Static, massive fields and vacuum polarization potential in Rindler space

B. Linet

Laboratoire de Mathématiques et Physique Théorique
CNRS/UPRES-A 6083, Université François Rabelais
Faculté des Sciences et Techniques
Parc de Grandmont 37200 TOURS, France

Abstract

In Rindler space, we determine in terms of special functions the expression of the static, massive scalar or vector field generated by a point source. We find also an explicit integral expression of the induced electrostatic potential resulting from the vacuum polarization due to an electric charge at rest in the Rindler coordinates. For a weak acceleration, we give then an approximate expression in the Fermi coordinates associated with the uniformly accelerated observer.

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1 Introduction

In the coordinate system \((\xi^0, \xi^1, \xi^2, \xi^3)\) with \(\xi^1 > 0\), the Rindler space is described by the metric

\[ ds^2 = -(g\xi^1)^2 d\xi^0)^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 \]

(1.1)

where \(g\) is a strictly positive constant. Metric (1.1) describes equivalently a constant, static, homogeneous gravitational field. There exists a horizon located at \(\xi^1 = 0\). These Rindler coordinates \((\xi^\mu)\) cover only a part of the Minkowski space-time; they are related to the Minkowskian coordinates \((x^\mu)\) for \(\xi^1 > 0\) by

\[ x^0 = \xi^1 \sinh g\xi^0, \quad x^1 = \xi^1 \cosh g\xi^0, \quad x^2 = \xi^2, \quad x^3 = \xi^3. \]

(1.2)

In the Rindler space, the world line characterized by \(\xi^i = \xi^i_g\) with

\[ \xi^1_g = \frac{1}{g}, \quad \xi^2 = 0, \quad \xi^3 = 0 \]

(1.3)
corresponds to the world line $z^\mu(\tau)$ of an observer undergoing a uniform acceleration $g$ which has the following equation in the Minkowskian coordinates

$$
\begin{align*}
z^0(\tau) &= \frac{1}{g} \sinh g\tau, \\
z^1(\tau) &= \frac{1}{g} \cosh g\tau, \\
z^2(\tau) &= 0, \\
z^3(\tau) &= 0
\end{align*}
$$

(1.4)

where $\tau$ is the proper time, the tangent vector being denoted $\dot{z}^\mu(\tau)$. The Fermi coordinates $(y^i)$ associated with this accelerated observer are defined by

$$
y^1 = \xi^1 - \frac{1}{g}, \quad y^2 = \xi^2, \quad y^3 = \xi^3.
$$

(1.5)

Metric (1.1) takes then the form

$$
ds^2 = -\left(1 + 2gy^1 + g^2(y^1)^2\right)(d\xi^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2
$$

(1.6)

allowing us to discuss the fields and the vacuum polarization potential in the neighborhood of the line $y^i = 0$ in an homogeneous gravitational field.

In electromagnetism, the electromagnetic field generated by a uniformly accelerated point charge has been a subject of considerable investigation. But as clearly showed by Boulware [1], there is no problem in the region delimited by $\xi^1$ strictly positive. The field generated by a point source having the world line (1.4) is calculated by using the retarded Green’s function in the Minkowskian coordinates. By means of the transformation of coordinates (1.2), this field can be thus expressed in coordinates $(\xi^0, \xi^1, \xi^2, \xi^3)$ for $\xi^1 > 0$. This field coincides with the electrostatic potential of a point charge in metric (1.1) which has been found by Whittaker [2] in slightly different coordinates. It corresponds to a charge at rest in the homogeneous gravitational field.

The first purpose of the present paper is to give in closed form the expression of the massive vector field generated by a point source having a uniform acceleration by using the retarded Green’s function $\triangle_R$. So far as we know, this determination has not been done. We will give also the result in the case of a massive scalar field where an analogous situation exists in the region $\xi^1 > 0$ [3].

In quantum electrodynamics, when the pair creation is neglected, the induced current resulting from the vacuum polarization can be determined at the first order in the fine structure constant $\alpha$ by the Schwinger’s formula [4]. He gave in the Minkowskian coordinates an integral expression with the aid of the half sum of the advanced and retarded Green’s functions $\triangle$. The second purpose of the present paper is to show that the vacuum polarization induced by a uniformly accelerated charge can be expressed in the Rindler space in terms of the vector Green’s function previously determined.

The plan of the work is as follows. In Sec. 2, we write down some preliminary formulas. We determine the massive scalar field in Sec. 3 and the massive vector field in Sec. 4. The vacuum polarization due to an electric charge at rest is treated in Sec. 5. We add in Sec. 6 some concluding remarks.
2 Preliminaries

In the Minkowskian coordinates, the retarded Green’s function (e.g. \[ \Delta_R(x, x') = \frac{\theta(x^0 - x^0')}{2\pi} \left[ \delta(\lambda) - \frac{m}{2\sqrt{\lambda}} \theta(\lambda) J_1(m\sqrt{\lambda}) \right] \] (2.1)

where \( J_1 \) is the Bessel function and the quantity \( \lambda \) is given by

\[ \lambda(x, x') = (x^0 - x^0')^2 - (x^1 - x^1')^2 - (x^2 - x^2')^2 - (x^3 - x^3')^2. \] (2.2)

The half sum of the advanced and retarded Green’s function is

\[ \overline{\Delta}(x, x') = \frac{1}{4\pi} \left[ \delta(\lambda) - \frac{m}{2\sqrt{\lambda}} \theta(\lambda) J_1(m\sqrt{\lambda}) \right]. \] (2.3)

We will apply these formulas when the point \( x' \) coincides with the point \( z(\tau) \) given by (1.4). By using (1.2), \( \lambda \) can be expressed in the Rindler coordinates

\[ \lambda(x, z(\tau)) = \frac{1}{g^2} \left( 2g\xi^1 \cosh g(\xi^0 - \tau) - (g\xi^1)^2 - (g\xi^2)^2 - (g\xi^3)^2 \right). \] (2.4)

The retarded time \( \tau_R \) is defined by \( \lambda(x, z(\tau_R)) = 0 \) and \( x^0 - z^0(\tau_R) > 0 \). From (2.4), we find

\[ \cosh g(\xi^0 - \tau_R) = \frac{1 + (g\xi^1)^2 + (g\xi^2)^2 + (g\xi^3)^2}{2g\xi^1} \] with \( \xi^0 > \tau_R \). (2.5)

The advanced time \( \tau_A \) is also given also by (2.3) but with \( \xi^0 < \tau_A \). In the next sections, it will be needed to know

\[ (x^\mu - z^\mu(\tau_R)z_\mu(\tau_R)) = -\xi^1 \sinh g(\xi^0 - \tau_R). \] (2.6)

We introduce the functions \( x \) and \( y \) of \( \xi^i \) and \( \xi_0^i \) by

\[ x(\xi^i, \xi_0^i) = \frac{m}{2} \left[ \sqrt{(\xi_0^1)^2 + (\xi^1)^2 + (\xi^2 - \xi_0^2)^2 + (\xi^3 - \xi_0^3)^2 + 2\xi_0^1\xi_0^1} \right. \\
- \left. \sqrt{(\xi_0^1)^2 + (\xi^1)^2 + (\xi^2 - \xi_0^2)^2 + (\xi^3 - \xi_0^3)^2 - 2\xi_0^1\xi_0^1} \right], \] (2.7)

\[ y(\xi^i, \xi_0^i) = \frac{m}{2} \left[ \sqrt{(\xi_0^1)^2 + (\xi^1)^2 + (\xi^2 - \xi_0^2)^2 + (\xi^3 - \xi_0^3)^2 + 2\xi_0^1\xi_0^1} \right. \\
+ \left. \sqrt{(\xi_0^1)^2 + (\xi^1)^2 + (\xi^2 - \xi_0^2)^2 + (\xi^3 - \xi_0^3)^2 - 2\xi_0^1\xi_0^1} \right]. \]

We notice that \( y \geq x > 0 \), the equality occuring if \( \xi^i = \xi_0^i \). It is easy to see that

\[ y^2 - x^2 = m^2 \sqrt{[(\xi_0^1)^2 + (\xi^1)^2 + (\xi^2 - \xi_0^2)^2 + (\xi^3 - \xi_0^3)^2]^2 - 4(\xi_0^1\xi^1)^2}, \]
\[ xy = m^2 \xi_0^1\xi_0^1. \] (2.8)
We note respectively $x_g$ and $y_g$ the functions of $\xi^i$ defined from the functions $x$ and $y$ for the value $\xi_0^i$ given by (3.3). Hence, we can rewrite (2.3) in the form
\[
\ln \frac{y_g}{x_g} = \frac{x_g^2 + y_g^2}{2x_g y_g} = \cosh g(\xi^0 - \tau_R), \quad \frac{y_g^2 - x_g^2}{2y_g} = \sinh g(\xi^0 - \tau_R). \quad (2.9)
\]

In terms of the Fermi coordinates (3.3), we have the relations
\[
x_g y_g = \frac{m^2}{g^2} (1 + g y^1), \quad y_g - x_g = m \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}. \quad (2.10)
\]

### 3 Determination of the massive scalar field

The covariant equation for a massive scalar field $\psi$ in a general metric $g_{\alpha\beta}$ is
\[
\frac{1}{\sqrt{-g}} \partial_\alpha \left( \sqrt{-g} g^{\alpha\beta} \partial_\beta \psi \right) - m^2 \psi = - \int_{-\infty}^{\infty} \frac{1}{\sqrt{-g}} \delta^{(4)}(x^\lambda - z^\lambda(s)) ds \quad (3.1)
\]
where $z^\lambda(s)$ is the world line of the point source of strength unit. In the Minkowski space-time, according to (2.4) we get the retarded solution in the general form
\[
\psi(x) = - \frac{1}{4\pi} \frac{m}{\sqrt{\mu - z^\mu(\tau_R)}} J_1 \left( \frac{m}{\sqrt{\mu(x, z(\tau))}} \right) d\tau. \quad (3.2)
\]

In the case of the uniformly accelerated point source characterized by (3.4), formula (3.2) yields
\[
\psi(\xi^i) = \frac{g}{2\pi \sqrt{[1 + (g\xi^1)^2 + (g\xi^2)^2 + (g\xi^3)^2]^2 - 4(g\xi^1)^2}} \\
- \frac{mg}{4\pi} \int_{-\infty}^{\tau_R} J_1 \left( \frac{mg}{2g\xi^1} \sinh g(\xi^0 - \tau) - 1 - (g\xi^1)^2 - (g\xi^2)^2 - (g\xi^3)^2 \right) d\tau. \quad (3.3)
\]

The change of variable $t = g(\xi^0 - \tau)$ in integral (3.3) gives
\[
\psi(\xi^i) = \frac{g}{2\pi \sqrt{[1 + (g\xi^1)^2 + (g\xi^2)^2 + (g\xi^3)^2]^2 - 4(g\xi^1)^2}} \\
+ \frac{m}{4\pi} \int_{-\infty}^{g(\xi^0 - \tau_R)} J_1 \left( \frac{mg}{2g\xi^1} \cosh t - 1 - (g\xi^1)^2 - (g\xi^2)^2 - (g\xi^3)^2 \right) dt. \quad (3.4)
\]

By considering the functions $x_g$ and $y_g$ defined in Sec. 2, expression (3.4) takes then the form
\[
\psi(\xi^i) = \frac{g}{2\pi \sqrt{[1 + (g\xi^1)^2 + (g\xi^2)^2 + (g\xi^3)^2]^2 - 4(g\xi^1)^2}} \\
- \frac{m^2}{4\pi g} \int_{\ln y_g/x_g}^{\infty} J_1 \left( \frac{2x_g y_g \cosh t - x_g^2 - y_g^2}{\sqrt{2x_g y_g \cosh t - x_g^2 - y_g^2}} \right) dt. \quad (3.5)
\]
where we have used (2.5) and (2.9). The definite integral with the Bessel function appearing in (3.5) is given by formula (A.2) of the appendix; after simplification we get

\[ \psi(\xi) = \frac{m^2 x g I_1(y_g) + y g I_0(x_g) K_1(y_g)}{y_g^2 - x_g^2} \]  

(3.6)

where \( I_\nu \) and \( K_\nu \) are the modified Bessel functions.

We are now in a position to determine the static scalar Green’s function in Rindler space. The static solution to equation (3.1) in metric (1.1) for a point source located at (1.3) obeys the following equation

\[ \frac{1}{\xi_1} \frac{\partial}{\partial \xi_1} \left( \xi_1 \frac{\partial}{\partial \xi_1} \psi \right) + \frac{\partial^2}{\partial (\xi^2)^2} \psi + \frac{\partial^2}{\partial (\xi^3)^2} \psi - m^2 \psi = -\delta(\xi_1 - \frac{1}{g})\delta(\xi^2)\delta(\xi^3). \] 

(3.7)

The scalar Green’s function \( G(\xi^i, \xi_0^i) \) is the solution to equation (3.7) with the source

\[ -\frac{1}{\xi_0^1} \delta(\xi_1 - \xi_0^1)\delta(\xi^2 - \xi_0^2)\delta(\xi^3 - \xi_0^3) \] 

with \( \xi_1^i > 0 \)  

(3.8)

since we must multiply by \( \xi_1 \) equation (3.7) to have a self-adjoint operator and in consequence a Green’s function symmetric in \( \xi^i \) and \( \xi_0^i \). So, we have established in closed form the expression of the static scalar Green’s function in Rindler space

\[ G(\xi^i, \xi_0^i) = \frac{m^2 x g I_1(x_0) + y g I_0(x) K_1(y)}{y^2 - x^2} \] 

(3.9)

where \( x \) and \( y \) are the functions (2.7). When \( m = 0 \), we obtain

\[ D(\xi^i, \xi_0^i) = \frac{1}{2\pi \sqrt{[\xi_0^1]^2 + ((\xi^1 - \xi_0^1)^2 + (\xi^2 - \xi_0^2)^2 + (\xi^3 - \xi_0^3)^2] - 4(\xi_0^1 \xi^1)^2}}. \]  

(3.10)

The Green’s function (3.9) is well defined at the horizon \( \xi_1 = 0 \).

4 Determination of the massive vector field

The Proca equations in covariant form for a massive vector field \( A^\mu \) in a general metric \( g_{\alpha\beta} \) are

\[ \frac{1}{\sqrt{-g}} \partial_\alpha \left[ \sqrt{-g} g^{\alpha\beta} g^{\gamma\delta}(\partial_\beta A_\gamma - \partial_\gamma A_\beta) \right] - m^2 A^\delta = \int_{\xi_1} \tilde{z}^\delta(s) \frac{1}{\sqrt{-g}} \delta^{(4)}(x^\lambda - z^\lambda(s)) ds \]  

and  

\[ \partial_\mu \left( \sqrt{-g} A^\mu \right) = 0 \]  

(4.1)

where \( z^\lambda(s) \) is the world line of the point source of strength unit.
According to (2.1), we obtain the expressions of the Minkowskian components $A^\mu$ in the general form

$$A^\mu(x) = \frac{\dot{z}^\mu(\tau_R)}{4\pi(x^\mu - z^\mu(\tau_R))\dot{z}_\mu(\tau_R)} + \frac{m}{4\pi} \int_{-\infty}^{\tau_R} \dot{z}^\mu(\tau) \frac{J_1 \left(m \sqrt{\lambda(x, z(\tau))} \right)}{\sqrt{\lambda(x, z(\tau))}} \, d\tau.$$  \hspace{1cm} (4.2)

We can now express Minkowskian components (4.2) in the Rindler coordinates

$$A^0(\xi^0, \xi^i) = -\frac{g \cosh g\tau_R}{2\pi \sqrt{1 + (g\xi^1)^2 + (g\xi^2)^2 + (g\xi^3)^2} - 4(g\xi^1)^2} + \frac{mg}{4\pi} \int_{-\infty}^{\tau_R} \cosh g\tau \frac{J_1 \left(m/g \sqrt{2g\xi^1 \cosh g(\xi^0 - \tau) - 1 - (g\xi^1)^2 - (g\xi^2)^2 - (g\xi^3)^2} \right)}{\sqrt{2g\xi^1 \cosh g(\xi^0 - \tau) - 1 - (g\xi^1)^2 - (g\xi^2)^2 - (g\xi^3)^2}} \, d\tau,$$  \hspace{1cm} (4.3)

$$A^1(\xi^0, \xi^i) = -\frac{g \sinh g\tau_R}{2\pi \sqrt{1 + (g\xi^1)^2 + (g\xi^2)^2 + (g\xi^3)^2} - 4(g\xi^1)^2} + \frac{mg}{4\pi} \int_{-\infty}^{\tau_R} \sinh g\tau \frac{J_1 \left(m/g \sqrt{2g\xi^1 \cosh g(\xi^0 - \tau) - 1 - (g\xi^1)^2 - (g\xi^2)^2 - (g\xi^3)^2} \right)}{\sqrt{2g\xi^1 \cosh g(\xi^0 - \tau) - 1 - (g\xi^1)^2 - (g\xi^2)^2 - (g\xi^3)^2}} \, d\tau$$  \hspace{1cm} (4.4)

the other components vanishing. By means of the transformation of coordinates (1.2), we obtain the components of the vector field in the Rindler coordinates

$$A_{\xi^0} = -g\xi^1 \cosh g\xi^0 A^0 + g\xi^1 \sinh g\xi^0 A^1, \quad A_{\xi^1} = -\sinh g\xi^0 A^0 + \cosh g\xi^0 A^1.$$  \hspace{1cm} (4.5)

From (4.3) the component $A_{\xi^0}$ can be written

$$A_{\xi^0}(\xi^i) = \frac{g^2\xi^1 \cosh g(\xi^0 - \tau_R)}{2\pi \sqrt{1 + (g\xi^1)^2 + (g\xi^2)^2 + (g\xi^3)^2} - 4(g\xi^1)^2} - \frac{mg^2\xi^1}{4\pi} \int_{-\infty}^{\tau_R} \cosh g(\xi^0 - \tau) \frac{J_1 \left(m/g \sqrt{2g\xi^1 \cosh g(\xi^0 - \tau) - 1 - (g\xi^1)^2 - (g\xi^2)^2 - (g\xi^3)^2} \right)}{\sqrt{2g\xi^1 \cosh g(\xi^0 - \tau) - 1 - (g\xi^1)^2 - (g\xi^2)^2 - (g\xi^3)^2}} \, d\tau$$  \hspace{1cm} (4.6)

and the component $A_{\xi^1}$

$$A_{\xi^1}(\xi^i) = \frac{g \sinh g(\xi^0 - \tau_R)}{2\pi \sqrt{1 + (g\xi^1)^2 + (g\xi^2)^2 + (g\xi^3)^2} - 4(g\xi^1)^2}$$
From formula (A.3) of the appendix, component (4.8) has then the explicit expression

\[ -\frac{mg}{4\pi} \int_{-\infty}^{\infty} \sinh g(\xi^0 - \tau) \]
\[ \times J_1 \left( \frac{m/g}{\sqrt{2g\xi^1}} \cosh g(\xi^0 - \tau) - 1 - (g\xi^1)^2 - (g\xi^2)^2 - (g\xi^3)^2 \right) \, d\tau. \]

By introducing the functions \( x_g \) and \( y_g \) defined in Sec. 2, we rewrite (4.6) in the form

\[ A_{\xi^0}(\xi^i) = \frac{g[1 + (g\xi^1)^2 + (g\xi^2)^2 + (g\xi^3)^2]}{4\pi \sqrt{1 + (g\xi^1)^2 + (g\xi^2)^2 + (g\xi^3)^2 - 4(g\xi^1)^2}} \]
\[ -\frac{m^2\xi^1}{4\pi} \int_{\ln y_g/x_g}^{\infty} J_1 \left( \frac{\sqrt{2x_gy_g \cosh t - x^2_g - y^2_g}}{2(x_gy_g \cosh t - x^2_g - y^2_g)} \right) \cosh td\tau. \]  

From formula (A.3) of the appendix, component (4.8) has then the explicit expression

\[ A_{\xi^0}(\xi^i) = \frac{m^2\xi^1}{2\pi} \frac{y_gI_1(x_g)K_0(y_g) + x_gI_0(x_g)K_1(y_g)}{y^2_g - x^2_g}. \]  

From formula (A.4) of the appendix, it is easy to see that \( A_{\xi^1} = 0 \).

On the other hand with the help of formula (A.4) of the appendix, we are now in a position to determine the static vector Green’s function in Rindler space. In the static case for a point source located at (4.3), equations (4.1) reduce to one equation for the only non-zero component \( A_{\xi^0} \)

\[ \xi^1 \frac{\partial}{\partial \xi^1} \left( \frac{1}{\xi^1} \frac{\partial}{\partial \xi^1} A_{\xi^0} \right) + \frac{\partial^2}{\partial(\xi^2)^2} A_{\xi^0} + \frac{\partial^2}{\partial(\xi^3)^2} A_{\xi^0} - m^2 A_{\xi^0} = -\delta(\xi^1 - 1/g) \delta(\xi^2) \delta(\xi^3) \]  

whose the solution is given by (4.9). The source of equation (4.10) for the static vector Green’s function \( G_{\xi^0}(\xi^i, \xi^i_0) \) is

\[ -\xi^1_0 \delta(\xi^1 - \xi^1_0) \delta(\xi^2 - \xi^2_0) \delta(\xi^3 - \xi^3_0) \]  

with \( \xi^1_0 > 0 \) because we must divide by \( \xi^1 \) equation (4.10) to have a self-adjoint operator and to obtain a Green’s function symmetric in \( \xi^i \) and \( \xi^i_0 \). Therefore we have

\[ G_{\xi^0}(\xi^i, \xi^i_0) = \frac{m^2\xi^1 \xi^i_0}{2\pi} \frac{yI_1(x)K_0(y) + xI_0(x)K_1(y)}{y^2 - x^2} \]  

where \( x \) and \( y \) are functions (2.7). When \( m = 0 \), i.e. in electrostatics, the massless Green’s function is

\[ D_{\xi^0}(\xi^i, \xi^i_0) = \frac{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 \delta(\xi^1 - \xi^1_0) \delta(\xi^2 - \xi^2_0) \delta(\xi^3 - \xi^3_0)}{4\pi \sqrt{[(\xi^1_0)^2 + (\xi^1_0)^2 + (\xi^1_0)^2 + (\xi^3)^2 \delta(\xi^1 - \xi^1_0) \delta(\xi^2 - \xi^2_0) \delta(\xi^3 - \xi^3_0)]^2 - 4(\xi^1\xi^1_0)^2}. \]
When $\xi^1 \to 0$, expression (4.12) has the asymptotic form

$$G_{\xi^0}(\xi^1, \xi^0_0) \sim \frac{(\xi^1_0 \xi^1)^2}{2\pi[(\xi^0_0)^2 + (\xi^2 - \xi^2_0)^2 + (\xi^3 - \xi^3_0)^2]}$$

$$[\frac{1}{2}K_0 \left( m\sqrt{(\xi^1_0)^2 + (\xi^2 - \xi^2_0)^2 + (\xi^3 - \xi^3_0)^2} \right) + \frac{m}{\sqrt{(\xi^1_0)^2 + (\xi^2 - \xi^2_0)^2 + (\xi^3 - \xi^3_0)^2}}K_1 \left( m\sqrt{(\xi^1_0)^2 + (\xi^2 - \xi^2_0)^2 + (\xi^3 - \xi^3_0)^2} \right) \bigg] (4.14)$$

since $I_0(0) = 0$ and $I_1(x) \sim x/2$ as $x \to 0$.

So, we have established in closed form expression (1.12) of the static vector Green’s function in Rindler space. By virtue of (4.14), the scalar invariant of the Proca field $A_\mu A^\mu$ in metric (1.1) has a regular behavior at the horizon $\xi^1 = 0$, likewise the associated field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

5 Vacuum polarization potential of a charge

We consider now a uniformly accelerated electric charge $e$ in the Minkowski space-time. We denote $a^\mu$ the vector potential generated by this charge. The Maxwell equations are equivalent to

$$\Box a^\mu = j^\mu \quad \text{and} \quad \partial_\mu a^\mu = 0 \quad (5.1)$$

where $j^\mu$ is the conserved electric current associated with this charge. The Proca equations (4.1) with $m = 0$ reduce to (5.1). The induced vector potential $<a^\mu>$ resulting from the vacuum polarization can be derived from the Schwinger formula

$$<a^\mu(x)> = \frac{-4\alpha e}{\pi} \int \int_0^1 \Delta \left[ \frac{2}{\sqrt{1-v^2}} (x-x') \right]$$

$$\times \frac{1-v^2/3}{(1-v^2)^2} v^2 j^\mu(x')d^4x' dv \quad (5.2)$$

where $\Delta$ is given by (2.3), $m$ being the mass of the electric charge.

We can apply formula (5.2) for a uniformly accelerated electric charge. Taking into account the specific property of the advanced time $\tau_A$ in the present case, the field determined with $\Delta$ coincides with the one calculated with $\Delta_R$. We proceed as in Sec. 4 and from (4.9) we obtain

$$<a_{\xi^0}(\xi^1) > = \frac{4\alpha e}{\pi} \frac{m^2 \xi^1_0}{2\pi} \int_0^1$$

$$\left[ y_g I_1 \left( \frac{2x_g}{\sqrt{1-v^2}} \right) K_0 \left( \frac{2y_g}{\sqrt{1-v^2}} \right) + x_g I_0 \left( \frac{2x_g}{\sqrt{1-v^2}} \right) K_1 \left( \frac{2y_g}{\sqrt{1-v^2}} \right) \right]$$

$$\times \frac{1}{y_g^2 - x_g^2} \frac{1-v^2/3}{(1-v^2)^2} v^2 dv \quad (5.3)$$

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and also \(< a_{\xi^1} >= 0 \).

Now, we can express formula (5.3) with the aid of the static vector Green’s function (4.12) under the form

\[
< a_{\xi^0}(\xi^i) > = g^2 \frac{\alpha e}{\pi} \int_0^1 G_{\xi^0} \left[ \frac{2}{\sqrt{1-v^2}} \xi^i, \frac{2}{\sqrt{1-v^2}} \xi^i_0 \right] \frac{1-v^2/3}{1-v^2} v^2 dv .
\] (5.4)

We have already found result (5.4) without rigour in our previous work \([3]\) by putting forward a development in power series in \(1/m^2\) of the Schwinger formula (5.2) in the Minkowskian coordinates. However in the present work, we have now an explicit expression for \(G_{\xi^0}\).

Integral expression (5.3) can be approximatively evaluated if we suppose that the points \(\xi^i\) and \(\xi^i_0\) are such that \(x_g \gg 1\) and \(y_g \gg 1\) with \(y > x\). This means that

\[
g \ll m \text{ and } |\xi^i - \xi^i_g| \ll \frac{1}{g} .
\] (5.5)

We treat only weak accelerations and we remain in the neighborhood of the world line of the charge. Taking into account the asymptotic behavior of the modified Bessel functions

\[
I_\nu(z) = \frac{1}{\sqrt{2\pi z}} e^z \left( 1 + O\left( \frac{1}{z} \right) \right) , \quad K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left( 1 + O\left( \frac{1}{z} \right) \right),
\]

integral (5.3) has the approximate expression

\[
< a_{\xi^0}(\xi^i) > \approx \frac{4\alpha e}{\pi} \frac{m^2 \xi^1}{16\pi \sqrt{x_g y_g (y_g - x_g)}} \int_0^1 \exp \left( \frac{2}{\sqrt{1-v^2}} (x_g - y_g) \right) \frac{1-v^2/3}{1-v^2} v^2 dv .
\] (5.6)

We recognize in (5.6) the Uehling potential \(U(mr)/4\pi r\) \([4]\) which is the vacuum polarization potential of a fixed charge in the Minkowski space-time

\[
U(mr) = \frac{\alpha}{\pi} \int_0^1 \exp \left( - \frac{2mr}{\sqrt{1-v^2}} \right) \frac{1-v^2/3}{1-v^2} v^2 dv ;
\] (5.7)

it can be expressed in terms of special functions. Because (5.3) the Fermi coordinates \((1.3)\) are well adapted to this case with \(|y^i| \ll 1/g\). By using (2.10), we can express (5.6) in the form

\[
< a_{\xi^0}(y^i) > \approx \frac{e \sqrt{1+gy^i}}{4\pi \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}} U \left( m \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} \right) .
\] (5.8)

By neglecting terms in \(|gy^i|^2\) in (5.8), we finally get

\[
< a_{\xi^0}(y^i) > \approx \frac{e}{4\pi \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}} \left( 1 + \frac{1}{2}gy^1 \right) U \left( m \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} \right) \] (5.9)

where \(U\) is given by (5.7).
We now return to the explicit expression (5.3) and we can discuss the regularity at the horizon $\xi^1 = 0$. From asymptotic form (4.14), we see immediately that the induced electrostatic potential is proportional to $(\xi^1)^2$ as $\xi^1 \to \infty$. The induced charge density is derived via the Maxwell equations which reduce in the static case to

$$<j^0> = \left[ \xi^1 \frac{\partial}{\partial \xi^1} \left( \frac{1}{\xi^1} \frac{\partial}{\partial \xi^1} \right) + \frac{\partial^2}{\partial (\xi^2)^2} + \frac{\partial^2}{\partial (\xi^3)^2} \right] <a^0>.$$  

So, the induced charge density (5.10) is proportional to $\xi^1$ as $\xi^1 \to 0$.

### 6 Conclusion

Probably due to the lack of physical motivation, the expression in terms of special functions of the static, massive scalar or vector field generated by a point source at rest in the Rindler metric (1.1) had not been determined. We have filled up this gap by giving formulas (3.6) and (4.9) and also the Green’s functions (3.9) and (4.12).

Furthermore, our method for determining these fields allows us to treat the vacuum polarization for a uniformly accelerated electric charge by using the Schwinger formula. In the Rindler coordinates, we have found the induced electrostatic potential as an explicit integral expression (5.3). In the Fermi coordinates $(y^i)$ and with the assumption of a weak acceleration $g$, we have derived approximate expression (5.9) which is just the Uehling potential multiplied by $1 + gy^1/2$. This potential corresponds equivalently to the vacuum polarization potential of an electric charge at rest in the homogeneous gravitational field described by metric (1.6).

### Appendix

The basic formula on the Bessel function $J_0$ which is necessary in our present work is the following definite integral [8]

$$\int_{\ln y/x}^{\infty} J_0 \left( \sqrt{2xy \cosh t - x^2 - y^2} \right) dt = 2I_0(x)K_0(y)$$  

(A.1)

for $y > x > 0$, where $I_\nu$ and $K_\nu$ are the modified Bessel functions. Now, we derive (A.1) with respect to $x$ and $y$ and we combine these results in order to obtain the two following relations

$$\int_{\ln y/x}^{\infty} J_1 \left( \sqrt{2xy \cosh t - x^2 - y^2} \right) dt =$$

$$\frac{2}{y^2 - x^2} \left( 2(xI_1(x)K_0(y) + yI_0(x)K_1(y)) \right),$$  

(A.2)
\[
\int_{\ln y/x}^{\infty} \frac{J_1 \left( \sqrt{2xy \cosh t - x^2 - y^2} \right)}{\sqrt{2xy \cosh t - x^2 - y^2}} \cosh t dt = \frac{x^2 + y^2}{xy(y^2 - x^2)} - \frac{2(yI_1(x)K_0(y) + xI_0(x)K_1(y))}{y^2 - x^2} \] (A.3)

where we have used \( J_0(0) = 1, J'_0 = -J_1, I'_0 = I_1 \) and \( K'_0 = -K_1 \).

By noticing that
\[
\int_0^{\infty} J_1(au)du = \frac{1}{a},
\]
it is easy to prove
\[
\int_{\ln y/x}^{\infty} \frac{J_1 \left( \sqrt{2xy \cosh t - x^2 - y^2} \right)}{\sqrt{2xy \cosh t - x^2 - y^2}} \sinh t dt = \frac{1}{xy}. \] (A.4)

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