On the problems of controllability and uncontrollability for some mechanical systems described by the equations of vibrations of plates and beams with integral memory

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Abstract. Controllability problems for some models of plates and beams with integral memory are considered. The vibrational equation of the plate contains an Abelian kernel in the integral term, and the vibrational equation of the beam contains a continuous kernel consisting of a finite sum of decreasing exponential functions. It is proved that by controlling the whole domain, the first system cannot be driven to a state of rest, and for the second system, controllability to rest is possible.

1. Introduction

Integro-differential equations with integral terms in the convolution form often arise in various applications in the mechanics of homogeneous media, the theory of viscoelasticity, thermal physics and the kinetic theory of gases. In some cases, the convolution kernel is a sum (finite or infinite) of exponentially decreasing functions. For example, in the theory of heterogeneous media, it has been proven that the model describing a two-phase medium (elastic medium and viscous fluid) is represented using a system of integro-differential equations with convolution kernels in the form of a finite or infinite sum of decreasing exponentials. The laws of heat conduction with integral memory have been studied in various works (see, for example, [1]). Much attention is paid to controllability issues for models of plates and beams in some sections of engineering.

A large number of works are devoted to controllability problems for similar systems, but not considering integral memory (see, for example, [2–6]). For the systems with memory, the articles [7–12] should be noted.

This research is devoted to the problems of the vibrations’ distributed control of a plate and a beam with integral memory. The question about the possibility of driving such systems to a state of rest is raised. It should be noted that, generally speaking, this concept for the systems with memory is not equivalent to driving the system to the zero state. As it will further be clear, controllability to rest is not always possible for such models. In this paper, we will not dwell in detail on the questions of the solvability of initial-boundary value problems, but we will consider the qualitative aspects of this theory.
Two cases will be considered. In the first, the integral term of the equation contains the Abel kernel, that is, a kernel with a singularity; in the second, a kernel, consisting of the sum of a finite number of decreasing exponential functions, is considered. Both types of kernels are used in viscoelasticity models. For example, a beam model with memory and a kernel consisting of the sum of exponential functions describes the process of viscoelastic vibrations.

2. The problem of irreducibility to a rest state of a system described by the equation of a plate with integral memory and Abel kernel

This section is of an overview nature and does not contain any fundamentally new results. Let us consider the initial-boundary value problem of the following form:

\[
\begin{align*}
\theta_a(t,x) + \alpha^2 \Delta^2 \theta(s,x) + \int_0^t K(t-s) \Delta^2 \theta(s,x) ds &= \nu(t,x), \quad t > 0, \quad x \in \Omega, \\
\theta|_{t=0} &= \xi_1(x), \quad \theta|_{t=0} = \xi_2(x), \\
\theta(t,x) &= \Delta \theta(t,x) = 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( K(t) \) – is a kernel and \( \nu(t,x) \) defines control. Here \( \Omega \) – is the bounded domain in \( \mathbb{R}^2 \) with a boundary consisting of a finite number of disjoint closed infinitely smooth curves, \( \Delta^2 \) – is the Laplace operator squared with domain

\[ D(\Delta^2) = \{ y \in H^4(\Omega) : y(x) = \Delta y(x) = 0, \quad x \in \partial \Omega \}. \]

In mechanics models, kernels of various types are used to describe the process of viscoelastic vibrations. For example, the kernel of the following form is quite widespread:

\[
K_j(t) = \sum_{j=1}^{N} d_j e^{-\gamma_j t}, \quad d_j, \gamma_j > 0, \quad j = 1, \ldots, N.
\]

For the equations “similar” to (1) and in a number of special cases (see, for example, [10] and [11]), it is possible to prove that the system oscillations can be completely stopped in finite time if the control is applied to the entire domain \( \Omega \). In [9], it is proved that the solution of the Gurtin-Pipkin equation for two-dimensional domains and a wide class of continuous kernels cannot be driven to rest by a control applied only to a subdomain which closure is contained in \( \Omega \). It should be noticed, that \( K_j(t) \) at \( N>1 \) is an example of such a kernel. In this regard, it would be interesting to investigate the question of controllability for the system (1) - (3) if control is applied only to a part of the domain.

Now let the kernel have the form:

\[
K_2(t) = \frac{1}{\Gamma(1-\sigma)} t^{-\sigma}, \quad \sigma \in (0,1),
\]

where \( \Gamma(\lambda) \) – is Gamma function. This is Abel kernel. It is proved in [8] that this system is uncontrollable to rest if control is applied to one end of the segment and the other end is fixed; for a one-dimensional heat equation with integral memory and kernel (5).

Without pretending to be a fundamental novelty of the result and using the methods of [8], we will show that in the problem of controlling vibrations of a plate with memory for a kernel \( K_2(t) \) it is impossible to bring the solution to rest. Namely, let us consider the control \( \nu(t) \in L_2((0,T); L_2(\Omega)) \), continued by zero at \( t > T \). Let us suppose the first initial condition (bias) is identically zero. Next, we will show that there is
such an initial velocity \( \zeta_1(x) \), at which the vibrations of the plate cannot be stopped. More precisely, there is an initial condition \( \zeta_2(x) \), stating that for each control \( \nu(t,x) \), which vanishes on the set \( \{t:t>T\} \) with some \( T>0 \), the corresponding solution cannot be identically zero outside the bounded segment (with respect to the variable \( t \)).

For the proof, let us consider the orthonormal system of eigenfunctions \( \{\psi_n\} \) and eigenvalues \( \alpha^2_n \) (\( n=1,2, \ldots \)) of the operator \( \Delta^2 \) with respect to the boundary conditions (3). Let us suppose, that

\[
\xi_2(x) = \xi_{2k}\psi_k(x),
\]

where \( k \) is some natural number and \( \xi_{2k} \neq 0 \). Let us expand the solution \( \theta(t,x) \) and the control action \( \nu(t,x) \) into the Fourier series with respect to the aforementioned system of eigenfunctions (this system is a basis in \( L_2(\Omega) \)). Then we obtain a countable system of integro-differential equations:

\[
\theta^{(2)}_n(t) + \alpha^2_n\theta_n(t) + a_n^2\int_0^t K_2(t-s)\theta_n(s)ds = \nu_n(t), \quad t>0, \quad n=1,2,\ldots
\]  \( (6) \)

It should be noted that only the \( k \)-th equation from the system (6) has a nonzero initial condition.

We apply the Laplace transform to both sides in (6) for \( n=k \) and express \( \hat{\theta}_k(\lambda) \) in terms of:

\[
\hat{\theta}_k(\lambda) = \frac{\hat{\nu}_k(\lambda) + \xi_{2k}}{\lambda^2 + a_k^2(\alpha^2 + \hat{K}_2(\lambda))}.
\]  \( (7) \)

Let \( PW_\infty \) be the linear space of the Laplace transforms of elements of \( L_2(0,\infty) \), so that they are equal to zero on the set \( \{t:t>T\} \) for some \( T>0 \). It is known that \( \phi(\lambda) \in PW_\infty \) if and only if it is an entire function such that:

1) there are real such numbers \( C \) and \( T \) that \( |\phi(\lambda)| \leq Ce^{\lambda T} \). It should be noted, that \( C \) and \( T \) depend on \( \phi(\lambda) \).

2) \( \sup_{x \geq 0} \int_0^{+\infty} |\phi(x+iy)|^2 \, dy < +\infty \).

Next, we will apply the method associated with the so-called “branch point” and described in [8] for solving another problem. Let us suppose that the solution to problem (1) - (3) is controllable to rest, then the functions \( \hat{\theta}_k(\lambda) \) and \( \hat{\nu}_k(\lambda) \) are the elements of space \( PW_\infty \). So these are the entire functions. Since the functions are holomorphic \( \hat{\theta}_k(\lambda) \) and \( \hat{\nu}_k(\lambda) \) it follows that the equalities:

\[
\lim_{\rho \to 0^+} \hat{\theta}_k(\rho e^{i\omega}) = \lim_{\rho \to 0^+} \hat{\theta}_k(\rho e^{i\omega}), \quad \lim_{\rho \to 0^+} \hat{\nu}_k(\rho e^{i\omega}) = \lim_{\rho \to 0^+} \hat{\nu}_k(\rho e^{i\omega}), \quad \rho>0.
\]  \( (8) \)

Then, using (7) and (8), we can state:

\[
\frac{\hat{\nu}_k(\rho) + \xi_{2k}}{\rho^2 + a_k^2(\alpha^2 + \rho^{\sigma-1})} = \frac{\hat{\nu}_k(\rho) + \xi_{2k}}{\rho^2 + a_k^2(\alpha^2 + \rho^{\sigma-1}e^{2i\sigma(\pi-1)})}.
\]  \( (9) \)

The equality (9) is obtained using the following expression: \( \hat{K}_2(\lambda) = \lambda^{\sigma-1} \). Let us suppose there is such a number \( \rho \), for which \( \hat{\nu}_k(\rho) + \xi_{2k} \neq 0 \). Then the equality (9) is impossible, since the number \( \sigma-1 \) takes
the values in the interval \((-1,0)\). If \(\hat{v}_{\lambda}(\rho) + \hat{\xi}_{\lambda} = 0\), for any \(\rho > 0\), then by the uniqueness theorem for holomorphic functions it follows that \(\hat{v}_{\lambda}(\lambda) + \hat{\xi}_{\lambda} = 0\), for any complex \(\lambda\). The last identity is impossible, since point 2 of the space definition \(PW\) is violated.

### 3. The problem of controlling vibrations of a beam with integral memory, in the case when the control action is distributed along the entire beam length

Let us consider the equation of vibration of a beam with integral memory:

\[
\theta_{\text{int}}(t,x) + G(0)\theta_{\text{ext}}(s,x) + \int_0^t \dot{G}(t-s)\theta_{\text{ext}}(s,x)ds = u(t,x), \quad t > 0, \quad x \in (0,\pi),
\]

(10)

where

\[
G(t) = \sum_{j=1}^{N} \frac{e^{-\gamma_j t}}{\gamma_j}
\]

(11)

and \(c_j, \gamma_j\) are given positive constants such that \(0 < \gamma_1 < \gamma_2 < ... < \gamma_N\). It should be noted that in this equation the coefficient of the function \(\theta_{\text{ext}}\), which stands outside the integral term, is consistent with the kernel. This equation has the initial conditions:

\[
\theta|_{t=0} = \xi_1(x), \quad \theta|_{t=0} = \xi_2(x),
\]

(12)

and also the conditions at the ends of the interval \((0,\pi)\):

\[
\theta(t,0) = \theta_{\text{int}}(t,0) = \theta(t,\pi) = \theta_{\text{int}}(t,\pi) = 0.
\]

(13)

The task is to drive the solution of the system (10) - (13) to a state of rest. This means that it is required to construct such a control action \(u(t,x)\), for which the corresponding solution \(\theta(t,x)\) will be equal to zero identically for \(t > T\), where \(T\) is some point in time. To drive the solution of the system to rest, it is necessary (but not enough) to satisfy the conditions:

\[
\theta(T,x) = \theta(t,x) = 0, \quad x \in (0,\pi).
\]

(14)

The distributed control problems for similar systems in the case when there is no integral term in the equation (classical plate) were considered in [3].

We note that for the systems with integral memory the fulfillment of conditions (14) is not sufficient to stop the oscillations of the system, since after reaching the zero state the system can leave it (see, for example, [8]). In this section, we will prove that under certain conditions imposed on the initial problem data, the system (10) - (13) can be driven to a state of rest in finite time by a distributed control action arbitrarily small in absolute value.

The research methods in this work are close to those used in [10] and [11].

Let us consider the operator \(A := \frac{d^2}{dx^2}\) with the domain

\[
D(A) = \{\varphi \in H^4(0,\pi) : \varphi(0) = \varphi_{\text{int}}(0) = \varphi(\pi) = \varphi_{\text{int}}(\pi) = 0\}.
\]

Let \(\{\psi_n(x)\}_{n=1}^{\infty}\) be the corresponding orthonormal system of eigenfunctions and \(\{n^4\}_{n=1}^{\infty}\) defines the corresponding eigenvalues, i.e.
\[ A\psi_n(x) = n^2\psi_n(x). \]

It should be noted that by virtue of the boundary conditions (13), the system of eigenfunctions \( \{\psi_n(x)\}_{n=1}^{\infty} \) coincides with the system of eigenfunctions of the Laplace operator with zero Dirichlet boundary condition.

Let us denote the linear function space \( f : \mathbb{R}_+ = (0, +\infty) \to D(A) \), by \( W^2_{2,1}(\mathbb{R}_+, A) \) standardized with the norm:

\[
\| \theta \|_{W^2_{2,1}(\mathbb{R}_+, A)} = \left( \int_0^{+\infty} e^{-2\rho t} \left( \| \theta^{(2)}(t) \|_{L^2_0(\mathbb{R}_+)}^2 + \| A\theta(t) \|_{L^2_0(\mathbb{R}_+)}^2 \right) dt \right)^{1/2}, \quad \rho > 0.
\]

The function \( \theta(t, x) \) is called a strong solution to the problem (10)–(13), if for some \( \rho \geq 0 \) this function is an element \( W^2_{2,1}(\mathbb{R}_+, A) \), satisfies the equation (10) almost everywhere in the variable \( t \) on the positive semiaxis \( \mathbb{R}_+ \) and satisfies the initial conditions (12).

We define the function of a complex variable \( \lambda : \)

\[ I_\nu(\lambda) := \lambda^2 + n^2\lambda \hat{G}(\lambda), \]

where

\[ \hat{G}(\lambda) = \sum_{k=1}^N \frac{c_k}{\gamma_k (\lambda + \gamma_k)}. \]

Let us now write down the solution of the problem (10) - (13) in explicit form. To do this, we use the well-known results of [13], in which the corresponding formulas are given. Let us present the formulations of two theorems, the first one refers to the formula for the solution of equation (10) with zero right-hand side, the second theorem to the solution with zero initial conditions. Due to the linearity of the problem, the sum of these two solutions will be the desired solution.

**Theorem 1.** Let us suppose, that \( u(t, x) = 0, \ t \in \mathbb{R}_+ \), the function \( \theta(t, x) \in W^2_{2,1}(\mathbb{R}_+, A), \ \rho > 0 \), the strong decision (10)–(13), then for any \( t \in \mathbb{R}_+ \) the formula is correct:

\[
\theta(t, x) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left( \frac{\xi_{2n}^+ + \lambda_n^+ \xi_{2n}}{l_n^{(1)}(\lambda_n^+)} \right) e^{\xi_{2n}^+ x} \psi_n(x) + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left( \frac{\xi_{2n}^- + \lambda_n^- \xi_{2n}}{l_n^{(1)}(\lambda_n^-)} \right) e^{\xi_{2n}^- x} \psi_n(x) + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left( \frac{\xi_{2n}^+ - q_{k,n} e^{\xi_{2n}}}{l_n^{(1)}(-q_{k,n})} \right) \psi_n(x), \quad (15)
\]

where \( -q_{k,n} \) denotes the real function zeros \( I_\nu(\lambda) \) ( \( q_{k,n} > 0, \ k = 1, 2, ..., N - 1 \), \( \lambda_n^+ \) – is a pair of complex conjugate zeros, \( \xi_{2n}^+, \xi_{2n}^- \) – are the Fourier coefficients of the initial data expansion in terms of eigenfunctions and series (15) converges in the norm of the space \( L_2(0, \pi) \).

**Theorem 2.** Let us suppose, that \( u(t, x) \in C([0, T], L^2(0, \pi)) \) for any \( T > 0 \), \( \theta(t, x) \in W^2_{2,1}(\mathbb{R}_+, A) \) – is the strong solution to the problem (10) - (13) for some \( \rho > 0, \ \xi_1 = \xi_2 = 0 \). Then for any \( t \in \mathbb{R}_+ \) the formula is correct:
\[
\theta(t,x) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \omega_n(t, \lambda_n^+) \psi_n(x) + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \omega_n(t, \lambda_n^-) \psi_n(x) + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left( \sum_{k=0}^{N-1} \omega_n(t, -q_{k,n}) \right) \psi_n(x),
\]

(16)

where \[ \omega_n(t, \lambda) = \frac{\int_0^t u_n(s)e^{f(t-s)} ds}{f^{(1)}(\lambda)}, \]

\(u_n(t)\) is the Fourier coefficient of the control action expansion in terms of eigenfunctions and series (16) converges in the space norm \(L_2(0, \pi)\).

It should be noted that by increasing the smoothness of the right-hand side of the equation and the initial data, it is possible to achieve the required smoothness of the solution (see details in [13]). All zeros of the \(f^{(1)}(\lambda)\) function (for each natural \(n\)) are known to be pairwise different; moreover, their asymptotics were studied in [14].

For further reasoning, we need the following lemma, the proof of which for a similar problem is given in the paper [11].

**Statement 1.** For any natural number \(n\), the following equality is true:

\[
\frac{1}{f^{(1)}(\lambda_n^+)} + \frac{1}{f^{(1)}(\lambda_n^-)} + \sum_{k=0}^{N-1} \frac{1}{f^{(1)}(-q_{k,n})} = 0.
\]

Let the control action \(u(t,x)\) satisfy the conditions of the theorem 2. As mentioned above, the solution to the problem (10) - (13) can be represented in the form:

\[
\theta(t,x) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left( \xi_{2n} + \lambda_n^+ \xi_{1n} \right) e^{\lambda_n^+ t} \psi_n(x) + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left( \xi_{2n} + \lambda_n^- \xi_{1n} \right) e^{\lambda_n^- t} \psi_n(x) +
\]

\[
+ \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left( \sum_{k=0}^{N-1} \left( \xi_{2n} - q_{k,n} \xi_{1n} \right) e^{-q_{k,n} t} \right) f^{(1)}(-q_{k,n}) \psi_n(x) + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \int_0^t u_n(s)e^{\lambda_n^+ (t-s)} ds f^{(1)}(\lambda_n^+) \psi_n(x) +
\]

\[
+ \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \int_0^t u_n(s)e^{\lambda_n^- (t-s)} ds f^{(1)}(\lambda_n^-) \psi_n(x) + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \sum_{k=0}^{N-1} \left( \int_0^t u_n(s)e^{-q_{k,n}(t-s)} ds f^{(1)}(-q_{k,n}) \psi_n(x) + \right)
\]

(17)

As \(q_{0,n} = 0\), then for the derivative of the Fourier coefficient of the function \(\theta\), taking into account statement 1, we can write (for all the natural numbers \(n\)):

\[
\theta'_n(t) = \frac{1}{\sqrt{2\pi}} \frac{\lambda_n^+ \left( \xi_{2n} + \lambda_n^+ \xi_{1n} \right) e^{\lambda_n^+ t} \psi_n(x)}{f^{(1)}(\lambda_n^+)} + \frac{1}{\sqrt{2\pi}} \frac{\lambda_n^- \left( \xi_{2n} + \lambda_n^- \xi_{1n} \right) e^{\lambda_n^- t} \psi_n(x)}{f^{(1)}(\lambda_n^-)} +
\]

\[
+ \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N-1} \frac{\left( -q_{k,n} \xi_{2n} - q_{k,n} \xi_{1n} \right) e^{-q_{k,n} t}}{f^{(1)}(-q_{k,n})} \psi_n(x) + \frac{1}{\sqrt{2\pi}} \frac{\lambda_n^+ \int_0^t u_n(s)e^{\lambda_n^+ (t-s)} ds}{f^{(1)}(\lambda_n^+)} \psi_n(x) +
\]

\[
+ \frac{1}{\sqrt{2\pi}} \frac{\lambda_n^- \int_0^t u_n(s)e^{\lambda_n^- (t-s)} ds}{f^{(1)}(\lambda_n^-)} \psi_n(x) +
\]
+ \frac{1}{\sqrt{2\pi}} \int_0^{r_n} (\lambda_n^r) \psi_n(x) + \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{N-1} \left( -q_{k,m} \right) \int_0^{r_n} e^{-q_{k,m}(r-s)} ds \right) \psi_n(x). \tag{18}

Let us formulate the control \( u(t,x) \), for which the conditions (14) are satisfied. We have

\[
- \left( \xi_{2n} + \lambda_n^r \xi_{1n} \right) e^{\lambda_n^r T} \int_0^{r_n} \frac{I_n^0(\lambda_n^r)}{I_n^1(\lambda_n^r)} ds + \left( \xi_{2n} + \lambda_n^r \xi_{1n} \right) e^{\lambda_n^r T} \int_0^{r_n} \frac{I_n^0(\lambda_n^r)}{I_n^1(\lambda_n^r)} ds \right) =
\]

\[
= \sum_{k=0}^{N-1} \left( -q_{k,m} \right) I_n^1(-q_{k,m}), \quad n = 1,2,\ldots, \tag{19}
\]

\[
\lambda_n^r \int_0^{r_n} \frac{I_n^0(\lambda_n^r)}{I_n^1(\lambda_n^r)} ds - \lambda_n^r \int_0^{r_n} \frac{I_n^0(\lambda_n^r)}{I_n^1(\lambda_n^r)} ds + \sum_{k=1}^{N-1} \left( -q_{k,m} \right) I_n^1(-q_{k,m}) =
\]

\[
= \sum_{k=1}^{N-1} \left( -q_{k,m} \right) I_n^1(-q_{k,m}), \quad n = 1,2,\ldots. \tag{20}
\]

Let us introduce the short notation:

\[
\alpha_n = -\left( \xi_{2n} + \lambda_n^r \xi_{1n} \right), \quad \beta_n = -\left( \xi_{2n} + \lambda_n^r \xi_{1n} \right), \quad h_n = -\left( q_{k,m} + \xi_{2n} \xi_{1n} \right), \quad k = 0,1,2,\ldots,N-1.
\]

Further, following the method of [10] and [11], we will equate the coefficients in front of the values

\[
\frac{1}{I_n^0(\lambda_n^r)}, \quad \frac{1}{I_n^1(\lambda_n^r)}, \quad \frac{1}{I_n^1(-q_{k,m})}, \quad k = 0,1,2,\ldots,N-1.
\]

on the left and right sides of the equations (19), (20). Therefore, for all natural \( n \) and \( k = 0,1,2,\ldots,N-1 \), we get the moment problem:

\[
\int_0^r u_n(s) e^{\lambda_n^r T} ds = \alpha_n e^{\lambda_n^r T}, \quad \int_0^r u_n(s) e^{\lambda_n^r T} ds = \beta_n e^{\lambda_n^r T}, \quad \int_0^r u_n(s) e^{-q_{k,m} T} ds = h_n e^{-q_{k,m} T}. \tag{21}
\]

Let us divide by a monomial factor in both sides of the equations (21). Then we get:

\[
\int_0^r u_n(s) e^{-\lambda_n^r T} ds = \alpha_n, \quad \int_0^r u_n(s) e^{-\lambda_n^r T} ds = \beta_n, \quad \int_0^r u_n(s) e^{q_{k,m} T} ds = h_n, \quad k = 0,1,2,\ldots,N-1. \tag{22}
\]

The system of integral equations (22) is a classical finite-dimensional moment problem for each natural \( n \). Let us define that \( \lambda_n = -\lambda_n^+ \) and \( \lambda_n = -\lambda_n^- \).

We will seek for the solution (22) in the following form:

\[
u_n(s) = D_{n,0} e^{\lambda_n^+ t} + D_{n,1} e^{\lambda_n^- t} + \sum_{j=0}^{N-1} C_{j,n} e^{q_{j,n} T}, \quad n = 1,2,\ldots, \tag{23}
\]


where \( D_{1,n} \), \( D_{2,n} \), and \( C_{j,n} \) are the unknown constants and \( s \in [0,T] \). Let us note, that for \( s > T \) control action \( u \) is equal to zero. Substituting (23) in (22), we obtain a system of algebraic equations for each natural \( n \):

\[
\begin{align*}
&\int_0^T e^{2\lambda_n t} ds + \int_0^T e^{(\lambda_n + \lambda_n) t} ds + \sum_{k=0}^{N-1} C_{k,n} \int_0^T e^{(\lambda_n + \lambda_{k,n}) t} ds = a_n, \\
&\int_0^T e^{(\lambda_n + \lambda_{k,n}) t} ds + \int_0^T e^{2\lambda_n t} ds + \sum_{k=0}^{N-1} C_{k,n} \int_0^T e^{(\lambda_n + \lambda_{k,n}) t} ds = a_n, \\
&\int_0^T e^{(\lambda_n + \lambda_{k,n}) t} ds + \int_0^T e^{(\lambda_n + \lambda_{j,n}) t} ds + \sum_{j=0}^{N-1} C_{j,n} \int_0^T e^{(\lambda_n + \lambda_{j,n}) t} ds = b_{k,n}, \quad k = 0,1,2,...,N-1
\end{align*}
\]  

(24)

Let \( \Delta_n \) be the system determinant (24). This is the Gram determinant and therefore in this case it is not zero for any moment in time \( T > 0 \). Then we obtain the unique solution of the linear system (24).

If we choose the initial data of the problem (10) - (13) in such a way that they contain only a finite number of nonzero Fourier coefficients, then we obtain a finite number of finite-dimensional moment problems. Each of which has a solution in the form (23). It should be noted that this choice of initial data is “natural”, since they can approximate the functions from more general classes. In this case, only a finite number of Fourier coefficients of the control \( u(t,x) \) will be nonzero. In this case, the nonzero coefficients given by formula (23) tend to zero in modulus at \( T \) tending to infinity (for the proof of this fact, see [10] and [11]). This means that \( u(t,x) \) can be made arbitrarily small in absolute value.

Let us now show that this choice of control brings the system to a state of rest, that is, the solution is identically zero starting from the moment \( t = T \). For this, we will use the formula (17). Let us suppose, that \( t = T \), then, as stated above, \( \theta_n(T) = 0 \). Let us prove that \( \theta_n(t) = 0 \), at \( t > T \). Let us suppose, that \( \lambda_0 \) is a zero of the function \( I_n(\lambda) \). Since the control \( u(t,x) \) is equal to zero identically zero for \( t > T \), then for such \( t \) it is possible to write:

\[
\int_0^T u_n(s)e^{-\lambda_0(s-t)} ds = e^{-\lambda_0(T)} \int_0^T u_n(s)e^{-\lambda_0 s} ds.
\]

Next, we use the obtained equality, the moment system (22) and the formula (17). It is not hard to see that the solution \( \theta_n(t) \), corresponding to the formulated control will be equal to zero identically for any \( t > T \).

Summary
The present paper is devoted to the problems of distributed controllability for some models of plates and beams with integral memory. The goal of the control is to drive these systems to rest. The integral term of the plate equation contains the Abel kernel and a kernel (of the integral term) in the beam equation consists of the sum of a finite number of decreasing exponential functions. It is proved that by controlling the whole domain, the first system cannot be driven to rest but the second system is controllable to rest in finite time.

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