Drinfel’d Twists and Functional Bethe Ansatz

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Abstract

Using Functional Bethe Ansatz technique, factorizing Drinfel’d twists for any finite dimensional irreducible representations of the Yangian \( \mathcal{Y}(sl_2) \) are constructed.

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1 Introduction

Quasi-triangular Hopf algebras provide a natural language for the study of low-dimensional integrable systems solvable by the quantum inverse scattering method. Especially, their representations produce particular $R$-matrices solution of the Yang-Baxter equation \[1–4\].

In \[2, 5, 6\], Drinfel’d introduced the notion of twisting for quasi-triangular quasi-Hopf algebras. Among these Drinfel’d twists, a particular interesting class is given by those connecting non-cocommutative coproducts to cocommutative one’s. Representation theory of particular examples of such Drinfel’d twists has been studied in \[7\], in terms of what is called there factorizing $F$-matrices in the case of unitary $R$-matrices associated to finite dimensional irreducible modules of quantum universal enveloping Hopf algebras. There, the corresponding $R$-matrices are completely factorized by these $F$-matrices. These objects have been constructed for irreducible finite tensor products of the fundamental evaluation representations (spin $1/2$) of the Yangian $\mathcal{Y}(sl_2)$ and the quantum affine algebra $U_q(sl_2)$. They have turned out to be useful for the explicit computation of form factors and the resolution of the quantum inverse problem for local spin operators (see \[8\]) in the case of the XXX and XXZ spin-$1/2$ Heisenberg chains. For the moment, no universal formula is known for such factorizing Drinfel’d twists for the Yangian.

The aim of this letter is to pursue the study of representation theory of these Drinfel’d twists, through the computation of factorizing $F$-matrices associated to any irreducible finite dimensional representations of $\mathcal{Y}(sl_2)$. This is a generalization, to higher spin representations, of what has been done in \[7\] for spin $1/2$, towards a possible universal formula. Besides the interest in elucidating the structure of factorizing Drinfel’d twists, another motivation of the present work is the computation of form factors, which has been achieved in the case of XXX or XXZ spin-$1/2$ Heisenberg chain thanks to the knowledge of these $F$-matrices, and that remains to be done for higher spin chains. In particular, one can wonder what happens in the limit of infinite spin and for the corresponding quantum field theories.

Our approach is based on the Functional Bethe Ansatz (FBA) technique developed by Sklyanin in \[9, 10\] for the XXX chain. Indeed, one can remark that the basis induced by FBA applied to operator $D$, that is such that it is factorized as $D(u) = \prod_i (u - \hat{x}_i)$ in terms of its operator roots $\hat{x}_i$, coincides with the $F$-basis of \[9\] for the XXX spin-$1/2$ Heisenberg chain. We will show that, using this FBA technique for higher spins, we are able to construct the corresponding factorizing $F$-matrices in this more general case by solving only linear equations.

2 Factorizing $F$-matrices and XXX Heisenberg chain

For $r_i \in \mathbb{N}$, we denote by $V_{r_i}$ an irreducible finite dimensional representation of the Lie algebra $sl_2$ ($\dim V_{r_i} = r_i + 1$), and by $V_{r_i}(z_i)$ the corresponding evaluation representation of $\mathcal{Y}(sl_2)$. It is known that every finite-dimensional irreducible $\mathcal{Y}(sl_2)$-module is isomorphic to a tensor product of such evaluation representations \[11\].

The purpose of this letter is to compute factorizing $F$-matrices associated to any irreducible finite dimensional representations of the Yangian $\mathcal{Y}(sl_2)$, and which factorize the
corresponding (unitary) $R$-matrices in these representations. The notion of factorizing $F$-matrices, inspired by the representation theory of a particular class of Drinfel’d twists, was defined in [7]. We recall here the basic definitions, and refer the reader to [7] for more details.

The notion of factorizing twist is essentially defined for triangular Hopf algebras $A$. It corresponds to invertible elements $F = \sum_i f^i \otimes f_i \in A \otimes A$ which factorize in particular the corresponding universal $R$-matrix:

$$R = F^{-1}_{21} F,$$

(2.1)

where $F_{21} = \sum_i f_i \otimes f^i$. Although Yangians are pseudo-triangular rather than triangular, it is possible to define, for the unitary $R$-matrices associated to their finite dimensional irreducible representations, the notion of factorizing $F$-matrix by analogy with Drinfel’d twists for triangular Hopf algebras. Such $F$-matrices are thus defined directly at the representation level as a transcription of what happens at the universal level for triangular Hopf algebras. To deal with such objects, we need to introduce some convenient notations that follow.

**Definition 1.** Let $n$ be an integer and $\sigma$ an arbitrary element of the permutation group $S_n$. Let $X \in \text{End}(V_q)$, $V_q = V_{r_1}(z_1) \otimes \cdots \otimes V_{r_n}(z_n)$, be denoted by

$$X_{r_1 \ldots r_n}(z_1, \ldots, z_n) = \sum_i x^{(i)}_{r_1} \otimes \cdots \otimes x^{(n)}_{r_n},$$

(2.2)

where $x^{(i)}_{r_j} \in \text{End}(V_{r_j}(z_j))$. We define the extended action of the symmetry group $S_n$ on $\text{End}(V_q)$ as

$$\sigma(X_{r_1 \ldots r_n}(z_1, \ldots, z_n)) = \sum_i x^{(\sigma^{-1}(i))}_{r_{\sigma^{-1}(1)}} \otimes \cdots \otimes x^{(\sigma^{-1}(n))}_{r_{\sigma^{-1}(n)}},$$

(2.3)

where $V_{\sigma(q)} = V_{r_{\sigma^{-1}(1)}}(z_{\sigma^{-1}(1)}) \otimes \cdots \otimes V_{r_{\sigma^{-1}(n)}}(z_{\sigma^{-1}(n)})$.

**Definition 2.** Let $n \in \mathbb{N}$, and consider a family $G_X$ of operators $X_{r_1 \ldots r_n}(z_1, \ldots, z_n)$ defined for all $r_i \in \mathbb{N}$, acting on irreducible tensor products of $n$ finite dimensional evaluation modules $V_{r_1}(z_1) \otimes \cdots \otimes V_{r_n}(z_n)$. To each element $\sigma$ of the symmetry group $S_n$, we associate the family $G_X^\sigma$ of operators $(X_\sigma)_{r_1 \ldots r_n}(z_1, \ldots, z_n) \in \text{End}(V_{r_1}(z_1) \otimes \cdots \otimes V_{r_n}(z_n))$ defined as

$$(X_\sigma)_{r_1 \ldots r_n}(z_1, \ldots, z_n) = \sigma(X_{r_{\sigma^{-1}(1)} \ldots r_{\sigma^{-1}(n)}}(z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)})).$$

(2.4)

Note that if the family $G_X$ corresponds to the different representations $(\rho_{r_1} \otimes \cdots \otimes \rho_{r_n})$ of a universal operator $X_{1 \ldots n} = \sum_i x^{(i)} \otimes \cdots \otimes x^{(n)} i \in \mathcal{Y}(sl_2)^{\otimes n}$ on $V_{r_1}(z_1) \otimes \cdots \otimes V_{r_n}(z_n)$, the above definition simply means that

$$(X_\sigma)_{r_1 \ldots r_n}(z_1, \ldots, z_n) = \sigma((\rho_{r_{\sigma^{-1}(1)}} \otimes \cdots \otimes \rho_{r_{\sigma^{-1}(n)}})(X_{1 \ldots n})),$$

(2.5)

where $X_{\sigma^{-1}(1) \ldots \sigma^{-1}(n)} = \sigma(X_{1 \ldots n})$ is the operator $\sum_i x^{(\sigma^{-1}(i))} \otimes \cdots \otimes x^{(\sigma^{-1}(n))} i \in \mathcal{Y}(sl_2)^{\otimes n}$.

These notations enable us to introduce the notions of generalized $R$-matrices associated to a given tensor product of modules and to a permutation $\sigma$, and of their factorizing $F$-matrices.
Proposition 1. For any integer \( n \), we can define a map from the permutation group \( \mathfrak{S}_n \) to \( \text{End}(V_q) \), \( V_q = V_{r_1}(z_1) \otimes V_{r_2}(z_2) \otimes \cdots \otimes V_{r_n}(z_n) \), which associates in a unique way an element \( R_q^\sigma \in \text{End}(V_q) \) to any permutation \( \sigma \in \mathfrak{S}_n \). It is defined recursively by its values for simple transpositions \((i, i + 1)\),

\[
R_q^{(i,i+1)} = R_{r_ir_{i+1}}(z_i, z_{i+1}),
\]

and by the following composition law for the product of two elements of \( \mathfrak{S}_n \):

\[
R_q^{\sigma_1 \sigma_2} = (R_q^{\sigma_2})^R_q \sigma_1, \quad \forall \sigma_1, \sigma_2 \in \mathfrak{S}_n.
\]

Here \( R_{r_ir_{i+1}}(z_i, z_{i+1}) \) is the \( R \)-matrix acting in \( V_{r_1}(z_i) \otimes V_{r_{i+1}}(z_{i+1}) \) as \( R \) and as the identity in all other modules in the tensor product \( V_q \), and \((R_q^{\sigma_2})^R_q \sigma_1 \) is defined from \( R_q^{\sigma_2} \) as in definition 3.

Remark 1. The consistency of this definition follows from the Yang-Baxter and unitary relations for the elementary \( R \)-matrices. It corresponds to representations of the intertwining relations in \( \mathcal{Y}(sl_2) \).

**Definition 3.** By factorizing \( F \)-matrices is meant a family of invertible operators \( F \) acting on irreducible finite dimensional modules \( V_q = V_{r_1}(z_1) \otimes V_{r_2}(z_2) \otimes \cdots \otimes V_{r_n}(z_n) \), defined for any integer \( n \) and any \( r_i \in \mathbb{N} \), and such that for any element \( \sigma \in \mathfrak{S}_n \),

\[
(F_\sigma)_{r_1 \ldots r_n}(z_1, \ldots, z_n) R_{\sigma}^{r_1 \ldots r_n}(z_1, \ldots, z_n) = F_{r_1 \ldots r_n}(z_1, \ldots, z_n),
\]

or in compact notations, \((F_\sigma)_q R_q^\sigma = F_q \), with \( q = (r_1 r_2 \ldots r_n) \) (\( F_\sigma \) being given from \( F \) by definition 3).

In the following, for given \( r_i \in \mathbb{N} \), we will use some simplified notations, denoting merely by \( X_{12 \ldots n} \) (instead of \( X_{r_1 r_2 \ldots r_n}(z_1, z_2, \ldots, z_n) \)) an operator \( X \) acting on the tensor product of evaluation representations \( V_{r_1}(z_1) \otimes V_{r_2}(z_2) \otimes \cdots \otimes V_{r_n}(z_n) \), and by \( X_\sigma(1 \ldots n) \) the operator \( X_\sigma (2.4) \) acting on the same tensor product (the order of spaces in the tensor product being given a priori).

Using these notations, we recall at last some characterizing properties of these factorizing \( F \)-matrices which will lead to their explicit computation in section 4.

**Proposition 2.** Let \( F_{1 \ldots n} \) be factorizing \( F \)-matrices and define partial \( F \)-matrices \( \widetilde{F}_{1 \ldots n} = F_{1 \ldots n} F_{2 \ldots n}^{-1} \) and \( \widetilde{F}_{1 \ldots n-1} = F_{1 \ldots n} F_{2 \ldots n-1}^{-1} \). They satisfy,

\[
\widetilde{F}_{1 \ldots n-1} \widetilde{F}_{2 \ldots n-1} = \widetilde{F}_{1 \ldots n-1} \widetilde{F}_{2 \ldots n-1},
\]

\[
\widetilde{F}_{0 \ldots 1} = F_{0, \sigma(1) \ldots \sigma(n)} \quad \forall \sigma \in \mathfrak{S}_n,
\]

\[
\widetilde{F}_{1 \ldots n, 0} = F_{\sigma(1) \ldots \sigma(n), 0} \quad \forall \sigma \in \mathfrak{S}_n,
\]

\[
\widetilde{F}_{1 \ldots n, 0} F_{1 \ldots n} R_{0_1} \ldots R_{0_{n-1}} = F_{0_1 \ldots n} F_{1 \ldots n}.
\]

Conversely, suppose we have defined sets of matrices \( \widetilde{F}_{1 \ldots n} \) and \( \widetilde{F}_{1 \ldots n-1} \) for any integer \( n \) satisfying the above properties with \( F_{1 \ldots n} = \widetilde{F}_{1 \ldots n-1} \ldots \widetilde{F}_{123} F_{12} \) or equivalently due to the cocycle relation (2.10), \( F_{1 \ldots n} = \widetilde{F}_{1 \ldots n} \ldots F_{n-1} \), then these sets of matrices define factorizing \( F \)-matrices.
In [4], such factorizing $F$-matrices have been computed for tensor products of spin-$\frac{1}{2}$ evaluation representations of $\mathcal{Y}(sl_2)$, and it has been shown that, when applied to the space of states of the XXX Heisenberg spin-$\frac{1}{2}$ chain, they induced a new basis in which the operator entries of the monodromy matrix have very simple forms. Here, we will reverse the process, computing factorizing $F$-matrices as matrices inducing a particular change of basis in the space of states of the associated XXX chain.

Thus, for a given finite tensor product of evaluation spin-$l_n$ representations of $\mathcal{Y}(sl_2)$ $\mathcal{H}_{1...N} = V_{2l_1}(\delta_1) \otimes \cdots \otimes V_{2l_N}(\delta_N)$, let us consider the general periodic inhomogeneous XXX Heisenberg chain of length $N$ whose quantum space of states is $\mathcal{H}_{1...N}$.

The quantum $L$-operator $L_n(u)$ at site $n$, which is linear in the spectral parameter $u$, is constructed in terms of the spin operators $S_n^+, S_n^-, S_n^z$ belonging to the irreducible finite dimensional representation $V_{2l_n}$ ($\dim V_{2l_n} = 2l_n + 1$) of the Lie algebra $sl_2$, and depends on the inhomogeneity parameter $\delta_n$:

$$L_n(u - \delta_n) = u - \delta_n + \eta \sum_{\alpha=1}^{3} S_n^\alpha \sigma^\alpha = \left(\begin{array}{cc} u - \delta_n + \eta S_n^+ & \eta S_n^- \\ \eta S_n^+ & u - \delta_n - \eta S_n^- \end{array}\right),$$

with

$$S_n^\pm = S_n^z \pm i S_n^y, \quad [S_n^+, S_n^-] = 2S_n^z,$$

$$S_n^2 = \frac{1}{2}(S_n^+ S_n^- + S_n^- S_n^+) + (S_n^z)^2 = l_n(l_n + 1), \quad [S_n^z, S_n^\pm] = \pm S_n^\pm.$$

The monodromy matrix of the chain,

$$T(u) \equiv T_{1...N}(u; \delta_1, \ldots, \delta_n) = L_N(u - \delta_N) \cdots L_1(u - \delta_1) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \quad (2.14)$$

is therefore a polynomial of degree $N$ in the spectral parameter $u$:

$$T(u) = \sum_{i=0}^{N} u^i \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}, \quad (2.15)$$

where $A_i$, $B_i$, $C_i$ and $D_i$ are quantum operators acting on the total quantum space $\mathcal{H}$ of the chain. Commutation relations of operators $A(u)$, $B(u)$, $C(u)$ and $D(u)$ are given by the relation:

$$R(u - v) (T(u) \otimes \text{Id}) (\text{Id} \otimes T(v)) = (\text{Id} \otimes T(v)) (T(u) \otimes \text{Id}) R(u - v), \quad (2.16)$$

where $R$ is the rational $R$-matrix associated to the fundamental representation of $\mathcal{Y}(sl_2)$:

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b(u) = \frac{u}{u + \eta}, \quad c(u) = \frac{\eta}{u + \eta}.$$  

As for the generalized $R$-matrices $R^\sigma$ defined on $\text{End} \mathcal{H}$ as in proposition [4] from the different $R$-matrices $R_{ij} \in V_{2l_i} \otimes V_{2l_j}$, they satisfy, for any element $\sigma \in \mathfrak{S}_n$,

$$R_{1...N}^\sigma T_{1...N} = T_{\sigma(1)...\sigma(N)} R_{1...N}^\sigma, \quad (2.17)$$
where $T_{\sigma(1)\ldots\sigma(N)} \in \text{End} \mathcal{H}$ is merely obtained by permutation of the ordered product of $L$-operators:

$$T_{\sigma(1)\ldots\sigma(N)}(u) = L_{\sigma(N)}(u - \delta_{\sigma(N)}) \ldots L_{\sigma(1)}(u - \delta_{\sigma(1)}).$$

This means that, if these $R$-matrices admit factorizing $F$-matrices, the latter induce a basis of $\mathcal{H}$ in which the expression of the monodromy matrix is symmetric under any permutation (2.18) of the sites. In [7], such a basis has been exhibited for spin 1/2, which happened to diagonalize $D$. In the following, $F$-basis for higher spin chains will be investigated directly as diagonalizing basis for the operator $D$.

### 3 Functional Bethe Ansatz for the chain

In this section, we apply to operator $D_{1\ldots N}(u)$ the Functional Bethe Ansatz method of Sklyanin in order to obtain explicit expressions for operators $A$, $B$, $C$, $D$ in a basis in which $D$ is diagonal. We merely give here the idea of this method applied to our special case, for the procedure is very similar to what is done in [9, 10].

The idea of Functional Bethe Ansatz, applied here to operator $D$ instead of $B$, is to define the operator roots $\hat{x}_n$, $1 \leq n \leq N$ of the polynomial of degree $N D_{1\ldots N}(u)$:

$$D_{1\ldots N}(u) = \prod_{n=1}^{N} (u - \hat{x}_n),$$

by constructing an isomorphism between Fun $\mathcal{Y}$ and the space of states $\mathcal{H}$ of the chain. Here $\mathcal{Y} \subset \mathbb{C}^N$ is the common spectrum of $\{\hat{x}_1, \ldots, \hat{x}_N\}$, and $\hat{x}_n$ are realized on Fun $\mathcal{Y}$ as the operators of multiplication by the corresponding coordinates (the $n$-th coordinate) in $\mathbb{C}^N$ (see [9] for details):

$$[\hat{x}_n f](y) = y_n f(y), \quad \forall f \in \text{Fun } \mathcal{Y}, \quad \forall y = (y_1, \ldots, y_N) \in \mathcal{Y}.$$

In our case, the spectrum $\mathcal{Y}$ can be determined directly from the diagonal elements of the lower triangular matrix $D_{1\ldots N}(u)$. Indeed, one can prove similarly as in [9] that the diagonal of $D(u)$ is given by

$$\text{Diag } D(u) = \prod_{i=1}^{N} (u - \delta_i - \eta S_i^z),$$

so that the associated spectrum is

$$\mathcal{Y} = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_N, \quad \Lambda_i = \{\delta_i + \eta k \mid k = -l_i, -l_i + 1, \ldots, l_i\}.$$  

**Remark 2.** FBA can be applied to $D(u)$ provided some conditions are satisfied (see [2]). They can be reduced here to the following non-intersection condition on the spectrum $\mathcal{Y}$:

$$\Lambda_i \cap \Lambda_j = \emptyset \quad \text{for } i \neq j.$$  

This ensures in particular that $D(u)$ is diagonalizable (since its spectrum is simple).
Operators \( \hat{X}_n^+, \hat{X}_n^- \) are then defined from polynomials \( B(u) \) and \( C(u) \) by substitution “from the left” (i.e. with some operator ordering) of \( u \) by \( \hat{x}_n \):

\[
\hat{X}_n^- = \sum_{p=0}^{N} \hat{x}_n^p B_p \equiv [B(u)]_{u=\hat{x}_n}, \quad \hat{X}_n^+ = \sum_{p=0}^{N} \hat{x}_n^p C_p \equiv [C(u)]_{u=\hat{x}_n}. \tag{3.22}
\]

The commutation relations for \( \hat{X}_n, \hat{X}_m^\pm \) follow from those of \( A, B, C, \) and \( D \):

\[
[\hat{x}_m, \hat{x}_n] = [\hat{X}_m^+, \hat{X}_n^-] = 0, \quad \forall m, n, \quad [\hat{X}_m^+, \hat{X}_n^-] = 0, \quad \forall m, n, m \neq n,
\]

\[
\hat{X}_m^\pm \hat{x}_n = (\hat{x}_n + \eta \delta_{mn}) \hat{X}_m^\pm, \quad \forall m, n, \quad \hat{X}_n^\pm \hat{X}_m^\pm = -\Delta(\hat{x}_n \pm \frac{\eta}{2}), \quad \forall n,
\]

where \( \Delta(u) = \prod_{n=1}^{N} (u - \delta_n - l_n \eta - \frac{\eta}{2})(u - \delta_n + l_n \eta + \frac{\eta}{2}) \) is the quantum determinant of the monodromy matrix \( T(u) \).

Conversely, \( B(u) \) and \( C(u) \) can be reconstructed in terms of the \( \hat{x}_n, \hat{X}_m^\pm \) by means of polynomial interpolation:

\[
B(u) = \sum_{n=1}^{N} \left\{ \prod_{i \neq n} \frac{u - \hat{x}_i}{\hat{x}_n - \hat{x}_i} \right\} \hat{X}_n^- \quad \text{and} \quad C(u) = \sum_{n=1}^{N} \left\{ \prod_{i \neq n} \frac{u - \hat{x}_i}{\hat{x}_n - \hat{x}_i} \right\} \hat{X}_n^+ \tag{3.23}
\]

which, with (3.19) and the fact that \( A \) can be obtained in terms of \( B, C, \) and \( D \) and the quantum determinant \( \Delta \), provides expressions of the matrix elements of the monodromy matrix only in terms of operators \( \hat{x}_n \) and \( \hat{X}_m^\pm \). Therefore, if we compute explicitly the action of \( \hat{X}_n^\pm \) in a basis which diagonalizes simultaneously all \( \hat{x}_n \), we will obtain explicit expressions for \( A, B, C, \) and \( D \) in a basis in which \( D \) is diagonal.

This can be done by considering the actions of \( \hat{x}_n, \hat{X}_m^\pm \) in an explicit basis of \( \text{Fun} \mathbb{Y} \) (which is isomorphic to \( \mathcal{H} \)). The point is thus to determine the action of \( \hat{X}_n^\pm \) on \( \text{Fun} \mathbb{Y} \).

Still following Sklyanin, let us define the function \( \Delta_n^\pm = \hat{X}_n^\pm \omega \), where \( \omega \) is the constant function \( \omega \equiv 1 \) of \( \text{Fun} \mathbb{Y} \). The action of \( \hat{X}_n^\pm \) on an arbitrary function \( f \in \text{Fun} \mathbb{Y} \) is then given in terms of \( \Delta_n^\pm \) (see [1]):

\[
[\hat{X}_n^\pm f](\mathbb{Y}) = \Delta_n^\pm(\mathbb{Y}) f(E_n^\mp \mathbb{Y}) \quad \forall \mathbb{Y} \in \mathbb{Y}, \tag{3.24}
\]

where \( E_n^\pm \) are the shift operators in \( \mathbb{C}^N \):

\[
E_n^\pm : (y_1, \ldots, y_n, \ldots, y_N) \mapsto (y_1, \ldots, y_n \mp \eta, \ldots, y_N).
\]

It is thus sufficient for our purpose to compute \( \Delta_n^\pm \).

The same way as in [1], commutation relations for \( \hat{x}_n, \hat{X}_m^\pm \) impose some conditions on \( \Delta_n^\pm \), and it is easy to see that these conditions determine \( \Delta_n^\pm \) up to multiplication by an arbitrary function:

\[
\Delta_n^\pm(\mathbb{Y}) = \xi_{\pm} \frac{\rho(\mathbb{Y})}{\rho(E_n^\mp \mathbb{Y})} \Delta^\pm(\mathbb{Y}) \quad \text{with} \quad \Delta^\pm(\mathbb{Y}) = \prod_{i=1}^{N} (y_n - \delta_i \pm \eta \xi_i), \tag{3.25}
\]

where \( \rho \) is an arbitrary function having no zeroes on \( \mathbb{Y} \), and \( \xi_+, \xi_- \in \mathbb{C} \) are such that \( \xi_+ \xi_- = 1 \). Note that the indetermination on \( \rho \) corresponds to the indetermination on
the isomorphism between $\mathcal{H}$ and $\text{Fun } \mathbb{Y}$. In the following we choose $\rho$ to be the constant function $\rho(y) \equiv 1$, which corresponds to fixing this isomorphism. Hence, in the basis

$$\{f_{1,k_1} \times f_{2,k_2} \times \cdots \times f_{N,k_N} \mid f_{i,k_i} \in \text{Fun } \Lambda_i, \ k_i = l_i, l_i - 1, \ldots, -l_i\} \quad (3.26)$$

of $\text{Fun } \mathbb{Y}$ defined by

$$f_{n,k_n}(\delta_n + \eta p_n) = \xi^{-l_n + k_n} \delta_{n,p_n} \quad \forall p_n \in \{-l_n, \ldots, l_n\}, \quad (3.27)$$

$\hat{x}_n$, $\hat{X}_n^\pm$ are given by

$$\hat{x}_n = \delta_n + \eta S_n^z, \quad \hat{X}_n^\pm = \eta \prod_{i=n}^{N} (\delta_n - \delta_i + \eta S_i^z \pm \eta l_i) S_n^\pm,$$

where $S_n^z$, $S_n^\pm$ are the following matrix representations (dimension $2l_n + 1$) of the spin operators $S_n^z$, $S_n^\pm$:

$$S_n^z = \begin{pmatrix} l_n & 0 & \cdots & 0 \\ 0 & l_n - 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -l_n + 1 \\ 0 & \cdots & 0 & -l_n \end{pmatrix}_{[n]}, \quad (3.28)$$

$$S_n^+ = \begin{pmatrix} 0 & 2l_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}_{[n]}, \quad S_n^- = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 2l_n & 0 \end{pmatrix}_{[n]} \quad (3.29)$$

The subscript $[n]$ means here that the corresponding operator acts as identity on all spaces of the tensor product but $\text{Fun } \Lambda_n \simeq V_{2l_n}$.

As a consequence, we have the following proposition:

**Proposition 3.** There exists a basis of $\mathcal{H}$ in which $D$, $B$, $C$ have the following expressions $\tilde{D}$, $\tilde{B}$ and $\tilde{C}$:

$$\tilde{D}_{1\ldots N}(u) = \prod_{n=1}^{N} (u - \delta_n - \eta S_n^z), \quad (3.30)$$

$$\tilde{B}_{1\ldots N}(u) = \sum_{n=1}^{N} \left\{ \prod_{i \neq n}^{N} (u - \delta_i - \eta S_i^z) \frac{\delta_n - \delta_i + \eta S_n^z - \eta l_i}{\delta_n - \delta_i + \eta S_n^z - \eta S_i^z} \right\} \eta S_n^-, \quad (3.31)$$

$$\tilde{C}_{1\ldots N}(u) = \sum_{n=1}^{N} \left\{ \prod_{i \neq n}^{N} (u - \delta_i - \eta S_i^z) \frac{\delta_n - \delta_i + \eta S_n^z + \eta l_i}{\delta_n - \delta_i + \eta S_n^z - \eta S_i^z} \right\} \eta S_n^+, \quad (3.32)$$

$\tilde{A}$ being given by the quantum determinant:

$$\tilde{A}(u) = \tilde{D}^{-1}(u - \eta)[\Delta(u - \frac{\eta}{2}) + \tilde{B}(u - \eta)\tilde{C}(u)]. \quad (3.33)$$
These expressions are to be compared to the expressions for operators $B$, $C$, $D$, $A$ in the $F$-basis obtained by J.M. Maillet and J. Sanchez de Santos in [1]. Indeed, in the case when all spins $l_i = 1/2$, these formulas coincide (up to a normalization factor). Moreover, as in [1], $\bar{D}$ is still here a pure tensor product of diagonal matrices and the expressions for $\bar{D}$, $\bar{B}$, $\bar{C}$, and thus $\bar{A}$, are completely symmetric under any permutation of the sites. So, the new basis obtained by Functional Bethe Ansatz technique appears to be a generalization of the $F$-basis obtained in [1]. In order to prove it, we will thus compute the matrix which induces this change of basis, and show that it is effectively a factorizing $F$-matrix in the sense of definition 3.

4 Determination of the $F$-matrix

In this section, we compute the matrix $F_{1\ldots N}$ which induces the previous change of basis for the chain of length $N$, that is such that, for $X = A$, $B$, $C$, or $D$,

$$ \tilde{X}_{1\ldots N}(u) = F_{1\ldots N} X_{1\ldots N}(u) F_{1\ldots N}^{-1}, $$

with $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ given by (3.30)–(3.33), and we show that the matrices $F_{1\ldots N}$ thus defined (for any spin chains) are factorizing $F$-matrices.

Remark 3. $F_{1\ldots N}$, which diagonalizes the lower triangular matrix $D_{1\ldots N}$ whose diagonal coefficients are all distinct, is itself lower triangular.

In order to compute $F_{1\ldots N}$ by induction on $N$, let us define, for all integer $n \geq 2$,

$$ \bar{F}_{1\ldots n} = F_{1\ldots n} F_{2\ldots n}^{-1}. $$

By definition, $\bar{F}_{1\ldots n}$ is a lower triangular matrix, and thus has to be of the form

$$ \bar{F}_{1\ldots n} = \bar{Q}_{1\ldots n} \left( 1 + \sum_{k=1}^{2|l_1|} \alpha_{1\ldots n}^{(k)} (S_1^-)^k \right), $$

where $\bar{Q}_{1\ldots n}$ and $\alpha_{1\ldots n}^{(k)}$ for all $k$ are diagonal on space (1).

Using the relations between $A$, $B$, $C$ and $D$ for $N$ and $N - 1$ sites,

$$ D_{1\ldots N}(u) = \eta S_1^- C_{2\ldots N}(u) + (u - \delta_1 - \eta S_1^z) D_{2\ldots N}(u), $$

$$ B_{1\ldots N}(u) = \eta S_1^- A_{2\ldots N}(u) + (u - \delta_1 - \eta S_1^z) B_{2\ldots N}(u), $$

$$ C_{1\ldots N}(u) = (u - \delta_1 + \eta S_1^z) C_{2\ldots N}(u) + \eta S_1^+ D_{2\ldots N}(u), $$

and applying the change of basis induced by $F_{1\ldots N} = \bar{F}_{1\ldots n} F_{2\ldots n}$, one obtains the following equations for $\bar{F}_{1\ldots n}$:

$$ \bar{D}_{1\ldots N}(u) \bar{F}_{1\ldots N} = \bar{F}_{1\ldots N} \left[ \eta S_1^- \bar{C}_{2\ldots N}(u) + (u - \delta_1 - \eta S_1^z) \bar{D}_{2\ldots N}(u) \right], $$

$$ \bar{B}_{1\ldots N}(u) \bar{F}_{1\ldots N} = \bar{F}_{1\ldots N} \left[ \eta S_1^- \bar{A}_{2\ldots N}(u) + (u - \delta_1 - \eta S_1^z) \bar{B}_{2\ldots N}(u) \right], $$

$$ \bar{C}_{1\ldots N}(u) \bar{F}_{1\ldots N} = \bar{F}_{1\ldots N} \left[ (u - \delta_1 + \eta S_1^z) \bar{C}_{2\ldots N}(u) + \eta S_1^+ \bar{D}_{2\ldots N}(u) \right]. $$

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These linear equations enable us to compute $\tilde{F}_{1,2...N}$. Indeed, decomposing (4.40) on all $(S^z_1)^k$, $0 \leq k \leq 2l_1$, we obtain:

$$\tilde{F}_{1,2...N} = \tilde{Q}_{1,2...N} \sum_{k=0}^{2l_1} \frac{1}{k!} [\tilde{C}_{2...N}(\delta_1 + \eta S^z_1) \tilde{D}_{2...N}^{-1}(\delta_1 + \eta S^z_1)]^k (S^z_1)^k, \quad (4.43)$$

where $\tilde{Q}_{1,2...N}$, which has to commute with $\tilde{D}_{1...N}(u)$ for all values of $u$, is necessarily a diagonal matrix. It is determined, up to a global numerical factor, using (4.41) and (4.42):

$$\tilde{Q}_{1,2...N} = \alpha(S^z_1; S^z_2, \ldots, S^z_N),$$

$$= \prod_{k=1}^{\infty} \left\{ \frac{\tilde{D}_{2...N}(\delta_1 + \eta S^z_1 + \eta k)}{\tilde{D}_{2...N}(\delta_1 + \eta l_1 + \eta k)} \prod_{i=2}^{N} \frac{\delta_1 - \delta_i + \eta l_1 + \eta l_i + \eta k}{\delta_1 - \delta_i + \eta S^z_i + \eta l_i + \eta k} \right\} \alpha(l_1; l_2, \ldots, l_N). \quad (4.44)$$

In the following, we choose the normalization $\alpha(l_1; l_2, \ldots, l_N) = 1$ so that $\tilde{F}_{1,2...N}|0\rangle = |0\rangle$ where $|0\rangle = (1, 0, \ldots, 0)$. Note that the product is actually finite for each matrix element of $S^z_1$.

The matrix $F_{1...N}$ inducing the change of basis is thus given by induction on the number of sites $N$ of the chain:

$$F_N = 1,$$

$$F_{12...N} = \tilde{F}_{1,2...N} F_{2...N},$$

with $\tilde{F}_{1,2...N}$ given by (4.43)-(4.44).

Its inverse can be computed by means of the linear equation for $\tilde{F}_{1,2...N}^{-1}$ equivalent to (4.40). One obtains:

$$\tilde{F}_{1,2...N}^{-1} = \left\{ \sum_{k=0}^{2l_1} \frac{(-1)^k}{k!} (S^z_1)^k [\tilde{D}_{2...N}^{-1}(\delta_1 + \eta S^z_1) \tilde{C}_{2...N}(\delta_1 + \eta S^z_1)]^k \right\} \tilde{Q}_{1,2...N}^{-1}. \quad (4.47)$$

Note that the set of matrices $F_{1...N}$ can also be determined similarly by computing the other partial matrices

$$\tilde{F}_{1...n-1,n} = F_{1...n} F_{1...n-1}^{-1}. \quad (4.48)$$

Linear equations similar to (4.40)-(4.42) lead to the following expressions for these partial matrices:

$$\tilde{F}_{1...n-1,N} = \tilde{Q}_{1...n-1,N} \sum_{k=0}^{2N} \frac{(-1)^k}{k!} [\tilde{B}_{1...n-1}(\delta_N + \eta S^z_N) \tilde{D}_{1...n-1}^{-1}(\delta_N + \eta S^z_N)]^k (S^z_N)^k, \quad (4.49)$$

$$\tilde{F}_{1...n-1,N}^{-1} = \left\{ \sum_{k=0}^{2N} \frac{1}{k!} (S^z_N)^k [\tilde{D}_{1...n-1}^{-1}(\delta_N + \eta S^z_N) \tilde{B}_{1...n-1}(\delta_N + \eta S^z_N)]^k \right\} \tilde{Q}_{1...n-1,N}^{-1}, \quad (4.50)$$

with

$$\tilde{Q}_{1...n-1,N} = \prod_{l=1}^{\infty} \left\{ \frac{\tilde{D}_{1...n-1}(\delta_N + \eta S^z_N - \eta k)}{\tilde{D}_{1...n-1}(\delta_N - \eta l_N - \eta k)} \prod_{i=1}^{N-1} \frac{\delta_N - \delta_i - \eta l_N - \eta l_i - \eta k}{\delta_N - \delta_i + \eta S^z_i - \eta l_i - \eta k} \right\}. \quad (4.51)$$
It has also to be mentioned that other sets of partial $F$-matrices, defined as $F_{1,2,...,n}^{-1} = F_{2,...,n}^{-1} F_{1,...,n}$ and $F_{1,...,n-1} = F_{1,...,n-1}^{-1} F_{1,...,n}$ are obtained from (4.43)-(4.44) and (4.49)-(4.51) by replacing in these formulas $\tilde{C}_{2,N}$, $\tilde{D}_{2,...,N}$ and $\tilde{B}_{1,...,N-1}$, $\tilde{D}_{1,...,N-1}$ respectively by $C_{2,N}$, $D_{2,...,N}$ and $B_{1,...,N-1}$, $D_{1,...,N-1}$.

**Remark 4.** In particular, $F_{12}$ is given by

$$F_{12} = Q_{12} \sum_{k=0}^{\infty} \frac{\eta^k}{k!} \prod_{j=1}^{k} [\delta_1 - \delta_2 + \eta S_1^z - \eta S_2^z + \eta j]^{-1} (S_1^-)^k (S_2^+)^k,$$

and its inverse is

$$F_{12}^{-1} = \left\{ \sum_{k=0}^{\infty} \frac{(-\eta)^k}{k!} (S_1^-)^k (S_2^+)^k \prod_{j=1}^{k} [\delta_1 - \delta_2 + \eta S_1^z - \eta S_2^z - \eta j]^{-1} \right\} Q_{12}^{-1}.$$

**Remark 5.** The non-diagonal part of the matrix $F_{12}^{-1}$, which satisfies the linear equation

$$F_{12}^{-1} \tilde{D}_{12} = D_{12} F_{12}^{-1},$$

can be directly obtained from it as an infinite formal product:

$$F_{12}^{-1} = \prod_{k=0}^{\infty} \tilde{D}_{12}(u) D_{12}(u) \tilde{D}_{12}^{-1}(k+1)(u) \cdot \tilde{Q}_{12}^{-1}.$$

Note that by computing this product explicitly, one finds again the expression (4.54) (in particular $F_{12}^{-1}$ does not depend on the spectral parameter $u$).

**Theorem 1.** The matrices $F_{1,...,N}$ given by induction on $N$ by (4.45), (4.46), (4.43) and (4.44) provide a set of factorizing $F$-matrices in the sense of definition 3.

**Proof** — The matrices $F_{1,...,N}$, which induce the change of basis (3.31), (3.32), (3.30), (3.33), being invertible by definition, we merely have to show that, for any permutation $\sigma \in \mathfrak{S}_N$,

$$F_{\sigma(1)...\sigma(N)} R_{1,...,N}^\sigma = F_{1,...,N}.$$

For $X = A$, $B$, $C$, or $D$, one knows from (2.17) that

$$R_{1,...,N}^\sigma X_{1,...,N} = X_{\sigma(1)...\sigma(N)} R_{1,...,N}^\sigma.$$

The formulas for $\tilde{X}_{1,...,N} = F_{1,...,N} X_{1,...,N} F_{1,...,N}^{-1}$ are given by (3.30), (3.31), (3.32) and (3.33), and are completely symmetric under any permutation of the sites:

$$\tilde{X}_{\sigma(1)...\sigma(N)} = \tilde{X}_{1,...,N}, \quad \forall \sigma \in \mathfrak{S}_N.$$

Thus

$$R_{1,...,N}^\sigma (F_{1,...,N}^{-1} \tilde{X}_{1,...,N} F_{1,...,N}) = (F_{\sigma(1)...\sigma(N)}^{-1} \tilde{X}_{\sigma(1)...\sigma(N)} F_{\sigma(1)...\sigma(N)}) R_{1,...,N}^\sigma,$$

which implies that the quantity $F_{\sigma(1)...\sigma(N)} R_{1,...,N}^\sigma F_{1,...,N}^{-1}$ commutes with $\tilde{A}(u)$, $\tilde{B}(u)$, $\tilde{C}(u)$ and $\tilde{D}(u)$ for all the values of $u$. Therefore, this is equal to the identity matrix times a numerical factor, which is 1 for the appropriate normalization of $R$. □
Remark 6. The expression for $F_{21}^{-1}F_{12}$ which follows from (4.52) and (4.53) coincides with the corresponding finite dimensional representation for the Gauss decomposition of the universal $R$-matrix of the Yangian double $DY(sl_2)$ obtained in [12] by Khoroshkin and Tolstoy:

$$R_{12} = F_{21}^{-1}F_{12} = R_+ R_0 R_-,$$

(4.56)

with

$$R_+ = \sum_{k=0}^{\infty} (S_1^+)^k (S_2^-)^k \left[ k! \prod_{j=1}^{k} (\lambda + S_1^z - S_2^z + j) \right]^{-1},$$

$$R_- = \sum_{k=0}^{\infty} \left[ k! \prod_{j=1}^{k} (\lambda + S_1^z - S_2^z + j) \right]^{-1} (S_1^-)^k (S_2^+)^k,$$

(4.57)

$$R_0 = \prod_{k=0}^{\infty} \frac{(\lambda + S_1^z - S_2^z + k)(\lambda - l_1 - l_2 + k)(\lambda + S_1^z - S_2^z + k + 1)(\lambda + l_1 + l_2 + k + 1)}{(\lambda - l_1 - S_2^z + k)(\lambda + S_1^z - l_2 + k)(\lambda + l_1 - S_2^z + k + 1)(\lambda + S_1^z + l_2 + k + 1)},$$

where $\lambda = \frac{\delta - \eta}{\eta}$. So the computation of $F$ leads to nice factorized expressions, in any finite dimensional representation, of the $R$-matrices associated to any permutation $\sigma \in S_n$. In particular, this gives a hint concerning an universal formula for $F$: the non-diagonal term of $F$ in our formula corresponds exactly to the representation $R_-$ of the non-diagonal part of the Gauss decomposition of the universal $R$-matrix of $DY(sl_2)$, and hence admits a universal formula; the question is thus if there exists an appropriate factorization of the diagonal part of this Gauss decomposition at the universal level.

Remark 7. Theorem 1 can also be easily proved by means of proposition 2 using remark 6 and the explicit expression obtained for $\tilde{F}_{1,2...N}$ and $\tilde{F}_{1...N-1,N}$.

5 Conclusion

In this letter we have computed the factorizing $F$-matrices representing Drinfel’d twists for all finite dimensional evaluation representations of Yangian $Y(sl_2)$. The FBA technique enables us to do this by solving only linear equations. The next step would be to generalize this to non-finite dimensional representations and to obtain an eventual universal form for this twist. Let us note here that, in our formula, only the diagonal part depends on the dimension of the representation, whereas the non-diagonal terms simply correspond to the representations of the non-diagonal parts of the Gauss decomposition of the universal $R$-matrix given in [12], and hence admit universal formulas. The point would be here to find an appropriate factorization of the diagonal part of this Gauss decomposition at the universal level, in order to obtain, as in [13], the universal $R$-matrix as a product $(F_{21})^{-1}F_{12}$, with $F^+$, $F^-$ satisfying the cocycle relation and $F^+ = F^-$ only for finite dimensional representations (in general, $R$ is not unitary, but pseudo-unitary).

The results obtained in this letter open furthermore the possibility to compute form factors for the XXX Heisenberg chain of spins $l$, by using the new basis given by this $F$-matrix, in the spirit of what has been done for spin-$1/2$ in [8]. Let also us mention here that the method we used to compute the factorizing $F$-matrices is most probably applicable to Yangians or quantum affine algebras associated to higher rank Lie algebras.
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