Reduction of Dual Theories

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Abstract

In view of the presence of a superpotential, the dual of a gauge theory like SQCD contains two coupling parameters. The method of the Reduction of Couplings is used in order to express the parameter \( \lambda \) of the superpotential in terms of the dual gauge coupling \( g \). In the conformal window and above it, a unique, isolated solution is obtained. It is given by \( \lambda(g^2) = g^2 f \). Here \( f \) is a function of the number of colors and the number of flavors, and it is known explicitly. This solution is valid to all orders in the asymptotic expansion, and it is the appropriate choice for the dual theory. The same solution exists in the free magnetic interval. A ‘general’ solution with non-integer powers is discussed, as are some exceptional cases.

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1. Introduction

In earlier publications [1, 2], we have shown that certain results about the phase structure of SQCD and similar supersymmetric gauge theories, which are obtained with the help of duality and holomorphy [3, 4, 5], are in quantitative agreement with the consequences of our earlier work [6, 7, 8] involving analyticity and superconvergence of the gauge field propagator [9, 10, 11], 12, 13. This connection is of considerable interest, because the superconvergence arguments are also valid for non-supersymmetric theories. A particular problem in this comparison, and generally in the application of duality, is the more detailed characterization of the dual theory beyond the anomaly matching conditions. In [1], we have already given a brief sketch of the use, for this purpose, of the method of reduction of couplings [14, 15, 16]. In dual SQCD, for example, the reduction method makes it possible to express the Yukawa coupling of the superpotential as a function of the coupling parameter associated with the dual gauge group. The result is a one-parameter magnetic theory, dual to the one-parameter electric gauge theory.

A priori, the reduction equations relate the original dual theory with two coupling parameters to a set of solutions which also has two parameters. There are two solutions with only the primary gauge coupling and no other parameters. They give rise to theories with renormalized power series expansions of their Green’s functions. Asymptotically associated with one of these solutions is a ‘general’ solution. It contains a free parameter besides the gauge coupling, and generally leads to asymptotic expansions with fractional powers of this coupling. These expansions do not correspond to renormalized perturbation theory in the usual sense.

Up to this point, only the renormalization group has been used. However, one of the power series solutions is excluded by the requirements of duality, like preservation of global symmetries. The general solution does not result in a conventional renormalizable theory. Hence, there remains one single-coupling theory with renormalized asymptotic power series expansion, which is the appropriate dual of the original SQCD.

As we will see, the reduction brings out essential features of the dual theory which are not apparent in the two-coupling formulation.

It is the purpose of this paper, to present the results of the reduction of couplings in the conformal window, where the original and the dual theory are asymptotically free at small distances. In addition, we perform the reduction in the free
magnetic region, where the correlation functions at large distances are those of the infrared-free magnetic gauge theory. We consider SQCD with the gauge group $G = SU(N_C)$ and $N = 1$ supersymmetry. There are $N_F$ superfields $Q_i$ and their antifields $\bar{Q}^i$, $i = 1, 2, \ldots, N_F$ in the fundamental representation. These fields are assumed to be massless. Otherwise, we suppose that there is a mass-independent renormalization scheme leading to renormalization group coefficients which are independent of mass parameters. The generalization to other gauge groups is certainly possible [17][18]. In the presence of matter superfields in the adjoint representation, the construction of dual theories requires a superpotential also on the electric side [19], and a corresponding reduction would be indicated already there. For a discussion of duality in general superconformal $N = 1$ models, we refer to [20].

2. Reduction

In order to fix the notation, and for the later discussion of fixed points in the conformal window, we reproduce the one- and two-loop $\beta$-function coefficients for SQCD:

$$\beta_e(g_e^2) = \beta_{e0} g_e^4 + \beta_{e1} g_e^6 + \cdots,$$

with

$$\beta_{e0} = (16\pi^2)^{-1}(-3N_C + N_F)$$
$$\beta_{e1} = (16\pi^2)^{-2} \left( 2N_C(-3N_C + N_F) + 4N_F \frac{N_F^2 - 1}{2N_C} \right).$$

The label $e$ indicates that these $\beta$-functions refer to the ‘electric’ theory. Later, we will omit a corresponding label for the functions of the ‘magnetic’ theory. The theory dual to SQCD involves the gauge group $G^d = SU(N_C^d)$, with $N_C^d = N_F - N_C$. There are $N_F$ quark superfields $q_i$, $\bar{q}^i$, $i = 1, 2, \ldots, N_F$ in the fundamental representation of $G^d$, as well as $N_F^2$ gauge singlet superfields $M_j^i$. The superfields $M$ are independent, and cannot be constructed from $q$ and $\bar{q}$. The number of flavors $N_F$ is the same for SQCD and for dual SQCD, because both theories must have equal global symmetries. The construction of the dual theory is done essentially on the basis of the anomaly matching conditions [3][21], which require the colorless fields $M_j^i$ and their coupling via the superpotential

$$W = \sqrt{\lambda}M_j^i q_i \bar{q}^j.$$
A priori, the dual theory has two parameters, the gauge coupling $g^2$ and the Yukawa coupling $\lambda$. The presence of the superpotential is of importance, not only for the coupling of the $M$-superfield, but also in order to remove a global U(1) symmetry acting on this field, which would otherwise be present. This symmetry has no counterpart in the electric theory. It would destroy the match between the physical symmetries of both theories, which is essential for duality. Consequently, we will accept only reductions which do not switch-off the superpotential.

We write the $\beta$-functions \[ \beta(g^2, \lambda) = \beta_0 g^4 + (\beta_1 g^6 + \beta_{1,\lambda} g^4\lambda) + \cdots \]
\[ \beta_\lambda(g^2, \lambda) = c_\lambda g^2\lambda + c_{\lambda\lambda}\lambda^2 + \cdots . \] (4)

The coefficients are given by

\begin{align*}
\beta_0 &= (16\pi^2)^{-1}(3N_C - 2N_F) \\
\beta_1 &= (16\pi^2)^{-2}\left(2(N_F - N_C)(3N_C - 2N_F) + 4N_F\frac{(N_F - N_C)^2 - 1}{2(N_F - N_C)}\right) \\
\beta_{1,\lambda} &= (16\pi^2)^{-2}\left(-2N_F^2\right) \\
c_\lambda &= (16\pi^2)^{-1}\left(-4\frac{(N_F - N_C)^2 - 1}{2(N_F - N_C)}\right) \\
c_{\lambda\lambda} &= (16\pi^2)^{-1}(3N_F - N_C) . \quad (5)
\end{align*}

We now want to express the Yukawa coupling $\lambda$ as a function of the gauge coupling $g^2$, which we choose as the primary coupling parameter: $\lambda = \lambda(g^2)$.

The method of reduction is based upon the requirement that the Green’s functions of the reduced one-parameter theory satisfy the appropriate renormalization group equations involving the single coupling parameter $g^2$. The corresponding $\beta$-function is then given by

$\beta(g^2) = \beta(g^2, \lambda(g^2))$. \quad (6)

Comparing the renormalization group relations for the one-paramter and the two-parameter theories, we obtain the reduction equations \[ \beta(g^2) \frac{d\lambda(g^2)}{dg^2} = \beta_\lambda(g^2) , \] (7)

\footnote{For recent surveys of the reduction method, see [15, 23]. The case of two couplings has been discussed in detail in [24].}
where $\beta(g^2)$ has been defined in Eq.(6) above, and

$$\beta_\lambda(g^2) = \beta_\lambda(g^2, \lambda(g^2)).$$

The reduction equation (7) is necessary and sufficient for the validity of the renormalization group equations for the Green's functions of the reduced one-parameter theory. It can also be obtained from the renormalization group equations for the effective couplings $g^2(u)$ and $\lambda(u)$ by elimination of the scaling parameter $u$ in favor of the function $\lambda(u)$, using proper caution. With the asymptotic expansions of the $\beta$-functions as given in Eq.(4), the reduction equation is singular at $g^2 = 0$. Uniformisation transformations can remove the singularity and show that all solution have asymptotic series expansions at the origin [14, 25]. These expansions may contain non-integer powers.

After these preliminaries, we return to the reduction of dual SQCD. At first we consider solutions [24, 26, 27] which have asymptotic power series expansions for $g^2 \to 0$. We can restrict ourselves to solutions where the ratio $\lambda(g^2)/g^2$ is bounded at $g^2 = 0$. As seen from the reduction equation (7), an Ansatz with $\lambda(0) \neq 0$ would require $\beta_\lambda(0, \lambda(0)) = 0$ since we have $\beta(0, \lambda(0)) = 0$. This constraint is generally not fullfilled.

We write

$$\lambda(g^2) = g^2 f(g^2), \quad \text{with} \quad f(g^2) = f_0 + \sum_{m=1}^{\infty} \chi^{(m)} g^{2m}. \quad (9)$$

Substitution into the reduction equation (7) yields the fundamental one-loop relation

$$\beta_0 f^0 = \left( c_{\lambda\lambda} f^0 + c_\lambda \right) f^0. \quad (10)$$

There are two solutions:

$$f^0 = f_{00} = 0 \quad \text{and} \quad f^0 = f_{01} = \frac{\beta_0 - c_\lambda}{c_{\lambda\lambda}}, \quad (11)$$

where $f_{01}$ is a function of $N_C$ and $N_F$, and is given by

$$f_{01}(N_C, N_F) = \frac{N_C (N_F - N_C - 2/N_C)}{(N_F - N_C)(3N_F - N_C)}. \quad (12)$$

Here and in the following, we do not consider possible additional terms which vanish exponentially or faster [27].
The one-loop equation (10) is the fundamental relation for reductions. One loop criteria also decide whether the higher order coefficients are determined by the reduction equation. Up to \( m + 1 \) loops, we have the relations:

\[
\left( M(f^0) - m \beta_0 \right) \chi^{(m)} = \left( \beta_m(f^0) f^0 - \beta^{(m)}(f^0) \right) + X^{(m)},
\]

(13)

where \( m = 1, 2, \ldots \). Here we have written the expansions of \( \beta(g^2) \) and \( \beta_\lambda(g^2) \) in a form, which will also turn out to be very useful later:

\[
\begin{align*}
\beta(g^2) &= \beta(g^2, g^2 f(g^2)) = \sum_{n=0}^{\infty} \beta_n(f)(g^2)^{n+2} \\
\beta_\lambda(g^2) &= \beta_\lambda(g^2, g^2 f(g^2)) = \sum_{n=0}^{\infty} \beta_\lambda^{(n)}(f)(g^2)^{n+2},
\end{align*}
\]

(14)

where

\[
\begin{align*}
\beta_0(f) &= \beta_0, \quad \beta_1(f) = \beta_1 + \beta_1 \lambda f, \quad \beta_\lambda^{(0)} = c_\lambda f + c_\lambda \lambda f^2, \quad \text{etc. .}
\end{align*}
\]

(15)

The coefficient \( M(f^0) \) in Eq.(13) given by

\[
M(f^0) = c_\lambda + 2c_\lambda \lambda f^0 - \beta_0.
\]

(16)

The rest term \( X^{(m)} \) depends only upon the coefficients \( \chi^{(1)}, \ldots, \chi^{(m-1)} \), and upon the \( \beta \)-function coefficients in Eqs.(14) of the order \( m - 1 \) and lower, evaluated at \( f = f^0 \). It vanishes for \( \chi^{(1)} = \ldots = \chi^{(m-1)} = 0 \). We see that the one–loop criteria

\[
\left( M(f^0) - m \beta_0 \right) \neq 0 \quad \text{for} \quad m = 1, 2, \ldots
\]

(17)

are sufficient to insure that all coefficients \( \chi^{(m)} \) in the expansion (9) are determined. Then the reduced theory has a renormalized power series expansion in \( g^2 \). All possible solutions of this kind are fixed by the one–loop equation (10) for \( f^0 \). In order to discuss the solutions for dual SQCD, it is convenient to consider characteristic intervals in \( N_F \) separately.

3. Conformal Window

We consider first the conformal window where \( \frac{3}{2} N_C < N_F < 3 N_C \). Here both SQCD and dual SQCD are asymptotically free at small distances, as seen from

\footnote{See page 450 of [8], where the existence of a phase transition at \( N_F = \frac{3}{2} N_C \) has already been derived on the basis of superconvergence relations, and [6] for the corresponding non-SUSY result.}
Eqs.(2) and (5). We consider first the solution with \( f^0 = f_{01} \) as given by Eq.(12). In the widow we have \( f_{01} > 0 \), as required by the superpotential. The factor in front of the coefficient \( \chi^{(m)} \) in Eq.(13) is of the form
\[
(M(f_{01}) - m\beta_0) = -\beta_0(\xi + m) ,
\]
where
\[
\xi(N_C, N_F) = \frac{N_C(N_F - N_C - 2/N_C)}{(N_F - N_C)(2N_F - 3N_C)} .
\]
In the widow, we have \( \xi > 0 \), and hence the coefficients \( \chi^{(m)} \) in the expansion (9) are all determined.

To go further, it is useful to consider regular reparametrisations of the theory [24]. These transformations leave the physics unchanged, but the \( \beta \)-functions of the original theory are generally not invariant. An important exception are the lowest order terms. We will use reparametrisation in order to transform the fixed coefficients \( \chi^{(m)} \) to zero. The transformations in question are of the form
\[
\begin{align*}
g'^2 &= g^2 + a^{(20)}g^4 + a^{(11)}g^2\lambda + \cdots , \\
\lambda' &= \lambda + b^{(20)}\lambda^2 + b^{(11)}\lambda g^2 + \cdots ,
\end{align*}
\]
and they leave the one-loop quantities \( \beta_0 \), \( \beta_\lambda^{(0)}(f^0) \), \( f^0 \) and \( M(f^0) \) invariant. On the other hand, there is a sufficient number of free parameters in the transformations (20), so that we can arrange for all coefficients \( \chi^{(m)} \) in the expansion (9) to be transformed to zero. Hence the power series solution (9) reduces to the simple form
\[
\lambda(g^2) = g^2 f_{01}(N_C, N_F) ,
\]
with \( f_{01} \) given by Eq.(12). The \( \beta \)-functions of the reduced theory, as defined by the solution (21), are now simply given by Eqs.(14) with the argument \( f \) of the coefficient functions replaced by \( f_{01}(N_C, N_F) \) , so that they are constants:
\[
\beta(g^2) = \beta(g^2, g^2 f_{01}) = \sum_{n=0}^{\infty} \beta_n(f_{01})(g^2)^{n+2} ,
\beta_\lambda(g^2) = f_{01} \beta(g^2) .
\]
The second relation follows from the reduction equation (7) with Eq.(21). The coefficient \( \beta_0 \) is as given in Eq.(5), and for \( \beta_1(f_{01}) \) we obtain explicitly
\[(16\pi^2)^2\beta_1(f_{01}) = 2(N_F - N_C)(3N_C - 2N_F) + 4N_F \frac{(N_F - N_C)^2 - 1}{2(N_F - N_C)}
- 4N_F^2 \frac{N_C(N_F - N_C - 2/N_C)}{2(N_F - N_C)(3N_F - N_C)}. \] (23)

These relations are used later in connection with the infrared fixed point of dual SQCD in the conformal window near \(N_F = \frac{3}{2}N_C\). We must note here, that for the expansion (22), in addition to \(\beta_0\), the two-loop coefficient \(\beta_1(f_{01})\) is reparametrization invariant. This result follows because \(f_{01}\) satisfies the reduction equation (10) \(^{27}\).

We now turn to the second solution of the reduction equation (7), which is associated with the one-loop result \(f_0 = f_{00} = 0\) in Eq.(11). Here the expansion coefficients \(\chi(m)\) in the series (9) have the factor

\[(M(0) - m\beta_0) = + \beta_0(\xi - m), \] (24)

with \(\xi(N_C, N_F)\) given in Eq.(19). In all cases where \(\xi\), which is positive in the conformal window, is not an integer, the coefficients \(\chi(m)\) are again determined. They all vanish, as seen from Eq.(13), and hence the corresponding solution is

\[\lambda(g^2) \equiv 0.\] (25)

This solution represents a well defined renormalized theory with an asymptotic power series expansion in the gauge coupling \(g^2\). As we have discussed before, it is not acceptable as a dual theory of SQCD because of the vanishing superpotential.

The situation described above prevails for most values of \(N_C\) and \(N_F\) in the window. An exception is the case \(N_C = 3, N_F = 5\), where we have \(\xi(3, 5) = +2\). Then the coefficient of the power \(g^4\) in the expansion (9) vanishes, and after reparametrization, we have the solution

\[\lambda(g^2) = Ag^6 + \chi^{(3)}g^8 + \cdots. \] (26)

Here the coefficient \(A\) is undetermined. Once \(A\) is given, the higher order coefficients are fixed. For \(A = 0\), they all vanish and we have again Eq.(25). For the exceptional situation mentioned here, and two similar cases encountered later in the free magnetic interval, the dual gauge group is \(G^d = SU(2)\), which has some special features not present for larger values of \(N_C^d\). Here we consider the solution (26) as a special case of the ‘general’ solution discussed below.
It remains to discuss the ‘general’ solutions of the reduction equation in
the conformal window. They generally involve non-integer powers of $g^2$. Trying an
Ansatz of this kind, we see that there are no such solutions associated with $f^0 = f_{01}$
in Eq.(11). On the other hand, for $f^0 = f_{00} = 0$, we have a solution of the form

$$\lambda(g^2) = B g^{2+2\xi} + \cdots,$$

(27)

where $\xi$ is again given by Eq.(19). It is non-integer in the window, at least for
$N_C < 16$. An exception is the case $N_C = 3$, $N_F = 5$ discussed above, where the dual
gauge group is $SU(2)$. Higher order terms are of the form $(g^2)^{n\xi+m}$, with appropriate
distinct powers. The coefficient $B$ is undetermined. If $B$ is fixed, the coefficients of
the higher terms are also determined, and they vanish if $B = 0$.

It is important to realize, that the two coupling parameters of the original version
of the dual theory are reflected, in the set of solutions for the reduction equations, in
the free coefficient $B$ of the ‘general’ solution (27) and the remaining primary coupling
$g$. Within the set of solutions, the power series $\lambda(g^2) = g^2 f_{01}$ stand out as being
appropriate for duality. The other power series solution $\lambda(g^2) \equiv 0$ is excluded. The
‘general’ solution is associated with this forbidden solution, since both approach
each other asymptotically. For these reasons, and because of the lack of standard
asymptotic power series expansions for the Green’s functions, we do not consider
theories involving the ‘general’ solution as appropriate duals to SQCD.

Since $\xi > 0$ in the window, the Yukawa coupling in (27) vanishes faster than $g^2$,
paticularly near the lower end of the widow considered here, where $\xi$ becomes very large. This behavior would have implications for the infrared fixed-point, as will be
discussed below.

We conclude that, in the conformal window, the solution $\lambda(g^2) = g^2 f_{01}$ represents
the unique and isolated one-coupling theory which is dual to SQCD. It is isolated
or ‘unstable’ [27], because there are no other solutions approaching it for $g^2 \to 0$.

5 Although they would not be appropriate for dual theories with renormalized asymptotic expansions, one may ask about the possibility of other ‘general’ solutions of the reduction equation, which do not approach the power series (21) or (25). If such solutions should exist, they would require more explicite, non-asymtotic knowledge of the $\beta$-functions. However, one can use the theorems of Ljapunov [28], as generalized by Malkin [29], in order to obtain information about the possible existence of such solutions on the basis of the linear part of the differential equation [30, 27]. In a finite neighborhood of the origin, no solutions of this kind are expected for the theories considered here. Stability or instability of the solutions (21) and (25), with $f(g^2)$ approaching $f^0$, is determined by the positive or negative sign of $\beta_0^{-1} M(f^0)$ respectively, as has been discussed above.
The theory has an asymptotic power series expansion, and the one-loop character of the anomalies is preserved.

**Infrared Fixed-Points**

Since a very long time, it is well known that QCD and SQCD appear to have non-trivial infrared fixed points for values of $N_F$ near the point where asymptotic freedom is lost [31]. We are interested in possible fixed points in the window $\frac{3}{2}N_C < N_F < 3N_C$ for the two theories considered here, which are dual to each other. Very near the point $N_F = 3N_C$, approaching from below, the electric theory is weakly coupled at large distances, and we may use the one- and two-loop $\beta$-function coefficients in Eq.(2) to find

$$\beta_e\left(g_e^2\right) = 0 \quad \text{for} \quad \frac{g_e^2}{16\pi^2} \approx \frac{3N_C - N_F}{6(N_C^2 - 1)}. \quad (28)$$

As mentioned, we have assumed that $(3N_C - N_F)$ is positive and sufficiently small compared to $N_C$ in order to get a small value of $g_e^2/16\pi^2$. We have neglected here all higher order terms in $g_e^2$, and evaluated the coefficient of $(3N_C - N_F)$ at $N_F = 3N_C$.

At the other end of the window, very near the point $N_F = \frac{3}{2}N_C$, the magnetic theory is weakly coupled in the infrared. We can use the unique reduced theory, as defined by Eq.(21), in order to obtain the fixed point there. With the $\beta$-function coefficients in Eqs.(5) and (23), we find

$$\beta\left(g^2\right) = 0 \quad \text{for} \quad \frac{g^2}{16\pi^2} \approx \frac{7}{3} \frac{N_F - \frac{3}{2}N_C}{N_C^2 - 1}, \quad (29)$$

with assumptions analogous to those described above, but now referring to $(N_F - \frac{3}{2}N_C)$. Certainly, larger values of $N_C$ are needed for this approximation to be useful.

It is relevant here, and has been pointed out before, that the two-loop coefficient $\beta_1(f_{01})$ is *reparametrization invariant* [27], as is $\beta_0$.

The important proposal by Seiberg [4, 5] is, that for given values of $N_C$ and $N_F$ in the conformal window, the electric and the magnetic theories flow to the *same* fixed point. For example, near the lower end of the window, with $N_F$ near $\frac{3}{2}N_C$, the magnetic theory is weakly coupled in the infrared, and we find that a fixed point is present. In contrast, the electric theory is strongly coupled there. Since both theories should be the same at the fixed point, we can obtain information about the electric theory by using the weakly coupled magnetic dual.
Although we did not consider the ‘general’ solution (27) to be appropriate for
the dual theory, it may be of interest to note, that with this solution, as well as with
the exceptional solution (26), the corresponding reduced theories flow to fixed points
which are different from those of the theory with \( \lambda(g^2) = g^2 f_{01} \). For \((N_F/N_C - 3/2)\)
sufficiently small, where we can obtain a crude estimate, these fixed points are more
near to the the Coulomb phase fixed point of the theory with \( \lambda(g^2) \equiv 0 \).

\[ N_F > 3N_C \]

We add here some remarks about the region above the window. For these values
of \( N_F \), the magnetic theory is asymptotically free at small distances. We again
have \( f_{01} > 0 \) and \( \xi > 0 \), as seen from Eqs.(12) and (19). The results of the
reduction method are the same as in the conformal window discussed above, with
the reparametrized power series solution given in Eq.(21) representing the dual one-
parameter theory. In contrast to the situation at the lower end of the conformal
window, for the larger values of \( N_F \) considered here, the magnetic theory is strongly
coupled, and the spectrum of the theory is that of the electric Lagrangian. It is the
electric theory which is infrared free in the region of \( N_F \) considered here.

4. Free Magnetic Phase

The free magnetic phase is the interval \( N_C + 2 \leq N_F < \frac{3}{2}N_C \). It is non-
empty for \( N_C > 4 \), which we assume in the following. For convenience, we consider
first \( N_F > N_C + 2 \), leaving the boundary case for later. The electric theory is
asymptotically free at small distances, but strongly coupled otherwise, while the
magnetic theory is infrared free. Hence, at low energies, the spectrum of the theory
is that of the magnetic Lagrangian. Although, in view of the lack of asymptotic
freedom at small distances, the theory may not exist as a strictly local field theory,
it can be considered as a large distance limit of an appropriate brane construction
in superstring theory, which can also reaffirm duality \[32\]. The same remarks apply
to the electric theory discussed above in the region \( N_F > 3N_C \).

For the reduction of the free magnetic theory, we consider again the asymptotic
expansion for \( g^2 \to 0 \). Now however, this limit corresponds to an approach to the
trivial infrared fixed point. The solutions of the reduction equation are similar to
those we have discussed above in the conformal window. The important change is,
that the function \( \xi(N_C, N_F) \) defined in Eq.(19) is now negative, while the coefficient
\( f_{01}(N_C, N_F) \) in Eq.(12) remains positive, as required by the reality condition for the
superpotential (3). Consider first the power series solution in Eq.(9) associated with the one-loop coefficient $f_{01}$ from Eq.(11). In calculating the expansion coefficients $\chi^{(m)}$ using Eq.(13), we now have the factor

\[(M(f_{01}) - m\beta_0) = -\beta_0(\xi + m),\]

which could possibly vanish. But inspection of $\xi(N_C, N_F)$ given in Eq.(19) shows, that it is not a negative integer in the free magnetic region considered here, at least for $N_C < 16$. Hence we have again a well determined power series solution. Using the regular reparametrizations (20), it can be transformed into the simple form given in Eq.(21): $\lambda(g^2) = g^2 f_{01}$.

With the other solution of the one-loop equation, $f^0 = f_{00} = 0$, we find for the coefficient of $\chi^{(m)}$ in Eq.(13)

\[(M(0) - m\beta_0) = -\beta_0(-\xi + m),\]

which cannot vanish since $\xi < 0$ in the free magnetic region. Hence all the coefficients $\chi^{(m)}$ are determined, and it follows from Eq. (13) that they all vanish, so that the solution $\lambda(g^2) \equiv 0$ is obtained. As we have pointed out before, because the presence of the superpotential (3) is required, this solution cannot be used as the magnetic theory.

We finally consider the ‘general’ solution of the reduction equation for the free magnetic region. It is easily seen to be of the form

\[\lambda(g^2) = g^2 f_{01} + C g^{2+2|\xi|} + \cdots,\]

with an undetermined coefficient $C$ and $|\xi| > 1$. In Eq.(32), we have implied a reparametrization transformation in order remove possible integer powers below the $C$-term. Higher order terms involve distinct powers of the form $(g^2)^n|\xi|+m$ with appropriate integers $n$ and $m$. Their coefficients are determined once $C$ is fixed, and the choice $C = 0$ leads back to the solution (21). In contrast to the situation in the conformal window, we see that here the power series solution (21) is stable in the sense that the general solution (32) approaches it near $g^2 = 0$. On the other hand, the null solution (25) is isolated or unstable.

As we have discussed in the case of the conformal window, the ‘general’ solution (32) would not give rise to a dual theory with a renormalized asymptotic power series expansion. Consequently, there remains the special power series solution (21)
which should be used as the magnetic theory in the free magnetic region considered. The one-loop character of the matching conditions is certainly valid for this solution. In addition, it connects with the solution appropriate for the conformal window.

It remains to mention the boundary case \( N_F = N_C + 2 \). It still belongs to the free magnetic phase provided \( N_C > 4 \), so that \( N_C + 2 < \frac{3}{2} N_C \). The magnetic gauge group \( G^d \) is \( SU(2) \), and there is some extra flavor symmetry.

For \( N_F = N_C + 2 \), the results of the reduction method are essentially the same as described above for the free magnetic region with \( N_F > N + 2 \). There are two exceptions for the power series solution (9) with \( f^0 = f_{01} \), namely \( N_C = 5 \) and \( N_C = 7 \). For these cases the corresponding coefficients of \( \chi^{(m)} \) in Eq.(13) vanish, because

\[
\xi(N_C, N_F = N_C + 2) = -1 - \frac{3}{N_C - 4}
\]

implies that \( \xi(5, 7) = -4 \) as well as \( \xi(7, 9) = -2 \). For these values of \( N_C \), we do not have the completely determined special solution (21) because of the undefined coefficient in the expansion (9). Rather there are the expressions

\[
\lambda(g^2) = g^2 f_{01} + D g^{2+2|\xi|} + \cdots,
\]

with \( |\xi| = 4 \) and \( |\xi| = 2 \) respectively. As in general solutions considered earlier, the coefficient \( D \) is undetermined, and all later terms are fixed once \( D \) is given, or vanish if we choose \( D = 0 \). The three exceptional cases we have encountered, all have \( G^d = SU(2) \) as the dual gauge group, and there are special features we plan to discuss elsewhere.

For smaller values of \( N_F \), like \( N_F = N_C + 1 \), there is no dual magnetic gauge theory. The spectrum is described by gauge invariant fields, massless baryons and mesons, which are then the elementary magnetic quanta.

\[
N_F = \frac{3}{2} N_C
\]

At this important transition point of the magnetic theory we have \( \beta_0 = 0 \), so that the function \( \xi(N_C, N_F) \), given in Eq.(19), is not defined. There are no general solutions, and we have only the two power series solutions, which can be brought into the forms (21) and (25) respectively. Again then the solution (21): \( \lambda(g^2) = g^2 f_{01} \) is the only choice for the magnetic theory. Here the coefficient \( f_{01} \), as defined in Eq.(12), is given by

\[
f_{01}(N_C, N_F = \frac{3}{2} N_C) = \frac{2}{7}(1 - \frac{4}{N_C^2}).
\]
We have assumed $N_C > 2$ throughout.

5. Conclusions

We summarize the results of the reduction method:

In the conformal window $\frac{3}{2}N_C < N_F < 3N_C$, at the transition point $N_F = \frac{3}{2}N_C$, as well as for $N_F > 3N_C$, we have the unique and isolated power series solution (21): $\lambda(g^2) = g^2 f_{01}(N_C, N_F)$, which is valid to all orders in the asymptotic expansion in $g^2$. Only possible contributions which vanish exponentially or faster are not written. It is the appropriate solution for the dual magnetic theory associated with SQCD.

The same solution exists for the free magnetic region $N_C + 2 \leq N_F < \frac{3}{2}N_C$. There this solution is accompanied by a ‘general’ solution, which approaches it for $g^2 \to 0$. Except for the two special cases mentioned, the general solution has non-integer powers and a free parameter. Since we want the magnetic theory to have asymptotic power series expansions for the Green’s functions, we select the solution (21), which then defines this theory for all $N_F > N_C + 2$. With the exception of two values of $N_C$, we can also include the boundary case $N_F = N_C + 2$. In the free magnetic region, the asymptotic expansion is a long distance expansion, because the magnetic theory is infrared-free, while in the conformal window and above, we have asymptotic freedom at small distances.

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