THE DECOMPOSITION OF LEVEL–1 IRREDUCIBLE HIGHEST WEIGHT MODULES WITH RESPECT TO THE LEVEL–0 ACTIONS OF THE QUANTUM AFFINE ALGEBRA $U'_q(\widehat{\mathfrak{sl}}_n)$

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Abstract. We decompose the level–1 irreducible highest weight modules of the quantum affine algebra $U'_q(\widehat{\mathfrak{sl}}_n)$ with respect to the level–0 $U'_q(\widehat{\mathfrak{sl}}_n)$-action defined in [11]. The decomposition is parameterized by the skew Young diagrams of the border strip type.

1. Introduction

In the papers [16] and [11] it was shown that the $q$-deformed Fock space module of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ admits an action of a new remarkable object – the so-called quantum toroidal algebra introduced in [1] and [12] as a $q$-deformation of the universal central extension of the $\mathfrak{sl}_n$-valued double-loop Lie algebra. The action of the quantum toroidal algebra on the $q$-Fock space depends on two parameters: the deformation parameter $q$ and an extra parameter $p$, when values of these parameters are taken to be generic complex numbers, the $q$-Fock space is known to be irreducible with respect to this action.

The quantum toroidal algebra has two subalgebras, $U'_q(\widehat{\mathfrak{sl}}_n)^{(1)}$ and $U'_q(\widehat{\mathfrak{sl}}_n)^{(2)}$, both isomorphic to $U'_q(\widehat{\mathfrak{sl}}_n)$. Accordingly, the $q$-Fock space admits two $U'_q(\widehat{\mathfrak{sl}}_n)$-actions.

The first of these actions has level 1, and coincides with the action originally introduced by Hayashi in [3]. The irreducible decomposition of the $q$-Fock space with respect to this action was given in [7] by using the semi-infinite $q$-wedge construction due to [13].

The second of the $U'_q(\widehat{\mathfrak{sl}}_n)$-actions has level 0, the irreducible decomposition of the $q$-Fock space with respect to this action was constructed in [12] at generic values of the parameters $q$ and $p$.

Kashiwara, Miwa and Stern [7] have shown, that the level-1 action of $U'_q(\widehat{\mathfrak{sl}}_n)$ on the $q$-Fock space is centralized by the action of the Heisenberg algebra. In the paper [11] it was proven that the proper ideal of the

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\(q\)-Fock space generated by the negative-frequency part of the Heisenberg algebra is invariant under the action of the quantum toroidal algebra provided the value of the parameter \(p\) in the latter is set to be equal to 1. The quotient of the \(q\)-Fock space by this ideal is isomorphic to one of the irreducible level-1 highest weight modules of \(U_q(\widehat{\mathfrak{sl}}_n)\). As a consequence, each of these modules admits an action of the quantum toroidal algebra.

The corresponding action of the subalgebra \(U'_q(\widehat{\mathfrak{sl}}_n)^{(1)}\) is irreducible, it is just the standard level-1 action on the highest weight irreducible module of \(U_q(\widehat{\mathfrak{sl}}_n)\). On the other hand, the action of the subalgebra \(U'_q(\widehat{\mathfrak{sl}}_n)^{(2)}\) has level 0 and is completely reducible. The construction of the irreducible decomposition of the level-1 \(U_q(\widehat{\mathfrak{sl}}_n)\)-modules relative to the level-0 action is the problem which we address in the present paper. To solve this problem we utilize, as our main tools, the semi-infinite wedges of \([7]\) and the Non-symmetric Macdonald polynomials of \([3]\).

As a result we obtain a parameterization of the irreducible components of the level-0 action by the skew Young diagrams of the border strip type, proving thereby a \(q\)-analogue of the conjecture made in \([8]\) in the classical setting.

2. The actions of the quantum affine algebra \(U'_q(\widehat{\mathfrak{sl}}_n)\)

2.1. Definition of the quantum affine algebra \(U'_q(\widehat{\mathfrak{sl}}_n)\).

**Definition 1.** The quantum affine algebra \(U'_q(\widehat{\mathfrak{sl}}_n)\) is the unital associative algebra over \(\mathbb{C}\) with generators \(E_i, F_i, K_i^{\pm 1} (i \in I := \{0, 1, \ldots, n-1\})\) and the following defining relations:

\[
K_i K_{i}^{-1} = 1 = K_{i}^{-1} K_{i}, \tag{2.1}
\]

\[
K_i K_j = K_j K_i, \tag{2.2}
\]

\[
K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \tag{2.3}
\]

\[
K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, \tag{2.4}
\]

\[
[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \tag{2.5}
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_q (E_i)^r (E_i)^{1-a_{ij}-r} = 0, \quad i \neq j. \tag{2.6}
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_q (F_i)^r (F_i)^{1-a_{ij}-r} = 0, \quad i \neq j. \tag{2.7}
\]
where \([n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}\), \([n]_q := \frac{[n][n-1]_q \ldots [n-r+1]_q}{[r]_q[r-1]_q \ldots [1]_q}\),

\[
a_{ij} = \begin{cases} 
2 & (i = j) \\
-1 & (|i - j| = 1, (i, j) = (1, n), (n, 1)) \\
0 & (\text{otherwise}) 
\end{cases} 
\]

\[a_{ij} = \begin{cases} 
2 & (i = j) \\
-2 & (i \neq j) 
\end{cases} \quad n = 2. \tag{2.10}
\]

The coproduct \(\Delta\) is given by

\[
\begin{align*}
\Delta(E_i) &= E_i \otimes K_i + 1 \otimes E_i, \\
\Delta(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i, \\
\Delta(K_i) &= K_i \otimes K_i.
\end{align*} \tag{2.11-2.13}
\]

We put \(c' := K_0 K_1 \ldots K_{n-1}\) in \(U'_q(\widehat{\mathfrak{sl}}_n)\), then \(c'\) is the central in \(U'_q(\widehat{\mathfrak{sl}}_n)\).

2.2. \(q\)-wedge product and semi–infinite \(q\)-wedge product. The affine Hecke algebra of type \(\mathfrak{gl}_N\), \(\widehat{H}_N(q)\) is a unital associative algebra over \(\mathbb{C}[q^{\pm 1}]\) with generators \(T_i^{\pm 1}, Y_j^{\pm 1}, i = 1, 2, \ldots, N - 1, j = 1, 2, \ldots, N\) and relations

\[
T_i T_i^{-1} = T_i^{-1} T_i = 1, \quad (T_i + 1)(T_i - q^2) = 0,
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_i,
\]

\[
T_i T_j = T_j T_i \quad \text{if } |j - i| > 1,
\]

\[
Y_i Y_j = Y_j Y_i, \quad T_i^{-1} Y_i T_i^{-1} = q^2 Y_i^{-1}
\]

\[
Y_i T_i = T_i Y_i \quad \text{if } j \neq i, i + 1.
\]

The subalgebra \(H_N(q)\) generated by \(T_i^{\pm 1}\) is isomorphic to the Hecke algebra of type \(\mathfrak{gl}_N\).

Let \(p \in \mathbb{C}^\times\) and consider the following operators in \(\text{End}(\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}])\)

\[
g_{i,j} = \frac{q^{z_i - z_j} - q^{-z_i} - q^{-z_j}}{z_i - z_j} (K_{i,j} - 1) + q, \quad 1 \leq i \neq j \leq N,
\]

\[
Y_i^{(N)} = g_{i,i+1}^{-1} K_{i,i+1} \cdots g_{i,N}^{-1} K_{i,N} p^{D_i} K_{1,i} g_{1,i} \cdots K_{i-1,i} g_{i-1,i}, \quad i = 1, 2, \ldots, N,
\]

where \(K_{i,j}\) acts on \(\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]\) by permuting variables \(z_i, z_j\) and \(p^{D_i}\) is the difference operator

\[
p^{D_i} f(z_1, \ldots, z_i, \ldots, z_N) = f(z_1, \ldots, p z_i, \ldots, z_N), \quad f \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}].
\]

Then the assignment

\[
T_i \mapsto \hat{T}_i = -q g_{i,i+1}^{-1}, \quad Y_i \mapsto q^{1-N} Y_i^{(N)} \tag{2.14}
\]

defines a right action of \(\widehat{H}_N(q)\) on \(\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]\).
The commuting difference operators $Y^{(N)}_1, \ldots, Y^{(N)}_N$ are called Cherednik’s operators. Moreover, the assignment
\[ T_i \mapsto \hat{T}_i = -qqg_{i,i+1}, \quad Y_i \mapsto z_i^{-1} \text{ (multiplication)} \]
(2.15)
defines another right action of $\hat{H}_N(q)$ on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$.

**Remark** The actions of $-qqg_{i,i+1}^{-1}, \ q^{1-N}Y_i^{(N)}, \ z_i^{-1}$ are related to the toroidal Hecke algebra [15] or the double affine Hecke algebra [3].

Let $V = \mathbb{C}^n$, with basis $\{v_1, \ldots, v_n\}$. Then $\otimes^N V$ admits a left $H_N(q)$-action given by
\[ T_i \mapsto \hat{T}_i = 1 \otimes \cdots \otimes T_i \otimes \cdots \otimes 1 \in \text{End}(\otimes^2 V) \]
(2.16)
and
\[ \hat{T}(v_{\epsilon_1} \otimes v_{\epsilon_2}) = \begin{cases} q^2 v_{\epsilon_1} \otimes v_{\epsilon_2} & \text{if } \epsilon_1 = \epsilon_2, \\ qv_{\epsilon_2} \otimes v_{\epsilon_1} & \text{if } \epsilon_1 < \epsilon_2, \\ qv_{\epsilon_1} \otimes v_{\epsilon_2} + (q^2 - 1)v_{\epsilon_1} \otimes v_{\epsilon_2} & \text{if } \epsilon_1 > \epsilon_2. \end{cases} \]
(2.17)

Let $V(z) = \mathbb{C}[z^{\pm 1}] \otimes V$, with basis $\{z^m \otimes v_{\epsilon}\}$, $m \in \mathbb{Z}$, $\epsilon \in \{1, 2, \ldots, n\}$. Often it will be convenient to set $k = \epsilon - nm$ and $u_k = z^m \otimes v_{\epsilon}$. Then $\{u_k\}$, $k \in \mathbb{Z}$ is a basis of $V(z)$. In what follows we will write $z^m v_{\epsilon}$ as a short-hand for $z^m \otimes v_{\epsilon}$, and use both notations: $u_k$ and $z^m v_{\epsilon}$ switching between them according to convenience. The two actions of the Hecke algebra are naturally extended on the tensor product $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^N V)$ so that $T_i$ acts trivially on $\otimes^N V$ and $\hat{T}_i$ acts trivially on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$. The vector space $\otimes^N V(z)$ is identified with $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^N V)$ and the $q$-wedge product [7] is defined as the following quotient space:
\[ \wedge^N V(z) = \otimes^N V(z)/ \sum_{i=1}^{N-1} \text{Ker} \left( \hat{T}_i + q^2 \hat{T}_i^{-1} \right). \]
(2.18)

Let $\Lambda : \otimes^N V(z) \to \wedge^N V(z)$ be the quotient map specified by (2.18). The image of a pure tensor $u_{k_1} \otimes u_{k_2} \otimes \cdots \otimes u_{k_N}$ under this map is called a wedge and is denoted by
\[ u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_N} := \Lambda(u_{k_1} \otimes u_{k_2} \otimes \cdots \otimes u_{k_N}). \]
(2.19)
A wedge is normally ordered if $k_1 > k_2 > \cdots > k_N$. In [7] it is proven that normally ordered wedges form a basis in $\wedge^N V(z)$. 
Let us now define the semi-infinite q–wedge product $\wedge^{\infty} V(z)$ and for any integer $M$ its subspace $F_M$, following [7].

Let $\otimes^{\infty} V(z)$ be the space spanned by the vectors $u_{k_1} \otimes u_{k_2} \otimes \ldots$, $(k_{i+1} = k_i - 1, \ i >> 1)$. We define the space $\wedge^{\infty} V(z)$ as the quotient of $\otimes^{\infty} V(z)$:

\[ \wedge^{\infty} V(z) := \otimes^{\infty} V(z) / \sum_{i=1}^{\infty} \text{Ker} \left( e T_i + q^2 (T_i)^{-1} \right). \] (2.20)

Let $\Lambda : \otimes^{\infty} V(z) \rightarrow \wedge^{\infty} V(z)$ be the quotient map specified by (2.20). The image of a pure tensor $u_{k_1} \otimes u_{k_2} \otimes \ldots$ under this map is called a semi-infinite wedge and is denoted by

\[ u_{k_1} \wedge u_{k_2} \wedge \cdots := \Lambda(u_{k_1} \otimes u_{k_2} \otimes \ldots). \] (2.21)

A semi-infinite wedge is normally ordered if $k_1 > k_2 > \cdots$ and $k_{i+1} = k_i - 1 (i >> 1)$. In [7] it is proven that normally ordered semi-infinite wedges form a basis in $\wedge^{\infty} V(z)$.

Let $U_M$ be the subspace of $\otimes^{\infty} V(z)$ spanned by the vectors $u_{k_1} \otimes u_{k_2} \otimes \ldots$, $(k_i = M - i + 1, \ i >> 1)$. Let $F_M$ be the quotient space of $U_M$ defined by the map (2.21). Then $F_M$ is a subspace of $\wedge^{\infty} V(z)$, and the vectors $u_{k_1} \wedge u_{k_2} \wedge \cdots$, $(k_1 > k_2 > \cdots, k_i = M - i + 1, \ i >> 1)$ form a basis of $F_M$. We will call the space $F_M$ the q–deformed Fock space.

2.3. Actions of the quantum affine algebra on the q–wedge product. We will define two actions of $U'_q(\mathfrak{sl}_n)$ on the space $\wedge^N V(z)$. 


The first one is defined as follows.

\[
E_i(m \otimes v) = \sum_{j=1}^{N} m \otimes E_j^{i,i+1} K_{j+1}^i \cdots K_N^i v, \tag{2.22}
\]

\[
F_i(m \otimes v) = \sum_{j=1}^{N} m \otimes (K_1^i)^{-1} \cdots (K_{j-1}^i)^{-1} E_j^{i+1,i} v, \tag{2.23}
\]

\[
K_i(m \otimes v) = m \otimes K_1^i K_2^i \cdots K_N^i v, \quad (i = 1, 2, \ldots, n - 1) \tag{2.24}
\]

\[
E_0(m \otimes v) = \sum_{j=1}^{N} m z_j^{-1} \otimes E_j^{n,1} K_{j+1}^0 \cdots K_N^0 v, \tag{2.25}
\]

\[
F_0(m \otimes v) = \sum_{j=1}^{N} m z_j^{-1} \otimes (K_1^0)^{-1} \cdots (K_{j-1}^0)^{-1} E_j^{1,n} v, \tag{2.26}
\]

\[
K_0 = (K_1 K_2 \cdots K_{n-1})^{-1}. \tag{2.27}
\]

Here \(E_j^{i,k} = 1^\otimes_{j-1} \otimes E_j^{i,k} \otimes 1^\otimes_{N-j}\), where \(E_j^{i,k} \in \text{End}(V)\) is the matrix unit in the basis \(v_1, \ldots, v_n\), and \(K_j^i = q^{E_j^{i,i}-E_j^{i+1,i+1}}, K_j^0 = (K_1^i K_2^i \cdots K_{n-1}^i)^{-1}\).

We will denote this action by \(U_0^{(N)}\). Note that it is well defined on the quotient space \(\wedge^N V(z)\) in view of the relations of the affine Hecke algebra.

The second one is defined as follows.

\[
E_0(m \otimes v) = \sum_{j=1}^{N} m z_j \otimes E_j^{n,1} K_{j+1}^0 \cdots K_N^0 v, \tag{2.28}
\]

\[
F_0(m \otimes v) = \sum_{j=1}^{N} m z_j^{-1} \otimes (K_1^0)^{-1} \cdots (K_{j-1}^0)^{-1} E_j^{1,n} v. \tag{2.29}
\]

The actions of other generators are the same as in (2.22–2.24, 2.27).

We will denote this action by \(U_1^{(N)}\). Again, this action is well defined on the quotient space \(\wedge^N V(z)\) in view of the relations of the affine Hecke algebra.

2.4. Level–0 action of the quantum affine algebra on the q–deformed Fock space. We will define a level-0 action of \(U'_q(\widehat{\mathfrak{sl}}_n)\) on \(F_M\ (M \in \mathbb{Z})\) following the paper [12, 11].

Let \(e := (\epsilon_1, \epsilon_2, \ldots, \epsilon_N)\) where \(\epsilon_i \in \{1, 2, \ldots, n\}\). For a sequence \(e\) we set

\[
\nu_e := \nu_{\epsilon_1} \otimes \nu_{\epsilon_2} \otimes \cdots \otimes \nu_{\epsilon_N} \quad (\in \otimes^N \mathbb{C}^n). \tag{2.30}
\]
A sequence \( \mathbf{m} := (m_1, m_2, \ldots, m_N) \) from \( \mathbb{Z}^N \) is called \( n \)-strict if it contains no more than \( n \) equal elements of any given value. Let us define the sets \( \mathcal{M}^n_N \) and \( \mathcal{E}(\mathbf{m}) \) by

\[
\mathcal{M}^n_N := \{ \mathbf{m} = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N \mid m_1 \leq m_2 \leq \cdots \leq m_N, \; \mathbf{m} \text{ is } n \text{-strict} \},
\]

and for \( \mathbf{m} \in \mathcal{M}^n_N \)

\[
\mathcal{E}(\mathbf{m}) := \{ \mathbf{e} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_N) \in \{1, 2, \ldots, n\}^N \mid \epsilon_i > \epsilon_{i+1} \text{ for all } i \text{ s.t. } m_i = m_{i+1} \}.
\]

In these notations the set \( \mathcal{E} \) is nothing but the base of the normally ordered wedges in \( \wedge^n V(\mathbf{z}) \). We will use the notation \( w(\mathbf{m}, \mathbf{e}) \) exclusively for normally ordered wedges.

Similarly for a semi-infinite wedge \( w = u_{k_1} \wedge u_{k_2} \wedge \cdots = z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \cdots \), such that \( w \in F_M \), the semi-infinite sequences \( \mathbf{m} = (m_1, m_2, \ldots) \) and \( \mathbf{e} = (\epsilon_1, \epsilon_2, \ldots) \) are defined by \( k_i = \epsilon_i - nm_i \), \( \epsilon_i \in \{1, 2, \ldots, n\} \), \( m_i \in \mathbb{Z} \). In particular the \( \mathbf{m} \)- and \( \mathbf{e} \)-sequences of the vacuum vector in \( F_M \) will be denoted by \( \mathbf{m}^0 \) and \( \mathbf{e}^0 \):

\[
|M\rangle = u_M \wedge u_{M-1} \wedge u_{M-2} \wedge \cdots = z^{m_0} v_{\epsilon_0} \wedge z^{m_2} v_{\epsilon_2} \wedge z^{m_3} v_{\epsilon_3} \wedge \cdots.
\]

The Fock space \( F_M \) is \( \mathbb{Z}_{\geq 0} \)-graded. For any semi-infinite wedge \( w = u_{k_1} \wedge u_{k_2} \wedge \cdots = z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \cdots \in F_M \) the degree \( |w| \) is defined by

\[
|w| = \sum_{i \geq 1} m_i^0 - m_i.
\]

Let us denote by \( F^k_M \subset F_M \) the homogeneous component of degree \( k \).

We will define a level-0 action of \( U_q(\widehat{\mathfrak{sl}_n}) \) on the Fock space \( F_M \) in such a way that each homogeneous component \( F^k_M \) will be invariant with respect to this action. Throughout this section we fix an integer \( M \) and \( s \in \{0, 1, 2, \ldots, n-1\} \) such that \( M = s \mod n \).

Let \( l \) be a non-negative integer and define \( V_M^{s+l} \subset \wedge^{s+l} V(\mathbf{z}) \) as follows:

\[
V_M^{s+l} = \bigoplus_{\mathbf{m} \in \mathcal{M}_s^{s+l}, \mathbf{e} \in \mathcal{E}(\mathbf{m})} \mathbb{C} w(\mathbf{m}, \mathbf{e}).
\]

(2.36)
The vector space $V_{s+nl}^{s+nl}$ has a grading similar to the grading of the Fock space $F_M$. In this case the degree $|w|$ of a wedge $w = u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_{s+nl}} = z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \cdots \wedge z^{m_{s+nl}} v_{\epsilon_{s+nl}} \in V_{s+nl}^{s+nl}$ is defined by

$$|w| = \sum_{i=1}^{s+nl} m_i - m_i.$$  (2.37)

The degree is a non-negative integer, and for $k \in \mathbb{Z}_{\geq 0}$ we denote by $V_{s+nl,k}^{s+nl}$ the homogeneous component of degree $k$.

The following result is contained in the paper [12]:

**Proposition 1.** For each $k \in \mathbb{Z}_{\geq 0}$ the homogeneous component $V_{s+nl,k}^{s+nl}$ is invariant under the $U_q(\hat{sl}_n)$-action $U_0^{(s+nl)}$ defined in section 2.3.

We have $|p_l^{M,k}(w)| = |w|$ and hence $p_l^{M,k} : V_{s+nl,k}^{s+nl} \to F_M^k$ for all $k \in \mathbb{Z}_{\geq 0}$. In the paper [11] the following propositions are shown.

**Proposition 2.** When $l \geq k$ the map $p_l^{M,k}$ is an isomorphism of vector spaces.

**Proposition 3.** For each triple of non-negative integers $k, l, m$ such that $k \leq l < m$ the map $p_{l,m}^{M,k} : V_{s+nl,k}^{s+nl} \to V_{s+nm,k}^{s+nl}$, defined for any $w \in V_{s+nl,k}^{s+nl}$ by

$$p_{l,m}^{M,k}(w) = w \wedge u_{M-s-nl} \wedge u_{M-s-nl-1} \wedge \cdots \wedge u_{M-s-nm+1},$$  (2.38)

is an isomorphism of the $U_q(\hat{sl}_n)$-modules.

We define on the vector space $F_M^k$ a level–0 action of $U_q(\hat{sl}_n)$ by using Propositions 2 and 3.

**Definition 2.** The vector space $F_M^k$ is a level-0 module of $U_q(\hat{sl}_n)$ with the action $U_0$ defined by

$$U_0 = p_l^{M,k} U_0^{(s+nl)} p_l^{M,k-1} \text{ where } l \geq k.$$  (2.39)

This definition does not depend on the choice of $l$ as long as $l$ is greater or equal to $k$. Since we have

$$F_M = \bigoplus_{k \geq 0} F_M^k$$  (2.40)

the level-0 action $U_0$ extends to the entire Fock space $F_M$. 

2.5. Level–1 action of the quantum affine algebra on the q–deformed Fock space. In this section we review the level–1 action of $U'_q(\hat{\mathfrak{sl}}_n)$ on the Fock space $F_M$ [7].

First we define the action of $U'_q(\hat{\mathfrak{sl}}_n)$ (generated by $E_i, F_i, K_i, i = 0, \ldots, n - 1$) on the vector $|M'\rangle$ as follows.

$$E_i|M'\rangle = 0,$$

(2.41)

$$F_i|M'\rangle = \begin{cases} u_{M'+1} \wedge u_{M'-1} \wedge u_{M'-2} \wedge \cdots & \text{if } i \equiv M' \mod n; \\ 0 & \text{otherwise}, \end{cases}$$

(2.42)

$$K_i|M'\rangle = \begin{cases} q|M'\rangle & \text{if } i \equiv M' \mod n; \\ |M'\rangle & \text{otherwise}. \end{cases}$$

(2.43)

For every element $v \in F_M$, there exists $N$ such that

$$v = v^{(N)} \wedge |M - N\rangle, \quad v^{(N)} \in \wedge^N V(z).$$

(2.44)

We define the actions of $E_i, F_i, K_i, i = 0, \ldots, n - 1$ on the vector $v$ as follows.

$$E_i v := E_i v^{(N)} \wedge K_i |M - N\rangle + v^{(N)} \wedge E_i |M - N\rangle,$$

(2.45)

$$F_i v := F_i v^{(N)} \wedge |M - N\rangle + K_i^{-1} v^{(N)} \wedge F_i |M - N\rangle,$$

(2.46)

$$K_i v := K_i v^{(N)} \wedge K_i |M - N\rangle.$$

(2.47)

The actions of $E_i, F_i, K_i, i = 0, \ldots, n - 1$ on $v^{(N)}$ are determined in Section 2.3. The definition of the actions on $v$ does not depend on $N$ and is well-defined, and we can easily check that the $U'_q(\hat{\mathfrak{sl}}_n)$-module defined in this section is level-1. We will use the notation $U_1$ for this $U'_q(\hat{\mathfrak{sl}}_n)$-action on the Fock space.

**Remark** The two actions $U_0$ and $U_1$ appear as the representations of the subalgebras of the quantum toroidal algebra. For details, see [11].

2.6. The $p = 1$ case. In the paper [4] it was demonstrated that the Fock space $F_M$ admits an action of the Heisenberg algebra $H$ which commutes with the level-1 action $U_1$ of the algebra $U'_q(\hat{\mathfrak{sl}}_n)$. The Heisenberg algebra is a unital $\mathbb{C}$-algebra generated by elements $1, B_a$ with $a \in \mathbb{Z}_{\neq 0}$ which are subject to relations

$$[B_a, B_b] = \delta_{a+b,0}a \frac{1 - q^{2na}}{1 - q^{2a}}.$$

(2.48)

The Fock space $F_M$ is an $H$-module with the action of the generators given by [4]

$$B_a = \sum_{i=1}^{\infty} z_i^a.$$

(2.49)
Let $\mathbb{C}[H_-]$ be the Fock space of $H$, i.e., $\mathbb{C}[H_-] = \mathbb{C}[B_{-1}, B_{-2}, \ldots]$. The element $B_{-a}$ $(a = 1, 2, \ldots)$ acts on $\mathbb{C}[H_-]$ by multiplication. The action of $B_a$ $(a = 1, 2, \ldots)$ is given by (2.48) together with the relation

$$B_a \cdot 1 = 0 \quad \text{for } a \geq 1.$$  

(2.50)

Let $\Lambda_i$ $(i \in \{0, 1, \ldots, n-1\})$ be the fundamental weights of $\widehat{\mathfrak{sl}}_n$. And let $V(\Lambda_i)$ be the irreducible (level-1) highest weight module of $U_q'(\widehat{\mathfrak{sl}}_n)$ with highest weight vector $V_{\Lambda_i}$ and highest weight $\Lambda_i$.

The following results are proven in [7]:

• The action of the Heisenberg algebra on $F_M$ and the action $U_1$ of $U_q'(\widehat{\mathfrak{sl}}_n)$ commute.

• There is an isomorphism

$$\iota_M : F_M \cong V(\Lambda_i) \otimes \mathbb{C}[H_-] \quad (M = i \mod n)$$  

(2.51)

of $U_q'(\widehat{\mathfrak{sl}}_n) \otimes H$-modules normalized so that $\iota_M(|M\rangle) = V(\Lambda_i) \otimes 1$.

In general the level-0 $U_q'(\widehat{\mathfrak{sl}}_n)$-action $U_0$ does not commute with the Heisenberg algebra. However if we choose the parameter $p$ in $U_0$ in a special way, then $U_0$ commute with the negative frequency part of $H$.

Precisely, we have the following proposition, proved in [11]:

**Proposition 4.** At $p = 1$ we have

$$[U_0, H_-] = 0.$$  

(2.52)

Let $H'_- \subseteq$ be the non-unital subalgebra in $H$ generated by $B_{-1}, B_{-2}, \ldots$.

Proposition 4 allows us to define a level-0 $U_q'(\widehat{\mathfrak{sl}}_n)$-module structure on the irreducible level-1 module $V(\Lambda_i)$ $(i \in \{0, 1, \ldots, n-1\})$. Indeed from this proposition it follows that the subspace

$$H'_- F_M \subset F_M$$  

(2.53)

is invariant with respect to the action $U_0$ at $p = 1$ and therefore a level-0 action of $U_q'(\widehat{\mathfrak{sl}}_n)$ is defined on the quotient space

$$F_M/(H'_- F_M)$$  

(2.54)

which in view of (2.51) is isomorphic to $V(\Lambda_i)$ with $i \equiv M \mod n$.

3. **Skew Young diagrams and the level–0 representations of $U_q'(\widehat{\mathfrak{sl}}_n)$**

3.1. **Skew Young diagrams.** Let us recall, following the book [10], the definitions of the skew (Young) diagrams, their semi-standard tableaux and the associated skew Schur functions.

Let $\lambda, \mu$ be partitions i.e. sequences of non–negative integers. We assume $\lambda_i \geq \mu_i$ for all possible $i$, and if $\mu_j < i \leq \lambda_j$ then we draw a
square whose edges are \((i - 1, j - 1), (i - 1, j), (i, j)\) and \((i, j - 1)\).
(For example, see Figure 1.) This diagram is called a skew (Young) diagram and is denoted as \(\lambda \setminus \mu\). We define the degree of the skew Young diagram \(\lambda \setminus \mu\) as 
\[|\lambda \setminus \mu| = \sum_i (\lambda_i - \mu_i).\]

A skew diagram is called a border strip if it is connected and contains no \(2 \times 2\) blocks of boxes. Let \(\langle m_1, \ldots, m_r \rangle\) denote the border strip of \(r\) columns such that the length of \(i\)-th column (from the right) is \(m_i\). (Figure 1)

A semi–standard tableau (s.s.t.) of the skew diagram \(\lambda \setminus \mu\) is obtained by inscribing integers \(1, 2, \ldots, n\) in each square of the skew diagram. The rule of the semi–standard tableau is as follows. The numbers are strictly increasing along the column and weakly increasing along the row. For each semi–standard tableau \(T\), let \(n_i(T)\) be the multiplicity of \(i\) in \(T\).

**Definition 3.** For each skew diagram \(\lambda \setminus \mu\), the skew Schur function \(s_{\lambda \setminus \mu}\) is defined as follows.
\[
s_{\lambda \setminus \mu}(z) = \sum_T z_1^{n_1(T)} z_2^{n_2(T)} \cdots z_N^{n_N(T)}. \tag{3.1}
\]
Here the summation is over the set of semi–standard tableaux of the skew diagram \(\lambda \setminus \mu\).

**3.2. The level–0 representations of \(U'_q(\hat{\mathfrak{sl}}_n)\) associated with the skew diagrams.** Fix a skew diagram \(\lambda \setminus \mu\) of the border strip type and degree \(N\). We put a number \((1, \ldots, N)\) on each box such that if \(l > k\) then \(x_l > x_k\) or \((x_l = x_k\) and \(y_l > y_k\)), where \((x, y)\) is a box contained in the skew diagram, and set \(a_l = -2x_l + 2y_l + a\) (\(a\) is fixed). (Figure 2)
On the space $\otimes^N V$, we define the evaluation action $\pi^{(N)}_{a_1, \ldots, a_N}$ of $U'_q(\widehat{sl}_n)$

\[
\pi^{(N)}_{a_1, \ldots, a_N}(E_i) = \sum_{j=1}^{N} K_{j+1}^{-1} \cdots K_N^{-1} E_{i}^{j+1, i},
\]

(3.2)

\[
\pi^{(N)}_{a_1, \ldots, a_N}(F_i) = \sum_{j=1}^{N} (K_{j}^{-1} \cdots (K_{j-1}^{-1}) E_{i}^{j+1, i},
\]

(3.3)

\[
\pi^{(N)}_{a_1, \ldots, a_N}(K_i) = K_i^{i} \cdots K_N^{i}, \quad (i = 1, 2, \ldots, n - 1)
\]

(3.4)

\[
\pi^{(N)}_{a_1, \ldots, a_N}(E_0) = \sum_{j=1}^{N} q^{a_j} E_j^{n, i} K_{j+1}^{0} \cdots K_N^{0},
\]

(3.5)

\[
\pi^{(N)}_{a_1, \ldots, a_N}(F_0) = \sum_{j=1}^{N} q^{-a_j} (K_{j}^{0})^{-1} \cdots (K_{j-1}^{0})^{-1} E_{j}^{1, n},
\]

(3.6)

\[
\pi^{(N)}_{a_1, \ldots, a_N}(K_0) = (\pi^{(N)}_{a_1, \ldots, a_N}(K_1K_2 \cdots K_{n-1}))^{-1},
\]

(3.7)

and on the same space, we consider the following operators:

\[
R_{i,j}(x) = \frac{xS_{i,j}^{-1} - S_{i,j}}{x-1} P_{i,j}, \quad \tilde{R}_{i,j}(x) = \frac{xS_{i,j}^{-1} - S_{i,j}}{x-1},
\]

(3.8)
where $P_{i,j} (\cdots \otimes \hat{u} \otimes \cdots \otimes \hat{v} \otimes \cdots) = \cdots \otimes \hat{v} \otimes \cdots \otimes \hat{u} \otimes \cdots$. We define
\begin{align}
R_{\lambda \lambda} &= \prod_{1 \leq i < j \leq N} R_{i,j}(q^{a_i-a_j}), \\
\hat{R}_{\lambda \lambda} &= \prod_{1 \leq i < j \leq N} R_{j,i}(q^{a_i-a_j}), \\
\hat{R}_{\lambda \lambda} &= \prod_{1 \leq i < j \leq N} \hat{R}_{N+i-j,N+i-j+1}(q^{a_i-a_j}),
\end{align}
where $(i,j)$ is on the right to $(i',j')$ in the product if $i < i'$ or $(j < j'$ and $i = i')$. As a special case of \cite{2} Proposition 1.5., we have

**Proposition 5** (\cite{2}). The subspace $\text{Im} R_{\lambda \lambda}(\otimes^N V) = \text{Im} \hat{R}_{\lambda \lambda}(\otimes^N V)$ with the action $\pi^{(N)}_{\ast_1,\ldots,\ast_N}$ is an irreducible $U_q'(\widehat{\mathfrak{sl}_n})$–module, and the map $\hat{R}_{\lambda \lambda} : (\pi^{(N)}_{\ast_1,\ldots,\ast_N}, \otimes^N V, \text{Ker} \hat{R}_{\lambda \lambda}) \to (\pi^{(N)}_{\ast_1,\ldots,\ast_N}, \text{Im} \hat{R}_{\lambda \lambda})$ is an isomorphism of the $U_q'(\widehat{\mathfrak{sl}_n})$–modules.

**Remark** In \cite{2}, this proposition is proved in the $U_q'(\widehat{\mathfrak{sl}_n})$–module case and the normalizations of $q$ and $x$ are different. The irreducibility as the $U_q'(\widehat{\mathfrak{sl}_n})$–module follows from the result of \cite{1}.

3.3. **Character formulas.** Let $\overline{\lambda}_i (i = 1, \ldots, n-1)$ be the fundamental weights of $\mathfrak{sl}_n$ and let $\epsilon_i = \overline{\lambda}_i - \overline{\lambda}_{i-1}$ $(i = 1, \ldots, n)$ with $\overline{\lambda}_0 = \overline{\lambda}_n := 0$.

The subalgebra of $U_q'(\widehat{\mathfrak{sl}_n})$ generated by $E_i, F_i, K_i^\pm$ $(i = 1, \ldots, n-1)$ is isomorphic to the algebra $U_q(\mathfrak{sl}_n)$. In the paper \cite{3} the $\mathfrak{sl}_n$–character of the irreducible $Y(\mathfrak{sl}_n)$–representation associated with a skew diagram was shown to be given by the corresponding skew Schur function. This result is immediately generalized to the $q$–deformed situation. Precisely we have

**Proposition 6** (\cite{3}). The skew Schur function $s_{\lambda \lambda}(z)$ where $z_i = e^{\epsilon_i}$ is equal to the $U_q(\mathfrak{sl}_n)$–character of the irreducible $U_q'(\widehat{\mathfrak{sl}_n})$–module described by Proposition 3.

As a corollary we get

**Corollary 1.** The dimension of the space $\text{Im} R_{\lambda \lambda} \subset (\otimes^N V)$ is equal to the total number of the semi–standard tableaux of the skew diagram $\lambda \backslash \mu$.

Let $V(\Lambda_k)$ be the level–1 irreducible module of $U_q(\widehat{\mathfrak{sl}_n})$ whose highest weight is the $k$–th fundamental weight $\Lambda_k$ of $\mathfrak{sl}_n$. We set $\text{ch}(V(\Lambda_k)) = \cdots$
∑_i,λ (dim V_{λ,i}) e^λ q^i, where V_{λ,i} is the weight subspace with U_q(\frak{sl}_n)-weight \lambda and homogeneous degree i. The following proposition is proved in [8].

**Proposition 7** ([8]). Setting z_i = e^{t_i} we have

\[ ch(V(Λ_k)) = q^{\frac{1-n}{2n} \cdot \frac{k(n-k)}{2n}} \sum_{\theta \in BS \ | \ \theta \ equiv k \ mod \ n} q^{\frac{1}{2n} |\theta| (n-|\theta|)} s_\theta(z). \]

where BS is the set of all the border strips \( \theta = \langle m_1, \ldots, m_r \rangle \) and \( t(\theta) = \sum_{i=1}^{r-1} (r-i)m_i \) with \( m_r < n \).

Note that if \( m_i > n \) for some \( i \), then the skew Schur function \( s_\theta \) is equal to 0, moreover, for the border strip of the form \( \theta_l = \langle m_1, \ldots, m_r, n, \ldots, n \rangle \) the number \( \frac{1}{2n} |\theta_l| (n-|\theta_l|) + t(\theta_l) \) does not depend on \( l \).

**4. Non–symmetric Macdonald polynomials and the decomposition**

**4.1. Non–symmetric Macdonald polynomials.** We will define the non–symmetric Macdonald polynomials as the joint eigenfunctions of the Cherednik’s operators \( Y_i^{(N)} \) \( (i = 1, \ldots, N) \). It will be convenient for our purposes to label these polynomials by the set of pairs \( (\lambda, \sigma) \) which we now describe.

Let \( \tilde{M}_N \) be the a set of all non–decreasing sequences of integers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \), and let \( \tilde{M}_N^n \) be the subset of \( \tilde{M}_N \) which consists of all \( n \)–strict non–decreasing sequences (cf. Section 2.4). For each \( \lambda \in \tilde{M}_N \) we set \( |\lambda| := \sum_{i=1}^{N} \lambda_i \). For \( \lambda, \mu \in \tilde{M}_N \) such that \( |\lambda| = |\mu| \) we define the dominance (partial) ordering:

\[ (4.1) \quad \lambda \succ \mu \quad \Leftrightarrow \quad \sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j \ (i \in \{1, 2, \ldots, N\}). \]

Let \( S^\lambda \subset S_N \) be the set of elements \( \sigma \) such that if \( \lambda_{\sigma(i)} = \lambda_{\sigma(j)} \) and \( \sigma(i) < \sigma(j) \) then \( i < j \). We define the total ordering on \( S^\lambda \):

\[ (4.2) \quad \sigma \succ \sigma' \quad \Leftrightarrow \quad \text{the last nonzero element of } (\lambda_{\sigma(i)} - \lambda_{\sigma'(i)})_{i=1}^{N} \text{ is } < 0. \]

Then the following properties are satisfied. (In what follows the \( \sigma(i, i+1) \) denotes the composition of \( \sigma \) and a transposition \( (i, i+1) \))

a) \( S^\lambda \) has the unique minimal element with respect to the ordering \( (4.2) \). We denote this element by \( \min \). Note that one has \( \lambda_{\min(i)} \leq \lambda_{\min(i+1)} \ (i = 1, \ldots, N - 1) \).

b) \( S^\lambda \) is connected, i.e. for any \( \sigma \in S^\lambda \), there exist \( i_1, \ldots, i_r \) such
that if we put \( \sigma_l = \sigma(i_1, i_1 + 1) \ldots (i_l, i_l + 1) \) then \( \sigma_r = \min_i, \sigma_l \in S^\lambda, \sigma_l \succ \sigma_{l+1} \) \((l = 1, \ldots, r)\).

c) Suppose \( \sigma \in S^\lambda \), then \( \sigma(i, i + 1) \in S^\lambda \iff \lambda_{\sigma(i)} \neq \lambda_{\sigma(i+1)} \).

d) If \( \lambda_{\sigma(i)} > \lambda_{\sigma(i+1)} \) and \( \sigma \in S^\lambda \) then \( \sigma \succ \sigma(i, i + 1) \).

We define the partial ordering of the set \( \{ (\lambda, \sigma) \mid \lambda \in \tilde{M}_N, \sigma \in S^\lambda \} \):

\[
(\lambda, \sigma) \succ (\tilde{\lambda}, \tilde{\sigma}) \iff |\lambda| = |\tilde{\lambda}| \quad \text{and} \quad \left\{ \begin{array}{ll}
\lambda \succ \tilde{\lambda} \\
\lambda = \tilde{\lambda}, \sigma \succ \tilde{\sigma}.
\end{array} \right.
\] (4.3)

Then \( Y_i^{(N)} \) act triangularly on \( \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \) with respect to this ordering [14]:

\[
Y_i^{(N)}z^{\lambda^e} = \xi_i^\lambda(\sigma)z^{\lambda^e} + \text{"lower terms"},
\] (4.4)

\[
\xi_i^\lambda(\sigma) = p^\lambda_{\sigma(i)}q^{2\sigma(i) - N - 1} \quad (\sigma \in S^\lambda).
\] (4.5)

In the above notation, we identify the ordering of monomials \( z^{\lambda^e} := z_1^{\lambda_{\sigma(1)}}z_2^{\lambda_{\sigma(2)}} \ldots z_N^{\lambda_{\sigma(N)}} \) with the ordering on the set of pairs \((\lambda, \sigma)\).

For generic \( q \) and \( p \) the pair \((\lambda, \sigma)\) is uniquely determined from the ordered set \((\xi_1^\lambda(\sigma), \xi_2^\lambda(\sigma), \ldots, \xi_N^\lambda(\sigma)):\)

\[
(\lambda, \sigma) \neq (\tilde{\lambda}, \tilde{\sigma}) \iff (\xi_1^\lambda(\sigma), \xi_2^\lambda(\sigma), \ldots, \xi_N^\lambda(\sigma)) \neq (\xi_1^{\tilde{\lambda}}(\tilde{\sigma}), \xi_2^{\tilde{\lambda}}(\tilde{\sigma}), \ldots, \xi_N^{\tilde{\lambda}}(\tilde{\sigma})).
\] (4.6)

Therefore one can simultaneously diagonalize the operators \( Y_i^{(N)} \) \((1 \leq i \leq N)\):

\[
Y_i^{(N)}\Phi^\lambda(\sigma)(z) = \xi_i^\lambda(\sigma)\Phi^\lambda(\sigma)(z), \quad \Phi^\lambda(\sigma)(z) = z^{\lambda^e} + \text{"lower terms".}
\] (4.7)

The Laurent polynomial \( \Phi^\lambda(\sigma)(z) \) is known as the non–symmetric Macdonald polynomial.

The action of \( g_{i,i+1} \) on the non–symmetric Macdonald polynomial is as follows [14].

\[
g_{i,i+1}\Phi^\lambda(\sigma)(z) = A_i(\sigma)\Phi^\lambda(\sigma)(z) + B_i(\sigma)\Phi^\lambda_{\sigma(i,i+1)}(z),
\] (4.8)

\[
A_i(\sigma) := \frac{(q - q^{-1})x}{x - 1}, \quad B_i(\sigma) := \left\{ \begin{array}{ll}
q^{-1}\{x\} & (\lambda_{\sigma(i)} > \lambda_{\sigma(i+1)}) \\
0 & (\lambda_{\sigma(i)} = \lambda_{\sigma(i+1)}) \\
q^{-1}\{x\} & (\lambda_{\sigma(i)} < \lambda_{\sigma(i+1)}).
\end{array} \right.
\] (4.9)

\[
\{x\} := \frac{(x - q^2)(q^2x - 1)}{(x - 1)^2}, \quad x := \frac{\xi_{i,i+1}^\lambda(\sigma)}{\xi_i^\lambda(\sigma)}.
\] (4.10)

The case \( p = 1 \) is not generic. However, from the results of [14] it follows that the coefficients of \( \Phi^\lambda(\sigma)(z) \) have no poles at \( p = 1 \). Therefore the non–symmetric Macdonald polynomials \( \Phi^\lambda(\sigma)(z) \) are still well–defined at \( p = 1 \) and the formulas (4.7)–(4.10) are still satisfied.
In what follows we will let \( \tilde{\Phi}_\sigma^\lambda(z) \) denote the non–symmetric Macdonald polynomial at \( p = 1 \). In virtue of the triangularity \([1.4]\) the non–symmetric Macdonald polynomials \( \tilde{\Phi}_\sigma^\lambda(z) (\lambda \in \tilde{\mathcal{M}}_N, \sigma \in S_{\lambda}^\lambda) \) form a base of \( \mathbb{C}[z_1^{\pm1}, \ldots, z_N^{\pm1}] \). We put

\[
E^\lambda = \bigoplus_{\sigma \in S_{\lambda}^\lambda} \mathbb{C}\tilde{\Phi}_\sigma^\lambda(z).
\]

Then \( \mathbb{C}[z_1^{\pm1}, \ldots, z_N^{\pm1}] = \oplus_{\lambda} E^\lambda \). In Section 4.2 we will use the following lemma.

**Lemma 1.** Let \( e_{-k} = \sum_{1 \leq n_1 < \cdots < n_k \leq N} z_{n_1}^{-1} \cdots z_{n_k}^{-1} \). Suppose that \( \lambda \in \tilde{\mathcal{M}}_N \) satisfies \( \lambda_i - \lambda_{i+1} = 0 \) or 1. Then we have

\[
e_{-k} \tilde{\Phi}_\varsigma^\lambda(z) = \tilde{\Phi}_\varsigma^\lambda(z).
\]

Here \( \lambda = (\lambda_1, \ldots, \lambda_{N-k}, \ldots, \lambda_{N-k+1} - 1, \ldots, \lambda_N - 1) \) and \( \varsigma(\in S_{\lambda}^\lambda, S_{\lambda}^\lambda) \) is the minimal element of \( S_{\lambda}^\lambda \).

**Proof.** By the triangularity of the non–symmetric Macdonald polynomial \([1.4]\), we have

\[
e_{-k} \tilde{\Phi}_\varsigma^\lambda(z) = e_{-k}(z_{\lambda_1}^{\lambda_1} + \sum_{\mu < \lambda, \sigma \in S_{\mu}^\mu} c_{\mu, \sigma} z_{\mu}^{\sigma})
\]

\[
= z_{\lambda_1}^{\lambda_1} + \sum_{(\mu, \sigma) < (\lambda, \varsigma)} c'_{\mu, \sigma} z_{\mu}^{\sigma} = \tilde{\Phi}_\varsigma^\lambda(z) + \sum_{(\mu, \sigma) < (\lambda, \varsigma)} c''_{\mu, \sigma} \tilde{\Phi}_{\sigma}^\mu(z).
\]

At \( p = 1 \), the operators \( Y_i^{(N)} \) commute with symmetric Laurent polynomials considered as multiplication operators on \( \mathbb{C}[z_1^{\pm1}, \ldots, z_N^{\pm1}] \). Hence we have

\[
Y_i^{(N)} e_{-k} \tilde{\Phi}_\varsigma^\lambda(z) = e_{-k} Y_i^{(N)} \tilde{\Phi}_\varsigma^\lambda(z) = q^{2\varsigma(i)-N-1} e_{-k} \tilde{\Phi}_\varsigma^\lambda(z).
\]

The ordered set of eigenvalues \( \{q^{2\varsigma(i)-N-1}\}_{i=1}^N \) determines the element \( \varsigma(\in S_{\lambda}^\lambda, S_{\lambda}^\lambda) \) uniquely. Hence \([1.13]\) and \([1.14]\) lead to

\[
e_{-k} \tilde{\Phi}_\varsigma^\lambda(z) = \tilde{\Phi}_\varsigma^\lambda(z) + \sum_{\mu < \lambda} c''_{\mu} \tilde{\Phi}_{\sigma}^\mu(z).
\]

Now let us consider any \( \mu \) which appears in the sum \([1.13]\). If there exists \( i < N - k \) such that \( \mu_i < \lambda_i \) then for \( j > i \) we necessarily have \( \mu_j < \lambda_j \) because of the assumption \( \lambda_i - \lambda_{i+1} = 0 \) or 1 and the fact that \( \lambda_i < \lambda_j \) implies \( \mu_i < \mu_j \), which follows since \( \varsigma(\in S_{\mu}^\mu) \). But \( \mu_j < \lambda_j \) \( (j > i) \) leads to \( |\lambda| > |\mu| \) which is in contradiction with \( \mu < \lambda \). Thus we have \( \mu_i \geq \lambda_i \) \( (i < N - k) \). If \( \mu_i > \lambda_i \) for some \( i \), then necessarily \( \mu \neq \lambda \) which is again a contradiction. Hence we get \( \mu_i = \lambda_i \) \( (i < N - k) \). By the
condition $\mu < \tilde{\lambda}$, we have $\mu_{N-k} = \tilde{\lambda}_{N-k}$. Because of the assumption on $\lambda$ and the fact that $\lambda_i < \lambda_j$ implies $\mu_i < \mu_j$, we get $\mu_j \leq \tilde{\lambda}_j$ ($j > N-k$). Combining this with $|\tilde{\lambda}| = |\mu|$ we conclude that $\mu_j = \tilde{\lambda}_j \forall j$. That is the second term in the r.h.s. of (4.15) is zero.

4.2. The decomposition. Let us consider the quotient space $F_M/H'F_M$ and for each $k \geq 0$ its subspace $F_M^k/(H'F_M \cap F_M^k)$.

It is straightforward to establish the necessary and sufficient condition for the vector $\omega = \sum_{\lambda} \sum_{\sigma \in S^\lambda} \bar{\Phi}_\sigma^\lambda(z) \otimes \psi_\sigma^\lambda (\in \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^N V))$ to be equivalent to 0 in the quotient space $\wedge^N V(z)$. The result is

$$\forall \lambda \left\{ \begin{array}{ll} \psi^\lambda_{\sigma(i,i+1)} = -\check{R}_{i,i+1} (q^{\sigma(i) - \sigma(i+1)}) \psi^\lambda_\sigma, & \forall \sigma \text{ s.t. } \lambda_{\sigma(i)} > \lambda_{\sigma(i+1)}; \\ (q^{-2} S_{i,i+1} - S_{i+1,i}) \psi^\lambda_\sigma = 0, & \forall \sigma \text{ s.t. } \lambda_{\sigma(i)} = \lambda_{\sigma(i+1)} (4.16) \end{array} \right. $$

where $\check{R}_{i,i+1}(x)$ is defined in (3.8). In view of the properties of the set $S^\lambda$, in the space $\wedge^N V(z)$ we have

$$\bar{\Phi}_\sigma^\lambda(z) \otimes \psi_\sigma \sim \bar{\Phi}_{\text{min}}^\lambda(z) \otimes \check{R}_{i,i+1} (q^{2(\sigma(i) - \sigma(i+1))}) \cdots \check{R}_{i+1,i+1} (q^{2(\sigma(i+1) - \sigma(i))}) \psi_\sigma. $$

Here we used the notations of Section 4.1.

By the triangularity of the non-symmetric Macdonald polynomial (4.4) and the relation (4.17), we get

$$V_{M}^{s+n,k} = \bigoplus_{\lambda} (E^\lambda \otimes (\otimes^N V))/\Omega \cap (E^\lambda \otimes (\otimes^N V)) \tag{4.18}$$

$$= \bigoplus_{\lambda} (\bar{\Phi}_{\text{min}}^\lambda(z) \otimes (\otimes^N V))/\Omega \cap (\bar{\Phi}_{\text{min}}^\lambda(z) \otimes (\otimes^N V)), $$

where the summation is over $\lambda \in \tilde{\mathcal{M}}_N^n$, such that $\lambda_1 \leq m^0_{s+n,k}$, $|m^0 - \lambda^{\text{min}}| = k$.

**Proposition 8.** Define the set $\tilde{\mathcal{M}}_{s+n,k}^n$ as

$$\tilde{\mathcal{M}}_{s+n,k}^n = \{ \lambda \in \tilde{\mathcal{M}}_N^n : \lambda_1 \leq m^0_{s+n,k}, |m^0 - \lambda^{\text{min}}| = k \text{ and } \lambda_i - \lambda_{i+1} = 0 \text{ or } 1 \}. \tag{4.19}$$

Every vector from the linear space $F_M^k/(H'F_M \cap F_M^k)$ can be expressed as a linear combination of vectors of the form $\wedge(\bar{\Phi}_{\text{min}}^\lambda(z) \otimes \psi^\lambda)|M - s - nk)$, where $\lambda \in \tilde{\mathcal{M}}_{s+n,k}^n$ and $\psi^\lambda \in \otimes^N V$. 

Proposition 9. Proof. By the equation (4.18), it is sufficient to show that
\[ \langle \Phi_{\min}^\lambda (z) \otimes \psi^\lambda \rangle | M - s - nk \rangle = 0 \]
for some \( \lambda \in \mathcal{M}_{s+nk} \), \( \lambda_1 \leq m_0^{s+nk} \), \( |m_0 - \lambda_{\min}| = k \), \( \psi^\lambda \in \otimes^N V \) is equivalent to 0 in the space \( F^k_M/(H'_F M \cap F^k_M) \) unless \( \lambda_i - \lambda_{i+1} = 0 \) or 1 for all \( i = 1, \ldots, N - 1 \). We will prove this by induction with respect to the ordering of the set \( \mathcal{M}_{s+nk} \). (Note that if \( \lambda \) is not \( n \)-strict then \( \langle \Phi_{\min}^\lambda (z) \otimes \psi^\lambda \rangle = 0 \).

Since \( \lambda_i - \lambda_{i+1} \neq 0,1 \) implies that \( \lambda = (\lambda_1, \ldots, \lambda_i - 1, \lambda_{i+1}, \ldots) \) is lower with respect to the ordering of \( \mathcal{M}_{s+nk} \), the minimal element satisfies the condition of Proposition 8.

Fix \( \lambda \) and assume that the proposition is proved for all \( \mu \) such that \( \mu < \lambda \). Define \( \bar{\lambda} \in \mathcal{M}_{s+nk} \) as follows:
\[ \lambda_i = \lambda_{i+1} \Leftrightarrow \bar{\lambda}_i = \bar{\lambda}_{i+1}, \quad (4.20) \]
\[ \lambda_1 = \bar{\lambda}_1, \quad (4.21) \]
\[ \bar{\lambda}_i \neq \bar{\lambda}_{i+1} \Rightarrow \bar{\lambda}_i = \bar{\lambda}_{i+1} + 1. \quad (4.22) \]

For each positive integer \( l \), define \( n_l = \# \{ i \mid \bar{\lambda}_i - \lambda_i = l \} \). If \( n_l = 0 \) for all \( l \geq 1 \) then either \( \langle \Phi_{\min}^\lambda (z) \otimes \psi^\lambda \rangle | M - s - nk \rangle \) itself satisfies the condition of the Proposition 3 or else the element \( \prod_{l \geq 1} B^m_{-l} \cdot \langle \Phi_{\min}^\lambda (z) \otimes \psi^\lambda \rangle | M - s - nk \rangle \) is in \( H'_F M \cap F^k_M \). Expanding the last element, in the space \( F^k_M/(H'_F M \cap F^k_M) \) we get
\[ \langle \Phi_{\min}^\lambda (z) \otimes \psi^\lambda \rangle | M - s - nk \rangle + \sum_{\mu < \lambda, \sigma \in S^\mu} \langle \Phi_{\mu}^\lambda (z) \otimes \psi_{\sigma}^\mu \rangle | M - s - nk \rangle \sim 0, \quad (4.23) \]
for some \( \psi_{\sigma}^\mu \). By (4.17) and the induction assumption the proposition is proven.

Proposition 9. For each \( \lambda \in \mathcal{N}^{n,k}_{s+nk} \), define \( J \) and \( r_j \) such that \( \lambda_1 = \cdots = \lambda_{r_j} > \lambda_{r_j+1} = \cdots = \lambda_{r_j+r_j-1} > \cdots \geq \lambda_{r_1 + \cdots + r_j = N} \), then in the space \( F^k_M/(H'_F M \cap F^k_M) \), for each \( \psi \in \otimes^N V \) we have
\[ \langle \Phi_{\min}^\lambda (z) \otimes R_{i,i+1}(q^2)\psi \rangle | M - s - nk \rangle \sim 0, \quad (4.24) \]
\[ \langle \Phi_{\min}^\lambda (z) \otimes \prod_{0 \leq a < r_j, 1 \leq a \leq r_j} R_{l_j+a,l_j+r_j+r_j+1}(q^{-2(a+b)})\psi \rangle | M - s - nk \rangle \sim 0, \quad (4.25) \]
where \( (a,b) \) is on the right to \( (a',b') \) in the product if \( a < a' \) or \( a = a' \) and \( b < b' \), \( l_j = \sum_{i=1}^{j-1} r_j \) and \( l_0 = 0 \).

Proof. The first relation follows from (4.16) and the identity
\[ \text{Im}(q^2 S^{-1}_{i,i+1} - S_{i,i+1}) = \text{Ker}(q^{-2} S^{-1}_{i,i+1} - S_{i,i+1}). \quad (4.26) \]
Consider the second relation. We define
\[ \bar{\lambda} = (\lambda_1, \ldots, \lambda_{r_j+1}, \ldots, \lambda_{1+r_j+1} + 1, \ldots, \lambda_N + 1) \] (4.27)

By Lemma 1, the definition of the space \( F^k_M / (H' \cap F^k_M) \) and the relation (6.51) in \([11]\), we get
\[ f(B_{-1}, \ldots, B_{-(r_j+1)}) \cdot \wedge(\tilde{\Phi}_{\min}(\bar{\lambda} \otimes \psi)|M - s - nk) | M - s - nk) \sim 0. \] (4.28)

Here \( f(x_1, \ldots, x_t) \) is a polynomial such that
\[ f(N \sum_{i=1}^N z_i, N \sum_{i=1}^N z_i^2, \ldots, N \sum_{i=1}^N z_i^t) = \sum_{1 \leq i_1 < \cdots < i_t} z_{i_1} \cdots z_{i_t}, \] (4.29)
and \( \varsigma \in S^{\bar{\lambda}} \) is the minimal element of \( S^{\bar{\lambda}} \).

If we apply the formula (4.17), we get
\[ \wedge(\tilde{\Phi}_{\min}(\bar{\lambda} \otimes \psi)|M - s - nk) \sim 0, \] (4.30)
where \((a, b)\) on the right to \((a', b')\) in the product if \( a < a' \) or \((a = a' \text{ and } b < b')\). Finally, taking into account the relation
\[ \prod_{0 \leq a \leq r_j, 0 \leq b \leq r_j+1} \hat{R}_{t_j+a,l_j+r_j+1-b}(q^{-2(a+b+1)}), \prod_{1 \leq a \leq r_j, 0 \leq b \leq r_j+1} P_{t_j+a,l_j+r_j+1-b}(q^{-2(a+b)}), \] (4.31)
we obtain (4.25).

With notations of Proposition 9, for each \( \lambda \in \tilde{M}_{s+nk}^{n,k} \) define the linear subspace of \( \otimes^N V \)
\[ V^\lambda = \sum_{\lambda_{\min(i)} = \lambda_{\min(i+1)}} \text{Im} R_{i,i+1}(q^2) + \sum_{j=1}^{J-1} \text{Im}(\prod_{1 \leq a \leq r_j, 1 \leq b \leq r_j+1} R_{t_j+a,l_j+r_j+1-b}(q^{-2(a+b)})). \] (4.32)

By Proposition 9, we get
Proposition 10. Consider the map

$$\psi_k : \bigoplus_{\lambda \in \mathcal{M}_{s+nk}^{n,k}} \tilde{\Phi}_{\min}^\lambda (z) \otimes (\otimes^{s+nk} V/V^\lambda) \rightarrow F_M^k/(H'_M F_M \cap F_M^k) \quad (4.33)$$

$$v \mapsto v \wedge |M - s - nk\rangle.$$ Define the action of $U_q' (\widehat{\mathfrak{sl}_n})$ on the l.h.s. of (1.34) by (2.22-2.27).

Then the map $\psi_k$ is well-defined, surjective and is a $U_q' (\widehat{\mathfrak{sl}_n})$-intertwiner.

Proposition 11. Let $\lambda \in \mathcal{M}_{s+nk}^{n,k}(=N)$. For any such $\lambda$ we define $J$, $r_j$ and $l_j$ in the same way as in Proposition [3]. Let $\theta$ be the border strip characterized by $\langle r_j, r_{j-1}, \ldots, r_1 \rangle$. We have

$$\otimes^N V/V^\lambda = \otimes^N V/\text{Ker} \tilde{R}_\theta. \quad (4.34)$$

Proof. First we will show that $\otimes^N V/V^\lambda \supset \otimes^N V/\text{Ker} \tilde{R}_\theta$. Applying repeatedly the Yang–Baxter equation $R_{a,b}(x)R_{a,c}(xy)R_{b,c}(y) = R_{b,c}(y)R_{a,c}(xy)R_{a,b}(x)$, we can move some special elements to the right in the product

$$\tilde{R}_\theta = \cdots R_{i+1,i} (q^{-2}) = \cdots \prod_{1 \leq a \leq r_{j+1}, 0 \leq b \leq r_j} R_{l_j + r_j + a, l_j + r_j - b} (q^{2(a+b)}),$$

where $(a, b)$ on the right to $(a', b')$ in the product if $a < a'$ or $(a = a'$ and $b < b'$), and $\lambda_{\min(i)} = \lambda_{\min(i+1)}$. By the formula

$$R_{b,a}(x)R_{a,b}(x) = \frac{(x - q^2)(x^{-1}q^{-2} - 1)}{(x - 1)(x^{-1} - 1)} \text{id}, \quad (4.36)$$

we get $\otimes^N V/V^\lambda \supset \otimes^N V/\text{Ker} \tilde{R}_\theta$.

Next we will show that $\otimes^N V/V^\lambda \subset \otimes^N V/\text{Ker} \tilde{R}_\theta$. We show that the vectors $\otimes_{i=1}^N v_{e_i}$ such that $e_i < e_{i+1}$ if $\lambda_{\min(i)} = \lambda_{\min(i+1)}$, $e_i \geq e_{i+1}$ if $\lambda_{\min(i)} \neq \lambda_{\min(i+1)}$ span the space $\otimes^N V/V^\lambda$.

Using the relations $R_{i,j+1}(q^2)(\otimes_i^N v_{e_i}) \sim 0$ ($\lambda_{\min(i')} = \lambda_{\min(i'+1)}$) we find that the set of vectors $\{\otimes_{i=1}^N v_{e_i} | \lambda_{\min(i)} = \lambda_{\min(i+1)} \}$ spans the space $\otimes^N V/V^\lambda$. For each vector of the form $\otimes_{i=1}^N v_{e_i}$ we define $\tilde{N}(\otimes_{i=1}^N v_{e_i}) = \#\{(i, j) | i < j, e_i \geq e_j \text{ and } \lambda_{\min(i)} \neq \lambda_{\min(j)}\}$. Consider the vector $\otimes_{i=1}^N v_{e_i}$ such that if $\lambda_{\min(i)} = \lambda_{\min(i+1)}$, then $e_i < e_{i+1}$. Assume that there is an $i$ such that $e_i < e_{i+1}$ (if $\lambda_{\min(i)} \neq \lambda_{\min(i+1)}$), then we get the relation

$$\prod_{1 \leq a \leq r_j, 1 \leq b \leq r_{j+1} - 1} R_{l_j + a, l_j + r_j + b}(q^{-2(a+b)}) (\otimes^N v_{e_i}) \sim 0, \quad (4.37)$$

for all possible $j$. By (4.37), the vector $\otimes^N v_{e_i}$ is equivalent to a linear combination of the vectors $\otimes^N v_{e_{i'}}$ such that $\tilde{N}(\otimes^N v_{e_{i'}}) < \tilde{N}(\otimes^N v_{e_i})$. 

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Because $\otimes^N v_{e_i}$ is invariant by the relations $R_{i',i'}(q^2)(\otimes^N v_{e_i})$ ($l_j + 1 \leq i' \leq l_{j+1} - 1$, $l_{j+1} + 1 \leq i' \leq l_{j+2} - 1$), if we use these relations we get that the vector $\otimes^N v_{e_i}$ ($e_i < e_{i+1}$ if $\lambda_{\min(i)} = \lambda_{\min(i+1)}$) is expressed by the sum of $\otimes^N v_{e_{i'}}$ such that $\tilde{N}(\otimes^N v_{e_{i'}}) < \tilde{N}(\otimes^N v_{e_i})$ and $e_{i'} < e_{i'+1}$ (if $\lambda_{\min(i')} \neq \lambda_{\min(i+1)}$).

By the induction on $\tilde{N}$ we find that the vectors $\otimes^N v_{e_i}$ ($e_i < e_{i+1}$ if $\lambda_{\min(i)} = \lambda_{\min(i+1)}$, $e_i \geq e_{i+1}$ if $\lambda_{\min(i)} \neq \lambda_{\min(i+1)}$) span the space $\otimes^N V/V^\lambda$.

The number of these vectors is equal to the number of semi–standard tableaux of $\theta$. By Corollary [4], we get $\otimes^N V/V^\lambda \subset \otimes^N V/Ker R_\theta$.

In what follows we identify the border strips $\langle m_1, \ldots, m_r \rangle$ and $\langle m_1, \ldots, m_r, n, \ldots, n \rangle$, and identify $\langle \lambda_1, \ldots, \lambda_{s+nk} \rangle$ and $\langle \lambda_1 + 1, \ldots, \lambda_1 + 1, \lambda_1, \ldots, \lambda_{s+nk} \rangle$ which are elements of $\bigsqcup_k \mathcal{M}_{s+nk}^{\lambda}$. Proposition [4] gives a one to one correspondence of $\bigsqcup_k \mathcal{M}_{s+nk}^{\lambda}$ and the set of all skew Young diagrams of the border strip type $\langle m_1, \ldots, m_r \rangle$ which satisfy $m_i \leq n$ for all $i$ and $\sum_{i=1}^r m_i \equiv s \mod n$. On this correspondence the degree of the semi–infinite wedge $\wedge(\Phi^\lambda_{\min}(z) \otimes \psi)|M - s - nk)$ is equal to $\frac{1}{2n} - \frac{k(n-k)}{2n} + \frac{1}{2n} |\theta|(n - |\theta|) + t(\theta)$, where $\theta$ is the border strip which corresponds to $\lambda$ and $t(\theta) = \sum_{i=1}^{r-1} (r - i)m_i$.

We define $ch(F_M/H'_F = \sum_{\mu,i} dim(V_{\mu,i})e^\mu q^i$, where $V_{\mu,i}$ is the subspace of $F_M/H'_F$ of the degree $[2.37]$ $i$ and of the $U_q(\mathfrak{sl}_n)$–weight $\mu$. We put $\sum_{\mu,i} a_{\mu,i} e^\mu q^i \leq \sum_{\mu,i} b_{\mu,i} e^\mu q^i$ iff $a_{\mu,i} \leq b_{\mu,i}$ for all $\mu$ and $i$. By Proposition [4] we have

$$ch(F_M/H'_F \leq q^{1\frac{1}{2n} - \frac{k(n-k)}{2n}} \sum_{\theta \in BS_{\overline{|\theta|} = k \mod n}} q^{\frac{1}{2n} |\theta|(n - |\theta|) + t(\theta)} s_{\theta}(z).$$ (4.38)

$F_M/H'_F$ with $U_1$–action is isomorphic to $V(\Lambda_k)$, this isomorphism is degree preserving with respect to the degree $[2.37]$ on $F_M/H'_F$ and the homogeneous degree on $V(\Lambda_k)$, and the character formula of $V(\Lambda_k)$ is given in Proposition [4]. Hence the inequality of (4.38) is, in fact, an equality and, therefore, the map (4.34) must be bijective. Thus have the following theorem

**Theorem 12.** We have the isomorphism of $U_q'(\mathfrak{sl}_n)$–modules:

$$F_M/H'_F \simeq \bigoplus_{\theta} V_{\theta},$$ (4.39)

where the sum is over all border strips $\langle m_1, \ldots, m_r \rangle$, $(m_i \leq n, m_r < n$ and $N \equiv M \mod n)$, the space $V_{\theta}$ and the level–0 $U_q'(\mathfrak{sl}_n)$–action is
defined by \((\pi_{a_1, \ldots, a_N}, R_\theta \otimes \mathbb{C}^2)\) where \(N = \sum_{i=1}^{r} m_i\) and \(a_l + \sum_{j=1}^{m_i} m_i = 2(l - 1 + \sum_{i=1}^{j-1} m_i)\).

4.3. \(\mathfrak{sl}_2\) case. In this section we will discuss the \(\mathfrak{sl}_2\) case in a somewhat more detail.

Let \(W_n\) be the \((n+1)\)-dimensional irreducible module of \(U_q(\mathfrak{sl}_2)\), and \(W_n(b)\) be the evaluation module with the parameter \(b\) whose \(U_q^\prime(\mathfrak{sl}_2)\)–module structure is given by

\[
E_0 = q^b F_1, \quad F_0 = q^{-b} E_1, \quad K_0 = K_1^{-1}.
\]  

(4.40)

It is known that every finite dimensional irreducible \(U_q^\prime(\mathfrak{sl}_2)\)–module is isomorphic to \(\otimes \mu W_{n_\mu}(b_\mu)\) for some \(n_\mu\) and \(b_\mu\). We will represent the \(U_q^\prime(\mathfrak{sl}_2)\)–module described by a skew Young diagram as the tensor product of the form \(\otimes \mu W_{n_\mu}(b_\mu)\).

**Proposition 13.** Let \(\theta\) be the skew Young diagram of border strip \(\langle m_1, \ldots, m_r \rangle\) such that \(m_i = 1\) or \(2\), and \(N = \sum_{i=1}^{r} m_i\), \(a_l + \sum_{i=1}^{m_i} m_i = 2(l - 1 + \sum_{i=1}^{j-1} m_i)\). We put \(I = \{i \mid m_i = 1\text{ and } m_{i-1} = 2\} = \{l_1, l_2, \ldots, l_r\}\) \((m_0 = 2, l_i < l_{i+1})\) and let \(n_i\) be the integer such that \(m_i = \cdots = m_{l_i+n_i-1} = 1\), \(m_{l_i+n_i} = 2\) and \(b_i = 2l_i + n_i - 3\).

The \(U_q^\prime(\mathfrak{sl}_2)\)–module \((\pi_{a_1, \ldots, a_N}, R_\theta \otimes \mathbb{C}^2)\) is isomorphic to \(W_{n_1}(b_1) \otimes W_{n_2}(b_2) \otimes \cdots \otimes W_{n_r}(b_r)\).

**Proof.** By Proposition 3, we find that \((\pi_{a_1, \ldots, a_N}, R_\theta \otimes \mathbb{C}^2)\) is isomorphic to \((\pi_{a_1, \ldots, a_N}, \otimes \mathbb{C}^2/\text{Ker} \tilde{R}_\theta)\).

As in the proof of Proposition 1, we get \(\text{Im} \tilde{R}_{i,i+1}(q^2) \subset \text{Ker} \tilde{R}_\theta\) if \(a_{i+1} = a_i - 2\) and \(\text{Im} \tilde{R}_{i,i+1}(q^{-2}) \subset \text{Ker} \tilde{R}_\theta\) if \(a_{i+1} = a_i + 2\).

We can directly confirm that the \(U_q^\prime(\mathfrak{sl}_2)\)–module \((\pi_{a,a+2, \ldots, a+2(l-1)}, \otimes \mathbb{C}^2/\sum_{i=1}^{l-1} \text{Im} \tilde{R}_{i,i+1}(q^{-2})\)) is 1–dimensional and the module \((\pi_{a,a+2, \ldots, a+2(l-1)}, \otimes \mathbb{C}^2/\sum_{i=1}^{l-1} \text{Im} \tilde{R}_{i,i+1}(q^{-2})\)) is isomorphic to \(W_{l}(a+2l-1)\), where \((l)\) is the Young diagram of degree \(l\) which has only one row.

If we put

\[
\tilde{V} = \sum_{i|a_{i+1}=a_{i+2}} \text{Im} \tilde{R}_{i,i+1}(q^2) + \sum_{i|a_{i+1}=a_{i-2}} \text{Im} \tilde{R}_{i,i+1}(q^{-2}),
\]  

(4.41)

then \((\pi_{a_1, \ldots, a_N}, \otimes \mathbb{C}^2/\tilde{V}) \simeq W_{n_1}(b_1) \otimes W_{n_2}(b_2) \otimes \cdots \otimes W_{n_r}(b_r)\).

Since the dimension of \(W_{n_1}(b_1) \otimes W_{n_2}(b_2) \otimes \cdots \otimes W_{n_r}(b_r)\) is equal to the dimension \(\otimes \mathbb{C}^2/\text{Ker} \tilde{R}_\theta\) the proof is finished.■
By Proposition 13, we can rewrite the decomposition (4.39) for \( \mathfrak{sl}_2 \) case. In fact we get the same decomposition as [6]. More precisely we get

**Proposition 14.** If we change the coproduct of the level-0 \( \mathcal{U}_q(\widehat{\mathfrak{sl}}_2) \)-module defined in [6] to fit our coproduct (2.11–2.13), The level-0 \( \mathcal{U}_q(\widehat{\mathfrak{sl}}_2) \)-module of \( V(\Lambda_s) \) \((s \equiv M \mod 2, \ s = 0 \text{ or } 1) \) defined in [6] is isomorphic to the level-0 \( \mathcal{U}_q(\widehat{\mathfrak{sl}}_2) \)-module of \( V(\Lambda_s) \) defined in this paper, and the degree is preserved under this isomorphism.

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