STABILITY OF RANKIN-SELBERG GAMMA FACTORS FOR $\text{Sp}(2n), \widetilde{\text{Sp}}(2n)$ AND $\text{U}(n,n)$

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Abstract. Let $F$ be a $p$-adic field and $E/F$ be a quadratic extension. In this paper, we prove the stability of Rankin-Selberg gamma factors for $\text{Sp}_{2n}(F), \widetilde{\text{Sp}}_{2n}(F)$ and $\text{U}_{E/F}(n,n)$ when the characteristic of the residue field of $F$ is not 2.

Introduction

Let $G_n$ be $\text{Sp}_{2n}, \widetilde{\text{Sp}}_{2n}$ and $\text{U}_{E/F}(n,n)$, where $E/F$ is a quadratic extension of local or global field. The global Rankin-Selberg zeta integrals for the generic irreducible cuspidal automorphic representations of $G_n$ twisted by generic irreducible cuspidal representations of $\text{GL}_{m}$ has been developed by Gelbart, Piatetski-Shapiro, Ginzburg, Rallis and Soudry, [GePS2, GiRS1, GiRS2]. Recently, the standard properties of such local $\gamma$-factors were established by Kaplan [Ka]. As a complimentary result of their work, in this paper, we prove the stability of the local gamma factor for a generic representation of $G_n(F)$ when twisted by a sufficiently highly ramified character of $\text{GL}_1$ for a $p$-adic field $F$, when the residue field of $F$ is not 2. More precisely, the main result of this paper is the following

Theorem 0.1. Let $F$ be a $p$-adic field such that the characteristic of its residue field is odd, $E/F$ be a quadratic extension. Let $\psi_U$ be a generic character of a maximal unipotent subgroup of $G_n(F)$ defined by a given nontrivial additive character $\psi$ of $F$. Let $\pi_1, \pi_2$ be two $\psi_U$-generic irreducible smooth representations of $G_n(F)$ with the same central character. If $\eta$ is a highly ramified quasi-character of $F^\times$, then

$$\gamma(s, \pi_1, \eta, \psi) = \gamma(s, \pi_2, \eta, \psi).$$

Here the $\gamma$-factors are the Rankin-Selberg gamma factors, see §1 for more details. We also notice that the main theorem also holds for $\text{U}_{E/F}$ if the residue characteristic of $F$ is 2 and $E/F$ is unramified.

Here we remark that in the $\text{Sp}_{2n}$ case, this result can be deduced from previous work. Cogdell, Kim, Piatetski-Shapiro and Shahidi proved the stability of gamma factors for classical groups (which at least includes $\text{Sp}_{2n}$ and $\text{SO}_n$) in [CKPSS], where the gamma factors are defined using Langlands-Shahidi method. In [Ka], Kaplan proved that Rankin-Selberg gamma factors agree with the Langlands-Shahidi gamma factor. Thus our result in the $\text{Sp}_{2n}$ case follows from the stability result in [CKPSS] and Kaplan’s result on the agreement of the two type gamma factors. In the $\text{U}_{E/F}(n,n)$-case, the stability of the Langlands-Shahidi gamma factors is proved in [KK]. Thus in principle, our result in the $\text{U}_{E/F}(n,n)$ case should follow from an agreement result of the two type $\gamma$-factors, which is unfortunately not included in [Ka].

In this paper, we prove the stability of gamma factors for $G_n$ in the Rankin-Selberg context. Although one can deduce this by pulling back the Langlands-Shahidi gamma factors via [Ka], it is still important to have a proof of stability that remains within the context of integral representations, since there are $L$-functions that we have integral representations for that are not covered by the Langlands-Shahidi method. So developing methods that work in the integral representation context have an intrinsic value.

Our proof of the stability of gamma factors follows the ideas of Baruch, [Ba1, Ba2] and is based on analysis of partial Bessel functions associated with Howe vectors, which can be viewed as a
continuation of the work [Zh1, Zh2]. One main ingredient of the proof is a result of stability properties of partial Bessel functions associated with Howe vectors, see Theorem 3.11, which might have some independent interest. For example, if a more general form of Theorem 3.11 is true, see the Remark after 3.11, it is possible to get a local converse theorem for \( \text{Sp}_{2n} \) and \( U(n, n) \), see [Zh1, Zh2] for the local converse theorem for the small rank case. To the author’s knowledge, there is no local converse theorem obtained from the Langlands-Shahidi’s gamma factors directly. We also expect that the method used here can be used to prove stability results for more groups and gamma factors.

Various results on stability of gamma factors were obtained in different settings, for example, [JS, Ba1, Ba2, CPS, CKPSS, CPSS, CST1] to list a few of them. Usually, the stability of gamma factors is used in conjunction of local global arguments. For example, in the proof of functoriality for classical groups [CKPSS], the stability of gamma factors is used to to resolve lack of local Langlands conjecture. In [CST1], the results on stability of gamma factors for exterior square for \( \text{GL}_n \) were used to show that the local Langlands correspondence for \( \text{GL}_n \) preserves \( \varepsilon \)-factors for exterior square and symmetric square.

The paper is organized as follows. In §1, we briefly review the definitions of local zeta integrals and \( \gamma \)-factors for generic representations of \( \text{Sp}_{2n} \times \text{GL}_1 \). In §2, we review the concept of Howe vectors following [Ba1] and prove several lemmas which will be used in the later calculations. In §3, we prove a stability result of partial Bessel functions associated with Howe vectors, which is the technical core of the proof of our main theorem. We prove our main theorem in the \( \text{Sp}_{2n} \) case in §4, and give a brief account in the \( \tilde{\text{Sp}}_{2n} \) and \( U_{E/F}(n, n) \) case in §5 and §6.

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NOTATIONS

Let \( F \) be a \( p \)-adic field, \( \mathcal{O} \) be the ring of integers, \( \mathcal{P} \) be the maximal ideal of \( \mathcal{O} \) and \( \varpi \) be a uniformizer of \( F \), i.e., a generator of \( \mathcal{P} \). Let \( q_F = |\mathcal{O}/\mathcal{P}| \), and \( | \cdot |_F \) be the standard valuation of \( F \) with \( |\varpi|_F = q_F^{-1} \).

The symplectic group \( \text{Sp}_{2n} \) and its subgroups. Let \( n > 1 \) be an integer and \( \text{Sp}_{2n} \) be the rank \( n \) symplectic group defined by the matrix

\[
\begin{pmatrix}
-J_n & J_n \\
J_n & -J_n
\end{pmatrix}
\]

where \( J_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \).

Explicitly,

\[
\text{Sp}_{2n}(F) = \left\{ g \in \text{GL}_{2n}(F) : g \begin{pmatrix} -J_n & J_n \\ J_n & -J_n \end{pmatrix} g^* = \begin{pmatrix} -J_n & J_n \\ J_n & -J_n \end{pmatrix} \right\}.
\]

Let \( P = MN \) be the Siegel Levi subgroup of \( \text{Sp}_{2n} \), where

\[
M = \left\{ m_n(g) := \begin{pmatrix} g & \\ g^* & g \end{pmatrix}, g \in \text{GL}_n(F), g^* = J_n \begin{pmatrix} g^{-1} \\ g \end{pmatrix} \right\},
\]

and

\[
N = \left\{ n_n(X) := \begin{pmatrix} I_n & X \\ I_n & I_n \end{pmatrix}, X \in \text{Mat}_{n \times n}(F), \begin{pmatrix} X \\ I_n \end{pmatrix} = J_n X J_n \right\}.
\]

Let \( U_M \) be the upper triangular unipotent subgroup of \( M \), and \( U = U_M N \), which is the maximal unipotent subgroup of the upper triangular Borel subgroup.

Let \( R \) be the subgroup of the Levi of \( P \) which consists elements of the form

\[
r(y, x) = m_n \begin{pmatrix} I_{n-2} & y \\ 1 & x \\ 1 \\ 1 \end{pmatrix}, y \in \text{Mat}_{(n-2) \times 1}(F) \cong F^{n-2}, x \in F.
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\]
Roots and Weyl group. Denote
\[ w_1 = \begin{pmatrix} 1 & \text{ } & \text{ } \\ \text{ } & \text{ } & I_{n-1} \\ \text{ } & I_{n-1} & \text{ } \end{pmatrix}, \]
and set \( j(g) = w_1 gw_1^{-1} \), for \( g \in \text{Sp}_{2n} \).

Let \( T \) be the maximal torus which consists elements of the form \( t = \text{diag}(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1}) \).
The simple roots of \( \text{Sp}_{2n} \) are \( \alpha_i, 1 \leq i \leq n-1, \beta \), where
\[ \alpha_i(t) = \frac{a_i}{a_{i+1}}, 1 \leq i \leq n-1, \beta(t) = a_n^2. \]

Let \( \Sigma^+ \) be the set of positive roots of \( \text{Sp}_{2n} \), and \( \Sigma \) be the set of roots of \( \text{Sp}_{2n} \). For \( \gamma \in \Sigma \), let \( U_{\gamma} \) be the root space of \( \gamma \) and let \( x_{\gamma} : F \rightarrow U_{\gamma} \) be the corresponding 1-parameter isomorphism.

Let \( \mathbf{W} \) be the Weyl group of \( \text{Sp}_{2n} \). For \( \gamma \in \Sigma^+ \), let \( s_{\gamma} \in \mathbf{W} \) be the simple reflection defined by \( \gamma \). Then \( s_{\gamma} \) acts on the set \( \Sigma \) by \( s_{\gamma}(\gamma') = \gamma' - \langle \gamma', \gamma \rangle / \langle \gamma, \gamma \rangle \gamma \), where \( \langle \gamma', \gamma \rangle \) is the coroot of \( \gamma \), and \( \langle \gamma', \gamma \rangle \) is the natural paring between roots and coroots.

The Weyl group \( \mathbf{W} \) is generated by \( s_{\alpha_i} \) and \( s_\beta \). We can take representative of \( s_{\alpha_i}, s_\beta \), by
\[ s_{\alpha_i} = m_n \begin{pmatrix} 1 & \text{ } & \text{ } & \text{ } \\ \text{ } & \text{ } & \text{ } & \text{ } \\ \text{ } & \text{ } & \text{ } & \text{ } \\ \text{ } & \text{ } & \text{ } & 1 \end{pmatrix}, 1 \leq i \leq n-1, \]
where the block \( \begin{pmatrix} 1 & \text{ } \\ \text{ } & 1 \end{pmatrix} \) is in the \((i, i+1) \times (i, i+1)\) position, and
\[ s_\beta = \begin{pmatrix} I_{n-1} & \text{ } \\ \text{ } & 1 \\ \text{ } & -1 \\ \text{ } & I_{n-1} \end{pmatrix}. \]

It is easy to check that \( w_1 = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_{n-1}} \). Let \( w_0 = w_1 s_\beta w_1^{-1} = j(s_\beta) \). In matrix form, we have
\[ w_0 = \begin{pmatrix} 1 & 0 \\ -1 & I_{2n-2} \end{pmatrix}. \]

The group \( \text{SL}_2 \). We will use the following notations for elements of \( \text{SL}_2(F) \):
\[ m_1(a) = \begin{pmatrix} a & \text{ } \\ -a & a^{-1} \end{pmatrix}, a \in F^\times, n_1(b) = \begin{pmatrix} 1 & b \\ \text{ } & 1 \end{pmatrix}, b \in F, \]
\[ \bar{n}_1(b) = \begin{pmatrix} 1 & \text{ } \\ b & 1 \end{pmatrix}, b \in F, w_1 = \begin{pmatrix} 1 & \text{ } \\ -1 & 1 \end{pmatrix}. \]

Denote \( U^1 = \{ n_1(b) : b \in F \} \) be the upper triangular unipotent subgroups and \( \bar{U}^1 = \{ \bar{n}_1(b) : b \in F \} \) be the lower triangular unipotent subgroups. Let \( A = \{ m_1(a), a \in F^\times \} \) be the torus of \( \text{SL}_2(F) \).

The metaplectic group \( \tilde{\text{Sp}}_{2n} \). Let \( \tilde{\text{Sp}}_{2n} \) be the metaplectic double cover of \( \text{Sp}_{2n} \). As a set, we have \( \tilde{\text{Sp}}_{2n} = \text{Sp}_{2n} \times \mu_2 \), where \( \mu_2 \) is the group \( \{ \pm 1 \} \). The group multiplication is given by
\[ (g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 c(g_1, g_2)), (g_i, \epsilon_i) \in \tilde{\text{Sp}}_{2n}, \]
where \( c(g_1, g_2) \) is the Rao cocycle defined in [Rao], cf [Sz1] for a brief review of the cocycle formulas.
1. Local zeta integrals and gamma factors for $\text{Sp}_{2n}$ and $\text{Mp}_{2n}$

In this section, we review the local zeta integrals for generic representations of $\text{Sp}_{2n} \times \text{GL}_1$ and $\text{Mp}_{2n} \times \text{GL}_1$ defined in [GiRS1], and the definition $\gamma$-factors. The paper [Ka] contains a nice review of these constructions.

1.1. Weil representations of $\widetilde{\text{Sp}}_2 \times \mathcal{H}$. Let $\mathcal{H}$ be the Heisenberg group of 3 variables, i.e., $\mathcal{H} = W \oplus F$, where $W$ is the symplectic space of dimension 2, with symplectic structure defined by $\langle w_1, w_2 \rangle = 2w_1 \begin{pmatrix} 1 & \ 0 \\ \ -1 & 1 \end{pmatrix} w_2^{-1}$. Here we view elements of $W$ as row vectors. A typical element of $\mathcal{H}$ is written as $[w, z]$, for $w \in W, z \in F$. The product in $\mathcal{H}$ is given by

$$[w, z] + [w', z'] = [w + w', z + z' + \frac{1}{2}(w, w')]$$

We identify $\text{Sp}(W)$ with $\text{SL}_2(F)$. Recall that $\widetilde{\text{SL}}_2$ denote the metaplectic double cover of $\text{SL}_2$. For later use, we recall the Rao cocycle and the product in $\mathcal{H}$. For an element $g \in \text{SL}_2(F) \times \text{SL}_2(F) \rightarrow \{ \pm 1 \}$ is defined by

$$c(g_1, g_2) = (\xi(g_1), \xi(g_2))_{\text{F}}(\xi(g_1)x(g_2), \xi(g_1g_2))_{\text{F}},$$

where

$$\xi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c, & c \neq 0, \\ d, & c = 0. \end{cases}$$

For these formulas, see [Sz2] for example.

For an element $g \in \text{SL}_2(F)$, we sometimes write $(g, 1) \in \widetilde{\text{SL}}_2(F)$ as $g$ by abuse notation.

A representation $\pi$ of $\widetilde{\text{SL}}_2(F)$ is called genuine if $\pi(\zeta g) = \zeta \pi(g)$ for all $g \in \widetilde{\text{SL}}_2(F)$ and $\zeta \in \mu_2$. Let $\psi$ be a nontrivial additive character of $F$, there is a Weil representation $\omega_{\psi}$ of $\widetilde{\text{SL}}_2(F) \times \mathcal{H}$ on $\mathcal{S}(F)$, the Bruhat-Schwartz functions on $F$. For $\phi \in \mathcal{S}(F), \xi \in F$, we have the familiar formulas:

$$\omega_{\psi}([[x, x', z]]\phi)(\xi) = \psi(z + 2\xi x' + x_0 x')\phi(\xi + x), [x, x', z] \in \mathcal{H},$$

$$\omega_{\psi}(w^1)\phi(\xi) = \gamma(\psi)\phi(\xi),$$

$$\omega_{\psi}(n_1(b))\phi(\xi) = \psi^{-1}(bx^2)\phi(\xi), b \in F$$

$$\omega_{\psi}(m_1(a))\phi(\xi) = |a|^{1/2}\frac{\gamma(\psi)}{\gamma(\psi_a)}\phi(a\xi), a \in F^\times,$$

and

$$\omega_{\psi}(\zeta)f(\xi) = \zeta f(\xi), \zeta \in \mu_2.$$ 

Here $\hat{f}(x) = \int_F f(y)\psi(2xy)dy$, where $dy$ is normalized so that $(\hat{f})'(x) = f(-x), \gamma(\psi)$ is the Weil index and $\psi_a(x) = \psi(ax)$. For these formulas, see [GePS1, GiRS2] for example. Note that the formula is affected by the factor 2 in the formula of $\langle w_1, w_2 \rangle$.

Let $\tilde{A}$ be the inverse image of the torus $A \subset \text{SL}_2(F)$ in $\widetilde{\text{SL}}_2(F)$. The product in $\tilde{A}$ is given by the Hilbert symbol, i.e.,

$$\langle m_1(a), \zeta_1 \rangle \langle m_1(b), \zeta_2 \rangle = \langle m_1(ab), \zeta_1 \zeta_2(a, b) \rangle_F,$$

where $(a, b)_F$ is the Hilbert symbol. The function

$$\mu_{\psi}(a) = \frac{\gamma(\psi)}{\gamma(\psi_a)}$$

satisfies

$$\mu_{\psi}(a)\mu_{\psi}(b) = \mu_{\psi}(ab)(a, b),$$

and thus defines a genuine character of $\tilde{A}$.

\footnote{The factor 2 is added to simplify some formulas.}
1.2. Genuine induced representation of $\text{SL}_2$. Recall that we denote $U^1$ the upper triangular unipotent of $\text{SL}_2(F)$. Then the Borel subgroup of $\text{SL}_2$ is $AU^1$. Let $\eta$ be a quasi-character of $F$, $s \in \mathbb{C}$, we consider the genuine induced representation $\hat{I}(s, \eta, \psi) = \text{Ind}_{AU^1}^{\text{SL}_2}((\mu_\psi)^{-1}\eta |^{|s+\frac{1}{2}})$ of $\hat{\text{SL}}_2$.

Since $\delta_{AU^1}((m_1(a), \zeta)) = |a|^2$, an element $f_s \in \hat{I}(s, \eta, \psi)$ satisfies the condition

$$f_s(n_1(b)(m_1(a), \zeta)) = \zeta |\mu_\psi(a)^{-1}\eta(a) | ^{|s+\frac{1}{2}}f_s(g), b \in F, a \in F^\times, \zeta \in \mu_2, g \in \hat{\text{SL}}_2(F).$$

1.3. The local zeta integral. Consider the embedding $\iota: \text{SL}_2 \to \text{Sp}_{2n}$

$$g \mapsto \iota(g) = \begin{pmatrix} I_{n-1} & g \\ 0 & I_{n-1} \end{pmatrix}.$$

Notice that $\iota\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = s_\beta$.

Recall that $U$ denote the standard maximal unipotent subgroup of $\text{Sp}_{2n}$. Given a nontrivial additive character $\psi$ of $F$, let $\psi_U$ be the generic character of $U$ defined by

$$\psi_U((u_{ij})) = \psi_U\left(\sum_{i=1}^n u_{i,i+1}\right), (u_{ij}) \in U.$$

Let $\pi$ be an irreducible $\psi_U$-generic representation of $\text{Sp}_{2n}(F)$, and let $W(\pi, \psi_U)$ be the space of $\psi_U$-Whittaker models of $\pi$. Let $\eta$ be a quasi-character of $F^\times$. For $W \in W(\pi, \psi_U)$, $\phi \in S(F)$ and $f_s \in I(s, \eta, \psi^{-1})$, we consider the local zeta integral

$$\Psi(W, \phi, f_s) = \int_{U^1 \backslash \text{SL}_2} \int_{F^{n-2}} \int_F W(j(r(y, x)g))|\omega_{\phi^{-1}}(g)\phi(x)f_s(g)|dydx dg.$$

Remark: (1) Recall that $j(g) = w_{11}gw_{11}^{-1}$ for $g \in \text{Sp}_{2n}$. In the above integral, we do not distinguish $g$ with $\iota(g)$ for $g \in \text{SL}_2(F)$ by abuse of notation.

(2) The above local zeta integral was first considered by Ginzburg, Rallis and Soudry in [GiRS1] when $\eta$ is trivial. In fact, Ginzburg, Rallis and Soudry defined a global zeta integral, proved it is Eulerian and did the unramified calculation in [GiRS1]. Later, similar constructions were generalized to the situation $\text{Sp}_{2n} \times \text{GL}_k$ for any $k$ in [GiRS2].

Proposition 1.1.  
(1) The integral $\Psi(W, \phi, f_s)$ is absolutely convergent when $\text{Re}(s) > 0$, defines a rational function of $q_F^{-\bullet}$.

(2) There exists a choice of datum such that $\Psi(W, \phi, f_s) = 1$.

Proof. It is not hard to show that the integral on the $y$-part has compact support. Thus the inner integral is absolutely convergent. The assertion of (1) follows from a gauge estimate of $W$, which is standard. We omit the details here. Part (2) is proved in Proposition 1.3, [GiRS1] when $\eta = 1$. In our case (when $F$ is $p$-adic), we will give a proof using Howe vectors later. □

Consider the standard intertwining operator $M_s: \hat{I}(s, \eta, \psi^{-1}) \to \hat{I}(1-s, \eta^{-1}, \psi^{-1})$ defined by

$$M_s(f_s)(\tilde{g}) = \int_F f_s(((w^1)^{-1}m_1(b), 1)\tilde{g}) db.$$

It is well-known that the above integral is absolutely convergent when $\text{Re}(s) > 0$ and can be meromorphically continued to all $\mathbb{C}$.

Proposition 1.2. Given a $\psi_U$-generic representation $\pi$ and a quasi-character $\eta$ on $F^\times$, there is a meromorphic function $\gamma(s, \pi, \eta, \psi)$ such that

$$\Psi(W, \phi, M_s(f_s)) = \gamma(s, \pi, \eta)\Psi(W, \phi, f_s),$$

for all $W \in W(\pi, \psi_U), \phi \in S(F)$, and $f_s \in \hat{I}(s, \eta, \psi^{-1})$.

Proof. This follows from the uniqueness of Fourier-Jacobi models, [GGP, Su]. For more details, see [Ka] for example. □
1.4. Local zeta integrals and gamma factors for generic representation of $\tilde{\text{Sp}}_{2n}$. The zeta integrals and gamma factors are defined similarly for $\tilde{\text{Sp}}_{2n}$ and we give a brief account of that. Let $\tilde{U}$ be the preimage of $U$ in $\tilde{\text{Sp}}_{2n}$. It is well-known that $\tilde{U} = U \times \mu_2$ as a group, see §3A of [Sz1] for example. The generic character $\psi_U$ of $U$ extends to a character $\psi_{\tilde{U}}$ of $\tilde{U}$ by $\psi_{\tilde{U}}((u, \epsilon)) = \psi_U(u)$ for $(u, \epsilon) \in \tilde{U}$. Let $\pi$ be a generic irreducible admissible $\psi_{\tilde{U}}$-generic representation of $\tilde{\text{Sp}}_{2n}$. By the main result of [Sz1], the Whittaker functional of $\pi$ is unique. Let $W(\pi, \psi_{\tilde{U}})$ be the space of $\psi_{\tilde{U}}$-Whittaker functional of $\pi$. Let $\eta$ be a quasi-character of $F^\times$ and let $I(s, \eta) = \text{Ind}_{\text{Ad}(F)}^{\text{SL}_2(F)}(\eta)^{\times s-1/2}$ be the induced representation of $\text{SL}_2(F)$. For $W \in W(\pi, \psi_{\tilde{U}})$, $\phi \in S(F)$ and $f_s \in I(s, \eta)$, one can consider the local zeta integral

$$
\psi(W, \phi, f_s) = \int_{U \setminus \text{SL}_2(F)} \int_{F^{n-2}} \int_F W(j(r(y, x)g)) (\omega_{\psi^{-1}}(g)\phi)(x)f_s(g)dydxdg,
$$

where an element $g \in \tilde{\text{Sp}}_{2n}$ is identified with the element $(g, 1) \in \tilde{\text{Sp}}_{2n}$. Similarly, there exists a gamma factor $\gamma(s, \pi, \eta, \psi)$ which satisfies similar property as in the $\text{Sp}_{2n}$ case. See [Ka] for more details.

2. Howe vectors for $\text{Sp}_{2n}$

In this section, we review the definition and basic properties of Howe vectors for $\text{Sp}_{2n}$ following [Ba1], and give some preliminary results which will be used in the proof of the stability of gamma factors.

Let $m > 0$ be a positive integer and $K_m = (I_{2n} + \text{Mat}_{2n \times 2n}(P^m)) \cap \text{Sp}_{2n}(F)$. Let $\psi$ be a fixed additive character of $F$ with conductor $O_F$. Consider the character $\tau_m$ of $K_m$ defined by

$$
\tau_m(k_{ij}) = \psi(-2m(\sum_{i=1}^n k_{i,i+1})).
$$

It is easy to check that $\tau_m$ is indeed a character. Let

$$
d_m = \text{diag}(\omega^{-m(2n-1)}, \omega^{-m(2n-3)}, \ldots, \omega^{-m}, \omega^m, \ldots, \omega^{m(2n-1)}) \in \text{Sp}_{2n}(F)
$$

and $H_m = d_mK_m(d_m)^{-1}$. Define a character $\psi_m$ on $H_m$ by $\psi_m(h) = \tau_m(d_mh^{-1}h)d_m$, $h \in H_m$. For a subgroup $S$ of $\text{Sp}_{2n}$, we will denote $S_m := S \cap H_m$.

**Lemma 2.1.**  
(1) The two characters $\psi_U$ and $\psi_m$ agree on $U_m = U \cap H_m$.

(2) For a positive root $\gamma$ of $\text{Sp}_{2n}$, then

$$
U_{\gamma, m} = \left\{ x_\gamma(r) : r \in P^{-(2ht(\gamma)-1)m} \right\},
$$

and

$$
U_{-\gamma, m} = \left\{ x_{-\gamma}(r) : r \in P^{(2ht(\gamma)+1)m} \right\}.
$$

Moreover, we have

$$
U_m = \prod_{\gamma \in \Sigma^+} U_{\gamma, m},
$$

where the product on the right side can be taken in any fixed order of $\Sigma^+$.

**Proof.** One can check (1) and the first part of (2) by direct calculation. The “moreover” part of (2) comes from the corresponding statement for $K_m \cap U$, cf [St].

Let $(\pi, V_\pi)$ be an irreducible smooth $\psi_U$-generic representation of $\text{Sp}_{2n}(F)$. We fix a Whittaker functional $\lambda_\pi \in \text{Hom}_U(\pi, \psi_U)$ and consider the Whittaker functions defined by $\lambda_\pi$, i.e., $W_v(g) = \lambda_\pi(\pi(g)v)$ for $v \in V_\pi$. We will write the identity element $I_{2n} \in \text{Sp}_{2n}(F)$ as 1 for simplicity. We fix a vector $v \in V_\pi$ such that $\lambda_\pi(v) = W_v(1) = 1$, and consider the vector

$$
v_m = \frac{1}{\text{vol}(U_m)} \int_{U_m} \psi_m(u)^{-1} \pi(u)vdudu.
$$

Let $C = C(v)$ be an integer such that $v$ is fixed by $\pi(K_C)$ (i.e., $C$ is bigger than the conductor of $v$), then a vector $v_m$ with $m \geq C$ is called a Howe vector as in [Ba1, Ba2].
Lemma 2.2. We have

1. \( W_{v_m}(1) = 1 \);
2. if \( m \geq C \), then \( \pi(h)v_m = \psi_m(h)v_m \), for all \( h \in H_m \);
3. for \( k \leq m \), we have
   \[
   v_m = \frac{1}{\text{vol}(U_m)} \int_{U_m} \psi_m^{-1}(u)\pi(u)v_k du.
   \]

Proof. Only (2) needs some work. The key ingredient of the proof of (2) is the Iwahori decomposition of \( K_m \) and hence of \( J_m \). The details can be found in Lemma 3.2 [Ba1], or Lemma 5.2 [Ba2] in the \( U(2,1) \) case. Baruch’s thesis [Ba1] is not published, but the proof in our case is the same as the proof in the \( U(2,1) \) case which is given in [Ba2].

By (2) of Lemma 2.2, for \( m \geq C \), the partial Bessel function \( W_{v_m}(g) \) satisfies the relation

\[ (2.1) \quad W_{v_m}(ugh) = \psi_U(u)\psi_m(h)W_{v_m}(g), \forall u \in U, h \in H_m, g \in \text{Sp}_{2n}(F). \]

Lemma 2.3. For \( m \geq C \) and \( t \in T \), if \( W_{v_m}(t) \neq 0 \), then \( \alpha_i(t) \in 1 + \mathcal{P}^m \) for all \( i \) with \( 1 \leq i \leq n-1 \) and \( \beta(t) \in 1 + \mathcal{P}^m \).

Proof. Write \( \gamma \) for a general simple root. Take an element \( r \in \mathcal{P}^{-m} \). We have the relation

\[ (2.2) \quad \{ x_{\gamma}(r) = x_{\gamma}(\gamma(t))t. \]

Since \( x_{\gamma}(r) \in U_m \subset H_m \), then by Eq.(2.1), we have

\[ \psi_m(x_{\gamma}(r))W_{v_m}(t) = \psi_U(x_{\gamma}(\gamma(t))t)W_{v_m}(t), \]

for \( m \geq C \). Thus if \( W_{v_m}(t) \neq 0 \), we get \( \psi_m(x_{\gamma}(r)) = \psi_U(x_{\gamma}(\gamma(t))t) \), or \( \psi(t) = \psi(\gamma(t))r \) for all \( r \in \mathcal{P}^{-m} \). Since \( \psi \) has conductor \( O_F \), we get \( \gamma(t) \in 1 + \mathcal{P}^m \), or \( \gamma(t) \in 1 + \mathcal{P}^m \). This proves the Lemma.

Lemma 2.4. Suppose that the residue characteristic of \( F \) is not 2, then the square map \( 1 + \mathcal{P}^m \to 1 + \mathcal{P}^m \) is well-defined and surjective.

Proof. This is a simple application of Newton’s Lemma, Proposition 2, Chapter II of [Lg]. We omit the details.

Note that the center \( Z \) of \( \text{Sp}_{2n}(F) \) is \( \{ \pm I_{2n} \} \). We will write the identity matrix \( I_{2n} \in \text{Sp}_{2n} \) as 1 for simplicity.

Corollary 2.5. Suppose the residue characteristic of \( F \) is not 2. Let \( t = \text{diag}(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1}) \in T \), and \( m \geq C \), then

\[ W_{v_m} = \begin{cases} \omega_\pi(e), & \text{if } t = e \cdot \text{diag}(a'_1, \ldots, (a'_1)^{-1}), \text{ for } a'_1 \in 1 + \mathcal{P}^m, e = \pm 1; \\ 0, & \text{otherwise}, \end{cases} \]

where \( \omega_\pi \) is the central character of \( \pi \).

Proof. Suppose that \( W_{v_m}(t) \neq 0 \), then \( a_i/a_{i+1} \in 1 + \mathcal{P}^m \) for \( 1 \leq i \leq n-1 \) and \( a_1^2 \in 1 + \mathcal{P}^m \) by Lemma 2.3. By Lemma 2.4, we can find an element \( a'_n \in 1 + \mathcal{P}^m \) such that \( (a'_n)^2 = a_n^2 \). Thus \( a_n = ea_n \) for some \( e \in \{ \pm 1 \} \). Since \( a_{n-1}/a_n \in 1 + \mathcal{P}^m \), we can write \( a_{n-1} = ea_{n-1} \) for \( a_{n-1} = a'_n \cdot \frac{2a_{n-1}}{a_n} \in 1 + \mathcal{P}^m \).

Inductively, we have \( a_i = ea'_i \) for some \( a'_i \in 1 + \mathcal{P}^m \). Then

\[ t = e \cdot \text{diag}(a'_1, \ldots, a'_n, (a'_n)^{-1}, \ldots, (a'_1)^{-1}), a'_i \in 1 + \mathcal{P}^m, \]

where we don’t distinguish \( e \) and \( \text{diag}(e, \ldots, e) \) by abuse of notation. Since \( \text{diag}(a'_1, \ldots, (a'_1)^{-1}) \in H_m \), then by Lemma 2.2 or Eq.(2.1), we get

\[ W_{v_m}(t) = W_{v_m}(e) = \omega_\pi(e)W_{v_m}(1) = \omega_\pi(e). \]

This completes the proof.

For \( a \in F^\times \), denote \( t(a) = \text{diag}(a, 1, 1, \ldots, 1, 1, a^{-1}). \)
Lemma 2.6. For $a \in F^\times$ and $y = \langle y_1, \ldots, y_{n-2} \rangle \in \text{Mat}_{(n-2) \times 1}(F)$, then for $m \geq C$, we have

$$W_v^m(t(a)j(r(y, 0))) = \begin{cases} W_v^m(t(a)), & \text{if } y_i \in \mathcal{P}^{(2i+1)m}, \text{ for all } i, 1 \leq i \leq n-2, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. We have

$$t(a)j(r(y, 0)) = m_n \begin{pmatrix} a & y & I_{n-2} \\ 0 & 0 & 0 \end{pmatrix},$$

(2.2)

If $y_i \in \mathcal{P}^{(2i+1)m}$, then $j(r(y, 0)) \in H_m \cap \bar{U}$, and thus $W_v^m(t(a)j(r(y, 0))) = W_v^m(t(a))$ by Lemma 2.2, or Eq.(2.1) for $m \geq C$. Now we suppose $y_i \notin \mathcal{P}^{(2i+1)m}$ for some $i$. Let $t$ be the biggest integer with this property, i.e., $i$ satisfies $1 \leq i \leq n-2$, $y_i \notin \mathcal{P}^{(2i+1)m}$ and $y_j \in \mathcal{P}^{(2j+1)m}$ for all $j$ with $n-2 \geq j > i$. By Eq.(2.1) and Eq.(2.2), we have

$$W_v^m(t(a)j(r(y, 0))) = W_v^m(m_n \begin{pmatrix} a & y^i & I_i \\ 0 & 0 & I_{n-i-1} \end{pmatrix}),$$

(2.3)

where $y^i = \langle y_1, \ldots, y_i \rangle \in \text{Mat}_{i \times 1}(F)$. Take $r \in \mathcal{P}^{-(2i+1)m}$ so that

$$X(r) := x_{a_1+\ldots+a_i+a_{i+1}}(r) = m_n \begin{pmatrix} I_n + re_1+i+2 \end{pmatrix} \in H_m,$$

where $e_1, i+2$ is the $n \times n$ matrix with 1 in the $(1, i+2)$ position, and zero elsewhere. We have the relation

$$X(-r)m_n \begin{pmatrix} a & y^i & I_i \\ 0 & 0 & I_{n-i-1} \end{pmatrix} X(r) = m_n \begin{pmatrix} I_i & y^i r & 1 \\ 1 & 0 \end{pmatrix} m_n \begin{pmatrix} a & y^i & I_{n-i} \\ 0 & 0 & I_{n-i-1} \end{pmatrix}.$$  

(2.4)

Note that $\psi_m(X(r)) = \psi_U(X(r)) = 1$, and

$$\psi_U \begin{pmatrix} 1 & y^i r & 1 \\ I_i & 0 \end{pmatrix} = \psi(y_r).$$

From Eq.(2.1), Eq.(2.3) and Eq.(2.4), we get

$$W_v^m(t(a)j(r(y, 0))) = \psi(y_r)W_v^m(t(a)j(r(y, 0))).$$

Since $y_i \notin \mathcal{P}^{(2i+1)m}$, we can take $r \in \mathcal{P}^{-(2i+1)m}$ such that $\psi(y_r) \neq 1$. Thus $W_v^m(t(a)j(r(y, 0))) = 0$. This completes the proof. □

3. A stability property of partial Bessel functions

In this section, we prove a stability property of partial Bessel functions associated with Howe vectors (Theorem 3.11 and Corollary 3.13), which is the key ingredient of the proof of the stability of the gamma factors.

3.1. Weyl elements and root spaces. We recall some notations on the roots and Weyl element of $\text{Sp}_{2n}$. For $t = \text{diag}(a_1, a_2, \ldots, a_n, a_{n-1}^{-1}, \ldots, a_1^{-1})$, the simple roots are given by

$$\alpha_i(t) = \frac{a_i}{a_{i+1}}, 1 \leq i \leq n-1, \beta(t) = a_n^2.$$ 

The Weyl group $W$ is generated by $s_{\alpha_i}$ and $s_{\beta}$.

We recall the notion of Bruhat order on $W$. For $w \in W$ with a minimal expression $s_{\xi_1} \ldots s_{\xi_t}$ where $\xi_i$ are simple roots. We say that $w' \leq w$ if $w'$ can be written as $w' = s_{\xi_1} \ldots s_{\xi_k}$ with $t_1, \ldots, t_k \in \{1, 2, \ldots, l\}$ and $t_1 < t_2 < \ldots$, i.e., $w'$ can be written as sub-expression of a minimal expression of $w$, see [Hu]. The definition of the Bruhat order does not depend on the choice of minimal expression of $w$. We will say that $w' \prec w$ if $w' \leq w$ and $w' \neq w$. 


Let $\Sigma^+$ be the set of positive roots. Let $U_M$ be the upper triangular unipotent of $GL_n(F)$ and $N$ be the Siegel unipotent of $Sp_{2n}(F)$. We will say a positive root $\gamma$ is in $U_M$ or $N$, if the root space of $\gamma$ is in $M$ or $N$. Suppose that $\gamma \in U_M$, then the root space of $\gamma$ is in the $(i,j)$-position, with $1 \leq i < j \leq n$, and the root $\gamma$ is $\sum_{i=1}^{j-1} (a_i).$ If $\gamma \in N$, then the root space of $\gamma$ is in the $(i,k)$ position with $1 \leq i \leq n < k \leq 2n$. By symmetry of the root spaces in $N$, we can assume that $i+k \leq 2n+1$. If $i+k = 2n+1$, i.e., the root space of $\gamma$ is in the skew diagonal, then $\gamma = 2\sum_{i=1}^{n-1} (a_i) + \beta$. If $i+k < 2n+1$, we put $j = 2n+1-i$, then $\gamma = \sum_{i=1}^{j-1} (a_i) + 2\sum_{i=j}^{n-1} (a_i) + \beta$.

**Lemma 3.1.** Let $\gamma_1, \gamma_2 \in \Sigma^+$, $\gamma_1 \neq \gamma_2$, $\frac{1}{2}\text{ht}(\gamma_2) < \text{ht}(\gamma_1) \leq \text{ht}(\gamma_2)$ and $\langle \gamma_2, \gamma_1 \rangle = 2$, then there exists integers $i,j$, with $1 \leq i < j \leq n$ such that $\gamma_2 = 2\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-1} + \beta$ and $\gamma_1 = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2\alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{n-1} + \beta$.

We will call a pair $(\gamma_1, \gamma_2)$ of positive roots which satisfies the condition of Lemma 3.1 a bad pair of positive roots. For the reason that such a pair is “bad”, see Lemma 3.4.

**Proof.** We first consider $\gamma_i \in U_M$, say, the root space of $\gamma_i$ is in the $(i,j)$ position with $1 \leq i < j \leq n$. Then $\gamma_i = \text{diag}(1, \ldots, 1, t, 1, \ldots, t^{-1}, 1, \ldots, 1)$ with $t$ in the $i$ and $2n+1-j$ position and $t^{-1}$ in the $j$ and $2n+1-j$ position. The only positive root $\gamma_2$ with $\gamma_2 \neq \gamma_1$ and $\langle \gamma_2, \gamma_1 \rangle = 2$ has root space in the $(i,2n+1-i)$ position, i.e., $\gamma_2 = 2(\alpha_1 + \cdots + \alpha_{n-1}) + \beta$. It is clear that $\text{ht}(\gamma_2) = 2(n+i-1)+1 > 2j-2(\gamma_1)$. Thus the pair $(\gamma_1, \gamma_2)$ does not satisfy the condition.

Next, consider the case $\gamma_i \in N$, say, the root space of $\gamma_i$ is in the $(i,k)$ position with $1 \leq i \leq n, n+1 \leq k \leq 2n$ and $i+k \leq 2n+1$. If $i+k = 2n+1$, then $\gamma_i = \text{diag}(1, \ldots, 1, t, 1, \ldots, 1, t^{-1}, 1, \ldots, 1)$, with $t$ in the $i$ position and $t^{-1}$ in the $j = 2n+1-j$ position. There is no $\gamma_2$ other than $\gamma_i$ itself such that $\langle \gamma_2, \gamma_i \rangle = 1$. If $i+k < 2n+1$, let $j = 2n+1-k$. Then $i < j \leq n$ and we have $\gamma_i = \alpha_1 + \cdots + \alpha_{i-1} + 2(\alpha_j + \cdots + \alpha_{n-1}) + \beta$. On the other hand $\gamma_i = \text{diag}(1, \ldots, 1, t, 1, \ldots, 1, t^{-1}, 1, \ldots, 1)$ is the diagonal element with $t$ in the $i$ and $j$ position, $t^{-1}$ in the $k = 2n+1-j$ and $2n+1-j$ position. There are two $\gamma_2 \neq \gamma_i$ with $\langle \gamma_2, \gamma_1 \rangle = 2$: i.e., $2(\sum_{i=1}^{n-1} \alpha_i) + \beta$ and $2(\sum_{i=1}^{n-1} \alpha_i) + \beta$. The second one has height $2(n-j) + 1$ which is smaller than the height of $\gamma_1$. Thus $\gamma_2 = 2(\sum_{i=1}^{n-1} \alpha_i) + \beta$. This finishes the proof.

For $\gamma \in \Sigma^+$, recall that we have an element $s_\gamma \in W$ which acts on $\Sigma^+$ by $s_\gamma(\gamma') = \gamma' - \langle \gamma', \gamma \rangle \gamma$. Let $w_0 = s_2(\alpha_1 + \cdots + \alpha_{n-1}) + \beta$. It is not hard to check that $w_0 = s_\alpha \cdots s_{\alpha_{n-1}} s_\beta s_{\alpha_{n-1}} \cdots s_\alpha$, which is a minimal expression of $w_0$. In general, we have $w_2(\alpha_1 + \cdots + \alpha_{n-1}) + \beta = s_\alpha \cdots s_{\alpha_{n-1}} s_\beta s_{\alpha_{n-1}} \cdots s_\alpha$, which is also a minimal expression.

We will say that a Weyl element $w$ is in $M$, if it has a representative in $M$, i.e., $w$ does not involve $s_\alpha$.

**Lemma 3.2.** If $w \in M$ and $\gamma \in N$, then $w(\gamma) \in N$, in particular $w(\gamma) > 0$.

**Proof.** This follows from the fact that $M$ normalizes $N$. □

**Proposition 3.3.** Given a bad pair of positive roots $(\gamma_1, \gamma_2) = (\alpha_1 + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{n-1}) + \beta, 2(\alpha_1 + \cdots + \alpha_{n-1}) + \beta)$ for $1 \leq i < j \leq n$ as in Lemma 3.1. Assume $w \leq w_0$, $w(\gamma_1) < 0$ and $w(\gamma_2) < 0$, then $w$ can be written in the form

$$w = w'_1 s_{\alpha_{j-1}} s_{\alpha_j} \cdots s_{\alpha_{n-1}} s_\beta s_{\alpha_{n-1}} \cdots s_{\alpha_1} w'_2,$$

with $w'_1 \leq s_{\alpha_1} \cdots s_{\alpha_{j-2}}$ and $w'_2 \leq s_{\alpha_{n-2}} s_{\alpha_{n-3}} \cdots s_{\alpha_1}$.

Here are some examples of $w$ with $w \leq w_0$, $w(\gamma_1) < 0$ and $w(\gamma_2) < 0$.

1. Suppose that $n = 3, i = 1, j = 2$, i.e., $(\gamma_1, \gamma_2) = (\alpha_1 + 2\alpha_2 + \beta, 2(\alpha_1 + \alpha_2) + \beta)$, then the only $w \leq w_0$ which satisfies $w(\gamma_1) < 0$ and $w(\gamma_2) < 0$ is $w = s_{\alpha_1} s_{\alpha_3} s_\beta s_{\alpha_2} s_{\alpha_1}$ itself.

2. Suppose that $n = 3, i = 1, j = 3$, i.e., $(\gamma_1, \gamma_2) = (\alpha_1 + \alpha_2 + \beta, 2(\alpha_1 + \alpha_2) + \beta)$, then $w = w_0 s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_\beta s_{\alpha_2} = 0$ or $w = s_{\alpha_1} s_\beta s_{\alpha_2}$.

3. Suppose that $n = 3, i = 2, j = 3$, i.e., $(\gamma_1, \gamma_2) = (\alpha_2 + \beta, 2(\alpha_2 + \beta)$, then $w = s_{\alpha_1} s_{\alpha_3} s_\beta s_{\alpha_2}$. $s_{\alpha_2} s_\beta s_{\alpha_2}$.

**Proof.** As a preparation, we compute the action of $s_{\alpha_k} (1 \leq k \leq n-1)$ on the positive roots $\beta, 2\alpha_{n-1} + \beta, \cdots, 2(\alpha_1 + \cdots + \alpha_{n-1}) + \beta$ which lies in the skew diagonal of $N$. For simplicity, denote
\[\beta_i = 2(\alpha_i + \cdots + \alpha_{n-1}) + \beta \text{ for } 1 \leq i \leq n-1, \text{ and } \beta_n = \beta \text{ temporarily. We have}\]
\[
\langle \beta_i, \alpha_k^\vee \rangle = \begin{cases} 
2, & i = k, \\
-2, & i = k + 1, \\
0, & \text{otherwise.}
\end{cases}
\]
Thus
\[
s_{\alpha_k}(\beta_i) = \beta_i - \langle \beta_i, \alpha_k^\vee \rangle \alpha_k = \begin{cases} 
\beta_{i+1}, & i = k, \\
\beta_{i-1}, & i = k + 1, \\
\beta_i, & \text{otherwise.}
\end{cases}
\]
In particular, \(s_{\alpha_k}\) preserves the set \(\{\beta_i\}_{1 \leq i \leq n}\). We also have
\[
s_\beta(\beta_i) = \beta_i, 1 \leq i \leq n - 1.
\]
Now we start the proof. Take a \(w \leq w_0\) such that \(w(\gamma_1) < 0\) and \(w(\gamma_2) < 0\).

First \(w\) must involve \(s_\beta\). In fact, if \(w\) does not involve \(s_\beta\), i.e., \(w = s_{\alpha_1} \cdots s_{\alpha_m} \in M\), then \(w(\gamma_1) > 0\) and \(w(\gamma_2) > 0\) from the above fact.

Since \(w \leq w_0\), we can assume that \(w = s_{\alpha_m} \cdots s_{\alpha_m} s_\beta s_{\alpha_1} \cdots s_{\alpha_k}\) for \(m_1 < m_2 < \cdots < m_t, l_k < l_{k-1} \cdots < l_1\). In our case, \(\gamma_2 = \beta_i\) for \(1 \leq i \leq n - 1\). Suppose that \(w(\gamma_1) < 0\) and \(w(\gamma_2) < 0\). We will prove \(w\) has the given form by the following claims.

Claim 1: We have \(s_{\alpha_1} \cdots s_{l_k}(\beta_i) = \beta_n = \beta\).

By \(\text{Eq.}(3.1)\), the expression \(s_{\alpha_1} \cdots s_{l_k}\) preserves the set \(\{\beta_j\}_{1 \leq i \leq n}\). Thus we can assume \(s_{\alpha_1} \cdots s_{l_k}(\beta_i) = \beta_p\) for some \(p\) with \(1 \leq p \leq n\). If \(p \neq n\), then \(s_\beta s_{\alpha_1} \cdots s_{l_k}(\beta_i) = s_\beta(\beta_p) = \beta_p\) by \(\text{Eq.}(3.2)\), and thus \(w(\beta_i) = s_{\alpha_m} \cdots s_{\alpha_m}(\beta_p) > 0\) by Lemma 3.2. Contradiction. This proves Claim 1.

Claim 2: If \(i > 1\), then the expression \(s_{\alpha_1} \cdots s_{l_k}\) does not involve \(s_{\alpha_{i-1}}\).

By contradiction, we assume that \(s_{\alpha_1} \cdots s_{l_k} = s_{\alpha_1} \cdots s_{l_p} s_{\alpha_{i-1}} s_{\alpha_{i+2}} \cdots s_{\alpha_k}\), with \(l_1 > \cdots l_p > i - 1 > l_{p+2} > \cdots > l_k\). By \(\text{Eq.}(3.1)\), we have
\[
s_{\alpha_1} \cdots s_{\alpha_p} s_{\alpha_{i-1}} s_{\alpha_{i+2}} \cdots s_{\alpha_k}(\beta_i) = s_{\alpha_1} \cdots s_{\alpha_p} s_{\alpha_{i-1}}(\beta_i) = s_{\alpha_1} \cdots s_{\alpha_p}(\beta_{i-1}) = \beta_{i-1} \neq \beta_n.
\]
This contradicts Claim 1. This proves Claim 2.

Claim 3: The expression \(s_{\alpha_{n-1}} \cdots s_{\alpha_i}\) must be a sub-expression of \(s_{\alpha_1} \cdots s_{l_k}\).

Write \(s_{\alpha_1} \cdots s_{l_k}(\beta_i) = s_{\alpha_1} \cdots s_{l_p} s_{\alpha_{i+1}} \cdots s_{\alpha_k}\) for \(l_1 > \cdots > l_p \geq i - 1 > l_{p+1} > \cdots > l_k\). We have
\[
s_{\alpha_1} \cdots s_{l_k}(\beta_i) = s_{\alpha_1} \cdots s_{l_p}(\beta_i).
\]
If \(l_p > i\), then \(s_{\alpha_1} \cdots s_{l_p}(\beta_i) = \beta_i\) from the above calculation. Thus \(l_p = i\). By induction, we have \(s_{\alpha_1} \cdots s_{l_p} = s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i}\). This proves Claim 3.

From the above 3 claims, we get \(w = s_{\alpha_m} \cdots s_{\alpha_m} s_\beta s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i} w'_2\) for some \(w'_2 \leq s_{\alpha_{i-2}} \cdots s_{\alpha_i}\).

We consider \(s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i} w'_2(\gamma_1)\).

Claim 4: We have \(s_\beta s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i} w'_2(\gamma_1) = \alpha_{j-1} + \alpha_j + \cdots + \alpha_{n-2} + \alpha_{n-1}\).

We have \(s_{\alpha_k}(\alpha_{k-1}) = \alpha_{k-1} + \alpha_k\) and \(s_{\alpha_k}(\alpha_{k+1}) = \alpha_k + \alpha_{k+1}\). Thus
\[
s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i} w'_2(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}) = s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i}(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}) = s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}}(\alpha_{i+1} + \cdots + \alpha_{j-1}) = \cdots = s_{\alpha_{n-1}} s_{\alpha_{j-1}}(\alpha_{j-1}) = s_{\alpha_{n-1}} s_{\alpha_{j-1}}(-\alpha_{j-1}) = \cdots = (\alpha_{j-1} + \cdots + \alpha_{n-2} + \alpha_{n-1}).
\]
From Claim 1, we have $s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i} w'_{2}(\gamma_2) = \beta$. Thus

$$
s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i} w'_2(\gamma_1)
$$

$$
= s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i} w'_2(\gamma_2 - (\alpha_i + \cdots + \alpha_{j-1})).
$$

$$
= \beta + \alpha_{j-1} + \cdots + \alpha_{n-2} + \alpha_{n-1}.
$$

Since $s_{\beta}$ preserves $\alpha_k$ for $k \leq n - 2$ and $s_{\beta}(\alpha_{n-1}) = \alpha_{n-1} + \beta$, we get

$$
s_{\beta} s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}} s_{\alpha_i} w'_2(\gamma_1)
$$

$$
= s_{\beta}(\beta + \alpha_{j-1} + \cdots + \alpha_{n-2} + \alpha_{n-1})
$$

$$
= \alpha_{j-1} + \cdots + \alpha_{n-1}.
$$

This proves Claim 4.

Claim 5: The expression $s_{\alpha_{j-1}} s_{\alpha_j} \cdots s_{\alpha_{n-1}}$ must be a sub-expression of $s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}}$.

By Claim 4, $w(\gamma_1) = s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}} (\alpha_{j-1} + \cdots + \alpha_{n-1})$. If $m_t \neq n - 1$, then $s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}} (\alpha_{j-1} + \cdots + \alpha_{n-1})$ must be a sum of $\alpha_{n-1}$ with another root, which cannot be negative. Thus $m_t = n - 1$.

By induction, we get Claim 5.

Now the proposition follows from the above Claims.

We call a tripe $(\gamma_1, \gamma_2, w)$ a bad tripe, if $(\gamma_1, \gamma_2)$ is a bad pair and $w \in W$ such that $w(\gamma_1) < 0$ and $w(\gamma_2) < 0$.

**Lemma 3.4.** For $\gamma_1, \gamma_2 \in \Sigma^+$ with $\gamma_1 \neq \gamma_2$, and $ht(\gamma_1) \leq ht(\gamma_2)$. If $(\gamma_1, \gamma_2)$ is not a bad pair as in Lemma 3.1, then $s_{\gamma_1}(\gamma_2) = \gamma_2 - \langle \gamma_2, \gamma_1' \rangle \gamma_1 \in \Sigma^+$.

**Proof.** We have $\langle \gamma_2, \gamma_1' \rangle = 0, \pm 1, \pm 2$. If $\langle \gamma_2, \gamma_1' \rangle \leq 0$, the assertion is clear. If $\langle \gamma_2, \gamma_1' \rangle = 1$, then $ht(s_{\gamma_1}(\gamma_2)) = ht(\gamma_2) - ht(\gamma_1) \geq 0$, and thus $s_{\gamma_1}(\gamma_2) > 0$. In fact, if we had $s_{\gamma_1}(\gamma_2) < 0$, we would have $ht(s_{\gamma_1}(\gamma_2)) < 0$. Now suppose that $\langle \gamma_2, \gamma_1' \rangle = 2$. Since the pair $(\gamma_1, \gamma_2)$ is not bad, we get $ht(\gamma_1) \leq \frac{1}{2}ht(\gamma_2)$, and thus

$$
ht(s_{\gamma_1}(\gamma_2)) = ht(\gamma_2) - 2ht(\gamma_1) \geq 0.
$$

The same argument as above shows that $s_{\gamma_1}(\gamma_2) > 0$.

**Lemma 3.5.** Given a bad tripe $(\gamma_1, \gamma_2, w)$. We assume that $(\gamma_1, \gamma_2) = (\sum_{t=1}^{j-1} \alpha_t + 2 \sum_{t=j}^{n-1} \alpha_t + \beta, 2 \sum_{t=1}^{n-1} \alpha_t + \beta)$ with $1 \leq i < j \leq n$, and $w = w'_1 cw'_2$ with $w'_1 \leq s_{\alpha_1} \cdots s_{\alpha_{j-2}}$, $w'_2 \leq s_{\alpha_{j-2}} \cdots s_{\alpha_1}$, and $\sigma = s_{\alpha_{n-1}} \cdots s_{\alpha_{n-2}} s_{\alpha_{n-1}} \cdots s_{\alpha_1}$. See Lemma 3.1 and Proposition 3.3.

(1) Given $\gamma \in \Sigma^+$ such that $ht(\gamma_1) \leq ht(\gamma) < ht(\gamma_2)$. If $s_{\gamma_2}(\gamma) < 0$, then there exists an integer $p$ with $i < p \leq j$ such that

$$
\gamma = \sum_{t=i}^{p-1} \alpha_t + 2 \sum_{t=p}^{n-1} \alpha_t + \beta.
$$

Moreover, for such a $\gamma$, if $\sigma(\gamma) < 0$, then $\gamma = \gamma_1$.

(2) Let $\gamma$ be a positive root such that $ht(\gamma) \geq ht(\gamma_1)$ and $s_{\gamma_2}(\gamma) > 0$, then $\sigma(\gamma) > 0$.

**Proof.** (1) By the formula $s_{\gamma_2}(\gamma) = \gamma - \langle \gamma, \gamma_2' \rangle \gamma_2$, we need to consider the pair $(\gamma, \gamma_2')$. We have $\gamma_2'(t) = \text{diag}(1, \ldots, 1, t, 1, \ldots, 1, t^{-1}, 1, \ldots, 1)$, where $t$ is in the $i$-th position and $t^{-1}$ in the $2n + 1 - i$ position. For $\gamma \in U_M$, suppose that $\gamma$ is in the $(k, l)$ position, with $1 \leq k < l \leq n$. Since $n - k \geq l - k = ht(\gamma) \geq ht(\gamma_1) = 2n - i + 1 \geq n - i + 1$, we get $k \leq i - 1$. Thus we have $\langle \gamma, \gamma_2' \rangle = 0$ or $-1$, and hence $s_{\gamma_2}(\gamma) > 0$.

Next, we consider $\gamma \in N$. Then it is easy to see that $\langle \gamma, \gamma_2' \rangle = 0, \pm 1$. If $\langle \gamma, \gamma_2' \rangle = 0$ or $-1$, then $s_{\gamma_2}(\gamma) > 0$. Thus we need to consider the $\gamma$ with $\langle \gamma, \gamma_2' \rangle = 1$. The root space of such $\gamma$ must be in the $i$-th row or $(2n + 1 - i)$-th column. In the latter case, we have $ht(\gamma) \geq ht(\gamma_2)$. In the former case, by the height condition of $\gamma$, we can choose a $p$ with $i < p \leq j$ such that

$$
\gamma = \sum_{t=i}^{p-1} \alpha_t + 2 \sum_{t=p}^{n-1} \alpha_t + \beta.
$$
This proves the first assertion of (1). To prove the moreover part, notice that we can write \( \sigma = s_{\alpha_{j-2}} \cdots s_{\alpha_1} s_{\gamma_2} \). Given a \( \gamma = \sum_{i=p}^{p-1} \alpha_i + 2 \sum_{i=p+1}^{n-1} \alpha_i + \beta \) with \( i < p \leq j \), we have \( s_{\gamma_2}(\gamma) = \gamma - \gamma_2 = - (\alpha_i + \cdots + \alpha_{p-1}) \). If \( p < j \), or equivalently, \( p-1 \leq j-2 \), we have \( \sigma(\gamma) = -s_{\alpha_{j-2}} \cdots s_{\alpha_1} (\alpha_i + \cdots + \alpha_{p-1}) > 0 \). This proves the moreover part of (1).

(2) Suppose that there is a \( \gamma \) such that \( s_{\gamma_2}(\gamma) > 0 \) but \( \sigma(\gamma) < 0 \). Denote \( \xi = s_{\gamma_2}(\gamma) \in \Sigma^+ \). Then \( \sigma(\gamma) = s_{\alpha_{j-2}} \cdots s_{\alpha_1}(\xi) < 0 \). Thus \( \xi \in \Sigma_{s_{\alpha_{j-2}} \cdots s_{\alpha_1}} \). It is not hard to check that

\[
\Sigma_{s_{\alpha_{j-2}} \cdots s_{\alpha_1}} = \left\{ \sum_{i=1}^p \alpha_i, i \leq p \leq j - 2 \right\}.
\]

Thus we can suppose that \( \xi = \sum_{i=p}^p \alpha_i \) for some \( p \) with \( i \leq p \leq j - 2 \). Note that \( \text{ht}(\xi) = p+1-i < \text{ht}(\gamma_1) \). By the definition of \( \xi \), we have

\[
\xi = s_{\gamma_2}(\gamma) = \gamma - (\gamma, \gamma_2^\vee)\gamma_2.
\]

If \( (\gamma, \gamma_2^\vee) < 0 \), then \( \xi = \gamma + \gamma_2 \), or \( \xi = \gamma + 2\gamma_2 \), which contradicts to \( \text{ht}(\xi) < \text{ht}(\gamma_1) \). If \( (\gamma, \gamma_2^\vee) = 0 \), then \( \gamma = \xi \), and thus \( \text{ht}(\gamma) < \text{ht}(\gamma_1) \). If \( (\gamma, \gamma_2^\vee) > 0 \), then \( \gamma = \gamma_2 + \xi \) or \( \gamma = 2\gamma_2 + \xi \). Note that neither \( \gamma_2 \) nor \( 2\gamma_2 + \xi \) is a root because it contains \( 3\alpha_i \). This proves (2). \( \square \)

Proposition 3.6. Let \( w \in \mathcal{W} \) and \( w \leq w_0 \). Let \( \xi_1, \ldots, \xi_k \in \Sigma_w \) ordered by \( \text{ht}(\xi_i) \leq \text{ht}(\xi_{i+1}) \).

1. Suppose that there is no \( l \) with \( 2 \leq l \leq k \) such that \( (\xi_1, \xi_l) \) is a bad pair as described in Lemma 3.1, i.e., \( \frac{1}{2} \text{ht}(\beta_l) < \text{ht}(\beta_l) \leq \text{ht}(\beta_l) \), then for all \( t \in T, r_i \in F, r_1 \neq 0 \), we have

\[
g := tw_{\xi_k}(r_k) \cdots x_{\xi_1}(r_1)x_{-\xi_1}(-r_1^{-1}) \in Bw'B
\]

for some \( w' \) with \( w' < w \).

2. Suppose that there exists an \( l \) with \( 2 \leq l \leq k \) such that \( (\xi_1, \xi_l) \) is a bad pair as described in Lemma 3.1, we have

\[
g := tw_{\xi_k}(r_k) \cdots \hat{x}_{\xi_l}(r_l) \cdots x_{\xi_1}(r_1)x_{-\xi_1}(r_1)x_{-\xi_1}(-r_1^{-1}) \in Bw'B,
\]

for some \( w' < w \), where \( \hat{x}_{\xi_l}(r_l) \) means the term \( x_{\xi_l}(r_l) \) is omitted.

Proof. For any \( \gamma \in \Sigma^+ \) and \( r \in F^x \), we have

\[
x_{\gamma}(r)x_{-\gamma}(-r^{-1}) \in s_{\gamma}B.
\]

This follows from a standard Chevalley relation, see [St]. We will also use the following fact on Bruhat order: given \( w', w \in \mathcal{W} \), then

\[ (*)\quad w' < w \text{ if and only if there exists positive roots } \xi_1, \ldots, \xi_k \text{ such that } w' = w_{\xi_k} \cdots w_{\xi_1} \text{ and } w_{\xi_1} \cdots w_{\xi_l}(\xi_{l+1}) \text{ is negative for all } i \text{ with } 1 \leq i \leq k - 1.\]

For a proof of this fact, see [Hu] for example.

1. We have \( x_{\xi_1}(r_1)x_{-\xi_1}(-r_1^{-1}) \in s_{\xi_1}B \) from the above Chevalley relation, Eq.(3.3). By Lemma 3.4, we have \( s_{\xi_1}(\xi_l) > 0 \) for all \( t \) with \( 2 \leq t \leq k \). Thus

\[
tw_{\xi_k}(r_k) \cdots x_{\xi_1}(r_1)x_{-\xi_1}(-r_1^{-1}) \in tw_{\xi_1}U_{s_{\xi_1}^{\xi_1}(\beta_2)}B \subset Bw'B.
\]

where \( w' = w_{\xi_1} \). Since \( \text{ht}(\xi_1) < 0 \), we have \( w' < w \), by the fact (*).

2. We suppose that \( (\xi_1, \xi_l) = (\alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{n-1}) + \beta, 2(\alpha_i + \cdots + \alpha_{n-1}) + \beta) \). By Proposition 3.3, we can assume

\[
w = w_1'\sigma w_2',
\]

for \( w_1' \leq s_{\alpha_1} \cdots s_{\alpha_{j-2}}, \sigma = s_{\alpha_{j-1}} \cdots s_{\alpha_{n-1}} s_{\beta} s_{\alpha_{n-1}} \cdots s_{\alpha_1} \) and \( w_2' \leq s_{\alpha_{i-2}} \cdots s_{\alpha_1} \). Note that \( w_2' \) commutes with \( \sigma \). In fact, each \( s_{\alpha_p} \) with \( p \leq i - 2 \) commutes with each element \( s_{\alpha_i}, \ldots, s_{\alpha_{n-1}}, s_{\beta} \). Then we can write

\[
w = w_1' w_2'.
\]
From the fact that \( x_{\xi_i}(r_i)x_{-\xi_i}(-r_i^{-1}) = s_{\xi_i}b = \sigma^{-1}s_{\alpha_j} \cdots s_{\alpha_2}b \) for some \( b \in B \), see Eq. (3.3), we get
\[
g = twx_{\xi_i}(r_k) \cdots \xi_{\beta_l}(r_l) x_{\xi_i}(r_1)x_{-\xi_i}(r_1)x_{-\xi_i}(-r_1^{-1})
= tw'_1w'_2\sigma x_{\xi_i}(r_k) \cdots \xi_{\beta_l}(r_l) x_{\xi_i}(r_1)x_{-\xi_i}(r_1)\sigma^{-1}s_{\alpha_j} \cdots s_{\alpha_3}b.
\]
From Lemma 3.4 and Lemma 3.5, we get \( \sigma(\xi_i) > 0 \) for all \( i > 1 \). Moreover, we have
\[
\sigma(\xi_i) = s_{\alpha_j} \cdots s_{\alpha_2}(s_{\xi_i}(\xi_i)) = -s_{\alpha_j} \cdots s_{\alpha_2}(\alpha_i + \cdots + \alpha_j) = -\alpha_j - 1.
\]
Thus we get
\[
g \in Bw'_1w'_2BU_{-\alpha_j} \cdots s_{\alpha_2}B.
\]
From the relation Eq. 3.3, we get \( U_{-\alpha_j} = Bs_{\alpha_j}B \), and thus
\[
g \in Bw'_1w'_2Bs_{\alpha_j} \cdots s_{\alpha_2}B.
\]
To proceed, we quote a general result from the structure theory of Chevalley group:

\((**)\) For \( w \in W \) and \( \gamma \) a simple root, we have
\[
BwBs_{\gamma}B = Bws_{\gamma}B, \quad \text{if } l(ws_{\gamma}) = l(w) + 1,
BwBs_{\gamma}B = BwB \cup Bws_{\gamma}B, \quad \text{if } l(ws_{\gamma}) = l(w) - 1.
\]
For a proof of this result, see Lemma 25 of [St] for example.

From 3.4 and the fact (**) , it is clear that
\[
g \in Bw'_1s_{\alpha_j} \cdots s_{\alpha_2}w'_2B.
\]
The assertion follows from the obvious relation \( w' := w'_1s_{\alpha_j} \cdots s_{\alpha_2}w'_2 < w \).

**Lemma 3.7.** Given a pair \( (\gamma_1, \gamma_2) = (\sum_{i=1}^{j-1} \alpha_i + 2 \sum_{i=n-1}^{j-1} \alpha_i + \beta, 2 \sum_{i=n-1}^{j-1} \alpha_i + \beta) \) with \( 1 \leq i < j \leq n \) as in Lemma 3.1, and an element \( w \in W \) such that \( w \leq w_0, w(\gamma_1) < 0 \) and \( w(\gamma_2) < 0 \). Let \( \xi \notin \Sigma^+ \) such that \( \text{ht}(\gamma_1) \leq \text{ht}(\xi) \leq \text{ht}(\gamma_2) \). If \( \gamma_2 - \xi = \sum t_\delta t_\delta \) is a sum of positive roots \( \delta_\xi \) then there is at least one \( t \) such that \( w(\delta_t) < 0 \).

**Proof.** We first claim that \( \xi \in N \). In fact, if \( \xi \notin U_M \), say the root space of \( \xi \) is in the \((k,l)\)-position with \( 1 \leq k < l \leq n \), then \( \xi = \sum_{i=1}^{l-1} a_i \). Since \( \text{ht}(\xi) = l - k \geq \text{ht}(\gamma_1) = 2n - i - j - 1 \geq n - i + 1 \), we get \( k \leq i - 1 \), (see the proof of Lemma 3.5). Thus in the expression \( \gamma_2 - \xi \), we have the term \( -\alpha_k \). Thus \( \gamma_2 - \xi \) cannot be a sum of positive roots. This proves the claim.

Then we can suppose that \( \xi = \sum_{i=m}^{p-1} a_i + 2 \sum_{i=p}^{m-1} \alpha_i + \beta \), for some integers \( m, p \) with \( 1 \leq m \leq p \leq n \). Since \( \gamma_2 - \xi \) is a sum of positive roots, we get \( m \geq i \). Thus we have
\[
\gamma_2 - \xi = 2 \sum_{i=1}^{m-1} a_i + \sum_{i=m}^{p-1} \alpha_i.
\]
Let \( \delta_t \) be the root which involves \( \alpha_t \). Then there exists a \( q \) with \( p - 1 \geq q \geq i \) such that \( \delta_t = \sum_{i=q}^{q-1} a_t \).

By Proposition 3.3, we can assume that
\[
w = w'_1s_{\alpha_j} \cdots s_{\alpha_1}w'_2s_{\alpha_1} \cdots w'_2,
\]
with \( w'_1 \leq s_{\alpha_1} \cdots s_{\alpha_{j-2}} \), and \( w'_2 \leq s_{\alpha_1} \cdots s_{\alpha_{j-1}} \). We can get \( w(\delta_t) \) by a simple calculation. In fact, we have
\[
s_{\alpha_j} \cdots s_{\alpha_1}w'_2(\delta_t) = -(\alpha_q + \cdots + \alpha_{n-1} + \beta),
\]
and \( w'_1s_{\alpha_j} \cdots s_{\alpha_1}(-(\alpha_q + \cdots + \alpha_{n-1} + \beta)) < 0 \) by Lemma 3.2.

**Lemma 3.8** (Chevalley relations). For \( r, s \in F \) and \( \gamma_1, \gamma_2 \in \Sigma^+ \), we have
\[
[x_{\gamma_1}(r), x_{\gamma_2}(s)] = \prod_{i \geq j \geq 1} x_{\gamma_1 + \gamma_2} (c_{ij}rs),
\]
for some \( c_{ij} \in F \).
For \( w \in W \), define \( U^+_w = \{ u \in U : wuw^{-1} \in U \} \) and \( U^-_w = \{ u \in U : wuw^{-1} \notin U \} \). Then
\[
U^+_w = \prod_{\gamma \in \Sigma_w^+} U_\gamma, \quad \text{and} \quad U^-_w = \prod_{\gamma \in \Sigma_w^-} U_\gamma,
\]
where \( \Sigma_w^+ = \{ \gamma \in \Sigma^+, w(\gamma) > 0 \} \) and \( \Sigma_w^- = \{ \gamma \in \Sigma^+, w(\gamma) < 0 \} \). Given \( w \in W \), suppose that \( \Sigma_w^- = \{ \xi_1, \ldots, \xi_l \} \). It is well-known that \( l = l(w) \), the length of \( w \). We now assume \( w \leq w_0 \). Inspired by Proposition 3.6, we give an order of the finite set \( \Sigma_w^- \) as follows.

**Definition 3.9** (Order of \( \Sigma_w^- \)). We order the set \( \Sigma_w^- = \{ \xi_1, \xi_2, \ldots, \xi_l \} \) as follows.

1. Suppose that there is no pair \( (\gamma_1, \gamma_2) \) of positive roots as in Lemma 3.1, such that \( \gamma_1, \gamma_2 \in \Sigma_w^- \), we ordered the set \( \{ \xi_i \} \) by \( \text{ht}(\xi_i) \leq \text{ht}(\xi_{i+1}) \). If two roots \( \xi, \xi' \in \Sigma_w^- \) have the same height, we do not mind what the order of \( \xi \) and \( \xi' \) is.

2. Suppose that there exists a pair \( (\gamma_1, \gamma_2) \) of positive roots as in Lemma 3.1 such that \( \gamma_1, \gamma_2 \in \Sigma_w^- \), we first order the set \( \Sigma_w^- \) by height as in (1), then we let \( \gamma_2 \) be the previous one adjacent to \( \gamma_1 \), i.e., if \( \gamma_1 = \xi_i \), then \( \gamma_2 = \xi_{i-1} \).

A general element in \( U^-_w \) can then be written as \( x_{\xi_1}(r_1) \ldots x_{\xi_l}(r_1) \). Recall that the notation \( U^+_{w,m} \) means \( U^-_w \cap U_m \), which is also \( U^-_w \cap U_m \).

**Lemma 3.10.** Given \( u^-_w = x_{\xi_1}(r_1) \ldots x_{\xi_l}(r_1) \in U^-_w - U^-_{w,m} \). Let \( q \) be an integer with \( 1 \leq q \leq l \) such that \( x_{\xi_q}(r_q) \in U_m \) for \( k < q \) but \( x_{\xi_q}(r_q) \notin U_m \). Let \( u = \prod_{\gamma \in \Sigma^+} x_\gamma(s_\gamma) \in U_m \). Then for \( t \in T \), we have
\[
tw x_{\xi_1}(r_1) \ldots x_{\xi_q}(r_q) u = \check{u}tw x_{\xi_1}(\check{r}_1) \ldots x_{\xi_q}(\check{r}_1),
\]
for some \( \check{u} \in U, \check{r}_1 \in F, 1 \leq t \leq l \), with \( |\check{r}_r| = |r_q| \).

**Proof.** We only consider the case that there exists a pair \( (\gamma_1, \gamma_2) \) as in Lemma 3.1 such that \( \gamma_1, \gamma_2 \in \Sigma_w^- \), and when \( \xi_q = \gamma_2 \). Actually one can check from the following argument that the proof of the remaining cases are easier than this exceptional case.

By our order on \( \Sigma_w^- \), we have \( \xi_{q+1} = \gamma_1 \) and \( \text{ht}(\xi_k) \geq \text{ht}(\xi_{q+1}) > \frac{1}{2} \text{ht}(\xi_k) \) for \( k \geq q \). We can write \( u = u^+ x_{\xi_1}(s_1) \ldots x_{\xi_q}(s_1) \), for \( s_q \in F \) with \( x_{\xi_q}(s_q) \in U_{\xi_{q,m}} \) for each \( k \), see the “moreover” part of Lemma 2.1 (2).

Claim 1: there exists \( u^+ \in U^+_{w,m}, u^- \in \prod_{t \geq q} U_{\xi_t} \) such that
\[
x_{\xi_1}(r_1) \ldots x_{\xi_q}(r_q) u^+ x_{\xi_k}(-r_k) \ldots x_{\xi_1}(r_1) = u^+_1 u^-_1.
\]

The idea is that we move \( u^+ \) to the left side step by step using Chevalley relations, Lemma 3.8. In each step, a commutator element will come out. By Lemma 3.7, the commutator does not involve elements in \( U_{\xi_q} \). We provide more details now. Write \( \Sigma^+ = \{ \delta_1, \ldots, \delta_v \} \). For \( k \) with \( q \leq k \leq l \) and \( u^+_k \in U^+_{w,m} \), we consider the conjugation
\[
x_{\xi_k}(r_k) u^+_k x_{\xi_k}(-r_k) = c_k u^+_k = u^+_k d_k,
\]
where \( c_k = [x_{\xi_k}(r_k), u^+_k] \), and \( d_k = c_k \cdot c_k^{-1} (u^+_k)^{-1} \). We have
\[
c_k, d_k \in \prod_{a_k \geq 1, \delta_1, \ldots, \delta_v \geq 0, b_1 + \cdots + b_v \geq 1} U_{a_k \xi_k + b_1 \delta_1 + \cdots + b_v \delta_v},
\]
by Lemma 3.8. We write \( d_k = d^+_k d^-_k \), where \( d^+_k \in U^+_{w,m} \) and \( d^-_k \in U^-_w \). Notice that we have
\[
(3.5) \quad a_k \xi_k + b_1 \delta_1 + \cdots + b_v \delta_v \neq \xi_q
\]
by Lemma 3.7. In fact, if \( a_k \geq 2 \), then we have \( \text{ht}(a_k \xi_k + b_1 \delta_1 + \cdots + b_v \delta_v) > \text{ht}(\xi_q) \), and thus Eq. (3.5) is clear. If \( a_k = 1 \), then we have \( \xi_k + b_1 \delta_1 + \cdots + b_v \delta_v \neq \xi_q \) by Lemma 3.7 and the fact that \( w(\delta_i) > 0 \) for each \( i \). Since \( \text{ht}(a_k \xi_k + b_1 \delta_1 + \cdots + b_v \delta_v) > \text{ht}(\xi_k) \geq \text{ht}(\xi_{q+1}) \), we get \( d^-_k \in \prod_{t > q} U_{\xi_t} \).

Thus we get
\[
x_{\xi_k}(r_k) u^+_k x_{\xi_k}(-r_k) = c_k u^+_k d^-_k := u^+_k d^-_k = u^+_{k+1} d^-_k,
\]
with \( u^+_{k+1} \in U^+_{w,m} \) and \( d^-_k \in \prod_{t > q} U_{\xi_t} \). If we start from \( u^+_q = u^+ \), repeat the above process and notice that the commutator \( \{ U_{\xi_q} U_{\xi_q'} \} \subset U_{a_k \xi_k + b_1 \delta_1 + \cdots + b_v \delta_v} \) has no nontrivial intersection with \( U_{\xi_q} \), because \( \text{ht}(\xi_k) + \text{ht}(\xi_k') \geq 2 \text{ht}(\xi_{q+1}) > \text{ht}(\xi_q) \), we get Claim 1.
By Claim 1, we have
\[ tw\xi_l(r_1) \ldots \xi_u(r_q)u = twu_1^{\xi_l(r_1)} \ldots \xi_u(r_q) \xi_i(s_l) \ldots \xi_i(s_1). \]
Next, we switch the order of the two elements \( \xi_l(r_1) \ldots \xi_u(r_q) \) and \( \xi_i(s_k) \) for \( k \geq q + 1 \) step by step. In each step, we get a commutator in
\[ \prod_{a,b \geq 1} U_{a\xi_k + b\xi_l} \] which has no nontrivial intersection with \( U_{\xi_k} \) because \( ht(\xi_{k'} + \xi_k) \geq ht(\xi_k) \). Thus finally, we get
\[ tw\xi_l(r_1) \ldots \xi_u(r_q)u = twu_1^{\xi_l(r_1)} \ldots \xi_u(r_q) \xi_i(s_{q-1}) \ldots \xi_i(s_1), \]
with \( \tilde{r}_q = r_q + s_q \). Since \( \xi_i(r_q) \notin U_{\xi_k,m} \) but \( \xi_i(s_k) \in U_{\xi_k,m} \), we get \( |\tilde{r}_q| = |r_q| \), see Lemma 2.1, (2).

The proof of this lemma is complete if we take \( u = twu_1^{+} w^{-1} t^{-1} \in U \), and \( \tilde{r}_k = s_k \) for \( k < q \). ☐

### 3.2. Stability property of partial Bessel functions associated with Howe vectors

In the following, we will fix two \( \psi_U \)-generic irreducible smooth representations \((\pi, V_{\pi}) \) and \((\pi', V_{\pi'}) \) of \( \text{Sp}_{2n}(F) \) with the same central character. We fix \( v \in V_{\pi} \) and \( v' \in V_{\pi'} \) such that \( W_\pi(1) = 1 = W_{\pi'}(1) \).

Let \( \text{C} = C(v, v') \) be a positive integer such that \( v \) is fixed by \( \pi(K_C) \) and \( v' \) is fixed by \( \pi'(K_C) \). Then we can consider the Howe vectors \( v_m, v'_m \) for \( m \geq C \) as defined in §2.

The main result of this section is the following

**Theorem 3.11.** Let \( w \leq w_0 \) be a Weyl element. Let \( a_t, 0 \leq t \leq l(w) \) be a sequence of integers with \( a_0 = 0 \) and \( a_t \geq t + a_{t-1} \) for all \( t \) with \( 1 \leq t \leq l(w) \). Let \( m \) be an integer such that \( m \geq 4^{a_l(w)}C \).

1. If \( W_{v_k}(tw') = W_{v_k'}(tw') \), for all \( w' < w, k \geq 4^{a_l(w)}C, \) and \( t \in T, \) then
\[ W_{v_m}(tww^{-m}) = W_{v'_m}(tww^{-m}), \]
for all \( u^{-m} \in U^{-m} - U_{w, m}^{-} \).

2. If \( W_{v_k}(tw') = W_{v_k'}(tw') \), for all \( w' \leq w, k \geq 4^{a_l(w)}C, \) and \( t \in T, \) then
\[ W_{v_m}(g) = W_{v'_m}(g), \]
for all \( g \in BwB \).

**Remark:**
1. We can take \( a_t = t^2 \) as Baruch did in [Ba1].
2. Baruch proved this result for the groups \( \text{GL}_n, \text{SL}_n, \text{SO}_{2n}, \text{U}(2,1), \) and \( \text{GSp}_4 \) for all \( w \in W \), see Lemma 6.2.2 and Lemma 6.2.6 of [Ba1], and Proposition 5.7 (c) of [Ba2]. Note that this result for \( \text{GSp}_{2n}, \text{Sp}_{2n} \) and \( \text{U}(n,n) \) case are the same because these groups have the same Weyl group structure. The proof of this result in the \( (2,2) \) case is also given in [Zh2], which justifies some ambiguity in the proof of Lemma 6.2.6 of [Ba1].

3. We expect this result holds for all \( w \in W \) for the group \( \text{Sp}_{2n} \) (without the restriction \( w \leq w_0 \)). By the previous work [Zh1, Zh2] in the low rank group case, if this is true, it is possible to prove a local converse theorem for \( \text{Sp}_{2n} \) and \( \text{U}(n,n) \).

**Proof of Theorem 3.11.** After the preparation in §3.1, in particular Proposition 3.6 and Lemma 3.10, the proof of this theorem follows from the method Baruch used to prove his Lemma 6.2.2 [Ba1] directly. Since [Ba1] is not published, we include a proof here.

First notice that (2) follows from (1) directly. In fact, any element \( g \in BwB \) can be written as \( g = u^+ tw^{-m} \) for \( u^+ \in U^+_w \) and \( u^- \in U^-_w \). Thus, if we take \( m \geq 4^{a_l(w)}C \), we have
\[ W_{v_m}(g) = \psi_U(u^+) W_{v_m}(tw^{-m}). \]

The same is true for \( W_{v'_m} \). If \( u^- \notin U^-_{w,m} \), then by (1), we get \( W_{v_m}(tw^{-m}) = W_{v'_m}(tw^{-m}) \), and thus \( W_{v_m}(g) = W_{v'_m}(g) \). If \( u^- \in U^-_{w,m} \), then by Eq. (2.1), we get \( W_{v_m}(g) = \psi_U(u^+ u^-) W_{v_m}(tw) \). By assumption of (2), we get \( W_{v_m}(tw) = W_{v'_m}(tw) \). Thus \( W_{v_m}(g) = W_{v'_m}(g) \).

We now prove (1) by induction. If \( w = 1 \), there is nothing to prove. For a general \( w \leq w_0 \), we assume that (1), and hence (2) hold for all \( w' \) with \( w' < w \leq w_0 \). Let \( m \) be an integer such that \( m \geq 4^{a_l(w)}C \) by assumption. Note that the induction hypothesis and the hypothesis of (1) implies that
\[ W_{v_k}(g) = W_{v'_k}(g), \]
for all \( t \in T, g \in Bu'B, k \geq 4^{a(\omega')}C \) and all \( w' < w \).

We assume \( \Sigma_{n} = \{ \xi_1, \xi_2, \ldots, \xi_q \} \), where the order of the index is defined in Definition 3.9. Given \( u_{-m} \in U_{w} - U_{m} \), we can write \( u = x_{\xi_1}(r_1) \ldots x_{\xi_q}(r_1) \). Let \( q \) be an integer with \( 1 \leq q \leq l \) such that \( x_{\xi_q}(r_1) \in U_{m} \) for all \( t < q \) but \( x_{\xi_q}(r_q) \notin U_{m} \). Then by Lemma 2.2 or Eq. (2.1), we have

\[
W_{v_{m}}(tu_{x_{\xi_q}(r_q)}) = \psi_{V}(x_{\xi_{q-1}(r_{q-1})} \ldots x_{\xi_1}(r_1))W_{v_{m}}(tu_{x_{\xi_q}(r_q)}).
\]

The same is true for \( W_{v'_{m}} \). Thus it suffices to show that

\[
W_{v_{m}}(tu_{x_{\xi_q}(r_q)}) = W_{v'_{m}}(tu_{x_{\xi_q}(r_q)}).
\]

We now take an integer \( k \) such that \( 3k \leq m < 4k \). By Lemma 2.2 (3), we have

\[
W_{v_{m}}(tu_{x_{\xi_q}(r_q)}) = \psi_{V}(u_{x_{\xi_q}(r_q)})W_{v_{m}}(tu_{x_{\xi_q}(r_q)})u_{\psi_{V}(u)^{-1}}du.
\]

The same is true for \( W_{v'_{m}} \). Now (1) of the theorem follows from:

**Claim 0:** we have

\[
W_{v_{k}}(tu_{x_{\xi_q}(r_q)}) = W_{v'_{k}}(tu_{x_{\xi_q}(r_q)}), \forall u \in U_{m}.
\]

By Lemma 3.10, we can write

\[
tu_{x_{\xi_q}(r_q)} = \tilde{u}tu_{x_{\xi_q}(r_q)} = \tilde{u}tu_{x_{\xi_q}(r_q)}u = \tilde{u}tu_{x_{\xi_q}(r_q)} \ldots x_{\xi_1}(r_1),
\]

for some \( \tilde{u} \in \tilde{U}, \tilde{r}_1 \in F \) and \( |q| = |r_q| \). To prove Claim 1, we consider two cases.

Case 1, if \( x_{\xi_q}(r_1) \in U_k \) for each \( t < q \). Then by Eq. (2.2), we have

\[
W_{v_{k}}(tu_{x_{\xi_q}(r_q)}) = \psi_{V}(u_{x_{\xi_q}(r_q)}(r_1) \ldots x_{\xi_q}(r_q))W_{v_{k}}(tu_{x_{\xi_q}(r_q)}).
\]

By assumption, we have \( x_{\xi_q}(r_q) \notin U_{m} \) and thus \( r_q \notin P - (2ht(\xi_q) - 1)m \) by Lemma 2.1. Since \( |q| = |r_q| \), we get \( r_q \notin P - (2ht(\xi_q) - 1)m \). Thus \( r_q - 1 \notin P(2ht(\xi_q) - 1)m \), which is \( P(2ht(\xi_q) - 1)m \). Since \( 3k \leq m \). Thus by Lemma 2.1, we get

\[
x_{-\xi_q}(-\tilde{r}_q^{-1}) \in U_{-\xi_q}k \subset H_k.
\]

By Lemma 2.2 or Eq. (2.1), we get

\[
W_{v_{k}}(tu_{x_{\xi_q}(r_q)}) = W_{v_{k}}(tu_{x_{\xi_q}(r_q)}).
\]

By Proposition 3.6, we get

\[
tu_{x_{\xi_q}(r_q)}(r_1) \ldots x_{\xi_q}(r_q)x_{\xi_q}(-\tilde{r}_q^{-1}) = Bw'B,
\]

for some \( w' < w \). Notice that \( k > \frac{1}{4} m \geq 4^{a(\omega')}C \geq 4^{a(\omega')}C \). Thus by the induction hypothesis and the hypothesis of (1), we get

\[
W_{v_{k}}(tu_{x_{\xi_q}(r_q)}(r_1) \ldots x_{\xi_q}(r_q))x_{\xi_q}(-\tilde{r}_q^{-1}) = W_{v'_{k}}(tu_{x_{\xi_q}(r_q)})x_{\xi_q}(-\tilde{r}_q^{-1}).
\]

By Eq. (3.6, 3.7) and their corresponding parts for \( W_{v'_{k}} \), we get Claim 0 in Case 1.

Case 2, it’s not true that \( x_{\xi_q}(r_1) \in U_k \) for each \( t < q \). We then take a \( q_1 \) such that \( x_{\xi_q}(r_1) \in U_{k} \) for all \( t < q_1 \) but \( x_{\xi_q}(r_{q_1}) \notin U_k \). Note that \( q_1 < q \) by assumption. We then take an integer \( k_1 \) such that \( 3k_1 \leq k < 4k_1 \), and write

\[
W_{v_{k_1}}(tu_{x_{\xi_q}(r_q)}) = \psi_{V}(u_{x_{\xi_q}(r_q)})W_{v_{k_1}}(tu_{x_{\xi_q}(r_q)})u_{\psi_{V}(u)^{-1}}du.
\]

Then we make the following

**Claim 1:** We have

\[
W_{v_{k_1}}(tu_{x_{\xi_q}(r_q)}) = W_{v_{k_1}}(tu_{x_{\xi_q}(r_q)}), \forall u \in U_{k_1}.
\]

Note that Claim 1 implies Claim 0. To prove Claim 1, we repeat the above process. The process will terminate after \( q \leq l(l(w)) \) steps. Note that in the \( t \)-th step, we need to take an integer \( k_t \) with \( k_t > \frac{1}{4}k_{t-1} \). Thus

\[
k_t \geq 4^{a(\omega')}m \geq 4^{a(\omega')}C \geq 4^{a(\omega')}C,
\]

for each \( w' < w \). Thus the induction hypothesis applies in each step. This completes the proof. \( \square \)

Before we state the consequences of Theorem 3.11, we need the following
Lemma 3.12. (1) If the residue field of $F$ has odd characteristic, then $W_{v,m}(g) = W'_{v,m}(g)$ for all $g \in B$, and $m \geq C$.

(2) We have $W_{v,m}(tw) = W'_{v,m}(tw)$ for all $t \in T$, $m \geq C$, $w < w_0$ and $w \neq 1$.

Proof. (1) This follows from Corollary 2.5 and the fact that $\pi$ and $\pi'$ have the same central character.

(2) We claim that for all $w < w_0$ and $w \neq 1$, there exists a simple root $\gamma$ such that $w(\gamma)$ is positive but not simple. We first show that this claim implies $W_{v,m}(tw) = W'_{v,m}(tw)$ for all $m \geq C, t \in T, w \leq w_0$ and $w \neq 1$. In fact, suppose that $\gamma$ is a simple root but $w(\gamma)$ is a positive but non-simple root. Take $r \in P^{-m}$, we have $x_w(r) \in U_m$. From the relation

$$twx_w(r) = x_w(\gamma(t)r)tw,$$

and Eq.(2.1), we get

$$\psi_m(x_w(\gamma(t)r))W_{v,m}(tw) = \psi_U(x_w(\gamma(t)r))W_{v,m}(tw).$$

Since $w(\gamma)$ is not simple, we get $\psi_U(x_w(\gamma(t)r)) = 1$. It is clear that $\psi_m(x_w(\gamma(t)r)) = \psi(r)$. Then we get $(\psi(r) - 1)W_{v,m}(tw) = 0$. Since $\psi$ is a nontrivial additive character with conductor $O$, we can choose $r \in P^{-m}$ such that $\psi(r) \neq 1$. Thus $W_{v,m}(tw) = 0$. The same argument shows that $W'_{v,m}(tw) = 0$.

Next, we prove the claim. By Proposition 3.2 of [CPS], it suffices to show that $w_tw_0$ is the long Weyl element of the Levi subgroup $M_{w_0}$ of a maximal parabolic subgroup $P_{w_0} \supset B$, where

$$w_t = \begin{pmatrix} -J_n \\ J_n \\ \end{pmatrix}$$

is the long Weyl element of $Sp_{2n}$. We have

$$w_tw_0 = \begin{pmatrix} -1 & J_{n-1} \\ -J_{n-1} & -1 \\ \end{pmatrix},$$

which is the long Weyl element of $M_{w_0} \cong GL_1 \times Sp_{2n-2}$. It is clear that the corresponding parabolic subgroup $P_{w_0}$ is a maximal parabolic subgroup. This proves the claim and hence the lemma. □

Corollary 3.13. Suppose that the field $F$ has odd residue characteristic.

(1) Given $w \in W$ with $w < w_0$, and $m \geq 4^{(w)^2} C$, we have

$$W_{v,m}(g) = W'_{v,m}(g),$$

for all $g \in BwB$.

(2) For $m \geq 4^{(w_0)^2} C$ and $u \in U_{w_0} - U_{w_0,m}$, we have

$$W_{v,m}(tw_0u) = W'_{v,m}(tw_0u),$$

for all $t \in T$.

Proof. This is a direct consequence of Theorem 3.11 and Lemma 3.12 □

We remark that Corollary 3.13 is the key to prove our main theorem, see the proof of Theorem 4.4.

4. Stability of $\gamma$-factors

4.1. Howe vectors for the Weil representations of $\widetilde{SL}_2(F)$. Given an unramified additive character $\psi$ of $F$, recall that we have a Weil representation $\omega_{\psi^{-1}}$ of $SL_2(F)$ on $S(F)$. For an integer $m > 0$, let $\phi_m \in S(F)$ be the characteristic function of $P^{(2n-1)m}$, which will play the role of Howe vectors for the Weil representations.

Lemma 4.1. Suppose the residue characteristic of $F$ is odd. We have

$$\omega_{\psi^{-1}}(n_1(b))\phi^m = \phi^m, \text{ for } b \in P^{-(4n-3)m}$$

and

$$\omega_{\psi^{-1}}(n_1(b))\phi^m = \phi^m, \text{ for } b \in P^{(4n-1)m}.$$
Here by abuse notation, we do not distinguish an element $g \in \text{SL}_2(F)$ with $(g, 1) \in \hat{\text{SL}}_2(F)$.

**Proof.** For $x \in F$, we have
\[
\omega_{\psi^{-1}}(n_1(b))\phi_m(x) = \psi(bx^2)\phi_m(x).
\]
For $x \in \text{Supp}(\phi') = \mathcal{P}^{(2n-1)m}$ and $b \in \mathcal{P}^{-(4n-3)m}$, we have $bx^2 \in \mathcal{P}^m \subset \mathcal{O}$, and thus $\psi(bx^2) = 1$. Now it is clear that $\omega_{\psi^{-1}}(u)\phi_m = \phi_m$.

To prove the second formula, we write $\bar{n}_1(b) = (w^1)^{-1}n_1(-b)w^1$, with $b \in \mathcal{P}^{(4n-1)m}$. Denote $\phi'_m = \omega_{\psi^{-1}}(w^1)\phi_m$. We have
\[
\phi'_m(x) = \omega_{\psi^{-1}}(w^1)\phi_m(x)
= \gamma(\psi^{-1}) \int_F \phi_m(y)\psi^{-1}(2xy)dy
= \gamma(\psi^{-1}) \int_{\mathcal{P}^{(2n-1)m}} \psi^{-1}(2xy)dy
= \gamma(\psi^{-1}) q_F^{-2(2n-1)m} \text{Char}_{\mathcal{P}^{-(2n-1)m}}(x),
\]
where $\text{Char}_{\mathcal{P}^{-(2n-1)m}}$ is the characteristic function of $\mathcal{P}^{-(2n-1)m}$. In the above A similar argument as above shows that $\omega_{\psi^{-1}}(n_1(-b))\phi'_m = \phi'_m$ for $b \in \mathcal{P}^{(4n-1)m}$. Thus we get
\[
\omega_{\psi^{-1}}(\bar{n}_1(b))\phi'_m = \omega_{\psi^{-1}}((w^1)^{-1})\omega_{\bar{n}_1(-b)}\phi'_m = \omega_{\psi^{-1}}((w^1)^{-1})\phi'_m = \phi_m.
\]
This finishes the proof of the lemma. 

4.2. **Sections of genuine induced representations of $\hat{\text{SL}}_2(F)$.** In this subsection, we construct some sections of genuine induced representations of $\hat{\text{SL}}_2(F)$, which will be used in the proof of stability of $\gamma$-factors for $\text{Sp}_{2n}$. The same constructions has been used in [ChZh] to get a local converse theorem for $\text{SL}_2$.

Note that $\bar{U}^1(F)$ and $U^1(F)$ splits in $\text{Mp}_2(F)$. Moreover, for $g_1 \in U^1$ and $g \in \bar{U}^1$ we have $c(g_1, g) = 1$. In fact, if $g_1 = n_1(y)$ and $g_2 = \bar{n}(x)$ with $x \neq 0$, we have $x(g_1) = 1$ and $x(g_2) = x$, and thus
\[
c(g_1, g_2) = (1, x)_F(-x, x)_F = 1.
\]
This shows that $\bar{U}^1 \cdot U^1 \subset \text{SL}_2(F)$, where $\text{SL}_2(F)$ denotes the subset of $\hat{\text{SL}}_2(F)$ which consists elements of the form $(g, 1)$ for $g \in \text{SL}_2(F)$.

For a positive integer $i$, we denote
\[
U^1_i = \begin{pmatrix} 1 & \mathcal{P}^{-i} \\ \mathcal{P}^i & 1 \end{pmatrix}, \quad \text{and} \quad \bar{U}^1_i = \begin{pmatrix} 1 & \mathcal{P}^{3i} \\ \mathcal{P}^{-3i} & 1 \end{pmatrix}.
\]
Note that $U^1_i = U^1 \cap H_i$ and $\bar{U}^1_i = \bar{U}^1 \cap H_i$, where we view $U^1_i$ and $\bar{U}^1_i$ as a subgroup of $\text{Sp}_{2n}$ by the standard embedding $\text{SL}_2 \hookrightarrow \text{Sp}_{2n}$.

Let $X$ be an open compact subgroup of $U^1$. For $x \in X$ and $i > 0$, we consider the set $A(x, i) = \{u \in \bar{U}^1 : u^x \in B^1 \cdot U^1_i\}$, where $B^1$ is the upper triangular Borel of $\text{SL}_2$. Note that the definition of $A(x, i)$ makes sense because $\bar{U}^1 \cdot U^1 \subset \hat{\text{SL}}_2$, as we showed above.

**Lemma 4.2.** (1) For any positive integer $c$, there exists an integer $i_1 = i_1(X, c)$ such that for all $i \geq i_1$, $x \in X$ and $\bar{u} \in A(x, i)$, we have
\[
\bar{u}^x = um_1(a)\bar{u}_0
\]
with $u \in U^1, \bar{u}_0 \in \bar{U}^1_i$ and $a \in 1 + \mathcal{P}^c$.

(2) There exists an integer $i_0 = i_0(X)$ such that for all $i \geq i_0$, we have $A(x, i) = \bar{U}^1_i$ for all $i \geq i_1$.

**Proof.** Since $X$ is compact, there is a constant $C$ such that $|x| < C$ for all $n_1(x) \in X \subset N$.

For $n_1(x) \in X, \bar{n}_1(y) \in A(n_1(x), i)$, we have $\bar{n}_1(y)n_1(x) \in B^1 \cdot \bar{U}^1_i$, thus we can assume that
\[
\bar{n}_1(y)n_1(x) = \begin{pmatrix} a & b \\ a & 1 \end{pmatrix} \bar{n}_1(y)
\]
for $a \in F^\times, b \in F$ and $\bar{y} \in P^{3i}$. Rewrite the above expression as

$$
\bar{n}_1(-y) \begin{pmatrix} a & b \\ a^{-1} \end{pmatrix} = n_1(x) \bar{n}_1(-\bar{y}),
$$
or

$$
\begin{pmatrix} a & b \\ -ay & a^{-1} - by \end{pmatrix} = \begin{pmatrix} 1 - x\bar{y} & x \\ -\bar{y} & 1 \end{pmatrix}.
$$

Thus we get

$$
a = 1 - x\bar{y}, ay = \bar{y}.
$$

Since $|x| < C$ and $\bar{y} \in P^{3i}$, it is clear that for any positive integer $c$, we can choose $i_1(X,c)$ such that $a = 1 - x\bar{y} \in 1 + P^c$ for all $n_1(x) \in X$ and $\bar{n}(y) \in A(n_1(x), i)$. This proves (1).

If we take $i_0(X) = i_1(X, 1)$, we get $a \in 1 + P \subset O^\times$ for $i \geq i_0$. From $ay = \bar{y}$, we get $y \in P^{3i}$. Thus we get that for $i \geq i_0(X)$, we have $\bar{n}_1(y) \in U^{1}_{1}$, i.e., $A(x, i) \subset U^{1}_{1}$. The other direction inclusion can be checked similarly if $i$ is large. We omit the details.

Now let $\eta$ be a quasi-character of $F^\times$. Given a positive integer $i$ and a complex number $s \in \mathbb{C}$, we consider the following function $f^i_s$ on $\widetilde{\text{SL}}_2(F)$:

$$
f^i_s((g, \zeta)) = \begin{cases} \\
\zeta \mu_{\psi^{-1}}(a)^{-1} \eta_{s+1/2}(a), & \text{if } g = \begin{pmatrix} a & b \\ a^{-1} \end{pmatrix}, \zeta \bar{n}_1(x), \text{ with } a \in F^\times, b \in F, \zeta \in \mu_2, x \in P^{3i}, \\
0, & \text{otherwise}. 
\end{cases}
$$

**Lemma 4.3.** Suppose that the residue characteristic of $F$ is odd.

1. There exists an integer $i_2(\eta)$ such that for all $i \geq i_2(\eta)$, $f^i_s$ defines a section in $\bar{f}(s, \eta, \psi^{-1})$.
2. Let $X$ be an open compact subset of $U^{1}$, then there exists an integer $I(X, \eta) \geq i_2(\eta)$ such that for all $i \geq I(X, \eta)$, we have

$$
\bar{f}^i_s(w^i x) = \text{vol}(U^{1}_{1}) = q^{-3i},
$$

for all $x \in X$, where $\bar{f}^i_s = M_s(f^i_s)$.

Recall that $w = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

**Proof.** (1) From the definition, it is clear that

$$
f^i_s \left( \begin{pmatrix} a & b \\ a^{-1} \end{pmatrix}, \zeta \bar{g} \right) = \zeta \mu_{\psi^{-1}}(a)^{-1} \eta_{s+1/2}(a)f^i_s(\bar{g}),
$$

for $a \in F^\times, b \in F, \zeta \in \mu_2$, and $\bar{g} \in \widetilde{\text{SL}}_2(F)$. It suffices to show that for $i$ large, there is an open compact subgroup $H_i \subset \text{SL}_2(F)$ such that $f^i_s(\bar{g}h) = f^i_s(\bar{g})$ for all $\bar{g} \in \text{SL}_2(F)$, and $h \in H_i$.

If $\psi$ is unramified and the residue characteristic is not 2 as we assumed, the character $\psi^{-1}$ is trivial on $O^\times_F$, see [Sz2] for example.

Let $c$ be a positive integer such that $\eta$ is trivial on $1 + P^c$. Denote $K^{1}_i = 1 + \text{Mat}_{2\times 2}(P^c)$, the standard congruence subgroup of $\text{SL}_2(F)$. Let $i_2(\eta) = \{c, i_0(U^{1} \cap K^{1}_i), i_{1}(U^{1} \cap K^{1}_i, c)\}$. For $i \geq i_2(\eta)$, we take $H_i = K^{1}_i = 1 + M_2(P^{4i})$. Note that the double cover map $\text{SL}_2 \rightarrow \text{SL}_2$ splits over $K^{1}_i$, and thus we can view $K^{1}_i$ as a subgroup of $\text{SL}_2$. We now check that for $i \geq i_2(\eta)$, we have $f^i_s(\bar{g}h) = f^i_s(\bar{g})$ for all $\bar{g} \in \text{SL}_2$ and $h \in K^{1}_i$. We have the Iwahori decomposition $K^{1}_i = (U^{1} \cap K^{1}_i)(A \cap K^{1}_i)(U^{1} \cap K^{4i})$.

For $h \in U^{1} \cap K^{1}_i \subset U^{1}_{1}$, it is clear that $f^i_s(\bar{g}h) = f^i_s(\bar{g})$ by the definition of $f^i_s$.

Now we take $h \in A \cap K^{1}_i$. Write $h = n_{1}(a_0)$, with $a_0 \in 1 + P^{4i}$. We have $\bar{n}_1(a_0)h = \bar{n}_1(a_0)^2 x$. It is clear that $x \in P^{3i}$ if and only if $a_0^{-2} x \in P^{3i}$. On the other hand, for any $a \in F^\times, b \in F$, we have

$$
c \begin{pmatrix} a & b \\ a^{-1} \end{pmatrix}, \bar{n}_1(a_0) \end{pmatrix} = (a^{-1}, a_0^{-1}) = 1,$$
since $a_0 \in 1 + \mathcal{P}_F^{3i} \subset F^{x,2}$ by Lemma 2.4. Thus we get

$$\left(\begin{pmatrix} a & b \\ a^{-1} & 0 \end{pmatrix}, \zeta\right) \hat{n}_1(x) h = \left(\begin{pmatrix} a a_0 & b a_0^{-1} \\ a^{-1} a_0^{-1} & 0 \end{pmatrix}, \zeta\right) \hat{n}_1(a_0^{-1} x).$$

By the definition of $f^i_s$, if $x \in \mathcal{P}_F^{3i}$, for $g = \left(\begin{pmatrix} a & b \\ a^{-1} & 0 \end{pmatrix}, \zeta\right)$, we get

$$f^i_s(gh) = \zeta \mu_{\psi^{-1}}(a a_0)^{-1} \eta_{s_1/2}(a a_0) = \zeta \mu_{\psi^{-1}}(a)^{-1} \eta_{s_1/2}(a) = f^i_s(g),$$

by the assumption on $i$.

Finally, we consider $h \in U^1 \cap K^1_{4i} \subset U^1 \cap K_c$. By assumption on $i$, we get

$$A(h, i) = A(h^{-1}, i) = \tilde{U}_i^1.$$  

In particular, for $\tilde{u} \in \tilde{U}_i^1$, we have $\tilde{u} h \in B^1 \cdot \tilde{U}_i^1$ and $\tilde{u} h^{-1} \in B^1 \cdot \tilde{U}_i^1$. Now it is clear that $\tilde{g} \in \tilde{B}^1 \cdot \tilde{U}_i^1$ if and only if $\tilde{g} h \in \tilde{B}^1 \cdot \tilde{U}_i^1$. Thus $f^i_s(\tilde{g}) = 0$ if and only if $f^i_s(\tilde{g} h) = 0$. Moreover, for $\tilde{u} \in \tilde{U}_i^1$, we have

$$\tilde{u} h = \begin{pmatrix} a_0 & b_0 \\ a_0^{-1} & 0 \end{pmatrix} \tilde{u}_0,$$

for $a_0 \in 1 + \mathcal{P}_c$, $b_0 \in F$ and $\tilde{u}_0 \in \tilde{U}_i^1$. Thus for $\tilde{g} = \left(\begin{pmatrix} a & b \\ a^{-1} & 0 \end{pmatrix}, \zeta\right)$ with $\tilde{u} \in \tilde{U}_i^1$, we get

$$\tilde{g} h = \left(\begin{pmatrix} a a_0 & a b_0 + a_0^{-1} b \\ a_0^{-1} a_0^{-1} & 0 \end{pmatrix}, \zeta\right) \tilde{u}_0.$$  

Here we used the fact that $a_0 \in 1 + \mathcal{P}_c$ is a square, and thus

$$c\left(\begin{pmatrix} a & b \\ a^{-1} & 0 \end{pmatrix}, \begin{pmatrix} a_0 & b_0 \\ a_0^{-1} & 0 \end{pmatrix}\right) = 1.$$  

Since $\mu_{\psi^{-1}}(a_0) = 1$, $(a, a_0) = 1$ and $\eta_{s_1/2}(a_0) = 1$, we get

$$f^i_s(\tilde{g} h) = f^i_s(g).$$

This finishes the proof of (1).

(2) As in the proof of (1), let $c$ be a positive integer such that $\eta$ is trivial on $1 + \mathcal{P}_c$. Take $I(X, \eta) = \max \{i_1(X, c), i_0(X)\}$. We have

$$f^i_s(w^1 x) = \int_F f^i_s((w^1)^{-1} n_1(b), 1) w^1 x) db = \int_F f^i_s((w^1)^{-1} n_1(b) w^1 x, 1) db.$$  

By the definition of $f^i_s$, $f^i_s((w^1)^{-1} n_1(b) w^1 x) \neq 0$ if and only if $(w^1)^{-1} n_1(b) w^1 x \in B^1 \tilde{U}_i^1$, if and only if $(w^1)^{-1} n_1(b) w^1 x \in A(x, i) = \tilde{U}_i^1$ for all $i \geq I(X)$, and $x \in X$. On the other hand, if $(w^1)^{-1} n_1(b) w^1 x \in A(x, i)$, we have

$$(w^1)^{-1} n_1(b) w^1 x = \begin{pmatrix} a & b_1 \\ a^{-1} & 0 \end{pmatrix} \tilde{u}_0,$$

with $a \in 1 + \mathcal{P}_c$ by Lemma 4.2. Thus

$$f^i_s((w^1)^{-1} n_1(b) w^1 x) = \eta_{s_1/2}(a) \mu_{\psi^{-1}}(a) = 1.$$  

Now it is clear that

$$\tilde{f}^i_s(w^1 x) = \text{vol}(\tilde{U}_i^1) = q_F^{-3i}.$$  

□
4.3. The stability of gamma factors for generic representations of $\text{Sp}_d(F)$ when the characteristic of $F$ is odd. Recall the notations from §3.2. We assume that $F$ is a $p$-adic field with odd residue characteristic, $(\pi, V_{\pi})$ and $(\pi', V_{\pi'})$ are two irreducible smooth $\psi_g$-generic representations of $\text{Sp}_d(F)$ with the same central character. We take $v \in V_{\pi}, v' \in V_{\pi'}$ such that $W_{\pi}(1) = 1 = W_{\pi'}$, and let $C = C(v, v')$ be an integer such that $v$ and $v'$ are fixed by $K_C$ under the action of $\pi$ and $\pi'$ respectively. Now we can state the main theorem of the paper:

**Theorem 4.4.** There is an integer $l = l(\pi, \pi')$ such that for any quasi-character $\eta$ of $F^\times$ with $\text{cond}(\eta) \geq l$, we have

$$\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi).$$

**Proof.** We take a quasi-character $\eta$ of $F^\times$ with conductor $\text{cond}(\eta)$. Let $m$ be an integer such that $m \geq \max \{\text{cond}(\eta), 4(\omega_0)^2\}$, and $i$ be an integer such that $i \geq \max \{i_2(\eta), I(U^i_{\pi}), (4n-1)m/3\}$, where $U^i_{\pi} = \{n(x), x \in \mathcal{P}^{-(4n-3)m}\}$, and $I(U^i_{\pi}, \eta)$ is defined in Lemma 4.3. By Lemma 4.3, we have a section $f^i_{\pi} \in I(s, \eta, \psi^{-1})$. Let $W_m$ be $W_{\pi_m}$ or $W_{\pi'_{\pi'}}$. We compute $\Psi(W_m, \phi_m, f^i_{\pi})$. We take the integral over $U^1 \setminus U^1 SL_2$ on the open dense subset $U^1 \setminus U^1 \setminus SL_2$

$$\Psi(W_m, \phi_m, f^i_{\pi}) = \int_{U^1 \setminus SL_2} \int_{U^1 \setminus SL_2} \int_{F^{n-2}} \int_{F^{n-2}} W_m(j(r(y, x)g))\omega_{\psi^{-1}}(g)\phi_m(x)f^i_{\pi}(g)dxdydg$$

$$= \int_{F^x \times F^x} \int_{F^{n-2}} \int_{F^{n-2}} W_m(j(r(y, x)))j(\mathbf{m}_1(a))j(\mathbf{n}_1(b)))$$

$$\cdot \omega_{\psi^{-1}}(\mathbf{m}_1(a)\mathbf{n}_1(b))\phi_m(x)f^i_{\pi}(\mathbf{m}_1(a)\mathbf{n}_1(b))dxdyb|a|^{-2}da.$$
By Lemma 2.6, we get
\[
\Psi(W_m, \phi_m, f_s^i) = q_F^{-3i-(2n-1)m} q_F^{\sum_{i=1}^{n-2}(2i+1)m} \int_{F \times F} W_m(t(a)) \eta_{s-n}(a) da
\]
\[
= q_F^{-3i-(n^2-1)m} \int_{F \times F} W_m(t(a)) \eta_{s-n}(a) da.
\]
Finally, by Corollary 2.5, we get \( W_m(t(a)) = 0 \) if \( a \notin 1 + \mathcal{P}^m \) and \( W_m(t(a)) = 1 \) if \( a \in 1 + \mathcal{P}^m \). Notice that \( m \geq \text{cond} (\eta) \) by assumption, we get
\[
\Psi(W_m, \phi_m, f_s^i) = q_F^{-3i-(2n-1)m} \int_{1 + \mathcal{P}^m} \eta_{s-n}(a) da = q_F^{-3i-(n^2-1)m} \text{vol}(1 + \mathcal{P}^m) = q_F^{-3i-n^2 m}.
\]
Note that this calculation works form both \( W_{v_m} \) and \( W_{v'_m} \), we then get
\[
(4.1) \quad \Psi(W_{v_m}, \phi_m, f_s^i) = \Psi(W_{v'_m}, \phi_m, f_s^i) = q_F^{-3i-n^2 m}.
\]
Next, we compute the other side of the functional equation, i.e., \( \Psi(W_m, \phi_m, f_s^i) \). We replace the domain \( U^1 \setminus \text{SL}_2 \) by its open dense subset \( U^1 \setminus U^1 A w^1 U^1 \). Thus
\[
\Psi(W_m, \phi_m, f_s^i) = \int_{F \times F} \int_{F_n-2} \int_F W_m(j(r(y, x) \mathbf{m}_1(a)w^1 \mathbf{n}_1(b))) \omega_{\phi^{-1}}(\mathbf{m}_1(a)w^1 \mathbf{n}_1(b)) \phi_m(x) \]
\[
\cdot \tilde{f}_s^i(\mathbf{m}_1(a)w^1 \mathbf{n}_1(b)) dx dy db |a|^{-2} da.
\]
Notice that \( j(w^1) = w_0 \) and
\[
j(r(y, x) \mathbf{m}_1(a)w^1 \mathbf{n}_1(b)) = j(\mathbf{m}_1(a)r(ya, xa)w^1 \mathbf{n}_1(b))
\]
\[
= j(\mathbf{m}_1(a)w^1(r(ya, xa)) \mathbf{n}_1(b))
\]
\[
= t(a)w_0 j(r'(ya, xa)) j(\mathbf{n}_1(b))
\]
where \( r'(y, x) = s_\beta^{-1}r(y, x)s_\beta \). By changing variables, we get
\[
\Psi(W_m, \phi_m, f_s^i) = \int_{F \times F} \int_{F_n-2} \int_F W_m(t(a)w_0 j(r'(y, x)) j(\mathbf{n}_1(b))) \]
\[
\cdot \mu_{\phi^{-1}}(a)|a|^{1/2} \omega_{\phi^{-1}}(w^1 \mathbf{n}_1(b)) \phi_m(x) \tilde{f}_s^i(\mathbf{m}_1(a)w^1 \mathbf{n}_1(b)) dx dy db |a|^{-1-n} da.
\]
We then get
\[
\Psi(W_{v_m}, \phi_m, f_s^i) - \Psi(W_{v'_m}, \phi_m, f_s^i)
\]
\[
= \int_{F \times F} \int_{F_n-2} \int_F \left( W_{v_m}(t(a)w_0 j(r'(y, x)) j(\mathbf{n}_1(b))) - W_{v'_m}(t(a)w_0 j(r'(y, x)) j(\mathbf{n}_1(b))) \right) \]
\[
\cdot \mu_{\phi^{-1}}(a)|a|^{1/2} \omega_{\phi^{-1}}(w^1 \mathbf{n}_1(b)) \phi_m(x) \tilde{f}_s^i(\mathbf{m}_1(a)w^1 \mathbf{n}_1(b)) dx dy db |a|^{-1-n} da.
\]
In matrix form, we have
\[
r'(y, x) = n_n \begin{pmatrix} y \\ x \\ 0 \\ x \\ y \end{pmatrix},
\]
and
\[
j(r'(y, x)) j(\mathbf{n}_1(b)) = n_n \begin{pmatrix} x & t y J_{n-2} & b \\ y & x \end{pmatrix}.
\]
We have \( j(r'(y, x)) j(\mathbf{n}_1(b)) \in U_{n_0}^- \). By Corollary 3.13, if \( j(r'(y, x)) j(\mathbf{n}_1(b)) \notin U_m \), we have
\[
W_{v_m}(t(a)w_0 j(r'(y, x)) j(\mathbf{n}_1(b))) - W_{v'_m}(t(a)w_0 j(r'(y, x)) j(\mathbf{n}_1(b))) = 0,
\]
and thus
\[ \Psi(W_{v_m}, \phi_m, \tilde{f}_s) - \Psi(W_{\nu m}, \phi_m, \tilde{f}_s) \]
\[ = \int_{F^\times} \int_{D_m} \left( W_{v_m}(t(a)w_0 j(r'(y, x)) j(n_1(b))) - W_{\nu m}(t(a)w_0 j(r'(y, x)) j(n_1(b))) \right) \]
\[ \cdot \mu_{\psi^{-1}}(a) |a|^{1/2} \omega_{\psi^{-1}}(1) \phi_m(x) \tilde{f}_s(m_1(a)w_1 n_1(b)) dx dy |a|^{-n} da, \]
where \( D_m = D \cap U_m \) with
\[ D = \left\{ j(r'(y, x)) j(n_1(b)) = n_n \begin{pmatrix} x & b J_{n-2} y \\ y & x \end{pmatrix}, \quad x, b \in F, y \in \text{Mat}_{(n-2) \times 1}(F) \right\}. \]

Now suppose that \( j(r'(y, x)) j(n_1(b)) \in D_m \subset U_m \), then by Eq.(2.1), we have
\[ W_{v_m}(t(a)w_0 j(r'(y, x)) j(n_1(b))) = W_{v_m}(t(a)w_0), \quad W_{\nu m}(t(a)w_0 j(r'(y, x)) j(n_1(b))) = W_{\nu m}(t(a)w_0). \]
For \( j(r'(y, x)) j(n_1(b)) \in D_m \subset U_m \), we have \( b \in P^{-(4n-3)m} \) and \( x \in P^{-(2n-1)m} \), and thus we get
\[ \omega_{\psi^{-1}}(1) \phi_m(x) = \omega_{\psi^{-1}}(1) \phi_m(x) = \gamma(\psi^{-1}) q_{F}^{-2(2n-1)m}, \]
see Lemma 4.1 and its proof. On the other hand, we have
\[ \tilde{f}_s(m_1(a)w_1 n_1(b)) = (\mu_{\psi^{-1}}(a))^{-1} \eta_{-s+3/2}(a) q_{F}^{-3i}, \]

Lemma 4.3 and the assumption that \( i > I(\widetilde{U}_{1, \eta}) \).

From the above discussions, we get
\[ \Psi(W_{v_m}, \phi_m, \tilde{f}_s) - \Psi(W_{\nu m}, \phi_m, \tilde{f}_s) = \gamma(\psi^{-1}) \text{vol}(D_m) q_{F}^{-3i-(2n-1)m} \int_{F^\times} \left( W_{v_m}(t(a)w_0) - W_{\nu m}(t(a)w_0) \right) \eta_{-s+n+1}(a) da. \]

Let \( k = 4^{(l(w_0)^2} C \). By Lemma 3.2, we have
\[ W_{v_m}(t(a)w_0) - W_{\nu m}(t(a)w_0) \]
\[ = \frac{1}{\text{vol}(U_m)} \int_{U_m} \left( W_{v_k}(t(a)w_0 u) - W_{\nu k}(t(a)w_0 u) \right) \psi^{-1}_U(u) du \]
\[ = \frac{1}{\text{vol}(U_m)} \int_{U_{w_0,m}} \int_{U_{w_0,m}} \left( W_{v_k}(t(a)w_0 u^+ u^-) - W_{\nu k}(t(a)w_0 u^+ u^-) \right) \psi^{-1}_U(u^+ u^-) du^+ du^- \]
\[ \cdot \]
\[ U_{w_0}^+ = \left\{ \begin{pmatrix} 1 & u \\ -u & 1 \end{pmatrix}, \quad u \in U^{(n-1)} \right\}, \]
where \( U^{(n-1)} \) is the upper triangular maximal unipotent of \( \Sp_{2(n-1)} \). It is clear that for \( u^+ \in U_{w_0,m} \), we have
\[ t(a)w_0 u^+ = u^+ t(a)w_0. \]
Since \( W_{v_k}(u^+ t(a)w_0 u^-) = \psi_U(u^+) W_{v_k}(t(a)w_0 u^-) \), we then get
\[ W_{v_m}(t(a)w_0) - W_{\nu m}(t(a)w_0) \]
\[ = \frac{\text{vol}(U_{w_0,m}^+)}{\text{vol}(U_m)} \int_{U_{w_0,m}} \left( W_{v_k}(t(a)w_0 u^-) - W_{\nu k}(t(a)w_0 u^-) \right) \psi^{-1}_U(u^-) du^- \]
\[ \cdot \]
By Corollary 3.13, we get
\[ W_{v_k}(t(a)w_0 u^-) - W_{v_k}(t(a)w_0 u^-) = 0, \]
for \( u^- \in U_{w_0,m}^- - U_{w_0,k}^- \). Thus we get

\[
W_{\gamma m}(t(a)w_0) - W_{\gamma m'}(t(a)w_0) = \frac{\text{vol}(U_{w_0,m})}{\text{vol}(U_m)} \int_{U_{w_0,k}^-} (W_{\gamma k}(t(a)w_0u^-) - W_{\gamma k'}(t(a)w_0u^-)) \psi^{-1}_{U'}(u^-) du^-.
\]

By Lemma 2.2 or Eq. (2.1), we get

\[
W_{\gamma k}(t(a)w_0u^-) = \psi_{U'}(u^-)W_{\gamma k}(t(a)w_0), W_{\gamma k'}(t(a)w_0u^-) = \psi_{U'}(u^-)W_{\gamma k'}(t(a)w_0).
\]

Thus we have

\[
W_{\gamma m}(t(a)w_0) - W_{\gamma m'}(t(a)w_0) = \frac{\text{vol}(U_{w_0,m})\text{vol}(U_{w_0,k}^-)}{\text{vol}(U_m)}(W_{\gamma k}(t(a)w_0u^-) - W_{\gamma k'}(t(a)w_0u^-))
\]

\[
= q_k^{(2n-1)^2k-(2n-1)^2m} W_{\gamma k}(t(a)w_0u^-) - W_{\gamma k'}(t(a)w_0u^-).
\]

Plug this into Eq. (4.2), we get

\[
\Psi(W_{\gamma m}, \phi_m, \bar{f}_s) - \Psi(W_{\gamma m'}, \phi_m, \bar{f}_s)
\]

\[
= \gamma(\psi^{-1})q_k^{(2n-1)^2k-(2n-1)^2m-i} \frac{\text{vol}(D_m)}{\text{vol}(U_{w_0,m})} \int_{F^x} (W_{\gamma k}(t(a)w_0) - W_{\gamma k'}(t(a)w_0)) \eta_{-s-n}^{-1}(a) da
\]

\[
= \gamma(\psi^{-1})q_k^{(2n-1)^2k-n^2m-3i} \int_{F^x} (W_{\gamma k}(t(a)w_0) - W_{\gamma k'}(t(a)w_0)) \eta_{-s-n}^{-1}(a) da.
\]

By Eq. (4.1, 4.3) and the local functional equation, we get

\[
\gamma(s, \pi, \eta, \psi) - \gamma(s, \pi', \eta, \psi) = \gamma(\psi^{-1})q_k^{(2n-1)^2k} \int_{F^x} (W_{\gamma k}(t(a)w_0) - W_{\gamma k'}(t(a)w_0)) \eta_{-s-n+1}^{-1}(a) da.
\]

Now we can prove the main theorem. Note that \( k \) only depends on the choices of \( v \) and \( v' \), which are fixed at the begining. Since the function \( a \mapsto W_{\gamma k}(t(a)w_0) \) and \( a \mapsto W_{\gamma k'}(t(a)w_0) \) are continuous, we can take an integer \( l = l(\pi, \pi') \) such that for \( c \geq l \), we have

\[
W_{\gamma k}(t(a_0)w) = W_{\gamma k}(t(a)w_0), \quad \text{and} \quad W_{\gamma k'}(t(a_0)) = W_{\gamma k'}(t(a)w_0),
\]

for all \( a_0 \in 1 + \mathcal{P}^c \). Now it is clear that of \( \eta \) is a quasi-character with \( \text{cond}(\eta) > l \), the right side of Eq. (4.4) vanishes, and hence

\[
\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi).
\]

\[\blacksquare\]

**Remark:** From Eq. (4.4) and the Mellin inversion, we can get that if \( \gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi) \) for all quasi-characters \( \eta \) of \( F^s \), then \( W_{\gamma k}(t(a)w_0) = W_{\gamma k'}(t(a)w_0) \) for all \( a \in F^s \). From this, it is easy to show that

\[
W_{\gamma m}(tw_0) = W_{\gamma m}(tw_0),
\]

for all \( t \in T \) and \( m \geq 4^{(w_0)^2} C \). By Theorem 3.11, we can get that \( W_{\gamma m}(g) = W_{\gamma m'}(g) \) for all \( g \in Bw_0B \). This should be the first step to get a local converse theorem for \( \text{Sp}_{2n} \) if Theorem 3.11 works for all \( w \in W \).

As a corollary of the stability of \( \gamma \)-factors, Theorem 4.4, and the multiplicativity of \( \gamma \)-factors [Kan], we have the following stable form for \( \gamma(s, \pi, \eta, \psi) \).
Proposition 4.5. Let \( \pi \) be a generic representation of \( \text{Sp}_{2n} \) and let \( \chi_1, \ldots, \chi_n \) be non-trivial characters of \( F^\times \). Then for sufficiently highly ramified character \( \eta \) of \( F^\times \), we have
\[
\gamma(s, \pi, \eta, \psi) = \gamma(s, \eta, \psi) \prod_{i=1}^{n} \gamma(s, \chi_i \eta, \psi) \gamma(s, \chi_i^{-1} \eta, \psi).
\]

5. Howe vectors and stability of gamma factors for metaplectic groups

In this section, we will extend the stability result to the \( \widetilde{\text{Sp}}_{2n}(F) \)-case. Throughout this section, we assume that \( F \) is a \( p \)-adic field with odd residue characteristic.

In \( \widetilde{\text{Sp}}_{2n}(F) \), we will frequently use the following relation,
\[
(p, \epsilon_1)(g, \epsilon)(p, \epsilon_1)^{-1} = (pgp^{-1}, \epsilon),
\]
for all \( g \in \text{Sp}_{2n}(F), p \in P, \) and \( \epsilon_1, \epsilon \in \mu_2 = \{ \pm 1 \} \), see Eq.(2-6) of [Sz1]. Recall that \( P \) is the Siegel parabolic subgroup of \( \text{Sp}_{2n}(F) \). Denote \( pr : \widetilde{\text{Sp}}_{2n}(F) \to \text{Sp}_{2n}(F) \) the natural projection.

Let \( K = \text{Sp}_{2n}(O_F) \), which is a maximal open compact subgroup of \( \text{Sp}_{2n}(F) \). It is known that there is a group homomorphism \( s : K \to \widetilde{\text{Sp}}_{2n}(F) \) such that \( pr \circ s = \text{id}_K \), see page 43 of [MVW]. This splitting \( s \) is known to be unique, see page 1662 of [GS] for example. Denote the splitting \( s \) by \( s(k) = (k, \epsilon(k)) \), where \( \epsilon(k) \in \{ \pm 1 \} \). It is easy to see that the splitting over \( K \cap U \) is also unique. In fact, any two such splittings differ by a quadratic character of \( K \cap U \) and it suffices to show that \( K \cap U \) has no nontrivial quadratic character. The latter statement follows from the fact that \( 2 \) is a unit in \( O_F^\times \) and (thus) the square map \( K \cap U \to K \cap U \) is surjective. Since there is a canonical splitting over \( U \) given by \( u \mapsto (u, 1) \), it follows that \( \epsilon(k) = 1 \) for all \( k \in K \cap U \).

Let \( m \) be a positive integer and \( K_m \) be the congruence subgroup \( (1 + \text{Mat}_{2n \times 2n}(F)) \cap \text{Sp}_{2n}(F) \subset K \) as in §2 and let \( \tilde{K}_m \) be the inverse image of \( K_m \) in \( \widetilde{\text{Sp}}_{2n} \). It is clear that \( \tilde{K}_m = s(K_m) \times \mu_2 \) as a group. Using the Iwahori decomposition, one can check that the square map \( \tilde{K}_m \to K_m \) is surjective and hence the splitting \( s \) restricted to \( \tilde{K}_m \) is also unique.

Let \( \psi \) be an unramified additive character of \( F \), recall that we defined a character \( \tau_m \) of \( K_m \) in §2. We now define a character \( \tilde{\tau}_m \) of \( \tilde{K}_m \) by
\[
\tilde{\tau}_m((k, \epsilon(k))) = \epsilon \tau_m(k), k \in K_m, \epsilon \in \{ \pm 1 \}.
\]
Since \( s \) is a group homomorphism, it is clear that \( \tilde{\tau}_m \) is indeed a character of \( \tilde{K}_m \).

Let
\[
d_m = \text{diag}(\omega^{-m(2n-1)}, \omega^{-m(2n-3)}, \ldots, \omega^{-m}, \omega^m, \ldots, \omega^{m(2n-1)}) \in \text{Sp}_{2n}(F)
\]
and \( H_m = d_m K_m d_m^{-1} \) be as in §2 and \( \tilde{d}_m = (d_m, 1) \in \widetilde{\text{Sp}}_{2n}(F) \). Define a group homomorphism \( s' : H_m \to \text{Sp}_{2n}(F) \), by
\[
s'(h) = \tilde{d}_m s(d_m h d_m) \tilde{d}_m^{-1}.
\]
We can check that \( s'(d_m k d_m^{-1}) = (d_m k d_m^{-1}, \epsilon(k)), k \in K_m \). Since \( s' \) is a group homomorphism, we have \( c(d_m k d_m^{-1}, d_m k' d_m^{-1}) = c(k, k') \).

Let \( \tilde{H}_m = \tilde{d}_m \tilde{K}_m \tilde{d}_m^{-1} = s'(H_m) \times \mu_2 \). We define a character \( \tilde{\psi}_m \) on \( \tilde{H}_m \) by
\[
\tilde{\psi}_m(h) = \tilde{\tau}_m(d_m^{-1} h d_m).
\]
If we write \( h = s'(h)(1, \epsilon) \) for \( h \in H_m \) and \( \epsilon \in \mu_2 \), we have
\[
\tilde{\psi}_m(h) = \epsilon \psi_m(h).
\]

Lemma 5.1. We have \( \tilde{\psi}_m|_{\tilde{U}_m} = \psi|_{\tilde{U}_m} \), where \( \tilde{U}_m = \tilde{U} \cap \tilde{H}_m \) and \( \psi \) is the generic character defined in §1, i.e., \( \psi_U(u, \epsilon) = \epsilon \psi_U(u) \).

Proof. By the above discussion, we have \( \epsilon(k) = 1 \) for \( k \in K \cap U \). Since \( \tilde{U}_m = \tilde{H}_m \cap \tilde{U} = \tilde{d}_m (s(K_m \cap U) \times \mu_2) \tilde{d}_m^{-1} \). A typical element \( \tilde{u} \in \tilde{U}_m \) is of the form \( \tilde{d}_m (u, \epsilon) \tilde{d}_m^{-1} = (d_m u d_m^{-1}, \epsilon) \) by Eq.(5.1), where \( u \in K_m \cap U \) and \( \epsilon \in \{ \pm 1 \} \). By the definition, we have \( \tilde{\psi}_m(\tilde{u}) = \tilde{\tau}_m((u, \epsilon)) = \epsilon \tau_m(u) = \epsilon \psi_m(d_m u d_m^{-1}) \), and \( \psi_U(\tilde{u}) = \epsilon \psi_U(d_m u d_m^{-1}) \). Thus it suffices to show that \( \psi_m(d_m u d_m^{-1}) = \psi_U(d_m u d_m^{-1}) \). This is Lemma 2.1 (1).
Note that the above calculation shows that $\tilde{U}_m = U_m \times \mu_2$ as a group and $\tilde{\psi}_m((u, \epsilon)) = \epsilon \psi_m(u)$ for $u \in U_m, \epsilon \in \mu_2$.

Let $(\pi, V_\pi)$ be a genuine irreducible $\psi_0$-generic smooth representation of $\tilde{\text{Sp}}_{2n}$, and $v \in V_\pi$, we can define the Howe vector $v_m$ similarly, i.e.,

$$v_m = \frac{1}{\text{vol}(U_m)} \int_{U_m} \tilde{\psi}_m(u)\psi(u)vd\tilde{u}.$$  

Note that $\tilde{U}_m = U_m \times \mu_2$ as a group, $\tilde{\psi}_m((u, \epsilon)) = \epsilon \psi_m(u)$ and $\pi((u, \epsilon))v = \epsilon \pi((u, 1))v$, for $u \in U_m$ and $\epsilon \in \mu_2$, we get

$$v_m = \frac{1}{\text{vol}(U_m)} \int_{U_m} \psi_m(u)^{-1} \pi((u, 1))vdu.$$  

As in the $\text{Sp}_{2n}$ case, we let $C = C(v)$ be a positive integer such that $v$ is fixed by $\pi(s(K_m))$. Then the analogue of Lemma 5.2 holds:

**Lemma 5.2.** We have

1. $W_{v_m}(1) = 1$;
2. if $m \geq C$, then $\pi(h)v_m = \tilde{\psi}_m(h)v_m$, for all $h \in \tilde{H}_m$;
3. for $k \leq m$, we have

$$v_m = \frac{1}{\text{vol}(U_m)} \int_{U_m} \tilde{\psi}_m(u)\psi(u)v_kdu = \frac{1}{\text{vol}(U_m)} \int_{U_m} \psi_m(u)^{-1} \pi((u, 1))v_kdu.$$  

**Proof.** (1) and (3) are clear and we show (2). Consider the vector

$$\tilde{v}_m = \frac{1}{\text{vol}(H_m)} \int_{H_m} \tilde{\psi}_m(h)^{-1} \pi(h)v_dh.$$  

It is clear that $\pi(h)\tilde{v}_m = \tilde{\psi}_m(h)\tilde{v}_m$. It suffices to show that $\tilde{v}_m = v_m$.

By Eq.(5.4) and the fact that $\pi$ is genuine, we have

$$v_m = \frac{1}{\text{vol}(H_m)} \int_{H_m} \psi_m(h)^{-1} \pi(s'(h))v dh.$$  

From the Iwahori decomposition of $K_m$, we have $H_m = B_m \cup U_m$, where $B_m = B \cap H_m$, $U_m = U \cap H_m$ and $B$ is the lower triangular Borel subgroup of $\text{Sp}_{2n}(F)$. For $h = u\bar{b} \in H_m$ with $u \in U_m, \bar{b} \in B_m$, we choose measures such that $dh = d\bar{b}du$. Thus

$$\tilde{v}_m = \frac{1}{\text{vol}(H_m)} \int_{U_m} \int_{B_m} \psi_m(u\bar{b})^{-1} \pi(s'(u)s'(\bar{b}))v d\bar{b}du.$$  

By the definition of $\psi_m$, we can get $\psi_m(\bar{b}) = 1$. Notice that $\bar{b} \in B_m \subset H_m \cap K_m$, see 2.1 for example. It is easy to see that the square map $H_m \cap K_m \to H_m \cap K_m$ is surjective and thus there is a unique splitting over $H_m \cap K_m$, i.e., $s(\bar{b}) = s'(\bar{b})$. For $m \geq C$, we have $\pi(s'(\bar{b}))v = \pi(s(\bar{b}))v = v$. Thus

$$\tilde{v}_m = \frac{1}{\text{vol}(U_m)} \int_{U_m} \psi_m(u)^{-1} \pi(s'(u))v du$$  

$$= \frac{1}{\text{vol}(U_m)} \psi_m(u)^{-1} \pi((u, 1))vdu$$  

$$= v_m.$$  

This proves (2). \qed

Using relation Eq.(5.1), one can check that all of the results in §2-4 for $\text{Sp}_{2n}$ have corresponding analogue for $\tilde{\text{Sp}}_{2n}$, with similar proof. We only check one of them to illustrate the idea how to modify the proof in the $\text{Sp}_{2n}$ case so that it is adapt to the $\tilde{\text{Sp}}_{2n}$ case.

**Lemma 5.3.** For $t \in T$ the torus of $\text{Sp}_{2n}$ and $m \geq C$. If $\epsilon \in \mu_2$ and $W_{v_m}((t, \epsilon)) \neq 0$, then $\alpha_i(t) \in 1 + P^m$ for $1 \leq i \leq n - 1$ and $\beta(t) \in 1 + P^m$. 
Proof. This is the analogue of Lemma 2.3, and in fact similar proof goes through. For a simple root \( \gamma \), we take \( r \in \mathcal{P}^{-m} \) so that \( x_\gamma(r) \in H_m \). Then \( (x_\gamma(r), 1) \in H_m \). By Eq. (5.1), we have
\[
(t, \epsilon)(x_\gamma(r), 1)(t, \epsilon)^{-1} = (tx_\gamma(r)t^{-1}, 1) = (x_\gamma(\gamma(t)r), 1),
\]
see the proof of Lemma 2.3. By Lemma 5.2, we have
\[
\bar{\psi}_m((x_\gamma(r), 1))W_{v_m}((t, \epsilon)) = \psi_U((x_\gamma(\gamma(t)r), 1))W_{v_m}((t, \epsilon)).
\]
By the definition of \( \bar{\psi}_m \) and \( \psi_U \), if \( W_{v_m}((t, \epsilon)) \neq 0 \), we have \( \psi(r) = \psi(\gamma(t)r) \) for all \( r \in \mathcal{P}^{-m} \). Since \( \psi \) is unramified, we get \( \gamma(t) \in 1 + \mathcal{P}^m \). \( \square \)

Similar consideration as in the \( \text{Sp}_{2n} \) case will give us the stability of gamma factors for \( \tilde{\text{Sp}}_{2n} \), i.e., we have the following

**Theorem 5.4.** Let \( (\pi, V_\pi) \) and \( (\pi', V_{\pi'}) \) be two genuine irreducible smooth \( \psi_U \)-generic representations of \( \text{Sp}_{2n}(F) \) with the same central character, where \( F \) is a \( p \)-adic field such that its residue characteristic is not \( 2 \). If \( \eta \) is a highly ramified quasi-character of \( F^\times \), then
\[
\gamma(s, \pi, \eta \psi) = \gamma(s, \pi', \eta \psi).
\]

6. Stability of \( \gamma \)-factors for \( U_{E/F}(n, n) \)

In this section, we assume \( E/F \) is a quadratic extension of \( p \)-adic fields, and denote \( x \mapsto \bar{x} \) the nontrivial Galois action in \( \text{Gal}(E/F) \). For objects corresponding to \( E \), we will add a subscript \( E \). For example, we denote \( \mathcal{O} \) the integer ring of \( F \) (as in the notation section) and \( \mathcal{O}_E \) the integer ring of \( E \).

The group \( U_{E/F}(n, n) \) is defined by
\[
U_{E/F}(n, n) = \left\{ g \in \text{GL}_2(E) | g \left( \begin{array}{cc} J_n & 0 \\ 0 & J_n \end{array} \right) g = \left( \begin{array}{cc} -J_n & 0 \\ 0 & -J_n \end{array} \right) \right\},
\]
where \( J_n \) is the same as in the notation section. We will use similar notations as in the \( \text{Sp}_{2n} \) case. For example,
\[
m_n(g) = \left( \begin{array}{cc} g & \ast \\ \ast & g^* \end{array} \right), \quad g \in \text{GL}_n(E), \quad g^* = J_n g^{-1} J_n,
\]
\[
m_n(b) = \left( \begin{array}{cc} I_n & b \\ I_n & b \end{array} \right), \quad b \in \text{Mat}_{n \times n}(E), \quad \bar{b} = J_n b J_n,
\]
and
\[
r(y, x) = m_n \left( \begin{array}{ccc} I_{n-2} & y & x \\ 1 & 1 & \end{array} \right), \quad y \in \text{Mat}_{(n-2) \times 1}(E), \quad x \in E.
\]
Let \( M \) and \( N \) be the subgroup which consists elements of the form \( m_n(g) \) and \( n_n(b) \) respectively. Let \( U_M \) be the standard maximal unipotent subgroup of \( M \). Let \( U = U_M \ltimes N \), which is a maximal unipotent subgroup of \( U_{E/F}(n, n) \). Let \( B \) be the standard upper triangular Borel subgroup and \( B = TU \) is the Levi decomposition.

Let \( \psi_E \) (resp. \( \psi \)) be a nontrivial additive character of \( E \) (resp. \( F \)). We consider the generic character \( \psi_U \) on \( U \) defined by
\[
\psi_U|_{U_M} \left( \left( \begin{array}{c} u_{ij} \end{array} \right) \right) = \psi_E \left( \sum_{i=1}^{n-1} u_{i,j+1}, (u_{ij}) \in U_M,
\]
and
\[
\psi_U|_{N} \left( u_{ij} \right) = \psi(u_{n,n+1}, (u_{ij}) \in N.
\]
Let \( w_1 \) be the same as in the \( \text{Sp}_{2n} \) case and define \( j(g) = w_1 g w_1^{-1} \) for \( g \in U_{E/F}(n, n) \).
6.1. Weil representations and induced representations on $U_{E/F}(1,1)$. The group $U_{E/F}(1,1)$ can be viewed as a subgroup of $\text{Sp}_4$. Let $\mu$ be a character of $E^\times$ such that $\mu|_{F^\times}$ is the class field theory character on $F^\times$ defined by $E/F$. Then it is know that $\mu$ defines a splitting $s_\mu : U_{E/F}(1,1) \hookrightarrow \text{Sp}_4$ of the double cover map $\text{Sp}_4 \rightarrow \text{Sp}_4$ over $U_{E/F}(1,1)$. Thus for a nontrivial additive character $\psi$ of $F$, we have a Weil representation $\omega_{\mu,\psi}$ of $U_{E/F}(1,1)$ on the space $S(E)$. For the splitting $s_\mu$ and the Weil representation, see [HKS] for example.

Given a quasi-character $\eta$ of $E^\times$ and $s \in C$, we can consider the induced representation $I(s, \eta) = \text{Ind}_{B^1}^{U_{E/F}(1,1)}(\eta_{s-1/2})$ of $U_{E/F}(1,1)$, where $B^1$ is the upper triangular Borel subgroup of $U_{E/F}(1,1)$. By [Ba2], we can parametrize the space $I(s, \eta)$ using the space $S(F)$, like the $GL_2$ case.

6.2. Local zeta integrals and $\gamma$-factors. For simplicity, we denote $G_n = U_{E/F}(n, n)$. We consider the embedding $\iota : G_1 \hookrightarrow G_n$

$$g \mapsto \begin{pmatrix} I_{n-1} & g \\ & I_{n-1} \end{pmatrix}.$$  

We will not distinguish an element $g \in G_1$ with its image $\iota(g) \in G_n$.

Let $\pi$ be a $\psi_U$-generic representation of $G_n = U_{E/F}(n, n)$, and $\eta$ be a quasi-character of $E^\times$. For $W \in \mathcal{W}(\pi, \psi_U), \phi \in S(E), f \in I(s, \eta), \eta$ a character of $E^\times$, we consider the local zeta integral

$$\Psi(W, \phi, f_s) = \int_{U_1 \backslash G_1} \int_{E(n-2)} \int_E W(j(r(y, x)g))\omega_{\mu, \psi}^{-1}(g)\phi(x)f_s(g)dx dy dg.$$  

There is a standard intertwining operator $M_s : I(s, \eta) \rightarrow I(1-s, \tilde{\eta}^{-1})$.

**Proposition 6.1.** The local zeta integral $\Psi(W, \phi, f_s)$ is absolutely convergent for $\text{Re}(s) >> 0$ and defines a rational function of $q_{E^s}^{-\alpha}$. Moreover, there exists a rational function $\gamma(s, \pi, \eta, \psi)$ such that

$$\Psi(W, \phi, M_s(f_s)) = \gamma(s, \pi, \eta, \mu, \psi)\Psi(W, \phi, f_s),$$  

for all $W \in \mathcal{W}(\pi, \psi_U), \phi \in S(E)$ and all $f_s \in I(s, \eta)$.

**Proof.** The convergence of the local zeta integral is standard, which comes from a standard gauge estimate of $W$. The existence of the $\gamma$ factors comes from the the uniqueness of the Fourier-Jacobi models in the unitary case, see [GGP, Su]. We omit the details. $\square$

**Remark:** The local integrals and the $\gamma$-factors in the unitary group case are analogues in the $Sp_{2n}$ case. But to the author’s knowledge, the local theory in the unitary group case is not studied in the literature.

6.3. Howe vectors. We can define the Howe vectors similarly. We provide a little bit details. Let $K_m = (1 + \text{Mat}_{2n \times 2n}(\mathcal{O}_E)^m))$, where $\mathcal{P}$ is the maximal ideal of $F$ and $\mathcal{O}_E$ is the ring of integers of $E$. Note that if $E/F$ is unramified, then $\mathcal{O}_E = \mathcal{P}^1$. Let $d_m = \text{diag}(\varpi^{-(2n-1)m}, \varpi^{-(2n-3)m}, \ldots, \varpi^{-m}, \varpi^m, \ldots, \varpi^{(2n-3)m}, \varpi^{(2n-1)m})$, where $\varpi$ is a uniformizer in $F$, and define

$$H_m = d_m K_m d_m^{-1}.$$  

Assume $\psi$ and $\psi_E$ are unramified additive characters of $F$ and $E$ respectively. We can define a character $\psi_m$ of $H_m$ similar to the $Sp_{2n}$ case. Let $(\pi, V_\pi)$ be a $\psi_U$-generic irreducible smooth representation of $G_n$. For $v \in V_\pi$ with $W_\pi(1) = 1$, we define

$$v_m = \frac{1}{\text{vol}(U_m)} \int_{U_m} \psi_U(u)^{-1} \pi(u)v du.$$  

Let $C$ be an integer such that $v$ is fixed by $\pi(K_C)$. Then the counterpart of Lemma 2.2 also holds in our case.

The counterpart of Lemma 2.3 becomes

**Lemma 6.2.** Let $m \geq C$ and $t \in T$. If $W_{m_\psi}(t) \neq 0$, then $\alpha(t) \in 1 + (\mathcal{O}_E)^m$ and $\beta(t) = 1 + \mathcal{P}^m$.

Denote $E^1$ the norm 1 elements in $E^\times$. The counterpart of Lemma 2.4 has the following form
Lemma 6.3. Suppose that $E/F$ is unramified, or $E/F$ is ramified but the residue characteristic of $F$ is not 2. For $a \in E^\times$, if $aa \in 1 + \mathcal{P}_E^m$, then $a \in E^1(1 + (\mathcal{P}_E)_E^m)$.

Proof. If $E/F$ is unramified, the result follows from the fact that the norm map $1 + \mathcal{P}_E^m \to 1 + \mathcal{P}_E^m, a \mapsto aa$ is surjective, see Proposition 3, Chapter V, §2 of [Se]. If $E/F$ is ramified, it needs a little bit more work. See Lemma 3.3 of [Zh1] for more details. \qed

Now we fix two $\psi_U$-generic irreducible smooth representations $(\pi, V_\pi)$ and $(\pi', V_{\pi'})$ with the same central character. We fix $v \in V_\pi, v' \in V_{\pi'}$ such that $W_v(1) = W_{v'}(1) = 1$, and an integer $C$ such that $v$ is fixed by $\pi(K_C)$ and $v'$ is fixed by $\pi'(K_C)$.

The counterpart of Lemma 3.12 also holds:

Lemma 6.4. (1) If $E/F$ is unramified, or $E/F$ is ramified but the residue characteristic of $F$ is not 2, then

$$W_{v_m}(g) = W_{v_m}(g),$$

for all $g \in B, m \geq C$.

(2) We have $W_{v_m}(tw) = 0 = W_{v'_m}(tw)$ for all $w < w_0; t \in T$ and $m \geq C$.

Proof. Notice that $T_m = T \cap H_m = \text{diag}(1 + (\mathcal{P}_E)_E^m, \ldots, 1 + (\mathcal{P}_E)_E^m)$. A simple calculation as in the Sp$_{2n}$ case shows (1) following Lemma 6.2 and Lemma 6.3.

The proof of (2) is the same as in the Sp$_{2n}$ case. \qed

Note that Theorem 3.11 and hence Corollary 3.13 also holds in the unitary group case. The same calculation as in §4 will show the stability of $\gamma$-factors in the unitary case. More precisely, we have

Theorem 6.5. Suppose that $E/F$ is unramified, or $E/F$ is ramified but the residue characteristic of $F$ is not 2. Let $\pi, \pi'$ be two $\psi_U$-generic irreducible smooth representations of $U_E/F(n, n)$. Then if $\eta$ is a highly ramified quasi-character of $F^\times$, we have

$$\gamma(s, \pi, \eta, \mu, \psi) = \gamma(s, \pi', \eta, \mu, \psi).$$

References

[Ba1] E. M. Baruch, Local factors attached to representations of $p$-adic groups and strong multiplicity one, Thesis, Yale University, 1995.

[Ba2] E. M. Baruch, On the Gamma factors attached to representations of $U(2, 1)$ over a $p$-adic field, Israel Journal of Math. 102 (1997), 317-345.

[ChZh] Jingsong Chai, and Qing Zhang, A strong multiplicity one theorem for SL(2), preprint, available at: http://arxiv.org/abs/1511.00354.

[CKPSS] J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi, Functoriality for the classical groups, Publ. Math. Inst. Hautes Études Sci. 99 (2004), 163-233.

[CPS] J. W. Cogdell, I. I. Piatetski-Shapiro, Stability of gamma factors for SO(2n + 1), manuscipta math. 95 (1998), 437-461.

[CPS] J. W. Cogdell, I. I. Piatetski-Shapiro, and F. Shahidi, Stability of gamma factors for quasi-split groups, J. Inst. Math. Jussieu 7 (2008), 27-66.

[CST1] J. W. Cogdell, F. Shahid, and T-L. Tsai, Local Langlands correspondence for GL$_m$ and the exterior and symmetric square $\varepsilon$-factors, preprint, available at http://arxiv.org/pdf/1412.1448.pdf.

[CST2] J. W. Cogdell, F. Shahid, and T-L. Tsai, On Stability of Root Numbers, Automorphic Forms and Related Geometry: Assessing the Legacy of I.I. Piatetski-Shapiro. AMS Contemporary Math., 2014.

[GGP] W. Gan, B. Gross and D. Prasa, Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups, Astérisque, 346, (2012), 1-109.

[GS] W. Gan, G. Savin, Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence, Compositio Mathematica, 148 (2012), 1655-1694.

[GePS1] S.Gelbart and I.Piatetski-Shapiro, Distinguished Representations and Modular forms of half-integral weight, Inventiones Math. 59 (1980), 145-188.

[GePS2] Gelbart, Piatetski-Shapiro, L-functions for $G \times GL(n)$, in “Explicit Constructions of Automorphic $L$-functions”, Springer Lecture Notes in Mathematics, 1254, (1987).

[GeRS] Gelbart, Rogawski, Soudry, Endoscopy, Theta-Liftings, and Period Integrals for the Unitary Group in Three Variables, Annals of Mathematics, 145, (1997), 419-476.

[GiRS1] D. Ginzburg, S. Rallis and D. Soudry, Periods, poles of $L$-functions and symplectic-orthogonal theta liftings, J. reine angew. Math. 487 (1997), 85-114.

[GiRS2] D.Ginzburg, S.Rallis and D.Soudry, L-functions for symplectic groups, Bull.Soc.Math.France, 126, (1998) 181-244.
[HKS] M. Harris, S. Kudla, W. Sweet, *Theta Dichotomy for Unitary Groups*, Journal of the American Mathematical Society, 9, (1996), 961-1004.

[Hu] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge studies in Advanced Mathematics, 29 (1990), Cambridge University Press.

[JL] H. Jacquet and R. Langlands, *Automorphic Forms on GL(2)*. Lecture Notes in Mathematics, 114, Springer-Verlag, New York, 1970.

[JS] H. Jacquet and J. Shalika, *A lemma on highly ramified $\varepsilon$-factors*, Math. Ann. 271 (1985), 319-332.

[Ka] Eyal Kaplan, *Complementary results on the Rankin-Selberg gamma factors of classical groups*, Journal of Number Theory, 146 (2015), 390-447.

[KK] Henry H. Kim and Muthukrishnan Krishnamurthy, *Stable base change lift from unitary groups to GL_N*, IMRP, 1 (2005), 1-52.

[La] S. Lang, *Algebraic Number Theory*, Second Edition, Graduate Texts in Mathematics, 110, Springer-Verlag.

[MVW] Moeglin, Vignéras, Waldspurger, *Correspondances de Howe sur un corps $p$-adiques*, Lecture Notes in Mathematics 1291, Springer-Verlag, 1987.

[Rao] R.R. Rao, *On some explicit formulas in the theory of Weil representation*, Pacific Journal of Math, 157 (1993), 335-371.

[Se] J-P. Serre, *Local Fields*, Graduate Texts in Mathematics, 67, Springer, 1979.

[Sh] F. Shahidi, *Local coefficients as Mellin transforms of Bessel functions: towards a general stability*, Int. Math. Res. Not. 2002 (2002), 2075-2119.

[St] Robert Steinberg, *Lectures on Chevalley groups*, Yale University, 1967.

[Su] B. Sun, *Multiplicity one theorems for Fourier-Jacobi models*, American Journal of Mathematics 134 (2012), 1655-1678.

[Sz1] Dani Szpruch, *Uniqueness of Whittaker model for the metaplectic group*, Pacific Journal of Mathematics, 232, (2007), 453-469.

[Sz2] Dani Szpruch, *Computation of the local coefficients for principal series representations of metaplectic double cover of SL_2(F)*, Journal of Number Theory 129, (2009), 2180-2213.

[Zh1] Qing Zhang, *A local converse theorem for U(1, 1)*, preprint, available at http://arxiv.org/abs/1508.07062

[Zh2] Qing Zhang, *A local converse theorem for U(2, 2)*, preprint, available at http://arxiv.org/abs/1509.00900

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