Inhomogeneous beam with two internal vertical lengthwise cracks: a fracture study

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Abstract. Fracture of an inhomogeneous cantilever beam with two internal parallel lengthwise cracks is analyzed. Cracks have different lengths and are located arbitrary along the beam width. The beam exhibits continuous material inhomogeneity in both width and length directions. The fracture is studied in terms of the strain energy release rate. For this purpose, the beam is treated as a structure with two degrees of internal static indeterminacy. The solutions to the strain energy release rate, derived in the present paper, are verified by the using the J-integral approach. A parametric investigation of the lengthwise fracture behaviour of the beam is performed.

1. Introduction
Inhomogeneous materials are extensively used in many load-bearing structural applications where high performances are required. The properties of inhomogeneous materials vary continuously (smoothly) in the solid [1, 2, 3]. A certain kind of inhomogeneous materials such as functionally graded materials can be built-up layer by layer [4] which is a premise for appearance of lengthwise cracks. It is therefore very important to analyze various aspects of lengthwise fracture behaviour of inhomogeneous structural members and components.

The goal of the present paper is to analyze lengthwise fracture of an inhomogeneous beam with two internal parallel cracks in contrast to previous papers [5, 6] which are focussed on inhomogeneous beams with one lengthwise crack located in the end of the beam.

2. Solution to the strain energy release rate for the two internal cracks
The present paper studies the lengthwise fracture in the inhomogeneous cantilever beam with two parallel lengthwise cracks (figure 1). The lengths of crack 1 and crack 2 are $a_1$ and $a_2$, respectively. The left-hand crack arm has length, $a_1$, and width, $b_1$. The central and right-hand crack arms have length, $a_2$. The widths of the central and right-hand crack arm are $b_2$ and $b_3$, respectively. The beam is clamped in section, $B$. The length of the beam is $l$. The cross-section of the beam is a rectangle of width, $b$, and height, $h$. The loading of the beam consists of one bending moment, $M$, applied at the free end. The beam exhibits continuous material inhomogeneity in width and length directions. The distribution of the modulus of elasticity, $E$, along the beam width is written as

$$E = E_0 e^{b \frac{h-\gamma}{2b}}$$

(1)
where $E_0$ is the value of the modulus of elasticity at the left-hand lateral surface of the beam, $f$ is a material property that controls the material gradient along the beam width, $y_1$ is the horizontal centroidal axis (figure 1).

![Figure 1. Inhomogeneous cantilever beam with two parallel lengthwise cracks.](image)

The distribution of $E_0$ along the beam length is expressed as

$$E_0 = E_{L0} e^{g x_1},$$

where $E_{L0}$ is the value of $E_0$ at the free end of the beam.

The fracture behaviour of the beam is analyzed in terms of the strain energy release rate, $G$. For this purpose, first, a small increase, $\Delta a_1$, of the length of crack 1 in crack tip, $D_1$, is assumed. By using linear-elastic fracture mechanics, the strain energy release rate is expressed as (figure 1)

$$G = \frac{1}{h \Delta a_1} \left\{ \Delta a_1 \int_{b_2}^{b_f} \int_{h_2}^{h_f} u_{0R_1} dy_2 dz_2 + \Delta a_1 \int_{b_2}^{b_f} \int_{h_2}^{h_f} u_{0P_1} dy_3 dz_3 - \Delta a_1 \int_{b_2}^{b_f} \int_{h_2}^{h_f} u_{0Q_1} dy_4 dz_4 \right\},$$

where $u_{0R_1}$ and $u_{0P_1}$ are the strain energy densities in the cross-sections, $R_1$ and $P_1$, of the two crack arms behind the crack tip, $u_{0Q_1}$ is the strain energy density in the beam cross-section ahead of the crack tip, $y_2$ and $z_2$, and $y_3$ and $z_3$ are the centroidal axes of the cross-sections, $R_1$ and $P_1$, of the crack arms, $y_4$ and $z_4$ are the centroidal axes of the beam cross-section ahead of the crack tip, $D_1$.

The strain energy density, $u_{0R_1}$, is written as
\[ u_{0 R_1} = \frac{1}{2} E \varepsilon_{R_1}^2 , \]  

where the distribution of the strain, \( \varepsilon_{R_1} \), is expressed as

\[ \varepsilon_{R_1} = \kappa_{R_1} z_2 . \]  

In (5), \( \kappa_{R_1} \) is the curvature of the crack arm in cross-section, \( R_1 \). It should be noted that the strain is distributed linearly in the cross-section since in the present paper beams of high length to height ratio are under consideration. Thus, validity of the Bernoulli’s hypothesis for plane sections is assumed.

The following equation of equilibrium of the elementary forces in the crack arm cross-section, \( R_1 \), is used to obtain the curvature, \( \kappa_{R_1} \):

\[ \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{R_1} dy_z d z_2 = M_H , \]  

where \( M_H \) is the bending moment in the left-hand crack arm. The stress, \( \sigma_{R_1} \), is found by the Hooke’s law

\[ \sigma_{R_1} = E \varepsilon_{R_1} , \]  

It is obvious that in order to obtain \( \kappa_{R_1} \) from (6), one needs \( M_H \). The bending moments in the crack arms are derived by treating the beam as a structure with two degrees of internal static indeterminacy by using the theorem of Menabrea

\[ \frac{\partial U}{\partial M_H} = 0 , \quad \frac{\partial U}{\partial M_T} = 0 , \]  

where \( U \) is the strain energy in the beam, \( M_T \) is the bending moment in the central crack arm. The strain energy is written as

\[
U = \int_{0}^{l_1} \int_{0}^{b} \int_{0}^{h} \frac{E \varepsilon_{a1}^2}{2} dx_i dy_j d z_4 + \int_{l_1}^{l_2} \int_{0}^{b} \int_{0}^{h} \frac{E \varepsilon_{a2}^2}{2} dx_i dy_j d z_5 + \int_{l_2}^{l_3} \int_{0}^{b} \int_{0}^{h} \frac{E \varepsilon_{a3}^2}{2} dx_i dy_j d z_6 + \int_{l_3}^{l_4} \int_{0}^{b} \int_{0}^{h} \frac{E \varepsilon_{a4}^2}{2} dx_i dy_j d z_7 \\
+ \int_{l_4}^{l_5} \int_{0}^{b} \int_{0}^{h} \frac{E \varepsilon_{a5}^2}{2} dx_i dy_j d z_8 + \int_{l_5}^{l_6} \int_{0}^{b} \int_{0}^{h} \frac{E \varepsilon_{a6}^2}{2} dx_i dy_j d z_9 + \int_{l_6}^{l_7} \int_{0}^{b} \int_{0}^{h} \frac{E \varepsilon_{a7}^2}{2} dx_i dy_j d z_{10},
\]  

where \( \varepsilon_{a1} \) is the strain in the un-cracked part of the beam, \( 0 \leq x_1 \leq l_1 , \varepsilon_{a2} \) is the strain in the beam part, \( l_1 \leq x_1 \leq l_2 , - (b_2 - b_3) / 2 \leq y_5 \leq (b_2 + b_3) / 2 \), \( \varepsilon_{a3} \) is the strain in the left-hand crack arm, \( \varepsilon_{a4} \) is the strain in the central crack arm, \( \varepsilon_{a5} \) is the strain in the right-hand crack arm, \( \varepsilon_{a6} \) is the strain in the beam part, \( l_2 + a_2 \leq x_1 \leq l_1 + a_1 , - (b_2 - b_3) / 2 \leq y_8 \leq (b_2 + b_3) / 2 \), \( \varepsilon_{a7} \) is the strain in the un-cracked part of the beam, \( l + a_5 \leq x_1 \leq l_i , y_j \) and \( z_i \) where \( i = 1, \ldots, 8 \) are the centroidal axes of the cross-sections of the corresponding portions of the beam.
The distribution of strain, $\varepsilon_{a1}$, in the cross-section of the un-cracked portion of the beam, $0 \leq x_1 \leq l_1$, is written as

$$\varepsilon_{a1} = k_{a1} z_1, \quad (10)$$

where the curvature, $k_{a1}$, is obtained by using the equation of equilibrium of the cross-section

$$\int_{-b/2}^{b/2} \int_{-h/2}^{b+h/2} E\varepsilon_{a1} dy_1 dz_1 = M, \quad (11)$$

After substituting of (1) and (10) in (11), the equation of equilibrium is solved with respect to $k_{a1}$. In this way, $k_{a1}$ and $\varepsilon_{a1}$ through (10) are expressed as functions of the bending moment. Analogically, the curvatures and strains in each part of the beam and in the three crack arms are obtained as functions of the corresponding bending moments. The bending moment, $M_C$, in the right-hand crack arm is expressed as function of $M_H$ and $M_T$ by using the equation for equilibrium of moments

$$M_C + M_H + M_T = M. \quad (12)$$

By substituting of strains, $\varepsilon_{a1}$, in (9), the strain energy in the beam is obtained as a function of $M_H$ and $M_T$. After substituting of the strain energy in (8), the two equations are solved with respect to $M_H$ and $M_T$.

After substituting of (5), (7) and $M_H$ in (6), the equation of equilibrium is solved with respect to $k_{R_i}$. Then the strain energy density, $u_{0R_i}$, is obtained by substituting of $k_{R_i}$ in (4). The strain energies, $u_{0R_i}$ and $u_{OQ_i}$, are found by considering the equilibrium of the elementary forces in the cross-sections of the right-hand crack arm behind the crack tip and the beam cross-section ahead of the crack tip, $D_1$. The strain energy release rate is derived by substituting of $u_{0R_i}$, $u_{0P_i}$ and $u_{0Q_i}$, in (3).

The fracture is analyzed also assuming a small increase, $\Delta a_1^*$, of the length of crack 1 at crack tip, $D_2$. The strain energy release rate is obtained as

$$G = \frac{1}{h\Delta a_1^*} \left( \Delta a_1^* \int \int u_{0R_2} dy_2 dz_2 + \Delta a_1^* \int \int u_{0P_2} dy_2 dz_2 - \Delta a_1^* \int \int u_{0Q_2} dy_2 dz_1 \right), \quad (13)$$

where the strain energy densities, $u_{0R_2}$, $u_{0P_2}$ and $u_{0Q_2}$, are derived by considering the equilibrium of the cross-sections of the crack arms behind the crack tip and the beam cross-section ahead of the crack tip, $D_2$.

A fracture analysis is carried-out also assuming a small increase, $\Delta a_2^*$, of the length of crack 2 at crack tip, $D_3$. The following expression for the strain energy release rate is derived:

$$G = \frac{1}{h\Delta a_2^*} \left( \Delta a_2^* \int \int u_{0R_6} dy_6 dz_6 + \Delta a_2^* \int \int u_{0P_6} dy_6 dz_6 - \Delta a_2^* \int \int u_{0Q_6} dy_6 dz_6 \right). \quad (14)$$

The equations for equilibrium of the cross-sections behind and ahead of the crack tip, $D_3$, are used to obtain the strain energy densities, $u_{0R_i}$, $u_{0P_i}$ and $u_{0Q_i}$. 


Finally, the fracture is analyzed assuming a small increase, $\Delta a_2^*$, of the length of crack 2 at crack tip, $D_4$. The strain energy release rate is written as

$$G = \frac{1}{h\Delta a_2^*} \left( \Delta a_2^* \int_{-b_2}^{b_2} \int_{-b}^{b} u_{00z_6} dy_6 dz_6 + \Delta a_4^* \int_{-b_4}^{b_4} \int_{-b}^{b} u_{00z_4} dy_4 dz_4 - \Delta a_2^* \int_{-b_2}^{b_2} \int_{-b}^{b} u_{00z_2} dy_2 dz_2 \right),$$

(15)

where the strain energy densities are obtained from equations for equilibrium of the cross-sections behind and ahead of the crack tip.

In order to verify solutions (3), (13), (14) and (15), the fracture is studied also by applying the $J$-integral approach. Solutions to the $J$-integral are derived for crack tips, $D_1$, $D_2$, $D_3$ and $D_4$.

![Figure 2](image.png)

**Figure 2.** The strain energy release rate in non-dimensional form plotted against $a_i/l$ ratio (curve 1 – at $b_1/b=0.2$, curve 2 – at $b_1/b=0.4$).

The fact that the $J$-integral values are exact matches of the strain energy release rates is a verification of the fracture analysis developed in the present paper.

3. **Numerical results**

Numerical results are obtained by using the solutions to the strain energy release rate in order to investigate the influence of locations and lengths of the two cracks and material inhomogeneity on the lengthwise fracture. The strain energy release rate is presented in non-dimensional form by using the formula $G_N = G/(E_l b)$. It is assumed that $l = 0.2$ m, $b = 0.008$, $h = 0.006$, $l_1 = 0.03$ m, $l_2 = 0.04$ m and $T = 15$ kNm.

The influence of length and location of crack 1 on the lengthwise fracture behaviour is illustrated in figure 2 where the strain energy release rate in non-dimensional form is plotted against $a_i/l$ ratio at two $b_1/b$ ratios (the strain energy release rate is obtained by (3)). The curves in figure 2 indicate that the strain energy release rate decreases with increasing of $a_i/l$ ratio (this behaviour is due to the fact that the modulus of elasticity of the beam cross-section in which the tip of crack 1 is located increases with increasing of $a_i/l$ ratio). One can observe in figure 2 also that the strain energy release rate at $b_1/b=0.2$ is higher than that at $b_1/b=0.4$. 

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The effect of material inhomogeneity on lengthwise fracture is investigated too. For this purpose, the strain energy release rate in non-dimensional form is plotted against $f$ in figure 3 (the strain energy release rate is found by (13) and (15)).

Figure 3. The strain energy release rate in non-dimensional form plotted against $f$ (curve 1 – at increase of crack 1 at crack tip $D_2$, curve 2 – at increase of crack 2 at crack tip $D_4$).

It can be observed in figure 3 that the strain energy release rate decreases with increasing of $f$. Besides, the strain energy release rate found by (13) is higher than that found by (15).

4. Conclusions
A lengthwise fracture analysis of an inhomogeneous cantilever beam with two internal parallel lengthwise cracks of different lengths is carried-out. The cracks are located arbitrary along the beam width. Thus, the widths of the cross-sections of the three crack arms are different. The beam exhibits continuous material inhomogeneity in width and length directions. The bending moments in the crack arms are derived by treating the beam as a structure with two degrees of internal static indeterminacy. Four solutions to the strain energy release rate are obtained by assuming increases of the cracks in the four crack tips. The solutions are verified by the $J$-integral approach. The influence of the lengths and locations of cracks on the fracture behaviour is evaluated. It is found that the strain energy release rate decreases with increasing of $a_l/l$ ratio. The increase of $b_l/b$ ratio leads to decrease of the strain energy release rate. The analysis reveals also that the strain energy release rate decreases with increasing of the material property, $f$.

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