Dynamics of viscoelastic orthotropic shallow shells of variable thickness

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Abstract. Thin-walled structural elements such as plates, panels, and shells of variable thickness are widely used at present in engineering, machine-building, and construction. Modern technologies allow creating any structural elements of a given shape, material, and the law of thickness variation. Therefore, the solution to the problems of the statics and dynamics of plates, panels, and shells of variable thickness, considering the real properties of the material, is relevant. Nonlinear parametric oscillations of viscoelastic orthotropic shallow shells of variable thickness are considered in the paper. Using the Kirchhoff-Love hypothesis, a mathematical model of the problem is constructed in a geometrically nonlinear statement. To describe the viscoelastic properties of a shallow shell, the hereditary Boltzmann-Volterra theory with the Koltunov-Rzhanitsyn relaxation kernel is used. To obtain resolving equations of the problem, the Bubnov-Galerkin method was used in combination with the numerical method. The effects of various physico-mechanical and geometrical parameters of a shallow shell of variable thickness were investigated.

1. Introduction
Plates and shells of variable thickness are widely used as structural elements in various technical units and construction structures. The easiest way to increase the stiffness of the unit is to increase its thickness. To give greater stiffness in the right places, the profile of thin plates or shells can have continuous thickenings. Therefore, the entire structure should be considered as a structure of variable thickness. Sufficiently large deflections can often occur in thin-walled elements. Under long-term loads, the viscoelasticity features can appear in the plates and shells materials, which will lead to a significant decrease in their bearing capacity [1]. Therefore, the calculations for strength, vibration, and stability of the described structures play an important role in the design of modern machines, units, and structures.

There are a number of studies devoted to the statics and dynamics of plates and shells considered at different laws of thickness variation.

Based on the simplified theory of nonlinear shells offered by Novozhilov, nonlinear vibrations of composite shells of variable thickness were studied in [2].

In [3], a new numerical-analytical method was proposed for studying nonlinear vibrations of shells with a variable layer thickness in a geometrically nonlinear statement. The problem was reduced to solving a sequence of linear problems, including those related to linear oscillations with a special type of elasticity.
The study of free vibrations of composite double-curved shells, panels, and plates of variable thickness was the subject of [4]. The equations obtained were solved by the proposed numerical method, the accuracy of which was validated by comparison with the available analytical and semi-analytical results found in the literature.

In [5], the stability of thin cylindrical shells of variable thickness under dynamic axial load was studied. The effect of the change in thickness and loading velocity on the critical load of stability loss was studied.

In [6], a methodology for optimizing the bending of thin-walled cylindrical shells of variable stiffness was described under the action of axial load.

The experimental and numerical study of bending and free vibrations of cylindrical shells of constant and variable stiffness was the subject of [7].

Based on the Bolotin method in combination with the Rayleigh-Ritz method, the problem of dynamic instability of composite plates of variable stiffness at various values of material parameters and geometry was considered in [8].

Numerical results of the study of free vibrations of cylindrical shells of variable thickness under various boundary conditions were given in [9]. The effects of thickness variation, boundary conditions, and material properties on the frequency and amplitude of oscillations were studied.

In [10], the nonlinear dynamics of circular cylindrical shells under axial compressive static and periodic loads were investigated experimentally.

Parametric vibrations of composite plates of variable stiffness were studied in [11]. The effect of various physical, mechanical, and geometrical parameters on the amplitude and frequency of oscillations of the plate was studied.

The dynamic instability of composite plates under harmonic axial loads was studied in [12].

Most problems of the theory of viscoelasticity were reduced to solving systems of integro-differential equations of Volterra type. Moreover, due to various features of the structures under consideration, such as inhomogeneity, geometrical nonlinearity, thickness variability, such systems can have a high order and variable coefficients. The development and implementation of improved numerical methods and computer programs allowed expanding the class of problems to be solved by the hereditary theory of viscoelasticity [13–20].

The studies devoted to nonlinear parametric vibrations of viscoelastic orthotropic plates and shells of variable thickness are almost not found in the literature. In this paper, we consider the problems of parametric vibrations of viscoelastic orthotropic shallow shells of variable thickness in a geometrically nonlinear statement.

2. Materials and methods

A viscoelastic orthotropic shallow shell of variable thickness \( h = h(x, y) \) rectangular in plan with dimensions \( a \times b \) and radii of curvature of the middle surface \( R_1 \) and \( R_2 \) was considered. Let the shell undergo dynamic periodic loading \( P(t) = P_0 + P_0 \cos \Theta t \) (\( P_0, R = \text{const} \); \( \Theta \) - is the frequency of the external periodic load). Assume that the shell has initial deflection \( w_0 \).

Using the results obtained in [21], considering periodic force \( P(t) \frac{\partial^2 w}{\partial x^2} \) and initial deflection, the following mathematical model of the problem is obtained with respect to transverse deflection \( w = w(x, y, t) \) and displacements \( u = u(x, y, t) \), \( v = v(x, y, t) \) under the corresponding initial and boundary conditions

\[
\begin{align*}
&h \left[ B_1 \left( 1 - \Gamma_{11}^* \right) \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right] - k_x B_1 \left( 1 - \Gamma_{11}^* \right) + k_y B_2 \left( 1 - \Gamma_{12}^* \right) \frac{\partial w}{\partial x} + \\
&+ \left[ B_2 \left( 1 - \Gamma_{12}^* \right) + 2B \left( 1 - \Gamma^* \right) \right] \left( \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right) + 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial y^2} \right) + \\
&+ \left[ B_3 \left( 1 - \Gamma_{13}^* \right) + 2B \left( 1 - \Gamma^* \right) \right] \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} \right) + 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x \partial y} \right) + \\
&+ \left[ B_4 \left( 1 - \Gamma_{14}^* \right) + 2B \left( 1 - \Gamma^* \right) \right] \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) + 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) + \\
&+ \left[ B_5 \left( 1 - \Gamma_{15}^* \right) + 2B \left( 1 - \Gamma^* \right) \right] \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} \right) + 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} \right) + \\
&+ \left[ B_6 \left( 1 - \Gamma_{16}^* \right) + 2B \left( 1 - \Gamma^* \right) \right] \left( \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right) + 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right) + \\
&+ \left[ B_7 \left( 1 - \Gamma_{17}^* \right) + 2B \left( 1 - \Gamma^* \right) \right] \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial y^2} \right) + 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial y^2} \right) + \\
&+ \left[ B_8 \left( 1 - \Gamma_{18}^* \right) + 2B \left( 1 - \Gamma^* \right) \right] \left( \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right) + 2B \left( 1 - \Gamma^* \right) \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right)
\end{align*}
\]
\[ \begin{align*}
+ \frac{\partial h}{\partial x} \left[ B_{12} (l - \Gamma_{12}^* + B_{12} (l - \Gamma_{12}^*) + k_{x} B_{12} (l - \Gamma_{12}^*) + k_{y} B_{22} (l - \Gamma_{22}^*) \right] \frac{\partial \gamma}{\partial y} & - \frac{1}{2} \frac{\partial w}{\partial x} \left( \frac{\partial \gamma}{\partial x} + \frac{\partial w}{\partial x} \right) \left( \frac{\partial \gamma}{\partial y} + \frac{\partial w}{\partial y} \right) \left( \frac{\partial \gamma}{\partial x} + \frac{\partial w}{\partial x} \right) \left( \frac{\partial \gamma}{\partial y} + \frac{\partial w}{\partial y} \right) - \rho h \frac{\partial^2 u}{\partial x^2} = 0, \\
\frac{h^2}{12} \left[ B_{11} (l - \Gamma_{11}^*) \frac{\partial^2 (w - w_0)}{\partial x^2} + \frac{1}{2} \frac{\partial w}{\partial x} \left( \frac{\partial \gamma}{\partial x} + \frac{\partial w}{\partial x} \right) \left( \frac{\partial \gamma}{\partial y} + \frac{\partial w}{\partial y} \right) \right] - \frac{1}{2} \frac{\partial w}{\partial x} \left( \frac{\partial \gamma}{\partial x} + \frac{\partial w}{\partial x} \right) \left( \frac{\partial \gamma}{\partial y} + \frac{\partial w}{\partial y} \right) \left( \frac{\partial \gamma}{\partial x} + \frac{\partial w}{\partial x} \right) \left( \frac{\partial \gamma}{\partial y} + \frac{\partial w}{\partial y} \right) & + 2 \frac{\partial h}{\partial x} \left( \frac{\partial \gamma}{\partial x} + \frac{\partial w}{\partial x} \right) \left( \frac{\partial \gamma}{\partial y} + \frac{\partial w}{\partial y} \right) + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} \left( \frac{\partial \gamma}{\partial x} + \frac{\partial w}{\partial x} \right) \left( \frac{\partial \gamma}{\partial y} + \frac{\partial w}{\partial y} \right) \left( \frac{\partial \gamma}{\partial x} + \frac{\partial w}{\partial x} \right) \left( \frac{\partial \gamma}{\partial y} + \frac{\partial w}{\partial y} \right) \left( \frac{\partial \gamma}{\partial x} + \frac{\partial w}{\partial x} \right) \left( \frac{\partial \gamma}{\partial y} + \frac{\partial w}{\partial y} \right) - \rho h \frac{\partial^2 \gamma}{\partial x^2} = 0,
\end{align*} \]
\[
+ \frac{\partial h}{\partial x} \left[ B_{11}(1 - \Gamma_{11}) \left( \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + B_{12}(1 - \Gamma_{12}) \left( \frac{\partial^2 v}{\partial y^2} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] - \\
-k \left[ B_{11}(1 - \Gamma_{11}) + k \left[ B_{12}(1 - \Gamma_{12}) \right] \right] \frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial h}{\partial y} B(1 - \Gamma)^s \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right) \right] - \\
-k \left[ \frac{\partial w}{\partial y} \right] \left[ B_{22}(1 - \Gamma_{22}) + 2 B(1 - \Gamma)^s \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \right) \right] + \\
+ B(1 - \Gamma)^s \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial w}{\partial x} \right) + \frac{\partial h}{\partial y} B(1 - \Gamma)^s \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right) + \\
+ \frac{\partial h}{\partial x} \left[ B_{22}(1 - \Gamma_{22}) \left( \frac{\partial^2 v}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + B_{21}(1 - \Gamma_{21}) \left( \frac{\partial^2 w}{\partial y^2} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] - \\
-k \left[ \frac{\partial w}{\partial y} \right] \left[ B_{21}(1 - \Gamma_{21}) + k \left[ B_{22}(1 - \Gamma_{22}) \right] \right] \right] - \\
-k \left[ \frac{\partial^2 w}{\partial x \partial y} \right] \frac{\partial h}{\partial x} B(1 - \Gamma)^s \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right) - P(t) \frac{\partial^2 w}{\partial t^2} + \rho h \frac{\partial^2 w}{\partial t^2} = q
\]

Note that the system (1) is a more general one. From it, in the particular case, it is possible to obtain mathematical models of problems on parametric vibrations of viscoelastic orthotropic plates, panels, and shells of variable thickness in both one and two directions.

In calculations, the Koltunov-Rzanisyn weakly singular kernel with three rheological parameters \(A, \beta, \alpha\) of the form [22] is used as the relaxation kernel:
\[\Gamma(t) = A e^{-\beta t^\alpha}, \quad (0 < \alpha < 1)\]

We approximate complete and initial deflections \(w\) and \(w_0\), as well as displacements \(u\), \(v\) in the resulting system using
\[
u(x, y, t) = \sum_{n=1}^{N} \sum_{m=1}^{M} u_{nm}(t) \varphi_{nm}(x, y), \quad w(x, y, t) = \sum_{n=1}^{N} \sum_{m=1}^{M} v_{nm}(t) \varphi_{nm}(x, y),
\]
\[
u_0(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} w_{0nm}(t) \varphi_{nm}(x, y), \quad w_0(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} w_{0nm}(t) \varphi_{nm}(x, y)
\]

Substituting (2) into the system of equations (1) and applying the Bubnov-Galerkin method, while introducing the following dimensionless quantities
and maintaining the previous notation, to determine the unknowns \( w_{nm} = w_{nm}(t) \), \( u_{nm} = u_{nm}(t) \), \( v_{nm} = v_{nm}(t) \), we obtain a system of basic resolving nonlinear integro-differential equations, where

\[
\frac{u}{h_0}, \frac{v}{h_0}, \frac{w}{h_0}, \frac{w_0}{a}, \frac{x}{a}, \frac{y}{b}, \frac{h}{h_0}, \frac{\delta}{h_0}, k_x = \frac{a^2}{R_1h_0}, k_y = \frac{b^2}{R_2h_0},
\]

\[
Q \left( \frac{b}{E(h_0)} \right)^4 \cdot \Theta \cdot \omega t
\]

3. Results and discussion

The integration of the obtained system was performed using the numerical method proposed in [23]. The calculation results are reflected in the graphs shown in Figs. 1-3. The dependence of the change in thickness is chosen in the form: \( h = 1 + \alpha^* x \) (\( \alpha^* \) is the parameter of the change in thickness). In calculations, the following data were taken as initial: \( \delta = 25 \); \( w_0 = 0.01 \); \( q = 0 \); \( \lambda = 1 \); \( \alpha^* = 0.5 \); \( k_x = 10 \); \( k_y = 10 \); \( \delta_0 = 0.3 \); \( \delta_1 = 0.5 \); \( \Theta = 1.1 \).

Figure 1 shows the results of a study of the effect of the thickness variation parameter \( \alpha^* \) on the behavior of a viscoelastic shallow shell.

![Figure 1](https://example.com/figure1.png)

**Figure 1.** Dependence of the deflection on time at \( \alpha^* = 0 \) (1); 0.3 (2); 0.5 (3).

The figure clearly shows that with an increase in this parameter, the amplitude and frequency of oscillations decrease. Note that at the beginning of the process of oscillations, the amplitudes slightly differ from the values obtained for the shells of constant thickness.

Figure 2 shows the results of studying shell behavior under various values of one of the curvatures \( k_x \). An analysis of results shows that an increase in this parameter leads to an increase in the amplitude of oscillations and a phase displacement.
The effect of inhomogeneous material properties on the behavior of a shell was studied (see Fig. 3). The results obtained show that an increase in the parameter $\Delta = \sqrt{E_3/E_2}$ determining the degree of anisotropy (curve 1 - $\Delta = 1$; curve 2 - $\Delta = 1.2$ and curve 3 - $\Delta = 1.5$) leads to a faster increase in the amplitude of the oscillations.

4. Conclusion

1. A mathematical model and a numerical method have been developed to solve the nonlinear problems of parametric oscillations of viscoelastic orthotropic shallow shells of variable thickness.
2. The parametric vibrations of viscoelastic orthotropic shallow shells of variable thickness were studied at various values of physico-mechanical and geometrical parameters.
3. The method and algorithm for solving the problem allow us to obtain the results for other types of thin-walled structures, such as plates, panels, and shells of variable thickness.
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