ALEKSANDROV–FENCHEL INEQUALITY FOR COCONVEX BODIES

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Abstract. We prove a version of the Aleksandrov–Fenchel inequality for mixed volumes of coconvex bodies. This version is motivated by an inequality from commutative algebra relating intersection multiplicities of ideals.

1. Introduction

We start by recalling the classical Aleksandrov–Fenchel inequality. The Minkowski sum of two convex sets $A, B \subset \mathbb{R}^d$ is defined as $A + B = \{a + b \mid a \in A, b \in B\}$. For a positive real number $\lambda$, we let $\lambda A$ denote the set $\{\lambda a \mid a \in A\}$. By definition, a convex body is a compact convex set, whose interior is nonempty. A linear family of convex bodies is a collection of the following objects: a real vector space $V$, an open subset $\Omega \subset V$, a map $f$ from $\Omega$ to the set of all convex bodies in $\mathbb{R}^d$ such that $f(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 f(v_1) + \cdots + \lambda_n f(v_n)$ whenever all $v_i \in \Omega$, all $\lambda_i$ are positive, and $\lambda_1 v_1 + \cdots + \lambda_n v_n \in \Omega$. A linear family of convex bodies with $m$ marked points is a linear family $(V, \Omega, f)$ of convex bodies, in which some $m$ elements of $\Omega$ are marked.

With every linear family $\alpha = (V, \Omega, f)$ of convex bodies, we associate the volume polynomial $\text{Vol}_\alpha$ as follows. For $v \in \Omega$, we define $\text{Vol}_\alpha(v)$ as the usual $d$-dimensional volume of the convex body $f(v)$. It is well known that the function $\text{Vol}_\alpha$ thus defined extends to a unique polynomial on $V$ that is homogeneous of degree $d$. For $v \in V$, we let $L_v$ denote the usual directional (Lie) derivative along $v$. Thus $L_v$ is a differential operator that acts on functions, in particular, degree $k$ polynomials on $V$ are mapped by this operator to degree $k - 1$ polynomials. If $\alpha = (V, \Omega, f)$ is a linear family of convex bodies with $(d - 2)$ marked points $v_1, \ldots, v_{d-2} \in \Omega$, then we define the Aleksandrov–Fenchel symmetric bilinear form $B_\alpha$ on $V$ by the formula

$$B_\alpha(u_1, u_2) = \frac{1}{d!} L_{u_1} L_{u_2} L_{v_1} \cdots L_{v_{d-2}} (\text{Vol}_\alpha).$$

Note that the expression in the right-hand side is a real number (called the mixed volume of the convex bodies $f(u_1), f(u_2), f(v_1), \ldots, f(v_{d-2})$). Indeed, this is the image of a homogeneous degree $d$ differential operator with constant coefficients on
a homogeneous degree $d$ polynomial. The corresponding quadratic form $Q_\alpha(u) = B_\alpha(u, u)$ is given by the formula $Q_\alpha = \frac{2}{d!}L_{v_1} \ldots L_{v_{d-2}}(\text{Vol}_\alpha)$. The expression in the right-hand side is the result of the action of a degree $d-2$ homogeneous differential operator with constant coefficients on a homogeneous degree $d$ polynomial, i.e. a quadratic form. The Aleksandrov–Fenchel inequality [A] states that, for all $u_1 \in \mathcal{V}$ and $u_2 \in \Omega$, we have

$$B_\alpha(u_1, u_2)^2 \geq B_\alpha(u_1, u_1)B_\alpha(u_2, u_2).$$

The Aleksandrov–Fenchel inequality is a far-reaching generalization of the classical isoperimetric inequality. See [Mc, VT] for generalizations of the Aleksandrov–Fenchel inequality for convex polytopes.

Let $C \subset \mathbb{R}^d$ be a strictly convex cone with the apex at 0 and a nonempty interior. A subset $A \subset C$ is called $(C)$-coconvex if the complement $A \setminus C$ is convex. Define a $(C)$-coconvex body as a compact coconvex subset of $C$ with a nonempty interior. If $A$ and $B$ are coconvex sets with respect to the same cone $C$, then we can define $A \oplus B$ as the complement in $C$ of the Minkowski sum of $A \setminus C$ and $C \setminus B$. Define a linear family of $C$-coconvex bodies as a collection of the following objects: a vector space $\mathcal{V}$, an open subset $\Omega \subset \mathcal{V}$, a map $g$ from $\Omega$ to the set of all $C$-coconvex bodies such that

$$g(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 g(v_1) \oplus \cdots \oplus \lambda_n g(v_n)$$

whenever all $v_i \in \Omega$, all $\lambda_i$ are positive, and $\lambda_1 v_1 + \cdots + \lambda_n v_n \in \Omega$. A linear family of $C$-coconvex bodies with $m$ marked points is a linear family $(\mathcal{V}, \Omega, g)$ of $C$-coconvex bodies, in which some $m$ elements of $\Omega$ are marked. With every linear family $\beta$ of $C$-coconvex bodies, we associate the volume function $\text{Vol}_\beta$ in the same way as with a linear family of convex bodies. The function $\text{Vol}_\beta$ thus defined is also a homogeneous degree $d$ polynomial (we will prove this below). Given a linear family $\beta$ of $C$-coconvex bodies with $d-2$ marked points $v_1, \ldots, v_{d-2}$, we define the coconvex Aleksandrov–Fenchel symmetric bilinear form as

$$B_\beta^C(u_1, u_2) = \frac{1}{d!}L_{u_1}L_{u_2}L_{v_1} \ldots L_{v_{d-2}}(\text{Vol}_\beta).$$

We will also consider the corresponding quadratic form $Q_\beta^C = \frac{2}{d!}L_{v_1} \ldots L_{v_{d-2}}(\text{Vol}_\beta)$. Our main result is the following

**Main Theorem.** The form $Q_\beta^C$ is non-negative, i.e. $Q_\beta^C(u) \geq 0$ for all $u \in \mathcal{V}$. In particular, the corresponding symmetric bilinear form satisfies the Cauchy–Schwartz inequality

$$B_\beta^C(u_1, u_2)^2 \leq B_\beta^C(u_1, u_1)B_\beta^C(u_2, u_2).$$

The inequality stated in the Main Theorem is called the coconvex Aleksandrov–Fenchel inequality. In recent paper [Fi], Theorem [I] is proved under the assumption that $C$ is a fundamental cone of some Fuchsian group $\Gamma$ acting by linear isometries of a pseudo-Euclidean metric, and $C \setminus A_k$ is the intersection of some convex $\Gamma$-invariant set with $C$. 


Theorem 1 is motivated by an Aleksandrov–Fenchel type inequality for (mixed) intersection multiplicities of ideals [KK]. There was an earlier preprint of ours, in which more sophisticated tools were used for proving Theorem 1. We plan a sequel to this paper that will deal with representations of coconvex bodies as virtual convex bodies in the sense of [KP].

The following inequalities follow from Theorem 1 in the same way as similar inequalities for convex bodies follow from the classical Alexandrov–Fenchel inequality (cf. [Fi]):

**Reversed Brunn–Minkowski inequality:** the function \( \text{Vol}^{\frac{1}{d}} \) is convex, i.e.

\[
(\text{Vol}_\beta(tu + (1-t)v))^\frac{1}{d} \leq t\text{Vol}_\beta(u)^\frac{1}{d} + (1-t)\text{Vol}_\beta(v)^\frac{1}{d}, \quad t \in [0, 1].
\]

**Generalized reversed Brunn–Minkowski inequality:** the function \( (L_{v_1} \ldots L_{v_k} \text{Vol}_\beta)^\frac{1}{d-k} \) is convex.

**First reversed Minkowski inequality:**

\[
\left( \frac{1}{d!} L_u L_{v_1}^{d-1} (\text{Vol}_\beta) \right)^d \leq \text{Vol}_\beta(u) \text{Vol}_\beta(v)^{d-1}.
\]

**Second reversed Minkowski inequality:** if all marked points coincide with \( u \), then

\[
B_C^C(u, v)^2 \leq \text{Vol}_\beta(u) B_C^C(v, v)
\]

2. **Proof of the Main Theorem**

Recall that every quadratic form \( Q \) on a finite dimensional real vector space can be represented in the form

\[
x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{k+\ell}^2
\]

for a suitable linear coordinate system \( (x_1, \ldots, x_m) \), \( m \geq k + \ell \). The pair \( (k, \ell) \) is called the *signature* of \( Q \). It is well known that \( Q \) has signature \( (1, \ell) \) for some \( \ell \) if and only if

1. there exists a vector \( v_0 \in V \) with \( Q(v_0) > 0 \);
2. the corresponding symmetric bilinear form \( B \) (such that \( B(u, u) = Q(u) \)) satisfies the reversed Cauchy–Schwartz inequality: \( B(u, v)^2 \geq Q(u) Q(v) \) for all \( u \in V \) and all \( v \in V \) such that \( Q(v) > 0 \).

Thus, the Aleksandrov–Fenchel inequality is equivalent to the fact that \( Q_\alpha \) has signature \( (1, \ell(\alpha)) \) for every finite-dimensional linear family \( \alpha \) of convex \( d \)-dimensional bodies with \( d - 2 \) marked points.

Fix a strictly convex cone \( C \) with the apex at 0 and a \( C \)-linear family \( \beta = (V, \Omega, g) \) of coconvex bodies. We may assume that \( g(\Omega) \) is bounded in the sense that there is a large ball in \( \mathbb{R}^d \) centered at 0 that contains all coconvex bodies \( g(v), v \in \Omega \). Then there is a linear functional \( \xi \) on \( \mathbb{R}^d \) and a number \( t_0 > 0 \) such that \( g(v) \) lies in the half-space

\[
W(t_0) = \{ x \in \mathbb{R}^d \mid \xi(x) \leq t_0 \},
\]
for every $v \in \Omega$. Choose any $t_1 > t_0$. We will now define a linear family $\alpha = (V \times \mathbb{R}, \Omega \times (t_0, t_1), f)$ of convex bodies as follows. For $v \in \Omega$ and $t \in (t_0, t_1)$, we set $f(v, t)$ to be the convex body $(C \cap W(t)) \setminus g(v)$. The proof of the coconvex Aleksandrov–Fenchel inequalities is based on the comparison between the linear families $\alpha$ and $\beta$.

Let $\pi : V \times \mathbb{R} \to V$ denote the natural projection. The polynomials $\Vol_\alpha$ and $\Vol_\beta \circ \pi$ are defined on the same space $V \times \mathbb{R}$. We have the following relation between the two polynomials:

$$\Vol_\alpha = \Vol(C \cap W(t)) - \Vol_\beta \circ \pi,$$

which is clear from the additivity of the volume. The first term in the right-hand side has the form $ct^d$, where $c$ is some positive constant. The second term in the right-hand side does not depend on $t$. It follows from $(V)$ that $\Vol_\beta$ is a homogeneous degree $d$ polynomial.

Let us mark some points $(v_1, s_1), \ldots, (v_{d-2}, s_{d-2})$ in $\Omega \times (t_0, t_1)$. Apply the differential operator $\frac{2}{d!} L_{(v_1, s_1)} \ldots L_{(v_{d-2}, s_{d-2})}$ to both sides of $(V)$. We obtain that

$$Q_\alpha = c't^2 - Q_\beta \circ \pi,$$

where $c'$ is some positive constant (equal to $cs_1 \cdots s_{d-2}$). The last term in the right-hand side of $(Q)$ is obtained using the chain rule and the fact that the differential of $\pi$ coincides with $\pi$. Consider the following linear family $\tilde{\beta} = (V \times \mathbb{R}, \Omega \times (t_0, t_1), \tilde{g})$ of $C$-coconvex polytopes: $\tilde{g} = g \circ \pi$. Then $Q^C_\beta = Q^C_\beta \circ \pi$, again by the chain rule. In the right-hand side of $(Q)$, we have the difference of two quadratic forms, moreover, these two forms depend on disjoint sets of variables.

If $q_1, q_2$ are quadratic forms depending on disjoint sets of variables, and $(k_1, \ell_1), (k_2, \ell_2)$, respectively, are signatures of these forms, then $q_1 + q_2$ is a quadratic form of signature $(k_1 + k_2, \ell_1 + \ell_2)$. We now apply this observation to identity $(Q)$. The first term of the right-hand side, $c't^2$, has signature $(1, 0)$. The signature of the left-hand side is equal to $(1, \ell)$ for some $\ell \geq 0$, by the classical Alexandrov–Fenchel inequality. It follows that the signature of $Q^C_\beta$ is $(\ell, 0)$, i.e. the form $Q^C_\beta$ is non-negative. Finally, since $Q^C_\beta = Q^C_\beta \circ \pi$, the quadratic form $Q^C_\beta$ is also non-negative.

References

[Al] A.D. Aleksandrov, *To the theory of mixed volumes of convex bodies II. New inequalities between mixed volumes and their applications* (Russian), Matem. Sb., 2 (1937), 6, pp. 1205–1238

[F] F. Fillastre, *Fuchsian convex bodies: basics of Brunn-Minkowski theory*, Geom. Funct. Anal., DOI: 10.1007/s00039-012-0205-4 (2013)

[KK] K. Kaveh, A.G. Khovanskii, *Convex bodies and multiplicities of ideals*, preprint (2013).

[KP] A. Khovanskiï, A. Pukhlikov, *Finitely additive measures of virtual polytopes*. St. Petersbg. Math. J. 4, No. 2 (1993), 337–356

[Mc] P. McMullen, *On simple polytopes*, Invent. math. 113 (1993), 419–444
[VT] V. Timorin, An analogue of the Hodge-Riemann relations for simple polytopes, Russian Mathematical Surveys, 54, No.2 (1999), 381–426

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