Multi-agent Optimal Consensus with Unknown Control Directions

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Abstract—In this paper, we consider an optimal consensus problem for a group of high-order agents with unknown control directions. Both the system orders and control directions of these agents are allowed to be nonidentical. To solve this problem, we first augment each agent with an optimal signal generator to reproduce the global optimal point of the given distributed optimization problem, and then solve the global optimal consensus problem by developing some adaptive tracking controllers for these augmented agents. The trajectories of all agents are shown to be well-defined and achieve a consensus on this optimal point. Two numerical examples are given to verify the effectiveness of our algorithms.

I. INTRODUCTION

During the last decade, distributed optimization of multi-agent systems has become a hot topic due to its wide applications in multi-robot networks, machine learning, and big data technologies. In a typical setting of this problem, each agent has a local cost function and all agents are expected to achieve a consensus on the optimal solution of the sum of these local cost functions. Many effective distributed algorithms have been proposed for single-integrator multi-agent systems to achieve this goal under various conditions (see [1], [10], [25] and references therein).

At the same time, there are plenty of optimization tasks implemented by or depending on engineering multi-agent systems of high-order dynamics, e.g., source seeking in mobile sensor networks [27], frequency control in power systems [28], and attitude formation control of rigid bodies [18]. Thus, some authors seek to solve the optimal consensus for non-single-integrator multi-agent systems, including second-order ones [13], [24], [29], general linear ones [20], and even special classes of nonlinear multi-agent systems [22], [23].

So far, most optimal consensus works were only devoted to the cases when we have a prior knowledge of the control directions of agents’ dynamics. Note that the control directions may not always be known beforehand in many applications. For example, under some steering conditions like a course-changing operation, the control direction of a ship may be unknown [3]. Even if it is known at first, the control direction of a plant may be changed by some structural damage [9]. Thus, it is important to consider the unknown control direction issue when dealing with the optimal consensus problem for high-order engineering multi-agent systems.

A standard way to handle the unknown control directions is the Nussbaum gain technique [11], which can dynamically generate oscillating control gains to ensure that both positive and negative control directions are tried. Some recent attempts have been further made to solve multi-agent coordination problems by extending the classical Nussbaum-type controls to decentralized and distributed cases. For example, [12] proposed a Nussbaum-type adaptive controller for single-integrator agent such that consensus for this multi-agent system can be achieved. Parameterized uncertainties were also considered for both first and second order agents by constructing a special type of Nussbaum-type functions in [2]. However, this result relied on the assumption that all high-frequency gains have the same sign. This limitation has been removed in [5] for multi-agent systems with more general dynamics. Nevertheless, when the control directions are unknown, the solvability of optimal consensus problem even for single-integrator agents is still unclear.

Based on the aforementioned observations, we consider the optimal consensus problem for a group of high-order multi-agent systems, which are allowed to have heterogeneous system orders and unknown control directions. Although some pure consensus and optimal consensus results have been partially done for this multi-agent system in literature [14], [15], [17], [20], [26], its optimal consensus problem without knowing the control directions is much more challenging. In fact, the gradient-based rules are basically nonlinear in light of the optimization requirement for the multi-agent system. More importantly, the unknown control directions of these agents bring many extra technical difficulties to the associated analysis and design.

Motivated by the given designs in [20], we aim to develop an embedded control to solve the formulated optimal consensus problem for these agents. Jointly with some Nussbaum-type arguments, we propose a distributed adaptive controller for each agent and the resultant trajectories of all agents are shown to be well-defined and achieve an optimal consensus on the minimizer of the global cost function. To the best knowledge of us, this might be the first attempt to solve such an optimal consensus problem for heterogeneous high-order agents with unknown control directions. Moreover, as pure consensus and average consensus can be completed by solving an optimal consensus problem, our algorithms naturally provide an alternative way to tackle such problems for these agents with unknown control directions extending the pure and average consensus results derived in [14], [15].

The rest of this paper is organized as follows. Some preliminaries are provided in Section II. The problem formulation part is given in Section III. Main results are presented in Section IV. Finally, simulations and our concluding remarks are presented at Sections V and VI.
II. PRELIMINARIES

We will use standard notations. Let $\mathbb{R}^N$ be the $N$-dimensional Euclidean space. Denote $|a|$ the Euclidean norm of a vector $a$ and $||A||$ the spectral norm of a matrix $A$. $I_N$ (or $0_N$) denotes an $N$-dimensional all-one (or all-zero) column vector, and $I_N$ denotes the $N$-dimensional identity matrix. We may omit the subscript when it is self-evident.

A weighted directed graph (digraph) is described by $G = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ with node set $\mathcal{N} = \{1, \ldots, N\}$ and edge set $\mathcal{E}$. $(i, j) \in \mathcal{E}$ denotes an edge from node $i$ to node $j$. The weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is defined by $a_{ii} = 0$ and $a_{ij} \geq 0$. Here $a_{ij} > 0$ iff there is an edge $(j, i)$ in the digraph. Node $i$’s neighbor set is defined as $\mathcal{N}_i = \{j \mid (j, i) \in \mathcal{E}\}$. We denote $\mathcal{N}_i^0 = \mathcal{N}_i \cup \{i\}$. A directed path is an alternating sequence $i_1i_2 \ldots i_k$ of nodes $i_l \in \mathcal{N}$ and edge $e_m = (i_m, i_{m+1}) \in \mathcal{E}$ for $l = 1, 2, \ldots, k$ and $m = 1, 2, \ldots, k-1$. If $a_{ij} = a_{ji}$ for any $i, j \in \mathcal{N}$, we say this graph is undirected. If there is a directed path between any two nodes, then the digraph is said to be strongly connected. A strongly connected undirected graph is said to be connected.

The in-degree and out-degree of node $i$ are defined by $d_i^\text{in} = \sum_{j=1}^{N} a_{ij}$ and $d_i^\text{out} = \sum_{j=1}^{N} a_{ji}$. The Laplacian of digraph $G$ is defined as $L = D^\text{in} - \mathcal{A}$ with $D^\text{in} = \text{diag}(d_1^\text{in}, \ldots, d_N^\text{in})$. Note that $L1_N = 0_N$ for any digraph. For an undirected graph, its Laplacian is symmetric and we can order its eigenvalues as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. Then, $\lambda_2 > 0$ if and only if it is connected.

A function $f: \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be convex if for $0 \leq a \leq 1$, we have
$$f(a\xi_1 + (1-a)\xi_2) \leq a f(\xi_1) + (1-a) f(\xi_2), \quad \forall \xi_1, \xi_2 \in \mathbb{R}^m$$
When the function $f$ is differentiable, it is verified that $f$ is convex if the following inequality holds,
$$f(\xi_1) - f(\xi_2) \geq \nabla f(\xi_2)^\top (\xi_1 - \xi_2), \quad \forall \xi_1, \xi_2 \in \mathbb{R}^m$$
It is $\omega$-strongly convex ($\omega > 0$) over $\mathbb{R}^m$ if we have
$$\langle \nabla f(\xi_1) - \nabla f(\xi_2), (\xi_1 - \xi_2) \rangle \geq \omega \|\xi_1 - \xi_2\|^2, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^m$$
A vector-valued function $\mathbf{f}: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is Lipschitz with constant $\vartheta > 0$ (or simply $\vartheta$-Lipschitz) if we have
$$\|\mathbf{f}(\xi_1) - \mathbf{f}(\xi_2)\| \leq \vartheta \|\xi_1 - \xi_2\|, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^m$$

III. PROBLEM FORMULATION

Consider a heterogeneous multi-agent system consisting of $N$ agents described by
$$y_i(t_n) = b_iu_i, \quad i = 1, \ldots, N \tag{1}$$
where $y_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are its output and input, respectively. Integer $n_i \geq 1$ is the order of system (1) and constant $b_i$ is assumed to be away from zero but unknown. This constant $b_i$ is often called the high-frequency gain of agent (1), which represents the motion direction of this agent in any control strategy. The parameters $n_i$ and $b_i$ of each agent are allowed to be different from each other.

We endow each agent with a local cost function $f_i: \mathbb{R} \rightarrow \mathbb{R}$, and define the global cost function as the sum of all local costs, i.e., $f(y) = \sum_{i=1}^{N} f_i(y_i)$. For multi-agent system (1), we aim to develop an algorithm such that all agent outputs achieve a consensus on the minimizer to this global cost function.

The following assumption is often made in literature [6], [8], [16], [22], which guarantees the existence and uniqueness of the minimal solution to function $f$.

Assumption 1: For $i = 1, \ldots, N$, function $f_i$ is $L$-strongly convex and its gradient is $\tilde{L}$-Lipschitz for two constants $\tilde{L} \geq L > 0$.

As usual, we assume this optimal solution is finite and denote it as $y^*$, i.e.,
$$y^* = \arg \min_{y \in \mathbb{R}} f(y) = \sum_{i=1}^{N} f_i(y) \tag{2}$$

Due to the privacy of local cost function $f_i$, no agent can unilaterally determine the global optimal solution $y^*$ by itself. Hence, our problem cannot be solved without cooperation and information sharing among these agents. For this purpose, we use a weighted undirected graph $G = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ to describe the information sharing topology with note set $\mathcal{N} = \{1, \ldots, N\}$. An edge $(i, j) \in \mathcal{E}$ between nodes $i$ and $j$ means that agent $i$ and agent $j$ can share information with each other.

To guarantee that any agent’s information can reach any other agents, we suppose the following assumption holds.

Assumption 2: The graph $G$ is undirected and connected.

Then, our optimal consensus problem is to design $u_i$ for each agent under the information constraint described by graph $G$, such that this multi-agent system achieves an optimal consensus determined by the global objective function $f$ in the sense that $y_i - y^* \rightarrow 0$ as $t \rightarrow \infty$ for any $i = 1, \ldots, N$, while all trajectories of the overall multi-agent system are maintained to be bounded.

Remark 1: This optimal consensus problem has been extensively studied in literature for multi-agent systems assuming the high-frequency gain is known. But in our work, this prior knowledge of each agents control direction is no longer necessary, which means that agents may have different and unknown control directions. To the best of our knowledge, no other works have studied the optimal consensus problem under these circumstances yet.

It is interesting to remark that when the local cost functions are chosen as $f_i(y) = c_i(y - y_i(0))^2$ with $c_i > 0$ for $i = 1, \ldots, N$, we actually solve a scaled consensus problem with the final consensus point $y^* = \sum_{i=1}^{N} c_i y_i(0)$. Thus, this formulation provides an applicable way to solve this type of multi-agent cooperation problems [14], [15] for high-order agents with unknown control directions.

IV. MAIN RESULT

In this section, we will present an embedded design to solve our formulated optimal consensus problem following the technical line developed in [20].
To this end, we first consider an auxiliary optimal consensus problem with the same requirement for agents in form of $\dot{r}_i = \mu_i$ and then convert our problem into some output tracking control problem for agent $\mathbf{1}$ with reference $r_i$. As the former subproblem is essentially a conventional optimal consensus problem for single-integrator multi-agent system with $b_i = 1$ and has been well-studied in existing literature, we use the following optimal signal generator to complete our design:

$$
\dot{r}_i = -\nabla f_i(r_i) - \sum_{j=1}^{N} a_{ij}(\lambda_i - \lambda_j) \\
\lambda_i = \sum_{j=1}^{N} a_{ij}(r_i - r_j)
$$

(3)

Generator (3) is a distributed variant of primal-dual dynamics to solve equality-constrained optimization problems. The effectiveness of (3) has already been established in [20]. With this optimal signal generator (3), each agent can get an asymptotic estimate $r_i$ of the global optimal solution $y^\star$. Thus, we are left to solve an output tracking problem for agent $\mathbf{i}$ with reference $r_i$.

When $b_i = 1$, a pole-placement based tracking control was presented in [20] for multi-agent system (1) to complete the whole design. Controllers with bounded constraints were also developed to achieve an optimal consensus in literature [13], [24]. However, the control directions are assumed to be unknown in our current case. Consequently, such methods are no longer applicable to agent (1) and we have to seek new tracking rules to solve our optimal consensus problem.

For clearance, we denote $y_{i1} = y_i - r_i$ and $y_{ij} = y_{i(j-1)}$ for $2 \leq j \leq n_i$. Choose constants $k_{ij}$ for $1 \leq j \leq n_i - 1$ such that the polynomial $p_i(\lambda) = \sum_{j=1}^{n_i-1} k_{ij}\lambda^{j-1} + \lambda^{n_i-1}$ is Hurwitz. Performing a sliding-mode transformation $\zeta_i = \sum_{j=1}^{n_i-1} k_{ij}y_{ij} + y_{in_i}$ gives a translated multi-agent system as follows.

$$
\dot{z}_i = A_{i1}z_i + A_{i2}\zeta_i + E_{i1}\dot{r}_i \\
\dot{\zeta}_i = A_{i3}z_i + A_{i4}\zeta_i + b_iu_i + E_{i2}\dot{r}_i
$$

(4)

where $z_i = \text{col}(y_{i1}, \ldots, y_{in_i})$ and the associated matrices are defined as follows:

$$
A_{i1} = \begin{bmatrix} 0_{n_i-2} & I_{n_i-2} \\ -k_1 & \cdots & -k_{n_i-1} \end{bmatrix}, \quad A_{i2} = \begin{bmatrix} 0_{n_i-2} \\ 1 \end{bmatrix} \\
A_{i3} = [-k_{n_i-1}k_1, k_1-k_{n_i-1}k_2, \cdots, k_{n_i-2}-k_{n_i-1}] \\
A_{i4} = k_{n_i-1}, \quad E_{i1} = \begin{bmatrix} 1 & 0_{n_i-2} \end{bmatrix}^T, \quad E_{i2} = -k_1
$$

From the above analysis, we can take $\dot{r}_i$ as an exponentially vanishing perturbation according to Lemma 3 in [20] and thus convert the original optimal coordination problem into a robust stabilization problem of the translated system (4).

Motivated by the adaptive controllers in [12] and [21], we use the following Nussbaum-type rule to serve the reference tracking purpose.

$$
u_i = -\overline{\mathcal{V}}(\theta_i)\xi_i, \quad \dot{\theta}_i = \xi_i^2$$

where $\xi_i$ is defined as above and $\overline{\mathcal{V}}(\theta_i) = \theta_i^2 \sin(\theta_i)$. The overall controller to solve our optimal consensus problem is then as follows.

$$
u_i = -\overline{\mathcal{V}}(\theta_i)\xi_i, \quad \dot{\theta}_i = \xi_i^2 \\
\dot{r}_i = -\nabla f_i(r_i) - \sum_{j=1}^{N} a_{ij}(\lambda_i - \lambda_j) \\
\lambda_i = \sum_{j=1}^{N} a_{ij}(r_i - r_j)
$$

(5)

This controller is distributed in the sense of using only agent $i$’s own and exchanging information with its neighbors.

It is time to present our main theorem of this paper.

Theorem 1: Consider the multi-agent system consisting of $N$ agents given by (1). Suppose Assumptions 1 and 2 hold. Then, the optimal consensus problem for this multi-agent system (1) and (2) is solved by the controller (5).

Proof. Before giving proofs, we recall the effectiveness of optimal signal generator (3) by Lemma 3 in [20]. Thus, it suffices for us to solve the tracking problem for each agent is solved. Note that this tracking problem for agent (1) with reference $r_i$ can be further converted to the robust stabilization problem for the translated agent $\mathbf{4}$. Hence, we only have to show the state of the translated system (1) is well-defined and converges to the origin from any initial condition subject to a vanishing perturbation $\dot{r}_i$.

To this end, we first prove the boundedness of $\zeta_i$ and $\dot{\zeta}_i$ and the trajectory of agents can be extended to $+\infty$. Note that the local error system for agent $i$ is

$$
\dot{z}_i = A_{i1}z_i + A_{i2}\zeta_i + E_{i1}\dot{r}_i \\
\dot{\zeta}_i = A_{i3}z_i + A_{i4}\zeta_i + b_iu_i + E_{i2}\dot{r}_i
$$

(6)

where $A_{i1}$ is Hurwitz according to the choice of $k_{ij}$. Thus, there must a positive definite matrix $P_i \in \mathbb{R}^{n_i-1 \times n_i-1}$ such that $A_{i1}P_i + P_iA_{i1} = -2I_{n_i-1}$. From the smoothness of related functions, the trajectory of each subsystem is well-defined on its maximal interval $[0, t_{ij}]$. We claim that $t_{ij} = +\infty$ for each $i$. In the following, we will prove this by seeking a contradiction.

Assume $t_{ij}$ is finite. We are going to prove that all involved signals are bounded over the time interval $[0, t_{ij}]$. Take $V_i(\zeta_i, \dot{\zeta}_i) = \zeta_i^2 P_i\dot{z}_i + \frac{1}{2}\dot{\zeta}_i^2$ as a sub-Lyapunov function for agent $i$. It is positive definite with a time derivative along the trajectory of the above error system as follows.

$$V_i \leq 2\zeta_i^2 P_i[A_{i1}z_i + A_{i2}\zeta_i + E_{i1}\dot{r}_i] \\
+ \zeta_i^2(A_{i3}z_i + A_{i4}\zeta_i + b_i\overline{\mathcal{V}}(\theta_i)\xi_i + E_{i2}\dot{r}_i) \\
\leq -2||z_i||^2 + \frac{1}{3}||z_i||^3 + 3||P_{i}A_{i2}||^2 \xi_i^2 + \frac{1}{3}||\dot{z}_i||^2 \\
+ 2||P_{i}E_{i1}||^2 ||\dot{r}_i||^2 + \frac{1}{3}||\dot{z}_i||^2 + 3||A_{i3}||^2 ||\dot{z}_i||^2 + A_{i4}\xi_i^2 \\
- b_i\overline{\mathcal{V}}(\theta_i)\xi_i^2 + \xi_i^2 + ||E_{i2}||^2 ||\dot{r}_i||^2 \\
= -||z_i||^2 - (b_i\overline{\mathcal{V}}(\theta_i) - C_{i1}\theta_i)\xi_i^2 + C_{i2}\theta_i^2 \\
= -||z_i||^2 - (b_i\overline{\mathcal{V}}(\theta_i) - C_{i1}\theta_i)\dot{\theta}_i + C_{i2}\theta_i^2
$$

(6)
where we use Young’s inequality to handle the cross terms with \( C_{i\theta_i} = 3|P_{A_{i2}}|^2 + 3|A_{i3}|^2 + A_{i4} + 1 \) and \( C_{i\theta_i} = 2|P_{E_{i1}}|^2 + ||E_{i2}||^2 \).

According to Lemma 3 in [20], the signals \( r_i(t) \) and \( r_i(t) \) exponentially converge to \( y^* \) and 0 under Assumptions 1 and 2. Thus, \( r_i(t) \) is square-integrable over \( [0, +\infty) \). We denote \( V_i(t) \equiv V_i(z_i(t), \xi_i(t)) \) for short. By integrating both sides of (6) from 0 to \( t \), we have the following inequality

\[
V_i(t) - V_i(0) \leq -\int_{\theta_i(0)}^{\theta_i(t)} [b_{it} E_i(s) - C_{i\theta_i}] \, ds + C_{i\theta_i}
\]

for some constant \( C_{i\theta_i} > 0 \). Computing the integral \( |L_i(t)| \equiv \int_{\theta_i(0)}^{\theta_i(t)} [b_{it} E_i(s) - C_{i\theta_i}] \, ds \) by parts, one can obtain that

\[
|L_i(t)| = -b_i[\dot{\theta}_i(t) \cos(\theta_i(t)) - 2\dot{\theta}_i(t) \sin(\theta_i(t)) - 2\cos(\theta_i(t))] - C_{i\theta_i} \dot{\theta}_i(t) + C_{i\theta_i}
\]

with \( C_{i\theta_i} = C_{i\theta_i}(0) + b_i[\dot{\theta}_i(t) \cos(\theta_i(t)) - 2\dot{\theta}_i(t) \sin(\theta_i(t)) - 2\cos(\theta_i(t))] \).

Combining these inequalities gives

\[
V_i(t) \leq b_i[\dot{\theta}_i(t)^2 \cos(\theta_i(t)) - 2\dot{\theta}_i(t) \sin(\theta_i(t)) - 2\cos(\theta_i(t))] + C_{i\theta_i} \dot{\theta}_i(t) + C_i
\]

(7)

where \( C_i = V_i(0) + |C_{i\theta_i}| + C_{i\theta_i} \) is a finite constant.

As \( \dot{\theta}_i(t) \) is monotonically increasing, it either has a finite limit or grows to infinity. If \( \dot{\theta}_i(t) \) tends to be unbounded, we can choose some time sequence \( \{t_i(m)\} \) by its continuity such that

\[
\dot{\theta}_i(t_i(m)) = \begin{cases} 
(2m + 1)\pi, & \text{if } b_i > 0, \\
2m\pi, & \text{if } b_i < 0
\end{cases}
\]

By direct calculations, one has

\[
V_i(t_i(m)) \leq b_i[-(2m + 1)^2\pi^2 + 2] + (2m + 1)\theta_i(t_i(m) + C_i)
\]

for the case when \( b_i > 0 \) and

\[
V_i(t_i(m)) \leq b_i[4m^2\pi^2 - 2] + 2m\theta_i(t_i(m) + C_i)
\]

for the case when \( b_i < 0 \). From this, one can deduce that \( V_i(t_i(m)) \) is bounded over the time interval \([0, t_f] \) for each \( i = 1, 2, \ldots, N \).

Recalling the equations (5) and (7), the signals \( z_i(t), \xi_i(t), \) and \( \dot{\theta}_i(t) \) are also bounded over the time interval \([0, t_f] \) for each \( i = 1, 2, \ldots, N \). This implies with a contradiction argument that \( \dot{\theta}_i(t) \) must be bounded over the time interval \([0, t_f] \), which contradicts the positive definiteness of \( V_i(z_i, \xi_i) \). Therefore, \( \dot{\theta}_i(t) \) must be bounded over the time interval \([0, t_f] \) for each \( i = 1, 2, \ldots, N \). Thus, one can conclude that \( t_f = +\infty \).

From the boundedness of \( \dot{\theta}_i(t) \), the function \( \dot{\theta}_i(t) \) is uniformly continuous with respect to time \( t \). Also note that

\[
\int_0^t \xi_i^2(s) \, ds = \int_0^t \dot{\theta}_i(s) \, ds \leq \dot{\theta}_i(t) + \infty - \dot{\theta}_i(0)
\]

Since \( \dot{\theta}_i(\infty) \) exists and is bounded, \( \xi_i^2(t) \) is thus integrable. By Lemma 8.2 in [7], we have \( \xi_i(t) \to 0 \) as \( t \to \infty \).

Considering the \( z_i \)-subsystem, it is input-state stable with input \( A_{2i} \xi_i + E_{i1} r_i \) and state \( z_i \). Since both \( \xi_i(t) \) and \( r_i(t) \) converge to zero when \( t \to \infty \), we recall Theorem 1 in [19] and obtain that \( ||y_i(t) - r_i(t)|| \to 0 \) as \( t \to \infty \). By the triangle inequality and also the convergence of \( r_i(t) \) to \( y^* \), we further have that \( ||y_i(t) - y^*|| \leq ||y_i(t) - r_i(t)|| + ||r_i(t) - y^*|| \to 0 \) as \( t \to \infty \). The proof is thus complete.

Remark 2: In the developed controller (5), multiple Nussbaum gains are employed to tackle the technical problem brought by unknown heterogeneous high-frequency-gain signs. Although we choose \( \dot{\theta}_i \cos(\theta(t)) \) here in our design, other Nussbaum-type functions such as \( \dot{\theta}_i \cos(\theta(t)) \) and \( e^{\dot{\theta}_i} \cos(\theta(t)) \) can also be verified as well to achieve the expected optimal consensus among this multi-agent system.

Remark 3: In contrast to most optimal consensus works, we remove the requirement of a prior knowledge of agents’ control directions. The obtained result definitely extends existing optimal consensus conclusions to allow such type of system uncertainties.

Remark 4: Compared with the previous pure consensus results for single-integrator agents with unknown control directions in [12], high-order dynamics and an optimization requirement are further considered in our formulation. At the same time, by letting \( f_i(y) = (y - y_i(0))^2 \), this theorem provides an alternative way to achieve an average consensus goal other than [14], [15] even these agents have heterogeneous uncertain control directions.

V. SIMULATION

In this section, we propose two numerical examples to verify the effectiveness of our design.

Example 1. Consider a four-agent network and each agent is described by double-integrator dynamics, that is,

\[
y = b_i u_i, \quad i = 1, \ldots, 4
\]
Assume their interconnection topology is depicted in Fig. 1 with unity weights. This graph satisfies Assumption 2 in Theorem 1. We are going to solve an average consensus for these agents.

According to Remark 4, it suffices for us to set $f_i(y) = (y - y_i(0))^2$ for $i = 1, \ldots, 4$ and use the controller (5) with $n_i = 2$ to complete the design.

For simulation, we let $b_1 = b_2 = -1$, $b_3 = b_4 = 1$ and choose $k_{21} = 1, k_{31} = 1, k_{32} = 2, k_{41} = 1, k_{42} = 3, k_{43} = 3$. The simulation result is depicted in Fig. 5 where outputs of all agents are observed to finally achieve a consensus on the global point $y^* = 3.24$. The profiles of agents’ control efforts and adaptive gains are showed in Figs. 3 and 4 respectively. One can observe that the trajectories of agents finally achieve a consensus on the average of their initial output values, while the adaptive gains remain bounded.

Example 2. Consider the optimal consensus problem for a heterogeneous multi-agent system with agents described by

\[ y^{(i)} = b_i u_i, \quad i = 1, 2, 3, 4 \]

Note that the system orders of these agents are different from each other. To make this problem more interesting, we take the local cost functions as follows.

\[ f_1(y) = (y - 8)^2 \]
\[ f_2(y) = \frac{y^2}{20\sqrt{y^2 + 1}} + y^2 \]
\[ f_3(y) = \frac{y^2}{80\ln(y^2 + 2)} + (y - 5)^2 \]
\[ f_4(y) = \ln \left( e^{-0.05y} + e^{0.05y} \right) + y^2 \]

Assumption 1 can be confirmed with $l = 1$ and $l = 3$ for $i = 1, \ldots, 4$ as that in [22]. Moreover, the global optimal point can be obtained numerically as $y^* = 3.24$.

Since all agents are of nonidentical system orders and unknown high-frequency gains, the rules proposed in [20], [24] fail to tackle this problem. Nevertheless, according to Theorem 1, we can utilize the controller (5) to solve our optimal consensus problem.

For simulation, we let $b_1 = b_2 = -1$, $b_3 = b_4 = 1$ and choose $k_{21} = 1, k_{31} = 1, k_{32} = 2, k_{41} = 1, k_{42} = 3, k_{43} = 3$. The simulation result is depicted in Fig. 6 where outputs of all agents are observed to finally achieve a consensus on the global point $y^* = 3.24$. The profiles of agents’ control efforts and adaptive gains are showed in Figs. 6 and 7 which are both maintained bounded. These observations verify the efficiency of our adaptive designs in handling both heterogeneous agent dynamics and also unknown control directions.

VI. CONCLUSION

An optimal consensus problem has been discussed for a high-order multi-agent system without a prior knowledge of their control directions. With the help of some Nussbaum-type control laws, we finally solve this problem by an
embedded design in a distributed manner. Further work will include the extensions for multi-agent agent systems with more general dynamics and possible disturbances.

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