The Daugavet equation for operators not fixing a copy of $C(S)$

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Abstract. We prove the norm identity $\|Id + T\| = 1 + \|T\|$, which is known as the Daugavet equation, for operators $T$ on $C(S)$ not fixing a copy of $C(S)$, where $S$ is a compact metric space without isolated points.

1. Introduction

An operator $T: X \to X$ on a Banach space is said to satisfy the Daugavet equation if

$$\|Id + T\| = 1 + \|T\|;$$

(1.1)

this terminology is derived from Daugavet’s theorem that a compact operator on $C[0,1]$ satisfies (1.1). Many authors have established the Daugavet equation for various classes of operators, e.g., the weakly compact ones, on various spaces; we refer to [1], [3], [4], [7], [10], [11], [13] and the references in these papers for more information.

The most far-reaching result for operators on $L^1[0,1]$ is due to Plichko and Popov [7, Th. 9.2 and Th. 9.8] who prove that an operator on $L^1[0,1]$ which does not fix a copy of $L^1[0,1]$ satisfies the Daugavet equation. (As usual, $T: X \to X$ fixes a copy of a Banach space $E$ if there is a subspace $F \subset X$ isomorphic to $E$ such that $T|_F$ is an (into-) isomorphism.) In this paper we shall establish the corresponding result for operators on spaces of continuous functions.

Theorem 1.1 Let $(S,d)$ be a compact metric space without isolated points. If $T: C(S) \to C(S)$ does not fix a copy of $C(S)$, then $T$ satisfies the Daugavet equation.
In the next section we shall provide a direct proof of this theorem which is measure theoretic in spirit, and in Section 3 we shall give another argument which relies on a deep result due to Rosenthal [8]. Though this proof is much shorter it is less revealing than the one in Section 2; therefore we found it worthwhile to present both arguments.

Either of our approaches depends on the analysis of the representing kernel of an operator $T$: $C(S) \to C(S)$; this is the family of Borel measures on $S$ defined by $\mu_s = T^* \delta_s$, $s \in S$. In [10] (see also [11]) a necessary and sufficient condition on the kernel of $T$ was established in order that $T$ satisfy the Daugavet equation. Here we record a special case which suffices for our needs.

**Lemma 1.2** If the representing kernel $(\mu_s)_{s \in S}$ of a bounded linear operator $T$: $C(S) \to C(S)$ satisfies

$$\inf_{s \in U} |\mu_s(\{s\})| = 0$$

(1.2)

for all nonvoid open subsets $U$ of $S$, then $T$ satisfies the Daugavet equation.

Our results are valid for real and complex spaces; we denote the scalar field by $\mathbb{K}$.

2. Proof of Theorem 1.1

We fix a diffuse probability measure $\lambda$ on $S$ whose support is $S$. To show the existence of such a measure, consider a countable basis of open sets $O_n \subset S$. Since each $O_n$ is perfect, we know from a theorem of Bessaga and Pełczyński (see [6, p. 52]) that there are diffuse probability measures $\lambda_n$ with $\text{supp} \lambda_n \subset O_n$; it remains to define $\lambda = \sum 2^{-n} \lambda_n$.

Now let $T: C(S) \to C(S)$ be a bounded linear operator and $(\mu_s)_{s \in S}$ its representing kernel. It is enough to show that $T$ fixes a copy of $C(S)$ if (1.2) of Lemma 1.2 fails for some open set $U$. Thus we assume, for some open $U \subset S$ and $\alpha > 0$, that

$$|\mu_s(\{s\})| \geq \alpha \quad \forall s \in U.$$  

(2.1)

We shall also assume without loss of generality that $\|T\| = 1$.

We shall decompose the $\mu_s$ into their atomic and diffuse parts. It is instrumental for our argument that this can be accomplished in a measurable
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fashion, as was proved by Kalton \[5\], Th. 2.10. More precisely we can write

$$\mu_s = \sum_{n=1}^{\infty} a_n(s) \delta_{\sigma_n(s)} + \nu_s$$

where

- each $a_n: S \to \mathbb{K}$ is measurable for the completion $\Sigma_\lambda$ of the Borel sets of $S$ with respect to $\lambda$,
- each $\sigma_n: S \to S$ is $\Sigma_\lambda$-Borel-measurable,
- each $\nu_s$ is diffuse, and $s \mapsto |\nu_s|$ is $\Sigma_\lambda$-Borel-measurable (we are considering the weak* Borel sets of the unit ball of $C(S)^*$),
- $|a_1(s)| \geq |a_2(s)| \geq \ldots$ for all $s \in S$,
- $\sigma_n(s) \neq \sigma_m(s)$ for all $s \in S$ whenever $n \neq m$,
- $\sum_{n=1}^{\infty} |a_n(s)| \leq 1$ for all $s \in S$.

Put $\beta = \lambda(U) > 0$. Applying the Egorov and Lusin theorems, we may find a compact subset $S_1 \subset S$ with $\lambda(S_1) \geq 1 - \beta/2$ such that

- each $a_n|_{S_1}$ is continuous on $S_1$,
- each $\sigma_n|_{S_1}$ is continuous on $S_1$,
- $\sum |a_n(s)|$ converges uniformly on $S_1$,
- $\lim_{n \to \infty} \sup_{s \in S_1} |\nu_s|(B(t, 1/n)) \to 0$ for all $t \in S$;

(2.2)

here $B(t, r)$ denotes the closed ball with centre $t$ and radius $r$. To prove (2.2) fix a countable dense subset $Q$ of $S$. Let $U_n$ be the countable set of all finite covers of $S$ by means of open balls of radius $1/n$ with centres in $Q$. Then

$$\varphi_n(s) = \inf_{C \in U_n} \sup_{B \in \mathcal{C}} |\nu_s|(B)$$

defines a sequence of $\Sigma_\lambda$-measurable functions, and we have $\varphi_n \to 0$ point-wise, since the $\nu_s$ are diffuse and $S$ is compact. By Egorov’s theorem ($\varphi_n$) tends to 0 uniformly on a large subset of $S$. This easily implies our claim.

After discarding relatively isolated points of $S_1$, if necessary, we come up with a perfect subset $S_1$ with the above properties; note that $U \cap S_1 \neq \emptyset$ since this intersection has positive $\lambda$-measure. Next we pick $N \in \mathbb{N}$ such that

$$\sum_{n>N} |a_n(s)| \leq \alpha/3 \quad \forall s \in S_1.$$  

(2.3)
Let \( s \in U \cap S_1 \). Then
\[
\alpha \leq |\mu_{s}(\{s\})| = \left| \sum_{n=1}^{\infty} a_n(s)\delta_{\sigma_n(s)}(\{s\}) \right|.
\]

Consequently \( \sigma_k(s) = s \) for some \( k \); in addition, we must have
\[
|a_k(s)| \geq \alpha \tag{2.4}
\]
for this \( k \) so that \( k \leq N \). This shows that
\[
U \cap S_1 = \bigcup_{l=1}^{N} \{ s \in U \cap S_1 : \sigma_l(s) = s \}.
\]

Each of these \( N \) sets is relatively closed in \( U \cap S_1 \), so one of them, with index \( k \) say, contains a relatively interior point \( s_0 \). Therefore, there exists some \( \delta > 0 \) with the following properties:

For \( B_l = \sigma_l(S_1 \cap B(s_0, \delta)) \), \( l = 1, \ldots, N \), we have

(a) \( B_k = S_1 \cap B(s_0, \delta) \subset S_1 \cap U \), and \( \sigma_k(s) = s \) for all \( s \in B_k \),

(b) \( B_l \cap B(s_0, \delta) = \emptyset \) for \( l \neq k \) (since \( s_0 = \sigma_k(s_0) \neq \sigma_l(s_0) \) and these functions are continuous on \( S_1 \)),

(c) \( |\nu_s|(B(s_0, \delta)) \leq \alpha/3 \) for all \( s \in S_1 \) (see (2.2)).

Now let \( V = S_1 \cap B(s_0, \delta/2) \). This is a compact perfect subset of \( S \) and thus \( C(V) \) is isomorphic to \( C(S) \) by Milutin’s theorem \[12\], Th. III.D.19]. Also let \( W = \{ s \in S : d(s, s_0) \geq \delta \} \). Again, this is a compact subset of \( S \), \( V \cap W = \emptyset \), and we have from (a) and (b) above

(a’) \( \sigma_k(s) = s \) for all \( s \in V \),

(b’) \( B_l \subset W \) for all \( l = 1, \ldots, N, l \neq k \).

By the Borsuk-Dugundji theorem (see e.g. \[12\], Cor. III.D.17]) there exists an isometric linear extension operator \( L : C(V \cup W) \to C(S) \). Let \( E = \{ f \in C(V \cup W) : f|_W = 0 \} \) and \( F = L(E) \); thus \( F \cong E \cong C(V) \), and \( F \) is isomorphic to \( C(S) \).

We finally prove that \( T|_F \) is an isomorphism. In fact, let us show that
\[
\|Tf\| \geq \frac{\alpha}{3} \|f\| \quad \forall f \in F.
\]
Suppose \( f \in F \) with \( \|f\| = 1 \). Then \( \|f|_V\| = 1 \) and \( f|_W = 0 \). Pick \( s \in V \) with \( |f(s)| = 1 \). Then we have (recall that \( V \subset S_1 \cap U \))

\[
\|Tf\| \geq |Tf(s)| = \left| \sum_{n=1}^{\infty} a_n(s)f(\sigma_n(s)) + \int_S f \, d\nu_s \right|
\]

\[
\geq |a_k(s)| - \sum_{n>N} |a_n(s)| - \int_S |f| \, d\nu_s |
\]

(by (a’) and (b’))

\[
\geq \alpha - \frac{\alpha}{3} - |\nu_s|(B(s_0, \delta))
\]

(by (2.4), (2.3) and since \( f|_W = 0 \))

\[
\geq \frac{\alpha}{3} \quad \text{(by (c)).}
\]

This completes the proof of the theorem. \( \square \)

3. Operators whose adjoints have separable ranges

In this section we discuss another, formally shorter, approach to Theorem 1.1 via the following proposition.

**Proposition 3.1** Suppose \( S \) is a compact Hausdorff space without isolated points. If \( T: C(S) \to C(S) \) is a bounded linear operator such that \( \text{ran}(T^*) \) is separable, then \( T \) satisfies the Daugavet equation.

**Proof.** Let \( (\mu_s)_{s \in S} \) be the representing kernel of \( T \), and let \( \{\mu_{s_n} : n \in \mathbb{N}\} \) be dense in \( \{\mu_s : s \in S\} \). We define \( \mu = \sum 2^{-n} |\mu_{s_n}| \); then all the \( \mu_s \) are absolutely continuous with respect to \( \mu \). Hence \( \mu_s(\{s\}) \neq 0 \) only if \( |\mu| (\{s\}) \neq 0 \). Therefore, \( \{s \in S : \mu_s(\{s\}) \neq 0\} \) is countable, and, since \( S \) is perfect, its complement is dense by Baire’s theorem. This implies that the condition of Lemma 1.2 is satisfied, and the proposition is proved. \( \square \)

Proposition 3.1 obviously covers the case of compact operators, operators factoring through a space with a separable dual, and, in the case of metric \( S \), weakly compact operators; for the latter observe that \( T^{**}(C(S)**) \subset C(S) \) so that \( T^{**} \) and, consequently, \( T^* \) have separable ranges. But even for non-metrizable perfect \( S \) the Daugavet equation for weakly compact operators on \( C(S) \) can be derived from the argument of Proposition 3.1; one only has
to recall that by a theorem of Bartle, Dunford and Schwartz [3, p. 306] also in this case all the \( \mu_s \) are absolutely continuous with respect to some finite measure \( \mu \). This seems to yield the easiest proof of the Daugavet equation for these classes of operators.

If \( S \) is a metric space then Proposition 3.1 clearly follows from Theorem 1.1. On the other hand, Rosenthal [8] (see also [9]) has shown that an operator \( T : C(S) \rightarrow C(S), S \) an uncountable compact metric space, with \( \text{ran}(T^*) \) nonseparable fixes a copy of \( C(S) \). Therefore Proposition 3.1 implies Theorem 1.1 so that both these results are in fact equivalent. Moreover we note that in this setting \( T \) does not fix a copy of \( C(S) \) if and only if \( T \) does not fix a copy of \( \ell^1 \) if and only if \( T \) is weakly sequentially precompact, i.e., if \( (f_n) \) is a bounded sequence in \( C(S) \), then \( (Tf_n) \) admits a weak Cauchy subsequence.

We finally extend Proposition 3.1 to the class of nicely embedded Banach spaces introduced in [11] so that we obtain a unified approach to several results of that paper.

Let \( S \) be a Hausdorff topological space, and let \( C^b(S) \) be the sup-normed Banach space of all bounded continuous scalar-valued functions. The functional \( f \mapsto f(s) \) on \( C^b(S) \) is denoted by \( \delta_s \). We say that a linear map \( J : X \rightarrow C^b(S) \) on a Banach space \( X \) is a nice embedding and that \( X \) is nicely embedded into \( C^b(S) \) if \( J \) is an isometry such that for all \( s \in S \) the following properties hold:

\( (\text{N1}) \) For \( p_s := J^*(\delta_s) \in X^* \) we have \( \|p_s\| = 1 \).

\( (\text{N2}) \) \( \text{lin}\{p_s\} \) is an \( L \)-summand in \( X^* \).

The latter condition means that there are projections \( \Pi_s \) from \( X^* \) onto \( \text{lin}\{p_s\} \) such that

\[
\|x^*\| = \|\Pi_s(x^*)\| + \|x^* - \Pi_s(x^*)\| \quad \forall x^* \in X^*.
\]

We will also need the equivalence relation

\[
s \sim t \text{ if and only if } \Pi_s = \Pi_t
\]

on \( S \). Then \( s \) and \( t \) are equivalent if and only if \( p_s \) and \( p_t \) are linearly dependent, which implies by (N1) that \( p_t = \lambda p_s \) for some scalar of modulus 1. The equivalence classes of this relation are obviously closed.

We will consider the following nondiscreteness condition.

\( (\text{N3}) \) None of the equivalence classes \( Q_s = \{t \in S: s \sim t\} \) contains an interior point.
If the set \( \{ p_s; s \in S \} \) is linearly independent, this simply means:

(N3') \( S \) does not contain an isolated point.

By (N2), the \( p_s \) are linearly independent as soon as they are pairwise linearly independent.

**Proposition 3.2** Let \( S \) be a Baire topological Hausdorff space and suppose that \( X \) is nicely embedded into \( C^b(S) \) so that additionally (N3) holds. If \( T: X \to X \) is a bounded linear operator with \( \text{ran}(T^*) \) separable, then \( T \) satisfies the Daugavet equation.

**Proof.** We stick to the above notation and put \( q_s = T^*(p_s) \). By [11, Prop. 2.1] it is sufficient to prove that

\[ S' = \{ t \in S: \Pi_t(q_s) = 0 \ \forall s \} \]

is dense in \( S \). If \( \{ q_{s_n}; n \in \mathbb{N} \} \) denotes a dense subset of \( \{ q_s; s \in S \} \), then \( t \notin S' \) if and only if \( \Pi_t(q_{s_n}) \neq 0 \) for some \( n \). This implies that \( S \setminus S' \) consists of at most countably many equivalence classes for the equivalence relation \( \sim \) (cf. [11, Lemma 2.3]), and by (N3) and Baire’s theorem \( S' \) must be dense.

\[ \Box \]

It is proved in [11] that the following classes of Banach spaces satisfy the assumptions of Proposition 3.2:

- \( X \) is a function algebra whose Choquet boundary does not contain isolated points,
- \( X \) is a real \( L^1 \)-predual space so that the set ex \( B_{X^*} \) of extreme functionals does not contain isolated points (for the weak* topology),
- \( X \) is a complex \( L^1 \)-predual space for which the quotient space ex \( B_{X^*}/\sim \) does not contain isolated points, where \( \sim \) means linear dependence,
- \( X \) is a space of type \( C_\Lambda \) for certain subsets \( \Lambda \) of abelian discrete groups.

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