Algebra extensions and derived-discrete algebras

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Abstract
Let \( \phi : A \rightarrow B \) be an algebra homomorphism between finite dimensional algebras. We prove that if \( \phi \) is split, the derived-discreteness of \( A \) implies the derived-discreteness of \( B \); if \( \phi \) is separable and the right \( A \)-module \( B_A \) is projective, the converse holds. We prove an analogous statement for piecewise hereditary algebras.

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1. Introduction
The notion of a derived-discrete algebra is introduced by [11]. This class of algebras plays a special role in the representation theory of algebras, since their derived categories are accessible [8]. The derived-discrete algebras over an algebraically closed field were classified by [11] up to Morita equivalence, and by [3] up to derived equivalence.

We plan to classify derived-discrete algebras over an arbitrary infinite field. One approach is to investigate the relation between derived-discreteness and field extensions. More generally, we study the relation between derived-discreteness and algebra extensions.

Let us describe our main results. Assume that \( k \) is an infinite field, and that \( A \) and \( B \) are finite dimensional algebras over \( k \). Let \( \phi : A \rightarrow B \) be an algebra extension, that is, an \( k \)-algebra homomorphism. We prove that if \( \phi \) is a split extension, the derived-discreteness of \( B \) implies the derived-discreteness of \( A \); if \( \phi \) is separable and the right \( A \)-module \( B_A \) is projective, the derived-discreteness of \( A \) implies the derived-discreteness of \( B \); see Theorem 4.1. The first statement strengthens [11, 3.3 Proposition]. The condition that \( B_A \) is projective as a right \( A \)-module in the second statement is necessary; see Example 4.4. We prove analogous statements for piecewise hereditary algebras; see Proposition 5.1.

Recall from [11, 2.1 Theorem] the classification of derived-discrete algebras: a connected algebra \( A \) over an algebraically closed field \( k \) is derived-discrete if and only if \( A \) is either piecewise hereditary of Dynkin type or Morita equivalent to a gentle one-cycle algebra with the clock condition. As an application of the above results, we prove that the dichotomic classification of derived-discrete algebras is compatible with skew group algebra extensions; see Proposition 6.2.

Throughout, we fix an infinite field \( k \), not algebraically closed in general. We require that all the algebras are finite dimensional over \( k \), and all the functors are \( k \)-linear.

2. Derived-discrete algebras
In this section, we recall derived-discrete algebras. Denote by \( \mathbb{N}^{(\mathbb{Z})} \) the set of vectors \( \underline{n} = (n_i)_{i \in \mathbb{Z}} \) of natural numbers with only finitely many nonzero entries.
Let $A$ be a finite dimensional algebra over $k$. Denote by $A\text{-mod}$ the abelian category of finitely generated left $A$-modules. Let $D^b(A\text{-mod})$ be its bounded derived category. For each $X$ in $D^b(A\text{-mod})$, let

$$\dim_k X = (\dim_k H^i(X))_{i \in \mathbb{Z}} \in \mathbb{N}^{|\mathbb{Z}|}$$

be its cohomology dimension vector. Denote by $[X]$ the isomorphism class of $X$ in $D^b(A\text{-mod})$.

**Definition 2.1.** A finite dimensional $k$-algebra $A$ is called **derived-discrete** over $k$, if for any vector $\underline{n} = (n_i)_{i \in \mathbb{Z}}$ in $\mathbb{N}^{|\mathbb{Z}|}$,

$$\{[X] \in D^b(A\text{-mod}) \mid \dim_k X = \underline{n}\}$$

is a finite set.

The lemma below shows that the definition above is equivalent to the one in [11, 1.1]. Denote by $K_0(A)$ the Grothendieck group of $A\text{-mod}$. For each $X$ in $D^b(A\text{-mod})$, set $K\dim X = (m_i)_{i \in \mathbb{Z}} \in K_0(A)^{|\mathbb{Z}|}$, where $m_i$ is the dimension vector of $H^i(X)$.

**Lemma 2.2.** A finite dimensional $k$-algebra $A$ is derived-discrete if and only if for each $(m_i)_{i \in \mathbb{Z}} \in K_0(A)^{|\mathbb{Z}|}$,

$$\{[X] \in D^b(A\text{-mod}) \mid X \text{ is indecomposable with } K\dim X = (m_i)_{i \in \mathbb{Z}}\}$$

is a finite set.

**Proof.** For each $(m_i)_{i \in \mathbb{Z}} \in K_0(A)^{|\mathbb{Z}|}$, let $n_i$ be the total dimension of $m_i$. For each object $X$ in $D^b(A\text{-mod})$ such that $H^i(X) = m_i$, $\forall i \in \mathbb{Z}$, we have $\dim_k X = (n_i)_{i \in \mathbb{Z}} = \underline{n}$. Hence the “only if” part holds.

Conversely, we assume that

$$\{[X] \in D^b(A\text{-mod}) \mid X \text{ is indecomposable with } K\dim X = (m_i)_{i \in \mathbb{Z}}\}$$

is a finite set for each $(m_i)_{i \in \mathbb{Z}}$. For each $(n_i)_{i \in \mathbb{Z}} \in \mathbb{N}^{|\mathbb{Z}|}$, there are only finitely many $(m_i)_{i \in \mathbb{Z}} \in K_0(A)^{|\mathbb{Z}|}$ such that the total dimension of $m_i$ equals $n_i$. Hence

$$\{[X] \in D^b(A\text{-mod}) \mid X \text{ is indecomposable with } \dim_k X = \underline{n}\}$$

is a finite set. Since $D^b(A\text{-mod})$ is Krull-Schmidt, the “if” part holds.

**Lemma 2.3.** Let $K/k$ be a finite field extension and $A$ be a finite dimensional $K$-algebra. Then $A$ is derived-discrete over $K$ if and only if it is derived-discrete over $k$.

**Proof.** Assume that $K/k$ is a finite field extension of degree $l$. For each $X \in D^b(A\text{-mod})$, we have

$$\dim_k X = (\dim_k H^i(X))_{i \in \mathbb{Z}} = (l \cdot \dim_K H^i(X))_{i \in \mathbb{Z}} = l \cdot \dim_K X.$$ 

Hence for each $\underline{n} = (n_i)_{i \in \mathbb{Z}}$ in $\mathbb{N}^{|\mathbb{Z}|}$,

$$\{[X] \in D^b(A\text{-mod}) \mid \dim_k X = \underline{n}\} = \{[X] \in D^b(A\text{-mod}) \mid \dim_k X = l \cdot \underline{n}\}.$$ 

Therefore, $A$ is derived-discrete over $K$ if it is derived-discrete over $k$.

Conversely, for each $m = (m_i)_{i \in \mathbb{Z}}$ in $\mathbb{N}^{|\mathbb{Z}|}$, the set

$$\{[X] \in D^b(A\text{-mod}) \mid \dim_k X = m\}$$

is non-empty only when $m_i$ is divisible by $l$ for any $i \in \mathbb{Z}$. By $(\star)$ again, $A$ is derived-discrete over $K$ if it is derived-discrete over $K$.

Denote by $K^-(A\text{-proj})$ the homotopy category of bounded-above complexes of finitely generated projective left $A$-modules. Let $K^b(A\text{-proj})$ (resp. $K^{-b}(A\text{-proj})$) be its full subcategory consisting of
bounded complexes (resp. complexes with bounded cohomologies). There is a well-known triangle equivalence

\[ p : \mathbf{D}^b(A\text{-mod}) \rightarrow \mathbf{K}^{\leq b}(A\text{-proj}) \]

sending \( X \) to its projective resolution \( pX \). For each \( P \) in \( \mathbf{K}^{\leq b}(A\text{-proj}) \), by \( P_{\geq t} \) in \( \mathbf{K}^b(A\text{-proj}) \) we denote the brutal truncation of \( X \) at degree \( t \).

**Lemma 2.4.** Let \( X, Y \) be in \( \mathbf{D}^b(A\text{-mod}) \), and \( t \) be an integer such that \( H^i(X) = H^i(Y) = 0 \) whenever \( i < t \). Then \( (pX)_{\geq t} \cong (pY)_{\geq t} \) in \( \mathbf{K}^b(A\text{-proj}) \) implies that \( X \cong Y \) in \( \mathbf{D}^b(A\text{-mod}) \).

**Proof.** Let \( (f^i)_{i \in \mathbb{Z}} : (pX)_{\geq t} \rightarrow (pY)_{\geq t} \) in \( \mathbf{K}^b(A\text{-proj}) \) be a homotopy equivalence. By assumption, \( H^t(pX) = H^t(pY) = 0 \). So \( (f^i)_{i \in \mathbb{Z}} \) induces an quasi-isomorphism \( (f^i)_{i \in \mathbb{Z}} \) from \( \overline{pX} \) to \( \overline{pY} \) as follows.

\[
\begin{array}{ccccccccc}
\overline{pX} : & \cdots & \rightarrow & 0 & \rightarrow & \text{Ker}d_{pX}^i & \rightarrow & (pX)^i & \rightarrow & (pX)^{i+1} & \rightarrow & \cdots \\
& & & 0 & \rightarrow & f^i & \rightarrow & pX^i & \rightarrow & pX^{i+1} & \rightarrow & \cdots \\
\overline{pY} : & \cdots & \rightarrow & 0 & \rightarrow & \text{Ker}d_{pY}^i & \rightarrow & (pY)^i & \rightarrow & (pY)^{i+1} & \rightarrow & \cdots \\
& & & 0 & \rightarrow & f^i & \rightarrow & pY^i & \rightarrow & pY^{i+1} & \rightarrow & \cdots \\
\end{array}
\]

So we have isomorphisms \( X \cong pX \cong \overline{pX} \cong \overline{pY} \cong pY \cong Y \) in \( \mathbf{D}^b(A\text{-mod}) \). \( \square \)

Recall that a complex \( (P^i, d^i) \) in \( \mathbf{K}^{\leq b}(A\text{-proj}) \) is called **homotopically-minimal** if \( \text{Im}d^i \subseteq \text{rad}P^{i+1} \) for each \( i \). Each \( X \) in \( \mathbf{D}^b(A\text{-mod}) \) has a homotopically-minimal projective resolution which is quasi-isomorphic to \( X \); see [2, Proposition B.1].

**Lemma 2.5.** Assume that we are given \( n = (n_i)_{i \in \mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}} \). Then the set

\[ p_i := \{ \dim_k P^i | P \in \mathbf{K}^{\leq b}(A\text{-proj}) \text{ homotopically-minimal with } \dim_k P = n \} \]

is bounded for each \( i \in \mathbb{Z} \).

**Proof.** By assumption, for each homotopically-minimal \( P \) in \( \mathbf{K}^{\leq b}(A\text{-proj}) \), we have

\[
\dim_k(P^i/\text{rad}P^i) \leq \dim_k(P^i/\text{Im}d^{i-1}) = \dim_k P^i - \dim_k \text{Im}d^{i-1} = \dim_k \text{Ker}d^i + \dim_k \text{Im}d^i - \dim_k \text{Im}d^{i-1} = \dim_k H^i(P) + \dim_k \text{Im}d^i \leq \dim_k H^i(P) + \dim_k P^{i+1}.
\]

For each homotopically-minimal \( P \) in \( \mathbf{K}^{\leq b}(A\text{-proj}) \) with \( \dim_k P = n \), let \( r \) be the largest integer such that \( n_r \neq 0 \). Then \( r \) is also the largest number such that \( P^r \neq 0 \). So \( \dim_k P^i = 0 \) for \( i > r \). Recall a fact that, given \( n \) in \( \mathbb{N} \), the set

\[
\{ \dim_k Q | Q \text{ a projective } A\text{-module with } \dim_k(Q/\text{rad}Q) \leq n \}
\]

is bounded. Hence \( p_r \) is bounded since \( \dim_k(P^i/\text{rad}P^i) \leq n_r \).

Once \( p_{t+1} \) is bounded for some \( t \leq r - 1 \), the set

\[
\{ \dim_k(P^i/\text{rad}P^i) | P \in \mathbf{K}^{\leq b}(A\text{-proj}) \text{ homotopically-minimal with } \dim_k P = n \}
\]

is bounded by the inequality above. Then \( p_t \) is bounded by the fact. Inductively, we can prove the statement. \( \square \)
Recall that the component dimension vector of a bounded complex $X$ is denoted by $c\text{-dim}_k X = (\dim_k X^i)_{i \in \mathbb{Z}} \in \mathbb{N}^{(\mathbb{Z})}$.

The following lemma is essentially contained in [11, Theorem 2.1 (ii) and (iii)] and [1, Theorem 2.3 a)]. We include a direct proof.

**Lemma 2.6.** The following statements are equivalent.

1. The algebra $A$ is derived-discrete over $k$.
2. For each $n \in \mathbb{N}^{(\mathbb{Z})}$, $\{[P] \in K^b(A\text{-proj}) \mid \dim_k P = n\}$ is a finite set.
3. For each $n \in \mathbb{N}^{(\mathbb{Z})}$, $\{[P] \in K^b(A\text{-proj}) \mid c\text{-dim}_k P = n\}$ is a finite set.

**Proof.** (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). For each $n \in \mathbb{N}^{(\mathbb{Z})}$, $n$ has finitely many partitions. By assumption, the set $\{[P] \in K^b(A\text{-proj}) \mid \dim_k P \leq n\}$ is finite. Since the cohomology dimension vector is not larger than the component dimension vector, the set $\{[P] \in K^b(A\text{-proj}) \mid c\text{-dim}_k P = n\}$ is finite.

(3) $\Rightarrow$ (1). For each $n = (n_i)_{i \in \mathbb{Z}} \in \mathbb{N}^{(\mathbb{Z})}$, let $t$ be the least number such that $n_{t+1} \neq 0$ and $r$ be the largest number such that $n_r \neq 0$. For each $X \in D^b(A\text{-mod})$, let $pX \in K^- (A\text{-proj})$ be the homotopically-minimal projective resolution. By Lemma 2.5, for each $i$, $\dim_k (pX)^{i}_{\geq t}$ is uniformly bounded, say by $m_i$.

We can assume that $m_i = 0$ for $i < t$ and $i > r$. Set $m = (m_i)_{i \in \mathbb{Z}} \in \mathbb{N}^{(\mathbb{Z})}$.

Notice that $m$ has finitely many partitions. By assumption, the set $\{[(pX)_{\geq t}] \in K^b(A\text{-proj}) \mid X \in D^b(A\text{-mod}) \text{ with } c\text{-dim}_k (pX)_{\geq t} \leq m\}$ is finite. By the argument in the above paragraph, the set $\{[(pX)_{\geq t}] \in K^b(A\text{-proj}) \mid X \in D^b(A\text{-mod}) \text{ with } \dim_k X = n\}$ is finite. By Lemma 2.4, the set $\{[X] \in D^b(A\text{-mod}) \mid \dim_k X = n\}$ is finite. $\square$

### 3. Separable functors and algebra extensions

In this section, we recall the notions of separable functors, separable extensions and split extensions.

According to [7], a functor $F \colon C \to D$ is called separable if for any $X, Y$ in $C$, there is a map $H_{X,Y} : \text{Hom}_D(F(X), F(Y)) \to \text{Hom}_C(X, Y)$ such that $H_{X,Y}(f) = f$, for any $f \in \text{Hom}_C(X, Y)$, and $H_{X,Y}$ is natural in $X$ and $Y$. It is called a cleaving functor in [11].

Let $\phi : A \to B$ be a $k$-algebra homomorphism (in some literature, it is called a $k$-algebra extension). It induces the restriction functor $\text{Hom}_B(B, -) : B\text{-mod} \to A\text{-mod}$, and its left adjoint functor $B \otimes_A - : A\text{-mod} \to B\text{-mod}$.
Definition 3.1 ([7, 1.3]). We call an algebra homomorphism $\phi: A \to B$ a split algebra extension if $B \otimes A \to A$-mod $\to B$-mod is separable, and a separable algebra extension if $\text{Hom}_B(B, -): B$-mod $\to A$-mod is separable.

The following theorem is the main result of this section. We recall some notations. We extend the adjoint pair $(F = B \otimes A, G = \text{Hom}_B(B, -))$ to an adjoint pair $(K(F), K(G))$ between $K^*(A$-mod) and $K^*(B$-mod) in a natural manner, where $*$ can be $b$, $-$ or $\cdot$. Since $G$ is exact, $D(G) = K(G): D^-(B$-mod) $\to D^-(A$-mod)
is its own right derived functor. Since $F$ is right exact and preserves projectives, the left derived functor of $F$ is

$$\mathbb{L}F = qK(F)p: D^-(A$-mod) $\to D^-(B$-mod),$$

where $q$ is the localization functor and sometimes we omit it on objects.

Theorem 3.2. Let $\phi: A \to B$ be a $k$-algebra extension between two finite dimensional $k$-algebras with $(F, G)$ the corresponding adjoint pair.

(1) The extension $\phi$ is a split extension if and only if $K(F)$ is separable if and only if $\mathbb{L}F$ is separable.

(2) The extension $\phi$ is a separable extension if and only if $K(G)$ is separable, which are implied by that $D(G)$ is separable. If further $F$ is exact, then $\phi$ is a separable extension if and only if $D(G)$ is separable.

To prove the theorem, we need some preparation. We first give some examples. The proof is in the end of this section.

Example 3.3. Let $A$ be a $k$-algebra.

(1) For each two-sided ideal $I$ of $A$, the canonical quotient $A \to A/I$ is separable.

(2) Let $G$ be a finite group acting on $A$ with its order $|G|$ invertible in $k$. Then the extension from $A$ to its skew group algebra $AG$ is separable and split; see [10, Section 1].

(3) Let $K/k$ be a finite field extension. We consider the extension

$$\phi: A \to A \otimes_k K, \psi(a) = a \otimes 1.$$

It has a retraction $a \otimes \lambda \mapsto a\pi(\lambda), \forall a \in A, \lambda \in K$, as an $A$-bimodule homomorphism, where $\pi : K \to k$ is a $k$-linear retraction of $k \leftarrow K$. Hence $\phi$ is a split extension by [7, Proposition 1.3 (2)].

If further $K/k$ is separable, then the multiplication map $K \otimes_k K \to K$ has a section $\psi$ as a $K$-bimodule homomorphism. It induces an $A \otimes_k K$-bimodule homomorphism

$$A \otimes_k K \xrightarrow{1 \otimes \psi} A \otimes_k K \otimes_k K \xrightarrow{\theta \otimes 1 \otimes k} A \otimes_A A \otimes_k K \otimes_k K \xrightarrow{\rho} (A \otimes_k K) \otimes_A (A \otimes_k K),$$

where $\theta(a) = a \otimes 1$ and $\rho(a_1 \otimes a_2 \otimes \lambda_1 \otimes \lambda_2) = a_1 \otimes \lambda_1 \otimes a_2 \otimes \lambda_2, \forall a_1, a_2 \in A, \lambda_1, \lambda_2 \in K$. This is a section of the multiplication map $(A \otimes_k K) \otimes_A (A \otimes_k K) \to A \otimes_k K$. By [7, Proposition 1.3 (1)], $\phi$ is also a separable extension as $k$-algebras.

Lemma 3.4. Let $F$ be a functor between two module categories.

(1) The functor $F$ is separable if and only if so is $K(F)$.

(2) If $F$ is exact and $D(F)$ is separable, then $F$ is separable.

Proof. (1). If $F$ is separable, it has a natural retraction $H_{M,N}$ on morphisms for any modules $M$ and $N$. For any complexes $X = (X^i)_{i \in \mathbb{Z}}$ and $Y = (Y^i)_{i \in \mathbb{Z}}$, we extend $H_{M,N}$ term-wise to a natural retraction

$$H_{X,Y}: (f^i)_{i \in \mathbb{Z}} \mapsto (H_{X^i,Y^i}(f^i))_{i \in \mathbb{Z}}$$
on chain maps between complex categories. Moreover, if $(f^i)_{i \in \mathbb{Z}}$ is null-homotopic with $f^i = F(d^{i-1}) \circ s^i + s^{i+1} \circ F(d^i)$, then $(H_{X^i,Y^i}(f^i))_{i \in \mathbb{Z}}$ is null-homotopic with

$$H_{X,Y}(f^i) = d^{i-1} \circ H_{X^i,Y^i}(s^i) + H_{X^{i+1},Y^i}(s^{i+1}) \circ d^i$$

(2). If $F$ is exact and $D(F)$ is separable, then $F$ is separable.
due to the naturality of $H_{X^i,Y}$. Thus we get a natural retraction on morphisms for $K(F)$ in homotopy categories.

Assume that $K(F)$ is separable. For any two $A$-modules $M$ and $N$, viewing as stalk complexes at degree zero, we identify $\text{Hom}_A(M,N)$ with $\text{Hom}_{K^*(A\text{-mod})}(M,N)$ and $\text{Hom}_B(F(M),F(N))$ with $\text{Hom}_{K^*(B\text{-mod})}(K(F)(M),K(F)(N))$. Hence a natural retraction
\[ \text{Hom}_{K^*(B\text{-mod})}(K(F)(M),K(F)(N)) \to \text{Hom}_{K^*(A\text{-mod})}(M,N), \]
for $K(F)$ gives a natural retraction
\[ \text{Hom}_B(F(M),F(N)) \to \text{Hom}_A(M,N) \]
for $F$.

(2). For any two modules $M$ and $N$ in $A\text{-mod}$, we have natural isomorphisms
\[ \text{Hom}_A(M,N) \cong \text{Hom}_{D^*(A\text{-mod})}(M,N) \]
and
\[ \text{Hom}_B(F(M),F(N)) \cong \text{Hom}_{D^*(B\text{-mod})}(F(M),F(N)), \]
where $M,N,F(M)=D(F)(M),F(N)=D(F)(N)$ in derived categories are viewed as stalk complexes at degree zero. Hence the statement holds.

When consider separable functors in adjoint pairs, the following lemma is used frequently; see [9, 1.2]

**Lemma 3.5.** Let $(F,G,\eta): \text{Id}_C \to GF,\epsilon: FG \to \text{Id}_D)$ be an adjoint pair between categories $C$ and $D$. Then the following statements hold.

1. The functor $F$ is separable if and only if there is a natural transformation $\delta: GF \to \text{Id}_C$ such that $\delta \circ \eta = \text{Id}$.
2. The functor $G$ is separable if and only if there is a natural transformation $\zeta: \text{Id}_D \to FG$ such that $\epsilon \circ \zeta = \text{Id}$.

Let $(F,G)$ be an adjoint pair between $A\text{-mod}$ and $B\text{-mod}$ with unit $\eta$ and counit $\epsilon$. Assume that $G$ is an exact functor. The unit and counit of $(K(F),K(G))$ are $K(\eta)$ and $K(\epsilon)$, where
\[ K(\eta)_X = (\eta_{XY})_{i \in \mathbb{Z}}, \forall X \in K^*(A\text{-mod}) \text{ and } K(\epsilon)_Y = (\epsilon_{XY})_{i \in \mathbb{Z}}, \forall Y \in K^*(B\text{-mod}). \]

It is well-known that $(\mathbb{L}F,D(G))$ is an adjoint pair; see [12, Section 10.7.1]. A natural isomorphism $\Psi$ of this adjoint pair can be given by the following commutative diagram for any complexes $X$ and $Y$.

\[ \begin{array}{ccc}
\text{Hom}_{D^*(B\text{-mod})}(\mathbb{L}F(X),Y) & \xrightarrow{\Psi} & \text{Hom}_{D^*(A\text{-mod})}(X,D(G)(Y)) \\
\downarrow f & & \downarrow q^{-1} \\
\text{Hom}_{D^*(B\text{-mod})}(K(F)(pX),Y) & \xrightarrow{q^{-1}\psi} & \text{Hom}_{K^*(A\text{-mod})}(pX,D(G)(Y)) \\
\end{array} \]

Here, since $pX$ and $K(F)(pX)$ are complexes of projectives, the localization functors $q$ are fully faithful. The isomorphism $f$ is induced by $a_X: pX \to X$, the projective resolution in $K^*(A\text{-mod})$, which is a quasi-isomorphism and natural on $X$. Finally, $\psi$ is the natural isomorphism of the adjoint pair $(K(F),K(G); K(\eta),K(\epsilon))$.

Using $\Psi$, we obtain the unit $D(\eta)$ and counit $D(\epsilon)$ of $(\mathbb{L}F,D(G))$, where
\[ D(\eta)_X = (K(\eta)_{pX})a_X^{-1}: X \to D(G)(\mathbb{L}F(X)), \forall X \in D^*(A\text{-mod}) \]
are given by fractions (see [12, Section 10.3]) and
\[
D(e)_Y = q(K(e)_Y \circ K(F)(a_{D(G)(Y)})): \mathbb{L}F(D(G)(Y)) \to Y, \forall Y \in D^{-}(B\text{-}mod).
\]
They are natural since they are constructed by functors and natural transformations.

The following lemma is well-known. We compare [11, 3.1].

**Lemma 3.6.** Let \((F, G)\) be an adjoint pair between module categories with \(G\) an exact functor. Then \(F\) is separable if and only if \(\mathbb{L}F\) is separable.

**Proof.** By Lemmas 3.4 and 3.5, we assume that \(K(F)\) is separable with \(K(\delta)\) a natural retraction of \(K(\eta)\). For each \(X \in D^{-}(A\text{-}mod), \)
\[
q(a_X \circ K(\delta)_p X) \circ (K(\eta)_p X)a_X^{-1} = (a_X \circ K(\delta)_p X) \circ K(\eta)_p Xa_X^{-1} = \text{Id}_X.
\]
Hence the unit \(D(\eta)\) of \((\mathbb{L}F, D(G))\) has a retraction \(q(a_X \circ K(\delta)_p X)\), which is natural on \(X\). Therefore, \(\mathbb{L}F\) is separable.

Conversely, if \(\mathbb{L}F\) is separable, \(D(\eta)\) has a natural retraction \(D(\delta)\). For each \(X \in A\text{-}mod, \) viewing as a stalk complex at degree zero, we have
\[
K(G)(K(F)(a_X)) \circ K(\eta)_p X = K(\eta)_p X \circ a_X \text{ and } H^0(q(K(\eta)_X)) = \eta_K,
\]
where \(H^0: D^{-}(A\text{-}mod) \to A\text{-}mod\) is the cohomology functor at degree zero. Since \(G\) and \(F\) are right exact and \(a_X\) is a quasi-isomorphism, \(H^0(qK(G)(K(F)(a_X)))\) and \(H^0(q(a_X))\) are invertible. Hence
\[
H^0(q(K(G)(K(F)(a_X))))^{-1} \circ H^0(q(K(\eta)_X)) = H^0(q(K(\eta)_p X)) \circ H^0(q(a_X))^{-1}.
\]
For each \(X \in A\text{-}mod, \)
\[
H^0(D(\delta)_X) \circ H^0(qK(G)(K(F)(a_X)))^{-1} \circ \eta_K
= H^0(D(\delta)_X) \circ H^0(qK(G)(K(F)(a_X)))^{-1} \circ H^0(q(K(\eta)_X))
= H^0(D(\delta)_X) \circ H^0(q(K(\eta)_p X)) \circ H^0(q(a_X))^{-1}
= H^0(D(\delta)_X \circ K(\eta)_p Xa_X^{-1}) = H^0(D(\delta)_X \circ D(\eta)_X)
= H^0(\text{Id}_X) = \text{Id}_X.
\]
Therefore, we obtain a retraction \(H^0(D(\delta)_X) \circ H^0(qK(G)(K(F)(a_X)))^{-1} \circ \eta_K\) which is natural on \(X\). So \(F\) is separable by Lemma 3.5. \(\square\)

Dually, if \(F\) is exact, \((D(F), \mathbb{R}G)\) is an adjoint pair between \(D^+(A\text{-}mod)\) and \(D^+(B\text{-}mod)\). We have the following result.

**Lemma 3.7.** Let \((F, G)\) be an adjoint pair between module categories with \(F\) an exact functor. Then \(G\) is separable if and only if \(\mathbb{R}G\) is separable.

**Remark 3.8.** In the above lemma, the condition that \(F\) is an exact functor is necessary; see Example 4.4.

**Proof of Theorem 3.2.** The statement (1) holds by Lemmas 3.4(1) and 3.6. The statement (2) is a consequence of Lemma 3.4(1), (2) and 3.7. \(\square\)

### 4. Derived-discreteness and split/separable extensions

We keep the notation as in Section 3. The following main result shows that derived-discreteness is compatible with split/separable extensions.
Theorem 4.1. Let \( \phi: A \rightarrow B \) be a \( k \)-algebra extension between two finite dimensional \( k \)-algebras. Then following statements hold.

1. If \( \phi \) is a split extension and \( B \) is derived-discrete over \( k \), then \( A \) is derived-discrete over \( k \).
2. If \( \phi \) is a separable extension with \( B_A \) a projective right \( A \)-module and \( A \) is derived-discrete over \( k \), then \( B \) is derived-discrete over \( k \).

Proof. For (1), if \( A \) is not derived-discrete, by Lemma 2.6 there is an \( n \in \mathbb{N}^\mathbb{Z} \) such that

\[
([P] \in K^b(A-proj) \mid c\dim_k P = n)
\]

is an infinite set. Since \( \dim_k K(F)(P) = \dim_k B \otimes_A P \) for each \( i \), the set

\[
\{c\dim_k K(F)(P) \mid \dim_k P = n\}
\]

is bounded, say by \( m \in \mathbb{N}^\mathbb{Z} \). By the derived-discreteness of \( B \) and Lemma 2.6,

\[
([K(F)(P)] \in K^b(B-proj) \mid c\dim_k K(F)(P) \leq m)
\]

is a finite set. Therefore,

\[
([K(F)(P)] \in K^b(B-proj) \mid c\dim_k P = n)
\]

is a finite set. Then

\[
([K(G)(K(F)(P))] \in K^b(A-mod) \mid c\dim_k P = n)
\]

is a finite set.

Since \( \phi \) is a split extension, \( K(F) \) is a separable functor by Theorem 3.2. By Lemma 3.5 each \( P \in K^b(A-proj) \) is a direct summand of \( K(G)(K(F)(P)) \) in \( K^b(A-mod) \). Then the first and last sets above imply that there is an object \( K(G)(K(F)(P)) \) in \( K^b(A-mod) \) with infinitely many pairwise non-isomorphic direct summands, which is impossible as \( K^b(A-mod) \) is Krull-Schmidt.

For (2), it is a consequence of Theorem 3.2(2) and the following lemma. \( \square \)

Lemma 4.2. If \( \phi: A \rightarrow B \) is a \( k \)-algebra extension with \( D(G) \) separable and \( A \) is derived-discrete over \( k \), then \( B \) is derived-discrete over \( k \).

Proof. If \( B \) is not derived-discrete, there is an \( n = (n_i)_{i \in \mathbb{Z}} \in \mathbb{N}^\mathbb{Z} \) such that

\[
([X] \in D^b(B-mod) \mid \dim_k X = n)
\]

is an infinite set. Since \( \dim_k D(G)(X) = \dim_k X \) and \( A \) is derived-discrete,

\[
([D(G)(X)] \in D^b(A-mod) \mid \dim_k X = n)
\]

is a finite set. Then

\[
([\mathbb{L}F(D(G)(X))] \in D^-(B-mod) \mid \dim_k X = n)
\]

is a finite set.

Since \( D(G) \) is separable, Lemma 3.5 implies that each \( X \in D^b(B-mod) \) is a direct summand of \( \mathbb{L}F(D(G)(X)) \) in \( D^-(B-mod) \). Then the first and last sets above imply that there is an object \( Y \in D^-(B-mod) \) with infinitely many pairwise non-isomorphic direct summands in the first set above. Let \( t \) be the least number such that \( n_t \neq 0 \). Denote by \( \tau_{\geq t} Y \in D^b(B-mod) \) the good truncation of \( Y \) at degree \( t \). Each direct summand of \( Y \) in the first set above is still a direct summand of \( \tau_{\geq t} Y \). So \( \tau_{\geq t} Y \) has infinitely many pairwise non-isomorphic direct summands. It is impossible since \( D^b(B-mod) \) is Krull-Schmidt. \( \square \)

Remark 4.3. One can consider the derived Brauer-Thrall conjecture on algebra extensions; see [6] for the field extension case. The above theorem may also hold for derived-tameness. If \( \text{gl.dim} A < \infty \), it has been proved that the derived-tameness of \( B \) implies that of \( A \); see [13, Theorem 3.1].
In (2) of the above theorem, the condition that $B$ is a projective right $A$-module is necessary.

**Example 4.4.** Let $k$ be algebraically closed, and $Q$ be a quiver as

```
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (1,1) {$2$};
\node (3) at (1,-1) {$3$};
\node (4) at (2,0) {$4$};
\node (5) at (0.5,0.5) {$a$};
\node (6) at (0.5,-0.5) {$c$};
\node (7) at (1.5,0.5) {$b$};
\node (8) at (1.5,-0.5) {$d$};
\draw (1) -- (2);
\draw (1) -- (3);
\draw (2) -- (4);
\draw (3) -- (4);
\end{tikzpicture}
```

Consider the quotient $kQ/(ba) \to kQ/(ba, dc)$. It is a separable extension. We have that $kQ/(ba)$ is derived-discrete. But $kQ/(ba, dc)$ is iterated tilted of $\tilde{A}$ type, which is not derived-discrete; see [11, 2.1 and 2.2]. In this case, $D(G)$ is not separable, otherwise $kQ/(ba, dc)$ is derived-discrete by the lemma above.

## 5. Piecewise hereditary algebras and split/separable extensions

We give an analogous statement for piecewise hereditary algebras in this section. For the field extension case, we have a refined result; see [5].

Recall that an algebra is called **piecewise hereditary** of type $H$ if it is derived equivalent to $D^b(H)$ for a hereditary abelian category $H$. When $k$ is algebraically closed, recall that one class of derived-discrete algebras is the piecewise hereditary algebras of Dynkin type. In view of Theorem 4.1, it is natural to expect that piecewise hereditary algebras is compatible with split/separable extension.

**Proposition 5.1.** Let $\phi: A \to B$ be a $k$-algebra extension between two finite dimensional $k$-algebras. The following statements hold.

1. If $\phi$ is a split extension and $B$ is piecewise hereditary, then $A$ is piecewise hereditary.
2. If $\phi$ is a separable extension with $AB$ a projective left $A$-module and $A$ is piecewise hereditary, then $B$ is piecewise hereditary.

Recall that the **strong global dimension** of a $k$-algebra $A$, denoted by $s.gl.\ dim A$, is given by

$$\sup\{l(P) \mid 0 \neq P \in K^b(A\text{-proj}) \text{ indecomposable and homotopically-minimal}\},$$

where $l(P) = \min\{b - a \mid a, b \in \mathbb{Z}, b \geq a, \text{ and } P^i = 0 \text{ for } i < a \text{ and } i > b\}$ is the length of $P \neq 0$.

We have a homological characterization of piecewise hereditary algebras saying that $A$ is piecewise hereditary if and only if $s.gl.\ dim A$ is finite; see [4, Theorem 3.2].

**Proof.** For (1), we claim that $s.gl.\ dim A \leq s.gl.\ dim B$. Indeed, for each indecomposable $P$ in $K^b(A\text{-proj})$, by Lemma 3.5, $P$ is a direct summand of $K(G)(K(F)(P))$ in $K^b(A\text{-mod})$. The length of each direct summand of $K(G)(P)$ in $K^b(B\text{-proj})$ is not larger than $s.gl.\ dim B$. As $l(K(G)(K(F)(P))) = l(K(F)(P))$ and each indecomposable direct summand will not have larger length, we have that $l(P) \leq s.gl.\ dim B$.

For (2), the condition that $B$ is a projective left $A$-module makes $G$ sending projectives to projectives. So we can prove that $s.gl.\ dim B \leq s.gl.\ dim A$ similarly as above.

In the proof of Proposition 5.1 (2), the condition that $AB$ is projective is necessary.

**Example 5.2.** Let $k$ be algebraically closed, and $Q$ be a quiver as

```
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (1,1) {$2$};
\node (3) at (1,-1) {$3$};
\node (4) at (2,0) {$4$};
\node (5) at (0.5,0.5) {$a$};
\node (6) at (0.5,-0.5) {$c$};
\node (7) at (1.5,0.5) {$b$};
\node (8) at (1.5,-0.5) {$d$};
\draw (1) -- (2);
\draw (1) -- (3);
\draw (2) -- (4);
\draw (3) -- (4);
\end{tikzpicture}
```

Consider the quotient $kQ \to kQ/(ba)$, which is a separable algebra extension. We have that $kQ$ is...
(piecewise) hereditary, but \( kQ/\langle ba \rangle \) is derived-discrete but not piecewise hereditary according to the classification of derived-discrete algebras in [11] (we recall it in the next section).

6. Applications

We give two applications of the results in Sections 4 and 5 for field extensions and skew group algebra extensions.

We recall from [11, 2.1 Theorem] the classification of derived-discrete algebras over an algebraically closed field \( k \): a connected \( k \)-algebra \( A \) is derived-discrete over \( k \) if and only if \( A \) is either piecewise hereditary of Dynkin type or \( A \) is Morita equivalent to \( kQ/I \) such that \( kQ/I \) is gentle one-cycle with the clock condition, that is, \( kQ/I \) is a gentle algebra containing exactly one cycle, and in the cycle the number of clockwise oriented relations does not equal the number of counterclockwise oriented relations.

**Proposition 6.1.** Let \( K/k \) be a finite separable field extension and \( A \) be a finite dimensional \( k \)-algebra. Then the following statements hold.

1. The algebra \( A \) is derived-discrete over \( k \) if and only if \( A \otimes_k K \) is derived-discrete over \( K \).
2. The algebra \( A \) is piecewise hereditary if and only if so is \( A \otimes_k K \).

**Proof.** By Example 3.3 (3), the extension \( A \rightarrow A \otimes_k K \) is both split and separable and \( A \otimes_k K \) is a left and right projective \( A \)-module.

1. By Theorem 4.1, \( A \) is derived-discrete over \( k \) if and only if \( A \otimes_k K \) is derived-discrete over \( k \). By Lemma 2.3, \( A \otimes_k K \) is derived-discrete over \( k \) if and only if \( A \otimes_k K \) is derived-discrete over \( K \).
2. By Proposition 5.1.

Let \( A \) be a finite dimensional algebra over an algebraically closed field \( k \). Assume that \( G \) is a finite group acting on \( A \) with its order \( |G| \) invertible in \( k \). The algebra extension from \( A \) to its skew group algebra \( AG \) is both split and separable extension with \( AG \) a both left and right projective \( A \)-module; see Example 3.3 (2).

**Proposition 6.2.** Let \( A \) be a connected algebra and \( A \rightarrow AG \) be a skew group extension as above. Then the following statements hold.

1. The algebra \( A \) is derived-discrete if and only if so is \( AG \).
2. The algebra \( A \) is piecewise hereditary of Dynkin type if and only if so is each connected component of \( AG \).
3. The algebra \( A \) is Morita equivalent to a gentle one-cycle algebra with the clock condition if and only if so is each connected component of \( AG \).

**Proof.** (1) is a consequence of Theorem 4.1.

An algebra is derived-discrete (or piecewise hereditary) if and only if so are its connected components. Notice that a gentle one-cycle algebra with the clock condition is not piecewise hereditary; see [11].

For the “only if” part of (2). By the classification of derived-discrete algebras, if \( A \) is piecewise hereditary of Dynkin type, then \( A \) is derived-discrete. Hence each connected component of \( AG \) is derived-discrete and piecewise hereditary by (1) and Proposition 5.1. Therefore, it must be piecewise hereditary of Dynkin type.

The “if” part of (2) and statement (3) can be proved in a similar argument.

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