State-efficient QFA Algorithm for Quantum Computers*

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Abstract. The study of quantum finite automata (QFA’s) is one of the possible approaches in exploring quantum computers with finite memory. Despite being one of the most restricted models, Moore-Crutchfield quantum finite automaton (MCQFA) is proven to be exponentially more succinct than classical finite automata models in recognizing certain languages such as \( \text{MOD}_p = \{ a^j \mid j \equiv 0 \mod p \} \), where \( p \) is a prime number. In this paper, we present a modified MCQFA algorithm for the language \( \text{MOD}_p \), the operators of which are selected based on the basis gates on the available real quantum computers. As a consequence, we obtain shorter quantum programs using less basis gates compared to the implementation of the original algorithm given in the literature.

Keywords: quantum finite automata · state-efficiency · quantum circuit · unary languages · bounded-error · Qiskit · IBMQ backends

1 Introduction

Quantum finite automaton (QFA) is a theoretical model for studying quantum computers with finite memory. The most natural extension from a classical automaton to QFA is obtained by replacing the transition matrices with unitary operators. The obtained model is called Moore-Crutchfield quantum finite automaton (MCQFA) \([8]\), and it is one of the earliest models. Even though some regular languages can not be recognized by MCQFAs, MCQFAs are proven to be exponentially more succinct for certain languages such as the language \( \text{MOD}_p = \{ a^j \mid j \equiv 0 \mod p \} \), where \( p \) is a prime number \([2]\).

The proposed MCQFA recognizing \( \text{MOD}_p \) language consists of sub-automata whose actions can be visualized as a sequence of rotations on the 2D plane, correspondingly around the \( y \)-axis on the Bloch Sphere. Several implementation ideas have been proposed in \([6]\) to realize the algorithms in gate-based quantum computers. Some new ideas and simulation results on IBMQ backends \([1]\) have recently appeared in \([5]\).

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When a quantum circuit is run on an IBMQ computer, the circuit is restructured to fit the topology of the computer, arbitrary gates are decomposed into basis gates, and the circuit is optimized to improve the performance. This process is known as transpilation. As the depth and the number of 2-qubit gates increase, the performance deteriorates due to noise. Hence, it becomes crucial to design quantum circuits with smaller depths and fewer gates.

With this motivation, we propose a modified MCQFA algorithm for the $\text{MOD}_p$ language. The proposed algorithm performs rotations around the $x$-axis on the Bloch Sphere and its correctness is proven through a change of basis. We show that the optimized implementation of the original algorithm from [6] can be adapted for our modified algorithm. When transpiled to be run on the real quantum backends, our algorithm algorithm is more efficient in the number of gates and has a smaller circuit depth. We also provide some experimental results from $\text{ibmq} \_\text{belem}$ backend.

2 Background

In this section, we provide information on quantum finite automata and IBMQ Qiskit framework.

2.1 Preliminaries

We denote the finite alphabet by $\Sigma$ not containing the left end-marker $\$ and the right end-marker $\$, and $\bar{\Sigma}$ denotes $\Sigma \cup \{\$, $\\}$. The length of any string $w \in \Sigma^*$ is denoted by $|w|$, and $w[i]$ denotes its $i$th symbol. The string $w$ is processed by an automaton as $ew\$ from left to right and symbol by symbol.

Among the many different quantum finite automaton (QFA) models proposed in the literature [4], we will focus on Moore-Crutchfield quantum finite automaton (MCQFA) [8], which is known as the most restricted model.

Formally, a MCQFA $M$ with $d$ states is a 5-tuple 

$$M = (Q, \Sigma, \{U_\sigma \mid \sigma \in \bar{\Sigma}\}, q_s, Q_A),$$

where $Q = \{q_1, \ldots, q_d\}$ is the set of states, $U_\sigma$ is the unitary operator for symbol $\sigma \in \bar{\Sigma}$, $q_s \in Q$ is the start state, and $Q_A \subseteq Q$ is the set of accepting state(s).

We can trace the computation of $M$ on a given input $w \in \Sigma^*$ with length $l$ by a $d$-dimensional column vector (quantum state or state vector), where the $j$th entry represents the amplitude of $q_j$. At the beginning of computation, $M$ starts in quantum state $|v_0\rangle = |q_s\rangle$, which has zeros except its $s$th entry that is 1. For each symbol $\sigma \in \bar{\Sigma}$, the unitary operator $U_\sigma$ is applied to the state vector. Thus, $M$ is in the following final state after reading the whole input:

$$|v_f\rangle = U_\$U_{w[l]}U_{w[l-1]} \cdots U_{w[1]}U_\epsilon |v_0\rangle.$$

Then, the final state is measured in the computational basis, and the input is accepted if and only if an accepting state is observed. The accepting probability
is calculated as
\[ \sum_{q_j \in Q_A} |\langle q_j | v_f \rangle|^2, \]
where \( \langle q_j | v_f \rangle \) gives the amplitude of \( q_j \) in \( |v_f\rangle \).

2.2 Qiskit

Qiskit is the quantum programming framework provided by IBM Quantum [1]. It allows designing quantum circuits that can be simulated in local simulators as well as real quantum computers. Before running a circuit in a real machine, the circuit is transpiled into basis gates which are defined as the set \( \{ CX, I, R_Z, S_X, X \} \). The matrices corresponding to the gates in the basis set are given in Figure 1.

The transpilation process also depends on the backend that is chosen and the selected optimization level. To run our experiments, we used \textit{ibmq.belem} backend and optimization level 2.

3 MCQFAs for the \( \text{MOD}_p \) Problem

Any bounded-error probabilistic finite automaton recognizing \( \text{MOD}_p \) requires at least \( p \) states while it was proven in [2] that there exists an MCQFA with \( O(\log p) \) states recognizing the language \( \text{MOD}_p \) with bounded error.

3.1 2-state MCQFA for \( \text{MOD}_p \)

Let us start by describing the well-known 2-state MCQFA construction for the language \( \text{MOD}_p \). Let \( M_p = (\{q_1, q_2\}, \{a\}, \{U_\sigma | \sigma \in \Sigma, q_1, \{q_1\}\}) \), where the unitary operators \( U_\epsilon \) and \( U_\delta \) are defined as the \( 2 \times 2 \) identity matrix, and operator \( U_a \)
is defined as \[
\begin{pmatrix}
\cos \left( \frac{2\pi}{p} \right) & -\sin \left( \frac{2\pi}{p} \right) \\
\sin \left( \frac{2\pi}{p} \right) & \cos \left( \frac{2\pi}{p} \right)
\end{pmatrix},
\]
and corresponds to a counter-clockwise rotation with angle \( \frac{2\pi}{p} \) on the 2D plane spanned by \( |q_1\rangle \) and \( |q_2\rangle \) as visualized in Figure 2.

![Figure 2: Visualization of the operator \( U_a \) on the 2D plane.](image)

After reading the string \( a^j \), the state of \( M_p \) is \( \cos \left( \frac{2\pi j}{p} \right) |q_1\rangle + \sin \left( \frac{2\pi j}{p} \right) |q_2\rangle \). It is easy to see that for member strings, that is when \( j \) is a multiple of \( p \), the probability of acceptance is equal to 1. For non-member strings, the acceptance probability is given by \( \cos^2 \left( \frac{2\pi |w|}{p} \right) \), which gets closer to 1 when \( |w| \) approaches an integer multiple of \( p \). Hence, the error for non-member strings can be arbitrarily large.

Next, we will describe an alternative MCQFA for the language \( \text{MOD}_p \), based on the following observation. Let
\[
U_a' = \begin{pmatrix} e^{-2\pi i/p} & 0 \\ 0 & e^{2\pi i/p} \end{pmatrix} \quad \text{and} \quad SX = \frac{1}{2} \begin{pmatrix} 1 + i & 1 - i \\ 1 - i & 1 + i \end{pmatrix}.
\]

**Lemma 1.** \( SX^\dagger U_a' SX = U_a \).

**Proof.** To start with note that
\[
SX = \frac{1}{2} \begin{pmatrix} 1 + i & 1 - i \\ 1 - i & 1 + i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} & e^{-i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{pmatrix} = \frac{1}{\sqrt{2}} e^{-i\pi/4} \begin{pmatrix} e^{i\pi/2} & 1 \\ 1 & e^{i\pi/2} \end{pmatrix}.
\]

We calculate \( SX^\dagger U_a' SX \)
\[
= \frac{1}{\sqrt{2}} e^{i\pi/4} \begin{pmatrix} e^{-i\pi/2} & 1 \\ 1 & e^{-i\pi/2} \end{pmatrix} \begin{pmatrix} e^{-2\pi i/p} & 0 \\ 0 & e^{2\pi i/p} \end{pmatrix} \frac{1}{\sqrt{2}} e^{-i\pi/4} \begin{pmatrix} e^{i\pi/2} & 1 \\ 1 & e^{i\pi/2} \end{pmatrix}
= \frac{1}{2} \begin{pmatrix} e^{-i\pi/2} & 1 \\ 1 & e^{-i\pi/2} \end{pmatrix} \begin{pmatrix} e^{i(\pi/2-2\pi/p)} & e^{-i2\pi/p} \\ e^{i2\pi/p} & e^{i(\pi/2+2\pi/p)} \end{pmatrix}
= \frac{1}{2} \begin{pmatrix} e^{-i2\pi/p} + e^{i2\pi/p} & e^{-i(\pi/2+2\pi/p)} + e^{i(\pi/2+2\pi/p)} \\ e^{i(\pi/2-2\pi/p)} + e^{-i(\pi/2-2\pi/p)} & e^{-i2\pi/p} + e^{i2\pi/p} \end{pmatrix}.
\]
We calculate each term of this matrix one by one by using the following trigonometric properties: (i) \( \cos(-\theta) = \cos(\theta) \), (ii) \( \sin(-\theta) = -\sin(\theta) \), (iii) \( e^{i\theta} + e^{-i\theta} = 2\cos(\theta) \), and (iv) \( \cos(\theta + \pi/2) = -\sin(\theta) \).

The top-left and the bottom-right terms are equal to
\[
e^{-i2\pi/p} + e^{i2\pi/p} = 2\cos(2\pi/p).
\]
The top-right term is equal to
\[
e^{-i(\pi/2+2\pi/p)} + e^{i(\pi/2+2\pi/p)} = 2\cos(\pi/2+2\pi/p) = -2\sin(2\pi/p).
\]
The bottom-left term is equal to
\[
\frac{1}{2}
\begin{pmatrix}
1 + i & 1 - i \\
1 - i & 1 + i
\end{pmatrix}.
\]
Thus, we obtain that
\[
SX^\dagger U_aSX = \frac{1}{2}
\begin{pmatrix}
\cos(2\pi/p) & -\sin(2\pi/p) \\
\sin(2\pi/p) & \cos(2\pi/p)
\end{pmatrix} = U_a.
\]

The above lemma suggests that the operator \( U_a \) for symbol \( a \) can be replaced with \( SX^\dagger U_aSX \). Now note that \( (SX^\dagger U_aSX)^j = SX^\dagger U_a^jSX \) for any \( j \geq 0 \). This suggests a new automaton which applies \( SX \) at the beginning of the computation, applies the unitary operator \( U_a \) for each scanned \( a \) and applies \( SX \) at the end of the computation. Based on this observation, we describe the following MCQFA.

Let \( M'_p = (Q, \Sigma, \{U'_\sigma \mid \sigma \in \tilde{\Sigma}\}, q_s, Q_A) \) where \( \Sigma, Q, q_s \) and \( Q_A \) are defined as before. The operator \( U'_c \) is defined as
\[
U'_c = SX = \frac{1}{2}
\begin{pmatrix}
1 + i & 1 - i \\
1 - i & 1 + i
\end{pmatrix}.
\]
The operator \( U'_a \) is defined as
\[
U'_a = \begin{pmatrix}
e^{-2\pi i/p} & 0 \\
0 & e^{2\pi i/p}
\end{pmatrix},
\]
\( \tilde{U}_s = \tilde{U}_c^\dagger \) and represented with the following matrix:
\[
U_s = SX^\dagger = \frac{1}{2}
\begin{pmatrix}
1 - i & 1 + i \\
1 + i & 1 - i
\end{pmatrix}.
\]

The relationship between \( M'_p \) and \( M_p \) can be also explained with a change of basis between \( V' = \{ |0'\rangle, |1'\rangle \} \) and \( V = \{ |0\rangle, |1\rangle \} \) where
\[
|0'\rangle = \frac{1}{2}
\begin{pmatrix}
1 + i \\
1 - i
\end{pmatrix},
|1'\rangle = \frac{1}{2}
\begin{pmatrix}
1 - i \\
1 + i
\end{pmatrix}.
\]

and with the change of basis matrix \( SX^\dagger \) that maps \( |0'\rangle \) to \( |0\rangle \) and \( |1'\rangle \) to \( |1\rangle \). In this manner, the linear transformation \( U'_a \) working on the vector space spanned by \( V' \) is mapped to the vector space spanned by \( V \) through the mapping \( SX^\dagger U'_aSX \).
3.2 \(O(\log p)\)-State MCQFA for \(\text{MOD}_p\)

In the above constructions, the rotation angle is selected as \(2\pi/p\). Note that one can use the rotation angle \(2\pi k/p\) for some \(k \in \{1, \ldots, p-1\}\), and depending on the value of \(k\), the acceptance probability for each non-member string varies. Nevertheless, the maximum acceptance probability of a non-member string is still not bounded. Combining multiple automata with different rotation angles, it is proven that for any \(p\), there exists an \(O(\log p)\)-state MCQFA recognizing the \(\text{MOD}_p\) language with bounded error in [2]. Let us start by recalling the construction.

2-state MCQFA \(M^p_k\) is defined similar to \(M^p\), except that it performs a rotation of angle \(2\pi k/p\) on the 2D plane. Let \(K = \{k_1, \ldots, k_d\}\) where each \(k_j \in \{1, \ldots, p-1\}\). We define \(2d\)-state MCQFA \(M^K_p = (\tilde{Q}, \Sigma, \{\tilde{U}_\sigma \mid \sigma \in \tilde{\Sigma}\}, \tilde{q}_s, \tilde{Q}_A)\). The state set \(\tilde{Q}\) is defined as \(\{q^1_1, q^1_2, q^2_1, q^2_2, \ldots, q^d_1, q^d_2\}\), where \(q^l_1, q^l_2\) belongs to state set of \(M^k_l\), \(\tilde{q}_s = q^1_1\) and \(\tilde{Q}_A = \{q^1_1\}\).

\(M^K_p\) simulates the individual \(M^k_l\)’s in parallel, through the following unitary transformations. \(\tilde{U}_e\) is defined as \(H^\otimes \log(2d) - 1 \otimes I\) and it creates the following equal superposition state:

\[
|q^1_1\rangle \xrightarrow{\tilde{U}_e} \frac{1}{\sqrt{d}} |q^1_1\rangle + \frac{1}{\sqrt{d}} |q^2_1\rangle + \cdots + \frac{1}{\sqrt{d}} |q^d_1\rangle .
\]

Upon reading each \(a\), the operator \(\tilde{U}_a\) is applied which is defined as

\[
\tilde{U}_a = \bigoplus_{l=1}^{d} R_l = \begin{pmatrix}
R_1 & 0 & \cdots & 0 \\
0 & R_2 & \cdots & 0 \\
: & : & \ddots & : \\
0 & 0 & \cdots & R_d
\end{pmatrix},
\]

where

\[
R_l = \begin{pmatrix}
\cos(2\pi k_l/p) & -\sin(2\pi k_l/p) \\
\sin(2\pi k_l/p) & \cos(2\pi k_l/p)
\end{pmatrix}
\]

The operator \(\tilde{U}_a\) helps executing each \(M^k_l\) where \(M^k_l\’s\) are counter-clockwise rotations by angle \(2\pi k_l/p\). The unitary operator \(\tilde{U}_e\) is defined as \(\tilde{U}_e^{-1}\).

In [3], they are proven that after reading \(a^j\), the acceptance probability is equal to 1 if \(j\) is divisible by \(p\) and otherwise equals \(\frac{1}{p} \sum_{i=1}^{d} \cos^2(2\pi k_i j/p)\); and, when \(j\) is not a multiple of \(p\), the acceptance probability is less than \(\varepsilon\) for \(d = 2\log_2 p\), resulting in an \(O(\log p)\)-state MCQFA recognizing the \(\text{MOD}_p\) language with bounded error.

Next we will describe an alternative \(O(\log p)\)-state MCQFA using the observation made in Lemma 1. To start with note that by Lemma 1 it follows that

\[
SX^l R'_l SX = R_l \text{ for any } l = 1, \ldots, d
\]

where

\[
R'_l = \begin{pmatrix}
e^{-2\pi i k_j/p} & 0 \\
0 & e^{2\pi i k_j/p}
\end{pmatrix}
\]
Let’s define
\[ \tilde{U}_a' = \bigoplus_{l=1}^{d} R'_l = \begin{pmatrix}
R'_1 & 0 & \cdots & 0 \\
0 & R'_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R'_d
\end{pmatrix}. \]

Since \( SX\dagger SX = I \), the following holds:
\[
\begin{pmatrix}
SX\dagger R'_1SX & 0 & \cdots & 0 \\
0 & SX\dagger R'_2SX & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & SX\dagger R'_dSX
\end{pmatrix}^j
= \begin{pmatrix}
SX\dagger & 0 & \cdots & 0 \\
0 & SX\dagger & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & SX\dagger
\end{pmatrix}
\begin{pmatrix}
R'_1 & 0 & \cdots & 0 \\
0 & R'_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R'_d
\end{pmatrix}^j
\begin{pmatrix}
SX & 0 & \cdots & 0 \\
0 & SX & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & SX
\end{pmatrix}
= (I \otimes \log(2^d) - 1 \otimes SX\dagger)(\tilde{U}_a')^j(I \otimes \log(2^d) - 1 \otimes SX) \]

Using the above equality, we present automaton \((\tilde{M}_p^K)' = (\tilde{Q}, \Sigma, \{\tilde{U}_\sigma' | \sigma \in \hat{\Sigma}\}, q_s, Q_A)\) where \(\Sigma, \tilde{Q}, \tilde{q}_s\), and \(Q_A = \{q^1\}\) are defined as above. The unitary operator \(\tilde{U}_c'\) is defined as
\[ \tilde{U}_c' = H^\otimes \log(2^d) - 1 \otimes U_c'. \]
the operator \(\tilde{U}_a'\) defined as above, and
\[ \tilde{U}_a' = \tilde{U}_c' = H^\otimes \log(2^d) - 1 \otimes U_c'. \]

4 Implementations

In this section, we will present the quantum circuit implementations for the automata given in the previous section

4.1 Single Qubit Implementation

Let us recall the implementation of single-qubit MCQFA \(M_{11}\) from [5]. The operator \(U_a\) is simply implemented by a \(R_y\) gate with the angle of rotation \(4\pi/p\), since in the Bloch sphere the angle between the states \(|0\rangle\) and \(|1\rangle\) is 180 degrees. (The matrix corresponding to \(R_y\) gate in Qiskit is given in Section 2). The circuit diagram for the string \(aa\) is given in Figure 3.

In Figure 4, the computation is visualized on the Bloch sphere for a string of length 11 and 4, respectively. The computation starts in state \(|0\rangle\) which is shown by the green vector. In the figure on the left hand side, the state obtained
Fig. 3: Single qubit MOD$_{11}$ implementation for string $aa$.

after applying $j$ $R_y$ gates is numbered as $j$ and shown using blue dots. After 11 rotations, the final quantum state is $|0\rangle$, resulting in the acceptance of the string with probability 1. On the right hand side, $R_y$ gate is applied for 4 times and the final vector is shown in red.

Fig. 4: Visualization of the original algorithm on the Bloch Sphere.

2-state MCQFA $\tilde{M}_{11}$ recognizing MOD$_{11}$ can be similarly realized using a single qubit. The operators $\tilde{U}_a$ and $\tilde{U}_s$ are implemented using the $Sx$-gate in Qiskit. The operator $\tilde{U}_a$ is implemented using a $Rz$-gate, providing two times the angle. In Figure 5, the circuit diagram is given for the input string $aa$.

Fig. 5: Alternative single qubit MOD$_{11}$ implementation for the string $aa$

Computation of the new algorithm is visualized in Figure 6. In (a), the initial state $|0\rangle$ is shown in green and the state obtained after reading the right end-marker is drawn in orange. In (b), the states obtained after reading $a^l$ and applying $j$ $R_y$ gates is visualized. After reading the end-marker, the final state becomes $|0\rangle$ which is the accept state. In (c), computation for a string of length 4 is visualized and the final state is drawn in red.

Since $R_y$ gate is not among the set of basis gates, the circuit for the original algorithm is transpiled as depicted in Figure 7 before running on the real ma-
chines. Each $R_y$ gate is decomposed into two $R_z$ gates and two $S_x$ gates, overall requiring $2j R_y$ gates and $2j S_x$ gates for an input string $a^j$.

$q_0 : |0\rangle \xrightarrow{S_x} R_z(\frac{15\pi}{11}) \xrightarrow{S_x} R_z(\pi) \xrightarrow{S_x} R_z(\frac{15\pi}{11}) \xrightarrow{S_x} R_z(\pi) \rightarrow c_0$

Since the new algorithm we have proposed uses only basis gates, the number of required gates does not change after the transpilation process requiring only two $S_x$ gates and $j R_z$-gates for the input string $a^j$.

Let us note that the $R_z$ gates in Qiskit are implemented virtually taking zero duration and with perfect accuracy [7]. Hence, running single qubit automata for any word length uses only two physical $S_x$ gates.

4.2 Optimized Implementation

Implementation of the operator $\tilde{U}$ is costly because it requires multiple controlled rotation gates as discussed in [6,5]. To overcome this issue, implementation of the following operator is proposed in [6] instead of implementing the unitary operator $\tilde{U}$.

$$\tilde{U}_a = \begin{pmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2R_1 & 0 & 0 \\ 0 & 0 & R_3R_1 & 0 \\ 0 & 0 & 0 & R_3R_2R_1 \end{pmatrix}$$

The proposed operator $\tilde{U}_a$ does not rotate each sub-automaton with a different angle but instead each sub-automaton rotates with a combination of the chosen 3 angles. The circuit diagram implementing $\tilde{U}_a$ is given in Figure 8.
Fig. 8: Optimized implementation for MOD$^{11}$ using the original algorithm.

To construct the optimized circuit for the new algorithm we have proposed, we can similarly implement the operator $\hat{U}^{'}_a$ instead of $\tilde{U}^{'}_a$:

$$\hat{U}^{'}_a = \begin{pmatrix} R_1' & 0 & 0 & 0 \\ 0 & R_2'R_1' & 0 & 0 \\ 0 & 0 & R_3'R_1' & 0 \\ 0 & 0 & 0 & R_3'R_2'R_1' \end{pmatrix}. $$

Let’s trace the computation step by step. Upon reading the left end-marker, the machine is in the following state:

$$|\psi_{\text{left}}\rangle = \frac{1}{2} \sum_{j=0}^{3} |j\rangle \otimes \left( \frac{1+i}{2} |0\rangle + \frac{1-i}{2} |1\rangle \right). $$

Recalling that $R_j'$ is implemented using an $R_z$ gate say with angle $\phi_j$, after reading the first $a$ the superposition becomes

$$|\psi_1\rangle = \frac{1}{2} |00\rangle \otimes R_z(\phi_1) \left( \frac{1+i}{2} |0\rangle + \frac{1-i}{2} |1\rangle \right) + $$ $$\frac{1}{2} |01\rangle \otimes R_z(\phi_2)R_z(\phi_1) \left( \frac{1+i}{2} |0\rangle + \frac{1-i}{2} |1\rangle \right) + $$ $$\frac{1}{2} |10\rangle \otimes R_z(\phi_3)R_z(\phi_1) \left( \frac{1+i}{2} |0\rangle + \frac{1-i}{2} |1\rangle \right) + $$ $$\frac{1}{2} |11\rangle \otimes R_z(\phi_3)R_z(\phi_2)R_z(\phi_1) \left( \frac{1+i}{2} |0\rangle + \frac{1-i}{2} |1\rangle \right). $$

Letting $\theta_1 = \phi_1$, $\theta_2 = \phi_2 + \phi_1$ and $\theta_3 = \phi_1 + \phi_2 + \phi_3$, we give the circuit diagram in Figure 9.

After the transpilation, the number of basis gates required by both implementations and the depth of the circuit for a string of length 11 is given in Table 1. Note that the transpilation of controlled rotation gates introduces CX gates. The difference in the number of $S_x$ and $R_z$ gates are due to different transpilation schemes for controlled $R_y$ and $R_z$ gates.

We ran both implementations on ibmq_belem letting $K = \{3, 5, 7\}$. The experiments are repeated for 3 times and the number of shots is set to 8192. The
average acceptance probabilities for each word length is plotted in Figure 10. The black lines above the bars show the minimum and maximum acceptance probabilities. Ideal probability shows the acceptance probability that is calculated analytically.

![Figure 10: Acceptance probabilities for various word lengths.](image)

Although the new implementation requires fewer gates and the circuit has a smaller depth, the experiment results do not look promising due to noise.

5 Conclusion and Future Work

In this paper, we presented a modified MCQFA algorithm for the $MOD_p$ problem, whose circuit implementation has smaller depth and uses fewer gates compared to the original algorithm when run on the IBMQ backends. Despite the
unsatisfying results from the real backend, the study contributes both to the field of QFA and the implementation of quantum algorithms on real devices. Further studies should be conducted to mitigate the effect of noise in the MC-QFA setting, which is not a trivial task since operators should be applied one at a time by definition of MCQFA.

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The code used for the experiments is available at [https://github.com/iitis/MCQFA](https://github.com/iitis/MCQFA).

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