S-Injective Modules and Rings

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ABSTRACT

We introduce and investigate the concept of s-injective modules and strongly s-injective modules. New characterizations of SI-rings, GV-rings and pseudo-Frobenius rings are given in terms of s-injectivity of their modules.

KEYWORDS

S-Injective Module; Kasch Ring; Pseudo Frobenius Ring

1. Introduction

In this paper we introduce and investigate the notion of s-injective Modules and Rings. A right R-module M is called right s-N-injective, where N is a right R-module, if every R-homomorphism f : K → M extends to N, where K is a submodule of the singular submodule Z(N). M is called strongly s-injective if M is s-N-injective for every right R-module N. The connection between this new injectivity condition and other injectivity conditions has been established, and examples are provided to distinguish s-injectivity from other injectivity concepts such as mininjectivity, soc-injectivity. Several properties of this new class of injectivity are highlighted.

Throughout this paper all rings are associative with identity, and all modules are unitary R-modules. For a right R-module M, we denoted J(M), soc(M), and Z(M) by the Jacobson radical, the socle and the singular submodule of M, respectively. S_r, S_s, Z_r, Z_s, and J are used to indicate the right socle, the left socle, the right singular ideal, the left singular ideal, and the Jacobson radical of R, respectively. For a submodule N of M, the notations N ≤^∞ M, N ≤^max M and N ≤^o M mean, respectively, that N is essential, maximal, and direct summand. If X is a subset of a right R-module M, right annihilators will be denoted by r(X) = r_k(X) = \{ r ∈ R : xr = 0 \ for all \ x ∈ X \} , with a similar definition of left annihilators l(X) = l_k(X). Multiplication maps x → ax and x → xa will be denoted by a* and a*, respectively. If M and N are right R-modules, then M is called N-injective if every R-homomorphism from a submodule of N into M can be extended to an R-homomorphism from N into M. Mod-R indicates the category of right R-modules. We refer to [1-3] for all the undefined notions in this paper.

2. Strongly S-Injective Modules

Definition 1 A right R-module M is called s-N-injective if every R-homomorphism f : K → M extends to N, where K is a submodule of the singular submodule Z(N). M is called s-injective if M is s-R-injective. M is called strongly s-injective, if M is s-N-injective for all right R-modules N.

For example every nonsingular R-module is strongly s-injective. In particular, the ring of integers Z is strongly s-injective, but not injective.

Proposition 1

1) Let N be a right R-module and \{ M_i : i ∈ I \} a family of right R-modules. Then the direct product \bigoplus_{i ∈ I} M_i...
is \(s\)-\(N\)-injective if and only if each \(M_i\) is \(s\)-\(N\)-injective, \(i \in I\).

2) Let \(M, N,\) and \(K\) be right \(R\)-modules with \(K \subseteq N\). If \(M\) is \(s\)-\(N\)-injective, then \(M\) is \(s\)-\(K\)-injective.

3) Let \(M, N,\) and \(K\) be right \(R\)-modules with \(K \subseteq N\). If \(K\) is \(s\)-\(M\)-injective, then \(N\) is \(s\)-\(M\)-injective.

4) Let \(N\) be a right \(R\)-module and \(\{A_i : i \in I\}\) a family of right \(R\)-modules. Then \(N\) is \(s\)-\(\bigoplus_{i \in I} A_i\)-injective if and only if \(N\) is \(s\)-\(A_i\)-injective, \(\forall i \in I\).

5) A right \(R\)-module \(M\) is \(s\)-injective if and only if \(M\) is \(s\)-\(P\)-injective for every projective right \(R\)-module \(P\).

6) Let \(M, N,\) and \(K\) be right \(R\)-modules with \(N \subseteq N\). If \(M\) is \(s\)-\(K\)-injective, then \(N\) is \(s\)-\(K\)-injective.

7) If \(A, B,\) and \(M\) are right \(R\)-modules, \(A \cong B\), and \(M\) is \(s\)-\(A\)-injective, then \(M\) is \(s\)-\(B\)-injective.

Proof. The proofs of 1) through 4) are routine.

5) This follows from 4).

6). Let \(f : L \rightarrow N\) be an \(R\)-homomorphism where \(L\) is singular submodule of \(N\). Then the map \(\iota \circ f : L \rightarrow M\) can be extended to an \(R\)-homomorphism \(g : K \rightarrow M\), where \(\iota : N \rightarrow M\) the inclusion map. Now, the map \(\pi \circ g : K \rightarrow N\) is an extension of \(f\), where \(\pi : M \rightarrow N\) the natural projection map into \(N\).

7). Let \(f : A \rightarrow B\) be an \(R\)-isomorphism, and \(g : K \rightarrow M\) an \(R\)-homomorphism where \(K\) is a singular submodule of \(B\). The restriction of \(f\) to \(Z(A)\) induces an isomorphism \(f^\# : Z(A) \rightarrow Z(B)\). By hypothesis, the map \(g \circ f : K \rightarrow M\) can be extended to an \(R\)-homomorphism \(\eta : A \rightarrow M\). Now, the map \(\eta \circ f^\# : B \rightarrow M\) is an extension of \(g\).

The next two corollaries are immediate consequences of the above proposition.

**Corollary 1** Let \(N\) be a right \(R\)-module. Then the following statements are true:

1) A finite direct sum of \(s\)-\(N\)-injective modules is again \(s\)-\(N\)-injective. In particular a finite direct sum of \(s\)-injective (strongly \(s\)-injective) modules is again \(s\)-injective (strongly \(s\)-injective).

2) A summand of \(s\)-\(N\)-injective \((s\)-injective, strongly \(s\)-injective) module is again \(s\)-\(N\)-injective \((s\)-injective, strongly \(s\)-injective) module.

**Corollary 2**

1) Let \(M\) be a right \(R\)-module and \(1 = e_1 + e_2 + \cdots + e_n\) in \(R\), where the \(e_i\) are orthogonal idempotents. Then \(M\) is \(s\)-\(N\)-injective if and only if \(M\) is \(s\)-\(e_i R\)-injective for each \(i, 1 \leq i \leq n\).

2) Assume that \(e\) and \(f\) are idempotents of \(R\), \(e R \subseteq f R\), and \(M\) is \(s\)-\(e R\)-injective. Then \(M\) is \(s\)-\(f N\)-injective.

**Proposition 2** If \(N\) is a finitely generated right \(R\)-module, then the following conditions are equivalent:

1) Any direct sum of \(s\)-\(N\)-injective modules is \(s\)-\(N\)-injective.

2) Any direct sum of injective modules is \(s\)-\(N\)-injective.

3) \(Z(N)\) is noetherian.

Proof. 1) \(\Rightarrow\) 2). Clear.

2) \(\Rightarrow\) 3). Consider a chain \(U_1 \subseteq U_2 \subseteq \cdots\) of singular submodules of \(N\) and \(U = \bigcup_{i \in I} U_i\). Let \(E(N/U_i)\) be the injective hull of \(N/U_i\), \(i \geq 1\), and \(f : U \rightarrow \bigoplus_{i \in I} E(N/U_i)\) be a map defined by \(f(n) = (n + U_i)\). Since, \(\bigoplus_{i \in I} E(N/U_i)\) is \(s\)-\(N\)-injective, \(f\) can be extended to an \(R\)-homomorphism \(\hat{f} : N \rightarrow \bigoplus_{i \in I} E(N/U_i)\). Since \(N\) is finitely generated, \(\hat{f}(N) \subseteq \bigoplus_{i \in I} E(N/U_i)\) for some \(n\), then \(f(U) \subseteq \bigoplus_{i \in I} E(N/U_i)\) and \(U = U_{j=0} \subseteq U_{j=0}\) for every \(j \geq 1\). Hence \(Z(N)\) is noetherian.

3) \(\Rightarrow\) 1). Let \(E = \bigoplus_{i \in I} E_i\) be a direct sum of \(s\)-\(N\)-injective modules, and \(f : U \rightarrow E_R\) be a homomorphism of right \(R\)-modules where \(U \subseteq Z(N)\). Since \(Z(N)\) is noetherian, \(f(U) \subseteq \bigoplus_{i \in I} E_i\) for some finite subset \(F \subseteq I\).

Since finite direct sums of \(s\)-\(N\)-injective modules is \(s\)-\(N\)-injective, \(f\) can be extended to an \(R\)-homomorphism \(f : N \rightarrow E\).

The second singular submodule of a right \(R\)-module \(M\), denoted by \(Z_2(M)\), is defined by the equality \(Z_2(M)/Z(M) = Z(M/Z(M))\). We see that \(Z_2(M)\) is closed submodule of \(M\) and \(M/Z_2(M)\) is non-singular. A right \(R\)-module is Goldie torsion if \(Z_2(G) = G\).

**Lemma 1** Let \(M\) and \(N\) be right \(R\)-modules such that \(Z_2(M)\) is injective. Then every homomorphism
Let $f : K \to M$ where $K \subseteq Z_2(N)$ extends to $N$.

**Proof.** Let $M = Z_2(M) \oplus T$ where $Z_2(M)$ is injective and $Z(T) = 0$. If $f : K \to M$ is a homomorphism where $K \subseteq Z_3(N)$ such that $f(Z(K)) = 0$, then $f(K) \subseteq K/Ker(f)$ is singular. So $f$ extendable to $N$. Now suppose that $0 \neq f(k) \in T$, so $f(kR) \cong kR/Ker(f)$ is singular which is a contradiction. Thus $f(K) \cap T = 0$. Suppose that $f(k) = a + b$ where $a \in Z_2(M)$ and $b \in T$. Since $r(a) \subseteq r(b)$ and the kernal of the map $ar \mapsto br$ is essential in $ar$ which is a contradiction. Then every homomorphism $f : K \to M$ where $K \subseteq Z_2(N)$ extends to $N$. \hfill \Box

**Proposition 3** The following statements are equivalent:

1) $M$ is strongly s-injective.
2) $M$ is $s-I(M)$-injective, where $I(M)$ is the injective hull of $M$.
3) $M = E \oplus K$, where $K$ is nonsingular and $E$ is injective with $Z(M) \subseteq_{ess} E$.
4) $Z_2(M)$ is injective.
5) $M$ is $G$-injective for every Goldie torsion module $G$.
6) $M$ is $I$-injective, where $I = I(Z_2(M))$ is the injective hull of $Z_2(M)$.

**Proof.**

2) $\Rightarrow$ 3). If $Z(M) = 0$, we are done. Assume that $Z(M) \neq 0$ and consider the following diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & Z(M) \\
\downarrow & & \downarrow i_2 \\
0 & \rightarrow & D \\
\end{array}
$$

where $i_2$ and $i_3$ are inclusion maps and $D$ is injective closure of $Z(M)$ in $I(M)$. Since $M$ is $s-I(M)$-injective, $M$ is $s-D$-injective. So there exists an $R$-homomorphism $\sigma : D \to M$, which extends $i_3$. Since $Z(M) \subseteq_{ess} D$, $\sigma$ is an embedding of $D$ in $M$. If we write $E = \sigma(D)$, then $M = E \oplus K$ for some submodule $K$ of $M$ because $E$ is injective. Finally $K$ is nonsingular because $Z(M) \subseteq E$.

3) $\Rightarrow$ 4). Since $E/Z(M)$ is singular and $Z_2(M)/Z(M)$ is singular submodule of $M/Z(M)$, so $E \subseteq Z_2(M)$ and $Z_2(M) = E \oplus L$ for some submodule $L$ of $M$. Then $Z_2(M) = E \{Z(M) \subseteq_{ess} Z_2(M)\}$ and $Z_2(M)$ is injective.

4) $\Rightarrow$ 5). Let $G$ be a Goldie torsion right $R$-module and $K$ a submodule of $G$. Using the above Lemma, every homomorphism $f : K \to M$ extends to $G$.

5) $\Rightarrow$ 6). If $I = I(Z_2(M))$ is the injective hull of $Z_2(M)$, then $Z_2(I) = I$ and $I$ is Goldie torsion.

6) $\Rightarrow$ 1). Let $N$ be a right $R$-module and $K$ a singular submodule of $N$. Consider the diagram

$$
\begin{array}{ccc}
K & \rightarrow & Z_2(M) \\
\downarrow i_1 & & \downarrow i_2 \\
N & \rightarrow & I(Z_2(M)) \\
\end{array}
$$

where $i_1$, $i_2$, and $i_3$ are the inclusion maps. Since $M$ is $I$-injective and $I$ is injective. So, there exist $R$-homomorphisms $h : I \to M$ and $g : N \to I$ such that $hi_1 = i_2$ and $gi_2 = i_3f$. Thus $i_2f = hi_1f = hg_i_1$.

Hence $M$ is strongly s-injective.

**Corollary 3** Let $M$ a Goldie torsion right $R$-module. Then $M$ is injective if and only if $M$ is strongly s-injective.

**Proposition 4** For a ring $R$, the following conditions are equivalent:

1) $R$ is a strongly s-injective.
2) $R$ is $s-I(R)$-injective, where $I(R)$ is the injective hull of $R$.
3) $R = E \oplus T$, where $E$ is injective and $T$ is nonsingular. Moreover, if $Z_i \neq 0$, then $Z_i \subseteq_{ess} E$, and in this case $E$ and $T$ are relatively injective.
4) $Z_2^i$ is injective.
5) $R$ is $G$-injective for every Goldie torsion right $R$-module $G$.
6) $R$ is $I$-injective, where $I = I(Z_2^i)$ is the injective hull of $Z_2^i$.
7) Every finitely generated projective right $R$-module is strongly s-injective.

**Proof.** The equivalence between 1), 2), 3), 4), 5) and 6) is from Proposition 3.
1) ⇒ 7) Since a finite direct sum of \( s \)-\( N \)-injective is \( s \)-\( N \)-injective for every right \( R \)-module \( N \) (Corollary 1), so every finitely generated free right \( R \)-module is strongly \( s \)-injective. But a direct summand of strongly \( s \)-injective is strongly \( s \)-injective (Corollary 1). Therefore every finitely generated projective module is strongly \( s \)-injective. The converse is clear. □

The following examples show that the two classes of strongly \( s \)-injective rings and of \( soc \)-injective rings are different.

**Example 1** Let \( F = \mathbb{Z}_2 \) be the field of two elements, \( F_n = F \) for \( n = 1, 2, \cdots \), \( Q = \prod_{i=1}^{\infty} F_i \), \( S = \bigoplus_{i=1}^{\infty} F_i \). If \( R \) is the subring of \( Q \) generated by \( 1 \) and \( S \), then \( R \) is a von Neumann regular ring with \( soc(R) = S \), and hence \( Z'_2 = 0 \) and \( R \) is strongly \( s \)-injective. However, the map \( f : S_R \rightarrow R_{S_R} \), given by \( (a_1, a_2, a_3, \cdots) \mapsto (a_1, 0, a_2, 0, \cdots) \), cannot be extended to an \( R \)-homomorphism from \( R \) into \( R \) (suppose that \( f = c \cdot \) for some \( c = (c_i) \in R \). Then for every \( (a_1, a_2, \cdots) \in S_R \), \( (a_1, 0, a_2, 0, \cdots) = (c_1)(a_1) \) which is impossible), and so \( R \) is not a \( soc \)-injective ring.

**Example 2** Let \( R = \mathbb{Z}_2[x_1, x_2, \cdots] \) where \( x_i^2 = 0 \) for all \( i \), \( x_i x_j = 0 \) for all \( i \neq j \) and \( x_i^2 = m \neq 0 \) for all \( i \) and \( j \). Then \( R \) is a commutative, semiprimary, local ring with \( J = \text{span}\{m, x_1, x_2, \cdots\} = \mathbb{Z}_2 \), and \( R \) has simple essential socle \( J^2 = \mathbb{Z}_2 m \). It is not difficult to see that \( R \) is right \( s \)-injective. However the \( R \)-homomorphism \( \gamma : \mathbb{Z}_2 \rightarrow R \) defined by \( \gamma(a) = a^2 \) for all \( a \in \mathbb{Z}_2 \) cannot be extended to an endomorphism of \( R \), and so \( R \) is not \( s \)-injective ring.

**Definition 2** A ring \( R \) is called a right generalized \( V \)-ring (right \( GV \)-ring) if every simple singular right \( R \)-module is injective.

**Proposition 5** A ring \( R \) is right \( GV \)-ring if and only if every simple right \( R \)-module is strongly \( s \)-injective.

**Proof.** Let \( R \) be a right \( GV \)-ring and \( M \) be a simple right \( R \)-module. The module \( M \) is either projective or singular, so \( M \) is strongly \( s \)-injective. Conversely, if \( M \) is a simple singular \( R \)-module, then \( M \) is strongly \( s \)-injective. Thus \( M \) is injective by Proposition 3. □

**Lemma 2** For a right \( R \)-module \( M \) the following conditions are equivalent:
1) \( M \) satisfies \( ACC \) on essential submodules.
2) \( M \oslash \text{Soc}(M) \) is noetherian.

**Proof.** Assume that \( M \) has \( ACC \) on essential submodules. Then \( B/A \) is noetherian for every submodule \( A \subset B \) of \( B \oslash (B \oplus L)/(A \oplus L) \) where \( L \) is an intersection complement of \( A \) and \( M/(A \oplus L) \) is noetherian. In particular, every uniform submodule of \( M \) is noetherian. Let \( H \) be an intersection complement of \( S \) (\( = \text{Soc}(M) \)) (see Kasch [2] p.112). Then \( M/(H \oplus S) \) is noetherian. So, to prove that \( M/S \) is noetherian it is enough to show that \( H \) is noetherian. Assume that \( H \) contains an infinite direct sum \( K = K_1 \oplus K_2 \oplus \cdots \) of nonzero submodules \( K_i \). Since \( K_i \cap S = 0 \), each \( K_i \) contains a proper essential submodule \( L_i \) and \( L = L_1 \oplus L_2 \oplus \cdots \) is essential in \( K \). But this gives that \( K/L \) is noetherian which is impossible because \( K/L \cong \bigoplus K_i/L_i \oplus \cdots \) with each \( K_i/L_i \) nonzero. Then \( H \) contains \( k \) independent uniform submodules \( U_i \) such that \( U = \bigoplus U_1 \oplus U_2 \oplus \cdots \oplus U_k \) is essential in \( H \). Thus \( U \) and \( H/U \) are noetherian. Hence \( H \) is noetherian.

It is well-known that, a ring \( R \) is right noetherian if and only if all direct sums of injective right \( R \)-modules are injective. In the next Proposition we obtain a characterization of ring which has \( ACC \) on essential right ideals in terms of strongly \( s \)-injective right \( R \)-modules.

**Proposition 6** The following conditions on a ring \( R \) are equivalent:
1) Every direct sum of strongly \( s \)-injective right \( R \)-modules is strongly \( s \)-injective.
2) Every direct sum of injective right \( R \)-modules is strongly \( s \)-injective.
3) Every finitely generated right \( R \)-module has \( ACC \) on essential submodules.
4) \( R/\text{Soc}(R) \) is noetherian.

**Proof.** 1) ⇒ 2). Clear.
2) ⇒ 3). Consider a chain \( K_1 \subset K_2 \subset \cdots \) of essential submodules of a finitely generated right \( R \)-module \( M \) and \( K = \bigcup_{i \in I} K_i \). Let \( I(M/K_i) \) be the injective hull of \( M/K_i \), \( i \geq 1 \), and \( f : K \rightarrow \bigoplus_{i \in I} I(M/K_i) \) be a map defined by \( f(k) = (k + I_i) \). Since \( \bigoplus I(M/K_i) \) is strongly \( s \)-injective and \( I(M/K_i) \) has an essential singular submodule, so \( \bigoplus I(M/K_i) \) is injective and \( f \) can be extended to an \( R \)-homomorphism.
\[ f : aR \to E \ni f(a) \text{ is simple right ideal, so } aR \text{ is a singular right ideal and } f(aR) \subseteq \bigoplus_{i \in I} E_i \text{ where } F \subseteq I \text{ is finite. Thus } f \text{ extends to } g : R \to \bigoplus_{i \in I} E_i. \text{ Then } 0 \neq g(a) = g(1) a = f(a) \text{ which is a contradiction with } ES_i = 0. \]

Hence any homomorphism \( h : U \to E \) where \( U \) is a right ideal of \( R \) induces a map \( h : (U + S_r)S_e \to E \) with \( h(u + S_r) = h(u) \). The map \( h \) extends to a homomorphism \( \hat{a} : R/S_r \to E \). Then the map \( \alpha = \hat{a} \pi : R \to E \) where \( \pi \) is the natural epimorphism \( \pi : R \to R/S_r \), extends \( h \). Hence \( E \) is injective. Therefore, every direct sum of strongly \( s \)-injective right \( R \)-modules is strongly \( s \)-injective.

If \( I \) is an ideal of \( R \), \( R \) is called \( I \)-semiperfect if for every right ideal \( K \), there is a decomposition \( K = eR \oplus U \) such that \( e^2 = e \) and \( U = K \cap (1-e)R \subseteq I \) \([4]\).

**Lemma 3** If \( R \) is \( Z \)-semiperfect, then the following statements hold:

1) A module \( M \) is \( s \)-injective if and only if \( M \) is injective.
2) \( K = rl(K) \) for all singular right ideals \( K \) of \( R \) if and only if \( K = rl(K) \) for all right ideals \( K \) of \( R \).

**Proof.** 1) Let \( M \) be \( s \)-injective, and \( f : T \to M \) be an \( R \)-homomorphism where \( T \) is a right ideal of \( R \). Then \( T = eR \oplus U \), where \( U = T \cap (1-e)R \subseteq Z \). Let \( g : R \to M \) be an extension of the restriction map \( f | U \). Define \( h : R \to M \) by \( h(x) = h((1-e)x) = f((1-e)x) + g((1-e)x) \) for all \( x \in R \). Clearly, \( h \) is an extension of \( f \), and so \( M \) is injective by the Baer’s Criterion.

2) Let \( T \) be a right ideal of \( R \). Since \( R \) is right \( Z \)-semiperfect, then \( T = eR \oplus U \), where \( e^2 = e \in R \) and \( U = T \cap (1-e)R \subseteq Z \). So \( I(T) = I(1-e) \cap I(U) \) and \( rl(T) = rl((1-e) \cap I(U)) \). If \( x \in r[(1-e) \cap I(U)] \), then \( I((1-e)U) \subseteq I(1-e) \) and so \( (1-e)x \in rl((1-e)U) \subseteq rl((1-e)U) = (1-e)U \). The last equality is because that \( (1-e)U \) is a singular right ideal of \( R \). Write \( (1-e)x = (1-e)u \) where \( u \in U \). Then \( x = e(x-u) + u \in T \). Therefore, \( T = rl(T) \). \( \square \)

**Proposition 7** Let \( M \) be a right \( R \)-module. \( Z(M) \) is semisimple summand of \( M \) if and only if every right \( R \)-module is \( s \)-\( M \)-injective.

**Proof.** If \( Z(M) \) is semisimple summand of \( M \), then every right \( R \)-module \( N \) is \( s \)-\( M \)-injective. Conversely, if every right \( R \)-module is \( s \)-\( M \)-injective, then every identity map \( i : K \to K \) where \( K \) is singular submodule of \( M \) extends to \( f : M \to K \). Thus \( K \) is a summand of \( M \). Hence \( Z(M) \) is a semisimple summand of \( M \). \( \square \)

**Corollary 4** A ring \( R \) is right nonsingular if and only if every right \( R \)-module is \( s \)-injective.

A ring \( R \) is called a right (left) \( SI \) ring if every singular right (left) \( R \)-module is injective. \( SI \) rings were initially introduced and investigated by Goodearl \([5]\).

**Theorem 1** The following statements are equivalent:

1) \( R \) is right \( SI \) ring.
2) Every right \( R \)-module is strongly \( s \)-injective.
3) Every singular right \( R \)-module is strongly \( s \)-injective.

**Proof.** Clear from Proposition 3. \( \square \)

A module \( M \) is said to satisfy the \( C2 \)-condition, if \( K \) and \( L \) are submodules of \( M \), \( K \trianglelefteq L \), and \( K \subseteq^{\oplus} M \). We also say \( M \) satisfies the \( C3 \)-condition if \( K \) and \( L \) are submodules of \( M \) with \( K \cap L = 0 \), \( K \subseteq^{\oplus} M \) and \( L \subseteq^{\oplus} M \), then \( K \oplus L \) is a summand of \( M \). It is a well-know fact that the \( C2 \)-condition implies the \( C3 \)-condition. In the next proposition we show that \( s \)-\( quasi \)-injective modules inherit a weaker version of these conditions.

**Proposition 8** Suppose \( M \) is a \( s \)-\( quasi \)-injective right \( R \)-module.

1) \( (s\text{-}C2) \) If \( K \) and \( L \) are singular submodules of \( M \), \( K \trianglelefteq L \), and \( K \subseteq^{\oplus} M \), then \( L \subseteq^{\oplus} M \).
2) \( (s\text{-}C3) \) Let \( K \) and \( L \) be singular submodules of \( M \) with \( K \cap L = 0 \). If \( K \subseteq^{\oplus} M \) and \( L \subseteq^{\oplus} M \), then \( K \oplus L \) is a summand of \( M \).
Proof. 1) Since $K \cong L$, and $K$ is $s$-injective, being a summand of the $s$-quasi-injective right $R$-module $M$, $L$ is $s$-injective. If $\iota: L \to M$ is the inclusion map, the identity map $Id_{L} : L \to L$ has an extension $\eta: M \to L$ such that $\iota \circ \eta = Id_{L}$, and so $K$ is a summand of $M$.

2) Since $K$ and $L$ are summands of $M$, and $M$ is $s$-quasi-injective, both $K$ and $L$ are $s$-$M$-injective. Thus the singular module $K \oplus L$ is $s$-$M$-injective, and so is a summand of $M$. □

It is a well-known fact that the $C2$-condition implies the $C3$-condition.

**Proposition 9** If a module $M$ has $s$-$C2$-condition, then $M$ has $s$-$C3$-condition.

Proof. Consider singular summands $M_1$ and $M_2$ of $M$ such that $M_1 \cap M_2 = 0$. Write $M = M_1 \oplus M_2$ and let $\pi$ denote the projection $M_1 \oplus M_2^\ast \to M_1^\ast$. Then $M_1 \oplus M_2 = M_1 \oplus \pi(M_2)$. If $b \in M_2$, $s = c + d$ where $c \in M_1$ and $d \in M_2^\ast$ and $\pi(b) = 0$, $\pi_{M_2}(c + d) = 0 = d$ and $b \in M_1 \cap M_2 = 0$. Then $\pi_{M_2}$ is a monomorphism; so $\pi(M_2)$ is a summand of $M$ by $S$-$C2$. As $\pi(M_2) \leq M_2^\ast$, $M_1 \oplus \pi(M_2) \subseteq M$.

**Proposition 10** Let $R$ and $S$ be Morita-equivalent rings with category equivalence $f : ModR \to ModS$. Let $M$, $N$, and $K$ be right $R$-modules. Then

1) $K_{s}$ is singular if and only if $f(K)_{s}$ is singular.

2) $M_{s}$ is $s$-$N$-injective if and only if $f(M)_{s}$ is $s$-injective.

Proof. There is a natural isomorphisms $\eta : GF \to 1_{modS}$ and $\zeta : FG \to 1_{modS}$. This means that for each $M_{s}$ there is an isomorphism $f_{M} : GF(M) \to M$ in $modR$ such that for each $M$, $M'$ in $modR$ and each $f : M \to M'$ in $modR$, the following diagram commutes

$$
\begin{array}{ccc}
M & \xrightarrow{f} & M' \\
\uparrow_{\eta_{M}} & & \uparrow_{\eta_{M'}}
\end{array}
\Rightarrow
\begin{array}{ccc}
GF(M) & \xrightarrow{GF(f)} & GF(M')
\end{array}
$$

1). The right $R$-module $K$ is singular if and only if there is an exact sequence of right $R$-modules $0 \to A \to B \to K \to 0$ with essential monomorphism $0 \to A \to B$. But using [6, Proposition 21.4 and Proposition 21.6] the sequence $0 \to A \to B \to K \to 0$ is exact with essential monomorphism $0 \to A \to B$ if and only if the sequence $0 \to F(A) \to F(B) \to F(K) \to 0$ of right $S$-modules is exact with essential monomorphism $0 \to F(A) \to F(B)$. So $K_{s}$ is singular if and only if $f(K)_{s}$ is singular.

2). Let $M$ be a $s$-$N$-injective and $K$ be a singular submodule of $F(N)$. Let $f : K \to F(M)$ be a homomorphism. Since $G(K)$ is singular, $G(1_{K})$ is a monomorphism and the maps $\eta_{N}$ and $\eta_{M}$ are isomorphisms (we may assume that $G(K)$ is a submodule of $N$), then we have the commutative diagram

$$
\begin{array}{ccc}
G(K) & \xrightarrow{\eta_{G}(f)} & M \\
\downarrow_{\eta_{G}(1_{K})} & & \uparrow_{\alpha}
\end{array}
\Rightarrow
\begin{array}{ccc}
N
\end{array}
$$

So the following diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\zeta_{K}(\eta_{G}(f))} & F(M) \\
\downarrow_{\zeta_{K}(\eta_{G}(1_{K}))} & & \uparrow_{\alpha}
\end{array}
\Rightarrow
\begin{array}{ccc}
F(N)
\end{array}
$$

is commutative where $f(M)_{s}$ is $s$-$F(N)_{s}$-injective. The converse is similarly. □

As for right self-injectivity, right strongly $s$-injectivity turns out to be a Morita invariant.

**Theorem 2** Right strong $s$-injectivity is a Morita invariant property of rings.

Proof. Let $R$ and $S$ be Morita-equivalent rings with category equivalences $f : modR \to modS$, and $G : modS \to modR$. Let $P$, and $N$ be right $R$-modules. $P_{R}$ is finitely generated projective $R$-module if and only if $F(P)_{S}$ is finitely generated projective $S$-module [6, Propositions 21.6 and 21.8]. Also $P_{R}$ is $s$-$N$-injective if and only if $f(P)_{S}$ is $s$-$F(N)$-injective (Proposition 10) and then $P_{R}$ is strongly $s$-injective if and only if $f(P)_{S}$ is strongly $s$-injective. Then, every finitely generated projective right $R$-module is strongly $s$-injective and if only if every finitely generated projective right $S$-module is strongly $s$-injective. Therefore right strong $s$-injectivity is a Morita invariant property of rings. □

**Proposition 11** For a projective right $R$-module $M$, the following conditions are equivalent:
1) Every homomorphic image of an $s$-$M$-injective right $R$-module is $s$-$M$-injective.
2) Every homomorphic image of an injective right $R$-module is $s$-$M$-injective.
3) Every singular submodule of $M$ is projective.

**Proof.** 1) $\Rightarrow$ 2) Obvious.
2) $\Rightarrow$ 3) Consider the following diagram:

$$
\begin{array}{ccc}
E & \xrightarrow{g} & N \\
\downarrow{f} & & \downarrow{\eta} \\
K & \rightarrow & M
\end{array}
$$

Where $K$ is a singular submodule of $M$, $E$ and $N$ are right $R$-modules with $E$ injective, $\eta$ an $R$-epimorphism, and $f$ an $R$-homomorphism. Since $N$ is $s$-injective, $f$ can be extended to an $R$-homomorphism $\tilde{g}: M \rightarrow E$ such that $\eta \circ \tilde{g} = g$. Now, define $\tilde{f}: K \rightarrow E$ by $\tilde{f} = \tilde{g}/K$. Clearly, $\eta \circ \tilde{f} = f$, and hence $K$ is projective.

3) $\Rightarrow$ 1) Let $N$ and $L$ be right $R$-modules with $\eta: N \rightarrow L$ an $R$-epimorphism, $K$ is a singular submodule of $M$ and $N$ is $s$-$M$-injective. Consider the following diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & K \\
& \downarrow{\eta} & \rightarrow \\
& N & \rightarrow L & \rightarrow 0
\end{array}
$$

Since $K$ is projective, $f$ can be lifted to an $R$-homomorphism $g: K \rightarrow N$ such that $\eta \circ g(x) = f(x)$, $\forall x \in K$. Since $N$ is $s$-injective, $g$ can be extended to an $R$-homomorphism $\tilde{g}: M \rightarrow N$. Clearly, $\eta \circ \tilde{g} = M \rightarrow L$ extends $f$.

**Corollary 5** The following conditions are equivalent:
1) Every quotient of $s$-injective right $R$-module is $s$-injective.
2) Every quotient of an injective right $R$-module is $s$-injective.
3) Every singular right ideal is projective.

**Proposition 12** The following conditions are equivalent:
1) Every strongly $s$-injective right $R$-module is injective.
2) Every nonsingular right $R$-module is semisimple injective.
3) For every right $R$-module $M$, $M = E \oplus Z_s(M)$ where $E$ semisimple injective.
4) $R$ is $Z_s$-semiperfect.

**Proof.** 1) $\Rightarrow$ 2) If $M$ is nonsingular right $R$-module, then $M$ is strongly $s$-injective. Thus $M$ is semisimple injective.
2) $\Rightarrow$ 3) Let $M$ be a right $R$-module. If $M$ is Goldie torsion, we are done. Now suppose that $Z(M)$ is not essential in $M$ and $K$ be a maximal submodule $M$ with respect to $K \cap Z(M) = 0$. Then $K$ is semisimple injective and $M = K \oplus L$. It is clear that $L = Z_s(M)$.
3) $\Rightarrow$ 4) Let $K$ be a right ideal of $R$. Then $K = E \oplus Z_s(K)$ where $E$ is semisimple injective. We have $Z_s(K) \subseteq Z_s^2(K)$ and $E$ is a summand of $R$ and generated by an idempotent. Hence $R$ is $Z_s$-semiperfect.
4) $\Rightarrow$ 1) Let $M$ be a strongly $s$-injective and $f: T \rightarrow M$ be an $R$-homomorphism where $T$ is a right ideal of $R$. By $T = eR \oplus U$, where $U = T \cap (1-e)R \subseteq Z_s^2$ and $e^2 = e$. Using Proposition 3 let $g: R_n \rightarrow M$ be an extension of the restriction map $f/\ker f$. Define $h: R \rightarrow M$ by $h(x) = h(ex + (1-e)x) = f(ex) + g((1-e)x)$ for all $x \in R$. Clearly, $h$ is an extension of $f$, and so $M$ is injective by the Baer’s Criterion.

**Theorem 3** The following are equivalent for a ring $R$:
1) $R$ is a right NF-ring.
2) $R$ is $Z_s$-semiperfect, right strongly $s$-injective ring with essential right socle.
3) $R$ is semiperfect, right min-C2, right strongly $s$-injective ring with essential right socle.
4) $R$ is semiperfect with $soc(J) = soc(Z_s)$, right strongly $s$-injective ring with essential right socle.
5) $R$ is right finitely cogenerated, right min-C2, right strongly $s$-injective ring.
6) $R$ is a right Kasch, right strongly $s$-injective ring.
7) $R$ is a right strongly $s$-injective ring and the dual of every simple left $R$-module is simple.

**Proof.** 1) $\Rightarrow$ 2). Clear.
2) \( \Rightarrow \) 1). Clear by Lemma 2.
1) \( \Rightarrow \) 3). Clear.
3) \( \Rightarrow \) 4). Suppose that \( 0 \neq aR \) is a non singular simple right ideal in \( J \), so \( r(a) \cap eR = 0 \) for some simple right ideal \( eR \) with \( e^2 = e \). Thus \( eR \subseteq aR \) and \( aR \) is a summand which is a contradiction. Then \( \text{soc}(J) \subseteq \text{soc}(Z) \). The other inclusion is clear.
4) \( \Rightarrow \) 1). Let \( R \) be semiperfect and right strongly \( s \)-injective ring with essential right socle and \( \text{soc}(J) = \text{soc}(Z) \). Then \( R = Z \oplus T \) where \( Z \) is injective and \( Z \subseteq \text{soc}(J) \). Thus \( J \subseteq Z \) and \( J(T) = 0 \).

The right ideal \( T \) may be considered as a \( R/J \)-module. Let \( f : L \to T \) be a map where \( L \) is a right ideal of \( R \), \( f \) induces a map \( h : (L+J)/J \to T \) given by \( h(l+J) = f(l) \). Since \( T \) is injective as an \( R/J \)-module so \( h \) extends to \( g : R/J \to T \). The map \( \pi : R \to T \), where \( \pi \) is the natural epimorphism \( \pi : R \to R/J \), extends \( f \) and \( T \) is injective. Therefore \( R \) is right selfinjective and \( R \) is right \( PF \).

1) \( \Rightarrow \) 5). Clear.
5) \( \Rightarrow \) 1). Since \( R \) is right strongly \( s \)-injective ring, it follows from Proposition 3 that \( R = Z \oplus T \), where \( Z \) is injective and \( T \) is nonsingular. If \( aR \) is simple right ideal in \( T \) such that \( (aR)^2 = 0 \), then, by the proof of 3) \( \Rightarrow \) 1) \( aR = 0 \) and every simple right ideal in \( T \) is a summand of \( T \). Since \( R \) is right finitely cogenerated, \( T \) has a finitely generated essential socle. Thus \( T \) is semisimple. Hence \( R \) is injective and \( R \) is right \( PF \)-ring.

1) \( \Rightarrow \) 6) Proposition 3 and [7, Theorem 5]
1) \( \Rightarrow \) 7). Assume that \( R \) is right strongly \( s \)-injective and the dual of every simple left \( R \)-module is simple.

If \( aR \) is a nonsingular simple right ideal, then \( r(a) \cap eR = 0 \) for some simple right ideal \( eR \) with \( e^2 = e \). But \( R \) is \( C2 \), so \( eR \subseteq aR \) and \( aR \) is a summand. Thus \( R \) is right \( min-CS \), so by [3, Theorem 4.8] \( R \) is semiperfect with essential right socle. Hence \( R \) is right \( PF \) by 3).

The following is an example of a right perfect, left Kasch ring and right strongly \( s \)-injective which is not right self-injective ring.

**Example 3** Let \( K \) be a field and let \( R \) be the ring of all upper triangular, countably infinite square matrices over \( R \) with only finitely many off-diagonal entries. Let \( S \) be the \( K \)-subalgebra of \( R \) generated by 1 and \( J(R) \). Then \( S \) is a right perfect, left Kasch \( (S \) has only one simple left \( R \)-module \( M \) up to isomorphism. So \( M \cong S/J(S) \cong K \) such that \( (Z_1)_S = 0 \) whereas \( S \) is neither left perfect nor right self-injective because it is not right finite dimensional.

**Remark 1** Note that the ring of integers \( \mathbb{Z} \) is an example of a commutative noetherian strongly \( s \)-injective ring which is not quasi-Frobenius.

**Definition 3** A ring \( R \) is called right \( CF \) (FGF-ring) if every cyclic (finitely generated) right \( R \)-module embeds in a free module. It is not known whether right \( CF \)-rings (FGF-rings) are right artinian (quasi-Frobenius). In the next result we provide a positive answer if we assume in addition that the ring \( R \) is strongly right \( s \)-injective.

**Proposition 13** Every right \( CF \) right strongly \( s \)-injective ring is quasi-Frobenius.

*Proof.* Theorem 3 and [7, Theorem 5] \( \square \)

### 3. S-CS Modules and Rings

A module \( M \) is said to satisfy \( C1 \)-condition or called \( CS \) module if every submodule of \( M \) is essential in a direct summand of \( M \).

**Definition 4** A right \( R \)-module \( M \) is called \( s-CS \) module if every singular submodule of \( M \) is essential in a summand of \( M \).

For example, every nonsingular module is \( s-CS \). In particular, the ring of integers \( \mathbb{Z} \) is \( s-CS \) but not \( CS \).

**Proposition 14** For a right \( R \)-module \( M \), the following statements are equivalent:

1) The second singular submodule \( Z_2(M) \) is \( CS \) and a summand of \( M \).
2) \( M \) is \( s-CS \).

*Proof.* 1) \( \Rightarrow \) 2). If the second singular submodule \( Z_2(M) \) of \( M \) is \( CS \) and a summand of \( M \), then every singular submodule of \( M \) is a summand of \( Z_2(M) \) and a summand of \( M \).

2) \( \Rightarrow \) 1). Let \( M \) be \( s-CS \) and \( K \) is a submodule of \( Z_2(M) \). Then \( Z(K) \subseteq L \) where \( L \) is a summand of \( M \) and \( L \subseteq K + L \). But \( L \) is closed, so \( K \subseteq L \). Since \( Z_1(M) \subseteq L + Z_2(M) \) and \( Z_1(M) \) is closed in \( M \), so \( L \subseteq Z_2(M) \) and \( Z_2(M) \) is \( CS \). In particular, \( Z_2(M) \) is the only closure of \( Z_2(M) \). Thus \( Z_2(M) \) is a summand of \( M \). \( \square \)
A module is called s-continuous if it satisfies both the s-C1- and s-C2-conditions, and a module is called quasi-s-continuous if it satisfies the s-C1- and s-C3-conditions, and \( R \) is called a right s-continuous ring (right quasi-s-continuous ring) if \( R_g \) has the corresponding property. Clearly every strongly s-injective is s-continuous.

**Proposition 15** If every singular simple right \( R \)-module embeds in \( M \) and \( M \) is s-CS, then \( Z_2(M) \) is finitely cogenerated.

**Proof.** Let \( M \) be a s-CS and every singular simple right \( R \)-module embeds in \( M \). Then \( Z_2(M) \) is a CS and summand of \( M \) by above Proposition. Also \( Z_2(M) \) cogenerates every simple quotient of \( Z_2(M) \) then by [3, Theorem 7.29], \( Z_2(M) \) is finitely cogenerated.

**Proposition 16** Let \( R \) be a ring. Then \( R \) is a right PF-ring if and only if \( R_a \) is a cogenerator and \( Z_2^2(R) \) is CS.

**Proof.** Every right PF-ring is right self-injective and is a right cogenerator by [3, 1.56]. Conversely, if \( Z_2^2 \) is a CS and \( R \) is cogenerator then \( Z_2^2 \) has finitely generated, essential right socle by Proposition 15. Since \( Z_2^2 \) is right finite dimensional and \( R_a \) is a cogenerator, let \( \text{Soc}(Z_2^2) = S_1 \oplus S_2 \oplus \cdots \oplus S_n \) and \( I_i = I(S_i) \) be the injective hull of \( S_i \), then there exists an embedding \( \sigma: I_i \to R' \) for some set \( I \). Then \( \pi \circ \sigma \neq 0 \) for some projection \( \pi: R' \to R \), so \( (\pi \circ \sigma) | S_i \neq 0 \) and hence is monic. Thus \( \pi \circ \sigma: I_i \to R \) is monic, and so \( R = E_1 \oplus \cdots \oplus E_n \oplus T \) where \( T \) is nonsingular. So \( R \) is a right PF-ring by Theorem 3.

**Proposition 17** If every singular simple right \( R \)-module embeds in \( R \) and \( Z_2^2(R) \) is continuous, then \( R \) is semiperfect.

**Proof.** Let \( \{ Z_2^2(R) \} \) be continuous and every simple singular right \( R \)-module embeds in \( R \). Then \( \{ Z_2^2(R) \} \) has a finitely generated essential socle by Proposition 15. Thus, by hypothesis, there exist simple submodules \( S_1, \ldots, S_n \) of \( \{ Z_2^2(R) \} \) such that \( \{ S_1, \ldots, S_n \} \) is a complete set of representatives of the isomorphism classes of simple singular right \( R \)-modules. Since \( \{ Z_2^2(R) \} \) is CS, there exist submodules \( Q_i, \ldots, Q_n \) of \( \{ Z_2^2(R) \} \) such that \( Q_i \) is a direct summand of \( \{ Z_2^2(R) \} \) and \( \{ S_i \} \subseteq \{ Q_i \} \) for \( i = 1, \ldots, n \). Since \( Q_i \) is an indecomposable continuous \( R \)-module, it has a local endomorphism ring; and since \( Q_i \) is projective, \( J(Q_i) \) is maximal and small in \( Q_i \) by [3, 1.54]. Then \( Q_i \) is a projective cover of the simple module \( Q_i/J(Q_i) \). Note that \( Q_i \cong Q_i \) clearly implies \( Q_i/J(Q_i) \cong Q_i/J(Q_i) \) and the converse also holds because every module has at most one projective cover up to isomorphism. But it is clear that \( Q_i \cong Q_i \) if and only if \( S_i \cong S_j \) if and only if \( i = j \). Moreover, every \( Q_i/J(Q_i) \) is singular. Thus, \( \{ Q_i/J(Q_i), \ldots, Q_i/J(Q_i) \} \) is a complete set of representatives of the isomorphism classes of simple singular right \( R \)-modules. Hence every simple singular right \( R \)-module has a projective cover. Since every non-singular simple right \( R \)-module is projective, we conclude that \( R \) is semiperfect.

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