Elastic scattering of protons and their structure

I.M. Dremin
Lebedev Physical Institute, Moscow 119991, Russia
E-mail: dremin@td.lpi.ru

Abstract. The experimental data about elastic scattering of protons and their theoretical interpretations are briefly reviewed. The time delay between the conference and preparation of the manuscript allowed me to modify it and include more recent important results. Some modern problems of elastic scattering of hadrons newly revealed by experimental data obtained at the LHC are discussed. Among them are the curious findings about the behavior of real and imaginary parts of the elastic scattering amplitude at non-forward scattering. The comparison with experiment at LHC energies shows that their ratio in this region can be drastically different from its values measured at low transferred momenta. Also, it is shown how the shape and the darkness of the interaction region of colliding protons change with increase of their energies. In particular, the collisions become fully absorptive at small impact parameters at LHC energies that results in some special features of inelastic processes. These problems ask for further careful treatment in a wide energy interval.

1. Few words about Alexei Kaidalov
Aliosha Kaidalov was a talented and generous person highly respected by physics community. He left important traces in high energy physics, especially, in the field of inelastic (in particular, diffractive) processes and elastic scattering of hadrons.

It is a honour for me to give a talk to his memory.

2. Introduction
Elastic scattering of protons is one of the sources to learn the strong interaction forces which were the main object of studies of Aliosha Kaidalov (e.g., see the papers [1, 2, 3]). This particular topic has been reviewed recently in my paper [4]. To shorten the presentation, I omit many details contained in there and presented during the talk but concentrate here on two problems which became quite actual nowadays. Namely, I’ll discuss the behavior of the elastic scattering amplitude at non-forward direction and our knowledge of the shape and opacity of the interaction region of two colliding protons.

3. Elastic scattering
At the first sight, the experimental data about elastic scattering seem to look very simplified. The only information about this process consists of the measurement of the differential cross section of the process related to the scattering amplitude $f(s,t)$ in a following way

$$\frac{d\sigma}{dt} = |f(s,t)|^2.$$
Fig. 1. The differential cross section of elastic proton-proton scattering at $\sqrt{s} = 7$ TeV measured by the TOTEM collaboration [5]. The region of the diffraction cone with the $|t|$-exponential decrease is shown.

It is a function of two variables: $s = 4E^2$, where $E$ is the energy of protons in the center of mass system, and the four-momentum transfer squared $-t = 2p^2(1 - \cos \theta)$ with $\theta$ denoting the scattering angle and $p$ the momentum in the center of mass system. The amplitude $f$ is normalized at $t = 0$ by the optical theorem such that

$$\text{Im}f(s, 0) = \sigma_t/\sqrt{16\pi}. \quad (2)$$

Moreover, one can find out the real part of the amplitude at very small $t$ (practically at $t = 0$) from experimental data using the interference of the Coulomb and nuclear contributions to $f$. Even though the real and imaginary parts are, in general, related at any $t$ by the dispersion relations as parts of a single analytic function, this treatment asks for some assumptions about their energy dependence, and the conclusions strongly depend on them.

Thus, from experiment, we get the knowledge only about the modulus of the amplitude at the available values of $s$ and $t$ and about the real and imaginary parts separately just in forward direction $t = 0$ but not at any other values of $t$. The theoretical approaches differ in ascribing different roles for their relative contributions at $t \neq 0$. Unfortunately, the available tools are rather moderate and can not exploit the power of QCD at full strength. Mostly, the phenomenological models and some insights from the unitarity relation are used.

The typical shapes of the differential cross section at small and larger values of $|t|$ are demonstrated in Figs 1 and 2. The most prominent feature of these plots is the fast decrease of the differential cross section with increasing transferred momentum $|t|$. As a first approximation at present energies, it can be described at comparatively small transferred momenta by the exponential shape with the slope $B$ such that

$$\frac{d\sigma}{dt} \propto \exp(-B|t|). \quad (3)$$

This region is called the diffraction peak. Its slope $B$ increases with energy approximately as $\ln s$. Moreover, it slightly depends on $t$ at more careful fits of experimental data as seen in Fig.
The differential cross section of elastic proton-proton scattering at \( \sqrt{s} = 7 \) TeV measured by the TOTEM collaboration [6].

The region beyond the diffraction peak is shown. The predictions of five models are demonstrated.

1. At larger values of \(|t|\) outside the peak we observe the dip and slower decrease of the plots (see Fig. 2).

We also know from experiment the energy behavior of real and imaginary parts of the amplitude (or their ratio \( \rho(s, 0) = \rho_0 \)) in forward direction \( t = 0 \). At high energies it is rather small (equal to about 0.1 as measured at 7 TeV). Most phenomenological models try to fit namely these features in a wide energy range. Up to now, we can not claim that the desired aim has been achieved as seen, in particular, in Fig. 2 where the failure of predictions of five theoretical models is shown. This situation is described in more details in the review paper [4]. Actually, my talk at the conference followed it quite closely and I will not repeat it here.

Instead, I concentrate on the following two important problems. First, we discuss what we can say about the behavior of real and imaginary parts of the elastic scattering amplitude at any value of the transferred momentum and not only for forward scattering at \( t = 0 \). Second, we discuss what we can say about the space-time region of the interaction of the two high energy protons.

The first problem can be approached directly in the \( s, t \)-variables discussed above. It was shown a long ago [7, 8] that the imaginary part of the amplitude \( f \) outside the diffraction cone can be derived from the general unitarity condition which is reduced there to the linear integral equation

\[
\text{Im} f(p, \theta) = \frac{p \sigma_t}{4 \pi \sqrt{2} \pi B} \int_{-\infty}^{+\infty} d\theta_1 \exp(-Bp^2(\theta - \theta_1)^2/2)g_\rho \text{Im} f(p, \theta_1) + G(p, \theta),
\]

where \( g_\rho = 1 + \rho(s, 0)\rho(s, \theta_1) \) and \( G(p, \theta) \) is the overlap function.

Let us assume that the overlap function is negligible at these transferred momenta\(^1\). Then,
the eigensolution of the homogeneous equation is

\[ \text{Im} f(p, \theta) = C_0 \exp \left( -\sqrt{2B \ln \left| \frac{Z}{g_\rho} \right| p\theta} \right) + \sum_{n=1}^{\infty} C_n \exp(-\text{Re}(b_n)p\theta) \cos(|\text{Im} b_n| p\theta - \phi_n) \] (5)

with

\[ Z = \frac{4\pi B}{\sigma_t}, \] (6)

\[ b_n \approx \sqrt{2\pi B|n|}(1 + i\text{sign} n) \quad n = \pm 1, \pm 2, \ldots \] (7)

This expression contains the exponentially decreasing with \( \theta \) (or \( \sqrt{|t|} \)) term (Orear regime!) with imposed on it oscillations strongly damped by their own exponential factors. These oscillating terms are responsible for the dip. Namely this formula was used in Refs [10, 11] for fits of experimental data in a wide energy range. The ratio \( \rho \) was approximated by its average values in and outside the diffraction cone so that \( g_{\rho} = 1 \pm \rho_0 \rho_l \) where \( \rho_l \) is treated as the average value of \( \rho \) in the Orear region. The fits at comparatively low energies [10] are consistent with \( g_\rho \approx 1 \), i.e., with small values of \( \rho_l \) close to zero. When \( Z = 1 \), as it happens at 7 TeV (see the Table below), the slope depends only on \( \rho \) and is very sensitive to its value. For the first time, that allowed to estimate the value of \( \rho \) in the Orear region at 7 TeV [11]. The great surprise of the fit of TOTEM data was necessity to use large in modulus negative value of \( \rho_l \approx -2.1 \) if \( \rho_0 = 0.14 \). It becomes larger in modulus \( \rho_l \approx -3 \) if the TOTEM value \( \rho_0 = 0.1 \) is used. Moreover, these values of \( \rho_l \) can be considered as upper limits because the effective value of \( \rho_0 \) inside the diffraction cone can be even smaller in view of its widely discussed zero there. No models have yet explained this finding. Further progress in solving the unitarity equation with proper dependence of \( \rho \) in and outside the diffraction cone is needed.

4. The structure function of proton

Let us come now to our second task. The structure of protons is one of the main problems in particle physics. It is successfully studied in electron-proton collisions. The point-like nature is ascribed to the colliding electron. The parton distribution functions of protons depending on the total energy and the virtuality of the exchanged photon are determined from experimental data. It is well known that they evolve with energy, i.e. the parton content of the proton evolves also. However, we do not get any knowledge about the space-time evolution of the interaction region.

In proton-proton collisions, both objects possess some complicated internal structure. The partons of one of them can interact with many partons from another one distributed somehow within some space volume. Therefore, it is hard to disentangle the individual contributions. Nevertheless, it becomes possible to study the space structure of the interaction region of the two protons formed by the combination of these effects. Here, we show what parameters obtained from experimental data about elastic scattering of protons influence such properties of this region as its size and opacity as well as its evolution with collision energy.

To define the geometry of the collision we must express all characteristics in terms of the transverse distance between the centers of the colliding protons called the impact parameter \( b \). It is easily done by the Fourier – Bessel transform of the amplitude \( f \) written as

\[ i\Gamma(s, b) = \frac{1}{2\sqrt{\pi}} \int_0^{\infty} dt |f(s, t)| J_0(b\sqrt{|t|}). \] (8)

The unitarity condition in the \( b \)-representation reads

\[ G(s, b) = 2\text{Re}\Gamma(s, b) - |\Gamma(s, b)|^2. \] (9)
The left-hand side (the overlap function) describes the impact-parameter profile of inelastic collisions of protons. It satisfies the inequalities \(0 \leq G(s, b) \leq 1\) and determines how absorptive is the interaction region depending on the impact parameter (with \(G = 1\) for full absorption). If integrated over the impact parameter, it leads to the cross section of inelastic processes. The terms on the right-hand side would produce the total cross section and the elastic cross section, respectively.

Also, the amplitude \(f(s, t)\) may be connected to the eikonal phase \(\delta(s, b)\) and to the opaqueness (or blackness \(\Omega(s, b)\)) at the impact parameter \(b\) by the Fourier – Bessel transformation

\[
f(s, t = -q^2) = \frac{1}{2i\sqrt{\pi}} \int d^2 b \exp(iqb)[\exp(2i\delta(s, b) - 1) = i\frac{1}{2\sqrt{\pi}} \int d^2 b \exp(iqb)[1 - \exp(-\Omega(s, b))].
\]

The integration is over the two-dimensional space of the impact parameter.

While the imaginary part of the amplitude in forward direction is defined by the relation (2), its real part can only be measured from interference with the Coulombic part of the amplitude. At high energies, its ratio to the imaginary part \(\rho(s, 0) = \rho_0\) happens to be rather small. Moreover, there are some theoretical arguments that this ratio is even smaller within the diffraction cone. The contribution from the diffraction peak prevails in Eq. (8). Therefore, in first approximation, the second term in Eq. (9) can be replaced by \([\text{Re} \Gamma(s, b)]^2\). Respectively, one can assume that \(\Omega(s, b)\) is real and use Eqs (2), (10) to get the following expressions for the total cross section

\[
\sigma_t = 4\pi \int_0^R [1 - \exp(-\Omega(s, b))] bdb,
\]

for the elastic cross section

\[
\sigma_{el} = 2\pi \int_0^R [1 - \exp(-\Omega(s, b))]^2 bdb,
\]

and for the inelastic cross section

\[
\sigma_{in} = 2\pi \int_0^R [1 - \exp(-2\Omega(s, b))] bdb.
\]

The black disk limit corresponds to \(\Omega = \infty\), i.e. \(\sigma_t = 2\sigma_{el} = 2\sigma_{in}\).

Using the above formulae, one can write the dimensionless \(\Gamma\) as

\[
i\Gamma(s, b) = \frac{\sigma_t}{8\pi} \int_0^\infty d|t| \exp(-B|t|/2)(i + \rho(s, t))J_0(b\sqrt{|t|}).
\]

Here, the diffraction cone approximation (3) is inserted. Herefrom, one calculates

\[
\text{Re} \Gamma(s, b) = \frac{1}{Z} \exp(-\frac{b^2}{2B}).
\]

This dependence on the impact parameter was used, in particular, in [12]. The differential cross section is quite small outside the diffraction peak and does not influence the impact parameter profile \(G\). Therefore, our second task happens to be practically independent of the first problem.

As was mentioned, the ratio \(\rho(s, t)\) is very small at \(t = 0\) and, at the beginning, we neglect it and get

\[
G(s, b) = \frac{2}{Z} \exp(-\frac{b^2}{2B}) - \frac{1}{Z^2} \exp(-\frac{b^2}{B}).
\]
The energy behavior of $Z$ and $G(s,0)$.

| $\sqrt{s}$, GeV | 2.70 | 4.11 | 4.74 | 7.62 | 13.8 | 62.5 | 546 | 1800 | 7000 |
|-----------------|------|------|------|------|------|------|-----|------|------|
| $Z$             | 0.64 | 1.02 | 1.09 | 1.34 | 1.45 | 1.50 | 1.20 | 1.08 | 1.00 |
| $G(s,0)$        | 0.68 | 1.00 | 0.993| 0.94 | 0.904| 0.89 | 0.97 | 0.995| 1.00 |

For central collisions with $b = 0$ it gives

$$G(s, b = 0) = \frac{2Z - 1}{Z^2}. \quad (17)$$

Thus, the darkness of the central region is fully determined by the ratio $Z$. It becomes completely absorptive [$G(s,0) = 1$ or $\Omega(s,0) = \infty$] in the center only at $Z = 1$ and diminishes for other values of $Z$. In the Table, we show the energy evolution of $Z$ and $G(s,0)$ for $pp$ and $p\bar{p}$ scattering.

The impact parameter distribution of the blackness $G(s, b) = 1 - \exp(-2\Omega(s, b))$ in Eq. (16) has the maximum at $b_m^2 = -2B\ln Z$ with full absorption $G(b_m) = 1$ or $\Omega(s, b_m) = \infty$. Its position depends both on $B$ and $Z$. Note, that, for $Z > 1$, one gets the incomplete absorption $G(s, b) < 1$ at any physical $b \geq 0$ with the largest value reached at $b = 0$ because the maximum appears at non-physical values of $b < 0$. The disk is semi-transparent. At $Z = 1$, the maximum is positioned exactly at $b = 0$, and $G(s,0) = 1$. The disk becomes black in the center. At $Z < 1$, the maximum shifts to positive physical impact parameters. The dip is formed at the center that leads to the concave shape of the interaction region. It becomes deeper at smaller $Z$. The limiting value $Z = 0.5$ leading to the complete transparency at the center $b = 0$ is considered in more details below.

The maximum absorption in central collisions $G(s,0) = 1$ is reached at the critical point $Z = 1$ which is the case at $\sqrt{s} = 7$ TeV considered first. Moreover, the strongly absorptive core of the interaction region grows in size as we see from expansion of Eq. (16) at small impact parameters:

$$G(s, b) = \frac{1}{Z^2}[2Z - 1 - \frac{b^2}{B}(Z - 1) - \frac{b^4}{4B^2}(2 - Z)]. \quad (18)$$

The second term proportional to $b^2$ vanishes at $Z = 1$, and $G(b)$ develops a plateau which extends to quite large values of $b$ about 0.4 - 0.5 fm. The plateau is very flat because the third term starts to play any role at 7 TeV (where $B \approx 20$ GeV$^{-2}$) only at even larger values of $b$. The structure of the interaction region with a central core at energies 7 - 8 TeV is also supported (see Fig. 3) by direct computation [13] using the experimental data of the TOTEM collaboration [5, 6] about the differential cross section in the region of $|t| \leq 2.5$ GeV$^2$. The results of analytical calculations and the computation practically coincide (see Fig. 1 in [14]). It was also shown in [14] that this two-component structure with the central black core and more transparent periphery is well fitted by the expression with the abrupt (Heaviside-like) change of the exponential. However, it is still pretty far from the black disk limit because the peripheral region at $b$ near 1 fm is very active and shows strong increase compared to ISR energies [13]. The diffraction cone contributes mostly to $G(s,b)$. Therefore, the large-$|t|$ elastic scattering can not serve as an effective trigger of the black core even though some models were proposed (see, e.g., [15, 16]) which try to elaborate some predictions.

Inelastic exclusive processes can be effectively used for this purpose. One needs such triggers which enhance the contribution due to the central black core. Following the suggestions of [12, 17], it becomes possible [14] to study the details of the central core using the experimental data of CMS collaboration at 7 TeV about inelastic collisions with high multiplicity triggered by the jet production [18] as well as some other related data. Usage of very high multiplicity events...
in combination with jet properties is crucial. Separating the core contribution with the help of these triggers, one comes to the important conclusion that the simple increase of the geometrical overlap area of the colliding protons does not account for properties of jet production at very high multiplicities. It looks as if the parton (gluon) density must strongly increase in central collisions and rare configurations (fluctuations) of the partonic structure of protons are involved.

It is interesting that the positivity of $G(s, b)$ imposes some limits on the relative role of $B$ and $\sigma_t$. Namely, it follows from Eq. (17) that

$$2Z = \frac{8\pi B}{\sigma_t} \geq 1.$$  \hspace{1cm} (19)

This relation implies that the slope $B$ should increase asymptotically at least as strong as the total cross section $\sigma_t$. This inequality must be fulfilled even at intermediate energies.

It is usually stated that the equality $2Z = 8\pi B/\sigma_t = 1$ corresponds to the black disk limit with equal elastic and inelastic cross sections $\sigma_{el} = \sigma_{in} = 0.5\sigma_t$. However, one sees that $G(s, b = 0) = 0$, i.e. the interaction region is completely transparent in central collisions. This paradox is resolved if we write the inelastic profile of the interaction region using Eq. (16). At $Z = 0.5$ it looks like

$$G(s, b) = 4[\exp(-\frac{b^2}{2B}) - \exp(-\frac{b^2}{B})].$$  \hspace{1cm} (20)

Recalling that $B = R^2/4$, we see that one should rename the black disk as a black torus (or a black ring) with full absorption $G(s, b_m) = 1$ at the impact parameter $b_m = R\sqrt{\ln 2} \approx 0.59R$, complete transparency at $b = 0$ and rather large half-width about $0.7R$. Thus, the evolution to values of $Z$ smaller than 1 at higher energies (this can happen if the decreasing tendency of $Z$ with energy shown in the Table persists) would imply quite special transition from the two-scale features at the LHC to theconcave torus-like configurations of the interaction region. The exponential shape of the diffraction cone at present energies must evolve to the typical for the black disk Bessel-like shape which is narrower than the exponent. The latter one acquires zero at quite small $|t| \approx 0.18$ GeV$^2$. No sign of it is seen at present energies beside a dip at $|t| = 0.53$ GeV$^2$ at 7 TeV. At the same time, the fit by a more steep exponential at the very edge of the diffraction peak used by TOTEM collaboration looks quite indicative to this tendency. The energy increase of the slope $B$ (or the radius $R$) is also to be accounted. Further implications of this peculiar structure (if it appears!) for inelastic processes are to be guessed and studied.
In principle, the positivity of the inelastic cross section

\[ \sigma_{\text{el}} = \pi B \frac{Z}{2Z} (4Z - 1) \geq 0 \]  

admits the value of \( Z \) as small as 0.25 which corresponds to \( \sigma_{\text{el}} = \sigma_t \) and \( \sigma_{\text{in}} = 0 \). However, this possibility looks unphysical and has no interpretation in terms of eikonal (blackness).

Another consequence of Eq. (17) follows from study of energy evolution of \( G(s,0) \) shown in the Table. In connection with the torus-like concave structure, it is interesting to point out the value of \( Z = 0.64 \) or \( G(s,0) = 0.68 \) at \( \sqrt{s} = 2.70 \text{ GeV} \) and maximum 1 at \( b^2_{\text{min}} = 4B \ln 2 \). One also notices that, in the energy interval 4 GeV < \( \sqrt{s} < 8 \text{ GeV} \), the values of \( Z \) are slightly larger than 1 so that the values of \( G(s,0) \) are smaller but very close to 1. It looks as if the interaction region becomes black at the center \( b = 0 \) but at higher energies up to ISR loses this property trying to restore it at the LHC. This fact asks for further studies in the energy interval 4 GeV < \( \sqrt{s} < 8 \text{ GeV} \) especially in view of proposed experiments in Protvino. The dark core must be smaller there than at LHC because of smaller values of \( B \). Moreover, the contribution due to the real part of the amplitude is larger at these energies as well as larger \( |t| \) beyond the diffraction cone can be important. One should also notice that \( Z \) becomes less than 1 at even smaller energies. As is easily shown, that does not pose any problem with the requirement \( G(s,b) \leq 1 \).

5. Conclusions

To conclude, we have shown that the absorption at the center of the interaction region of protons is determined by a single energy-dependent parameter \( Z \). The region of full absorption extends to quite large impact parameters if \( Z \) tends to 1. This happens at \( \sqrt{s} = 7 \text{ TeV} \) where the two-scale structure of the interaction region of protons becomes well pronounced. That leads to special consequences both for elastic and inelastic processes. The value of \( Z = 1 \) attained at the LHC energies is crucial for the behavior of the differential cross section outside the diffraction cone. Its slope there becomes fully defined by the ratio of the real part of the amplitude to its imaginary part yet unmeasured in this range of the transferred momenta. Analysis of experimental data at 7 TeV about the slope of the differential cross section inside the Orear region reveals that this ratio is negative and large in modulus. The behavior of \( Z \) at higher energies is especially important for the evolution of the geometry of the interaction region.

I am grateful for support by the RFBR grants 12-02-91504-CERN-a, 14-02-00099 and the RAS-CERN program.

[1] A.B. Kaidalov, Sov. J. Nucl. Phys. 13, 401 (1971).
[2] A.B. Kaidalov, K.A. Ter-Martirosian, Phys. Lett. B 117, 247 (1982).
[3] I.M. Dremin, A.B. Kaidalov, UFN 176, 275 (2006); Physics-Uspekhi 49, 263 (2006).
[4] I.M. Dremin, UFN 183, 3 (2013); Physics-Uspekhi 56, 3 (2013).
[5] G. Antchev et al. TOTEM Collaboration, Europhys. Lett. 95, 41001 (2011).
[6] G. Antchev et al. TOTEM Collaboration, Europhys. Lett. 96, 21002 (2011).
[7] I.V. Andreev, I.M. Dremin, ZhETF Pis’ma 6, 810 (1967).
[8] I.V. Andreev, I.M. Dremin, Sov. J. Nucl. Phys. 8, 473 (1969).
[9] I.V. Andreev, I.M. Dremin, D.N. Steinberg, Sov. J. Nucl. Phys. 11, 261 (1970).
[10] I.V. Andreev, I.M. Dremin, I.M. Gramenitskii, Nucl. Phys. 10 137 (1969).
[11] I.M. Dremin, V.A. Nechitailo, Phys. Rev. D 85, 074009 (2012).
[12] L. Frankfurt, M. Strikman, C. Weiss, Phys. Rev. D 83, 054012 (2004).
[13] I.M. Dremin, V.A. Nechitailo, Nucl. Phys. A 916, 241 (2013).
[14] M.Yu. Azarkin, I.M. Dremin, M. Strikman, Phys. Lett. B (2014) arXiv:1401.1973.
[15] B.Z. Kopeliovich, I.K. Potashnikova, B. Povh, E. Predazzi, Phys. Rev. Lett. 85, 507 (2000).
[16] M.M. Islam, R.J. Lukki, A.V. Prokudin, Mod. Phys. Lett. A 18, 743 (2003).
[17] L. Frankfurt, M. Strikman, C. Weiss, Phys. Rev. D 69, 114010 (2004).
[18] CMS Collaboration, Eur. Phys. J. C 73, 2674 (2013).