The $k$-Metric Dimension of $N_k + P_n$ Graph and Starbarbell Graph

Citra Ayu Ratna Saidah$^1$, Tri Atmojo Kusmayadi$^2$

$^1$Department of Mathematics, Faculty of Mathematics and Natural Sciences, Sebelas Maret University, Surakarta, Indonesia
Email: citraratna2603@gmail.com

$^2$Department of Mathematics, Faculty of Mathematics and Natural Sciences, Sebelas Maret University, Surakarta, Indonesia

Abstract

Let $G$ be a simple connected graph with a set of vertices $V(G)$ and set of edges $E(G)$. The distance between two vertices $u$ and $v$ in a graph $G$ are the shortest path length between two vertices $u$ and $v$ denoted by $d(u, v)$. Let $k$ be a positive integer, $S \subseteq V$ with $S$ is a $k$-metric generator if and only if for each different vertex pair $u, v \in V$ there are at least $k$ vertices $w_1, w_2, ..., w_k \in S$ and fulfill $d(u, w_i) \neq d(v, w_i)$ with $i \in \{1, 2, ..., k\}$. Minimum cardinality of a $k$-metric generator of a graph $G$ is called the basis $k$-metric of graph $G$. The number of elements on the basis of $k$-metric graph $G$ are called $k$-metric dimension of graph $G$ and denoted by $\text{dim}_k(G)$. $N_k + P_n$ is the result of a join operation between null graph $N_k$ and path graph $P_n$ with $k, n \geq 2$. Starbarbell graph denoted by $SB_{m_1, m_2, ..., m_n}$ is a graph formed from a star graph $K_{1,n}$ and $n$ complete graph $K_{m_i}$ then merge one vertex from each $K_{m_i}$ with $i$th leaf of $K_{1,n}$ with $m_i \geq 3, 1 \leq i \leq n$, and $n \geq 2$. In this paper, we determine the $k$-metric dimension of $N_k + P_n$ graph and starbarbell graph.

Keywords: $k$-metric dimension, $k$-metric generator, basis of $k$-metric, $N_k + P_n$ graph, starbarbell graph

1. Introduction

The branch of mathematics that is now developing rapidly is graph theory. According to Chartrand [5], a graph $G$ is a finite non-empty set with $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices and $E(G) = \{e_1, e_2, ..., e_n\}$ is the set of edges that connects members of $V(G)$ in sequence.

One topic of graph theory is the $k$-metric dimension. Dimension of $k$-metric is one of the concepts in graph theory obtained from the expansion of metric dimension. The metric dimension was introduced by Slater [10] in 1975, then Harary and Melter [4] in 1976 also introduced the same concept. While the $k$-metric dimension was first developed by Estrada-Moreno et al. [1] in 2015. Let $G$ be a simple connected graph with a set of vertices $V(G)$ and set of edges $E(G)$. The distance between two vertices $u$ and $v$ in a graph $G$ are the shortest path length between two vertices $u$ and $v$ denoted by $d(u, v)$. Let $k$ be a positive integer, $S \subseteq V$ with $S$ is a $k$-metric generator if and only if for each different vertex pair $u, v \in V$ there are at least $k$ vertices $w_1, w_2, ..., w_k \in S$ and fulfill $d(u, w_i) \neq d(v, w_i)$ with $i \in \{1, 2, ..., k\}$. Minimum cardinality of a $k$-metric generator of a graph $G$ is called the basis $k$-metric of graph $G$. The number of elements on the basis of $k$-metric graph $G$ are called $k$-metric dimension of graph $G$ and denoted by $\text{dim}_k(G)$. In 2015 Estrada-Moreno et al. [1] have found the $k$-metric dimension in the path graph, cycle graph, tree graph, and join operation between two graphs. Then in 2016 Estrada-Moreno et al. [2] have found $k$-metric dimension on the corona operation between two graphs. In 2017 Geetha and Sooryanarayana [9] have found the...
2. Main Results

Before giving the main results, we give the following definition and lemma due to Estrada-Moreno et al. [1].

Definition 1. Let G be a graph. Two vertices x, y are called false twins if N(x) = N(y) and x, y are called true twins if N[x] = N[y]. Two vertices x, y are twins if they are false twins or true twins. A vertex x is said to be a twin if there exists a vertex y ∈ V(G) − {x} such that x and y are twins in G.

Lemma 1. A connected graph G with orde n ≥ 2 is k-metric dimension if and only if G has twin vertices.

2.1. The k-metric dimension of N_k + P_n graph

N_k + P_n is a graph obtained from join operation between null graph N_k and path graph P_n with k, n ≥ 2. So, N_k + P_n graph has k + n vertices. Chartrand and Lesniak [6] define a null graph, path graph, and join operation between two graphs. Null graph N_k is a graph whose set of edges are an empty set and the set of vertices are k vertices. Null graph is called an empty graph. While the path graph P_n is walk that does not repeat any vertices and the set of vertices are n vertices. Next, the definition of join operation in a graph. Let G_1 = (V(G_1), E(G_1)) and G_2 = (V(G_2), E(G_2)) be a graph. Join operation between G_1 = (V(G_1), E(G_1)) and G_2 = (V(G_2), E(G_2)) denoted by G_1 + G_2 is a graph with V(G_1 + G_2) = V(G_1) ∪ V(G_2) and E(G_1 + G_2) = E(G_1 ∪ G_2) ∪ {uv | u ∈ V(G_1), v ∈ V(G_2)}). The N_k + P_n graph can be depicted as in Figure 1.

The following is given lemma of k-metric dimension on N_k + P_n graph.

Lemma 2. Let N_k + P_n be a graph from join operation between null graph N_k and path graph P_n, with k, n ≥ 2, then N_k + P_n is a 2-metric dimension graph. Proof. It is known that N_k + P_n is a graph from join operation between null graph N_k and path graph P_n. N_k + P_n graph has k + n vertices.

Based on Figure 1, obtained that N_k+P_n(u_1) = N_{N_k+P_n}(u_2) = N_{N_k+P_n}(u_3) = ... = N_{N_k+P_n}(u_k), so u_1, u_2, ..., u_k are twin vertices. Based on Lemma 1, N_k + P_n is a 2-metric dimension graph.

Lemma 3. Let N_k + P_n be a graph from join operation between null graph N_k and path graph P_n with k, n ≥ 2. If S is a 2-metric generator of N_k + P_n, then |S| ≥ k + \left[\frac{n+1}{2}\right].

Proof. It is known that S is a 2-metric generator, it means that for every u, v ∈ V(N_k + P_n) there are W ⊂ S such that r(u|W) ≠ r(v|W) with |W| = 2. Suppose S is a 2-metric generator with |S| < k + \left[\frac{n+1}{2}\right]. Assume that V_1 = \{u_1, u_2, u_3, ..., u_k\} and V_2 = \{v_1, v_2, v_3, ..., v_n\}. Defined S_1 = S ∩ V_1 and S_2 = S ∩ V_2 Therefore |S_1| + |S_2| < k + \left[\frac{n+1}{2}\right]. there are u, v ∈ V_1 \ S such that r(u|W) = r(v|W) for each W ⊂ S with |W| = 2. Therefore S is not a 2-metric generator, a contradiction. So, it is obtained that |S| ≥ k + \left[\frac{n+1}{2}\right].

Theorem 1. Let N_k + P_n is a graph with k, n ≥ 2 then,
\[\dim_2(N_k + P_n) = k + \left[\frac{n+1}{2}\right].\] (1)

Proof. It is known that N_k + P_n is a 2-metric dimension graph for k, n ≥ 2, it means that there is a basis of 2-metric on N_k + P_n. In this case, the proof is divided into two cases according the value of n.
a. For $n$ odd numbers.

Assume that $S = \{u_1, u_2, ..., u_k, v_1, v_2, v_3, ..., v_{n}\}$, it will be shown that $S$ is a basis of 2-metric. The representations of every vertex in $N_k + P_n$ with respect to $S$ are

$$
\begin{align*}
    r(u_1|S) &= (0, 2, ..., 2, 1, 1, ..., 1, 1); \\
    r(u_2|S) &= (2, 0, ..., 2, 1, 1, ..., 1, 1); \\
    &\vdots \\
    r(u_k|S) &= (2, 2, ..., 0, 1, 1, 1, ..., 1, 1); \\
    r(v_1|S) &= (1, 1, ..., 1, 0, 2, 2, ..., 2, 2); \\
    r(v_2|S) &= (1, 1, ..., 1, 1, 1, 2, ..., 2, 2); \\
    &\vdots \\
    r(v_{n}|S) &= (1, 1, ..., 1, 2, 2, 2, ..., 1, 0).
\end{align*}
$$

Based on this representation, if taken a $W \subset S$ with $|W| = 2$, then for every $u, v \in V (N_k + P_n)$ applies $r(u|W) \neq r(v|W)$.

b. For $n$ even numbers.

Assume that $S = \{u_1, u_2, ..., u_k, v_1, v_2, v_3, ..., v_{n-1}, v_n\}$, it will be shown that $S$ is a basis of 2-metric. The representations of every vertex in $N_k + P_n$ with respect to $S$ are

$$
\begin{align*}
    r(u_1|S) &= (0, 2, ..., 2, 1, 1, ..., 1, 1); \\
    r(u_2|S) &= (2, 0, ..., 2, 1, 1, ..., 1, 1); \\
    &\vdots \\
    r(u_k|S) &= (2, 2, ..., 0, 1, 1, 1, ..., 1, 1); \\
    r(v_1|S) &= (1, 1, ..., 1, 0, 2, 2, ..., 2, 2); \\
    r(v_2|S) &= (1, 1, ..., 1, 1, 1, 2, ..., 2, 2); \\
    &\vdots \\
    r(v_{n}|S) &= (1, 1, ..., 1, 2, 2, 2, ..., 1, 0).
\end{align*}
$$

Based on this representation, if taken a $W \subset S$ with $|W| = 2$, then for every $u, v \in V (N_k + P_n)$ applies $r(u|W) \neq r(v|W)$.

From (a) and (b), it is obtained that $S$ is a 2-metric generator. Based on Lemma 3, it is obtained that $S$ is a basis of 2-metric. So, it concludes that $\dim_2(N_k + P_n) = k + \left\lceil \frac{n+1}{2} \right\rceil$.

\[
2.2. \text{The k-metric dimension of starbarbell graph}
\]

Budianto and Kusmayadi [11] define the starbarbell graph. Starbarbell graph denoted by $SB_{m_1, m_2, ..., m_n}$ is a graph formed from a star graph $K_{1,n}$ and $n$ complete graph $K_{m_i}$ then merge one vertex from each $K_{m_i}$ with $i^{th}$ leaf of $K_{1,n}$ with $m_i \geq 3, 1 \leq i \leq n$, and $n \geq 2$. According to Chartrand et al. [8], complete graph is a graph which every pair of distinct vertices is connected by an edge. According to Chartrand and Zhang [7] complete bipartite graph $K_{m,n}$ with $m = 1$ is called star graph $K_{1,n}$ with orde $n + 1$ and size $n$. The starbarbell graph can be depicted as in Figure 2. It looks that the starbarbell graph has $\sum_{i=1}^{n} m_i + 1$ vertices.

![Fig. 2. Starbarbell graph $SB_{m_1, m_2, ..., m_n}$](image)

The distance for each of the two vertices in the starbarbell graph presented in Table 1 (in Appendix).

The following is given lemma of $k$-metric dimension on starbarbell graph.

Lemma 4. The central vertex $u$ is not contained in any basis on a starbarbell graph $SB_{m_1, m_2, ..., m_n}$.

Proof. Proven by contradiction. Suppose that $P$ is the basis of the starbarbell graph $SB_{m_1, m_2, ..., m_n}$ which contains the center vertex $u$. Because $P \setminus \{u\}$ is not a basis, there are vertices $v, v' \in V(SB_{m_1, m_2, ..., m_n})$ such that $d(v, x) = d(v', x)$ for every $x \in P \setminus \{u\}$. It is clear that $P = \{u\}$ is not a basis, so $P \setminus \{u\} \neq \emptyset$. If it does not apply $v = u$ and $v' = u$ then $d(v, u) = d(v', u)$ and $P$ are not basis on the starbarbell graph $SB_{m_1, m_2, ..., m_n}$. Without reducing the generality, we assume that $v' = u$ and $v$ is vertex $v_2$. In this case, we obtain $d(v_2, x) = d(u, x)$ for each $x \in P$. This means that $P$ is not a basis. So it is proven that the central vertex $u$ is not contained in any basis.
Lemma 5. Let $SB_{m_1,m_2\ldots m_n}$ be a starbarbell graph with $m_i \geq 3$ for every $1 \leq i \leq n$ and $n \geq 2$, then $SB_{m_1,m_2\ldots m_n}$ is a 2-metric dimension graph.

Proof. It is known that $SB_{m_1,m_2\ldots m_n}$ is a starbarbell graph with orde $\sum_{i=1}^{n} m_i + 1$. Based on Figure 2, obtained that $N_{SB_{m_1,m_2\ldots m_n}}(v_s^i) = N_{SB_{m_1,m_2\ldots m_n}}(v_i^r)$ with $x = s = 1, 2, \ldots, n$ and $y \neq t$ for $y, t = 2, 3, \ldots, m_i$. So, $v_s^i$ and $v_r^i$ are twin vertices. Based on Lemma 1, starbarbell graph $SB_{m_1,m_2\ldots m_n}$ is a 2-metric dimension graph.

Lemma 6. Let $SB_{m_1,m_2\ldots m_n}$ be a starbarbell graph with $m_i \geq 3$ for every $1 \leq i \leq n$ and $n \geq 2$. If $S$ is a 2-metric generator of $SB_{m_1,m_2\ldots m_n}$, then $|S| \geq \sum_{i=1}^{n} (m_i - 1)$.

Proof. It is known that $S$ is a 2-metric generator, it means that for every $u, v \in V(SB_{m_1,m_2\ldots m_n})$ there are $W \subset S$ such that $r(u|W) \neq r(u|V)$ with $|W| = 2$. Suppose $S$ is a 2-metric generator with $|S| < \sum_{i=1}^{n} (m_i - 1)$. Assume that $V_1 = \{v_1^1, v_1^2, v_1^3, \ldots, v_{m_1}^1\}$, $V_2 = \{v_2^1, v_2^2, v_2^3, \ldots, v_{m_2}^1\}$, $\ldots$, $V_n = \{v_n^1, v_n^2, v_n^3, \ldots, v_{m_n}^1\}$. Defined $S_1 = S \cap V_1$, $S_2 = S \cap V_2$, $\ldots$, $S_n = S \cap V_n$. Therefore $|S_1| + |S_2| + \ldots + |S_n| < \sum_{i=1}^{n} (m_i - 1)$, there are $u, v \in V_i \setminus S$ such that $r(u|W) = r(v|W)$ for each $W \subset S$ with $|W| = 2$. Therefore $S$ is not a 2-metric generator, a contradiction. So, it is obtained that $|S| \geq \sum_{i=1}^{n} (m_i - 1)$.

Theorem 2. Let $SB_{m_1,m_2\ldots m_n}$ be a starbarbell graph with $m_i \geq 3$ for every $1 \leq i \leq n$ and $n \geq 2$ then $dim_2(SB_{m_1,m_2\ldots m_n}) = \sum_{i=1}^{n} (m_i - 1)$.

Proof. Let starbarbell graph $SB_{m_1,m_2\ldots m_n}$ be a 2-metric dimension graph with $m_i \geq 3$ for every $1 \leq i \leq n$ and $n \geq 2$.

a. We will show that $S \leq \sum_{i=1}^{n} (m_i - 1)$.

Assume, the set $S = \{v_j^i\}$ with $1 \leq j \leq n$ and $2 \leq j \leq m_i$, cardinality of $S$ is $\sum_{i=1}^{n} (m_i - 1)$. The following are given a representation of each vertex in $SB_{m_1,m_2\ldots m_n}$ with respect to $S$.

\[
\begin{align*}
  r(v_1^1|S) &= (1, 1, \ldots, 1, 3, 3, \ldots, 3, 3, \ldots, 3); \\
  r(v_2^1|S) &= (0, 1, \ldots, 1, 4, 4, \ldots, 4, 4, 4, \ldots, 4); \\
  r(v_3^1|S) &= (1, 1, \ldots, 1, 4, 4, \ldots, 4, 4, 4, \ldots, 4); \\
  \vdots \\
  r(v_{m_1}^1|S) &= (3, 3, \ldots, 3, 3, \ldots, 3, 3, 1, 1, \ldots, 1); \\
  r(v_1^i|S) &= (4, 4, \ldots, 4, 4, 4, \ldots, 4, 0, 1, \ldots, 1); \\
  \vdots
\end{align*}
\]

Based on this representation, if taken a $W \subset S$ with $|W| = 2$, then for every $u, v \in V(SB_{m_1,m_2\ldots m_n})$ applies $r(u|W) \neq r(v|W)$.

b. We will show that $S \geq \sum_{i=1}^{n} (m_i - 1)$.

Assume, the set $S = \{v_j^i\}$ with $1 \leq j \leq n$ and $2 \leq j \leq m_i$, cardinality of $S$ is $\sum_{i=1}^{n} (m_i - 1)$. Suppose $S$ is a 2-metric generator of the starbarbell graph $SB_{m_1,m_2\ldots m_n}$ with $S < \sum_{i=1}^{n} (m_i - 1)$. If the set $S = \{v_2^3, v_2^4\}$ with $2 \leq x \leq m_1 - 1$, $2 \leq y \leq n$, and $2 \leq z \leq m_y$. Note that for each $u, v \in V(SB_{m_1,m_2\ldots m_n})$, there are set of $W \subset S$ with $|W| = 2$ which must fulfill $r(u|W) \neq r(v|W)$ for $S$ to be 2-metric generator. Based on Table 1 (in Appendix), it is found there are $u, v \in V(SB_{m_1,m_2\ldots m_n})$ such that $r(u|W) = r(v|W)$ for each $W \subset S$ with $|W| = 2$. Therefore $S$ is not a metric generator, a contradiction. So, it is obtained that $S \geq \sum_{i=1}^{n} (m_i - 1)$. Based on Lemma 6, it is obtained that $S$ is a basis of 2-metric.

From (a) and (b), it concludes that $dim_2(SB_{m_1,m_2\ldots m_n}) = \sum_{i=1}^{n} (m_i - 1)$.

3. Conclusion

It can be concluded that the $k$-metric dimension of $N_k + P_n$ graph and starbarbell graph $SB_{m_1,m_2\ldots m_n}$ are as stated in Theorem 1 and Theorem 2.

Acknowledgment

The authors would like to thank for the support from Department of Mathematics and Natural Sciences Faculty, Sebelas Maret University.

References

[1] Estrada-Moreno, J. A., Rodriguez-Velazquez, and I. G. Yero, “The k-metric dimension of a graph”, Applied Mathematics and Information Sciences, 9 (2015), 2829-2840.

[2] A. Estrada-Moreno, J. A. Rodriguez-Velazquez, and I. G. Yero, “The k-metric dimension of corona product graph”, Bull. Malays. Math. Sci., 39 (1) (2016), 135-136.

[3] D. Rahmadi and Y. Susanti, “Dimension of k-metric on double fan graph and multiple graph related”, Thesis Department of Mathematics, Faculty of Mathematics and Natural Sciences UGM, Yogyakarta, 2018.

[4] F. Harary and R. A. Melter, “On the metric dimension of a graph”, Ars Combinatoria, 2 (1976), 191-195.
Appendix

Table 1. The Distance From Each of the Two Different Vertices on the Starbarbell Graph

| Distance | $v_1^1$ | $v_2^1$ | $v_{m1}^1$ | $v_1^2$ | $v_2^2$ | $v_{m2}^2$ | $v_1^n$ | $v_2^n$ | $v_{mn}^n$ | $u$ |
|----------|--------|--------|------------|--------|--------|------------|--------|--------|------------|-----|
| $v_1^1$  | 0      | 1      | …          | 1      | 2      | 3          | 3      | 2      | 3          | 1   |
| $v_2^1$  | 1      | 0      | …          | 1      | 3      | 4          | 4      | 3      | 4          | 2   |
| $v_{m1}^1$ | …      | …      | …          | …      | …      | …          | …      | …      | …          | …   |
| $v_1^2$  | 1      | 1      | …          | 0      | 3      | 4          | 4      | 3      | 4          | 2   |
| $v_2^2$  | 2      | 3      | …          | 3      | 0      | 1          | 1      | 2      | 3          | 3   |
| $v_{m2}^2$ | …      | …      | …          | …      | …      | …          | …      | …      | …          | …   |
| $v_1^n$  | 3      | 4      | …          | 4      | 1      | 0          | 1      | 3      | 4          | 2   |
| $v_2^n$  | 2      | 3      | …          | 3      | 2      | 3          | 3      | 0      | 1          | 1   |
| $v_{mn}^n$ | …      | …      | …          | …      | …      | …          | …      | …      | …          | …   |
| $u$      | 1      | 2      | …          | 2      | 1      | 2          | 2      | 1      | 2          | 0   |