The firing squad synchronization problem for squares with holes

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Abstract

The firing squad synchronization problem (FSSP, for short) is a problem in automata theory introduced in 1957 by John Myhill. Its goal is to design a finite automaton $A$ such that, if copies of $A$ are placed in a line and connected and are started at time 0 with their leftmost copy in a special triggering state, then at some time (the “firing time”) all copies enter a special “firing state” simultaneously for the first time. FSSP has many variations and for many of them we know minimal-time solutions (solutions having shortest firing time). One of such variations is the FSSP for squares (denoted by SQ) in which copies are placed in a square. In this paper we introduce a variation which we call the FSSP for squares with $k$ holes and denote by SH[$k$] by slightly modifying SQ ($k \geq 1$). In the variation, copies of a finite automaton are placed in a square but there are $k$ positions (“holes”) in the square where no copies are placed. We show that SH[$1$] has a minimal-time solution. Moreover, for each problem instance (a placement of copies in a square) $C$ of SH[$2$], we determine the minimum firing time of $C$ (the minimum value of firing times of $C$ by $A$ where $A$ ranges over all solutions of SH[$2$]). The variation SQ was introduced and its minimal-time solutions were found in 1970’s. However, to find minimal-time solutions of SH[$k$], a very simple modification of SQ, seems to be a very difficult and challenging problem for $k \geq 2$.

Keywords: firing squad synchronization problem, square, minimal-time solution, distributed computing

1 Introduction

The firing squad synchronization problem is a puzzle in automata theory. According to [17] it was devised in 1957 as a problem of how to turn on all parts of a self-reproducing machine simultaneously. The problem continues to attract interests of researchers even now. The problem is formulated as follows.

In the formulation we use a finite automaton $A$ that has two inputs, one from the left and another from the right and two outputs, one to the left and another to the right. The state of $A$ at a time $t+1$ is completely determined by the state $s$ of $A$, the value $x$ of its left input and the value $y$ of its right input at the time $t$ by a state transition function $\delta(s, x, y)$. The values of the two outputs of $A$ at a time $t$ are the state $s$ of $A$ at the time $t$. The set of states
of $A$ contains three special states $G$, $Q$, $F$ called the general state, the quiescent state and the firing state respectively.

We place copies of $A$ in a line and connect their inputs and outputs as shown in Fig. 1.1. When the number of copies is $n$, we call this line the configuration $C_n$ of size $n$ and denote it by $C_n$. We call each copy of $A$ in $C_n$ a node of $C_n$. The values of the left input of the leftmost node and the right input of the rightmost node are special value # meaning that there are no nodes there.

We call the leftmost node of $C_n$ the general of $C_n$. At time 0, the general of $C_n$ is in the general state $G$ and other nodes are in the quiescent state $Q$. Then the state of a node $v$ in $C_n$ at a time $t$ is completely determined by the state transition function $\delta$ of $A$. We denote this state by $st(v, t, C_n, A)$. The transition function $\delta$ must satisfy the following condition: if both of $x$, $y$ are either $Q$ or # then $\delta(Q, x, y)$ must be $Q$. Intuitively, a node in the quiescent state $Q$ remains in the state until at least one of its adjacent nodes is in a non-quiescent state.

The firing squad synchronization problem, or FSSP for short, is the problem to design $A$ so that for any number $n$ of nodes, all nodes in $C_n$ enter the firing state $F$ for the first time simultaneously at a time. Formally stated, for any $n$ there must exist a time $t_n$ such that the following statement is true:

$$(\forall t < t_n)(\forall v \in C_n)[st(v, t, C_n, A) \neq F] \land (\forall v \in C_n)[st(v, t_n, C_n, A) = F]. \quad (1.1)$$

We call a finite automaton $A$ that satisfies this condition a solution of FSSP. We call the time $t_n$ mentioned in (1.1) the firing time of the solution $A$ for the configuration $C_n$ and denote it by $ft(C_n, A)$. Intuitively, $C_n$ is a firing squad and each node of $C_n$ is a soldier. The special state $G$ represents the general (the leftmost soldier) giving the order “Fire” and the special state $F$ represents a firing soldier.

This is an interesting puzzle and we can easily find a solution using the well-known “Divide and Rule” strategy in the design of algorithms. We usually find a solution $A$ with the firing time $ft(C_n, A) = 3n + O(\log n)$.

One of the most interesting problems is to find fast solutions (that is, solutions with small values of $ft(C_n, A)$). We call a solution $\tilde{A}$ a minimal-time solution if it is fastest among all solutions, or more precisely, if the following statement is true:

$$(\forall A)(\forall C_n)[ft(C_n, \tilde{A}) \leq ft(C_n, A)]. \quad (1.2)$$
Here $A$ ranges over all solutions.

It is not obvious that such solutions exist. However, one such solution was found by E. Goto (7), reconstructed by H. Umeo (28) and later by A. Waksman (35) and R. Balzer (11) using ideas that are different from Goto’s. By these minimal-time solutions we know that the firing time $ft(C_n, \tilde{A})$ of a minimal-time solution $\tilde{A}$ is $2n - 2$ for $n \geq 2$.

Many variations of FSSP have been also studied. The following is a list of some of the variations: (1) lines (the general is the leftmost node) (the original FSSP, [1, 7, 35]), (2) lines (the general may be an arbitrary node, [18]), (3) two-way rings (2, 8, 9), (4) one-way rings (2, 8, 9, 20), (5) squares (the general is the left down corner node, 24), (6) cubes (the general is the left down bottom corner node, 25), (7) rectangles (the general is the left down corner node, 25), (8) rectangles (the general may be an arbitrary node, 26), (9) cuboids (that is, rectangular parallelepiped) (the general may be an arbitrary node, 26), (10) two-way tori constructed from squares (8), (11) one-way tori constructed from squares (14), (12) two-way tori constructed from rectangles (33), (13) undirected networks (20, 23, 24), (14) directed networks (4, 10, 21).

For all of these variations except (13), (14) we know minimal-time solutions (see also 6). Variations of FSSP for Cayley graphs have been also studied (22).

The main motivation for studying variations of FSSP is that their solutions can be used to synchronize networks composed of large numbers of identical computing devices. Another motivation is that it is a mathematical formulation of one case of the general problem: how to control the global behavior of a large network using only local information exchanges. This general problem is one of the most fundamental problems in the theory of distributed computing.

Concerning FSSP, mainly three research themes have been studied. First is to find fast solutions. Especially interesting is to determine whether a variation has minimal-time solutions or not, and to find one if they exist. If we cannot find them, it is desirable to formally prove that they do not exist. The second is to find small solutions (that is, solutions with small numbers of states). For example, for the original FSSP a six state minimal-time solution is known (15) and it is known that four state minimal-time solutions do not exist (see, for example, 33). The third is to find “good” solutions. There are many criteria for “goodness.” The followings are among what we mean by “good” solutions: easy to understand solutions, solutions that are easy to prove correctness, solutions using interesting or useful ideas, solutions having interesting or useful features, and solutions especially suited for some specific hardware implementation.

There are many surveys on FSSP and we refer the reader to them. Mazoyer [16] provides a survey of the problem up to 1986 and Napoli and Parente [19] give a survey of recent developments. Goldstein and Kobayashi [5, 6] give surveys concentrating on the problem of existence/nonexistence of minimal-time solutions. Umeo, Hisaoka and Sogabe [29] give a survey of the minimal-time solutions of the original FSSP.

Before proceeding to explain the main results of this paper we give one more definition. Let $\Gamma$ be a variation of FSSP that has at least one solution. For
this Π too we can define “minimal-time solutions” of Π as solutions ˜A of Π that satisfy the condition (1.2). However there is another way to define minimal-time solutions of Π. For each configuration (a problem instance) C of Π, we define the minimum firing time of C (denoted by mft(Π)) by the following formula:

\[ mft(Π) = \min \{ \text{ft}(C, A) \mid A \text{ a solution of } Π \} \]

This value is well-defined because we assume that Π has at least one solution. We say that a solution A of Π is a minimal-time solution of Π if it satisfies the following condition:

\[ (\forall C) [\text{ft}(C, ˜A) = mft(Π)] \] (1.3)

Now we have two definitions of minimal-time solutions, one using (1.2) and another using (1.3). However we can easily show that these two definitions are equivalent. The first definition has a clear intuitive meaning. The second is technical but is useful for the study of minimal-time solutions as we see in the following. Moreover, we can determine the value mft(Π) even when we do not know whether Π has minimal-time solutions or not. In such cases the problem to find minimal-time solutions is reduced to the problem to find solutions ˜A that satisfy (1.3).

We are ready to explain the main results of this paper. One of the most basic variations of FSSP is the FSSP for squares (the variation (5) in our previous list). We denote it by SQ. In the variation, for each w (≥ 0) we have a configuration Cw consisting of (w+1)^2 nodes that are placed as a square of w+1 rows and w+1 columns. Each node is a copy of a finite automaton A that has four inputs and four outputs corresponding to the four directions the east, the north, the west and the south. (From now on, we use these four directions instead of “right,” “up,” “left” and “down.”) The nodes are connected as shown in Fig. 1.2 (The value of w is 6 for this example.) The general is at the southwest corner of the square. A minimal-time solution was constructed by Shinahr (25). The

![Figure 1.2: A configuration C_w of SQ (w = 6).](image)

minimum firing time mft_{SQ}(C_w) is 2w for w ≥ 1.
In this paper we consider a variation of FSSP which we call the FSSP for squares with \( k \) holes. Here, \( k \geq 0 \) is an integer and is a parameter of the variation. We denote this variation by \( \text{SH}[k] \). A configuration \( C \) of size \( w \) of \( \text{SH}[k] \) is obtained by removing \( k \) nodes from the configuration \( C_w \) of SQ so that

- nodes on the boundary of the square are not removed, and
- the set of the remaining \((w + 1)^2 - k\) nodes must be connected.

We call the position of a removed node of a configuration a hole of the configuration. Each configuration of \( \text{SH}[k] \) has \( k \) holes. In Fig. 1.3 (a) we show an example of configurations of \( \text{SH}[k] \). The size \( w \) of this configuration is 6

![Figure 1.3](image)

and the number of holes \( k \) of the configuration is 7. (Note that by a hole we mean a position that is not occupied by a node, not a set (a region) of adjacent nonoccupied positions such as \{\((1, 5), (2, 5), (3, 4), (3, 5)\) or \{\((4, 1), (5, 1)\)\}.) To save space we represent the configuration shown in Fig. 1.3 (a) by the figure shown in Fig. 1.3 (b).

We explain our motivations to study this variation \( \text{SH}[k] \) later and here we show the results concerning \( \text{SH}[k] \) obtained in this paper. They are as follows:

1. \( \text{SH}[1] \) has a minimal-time solution. The minimum firing time of a configuration \( C \in \text{SH}[1] \) of size \( w \) is \( \text{mft}_{\text{SH}[1]}(C) = 2w \). (Theorem 3.1)

2. For each \( k \geq 0 \) there is a constant \( c_k \) such that
   - \( 2w \leq \text{mft}_{\text{SH}[k]}(C) \leq 2w + c_k \) for any \( C \in \text{SH}[k] \) of size \( w \), and
   - for all sufficiently large \( w \), there are \( C, C' \in \text{SH}[k] \) of size \( w \) such that \( \text{mft}_{\text{SH}[k]}(C) = 2w, \text{mft}_{\text{SH}[k]}(C') = 2w + c_k \).
(Theorem 4.1, Corollary 4.14)

(3) \( c_0 = c_1 = 0, c_2 = 1 \) and \( k - 2 \leq c_k \leq k^2 + 4k \) for all \( k \geq 3 \). (Theorem 4.15)

(4) The definition of \( c_k \) itself gives an algorithm to compute the values \( c_k \). Using this algorithm we determined the values of \( c_k \) for \( 2 \leq k \leq 9 \) with computers. The results are \( c_2 = 1 \) and \( c_k = k - 2 \) for \( 3 \leq k \leq 9 \). (In Table 1.1 we summarize known values of \( c_k \).) (Subsection 4.5)

(5) We give a characterization of the minimum firing time \( \text{mft}_{\text{SH}[2]}(C) \) of \( \text{SH}[2] \) (Theorem 5.5). Although the characterization itself is simple, its derivation is very lengthy and tedious. (All of Section 5 and Appendix A are devoted to the derivation.)

| \( k \) | \( c_k \) |
|-------|-------|
| 0     | 0     |
| 1     | 0     |
| 2     | 1     |
| 3     | 1     |
| 4     | 2     |
| 5     | 3     |
| 6     | 4     |
| 7     | 5     |
| 8     | 6     |
| 9     | 7     |

Table 1.1: Known values of \( c_k \).

The following is a summary of the results we have at present on \( \text{SH}[k] \).

- For \( \text{SH}[1] \) we know a minimal-time solution but for \( k \geq 2 \) we do not know whether \( \text{SH}[k] \) has minimal-time solutions or not.
- For \( \text{SH}[2] \) we know the minimum firing time \( \text{mft}_{\text{SH}[2]}(C) \) but for \( k \geq 3 \) we do not know it.

As we mentioned above, the characterization of the minimum firing time \( \text{mft}_{\text{SH}[2]}(C) \) of a configuration \( C \) of \( \text{SH}[2] \) is simple. We show it using the case where the size \( w \) of \( C \) is 12 as an example. (The characterization is slightly different for even \( w \) and odd \( w \).) Suppose that \( C \) is a configuration of size 12.

There are \( 13^2 = 169 \) positions in the square of \( C \) and two of them are holes.

We classify these 169 positions into the following four disjoint sets \( U, V, W, X \) (see Fig. 1.4).

- \( U = \{(x, y) \mid 0 \leq x \leq 5, 0 \leq y \leq 5\} \),
- \( V = \{(x, y) \mid 0 \leq x \leq 5, y = 6 \text{ or } x = 6, 0 \leq y \leq 5 \text{ or } x = 6, y = 6\} \),
- \( W = \{(x, y) \mid 0 \leq x \leq 6, y = 7 \text{ or } x = 7, 0 \leq y \leq 6\} \),
- \( X = \{(x, y) \mid 0 \leq x \leq 12, 0 \leq y \leq 12\} - U \cup V \cup W \).
Then, for the following three cases we have $m_{fS_{H[2]}}(C) = 25 (= 2w + 1)$ and for other cases we have $m_{fS_{H[2]}}(C) = 24 (= 2w)$.

- $C$ has no holes in $U \cup V \cup W$.
- $C$ has no holes in $U \cup V$, has one hole in $W$, and the position $v = (x, y)$ of the hole in $W$ satisfies $|x - y| = 2$.
- $C$ has two holes in $U \cup V$ and their positions $v = (x, y), v'$ satisfy $|x - y| = 2$ and $v' = v + (1, 1)$.

In Fig. 1.4 we show four sets $U, V, W, X$.

Now we explain our main motivation for studying $SH[k]$. In these two decades, some pairs of variations of FSSP ($\Gamma, \Gamma'$) having the following properties have been discovered.

- $\Gamma$ is one of the basic variations that were extensively studied in the early days of the research of FSSP (that is, in 1960s and 1970s) and their minimal-time solutions were obtained at that time.
- $\Gamma'$ is a natural and simple modification of $\Gamma$.
- At present we do not know whether $\Gamma'$ has minimal-time solutions or not and moreover the problem to know it seems to be very difficult.

The first example is the pairs (ORG, 2PATH) and (ORG, 3PATH). Here ORG denotes the original FSSP (the variation (1) in our previous list), 2PATH denotes the FSSP of paths in the two-dimensional grid space and 3PATH denotes the same problem for the three-dimensional grid space. In both of 2PATH and 3PATH the general of a path is one of the two terminal nodes of the path. In Fig. 1.5(a) and (b) we show examples of configurations of 2PATH and 3PATH respectively.
Figure 1.5: (a) A configuration of 2PATH. (b) A configuration of 3PATH.

In all of ORG, 2PATH, 3PATH, configurations are lines of nodes. The difference is that in ORG they must be straight but in 2PATH, 3PATH they may be bent in the grid spaces. Therefore, 2PATH, 3PATH are natural and simple modifications of ORG. Minimal-time solutions of ORG were obtained by [1, 7, 35].

Both of 2PATH and 3PATH were studied in [11] and [5] respectively. For them we know the minimum firing times \( mft_{2\text{PATH}}(C) \), \( mft_{3\text{PATH}}(C) \) but at present we do not know whether they have minimal-time solutions or not. Nevertheless, we have the following circumstantial evidences that they have no minimal-time solutions.

- If the problem 2PEP has no polynomial-time algorithms then 2PATH has no minimal-time solutions ([11]).
- If \( P \neq \text{NP} \) then 3PATH has no minimal-time solutions ([5]).

Here, 2PEP (the two-dimensional path extension problem) is a purely combinatorial problem on paths in the two-dimensional grid space such that (1) it is in NP, (2) at present we know only exponential-time algorithms for it, but (3) at present we cannot prove that it is NP-complete ([11]). The first of the above two results implies that to find a minimal-time solution of 2PATH is at least as difficult as finding a polynomial-time algorithm for 2PEP. The second result is a sufficiently convincing evidence that 3PATH has no minimal-time solutions.

The second example is the pair (SQ, gSQ). Here gSQ denotes the variation obtained from SQ by modifying so that the general of a configuration may be an arbitrary node in it. We call gSQ the “generalized FSSP for squares.” As we mentioned previously a minimal-time solution of SQ was obtained by Shinahr ([25]). In [32], Umeo and Kubo noted that we do not know whether gSQ has minimal-time solutions or not. By [12], at least we know the minimum firing time \( mft_{g\text{SQ}}(C) \) of gSQ.

The pairs (SQ, SH\([k]\)) for \( k \geq 2 \) are the third example of such pairs. As we mentioned above, we do not know whether SH\([k]\) has minimal-time solutions or not for \( k \geq 2 \). For SH\([2]\) we know the minimum firing time. However, its lengthy and tedious derivation given in Section 5 and Appendix A suggests that
the problem to determine the minimum firing time of \( SH[k] \) seems to be very difficult for \( k \geq 3 \).

These examples of \((\Gamma, \Gamma')\) give us the following impression concerning variations of FSSP.

- Most variations of FSSP are very difficult.
- In our previous list (1) – (14) of variations of FSSP, the variations (1) – (12) for which we know minimal-time solutions are exceptionally simple and are isolated in difficult variations.

If this impression is correct, to construct a general theory of FSSP that includes a much broader class of variations of FSSP is a very interesting open problem. In that case, 2PATH, 3PATH, gSQ, \( SH[k] \) are good variations to start with for the study of such general theory. This is our main motivation for introducing and studying \( SH[k] \).

Another motivation for studying \( SH[k] \) is that it is one of the formulations of the problem to synchronize networks of computing devices in situations where the networks may have faulty devices \([3, 13, 27, 31, 36]\).

This paper is organized as follows. In Section 2 we explain basic notions and notations. In Section 3 we prove that \( SH[1] \) has a minimal-time solution (Result (1)). In Section 4 we show many results on the function \( \text{mft}_{SH[k]}(C) \) for general values of \( k \) (Results (2), (3), (4)). In Section 5 we concentrate on \( SH[2] \) and give a characterization of the value \( \text{mft}_{SH[2]}(C) \) (Result (5)). Section 6 is for discussions and conclusion.

### 2 Preliminaries

Let \( v = (x, y) \) and \( v' = (x', y') \) be positions in the two-dimensional grid space \( \mathbb{Z}^2 \) (\( \mathbb{Z} \) denotes the set \( \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \) of integers). We say that \( v, v' \) are adjacent if \( x = x' \) and \( |y - y'| = 1 \) or \( |x - x'| = 1 \) and \( y = y' \), and \( v, v' \) touch with corners if \( |x - x'| = 1 \) and \( |y - y'| = 1 \). By the Manhattan distance (or the MH distance for short) between \( v \) and \( v' \), we mean the value \( |x - x'| + |y - y'| \) and denote it by \( d_{\text{MH}}(v, v') \).

By a path we mean a sequence of positions \( P = v_0, \ldots, v_n \) in \( \mathbb{Z}^2 \) such that \( v_i, v_{i+1} \) are adjacent for each \( i \) (\( 0 \leq i \leq n - 1 \)). We call \( P \) a path from \( v_0 \) to \( v_n \) or a path between \( v_0 \) and \( v_n \). We call the value \( n \) the length of the path \( P \) and denote it by \( |P| \). When \( P, P' \) are paths such that the end position \( \tilde{v} \) of \( P \) and the start position \( \tilde{v} \) of \( P' \) are the same, by \( P + P' \) we mean the path obtained from \( P \) and \( P' \) by concatenating them (but deleting one of the overlapping \( \tilde{v}, \tilde{v} \)).

For each \( w \) (\( \geq 1 \)), by \( S_w \) we denote the square

\[
S_w = \{(x, y) \mid 0 \leq x \leq w, 0 \leq y \leq w\}
\]

in \( \mathbb{Z}^2 \). By the main diagonal of \( S_w \) we mean the set of positions \( \{(u, u) \mid 0 \leq u \leq w\} \). By the boundary of \( S_w \) we mean the set of positions \( \{(x, y) \in S_w \mid \text{either } x = 0, x = w, y = 0, \text{ or } y = w\} \).
We give a formal definition of SH[$k$]. A configuration $C$ is obtained as follows. First we select an integer $w$ such that $(w-1)^2 \geq k$. Next we select $(w+1)^2 - k$ positions from the $(w+1)^2$ positions in $S_w$ so that two conditions are satisfied. The first condition is that all positions in the boundary of $S_w$ are selected. We assume that $(w-1)^2 \geq k$ and hence there is at least one way to select positions so that this condition is satisfied. The second condition is that, for any two selected positions $v$, $v'$ there is a path of selected positions between $v$ and $v'$. Finally copies of a finite automaton $A$ are placed on the selected positions. The placement of these copies of $A$ is a configuration $C$ of SH[$k$] of size $w$. The general of $C$ is the copy placed at the position $(0,0)$. By $v_{\text{gen}}$ we denote the general of a configuration.

We call each copy a node of $C$. When there is a node at a position $v$ we may say “a node $v$" instead of “the node at a position $v$.” For a position $v$, by the expression “$v \in C$" we mean that $v$ is a node of $C$ (not that $v$ is a position in $S_w$). We call a position in $S_w$ that is not selected for $C$ a hole of $C$. There are $k$ holes of $C$.

We say that a path is a path in $C$ when all positions in it are nodes of $C$. For nodes $v$, $v'$ of $C$, by the distance between $v$ and $v'$ we mean the minimum value of $|P|$ when $P$ ranges over all paths in $C$ between $v$ and $v'$, and denote it by $d_C(v, v')$ or $d(v, v')$ when $C$ is understood. By $d_{\text{MH}}(v, v''; v')$ and $d_C(v, v''; v')$ we mean $d_{\text{MH}}(v, v') + d_{\text{MH}}(v', v'')$ and $d_C(v, v') + d_C(v', v'')$ respectively.

When the length $|P|$ of a path $P$ from $v$ to $v'$ is $d_{\text{MH}}(v, v')$ we say that the path is of the MH distance length. When an event occurs at a node $v$ at time $d_{\text{MH}}(v_{\text{gen}}, v)$ we say that the event occurs at the MH distance length time.

We call the direction in $\mathbb{Z}^2$ in which the $x$-variable value increases the east, and define the directions the north, the west and the south similarly.

By the boundary condition of a node $v = (x, y)$ in a configuration $C$ we mean the vector $(b_0, b_1, b_2, b_3)$. Here, $b_0$ is 1 if the position $v' = (x + 1, y)$ east of $v$ is a node of $C$. Otherwise (that is, either $v'$ is a hole of $C$ or $v'$ is out of the square $S_w$), $b_0$ is 0. We define $b_1$, $b_2$, $b_3$ similarly for the directions the north, the west, the south. By $b_C(v)$ we denote the boundary condition of $v$ in $C$.

We defined “solutions” of the original FSSP in Section 1 and this definition can be modified for any variation $\Gamma$ of FSSP by replacing configurations $C_n$ of the original FSSP with configurations $C$ of $\Gamma$ and replacing the times $t_n$ that may depend on $n$ with times $t_C$ that may depend on $C$.

For each variation $\Gamma$, we define a partial solution of $\Gamma$ as a finite automaton $A$ such that for any configuration $C$ of $\Gamma$, either (1) each node of $C$ never fires (that is, the statement $\forall t (\forall v \in C) [\text{st}(v, t, C, A) \neq F]$ is true) or (2) there is a time $t_C$ such that all nodes in $C$ fire for the first time simultaneously at the time (that is, the statement (1.1) is true with $C_n$ and $t_n$ replaced with $C$ and $t_C$ respectively).

When $A$ is a partial solution of $\Gamma$, by the domain of $\Gamma$ we mean the set of configurations $C$ of $\Gamma$ for which the case (2) of the definition holds true.

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1The term “a partial solution of a variation of FSSP” is also used for a different meaning (\textit{[34]}).
Moreover, for each $C$ in the domain of $A$, by $ft(C, A)$ we denote the time $t_C$ mentioned in (2).

Suppose that $\Gamma$ is a variation that has a solution. Then we have

$$\text{mft}_\Gamma(C) \leq ft(C, A)$$

for any partial solution $A$ and any configuration $C$ in the domain of $A$. The proof is as follows.

Let $A_0$ be a solution of $\Gamma$ and $A_1$ be any partial solution of $\Gamma$. Let $A_2$ be the finite automaton that simulates both of $A_0$, $A_1$ and fires as soon as at least one of $A_0$, $A_1$ fires. Then $A_2$ is a solution of $\Gamma$ and

$$\text{mft}_\Gamma(C) \leq ft(C, A_2)$$

$$= \begin{cases} \min\{ft(C, A_0), ft(C, A_1)\} & \text{if } C \text{ is in the domain of } A_1, \\ ft(C, A_0) & \text{otherwise} \end{cases}$$

for any configuration $C$ of $\Gamma$. Therefore, if $C$ is in the domain of $A_1$ we have $\text{mft}_\Gamma(C) \leq ft(C, A_1)$.

### 3 The variation SH[1]

In this section we show that SH[1] has a minimal-time solution.

**Theorem 3.1** (1) For a configuration $C$ of size $w$ of SH[1], $\text{mft}_{\text{SH}[1]}(C) = 2w$.

(2) SH[1] has a minimal time solution.

**Proof.** We construct a solution of SH[1] that fires a configuration $C$ of size $w$ at time $2w$. This shows both of (1), (2) of the theorem because we have $d_C(v_{\text{gen}}, (w, w)) = 2w$ and hence the firing time of $C$ of any solution cannot be smaller than $2w$. We modify the idea by Shinahr [25] used to construct a minimal-time solution of SQ.

We construct four finite automata $A_0$, $A_1$, $A_2$, $A_3$. The last one $A_3$ is the desired solution. First we explain the finite automaton $A_0$.

To locate nodes $(i, i)$ in the main diagonal $A_0$ uses nine signals $A$, $B$, $\ldots$, $I$. These signals are generated by the following rules.

- A signal $A$ is generated at $v_{\text{gen}}$ at time $0$.
- A signal $A$ generates other signals as shown in Fig. 3.1 (if the generation is not blocked by holes).
- When a signal $E$ or $I$ is generated at a node, a signal $A$ is simultaneously generated at the node.

For example, the arrow from $B$ to $E$ in Fig. 3.1 means that if a signal $B$ is generated at a node $(x, y)$ at time $t$ and the position $(x + 1, y)$ is a node then a signal $E$ is generated at the node $(x + 1, y)$ at time $t + 1$. In Fig. 3.2 we show three examples of the generation of signals.
For any node \((i, i)\) \(0 \leq i \leq w\) in the main diagonal the signal A is generated at the node at time \(2i\). We can prove this by the induction on \(i\) using our assumption that there is exactly one hole.

When a signal A is generated at a node \((i, i)\) at time \(2i\) \(0 \leq i \leq w - 1\), it activates the original FSSP for the horizontal sequence of positions \((i + 1, i), (i + 2, i), \ldots\) at time \(2i + 1\) assuming that the general is at the left end position \((i + 1, i)\) of the sequence. We use a minimal-time solution of the original FSSP \((1, 35)\) that fires a sequence of \(n\) nodes at time \(2n - 2\). There are three cases.

Case 1. The position \((i + 1, i)\) is a hole. In this case there is no node at the position supposed to be the general. Hence no nodes \((i + 2, i), (i + 3, i), \ldots, (w, i)\) fire.

Case 2. A position \((i + s, i)\) for some \(2 \leq s \leq w - i - 1\) is a hole. In this case the activated original FSSP fires the \(s - 1\) nodes \((i + 1, i), (i + 2, i), \ldots, (i + s - 1, i)\) at time \((2i + 1) + \{2(s - 1) - 2\} = 2i + 2s - 3 \leq 2w - 5\). The remaining \(w - i - s\) nodes \((i + s + 1, i), \ldots, (w, i)\) do not fire because the hole \((i + s, i)\) blocks the signals from the node \((i + s - 1, i)\).

Case 3. There is no hole in the \(w - i\) positions \((i + 1, i), (i + 2, i), \ldots, (w, i)\). In this case the activated original FSSP fires all of these nodes at time \((2i + 1) + \{2(w - i) - 2\} = 2w - 1\).

For the vertical sequence of positions \((i, i + 1), (i, i + 2), \ldots\) too, the signal A activates the original FSSP at time \(2i + 1\) similarly. This completes the definition of \(A_0\).

In Fig. 3.3 we show four examples of configurations of size \(w = 7\). At each
We modify $A_0$ to another finite automaton $A_1$. By the modification, each node that fires with $A_0$ before or at time $2w - 5$ by Case 2 never fires with $A_1$. For this modification $A_1$ uses six signals J, K, L, M, N, O. The three signals J, K, L are generated and travel as follows.

- A signal J is generated at $v_{\text{gen}}$ at time 0 and proceeds to the east to the node $(w, 0)$.
- When the signal J arrives at the node $(w, 0)$ it changes to a signal K. The signal K proceeds to the north to the node $(w, w - 1)$.
- At each node $(w, j)$ ($0 \leq j \leq w - 1$), the signal K generates a signal L. The signal L proceeds to the west to the node $(j + 1, j)$ unless it is blocked by a hole.

The generation and the travel of the three signals M, N, O are similar, replacing the directions the east, the north and the west with the north, the east and the south respectively. In Fig. 3.4 we show all the generated signals in an example configuration.

We define the firing rule of $A_1$ as follows: a node fires at a time $t$ with $A_1$ if and only if the node fires at the time $t$ with $A_0$ and it has received the signal L or O before or at the time $t$.
It is obvious that any node \((i, i)\) on the main diagonal never fires. Consider a node \((i, j)\) such that \(i \geq j + 1\). If the node fires at time \(2w - 1\) by Case 3 with \(A_0\) then the signal \(L\) arrives at the node at time \(w + j + (w - i) = 2w - (i - j) \leq 2w - 1\) and the node fires at time \(2w - 1\) with \(A_1\). If the node fires before or at time \(2w - 5\) by Case 2 with \(A_0\) then the signal \(L\) does not arrive at the node blocked by a hole, and hence the node never fires with \(A_1\). The same is true also for a node \((i, j)\) such that \(i \leq j - 1\). Hence, for any node \(v = (i, j)\) the following statement is true with \(A_1\): either (1) \(v\) fires at time \(2w - 1\) or (2) \(v\) never fires. In Fig. 3.3 we show the firing times of nodes in the four configurations shown in Fig. 3.3 with \(A_1\).

We will prove that the following stronger statement is true with \(A_1\) except the special case where \(v = (w, w)\) and the position \((w - 1, w - 1)\) is a hole: either (1) \(v\) fires at time \(2w - 1\) or (2) \(v\) never fires but \(v\) is adjacent to a node \(v'\) that fires at time \(2w - 1\). We assume that \(i \geq j\). There are three cases.

(Case 1) \(j \leq w - 1\) and there are no holes at positions \((j, j), (j + 1, j), \ldots, (w, j)\). In this case all nodes \((j + 1, j), \ldots, (w, j)\) fire at time \(2w - 1\). Although the node \((j, j)\) never fires it is adjacent to the node \((j + 1, j)\) that fires at time \(2w - 1\). (Note that we assume \(j \leq w - 1\) and hence there is really a node at \((j + 1, j)\).) Therefore, if \(i \geq j + 1\) then \(v\) fires at time \(2w - 1\) and if \(i = j\) then \(v = (j, j)\) is adjacent to a node \(v' = (j + 1, j)\) that fires at time \(2w - 1\).

(Case 2) \(j \leq w - 1\) and there is a hole at one of the positions \((j, j), (j + 1, j), \ldots, (w, j)\). In this case any of the nodes in these positions (including the node \(v = (i, j)\)) never fires. We have \(j \geq 1\) because the boundary of \(C\) has no holes in it. There are no holes in positions \((j - 1, j - 1), (j, j - 1), \ldots, (w, j - 1)\) because there is only one hole. All nodes \((j, j - 1), (j + 1, j - 1), \ldots, (w, j - 1)\) (including the node \(v - (0, 1) = (i, j - 1)\) fire at time \(2w - 1\). Therefore, \(v = (i, j)\) never fires and it is adjacent to a node \(v' = (i, j - 1)\) that fires at time \(2w - 1\).

(Case 3) \(j = w\). In this case we have \(v = (w, w)\) and the position \((w - 1, w - 1)\) is a node because we exclude the case where \(v = (w, w)\) and the position \((w - 1, w - 1)\) is a hole. By the definition of \(A_1\) the node \(v\) never fires. At least one of the two positions \((w - 1, w), (w, w - 1)\) is a node and it fires at time \(2w - 1\). Therefore, \(v\) never fires and \(v\) is adjacent to a node that fires at time \(2w - 1\).
In Fig. 3.5 (b) we show an example of the exceptional case. In this case the node \( v = (w, w) \) never fires and the two nodes adjacent to \( v \) (that is, nodes \((w - 1, w)\) and \((w, w - 1)\)) also do not fire at \( 2w - 1 \).

We define the third finite automaton \( A_2 \). \( A_2 \) simulates the behavior of \( A_1 \). A node \( v \) in \( C \) fires at a time \( t \) with \( A_2 \) if and only if either \( v \) fires at the time \( t - 1 \) with \( A_1 \) or \( v \) is adjacent to a node \( v' \) that fires at the time \( t - 1 \) with \( A_1 \). Then, by what we have proved above, any node \( v \) of \( C \) fires at time \( 2w \) except the case where \( v = (w, w) \) and the position \((w - 1, w - 1)\) is a hole.

We modify \( A_2 \) by adding the following ad hoc rule to let the node \((w, w)\) fire at \( 2w \): if the signal \( A \) (see Fig. 3.1) arrives at a node \( v \) having the boundary condition \((0, 0, 1, 1)\) (that is, the boundary condition of the node \((w, w)\)) at a time \( t \) the node \( v \) fires at the time \( t \). Let \( A_3 \) be the finite automaton obtained by this modification. Then all nodes in \( C \) fire at time \( 2w \) with \( A_3 \) and \( A_3 \) is the desired solution.

### 4 Some results on minimum firing times

\( mft_{SH[k]}(C) \) of \( SH[k] \)

For \( k \geq 2 \), we do not know whether \( SH[k] \) has minimal-time solutions or not. However the author has the conjecture that \( SH[k] \) has minimal-time solutions for all values of \( k \). A first step to prove this is to know the exact value of \( mft_{SH[k]}(C) \). In this section we consider the minimum value and the maximum value of \( mft_{SH[k]}(C) \) when \( k, w \) are fixed and \( C \) ranges over all configurations of \( SH[k] \) of size \( w \). Our results were summarized in Section 1. Here we give a more detailed outline of this section.

In Subsection 4.1 we show that \( 2w \) is the smallest value of \( mft_{SH[k]}(C) \) for all \( k \) and all sufficiently large \( w \). In Subsection 4.2 we define a value \( H_{k,w} \) and show that \( H_{k,w} \) is the maximum value of \( mft_{SH[k]}(C) \) for all \( k \) and all \( w \). In Subsection 4.3 we introduce a notion “maximal barriers of configurations” and show one property of this notion. In Subsection 4.4 we define a value \( c_k \) using this notion and show that \( H_{k,w} = 2w + c_k \) for all \( k \) and all sufficiently large \( w \). Therefore \( 2w + c_k \) is the maximum value of \( mft_{SH[k]}(C) \) for all \( k \) and all sufficiently large \( w \). The definition of \( c_k \) itself gives an algorithm for computing \( c_k \). Using this algorithm we determined the value of \( c_k \) for \( 2 \leq k \leq 9 \). In Subsection 4.5 we show the result.

#### 4.1 The minimum value of \( mft_{SH[k]}(C) \)

By \( C_{k,w} \) we denote the set of all configurations \( C \) of size \( w \) of \( SH[k] \). First we show a result on the minimum value of \( mft_{SH[k]}(C) \) for \( C \in C_{k,w} \).

**Theorem 4.1** (1) For any \( k \) and any \( w \), \( 2w \leq mft_{SH[k]}(C) \) for any \( C \in C_{k,w} \).

(2) For any \( k \) and any \( w \geq 2k + 1 \), there exists \( C \in C_{k,w} \) such that \( mft_{SH[k]}(C) = 2w \).
Proof. (1) The lower bound is obvious because \( 2w = d_C(v_{\text{gen}}, (w, w)) \leq \text{mft}(C) \).

(2) As an example we consider the case \( k = 3 \) and show a configuration \( \tilde{C} \) of size \( \tilde{w} = 7 \) of \( \text{SH}[3] \) such that \( \text{mft}_{\text{SH}[3]}(\tilde{C}) \leq 14 \) (and hence \( \text{mft}_{\text{SH}[3]}(\tilde{C}) = 14 \) by (1) of this theorem). This \( \tilde{C} \) is shown in Fig. 4.1. We will construct a partial solution \( A \) of \( \text{SH}[3] \) that fires \( \tilde{C} \) at time 14. This shows \( \text{mft}(\tilde{C}) \leq 14 \) by (2.2).

Figure 4.1: A configuration \( \tilde{C} \) in \( C_{3,7} \) that has the minimum firing time 14.

Suppose that copies of \( A \) are placed in a configuration \( C \) of \( \text{SH}[3] \) of size \( w \).

\( A \) uses two signals to check the condition “\( w = 7 \).” The first signal starts at \( v_{\text{gen}} \) at time 0 and proceeds to the node \((0,7)\) by going 7 steps to the north. If the signal falls off the north boundary of \( C \) before arriving at the node \((0,7)\) (and hence \( w < 7 \)) the signal vanishes. If the signal arrives at the node \((0,7)\) but its boundary condition is not \((1,0,0,1)\) (the boundary condition of the northwest corner) (and hence \( w > 7 \)) the signal vanishes. If the signal arrives at the node \((0,7)\) and its boundary condition is \((1,0,0,1)\) then it knows that the condition “\( w = 7 \)” is true and it generates a message \( W_0 \) at the node \((0,7)\) at time 7.

Similarly the second signal starts at \( v_{\text{gen}} \) at time 0 and proceeds to the node \((7,0)\) by going 7 steps to the east. If \( w < 7 \) or \( w > 7 \) then the signal vanishes. If \( w = 7 \) then the signal generates a message \( W_1 \) at the node \((7,0)\) at time 7.

\( A \) also uses a signal that checks the condition “there are holes at the positions \((1,1), (2,1), (3,1)\).” It starts at \( v_{\text{gen}} \) at time 0, proceeds to the east, and checks the above condition by checking the boundary conditions of the three nodes \((1,0), (2,0), (3,0)\). If the condition is satisfied the signal knows it at the node \((3,0)\) at time 3, and the signal generates a message \( M \) at the node \((3,0)\) at time 3. If the condition is not satisfied the signal vanishes.

Messages \( W_0, W_1, M \) propagate to all nodes in \( C \) as soon as they are generated.

\( A \) uses the following rule to fire: a node fires at a time \( t \) if and only if \( t = 14 \) and the node has received at least one of the two messages \( W_0, W_1 \) and also the message \( M \) before or at the time \( t \). (We may assume that each message keeps the current time up to 14 and hence each node knows the current time when it receives a message before or at time 14.) We show that \( A \) is a partial solution that has the set \( \{C\} \) as its domain and that fires configurations in the domain at time 14.

Suppose that \( C = \tilde{C} \). Then all of the three messages are generated, \( W_0 \) at \((0,7)\) at time 7, \( W_1 \) at \((7,0)\) at time 7, and \( M \) at \((3,0)\) at time 3. (In Fig. 4.1 the three “*” denote the nodes where these messages are generated.) Therefore, any node \( v = (x,y) \) in \( C \) receives at least one of \( W_0, W_1 \) and also \( M \) before or at time 14.
14 and hence fires at 14. This follows from the following observation. As for \(W_0, W_1\), we have \(\min\{7 + d_C((0, 7), v), 7 + d_C((7, 0), v)\} \leq 14\) (the equality is true for \(v = (0, 0), (2, 2), (3, 3), \ldots, (7, 7)\)). As for \(M\), we have \(3 + d_C((3, 0), v) \leq 14\) (the equality is true for \(v = (1, 7), (7, 7)\)).

Conversely suppose that a node \(v\) in \(C\) fires at some time. Then at least one of \(W_0\) and \(W_1\) was generated. Hence \(w = 7\) is true. Moreover \(M\) was generated. Hence there are holes at \((1, 1), (2, 1), (3, 1)\). Therefore \(C = \bar{C}\).

Thus we have proved that \(A\) is a partial solution that has the domain \(\{\bar{C}\}\) and that fires configurations in the domain at time 14. \(\Box\)

In the proof of Theorem 4.1 we used two messages \(W_0, W_1\) that imply \(w = \bar{w}\). We call these two messages the size check messages and from now on we use them repeatedly. We summarize the situations where these messages are used as follows.

\(\bar{w}\) is some fixed value and \(C\) is an arbitrary configuration of \(SH[k]\) of an arbitrary size \(w\). Nodes of \(C\) are copies of a finite automaton \(A\). The two size check messages \(W_0, W_1\) are used for nodes in \(C\) to know whether \(w = \bar{w}\) or not.

\(W_0, W_1\) are generated and propagate to all nodes in \(C\) as explained in the proof of Theorem 4.1. If \(w = \bar{w}\) then \(W_0\) and \(W_1\) are generated at time \(w\) at \((0, w)\) and \((w, 0)\) respectively and propagate to all nodes in \(C\). A node \(v\) in \(C\) receives at least one of \(W_0\), \(W_1\) (and hence knows that \(w = \bar{w}\)) at time \(w + \min\{d_C((0, w), v), d_C((w, 0), v)\}\). If \(w \neq \bar{w}\) then \(W_0, W_1\) are not generated and nodes in \(C\) never receive them.

### 4.2 A characterization of the maximum value of \(mft_{SH[k]}(w)\)

In this subsection we concentrate our attention on the maximum value of \(mft(C)\) for \(C \in C_{k,w}\). First we show a result (Corollary 4.4) that is used repeatedly to show lower bounds of \(mft(C)\).

Suppose that \(C, C'\) are configurations of \(SH[k]\), \(t \geq 0\) is a number, and \(v\) is a node in both of \(C, C'\) (or more precisely, \(v\) is a position in \(Z^2\) that is a node in both of \(C, C'\)). By \(C \equiv t,v C'\) we mean that the following two statements are true.

- If \(P\) is a path in \(C\) from \(v_{gen}\) to \(v\) of length at most \(t\) then \(P\) is also a path in \(C'\) and \(bc_C(u) = bc_{C'}(u)\) for any node \(u\) in \(P\).
- The same statement with \(C, C'\) exchanged.

**Theorem 4.2** If \(C \equiv t,v C'\) then \(mft_{SH[k]}(C) \geq t+1\) if and only if \(mft_{SH[k]}(C') \geq t+1\).

**Proof.** Suppose that \(C \equiv t,v C'\). Let \(A\) be any solution of \(SH[k]\). First we prove that \(st(v, s, C, A) = st(v, s, C', A)\) for any \(s\) such that \(s \leq t\). To prove it we assume \(st(v, s, C, A) \neq st(v, s, C', A)\) for a value \(s \leq t\) and derive a contradiction.

If \(st(u, r, C, A) \neq st(u, r, C', A)\) for a node \(u\) in both of \(C, C'\) and a time \(r\), then one of the following is true.
(1) \( \text{bcc}_C(u) \neq \text{bcc}_{C'}(u) \).

(2) \( \text{bcc}_C(u) = \text{bcc}_{C'}(u) \) and \( r = 0 \).

(3) \( \text{bcc}_C(u) = \text{bcc}_{C'}(u), \ 0 < r \) and \( \text{st}(u, r - 1, C, A) \neq \text{st}(u, r - 1, C', A) \).

(4) \( \text{bcc}_C(u) = \text{bcc}_{C'}(u), \ 0 < r \) and \( \text{st}(w, r - 1, C, A) \neq \text{st}(w, r - 1, C', A) \) for a node \( w \) that is in both of \( C, C' \) and is adjacent to \( u \).

Repeatedly using this property starting with our assumption \( \text{st}(v, s, C, A) \neq \text{st}(v, s, C', A) \) we know that there is a sequence \( u_0, u_1, \ldots, u_s \) of nodes in both of \( C, C' \) such that

- \( u_s = v \),
- \( \text{st}(u_r, r, C, A) \neq \text{st}(u_r, r, C', A) \) for all \( r \) \( \leq s \),
- \( \text{bcc}_C(u_r) = \text{bcc}_{C'}(u_r) \) for all \( r \) \( < s \),
- either \( u_{r-1} = u_r \) or \( u_{r-1} \) and \( u_r \) are adjacent for all \( r \) \( < s \),

and moreover one of the following is true:

(5) \( \text{bcc}_C(u_{r_0}) \neq \text{bcc}_{C'}(u_{r_0}) \),

(6) \( \text{bcc}_C(u_{r_0}) = \text{bcc}_{C'}(u_{r_0}) \) and \( r_0 = 0 \).

However, from each of (5), (6) we can derive a contradiction.

Suppose the case (5). One of \( \text{st}(u_{r_0}, r_0, C, A) \), \( \text{st}(u_{r_0}, r_0, C', A) \) is not the quiescent state \( Q \). We consider the case where the former is not \( Q \). Then \( d_C(v_{gen}, u_{r_0}) \leq r_0 \), there is a path \( P \) in \( C \) from \( v_{gen} \) to \( v \) of length at most \( s \) \((\leq t)\), and there is a node \( u_{r_0} \) in it such that \( \text{bcc}_C(u_{r_0}) \neq \text{bcc}_{C'}(u_{r_0}) \). This contradicts our assumption that \( C \equiv^{t}_{v} C' \).

Suppose the case (6). In this case, \( \text{st}(u_{r_0}, r_0, C, A) = \text{st}(u_{r_0}, r_0, C', A) = G \) if \( u_{r_0} = v_{gen} \) and \( \text{st}(u_{r_0}, r_0, C, A) = \text{st}(u_{r_0}, r_0, C', A) = Q \) otherwise. Therefore, \( \text{st}(u_{r_0}, r_0, C, A) \neq \text{st}(u_{r_0}, r_0, C', A) \) cannot be true.

Thus we have proved that \( \text{st}(v, s, C, A) \neq \text{st}(v, s, C', A) \) for any \( s \) \((\leq t)\). Next we assume that \( \text{mft}(C) \geq t + 1 \), \( \text{mft}(C') \leq t \) and derive a contradiction. (The derivation of a contradiction for the case where \( \text{mft}(C) \leq t \), \( \text{mft}(C') \geq t + 1 \) is similar.) We select a solution that fires \( C' \) at time \( \text{mft}(C') \leq t \) as \( A \). Then we have \( F = \text{st}(v, \text{mft}(C'), C', A) = \text{st}(v, \text{mft}(C'), C, A) \). This means that \( A \) is a solution that fires \( C \) at the time \( \text{mft}(C') \leq t \). This contradicts our assumption that \( \text{mft}(C) \geq t + 1 \). \( \square \)

For \( C \in \mathcal{C}_{k,w}, v \in C \), let \( T(v, C) \) be defined by

\[ T(v, C) = \min \{ d_C(v_{gen}, v; (0, w)), d_C(v_{gen}, v; (w, 0)) \}. \quad (4.1) \]

**Theorem 4.3** For any configuration \( C \) of \( \text{SH}[k] \), \( \max_{v \in C} T(v, C) \leq \text{mft}_{\text{SH}[k]}(C) \).
Proof. It is sufficient to prove $T(v, C) \leq \text{mft}(C)$ for any $v \in C$. Suppose that $v$ is a fixed node of $C$.

Let $w$ be the size of $C$. We define another configuration $C'$ of size $w'$ of $\text{SH}[k]$ (see Fig. 4.2) such that

- $w < w'$ and $T(v, C) \leq 2w'$.
- The distribution of holes in the square $S_w$ is the same in $C$ and in $C'$ (and hence $C'$ has no holes in $S_{w'} - S_w$). (For the definition of $S_w$ and so on, see (2.1) in Section 2.)

![Figure 4.2: Two configurations $C, C'$ used in the proof of Theorem 4.3.](image)

We assume that $P$ is a path in $C$ from $v_{\text{gen}}$ to $v$ of length at most $T(v, C) - 1$ and $P$ passes through the set $S_w - S_{w-1}$ (the north and the east boundaries of $C$) and derive a contradiction.

Let $v'$ be a node in $P$ that is in $S_w - S_{w-1}$. We consider the case where $v'$ is in the horizontal part of $S_w - S_{w-1}$ (the north boundary of $C$). Then we have the following contradiction:

$$T(v, C) - 1 \geq |P| \geq d_C(v_{\text{gen}}, v; v')$$
$$= d_C(v_{\text{gen}}, v') + d_C(v', v)$$
$$\geq d_{\text{MH}}(v_{\text{gen}}, v') + d_C(v', v)$$
$$= d_C(v_{\text{gen}}, (0, w)) + d_C((0, w), v') + d_C(v', v)$$
$$\geq d_C(v_{\text{gen}}, (0, w)) + d_C((0, w), v)$$
$$= d_C(v_{\text{gen}}, v; (0, w))$$
$$\geq T(v, C).$$
This means that if \( P \) is a path from \( v_{gen} \) to \( v \) in \( C \) of length at most \( T(v, C) - 1 \) then \( P \) is in \( S_{w-1} \). Therefore, \( P \) is also in \( C' \) and \( bc_C(u) = bc_{C'}(u) \) for any \( u \) in \( P \).

Similarly, we can show that if \( P \) is a path from \( v_{gen} \) to \( v \) in \( C' \) of length at most \( T(v, C) - 1 \) then \( P \) is also in \( C \) and \( bc_C(u) = bc_{C'}(u) \) for any \( u \) in \( P \). In the proof we use the fact that \( S_{w'} - S_{w-1} \) has no holes of \( C' \) in it and hence for any nodes \( v' \), \( v'' \) of \( C' \) in \( S_w - S_{w-1} \) there is a path that is from \( v' \) to \( v'' \), is a path in \( C' \), is a path in \( S_w - S_{w-1} \), and is of length \( d_{SH}(v', v'') \).

Thus we have proved that \( C \equiv'_{t,v} C' \) with \( t = T(v, C) - 1 \). By Theorem 4.2, \( T(v, C) \leq mft(C) \) is true if and only if \( T(v, C) \leq mft(C') \) is true. However the latter is true because \( T(v, C) \leq 2w' = d_{C'}(v_{gen}, (w', w')) \) \( \leq mft(C') \). Therefore we have \( T(v, C) \leq mft(C) \).

We simply write \( C \equiv' t C' \) if there exists \( v \) such that \( C \equiv'_{t,v} C' \). Let \( \equiv' \) be the reflexive and transitive closure of the relation \( \equiv' \). More precisely, \( C \equiv' C' \) is true if and only if there exists a sequence \( C_0, \ldots, C_n \) of configurations (\( 0 \leq n \)) such that \( C = C_0 \), \( C' = C_n \), and \( C_i \equiv'_{t,v} C_{i+1} \) for any \( 0 \leq i \leq n - 1 \).

**Corollary 4.4** For any configuration \( C \) of \( SH[k] \) and any \( t \), if there exists a configuration \( C' \) of \( SH[k] \) such that \( C \equiv'_{t,v} C' \) and \( t + 1 \leq \max_{v \in C} T(v, C') \) then \( t + 1 \leq mft(C) \).

**Proof.** By Theorem 4.2 the definition of the relation \( \equiv' \) and \( C \equiv'_{t,v} C' \), \( t + 1 \leq mft(C) \) is true if and only if \( t + 1 \leq mft(C') \) is true. However the latter is true because \( t + 1 \leq \max_{v \in C'} T(v, C') \). By Theorem 4.3 we have proved that \( T(v, C') \leq mft(C) \).

**Theorem 4.5** Let \( C, C' \) be configurations of \( SH[k] \) and let \( t \geq 0 \) be a number. If \( C \equiv'_{t,v} C' \) and \( \max_{v \in C} T(v, C) \leq t \) then \( C \) and \( C' \) have the same size.

**Proof.** Let \( v \) be a node in both of \( C, C' \) such that \( C \equiv'_{t,v} C' \) and \( w \) be the size of \( C \). Then we have either \( d_{C}(v_{gen}, v; (0, w)) \leq t \) or \( d_{C}(v_{gen}, v; (w, 0)) \leq t \) because \( \max_{v \in C} T(v, C) \leq t \). We assume the former is true. Then there is a path in \( C \) from \( v_{gen} \) to \( v \) via \((0, w)\) of length at most \( t \). By \( C \equiv'_{t,v} C' \), this path is also a path in \( C' \) and \( bc_{C}(u) = bc_{C'}(u) \) for any node \( u \) on the path. This means that the size of \( C' \) is \( w \).

**Corollary 4.6** Let \( C \) be a configuration of \( SH[k] \) and \( t \geq 0 \) be a number. If there exists a configuration \( C' \) of \( SH[k] \) such that \( C \equiv'_{t,v} C' \) and \( t + 1 \leq \max_{v \in C'} T(v, C') \) then there exist configurations \( C_0, C_1, \ldots, C_n \) (\( n \geq 0 \)) of \( SH[k] \) of the same sizes as \( C \) such that \( C = C_0 \equiv'_{t} C_1 \equiv'_{t} \ldots \equiv'_{t} C_n \) and \( t + 1 \leq \max_{v \in C_n} T(v, C_n) \).
Proof. Let $C_0, C_1, \ldots, C_m$ be configurations of $\text{SH}[k]$ such that $C = C_0 \equiv'_t C_1 \equiv'_t \ldots \equiv'_t C_m = C'$. Let $i_0$ be the smallest value of $i$ (possibly 0) such that $t + 1 \leq \max_{v \in C_i} T(v, C_i)$. Then if we set $n = i_0$ then the configurations $C_0, C_1, \ldots, C_n$ satisfy the condition stated in the corollary by Theorem 4.5. □

Although $\max_{v \in C} T(v, C) \leq \text{mft}(C)$ is true, $\text{mft}(C) \leq \max_{v \in C} T(v, C)$ is not necessarily true. However we have a weaker result. Let $H_{k,w}$ be defined by

$$H_{k,w} = \max_{C \in \mathcal{C}_{k,w}, v \in C} T(v, C). \quad (4.2)$$

**Theorem 4.7**

1. For any $k$, any $w$ and any $C \in \mathcal{C}_{k,w}$, $\text{mft}_{\text{SH}[k]}(C) \leq H_{k,w}$.

2. For any $k$ and any $w$ there exists $C \in \mathcal{C}_{k,w}$ such that $\text{mft}_{\text{SH}[k]}(C) = H_{k,w}$.

**Proof.** (1) Let $\hat{w}$ be some fixed value and let $\hat{\mathcal{C}}$ be some fixed configuration of size $\hat{w}$ of $\text{SH}[k]$. We construct a partial solution $A$ that has $\mathcal{C}_{k,\hat{w}}$ as its domain and that fires configurations in the domain at time $H_{k,\hat{w}}$. Suppose that $C$ is a configuration of size $w$ of $\text{SH}[k]$ and that copies of $A$ are placed on $C$.

A uses the size check messages $W_0, W_1$ (see the comment after Theorem 4.1). $W_0, W_1$ are generated if and only if $w = \hat{w}$ and if they are generated it is at the nodes $(0, \hat{w}), (\hat{w}, 0)$ and at time $\hat{w}$. They propagate in $C$ as soon as they are generated.

A node in $C$ fires at a time $t$ if and only if $t = H_{k,\hat{w}}$ and it has received at least one of $W_0, W_1$ before or at the time $t$. We show that $A$ is a desired partial solution.

Suppose that $C \in \mathcal{C}_{k,\hat{w}}$. Then $w = \hat{w}$ is true and $W_0, W_1$ are generated. Therefore any node $v$ in $C$ receives at least one of $W_0, W_1$ at time $\hat{w} + \min\{d_C((0, \hat{w}), v), d_C((\hat{w}, 0), v)\} = T(v, C) \leq H_{k,\hat{w}}$. Hence $v$ fires at time $H_{k,\hat{w}}$. Therefore, $A$ fires $C$ at time $H_{k,\hat{w}}$.

Conversely, suppose that a node in $C$ fires at some time. Then at least one of $W_0, W_1$ was generated, $w = \hat{w}$ is true, and hence $C \in \mathcal{C}_{k,\hat{w}}$.

Thus we constructed a partial solution $A$ that has $\mathcal{C}_{k,\hat{w}}$ as its domain and that fires configurations in the domain (including $\hat{\mathcal{C}}$ itself) at time $H_{k,\hat{w}}$. This shows $\text{mft}(\hat{\mathcal{C}}) \leq H_{k,\hat{w}}$.

(2) Let $C \in \mathcal{C}_{k,w}$ and $v \in C$ be such that $T(v, C) = H_{k,w}$. Then by Theorem 4.3 $H_{k,w} = T(v, C) \leq \text{mft}(C)$. However, by (1) of the present theorem, $\text{mft}(C) \leq H_{k,w}$. Hence $\text{mft}(C) = H_{k,w}$. □

### 4.3 Maximal barriers of configurations

We have proved that $H_{k,w}$ is the largest value of $\text{mft}_{\text{SH}[k]}(C)$ for $C \in \mathcal{C}_{k,w}$. To determine the exact value of $H_{k,w}$, in this subsection we introduce a notion “maximal barriers” and study its properties. Suppose that a configuration $C$ of size $w$ of $\text{SH}[k]$ is given and is fixed. Let the square $S_w$ defined by (2.1) be the set of positions in $C$. 

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Let $R$ be a nonempty subset of $S_w$ of the form of a rectangle $\{(x, y) \mid x_0 \leq x \leq x_1, y_0 \leq y \leq y_1\}$ ($0 \leq x_0 \leq x_1 \leq w, 0 \leq y_0 \leq y_1 \leq w$). We call $R$ a barrier of $C$ if each column and each row of $R$ contain at least one hole. Moreover, we say that a barrier $R$ is maximal if it is not properly contained in another barrier. As an example, in Fig. 4.3 we show a configuration of size 14 of $\text{SH}[29]$. It has 7 maximal barriers (regions shown by dotted lines) and 70 barriers. For example, the northwest maximal barrier (with three columns and four rows) has 20 barriers in it.

![Figure 4.3: An example of a configuration of size 14 of $\text{SH}[29]$ and its maximal barriers.](image)

**Theorem 4.8**

(1) Suppose that $R, R'$ are different barriers of $C$ and one of the following three statements is true.

- $R \cap R' \neq \emptyset$ (Fig. 4.4(a1), (a2), (a3)).
- $R \cap R' = \emptyset$ but there are positions $v \in R, v' \in R'$ that are adjacent (the figure (b)).
- $R \cap R' = \emptyset$, there are no positions $v \in R, v' \in R'$ that are adjacent but there are positions $v \in R, v' \in R'$ that touch with corners (the figure (c)).

Then the smallest rectangle $R''$ that includes both of $R, R'$ is also a barrier.

(2) If $R, R'$ are different maximal barriers of $C$ then none of the three statements in (1) are true.

(3) Any barrier $R$ of $C$ is included in exactly one maximal barrier of $C$. Especially, any hole $v$ of $C$ is contained in exactly one maximal barrier of $C$ because $\{v\}$ is a barrier.

(4) A barrier of $C$ does not contain a position in the boundary of $S_w$. 

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Suppose that \( R \) is a maximal barrier of \( C \) and a position \( v \) is out of \( R \) but either \( v \) is adjacent to a position in \( R \) or \( v \) touches a position in \( R \) with corners. Then \( v \) is a node of \( C \) and \( v \) is not in maximal barriers of \( C \).

![Diagram](image)

Figure 4.4: Three cases mentioned in (2) of Theorem 4.8.

Proof. (1) Any column of \( R'' \) includes a column of \( R \) or \( R' \). Therefore it contains at least one holes. The same is also true for rows.

(2) Suppose that \( R, R' \) are different maximal barriers of \( C \) and one of the three statements is true. Then the smallest rectangle \( R''' \) that includes both of \( R, R' \) is a barrier by (1). Therefore we have \( R = R'', R' = R'' \) because \( R, R' \) are maximal barriers. However this contradicts our assumption that \( R, R' \) are different.

(3) Let \( \mathcal{R} \) be the set of all barriers that include \( R \). \( \mathcal{R} \) is not empty because \( R \) itself is a barrier. If \( R', R'' \) are two different barriers in \( \mathcal{R} \) then the smallest rectangle \( R''' \) that includes both of \( R', R'' \) is a barrier by \( R' \cap R'' \neq \emptyset \) and (1). Moreover \( R''' \) includes \( R \). Hence \( R''' \) is in \( \mathcal{R} \). This means that there is one barrier \( \bar{R} \) in \( \mathcal{R} \) that is maximum in \( \mathcal{R} \) with respect to the inclusion relation. It is obvious that this \( \bar{R} \) is a maximal barrier and it includes \( R \). Moreover we can show that there is at most one maximal barrier that includes \( R \) using (2).

(4) Suppose that \( v \) is a position in the boundary of \( S_w \) and that \( v \) is in a rectangle \( R \) in \( S_w \). Then either the column of \( R \) containing \( v \) or the row of \( R \) containing \( v \) has no holes in it. Therefore \( R \) cannot be a barrier.

(5) Suppose that \( v \) is in a maximal barrier \( R' \). \( R \) and \( R' \) are different because \( R \) does not contains \( v \) but \( R' \) contains \( v \). Moreover it is obvious that one of the three statements in (1) is true. This is a contradiction by (2). Therefore \( v \) is not in maximal barriers of \( C \). This implies that \( v \) is a node of \( C \) because a hole is in a maximal barrier by (4). \( \square \)

The definition of maximal barriers itself gives the following algorithm to enumerate all maximal barriers of \( C \): enumerate all rectangles in \( S_w \) having at most \( k \) columns and at most \( k \) rows, delete all rectangles that are not barriers, and select maximal rectangles. However this algorithm is not efficient. We show a more efficient algorithm.

Let \( \mathcal{R} \) be the set consisting of one large rectangle \( \{(x, y) \mid 1 \leq x \leq w - 1, 1 \leq y \leq w - 1\} \). Starting with this \( \mathcal{R} \), repeat the following. If all rectangles in \( \mathcal{R} \) are barriers then finish the algorithm with \( \mathcal{R} \) as its result. Otherwise, select a
rectangle $R$ that is not a barrier from $\mathcal{R}$. Suppose that a column of $R$ contains no holes. If there is such a column that is also a side boundary column of $R$ then replace $R$ in $\mathcal{R}$ with the rectangle $R'$ that is obtained from $R$ by deleting that side boundary column. Otherwise the column must be an inner column. In this case replace $R$ with the two rectangles $R', R''$ that are obtained from $R$ by deleting that inner column. Similarly for the case when a row of $R$ contains no holes. It is evident that at some step all rectangles in $\mathcal{R}$ are barriers and the algorithm finishes. Let $\tilde{\mathcal{R}}$ denote the resulting set $\mathcal{R}$.

**Theorem 4.9** $\tilde{\mathcal{R}}$ is the set of all maximal barriers of $C$.

*Proof.* First we show that any barrier $R$ is included in a barrier in $\tilde{\mathcal{R}}$. At the start of the algorithm $R$ is included in the unique large rectangle in $\mathcal{R}$ by Theorem 4.8 (4). Moreover, as the algorithm is executed $R$ continues to be in a set in $\mathcal{R}$. This is because $R$ is a barrier and hence any deleted column or row does not include columns or rows of $R$. Therefore, $R$ must be in a barrier in $\tilde{\mathcal{R}}$.

Suppose that $R$ is a maximal barrier but $R$ is not in $\tilde{\mathcal{R}}$. Then $R$ is included properly in a barrier $R'$ in $\tilde{\mathcal{R}}$. But this contradicts the assumption that $R$ is maximal. Therefore any maximal barrier is in $\tilde{\mathcal{R}}$.

Suppose that $R$ is a barrier in $\tilde{\mathcal{R}}$ but $R$ is not maximal. Then there is a barrier $R'$ that includes $R$ properly. But then there is another barrier $R''$ in $\tilde{\mathcal{R}}$ that includes $R'$. This contradicts our assumption that $R$ is in $\tilde{\mathcal{R}}$. Therefore, any barrier in $\tilde{\mathcal{R}}$ is maximal. $\square$

Fig. 4.5 shows an example of application of the above algorithm. We show the change of $\mathcal{R}$ from left to right. We obtain the final $\tilde{\mathcal{R}}$ in 4 steps.

![Figure 4.5: An example of application of the algorithm to obtain all maximal barriers.](image)

The usefulness of maximal barriers in the analysis of $H_{k,w}$ comes from the following theorem.

**Theorem 4.10** Let $v$ be one of the four corners $(0,0)$, $(0,w)$, $(w,0)$, $(w,w)$ of $C$ and $v'$ be any node of $C$ that is not in maximal barriers of $C$. Then $d_{C}(v,v') = d_{MH}(v,v')$, and hence there is a path in $C$ from $v$ to $v'$ of the MH distance length.

*Proof.* We prove this only for the case $v = (0,0)$. Proofs for other cases are similar.
Suppose that a node \((x, y)\) is not in maximal barriers and \(0 < x, 0 < y\). We assume that both of the two positions \((x-1, y)\), \((x, y-1)\) are in maximal barriers and derive a contradiction. Let \(R\) and \(R'\) respectively be the maximal barriers that contain \((x-1, y)\) and \((x, y-1)\) respectively.

It is not possible that \(R = R'\) because if two positions \((x-1, y)\), \((x, y-1)\) are in one maximal barrier then the position \((x, y)\) is also in the maximal barrier, contradicting our assumption that \((x, y)\) is not in maximal barriers. Hence \(R\), \(R'\) are different maximal barriers.

Both of \(R\) and \(R'\) do not contain \((x, y)\) and hence \((x-1, y)\) is in the east boundary of \(R\) and \((x, y-1)\) is in the north boundary of \(R'\). Therefore, one of the three statements mentioned in Theorem 4.8 (1) must be true and we have a contradiction by Theorem 4.8 (2).

We have proved that if a node \((x, y)\) is not in maximal barriers and \(x > 0, y > 0\) then at least one of the positions \((x-1, y)\), \((x, y-1)\) is not in maximal barriers (and hence is a node). This, together with our assumption that \(v'\) is not in maximal barriers, means that there is a path \(P\) in \(C\) from \(v'\) to a node \(v''\) that is either in the west boundary or in the south boundary of \(C\) satisfying the condition: the path proceeds only to the west or to the south. It is obvious that there is a similar path \(P'\) in \(C\) from \(v''\) to \(v\). Therefore, there is a similar path \(P + P'\) in \(C\) from \(v'\) to \(v\). Hence we have \(d_C(v, v') = d_{MH}(v, v')\). \(\square\)

### 4.4 Maximal barriers and \(H_{k,w}\)

The following lemma is used repeatedly.

**Lemma 4.11** Let \(C\) be a configuration of size \(w\) of \(SH[k]\). Then

\[
\max\{\min\{d_{MH}(v_{gen}, v; (0, w)), d_{MH}(v_{gen}, v; (w, 0))\} \mid v \in C\} = 2w.
\]

**Proof.** Let \(v\) be \((x, y)\) and let \(\delta\) denote \(x - y\). Then \(-w \leq \delta \leq w\) and \(\delta = 0\) for at least two nodes \(v = (0, 0), (w, w)\) in \(C\). Therefore,

\[
\begin{align*}
\max\{\min\{d_{MH}(v_{gen}, v; (0, w)), d_{MH}(v_{gen}, v; (w, 0))\} \mid v \in C\} &= w + \max\{\min\{x + (w - y), (w - x) + y\} \mid v \in C\} \\
&= w + \max\{\min\{w + \delta, w - \delta\} \mid v \in C\} \\
&= 2w.
\end{align*}
\]

\(\square\)

Suppose that a node \(v\) in \(C \in C_{k,w}\) is given and is fixed. We derive a formula for the value \(T(v, C)\). If \(v\) is not in maximal barriers of \(C\), we have

\[
T(v, C) = \min\{d_C(v_{gen}, v; (0, w)), d_C(v_{gen}, v; (w, 0))\}
= \min\{d_{MH}(v_{gen}, v; (0, w)), d_{MH}(v_{gen}, v; (w, 0))\}
\leq 2w.
\] (4.3)
by (4.1), Theorem 4.10 and Lemma 4.11. From now on, we consider the case when \( v \) is in a maximal barrier \( R \) of \( C \). We will show that \( T(v, C) \) is expressed as \( T(v, C) = 2w + E(S, p, \delta) \) (see (4.5)). This equation implies that \( T(v, C) \) is determined by the three factors: (1) the form of \( R \) (represented by \( S \)), (2) the position of \( v \) in \( R \) (represented by \( p \)), (3) the position of \( R \) in \( C \) relative to the main diagonal (represented by \( \delta \)).

When \( R \) is a rectangle (a maximal barrier, for example) \( \{(x, y) \mid x_0 \leq x \leq x_1, y_0 \leq y \leq y_1 \} \) and \( 1 \leq x_0, x_1 \leq w - 1, 1 \leq y_0, y_1 \leq w - 1 \), by the enlarged rectangle of \( R \) we mean the rectangle \( X = \{(x, y) \mid x_0 - 1 \leq x \leq x_1 + 1, y_0 - 1 \leq y \leq y_1 + 1 \} \).

**Theorem 4.12** Suppose that a node \( v \) of \( C \in C_{k,w} \) is in a maximal barrier \( R \) of \( C \). Let \( W, H, z, \delta, d_0, d_1 \) be defined as follows (see Fig. 4.6):

- \( W \) is the number of columns of \( R \).
- \( H \) is the number of rows of \( R \).
- \( z = (z_x, z_y) \) is the position of the southwest corner of \( R \).
- \( \delta = z_x - z_y \).
- \( d_0 = d_C(z + (-1, H), v), d_1 = d_C(z + (W, -1), v) \).

Then

\[
T(v, C) = 2w + \min\{\delta - H - 1 + d_0, -\delta - W - 1 + d_1\}. 
\]

(4.4)

**Proof.** First we represent the value \( d_C(v_{\text{gen}}, v; (0, w)) \) with \( W, H, z = (z_x, z_y), \delta, d_0, d_1 \). Let \( X \) be the enlarged rectangle of \( R \),

\[
X = \{(x, y) \mid z_x - 1 \leq x \leq z_x + W, z_y - 1 \leq y \leq z_y + H \}
\]

(see Fig. 4.7).

\( R \) is a maximal barrier. Hence by Theorem 4.8 (5), \( X - R \) has no holes in it and any node in \( X - R \) is not in maximal barriers.

Let \( P \) be a shortest path in \( C \) from \((0, w)\) to \( v \), \( v' \) be the last node in \( P \) that is not in \( R \), and \( P_1 \) and \( P_2 \) be the parts of \( P \) from \((0, w)\) to \( v' \) and from \( v' \) to \( v \) respectively. Let \( v'' \) be the node \( z + (-1, H) \) and let \( P_{11} \) and \( P_{12} \) be shortest paths from \((0, w)\) to \( v'' \) and from \( v'' \) to \( v' \) respectively.

Then we have \( |P_1| = d_{\text{MH}}((0, w), v'), |P_{11}| = d_{\text{MH}}((0, w), v'') \) and \( |P_{12}| = d_{\text{MH}}(v'', v) \). The first and the second equalities are true because \( v', v'' \) are in \( X - R \) and hence are not in maximal barriers (Theorem 4.10). The third equality is true because both of \( v', v'' \) are in \( X - R \), \( X - R \) has no holes, and \( v'' \) is the northwest corner of \( X - R \). From this we have \( |P_1| = |P_{11}| + |P_{12}| \). This implies that the path \( P_{11} + P_{12} + P_2 \) is a shortest path in \( C \) from \((0, w)\) to \( v \). Therefore, \( P_{12} + P_2 \) is a shortest path from \( v'' \) to \( v \) in \( C \) and hence \( |P_{12} + P_2| = d_0 \) by the definition of \( d_0 \).
Figure 4.6: A configuration $C$ in $C_{k,w}$ and node $v$ that is in a maximal barrier $R$ of $C$.

Figure 4.7: Paths and nodes used in the estimation of $d_C(v_{\text{gen}}, v; (0, w))$. 
Hence we have
\[
d_C(v_{gen}, v; (0, w)) = w + |P|
= w + |P_{11} + P_{12}| + d_0
= w + d_{MH}((0, w), z + (1, H)) + d_0
= w + (z_x - 1) + (w - (z_y + H)) + d_0
= 2w + \delta - H - 1 + d_0.
\]

Similarly we have \(d_C(v_{gen}, v; (w, 0)) = 2w - \delta - W - 1 + d_1\). Hence we have

\[
T(v, C) = 2w + \min\{\delta - H - 1 + d_0, -\delta - W - 1 + d_1\}.
\]

In the above proof we proved that \(P_{12} + P_2\) is a shortest path in \(C\) from \(v''\) to \(v\). Moreover, this path is in \(X\). Therefore, \(d_0\) is the length of a shortest of all paths from \(v''\) to \(v\) that are in \(X\) and \(d_0\) is completely determined by the distribution of holes of \(C\) in \(X\) irrespective of the distribution out of \(X\). The same is also true for \(d_1\).

The value \(\delta\) represents the relative position of \(R\) with respect to the main diagonal of \(C\). If \(\delta \geq 0\), the southwest corner of \(R\) is to the east of the main diagonal by \(\delta\) positions. If \(\delta < 0\), the position is to the west by \(-\delta\) positions.

Using the equation \(4.4\) we can determine the value \(H_{k,w}\) for all \(k\) and for all sufficiently large \(w\). Let \(E\) denote the value \(E = \min\{\delta - H - 1 + d_0, -\delta - W - 1 + d_1\}\). Then \(T(v, C) = 2w + E\). This value \(E\) is a function of \(C\) and \(v\).

However, \(E\) can be determined by the following three factors:

- The shape \(S\) of the maximal barrier \(R\). More precisely, it is the triple:
  - \(W\),
  - \(H\),
  - whether the position \(z + (i, j)\) in \(C\) is a hole or not for each \((i, j)\) such that \(0 \leq i \leq W - 1, 0 \leq j \leq H - 1\).

We regard \(S\) as a barrier.

- \(p = v - z\).
- \(\delta = z_x - z_y\).

Although the two values \(d_0, d_1\) are not included, we can determine them from these three factors. For example, as we mentioned above, \(d_0\) is the length of a shortest of all paths in \(C\) from the northwest corner of the enlarged rectangle \(X\) of \(R\) to \(v\) that are in \(X\). This can be determined from \(S\) and \(p\).

Therefore, we will denote \(E\) as a function \(E(S, p, \delta)\) of these three factors \(S, p, \delta\). Then the equation \(4.4\) is written as

\[
T(v, C) = 2w + E(S, p, \delta).  \tag{4.5}
\]
We continue to call $S$ a “barrier” of $SH[k]$ and write “$p \in S$” to mean that a position $p$ in $S$ is a node.

Let $S_k$ denote the set of all barriers having at most $k$ holes. Then, for any maximal barrier $R$ in any configuration $C \in C_{k,w}$ the corresponding barrier $S$ is in $S_k$. Conversely, for any barrier $S$ in $S_k$ and any sufficiently large $w$, there are $C \in C_{k,w}$ and a maximal barrier $R$ in $C$ such that $S$ corresponds to $R$. When a maximal barrier $R$ in $C \in C_{k,w}$ and a barrier $S \in S_k$ correspond, there is also a one-to-one correspondence between nodes $v$ in $R$ and nodes $p$ in $S$.

In Fig. 4.8 we show the barrier $S$ and the node $p \in S$ that correspond to the maximal barrier $R$ and the node $v \in R$ in Fig. 4.6.

![Figure 4.8: The barrier $S$ and the node $p \in S$ corresponding to the maximal barrier $R$ and the node $v \in R$ in Fig. 4.6.](image)

We define several values. We defined $E(S, p, \delta)$ by

$$E(S, p, \delta) = \min\{\delta - H - 1 + d_0, -\delta - W - 1 + d_1\} \quad (4.6)$$

for $S \in S_k$, $p \in S$, $\delta \in \mathbb{Z}$. We define $E_{\max}(S, p)$ by

$$E_{\max}(S, p) = \max_{\delta \in \mathbb{Z}} E(S, p, \delta) \quad (4.7)$$

for $S \in S_k$, $p \in S$. Finally, we define $c_k$ by

$$c_k = \max_{S \in S_k, p \in S} E_{\max}(S, p) \quad (4.8)$$

for $k \geq 2$. By the assumption $k \geq 2$ there is at least one pair of $S$ and $p$ such that $p \in S$ (see Fig. 4.14).

Let $X = \{(i, j) \mid -1 \leq i \leq W, -1 \leq j \leq H\}$ be the enlarged rectangle of $S$ and let $v_0 = (-1, H), v_1 = (W, -1)$ be its northwest and southeast corners respectively. Then we have $W + H + 2 = d_{HM}(v_0, v_1), d_0 + d_1 = d_X(v_0, p) + d_X(p, v_1)$. Hence both of $W + H + 2$ and $d_0 + d_1$ are lengths of paths in $X$ from $v_0$ to $v_1$. This means that $W + H + 2 \equiv d_0 + d_1 \pmod{2}$. (Regard $X$ as a checkerboard.)

Using this we obtain the following simplified expression for $E_{\max}(S, p)$.

$$E_{\max}(S, p) = \max\{\min\{\delta - H - 1 + d_0, -\delta - W - 1 + d_1\} \mid \delta \in \mathbb{Z}\}$$

$$= (-W - H - 2 + d_0 + d_1) / 2. \quad (4.9)$$
The value \( \delta_{\text{opt}}(S, p) \) such that \( E_{\text{max}}(S, p) = E(S, p, \delta_{\text{opt}}(S, p)) \) is given by

\[
\delta_{\text{opt}}(S, p) = \left(-W + H - d_0 + d_1\right)/2.
\]

We can also show \( E_{\text{max}}(S, p) \geq 0 \) as follows.

\[
E_{\text{max}}(S, p) = \frac{d_0 + d_1 - (W + H + 2)}{2} \geq \frac{d_X(v_0, v_1) - d_{\text{MH}}(v_0, v_1)}{2} \geq 0.
\]

This implies \( c_k \geq 0 \).

Fig. 4.9 shows a barrier \( S \) that has 9 holes. It is in \( S_k \) for \( 9 \leq k \). For this \( S \), we have \( W = 4, H = 5, E(S, p, \delta) = \min\{\delta - 6 + d_0, -\delta - 5 + d_1\} \), \( E_{\text{max}}(S, p) = (-11 + d_0 + d_1)/2 \), and \( \delta_{\text{opt}}(S, p) = (1 - d_0 + d_1)/2 \). In Table 4.1 we show values \( d_0, d_1, E(S, p, \delta), E_{\text{max}}(S, p), \delta_{\text{opt}}(S, p) \) for each node \( p \) of the 11 nodes in \( S \).

| \( p \) | \( d_0 \) | \( d_1 \) | \( E(S, p, \delta) \) | \( E_{\text{max}}(S, p) \) | \( \delta_{\text{opt}}(S, p) \) |
|-----|----|----|----------------|----------------|----------------|
| (0, 0) | 6  | 5  | \( \min\{\delta, -\delta\} \) | 0 | 0 |
| (0, 1) | 5  | 6  | \( \min\{\delta - 1, -\delta + 1\} \) | 0 | 1 |
| (0, 2) | 4  | 7  | \( \min\{\delta - 2, -\delta + 2\} \) | 0 | 2 |
| (1, 1) | 6  | 7  | \( \min\{\delta, -\delta + 2\} \) | 1 | 1 |
| (1, 2) | 5  | 8  | \( \min\{\delta - 1, -\delta + 3\} \) | 1 | 2 |
| (1, 4) | 3  | 10 | \( \min\{\delta - 3, -\delta + 5\} \) | 1 | 4 |
| (2, 1) | 7  | 8  | \( \min\{\delta + 1, -\delta + 3\} \) | 2 | 1 |
| (2, 3) | 7  | 6  | \( \min\{\delta + 1, -\delta + 1\} \) | 1 | 0 |
| (3, 2) | 7  | 4  | \( \min\{\delta + 1, -\delta - 1\} \) | 0 | -1 |
| (3, 3) | 6  | 5  | \( \min\{\delta, -\delta\} \) | 0 | 0 |
| (3, 4) | 5  | 6  | \( \min\{\delta - 1, -\delta + 1\} \) | 0 | 1 |

Table 4.1: Values of \( d_0, d_1 \) and so on for the barrier \( S \) in Fig. 4.9 and \( p \in S \).

This table shows that the maximum value of \( E_{\text{max}}(S, p) \) for \( p \in S \) is 2 and this value 2 is realized only by \( p = (2, 1), \delta = 1 \). In Fig. 4.10 (a) we show the position of \( p = (2, 1) \) in \( S \). In Fig. 4.10 (b) we show a configuration \( C \) of
size \( w = 7 \) of \( SH[10] \) in which \( S \) appears as a maximal barrier \( R \) at a position \( z = (z_x, z_y) = (3, 2) \) such that \( \delta = z_x - z_y = 3 - 2 = 1 \). The node \( v \) in \( C \) that corresponds to \( p = (2, 1) \) in \( S \) is \( v = z + p = (5, 3) \). This pair \( C, v \in C \) realizes \( T(v, C) = 2w + 2 = 16 \). In Fig. 4.10 (c) we show a shortest path \( P_0 \) from \( v_{gen} \) to \( v \) via \( (0, w) \) \( (= (0, 9)) \) and a shortest path \( P_1 \) from \( v_{gen} \) to \( v \) via \( (w, 0) \) \( (= (9, 0)) \). We have \( T(v, C) = \min\{|P_0|, |P_1|\} = \min\{16, 16\} = 16 \) confirming the above equation. For this \( C \) we have \( mft_{SH[10]}(C) \geq \max_{v \in C} T(v, C) \geq 16 \).}

![Diagram](image)

Figure 4.10: (a) The position of \( p = (2, 1) \) in \( S \). (b) A configuration \( C \) in which \( S \) appears as a maximal barrier \( R \) so that \( E(S, p, \delta) = E_{\max}(S, p) = 2 \) with \( p = (2, 1), \delta = \delta_{\text{opt}}(S, p) = 1 \). (C) Two paths \( P_0, P_1 \) from \( v_{gen} \) to \( v \) via \( (0, w) \) and via \( (w, 0) \).

**Theorem 4.13**

1. For any \( k \geq 2 \) and any \( w \), \( H_{k,w} \leq 2w + c_k \).

2. For any \( k \geq 2 \) and any \( w \) such that \( (k^2 + 7k + 5)/2 \leq w \), \( H_{k,w} = 2w + c_k \).

**Proof.** (1) We assume that \( C \in \mathcal{C}_{k,w}, v \in C \) and prove \( T(v, C) \leq 2w + c_k \). This implies \( H_{k,w} = \max_{C \in \mathcal{C}_{k,w}, v \in C} T(v, C) \leq 2w + c_k \).

If \( v \) is not in maximal barriers of \( C \) then \( T(v, C) \leq 2w \leq 2w + c_k \) by \((4.3)\). (Note that \( c_k \geq 0 \) by \((4.11)\).) Suppose that \( v \) is in a maximal barrier \( R \) of \( C \). Let \( S \in \mathcal{S}_k, p \in S, \delta \) be the barrier and so on that are determined from \( C, R, v \) by the correspondence explained in the proof of Theorem 4.12. Then by \((4.5)\), we have \( T(v, C) = 2w + E(S, p, \delta) \leq 2w + E_{\max}(S, p) \leq 2w + c_k \).

(2) We assume that \( S \in \mathcal{S}_k, p \in S, (k^2 + 7k + 5)/2 \leq w \) and prove \( 2w + E_{\max}(S, p) \leq H_{k,w} \). This implies that if \( (k^2 + 7k + 5)/2 \leq w \) then \( 2w + c_k = 2w + \max_{S \in \mathcal{S}_k, v \in S} E_{\max}(S, p) \leq H_{k,w} \) and hence \( H_{k,w} = 2w + c_k \).

We assume \( \delta_{\text{opt}}(S, p) \geq 0 \). The proof for the other case is similar. Let \( S \) has \( k' \) holes and let \( W \) and \( H \) be the number of columns and that of rows of \( S \) respectively (see Fig. 4.11 (a)). We have \( W \leq k, H \leq k \) by the definition of barriers.
It is obvious that $d_1 \leq (W + 2)(H + 2) - 1 \leq k^2 + 4k + 3$ because the enlarged rectangle $X$ of $S$ has $(W + 2)(H + 2)$ positions and $d_1$ is the length of a shortest path from $(W, -1)$ to $p$ in $X$. Using this we have an upper bound of $\delta_{\text{opt}}(S, p)$:

$$
\delta_{\text{opt}}(S, p) = (-W + H - d_0 + d_1) / 2 \\
\leq (H + d_1) / 2 \\
\leq (k^2 + 5k + 3) / 2.
$$

We define a configuration $C \in C_{k, w}$ as shown in Fig. 4.11 (b). In $C$, the barrier $S$ is placed at the position $z = (z_x, z_y) = (\delta_{\text{opt}}(S, p) + 1, 1)$ as a barrier $R$ in $C$. Moreover, $k - k'$ holes are placed in $C$ at positions $(1 + \delta_{\text{opt}}(S, p), H + 2)$, $\ldots$, $(\delta_{\text{opt}}(S, p) + k - k', H + 2)$ so that they constitute another barrier (if $k' < k$). We can show that these two barriers are two different maximal barriers in $C$ using our assumption $(k^2 + 7k + 5)/2 \leq w$, $k \geq 2$ and the above mentioned upper bound of $\delta_{\text{opt}}(S, p)$.

For example, we can prove that the $x$-coordinate $(\delta_{\text{opt}}(S, p) + k - k')$ of the easternmost hole of the $k - k'$ holes (if $k' < k$) is at most $w - 1$ as follows:

$$
\delta_{\text{opt}}(S, p) + k - k' \leq (k^2 + 5k + 3)/2 + k \\
= (k^2 + 7k + 3)/2 \\
\leq w - 1.
$$
Let \( v = z + p \) be the node in \( R \) that corresponds to \( p \) in \( S \) and \( \delta \) be the value \( \delta = \delta_{\text{opt}}(S, p) \). Then

\[
2w + E_{\max}(S, p) = 2w + E(S, p, \delta_{\text{opt}}(S, p))
= 2w + E(S, p, \delta)
= T(v, C) \quad \text{(by (4.5))}
\leq H_{k, w}.
\]

\[\Box\]

**Corollary 4.14**

1. For any \( k \geq 2 \), any \( w \) and any \( C \in C_{k, w} \), \( mft_{SH[k]}(C) \leq 2w + c_k \).
2. For any \( k \geq 2 \) and any \( w \) such that \((k^2 + 7k + 5)/2 \leq w\), there exists \( C \in C_{k, w} \) such that \( mft_{SH[k]}(C) = 2w + c_k \).

**Proof.** This corollary follows from Theorems 4.7, 4.13.

Below we show a lower bound and an upper bound of \( c_k \). As we explain later the lower bound \( k - 2 \) is the correct value of \( c_k \) for \( 3 \leq k \leq 9 \). The upper bound \( k^2 + 4k \) was obtained by a very simple estimation and we expect to be able to improve it considerably.

**Theorem 4.15** For any \( k \geq 3 \),

\[ k - 2 \leq c_k \leq k^2 + 4k. \]

**Proof.** Upper bound: In the proof of Theorem 4.13 we showed \( d_1 \leq k^2 + 4k + 3 \) and we have a similar result also for \( d_0 \). Hence, using \( 2 \leq W, 2 \leq H \) (a barrier \( S \) with either \( W = 1 \) or \( H = 1 \) has no nodes \( p \) in it), we have

\[
E_{\max}(S, p) = (-W - H - 2 + d_0 + d_1)/2
\leq (-6 + 2(k^2 + 4k + 3))/2
= k^2 + 4k.
\]

Therefore, \( c_k = \max_{S \in S_k, \, p \in S} E_{\max}(S, p) \leq k^2 + 4k \).

Lower bound: In Fig. 4.12 we show barriers \( S \in S_k \) and \( p \in S \) such that \( E_{\max}(S, p) = k - 2 \) implying \( c_k \geq k - 2 \). The figure (a) is for even \( k \) (\( \geq 4 \)) and the figure (b) is for odd \( k \) (\( \geq 3 \)). The marks "" denote the nodes \( p \). The dotted lines are main diagonals.

For (a), we have \( k = 20, W = H = (k + 2)/2 = 11, d_0 = k = 20, d_1 = 2k = 40, E_{\max}(S, p) = (-k + 2)/2 - (k + 2)/2 + k + 2k - 2)/2 = k - 2 = 18, \delta_{\text{opt}} = (-k + 2)/2 + (k + 2)/2 - k + 2k)/2 = k/2 = 10 \).

For (b), we have \( k = 21, W = H = (k + 3)/2 = 12, d_0 = k = 21, d_1 = 2k + 1 = 43, E_{\max}(S, p) = (-k + 3)/2 - (k + 3)/2 + k + (2k + 1)/2 = k - 2 = 19, \delta_{\text{opt}} = (-k + 3)/2 + (k + 3)/2 - k + (2k + 1))/2 = (k + 1)/2 = 11. \]
4.5 Determination of values $c_k$ by an algorithm

Definition (4.8) of $c_k$ itself gives an algorithm for computing $c_k$. We enumerate all $S$ in $S_k$. For each pair $(S, p)$ of $S \in S_k$ and $p \in S$ we compute the value $E_{\text{max}}(S, p)$. Then the maximum value of this value $E_{\text{max}}(S, p)$ over all pairs $(S, p)$ is the desired $c_k$.

Before carrying out this computation we define one value which we will denote by $\epsilon_{\text{opt}}(S, p)$. Let a barrier $S$ and a node $p = (p_x, p_y)$ in $S$ correspond to a maximal barrier $R$ in a configuration $C$ and a node $v = (v_x, v_y)$ in $R$ respectively. Let $z = (z_x, z_y)$ be the southwest corner of $R$ and let $\delta$ denote $z_x - z_y$. Let $\epsilon$ denote $v_x - v_y$. Then we have the relation $\epsilon = v_x - v_y = (z_x + p_x) - (z_y + p_y) = \delta + p_x - p_y$ between $\epsilon$ and $\delta$. We defined the value $\delta_{\text{opt}}(S, p)$ to be the value of $\delta$ such that $E(S, p, \delta)$ is maximum. Let $\epsilon_{\text{opt}}(S, p)$ be defined by $\epsilon_{\text{opt}}(S, p) = \delta_{\text{opt}}(S, p) + p_x - p_y$. Then $\epsilon_{\text{opt}}(S, p)$ is the value of $\epsilon$ such that the value $E(S, p, \delta)$ is maximum as a function of $\epsilon$. Intuitively, $\delta_{\text{opt}}(S, p)$ and $\epsilon_{\text{opt}}(S, p)$ represent the positions of $z$ and $v$ respectively relative to the main diagonal when $S$ is placed in $C$ so that the value $E(S, p, \delta)$ is maximum (Fig. 4.13).

Now we determine the value of $c_2$ using the above algorithm. $S_2$ has 5 barriers $S_1, \ldots, S_5$ shown in Fig. 4.14. There are 4 pairs $(S, p)$ such that $S \in S_2, p \in S$. In Table 4.2 we show values $W, H, d_0, d_1, E_{\text{max}}(S, p), \delta_{\text{opt}}(S, p), \epsilon_{\text{opt}}(S, p)$ for each of these four pairs. From this table we have $c_2 = \max_{S \in S_2, p \in S} E_{\text{max}}(S, p) = 1$.

Two pairs $(S_5, (0, 1)), (S_5, (1, 0))$ realize the value $E_{\text{max}}(S, p) = 1 (= c_2)$. In Fig. 4.15 we show these two pairs. The left represents $(S_5, (0, 1))$ with the $\epsilon_{\text{opt}}$ value 1 and the right represents $(S_5, (1, 0))$ with the $\epsilon_{\text{opt}}$ value $-1$. Marks '*' represent positions of $p$. The two pairs are symmetric with respect to the main diagonal represented by dotted lines.

Next we determine the value of $c_3$. There are 29 barriers in $S_3$ and there
are 80 pairs \((S,p)\) such that \(S \in S_1, p \in S\). Of these 80 pairs, the value of \(E_{\text{max}}(S,p)\) is 0 for 46 pairs and 1 for 34 pairs. Therefore \(c_3 = 1\). Of the 34 pairs with \(E_{\text{max}}(S,p) = 1\), the value of \(\epsilon_{\text{opt}}(S,p)\) is 1 for 17 pairs and \(-1\) for 17 pairs. The former 17 pairs and the latter 17 pairs are symmetric with respect to the main diagonal. In Fig. 4.16 we show the 17 pairs having the \(\epsilon_{\text{opt}}\) value 1.

These 17 pairs have also symmetry with respect to the direction from the northwest to the southeast (the direction that is orthogonal to the direction of the main diagonal). Each of the 5 pairs \((a), (b), \ldots, (e)\) is symmetric with itself. The six pairs \((f0), (g0), \ldots, (k0)\) are symmetric to the six pairs \((f1), (g1), \ldots, (k1)\) respectively.

The pair \((a)\) has only 2 holes but it realizes \(E_{\text{opt}}(S,p) = 1 = c_3\). The 13 pairs \((b), (c), (d), (f0), (g0), (h0), (i0), (k0), (f1), (g1), (h1), (i1), (k1)\) are obtained by adding one hole to the two hole pair \((a)\) but realize the same \(E_{\text{opt}}(S,p)\) value

| \((S,p)\) | \(W\) | \(H\) | \(d_0\) | \(d_1\) | \(E_{\text{max}}(S,p)\) | \(\delta_{\text{opt}}(S,p)\) | \(\epsilon_{\text{opt}}(S,p)\) |
|----------|------|------|------|------|----------------|----------------|----------------|
| \((S_4,(0,0))\) | 2 | 2 | 3 | 3 | 0 | 0 | 0 |
| \((S_4,(1,1))\) | 2 | 2 | 3 | 3 | 0 | 0 | 0 |
| \((S_5,(0,1))\) | 2 | 2 | 6 | 1 | 2 | 1 | 1 |
| \((S_5,(1,0))\) | 2 | 2 | 6 | 2 | 1 | -2 | -1 |

Table 4.2: Values of \(W\), \(H\) and so on for the four pairs \((S,p)\) such that \(S \in S_2, p \in S\).
as (a). Hence they are essentially the two hole pair (a). For the remaining three pairs (c), (j0), (j1), the three holes are essentially used.

We computed the value of $c_k$ for up to $k = 9$ by computer. We show the result in Table 4.3. In the table we also show the number of barriers $S \in S_k$ (that is, the number of barriers $S$ having at most $k$ holes), the number of pairs $(S, p)$ such that $S \in S_k$, $p \in S$, and the number of pairs $(S, p)$ such that $S \in S_k$, $p \in S$, $E_{\text{max}}(S, p) = c_k$.

For $k = 9$ we have $c_9 = 7$. In Fig. 4.17 we show the 16 pairs $(S, p)$ that realize the value $E_{\text{max}}(S, p) = 7$ and that have nonnegative $\epsilon_{\text{opt}}$ values. The value $\epsilon_{\text{opt}}(S, p)$ is 5 for the 4 pairs (a0), (b0), (a1), (b1) and 7 for the remaining 12 pairs (c0), (d0), . . . , (h0), (c1), (d1), . . . , (h1). The forms of the barriers of the former 4 pairs are curved caves and those of the latter 12 pairs are straight caves, both with the positions of $p$ at their dead ends. The 8 pairs (a0), (b0), . . . , (h0) and the 8 pairs (a1), (b1), . . . , (h1) are symmetric with respect to the direction from the northwest to the southeast.
Table 4.3: The values of $c_k$ for $k = 2, \ldots, 9$ obtained by computer. The table also shows the numbers of barriers $S$ in $S_k$, the numbers of pairs $(S, p)$, and the numbers of pairs $(S, p)$ such that $E_{\text{max}}(S, p) = c_k$.

5 Determination of the minimum firing time $\text{mft}_{\text{SH}[2]}(C)$ of $\text{SH}[2]$

In this section we determine the value of the minimum firing time $\text{mft}_{\text{SH}[2]}(C)$ of $\text{SH}[2]$.

Suppose that $C$ is a configuration of size $w$ of $\text{SH}[2]$. We know that $\text{mft}(C)$ is either $2w$ or $2w + 1$. Therefore to determine $\text{mft}(C)$ it is only necessary to prove either the lower bound $\text{mft}(C) \geq 2w + 1$ or the upper bound $\text{mft}(C) \leq 2w$. To show the lower bound $\text{mft}(C) \geq 2w + 1$ we use Corollary 4.4. In Subsection 5.1 we show some results that are used in proving lower bounds. In Subsection 5.2 we define a division of a square $S_w$ into four nonoverlapping subsets $U, V, W, X$ and show one theorem on this division. To show the lower bound $\text{mft}(C) \leq 2w$ we construct a partial solution that fires $C$ at time $2w$. To construct such a partial solution we use one unified strategy. In Subsection 5.3 we explain it. Finally, in Subsection 5.4 we state the main result and prove it.

5.1 Some results used for proving lower bounds of $\text{mft}_{\text{SH}[2]}(C)$

We use Corollary 4.4 to prove a lower bound $2w + 1 \leq \text{mft}_{\text{SH}[2]}(C)$ for a configuration $C$ of $\text{SH}[2]$ of size $w$. By Corollary 4.6 we may assume that the size of $C'$ mentioned in Corollary 4.4 is $w$. In this subsection we show some results that are useful for finding such $C'$.

By a pattern $\pi$ we mean a partial function from $\mathbb{Z}^2$ to the two element set $\{N, H\}$. The letters "N" and "H" are abbreviations of nodes and holes respectively. We say that a configuration $C$ has a pattern $\pi$ if for any $v \in \mathbb{Z}^2$, if $\pi(v) = N$ then the position $v$ is a node in $C$ and if $\pi(v) = H$ then there
is a hole at the position \( v \) in \( C \). For a configuration \( C \) of size \( w \) and a set \( X \subseteq \{0, 1, \ldots, w\}^2 \), by \( \pi(C, X) \) we denote the pattern having \( X \) as its domain such that, for any \( v \in X \), if \( v \) is a node of \( C \) then \( \pi(C, X)(v) = N \) and if there is a hole at \( v \) in \( C \) then \( \pi(C, X)(v) = H \).

Let \( H_0, H_1, H_2 \) be the following subsets of \( S_w \) (see (2.1) for the definition of \( S_w \)):

\[
H_0 = \{(x, y) \mid 0 \leq x \leq w, 0 \leq y \leq w, x + y \leq w + 1\},
\]

\[
H_1 = \{(x, y) \mid 0 \leq x \leq \lfloor w/2 \rfloor + 1, 0 \leq y \leq w\},
\]

\[
H_2 = \{(x, y) \mid 0 \leq x \leq w, 0 \leq y \leq \lfloor w/2 \rfloor + 1\}.
\]

In Fig. 5.1 (a), (b), (c) we show examples of these sets. For each of them we show an example for an even \( w \) and for an odd \( w \). Dotted lines in the figures are vertical lines \( x = \lfloor w/2 \rfloor \) and horizontal lines \( y = \lfloor w/2 \rfloor \). In the determination of \( mft(C) \) of \( C \) of size \( w \), these special vertical and horizontal lines play important roles. Therefore, when we show \( S_w \) by figures we write dotted lines in these
Theorem 5.1 Suppose that \( C, C' \) are configurations of size \( w \) of \( SH[2] \) and \( \pi(C, H) = \pi(C', H) \) for one of \( H = H_0, H_1, H_2 \). Then we have \( C \equiv_{2w} C' \).

Proof. We consider the case \( H = H_0 \) and prove \( C \equiv_{2w} C' \). Let \( H'_0 \) be the set of positions

\[ H'_0 = \{ (x, y) \in H_0 \mid x + y \leq w \}. \]

Then \( H'_0 \subseteq H_0 \) and any position in \( S_w \) adjacent to a position in \( H'_0 \) is in \( H_0 \).

Let \( P \) be an arbitrary path in \( C \) of length at most \( 2w \) from \( v_{\text{gen}} (= (0, 0)) \) to the node \( (0, 0) \). Then for any node \( u = (x, y) \) on this path we have \( 2(x + y) = d_{\text{MH}}(v_{\text{gen}}, u) + d_{\text{MH}}(u, (0, 0)) \leq d_C(v_{\text{gen}}, u) + d_C(u, (0, 0)) \leq |P| \leq 2w \) and hence \( x + y \leq w \). Therefore the node \( u \) is in \( H'_0 \). Then, by \( \pi(C, H_0) = \pi(C', H_0) \), \( u \) is also a node in \( C' \). Let \( u' \) be any position in \( S_w \) adjacent to \( u \). Then \( u' \) is in \( H_0 \) and by \( \pi(C, H_0) = \pi(C', H_0) \), \( u' \) is a node in \( C \) if and only if \( u' \) is a node in \( C' \). This means that \( \text{bc}_C(u) = \text{bc}_{C'}(u) \).

Therefore, any path in \( C \) of length at most \( 2w \) from \( v_{\text{gen}} \) to \( (0, 0) \) is also a path in \( C' \) and the boundary condition of any node in the path is the same in \( C \) and \( C' \). Similarly we can prove the same statement with \( C, C' \) interchanged. Hence we have \( C \equiv_{2w, (0,0)} C' \).

For the cases \( H = H_1 \) and \( H = H_2 \) we show \( C \equiv_{2w, (w,0)} C' \) and \( C \equiv_{2w, (w,0)} C' \) respectively. Instead of \( 2(x + y) \leq 2w \) we use \( 2x + w \leq 2w \) (and hence \( x \leq \lfloor w/2 \rfloor \)) and \( w + 2y \leq 2w \) (and hence \( y \leq \lfloor w/2 \rfloor \)) respectively. \( \square \)
This theorem is useful for finding $C'$ such that $C \equiv 2w + 1 \ C'$ in applying Corollary 4.4. Next we show a characterization of configurations $C'$ of size $w$ such that $2w + 1 \leq \max_{v \in C'} T(v, C')$.

We call a hole at $v = (x_0, y_0)$ a critical hole if $|x_0 - y_0| = 2$ and by a critical pair (of holes) we mean a pair $v_0, v_1$ of two critical holes such that $v_1 = v_0 + (1, 1)$. In Fig. 5.2 we show examples of critical pairs.

![Figure 5.2: Examples of critical pairs](image)

$Figure 5.2: Examples of critical pairs $v_0 = (x_0, y_0), v_1 = (x_1, y_1)$.

**Theorem 5.2** Let $C$ be a configuration of size $w$ of $SH[2]$. The following two statements are equivalent.

1. $2w + 1 \leq \max_{v \in C} T(v, C)$.

2. $C$ has a critical pair of holes.

**Proof.** When a node $v$ of $C$ is in a maximal barrier $R$, by $S, p, \delta, z (= (z_x, z_y))$ we denote the items determined from $v, R$ by Figures 4.6, 4.8.

Suppose that the statement (1) is true. Let $v$ be a node in $C$ such that $T(v, C) > 2w$. Then by 4.14 there are four pairs $(S, p)$ such that $p \in S$; that is, $(S_4, (0, 0)), (S_4, (1, 1)), (S_5, (0, 1))$ and $(S_5, (1, 0))$. We have $E(S, p, \delta) = T(v, C) - 2w > 0$. Table 4.2 and a simple calculation show that there are two triples $(S, p, \delta)$ such that $p \in S$ and $E(S, p, \delta) > 0$ and they are $(S_5, (0, 1), 2)$ and $(S_5, (1, 0), -2)$.

Suppose that $S = S_5, p = (0, 1), \delta = 2$ correspond to $v, R$. Then there are two holes at $v_0 = z, v_1 = z + (1, 1)$ and $z_x - z_y = 2$. Therefore holes $v_0, v_1$ are critical and the pair $v_0, v_1$ is a critical pair. Similarly, for the case $S = S_5, p = (1, 0), \delta = -2$ too there are two holes at $v_0 = z, v_1 = z + (1, 1)$ and $z_x - z_y = -2$ and the pair $v_0, v_1$ is a critical pair. Hence the statement (2) is true for both cases.

Next, suppose that the statement (2) is true. Let a pair $v_0 = (x_0, y_0), v_1 = v_0 + (1, 1)$ be a critical pair. We consider only the case $x_0 - y_0 = 2$. Let $v$ be the position $(x_0, y_0 + 1)$. Then $v$ is in the maximal barrier $R$ consisting of the four positions $(x_0, y_0)$ (the hole $v_0), (x_0, y_0 + 1)$ (the node $v), (x_0 + 1, y_0)$ (a
node), $(x_0 + 1, y_0 + 1)$ (the hole $v_1$). Then we have $S = S_5$, $p = v - v_0 = (0, 1)$, 
$\delta = x_0 - y_0 = 2$. Therefore, by Table 4.2 we have $T(v, C) = 2w + E(S, p, \delta) = 2w + E(S_5, (0, 1), 2) = 2w + \min\{\delta - H - 1 + d_0, -\delta - W - 1 + d_1\} = 2w + \min\{2 - 2 - 1 + 2, -2 - 2 - 1 + 6\} = 2w + 1$ and the statement (1) is true. \[\square\]

When we prove $2w + 1 \leq \text{mft}(C)$ in the proof of the main theorem (Theorem 5.5) we prove this by showing existence of a sequence $C_0, \ldots, C_n$ of configurations of size $w$ $(n \geq 0)$ such that $C_0 = C$, $C_n$ has a critical pair of holes, and $\pi(C_i, H) = \pi(C_{i+1}, H)$ ($H$ is one of $H_0, H_1, H_2$) for any $0 \leq i \leq n - 1$.

5.2 Subsets $U$, $V$, $W$, $X$ of squares $S_w$

We define four nonoverlapping subsets $U$, $V$, $W$, $X$ of $S_w$ as follows. First we define $U \cup V$ and $U \cup V \cup W$ by

$$U \cup V = \{v \in S_w \mid \text{for any } v' \in S_w, d_{\text{MH}}(v_{\text{gen}}, v) + d_{\text{MH}}(v, v') \leq 2w\}, \quad (5.1)$$

$$U \cup V \cup W = \{v \in S_w \mid \text{for any } v' \in S_w \text{ there is } v'' \in S_w \text{ such that } d_{\text{MH}}(v, v'') \leq 1 \text{ and } d_{\text{MH}}(v_{\text{gen}}, v'') + d_{\text{MH}}(v'', v') \leq 2w\}. \quad (5.2)$$

We can define these two sets more explicitly as follows (see Fig. 5.3). For $U \cup V$ we define

$$U \cup V = \{(x, y) \in S_w \mid 0 \leq x \leq \lfloor w/2 \rfloor, 0 \leq x \leq \lfloor w/2 \rfloor\}. \quad (5.3)$$

For $U \cup V \cup W$ we have different definitions for an even $w$ and for an odd $w$. If $w$ is even then

$$U \cup V \cup W = \{(x, y) \in S_w \mid \text{either } 0 \leq x \leq \lfloor w/2 \rfloor \text{ and } 0 \leq y \leq \lfloor w/2 \rfloor, \text{ or } 0 \leq x \leq \lfloor w/2 \rfloor \text{ and } y = \lfloor w/2 \rfloor + 1, \text{ or } x = \lfloor w/2 \rfloor + 1 \text{ and } 0 \leq y \leq \lfloor w/2 \rfloor\} \quad (5.4)$$

and if $w$ is odd then

$$U \cup V \cup W = \{(x, y) \in S_w \mid 0 \leq x \leq \lfloor w/2 \rfloor + 1, 0 \leq y \leq \lfloor w/2 \rfloor + 1\}. \quad (5.5)$$

We define $U$ explicitly by

$$U = \{(x, y) \in S_w \mid 0 \leq x \leq \lfloor w/2 \rfloor - 1, 0 \leq y \leq \lfloor w/2 \rfloor - 1\} \quad (5.6)$$

and define $X$ by

$$X = S_w - U \cup V \cup W. \quad (5.7)$$

The equivalence of the two definitions (5.1), (5.3) of $U \cup V$ is easy to show. We show an outline of the proof of the equivalence of the first definition (5.2) and the second definition (5.4), (5.5) of $U \cup V \cup W$.

Suppose that $v = (x, y)$ is in $U \cup V \cup W$ of the first definition. We show that $v$ is in $U \cup V \cup W$ of the second definition. In this case $y \geq \lfloor w/2 \rfloor + 2$ is not possible
because for $v' = (w, 0)$ there does not exist $v''$ such that the statement of the first definition is true. Similarly $x \geq \lfloor w/2 \rfloor + 2$ is not possible. Moreover, when $w$ is even $x = y = \lfloor w/2 \rfloor + 1$ is not possible because for $v' = (0, 0)$ there does not exist $v''$ such that the statement of the first definition is true. Therefore, $v$ must be in $U \cup V \cup W$ of the second definition.

Next suppose that $v = (x, y)$ is in $U \cup V \cup W$ of the first definition. We show that $v$ is in $U \cup V \cup W$ of the first definition. We use the equivalence of the two definitions (5.1), (5.3) of $U \cup V$ to show it. If $v$ is in $U \cup V$ then the statement of the first definition is true because for any $v'$ we can use $v$ itself as $v''$. If $v$ is in $W$ and is adjacent to a position in $U \cup V$ then the statement of the first definition is true because for any $v'$ we can use the position in $U \cup V$ that is adjacent to $v$ as $v''$. Finally, if $v$ is odd and $v = (\lfloor w/2 \rfloor + 1, \lfloor w/2 \rfloor + 1)$ then the statement of the first definition is true because for any $v'$ we can use $v - (1, 0)$, $v - (0, 1)$, or $v$ respectively as $v''$ according as $x' \leq \lfloor w/2 \rfloor$, $y' \leq \lfloor w/2 \rfloor$, or otherwise respectively. Therefore, in any case $v$ is in $U \cup V \cup W$ of the first definition.

By $v_{\text{cut}}$ we denote the position $([\hat{w}/2], [\hat{w}/2])$ (“cnt” is for center). It is at the corner of $V$. We have the following relations among $U \cup V \cup W$, $H_0$, $H_1$, $H_2$.

- $U \cup V \cup W \subseteq H_0$ for both of even $w$ and odd $w$.
- $U \cup V \cup W = H_1 \cap H_2 - \{v_{\text{cut}} + (1, 1)\}$ for even $w$ and $U \cup V \cup W = H_1 \cap H_2$ for odd $w$.

$U \cup V$ is the set of positions $v$ such that $d_{\text{MH}}(v_{\text{gen}}, v) + d_C(v, v') \leq 2w$ for any position $v'$. Therefore, if $v$, $v'$ are nodes in a configuration $C$ such that $v \in U \cup V$ then we expect that $d_{\text{MH}}(v_{\text{gen}}, v) + d_C(v, v') \leq 2w$ is true except the case where $v'$ is near the four corners $(0, 0)$, $(0, w)$, $(w, 0)$, $(w, w)$. The following theorem is a precise statement of this intuitive statement.

**Theorem 5.3** Suppose that $w \geq 5$, $C$ is a configuration of size $w$ of $\text{SH}[2]$, and $v = (x, y)$, $v' = (x', y')$ are nodes in $C$ such that $v \in U \cup V$. Then $d_{\text{MH}}(v_{\text{gen}}, v) + d_C(v, v') \leq 2w$ except the following cases.
(1) Two holes are at $v+(0,1)$, $v+(1,0)$ and $v'$ is one of $(w−1,w)$, $(w,w−1)$, $(w,w)$.

(2) $x = \lfloor w/2 \rfloor$, two holes are at $v−(1,0)$, $v+(0,1)$, and $v'$ is one of $(0,w−1)$, $(1,w)$, $(0,w)$ if $w$ is even and is $(0,w)$ if $w$ is odd.

(3) $y = \lfloor w/2 \rfloor$, two holes are at $v−(0,1)$, $v+(1,0)$, and $v'$ is one of $(w−1,0)$, $(w,1)$, $(w,0)$ if $w$ is even and is $(w,0)$ if $w$ is odd.

(4) $w$ is even, $x = y = \lfloor w/2 \rfloor$, two holes are at $v−(0,1)$, $v−(1,0)$, and $v'$ is one of $(1,0)$, $(0,1)$, $(0,0)$.

In Fig. 5.4 we show examples of these four exceptional cases for an even $w$. A bullet denotes a position of $v$ and a small circle denotes a position of $v'$ of the four exceptions.

Figure 5.4: The four exceptions in Theorem 5.3 for an even $w$.

The proof of this theorem is not essentially difficult but is tedious. Therefore we show it in Appendix A.

5.3 An idea to construct partial solutions for proving upper bounds of $\text{mft}_{\text{SH}[k]}(C)$

Suppose that $\tilde{C}$ is a fixed configuration of $\text{SH}[k]$ of size $\tilde{w}$, $s (\geq 0)$ is a fixed constant, and we want to prove an upper bound $\text{mft}_{\text{SH}[k]}(\tilde{C}) \leq 2\tilde{w} + s$ of $\text{mft}_{\text{SH}[k]}(\tilde{C})$. 

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We prove it if we can construct a partial solution \( A \) of \( \text{SH}[k] \) that fires \( \tilde{C} \) at time \( 2\hat{w} + s \). Below we show one idea to construct such a partial solution \( A \). Assume that \( C \) is a configuration of \( \text{SH}[k] \) of size \( w \) and nodes of \( C \) are copies of \( A \).

\( A \) uses the size check messages \( W_0, W_1 \) that were used in the proofs of Theorems 4.1, 4.7 (see also the comment after Theorem 4.1). These messages are generated if and only if \( w = \hat{w} \). If \( W_0 \) or \( W_1 \) is generated then it is generated at \((0, w)\) or \((w, 0)\) respectively and at the time \( w \).

We define a pattern \( \pi \) such that \( C \) has \( \tilde{\pi} \). Moreover we define patterns \( \pi_{i,j} \), positions \( v_{i,j} \), values \( r_{i,j} \) \((\geq 0)\) for \( 0 \leq i \leq n - 1, 0 \leq j \leq m_i - 1 \) \((n \geq 0, m_i \geq 1)\) and design \( A \) so that all of the following five statements are true under the assumption that \( w = \hat{w} \).

\[
\begin{align*}
\text{(C1)} & \text{ If } C \text{ has } \tilde{\pi} \text{ then } \max_{v \in C} T(v, C) \leq 2\hat{w} + s. \\
\text{(C2)} & \text{ For any } i, C \text{ has } \tilde{\pi} \text{ if and only if } C \text{ has all of } \pi_{i,0}, \ldots, \pi_{i,m_i - 1} \((0 \leq i \leq n-1)\). \\
\text{(C3)} & \text{ For any } i, j, C \text{ has } \pi_{i,j} \text{ if and only if the message } M_{i,j} \text{ is generated } \((0 \leq i \leq n - 1, 0 \leq j \leq m_i - 1)\). \\
\text{(C4)} & \text{ For any } i, j, \text{ if } M_{i,j} \text{ is generated then it is generated at the node } v_{i,j} \text{ at time } d_{\text{MH}}(v_{\text{gen}}, v_{i,j}) + r_{i,j} \((0 \leq i \leq n - 1, 0 \leq j \leq m_i - 1)\). \\
\text{(C5)} & \text{ If } C \text{ has } \tilde{\pi} \text{ then for any node } v \text{ in } C \text{ there exists } i \((0 \leq i \leq n - 1)\) \text{ such that } d_{\text{MH}}(v_{\text{gen}}, v_{i,j}) + r_{i,j} + d_C(v_{i,j}, v) \leq 2\hat{w} + s \text{ for any } j \((0 \leq j \leq m_i - 1)\).
\end{align*}
\]

A node \( v \) of \( C \) fires if and only if the current time is \( 2\hat{w} + s \) and the following statement is true: the node has received at least one of \( W_0 \), \( W_1 \) before or at time \( 2\hat{w} + s \) and there is \( i \((0 \leq i \leq n - 1)\) \) such that the node has received all of \( M_{i,0}, \ldots, M_{i,m_i - 1} \) before or at \( 2\hat{w} + s \). From now on, we denote the above statement by a logical formula-like expression

\[
(W_0 \lor W_1) \land ((M_{0,0} \land \ldots \land M_{0,m_0-1}) \lor \ldots \lor (M_{n-1,0} \land \ldots \land M_{n-1,m_{n-1}-1})). \quad (5.8)
\]

This completes the explanation of our idea for constructing \( A \). In the following theorem we show that if we can successfully construct a finite automaton \( A \) using the above idea then \( A \) is a desired partial solution.

**Theorem 5.4** If \( A \) is a finite automaton constructed by the above idea for a configuration \( \tilde{C} \) of \( \text{SH}[k] \) of size \( \hat{w} \) then \( A \) is a partial solution of \( \text{SH}[k] \) that has the set

\[
X = \{ C \mid C \text{ is of size } \hat{w} \text{ and has } \tilde{\pi} \}
\]

as its domain and that fires any configuration \( C \) in the domain \( X \) (including \( \tilde{C} \)) at time \( 2\hat{w} + s \).

**Proof.** (1) We show that if \( C \) is in \( X \) then any node in it fires at time \( 2\hat{w} + s \). \( C \) is of size \( \hat{w} \). Hence all of (C1) – (C5) are true. Both of the two messages \( M_0, M_1 \) are generated because \( w = \hat{w} \). Moreover, by (C1) \( T(v, C) \leq 2\hat{w} + s \) for
any node \( v \) in \( C \). Hence any node \( v \) in \( C \) receives at least one of \( M_0, M_1 \) before or at \( 2 \hat{w} + s \).

\( C \) has \( \hat{\pi} \). Therefore by (C2) \( C \) has \( \pi_{i,j} \) for all \( i, j \) and hence by (C3), (C4) the message \( M_{i,j} \) is generated at \( v_{i,j} \) at time \( d_{MH}(v_{gen}, v_{i,j}) + r_{i,j} \) for all \( i, j \). Then, by (C5), for any node in \( C \) there is \( i \) such that the node receives all of the messages \( M_{i,0}, \ldots, M_{i,m_i-1} \) before or at time \( 2 \hat{w} + s \). Therefore, any node in \( C \) fires at time \( 2 \hat{w} + s \).

(2) We show that if a node \( v \) in \( C \) fires at some time then \( C \) is in \( X \). (This means that if \( C \) is not in \( X \) then any node in \( C \) never fires.)

The node \( v \) received at least one of \( M_0, M_1 \). Hence \( w = \hat{w} \) is true. Hence all of (C1) – (C5) are true. Moreover, for some \( i, v \) received all of \( M_{i,0}, \ldots, M_{i,m_i-1} \). \( C \) has all of \( \pi_{i,0}, \ldots, \pi_{i,m_i-1} \) by (C3), and \( C \) has \( \hat{\pi} \) by (C2). Therefore \( C \) is in \( X \).

In the following subsection we prove all the upper bounds \( \text{mft}_{\text{SH}[2]}(\tilde{C}) \leq 2 \hat{w} \) in the proof of the main result (Theorem 5.5) using the above idea with \( k = 2, s = 0 \).

### 5.4 The statement of the result and its proof

Now we are ready to state our main result.

**Theorem 5.5** Let \( \tilde{C} \) be a configuration of size \( \hat{w} \) of \( \text{SH}[2] \) and assume that \( \hat{w} \geq 11 \).

1. If one of the following is true then \( \text{mft}_{\text{SH}[2]}(\tilde{C}) = 2 \hat{w} + 1 \).
   - \( \tilde{C} \) has no holes in \( U \cup V \cup W \).
   - \( \tilde{C} \) has no holes in \( U \cup V \), has one hole in \( W \), and the hole is critical.
   - \( \tilde{C} \) has a critical pair of holes in \( U \cup V \cup W \).

   (See the five examples of such configurations for an even value of \( \hat{w} \) shown in Fig. 5.5)

2. Otherwise, \( \text{mft}_{\text{SH}[2]}(\tilde{C}) = 2 \hat{w} \).

**Proof.**

The statement of this theorem is very simple. However our proof for it is by a very detailed and tedious case analysis.

Part I: Proof of the statement (1).

By Corollary 4.14 and \( c_2 = 1 \) (Table 4.3), to prove \( \text{mft}_{\text{SH}[2]}(\tilde{C}) = 2 \hat{w} + 1 \) it is sufficient to prove \( \text{mft}_{\text{SH}[2]}(\tilde{C}) \geq 2 \hat{w} + 1 \).

(Case 1) \( \tilde{C} \) has no holes in \( U \cup V \cup W \).

(Case 1.1) There is no hole at \( v_{\text{cnt}} + (1,1) \) (this is always true for odd \( \hat{w} \)).
Figure 5.5: Five examples of configurations mentioned in the statement (1) of Theorem 5.5 for an even value of $\tilde{w}$ ($\tilde{w} = 12$). (a) $\tilde{C}$ has no holes in $U \cup V \cup W$. (b), (c) $\tilde{C}$ has no holes in $U \cup V$, has one hole in $W$ and the hole $((5,7)$ for (b) and $(7,5)$ for (c)) is critical. (d), (e) $\tilde{C}$ has a critical pair of holes $((4,6), (5,7)$ for (d) and $(4,2), (5,3)$ for (e)) in $U \cup V \cup W$. 
In this case the two holes are in $H_1 \cup H_2$. We consider the case where a hole $v_0$ is in $H_1 \cap H_2$ and another hole $v_1$ is in $H_1 \cap H_2$. (The proofs for other cases are simpler.) In Fig. 5.6 (a) we show an example of such $\tilde{C}$. For this $\tilde{C}$ we define three configurations $C_0$, $C_1$, $C_2$ shown in the figure (a).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.6.png}
\caption{Examples of configurations used in Case 1.1 and Case 1.2.}
\end{figure}

$C_0$ is $\tilde{C}$ itself. $C_1$ is obtained from $C_0$ by moving $v_0$ to $(\tilde{w} - 3, \tilde{w} - 1)$. Both of the old and the new positions of $v_0$ are in $H_2$ and hence we have $\pi(C_0, H_2) = \pi(C_1, H_2)$ and $C_0 \equiv_{2w} C_1$ by Theorem 5.1. $C_2$ is obtained from $C_1$ by moving $v_1$ to $(\tilde{w} - 4, \tilde{w} - 2)$. Both of the old and the new positions of $v_1$ are in $H_1$ because we assume $\tilde{w} \geq 11$, and hence we have $\pi(C_1, H_1) = \pi(C_2, H_1)$ and $C_1 \equiv_{2w} C_2$ by Theorem 5.1. For the last configuration $C_2$, it has a critical pair of holes and hence $2\tilde{w} + 1 \leq \max_{v \in C_2} T(v, C_2)$ by Theorem 5.2. Therefore we have $2\tilde{w} + 1 \leq \text{mft}(C_0) = \text{mft}(\tilde{C})$ by Corollary 4.4.

For other cases too we use the same reasoning. We define a sequence of configurations $C_0, \ldots, C_{n-1}$ such that $\tilde{C} = C_0$, $\pi(C_i, H) = \pi(C_{i+1}, H)$ for each $i$ ($0 \leq i \leq n-2$) and for some $H = H_0, H_1, H_2$, and $C_{n-1}$ has a critical pair of holes. Then using Theorem 5.1, Theorem 5.2, and Corollary 4.4 we can derive $2\tilde{w} + 1 \leq \text{mft}(\tilde{C})$. Therefore, we will show only the sequence $C_0, \ldots, C_{n-1}$.

(Case 1.2) There is a hole at $v_0 = v_{cm} + (1, 1)$ (this is possible only for even $\tilde{w}$).

We consider the case where the other hole $v_1$ is in $H_1 \cap H_2$. (The proofs for other cases are similar.)

In Fig. 5.6 (b) we show an example of such $\tilde{C}$ ($= C_0$) and the corresponding $C_1$, $C_2$. In this case we move $v_0$ to $(\tilde{w} - 3, \tilde{w} - 1)$ and move $v_1$ to $(\tilde{w} - 4, \tilde{w} - 2)$.
to obtain $C_1$ and $C_2$. We have $\pi(C_0, H_0) = \pi(C_1, H_0)$, $\pi(C_1, H_1) = \pi(C_2, H_1)$ and $C_2$ has a critical pair of holes.

(Case 2) $ \tilde{C} $ has no holes in $U \cup V$ but has one hole in $W$ and the hole is critical.

(Case 2.1) There is no hole at $v_{cnt} + (1, 1)$ (this is always true for odd $\tilde{w}$).

In Fig. 5.7 (a) we show an example of such $\tilde{C} (= C_0)$ and the corresponding $C_1, C_2$. We use $\pi(C_0, H_1) = \pi(C_1, H_1)$, $\pi(C_1, H_2) = \pi(C_2, H_2)$.

(Case 2.2) There is a hole at $v_{cnt} + (1, 1)$ (this is possible only for even $\tilde{w}$).

In Fig. 5.7 (b) we show an example of such $\tilde{C} (= C_0)$ and the corresponding $C_1$. We use $\pi(C_0, H_0) = \pi(C_1, H_0)$.

(Case 3) $ \tilde{C} $ has a critical pair of holes in $U \cup V \cup W$.

In this case we have $2\tilde{w} + 1 \leq \max_{v \in \tilde{C}} T(v, \tilde{C}) \leq mft(\tilde{C})$ by Theorem 5.2.

Part II: Proof of the statement (2).

In the statement of the theorem we include the assumption “$\tilde{w} \geq 11$.” In Part I this assumption was essentially used. However in the proofs in Part II we need only a weaker assumption “$\tilde{w} \geq 5$.” (We need it when we use Theorem 5.3.) Therefore, as example configurations $\tilde{C}$ we may use configurations of sizes $\tilde{w}$ smaller than 11 to save space.

By Theorem 1.1 to prove $mft_{SH[2]}(\tilde{C}) = 2\tilde{w}$ it is sufficient to prove $mft_{SH[2]}(\tilde{C}) \leq 2\tilde{w}$. We prove this by constructing a partial solution $A$ that fires $\tilde{C}$ at time $2\tilde{w}$ using the idea explained previously with $k = 2, s = 0$. 48
Let \( \#U \) denote the number of holes of \( \tilde{C} \) in \( U \), and similarly for \( \#V, \#W, \#X \). For a configuration \( \tilde{C} \), we call the 4-tuple \( (\#U, \#V, \#W, \#X) \) the type of \( \tilde{C} \). A type is a 4-tuple \( (a, b, c, d) \) of nonnegative integers such that \( a+b+c+d = 2 \). Hence there are ten types \((2, 0, 0, 0), (1, 1, 0, 0), \ldots, (0, 0, 0, 2)\).

Using this notation we can represent the three conditions in the statement (1) of the theorem as follows:

- The type of \( \tilde{C} \) is \((0, 0, 0, 2)\).
- The type of \( \tilde{C} \) is \((0, 0, 1, 1)\) and the unique hole in \( W \) is critical.
- The type of \( \tilde{C} \) is of the form \((a, b, c, 0)\) and \( \tilde{C} \) has a critical pair in \( U \cup V \cup W \).

Therefore, the conditions in the statement (2) (that is, “otherwise”) are as follow:

- The type of \( \tilde{C} \) is \((0, 0, 1, 1)\) and the unique hole in \( W \) is not critical.
- The type of \( \tilde{C} \) is \((0, 1, 1, 0)\) and \( \tilde{C} \) has no critical pairs in \( V \cup W \).
- The type of \( \tilde{C} \) is \((1, 1, 0, 0)\) and \( \tilde{C} \) has no critical pairs in \( U \cup V \).
- The type of \( \tilde{C} \) is \((2, 0, 0, 0)\) and \( \tilde{C} \) has no critical pairs in \( U \).
- The type of \( \tilde{C} \) is one of \((0, 1, 0, 1), (1, 0, 0, 1), (0, 0, 2, 0), (1, 0, 1, 0), (0, 2, 0, 0)\). (Note that in these cases \( \tilde{C} \) has no critical pairs.)

We merge these conditions to the following conditions:

- The type of \( \tilde{C} \) is of the form \((a, 0, c, d)\) \( (a \geq 1) \) and \( \tilde{C} \) has no critical pairs in \( U \).
- The type of \( \tilde{C} \) is of the form \((0, 0, c, d)\) \( (c \geq 1) \) and “\( c = 1 \) and the unique hole in \( W \) is critical” is false.
- The type of \( \tilde{C} \) is of the form \((0, b, c, d)\) \( (b \geq 1) \) and \( \tilde{C} \) has no critical pair in \( V \cup W \).
- The type of \( \tilde{C} \) is \((1, 1, 0, 0)\) and \( \tilde{C} \) has no critical pairs in \( U \cup V \).

(Case 1) The type of \( \tilde{C} \) is of the form \((a, 0, c, d)\) \( (a \geq 1) \) and \( \tilde{C} \) has no critical pairs in \( U \).

We construct a partial solution \( A \) that fires \( \tilde{C} \) at time \( 2\tilde{w} \) using \( \tilde{\pi} = \pi(\tilde{C}, U \cup V) \), \( r_{0,0} = \tilde{r}, v_{0,0} = v_{\text{cnt}}, r_{0,0} = 0 \). The firing rule is \( (W_0 \lor W_1) \land M_{0,0} \). More precisely, a node in a configuration \( C \) fires at a time if and only if the current time is \( 2\tilde{w} \) and it has received at least one of the two size check messages \( W_0, W_1 \) and also the message \( M_{0,0} \) before or at that time.

It is obvious that \( \tilde{C} \) has \( \tilde{\pi} \) because \( \tilde{\pi} = \pi(\tilde{C}, U \cup V) \). We prove that all of the five statements \((\text{C}1), \ldots, \text{(C}5)\) mentioned in the explanation of our idea for constructing partial solutions are true. The condition \( \text{(C}2) \) is true by our
definition of $\tilde{\pi}$, $\pi_{0,0}$. To prove (C1), (C3), (C4), (C5) we assume that $C$ is a configuration of size $\tilde{w}$.

The proof of (C1) is as follows. Let $(a,0,c,d)$ ($a \geq 1$) be the type of $\tilde{C}$. Suppose that $C$ has $\tilde{\pi}$. Then, by the definition $\tilde{\pi} = \pi(\tilde{C}, U \cup V)$, for any position $v$ in $U \cup V$ whether $v$ is a node or a hole is the same in $\tilde{C}$ and $C$. Therefore the type of $C$ is of the form $(a,0,c',d')$. If $a = 1$ then $C$ has one hole in $U$ and no holes in $V$. Therefore $C$ has an isolated hole and consequently $C$ has no critical pairs. If $a = 2$ then $C$ has two holes in $U$ but the pair of these two holes is not a critical pair because $\tilde{C}$ has no critical pairs in $U$. Therefore $C$ has no critical pairs. In both cases $C$ has no critical pairs and hence $\max_{v \in C} T(v,C) \leq 2 \tilde{w}$ by Theorem 5.2. This shows (C1).

Next we explain how to generate the message $M_{0,0}$ so that (C3), (C4) are true.

Suppose that $C$ has $\pi_{0,0}$ and copies of $A$ are placed in $C$ as nodes. Then $C$ has no holes in $V$. Therefore the part of $C$ in $U \cup V$ satisfies the definition of configurations of SH$[k]$ ($k \leq 2$). We can apply Theorem 4.10 to this configuration-like region and show the following for any position $v$ in $U \cup V$: (1) if $v$ is a node that is not in maximal barriers then it is on a path from $v_{\text{gen}}$ to $v_{0,0}$ of the MH distance length, and (2) if $v$ is a node or a hole in a maximal barrier then it is adjacent to a node $v'$ that is on such a path. (To prove (2) we essentially use the fact that maximal barriers of SH$[2]$ are of the forms $S_1, \ldots, S_5$ shown in Fig. 4.14. We cannot prove (2) for general SH$[k]$.)

Using the above fact we can design the finite automaton $A$ so that $A$ generates a finite number of signals such that

- each signal starts at $v_{\text{gen}}$ at time 0 and proceeds along one specific path from $v_{\text{gen}}$ to $v_{0,0}$ of the MH distance length with speed 1,
- if $C$ has $\pi_{0,0}$ then all of the signals arrive at $v_{0,0}$, and
- if $C$ has not $\pi_{0,0}$ then at least one of the signals vanishes and fails to arrive at $v_{0,0}$.

We define the rule to generate the message $M_{0,0}$ as follows: $M_{0,0}$ is generated at a node at a time if and only if the time is $2\tilde{w}$ and all of the above mentioned signals arrive at the node at the time. Then we can easily prove (C3), (C4) using $d_{\text{MH}}(v_{\text{gen}}, v_{0,0}) = 2\tilde{w}$.

We explain what signals to use for generating $M_{0,0}$ using the configuration $\tilde{C}$ shown in Fig. 5.8 as an example. This is a configuration of size 12 of type $(2,0,0,0)$ and the region $U \cup V$ has one maximal barrier $R$ consisting of the four positions $(1,3)$ (a node), $(1,4)$ (a hole), $(2,3)$ (a hole), $(2,4)$ (a node). The barrier is of the form $S_4$ in Fig. 4.14.

Fig. 5.9 (a) shows the pattern $\pi_{0,0}$ ($= \tilde{\pi} = \pi(\tilde{C}, U \cup V)$) and the position $v_{0,0}$ ($= v_{\text{cnt}}$). (From now on, to represent a pattern $\pi_{i,j}$ by a figure, we write a
Figure 5.8: An example configuration $\tilde{\mathcal{C}}$.

Figure 5.9: (a) The pattern $\pi_{0,0}$ ($= \tilde{\pi} = \pi(\tilde{\mathcal{C}}, U \cup V)$) and the position $v_{0,0}$ ($= v_{\text{cnt}}$). (b) The eight paths along which the eight signals travel.
circle or a cross at a position if the value of the pattern at the position is “N” (a node) or “H” (a hole) respectively. We also show the position $v_{i,j}$ by a bullet.

In Fig. 5.9 (b) we show eight paths from $v_{gen}$ to $v_{0,0}$ by arrow lines. We selected these paths so that (1) for any position in $U \cup V - R$ there is at least one path that passes it, (2) no paths enter $R$, and (3) each path goes only to the north and to the east so that it is of the MH distance length.

For each of these path a signal proceeds from $v_{gen}$ to $v_{0,0}$ along it. If a hole is on the path the signal vanished at the position. There are 8 positions that are not in $R$ but are adjacent to positions in $R$. They are $(0, 3)$, $(0, 4)$, $(1, 2)$, $(2, 2)$, $(1, 5)$, $(2, 5)$, $(3, 3)$, $(3, 4)$. If a signal is at one of these positions it checks that the position in $R$ adjacent to itself is a node or a hole according as the position in the pattern $\pi_{0,0}$ is a node or a hole. If the check fails the signal vanishes.

It is evident that if $C$ has $\pi_{0,0}$ then no signals vanish. If $C$ has not $\pi_{0,0}$ then either (1) there is at least one hole in $U \cup V - \{(x, y) \mid x = 0 \text{ or } y = 0\} - R$ or (2) the part $R$ is not a barrier of the form $S_4$. In both of the cases at least one signal vanishes.

Finally we prove the statement (C5). Suppose that $C$ has $\tilde{\pi}$. Let $\tilde{v}_0 = v_{0,0} + (1, 0)$, $\tilde{v}_1 = v_{0,0} + (0, 1)$, $\tilde{v}_2 = v_{0,0} + (-1, 0)$, $\tilde{v}_3 = v_{0,0} + (0, -1)$ be the four positions that are adjacent to $v_{0,0}$. Both of $v_2$, $v_3$ are nodes in $C$ because they are in $V$. At least one of $v_0$, $v_1$ is a node in $C$ because there is at least one holes in $U$ and there are exactly two holes. Therefore, we have none of the four exceptions when we use Theorem 5.3 for $C$ with $v = v_{0,0}$. Hence by this theorem we have $d_{\text{MH}}(v_{gen}, v_{0,0}) + d_C(v_{0,0}, v') \leq 2 \tilde{w}$ for any node $v'$ in $C$. Therefore (C5) is true.

This completes the proof of Case 1 of Part II. In the remainder of the proof of Part II we have many cases. However the proofs for these cases are similar to that of Case 1. Hence, from now on we only show

- an example configuration $C$,
- the pattern $\tilde{\pi}$ for that $C$,
- the patterns $\pi_{i,j}$ and the positions $v_{i,j}$ for that $C$, and
- the firing rule

To explain an idea to construct a partial solution. The value $r_{i,j}$ is 0 for all cases.

In all cases we define $\tilde{\pi}$ to be $\pi(C, Z)$ for some $Z \subseteq S_{\tilde{w}}$ and hence $\tilde{C}$ has $\tilde{\pi}$. We give proofs for the statements (C1), . . . , (C5) only when it is necessary. Usually we can prove them as follows. For (C1), we can easily prove that if $C$ has $\tilde{\pi}$ then $C$ has no critical pairs of holes. This proves (C1) by Theorem 5.2.

The “only if” part of (C2) is obvious. We can prove the “if” part of (C2) using our assumption that each configuration has exactly two holes. The design of signals to generate messages $M_{i,j}$ is obvious and the proofs of (C3), (C4) follow from the design of the signals.

In the proof of (C5) we use a value $D_{i,j}(C, v)$ defined by

$$D_{i,j}(C, v) = d_{\text{MH}}(v_{gen}, v_{i,j}) + d_C(v_{i,j}, v).$$
Suppose that the firing rule is \((W_0 \lor W_1) \land ((M_0 \land \ldots \land M_{0,m_0-1}) \lor \ldots \lor (M_{n-1,0} \land \ldots \land M_{n-1,m_{n-1}-1}))\). Then, for (C5) we must prove that if a configuration \(C\) of size \(\tilde{w}\) has \(\tilde{\pi}\) then for any node \(v\) in \(C\) there is \(i\) such that \(\max_{0 \leq j \leq m_i-1} D_{i,j}(C, v) \leq 2\tilde{w}\). Usually we can prove this by Theorem 5.3. However, in some cases (for example, in cases where \(v_{i,j}\) is not in \(U \cup V\)) we cannot use that theorem and we need ad hoc analyses of the value \(D_{i,j}(C, v)\).

(Case 2) The type of \(\tilde{C}\) is of the form \((0,0,c,d)\) \((c \geq 1)\) and “\(c = 1\) and the unique hole in \(W\) is critical” is false.

(Case 2.1) In \(\tilde{C}\), at least one of \(v_{\text{cnt}} + (0,1)\), \(v_{\text{cnt}} + (1,0)\) is a node.

(Case 2.1.1) \(\tilde{w}\) is even.

As the type of a configuration \(C\) we use \((a,b,(c_0,c_1),d)\) instead of \((a,b,c,d)\).

Here \(c_0\) and \(c_1\) are the number of critical holes of \(C\) in \(W\) and the number of noncritical holes of \(C\) in \(W\), respectively. The possible pairs \((c_0,c_1)\) are \((0,1)\), \((1,0)\), \((0,2)\), \((1,1)\), \((2,0)\). However at present we assume that “\(c = 1\) and the unique hole in \(W\) is critical” is false. Hence \((1,0)\) is excluded. Therefore the four pairs \((0,1), (0,2), (1,1), (2,0)\) are possible.

Suppose that the type of \(\tilde{C}\) is \((0,0,(c_0,c_1),d)\) and \((c_0,c_1)\) is one of \((0,1), (0,2), (1,1), (2,0)\). Then we can construct a partial solution \(A\) that fires \(\tilde{C}\) at \(2\tilde{w}\) by \(\tilde{\pi} = \pi(C, U \cup V \cup W)\), \(\pi_{0,0} = \tilde{\pi}, v_{0,0} = v_{\text{cnt}}\) and the firing rule \((W_0 \lor W_1) \land M_{0,0}\). In Fig. 5.10 we show an example configuration \(\tilde{C}\) of size 12 and \(\tilde{\pi}, \pi_{0,0}, v_{0,0}\) for this \(\tilde{C}\). \(C\) has one hole \((3,7)\) in \(W\) and it is a noncritical hole. Therefore the type of \(\tilde{C}\) is \((0,0,(0,1),1)\).

(C1) is true as follows. If \((c_0,c_1)\) is one of \((0,1), (0,2), (1,1)\) then \(C\) has at least one noncritical hole in \(W\). If \((c_0,c_1)\) is \((2,0)\) then \(C\) has two critical holes in \(W\) (that is, \(v_{\text{cnt}} + (-1,1), v_{\text{cnt}} + (1,-1)\)) but they cannot constitute a critical pair. Therefore, in both cases \(C\) has no critical pairs and hence \(\max_{v \in C} T(v, C) \leq 2\tilde{w}\) by Theorem 5.2.

It is easy to generate a message \(M_{0,0}\) so that (C3), (C4) are true. A finite number of signals like those shown in Fig. 5.9 (b) can check that \(C\) has no holes in \(U \cup V\). Let \(F_0, P_1\) be the paths from \(v_{\text{gen}}\) to \(v_{0,0}\) via \((0,\lfloor \tilde{w}/2 \rfloor)\) or via \((\lfloor \tilde{w}/2 \rfloor, 0)\) respectively of the MH distance length. Then each position in \(W\) is
adjacent to a position in \( P_0 \) or \( P_1 \). Therefore two signals that travel on these two paths can check that the distribution of holes in \( W \) in \( C \) is consistent with that specified by \( \tilde{\pi} \). Here we essentially use our assumption of Case 2.1.1 that \( \tilde{w} \) is even and hence the position \( v_{\text{cnt}} + (1, 1) \) is not in \( W \).

The proof of (C5) is as follows. \( C \) has no holes at \( v_{0,0} = (0, 1), v_{0,0} = (1, 0) \) because they are in \( V \). \( C \) has a node at at least one of \( v_{0,0} + (0, 1), v_{0,0} + (1, 0) \) by our assumption of Case 2.1. Therefore, we have none of the four exceptions when we apply Theorem 5.3 to \( C \) with \( v = v_{0,0} \) and this shows (C5).

(Case 2.1.2) \( \tilde{w} \) is odd.

When \( \tilde{w} \) is odd, \( W \) contains the position \( v_{\text{cnt}} + (1, 1) \) (see Fig. 5.3) and this position is not adjacent to any position in \( U \cup V \). This makes the design of the partial solution a little complicated.

By \( W' \) we denote the set \( W - \{ v_{\text{cnt}} + (1, 1) \} \). As the type of a configuration \( C \) we use \( (a, b, (c_0, c_1, c_2), d) \) instead of \( (a, b, c, d) \). Here, \( c_0 \) is the number of critical holes in \( W' \), \( c_1 \) is the number of noncritical holes in \( W' \), and \( c_2 \) is the number of noncritical holes in the set \( \{ v_{\text{cnt}} + (1, 1) \} \) (that is, \( c_2 \) is 0 or 1 according as the position \( v_{\text{cnt}} + (1, 1) \) is a node or a hole).

Suppose that the type of \( \tilde{C} \) is \((0, 0, (c_0, c_1, c_2), d)\). We have \( 0 \leq c_0 \leq 2, 0 \leq c_1 \leq 2, 0 \leq c_2 \leq 1, 1 \leq c_0 + c_1 + c_2 \leq 2 \). The possible triples \((c_0, c_1, c_2)\) are \((0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (0, 2, 0), (1, 1, 0), (2, 0, 0)\). However, \((1, 0, 0)\) is excluded by the same reason as in Case 2.1.1. Therefore, the seven triples \((0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (0, 2, 0), (1, 1, 0), (2, 0, 0)\) are possible.

We construct a partial solution \( \tilde{A} \) that fires \( \tilde{C} \) at \( 2\tilde{w} \) by \( \tilde{\pi} = \pi(\tilde{C}, U \cup V \cup W') \), \( \pi_{0,0} = \tilde{\pi}, v_{0,0} = v_{\text{cnt}} \) and the firing rule \((W_0 \lor W_1) \land M_{0,0}\). In Fig. 5.11 we show an example configuration \( \tilde{C} \) of size 13 and \( \tilde{\pi}, \pi_{0,0}, v_{0,0} \) for this \( \tilde{C} \). The type of \( \tilde{C} \) is \((0, 0, (0, 1, 0), 1)\). The hole at \((3, 7)\) is the unique hole in \( W' \) and it is not a critical hole.

Figure 5.11: An example configuration \( \tilde{C} \) and \( \tilde{\pi}, \pi_{0,0}, v_{0,0} \) for Case 2.1.2.1.

(C1) is shown as follows. Let \((0, 0, (c_0, c_1, c_2), d)\) be the type of \( \tilde{C} \). Suppose that \( C \) of size \( \tilde{w} \) has \( \tilde{\pi} \). If \((c_0, c_1, c_2)\) is one of \((0, 1, 0), (0, 1, 1), (0, 2, 0), (1, 1, 0)\)
then \( C \) has at least one noncritical hole in \( W' \) and hence \( C \) has no critical pairs. If \((c_0, c_1, c_2) = (2, 0, 0)\) then \( C \) has two critical holes in \( W' \) and hence \( C \) has no critical pairs. Therefore, in both cases we have (C1) by Theorem 5.2.

The generation of the message \( M_{0,0} \) and the proof of (C5) are completely the same as for Case 2.1.1.

Now there remain two cases \((c_0, c_1, c_2) = (1, 0, 1), (0, 0, 1)\). We already know that if \((c_0, c_1, c_2) = (1, 0, 0), (0, 0, 0)\) then \( \text{mft}(C) \geq 2\bar{w} + 1 \) (Part I of the proof). Therefore, in any partial solution for these two cases some signal must check that \( v\text{cnt} + (1, 1) \) is really a hole and hence \( v_{i,j} \) must be out of \( U \cup V \) for some \( i, j \).

(Case 2.1.2.2) \((c_0, c_1, c_2) = (1, 0, 1)\).

We construct a partial solution \( A \) by \( \tilde{\pi} = \pi(C, U \cup V \cup W) \), \( \pi_{0,0} = \pi_{1,0} = \tilde{\pi} \), \( v_{0,0} = v_{\text{cnt}} + (0, 1) \), \( v_{1,0} = v_{\text{cnt}} + (1, 0) \) and the firing rule \((W_0 \vee W_1) \land (M_{0,0} \vee M_{1,0})\).

In Fig. 5.12 we show \( \tilde{\pi}, \pi_{i,j}, v_{i,j} \) for an example configuration \( \tilde{C} \) of size 11.

![Diagram](https://example.com/diagram.png)

Figure 5.12: An example configuration \( \tilde{C} \) and \( \tilde{\pi}, \pi_{0,0}, v_{0,0}, \pi_{1,0}, v_{1,0} \) for \( \tilde{C} \) for Case 2.1.2.2.

(C1) is obvious because \( \tilde{\pi} \) has one critical hole \( (v_{\text{cnt}} + (-1, 1) \) or \( v_{\text{cnt}} + (1, -1) \) and one noncritical hole \( (v_{\text{cnt}} + (1, 1)) \) and hence if \( C \) has \( \tilde{\pi} \) then \( C \) has no critical pairs of holes.

To generate the message \( M_{0,0} \) satisfying (C3), (C4) we use the fact that each position in \( W \) (including \( v_{\text{cnt}} + (1, 1) \)) has an adjacent position in one of the following two paths: (1) \( v_{\text{gen}} \rightarrow (0, \lceil \bar{w}/2 \rceil) \rightarrow v_{\text{cnt}} \rightarrow v_{0,0} \), (2) \( v_{\text{gen}} \rightarrow (\lceil \bar{w}/2 \rceil, 0) \rightarrow v_{\text{cnt}} \rightarrow v_{0,0} \). The idea for generating \( M_{1,0} \) is similar. (The destinations of the two paths are \( v_{1,0} \) instead of \( v_{0,0} \).)

(C5) is the only step we need a proof. For the proof we cannot use Theorem 5.3 because \( v_{0,0}, v_{1,0} \) are not in \( U \cup V \). The firing rule is \((W_0 \vee W_1) \land (M_{0,0} \vee M_{1,0})\). Therefore we must prove that if \( C \) has \( \tilde{\pi} \) then for any \( v \in C \) either \( D_{0,0}(C, v) \leq 2\bar{w} \) or \( D_{1,0}(C, v) \leq 2\bar{w} \). (See 5.9 for the definition of \( D_{i,j}(C, v) \).) However, if \( C \) has \( \tilde{\pi} \) then \( C = \tilde{C} \) because \( \tilde{\pi} \) has two holes. Therefore the value of the two argument function \( D_{i,j}(C, v) \) depends only on \( v \). Using this, for example we can determine the value \( D_{0,0}(C, (\bar{w}, 0)) \) by \( D_{0,0}(C, (\bar{w}, 0)) = \lceil \bar{w}/2 \rceil + (\lceil \bar{w}/2 \rceil + 1) + (\lceil \bar{w}/2 \rceil + 1) + (\bar{w} - \lceil \bar{w}/2 \rceil) \) = \( 2\bar{w} + 1 \). By similar elementary calculation we have the following conclusion.

(a1) \( D_{0,0}(v) \leq 2\bar{w} \) except for \( v = (\bar{w}, 0) \) and
(a2) $D_{1,0}(v) \leq 2\tilde{w}$ except for $v = (0, \tilde{w})$.

This shows (C5).

(Case 2.1.2.3) $(c_0, c_1, c_2)$ is $(0, 0, 1)$.

We construct a partial solution $A$ by $\tilde{\pi} = \pi(\tilde{C}, U \cup V \cup W)$, $\pi_{0,0} = \pi_{1,0} = \tilde{\pi}$, $v_{0,0} = v_{\text{cnt}} + (0, 1)$, $v_{1,0} = v_{\text{cnt}} + (1, 0)$ and the firing rule $(W_0 \lor W_1) \land (M_{0,0} \lor M_{1,0})$. We show an example in Fig. 5.13.

Figure 5.13: An example configuration $\tilde{C}$ and $\tilde{\pi}, \pi_{0,0}, v_{0,0}, \pi_{1,0}, v_{1,0}$ for $\tilde{C}$ for Case 2.1.2.3.

For (C5) we must prove that if a configuration $C$ has $\tilde{\pi}$ then for any $v \in C$ either $D_{0,0}(C, v) \leq 2\tilde{w}$ or $D_{1,0}(C, v) \leq 2\tilde{w}$. In this case $D_{i,0}(C, v)$ depends on $C$. If $C$ has $\tilde{\pi}$ then $C$ has one hole at $v_{\text{cnt}} + (1, 1)$ and another hole at another position in $S_w - U \cup V \cup W$ which we will denote by $v_C$. We can determine the value $D_{i,0}(C, v)$ for each of the five possible relative relations between $v_C$ and $v_{i,0}$: (1) $v_C$ is northwest of $v_{i,0}$, (2) $v_C$ is north of $v_{i,0}$, (3) $v_C$ is northeast of $v_{i,0}$, (4) $v_C$ is east of $v_{i,0}$, (5) $v_C$ is southeast of $v_{i,0}$. Such analysis gives the same conclusion (a1), (a2) of Case 2.1.2.2 except two cases. The first exceptional case is when $v_C = v_{\text{cnt}} + (0, 2)$ (Fig. 5.14 (a)) and we have the following conclusion.

(b1) $D_{0,0}(C, v) \leq 2\tilde{w}$ except for $v = (\tilde{w}, 0), (\tilde{w} - 1, \tilde{w}), (\tilde{w}, \tilde{w} - 1), (\tilde{w}, \tilde{w})$ and $v_{i,0}$.

(b2) $D_{1,0}(C, v) \leq 2\tilde{w}$ except for $v = (0, \tilde{w})$.

Figure 5.14: Two exceptional cases in the proof of (C6) of Case 2.1.2.3.

The second exceptional case is when $v_C = v_{\text{cnt}} + (2, 0)$ (Fig. 5.14 (b)) and we have the following conclusion.
(c1) \(D_{0,0}(C, v) \leq 2\bar{w}\) except for \(v = (\bar{w}, 0)\) and

(c2) \(D_{1,0}(C, v) \leq 2\bar{w}\) except for \(v = (0, \bar{w}), (\bar{w} - 1, \bar{w}), (\bar{w}, \bar{w} - 1), (\bar{w}, \bar{w})\).

All of the conclusions (a1), (a2), (b1), (b2), (c1), (c2) show (C5).

(Case 2.2) \(\bar{C}\) has the two holes at \(v_{\text{cut}} + (0, 1), v_{\text{cut}} + (1, 0)\).

First we consider the case where \(\bar{w}\) is even and later we explain how to modify the proof for odd \(\bar{w}\). For each \(\bar{w}\) there is only one configuration \(\bar{C}\) of this form. Its type is \((0, 0, 2, 0)\). In Fig. 5.15 (a) we show the configuration for \(\bar{w} = 10\).

\[\begin{align*}
\bar{C} \\
\bar{\pi} = \pi_{0,0} = \pi(\bar{C}, U \cup V \cup W), \\
v_{0,0} = v_{\text{cut}}
\end{align*}\]

Figure 5.15: (a) An example configuration \(\bar{C}\) of Case 2.2. (b) One idea for constructing a partial solution \(A\) for \(\bar{C}\) uses these patterns and so on but it fails.

One idea to construct a partial solution \(A\) for \(\bar{C}\) is to use \(\bar{\pi} = \pi(\bar{C}, U \cup V \cup W)\), \(\pi_{0,0} = \bar{\pi}\), \(v_{0,0} = v_{\text{cut}}\) and the firing rule \((W_0 \lor W_1) \land M_{0,0}\). We show these patterns and so on for the example configuration \(\bar{C}\) in Fig. 5.15 (b). However, for this \(A\) we cannot prove (C5) because for a configuration \(C\) of size \(\bar{w}\) that has \(\bar{\pi}\), we have \(D_{0,0}(C, v) > 2\bar{w}\) for the three nodes \(v = (\bar{w} - 1, \bar{w}), (\bar{w}, \bar{w} - 1), (\bar{w}, \bar{w})\) (the small circles in Fig. 5.15 (b)). (We have the exception (1) in Theorem 5.3.)

Instead of the above idea we use another idea for constructing the desired partial solution \(A\). In Fig. 5.16 we show patterns and nodes \(\bar{\pi} (= \pi(C, U \cup V \cup W)), \pi_{0,0} (= \bar{\pi}), \pi_{1,0}, \pi_{1,1}, v_{0,0} (= v_{\text{cut}}), v_{1,0} (= v_{\text{cut}} + (-1, 1)), v_{1,1} (= v_{\text{cut}} + (1, -1))\) used for \(A\). The firing rule is \((W_0 \lor W_1) \land (M_{0,0} \lor (M_{1,0} \land M_{1,1}))\).

The message \(M_{0,0}\) implies that \(C\) has \(\bar{\pi}\). Each of the two messages \(M_{1,0}, M_{1,1}\) has only a partial information on \(C\). However the two messages as a whole imply that \(C\) has \(\bar{\pi}\). Therefore a node knows that \(C\) has \(\bar{\pi}\) if either (i) it receives \(M_{0,0}\) or (ii) it receives both of \(M_{1,0}, M_{1,1}\). Hence the main part of the firing rule is \(M_{0,0} \lor (M_{1,0} \land M_{1,1})\).

We can prove the “if” part of (C2) for \(\pi_{1,0}, \pi_{1,1}\) using the fact that any \(C\) has exactly two holes. The message \(M_{1,0}\) is generated by the signal that goes from \(v_{\text{gen}}\) to \(v_{1,0}\) via \((0, \lfloor\bar{w}/2\rfloor + 1)\) and similarly for \(M_{1,1}\).
Figure 5.16: The patterns and so on we use for the example configuration $\tilde{C}$ shown in Fig. 5.15 (a).

The proof of (C5) is as follows. We must prove that if $C$ of size $\tilde{w}$ has $\tilde{\pi}$ then for any $v \in C$ either (1) $D_{0,0}(C, v) \leq 2\tilde{w}$ or (2) $D_{1,0}(C, v) \leq 2\tilde{w}$ and $D_{1,1}(C, v) \leq 2\tilde{w}$. However, if $C$ has $\tilde{\pi}$ then $C = \tilde{C}$. Using this we can show the following:

- $D_{0,0}(C, v) \leq 2\tilde{w}$ except for $v = (\tilde{w} - 1, \tilde{w})$ (Theorem 5.3, exception (1));
- $D_{1,0}(C, v) \leq 2\tilde{w}$ and $D_{1,1}(C, v) \leq 2\tilde{w}$ for $v = (\tilde{w} - 1, \tilde{w})$, $(\tilde{w}, \tilde{w} - 1)$, $(\tilde{w}, \tilde{w})$ (among others) (by elementary calculation and our assumption $\tilde{w} \geq 11$).

This shows (C5).

The proof for odd $\tilde{w}$ is obtained from the above proof by changing the definition of $\tilde{\pi}$ to $\tilde{\pi} = \pi(C, U \cup V \cup (W - \{v_{\text{cnt}} + (1, 1)\}))$ (see $\tilde{\pi}$ shown in Fig. 5.11).

(Case 3) The type of $\tilde{C}$ is of the form $(0, b, c, d)$ ($b \geq 1$) and $\tilde{C}$ has no critical pairs in $V \cup W$.

Case 3 is a “shrunk” version of Case 2 for odd $\tilde{w}$. In Case 3, $U$, $V$, $v_{\text{cnt}}$ (the corner of $V$) play the roles of $U \cup V$, $W$, $v_{\text{cnt}} + (1, 1)$ (the corner of $W$) respectively of Case 2 for odd $\tilde{w}$. However the proofs for Case 3 are easier than those for Case 2 for odd $\tilde{w}$ because $v_{\text{cnt}} + (1, 1)$ is out of $U \cup V$ but $v_{\text{cnt}}$ is in $U \cup V$.

The proofs in Case 3 are independent of whether $\tilde{w}$ is even or odd. As example configurations we show configurations with even $\tilde{w}$.

(Case 3.1) In $\tilde{C}$, at least one of $v_{\text{cnt}} - (1, 0)$, $v_{\text{cnt}} - (0, 1)$ is a node.

By $V'$ we denote the set $V - \{v_{\text{cnt}}\}$. As the type of a configuration $C$ we use $(a, (b_0, b_1, b_2), c, d)$ instead of $(a, b, c, d)$. Here, $b_0$ is the number of critical holes in $V'$, $b_1$ is the number of noncritical holes in $V'$, and $b_2$ is the number of noncritical holes in the set $\{v_{\text{cnt}}\}$ (that is, $b_2$ is 0 or 1 according as the position $v_{\text{cnt}}$ is a node or a hole). As was in Case 2.1.2, the eight triples $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, $(0, 2, 0)$, $(1, 1, 0)$, $(2, 0, 0)$ are possible as $(b_0, b_1, b_2)$.

(We do not exclude $(1, 0, 0)$.)
Suppose that the type of $\tilde{C}$ is $(0, (b_0, b_1, b_2), c, d)$ and $(b_0, b_1, b_2)$ is one of $(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (0, 2, 0), (1, 1, 0), (2, 0, 0)$ (that is, we ignore $(1, 0, 0)$). Then we can construct a desired partial solution using the idea we used in Case 2.1.2 for the type $(0, (b_0, b_1, b_2), d)$. The idea used in Fig. 5.11 can be used for $(0, 1, 0), (0, 1, 1), (0, 2, 0), (1, 1, 0), (2, 0, 0)$ (that is, cases where either $V'$ contains at least one noncritical hole or $V''$ contains two critical holes). The idea used in Fig. 5.12 can be used for $(1, 0, 1)$ and that used in Fig. 5.13 can be used for $(0, 0, 1)$.

We show an example for the type $(0, (1, 0, 1), c, d)$ in Fig. 5.17. (Compare this figure with Fig. 5.12, an example for the type $(0, 0, (1, 0, 1), 0)$.) In this case we use $\tilde{\pi} = \pi(\tilde{C}, U \cup V), \pi_{0,0} = \tilde{\pi}, v_{0,0} = v_{\text{cnt}} - (1, 0), v_{1,0} = v_{\text{cnt}} - (0, 1)$. The firing rule is $(W_0 \lor W_1) \land (M_{0,0} \lor M_{1,0})$. The proof of (C5) is simpler. (We can use Theorem 5.3 because both of $v_{0,0}, v_{1,0}$ are in $U \cup V$.)

Now there remains the case $(b_0, b_1, b_2) = (1, 0, 0)$. The following two statements are equivalent:

- The type of $\tilde{C}$ is $(0, (1, 0, 0), c, d)$ and $\tilde{C}$ has no critical pairs in $V \cup W$.
- The type of $\tilde{C}$ is $(0, (1, 0, 0), c, d)$ and $v_0 + (1, 1)$ is a node. Here $v_0$ is the unique critical hole in $V'$.

Using the latter statement as the definition of the present case we can construct a partial solution. In Fig. 5.18 we show an example. In this case we use $\tilde{\pi} = \pi(\tilde{C}, U \cup V \cup \{v_{\text{cnt}} + (-1, 1)\}), \pi_{0,0} = \tilde{\pi}, v_{0,0} = v_{\text{cnt}} - (1, 0)$. (We assume that $v_{\text{cnt}} - (2, 0)$ is the unique critical hole in $V'$.) The firing rule is $(W_0 \lor W_1) \land M_{0,0}$. 

Figure 5.17: An example for the type of the form $(0, (1, 0, 1), c, d)$.

Figure 5.18: An example for the type $(0, (1, 0, 0), c, d)$. 

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(Case 3.2) \( \bar{C} \) has the two holes at \( v_{cnt} - (1, 0), v_{cnt} - (0, 1) \).

The idea for constructing a partial solution is completely the same as for Case 2.2. (Consider to modify the idea shown in Fig. 5.16.)

(Case 4) The type of \( \bar{C} \) is \((1, 1, 0, 0)\) and \( \bar{C} \) has no critical pairs in \( U \cup V \).

Let \( v_0 = (x_0, y_0) \) be the unique hole in \( U \) and \( v_1 = (x_1, y_1) \) be the unique hole in \( V \). We assume that \( v_1 \) is in the horizontal part of \( V \) (that is, \( y_1 = \lfloor \tilde{w}/2 \rfloor \)).

(Case 4.1) \( v_1 \neq v_0 + (1, 1) \).

In Fig. 5.19 we show an example configuration \( \bar{C} \) of this case and \( \bar{\pi}, \pi_{0,0}, v_{0,0} (= v_0 - (0, 1)), \pi_{0,1}, v_{0,1} (= v_1 - (1, 0)) \) used to construct a partial solution \( A \) for \( \bar{C} \). The firing rule is \((W_0 \lor W_1) \land (M_{0,0} \land M_{0,1})\). The message \( M_{0,0} \) is generated by a signal that goes from \( v_{gen} \) to \( v_{0,0} \) via \((x_0, 0)\) and the message \( M_{0,1} \) is generated by a signal that goes from \( v_{gen} \) to \( v_{0,1} \) via \((0, \lfloor \tilde{w}/2 \rfloor)\). We can prove all of the statements \((C1), \ldots, (C5)\) for \( A \). (Note that if \( v_1 = v_0 + (1, 1) \) we cannot prove \((C5)\) because we have the exception (3) when we use Theorem 5.3 with \( v = v_{0,1} \).)

![Figure 5.19: An example configuration \( \bar{C} \) of Case 4.1 and \( \bar{\pi}, \pi_{0,0}, v_{0,0}, \pi_{0,1}, v_{0,1} \) for it.](image)

(Case 4.2) \( v_1 = v_0 + (1, 1) \).

In Fig. 5.20 we show an example configuration \( \bar{C} \) of this case and \( \bar{\pi}, \pi_{0,0}, v_{0,0} (= v_0 + (1, 0)), \pi_{1,0}, v_{1,0} (= v_1 - (1, 0)) \) used to construct a partial solution \( A \) for \( \bar{C} \). The firing rule is \((W_0 \lor W_1) \land (M_{0,0} \lor M_{1,0})\). We can easily prove \((C1), \ldots, (C4)\).

The proof of \((C5)\) is as follows. We must show that either \( D_{0,0}(C,v) \leq 2\tilde{w} \) or \( D_{1,0}(C,v) \leq 2\tilde{w} \) for any \( v \in C \). Using Theorem 5.3 we have the following estimations of \( D_{i,0}(C,v) \).

- \( D_{0,0}(C,v) \leq 2\tilde{w} \) except for \((0, \tilde{w} - 1), (1, \tilde{w}), (0, \tilde{w})\) when \( \tilde{w} \) is even and except for \((0, \tilde{w})\) when \( \tilde{w} \) is odd.
- \( D_{1,0}(C,v) \leq 2\tilde{w} \) except for \((\tilde{w} - 1, 0), (\tilde{w}, 1), (\tilde{w}, 0)\) when \( \tilde{w} \) is even and except for \((\tilde{w}, 0)\) when \( \tilde{w} \) is odd.

Therefore \((C5)\) is true. \( \square \)
Figure 5.20: An example configuration $\tilde{C}$ of Case 4.2 and $\tilde{\pi}$, $\pi_{0,0}$, $v_{0,0}$, $\pi_{1,0}$, $v_{1,0}$ for it.

6 Discussions and conclusion

We introduced a variation $\text{SH}[k]$ of FSSP and showed some results on it. We summarized the results in Section 4. The most important of them are the following:

- We showed a minimal-time solution of $\text{SH}[1]$ (Theorem 3.1).
- We determined the value $\text{mft}_{\text{SH}[2]}(C)$ (Theorem 5.5).

In the remainder of this section we list open problems and give some comments on them.

First we list two rather technical open problems on $\text{SH}[k]$ for the general values of $k$.

- Prove $c_k = k - 2$ for $k \geq 10$. To prove this is equivalent to prove that the barriers shown in Fig. 4.12 (a), (b) give the maximum value of $E_{\text{max}}(S, p)$ for $S \in S_k$, $p \in S$.
- For any $k \geq 4$, any $s$ such that $0 \leq s \leq c_k$ and any sufficiently large $\tilde{w}$, prove that there exists a configuration $\tilde{C}$ of size $\tilde{w}$ of $\text{SH}[k]$ such that $\text{mft}_{\text{SH}[k]}(\tilde{C}) = 2\tilde{w} + s$. We proved this only for $s = 0$ (Theorem 4.1 (2)) and $s = c_k$ (Theorem 4.7 (2), Theorem 4.13 (2)).

Our ultimate goal is either to construct a minimal-time solution of $\text{SH}[k]$ or to prove its nonexistence for each $k \geq 2$. The author has the conjecture that $\text{SH}[k]$ has a minimal-time solution for all $k$.

We have two problems to start with to realize the above goal. The first is to construct a minimal-time solution of $\text{SH}[2]$. We already know the value of $\text{mft}_{\text{SH}[2]}(C)$. Hence our goal is reduced to construct a solution that fires $C$ before or at this time.

The second is to determine the value of $\text{mft}_{\text{SH}[k]}(C)$ for general values of $k \geq 3$ or for some specific values $k = 3$, $k = 4$, $\ldots$. We determined this value for $k = 2$ in Theorem 5.5. However its derivation essentially used the fact $k = 2$ and it cannot be readily modified for $k \geq 3$.

By analyzing the proof of Theorem 5.5 we notice that one of the difficult parts in it is to estimate the range of the value $d_C(v, v')$ for two positions $v, v'$.
in a situation where we have only partial information on $C$ (such as “$C$ has a pattern $\pi$”). For the proof of Theorem 5.5 we derived the range by an ad hoc analysis essentially using the fact $k = 2$ exemplified in Fig. [A.4]. However this type of ad hoc analyses becomes very difficult even for $k = 3$ and nearly impossible for $k = 4$. Hence, in order to derive the value of $\text{mft}_{\text{SH}[k]}(C)$ for $k \geq 3$ using our idea, it is mandatory to construct a theory for estimating $d_C(v,v')$ with partial information on $C$. This is itself an interesting problem in discrete mathematics.

Although the author has the conjecture that $\text{SH}[k]$ has minimal-time solutions for all $k$, we should include the problem to prove nonexistence of minimal-time solutions of $\text{SH}[k]$ for some $k$ in the list of open problems.

This completes our list of open problems and comments on $\text{SH}[k]$. Finally we give more general comments on variations of FSSP restricting ourselves to the problem to know existence/nonexistence of minimal-time solutions and to find one if they exist.

For the two variations ORG, SQ we found minimal-time solutions long ago and by slightly modifying them we obtain very difficult variations 2PATH, 3PATH, gSQ, $\text{SH}[k]$. Hence generally speaking variations are very difficult and we know minimal-time solutions only for some rare simple cases. This observation leads us to the following research themes.

(1) To study these and other difficult variations and develop basic tools to solve these variations.

(2) To construct a general theory of variations of FSSP.

These two research themes are not independent and development in one will contribute to development of the other.

As for (2) we are not sure whether such a general theory exists. However we have at least one encouraging result. We know that there is an algorithm for computing $\text{mft}_\Gamma(C)$ for any natural variation $\Gamma$ and it is based on one simple idea “if a node of a configuration $C$ fires as soon as it finds no reasons not to fire then the node fires at the time $\text{mft}_\Gamma(C)$” ([12]). This means that the value $\text{mft}_\Gamma(C)$ is determined by a simple mechanism that is common for all natural variations $\Gamma$. This result is one that should be in the above mentioned general theory.

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Appendix

A Proof of Theorem 5.3

In this section we prove Theorem 5.3. First we prove two lemmas.

Lemma A.1 Suppose that $w \geq 5$, $C$ is a configuration of $SH[2]$, $v = (x, y)$, $v' = (x', y')$ are nodes of $C$ such that $v \in U \cup V$, either $|x - x'| \leq 1$ or $|y - y'| \leq 1$, and $d_C(v, v') \leq d_{MH}(v, v') + 2$. Then $d_{MH}(v_{gen}, v) + d_C(v, v') \leq 2w$.

Proof. In Fig. A.1 we show possible positions of $v'$ by shadow. We prove only the case $|x - x'| \leq 1$. We have $y + |y' - y| \leq w$ because if $y \leq y'$ then

![Figure A.1: Possible positions of $v'$ in Lemma A.1](attachment:image.png)
Lemma A.2 Suppose that the followings are true.

1. $C$ is a configuration of size $w$ of SH[2].

2. $v_0 = (x_0, y_0)$, $v_1 = (x_1, y_1)$ are nodes in $C$ such that $v_0 \in U \cup V$.

3. There is a path in $C$ from $v_0$ to $v_1$ of the MH distance length.

4. $H$ is a set of positions in $S_w$ of the form of a rectangle (that is, there are two positions $v_2 = (x_2, y_2)$, $v_3 = (x_3, y_3)$ in $S_w$ such that $x_2 \leq x_3$, $y_2 \leq y_3$, $H = \{(x, y) \mid x_2 \leq x \leq x_3, y_2 \leq y \leq y_3\}$).

5. There are no holes on the boundary of the rectangle $H$.

6. One of the followings is true:
   - $v_0$ is southwest of $v_1$ and $v_1$ is the southwest corner $(x_2, y_2)$ of $H$.
   - $v_0$ is southeast of $v_1$ and $v_1$ is the southeast corner $(x_3, y_2)$ of $H$.
   - $v_0$ is northeast of $v_1$ and $v_1$ is the northeast corner $(x_3, y_3)$ of $H$.
   - $v_0$ is northwest of $v_1$ and $v_1$ is the northwest corner $(x_2, y_3)$ of $H$.

Then $d_{MH}(v_{gen}, v_0) + d_C(v_0, v) \leq 2w$ for any node $v$ in $H$.

Proof. Let $v_4 = (x_4, y_4)$ be the corner of $H$ that is the opposite of $v_1$ (that is, $|x_1 - x_4| = |x_2 - x_3|$, $|y_1 - y_4| = |y_2 - y_3|$). Fig. A.2 shows an example of $C$, $v_0$, $v_1$, $v_4$, $H$ and a path from $v_0$ to $v_1$ in $C$ of the MH distance length (the bent line). In this figure, a circle represents a node of $C$. First we prove $d_C(v_1, v) \leq d_{MH}(v_1, v_4)$ for any node $v$ in $H$.

By (4), (5), $H$ is a configuration of SH[k] with $k \leq 2$ except that its shape is not necessarily a square. We can naturally define maximal barriers of this configuration-like region $H$ and apply Theorem 4.10 to them.

Suppose that $v$ is a node in $H$. By Theorem 4.10, if $v$ is not in maximal barriers of $H$ then we have $d_H(v_1, v) = d_{MH}(v_1, v) \leq d_{MH}(v_1, v_4)$. If $v$ is in a maximal barrier $R$ of $H$ then $v$ is adjacent to a node $v'$ that is in $H$ but is not in maximal barriers of $H$. This is because the maximal barrier $R$ must be of the forms $S_4$ or $S_5$ shown in Fig. 4.14. Moreover this node $v'$ cannot be the corner $v_4$ by (5). Therefore we have $d_H(v_1, v) \leq d_H(v_1, v') + 1 = d_{MH}(v_1, v') + 1 \leq d_{MH}(v_1, v_4)$. Therefore we have proved $d_C(v_1, v) \leq d_H(v_1, v) \leq d_{MH}(v_1, v_4)$ for any node $v$ in $H$.

We have $x_0 + |x_4 - x_0| \leq w$ because if $x_0 \leq x_4$ then $x_0 + |x_4 - x_0| = x_4 \leq w$ and if $x_4 < x_0$ then $x_0 + |x_4 - x_0| = 2x_0 - x_4 \leq 2x_0 \leq 2|w/2| \leq w$. Similarly we have $y_0 + |y_4 - y_0| \leq w$.

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Then, for any node $v$ in $H$,
\[
\begin{align*}
d_{MH}(v_{gen}, v_0) + d_C(v_0, v) &\leq d_{MH}(v_{gen}, v_0) + d_C(v_0, v_1) + d_C(v_1, v) \\
&\leq d_{MH}(v_{gen}, v_0) + d_{MH}(v_0, v_1) + d_{MH}(v_1, v_4) \\
&= d_{MH}(v_{gen}, v_0) + d_{MH}(v_0, v_4) \quad \text{(by (6))} \\
&= (x_0 + y_0) + (|x_4 - x_0| + |y_4 - y_0|) \\
&= (x_0 + |x_4 - x_0|) + (y_0 + |y_4 - y_0|) \\
&\leq 2w.
\end{align*}
\]

Now we prove Theorem 5.3. Suppose that $w \geq 5$, $C$ is a configuration of size $w$ of $SH[2]$, $v = (x, y) \in U \cup V$, and $v' = (x', y') \in C$. We denote the value $d_{MH}(v_{gen}, v) + d_C(v, v')$ by $F$. Our goal is to prove $F \leq 2w$ except the four cases mentioned in the theorem.

We divide $S_w$ into four partially overlapping regions $Q_I, \ldots, Q_{IV}$ which we call quadrants by analogy with quadrants in plane geometry: $Q_I = \{(x', y') \in S_w \mid x \leq x', y \leq y'\}$, $Q_{II} = \{(x', y') \in S_w \mid x' \leq x, y \leq y'\}$, $Q_{III} = \{(x', y') \in S_w \mid x \leq x', y' \leq y\}$, $Q_{IV} = \{(x', y') \in S_w \mid x \leq x', y' \leq y\}$.

First we prove the theorem for the case $v' \in Q_I$. Later we explain how to modify the proof for other cases.

We define five subsets $I_0$, $J_0$, $I_1$, $J_1$, $K$ of $Q_I$. We show them in Fig. A.3 assuming $Q_I$ has six columns and seven rows. We assume that $x \leq w - 2$, $y \leq w - 2$ so that the set $Q_I$ has at least three columns and at least three rows and hence all of these five subsets are well-defined. We can easily prove the theorem for the case $w - 1 \leq x \leq w$ and/or $w - 1 \leq y \leq w$ by modifying the ideas used in the following proof.

We consider five cases. These cases are not necessarily disjoint. By $\#I_0$ and so on we denote the number of holes in $I_0$ and so on. In Fig. A.4 (a) – (e)
we show situations for these five cases. In these figures a bullet represents the position of $v$ and a circle represents a node.

(Case 1) $\#I_0 = \#J_0 = 0$. In this case $Q_I$ itself is a rectangle whose boundary has no holes. Therefore, if we select $H = Q_I$, $v_0 = v_1 = v$ then we can use Lemma A.2 to show $F \leq 2w$ for $v' \in Q_I$.

(Case 2) Either $\#I_1 = \#J_0 = 0$ or $\#I_0 = \#J_1 = 0$. The two sub cases are symmetric and we consider the first sub case.

In this sub case, if we select $H = I_1 \cup J_0 \cup J_1 \cup K$, $v_0 = v$, $v_1 = v + (1, 0)$ then we can use Lemma A.2 to show $F \leq 2w$ for $v' \in H$. For $v' = v + (0, s) \in I_0$ ($1 \leq s \leq w - y$), we have a path $v \rightarrow v + (1, 0) \rightarrow v + (1, s) \rightarrow v + (0, s) = v'$ from $v$ to $v'$ having length $d_{\text{MH}}(v, v') + 2$ (shown by arrow lines in Fig. A.4 (b)) and hence $d_C(v, v') \leq d_{\text{MH}}(v, v') + 2$. Moreover we have $|x - x'| = 0$. Therefore we can use Lemma A.1 to show $F \leq 2w$.

(Case 3) $\#I_1 = \#J_1 = 0$ and either $v + (0, 1)$ is a node or $v + (1, 0)$ is a node. We consider the first sub case.
Similarly, for any \( F \) we can use Lemma A.2 to show we have \( d \) implies that for any \( v \) that \( x \) \( y \) we have \( |x - x'| = 0 \) or \( |y - y'| = 0 \). Therefore we can use Lemma A.1 to show \( F \leq 2w \).

(Case 4) The two positions \( v + (0, 1), v + (1, 0) \) have holes. In this case these two holes are all the holes in \( C \).

There is a column west of the set \( I_0 \) because \( I_0 \) contains a hole. Therefore we have \( d_C(v, v + (0, 2)) = 4 = d_{\text{MH}}(v, v + (0, 2)) + 2 \) (see the path from \( v \) to \( v + (0, 2) \) of length 4 in Fig. A.4 (d)). From this we know that for any \( v' \) such that \( y + 2 \leq y' \) we have \( F \leq d_{\text{MH}}(v, v') + 2 = d_{\text{MH}}(v, v') + 2 \). Similarly, for any \( v' \) such that \( x + 2 \leq x' \) we have \( F \leq d_{\text{MH}}(v, v') + 2 \). This implies that for any \( v' \) such that \( x + 2 \leq x' \) or \( y + 2 \leq y' \) we have \( F \leq d_{\text{MH}}(v, v') + 2 \leq 2w \) except the three cases \( v' = (w - 1, w), (w, w - 1), (w, w) \). A node \( v' \) such that \( x' \leq x + 1 \) and \( y' \leq y + 1 \) is \( v' = v + (1, 1) \). For this \( v' \), \( d_C(v, v') = 6 \) (see the path from \( v \) to \( v + (1, 1) \) of length 6 shown in Fig. A.4 (d)), and we have \( F \leq d_{\text{MH}}(v, v') + 6 = (x + y) + 6 \leq 2w/2 \) + 6 \leq 2w \) using our assumption \( w \geq 5 \).

Summarizing, we have \( F \leq 2w \) except the cases \( v' = (w - 1, w), (w, w - 1), (w, w) \).

We consider what cases remain here. The cases \# I_0 = 2 \) and \# J_0 = 2 \) are contained in Case 2. The case \# I_0 = \# J_0 = 1 \) is contained in Case 3 and Case 4. The case \# I_0 = \# J_0 = 0 \) is contained in Case 1. The cases \# I_0 = 1 and \# J_0 = \# I_1 = 0 \) and \# I_0 = \# J_0 = \# J_1 = 0 \) are contained in Case 2. Therefore only the following case remains.

(Case 5) Either \# I_0 = \# I_1 = 1 or \# J_0 = \# J_1 = 1. We consider the first sub case. The two holes in \( I_0, I_1 \) are all the holes in \( C \). Let \( H \) be the rectangular subset of \( Q_1 \) having the nodes \( v + (2, 0), (w, w) \) as corners and let \( v_0 \) and \( v_1 \) be \( v \) and \( v + (2, 0) \) respectively. Then we can use Lemma A.2 to show \( F \leq 2w \) for \( v' \in H \).

As for \( v' \) in \( I_0 \cup I_1 \cup \{ v + (1, 0) \} \), we can show \( d_C(v, v') \leq d_{\text{MH}}(v, v') + 2 \) using arrow lines shown in Fig. A.4 (e). (We use the fact that there is at least one column west of \( I_0 \).) Moreover we have \( |x - x'| \leq 1 \). Using Lemma A.1 we can show \( F \leq 2w \).

This completes the proof for \( Q_1 \). We explain how to modify the proof for other quadrants. The two quadrants \( Q_{11}, Q_{11} \) are symmetric. Therefore we consider only \( Q_{11}, Q_{31} \).

We define the five subsets \( I_0, \ldots, K \) similarly for \( Q_{11}, Q_{31} \). For example, we define \( I_1 \) to be the set \( \{ (x', y') \mid x' = x - 1, y + 1 \leq y' \leq w \} \) for \( Q_{11} \) and the set \( \{ (x', y') \mid x' = x - 1, 0 \leq y' \leq y - 1 \} \) for \( Q_{31} \). With these modifications of the definitions of \( I_0, \ldots, K \), all the proofs of Cases 1, 2, 3, 5 for \( Q_1 \) are valid also for \( Q_{11}, Q_{31} \). Only the proof for Case 4 needs essential modifications.

We show the situations of Case 4 for \( Q_{11} \) and \( Q_{31} \) in Fig. A.5 (a), (b). With the ideas used for \( Q_1 \) we can show the followings.
Figure A.5: The situations of Case 4 for $Q_{II}$ and $Q_{III}$.

- For $Q_{II}$ and $v' = v + (-1, 1)$, $F \leq x + y + 6$.
- For $Q_{II}$ and $v' \neq v + (-1, 1)$, $F \leq (x + y) + \{(x - x') + (y' - y)\} + 2 = 2x - x' + y' + 2$.
- For $Q_{III}$ and $v' = v + (-1, -1)$, $F \leq x + y + 6$.
- For $Q_{III}$ and $v' \neq v + (-1, -1)$, $F \leq (x + y) + \{(x - x') + (y - y')\} + 2 = 2x + 2y - x' - y' + 2$.

With these and our assumption $w \geq 5$ we can show that $F \leq 2w$ except the following cases by elementary calculation.

- The quadrant is $Q_{II}$, $x = \lfloor w/2 \rfloor$, $w$ is even, $v' = (0, w - 1), (1, w), (0, w)$.
- The quadrant is $Q_{II}$, $x = \lfloor w/2 \rfloor$, $w$ is odd, $v' = (0, w)$.
- The quadrant is $Q_{III}$, $x = y = \lfloor w/2 \rfloor$, $w$ is even, $v' = (0, 1), (1, 0), (0, 0)$.

This, together with our previous result for $Q_{I}$, completes the proof of Theorem 5.3.