Probabilistic pointwise convergence problem of some dispersive equations

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Abstract. In this paper, we investigate the almost surely pointwise convergence problem of free KdV equation, free wave equation, free elliptic and non-elliptic Schrödinger equation respectively. We firstly establish some estimates related to the Wiener decomposition of frequency spaces which is just Lemmas 2.1-2.6 in this paper. Secondly, by using Lemmas 2.1-2.6, 3.1, to establish the probabilistic estimates of some random series which is just Lemmas 3.2-3.11 in this paper. Finally, combining the density theorem in $H^s$ with Lemmas 3.2-3.11, we obtained almost surely pointwise convergence of the solutions to corresponding equations with randomized initial data in $L^2$, which require much less regularity of the initial data than the rough data case. We also present the probabilistic density theorem.

Keywords: Probabilistic pointwise convergence; KdV equation, Wave equation, Elliptic and non-elliptic Schrödinger equation, Random data

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1. Introduction

In this paper, we investigate the pointwise convergence problem of the free KdV equation in $\mathbb{R}$

\[
\begin{aligned}
&\left\{\begin{array}{l}
  u_t + \partial_x^3 u = 0, \ (x, t) \in \mathbb{R} \times \mathbb{R}, \\
  u(x, 0) = f(x), \ x \in \mathbb{R},
\end{array}\right. \\
&(1.1)
\end{aligned}
\]

free wave equation in $\mathbb{R}^n$, $n \geq 2$,

\[
\begin{aligned}
&\left\{\begin{array}{l}
  u_{tt} + \Delta u = 0, \ (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\
  u(x, 0) = f(x), \ u_t(x, 0) = 0, \ x \in \mathbb{R}^n,
\end{array}\right. \\
&(1.2)
\end{aligned}
\]

and free Schrödinger equation in $\mathbb{R}^n$, $n \geq 1$,

\[
\begin{aligned}
&\left\{\begin{array}{l}
  iu_t + \Delta \pm u = 0, \ (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\
  u(x, 0) = f(x), \ x \in \mathbb{R}^n.
\end{array}\right. \\
&(1.3)
\end{aligned}
\]

Here $\Delta = \sum_{j=1}^n \epsilon_j \partial_{x_j}^2$, $\epsilon_j = \pm 1$. The formal solutions to the free KdV (1.1), the free wave equation (1.2) and the free Schrödinger equation (1.2) are given respectively by

\[
S_1(t) f(x) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{ix\xi + i\xi^3 t} \mathcal{F}_x f(\xi) d\xi, \\
(1.4)
\]

\[
S_{2\pm}(t) f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\pm i|\xi| t} \mathcal{F}_x f(\xi) d\xi_1 d\xi_2 \cdots d\xi_n, \\
(1.5)
\]

and

\[
S_3(t) f(x_1, x_2, \cdots, x_n) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi + i|\xi| t} \left[ \sum_{j=1}^n \epsilon_j \xi_j \right] \mathcal{F}_x f(\xi_1) d\xi_1 d\xi_2 \cdots d\xi_n, \epsilon_j = \pm 1, \\
(1.6)
\]

where

\[
\mathcal{F}_x f(\xi) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx,
\]

\[
\mathcal{F}_x f(\xi_1, \xi_2, \cdots, \xi_n) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i \sum_{j=1}^n x_j \xi_j} f(x) dx_1 dx_2 \cdots dx_n.
\]

The pointwise problem was originally studied by Carleson [12], who showed pointwise convergence problem of the one dimensional Schrödinger equation in $H^s(\mathbb{R})$, $s \geq 1/4$. The necessary condition and sufficient condition for the pointwise convergence problem of the Schrödinger equation attracts much attentions. For instance, Dahlberg
and Kenig [20] showed that \( s \geq \frac{1}{4} \) is the necessary condition for the pointwise convergence problem of the Schrödinger equation in any dimension. Dahlberg, Kenig [20] and Kenig et al. [29, 30] have showed the pointwise convergence problem of KdV equation in \( H^s(\mathbb{R}) \) if and only if \( s \geq \frac{1}{4} \). Bourgain [9] presented counterexamples about Schrödinger equation showing that convergence can fail if \( s < \frac{n}{2(n+1)} \). Du et al. [23] proved that the pointwise convergence problem of two dimensional Schrödinger equation in \( H^s(\mathbb{R}) \) with \( s > \frac{1}{3} \). Du and Zhang [25] proved the pointwise convergence problem of \( n \) dimensional Schrödinger equation in \( H^s(\mathbb{R}) \) with \( s > \frac{n}{2(n+1)} \), \( n \geq 3 \). Thus, \( \frac{n}{2(n+1)} \), \( n \geq 2 \) is optimal for the pointwise convergence problem of the Schrödinger equation. Associated to the wave equation, Rogers and Villarroya [49] have proved that \( \frac{1}{2} \left[ e^{it\sqrt{-\Delta}} + e^{-it\sqrt{-\Delta}} \right] f \to f \) with \( f \in H^s(\mathbb{R}^n) \) if and only if \( s > \max \left\{ \frac{n}{2} - \frac{1}{q}, \frac{n+1}{4} - \frac{n-1}{2q}, \frac{1}{2} \right\} (q \geq 1) \). For the pointwise convergence problem of the Schrödinger equation in higher dimension and other dispersive equations, we also refer the readers to [4, 6, 8, 16, 18, 20, 21, 26, 27, 29, 30, 34, 36–40, 48, 50, 51, 54–56].

Recently, Compaan et al. [17] applied randomized initial data to study pointwise convergence of the Schrödinger flow, and then prove almost everywhere convergence with less regularity of the initial data. The method of the suitably randomized initial data originated from Lebowitz-Rose-Speer [33] and Bourgain [5, 7] and Burq-Tzvetkov [10, 11]. Many authors applied the method to study nonlinear dispersive equations and hyperbolic equations in scaling super-critical regimes, for example, see [1–3, 13–15, 19, 21, 22, 28, 32, 35, 41–45, 47, 59, 60].

In this paper, inspired by [17, 57], we mainly investigate the almost surely pointwise convergence problem of free KdV equation, free wave equation and elliptic and non-elliptic Schrödinger equation with randomized initial data in \( L^2 \), respectively. The main tools that we used are the density theorem and some estimates related to the Wiener decomposition of the frequency spaces and Lemma 3.1. The crucial ingredients introduced in this paper are the probabilistic estimates of some random series which are just Lemmas 3.2–3.11 in this paper.

We give some notations before presenting our main results. For \( x \in \mathbb{R}^n \), we define \( x^\alpha = \prod_{j=1}^n x_j^{\alpha_j} \), \( \partial^\beta \phi = \prod_{j=1}^n (\partial/\partial x_j)^{\beta_j} \phi \), where \( \alpha = \sum_{j=1}^n \alpha_j \), \( \beta = \sum_{j=1}^n \beta_j \). For \( \xi \in \mathbb{R}^n \), we have
\[ |\xi| = \sqrt{\sum_{j=1}^{2} \xi_j^2}. \] We define

\[
D_x^a S_1(t) f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi + it|\xi|^3} |\xi|^a \mathcal{F}_x f(\xi) d\xi,
\]

\[
D_x^a S_2(t) f = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\xi + it|\xi|^3} \mathcal{F}_x f(\xi) d\xi,
\]

\[
D_x^a S_3(t) f = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\xi + it|\xi|^3} \mathcal{F}_x f(\xi) d\xi,
\]

\[
D_t^a S_1(t) f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi + it|\xi|^3} \mathcal{F}_x f(\xi) d\xi,
\]

\[
D_t^a S_2(t) f = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\xi + it|\xi|^3} \mathcal{F}_x f(\xi) d\xi,
\]

\[
D_t^a S_3(t) f = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\xi + it|\xi|^3} \mathcal{F}_x f(\xi) d\xi.
\]

Now we introduce the randomization procedure for the initial data, which can be seen in [1, 2, 35, 60]. Let \( B(0,1) \) be a unit ball centered in zero with radius equal to 1. Let \( \psi \in C_c^\infty(\mathbb{R}^n) \) be a real-valued, even, non-negative bump function with \( \text{supp } \psi \subset B(0,1) \) such that for all \( k \in \mathbb{Z}^n \), \( \sum_{k \in \mathbb{Z}^n} \psi(\xi - k) = 1 \) for all \( \xi \in \mathbb{R}^n \), which is known as Wiener decomposition of the frequency space. For \( s \in \mathbb{R} \) and \( f \in H^s(\mathbb{R}^n) \). For every \( \xi \in \mathbb{Z}^n \), we define the function \( \psi(D - k) f : \mathbb{R}^n \to \mathbb{C} \) by

\[
(\psi(D - k) f)(x) = \mathcal{F}^{-1}((\psi(\xi - k)) \mathcal{F} f)(x), \quad x \in \mathbb{R}^n.
\] (1.7)

If \( f \in H^s \) for some \( s \in \mathbb{R} \), then \( P(D - k) f \in H^s \) and

\[
f = \sum_{k \in \mathbb{Z}^n} P(D - k) f
\]

in \( H^s \) with

\[
\| f \|_{H^s} \sim \left[ \sum_{k \in \mathbb{Z}^n} \| P_k f \|_{H^s}^2 \right]^{\frac{1}{2}} = \left[ \sum_{k \in \mathbb{Z}^n} \| P(D - k) f \|_{H^s}^2 \right]^{\frac{1}{2}}.
\]

We will crucially exploit that these projections satisfy a unit-scale Bernstein inequality, namely that for all \( 2 \leq p_1 \leq p_2 \leq \infty \) there exists \( C \equiv C(p_1,p_2) > 0 \) such that for all \( f \in L^2(\mathbb{R}^n) \) and for all \( k \in \mathbb{Z}^n \)

\[
\| \psi(D - k) f \|_{L^{p_2}}(\mathbb{R}^n) \leq C \| \psi(D - k) f \|_{L^{p_1}}(\mathbb{R}^n) \leq C \| \psi(D - k) f \|_{L^2(\mathbb{R}^n)}.
\] (1.9)

Let \( \{ g_k \}_{k \in \mathbb{Z}^n} \) be a sequence of independent, zero-mean, complex-valued Gaussian random variables on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\), where the real and imaginary parts of \( g_k \) are
independent and endowed with probability distributions $\mu^1_k$ and $\mu^2_k$, respectively. Assume that there exists $c > 0$ such that

$$\left| \int_{-\infty}^{+\infty} e^{\gamma x} d\mu^j_k(x) \right| \leq e^{c\gamma^2}, \quad (1.10)$$

for all $\gamma \in \mathbb{R}$, $k \in \mathbb{Z}^n$, $j = 1, 2$. Thereafter for a given $f \in H^s(\mathbb{R}^n)$, $n \geq 1$, we define its randomization by

$$f^\omega := \sum_{k \in \mathbb{Z}^n} g_k(\omega) \psi(D - k)f. \quad (1.11)$$

Lemma B.1 in [10] showed that there is no smoothing upon randomization in terms of differentiability. This randomization improved the integrability of $f$, see Lemma 2.3 of [2]. Such results for random Fourier series are known as Paley-Zygmund’s theorem [46]. We define

$$\|f\|_{L^p_\omega(\Omega)} = \left[ \int_{\Omega} |f(\omega)|^p dP(\omega) \right]^{\frac{1}{p}}.$$

Obviously, $\|f^\omega\|_{H^s_\omega} = \|f\|_{H^s}$. We will restrict ourselves to a subset $\sum \subset \Omega$ with $P(\sum) = 1$ such that $f^\omega \in H^s$ for every $\omega \in \Omega$.

Then we show the main results of this paper as following:

**Theorem 1.1.** Let $f \in L^2(\mathbb{R})$ and $f^\omega$ be a randomization of $f$ as defined in (1.11). Then, $\forall \alpha > 0$, we have

$$\lim_{t \to 0} P(\omega \in \Omega : |S_1(t)f^\omega - f^\omega| > \alpha) = 0. \quad (1.12)$$

**Remark 1.** Dahlberg, Kenig [20] and Kenig et al. [29, 30] have showed the pointwise convergence problem of KdV equation in $H^s(\mathbb{R})$ if and only if $s \geq \frac{1}{4}$. Obviously,

$$\lim_{\epsilon \to 0} \alpha = \lim_{\epsilon \to 0} C\epsilon \epsilon \left[ \ln \frac{C_2}{\epsilon} \right]^{\frac{1}{2}} = 0 \quad (1.13)$$

and $\alpha = o(\epsilon^{\frac{3}{2}})$. From [20, 29, 30] and Theorem 1.1, we know that the pointwise convergence problem of KdV equation with random data requires less regularity of the initial data than the pointwise convergence problem of KdV equation with rough data.

**Theorem 1.2.** Let $f \in L^2(\mathbb{R}^n)$ and $f^\omega$ be a randomization of $f$ as defined in (1.11). Then, $\forall \alpha > 0$, we have

$$\lim_{t \to 0} P(\omega \in \Omega : \frac{1}{2} |S_{2+}(t)f^\omega(x) + S_{2-}f^\omega(x) - f^\omega(x)| > \alpha) > 0. \quad (1.14)$$
Remark 2. Rogers and Villarroya [49] have proved that \( \frac{1}{2} \left[ e^{it\sqrt{-\Delta}} + e^{-it\sqrt{-\Delta}} \right] f \rightarrow f \) with \( f \in H^s(\mathbb{R}^n) \) if and only if \( s > \max \left\{ n\left(\frac{1}{2} - \frac{1}{q}\right), \frac{n+1}{4} - \frac{n-1}{2q}, \frac{1}{2} \right\} \) \((q \geq 1)\). From [49] and Theorem 1.2, we know that the pointwise convergence problem of wave equation with random data requires less regularity of the initial data than the pointwise convergence problem of wave equation with rough data.

**Theorem 1.3.** Let \( f \in L^2(\mathbb{R}^n) \) and \( f^\omega \) be a randomization of \( f \) as defined in (1.11). Then, \( \forall \alpha > 0 \), we have

\[
\lim_{t \to 0} P(\omega \in \Omega : |S_3(t) f^\omega - f^\omega| > \alpha) = 0.
\] (1.15)

**Remark 3.** Compaan et al. [17] have proved the almost surely pointwise convergence problem in \( H^s(s > 0) \) for elliptic Schrödinger equation with random data. Thus, our result improves the result of [17] to elliptic and non-elliptic Schrödinger equation. From [9, 23–25] and Theorem 1.3, we know that the pointwise convergence problem of elliptic Schrödinger equation with random data requires less regularity of the initial data than the pointwise convergence problem of elliptic Schrödinger equation with rough data. Rogers et al. [48] showed that the solution to the two dimensional non-elliptic Schrödinger equation converges to its initial datum \( f \), for all \( f \in H^s(\mathbb{R}^2) \) if and only if \( s \geq \frac{1}{2} \). Thus, from [48] and Theorem 1.3, we know that the pointwise convergence problem of two dimensional non-elliptic Schrödinger equation with random data requires less regularity of the initial data than the pointwise convergence problem of two dimensional non-elliptic Schrödinger equation with rough data.

**Theorem 1.4.** *(Probabilistic density Theorem)* For \( f \in L^2(\mathbb{R}^n) \) and \( \forall \epsilon > 0 \), there exist a rapidly decreasing function \( g \) and \( h \in L^2(\mathbb{R}) \) with \( \|h\|_{L^2(\mathbb{R})} < \epsilon \) such that

\[
\forall \omega \in \Omega_{LM} := \{\omega \in \Omega : \|h^\omega\|_{L^2} \leq \lambda\} \cap \{\omega \in \Omega : |x^\alpha \partial^\beta g^\omega| \leq M\},
\]

we have \( f^\omega = g^\omega + h^\omega \). Here,

\[
\lambda := C\epsilon \left( \ln \frac{C_1}{\epsilon} \right)^{\frac{1}{2}}, \quad M := C\epsilon \left[ \ln \frac{C_1}{\epsilon} \right]^{\frac{1}{2}}.
\]

and

\[
\mathbb{P}(\Omega_{LM}) \geq 1 - 2\epsilon.
\]

Now, we present the outline of proof of Theorem 1.1 to explain the main idea of this paper since the Theorem 1.2, 1.3 can be proved similarly to Theorem 1.1.
More precisely, \( f \in L^2 \) and since rapidly decreasing functions are dense in \( L^2 \) (the density theorem which is just Lemma 2.2 in [24]), we write \( f = g + h \), where \( g \) is a rapidly decreasing function and \( \|h\|_{L^2} < \epsilon \). Since \( f^\omega = g^\omega + h^\omega \), then we get

\[
S_1(t)f^\omega - f^\omega = S_1(t)g^\omega - g^\omega + S_1(t)h^\omega - h^\omega.
\] (1.16)

Here, \( f^\omega \) is defined as in (1.11).

Then, when \( |t| < \epsilon \), \( \alpha = Cee \left[ \ln \frac{3C}{\epsilon} \right] \frac{1}{2} \), we have

\[
\mathbb{P} \left( \left\{ \omega \in \Omega : |S_1(t)f^\omega - f^\omega| > \alpha \right\} \right) \leq \mathbb{P} \left( \left\{ \omega \in \Omega : |S_1(t)g^\omega - g^\omega| > \frac{\alpha}{2} \right\} \right) + \mathbb{P} \left( \left\{ \omega \in \Omega : |S_1(t)h^\omega| > \frac{\alpha}{4} \right\} \right) + \mathbb{P} \left( \left\{ \omega \in \Omega : |h^\omega| > \frac{\alpha}{4} \right\} \right).
\]

Hence, we only need to deal with the right-hand side terms of the above inequality one by one. Note that \( g \) is a rapidly decreasing function and \( \|h\|_{L^2} < \epsilon \), and then combining the probabilistic estimate Lemma 3.1 with Strichartz estimates related to the uniform partition to the frequency spaces, we obtained the following estimates, the proofs are given in Lemma 3.2, Lemma 3.3 and Lemma 3.8, respectively.

\[
\mathbb{P} \left( \left\{ \omega \in \Omega : |S_1(t)g^\omega - g^\omega| > \frac{\alpha}{2} \right\} \right) \leq C_1 e^{-\left(\frac{\alpha}{Cee}\right)^2},
\] (1.17)

\[
\mathbb{P} \left( \left\{ \omega \in \Omega : |S_1(t)h^\omega| > \frac{\alpha}{4} \right\} \right) \leq C_1 e^{-\left(\frac{\alpha}{Cee \|h^\omega\|_{H^1}}\right)^2} \leq C_1 e^{-\left(\frac{\alpha}{Cee}\right)^2},
\] (1.18)

and

\[
\mathbb{P} \left( \left\{ \omega \in \Omega : |h^\omega| > \frac{\alpha}{4} \right\} \right) \leq C_1 e^{-\left(\frac{\alpha}{Cee \|h^\omega\|_{H^1}}\right)^2} \leq C_1 e^{-\left(\frac{\alpha}{Cee}\right)^2}.
\] (1.19)

Thus, when \( |t| < \epsilon \), \( Cee \left[ \ln \frac{3C}{\epsilon} \right] \frac{1}{2} \leq \alpha \), we have

\[
\mathbb{P} \left( \left\{ \omega \in \Omega : |S_1(t)f^\omega - f^\omega| > \alpha \right\} \right) \leq \mathbb{P} \left( \left\{ \omega \in \Omega : |S_1(t)g^\omega - g^\omega| > \frac{\alpha}{2} \right\} \right) + \mathbb{P} \left( \left\{ \omega \in \Omega : |S_1(t)h^\omega| > \frac{\alpha}{4} \right\} \right) + \mathbb{P} \left( \left\{ \omega \in \Omega : |h^\omega| > \frac{\alpha}{4} \right\} \right) \leq C_1 e^{-\left(\frac{\alpha}{Cee}\right)^2} + 2C_1 e^{-\left(\frac{\alpha}{Cee}\right)^2} \leq 3C_1 e^{-\left(\frac{\alpha}{Cee}\right)^2} \leq \epsilon.
\] (1.20)

The proof of the remainder of Theorem 1.1 can be seen in Lemma 3.11.

This completes the proof of Theorem 1.1.
2. Preliminaries

In this section, we give some estimates related to the Wiener decomposition of the frequency spaces.

**Lemma 2.1.** For \( f \in L^2(\mathbb{R}^n) \), we have

\[
\left[ \sum_{k \in \mathbb{Z}^n} |\psi(D - k)f|^2 \right]^{\frac{1}{2}} \leq \|f\|_{L^2(\mathbb{R}^n)}.
\]

**(Proof.** To obtain (2.1), it suffices to prove

\[
\sum_{k \in \mathbb{Z}^n} |\psi(D - k)f|^2 \leq \|f\|_{L^2(\mathbb{R}^n)}^2.
\]

By using the Cauchy-Schwarz inequality with respect to \( \xi \), since \( \text{supp} \, \psi \subset B(0, 1) \) we have

\[
\left[ \sum_{k \in \mathbb{Z}^n} \int_{|\xi - k| \leq 1} |e^{i \sum_{j=1}^n x_j \xi_j} \psi(\xi - k) \mathcal{F}_x f(\xi) d\xi|^2 \right]^{\frac{1}{2}} \leq \sum_{k \in \mathbb{Z}^n} \left( \int_{|\xi - k| \leq 1} |\psi(\xi - k) \mathcal{F}_x f(\xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]

Combining (2.3) with (2.5), we derive (2.2).

From

\[
\mathcal{F}_x f(\xi) = \sum_{k \in \mathbb{Z}^n} \psi(\xi - k) \mathcal{F}_x f(\xi),
\]

by using \( \text{supp} \, \psi \subset B(0, 1) \), we have

\[
\|\mathcal{F}_x f(\xi)\|_{L^2}^2 = \sum_{k \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |\psi(\xi - k) \mathcal{F}_x f(\xi)| \left| \psi(\xi - l) \mathcal{F}_x f(\xi) \right| d\xi
\]

\[
= \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |\psi(\xi - k) \mathcal{F}_x f(\xi)|^2 d\xi.
\]

Combining (2.3) with (2.5), we derive (2.2).

This completes the proof of Lemma 2.1. \( \square \)
Lemma 2.2. For $f \in L^2(\mathbb{R})$, we have

\[
\left[ \sum_{k \in \mathbb{Z}} |\psi(D-k)S_1(t)f|^2 \right]^{\frac{1}{2}} \leq \| f \|_{L^2(\mathbb{R})}.
\]  \hfill (2.6)

**Proof.** To obtain (2.6), it suffices to prove

\[
\sum_{k \in \mathbb{Z}} |\psi(D-k)S_1(t)f|^2 \leq \| f \|_{L^2(\mathbb{R})}^2.
\]  \hfill (2.7)

By using the Cauchy-Schwarz inequality with respect to $\xi$, since $\text{supp} \, \psi \subset B(0, 1)$, we have

\[
\sum_{k \in \mathbb{Z}} |\psi(D-k)S_1(t)f|^2 = \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k \in \mathbb{Z}^n} \left| \int_{\mathbb{R}^n} e^{ix\xi}e^{it\xi} \psi(\xi-k) \hat{F}_x f(\xi) d\xi \right|^2 \\
= \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k \in \mathbb{Z}^n} \left| \int_{|\xi-k| \leq 1} e^{ix\xi}e^{it\xi} \psi(\xi-k) \hat{F}_x f(\xi) d\xi \right|^2 \\
\leq \left[ \sum_{k \in \mathbb{Z}^n} \int_{|\xi-k| \leq 1} |\psi(\xi-k)\hat{F}_x f(\xi)|^2 d\xi \right] \left[ \int_{|\xi-k| \leq 1} d\xi \right] \\
\leq \left[ \sum_{k \in \mathbb{Z}^n} \int_{|\xi-k| \leq 1} |\psi(\xi-k)\hat{F}_x f(\xi)|^2 d\xi \right] \\
= \sum_{k \in \mathbb{Z}^n} \| \psi(\xi-k)\hat{F}_x f(\xi) \|_{L^2}^2.
\]  \hfill (2.8)

From

\[
\hat{F}_x f(\xi) = \sum_{k \in \mathbb{Z}^n} \psi(\xi-k)\hat{F}_x f(\xi),
\]  \hfill (2.9)

by using $\text{supp} \, \psi \subset B(0, 1)$, we have

\[
\| \hat{F}_x f(\xi) \|_{L^2}^2 = \sum_{k \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} \int_{\mathbb{R}^n} [\psi(\xi-k)\hat{F}_x f(\xi)] [\psi(\xi-l)\hat{F}_x f(\xi)] d\xi \\
= \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |\psi(\xi-k)\hat{F}_x f(\xi)|^2 d\xi.
\]  \hfill (2.10)

Combining (2.8) with (2.10), we derive (2.7).

This completes the proof of Lemma 2.2. \hfill \Box

Lemma 2.3. For $f \in L^2(\mathbb{R}^n)$, we have

\[
\left[ \sum_{k \in \mathbb{Z}^n} |\psi(D-k)S_2(t)f|^2 \right]^{\frac{1}{2}} \leq \| f \|_{L^2(\mathbb{R}^n)}.
\]  \hfill (2.11)
Lemma 2.3 can be proved similarly to Lemma 2.2.

**Lemma 2.4.** For \( f \in L^2(\mathbb{R}^n) \), we have

\[
\left[ \sum_{k \in \mathbb{Z}^n} |\psi(D - k)S_3(t)f|^2 \right]^\frac{1}{2} \leq \|f\|_{L^2(\mathbb{R}^n)}.
\]

(2.12)

Lemma 2.4 can be proved similarly to Lemma 2.2.

**Lemma 2.5.** Let \( g \) be a rapidly decreasing function and we denote by \( \psi^{(\beta)} \) the \( \beta \) order derivative of \( \psi \), we have

\[
\sum_{|k| \geq 3} \int_{\mathbb{R}^n} |\xi^{\alpha} \mathcal{F}xg(\xi)\psi^{(\beta)}(\xi - k)|^2 d\xi \leq C.
\]

(2.13)

**Proof.** Since \( \text{supp} \, \psi \subset B(0, 1) \), we have \( \text{supp} \, \psi^{(\beta)} \subset B(0, 1) \). Let \( \xi - k = \eta \), then, \( \xi = k + \eta \), since \( g \) is a rapidly decreasing function, we have

\[
\sum_{|k| \geq 3} \int_{|\eta| \leq 1} \frac{1}{1 + |\eta + k|^2} d\eta \leq \sum_{|k| \geq 3} \frac{C}{k^2} \leq C.
\]

This completes the proof of Lemma 2.5.

**Lemma 2.6.** Let \( g \) be a rapidly decreasing function, we have

\[
\sum_{|k| \geq 3} \int_{\mathbb{R}^n} |\xi^{\alpha} \mathcal{F}xg(\xi)\partial^\beta \psi(\xi - k)|^2 d\xi \leq C.
\]

(2.15)

Lemma 2.6 can be proved similarly to Lemma 2.5.

### 3. Probabilistic estimates of some random series

In this section, we establish probabilistic estimates of some random series. More precisely, we apply Lemmas 2.1-2.6 and 3.1 to establish Lemmas 3.2-3.11 which play crucial role in establishing Theorems 1.1-1.3. In particular, we apply Lemma 3.1 to establish Lemmas 3.9, 3.10 which can be used to establish Lemma 3.11 which is called as the probabilistic density theorem.
Lemma 3.1. Assume (1.10). Then, there exists $C > 0$ such that
\[
\left\| \sum_{k \in \mathbb{Z}^n} g_k(\omega)c_k \right\|_{L_p^p(\Omega)} \leq C \sqrt{p} \|c_k\|_{l^2(\mathbb{Z}^n)}.
\]  
(3.1)
for all $p \geq 2$ and $\{c_k\} \in l^2(\mathbb{Z}^n)$.

For the proof of Lemma 3.1, we refer the readers to Lemma 3.1 of [10].

Lemma 3.2. \(\forall \alpha > 0\). Let $g$ be a rapidly decreasing function and we denote by $g^\omega$ the randomization of $g$ as defined in (1.11). Then, there exist $C > 0, C_1 > 0$ such that
\[
\mathbb{P}(\Omega_1^\alpha) \leq C_1 e^{-\left(\frac{\alpha}{C \sqrt{p}}\right)^2}.
\]  
(3.2)
where
\[
\Omega_1^\alpha = \{ \omega \in \Omega : |S_1(t)g^\omega - g^\omega| > \alpha \}.
\]

Proof. By using Lemma 3.1 and the Cauchy-Schwartz inequality, since $g$ is a rapidly decreasing function and $|e^{it\xi^3} - 1| \leq |t\xi^3|$, we have
\[
\|S_1(t)g^\omega - g^\omega\|_{L_p^p(\Omega)} \leq C \sqrt{p} \left[ \sum_k \left| \int_{\mathbb{R}} (e^{it\xi^3} - 1)e^{ix\xi} \psi(\xi - k)Fg(\xi) d\xi \right|^2 \right]^{\frac{1}{2}}
\leq C|t|\sqrt{p} \left[ \sum_k \int_{|\xi - k|\leq 1} |\xi^3| \psi(\xi - k)Fg(\xi) |^2 d\xi \right]^{\frac{1}{2}}
\leq C|t|\sqrt{p} \left[ \sum_k \int_{|\xi - k|\leq 1} |\xi^6| \psi(\xi - k)Fg(\xi) |^2 d\xi \right]^{\frac{1}{2}} \left[ \int_{|\xi - k|\leq 1} d\xi \right]^{\frac{1}{2}}
\leq C|t|\sqrt{p} \left[ \sum_k \left| \psi(D - k)g \right|_{H^3}^2 \right]^{\frac{1}{2}}
= C|t|\sqrt{p} \left\| g \right\|_{H^3} \leq C \sqrt{p}|t|.
\]  
(3.3)
Thus, by using Chebyshev inequality, from (3.3), we have
\[
\mathbb{P}(\Omega_1^\alpha) \leq \int_{\Omega_1^\alpha} \left[ \frac{|S_1(t)g^\omega - g^\omega|}{\alpha} \right]^p d\mathbb{P}(\omega) \leq \left( \frac{C \sqrt{p}|t|}{\alpha} \right)^p.
\]  
(3.4)
Take
\[
p = \left( \frac{\alpha}{Ce|t|} \right)^2.
\]  
(3.5)
If \( p \geq 2 \), from (3.4), then we have
\[
P(\Omega_1^c) \leq e^{-p} = e^{-\left(\frac{\alpha}{\sqrt[p]{|t|}}\right)^2}. \tag{3.6}\]
If \( p \leq 2 \), from (3.4), we have
\[
P(\Omega_1^c) \leq 1 \leq e^{2e^{-2}} \leq C_1 e^{-\left(\frac{\alpha}{\sqrt[p]{|t|}}\right)^2}. \tag{3.7}\]
Here \( C_1 = e^2 \). Thus, from (3.6), (3.7), we have
\[
P(\Omega_1^c) \leq C_1 e^{-\left(\frac{\alpha}{\sqrt[p]{|t|}}\right)^2}. \tag{3.8}\]
This completes the proof of Lemma 3.2. \( \square \)

**Lemma 3.3.** Let \( h \in L^2(\mathbb{R}) \) and we denote by \( h^\omega \) the randomization of \( h \) as defined in (1.11). Then, \( \forall \alpha > 0 \) there exist \( C > 0, C_1 > 0 \) such that
\[
P(\Omega_2^c) \leq C_1 e^{-\left(\frac{\alpha}{\|h\|_{L^2}}\right)^2}, \tag{3.9}\]
where \( \Omega_2^c = \{ \omega \in \Omega : |S_1(t)h^\omega| > \alpha \} \).

**Proof.** By using Lemmas 3.1, 2.2, we have
\[
\|S_1(t)h^\omega\|_{L^p(\Omega)} = \left\| \sum_{k \in \mathbb{Z}^n} g_k(\omega)\psi(D-k)S_1(t)h \right\|_{L^p(\Omega)} \\
\leq C \sqrt[p]{\mathbb{E}} \left( \sum_{k \in \mathbb{Z}^n} |\psi(D-k)S_1(t)h|^2 \right)^{\frac{1}{p}} \leq C \sqrt[p]{\mathbb{E}} \|h\|_{L^2}. \tag{3.10}\]
Thus, by using Chebyshev inequality, we have
\[
P(\Omega_2^c) \leq \int_{\Omega_2^c} \left[ \frac{|S_1(t)h^\omega|}{\alpha} \right]^p \, d\mathbb{P}(\omega) \leq \left( \frac{C \sqrt[p]{\mathbb{E}} \|h\|_{L^2}}{\alpha} \right)^p. \tag{3.11}\]
By using a proof similar to (3.8), we obtain (3.9).

This completes the proof of Lemma 3.3. \( \square \)

**Lemma 3.4.** Let \( g \) be a rapidly decreasing function and we denote by \( g^\omega \) the randomization of \( g \) as defined in (1.11). Then, \( \forall \alpha > 0 \), there exist \( C > 0, C_1 > 0 \) such that
\[
P(\Omega_3^c) \leq C_1 e^{-\left(\frac{\alpha}{\sqrt[p]{|t|}}\right)^2}, \tag{3.12}\]
where \( \Omega_3^c = \{ \omega \in \Omega : \left| \frac{1}{2} [S_{2+}(t) + S_{2-}(t)] g^\omega - g^\omega \right| > \alpha \} \).
Proof. By using Lemma 3.1 and the Cauchy-Schwarz inequality with respect to $\xi$, since $g$ is a rapidly decreasing function and $|\frac{1}{2} [e^{it|\xi|} + e^{-it|\xi|}] - 1| \leq |t||\xi|$, we have

$$
\| \frac{1}{2} [S_{2+}(t) + S_{2-}(t)] g^\omega - g^\omega \|_{L^p_c(\Omega)} \\
\leq C \sqrt{p} \left[ \sum_k \left| \int_{\mathbb{R}^n} \left( \frac{1}{2} [e^{it|\xi|} + e^{-it|\xi|}] - 1 \right) e^{ix\xi}(\xi - k) \mathcal{F} g(\xi) d\xi \right|^2 \right]^{\frac{1}{2}} \\
= \sqrt{p} \left[ \sum_k \left| \int_{|\xi - k| \leq 1} \left( \frac{1}{2} [e^{it|\xi|} + e^{-it|\xi|}] - 1 \right) e^{ix\xi}(\xi - k) \mathcal{F} g(\xi) d\xi \right|^2 \right]^{\frac{1}{2}} \\
\leq C |t| \sqrt{p} \left[ \sum_k \left| \int_{|\xi - k| \leq 1} |\xi\psi(\xi - k) \mathcal{F} g(\xi)|^2 d\xi \right| \right]^{\frac{1}{2}} \\
\leq C |t| \sqrt{p} \left[ \sum_k \left| \int_{|\xi - k| \leq 1} \left| \xi\psi(\xi - k) \mathcal{F} g(\xi) \right|^2 d\xi \right| \right]^{\frac{1}{2}} \\
\leq C |t| \sqrt{p} \left[ \sum_k \left| \int_{|\xi - k| \leq 1} \left| \psi(D - k) \mathcal{F} g(\xi) \right|^2 d\xi \right| \right]^{\frac{1}{2}} \\
= C |t| \sqrt{p} \left[ \sum_k \| \psi(D - k) \|^2_{H^1} \right]^{\frac{1}{2}} \\
= C |t| \sqrt{p} \| g \|_{H^1} \leq C |t| \sqrt{p}. \quad (3.13)
$$

Thus, from (3.13), by using Chebyshev inequality, from (3.13), we have

$$
\mathbb{P}(\Omega_3^c) \leq \frac{\| \frac{1}{2} [S_{2+}(t) + S_{2-}(t)] g^\omega - g^\omega \|_{L^p_c(\Omega)}^p}{\alpha^p} \leq \frac{C \sqrt{p}}{\alpha^p}. \quad (3.14)
$$

By using a proof similar to (3.8), from (3.14), we have

$$
\mathbb{P}(\Omega_3^c) \leq C_1 \exp \left[ - \left( \frac{\alpha}{C \sqrt{p}|t|} \right)^2 \right]. \quad (3.15)
$$

This completes the proof of Lemma 3.4. \hfill \square

Lemma 3.5. Let $h \in L^2(\mathbb{R}^n)$ and we denote by $h^\omega$ the randomization of $h$ as defined in (1.11). Then, $\forall \alpha > 0$, there exist $C > 0$ and $C_1 > 0$ such that

$$
\mathbb{P}(\Omega_4^c) \leq C_1 e^{-\left( \frac{\alpha}{C \sqrt{p}|t|} \right)^2}, \quad (3.16)
$$

where

$$
\Omega_4^c = \{ \omega \in \Omega : |S_{2\pm}(t)h^\omega| > \alpha \}. \quad (3.17)
$$
Proof. By using Lemmas 3.1, 2.2, we have
\[ \| S_1(t)h^\omega \|_{L^p_c(\Omega)} = \left\| \sum_{k \in \mathbb{Z}^n} g_k(\omega)\psi(D - k)S_2(t)h \right\|_{L^p_c(\Omega)} \]
\[ \leq C\sqrt{p} \left( \sum_{k \in \mathbb{Z}^n} |\psi(D - k)S_2(t)h|^2 \right)^{\frac{1}{2}} \leq C\sqrt{p}\| h \|_{L^2}. \tag{3.18} \]

Thus, by using Chebyshev inequality, from (3.18), we have
\[ \mathbb{P}(\Omega^c_\epsilon) \leq \int_{\Omega^c_\epsilon} \left[ \frac{|S_2(t)h^\omega|}{\alpha} \right]^p d\mathbb{P}(\omega) \leq \left( \frac{C\sqrt{p}\| h \|_{L^2}}{\alpha} \right)^p. \tag{3.19} \]

By using a proof similar to (3.8), from (3.19), we have
\[ \mathbb{P}(\Omega^c_\epsilon) \leq C_1\exp \left[ - \left( \frac{\alpha}{Ce|t|} \right)^2 \right]. \tag{3.20} \]

This completes the proof of Lemma 3.5. \(\square\)

Lemma 3.6. Let \( g \) be a rapidly decreasing function and we denote by \( g^\omega \) the randomization of \( g \) as defined in (1.11). Then, \( \forall \alpha > 0 \), there exist \( C > 0, C_1 > 0 \) such that
\[ \mathbb{P}(\Omega^c_\epsilon) \leq C_1e^{-\left( \frac{\alpha}{Ce|t|} \right)^2}. \tag{3.21} \]

where
\[ \Omega^c_\epsilon = \{ \omega \in \Omega : |S_3(t)g^\omega - g^\omega| > \alpha \}. \]

Proof. By using Lemma 3.1 and the Cauchy-Schwartz inequality with respect to \( \xi \), since \( g \) is a rapidly decreasing function and \( \left| e^{-it\sum_{j=1}^n \epsilon_j \xi_j} - 1 \right| \leq t|\xi|^2 \). we have
\[ \| S_3(t)g^\omega - g^\omega \|_{L^p_c(\Omega)} \leq C\sqrt{p} \left[ \sum_k \left| \int_{\mathbb{R}^n} (e^{-it\sum_{j=1}^n \epsilon_j \xi_j} - 1)e^{ix\xi}\psi(\xi - k)g(\xi) d\xi \right|^2 \right]^{\frac{1}{2}} \]
\[ = C\sqrt{p} \left[ \sum_k \left| \int_{|\xi - k| \leq 1} (e^{-it\sum_{j=1}^n \epsilon_j \xi_j} - 1)e^{ix\xi}\psi(\xi - k)g(\xi) d\xi \right|^2 \right]^{\frac{1}{2}} \]
\[ \leq C|t|\sqrt{p} \left[ \sum_k \int_{|\xi - k| \leq 1} |\xi|^2 |\psi(\xi - k)g(\xi)|^2 d\xi \right]^{\frac{1}{2}} \left[ \int_{|\xi - k| \leq 1} d\xi \right]^{\frac{1}{2}} \]
\[ = C|t|\sqrt{p} \left[ \sum_k \int_{\mathbb{R}^n} |\xi|^2 |\psi(\xi - k)g(\xi)|^2 d\xi \right]^{\frac{1}{2}} \]
\[ \leq C|t|\sqrt{p} \left[ \sum_k \|\psi(D - k)g\|_{H^1}^2 \right]^{\frac{1}{2}} \]
\[ = C|t|\sqrt{p}\| g \|_{H^1} \leq C\sqrt{p}|t|. \tag{3.22} \]
From (3.20), by using Chebyshev inequality, from (3.22), we have
\[ \mathbb{P}(\Omega_{c5}^c) \leq \frac{(C\sqrt{p}|t|)^p}{\alpha^p}. \]  
(3.23)

Thus, by using a proof similar to (3.8), from (3.23), we have
\[ \mathbb{P}(\Omega_{c5}^c) \leq C_1 \exp \left[ -\left( \frac{\alpha}{C|t|e} \right)^2 \right]. \]  
(3.24)

This completes the proof of Lemma 3.6. □

**Lemma 3.7.** Let \( h \in L^2(\mathbb{R}^n) \) and we denote by \( h^\omega \) the randomization of \( h \) as defined in (1.11). Then, \( \forall \alpha > 0 \), there exist \( C > 0 \) and \( C_1 > 0 \) such that
\[ \mathbb{P}(\Omega_{c6}^c) \leq C_1 \exp \left[ -\left( \frac{\alpha}{C|t|e} \right)^2 \right]. \]  
(3.25)

where
\[ \Omega_{c6}^c = \{ \omega \in \Omega : |S_3(t)h^\omega| > \alpha \}. \]  
(3.26)

**Proof.** By using Lemmas 3.1, 2.2, we have
\[ \|S_3(t)h^\omega\|_{L^p(\Omega)} = \left\| \sum_{k \in \mathbb{Z}^n} g_k(\omega)\psi(D-k)S_2(t)h \right\|_{L^p(\Omega)} \]
\[ \leq C \sqrt{p} \left( \sum_{k \in \mathbb{Z}^n} |\psi(D-k)S_2(t)h|^2 \right)^{\frac{1}{2}} \leq C \sqrt{p}\|h\|_{L^2}. \]  
(3.27)

Thus, by using Chebyshev inequality, from (3.27), we have
\[ \mathbb{P}(\Omega_{c6}^c) \leq \int_{\Omega_{c6}} \left[ \frac{|S_3(t)h^\omega|}{\alpha} \right]^p d\mathbb{P}(\omega) \leq \left( \frac{C \sqrt{p}\|h\|_{L^2}}{\alpha} \right)^p. \]  
(3.28)

By using a proof similar to (3.8), from (3.28), we have
\[ \mathbb{P}(\Omega_{c6}^c) \leq C_1 \exp \left[ -\left( \frac{\alpha}{C|t|e} \right)^2 \right]. \]  
(3.29)

This completes the proof of Lemma 3.7. □

**Lemma 3.8.** Let \( h \in L^2(\mathbb{R}^n) \) and we denote by \( h^\omega \) the randomization of \( h \) as defined in (1.11). Then, \( \forall \alpha > 0 \), there exist \( C > 0 \) and \( C_1 > 0 \) such that
\[ \mathbb{P}(\Omega_{c7}^c) \leq C_1 e^{-\left( \frac{\alpha}{C|t|e} \right)^2}. \]  
(3.30)

where
\[ \Omega_{c7}^c = \{ \omega \in \Omega : |h^\omega| > \alpha \}. \]  
(3.31)
Proof. By using Lemmas 3.1, 2.2, we have

\[
\|h^\omega\|_{L^p(\Omega)} = \left\| \sum_{k \in \mathbb{Z}^n} g_k(\omega) \psi(D - k)h \right\|_{L^p(\Omega)} \\
\leq C \left( \sum_{k \in \mathbb{Z}^n} |\psi(D - k)h|^2 \right)^{\frac{1}{2}} \leq C \sqrt{p} \|h\|_{L^2}.
\]

(3.32)

Thus, by using Chebyshev inequality, from (3.32), we have

\[
\mathbb{P}(\Omega_7^c) \leq \int_{\Omega_7^c} \left[ \frac{|h^\omega|}{\alpha} \right]^p d\mathbb{P}(\omega) \leq \left( \frac{C \sqrt{p} \|h\|_{L^2}}{\alpha} \right)^p.
\]

(3.33)

By using a proof similar to (3.8), we obtain (3.30).

This completes the proof of Lemma 3.8. \( \Box \)

Lemma 3.9. \( \forall \epsilon > 0. \) Let \( g \) be a rapidly decreasing function satisfying \( \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta g| < \infty. \) We denote by \( g^\omega \) the randomization of \( g \) as defined in (1.11). Then, there exist \( C > 0 \) and \( C_1 > 0 \) such that

\[
\mathbb{P} \left\{ \{\omega \in \Omega : |x^\alpha \partial^\beta g^\omega| > M \} \right\} \leq C_1 e^{-\left( \frac{M}{\epsilon} \right)^2}.
\]

(3.34)

In particular, take \( M = Ce \left[ \ln \frac{\epsilon}{C_1} \right]^\frac{1}{2} \), then, we have

\[
\mathbb{P} \left\{ \{\omega \in \Omega : |x^\alpha \partial^\beta g^\omega| > M \} \right\} \leq \epsilon.
\]

(3.35)

Proof. We firstly show

\[
\mathbb{P} \left\{ \{\omega \in \Omega : |x^\alpha \partial^\beta g^\omega| > M \} \right\} \leq C_1 e^{-\left( \frac{M}{\epsilon} \right)^2}.
\]

(3.36)

By using Lemmas 3.1, 2.5, since \( g \) is a rapidly decreasing function which yields

\[
\sum_{|k| \leq 2} \int_{|\xi - k| \leq 1} \left[ |(\partial^\alpha [\psi(\xi - k))\xi^\beta g^\omega])|^2 d\xi \leq C,
\]

\[
\sum_{|k| \leq 2} \int_{|\xi - k| \leq 1} \left[ |(\partial^\alpha \psi(\xi - k))\xi^\beta g^\omega])|^2 d\xi \leq C,
\]

\[
\sum_{|k| \leq 2} \int_{|\xi - k| \leq 1} \left[ |(\partial^\alpha [\psi(\xi - k))\xi^\beta g^\omega])|^2 d\xi \leq C.
\]

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thus, we have

\[ \| x^\alpha \partial^\beta g^\omega \|_{L_p^p(\Omega)} = \left\| \sum_{k \in \mathbb{Z}^n} g_k(\omega) x^\alpha \partial^\beta \psi(D - k) h \right\|_{L_p^p(\Omega)} = \left\| \sum_{k \in \mathbb{Z}^n} g_k(\omega) \int_{\mathbb{R}^n} e^{ix\xi} \left[ -(i\partial^\alpha) \left[ \psi(\xi - k)(i\xi)^\beta \mathcal{F}_x g(\xi) \right] \right] \right\|_{L_p^p(\Omega)} \leq C \sqrt{p} \sum_{k \in \mathbb{Z}^n} \left( \int_{\mathbb{R}^n} e^{ix\xi} \left[ -(i\partial^\alpha) \left[ \psi(\xi - k)(i\xi)^\beta \mathcal{F}_x g(\xi) \right] \right] d\xi \right)^2 \leq C \sqrt{p} \sum_{k \in \mathbb{Z}^n} \int_{|\xi - k| \leq 1} \left| \left[ (\partial^\alpha \left[ \psi(\xi - k)\xi^\beta \mathcal{F}_x g(\xi) \right] \right] \right|^2 d\xi + C \sqrt{p} \leq C \sqrt{p}. \]

(3.37)

Thus, by Chebyshev inequality and (3.37), we have

\[ \mathbb{P} \left( \omega \in \Omega : |x^\alpha \partial^\beta g^\omega| > M \right) \leq \frac{\| x^\alpha \partial^\beta g^\omega \|_{L_p^p(\Omega)}^p}{M^p} \leq \frac{(C \sqrt{p})^p}{M^p}. \] (3.38)

By using a proof similar to (3.8), from (3.41), we obtain (3.34).

This completes the proof of Lemma 3.9.

Remark 4. From Lemma 3.9, we know that, if \( g \) is a rapidly decreasing function, then the randomized function \( g^\omega \) is almost surely a rapidly decreasing function.

Lemma 3.10. \( \forall \epsilon > 0 \) and \( \forall \lambda > 0 \) and \( \| h \|_{L^2(\mathbb{R}^n)} < \epsilon \) and we denote by \( h^\omega \) the randomization of \( h \) as defined in (1.11). Then, there exist \( C > 0 \) and \( C_1 > 0 \) such that

\[ \mathbb{P} \left( \{ \omega \in \Omega : \| h^\omega \|_{L^2} > \lambda \} \right) \leq C_1 e^{-\left( \frac{\lambda}{C \| h \|_{L^2}} \right)^2} \leq C_1 e^{-\left( \frac{\lambda}{\epsilon} \right)^2}. \] (3.39)

In particular, take \( \lambda = C_\epsilon \left( \ln \left( \frac{C}{\epsilon} \right) \right)^\frac{1}{2} \), and

\[ \mathbb{P} \left( \{ \omega \in \Omega : \| h^\omega \|_{L^2} > \lambda \} \right) \leq C_1 e^{-\left( \frac{\lambda}{C \| h \|_{L^2}} \right)^2} \leq \epsilon. \] (3.40)
Proof. For the proof of (3.39), we refer the readers to Lemma 2.2 of [2]. When \( \lambda = C e\left(\ln \left(\frac{C_1}{\epsilon}\right)^\frac{1}{2}\right)\), we have \( C_1 e^{-\left(\frac{\alpha}{\epsilon}\right)^2} = \epsilon \). We have

\[
\lim_{\epsilon \to 0} \frac{C e\left(\ln \left(\frac{C_1}{\epsilon}\right)^\frac{1}{2}\right)}{\epsilon^{\frac{1}{2}}} = C e \lim_{\epsilon \to 0} \frac{\left(\ln \left(\frac{C_1}{\epsilon}\right)^\frac{1}{2}\right)}{\epsilon^{\frac{1}{2}}} = 0 \quad (3.41)
\]

This completes the proof of Lemma 3.10.

Remark 5. From Lemma 3.10, we know that if \( f \in L^2(\mathbb{R}^n) \), \( n \geq 1 \), then the randomized function \( f^\omega \) is almost surely in \( L^2(\mathbb{R}^n) \), \( n \geq 1 \).

4. Proof of Theorem 1.1

Proof of Theorem 1.1. When \( f \in L^2(\mathbb{R}) \), by density theorem which is just Lemma 2.2 in [24], there exists a rapidly decreasing function \( g \) such that \( f = g + h \), where \( \|h\|_{H^{\alpha}} < \epsilon \). We define

\[
\Omega_8^c = \{ \omega \in \Omega : |S_1(t)f^\omega - f^\omega| > \alpha \}. \quad (4.1)
\]

Thus, we have

\[
\Omega_8^c \subset \Omega_9^c \cup \Omega_{10}^c, \quad (4.2)
\]

where

\[
\Omega_9^c = \{ \omega \in \Omega : |S_1(t)g^\omega - g^\omega| > \frac{\alpha}{2} \}, \quad (4.3)
\]

\[
\Omega_{10}^c = \{ \omega \in \Omega : |S_1(t)h^\omega - h^\omega| > \frac{\alpha}{2} \}. \quad (4.4)
\]

Obviously,

\[
\Omega_{10}^c \subset \Omega_{11}^c \cup \Omega_{12}^c, \quad (4.5)
\]

where

\[
\Omega_{11}^c = \{ \omega \in \Omega : |S_1(t)h^\omega| > \frac{\alpha}{4} \}, \quad (4.6)
\]

\[
\Omega_{12}^c = \{ \omega \in \Omega : |h^\omega| > \frac{\alpha}{4} \}. \quad (4.7)
\]

From Lemma 3.2, we have

\[
\mathbb{P}(\Omega_8^c) \leq C_1 e^{[-\left(\alpha \sqrt{\|f^\omega\|} \right)^2] \leq C_1 e^{-\left(\alpha \sqrt{\|f^\omega\|} \right)^2}. \quad (4.8)
\]

From Lemma 3.3, we have

\[
\mathbb{P}(\Omega_{11}^c) \leq C_1 e^{[-\left(\alpha \sqrt{\|h^\omega\|} \right)^2] \leq C_1 e^{-\left(\alpha \sqrt{\|h^\omega\|} \right)^2}. \quad (4.9)
\]
From Lemma 3.8, we have
\[ \mathbb{P}(\Omega_{12}^c) \leq C_1 e^{-\left[\frac{\alpha}{C_1 e}\right]^2} \leq C_1 e^{-\left[\frac{\alpha}{C_1 e}\right]^2}. \] (4.10)

From (4.8)-(4.10), we have
\[ \mathbb{P}(\Omega_{8}^c) \leq \mathbb{P}(\Omega_{9}^c) + \mathbb{P}(\Omega_{10}^c) \leq \mathbb{P}(\Omega_{8}^c) + \mathbb{P}(\Omega_{11}^c) + \mathbb{P}(\Omega_{12}^c) \leq C_1 e^{-\left[\frac{\alpha}{C_1 e}\right]^2} + 2C_1 e^{-\left[\frac{\alpha}{C_1 e}\right]^2}. \] (4.11)

When \(|t| \leq \epsilon\), from (4.11), we have
\[ \mathbb{P}(\Omega_{8}^c) \leq C_1 e^{-\frac{\alpha^2}{(C_1 e)^2}}. \] (4.12)

Here, \(\epsilon\) satisfies \(Cee \left(\ln \frac{C_1}{\epsilon}\right)^\frac{1}{2} \leq \alpha\). From (4.12), we have
\[ \mathbb{P}(\Omega_{8}^c) \leq C_1 e^{-\frac{\alpha^2}{(C_1 e)^2}} \leq \epsilon. \] (4.13)

From (4.13), we have
\[ \mathbb{P}(\Omega_{8}) \geq 1 - \epsilon. \] (4.14)

For the proof of the remainder of Theorem 1.1 can be seen in Lemma 3.11.
This completes the proof of Theorem 1.1. \(\square\)

5. Proof of Theorem 1.2

**Proof of Theorem 1.2.** When \(f \in L^2(\mathbb{R}^n)\), by density theorem which is just Lemma 2.2 in [24], there exists a rapidly decreasing function \(g\) such that \(f = g + h\), where \(\|h\|_{L^2} < \epsilon\). We define
\[ \Omega_{13}^c = \{\omega \in \Omega : |S_2(t)f^{\omega} - f^{\omega}| > \alpha\}. \] (5.1)

Thus, we have
\[ \Omega_{13}^c \subset \Omega_{14}^c \cup \Omega_{15}^c, \] (5.2)
where
\[ \Omega_{14}^c = \{\omega \in \Omega : |S_2(t)g^{\omega} - g^{\omega}| > \frac{\alpha}{2}\}, \] (5.3)
\[ \Omega_{15}^c = \{\omega \in \Omega : |S_2(t)h^{\omega} - h^{\omega}| > \frac{\alpha}{2}\}. \] (5.4)
Obviously,

\[ \Omega_{i5}^c \subset \Omega_{i6}^c \cup \Omega_{i7}^c, \tag{5.5} \]

where

\[ \Omega_{i6}^c = \left\{ \omega \in \Omega : |S_2(t)h^\omega| > \frac{\alpha}{4} \right\}, \tag{5.6} \]
\[ \Omega_{i7}^c = \left\{ \omega \in \Omega : |h^\omega| > \frac{\alpha}{4} \right\}. \tag{5.7} \]

From Lemma 3.4, we have

\[ \mathbb{P}(\Omega_{i3}^c) \leq C_1 e^{-\left[\frac{\alpha}{C2\epsilon}\right]^2} \leq C_1 e^{-\left[\frac{\alpha}{C2\epsilon}\right]^2}. \tag{5.8} \]

From Lemma 3.5, we have

\[ \mathbb{P}(\Omega_{i6}^c) \leq C_1 e^{-\left[\frac{\alpha}{C2\epsilon}\right]^2} \leq C_1 e^{-\left[\frac{\alpha}{C2\epsilon}\right]^2}. \tag{5.9} \]

From Lemma 3.8, we have

\[ \mathbb{P}(\Omega_{i7}^c) \leq C_1 e^{-\left[\frac{\alpha}{C2\epsilon}\right]^2} \leq C_1 e^{-\left[\frac{\alpha}{C2\epsilon}\right]^2}. \tag{5.10} \]

From (5.8)-(5.10), we have

\[ \mathbb{P}(\Omega_{i3}^c) \leq \mathbb{P}(\Omega_{i4}^c) + \mathbb{P}(\Omega_{i5}^c) \leq \mathbb{P}(\Omega_{i4}^c) + \mathbb{P}(\Omega_{i6}^c) + \mathbb{P}(\Omega_{i7}^c) \]
\[ \leq C_1 e^{-\left[\frac{\alpha}{C2\epsilon}\right]^2} + 2C_1 e^{-\left[\frac{\alpha}{C2\epsilon}\right]^2}. \tag{5.11} \]

When \(|t| \leq \epsilon\), from (5.11), we have

\[ \mathbb{P}(\Omega_{i3}^c) \leq C_2 e^{-\left(\frac{\alpha^2}{(C2\epsilon)^2}\right)}. \tag{5.12} \]

Here, \(\epsilon\) satisfies \(C2\epsilon \left(\ln \frac{C2\epsilon}{\alpha}\right)^{\frac{1}{2}} \leq \alpha\). From (5.12), we have

\[ \mathbb{P}(\Omega_{i3}^c) \leq C_2 e^{-\left(\frac{\alpha^2}{(C2\epsilon)^2}\right)} \leq \epsilon. \tag{5.13} \]

From (5.13), we have

\[ \mathbb{P}(\Omega_{i3}) \geq 1 - \epsilon. \tag{5.14} \]

For the proof of the remainder of Theorem 1.2 can be seen in Lemma 3.11.

This completes the proof of Theorem 1.2. \(\square\)
6. Proof of Theorem 1.3

Proof of Theorem 1.3. When \( f \in L^2(\mathbb{R}^n) \), by density theorem which is just Lemma 2.2 in [24], there exists a rapidly decreasing function \( g \) such that \( f = g + h \), where \( \|h\|_{L^2(\mathbb{R}^n)} < \epsilon \). We define

\[ \Omega_{18}^c = \{ \omega \in \Omega : |S_3(t)f^\omega - f^\omega| > \alpha \}. \] (6.1)

Thus, we have

\[ \Omega_{18}^c \subset \Omega_{19}^c \cup \Omega_{20}^c, \] (6.2)

where

\[ \Omega_{19}^c = \{ \omega \in \Omega : |S_3(t)g^\omega - g^\omega| > \frac{\alpha}{2} \}, \] (6.3)
\[ \Omega_{20}^c = \{ \omega \in \Omega : |S_3(t)h^\omega - h^\omega| > \frac{\alpha}{2} \}. \] (6.4)

Obviously,

\[ \Omega_{20}^c \subset \Omega_{21}^c \cup \Omega_{22}^c, \] (6.5)

where

\[ \Omega_{21}^c = \{ \omega \in \Omega : |S_3(t)h^\omega| > \frac{\alpha}{4} \}, \] (6.6)
\[ \Omega_{22}^c = \{ \omega \in \Omega : |h^\omega| > \frac{\alpha}{4} \}. \] (6.7)

From Lemma 3.6, we have

\[ P(\Omega_{19}^c) \leq C_1 e^{-\frac{\alpha}{2c_{	ext{en}}}} \leq C_1 e^{-\frac{\alpha}{2c_{	ext{en}}}}. \] (6.8)

From Lemma 3.7, we have

\[ P(\Omega_{21}^c) \leq C_1 e^{-\frac{\alpha}{C_1 \|h\|_{L^2}}} \leq C_1 e^{-\frac{\alpha}{C_1 \|h\|_{L^2}}}. \] (6.9)

From Lemma 3.8, we have

\[ P(\Omega_{22}^c) \leq C_1 e^{-\frac{\alpha}{C_1 \|h\|_{L^2}}} \leq C_1 e^{-\frac{\alpha}{C_1 \|h\|_{L^2}}}. \] (6.10)

From (4.8)-(4.10), we have

\[ P(\Omega_{18}^c) \leq P(\Omega_{19}^c) + P(\Omega_{20}^c) \leq P(\Omega_{19}^c) + P(\Omega_{21}^c) + P(\Omega_{22}^c) \leq C_1 e^{-\frac{\alpha}{C_1 \|h\|_{L^2}}} + 2C_1 e^{-\frac{\alpha}{C_1 \|h\|_{L^2}}}. \] (6.11)
When $|t| \leq \epsilon$, from (6.11), we have
\[ \mathbb{P}(\Omega_{18}^t) \leq C_2 e^{-\frac{\alpha^2}{(C_\epsilon)^2}}. \] (6.12)

Here $\epsilon$ satisfies $Ce \left( \ln \frac{C_2}{\epsilon} \right)^{\frac{1}{2}} \leq \alpha$. From (6.12), we have
\[ \mathbb{P}(\Omega_{18}^t) \leq C_2 e^{-\frac{\alpha^2}{(C_\epsilon)^2}} \leq \epsilon. \] (6.13)

From (4.13), we have
\[ \mathbb{P}(\Omega_{18}) \geq 1 - \epsilon. \] (6.14)

For the proof of the remainder of Theorem 1.3 can be seen in Lemma 3.11. This completes the proof of Theorem 1.3.

7. Proof of Theorem 1.4

In this section, we prove Theorem 1.4.

**Proof.** For $f \in L^2(\mathbb{R}^n)$, from the density theorem which is just Lemma 2.2 in [24], we know that there exists a decreasing rapidly function $g$ and $h \in L^2(\mathbb{R}^n)$ with $\|h\|_{L^2(\mathbb{R}^n)} < \epsilon$ such that $f = g + h$. Thus, we have $f^\omega = \sum_{k \in \mathbb{Z}^n} g_k(\omega)P(D - k)f = \sum_{k \in \mathbb{Z}^n} g_k(\omega)P(D - k)(g + h) = \sum_{k \in \mathbb{Z}^n} g_k(\omega)P(D - k)g + \sum_{k \in \mathbb{Z}^n} g_k(\omega)P(D - k)h = g^\omega + h^\omega$. By using a direct computation, we have
\[ \mathbb{P} \left\{ \{ \omega \in \Omega : \|h^\omega\|_{L^2} \leq \lambda \} \cap \{ \omega \in \Omega : |x^\alpha \partial^\beta g^\omega| \leq M \} \right\} \]
\[ = \mathbb{P} \left\{ \{ \omega \in \Omega : \|h^\omega\|_{L^2} \leq \lambda \} \right\} - \mathbb{P} \left\{ \{ \omega \in \Omega : \|h^\omega\|_{L^2} \leq \lambda \} \cap \{ \omega \in \Omega : |x^\alpha \partial^\beta g^\omega| > M \} \right\} \]
\[ \geq \mathbb{P} \left\{ \{ \omega \in \Omega : \|h^\omega\|_{L^2} \leq \lambda \} \right\} - \mathbb{P} \left\{ \{ \omega \in \Omega : |x^\alpha \partial^\beta g^\omega| > M \} \right\} \]
\[ \geq 1 - \epsilon - \epsilon = 1 - 2\epsilon. \] (7.1)

This completes the proof of Theorem 1.4.

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References
References

[1] A. Bényi, T. Oh and O. Pocovnicu, On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^d, d \geq 3$, *Trans. Amer. Math. Soc. Ser. B* 2(2015), 1-50.

[2] A. Bényi, T. Oh and O. Pocovnicu, Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS, in: Excursions in Harmonic Analysis, Vol. 4, Birkhäuser/Springer, Cham (2015), pp. 3-25.

[3] A. Bényi, T. Oh and O. Pocovnicu, Higher order expansions for the probabilistic local cauchy theory of the cubic nonlinear schrödinger equation on $\mathbb{R}^3$, arXiv:1709.01910.

[4] J. Bourgain, A remark on Schrödinger operators, *Israel J. Math.* 77(1992), 1-16.

[5] J. Bourgain, Periodic nonlinear Schrödinger equation and invariant measures, *Comm. Math. Phys.* 166(1994), 1-26.

[6] J. Bourgain, Some new estimates on oscillatory integrals, In: Essays on Fourier Analysis in Honor of Elias M. Stein, Princeton, NJ 1991. Princeton Mathematical Series, vol. 42, pp. 83.112. Princeton University Press, New Jersey (1995).

[7] J. Bourgain, Invariant measures for the 2D-defocusing nonlinear Schrödinger equation, *Comm. Math. Phys.* 176(1996), 421-445.

[8] J. Bourgain, On the Schrödinger maximal function in higher dimensions, *Proc. Steklov Inst. Math.* 280(2013), 46-60.

[9] J. Bourgain, A note on the Schrödinger maximal function, *J. Anal. Math.* 130(2016), 393-396.

[10] N. Burq and N. Tzvetkov, Random data Cauchy theory for supercritical wave equations, I. Local theory, *Invent. Math.* 173(2008), 449-475.

[11] N. Burq and N. Tzvetkov, Random data Cauchy theory for supercritical wave equations. II. A global existence result, *Invent. Math.* 173(2008), 477-496.

[12] L. Carleson, Some analytical problems related to statistical mechanics. Euclidean Harmonic Analysis. Lecture Notes in Mathematics, vol. 779, pp. 5.45, Springer, Berlin, (1979).

[13] Y. Chen and H. Gao, The Cauchy problem for the Hartree equations under random influences, *J. Diff. Eqns.* 259(2015), Pages 5192-5219.
[14] J. Colliander and T. Oh, Almost sure well-posedness of the cubic nonlinear Schrödinger equation below \( L^2(T) \), Duke Math. J. 161(2012), 367-414.

[15] M. Chen and S. Zhang, Random data Cauchy problem for the fourth order Schrödinger equation with the second order derivative nonlinearities, Nonl. Anal. 190(2020), 111608.

[16] C. Cho, S. Leea and A. Vargas, Problems on pointwise convergence of solutions to the Schrödinger equation, J. Fourier Anal. Appl. 18(2012), 972-994.

[17] E. Compaan, R. Lucá and G. Staffilani, Pointwise convergence of the Schrödinger flow, arXiv:1907.11192v1 [math.AP] 25 Jul 2019.

[18] M. Cowling, Pointwise behavior of solutions to Schrödinger equations. In: Harmonic Analysis (Cortona, 1982). Lecture Notes in Mathematics, vol. 992, pp. 83-90. Springer, Berlin, (1983).

[19] C. Deng, S. Cui, Random-data Cauchy problem for the Navier-Stokes equations on \( T^3 \), J. Diff. Eqns. 251(2011), 902-917.

[20] B. E. Dahlberg and C. E. Kenig, Anote on the almost everywhere behavior of solutions to the Schrödinger equation. In: Proceedings of Italo-American Symposium in Harmonic Analysis, University of Minnesota. Lecture Notes in Mathematics, vol. 908, pp. 205-209. Springer, Berlin, (1981).

[21] C. Demeter and S. Guo, Schrödinger maximal function estimates via the pseudo-conformal transformation, arXiv: 1608.07640.

[22] B. Dodsona and J. Lührmann, D. Mendelson, Almost sure local well-posedness and scattering for the 4D cubic nonlinear Schrödinger equation, Adv. Math. 347(2019), 619-676.

[23] X. Du, L. Guth and X. Li, A sharp Schrödinger maximal estimate in \( \mathbb{R}^2 \), Ann. Math. 188(2017), 607-640.

[24] X. Du, A sharp Schrödinger maximal estimate in \( \mathbb{R}^2 \), Dissertation 2017.

[25] X. Du and R. Zhang, Sharp \( L^2 \) estimates of the Schrödinger maximal function in higher dimensions, Ann. Math. 189(2019), 837-861.

[26] X. Du, L. Guth, X. Li and R. Zhang, Pointwise convergence of Schrödinger solutions and multilinear refined Strichartz estimates, Forum Math.Sigma, 6(2018).
[27] G. Gigante and F. Soria, On the the boundedness in $H^{1/4}$ of the maximal square function associated with the Schrödinger equation, J. Lond. Math. Soc. 77(2008), 51-68.

[28] H. Hirayama and M. Okamoto, Random data Cauchy problem for the nonlinear Schrödinger equation with derivative nonlinearity, Discrete Conti. Dyn. Sys. A 36(2016), 6943-6974.

[29] C. E. Kenig, G. Ponce and L. Vega, Oscillatory integrals and regularity of dispersive equations. India. Uni. Math. J. 40(1991), 33-69.

[30] C. E. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Comm. Pure Appl. Math. XLVI(1993), 527-620.

[31] C. E. Kenig, G. Ponce and L. Vega, The Cauchy Problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices, Duke Math. J. 71(1993), 1-21.

[32] R. Killip, J. Murphy and M. Visan, Almost sure scattering for the energy-critical NLS with radial data below $H^1(\mathbb{R}^4)$, arXiv:1707.09051.

[33] J. Lebowitz, H. Rose and E. Speer, Statistical mechanics of the nonlinear Schrödinger equation, J. Statist. Phys. 50(1988), 657-687.

[34] S. Lee, On pointwise convergence of the solutions to Schrödinger equation in $\mathbb{R}^2$. Int. Math. Res. Not. 2006, 32597.

[35] J. Lührmann and D. Mendelson, Random data Cauchy theory for nonlinear wave equations of power-type on $\mathbb{R}^3$, Comm. Partial Diff. Eqns. 39(2014), 2262-2283.

[36] R. Luca and M. Rogers, An improved neccessary condition for Schrödinger maximal estimate, arXiv: 1506.05325.

[37] R. Luca and M. Rogers, Coherence on fractals versus pointwise convergence for the Schrödinger equation, Commun. Math. Phys. 351(2017), 341-359.

[38] C. Miao, J. Yang and J. Zheng, An improved maximal inequality for 2D fractional order Schrödinger operators, Stud. Math. 230(2015), 121-165.

[39] C. Miao, J. Zhang and J. Zheng, Maximal estimates for Schrödinger equation with inverse-square potential, Pac. J. Math. 273(2015), 1-19.

[40] A. Moyua, A. Vargas and L. Vega, Schrödinger maximal function and restriction properties of the Fourier transform, IMRN, 1996(1996), 793-815.
[41] A. Nahmod, T. Oh, L. Rey-Bellet and G. Staffilani, Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS, *J. Eur. Math. Soc.* 14(2012), 1275-1330.

[42] A. Nahmod, N. Pavlovic and G. Staffilani, Almost sure existence of global weak solutions for supercritical Navier-Stokes equations, *SIAM J. Math. Anal.* 45(2013), 3431-3452.

[43] A. Nahmod and G. Staffilani, Almost sure well-posedness for the periodic 3D quintic nonlinear Schrödinger equation below the energy space, *J. Eur. Math. Soc.* 17(2015), 1687-1759.

[44] T. Oh, M. Okamoto and O. Pocovnicu, On the probabilistic well-posedness of the nonlinear Schrödinger equations with non-algebraic nonlinearities, arXiv:1708.01568.

[45] T. Oh and O. Pocovnicu, Probabilistic global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on $\mathbb{R}^3$, *J. Math. Pures Appl.* 105(2016), 342-366.

[46] R. Paley, A. Zygmund, On some series of functions (1), (2), (3), *Proc. Camb. Philos. Soc.* 26(1930), 337-357, 458-474; 28(1932), 190-205.

[47] O. Pocovnicu, Almost sure global well-posedness for the energy-critical defocusing nonlinear wave equation on $\mathbb{R}^d$, $d = 4$ and $5$, *J. Eur. Math. Soc.* 19(2017), 2521-2575.

[48] K. Rogers, A. Vargas and L. Vega, Pointwise convergence of solutions to the nonelliptic Schrödinger equation, *Indiana Univ. Math. J.* 55(2006), 1893-1906.

[49] K. Rogers and P. Villarroya, Sharp estimates for maximal operators associated to the wave equation, Ark. Mat. 46(2008), 143-151.

[50] P. Sjölin, Regularity of solutions to the Schrödinger equation, *Duke Math. J.* 55(1987), 699-715.

[51] S. Shao, On localization of the Schrödinger maximal operator, arXiv: 1006.2787v1.

[52] C. D. Sogge, Fourier integrals in classical analysis, Second Edition, 2017.

[53] R. Strichartz, Restrictions of Fourier transforms to surfaces and decay of solutions of wave equation, *Duke Math. J.* 44(1977), 705-714.

[54] T. Tao, A sharp bilinear restriction estimate for paraboloids, *Geom. Funct. Anal.* 13(2003), 1359-1384.
[55] T. Tao and A. Vargas, A bilinear approach to cone multipliers, II. Appl. Geom. Funct. Anal. 10(2003), 216-258.

[56] L. Vega, Schrödinger equations: pointwise convergence to the initial data, Proc. Am. Math. Soc. 102(1988), 874-878.

[57] B. Wang and C. Huang, Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations, J. Diff. Eqns. 239(2007), 213-250.

[58] Y. Wang, Global well-posedness and scattering for derivative Schrödinger equation, Comm. Partial Diff. Eqns. 36(2011), 1694-1722.

[59] T. Zhang and D. Fang, Random data Cauchy theory for the incompressible three dimensional Navier-Stokes equations, Proc. Amer. Math. Soc. 139(2011), 2827-2837.

[60] T. Zhang and D. Fang, Random data Cauchy theory for the generalized incompressible Navier-Stokes equations, J. Math. Fluid Mech. 14(2012), 311-324.