ON ENDOMORPHISMS OF THE EINSTEIN GYROGROUP
IN ARBITRARY DIMENSION

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Abstract. We determine the automorphisms and the continuous
endomorphisms of the Einstein gyrogroup in arbitrary dimension. This
generalizes a recent result of L. Molnár and D. Virosztek, who have de-
dtermined the continuous endomorphisms in the three-dimensional case.

1. The Einstein gyrogroup

The $n$-dimensional Einstein gyrogroup is the open unit ball $B^n$ in $\mathbb{R}^n$,
endowed with the binary operation of velocity addition from the special
theory of relativity (with the speed of light taken to be 1):

$$u \oplus v = \frac{1}{1 + (u, v)} \left( u + \sqrt{1 - |u|^2} \cdot v + \frac{(u, v)}{1 + \sqrt{1 - |u|^2}} \cdot u \right).$$

Here $(u, v)$ is the inner product of $u$ and $v$, and $|u| = \sqrt{(u, u)}$ is the usual
Euclidean norm.

Note that $|u \oplus v| < 1$ if $|u| < 1$ and $|v| < 1$, so $(B^n, \oplus)$ is an algebraic
structure. It satisfies certain axioms that make it a gyrogroup [7]. We shall
not need all of the axioms, but let us observe that $u \oplus 0 = 0 \oplus u = u$
and $u \oplus (-u) = 0$ for all $u \in B^n$. The operation $\oplus$ is not associative, but
$(-u) \oplus (u \oplus v) = v$ holds for all $u$ and $v$ in $B^n$.

The Einstein gyrogroup is closely related to hyperbolic geometry. If we
think of $B^n$ as the Cayley–Klein–Beltrami model of hyperbolic $n$-space, then
the map $v \mapsto u \oplus v$ is an isometry of hyperbolic $n$-space for any fixed $u$. When
$u \neq 0$, this isometry maps the halfline starting at 0 and passing through $u$
(henceforth referred to as halfline $0u$) onto its sub-halfline starting at $u$.

This implies a well-known fact about commutativity in the Einstein gy-
rogroup:

Proposition 1. Let $u, v \in B^n$. Then the equality $u \oplus v = v \oplus u$ holds if and
only if $u$ and $v$ are linearly dependent (in the usual sense of vector algebra
in $\mathbb{R}^n$).

Proof. If $u$ and $v$ are linearly dependent, then they belong to a diameter
of the ball $B^n$. This diameter represents a line $L$ in hyperbolic space. The
$L \to L$ maps $w \mapsto u \oplus w$ and $w \mapsto v \oplus w$ are translations of $L$, so they
commute. Hence, $u \oplus v = u \oplus (v \oplus 0) = v \oplus (u \oplus 0) = v \oplus u$ as claimed.

If $u$ and $v$ are linearly independent, then they span a two-dimensional
plane, which intersects $B^n$ in a disc. This disc represents a hyperbolic plane.

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Hyperbolic isometries preserve angles. Thus, the halfline \( u(u \oplus v) \) forms the same angle with the halfline \( 0u \) as \( 0v \) does. Hence,
\[
\angle 0u(u \oplus v) = \pi - \angle u0v.
\]
Similarly,
\[
\angle 0v(v \oplus u) = \pi - \angle u0v.
\]
If, by way of contradiction, we have \( u \oplus v = v \oplus u = w \), then corresponding sides of the triangles \( u0v \) and \( vuw \) have equal length, making the two triangles congruent and implying
\[
\angle u0v = \angle vuw.
\]
But then the four angles of the quadrilateral \( u0vw \) sum to \( 2\pi \), which is impossible in the hyperbolic plane. □

**Corollary 2.** The points \( x, y, z \in \mathbb{B}^n \) are collinear if and only if
\[
((x) \oplus y) \oplus ((x) \oplus z) = ((x) \oplus z) \oplus ((x) \oplus y).
\]

*Proof.* In the Cayley–Klein–Beltrami model, lines of hyperbolic space are represented by chords of the ball \( \mathbb{B}^n \). Thus, \( x, y, z \) are collinear in the ordinary sense of Euclidean geometry if and only if they are collinear as points of hyperbolic space.

The map \( w \mapsto (x) \oplus w \) is an isometry of hyperbolic space, so it preserves collinearity. Thus \( x, y \) and \( z \) are collinear if and only if \( 0, (x) \oplus y \) and \( (x) \oplus z \) are. The claim now follows from Proposition 1. □

### 2. Endomorphisms and Automorphisms

An endomorphism of the \( n \)-dimensional Einstein gyrogroup is a map \( f : \mathbb{B}^n \rightarrow \mathbb{B}^n \) such that
\[
f(u \oplus v) = f(u) \oplus f(v)
\]
for all \( u, v \in \mathbb{B}^n \). An automorphism is a bijective endomorphism.

Note that any endomorphism \( f \) satisfies \( f(0) = 0 \). Indeed, \( 0 \oplus 0 = 0 \), whence \( f(0) \oplus f(0) = f(0) = f(0) \oplus 0 \). But \( v \mapsto f(0) \oplus v \) is bijective, so \( f(0) = 0 \).

When \( n = 1 \), the Einstein gyrogroup is a group. It is isomorphic to the additive group \((\mathbb{R}, +)\) of real numbers. Endomorphisms of this group have been extensively studied, they go under the name of additive functions. Most of them are non-continuous. Moreover, most of the automorphisms of \((\mathbb{R}, +)\) are also non-continuous. In fact, the continuous endomorphisms are precisely the linear functions \( x \mapsto ax : \mathbb{R} \rightarrow \mathbb{R} \) with fixed \( a \in \mathbb{R} \), and there are many further automorphisms, let alone endomorphisms.

Henceforth, we assume \( n \geq 2 \).

**Theorem 3.** For \( n \geq 2 \), automorphisms of the Einstein gyrogroup \((\mathbb{B}^n, \oplus)\) are precisely the restrictions to \( \mathbb{B}^n \) of the orthogonal transformations of \( \mathbb{R}^n \).

*Proof.* Orthogonal transformations of \( \mathbb{R}^n \) preserve the inner product and therefore the Euclidean norm, so they map \( \mathbb{B}^n \) bijectively onto itself and satisfy \( f(0) = 0 \) for all \( u \) and \( v \).

Conversely, if \( f \) is an automorphism, then so is its inverse \( f^{-1} \). By Corollary 2, both \( f \) and \( f^{-1} \) map collinear points to collinear points. I.e., \( f \) — as
a self-map of hyperbolic space — maps any line onto a line. In other words, 
f is a collineation of hyperbolic space. By a well-known result sometimes
referred to as the fundamental theorem of hyperbolic geometry [11, 3, 6, 5],
y any collineation is an isometry for \( n \geq 2 \). So \( f \) is an isometry. It is well
known that in the Cayley–Klein–Beltrami model, any isometry of hyperbolic
\( n \)-space fixing 0 is represented by the restriction of an orthogonal transfor-
mation.

We now turn to endomorphisms. We urge the reader to solve

**Problem 4.** For \( n \geq 2 \), is every endomorphism of the Einstein gyrogroup
continuous?

Meanwhile, we wish to classify continuous endomorphisms. For \( n = 3 \),
which is the most relevant to physics, this was done by L. Molnár and D.
Virozskét [5], while the general case was posed by them as an open problem.
Their result relies on a chain of reinterpretations of \((\mathcal{B}_3, \oplus)\). The first step in
the chain is an observation of S. Kim [2]: \((\mathcal{B}_3, \oplus)\) is bicontinuously isomorphic
to \((\mathbb{D}, \circ)\), where \( \mathbb{D} \) is the set of 2-square regular density matrices and \( A \circ B \)
is \( \sqrt{AB} \sqrt{A} \) divided by its trace. Molnár and Virozskét show that this in-
turn is bicontinuously isomorphic to \((\mathbb{P}_2, \boxtimes)\), where \( \mathbb{P}_2 \) is the set of 2-square
positive definite matrices with determinant 1, and \( A \boxtimes B = \sqrt{AB} \sqrt{A} \). Then
they invoke a result from their previous paper [4, Theorem 1] and deduce
from it the classification of the continuous endomorphisms of \((\mathbb{P}_2, \boxtimes)\).

From Theorem 3 of the present paper, using the bicontinuous iso-
morphisms mentioned above (but in the opposite direction), we infer

**Corollary 5.** Every automorphism of \((\mathbb{D}, \circ)\) or \((\mathbb{P}_2, \boxtimes)\) is continuous.

Turning to arbitrary dimension, we have

**Theorem 6.** For \( n \geq 2 \), continuous endomorphisms of the Einstein gy-
rogroup \((\mathcal{B}^n, \oplus)\) are precisely the restrictions to \( \mathbb{B}^n \) of orthogonal transfor-
mations of \( \mathbb{R}^n \) and the map that sends everything to 0.

For \( n = 3 \), this recovers the classifications of continuous endomorphisms
of \((\mathcal{B}^3, \oplus), (\mathbb{D}, \circ)\) and \((\mathbb{P}_2, \boxtimes)\) given by Molnár and Virozskét in [5].

**Proof.** It is clear that orthogonal transformations and the identically zero
map are continuous endomorphisms.

Conversely, let \( f \) be an arbitrary continuous endomorphism.

If \( f \) is injective, then it is an open map, so its image contains a neighbour-
hood of 0. But this neighbourhood generates \( \mathbb{B}^n \) under \( \oplus \), and the image
of \( f \) is closed under \( \oplus \), so \( f \) must be surjective, i.e., \( f \) is an automorphism.
The claim now follows from Theorem 3

If \( f \) is not injective, then we have a pair \( u \neq v \) with \( f(u) = f(v) = f(u \oplus ((-u) \oplus v)) = f(u) \oplus f((-u) \oplus v) \). Let \( x = (-u) \oplus v \), then \( f(x) = 0 \)
but \( x \neq 0 \). The diameter \( L \) passing through \( x \) is a subgroup isomorphic to
\((\mathbb{R}, +)\). We may choose an isomorphism such that \( x \) corresponds to 1. It is
easy to see that \( f(y) = 0 \) for every point of the diameter \( L \) that corresponds
to a rational number. But then, by continuity, \( f(y) = 0 \) for all \( y \) on the
diameter \( L \). Thus, \( f \) is constant on sets of the form \( a \oplus L \) and \( L \oplus b \).
The former sets are lines in hyperbolic \( n \)-space, i.e., chords of the ball \( \mathbb{B}^n \).
The chord \(a \oplus L\) passes through \(a\) and is parallel to \(L\) if \(a\) is orthogonal to \(L\). The latter sets, when \(b \notin L\), are hypercycles in hyperbolic \(n\)-space, or half-ellipses in \(\mathbb{B}^n\). The half-ellipse \(L \oplus b\) connects the two ends of its major axis \(L\) and passes through \(b\). It follows that \(f\) is constant on any two-dimensional open half-disk whose boundary diameter is \(L\). By continuity, \(f = 0\) everywhere.

\[\begin{align*}
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\text{References} \\
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