On WZ and RR Couplings of BPS Branes and their all order $\alpha'$ Corrections in IIB, IIA

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Abstract

We compute all three and four point couplings of BPS $D_p$-branes for all different nonzero $p$-values on the entire world volume and transverse directions. We start finding out all four point function supersymmetric Wess-Zumino (WZ) actions of one closed string Ramond-Ramond (RR) field with two fermions, either with the same (IIB) or different chirality (IIA) as well as their all order $\alpha'$ corrections. The closed form of S-matrices of two closed string RR in both IIB, IIA, including their all order $\alpha'$ corrections have also been addressed. Our results confirm that, not only the structures of $\alpha'$ corrections but also their coefficients of IIB are quite different from their IIA ones.

The S-matrix of an RR and two gauge (scalar) fields and their all order corrections in antisymmetric picture of RR have been carried out as well. Various remarks on the restricted Bianchi identities as well as all order $\alpha'$ corrections to all different supersymmetric WZ couplings in both type IIA and IIB superstring theory are also released. Lastly, different singularity structures as well as all order contact terms for all non-vanishing traces in type II have also been constructed.

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1 Introduction

Dp-branes have been taken to be the known sources for Ramond-Ramond (RR) closed string for all kinds of BPS branes [1, 2]. RR couplings have been playing the major roles in many areas, for instance the phenomenon dissolving brane within branes [3], K-theory (through D-brane approach) [4, 5], Myers effect [6] and some of their $\alpha'$ corrections [7, 8] are revealed. To describe the dynamics of D-branes, their effective actions must be established and almost all relevant literatures have been pointed out in the introduction of [9]. By dealing with Conformal Field Theory (CFT) and evaluating scattering amplitudes, we hope to enhance our knowledge of knowing new Effective Actions, more crucially, given S-Matrix formalism, among the applications to this formalism, new approaches to Effective Field Theory (EFT) will be discovered. In this note, we just highlight some of the applications that have been worked out such as $N^3$ phenomenon for M5 branes, dS solutions and the entropy growth [10]. RR Couplings with non-BPS branes have also been figured out by [11, 12] and the three ways of deriving EFT couplings are clarified in detail [13]. As last remark, we emphasize that by just going through S-matrix calculations, not only one is able to construct new string couplings but also exactly gain the coefficients of all the higher derivative corrections to all orders in $\alpha'$. One may find out some partial results of BPS string amplitude computations in [14].

Here is the outline of the paper. We first explore the supersymmetric Wess-Zumino (WZ) couplings of an RR and two fermion fields with either the same or different chirality of both IIB and IIA and then start building their all order $\alpha'$ corrections. Our computations clarify that not only the structures of $\alpha'$ corrections but also their coefficients of IIB are quite different from their IIA ones. We then deal with all symmetric and asymmetric amplitudes of three and four point functions of an RR and two gauge (scalar) fields and reconstruct their all order corrections as well.

In [15] the role of picture-changing operators for perturbative string computations has been discussed, more importantly, in section three the whole setup was made. It was also argued that to calculate scattering amplitudes at each order not only one must take into account all pieces together with local descriptions but also one needs to consider the vertical integration method which actually avoids all spurious singularities and a clear example was given in section 3.2 as well. Potentially its analysis has something to do with disc string computations, however, in [15] it is not discussed how to find out RR bulk momenta. Note
that the correlation function \( < \partial^i X(x_1)e^{ip.x(z)} > \) (between scalar field vertex operator in zero picture and exponential part of RR vertex operator) has non-zero contribution to our S-matrices and therefore there will be non-zero terms in S-matrices such as \( p.\xi_1 \) and \( p.\xi_2 \) terms. These terms are related to RR bulk momenta as they clearly carry momentum of RR in transverse directions. Indeed, unlike [15], in this paper we clearly keep track of all S-matrix elements, including the terms that carry RR’s momenta in the bulk, for instance \( p.\xi_1 \) and \( p.\xi_2 \) terms.

To make sense of all supersymmetric WZ actions in antisymmetric picture of RR, we make various remarks on the restricted Bianchi identities. Ultimately all contact terms for various field content and their all order \( \alpha' \) corrections for different WZ couplings in both type IIA and IIB and for all non-vanishing traces will be constructed out.

2 The \( \bar{\Psi} - \Psi - C \) of type IIB

In this section we would like to directly apply all the CFT techniques [16] to actually derive entirely all supersymmetric WZ actions including their all order \( \alpha' \) corrections. All four point functions of a closed string RR and two fermion vertex operators with either the same or different chirality in ten dimensional space time of type IIB (IIA) superstring theory are going to be explored. Hence, this four point function in IIB of BPS \( D_p \)-branes is given by the following correlation function

\[
\mathcal{A}^{\bar{\Psi}\Psi RR} \sim \int dx_1 dx_2 d^2 z \langle V^{(-1/2)}_\Psi(x_1)V^{(-1/2)}_\Psi(x_2)V^{(-1)}_{RR}(z, \bar{z}) \rangle,
\]

where all the vertex operators are given by

\[
\begin{align*}
V^{(-1/2)}_\Psi(x) &= \bar{u}^\gamma e^{-\phi(x)/2} S_\gamma(x) e^{i\alpha' q.X(x)} \\
V^{(-1/2)}_\Psi(x) &= u^\delta e^{-\phi(x)/2} S_\delta(x) e^{i\alpha' q.X(x)} \\
V^{(-1/2)}_{C}(z, \bar{z}) &= (P_- \mathcal{H}(n) M_p)_{\alpha\beta} e^{-\phi(z)/2} S_\alpha(z) e^{i\beta' p.X(z)} e^{-\phi(\bar{z})/2} S_\beta(\bar{z}) e^{i\alpha' p.D.X(\bar{z})},
\end{align*}
\]

Our notation is such that \( \mu, \nu = 0, 1, ..., 9 \), world volume directions are shown by \( a, b, c = 0, 1, ..., p \) and transverse indices are give by \( i, j = p + 1, ..., 9 \). All the objects are massless, where the projector, RR’s field strength and spinor are shown by

\[
P_- = \frac{1}{2}(1 - \gamma^{11}), \quad \mathcal{H}(n) = \frac{a_n}{n!} H_{\mu_1...\mu_n} \gamma^{\mu_1} \cdots \gamma^{\mu_n}, \quad (P_- \mathcal{H}(n))^{\alpha\beta} = C^{\alpha\delta} (P_- \mathcal{H}(n))_{\delta\beta}.
\]
For type IIA (type IIB) $n = 2, 4, a_n = i$ ($n = 1, 3, 5, a_n = 1$). We would like to deal with just the holomorphic components of the world-sheet fields, therefore, we are going to employ the so called doubling trick, that is, the following change of variables have to be taken into consideration

$$\tilde{X}^\mu(\tilde{z}) \to D_\nu^\mu X^\nu(\tilde{z}), \quad \tilde{\psi}^\mu(\tilde{z}) \to D_\nu^\mu \psi^\nu(\tilde{z}), \quad \tilde{\phi}(\tilde{z}) \to \phi(\tilde{z}), \quad \text{and} \quad \tilde{S}_\alpha(\tilde{z}) \to M_\alpha^\beta S_\beta(\tilde{z}),$$

with the following matrices

$$D = \begin{pmatrix} -1 & -p & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad M_p = \begin{cases} \frac{\gamma^{i_1} \gamma^{i_2} \ldots \gamma^{i_p+1} \epsilon_{i_1 \ldots i_{p+1}}}{(p+1)!} & \text{for } p \text{ even} \\ \frac{\gamma^{i_1} \gamma^{i_2} \ldots \gamma^{i_p+1} \gamma^{i_{p+1}}}{(p+1)!} & \text{for } p \text{ odd} \end{cases}$$

Having set the above matrices, we would be able to make use of holomorphic part of the two point functions or standard propagators for all the fields of $X^\mu, \psi^\mu, \phi$, as below

$$\langle X^\mu(z) X^\nu(w) \rangle = -\frac{\alpha'}{2} \eta^{\mu\nu} \log(z - w),$$

$$\langle \psi^\mu(z) \psi^\nu(w) \rangle = -\frac{\alpha'}{2} \eta^{\mu\nu} (z - w)^{-1},$$

$$\langle \phi(z) \phi(w) \rangle = -\log(z - w). \quad (2)$$

By considering the above vertex operators, the S-matrix can be written down as follows

$$\int dx_1 dx_2 dx_3 dx_4 dx_5 \tilde{u}^\gamma \tilde{u}^\delta (x_{12} x_{14} x_{15} x_{24} x_{25} x_{45})^{-1} (P_\gamma H_{(n)} M_p)^{\alpha\beta}$$

$$\times |x_1|^{-2u} |x_{14} x_{15} x_{24} x_{25}|^{-u} |x_{45}|^{-2u} <: S_\alpha(x_4) : S_\beta(x_5) : S_\gamma(x_1) : S_\delta(x_2) : >,$$

where $x_4 = z = x + iy, x_5 = \bar{z} = x - iy$ and $u = -\frac{\alpha'}{2} (k_1 + k_2)^2$. The correlation function of four spin operator (with the same chirality) in type IIB is as follows

$$<: S_\gamma(x_1) : S_\delta(x_2) : S_\alpha(x_4) : S_\beta(x_5) : > = \left[ (\gamma^\mu C)_{\alpha\beta} (\gamma^\mu C)_{\gamma\delta} x_{15} x_{24} - (\gamma^\mu C)_{\gamma\delta} (\gamma^\mu C)_{\alpha\beta} x_{12} x_{45} \right]$$

$$\times \frac{1}{2(x_{12} x_{14} x_{15} x_{24} x_{25} x_{45})^{-3/4}} \quad (3)$$

If we apply the above correlator to the amplitude, then we could readily check the $SL(2, R)$ invariance of the S-matrix. To actually remove the volume of conformal killing group we choose the gauge fixing as $(x_1, x_2, z, \bar{z}) = (x, -x, i, -i)$ and the Jacobian becomes $J = -2i(1 + x^2)$. Having set the above gauge fixing, we find out the final form of S-matrix as follows

3
\[ \int_{-\infty}^{\infty} dx (2x)^{-2u} \left( (\gamma^\mu C)_{\alpha\beta} (\gamma^\mu C)_{\gamma\delta} \left( \frac{-1 + x^2}{2x} + i \right) + 2i (\gamma^\mu C)_{\gamma\beta} (\gamma^\mu C)_{\alpha\delta} \right) (1 + x^2)^{-1+2u} \times \frac{(2i)^{-2u}}{2} (P - \hat{H}(n) M_p)^{\alpha\beta} \bar{u}^\gamma u^\delta, \]

Momentum conservation on brane’s world volume is \( k_1^a + k_2^a + p^a = 0 \), obviously the first term in the amplitude has zero contribution to the S-matrix as the integrand is odd function and the interval is symmetric. The second and third term of the above amplitude are contact interactions and the integral for the 2nd term (likewise the result for 3rd term) can be derived. The ultimate result for the amplitude becomes

\[ A_{\bar{\Psi} \Psi, \text{RR}} = (\mu_p/2)^{2-2u-1} \sqrt{\frac{\Gamma(-u + 1/2)}{\Gamma(1 - u)}} \bar{u}^\gamma (\gamma^\mu C)_{\gamma\delta} u^\delta \text{Tr} (P - \hat{H}(n) M_p \gamma^\mu) \text{Tr} (\lambda_1 \lambda_2). \quad (4) \]

where \((\mu_p/2)\) is a normalisation constant and \(\mu_p\) is RR’s brane charge. If \(\mu\) picks the world volume indices up \((\mu = a)\), we then get to know that the trace is non-zero for \(p = n\) case and it can be extracted out as

\[ \text{Tr} \left( \hat{H}(n) M_p \gamma^a \right) \delta_{p,n} = \pm \frac{32}{(p)!} \epsilon^{a_0...a_{p-1}a} H_{a_0...a_{p-1}} \delta_{p,n} \]

Notice that inside the trace the term including \(\gamma^{11}\) confirms that all results are being held for the following as well

\[ p > 3, H_n = *H_{10-n}, n \geq 5. \]

The expansion is low energy expansion which is \( u = -p^a p_a \to 0 \), expanding the Gamma functions inside the the amplitude we then would clearly gain all infinite higher derivative corrections of a field strength RR potential \( p \) form field and two fermions with the same chirality. The momentum expansion is

\[ 2^{(-2u-1)} \sqrt{\pi} \frac{\Gamma(-u + 1/2)}{\Gamma(1 - u)} = \pi \sum_{m=-1}^{\infty} c_m(u)^{m+1}. \]

with some of the coefficients to be

\[ c_{-1} = 1/2, c_0 = 0, c_1 = \frac{\pi^2}{12}, c_2 = \zeta(3), \]

\[ c_3 = \frac{19\pi^4}{720}, c_4 = \frac{\pi^2}{6} \zeta(3) + 3\zeta(5), \quad (5) \]
Note that these coefficients are different from the coefficients that have shown up in the expansion of non-BPS amplitude of an RR, a tachyon and a scalar field $CT\phi$. This clearly confirms that the normalisation of WZ action of BPS branes is different from non-BPS branes. The first term in the expansion is contact interaction and can be produced by the following supersymmetric Wess-Zumino coupling

$$(2\pi\alpha')^{\frac{\mu p}{p!}}\varepsilon^{a_0...a_{p-1}a} H_{a_0...a_{p-1}} \bar{\Psi}^\gamma (\gamma^a)_{\gamma \delta} \Psi^\delta$$

consequently all order $\alpha'$ higher derivative corrections can be constructed by applying the proper higher derivative corrections to the above EFT coupling and also by comparing each term with its string theory elements so that the closed form of corrections to all orders in IIB is demonstrated by

$$(2\pi\alpha')^{\frac{\mu p}{p!}}\varepsilon^{a_0...a_{p-1}a} H_{a_0...a_{p-1}} \sum_{m=-1}^{\infty} c_m (\alpha')^{m+1} D_{a_1} \cdots D_{a_{m+1}} \bar{\Psi}^\gamma (\gamma^a)_{\gamma \delta} \Psi^\delta$$

Note that if $\mu$ takes the value from transverse directions ($\mu = i$), then the amplitude is non zero for $n = p + 2$ case. One can show that all the higher derivative corrections can be looked for by the following coupling in an EFT

$$(2\pi\alpha')^{\frac{\mu p}{(p+1)!}} \varepsilon^{a_0...a_p} H_{a_0...a_p} \sum_{m=-1}^{\infty} c_m (\alpha')^{m+1} D_{a_1} \cdots D_{a_{m+1}} \bar{\Psi}^\gamma (\gamma^i)_{\gamma \delta} D^a_1 \cdots D^{a_{m+1}} \Psi^\delta \quad (6)$$

Note that the computations in this section give only the derivative pieces of (6) and not the full covariant derivatives and indeed using gauge invariance one can covariantize the action. It is worthwhile to point out a remark. Likewise the result for supersymmetric amplitude, the result and corrections for asymmetric amplitude of $<V^{(1/2)}(x_1)V^{(-1/2)}(x_2)V^{(-2)}(z,\bar{z})>$ are also the same, as there is no picture dependence of supersymmetric fermionic amplitudes. Let us look at its IIA version to explore whether or not the structures and coefficients of $\alpha'$ corrections of IIA are different from their IIB ones.

2.1 The entire form of $(RR\bar{\Psi}^\gamma \Psi^\delta)$ in type IIA

In this section we would like to see whether or not there are some singularities for a particular one RR and two fermion fields with different chirality in IIA. More crucially, our aim is to construct all order $\alpha'$ higher derivative corrections to these elements as well. There is no issue of picture dependence for mixed closed RR and fermion fields, hence, for simplicity we just deal with RR in its symmetric picture.
The correlation function of four spin operators with different chirality in IIA was given by [16, 17] as follows

\[ <S_\alpha(x_4)S_\beta(x_5)S^\gamma(x_1)S^\delta(x_2)> = \left( \frac{x_{45}x_{12}}{x_{41}x_{42}x_{51}x_{52}} \right)^{1/4} \left[ \frac{C^\delta_{\alpha} C^\gamma_{\beta}}{x_{42}x_{51}} - \frac{C^\delta_{\alpha} C^\delta_{\beta}}{x_{41}x_{52}} + \frac{1}{2} \left( \gamma^\alpha C^\alpha_{\alpha} (\bar{\gamma}^\mu C)^{\delta\beta} \right) \right] \] (7)

Having replaced the above correlation function into the amplitude, we would be able to obtain the final form of IIA amplitude in a manifest way as follows

\[ A = \int dx_1 dx_2 dx_3 dx_5 (P_{-H}(n) M_p)^{\alpha\beta} \bar{u}^\gamma u^\delta (x_{12}x_{14}x_{15}x_{24}x_{25}x_{45})^{-1/4} |x_{12}|^{-2u} |x_{14}x_{15}x_{24}x_{25}|^u |x_{45}|^{-2u} \]

\[ \times \left( \frac{x_{45}x_{12}}{x_{41}x_{42}x_{51}x_{52}} \right)^{1/4} \left[ \frac{C^\delta_{\alpha} C^\gamma_{\beta}}{x_{42}x_{51}} - \frac{C^\delta_{\alpha} C^\delta_{\beta}}{x_{41}x_{52}} + \frac{1}{2} \left( \gamma^\alpha C^\alpha_{\alpha} (\bar{\gamma}^\mu C)^{\delta\beta} \right) \right] \] (8)

We used the same gauge fixing as in IIB one and the final form of amplitude can be packed as follows

\[ A = (P_{-H}(n) M_p)^{\alpha\beta} \bar{u}^\gamma u^\delta \int_{-\infty}^{\infty} dx (2x)^{-2u} (x^2 + 1)^{2u-2u+1} \]

\[ \times \left[ (C^\delta_{\alpha} C^\gamma_{\beta} - C^\delta_{\alpha} C^\delta_{\beta})(x^2 - 1) + 2ix (C^\delta_{\alpha} C^\gamma_{\beta} + C^\delta_{\alpha} C^\delta_{\beta}) - \frac{1}{8ix} (\gamma^\alpha C^\alpha_{\alpha} (\bar{\gamma}^\mu C)^{\delta\beta}) \right] , \] (9)

where the 2nd and 3rd term have zero contribution to our S-matrix. Having evaluated the integrals, the final form of the amplitude in IIA would be given by

\[ A_{\bar{\Psi} \Psi, RR} = \mu_p \frac{32}{32} (P_{-H}(n) M_p)^{\alpha\beta} 2^{-2u+3} \sqrt{\pi} \Gamma(-u - 3/2) \bar{u}^\gamma u^\delta (C^\delta_{\alpha} C^\gamma_{\beta} - C^\delta_{\alpha} C^\delta_{\beta}) . \] (10)

The trace is non-zero for \( p + 1 = n \) case, and it can be found out through the way we did in the previous section. The expansion is

\[ 2(-2u) \sqrt{\pi} \frac{\Gamma(-u - 3/2)}{\Gamma(-u)} = \pi \sum_{m=-1}^{\infty} c_m(u)^{m+1} . \]

with the following coefficients

\[ c_{-1} = 0, c_0 = -\frac{4}{3}, c_1 = \frac{32}{9}, c_2 = -\frac{2}{27} (3\pi^2 + 104), \]

\[ c_3 = -\frac{8}{81} (-6\pi^2 + 27\zeta(3) - 160) . \] (11)

It is now clarified that these coefficients are different from the coefficients that have appeared in the expansion of \( C_{\bar{\Psi} \Psi} \) of type IIB of the previous section. The first contact interaction can be generated by the following supersymmetric Wess-Zumino coupling of IIA

\[ \frac{2\pi \alpha'^2 c_0 \mu_p}{(p + 1)!} D^a \bar{\Psi}^i D_a \Psi^\delta \epsilon^{a_0 a_p} H_{a_0 a_p} \]
and all order $\alpha'$ higher derivative corrections can be derived by comparing with string amplitudes as follows

$$\frac{2\pi\alpha'^2\mu_p}{(p+1)!} \sum_{m=-1}^{\infty} c_m (\alpha')^{m+1} D_a D_{a_1} \cdots D_{a_{m+1}} \bar{\Psi}^\gamma D^\alpha D^{a_1} \cdots D^{a_{m+1}} \Psi^\delta \epsilon^{a_1 \cdots a_p} H_{a_0 \cdots a_p}$$  \hspace{1cm} (12)

where the first correction for IIA couplings (unlike IIB) appears to be at $\alpha'^2$ order. As it is evident not only the structures but also the coefficients of the corrections of IIA are very different from IIB ones (6). The reasons and intuitions for this conclusion are as follows. Indeed not only $\alpha'$ corrections keep changing at each order but also there is no definite rule for finding $\alpha'$ corrections of fermionic couplings, note also that they obviously couple to different RR forms as well.

3 RR couplings of type IIB and IIA

In this section we would like to use CFT to build not only singularities but also all the infinite contact interactions as well as $\alpha'$ corrections of two closed string RR at disk level. Clearly all spin operators in type IIB carry the same chirality, hence, this four point function in IIB of BPS branes $\langle V_C^{(-1)}(x_1, x_2) V_C^{(-1)}(x_4, x_5) \rangle$ can be found by exploring all the correlation functions where C-vertex has already been given. The definitions for projection operator and the other matrices as well as notations kept held here as well.

Consider $n_o$ open strings and $n_c$ closed strings, we then have $(n_o + n_c)(n_o + n_c - 3)/2$ independent variables from the tangent to the brane momenta. This takes into account the momentum conservation constrain along the brane, also we will have $n_c(n_c - 1)/2$ variables of the $p_i N p_j$ so that $i \neq j$ and $n_c$ variables of the type $p_i N p_i = -p_i V p_i$ since $p^2 = 0$.

Therefore, in general we have $(n_o + n_c)(n_o + n_c - 3)/2 + n_c(n_c + 1)/2$ independent variables. Indeed we might think of having 3 independent Mandelstam variables for this world sheet four point function of $CC$ amplitude, however, $t + s + u = 0$ and therefore $u$ can be removed in terms of $s$ and $t$. Thus, we define $s = -\frac{\alpha'}{2}(p_1 + D.p_1)^2$ and $t = -\frac{\alpha'}{2}(p_1 + p_2)^2$ so that $p_2.D.p_2 = p_1.D.p_1$ and $p_1.D.p_2 = \frac{s+t}{2}$. The correlation function of four spin operator in IIB has been given in the previous section so the S-matrix is got to be

$$\int dx_1 dx_2 dx_4 dx_5 (P \cdot H_{(1n)} M_p)^{\alpha \beta} (P \cdot H_{(2n)} M_p)^{\gamma \delta} (x_{12} x_{14} x_{15} x_{24} x_{25} x_{45})^{-1}$$
$$\times \frac{1}{2} \left[ (\gamma^\mu C)_{\alpha \beta} (\gamma^\mu C)_{\gamma \delta} x_{15} x_{24} - (\gamma^\mu C)_{\gamma \beta} (\gamma^\mu C)_{\alpha \delta} x_{12} x_{45} \right]$$
$$\times \left| x_{12} x_{45} \right|^{-s/2} \left| x_{14} x_{25} \right|^{-t/2} \left| x_{15} x_{24} \right|^{(s+t)/2}$$
We could check the \( SL(2, R) \) invariance of the S-matrix as well. To actually remove the volume of conformal killing group, we have chosen the gauge fixing as follows

\[
(x_1, x_2, x_4, x_5) = (iy, -iy, i, -i), \quad \text{Jacobian} = -2i(1 - y^2), \quad 0 \leq y \leq 1
\]

Indeed we map it to disk, setting the above gauge fixing, we reveal the final form of S-matrix as follows

\[
\mathcal{A}_{IIB}^{C-1C^{-1}} = \int_0^1 dy (y)^{-s/2-1}(1 - y)^{-t-1}(1 + y)^{s+t-1} \left( (y + 1)^2(\gamma^\mu C)_{\alpha\beta}(\gamma^\mu C)_{\gamma\delta}
+ 4y(\gamma^\mu C)_{\gamma\beta}(\gamma^\mu C)_{\alpha\delta} \right) 2^{-s-2}i(P - \mathcal{H}(1n)M_p)^{\alpha\beta}(P - \mathcal{H}(2n)M_p)^{\gamma\delta},
\]

Now in order to actually obtain the solutions for integrals in terms of Euler functions, the best way is to deal with the following transformation

\[
y = \frac{1 - x^{1/2}}{1 + x^{1/2}}
\]

which maps all the integrals to radial integrals on the unit disk. So the whole above S-matrix will be divided to two distinct parts and the solutions after coordinate transformation will be given by

\[
I_1 = -\int_0^1 dx (x)^{-t/2-1}(1 - x)^{-s/2-1} = -\frac{\Gamma(-s/2)\Gamma(-t/2)}{\Gamma(-s/2 - t/2)}
\]

\[
I_2 = -\int_0^1 dx (x)^{-t/2-1}(1 - x)^{-s/2} = -\frac{\Gamma(-s/2 + 1)\Gamma(-t/2)}{\Gamma(-s/2 - t/2 + 1)}
\]

Since the expansion is low energy expansion, one can send off \( \alpha' \) to zero and start discovering, singularity and contact terms related to two closed string RR of type IIB. The expansions of \( I_1 \) and \( I_2 \) are accordingly

\[
I_1 = \frac{2(s + t)}{ts} - \frac{\pi^2(s + t)}{12} - \frac{1}{4}\zeta(3)(s + t)^2 - \frac{1}{2880}(s + t)\pi^4(4s^2 + st + 4t^2) + ...
\]

\[
I_2 = \frac{2}{t} - \frac{\pi^2s}{12} - \frac{1}{4}\zeta(3)s(s + t) - \frac{1}{2880}\pi^4s(4s^2 + st + 4t^2) + ...
\]

Note that one could write down the compact form of the above series, for instance the closed form of \( I_2 \) is given by

\[
I_2 = \frac{2}{t} - s \sum_{n,m=0}^{\infty} h_{n,m}(ts)^n(t + s)^m
\]
Having extracted the traces and further simplifications the ultimate and closed form of two closed string RR amplitude in IIB to all orders in $\alpha'$ would be written down by

$$
A_{IIB}^{CC} = \frac{i\mu_1\mu_2}{p!p!} \left( \frac{2}{s} - t \sum_{n,m=0}^{\infty} h_{n,m}(ts)^n(t+s)^m \right)
\times \epsilon_1^{a_2\ldots a_{p-1}a} H_{1a_0\ldots a_p} \epsilon_2^{a_2\ldots a_{p-1}a} H_{2a_0\ldots a_p} (13)
$$

where $(\mu_1, \mu_2)$ are the first and the second RR charge of branes. We have chosen $\mu$ to take values on world volume directions $(\mu = a)$, so that all the traces are non-zero for $p = n$.

The presence of the first singularity clearly shows that we do have just a simple gauge field singularity that propagates between two $p$-form closed string RR as well as all infinite $\alpha'$ higher derivative corrections to two RR’s of IIB.

Note that if $\mu$ takes value on transverse directions $(\mu = i)$, then traces make sense for $p + 2 = n$ case and evidently we would get just first simple scalar field singularity structure that propagates between two $p + 1$-form closed string RR as well as all the same (but with different $H$) infinite $\alpha'$ higher derivative corrections to two RR’s as follows

$$
A_{IIB}^{CC} = \frac{i\mu_1\mu_2}{(p+1)!(p+1)!} \epsilon_1^{a_2\ldots a_p} H_{1a_0\ldots a_p} \epsilon_2^{a_2\ldots a_p} H_{2a_0\ldots a_p} \left( \frac{2}{s} - t \sum_{n,m=0}^{\infty} h_{n,m}(ts)^n(t+s)^m \right)
$$

We normalised the amplitude by $\frac{1}{2^6}$, let us now reconstruct the simple gauge (scalar) pole and continue explaining all order $\alpha'$ corrections.

$$
A = V^a_\alpha(C_{1p-1}, A) G^{ab}_{\alpha\beta}(A) V^b_\beta(C_{2p-1}, A),
$$

and scalar pole by

$$
A = V^i_\alpha(C_{1p+1}, \phi) G^{ij}_{\alpha\beta}(\phi) V^j_\beta(C_{2p+1}, \phi),
$$

where $V^a_\alpha(C_{1p-1}, A)$ is obtained from the Chern-Simons coupling as

$$
i(2\pi\alpha')\mu_1 \int_{\Sigma_{p+1}} C_{1p-1} \wedge F
$$

and accordingly $V^i_\alpha(C_{1p+1}, \phi)$ can be gained from the Taylor expansion of scalar field with mixed RR in an effective field theory coupling

$$
i(2\pi\alpha')\mu_1 \int_{\Sigma_{p+1}} \partial^i C_{1p+1} \phi_i
$$

The gauge field and scalar field propagators are produced from their kinetic term in DBI action as $(2\pi\alpha')^2 F_{ab} F^{ab}$ and $(2\pi\alpha')^2 \frac{1}{2} Tr (D_\alpha \phi^i D^a \phi_i)$. We do not have any other gauge
(scalar) poles, simply because the kinetic terms of gauge fields and scalars have already been fixed and there are no correction to them any more. Notice that there is no correction to the Chern-Simons and (WZ) couplings of an RR and a gauge (scalar) field either. The vertices could be easily established by

\[ V_\alpha^a(C_{1p-1}, A) = i(2\pi\alpha') \frac{\mu_{1p}^a}{p!} \epsilon_1^{a_0...a_{p-1}a} H_{1a_0...a_{p-1}} \text{Tr} \left( \lambda_\alpha \right) \]

\[ V_\phi^i(C_{1p+1}, \phi) = i(2\pi\alpha') \frac{\mu_{1p}^i}{(p+1)!} \epsilon_1^{a_0...a_p} H_{1a_0...a_p} \text{Tr} \left( \lambda_\alpha \right) \]

\[ G_{\alpha\beta}^{ab}(A) = \frac{-1}{(2\pi\alpha')^2} \frac{\delta^{ab} \delta_{\alpha\beta}}{k^2} , \]

\[ V_\beta^b(C_{2p-1}, A) = i(2\pi\alpha') \frac{\mu_{2p}^b}{p!} \epsilon_2^{a_0...a_{p-1}b} H_{2a_0...a_{p-1}} \text{Tr} \left( \lambda_\beta \right) \]

\[ k^2 = -(p_1 + D_p_1)^2 = -s \] should also be substituted in the propagator.

Replacing (18) into (14) and (15) appropriately, we are precisely able to find out the gauge field (scalar field) singularity of string amplitude in an EFT. Given the closed and all order form of the amplitude in (13), now one starts to apply properly all order \( \alpha' \) higher derivative corrections to two closed string RR of type IIB as follows

\[ \sum_{n,m=0}^\infty h_{n,m}(\alpha')^{m+2n+1} \left( D_a D_a \right)^m \left( D_{a_1}...D_{a_{n+1}} D_{a_{n+2}} C_{1a_0...a_{n+2}} D_{a_1}...D_{a_{n+1}} C_{2a_0...a_{n+2}} \right) \times \frac{\mu_{1p}^a \mu_{2p}^b}{(p-1)!(p-1)!} \epsilon_1^{a_0...a_{p-2}a} \epsilon_2^{a_0...a_{p-2}a} \]

\[ (19) \]

Let us find RR couplings and their corrections in type IIA. Clearly here just the correlation function of four spin operator gets changed and the same definitions for Mandelstam variables as well as the same notations are being held. Therefore the amplitude in IIA is given by

\[ \int dx_1 dx_2 dx_4 dx_5 (\gamma_{(1n)} \gamma_{(2n)} \gamma_{(1n)})^{-1/4} \times \left( \frac{x_{12} x_{24} x_{52}}{x_{14} x_{25} x_{52}} \right)^{1/2} \left[ C_{\alpha\beta}^{\gamma\delta} C_{\beta\gamma}^{\delta\alpha} + \frac{1}{2} \frac{(\gamma_C)_{\alpha\beta} (\gamma_C)_{\beta\gamma}}{x_{15} x_{25} (s-t)/2} \right] \]

\[ \times 2^{-s} \left( P_{-H_{(1n)}} M_p \right)^{a_1} \left( P_{-H_{(2n)}} M_p \right)^{a_2} \gamma_{(s+t)} \]

\[ (20) \]

We carry out the same gauge fixing as \( (x_1, x_2, x_4, x_5) = (iy, -iy, i, -i) \), \( J = -2i(1 - y^2) \) and eventually the amplitude gets reduced to

\[ \mathcal{A}_{IIA}^{C^{-1}} = -2i \int_0^1 dy (1 - y)^{-s/2} (1 - y)^{-t} (1 + y)^{s+t} \left[ C_{\alpha\beta}^{\gamma\delta} C_{\beta\gamma}^{\delta\alpha} \left( \frac{C_{\alpha\beta}^{\gamma\delta}}{(y+1)^2} + \frac{C_{\alpha\beta}^{\gamma\delta}}{(1-y)^2} - \frac{1}{8y} (\gamma_C)_{\alpha\beta} (\gamma_C)_{\beta\gamma} \right) \right] \]

\[ \times 2^{-s} \left( P_{-H_{(1n)}} M_p \right)^{a_1} \left( P_{-H_{(2n)}} M_p \right)^{a_2} \gamma_{(s+t)} \]

\[ (21) \]
We also map the integrals to radial integrals on the unit disk and use the same change of variable as \( y = \frac{1 - s^{1/2}}{1 + x^{1/2}} \). Hence, the whole S-matrix is going to be divided to three different parts. One needs to evaluate the integrals where we just illustrate the ultimate result as follows

\[
I_1 = -2^{-s} \int_0^1 2^{s-2} dx(x)^{-t/2 - 1/2}(1 - x)^{-s/2} = -\frac{\Gamma(-s/2 + 1)\Gamma(-t/2 + 1/2)}{4\Gamma(-s/2 - t/2 + 3/2)}
\]

\[
I_2 = -2^{s-2} - 2^{-s} \int_0^1 dx(x)^{-t/2 - 3/2}(1 - x)^{-s/2} = -\frac{\Gamma(-s/2 + 1)\Gamma(-t/2 - 1/2)}{4\Gamma(-s/2 - t/2 + 1/2)}
\]

\[
I_3 = -2^{s-3} - 2^{-s} \int_0^1 dx(x)^{-t/2 - 1/2}(1 - x)^{-s/2 - 1} = -\frac{\Gamma(-s/2)\Gamma(-t/2 + 1/2)}{8\Gamma(-s/2 - t/2 + 1/2)}
\]

The expansion is again low energy expansion so we send \( \alpha' \) to zero and start to reveal singularity and contact terms related to two closed string RR of type IIA. Obviously, \( I_1, I_2 \) just include all the contact interactions, meanwhile \( I_3 \) has just a simple s-channel pole and some of the expansions are given by

\[
I_2 = -\frac{1}{4} \left( -2 + 2ln2s + 2t + s^2\left(\frac{\pi^2}{12} - 2ln2\right) + st(-2ln2 + \frac{\pi^2}{4}) + ... \right),
\]

\[
I_3 = -\frac{1}{8} \left( \frac{-2}{s} + 2ln2 + \frac{\pi^2 s}{12} - 2ln2s + \frac{\pi^2 t}{4} + s^2\left(\frac{1}{2}\zeta(3) + ln2 - \frac{\pi^2 ln2}{12}\right) + st\left(\frac{7}{4}\zeta(3) - \frac{\pi^2 ln2}{4}\right) + \frac{7}{4}\zeta(3)t^2 + .... \right)
\]

The closed form of \( I_3 \) is given by

\[
I_3 = -\frac{1}{8} \left( \frac{-2}{s} + \sum_{n,m=0}^{\infty} n_m s^n \right)
\]

Extracting the traces and further simplifications, one might explore the closed form of the third term of (21) to all orders in \( \alpha' \) by the following algebraic function

\[
\mathcal{A}_{IIA}^{CC} = \frac{i\mu_1\mu_2 p!}{p!} \epsilon_1^{a_0 \cdots a_{p-1}} H_1^{a_0 \cdots a_{p-1}} \epsilon_2^{a_0 \cdots a_{p-1}} H_2^{a_0 \cdots a_{p-1}} \left( -\frac{2}{s} + \sum_{n,m=0}^{\infty} l_n s^n \right) \quad (22)
\]

We have chosen \( \mu \) to take values on world volume directions (\( \mu = a \)), thus all the traces are non-zero for \( p = n \). The presence of the first singularity clearly shows that we do have just a simple gauge field singularity that propagates between two \( p \)-form closed string RR as well as all infinite \( \alpha' \) higher derivative corrections to two RR’s of IIA. Note that if \( \mu \) takes value on transverse directions (\( \mu = i \)), then the traces will have non vanishing values
for \( p + 2 = n \) case. Evidently we would also get just a simple scalar field singularity that propagates between two \( p + 1 \)-form closed string RR as well as contact terms as follows

\[
A_{\text{IA}}^{\text{CC}} = \frac{i\mu_1\mu_2 p}{(p + 1)!(p + 1)!} \epsilon_1^{a_0 \cdots a_p} H_{1a_0 \cdots a_p}^{i_1} \epsilon_2^{a_0 \cdots a_p} H_{2a_0 \cdots a_p}^{i_2} \left( -\frac{2}{s} + \sum_{n,m=0}^{\infty} l_{n,m} s^n t^m \right) \tag{23}
\]

Looking carefully at (23), we come to know that unlike the structures of corrections, the coefficients of all infinite \( \alpha' \) higher derivative corrections to two RR’s of IIA for this case would be the same as appeared in (22).

It is worth to highlight the fact that both simple s-channel gauge field and scalar field can be precisely reconstructed in an effective field theory by the same rules of (14) and (15) appropriately, where all the vertices have also been pointed out in (18). Let us fully address the point of this section, which is finding out not only structures but also compact and the closed form of the coefficients of all order \( \alpha' \) higher derivative corrections of 2 RR of IIA one. Indeed one can start to compare order by order the elements of string amplitude with effective field theory couplings and eventually provide all the corrections involving their structures of IIA as follows

\[
\sum_{n,m=0}^{\infty} l_{n,m} (\alpha')^{m+n} \left( (D^b D_b)^n D_{a_1} \cdots D_{a_m} C_{1a_0 \cdots a_{p-2}} D^{a_1} \cdots D^{a_m} C_{2a_0 \cdots a_{p-2}} \right) \\
\times \frac{\mu_1\mu_2 p}{(p - 1)!(p - 1)!} \epsilon_1^{a_0 \cdots a_{p-2} a} \epsilon_2^{a_0 \cdots a_{p-2} a} \tag{24}
\]

If one considers (23), then one is able to generate all the corrections including their structures for the only non vanishing particular elements of \( n = p + 2 \) as below

\[
\sum_{n,m=0}^{\infty} l_{n,m} (\alpha')^{m+n} \left( (D^b D_b)^n D_{a_1} \cdots D_{a_m} C_{1a_0 \cdots a_{p-1}} D^{a_1} \cdots D^{a_m} C_{2a_0 \cdots a_{p-1}} \right) \\
\times \frac{\mu_1\mu_2 p}{p!p!} \epsilon_1^{a_0 \cdots a_{p-1}} \epsilon_2^{a_0 \cdots a_{p-1}} \tag{25}
\]

To end this section, we provide all the other contact interactions that are produced by \( I_1, I_2 \) as well. By extracting the related traces and carrying out some further algebraic simplifications, one can establish the closed form of the contact interactions of two closed string RR amplitude to all order \( \alpha' \) in IIA as follows

\[
\frac{i\mu_1\mu_2 p}{(p + 1)!(p + 1)!} \epsilon_1^{a_0 \cdots a_p} H_{1a_0 \cdots a_p}^{i_1} \epsilon_2^{a_0 \cdots a_p} H_{2a_0 \cdots a_p}^{i_2} \left( \sum_{n,m=0}^{\infty} k_{n,m} s^n t^m \right), \tag{26}
\]

where some of the coefficients are

\[
k_{0,0} = -4, k_{1,0} = 4ln2 - 2, k_{2,0} = -2ln2 - 2 + \frac{\pi^2}{6}, k_{1,1} = \frac{\pi^2}{2} - 4, k_{0,2} = -4
\]
Given the prescription for the corrections, now one explores the leftover $\alpha'$ higher derivative corrections of $I_1, I_2$ to be

$$
\sum_{n,m=0}^{\infty} k_{n,m}(\alpha')^{m+n} \left( D_{a_1} \ldots D_{a_m} (D_b D_b)^n C_{(aq \ldots ap-1)} D^{a_1} \ldots D^{a_m} C_{(aq \ldots ap-1)} \right) \\
\times \mu_1 \mu_2 a_0 \ldots a_{p-1} \epsilon_1 \epsilon_2 \alpha' (27)
$$

We are now at the steps to conclude. By comparisons of the corrections in both IIB,IIA now it becomes evident that not only the coefficients but also the structures of $\alpha'$ corrections of type IIB are quite different from their IIA ones and this fact becomes known by evaluating direct CFT techniques and performing all world sheet calculations to all orders.

### 4 Other world-sheet 4 point functions

For warm-up, we start addressing three point function of BPS branes with their restricted Bianchi identities in both symmetric picture of RR and in terms of potential C-field (antisymmetric picture of RR). To start with, we highlight the needed vertex operators in both symmetric and asymmetric pictures as follows

$$
V_{\phi}^{(-1)}(x) = e^{-\phi(x)} \xi_{i} \psi^{i}(x) e^{x' i q x(x)} \\
V_{A}^{(-1)}(x) = e^{-\phi(x)} \xi_{a} \psi^{a}(x) e^{x' i q x(x)} \\
V_{\phi}^{(-2)}(x) = e^{-2\phi(x)} V_{\phi}^{(0)}(x) \\
V_{A}^{(0)}(x) = \xi_{1a} (\partial^{a} x(x) + i \alpha' k . \psi \psi^{a}(x)) e^{x' i k . X(x)} \\
V_{RR}^{(-2)}(z, \bar{z}) = (P - C_{(n-1)}) M^{(\alpha \beta)} e^{-3\phi(z)/2} S_{\alpha}(z) e^{i p \cdot X(z)} e^{-\phi(\bar{z})/2} S_{\beta}(\bar{z}) e^{i p \cdot \bar{X}(\bar{z})} (28)
$$

where the C-vertex operator in antisymmetric picture was earlier pointed out in $[18]$ and later on was built in $[19]$. The world-sheet 3-point function of an antisymmetric closed string RR and a gauge field on the whole ten dimensional spacetime can be derived by $\langle V_{A}^{(0)}(x) V_{C}^{(-2)}(z, \bar{z}) \rangle$, where at disk level the world volume gauge field is located on the boundary, while the closed string would be located in the middle of disk and on-shell conditions are $k^2 = p^2 = 0, \quad k. \xi_1 = 0$. Carrying out the correlators, one figures out the S-matrix as follows

$$
\int dx_1 dx_4 dx_5 (P_{- C}^{(n-1)} M^{\alpha \beta}) (x_{45})^{-3/4} \xi_{1a} \left( -i p^a \frac{x_{45}}{x_{14} x_{15}} + (2ik_{1b}) I_2 \right) |x_{14} x_{15} |^{2 k_{1b} p} |x_{45}|^{2 p . D . p}
$$
where $x_4 = z = x + iy, x_5 = \bar{z} = x - iy$. $I_2$ is related to two spinor and a current correlation function that can be accommodated by the Wick-like rule \[19, 20\] as below

$$I_2 = \langle S_\alpha(x_4) : S_\beta(x_5) : \psi^\dagger \psi^a(x_1) : \rangle = 2^{-1} (x_{14} x_{15})^{-1} (x_{45})^{-1/4} (\Gamma^{ab} C^{-1})_{\alpha\beta}$$

We make use of $(x_1, z, \bar{z}) = (\infty, i, -i)$ as gauge fixing and the final result for the amplitude in both type IIB, IIA can be written as follows

$$\mathcal{A}^{A0, C^{-2}} = (2i)^{-1} \xi_1^a \left[ - ip^a \text{Tr} (P^C_{(n-1)} M_p) + ik_{1b} \text{Tr} (P^C_{(n-1)} M_p \Gamma^{ab}) \right]$$

We just hint out to the final result of the same S-matrix in symmetric picture as well

$$\mathcal{A}^{A^{-1}, C^{-1}} = 2^{-1/2} (2i)^{-1} \xi_1^a \text{Tr} (P^C_{(n)} M_p \gamma^a)$$

Extracting the trace, the result for the symmetric S-matrix is given by

$$\mathcal{A}^{A^{-1}, C^{-1}} = 2^{-1/2} (2i)^{-1} \xi_1^a \frac{16}{p!} \epsilon^{a_0 \ldots a_{p-1} a} H_{a_0 \ldots a_{p-1}}$$

Now if we multiply the amplitude by $2^{-1/2} \pi \mu_p$ then one realizes that the S-matrix can be regenerated by (16).

If we simultaneously apply the momentum conservation along the world volume of brane $(k_1 + p)^b = 0$ and on-shell condition for the gauge field $p^a \xi_1 = -k_1 \cdot \xi_1 = 0$ to the first term (29), then we come to know that the 1st term of (29) has no physical contribution to the asymmetric S-matrix. Eventually, if we extract the trace for the 2nd term of asymmetric amplitude and apply $(k_1 + p)^b = 0$ relation, we are then able to precisely reproduce the asymmetric amplitude by the Chern-Simons coupling as well. Therefore, in order to make sense of non-vanishing asymmetric amplitude, we also come to the conclusion that the following term

$$p_b \epsilon^{a_0 \ldots a_{p-2} a} C_{a_0 \ldots a_{p-2}}$$

is non-zero and therefore $p_b \epsilon^{a_0 \ldots a_{p-2} a}$ is non-zero for BPS branes.

Note that unlike above, for the mixed RR, scalar fields, one needs to explore the restricted Bianchi identity to actually make sense of asymmetric S-Matrix. The symmetric 3 point function of one RR and a scalar field is

$$\mathcal{A}^{C^{-1} \phi^{-1}} = 2^{-1/2} \text{Tr} (P^C_{(n)} M_p \gamma^i) \xi_{1i}$$

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where the amplitude can be reconstructed in an EFT by \( \mu_p(2\pi\alpha') \int \partial_i C_{p+1} \phi^i \) where the Taylor expansion has been used. The S-matrix in asymmetric picture was found to be

\[
A^{\phi, C^{-2}} = \left[ -ip^i \text{Tr} (P_+ \mathcal{C}_{(n-1)} M_p) + ik_1a \text{Tr} (P_+ \mathcal{C}_{(n-1)} M_p \Gamma^a) \right] \xi_i \tag{34}
\]

If we apply the momentum conservation to the 2nd term of asymmetric amplitude \( k_1^a + p^a = 0 \) and then extract its trace we then come to conclusion that, to be able to get to the same result as appeared in symmetric S-matrix in (33) the following strong restricted Bianchi identity should be valid for a transverse scalar field in the presence of RR

\[
p^a \epsilon^{a_0 \ldots a_{p-1} a} = 0 \tag{35}
\]

thus the 2nd term of (34) has no contribution to asymmetric S-matrix and the 1st term in (33) builds exactly the same contact term of 3 point function. Let us deal with 4-point world sheet S-matrix.

### 4.1 \( \langle V_A^{(0)} V_A^{(0)} V_{RR}^{(-2)} \rangle \) to all orders

The four point function of an antisymmetric RR closed string and two world volume gauge fields or \( \langle V_A^a V_A^a V_{C^{-2}} \rangle \) can be obtained by finding the correlators of \( \langle V_A^{(0)} (x_1) V_A^{(0)} (x_2) V_{RR}^{(-2)} (z_1, z_2) \rangle \). Here we have just one Mandelstam variable that can be introduced to be \( u = -\frac{\alpha'}{2}(k_1 + k_2)^2 \) and using Wick theorem, the amplitude is derived as below

\[
\int dx_1 dx_2 dx_3 dx_4 (P_+ \mathcal{C}_{(n-1)} M_p)^{a\beta}(x_45)^{-3/4} \xi_{a_1} \xi_{a_2} (I_1 + I_2 + I_3 + I_4) |x_{12}|^{-2u} |x_{14}x_{15}|^u |x_{24}x_{25}|^u |x_{45}|^{-2u}
\]

If we start applying the generalized form of Wick-like rule then we are able to find all the fermionic correlators as

\[
I_1 = \left( -\eta^{ab}(x_{12})^{-2} + a_{1a}a_{2b}(x_{45})^{-5/4}(C^{-1})_{a\beta} \right),
\]

\[
a_{1a} = ik_a \left[ \frac{x_{42}}{x_{12}x_{14}} + \frac{x_{52}}{x_{12}x_{15}} \right]
\]

\[
a_{2b} = ik_b \left[ \frac{x_{14}}{x_{12}x_{24}} + \frac{x_{15}}{x_{12}x_{25}} \right]
\]

\[
I_2 = ik_{2d}a_{1a}(x_{24}x_{25})^{-1}(x_{45})^{-1/4}(\Gamma^{bd}C^{-1})_{a\beta}
\]

\[
I_3 = ik_{1c}a_{2b}(x_{14}x_{15})^{-1}(x_{45})^{-1/4}(\Gamma^{ac}C^{-1})_{a\beta}
\]

\[
I_4 = -ik_{1c}k_{2d}(x_{14}x_{15}x_{24}x_{25})^{-1}(x_{45})^{3/4}
\]

\[
\times \left[ (\Gamma^{bdc}C^{-1})_{a\beta} + 2Re\left( x_{14}x_{25} \right) \eta^{cd}(\Gamma^{ba}C^{-1})_{a\beta} - \eta^{ch}(\Gamma^{da}C^{-1})_{a\beta} - \eta^{ad}(\Gamma^{bc}C^{-1})_{a\beta} + \eta^{ab}(\Gamma^{dc}C^{-1})_{a\beta} \right]^{2} \tag{36}
\]

\[
+ 4Re\left( x_{14}x_{25} \right) (\eta^{ab} \eta^{cd} + \eta^{ad} \eta^{bc})(C^{-1})_{a\beta}
\]

\[
+ 4Re\left( x_{14}x_{25} \right) (\eta^{ab} \eta^{cd} + \eta^{ad} \eta^{bc})(C^{-1})_{a\beta}
\]
We wrote all the elements of the amplitude in such a way that, the $SL(2, R)$ invariance of them becomes manifest. Using the gauge fixing as $(x_1, x_2, z, \bar{z}) = (x, -x, i, -i)$, Jacobian turns out to be $-2i(1 + x^2)$. Lastly, one could gain the final form of S-matrix as

\[
\mathcal{A}^{A0A0C^{-2}} = \xi_{1a} \xi_{2b} \int_{-\infty}^{\infty} dx (1 + x^2)^{2u-1}(2x)^{-2u}(2i)^{-2u}(\mathcal{P}_- \mathcal{Q}_{(n-1)}M_p)^{\alpha\beta}\left[1 - \frac{x^2}{x}\left(k_{2d}k_{2a}(\Gamma^{bdC^{-1}})_{\alpha\beta} - k_{1c}k_{1b}(\Gamma^{acC^{-1}})_{\alpha\beta} - k_{1c}k_{2d}(\eta^{cd}(\Gamma^{baC^{-1}})_{\alpha\beta} - \eta^{cb}(\Gamma^{daC^{-1}})_{\alpha\beta} - \eta^{ab}(\Gamma^{bcC^{-1}})_{\alpha\beta}ight) + \frac{k_{1b}k_{2a}}{2i}(1 - \frac{x^2}{x})^2(C^{-1})_{\alpha\beta} - \eta^{ab}(2x)^{-2}(\frac{x^2 + 1}{2})^2(C^{-1})_{\alpha\beta} - 2ik_{1c}k_{2d}(\Gamma^{bdacC^{-1}})_{\alpha\beta} + 4(1 - \frac{x^2}{4ix})^2(\eta^{ad}\eta^{bc} - \eta^{ab}\eta^{cd})(C^{-1})_{\alpha\beta}\right] (37)
\]

Now if we simplify the amplitude further we then realize that the 7th and 8th terms of the above amplitude will be cancelled by the 10th and 11th terms of (37) accordingly, meanwhile the 1st up to the 6th term of (37) have also zero contribution to asymmetric S-matrix due to the following reason. Indeed the integrand is odd function while the interval is symmetric and therefore the outcome is zero. The ultimate result for the amplitude in asymmetric picture is given by

\[
\mathcal{A}^{C^{-2}A0A0} = \pm \mu_p \frac{8}{(p-2)!}k_{1c}k_{2d}\xi_{1a}\xi_{2b}\epsilon^{a_0\cdots a_{p-4}}(\Gamma^{bdacC_{a_{0}\cdots a_{p-4}}}C_{a_{0}\cdots a_{p-4}}(2))^{-2u}\frac{\pi^{1/2}\Gamma(-u + 1/2)}{\Gamma(-u + 1)},
\]

where $\frac{4\mu\pi}{4}$ is a normalisation constant. The amplitude is antisymmetric under interchanging the gauge fields, it is non zero for just non abelian case $p = n + 2$ and respects the Ward identity. The expansion is low energy expansion, that is, $u = -p\alpha \rightarrow 0$ and the function is expanded around it to be

\[
(2)^{-2u}\frac{\pi^{1/2}\Gamma(-u + 1/2)}{\Gamma(-u + 1)} = \pi \sum_{n=-1}^{\infty} b_n u^{n+1}.
\]

where some of the coefficients $b_n$ are

\[
b_{-1} = 1, b_0 = 0, b_1 = \frac{1}{6}\pi^2, b_2 = 2\zeta(3) b_3 = \frac{19}{360}\pi^4, b_4 = \frac{1}{3}(\pi^2\zeta(3) + 18\zeta(5)).
\]

The first term in string amplitude can be regenerated by the following Chern-Simons coupling

\[
\frac{1}{2!} \mu_p (2\pi\alpha')^2 \text{Tr} \left(C_{p-3} \wedge F \wedge F\right). \tag{38}
\]

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and all the other contact terms are related to an infinite higher derivative corrections to the above coupling. Thus one can start to apply all order $\alpha'$ corrections to the above coupling and find out the closed form of all order $\alpha'$ higher derivative corrections as follows. The non-leading terms are corresponded to the higher derivative correction of (38). Thus, the corrections to all orders turned out as the closed form to be

$$\frac{1}{2!} \mu_p (2\pi \alpha')^2 C_{p-3} \wedge \text{Tr} \left( \sum_{n=-1}^{\infty} b_n (\alpha')^{n+1} D_{a_0} ... D_{a_n} F \wedge D^{a_0} ... D^{a_n} F \right).$$  \hspace{1cm} (39)$$

Finally we just illustrate the restricted Bianchi identity for the scalar field that has to be worked out in the presence of an RR and a gauge field. In order to get the consistent result for the four point function of an RR, a scalar and a gauge field, in [21] we have derived all possible ways of distributing super ghost charge and explored the results as follows

$$A_0 A \xi_1 \xi_2 a 2ik_{2c} p^j (2i)^{-2u} \text{Tr} \left( P_- \xi_{i(n-1)} M_p \Gamma^{abc} \right) \frac{\Gamma(-u + 1/2)}{\Gamma(1 - u)} \pi^{1/2} \frac{\Gamma(1 - u)}{\Gamma(1 - u)}$$ \hspace{1cm} (40)$$

while with symmetric case the result gets deduced to

$$A_0 A^{-1} \xi_1 \xi_2 a \int_{-\infty}^{\infty} dx (1 + x^2)^{2u-1} (2x)^{-2u} \left[ -2ik_{2b} \text{Tr} \left( P_- \xi_{i(n)} M_p \Gamma^{ab} \right) \right]$$

and also the other case $A_0 A^{-1} C^{-1}$ would have become

$$2^{1/2} \xi_1 \xi_2 a \int_{-\infty}^{\infty} dx (1 + x^2)^{2u-1} (2x)^{-2u} \left[ k_{1b} \text{Tr} \left( P_- \xi_{i(n)} M_p \Gamma^{bai} \right) - p^{i} \text{Tr} \left( P_- \xi_{i(n)} M_p \gamma^a \right) \right]$$ \hspace{1cm} (41)$$

Suppose we apply momentum conservation to the 1st term of (41), and try to make use of the following restricted Bianchi identity

$$p \epsilon^{a_0 ... a_{p-2} b a} H_{a_0 ... a_{p-2}} + p^i \epsilon^{a_0 ... a_{p-1} a} H_{a_0 ... a_{p-1}} = 0$$ \hspace{1cm} (42)$$

By doing so, we are precisely able to remove the 2nd term (41), more significantly, the derivation of effective action as (40) is held. On the other hand, from the effective field theory side, one can start to construct all order $\alpha'$ higher derivative corrections of an RR,
a gauge field and an scalar field in both IIB,IIA through applying the same pattern (as discussed for an RR and 2-gauge fields). Hence, all order higher derivative corrections can be constructed out without any ambiguity by the following coupling

\[
(2\pi\alpha')^2 \mu_p \sum_{n=-1}^{\infty} b_n (\alpha')^{n+1} \int_{H^+} \partial_i C_{p-1} \wedge D_{a_0}...D_{a_n} F D^{a_0}...D^{a_n} \phi^i
\]

where the mixed combination of Taylor expansion and Chern-Simons coupling was made.

5 Conclusion

We calculated all three and four point couplings of BPS $D_p$-branes for all different cases, including an RR and two fermion fields with the same or different chirality of IIB and IIA, as well as two closed string RR and an RR and two gauge (scalar) fields in asymmetric case. Their all order $\alpha'$ higher derivative corrections have also been explored. We also obtained the closed form of supersymmetric Wess-Zumino (WZ) actions, clarifying that not only the structures of $\alpha'$ corrections but also their coefficients of IIB are quite different from their IIA ones. Eventually, we made some remarks on the restricted Bianchi identities for several supersymmetric amplitudes of different field content.

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