On the comparison of different notions
of geometric categories

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Abstract

In this paper we explain our notion of a 'Nash geometric category',
which allows an easy comparison between the following different ax-
omatic notions of geometric categories:

1. The o-minimal structures $\mathcal{G}$ on the real field $(\mathbb{R}, +, \cdot)$, as defined
   by van den Dries [vDr].

2. The analytic geometric categories of van den Dries and Miller
   [vDrMi].

3. The $\mathcal{X}$-sets of Shiota [Shiota].

Introduction

In the last years different axiomatic generalizations of the theory of semialge-
braic- and subanalytic sets where developed.

First we have the theory of o-minimal structures, and especially of o-
minimal structures on the real field $(\mathbb{R}, +, \cdot)$, as defined by van den Dries
and explained in his book [vDr]. This theory is an abstraction and general-
ization of the theory of semialgebraic sets of affine spaces.

Another abstraction and generalization of the theory of subanalytic sets
of real analytic manifolds is the theory of analytic geometric categories as
introduced and developed by van den Dries and Miller [vDrMi]. This pa-
per was motivated by the work and the applications of Schmid and Vilonen
about characteristic cycles of constructible complexes of sheaves [SchVi] (as
remarked by van den Dries and Miller in the beginning of the introduction

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Moreover, van den Dries and Miller explain a one to one correspondence between their analytic geometric categories and those o-minimal structures on the real field \((\mathbb{R}, +, \cdot)\), which contain all globally subanalytic subsets. They use this correspondence to transfer results from the o-minimal context into the context of analytic geometric categories.

Finally, Shiota introduced another axiomatic notion of \(X\)-sets of real affine spaces in his book [Shiota]. This theory is a simultaneously generalization of the theory of semialgebraic- and subanalytic subsets of real affine spaces.

There are some results (e.g. results about curve selection, dimension, Whitney stratifications and triangulability), which are quite similarly in these categories (compare also with the article of Teissier [Te]). But there are also some important specific results, which are not yet explained in all these different axiomatic generalizations:

1. The notion of a polynomially bounded o-minimal structure on the real field, related to the Łojasiewicz inequality, and the corresponding growth dichotomy result (compare [vDrMi, p.510/511]).

2. The generic triviality results for o-minimal structures ([vDrMi, 4.11] and [vDr, chap. 9]).

3. The results of van den Dries ([vDr, chap. 10]) about definable spaces and quotients.

4. The description of closed definable sets as the zero-set of a definable \(C^p\)-function \((1 \leq p < \infty)\), as given by [vDrMi, 1.20,d.19].

5. The results in [Shiota] about:
   - Triangulability of definable functions and maps.
   - Uniqueness results about this type of triangulations.
   - \(X\)-versions of Thom’s first and second isotopy lemmas.

We introduced in our paper [Sch] a variant of the analytic geometric categories of van den Dries and Miller [vDrMi], by restricting ourselves to (analytic) Nash manifolds. In this way, every o-minimal structure \(S\) on the real field \((\mathbb{R}, +, \cdot)\) corresponds uniquely to a category, which we call a Nash geometric category. This notion allows an easy comparison (contrary to a remark in [vDrMi, p.498]) between the different notions of 'geometric categories' as before. Moreover, it can easily be used to extend the above results...
to all these different ‘geometric categories’.

But we used this result in [Sch] for the development of the theory of constructible sheaves in these ‘geometric categories’. We recall in this paper our comparison result and its proof, since it is not related to this abstract sheaf theory (so this abstract language is not used in this paper). Some people explained to me, that this should be useful. Let us now recall the basic notions.

An o-minimal structure $\mathcal{S}$ on $\mathbb{R}$ is a sequence $\mathcal{S}_n$ $(n \in \mathbb{N})$ such that for each $n$:

1. $\mathcal{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$.
2. $A \in \mathcal{S}_n \Rightarrow A \times \mathbb{R}, \mathbb{R} \times A \in \mathcal{S}_{n+1}$.
3. $\{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_i = x_j\} \in \mathcal{S}_n$ for $1 \leq i < j \leq n$.
4. $A \in \mathcal{S}_{n+1} \Rightarrow \pi(A) \in \mathcal{S}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first coordinates.
5. $\{r\} \in \mathcal{S}_1$ for each $r \in \mathbb{R}$, and $\{(x, y) \in \mathbb{R}^2 \mid x < y\} \in \mathcal{S}_2$.
6. The only sets in $\mathcal{S}_1$ are the finite unions of open intervals and points.

This notion is an elegant generalization of semialgebraic geometry (a standard reference for this theory is [BCR]) and the semilinear or semialgebraic sets give the simplest examples of an o-minimal structure.

$\mathcal{S}$ is called an o-minimal structure on the real field $(\mathbb{R}, +, \cdot)$, if it contains the graph of addition and multiplication on $\mathbb{R}$ (and therefore all semialgebraic sets).

For the construction of other o-minimal structures, see [KaraMac, LiRe, PeSpSt, Speiss, RoSpWi, vDrSp, vDrSp2, Wilkie] and the references in [vDrMi]. We use the notation $\mathbb{R}_{an}$ for the o-minimal structure of globally subanalytic sets (i.e. subsets of $\mathbb{R}^n$, that are subanalytic as subsets of the larger projective space $\mathbb{P}^n(\mathbb{R})$).

An abstraction of the theory of o-minimal structures on the real field to the more general context of real analytic manifolds is the theory of ‘analytic geometric categories’, as introduced and studied by van den Dries and Miller [vDrMi]. This is a generalization of the theory of subanalytic subsets of real analytic manifolds.
An analytic geometric category $\mathcal{S}$ is given if each real analytic manifold $M$ is equipped with a collection $\mathcal{S}(M)$ of subsets of $M$ such that the following conditions are satisfied (for each such manifold):

\begin{enumerate}[AG1.]
\item $\mathcal{S}(M)$ is a boolean algebra of subsets of $M$, with $M \in \mathcal{S}(M)$.
\item If $A \in \mathcal{S}(M)$, then $A \times \mathbb{R} \in \mathcal{S}(M \times \mathbb{R})$.
\item If $f : M \rightarrow N$ is a proper analytic map and $A \in \mathcal{S}(M)$, then $f(A) \in \mathcal{S}(N)$.
\item If $A \subseteq M$ and $(U_i)_{i \in I}$ is an open covering of $M$, then $A \in \mathcal{S}(M)$ if and only if $A \cap U_i \in \mathcal{S}(U_i)$ for all $i \in I$.
\item Every bounded set in $\mathcal{S}(\mathbb{R})$ has a finite boundary.
\end{enumerate}

$\mathcal{S}$ corresponds uniquely (\cite[D.10]{DeM}) to an o-minimal structure $\mathcal{G}$ on $\mathbb{R}_{an}$ (i.e. an o-minimal structure on the real field containing the structure $\mathbb{R}_{an}$ of globally subanalytic subsets). $\mathcal{G}$ is defined by the subsets of $\mathbb{R}^n$, which belong to $\mathcal{S}$ as a subset of $\mathbb{P}^n(\mathbb{R})$ (for the standard inclusion $\mathbb{R}^n \hookrightarrow \mathbb{P}^n(\mathbb{R})$). Moreover, $\mathcal{S}$ can be recovered as the subsets of real analytic manifolds, which are locally (at each point of the ambient manifold) real analytic isomorphic to $\mathcal{G}$-sets. Note that the last step is well defined, since the real analytic isomorphisms between bounded open subanalytic subsets of affine spaces are definable in $\mathcal{G}$.

We recall in this paper our variant of this notion, by restricting ourselves to (analytic) Nash manifolds (so in the above constructions, one should only look at Nash manifolds and semialgebraic maps). In this way, every o-minimal structure $\mathcal{G}$ on the real field $(\mathbb{R}, +, \cdot)$ (and not only those on $\mathbb{R}_{an}$) corresponds uniquely to a category, which we call a Nash geometric category. This notion is a generalization of the theory of locally semialgebraic subsets of Nash manifolds.

A Nash geometric category $\mathcal{S}$ is given if each (real analytic) Nash manifold $M$ is equipped with a collection $\mathcal{S}(M)$ of subsets of $M$ such that the conditions $\text{NG}i. := \text{AG}i., i = 1, 2, 5$ and the following conditions are satisfied (for each such manifold):

\begin{enumerate}[NG3.]
\item If $f : M \rightarrow N$ is a proper Nash map (i.e. analytic with a semialgebraic graph) and $A \in \mathcal{S}(M)$, then $f(A) \in \mathcal{S}(N)$.
\end{enumerate}
NG4. If $A \subseteq M$ and $(U_i)_{i \in I}$ is a covering of $M$ by open semialgebraic subsets, then $A \in S(M)$ if and only if $A \cap U_i \in S(U_i)$ for all $i \in I$.

Then all results of \[vDrMi\] extend to this context, if one makes the obvious modifications (which way state and explain in the next section). Especially, an 'analytic geometric category' induces by restriction a 'Nash geometric category', and this correspondence is injective, since the associated o-minimal structure is unique. A Nash geometric category $S$ induces an o-minimal structure $S(S)$ on $(\mathbb{R}, +, \cdot)$, by the same definition as for analytic geometric categories ($\mathbb{R}^n$ is an open semialgebraic subset of the Nash manifold $\mathbb{P}^n(\mathbb{R})$). Moreover, $S$ can be recovered as the subsets of Nash manifolds, which are locally (at each point of the ambient manifold) Nash isomorphic to $S$-sets. Note that the last step is well defined, since the Nash isomorphisms between bounded open semialgebraic subsets of affine spaces are definable in $S$.

The advantage of the notion of a Nash geometric category is the fact, that this last step is also well defined, if one starts with a category of $\mathcal{X}$-sets, as defined by Shiota \[Shiota\], who uses a different axiomatic setting.

A family of subsets of all affine spaces $\mathbb{R}^n$ is called $\mathcal{X}$ (\[Shiota\], p.viii, p.95,96), if it satisfies the axioms:

\begin{itemize}
  \item $\mathcal{X}(i)$ Every algebraic set in any Euclidean space is an element of $\mathcal{X}$.
  \item $\mathcal{X}(ii)$ If $X_1 \subseteq \mathbb{R}^n$ and $X_2 \subseteq \mathbb{R}^n$ are elements of $\mathcal{X}$, then $X_1 \cap X_2, X_1 \setminus X_2$ and $X_1 \times X_2$ are elements of $\mathcal{X}$.
  \item $\mathcal{X}(iii)$ If $X \subseteq \mathbb{R}^n$ is an element of $\mathcal{X}$ and $p : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map such that the restriction of $p$ to $X$ is proper, then $p(X)$ is an element of $\mathcal{X}$.
  \item $\mathcal{X}(iv)$ If $X \subseteq \mathbb{R}$ and $X \in \mathcal{X}$, then each point of $X$ has a neighborhood in $X$, which is a finite union of points and intervals.
\end{itemize}

Note that this a generalization of the notion of an o-minimal structure on $(\mathbb{R}, +, \cdot)$ (which is the same as a family $\mathcal{X}_0$ in the notation of Shiota). Moreover, a Nash isomorphisms $h$ between bounded open semialgebraic subsets $U_i$ ($i = 1, 2$) of affine spaces is definable in $\mathcal{X}$, and $X \subseteq U_1$ belongs to $\mathcal{X}$ if and only if $h(X) \subseteq U_2$ belongs to $\mathcal{X}$ (by axiom $\mathcal{X}(iii)$ and the graph embedding, as in \[Shiota\], III.6]). So we get a well defined notion of subsets of (analytic) Nash manifolds, which are locally Nash isomorphic to some $\mathcal{X}$-set. Note, that this argument doesn’t work, if we do not use 'bounded
charts’ (so it doesn’t make sense, to introduce the notion of \(X\)-sets of Nash manifolds. This notion works only for an o-minimal structure on the real field, compare \[vDrMi\], p.507/508).}

Sometimes, Shiota assumes in addition the axiom (\[Shiota\], p.97):

\(X(v)\) If a subset \(X\) of \(\mathbb{R}^n\) is an \(X\)-set locally at each point of \(\mathbb{R}^n\), then \(X\) is an \(X\)-set.

Note that an o-minimal structure never satisfies this axiom. But if we restrict a Nash geometric category \(\mathcal{S}\) to all affine spaces \(\mathbb{R}^n\), we get a family \(X\) satisfying this axiom. Indeed, the notion of a Nash geometric category is equivalent to a family \(X\), satisfying the axiom \(X(v)\). More precisely, the arguments of \[vDrMi\] imply the following (see the next section):

**Theorem 0.1.** 1. A family \(X\) induces a Nash geometric category \(\mathcal{S}(X)\), whose sets are the subsets of Nash manifolds, which are locally (at each point of the ambient manifold) Nash-isomorphic to \(X\)-sets.

2. A Nash geometric category \(\mathcal{S}\) induces:

   (a) An o-minimal structure \(\mathcal{S}(\mathcal{S})\) on \((\mathbb{R},+,\cdot)\), whose sets are the subsets of \(\mathbb{R}^n\), which belong as a subset of \(\mathbb{P}^n(\mathbb{R})\) (for the standard inclusion \(\mathbb{R}^n \hookrightarrow \mathbb{P}^n(\mathbb{R})\)) to \(\mathcal{S}\).

   (b) By restriction to \(\mathbb{R}^n\) \((n \in \mathbb{N})\) a family \(X(\mathcal{S})\) satisfying the axiom \(X(v)\).

3. \(\mathcal{S}(\mathcal{S}(\mathcal{S})) = \mathcal{S}(\mathcal{S}(X))\).

4. \(\mathcal{S}(\mathcal{S}(X)) \subseteq X \subseteq X(\mathcal{S}(X))\), with \(\mathcal{S}(\mathcal{S}(X)) = X\), if \(X\) is an o-minimal structure on \((\mathbb{R},+,\cdot)\), and \(X = X(\mathcal{S}(X))\), if \(X\) satisfies the axiom \(X(v)\).

Remark, that this theorem allows an easy comparison between the different notions of ‘geometric categories’:

We get a one to one correspondence between o-minimal structures \(\mathcal{S}\) on \((\mathbb{R},+,\cdot)\), Nash geometric categories \(\mathcal{S}\) and families \(X\), satisfying the axiom \(X(v)\). Moreover, the analytic geometric categories (as a subset of the Nash geometric categories) correspond in this way to the o-minimal structures on \(\mathbb{R}^n\), and to the families \(X\), containing the subanalytic subsets of affine spaces and satisfying the axiom \(X(v)\) (since the above correspondence is
compatible with the natural partial order on the families, which is induced by the inclusions of the subsets of the families).

Notice that \( S(S(\mathcal{X})) \) is the greatest o-minimal structure contained in \( \mathcal{X} \). Every bounded (especially every compact) \( \mathcal{X} \)-set belongs to it, and every \( \mathcal{X} \)-map between such sets is definable in this structure. More precisely, the proof of theorem 0.1 gives the following characterization:

\[ (*) \text{ For each } A \subseteq \mathbb{R}^n, A \in S(S(\mathcal{X})) \text{ if and only if } \tau_n(A) \text{ is an } \mathcal{X} \text{-set locally at each point of } \mathbb{R}^n. \]

Here we use the semialgebraic map \( \tau_n : \mathbb{R}^n \rightarrow \mathbb{R}^n \), given by

\[ \tau_n(x_1, ..., x_n) := (x_1(1 + x_1^2)^{-1/2}, ..., x_n(1 + x_n^2)^{-1/2}). \]

Note that \( \tau_n \) is a Nash isomorphism of \( \mathbb{R}^n \) onto \([-1, 1]^n\).

Similarly \( \mathcal{X}(S(\mathcal{X})) \) is the smallest family of this type, which contains \( \mathcal{X} \) and satisfies the axiom \( \mathcal{X}(v) \). Moreover, it can be directly constructed as the family of sets, which are locally an \( \mathcal{X} \)-set at each point of the ambient affine space (compare \cite[Shiota, p.268]{Shiota}). For most of the results of \cite{Shiota} one can therefore assume that one works with an o-minimal structure, or with a family \( \mathcal{X} \) satisfying the axiom \( \mathcal{X}(v) \). Moreover, one can extend these results into the framework of analytic or Nash geometric categories, by using a suitable affine embedding of the ambient manifolds.

\section{Geometric categories}

In this section, we prove theorem 0.1 and explain the modifications compared to \cite{vDrMi}, which are necessary to translate their results into the context of Nash geometric categories. We assume that the reader is familiar with \cite{vDrMi}. We follow the notation and numbering of \cite[sec.1,app.D]{vDrMi}.

For example, the fact that the family of subanalytic subsets of a real analytic manifold is the 'smallest' analytic geometric category (\cite[p.499]{vDrMi}), translates of course into the fact that the family of locally semialgebraic subsets of Nash manifolds is the 'smallest' Nash geometric category.

Fix a Nash geometric category \( \mathcal{S} \), let \( M, N \) be (analytic) Nash manifolds, and let \( A \in \mathcal{S}(M), B \in \mathcal{S}(N) \).

1.1’ Every Nash map \( f : M \rightarrow N \) (i.e. \( f \) is analytic with semialgebraic graph) is an \( \mathcal{S} \)-map.
1.2’ Given an covering \((U_i)\) of \(M\) by open semialgebraic subsets, a map \(f : A \to N\) is an \(S\)-map if and only if each restriction \(f|_{U_i \cap A} : U_i \cap A \to N\) is an \(S\)-map.

Moreover, the statements 1.\(i\) for \(i = 3, \ldots, 20\) of [vDrMi] remain true in the Nash geometric context without any modification (for example 1.7 implies \(cl(A) := \overline{A} \in S(M)\)).

For their proof, we modify [vDrMi, app.D] in the following way.

D.1’ All locally semialgebraic sets are \(S\)-sets; in particular \(\mathbb{R}^n \in S(\mathbb{P}^n(\mathbb{R}))\).

Since this is a local statement, it suffices to show that the sets \(\{x \in M| f(x) = 0\}\) and \(\{x \in M| f(x) > 0\}\) belong to \(S(M)\), if \(M\) is an open semialgebraic subset of some affine space and \(f : M \to \mathbb{R}\) is a polynomial map. Then the proof of [vDrMi, p.530] applies (because the embeddings \(M \to M \times \mathbb{P}^1(\mathbb{R}), x \mapsto (x, f(x)), x \mapsto (x, 0)\) and the projection \(M \times \mathbb{P}^1(\mathbb{R}) \to M\) are proper Nash maps, and \(\{(x, y) \in M \times \mathbb{P}^1(\mathbb{R})| f(x) \cdot y_1^2 - y_2^2 = 0\}\) is locally given by the vanishing of a polynomial).

This implies also D.2’:=1.1’, since the graph of a Nash map is (locally) semialgebraic, and D.3’:=1.2’ follows directly from the axiom NG4. Moreover, the statements D.\(i\) for \(i = 4, \ldots, 9\) of [vDrMi] remain true in the Nash geometric context, with the same proof. For later applications let us just recall that D.4 implies the stability of \(S\)-sets under products, und D.6 implies the stability under proper images:

Let \(A, A' \in S(M)\) with \(A' \subseteq A\), \(A\) locally closed and let \(f : A \to N\) be a proper \(S\)-map. Then \(f(A') \in S(N)\).

Now we come to the proof of theorem D.1 (which is a generalization of [vDrMi, D.10]).

Proof. (1) The statement 1. corresponds to [vDrMi, D.10.3]. We already explained in the introduction, that \(S(\mathfrak{X})\) is well defined by the axioms \(\mathfrak{X}(i-iii)\). Moreover, NG4. follows from the definiton and NG.1,2 follow from \(\mathfrak{X}(i,ii)\). For NG3., we start with a proper Nash map \(f : M \to N\) and an \(S(\mathfrak{X})\)-set \(A\) in \(M\). Then the proof of [vDrMi, p.533,534] applies, if we take for \(y \in N\) an open semialgebraic neighborhood \(V\) of \(y\), with a Nash isomorphism \(h : V \to h(V)\) onto an open bounded (!) semialgebraic subset of \(\mathbb{R}^n\) containing \([-1,1]^n\) (and similarly for \(g_x : U_x \to g_x(U_x)\), with \(x \in f^{-1}h^{-1}([-1,1]^n), f(U_x) \subset V\) and such that \(g_x(A \cap U_x)\) belongs to \(\mathfrak{X}\)). Then the map \(g_x(U_x) \to h(V), a \mapsto h(f(g_x^{-1}(a)))\)
is a Nash map between open bounded (!) semialgebraic subsets of affine spaces.

\[ g_x(A \cap U_x) \cap [-1, 1]^n \subset g_x(U_x) \]

belongs to \( \mathcal{X} \) by \( \mathcal{X}(i,ii) \). Therefore its image under the above map belongs to \( \mathcal{X} \) by \( \mathcal{X}(iii) \) (and the graph embedding). The rest of the proof works without changes and uses then only \( \mathcal{X}(i,ii) \). Similarly, the proof of AG5. in [DrMi, p.534] applies also to the proof of NG5. in our situation, and uses \( \mathcal{X}(iv) \) (instead of condition (6) for o-minimal structures).

(2) The statement 2.(a) corresponds to [DrMi, D.10.2] and their proof applies without changes. The proof of 2.(b) goes as follows:
\( \mathcal{X}(i) \) follows from D.1' and \( \mathcal{X}(ii) \) follows from NG1. and D.4 ('stability under products'). Moreover, every bounded \( \mathcal{S} \)-set in \( \mathbb{R} \) belongs by 2.(a) to an o-minimal structure and is therefore a finite union of intervals and points. This implies \( \mathcal{X}(iv) \) (by D.1' and NG1.). With \( \mathcal{X} \) belongs also the closure \( \bar{X} \) to \( S \) (this is a special case of 1.7, which will follow from the first part of the statement 3. of the theorem). Then D.2' := 1.1' and D.6 imply the condition \( \mathcal{X}(iii) \).

(3) The first equality in the statement 3. corresponds to the first part of [DrMi, D.10.3]. The proof in [DrMi, p.534] applies (if one works with Nash isomorphisms instead of analytic isomorphisms).

Let \( A \in \mathcal{S}(\mathcal{X}(\mathcal{S}))(M) \). By definition, there exists for all \( x \in M \) a Nash isomorphism \( h_x : U_x \rightarrow V_x \), with \( U_x \) an open semialgebraic neighborhood of \( x \) in \( M \) and \( V_x \) an open semialgebraic subset of some affine space such that \( h_x(A \cap U_x) \) belongs to \( \mathcal{S}(V_x) \). But then \( A \cap U_x \) belongs to \( \mathcal{S}(U_x) \) (by NG3.) for all \( x \in M \), and therefore \( A \in \mathcal{S}(M) \) (by NG4.).

Altogether we get \( \mathcal{S}(M) \subseteq \mathcal{S}(\mathcal{S}(\mathcal{S}))(M) \subseteq \mathcal{S}(\mathcal{X}(\mathcal{S}))(M) \subseteq \mathcal{S}(M) \).

(4) First note, that [DrMi, D.10.1] gives the characterization (*) of \( \mathcal{S}(\mathcal{S}(\mathcal{X})) \)-sets (the proof applies without changes, since \( \tau_n \) has a semialgebraic graph and \( N \times \mathbb{P}^n(\mathbb{R}) \rightarrow N \) is a proper Nash map for each Nash manifold \( N \)). Here we also use the obvious fact, that \( A \subset \mathbb{R}^n \) belongs to \( \mathcal{S}(\mathcal{X}) \), if and only if \( A \) is an \( \mathcal{X} \)-set locally at each point of \( \mathbb{R}^n \). This fact implies already the second part of statement 4.

Let \( A \subset \mathbb{R}^n \) belong to \( \mathcal{X} \). Then \( \tau_n(A) \) belongs to \( \mathcal{X} \) and therefore, \( A \) belongs to \( \mathcal{S}(\mathcal{S}(\mathcal{X})) \) by condition (*), if \( \mathcal{X} \) is an o-minimal structure (this argument breaks down for more general \( \mathcal{X} \)). Conversely, suppose \( A \subset \mathbb{R}^n \) belongs to \( \mathcal{S}(\mathcal{X}) \) so that \( \tau_n(A) \) is an \( \mathcal{X} \)-set locally at each point of \( \mathbb{R}^n \).
For each $x \in \mathbb{R}^n$, there exists $U_x$ open in $\mathbb{R}^n$ such that $\tau_n(A) \cap U_x$ belongs to $\mathcal{X}$. Since $\tau_n(A)$ is bounded, finitely many of the $U_x$ cover $\tau_n(A)$ so that $\tau_n(A)$ belongs to $\mathcal{X}$ (by $\mathcal{X}$ (iii)). Hence, $A$ belongs to $\mathcal{X}$ (by $\mathcal{X}$ (iii), since $\tau_n(A)$ is bounded. Compare with [Shiota, II.1.6]). This proves the first part of statement 4., and the proof of theorem 0.1 is finished.

We finally remark, that the statements $D.i$ for $i = 11, \ldots, 19$ of $[vDrMi]$ remain true in the Nash geometric context, with the same proof (if one uses in the proof of D.11 (or D.19) an open covering $(U_i)_{i \in N}$ with $U_i$ open semialgebraic subsets of $M$ and $\phi_i : U_i \to \mathbb{R}^m$ (or $h_i : U_i \to \mathbb{R}^m$) analytic Nash isomorphisms (with $h_i(U_i \cap A) \in \mathcal{S}_m$).

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