Order Selection Prophet Inequality: From Threshold Optimization to Arrival Time Design

Bo Peng*
\*ITCS, Shanghai University of Finance and Economics
Email: ahqspb@163.sufe.edu.cn

Zhihao Gavin Tang†
†ITCS, Shanghai University of Finance and Economics
Email: tang.zhihao@mail.shufe.edu.cn

Abstract—In the classical prophet inequality, a gambler faces a sequence of items, whose values are drawn independently from known distributions. Upon the arrival of each item, its value is realized and the gambler either accepts it and the game ends, or irrevocably rejects it and continues to the next item. The goal is to maximize the value of the selected item and compete against the expected maximum value of all items. A tight competitive ratio of \( \frac{1}{2} \) is established in the classical setting and various relaxations have been proposed to surpass the barrier, including the i.i.d. model, the order selection model, and the random order model.

In this paper, we advance the study of the order selection prophet inequality, in which the gambler is given the extra power for selecting the arrival order of the items. Our main result is a \( 0.725 \)-competitive algorithm, that substantially improves the state-of-the-art \( 0.669 \) ratio by Correa, Saona and Ziliotto (Math. Program. 2021), achieved in the harder random order model. Recently, Agrawal, Sethuraman and Zhang (EC 2020) proved that the task of selecting the optimal order is NP-hard. Despite this fact, we introduce a novel algorithm design framework that translates the discrete order selection problem into a continuous arrival time design problem. From this perspective, we can focus on the arrival time design without worrying about the threshold optimization afterwards. As a side result, we achieve the optimal \( 0.745 \) competitive ratio by applying our algorithm to the i.i.d. model.

Index Terms—Online algorithms; Prophet inequality

I. INTRODUCTION

Prophet inequality has been a cornerstone of optimal stopping theory, since the classical result of Krengel and Sucheston [2], [3]. Consider a gambler facing a sequence of items, whose values are drawn independently from known distributions. After seeing an item, the gambler observes its realized value, and either accepts it and the game ends, or irrevocably rejects it and continues to the next item. The classical prophet inequality states that the gambler can achieve at least half of the expected maximum value. The latter is referred to as a prophet, who knows the realization of all values beforehand. Furthermore, the ratio of half is proved to be the best possible in the worst case. Later, Samuel-Cahn [4] showed that the competitive ratio\(^1\) of \( \frac{1}{2} \) can be achieved using a single-threshold algorithm.

In the past fifteen years, there has been an increased interest of prophet inequality related problems in the algorithmic game theory and online algorithms literature, due to its close connection to mechanism design and posted pricing mechanisms [5]. Among the fruitful extensions of the classical prophet inequality, a remarkable line of research focuses on surpassing the \( \frac{1}{2} \) impossibility result by relaxing the worst case model. Consider the following three variants in progressive order of difficulty.

a) I.I.D. Model: Hill and Kertz [6] studied the case when the value distributions are identical and designed a \( 1 - \frac{1}{e} \approx 0.632 \)-competitive algorithm. They also constructed a family of instance showing that no algorithm can be better than 0.745-competitive\(^2\). The \( 1 - \frac{1}{e} \) ratio is improved to 0.738 by Abolhassan et al. [7]. Recently, Correa et al. [8] designed an optimal 0.745-competitive algorithm, matching the hardness of Hill and Kertz.

b) Order Selection Model: In this variant, the gambler is given an extra power for selecting the arrival order of each item. This assumption is natural in the application of sequential posted pricing mechanisms [9], as the mechanism designer plays the role of the gambler. Chawla et al. [9] proposed an \( 1 - \frac{1}{e} \)-competitive algorithm and the ratio is later improved to \( 1 - \frac{1}{e} + 0.022 \approx 0.654 \) by Beyhaghi et al. [10]. This variant subsumes the i.i.d. model as a special case. Indeed, when the value distributions are identical, the extra power of order selection is useless. Recently, Agrawal, Sethuraman and Zhang [11] established a negative result, showing that the task of selecting the optimal order is NP-hard, even when the support of each distribution is of size 3. They also provide a 0.8-competitive algorithm when the support of each distribution is of size at most 2.

c) Random Order Model: This variant is also known as prophet secretary, in which items arrive in a random order. This model can be viewed as a generalization of the i.i.d. model and is no easier than the order selection model. Esfandiari et

\(^1\)We choose to use the terminology competitive ratio, due to the online nature of prophet inequality.

\(^2\)The constant \( \Gamma = 0.745 \) is the unique solution to \( \int_0^1 \frac{1}{y(y-1)+1} dy = 1. \)
al. [12] initiated the study of this variant and designed a $1 - \frac{1}{e}$-competitive algorithm. Later, the same $1 - \frac{1}{e}$ ratio is achieved using different strategies, including using personalized but time-invariant thresholds [8], and a single-threshold algorithm with randomized tie-breaking [13]. Later, the ratio is improved to $1 - \frac{1}{2} + \frac{1}{2e}$ by Azar, Chiplunkar, and Kaplan [14] and to 0.669 by Correa, Saona, and Ziliotto [15]. The latter work also establishes a hardness of $\sqrt{3} - 1 \approx 0.732$, showing a separation between the random order model and the i.i.d. model.

A. Our Contributions

In this work, we focus on the order selection model. Despite the NP-hardness of selecting the optimal order, we strongly exploit the power of order selection and design a 0.725-competitive algorithm, that substantially improve the state-of-the-art 0.669 ratio from the random order model. As a side result, our algorithm is 0.745-competitive for the i.i.d. prophet inequality.

Previous Approaches. We briefly summarize the previous techniques. Naturally, an algorithm is consisted of two parts: selecting the order and setting the thresholds. Each step is easy to optimize on its own. Specifically, when the arrival order is fixed, the optimal thresholds can be calculated through backward induction; when the thresholds are fixed for each item, we can calculate the expected value of each item conditioning on that its value exceeds the threshold, and then set the arrival order to be a descending order of the calculated values.

Chawla et al. [9] and Beyhaghi et al. [10] applied a two-step approach of first designing the thresholds, and then selecting the order. Both works studied the order selection prophet inequality from the perspective of sequential posted pricing mechanisms. It is implicitly shown by Chawla et al. [9] that the latter setting reduces to the first setting.

The line of work studying prophet secretary [12], [14], [15] can be viewed as a two-step approach of first selecting the order, and then designing the thresholds. More accurately, the algorithm selects the uniform distribution over all permutations and then focuses on designing the thresholds. Remarkably, prior to our work, the state-of-the-art 0.669 ratio for the order selection prophet inequality is established in the random order setting by Correa, Saona, and Ziliotto [15].

Our Perspective: Arrival Time Design. Recall a folklore continuous formulation of the prophet secretary problem. Let the time horizon be $[0, 1]$ and assume that each item $i$ arrives at time $t_i \sim \text{Unif}(0, 1)$ (i.e., the uniform distribution over $[0, 1]$). This formulation is equivalent to the random arrival order and often eases the analysis. Specifically, under this formulation, Correa, Saona and Ziliotto [15] carefully set time-dependent thresholds and accept the first item whose value exceeds the threshold on its arrival time.

We provide a novel point of view by re-scaling the time horizon. We first fix the time-dependent thresholds. Specifically, at time $t$, we set the threshold to be the value $\tau(t)$ so that the maximum value of all items is larger than it with probability exactly $t$. Then, we design an arrival time distribution $F_i$ for each item $i$ and let the items arrive at a random time with respect to $F_i$. In principle, this formulation is without loss of generality, since we can choose the distributions to be deterministic. Under this formulation, we only need to optimize for the arrival times. Moreover, the continuous formulation allows us to adapt the analysis framework from the i.i.d. setting [8] and the random order setting [15]. Noticeably, if the distributions $F_i$ are identical, our algorithm can be implemented in the prophet secretary setting. See Section II for a more detailed discussion.

Our Results. We explicitly construct arrival time distributions and achieve a competitive ratio of $\Gamma = \frac{\ln(\alpha+1)}{\ln\alpha} \approx 0.725$, where $\alpha \approx 0.211$ is the unique solution
d to $\int_{\frac{1}{\ln\alpha}}^{\frac{1}{\ln(\alpha+1)}} \frac{\ln x + x}{\ln x + 1} dx + \frac{1}{\ln\alpha} = 0$.

Furthermore, our algorithm serves as an alternative optimal $\Gamma \approx 0.745$-competitive algorithm for the i.i.d. setting, with only one parameter modified, compared to our algorithm in the order selection setting. Our unified analysis bridges the i.i.d. setting and the order selection setting, and suggests that our novel arrival time design perspective to be the right framework.

B. Related Work

There is a vast literature on prophet inequalities. We refer interested readers to the survey of Hill and Kertz [16] for the classical results, the suvery of Lucier [17] for the economic perspective of prophet inequalities, and the survey of Correa et al. [18] for more recent developments. Below, we review the most related works.

Hajiaghayi et al. [5], and Chawla et al. [9] observed a close relation between prophet inequalities and sequential posted pricing. They showed that designing posted pricing mechanisms can be reduced to the prophet inequality problem. Recently, Correa et al. [19] proved that the two settings are indeed equivalent.

Besides the results that we have discussed before, there are a few special cases in which better competitive ratios are known to the order selection prophet inequality problem. When the number of items is a small constant, Beyhaghi et al. [10] obtained a better competitive ratio than their general bound of 0.654. If each type of distribution occurs at least

\[\text{for completeness, we provide a proof of the uniqueness of } \alpha \text{ in the full version of our paper.}\]
\(\Omega(\log n)\) times, Abolhassani et al. [7] improved the competitive ratio to 0.738 for the order selection model. Liu et al. [20] relaxed the problem by allowing the algorithm to remove a constant number of items. After so, they showed that the competitive ratio can be arbitrary close to 0.745 against the relaxed prophet.

Closely related to the order selection prophet inequality is the optimal ordering problem. This problem shares the same input model as the order selection prophet inequality, while the benchmark is changed to the optimal online algorithm instead of the expected maximum value. Agrawal, Sethuraman and Zhang [11] proved that the problem is NP-hard, and designed a FPTAS when the support of each distribution is of size 3. Fu et al. [21] gave a PTAS when each distribution has a constant support size. Chakraborty et al. [22] obtained a PTAS without any assumption on the support of the distribution. Their original results were stated in the setting of sequential posted pricing mechanisms, that can be translated to the optimal ordering problem by the reduction of Correa et al. [19]. Liu et al. [20] improved the results to an EPTAS based on a novel decomposition technique.

II. PRELIMINARIES

Let there be \(n\) items, whose values \(\tilde{V} = (v_1, v_2, \ldots, v_n)\) are drawn independently from known distributions \(D = D_1 \times D_2 \times \cdots \times D_n\). The algorithm first selects an arrival order of the items. Then, the items arrive in a sequence according to the selected order. Upon the arrival of an item, its value is realized and the algorithm either accepts the item and stops, or rejects the item and continues to the next. Our goal is to maximize the expected value of the selected item and compare against the prophet

\[
\text{OPT} \overset{\text{def}}{=} E \left[ \max_i v_i \right].
\]

Absolute Continuity. For the ease of presentation, we assume the cumulative distribution functions (CDF) of value distributions \(D_i\)’s are absolutely continuous. For general random variables whose cumulative distribution functions are not necessarily absolutely continuous, consider perturbing each random variable by adding a noise of uniform distribution supported on \([0, \epsilon]\). Then it is straightforward to verify the following.

- The CDF of each perturbed variable, i.e., the summation of an arbitrary random variable and a uniform random variable, must be absolutely continuous.
- We apply our algorithm for the perturbed instance. The expected reward of our algorithm on the original instance is at most \(\epsilon\) smaller than the expected reward of our algorithm on the perturbed instance. Consequently, the competitive ratio decrease by at most \(\epsilon\). Here, we assume the value of the prophet equals to 1 and notice that \(\epsilon\) can be made arbitrarily small.

Our algorithm is parameterized by \(n\) distributions \(F_i\) for each \(i \in [n]\), supported on \([0, 1]\). Consider the following algorithm:

**Independent Arrival Time (\((F_i)\)).**

- Sample independently \(t_i \sim F_i\) for each \(i\). We refer to \(t_i\) as the arrival time of item \(i\).
- Let the items arrive in ascending order according to their arrival times.
- We accept the first item \(i\) with \(v_i > \tau(t_i)\), where \(\tau(t)\) is the threshold that

\[
\Pr \left[ \max_i v_i > \tau(t) \right] = t
\]

**Remark.** Before we go to the detailed analysis of our algorithm, it is worthwhile to make a comparison with the algorithm by Correa, Saona, and Ziliotto [15] for the prophet secretary problem. In the prophet secretary problem (and other online optimization problems with random arrival), a folklore formulation is to assume that each item \(i\) arrives at time \(t_i \sim \text{Unif}[0, 1]\) (i.e., the uniform distribution over \([0, 1]\)). Correa, Saona, and Ziliotto first set time-dependent thresholds \(\tau(\alpha(t))\) at time \(t\), with an appropriate function \(\alpha\), and then accept the first item whose value exceeds the threshold.

Alternatively, we re-scale the time horizon by fixing the threshold to be \(\tau(t)\) at time \(t\), and then let the items arrive according to carefully chosen distributions. Indeed, if all the distributions \(F_i\)’s are identical, our algorithm can be implemented in the prophet secretary setting. Specifically, for any function \(\alpha\), by setting \(F_i(\alpha^{-1}(t)) = t\) for every item \(i\), our algorithm is equivalent to the algorithm of Correa, Saona, and Ziliotto [15]. On the other hand, our formulation admits a natural generalization to the order selection setting by allowing non-identical \(F_i\)’s.

**Analysis.** Our analysis is similar to the framework of [15]. We abuse \(\text{ALG}\) to denote our algorithm and to denote the (random) value of the accepted item of our algorithm. We show the competitive ratio of our algorithm through the following stronger statement.

**Theorem II.1.** For the order selection prophet inequality, there exists distributions \(\{F_i\}_{i \in [n]}\), so that for every \(t \in [0, 1]::

\[
\Pr[\text{ALG} > \tau(t)] \geq \Gamma \cdot t = \Gamma \cdot \Pr[\max_i v_i > \tau(t)],
\]

where \(\Gamma = \frac{\log(\tau^{-1}(t))}{\log(\tau^{-1}(0))} \approx 0.725\) and \(\alpha \approx 0.211\) is the unique solution to \(\int_0^1 \frac{\log(1+1/z)}{\log(1+1/\alpha)} dz + \frac{\log(1+1/\alpha)}{\log(1+1/\alpha)} = 0\).

Observe that for any non-negative random variable \(V\), we have \(E[V] = \int_0^\infty \Pr[V > \tau] d\tau\). The above theorem immediately concludes the competitive ratio of our algorithm.

**Corollary 1.** The independent arrival time algorithm with functions \(\{F_i\}\) chosen in Theorem II.1 is \(\Gamma \approx 0.725\)-competitive for the order selection prophet inequality. I.e., \(E[\text{ALG}] \geq \Gamma \cdot \text{OPT}\).

As a side result, for the i.i.d. prophet inequality, i.e., when the distributions \(D_1, D_2, \ldots, D_n\) are identical, our construction in Theorem II.1 works with a different parameter \(\Gamma \approx 0.745\).
Thus, we give an alternative optimal competitive algorithm for the i.i.d. prophet inequality. Formally, we prove the following theorem and corollary.

**Theorem II.2.** For the i.i.d. prophet inequality, there exists distributions $\{F_i\}_{i \in [n]}$, so that for every $t \in [0, 1]$:

$$\Pr[\text{ALG} > \tau(t)] \geq \Gamma \cdot t = \Gamma \cdot \Pr\left[ \max_i v_i > \tau(t) \right],$$

where $\Gamma \approx 0.745$ is the unique solution to

$$\int_0^1 \frac{1}{y(1-\ln y)+t-1} \, dy = 1.$$

**Corollary 2.** The independent arrival time algorithm with functions $\{F_i\}$ chosen in Theorem II.2 is $\Gamma \approx 0.745$-competitive for the i.i.d. prophet inequality. I.e., $E[\text{ALG}] \geq \Gamma \cdot OPT$.

### III. Analysis

In this section, we prove Theorem II.1 and II.2. We provide the construction of the distributions $\{F_i\}$ and prove the stated inequalities in Section III-A. Without specifying the constant $\Gamma$ and assuming that our distributions $\{F_i\}$ are well-defined, our constructions and analysis are unified for the non-i.i.d. case and the i.i.d. case. In Section III-B, we find the largest possible constants $\Gamma$ for our algorithm to be well-defined for the non-i.i.d. case and the i.i.d. case, respectively.

#### A. Construction of $\{F_i\}$

We start with analyzing $\Pr[\text{ALG} > \tau(t)]$ for general arrival time distributions $\{F_i\}_{i \in [n]}$. We use $\{f_i\}_{i \in [n]}$ to denote the probability density functions. We introduce some notations. For every $t \in [0, 1]$ and $i \in [n]$, let

$$p_i(t) \overset{\text{def}}{=} \Pr[v_i > \tau(t)] \quad \text{and} \quad q_i(t) \overset{\text{def}}{=} \Pr\left[ \max_{j \neq i} v_j > \tau(t) \right].$$

With the assumption that the value distributions $\{D_i\}$ are absolutely continuous, we have that $p_i(t)$, $q_i(t)$ are non-decreasing continuous functions and differentiable almost everywhere. We use $p'_i(t)$, $q'_i(t)$ to denote the derivatives. We have the following simple observation according to the definition of $p_i(t), q_i(t)$:

$$\prod_i (1 - p_i(t)) = 1 - t, \quad \forall t \in [0, 1] \quad (1)$$

$$\prod_{j \neq i} (1 - p_j(t)) = 1 - q_i(t), \quad \forall t \in [0, 1], \forall i \in [n] \quad (2)$$

By taking derivatives on both sides of (1), we have

$$\sum_i p'_i(t) \cdot (1 - q_i(t)) = \sum_i p'_i(t) \cdot \prod_{j \neq i} (1 - p_j(t)) = 1. \quad (3)$$

Consider equation (1) when $t = 1$, we have $\prod_i (1 - p_i(1)) = 0$. Hence, there exists at least one index $i$ with $p_i(1) = 1$. Without loss of generality, let it be the index 1. Consequently, $q_1(1) = 1 - \prod_{j \neq 2} (1 - q_j(1)) = 1$ for all $i \neq 1$.

Fixing an arbitrary time $t$, the event that our algorithm accepts an item with value larger than $\tau(t)$ can be partitioned into the following $n + 1$ possibilities:

- Our algorithm stops before time $t$. In this case, the accepted value must be larger than $\tau(t)$, since the threshold function $\tau$ is decreasing.
- For some $i \in [n]$, our algorithms accepts item $i$ at time $t_i \geq t$ and $v_i > \tau(t)$.

We introduce notations $A_i(t), B_i(t)$ for each $i \in [n]$ to denote the following events:

- $A_i(t)$: item $i$ arrives at time $t_i < t$ and $v_i > \tau(t_i)$.
- $B_i(t)$: item $i$ is accepted by our algorithm at time $t_i \geq t$ and $v_i > \tau(t_i)$.

Observe that our algorithm stops before time $t$ if and only if at least one of the events $\{A_i(t)\}_{i \in [n]}$ happens. Moreover, the events $A_i(t)$ are independent from each other for different $i$’s. Consequently,

$$\Pr[\text{ALG} \text{ stops before time } t] = \Pr[\cup_i A_i(t)] = 1 - \prod_{i} (1 - \Pr[A_i(t)]) = 1 - \prod_{i} \left( 1 - \int_0^t p_i(t_i) \cdot f_i(t_i) \, dt_i \right). \quad (4)$$

Next, we study the events $\{B_i(t)\}$. For any $i$, fixing the arrival time $t_i \in (t, 1)$ of $i$ and conditioning on that its realized value $v_i$ is larger than $\tau(t)$, our algorithm accepts it as long as we haven’t stopped before time $t_i$. Specifically, the last event happens when none of the $\{A_j(t)\}_{j \neq i}$ happens. Thus,

$$\Pr[B_i(t)] \geq \int_t^1 f_i(t_i) \cdot \Pr[v_i > \tau(t)] \cdot \Pr[i \text{ is accepted by ALG } | t_i, v_i > \tau(t)] \, dt_i \overset{\text{def}}{=} \int_t^1 f_i(t_i) \cdot \prod_{j \neq i} (1 - \Pr[A_j(t_i)]) \, dt_i = \int_t^1 f_i(t_i) \cdot \prod_{j \neq i} \left( 1 - \int_0^{t_i} p_j(t_j) \cdot f_j(t_j) \, dt_j \right) \, dt_i. \quad (5)$$

Putting the two bounds together, we have

$$\Pr[\text{ALG} \geq \tau(t)] = \Pr[\cup_i A_i(t)] + \sum_i \Pr[B_i(t)] \geq 1 - \prod_{i} \left( 1 - \int_0^t p_i(t_i) \cdot f_i(t_i) \, dt_i \right) + \sum_i \left( p_i(t) \cdot \int_t^1 f_i(t_i) \cdot \prod_{j \neq i} \left( 1 - \int_0^{t_i} p_j(t_j) \cdot f_j(t_j) \, dt_j \right) \, dt_i \right).$$

**Informal Argument.** We aim at designing functions $\{f_i\}$ so that the right hand side of the above equation equals $\Gamma \cdot t$ for
every $t \in [0,1]$. Denote the right hand side by $H(t)$. Notice that,
\[
H'(t) = \sum_i p_i(t) f_i(t) \cdot \prod_{j \neq i} \left(1 - \int_0^t p_j(t_j) f_j(t_j) dt_j \right) \\
= \sum_i p_i(t) f_i(t) \cdot \prod_{j \neq i} \left(1 - \int_0^t p_j(t_j) f_j(t_j) dt_j \right) \\
+ \sum_i \left( p'_i(t) \cdot \int_0^t \prod_{j \neq i} \left(1 - \int_0^{t'} p_j(t_j) f_j(t_j) dt_j \right) f_i(t_i) dt_i \right) \\
= \sum_i \left( p'_i(t) \cdot \int_0^t \prod_{j \neq i} \left(1 - \int_0^{t'} p_j(t_j) f_j(t_j) dt_j \right) f_i(t_i) dt_i \right).
\]

Recall equation (3), $\sum_i p'_i(t_j) (1-q_i(t)) = 1$ for all $t$. If we set $\{f_i\}$ to satisfy the following equations, we shall automatically have that $H'(t) = \Gamma$ for all $t \in [0,1]$.
\[
\int_0^t \prod_{j \neq i} \left(1 - \int_0^{t'} p_j(t_j) f_j(t_j) dt_j \right) f_i(t_i) dt_i = \Gamma \cdot (1-q_i(t)) \quad (6)
\]
\[
\prod_{j \neq i} \left(1 - \int_0^{t'} p_j(t_j) f_j(t_j) dt_j \right) f_i(t_i) = -\Gamma \cdot q'_i(t) \quad (7)
\]
A caveat is that (6) might be infeasible for $i = 1$, since when $t = 1$, the left hand side equals 0 while $1-q_1(1)$ is not necessarily zero. Nevertheless, we solve the set of differential equations (7) and shall fix the problem by applying a stronger lower bound than (5) for $i = 1$.

**Construction of $\{F_i\}$**.
- Let $g(t) \overset{\text{def}}{=} \Gamma \cdot (\sum_i (1-q_i(t)) \cdot p_i(t) - t) + 1$ be an auxiliary function.
- For every item $i$, let $f_i(t) \overset{\text{def}}{=} \frac{\Gamma \cdot q'_i(t)}{g(t)}$ be the probability density function for its arrival time $t_i \in [0,1)$ and let $t_i = 1$ with probability $1 - \int_0^1 f_i(t) dt$.

**Remark.** Our construction is unified for the non-i.i.d. case and the i.i.d. case, except for a different choice of the constant $\Gamma$. We need to be careful when multiple items arrive at time $t = 1$, since the distributions $F_i$’s might have point masses on $t_i = 1$ according to our construction. We resolve this issue by doing a special treatment for item 1: 1) if $t_i = 1$ for $i \neq 1$, we reject item $i$ without looking at its realized value; 2) if $t_i = 1$, we accept it without looking at its realized value. Recall that $p_1(1) = \Pr[y_1 > \tau(1)] = 1$, there is no difference between always accepting item 1 and setting a threshold of $\tau(1)$ to item 1.

For our algorithm to be well-defined, we need to verify that the distributions are valid, i.e. $\int_0^1 f_i(t) dt \leq 1$ for all $i \in [n]$. This is the crucial place where we have different constants $\Gamma$ for the non-i.i.d. case and i.i.d. case respectively.

**Lemma III.1.** For the i.i.d. case, for $\Gamma \approx 0.745$ (the unique solution to $\int_0^1 \frac{1}{\sqrt{1 - \ln \alpha - \frac{\ln \alpha + 1}{\ln \alpha + 1}} \cdot \int_0^1 (1 - q_i(t)) dt = 1$, and for each $i \in [n]$, we have
\[
\int_0^1 f_i(t) dt \leq 1.
\]
For the non-i.i.d. case, we prove the following stronger statement that automatically implies the validity of our algorithm.

**Lemma III.2.** For the non-i.i.d. case, for $\Gamma = \frac{\ln n + 1}{\ln n + 1} \approx 0.725$ where $\alpha \approx 0.211$ is the unique solution to $\int_0^1 \frac{1}{\ln \alpha + 1 - \frac{\ln \alpha + 1}{\ln \alpha + 1}} \cdot \int_0^1 (1 - q_i(t)) dt = 1$, and for each $i \in [n]$, we have
\[
(1 - \Gamma \cdot q_i(t)) \left(1 - \int_0^1 f_i(t) dt \right) \exp \left(\Gamma \cdot \int_0^1 \frac{q'_i(s) \cdot p_i(s)}{g(s)} ds \right) \geq \Gamma \cdot (1 - q_i(t)).
\]
We defer the proofs of the above lemmas to the next subsection and continue proving the stated inequality of Theorem II.1 and II.2, assuming the validity of our algorithm.

We first prove two useful mathematical properties of the functions $\{f_i(t)\}$ and $g(t)$.

**Lemma III.3.** The functions $f_i(t), g(t)$ satisfy
1) $1 - \int_0^1 p_i(s) f_i(s) ds = \exp \left(-\Gamma \cdot \int_0^1 \frac{q_i(s) \cdot p_i(s)}{g(s)} ds \right), \forall t \in [0,1], \forall i \in [n];$
2) $g(t) = \prod_i \left(1 - \int_0^1 f_i(s) p_i(s) ds \right), \forall t \in [0,1].$

**Proof.** We verify the first equation by plugging in the definition of $f_i$ to the left hand side:
\[
\int_0^t f_i(s) ds = 1 - \int_0^t p_i(s) \cdot \frac{q'_i(s)}{g(s)} \cdot \exp \left(-\Gamma \cdot \int_0^s \frac{q'_i(s) \cdot p_i(s)}{g(s)} ds \right) ds \\
= 1 + \int_0^t \frac{g(s)}{g(x)} \cdot q'_i(s) \cdot p_i(s) \cdot \exp \left(-\Gamma \cdot \int_0^x \frac{q'_i(x) \cdot p_i(x)}{g(x)} dx \right) ds \\
= \exp \left(-\Gamma \cdot \int_0^t \frac{q'_i(s) \cdot p_i(s) \cdot g(x)}{g(s)} ds \right). \\
\]
Next, we prove the second statement. We first calculate the derivative of function $g(t)$:
\[
g'(t) = \Gamma \cdot \left(\sum_i (1 - q_i(t)) \cdot p_i(t) - t\right) \\
= -\Gamma \cdot \sum_i q'_i(t) \cdot p_i(t) + \Gamma \cdot \sum_i (1 - q_i(t)) \cdot p'_i(t) - \Gamma \\
\overset{(\text{def})}{=} -\Gamma \cdot \sum_i q'_i(t) \cdot p_i(t). \\
\]
Then, by applying the first stated equation, we have:

\[
\prod_i \left(1 - \int_0^t p_i(s)f_i(s)\,ds\right)
= \prod_i \exp\left(-\Gamma \cdot \int_0^t q_i(s) \cdot p_i(s)\,ds\right)
= \exp\left(\int_0^t \frac{\Gamma}{g(s)} \,ds\right) \cdot \exp\left(\int_0^t g(s)\,ds\right)
= \exp\left(\ln(g(s))\bigg|_{s=0}^1 = \frac{g(t)}{g(0)} = g(t)\right).
\]

\(\square\)

Next, we formalize the lower bound of \(\Pr[\text{ALG} > \tau(t)]\).

**Lemma III.4.** For any \(t \in [0, 1]\),

\(\Pr[\text{ALG stops before time } t] = 1 - g(t)\).

**Proof.** By equation (4) and Lemma III.3, we have

\[
\Pr[\text{ALG stops before time } t] = 1 - \prod_i \left(1 - \int_0^t p_i(t_i) \cdot f_i(t_i)\,dt_i\right)
= 1 - g(t),
\]

where the last equation follows from the second statement of Lemma III.3. \(\square\)

**Lemma III.5.** For any \(t \in [0, 1]\) and \(i \in [n]\), \(\Pr[B_i(t)] \geq \Gamma \cdot p_i(t) \cdot (1 - q_i(t))\).

**Proof.** By equation (5) and Lemma III.3, we derive

\[
\Pr[B_i(t)] \\
\geq p_i(t) \cdot \int_t^1 f_i(t) \cdot \prod_{j \neq i} \left(1 - \int_0^t p_j(t_j) \cdot f_j(t_j)\,dt_j\right)\,dt_i
= p_i(t) \cdot \int_t^1 f_i(t) \cdot \frac{\prod_{j \neq i} \left(1 - \int_0^t p_j(t_j) \cdot f_j(t_j)\,dt_j\right)}{1 - \int_0^t p_i(s)\,f_i(s)\,ds}\,dt_i
= p_i(t) \cdot \int_t^1 f_i(t) \cdot \frac{g_i(t)}{g(t)} \exp\left(-\Gamma \int_0^t \frac{q_i(s)\cdot p_i(s)}{g(s)}\,ds\right)\,dt_i
= p_i(t) \cdot \int_t^1 \Gamma \cdot q_i(t)\,dt_i
= \Gamma \cdot p_i(t) \cdot (1 - q_i(t)).
\]

For \(i \neq 1\), we conclude the proof of the statement by noticing that \(q_1(1) = 1\). However, \(q_1(1)\) not necessarily equals 1. We remark that for the i.i.d. case, all distributions are symmetric and the above analysis is sufficient since \(q_i(1) = 1\) for all \(i \in [n]\). The rest of our proof is only for the non-i.i.d. case.

Note that the previous analysis ignores the point mass of \(F_1\) on \(t_1 = 1\) and recall that we have a special treatment of item 1 when it arrives at time 1. It suffices to calculate the extra probability when item 1 is accepted at time 1 and \(v_1 > \tau(t)\).

\[
\Pr[\text{ALG accepts item 1 at time 1 and } v_1 > \tau(t)]
= \Pr[t_1 = 1] \cdot \Pr[v_1 > \tau(t)] \cdot \prod_{j \neq 1} \left(1 - \Pr[A_j(1)]\right)
= \left(1 - \int_0^1 f_1(t)\,dt\right) \cdot p_1(t) \cdot \left(1 - \int_0^1 \frac{q_1(s)\cdot p_1(s)}{g(s)}\,ds\right)
\]

\[
\geq (1 - \Gamma \cdot q_1(1)) \cdot p_1(t) \cdot \left(1 - \int_0^1 f_1(t)\,dt\right)
\]

\[
\geq \Gamma \cdot p_1(t) \cdot (1 - q_1(1)).
\]

Here, the third equality follows from the fact that \(g(1) = \Gamma \cdot (\sum_j (1 - q_j(1)) \cdot p_j(1) - 1) = 1 - \Gamma \cdot q_1(1)\); the last inequality follows from Lemma III.2. This concludes the proof of the lemma. \(\square\)

Equipped with the above lemmas, we conclude the proof of Theorem II.1 and II.2.

\[
\Pr[\text{ALG} > \tau(t)]
= \Pr[\text{ALG stops before time } t] + \sum_i \Pr[B_i(t)]
\geq 1 - g(t) + \sum_i \Gamma \cdot p_i(t) \cdot (1 - q_i(t))
\]

(by Lemma III.4 and III.5)

\[
= 1 - \left(\sum_i (1 - q_i(t)) \cdot p_i(t) - t\right) + 1
\]

\[
= \Gamma \cdot t .
\]

**B. Maximization of \(\Gamma\)**

Finally, we prove Lemma III.1 and III.2. Recall the definition of \(f_i(t)\). We have that

\[
\int_0^1 f_i(t)\,dt = \int_0^1 \frac{\Gamma \cdot q_i(t)}{g(t)} \exp\left(\int_0^t \frac{q_i(s)\cdot p_i(s)}{g(s)}\,ds\right)\,dt .
\]

Since \(q_i(t)\) is continuous and non-decreasing, we do the following change of variables: for \(x \in [0, q_i(1)]\),

- \(q_i^{-1}(x) \equiv \sup \{t \mid q_i(t) \leq x\}\);\n- \(\tilde{p}(x) \equiv p_1(q_i^{-1}(x))\) and \(\tilde{g}(x) \equiv g(q_i^{-1}(x))\).

Then,

\[
\int_0^1 f_i(t)\,dt = \int_0^{q_i(1)} \frac{\Gamma \cdot q_i(t)}{g(t)} \exp\left(\int_0^t \frac{p_1(s)}{g(s)}\,ds\right)\,dx
= \int_0^{q_i(1)} \frac{\Gamma \cdot \tilde{p}(x)}{\tilde{g}(x)} \exp\left(\int_0^x \frac{\tilde{p}(y)}{\tilde{g}(y)}\,dy\right)\,dx .
\]
Remark. If \( q_i(t) \) is strictly monotonically increasing, our definition of \( q_i^{-1}(x) \) is the standard inverse function of \( q_i(t) \). The above change of variables works for arbitrary absolute continuous non-decreasing function \( q_i(t) \). Indeed, for any Lebesgue measurable function \( h \geq 0 \) and \( a \leq b, \)
\[
\int_a^b h(t)q_i(t)dt = \int_a^b h(q_i^{-1}(q_i(t)))q_i(t)dt = \int_{q_i(a)}^{q_i(b)} h(q_i^{-1}(x))dx,
\]
where the first equation follows from the fact that the Lebesgue measure of \( \{ t \mid h(q_i^{-1}(q_i(t)))q_i(t) \neq h(t)q_i(t) \} \) equals 0.

1) I.I.D.: Proof of Lemma III.1: We start with the case of i.i.d. distributions. Within this subsection, \( \Gamma \approx 0.745 \) is the unique solution to \( \int_0^1 \frac{1}{y(1-ln y)\Gamma} dy = 1 \). By symmetry, all functions \( p_i(t) \) are the same. Since \( \prod_i (1 - p_i(t)) = 1 - t \), we have that for all \( i \)
\[
p_i(t) = 1 - (1 - t)^{\frac{1}{n}}
\]
and
\[
q_i(t) = 1 - \prod_{j \neq i} (1 - p_j(t)) = 1 - (1 - t)^{\frac{n-1}{n}}.
\]
Consequently, we have
\[
\tilde{p}_i(x) = 1 - (1 - q_i^{-1}(x))^{\frac{1}{n}} = 1 - (1 - x)^{\frac{n}{n-1}}
\]
and
\[
\tilde{g}_i(x) = \Gamma \cdot \left( \sum_j (1 - q_j(q_i^{-1}(x))) \cdot p_j(q_i^{-1}(x)) - q_i^{-1}(x) \right) + 1
\]
\[
= \Gamma \cdot \left( \sum_j (1 - x) \cdot \tilde{p}_j(x) - q_i^{-1}(x) \right) + 1
\]
\[
(\text{since } p_i, q_i \text{ are the same for all } i)
\]
\[
= \Gamma \cdot \left( n \cdot (1 - x - (1 - x)^{\frac{n}{n-1}}) - 1 + (1 - x)^{\frac{n}{n-1}} \right) + 1.
\]
We have the following mathematical fact, whose proof involves tedious calculations and is given in the full version of this paper.

Claim III.1. For any \( x \in [0,1] \), we have
\[
\tilde{g}_i(x) \cdot \exp \left( \Gamma \cdot \int_0^x \frac{\tilde{p}_j(y)}{\tilde{g}_i(y)} dy \right) \geq \Gamma \cdot (-1 - x) \ln(1 - x) + 1.
\]
Applying the above claim and recalling that \( q_i(1) = 1 \), we have that
\[
\int_0^1 f_i(t)dt = \int_0^1 \Gamma \cdot \frac{\tilde{g}_i(x) \cdot \exp \left( \Gamma \cdot \int_0^x \frac{\tilde{p}_j(y)}{\tilde{g}_i(y)} dy \right)}{\tilde{g}_i(x) \cdot \exp \left( \Gamma \cdot \int_0^x \frac{\tilde{p}_j(y)}{\tilde{g}_i(y)} dy \right)} dx
\]
\[
\leq \int_0^1 \Gamma \cdot \left( \frac{\Gamma \cdot (-1 - x) \ln(1 - x) + 1}{\Gamma} \right) dx
\]
\[
= \int_0^1 \Gamma \cdot \left( -y \cdot \ln y - (1 - y) \right) dy + 1
\]
\[
= \int_0^1 \frac{1}{y(1 - \ln y) + \frac{1}{\Gamma} - 1} dy = 1,
\]
where the last equality follows from the definition of \( \Gamma \).

2) Non-I.I.D.: Proof of Lemma III.2: Finally, we derive the constant \( \Gamma \) for the non-i.i.d. case. In contrast to the analysis for the i.i.d. case, we no longer have explicit expressions for functions \( p_i(t), q_i(t) \). The challenge is to prove that for all possible \( p_i(t), q_i(t) \), the stated inequality holds. Within this subsection, \( \Gamma = \frac{\ln a + 1}{\ln a + 1 - a} \approx 0.725 \) and \( a \approx 0.211 \) is the unique solution of the following equation on \( (0, 1) \)
\[
\int_a^1 \frac{\ln x + 1}{\ln x} (1 - x) dx - a \cdot \int_a^1 \frac{\ln x + 1}{\ln x} (1 - x) dx = 0.
\]
We first observe the following property regarding functions \( \tilde{p}_i(x) \) and \( \tilde{g}_i(x) \).

Claim III.2. For each \( x \in [0, q_i(1)] \), we have
\[
\tilde{g}_i(x) \geq \Gamma \cdot (-1 - x) \cdot \ln(1 - x) \cdot (1 - \tilde{p}_i(x)) - x + 1.
\]
Proof. For notation simplicity, let \( t = q_i^{-1}(x) \). Since \( (1 - q_i(t)) - (1 - p_i(t)) = 1 - t \), we have that \( p_i(t) = 1 - \frac{1}{1 - q_i(t)} \). Then,
\[
\tilde{g}_i(x) = \Gamma \cdot \left( \sum_j (1 - q_j(t)) \cdot p_j(t) - t \right) + 1
\]
\[
= \Gamma \cdot \left( \sum_j (1 - q_j(t)) \cdot p_j(t) + (1 - q_i(t)) \cdot p_i(t) - t \right) + 1
\]
\[
= \Gamma \cdot \left( 1 - t \right) \cdot \left( \sum_j \frac{p_j(t)}{1 - p_j(t)} + (1 - x) \cdot t - 1 \right) + 1
\]
\[
= \Gamma \cdot \left( 1 - t \right) \cdot \left( \sum_j \ln \left( \frac{1 - q_j(t)}{1 - p_j(t)} \right) - x \right) + 1
\]
\[
= \Gamma \cdot \left( 1 - t \right) \cdot \ln \left( \frac{1}{1 - q_i(x)} \right) - x + 1
\]
\[
= \Gamma \cdot \left( -1 - q_i(t) \right) \cdot (1 - p_i(t)) \cdot \ln(1 - q_i(t)) + 1
\]
\[
= \Gamma \cdot (-1 - x) \cdot \ln(1 - x) \cdot (1 - \tilde{p}_i(x)) - x + 1.
\]
Here, the third equality holds since \( (1 - q_i(t)) - (1 - p_i(t)) = 1 - t \) for all \( j \); the inequality holds since \( \frac{1}{1 - q_i(t)} \geq \ln \frac{1}{1 - q_i(t)} \) for all \( p \in [0, 1] \). \( \square \)

Observe that functions \( \tilde{p}_i(x), \tilde{g}_i(x) \) are only defined on \( [0, q_i(1)] \). We further extend the two functions by defining \( \tilde{p}_i(x) = 1 \) and \( \tilde{g}_i(x) = 1 - \Gamma \cdot x \) for \( x \in (q_i(1), 1] \). It is straightforward to verify that the extended functions satisfy (9) for all \( x \in [0, 1] \). This condition is the only property that we are going to use for functions \( \tilde{p}_i \) and \( \tilde{g}_i \). Specifically, we prove the following technical lemma.

Lemma III.6. Suppose functions \( \tilde{p}, \tilde{g} : [0, 1] \rightarrow [0, 1] \) satisfies that
\[
\tilde{g}(x) \geq \Gamma \cdot (-1 - x) \cdot \ln(1 - x) \cdot (1 - \tilde{p}(x)) - x + 1.
\]
Then
\[
\int_0^1 \frac{\Gamma}{\tilde{g}(x) \cdot \exp \left( \Gamma \cdot \int_0^x \frac{\tilde{p}(y)}{\tilde{g}(y)} dy \right)} dx \leq 1.
\]
The proof of the above lemma is provided in the full version of the paper. By applying it to the extended functions \( \tilde{p}_i(x), \tilde{g}_i(x) \), we conclude the proof of Lemma III.2:

\[
(1 - \Gamma \cdot q_i(1)) \left( 1 - \int_0^1 f_i(t) \, dt \right) \exp \left( \Gamma \int_0^1 q_i'(s) \cdot p_i(s) \, ds \right) = (1 - \Gamma \cdot q_i(1)) \cdot \left( 1 - \int_{q_i(1)}^1 \frac{\Gamma}{\tilde{g}_i(x)} \exp \left( \Gamma \cdot \int_0^x \frac{\tilde{p}_i(y)}{\tilde{g}_i(y)} \, dy \right) \, dx \right) \cdot \exp \left( \Gamma \int_0^{q_i(1)} \frac{\tilde{p}_i(x)}{\tilde{g}_i(x)} \, dx \right) \geq (1 - \Gamma \cdot q_i(1)) \cdot \int_{q_i(1)}^1 \frac{\Gamma}{\tilde{g}_i(x)} \exp \left( \Gamma \cdot \int_0^x \frac{\tilde{p}_i(y)}{\tilde{g}_i(y)} \, dy \right) \, dx \cdot \exp \left( \Gamma \int_0^{q_i(1)} \frac{\tilde{p}_i(x)}{\tilde{g}_i(x)} \, dx \right) \quad \text{(by Lemma III.6)}
\]

\[
= (1 - \Gamma \cdot q_i(1)) \cdot \int_{q_i(1)}^1 \frac{\Gamma}{(1 - \Gamma \cdot x) \exp \left( \Gamma \cdot \int_{q_i(1)}^x \frac{1 - 1/y}{1/y} \, dy \right) \, dx} \quad \text{(by our extension of } \tilde{p}_i, \tilde{g}_i) \)
\]

\[
= (1 - \Gamma \cdot q_i(1)) \int_{q_i(1)}^1 \frac{\Gamma}{(1 - \Gamma \cdot x) \exp \left( - \ln(1 - \Gamma \cdot y) \right)_{q_i(1)}} \, dx \]

\[
= (1 - \Gamma \cdot q_i(1)) \int_{q_i(1)}^1 \frac{\Gamma}{1 - \Gamma \cdot q_i(1)} \, dx \]

\[
= \Gamma \cdot (1 - q_i(1))
\]

REFERENCES

[1] B. Peng and Z. G. Tang, “Order selection prophet inequality: From threshold optimization to arrival time design,” CoRR, vol. abs/2204.01425, 2022.

[2] U. Krengel and L. Sucheston, “Semimartingales and finite values,” Bulletin of the American Mathematical Society, vol. 83, no. 4, pp. 745–747, 1977.

[3] ———. “On semimartingales, amarts, and processes with finite value,” Probability on Banach spaces, vol. 4, pp. 197–266, 1978.

[4] E. Samuel-Cahn, “Comparison of threshold stop rules and maximum for independent nonnegative random variables,” the Annals of Probability, pp. 1213–1216, 1984.

[5] M. T. Hajiaghayi, R. D. Kleinberg, and T. Sandholm, “Automated online mechanism design and prophet inequalities,” in AAAI. AAAI Press, 2007, pp. 58–65.

[6] T. P. Hill and R. P. Kertz, “Comparisons of stop rule and supremum expectations of iid random variables,” The Annals of Probability, pp. 336–345, 1982.

[7] M. Abolhassani, S. Ehsani, H. Esfandiari, M. Hajiaghayi, R. D. Kleinberg, and B. Lucier, “Beating 1/ε for ordered prophets,” in STOC. ACM, 2017, pp. 61–71.

[8] J. R. Correa, P. Foncea, R. Hoeksma, T. Oosterwijk, and T. Vredeveld, “Posted price mechanisms and optimal threshold strategies for random arrivals,” Math. Oper. Res., vol. 46, no. 4, pp. 1452–1478, 2021.

[9] S. Chawla, J. D. Hartline, D. L. Malec, and B. Sivan, “Multi-parameter mechanism design and sequential posted pricing,” in STOC. ACM, 2010, pp. 311–320.

[10] H. Beyhaghi, N. Golrezaei, R. P. Leme, M. Páll, and B. Sivan, “Improved revenue bounds for posted-price and second-price mechanisms,” Oper. Res., vol. 69, no. 6, pp. 1805–1822, 2021.

[11] S. Agrawal, J. Sethuraman, and X. Zhang, “On optimal ordering in the optimal stopping problem,” in EC. ACM, 2020, pp. 187–188.

[12] H. Esfandiari, M. Hajiaghayi, V. Liaghat, and M. Monemizadeh, “Prophet secretary,” SIAM J. Discret. Math., vol. 31, no. 3, pp. 1685–1701, 2017.

[13] S. Ehsani, M. Hajiaghayi, T. Kesselheim, and S. Singla, “Prophet secretaries for combinatorial auctions and matroids,” in SODA. SIAM, 2018, pp. 700–714.

[14] Y. Azar, A. Chipfunkur, and H. Kaplan, “Prophet secretary: Surpassing the 1-1/ε barrier,” in EC. ACM, 2018, pp. 303–318.

[15] J. R. Correa, R. Saona, and B. Ziloott, “Prophet secretary through blind strategies,” Math. Program., vol. 190, no. 1, pp. 483–521, 2021.

[16] T. P. Hill and R. P. Kertz, “A survey of prophet inequalities in optimal stopping theory,” Contemp. Math., vol. 125, pp. 191–207, 1992.

[17] B. Lucier, “An economic view of prophet inequalities,” SIGecom Exch., vol. 16, no. 1, pp. 24–47, 2017.

[18] J. R. Correa, P. Foncea, R. Hoeksma, T. Oosterwijk, and T. Vredeveld, “Recent developments in prophet inequalities,” SIGecom Exch., vol. 17, no. 1, pp. 61–70, 2018.

[19] J. R. Correa, P. Foncea, D. Pizarro, and V. Verdugo, “From pricing to prophets, and back!” Oper. Res. Lett., vol. 47, no. 1, pp. 25–29, 2019.

[20] A. Liu, R. P. Leme, M. Páll, J. Schneider, and B. Sivan, “Variable decomposition for prophet inequalities and optimal ordering,” in EC. ACM, 2021, p. 692.

[21] H. Fu, Ji. Li, and P. Xu, “A PTAS for a class of stochastic dynamic programs,” in ICALP, ser. LIPIcs, vol. 107. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018, pp. 56:1–56:14.

[22] T. Chakraborty, E. Even-Dar, S. Guha, Y. Mansour, and S. Muthukrishnan, “Approximation schemes for sequential posted pricing in multi-unit auctions,” in WINE, ser. Lecture Notes in Computer Science, vol. 6484. Springer, 2010, pp. 158–169.