dz_2,$$

$$p_{s_3 + \delta}(x_2, x_3) = \int_{D} \int_{D} p_{\frac{1}{2}}(x_2, y_3) p_{\frac{1}{2} + s_3}(y_3, x_3) \, dy_3$$

give that

$$H^{(i)}_{t}(\emptyset; x_1, x_2, x_3)$$

$$= \int_{D} \nu^{(i)}(dx_0) \int_{[0, \infty)^3} ds \int_{D^2} dy_2 dy_3 \int_{D^2} d\mathbf{z} \ 1_{\{s_1 + s_2 + s_3 \leq \delta\}}$$

$$\left( p_{\frac{1}{2}}(z_1, x_1) 1_U(x_1) \right) \left( p_{\frac{1}{2}}(z_1, x_1) 1_U(x_1) \right) \left( p_{\frac{1}{2}}(z_2, x_2) 1_U(x_2) \right) \left( p_{\frac{1}{2}}(z_2, x_2) 1_U(x_2) \right)$$

As we mentioned at the beginning of Section 3, the point is that the integrand of the above equality is separated as the functions of $x_1$, $x_2$ and $x_3$. We will obtain the contribution of $C_1(\varepsilon, \delta)$ from the functions of $x_1$ and $x_2$ by applying the operator $(T_{\varepsilon} - \text{id})$. On the other hand, the function of $x_3$ does not contribute to the super-exponential estimate but it is bounded from above by $C_3$, which is
independent of $\varepsilon$. For this reason, we first apply $(T_\varepsilon - id) \otimes (T_\varepsilon - id) \otimes id$ to $H_t^{(i)}(\emptyset; \cdot)$ to get

$$(T_\varepsilon - id) \otimes (T_\varepsilon - id) \otimes id[H_t^{(i)}(\emptyset; \cdot)](x_1, x_2, x_3)$$

$$= \int_D \nu^{(i)}(dx_0) \int_{[0, \infty)^3} ds \int_{D^2} dy_2 dy_3 \int_{D^2} dz_1 dz_2 \ 1 \{s_1 + s_2 + s_3 \leq t - 3\} \ p^{s_1 + s_2}(x_0, z_1)$$

$$\left( (T_\varepsilon - id)[p_2^i(z_1, \cdot)1U(\cdot)p_2^i(\cdot, y_2)](x_1) \right) p_{s_2}(y_2, z_2)$$

$$\left( (T_\varepsilon - id)[p_2^i(z_2, \cdot)1U(\cdot)p_2^i(\cdot, y_3)](x_2) \right) p_{s_2 + s_3}(y_3, x_3)1U(x_3)$$

and then bound it as

$$|\langle (T_\varepsilon - id) \otimes (T_\varepsilon - id) \otimes id[H_t^{(i)}(\emptyset; \cdot)](x_1, x_2, x_3) \rangle| \leq e^t \int_D \nu^{(i)}(dx_0) \int_{D^2} dy_2 dy_3 \int_{D^2} dz_1 dz_2 \ r_1(x_0, z_1)$$

$$\left| (T_\varepsilon - id)[p_2^i(z_1, \cdot)1U(\cdot)p_2^i(\cdot, y_2)](x_1) \right| r_1(y_2, z_2)$$

$$\left| (T_\varepsilon - id)[p_2^i(z_2, \cdot)1U(\cdot)p_2^i(\cdot, y_3)](x_2) \right| r_1(y_3, x_3).$$

By taking $p$-th power and integrating $(x_1, x_2, x_3)$ over $D^3$, we have

$$\| (T_\varepsilon - id) \otimes (T_\varepsilon - id) \otimes id[H_t^{(i)}(\emptyset; \cdot)] \|^p_{L^p(D^3)}$$

$$\leq e^{pt} \int_{D^p} \nu^{(i)}(dx_0) \int_{D^{2p}} dy_2 dy_3 \int_{D^{2p}} dz_1 dz_2 \ \prod_{l=1}^p r_1(x_l^{(l)}, z_l^{(l)})$$

$$\left( \int_D \prod_{l=1}^p (T_\varepsilon - id)[p_2^i(z_1^{(l)}, \cdot)1U(\cdot)p_2^i(\cdot, y_2^{(l)})](x_1) \right) \prod_{l=1}^p r_1(y_2^{(l)}, z_2^{(l)})$$

$$\left( \int_D \prod_{l=1}^p (T_\varepsilon - id)[p_2^i(z_2^{(l)}, \cdot)1U(\cdot)p_2^i(\cdot, y_3^{(l)})](x_2) \right) \left( \int_D \prod_{l=1}^p r_1(y_3^{(l)}, x_3) dx_3 \right).$$

We apply Hölder’s inequality to $(\int_D \prod_{l=1}^p r_1(y_3^{(l)}, x_3) dx_3)$ and recall the notation $C_3$ introduced in (3.11) to bound (3.10) by

$$e^{pt} C_3^p \int_{D^p} \nu^{(i)}(dx_0) \int_{D^{2p}} dy_2 dy_3 \int_{D^{2p}} dz_1 dz_2 \ \prod_{l=1}^p r_1(x_l^{(l)}, z_l^{(l)})$$

$$\left( \int_D \prod_{l=1}^p (T_\varepsilon - id)[p_2^i(z_1^{(l)}, \cdot)1U(\cdot)p_2^i(\cdot, y_2^{(l)})](x_1) \right) \prod_{l=1}^p r_1(y_2^{(l)}, z_2^{(l)})$$

$$\left( \int_D \prod_{l=1}^p (T_\varepsilon - id)[p_2^i(z_2^{(l)}, \cdot)1U(\cdot)p_2^i(\cdot, y_3^{(l)})](x_2) \right) \left( \int_D \prod_{l=1}^p r_1(y_3^{(l)}, x_3) dx_3 \right).$$

We estimate the integral of (3.11) with respect to $dy_3 dz_2$. Regarding the integral with respect to $dy_3$, we apply Hölder’s inequality to $(\int_D \prod_{l=1}^p (T_\varepsilon - id)[p_2^i(z_2^{(l)}, \cdot)1U(\cdot)p_2^i(\cdot, y_3^{(l)})](x_2) dx_2)$ and recall the notation $C_1(\varepsilon, \delta)$ introduced in (2.3). Regarding the integral with respect to $dz_2$, we use the trivial inequality $\int_D \prod_{l=1}^p r_1(y_2^{(l)}, z_2^{(l)}) dx_2 \leq 1$ for all $y_2^{(l)}$. Then (3.11) is bounded from above by

$$e^{pt} C_3^p C_1(\varepsilon, \delta)^p \int_{D^p} \nu^{(i)}(dx_0) \int_{D^p} dy_2 \int_{D^p} dz_1$$

$$\prod_{l=1}^p r_1(x_l^{(l)}, z_l^{(l)}) \left( \int_D \prod_{l=1}^p (T_\varepsilon - id)[p_2^i(z_1^{(l)}, \cdot)1U(\cdot)p_2^i(\cdot, y_2^{(l)})](x_1) \right) dx_1.$$
and hence, by the \(L^p\)-contractivity of the operator \(T_\varepsilon\) we conclude
\[
\| (T_\varepsilon - \text{id})^\otimes (\mathcal{H}_t^{(i)}(\varnothing; \cdot)) \|_{L^p(D^3)} \leq 2 \| (T_\varepsilon - \text{id}) \otimes \text{id}[\mathcal{H}_t^{(i)}(A; \cdot)] \|_{L^p(D^3)} \\
\leq 2e^\varepsilon C_3 C_4(\varepsilon, \delta)^2 \\
\leq 2^k e^\varepsilon (C_3 + 1)^k C_4(\varepsilon, \delta)^\frac{3}{2},
\]
where in the last inequality, recall that we take small \(\varepsilon\) and \(\delta\) so that \(C_1(\varepsilon, \delta) < 1\). Therefore we complete the proof of Proposition 3.2 (ii) in this case.

3.3 Proof of Proposition 3.2 (ii); general case

Now we prove Proposition 3.2 in general case. Fix \(A \subset \{1, \ldots, k\} \) with \(#A \leq \frac{k}{4}\). We decompose \(A^c\) into the following four disjoint parts \(A^c = F_1 \cup F_2 \cup F_3 \cup F_4:\)
\[
F_1 := \{1 \leq j \leq k : j - 1 \notin A^c, j \in A^c, j + 1 \notin A^c\}, \\
F_2 := \{1 \leq j \leq k : j - 1 \notin A^c, j \in A^c, j + 1 \in A^c\}, \\
F_3 := \{1 \leq j \leq k : j - 1 \in A^c, j \in A^c, j + 1 \in A^c\}, \\
F_4 := \{1 \leq j \leq k : j - 1 \in A^c, j \in A^c, j + 1 \notin A^c\}.
\]
For example, if \(A^c = \{1, 3, 4, 6, 7, 8, 9\}\), then \(F_1 = \{1\}\), \(F_2 = \{3, 6\}\), \(F_3 = \{7, 8\}\), and \(F_4 = \{4, 9\}\). The previous section 3.2 is the case of \(A = F_1 = \varnothing, F_2 = \{1\}, F_3 = \{2\}\) and \(F_4 = \{3\}\). The indices in \(F_1\) and \(F_4\) do not contribute to the super-exponential estimate, and the corresponding factors are bounded from above by \(C_3\). On the other hand, from each index in \(F_2\) and \(F_3\), we obtain the contribution of \(C_1(\varepsilon, \delta)\) as in the previous section.

We repeat the argument in the previous section 3.2. Recall the definition of \(\mathcal{H}_t^{(i)}(A; x_1, x_2, x_3)\) in (3.10). By the change of variables, we have
\[
\mathcal{H}_t^{(i)}(A; x_1, \ldots, x_k) = \int_D \nu^{(i)}(dx_0) \int_{[0, \infty)^k} ds \sum_{j=1}^{t} \mathbf{1}_{\{s_j \leq t - \delta(#F_2 + #F_3 + #F_4)\}} \prod_{j \in A} \mathbf{1}_{[0, \delta]}(s_j) p_{s_j}(x_{j-1}, x_j) 1_U(x_j) \\
\prod_{j \in F_1} \mathbf{1}_{[s_j, \infty]}(s_j) p_{s_j}(x_{j-1}, x_j) 1_U(x_j) \prod_{j \in F_2 \cup F_3 \cup F_4} p_{s_j + \delta}(x_{j-1}, x_j) 1_U(x_j)
\]
and then, the Chapman-Kolmogorov equations
\[
p_{s_j + \delta}(x_{j-1}, x_j) = \int_D p_{s_j + \delta}(x_{j-1}, z_j) p_{\frac{\delta}{2}}(z_j, x_j) dz_j \quad \text{for } j \in F_2, \\
p_{s_j + \delta}(x_{j-1}, x_j) = \int_D \int_D p_{\frac{\delta}{2}}(x_{j-1}, y_j) p_{s_j}(y_j, z_j) p_{\frac{\delta}{2}}(z_j, x_j) dy_j dz_j \quad \text{for } j \in F_3, \\
p_{s_j + \delta}(x_{j-1}, x_j) = \int_D p_{\frac{\delta}{2}}(x_{j-1}, y_j) p_{s_j + \delta}(y_j, x_j) dy_3 \quad \text{for } j \in F_4
\]
give that
\[
\mathcal{H}_t^{(i)}(A; x_1, \ldots, x_k)
\]
\[
= \int_D \nu^{(i)}(dx_0) \int_{[0,\infty)^k} ds \int_{D F_3 \cup F_4} dy \int_{D F_2 \cup F_3} dz \ 1_{\{\sum_{j=1}^k s_j \leq t - \delta (#F_2 + #F_3 + #F_4)\}} \\
\prod_{j \in A} 1_{[0,\delta]}(s_j)p_s(x_{j-1}, x_j)1_U(x_j) \prod_{j \in F_1} 1_{[\delta, \infty)}(s_j)p_s(x_{j-1}, x_j)1_U(x_j) \\
\prod_{j \in F_2} p_{\frac{1}{2} + s_j}(x_{j-1}, z_j)\left( p_{\frac{1}{2}}(z_j, x_j)1_U(x_j)p_{\frac{1}{2}}(x_j, y_{j+1}) \right) \\
\prod_{j \in F_3} p_s(y_j, z_j)\left( p_{\frac{1}{2}}(z_j, y_{j+1})1_U(x_j)p_{\frac{1}{2}}(x_j, y_{j+1}) \right) \\
\prod_{j \in F_4} p_{\frac{1}{2} + \frac{1}{2}}(y_j, x_j)1_U(x_j).
\]

Here again, the point is that the integrand of the above equality is separated as the functions of \((x_j)_{j \in A \cup F_2}\), and \(x_j, j \in F_2 \cup F_3 \cup F_3\) because of the Chapman-Kolmogorov equations. As we mentioned at the beginning of this section, we obtain the contribution \(C_1(\epsilon, \delta)\) by applying the operator \((T_\epsilon - \text{id})\) to each function of the indices in \(F_2 \cup F_3\). On the other hand, the indices \(A, F_1\) and \(F_4\) do not contribute to the super-exponential estimate, and the factor is bounded above by \(C_3\), which is independent of \(\epsilon\). Note that \(F_2 \cup F_3 \subset \{1, \ldots, k - 1\}\). By setting

\[
U_j := \begin{cases} 
(T_\epsilon - \text{id}) & \text{when } j \in F_2 \cup F_3, \\
\text{id} & \text{otherwise},
\end{cases}
\]

we have

\[
(U_1 \otimes \cdots \otimes U_k)[H_t^{(i)}(A; \cdot)](x_1, \ldots, x_k)
= \int_D \nu^{(i)}(dx_0) \int_{[0,\infty)^k} ds \int_{D F_3 \cup F_4} dy \int_{D F_2 \cup F_3} dz \ 1_{\{\sum_{j=1}^k s_j \leq t - \delta (#F_2 + #F_3 + #F_4)\}} \\
\prod_{j \in A} 1_{[0,\delta]}(s_j)p_s(x_{j-1}, x_j)1_U(x_j) \prod_{j \in F_1} 1_{[\delta, \infty)}(s_j)p_s(x_{j-1}, x_j)1_U(x_j) \\
\prod_{j \in F_2} p_{\frac{1}{2} + s_j}(x_{j-1}, z_j)\left( (T_\epsilon - \text{id})[p_{\frac{1}{2}}(z_j, \cdot)p_{\frac{1}{2}}(\cdot, y_{j+1})1_U(\cdot)](x_j) \right) \\
\prod_{j \in F_3} p_s(y_j, z_j)\left( (T_\epsilon - \text{id})[p_{\frac{1}{2}}(z_j, \cdot)p_{\frac{1}{2}}(\cdot, y_{j+1})1_U(\cdot)](x_j) \right) \\
\prod_{j \in F_4} p_{\frac{1}{2} + \frac{1}{2}}(y_j, x_j)1_U(x_j)
\]

and then, since \(\sum_{j=1}^k s_j + \frac{t}{2} (#F_2 + #F_4) \leq t\), we have

\[
\left\| (U_1 \otimes \cdots \otimes U_k)[H_t^{(i)}(A; \cdot)](x_1, \ldots, x_k) \right\|
\leq e^t \int_D \nu^{(i)}(dx_0) \int_{D F_3 \cup F_4} dy \int_{D F_2 \cup F_3} dz \ |r_1(x_{j-1}, x_j)\prod_{j \in A} r_1(x_{j-1}, x_j)| \\
\prod_{j \in F_2} r_1(x_{j-1}, z_j)\left| (T_\epsilon - \text{id})[p_{\frac{1}{2}}(z_j, \cdot)p_{\frac{1}{2}}(\cdot, y_{j+1})1_U(\cdot)](x_j) \right| \\
\prod_{j \in F_3} r_1(y_j, z_j)\left| (T_\epsilon - \text{id})[p_{\frac{1}{2}}(z_j, \cdot)p_{\frac{1}{2}}(\cdot, y_{j+1})1_U(\cdot)](x_j) \right| \\
\prod_{j \in F_4} r_1(y_j, x_j).
\]

We repeat the argument from (3.8) to (3.10). Then we have

\[
\left\| (U_1 \otimes \cdots \otimes U_k)[H_t^{(i)}(A; \cdot)] \right\|_{L^p(D^k)} \leq e^t C_3^{#A + #F_1 + #F_4} C_1(\epsilon, \delta) (#F_2 + #F_3)
\leq e^t (C_3 + 1)^k C_1(\epsilon, \delta)^k,
\]

where we used \(C_1(\epsilon, \delta) < 1\). In the second line (3.10), we used the estimate \(#F_2 + #F_3 \geq \frac{t}{6}\) which is obtained as follows: the minimum of \(#F_2 + #F_3\) over \(#A \leq \frac{t}{4}\) is attained for \(A = \{2l : 1 \leq l \leq \#A\},\)
and in this case, we have \( \#F_2 + \#F_3 = \{2\#A + 1, 2\#A + 2, \ldots, k - 1\} = k - 1 - 2\#A \geq \frac{k}{6} \) since \( k \geq 3 \).

Set

\[
V_j := \begin{cases} 
\text{id} & \text{when } j \in F_2 \cup F_3, \\
(T_\varepsilon - \text{id}) & \text{otherwise.}
\end{cases}
\]

By combining (3.13) with the \( L^p \)-contractivity of the operator \( T_\varepsilon \), we have

\[
\| (T_\varepsilon - \text{id})^k [H_i^{(i)}(A; \cdot)] \|_{L^p(D^k)} = \| (V_1 \otimes \cdots \otimes V_k)(U_1 \otimes \cdots \otimes U_k)[H_i^{(i)}(A; \cdot)] \|_{L^p(D^k)}
\leq 2^k \| (U_1 \otimes \cdots \otimes U_k)[H_i^{(i)}(A; \cdot)] \|_{L^p(D^k)}
\leq 2^k e^k (C_3 + 1)^k C_1(\varepsilon, \delta)^k,
\]

which concludes the proof of Proposition 3.2 (ii).

### 3.4 Estimate of \( C_1(\varepsilon, \delta) \)

In this section, we estimate the constant introduced in (3.2):

\[
C_1(\varepsilon, \delta) := \sup_{z \in D} \left\{ \int_D \left\| (T_\varepsilon - \text{id}) \left[ p_\frac{1}{2}(z, \cdot)p_\frac{1}{2}(\cdot, y)1_{U(\cdot)} \right] \right\|_{L^p(D)} \, dy \right\}.
\]

**Lemma 3.4.** It holds that

\[
\lim_{\varepsilon \to 0} C_1(\varepsilon, \delta) = 0 \quad \text{for every } \delta > 0.
\]

**Proof.** We easily find that the transition density function \( p_t(x, y) \) of a killed Brownian motion in \( D \) has the following properties (see [CZ82, Theorem 2.4] for example):

\[
p_\frac{1}{2}(\cdot, \cdot) \text{ is continuous on } D \times D, \tag{3.14}
\]

\[
\int_D \left\| p_\frac{1}{2}(\cdot, y)1_{U(\cdot)} \right\|_{L^p(D)} \, dy < \infty, \tag{3.15}
\]

\[
\lim_{z \to \delta} p_\frac{1}{2}(z, x) = 0 \quad \text{for each } x \in D. \tag{3.16}
\]

Now, we can see that (3.13) implies \( \lim_{\varepsilon \to 0} \| (T_\varepsilon - \text{id})[p_\frac{1}{2}(z, \cdot)p_\frac{1}{2}(\cdot, y)1_{U(\cdot)}] \|_{L^p(D)} = 0 \) for each \( y, z \in D \). By taking a relatively compact neighborhood \( V \) of \( z \), we can also see the bound

\[
\| (T_\varepsilon - \text{id})[p_\frac{1}{2}(z, \cdot)p_\frac{1}{2}(\cdot, y)1_{U(\cdot)}] \|_{L^p(D)} \leq 2 \left( \sup_{V \times U} p_\frac{1}{2}(\cdot, \cdot) \right) \| [p_\frac{1}{2}(\cdot, y)1_{U(\cdot)}] \|_{L^p(D)} \tag{3.17}
\]

for each \( \varepsilon > 0 \) and each \( y \in D \). Note that the upper bound in (3.17) is an integrable function of \( y \) because of (3.13) and (3.14). By the dominated convergence theorem, we have

\[
\lim_{\varepsilon \to 0} \int_D \| (T_\varepsilon - \text{id})[p_\frac{1}{2}(z, \cdot)p_\frac{1}{2}(\cdot, y)1_{U(\cdot)}] \|_{L^p(D)} \, dy = 0 \quad \text{for fixed } z \in D \tag{3.18}
\]

and find that

\[
D \ni z \mapsto \int_D \| (T_\varepsilon - \text{id})[p_\frac{1}{2}(z, \cdot)p_\frac{1}{2}(\cdot, y)1_{U(\cdot)}] \|_{L^p(D)} \, dy \quad \text{is continuous} \tag{3.19}
\]

for each \( \varepsilon > 0 \).

Next, (3.13) and (3.14) give the estimate

\[
\sup_{\varepsilon > 0} \int_D \| (T_\varepsilon - \text{id})[p_\frac{1}{2}(z, \cdot)p_\frac{1}{2}(\cdot, y)1_{U(\cdot)}] \|_{L^p(D)} \, dy
\leq 2 \left( \sup_{x \in U} p_\frac{1}{2}(z, x) \right) \int_D \| p_\frac{1}{2}(\cdot, y)1_{U(\cdot)} \|_{L^p(D)} \, dy \tag{3.20}
\]

and the right-hand side goes to 0 as \( z \to \partial \). Therefore, the desired uniform convergence \( \lim_{\varepsilon \to 0} C_1(\varepsilon, \delta) = 0 \) follows from (3.13), (3.14) and (3.20). \( \square \)
4 Large deviation lower bound

In this section we prove Theorem 2.1 (i), the LDP lower bound. Let $X$ be a killed Brownian motion in a domain $D \subset \mathbb{R}^d$ with smooth boundary. Define the occupation measure $\ell_t$ of $X$ up to $t$ by $\langle f, \ell_t \rangle = \int_0^t f(X_s) \, ds$ for bounded Borel functions $f$ on $D$. We first recall the well known Donsker-Varadhan type large deviation lower bound for the normalized occupation measure $t^{-1}\ell_t$ on $(\mathcal{M}_1(D), \tau_w)$.

**Theorem 4.1** ([10], Proposition 4.1]). Define the function $I : \mathcal{M}_1(D) \rightarrow [0, +\infty]$ by

$$I(\mu) = \begin{cases} \frac{1}{2} \int_D |\nabla \psi|^2 \, dx & \text{if } \mu = \psi^2 \, dx, \psi \in W^{1,2}_0(D), \psi \geq 0, \\ \infty & \text{otherwise} \end{cases}$$

for $\mu \in \mathcal{M}_1(D)$. Then, on the space $(\mathcal{M}_1(D), \tau_w)$, the family of occupation measures $\{t^{-1}\ell_t\}_t$ satisfies the LDP lower bound as $t \rightarrow \infty$ under $\mathbb{P}(\cdot, t < \tau_D)$ with the rate function $I$.

For each $\varepsilon > 0$, we define the function $\Phi_\varepsilon : (\mathcal{M}_1(D), \tau_w)^p \rightarrow (\mathcal{M}(D), \tau_v) \times (\mathcal{M}_1(D), \tau_w)^p$ by

$$\Phi_\varepsilon(\mu^{(1)}, \ldots, \mu^{(p)}) := \left( \prod_{i=1}^p q_\varepsilon[\mu^{(i)}](x) \, dx ; \mu^{(1)}, \ldots, \mu^{(p)} \right).$$

Since the function $\Phi_\varepsilon$ is continuous, the contraction principle of LDP gives the following.

**Lemma 4.2.** Define the function $I_\varepsilon : \mathcal{M}(D) \times (\mathcal{M}_1(D))^p \rightarrow [0, +\infty]$ by

$$I_\varepsilon(\mu; \mu^{(1)}, \ldots, \mu^{(p)}) := \inf \left\{ \sum_{i=1}^p I(\nu^{(i)}) \right\} \nu^{(1)}, \ldots, \nu^{(p)} \in \mathcal{M}_1(D), \Phi_\varepsilon(\nu^{(1)}, \ldots, \nu^{(p)}) = (\mu; \mu^{(1)}, \ldots, \mu^{(p)})$$

for $(\mu; \mu^{(1)}, \ldots, \mu^{(p)}) \in \mathcal{M}(D) \times (\mathcal{M}_1(D))^p$.

Then, for any open set $G \subset (\mathcal{M}(D), \tau_v) \times (\mathcal{M}_1(D), \tau_w)^p$, it holds that

$$\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P}\left( (t^{-p}\ell^{(1)}_t, \ldots, t^{-p}\ell^{(p)}_t) \in G, t < \tau_D^{(1)} \wedge \cdots \wedge \tau_D^{(p)} \right) \geq - \inf_{\mu \in G} I_\varepsilon(\mu).$$

We will show that $I$ defined in (4.1) is a rate function later, so we don’t check whether $I_\varepsilon$ is a rate function or not.

We next give the relation between $I$ and so-called the $\Gamma$-lower limit of $I_\varepsilon$.

**Proposition 4.3.** For every $\mu \in \mathcal{M}(D) \times (\mathcal{M}_1(D))^p$, it holds that

$$I(\mu) \geq \sup_{\delta > 0} \liminf_{\varepsilon \downarrow 0} \inf_{\nu \in B_\delta(\mu)} I_\varepsilon(\nu),$$

where $B_\delta(\mu)$ is the open ball with center $\mu$ and radius $\delta$, with respect to a metric of $(\mathcal{M}(D), \tau_v) \times (\mathcal{M}_1(D), \tau_w)^p$.

**Proof.** The following is based on the proof of [10], Proposition 1.2. Let $\mu = (\mu; \mu^{(1)}, \ldots, \mu^{(p)}) \in \mathcal{M}(D) \times (\mathcal{M}_1(D))^p$ with $I(\mu) < \infty$ be given. Take nonnegative $\psi^{(i)} \in W^{1,2}_0(D)$ such that $\mu^{(i)}(dx) = (\psi^{(i)})^2 \, dx$ and $\mu(dx) = \prod_{i=1}^p (\psi^{(i)})^2 \, dx$.

Fix $\delta > 0$ and take $\varepsilon > 0$ so small such that $\prod_{i=1}^p q_\varepsilon[\mu^{(i)}] \in B_{\delta^2/2p}(\mu)$. This is possible, since the triangle inequality, Hölder’s inequality and the $L^p(\mathbb{R}^d)$-contractivity of $q_\varepsilon$ give that

$$\left\| \prod_{i=1}^p q_\varepsilon[\psi^{(i)}]^2 - \prod_{i=1}^p (\psi^{(i)})^2 \right\|_{L^1(D)}$$
\[
\leq \left\| \prod_{i=1}^{p} q_\varepsilon[(\psi^{(i)})^2] - (\psi^{(1)})^2 \prod_{i=2}^{p} q_\varepsilon[(\psi^{(i)})^2] \right\|_{L^1(D)} \\
+ \left\| (\psi^{(1)})^2 \prod_{i=2}^{p} q_\varepsilon[(\psi^{(i)})^2] - (\psi^{(1)})^2(\psi^{(2)})^2 \prod_{i=3}^{p} q_\varepsilon[(\psi^{(i)})^2] \right\|_{L^1(D)} \\
+ \cdots + \left\| \prod_{i=1}^{p-1} (\psi^{(i)})^2 q_\varepsilon[(\psi^{(p)})^2] - \prod_{i=1}^{p-1} (\psi^{(i)})^2 \right\|_{L^1(D)} \\
\leq \sum_{i=1}^{p} \left( \prod_{l<i} \|\psi^{(l)}\|_{L^2p(D)}^2 \right) \left\| q_\varepsilon[(\psi^{(i)})^2] - (\psi^{(i)})^2 \right\|_{L^p(\mathbb{R}^d)} \left( \prod_{l>i} \|q_\varepsilon[(\psi^{(l)})^2]\|_{L^p(\mathbb{R}^d)} \right) \\
\leq \sum_{i=1}^{p} \left\| q_\varepsilon[(\psi^{(i)})^2] - (\psi^{(i)})^2 \right\|_{L^p(\mathbb{R}^d)} \left( \prod_{l\neq i} \|\psi^{(l)}\|_{L^2p(D)}^2 \right).
\]

(4.1)

The last line (4.1) goes to 0 as \( \varepsilon \to 0 \), because of the Sobolev embedding theorem \( W^{1,2}(D) \subset L^{2p}(D) \) (recall the assumption \( \dim(D) - p(d-2) > 0 \), i.e., \( 2p < 2d/(d-2) \)) and the \( L^p(\mathbb{R}^d) \)-continuity of \( q_\varepsilon \). Hence we have for any \( f \in C_K(D) \)

\[
\left| \left\langle f, \prod_{i=1}^{p} q_\varepsilon[\mu^{(i)}] \right\rangle - \langle f, \mu \rangle \right| = \left| \left\langle f, \prod_{i=1}^{p} q_\varepsilon[(\psi^{(i)})^2] \right\rangle - \left\langle f, \prod_{i=1}^{p} (\psi^{(i)})^2 \right\rangle \right| \leq \|f\|_\infty \left\| \prod_{i=1}^{p} q_\varepsilon[(\psi^{(i)})^2] - \prod_{i=1}^{p} (\psi^{(i)})^2 \right\|_{L^1(D)}
\]

and the right-hand side goes to 0 as \( \varepsilon \to 0 \). We thus obtain \( \left( \prod_{i=1}^{p} q_\varepsilon[\mu^{(i)}] \right) dx; \mu^{(1)}, \ldots, \mu^{(p)} \in B_3(\mu) \) and hence

\[
\inf_{B_3(\mu)} I \leq \left\| \prod_{i=1}^{p} q_\varepsilon[\mu^{(i)}] \right\|_{L^1(D)} \leq \left\| \prod_{i=1}^{p} q_\varepsilon[\mu^{(i)}] \right\|_{L^1(D)},
\]

which concludes the proof. \( \square \)

**Proof of Theorem**. \( \square \) (i). We first prove that \( I \) is a rate function. Let \( \alpha > 0 \) be fixed. Suppose a sequence \( \{\mu_n; \mu^{(1)}_n, \ldots, \mu^{(p)}_n\} \subset \{1 \leq \alpha\} \) and take nonnegative \( \psi_n^{(i)} \in W^{1,2}_0(D) \) such that \( \mu_n^{(i)}(dx) = (\psi_n^{(i)})^2dx \) and \( \mu_n(dx) = \left[ \prod_{i=1}^{p} (\psi_n^{(i)})^2 \right] dx \). We assume that \( \mu_n \) converges to \( \mu \) in \( (M(D), \tau_\varepsilon) \) and \( \mu^{(i)}_n \) converges to \( \mu^{(i)} \) in \( (M_1(D), \tau_\varepsilon) \). Since \( \{\psi^{(i)}_n\} \) is bounded in \( W^{1,2}_0(D) \) for each \( i \), by taking a subsequence we may assume that \( \psi^{(i)}_n \) converges weakly to \( \psi^{(i)} \in W^{1,2}_0(D) \) for all \( i \).

For \( f \in C_K(D) \), take a bounded open set \( U \subset D \) with smooth boundary such that \( \text{supp}(f) \subset U \) and \( \bar{U} \subset D \). The Rellich-Kondrashov theorem gives that \( \{\psi^{(i)}_n 1_U\} \) converges strongly to \( \psi^{(i)} 1_U \) in \( W^{1,2}(U) \) for each \( i \) (it hold that \( 2p < 2d/(d-2) \) as we wrote below (4.2)). Then we have

\[
\int_D f(\psi^{(i)})^2 dx = \int_U f(\psi^{(i)} 1_U)^2 dx = \lim_{n \to \infty} \int_U f(\psi^{(i)}_n 1_U)^2 dx = \lim_{n \to \infty} \langle f, \mu^{(i)}_n \rangle
\]

(4.2)

and have

\[
\frac{1}{2} \sum_{i=1}^{p} \int_D |\nabla \psi^{(i)}|^2 dx = \frac{1}{2} \sum_{i=1}^{p} \int_U |\nabla (\psi^{(i)} 1_U)|^2 dx = \lim_{n \to \infty} \frac{1}{2} \sum_{i=1}^{p} \int_U |\nabla (\psi^{(i)}_n 1_U)|^2 dx \leq \|f\|_\infty \alpha.
\]

(4.3)

By the same way as to obtain (4.2), we also have

\[
\left\| \prod_{i=1}^{p} (\psi^{(i)}_n 1_U)^2 - \prod_{i=1}^{p} (\psi^{(i)} 1_U)^2 \right\|_{L^1(D)} \leq \sum_{i=1}^{p} \left\| (\psi^{(i)}_n 1_U)^2 - (\psi^{(i)} 1_U)^2 \right\|_{L^p(D)} \prod_{l\neq i} \left( \sup_n \|\psi^{(i)}_n\|^2_{L^{2p}(D)} + \|\psi^{(i)}\|^2_{L^{2p}(D)} \right).
\]
The right-hand side of the above inequality goes to 0 as \( \varepsilon \to 0 \), because of the Sobolev and Rellich-Kondrashov embedding theorem. Hence \( \mu_n \) converges to \( \left[ \prod_{i=1}^p (\psi^{(i)})^2 \right] \) dx in \( (\mathcal{M}(D), \tau_v) \) and therefore \( \mu(dx) = \left[ \prod_{i=1}^p (\psi^{(i)})^2 \right] dx, \mu^{(i)}(dx) = (\psi^{(i)})^2 dx \) and \( I(\mu; \mu^{(1)}, \ldots, \mu^{(p)}) \leq \alpha \). Hence \( I \) is a rate function.

As we have seen below Theorem \( \text{(ii)} \), the tuple of random measures \( \left\{ (t^{-p}(\delta_{t}^{1})^{(1)}, \ldots, -1, t^{-1}) \right\} \) are exponentially good approximations of \( \left\{ (t^{-p}(\delta_{t}^{1}), \ldots, -1, \delta_{t}^{p}) \right\} \). Then it is straightforward to get the desired LDP lower bound from Proposition \( \text{(ii)} \). See, for example, the lower bound of the proof of \( \text{[DZ98, Theorem 4.2.16 (a)]} \).

\[ \square \]

5 Large deviation upper bound

In this section we prove Theorem \( \text{(ii)} \), the LDP upper bound. We also prove Proposition \( \text{(ii)} \) at the end of this section. As we mentioned in Section \( 2 \), when the domain \( D \) is unbounded, we need to consider the case that some mass of the (normalized) occupation measure of a Brownian motion escapes to infinity. Hence, for the occupation measure, it is natural to consider the full LDP on the space \( \mathcal{M}_1(D_0) \) even when \( D = \mathbb{R}^d \).

Let \( \ell \) be the occupation measure of a killed Brownian motion up to \( t \) and regard this as a measure on \( D_0 \). Just as in the previous section, we have the Donsker-Varadhan type large deviation upper bound for the normalized occupation measure \( t^{-1} \ell \) on the compactified space \( \mathcal{M}_1(D_0), \tau_v \).

Lemma 5.1. Define the function \( I^0 : \mathcal{M}_1(D_0) \to [0, +\infty] \) by

\[
I^0(\mu) = \begin{cases}
\frac{1}{2} \int_D |\nabla \psi|^2 dx & \text{if } \mu = \psi^2 dx + \rho \delta_0, \psi \in W^{1,2}_0(D), \psi \geq 0, c \geq 0, \\
\infty & \text{otherwise}
\end{cases}
\]

for \( \mu \in \mathcal{M}_1(D_0) \). Then, on the space \( (\mathcal{M}_1(D_0), \tau_v) \), the family of occupation measures \( \{ t^{-1}\ell \} \) satisfies the LDP upper bound as \( t \to \infty \) under \( \mathbb{P} \left( \cdot, t < \tau_D \right) \) with the good rate function \( I^0 \).

Proof. We first prove the compactness of the level set. Let \( \{ \mu_n \} \subseteq \{ I^0 \leq \alpha \} \) and take \( \psi_n \in W^{1,2}_0(D) \) such that \( \mu_n|_D(dx) = \psi^2_n dx \). By the same way as to obtain \( \text{(ii)} \) and \( \text{(ii)} \), there exists \( \psi_0 \in W^{1,2}_0(D) \) with \( \| \psi \|_2 \leq 1 \) such that, by taking a subsequence, \( \mu_n \) converges to \( \mu := \psi^2 dx + (1 - \| \psi \|_2^2) \delta_0 \) in \( (\mathcal{M}_1(D_0), \tau_v) \) and \( I^0(\mu) \leq \alpha \). Hence \( I^0 \) is a good rate function.

The LDP upper bound is proved in \( \text{[DV73]} \) in the case of \( D = \mathbb{R}^d \). More general case, we use the same argument as that to obtain \( \text{[KMT10, (3.7)]} \) or \( \text{[KMT10, (4.5)]} \).

For each \( \varepsilon > 0 \) we define the function \( \Phi^0 : (\mathcal{M}_1(D_0), \tau_v)^p \to (\mathcal{M}(D), \tau_v) \times (\mathcal{M}_{\leq 1}(D), \tau_v)^p \) by

\[
\Phi^0(\mu^{(1)}, \ldots, \mu^{(p)}) := \left( \prod_{i=1}^p q_x[\mu^{(i)}](x) \right) dx; \mu^{(1)}|_D, \ldots, \mu^{(p)}|_D.
\]

Since the function \( \Phi^0 \) is continuous, the contraction principle of LDP gives the following.

Lemma 5.2. Define the function \( I^0 : \mathcal{M}(D) \times \mathcal{M}_{\leq 1}(D)^p \to [0, +\infty] \) by

\[
I^0(\mu, \mu^{(1)}, \ldots, \mu^{(p)}) := \inf \left\{ \sum_{i=1}^p I^0(\psi^{(i)}) \left| \psi^{(1)}, \ldots, \psi^{(p)} \in \mathcal{M}_1(D_0), \Phi^0(\psi^{(1)}, \ldots, \psi^{(p)}) = (\mu, \mu^{(1)}, \ldots, \mu^{(p)}) \right. \right\}
\]

\[
= \begin{cases}
\frac{1}{2} \sum_{i=1}^p \int_D |\nabla \psi^{(i)}|^2 dx, & \text{if } \psi^{(i)} = \sqrt{\frac{d\mu^{(i)}}{dx}} \in W^{1,2}_0(D) \text{ and } d\mu = \prod_{i=1}^p q_x[\mu^{(i)}], \\
\infty, & \text{otherwise}
\end{cases}
\]

for \( (\mu, \mu^{(1)}, \ldots, \mu^{(p)}) \in \mathcal{M}(D) \times (\mathcal{M}_{\leq 1}(D))^p \).
Then, on the space (\(M(D), \tau_v\)) \(\times (\mathcal{M}_{\leq 1}(D), \tau_v)^p\), the law of the tuple \((t^{-p}t^1_{t, \varepsilon}, t^{-1}t^1_{t}, \ldots, t^{-1}t^p_{t})\) satisfies the LDP upper bound as \(t \to \infty\) under \(\mathbb{P}(\cdot, t < \tau^1_D \land \cdots \land \tau^p_D)\), with the good rate function \(\overline{\mathbf{I}}_v\).

Recall the function \(\overline{\mathbf{I}}\) defined in Section 2.

**Proposition 5.3.** For every closed set \(F \subset (\mathcal{M}(D), \tau_v) \times (\mathcal{M}_{\leq 1}(D), \tau_v)^p\), it holds that

\[
\inf_{\mu \in F} \overline{\mathbf{I}}(\mu) \leq \liminf_{\varepsilon \to 0} \inf_{\mu \in F} \overline{\mathbf{I}}_v(\mu). \tag{5.1}
\]

**Proof.** Without loss of generality, we may assume that

\(R := \liminf_{\varepsilon \to 0} \inf_{\mu \in F} \overline{\mathbf{I}}_v(\mu) < \infty\).

Fix \(\eta > 0\). Then for each \(\varepsilon > 0\) we can pick \(\mu_\varepsilon = (\mu_{\varepsilon}^{(1)}, \ldots, \mu_{\varepsilon}^{(p)}) \in F\) with \(\overline{\mathbf{I}}_v(\mu_\varepsilon) \leq R + \eta\). By the definition of \(\overline{\mathbf{I}}_v\), there are nonnegative \(\psi^{(i)} \in W^{1,2}(D)\) such that \(\mu_{\varepsilon}^{(i)}(dx) = (\psi^{(i)}_\varepsilon)^2 dx\) and \(\mu_{\varepsilon}(dx) = \left(\prod_{i=1}^p q_{\varepsilon}[(\psi^{(i)}_\varepsilon)^2]\right) dx\). In particular, \(\frac{1}{2} \sum_{i=1}^p \int_D |\nabla \psi^{(i)}_\varepsilon|^2 dx \leq R + \eta\) and hence \(\{\psi^{(i)}_\varepsilon\}_\varepsilon\) is bounded in \(W^{1,2}(D)\).

Set \(\mu_\varepsilon' = \left(\prod_{i=1}^p (\psi^{(i)}_\varepsilon)^2\right) dx\) and \(\mu_\varepsilon'' = (\mu_{\varepsilon}^{(1)}, \ldots, \mu_{\varepsilon}^{(p)})\). By the same way as the proof of Theorem 4.2 (i), for some sequence \(\varepsilon_n \downarrow 0\) (in the following, we write this as \(\varepsilon \downarrow 0\) with some abuse of notations), \(\mu_\varepsilon''\) converges to some \(\mu = (\mu^{(1)}, \ldots, \mu^{(p)})\) in \((\mathcal{M}(D), \tau_v) \times (\mathcal{M}_{\leq 1}(D), \tau_v)^p\). By the same way as to obtain (4.1), we also have for any a bounded open set \(U \subset D\) with smooth boundary and \(\overline{U} \subset D\),

\[
\left|\prod_{i=1}^p q_{\varepsilon}[(\psi^{(i)}_\varepsilon)^2] - \prod_{i=1}^p (\psi^{(i)}_\varepsilon)^2\right|_{L^1(U)} \leq \sum_{i=1}^p \|q_{\varepsilon}[(\psi^{(i)}_\varepsilon)^2] - (\psi^{(i)}_\varepsilon)^2\|_{L^p(U)} \sum_{i \neq k} \left(\sup_{\varepsilon} \|\psi^{(i)}_\varepsilon\|_{L^2(D)}\right),
\]

which goes to 0 as \(\varepsilon \to 0\) because of the Sobolev and Rellich-Kondrashov embedding theorems. Hence \(\mu_\varepsilon''\) converges to \(\mu\) in \((\mathcal{M}(D), \tau_v) \times (\mathcal{M}_{\leq 1}(D), \tau_v)^p\).

Therefore, we have \(\mu \in F\) and \(\inf_{\mu \in F} \overline{\mathbf{I}} \leq \overline{\mathbf{I}}(\mu) \leq R + \eta\). The conclusion (5.1) follows by letting \(\eta \to 0\).

**Proof of Theorem 4.2 (ii).** We can show that \(\overline{\mathbf{I}}\) is a good rate function by a similar way to the proof of Theorem 4.2 (i). As we have seen in below Theorem 4.1, \(\{(t^{-p}t^1_{t, \varepsilon}, t^{-1}t^1_{t}, \ldots, t^{-1}t^p_{t})\}_t\varepsilon\) are exponentially good approximations of \(\{(t^{-p}t^1_{t}, t^{-1}t^1_{t}, \ldots, t^{-1}t^p_{t})\}_t\). By combining this with Proposition 5.1, it is straightforward to get the desired LDP upper bound. See, for example, the lower bound of the proof of \([DZ98]\), Theorem 4.2.16 (b)].

**Proof of Proposition 5.3.** When the domain \(D\) is bounded, the rate function \(I\) defined in Theorem 4.1 is indeed a good rate function and the upper LDP also holds for the normalized occupation measure \(t^{-1}t^1_{t}\) (see [IY11] Theorem 1.1 for example). We repeat the arguments in this section with replacing \(I^0, I_v, \text{and } I\) by \(I, I_v, \text{and } I\), respectively. Then the desired LDP for the intersection measure follows.

### 6 Large deviation principle for the intersection measure of stable processes

In this section, we discuss the LDP for the intersection measure of stable processes. Throughout this section, let \(\alpha \in (0, 2)\). We consider a rotationally symmetric \(\alpha\)-stable process killed upon leaving a domain \(D\) with smooth boundary. It is known that (see [CKS11] for example) the process has a transition density function \(p_t(x, y)\) with respect to the Lebesgue measure that is jointly continuous and \(p_t(x, y)\) has a following upper estimate: for every \(T > 0\) there exists a constant \(C > 0\) such that

\[
p_t(x, y) \leq C \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right)
\]

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for all \((t, x, y) \in (0, T] \times D \times D\), where \(d_D(x)\) is the Euclidean distance between \(x\) and \(D^c\). The following embedding theorem and compact embedding theorem for the fractional Sobolev space \(W^{\frac{\alpha}{2}, 2}(\mathbb{R}^d)\) are also known (see [DNPV12] for example):

**Theorem 6.1.** Suppose \(\alpha < d\). The Banach space

\[
W^{\frac{\alpha}{2}, 2}(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+\alpha}} \, dx \, dy < \infty \right\}
\]

equipped with the norm

\[
\|u\|_{W^{\frac{\alpha}{2}, 2}} = \|u\|_{L^2}^2 + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+\alpha}} \, dx \, dy, \quad u \in W^{\frac{\alpha}{2}, 2}(\mathbb{R}^d)
\]

is continuously embedded in \(L^{2p}(\mathbb{R}^d)\) for \(p \geq 1\) with \(d - p(d - \alpha) > 0\).

**Theorem 6.2.** Suppose \(\alpha < d\) and \(U \subset \mathbb{R}^d\) be a bounded open set with smooth boundary. If \(\mathcal{J}\) is a bounded subset of \(L^2(U)\) satisfying

\[
\sup_{u \in \mathcal{J}} \int_{U} \int_{U} \frac{|u(x) - u(y)|^2}{|x - y|^{d+\alpha}} \, dx \, dy < \infty,
\]

then \(\mathcal{J}\) is relatively compact in \(L^{2p}(U)\) for \(p \geq 1\) with \(d - p(d - \alpha) > 0\).

Now suppose \(\alpha < d\) and \(d - p(d - \alpha) > 0\). We can find that until the previous section we used only the following conditions as properties of a killed Brownian motion:

- \(\lim_{\alpha \downarrow 0} C_2(\delta) = 0\) and \(C_3 < \infty\) (recall (13) and (14) for notation),
- conditions (12.13), (12.15) and (12.10) (and hence \(\lim_{\varepsilon \to 0} C_1(\varepsilon, \delta) = 0\)),
- the Sobolev and Rellich-Kondrashov embedding theorems.

Then our main result Theorem 13 also holds for the intersection measure of killed stable processes by replacing \(W^{1, 2}_0(D)\) and \(\int_D |\nabla \psi|^2 \, dx\) with \(W^{\frac{\alpha}{2}, 2}_0(D) := \frac{C_0^\infty}{C_0^\infty}(D)\) and \(\int_D \int_D \frac{\psi(x) - \psi(y)^2}{|x - y|^{d+\alpha}} \, dx \, dy\), respectively.

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**References**

[ABK06] S. Adams, J.-B. Bru, and W. König. Large deviations for trapped interacting Brownian particles and paths. *Ann. Probab.*, 34(4):1370–1422, 2006.

[Che04] X. Chen. Exponential asymptotics and law of the iterated logarithm for intersection local times of random walks. *Ann. Probab.*, 32(4):3248–3300, 2004.

[Che10] X. Chen. *Random walk intersections*, volume 157 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[CKS10] Z.-Q. Chen, P. Kim, and R. Song. Heat kernel estimates for the Dirichlet fractional Laplacian. *J. Eur. Math. Soc. (JEMS)*, 12(5):1307–1329, 2010.
[MV16] C. Mukherjee and S. R. S. Varadhan. Brownian occupation measures, compactness and large deviations. *Ann. Probab.*, 44(6):3934–3964, 2016.

[Tak98] M. Takeda. Asymptotic properties of generalized Feynman-Kac functionals. *Potential Anal.*, 9(3):261–291, 1998.

[Tak07] M. Takeda. $L^p$-independence of spectral bounds of Schrödinger type semigroups. *J. Funct. Anal.*, 252(2):550–565, 2007.

[Tak11] M. Takeda. A large deviation principle for symmetric Markov processes with Feynman-Kac functional. *J. Theoret. Probab.*, 24(4):1097–1129, 2011.