ON A HYPERSPACE OF COMPACT SUBSETS WHICH IS HOMEOMORPHIC TO A NON-SEPARABLE HILBERT SPACE

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Abstract. Let $X$ be a metrizable space and $\text{Comp}(X)$ be the hyperspace consisting of non-empty compact subsets of $X$ endowed with the Vietoris topology. In this paper, we give a necessary and sufficient condition on $X$ for $\text{Comp}(X)$ to be homeomorphic to a non-separable Hilbert space. Moreover, we consider the topological structure of pair $(\text{Comp}(X), \text{Fin}(X))$ of hyperspaces of $X$ and its completion $\overline{X}$, where $\text{Fin}(X)$ is the hyperspace of non-empty finite sets in $X$.

1. Introduction

Throughout this paper, spaces are metrizable, maps are continuous and $\kappa$ is an infinite cardinal. Given a space $X$, let $\text{Comp}(X)$ be the hyperspace of non-empty compact sets in $X$ with the Vietoris topology. The hyperspace $\text{Comp}(X)$ is a classical object and has been studied in infinite-dimensional topology, see, for instance, [2, 3]. D.W. Curtis [2] gave a necessary and sufficient condition on $X$ for $\text{Comp}(X)$ to be homeomorphic to the separable Hilbert space as follows:

**Theorem 1.1.** A space $X$ is separable, connected, locally connected, topologically complete and nowhere locally compact if and only if $\text{Comp}(X)$ is homeomorphic to the separable Hilbert space.

We denote the Hilbert space of density $\kappa$ by $\ell_2(\kappa)$. For a space $X$, an $X$-manifold is a topological manifold modeled on $X$. In the non-separable case, combining the result of [15] with the open embedding theorem and the classification theorem of Hilbert manifolds, refer to [6], we can establish the following:

**Theorem 1.2.** If a space $X$ is a connected $\ell_2(\kappa)$-manifold, then $\text{Comp}(X)$ is homeomorphic to $\ell_2(\kappa)$.

In this paper, we characterize a space $X$ whose hyperspace $\text{Comp}(X)$ is homeomorphic to a Hilbert space of density $\kappa$ as follows:

**Main Theorem.** A space $X$ is connected, locally connected, topologically complete, nowhere locally compact, and for each point $x \in X$, any neighborhood of $x$ in $X$ is of density $\kappa$ if and only if $\text{Comp}(X)$ is homeomorphic to $\ell_2(\kappa)$.

Let $\text{Fin}(X) \subset \text{Comp}(X)$ be the hyperspace of non-empty finite subsets of a space $X$. By $\ell_2^f(\kappa)$, we mean the linear subspace spanned by the canonical orthonormal basis of $\ell_2(\kappa)$. D.W. Curtis and N.T. Nhu [3] characterized a space $X$ whose hyperspace $\text{Fin}(X)$ is homeomorphic to $\ell_2^f(\omega)$, and the author [8] generalized it as follows:

**Theorem 1.3.** A space $X$ is connected, locally path-connected, strongly countable-dimensional, $\sigma$-locally compact, and for every point $x \in X$, any neighborhood of $x$ in $X$ is of density $\kappa$ if and only if $\text{Fin}(X)$ is homeomorphic to $\ell_2^f(\kappa)$.

2010 Mathematics Subject Classification. Primary: 54B20, Secondary: 54F65, 57N20.

Key words and phrases. hyperspace, the Vietoris topology, Hilbert space, the Hausdorff metric.

This work was supported by JSPS KAKENHI Grant Number 15K17530.
For spaces $X$ and $Y$, writing $(X, Y)$, we understand $Y$ is a subspace of $X$. A pair $(X, Y)$ of spaces is homeomorphic to $(X', Y')$ if there exists a homeomorphism $f : X \rightarrow X'$ such that $f(Y) = Y'$. A subset $A$ of a space $X$ is locally non-separating in $X$ if for every non-empty connected open set $U$ in $X$, $U \setminus A$ is non-empty and connected. As a corollary of the main theorem and the paper [8], we can establish the following:

**Corollary 1.4.** Let $X$ be a connected, locally path-connected, strongly countable-dimensional and $\sigma$-locally compact space such that for each point $x \in X$, any neighborhood of $x$ in $X$ is of density $\kappa$. Suppose that $X$ has a locally connected and nowhere locally compact completion $\overline{X}$. Then the pair $(\text{Comp}(X), \text{Fin}(X))$ is homeomorphic to $(\ell_2(\kappa), \ell_2(\kappa))$ if and only if $X \setminus X$ is locally non-separating in $\overline{X}$.

2. Notation and Toruńczyk’s characterization of Hilbert manifolds

In this section, we fix some notation and introduce Toruńczyk’s characterization of Hilbert manifolds that will be used to prove the main theorem. We denote the closed unit interval $[0, 1]$ by $I$. Let $X = (X, d)$ be a metric space. For a point $x \in X$ and subsets $A, B \subseteq X$, we define the distance $d(x, A)$ between $x$ and $A$ by $d(x, A) = \inf\{d(x, a) \mid a \in A\}$ and the distance $d(A, B)$ between $A$ and $B$ by $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$. For $\epsilon > 0$, let $B_d(x, \epsilon) = \{x' \in X \mid d(x, x') < \epsilon\}$ and $N_d(A, \epsilon) = \{x' \in X \mid d(x', A) < \epsilon\}$. The diameter of $A \subseteq X$ is denoted by $\text{diam}_d A$. The topology of $\text{Comp}(X)$ is induced by the Hausdorff metric $d_H$ defined as follows:

$$d_H(A, B) = \inf\{r > 0 \mid A \subset N_d(B, r), B \subset N_d(A, r)\}.$$

It is said that a space $X$ has the countable locally finite approximation property if for each open cover $\mathcal{U}$ of $X$, there exists a sequence $\{f_n : X \rightarrow X\}_{n<\omega}$ of maps such that every $f_n$ is $\mathcal{U}$-close to the identity map on $X$ and the family $\{f_n(X)\}_{n<\omega}$ of the images is locally finite in $X$. Recall that for maps $f : X \rightarrow Y$ and $g : X \rightarrow Y$, and for an open cover $\mathcal{U}$ of $Y$, $f$ is $\mathcal{U}$-close to $g$ if for each point $x \in X$, there exists a member $U \in \mathcal{U}$ such that $f(x)$ and $g(x)$ are contained in $U$. For $n < \omega$, a space $X$ has the $\kappa$-discrete $n$-cells property provided that the following condition holds:

- For every open cover $\mathcal{U}$ of $X$ and every map $f : \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow X$, where each $A_\gamma = I^\alpha$, there is a map $g : \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow X$ such that $g$ is $\mathcal{U}$-close to $f$ and $\{g(A_\gamma)\}_{\gamma < \kappa}$ is discrete in $X$.

H. Toruńczyk [11] [12] gave the following celebrated characterization to an $\ell_2(\kappa)$-manifold (cf. [11] Theorem 3.1):

**Theorem 2.1.** A connected space $X$ is an $\ell_2(\kappa)$-manifold if and only if the following conditions are satisfied:

1. $X$ is a topologically complete ANR of density $\kappa$;
2. $X$ has the countable locally finite approximation property;
3. $X$ has the $\kappa$-discrete $n$-cells property for every $n < \omega$.

In the above, replacing “ANR” with “AR”, we can obtain a characterization of a Hilbert space $\ell_2(\kappa)$.

3. Basic topological properties of $\text{Comp}(X)$

In this section, some basic properties of $\text{Comp}(X)$ are collected. On the density of $\text{Comp}(X)$, the following holds [15] Corollary 4.2:

**Proposition 3.1.** The hyperspace $\text{Comp}(X)$ has the same density as a space $X$.

**Proposition 3.2.** Let $X$ be a space and $x \in X$. If any neighborhood of $\{x\}$ in $\text{Comp}(X)$ is of density $\kappa$, then every neighborhood of $x$ in $X$ is of density $\kappa$. 
Proof. Since \( \text{Comp}(X) \) is of density \( \kappa \), \( X \) is also of density \( \kappa \) by Proposition 3.1. Suppose that the point \( x \in X \) has a neighborhood \( U \) of density \( < \kappa \). Observe that \( \text{Comp}(U) \subset \text{Comp}(X) \) is a neighborhood of \( \{x\} \). Due to Proposition 3.1, the density of \( \text{Comp}(U) \) is less than \( \kappa \), which is a contradiction. Hence all neighborhoods of \( x \) are of density \( \kappa \). \( \square \)

We can easily observe the following (cf. [9, Theorem 5.12.5. (2)]):

**Proposition 3.3.** For every space \( X \), \( X \) is topologically complete if and only if \( \text{Comp}(X) \) is topologically complete.

Concerning the ANR-property of \( \text{Comp}(X) \), the following holds [14,10] (cf. [3, Theorem 1.6]):

**Proposition 3.4.** Let \( X \) be topologically complete. Then \( X \) is connected and locally connected if and only if \( \text{Comp}(X) \) is an AR.

4. **The countable locally finite approximation property of \( \text{Comp}(X) \)**

In this section, using the similar strategy in the proof of Theorem E of [2], we shall verify the countable locally finite approximation property of the hyperspace \( \text{Comp}(X) \). Let \( K \) be a simplicial complex. We denote the polyhedron\(^3\) of \( K \) by \( |K| \) and the \( n \)-skeleton of \( K \) by \( K^{(n)} \) for each \( n < \omega \). We often regard \( \sigma \in K \) as a simplicial complex consisting of its faces. The next two lemmas concerning nice subdivisions of simplicial complexes are used in the proof of Theorem E of [2] and the details of the proofs is given in [8].

**Lemma 4.1.** Let \( X = (X,d) \) be a metric space, \( K \) be a simplicial complex and \( f : |K| \to X \) be a map. For every map \( \alpha : X \to (0,\infty) \), there is a subdivision \( K' \) of \( K \) such that \( \text{diam}_d f(\sigma) < \inf_{x \in \sigma} \alpha(f(x)) \) for any \( \sigma \in K' \).

**Lemma 4.2.** For each map \( \alpha : |K| \to (0,\infty) \) of the polyhedron of a simplicial complex \( K \) and \( \beta > 1 \), \( K \) has a subdivision \( K' \) such that \( \sup_{x \in \sigma} \alpha(x) < \beta \inf_{x \in \sigma} \alpha(x) \) for all \( \sigma \in K' \).

The following two lemmas are also used in the proof of Theorem E of [2]. The analogues of these lemmas for hyperspaces of finite subsets are proved in [8].

**Lemma 4.3.** Let \( X = (X,d) \) be a metric space. Suppose that \( \{A_n\}_{n<\omega} \) is a sequence in \( \text{Comp}(X) = (\text{Comp}(X),d_H) \) converging to \( A \in \text{Comp}(X) \). Then for each closed set \( B_n \subset A_n \), \( \{B_n\}_{n<\omega} \) has a subsequence that converges to some compact subset \( B \subset A \).

**Lemma 4.4.** Let \( X = (X,d) \) be a locally path-connected metric space and \( \alpha : \text{Comp}(X) \to (0,\infty) \) be a map. Then there is a map \( \beta : \text{Comp}(X) \to (0,\infty) \) such that for any \( A \in \text{Comp}(X) \), each point \( x \in N_d(A,\beta(A)) \) has an arc connecting to some point of \( A \) of diameter \( < \alpha(A) \).

Now, we show the following:

**Proposition 4.5.** Let \( X \) be a locally path-connected and nowhere locally compact space. Then \( \text{Comp}(X) \) has the countable locally finite approximation property.

**Proof.** Let \( U \) be an open cover of \( \text{Comp}(X) \). Take an open cover \( V \) of \( \text{Comp}(X) \) that is a star-refinement of \( U \). Since \( \text{Comp}(X) \) is an ANR by Theorem 1.6 of [3], there are a simplicial complex \( K \) and maps \( f : \text{Comp}(X) \to |K|, g : |K| \to \text{Comp}(X) \) such that \( gf \) is \( V \)-close to the identity map on \( \text{Comp}(X) \), refer to [9, Theorem 6.6.2]. It suffices to construct a map \( g_i : |K| \to \text{Comp}(X) \) for each \( i < \omega \) so that \( g_i \) is \( V \)-close to \( g \) and \( \{g_i(|K|)\}_{i<\omega} \) is locally finite in \( \text{Comp}(X) \). Then \( \{g_i f\}_{i<\omega} \) will be the desired sequence of maps.

\(^3\)In this paper, we do not need polyhedra to be metrizable.
Take an admissible metric $d$ on $X$ and a map $\alpha : \text{Comp}(X) \to (0, 1)$ so that the family \(\{B_{d_{\alpha}}(A, 2\alpha(A)) \mid A \in \text{Comp}(X)\}\) refines $V$. Since $X$ is locally path-connected, according to Lemma 4.4, there exists a map $\beta : \text{Comp}(X) \to (0, 1)$ such that for every $A \in \text{Comp}(X)$, each point $x \in N_d(A, \beta(A))$ has an arc from some point of $A$ to $x$ of diameter $< \alpha(A)/2$. We may assume that $\beta(A) \leq \alpha(A)/2$ for every $A \in \text{Comp}(X)$. Combining Lemmas 4.1 with 4.2, we can replace $K$ with a subdivision so that for every $\sigma \in K$,

1. \(\text{diam}_{d_{\beta}}(\sigma) < \inf_{y \in \sigma} \beta g(y)/2\),
2. \(\sup_{y \in \sigma} \beta g(y) < 2 \inf_{y \in \sigma} \beta g(y)\),
3. \(\sup_{y \in \sigma} \alpha g(y) < 4 \inf_{y \in \sigma} \alpha g(y)/3\).

For each $n < \omega$, we can find a locally finite open cover $V_n$ of $X$ of mesh $< 1/n$. Since $X$ is nowhere locally compact, for every $n < \omega$ and $\emptyset \neq V \in \mathcal{V}_n$, $V$ contains an infinite subset $Z(V) = \{z_i^V \mid i < \omega\}$ that is discrete in $X$. Let $Z^i(n) = \{z_i^V \mid V \in \mathcal{V}_n\}$, $i < \omega$. Here we may assume that $Z^i(n) \cap Z^j(n) = \emptyset$ if $i \neq j$. Indeed, it will be shown by induction. Suppose that for some $i < \omega$, $Z^i(n) \cap Z^j(n) = \emptyset$ if $j < k < i$. By the local finiteness of $\mathcal{V}_n$, for every $z_i^V \in Z^i(n)$, the family $\{V \in \mathcal{V}_n \mid z_i^V \in V\}$ is finite. Since $X$ is nowhere locally compact and each $Z(V)$ is discrete, we can find a point $x_i^V \in V$ sufficiently close to $z_i^V$ so that $x_i^V \notin \bigcup_{j < i} Z^j(n)$ and even if $z_i^V$ is substituted by $x_i^V$, $Z(V)$ is still discrete. Due to this substitution, we have that $Z^i(n) \cap Z^i(n) = \emptyset$ for all $j < i$. Put $Z(n) = \bigcup_{\emptyset \neq V \in \mathcal{V}_n} Z(V) = \bigoplus_{i < \omega} Z^i(n)$, so it is locally finite in $X$.

First, we shall construct the restriction $g_i|_{K^{(0)}}$, $i < \omega$. For every $v \in K^{(0)}$, there is $n_v \geq 2$ such that $1/n_v < \beta g(v)/4 \leq 1/(n_v - 1)$. Then we can find a point $z^i(v) \in Z^i(n_v) \subset Z(n_v)$, $i < \omega$, so that $d(z^i(v), g(v)) < 1/n_v$. Note that for any $v', v'' \in K^{(0)}$ with $n_{v'} = n_{v''}$, $z^i(v') \neq z^i(v'')$ if $i \neq j$. Let $g_i(v) = g(v) \cup \{z^i(v)\} \in \text{Comp}(X)$, so

$$d_H(g(v), g_i(v)) < 1/n_v < \beta g(v)/4 \leq \alpha g(v)/2.$$  

Next, we will extend each $g_i$ over the 1-skeleton $[K^{(1)}]$. Let $\sigma \in K^{(1)} \setminus K^{(0)}$, $\sigma^{(0)} = \{v_1, v_2\}$ and $\hat{\sigma}$ be its barycenter. According to conditions (1) and (2), we get for any $m = 1, 2$,

$$d(z^i(m, \sigma), g(\hat{\sigma})) \leq d(z^i(v_m), g(v_m)) + d_H(g(v_m), g(\hat{\sigma})) < \beta g(v_m)/4 + \text{diam}_{d_{\beta}} g(\sigma)$$  

$$< \sup_{y \in \sigma} \beta g(y)/4 + \inf_{y \in \sigma} \beta g(y)/2 < \inf_{y \in \sigma} \beta g(y) \leq \beta g(\hat{\sigma}).$$

Applying Lemma 4.3, we can take an arc $\gamma_m : I \to X$ from some point of $g(\hat{\sigma})$ to $z^i(v_m)$ of $\text{diam}_{d_{\beta}} \gamma_m(I) < \alpha g(\hat{\sigma})/2$. Put $g_i(\hat{\sigma}) = g(\hat{\sigma}) \cup \{z^i(v_m) \mid m = 1, 2\}$. Then $d_H(g(\hat{\sigma}), g_i(\hat{\sigma})) \leq \beta g(\hat{\sigma}) \leq \alpha g(\hat{\sigma})/2$. Let $\phi : I \to \text{Comp}(X)$ be a map defined by $\phi(t) = g(\hat{\sigma}) \cup \{\gamma_m(t) \mid m = 1, 2\}$, which is a path from $g(\hat{\sigma})$ to $g_i(\hat{\sigma})$. For each $m = 1, 2$, define $g_i : \langle v_m, \hat{\sigma} \rangle \to \text{Comp}(X)$, where $\langle v_m, \hat{\sigma} \rangle$ is the segment between $v_m$ and $\hat{\sigma}$, as follows:

$$g_i((1-t)v_m + t\hat{\sigma}) = \begin{cases} g((1-2t)v_m + 2t\hat{\sigma}) \cup \{z^i(v_m)\} & \text{if } 0 \leq t \leq 1/2, \\
\phi(2t-1) \cup \{z^i(v_m)\} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Then for each $y \in \sigma$, if $y = (1-t)v_m + t\hat{\sigma}$, $0 \leq t \leq 1/2$,

$$d_H(g(\hat{\sigma}), g_i(y)) \leq \max\{d_H(g(\hat{\sigma}), g((1-2t)v_m + 2t\hat{\sigma})), d(g(\hat{\sigma}), z^i(v_m))\}$$  

$$\leq \max\{\text{diam}_{d_{\beta}} g(\sigma), \beta g(\hat{\sigma})\} \leq \max\{\inf_{y' \in \sigma} \beta g(y')/2, \beta g(\hat{\sigma})\}$$  

$$\leq \beta g(\hat{\sigma}) \leq \alpha g(\hat{\sigma})/2,$$
and if \( y = (1 - t)v_m + t\hat{\sigma}, \) \(1/2 \leq t \leq 1,\)
\[
d_{H}(g(\hat{\sigma}), g_i(y)) \leq \max\{d_{H}(g(\hat{\sigma}), \phi(2t - 1)), d(\hat{\sigma}, z_i(v_m))\}
\leq \max\{\max\{\text{diam}_H \gamma_n(I) \mid n = 1, 2\}, \beta g(\hat{\sigma})\}
\leq \max\{\alpha g(\hat{\sigma})/2, \beta g(\hat{\sigma})\} \leq \alpha g(\hat{\sigma})/2.
\]
It follows from condition (3) that
\[
d_{H}(g(y), g_i(y)) \leq d_{H}(g(y), g(\hat{\sigma})) + d_{H}(g(\hat{\sigma}), g_i(y)) \leq \text{diam}_H g(\sigma) + \alpha g(\hat{\sigma})/2
< \inf_{y' \in \sigma} \beta g(y')/2 + \alpha g(\hat{\sigma})/2 \leq \beta g(\hat{\sigma})/2 + \alpha g(\hat{\sigma})/2 \leq 3\alpha g(\hat{\sigma})/4.
\]
\[
\leq 3\sup_{y' \in \sigma} \alpha g(y')/4 < \inf_{y' \in \sigma} \alpha g(y') \leq \alpha g(y).
\]
Note that each \( g_i(y) \) contains \( z_i'(v_1) \) or \( z_i'(v_2) \).

By induction, we shall construct a map \( g_i : \vert K \vert \to \text{Comp}(X), i < \omega, \) such that for each \( \sigma \in K \setminus K(0) \) and each \( y \in \sigma, \) \( g_i(y) = \bigcup_{a \in A(y)} g_i(a) \) for some \( A(y) \in \text{Fin}(\vert \sigma(1) \vert) \). Assume that \( g_i \) has extended over \( \vert K(0) \vert \) for some \( n < \omega \) such that for every \( \sigma \in K(0) \setminus K(0) \) and \( y \in \sigma, \)
\( g_i(y) = \bigcup_{a \in A(y)} g_i(a) \) for some \( A(y) \in \text{Fin}(\vert \sigma(1) \vert) \). Take any \( n + 1 \)-simplex \( \sigma \in K(n+1) \setminus K(n). \)
Due to Lemma 3.3 of \( \mathbb{[3]} \), there exists a map \( r : \sigma \to \text{Fin}(\partial \sigma) \) such that \( r(y) = \{ y \} \) for all \( y \in \partial \sigma, \) where \( \partial \sigma \) means the boundary of \( \sigma. \) The restriction \( g_i|_{\partial \sigma} \) induces \( \tilde{g}_i : \text{Fin}(\partial \sigma) \to \text{Comp}(X) \) defined by \( \tilde{g}_i(A) = \bigcup_{a \in A} g_i(a). \) Then the composition \( g_i,\sigma = \tilde{g}_i r : \sigma \to \text{Comp}(X) \) satisfies that \( g_i,\sigma|_{\partial \sigma} = g_i|_{\partial \sigma}. \) Observe that for each \( y \in \sigma, \)

\[
g_i,\sigma(y) = \tilde{g}_i r(y) = \bigcup_{y' \in r(y)} g_i(y') = \bigcup_{y' \in r(y)} \bigcup_{a \in A(y')} g_i(a) = \bigcup_{a \in A(y')} g_i(a),
\]
where \( g_i(y') = \bigcup_{a \in A(y')} g_i(a) \) for some \( A(y') \in \text{Fin}(\vert \sigma(1) \vert) \) by the inductive assumption. Therefore \( g_i \) can be extended over \( \vert K(n+1) \vert \) by \( g_i|_{\sigma} = g_i,\sigma \) for all \( \sigma \in K(n+1) \setminus K(n). \)

Completing this induction, we can obtain a map \( g_i : \vert K \vert \to \text{Comp}(X) \) for every \( i < \omega. \) For each \( y \in \sigma \in K \setminus K(0) \) and each \( a \in \vert \sigma(1) \vert, \) by conditions (1) and (3), we have
\[
d_{H}(g(y), g_i(a)) \leq d_{H}(g(y), g(a)) + d_{H}(g(a), g_i(a)) \leq \text{diam}_H g(\sigma) + \alpha g(a)
< \inf_{y' \in \sigma} \beta g(y')/2 + \sup_{y' \in \sigma} \alpha g(y') \leq \inf_{y' \in \sigma} \alpha g(y')/4 + 4\inf_{y' \in \sigma} \alpha g(y')/3
= 19\inf_{y' \in \sigma} \alpha g(y')/12 < 2\alpha g(y).
\]
It follows that
\[
d_{H}(g(y), g_i(y)) = d_{H} \left( g(y), \bigcup_{a \in A(y')} g_i(a) \right) \leq \max_{a \in A(y')} d_{H}(g(y), g_i(a)) < 2\alpha g(y),
\]
which implies that \( g_i \) is \( \mathcal{V} \)-close to \( g. \) Remark that \( z_i'(v) \in g_i(y) \) for some vertex \( v \in \sigma(0). \) Here we may replace \( g_i(y) \) with the union \( g(y) \cup g_i(y) \) for every \( y \in \vert K \vert, \) so we get \( g(y) \subset g_i(y). \) It remains to prove that \( \{g_i(\vert K \vert)\}_{i < \omega} \) is locally finite in \( \text{Comp}(X). \)
Suppose the contrary. Then we can find a subsequence \( \{g_{n_i}\}_{i < \omega} \) of \( \{g_i\}_{i < \omega} \) such that \( \{g_{n_i}(y_i)\}_{i < \omega}, y_i \in \vert K \vert, \) is converging to some \( A \in \text{Comp}(X). \) For simplicity, replace each \( g_{n_i} \) with \( g_i. \) Take the carrier \( \sigma_i \in K \) of \( y_i \) and choose a vertex \( v_i \in \sigma_i(0) \) so that \( z_i'(v_i) \in g_i(y_i). \) Since \( g(y_i) \subset g_i(y_i) \), replacing \( \{g(y_i)\}_{i < \omega} \) with a subsequence, we can obtain a compact subset \( B \subset A \) to which \( \{g(y_i)\}_{i < \omega} \) converges by Lemma 3.3. Thus \( \{\beta g(y_i)\}_{i < \omega} \) converges to \( \beta(B) > 0. \) On the other hand, since \( \{g_i(y_i)\}_{i < \omega} \) converges to \( A, \) any subsequence of \( \{z_i'(v_i)\}_{i < \omega} \) has an accumulation point in \( A. \)
Then \( \{n_i\}_{i<\omega} \) diverges to \( \infty \). Indeed, supposing the contrary, we can find \( n_0 < \omega \) and replace \( \{n_i\}_{i<\omega} \) with a subsequence so that \( n_i = n_0 \) for all \( i < \omega \). By the choice of \( z^i(v_i) \), \( \{z^i(v_i)\}_{i<\omega} \) is pairwise distinct and contained in the locally finite subset \( Z(n_0) \), which is a contradiction. Hence \( \{\beta g(v_i)\}_{i<\omega} \) converges to \( 0 \) because \( 1/n_i < \beta g(v_i)/4 \leq 1/(n_i - 1) \). Moreover, by condition (2), for every \( i < \omega \),

\[
\beta g(v_i) \leq \sup_{y \in \Sigma_i} \beta g(y) < 2 \inf_{y \in \Sigma_i} \beta g(y) \leq 2 \beta g(v_i),
\]

so \( \{\beta g(v_i)\}_{i<\omega} \) also converges to \( 0 \). This is a contradiction. Consequently, \( \{g_i(|K|)\}_{i<\omega} \) is locally finite in \( \text{Comp}(X) \). \( \Box \)

5. The \( \kappa \)-discrete \( n \)-cells property of \( \text{Comp}(X) \)

This section is devoted to the verification of the \( \kappa \)-discrete \( n \)-cells property in \( \text{Comp}(X) \). To detect the \( \kappa \)-discrete \( n \)-cells property in a space, the following two lemmas are useful.

**Lemma 5.1.** \( \Box \) [Lemma 3.1] Let \( n < \omega \). A space \( X \) has the \( \kappa \)-discrete \( n \)-cells property if and only if for each open cover \( \mathcal{U} \) of \( X \), and each map \( f : \bigoplus_{\gamma < \kappa} A_\gamma \to X \), where each \( A_\gamma = \Gamma^n \), there is a map \( g : \bigoplus_{\gamma < \kappa} A_\gamma \to X \) such that \( g \) is \( \mathcal{U} \)-close to \( f \) and \( \{g(A_\gamma)\}_{\gamma < \kappa} \) is locally finite in \( X \).

**Lemma 5.2.** \( \Box \) [Lemma 3.2] Let \( X \) be a space with the countable locally finite approximation property and \( n < \omega \). The space \( X \) has the \( \kappa \)-discrete \( n \)-cells property if and only if it has the \( \lambda \)-discrete \( n \)-cells property for every \( \lambda \leq \kappa \) of uncountable cofinality.

The following lemma can be easily observed, refer to the proof of \( \Box \) [Lemma 6.2]:

**Lemma 5.3.** Let \( X \) be a space and \( \kappa \) be of uncountable cofinality. If a subset \( A \subset X \) is of density \( \geq \kappa \), then \( A \) contains a discrete subset of cardinality \( \geq \kappa \).

Now, we get the following:

**Proposition 5.4.** Let \( X \) be locally path-connected and nowhere locally compact. Suppose that any neighborhood of each point in \( X \) is of density \( \geq \kappa \). Then the hyperspace \( \text{Comp}(X) \) has the \( \kappa \)-discrete \( n \)-cells property for every \( n < \omega \).

**Proof.** According to Proposition 4.5, the hyperspace \( \text{Comp}(X) \) has the countable locally finite approximation property. Hence Lemma 5.2 guarantees that we may only consider \( \kappa \) be of uncountable cofinality. Let \( n < \omega \) and \( \mathcal{V} \) be an open cover of \( \text{Comp}(X) \). By virtue of Lemma 5.1, it is sufficient to show that for any map \( g : \bigoplus_{\gamma < \kappa} A_\gamma \to \text{Comp}(X) \), where each \( A_\gamma = \Gamma^n \), there is a map \( g_\gamma : A_\gamma \to \text{Comp}(X) \), \( \gamma < \kappa \), such that \( g_\gamma \) is \( \mathcal{V} \)-close to \( g|_{A_\gamma} \) and \( \{g_\gamma(A_\gamma)\}_{\gamma < \kappa} \) is locally finite in \( \text{Comp}(X) \).

Take an admissible metric \( d \) on \( X \) and the same maps \( \alpha, \beta : \text{Comp}(X) \to (0,1) \) as in Proposition 4.5. Moreover, combining Lemmas 4.1 with 4.2, we can triangulate each \( A_\gamma \) into a simplicial complex \( K_\gamma \) satisfying the same conditions (1), (2) and (3) as in Proposition 4.5. For each \( m < \omega \), choose a locally finite open cover \( \mathcal{V}_m \) of \( X \) of mesh \( < 1/m \). By the assumption and Lemma 5.3 for each \( m < \omega \) and each non-empty \( V \in \mathcal{V}_m \), there is a discrete subset \( Z(V) \subset V \) of cardinality \( \geq \kappa \). By the local finiteness of \( \mathcal{V}_m \), the subset \( Z(m) = \bigcup_{\emptyset \neq V \in \mathcal{V}_m} Z(V) \) is locally finite in \( X \).

We construct the desired map \( g_\gamma \) for each \( \gamma < \kappa \). For every \( v \in K_\gamma \), there is \( m_v \geq 2 \) such that \( 1/m_v < \beta g(v)/4 \leq 1/(m_v - 1) \). Then we can choose a point \( z^{\gamma}(v) \in Z(m_v) \), \( \gamma < \kappa \), so that \( d(z^{\gamma}(v),g_\gamma(v)) < 1/m_v \) and for any \( v', v'' \in \bigcup_{\gamma < \kappa} K_\gamma \) with \( m_{v'} = m_{v''} \), \( z^{\gamma}(v') \neq z^{\gamma}(v'') \) if \( \gamma < \gamma' < \kappa \), where we may adjust each \( Z(m_v) \) as in the proof of Proposition 4.5 if necessary. Define \( g_\gamma(v) = g(v) \cup \{z^{\gamma}(v)\} \in \text{Comp}(X) \). The rest of the proof follows from the same argument as in Proposition 4.5. \( \Box \)
6. Proof of the main theorem

In this final section, we shall prove the main theorem.

Proof of the main theorem. In the case that \(X\) is separable, the proof follows from Theorem 1.1. Let \(\kappa\) be uncountable.

(The “only if” part) According to Propositions 3.1, 3.2 and 3.3, the hyperspace \(\text{Comp}(X)\) is a topologically complete AR of density \(\kappa\). Since \(X\) is connected, locally connected and topologically complete, it is locally path-connected. Due to Proposition 1.5, \(\text{Comp}(X)\) has the countable locally finite approximation property. Moreover, the \(\kappa\)-discrete \(n\)-cells property, \(n < \omega\), of \(\text{Comp}(X)\) follows from Proposition 5.4. Using Toruńczyk’s characterization [2.1], we have that \(\text{Comp}(X)\) is homeomorphic to \(\ell_2(\kappa)\).

(The “if” part) Since \(\text{Comp}(X)\) is homeomorphic to \(\ell_2(\kappa)\), it is a topologically complete AR, and hence, by Propositions 3.2 and 3.3, \(X\) is connected, locally connected and topologically complete. Remark that for each \(A \in \text{Comp}(X)\), all neighborhoods of \(A\) are of density \(\kappa\). It follows from Proposition 3.2 that any neighborhood of each point in \(X\) is also of density \(\kappa\), and hence \(X\) is nowhere locally compact. Thus the proof is complete. \(\square\)

7. Pair of hyperspaces

In this final section, we will discuss the topological structure of pair of hyperspaces. A subset \(A\) of a space \(X\) is said to be homotopy dense in \(X\) if there exists a homotopy \(h : X \times I \to X\) such that \(h(x, 0) = x\) for all \(x \in X\) and \(h(X \times (0, 1)) \subset A\). To show Corollary 1.4, we will use the following characterization of the pair \((\ell_2(\kappa), \ell_2^I(\kappa))\) [13, 7]:

Theorem 7.1. A pair \((X, Y)\) of spaces is homeomorphic to \((\ell_2(\kappa), \ell_2^I(\kappa))\) if and only if \(X\) is homeomorphic to \(\ell_2(\kappa)\), \(Y\) is homeomorphic to \(\ell_2^I(\kappa)\) and \(Y\) is homotopy dense in \(X\).

We denote the \(n\)-dimensional unit sphere by \(S^n\) and the \(n\)-dimensional unit ball by \(B^n\). The homotopy density between ANRs is characterized as follows [9, Corollary 7.4.6]:

Lemma 7.2. Suppose that \(X\) and \(Y\) are ANRs and \(Y\) is dense in \(X\). Then \(Y\) is homotopy dense in \(X\) if and only if the following condition holds:

- For each point \(x \in X\) and each neighborhood \(U\) of \(x\) in \(X\), there is a neighborhood \(V \subset U\) of \(x\) such that any map \(f : S^n \to V \cap Y\) can extend to a map \(\tilde{f} : B^{n+1} \to U \cap Y\) for all \(n < \omega\).

Using this lemma, we shall prove Corollary 1.4.

Proof of Corollary 1.4. The main theorems of this paper and the paper [8] guarantee that \(\text{Comp}(\overline{X})\) is homeomorphic to \(\ell_2(\kappa)\) and \(\text{Fin}(X)\) is homeomorphic to \(\ell_2^I(\kappa)\). By virtue of Theorem 7.1 it remains to prove that \(\text{Fin}(X)\) is homotopy dense in \(\text{Comp}(\overline{X})\) if and only if \(\overline{X} \setminus X\) is locally non-separating in \(\overline{X}\).

The “only if” part follows from the same argument as the proof of implication (ii) \(\Rightarrow\) (iii) in [4, Theorem 3.2]. We shall prove the “if” part. As is easily observed, \(\text{Fin}(X)\) is dense in \(\text{Comp}(\overline{X})\). Due to Lemma 7.2, we need only to show that for each point \(A \in \text{Comp}(\overline{X})\) and each neighborhood \(U\) of \(A\) in \(\text{Comp}(\overline{X})\), there is a neighborhood \(V \subset U\) of \(A\) such that any map \(f : S^n \to V \cap \text{Fin}(X)\) can extend to a map \(\tilde{f} : B^{n+1} \to U \cap \text{Fin}(X)\) for all \(n < \omega\). Let \(A \in \text{Comp}(\overline{X})\) and \(U\) be a neighborhood of \(A\) in \(\text{Comp}(\overline{X})\). By the local connectedness of \(\overline{X}\) and the compactness of \(A\), there exist a finite number of points \(a_1, \ldots, a_n \in A\) and connected open neighborhoods \(U_i\) of \(a_i\),
Moreover, for every $\emptyset \subseteq \bigcup_{i=1}^{n} U_i$ and for any $B \in \text{Comp}(\overline{X})$, $B \in \mathcal{U}$ if $B \subseteq \bigcup_{i=1}^{n} U_i$ and $B \cap U_i \neq \emptyset$ for all $i \in \{1, \ldots, n\}$. Let

$$\mathcal{V} = \left\{ B \in \text{Comp}(\overline{X}) \mid B \subseteq \bigcup_{i=1}^{n} U_i \text{ and } B \cap U_i \neq \emptyset \text{ for every } 1 \leq i \leq n \right\}.$$ 

To show that $\mathcal{V}$ is the desired neighborhood of $A$, fix an arbitrary map $f : S^n \to \mathcal{V} \cap \text{Fin}(X)$.

(1) $n = 0$. Note that $S^0 = \{0, 1\}$. It suffices to construct a path between $f(0)$ and $f(1)$ in $\mathcal{V} \cap \text{Fin}(X)$. Since $f(0), f(1) \in \mathcal{V} \cap \text{Fin}(X)$, for each $x \in f(1)$, we can choose $i(x) \in \{1, \ldots, n\}$ and $y(x) \in f(0)$ so that $x, y(x) \in U_{i(x)} \cap X$. Since $\overline{X} \setminus X$ is locally non-separating in $\overline{X}$, $U_{i(x)} \cap X$ is connected. Hence the open subset $U_{i(x)} \cap X$ is path-connected because $X$ is locally path-connected. So there is a path $\gamma(x)$ from $y(x)$ to $x$ in $U_{i(x)} \cap X$. Then we can define a path $\phi : I \to \text{Fin}(X)$ from $f(0)$ to $f(0) \cup f(1)$ by $\phi(t) = f(0) \cup \{ \gamma(x)(t) \mid x \in f(1) \}$. Observe that $\phi(I) \subseteq \mathcal{V}$. By the same argument, we can obtain a path $\psi : I \to \mathcal{V} \cap \text{Fin}(X)$ from $f(1)$ to $f(0) \cup f(1)$. Join the paths $\phi$ and $\psi$, so we can take a path between $f(0)$ and $f(1)$ that is contained in $\mathcal{V} \cap \text{Fin}(X)$.

(2) $n \geq 1$. The map $f$ induces $\tilde{f} : \text{Fin}(S^n) \to \text{Fin}(X)$ defined by $\tilde{f}(B) = \bigcup_{b \in B} f(b)$. Due to Lemma 3.3 of [5], there exists a map $r : B^{n+1} \to \text{Fin}(S^n)$ such that $r(y) = \{y\}$ for all $y \in S^n$. Let $\tilde{f} = fr$, that is the desired extension of $f$. Indeed, for each $x \in S^n$, we have

$$\tilde{f}(x) = fr(x) = \tilde{f}(\{x\}) = f(x).$$

Moreover, for every $x \in B^{n+1}$, it follows that

$$\tilde{f}(x) = fr(x) = \bigcup_{b \in r(x)} f(b) \in \mathcal{V} \cap \text{Fin}(X)$$

because each $f(b) \in \mathcal{V} \cap \text{Fin}(X)$. Consequently, $\mathcal{V}$ is the desired neighborhood, so $\text{Fin}(X)$ is homotopy dense in $\text{Comp}(\overline{X})$. The proof is complete. $\square$

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