Apollonian circle packings

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Apollonius of Perga:

- Lived from about 262 BC to about 190 BC.
- Known as “The Great Geometer”.
- His famous book on Conics introduced the terms parabola, ellipse and hyperbola.
Apollonius’ theorem

Theorem (Apollonius of Perga, 262-190 BC)

Given 3 mutually tangent circles, there exist exactly two circles tangent to all three.
Proof of Apolloninus’ theorem

We give a modern proof of this ancient theorem using Mobius transformations: For \( a, b, c, d \in \mathbb{C}, \ ad - bc = 1 \),

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}, \quad z \in \mathbb{C} \cup \{\infty\}.
\]

A Mobius transformation maps circles (including lines) to circles, preserving angles between them.

In particular, it maps tangent circles to tangent circles.
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A Mobius transformation maps circles (including lines) to circles, preserving angles between them.

In particular, it maps **tangent circles to tangent circles**.
Proof of Apollonius’ theorem

\[ g(p) = \infty \]
Apollonian circle packing

Start with 4 mutually tangent circles.

Four possible configurations:

(a) 
(b) 
(c) 
(d)
Apollonian circle packing

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Four possible configurations

(a) (b)

(c) (d)
Using Apollonius theorem, we can keep adding newer circles tangent to three of the previous circles, and obtain an infinite circle packing called an

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We’ll show the first few generations of this process:
Each circle is labeled with the curvature (=reciprocal of its radius) and the greatest circle has radius one and hence the curvature $-1$ (oriented to have disjoint interiors).
First generation
Second generation
Third generation
Example of bounded Apollonian circle packing

The outermost circle has curvature \(-1\)
Example of bounded Apollonian circle packing

The outermost circle has curvature $-6$
Example of unbounded Apollonian circle packing

There are also other **unbounded** Apollonian packings containing either only one line or no line at all.
Counting question

For a bounded Apollonian packing $\mathcal{P}$, there are only finitely many circles of radius bigger than a given number.

Setting,

$$N_\mathcal{P}(T) := \#\{ C \in \mathcal{P} : \text{curv}(C) < T \} < \infty,$$

**Question**

- Is there an asymptotic of $N_\mathcal{P}(T)$ as $T \to \infty$?
- If so, can we compute?
Apollonian circle packing

Clearly, $N_P(T) \to \infty$ as $T \to \infty$. 
The study of this question involves notions related to metric properties of the underlying fractal set called residual set.
Residual set

Definition (Residual set of $\mathcal{P}$)

$$\text{Res}(\mathcal{P}) := \bigcup_{C \in \mathcal{P}} C.$$  

Equivalently, the residual set of $\mathcal{P}$ is the fractal set which is left in the plane after removing all the open disks enclosed by circles in $\mathcal{P}$. 
The Hausdorff dimension of the residual set of $\mathcal{P}$ is called the \textbf{Residual dimension of $\mathcal{P}$}, which we denote by $\alpha$. 
Hausdorff dim. (Hausdorff and Carathéodory (1914))

**Definition (s-dim. Hausdorff measure)**

Let $s \geq 0$. $F \subset \mathbb{R}^n$.

$$\mathcal{H}^s(F) := \lim_{\epsilon \to 0} \left( \inf \left\{ \sum \text{diam}(B_i)^s : F \subset \bigcup_i B_i, \text{diam}(B_i) < \epsilon \right\} \right)$$

As $s$ increases, the $s$-dim Haus measure of $F$ will be $\infty$ up to a certain value and then jumps down to 0.

**Definition**

The **Hausdorff dim** of $F$ is this critical value of $s$:

$$\dim_{\mathcal{H}}(F) = \sup\{s : \mathcal{H}^s(F) = \infty\} = \inf\{s : \mathcal{H}^s(F) = 0\}.$$ 

Roughly, min # of $r$-balls required to cover $F \sim r^{-\dim_{\mathcal{H}}(F)}$.
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We observe

- $1 \leq \alpha \leq 2$

- $\alpha$ is independent of $\mathcal{P}$: any two Apollonian packings are equivalent to each other by a Mobius transformation.

- The precise value of $\alpha$ is unknown, but approximately, $\alpha = 1.30568(8)$ (McMullen 1998)

In particular, $\text{Res}(\mathcal{P})$ is much bigger than a c’ble union of circles of $\mathcal{P}$, but not too big in the sense that its Leb. area ($=2$ dimensional Haus. measure) is zero.
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Residual dimension: $\alpha$

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Confirming Wilker’s prediction based on computer experiments, Boyd showed: \( N_P(T) := \#\{C \in P : \text{curv}(C) < T\} \)

**Theorem (Boyd 1982)**

\[
\lim_{T \to \infty} \frac{\log N_P(T)}{\log T} = \alpha.
\]

Boyd asked whether \( N_P(T) \sim cT^\alpha \) as \( T \to \infty \), and wrote that his numerical experiments suggest this may be false and perhaps

\[ N_P(T) \sim c \cdot T^\alpha (\log T)^\beta \]

might be more appropriate.
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might be more appropriate.
Theorem (Kontorovich-O. 11)

For a bounded Apollonian packing $\mathcal{P}$, there exists a constant $c_\mathcal{P} > 0$ such that

$$N_\mathcal{P}(T) \sim c_\mathcal{P} \cdot T^\alpha$$

where $\alpha = 1.30568(8)$ is the residual dimension of $\mathcal{P}$. 
Theorem (Lee-O. 13)

There exists $\eta > 0$ such that for any bounded Apollonian packing $\mathcal{P}$,

$$N_{\mathcal{P}}(T) = c_{\mathcal{P}} \cdot T^{\alpha} + O(T^{\alpha-\eta})$$

where $\alpha = 1.30568(8)$ is the residual dimension of $\mathcal{P}$. 
Counting inside Triangle

For unbounded Apollonian packing $\mathcal{P}$, $N_{\mathcal{P}}(T) = \infty$.

Consider a curvilinear triangle $\mathcal{R}$ whose sides are given by three mutually tangent circles in any Apollonian packing (either bounded or unbounded):

Set

$$N_{\mathcal{R}}(T) := \#\{C \in \mathcal{R} : \text{curv}(C) < T\} < \infty.$$
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Consider a curvilinear triangle $\mathcal{R}$ whose sides are given by three mutually tangent circles in any Apollonian packing (either bounded or unbounded):

Set $N_{\mathcal{R}}(T) := \#\{\mathcal{C} \in \mathcal{R}: \text{curv}(\mathcal{C}) < T\} < \infty$. 
Theorem (O.-Shah 12)

For a curvilinear triangle $\mathcal{R}$ of any Apollonian packing $\mathcal{P}$,

$$N_{\mathcal{R}}(T) \sim c_{\mathcal{R}} \cdot T^\alpha.$$
Question
Can we describe the asymptotic distribution of circles in $\mathcal{P}$ of curvature at most $T$ as $T \to \infty$?

For a bounded Borel set $E$, set

$$N_T(\mathcal{P}, E) := \#\{C \in \mathcal{P} : C \cap E \neq \emptyset, \text{curv}(C) < T\}.$$ 

As we vary $E \subset \mathbb{C}$, how does $N_T(\mathcal{P}, E)$ depend on $E$?
Distribution of circles in Apollonian packing

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For a bounded Borel set $E$, set

$$N_T(\mathcal{P}, E) := \#\{C \in \mathcal{P} : C \cap E \neq \emptyset, \text{curv}(C) < T\}.$$ 

As we vary $E \subset \mathbb{C}$, how does $N_T(\mathcal{P}, E)$ depend on $E$?
Does there exist a measure $\omega_\mathcal{P}$ on $\mathbb{C}$ s.t. for any bdd Borel $E \subset \mathbb{C}$,

$$\lim_{T \to \infty} \frac{N_T(\mathcal{P}, E)}{T^\alpha} = \omega_\mathcal{P}(E)?$$
Note that all the circles in $\mathcal{P}$ lie on the residual set of $\mathcal{P}$. Hence any measure describing the asymptotic distribution of circles of $\mathcal{P}$ must be supported on the residual set of $\mathcal{P}$. What measure could that be?
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What measure could that be?
We show that the $\alpha$-dim. Haus. measure $\mathcal{H}^\alpha$ of $\text{Res}(\mathcal{P})$ does the job.

**Theorem (O.-Shah, 10)**

*For any bdd. Borel $E \subset \mathbb{C}$ with smooth bdry,*

$$N_T(\mathcal{P}, E) \sim c_A \cdot \mathcal{H}^\alpha(E) \cdot T^\alpha$$

*where $0 < c_A < \infty$ is independent of $\mathcal{P}$.*
Thm says that circles in an Apollonian packing are uniformly distributed w.r.t the $\alpha$-dim. Hausdorff meas. on its residual set:

\[
\frac{N_T(\mathcal{P}, E_1)}{N_T(\mathcal{P}, E_2)} \sim \frac{\mathcal{H}^\alpha(E_1 \cap \text{Res}(\mathcal{P}))}{\mathcal{H}^\alpha(E_2 \cap \text{Res}(\mathcal{P}))}.
\]
We call an Apollonian packing $\mathcal{P}$ integral if every circle in $\mathcal{P}$ has integral curvature.

Does there exist any integral $\mathcal{P}$?

The answer is positive thanks to the following beautiful thm of Descartes:
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Theorem (In a letter to Princess Elisabeth of Bohemia)

A quadruple \((a, b, c, d)\) is the curvatures of four mutually tangent circles if and only if it satisfies the quadratic equation:

\[
2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.
\]
E.g: $2((-1)^2 + 2^2 + 2^2 + 3^2) = 36 = (-1 + 2 + 2 + 3)^2$

E.g: $2(2^2 + 6^2 + 3^2 + 23^2) = 1156 = (2 + 6 + 3 + 23)^2$
Given three mutually tangent circles of curvatures $a, b, c$, if we denote by $d$ and $d'$ for the curvatures of the two circles tangent to all three, then

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$$

and

$$2(a^2 + b^2 + c^2 + (d')^2) = (a + b + c + (d'))^2.$$

By subtracting one from the other, we obtain

$$d + d' = 2(a + b + c).$$

So, if $a, b, c, d$ are integers, so is $d'$. 
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So, if \(a, b, c, d\) are integers, so is \(d'\).
Theorem (Soddy 1936)

If the initial 4 circles in an Apollonian packing $\mathcal{P}$ have integral curvatures, $\mathcal{P}$ is integral.

Combined with Descartes’ theorem, for any integral solution of

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2,$$

$\exists$ an integral Apollonian packing!
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Integral Apollonian circle packings

Any integral Apollonian packing is either bounded or lies between two parallel lines:
For a given integral Apollonian packing $\mathcal{P}$, it is natural to inquire about its the Diophantine properties such as

**Question**

- Are there infinitely many circles with prime curvatures?
- Which integers appear as curvatures?

Assume that $\mathcal{P}$ is primitive, i.e., $\gcd_c \{\text{curv}(C) \in \mathcal{P}\} = 1$.

**Definition**

1. A circle is **prime** if its curvature is a prime number.
2. A pair of tangent prime circles is a **twin prime**.
Diophantine questions

For a given integral Apollonian packing \( \mathcal{P} \), it is natural to inquire about its the Diophantine properties such as

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2. A pair of tangent prime circles is a twin prime.
prime circles: 2, 3, 11, 23,... Twin prime circles: (2, 3), (2, 11), (3, 23), ...
Theorem (Sarnak 07)

There are infinitely many prime circles as well as twin prime circles in any primitive integral \( \mathcal{P} \).

Set

\[
\Pi_T(\mathcal{P}) := \# \{ \text{prime } C \in \mathcal{P} : \text{curv}(C) < T \}
\]

and

\[
\Pi^{(2)}_T(\mathcal{P}) := \# \{ \text{twin primes } C_1, C_2 \in \mathcal{P} : \text{curv}(C_i) < T \}.
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Analogue of Prime number theorem?

Heuristics using the circle method based on the randomness conjecture of Mobius function:

**Conjecture (Fuchs-Sanden)**

\[ \Pi_T(\mathcal{P}) \sim c_1 \frac{N_T(\mathcal{P})}{\log T}; \quad \Pi_T^{(2)}(\mathcal{P}) \sim c_2 \frac{N_T(\mathcal{P})}{(\log T)^2} \]

where \( c_1 \) and \( c_2 \) can be given explicitly.
Based on the breakthrough work of Bourgain, Gamburd, Sarnak on expanders together with Selberg’s upper bound sieve, we obtain upper bounds of true order of magnitude:

Theorem (Kontorovich-O. 09)

1. $\Pi_T(P) \ll \frac{T^\alpha}{\log T}$
2. $\Pi_T^{(2)}(P) \ll \frac{T^\alpha}{(\log T)^2}$.

The lower bounds are still open and seem very challenging.
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Question

How many integers appear as curvatures?

I.e., How big is \( \#\{\text{curv}(C) \leq T : C \in \mathcal{P} \} \) compared to \( T \)?

Our counting result for circles says:

\[ \#\{\text{curv}(C) \leq T \text{ counted with multiplicity } : C \in \mathcal{P} \} \sim c \cdot T^{1.305..} \]

So we may hope that a positive density (=proportion) of integers arise as curvatures, as conjectured by Graham, Lagarias, Mallows, Wilkes, Yan (Positive density conjecture)
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Positive density conjecture

\( \mathcal{P} \): primitive integral Apollonian packing

**Theorem (Bourgain-Fuchs 10)**

\[
\# \{ \text{curv} (C) < T : C \in \mathcal{P} \} \gg T.
\]

**Theorem (Bourgain-Kontorovich 12)**

\[
\# \{ \text{curv} (C) < T : C \in \mathcal{P} \} \sim \frac{\kappa(\mathcal{P})}{24} \cdot T
\]

where \( \kappa(\mathcal{P}) = \# \text{residue classes (mod 24) of curvatures of } \mathcal{P} \).

▶ as predicted by Lagarias et al, based on the local-global principle
Positive density conjecture

\[ P: \text{primitive integral Apollonian packing} \]

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Improving Sarnak's result on the infinitude of prime circles, Bourgain showed that a positive fraction of prime numbers appear as curvatures in \( \mathcal{P} \).

**Theorem (Bourgain, 2011)**

\[
\#\{\text{prime curv } (C) \leq T : C \in \mathcal{P}\} \gg \frac{T}{\log T}.
\]
Hidden symmetries

**Question**

How are we able to count circles in an Apollonian packing?

We exploit the fact that an Apollonian packing has lots of hidden symmetries.

Explaining these hidden symmetries will lead us to explain the relevance of the packing with a Kleinian group, called the Apollonian group.
Hidden symmetries

**Question**

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an Apollonian packing has lots of hidden symmetries.

Explaining these hidden symmetries will lead us to explain the relevance of the packing with a Kleinian group, called the Apollonian group.
Consider the dual circles of fixed 4 mutually tangent circles in $\mathcal{P}$ determined by 6 tangent points.
Inverting w.r.t a dual circle fixes the three circles that it meets perpendicularly and interchanges the two circles which are tangent to the three circles.
The **Apollonian group** $\mathcal{A}$ associated to $\mathcal{P}$ is generated by 4 inversions w.r.t those dual circles:

$$\mathcal{A} = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle < \text{Mob}(\hat{\mathbb{C}})$$

where $\text{Mob}(\hat{\mathbb{C}}) = \text{PSL}_2(\mathbb{C})^\pm$ the gp of Mobius transf. of $\hat{\mathbb{C}}$. 
The Apollonian group $\mathcal{A}$ is a Kleinian group (= discrete subgroup of $\text{PSL}_2(\mathbb{C})^\pm$) and satisfies

- $\mathcal{P} = \bigcup_{i=1}^{4} \mathcal{A}(C_i)$: inverting the initial four (black) circles in $\mathcal{P}$ w.r.t the (red) dual circles generate the whole packing $\mathcal{P}$;

- $\text{Res}(\mathcal{P}) = \Lambda(\mathcal{A})$: limit set of $\mathcal{A}$ (=the set of all accumulation pts of an orbit $\mathcal{A}(z)$ for $z \in \hat{\mathbb{C}}$).
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- \(\text{Res}(\mathcal{P}) = \Lambda(\mathcal{A})\): **limit set of \(\mathcal{A}\)** (\(=\)the set of all accumulation pts of an orbit \(\mathcal{A}(z)\) for \(z \in \hat{\mathbb{C}}\)).
The upper-half space model for hyp. 3 space $\mathbb{H}^3$:

$$\mathbb{H}^3 = \{(x_1, x_2, y) : y > 0\} \text{ with } \quad ds = \frac{\sqrt{dx_1^2 + dx_2^2 + dy^2}}{y}$$

and $\partial_\infty(\mathbb{H}^3) = \hat{C}$. 
Via the Poincare extension thm,

$$\text{PSL}_2(\mathbb{C})^\pm = \text{Isom}(\mathbb{H}^3).$$

Note $\text{PSL}_2(\mathbb{C})^\pm$ acts on $\hat{\mathbb{C}}$ by linear fractional transformations and an inversion w.r.t a circle $C$ in $\hat{\mathbb{C}}$ corresponds to the inversion w.r.t the vertical hemisphere in $\mathbb{H}^3$ above $C.$
The Apollonian gp \( \mathcal{A} \) (now considered as a discrete subgp of Isom(\( \mathbb{H}^3 \))) has a fund. domain in \( \mathbb{H}^3 \), given by the exterior of the hemispheres above the dual circles to \( \mathcal{P} \):

In particular, \( \mathcal{A} \setminus \mathbb{H}^3 \) is an \textit{infinite vol.} hyperbolic 3-mfld and has finitely many sided fund. domain (called geometrically finite mfld).
Orthogonal translates of geodesic surface

Note counting circles in $\mathcal{P}$ of curvature at most $T$ is same as counting the vertical hemispheres (=geodesic planes) above circles in $\mathcal{P}$ of height at at most $T^{-1}$.

We relate the counting problem for geodesic planes with the equidistribution of translates of a properly immersed geodesic plane in $\mathcal{A}\backslash\mathbb{H}^3$. 
For a properly immersed geodesic plane $S$ of $T^1(\mathcal{A}/\mathbb{H}^3)$, what is the asymptotic dist. of its orthogonal translates $g^t(S)$ as $t \to \infty$?

For acpt $\Omega \subset T^1(\mathcal{A}/\mathbb{H}^3)$, how much proportion of $g^t(S)$ intersects $\Omega$ as $t \to \infty$?
Difficulties lie in the fact that the Apollonian manifold is of infinite volume, as the dynamics of flows in infinite volume hyperbolic manifolds were much less understood (if it were of finite volume, everything is well-understood due to Margulis, Duke-Rudnick-Sarnak, and Eskin-McMullen).
We show that this distribution in $\mathbb{T}^1(\mathcal{A}\backslash \mathbb{H}^3)$ is described by a singular measure, called the **Burger-Roblin measure**, whose conditional measures on horospherical foliations are $\alpha$-dim’l Haus. measures, and this is why we have the $\alpha$-dim’l Haus. measure in our counting thm.
More circle packings

This viewpoint via the study of Kleinian groups allows us to deal with more general circle packings, provided they are invariant under a geometrically finite Kleinian group $\Gamma$. 
General circle packings

Let $\mathcal{P}$ be a circle packing invariant under a g.f. Kleinian group $\Gamma$.

**Theorem (O.-Shah, 2013)**

For any bdd. Borel $E \subset \mathbb{C}$ with nice boundary,

$$\#\{C \in \mathcal{P} : C \cap E \neq \emptyset, \text{curv}(C) < T\} \sim \omega_\Lambda(E) \cdot T^\delta$$

where $\Lambda$ is the limit set of $\Gamma$, $\delta = \dim_H \Lambda$ and $\omega_\Lambda$ is a locally finite Borel measure on $\Lambda$. 

![Diagram of circle packing with some highlighted regions.](image-url)
Thank you!