Structure and convergence of Poincaré-like normal forms.

S.Louies and L.Brenig

Université Libre de Bruxelles. Service de physique statistique CP231. Campus de la Plaine. 1050 Brussels-Belgium.

Abstract

The general term of the Poincaré normalizing series is explicitly constructed for non-resonant systems of ODE’s in a large class of equations. In the resonant case, a non-local transformation is found, which exactly linearizes the ODE’s and whose series expansion always converges in a finite domain. Examples are treated.

PACS code: 47.20.Ky.

1 Normal forms and Poincaré transformation.

Let us consider a system of nonlinear ordinary differential equations:

$$\dot{x}_i = \lambda_i x_i + f_i(x), \quad (i = 1, ..., N)$$

(1)

where the (purely nonlinear) functions $f_i(x)$ are analytic in the real variables $x_1...x_N$. We suppose, for the simplicity of the purpose, that the linear part has been diagonalised.

The study of the qualitative behavior of the system starts with a linear stability analysis of the fixed points [1]. This determines the values of the control parameters for which the system bifurcates (i.e. for which some stable fixed point becomes unstable, or vice-versa.). A further analysis gives the nonlinear behavior of the solutions in the neighborhood of the fixed points. Among the methods to compute this last step, the most often used is the normal form
analysis [4] that we shortly describe here: consider a particular form of the system (1) for which the origin is a fixed point; this implies the following form:

$$\dot{x}_i = \lambda_i x_i + \sum m a_i(m) x^m, \quad (i = 1, ..., N)$$ (2)

where we used the multi-index notation, $m \equiv (m_1, ..., m_N)$, $a_i(m) \equiv a_i(m_1, ..., m_N)$ and $x^m \equiv x_1^{m_1} ... x_N^{m_N}$. The multiple sum is taken over the integers $m_1, ..., m_N$, such that $\sum_{j=1}^{N} m_j = |m| \geq 2$, with $m_i \geq 0$. Consider now the near identity change of variables given by the following formal series:

$$x_i = y_i + \sum_{m, |m| \geq 2} b_i(m) y^m$$ (3)

This gives rise to a new system for the $y_i$:

$$\dot{y}_i = \lambda_i y_i + \sum_{m, |m| \geq 2} c_i(m) y^m, \quad (i = 1, ..., N)$$ (4)

Now, let us choose the $b_i(m)$ in such a way that this last system is as simple as possible. The ideal case would be $c_i(m) = 0$ for all $i$ and all $m$, but it is easy to see that this will be possible only if no resonance condition is satisfied, i.e. if there exist no set of $N$ positive integers $r_i$ with $\sum_{j=1}^{N} r_j \geq 2$, such that:

$$\sum_{j=1}^{N} r_j \lambda_j - \lambda_i = 0$$ (5)

for some $i$. The reason is that, substituting (3) in (2) and requiring $c_i(m) = 0$, for all $i$ and all $m$, we determine $b_i(m)$ by a relation where the factor of $b_i(m)$ is precisely $(\sum_{j=1}^{N} m_j \lambda_j - \lambda_i)$. If this quantity is equal to zero, $b_i(m)$ remains undetermined, and the system (4) will still be nonlinear, with:

$$c_i(m) \neq 0, \quad \text{if } \sum_{j=1}^{N} m_j \lambda_j - \lambda_i = 0$$

$$= 0, \quad \text{otherwise}$$ (6)

Equations (4) with conditions (6) are called normal form. They form, in fact, a system equivalent to (2), where all the non-resonant monomial have been eliminated. The transformation (3) is then called Poincaré transformation, and the monomials remaining in (4) are called resonant monomials.

In most cases, the general term of the series (3) giving the Poincaré transformation is not known, and, usually, it is computed order by order. When no resonance condition is satisfied, the solutions of (4) are simply exponentials of time, and (3) provides a formal series of exponentials which is solution of (2). Moreover, the convergence of (3) is guaranteed in two cases (3): if the real part
of all the $\lambda_i$ are all of the same sign (Poincaré domain). Or if the $\lambda_i$ belong to the Siegel domain, i.e. zero lies within the convex hull of $\lambda_1, ..., \lambda_N$ in the complex $\lambda$-plane, along with:

$$| \lambda_i - \sum_{j=1}^{N} \lambda_j m_j | \geq \frac{c}{|m|^\nu}$$

for some $c > 0$, $\nu \geq \frac{N-2}{2}$.

Nonetheless, there are domains of the complex $\lambda$-plane for which the convergence of the Poincaré series is not guaranteed.

Even if the series is convergent, still one problem remains: the initial conditions of the new system (4) have to be calculated in terms of the initial conditions of the old system (2) and of its parameters $\lambda_i$ and $a_i(m)$. To investigate this, we have to invert the Poincaré series, which is not an easy task. In practice, one inverts a truncated series. This leads to an error on the new initial conditions which may be excessively large and not controlled.

In the resonant case, equation (4) is no longer linear, and contains in fact an infinity of monomials (since if one resonance condition is satisfied, so are an infinity of others). The use of the Poincaré series to approximate the solution of (4) is even more difficult, and suffers from the absence of general convergence theorems.

From a geometrical point of view, the Poincaré transformation defines a diffeomorphic transformation to coordinates in the phase-space, in which the solutions are, in the non-resonant case, exponentials. The possible divergence in the non-resonant case, and the impossibility of eliminating the resonant monomials express the fact that the transformation leading to such a coordinate system is not always a diffeomorphism.

The main goal of this paper is first to derive the general term of the Poincaré series for an $n$-dimensional Lotka-Volterra (L.V.) system in the non-resonant case. That is, we find an exact solution to the recursion relation for the coefficient of the Poincaré series. In a second step, we build a Poincaré-like transformation such that:

1. The system for the new variables is always linear, no matter if resonance conditions are satisfied.
2. The initial conditions for the new system are known (we show that they may be chosen equal to the initial conditions for the original system.).
3. The convergence of the series giving the transformation is always guaranteed in a domain around the initial time $t_0$.

The price to pay for such nice properties is to be found in the fact that this transformation is non-local as will be discussed later: it does not define a diffeomorphism. But still, it has many attractive features ranging from explicit computation of solutions to more theoretical questions in nonlinear dynamics.

The restriction to the L.V. format may look rather restrictive. This is not
the case, since it is known that a large class of systems can be transformed into L.V. systems \[3\] \[4\] \[5\] \[6\]. We show in the next section how this can be done.

2 The quasi-monomial formalism.

We restrict our attention to nonlinear systems that can be written as:

\[
\dot{x}_i = x_i(\alpha_i + \sum_{j=1}^{M} A_{ij} \prod_{k=1}^{N} x_{B_{jk}^k}), \quad (i = 1, \ldots, N) \quad (7)
\]

No particular restriction is made neither on the \(A_{ij}\) nor on the \(B_{jk}\): they may be real or complex numbers. \(M\) is the number of quasi-monomials \(\prod_{k=1}^{N} x_{B_{jk}^k}\) appearing in (7). In what follows, we suppose that \(M\) is finite. The results proposed in this work may be extended to the case of an infinite number \(M\) of quasi-monomials, provided some convergence conditions on the \(A_{ij}\) are satisfied.

The class (7) is thus quite general: many of the systems of interest in physics belong to this category. Systems of the type (7) are called Quasi-Monomial (Q.M.) systems.

The transformation to the L.V. format is made by adding \(M\) new variables to the system (7). These are defined as follow:

\[
x_{N+k} = \prod_{l=1}^{N} x_{B_{kl}^l}, \quad (k = 1, \ldots, M) \quad (8)
\]

Taking the derivative with respect to time, we find the \((N + M)\) dimensional system:

\[
\dot{x}_i = \lambda_i x_i + \sum_{j=1}^{N+M} M_{ij} x_j \quad (9)
\]

with:

\[
\lambda_i = \alpha_i, \quad \text{for } 1 \leq i \leq N \quad (10)
\]

\[
\sum_{k=1}^{N} B_{(i-N)k}\alpha_k, \quad \text{for } N+1 \leq i \leq N+M
\]

and:

\[
M_{ij} = 0, \quad \text{for } 1 \leq i \leq N+M; \quad 1 \leq j \leq N
\]

\[A_{(j-N)}, \quad \text{for } 1 \leq i \leq N; \quad N+1 \leq j \leq N+M \quad (11)
\]

\[
\sum_{k=1}^{N} B_{(i-N)k} A_{(j-N)}, \quad \text{for } N+1 \leq i \leq N+M; \quad N+1 \leq j \leq N+M
\]
The system (9) is of the form of the well known Lotka-Volterra equations, first introduced in theoretical ecology [7].

Note that the linear term can be eliminated by adding one more variable which is put equal to one ($\dot{x}_{N+M+1} = 0; x_{N+M+1}(t_0) = 1$). We then have:

$$\dot{x}_i = x_i \sum_{j=1}^{N+M+1} \tilde{M}_{ij} x_j$$

(12)

where:

$$\tilde{M}_{ij} = M_{ij}, \quad \text{for } 1 \leq i, j \leq M + N$$

$$\lambda_i, \quad \text{for } 1 \leq i \leq M + N; j = M + N + 1$$

$$0, \quad \text{for } i = M + N + 1; 1 \leq j \leq M + N + 1$$

The main characteristic of the systems (9) and (12) is that they present the lowest nonlinearity, i.e. the quadratic one. Moreover, the quadratic term is not the most general one (which would be of the form $\sum_{j,k} N_{ijk} x_j x_k$) but still permits complex behavior like chaos.

It has been shown [4] that the general structure for the Taylor series coefficient can be obtained for the solution of (12). One finds:

$$x_i(t) = \sum_{n=0}^{\infty} c_i(n) \frac{(t - t_0)^n}{n!}$$

$$c_i(0) = x_i(t = t_0) = x_{i0}$$

$$c_i(n) = x_{i0} \sum_{i_1...i_n} \tilde{M}_{i_1i_1} (\tilde{M}_{i_2i_2} + \tilde{M}_{i_1i_2})...(\tilde{M}_{i_i i_n} + \tilde{M}_{i_1i_1} + ...$$

$$+ \tilde{M}_{i_n i_1}) x_{i_10} x_{i_20} ... x_{i_n0}$$

We will show, in the next section, that the general structure of the Poincaré series coefficient is very close to this one. In the last part of this paper, we also show that the Taylor series is a particular case of a class of series expansions of the solution which includes also the Poincaré series.

3 Structural analysis of the Poincaré transformation.

We take as starting point (since the linear part plays an important role in the normal form analysis) the system:

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{j=1}^{N} M_{ij} x_j, \quad (i = 1, ..., N)$$

(13)
Where, from now on, \( N \) stands for \( N + M \) in the previous expressions.

It is known, since the work of Carleman [8] [9], that a nonlinear system can be viewed as an infinite-dimensional linear system. This can be realized by considering as new variables all the monomials one can build with products of positive integer powers of the \( x_i \). Using the multi-index notation:

\[
X_m \equiv x^m = x_1^{m_1} x_2^{m_2} \ldots x_N^{m_N}
\]

and taking the derivative of (14), one finds:

\[
\dot{X}_m = (\sum_{k=1}^{N} m_k \lambda_k) X_m + \sum_{p=1}^{N} (\sum_{l=1}^{N} m_l M_{lp}) X_{m+e_p}
\]

(15)

where \( e_p \) is a unit vector such that \((e_p)_s = \delta_{p,s}\) (i.e. \( X_{m+e_p} = X_{m_1,\ldots,m_p+1,\ldots,m_N} \)).

The infinite-dimensional linear system (15) is characterized by a triangular matrix \( R \) which is given by:

\[
R_{mp} = \sum_{o=1}^{N} m_o \delta_{m,p} + \sum_{k=1}^{N} \sum_{l=1}^{N} m_k M_{kl} \delta_{m+e_1,p}
\]

For an original system (13) that would be linear (that is, the matrix \( M \) vanishes) the system (15) would be purely diagonal. This implies that the Poincaré transformation on (13), for non vanishing matrix \( M \), but in absence of any resonance, corresponds to the diagonalisation of the infinite-dimensional matrix defined by the system (13).

Consider now the relation:

\[
L_{mp} = \delta_{m,p} + \sum_{k} R_{mk} (1 - \delta_{mk}) L_{kp}
\]

(16)

where, once again, the indices are multiple (the sum over \( k \) is a multi-sum over the \( N \) indices \( k_1, \ldots, k_N \) running from 0 to \( \infty \), and \( \delta_{m,p} \) stands for \( \delta_{m_1,p_1} \ldots \delta_{m_N,p_N} \)).

We claim that since \( R_{mp} \) is triangular (that is: \( R_{mp} = 0 \) if there exists at least one integer \( k \) between 1 and \( N \), such that \( m_k > p_k \)) and if \( R_{mm} \neq R_{pp} \) for all \( m,p \) with \( m \neq p \), then \( (L_{mp}) \) diagonalises \( (R_{mp}) \).

The condition \( R_{mm} \neq R_{pp} \) for \( m \neq p \) implies that the relation

\[
\sum_{k=1}^{N} \lambda_k r_k = 0
\]

is satisfied only if \( r_k = 0 \) for all \( k \) and restricts thus the system (13) to the non-case. That \( (L_{mp}) \) diagonalises \( (R_{mp}) \) means here that, considering the inverse operator \( L_{mp}^{-1} \) defined by:

\[
\sum_{k} L_{mk}^{-1} L_{kp} = \sum_{k} L_{mk} L_{kp}^{-1} = \delta_{mp},
\]

(17)
we have:

$$\sum_{k,o} L^{-1}_{mk} R_{ko} L_{op} = R_{mp} \delta_{mp} \tag{18}$$

The question of the existence of the inverse operator $L^{-1}$ is obvious. This operator represents the inverse of the Poincaré transformation. The latter is a diffeomorphism. Hence, when it exists, so does its inverse.

Let us now prove this proposition:

Multiplying both sides of (16) by $(R_{pp} - R_{mm} + \delta_{m,p})$ one finds:

$$\sum_k R_{mk} L_{kp} = L_{mp} R_{pp} + \delta_{mp}(L_{mp} - 1) - \delta_{mp}(R_{pp} - R_{mm}) \tag{19}$$

We will now show that the last two terms of (19) are equal to zero. It is clearly the case for $\delta_{mp}(R_{pp} - R_{mm})$. For $\delta_{mp}(L_{mp} - 1)$ we now demonstrate that, thanks to the fact that $R_{mp}$ is triangular, so is also $L_{mp}$, and all the elements $L_{mm}$ are equal to one. To show this, consider the series obtained by iterating (16). This will be a series of powers of $R_{mp}(1 - \delta_{mp})$ (with coefficients depending on $m$ and $p$). Now, the elements of the $k^{th}$ power of $R_{mp}(1 - \delta_{mp})$ for which $p \leq m + k$ are all equal to zero. This implies that the only contribution to $L_{mn}$ comes from the first term of the series, that is $\delta_{mp}$. So, $L_{mm} = 1$, and $\delta_{mp}(L_{mp} - 1) = 0$. Multiplying then (19) by $L_{om}^{-1}$ and summing over $m$, we find the announced result.

Taking this result back to the original L.V. system, we can build the Poincaré series. We insert in (16), the matrix $R$ defined by (15):

and, writing (16) for $m = e_i$, we find after some simple algebra:

$$x_i = y_i \sum_{n=0}^{\infty} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_n=1}^{N} \left[ M_{i_1i_2}(M_{i_1i_2} + M_{i_1i_3}) \cdots (M_{i_1i_n} + M_{i_1i_n+1} + \cdots + M_{i_1i_n-1}) \right]^{-1} y_{i_1} y_{i_2} \cdots y_{i_n} \tag{20}$$

where the term corresponding to $n = 0$ is, by convention, set equal to one.

This result can be readily extended to the system (7), by using the definitions (10) and (11). If the system (7) is polynomial, the obtained series is the Poincaré series. But the class of systems given by (7) also contains non-polynomial systems. In these cases, the series which is derived is more general than the Poincaré transformation of (6).

Comparing this to the Taylor series, we see that the two structures are very close. This is not surprising: the Poincaré series is, in fact, the Taylor series in which an infinity of resummations have been performed.

Accordingly with what we said about the Poincaré series, (20) is not always convergent, as we shall see in the examples treated later. Moreover, this result is not valid in the resonant case, but in some sense, can be extended to it, as we now see.
4 The Poincaré transformation revisited.

Let us define the \( N \) variables:
\[
    u_i(t) = x_i(t)e^{-\int_t^\gamma M_{ij}x_j(\beta) d\beta} \quad (i = 1, \ldots, N)
\]  
(21)

where the \( x_i \) satisfies (13). Taking the derivative of (21) with respect to time, we see that the differential system for \( u_i \) is always linear, \( \dot{u}_i = \lambda_i u_i \), no matter if resonance conditions are satisfied, and independently of the value of \( \gamma \). As announced, the definition (21) is non-local. This is expressed by the presence of the integral: the value of \( u_i(t) \) depends on the values of \( x_i(t') \) for \( t' \) between \( \gamma \) and \( t \). The inverse transformation (22) reveals this nonlocality in an even clearer way.

Choosing \( \gamma \) equal to the initial time \( t_0 \), \( u_i(t) \) and \( x_i(t) \) have the same initial conditions. Moreover, (21) is a recursion relation defining \( x_i(t) \) in terms of \( u_i(t) \).

It is possible to prove that (21) generates the series:
\[
    x_i(t) = x_i(t_0)e^{\lambda_i(t-t_0)}\sum_{n=0}^{\infty} \sum_{i_1=1}^{N} \ldots \sum_{i_n=1}^{N} M_{ii_1}(M_{ii_2} + M_{i_1i_2}) \ldots (M_{ii_n} + M_{i_1i_n}) + \ldots \\
    \\
    + M_{i_n-1i_n}) \int_{t_0}^{t} d\alpha_1 \int_{t_0}^{\alpha_1} d\alpha_2 \ldots \int_{t_0}^{\alpha_{n-1}} d\alpha_n \left[ u_{i_1}(\alpha_1)u_{i_2}(\alpha_2)\ldots u_{i_n}(\alpha_n) \right]
\]  
(22)

Here again, the explicit expressions of (21) and (22) for the system (7) are obtained using (10) and (11).

Comparing this to (20), we see that the \( \lambda_i \) appearing in the denominator, disappeared in the integrals of (22). Moreover these integrals are finite, since the integrands are exponentials, and the domains of integration are finite. Using the explicit form of \( u_i(t) \), we have:
\[
    x_i = x_i(t_0)e^{\lambda_i(t-t_0)}\sum_{n=0}^{\infty} \sum_{i_1=1}^{N} \ldots \sum_{i_n=1}^{N} M_{ii_1}(M_{ii_2} + M_{i_1i_2}) \ldots (M_{ii_n} + M_{i_1i_n})x_{i_10} \ldots x_{i_n0} \\
    \int_{t_0}^{t} d\alpha_1 \int_{t_0}^{\alpha_1} d\alpha_2 \ldots \int_{t_0}^{\alpha_{n-1}} d\alpha_n e^{\sum_{p=1}^{n} \lambda_{ip}(\alpha_p-t_0)}
\]  
(23)

The integrals appearing in (23) can be performed in closed form by using Carlson special functions [12], but we do not need this result for our present purpose. Moreover, it can be proven that this series always converges on a finite domain around \( t_0 \), even in the resonant case. These results will be detailed in a forthcoming publication [13].

We can ask under which conditions on \( \gamma \) the recursion (21) generates the series (20). To answer this question, let us substitute (20) in (21). The integral
can then be performed order by order, and we see that \( u_i(t) \) will be equal to \( y_i(t) \) if \( \gamma \) is fixed in such a way that the constant term appearing after performing the integral (i.e. the primitive of \( \sum_{j=1}^{N} M_{ij} x_j \) evaluated at \( t = \gamma \)), is equal to zero. If the real parts of the \( \lambda_i \)'s are all positive (resp. negative) we have to put \( \gamma = -\infty \) (resp. \( +\infty \)), since the constant term is a sum of exponentials of \( \gamma \). But if there exist \( \lambda_i \)'s whose real parts are of opposite sign, there will be exponentials with positive arguments, and others with negative arguments. In this case, if \( \gamma \) tends to infinity (no matter if it is \( +\infty \) or \( -\infty \)), some exponentials will be divergent. Identically, in the resonant case, some monomials built on the variables \( y_i \)'s are constant, and no value of \( \gamma \) will fulfill the requirement.

In fact, the correspondence between (20) and (23) is more subtle. A bijective correspondence can be established between the terms appearing in (20) and rooted tree graphs. In the graph formalism, (21) represents a relation between these graphs [11].

An easy way to see why (20) can be divergent even in the non-resonant case, and in the same blow, why (21) is always convergent in a certain domain, is the following: series (21) is a Taylor series expansion (in powers of \( y_i \)) of the Poincaré transformation around the origin in the \( y \)-space. But nothing secures that the trajectory of the solution will go through the origin. On the other hand, by definition, (21) guarantees that the series reduces to its order zero at \( t = t_0 \). In other words, the solution in the \( y \)-space is, at \( t = t_0 \), precisely at the point around which the series is built.

Finally, as announced, when all the \( \lambda_i \) tend to zero, the integrals in (23) generate the factors \( (t - t_0)^{n}/n! \), and (23) is precisely the Taylor series given above, whereas series (20) for \( \lambda_i \) tending to zero diverges.

Thus, although the transformation (21) is non-local, its structure explains many aspects related to the existence and convergence of the Poincaré transformations. Moreover, it leads to useful algorithms for explicit calculation of solutions.

5 Example: Riccati projective systems.

As an example, we now treat a particular case of the system (13): a class of \( N \)-dimensional, integrable systems, called projective Riccati systems [12] [13]:

\[
\dot{x}_i = \lambda_i x_i + x_i \sum_{j=1}^{N} c_j x_j \quad (i = 1, ..., N)
\]

The usual Poincaré series (20) gives:

\[
x_i(t) = y_i(t) \sum_{n=0}^{\infty} \left( \sum_{j=1}^{N} c_j y_j(t) / \lambda_j \right)^n
\]
Thanks to the fact that the general solution of the system is known, we can determine the initial conditions for the $y_i$. This leads to:

$$x_i(t) = \frac{x_{i0}}{1 - \sum_{j=1}^{N} \frac{c_j x_{j0}}{\lambda_j}} e^{\lambda_i (t-t_0)} \sum_{n=0}^{\infty} \left( \frac{\sum_{j=1}^{N} \frac{c_j x_{j0}}{\lambda_j} e^{\lambda_j (t-t_0)}}{1 - \sum_{k=1}^{N} \frac{c_k x_{k0}}{\lambda_k}} \right)^n$$

It is clear that, for some choice of the $c_i$, $\lambda_i$, and $x_{i0}$, this series will be divergent for any $t$. For example, in two dimensions, if we choose $\lambda_1$ real and positive ($\lambda_1 = \ell$), $\lambda_2 = -\lambda_1$, $x_{10} = x_{20} = s$, and $c_1 = 1 = -c_2$, the series becomes:

$$x_i(t) = \frac{s}{1 - 2s/\ell} e^{\lambda_i (t-t_0)} \sum_{n=0}^{\infty} \left( \frac{s/\ell (e^{\ell (t-t_0)} + e^{-\ell (t-t_0)})}{1 - 2s/\ell} \right)^n$$

And the radius of convergence is given by the inequality:

$$\left| \frac{2}{s - 2} \right| < 1$$

If $\ell > 4s$, there exist no $t$ for which the above relation is satisfied.

Consider now the series (23): in this case, it gives:

$$x_i(t) = x_{i0} e^{\lambda_i (t-t_0)} \sum_{n=0}^{\infty} \left( \sum_{j=1}^{N} \frac{c_j x_{j0}}{\lambda_j} e^{\lambda_j (t-t_0)} - 1 \right)^n$$

which is always convergent on a domain $|t_0 - \tau, t_0 + \tau|$. Moreover, if one of the $\lambda_i$’s tends to zero, the limit can be taken without difficulty (and is finite), which is not the case for the usual Poincaré series given above.

Acknowledgment:
One of us (L.B.) benefitted of the financial support of the Euratom-Belgium Association on Fusion.

We thank A. Figueiredo from the Universidade National de Brazilia for fruitful discussions.

References
[1] Guckenheimer J., Holmes P., Nonlinear oscillations, Dynamical systems, and bifurcations of vector fields. Springer-Verlag, 1983.

[2] Arnold, V.I., and Ilyaschenko Y.S., Ordinary differential equations in Dynamical systems I, D.V. Anosov and V.I. Arnold Eds, Springer Verlag, 1988.
[3] Peschel M., Mende W., The predator-prey model. Springer-Verlag, Berlin, 1986.

[4] Brenig, L. and Goriely, A., Normal forms and Painlevé analysis, in Computer Algebra and Differential equations., London Mathematical Society Lecture Notes Series., (Cambridge University Press, Cambridge), 1994.

[5] Brenig, L., Phys. Letters A 133 (7,8), pp. 378, 1988.

[6] Hernández-Bermejo, B. and Fairén, V., Phys. Letters A 206 (1,2), pp. 31, 1995.

[7] May R.M., Stability an complexity in Model Ecosystems. Princeton University Press, N.J., 1973.

[8] Carleman, T., Ark. Mat., Astron. Fyzy., 228, 63, 1931.

[9] Brenig, L. and Fairén, V., J. Math. Phys., 22 (4), pp. 649, 1981.

[10] Carlson, B.C., Special functions of applied mathematics., Academic Press, N.Y., 1977.

[11] Louies S. and Brenig L., in preparation.

[12] Bountis, T.C. et al, J. Math. Phys. 27 (5), pp. 1215, 1986.

[13] Reid, L., Phys. Letters. A 103 (9), pp. 403, 1984.