The Stefan-Boltzmann law: $SU(2)$ versus $SO(3)$ lattice gauge theory

Kurt Langfeld, Hugo Reinhardt
Institut für Theoretische Physik, Universität Tübingen, D–72076 Tübingen, Germany
(UNITU-THEP-17/99, November 30, 2021)

We investigate the high temperature limit of $SU(2)$ and $SO(3)$ lattice gauge theory, respectively. In particular, we study the Stefan-Boltzmann constant in both cases. As is well known, the Stefan-Boltzmann constant extracted from $SU(2)$ lattice gauge theory by incorporating finite size effects is smaller than the continuum value which assumes three gluon degrees of freedom. On the other hand, the extrapolation of our $SO(3)$ lattice data comes much closer to the continuum value. This rises the question whether $SU(2)$ and $SO(3)$ lattice gauge theories represent different quantum theories in the continuum limit.

Understanding the high temperature phase of Yang-Mills theory is essential for a wide span of physics, ranging from the evolution of the early universe and the description of compact stars. With the advent of large scale numerical simulations of lattice gauge theories, it became evident that $SU(2)$ and $SU(3)$ gauge theories undergo a phase transition at a critical temperature $T_c$ of a few hundred MeVs and that the high temperature phase is non-confining. The fact that the effective ("running") coupling constant becomes small at high energy scales in non-Abelian Yang-Mills theories indicates that the high temperature phase is described in terms of a gas of weakly interacting quarks and gluons forming a plasma. At temperatures well above the intrinsic energy scales, temperature is the only relevant scale. One therefore expects on general grounds that the vacuum energy density $\epsilon$ is related to the temperature $T$ by the Stefan-Boltzmann law

$$ \epsilon = \kappa T^4 , \quad (T \text{ large}) . \tag{1} $$

The Stefan-Boltzmann constant $\kappa$ only depends on the number of degrees of freedom constituting the high temperature phase. In the case of a pure $SU(N)$ continuum gauge theory, a gas of $N^2 - 1$ gluons would imply $\kappa = (N^2 - 1) \pi^2 / 15$.

The remarkable finding of recent investigations of $SU(2)$ $\text{and SU}(3)$ pure gauge theory is that at temperatures $T = 2 \ldots 3 T_c$ where the Stefan Boltzmann law is realized to good accuracy the ratio $\epsilon / T^4$ strongly underestimates the asymptotic value $\kappa$ corresponding to a plasma made out of gluons. In particular for the $SU(2)$ case, one finds at twice the critical temperature that the ratio $\epsilon / T^4$ only reaches 70% of the asymptotic gluon plasma value.

A possible explanation of this discrepancy comes to mind: the continuum limit of $SU(2)$ lattice theory is not the same as the usual continuum Yang-Mills theory defined in terms of the gauge connection. In fact, lattice gauge theory is formulated in terms of link variables living in the gauge group, while the gauge potential of the continuum theory is defined in the algebra, which is the same for the $SU(2)$ and the $SO(3)$ group, respectively. Since furthermore the $SU(2)$ and $SO(3)$ lattice actions both reproduce the continuum action for zero lattice spacing, one would therefore expect that $SU(2)$ and $SO(3)$ lattice theory approach the same fix-point in the continuum limit. However, to our knowledge there is no rigorous proof that this is indeed the case. By contrast, since $SU(2) \simeq SO(3) \times Z_2$, in addition to the $SO(3)$ degrees of freedom, the $SU(2)$ lattice theory contains $Z_2$ center degrees of freedom which, in principle, could survive the continuum limit and hence contribute to the Stefan-Boltzmann constant. The fact that the discrete degrees of freedom of a $Z_2$ theory can contribute to physical quantities is observed in the so-called Maximal Center Gauge: the effective $Z_2$ gauge theory is determined from the full $SU(2)$ gauge theory by center projection and is formulated in terms of vortices. It was observed that these vortices survive the continuum limit and are relevant infrared degrees of freedom. In fact, if these vortices are eliminated by hand, quark confinement and spontaneous breaking of chiral symmetry are lost.

In this letter, we study the Stefan-Boltzmann constant in $SU(2)$ and $SO(3)$ lattice theories. The Stefan-Boltzmann constant measures the number of degrees of freedom forming the heat bath. Since the continuum extrapolation of $SO(3)$ lattice gauge theory can be formulated employing three gluon fields as in the case of continuum Yang-Mills theory, we expect that their Stefan-Boltzmann constant match, while a deviation from the continuum value should occur for the $SU(2)$ case if center degrees of freedom survive at the continuum fixed point.

The degrees of freedom of $SU(2)$ lattice gauge theory are defined by the link variables $U_{x}(\mu) = Z_{\mu}(x) O_{\mu}(x)$ while the link variables of $SO(3)$ lattice gauge theory, $O_{\mu}(x)$, can be constructed from the link variables $U_{x}(\mu)$ by enforcing the constraint $Z_{\mu}(x) = 1$, i.e., for the $SO(3)$ case the link variables $U_{x}(\mu)$ are restricted to $\text{tr} U_{x}(\mu) > 0$. The actions for $SU(2)$ and $SO(3)$ gauge theories are given in terms of the plaquette variables.
where

\[ P^F_{\mu\nu}(x) := \frac{1}{2} \text{tr} \left[ U_\mu(x) U_\nu(x) U_\nu^\dagger(x + \mu) U_\mu^\dagger(x + \nu) \right] \]

\[ P^A_{\mu\nu}(x) = \frac{4}{3} \left( P^F_{\mu\nu}(x) \right)^2 \]

The SU(2) action is the standard Wilson action while the SO(3) action is a special case of the Bhanot-Creutz action \[ \text{f} \]. Either gauge theory is defined by its partition function

\[ Z_{su2/sol} = \int DU_\mu(x) \exp \left\{ S_{su2/sol} \right\} \]

Finite temperature simulations can be performed by using asymmetric lattices with \( N_\tau \) and \( N_\sigma \) lattice points in time and spatial directions, respectively. The actual temperature is given by \( T = 1/N_\tau a(\beta) \) where \( a(\beta) \) is the lattice spacing. In order to retain the Casimir effect from distorting the energy density, a sufficiently large ratio \( N_\sigma/N_\tau \) must be chosen. It was found in \[ 3 \] that \( N_\sigma/N_\tau = 4 \) already yields reasonable results for \( N_\tau \geq 4 \) and \( \beta \leq 2.8 \).

\[ \epsilon^{su2/sol} = 3 \beta F/A N^2_\tau \left\{ \left[ 1 - \frac{f^{F/A}(\beta F/A)}{\beta F/A} \right] \left( P^F_\tau - P^A_\sigma \right) - \beta F/A f^{F/A}_2(\beta F/A) \left[ 2 P^F_0 - \left( P^F_\tau + P^A_\sigma \right) \right] \right\} , \]

where \( P_\tau \) and \( P_\sigma \) denote the expectation values of temporal and spatial plaquettes (in the asymmetric lattice) and \( P^{F/A}_0 \) is the plaquette expectation value for the symmetric lattice \( N_\tau = N_\sigma \). The important observation is that the functions \( f^{F/A}(\beta F/A) \) approach finite values in the continuum limit \( \beta \rightarrow \infty \). In particular, the function \( f^F(\beta F) \) can be deduced for several \( \beta F \) values from data reported in \[ 3 \]. The function \( F^A(\beta A) \) can be calculated for \( \beta A \gg 1 \) by expanding the link variables \( U_\mu(x) = \exp \{ i A_\mu(x) a \} \) in powers of the lattice spacing \( a \) around the unit element. The result of this calculation can be also found in \[ 3 \] and is referred to as the "weak coupling regime" of SU(2) gauge theory. Note, however, that this calculation, which relies on the expansion of the link variables near \( U_\mu = 1 \), is only justified for the SO(3) case where the link variables are sufficiently close to the unit element for large \( \beta A \). This is because in the SU(2) case, this calculation does not properly take into account the non-trivial center elements \( Z_\mu(x) \) which we consider the progenitor of vortices (see discussion below). In the continuum limit, one finally obtains

\[ \epsilon^{su2/sol} = E^{su2/sol} (\beta F/A \rightarrow \infty) \]

\[ E^{su2/sol}/T^4 = 3 \beta F/A N^2_\tau \left( P^F_\tau - P^A_\sigma \right) . \]
In fact, one observes that the term in (3) proportional to $f_2(\beta)$ exponentially decreases with increasing $\beta^F$ in the $SU(2)$ case and is for $\beta^F > 2.7$ orders of magnitude smaller than the dominant term proportional to $P_{\tau} - P_{\sigma}$. However, one observes significant corrections to (3) from the term proportional to $f_1^F(\beta_F)$ if $\beta_F \in [2.5, 3]$. Thus a suitable approximation to the Stefan-Boltzmann constant is

$$
\kappa := \frac{e^{s n_2/s o_3}}{T^4} \approx 3 \beta^F/A N^2_\tau \left[ 1 - \frac{f_1^F(\beta_F)}{\beta^F} \right] \left( P_{\tau} - P_{\sigma} \right).
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Our numerical data for the $SU(2)$ case were obtained using the standard algorithm proposed by Creutz 11, while a novel heat bath algorithm was used for the study of $SO(3)$ gauge theory 12. Figures 1 and 2 show the raw data $E/T^4$ as function of $\beta^F/A$. One observes a clear signal of the deconfinement phase transition in the $SU(2)$ case. Since the $SO(3)$ lattice gauge theory possesses an un-physical phase transition at $\beta^A = 2.5$ (independent of the lattice size) 13, only data corresponding to the physical regime $\beta^A > 2.5$ are shown. In either case, a plateau value seems to be reached for $\beta^A/F > 2.7$.

For an investigation of the Stefan-Boltzmann constant in either field theory (being defined as the continuum limit of the lattice formulation), a thorough study of the limit $N_\tau \to \infty$ is requested. Assuming eq.(6) and that the plateau value is reached for $\beta^A/F = 2.8$, we studied the $N_\tau$ dependence of $\kappa$ for fixed ratio $N_\sigma/N_\tau = 4$. We estimated $f_1^F(\beta_F)/\beta^F = 0.2$ at $\beta^F = 2.8$ with the help of the data tabulated in 6. Since a detailed study of the non-perturbative $\beta$-function is not available for the $SO(3)$ case so far, we assume that the finite $\beta^A$ correction to the continuum result is of the same order of magnitude (as suggested by lattice perturbation theory) and approximate $f_1^A(2.8) \approx f_1^F(2.8)$. Our numerical data are presented in figure 3. We note that this approximation can lead to an absolute error of 10% for the absolute value of $\kappa$. Also shown is one data point for the $SU(2)$ case at $N_\tau = 8$ which is constructed with the help of the tabulated values in 6. For guiding the eye, we have fitted the data points to the ansatz

$$
\kappa = \kappa_\infty + \frac{c_1}{N_\tau^2} + \frac{c_2}{N_\tau^4},
$$

which was investigated in 6. Note that $\kappa_{SO(3)}/\kappa_{SU(2)}$ is insensitive to the absolute values of functions $f_1^A(\beta^A)$ and $f_1^F(\beta^F)$ as long as $f_1^A(\beta^A) \approx f_1^F(\beta^F)$. Our result for this ratio is tabulated in table I.

Our results indicate that the Stefan-Boltzmann constant which emerges from the continuum extrapolation is larger in the $SO(3)$ than in the $SU(2)$ case. The explanation at hand is that in the $SU(2)$ case certain correlations survive even in the high temperature phase and prevent gluonic degrees of freedom from contributing to the Stefan-Boltzmann constant. In fact, lattice calculations performed in the Maximum Center Gauge 5 show that center vortices percolate in the confined phase implying strong gluonic correlations. As a result, the energy density vanishes in this regime. Furthermore, vortex dominance for the string tension is not only observed in the confinement regime, but also above the deconfinement phase transition for the spatial string tension 5. In the deconfined phase, vortices partially align along the time axis but are still percolating in the 3-dimensional spatial universe resulting in a spatial string tension which is even larger than the string tension at zero temperature. This vortex scenario is compatible with dimensional reduction 13 which support strong correlations in $SU(2)$ lattice gauge theory even in the high temperature limit, thus effectively reducing the number of degrees of freedom participating in the gluonic heat bath.

**FIG. 3.** The Stefan-Boltzmann constant $\kappa$ as function of $1/N_\tau^2$ for fixed ration $N_\sigma/N_\tau = 4$.

| $N_\tau \times N_\sigma^2$ | $3 \times 10^2$ | $4 \times 10^3$ | $5 \times 10^3$ | $6 \times 24^4$ |
|--------------------------|-----------------|-----------------|-----------------|-----------------|
| $\kappa_{SO(3)}/\kappa_{SU(2)}$ | $1.26 \pm 0.03$ | $1.24 \pm 0.07$ | $1.27 \pm 0.08$ | $1.21 \pm 0.1$ |

**TABLE I.** The ratio of the estimates for the Stefan-Boltzmann constants of $SO(3)$ and $SU(2)$ gauge theory.
In conclusions, we have studied for the first time the Stefan-Boltzmann constant of $SO(3)$ gauge theory by an extrapolation of lattice Monte-Carlo data to the continuum and infinite volume limit. We find preliminary evidence that this constant is about 20% larger than the corresponding constant of $SU(2)$ gauge theory. Given the fact that the Stefan-Boltzmann constant which arises from the continuum extrapolation of $SU(2)$ lattice gauge theory is roughly 30% smaller than the expectation provided by three gluon degree's of freedom [3], our results indicate that the Stefan-Boltzmann constant of $SO(3)$ gauge theory comes closer to the continuum expectation than the $SU(2)$ one. In our opinion, a large scale numerical analysis (comparable with [3]) of the $SO(3)$ theory is highly desirable for a more detailed study of the important question whether $SO(3)$ and $SU(2)$ lattice gauge theories give rise to different continuum field theories.

Acknowledgements: Helpful discussions with M. Ilgenfritz are greatly acknowledged. We thank M. Engelhardt and R. Alkofer for comments on the manuscript. This work is supported in part by Deutsche Forschungsgemeinschaft under contract DFG-Re 856/4-1.

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