New function classes of Morrey–Campanato type and their applications

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Abstract
The aim of this paper is to introduce and investigate some new function classes of Morrey–Campanato type. The domains \( \Omega \subset \mathbb{R}^n \) in this paper are supposed to satisfy the following property: there exists a constant \( A > 0 \) such that for all \( x_0 \in \Omega, \rho < \text{diam } \Omega \), we have \( |Q(x_0, \rho) \cap \Omega| \geq A \rho^n \). Let \( 0 < p < \infty \) and \( 0 \leq \lambda \leq n + p \). We say that \( f \in \mathring{L}^{p,\lambda}(\Omega) \) if

\[
\sup_{x_0 \in \Omega, \rho > 0} \rho^{-\lambda} \int_{Q(x_0, \rho)} |f(x) - |f|_{\Omega(x_0, \rho)}|^p \, dx < \infty,
\]

where \( \Omega(x_0, \rho) = Q(x_0, \rho) \cap \Omega \) and \( Q(x, \rho) \) is denote the cube of \( \mathbb{R}^n \). Some basic properties and characterizations of these classes are presented. If \( 0 \leq \lambda < n \), the space is equivalent to related Morrey space under certain conditions. If \( \lambda = n \), then \( f \in \mathring{L}^{p,n}(\Omega) \) if and only if \( f \in \text{BMO}(\Omega) \) with \( f^- \in L^\infty(\Omega) \), where \( f^- = - \min\{0, f\} \). If \( n < \lambda \leq n + p \), the \( \mathring{L}^{p,\lambda}(\Omega) \) functions establish an integral characterization of the nonnegative Hölder continue functions. As applications, this paper gives unified criteria on the necessity of bounded commutators of maximal functions on ball Banach function spaces.

Keywords Morrey–Campanato space · BMO function · Hölder continue functions · Commutator · Maximal operators

Mathematics Subject Classification 46E36 · 46E35
1 Introduction

The domains $\Omega \subset \mathbb{R}^n$ in this paper are supposed to satisfy the following property: there exists a constant $A > 0$ such that for all $x_0 \in \Omega$, $\rho < \text{diam} \Omega$ we have

$$|Q(x_0, \rho) \cap \Omega| \geq A\rho^n,$$

where $Q(x, \rho)$ is denote the cube of $\mathbb{R}^n$, with sides parallel to the coordinate axes, having a center at $x$ and side $2\rho$. We set $\Omega(x_0, \rho) := Q(x_0, \rho) \cap \Omega$. Note that every domain of class $C^1$ or Lipschitz has the above property.

Let $0 < p < +\infty$ and $\lambda \geq 0$. Define the Morrey–Campanato space

$$L^{p,\lambda}(\Omega) := \left\{ f \in L^p(\Omega) : \sup_{x_0 \in \Omega, \rho > 0} \rho^{-\lambda} \int_{\Omega(x_0, \rho)} |f(x) - f_{\Omega(x_0, \rho)}|^p \, dx < \infty \right\},$$

endowed with the norm defined by

$$\|f\|_{L^{p,\lambda}(\Omega)} := \sup_{x_0 \in \Omega, \rho > 0} \left( \rho^{-\lambda} \int_{\Omega(x_0, \rho)} |f(x) - f_{\Omega(x_0, \rho)}|^p \, dx \right)^{1/p} + \|f\|_{L^p(\Omega)}.$$

We list some known results on the structure of the Morrey–Campanato spaces.

(i) $\lambda = 0$. It is obvious that $L^{p,0}(\Omega) = L^p(\Omega)$.

(ii) $0 < \lambda < n$. $L^{p,\lambda}(\Omega)$ is equivalent to the Morrey space $L^{p,\lambda}(\Omega)$, i.e.

$$L^{p,\lambda}(\Omega) := \left\{ f \in L^p(\Omega) : \sup_{x_0 \in \Omega, \rho > 0} \rho^{-\lambda} \int_{\Omega(x_0, \rho)} |f(x)|^p \, dx < \infty \right\},$$

endowed with the norm defined by

$$\|f\|_{L^{p,\lambda}(\Omega)} := \sup_{x_0 \in \Omega, \rho > 0} \rho^{-\lambda} \int_{\Omega(x_0, \rho)} |f(x)|^p \, dx.$$

This was proved by Campanato [5].

(iii) $\lambda = n$. $L^{1,n}(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$, the spaces of bounded mean oscillation. The crucial property of BMO functions is the John–Nirenberg inequality [15],

$$|\{ x \in Q : |f(x) - f_Q| > \lambda \}| \leq c_1 |Q| e^{-\frac{\lambda^2}{\|f\|_{\text{BMO}(\mathbb{R}^n)}}}, \lambda > 0,$$

where $c_1$ and $c_2$ depend only on the dimension. A well-known immediate corollary of the John–Nirenberg inequality as follows:

$$\|f\|_{\text{BMO}(\mathbb{R}^n)} \approx \sup_Q \frac{1}{|Q|} \left( \int_Q |f(x) - f_Q|^p \, dx \right)^{1/p},$$

for all $1 < p < \infty$. In fact, the equivalence also holds for $0 < p < 1$. See, for example, the work of Strömberg [25](or [10] and [33] for the general case).
(iv) \( n < \lambda \leq n + p \). \( \mathcal{L}^{\lambda,p}(\Omega) = C^{0,\alpha}(\Omega) \) with \( \alpha = (\lambda - n)/p \). For \( 0 < \alpha \leq 1 \), the Hölder continuous functions \( C^{0,\alpha}(\Omega) \) is the set of functions \( f \) such that

\[
\|f\|_{C^{0,\alpha}(\Omega)} := \sup_{x, y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < \infty.
\]

This was shown independently by Campanato [4] and by Meyers [19] for \( 1 \leq p < \infty \), and by the first author, Zhou and Teng [30] for \( 0 < p < 1 \).

The Morrey–Campanato spaces on Euclidean spaces play an important role in the study of partial differential equation; see [20, 22]. Campanato spaces are useful tools in the regularity theory of PDEs as a result of their better structures, which allow us to give an integral characterization of the spaces of Hölder continuous functions. This also allows generalization of the classical Sobolev embedding theorems [16–18]. It is also well-known result that Campanato space is the dual space of related Hardy space [29]. Other types of Morrey–Campanato function have also received attention. Duong and Yan [9] introduced new function spaces of BMO type, which are defined by variants of maximal functions associated with generalized approximations to the identity. Later, Tang [27] generalized the results to the new function spaces of Morrey–Campanato type. In 2000, Bastero, Milman and Ruiz [1] studied new function classes of BMO type via replacing \( f_Q \) by \( M_Q(f) \), where

\[
M_Q(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy.
\]

They proved that for \( 1 \leq p < \infty \),

\[
\sup_Q \frac{1}{|Q|} \int_Q |f(x) - M_Q(f)(x)|^p dx < \infty,
\]

if and only if \( f \in \text{BMO}(\mathbb{R}^n) \) with \( f^- \in L^\infty(\mathbb{R}^n) \).

Inspired by the above, we investigate some new function classes of Morrey–Campanato type via replacing the mean value of \( f \) in the Campanato norm by the the mean value of \(|f|\). Let \( 0 < p < +\infty \) and \( \lambda \geq 0 \). Define the variant of the Morrey–Campanato class \( \mathcal{L}^{\lambda,p,\Delta}(\Omega) \) with

\[
\|f\|_{\mathcal{L}^{\lambda,p,\Delta}(\Omega)} := \sup_{x_0 \in \Omega, \rho > 0} \rho^{-\Delta} \int_{\Omega(x_0, \rho)} |f(x) - [f]_{\Omega(x_0, \rho)}|^p dx < \infty,
\]

where \([f]_{\Omega(x_0, \rho)} = \frac{1}{|\Omega(x_0, \rho)|} \int_{\Omega(x_0, \rho)} |f(y)|dy\). Some properties and characterizations will be shown.

Meanwhile, the BMO space is special case of Morrey–Campanato spaces, which is one of the important function spaces in harmonic analysis. For example, the singular integral operator maps from \( L^\infty(\mathbb{R}^n) \) to \( \text{BMO}(\mathbb{R}^n) \) space and the dual theory of the classical Hardy space. Moreover, the foundational paper of Coifman, Rochberg and Weiss [8] proved that the commutator...
is bounded on some Lebesgue spaces if and only if $b$ belongs to $\text{BMO}(\mathbb{R}^n)$, where $T$ is the Riesz transforms. The theory was then generalized to several directions, and on the theory of the boundedness of commutators, many results show that BMO function is the right set. Specially, Hu and Yang [13] studied maximal commutators of BMO functions on spaces of homogeneous type. Is BMO still necessary for the boundedness of maximal commutators? In this paper, applying the properties of $\mathcal{L}^{p,\lambda}(\Omega)$, we will studied the necessary condition for the boundedness of maximal commutators on ball Banach function spaces. Ball Banach function spaces were first introduced in [24], which generalizes the Banach function space in [2], and further developed in, for instance, [2, 26, 28, 32, 35–37, 41]. Compared with Banach function spaces, ball Banach function function spaces contain more function spaces. For instance, Morrey spaces, mixed-norm Lebesgue spaces, weighted Lebesgue spaces, and Orlicz-slice spaces are all ball Banach function spaces. Especially, the recent reference [7, 32, 34] are devoted to the characterizations and their applications of (weak or local) Hardy spaces associated with ball Banach function spaces, and [28] studied the compactness characterizations of commutators on ball Banach function spaces.

This paper is organized as follows. In Sect. 2, for $0 < \lambda < n$ and $1 \leq p < \infty$, we obtain the equivalence relationship between $\mathcal{L}^{p,\lambda}(\Omega)$ and certain classical Morrey spaces, applying the methods used in most of the previous works. Section 3 is concerned with BMO function with its negative part bounded. There we settle some technical theorems for the equivalent definitions and characterizations. In Sect. 4, some characterization of $\mathcal{L}^{p,\lambda}(\Omega)$ will be given. Finally, Section 5 contains some further results for the new classes of Morrey–Campanato function, and we establish a general criterion for the necessity of bounded commutators of maximal functions for general Banach spaces.

2 The case $0 \leq \lambda < n$

For $1 \leq p < \infty$ and $0 \leq \lambda < n$, it is obvious that there holds a continuous embedding $L^{p,\lambda}(\Omega) \hookrightarrow \mathcal{L}^{p,\lambda}(\Omega)$. Now, we show that $\|f\|_{L^{p,\lambda}(\Omega)} \approx \|f\|_{\mathcal{L}^{p,\lambda}(\Omega)}$ when $f \in L^p(\Omega)$. The approach for this part is similar to that in [5] but we need to carefully use the properties of $|f|_Q$.

**Theorem 2.1** For $1 \leq p < \infty$, $0 \leq \lambda < n$ and $f \in L^p(\Omega)$, we have

$$
\|f\|_{L^{p,\lambda}(\Omega)} \approx \|f\|_{\mathcal{L}^{p,\lambda}(\Omega)}.
$$

**Proof** We will freely use the inequality

$$(a + b)^p \leq 2^{p-1}(a^p + b^p)$$

valid for every $p \geq 1$. Then
\[
\int_{\Omega(x_0, \rho)} |f(x) - |f|_{\Omega(x_0, \rho)}|^p \, dx \\
\leq 2^{p-1} \left\{ \int_{\Omega(x_0, \rho)} |f(x)|^p \, dx + |\Omega(x_0, \rho)| \left( |f|_{\Omega(x_0, \rho)} \right)^p \right\}
\]

and by Hölder’s inequality

\[
(|f|_{\Omega(x_0, \rho)}|^p \leq \frac{1}{|\Omega(x_0, \rho)|} \int_{\Omega(x_0, \rho)} |f(x)|^p \, dx.
\]

Inserting (3) into (2), divide by \( \rho^d \) to obtain

\[
\|f\|_{L^{p,\lambda}(\Omega)}^p \leq 2^p \|f\|_{L^{p,\lambda}(\Omega)}^p,
\]

thus concluding \( L^{p,\lambda}(\Omega) \subset \mathbb{L}^{p,\lambda}(\Omega) \).

On the other hand, for any fixed cube \( Q(x_0, \rho) \), it is easy to see that

\[
\int_{\Omega(x_0, \rho)} |f(x)|^p \, dx \leq 2^{p-1} \left\{ \int_{\Omega(x_0, \rho)} |f(x) - |f|_{\Omega(x_0, \rho)}|^p \, dx + |\Omega(x_0, \rho)| \left( |f|_{\Omega(x_0, \rho)} \right)^p \right\}.
\]

We need to estimate the term \(|f|_{\Omega(x_0, \rho)}|^p \) only. For \( 0 < r < R \) and \( x_0, x \in \Omega \) we have

\[
|f|_{\Omega(x_0, r)} - |f|_{\Omega(x_0, R)}|^p \leq 2^{p-1} \left\{ |f(x) - |f|_{\Omega(x_0, R)}|^p + |f(x) - |f|_{\Omega(x_0, r)}|^p \right\}.
\]

Integrating with respect to \( x \) on \( \Omega(x_0, r) \), we obtain

\[
|f|_{\Omega(x_0, r)} - |f|_{\Omega(x_0, R)}|^p \leq 2^{p-1} \left( \int_{\Omega(x_0, R)} |f(x) - |f|_{\Omega(x_0, R)}|^p \, dx + \int_{\Omega(x_0, r)} |f(x) - |f|_{\Omega(x_0, r)}|^p \, dx \right),
\]

thus

\[
|f|_{\Omega(x_0, r)} - |f|_{\Omega(x_0, R)}|^p \leq C \frac{1}{r^n} (R^\lambda + r^\lambda) \|f\|_{L^{p,\lambda}(\Omega)}^p \leq C \frac{1}{r^n} R^\lambda \|f\|_{L^{p,\lambda}(\Omega)}^p,
\]

we arrive at

\[
|f|_{\Omega(x_0, r)} - |f|_{\Omega(x_0, R)}|^p \leq C \|f\|_{L^{p,\lambda}(\Omega)}^p R^\lambda r^{-n}.
\]

Set \( R_k = \frac{R}{2^k} \). The inequality (5) gives us that

\[
|f|_{\Omega(x_0, R_k)} - |f|_{\Omega(x_0, R_{k+1})}|^p \leq C \|f\|_{L^{p,\lambda}(\Omega)} R^\lambda \frac{2^k - 2^{k+1}}{r^p}.
\]

It follows from (6) that
We write $R_{h+1} = \rho$ and obtain
\[ |f\|_{\Omega(x_0, R)} - |f\|_{\Omega(x_0, R_{h+1})} | \leq C \|f\|_{L^{p,\lambda}(\Omega)} R_{h+1}^{\frac{\lambda-n}{n}}. \] (7)

We write $R_{h+1} = \rho$ and obtain
\[ |f|_{\Omega(x_0, \rho)}^p \leq 2^{p-1} \left( |f|_{\Omega(x_0, R)}^p + |f|_{\Omega(x_0, \rho)}^p - |f|_{\Omega(x_0, R)}^p \right) \]
\[ \leq 2^{p-1} \left( |f|_{\Omega(x_0, R)}^p + \rho^{\lambda-n} \|f\|_{L^{p,\lambda}(\Omega)}^p \right). \] (8)

If $\text{diam} \Omega < \infty$, choose $R$ such that $\text{diam} \Omega \leq R \leq 2 \text{diam} \Omega$, then
\[ \rho^{n-\lambda} |f|_{\Omega(x_0, R)}^p \leq C |f|_{\Omega(x_0, R)}^p \leq C \|f\|_{L^p(\Omega)}^p, \] (9)

If $\text{diam} \Omega = \infty$, let $R$ big enough such that $R > \rho^{n-\lambda}$, then
\[ \rho^{n-\lambda} |f|_{\Omega(x_0, R)}^p \leq C \frac{\rho^{n-\lambda}}{R^n} \|f\|_{L^p(\Omega)}^p \leq C \|f\|_{L^p(\Omega)}^p. \] (10)

Combining with (4), (8), (9) and (10), we conclude that
\[ \frac{1}{\rho^\lambda} \int_{\Omega(x_0, \rho)} |f(x)|^p \, dx \leq C \left( \|f\|_{L^{p,\lambda}(\Omega)}^p + \rho^{n-\lambda} |f|_{\Omega(x_0, R)}^p \right) \]
\[ \leq C \left( \|f\|_{L^{p,\lambda}(\Omega)}^p + \|f\|_{L^p(\Omega)}^p \right). \]

Then, $f \in L^p(\Omega) \cap L^{p,\lambda}(\Omega)$ implies that $f \in L^{p,\lambda}(\Omega)$ and the proof is completed. \qed

3 The case $\lambda = n$

The notion of functions of bounded mean oscillation was introduced and studied by John and Nirenberg [15] in connection with the work of John on quasi-isometric maps and of Moser on Harnack inequality. Let $Q_0$ be a cube in $\mathbb{R}^n$. We say that a function $f \in L^1(Q_0)$ belongs to the space of functions with bounded mean oscillation $\text{BMO}(Q_0)$ if
\[ |f|_* := \sup_{|Q|} \frac{1}{|Q|} \int_Q |f - f_Q| \, dx < +\infty, \] (11)

where the supremum it taken over all the cubes $Q \subset Q_0$.

Commonly BMO is defined in the whole of $\mathbb{R}^n$ by requiring $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and the supremum in (11) to be taken over all cubes in $\mathbb{R}^n$. It is easy to see that $\text{BMO}(Q_0) = L^{1,\infty}(Q_0)$. In this section, we shall in fact see later that the function $f \in L^{p,\lambda}(Q_0)$ for all $p, 0 < p < \infty$ if and only if $f \in \text{BMO}(Q_0)$ with $f^- \in L^\infty(Q_0)$. 

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3.1 John–Nirenberg Lemma

To obtain the desired results, we need the following John–Nirenberg inequality for \( \mathcal{L}^{p,n}(Q_0) \) with \( 0 < p \leq 1 \).

**Theorem 3.1** Let \( 0 < p \leq 1 \). There are constants \( a_1, a_2 > 0 \), depending only \( n \), such that

\[
\left| \{ x \in Q : |f(x) - |f||_{Q} > t \} \right| \leq a_1 \exp \left( -\frac{a_2 t}{\| f \|_{\mathcal{L}^{p,n}(Q_0)}} \right) |Q| \quad (12)
\]

for all \( Q \subset Q_0 \) with sides parallel to those of \( Q_0 \), all \( f \in \mathcal{L}^{p,n}(Q_0) \) and all \( t > 0 \).

**Proof** By \( f \in \mathcal{L}^{p,n}(Q_0) \Rightarrow f \in \mathcal{L}^{p,n}(Q) \), it is enough to prove (12) for \( Q = Q_0 \) only. Let \( \| f \|_{\mathcal{L}^{p,n}(Q_0)} = 1 \). Take

\[
\alpha > 1 \geq \frac{1}{|Q_0|} \int_{Q_0} |f(x) - |f||_{Q_0}|^p \, dx
\]

and applying the Calderón–Zygmund decomposition with \( f = |f - |f||_{Q_0}|^p \) and parameter \( \alpha \), we can obtain a sequence \( \{ Q_k^i \}_{k \in K_i} \) satisfy

\[
\alpha < \frac{1}{|Q_k^i|} \int_{Q_k^i} |f(x) - |f||_{Q_0}|^p \, dx \leq 2^n \alpha, \quad \text{for all } k \in K_1, \quad (13)
\]

and

\[
|f - |f||_{Q_0}|^p \leq \alpha \quad \text{a.e. on } Q_0 \setminus \bigcup_{k \in K_1} Q_k^1. \quad (14)
\]

By (13), we have

\[
|f||_{Q_k^i} - |f||_{Q_0}|^p = \frac{1}{|Q_k^i|} \int_{Q_k^i} |f||_{Q_k^i} - |f||_{Q_0}|^p \, dx \leq \frac{1}{|Q_k^i|} \int_{Q_k^i} |f(x) - |f||_{Q_0}|^p \, dx + \frac{1}{|Q_k^i|} \int_{Q_k^i} |f(x) - |f||_{Q_k^i}|^p \, dx \leq 1 + 2^n \alpha \leq 2^{n+1} \alpha,
\]

and

\[
\sum_{k \in K_1} |Q_k^i| \leq \frac{1}{\alpha} \sum_{k \in K_1} \int_{Q_k^i} |f(x) - |f||_{Q_0}|^p \, dx \leq \frac{1}{\alpha} \int_{Q_0} |f(x) - |f||_{Q_0}|^p \, dx \leq \frac{1}{\alpha} |Q_0|. \quad (15)
\]

We can apply again the Calderón-Zygmund decomposition with \( Q_k^1, f = |u - |u||_{Q_k^i} \) and parameter \( \alpha \), we also can find a sequence of cubes \( \{ Q_{k,j}^{1,1} \}_{i \in J(k)} \) such that
\[ \alpha < \frac{1}{|Q_{k,j}^i|} \int_{Q_{k,j}^i} |f(x) - |f|_{Q_0}|^p \, dx \leq 2^n \alpha, \quad \text{for all } j \in J_k, \]  

(16)

and

\[ |f - |f|_{Q_0}|^p \leq \alpha \quad \text{a.e. on } Q_k^1 \setminus \bigcup_{j \in J_k} Q_{k,j}^i. \]  

(17)

Write

\[ \{Q_{k,j}^i\}_{j \in J(k), k \in K_1} = \{Q_k^2\}_{k \in K_2}. \]

From (15) and (17), for \( x \in Q_0 \setminus \bigcup_{k \in K_2} Q_k^2 \), there is a unique index \( k = k(x) \in K_1 \) such that \( x \in Q_k^1 \), we get

\[ |f(x) - |f|_{Q_0}|^p \leq |f(x) - |f|_{Q_k^1}|^p + |f|_{Q_0} - |f|_{Q_k^1}|^p \leq 2^{n+2} \alpha. \]

Moreover, it follows from (15) that

\[ \sum_{j \in K_2} |Q_j^2|^p \leq \frac{1}{\alpha} \sum_{k \in K_1} \int_{Q_k^1} |f(x) - |f|_{Q_k^1}|^p \, dx \leq \frac{1}{\alpha} \sum_{k \in K_1} |Q_k^1| \leq \frac{1}{\alpha^2} |Q_0|. \]

Repeating this procedure inductively, for every \( k \in \mathbb{N} \), we can obtain a sequence of cubes \( \{Q_k^i\}_{k \in K_i} \) such that

\[ |f - |f|_{Q_0}|^p \leq i2^{n+1} \alpha \quad \text{a.e. on } Q_0 \setminus \bigcup_{k \in K_i} Q_k^i. \]

and

\[ \sum_{k \in K_i} |Q_k^i| \leq \frac{1}{\alpha} |Q_0|. \]

For \( t > 2^{n+1} \alpha \), choose \( i \in \mathbb{N} \) in such a way that \( i2^{n+1} \alpha < t \leq (i + 1)2^{n+1} \alpha \). We set \( a_1 = \alpha \) and \( a_2 = \frac{\log a}{2^{n+1} \alpha} \), then

\[ |\{x \in Q_0 : |f(x) - |f|_{Q_0}| > t\}| \leq |\{x \in Q_0 : |f(x) - |f|_{Q_0}| > i2^{n+1} \alpha\}| \]

\[ \leq \sum_{k \in K_i} |Q_k^i| \leq \frac{1}{\alpha} |Q_0| \]

\[ \leq a_1 e^{-a_2 t}|Q_0|. \]

For \( 0 < t \leq 2^{n+1} \alpha \), the result is obtained directly. This implies the desired conclusion and completes the proof of the Theorem 3.1.

\[ \square \]

**Theorem 3.2** Let \( 0 < p < 1 \) and \( Q_0 \) be a cube in \( \mathbb{R}^n \). Then,

\[ \mathcal{L}^{p,n}(Q_0) = \mathcal{L}^{1,n}(Q_0). \]
with equivalence of the corresponding norms.

**Proof** Using (3.1), for any $Q \subset Q_0$ and $f \in \tilde{L}^{p,n}(Q_0)$,

$$
\int_Q |f(x) - |f||_Q| \, dx = \int_0^\infty \left| \{ x \in Q_0 : |f(x) - |f||_Q > t \} \right| \, dt \\
\leq c_1 \int_0^\infty \exp \left( \frac{-c_2 t}{\|f\|_{\tilde{L}^{p,n}(Q_0)}} \right) |Q| \, dt \\
\leq C \|f\|_{\tilde{L}^{p,n}(Q_0)} |Q|,
$$

then $f \in \tilde{L}^{1,n}(Q_0)$.

Conversely, it is immediately that $\|f\|_{\tilde{L}^{p,n}(Q_0)} \leq \|f\|_{\tilde{L}^{1,n}(Q_0)}$ by Hölder inequality.

\[ \square \]

Similarly, we also have the result for $1 < p < \infty$ as follows.

**Theorem 3.3** Let $1 < p < \infty$ and $Q_0$ be a cube in $\mathbb{R}^n$. Then

$$\tilde{L}^{p,n}(Q_0) = \tilde{L}^{1,n}(Q_0)$$

with equivalence of the corresponding norms.

In addition, (12) is in fact equivalent to $f$ being a $\tilde{L}^{1,n}(Q_0)$ function.

**Theorem 3.4** The following facts are equivalent:

(i) $f \in \tilde{L}^{1,n}(Q_0)$;

(ii) there are $a_1, a_2$ such that for all $Q \subset Q_0$ and $t > 0$,

$$\left| \{ x \in Q : |f(x) - |f||_Q > t \} \right| \leq a_1 e^{-a_2 t} |Q|;$$

(iii) there are $a_3, a_4$ such that for all $Q \subset Q_0$

$$\frac{1}{|Q|} \int_Q e^{a_1 t} |f(x) - |f||_Q| - 1 \, dx \leq a_3.$$

**Proof** John–Nirenberg lemma yields the fact that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Set $a_4 := \frac{a_2}{2}$ and $s := e^{a_1 t}$. We get
\[
\frac{1}{|Q|} \int_Q e^{a_4|f(x)-|f||_Q|} - 1 \, dx = \int_1^{\infty} \left| \left\{ x \in Q : e^{a_4|f(x)-|f||_Q|} > s \right\} \right| ds \\
= \int_0^{\infty} a_4 e^{a_4t} \left| \left\{ x \in Q : |f(x) - |f||_Q| > t \right\} \right| dt \\
\leq a_1 |Q| \int_0^{\infty} a_4 e^{a_4t} e^{-a_4t} \, dt \\
= a_1 |Q| \int_0^{\infty} \frac{a_4}{2} e^{-\frac{a_4}{2}t} \, dt = a_1 |Q|.
\]

(iii) \Rightarrow (i). As \( t \leq e^t - 1 \), hence \( |f - |f||_Q| \leq \frac{1}{a_4} e^{a_4|f - |f||_Q|} - 1 \), then
\[
\frac{1}{|Q|} \int_Q |f(x) - |f||_Q| \, dx \leq \frac{1}{a_4} \frac{1}{|Q|} \int_Q \left( e^{a_4|f - |f||_Q|} - 1 \right) \, dx \leq \frac{a_3}{a_4}.
\]
Thus, we complete the proof of Theorem 3.4. \( \square \)

### 3.2 Some equivalent definitions of \( \tilde{\mathcal{L}}^{1,n}(Q_0) \) function

Next, we give some equivalent definitions of \( \tilde{\mathcal{L}}^{1,n}(Q_0) \) function.

**Theorem 3.5** Let \( Q_0 \) be a cube in \( \mathbb{R}^n \). Then, the following statements are equivalent:

1. \( f \in \tilde{\mathcal{L}}^{1,n}(Q_0) \),
2. \( f \in \text{BMO}(Q_0) \) with \( f^- \in L^\infty(Q_0) \);
3. For any \( Q \subset Q_0 \), there is a constant \( C \) such that
   \[
   \frac{1}{|Q|} \int_Q ||f(x)| - f_Q| \, dx \leq C.
   \]

**Proof** (i) \( \Rightarrow \) (ii). We write \( E_Q = \{ x \in Q : f(x) \geq f_Q \} \) and \( F_Q = Q \setminus E_Q \). From the fact
\[
\int_Q (f(x) - f_Q) \, dx = 0,
\]
it follows that
\[
\frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx = \frac{2}{|Q|} \int_{F_Q} (f_Q - f(x)) \, dx \\
\leq \frac{2}{|Q|} \int_{F_Q} |f| \, dx \\
\leq \frac{2}{|Q|} \int_Q |f|_Q - f(x) \, dx \leq 2\|f\|_{\tilde{L}^{1,n}(Q_0)}.
\]

Meanwhile,
\[
\frac{2}{|Q|} \int_Q f^{-}(x) \, dx = |f|_Q - f_Q \leq \frac{1}{|Q|} \int_Q |f(x) - |f|_Q| \, dx \leq \|f\|_{\tilde{L}^{1,n}(Q_0)}.
\]

which implies that \( f^- \in L^\infty(Q_0) \) by Lebesgue different theorem.

\( (ii) \Rightarrow (iii) \). Let \( f \in \text{BMO}(Q_0) \) with \( f^- \in L^\infty(Q_0) \). For any \( Q \subset Q_0 \),
\[
\frac{1}{|Q|} \int_Q |f(x)| - f_Q \, dx \leq \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx + 2(f^-)_Q \\
\leq |f|_* + 2\|f^-\|_{L^\infty(Q_0)}.
\]

\( (iii) \Rightarrow (i) \). Using the fact
\[
\frac{2}{|Q|} \int_Q f^{-}(x) \, dx = |f|_Q - f_Q,
\]
we arrive at
\[
\frac{1}{|Q|} \int_Q |f(x) - |f|_Q| \, dx \\
\leq \frac{1}{|Q|} \int_Q |f(x) - |f|_Q| \, dx + (|f|_Q - f_Q) + \frac{1}{|Q|} \int_Q |f_Q - |f|_Q| \, dx \\
\leq \frac{3}{|Q|} \int_Q |f(x)| - f_Q \, dx.
\]

Therefore, we complete the proof of Theorem 3.5.

On the other hand, it is well known that \( f \in \text{BMO}(Q_0) \) if and only if
\[
\sup_{Q \subset Q_0} \inf_{c \in \mathbb{R}^n} \rho^{-n} \int_Q |f(x) - c| \, dx < \infty.
\]

A similar conclusion can be obtained as follows.

**Theorem 3.6** Let \( Q_0 \) be an \( n \)-dimensional cube in \( \mathbb{R}^n \). Then \( f \in \tilde{L}^{1,n}(Q_0) \) if and only if
\[
\sup_{Q \subset Q_0} \inf_{c \geq 0} \rho^{-n} \int_Q |f(x) - c| \, dx < \infty.
\]

**Proof** \((\Rightarrow)\) is obviously.

\((\Leftarrow)\). Let \(c_Q \geq 0\) be the value which minimizes \(\int_Q |f(x) - c| \, dx\) with \(c \geq 0\), then
\[
\frac{1}{|Q|} \int_Q |f(x) - |f||_Q| \, dx \leq \frac{1}{|Q|} \int_Q |f(x) - c_Q| \, dx + \frac{1}{|Q|} \int_Q |c_Q - |f||_Q| \, dx.
\]

By \(c_Q \geq 0\), we have
\[
||f||_Q - c_Q| \leq \frac{1}{|Q|} \int_Q |f(x) - c_Q| \, dx.
\]

Therefore,
\[
\frac{1}{|Q|} \int_Q |f(x) - |f||_Q| \, dx \leq C
\]
and \(f \in \mathcal{L}^{1,n}(Q_0)\).

In fact, we can replace (18) with the following (19), for \(0 < p < \infty\)
\[
||f||^p \mathcal{L}^{p,n}(Q_0) = \sup_{Q \subset Q_0} \inf_{c \geq 0} \rho^{-n} \int_Q |f(x) - c|^p \, dx < \infty,
\]
(19)

Now, we establish a version of John–Nirenberg inequality suitable for \(\mathcal{L}^{p,n}(Q_0)\) with \(0 < p < 1\).

**Theorem 3.7** Let \(0 < p < 1\) and \(||f||^{p,n}(Q_0) = 1\) and for each cube \(Q\) let \(c_Q\) be the positive constant which minimizes \(\int_Q |f(x) - c|^p \, dx\). Then
\[
\left| \left\{ x \in Q : |f(x) - c_Q| > t \right\} \right| \leq a_1 e^{-a_2 t |Q|},
\]
where \(a_1\) and \(a_2\) are positive constants.

**Proof** Take any cube \(Q\), write \(E_Q = \{ x \in Q : |f(x) - c_Q| > t \}\). Then
\[
|E_Q| \leq \int_{E_Q} \frac{|f(x) - c_Q|^p}{t^p} \, dx
\]
\[
\leq \frac{1}{t^p} \int_Q |f(x) - c_Q|^p \, dx
\]
\[
\leq \frac{1}{t^p} |Q|.
\]

Write \(F_1(t) = \frac{1}{t^p}\), then
Let $s > 1$ and $t \in (0, \infty)$ such that $2^{\frac{n+1}{p}} s \leq t$. Fix a cube $Q$, there is a Calderón-Zygmund decomposition to $|f(x) - c_Q|^p$ of disjoint cubes $\{Q_j\}$ such that $Q_j \subset Q$ and

(i) $s^p < \frac{1}{|Q_j|} \int_{Q_j} |f(x) - c_Q|^p \, dx \leq 2^n s^p,$

(ii) $|f(x) - c_Q| \leq s$ for $x \in Q \setminus \bigcup_j Q_j$.

Notice that

$\int_{Q_j} |f(y) - c_Q|^p \, dy \leq \int_{Q_j} |f(y) - c_Q|^p \, dy.$

Therefore, by (i), we have

$|c_{Q_j} - c_Q|^p = \frac{1}{|Q_j|} \int_{Q_j} |c_{Q_j} - c_Q|^p \, dy$

$\leq \frac{1}{|Q_j|} \int_{Q_j} |f(y) - c_Q|^p \, dy + \frac{1}{|Q_j|} \int_{Q_j} |f(y) - c_Q|^p \, dy$

$\leq \frac{2}{|Q_j|} \int_{Q_j} |f(y) - c_Q|^p \, dy$

$\leq 2^{n+1} s^p.$

Since $2^{\frac{n+1}{p}} s \leq t$, then $E_Q \subset \bigcup_j Q_j$ and

$|E_{Q_j}| = \sum_j |\{x \in Q_j : |f(x) - c_Q| > t\}|$

$\leq \sum_j |\{x \in Q_j : |f(x) - c_Q| + |c_{Q_j} - c_Q| > t\}|$

$\leq \sum_j |\{x \in Q_j : |f(x) - c_Q| > t - 2^{\frac{n+1}{p}} s\}|$

$\leq \sum_j F_1(t - 2^{\frac{n+1}{p}} s) \cdot |Q_j|$

$\leq F_1(t - 2^{\frac{n+1}{p}} s) \sum_j \frac{1}{s^p} \int_{Q_j} |f(x) - c_Q|^p \, dx$

$\leq \frac{F_1(t - 2^{\frac{n+1}{p}} s)}{s^p} \int_{Q_0} |f(x) - c_Q|^p \, dx$

$\leq \frac{F_1(t - 2^{\frac{n+1}{p}} s)}{s^p} |Q|.$
\[
F_2(t) = \frac{F_1(t - 2^{\frac{n+1}{p}} s)}{s^p}.
\]

Continue this process indefinitely, we obtain for any \( k \geq 2 \),
\[
F_k(t) = \frac{F_{k-1}(t - 2^{\frac{n+1}{p}} s)}{s^p}.
\]

and
\[
|E_Q| \leq F_k(t)|Q|.
\]

We fix a constant \( t > 0 \). If
\[
k \cdot 2^{\frac{n+1}{p}} s < t \leq (k + 1) \cdot 2^{\frac{n+1}{p}} s.
\]

for some \( k \geq 1 \), thus
\[
|E_Q| \leq \left| \left\{ x \in Q_0 : |f(x) - c_Q| > t \right\} \right| \\
\leq \left| \left\{ x \in Q : |f(x) - c_Q| > k \cdot 2^{\frac{n+1}{p}} s \right\} \right| \\
\leq F_k(k \cdot 2^{\frac{n+1}{p}} s)|Q| \\
= \frac{F_1(2^{\frac{n+1}{p}} s)}{s^{(k-1)p}}|Q| \\
= \frac{1}{2^{n+1} \cdot s^p}|Q| \\
\leq \frac{e^{-kp \log s}}{2^{n+1}}|Q| \\
\leq \frac{e^{-\frac{kp}{2} \log s}}{2^{n+1}}|Q|.
\]

Since \(-k \leq 1 - \frac{t}{2^{\frac{n+1}{p}} s}\). If \( t \leq 2^{\frac{n+1}{p}} s \), then use the trivial estimate
\[
|E_Q| \leq |Q| \leq e^{-l} e^{\frac{n+1}{p} \log s}|Q|.
\]

Recall that \( s \) is any real number greater that 1. Choosing \( s = e \), this yields
\[
\left| \left\{ x \in Q : |f(x) - c_Q| > t \right\} \right| \leq a_1 e^{-a_2 t}|Q|,
\]

for some positive constants \( a_1 \) and \( a_2 \), which proves the inequality of the Proposition 3.7. \( \square \)

From Theorems 3.2, 3.6 and 3.7, it is immediately that

**Theorem 3.8** Let \( 0 < p < \infty \) and \( Q_0 \) be an \( n \)-dimensional cube in \( \mathbb{R}^n \). Then \( f \in L^{1,p}(Q_0) \) if and only if
Remark 3.9 It is worth remarking that

(a) \( f \in \overline{L}^{1,n}(Q_0) \) if and only if for every \( Q \subset Q_0 \) there is a constant \( c_Q \geq 0 \) such that

\[
\sup_Q \frac{1}{|Q|} \int_Q |f(x) - c_Q| \, dx < \infty.
\]

Indeed for \( x \in Q \),

\[
|f(x) - |f|_Q| \leq |f(x) - c_Q| + |c_Q - |f|_Q|
\]

\[
\leq |f(x) - c_Q| + \frac{1}{|Q|} \int_Q |f(y) - c_Q| \, dy,
\]

and averaging over \( Q \), we get

\[
\frac{1}{|Q|} \int_Q |f(x) - |f|_Q| \, dx \leq \frac{2}{|Q|} \int_Q |f(x) - c_Q| \, dx.
\]

(b) \( f \in \overline{L}^{1,n}(Q_0) \) if and only if for every \( Q \subset Q_0 \) there is a constant \( c_Q \leq 0 \) such that

\[
\sup_Q \frac{1}{|Q|} \int_Q ||f(x)| - c_Q| \, dx < \infty.
\]

Indeed, for \( x \in Q \)

\[
||f(x)| - f_Q| \leq ||f(x)| - c_Q| + |c_Q - f_Q|
\]

\[
\leq |f(x) - c_Q| + \frac{1}{|Q|} \int_Q |f(y) - c_Q| \, dy
\]

\[
\leq |f(x) - c_Q| + \frac{1}{|Q|} \int_Q ||f(y)|| - c_Q| \, dy,
\]

and averaging over \( Q \), we get

\[
\frac{1}{|Q|} \int_Q ||f(x)| - f_Q| \, dx \leq \frac{2}{|Q|} \int_Q ||f(x)| - c_Q| \, dx.
\]

3.3 Characterizations of \( \overline{L}^{1,n}(\mathbb{R}^n) \) function associated to maximal functions

In 2000, Bastero et al. [1] studied the class of functions for which the boundedness of commutator with the Hardy–Littlewood maximal function

\[
M(f)(x) := \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy.
\]
They proved that $[b, M]$ is bounded on $L^p(\mathbb{R}^n)$ if and only if $b \in \text{BMO}(\mathbb{R}^n)$ with $b^- \in L^\infty(\mathbb{R}^n)$, where $b^-(x) = -\min\{b(x), 0\}$ and $1 < p < \infty$. They also showed that $f \in \text{BMO}(\mathbb{R}^n)$ with $f^- \in L^\infty(\mathbb{R}^n)$ if and only if

$$\sup_Q \left( \frac{1}{|Q|} \int_Q |f(x) - M_Q(f)(x)|^p dx \right)^{1/p} < \infty,$$

where $1 \leq p < \infty$ and

$$M_Q(f)(x) = \sup_{Q \supseteq Q'} \frac{1}{|Q'|} \int_{Q'} |f(y)| dy.$$ 

Later, Zhang and Wu obtained similar results for the fractional maximal function in [38] and extended the above results to variable exponent Lebesgue spaces in [39, 40]. For $0 < \alpha < n$, the fractional maximal function is defined by

$$M_\alpha(f)(x) = \sup_\{x \in Q\} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy.$$ 

It is proved that for $1 < p, q < \infty$ and $1/p - 1/q = \alpha/n$, the commutator $[b, M_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if

$$\sup_Q \left( \frac{1}{|Q|} \int_Q |f(x) - |Q|^{-\alpha/n}M_\alpha Q(f)(x)|^q dx \right)^{1/q} < \infty,$$

where

$$M_\alpha Q(f)(x) = \sup_{Q \supseteq Q'} \frac{1}{|Q'|^{1-\alpha/n}} \int_{Q'} |f(y)| dy.$$ 

The above results show that the (20) is equivalent in the sense of norm, when $1 \leq p < \infty$. It is a natural question: can we establish the John–Nirenberg inequality suitable for the function $f \in \text{BMO}(\mathbb{R}^n)$ with $f^- \in L^\infty(\mathbb{R}^n)$? In [12], we solved the problem as follows.

**Theorem 3.10** [12] Suppose $f \in \text{BMO}(\mathbb{R}^n)$ with $f^- \in L^\infty(\mathbb{R}^n)$, then for any cube $Q$ and $t > 0$, we have

$$|\{x \in Q : |f(x) - M_Q(f)(x)| > t\}| \leq a_1 e^{-a_2 t} |Q|,$$

where $a_1$ and $a_2$ are positive constants.

Meanwhile, we now establish the John–Nirenberg inequality associated to fractional maximal functions.

**Theorem 3.11** Suppose $f \in \text{BMO}(\mathbb{R}^n)$ with $f^- \in L^\infty(\mathbb{R}^n)$, then for any cube $Q$ and $t > 0$, we have
where $a_1$ and $a_2$ are positive constants.

**Proof** For any cube $Q$ and $x \in Q$, it follows from the definitions of $M_Q$ and $M_{a,Q}$ that

$$|f|_Q \leq |Q|^{-a/n} M_{a,Q}(f) \leq M_Q(f(x)).$$

Set $E := \{ y \in Q : f(y) \geq |f|_Q \}$ and $F := Q \setminus E$, then

$$\left| \left\{ x \in Q : |f(x) - |Q|^{-a/n} M_{a,Q}(f)| > t \right\} \right|$$

$$= \left| \left\{ x \in E : |f(x) - |Q|^{-a/n} M_{a,Q}(f)| > t \right\} \right|$$

$$+ \left| \left\{ x \in F : |Q|^{-a/n} M_{a,Q}(f) - f(x) > t \right\} \right|$$

$$\leq \left| \left\{ x \in E : f(x) - |f|_Q > t \right\} \right| + \left| \left\{ x \in F : M_Q(f(x)) - f(x) > t \right\} \right|$$

$$\leq a_1 e^{-a_2 t} |Q|.$$

We complete the proof of Theorem 3.11. $\square$

### 4 The case $n < \lambda \leq n + p$

In the theory of PDEs, one encounters two types of a priori estimates: $L^p$ norm estimates and Schauder estimates (estimates in the space of the Hölder continuous function). An attempt to achieve this is provided by the integral characterization of the Hölder continue functions.

#### 4.1 A characterization of nonnegative Hölder continuous functions

In this section, we give the equivalent definitions of nonnegative Hölder continuous functions.

**Theorem 4.1** Let $1 \leq p < \infty$ and $0 < \beta \leq 1$. For the function $f \in L^1_{\text{loc}}(\Omega)$, the following three statements are equivalent:

1. $f \in C^{0,\beta}(\Omega)$ and $f \geq 0$.
2. There exists a constant $C_1$ such that
   $$|f(x) - f(y)| \leq C_1 |x - y|^\beta$$
   for almost every $x$ and $y$.
(3) There exists a constant $C_2$ such that for any $x_0 \in \Omega$ and $0 < \rho < \text{diam}\Omega$

$$\left( \frac{1}{|\Omega(x_0, \rho)|} \int_{\Omega(x_0, \rho)} |f(x) - |f|_{\Omega(x_0, \rho)}|^p \, dx \right)^{1/p} \leq C_2 \rho^\beta.$$ 

**Proof** \( (1) \implies (2). \) Assume \( f \in C^{0,\beta}(\Omega) \) and \( f \geq 0. \) For \( x, y \in \Omega, \) we conclude that

\[ |f(x) - f(y)| = |f(x) - f(y)| \leq \|f\|_{C^{0,\beta}(\Omega)} |x - y|^\beta. \]

\( (2) \implies (3). \) If \( x, y \in \Omega(x_0, \rho), \) from the fact that \( |f(x) - f(y)| \leq C_1 |x - y|^\beta \) we have

\[ |f(x) - f(y)| \leq C_1 \rho^\beta, \]

hence

\[ |f(x) - f|_{\Omega(x_0, \rho)}| \leq C \rho^\beta. \]

Consequently,

\[ \int_{\Omega(x_0, \rho)} |f(x) - f|_{\Omega(x_0, \rho)}|^p \, dx \leq C \rho^{p\beta+n}. \]

Now we give the proof of \( (3) \implies (1). \) For any \( x, y \in \Omega, \) take \( \Omega_0 = \Omega(x, \rho) \) with \( \rho \leq |x - y| \) and \( U = \Omega(x, 2|x - y|), \) define \( \Omega_k = \Omega(x, 2^k \rho) \) for \( 0 \leq k \leq \tilde{k}, \) where \( \tilde{k} \) is the first integer such that \( 2^k \rho \geq |x - y|. \)

Notice that for any \( R_1 = \Omega(x_1, \rho_1), R_2 = \Omega(x_2, \rho_2) \) with \( R_1 \subset R_2 \) and \( \rho_2 \leq 2\rho_1, \) we have

\[ |f_{R_1} - |f|_{R_2}| = \frac{1}{|R_1|} \int_{R_1} |f(z) - |f|_{R_2}| \, dz \leq C \rho_1^\beta, \]

and

\[ \left| |f|_{R_1} - |f|_{R_2} \right| = \frac{1}{|R_1|} \int_{R_1} |f|_{R_1} - |f|_{R_2} \, dz \]

\[ \leq \frac{1}{|R_1|} \int_{R_1} |f(z) - |f|_{R_1}| \, dz + \frac{1}{|R_1|} \int_{R_1} |f(z) - |f|_{R_2}| \, dz \leq C \rho_1^\beta. \]

Therefore,

\[ |f_{\Omega_0} - |f|_U| \leq |f_{\Omega_0} - |f|_{\Omega_1}| + \sum_{k=1}^{\tilde{k}-1} |f_{\Omega_k} - |f|_{\Omega_{k+1}}| + |f_{\Omega_{\tilde{k}}} - |f|_U| \]

\[ \leq C \sum_{k=0}^{\tilde{k}} (2^k \rho)^\beta \leq C |x - y|^\beta. \]
A similar argument can be made for the point $y$ with $\Omega'_0 = \Omega(x, \rho')$ and $V = \Omega(y, 3|x - y|)$. Thus,

$$|f_{\Omega_0} - f_{\Omega'_0}| \leq |f_{\Omega_0} - |f|_V| + |f|_V - |f|_V + |f|_V - f_{\Omega'_0}| \leq C|x - y|^\beta.$$ 

From the differentiation theorem of Lebesgue we know that $f_{\Omega_0} \to f(x)$ as $\rho \to 0$, and $f_{\Omega'_0} \to f(y)$ in a.e. $x, y \in \Omega$ as $\rho' \to 0$. It follows that

$$|f(x) - f(y)| \leq C|x - y|^\beta.$$ 

Therefore, we can conclude that $f \in C^{0, \beta}(\Omega)$. Meanwhile,

$$\frac{2}{|\Omega(x, \rho)|} \int_{|\Omega(x, \rho)|} f^-(z)dz \leq \frac{1}{|\Omega(x, \rho)|} \int_{|\Omega(x, \rho)|} |f(z) - f(z)|dz,$$

$$\leq \frac{1}{|\Omega(x, \rho)|} \int_{|\Omega(x, \rho)|} |f(z)|dz + \frac{1}{|\Omega(x, \rho)|} \int_{|\Omega(x, \rho)|} |f(z) - f|_{|\Omega(x, \rho)|}dz,$$

$$\leq C\rho^\beta \to 0, \text{ as } \rho \to 0,$$

which shows that $f^-(x) = 0, \text{ a.e. } x \in \Omega.$

In fact, we can also obtain the following results. The remaining proofs are similar to the ones in Proposition 3.8 and we leave the details to the interested reader.

**Theorem 4.2** Let $0 < p < \infty$ and $0 < \beta \leq 1$. Then $f \in C^{0, \beta}(\Omega)$ and $f \geq 0$ if and only if

$$\sup_{Q \subset \Omega} \inf_{c \geq 0} \rho^{-\beta-\beta} \int_Q |u(x) - c|^pdx < \infty.$$ 

### 4.2 Characterizations of $\tilde{L}^{1,\lambda}(\mathbb{R}^n)$ function associated to maximal functions

To obtain the result of maximal function characterizations of $\tilde{L}^{1,\lambda}(\mathbb{R}^n)$, we need the locally boundedness of $M_Q$ on $\tilde{L}^{1,\lambda}(Q)$.

**Theorem 4.3** If $f \in \tilde{L}^{1,\lambda}(Q)$ with $n < \lambda \leq n + 1$, then so does $M_Q(f)$ and there exists a positive constant $C_0$ such that

$$\|M_Qf\|_{\tilde{L}^{1,\lambda}(Q)} \leq C_0\|f\|_{\tilde{L}^{1,\lambda}(Q)}.$$  \hspace{1cm} (21)

**Proof** It follows from Theorem 4.1 that $f$ is nonnegative. Writing $F$ for the maximal function $M_Qf$ of $f$, we thus need to prove

$$\frac{1}{|R|^{\lambda/n}} \int_R |F(x) - F_R|dx \leq C\|f\|_{\tilde{L}^{1,\lambda}(Q)}$$  \hspace{1cm} (22)

for every subcubes $R$ of $Q$. 

\[ \text{Birkhäuser} \]
Fix $R$ and let $3R$ denote the cube that is concentric with $R$ and has three times the diameter. Let $\tilde{R}$ be the smallest subcube of $Q$ containing $(3R) \cap Q$, and for each $x \in R$ let

\[
F_1(x) = \sup \{ f_{\tilde{R}} : \tilde{R} \subset \tilde{R} \text{ and } x \in \tilde{R} \},
\]
\[
F_2(x) = \sup \{ f_{\tilde{R}} : \tilde{R} \subset Q, x \in \tilde{R} \text{ and } \tilde{R} \cap (Q \setminus \tilde{R}) \neq \emptyset \}.
\]

Meanwhile, if

\[
D = \{ x \in R : F(x) > F_R \}, \quad D_1 = \{ x \in D : F_1(x) \geq F_2(x) \}
\]

and $D_2 = D \setminus D_1$, then

\[
\frac{1}{|R|^{\lambda/n}} \int_R |F(x) - F_R| \, dx = \frac{2}{|R|^{\lambda/n}} \int_D |F(x) - F_R| \, dx
\]

\[= \frac{2}{|R|^{\lambda/n}} \sum_{i=1}^{2} \int_{D_i} |F_i(x) - F_R| \, dx.
\]

Hence, we can establish the inequality (22) by

\[
\frac{1}{|R|^{\lambda/n}} \int_{D_i} |F_i(x) - F_R| \, dx \leq C \| f \|_{L^{1,\lambda}(Q)},
\]

(23) Consider first the case $i = 1$. Since $f_{\tilde{R}} \leq F(x)$ for all $x \in R$, then $f_{\tilde{R}} \leq F_R$ so we may construct the Calderón–Zygmund decomposition for $f$ and $\tilde{R}$ with respect to the constant $F_R$. If the resulting sequence of pairwise disjoint cubes is denoted by $\{R_k\}_{k=1}^\infty$ and if $\bar{R}_k$ denotes the "parent" cube of $R_k$, then the following properties hold:

(i) $\bigcup R_k \subset \tilde{R}$;
(ii) $f_{\bar{R}_k} \leq F_R < f_{R_k}$;
(iii) $|\bar{R}_k| = 2^n |R_k|$;
(iv) $f \leq F_R$ almost everywhere on $E = \tilde{R} \setminus (\bigcup_k R_k)$.

Define function $b$ and $g$ on $Q$ by

\[
b = \sum_k (f - f_{\bar{R}_k}) \chi_{R_k}, \quad g = \sum_k f_{\bar{R}_k} \chi_{R_k} + f \chi_E.
\]

So $f \chi_R = b + g$. It follows from (ii) and (iv) that

\[
\| g \|_{L^\infty(Q)} \leq F_R,
\]

(24) while on the other hand the John–Nirenberg lemma and (i) and (iii) give
\[ \|b\|_{L^2(Q)} = \left\{ \sum_k \int_{R_k} |f(x) - f_{R_k}|^2 \, dx \right\}^{1/2} \]
\[ \leq \left\{ \sum_k |R_k|^{1/2} \frac{1}{|R_k|^{1/2}} \int_{R_k} |f(x) - f_{R_k}|^2 \, dx \right\}^{1/2} \]
\[ \leq C |R|^{1/2} \|f\|_{L^{2,i}(Q)}. \tag{25} \]

Now it follows from the definition of \(F_1\) that
\[ F_1 \leq M_Q(f \chi_R) = M_Q(b + g) \leq M_Q b + M_Q g, \]
so applying the Cauchy–Schwarz inequality we obtain
\[ \int_{D_1} F_1(x) \, dx \leq |D_1|^{1/2} \|M_Q b\|_{L^2(Q)} + |D_1\| \|M_Q b\|_{L^\infty(Q)} \]
\[ \leq C |R|^{1/2} \|b\|_{L^2(Q)} + |D_1\| \|g\|_{L^\infty(Q)}. \]

Combining this with (24) and (25), we obtain (23) for \(i = 1\).

Fix \(x \in D_2\) and let \(P\) be any subcube of \(Q\) that contains \(x\) and has nonempty intersection with \(Q \setminus R\). Clearly \(|P| \geq |R|\). Let \(P'\) be the smallest subcube of \(Q\) containing both \(P\) and \(R\). Then \(|P'| \leq 2^n |P|\). Arguing as before, we note that \(f_{P'} \leq f_R\). Hence
\[ f_p - f_R \leq f_p - f_{P'} \leq \frac{1}{|R|} \int_P |f(y) - f_{P'}| \, dy \leq 2^n \|f\|_{L^{2,i}(Q)}, \]
so taking the supremum over all such cubes \(P\) we obtain
\[ F_2(x) - F_R \leq C \|f\|_{L^{2,i}(Q)}. \]
This establishes the case \(i = 2\) and the proof of the Theorem is completed. \(\square\)

Similar to [12, 30], it is easy to obtain that

**Theorem 4.4** Let \(n < \lambda \leq n + 1, \beta = \lambda - n\) and \(\|f\|_{L^{1,i}({\mathbb{R}}^n)} = 1\). There are constants \(a_1, a_2 > 0\), depending only \(n\), such that
\[ \left| \left\{ x \in Q : |f(x) - (M_Q f)_Q| > t|Q|^\beta \right\} \right| \leq a_1 e^{-a_2 t} |Q| \tag{26} \]
for all \(Q \subset {\mathbb{R}}^n\) and all \(t > 0\).

Then, we conclude that

**Theorem 4.5** Let \(n < \lambda \leq n + 1, \beta = \lambda - n\) and \(\|f\|_{L^{1,i}({\mathbb{R}}^n)} = 1\). There are constants \(c_1, c_2 > 0\), depending only \(n\), such that
\[ \left| \left\{ x \in Q : |f(x) - M_Q f(x)| > t|Q|^\beta \right\} \right| \leq c_1 e^{-c_2 t} |Q| \tag{27} \]
for all $Q \subset \mathbb{R}^n$ and all $t > 0$.

**Proof** Theorem 4.3 shows that $M_Q (f) \in \mathcal{L}^{1, \lambda} (Q)$ with $\| M_Q (f) \|_{\mathcal{L}^{1, \lambda} (Q)} \leq C_0 \| f \|_{\mathcal{L}^{1, \lambda} (Q)} \leq C_0$. In [30], Wang, Zhou and Teng gave a version of John–Nirenberg inequality suitable for $\mathcal{L}^{1, \lambda} (\mathbb{R}^n)$. In fact, it is easy to obtain the similar result for $\mathcal{L}^{1, \lambda} (Q)$. Then, the version of John–Nirenberg inequality suitable for $\mathcal{L}^{1, \lambda} (Q)$ applied to $M_Q (f)$ gives

$$\left| \left\{ x \in Q : |M_Q (f) (x) - (M_Q (f) )_Q | > t |Q|^{\beta} \right\} \right| \leq b_1 e^{\frac{-b_2}{\| M_Q (f) \|_{\mathcal{L}^{1, \lambda} (Q)} t}} |Q|$$

for some positive constants $b_1$ and $b_2$. Combining Theorems 4.3 and 4.4, and choose $c_1 = \max \{ 2a_1, 2b_1 \}$ and $c_2 = \min \{ a_2, b_2 / C_0 \}$, we arrive at

$$\frac{1}{c_1 |Q|} e^{c_2 t} \left| \left\{ x \in Q : |f(x) - M_Q (f) (x) | > t |Q|^{\beta} \right\} \right| \leq \frac{1}{c_1 |Q|} e^{c_2 t} \left| \left\{ x \in Q : |f(x) - (M_Q (f) )_Q | > t |Q|^{\beta} \right\} \right|$$

$$+ \frac{1}{c_1 |Q|} e^{c_2 t} \left| \left\{ x \in Q : |(M_Q (f) )_Q - M_Q (f) (x) | > t |Q|^{\beta} \right\} \right| \leq 1.$$

Thus, we complete the proof of Theorem 4.5. $\square$

Furthermore, we have

**Theorem 4.6** Suppose $f \in C^{0, \beta} (\mathbb{R}^n)$ with $f \geq 0$ and $0 < \beta \leq 1$, then for any cube $Q$ and $t > 0$, we have

$$\left| \left\{ x \in Q : |f(x) - |Q|^{-\beta/n} M_{d, Q} (f) (x) | > t |Q|^{\beta} \right\} \right| \leq c_1 e^{-c_2 t} |Q|.$$

where $c_1$ and $c_2$ are positive constants.

## 5 The necessity of the boundedness of commutators on ball Banach function spaces

To state our results we recall some basic facts about Muckenhoupt weights and ball Banach function spaces.

We first recall the definition of $A_p$ weight introduced by Muckenhoupt in [21], which give the characterization of all weights $\omega(x)$ such that the Hardy–Littlewood maximal operator is bounded on $L^p (\omega)$. For $1 < p < \infty$ and a nonnegative locally integrable function $\omega$ on $\mathbb{R}^n$, $\omega$ is in the Muckenhoupt $A_p$ class if it satisfies the condition
And a weight function $\omega$ belongs to the class $A_1$ if
\[
[\omega]_{A_1} := \frac{1}{|Q|} \int_Q \omega(x)\,dx \left( \frac{1}{|Q|} \int_Q \omega(x)^{-1} \,dx \right)^{p-1} < \infty.
\]

We write $A_{\infty} = \bigcup_{1 \leq p < \infty} A_p$. In fact, the reverse Hölder inequality holds for $A_p$, that is, there exist constants $q > 1$ and $C$ such that for any cube $Q$ and $\omega \in A_p$,
\[
\left( \frac{1}{|Q|} \int_Q (\omega(x)^q \,dx) \right)^{1/q} \leq \frac{1}{|Q|} \int_Q \omega(x)\,dx.
\]

Let $1 < p, q < \infty$ and $1/p + 1/p' = 1$. For a nonnegative locally integrable function $\omega$ on $\mathbb{R}^n$, $\omega$ is in the Muckenhoupt $A_{p,q}$ class if
\[
\sup_Q \left( \frac{1}{|Q|} \int_Q (\omega(x)^q \,dx) \right)^{1/q} \left( \frac{1}{|Q|} \int_Q (\omega(x)^{-p'} \,dx) \right)^{1/p'} < \infty.
\]

For $\omega \in A_{\infty}$, there exists $0 < \epsilon, L < \infty$ such that for all measurable subsets $S$ of cube $Q$,
\[
\frac{\omega(S)}{\omega(Q)} \leq C \left( \frac{|S|}{|Q|} \right)^{\epsilon}
\]
\[\tag{29}\]
and
\[
\left( \frac{|S|}{|Q|} \right)^L \leq C \frac{\omega(S)}{\omega(Q)}.
\]
\[\tag{30}\]

By a ball Banach function space $X$ whose norm $\| \cdot \|_X$ satisfies the following for all $f, g \in X$:

1. $\|f\|_X = \|f\|_X$;
2. if $|f| \leq |g|$ a.e., then $\|f\|_X \leq \|g\|_X$;
3. if $\{f_n\} \subset X$ is a sequence such that $|f_n|$ increases to $|f|$ a.e., then $\|f_n\|_X$ increases to $\|f\|_X$;
4. if $E \subset \mathbb{R}^n$ is bounded, then $\|X_E\|_X < \infty$;
5. if $E$ is bounded, then $\int_E |f(x)|\,d\mu \leq C\|f\|_X$, where $C = C(E, X)$.

Given a ball Banach function space $X$, there exists another ball Banach function space $X'$, called the associate space of $X$, such that for all $f \in X$,
\[
\|f\|_X \approx \sup_{g \in X', \|g\|_{X'} \leq 1} \int_{\mathbb{R}^n} f(x)g(x)\,dx.
\]
The associate space is equal to the dual space \( X^* \) and always reflexive in many cases. Moreover, we have

\[
\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \|f\|_X \|g\|_{X'}.
\]

(31)

Let \( X \) be a Banach function and define

\[
\|f\|_{\text{BMO}_X} := \sup_Q \frac{\|(b - b_Q)\chi_Q\|_X}{\|\chi_Q\|_X}.
\]

When the Hardy–Littlewood maximal operator \( M \) is bounded on \( X \), Ho [11] first proved that \( \|f\|_{\text{BMO}_X} \) is equivalent to \( \text{BMO}(\mathbb{R}^n) \). Izuki and Sawano [14] gave another proof on Banach function space using the Rubio de Francia algorithm.

One can show that Lebesgue spaces, Morrey spaces, Lorentz spaces, variable Lebesgue spaces, weighted Lebesgue spaces and Orlicz spaces are Banach function spaces. In this section, Our results relax the restriction of Banach spaces in previous to quasi-Banach spaces and extend \( \text{BMO}(\mathbb{R}^n) \) to the class of \( \mathcal{L}^{1,n}(\mathbb{R}^n) \).

**Theorem 5.1** Let \( 0 < s < \infty \) and \( X \) be a ball Banach function space such that the Hardy–Littlewood maximal operator \( M \) is bounded on the associate space \( X' \). Then \( f \in \mathcal{L}^{1,n}(\mathbb{R}^n) \) if and only if

\[
\sup_Q \frac{\|(f - f|_Q)\chi_Q\|_{X'}}{\|\chi_Q\|_{X'}} < \infty,
\]

where \( X^s := \{f : \|f\|_{X'} := \|f^s\|_{X}^{1/s} < \infty\} \).

**Proof** \((\Rightarrow)\). Let \( B := \|M\|_{X' \to X'} \). Take \( g \in X' \) with \( \|g\|_{X'} = 1 \) and define a function

\[
R(g)(x) := \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2B)^k}, \quad g \in X',
\]

(32)

where \( M^k(g) = M \circ (M^{k-1}g) \) and \( M^0 g = |g| \). Then, the function \( Rg \) satisfies the following properties:

1. \( |g(x)| \leq Rg(x) \) for any \( x \in \mathbb{R}^n \);
2. \( \|Rg\|_{X'} \leq 2\|g\|_{X'} \leq 2 \);
3. \( M(Rg)(x) \leq 2BRg(x) \), that is, \( Rg \) is a Muckenhoupt \( A_1 \) weight.

By the property of Muckenhoupt \( A_1 \), we know that there exist positive \( q > 1 \) such that for any cubes \( Q \),

\[
\left( \frac{1}{|Q|} \int_Q Rg(x)^q \, dx \right)^{1/q} \leq \frac{C}{|Q|} \int_Q Rg(x) \, dx.
\]
By the generalized Hölder inequality (31), we obtain
\[ \|Rg\chi_Q\|_{L^s(\mathbb{R}^n)} = \left( \int_Q Rg(x)^s \, dx \right)^{1/q} \leq |Q|^{1/q-1} \int_Q Rg(x) \, dx \leq C|Q|^{1/q-1} \|Rg\|_{X'} \|\chi_Q\|_X \leq C|Q|^{1/q-1} \|\chi_Q\|_X. \]

Thus, we have
\[ \| (f - |f|_Q)^s \chi_Q \|_X \leq C \sup \left\{ \left| \int_Q (f(x) - |f|_Q)^s g(x) \, dx \right| : g \in X', \|g\|_{X'} \leq 1 \right\} \]
\[ \leq C \sup \left\{ \int_Q |f(x) - |f|_Q|^s Rg(x) \, dx : g \in X', \|g\|_{X'} \leq 1 \right\} \]
\[ \leq C \sup \left\{ \| (f - |f|_Q)^s \chi_{Q'} \|_{L^s(\mathbb{R}^n)} \| Rg \chi_Q \|_{L^{s'}(\mathbb{R}^n)} : g \in X', \|g\|_{X'} \leq 1 \right\} \]
\[ \leq C \left( \frac{1}{|Q|} \int_Q |f - |f|_Q|^s \, dx \right)^{1/s} \chi_Q \|_X. \]

This yields that
\[ \frac{\| (f - |f|_Q)^s \chi_Q \|_{X'}^{1/s}}{\|\chi_Q\|_X^{1/s}} \leq C \left( \frac{1}{|Q|} \int_Q |f - |f|_Q|^s \, dx \right)^{1/(sq')} \leq C\|f\|_{\mathcal{L}^{1,s}(\mathbb{R}^n)}. \]

(\Leftarrow). For any cube $Q$,
\[ \int_Q |f(x) - |f|_Q|^s \, dx \leq C\|f - |f|_Q\|_X \|\chi_Q\|_{X'}. \]

The boundedness of $M$ on $X'$ gives us that
\[ \|\chi_Q\|_X \|\chi_Q\|_{X'} \leq C|Q|. \]

it follows that $f \in \mathcal{L}^{1,s}(\mathbb{R}^n)$ by Propositions 3.2 and 3.2. \hfill \Box

Unfortunately, if $0 < s < 1$, we do not know whether or not the condition $f \in \mathcal{L}^{1,s}(\mathbb{R}^n)$ is necessary for
\[ \sup_Q \int_Q |f(x) - |f|_Q|^s \, dx \leq C. \]

Then, we only obtain partly results about characterizations of $\mathcal{L}^{1,s}(\mathbb{R}^n)$ function associated to maximal functions on ball Banach function space.
Theorem 5.2 Let $1 \leq s < \infty$ and $X$ be a ball Banach function space such that the Hardy–Littlewood maximal operator $M$ is bounded on the associate space $X'$. Then $f \in \dot{L}^{1,\infty}(\mathbb{R}^n)$ if and only if
\[
\sup_Q \frac{\|(f - M_Q f)\chi_Q\|_{X'}}{\|\chi_Q\|_{X'}} < \infty.
\]

Theorem 5.3 Let $1 \leq s < \infty$ and $X$ be a ball Banach function space such that the Hardy–Littlewood maximal operator $M$ is bounded on the associate space $X'$. Then $f \in \dot{L}^{1,\infty}(\mathbb{R}^n)$ if and only if
\[
\sup_Q \frac{\|\langle f \rangle^{\alpha} M_{a,Q}(f)\chi_Q\|_{X'}}{\|\chi_Q\|_{X'}} < \infty.
\]

In [28], Tao, Yang, Yuan and Zhang showed the compactness characterizations of commutators of convolutional singular integral operator on ball Banach function spaces. Now, we consider the similar results for commutators of maximal functions.

Theorem 5.4 Let $X$ be a ball Banach function space such that $M$ is bounded on the associate space $X'$. If the commutator satisfy $[b,M] : X \to X$, then $b \in \text{BMO}(\mathbb{R}^n)$ with $b^- \in L^\infty(\mathbb{R}^n)$.

Proof For any cube $Q$, we write $f = \chi_Q$, then for any $x \in Q$
\[
M(f)(x) = 1, M(bf)(x) = M_Q(b)(x).
\]
This shows that $[b,M](f)(x) = b(x) - M_Q(b)(x)$ and
\[
\|[b,M](f)\|_X = \|(b - M_Q(b))\chi_Q\|_X \leq \|[b,M]\|_{X \to X} \|\chi_Q\|_X.
\]
Since this is true for every cube $Q, b \in \text{BMO}(\mathbb{R}^n)$ by Theorem 5.2. \qed

Theorem 5.5 Let $X$ and $Y$ be the ball Banach function space. Suppose that $M$ is bounded on the associate space $Y'$ and $M_a$ is bounded from $Y'$ to $X'$. If the commutator satisfy $[b,M_a] : X \to Y$, then $b \in \text{BMO}(\mathbb{R}^n)$ with $b^- \in L^\infty(\mathbb{R}^n)$.

Proof For any cube $Q$, we write $f = \langle |Q|^{-a/n} \chi_Q \rangle$, then for any $x \in Q$
\[
M(f)(x) = 1, M_a(bf)(x) = \langle |Q|^{-a/n} M_{a,Q}(b) \rangle(x).
\]
This shows that $[b,M_a](f)(x) = b(x) - \langle |Q|^{-a/n} M_{a,Q}(b) \rangle(x)$ and
\[
\|[b,M](f)\|_Y = \|(b - \langle |Q|^{-a/n} M_{a,Q}(b) \rangle)\chi_Q\|_Y \leq \langle |Q|^{-a/n} \|[b,M_a]\|_{X \to Y} \|\chi_Q\|_X.
\]
From the $(Y',X')$ boundedness of $M_a$ and [3, Lemma 2.1], we have
\[
\|\chi_Q\|_{Y'} \|\chi_Q\|_X \leq C\langle |Q|^{1-a/n} \rangle.
\]
We can now continue the above estimate:

\[
\| (b - |Q|^{-a/n} M_{a,Q}(b)) \chi_Q \|_Y \leq \frac{C|Q|}{\| \chi_Q \|_Y} \leq C \| \chi_Q \|_Y.
\]

Then, \( b \in \text{BMO}(\mathbb{R}^n) \) and \( b^- \in L^\infty(\mathbb{R}^n) \) by Theorem 5.3.

As we discuss above, the assumption of a geometric condition on the underlying spaces that is closely related to the boundedness of the Hardy–Littlewood maximal operator, and which holds in a large number of important special cases, such as the Morrey and variable Lebesgue spaces. However, Theorem 5.5 cannot generalized to the weighted Lebesgue spaces. For example, the maximal function may not bounded on \( L^q(\omega^{-q'}) \) for \( \omega \in A_{p,q} \). In fact, for the weighted Lebesgue spaces, we have

**Theorem 5.6** Let \( 1 < p < q < \infty, \frac{1}{p} - \frac{1}{q} = \frac{a}{n} \) and \( \omega \in A_{p,q} \). Then \( f \in \mathcal{L}^{1,n}(\mathbb{R}^n) \) if and only if

\[
\sup_Q \frac{|Q|^{a/n} \|f - |Q|^{-a/n} M_{a,Q}(f)\|_{L^q(\omega^{-q'})}}{\| \chi_Q \|_{L^p(\omega)}} < \infty.
\]

**Proof** (\( \Rightarrow \)): By Theorem 3.11, we have

\[
\left| \left\{ x \in Q : |f(x) - |Q|^{-a/n} M_{a,Q}(f)(x)| > t \right\} \right| \leq a_1 e^{-a_2 t} |Q|.
\]

Since \( \omega \in A_{p,q} \), we have \( \mu := \omega^q \in A_{1+q'/p'} \subset A_{\infty} \). Then, for any cube \( Q \) and any measurable set \( E \) contained in \( Q \), there are positive constants \( C_0 \) and \( L \) such that

\[
\left( \frac{|E|}{|Q|} \right)^N \leq C \frac{\mu(E)}{\mu(Q)}.
\]

This implies that

\[
\mu \left( \left\{ x \in Q : |f(x) - |Q|^{-a/n} M_{a,Q}(f)(x)| > \lambda \right\} \right) \leq C e^{-ct} \mu(Q).
\]

Hence, for any ball \( Q \),

\[
\| (f(x) - |Q|^{-a/n} M_{a,Q}(f)(x)) \chi_Q \|_{L^q(\mu)}^q = q \int_0^\infty \lambda^{q-1} \mu \left( \left\{ x \in Q : |f(x) - |Q|^{-a/n} M_{a,Q}(f)(x)| > \lambda \right\} \right) d\lambda \\
\leq C \int_0^\infty \lambda^{q-1} e^{-ct} \mu(Q) d\lambda \\
\leq C \mu(Q).
\]

By Hölder inequality, we have
\[ |Q| \leq \left( \int_Q \omega(x)^p \, dx \right)^{1/p} \left( \int_Q \omega(x)^{-p'} \, dx \right)^{1/p'} . \]

Then, it follows from \( \omega \in A_{p,q} \) that
\[
\frac{\mu(Q)^{1/q} |Q|^{a/n}}{\omega^p(Q)^{1/p}} \leq |Q|^{1/p-1/q-1} \left( \int_Q \omega(x)^q \, dx \right)^{1/q} \left( \int_Q \omega(x)^{-p'} \, dx \right)^{1/p'} \leq \left( \frac{1}{|Q|} \int_Q \omega(x)^q \, dx \right)^{1/q} \left( \frac{1}{|Q|} \int_Q \omega(x)^{-p'} \, dx \right)^{1/p'} \leq C.
\]

Thus, \( f \in \mathcal{L}^{1,q}(\mathbb{R}^n) \) implies that
\[
\frac{|Q|^{a/n}}{\omega^p(Q)^{1/p}} \left( \int_Q |f(x) - |Q|^{-a/n} M_{a,q}(f)(x)|^q \omega(x)^q \, dx \right)^{1/q} \leq C.
\]

(\( \Rightarrow \)): Now, we prove that if there exists a constant \( C \) such that for any cube \( Q \),
\[
\frac{1}{\omega^p(Q)^{1/p}} \left( \int_Q |f(x) - |Q|^{-a/n} M_{a,q}(f)(x)|^q \omega(x)^q \, dx \right)^{1/q} \leq C|Q|^{-a/n}.
\]
then \( f \in \mathcal{L}^{1,q}(\mathbb{R}^n) \).

When \( p > 1 \), Hölder inequality gives us that
\[
\int_Q |f(x) - |Q|^{-a/n} M_{a,q}(f)(x)| \, dx \leq \left( \int_Q |f(x) - |Q|^{-a/n} M_{a,q}(f)(x)|^p \omega(x)^p \, dx \right)^{1/p} \left( \int_Q \omega(x)^{-p'} \, dx \right)^{1/p'} \leq C|Q|^{a/n} \left( \int_Q |f(x) - |Q|^{-a/n} M_{a,q}(f)(x)|^q \omega(x)^q \, dx \right)^{1/q} \left( \int_Q \omega(x)^{-p'} \, dx \right)^{1/p'} \leq C \left( \int_Q \omega(x)^{-p'} \, dx \right)^{1/p'} \left( \int_Q \omega(x)^p \, dx \right)^{1/p} \leq C \left( \frac{1}{|Q|} \int_Q \omega(x)^{-p'} \, dx \right)^{1/p'} \left( \frac{1}{|Q|} \int_Q \omega(x)^q \, dx \right)^{1/q} \leq C|Q|.
\]

When \( p = 1 \), applying the definition of \( A_{1,q} \), we have
Therefore, we conclude that $f \in \overline{L}^{1,n}(\mathbb{R}^n)$. \hfill \Box

Finally, we can establish the similar results for $n < \lambda \leq n + p$ and we omitted the detail.

6 Remarks

In the bilinear setting, the linear commutator is defined by

$$[b_1, T](f_1, f_2)(x) := b_1 T(f_1, f_2)(x) - T(b_1 f_1, f_2)(x),$$

$$[b_2, T](f_1, f_2)(x) := b_2 T(f_1, f_2)(x) - T(f_1, b_2 f_2)(x)$$

and

$$[\Sigma \vec{b}, T](f_1, f_2)(x) := [b_1, T](f_1, f_2)(x) + [b_2, T](f_1, f_2)(x).$$

The boundedness result of linear commutators of multilinear Calderón–Zygmund operators $[\Sigma \vec{b}, T]$ was shown in [23]. The necessity conclusion lasted a long time and the proofs in [6] treat the term $[b_1, T]_1$ only. However, the boundedness of $[\Sigma \vec{b}, T]$ can not implies that $[b_i, T]_i$ is a bounded operator. Using some tedious calculations in applications, the linear characterization result was obtained in [31]. However, it is easy to obtain the linear characterization result related to multilinear maximal operator

$$\mathcal{M}(f_1, f_2)(x) = \sup_{Q \ni x} \prod_{i=1}^2 \frac{1}{|Q|} \int_Q |f_i(y_i)|dy_i.$$

**Theorem 6.1** Let $b = (b_1, b_2), 1 < p, p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. If the linear commutator $[\Sigma \vec{b}, \mathcal{M}]$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, then $b_i \in \text{BMO}(\mathbb{R}^n)$ with $b_i^{-} \in L^{\infty}(\mathbb{R}^n)$ with $i = 1, 2$.

**Proof** From the fact that
\[ M(\chi_Q, \chi_Q)(x) = 1, \]
\[ M(b_1 \chi_Q, \chi_Q)(x) = M_Q(b_1)(x), \]
\[ M(\chi_Q, b_2 \chi_Q)(x) = M_Q(b_2)(x), \quad x \in Q. \]

It is obvious that
\[ b_1(x) \leq M_Q(b_1)(x) \quad \text{and} \quad b_2(x) \leq M_Q(b_2)(x). \]

Therefore,
\[
\frac{1}{|Q|} \int_Q |b_1(x) - M_Q(b_1)(x)| \, dx
\leq \frac{1}{|Q|} \int_Q |b_1(x) + b_2(x) - M_Q(b_1)(x) - M_Q(b_2)(x)| \, dx
\leq \left( \frac{1}{|Q|} \int_Q \left| b_1(x) + b_2(x) - M_Q(b_1)(x) - M_Q(b_2)(x) \right|^p \, dx \right)^{1/p}
\leq \| [\Sigma \hat{h}, M] \|_{L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)},
\]

Which shows that \( b_1 \in \text{BMO}(\mathbb{R}^n) \) with \( b_1 \in L^\infty(\mathbb{R}^n) \). So does \( b_2 \). \( \square \)

In addition, for \( 0 < p < \infty \) and \( 0 < \lambda \leq n + p \), define the variant of the Morrey–Campanato class \( \tilde{L}^{p,\lambda}(\Omega) \)

\[
\| f \|_{L^{p,\lambda}(\Omega)}^p := \sup_{x \in \Omega, \rho > 0} \rho^{-\lambda} \int_Q |f(x) + |f||_Q \|^p \, dx < \infty.
\]

We can see that \( f \in \tilde{L}^{p,\lambda}(Q_0) \) if and only if \( -f \in \tilde{L}^{p,\lambda}(Q_0) \). Therefore

**Theorem 6.2** \textit{For} \( 1 \leq p < \infty \) \textit{and} \( 0 \leq \lambda < n \), \textit{we have} \( L^{p,\lambda}(\Omega) \approx \tilde{L}^{p,\lambda}(\Omega) \).

**Theorem 6.3** \textit{Let} \( Q_0 \) \textit{be a cube in} \( \mathbb{R}^n \). \textit{Then, the following statements are equivalent:}

\begin{enumerate}
  \item \( f \in \tilde{L}^{1,n}(Q_0) \);
  \item \( f \in \text{BMO}(Q_0) \) \textit{with} \( f^+ \in L^\infty(Q_0) \);
  \item \textit{For any} \( Q \subset Q_0 \) \textit{and} \( 0 < p < \infty \), \textit{there is a constant} \( C \) \textit{such that}
  \[
  \frac{1}{|Q|} \int_Q |f(x)| + f_Q|^p \, dx \leq C;
  \]
  \item \textit{For every} \( Q \subset Q_0 \) \textit{there is a constant} \( c_Q \geq 0 \) \textit{such that}
  \[
  \frac{1}{|Q|} \int_Q |f(x) + c_Q| \, dx < \infty;
  \]
\end{enumerate}
(v) For every $Q \subset Q_0$, we have

$$\inf_{c \geq 0} \frac{1}{|Q|} \int_Q |f(x) + c| \, dx < \infty;$$

**Theorem 6.4** Let $0 < \beta \leq 1$. For the function $f \in L^1_{\text{loc}}(\Omega)$, the following three statements are equivalent:

(i) $f \in C^{0,\beta}(\Omega)$ and $f \leq 0$.

(ii) There exists a constant $C_1$ such that

$$|f(x) + |f(y)| \leq C_1 |x - y|^\beta$$

for almost every $x$ and $y$.

(iii) There exists a constant $C_2$ such that for any $0 < p < \infty$ and any $x_0 \in \Omega$ and $0 < \rho < \text{diam} \Omega$

$$\left( \frac{1}{|\Omega(x_0, \rho)|} \int_{\Omega(x_0, \rho)} |f(x) + |f|_{\Omega(x_0, \rho)}|^p \, dx \right)^{1/p} \leq C_2 \rho^\beta.$$

(iv) There exists a constant $C_3$ such that for any $0 < p < \infty$ and any $x_0 \in \Omega$ and $0 < \rho < \text{diam} \Omega$

$$\left( \frac{1}{|\Omega(x_0, \rho)|} \int_{\Omega(x_0, \rho)} |f(x)| + |f|_{\Omega(x_0, \rho)}|^p \, dx \right)^{1/p} \leq C_3 \rho^\beta.$$

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