Constructible Sheaves
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Abstract
This article contains a proof of the basic lemma, which yields a motivic proof of the Andreotti-Frankel theorem for affine varieties. Next, it is shown that the triangulated category of "Cohomologically Constructible Sheaves" (as it is referred to in the Riemann-Hilbert correspondence) coincides with the derived category of bounded complexes of constructible sheaves. It is also shown that higher direct images and the sheaf-Ext groups are effaceable in the category of constructible sheaves.

Introduction
The Andreotti-Frankel theorem asserts that a closed complex manifold $X$ of $\mathbb{C}^N$ of dimension $n$ has the homotopy type of a cell complex of dimension at most $n$ (see [AF] or [Mil]). This is achieved by constructing a Morse function $f : X \to \mathbb{R}$ with critical points of index at most $n$. Put

$$X_a = \{ x \in X : f(x) \leq a \}.$$

The cohomology of the pair $(X_a, X_b)$ when $b < a$ now enters the picture. But the cohomology of these pairs do not inherit the rich structure (Galois action, mixed Hodge structure, for instance) that the cohomology of $X$ carries when $X$ is an algebraic variety. This is the reason for formulating the basic lemma below.

Basic Lemma – first form (A. Beilinson) Let $k$ be a subfield of $\mathbb{C}$. Let $W$ be Zariski closed in an affine variety $X$ defined over $k$. Assume $\dim(W) < \dim(X)$. Then there is a Zariski closed $Z$ in $X$ so that $\dim(Z) < n$ with $W \subset Z$, and $H^q(X, Z) = 0$ whenever $q \neq \dim(X)$.

In the above lemma, $H^q(X, Z)$ denotes the singular cohomology of the pair $(X(\mathbb{C}), Z(\mathbb{C}))$. The cohomological version of the Andreotti-Frankel theorem for affine varieties is of course an immediate consequence. Indeed if $X$ is affine of dimension $n$, from the above lemma, we deduce an increasing sequence of Zariski closed sets $X_i$ of dimension at most $i$ so that $H^j(X_i, X_{i-1}) = 0$ whenever $j \neq i$. Consequently, the $j$-th cohomology of the complex $D^\bullet$, where $D^i = H^i(X_i, X_{i-1})$ coincides with the $j$-th cohomology of $X$, which therefore vanishes whenever $j > n$. Furthermore $D^\bullet$
may be regarded as a complex of “motives”. See also [Be1] and [Be2] for such statements for mixed Hodge structures and Galois representations.

The proof given here of the basic lemma is geometric. It is not hard to see that for our choice of \( Z \) in the basic lemma, \((X(\mathbb{C}), Z(\mathbb{C}))\) is in fact, \(\text{upto homotopy, a relative CW pair obtained by attaching cells of dimension } n\). Furthermore, the same method gives a triangulation of real affine semi-algebraic sets.

Our proof is valid only in characteristic zero. The proof of the intermediate Proposition 1.3 is not valid in positive characteristic due to the presence of wild ramification.

The only proof of the basic lemma in positive characteristic is that of Beilinson ([Be2, Lemma 3.3]). In fact, Beilinson’s Lemma 3.3 is a vast generalisation of the basic lemma to perverse sheaves, in the sense that it produces plenty of perverse sheaves with at most one nonvanishing hypercohomology. With \( X \) and \( W \) as in the basic lemma, the proof of Lemma 3.3, [Be2], as Beilinson explained to the author, yields \( Z \) in the following shape: first enlarge \( W \) so that its complement \( V = X - W \) is affine and smooth of pure dimension \( n \), where \( n = \text{dim}(X) \). Next, embed \( X \) as a Zariski locally closed set in projective space, and let \( H \) be a general hyperplane section of \( X \). Then \( H^q(X, W \cup H) = 0 \) for \( q \neq n \), and thus we may take \( Z = W \cup H \) in the basic lemma. Beilinson’s Lemma 3.3 also includes a very general cohomological (rather than homotopy theoretic) version of the “Lefschetz hyperplane section theorem for complements”: with \( V = X - W \) and \( H \) as above, his result shows that \( H^q(V, V \cap H) = 0 \) for \( q \neq n \). However, Beilinson relies on M. Artin’s sheaf-theoretic version of the Andreotti-Frankel theorem, whereas we deduce M. Artin’s theorem (in characteristic zero).

We now turn to constructible sheaves. A ring \( R \) will be fixed once and for all. All sheaves considered are sheaves of left \( R \)-modules. A subfield \( k \) of the complex numbers will remain fixed throughout the paper. All varieties and morphisms considered are defined over this field \( k \). By a “sheaf on \( X \)” we mean a sheaf of \( R \)-modules on the set \( X(\mathbb{C}) \) of \( \mathbb{C} \)-rational points equipped with the usual topology. A sheaf \( F \) on a variety \( X \) is said to be weakly constructible if \( X \) is the disjoint union of a finite collection of locally closed subschemes \( Y_i \) so that the restrictions \( F|Y_i \) are all locally constant sheaves. We also fix a full Abelian subcategory \( \mathcal{N} \) of the category of all \( R \)-modules so that every \( R \)-module \( M \) that is isomorphic to an object of \( \mathcal{N} \) is actually an object of \( \mathcal{N} \). We call \( F \) constructible if, in addition, all the stalks of \( F \) are objects of \( \mathcal{N} \). We warn the reader that, in the literature (e.g., [KS, Chapter VIII]), when constructible sheaves are discussed, \( \mathcal{N} \) is the category of all Noetherian modules. But we do not place such restrictions on \( \mathcal{N} \).

For constructible or weakly constructible sheaves \( F \), the sheaf cohomology \( H^q(X(\mathbb{C}), F) \) will be denoted simply by \( H^q(X, F) \). To compute sheaf
cohomology, one takes a resolution $K^\bullet$ of the given sheaf $F$ and considers the cohomology of the complex: $\Gamma(X, K^\bullet)$. If $K^\bullet$ is an injective, or even a flasque, resolution (e.g., the Godement resolution), we get:

$$H^q(\Gamma(X, K^\bullet)) = H^q(X, F).$$

Constructible sheaves are rarely flasque (if so, they are supported on a finite set). Nevertheless, it is true that every constructible sheaf $F$ on $\mathbb{A}^n$ has a constructible resolution $K^\bullet$ so that $H^q(\Gamma(\mathbb{A}^n, K^\bullet)) = H^q(\mathbb{A}^n, F)$ for all $q \geq 0$. Furthermore, we may assume that $K^q = 0$ for all $q > n$. This is a consequence of Theorem 1 below. The comparison theorem of Artin-Grothendieck for étale cohomology (see [SGA4, Expose XI]) is the special case of Theorem 2 for $R = \mathbb{Z}/n\mathbb{Z}$.

**Theorem 1** Every constructible sheaf $F$ on affine $n$-space $\mathbb{A}^n$ is a subsheaf of a constructible $G$ so that $H^q(\mathbb{A}^n, G) = 0$ for all $q > 0$.

**Theorem 2** For every constructible sheaf $F$ on a variety $X$, there is a monomorphism $a : F \rightarrow G$ with $G$ constructible so that

$$H^q(X, a) : H^q(X, F) \rightarrow H^q(X, G)$$

is zero, for all $q > 0$.

Further effaceability results and their consequences are formulated below. We shall denote by $Sh(X)$ the category of sheaves of $R$-modules on $X$ and by $C(X)$ the full subcategory of $Sh(X)$ with Obj $C(X)$= constructible sheaves. An additive functor $T : C(X) \rightarrow A$, where $A$ is an Abelian category is effaceable, if for all objects $F$ of $C(X)$, there is a monomorphism $a : F \rightarrow G$ in $C(X)$ so that $Ta : TF \rightarrow TG$ is the zero morphism.

**Theorem 3** (a) Assume that $R$ is commutative. For every weakly constructible sheaf $P$ on $X$ such that all the stalks of $P$ are finitely generated projective modules, and for every $q > 0$, the functor

$$\text{Ext}_{Sh(X)}^q(P, \cdot)(C(X))$$

is effaceable.

(b) Let $R$ be a field, and let $\mathcal{N}$ be the category of all finite dimensional vector spaces over $R$. Let $F, G$ be constructible sheaves on $X$. Then $\text{Ext}_{C(X)}^q(F, G) \rightarrow \text{Ext}_{Sh(X)}^q(F, G)$ is an isomorphism for all $q \geq 0$. 
Throughout the paper, \( \text{Ext}^q_A(F, G) \) denotes the Yoneda Ext group for objects \( F \) and \( G \) of an Abelian category \( A \). That Theorem 3(a) implies Theorem 3(b) is the standard application of effaceability (see [Toh Prop. 2.2.1, page 141]). Theorem 3(b) implies (see e.g., [Be2 Lemma 1.4]) that the derived category \( D^b(C(X)) \), with \( R = k = \mathbb{C} \) and \( N = \) finite dimensional vector spaces, is equivalent to the triangulated category of “cohomologically constructible sheaves” as it is referred to in the Riemann-Hilbert correspondence. In view of Beilinson’s result ([Be2 Main theorem 1.3, page 29]), one may re-state the Riemann-Hilbert correspondence ([Bo Thm. 14.4]) as follows:

The derived category of bounded complexes of regular holonomic modules is equivalent to the derived category of bounded complexes of constructible sheaves of complex vector spaces.

**Theorem 4** Assume that \( R \) is a field, and that \( N = \) all finite dimensional vector spaces. Let \( f : X \to Y \) be a morphism. Then the functors \( R^q f_*|C(X) \) are effaceable for all \( q > 0 \).

For the next theorem, assume that \( R \) is commutative and recall that for sheaves \( F, G \) on \( X \), we have the sheaf (of \( R \)-modules) \( \text{Hom}(F, G) \) on \( X \) and the derived functors \( R^q \text{Hom}(F, \cdot) = \text{Ext}^q(F, \cdot) \).

**Theorem 5** With \( R \) and \( N \) as in Theorem 4 above, the functors \( \text{Ext}^q(P, \cdot)|C(X) \) are effaceable for all \( q > 0 \) and for all constructible sheaves \( P \) on \( X \).

The minimal hypotheses on \( R \) and \( N \) under which Theorems 3(b), 4 and 5 are valid remain unclear. It is hoped that Theorems 1–5, or at least the method of proof, will help in constructing a category of motivic sheaves. The reader will observe that the proofs of Theorems 1 and 2 require nothing more than the formal properties of the operations \( f^* \) and \( R^q f_* \) while Theorems 3, 4 and 5 require the internal \( \text{Hom} \). That the analogues of Theorems 1–5 hold for \( \mathbb{Q}_l \)-sheaves (see [SGA4 page 84]) is immediate from the proofs given here.

The Noether normalisation lemma plays a major role in the proofs of the sheaf-theoretic version of the basic lemma and Theorem 1. These proofs appear in the first two sections. Some care has been taken to keep these sections self-contained and elementary (modulo the use of the Leray spectral sequence and the proper base change theorem). For this purpose, direct proofs of weak constructibility appear in these sections. In the third section, the remaining theorems are essentially deduced from Theorem 1, Whitney
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stratifications, devissage, and a reduction from \([\text{SGA4}], \text{SGA4}_2\) (embedding a sheaf \(F\) on \(X\) in \(i_* i^* F \oplus j_* j^* F\) when the images of \(i\) and \(j\) cover \(X\).

The first proof of Theorem 3(b), not the one given here, was found with some aid in homological algebra from P. Deligne. The version of Lemma 3.2(b) given here is based on a remark of V. Srinivas. The author thanks both P. Deligne and V. Srinivas for useful discussions, and most of all, A. Beilinson for taking the trouble to explain in detail the very first consequences of his far-reaching Lemma 3.3.

1 The basic lemma for sheaves

A subfield \(k \subset \mathbb{C}\) remains fixed throughout the paper. The phrases “sheaf on \(X\)”, “weakly constructible sheaf”, etc. are as in the introduction. All varieties and morphisms considered are defined over \(k\). All sheaves considered are sheaves of \(R\)-modules. The cohomology group \(H^q(X(\mathbb{C}), F)\) will always be denoted simply by \(H^q(X,F)\). The lemma below evidently implies the theorem of M. Artin : “\(H^q(X,F) = 0\) for \(q > \dim(X)\), where \(X\) is affine ” (see \([\text{SGA4} \text{ Expose XIV, Corollary 3.2}]\)) and the analogous results of Hamm and Le (see \([\text{HL1]}\) and \([\text{HL2}]\)) for weakly constructible sheaves.

It is unclear whether or not the powerful relative version, due to M. Artin (\([\text{SGA4} \text{ Thm. 3.1, Expose XIV}]\)), can also be obtained directly.

Beilinson’s Lemma 3.3 of \([\text{Be2}]\) covers (at least) the case \(F\) constructible, \(R\) commutative Noetherian, \(N=\) all finitely generated modules, in the lemma below. Beilinson works in fields of all characteristics. Remark 1.1 below holds for his proof as well. It would be nice to get a proof of this special case of Beilinson’s very general lemma via Morse theory or the method of pencils. This would have the double advantage of covering the general case and proving the theorems of M. Artin and Andreotti-Frankel as well.

Basic Lemma – second form  (A. Beilinson) Let \(F\) be a weakly constructible sheaf on an affine variety \(X\). Let \(n = \dim(X)\). Then there is a Zariski open \(U \subset X\) with the properties below, where \(j : U \rightarrow X\) denotes the inclusion, and \(F' = j_! j^* F \subset F\).

1. \(\dim(Y) < n\), where \(Y\) is the complement of \(U\) in \(X\),
2. \(H^q(X,F') = 0\) for \(q \neq n\).
3. There is a finite subset \(E \subset U(\mathbb{C})\) and an \(R\)-module isomorphism of \(H^n(X,F')\) with \(\bigoplus\{F_x : x \in E\}\).
Remark 1.1 With \( F \) on \( X \) as in the lemma above, let \( X' \) be the largest Zariski open subset of \( X \) so that \( X' \) is smooth and \( F|X' \) is locally constant. The open \( U \) in the statement of the lemma \textit{depends only on} \( X' \) \textit{and not on} \( F \), as can be seen from the proof.

The proof given uses the Noether normalisation lemma twice. If in both these places, one uses general linear projections instead, one sees that the \( U \) in the lemma can be chosen to contain any given finite subset of \( X'(\mathbb{C}) \). In particular, the \( U \) \textit{for which the lemma holds cover} \( X \), \( F \) \textit{is locally constant} and \( F \) \textit{is smooth}.

Remark 1.2 The first form of the basic lemma, as given in the introduction, is an immediate consequence. If \( R_X \) denotes the constant sheaf on \( X \) and \( v : V \to X \) is a Zariski open immersion with \( W \) as its complement, then the sheaf cohomology \( H^q(X, F) \), where \( F = v_*v^*R_X \) coincides with the singular cohomology \( H^q(X, W; R) \). So, the above lemma applied to \( F \) and \( R = \mathbb{Z} \) yields \( Y \) with \( \dim(Y) < n \) and \( H^q(X, Y \cup W) = 0 \) for \( q \neq n \), and \( H^n(X, Y \cup W) = 0 \) is a free Abelian group. The universal coefficient theorem gives the same result for all \( R \) and for homology as well.

Proof (of the basic lemma (second form)) Because the direct image of a locally constant sheaf under a covering projection is locally constant, it follows easily that the direct image of a weakly constructible sheaf under a finite morphism is weakly constructible. With \( X \) and \( F \) as in the lemma, by Noether normalisation, we have a finite morphism \( \pi : X \to \mathbb{A}^n \). Assuming the lemma for the sheaf \( \pi_*F \) on \( \mathbb{A}^n \), we get a non-empty Zariski open \( V \subset \mathbb{A}^n \) with the desired properties.

Putting \( U = \pi^{-1}V \) and denoting the inclusions of \( U \) in \( X \) and \( V \) in \( \mathbb{A}^n \) by \( j \) and \( v \) respectively, we see that \( \pi_*j_*j^*F = v_*v^*\pi_*F \). Consequently \( H^q(X, j_*j^*F) = H^q(\mathbb{A}^n, v_*v^*\pi_*F) \), and by the very choice of \( V \), the latter vanishes for \( q \neq n \). This proves part (2) of the lemma. Part (3) follows because \( (\pi_*F)_y \) is the direct sum of \( F_x \) taken over \( x \in \pi^{-1}y \).

It remains to prove the lemma for affine space. We will proceed by induction on dimension. Given a weakly constructible sheaf \( F \) on \( \mathbb{A}^n \), choose a nonconstant \( f \) in the coordinate ring of \( \mathbb{A}^n \) so that the restriction \( F|D(f) \) is locally constant, where \( D(f) \) and \( V(f) \), are, as usual, \( \{x \mid f(x) \neq 0\} \) and \( \{x \mid f(x) = 0\} \) in \( \mathbb{A}^n \) respectively. Clearly, \( F \) may be replaced by its subsheaf \( dd^*F \) where \( d \) denotes the inclusion of \( D(f) \) in \( \mathbb{A}^n \). \textit{Thus we will assume that} \( F|V(f) = 0 \). Next, after a linear change of variables, we may assume that \( f \) is monic in the last variable. Denote by \( \pi : \mathbb{A}^n \to \mathbb{A}^{n-1} \) the projection \( (x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{n-1}) \). The restriction \( \pi|V(f) \) is now both \textit{finite} and \textit{surjective}. For \( y \in \mathbb{A}^{n-1}(\mathbb{C}) \), we consider the cohomology groups \( H^q(\pi^{-1}y, F|\pi^{-1}y) \). That this vanishes for \( q > 1 \) is standard (e.g.,
For every point \((H_2)\)
Every point \((H_3)\)

Proposition 1.3A (Variation of proper base-change)
Let \(F\) be a proper continuous map, where \(L\) and \(M\) are second countable locally compact Hausdorff spaces. Let \(L \subset \overline{L}\) be open and let \(A\) be a sheaf on \(L\). Put \(f = \overline{f}\mid L\). Assume \((H_1)\), \((H_2)\) and \((H_3)\) below.

\((H_1)\) There is a closed subset \(L_1 \subset L\) so that \(f\mid L_1\) is proper and \(A\mid L - L_1\) is a locally constant sheaf.

\((H_2)\) For every point \(x \in \overline{L} - L\) there is a neighbourhood \(U\) of \(x\) in \(\overline{L}\) so that \(\overline{f}(U)\) is open and \(\overline{f}(U, U \cap L) \to \overline{f}(U)\) is a fiber bundle pair.

\((H_3)\) Every point \(m \in M\) has a fundamental system of simply connected neighbourhoods.

Then \(R^q f_* A_m \to H^q(f^{-1} m, A|f^{-1} m)\) is an isomorphism for all \(m \in M\).

Corollary 1.3B With notation and assumptions as in Proposition 1.3A above, let \(g : M' \to M\) be continuous with \(M'\) second countable locally compact Hausdorff spaces. Let \(U\) be open and let \(U'\) be a neighborhood of \(U\) in \(M\). Put \(j : A' = A|U'\). Assume \((H_4)\), \((H_5)\) and \((H_6)\) below.

\((H_4)\) \((H_5)\) \((H_6)\)
compact Hausdorff. Put \( L' = M' \times_M L \). Let \( f' : L' \to M' \) and \( g' : L' \to L \) denote the projections. Then \( g^* R^q f_* A \to R^q f'_* g'^* A \) is an isomorphism.

**Corollary 1.3C**  With notation and assumptions as in Proposition 1.3A above, the homomorphism \( B \otimes R^q f_* A \to R^q (A \otimes f^* B) \) is an isomorphism for all sheaves \( B \) on \( M \) of the form \( j! j^* (Z_M) \) for an open immersion \( j : V \to M \).

**Proof (of Proposition 1.3A)**  Let \( L_m \) and \( L'_m \) be the fibers over \( m \in M \) in \( L \) and \( L' \) respectively. Let \( j : L \to \overline{L} \) and \( j_m : L_m \to \overline{L}_m \) denote the inclusions. We have the spectral sequences:

\[
E_2^{p,q} = (R^p j_* R^q f_* A)_m \Rightarrow (R^{p+q} f_* A)_m \quad \text{and} \quad E_2^{p,q} = H^p(\overline{L}_m, R^q j_m^* A|_{L_m}) \Rightarrow H^{p+q}(L_m, A|_{L_m})
\]

and a homomorphism from the first to the second. The second is the Leray spectral sequence for the sheaf \( A|_{L_m} \) and the inclusion \( j_m \). The first is obtained by taking stalks at \( m \) of the Leray spectral sequence for the composite \( L \to \overline{L} \to M \) and the sheaf \( A \). By the proper base-change theorem, we see that the \( E_2^{p,q} \) terms of the first spectral sequence coincide with \( H^p(\overline{L}_m, R^q j_m^* A|_{L_m}) \). To prove the proposition, it suffices to check that \( R^q j_m^* A|_{L_m} \to R^q (j_m)_* (A|_{L_m}) \) is an isomorphism. This arrow evidently induces an isomorphism of stalks at \( x \in L_m \). For the rest, one observes that \( (H1) \), \( (H2) \) and \( (H3) \) imply \( (H4) \) below and then notes that for \( x \in \overline{L}_m - L_m \) the required isomorphism on stalks is a consequence. In any “geometric situation” where \( M \) is triangulable, this is immediate. That \( (H4) \) and our assumptions on the topology of \( L \) and \( M \) are adequate to draw the same conclusion is left to the reader. This may be seen, for instance, by using the Kunneth formula.

\( (H4) \): Every \( x \in \overline{L} - L \) has a neighbourhood \( U \subset \overline{L} \) with a continuous \( h : (U, U \cap L) \to (V, W) \) so that

\begin{enumerate}
  \item \( \overline{f}(U) \) is open in \( M \),
  \item \((h, \overline{f}(U)) \) induce a homeomorphism \((U, U \cap L) \to (V, W) \times \overline{f}(U)\), and
  \item there is a sheaf \( C \) on \( W \) and an isomorphism \((h|U \cap L)^* C \to A|U \cap L\).
\end{enumerate}

\( \square \)

**Proof (of Corollary 1.3B)**  Once it is noted that the hypothesis \( (H4) \) in the proof of Proposition 1.3A is stable under base-change, we see that stalks of both sheaves at \( m' \in M' \) are compatibly isomorphic to \( H^q(f^{-1}m, A|f^{-1}m) \) where \( g(m') = m \) (this argument is standard, see e.g., the proof of the operation \( R^q f! \) being stable under base change in [SGA4\textsuperscript{+}, page 49]).

\( \square \)
Proof (of Corollary 1.3C) Denote by $S^q B$ the natural homomorphism $B \otimes R^q f_* A \rightarrow R^q (A \otimes f^* B)$. Let $i : F \rightarrow M$ be the the inclusion of the complement of the given open subset $V$. Now $S^q Z_M$ is tautologically an isomorphism. Putting $g = i$ in Corollary 1.3B, we see that $S^q i_* Z_F$ is an isomorphism. That $S^q j_* Z_V$ is an isomorphism follows from the long exact sequence of $S^q$ obtained from the short exact sequence:

$$0 \rightarrow i_* Z_F \rightarrow Z_M \rightarrow j_* Z_V \rightarrow 0.$$ 

□

Remark 1.4 Consider a weakly constructible sheaf $F$ on $\mathbb{A}^1$ whose restriction to the complement of a non-empty finite subset $S \subset \mathbb{C}$ is locally constant. Let $T$ be a tree embedded in $\mathbb{C}$ with $S$ as its set of vertices, and $E$ as its set of edges. Let $b(e)$ denote the barycenter of an edge $e \in E$. One checks easily that

(a) $H^q(\mathbb{A}^1, F) \rightarrow H^q(T, F|T)$ is an isomorphism, and

(b) the open covering $\{st(v) : v \in S\}$ of $T$ give a Leray covering for $F|T$ (see e.g., [KS, Chapter VIII, Prop. 8.1.4.(ii), page 323]), for a more general statement).

The Cech complex for this covering is the arrow $\bigoplus_{s \in S} F_s \rightarrow \bigoplus_{e \in E} F_{b(e)}$. Thus, if $F|S = 0$, then $\bigoplus_{e \in E} F_{b(e)} \rightarrow H^1(T, F|T)$ is an isomorphism. This gives part (3) of the basic lemma for the affine line.

Remark 1.5 Let $\pi : X \times \mathbb{A}^1 \rightarrow X$ denote the projection. Let $V \subset X \times A^1$ be a Zariski closed subset so that $\pi|V$ is a finite surjective morphism. Let $A$ be a sheaf on $X \times \mathbb{A}^1$ so that $A|V = 0$ and the restriction of $A$ to the complement $V$ is locally constant. We sketch briefly a proof that $R^1 \pi_* A$ is weakly constructible.

If $\pi|V$ is a finite etale morphism, then $(X \times \mathbb{A}^1, V) \rightarrow X$ is a fiber bundle pair. Consequently $R^1 \pi_* A$ is a locally constant sheaf. For the general case, express $X$ as a finite disjoint union of Zariski locally closed smooth subvarieties $X_i$ so that $(\pi^{-1} X_i \cap V)_{red} \rightarrow X_i$ is a finite etale morphism. By Corollary 1.3B, we see that $R^1 \pi_* A|X_i$ maps isomorphically to the locally constant sheaf $R^1(\pi|X_i \times \mathbb{A}^1)_* A|X_i \times \mathbb{A}^1)$. This proves the weak constructibility of $R^1 \pi_* A$.

Proposition 1.6 Let $F$ be a constructible sheaf on a variety $X$. Then $H^q(X, F)$ is an object of $\mathcal{N}$ for all $q \geq 0$. 


Proof Part (3) of the second form of the basic lemma, combined with induction on dimension, proves the proposition when $X$ is affine. The general case now follows by induction on the number $n$ of affine open sets required to cover $X$, and the Mayer-Vietoris sequence (the same technique is used in the proof of Proposition 3.6).

\[\square\]

2 The cohomology of affine space

This section begins with the proof of Theorem 1 as stated in the introduction. The conventions and notation are as in the previous section. The proof is by induction on $n$, the inductive step relying crucially on a canonical proof for the case $n = 1$ that works with parameters (Proposition 2.2 below). We then deduce the effaceability of cohomology for affine varieties (Corollary 2.4).

We recall the statement of the theorem.

**Theorem 1** Every constructible $F$ on affine space is a subsheaf of a constructible $G$ so that $H^q(\mathbb{A}^n, G) = 0$ for all $q > 0$.

**Proof** We begin as in the proof of the basic lemma. Given a constructible sheaf $F$ on $\mathbb{A}^n$, choose a nonconstant $f$ in the co-ordinate ring of $\mathbb{A}^n$ so that the restriction $F|D(f)$ is locally constant, where $D(f)$ and $V(f)$, are, as before, $\{ x | f(x) \neq 0 \}$ and $\{ x | f(x) = 0 \}$ in $\mathbb{A}^n$. The projection $\pi : \mathbb{A}^n \to \mathbb{A}^{n-1}$ is, once again, arranged so that $\pi|V(f)$ is a finite (and surjective) morphism. Denote by $j : D(f) \to \mathbb{A}^n$ and $i : V(f) \to \mathbb{A}^n$ the inclusions.

From Proposition 2.2 below (with $X = \mathbb{A}^{n-1}$ and $A = j j^* F$) we see that $j j^* F$ is a subsheaf of a constructible $H'$ on $\mathbb{A}^n$ so that $R^q \pi_* H' = 0$ for all $q \geq 0$. Taking push-outs, we obtain the commutative diagram of exact sequences below:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & j j^* F & \rightarrow & F & \rightarrow & i i^* F & \rightarrow & 0 \\
& & \downarrow \text{mono} & & \downarrow \text{mono} & & \downarrow \text{iso} & & \\
0 & \rightarrow & H' & \rightarrow & H & \rightarrow & H'' & \rightarrow & 0.
\end{array}
$$

We will say that a homorphism $P \rightarrow Q$ of sheaves on $\mathbb{A}^n$ is a $\pi$-isomorphism if $R^q \pi_* P \rightarrow R^q \pi_* Q$ is an isomorphism for all $q \geq 0$. We shall see that all
arrows in the commutative diagram below are $\pi$-isomorphisms.

\[
\begin{array}{ccc}
\pi^*\pi_* H & \rightarrow & H \\
\downarrow & & \downarrow \\
\pi^*\pi_* H'' & \rightarrow & H''.
\end{array}
\]

We have the implications:

\[
R^q \pi_* H' = 0 \text{ for all } q \geq 0 \\
\Rightarrow H \rightarrow H'' \text{ is a } \pi \text{-isomorphism} \\
\Rightarrow \pi_* H \rightarrow \pi_* H'' \text{ is an isomorphism} \\
\Rightarrow \pi^*\pi_* H \rightarrow \pi^*\pi_* H'' \text{ is an isomorphism.}
\]

Next, because $\pi|V(f)$ is a finite morphism, we see that $H''$ is constructible and that $R^q \pi_* H''$ vanishes for $q > 0$. For any sheaf $K$ on $\mathbb{A}^{n-1}$, $R^q \pi_* \pi^* K$ vanishes for $q > 0$. Putting $K = \pi_* H''$, we deduce that $\pi^*\pi_* H'' \rightarrow H''$ is a $\pi$-isomorphism. We conclude that the only remaining arrow in the square $\pi^*\pi_* H \rightarrow H$ is itself a $\pi$-isomorphism.

Now let $\pi_* H \subset J$ with $J$ a sheaf on $\mathbb{A}^{n-1}$. Taking push-outs once again yields the commutative diagram of exact sequences below:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \pi^*\pi_* H & \rightarrow & \pi^* J & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H & \rightarrow & G & \rightarrow & M & \rightarrow & 0.
\end{array}
\]

The vertical arrows on the sides being $\pi$-isomorphisms, it follows that $\pi^* J \rightarrow G$ is a $\pi$-isomorphism as well. It follows that

\[H^q(\mathbb{A}^{n-1}, J) \rightarrow H^q(\mathbb{A}^n, \pi^* J) \rightarrow H^q(\mathbb{A}^n, G)\]

are isomorphisms, for all $q \geq 0$. By the induction hypothesis, we may choose $J$ satisfying $H^q(\mathbb{A}^{n-1}, J) = 0$ for all $q > 0$ with $J$ constructible. We now have $F \subset H \subset G$ with $G$ constructible and $H^q(\mathbb{A}^n, G) = 0$ for all $q > 0$. The proof of the theorem is now complete modulo Proposition 2.2.

\[\square\]

\textbf{Notation 2.1} We fix a variety $X$ and work in the category of $X$-schemes. The product $\mathbb{A}^r \times X$ is denoted by $\mathbb{A}^r_X$. Let $p_i : \mathbb{A}^r_X \rightarrow \mathbb{A}^1_X$, $\pi : \mathbb{A}^1_X \rightarrow X$ and $\Pi : \mathbb{A}^2_X \rightarrow X$ denote the given projections, and let $\Delta : \mathbb{A}^1_X \rightarrow \mathbb{A}^2_X$ denote the diagonal morphism.
For any sheaf $A$ on $\mathbb{A}_X^1$, we define $B$ on $\mathbb{A}_X^2$ by the exact sequence:

$$0 \to B \to p_1^*A \to \Delta_*A \to 0.$$ 

Applying $R^q(p_2)_*$ to the above exact sequence gives the arrow:

$$\delta: A = (p_2)_*\Delta_*A \to C = R^1(p_2)_*B.$$ 

**Proposition 2.2** Let $V \subset \mathbb{A}_X^1$ be a Zariski closed subset so that $\pi|V$ is a finite surjective morphism. Let $A$ be a sheaf on $\mathbb{A}_X^1$ so that $A|V = 0$ and the restriction of $A$ to the complement $\mathbb{A}_X^1 - V$ is locally constant. With $\delta: A \to C$ as above, we have:

1. The above $\delta: A \to C$ is a monomorphism,
2. $R^q\pi_*C = 0$ for all $q \geq 0$,
3. $C$ is weakly constructible. If $A$ is constructible, so is $C$.

**Proof** Applying $R^q(p_1)_*$ to the short exact sequence that defines $B$, we see that $R^q(p_1)_*B = 0$ for all $q$. From the Leray spectral sequence applied to $\Pi = \pi \circ p_1$ we see that $R^q\Pi_*B = 0$ for all $q$.

Put $Y = \Delta(\mathbb{A}_X^1) \cup p_1^{-1}V$. Then $B|Y = 0$ and $B$ is locally constant on the complement of $Y$. Now $p_2|Y$ is a finite surjective morphism. So, substituting $(\mathbb{A}_X^2, \mathbb{A}_X^1, p_2, B)$ for $(L, M, f, A)$ in Proposition 1.3A, we get $R^q(p_2)_*B = 0$ for $q \neq 1$. By the Leray spectral sequence for $\Pi = \pi \circ p_2$ we deduce that $R^q\pi_*C = R^{q+1}\Pi_*B = 0$. This proves part (2) of the proposition. That $C$ is weakly constructible follows from Remark 1.5. By Proposition 1.3A and Remark 1.4, any stalk of $C$, being isomorphic to a finite direct sum of stalks of $B$, is indeed an object of $\mathcal{N}$ if $A$ is constructible. This proves part (3). Finally, $\pi^*R^0\pi_*A \to R^0(p_2)_*p_1^*A$ is an isomorphism by Corollary 1.3B, because $V \to X$ is a finite morphism. Furthermore, $V \to X$ being surjective, the stalks of these sheaves vanish for $q = 0$, and this proves part (1) of the proposition.

**Remark 2.3** From the above proof, we see that $A \to C$ is, in fact, an exact functor for $A$ as in the proposition. Furthermore $R^q p_2*$ gives the short exact sequence:

$$0 \to A \to C \to \pi^*R^1\pi_*A \to 0.$$ 

**Corollary 2.4** Every constructible sheaf $F$ on an affine variety $X$ is a subsheaf of a constructible sheaf $G$ on $X$ so that the induced homomorphism $H^0(X, F) \to H^0(X, G)$ is zero for all $q > 0$. 
Proof Let $i: X \to \mathbb{A}^n$ be a closed immersion. From Theorem 1, we have $i_* F \subset T$ with $T$ constructible so that $H^q(\mathbb{A}^n, T) = 0$ for all $q > 0$. The commutative diagram:

\[
\begin{array}{ccc}
H^q(\mathbb{A}^n, i_* F) & \longrightarrow & H^q(\mathbb{A}^n, T) \\
\downarrow & & \downarrow \\
H^q(X, i^* i_* F) & \longrightarrow & H^q(X, i^* T)
\end{array}
\]

shows that $G = i^* T$ is the desired sheaf.

\[\square\]

3 Ext and higher direct images

Definition and notation 3.1: Admissibility, $F[U]$ and $R_X$.

An object $F$ of $\text{Sh}(X)$ is admissible if the functor $\text{Ext}^q_{\text{Sh}(X)}(F, \cdot)|C(X)$ is effaceable for every $q > 0$.

Let $F$ be a sheaf on $X$ and let $U$ be Zariski open in $X$. If $j: U \to X$ denotes the inclusion, then $F[U] = j_! j^* F$.

$R_X$ always denotes the constant sheaf on $X$ with all stalks equal to $R$.

Lemma 3.2 Let

\[
\begin{array}{ccc}
0 & \longrightarrow & F' \\
\longrightarrow & F & \longrightarrow F'' \\
\longrightarrow & 0
\end{array}
\]

be an exact sequence in $\text{Sh}(X)$.

(a) Assume $F''$ is admissible and at least one of $F', F$ is admissible. Then all three are admissible.

(b) Assume $F$ and $F'$ are admissible, and also that

\[
\text{coker}(\text{Hom}_{\text{Sh}(X)}(F, \cdot) \to \text{Hom}_{\text{Sh}(X)}(F', \cdot))|C(X)
\]

is effaceable. Then $F''$ is admissible.

Proof Consider the long exact sequence of Ext:

\[
\begin{array}{ccc}
\text{Ext}^{q-1}_{\text{Sh}(X)}(F', \cdot) & \longrightarrow & \text{Ext}^q_{\text{Sh}(X)}(F'', \cdot) \\
\longrightarrow & \text{Ext}^q_{\text{Sh}(X)}(F', \cdot) & \longrightarrow \text{Ext}^q_{\text{Sh}(X)}(F'', \cdot)
\end{array}
\]

Let $P$ be a constructible sheaf on $X$ and let $q > 0$. For part (b), there is a monomorphism $P \to Q$ with $Q$ constructible that effaces $\text{Ext}^q_{\text{Sh}(X)}(F, P)$. Next choose a monomorphism $Q \to S$ with $S$ constructible that effaces
Ext^{q-1}_{Sh(X)}(F', Q) if q > 1. If q = 1, then Q \to S is chosen so as to efface the cokernel of Hom_{Sh(X)}(F, Q) \to Hom_{Sh(X)}(F', Q). The long exact sequence of Ext shows that P \to S effaces Ext^{q}_{Sh(X)}(F'', P). Part (a) follows in the same manner.

Corollary 3.3 below is an immediate consequence of Lemma 3.2(a).

Corollary 3.3 If 0 = F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 = F and if all the F^i/F^{i+1} are admissible, then F is admissible.

Proposition 3.4 If F is constructible on X and f : X \to Y is a morphism, then R^qf_*F is constructible for all q \geq 0.

Proof This proposition is by now well known; the conventional assumption \(\mathcal{N}= all Noetherian R\text{-modules}\) plays no role in the proof. It suffices to prove the assertion in the two cases: (i) f is proper, and (ii) f is an open immersion.

When f is proper, the proper base-change theorem, the existence of Whitney stratifications, and Thom’s isotopy lemma (see [GM, page 41]) together show that R^qf_*F is weakly constructible. By proper base-change, the stalk of this sheaf at \(y \in Y(\mathbb{C})\) coincides with H^q(f^{-1}y, F|_{f^{-1}y}), which is an object of \(\mathcal{N}\), by Proposition 1.6. Thus R^qf_*F is constructible.

When f is an open immersion, one chooses a Whitney stratification: Y = finite disjoint union of Y_i so that f_iF|_{Y_i} is locally constant for all i. It follows that R^qf_*F|_{Y_i} is locally constant, because we are in a “product situation”. The links L(p) (see [GM, page 41]) being finite simplicial complexes, we’re assured (e.g., by [KS, Prop. 8.1.4(ii)]) that the stalks of R^qf_*F|_{Y_i} are objects of \(\mathcal{N}\). This completes the proof.

Lemma 3.5 Let F be an object of Sh(X).

(a) If F and F[U] are both admissible where U \subset X is Zariski open, then so is F/F[U].

(b) Let V and W be Zariski open in X. Assume that F[V], F[W] and F[V \cap W] are admissible. Then F[V \cup W] is also admissible (see 3.1 for notation).

Proof Both parts follow from Lemma 3.2(b). For (a), it is sufficient to show that every constructible P is contained in a constructible Q with Hom_{Sh(X)}(F, Q) \to Hom_{Sh(X)}(F[U], Q) surjective. Denoting by j and i
the inclusions of $U$ and its complement $Y$ in $X$ respectively, the natural
arrow $P \to Q = j_*j^*P \oplus i_*i^*P$ is a monomorphism that has the desired
property because $\text{Hom}_{\text{Sh}(X)}(F[U], i_*i^*P) = 0$ and

$$\text{Hom}_{\text{Sh}(X)}(F[U], j_*j^*P) = \text{Hom}_{\text{Sh}(U)}(j^*F, j^*P).$$

That $Q$ is constructible has been shown in Proposition 3.4 above.

For (b), consider the Mayer-Vietoris exact sequence:

$$0 \longrightarrow F[V \cap W] \longrightarrow F[V] \oplus F[W] \longrightarrow F[V \cup W] \longrightarrow 0.$$

We obtain $Q$ from $P$ as in the proof of part (a) by putting $U = V \cap W$.
The surjectivity of $\text{Hom}_{\text{Sh}(X)}(F, Q) \to \text{Hom}_{\text{Sh}(X)}(F[V \cap W], Q)$ certainly
implies the surjectivity of

$$\text{Hom}_{\text{Sh}(X)}(F[V], Q) \oplus \text{Hom}_{\text{Sh}(X)}(F[W], Q) \to \text{Hom}_{\text{Sh}(X)}(F[V \cap W], Q).$$

This completes the proof of the lemma. \qed

**Proposition 3.6** Let $U \subset X$ be Zariski open. Then $R_X[U]$ is admissible (see 3.1 for notation).

**Proof** We proceed by induction on the number $n$ of affine open subsets
required to cover $U$. Now assume that $U$ is affine (the case: $n = 1$).
Corollary 2.4 says that $R_U$ is admissible in $\text{Sh}(U)$. It follows that $R_X[U]$ is
admissible in $\text{Sh}(X)$ (see e.g., Remark 3.8(a)). For the general case, write
$U = V \cup W$ where $V$ is covered by $n - 1$ affine opens and $W$ is affine open.
Because $X$ is separated, by Chevalley, $V \cap W$ is covered by $n - 1$ affine
open subsets in $X$, so we may assume the result for $V, W$ and $V \cap W$. The
admissibility of $R_X[U]$ follows from Lemma 3.5(b). \qed

**Proof (of Theorem 2)** The proof of the admissibility of $R_X$ in Proposition 3.6 above actually proves Theorem 2. In any case, the admissibility
plus the vanishing of $H^q(X, F)$ for $q > 2 \dim(X)$ implies the theorem. \qed

**Remark 3.7** Let $F$ be a constructible sheaf on $X$ with the additional
property that $X$ is the union of a finite collection of Zariski locally closed
subsets $Y_i$ so that for each $i$, $F|Y_i$ is a constant sheaf. Assume that $R$
is Noetherian and that $\mathcal{N}$ is the category of all finitely generated $R$-modules.
In this case, one can check that $F$ is admissible. We will not use this
however in the sequel.
Remark 3.8 (Avoiding the use of injective objects) Let $F : A \to B$ and $G : B \to A$ be adjoint functors, viz.,

$$\phi(A, B) : \text{Hom}(FA, B) \to \text{Hom}(A, GB)$$

for all objects $A$ of $A$ and $B$ of $B$, is an isomorphism functorial in $A$ and $B$ (see [Mac] Chapter IV, page 80). Assume that $A$ and $B$ are Abelian categories, and that $F$ is left exact. Then it is standard (see [Toh]) that $G$ takes injectives to injectives. We will not assume that $B$ possesses injective objects. We will observe instead that the left exactness of $F$ ensures:

For every monomorphism $u : GB \to A$, there is a monomorphism $w : A \to GB'$ so that $Gw = w \circ u$. In particular, if $H : A \to C$ is effaceable, then $H \circ G$ is also effaceable. The $v : B \to B'$ is obtained simply as the push-out of the natural arrow $\epsilon(B) : FGB \to B$ with the monomorphism $Fu : FGB \to FA$. Denote by $t : FA \to B'$ the resulting arrow. The $w$ is then obtained as $Gt \circ \eta(A)$ where $\eta(A) : A \to GFA$ is the natural arrow. For the dfns of $\epsilon(B)$ and $\eta(A)$ see [Mac] Thm. 1, page 82. The $(F, G)$ we are concerned with are:

(a) $(j_!, j^*)$ where $j : U \to X$ is an open immersion and $A = C(U)$ and $B = C(X)$. In particular, the admissibility of $Q$ on $U$ implies the admissibility of $j_!Q$ on $X$.

(b) $(f^*, f_*)$ where $f : X \to Y$ is a morphism and $A = C(Y)$ and $B = C(X)$.

(c) $(P \otimes - , \text{Hom}(P, -))$. Here $R$ is assumed to be commutative, and $P$ is weakly constructible on $X$ with all stalks finitely generated projective modules and $A = B = C(X)$.

Definition 3.9 (elementary, projectively elementary) A sheaf $F$ on $X$ is elementary if there are Zariski open $V \subset U \subset X$ and a locally constant sheaf $L$ on $U$ so that $F$ is isomorphic to $j_!(L/L[1])$, where $j : U \to X$ denotes the inclusion.

If all the stalks of $L$ are finitely generated projective $R$-modules, then $F$ will be called projectively elementary.

Proposition 3.10 Every weakly constructible sheaf $F$ on $X$ has a finite filtration

$$0 = F_n \subset F_{n-1} \subset \cdots \subset F^1 \subset F^0 = F$$

so that $F^r/F^{r+1}$ is a direct summand of an elementary sheaf $T^r$ on $X$ for all $r$. Every stalk of $T^r$ is a finite direct sum of stalks of $F$. In particular,
if all the stalks of \( F \) are finitely generated projective \( R \)-modules, then the \( T'' \) above are projectively elementary.

We will assume the above proposition and proceed. Proposition 3.10 is proved at the end of this section.

**Proof (of Theorem 3)** We first assume that \( F \) is projectively elementary.

Let \( L, U, V, F, j \) be as in the dfn of “projectively elementary”. Because \( L \) is a locally constant sheaf on a variety \( U \) with all stalks finitely generated projective, we see that \( \text{Ext}^q(L, D) = 0 \) for all \( q > 0 \) and for all \( D \) on \( U \). The spectral sequence (see [Toh, Thm. 4.2.1, page 188]) shows that \( \text{Ext}^q_{\text{Sh}(U)}(L, D) = H^q(U, \text{Hom}(L, D)) \). From Theorem 2 and Remark 3.8(c), it follows that \( L \) is admissible on \( U \). Replacing \((L, U)\) by \((L|V, V)\) we see that \( L|V \) is admissible on \( V \). From Remark 3.8(a), we see that \( L|V \) is admissible on \( U \). By Lemma 3.5(a), \( L/L[V] \) is admissible on \( U \). By Remark 3.8(a) once again, \( F = j!(L/L[V]) \) is also admissible on \( X \).

Evidently, any direct summand of an admissible is admissible. The admissibility of \( P \) as in Theorem 3(a) now follows from Proposition 3.10 and Corollary 3.3.

\( \square \)

**Theorem 3.11** Assume that \( \mathcal{A}', \mathcal{A}, \mathcal{B}', \mathcal{B}, F \) and \( G \) satisfy the properties (1),(2),(3) and (4) below. It then follows that the functors \( R^eG|\mathcal{B}' \) are effaceable for all \( e > 0 \).

1. \( \mathcal{A}' \) is a full Abelian subcategory of an Abelian category \( \mathcal{A} \). Every object of \( \mathcal{A} \) that is isomorphic to an object of \( \mathcal{A}' \) is an object of \( \mathcal{A}' \). The category \( \mathcal{A} \) possesses enough injectives. For every object \( A' \) of \( \mathcal{A}' \) and for every \( q > 0 \), the functor \( \text{Ext}^q_{\mathcal{A}}(A', \cdot)|\mathcal{A}' \) is effaceable.

2. \( \mathcal{B}' \) is a full Abelian subcategory of an Abelian category \( \mathcal{B} \). All the properties in (1) above hold for \( \mathcal{B}' \) and \( \mathcal{B} \).

3. \( G : \mathcal{B} \to \mathcal{A} \) is a left exact functor so that \( R^q\mathcal{G}\mathcal{B}' \) is an object of \( \mathcal{A}' \) for all objects \( \mathcal{B}' \) of \( \mathcal{B}' \) and for all \( q \geq 0 \).

4. \( F : \mathcal{A} \to \mathcal{B} \) is a left adjoint of \( G \). In addition, \( F \) is left exact. Furthermore, \( FA' \) is an object of \( \mathcal{B}' \) for all objects \( A' \) of \( \mathcal{A}' \).

**Proof** Proceeding by induction on \( e \), we assume the effaceability of \( R^q\mathcal{G}|\mathcal{B}' \) for all \( q \) such that \( 0 < q < e \). Let \( f : A' \to R^e\mathcal{G}|\mathcal{B}' \) be a homomorphism in
\(A\), where \(A'\) and \(B'\) are objects of \(A'\) and \(B'\) respectively. We will find a monomorphism \(B' \to B'_{c+1}\) in \(B'\) so that the composite

\[
A' \longrightarrow R^c GB' \longrightarrow R^c GB'_{c+1}
\]

is zero. Consider the functor \(\text{Hom}(A', \cdot) : A \to Ab\) where \(Ab\) is the category of Abelian groups. By (4) above, note that \(\text{Hom}(A', \cdot) \circ G = \text{Hom}(FA', \cdot)\) and that \(G\) takes injectives to injectives. From [Tol], for every object \(B\) of \(B\), we get the spectral sequence below, functorial in \(B\):

\[
E^{q,q}_2(B) = \text{Ext}^q_A(A, R^q GB) \Rightarrow \text{Ext}^{q+q}_B(FA', B).
\]

We put \(f = f_0\) and \(B' = B_0'\). The given \(f_0\) belongs to \(E^{0,e}_2(GB)\). Its differential \(\delta f_0\) belongs to \(E^{2,e-1}_2(B_0')\). By the induction hypothesis, there is a monomorphism \(B_0' \to B_1'\) in \(B'\) that effaces \(R^{e-1} GB_0'\). Thus \(B_0' \to B_1'\) induces zero on \(E^{2,e-1}_2\). It follows that \(\delta f_0 = 0\) under \(B_0' \to B_1'\). Denoting the image of \(f_0\) in \(E^{0,e}_2(B_1')\) by \(f_1\), we see that \(f_1 \in E^{0,e}_3(B_1') \subset E^{0,e}_2(B_1')\). Continuing in this manner, using the induction hypothesis alone, we obtain a chain of elements \(f_i \in E^{0,e}_i(B_i')\) for \(0 \leq i < e\) and monomorphisms \(B_i' \to B_{i+1}'\) in \(B'\) for \(0 \leq i < e - 1\) that take \(f_i\) to \(f_{i+1}\). Note that this is meaningful because \(E^{0,e}_{i+3}\) is zero. Next note that

\[
\delta f_{e-1} \in E^{e+1,0}_{e+1}(B'_{e-1}) = \text{Ext}^{e+1}_A(A', GB'_{e-1}).
\]

Assumptions (1) and (4), combined with Remark 3.8, imply that the functor \(\text{Ext}^{e+1}_A(A', \cdot) \circ G\) is effaceable. Thus we get a monomorphism \(B_{e-1}' \to B_e'\) in \(B'\) that effaces \(\text{Ext}^{e+1}_A(A', B'_{e-1})\), and as before, this monomorphism takes \(f_{e-1}\) to \(f_e\) with \(f_e \in E^{0,e}_{e+2}(B_e') = E^{0,e}_{e+2}(B_e')\). Finally, because \(e > 0\), by (2), there is a monomorphism \(B_e' \to B_{e+1}'\) in \(B'\) that effaces \(\text{Ext}_B(FA', B_e')\). Because \(E^{0,e}_e(B_e')\) is a quotient of \(\text{Ext}^e(FA', B_e')\), it follows that the induced homomorphism \(E^{0,e}_e(B_e') \to E^{0,e}(B_{e+1}')\) is zero. Consequently, \(f_e = 0\) in \(\text{Hom}(A', R^e GB'_{e+1}) \to \text{Hom}(A', R^e GB'_{e+1})\). It follows that \(f = 0\) in \(\text{Hom}(A', R^e GB') \to \text{Hom}(A', R^e GB'_{e+1})\). This proves our claim.

Now let \(f\) be the identity homomorphism of \(R^e GB'\). The theorem follows.

\[\square\]

Proof (of Theorem 4) Given a morphism \(f : X \to Y\), we substitute

\[
(f^*, f_*, Sh(Y), C(Y), Sh(X), C(X)) = (F, G, A, A', B, B')
\]

in Theorem 3.11. The assumptions (1) and (2) of Theorem 3.11 are valid by Theorem 3(b); assumption (3) is valid by Proposition 3.4, and assumption (4) is evident. So Theorem 4 is a consequence of Theorem 3.11.

\[\square\]
Proof (of Theorem 5) Once again, we apply Theorem 3.11 to

\[ \mathcal{A} = \mathcal{B} = \text{Sh}(X), \quad \mathcal{A}' = \mathcal{B}' = C(X), \quad \text{and} \quad (F, G) = (P \otimes \cdot, \text{Hom}(P, \cdot)), \]

for any constructible sheaf \( P \) on \( X \). To check that the assumptions of Theorem 3.11 are valid, it remains to show that \( \text{Ext}^q(P, Q) \) is constructible if both \( P \) and \( Q \) are constructible. By Proposition 3.10, it suffices to check this for \( P = j_!L \) where \( L \) is locally constant on \( U \) and \( j : U \to X \) denotes the inclusion. But \( \text{Ext}^q(j_!L, Q) = R^qj_*\text{Hom}(L, j^*Q) \) which is constructible by Proposition 3.4. This completes the proof of Theorem 5.

\[ \square \]

Proof (of Proposition 3.10) Let \( F \) be a weakly constructible sheaf on a quasi-projective variety \( X \). Let \( \dim \text{supp.} F = d \). We will first find Zariski open \( V \subset U \subset X \), a locally constant sheaf \( L \) on \( U \), and an isomorphism of \( F|U \) with a direct summand of \( L/L[V] \) (see 3.1 and 3.9 for notation).

Let \( Z_1 \) be the support of \( F \). Let \( Z_2 \) be the union of all the \( d \)-dimensional irreducible components of \( Z_1 \). Let \( Z_3 \) be the largest Zariski open subset of \( Z_2 \) so that \( F|Z_3 \) is locally constant. Let \( U' \subset X \) be an affine open subset so that \( U' \cap Z_1 = U' \cap Z_3 \) and \( U' \cap Z_3 \) is Zariski dense in \( Z_3 \). Choose a finite surjective morphism \( U' \cap Z_1 \to \mathbb{A}^d \) and extend it to a morphism \( h' : U' \to \mathbb{A}^d \). Let \( W \subset \mathbb{A}^d \) be non-empty Zariski open. Put \( U = h'^{-1}W, Z = U \cap Z_1 \) and \( V = U - Z \). Denote the restriction of \( h' \) to \( U \) by \( h : U \to W \). The sheaf \( G = F|Z \) is locally constant by assumption. The \( W \) is chosen so that \( f : Z \to W \) is an etale morphism (note that \( f \) is already assumed to be a finite morphism); this ensures that \( f_*G \) is locally constant. It follows that \( L = h^*f_*G \) is also a locally constant sheaf. Note that \( f^*f_*G = L|Z \) has \( G \) as a direct summand because \( f \) is etale. Denoting by \( i : Z \to U \) the inclusion, we see that \( i_*G = F|U \) is a direct summand of \( i_*L|Z = L/L[V] \).

From the above, we see that \( F|U \) is a direct summand of an elementary sheaf. Put \( QF = F/F|U \). Because \( \dim \text{supp.} QF < d \), induction on \( \dim \text{supp.} F \) proves the proposition with the length of the desired filtration bounded above by \( 1 + \dim \text{supp.} F \), when \( X \) is quasi-projective. In the general case, one proceeds by Noetherian induction, replacing the hypothesis “\( U' \cap Z_3 \) is Zariski dense in \( Z_3 \)” by “\( U' \cap Z_3 \) is non-empty”.

\[ \square \]

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