UNITARITY AND SINGULAR BACKGROUNDS

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Abstract

We compute the graviton Kaluza-Klein spectrum on a gravity-dilaton background with a naked singularity for all possible boundary conditions at the singularity which are consistent with unitary evolution. We apply methods from non-relativistic quantum mechanics with singular Schrödinger potentials. In general the spectrum contains a tachyon, a sign of instability. Only for a particular boundary condition at the singularity is the spectrum free of tachyons. In this case the lowest-lying graviton mode is massless. We argue that this result will also hold for other backgrounds with similar geometry near the curvature singularity. We complete our study with a brief discussion on radion perturbations and the Higgs mechanism on this singular background.
1 Introduction

Models with a warped extra dimension, such as those introduced by Randall and Sundrum [1], offer a geometric solution to the hierarchy problem. This setup (RS1) originally consisted of ultraviolet (UV) and infrared (IR) branes both embedded in a five-dimensional (5D) space with Standard Model fields localized on the IR brane. The main feature of RS1 is that the electroweak scale gets suppressed as the volume element on the IR brane becomes exponentially small.

With the development of the AdS/CFT correspondence [2,3], it was realized that the strong coupling limit of gauge theories can be described using a perturbative higher-dimensional gravitational model. In this context, the Randall-Sundrum geometry is associated with the $\mathcal{N} = 4$ SYM theory. The holographic dual of QCD was constructed [4] following the inverse argument, an approach widely known as AdS/QCD. The conformal symmetry of the $AdS_5$ geometry corresponds to the conformal limit of QCD at high energies. The usual IR brane is introduced in order to break conformal symmetry at low energies.

An interesting modification of standard AdS/QCD is to deviate from the $AdS_5$ metric using a (dilaton) scalar field. Conformal symmetry is required at high energies. Thus the metric is chosen to be asymptotically $AdS_5$ near the UV boundary. On the other hand the geometry becomes significantly different from $AdS_5$ near the IR. In these soft-wall models conformal symmetry is gradually broken giving a more elaborate description of strong coupling dynamics and confinement [5]. This situation is reminiscent of the Goldberger-Wise mechanism [6], where a scalar field is used for the stabilization of the position of the IR brane.

Solutions of gravity coupled to the dilaton typically have a naked singularity at a finite distance from the UV brane. The introduction of the IR brane can thus be avoided in this case as the geometry ends naturally at the position of the singularity. These singular models were initially introduced in an effort to explain the smallness of the cosmological constant [7]. It was subsequently understood that this singularity had to be resolved by a yet unknown stringy configuration, possibly equivalent to a 3-brane, in a way that spoils the self-tuning of the cosmological constant [8]. One reason for using soft-wall models is that, in contrast to standard AdS/QCD, they can predict a linear Regge spectrum for the masses of hadrons and glueballs [9–11]. This property is intimately related to the behaviour of the warp factor near the singularity.

The application of soft-wall models is not only restricted to QCD. The hierarchical flavour problem in extra-dimensional models can be addressed by allowing Standard Model fields to propagate in the bulk [12]. In this context it is also possible to describe electroweak symmetry breaking by a strongly coupled sector using AdS/CFT correspondence (for reviews see [13,14]). Gauge bosons propagating in five dimensions acquire a Kaluza-Klein (KK) spectrum but the mass of the first excited KK mode is constrained by electroweak precision tests to be very high [15]. The use of soft-wall backgrounds made it possible to relax this stringent constraint allowing the mass of KK excitations to get as low as $O$(TeV) [16–20].

Some difficulties in soft-wall models arise because of the naked singularity. Imposing
boundary conditions at the singularity is not straightforward. This situation is similar to non-relativistic quantum mechanics when the Schrödinger potential is singular, e.g. for a Coulomb potential [21]. The usual approach in this case is to demand boundary conditions that preserve unitarity. Since wave functions, or their derivatives, generally diverge at the singularity it is impossible to impose boundary conditions in the usual Robin form ($\phi' = a\phi$), which guarantees unitarity. Alternatively it is possible to fix the ratio of the linear combination of solutions ($c_1/c_2$) of the time-independent Schrödinger-like equation that describes the various modes [22,23]. This procedure is also used in [24–26] to study waves on singular gravitational backgrounds.

In this article we will reanalyze the gravitational and Higgs spectra in the specific soft-wall model described in Ref. [27] by imposing unitarity on the bulk solutions. In section 2 we review the gravitational and dilaton background we will be using in the rest of the paper. In section 3 we study the gravitational spectrum for all possible boundary conditions at the position of the singularity which are consistent with unitarity. In order to do this we use a convenient method to include all possible KK spectra in the same plot. Using this tool we see that a typical KK spectrum contains a tachyon. This is possible as, contrary to the RS1 model, the Schrödinger operator for gravitons is not positive definite in this case. In fact, we find that only for one specific boundary condition are there no tachyons, in which case the KK spectrum contains a massless mode! This is an attractive feature since we can explain the tuning required in order to obtain a massless graviton. The essence of the argument is that every other choice is excluded because of the existence of a tachyon which renders the geometrical background unstable. The gravitational background we study depends on a free parameter $\nu$. Depending on $\nu$ it is possible to have zero, one, or two independent normalizable eigenfunctions of the Schrödinger operator that describes gravitons. The usual approach is to discard non-normalizable modes from the spectrum. This is equivalent to specific boundary conditions. In Ref. [21] it is explained how the existence of non-normalizable solutions is a signal that the actual KK spectrum depends on the details of the resolution of the singularity. Given such a resolution the originally non-normalizable modes will be smoothed out. From this point of view constructing the spectrum using only normalizable solutions is not mandatory, but rather a choice not to include modes that depend on the details of the resolution. Of course the most straightforward way to regularize the non-normalizable solutions is introducing a second IR brane at a small but finite distance from the position of the singularity [18]. In this article we mainly focus on the case where both solutions are normalizable, with the exception of radion perturbations, where only one solution is normalizable for all the values of $\nu$ that we consider. In section 4 we give a brief description of the Higgs mechanism on the singular soft-wall background. It is shown that Higgs fluctuations can be treated in the same way as gravitons. In this case it is possible to have a KK spectrum that is free of tachyons and which has a non-zero lowest mass. Finally section 5 is devoted to our conclusions and outlook. A somewhat technical review of some facts concerning unbounded differential operators and their spectrum is given in appendix A.
2 The Gravitational Background

We consider the 5D action of gravity, a 3-brane and a dilaton field. The tension of the
3-brane depends on the value of the dilaton field $\phi$. In the Einstein frame we have

$$S = M^3 \int d^5x \sqrt{-\tilde{g}} \left( R - \frac{1}{2} \partial_M \phi \partial^M \phi - V(\phi) \right) - M^3 \int d^4x \sqrt{-\bar{g}} \lambda(\phi),$$

(2.1)

where $\tilde{g}$ is the determinant of the induced metric on the 3-brane and $M$ is the 5D Planck
scale. The dilaton field $\phi$ in the above action is dimensionless. We assume that the metric
is of the form

$$ds^2 = e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,$$

(2.2)

with $\eta_{MN} = \text{diag}\{-1, 1, 1, 1, 1\}$. The brane is located at the $y = 0$ hypersurface and we
impose a $\mathbb{Z}_2$ orbifold symmetry $y \rightarrow -y$. A variation of the action with respect to $g_{MN}$ gives

$$G_{MN} = \frac{1}{2} T_{MN} - \frac{1}{2} \sqrt{-\tilde{g}} g_{\mu\nu} \delta^\mu_M \delta^\nu_N \lambda(\phi) \delta(y).$$

(2.3)

The equations of motion for the ansatz of Eq. (2.2) are

$$3A''(y) = \frac{\phi'(y)^2}{2} + \frac{\lambda(\phi) \delta(y)}{2},$$

(2.4)

$$6A'(y)^2 = -\frac{V(\phi)}{2} + \frac{\phi'(y)^2}{4},$$

(2.5)

$$\phi''(y) - 4A'(y) \phi'(y) = V'(\phi) + \lambda'(\phi) \delta(y).$$

(2.6)

where $f'$ denotes derivative with respect to the function argument. Boundary conditions on
the UV brane are calculated by integrating in a small interval around $y = 0$,

$$A'|_{0^+} - A'|_{0^-} = \frac{\lambda[\phi(0)]}{6},$$

(2.7)

$$\phi'|_{0^+} - \phi'|_{0^-} = \lambda'[\phi(0)].$$

(2.8)

Taking into account the $\mathbb{Z}_2$ symmetry around the brane we have

$$A'|_{0^+} = \frac{\lambda[\phi(0)]}{12},$$

(2.9)

$$\phi'|_{0^+} = \frac{\lambda'[\phi(0)]}{2}.$$ (2.10)

The system of Eqs. (2.4)-(2.6) with the above boundary conditions has a unique solution,
given the potentials $\lambda(\phi), V(\phi)$ and the value of $\phi(0)$. Following [28] we introduce the
"superpotential" $W(\phi)$, defined by

$$V = 18 \left( \frac{\partial W}{\partial \phi} \right)^2 - 12W^2.$$

(2.11)

$^1$We can choose $A(0) = 0$ without loss of generality.
Background solutions can be generated using $W$ as
\begin{align}
A'(y) &= W'(\phi), \\
\phi'(y) &= 6W'(%20\phi),
\end{align}
while the boundary conditions are satisfied if
\begin{align}
W(\phi(0)) &= \frac{\lambda[\phi(0)]}{12}, \\
W'(\phi(0)) &= \frac{\lambda'[\phi(0)]}{12},
\end{align}
Choosing $W$ is not completely arbitrary. The asymptotic form of the metric near the UV brane must approach AdS_5. Additional pathologies appear when the scalar potential $V(\phi)$ is not bounded above for the background solution [29].

Considering the above restrictions, we will consider the superpotential in Ref. [27] which reads as
\[ W = k \left( 1 + e^{\nu \phi/\sqrt{6}} \right). \]
The resulting background has an asymptotic form near the singularity which has been often considered in the literature [9–11]. The background generated in this case reads as
\begin{align}
A(y) &= ky - \frac{1}{\nu^2} \log \left( 1 - \frac{y}{y_s} \right), \\
\phi(y) &= -\frac{\sqrt{6}}{\nu} \log \left[ k \nu^2 (y_s - y) \right],
\end{align}
for $0 < y < y_s$. The above expressions also hold in the $-y_s < y < 0$ interval, with the replacement $y \to -y$, due to the $\mathbb{Z}_2$ symmetry. There is a naked curvature singularity at a finite coordinate distance $y_s$ from the UV brane. This is to be considered as the end of spacetime. The dilaton also blows up at $y_s$. The location of the singularity $y_s$ depends exponentially on the brane value of $\phi$ as $ky_s = \frac{1}{\nu^2} \exp[-\nu \phi(0)/\sqrt{6}]$. All we need to create the weak hierarchy is $\phi(0) < 0$ and otherwise $|\phi(0)| = \mathcal{O}(1)$ which can be achieved with a fairly generic brane potential [27].

The low energy 4D Lagrangian is calculated by integrating the action along the fifth dimension for the background configuration. This is equivalent to an effective 4D cosmological constant,
\[ \Lambda_{\text{eff}} = \frac{2}{3} \int_{-y_s}^{y_s} dy e^{-4A(y)} V(\phi) + \frac{1}{3} e^{-4A(0)} \lambda[\phi(0)], \]
where we have used the equations of motion (2.4)-(2.6) in the scalar kinetic and curvature terms of the action and we are omitting the global $M^3$ factor. The above formula can be simplified by using the identity
\[ \int dy e^{-4A(y)} V(\phi) = 3 \int \frac{d}{dy} \left( e^{-4A(y)} W[\phi(y)] \right), \]
This is known in the literature as the Gubser criterion.
and condition \((2.14)\). The final result is
\[
\Lambda_{\text{eff}} = 4e^{-4A(y_s)}W[\phi(y_s)].
\] (2.21)

Substituting \(A(y)\) and \(\phi(y)\) from Eq. \((2.17)\) we see that \(\Lambda_{\text{eff}} = 0\) for \(\nu < 2\). In this case the background is consistent without a contribution to the 4D cosmological constant from the singularity. For \(\nu = 2\) we have
\[
\Lambda_{\text{eff}} = k \frac{e^{-4ky_s}}{4ky_s}.
\] (2.22)

This is in conflict with the metric ansatz of Eq. \((2.2)\) which is flat from the 4D perspective. Consistency requires a contribution to the cosmological constant from the position of the singularity that would exactly cancel \(\Lambda_{\text{eff}}\). This corresponds to the usual tuning of the cosmological constant in extra-dimensional models \([8]\). It is interesting to notice that when the Gubser criterion gets violated for \(\nu > 2\), \(\Lambda_{\text{eff}}\) is infinite. It is hard to imagine how such a contribution could be canceled in order to have a consistent background \([30]\). For this reason in this article we will consider backgrounds with \(\nu \leq 2\)\(^3\).

### 3 Gravitational Perturbations

In the previous section we have introduced two scalars, the metric \(A\) and the stabilizing scalar field \(\phi\), whose background values define the geometry of the 5D space time, as well as the constant graviton background \(\eta_{\mu\nu}\). In this section we will study fluctuation of the quantum fields around the previous background values which give rise to the respective KK modes which define the 4D spectrum of the gravitational sector.

#### 3.1 Graviton spectrum

In this section we proceed with the calculation of the KK spectrum of gravitational fluctuations in the background given by Eqs. \((2.17)-(2.18)\) and whose massive modes can be interpreted as the spectrum of excited composite states of the strongly coupled dual field theory. Metric perturbations can be decomposed, according to their transformation properties, to a transverse traceless tensor, a four-vector and two scalars. In this section we focus on the tensor (graviton) part of the spectrum. Transverse traceless graviton perturbations are defined by
\[
ds^2 = e^{-2A} (\eta_{\mu\nu} + h_{\mu\nu}) \, dx^\mu dx^\nu + dy^2,
\] (3.1)
with \(\partial^\mu h_{\mu\nu} = 0\) and \(h^\mu_\mu = 0\). A separation of variables is performed assuming solutions of the form \(h_{\mu\nu}(x^\mu, y) = h_{\mu\nu}(x^\mu) h(y)\) where the sum over KK modes is left implicit.

\(^3\)Of course the \(\nu > 2\) case can be made consistent by assuming that the singularity is resolved by introducing a second brane at \(y_1 = y_s - \epsilon\), with \(\epsilon > 0\) a small but finite coordinate distance from \(y_s\) \([31]\).
Following [32][33] the KK spectrum of gravitons is described by the eigenvalue problem

\[ \hat{O} h(y) = m^2 h(y). \] (3.2)

where we have defined the operator

\[ \hat{O} \equiv -e^{-2A} \frac{d^2}{dy^2} + 4e^{-2A} A' \frac{d}{dy}. \] (3.3)

which can be self-adjoint on a Hilbert space with norm

\[ (f, g) = \int e^{-2A} f^* g \, dy. \] (3.4)

Defining \( \tilde{h} = he^{-2A} \) we can write Eq. (3.2) as

\[ -\tilde{h}'' + V(y) \tilde{h} = m^2 e^{2A} \tilde{h}, \] (3.5)

where the potential \( V(y) \) is given by

\[ V(y) = 4A'^2(y) - 2A''(y). \] (3.6)

and for the metric (2.17) it takes the form

\[ V(y) = 4k \left( k + \frac{2}{\nu^2(y_s - y)} \right) - \frac{2(\nu^2 - 2)}{\nu^4(y - y_s)^2}. \] (3.7)

The criterion of normalizability becomes in this case

\[ \int_0^{y_s} e^{2A} |\tilde{h}|^2 \, dy < \infty. \] (3.8)

It is useful to examine the form of Eq. (3.5) dropping subdominant terms near the singularity as

\[ \tilde{h}''(y) + \frac{2(\nu^2 - 2)}{\nu^4(y - y_s)^2} + m^2 \tilde{h}(y) \left( 1 - \frac{y}{y_s} \right)^{-2} = 0. \] (3.9)

When \( \nu \leq 1 \) the dominant terms of Eq. (3.9) near the singularity are the first and the third. Both linearly independent solutions are non-normalizable near the singularity. This is in agreement with Ref. [27]. There it is shown that, working in conformal coordinates defined as

\[ dz = e^{A(y)} dy \] (3.10)

and with \( \nu < 1 \), the position of the singularity moves to infinity. The Schrödinger potential for the graviton in this frame is almost zero for sufficiently large \( z \). The eigenfunctions behave asymptotically like plane waves. These are non-normalizable functions in an infinite interval that therefore do not belong to the physical spectrum. On the other hand, it is possible to build normalizable functions as linear combinations of plane waves because they
form a complete basis in function space. In complete analogy with the spreading of the Gaussian wave packet in quantum mechanics, their evolution is unitary but the probability distribution spreads with time. Since there is no discrete spectrum of mass eigenfunctions, fluctuations in the $\nu < 1$ case can be interpreted as unparticles.

When $\nu > 1$ the dominant balance near the singularity is between the first and the second term. The general solution can be written in general as a linear combination $\tilde{h}(y) = c_1g_1(y) + c_2g_2(y)$ of two linearly independent solutions $g_1, g_2$, with the behaviour

$$\tilde{h} \sim c_1 \left[k(y_s - y)\right]^{1-2/\nu^2} + c_2 \left[k(y_s - y)\right]^{2/\nu^2},$$

(3.11)

near the singularity. For $1 < \nu < \sqrt{2}$ only the second term of (3.11) is normalizable. The spectrum of $\hat{O}$ can be restricted to contain only functions with $c_1 = 0$. This condition is equivalent to the requirement that $e^{-4A(y_s)}h'(y_s) = 0$ and the lowest KK mode is massless. One can notice that the Schrödinger potential $V(y)$ is in this case repulsive and it behaves like $V(y) \sim c/(y_s - y)^2$ with $0 < c < 2$ near $y = y_s$.

It is important to realize that the resolution of the singularity would regularize the singular Schrödinger potential near $y = y_s$ normalizing the previously non-normalizable solution. It is then possible to argue that the choice of $c_1/c_2$ is actually arbitrary even for $1 < \nu < \sqrt{2}$, but only for $c_1 = 0$ can we compute the mass spectrum without knowing the details of the singularity resolution. Notice that eigenfunctions that contain the non-normalizable term will behave like $(y_s - y)^{1-2/\nu^2}$ near the singularity. Let us now assume that we wish to normalize such eigenfunctions e.g. by introducing a normalizing factor $N$. Such a normalizing factor would behave like $N \sim \ell^{(3/\nu^2-3/2)}$, where $\ell$ is the cutoff scale of the resolution. For $\nu < \sqrt{2}$ the normalizing factor $N$ goes to zero when $\ell$ goes to zero. As a result non-normalizable eigenfunctions would correspond to states that are highly localized towards the singularity when the resolution is introduced. If this resolution is artificial then non-normalizable states would also be artifacts of the cutoff.

Things become different when $\nu > \sqrt{2}$. Both linearly independent solutions of Eq. (3.5) are normalizable. The $y$ derivative of the $c_1$ term is divergent at $y_s$, so it is impossible to impose boundary conditions using the Robin form. What can be done is to choose a constant value for the ratio $c_1/c_2$. This condition is consistent with unitarity and will provide us with a discrete mass spectrum for $\hat{O}$. The proof of this statement can be found in the Appendix. There are two independent ways to calculate the KK spectrum when $\nu > \sqrt{2}$.

### 3.1.1 Method I

The first method uses the fact that self-adjoint operators must have an orthonormal set of eigenfunctions which form a complete basis. This can be used in order to build an algorithm to compute the mass spectrum. Initially an arbitrary value for $m^2$ is chosen, let us say $m^2_{\text{in}}$. The Neumann condition $h'(0) = 0$, or the equivalent one

$$\tilde{h}'(0) = -2 \left(k + \frac{1}{\nu^2 y_s}\right) \tilde{h}(0)$$

(3.12)
Figure 1: On the left panel we plot $(\tilde{h}_{in}, \tilde{h})$ as a function of $m^2/k^2$ with initial conditions $h(0) = 1, h'(0) = 0$. We have chosen $ky_s = 1$ and $\nu = 1.5$. The eigenfunction $\tilde{h}_{in}$ corresponds to $m^2_{in} = 0$. The zeroes $m^2_i$ of this plot correspond to the spectrum of mass eigenvalues. On the right panel we plot the same but for $m^2_{in} = 14k^2$.

on the regular brane, along with $\tilde{h}(0) = 1$, is sufficient to uniquely determine a solution $\tilde{h}_{in}$ of Eq. (3.5). As in most cases an analytical form is usually difficult to obtain, so it is convenient to use a numerical algorithm. Solution $\tilde{h}_{in}$ corresponds to a definite value for $c_1/c_2$ or equivalently to a boundary condition at the singularity. All the modes in the spectrum that correspond to $m^2_{in}$ must then be consistent with this boundary condition.

The value of the mass is then increased (and/or decreased) until we find an $m^2 = m^2_n$ with a corresponding eigenfunction $\tilde{h}_{in}$ that satisfies

$$ (\tilde{h}_{in}, \tilde{h}_n) = \int e^{2A(y)}\tilde{h}_{in}^* \tilde{h}_n dy = 0. $$

(3.13)

This is a new eigenfunction with eigenvalue $m^2_n$. Repeating this procedure for higher (and/or lower) values of the mass, the whole spectrum of $\mathcal{O}$ can be calculated. There is no need to estimate the ratio $c_1/c_2$ at any point as the self-adjointness of the operator is established through the orthogonality of the eigenfunctions. As an example in Fig. 1 we can see the spectrum for $ky_s = 1$ and $\nu = 1.5$ assuming an initial mass $m^2_{in} = 0$ (left panel) and $m^2_{in} = 14k^2$ (right panel). The mass eigenvalues correspond to the zeroes of $(\tilde{h}_{in}, \tilde{h})$.

Throughout this article the values of $ky_s$ for the numerical calculations have been chosen arbitrarily since we did not aim to perform any phenomenological analysis. As far as we can tell, the main conclusions of this article hold for every $k \geq 0$ and $y_s > 0$. Further discussion on this can be found below.

3.1.2 Method II

A second method to calculate the KK spectrum is to solve Eq. (3.5) numerically for a range of values of $m^2$ with the same initial conditions $h(0) = 1$ and $h'(0) = 0$ at $y = 0$ and then
calculate the ratio $c_1/c_2$ from the asymptotic behaviour of these solutions. In order to do this we focus on a small interval very close to $y_s$ and perform a fit of our solution using Eq. (3.11). The result of this procedure is shown in the left panel of Fig. 2 where we see $c_1/c_2$ as a function of $m^2/k^2$. In order to compute the spectrum, for fixed $c_1/c_2 \equiv c_{12}$, we have to draw a horizontal line on the left panel of Fig. 2 and compute the solution of the equation $c_1/c_2[m^2] = c_{12}$. The $m^2 \sim 0$ region of Fig. 2 is not clearly visible. In the right panel of Fig. 2 we zoom it in order to demonstrate that the function $c_1/c_2[m^2]$ crosses the origin. Thus for $c_{12} = 0$ the spectrum contains a massless mode.

Figure 3: Plot of $(\tilde{h}_{in}, \tilde{h})$ for $m^2_{in} = 0$, $ky_s = 1$ and $\nu = 1.8$.

The validity of this numerical procedure can be confirmed by a cross-check using the first method we discussed. For example in Fig. 3 we plot $(\tilde{h}_{1}, \tilde{h})$ for $m^2_{in} = 0$, which corresponds
to $c_{12} = 0$. The zeroes of Fig. 3 match those of Fig. 2 implying that the spectra coincide. An advantage of studying Fig. 2 is that one can inspect every possible spectrum of $\hat{O}$ for every consistent choice of boundary conditions on the singularity.

The structure of the $c_1/c_2$ curves around $m^2 = 0$ is of particular interest. In the right panel of Fig. 2 we have zoomed in the $m^2 \sim 0$ part of its left panel in order to reveal some hidden features. There is a branch that crosses the origin and has negative $m^2$ with $c_{12} > 0$. The negative $m^2$ modes correspond to tachyons which render the particular background unstable. A second branch with $m^2 \lesssim -0.01k^2$ also exists. If $c_{12} \to 0$ as $m^2 \to -\infty$ then the spectrum is tachyon-free only for $c_{12} \to 0$. In this case the lowest mass is $m^2 = 0$.

Performing numerical fits, we indeed verify that the ratio $c_1/c_2$ behaves as $(-k^2/m^2)^{1/4}$ for $m^2/k^2 \to -\infty$. As the numerical approach could be debatable, since numerics become untrustworthy for large values of $|m^2|$, we will confirm this conclusion in a case where there exists an analytical solution. To this end we will consider the simplified metric

$$A(y) = -\frac{1}{\nu^2} \log \left( 1 - \frac{y}{y_s} \right)$$

stemming from the superpotential

$$W = ke^{\nu \phi / \sqrt{3}}$$

The metric (3.14) retains the main features of the complete metric (2.17) near the singularity, although it departs from it near the UV brane and thus it would not be a good candidate to explain the weak/Planck hierarchy. The potential in Eq. (3.5) corresponding to the metric in Eq. (3.14) is given by

$$V(y) = -\frac{2(\nu^2 - 2)}{\nu^4(y - y_s)^2}.$$  

which formally corresponds to keeping only the last term of the full potential (3.7) or the zeroth order in the expansion in powers of $k$. Now Eq. (3.5) for $\nu = \sqrt{3}$ can be solved analytically. The result is a linear combination of Bessel functions as

$$h = c_1 \frac{\sqrt{x} I_{-\frac{1}{4}} \left( \frac{3x^{2/3}}{2\sqrt{\epsilon}} \right)}{e^{3/8}} + c_2 \frac{\sqrt{x} I_{\frac{1}{4}} \left( \frac{3x^{2/3}}{2\sqrt{\epsilon}} \right)}{e^{3/8}},$$  

where $x = k^{3/2}(y_s - y)y_s^{1/2}$ and $\epsilon = -k^2/m^2$. Applying the usual conditions on the regular brane and after a Taylor expansion of the solutions for $\epsilon \to 0$, we confirm that the leading behaviour is $c_1/c_2 \sim \epsilon^{1/4}$. Moreover in Fig. 4 $c_1/c_2$ is calculated analytically for $\nu = \sqrt{3}$ as a function of the mass. One can easily check that the branch structure in Fig. 4 is similar to the one in the left panel of Fig. 2.

We have repeated the numerical calculation of the $c_1/c_2$ plot for various values of $0 \leq ky_s \leq 8$, where our numerical methods are accurate, and $\sqrt{2} < \nu < 2$ getting the same result: only for $c_1/c_2 = 0$ the spectrum is tachyon free. This is not a surprise as stability is related to the low-lying eigenvalues of Eq. (3.5), and the low-lying modes are dominated by the most negative term of $V(y)$, which is the last, $k$-independent, term of Eq. (3.7) near the
Figure 4: $c_1/c_2$ as function of $m^2/k^2$ calculated analytically for the simplified metric $\eqref{3.14}$, $ky_s = 1$ and $\nu = \sqrt{3}$. Horizontal lines mark the position of poles.

singularity. The existence of tachyons relies on the asymptotic form of the potential and the asymptotic behaviour of the solutions near the singularity, which is given by the ratio $c_1/c_2$. The $k$-dependent term in the potential $V(y)$ is only important for higher KK modes. An additional $k$-independent feature is that the $c_1/c_2$ curves cross the origin. For $m^2 = 0$ and $h'(0) = 0$ the solution of Eq. $\eqref{3.2}$ is $h(y) = \text{constant}$, which leads to $c_1 = 0$. Taking into account the accumulated evidence, we believe that it is safe to conclude that the graviton KK spectrum contains a tachyon for $c_1 \neq 0$ and any value of $ky_s$.

3.2 Scalar perturbations

The analysis of perturbations on the soft-wall background is completed with the study of scalar radion and dilaton perturbations. Vector perturbations can be gauged away except for a possible zero mode $\eqref{33}$. Following the study of Ref. $\eqref{32}$ (see also Ref. $\eqref{27}$) scalar perturbations are defined by

$$
\phi(x, y) = \phi(y) + \varphi(x, y),
$$

$$
ds^2 = e^{-2A(y)-2F(x,y)}n_{\mu\nu}dx^\mu dx^\nu + (1 + G(x,y)^2)dy^2,
$$

where $\varphi(x, y)$ is the dilaton perturbation and $F(x, y)$ and $G(x, y)$ are the scalar gravitational perturbations. Not all of the above quantities are dynamically independent. The equation of motion of $F$ is given, after a separation of variables, by

$$
F'' - 2A'F' + 4A''F - 2\frac{\phi''}{\phi'}F' + 4A'\frac{\phi''}{\phi'}F = -m^2e^{2A}F.
$$

where $\varphi(x, y)$ is the dilaton perturbation and $F(x, y)$ and $G(x, y)$ are the scalar gravitational perturbations. Not all of the above quantities are dynamically independent. The equation of motion of $F$ is given, after a separation of variables, by

$$
F'' - 2A'F' + 4A''F - 2\frac{\phi''}{\phi'}F' + 4A'\frac{\phi''}{\phi'}F = -m^2e^{2A}F.
$$

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$$

Two constraints fix the nondynamical quantities $\varphi$ and $G$

\begin{align}
\phi' \varphi &= 6(F' - 2A'F), \\
G &= 2F.
\end{align} 

(3.21) 

(3.22)

In order to define a self-adjoint operator from Eq. (3.20) it is necessary to use the following inner product

\begin{equation}
(F_1, F_2) = \int_0^{y_s} e^{-A - \log \phi'} F_1^* F_2 \, dy.
\end{equation}

(3.23)

With the redefinition $\tilde{F} = e^{-A - \log \phi'} F$, Eq. (3.20) can be written in the form

\begin{equation}
- \tilde{F}''' + V_F(y) \tilde{F} = m^2 e^{2A} \tilde{F}.
\end{equation}

(3.24)

Near the singularity the above equation is approximated by

\begin{equation}
- \tilde{F}''' + \frac{1 + \nu^2}{\nu^4(y_s - y)^2} \tilde{F} = m^2(y_s - y)^{-2/\nu^2} y_s^2/\nu^2 \tilde{F}.
\end{equation}

(3.25)

For $1 < \nu < 2$, the $m^2$ term is subdominant and the asymptotic form of solutions is

\begin{equation}
\tilde{F} = c_1(y_s - y)^{-1/\nu^2} + c_2(y_s - y)^{1+1/\nu^2}.
\end{equation}

(3.26)

Taking into account the normalizability condition for $\tilde{F}$

\begin{equation}
\int_0^{y_s} e^{A + \log \phi'} |\tilde{F}|^2 \, dy < \infty,
\end{equation}

(3.27)

it can be seen that the $c_1$ term of Eq. (3.26) is non-normalizable for every $\nu$. The spectrum can be computed, without resolving the singular potential $V_F$, only for $c_1 = 0$ [27].

The most straightforward way to resolve the singularity is to introduce a brane at $y = y_s - \ell$, where $\ell$ is a small but finite coordinate distance. The resulting spectrum will depend on $\ell$ in addition to $k$ and $\nu$. A complete understanding of all possible spectra in this case is complicated and beyond the scope of this article. Nevertheless we present a typical calculation with an IR brane resolution in order to exhibit how to apply the $c_1/c_2$ plot method in this case.

Indeed it is not necessary to use the $c_1/c_2$ ratio in this case since it is possible to define a boundary condition of the form $\tilde{F}'(y_s - \ell) = \alpha \tilde{F}(y_s - \ell)$ at $y = y_s - \ell$ with $\alpha \in \mathbb{R}$. Following the same steps as for the $c_1/c_2$ method above, a solution of Eq. (3.20) with initial conditions $F(0) = 0$, $F'(0) - 2A'(0)F(0) = 0$ is found for a number of $m^2$ values in an interval. The boundary conditions above correspond to the stiff potential limit ($\lambda''(\phi) \gg 1$). Given a set of solutions, $\alpha$ is calculated in order to construct an $\alpha = \alpha(m^2)$ plot. The spectrum is given by the intersection points of this plot with the horizontal line corresponding to a given value of $\alpha$. For example in Fig. 5 we see the $\alpha = \alpha(m^2)$ plot for the simplified metric of Eq. (3.14) which yields the potential

\begin{equation}
V_F = \frac{1 + \nu^2}{\nu^4(y_s - y)^2}
\end{equation}

(3.28)

for $ky_s = 1$, $\nu = \sqrt{3}$ and $\ell = 10^{-3}$. In accordance with the case of gravitons the solution is obtained analytically as a combination of Bessel functions. We see that for $\alpha \lesssim -33$ a tachyon appears in the spectrum rendering the background unstable.
4 The Higgs Mechanism

In this section we present some aspects of EWSB on the soft-wall background with a Higgs field propagating in the bulk. The strategy, which is usually followed in the literature in order to trigger electroweak symmetry breaking (EWSB), is to introduce a new infrared (IR) brane with a localized Higgs potential. The Higgs then acquires a vacuum expectation value (VEV) which is the coordinate dependent classical solution. Following the method developed in the previous section, it is possible to avoid the introduction of the IR brane. Assuming unitarity, the ratio $c_1/c_2$ of Higgs fluctuations around a given background is fixed and the KK spectrum of the Higgs field is now well defined.

A 5D Higgs transforming as the $(2,1/2)$ representation of $SU(2)_L \times U(1)_Y$ is introduced as

$$H = e^{i\vec{\chi}(x,y)\vec{\sigma}} \begin{pmatrix} 0 \\ h(y) + \xi(x, y) \end{pmatrix},$$

where the vector $\vec{\chi}$ corresponds to the three 5D Goldstone bosons. The Higgs action contains a gauge invariant kinetic term and a potential $V(\phi, h)$ that couples the Higgs field with the dilaton.

The classical Higgs background $h(y)$ can be generated by using the superpotential formalism. In particular we can assume a superpotential of the form

$$W_H = \frac{1}{12} a k |H|^2,$$

Figure 5: Radion KK modes: $\alpha$ as function of $m^2/k^2$ calculated analytically for the simplified metric of Eq. (3.14), $ky_s = 1$, $\nu = \sqrt{3}$ and $\ell = 10^{-3}$.
where \( a \) is a dimensionless constant, which is added to the dilaton superpotential as

\[
W = W_\phi + W_H = k \left( 1 + e^{\nu \phi/\sqrt{6}} \right) + \frac{1}{12} a k h^2. \tag{4.3}
\]

The superpotential is related to the Higgs-dilaton potential by

\[
V(\phi, h) = 18 \left( \left( \frac{\partial W}{\partial \phi} \right)^2 + \left( \frac{\partial W}{\partial h} \right)^2 \right) - 12 W^2. \tag{4.4}
\]

The background solution of the Higgs field is

\[
h'(y) = 6 \frac{\partial W}{\partial h}, \tag{4.5}
\]

which can be easily solved for \( h(y) \) as

\[
h(y) = h_0 e^{aky}. \tag{4.6}
\]

The dimensionless integration constant \( h_0 \) can be considered to be small in order to avoid backreaction to the metric. Nevertheless we will include the backreaction in our description as it could possibly result in interesting models. The background solution for the dilaton is given in Eq. (2.18) and the warp factor is then calculated from \( A' = W \) as

\[
A(y) = \frac{1}{24} h_0^2 (e^{2aky} - 1) + ky - \frac{1}{\nu^2} \log \left( 1 - \frac{y}{y_s} \right). \tag{4.7}
\]

The properties of Higgs particles result from the study of fluctuations \( \xi(x, y) \), that occur around the classical solution. A separation of variables for \( \xi \) is needed in order to compute the KK spectrum: \( \xi(x, y) = e^{A(x)} \xi(y) \) with

\[
\xi'' - 2A'\xi' = \left( \frac{\partial^2 V(\phi, h)}{\partial h^2} - m^2 e^{2A} + 3A'^2 - A'' \right) \xi + \frac{\delta^2 \lambda(\phi, h)}{\delta h^2} \delta(y) \xi, \tag{4.8}
\]

where \( m^2 \) is the 4D mass eigenvalue. It is convenient to define

\[
m_\xi^2(y) \equiv \frac{\partial^2 V(\phi, h)}{\partial h^2} = a^2 k^2 \left( 1 - h_0^2 e^{2aky} \right) - 4ak - \frac{4ak}{\nu^2(y_s - y)}, \tag{4.9}
\]

so that

\[
\xi'' - 2A'\xi' = (m_\xi^2(y) - m^2 e^{2A} + 3A'^2 - A'') \xi + \frac{\delta^2 \lambda(\phi, h)}{\delta h^2} \delta(y) \xi, \tag{4.10}
\]

As in the case of gravitons, the inner product for \( \xi \) is given by

\[
(\xi_1, \xi_2) = \int_0^{y_s} \xi_1^* \xi_2 \, dy. \tag{4.11}
\]
The boundary conditions for the fluctuations $\xi$ are given by integrating (4.10) around $y = 0$

$$\left. \frac{\xi'(0)}{\xi(0)} \right|_{y=0} = \frac{\delta^2 \lambda(\phi, h)}{\delta h^2} = 2ak$$

(4.12)

Equation (4.10) can be brought into a Schrödinger form by defining $\tilde{\xi} = \xi \exp(-A)$. The inner product in this case becomes

$$(\tilde{\xi}_1, \tilde{\xi}_2) = \int_0^{y_s} e^{2A} \tilde{\xi}_1^* \tilde{\xi}_2 dy.$$ (4.13)

The dominant terms of Eq. (4.10) near the singularity for $1 < \nu < 2$ are

$$\check{\xi}''(y) + \frac{2(\nu^2 - 2) \check{\xi}(y)}{\nu (y - y_s)^2} = 0,$$ (4.14)

and the asymptotic form of the solution is

$$\check{\xi}(y) = c_1(y_s - y)^{1-2/\nu^2} + c_2(y_s - y)^{2/\nu^2},$$ (4.15)

which is identical to the asymptotic behaviour of graviton fluctuations. As a result the $c_1/c_2$ plot in order to compute the KK spectrum can be done numerically. In Fig. 6 we give two examples of $c_1/c_2$ plots that correspond to qualitatively different situations. Both plots are made for $ky_s = 1$, $\nu = 1.8$ and $h_0 = 10^{-4}$. The value of $h_0$ is taken small enough in order to have negligible backreaction to the metric. In the left panel of Fig. 6 the $c_1/c_2$ plot is computed with $a = 3.6$. In this case we see that there is a tachyon-free regime for $0 \leq c_{12} \lesssim 0.2$. In the right panel of Fig. 6 for $a = 0.6$ the tachyon-free regime is for $c_{12} \lesssim -0.65$ and $c_{12} \geq 0$.

![Figure 6: $c_1/c_2$ as a function of $m^2/k^2$ for $ky_s = 1$ and $\nu = 1.8$. Left panel: $a = 3.6$. Right panel: $a = 0.6$.](image)

Of course, the above discussion for EWSB is far from complete. This is mainly due to the fact that we have three free parameters $a$, $\nu$ and $h_0$. It is quite a complex task to understand
the behavior of the KK spectrum as we vary those three parameters independently. The plots in Fig. 6 represent typical cases where the KK spectrum of the Higgs can be ghost free. An additional shortcoming is that we are not trying, by choosing $k y_s = 1$, to address the hierarchy problem which would require values $k y_s \sim 30$. The reason we did not use such large values of $k y_s$ is that the numerical algorithms we use to construct $c_1/c_2$ plots are not accurate enough in this case due to the large exponential factors which appear in Eq. (4.10). However, the main effect of large $k y_s$ is to scale down the $m^2$ values of the KK modes as a result of the warp factor. In fact we can see from Eq. (4.11) that $\xi$ provides a direct physical interpretation of localization properties along the extra dimension. The profile of normalized KK modes for $\xi$, see e.g. the plot in Fig. 7, is always localized near the singularity, say at $y_1 = y_s - \ell$ for $k \ell \ll k y_s$. It turns out that in the support of $\xi$ the term $m^2 e^{2A} \simeq m^2 e^{2A(y_1)}$ and for values of $m^2 e^{2A(y_1)} \sim k^2$, as the solution of Eq. (4.10) requires all terms to have the same order of magnitude, the eigenvalues are warped down as $m \sim e^{-A(y_1)} k$ which can accommodate an electroweak Higgs from the Planckian value of $k$. Although our results for EWSB cannot be used directly for a phenomenological analysis, we believe that they make up a useful guide for future efforts towards a viable EWSB model.

\footnote{Nevertheless we have been able to check that for $k y_s \in \{0, 8\}$ the qualitative structure of the $c_1/c_2$ plots remain unchanged.}
5 Discussion and Outlook

The problem of specifying boundary conditions in soft-wall models at the position of the singularity has been addressed in the past in the literature. When the warp factor has a logarithmic divergence it is possible to have two independent normalizable solutions for the graviton eigenvalue equation. The approach followed in Refs. [10, 33] is that one has to fix the value of the ratio $c_1/c_2$, where $c_1$ and $c_2$ are the coefficients of the two independent graviton solutions, in order to be consistent with unitarity. For a specific value $c_1/c_2 = 0$ the lowest KK mode is massless. An equivalent proposal [27] is to assume the $e^{-4A(y_s)h'(y_s)} = 0$ boundary condition. We have supplemented previous works by finding that only the value $c_1/c_2 = 0$ is acceptable as every other choice is plagued by an instability of the background. For this particular value the lowest graviton KK mode is massless. We presented arguments showing that this result does not depend on the exact form of the warp factor, but rather on its asymptotic behaviour near the singularity.

Along with the spectrum of gravitons we studied scalar and dilaton perturbations on the soft-wall background. For the gauge choice we used the only dynamical scalar perturbation that corresponds to the radion. In this case one of the independent solutions of the eigenvalue problem is non-normalizable. It is common throughout the literature to exclude the non-normalizable solution from the spectrum. In analogy to the treatment of singular potentials in non-relativistic quantum mechanics [21,22], we adopted the point of view of not rejecting the non-normalizable solution altogether, but assuming that it will become normalizable when unknown physics related to the curvature singularity is taken into account. The resulting spectrum will strongly depend on the details of the resolution of the singularity. The most economical regularization is to introduce an IR brane near the position of the singularity. As an example we calculated the radion spectrum in this case finding that it is tachyon free for a wide range of boundary conditions.

The approach developed in this work can be applied to the study of massive scalar, vector and spinor fields on soft-wall backgrounds within studies beyond the Standard Model, using extra dimensions. A particular application has been done in this paper to the case of the Standard Model Higgs propagating in the bulk of the singular metric defining the soft wall assuming EWSB. In this case it was understood that it is possible to have a Higgs KK spectrum that is free of tachyons for a range of $c_1/c_2$ values without introducing an ad hoc IR brane. The mass of the lowest KK mode depends on the value of the parameter $c_1/c_2$. From the point of view of AdS/CFT it would be interesting to better understand the role of non-normalizable graviton solutions and to relate their regularization to nontrivial IR dynamics.

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A Appendix

In this appendix we give a brief review of some facts regarding unbounded differential operators and their spectrum. For a more precise exposition the reader can consult Refs. [34] and [35]. We will also argue that fixing $c_1/c_2$ is sufficient to ensure that the evolution is unitary.

Our goal is to study the mass spectrum described by Eq. (3.5). The corresponding operator is

$$\hat{O}_s = -e^{-2A(y)} \frac{d^2}{dy^2} + e^{-2A(y)}V(y),$$

(A.1)

where $V(y)$ is given by Eq. (3.6). This operator acts on a subset of the Hilbert space $L^2[0,y_s)$ with weight function $w(x) = e^{2A(y)}$. Thus the norm on this Hilbert space is defined by

$$(f,g) = \int_0^{y_s} e^{2A(y)} f^*(y)g(y)dy.$$

(A.2)

The asymptotic form of eigenfunctions of $\hat{O}_s$ is given by Eq. (3.11). We assume that the domain of the operator $D(\hat{O}_s)$ consists of functions with this asymptotic behaviour. An operator $\hat{O}_s$ is called self-adjoint if:

1. It is symmetric when acting on functions $f, g \in D(\hat{O}_s)$,

$$(f, \hat{O}_s g) = (\hat{O}_s f, g),$$

(A.3)

2. The domain of the adjoint operator $D(\hat{O}_s^*) \equiv D(\hat{O}_s)$.

The domain of the adjoint is defined to consist of all functions $f \in L^2[0,y_s)$ that satisfy Eq. (A.3) for every $g \in D(\hat{O}_s)$. Notice that, even if $\hat{O}_s^*$ is formally identical to $\hat{O}_s$, in reality they can be distinct since it is possible that they act on different domains. The second requirement is needed to ensure that the operator is symmetric also when it acts on functions of the form

$$\psi(t) = e^{it\hat{O}_s} \psi(0) \quad \psi(0) \in D(\hat{O}_s),$$

(A.4)

that will be generated by the time evolution.

If $\hat{O}_s$ is self-adjoint then it has the following properties:

- A spectrum of real eigenvalues.
- Eigenfunctions that form an orthonormal basis which is complete in $L^2[0,y_s)$. 

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It generates unitary evolution.

Let us consider $f, g$ to be functions that satisfy the boundary condition of Eq. (3.12) and having an asymptotic behaviour near the singularity of the form of Eq. (3.11)

\[
\begin{align*}
  f(y) &\sim a_1(y_s - y)^{1-2/\nu^2} + a_2(y_s - y)^{2/\nu^2}, \\
  g(y) &\sim b_1(y_s - y)^{1-2/\nu^2} + b_2(y_s - y)^{2/\nu^2}.
\end{align*}
\]

(A.5) \hspace{1cm} (A.6)

Now we can check whether $\hat{O}_s$ is symmetric with respect to $f, g$ or not. Performing two consecutive integrations by parts in the interval $[0, y_s)$, we can see that

\[
(f, \hat{O}_s g) = (\hat{O}_s f, g) + \left[ f^*(y) g'(y) - f'(y) g(y) \right]_{y=0}^{y=y_s}.
\]

(A.7)

The boundary term at $y = 0$ vanishes due to the boundary condition of Eq. (3.12). Taking into account the asymptotic form of $f, g$ we see that

\[
(f, \hat{O}_s g) = (\hat{O}_s f, g) + (a_1^* b_2 - a_2^* b_1) \left( 1 - \frac{4}{\nu^2} \right).
\]

(A.8)

The operator $\hat{O}_s$ is then symmetric provided that $a_1, a_2 \in \mathbb{R}$ and

\[
\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.
\]

(A.9)

We have proven that if we choose a $\mathcal{D}(\hat{O}_s)$ that consists of functions with fixed $c_1/c_2$, the operator is symmetric and the domain of the adjoint is identical with the domain of $\hat{O}_s$: in this case the operator is self-adjoint. It is possible to see that time evolution with $O_s$ will not change the asymptotic behaviour and the value of $c_1/c_2$.

Another mathematically rigorous proof of the $c_1/c_2$ statement can be given using von Neumann’s theory of self-adjoint extensions (see [34]). One can start assuming that $\mathcal{D}(\hat{O}_s)$ consists of functions satisfying Eq. (3.12) and having a compact support at the singularity. The deficiency indices in this case are

\[
n_{\pm} = \dim(Ker(\hat{O}_s \pm i)) = 1,
\]

(A.10)

Thus we can have a self-adjoint extension of $\hat{O}_s$ by adding to the domain of $\hat{O}_s$ functions of the form

\[
\psi_c = \psi_+ + e^{i\gamma} \psi_-,
\]

(A.11)

where $\psi_\pm$ satisfy $\hat{O}_s \psi_\pm = \pm i \psi_\pm$ and $\gamma$ is a fixed number in $\mathbb{R}$. Since $\psi_\pm$ are eigenfunctions of $\hat{O}_s$ they have the usual asymptotic behaviour mentioned above:

\[
\psi_+ = c_{1+}(y_s - y)^{1-2/\nu^2} + c_{2+}(y_s - y)^{2/\nu^2}
\]

(A.12)

\[
\psi_- = c_{1-}(y_s - y)^{1-2/\nu^2} + c_{2-}(y_s - y)^{2/\nu^2}.
\]

(A.13)
Since $\psi_{\pm}$ are unique solutions, $\{c_{1+}, c_{1-}, c_{2+}, c_{2-}\}$ are fixed numbers satisfying $c_{1,2+} = c_{1,2-}$. The asymptotic form of the functions $\psi_e$ that extend the domain of the operator is

$$
\psi_e \sim \left( c_{1+} + e^{i\gamma}c_{1-} \right) (y_s - y)^{1-2/\nu^2} + \left( c_{2+} + e^{i\gamma}c_{2-} \right) (y_s - y)^{2/\nu^2}.
$$

(A.14)

It is trivial to check that the ratio

$$
\frac{c_{1+} + e^{i\gamma}c_{1-}}{c_{2+} + e^{i\gamma}c_{2-}} \equiv \frac{c_1}{c_2},
$$

(A.15)

is an arbitrary real number parametrized by the angle $\gamma$.

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