Ruin probabilities for risk process in a regime-switching environment

Zbigniew Palmowski

Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wrocław, Poland

**ABSTRACT**

In this paper we give a few expressions and asymptotics of ruin probabilities for a Markov modulated risk process for various regimes of a time horizon, initial reserves and a claim size distribution. We also consider a few versions of the ruin time.

**ARTICLE HISTORY**

Received 12 June 2021
Accepted 23 October 2021

**KEYWORDS**

Ruin time; asymptotics; change of measure; Cramér asymptotics; subexponential distribution; central limit theorem

1. Introduction

This paper concerns the ruin probabilities in the context of regime-switching environment. The risk theory is substantial for insurance mathematics and has been the subject of many interests since Lundberg. There is a good deal of work related to this theory and there are many great books describing it; see e.g. Asmussen & Albrecher (2010), Rolski et al. (1999) and Kyprianou (2013). The understanding of so-called Markov modulated risk process is also deep; see Chap. VII of Asmussen & Albrecher (2010). The need for Markov modulation is not questionable nowadays. It can model the state of the roads in the case of car insurance, economic environments or insurance claims with a higher degree of complexity insofar as claim frequencies and/or severities are concerned; see e.g. Cheung & Landriault (2009b) and references therein.

Still, we believe that there is a need for the short survey related to these results and with a few additional new facts that have not been proved yet. In particular, we give the counterpart of Arfwedson Theorem for the Markov modulated risk process. We also give a subexponential asymptotics of the Parisian probability of ruin. This is our first goal. Second, main goal is the ordering of a few key results with the unified use of so-called scale matrices introduced by Ivanovs & Palmowski (2012). This part of the paper can be then treated as a Markov modulated version of some facts presented in Kyprianou (2013). Due to Markov modulation, some additional care is required here though.

The paper is organized as follows. In the next section, we formally introduce the risk process that we work with. In Section 3, we give a few main facts concerning the scale matrices. In Section 4 we present the key compensation formula and related representations of Gerber-Shiu function. Section 5 gives the ordinary differential equation that Gerber-Shiu function solves. In Sections 6 and 7, we present the Cramér and subexponential asymptotics, respectively. In Sections 8 and 9, we focus on Segerdahl’s and Höglund’s approximations of the finite-time ruin probability. The last section is dedicated to Parisian ruin probability.
2. Gerber-Shiu function

The goal of this paper is to analyse various ruin probabilities of a Markov modulated risk process. To describe this process properly we start by introducing a random environment that is given by a continuous-time Markov chain $J_t$ living on the state space $E = \{1, 2, \ldots N\}$. Let $T_k$ denote successive jump epochs of the Markov chain $J_t$. Then the risk process under consideration is given by

$$X_t = x + \int_0^t p_{J_u} \, du - \sum_{k=1}^{N_t} C_k^{(J_t)} - \sum_{t_k \leq t} C_k^{(J_{t_k} - J_{t_k})}. \quad (1)$$

Above, $x$ describes an initial capital, $N_t$ is a Markov modulated Poisson process with an arrival intensity $\lambda_i$ at time $t$ when $J_t = i$ determining the arrival of i.i.d. claims $\{C_k^{(i)}\}$ which are conditionally independent of $N_t$ and having distribution $F^{(i)}$ depending on the state $i \in E$ of the environmental Markov chain $J$ at time $t$. Apart from it, we have possible claims $C_k^{(i, j)}$, $i, j \in E$, appearing when the environmental Markov chain changes its state. The vector $(p_1, p_2, \ldots, p_N)$ is a vector of premium intensities. In the following we assume that the processes $X$ and $J$ are defined on a common filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and we use $\mathbb{P}_{x, j}$ to denote the law of $(X, J)$ given $\{X_0 = x, J_0 = i\}$ and by $\mathbb{P}$ and $\mathbb{P}_x$ we denote the law of $(X, J)$ given $\{X_0 = 0, J_0 = i\}$ and $\{X_0 = x\}$, respectively. The appropriate expectations we will denote by $\mathbb{E}_{x, i}$, $\mathbb{E}_i$ and $\mathbb{E}_x$. We also write $\mathbb{E}[Z; J_t]$ where $Z$ is some random variable, to denote the $N \times N$ matrix with entries $\mathbb{E}[Z; J_t = j] = \mathbb{E}_i[\mathbb{E}[Z_{(j_{n=j})}]]$. Finally, we denote $\mathbb{P}(\cdot) = \mathbb{P}_0(\cdot)$ and $\mathbb{E}[] = \mathbb{E}_0[\cdot]$.

We will assume throughout this work the following net-profit condition

$$\mathbb{E}_i X_1 > 0, \quad \text{for all } i \in E, \quad (2)$$

under which risk process $X$ tends to infinity a.s.

Observe that $X_t$ is a spectrally negative Markov Additive Process (MAP) with the matrix exponent

$$F(\alpha) = \text{diag} \left( p_i \alpha + \lambda_i (\mathbb{E} e^{-C^{(i)} \alpha} - 1) \right)_{i \in E} + (q_{ij} \mathbb{E} e^{-C^{(j)} \alpha})_{i, j \in E}, \quad (3)$$

where $\alpha \geq 0$, $C^{(i)}$ and $C^{(j)}$ are generic random variables representing claims $C_k^{(i)}$ and $C_k^{(j)}$, respectively, and $Q = (q_{ij})_{i, j \in E}$ is an intensity matrix of $J$. In other words,

$$\mathbb{E}[e^{\alpha X_i}; J_t] = e^{F(\alpha)t}. \quad \text{The main object of the study is expected discounted penalty function (EDPF) called a Gerber-Shiu function as well and defined by }$$

$$\phi_{w, ij}^{(q)}(x) = \mathbb{E}_{x, i} [e^{-q \tau^-_0} w(X_{\tau^-_0}, |X_{\tau^-_0}|); J_{\tau^-_0} = j, \tau^-_0 \leq \infty] \quad (4)$$

for the ruin time

$$\tau^-_0 = \inf \{ t \geq 0 : X_t < 0 \}. \quad (5)$$

Above, $q \geq 0$ is a discounting factor and $w$ is a bivariate non-negative penalty function. This function describes the penalty that is paid at the moment of ruin. It depends on the position prior ruin $X_{\tau^-_0}$ and the deficit $|X_{\tau^-_0}|$ at the ruin moment. If $w \equiv 1$ then Gerber-Shiu function is a discounted ruin
probability:
\[
\phi_{1,ij}^{(q)} = \mathbb{E}_{x,i}[e^{-q\tau_0^+}; J_{\tau_0^+} = j, \tau_0^- < \infty].
\]  
(6)

The most common case is when \( q = 0 \) and then
\[
\phi_{ij}(x) = \phi_{1,ij}^{(0)}(x) = \mathbb{P}_{x,i}(\tau_0^- < \infty, J_{\tau_0^-} = j)
\]
(7)
is the ruin probability. We will write \( \phi_{w,i}(x) = \sum_{j \in E} \phi_{w,ij}(x) \), \( \phi_{1,i}(x) = \sum_{j \in E} \phi_{1,ij}^{(q)}(x) \) and
\[
\psi(\alpha) = q \left( \int_{0}^{\infty} e^{-\alpha x} W^{(q)}(x) dx = (F(\alpha) - q\mathbb{I})^{-1}, \quad \alpha > \max\{\text{Re}(\lambda) : \det(F(\lambda) - \lambda q\mathbb{I}) = 0\}, \right)
\]
(10)
where \( \mathbb{I} \) is the identity matrix. The matrix \( W^{(q)}(x) \) is invertible for \( x > 0 \) and satisfies
\[
\mathbb{E}_x[e^{-q\tau_0^+}; \tau_0^+ < \tau_0^-] = W^{(q)}(x)W^{(q)}(a)^{-1}
\]
for \( 0 \leq x \leq a \), where
\[
\tau_0^+ = \inf\{t \geq 0 : X_t > a\}.
\]
In the case of \( N = 1 \) this result corresponds to Kyprianou & Palmowski (2005, Thm. 1).

We also define
\[
Z^{(q)}(\alpha, x) = e^{\alpha x} \left( \mathbb{I} - \int_{0}^{x} e^{-\alpha y} W^{(q)}(y) dy (F(\alpha) - q\mathbb{I}) \right) \quad \text{for} \quad \alpha, q, x \geq 0
\]
and when \( \alpha = 0 \) we denote
\[
Z^{(q)}(x) = \mathbb{I} - \int_{0}^{x} W^{(q)}(y) dy (F(0) - q\mathbb{I}) = \mathbb{I} - \int_{0}^{x} W^{(q)}(y) dy (Q - q\mathbb{I}).
\]
(12)
It is so-called second scale matrix. We will also need a matrix \( R \) being a left solution of
\[
F(-R^{(q)}) = q\mathbb{I}.
\]
In case when \( N = 1 \) we have \( R = -\Phi(q) \) where \( \Phi(q) \) is a right-inverse of the Laplace exponent \( \psi(\alpha) = F(\alpha) \) of spectrally negative Lévy process \( X \) (hence \( \psi(\Phi(q)) = q \)). Then for all \( x, \alpha \geq 0 \) and

### 3. Preliminaries

We follow here mainly Ivanovs & Palmowski (2012). Define an \( N \times N \) matrix-valued function \( W^{(q)}(x) \) which is continuous for \( x \geq 0 \) and is identified by the following Laplace transform:
\[
\int_{0}^{\infty} e^{-\alpha x} W^{(q)}(x) dx = (F(\alpha) - q\mathbb{I})^{-1}, \quad \alpha > \max\{\text{Re}(\lambda) : \det(F(\lambda) - \lambda q\mathbb{I}) = 0\},
\]
(10)
where \( \mathbb{I} \) is the identity matrix. The matrix \( W^{(q)}(x) \) is invertible for \( x > 0 \) and satisfies
\[
\mathbb{E}_x[e^{-q\tau_0^+}; \tau_0^+ < \tau_0^-] = W^{(q)}(x)W^{(q)}(a)^{-1}
\]
for \( 0 \leq x \leq a \), where
\[
\tau_0^+ = \inf\{t \geq 0 : X_t > a\}.
\]
In the case of \( N = 1 \) this result corresponds to Kyprianou & Palmowski (2005, Thm. 1).

We also define
\[
Z^{(q)}(\alpha, x) = e^{\alpha x} \left( \mathbb{I} - \int_{0}^{x} e^{-\alpha y} W^{(q)}(y) dy (F(\alpha) - q\mathbb{I}) \right) \quad \text{for} \quad \alpha, q, x \geq 0
\]
and when \( \alpha = 0 \) we denote
\[
Z^{(q)}(x) = \mathbb{I} - \int_{0}^{x} W^{(q)}(y) dy (F(0) - q\mathbb{I}) = \mathbb{I} - \int_{0}^{x} W^{(q)}(y) dy (Q - q\mathbb{I}).
\]
(12)
It is so-called second scale matrix. We will also need a matrix \( R \) being a left solution of
\[
F(-R^{(q)}) = q\mathbb{I}.
\]
In case when \( N = 1 \) we have \( R = -\Phi(q) \) where \( \Phi(q) \) is a right-inverse of the Laplace exponent \( \psi(\alpha) = F(\alpha) \) of spectrally negative Lévy process \( X \) (hence \( \psi(\Phi(q)) = q \)). Then for all \( x, \alpha \geq 0 \) and
where \( q > 0 \)
\[
\mathbb{E}_x[e^{-q\tau_0^+ + \alpha X_{\tau_0^-}}; \tau_0^- < \tau_0^+, J_{\tau_0^-}] = Z^{(q)}(\alpha, x) - W^{(q)}(x)W^{(q)}(a)^{-1}Z^{(q)}(\alpha, a).
\] (13)
Hence by taking \( a \to +\infty \) and assuming that \( R^{(q)} + \alpha \mathbb{I} \) is non-singular, it holds that
\[
\mathbb{E}_x[e^{-q\tau_0^+ + \alpha X_{\tau_0^-}}; \tau_0^- < \tau_0^+] = Z^{(q)}(\alpha, x) - W^{(q)}(x)(R^{(q)} + \alpha \mathbb{I})^{-1}(F(\alpha) - q\mathbb{I}).
\]
The above result can be extended to \( q = 0 \) by taking the limits as \( q \downarrow 0 \). Moreover, it is noted that
\[
(R^{(q)} + \alpha)^{-1}(F(\alpha) - q\mathbb{I}) \text{ reduces to } (\psi(\alpha) - q)/(\alpha - \Phi^{(q)}) \text{ in the Lévy case. This leads to the known identity with } \alpha = 0 \text{ for a Lévy process, see Kyprianou (2014, Thm. 8.1).}
\]

We can consider process \( X \) killed on exiting positive half-line and the corresponding potential measure (a matrix of measures)
\[
U^{(q)}(x, A) = \int_0^{\infty} e^{-qt} \mathbb{P}_x[X_t \in A; t > \tau_0^- , J_t] dt,
\]
where \( A \) is a Borel set. It turns out that the measure \( U^{(q)}(x, A) \) has a matrix density \( u^{(q)}(x, z) \) on \((0, +\infty)\) with respect to Lebesgue measure and by Ivanovs (2014) it is given by
\[
u^{(J_t-J_s)}(dy) = \lambda f^{(J_t-J_s)}(dy) + q_{J_t-J_s} F^{(J_t-J_s)}(dy). (16)
\]
is a compensator of \( NC \), that is, for any measurable and bounded \( g \) we have
\[
\mathbb{E}_t \int_0^t g(s, y, J_s) dC(s, dy) = \mathbb{E}_t \int_0^t g(s, y, J_s) \nu^{(J_t-J_s)}(dy) ds.
\]
We denote by \( T^C_\tau \) the jump epoch of the risk process (1). This means that \( T^C_\tau \) is either \( T^N_\tau \) (the jump epoch of the Poisson process) or \( T_\tau \) (the moment when Markov chain \( J \) changes a state). This gives the following compensation formula.

**Theorem 4.1:** For any measurable and bounded function \( f : [0, +\infty)^2 \times \mathbb{R} \times E^2 \to \mathbb{R} \) we have
\[
\mathbb{E}_{x,i} \left[ \sum_{T^C_\tau \leq t} f(T^C_\tau, X_{T^C_\tau}, X_{T^C_\tau}^- J^C_\tau, J^C_\tau^-) \right] = \sum_{k,j \in E} \int_0^{\infty} \mathbb{E}_{x,i} \int_0^t f(s, x_s, x_s - y, k, j) ds \, \nu^{(k,j)}(dy).
\]
**Proof:** Note that

\[
\mathbb{E}_{x,i} \left[ \sum_{T_k^c \leq t} f(T_k^c, X_{T_k^c-}, X_{T_k^c}, J_{T_k^c-}, J_{T_k^c}) \right] 
\]

\[
= \mathbb{E}_{x,i} \left[ \sum_{T_k^c \leq t} f(T_k^c, X_{T_k^c-}, X_{T_k^c}, C_k^{(j_{T_k^c})} 1_{j_{T_k^c-} = j_{T_k^c}} - C_k^{(j_{T_k^c} - j_{T_k^c})} 1_{j_{T_k^c-} = j_{T_k^c}}, J_{T_k^c-}, J_{T_k^c}) \right] 
\]

\[
= \mathbb{E}_{x,i} \int_0^t f(s, X_{s-}, X_s - y, J_{s-}, J_s) N_C(ds, dy) 
\]

\[
= \sum_{k,j \in E} \int_0^\infty \mathbb{E}_{x,i} \int_0^t f(s, X_{s}, X_s - y, k, j) ds \nu^{(k,j)}(dy) 
\]

which completes the proof. 

**Corollary 4.1:** We have

\[
\phi^{(q)}_{w,i}(x) = \sum_{k \in E} \int_0^\infty \int_0^y w(z, y - z) u^{(q)}_{ik}(x, z) dz \nu^{(k,j)}(dy), 
\]

where \( i, j \in E \) and \( u^{(q)}(x, z) \) is a \( q \)-potential density of the process \( X \) starting at \( x \) and killed on exiting from \([0, +\infty)\).

**Proof:** Observe that ruin can happen only at the moments of claim arrivals. Hence from Theorem 4.1,

\[
\phi^{(q)}_{w,i}(x) = \mathbb{E}_{x,i} \left[ \sum_{k=1}^\infty e^{-q x_{T_k^c}} w(X_{T_k^c-}, X_{T_k^c}) 1_{X_{T_k^c-} \geq 0, X_{T_k^c} < 0, J_{T_k^c-} = j} \right] 
\]

\[
= \sum_{k \in E} \int_0^\infty \left( \mathbb{E}_{x,i} \int_0^\infty \int_0^y e^{-q s} w(z, y - z) 1_{X_{s-} \in dz, \tau_0^- > s} 1_{J_{s-} = k, J_s = j} ds \right) \nu^{(k,j)}(dy) 
\]

\[
= \sum_{k \in E} \int_0^\infty \left( \int_0^\infty \int_0^y e^{-q s} w(z, y - z) \mathbb{P}_{x,i}(X_{s-} \in dz, \tau_0^- > s, J_{s-} = k) ds \right) \nu^{(k,j)}(dy). 
\]

This completes the proof.

**Remark 4.1:** For the above considerations, the downward jumps are not crucial and the above corollary could be easily adopted for the general direction of jumps of the process \( X \).
A similar result was derived by Salah & Morales (2012) and Kyprianou (2013, Thm. V.5.5) in the context of spectrally negative Lévy processes. From Corollary 4.1 and (15) we have the first main result of this section.

**Theorem 4.2 (Compensation representation):** We have

\[ \phi^{(q)}_w(x) = \int_0^\infty \int_0^\infty w(z,y) u^{(q)}(x,z) \nu(z + dy)dz, \]

where \( u^{(q)}(x,z) \) is given in (15).

We recall that the matrix \( \phi^{(q)}_1(x) = (\phi^{(q)}_{1,i}(x))_{i,j \in E} \) describes the Laplace transform of the ruin time on the event when ruin happens and it is defined formally in (6). From (13) by taking \( \alpha = 0 \) and \( a \to +\infty \) the following its representation holds true (compare with Kyprianou 2013, Thm. IV.4.3).

**Theorem 4.3 (Discounted ruin probability):** We have,

\[ \phi^{(q)}_1(x) = Z^{(q)}(x) - W^{(q)}(x)C_{W(\infty)^{-1}Z(\infty)}, \] (18)

for

\[ C_{W(\infty)^{-1}Z(\infty)} = \lim_{a \to \infty} W^{(q)}(a)^{-1}Z^{(q)}(a) < \infty. \]

Observe that \( C_{W(\infty)^{-1}Z(\infty)} \) is well-defined since \( \phi^{(q)}_1(x) \) as a limit is well-defined. Note that identity (18) gives the Laplace transform of the finite-time ruin time as well, that is,

\[ \phi^{(q)}_1(x) = \int_0^\infty e^{-qt}dP_x(\tau_0^- \leq t). \]

In other words, inverting (18) gives the finite-time ruin probability

\[ \phi(x,t) = P_x(\tau_0^- \leq t). \] (19)

Multiplying by 1 from the right and taking \( q \downarrow 0 \) in (18) gives also the representation of the ruin probability \( \phi_i(x) \) defined in (8). Indeed, recalling that \( \phi(x) = (\phi_{ij}(x))_{i,j \in E} \) is defined in (7), note that

\[ \phi_i(x) = (\phi(x)1)_i = P_{x,i}(\tau_0^- < \infty) \]

\[ = \left( \mathbb{I}1 - \int_0^x W(y)dyF(0)1 - W(x)C_{W(\infty)^{-1}Z(\infty)}1 \right)_i = 1 - (W(x)C_{W(\infty)^{-1}Z(\infty)}1)_i. \]

Above we have used the definition of the second scale matrix \( Z^{(q)}(x) \) given in (12) and fact that \( F(0)1 = Q1 = 0 \). We introduce now the survival probability

\[ \phi_i(x) = 1 - \phi_i(x). \]

Using fact that matrix \( C_{W(\infty)^{-1}Z(\infty)} \) is well-defined we can conclude the following representation of survival probability.

**Theorem 4.4 (Survival probability):** Under net-profit condition (2) we have

\[ \phi_i(x) = (W(x)C_{W(\infty)^{-1}Z(\infty)}1)_i, \quad i \in E. \] (20)
Let now $N = 1$. Then $Q = 0$. Recall that  
\[
\int_0^\infty e^{-\alpha x} W(x) \, dx = F(\alpha)^{-1}.
\]
In our case by (3),
\[
F(\alpha)^{-1} = \frac{1}{p_1 \alpha} \frac{1}{1 - \frac{\lambda_1 \mathbb{E}C_1^{(1)}}{p_1} \int_0^\infty e^{-\alpha y} F_{C_1}^{(1)}(y) \mathbb{E}C_1^{(1)} \, dy} = \frac{1}{p_1 \alpha} \sum_{k=1}^\infty \rho_1^k \left( \int_0^\infty e^{-\alpha y} F_{C_1}^{(1)}(y) \, dy \right)^k
\]
where
\[
\rho_1 = \frac{\lambda_1 \mathbb{E}C_1^{(1)}}{p_1},
\]
\[
F_{C_1}^{(1)}(x) = \frac{1}{\mathbb{E}C_1^{(1)}} \int_0^x F_{C_1}^{(1)}(y) \, dy,
\]
for
\[
F_{C_1}^{(1)}(x) = 1 - F_{C_1}(x).
\]
Thus
\[
W(x) = \sum_{k=1}^\infty \rho_1^k (F_{C_1}^{(1)})^* k(x),
\]
where $(F_{C_1}^{(1)})^* k$ denotes the $k$th convolution of distribution $F_{C_1}^{(1)}$. To sum up,
\[
\bar{\phi}_1(x) = A_1 \sum_{k=1}^\infty \rho_1^k (F_{C_1}^{(1)})^* k(x)
\]
for some constant $A_1$. From the fact that $\lim_{x \to +\infty} \bar{\phi}_1(x) = 1$ we can get identify constant $A_1$ and derive the seminal Pollaczek-Khintchine formula for the survival probability.

**Theorem 4.5 (Pollaczek-Khintchine formula):** If $N = 1$ then
\[
\bar{\phi}_1(x) = (1 - \rho_1) \sum_{k=1}^\infty \rho_1^k (F_{C_1}^{(1)})^* k(x).
\]

We denote by
\[
\xi_k = X_{T_{k-1}^{C_1}} - X_{T_k^{C_1}}
\]
the negative of the increments of the risk process between consecutive jumps. Note that, conditionally on $J_{I_1^{C_1}}$, random variables $\xi_k$ are independent of each other. Moreover, conditionally on $\{J_{I_k^{C_1}} = j, J_{I_{k-1}^{C_1}} = i\}$, $-\xi_k$ has the same law as
\[
p_1 \text{Exp}(\lambda_i - q_{ii}) - \frac{\lambda_i}{\lambda_i - q_{ii}} C_{i}^{(i)} - \frac{q_{ij}}{\lambda_i - q_{ii}} C_{ij}^{(i)}
\]
because waiting time for the next jump has exponential law $\text{Exp}(\lambda_i - q_{ii})$ with parameter $q_{ij} + \lambda_i$. Then we have to take into account which jump is first: the one coming from Markov modulated
Poisson process $N$ or the one that Markov chain change of the state brings. In the first scenario Markov chain $J_t$ remains in the state $i$. Note that the law of $\xi_k$ depends on the states $J_{k-1}^C$ and $J_k^C$ of the discrete-time Markov chain

$$J_0^C = J_0, \quad J_k^C = J_{\tau_k^C}, \quad k = 1, 2, \ldots$$

with the transition matrix

$$\mathcal{P} = (p_{ij}) \quad \text{with} \quad p_{ii} = \frac{\lambda_i}{\lambda_i - q_{ii}} \quad \text{and} \quad p_{ij} = \frac{q_{ij}}{\lambda_i - q_{ii}} \quad \text{for} \ i \neq j. \quad (22)$$

We define the following discrete-time Markov modulated random walk:

$$S_0 = 0, \quad S_n = \sum_{k=1}^{n} \xi_k. \quad (23)$$

We recall the key observation that ruins of the risk process (9) only at the claim arrivals. By shifting a generic trajectory of $X_t$ by $x$ units downward and then reflecting it across the horizontal axis, we can observe that the risk process $X$ gets below zero level if and only if $S_k$ will ever cross level $x$. Moreover, the net-profit condition (2) is equivalent to the requirements that $S_n \to +\infty$ a.s. This gives us the last representation.

**Theorem 4.6 (Maximum random walk):** We have

$$\phi_i(x) = \mathbb{P}_i \left( \max_{k \geq 0} S_k > x \right) < 1. \quad (24)$$

**Remark 4.2:** Note that by enriching the state space to the pairs $(i,j) \in \mathbb{L}^2$ and by taking the Markov chain $\tilde{J}_t = (J_t, \tilde{J}_t)$, we can assume without loss of generality that the distribution of generic increment $\xi$, depends only on the state of the Markov chain just prior jump. In this case $X$ becomes so-called non-anticipative MAP. In other words, without loss of generality we can assume the distribution of $\xi$ (being the negative of r.v. given in (21)) can be indexed by $F^{(i)}_{\xi}(x)$. To simplify this analysis we will assume from now on that this condition holds whenever we work with random walk $S_n$. In this case, denoting

$$a_i = \int_{\mathbb{R}} x F^{(i)}_{\xi}(dx)$$

which we assume to be finite, the net-profit condition (2) is equivalent to

$$a = -\sum_{i=1}^{N} \pi_i a_i > 0. \quad (25)$$

**Remark 4.3:** The representation (24) of the ruin probability via maximum random walk remains true even if the generic time between arrivals of the claims has a general distribution possibly depending on the discrete-time Markov chain $J^C$.

**5. Ordinary differential equation**

In this section, we derive an ordinary differential equation for Gerber-Shiu functions. We start by proving some crucial martingale property.
Theorem 5.1: For each \( i \in E \), the processes:
\[
e^{-qt \wedge \tau_0^- \wedge \tau_a^+} I_{[\tau_a^-,\tau_a^+]}(X_{t \wedge \tau_0^- \wedge \tau_a^+}), \quad e^{-qt \wedge \tau_0^- \wedge \tau_a^+} \mathcal{F}_{t \wedge \tau_0^- \wedge \tau_a^+} (X_{t \wedge \tau_0^- \wedge \tau_a^+})
\]
are uniformly integrable martingales with respect to the natural filtration \( \mathcal{F}_t \) of MAP \( (X, I) \).

Proof: From (11) and the strong Markov property it follows that
\[
\mathbb{E}_{x, i} \left[ e^{-qt \wedge \tau_0^- \wedge \tau_a^+} 1_{\{\tau_a^- < \tau_a^+, J_{i,j} = j\}} | \mathcal{F}_{t \wedge \tau_0^- \wedge \tau_a^+} \right] = e^{-qt \wedge \tau_0^- \wedge \tau_a^+} W_{i, J_{i,j}}^q (X_{t \wedge \tau_0^- \wedge \tau_a^+}) [W^q(a)^{-1}]_{t \wedge \tau_0^- \wedge \tau_a^+}
\]
where we used the fact that \( W^q(X_{t_0^-}) = 0 \) and \( W^q(X_{t_0^+}) W^q(a)^{-1} = 1 \). This completes the proof of the first martingale property since \([W^q(a)^{-1}]_{t \wedge \tau_0^- \wedge \tau_a^+} \) is \( \mathcal{F}_{t \wedge \tau_0^- \wedge \tau_a^+} \)-measurable.

Similarly, from (13) we have
\[
\mathbb{E}_{x, i} \left[ e^{-qt \wedge \tau_0^- \wedge \tau_a^+} 1_{\{\tau_a^- < \tau_a^+, J_{i,j} = j\}} | \mathcal{F}_{t \wedge \tau_0^- \wedge \tau_a^+} \right] = e^{-qt \wedge \tau_0^- \wedge \tau_a^+} \left( Z_{i, J_{i,j}}^q (X_{t \wedge \tau_0^- \wedge \tau_a^+}) - W_{i, J_{i,j}}^q (X_{t \wedge \tau_0^- \wedge \tau_a^+}) A_{t \wedge \tau_0^- \wedge \tau_a^+} \right)
\]
for some matrix \( A \). Using similar arguments as above we derive the martingale property of the second process.

From the compensation representation of the Gerber-Shiu function given in Theorem 4.2, we can conclude that
\[
e^{-qt \wedge \tau_0^- \wedge \tau_a^+} \phi_{w,i}^q (X_{t \wedge \tau_0^- \wedge \tau_a^+})
\]
is a martingale as well.

Moreover, from Theorem 5 of Ivanovs & Palmowski (2012) (see remark just after this theorem as well) and by Lemma 2.4 of Kyprianou et al. (2013) we know that if \( F_C^{i,j} \) and \( F_C^{i,j} \) \((i, j \in E)\) are absolutely continuous, then \( W^q \in C^1(0, \infty) \). Which means from Theorem 4.2 that \( \phi_w \in C^1(0, \infty) \). Thus \( \phi_w \) is sufficiently smooth to apply an infinitesimal generator.

Theorem 5.2: Assume that for each \( i, j \in E \) the distribution functions \( F_C^{i,j} \) and \( F_C^{i,j} \) have continuous densities \( f_C^{i,j} \) and \( f_C^{i,j} \). Then the Gerber-Shiu function \( \phi_w^q \) is in \( C^1(0, \infty) \) and for \( x \geq 0 \) it satisfies the following differential equation
\[
p_i \left( \phi_{w,i}^q \right)'(x) + \int_0^\infty (\phi_{w,i}(x, y) - \phi_{w,i}(x)) f_C^{i,j}(y) dy
+ \sum_{k \in E \setminus \{i\}} q_{ik} \int_0^\infty (\phi_{w,i}^q(x, y) - \phi_{w,i}^q(x)) f_C^{i,k}(y) dy - q \phi_{w,i}^q(x) = 0
\]
with the boundary conditions
\[
\phi_{w,i}^q(x) = w(x) \quad \text{for} \ x < 0,
\]
\[
\lim_{x \to +\infty} \phi_{w,i}^q(x) = 0.
\]
Proof: Since \( a > 0 \) in (26) is general then (27) follows straightforward from (26) and Dynkin formula since it is equivalent to the requirement that

\[
\mathcal{A} \Phi_{w,i}^{(q)}(x) - q \Phi_{w,i}^{(q)}(x) = 0, \quad i \in E,
\]

where \( \mathcal{A} \) is the infinitesimal generator of \( X \) with the domain included in \( C^1(0, \infty) \). It suffices to prove now only the boundary conditions. The first one follows straightforward from definition of the Gerber-Shiu function. The second one is a consequence of net-profit condition (2).

The equation given in Theorem 5.2 is well-known for the classical risk process; see e.g. Kyprianou (2013) and references therein. In the context of Markov modulated risk process it appears e.g. in Asmussen (1989), Jacobsen (2005), Ng & Yang (2006), Badescu & Landriault (2009) and Cheung & Landriault (2009a).

6. Cramér asymptotics

We will follow Asmussen & Albrecher (2010) and Asmussen (1989).

We recall that for \( \alpha \geq 0 \) the matrix \( F(\alpha) \) has a real simple eigenvalue \( k(\alpha) \), which is larger than the real part of any other eigenvalue. The corresponding left-eigenvector \( v(\alpha) \) and right-eigenvector \( h(\alpha) \) can be chosen so that \( v_i(\alpha) > 0 \) and \( h_i(\alpha) > 0 \) for all \( i \). The normalization requirement

\[
\pi h(\alpha) = 1, \quad v(\alpha) h(\alpha) = 1
\]

results in the unique choice of \( v(\alpha) \) and \( h(\alpha) \), where \( \pi = (\pi_1, \pi_2, \ldots, \pi_N) \) is a stationary distribution of \( J \). Observe that \( k(0) = 0, h(0) = 1 \) and \( v(0) = \pi \).

We assume in this section that there exists solution \( \gamma > 0 \) of so-called Cramér-Lundberg equation

\[
k(-\gamma) = 0.
\]

Note that a necessary condition for this existence is that \( \int_0^\infty e^{\gamma x} P(C(i) \in dx) < +\infty \) and that \( \int_0^\infty e^{\gamma x} P(C^{(ij)} \in dx) < +\infty \), hence both \( C(i) \) and \( C^{(ij)} \) must be light-tailed. This solution \( \gamma \) is called an adjustment coefficient.

We consider now the exponential change of measure

\[
\frac{d\tilde{P}|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}} = e^{-\gamma(X_t-x) - k(-\gamma)t} \frac{h_{f}(\gamma)}{h_{f_0}(\gamma)} = e^{-\gamma(X_t-x)} \frac{h_{f}(\gamma)}{h_{f_0}(\gamma)}. \tag{29}
\]

From Palmowski & Rolski (2002) it follows that our risk process \((X, J)\) under \( \tilde{P} \) is again MAP with the matrix exponent

\[
\tilde{F}(\alpha) = \Delta_{h(-\gamma)}^{-1} F(\alpha - \gamma) \Delta_{h(-\gamma)}.
\]

Thus the largest eigenvalue under \( \tilde{P} \) equals \( \tilde{k}(\alpha) = k(\alpha - \gamma) \) and hence \( \tilde{k}'(0) = k'(-\gamma) \). Moreover, from the martingale property of the density process used in the above change of measure, we can
conclude that

\[ E \left[ e^{\alpha X_t} \frac{1}{h_t(\alpha)} \right] = e^{k(\alpha)t} \frac{1}{h_t(\alpha)}. \]

By twice differentiation, we derive

\[ \text{Var}_\pi X_1 = k''(0) \geq 0 \]

(see e.g. Asmussen 2003, Cor. 2.6), where \( \text{Var}_\pi \) is a variance taken with respect of \( \mathbb{E}_\pi := \sum_{i=1}^N \pi_i \mathbb{E}_i \). Thus the function \( \beta \rightarrow k(\beta) \) is convex. We have

\[ k'(0) = \mathbb{E}_\pi X_1. \tag{30} \]

By net-profit condition (2) we have \( k'(0) > 0 \) and therefore this means that \( \tilde{k}'(0) = k'(-\gamma) < 0 \). From (30) we know that \( \tilde{k}'(0) \) equals the asymptotic drift of \( X \) under \( \tilde{P} \) which is negative, that is, \( \lim_{t \to +\infty} X_t = -\infty \tilde{P}\text{-a.s.} \) In other words, under a new measure the ruin is certain:

\[ \tilde{P}(\tau^-_0 < \infty) = 1. \tag{31} \]

This allows us to prove the main theorem of this section.

**Theorem 6.1:**

\[ \lim_{x \to +\infty} \frac{\phi_i(x)}{e^{-\gamma x}} = C_i \]

for some finite constant \( C_i > 0 \) and \( i \in E \).

**Proof:** Denoting by \( \tilde{\mathbb{E}} \) the expectation with respect of \( \tilde{P} \) note that by Optional Stopping Theorem we have

\[ \phi_i(x) = e^{-\gamma x} \tilde{\mathbb{E}}_{x,i} \left[ e^{\gamma X_{\tau^-_0}} \frac{h_i(\gamma)}{h_{\tau^-_0}(\gamma)}; \tau^-_0 < \infty \right] = e^{-\gamma x} \tilde{\mathbb{E}}_{x,i} \left[ e^{\gamma X_{\tau^-_0}} \frac{h_i(\gamma)}{h_{\tau^-_0}(\gamma)} \right], \]

where the last equality follows from (31). Moreover, using dual-process \( \hat{X} = -X \), we have

\[ \frac{\phi_i(x)}{e^{-\gamma x}} = \tilde{\mathbb{E}}_{0,i} \left[ e^{\gamma (\hat{X}_{\tau^+_x} - x)} \frac{h_i(\gamma)}{h_{\tau^+_x}(\gamma)} \right], \]

where

\[ \hat{\tau}^+_x = \inf\{t \geq 0 : \hat{X}_t > x\}. \]

From the Renewal Theorem 28 of Dereich et al. (2017) (see also Kesten 1974, Athreya et al. 1978, Lalley 1984, Alsmeyer 1994), we can conclude that right-hand side of the above identity tends to be constant.

**Remark 6.1:** In the context of general Lévy processes above Cramér asymptotics was proved by Bertoin & Doney (1994).
7. Subexponential asymptotics

We recall that by Theorem 4.6 the ruin probability $\phi_i(x)$ defined formally in (7) equals the tail of the distribution of the maximum

$$M = \sup_{k \geq 0} S_k$$

of a Markov modulated random walk

$$S_k = \sum_{i=1}^{k} \xi_k$$

with negative drift defined in (23) where the distribution $F^{(i)}(\xi)$ of $\xi$ is determined by the random variable (21) and discrete-time Markov chain with the transition matrix (22). This section deals with the study of the asymptotic distribution of the tail of the distribution of $M$ when the increments $\xi_k$ have heavy-tailed distributions, that is, when solution of Cramér-Lundberg Equation (28) does not exist. By a heavy-tailed distribution we mean a distribution (function) $G$ on $\mathbb{R}$ possessing no exponential moments: $\int_{0}^{\infty} e^{sy} G(dy) = \infty$ for all $s > 0$. We will use the principle of a single big jump, which says that the maximum of the random variable is essentially due to a single very large jump. More precisely, in this section, we will model the claim size by a subexponential distribution. This family of distributions is used to model many catastrophic events like earthquakes, storms, terrorist attacks etc. Additionally, insurance companies use e.g. the lognormal distribution (which is subexponential) to model car claims.

For any distribution function $G$ on $\mathbb{R}$, we set $\overline{G}(x) = 1 - G(x)$ and denote by $G^{*n}$ the $n$-fold convolution of $G$ by itself. A distribution $G$ on $\mathbb{R}_+$ belongs to the class $\mathcal{S}$ of subexponential distributions if and only if, for all $n \geq 2$, we have

$$\lim_{x \to \infty} \frac{G^{*n}(x)}{\overline{G}(x)} = n.$$ 

It is sufficient to verify this condition in the case $n = 2$ – see Chistyakov (1964). This statement is easily shown to be equivalent to the condition that, if $\xi_1, \ldots, \xi_n$ are i.i.d. random variables with common distribution $G$, then

$$\mathbb{P}(\xi_1 + \cdots + \xi_n > x) \sim \mathbb{P}(\max(\xi_1, \ldots, \xi_n) > x),$$

a statement which already exemplifies the principle of a single big jump. Here for any two functions $f$, $g$ on $\mathbb{R}$, by $f(x) \sim g(x)$ as $x \to \infty$ we mean $\lim_{x \to \infty} f(x)/g(x) = 1$; we also say that $f$ and $g$ are tail-equivalent. The class $\mathcal{S}$ includes all the heavy-tailed distributions commonly found in applications, in particular regularly varying, lognormal and Weibull distributions.

For any distribution $G$ on $\mathbb{R}$ with finite mean, we define the integrated (or second) tail distribution (function) $G^1$ by

$$\overline{G^1}(x) = 1 - G^1(x) = \min\left(1, \int_{x}^{\infty} \overline{G}(z) \, dz\right).$$

Good surveys of the basic properties of heavy-tailed distributions, in particular long-tailed and subexponential distributions, may be found in Foss et al. (2013), Embrechts et al. (1997) and in Asmussen & Albrecher (2010).
In this section, we assume that there exists some reference distribution $F$ with finite mean and some constants $c_i \geq 0$ ($i \in E$) such that

$$F_{\xi_i}(x) \leq F_i(x), \quad \text{for all } x \in \mathbb{R}, \ i \in E, \quad (D1)$$

$$F_{\xi_i}(x) \sim c_i F_i(x) \quad \text{as } x \to \infty, \ i \in E, \quad (D2)$$

$$F^i \in S. \quad (D3)$$

The condition (D1) is no less restrictive than the condition

$$\lim_{x \to \infty} \max_{i \in E} \frac{F_{\xi_i}(x)}{F_i(x)} < \infty,$$

in which case it is straightforward to redefine $F$, and then $c$, so that (D1) and (D2) hold as above. Further, the condition (D3) holds for example when the integrated tail distribution of claim size at some state $i \in E$ is subexponential, that is there exists $i \in E$ such that

$$F_{\xi i}^i \in S.$$

Define

$$C_S = \sum_{i \in E} c_i \pi_i. \quad (33)$$

In Foss et al. (2007) the following result is proved.

**Theorem 7.1:** Suppose that (D1)–(D3) hold. Then

$$\lim_{x \to \infty} \frac{\phi_i(x)}{F^i(x)} = \lim_{x \to \infty} \frac{\mathbb{P}_i(M > x)}{F^i(x)} = \frac{C_S}{\bar{a}},$$

where $\bar{a}$ is given in (25) and $i \in E$.

Most of the papers concern simple random walk hence the case when $N = 1$ and there is no Markov modulation. Then this problem has been very well understood. In this context taking $\bar{a} = -E\xi_1 > 0$ and assuming that $F^i \in S$ for distribution function $F$ of $\xi_1$ from Theorem 7.1 we derive classical the Pakes-Veraverbeke’s Theorem:

$$\mathbb{P}(M > x) \sim \frac{1}{\bar{a}} F^i(x) \quad \text{as } x \to \infty; \quad (34)$$

see e.g. Pakes (1975), Embrechts & Veraverbeke (1982). The intuitive idea underlying this result is the following: the maximum $M$ will exceed a large value $x$ if the random walk and modulating Markov chain follow the typical behaviour specified by the law of large numbers and stationary law $\pi$, respectively, except that at some time $n$ a jump occurs of size greater than $x + n\bar{a}$; this has probability $F_{\xi i}(x + n\bar{a})$ if Markov chain is in a state $i$; replacing the sum over all $n$ of these probabilities by an integral yield (34). That is, informally, $\mathbb{P}(M > x)$ is equivalent (for large initial capital $x$) to the
following sum

\[ P(M > x) \sim \sum_{n \geq 1} \sum_{i=1}^{N} \pi_i \mathbb{P}(\xi_n > x + \bar{a}n, \max_{k \leq n-1} S_k < x, \xi_{n-1} \in ((\bar{a} - \epsilon)(n - 1), (\bar{a} + \epsilon)(n - 1)) \sim \sum_{n \geq 1} \sum_{i=1}^{N} \pi_i \mathbb{P}(\xi_n > x + \bar{a}n) \]

\[ \sim \sum_{i=1}^{N} \pi_i \int_{0}^{\infty} \mathbb{P}(x + \bar{a}t) dt \sim \sum_{i=1}^{N} \pi_i c_i \int_{0}^{\infty} \mathbb{P}(x + \bar{a}t) dt = \frac{C_S}{\bar{a}} \Phi_1(x) \]

for the random walk \( S_n \) defined in (32) and \( \epsilon > 0 \). This again is the principle of a single big jump. See Zachary (2004) for a short proof of (34) based on this idea (see also Foss & Zachary 2002).

8. Segerdahl’s approximation of finite-time ruin probability

Our goal in this section is generalizing the seminal result of Segerdahl (1959) to Markov modulated risk process, that is, to get the asymptotics of the finite-time ruin probability \( \phi(x, t) \) introduced in (19) when time horizon \( t \) is of order \( x/m + yc\sqrt{x/m} \) for

\[ m = \mathbb{E}_\pi \hat{X}_1 = -\mathbb{E}_\pi X_1 > 0 \quad \text{and} \quad c^2 = \mathbb{V}_\pi \hat{X}_1 = \mathbb{V}_\pi X_1. \]  

(35)

Above \( \mathbb{V}_\pi \) denotes the variance calculated under \( \tilde{P} \) defined in (29) where \( I_0 \) starts at stationary distribution \( \pi \). Let \( \Phi_N \) be a cumulant distribution function of a standard Gaussian random variable.

**Theorem 8.1:** We have

\[ \lim_{x \to +\infty} \frac{\phi_t(x, x/m + yc\sqrt{x/m}^{3/2})}{e^{-\gamma x}} = C_i \Phi_N(y), \]

where constant \( C_i \) and the adjustment coefficient \( \gamma > 0 \) are given in (28).

**Proof:** Using the same arguments as in the proof of Theorem 6.1 we have

\[ \phi_t(x, t) = e^{-\gamma x \mathbb{E}_{0,i}^{\tilde{P}}} \left[ e^{\gamma (\hat{X}_{t_+}^+ - x)} ; t_+ < t \right]. \]  

(36)

Using Markov modulated random walk Central Limit Theorem of Keilson & Wishart (1964) we have the following result.

**Lemma 8.1:** Under \( \tilde{P} \), \( \frac{\hat{X}_1 - m_1}{c\sqrt{S}} \) converges weakly to standard Gaussian random variable \( N(0, 1) \) as \( s \to +\infty \).
From Anscombe’s theorem, it follows that \( \frac{\hat{X}_{t^+} - mt^+}{c\sqrt{t^+}} \) converges weakly to \( N(0, 1) \) as well. Moreover, similarly by the Law of Large Numbers and Anscombe’s theorem we get

\[
\frac{X}{\hat{X}_{t^+}} \to m \quad \tilde{P} \quad \text{a.s. as } x \to +\infty.
\]

Further, by Renewal Theorem 28 of Dereich et al. (2017) the overshoot \( \hat{X}_{t^+} - x \) converges weakly to some random variable and hence

\[
\frac{\hat{X}_{t^+} - x}{m \sqrt{x/m^{3/2}}} \to N(0, 1)
\]

converges weakly to \( N(0, 1) \) as well. The following Stam’s lemma states that ruin time and deficit are asymptotically independent. Its proof is the same as the proof of Proposition 4.4, p. 130 of Asmussen & Albrecher (2010).

**Lemma 8.2:** For bounded and continuous functions \( f \) on \( [0, \infty) \) and \( g \) on \( \mathbb{R} \) for \( x > 0 \), we have that

\[
\tilde{\mathbb{E}}_i \left( f(\xi(x)) \right) \sim \tilde{\mathbb{E}}_i \left( f(\xi(\infty)) \right) \mathbb{P}(N(0, 1)),
\]

where \( \xi(x) = \hat{X}_{t^+} - x \).

Now, from (36) we get

\[
\phi_i(x, x/m + yc\sqrt{x/m^{3/2}}) = e^{-\gamma x} \tilde{\mathbb{E}}_{0,i} \left[ e^{\gamma (\hat{X}_{t^+} - x)} ; \hat{X}_{t^+} < x/m + yc\sqrt{x/m^{3/2}} \right]
\]

\[
\sim e^{-\gamma x} \tilde{\mathbb{E}}_{i} \left[ e^{\gamma (\xi(\infty))} \right] \tilde{\mathbb{P}} \left( \frac{\hat{X}_{t^+} - x}{c\sqrt{x/m^{3/2}}} \leq y \right) \sim C_i e^{-\gamma x} \Phi_N(y)
\]

which completes the proof of Theorem 8.1.

**9. Höglund’s asymptotics of Markov modulated renewal function and finite-time ruin probability**

Recall that \( E = \{1, 2, \ldots, N\} \). In this section, we will consider the solution

\[
U \ast f(x) = \sum_{n=0}^{\infty} \gamma^{*n} \ast f(x), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}
\]

of the renewal equation \( U - \gamma \ast U = f \) where \( \gamma = (\gamma_{ij})_{i,j \in E} \) is a matrix of positive measures and \( f \) is a vector of measurable functions for which series (38) converges. Moreover, \( \ast \) denotes convolution in the Markov modulation set-up, that is,

\[
(\gamma^{*k}f(x))_i = \sum_{i_1, i_2, \ldots, i_{k-1}, i_k \in E} \int f_{i_k}(x - y_1 - y_2 - \ldots - y_k) \gamma_{i_1 i_2}(dy_1) \gamma_{i_2 i_3}(dy_2) \ldots \gamma_{i_{k-1} i_k}(dy_k).
\]

We assume that support of each measure \( \gamma_{ij} \) is \( d \)-dimensional. Let \( \lambda_{\mathbb{R}^d} \) denotes the Lebesgue measure in \( \mathbb{R}^d \) and by \( \langle \cdot, \cdot \rangle \) we denote the scalar product in \( \mathbb{R}^d \). Finally, let \( \pi^\gamma = (\pi_1^\gamma, \ldots, \pi_N) \) be a stationary
distribution of the discrete-time Markov chain with the transition matrix \(\{\gamma_{ij}(\mathbb{R}^d)\}_{i,j \in E}\). We denote
\[
\Theta = \left\{ \theta \in \mathbb{R}^d : \int_{\mathbb{R}^d} |x|^2 e^{(\theta,x)} \gamma_{ij}(dx) < \infty, \quad i,j \in E \right\}
\]
and let for \(\theta = (\theta_1, \ldots, \theta_d) \in \Theta\) the function \(\varphi(\theta)\) be a Perron-Frobenius eigenvalue of the matrix \((\int_{\mathbb{R}^d} e^{(\theta,x)} \gamma_{ij}(dx))_{i,j \in E}\). We also define
\[
\varphi'(\theta) = \left( \frac{\partial}{\partial \theta_1} \varphi(\theta), \ldots, \frac{\partial}{\partial \theta_d} \varphi(\theta) \right), \quad \varphi''(\theta) = \left\{ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \varphi(\theta) \right\}_{i,j \in E},
\]
\[
\beta = \max \left\{ 2, \frac{d-1}{2} \right\}, \quad C = \varphi'(\theta)(\varphi''(\theta))^{-1} \varphi'(\theta) \det \varphi''(\theta),
\]
\[
\sigma = x(\varphi''(\theta))^{-1} \varphi'(\theta)/\varphi'(\theta)(\varphi''(\theta))^{-1} \varphi'(\theta).
\]
We will look for the solution of the equation
\[
\varphi(\theta) = 1. \quad (39)
\]
If this solution exists, we have either \(\varphi'(\theta) \neq 0\) for all \(\theta\) or it is one-point set otherwise (compare with Hoglund 1988, Lem. 1). We recall that function \(g\) is directly Riemann integrable if \(\int g(x) - \int g_h\) tends to 0 as \(|h| \to 0\) for \(g_h(x) = \sup_{y \in (nh,(n+1)h]} g(y)\) and \(g_h(x) = \inf_{y \in (nh,(n+1)h]} g(x)\) for \(x \in (nh, (n+1)h]\).

**Theorem 9.1:** Assume that \(\varphi'(\theta) \neq 0\) and \(\theta\) satisfies (39). Let \(\sigma \geq 0\). Suppose that \(\int (|x|^2 + |\langle c, x \rangle|^d e^{(\theta,x)} \gamma_{ij}(dx))\) and \(\int (1 + |\langle c, x \rangle|^d e^{(\theta,x)} f(x)|\lambda_{\mathbb{R}^d}(dx))\) are finite for all \(i,j \in E\) and that integrand in the last integral is directly Riemann integrable for some \(0 \neq c \in \mathbb{R}^d\). In addition, we assume that \((c, \varphi(\theta)) \neq 0\) if \(d = 2\). Then
\[
e^{(\theta,y)} (U \ast f(x))_i = (2\pi \sigma)^{-(d-1)/2} C^{-1/2} \exp\{-(x - \sigma \varphi'(\theta))(\varphi''(\theta))^{-1}(x - \sigma \varphi'(\theta))\}
\]
\[
\times \int e^{(\theta,y)} \pi^T f(y) \lambda_{\mathbb{R}^d}(dy) + o(1 + |\langle c, x \rangle|)^{-(d-1)/2}
\]
uniformly in \(x\), as \(|x| \to +\infty\). In particular, if \(|\varphi'(\theta)/\varphi'(\theta)| = x/|x|\) we have
\[
\lim_{|x| \to +\infty} \frac{(U \ast f(x))_i}{e^{-\langle \theta, x \rangle}} = (2\pi \sigma)^{-(d-1)/2} C^{-1/2} \int e^{(\theta,y)} \pi^T f(y) \lambda_{\mathbb{R}^d}(dy). \quad (40)
\]
**Proof:** Basically, the proof is the same as the proof of Hoglund (1988, Thm. 1.2 and Thm. 1.4). There are some minor differences though. In any estimate, one has to take all algebraic operations entrywise for matrices and then maximum of minimum should be applied over \(E\) should be applied. The crucial difference is in the proof of equations (2.43) where Central Limit Theorem is applied to a random walk associated with measure \(\gamma\). In this place, one has to use Keilson & Wishart (1964, p. 552) instead. 

To identify the finite-time ruin probability at the beginning we follow Hoglund (1990, Prop. 3.2). Let \(S = (S^1, S^2) = \{(S^1_n, S^2_n), n = 1, 2, \ldots\}\) be a (possibly killed) Markov modulated random walk starting from \((0, 0)\) whose components \(S^1\) and \(S^2\) have non-negative increments. Let \(J^T\) be a Markov chain modulating this random walk living in the state space \(E = \{1, 2, \ldots, N\}\). Consider the crossing
Now observe that
probability and

Assume that (Proposition 9.1: Cor. 2.7, p. 313 of Asmussen 2003).

Let \( \pi^\gamma \) be the stationary distribution of environmental Markov chain \( J_0^\gamma \) and let

\[
V(\zeta) = \mathbb{E}^{\zeta}_\pi \left[ (S_1^2 \mathbb{E}^{\zeta}_\pi [S_1^4] - S_1^4 \mathbb{E}^{\zeta}_\pi [S_1^2])^2 / \mathbb{E}^{\zeta}_\pi [S_1^3] \right]^3
\]

for \( \zeta = (\eta_1, \eta_2) \) where \( \mathbb{E}^{\zeta}_\pi \) denotes the expectation w.r.t.

\[
\mathbb{P}^{\zeta}_\pi (\cdot) = \sum_{i,j=1}^N \pi_i^\gamma \mathbb{E}[e^{(S_1, \zeta)}, S_1 \in \cdot, J_1^\gamma = j, J_0^\gamma = i].
\]

For our purposes it will suffice to consider random walks that satisfy the following assumption (the analogue of the non-lattice assumption in one dimension):

The additive group spanned by the support of \( F^\zeta \) contains \( \mathbb{R}^2_+ \). (G)

Now observe that \( G_{a,b,i}(x,y) \) and \( K_{a,b,i}(x,y) \) are of the form \( (\sum_{n=0}^{\infty} \gamma^n f(x) \mathbf{1})_i \) with \( d = 2 \),

\[
\gamma (dx, dy) = \mathbb{P}(S^1_1 \in dx, S^2_2 \in dy)
\]

and \( f(x,y) = \gamma ((x+a,+,\infty), (-\infty, x+b)) \mathbf{1} \) or \( f(x,y) = \gamma ((x+a,+,\infty), [x+b,+,\infty)) \mathbf{1} \), respectively. For \( \theta = (\theta_1, \theta_2) \in \Theta^\circ \) let \( \varphi(\theta) \) be a Perron-Frobenius eigenvalue of the matrix \( (\int_{\mathbb{R}^2} e^{\theta_1 x + \theta_2 y} \gamma(dx,dy))_{ij} \in E \). One can easily check that all moments and direct Riemann integrability conditions of Theorem 9.1 are satisfied. We will also write \( f \asymp g \) if \( \lim_{x,y \rightarrow \infty, x=y+o(y^{1/2})} f(x,y)/g(x,y) = 1 \). Applying (40) produces the following key proposition (see Theorem 2.1 of Hoglund 1990 and Cor. 2.7, p. 313 of Asmussen 2003).

**Proposition 9.1:** Assume that (G) holds, and that there exists a \( \zeta = (\eta_1, \eta_2) \) with \( \varphi(\zeta) = 1 \) such that \( \nu = \mathbb{E}_{\pi}^{\zeta} [S_1^2] / \mathbb{E}_{\pi}^{\zeta} [S_1^4] \), where \( \varphi \) is finite in a neighbourhood of \( \zeta \) and \( (0, \eta_2) \). If \( x, y \) tend to infinity such that \( x = vy + o(y^{1/2}) > 0 \) then it holds that

\[
G_{a,b,i}(x,y) \asymp D(a,b)x^{-1/2}e^{\gamma \eta_1 + \eta_2} \quad \text{if } \eta_2 > 0,
\]

\[
K_{a,b,i}(x,y) \asymp D(a,b)x^{-1/2}e^{\gamma \eta_1 + \eta_2} \quad \text{if } \eta_2 < 0,
\]

for \( a \geq 0, b \in \mathbb{R} \), where \( D(a,b) = C(a,b) \cdot (2\pi V(\zeta))^{-1/2} \), with \( V(\zeta) > 0 \) and

\[
C(a,b) = \frac{1}{|\eta_2|\mathbb{E}^{\zeta}_\pi [S_1^2]} e^{b\eta_2} \int_a^\infty \mathbb{P}_\pi^{\zeta} (S_1^1 \geq x)e^{\gamma \eta_1 x}dx.
\]

In the last step, we follow Palmowski & Pistorius (2009) and prove the following main result satisfied for general MAP \( \hat{X}_t = -X_t \). The result below concerns the asymptotics of the finite-time ruin probability

\[
\phi_t(x) = \mathbb{P}(\hat{\xi}_x^+ \leq t)
\]

when \( x, t \) jointly tend to infinity in a fixed proportion. For a given proportion \( \nu \) the rate of decay is either equal to \( \nu \nu t \) or to \( k^* (\nu) t \), where \( k^* \) is the convex conjugate of the Perron-Frobenius eigenvalue.
\( \hat{k} \) of the matrix exponent \( \hat{F} \) of the dual MAP \( \hat{X} = -X \):

\[
\hat{k}^*(x) = \sup_{\alpha \in \mathbb{R}} (\alpha x - \hat{k}(\alpha)).
\]

We restrict ourselves to the risk process in the regime-switching environment satisfying the following condition

\[
u^{(ij)} \text{ defined in (16) are non-lattice for all } i, j \in E.\tag{H}\]

Recall that a measure is called non-lattice if its support is not contained in a set of the form \( \{a + bh, h \in \mathbb{Z}\} \), for some \( a, b > 0 \).

To present the main result we need a few additional notations.

Let \( L_i \) be a local time of \( X_t - \inf_{s \leq t} X_s \) which in our case increases only at epochs \( \tau_k \) defined via \( \tau_0 = 0, \tau_k = \inf\{t > \tau_{k-1} : X_t - \inf_{s \leq t} X_s = 0\} \). Let \( L_i^{-1} \) be the inverse local time and as usual let \( L_i^{-1} \) denote the vector of total times spent in different phases up to \( L_i^{-1} \). Observe that the ladder process \( (H_i, L_i^{-1}, J_i^{-1}) \) for \( H_i = X_{L_i^{-1}} \) is a bivariate Markov additive process. Thus for \( \alpha \geq 0 \) and \( b = (\beta_1, \ldots, \beta_N) \) with \( \beta_i \geq 0 \) there exists matrix-valued function \( K(b, \alpha) \) such that

\[
\mathbb{E}[e^{\alpha H_t - (b L_t^{-1}), J_t^{-1}}] = e^{K(b, \alpha)t},
\]

because \( L_0^{-1} = 0 \).

Let

\[
((H_t, L_t^{-1}), J_t) = ((H_t, L_t^{-1}), J_i)
\]

with

\[
L_i = J_i^{-1}
\]

be a ladder process related with supremum of the dual risk process \( \hat{X} = -X \) and \( \hat{K}(\alpha, b) \) is its matrix Laplace exponent. Note that \( ((H_t, L_t^{-1}), J_t) \) is the bivariate MAP. We denote by \( \hat{\pi} \) the stationary distribution of \( L_t \). The assumption (H) and Vigon’s formula (see Kyprianou 2014, Thm. 6.22 and Sec. 6.6.2) implies that the jump measures associated with \( \hat{H} \) are non-lattice as well. To simplify notation we will write in this section

\[
(H_t, L_t^{-1}, 1) = (H_t, L_t^{-1}).
\]

Let

\[
\Theta_{< \infty} = \{\theta \in \mathbb{R} : \hat{k}(\theta) = k(-\theta) < \infty\}
\]

and \( \Theta_{< \infty} \) be its interior.

**Theorem 9.2:** Assume that (H) holds. Suppose that \( 0 < \hat{k}'(\gamma) = k'(-\gamma) < \infty \) and that there exists a \( \Gamma(\gamma) \in \Theta_{< \infty} \) such that \( \hat{k}'(\Gamma(\gamma)) = \nu \). If \( x \) and \( t \) tend to infinity such that \( x = vt + o(t^{1/2}) \) then

\[
\phi_t(x) = \mathbb{P}_1(\hat{\tau}_x^+ \leq t) \propto \begin{cases} C_t e^{-\nu x}, & \text{if } 0 < \nu < \hat{k}'(\gamma), \\ \frac{D_t t^{1/2} e^{-\hat{k}(\nu)t}}{\nu}, & \text{if } \nu > \hat{k}'(\gamma), \end{cases}
\]

with \( C_t \) given in Theorem 6.1 and

\[
D_{\nu} = \frac{1}{\eta_{\nu} \mathbb{E}[e^{\nu H_{1 - H_{L_1}^{-1}} H_1]}] \times \frac{\nu}{\Gamma(\nu) \sqrt{2\pi \hat{k}''(\Gamma(\gamma))}} \times \hat{\nu} \kappa(\hat{k}(\Gamma(\gamma)), 0)(q^\perp + \kappa(\hat{k}(\Gamma(\gamma)), 0))^{-1} 1,
\]

where \( \eta_{\nu} = \hat{k}(\Gamma(\gamma)) \).
Remark 9.1: In the case of risk process (1) with \( N = 1 \) (there is no Markov modulation) we can find that
\[
D_{v} = \frac{\Gamma(v) + \hat{\Gamma}(v)}{\Gamma(v)\hat{\Gamma}(v)} \frac{1}{\sqrt{2\pi \hat{k}''(\Gamma(v))}}, \quad C_{1} = \frac{\hat{k}'(0)}{\hat{k}'(\gamma)},
\]
where \( \hat{\Gamma}(v) = \sup\{\theta : \hat{k}(-\theta) = \hat{k}(\Gamma(v))\} \), recovering formulas that can be found in Arfwedson (1955) and Feller (1966) respectively, for the case of a classical risk process.

Remark 9.2: Note that asymptotics given in Theorem 9.2 works for different range of time horizons than Segerdahl’s approximation identified in Theorem 8.1.

Proof: In the beginning, we will use the exponential change of measure (29) defining \( \widetilde{P}_{i} \). In the case \( 0 < v < \hat{k}'(\gamma) \), the asymptotics in Theorem 9.2 are a consequence of the law of large numbers. To see why this is the case, note that
\[
e^{\gamma x}P_{i}(\hat{\tau}_{x}^{+} < t) = e^{\gamma x}P_{i}(\hat{\tau}_{x}^{+} < \infty) - e^{\gamma x}P_{i}(t < \hat{\tau}_{x}^{+} < \infty),
\]
where the first term tends to \( C_{i} \) in view of Theorem 6.1, while for the second term the Markov property implies that
\[
e^{\gamma x}P_{i}(t < \hat{\tau}_{x}^{+} < \infty) \leq \sum_{j \in E} \int_{-\infty}^{x} P_{i}(\hat{\tau}_{x}^{+} > t, \hat{X}_{t} \in dy)e^{\gamma y}e^{\gamma(y-x)}P_{j}(\hat{\tau}_{y-x}^{+} < \infty)
\]
\[
\leq \int_{-\infty}^{x} P_{i}(\hat{X}_{t} \in dy)e^{\gamma y} = \widetilde{P}_{i}(\hat{X}_{t} \leq x),
\]
which tends to 0 as \( t \) tends to infinity in view of the law of large numbers since \( \widetilde{E}[\hat{X}_{t}] = it \hat{k}'(\gamma) > x = vt + o(t^{1/2}) \). We will now consider the case of \( v > \hat{k}'(\gamma) \).

We start from construction of the embedded Markov modulated random walk at Poisson epochs. Denote by \( e_{1}, e_{2}, \ldots \) a sequence of independent identically exponentially distributed with parameter \( q \) random variables and by \( \sigma_{n} = \sum_{i=1}^{n} e_{i} \), with \( \sigma_{0} = 0 \), the corresponding partial sums. Consider the two-dimensional (killed) Markov modulated random walk \( \{(S^{1}_{n}, S^{2}_{n}) \}, n = 1, 2, \ldots \) starting from \((0,0)\) with step-sizes distributed according to
\[
\Upsilon^{(q)}(dx, dt) = P(S^{1}_{1} \in dx, S^{2}_{1} \in dt) = P(H_{\sigma_{1}} \in dx, L_{\sigma_{1}}^{-1} \in dt),
\]
and write \( G^{(q)} \) for the corresponding crossing probability
\[
G^{(q)}(x, t) = G_{0,0;i}(x, t) = P_{i}(N(x) < \infty, S^{2}_{N(x)} \leq t)
\]
for \( N(x) \) defined in (42) and \( G_{a,b;i}(x,y) \) defined in (41). Note that
\[
\int \int e^{-ux-\theta t}P(S^{1}_{1} \in dx, S^{2}_{1} \in dt) = q(qI - \kappa(v, u))^{-1}, \quad (44)
\]
where
\[
\kappa(v, u) = \overline{K}(v 1^{T}, u), \quad (45)
\]
and \( \overline{K} \) is a matrix Laplace exponent of a ladder process related with supremum of the dual risk \( \hat{X} = -X \) defined formally below (43).

The key step in the proof is to derive bounds for \( P(\hat{\tau}_{x}^{+} \leq t) \) in terms of crossing probabilities involving the random walk \( (S^{1}, S^{2}) \).
Lemma 9.1: Let \( M, q > 0 \). For \( x, t > 0 \) it holds that

\[
G^{(q)}(x, t) \leq \mathbb{P}_i(\hat{\tau}_x^+ \leq t) \leq G^{(q)}(x, t + M)/\min_{i \in E} h_i(0-, M),
\]

where \( h(0-, M) = \lim_{x \uparrow 0} h(x, M) \), with \( h(x, t) = \mathbb{P}(H_{\sigma_1} > x, L_{\sigma_1}^{-1} \leq t) \).

Proof: Let \( T(x) = \inf\{t \geq 0 : H_t > x\} \) and note that \( \hat{\tau}_x^+ = L_{\hat{T}(x)}^{-1} \). By applying the Markov property it follows that

\[
\mathbb{P}_i(\hat{\tau}_x^+ \leq t) = \mathbb{P}_i(T(x) < \infty, L_{\hat{T}(x)}^{-1} \leq t)
\]

\[
= \sum_{n=1}^{\infty} \mathbb{P}(\sigma_{n-1} \leq T(x) < \sigma_n, L_{\hat{T}(x)}^{-1} \leq t)
\]

\[
= \sum_{n=1}^{\infty} \mathbb{P}(H_{\sigma_{n-1}} \leq x, H_{\sigma_n} > x, L_{\hat{T}(x)}^{-1} \leq t)
\]

\[
= \sum_{n=1}^{\infty} \int \mathbb{P}(H_{\sigma_{n-1}} \in dy, L_{\sigma_{n-1}}^{-1} \in ds)
\]

\[
\quad \times \mathbb{P}(H_{\sigma_1} > x - y, L_{\hat{T}(x-y)}^{-1} \leq t - s)
\]

\[
= \left( \sum_{n=0}^{\infty} (\Upsilon^{(q)})^n * f(x, t) \right)_{i} = ((U * f)(x, t))_{i},
\]

where \( U = \sum_{n=0}^{\infty} (\Upsilon^{(q)})^n, f(x, t) = \mathbb{P}(H_{\sigma_1} > x, L_{\hat{T}(x)}^{-1} \leq t) \) and \( \star \) denotes convolution. Following a similar reasoning it can be checked that

\[
G^{(q)}(x, t) = (U * h(x, t))_{i}.
\]

In view of (49) and (50), the lower bound in (46) follows since

\[
f(x, t) \geq h(x, t),
\]

taking note of the fact that \( H_{\sigma_1} > x \) precisely if \( T(x) < \sigma_1 \), while the upper bound in (46) follows by observing that for fixed \( M > 0 \),

\[
h(x, t + M) \geq \mathbb{P}(H_{\sigma_1} > x, L_{\hat{T}(x)}^{-1} \leq t, L_{\sigma_1}^{-1} - L_{\hat{T}(x)}^{-1} \leq M) \mathbb{1}
\]

\[
= \mathbb{P}(H_{\sigma_1} > x, L_{\hat{T}(x)}^{-1} \leq t) \mathbb{P}(L_{\sigma_1}^{-1} \leq M) \mathbb{1}
\]

\[
\geq f(x, t) \min_{i \in E} h_i(0-, M),
\]

where we used the strong Markov property of \( L^{-1} \) and the lack of memory property of \( \sigma_1 \).

Applying Högland’s asymptotics in Proposition 9.1 yields the following result.

Lemma 9.2: Let the assumptions of Proposition 9.1 hold true. If \( x, t \to \infty \) such that for \( v > \hat{k}'(\gamma) \) we have \( x = v t + o(t^{1/2}) \) then

\[
G^{(q)}(x, t + M) \sim D_{q,M} t^{-1/2} e^{-\hat{k}'(\gamma) t}, \quad M \geq 0,
\]

where \( D_{q,M} = \frac{v}{\sqrt{2\pi k''(\Gamma(\gamma))}} C_{q,M} \) with
\[
C_{q,M} = \frac{q e^{\hat{k}(\Gamma(v))M}}{c_v \hat{k}(\Gamma(v)) \Gamma(v)} \mathbb{E}_{\pi} \left[ \hat{k}(\Gamma(v)), 0 \right] \left( q \hat{\pi} + \kappa \hat{k}(\Gamma(v)), 0 \right)^{-1} \mathbf{1},
\]

where \( c_v = \mathbb{E}_\pi \left[ e^{\Gamma(v)H_1 - \hat{k}(\Gamma(v))L_1^{-1}H_1 L_1^{-1} \mathbf{1}_{\{L_1 < \infty\}} \right] \) and \( \kappa \) is defined in (45).

For \( u > \gamma \) and \( u \in \Theta_{<\infty} \), we denote by \( \mathbb{P}^{(u)} \) the measure

\[
\frac{d\mathbb{P}^{(u)}}{d\mathbb{P}^{(\gamma)}} = e^{\mu(X_{\gamma} - \pi \eta)} \left. \frac{h_f(u)}{h_0(u)} \right. .
\]

Let \( \mathbb{E}^{(u)} \) be the expectation with respect to \( \mathbb{P}^{(u)} \). Lemma 9.2 is a consequence of the following auxiliary identities given in Palmowski & Pistorius (2009):

\[
\varphi(z, -u) = 1 \quad \text{iff} \quad \kappa(z, -u) = 0 \quad \text{iff} \quad \hat{k}(u) = z,
\]

\[
\hat{k}'(u) = \mathbb{E}_{\pi}^{(u)} [\hat{X}_1] = \mathbb{E}_{\pi}^{(u)} [H_{\sigma_1}] \cdot (\mathbb{E}_{\pi}^{(u)} [L_{\sigma_1}^{-1}])^{-1},
\]

\[
\hat{k}''(u) = \mathbb{E}_{\pi}^{(u)} [(H_{\sigma_1} - \hat{k}'(u)L_{\sigma_1}^{-1})^2] \cdot (\mathbb{E}_{\pi}^{(u)} [L_{\sigma_1}^{-1}])^{-1},
\]

\[
= \hat{k}'(u) \mathbb{E}_{\pi}^{(u)} [(H_{\sigma_1} - \hat{k}'(u)L_{\sigma_1}^{-1})^2] \cdot (\mathbb{E}_{\pi}^{(u)} [H_{\sigma_1}])^{-1}, \tag{54}
\]

\[
\hat{k}^\varsigma(v) = v \Gamma(v) - \hat{k}(\Gamma(v)) \quad \text{for} \quad v > 0 \quad \text{with} \quad \Gamma(v) \in \Theta_{<\infty} \tag{55}.
\]

In particular, the first identity follows from (44) and (45) and Wiener-Hopf factorization given in Theorem 26 of Dereich et al. (2017).

**Proof of Lemma 9.2:** The proof follows by an application of Proposition 9.1 to \( G^{(q)}(x, t + M) \) with

\[
(S_1, S^2_1) = (H_{\sigma_1}, L_{\sigma_1}^{-1}) \quad \text{and} \quad \varsigma = (-\Gamma(v), \eta_v).
\]

Note that, by (52) with \( u = \Gamma(v) \), \( \varphi(\varsigma) = 1 \), and that \( \eta_v = \hat{k}((\Gamma(v)) > 0 \) if \( v > \hat{k}'(\gamma) \). For this choice of the parameters, \( \mathbb{E}_{\pi}^{(u)} [S^1_1] = \mathbb{E}_{\pi}^{(\Gamma(v))} [H_{\sigma_1}] = c_v / q \), and Equations (53), (54) and (55) imply that \( \xi x + \eta t = -\hat{k}^\varsigma(v)t \) and

\[
V(\varsigma) = \hat{k}'(\Gamma(v)) / \hat{k}'(\Gamma(v)) = \hat{k}'(\Gamma(v)) / v. \tag{56}
\]

To complete the proof we are left to verify the form of the constants. The calculation of the \( C_{q,M} = C(0, 0) e^{h_0,M} \) goes as follows:

\[
C_{q,M} = \frac{q e^{\hat{k}(\Gamma(v))M}}{\hat{k}(\Gamma(v)) \Gamma(v)} \mathbb{E}_{\pi} \left[ \hat{k}(\Gamma(v)), 0 \right] \left( q \hat{\pi} + \kappa \hat{k}(\Gamma(v)), 0 \right)^{-1} \mathbf{1},
\]

\[
= \frac{q e^{\hat{k}(\Gamma(v))M}}{\hat{k}(\Gamma(v)) \Gamma(v) c_v} \mathbf{1} - \mathbb{E}_{\pi} \left[ e^{\hat{k}(\Gamma(v))L_{\sigma_1}^{-1}1_{\{1_{\{L_1 < \infty\}}} \mathbf{1} \right] \right)
\]

\[
= \frac{q e^{\hat{k}(\Gamma(v))M}}{\hat{k}(\Gamma(v)) \Gamma(v) c_v} \mathbf{1} - q \hat{\pi} (q \hat{\pi} + \kappa \hat{k}(\Gamma(v)), 0)^{-1} \mathbf{1}. \tag{57}
\]

Combining all results completes the proof.
Now we can complete the proof of Theorem 9.2. Writing \( l(t,x) = t^{1/2} e^{\hat{K}^*(v)t} \mathbb{P}(\hat{\tau}_x^+ \leq t) \), Lemmas 9.1 and 9.2 imply that

\[
\begin{align*}
    s &= \limsup_{x,t \to \infty, x=tv+o(t^{1/2})} l(t,x) \leq D_{q,M} / \min_{i \in E} h_i(0-, M), \\
    i &= \liminf_{x,t \to \infty, x=tv+o(t^{1/2})} l(t,x) \geq D_{q,0}.
\end{align*}
\]

By definition of \( h \) and \( D_{q,M} \) it directly follows that, as \( q \to \infty \),

\[
D_{q,0} \to D_v, \quad D_{q,M} \to D_v e^{\hat{K}(v)M} \quad \text{and} \quad h(0-, M) = \mathbb{P}(L_{\sigma_1}^{-1} \leq M) \mathbf{1} \to \mathbf{1}.
\]

Letting \( M \downarrow 0 \) yields that \( s = i = D_v \), and the proof is complete. \( \blacksquare \)

10. Parisian ruin

In this section, we follow Czarna & Palmowski (2011), Loeffen et al. (2013) and Dassios & Wu (2008) considering so-called Parisian ruin probability, that occurs if the risk process \( X \) defined in (1) stays below zero for a longer period than a fixed \( \zeta > 0 \). Formally, we define Parisian time of ruin by:

\[
\tau^\zeta = \inf \{ t > 0 : t - \sup \{ s < t : X_s \geq 0 \} \geq \zeta, X_t < 0 \}
\]

and Parisian ruin probability is then given by:

\[
\mathbb{P}_{x,i}(\tau^\zeta < \infty).
\]

The case \( \zeta = 0 \) corresponds to the classical ruin problem and we do not deal with this case in this section. The name for this problem is borrowed from the Parisian option. Depending on the type of such option the prices are activated or cancelled if the underlying asset stays above or below the barrier long enough in a row. It is a common belief that Parisian ruin probability is a better measure of risk than classical ruin probability in many situations giving the possibility for the insurance company to get solvency. We consider here the fixed delay \( \zeta \). In other papers the deterministic and fixed delay \( \zeta \) is replaced by an independent exponential random variable; see e.g. Landriault et al. (2014) and Baurdoux et al. (2016). In this case, as pointed in Albrecher & Ivanovs (2017) and Bin et al. (2018), the ruin probability is closely related to Poisson observed ruin probability. The paper that deals with this probability in the context of MAP risk processes is Zhao & Dong (2018).

We give in next theorem a different main representation of the Parisian survival probability though. Recall that by

\[
\tau_x^+ = \inf \{ t \geq 0 : X_t > x \}
\]

we denote the first passage time over a level \( x \geq 0 \) and observe that \( X_{\tau_x^+} = x \) a.s., because of the absence of positive jumps. The Markov additive property applied at \( \tau_x^+ \) implies that \( J_{\tau_x^+} \), the phase observed at the first passage times, is a Markov chain indexed by the level \( x \geq 0 \). Hence there is an identity:

\[
\mathbb{E}[e^{-q \tau_x^+}, J_{\tau_x^+}] = e^{G^{(q)}x}, \quad x \geq 0
\]

for some transition rate matrix \( G^{(q)} \) which is of size \( N \times N \). Moreover, by Ivanovs & Palmowski (2012) the matrix \( G^{(q)} \) is a right solution of

\[
F(-G^{(q)}) = q \mathbb{I}.
\]
Theorem 10.1: Parisian survival probability for a MAP risk process equals:

\[
\mathbb{P}_{x,i}(\tau^\xi = +\infty) = \mathbb{P}_{x,i}(\tau^- = +\infty) + \int_0^\infty \mathbb{P}_{x,i}(\tau^-_0 < \infty, -X_{\tau^-_0} \in dz) \mathbb{P}(\tau^+_z \leq \xi) \mathbb{P}(\tau^\xi = +\infty),
\]

where \( \mathbb{P}_{x,i}(\tau^- = +\infty) = \overline{\phi}_j(x) = (W(x) W(\infty)^{-1})_i \) by (20), and the vector \( \mathbb{P}(\tau^\xi = +\infty) = (\mathbb{P}_j(\tau^\xi = +\infty))_{i \in E} \) solves the following system of equations

\[
\mathbb{P}_i(\tau^\xi = +\infty) = \mathbb{P}_i(\tau^- = +\infty) + \sum_{j,k \in E} \int_0^\infty \mathbb{P}_i(\tau^-_0 < \infty, -X_{\tau^-_0} \in dz, I_{\tau^-_0} = k) \mathbb{P}_k(\tau^+_z \leq \xi, I_{\tau^+_z} = j) \mathbb{P}_j(\tau^\xi = +\infty).
\]

Moreover,

\[
\int_0^\infty e^{-\theta s} ds \int_0^\infty \mathbb{P}_{x,i}(\tau^-_0 < \infty, -X_{\tau^-_0} \in dz) \mathbb{P}(\tau^+_z \leq s) = \frac{1}{\theta} \int_0^\infty \int_0^\infty u^{(0)}(x, z) v(z + dy) dz e^{G(\xi)y},
\]

where

\[
u^{(0)}(x, z) = W(x) e^{Rz} - W(x - z).
\]

Proof: On the event \( \{\tau^\xi = \infty\} \) we decompose a possible trajectory that goes below zero into two parts. The first one starts at the undershoot of 0 of size, say, \(-z < 0\) visiting zero in a continuous way because of the spectral negativity of \(X\) in a shorter period than \(\xi\). The second part starts at 0 after this excursion below 0. Using the strong Markov property it will produce:

\[
\mathbb{P}_{x,i}(\tau^\xi = \infty) = \mathbb{P}_{x,i}(\tau^- = \infty) + \int_0^\infty \mathbb{P}_{x,i}(\tau^-_0 < \infty, -X_{\tau^-_0} \in dz) \mathbb{P}(\tau^+_z \leq \xi) \mathbb{P}(\tau^\xi = \infty).
\]

This justifies the Equation (57). System of Equation (58) follows straightforward from (57) by taking \(x = 0\) there. Finally, note that by (56),

\[
\int_0^\infty e^{-\theta s} \mathbb{P}(\tau^+_z \leq s) ds = \frac{1}{\theta} \mathbb{E}_{x,i} \left(e^{-\theta \tau^+_z}, \tau^+_z < \infty\right) = \frac{1}{\theta} e^{G(\xi)z}.
\]

Further, from the compensation formula (4.1) and (4.2) we have

\[
\int_0^\infty e^{-\theta s} ds \int_0^\infty \mathbb{P}_{x,i}(\tau^-_0 < \infty, -X_{\tau^-_0} \in dz) \mathbb{P}(\tau^+_z \leq s) = \frac{1}{\theta} \int_0^\infty \mathbb{P}_{x,i}(\tau^-_0 < \infty, -X_{\tau^-_0} \in dy) e^{G(\xi)y} = \frac{1}{\theta} \int_0^\infty \int_0^\infty u^{(0)}(x, z) v(z + dy) dz e^{G(\xi)y}
\]

which completes the proof.

We will derive now the Cramér’s estimate of the Parisian ruin probability.
Theorem 10.2: Under Cramér condition (28),
\[
\lim_{x \to +\infty} \frac{\mathbb{P}(\tau^x \leq +\infty)}{e^{-\gamma x}} = C^x
\]
for some finite constant \( C^x > 0 \).

Proof: We follow the same idea like in the proof of Theorem 6.1. That is, from (57)
\[
\mathbb{P}(\tau^x \leq +\infty) = \mathbb{P}(\tau^-_0 < +\infty) \mathbb{P}(\tau^+_z \leq \zeta) \mathbb{P}(\tau^x = +\infty) \\
= e^{-\gamma x} \mathbb{E}_0 \left[ e^{\gamma (\hat{X}^+_\tau - x)} \frac{h_1(\gamma)}{h^{+}_{\hat{X}^+_\tau}(\gamma)} \right] \\
- e^{-\gamma x} \int_0^\infty \mathbb{E}_0 \left[ e^{\gamma (\hat{X}^+_\tau - x)} \frac{h_1(\gamma)}{h^{+}_{\hat{X}^+_\tau}(\gamma)}, (\hat{X}^+_\tau - x) \in dz \right] \mathbb{P}(\tau^+_z \leq \zeta) \mathbb{P}(\tau^x = +\infty) \text{(64)}
\]
Using Renewal Theorem 28 of Dereich et al. (2017) and Tonelli theorem complete the proof. ■

We will move now to the MAP risk process considered in Section 7 in which \( X \) is a Markov modulated drift minus compound Poisson process and claim size are subexponential.

Theorem 10.3: Under assumptions of Theorem 7.1 we have
\[
\lim_{x \to +\infty} \frac{\mathbb{P}(\tau^x < +\infty)}{\mathbb{P}(\tau^x)} = \frac{C}{\alpha}
\]
for \( C \) and \( \alpha \) given in (33) and (25), respectively.

Proof: Note that from the definition of Parisian ruin time it follows that
\[
\mathbb{P}(\tau^x < +\infty) \leq \mathbb{P}(\tau^-_0 < +\infty).
\]
Moreover, conditioned that risk process \( X \) got ruined, the deficit cannot be larger than \( p = \max_{i \in E} p_i \xi \), otherwise it will not manage to return to zero. Thus by Theorem 10.1
\[
\mathbb{P}(\tau^x = +\infty) = \mathbb{P}(\tau^-_0 = +\infty) \\
+ \int_0^p \mathbb{P}(\tau^-_0 < +\infty, -X^-_{\tau^-_0} \in dz) \mathbb{P}(\tau^+_z \leq \zeta) \mathbb{P}(\tau^x = +\infty)
\]
and hence
\[
\mathbb{P}(\tau^x < +\infty) \geq \mathbb{P}(\tau^-_0 < +\infty) \\
- \mathbb{P}(\tau^-_0 < +\infty, -X^-_{\tau^-_0} \leq p).
\]
Further,
\[
\mathbb{P}(\tau^-_0 < +\infty, -X^-_{\tau^-_0} \leq p) = \mathbb{P}(\tau^-_0 < +\infty) - \mathbb{P}(\tau^-_0 (\infty, -X^-_{\tau^-_0} p) \\
\leq \mathbb{P}(M > x) \\
- \sum_{n=1}^\infty \mathbb{P}(\max_{k \leq n-1} S_k < x, S_{n-1} \geq -a_{n-1}, \xi_n > x + a_{n-1} + p),
\]
where $S_k$ is defined in (32) and $a_n = a_0 + (\bar{a} + \epsilon)n$ for some $a_0, \epsilon > 0$. Now using principle of one big jump (see inequalities (110)–(114) of Foss et al. 2007) and fact that any subexponential distribution is long-tailed one get that

$$
\sum_{n=1}^{\infty} P \left( \max_{k \leq n-1} S_k < x, S_{n-1} \geq -a_{n-1}, \xi_n > x + a_{n-1} + p \right) \geq \left( 1 + o(1) \right) \frac{C}{\bar{a}} F(x).
$$

Thus

$$
P_{x,i}(\tau^-_0 < \infty, -X_{\tau^-_0} \leq p) = o(F(x))
$$

and this gives the assertion of the theorem.

\section*{Disclosure statement}
No potential conflict of interest was reported by the author.

\section*{Funding}
This work is partially supported by the Polish National Science Centre [grant number 2018/29/B/ST1/00756, 2019–2022].

\section*{ORCID}
Zbigniew Palmowski \(\text{http://orcid.org/0000-0001-9257-1115\)}

\section*{References}
Albrecher H. & Ivanovs J. (2017). Strikingly simple identities relating exit problems for Lévy processes under continuous and poisson observations. Stochastic Processes and Their Applications 127(2), 643–656.
Alsmeyer G. (1994). On the Markov renewal theorem. Stochastic Processes and Their Applications 50(1), 37–56.
Arfwedson G. (1955). Research in collective risk theory. Skand. Aktuarietidskr. 38, 53–100.
Asmussen S. (1989). Risk theory in a Markovian environment. Scandinavian Actuarial Journal 2, 69–100.
Asmussen S. (2003). Applied probability and queues, 2nd ed. New York: Springer-Verlag.
Asmussen S. & Albrecher H. (2010) Ruin Probabilities. 2nd ed. Singapore: World Scientific Publishing.
Athreya K. B., McDonald D. & Ney P. (1978). Limit theorems for semi-Markov processes and renewal theory for Markov chains. Annals of Probability 6(5), 788–797.
Badescu A. L. & Landriault D. (2009). Applications of fluid flow matrix analytic methods in ruin theory – a review. RACSAM-Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas 103(2), 353–372.
Baurdoux E. J., Pardo J. C., Pérez J. L. & Renaud J.-F. (2016). Gerber–Shiu distribution at Parisian ruin for Lévy insurance risk processes. Journal of Applied Probability 53(2), 572–584.
Bertoin J. & Doney R. (1994). Cramér’s estimate for Lévy processes. Statistics & Probability Letters 21(5), 363–365.
Bin L., Willmot G. E. & Wong J. T. Y. (2018). A temporal approach to the Parisian risk model. Journal of Applied Probability 55, 302–317.
Cheung E. & Landriault D. (2009a). Perturbed map risk models with dividend barrier strategies. Journal of Applied Probability 46(2), 521–541.
Cheung E. & Landriault D. (2009b). Analysis of a generalized penalty function in a semi-Markovian risk model. North American Actuarial Journal 13(4), 497–513.
Chistyakov V. P. (1964). A theorem on sums of independent positive random variables and its applications to branching random processes. Theory of Probability & Its Applications 9(4), 640–648.
Çınlar E. (1972). Markov additive processes. I. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 24, 85–93.
Czarna I. & Palmowski Z. (2011). Ruin probability with parisian delay for a spectrally negative Lévy risk process. Journal of Applied Probability 48(4), 984–1002.
Dassios A. & Wu S. (2008). Parisian ruin with exponential claims. Unpublished manuscript. Available at http://stats.lse.ac.uk/angelos/
Dereich S., Döring L. & Kyprianou A. (2017). Real self-similar processes started from the origin. Annals of Probability 45(3), 1952–2003.
Embrechts P., Klüppelberg C. & Mikosch T. (1997). Modelling extremal events. Heidelberg: Springer-Verlag.

Embrechts P. & Veraverbeke N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. Insurance: Mathematics and Economics 1(1), 55–72.

Feller W. (1966). An introduction to probability theory and its applications, Vol. II. New York, London, Sydney: John Wiley & Sons.

Foss S., Konstantopoulos T. & Zachary S. (2007). Discrete and continuous time modulated random walks with heavy-tailed increments. Journal of Theoretical Probability 20(3), 581–612.

Foss S., Korshunov D. & Zachary S. (2013). An introduction to heavy-tailed and subexponential distributions. New York: Springer-Verlag.

Foss S. & Zachary S. (2002). Asymptotics for the maximum of a modulated random walk with heavy-tailed increments. In Analytic Methods in Applied Probability (In Memory of Fridrih Karpelevich), Vol. 207. P. 37–52.

Hoglund T. (1988). A multidimensional renewal theorem. Bulletin des Sciences Mathématiques 2e série 112, 111–138.

Hoglund T. (1990). An asymptotic expression for the probability of ruin within finite time. Annals of Probability 18(1), 378–389.

Ivanovs J. (2014). Potential measures of one-sided Markov additive processes with reflecting and terminating barriers. Journal of Applied Probability 51(4), 1154–1170.

Ivanovs J. & Palmowski Z. (2012). Occupation densities in solving exit problems for Markov additive processes and their reflections. Stochastic Processes and Their Applications 122(9), 3342–3360.

Jacobsen M. (2005). The time to ruin for a class of markov additive risk process with two-sided jumps. Advances in Applied Probability 37(4), 963–992.

Kyprianou A. (2013). Gerber–Shiu risk theory. New York: Springer.

Kyprianou A. (2014). Introductory lectures on fluctuations of Lévy processes with applications, 2nd ed). New York: Springer-Verlag.

Kyprianou A. E., Kuznetsov A. & Rivero V. (2013). The theory of scale functions for spectrally negative Lévy processes. In Lévy Matters II, Springer Lecture Notes in Mathematics. P. 97–186.

Kyprianou A. & Palmowski Z. (2005). A martingale review of some fluctuation theory for spectrally negative Lévy processes. In Séminaire de Probabilité XXXVIII. Springer. P. 16–29. Berlin, Heidelberg.

Lalley S. P. (1984). Conditional Markov renewal theory I. Finite and denumerable state space. Annals of Probability 12(4), 1113–1148.

Landriault D., Renaud J.-F. & Zhou X. (2014). Insurance risk models with Parisian implementation delays. Methodology and Computing in Applied Probability 16(3), 583–607.

Loeffen R., Czarna I. & Palmowski Z. (2013). Parisian ruin probability for spectrally negative Lévy processes. Bernoulli 19(2), 599–609.

Ng A. C. Y. & Yang H. (2006). On the joint distribution of surplus before and after ruin under a markovian regime switching model. Stochastic Processes and Their Applications 116(2), 244–266.

Pakes A. (1975). On the tails of waiting time distributions. Journal of Applied Probability 7, 745–789.

Palmowski Z. & Pistorius M. (2009). Cramér asymptotics for finite time first passage probabilities of general Lévy processes. Statistics & Probability Letters 79(16), 1752–1758.

Palmowski Z. & Rolski T. (2002). A technique for the exponential change of measure for Markov processes. Bernoulli 8(6), 767–785.

Rolski T., Schmidt H., Schmidt V. & Teugels J. (1999). Stochastic processes for insurance and finance. Chichester: Wiley.

Salah Z. B. & Morales M. (2012). Lévy systems and the time value of ruin for Markov additive processes. European Actuarial Journal 2(2), 289–317.

Segerdahl C.-O. (1959). A survey of results in the collective theory of risk. In Probability and statistics: The Harald Cramér Volume. P. 276–299.

Zachary S. (2004). A note on Veraverbeke’s theorem. Queueing Systems 46(1/2), 9–14.

Zhao X. & Dong H. (2018). Parisian ruin probability for Markov additive risk processes. Advances in Difference Equations 2018, 179.