Fractal non-Fermi liquids from Moire-Hofstadter phonons

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We theoretically explore 2d Moire heterostructures in lattice-commensurate magnetic fields as platforms for quantum simulation of a paradigmatic model of non-Fermi liquid physics: a Fermi-surface coupled to a fluctuating gauge field. In these Moire-Hofstadter (MH) systems, long-wavelength acoustic phonons exhibit singular interactions with electrons analogous to those of electrons with 2d gauge fields. This leads to a breakdown of Fermi-liquid theory at low temperatures. We show that a combination of large Moire-unit cell size, tunable Fermi-surface topology, and enhanced coupling to interlayer sliding modes, enhance these effects by over many orders-of-magnitude compared to bulk crystals, placing them within experimental reach. Though we find that the asymptotic low-temperature non-Fermi liquid regime remains at prohibitively low temperatures, striking precursor non-Fermi liquid signatures can be observed, and we propose surface acoustic wave attenuation and quantum oscillation transport experiments. We also study the motion of MH acoustic-polarons, which we predict exhibit logarithmically diverging effective mass and unconventional magnetic field scaling for scaling of cyclotron resonance frequency and quantum oscillation amplitude.

The problem of metallic electrons strongly coupled to fluctuating gapless (bosonic) collective modes is believed to underly some of the least-understood quantum phenomena, from strange-metal phases of high-temperature superconductors [1][3], to metallic quantum critical systems [4][5], composite fermion liquids [6], and gapless spin-liquids [7][10]. These systems lack well-defined quasi-particles and are not captured by conventional Fermi liquid theory paradigm. The detailed behavior of these non-Fermi liquid (NFL) systems remain poorly understood due to the absence of naturally controlled theoretical calculations [10][12] or efficient numerical methods, complex materials chemistry, relatively high impurity concentrations, and limited ability to tune the electron density or interactions in the underlying host materials. To this end alternative platforms to explore NFL behavior in simpler, cleaner, and more tunable material platforms are highly desirable.

A common and reliable source of gapless bosonic collective modes are: Nambu-Goldstone (NG) modes, such as acoustic phonons and magnons, arising due to spontaneously broken continuous symmetries. However, the same mechanism that ensures their masslessness ordinarily causes NG-modes to decouple from electrons at low temperatures, producing conventional Fermi-liquid behavior. Exceptions occur for rotational NG-modes, in which we predict exhibit logarithmically diverging effective mass and unconventional magnetic field scaling for scaling of cyclotron resonance frequency and quantum oscillation amplitude to phonon fluctuations in much the same way as they would couple to dynamically fluctuating magnetic fields. An important caveat is that magnetic fields tend to produce non-dispersing Landau levels, destroying the Fermi-surface at the single-particle level. In a crystal, the bandwidth of Landau levels, which sets an upper bound on the energy scale for observing NFL behavior, scales as $e^{-1/\nu}$ where $\nu$ is the number of flux per unit cell. For atomic-scale lattices, reaching $\nu \sim 1$, would require astronomical $B \gtrsim 10^4 \text{T}$. For this reason, the prediction of [10] has not been experimentally tested.

In contrast, Moire-superlattice potentials of small-angle twisted structures can have sufficiently large unit-cells to reach $\nu \sim O(1)$ with laboratory magnetic fields. In this paper we explore these systems as platforms for exploring NFL physics, focusing on twisted bilayer graphene (TBG) as a particularly promising example due to its large Moire-potential.

Moire-Hofstadter (MH) bands — Our analysis begins from Bistritzer and MacDonald’s (BM) continuum model for twisted bilayer graphene (TBG) [17][19], with single-electron Hamiltonian:

$$H(r) = \begin{pmatrix} h(-\theta/2) & T(r) \\ T^\dagger(r) & h(\theta/2) \end{pmatrix},$$

where $h(\theta) = v (\Pi + \text{sgn}(\theta) \frac{k_z}{2}) \cdot \mathbf{\sigma}_\theta$ is the Hamiltonian for a single graphene layer twisted by angle $\theta$, $\Pi = \mathbf{p} + e\mathbf{A}$ is the canonical momentum with in magnetic field $\mathbf{B} = \nabla \times \mathbf{A} = B\hat{z}$, $k_\theta = -k_0 \hat{y} = -\frac{e\mathbf{A}}{2} \sin \frac{\pi}{\sqrt{3}} \hat{y}$ is the vector connecting the Dirac points in the two layers, and $a$ is the graphene lattice spacing. $T(r)$ represent interlayer tunneling with the (approximate) spatial periodicity of the Moire lattice

$$T(r) = w \sum_j e^{-i\mathbf{g}_j \cdot \mathbf{r}} T_j.$$

Here $\mathbf{g}_0 = 0$, $\mathbf{g}_{1,2} = \frac{\sqrt{3}a}{2} (\mp \hat{x} + \sqrt{3} \hat{y})$ are Moire-reciprocal-lattice vectors, and $T_j = \eta_j + \sigma_{2\pi j/3}$ are sub-lattice matrices. The real-space lattice vectors for the

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and ii) the conduction (valence) bands exhibits a pair of destructive interference responsible for that phenomena, B-field. In particular, i) there is no magic-angle suppression of the bands resemble those of single-layer graphene in zero-charge-neutrality for $\theta$. Apart from having lower bandwidth and spin-splitting, cyclotron motion near commensurate filling – Upon detuning the magnetic field slightly away from commensurate filling $B = B_\nu + \Delta B$, with $\epsilon\Delta B \ll 1/|a_1,2|^2$, electrons will exhibit cyclotron motion with effective cyclotron frequency, $\omega_c = \frac{e\Delta B}{m}$ set by the detuning field $\Delta B$ rather than the full field $B$ (as can be seen by semiclassical motion of a wave-packet made from Bloch states of the bands at commensurate filling $B_\nu$, moving in effective field, $\Delta B$). Oscillations with the reduced field $\Delta B$, rather than the full field $B$ could be measured by standard Shubnikov-de-Haas, cyclotron resonance, or current-focusing techniques, but to our knowledge have not yet been explored.

Electron-phonon coupling – To describe phonons, we introduce displacement fields: $u_{\ell}(r)$ for each layer $\ell = 1, 2$, which are conveniently re-expressed in terms of the mean-displacement of the bilayer: $u = \frac{1}{2} (u_1 + u_2)$ and relative displacements of the layers, $d = u_1 - u_2$. The effect of both types of phonon displacements on the electronic Hamiltonian can be accounted by displacing the tunneling operators $T(r) \rightarrow T(r - u)$ by a single effective displacement field $u_{\ell}(r)$:

$$u = \bar{u} - \frac{\hat{c} \times d}{2\tan \theta/2},$$

since the relative sliding of the two graphene sheets is equivalent to a translation of the Moire pattern perpendicular to the sliding direction. This has two important consequences: First, at small twist angles, the coupling to $d$ is effectively enhanced by a factor of $\approx \theta^{-1}$ (which has been previously highlighted as a possible mechanism.
for magic-angle superconductivity \[26\]. Second, for a commensurate crystal relative displacements would be gapped optical modes. Instead, for incommensurate twist angles uniform \(d\)-displacements have no energy cost \[27\] and result in gapless acoustic modes (in analogy to the sliding “phason” mode of incommensurate charge density wave orders \[28\]). As our focus is on universal aspects, and extraction of approximate cross-over energy scales, we ignore anisotropy and polarization dependence of the speed-of-sounds, \(c_s\).

The electron-phonon coupling is conveniently identified by changing coordinates to a co-moving frame of the lattice: \(\mathbf{r} \leftarrow (\mathbf{r} - \mathbf{u}(\mathbf{r}))\) \[23\], in which the lattice is restored to its undistorted form. To properly account for the non-commutivity of magnetic translations, while manifestly preserving gauge invariance, we implement this frame transformation with the unitary operator:

\[
W_u = e^{-i\mathbf{u}(\mathbf{R})\cdot \mathbf{Π}}
\]

where \(\mathbf{R} = \mathbf{r} - \ell_B^2 \hat{z} \times \mathbf{Π}\) is the guiding center coordinate (whose components all commute with \(\mathbf{Π}\)). Neglecting subleading terms of order \(O(\nabla u, u^2)\), this transformation effectively restores the Moire tunneling potential: \(W_u^2 T(\mathbf{r} - \mathbf{u}(\mathbf{r})) W_u \approx T(\mathbf{r})\), while transforming the kinetic energy by: \(W_u^2 \Pi W_u \approx \Pi - e\mathbf{B} \times \mathbf{u}(\mathbf{R})\), yielding a direct (gradient-free) electron-phonon interaction:

\[
H_{e-ph} = ev\hat{u} \cdot \hat{B} \times \left( \begin{array}{cc} \sigma^{-\theta/2} & 0 \\ 0 & \sigma^{\theta/2} \end{array} \right) \equiv \hat{u} \cdot \mathbf{Γ}_m + \mathbf{d} \cdot \mathbf{Γ}_r
\]

For future convenience, we define the electron phonon vertices:

\[
\Gamma_{n,\alpha,\lambda}(\mathbf{k}, \mathbf{q}) = \langle n, \mathbf{k} + \mathbf{q}|\hat{e}_\lambda(\mathbf{q}) \cdot \mathbf{Γ}_\alpha|n, \mathbf{k}\rangle
\]

for the \(n\)th magneto-Bloch band, with wave-vector \(\mathbf{k}\) state \(|n, \mathbf{k}\rangle\), where \(\alpha \in \{m, r\}\) labels the phonon type, and \(\lambda\) labels longitudinal (L) or transverse (T) polarizations corresponding to polarization vector \(\hat{e}_\lambda(\mathbf{q})\). For notational simplicity we omit the band-index \(n\) in subsequent expressions.

Crucially, these interaction vertices are generically non-vanishing in the limit of zero phonon momentum transfer \[16\] (\(\mathbf{q} \to 0\)), as demonstrated numerically in Fig. 2. At asymptotically low-temperatures, this direct-coupling is expected to produce a complex non-Fermi liquid state \[13\] \[10\], whose properties cannot be reliably calculated except in artificial limits \[11\] \[12\]. We will make predictions based only on leading order perturbative calculations, which reliably predict the onset of the non-Fermi liquid behavior approached from higher-temperatures or energy-scales, but are merely suggestive of the possible asymptotic behavior at low-temperatures.

**Landau damping and SAW attenuation** — In the presence of a Fermi-surface, this direct coupling causes phonons to decay into the electron-hole continuum, and become soft and overdamped. A standard one-loop perturbative calculation predicts a singular Landau-damping (LD) form of the phonon self-energy:

\[
\Pi_{1,\text{loop}}(\Omega, \mathbf{q}) = \frac{1}{2\Omega_q} \gamma_{\alpha,\lambda}(\mathbf{q})|\Omega\rangle_q\langle q|\]

where \(q||\) is the component of \(q\) parallel to the Fermi-surface, and we define the LD-parameters:

\[
\gamma_{\alpha,\lambda}(\mathbf{q}) = \frac{|m|}{2\pi\rho V_F} \Gamma_{\alpha,\lambda}(\mathbf{k}, \mathbf{q})|q||\text{FS}
\]

with \(\rho_m = 2\rho, \rho_r = \rho/2\), where \(\rho\) is the mass density for SLG. For energy (or temperature) scales \(E, T \ll E_{LD} = \sqrt{c_s^2}\), the LD self-energy overwhelms the bare phonon dynamics \(\Omega_q = c_s q\), resulting in strongly overdamped phonons.

The crossover scale, \(E_{LD}\), will play an important role in our following discussion. Its numerically computed values are shown in Fig. 3 for representative set of parameters. \(E_{LD}\) is generally deeply sub-Kelvin except at the FFS fillings where the vanishing Fermi-surface curvature results in divergent effective mass and Landau damping coefficient for select directions along the Fermi-surface. This divergence will be cutoff in practice by disorder and non-perturbative effects, but will still result in a strongly peaked \(E_{LD}\). For \(\theta = 0.8^\circ\), \(E_{LD}/k_B\) exceeds 1K over appreciable ranges of chemical potential (\(\pm 0.3\text{meV}\)) and density (\(\pm 10^{-13}\text{cm}^{-2}\)).

Surface acoustic wave (SAW) attenuation provides a possible experimental probe of this unconventional phonon damping. SAWs can be injected and detected by piezoelectric contacts, which we model as semi-infinite in the \(y\)-direction, and separated by finite distance \(L\) in the \(x\) direction. We extract the SAW attenuation length, \(\xi\), by computing the fixed-frequency phonon propagator between source to detector: \(\int dy D(\omega, x = L, y) \propto e^{-L/\xi}\), and find (see Appendix C 2):

\[
\xi = \left\{ \begin{array}{ll} \frac{2\nu}{\gamma} & \omega \gg E_{LD} \\ \frac{2\nu}{E_{LD}} \left( \frac{E_{LD}}{\omega} \right)^{1/3} & \omega \ll E_{LD} \end{array} \right.
\]

with numerical results shown in Fig. 3. The FFS points near VH-fillings produce a singular suppression of SAW propagation, providing a characteristic fingerprint of the
anomalous power-law scaling of AC electronic compressibility $\chi(\omega) \sim \omega^{2/3}$, and spatial decay of quasi-particle interference patterns.

**MHA Polarons** — Even for energy and temperature scales above $E_{LD}$, electrons and MHA-phonons are strongly coupled into unusual polaronic degrees of freedom, which we dub “MHA polarons” whose dynamics differ dramatically from conventional Fermi-liquid quasi particles. Non-equilibrium electron motion results in divergent rate of soft-collinear phonons, in direct analogy to cyclotron radiation and soft-collinear divergences in quantum electrodynamics. These soft-collinear singularities do not alter temperature dependence of DC-resistivity (which is insensitive to small-angle scattering), but can have qualitative impact on the scaling of quantum oscillatory and cyclotron resonance phenomena, in an excess field $\Delta B$. We analyze the MHA-polaron motion and phonon-radiation within a semiclassical framework (Appendix D), which amounts to an infinite-order resummation of soft-collinear divergences in the eikonal approximation.

Following standard techniques, we derive a semiclassical equation of motion for the MHA-polaron coordinate, $r$ (see Appendix D1): 

$$\dot{r}(t) = \frac{F_{\text{ext}}}{m} - g^2 \int_{-\infty}^{t} dt' \dot{r}(t') \left( \frac{r(t')}{(t-t')^2} \right)$$

where $g = \frac{m}{2\pi\epsilon_c^2} \frac{d^2}{d k} (k_{g=0})$ is a dimensionless measure of electron-phonon coupling strength, and $F_{\text{ext}}$ represent external forces. The second term represents the non-Markovian effects due to the gapless phonons. Solving this equation for oscillatory cyclotron motion $r(t) = r_0 (\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y})$, in an excess field: $B - B_0 = \Delta B$, we find that the MHA-polaron exhibits a scale-dependent logarithmically diverging effective mass: $m_{\text{eff}}(\omega) = m (1 + g^2 \log \frac{\omega}{\Lambda})$, and decay rate: $\Gamma_\omega = g^2 \Lambda$.

For cyclotron motion in excess field, $\Delta B$ the cyclotron frequency scales with an unconventional power of $\Delta B$:

$$\omega_{c,p} = \omega_{c,0} \left( \frac{\omega_{c,0}}{\Lambda} \right) g^2 \sim (\Delta B)^{1+g^2}$$

which could be observed in standard AC-conductivity cyclotron resonance measurements. Here $\omega_{c,0} = \frac{e\Delta B}{m}$ is the bare (non-interacting) electron cyclotron frequency of the MH-bands. For TBG we find that the $g^2 \lesssim 10^{-2}$ (even close to the VH-fillings), so these effects may be challenging to observe. We note, however, that these predictions equally apply to a variety of analog non-Fermi liquid systems where stronger coupling could be achieved.

In addition to energy dissipation by phonon radiation, MH-phonons cause characteristic dephasing of phase-sensitive measurements such as periodic-in-$1/\Delta B$ quantum oscillations in density of states and resistivity. As electrons experience MH-phonon fields $\nu$ as an effective electromagnetic vector potential, zero-point fluctuations...
of MH-phonons give rise to quantum-fluctuating geometric (Aharonov-Bohm-like) phases that decohere quantum oscillations. We compute the quantum oscillatory contribution to density of states at energy $\varepsilon$: $N(\varepsilon)$ via a semiclassical sum of the return amplitude for multiple classical cyclotron orbits of the MHA-polaron, averaged over fluctuating geometric phases due to MH-phonons (Appendix 17.2), to find:

$$N(\varepsilon) = \sum_n A_n \exp \left[-\frac{1}{2L^2} (\varepsilon - n\omega_c n)^2\right]$$  (13)

This result predicts a comb of Gaussian peaks centered over fluctuating geometric phases due to MHA-phonons. This contrasts the usual $A_n \sim \sqrt{\Delta B}$ scaling expected from impurity scattering, providing a qualitatively distinct scaling.

**Outlook** – We have explored Moire super-lattice structures in high magnetic fields as potential platforms for simulating non-Fermi liquid (NFL) physics, due to unusually singular electron-phonon coupling. While the asymptotic low-temperature NFL fixed point remains at inaccessibly low temperature, intermediate scale non-Fermi liquid precursor behavior and unconventional polaron dynamics can be observed at by tuning carrier density near a van-Hove singularity. At or very close to the van Hove filling, the non-Fermi liquid physics will compete (or intertwine) with enhanced tendency to order with a nested Fermi surface. This competition appears to be controllable by twist-engineering, and a more detailed study of the interplay between exotic interaction-driven orders, and non-Fermi liquid physics in these systems will be a compelling subject for future investigation.

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In this appendix, we review the derivation of the magnetic Bloch bands for twisted bilayer graphene. Define the magnetic lattice translational operators:

$$T_i = e^{iK \cdot a_i},$$

where $i = 1, 2$, which perform translation by Moire lattice-vectors: $a_1 = \frac{a}{\sqrt{3}} \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $a_2 = \frac{a}{\sqrt{3}} \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Further, define guiding-center momenta:

$$K_x = \Pi_x + \frac{y}{l_B^2}, \quad K_y = \Pi_y - \frac{x}{l_B^2},$$

which satisfy $[K_x, K_y] = \frac{i}{l_B^2}$, $[K_x, \Pi_y] = 0$, and $[T_i, H] = 0$.

Unlike ordinary lattice-translations, magnetic translations do not generically commute except at special commensurate magnetic fields. Generically,

$$T_1 T_2 = T_2 T_1 \exp \left(i \frac{\sqrt{3} a^2}{2 \theta l_B^2}\right),$$

which vanishes if and only if:

$$\frac{\sqrt{3} a^2}{2 \theta l_B^2} = \frac{2 \pi p}{q} \text{ or } k_{\theta l_B}^2 = \frac{4 \pi q}{3 \sqrt{3} p},$$

which gives the condition for commensuration of the Moire lattice and magnetic field.

At commensurate filling, $[T_1, T_2] = 0$, so that we can construct the magnetic Bloch states $|\alpha k\rangle$ with band index $\alpha$, as the simultaneous eigenstates of $H$, $T_1$, and $T_2$:

$$H|\alpha k\rangle = E_{\alpha}(k)|\alpha k\rangle,$$

$$T_1|\alpha k\rangle = e^{i k \cdot a_1}|\alpha k\rangle,$$

$$T_2^\dagger|\alpha k\rangle = e^{-i q k \cdot a_2}|\alpha k\rangle.$$

### Appendix A: Moire-Hofstadter bands

In this appendix, we review the derivation of the magnetic Bloch bands for twisted bilayer graphene. Define the magnetic lattice translational operators:

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$$T_2^\dagger|\alpha k\rangle = e^{-i q k \cdot a_2}|\alpha k\rangle.$$
An extended magnetic Brillouin zone (MBZ) is spanned by magnetic-reciprocal lattice vectors: \( \mathbf{g}_1 \) and \( \mathbf{g}_2/q \) (where \( \mathbf{g}_i \) are the original Moire reciprocal lattice vectors). As we now show, the commensurability integers \((p, q)\), determine the number of sub-bands, and size of the reduced MBZ. The commutation relations for \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) give:

\[
\mathcal{T}_1 \mathcal{T}_2^\dagger |\alpha \mathbf{k}\rangle = e^{i(\mathbf{k} + \mathbf{g}_j)p/q} \mathcal{A}_k \mathcal{T}_2^\dagger |\alpha \mathbf{k}\rangle
\]

(A8)

implying a set of degenerate states \( \mathcal{T}_2^\dagger |\alpha \mathbf{k}\rangle \sim |\alpha, \mathbf{k} + \frac{2\pi}{q} \mathbf{g}_j\rangle \) \((j = 1, 2, \ldots, q - 1)\), which have identical energy since \([H, \mathcal{T}(a_2)] = 0\). Thus there is a \(q\)-fold degeneracy along the \( \mathbf{g}_1 \) direction. Equivalently, we can consider a reduced magnetic Brillouin zone spanned by \( \mathbf{g}_1/q \) and \( \mathbf{g}_2/q \) with \(q\)-fold degeneracy. In addition, Eq. (A8) implies: \( E_\alpha (\mathbf{k}) = E_\alpha \left( \mathbf{k} + \frac{2\pi}{q} \mathbf{g}_j \right) \), that is, one period in a magnetic band extends over \(p\) MBZ’s in the direction of \( \mathbf{g}_j \). The reduction of energy bands to one MBZ will yield \(p\) different subbands. For convenience, we extend \( E_\alpha \left( \mathbf{k} + l \mathbf{g}_1 \right) \equiv E_{al}(\mathbf{k}) \) where \( l = 0, 1, \ldots, p - 1 \) represents subbands and \( \mathbf{k} \) is restricted to the reduced MBZ.

Having identified the distinct magnetic sub-bands and reduced MBZ, we can label states of TBG by: \( |\tau, \sigma, n, l, \mathbf{k}\rangle \), where \( \tau = 1, 2 \) represents layers, \( \sigma = A, B \) represents sublattices, \( n \) represents LL index, and \( l \) represents subbands. Under this basis, the single-layer Hamiltonian reads

\[
h(\theta/2) = \frac{\sqrt{2}v}{l_B} \left( e^{i\theta/2}\sqrt{n + 1} |2, B, n + 1, l, \mathbf{k}\rangle \langle 2, A, n, l, \mathbf{k}| + \text{h.c.} \right)
\]

(A9)

To calculate matrix elements of the interlayer hopping term under this basis, we first split \( e^{-ig_j \cdot R} \) as

\[
e^{-ig_j \cdot R} = e^{-ig_j \cdot \eta} e^{-ig_j \cdot \eta R}.
\]

(A10)

where

\[
\eta = \begin{pmatrix} R_x \\ R_y \end{pmatrix} = \begin{pmatrix} x - \Pi_2 l_B^2 \\ y + \Pi_1 l_B^2 \end{pmatrix} = \begin{pmatrix} x - \eta_x \\ y - \eta_y \end{pmatrix} = \mathbf{R} - \eta
\]

(A11)

Notice that \( \mathbf{R} = l_B^2 (\hat{z} \times \mathbf{K}) \) only acts on \(|l, \mathbf{k}\rangle \) and \( \eta = -l_B^2 (\hat{z} \times \mathbf{K}) \) only acts on \(|n\rangle \), so we have

\[
\langle n, l, \mathbf{k} | e^{-ig_j \cdot R} | n', l', \mathbf{k}\rangle = \langle n | e^{-ig_j \cdot \eta} | n'\rangle \langle l, \mathbf{k} | e^{-ig_j \cdot \eta R} | l', \mathbf{k}\rangle.
\]

(A12)

The first term on the right hand side of Eq. (A12) is given by \( \langle n | e^{-ig_j \cdot \eta} | n'\rangle = F_{nn'} (\mathbf{g}_2 l_B / \sqrt{2}) \), where

\[
F_{nn'}(z) = \begin{cases} 
\sqrt{n!} / n! (-z_x + iz_y)^{n-n'} e^{-z_x^2} L_{n-n'}^2 (z^2) & n \geq n' \\
\sqrt{n!} / n! (z_x + iz_y)^{n-n'} e^{-z_x^2} L_{n-n}^2 (z^2) & n < n'
\end{cases}
\]

(A13)

with \( L \) being the associated Laguerre polynomial.

The second term on the right hand side of Eq. (A12) can be converted to matrix elements of magnetic translation operators by using \( e^{-ig_j \cdot \eta R} = e^{-i\theta_R} K (\eta \times \hat{z}) \). Specifically:

\[
e^{-ig_j \cdot \eta R} = \mathcal{T}_2^{-\frac{\theta}{2}} \quad \text{and} \quad e^{-ig_j \cdot \eta R} = \mathcal{T}_1^{\frac{\theta}{2}}.
\]

(A14)

Considering \( \mathcal{T}_2^{-\frac{\theta}{2}} |\alpha \mathbf{k}\rangle \sim |\alpha, \mathbf{k} - \mathbf{g}_1\rangle \) and applying the commutation relation between \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \), we can obtain

\[
\mathcal{T}_2^{-\frac{\theta}{2}} |l + 1, \mathbf{k}\rangle = e^{-i\frac{\theta}{2} k \cdot \mathbf{a}_2} |l, \mathbf{k}\rangle.
\]

(A15)

\[
\mathcal{T}_1^{\frac{\theta}{2}} |l, \mathbf{k}\rangle = e^{i\frac{\theta}{2} (k \cdot \mathbf{a}_1 + 2\pi l)} |l, \mathbf{k}\rangle.
\]

(A16)

Hence, the interlayer hopping term in the MBZ reads

\[
T(\mathbf{k}) = w \mathcal{T}_0 |2, \sigma, n, l, \mathbf{k}\rangle \langle 1, \sigma', n', l, \mathbf{k}| + w \mathcal{T}_1 F_{nn'} (g_1 l_B / \sqrt{2}) e^{-i\frac{\theta}{2} k \cdot \mathbf{a}_2} |2, \sigma, n, l + 1, \mathbf{k}\rangle \langle 1, \sigma', n', l, \mathbf{k}| + w \mathcal{T}_2 F_{nn'} (g_2 l_B / \sqrt{2}) e^{i\frac{\theta}{2} (k \cdot \mathbf{a}_1 + 2\pi l)} |2, \sigma, n, l, \mathbf{k}\rangle \langle 1, \sigma', n', l, \mathbf{k}|.
\]

(A17)

The numerical plots shown in the main text were obtained by diagonalizing this inter-layer hopping matrix truncated to a sufficiently large range of Landau indices \( n \) to achieve well-converged cutoff-independent results. Notice that the direct truncation in LL basis will introduce an artificial zero mode in Eq. (A9). To remove it, we add a penalty term \( E_P |2, A, N_{\text{cutoff}}\rangle |2, A, N_{\text{cutoff}}| \) with \( E_P \gg \frac{\sqrt{2N_{\text{cutoff}}}}{l_B} \) to Eq. (A9) in the numerical calculation.
Appendix B: MHA electron-phonon vertex

In this appendix, we derive the MHA electron-phonon coupling. For the convenience, we can write Hamiltonian under the lattice distortion $H_u$ as $H_u = H_0(\Pi) + H_t(r - u(r))$, where

$$H_0(\Pi) = \begin{pmatrix} h(-\theta/2) & 0 \\ 0 & h(\theta/2) \end{pmatrix}$$  \hspace{1cm} (B1)

and

$$H_t(r - u(r)) = \begin{pmatrix} 0 \\ T^\dagger (r - u(r)) \\ 0 \end{pmatrix}.$$  \hspace{1cm} (B2)

Then we implement the co-moving frame transformation with $W_u = e^{-i u(R) \cdot \Pi}$ on $H_0(\Pi)$ and $H_t(r - u(r))$ respectively. Since $[R, \Pi] = 0$, for small $u$ we have

$$W_u^\dagger \Pi_i W_u = \Pi_i + i [u(R) \cdot \Pi, \Pi_i] = \Pi_i - eB \varepsilon_{ij} u_j(R) + O(u^2)$$  \hspace{1cm} (B3)

and corresponding transformed $H_0(\Pi)$:

$$W_u^\dagger H_0(\Pi) W_u = H_0(\Pi) - eB \times u(R)$$  \hspace{1cm} (B4)

$$= H_0(\Pi) + ve u \cdot \begin{pmatrix} B \times \sigma_{-\theta/2} & 0 \\ 0 & B \times \sigma_{\theta/2} \end{pmatrix} + O(u^2)$$  \hspace{1cm} (B5)

where $\sigma_{\theta/2} \equiv R_{\theta/2} \sigma$ are rotated Pauli matrices.

Similarly, for the tunneling term, we have

$$W_u^\dagger T(r - u(r)) W_u = T(r) + O(\nabla u, u^2)$$  \hspace{1cm} (B6)

and therefore the transformed total Hamiltonian is

$$W_u^\dagger H_u W_u = H_{u=0} + ve u \cdot \begin{pmatrix} B \times \sigma_{-\theta/2} & 0 \\ 0 & B \times \sigma_{\theta/2} \end{pmatrix} + O(\nabla u, u^2).$$  \hspace{1cm} (B7)

which gives electron-phonon coupling for both the mean displacement and the relative displacement in the transformed coordinate:

$$H_{e-ph} = \bar{u} \cdot \Gamma_m + d \cdot \Gamma_r$$  \hspace{1cm} (B8)

with

$$\Gamma_m = veB \begin{pmatrix} \hat{z} \times \sigma_{-\theta/2} & 0 \\ 0 & \hat{z} \times \sigma_{\theta/2} \end{pmatrix}$$  \hspace{1cm} (B9)

and

$$\Gamma_r = -\frac{veB}{2 \tan \frac{\theta}{2}} \begin{pmatrix} \sigma_{-\theta/2} & 0 \\ 0 & \sigma_{\theta/2} \end{pmatrix}$$  \hspace{1cm} (B10)

Appendix C: Computation of non-Fermi liquid scales

In this appendix, we estimate the relevant energy scales for the onset of Landau damping for the phonons, and non-Fermi liquid behavior of electrons, using self-consistent one-loop propagators (random phase approximation). While uncontrolled at asymptotically low-temperatures in the non-Fermi liquid regime, this approximation produces a reliable estimate of the onset for non-Fermi liquid behavior approaching from the high-temperature perturbative regime. Unless otherwise specified, all propagators are specified in imaginary time (Matsubara frequency).
1. One-loop self-energies and propagators

The (imaginary time/Matsubara) bare phonon propagator and bare electron propagator are given respectively by

\[ D^{-1}_0(q, \Omega) = -\frac{\Omega^2 + \omega_q^2}{2\omega_q} \] (C1)

\[ G^{-1}_0(k, \omega) = i\omega - \varepsilon_k \] (C2)

where for the long wave limit the phonon dispersion is \( \omega_q \approx c_S |q| \) and \( \varepsilon_k \) is the electron dispersion.

With electron-phonon interaction, the full phonon propagator depends on the phonon self-energy as \( D^{-1} = D^{-1}_0 - \Pi \).

At the lowest order of electron-phonon interaction, the one-loop phonon self-energy can be calculated by

\[ \Pi_1(q, \Omega) = \frac{1}{2\rho_\lambda\omega_q} \int \frac{d^2k d\omega}{(2\pi)^3} G_0(k, \omega) G_0(k + q, \omega + \Omega) \Gamma^2_{\alpha,\lambda}(k, q) \] (C3)

where \( \lambda = m/r \) is the label for mean and relative displacement, \( \alpha = L/T \) is the label for longitudinal and transverse mode, \( \rho_m = 2\rho, \rho_r = \frac{\rho}{2} \) (\( \rho \) is the mass density of single layer graphene). At the limit \( q \to 0 \), only the nonvanished coupling \( \Gamma^2_{\alpha,\lambda}(k, \hat{q}) \) is left. After performing the integrals for the patch dispersion \( \varepsilon_k = v_F k_\perp + k_\parallel^2 m \), we can obtain

\[ \Pi(q, \Omega) = \frac{1}{2\omega_q} \gamma_{\alpha,\lambda} \frac{|\Omega|}{|q||} \] (C4)

where \( q_\parallel \) is the component parallel to the fermi surface and the Landau damping coefficient \( \gamma_{\alpha,\lambda} \) is given by

\[ \gamma_{\alpha,\lambda}(\hat{q}) = \left. \frac{|m|}{2\pi \rho_\lambda v_F} \Gamma^2_{\alpha,\lambda}(k, \hat{q}) \right|_{q_\parallel FS} \] (C5)

where \( m \) is the effective mass of electron. Then, the full phonon propagator reads

\[ D(q, \Omega) = -\frac{2\omega_q}{\Omega^2 + \omega_q^2 + \gamma_{\alpha,\lambda} \frac{|\Omega|}{|q||}} \] (C6)

Using \( \Omega \sim c_S q \), the energy scale below which Landau damped modes dominate is

\[ E_{LD,\alpha,\lambda} \sim \sqrt{\gamma_{\alpha,\lambda} c_S} = \sqrt{\frac{c_S|m|}{2\pi \rho_\lambda v_F} \Gamma^2_{\alpha,\lambda}(k, \hat{q}) \left|_{q_\parallel FS} \right.} \] (C7)

On the other hand, using the Landau damped phonon propagator

\[ D(q, \Omega) \approx -\frac{2\omega_q}{\omega_q^2 + \gamma_{\alpha,\lambda} \frac{|\Omega|}{|q||}}, \] (C8)

we can obtain the one-loop electron self-energy at the Fermi level defined by \( G^{-1} = G^{-1}_0 - \Sigma \)

\[ \Sigma(\omega) = \frac{1}{2\omega_q \rho_\lambda} \int \frac{d^2q d\Omega}{(2\pi)^3} D(q, \Omega) G_0(q + k_F, \Omega + \omega) \Gamma^2_{\alpha,\lambda}(k, \hat{q}) \]

\[ = i \left( \frac{\Gamma^4_{\alpha,\lambda}}{12\sqrt{3}\pi^2 \rho_\lambda^2 v_F^2 |m| c_S^4} \right)^{\frac{1}{2}} \omega^2 \text{sgn}(\omega) \] (C9)

which gives the Non-Fermi liquid energy scale

\[ E_{NFL} \sim \frac{\Gamma^4_{\alpha,\lambda}}{12\sqrt{3}\pi^2 \rho_\lambda^2 v_F^2 |m| c_S^4} \] (C10)

below which the quasiparticle lifetime goes as \( \tau^{-1} \propto \omega^2 \).
2. SAW attenuation length

In this appendix, we give the propagation intensity of a surface acoustic wave on the twist bilayer graphene is attenuated by the electron-phonon scattering and decaying as $e^{-r/\xi}$. Here we consider the surface acoustic wave generated by a line source and the phonon propagator in the direction perpendicular to the source is given by, the retarded phonon propagator:

$$D^R(r, \hat{q}, \omega) = \int dq e^{iqr} \frac{2csq}{\omega^2 - cs^2q^2 - i\gamma(\hat{q})\frac{\omega}{|\hat{q}|}}$$  \hspace{1cm} (C11)

(where we have analytically continued from Matsubara to retarded frequency: $i\Omega \rightarrow \omega + i0^+$).

For further convenience, we first define dimensionless quantities: $x \equiv qcs/\omega$, $y \equiv r\omega/cs$, and $\lambda \equiv E_{LD}/\omega$ and then the integral becomes

$$D^R = \frac{4i}{cs} \int_0^\infty dx \frac{x^2 \sin(xy)}{x - x^3 - i\lambda^2}$$

$$= \frac{4i}{cs} \int_0^\infty dx \frac{(x^3 - x^5)\sin(xy)}{(x - x^3)^2 + \lambda^4} - \frac{4i}{cs} \int_0^\infty dx \frac{\lambda^2 x^2 \sin(xy)}{(x - x^3)^2 + \lambda^4}$$ \hspace{1cm} (C12)

We focus on the first term $I_1$ which is the imaginary part of $D$ and dictates the attenuation of SAWs

$$I_1 = \frac{2i}{cs} \text{Im} \int_{-\infty}^{\infty} dx \frac{(x^3 - x^5)e^{ixy}}{(x - x^3)^2 + \lambda^4}$$ \hspace{1cm} (C13)

Note that for larger imaginary part of poles, the decaying rate of the integral is larger, thus we want to find the pole with the smallest positive imaginary part which dominates the attenuation. Solving $(x - x^3)^2 + \lambda^4 = 0$ gives 6 roots: $\beta \pm i\alpha, -\beta \pm i\alpha$ and $\pm 2i\alpha$, where

$$\alpha = \frac{2^{\frac{1}{4}}\eta^{\frac{3}{2}} - 2 \cdot 3^{\frac{1}{2}}}{2 \cdot 6^{\frac{1}{2}}\eta^{\frac{3}{2}}}, \quad \beta = \frac{2^{\frac{1}{4}}3^{\frac{1}{2}}\eta^{\frac{3}{2}} + 2 \cdot 3^{\frac{1}{2}}}{2 \cdot 6^{\frac{1}{2}}\eta^{\frac{3}{2}}}$$ \hspace{1cm} (C14)

with $\eta = \sqrt{12 + 81\lambda^2 + 9\lambda^2}$. Hence the smallest positive imaginary part of poles is $\alpha$ and correspondingly the typical decaying length $\xi$ is given by

$$\xi = \frac{r}{y\alpha} = \frac{cs\lambda}{E_{LD}\alpha}$$ \hspace{1cm} (C15)

When energy scale of acoustic wave is much larger than Landau damping scale, i.e., $\lambda \ll 1$, $\alpha = \frac{\lambda^2}{2} + O(\lambda^6)$, thus

$$\xi \approx \frac{2cs}{E_{LD}\lambda} = \frac{2\omega}{\gamma}$$ \hspace{1cm} (C16)

On the other hand, for energy scale much smaller than Landau damping scale, i.e., $\lambda \gg 1$, $\alpha = \frac{1}{2}\lambda^{\frac{3}{2}} + O\left(\lambda^{-\frac{5}{2}}\right)$, thus

$$\xi \approx \frac{2cs}{E_{LD}} \left(\frac{E_{LD}}{\omega}\right)^{\frac{1}{4}}$$ \hspace{1cm} (C17)

3. Perturbative electron lifetime in intermediate-temperature regime

To compute the electronic life-time for temperatures or frequencies above the Landau-damping scale $E_{LD}$, we compute the tree-level electron-phonon quantum and transport scattering rates using the bare (undamped) phonon propagator, and focus on the relative (interlayer-sliding) phonons which have two orders of magnitude stronger interactions with electrons.

We work in a patch description of the Fermi-surface, which captures the universal aspects of the leading-order singularities in the electron-phonon scattering rate, while neglecting smooth, non-singular “background” scattering processes. We consider an electron at initial momentum $k = k_Fx$ above the Fermi surface in a particular (x) direction (i.e. $k$ is measured relative to the Fermi-momentum $k_Fx$). We linearize the electron dispersion near the Fermi energy,
and neglect the curvature of the Fermi-surface, $e_p \approx v_F p_e$. In this computation, the electron dispersions dominate the energetics in the $x$ direction parallel to the Fermi-velocity, and the phonon dispersion controls the energetics in the $y$ (perpendicular) direction, allowing us to neglect the dispersion of the phonons along $x$: $\omega_q \approx c_s |q_y|$ (formally, this is justified for $c_s \ll v_F$). For tree-level scattering rates, we may consider the Fermi-sea as Pauli-blocked, since at this order in perturbation theory, additional virtual electron-hole pairs cannot be excited. These approximations capture the leading singular behavior of scattering rates in the perturbative intermediate-temperature scale regime.

Writing the relative layer displacement field $d$ in terms of annihilation operators $a_{\lambda,q}$ that destroy a phonon with polarization $\lambda$ and wave-vector $q$:

$$d = \sum_{\lambda,q} \frac{1}{\sqrt{2\rho_c c_s |q|}} (\epsilon_{\lambda,q} a_{\lambda,q} e^{iqr} + h.c.), \quad \text{(C18)}$$

In the above-described setup, the electron couples only to a single polarization of the phonons, so we may drop the polarization indices, and write the corresponding electron-phonon coupling interaction coefficient as $\Gamma_{r,\lambda,k} := \Gamma$.

We first compute the total electron scattering-rate, which enters the so-called “quantum” scattering rate that effects quantum-coherent processes such as quantum-oscillations:

$$\tau_Q^{-1} = 2\pi \int_0^k \frac{dq_x}{2\pi} \int \frac{dq_y}{2\pi} \frac{\Gamma^2}{2\rho_c c_s |q|} \delta(v_F q_x - c_s q_y) = \frac{\Gamma^2}{2\pi \rho_c c_s v_F} \int_0^k dq_x q_x, \quad \text{(C19)}$$

where the $\delta$-function enforces energy conservation, and the integration limits $[0,k]$ on $q_x$ reflect momentum conservation and Pauli exclusion. The integral is logarithmically divergent in the infrared (IR, small $q$). This effect is directly analogous to similar infrared singularities arising in soft-photon emission in quantum electrodynamics (QED), where the logarithmic divergence is physically cutoff by an external IR scale such as the energy resolution of the detector. In this regime we see that the quantum lifetime is dominated by very small angle, low-energy scattering processes. To properly account for these infra-red divergences, in the following Appendix, we implement a semiclassical treatment which is equivalent to resumming the divergent soft-colinear radiation processes to infinite order [33].

For transport, small angle scatterings are ineffective at relaxing momentum. A standard cheap way of accounting for this is to weight the scattering processes by the change in angle, which for small angle scattering can be approximated by inserting a factor of $\left(\frac{q_x}{k_F}\right)^2$ into the above integrals. This angle-weighting results in a non-diverging transport scattering rate:

$$\tau_{tr}^{-1} = \frac{\Gamma^2 v_F}{4\pi \rho_c c_s^2} \left(\frac{k}{k_F}\right)^2 \sim T^2, \quad \text{(C20)}$$

At temperature $T$, electrons typically have momentum $k \sim T/v_F$ relative to the Fermi surface, and we see that the above expression gives a $\rho(T) \sim T^2$ contribution to resistivity. We emphasize that, despite the Fermi-liquid like scaling of resistivity, the logarithmically divergent quantum-lifetime signals a breakdown of Fermi-liquid quasi-particles.

**Appendix D: MHA Polaron dynamics**

In this section, we study the problem of an electron directly coupled to MH-phonons for intermediate to high energy regime where the phonons are not yet overdamped (the analogous problem in the asymptotic low-temperature non-Fermi liquid regime was previously studied using a quantum Boltzmann equation approach [37]). This situation is relevant not only for MH-polaron problems involving a single electron, but also allows a non-perturbative effective resummation of divergent radiative corrections due to emission of soft colinear phonons at energy scales exceeding $E_{LD}$. By numerical computation, we find that $\langle n,k |[\sigma_i]|n,k \rangle \approx \beta k_i k_0$ for small $k_i/k_0$ with $\beta$ being a constant. The electron-phonon vertex is then,

$$\Gamma_{n,m,\lambda}(k,q = 0) = \frac{\beta ev_F B}{k_0} \hat{z} \times k, \quad \Gamma_{n,r,\lambda}(k,q = 0) = \frac{\beta ev_F B}{\theta k_0} k \quad \text{(D1)}$$

Because of its enhanced coupling by a factor of $1/\theta$, we focus on the layer-antisymmetric phonon mode, in the following. The electron-phonon coupling, using first-quantized notation for the electron coordinate $r$ and momentum $k$, and second quantized for the phonon field $u$, and keeping only the leading term at small phonon momentum $q$ (dropping gradient coupling terms) is,

$$H_{e-ph}(r,k) = \frac{\alpha}{m} \sum_q e^{iqr} u(q) \cdot k = \frac{\alpha}{m} u(r) \cdot k \quad \text{(D2)}$$
with \( \alpha = \frac{3\nu_{\text{ph}} B_m}{m_c} \). The coupling is analogous to the paramagnetic coupling of the particle to a \( U(1) \) gauge field. Our problem is therefore, equivalent to that of a charged particle coupled minimally to a fluctuating gauge field.

For simplicity, we consider the phonon spectrum to be isotropic with a polarization independent velocity, as it will not effect the universal physics. The effective Feynman Lagrangian for the particle in the presence of a uniform excess magnetic field discussed in the main text \( \Delta B = \nabla \times A(r) \) is then:

\[
L[r, u] = L_e[r] + L_{e-ph}[r, u(r)] + \int d^2 r L_{ph}[u]
\]

\[
L_e[r] = \frac{1}{2} m \ddot{r}^2 - e \dot{r} \cdot A(r)
\]

\[
L_{ph}[u] = \frac{\rho}{2} \left[ (\partial_t u)^2 - c_s^2 (\nabla u)^2 \right]
\]

\[
L_{e-ph}[r, u(r)] = \alpha \dot{r} \cdot u(r)
\]  

(D3)

The dynamics of the electron is naturally described in the Schwinger-Keldysh path integral formulation [35]. The corresponding action on the closed-time Keldysh contour \( C \) is:

\[
S[r, u] = \int_C dt L[r, u]
\]  

(D4)

Decomposing the fields \( u(r,t) \) and \( r(t) \) in terms of fields residing on the forward time contour \( -u^+(r,t) \) and \( r^+(t) \), and the backward time contour \( -u^-(r,t) \) and \( r^-(t) \), we rewrite the action:

\[
S[r, u] = \int_{-\infty}^\infty dt \left( L[r^+, u^+] - L[r^-, u^-] \right)
\]  

(D5)

We now perform standard rotation for the fields:

\[
r^{cl}(t) = \frac{1}{2} \left[ r^+(t) + r^-(t) \right], \quad u^{cl}(r, t) = \frac{1}{2} \left[ u^+(r, t) + u^-(r, t) \right]
\]

\[
r^{q}(t) = \frac{1}{2} \left[ r^+(t) - r^-(t) \right], \quad u^{q}(r, t) = \frac{1}{2} \left[ u^+(r, t) - u^-(r, t) \right]
\]  

(D6)

and obtain the following action:

\[
S = S_e[r] + S_{ph}[u] + S_{e-ph}[r, u]
\]

\[
S_e[r] = \int_{-\infty}^\infty dt \left[ -2m r^q \cdot \dot{r}^{cl} - \left( \dot{r}^{cl} + \dot{r}^{q} \right) \cdot A \left( r^{cl} + r^{q} \right) + \left( \dot{r}^{cl} - \dot{r}^{q} \right) \cdot A \left( r^{cl} - r^{q} \right) \right]
\]

\[
S_{ph}[u] = \frac{1}{2} \int_{-\infty}^\infty dtd^2 r \dot{u}^T \hat{D}^{-1} \dot{u}
\]

\[
S_{e-ph}[r, u] = \alpha \int_{-\infty}^\infty dt \left( u^{cl}(r^+) + u^q(r^+) \right) \cdot u^{cl}(r^-) - u^{q}(r^-) \left( \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right) \left( \begin{array}{c} \dot{r}^{cl} \\ \dot{r}^{q} \end{array} \right)
\]  

(D7)

where we express the field \( u \) by a vector in the Keldysh cl-q space:

\[
\hat{u}(t) = \left( \begin{array}{c} u^{cl}(t) \\ u^q(t) \end{array} \right), \quad \hat{D}^{-1} = \left( \begin{array}{cc} 0 & [D^{-1}]^R \\ [D^{-1}]^R & [D^{-1}]^K \end{array} \right)
\]  

(D8)

Here, \( \frac{1}{2} [D^{-1}]^{R(A)} = \rho \left((\partial_t \pm i0^+)^2 - c_s^2 \nabla^2\right) \). To study the motion of the electron coupled to the phonon bath, we first integrate out the phonons, and further obtain the classical equation of motion for the electron. Integrating out \( u \), the effective action for the electron is,

\[
S_{\text{eff}}[r] = S_{e}[r] + \frac{\alpha^2}{2} \int_{-\infty}^\infty dt dt' \partial_t \bar{r}(t)
\]

\[
\left( \left( [D^K + D^R + D^A] (r^+(t) - r^+(t'), t - t') \right) \left( D^K - D^R + D^A \right) (r^+(t) - r^+(t'), t - t') \right) \partial_t \bar{r}(t')
\]  

(D9)
Where
\[ \tilde{r}(t) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} r^{cl}(t) \\ r^{q}(t) \end{pmatrix}, \]
\[ (D10) \]

We now make two simplifying approximations on the effective action. Following Ref. [34], we are interested in electron motion where its position is changing slowly compared to the relevant wavelengths of the phonons being integrated out (Lamb-Dicke approximation). This corresponds to replacing \( e^{-i \Delta t} \approx 1 \) in the following expression for the advanced and the retarded Green’s function components:
\[ D^{R/A}(\Delta t, \Delta t) = \frac{1}{\rho} \int \frac{d\Omega d^2q}{(2\pi)^3} \frac{e^{-i q \Delta r} e^{-i s \Omega \Delta t}}{(\Omega \pm i0^+)^2 - c_s^2 q^2} \]
\[ \approx \pm \frac{1}{2\pi \rho c_s^2} \frac{\theta(\pm \Delta t)}{\Delta t} \]
\[ (D11) \]

with \( \Delta r = \tilde{r}_r - \tilde{r}_l \) and \( \Delta t = t' - t \). The effective action \[ (D9) \] is then quadratic in \( r \):
\[ S_{\text{eff}} = S_e[r] + \frac{\alpha^2}{2} \int_{-\infty}^{\infty} dt dt' \tilde{r}(t) \left( \frac{0}{D^R} \frac{D^A}{D^R} \right) \tilde{r}(t') \]
\[ (D12) \]

Furthermore, we restrict ourselves to semiclassical dynamics of the electron at zero temperature; thus ignoring the role of thermally excited phonons. In this limit, we can ignore the Keldysh component of the Green’s function \[ D^K \], in which case, the effective action takes the form:
\[ S_{\text{eff}} = S_e[r] + \frac{\alpha^2}{2\pi \rho c_s^2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \tilde{r}^{cl}(t) \cdot \tilde{r}^{q}(t') \]
\[ (D13) \]

Expanding the vector potential \( A \) to first order in \( r^{cl} \), and integrating the non-local in time term by parts we get,
\[ S_{\text{eff}} = \int_{-\infty}^{\infty} dt \left[ -2m \mathbf{r} \cdot \tilde{r}^{cl} - 2r^{q} \cdot \left( \tilde{r}^{cl} \nabla A_i(r^{cl}) - \frac{dA_i(r^{cl})}{dt} \right) - \frac{\alpha^2}{2\pi \rho c_s^2} r^{q}(t) \cdot \int_{-\infty}^{\infty} dt' \tilde{r}^{cl}(t') \frac{(t - t')^2}{(t - t')^2} \right] \]
\[ (D14) \]

1. Polaron cyclotron motion

We obtain the classical equation of motion of the polaron by integrating over \( r^{q} \), and expanding \( \frac{dA(r^{cl})}{dt} = \tilde{r}^{cl} \partial_i A_i(r^{cl}) \):
\[ \partial_t^2 r^{cl} = -\frac{e \tilde{r}^{cl} \times \Delta B}{m} - g^2 \int_{-\infty}^{t} dt' \tilde{r}^{cl}(t') \frac{(t - t')^2}{(t - t')^2} \]
\[ (D15) \]

where the dimensionless coupling constant
\[ g^2 = \frac{\alpha^2}{2\pi \rho mc_s^2} = \frac{\beta^2 m}{8\sqrt{3\pi M}} \left( \frac{v_F \nu}{c_s} \right)^2 \]
\[ (D16) \]

We now solve for the cyclotron motion of the polaron: \( r(t) = r_0 (\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y}) \) where we have suppressed the \( cl \) label, for convenience. We take into account damping effects by allowing \( \omega \) to be complex. To solve for \( \omega \), we first express \( r(t) \) as a complex variable with the \( x \) and \( y \) components denoting the real and imaginary parts respectively. Now, using the polaron equation of motion (Eq. (D15)), we get:
\[ \omega = \omega_{c,0} - ig^2 \int_{-\infty}^{\infty} dt' e^{i\omega(\omega - t')} \frac{(t - t')^2}{(t - t')^2} \]
\[ (D17) \]

where \( \omega_{c,0} \equiv \frac{eB}{m} \). Defining an effective frequency-dependent mass for the polaron: \( m_p(\omega) \equiv \frac{eB}{\omega} \) and introducing a UV-cutoff the small time differences, we obtain an expression for \( m_p \):
\[ m_p(\omega) = \frac{m}{1 + g^2 \log \left( \frac{\Lambda}{\omega} \right) - ig^2 \frac{\Lambda}{\omega}} \]
\[ (D18) \]
which is enhanced at small frequencies, and furthermore, diverges at $\omega = 0$. A signature of this effect is seen in the unconventional power law dependence of the cyclotron frequency of the polaron $\omega_{c,p}$ on $\Delta B$, discussed in the main text, which we obtain by solving for $\omega$ using $\omega = \frac{eB}{m_p(\omega)}$:

$$\omega = \omega_{c,0} - g^2\omega \log \frac{\Lambda}{\omega} + ig^2\Lambda$$  \hspace{1cm} (D19)

The real and imaginary solutions of $\omega$ give $\omega_{c,p}$ and $\Gamma_c$ respectively, and have been stated in the main text.

2. Dephasing of quantum oscillations

The previous section focused on the energy-damping of oscillatory MHA-polaron motion due to radiation of soft-collinear MHA phonons. The cyclotron line-width computed there is relevant for phase-insensitive transport and optics cyclotron resonance measurements. For quantum oscillations measurements (e.g. Shubnikov-deHaas (SdH) oscillations in resistivity or related oscillations in tunneling density of states), the phase-coherence of the electronic orbit is also important, and these phenomena are affected more strongly by MHA phonons.

To analyze this effect, we employ a semiclassical computation of the density of states for an MHA polaron in an excess magnetic field $\Delta B$ away from commensurate filling, by summing up the “return” amplitudes for a particle to start at some position, and return to that position time $t$ later, and then Fourier transforming with respect to $t$.

For cyclotron motion of an electron coupled to MH-phonons, the electron picks up a Berry phase equal to:

$$e^{i\theta_B} = e^{i\alpha} \int dt \dot{\vec{r}} \cdot \vec{u}(\vec{r}(t))$$  \hspace{1cm} (D20)

This phase has quantum fluctuations due to the quantum fluctuations of $u$, which give a suppression factor for the quantum-oscillation amplitude $\sim e^{-\frac{\alpha^2}{2} \left(\int dt \dot{\vec{r}} \cdot \vec{u}(\vec{r}(t))\right)^2}$, where the average is taken over the thermal ensemble of phonons (we will restrict our attention to zero temperature, as finite temperature effects with $T \ll \omega_c$ do not effect the scaling form that we identify).

Since the phonons are much slower than the electrons, we can approximate the phonon configuration during a cyclotron orbit as being static (fluctuations of phonon field are approximately quenched on the time scale of cyclotron motion). In that limit, we can replace $\int dt \dot{\vec{r}} \cdot \vec{u}(\vec{r}(t)) \to \Phi$ where (by Stoke’s theorem) $\Phi$ is the flux of the effective “magnetic field”

$$b = \nabla \times u$$  \hspace{1cm} (D21)

through the cyclotron orbit. This approximation is valid for up to $n \sim \frac{v_F}{c_s \omega_c}$ cyclotron orbits, on longer time scales than $\frac{2\pi \rho c_s}{c_s \omega_c}$, then we would have to account for the dynamics of the phonons.

Let’s begin by computing the $\langle uu \rangle$ correlator in real-space. We have a Matsubara action:

$$\frac{\rho^2}{2} \left( \partial^2 \tau u + c_s^2 (\nabla u)^2 \right)$$

in 2+1d Euclidean time. At zero temperature, the Green’s function of the phonon will just be the 3d Coulomb potential

$$D_M(\tau, r) = \frac{1}{2\pi \rho c_s \sqrt{c_s^2 \tau^2 + r^2}} \approx \frac{1}{2\pi \rho c_s} \frac{1}{r}.$$  \hspace{1cm} (D22)

The $\langle bb \rangle$ correlator can be obtained by taking spatial derivatives of this propagator, giving: $\langle bb \rangle = -\nabla^2 \langle uu \rangle = \frac{1}{2\pi \rho c_s r^2}$. Integrating over the cyclotron orbit, we get a flux-flux correlator:

$$\langle \Phi \Phi \rangle = \frac{1}{2\pi \rho c_s} \int_{r, r' \subset \text{cyc. orbit}} \frac{1}{|r - r'|^3} \approx \frac{A_c}{4\pi \rho c_s} \int_a^{R_c} \frac{d\delta r}{\delta r^2} \approx \frac{A_c}{\rho c_s a}$$  \hspace{1cm} (D23)

where $a$ is a short-distance (lattice-scale) cutoff, and

$$A_c = \frac{S_k}{(eB)^2}$$  \hspace{1cm} (D24)

is the real-space area of the cyclotron orbit, and $S_k$ is the momentum-space area of the Fermi-surface.
Inserting this expression into the semiclassical version of the density of states, computed as a sum over return probabilities for various number of cyclotron oscillations [38], gives:

\[ \nu(\varepsilon) = \frac{-1}{\pi} \int_0^t dt \text{Im} G^R(r = r'; t) e^{-i e t} \approx \sum_n e^{-i \frac{2\pi n}{\omega_c}} n e^{-i \alpha n \Phi(u)} = \sum_n e^{-i \frac{2\pi n}{\omega_c}} n e^{-i \frac{1}{2} n^2 \tau} (\Phi \Phi) \]

where in the last step, we have used the Poisson summation formula.

This result predicts a comb of Gaussian peaks centered at integer multiples of the cyclotron frequency with peak-amplitude:

\[ A_c \approx \sqrt{\frac{\pi \rho c_s a}{\alpha^2 A_c}} \propto \Delta B \]

and width:

\[ \Gamma \sim \frac{\omega_c^2 A_c \alpha^2}{\pi^2 \rho c_s a} \propto \Delta B^{1+2g^2}. \]

By contrast, for elastic impurity scattering, the return probability for n-orbits is \( e^{-n^2(\omega_c \tau)} \), and the quantum oscillation amplitude would scale like \( \sqrt{\frac{1}{1/(\omega_c \tau)}} \sim \sqrt{\Delta B} \). (Note that the amplitude suppression of multiple scattering mechanisms combine in parallel not in series, so that the linear-\( \Delta B \) dependence of the MHA-phonon dephasing would dominate at small \( \Delta B \) over impurity contributions). Hence, the above linear-\( \Delta B \) scaling provides a signature of the unconventional MHA polaron dynamics that is qualitatively distinct from conventional contributions from impurities.

We also remark that essentially identical expressions apply for analog systems of electrons coupled to emergent gauge-like fields, such as the composite fermion liquid of the half-filled Landau level, spinon Fermi surfaces in gapless \( U(1) \) spin liquids, and zero-wavevector quantum critical points in metals.