On non-abelian Lubin-Tate theory for GL(2) in the odd equal characteristic case

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Abstract

In this paper, we define a family of affinoids in the tubular neighborhoods of CM points in the Lubin-Tate curve with suitable level structures, and compute the reductions of them in the equal characteristic case. By using étale cohomology theory of adic spaces due to Huber, we show that the cohomology of the reductions contributes to the cohomology of the Lubin-Tate curve. As an application, with the help of explicit descriptions of the local Langlands correspondence and the local Jacquet-Langlands correspondence due to Bushnell-Henniart via the theory of types, we prove the non-abelian Lubin-Tate theory for GL(2) in the odd equal characteristic case in a purely local manner. Conversely, if we admit the non-abelian Lubin-Tate theory, we can recover the explicit descriptions of the two correspondences geometrically.

1 Introduction

It is known that the cohomology of Lubin-Tate spaces simultaneously realizes the local Langlands correspondence (LLC, shortly) and the local Jacquet-Langlands correspondence (LJLC, shortly) for general linear groups over non-archimedean local fields. This is conjectured in [Ca2]. This is called the Deligne-Carayol conjecture or the non-abelian Lubin-Tate theory (NALT, for shortly). This has been proved in [Bo] in the equal characteristic case and [HT] in the mixed characteristic case respectively. The proofs depend on global automorphic representations, and Shimura variety or Drinfeld modular variety. In [St2], [Mi] and [Mi3], by a purely local and geometric method, the LJLC is proved to be realized in the cohomology of the Lubin-Tate space. A purely local and geometric proof of the corresponding assertion for the LLC is not known. Since the NALT is a generalization of the Lubin-Tate theory, it is hoped that it is proved in a local approach.

To obtain such a proof, it is necessary to understand the cohomology of the Lubin-Tate space with respect to the cohomology of the reductions of some affinoid subdomains in it (cf. [Ha]). In this paper, in the odd equal characteristic case, we define a family of affinoids near CM points in the Lubin-Tate tower, determine the reductions of them, and analyze the middle cohomology of them. By relating such analysis to the cohomology of the Lubin-Tate curve using étale cohomology theory of rigid analytic varieties given in [Hu2], we obtain a new geometric proof of the NALT for GL(2) without depending on global automorphic representations and geometry of Drinfeld modular curves.

In [We4], Weinstein defines a family of affinoids in the Lubin-Tate perfectoid curve, and determines the reductions of them. As a result, with the help of the NALT, he classifies

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irreducible components into four types in the stable reductions of the Lubin-Tate curves with Drinfeld level structures up to purely inseparability.

In this paper, without using perfectoid space, we explicitly compute the reductions of affinoids in the Lubin-Tate curves with finite level structures of two types in the equal characteristic case. In the unramified case, we use the Lubin-Tate curve with Drinfeld level structures. This curve has a nice simple formal model (2.2), which is very compatible with the points which have complex multiplication by the ring of integers in the quadratic unramified extension. In the ramified case, we use the Lubin-Tate curve with Iwahori level structures whose formal model is given in [GL]. Its natural formal model (2.31) is very compatible with the points which have complex multiplication by the rings of integers in quadratic ramified extensions. In the equal characteristic case, the formal model (2.31) is described through the theory of coordinate modules, which is developed in [Ge]. A group action on this model is very explicitly described in [GL]. This model is originally used to explicitly compare Lubin-Tate tower with Drinfeld tower. Also for our purpose, this model plays a very nice role.

The image of the affinoid by the level lowering map between Lubin-Tate curves equals a closed disk. This fact implies that the reductions are considered to be “new components” in each level. It seems difficult to determine all “old components” in the stable reduction of Lubin-Tate tower without cohomological understanding of Lubin-Tate tower. In our proof of the NALT for GL(2), it is unnecessary to compute all irreducible components in the stable reduction of Lubin-Tate tower. This is a key point in our proof.

In §2, we define affinoids near CM points and compute the reductions of the affinoids. The reductions are classified into three types (cf. Lemma 2.9, Proposition 2.10 and Proposition 2.12). This classification fits into the one given in [We4, Theorem 1.0.1]. The other one except for the three is a projective line. Since the middle cohomology of it is trivial, we will not consider. As for several previous works on types of irreducible components in the stable reductions of modular curves or Lubin-Tate curves, see [We4, Introduction]. The idea to study affinoids near CM points in order to understand stable coverings of modular curve or Lubin-Tate curve appears in [CMc, Corollary 4.2] first in the literature. This idea is used also in [We4] at infinity level in a general setting. Originally, CM points on Lubin-Tate spaces are studied in [Gi] and CH (cf. [Fa]).

In §3, we calculate the group action on the reductions in §2 which is induced by the group action on the Lubin-Tate tower. Similar descriptions are predicted in [We3]. In [We3], he constructs a stable curve over a finite field with appropriate group action, whose middle cohomology realizes the NALT for GL(2).

In §4, we collect some known facts on the first cohomology of some curves. To relate the cohomology of the reductions of the affinoids to the cohomology of the Lubin-Tate curve, Lemma 4.5 plays a key role. Let $K$ be a non-archimedean local field. Let $p$ be the residue characteristic of $K$. By this lemma, for a rigid analytic variety $X$ over $K$ and its affinoid subdomain $W$, if $W$ has good reduction, under some condition on the cohomology of the canonical reduction $\overline{W}$ of $W$, we can relate the cohomology of the reduction $\overline{W}$ to the cohomology of the total space $X$. This fact is an immediate consequence of a comparison theorem between formal nearby cycles and usual nearby cycles in [Hu2, Theorem 0.7.7] (cf. [12]). Note that Lemma 4.5 works for any dimensional case. It could be possible to prove it also by using Berkovich’s results in [Be] and [Be2]. Let $\ell \neq p$ be a prime number. For the reductions $\overline{W}$ of the affinoids $W$ in the Lubin-Tate curve in §2 except for the level zero case, the canonical map $f : H^1_c(\overline{W}_p, \overline{\mu}_\ell) \to H^1(\overline{W}_p, \overline{\mu}_\ell)$ is an isomorphism. This follows from the fact that the reduction is isomorphic to a curve of Artin-Schreier type associated to a monomial (cf. Corollary 4.11). For the level zero case, the map $f$ becomes an injection on the cuspidal part of $H^1(\overline{W}_p, \overline{\mu}_\ell)$ by Lemma 4.18.1. The cuspidal part is well-understood through a small part of the Deligne-Lusztig theory in [DL].
To apply Lemma 4.5, we use these properties of the reductions. The depth zero case, in any dimensional case, is studied in [Yo].

Let $W_K$ denote the Weil group of $K$, and let $D$ denote the quaternion division algebra over $K$. In the following, we assume that $K$ has odd characteristic. In Theorem 5.17.2, on the basis of the theory of types for GL(2) and results in [Mi] and [St], we give explicit and geometric one-to-one correspondences between the following three sets:

- $G^0(K)$: the set of isomorphism classes of two-dimensional irreducible smooth representations of $W_K$,
- $A_1^0(D)$: the set of isomorphism classes of irreducible smooth representations of $D^\times$ of degree $> 1$, and
- $A^0(K)$: the set of isomorphism classes of irreducible cuspidal representations of $GL_2(K)$,

in the first cohomology of the Lubin-Tate curve. This is our main result in this paper. Theorem 5.17 is reduced to the assertion for special cases (Proposition 5.16). To do so, we need to understand group action on the set of geometrically connected components of Lubin-Tate tower. In §5.2 this is done by using results on the Lubin-Tate side in [GL V.5]. Roughly speaking, the result, which we use, asserts that the action of $G = GL_2(K) \times D^\times \times W_K$ on the set of geometrically connected components of the Lubin-Tate tower realizes the Lubin-Tate theory (cf. the proof of Corollary 5.15). Note that the action of an open compact subgroup of $G$ on $\pi_0$ of the Lubin-Tate space with Drinfeld level structures is studied in [St2] in a purely local manner. Since we need an action on the set of geometrically connected components of the Lubin-Tate tower in §5.2, this is done by using results on the Lubin-Tate side in [GL V.5]. Roughly speaking, the result, which we use, asserts that the action of $G = GL_2(K) \times D^\times \times W_K$ on the set of geometrically connected components of the Lubin-Tate tower realizes the Lubin-Tate theory (cf. the proof of Corollary 5.15). Note that the action of an open compact subgroup of $G$ on $\pi_0$ of the Lubin-Tate space with Drinfeld level structures is studied in [St2] in a purely local manner. Since we need an action on the set of geometrically connected components of the Lubin-Tate tower, we need to use determinant morphisms explicitly constructed in [GL]. It is known that the cohomology of Lubin-Tate curve realizes the LJLC with multiplicity two by [Mi] and [St] (cf. the proof of Theorem 5.18). Their proofs do not depend on any global method. To prove Theorem 5.17.2, their results play an important role. On the basis of the analysis in §2, §3 and §4, Proposition 5.16 will be proved in a purely local manner in §6. In §6 we describe representations appearing in the middle cohomology of the reductions. To describe them in the unramified case, we use linking orders studied in [We] and [We4].

By the work of Bushnell-Henniart in [BH], the three sets $G^0(K)$, $A_1^0(D)$, and $A^0(K)$ are simply parametrized by admissible pairs $(L/K, \chi)$, where $L/K$ is a quadratic separable extension and an appropriate character $\chi$ of $L^\times$ (cf. (5.7)). For example, starting from an admissible pair $(L/K, \chi)$, we can construct a smooth representation $\pi_{\chi}$ of an open compact-mod-center subgroup $J_{\chi}$ of $GL_2(K)$ through representation theory of a finite Heisenberg group. Then, the compact induction of $\pi_{\chi}$ from $J_{\chi}$ to $GL_2(K)$ is an irreducible cuspidal representation of $GL_2(K)$ (cf. §5.1). As above, the theory of types gives a recipe to construct irreducible cuspidal representations. An explicit description of the LLC and the LJLC via the theory of types is given in [BH]. We call the description the explicit LLC and the LJLC. See Theorem 5.8 for precise statements of them. This theory is established in a purely local and representation-theoretic method without geometry. See [Hen] for more developments in this direction.

In §5.2 we introduce a direct consequence of Theorem 5.17. In Theorem 5.19, we show that under Theorem 5.17 the explicit LLC and the LJLC is equivalent to the NALT for GL(2). In this sense, a new proof of the NALT for GL(2) is obtained in a local approach.

We emphasize that, in the proof of our main theorem, it is unnecessary to understand a whole shape of the stable reduction of the Lubin-Tate curve with each finite level structure. To prove it, it is enough to understand the first cohomology of the affinoids in this paper. To justify this, the étale cohomology theory in [Hu2] is needed. To apply this theory, we need to work at finite levels. It has been an anxious problem for us to relate the cohomology of the
reductions of affinoids to the one of Lubin-Tate tower. As explained above, this is settled by understanding the shape of the reductions of affinoids and by just applying Huber’s theory. It makes possible for us to obtain a geometric proof of NALT for GL(2) without depending on global automorphic representations. Our approach will be applied to higher dimensional case in a subsequent paper. We note that the analysis given in [2] and [3] is elementary and explicit.

In the case where the residual characteristic equals two, it seems unknown to define a family of affinoids in a systematic or conceptual way. An example of a semi-stable model of some Lubin-Tate curve in the residual characteristic two case is found in [IT]. In [IT2], the LLC for primitive representations of conductor three over dyadic fields is proved.

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Notation. For a non-archimedean valued field $K$, let $\mathcal{O}_K$ denote the valuation ring of $K$, and let $p_K$ denote the maximal ideal of $\mathcal{O}_K$. We set $\mathbb{F}_K = \mathcal{O}_K/p_K$. For a non-archimedean local field $K$, let $U^0_K = \mathcal{O}_K^\times$ and, for a positive integer $n \geq 1$, let $U^n_K = 1 + p^n_K$. Let $v_K(\cdot)$ denote the normalized valuation of $K$. For a prime number $p$ and a positive integer $r \geq 1$, let $\mathbb{F}_{p^r}$ denote the finite field of cardinality $p^r$.

2 Reductions of affinoids

In this section, we define affinoids of two types in the Lubin-Tate curve and determine the reductions of them over some finite extension of a base field. One is contained in tubular neighborhoods of points which have complex multiplication by the ring of integer in the unramified quadratic extension of the base field. The other is contained in tubular neighborhoods of points which have complex multiplication by the rings of integers in ramified quadratic extensions. Except for the depth zero case, their reductions are isomorphic to some Artin-Schreier curves as in Proposition 2.10 and Proposition 2.12. In the depth zero case, the reduction of the affinoid is isomorphic to the Deligne-Lusztig curve for a special linear group of degree two over finite fields.

2.1 Preliminary on the canonical reduction

In this subsection, we recall several known facts on the canonical reduction of an affinoid. Let $K$ be a complete non-archimedean valued field of height one. Let $X = \text{Sp} A$ be a reduced affinoid variety over $K$. Let $|\cdot|_{\text{sup}}$ be the supremum norm on $A$. We set

$$A^\circ = \{x \in A \mid |x|_{\text{sup}} \leq 1\} : \text{the set of all power-bounded elements},$$

$$A^{\circ\circ} = \{x \in A \mid |x|_{\text{sup}} < 1\} : \text{the set of all topologically nilpotent elements}$$

(cf. [BGR] Propositions 1 and 2 in §6.2.3]). Then, $A^\circ$ is a subring of $A$, and $A^{\circ\circ}$ is an ideal of $A^\circ$. Then, we set

$$\overline{A} = A^\circ/A^{\circ\circ}.$$
This is called the canonical reduction of $A$ (cf. [BGR] §6.3 and [BL] §1]). We write $\overline{X}$ for $\text{Spec} \, \overline{A}$, which we simply call the reduction of $X$. This is reduced, because $| \cdot |_{\sup}$ is a power-multiplicative norm. Let

$$T_{n,K} = K(X_1, \ldots, X_n)$$

be the free Tate algebra in $n$ indeterminates over $K$ (cf. [BGR] §5.1.1]). Then, we have

$$T_{n,K}^\circ = \mathcal{O}_K \langle X_1, \ldots, X_n \rangle,$$

where the right hand side denotes the $p_K$-adic completion of $\mathcal{O}_K[X_1, \ldots, X_n]$. We take a surjective morphism of $K$-affinoid algebras $\alpha : T_{n,K} \to A$. Let $| \cdot |_\alpha$ be the residue norm on $A$ associated to $\alpha$. We write $A_\alpha$ for the image of $T_{n,K}^\circ$ by $\alpha$. Then, we have

$$A_\alpha = \{ x \in A \mid |x|_\alpha \leq 1 \}.$$

This is a subring of $A^\circ$, because we have $|x|_{\sup} \leq |x|_\alpha$ for any $x \in A$.

We keep the following lemma in mind whenever we compute the reductions of affinoids in the proceeding sections.

**Lemma 2.1.** We assume that $A_\alpha \otimes_{\mathcal{O}_K} \mathbb{F}_K$ is reduced. Then, we have $| \cdot |_{\sup} = | \cdot |_\alpha$ on $A$. Furthermore, we have

$$A^\circ = A_\alpha \supset A^{\circ \circ} = p_K A_\alpha,$$

$$\overline{A} = A_\alpha \otimes_{\mathcal{O}_K} \mathbb{F}_K = A^\circ \otimes_{\mathcal{O}_K} \mathbb{F}_K.$$

**Proof.** By [BLR] Proposition 1.1], we obtain $| \cdot |_{\sup} = | \cdot |_\alpha$ on $A$, and hence $A^\circ = A_\alpha$. By [BGR] Proposition 3 (i) in §6.4.3], we have $A^\circ = \alpha(T_{n,K}^\circ)$ and $A^{\circ \circ} = \alpha(T_{n,K}^{\circ \circ})$. Hence, we obtain $A^{\circ \circ} = \alpha(p_K T_{n,K}^\circ) = p_K A_\alpha$. Hence, the claims follow. \qed

**Remark 2.2.** By the reduced fiber theorem in [BLR] Theorem 1.3], for any geometrically reduced affinoid $K$-algebra $A$, there exist a finite separable extension $K'$ over $K$, and an epimorphism $\alpha : T_{n,K'} \to A \otimes_K K'$ such that $A_\alpha \otimes_{\mathcal{O}_{K'}} \mathbb{F}_{K'}$ is reduced.

### 2.2 Morphisms between formal schemes

In this subsection, we fix some terminology on formal geometry (cf. [Be] and [Be2]).

Let $K$ be a complete non-archimedean valued field. A morphism of affine formal schemes $\text{Spf} \, A \to \mathcal{S} = \text{Spf} \, \mathcal{O}_K$ is topologically finitely generated if $A$ is $\mathcal{O}_K$-isomorphic to $T_{n,K}^\circ/J$, where $J$ is a finitely generated ideal of $T_{n,K}^\circ$. A morphism of formal schemes $\mathcal{X} \to \mathcal{S}$ is locally finitely presented if it is locally isomorphic to $\text{Spf} \, A$, where $\text{Spf} \, A \to \mathcal{S}$ is topologically finitely presented.

Let $\mathcal{O}_{K-Fsch}$ denote the category of formal schemes which are locally finitely presented over $\mathcal{S}$. Assume that the valuation of $K$ is non-trivial. Let $a \in p_K \setminus \{0\}$. For an object $\mathcal{X} \in \mathcal{O}_{K-Fsch}$ and a positive integer $n$, let $\mathcal{X}_n$ denote the scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/a^n \mathcal{O}_{\mathcal{X}})$, which is locally finitely presented over $\text{Spec}(\mathcal{O}_K/a^n)$.

**Definition 2.3.** A morphism $\mathcal{Y} \to \mathcal{X}$ in $\mathcal{O}_{K-Fsch}$ is said to be étale if for any positive integer $m$, the induced morphism of schemes $\mathcal{Y}_m \to \mathcal{X}_m$ is étale.

Clearly, this notion is independent of the choice of $a$.

Let $\hat{A}_S^n$ be an $n$-dimensional formal affine space $\text{Spf} \, T_{n,K}^\circ$. 

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Definition 2.4. A morphism \( \mathcal{Y} \to \mathcal{X} \) in \( \mathcal{O}_K \text{-} \mathcal{F} \text{sch} \) is said to be smooth if locally the map factors through an étale morphism \( \mathcal{Y} \to \hat{\mathbb{A}}^n_S \times_S \mathcal{X} \).

Assume that \( K \) has a discrete valuation.

Definition 2.5. Let \( \mathcal{X} \in \mathcal{O}_K \text{-} \mathcal{F} \text{sch} \). We say that \( \mathcal{X} \to \text{Spf} \mathcal{O}_K \) is smoothly algebraizable if there exists a scheme \( X \) which is smooth, separated and of finite type over \( \text{Spec} \mathcal{O}_K \), and whose formal completion along the special fiber \( X_s = X \otimes_{\mathcal{O}_K} \mathbb{F}_K \) is isomorphic to \( \mathcal{X} \) over \( S \).

A smoothly algebraizable formal scheme \( \mathcal{X} \to \text{Spf} \mathcal{O}_K \) is smooth in the sense of Definition 2.4.

2.3 Unramified components

In the following, we fix a non-archimedean local field \( F \). We simply write \( p \) for \( p_F \). We write \( q \) for the cardinality of \( \mathbb{F}_q \). Assume that the characteristic of \( F \) equals \( p \). We fix a separable algebraic closure of \( F \), for which we write \( \overline{F} \). Let \( F^{ur} \) be the maximal unramified extension of \( F \) in \( \overline{F} \). We write \( F^{ur} \) for the completion of \( F^{ur} \). We write \( F \) for the residue field of \( O_{F^{ur}} \).

Every formal module considered in this paper is assumed to be one-dimensional. Let \( q = \text{length} \cdot \text{rank} \). Then \( q \) is a Drinfeld level for an \( X \), and \( \mathcal{O}_X \) is a \( \mathcal{O}_F \text{-} \mathcal{F} \text{sch} \)-algebra.

We set \( \mathfrak{m} = (O_F) \). We write \( \mathfrak{m}^0 \) for the completion of \( \mathcal{O}_F \text{-} \mathcal{F} \text{sch} \) for \( \mathfrak{m} \).

Let \( \mathcal{C} \) be the category of complete noetherian \( \mathcal{O}_{F^{ur}} \text{-} \mathcal{F} \text{sch} \) algebras \( A \) with residue field \( F \). Let \( \mathfrak{p}_A \) denote the maximal ideal of \( A \). For \( n \geq 0 \), let \( \mathcal{R}(\mathfrak{p}^n) \) denote the functor which associates to \( A \in \mathcal{C} \) the set of isomorphism classes of triples \( (\mathcal{F}_A, \rho, \phi) \), where \( \mathcal{F}_A \) is a formal \( \mathcal{O}_F \text{-} \mathcal{F} \text{sch} \) module over \( A \) with an isomorphism of \( \mathcal{O}_F \text{-} \mathcal{F} \text{sch} \) modules \( \rho : \mathcal{F}_A \to A \) \( \otimes \mathcal{F} \text{sch} \) and the \( \mathcal{O}_F \text{-} \mathcal{F} \text{sch} \) homomorphism \( \phi \). Let \( \mathcal{R}_0(\mathfrak{p}^n) = \mathcal{R}(\mathfrak{p}^n) \uparrow_{\mathcal{F}_A} A \).

We can choose an isomorphism \( \mathcal{O}_{F^{ur}}[[u]] \simeq R(1) \) such that the universal formal \( \mathcal{O}_F \text{-} \mathcal{F} \text{sch} \) module \( \mathcal{F}^{univ} \) over \( \text{Spf} R(1) \) has the form:

\[
([\varpi])_{\mathcal{F}^{univ}}(X) = X^{q^2} + aX^q + \varpi X,
X + \mathfrak{m}^{univ} Y = X + Y, \quad [a]_{\mathcal{F}^{univ}}(X) = aX \text{ for } a \in \mathbb{F}_q
\]

(cf. [Sl] Proposition 5.1.1 (iii) and [Wc] (2.2.11)). We simply write \( [a]_{u} \) for \( [a]_{\mathcal{F}^{univ}} \) for \( a \in \mathcal{O}_F \).

We set \( \mu_1(S, T) = S^qT - ST^q \in \mathbb{Z}[S, T] \).

For a formal scheme \( \mathcal{M} \), we write \( \mathcal{M}^{rig} \) for the rigid analytic variety associated to \( \mathcal{M} \). The rigid analytic variety \( \mathcal{M}^{rig} \) is called the generic fiber of \( \mathcal{M} \). For an integer \( n \geq 0 \), we write \( X(\mathfrak{p}^n) \) for \( \text{Spf} R(\mathfrak{p}^n)^{rig} \) (cf. [Ca2] §1.2). For a positive integer \( n \geq 1 \), we have the natural explicit description of \( R(\mathfrak{p}^n) \):

\[
R(\mathfrak{p}^n) = \mathcal{O}_{F^{ur}}[[u, X_n, Y_n]]/\mathcal{I}_n,
\]

where \( \mathcal{I}_n \) is generated by

\[
[\varpi^n]_u(X_n), \quad [\varpi^n]_u(Y_n), \quad \mu_1(([\varpi^{n-1}]_u(X_n), [\varpi^{n-1}]_u(Y_n)))^{n-1} - \varpi.
\]
The parameters $X_n$ and $Y_n$ are regarded as sections of $\mathcal{F}^{\text{univ}}[p^n]$. The final relation in (2.3) comes from the condition that $(X_n, Y_n)$ makes a Drinfeld basis of $\mathcal{F}^{\text{univ}}[p^n]$. We simply write $X_1$ and $Y_1$ for $[\pi^{n-1}]_u(X_n)$ and $[\pi^{n-1}]_u(Y_n)$ respectively. Then, we have

$$\mu_1(X_1, Y_1) - \pi\mu_1(X_1, Y_1) \equiv ([\pi]_u(X_1), [\pi]_u(Y_1)) = ([\pi^n]_u(X_n), [\pi^n]_u(Y_n))$$

in $\mathcal{O}_{\bar{F}}[[u, X_n, Y_n]]$. Hence, we have an isomorphism

$$R(p^n) \cong \mathcal{O}_{\bar{F}}[[u, X_n, Y_n]]/[\mu_1(X_1, Y_1)^{-1}] / ([\pi^n]_u(X_n), [\pi^n]_u(Y_n)).$$

The formal model (2.2) is used in [We2]. In [We2, 3.3], the formal model is described via an explicit description of determinant of higher level structure.

Note that (2.2) is the base change of $\text{Spf } \mathcal{O}_F[[u, X_n, Y_n]]/\mathcal{L}_n$ to $\text{Spf } \mathcal{O}_{\bar{F}}$. For $1 \leq i \leq n$, we set

$$[\pi]_u(X_i) = X_{i-1}, \quad [\pi]_u(Y_i) = Y_{i-1}, \quad (2.4)$$

where we set $X_0 = Y_0 = 0$.

Let $F_2$ denote the quadratic unramified extension of $F$. Let $\mathcal{F}$ be the formal $\mathcal{O}_{F_2}$-module over $\mathcal{O}_{F_2}$ such that

$$[\pi]_\mathcal{F}(X) = X^{q^2} + \pi X, \quad X + \pi Y = X + Y, \quad [a]_\mathcal{F}(X) = aX \text{ for } a \in F_{q^2} \quad (2.5)$$

(cf. [Gr] or [GH, §13]). Note that the isomorphism class of the pair $(\mathcal{F} \otimes_{\mathcal{O}_{F_2}} \mathcal{O}_{\bar{F}})$ with the natural isomorphism $\rho : \mathcal{F}_0 \to \mathcal{F} \otimes_{\mathcal{O}_{F_2}} F$ corresponds to the $\mathcal{O}_{\bar{F}}$-valued point $\text{Spf } \mathcal{O}_{\bar{F}} \to \text{Spf } \mathcal{O}_{\bar{F}}[[u]]$ defined by $u \mapsto 0$. We consider the set of primitive $\pi^n$-torsion points of $\mathcal{F}$:

$$\mathcal{F}[p^n]_{\text{prim}} = \{ \pi \in \mathcal{F} \mid [\pi]_{\mathcal{F}}(\pi) = 0, \quad [\pi^{n-1}]_{\mathcal{F}}(\pi) \neq 0 \}. \quad (2.6)$$

Let $\pi \in \mathcal{F}[p^n]_{\text{prim}}$. We set $\pi_i = [\pi^{n-1}]_{\mathcal{F}}(\pi)$ for $1 \leq i \leq n - 1$. Let $\zeta \in F_{q^2} \setminus F_q$. For $1 \leq i \leq n$, we set

$$X_i = \pi_i + S_{\pi_i}, \quad Y_i = \zeta \pi_i + T_{\pi_i}. \quad (2.7)$$

The parameters $S_{\pi_i}$ and $T_{\pi_i}$ depend on $\pi_i$. However, we simply write $S_i$ and $T_i$ for $S_{\pi_i}$ and $T_{\pi_i}$ respectively. By (2.4) and (2.6), for $1 \leq i \leq n$, we obtain

$$S_i^q + \pi S_i + u(\pi_i^q + S_i^q) = S_{i-1},$$

$$T_i^q + \pi T_i + u((\zeta \pi_i)^q + T_i^q) = T_{i-1}, \quad (2.8)$$

where $S_0 = T_0 = 0$. We put $\zeta_i = \zeta^q - \zeta$. We have $\zeta_i^q + \zeta_i = 0$. By $\zeta \notin F_q$, we have $\zeta_i \neq 0$ and $\zeta_i^{q-1} = -1$. We set

$$U_{\pi_i} = \zeta^q S_i - T_i \quad \text{for any } 1 \leq i \leq n. \quad (2.9)$$

We simply write $U_i$ for $U_{\pi_i}$. By using (2.8), for $1 \leq i \leq n$, we acquire

$$S_i^q + \pi S_i + u(\pi_i^q + S_i^q) = S_{i-1},$$

$$U_i^q + \pi U_i + u((\zeta_i S_i^q + U_i^q) = U_{i-1}, \quad (2.10)$$

where $U_0 = 0$.

Let $k$ be a positive integer. We set $m = \left\lceil \frac{k+1}{2} \right\rceil$ and $h = (q^2 - 1)^{-1}$. Let $C$ be the completion of $F$. We simply write $v$ for the normalized valuation $v_C$ on $C$. We write also $v$ for the unique extension of $v$ to $C$. Let $X_{n,k,\zeta,\pi_n} \subset X(p^n)$ be the affinoid defined by

$$v(u) \geq m, \quad v(S_k) \geq h/q^{k-1}, \quad v(U_k) \geq h \quad \text{if } k \text{ is even},$$

$$v(u) \geq m - \frac{1}{q^k}, \quad v(S_k) \geq h/q^{k-1}, \quad v(U_k) \geq h \quad \text{if } k \text{ is odd}. \quad (2.12)$$
Lemma 2.6. On $X_{n,k,ζ,ω}$, we have

$$v(u) \geq m, \quad v(S_{m+i}) \geq h/\eta^{2i-1} \quad \text{for } 1 \leq i \leq m, \quad v(S_{m+i}) \geq i + hq \quad \text{for } 0 \leq i \leq m - 1,$$

$$v(U_{k-i}) \geq i + h \quad \text{for } 0 \leq i \leq m, \quad v(U_i) = v(uS_i^{q}/\omega) \quad \text{for } 1 \leq i \leq m$$

(2.13)

if $k$ is even, and

$$v(u) \geq m - \frac{1}{\eta + 1}, \quad v(S_{m+i}), \quad v(U_{k-i}) \geq i + h \quad \text{for } 0 \leq i \leq m - 1,$$

$$v(U_i) = v(uS_i^{q}/\omega), \quad v(S_{m+i}) \geq h/\eta^{2i} \quad \text{for } 1 \leq i \leq m - 1$$

(2.14)

if $k$ is odd.

Proof. These assertions immediately follow from (2.10) and (2.11).

In the following, we focus on the case where $n = k$.

Lemma 2.7. Let $k \geq 1$ be a positive integer.
1. Assume that $k = 1$. The affinoid $X_{1,1,ζ,ω_1}$ is independent of $ζ$ and $ω_1$.
2. Assume that $k = 2$. Let $ω_k, ω'_k \in \mathcal{F}[p^k_{\mathbb{F}_2}]_{\text{prim}}$.
3. If $v(ω_k - ω'_k) < h$, we have $X_{k,k,ζ,ω_k} \cap X_{k,k,ζ,ω'_k} = \emptyset$ in $X(p^k)$.
4. If $v(ω_k - ω'_k) \geq h$, we have $X_{k,k,ζ,ω_k} = X_{k,k,ζ,ω'_k}$.

Proof. We prove the first assertion. The affinoid $X_{1,1,ζ,ω_1}$ is defined only by $v(u) \geq \frac{a}{q+1}$. Hence, the required assertion follows.

We set $ω'_i = [ω^{k-i}] \mathcal{F}(ω'_k)$ for $1 \leq i \leq k - 1$. By (2.3), we have $U_{ω_i} = -ζ_1ω_i + ζ^qX_i - Y_i$ for $1 \leq i \leq k$. Hence, in particular, we have

$$U_{ω_k} - U_{ω'_k} = -ζ_1(ω_k - ω'_k).$$

(2.15)

On $X_{k,k,ζ,ω_k} \cap X_{k,k,ζ,ω'_k}$, we have $v(U_{ω_k} - U_{ω'_k}) \geq h$. Hence, the second assertion follows from (2.15).

Assume that $v(ω_k - ω'_k) \geq h$. This implies that $ω_i = ω'_i$ for $1 \leq i \leq k - 1$. Hence, the third assertion follows from the assumption, (2.7), (2.12) and (2.15).

Remark 2.8. Let $ω_k, ω'_k \in \mathcal{F}[p^k_{\mathbb{F}_2}]_{\text{prim}}$. There exists a unique element $a \in U^{k}_{F_2}/U^{k}_{F_2}$ such that $ω'_k = [a] \mathcal{F}(ω_k)$. Note that

$$v(ω_k - [a] \mathcal{F}(ω_k)) \geq h \iff a \in U^{k-1}_{F_2}/U^{k}_{F_2}.$$

For $k \geq 1$, we put $K_k = \hat{F}^{ur}(ω_k)$. By Lemma 2.7, we simply write $X_{1,1}$ for $X_{1,1,ζ,ω_1}$. For a field extension $L/\hat{F}^{ur}$ in $\mathbb{C}$ and a rigid analytic variety $X$ over $\hat{F}^{ur}$, we write $X_L$ for the base change of $X$ to $L$.

Lemma 2.9. We set

$$A_1 = \Gamma \left( X_{1,1,K_1}, \mathcal{O}_{X_{1,1,K_1}} \right).$$

Then, the formal scheme $Spf A^\wedge_1 \rightarrow Spf \mathcal{O}_{K_1}$ is smoothly algebraizable in the sense of Definition 2.5. Furthermore, we have isomorphisms

$$X_{1,1,K_1} \simeq Spec (A^\wedge_1 \otimes_{\mathcal{O}_{K_1}} \mathbb{F}) \simeq Spec \mathbb{F}[X,Y]/((X^qY - XY^q)^{q-1} + 1).$$
Proof. In the sequel, we discuss on $X_{1,1,K_1}$. By setting $u = \omega_1^{q(q-1)} u_0$, $X_1 = \omega_1 X$ and $Y_1 = \omega_1 Y$ with $v(u_0), v(X), v(Y) \geq 0$, we have

$$(X^q Y - XY^q)^{q-1} = -1$$

by $\mu_1(X_1, Y_1)^{q-1} = \omega$. Hence, we obtain $v(X) = v(Y) = 0$. Furthermore, we have

$$u_0 = X^{-(q-1)} - X^{q(q-1)} = Y^{-(q-1)} - Y^{q(q-1)}$$

by $[\omega]_u(X_1) = [\omega]_u(Y_1) = 0$. We set $f = (X^q Y - XY^q)^{q-1} + 1$. Let

$$A'_1 = \mathcal{O}_{K_1}((u_0, X_1^\pm 1, Y_1^\pm 1)/ (f, u_0 - X^{-(q-1)} + X^{q(q-1)}, u_0 - Y^{-(q-1)} + Y^{q(q-1)})$$

Then, $A'_1 \otimes_{\mathcal{O}_{K_1}} \mathbb{F}$ is reduced and $A_1 = A'_1 \otimes_{\mathcal{O}_{K_1}} K_1$. Hence, by Lemma 2.1, we obtain

$$A_1 = A'_1 \otimes_{\mathcal{O}_{K_1}} \mathbb{F} \simeq \mathbb{F}[X_1^{\pm 1}, Y_1^{\pm 1}]/(f). \tag{2.16}$$

We set

$$X = \text{Spec} \mathcal{O}_{K_1}[X_1^{\pm 1}, Y_1^{\pm 1}]/(f).$$

Then, $X$ is smooth over $\text{Spec} \mathcal{O}_{K_1}$, and $\text{Spf} A_1^0$ is isomorphic to the formal completion of $X$ along the special fiber $X_s = X \otimes_{\mathcal{O}_{K_1}} \mathbb{F}_{K_1}$. Hence, the required assertions follow from (2.16). □

**Proposition 2.10.** Assume that $k \geq 2$. We set

$$A_k = \Gamma\left(X_{k,k,\mathbb{C},w_k,K_k}, \mathcal{O}_{X_{k,k,\mathbb{C},w_k,K_k}} \right).$$

Then, the formal scheme $\text{Spf} A_k^0 \to \text{Spf} \mathcal{O}_{K_k}$ is smoothly algebraizable. Furthermore, we have isomorphisms

$$\mathfrak{X}_{k,k,\mathbb{C},w_k,K_k} \simeq \text{Spec} \left(A_k^0 \otimes_{\mathcal{O}_{K_k}} \mathbb{F}\right) \simeq \text{Spec} \mathbb{F}[X, Y]/ \left(X^{q^2} - X - (Y^{q(q+1)} - Y^{q+1})^{q^{-1}}\right). \tag{2.17}$$

**Proof.** For any $\alpha \in \mathbb{Q}_{\geq 0}$, we write $f \equiv g \mod \alpha + 1$. In the following, we always consider on $X_{k,k,\mathbb{C},w_k,K_k}$. Assume that $k$ is even. We write $k = 2m$. By (2.10), (2.11) and (2.13), if $m = 1$, we have

$$U_2^q + \omega U_2 + u_1 S_2^q \equiv U_1 \mod (1 + h),$$

$$\omega U_1 + u_1 S_1^q \equiv 0 \mod (2 + h),$$

$$S_2^q \equiv S_1 \mod hq,$$

$$\omega S_1 + \omega_1 u \equiv 0 \mod (1 + h), \tag{2.18}$$

and, if $m > 1$,

$$U_k^q + \omega U_k \equiv U_{k-1} \mod (1 + h),$$

$$\omega U_{k-i} \equiv U_{k-i} \mod (i + 1 + h) \quad \text{for} \ 1 \leq i \leq m - 2,$$

$$\omega U_{m+1} + u_1 S_{m+1}^q \equiv U_m \mod (m + h),$$

$$\omega U_i + u_1 S_i^q \equiv 0 \mod v(\omega U_i) \quad \text{for} \ 1 \leq i \leq m,$$

$$S_{m+1}^q \equiv S_m \mod hq,$$

$$\omega^m S_m + \omega_1^q u \equiv 0 \mod (m + h). \tag{2.19}$$
By (2.18) and (2.19), we obtain

\[ U_k^{q^2} + \varpi U_k \equiv \begin{cases} U_1 - u_0 S_2^q & \text{if } m = 1, \\ U_{k-1} \equiv U_{m+1}/\varpi^{m-2} & \text{if } m > 1 \end{cases} \]

\[ = -u_0 S_{m+1}^q + U_m \equiv -u_0 \varpi (\varpi S_{m+1}^q + S_m^q) \]

\[ = \frac{\zeta_1}{\varpi^q} \left( \varpi S_{m+1}^{q(q+1)} + S_m^{q+1} \right) \equiv \frac{\zeta_1}{\varpi^q} \left( \varpi S_k^{q^{k-1}(q+1)} + S_k^{q^k(q+1)} \right) \pmod{(1 + h) +} \]

(2.20)

We set

\[ u = \varpi^m u_0 \quad \text{with } v(u_0) \geq 0, \]

\[ U_k = \zeta_1 \varpi_1 X, \quad S_k = \varpi_1^{q+1} Y \quad \text{with } v(X), v(Y) \geq 0. \]

(2.21)

By (2.18) and (2.19), we have

\[ u_0 \equiv \frac{S_m}{\varpi^q} \equiv \frac{S_k^{q^k}}{\varpi^q} \equiv -Y^q \pmod{0+}, \]

where we use \( \varpi_{m+1}^{q^k} \equiv \varpi_1 \pmod{h^+} \) at the third congruence. Hence, by (2.21), we obtain

\[ f_{1,k}(u_0, X, Y) = u_0 + Y^{q^k} - F_k(u_0, X, Y) = 0 \]

(2.22)

with some polynomial \( F_k(u_0, X, Y) \in \mathcal{O}_{K_k}[u_0, X, Y] \) such that \( v(F_k(u_0, X, Y)) > 0 \). By substituting (2.21) to (2.20) and dividing it by \( \varpi_1^{q^k} \), we obtain

\[ f_{2,k}(u_0, X, Y) = X^{q^2} - X - (Y^{q(q+1)} - Y^{q+1})^{q^{k-1}} - G_k(u_0, X, Y) = 0 \]

(2.23)

with some polynomial \( G_k(u_0, X, Y) \in \mathcal{O}_{K_k}[u_0, X, Y] \) such that \( v(G_k(u_0, X, Y)) > 0 \). We consider the subring in \( K_k \):

\[ A_k' = \mathcal{O}_{K_k}(u_0, X, Y)/\langle f_{1,k}(u_0, X, Y), f_{2,k}(u_0, X, Y) \rangle. \]

(2.24)

Then, \( A_k' \otimes_{\mathcal{O}_{K_k}} \mathbb{F} \) is reduced, and \( A_k = A_k' \otimes_{\mathcal{O}_{K_k}} K_k \). Hence, by Lemma 2.1 we obtain

\[ A_k^0 = A_k', \]

\[ \overline{A_k} \simeq A_k^0 \otimes_{\mathcal{O}_{K_k}} \mathbb{F} \simeq \mathbb{F}[X, Y]/\left( X^{q^2} - X - (Y^{q(q+1)} - Y^{q+1})^{q^{k-1}} \right). \]

(2.25)

Assume that \( k \) is odd. We write \( k = 2m - 1 \). Then, by (2.10), (2.11) and (2.14), we have

\[ U_k^{q^2} + \varpi U_k \equiv U_{k-1} \pmod{(1 + h) +}, \]

\[ \varpi U_{k-1} \equiv U_{k-2} \pmod{(i + 1 + h) +} \quad \text{for } 0 \leq i \leq m - 2, \]

\[ \varpi U_i + u_0 S_i^q \equiv 0 \pmod{\varpi U_i} \quad \text{for } 1 \leq i \leq m, \]

\[ S_{m+i}^{q^2} \equiv S_m \pmod{h} \quad \text{for } 1 \leq i \leq m - 1, \]

\[ S_m^{q^2} + \varpi S_m \equiv S_{m-1} \pmod{(1 + h) +}, \]

\[ \varpi S_{m-1} \equiv S_{m-2} \pmod{(i + 1 + h) +} \quad \text{for } 0 \leq i \leq m - 2, \]

\[ \varpi S_1 + \varpi^i u \equiv 0 \pmod{(m + h) +}. \]

(2.26)
Hence, we obtain
\[ U_k^2 + \varpi U_k \equiv \frac{U_m}{\varpi^{m-2}} \equiv -u \zeta_1 S_m' \equiv \frac{\zeta_1 S_{m-1} S_m'}{\varpi_1^m} \]
\[ \equiv \frac{\zeta_1}{\varpi_1} \left( S_{m}^q + \varpi S_{m} \right) S_m' \equiv \frac{\zeta_1}{\varpi_1} \left( S_k^{q^k(q+1)} + \varpi S_k^{q^k(q+1)} \right) \pmod{(1+h) + \varpi}. \] (2.27)

We set
\[ u = \varpi^{m-1} \varpi_1^q u_0 \text{ with } v(u_0) \geq 0, \]
\[ U_k = \zeta_1 \varpi_1 X, \quad S_k = \varpi_m Y \text{ with } v(X), v(Y) \geq 0. \] (2.28)

By (2.26) and (2.27), we have
\[ f_{1,k}(u_0, X, Y) = u_0 + (Y^q - Y)^{q^k - 1} - F_k(u_0, X, Y) = 0, \]
\[ f_{2,k}(u_0, X, Y) = X^q - X - (Y^{q(q+1)} - Y^{q+1})^{q^k - 1} - G_k(u_0, X, Y) = 0 \]
with some elements \( F_k(u_0, X, Y), G_k(u_0, X, Y) \in O_{K_k}[u_0, X, Y] \) such that
\[ v(F_k(u_0, X, Y)), \ v(G_k(u_0, X, Y)) > 0. \]

Let
\[ A'_k = O_{K_k}(u_0, X, Y)/\langle f_{1,k}(u_0, X, Y), f_{2,k}(u_0, X, Y) \rangle \subset A_k. \] (2.29)

Since \( A'_k \otimes_{O_{K_k}} \mathbb{F} \) is reduced, and \( A'_k \otimes_{O_{K_k}} K_k = A_k \), we obtain
\[ A'_k = A_k^0, \]
\[ A_k = A_k^0 \otimes_{O_{K_k}} \mathbb{F} \simeq \mathbb{F}[X, Y]/\left( X^{q^k - X - (Y^{q(q+1)} - Y^{q+1})^{q^k - 1}} \right) \] (2.30)
by Lemma 2.7. We set
\[ \Delta_k = \left| \begin{array}{cc} \frac{\partial f_{1,k}}{\partial u_0} & \frac{\partial f_{1,k}}{\partial X} \\ \frac{\partial f_{2,k}}{\partial u_0} & \frac{\partial f_{2,k}}{\partial X} \end{array} \right| \in O_{K_k}[u_0, X, Y] \]
and
\[ V_k = \text{Spec } O_{K_k}[u_0, X, Y, \Delta_k^{-1}] / \langle f_{1,k}, f_{2,k} \rangle. \]

Note that \( \Delta_k \equiv -1 \mod p_{K_k} \). Then, \( V_k \to \text{Spec } O_{K_k}[Y] \) is étale and hence \( V_k \to \text{Spec } O_{K_k} \) is smooth. By (2.24) and (2.29), the formal scheme \( \text{Spf } A_k^0 \) is isomorphic to the formal completion of \( V_k \) along \( (V_k)^0 \) over \( \text{Spf } O_{K_k} \). Hence, the required assertions follow from (2.25) and (2.30). \( \square \)

### 2.4 Ramified components

In this subsection, we assume that \( p \neq 2 \). Whenever we treat the ramified case, we always assume this. We simply write \( \mathbb{Z}/2\mathbb{Z} \) for \( \mathbb{Z}/2\mathbb{Z} \). We consider the following formal scheme in [GL II.2.1]:
\[ \text{Spf } O_{\overline{\mathbb{F}}}[[x_1, x_2]][(s_{i,j})_{i \in \mathbb{Z}, 0 \leq j \leq n}]/J_n, \] (2.31)
where \( J_n \) is generated by
\[ x_1 x_2 - \varpi, \quad s_{i,0}^{q-1} - x_i, \quad s_{i,j}^q - x_{i-j} s_{i,j} - s_{i,j-1} \quad \text{for } i \in \mathbb{Z}, \ 1 \leq j \leq n. \] (2.32)
The model (2.31) is the base change of \( \mathcal{O}_F[[x_1, x_2]]/(s_{i,j})_{i \in \mathbb{Z}, 0 \leq j \leq n}/J_n \) to \( \mathcal{O}_{F_{u_2}} \). We write \( Y(p^n) \) for the generic fiber of this formal scheme (2.31). If \( n = 2m - 1 \) is odd, the formal scheme (2.31) equals \( \mathcal{M}_{\mathcal{E}, \mathcal{B}_{m}^{\times}} \) in the notation of [Gr], Remarque II.2.3. Hence, the rigid analytic curve \( Y(p^n) \) is the quotient of \( X(p^{m+1}) \) by \( \mathcal{B}_{m}^{\times} = U_3^{m+1} \) in the notation of [3.3].

Let \(((x_i)_{i \in \mathbb{Z}}, (s_{i,j})_{i \in \mathbb{Z}, 0 \leq j \leq n}) \in Y(p^n) \). We set

\[
t_{i,0} = \frac{s_{i,0}}{s_{i-1,0}}, \quad t_{i,j} = \frac{s_{i,j}}{s_{i,j-1}} \quad \text{for } i \in \mathbb{Z} \text{ and } 1 \leq j \leq n.
\] (2.33)

Then, for \( i \in \mathbb{Z} \), we have

\[
t_{1,0}t_{2,0} = (s_{1,0}s_{2,0})^{-(q-1)} = \varpi^{-1},
\]

\[
t_{i,j}^q - t_{i,j} = \begin{cases} t_{i,0} & \text{if } j = 1, \\ t_{i,j-1}t_{i,j+1,0} & \text{if } 2 \leq j \leq n. \end{cases}
\] (2.34)

We take a second root of \( \varpi \), for which we write \( \varpi_E \in \mathcal{P} \). We set \( E = F(\varpi_E) \). Let \( \mathcal{G} \) be the formal \( \mathcal{O}_E \)-module over \( \mathcal{O}_E \) such that

\[
[\varpi_E]_{\mathcal{G}}(X) = X^q + \varpi_E X, \quad X + \varpi Y = X + Y, \quad [\xi]_{\mathcal{G}}(X) = \xi X \text{ for } \xi \in \mathcal{F}_q.
\] (2.35)

(cf. [Gr]). For any positive integer \( i \), we set

\[
[\varpi_E^i]_{\mathcal{G}}(\varpi_E, i \in \mathcal{P} \mid [\varpi_E^i]_{\mathcal{G}}(\varpi_E, i) = 0, \ [\varpi_E^{i+1}]_{\mathcal{G}}(\varpi_E, i) \neq 0 \}.
\] (2.36)

Let \( \varpi_{E,n+1} \in \mathcal{G}[\varpi_E^{n+1}]_{\mathcal{G}} \). We put

\[
\theta_i = \begin{cases} \varpi_E, & \text{if } i = 1, \\ [\varpi_E^{n+1}]_{\mathcal{G}}(\varpi_{E,n+1})/\varpi_E, & \text{if } 2 \leq i \leq n. \end{cases}
\]

Then, we have

\[
\theta_1^{-1} = -\varpi_E, \quad \theta_2^2 - \theta_2 = -\varpi_E^{-1}
\]

\[
\theta_3^2 - \theta_3 = -\theta_2\varpi_E^{-1} \quad \text{for } 3 \leq i \leq n + 1.
\] (2.37)

The extension \( E_{n+1} = E(\varpi_{E,n+1}) = E(\theta_1, \theta_{n+1}) \) is a Lubin-Tate extension of \( E \) of degree \( q^n(q-1) \). Assume that \( s_{1,0}s_{2,0} = \theta_1^2 \). Then, for \( i \in \mathbb{Z} \), we have

\[
t_{i,0} = \frac{s_{i,0}^{q+1}}{\theta_1^q}.
\] (2.38)

Let \( k \) be a positive integer. For \( i \in \mathbb{Z} \), we set

\[
s_{i,0} = \theta_1 + \theta_1^{q-1}u_1,
\]

\[
t_{i,j} = \begin{cases} -\varpi_E^{-1} + u_i & \text{if } j = 0, \\ \theta_{j+1} + u_{i,j} & \text{if } 1 \leq j \leq n. \end{cases}
\] (2.39)

By (2.31), (2.37), (2.38) and (2.34), for \( i \in \mathbb{Z} \), we have

\[
u_i = u_i^0 + \varpi_E^{-1}(u_i^0)^q(1 - \varpi_E u_i^0),
\] (2.40)

\[
u_1 + u_2 = \varpi_E u_1 u_2,
\] (2.41)

\[
u_i^q - u_{i,j} = \begin{cases} u_i & \text{if } j = 1, \\ -\varpi_E^{-1}u_{i,j-1} + (\theta_j + u_{i,j-1})u_{i,j+1} & \text{if } 2 \leq j \leq n. \end{cases}
\] (2.42)
We put
\[
U_i = \begin{cases} 
  u_1 + u_2 & \text{if } i = 0, \\
  u_{i,j} + u_{i,j} & \text{if } 1 \leq i \leq n,
\end{cases} 
\]
(2.43)
\[
Q_i = \begin{cases} 
  w_E u_1 u_2 & \text{if } i = 1, \\
  u_{i-1} u_i + u_{i-1} u_{i-1} & \text{if } 2 \leq i \leq n.
\end{cases} 
\]
By summing up equations (2.42) through \( i \in \mathbb{Z} \) respectively, and using (2.41) and (2.43), we obtain
\[
U_i^q - U_i = \begin{cases} 
  U_0 = Q_0 & \text{if } i = 1, \\
  -w_E^{-1} U_{i-1} + \theta_i U_0 + Q_i & \text{if } 2 \leq i \leq n.
\end{cases} 
\]
(2.44)
We define an affinoid subdomain \( Z_{n,k,w_E,k+1} \) by
\[
v(u_i^q) \geq \frac{k - 2}{4}, \quad v \left( u_1 \left( \frac{1}{2} \right) \right) \geq 4^{-1} \left( k - 2 \left[ \frac{k}{2} \right] \right), 
\]
\[
v(u_{i,j}) = q^{-1} v \left( u_{i,j-1} / w_E \right) \quad \text{for } [k/2] + 1 \leq j \leq k, \quad v(U_k) \geq 0.
\]
By (2.38)--(2.44), on \( Z_{n,k,w_E,k+1} \), we can check that
\[
\bullet \text{ for each } i \in \mathbb{Z}, \quad v(x_i + w_E) \geq (k+2)/4, \quad v(u_i^q) = v(u_i) \geq (k-2)/4, 
\]
\[
v(u_{i,j}) \geq (k-2j)/4 \quad \text{for } 1 \leq j \leq [k/2], v(u_{i,j}) = q^{-1} v \left( u_{i,j-1} / w_E \right) \quad \text{for } [k/2] + 1 \leq j \leq k, \quad \text{and}
\]
\[
\bullet \text{ for } U_0 \geq (k-1)/2, \quad v(U_i) \geq (k-i)/2 \quad \text{for } 1 \leq i \leq k,
\]
and
\[
v(Q_i) \geq \frac{k - i}{2} \quad \text{for } 1 \leq i \leq \left[ \frac{k}{2} + 1 \right], \quad v(Q_i) > \frac{k - i}{2} \quad \text{for } \left[ \frac{k}{2} + 1 \right] < i \leq k.
\]
(2.45)
In the following, we consider only the case where \( n = k \).

Lemma 2.11. Assume that \( p \neq 2 \). Let \( w_E,k+1, w_E',k+1 \in \mathcal{O}^k_{E,k+1} \).

1. If \( v(w_E,k+1 - w_E',k+1) < (q-1)^{-1} \), we have \( Z_{k,k,w_E,k+1} \cap Z_{k,k,w_E',k+1} = \emptyset \) in \( Y(p^k) \).

2. If \( v(w_E,k+1 - w_E',k+1) \geq (q-1)^{-1} \), we have \( Z_{k,k,w_E,k+1} = Z_{k,k,w_E',k+1} \).

Proof. The parameter \( U_i \) on \( Z_{k,k,w_E,k+1} \) actually depends on \( w_E,i+1 \). So, in this proof, we write \( U w_E,i+1 \) for \( U_i \). We have \( U w_E,i+1 = t_{1,i} + t_{2,i} - 2\theta_i+1 \). Hence, we have
\[
U w_E,k+1 - U w_E',k+1 = -2(\theta_{k+1} - \theta'_{k+1}).
\]
(2.46)
On \( Z_{k,k,w_E,k+1} \cap Z_{k,k,w_E',k+1} \), we have \( v(U w_E,k+1 - U w_E',k+1) \geq 0 \). Hence, the first assertion follows. Assume that \( v(w_E,k+1 - w_E',k+1) \geq (q-1)^{-1} \). This implies that \( \theta_i = \theta'_i \) for \( 1 \leq i \leq k \) and \( v(\theta_{k+1} - \theta'_{k+1}) \geq 0 \). Therefore, the second assertion follows from the definition of \( Z_{k,k,w_E,k+1} \) and (2.46).

\[\square\]

Proposition 2.12. Let \( k = 2m - 1 \) be an odd positive integer. We set \( L_k = \hat{F}^{ur}(w_E,k+1) \). We set
\[
B_k = \Gamma \left( Z_{k,k,w_E,k+1}, L_k, O_{Z_{k,k,w_E,k+1}, t_k} \right).
\]
Then, \( Spf B_k \to Spf O_{L_k} \) is smoothly algebraizable. Furthermore, we have isomorphisms
\[
Z_{k,k,w_E,k+1}, t_k \cong \text{Spec} \left( B_k \otimes O_{L_k} \mathbb{F} \right) \cong \text{Spec} \mathbb{F}[a,s]/(a^n - a - s^{2n})
\].

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Proof. In the following, we always consider on $Z_{k,k+1,L_k}$. By (2.41), we have

$$U_i \equiv -U_0 \equiv -Q_1 \mod \frac{k-1}{2} +,$$
$$U_i \equiv \frac{U_{i-1}}{\omega_E} - Q_i \mod \frac{k-i}{2} + \quad \text{for } 2 \leq i \leq m,$$
$$U_i \equiv \frac{U_{i-1}}{\omega_E} \mod \frac{k-i}{2} + \quad \text{for } m+1 \leq i \leq k-1,$$
$$U_k^q - U_k \equiv -\frac{U_{k-1}}{\omega_E} \mod 0 + .$$

By (2.41), (2.40) and (2.42), for $i \in 2$, we have

$$-u_{i,1} \equiv u_i \equiv u_i^0 \mod \frac{k-2}{4} + ,$$
$$u_{i,j} \equiv \frac{u_{i,j-1}}{\omega_E} \mod \frac{k-2j}{4} + \quad \text{for } 2 \leq j \leq m - 1,$$
$$u_{i,j}^q \equiv -\frac{u_{i,j-1}}{\omega_E} \mod qv(u_{i,j}) + \quad \text{for } m \leq j \leq k. $$

We choose a second root $(-1)^{(m-1)/2} \in \mathbb{F}_q^+$ of $(-1)^{m-1}$. We set as follows:

$$U_0 = \omega_E^{k-1} U_{0,0}, \quad U_i = \omega_E^{k-i} U_{i,0} \quad \text{for } 1 \leq i \leq k,$$
$$u_i = (-1)^{\frac{k-1}{2}} \omega_E^{-2} \theta_1^{\frac{m}{2}} b_i, \quad u_i^0 = (-1)^{\frac{m}{2}} \omega_E^{-2} \theta_1^{\frac{m}{2}} b_i^0,$$
$$u_{i,j} = (-1)^{\frac{k-1}{2}} \omega_E^{-m-2} \theta_1^{\frac{m}{2}} b_{i,j} \quad \text{for } 1 \leq j \leq m - 1,$$
$$u_{i,m} = (-1)^{\frac{m}{2}} \omega_E^{-m-2} \theta_1^{\frac{m}{2}} b_{i,m},$$

$$u_{i,m+j} = (-1)^{\frac{m}{2}} \left( \omega_E^{-2} \prod_{l=2}^{j+1} \omega_E^{2} \right) b_{i,m+j} \quad \text{for } 1 \leq j \leq m - 1$$

with $v(b_i^0), v(b_i), v(b_{i,j}), v(U_{i,0}) \geq 0$. By using (2.48) and (2.49), we have

$$Q_1 \equiv \omega_E^{k-2} u_{1,m-1} u_{2,m-1} = (-1)^m \omega_E^{k-1} b_{1,m-1} b_{2,m-1} \mod \frac{k-1}{2} + ,$$
$$Q_i = -\omega_E^{k-i} \sum_{j=1}^{2} u_{j,m-1} u_{j-i+1,m-1} = (-1)^{m-1} \omega_E^{k-i} \sum_{j=1}^{2} b_{j,m-1} b_{j-i+1,m-1} \mod \frac{k-i}{2} +$$

for $2 \leq i \leq m$. Hence, by (2.47), (2.48) and (2.49), we obtain

$$h_1 = U_{0,0} + U_{1,0} - G_0 = 0,$$
$$h_{i+1} = U_i - (-1)^m \left( -b_{1,m-1} b_{2,m-1} + \sum_{j=2}^{i} \sum_{l=1}^{2} b_{l,m-1} b_{l-j+1,m-1} \right) - G_i = 0 \quad \text{for } 1 \leq i \leq m,$$
$$h_{i+1} = U_{i,0} - U_{i-1,0} - G_i = 0 \quad \text{for } m+1 \leq i \leq k-1,$$
$$h_{k+1} = U_{k,0} - U_{k-1,0} - G_k = 0,$$
$$h_{k+2} = b_1^0 - b_1 - H_0 = 0, \quad h_{k+3} = b_{1,1} + b_1 + H_1 = 0,$$
$$h_{k+i+2} = b_{1,i} - b_{1,i-1} - H_i = 0 \quad \text{for } 2 \leq i \leq m - 1,$$
$$h_{k+i+2} = b_{1,i}^g - b_{1,i-1} - H_i = 0 \quad \text{for } m \leq i \leq k,$$

(2.50)
with some polynomials
\[ G_i, H_i \in \mathcal{O}_{L_k} \left[ (U_{i,0})_{0 \leq i \leq k}, b_1^0, b_1, (b_{1,j})_{1 \leq j \leq k} \right] \]
such that \( v(G_i), v(H_i) > 0 \). We consider the subring in \( B_k \):
\[ B'_k = \mathcal{O}_{L_k} \left[ (U_{i,0})_{0 \leq i \leq k}, b_1^0, b_1, (b_{1,j})_{1 \leq j \leq k} \right] / J_k, \]
where \( J_k \) is generated by \( \{2.50\} \). Since \( B'_k \otimes_{\mathcal{O}_{L_k}} \mathbb{F} \) is reduced, and \( B'_k \otimes_{\mathcal{O}_{L_k}} L_k \simeq B_k \), we obtain
\[ B'_k \simeq B_k^0, \quad \overline{B_k} = B'_k \otimes_{\mathcal{O}_{L_k}} \mathbb{F} \quad (2.51) \]
by Lemma 2.1. We write \( a \) and \( s \) for \( U_{k,0} \) and \( b_{1,k} \) respectively. By \( b_{1,m-1} \equiv -b_{2,m-1} \equiv s^{q^m} \) mod 0+ and \( \{2.50\} \), we can check that
\[ a^q - a \equiv -U_{m,0} \equiv (-1)^m \left( -b_{1,m-1}b_{2,m-1} + \sum_{i=2}^{m} \sum_{j=1}^{2} b_{j,m-1}b_{j-i+1,m-1} \right) \equiv s^{2q^m} \mod 0+ . \]
Therefore, the reduction \( B'_k \otimes_{\mathcal{O}_k} \mathbb{F}_k \) is isomorphic to the affine curve defined by \( a^q - a = s^{2q^m} \) by \( \{2.50\} \). Hence, by \( \{2.51\} \), we obtain
\[ \overline{B_k} = B_k^0 \otimes_{\mathcal{O}_k} \mathbb{F} \simeq \mathbb{F}[a, s]/\left( a^q - a - s^{2q^m} \right) . \quad (2.52) \]
We write \( (z_i)_{1 \leq i \leq 2k+3} \) for \( ((U_{i,0})_{0 \leq i \leq k}, b_1^0, b_1, (b_{1,j})_{1 \leq j \leq k}) \). Let
\[ \Delta'_k = \det \left( \frac{\partial h_i}{\partial z_j} \right)_{1 \leq i, j \leq 2k+2} \in \mathcal{O}_{L_k} \left[ z_1, \ldots, z_{2k+3} \right]. \]
Note that \( \Delta'_k \equiv -1 \mod p_{L_k} \). We put
\[ Z_k = \operatorname{Spec} \mathcal{O}_{L_k} \left[ z_1, \ldots, z_{2k+3}, \frac{1}{\Delta_k^{t-1}} \right]/(h_1, \ldots, h_{2k+2}). \]
Then, the natural map \( Z_k \to \operatorname{Spec} \mathcal{O}_{L_k}[z_{2k+3}] \) is étale and hence \( Z_k \) is smooth over \( \operatorname{Spec} \mathcal{O}_{L_k} \). Furthermore, the formal completion of \( Z_k \) along \( (Z_k)_s \) is isomorphic to \( \operatorname{Spf} B_k \) over \( \operatorname{Spf} \mathcal{O}_{L_k} \). Hence, the required assertions follow from \( \{2.52\} \).

**Remark 2.13.** We can prove that, if \( k \) is even, the affinoid \( Z_{k,k,\varpi_{E,n+k},L_k} \) reduces to a disjoint union of \( q \) copies of an affine line. Since we will not use this fact later, we omit the details of the proof.

**Remark 2.14.** It is difficult to compute the reductions of \( X_{n,k,\varpi_n} \) and \( Z_{n,k,\varpi_{E,n+1}} \) for \( k < n \) in general. In a representation-theoretic view point, the cohomology of \( X_{n,n,\varpi_n} \) and \( Z_{n,n,\varpi_{E,n+1}} \) is most interesting among \( \{X_{n,k,\varpi_n}\}_{k \leq n} \) and \( \{Z_{n,n,\varpi_{E,n+1}}\}_{k \leq n} \) respectively. These realize “new parts” in the cohomology of the Lubin-Tate curve. On the other hand, the cohomology of \( X_{n,k,\varpi_n} \) and \( Z_{n,k,\varpi_{E,n+1}} \) for \( k < n \) is contained in “old parts” in the cohomology of the Lubin-Tate curve. These things will be understood in this paper. In \([1]\), we have computed the stable reduction of \( X(p^3) \). Actually, in the stable reduction of \( X(p^2) \), many old irreducible components appear. Similar examples in the mixed characteristic case can be found in \([C2]\) and \([2]\).

For \( n > k \), we expect that \( X_{n,k,\varpi_n} \) has several connected components and, the restriction of the canonical level lowering map \( X(p^n) \to X(p^k) \) to each connected component induces a purely inseparable map between their reductions. We expect that similar things happen for \( Z_{n,k,\varpi_{E,n+1}} \). In the tower of modular curves \( \{X_0(p^n)\}_{n \geq 0} \), such a phenomenon in a special case is observed in \([3]\).
2.4.1 Relation with stable reduction of Lubin-Tate curve

The following corollary is a direct consequence of the analysis in the previous subsections. By this, we obtain a partial information on the stable reduction of the Lubin-Tate curve. However, we will not use this fact later.

Corollary 2.15. For each \( n \geq 1 \), \( \varpi_n \in \mathcal{F}[p^n]_{\text{prim}} \) and \( \varpi_{E,n+1} \in \mathcal{G}[p^n]_{\text{prim}} \), the reductions of \( X_{n,n,\xi,\varpi_n} \) and \( Z_{n,n,\varpi_{E,n+1}} \) appear as open dense subschemes of irreducible components in the stable reductions of \( X(p^n) \) and of \( Y(p^n) \) respectively.

Proof. By Propositions 2.10 and 2.12, the reductions of \( X_{n,n,\xi,\varpi_n} \) and \( Z_{n,n,\varpi_{E,n+1}} \) are smooth, and their smooth compactifications have positive genera. Hence, by [IT, Proposition 7.11], the required assertion follows. \( \square \)

3 Group action on the reductions

In this section, we give an explicit description of some group action on the reductions of the affinoids in \( \mathfrak{C} \) which is induced by the group action on the Lubin-Tate tower summarized in \( \S 3.1 \).

Let \( K \) be a non-archimedean local field, and \( \overline{K} \) its algebraic closure. Let \( W_K \) denote the Weil group of \( K \). Let \( a_K : W_K^{ab} \rightarrow K^\times \) be the Artin reciprocity map normalized such that a geometric Frobenius is sent to a prime element in \( \sigma \) of the Weil group of \( K \). We will not use this fact later.

The following corollary is a direct consequence of the analysis in the previous subsections. By this, we obtain a partial information on the stable reduction of the Lubin-Tate curve. However, we will not use this fact later.

3.1 Review on group action on Lubin-Tate curve

We briefly recall a group action on Lubin-Tate curves. In the following, we always consider a group action on spaces. Our main references are [Ca2], [Da], [Fa], [GL] and [St].

For any integer \( h \in \mathbb{Z} \), let \( \mathcal{R}^{(h)}(p^n) \) denote the functor which associates to \( A \in \mathcal{C} \) the set of isomorphism classes of triples \( (\mathcal{F}_A, \rho, \phi) \), where \( \mathcal{F}_A \) is a formal \( \mathcal{O}_F \)-module over \( A \), \( \rho : \mathcal{F}_0 \rightarrow \mathcal{F}_A \otimes_A \mathbb{F}_q \) is a quasi-isogeny of height \( h \), and \( \phi : (\mathcal{O}_F/p^n)^2 \rightarrow \mathcal{F}_A[p^n](A) \) is a Drinfeld level \( p^n \)-structure. This is representable by a regular local ring \( R^{(h)}(p^n) \) by [Dr, Proposition 4.3]. Let \( X^{(h)}(p^n) \) denote the generic fiber of \( \mathcal{M}^{(h)}(p^n) = \text{Spf} R^{(h)}(p^n) \). We set

\[
LT(p^n) = \bigsqcup_{h \in \mathbb{Z}} X^{(h)}(p^n).
\]

Let \( \mathcal{O}_D = \text{End}_F(\mathcal{F}_0) \). Let \( \varphi \in \mathcal{O}_D \) be the endomorphism of \( \mathcal{F}_0 \) defined by \( X \mapsto X^q \). Then, \( D = \mathcal{O}_D[\varphi^{-1}] \) is the quaternion division algebra over \( F \). Let \( D^\times \) act on \( LT(p^n) \) by \( d : X^{(h)}(p^n) \rightarrow X^{(h+e(\text{Nrd}_D/d(p^n))}(p^n) ; \ (\mathcal{F}_A, \rho, \phi) \mapsto (\mathcal{F}_A, \rho \circ d, \phi) \) for any \( d \in D^\times \). Let \( \text{GL}_2(\mathcal{O}_F) \) act on \( LT(p^n) \) by \( g : X^{(h)}(p^n) \rightarrow X^{(h)}(p^n) ; \ (\mathcal{F}_A, \rho, \phi) \mapsto (\mathcal{F}_A, \rho, \phi \circ g) \) for any \( g \in \text{GL}_2(\mathcal{O}_F) \). This action extends to an action of \( \text{GL}_2(F) \) on the projective system \( \{LT(p^n)\}_{n \geq 0} \) (cf. [Da, §3.2.4], [GL II.2.2] or [St, §2.2]). Let \( U^+_n = \ker(\text{GL}_2(\mathcal{O}_F) \rightarrow \)
GL$_2(O_F/p^n)$ for $n \geq 0$. For an open compact subgroup $M \subset$ GL$_2(O_F)$, we take $n$ such that $U_{3n}^n \subset M$. We write $X^{(h)}_M$ for the quotient $X^{(h)}(p^n)/M$. Furthermore, we set

$$\text{LT}_M = \coprod_{h \in \mathbb{Z}} X^{(h)}_M.$$ 

The extended action of $g \in$ GL$_2(F)$ induces morphisms

$$g: \text{LT}_M \to \text{LT}_{g^{-1}Mg},$$

$$g: X^{(h)}_M \to X^{(h-v(\det(g)))}_{g^{-1}Mg}.$$ 

The subgroup $\{ (x, x, 1) | x \in F^\times \} \subset$ GL$_2(F) \times D^\times$ acts trivially on LT($p^n$).

Finally, we recall the action of $W_F$ on LT($p^n$)$_C$ (cf. [Da, §3], [Fa I.1.4] and [RZ, Theorem 3.49]). We fix an isomorphism

$$\mathcal{M}^{(h)}(p^n) \xrightarrow{\sim} \mathcal{M}^{(0)}(p^n); \quad (\mathcal{F}_A, \rho, \phi) \mapsto (\mathcal{F}_A, \rho \circ \varphi^{-h}, \phi). \quad \text{(3.1)}$$

There exists a formal scheme $\mathcal{M}^{(0)}(p^n)^0$ over Spf $O_F$ such that

$$\mathcal{M}^{(0)}(p^n) \equiv \mathcal{M}^{(0)}(p^n)^0 \circ \circ \circ O_F \circ \circ \circ F.$$ \text{(3.2)}

The formal scheme $\mathcal{M}^{(0)}(p^n)^0$ equals $\mathcal{M}_{LT,n}$ in the notation of [GL, p. 335]. We write $X(p^n)^0$ for $(\mathcal{M}^{(0)}(p^n)^0)_{rig}$. Let $\sigma \in W_F$. By (3.1) and (3.2), we identify as follows:

$$\text{LT}(p^n)_{C} = \coprod_{h \in \mathbb{Z}} X^{(h)}(p^n)_{C} \simeq \coprod_{h \in \mathbb{Z}} X^{(0)}(p^n)_{C} \simeq \coprod_{h \in \mathbb{Z}} X(p^n)^0_h \times_{Sp F} Sp C,$$

where $X(p^n)^0_h = X(p^n)^0$. Let $\sigma \in W_F$. This gives the automorphism $\sigma^*: Sp C \to Sp C$. Let $\sigma$ denote the automorphism of LT($p^n$)$_C$ defined by

$$1 \times \sigma^*: X(p^n)^0_h \times_{Sp F} Sp C \to X(p^n)^0_h \times_{Sp F} Sp C$$

for each $h \in \mathbb{Z}$. Then, let $\sigma \in W_F$ act on LT($p^n$)$_C$ as the composite

$$\varphi^{n\sigma} \circ \sigma: \text{LT}(p^n)_{C} \to \text{LT}(p^n)_{C}.$$ 

We set

$$G = \text{GL}_2(F) \times D^\times \times W_F.$$

Let

$$\delta: G \to \mathbb{Z}; \quad (g, d, \sigma) \mapsto v(\det(g) \text{Nrd}_{D/F}(d)^{-1}a_F(\sigma)^{-1})$$

and $G^0 = \text{ker} \delta$. As above, the tower $\{\text{LT}(p^n)_{C}\}_{n \geq 0}$ admits a $G$-action. The subgroup $G^0$ is the stabilizer of the tower $\{X(p^n) = X^{(0)}(p^n)\}_{n \geq 0}$.

### 3.2 Action of Weil group on the reductions

#### 3.2.1 Unramified case

Let $n \geq 2$. Let $\mathcal{F}$ be as in [2.5]. We choose an element $\varpi_n \in \mathcal{F}[p^n_{F_2}]_{prim}$. By the Lubin-Tate theory, we have the isomorphism $\iota_{\varpi_n}: U_0^n_{F_2}/U_1^n_{F_2} \xrightarrow{\sim} \mathcal{F}[p^n_{F_2}]_{prim}; a \mapsto [a]_{\varpi}^{\varpi_n}$ and the commutative diagram

$$\begin{array}{ccc}
U_0^n_{F_2}/U_1^n_{F_2} & \xrightarrow{\sim} & \mathcal{F}[p^n_{F_2}]_{prim} \\
\downarrow \text{can.} & & \downarrow [\varpi]_{\varpi} \\
U_0^n_{F_2}/U_{n-1}^n_{F_2} & \xrightarrow{\sim} & \mathcal{F}[p^{n-1}_{F_2}]_{prim}.
\end{array}$$
Let $\zeta \in F_{q^2} \setminus F_q$. For $a \in U_{F_2}/U_{F_2}^{n-1}$, we take a lift $\tilde{a}$ in $U_{F_2}/U_{F_2}^n$. Then, the affinoid $X_{n,n,\zeta,\tau_n}(\tilde{a})$ is independent of the choice of the lift $\tilde{a}$ by Lemma $2.7$ and Remark $2.8$. Hence, we write $X_{n,n,\zeta,a,\tau_n}$ for this affinoid. We consider the union of the affinoids

$$X_{n,\zeta} = \bigcup_{\varpi' \in \mathcal{F}[pF_2]_{\text{prim}}} X_{n,n,\zeta,\varpi_n'} = \bigcap_{a \in U_{F_2}^n/U_{F_2}^{n-1}} X_{n,n,\zeta,a,\tau_n} \subset X(p^n). \quad (3.3)$$

We write $\mathcal{S}_n$ for the index set $U_{F_2}^n/U_{F_2}^{n-1}$ in $\mathfrak{R}_n$. In the following, we compute the action of

$$W_{F_2}' = \{(1, \varpi^{-n}, \sigma) \in G \mid \sigma \in W_{F_2}\}$$

on the reduction $\mathfrak{X}_{n,n,\zeta,\varpi_n}$. We identify $W_{F_2}'$ with $W_{F_2}$ by $(1, \varpi^{-n}, \sigma) \mapsto \sigma$. For an integer $n \geq 1$, let $\varpi_n \in \mathcal{F}[p^n]_{\text{prim}}$, and set $F_{2,0} = F_2$ and $F_{2,n} = F_2(\varpi_n)$ for $n \geq 1$. We consider the homomorphism

$$a_{F_{2,n}} : \text{Gal}(F_{2,n}/F_2) \to U_{F_2}^n/U_{F_2}^{n-1} : \sigma \mapsto a_{\sigma},$$

where $a_{\sigma}$ is characterized by $\sigma(\varpi_n) = [a_{\sigma}]_{\varpi}(\varpi_n)$. By the Lubin-Tate theory (cf. [1w]), the map $a_{F_{2,n}}$ is an isomorphism. We have the commutative diagram

$$\begin{array}{cccc}
W_{F_2} & \to & W_{ab}^{F_2} & \to & \text{Gal}(F_{2,n}/F_2) \to U_{F_2}^n/U_{F_2}^{n-1} \\
\text{can} & & \text{res} & & \text{can} & \Rightarrow & U & \to & U & \to & U
\end{array} \quad (3.4)$$

Note that the composite of the homomorphisms on the upper line equals the composite of the canonical map $W_{F_2} \to W_{ab}^{F_2}$, $a_{F_2} : W_{ab}^{F_2} \cong F_2^x$ and $F_2^x \to U_{F_2}^n/U_{F_2}^{n-1}$ by $x \mapsto x/\varpi^{-n}(x)$ by the Lubin-Tate theory. We identify $U_{F_2}^n/U_{F_2}^{n-1}$ with $F_{q^2}$ by $x \mapsto \tilde{x}$ for $x \in U_{F_2}$ if $n = 1$, and $F_{q^2}$ by $1 + \varpi^{-1}x \mapsto \tilde{x}$ for any $x \in O_{F_2}$ if $n \geq 2$. We write $\pi_{n-1}$ for the composite of the three homomorphisms on the lower line in $\mathfrak{R}_n$.

**Lemma 3.1.** 1. The group $W_{F_2}$ acts on $\mathfrak{X}_{1,1}$ as follows:

$$\sigma : \mathfrak{X}_{1,1} \to \mathfrak{X}_{1,1} ; (X,Y) \mapsto (\pi_0(\sigma^{-1})X^q^{a_{n}}, \pi_0(\sigma^{-1})Y^q^{a_{n}})$$

for $\sigma \in W_{F_2}$.

2. Let $n \geq 2$ be a positive integer. The induced action of $W_{F_2}$ on the set $\mathcal{S}_n$ is transitive. The stabilizer of $X_{n,n,\zeta,\varpi_n}$ in $W_{F_2}$ equals the subgroup $W_{F_{2,n-1}}$. Let $\sigma \in W_{F_{2,n-1}}$. Then, $\sigma$ acts on $X_{n,n,\zeta,\varpi_n}$ as follows:

$$\sigma : X_{n,n,\zeta,\varpi_n} \to X_{n,n,\zeta,\varpi_n} ; (X,Y) \mapsto (X^{q^{a_{n}}} + \pi_{n-1}(\sigma^{-1}), Y^q^{a_{n}}).$$

**Proof.** The assertion 1 is easily checked on the basis of the computations in the proof of Lemma $2.9$. We omit the details.

We prove the assertion 2. The first assertion follows from the Lubin-Tate theory over $F_2$. We have

$$v(\sigma^{-1}(\varpi_n) - \varpi_n) \geq h \iff \sigma \in W_{F_{2,n-1}}.$$ 

The second assertion follows from this and Lemma $2.7$. Finally, we prove the third assertion.

Let $\sigma \in W_{F_{2,n-1}}$ and $P \in X_{n,n,\varpi_n}(C)$. Then we have $\pi_{n-1}(\sigma^{-1}) = \sigma^{-1}(\varpi_n) - \varpi_n$ in $F_{q^2}$. We have $X_n(P\sigma) = \sigma^{-1}(X_n(P))$ and $Y_n(P\sigma) = \sigma^{-1}(Y_n(P))$. Therefore, by $(2.7)$, we have

$$S_n(P\sigma) = \pi_{n-1}(\sigma^{-1}) + \sigma^{-1}(S_n(P)),$$

$$T_n(P\sigma) = \zeta \pi_{n-1}(\sigma^{-1}) + \sigma^{-1}(T_n(P)).$$
Hence, by (2.20), we obtain

$$U_n(\pi) = \zeta_1 \pi_{n-1}(\sigma^{-1}) + \pi^{-1}(U_n(\pi)).$$

By (2.21), we have $\chi(\pi) = \zeta_1 \pi_{n-1}(\sigma^{-1}) + \pi^{-1}(\chi(\pi)) \equiv \zeta_1 \pi_{n-1}(\sigma^{-1}) + \pi(\chi)^n \mod 0^+$. We can check that $\chi(\pi) \equiv \pi^{-1}(\chi(\pi)) \equiv \chi(\pi)^n \mod 0^+$.

### 3.2.2 Ramified case

Let $n$ be an odd positive integer. As in (3.3), by using Lemma 2.11 and choosing $\varpi_{E,n+1} \in \mathcal{O}[\mathfrak{p}_{E,n+1}]_{\text{prim}}$, we can define $Z_{n,n,a,\varpi_{E,n+1}}$ for any $a \in U_{E}^0/U_{E}$. We consider the affinoid

$$Z_{\varpi_{E,n}} = \bigcup_{\varpi_{E,n+1} \in \mathcal{O}[\mathfrak{p}_{E,n+1}]_{\text{prim}}} Z_{n,n,\varpi_{E,n+1}} = \coprod_{a \in U_{E}^0/U_{E}} Z_{n,n,a,\varpi_{E,n+1}} \subset Y(\mathfrak{p}^n). \quad (3.5)$$

We write $\Sigma_n$ for the index set $U_{E}^0/U_{E}$ in (3.5). In the following, we determine the action of

$$W_E = \{(1, \varphi^{-n\sigma}, \sigma) \in G \mid \sigma \in W_E\}$$

on the reduction $Z_{\varpi_{E,n}}$. We identify $W_E$ with $W_E$ by $(1, \varphi^{-n\sigma}, \sigma) \mapsto \sigma$.

Let $\varphi$ be as in (2.35). For an integer $n \geq 1$, let $\varpi_{E,n} \in \mathcal{O}[\mathfrak{p}_{E,n}]_{\text{prim}}$ and set $E_n = E(\varpi_{E,n})$. We consider the homomorphism

$$a_{E,n} : \text{Gal}(E_n/E) \to U_{E}^0/U_{E}; \sigma \mapsto a_\sigma,$$

where $a_\sigma$ is characterized by $\sigma(\varpi_{E,n}) = [a_\sigma]_{\varphi}(\varpi_{E,n})$. By the Lubin-Tate theory, this is an isomorphism. For $n \geq 2$, we consider the commutative diagram

$$
\begin{array}{ccc}
W_E & \xrightarrow{\text{can.}} & W_E^{\text{ab}} \\
\Downarrow \text{can.} & & \Downarrow \text{res.} \\
W_{E_{n-1}} & \xrightarrow{\text{can.}} & W_{E_{n-1}}^{\text{ab}} \\
& & \Downarrow \text{res.} \\
& \text{Gal}(E_n/E) & \xrightarrow{\text{can.}} \\
& \xrightarrow{\text{can.}} & U_{E}^0/U_{E} \\
& \xrightarrow{\text{can.}} & U_{E}^{n-1}/U_{E} \\
\end{array}
$$

We identify $U_{E}^{n-1}/U_{E}$ with $\mathbb{F}_q$ by $1 + \varpi_{E,n}^{-1}x \mapsto x$ for any $x \in \mathbb{F}_q$. For any $n \geq 2$, we write $\pi_{E,n-1}$ for the composite of the three homomorphisms on the lower line in (3.6).

As before, we set

$$\theta_1 = \varpi_{E,1}, \quad \theta_n = \varpi_{E,n}/\varpi_{E,1} \text{ for } n \geq 2.$$ 

Then, we have the equality $\pi_{E,n-1}(\sigma) = \sigma(\theta_n) - \theta_n$ in $\mathbb{F}_q$ for $n \geq 2$ and $\sigma \in W_{E_{n-1}}^{\text{ab}}$.

**Lemma 3.2.** Let $n = 2m - 1 \geq 1$ be an odd integer. The induced action of $W_E$ on the set $\Sigma_n$ is transitive. The stabilizer of $Z_{n,n,\varpi_{E,n+1}}$ in $W_E$ equals $W_{E_n}$. Let $\sigma \in W_{E_n}$. Then, $\sigma$ acts on $Z_{n,n,\varpi_{E,n+1}}$ as follows:

$$\sigma : Z_{n,n,\varpi_{E,n+1}} \to Z_{n,n,\varpi_{E,n+1}}; \quad (a, s) \mapsto \left(q^n \sigma + 2\pi_{E,n}(\sigma^{-1}), \left(-\frac{1}{\mathbb{F}_q}\right)^{(m-1)n_\sigma} s q^n \sigma \right).$$

**Proof.** The first assertion follows from the Lubin-Tate theory over $E$. We have $v(\sigma^{-1}(\theta_{n+1}) - \theta_{n+1}) \geq 0 \iff \sigma \in W_{E_n}$. 

\[ \Box \]
The second assertion follows from this and Lemma 2.112 and 3. We prove the third assertion. Let \( P \in \mathbb{Z}_{n,n,\varpi E,n+1}(\mathbb{C}) \). For \( i \in 2 \), we have

\[
u_{i,n}(P\sigma) = \pi_{E,n}(\sigma^{-1}) + \sigma^{-1}(\nu_{i,n}(P)).
\]

Hence, by (2.49), we can check that

\[
u_{n,n,\varpi E,n+1}(\mathbb{C}).
\]

Let \( \mathbb{C} \) we have

\[
U_n(P\sigma) = 2\pi_{E,n}(\sigma^{-1}) + \sigma^{-1}(U_n(P)) \equiv 2\pi_{E,n}(\sigma^{-1}) + U_n(P)q^n \mod 0 + .
\]

By (2.49), we can check that

\[
s(P\sigma) = \frac{1}{(n-1)^{m-1}}\sigma^{-1}(s(P)) \equiv \left(\frac{1}{F_q}\right)^{(m-1)n\sigma} s(P)q^n \mod 0 + .
\]

Hence, the required assertion follows.

\[
\square
\]

### 3.3 Action of GL(2) on unramified components

In the following, we describe the action of some elements, which stabilizes the affinoid \( X_{n,n,\varpi E,n} \).

Let \( n \geq 2 \) be a positive integer until the end of (3.4.1). Let \( \zeta \in \mathbb{F}_q^2 \setminus \mathbb{F}_p \). We consider the \( F \)-embedding

\[
\iota_\zeta: F_2 \hookrightarrow M_2(F); \quad a + b\zeta \mapsto \begin{pmatrix}
a & -b\zeta + 1 \\
b & a + b(\zeta + \zeta^q)
\end{pmatrix}
\]

for \( a, b \in F \). We consider the non-degenerate symmetric \( F \)-bilinear form \( (\cdot, \cdot) \) on \( M_2(F) \) defined by \( (g_1, g_2) = \text{Tr}(g_1 g_2) \) for \( g_1, g_2 \in M_2(F) \). We set

\[
C_1 = \{ g \in M_2(F) \mid (x, g) = 0 \text{ for } x \in F_2 \}, \quad \mathfrak{C}_1 = C_1 \cap M_2(\mathcal{O}_F).
\]

Then \( C_1 \) and \( \mathfrak{C}_1 \) are stable under the left and right actions of \( F_2 \) and \( \mathcal{O}_F \), respectively. Explicitly, we have

\[
C_1 = \left\{ g(a, b) = \begin{pmatrix}
a & a(\zeta + \zeta^q) + b\zeta + 1 \\
b & -a
\end{pmatrix} \in M_2(F) \mid a, b \in F \right\}.
\]

We write \( \mathfrak{M} \) for \( M_2(\mathcal{O}_F) \). Then, we have decompositions

\[
M_2(F) = F_2 \oplus C_1, \quad \mathfrak{M} = \mathcal{O}_{F_2} \oplus \mathfrak{C}_1
\]

as \( F_2 \)-vector spaces and \( \mathcal{O}_{F_2} \)-modules respectively. We can check that \( \mathfrak{C}_1 \mathfrak{C}_1 = \mathcal{O}_{F_2} \) (cf. [We, Lemma 4.1]). For \( m \geq 1 \), we set \( U_{2m} = 1 + p^m\mathfrak{M} \) and

\[
H^{n}_\zeta = 1 + p^{m-1} + p^{\lceil \frac{m}{2} \rceil} \mathfrak{C}_1 \subset U_{2m}^{\lceil \frac{m}{2} \rceil}.
\]

This is an open compact subgroup of \( \text{GL}_2(F) \), because it contains \( U_{2m}^{m-1} \). Clearly, \( F_2^\times \) normalizes \( H^{n}_\zeta \). We consider the action of the group \( U_{p}^{n} \mathcal{H}^n \) on the reduction of the affinoid \( X_{n,\zeta} \) in (3.3). For an element \( \alpha = \sum_{i=0}^{\infty} a_i \varpi^i \in \mathcal{O}_F^\times \) with \( a_i \in \mathbb{F}_q \), we set

\[
[a](X) = \sum_{i=0}^{\infty} a_i X^{q^i}
\]

For \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_F) \), we have

\[
g^* X_n \equiv [a](X_n) + [c](Y_n) \mod (\varpi, u),
g^* Y_n \equiv [b](X_n) + [d](Y_n) \mod (\varpi, u),
\]

(3.10)
because \([\varpi]_u(X) \equiv X^{q^2} \mod (\varpi, u)\). Let \(\zeta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q\) and set \(\zeta_1 = \zeta^q - \zeta\). It is not difficult to check that \(U_{F_2}^0 H_\zeta^a\) acts on the affinoid \(X_{n,\zeta}\) by using (3.10). For an element \(x \in \mathcal{O}_\mathcal{F}\), we write \(\bar{x}\) for its image by the canonical map \(\mathcal{O}_\mathcal{F} \to \mathbb{F}\).

**Proposition 3.3.** The subgroup \(U_{F_2}^0 H_\zeta^a\) acts on the index set \(\mathcal{S}_n\) transitively. Let \(\varpi_n \in \mathcal{F}[\mathfrak{p}_{F_2}]_{\text{prim}}\). The stabilizer of \(X_{n,n,\zeta,\varpi_n}\) in \(U_{F_2}^0 H_\zeta^a\) equals \(H_\zeta^a\). Let \(1 + g = 1 + \varpi(\varpi) g(a, b) + \varpi^{-1} (a_2 \ b_2 \ c_2 \ d_2) \in H_\zeta^a\). We define elements \(\beta(g), \gamma(g) \in \mathbb{F}_{q^2}\) as follows:

\[
\beta(g) = a + b \zeta, \quad \gamma(g) = \zeta^{-1} (\zeta^q(a_2 + c_2 \zeta) - (b_2 + d_2 \zeta)).
\]

Then, \(1 + g\) acts on \(X_{n,n,\zeta,\varpi_n}\) as follows:

\[
1 + g: X_{n,n,\zeta,\varpi_n} \to X_{n,n,\zeta,\varpi_n};
\]

\[
(X, Y) \mapsto \begin{cases} (X + \beta(g)Y q^{-1} + \gamma(g), Y + \beta(g)) & \text{if } n \text{ is odd}, \\ (X + \gamma(g), Y) & \text{if } n \text{ is even}. \end{cases}
\]

In particular, if \(g = \varpi n^{-1} x\) for \(x \in \mathcal{O}_{F_2}\), we have \(\gamma(g) = \bar{x}\).

**Proof.** We put \(m_0 = [n/2]\). We write \(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\) for the matrix \(g(a, b)\). By (3.10), we have

\[
(1 + g)^* X_n \equiv X_n + ([a_1](X_{n-m_0}) + [c_1](Y_{n-m_0})) + ([a_2](X_1) + [c_2](Y_1)) \mod h^+, \\
(1 + g)^* Y_n \equiv Y_n + ([b_1](X_{n-m_0}) + [d_1](Y_{n-m_0})) + ([b_2](X_1) + [d_2](Y_1)) \mod h^+.
\]

Hence, by (2.9) and (2.19), we obtain

\[
(1 + g)^* S_n \equiv S_n + [a_1 + c_1 \zeta](\varpi_{n-m_0}) \mod h / q^{n-1}^+, \\
(1 + g)^* U_n \equiv U_n + \zeta_1 (a_1 + c_1 \zeta^q) S_{n-m_0} + (\zeta^q(a_2 + c_2 \zeta) - (b_2 + d_2 \zeta)) \mod h^+.
\]

Therefore, the required assertion follows from (2.21), (2.28) and \(\beta(g) = a + b \zeta\). \(\square\)

### 3.4 Action of \(D^\times\) on unramified components

Let \(\mathcal{F}_0\) and \(\mathcal{F}\) be as in (2.1) and (2.5) respectively. Since we have \(\mathcal{F}_0 = \mathcal{F} \otimes \mathcal{O}_{F_2} \mathbb{F}_{q^2}\), the reduction mod \(\mathfrak{p}_{F_2}\) induces the injective \(\mathcal{O}_\mathcal{F}\)-homomorphism

\[
\mathcal{O}_{F_2} = \text{End}_{\mathcal{O}_{F_2}}(\mathcal{F}) \to \text{End}_{\mathcal{F}}(\mathcal{F}_0) = \mathcal{O}_D
\]

by [GH] Proposition 4.2]. We have \(\mathcal{O}_D = \mathcal{O}_{F_2} \oplus \varphi \mathcal{O}_{F_2}\) by [GH] Proposition 13.10]. We have \(\varphi^2 = \varpi\) and \(\varphi \alpha = \alpha^q \varphi\) for \(\alpha \in \mathbb{F}_{q^2}\). Let \(\mathfrak{p}_D = (\varphi)\) be the two-sided maximal ideal of \(\mathcal{O}_D\). We consider the closed subscheme \(\text{Spf} \mathcal{O}_{\mathcal{F}_{\text{ur}}} \to \text{Spf} \mathcal{O}_{\mathcal{F}_{\text{ur}}} [\mathfrak{u}]\): \(u \mapsto 0\) corresponding to \((\mathcal{F} \otimes \mathcal{O}_{F_2} \mathcal{O}_{\mathcal{F}_{\text{ur}}}, \rho)\). The \(\mathcal{O}_{\mathcal{F}_{\text{ur}}}\)-action, which is induced by (3.13) and the \(\mathcal{O}_{\mathcal{F}}^\times\)-action on \(\text{Spf} \mathcal{O}_{\mathcal{F}_{\text{ur}}} [\mathfrak{u}]\), fixes the closed subscheme \(\text{Spf} \mathcal{O}_{\mathcal{F}_{\text{ur}}}\).

We set \(C_2 = \varphi F_2 \subset D\) and \(\mathfrak{c}_2 = C_2 \cap \mathcal{O}_D\). We consider the non-degenerate \(F\)-bilinear form \((\cdot, \cdot)_D\) on \(D\) defined by \((d_1, d_2)_D = \text{Trd}_{D/F}(d_1 d_2)\) for \(d_1, d_2 \in D\). Then, we have

\[
C_2 = \{ d \in D \mid (d, x)_D = 0 \text{ for any } x \in F_2 \}.
\]

Further we have natural decompositions

\[
D = F_2 \oplus \varphi F_2, \quad \mathcal{O}_D = \mathcal{O}_{F_2} \oplus \mathfrak{c}_2.
\]

(3.14)
Let $U^n_D = 1 + p^n_D$ for a positive integer $m$. We set
\[ H^n_{\zeta,D} = 1 + p^{n-1}_D + p^{(n-1)} \mathfrak{c}_2 \subset U^{2(n-1)+1}_D. \] (3.15)
For an element $d = \sum_{i=0}^{\infty} d_i \varphi^i \in \mathfrak{O}_D^\times$ with $d_i \in \mathbb{F}_q^\times$, we set
\[ [d](X) = \sum_{i=0}^{\infty} d_i X^{q^i}. \]

Then, we have
\[ d^* X_n \equiv [d^{-1}](X_n) \mod (\varpi, u), \quad d^* Y_n \equiv [d^{-1}](Y_n) \mod (\varpi, u) \] (3.16)
(see [GH, Proposition 14.7]). We describe the action of $U^n_{F_2} H^n_{\zeta,D}$ on the reduction of $X_{n,\zeta}$ in (3.3). For an element $d \in \mathfrak{O}_D$, we write $d$ for its image by the canonical map $\mathfrak{O}_D \to \mathbb{F}_q^\times$.

**Proposition 3.4.** The induced action of $U^n_{F_2} H^n_{\zeta,D}$ on $\mathfrak{S}_n$ is transitive. Let $\varpi_n \in \mathcal{F}[p^n_{F_2}]_{\text{prim}}$. The stabilizer of $\overline{X}_{n,n,\zeta,\varpi_n}$ in $U^n_{F_2} H^n_{\zeta,D}$ equals $H^n_{\zeta,D}$. Let $(1 + d)^{-1} = 1 + \varpi^{(n-1)} \varphi x + \varpi^{n-1} y \in H^n_{\zeta,D}$. We set
\[ \beta(d) = \bar{x}, \quad \gamma(d) = \bar{y} \in \mathbb{F}_q^\times. \]

Then, $1 + d$ acts on $\overline{X}_{n,n,\zeta,\varpi_n}$ as follows:
\[ 1 + d : \overline{X}_{n,n,\zeta,\varpi_n} \to \overline{X}_{n,n,\zeta,\varpi_n}; \]
\[ (X, Y) \mapsto \begin{cases} 
(X + \beta(d) Y q^{n-1} + \gamma(d) Y + \beta(d) Y) & \text{if } n \text{ is even}, \\
(X + \gamma(d) Y) & \text{if } n \text{ is odd}.
\end{cases} \]

**Proof.** We set $m_1 = [(n - 1)/2]$. By (3.16), we have
\[ (1 + d)^* X_n \equiv X_n + (x X_{n-m_1})^q + y X_1 \mod h^+, \]
\[ (1 + d)^* Y_n \equiv Y_n + (x Y_{n-m_1})^q + y Y_1 \mod h^+. \] (3.17)
Hence, by (2.7) and (2.9), we have
\[ (1 + d)^* S_n \equiv S_n + (x \varpi_{n-m_1})^q \mod h/q^{n-1}+, \]
\[ (1 + d)^* U_n \equiv U_n + \zeta_1 (x S_{n-m_1})^q + \zeta_1 y \varpi_1 \mod h+. \] (3.18)
The required assertion follows from (2.21), (2.28), (3.17) and (3.18). \qed

### 3.4.1 Action of some diagonal elements on unramified components

We simply write $G_D$ for $\text{GL}_q(F) \times D^\times$. In this subsection, we describe the actions of some diagonal elements in $G_D$ and $W_F \times D^\times$ on $\overline{X}_{n,n,\zeta,\varpi_n}$ in (3.3).

**Lemma 3.5.** We consider the $F$-embedding $\Delta_\zeta : F_2^\times \hookrightarrow G_D$; $\alpha \mapsto (i_\zeta(\alpha), \alpha)$. Then, $\Delta_\zeta(F_2^\times)$ stabilizes $\overline{X}_{n,n,\zeta,\varpi_n}$. Let $\alpha \in F_2^\times$. Then, $\Delta_\zeta(\alpha)$ acts on $\overline{X}_{n,n,\zeta,\varpi_n}$ as follows:
\[ \overline{X}_{n,n,\zeta,\varpi_n} \to \overline{X}_{n,n,\zeta,\varpi_n}; \quad (X, Y) \mapsto \left( X, \frac{\alpha / \varpi F_2(\alpha)}{\varpi^{q-1} Y} \right). \]
Hence, by (2.9), we have
\[
\alpha^*X_n \equiv [\alpha^{-1}][a(X_n) + [b](Y_n)] \equiv \varpi_n + [\alpha^*/\alpha](S_n) \mod h, \\
\alpha^*Y_n \equiv [\alpha^{-1}][1-b\zeta^{q+1}](X_n) + [a + b(\zeta + \zeta)](Y_n) \mod h.
\]
Hence, by (2.9), we have
\[
\alpha^*S_n \equiv [\alpha^*/\alpha](S_n) \mod h/q^{n-1} +, \quad \alpha^*U_n \equiv U_n \mod h +.
\]
Therefore, the required assertion follows from (2.21) and (2.28).

Let \( L \) be a non-archimedean local field. We take a uniformizer \( \varpi_L \) of \( L \). For \( \sigma \in W_L \), we set \( a^0_{L,\varpi_L} = a_L(\sigma)/\varpi_L^{\nu_L(a_L(\sigma))} \in O_L^\times \). We write \( I_{L} \) for the inertia subgroup of \( L \). For \( \sigma \in I_L \), the value \( a^0_{L,\varpi_L} \) is independent of the choice of the uniformizer, for which we write \( a^0_{L}(\sigma) \). We consider the subgroup
\[
I_{F_2}^{(n)} = \begin{cases} 
\{ (d, 1, \sigma) \in G \mid d \in O_{F_2}^\times, \sigma \in I_{F_2}, a^0_{F_2}(\sigma)d = 1 \} & \text{if } n \text{ is odd}, \\
\{ (g, 1, \sigma) \in G \mid g \in O_{F_2}^\times, \sigma \in I_{F_2}, a^0_{F_2}(\sigma) = g \} & \text{if } n \text{ is even}.
\end{cases}
\] (3.19)

**Lemma 3.6.** The subgroup \( I_{F_2}^{(n)} \) acts on \( \mathbf{X}_{n,n,\zeta,\varpi_n} \) trivially.

**Proof.** Let \( (d, 1, \sigma) \in G \) be an element such that \( d \in O_{F_2}^\times, \sigma \in I_{F_2} \) and \( a^0_{F_2}(\sigma)d = 1 \). Let \( P \in \mathbf{X}_{n,n,\zeta,\varpi_n}(C) \). By (3.16), we have
\[
(d^*X_n)(P\sigma) \equiv [a^0_{F_2}(\sigma)](\sigma^{-1}(X_n(P))) \mod h +. \tag{3.20}
\]
We have \( \sigma^{-1}(\varpi_n) = [a^0_{F_2}(\sigma^{-1})]/_\varpi(a^0_{F_2}(\sigma^{-1})(\varpi_n)) \equiv [a^0_{F_2}(\sigma^{-1})](\varpi_n) \mod 1 \) by the Lubin-Tate theory. Hence, by (2.7) for \( i = n \) and (3.20), we obtain
\[
(d^*S_n)(P\sigma) \equiv [a^0_{F_2}(\sigma)](\sigma^{-1}(S_n(P))) \mod h +. \tag{3.21}
\]
We consider the case where \( n = 2m - 1 \) is odd. By (2.28), (3.21) and the Lubin-Tate theory, we obtain
\[
(d^*Y)(P\sigma) \equiv a^0_{F_2}(\sigma)[\sigma^{-1}(\varpi_m)]_{\varpi_m}(\varpi_m)^{-1}(Y(P)) \equiv Y(P) \mod 0 +.
\]
In the same way, we have \( (d^*X)(P\sigma) \equiv X(P) \mod 0+ \).

We consider the case where \( n = 2m \) is even. By (2.21), we have
\[
(d^*Y)(P\sigma) \equiv a^0_{F_2}(\sigma)[\sigma^{-1}(\varpi_{m+1}^q)]_{\varpi_m}^{q^{-1}}(Y(P)) \equiv d^{q-1}Y(P) \mod 0 +. \tag{3.22}
\]
For an element \( (g, 1, \sigma) \in I_{F_2}^{(n)} \), we write \( (g, 1, \sigma) = \Delta(\zeta)(1, g^{-1}, \sigma) \). Then, by Lemma 3.5 and (3.22), the element \( (g, 1, \sigma) \) acts on the parameter \( Y \) trivially. In the same way, we can check that the action of it on the parameter \( X \) trivially. \( \square \)
3.5 Action of $GL(2)$ on ramified components

As in [GL, II.2.2], an action of $G_D$ on the tower $\{Y(p^n)\}_{n \geq 0}$ is explicitly described. Let $E = F(\varpi_E)$ be as before. We consider the $F$-embedding

$$\Delta_E^{(1)} : E \hookrightarrow M_2(F); \quad a + b\varpi_E \mapsto \begin{pmatrix} a & b \\ b\varpi & a \end{pmatrix}$$

for $a, b \in F$. (3.23)

Let

$$C_{1,E} = \{g \in M_2(F) \mid (g, x) = 0 \text{ for } x \in E\} = \left\{ h(x, y) = \begin{pmatrix} x & y \\ -\varpi y & -x \end{pmatrix} \in M_2(F) \mid x, y \in F \right\},$$

$$\mathfrak{c}_{1,E} = C_{1,E} \cap M_2(\mathcal{O}_F).$$

The former is a one-dimensional left and right $E$-vector space, and the latter is a free left and right $\mathcal{O}_E$-module of rank one. We set

$$\mathfrak{B} = \begin{pmatrix} p & \mathcal{O}_F \\ p & p \end{pmatrix} \subset \mathfrak{J} = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ p & \mathcal{O}_F \end{pmatrix} \subset M_2(F).$$

Then, $\mathfrak{J}$ is an $\mathcal{O}_F$-order in $M_2(F)$ and $\mathfrak{B}$ is its Jacobson radical. The order $\mathfrak{J}$ is called the standard Iwahori order. We have natural decompositions

$$M_2(F) \simeq E \oplus C_{1,E}, \quad \mathfrak{J} \simeq \mathcal{O}_E \oplus \mathfrak{c}_{1,E}$$

as $E$-vector spaces and $\mathcal{O}_E$-modules respectively. For $m \geq 1$, we set $U^m_1 = 1 + \mathfrak{B}^m$. Let $n = 2m - 1$ be an odd positive integer until the end of §3.6.1. We set

$$H^m_E = 1 + \mathfrak{p}_E^n + \mathfrak{p}_E^m \mathfrak{c}_{1,E} \subset U^m_1.$$

Then $E^\times$ normalizes $H^n_E$. We simply write $\text{Spf } R_n$ for the formal scheme $(2.31)$. We set $R = (\lim_{\to n} R_n)$, where $(\cdot)$ denote the $(x_1, x_2)$-adic completion. We set $z = \varpi \otimes 1 \in \mathcal{O}_F \otimes_{\mathbb{F}_q} R$ and

$$\varpi_E, z = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \in \hat{\mathcal{J}} \otimes_{\mathbb{F}_q} R.$$

We consider

$$S = \sum_{i=0}^{\infty} \text{diag}(s_{1,i}, s_{2,i})^t \varpi_E^i z^i = \sum_{i=0}^{\infty} \sum_{z^i}^{s_{1,2i}z^i} s_{1,2i} \sum_{z^i}^{s_{2,2i-1}z^i} s_{2,2i} \in \mathcal{T} \hat{\otimes}_{\mathbb{F}_q} R.$$

For an element $g = \sum_{i=0}^{\infty} \text{diag}(g_{1,i}, g_{2,i}) \varpi_E^i \in \mathcal{J}^\times = (\text{diag}(\mathbb{F}_q)((\varpi_E)))^\times$ with $g_{k,i} \in \mathbb{F}_q$, we regard it as an element of $\hat{\mathcal{J}} \otimes_{\mathbb{F}_q} R$, for which we write $g_z$. As in [GL, p. 347], we define an action of $\mathcal{J}^\times$ on $R$ by

$$g^* S = S^t g_z.$$

(3.26)

By regarding $S$ modulo $z^m (\mathcal{T} \hat{\otimes}_{\mathbb{F}_q} R)$, we obtain an action of $\mathcal{J}^\times$ on $R_n$.

For an element $\alpha = \sum_{i=0}^{\infty} \alpha_i \varpi^i$ with $\alpha_i \in \mathbb{F}_q$, we set

$$[\alpha](s_{i,j}) = \sum_{k=0}^{[j/2]} \alpha_k s_{i,j-2k}$$

for $i \in \mathbb{Z}$. Note that $E^\times$ normalizes $H^n_E$. We describe the action of $U^0_E H^n_E$ on the reduction of $\mathbb{Z}_{\varpi_E,n}$ in (3.5).
Lemma 3.7. The induced action of $U_E^0 H_E^0$ on the index set $\Sigma_n$ is transitive. Let $\varpi_{E,n+1} \in \mathcal{G}[p_{E,n+1}^{n+1}]_{\text{prim}}$. The stabilizer of $\Omega_{n,n,\varpi_{E,n+1}}$ in $U_E^0 H_E^0$ is $H_E^0$. Let $1 + g = 1 + \varpi_E^n h(a_1, b_1) + \varpi_E^n \left( \begin{array}{cc} a_{1,1}^{(2)} & a_{1,2}^{(2)} \\ \varepsilon a_{2,1}^{(2)} & a_{2,2}^{(2)} \end{array} \right) \in H_E^0$. We set $\gamma_E(g) = \frac{a_{1,1}^{(2)} + a_{2,2}^{(2)}}{a_{1,2}^{(2)}}$. Then, $1 + g$ acts on $\Omega_{n,n,\varpi_{E,n+1}}$ as follows:

$$1 + g: \Omega_{n,n,\varpi_{E,n+1}} \to \Omega_{n,n,\varpi_{E,n+1}^\prime}; \ (a, s) \mapsto (a + \gamma_E(g), s).$$

In particular, for $g = \varpi_E^n x$ with $x \in \mathcal{O}_E$, we have $\gamma_E(g) = 2\bar{x}$.

Proof. Set $h_0 = (2(q - 1))^{-1}$. Note that

$$\text{diag}(x, y)^t \varpi_E x = t \varpi_E x \text{diag}(y, x).$$

(3.27)

Let $g' = \sum_{i=0} \varpi_E^i \text{diag}(g_{1,i}, g_{2,i})$. By using (3.28), for $i \in \mathbb{Z}$, we can check that

$$g^* s_{i,k} = \sum_{j=0}^k g_{i+j,k-j} s_{i,j}$$

for any $0 \leq k \leq n$. Hence, we obtain

$$(1 + g)^* s_{i,n} \equiv s_{i,n} + s_{i,0} a_{i,i}^{(2)} + (-1)^{m+i} (a_1(s_{i,m-1}) + b_1(s_{i,m-2})) \mod h_0 +$$

for $i \in \mathbb{Z}$. By dividing this by $s_{i+1,0}$, we acquire

$$(1 + g)^* t_{i,n} \equiv t_{i,n} + \frac{s_{i,0}}{s_{i+1,0}} a_{i,i}^{(2)} + (-1)^{m+i} \frac{a_1(s_{i,m-1}) + b_1(s_{i,m-2})}{s_{i+1,0}} \mod 0+$$

for $i \in \mathbb{Z}$. Therefore, by (3.30) and (3.31), we have

$$
(1 + g)^* u_{1,n} \equiv u_{1,n} \mod v(u_{1,n}) +, \\
(1 + g)^* U_n \equiv U_n + \frac{s_{1,0}}{s_{2,0}} a_{1,1}^{(2)} + \frac{s_{2,0}}{s_{1,0}} a_{2,2}^{(2)} \mod 0+. 
$$

The required assertion follows from $s_{1,0}/s_{2,0} \equiv 1 \mod 0+$.

\[\square\]

3.6 Action of $D^\times$ on ramified components

We consider the $F$-embedding

$$\Delta_E^{(2)}: E \hookrightarrow D; \ a + b \varpi_E \mapsto a + b \varphi$$

(3.28)

for $a, b \in F$. We take an element $\zeta_0 \in F_{q^2}^\times$ satisfying $\zeta_0^{q-1} = -1$. We consider an $E$-vector space $C_{2,E} = \zeta_0 E \subset D$. Set $C_{2,E} = C_{2,E} \cap \mathcal{O}_D$, which is a free $\mathcal{O}_E$-module of rank one. Note that

$$C_{2,E} = \{ d \in D \mid (d, x)_D = 0 \text{ for any } x \in E \}.$$

We have decompositions

$$D = E \oplus C_{2,E}, \ \mathcal{O}_D = \mathcal{O}_E \oplus C_{2,E}$$

(3.29)

as $E$-vector spaces and $\mathcal{O}_E$-modules respectively. We set

$$H_{E,D}^n = 1 + p_E^n + p_E^n C_{2,E} \subset U_D^n.$$  

(3.30)
Let $t: \mathcal{O}_{\text{Fur}} \hookrightarrow R$ be the natural inclusion. For $d = \sum_{i=0}^{\infty} d_i \varphi^i \in \mathcal{O}_D^\times$ with $d_i \in \mathbb{F}_q$, we set

$$d_z = \sum_{i=0}^{\infty} \text{diag}(\varphi(d_i), \varphi(d'_i)) \varpi_E \in \mathcal{I} \hat{\otimes}_{\mathbb{F}_q} R.$$  

We briefly recall [GL, II.2]. Let $\text{Nilp} \mathcal{O}_{\text{Fur}}$ be the category of $\mathcal{O}_{\text{Fur}}$-schemes on which $\varpi$ is locally nilpotent. For $S \in \text{Nilp} \mathcal{O}_{\text{Fur}}$, let $S_0$ denote the closed subscheme $S \otimes_{\mathcal{O}_{\text{Fur}}} \mathbb{F}$. Let $\tilde{\mathcal{M}}_{LT, 2^h}^\times$ be the functor which associates to $S \in \text{Nilp} \mathcal{O}_{\text{Fur}}$ the set of isomorphism classes of triples $(\mathcal{F}_S, \rho, H)$, where $\mathcal{F}_S$ is a formal $\mathcal{O}_\mathbb{F}$-module over $S$ with a quasi-isogeny $\rho: \mathcal{F}_0 \times_{\mathbb{F}} S_0 \to \mathcal{F}_S$, and $H$ is a finite flat group subscheme of $\mathcal{F}_S$ of degree $\mathfrak{f}$ with $\mathbb{F}_q$-action. Then, by [DR], this functor is pro-representable. For $h \in \mathbb{Z}$, let $\tilde{\mathcal{M}}_{LT, 2^h}^{(h)}$ denote the open and closed subscheme of $\tilde{\mathcal{M}}_{LT, 2^h}^\times$, on which the universal quasi-isogeny has height $h$. We set

$$R_0 = \mathcal{O}_{\text{Fur}}[[x_1, x_2]]/(x_1x_2 - \varpi).$$

Then, we have $\tilde{\mathcal{M}}_{LT, 2^h}^{(0)} \simeq \text{Spf} R_0$. We consider the $\text{Spf} \mathcal{O}_{\text{Fur}}$-valued point:

$$\text{Spf} \mathcal{O}_{\text{Fur}} \to \text{Spf} R_0; \quad x_i \mapsto -\varpi_E \quad \text{for } i \in \mathbb{Z}. \quad (3.31)$$

As a point of $\tilde{M}_{LT, 2^h}^\times$, this corresponds to the isomorphism class of the triple $(\mathcal{G}', \rho, H_1)$, where $\mathcal{G}'$ is the base change of the formal $\mathcal{O}_\mathbb{F}$-module in (2.35) to $\text{Spf} \mathcal{O}_{\text{Fur}}$, $\rho: \mathcal{F}_0 \to \mathcal{G}' \otimes_{\mathcal{O}_{\text{Fur}}} \mathbb{F}$ is an isomorphism as formal $\mathcal{O}_\mathbb{F}$-modules over $\mathbb{F}$, and $H_1$ is the closed subscheme $\text{Spf} \mathcal{O}_{\text{Fur}}[[X]]/([\varpi_E]_{\mathcal{G}'(X)})$ of $\mathcal{G}'$. The isomorphism $\rho$ induces the embedding $\mathcal{O}_{\mathbb{F}}^\times \hookrightarrow \mathcal{O}_{\mathbb{F}}^\times$ in (3.28). Then, we consider the action of $\mathcal{O}_E^\times$ which is the restriction of the action of $\mathcal{O}_D^\times$ on $\tilde{M}_{LT, 2^h}^{(0)}$. Then, $\mathcal{O}_E^\times$ fixes the point (3.31), because the subscheme $H_1$ is stable under the $\mathcal{O}_E^\times$-action. Let $\mathcal{O}_{\text{Fur}}[[u]] = R(1)$ be as in (2.3)

Then, we have the natural map induced by forgetting the level structure of $\tilde{M}_{LT, 2^h}^{(0)}$:

$$R(1) = \mathcal{O}_{\text{Fur}}[[u]] \to R_0; \quad u \mapsto -(x_1^2 + x_2).$$

Let $R_{\varpi, \text{univ}}$ be the matrix $R_X$ when $h = 0$ in the notation of [GL, Théorème II.2.1]. Let $d \in \mathcal{O}_D^\times$ and let $\Delta_d \in (\text{Spf} \mathcal{O}_{\text{Fur}}) \times \mathfrak{f}(R_0^\times)/([\varpi_E, z]) \times$ be the unique element $\Delta$ in the notation of [GL] p. 348. Then, by the description of the action of $\mathcal{D}^\times$ given in [GL] p. 348, the element $d$ induces

$$d^* R_{\varpi, \text{univ}} = \Delta_d R_{\varpi, \text{univ}}' d_z, \quad (3.32)$$

$$d^* S = \Delta_d S. \quad (3.33)$$

By [GL Théorème II.2.1], we have

$$\Delta_d \equiv d_z^{-1} \mod (x_1, x_2). \quad (3.34)$$

For an element $\alpha = \sum_{i=0}^{\infty} \alpha_i \varpi_E \in \mathcal{O}_E^\times$ with $\alpha_i \in \mathbb{F}_q$, we set

$$[\alpha](s_{i,j}) = \sum_{k=0}^{j} \alpha_k s_{i,j-k}$$

for $i \in \mathbb{Z}$. We describe the action of $U_{E,D}^{1,2, h}$ on $\mathbb{Z}_{\varpi, m}$ in (3.35).
Lemma 3.8. The induced action of $U_E^0H_{E,D}^n$ on the index set $\Sigma_n$ is transitive. Let $\varpi_{E,n+1} \in \mathcal{G}[p_{E,n+1}]_{\text{prim}}$. The stabilizer of $\mathcal{Z}_{n,n,\varpi_{E,n+1}}$ is $H_{E,D}^0$. Let $(1 + d)^{-1} = 1 + \varphi^m \zeta_0 x + \varphi^n y \in H_{E,D}^0$ with $x \in \mathcal{O}_E$ and $y \in \mathcal{O}_D$. Then, $1 + d$ acts on $\mathcal{Z}_{n,n,\varpi_{E,n+1}}$ as follows:

$$1 + d: \mathcal{Z}_{n,n,\varpi_{E,n+1}} \to \mathcal{Z}_{n,n,\varpi_{E,n+1}}; \ (a, s) \mapsto \left( a + \text{Tr}_{F_q/F_q}(\bar{y}), s \right).$$

Proof. For $i \in \mathbb{Z}$, we have $v(x_i) = 1/2$ on $\mathcal{Z}_{n,n,\varpi_{E,n+1}}$. Let $\sigma \in \text{Gal}(F_2/F)$ be the non-trivial element. Let $i \in \mathbb{Z}$. By (3.33) and (3.34), we can check that

$$(1 + d)^s_i = s_i + (-1)^{i-1} \zeta_0 (s_{i-1}) + y^{\sigma - 1} s_i \mod h_0 +$$

By dividing it by $s_i$, we obtain

$$(1 + d)^s_i t_i = t_i + (-1)^{i-1} \zeta_0 (s_{i-1}) + s_i y^{\sigma - 1} \mod 0 + .$$

Therefore, by (2.39) and (2.43), we have

$$(1 + d)^s_i u_{1,n} \equiv u_{1,n} \mod v(u_{1,n}) +,$n

$$(1 + d)^s_i U_n \equiv U_n + \frac{s_{1,0}}{s_{2,0}} y + \frac{s_{2,0}}{s_{1,0}} y \mod 0 + .$$

The required assertion follows from $s_{1,0}/s_{2,0} \equiv 1 \mod 0 +$ and $y^x \equiv y^x \mod 0 +$. \qed

3.6.1 Action of some diagonal elements on ramified components

We consider the diagonal $F$-embedding

$$\Delta_E: E \hookrightarrow M_2(F) \times D; \ x \mapsto \left( \Delta_E^{(1)}(x), \Delta_E^{(2)}(x) \right),$$

(3.35)

where $\Delta_E^{(1)}$ and $\Delta_E^{(2)}$ are in (3.23) and (3.28) respectively. Let $\varpi_{E,n+1} \in \mathcal{G}[p_{E,n+1}]_{\text{prim}}$.

Lemma 3.9. Let $x \in E^\times$. The element $\Delta_E(x)$ acts on $\mathcal{Z}_{n,n,\varpi_{E,n+1}}$ by $(a, s) \mapsto (a, (-1)^{v_E(x)} s)$.

Proof. By using (3.27), we can directly check that

$$\Delta_E(\varpi_E^{i})^s_i \equiv s_i \mod h_0 .$$

Hence, by (2.33), (2.35), (2.43) and (2.49), we have

$$\Delta_E(\varpi_E^{i})^s_i \equiv (-1)^s \mod 0 + ,$$

(3.36)

Let $a = \sum_{i=0}^{\infty} a_i \varpi_E^i \in \mathcal{O}_E^\times$ with $a_i \in F_q$. We set $a^{-1} = \sum_{i=0}^{\infty} b_i \varpi_E^i$ with $b_i \in F_q$. By (3.20), (3.33) and (3.31), for any $i \in \mathbb{Z}$ and $l \geq 1$, we have

$$\Delta_E(a)^s_i \equiv \sum_{i+j+k=l} a_j b_k s_{i+k,i} \mod h_0 .$$

(3.37)

Note that $s_{1,0}/s_{2,0} \equiv 1 \mod (k/4)$. By (2.33), (2.35), (2.43), (2.49) and (3.37), the element $\Delta_E(a)$ acts on $\mathcal{Z}_{n,n,\varpi_{E,n+1}}$ trivially. Hence, the required assertion follows from (3.36). \qed

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We consider the subgroup of $G$:

$$I_E' = \{(1, d, \sigma) \in G \mid \sigma \in I_E, \; d \in \mathcal{O}_E^\times, \; a_E^0(\sigma)d = 1\}. \quad (3.38)$$

For $x \in \mathbb{F}_q^\times$, let $(\frac{x}{\mathbb{F}_q}) \in \{\pm 1\}$ denote the quadratic residue symbol.

**Lemma 3.10.** The $I_E'$ acts on $\mathbb{Z}_{n,n,\varphi_E,n+1}$ by

$$(1, d, \sigma) : \mathbb{Z}_{n,n,\varphi_E,n+1} \rightarrow \mathbb{Z}_{n,n,\varphi_E,n+1}; \; (a, s) \mapsto (a, (\frac{\tilde{d}}{\mathbb{F}_q})^s)$$

for any $(1, d, \sigma) \in I_E'$.

**Proof.** Let $P \in \mathbb{Z}_{n,n,\varphi_E,n+1}(\mathbb{C})$ and $(1, d, \sigma) \in I_E'$. We write $d^{-1} = \sum_{i=0}^{\infty} a_i \varphi^i \in \mathcal{O}_E^\times$ with $a_i \in \mathbb{F}_q^\times$. Let $\iota \in 2$. We have

$$d^*s_{t,n}(P\sigma) \equiv \sum_{i=0}^{n} a_i \sigma^{-1}(s_{t+i,n-i}(P)) \mod h_0 + . \quad (3.39)$$

By the Lubin-Tate theory and $a_E^0(\sigma)d = 1$, we have $\sigma^{-1}(\varphi_E,n-i+1) = [d]_\sigma(\varphi_E,n-i+1)$ for $0 \leq i \leq n$. Hence, by (2.39) and (3.39), we obtain

$$d^*u_{t,n}(P\sigma) \equiv \sum_{i=0}^{n} a_i \sigma^{-1}(u_{t+i,n-i}(P)) \mod 0 + . \quad (3.40)$$

By (2.43) and $\sigma \in I_E$, we acquire $d^*U_n(P\sigma) \equiv \sigma^{-1}(U_n(P)) \equiv U_n(P) \mod 0+$. By (2.49) and (3.40), we have

$$d^*b_{t,n}(P\sigma) \equiv \left(\frac{\sigma^{-1}(\vartheta_1)}{\vartheta_1}\right)^{(q-1)/2} b_{t,n}(P) \mod 0 + .$$

Hence, the required assertion follows from $\overline{\sigma^{-1}(\vartheta_1)/\vartheta_1} = \tilde{d}$. \qed

## 4 Preliminaries on étale cohomology

In this section, we collect several known facts on the cohomology in the form needed in this paper. In §4.1, we give a simple criterion whether the cohomology of the reductions of affinoids in a rigid analytic variety contributes to the cohomology of the rigid analytic variety. This criterion is a direct consequence of étale cohomology theory of adic spaces in [Hu2]. In §4.2 and §4.3, we collect classically well-known facts on the cohomology of Artin-Schreier curves and the Deligne-Lusztig curve for $GL_2(\mathbb{F}_q)$ respectively.

### 4.1 Preliminary on étale cohomology of rigid analytic varieties

We recall several general facts on étale cohomology of rigid analytic varieties from [Hu2].

Let $K$ be a non-archimedean complete discrete valued field. Assume that its residue field is separably closed field of characteristic $p > 0$. We fix a separable algebraic closure $\overline{K}$ of $K$. Let $\mathcal{C}$ denote the completion of $\overline{K}$. Let $\ell \neq p$ be a prime number. In the following, we consider $\ell$-adic étale cohomology (with compact support) of taut and separated rigid analytic varieties (cf. [Hu2] §0 and §5.6). We regard a rigid analytic variety as an adic space as in [Hu2] (1.1.11).

Let $X$ be a taut and separated rigid analytic variety over $K$. For a taut and separated morphism of rigid analytic varieties $f : X \rightarrow Y$, we write $Rf_!$ for the functor $R^+f_!$ in the
notation of \([Hu2, (5.6.10)]\). We briefly recall the definition of \(Rf\). Let \(f : X \to Y\) be a taut and separated morphism of rigid analytic varieties. Then, by \([Hu2, Proposition 0.4.9 and Corollary 5.1.12]\), there exists a commutative diagram of adic spaces

\[
\begin{array}{ccc}
X & \xrightarrow{j} & X' \\
\downarrow{f} & & \downarrow{f'} \\
Y & \xrightarrow{} & Y',
\end{array}
\]

(4.1)

where \(j\) is an open immersion and \(f'\) is partially proper (cf. \([Hu2, Definition 1.3.3 ii)\]). Note that partially proper is taut (cf. \([Hu2, Lemma 5.1.10 i)\]). Let \(Rf'\) be the right derived functor of the left exact functor \(f'_!\) (cf. \([Hu2, \S 0.4 A, \S 5.2 and \S 5.3]\)). Then, we set

\[
Rf = Rf' \circ j_!.
\]

(4.2)

This definition is independent of the choice of the compactification \(j : X \hookrightarrow X'\). Note that \(Rf\) does not correspond to the right derived functor of \(f!\) in general (cf. \([Hu2, \S 0.4 B]\)). The natural transformations \(Rf'_! \to Rf'_*\) and \(j_! \to Rj_*\) induce \(Rf! \to Rf_*\).

We set \(\Lambda_n = \mathbb{Z}/\ell^n\mathbb{Z}\) for \(n \geq 1\). For a taut and separated rigid analytic variety \(f : X \to \text{Spa}(K, \mathcal{O}_K)\), we set

\[
\begin{align*}
H^n_c(X_C, \overline{\mathbb{Q}}_\ell) &= \left(\lim_{\leftarrow n}(Rf_*\Lambda_n)_{\text{Spa}(C, \mathcal{O}_C)}\right) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell, \\
H^n_c(X_C, \overline{\mathbb{Q}}_\ell) &= \left(\lim_{\leftarrow n}(Rf_!\Lambda_n)_{\text{Spa}(C, \mathcal{O}_C)}\right) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell
\end{align*}
\]

(cf. \([Hu2, Example 2.6.2 and Corollary 5.4.8]\)). By the natural transformation \(Rf! \to Rf_*\), we have the canonical map

\[
\text{can.}: H^n_c(X_C, \overline{\mathbb{Q}}_\ell) \to H^n(X_C, \overline{\mathbb{Q}}_\ell).
\]

We consider a commutative diagram of taut and separated rigid analytic varieties

\[
\begin{array}{ccc}
W & \xrightarrow{j} & X \\
\downarrow{f_W} & & \downarrow{f} \\
\text{Spa}(K, \mathcal{O}_K) & \xrightarrow{} & Y,
\end{array}
\]

where \(j\) is an open immersion. By applying \(Rf_W\) to the adjunction map \(j_!\Lambda_n \to \Lambda_n\) and using a natural isomorphism \(Rf_W! \Lambda_n \xrightarrow{\sim} Rf_Wj_!\Lambda_n\) in \([Hu2, Theorem 5.4.3]\), we have \(Rf_W!\Lambda_n \to Rf_Wj_!\Lambda_n\). This induces the canonical map

\[
H^i_c(W_C, \overline{\mathbb{Q}}_\ell) \to H^i_c(X_C, \overline{\mathbb{Q}}_\ell).
\]

(4.3)

By applying \(Rf_*\) to the adjunction map \(\Lambda_n \to Rj_*\Lambda_n\), we have \(Rf_*\Lambda_n \to Rf_*j_!\Lambda_n\). This induces the restriction map

\[
H^i(X_C, \overline{\mathbb{Q}}_\ell) \to H^i(W_C, \overline{\mathbb{Q}}_\ell).
\]

(4.4)

**Lemma 4.1.** For each \(i\), we have the commutative diagram

\[
\begin{array}{ccc}
H^i_c(W_C, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\text{can.}} & H^i_c(X_C, \overline{\mathbb{Q}}_\ell) \\
\downarrow{\text{can.}} & & \downarrow{\text{can.}} \\
H^i(W_C, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\text{can.}} & H^i(X_C, \overline{\mathbb{Q}}_\ell).
\end{array}
\]

(4.5)
\textbf{Proof.} By [Mii2 Proposition 4.16], we have the commutative diagram
\begin{equation}
\begin{array}{ccc}
Rf_{j!} & \xrightarrow{(1)} & Rf_{j!}R_{j!}
\end{array}
\begin{array}{ccc}
Rf_{j!} & \xrightarrow{(2)} & Rf_{j!}R_{j!}
\end{array}
\begin{array}{ccc}
\simeq & & \\
Rf_{w_1!} & \xrightarrow{(3)} & Rf_{w_1!}
\end{array}
\end{equation}

where (1), (2) and (3) are induced by the natural transformations $j_! \rightarrow R_{j!}$, $Rf_{j!} \rightarrow Rf_*$ and $Rf_{w_1!} \rightarrow Rf_{w_1!}$ respectively. Note that the composite of the two adjoint maps $j_!j^* \rightarrow \text{id}$ and $\text{id} \rightarrow R_{j!}j^*$ equals the natural transformation $j_!! \rightarrow j^*_!$. We have the commutative diagram
\begin{equation}
\begin{array}{ccc}
Rf_{j!} & \xrightarrow{(1)} & Rf_{j!}R_{j!}
\end{array}
\begin{array}{ccc}
Rf_{j!} & \xrightarrow{(2)} & Rf_{j!}R_{j!}
\end{array}
\begin{array}{ccc}
\xrightarrow{(a)} & & \\
Rf_{j!} & \xrightarrow{(b)} & Rf_{j!}R_{j!}
\end{array}
\begin{array}{ccc}
\xrightarrow{(b')}& & \\
Rf_{j!} & \xrightarrow{(2')}& Rf_{j!}R_{j!}
\end{array}
\end{equation}

where $(a)$ is induced by $j_!j^* \rightarrow \text{id}$, $(b)$ and $(b')$ are induced by $\text{id} \rightarrow R_{j!}j^*$, and $(2')$ is induced by $Rf_{j!} \rightarrow Rf_*$. By considering the stalk at $\text{Spa}(C, \mathcal{O}_C)$ of the diagrams (4.6) and (4.7), we obtain the commutative diagram
\begin{equation}
\begin{array}{ccc}
H^i(\mathcal{W}_C, \Lambda) & \longrightarrow & H^i(\mathcal{X}_C, \Lambda)
\end{array}
\begin{array}{ccc}
\xrightarrow{\text{can.}} & & \\
H^i(\mathcal{W}_C, \Lambda) & \longrightarrow & H^i(\mathcal{X}_C, \Lambda)
\end{array}
\end{equation}

By taking $\lim_{\to \Lambda_n}$ of this diagram and applying $(-) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, we obtain the claim. \hfill $\square$

\subsection{4.1.1 Formal nearby cycle functor}

Let $K$ be a non-archimedean valued field of height one. Let $\widehat{K}$ denote the completion of $K$. Let $X$ be a formal scheme which is locally finitely presented over $S = \text{Spf} \mathcal{O}_K$. We set $\mathcal{X} = (X, \mathcal{O}_X/p_K \mathcal{O}_X)$. Then, we have the morphism of étale sites
\begin{equation}
\lambda_X : (\mathcal{X}_{\text{rig}})_{\text{ét}} \rightarrow X_{\text{ét}} \simeq (X_{\text{s}})_{\text{ét}},
\end{equation}

which is given in [Hn2] (0.7.1) and Lemma 3.5.1.

\textbf{Definition 4.2.} We write $R\Psi_X^{\text{ad}}$ for $R\lambda_X^*$, which we call the formal nearby cycle functor.

Let $K$ be as in the beginning of 4.1.1. Let $W = \text{Sp} A$ be an affinoid variety over $K$. We consider the formal scheme $\mathcal{W} = \text{Spf} A^\circ \rightarrow S$. We regard it as an object in $\mathcal{O}_K-\mathcal{F}_{\text{sch}}$, because $K$ has a discrete valuation (cf. [BLR Introduction]).

We assume that the formal scheme $\mathcal{W}$ is isomorphic to the formal completion of a scheme $W$, which is separated and of finite type over $S = \text{Spec} \mathcal{O}_K$, along the special fiber $W_s$. Let $W_{\overline{S}}$ denote the base change of $W$ to $\overline{S} = \text{Spec} \mathcal{O}_{\overline{K}}$. Let $W_{\overline{h}}$ denote the generic fiber of $W_{\overline{S}}$. Let $\mathcal{W}_{\overline{S}}$ denote the completion of $W_{\overline{S}}$ along the special fiber $W_s$. Then we have $\mathcal{W} = \mathcal{W}_{\overline{S}}$ and $\mathcal{W}_C = \mathcal{W}_{\overline{S}}^{\text{rig}}$.

Let $W_{\overline{S}} \subset W_{\overline{C}}$ be a compactification of $W_{\overline{S}}$ over $\overline{S}$. Let $\mathcal{W}_{\overline{S}}$ be the formal completion of $W_{\overline{S}}$ along the special fiber. Then, we have the commutative diagram of formal schemes
\begin{equation}
\begin{array}{ccc}
\mathcal{W}_{\overline{S}} & \xrightarrow{j} & \mathcal{W}_{\overline{C}}
\end{array}
\begin{array}{ccc}
\xrightarrow{f} & & \\
\text{Spf} \mathcal{O}_C, & \xrightarrow{f^c} & \mathcal{W}_{\overline{S}}
\end{array}
\end{equation}
where $j$ is an open immersion (cf. [Hu2, the proof of Corollary 0.7.9] and [Mi2, Example 4.22 ii]). We write $W^c_C$ for the rigid analytic variety $(W^c_S)^{\text{rig}}$ over $C$. The diagram (4.8) induces the commutative diagram of rigid analytic varieties

$$
\begin{array}{ccc}
W^c_C & \xrightarrow{j^c} & W^c_C \\
\downarrow f^c & & \downarrow f^c \\
\text{Spa}(C, \mathcal{O}_C), & & \\
\end{array}
$$

where $f^c$ is proper. By (4.8), we have the commutative diagram of schemes

$$
\begin{array}{ccc}
\mathcal{W}_s & \xrightarrow{j^s} & \mathcal{W}_s^c \\
\downarrow f^s & & \downarrow f^c \\
\text{Spec } \mathbb{F}. & & \\
\end{array}
$$

We have

$$H^i(W_s, R\Psi_{W^c_S}^\text{ad}(\Lambda_n)) = H^i(W^c_C, \Lambda_n). \tag{4.9}$$

We recall a natural isomorphism

$$\xi: H^i_c(W_s, R\Psi_{W^c_S}^\text{ad}(\Lambda_n)) \xrightarrow{\sim} H^i_c(W^c_C, \Lambda_n).$$

We define $\xi$ to be the composite of the following isomorphisms:

$$H^i_c(W_s, R\Psi_{W^c_S}^\text{ad}(\Lambda_n)) = H^i(W^c_s, j^s_! R\Psi_{W^c_S}^\text{ad}(\Lambda_n)) \xrightarrow{G} H^i(W^c_s, j^c_! \Lambda_n) = H^i(W^c_C, j^c_! \Lambda_n) = H^i_c(W^c_C, \Lambda_n),$$

where $G$ is induced by the natural isomorphism $j^s_! R\Psi_{W^c_S}^\text{ad}(\Lambda_n) \xrightarrow{\sim} R\Psi_{W^c_S}^\text{ad}(j^c_! \Lambda_n)$ in [Hu2, Corollary 0.7.5 and Corollary 3.5.11], and the last equality follows from properness of $f^c$ and (4.2). The isomorphism $\xi$ is independent of the choice of the compactification $j: W^c_S \hookrightarrow W_S^c$ by [Mi2, Lemma 4.25 i)]. In [Mi2, Definition 4.24], the map $\xi$ is defined for a more general formal scheme.

**Lemma 4.3.** We have the commutative diagram

$$
\begin{array}{ccc}
H^i_c(W_s, R\Psi_{W^c_S}^\text{ad}(\Lambda_n)) & \xrightarrow{\sim} & H^i_c(W^c_C, \Lambda_n) \\
\downarrow \text{can.} & & \downarrow \text{can.} \\
H^i(W_s, R\Psi_{W^c_S}^\text{ad}(\Lambda_n)) & \xrightarrow{(4.9)} & H^i(W^c_C, \Lambda_n).
\end{array}
$$

**Proof.** We have the commutative diagram of étale sites

$$
\begin{array}{ccc}
(W^c_C)_{\text{ét}} & \xrightarrow{j^c} & (W^c_C)_{\text{ét}} \\
\downarrow \lambda_{W^c} & & \downarrow \lambda_{W^c} \\
(W^c_S)_{\text{ét}} & \xrightarrow{f^c} & (\text{Spec } \mathbb{F})_{\text{ét}}.
\end{array}
$$

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Hence, we have the commutative diagram

\[
\begin{array}{ccc}
  j_{s!}R\Psi_{W_{\mathfrak{p}}}^{ad}(\Lambda_n) & \xrightarrow{\sim} & R\Psi_{W_{\mathfrak{p}}}^{ad}(j_{\bar{\eta}!}\Lambda_n) \\
  \downarrow & & \downarrow \\
  Rj_{ss}R\Psi_{W_{\mathfrak{p}}}^{ad}(\Lambda_n) & \xrightarrow{\sim} & R\Psi_{W_{\mathfrak{p}}}^{ad}(Rj_{\bar{\eta}!}\Lambda_n),
\end{array}
\]

where the above horizontal isomorphism follows from \([Hu2\text{ Corollary }0.7.5]\), and the left vertical and the right vertical morphisms are induced by \(j_{s!} \rightarrow Rj_{ss}\) and \(j_{\bar{\eta}!} \rightarrow Rj_{\bar{\eta}!}\) respectively. By applying \(Rf_{s!}\) to the diagram (4.11) and using (4.10), we obtain the commutative diagram

\[
\begin{array}{ccc}
  Rf_{s!}R\Psi_{W_{\mathfrak{p}}}^{ad}(\Lambda_n) & \xrightarrow{\sim} & Rf_{ss}^cR\Psi_{W_{\mathfrak{p}}}^{ad}(j_{\bar{\eta}!}\Lambda_n) \\
  \downarrow & & \downarrow \\
  Rf_{ss}R\Psi_{W_{\mathfrak{p}}}^{ad}(\Lambda_n) & \xrightarrow{\sim} & Rf_{ss}^cR\Psi_{W_{\mathfrak{p}}}^{ad}(Rj_{\bar{\eta}!}\Lambda_n) \\
  \downarrow & & \downarrow \\
  R\lambda_{C_s}Rj_{\bar{\eta}!}\Lambda_n & = & R\lambda_{C_s}Rf_{\bar{\eta}!}\Lambda_n \\
  \downarrow & & \downarrow \\
  R\lambda_{C_s}Rj_{\bar{\eta}!}\Lambda_n & = & R\lambda_{C_s}Rf_{\bar{\eta}!}\Lambda_n.
\end{array}
\]

Hence, the required assertion follows.

Let \(R\Psi_W(\Lambda_n)\) denote the nearby cycle complex of the constant sheaf \(\Lambda_n\) on the scheme \(W\) over \(S\) in \([Del2, \S 2.1]\). By \([Hu2\ Theorem 0.7.7 or Theorem 3.5.13]\), we have the natural isomorphism

\[
R\Psi_W(\Lambda_n) \xrightarrow{\sim} R\Psi_{W_{\mathfrak{p}}}^{ad}(\Lambda_n).
\]

Hence, we have natural isomorphisms induced by (4.9) and \(\xi\):

\[
H^i(W_s, R\Psi_W(\Lambda_n)) \simeq H^i(W_C, \Lambda_n),
\]

\[
H^i_c(W_s, R\Psi_W(\Lambda_n)) \simeq H^i_c(W_C, \Lambda_n)
\]

as in \([Hu2\ Corollary 0.7.9, Corollary 3.5.14 and Theorem 5.7.6]\). Note that the latter isomorphism is generalized in \([Hu3, Lemma 2.13]\).

**Corollary 4.4.** We have the commutative diagram

\[
\begin{array}{ccc}
  H^i_c(W_s, R\Psi_W(\Lambda_n)) & \xrightarrow{\sim} & H^i_c(W_C, \Lambda_n) \\
  \downarrow \text{can.} & & \downarrow \text{can.} \\
  H^i(W_s, R\Psi_W(\Lambda_n)) & \xrightarrow{\sim} & H^i(W_C, \Lambda_n).
\end{array}
\]

**Proof.** The required assertion follows from Lemma 4.3 and (4.13). \(\Box\)

### 4.1.2 Key lemma

We state a key lemma to relate the cohomology of the reductions of the affinoids in the Lubin-Tate curve in \([2]\) to the cohomology of the Lubin-Tate curve. We consider the following situation. Let \(K\) be as in the beginning of \([4.1]\). Let \(X\) be a taut and separated rigid analytic variety over \(K\). Let \(W = \text{Sp } A \subset X\) be an affinoid subdomain. Assume that

\[
\text{can.}
\]

\[
\text{can.}
\]
1. the morphism \( \text{Spf} A^o \to \text{Spf} \mathcal{O}_K \) is smoothly algebraizable, and

2. \( \overline{W} = \text{Spec} (A^o \otimes_{\mathcal{O}_K} \mathbb{F}) \).

By the first assumption, we take a scheme \( W \) which is separated, smooth and of finite type over \( S \) and whose formal completion along \( W_s \) is isomorphic to \( W = \text{Spf} A^o \). Since \( W \to S \) is smooth, the natural morphism \( \Lambda \to \mathcal{R} \Psi W(\Lambda) \) is an isomorphism by the smooth base change theorem as in [Del2, Reformation 2.1.5]. By the second assumption, we have \( \overline{W} \simeq W_s \). Hence, by Corollary 4.4, we have the commutative diagram

\[
\begin{array}{ccc}
H^i_c(W, \mathbb{Q}_\ell) & \xrightarrow{\text{can.}} & H^i_c(W_C, \overline{\mathbb{Q}}_\ell) \\
\downarrow & & \downarrow \\
H^i(W, \mathbb{Q}_\ell) & \xrightarrow{\text{can.}} & H^i(W_C, \overline{\mathbb{Q}}_\ell).
\end{array}
\]

(4.14)

**Lemma 4.5.** Let the notation and the assumption be as above.

1. We have the following commutative diagram:

\[
\begin{array}{ccc}
H^i_c(W, \mathbb{Q}_\ell) & \xrightarrow{\text{can.}} & H^i_c(W_C, \overline{\mathbb{Q}}_\ell) \\
\downarrow & & \downarrow \\
H^i(W, \mathbb{Q}_\ell) & \xrightarrow{\text{can.}} & H^i(W_C, \overline{\mathbb{Q}}_\ell).
\end{array}
\]

(4.15)

Let

\[
H^i_c(W, \mathbb{Q}_\ell) \to H^i_c(X_C, \overline{\mathbb{Q}}_\ell)
\]

(4.16)

be the composite of the maps in the above horizontal maps in (4.15).

2. Assume that the canonical map \( H^i_c(W, \mathbb{Q}_\ell) \to H^i(W, \mathbb{Q}_\ell) \) is injective on a subspace \( W \subset H^i_c(W, \mathbb{Q}_\ell) \). Then, the restriction map \( W \to H^i_c(X_C, \overline{\mathbb{Q}}_\ell) \) of the canonical map (4.16) to \( W \) is an injection.

**Proof.** A quasi-compact and quasi-separated rigid analytic variety is taut by [Hu2, Lemma 5.1.3 iv)]. Hence, any affinoid rigid analytic variety is taut. The first assertion follows from Lemma 4.1 and the commutative diagram (4.14). The second assertion immediately follows from (4.15). \( \square \)

**Remark 4.6.** Let the assumption be as in Lemma 4.5. The composite of the map \( W \to H^i_c(X_C, \overline{\mathbb{Q}}_\ell) \) in Lemma 4.5 and the canonical map \( H^i_c(X_C, \overline{\mathbb{Q}}_\ell) \to H^i(X_C, \overline{\mathbb{Q}}_\ell) \) is also injective.

**Remark 4.7.** One could probably rewrite results in this subsection in Berkovich’s language in [Be] and [Be2] through comparison theorems in [Hu2, §8.3].

**Remark 4.8.** An open unit polydisk in rigid geometry is taut and separated. Note that a finite morphism is taut and separated, and tautness and separatedness are stable under composition. Since a Lubin-Tate space is a finite étale covering of an open unit polydisk, it is taut and separated.

**Remark 4.9.** We note that the reductions of formal models of affinoids in the Lubin-Tate perfectoid space in [IT3] and [IT4] also satisfy the property in Lemma 4.5.2. Recently, in [To], Tokimoto generalizes [IT3]. The reductions of formal models of affinoids in the Lubin-Tate perfectoid space in [To] also satisfy it. In a subsequent paper, we will study corresponding affinoids in the Lubin-Tate space and prove the NALT for ramified essentially tame representations in some case.
4.2 Review on $\ell$-adic cohomology of Artin-Schreier curves

Let $p$ be a prime number, and let $q$ be a power of $p$. For a finite abelian group $A$, we write $A^\vee$ for $\text{Hom}_\mathbb{Z}(A, \mathbb{Q}_\ell^\vee)$.

Let $A^1$ be an affine line over $F_q$. For $\psi \in F_q^\vee$, let $\mathcal{L}_\psi$ denote the smooth $\mathbb{Q}_\ell$-sheaf of rank one on $A^1$ defined by the Artin-Schreier covering $a^q = a - x$ and $\psi$. Note that $\mathcal{L}_\psi$ is equal to $\mathfrak{S}(\psi)$ in the notation of [Del, Proposition 4.3 in Sommes trig.]. For a variety $Y$ over $\mathbb{F}_q$ and a function $f: Y \to A^1$, let $\mathcal{L}_\psi(f)$ denote the pull-back $f^* \mathcal{L}_\psi$ to $Y$.

We set $\mathbb{G}_m = A^1 \setminus \{0\}$. Let $n$ be a positive integer which is prime to $p$. Let $m$ be a positive integer such that $\mu_n(F) \subset F_q^\times$. We simply write $\mu_n$ for $\mu_n(F)$. For $c_0 \in F_q^\times$ and $\chi \in \mu_q^\times$, let $\mathcal{K}_{\chi,c_0}$ be the smooth $\mathbb{Q}_\ell$-sheaf of rank one on $\mathbb{G}_m$ defined by the Kummer torsor $K_{n,c_0} = \mathbb{G}_m \to \mathbb{G}_m; y \mapsto c_0y^n$ and $\chi$. Note that $\mathcal{K}_{\chi,c_0}$ equals $\chi^{-1}(K_{n,c_0})$ in the notation of [Del, 1.2 in Sommes trig.].

We consider the Gauss sum
\[
G_{m,n,c_0}(\chi, \psi) = - \sum_{x \in F_q} \chi\left(\frac{x}{c_0}\right)\mu_n^{-1} \psi\left(\text{Tr}_{F_q/m}(x)\right).
\] (4.17)

**Lemma 4.10.** 1. We have $H_c^1(A^1, \mathcal{L}_\psi(c_0y^n)) = 0$ except for $i = 1$, and an isomorphism
\[
H_c^1(A^1, \mathcal{L}_\psi(c_0y^n)) \simeq \bigoplus_{\chi \in \mu_n^\times \setminus \{1\}} H_c^1(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{K}_{\chi,c_0}).
\] (4.18)

Furthermore, we have $\dim_{\mathbb{F}_q} H_c^1(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{K}_{\chi,c_0}) = 1$. The geometric Frobenius element over $\mathbb{F}_q$ acts on $H_c^1(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{K}_{\chi,c_0})$ as multiplication by $G_{m,n,c_0}(\chi, \psi)$.

2. The canonical map $H_c^1(A^1, \mathcal{L}_\psi(c_0y^n)) \to H^1(A^1, \mathcal{L}_\psi(c_0y^n))$ is an isomorphism.

**Proof.** By [Del, Remarques 1.8 b), c) in Sommes trig.], we have
\[
H_c^i(A^1, \mathcal{L}_\psi(c_0y^n)) = 0
\]
for $i = 0, 2$. We prove (4.18). Let $f: A^1 \to A^1$ be the morphism defined by $y \mapsto c_0y^n$. Since $f$ is étale over $\mathbb{G}_m$, by the projection formula, we have a decomposition
\[
f_\ast \mathcal{L}_\psi(c_0y^n) \simeq \mathcal{L}_\psi \otimes f_\ast \mathcal{O}_\ell \simeq \bigoplus_{\chi \in \mu_n^\times \setminus \{1\}} \mathcal{L}_\psi \otimes \mathcal{K}_{\chi,c_0} \quad \text{on } \mathbb{G}_m.
\] (4.19)

We have short exact sequences
\[
0 \to H_c^0(\{0\} \setminus \mathbb{G}_m, \mathcal{O}_\ell) \simeq \mathcal{O}_\ell \to H_c^1(\mathbb{G}_m, \mathcal{L}_\psi(c_0y^n)) \to H_c^1(A^1, \mathcal{L}_\psi(c_0y^n)) \to 0,
\]
\[
0 \to H_c^0(\{0\} \setminus \mathbb{G}_m, \mathcal{O}_\ell) \simeq \mathcal{O}_\ell \to H_c^1(\mathbb{G}_m, \mathcal{L}_\psi) \to H_c^1(A^1, \mathcal{L}_\psi) = 0 \to 0.
\] (4.20)

Hence, the required assertion for the cohomology with compact support follows from (4.19) and (4.20). We prove the second and the third assertions in 1. These follow from the Grothendieck-Ogg-Shafarevich formula in [SGA 5, Théorème 7.1 in Exposé X] and the Grothendieck trace formula respectively (cf. [Del, Proposition 4.3 in Sommes trig.]).

We prove 2. In the same way as above, we have an isomorphism
\[
H^1(A^1, \mathcal{L}_\psi(c_0y^n)) \simeq \bigoplus_{\chi \in \mu_n^\times \setminus \{1\}} H^1(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{K}_{\chi,c_0}),
\]
and the commutative diagram
\[
\begin{array}{ccc}
H^1_c(A_F^1, \mathcal{L}_\psi(c_0y^n)) & \overset{\sim}{\longrightarrow} & \bigoplus_{\chi \in \mu_n^\vee \setminus \{1\}} H^1_c(G_m, \mathcal{L}_\psi \otimes \mathcal{K}_{X,c_0}) \\
\text{can.} & & \text{can.} \\
H^1(A_F^1, \mathcal{L}_\psi(c_0y^n)) & \overset{\sim}{\longrightarrow} & \bigoplus_{\chi \in \mu_n^\vee \setminus \{1\}} H^1(G_m, \mathcal{L}_\psi \otimes \mathcal{K}_{X,c_0}).
\end{array}
\] (4.21)

By [Del, Proposition 4.3 in Sommes trig.], for any $\chi \in \mu_n^\vee \setminus \{1\}$, the canonical map
\[
H^1_c(G_m, \mathcal{L}_\psi \otimes \mathcal{K}_{X,c_0}) \to H^1(G_m, \mathcal{L}_\psi \otimes \mathcal{K}_{X,c_0})
\] (4.22)
is an isomorphism. Hence, by (4.21) and (4.22), the canonical map $H^1_c(A_F^1, \mathcal{L}_\psi(c_0y^n)) \to H^1(A_F^1, \mathcal{L}_\psi(c_0y^n))$ is an isomorphism. Therefore, the required assertion 2 follows.

**Corollary 4.11.** Let $X_{m,n,c_0}$ be the affine smooth curve defined by $a^q - a = c_0y^n$ over $\mathbb{F}_q^m$.

1. We have an isomorphism
\[
H^1_c(X_{m,n,c_0,F}, \overline{\mathbb{Q}_q}) \simeq \bigoplus_{\psi \in \mathbb{F}_q^\vee \setminus \{1\}} H^1_c(G_m, \mathcal{L}_\psi \otimes \mathcal{K}_{X,c_0}).
\]
Furthermore, we have $\dim_{\mathbb{Q}_q} H^1_c(X_{m,n,c_0,F}, \overline{\mathbb{Q}_q}) = (q - 1)(n - 1)$.

2. The canonical map $H^1_c(X_{m,n,c_0,F}, \overline{\mathbb{Q}_q}) \to H^1(X_{m,n,c_0,F}, \overline{\mathbb{Q}_q})$ is an isomorphism.

**Proof.** We have an isomorphism
\[
H^1_c(X_{m,n,c_0,F}, \overline{\mathbb{Q}_q}) \simeq \bigoplus_{\psi \in \mathbb{F}_q^\vee \setminus \{1\}} H^1_c(A_F^1, \mathcal{L}_\psi(c_0y^n)).
\] (4.23)

Hence, the required assertions follow from Lemma 4.10.

We consider the case $(m, n, c_0) = (1, 2, 1)$ in the notation of Corollary 4.11.

**Lemma 4.12.** We consider the automorphism $i: X_{1,2,1} \to X_{1,2,1}$ defined by $(a, y) \mapsto (a, -y)$. Then, $i$ acts on $H^1_c(X_{1,2,1,F}, \overline{\mathbb{Q}_q})$ as scalar multiplication by $-1$.

**Proof.** Let $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$. By Lemma 4.10, we have
\[
\dim_{\mathbb{Q}_q} H^1_c(A_F^1, \mathcal{L}_\psi(y^2)) = 1.
\]
Hence, by the Grothendieck trace formula, $i$ acts on $H^1_c(A_F^1, \mathcal{L}_\psi(y^2))$ as scalar multiplication by $-1$. Therefore, the required assertion follows from (4.23) for $(m, n, c_0) = (1, 2, 1)$.

It is easy to directly calculate the Gauss sum in some special cases.

**Lemma 4.13.** Let $\zeta_1 \in \mathbb{F}_q^\ast$ satisfying $\zeta_1^{-1} = -1$. Then, we have $G_{2,q+1,\zeta_1}(\chi, \psi) = -q$.

**Proof.** We have equalities
\[
G_{2,q+1,\zeta_1}(\chi, \psi) = - \sum_{x^{q-1} = -1} \chi((x/\zeta_1)^{q-1}) - \sum_{\mu \in \mathbb{F}_q^\ast} \sum_{x^{q} + x = \mu} \chi((x/\zeta_1)^{q-1}) \psi(\mu)
\]
\[
= -(q - 1) - \sum_{\mu \in \mathbb{F}_q^\ast} \sum_{y^{q-1} = 1} \chi(-y^{q-1}) \psi(\mu),
\] (4.24)
where we change a variable $y = x/\mu$ and use $\zeta_q^{-1} = -1$ at the second equality. The map $f: \mathbb{F}_{q^2}^{\times} \to \mu_{q+1}; \ x \mapsto x^{q-1}$ is injective on the subset $S = \{y \in \mathbb{F}_{q^2}^{\times} \mid y^q + y = 1\}$. Since $S$ consists of $q$ elements satisfying $f(S) \cap \{-1\} = 0$, we obtain $f(S) = \mu_{q+1} \setminus \{-1\}$. Hence, we have $\sum_{\eta \in S} \chi(-\eta^{q-1}) = -1$. Since we have $\sum_{\mu \in \mathbb{F}_q} \psi(\mu) = -1$, we obtain the required assertion by \cite{BH, Lemma 2 (1) in §22.1}. \hfill \Box

We consider the affine curve $X_0$ defined by $X^q - X = Y^{q(q+1)} - Y^{q+1}$ over $\mathbb{F}_{q^2}$. Let

$$Q = \left\{ g(\alpha, \beta, \gamma) = \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha^q & \beta^q & \gamma \end{pmatrix} \in \text{GL}_3(\mathbb{F}_{q^2}) \right\},$$

which is a subgroup of $\text{GL}_3(\mathbb{F}_{q^2})$. This group appears also in \cite{We} the proof of Proposition 4.3.4. We identify the center $Z = \{g(1, 0, \gamma) \mid \gamma \in \mathbb{F}_{q^2}\} \subset Q$ with $\mathbb{F}_{q^2}$ by $g(1, 0, \gamma) \mapsto \gamma$. Let $Q$ act on $X_0$ by

$$g(\alpha, \beta, \gamma): X_0 \to X_0; \ (X, Y) \mapsto (X + \beta Y + Y^{q-1} + \beta, \frac{\gamma}{\alpha}).$$ \hfill (4.25)

We regard $\mathbb{F}_{q^2}^\times$ as a normal subgroup of $Q$ by $\alpha \mapsto g(\alpha, 0, 0)$ for $\alpha \in \mathbb{F}_{q^2}^\times$. By \cite{BH, §22.2}, the group $Q$ acts on $X_0$ factoring through $Q \to Q/\mathbb{F}_{q^2}^\times$. We regard $\mathbb{F}_{q^2}^\times$ as a subset of $\mathbb{F}_{q^2}^\times$ via $\text{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}$. We simply write $\mathcal{C}$ for $\mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times$. Let $Q_0$ be the subgroup of $Q$ consisting of all elements of the form $g(1, \beta, \gamma)$. The subgroup $\mathcal{Z}$ is the center of $Q_0$, and the quotient $V = Q_0/\mathcal{Z}$ is isomorphic to $\mathbb{F}_{q^2}$. Therefore, $Q_0$ is a finite Heisenberg group. The following lemma is a well-known fact in a representation theory of finite groups.

**Lemma 4.14.** Let $\psi \in \mathcal{C}$. There exists a unique representation $\tau_\psi^0$ of $Q$ such that

$$\tau_\psi^0|_{\mathbb{F}_{q^2}^\times} = 1^\oplus q, \quad \tau_\psi^0|_{\mathcal{Z}} \simeq \psi^{\oplus q}, \quad \text{Tr} \tau_\psi^0(g(\alpha, 0, 0)) = -1 \text{ for any } \alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q,$$

where 1 denotes the trivial representation of $Q$. Furthermore, $\tau_\psi^0$ is irreducible.

**Proof.** Let $\mu_{q+1}$ act on $Q_0$ by

$$\zeta: g(1, \beta, \gamma) \mapsto g(\tilde{\zeta}, 0, 0)\zeta^{-1} g(1, \beta, \gamma) g(\tilde{\zeta}, 0, 0) \quad \text{for } \zeta \in \mu_{q+1} \text{ and } g(1, \beta, \gamma) \in Q_0,$$

where $\tilde{\zeta} \in \mathbb{F}_{q^2}^\times$ is an element such that $\tilde{\zeta}^{q-1} = \zeta$. The center $\mathcal{Z}$ is fixed by this action. For each non-trivial element $a \in \mu_{q+1}$, then $a$ has only the trivial fixed point in $V$. We have an isomorphism $Q/\mathbb{F}_{q^2}^\times \simeq \mu_{q+1} \setminus Q_0$. Let $\psi \in \mathcal{C}$. By taking $(\mu_{q+1}, Q_0/\ker \psi)$ as $(A, G)$ in the notation of \cite{BH, §22.1} and applying \cite{BH} Lemma in §22.2 to this situation, we obtain the $\mu_{q+1} \ltimes Q_0/\ker \psi$-representation $\eta_1$ in the notation of \cite{BH} Lemma in §22.2. The inflation of $\eta_1$ to $Q$ by the composite

$$Q \to Q/\mathbb{F}_{q^2}^\times \simeq \mu_{q+1} \ltimes Q_0 \to \mu_{q+1} \ltimes (Q_0/\ker \psi)$$

satisfies \cite{BH, §22.2} by \cite{BH} Lemma 2 (1) in §16.4. The uniqueness and irreducibility of $\tau_\psi^0$ follows again from \cite{BH} Lemma 2 in §16.4. \hfill \Box

**Remark 4.15.** The principle in \cite{BH} Lemma 2 in §16.4 and \cite{BH} Lemma in §22.2 plays an important role in the theory of types for $\text{GL}(2)$ (cf. \cite{BH} Proposition 19.4 and §22.4 or §25.1).
Lemma 4.16. 1. We have an isomorphism
\[ H_1^c(X_{0,F}, \overline{Q}_\ell) \cong \bigoplus_{\psi \in \mathcal{C}} \tau^0_\psi \]  
(4.27)
as $Q$-representations. Furthermore, we have $\dim_{\overline{Q}_\ell} H_1^c(X_{0,F}, \overline{Q}_\ell) = q^2(q - 1)$.
2. The geometric Frobenius element over $\mathbb{F}_{q^2}$ acts on $H_1^c(X_{0,F}, \overline{Q}_\ell)$ as scalar multiplication by $-q$.
3. The canonical map $H_1^c(X_{0,F}, \overline{Q}_\ell) \to H^1(X_{0,F}, \overline{Q}_\ell)$ is an isomorphism.

Proof. We prove the first assertion. The curve $X_0$ is isomorphic to $\bigsqcup_{\mu \in \mathbb{F}_q} Y_{\mu}$ over $\mathbb{F}_{q^2}$, where $Y_{\mu}$ is defined by $X^q + X = Y^{q+1} + \mu$. Let $\zeta, \xi \in \mathbb{F}_{q^2}^*$ be elements such that $\zeta^{q-1} = -1$ and $\xi^{q} = \xi = \mu$ respectively. By setting $a = -\zeta_1(X - \xi)$, the curve $Y_0$ is defined by $a^q - a = \zeta_1Y^{q+1}$. The stabilizer of $Y_0$ in $Z \cong \mathbb{F}_{q^2}$ equals $\zeta_1\mathbb{F}_q$. Hence, we have an isomorphism
\[ H_1^c(X_{0,F}, \overline{Q}_\ell) \cong \mathrm{Ind}_{\zeta_1\mathbb{F}_q}^\mathbb{F}_{q^2} H_1^c(Y_0,F, \overline{Q}_\ell) \]  
(4.28)
as $\mathbb{F}_{q^2}$-representations. By Corollary 4.11, we have an isomorphism
\[ H_1^c(Y_0,F, \overline{Q}_\ell) \cong \bigoplus_{\psi \in (\zeta_1\mathbb{F}_q)^\circ \setminus \{1\}} \psi^{\mathbb{F}_q} \]
as $\zeta_1\mathbb{F}_q$-representations. For $\psi \in \mathbb{F}_{q^2}^*$, we have $\psi|_{\zeta_1\mathbb{F}_q} \neq 1 \iff \psi \in \mathcal{C}$. Hence, we have an isomorphism $H_1^c(X_{0,F}, \overline{Q}_\ell) \cong \bigoplus_{\psi \in \mathcal{C}} \psi^{\mathbb{F}_q}$ as $\mathbb{F}_{q^2}$-representations. Let $H_1^c(X_{0,F}, \overline{Q}_\ell)_{\psi}$ denote the $\psi$-part, on which $\mathbb{F}_{q^2}$ acts by $\psi$. We consider the subgroup $A = \mathbb{F}^\times_{q^2}/\mathbb{F}^\times_q \subset Q/\mathbb{F}^\times_q$. By Corollary 4.11, as $A$-representations, we have an isomorphism $H_1^c(X_{0,F}, \overline{Q}_\ell)_{\psi} \cong \bigoplus_{\chi \in A^\circ \setminus \{1\}} \chi$. Since $Z$ is the center of $Q$, we can regard $H_1^c(X_{0,F}, \overline{Q}_\ell)_{\psi}$ as a $Q$-representation. The $Q$-representation $H_1^c(X_{0,F}, \overline{Q}_\ell)_{\psi}$ satisfies (4.26). Hence, this is isomorphic to $\tau^0_\psi$. Therefore, (4.27) follows. The latter assertion follows from $\dim_{\overline{Q}_\ell} \tau^0_\psi = q$ and $|\mathcal{C}| = q(q - 1)$.

The second assertion follows from Lemma 4.10, Corollary 4.11, Lemma 4.13 and (4.28). The third assertion follows from Corollary 4.11.2. 

Remark 4.17. A similar analysis to the one in this subsection is found in [We3, §7] or [IT, §7.1].

4.3 Review on $\ell$-adic cohomology of Deligne-Lusztig curve for GL$2(\mathbb{F}_q)$

We keep the same notation in §4.2. We take an element $\zeta_1 \in \mathbb{F}^\times_q$ such that $\zeta_1^{q-1} = -1$. Let $W$ be the affine smooth curve over $\mathbb{F}_{q^2}$ defined by $S^qT - ST^q = \zeta_1$, which is called the Drinfeld curve (cf. [DL, p.117]). The isomorphism class of $W$ over $\mathbb{F}_{q^2}$ does not depend on the choice of $\zeta_1$. Let $\overline{W}$ be the smooth compactification of $W$, which is defined by $X^qY - XY^q = \zeta_1Z^{q+1}$ in $\mathbb{P}_{\mathbb{F}_{q^2}}^2$. We consider the open immersion $j : W \hookrightarrow \overline{W}$; $(S, T) \mapsto (S : T : 1)$. Let $\mu_{q+1}$ act on $\overline{W}$ by $[X : Y : Z] \mapsto [\xi X : \xi Y : Z]$ for $\xi \in \mu_{q+1}$. The open subscheme $W$ is stable under this $\mu_{q+1}$-action. The closed subscheme $D_W = \overline{W} \setminus W$ with reduced scheme structure consists of $q + 1$ closed points, and it is fixed by the action of $\mu_{q+1}$. Let $1$ be the trivial character of $\mu_{q+1}$ valued in $\overline{Q}_\ell$. We have a $\mu_{q+1}$-equivariant short exact sequence
\[ 0 \to 1^{\mathbb{F}_q} \to H_1^c(W_F, \overline{Q}_\ell) \to H^1(\overline{W}_F, \overline{Q}_\ell) \to 0. \]
By this and the Riemann-Hurwitz formula, we have
\[
\begin{align*}
\dim_{\mathbb{Q}_\ell}H^1(W, \mathcal{T}_\ell) &= q(q-1), \\
\dim_{\mathbb{Q}_\ell}H^1_c(W, \mathcal{T}_\ell) &= q^2.
\end{align*}
\] (4.29)

Let \( i: D_W \hookrightarrow W \) be the closed immersion. By the distinguished triangle \( i_* R^1 j_! \mathcal{T}_\ell \to j_! \mathcal{T}_\ell \to R j_* \mathcal{T}_\ell \to \) on \( W \), we have a \( \mu_{q+1} \)-equivariant long exact sequence
\[
0 \to 1 \cong H^0(W, \mathcal{T}_\ell) \to H^1_{D_W, \mathcal{T}}(W, j_! \mathcal{T}_\ell) \to H^1_c(W, \mathcal{T}_\ell) \\
\to H^1(W, \mathcal{T}_\ell) \to H^2_{D_W, \mathcal{T}}(W, j_! \mathcal{T}_\ell) \to H^2_c(W, \mathcal{T}_\ell) \cong 1(-1) \to 0.
\] (4.30)

By the distinguished triangle \( j_! \mathcal{T}_\ell \to \mathcal{T}_\ell \to i_* \mathcal{T}_\ell \) on \( W \), we have a \( \mu_{q+1} \)-equivariant long exact sequence
\[
0 \to H^0(D_{W, \mathcal{T}}) \to H^1_{D_W, \mathcal{T}}(W, j_! \mathcal{T}_\ell) \to H^1_{D_W, \mathcal{T}}(W, \mathcal{T}_\ell) \\
\to H^1(D_{W, \mathcal{T}}) = 0 \to H^2_{D_W, \mathcal{T}}(W, j_! \mathcal{T}_\ell) \to H^2_{D_W, \mathcal{T}}(W, \mathcal{T}_\ell) \to 0.
\]

Since we have \( H^1_{D_W, \mathcal{T}}(W, \mathcal{T}_\ell) = 0 \) by the purity theorem, we obtain isomorphisms
\[
\begin{align*}
H^1_{D_W, \mathcal{T}}(W, j_! \mathcal{T}_\ell) &\cong H^0(D_{W, \mathcal{T}}) \cong 1^{\oplus (q+1)}, \\
H^2_{D_W, \mathcal{T}}(W, j_! \mathcal{T}_\ell) &\cong H^2_{D_W, \mathcal{T}}(W, \mathcal{T}_\ell) \cong 1(-1)^{\oplus (q+1)}
\end{align*}
\] (4.31)
as \( \mu_{q+1} \)-representations again by the purity theorem. By (4.30) and (4.31), we obtain a \( \mu_{q+1} \)-equivariant long exact sequence
\[
0 \to 1^{\oplus q} \to H^1_c(W, \mathcal{T}_\ell) \to H^1(W, \mathcal{T}_\ell) \to 1(-1)^{\oplus q} \to 0.
\] (4.32)

Let \( X_{DL} \) be the affine curve over \( \mathbb{F}_q \) defined by \((S^q T - ST^q)^{q-1} = 1\), which is called the Deligne-Lusztig curve for \( GL_2(\mathbb{F}_q) \). Let \( \mathbb{F}_{q^2}^\times \) act on \( X_{DL} \) by \( \xi: (S, T) \mapsto (\xi^{-1} S, \xi^{-1} T) \) for \( \xi \in \mathbb{F}_{q^2}^\times \).

Let \( GL_2(\mathbb{F}_q) \) act on \( X_{DL} \) by \( g: (S, T) \mapsto (aS + cT, bS + dT) \) for \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_q) \). Clearly, the actions of \( \mathbb{F}_{q^2}^\times \) and \( GL_2(\mathbb{F}_q) \) commute. The curve \( X_{DL} \) is isomorphic to a disjoint union of \( q-1 \) copies of \( W \) over \( \mathbb{F}_{q^2} \). The stabilizer of each connected component of \( X_{DL} \) in \( \mathbb{F}_{q^2}^\times \) equals \( \mu_{q+1} \). Hence, by inducing (1.32) from \( \mu_{q+1} \) to \( \mathbb{F}_{q^2}^\times \), we acquire an \( \mathbb{F}_{q^2}^\times \)-equivariant long exact sequence
\[
0 \to \left( \text{Ind}_{\mu_{q+1}}^{\mathbb{F}_{q^2}^\times} 1 \right)^{\oplus q} \to H^1_c(X_{DL, \mathcal{T}}) \to H^1(X_{DL, \mathcal{T}}) \to \left( \text{Ind}_{\mu_{q+1}}^{\mathbb{F}_{q^2}^\times} 1 \right)^{\oplus q} \to 0.
\] (4.33)

Let \( H^1_c(X_{DL, \mathcal{T}})_{\text{cusp}} \subset H^1_c(X_{DL, \mathcal{T}}) \) be the cuspidal part regarded as a \( GL_2(\mathbb{F}_q) \)-representation. We say that a character \( \chi \in (\mathbb{F}_{q^2}^\times)^\vee \) is in general position if \( \chi \) does not factor through \( \text{Nm}_{\mathbb{F}_q}^{\mathbb{F}_{q^2}}:\mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times \). We write \( C \subset (\mathbb{F}_{q^2}^\times)^\vee \) for the set of all characters in general position.

By the Deligne-Lusztig theory for \( GL_2(\mathbb{F}_q) \) in [DL Theorem 6.2], we have a decomposition
\[
H^1_c(X_{DL, \mathcal{T}})_{\text{cusp}} = \bigoplus_{\chi \in C} H^1_c(X_{DL, \mathcal{T}}) \chi
\] (4.34)
as \( GL_2(\mathbb{F}_q) \times \mathbb{F}_{q^2} \)-representations, and the \( \chi \)-part \( H^1_c(X_{DL, \mathcal{T}}) \chi \) is an irreducible and cuspidal \( GL_2(\mathbb{F}_q) \)-representation (cf. [BH §6.4] and [Yo Corollary 6.9]).
Lemma 4.18. 1. The canonical map $H^1_c(X_{\text{DL}, F}, \mathbb{Q}_\ell) \to H^1(X_{\text{DL}, F}, \mathbb{Q}_\ell)$ is injective on the cuspidal part.

2. Let $\chi \in C$. Then, the geometric Frobenius element over $\mathbb{F}_{q^2}$ acts on $H^1_c(X_{\text{DL}, F}, \mathbb{Q}_\ell)_\chi$ as scalar multiplication by $-q$.

Proof. Since $H^1_c(X_{\text{DL}, F}, \mathbb{Q}_\ell)_\chi$ is irreducible, the geometric Frobenius element over $\mathbb{F}_{q^2}$ acts on it as scalar multiplication by Schur’s lemma. Hence, the second assertion follows from the Grothendieck trace formula.

We prove the first assertion. By [BH, Theorem (1) in §6.4], any irreducible and cuspidal representation of $\text{GL}_2(\mathbb{F}_q)$ is $(q - 1)$-dimensional. Clearly, we have $|C| = q(q - 1)$. Hence, by (4.31), we have

$$\dim_{\mathbb{F}_q} H^1_c(X_{\text{DL}, F}, \mathbb{Q}_\ell)_{\text{cusp}} = q(q - 1)^2.$$  

(4.35)

Since $\text{Ind}_{\mathbb{F}_q}^{\mathbb{F}_{q^2}} 1$ contains no character of $\mathbb{F}_{q^2}$ in general position, we have

$$\text{Im} \varphi \cap H^1_c(X_{\text{DL}, F}, \mathbb{Q}_\ell)_{\text{cusp}} = \{0\}$$

(4.36)

by (4.34). By (4.35) and (4.36), we obtain $H^1_c(X_{\text{DL}, F}, \mathbb{Q}_\ell) \simeq \text{Im} \varphi \oplus H^1_c(X_{\text{DL}, F}, \mathbb{Q}_\ell)_{\text{cusp}}$. Hence, the required assertion follows from (4.33).

Remark 4.19. By setting $a = S/T$ and $t = 1/T$, the curve $X_{\text{DL}}$ is isomorphic to the affine curve defined by $a^q - a = \zeta_{1/q}^{q+1}$ with $t \neq 0$. By using this fact, Lemma 4.10.1 and Lemma 4.13, we can deduce Lemma 4.18.2.

5 Statement of main theorem

In this section, we state our main theorem in Theorem 5.17. This theorem is reduced to Proposition 5.16. A proof of Proposition 5.16 will be given in §6 in a purely local manner. In §5.1, we give a summary of the theory of types for $\text{GL}(2)$, and introduce a statement of the explicit LLC and LJLC. In §5.2, admitting Proposition 5.16, we deduce Theorem 5.17. To do so, we need to show Corollary 5.15. To prove this, we need to understand some group action on $\pi_0$ of the Lubin-Tate tower. We can do this by using determinant morphisms given in [GL, V.4]. These are done in §5.2.3. Furthermore, in §5.2.3 by using Proposition 5.16 and Theorem 5.17, we will show the equivalence of the explicit LLC and LJLC, and the NALT for $\text{GL}(2)$ (cf. Corollaries 5.18 and 5.19).

For a character $\chi$ of $L^\times$, we often identify it with a character of $W_L$ via the Artin reciprocity map $a_L$.

5.1 Theory of types for $\text{GL}(2)$

For any admissible pair $(L/F, \chi)$, one can explicitly construct an irreducible cuspidal representation of $\text{GL}_2(F)$, an irreducible smooth representation of $D^\times$, and a two-dimensional irreducible smooth representation of $W_F$ so that they match under the LLC and the LJLC. In this subsection, in the odd residue characteristic case, we briefly recall the construction in [BH, §19, §34.1, §56] and a main statement of explicit LLC and LJLC. Under this assumption, this theory is described with respect to admissible pairs.
5.1.1 Admissible pair

We briefly recall admissible pairs from [BH, Definition in §18.2]. We consider a pair $(L/F, \chi)$, where $L/F$ is a tamely ramified quadratic field extension and $\chi$ is a smooth character of $L^\times$.

**Definition 5.1.** ([BH, Definition 18.2]) The pair $(L/F, \chi)$ is admissible if

- $\chi$ does not factor through the norm map $\text{Nr}_{L/F} : L^\times \to F^\times$, and
- if $\chi|_{U_L}$ factors through $\text{Nr}_{L/F}$, then $L/F$ is unramified.

Admissible pairs $(L/F, \chi)$ and $(L'/F, \chi')$ are said to be $F$-isomorphic if there is an $F$-isomorphism $\phi : L \sim L'$ such that $\chi = \chi' \circ \phi$. We write $\mathcal{P}_2(F)$ for the set of $F$-isomorphism classes of admissible pairs.

The level of $\chi$ means the least integer $n \geq 0$ such that $\chi|_{U_L^{n+1}}$ is trivial, which is denoted by $l(\chi)$. We say that an admissible pair $(L/F, \chi)$ such that $l(\chi) = n$ is minimal if $\chi|_{U_L^n}$ does not factor through $\text{Nr}_{L/F}$.

By the local class field theory, the level of $\chi$ equals the Swan conductor exponent of $\chi$ regarded as a character of $W_L$.

For a character $\phi$ of $F^\times$, we write $\phi_L$ for the composite $\phi \circ \text{Nr}_{L/F}$. For any admissible pair $(L/F, \chi)$ is $F$-isomorphic to one of the form $(L/F, \chi' \otimes \phi_L)$, with a character $\phi$ of $F^\times$ and a minimal admissible pair $(L/F, \chi')$.

5.1.2 Level and conductor

For an irreducible smooth representation $\pi$ of $\text{GL}_2(F)$, let $l(\pi) \in 2^{-1} \mathbb{Z}$ denote the normalized level of $\pi$ in [BH, §12.6].

**Definition 5.2.** For an irreducible cuspidal representation $\pi$ of $\text{GL}_2(F)$, we define a conductor of $\pi$ to be $2(l(\pi) + 1) \in \mathbb{Z}$, which we denote by $c(\pi)$.

For an irreducible smooth representation $\rho$ of $D^\times$, let $m(\rho)$ be the largest integer $i$ such that $\rho|_{U_D^i}$ is non-trivial. If $\rho|_{O_D^\times}$ is trivial, we set $m(\rho) = -1$. If $\dim \rho > 1$, the integer $m(\rho)$ equals the level of $\rho$ in [BH, §5.4.1].

**Definition 5.3.** ([Tu, p. 185]) We define a conductor of $\rho$ to be $m(\rho) + 2$, which we denote by $c(\rho)$.

5.1.3 Unramified case

Let $n \geq 1$ be a positive integer. Let $(F_2/F, \chi)$ be a minimal admissible pair such that $l(\chi) = n - 1$. We take $F$-embeddings $F_2 \hookrightarrow \text{M}_2(F)$ and $F_2 \hookrightarrow D$. We set

\[ J_{1,n} = F_2^\times U_{\mathfrak{m}_1}^{[\frac{n}{2}]} \subset \text{GL}_2(F), \]
\[ J_{2,n} = F_2^\times U_{D}^{-1} \subset D^\times, \]

with $U_{\mathfrak{m}_1} = \text{GL}_2(O_F)$ and $U_D^0 = O_D^\times$. Let $\psi_0 \in F^\times \setminus \{1\}$. Let $\psi_F$ be a character of $F$ such that $\psi_F(x) = \psi_0(\bar{x})$ for $x \in O_F$. For a finite extension $L/F$, let $\psi_L$ denote the composite $\psi_F \circ \text{Tr}_{L/F}$. For $n \geq 2$, let $\alpha \in \mathfrak{p}_F^{-(n-1)}$ be an element such that $\chi(1 + x) = \psi_F(\alpha x)$ for $x \in \mathfrak{p}_F^{([n-1]/2)+1}$. 

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Level zero case. First, we consider the case $n = 1$. We naturally identify $U_{F_2}^0/U_{F_2}^1$ with $\mathbb{F}_{q^2}^\times$. Let $C$ be as in [41, §4.3]. Then, $\chi$ is a tamely ramified character of $F_2^\times$. We take an $\mathbb{F}_q$-embedding $\mathbb{F}_{q^2} \hookrightarrow M_2(\mathbb{F}_q)$. There exists a unique irreducible cuspidal representation $\pi_{\chi_0}$ of $GL_2(\mathbb{F}_q)$ which satisfies

$$\text{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2(\mathbb{F}_q)} \chi_0 \simeq \overline{\text{St}} \otimes \pi_{\chi_0}, \quad (5.1)$$

where $\overline{\text{St}}$ is the Steinberg representation of $GL_2(\mathbb{F}_q)$. Note that $\dim \pi_{\chi_0} = q - 1$ and $\dim \overline{\text{St}} = q$. The isomorphism class of the representation \(5.1\) is independent of the choice of the $\mathbb{F}_{q^2}$-embedding $\mathbb{F}_{q^2} \hookrightarrow M_2(\mathbb{F}_q)$, and depends only on $\chi_0$.

Let $\Lambda_\chi$ be the irreducible smooth representation of $J_{1,1}$ such that

- the restriction $\Lambda_\chi|_{GL_2(\mathcal{O}_F)}$ is isomorphic to the inflation of $\pi_{\chi_0}$ by $GL_2(\mathcal{O}_F) \to GL_2(\mathbb{F}_q)$, and
- $\Lambda_\chi|_{F_2^\times}$ is a multiple of $\chi$.

Let $\Lambda_\chi'$ be the character of $J_{2,1}$ satisfying $\Lambda_\chi'|_{U_2^b} = 1$ and $\Lambda_\chi'|_{F_2^\times} = \chi$. Let $\Delta_0$ denote the unramified character of $F_2^\times$ of order two.

Unramified case of positive level. We consider the case $n \geq 2$. We define smooth representations $\Lambda_\chi$ and $\Lambda_\chi'$ of $J_{1,n}$ and $J_{2,n}$ respectively as follows. First, we define $\Lambda_\chi$ (cf. [BH §19.4]). Assume that $n$ is even. We define $\Lambda_\chi$ by

$$\Lambda_\chi(x(1 + y)) = \chi(x)\psi_F(\text{Tr}(\alpha y)) \quad \text{for } x \in F_2^\times \text{ and } 1 + y \in U_{2n}^1.$$

Assume that $n$ is odd. We write $n = 2m - 1$. Let $\theta$ be the character of $U_{F_2}^1U_{2n}^{m-1}$ which satisfies

$$\theta(x(1 + y)) = \chi(x)\psi_F(\text{Tr}(\alpha y)) \quad \text{for } x \in U_{F_2}^1 \text{ and } 1 + y \in U_{2n}^m.$$

There exists a unique $q$-dimensional irreducible representation $\eta_\theta$ of $U_{F_2}^1U_{2n}^{m-1}$ such that the restriction to $U_{F_2}^1U_{2n}^{m-1}$ is a multiple of $\theta$ by [BH Lemma 15.6] (cf. [BH §16.4]). We define $\Lambda_\chi$ to be the irreducible representation of $J_{1,n}$ such that

- $\Lambda_\chi|_{U_{F_2}^1U_{2n}^{m-1}} = \eta_\theta$,
- $\Lambda_\chi|_{F_2^\times}$ is a multiple of $\chi|_{F_2^\times}$, and
- $\text{Tr} \Lambda_\chi(\mu) = -\chi(\mu)$ for $\mu \in \mathfrak{m}_{q^2-1}(F_2) \setminus \mathfrak{m}_{q-1}(F)$

as in [BH Corollary 19.4] (cf. [BH §22.4]).

Secondly, we define $\Lambda_\chi'$ (cf. [BH §56]). Assume that $n$ is odd. We define $\Lambda_\chi'$ by

$$\Lambda_\chi'(x(1 + y)) = \chi(x)\psi_F(\text{Tr}_{D/F}(\alpha y)) \quad \text{for } x \in F_2^\times \text{ and } 1 + y \in U_{D}^{n-1}.$$

Assume that $n$ is even. We define a character $\theta'$ of $U_{F_2}^1U_{D}^{n}$ by

$$\theta'(x(1 + y)) = \chi(x)\psi_F(\text{Tr}_{D/F}(\alpha y)) \quad \text{for } x \in U_{F_2}^1 \text{ and } 1 + y \in U_{D}.$$

There exists a unique $q$-dimensional irreducible representation $\eta_{\theta'}$ of $U_{F_2}^1U_{D}^{n-1}$ whose restriction to $U_{F_2}^1U_{D}^{n}$ is a multiple of $\theta'$. We define $\Lambda_\chi'$ to be the unique $q$-dimensional irreducible representation of $J_{2,n}$ such that
\[\Lambda'_{\chi}|_{U_{2}^{\infty}} = \eta^{\psi};\]

- \(\Lambda'_{\chi}|_{F^{\times}}\) is a multiple of \(\chi|_{F^{\times}}\), and

- \(\text{Tr} \Lambda'_{\chi}(\mu) = -\chi(\mu)\) for \(\mu \in \mu_{q^{2}-1}(F_{2}) \setminus \mu_{q-1}(F)\).

For a positive integer \(n\), we set

\[
\pi_{\chi} = \text{c-Ind}_{\GL_{2}(F)}^{\GL_{2}(F_{2})} J_{1,n} \Lambda_{\chi},
\]

\[
\rho_{\chi} = \text{Ind}_{D^{\times}}^{D_{2}} J_{2,n} \Lambda'_{\chi},
\]

\[
\tau_{\chi} = \text{Ind}_{F_{2}/F}^{D_{2}} (\Delta_{0,\chi}).
\]

These are irreducible (cf. \cite[Theorems 11.4, 15.1 and 54.4]{BH}). They are independent of the choice of the \(F\)-embeddings \(F_{2} \hookrightarrow \text{M}_{2}(F)\) and \(F_{2} \hookrightarrow D\). The representation \(\pi_{\chi}\) is cuspidal by \cite[Theorem 14.5]{BH}. By the conductor-discriminant formula in \cite[Corollary in IV §2]{Se}, the Artin conductor exponent of \(\tau_{\chi}\) equals 2\(n\).

### 5.1.4 Ramified case

Let \(E\) be a totally tamely ramified separable quadratic extension of \(F\). Let \(n = 2m - 1\) be a positive odd integer. Let \((E/F, \chi)\) be a minimal admissible pair such that \(l(\chi) = n\). Let \(\alpha \in p_{E}^{-n}\) be an element such that \(\chi(1 + x) = \psi_{E}(\alpha x)\) for \(1 + x \in U_{m}^{m}\). We choose \(F\)-embeddings \(E \hookrightarrow \text{M}_{2}(F)\) and \(E \hookrightarrow D\). Let

\[
J_{E,1,n} = E^{\times} U_{3}^{m} \subset \GL_{2}(F),
\]

\[
J_{E,2,n} = E^{\times} U_{D}^{m} \subset D^{\times}.
\]

We define a character \(\Lambda_{E,\chi}\) of \(J_{1,E,n}\) by

\[
\Lambda_{E,\chi}(x(1 + y)) = \chi(x) \psi_{F}(\text{Tr}(\alpha y))\]

for \(x \in E^{\times}\) and \(1 + y \in U_{3}^{m}\).

Similarly, we define a character \(\Lambda'_{E,\chi}\) of \(J_{2,E,n}\) by

\[
\Lambda'_{E,\chi}(x(1 + y)) = (-1)^{v_{E}(x)} \chi(x) \psi_{F}(\text{Trd}_{D/F}(\alpha y))\]

for \(x \in E^{\times}\) and \(1 + y \in U_{D}^{m}\).

Let \(\kappa_{E/F}\) be the non-trivial character of \(F^{\times}\) factoring through \(F^{\times}/\text{Nr}_{E/F}(E^{\times}) \simeq U_{F}^{0}/\text{Nr}_{E/F}(U_{E}^{0})\). This character is tamely ramified of order two. Hence, we have

\[
\kappa_{E/F}(x) = \left( \frac{x}{F_{q}} \right)\]

for any \(x \in U_{F}^{0}\). (5.2)

We consider the quadratic Gauss sum

\[
\tau(\kappa_{E/F}, \psi_{F}) = \sum_{x \in (O_{F}/p)^{\times}} \kappa_{E/F}(x) \psi_{F}(x) = \sum_{x \in q^{m}} \left( \frac{x}{F_{q}} \right) \psi_{0}(x),
\]

where we use (5.2) at the second equality. Recall that

\[
\tau(\kappa_{E/F}, \psi_{F})^{2} = \left( \frac{-1}{F_{q}} \right) q.\]

(5.3)
Let $\lambda_{E/F}(\psi_F)$ denote the Langlands constant of the extension $E/F$ (cf. [BH §34.3]). Then, we have
\[
\begin{align*}
\lambda_{E/F}(\psi_F) &= \tau(\kappa_{E/F}, \psi_F)q^{-\frac{1}{2}}, \\
\lambda_{E/F}^2(\psi_F) &= \kappa_{E/F}(-1) \\
\end{align*}
\] (5.4)
by [BH Proposition 34.3 (2)] and (5.3). We choose a uniformizer $\varpi_E$ of $E$. Let $\zeta(\alpha, \chi) = \frac{\varpi_E^n}{\alpha} \in \mathbb{F}_q$. As in [BH §34.4], we define a tamely ramified character $\Delta_{E,\chi}$ of $E^\times$ by
\[
\begin{align*}
\Delta_{E,\chi}(x) &= \left(\frac{\overline{x}}{q}\right)_{\mathbb{F}_q} \quad \text{for} \quad x \in U_0^E, \\
\Delta_{E,\chi}(\varpi_E^n) &= \kappa_{E/F}(\zeta(\alpha, \chi)) \lambda_{E/F}(\psi_F)^n.
\end{align*}
\]
The order of $\Delta_{E,\chi}$ is divisible by 4. By (5.2) and (5.4), we have
\[
\lambda_{E/F}(\psi_F)^n = \kappa_{E/F}(-1)^{m-1} \lambda_{E/F}(\psi_F) = \left(\frac{-1}{\mathbb{F}_q}\right)^{m-1} \lambda_{E/F}(\psi_F).
\] (5.5)
We set
\[
\begin{align*}
\pi_\chi &= c\text{-Ind}_{E,1,n}^{\text{GL}_2(F)} \Lambda_{E,\chi}, \\
\rho_\chi &= \text{Ind}_{E,2,n}^{D_{\chi}} \Lambda_{E,\chi}', \\
\tau_\chi &= \text{Ind}_{E/F}(\Delta_{E,\chi} \chi).
\end{align*}
\]
These are irreducible. They are independent of the choice of the $F$-embeddings $E \hookrightarrow M_2(F)$ and $E \hookrightarrow D$. The representation $\pi_\chi$ is cuspidal by [BH Theorem 14.5]. By the conductor-discriminant formula, the Artin conductor exponent of $\tau_\chi$ is equal to $n + 2$.

Let
\[
\begin{align*}
\mathcal{A}^0(F) \quad &\text{be the set of equivalent classes of irreducible cuspidal representations of} \ \text{GL}_2(F), \\
\mathcal{A}^0_1(D) \quad &\text{the set of equivalent classes of irreducible smooth representations of} \ D^\times \ \text{whose dimension is greater than one, and} \\
\mathcal{G}^0(F) \quad &\text{the set of equivalent classes of irreducible smooth representations of} \ W_F \ \text{of degree two.}
\end{align*}
\]
Let $(L/F, \chi)$ be an admissible pair. We choose a minimal admissible pair $(L/F, \chi')$ and a smooth character $\phi$ of $F^\times$ such that $\chi = \chi' \otimes \phi_L$. Then, we set
\[
\begin{align*}
\pi_\chi &= \pi_{\chi'} \otimes (\phi \circ \text{det}), \\
\rho_\chi &= \rho_{\chi'} \otimes (\phi \circ \text{Nrd}_{D/F}), \\
\tau_\chi &= \tau_{\chi'} \otimes \phi
\end{align*}
\] (5.6)
The equivalent classes of them are independent of the choice of the pair $(\chi', \phi)$. Under the assumption $p \neq 2$, it is known that the above construction induces bijections
\[
\begin{align*}
\mathbb{P}_2(F) &\sim \mathcal{A}^0(F); \ (L/F, \chi) \mapsto \pi_\chi, \\
\mathbb{P}_2(F) &\sim \mathcal{A}^0_1(D); \ (L/F, \chi) \mapsto \rho_\chi, \\
\mathbb{P}_2(F) &\sim \mathcal{G}^0(F); \ (L/F, \chi) \mapsto \tau_\chi
\end{align*}
\] (5.7)
(cf. [BH Theorem 20.2, §54, Theorem 34.1]). The last bijection is different from that in [BH Theorem 34.1]. The correspondences (5.7) are modified so that they are compatible with the LLC and the LJLC as in [BH p. 219, §56]. For a smooth representation $\pi$ of a locally profinite group, let $\pi^\vee$ denote its contragredient.
Lemma 5.4. For any \((L/F, \chi) \in \mathbb{P}_2(F)\), we have
\[
\begin{align*}
\pi^\vee = \pi^\vee, & \quad \rho^\vee = \rho^\vee, \quad \tau^\vee = \tau^\vee, \\
\pi \otimes \phi_L & \simeq \pi \otimes (\phi \circ \det), \quad \rho \otimes \phi_L \simeq \rho \otimes (\phi \circ \text{Nrd}_{D/F}), \quad \tau \otimes \phi_L \simeq \tau \otimes \phi
\end{align*}
\]
for any character \(\phi\) of \(F^\times\).

Proof. We prove (5.8). The required assertion for \(\pi\) follows from [BH, Theorem 20.2 (3)]. Similarly, the claim for \(\rho\) is proved by using results in [BH, §54.4]. The claim for \(\tau\) is clear by the construction of it.

The claim (5.9) follows from the construction (5.6).

Lemma 5.5. For \((L/F, \chi) \in \mathbb{P}_2(F)\), we have \(c(\pi) = c(\rho)\).

Proof. We can directly compute the both sides by using [BH, (19.6.2)].

5.1.5 Explicit LLC and LJLC

Let
\[
\begin{align*}
\text{LL}: & \mathcal{A}^{0}(F) \xrightarrow{\sim} \mathcal{C}^0(F); \pi \mapsto \text{LL} \pi, \\
\text{JL}: & \mathcal{A}^{0}(F) \xrightarrow{\sim} \mathcal{A}^{1}(D); \pi \mapsto \text{JL} \pi
\end{align*}
\]
declare the LLC and the LJLC respectively (cf. [BH p. 219, §56.1]).

Lemma 5.6. For any irreducible cuspidal representation \(\pi\) of \(\text{GL}_2(F)\), we have \(c(\pi) = c(\text{JL}(\pi))\).

Proof. For an irreducible smooth representation \(\rho\) of \(D^\times\), let \(l(\rho)\) denote the level of \(\rho\) in [BH §54.1]. By [BH §56.1], we have \(l(\text{JL}(\pi)) = 2l(\pi)\). Hence, the required assertion follows.

Remark 5.7. For any \(\pi \in \mathcal{A}^0(F)\), the Artin conductor exponent of LL(\(\pi\)) equals \(c(\pi)\). We will not use this fact later.

The following theorem is stated in [BH p. 219, p. 334], which we call the explicit LLC and LJLC. This theorem is due to Bushnell and Henniart.

Theorem 5.8. (Explicit LLC and LJLC) Assume that \(p \neq 2\). Let the notation be as in (5.7). For any admissible pair \((L/F, \chi)\), we have
\[
\text{JL}(\pi) = \rho, \quad \text{LL}(\pi) = \tau.
\]

5.2 Main theorem and its application

5.2.1 Local fundamental representation

We have introduced Lubin-Tate curves in the previous sections. In the following, only in §5.2.1 and §5.2.2 we consider any dimensional Lubin-Tate spaces. We introduce a main subject, which is called local fundamental representation, in the non-abelian Lubin-Tate theory. Main references are [Bo, §2.4], [Ca2, §1.5], [Da, §3.5], [Fa2, §4.5] and [St, §2.5].

We choose an isomorphism \(\iota: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell\). Let \(q^{\frac{1}{2}} \in \overline{\mathbb{Q}}_\ell\) denote the second root of \(q\) such that \(q^{-1}(q^{\frac{1}{2}})\) is positive in \(\mathbb{R}\). Let \(d \geq 1\). For each \(n \geq 1\), let \(X^d(p^n)\) be the \((d-1)\)-dimensional Lubin-Tate space with Drinfeld level \(p^n\)-structure. We consider
\[
\mathcal{H}^d_c = \lim_{n \rightarrow \infty} \mathcal{H}^{d-1}_{c} (X^d(p^n)_{C}, \overline{\mathbb{Q}}_\ell) \left( \frac{d-1}{2} \right),
\]
We consider the homomorphism
\[ \delta_d : G_d = \text{GL}_d(F) \times D^\times \times W_F \to \mathbb{Z}; \quad (g, x, \sigma) \mapsto v \left( \det(g) \text{Nrd}_{D/F}(x)^{-1} a_F(\sigma)^{-1} \right). \] (5.10)

Let \( G_0^d \) be the kernel of this homomorphism. Then, there exists an action of \( G_0^d \) on the tower \( \{ X^d(p^n)c \}_{n \geq 0} \) by [Ca2 §1.3]. Hence, \( \mathcal{H}_c^d \) can be regarded as a representation of \( G_0^d \). This is a smooth/continuous representation of \( G_0^d \) (cf. [Bo] §3.1, [HT] Lemma II.2.8 and [St] Lemma 2.5.1 and Remark after it). In the below definition, we consider the usual topology on \( \text{GL}_d(F) \times D^\times \), the discrete topology on \( W_F \), and the product topology on \( G_d \). Then, we regard \( \mathcal{H}_c^d \) as a smooth representation of \( G_0^d \).

**Definition 5.9.** We define a \( G_d \)-representation
\[ \mathcal{U}_c^d = c-\text{Ind}_{G_0^d}^{G_d} \mathcal{H}_c^d, \]
which we call the local fundamental representation.

### 5.2.2 Geometrically connected components of Lubin-Tate spaces

We recall group action on \( \pi_0 \) of Lubin-Tate spaces. To do so, we give complements on [GL]. As a result, we show Corollary 5.13. To show this, we use also a result in [St2].

We recall notations and results in [GL, V]. Let \( d \geq 1 \). We simply write \( \mathfrak{d} \) for \( \mathbb{Z}/d\mathbb{Z} \). Let
\[ B_{\mathfrak{L}T, m}^d = \mathcal{O}_{\mathbb{F}_m}[[[x_i]_{i \in \mathfrak{d}}]] \times \mathcal{O}_{\mathbb{F}_m}[[[y_{i,j}]_{i,j \in \mathfrak{d}}; 0 \leq j \leq md-1]]/\mathcal{I}_{m, d}, \]
where \( \mathcal{I}_{m, d} \) is generated by
\[ \prod_{k=1}^d x_k - (-1)^d \varpi, \quad s_{i,0}^{q-1} - x_i, \quad s_{i,j}^q - x_i - j s_{i,j} - s_{i,j-1} \quad \text{for } i \in \mathfrak{d} \text{ and } 1 \leq j \leq md - 1. \]

We set \( Z^d(p^m) = (\text{Spf} B_{\mathfrak{L}T, m}^d)^{\text{rig}} \), which is a \( (d-1) \)-dimensional rigid analytic variety over \( \hat{\mathbb{F}}_{\text{ur}} \). Note that \( Z^2(p^m) \) equals \( Y(p^{2m-1}) \) in [GL Remarque II.2.3], \( Z^d(p^m) \) is an intermediate covering between \( X^d(p^m) \) and \( X^d(p^{m+1}) \). Then, the tower \( \{ Z^d(p^m)c \}_{m \geq 0} \) admits an action of \( G_0^d \). Let \( \mathfrak{J} \) be the inverse image of the subring consisting of all upper triangular matrices in \( M_d(\mathbb{F}_q) \) by the canonical map \( M_d(\mathcal{O}_E) \to M_d(\mathbb{F}_q) \), which is \( \mathfrak{J} \) in the notation of [GL I.1]. Then, \( Z^d(p^m) \) is the quotient of \( X^d(p^{m+1}) \) by \( \mathfrak{J}_{m-1} = 1 + \varpi^m \mathfrak{J} \). Let \( \mathcal{O}_{\mathbb{F}_m} \hat{\otimes}_{\mathbb{F}_q} B_{\mathfrak{L}T, m}^d \) denote the completion of \( \mathcal{O}_{\mathbb{F}_m} \hat{\otimes}_{\mathbb{F}_q} B_{\mathfrak{L}T, m}^d \) for the \( \varpi \otimes 1 \)-adic topology. We set \( z = \varpi \otimes 1 \in \mathcal{O}_{\mathbb{F}_m} \hat{\otimes}_{\mathbb{F}_q} B_{\mathfrak{L}T, m}^d \) and \( P_z = \begin{pmatrix} 0_{d-1} & E_{d-1} \\ z & 0_{d-1} \end{pmatrix} \in M_d \left( \mathcal{O}_{\mathbb{F}_m} \hat{\otimes}_{\mathbb{F}_q} B_{\mathfrak{L}T, m}^d \right) \). Let
\[ S_m = \sum_{j=0}^{md-1} \text{diag}(s_{1,j}, \ldots, s_{d,j})^t P_z^j \in M_d \left( \mathcal{O}_{\mathbb{F}_m} \hat{\otimes}_{\mathbb{F}_q} B_{\mathfrak{L}T, m}^d \right). \]

We set
\[ \sum_{i=0}^\infty s_{i,0} z^i = \det S_m. \]

We take an element \( \mu \in \mathbb{F}_q^\times \) such that \( \mu^{q-1} = -1 \). We have the injective ring homomorphism
\[ \det_{m, \mu} : B_{\mathfrak{L}T, m}^1 \hookrightarrow B_{\mathfrak{L}T, m}^d; \quad s_{0,i} \mapsto \mu s_i \quad \text{for each } 1 \leq i \leq m - 1. \] (5.11)
For $m \geq 1$, we have the trivial commutative diagram
\[
\begin{array}{c}
B_{LT,m}^d \xrightarrow{\text{can}} B_{LT,m+1}^d \\
\downarrow \text{det}_{m,\mu} \downarrow \quad \downarrow \text{det}_{m+1,\mu} \\
B_{LT,m}^1 \xrightarrow{\text{can}} B_{LT,m+1}^1.
\end{array}
\]

(5.12)

Let $\mathcal{L} \mathcal{F}$ denote the formal $\mathcal{O}_F$-module over $\mathcal{O}_F$ defined by
\[
[\varpi]_{\mathcal{L} \mathcal{F}}(X) = X^q + \varpi X, \quad X + \mathcal{L} \mathcal{F} Y = X + Y, \quad [\zeta]_{\mathcal{L} \mathcal{F}}(X) = \zeta X \quad \text{for } \zeta \in \mathbb{F}_q.
\]

Let $\mathcal{L} \mathcal{F}[p^m]_{\text{prim}} = \{ x \in \mathcal{T} \mid [\varpi^m]_{\mathcal{L} \mathcal{F}}(x) = 0, \ [\varpi^{m-1}]_{\mathcal{L} \mathcal{F}}(x) \neq 0 \}$, and $F_{\mathcal{L} \mathcal{F},m} = \widehat{\text{Fur}}(\mathcal{L} \mathcal{F}[p^m]_{\text{prim}})$. By the Lubin-Tate theory, we have the isomorphism
\[
\text{Gal} \left( F_{\mathcal{L} \mathcal{F},m}/\widehat{\text{Fur}} \right) \simeq (\mathcal{O}_F/p^m)^\times; \quad \sigma \mapsto a_\sigma,
\]
where $[a_\sigma]_{\mathcal{L} \mathcal{F}}(x) = \sigma(x)$ for $x \in \mathcal{L} \mathcal{F}[p^m]_{\text{prim}}$. Note that we have an isomorphism
\[
B_{LT,m}^1 = \mathcal{O}_{\text{Fur}}[s_{1,m-1}]/([\varpi^m]_{\mathcal{L} \mathcal{F}}(s_{1,m-1})/([\varpi^{m-1}]_{\mathcal{L} \mathcal{F}}(s_{1,m-1})) \simeq \mathcal{O}_{F_{\mathcal{L} \mathcal{F},m}}
\]
for $m \geq 1$. The map (5.11) induces the morphism of rigid analytic varieties
\[
\mathbb{Z}^d(p^m) \to \text{Sp} F_{\mathcal{L} \mathcal{F},m}.
\]
(5.14)

For a rigid analytic variety $Y$ over $\mathbb{C}$, let $\pi_0(Y)$ be the set of the connected components of $Y$. The map (5.14) induces
\[
\pi_0(\mathbb{Z}^d(p^m)_C) \to \pi_0 \left( \text{Sp} \left( F_{\mathcal{L} \mathcal{F},m} \times \text{Fur} \mathbb{C} \right) \right) \simeq (\mathcal{O}_F/p^m)^\times,
\]
where the isomorphism is given by (5.13). By [St2] Theorem 4.4 (i) and $\det U^m_{\text{ind}} = U^m_F$, the cardinality of $\pi_0(\mathbb{Z}^d(p^m)_C)$ equals $q^{m-1}(q - 1)$. Hence, the map (5.15) is bijective. Therefore, by taking the projective limit of (5.15), we have
\[
\lim_{\longrightarrow} \pi_0(\mathbb{Z}^d(p^m)_C) \to O_F^\times.
\]

As in [GL, p. 398], we set
\[
\widehat{B}_{LT,\infty}^d = \left( \bigcup_{m=1}^{\infty} B_{LT,m}^d \right)^\sim,
\]
where $(\cdot)^\sim$ denotes the $(x_1, \ldots, x_d)$-adic completion. By (5.11) and (5.12), we have the injective homomorphism
\[
\det_{\infty,\mu}: \widehat{B}_{LT,\infty}^1 \to \widehat{B}_{LT,\infty}^d.
\]
(5.16)

Let $A^{d}_{\text{Int}}$ be as in [GL] V.1. By [GL] pp. 398–399, we have the injective ring homomorphism
\[
(*) : \widehat{B}_{LT,\infty}^d \hookrightarrow A^{d}_{\text{Int}}.
\]
(5.17)

By [GL] V.4, we have the determinant map
\[
\det_{\mu}: A^1_{\text{Int}} \to A^{d}_{\text{Int}}.
\]
(5.18)
Proposition 5.10. For $m \geq 1$, we have the commutative diagram

\[
\begin{array}{ccc}
B^d_{LT,m} & \xrightarrow{\text{can}} & \hat{B}^d_{LT,\infty} \xrightarrow{(\ast)} A^d_{\text{Int}} \\
\downarrow \det_{\mu,m} & & \downarrow \det_{\mu} \\
\hat{B}^1_{LT,m} & \xrightarrow{\text{can}} & \hat{B}^1_{LT,\infty} \cong A^1_{\text{Int}}.
\end{array}
\] (5.19)

Proof. The left commutativity in (5.19) is clear. Hence, we prove the right commutativity. By definition, we have $A^1_{\text{Int}} = \mathbb{F}((t^{1/q^\infty}))$ as in [GL, V.1]. The inverse map of $\hat{B}^1_{LT,\infty} \to A^1_{\text{Int}}$ is given by $t \mapsto \lim_{j \to +\infty} s^q_{0,j}$ (cf. [GL, p. 384]). Let $\hat{A}^d_{LT,\infty}$ be as in [GL, pp. 370–371]. We have the natural injective map

\[
\hat{B}^d_{LT,\infty} \hookrightarrow \hat{A}^d_{LT,\infty},
\] (5.20)

and the isomorphism

\[
(\text{décomposition})^*: \hat{A}^d_{LT,\infty} \sim \to A^d_{\text{Int}}
\] (5.21)
in [GL] p. 398 (cf. The inverse map of (5.21) is given in [GL, V.2.1]). The composite of (5.20) and (5.21) equals the injective map (5.17). By [GL, V.4], we have the determinant map

\[
\det_{\mu}: \hat{A}^1_{LT,\infty} \hookrightarrow \hat{A}^1_{LT,\infty}.
\]

By the definition of this and the definition of $\hat{\det}_{\infty,\mu}$ in (5.16), we have the commutative diagram

\[
\begin{array}{ccc}
\hat{B}^d_{LT,\infty} & \xrightarrow{\text{(décomposition)}} & \hat{A}^d_{LT,\infty} \\
\downarrow \hat{\det}_{\infty,\mu} & & \downarrow \hat{\det}_{\mu} \\
\hat{B}^1_{LT,\infty} & \xrightarrow{\text{(décomposition)}} & \hat{A}^1_{LT,\infty}.
\end{array}
\]

Furthermore, by [GL, p. 404], we have the commutative diagram

\[
\begin{array}{ccc}
\hat{A}^d_{LT,\infty} & \xrightarrow{(\text{décomposition})^*} & A^d_{\text{Int}} \\
\downarrow \det_{\mu} & & \downarrow \det_{\mu} \\
\hat{A}^1_{LT,\infty} & \xrightarrow{(\text{décomposition})^*} & A^1_{\text{Int}}.
\end{array}
\]

The required assertion follows from the above two commutative diagrams. \qed

Remark 5.11. A similar determinant morphism is studied in [We4, (2.7.3)] by using [He], which is based on the theory of displays due to Zink in [Zi]. In [GL], formal models of Lubin-Tate tower are described on the basis of the theory of coordinate modules, because the characteristic of $F$ is positive.

Let

\[
\delta^1_d: G_d \to F^\times; \quad (g, d, \sigma) \mapsto \det(g) \operatorname{Nrd}_{D/F}(d)^{-1} a_F(\sigma)^{-1}
\]
and $G^1_d = \ker \delta^1_d$. The restriction of $\delta^1_d$ to $G^0_d$ induces the surjective homomorphism

\[
\delta^1_d: G^0_d \to O_F^\times.
\]
Corollary 5.12. 1. The subgroup $G_d^1$ acts on $\lim_m \pi_0(Z^d(p^m)c)$ trivially.
2. Let
$$H_d = \text{GL}_d(O_F) \times O_D^\times \times I_F \subset G_d^0.$$ Then, $H_d$ acts on $\lim_m \pi_0(Z^d(p^m)c) \simeq O_F^\times$ via $\delta_d^1$.

Proof. We prove the first assertion. Let $(H)$. Then, Corollary 5.12.2 is proved in [St2, Theorem 4.4 (i)].

For an element $d' = \sum_{i \in \mathbb{Z}} a_i \varphi^i \in D^\times$ with $a_i \in \mathbb{F}_{q^d}$, we set
$$d'_z = \sum_{i \in \mathbb{Z}} \text{diag}(1 \otimes a_i, 1 \otimes a_i^q, \ldots, 1 \otimes a_i^{qz-1}) P_z \in M_d(F \otimes \mathbb{F}_q \hat{B}_{LT, \infty}^d).$$
Note that $Nrd_{D/F}(d'_z) = \text{det}(d'_z) \in \hat{B}_d_{LT, \infty}^1$. For an element $g \in M_d(F)$, let $g_z$ denote the image of $g$ by the natural map $\text{id} \otimes 1 : M_d(F) \to M_d(F \otimes \mathbb{F}_q \hat{B}_{LT, \infty}^d)$. Let $T_{LT}$ be as in [GL, Remarque II.2.4 or the proof of Corollaire IV.2.5], the group $\text{GL}_d(F) \times D^\times$ acts on the matrix $T_{LT}$ by
$$T_{LT} \mapsto t d'_{z^{-1}} T_{LT} t g_z \quad \text{for} \quad (g, d') \in \text{GL}_d(F) \times D^\times.$$ (5.22)

The determinant morphism (5.18) is given by
$$\text{Spf } A^d_{LT, \text{int}} \to \text{Spf } A^1_{LT, \text{int}}; \quad T_{LT} \mapsto \mu \text{ det } T_{LT}.$$ Hence, by (5.22), the element $(g, x_1, 1)$ acts on $A^1_{LT, \text{int}}$ trivially. By the diagram (5.19), the element $(g, x_1, 1)$ acts on $\hat{B}_d^1_{LT, \infty}$ trivially. We write $(1, x_2, \sigma) = (1, x_2', 1)(1, \varphi^{-n_{x_2}}, \sigma)$ with $x_2' \in O_D^\times$. Let $a_{F, \varphi}^0(\sigma) = a_{F}(\sigma)/\varphi^{-n_x} \in O_F^\times$. By the Lubin-Tate theory, $(1, \varphi^{-n_x}, \sigma)$ acts on $\hat{B}_d^1_{LT, \infty}$ as scalar multiplication by $a_{F, \varphi}^0(\sigma)^{-1}$. Hence, by (5.22), the element $(1, x_2, \sigma)$ acts on $\hat{B}_d^1_{LT, \infty}$ trivially. Therefore, the required assertion follows.

The second assertion follows from (5.22) and the proof of the first assertion.

Remark 5.13. By $X^d(p^{m+1}) \to Z^d(p^m) \to X^d(p^m)$ for each $m \geq 1$, we have an isomorphism
$$\lim_m \pi_0(Z^d(p^m)c) \simeq \lim_m \pi_0(X^d(p^m)c).$$ Then, Corollary 5.12 is proved in [St2, Theorem 4.4 (i)].

Corollary 5.14. The group $G_d^0$ acts on
$$\lim_m \pi_0(Z^d(p^m)c) \sim \pi_0(X^d(p^m)c).$$ via $\delta_d^1$.

Proof. We have $G_d^0 = G_d^1 H_d$, because the restriction $\delta_d^1|H_d : H_d \to O_F^\times$ is surjective. Hence, by Corollary 5.12, the required assertion follows.

Corollary 5.15. The $G_d$-representation $U^d$ is invariant under twisting by any character of $G_d$ factoring through $\delta_d^1 : G_d \to F^\times$. 48
Proof. For each \( m \geq 1 \), we have the natural morphisms of rigid analytic varieties \( X^d(p^{m+1}) \rightarrow Z^d(p^m) \rightarrow X^d(p^m) \). This induces the isomorphism

\[
\mathcal{H}_c^d = \lim_{\to} H_{c}^{d-1}(X^d(p^m)_C, \overline{\mathbb{Q}_\ell}) \left( \frac{d-1}{2} \right) \overset{\sim}{\rightarrow} \lim_{\to} H_{c}^{d-1}(Z^d(p^m)_C, \overline{\mathbb{Q}_\ell}) \left( \frac{d-1}{2} \right). \tag{5.23}
\]

Let \( \alpha \in \mathcal{O}_F^\times \). For an integer \( m \geq 1 \), let \( \alpha_m \) denote the image of \( \alpha \) by the canonical map \( \mathcal{O}_F^\times \rightarrow \mathcal{O}_F^\times / U_m^m \). Let \( Z_d^{d,\alpha_m}(p^m) \) denote the connected component of \( Z^d(p^m) \) corresponding to \( \alpha_m \) in \( \pi_0(Z^d(p^m)) \cong \mathcal{O}_F^\times / U_m^m \). Then, \( \{ Z_d^{d,\alpha_m}(p^m) \}_{m \geq 1} \) makes a projective system. Let \( \xi \) be a character of \( F^\times \). Let \( \xi_0 = \xi|_{\mathcal{O}_F^\times} \). The image of \( \xi_0 \) equals \( \mu_n(\overline{\mathbb{Q}_\ell}) \) with some integer \( n \geq 1 \). We put \( U_\xi = \ker \xi_0 \). We consider the composite

\[
\xi': G_0^d \xrightarrow{\delta_1^\ell} \mathcal{O}_F^\times \xrightarrow{\xi_0} \mu_n(\overline{\mathbb{Q}_\ell}).
\]

Let \( G_\xi^0 = \ker \xi' \). For \( m \geq 1 \), we write \( U_{\xi,m} \) for the image of \( U_\xi \) by \( \mathcal{O}_F^\times \rightarrow \mathcal{O}_F^\times / U_m^m \). We consider the projective system \( \{ Z_{\xi,m}^d \}_{m \geq 1} = \left\{ \bigcup_{\alpha_m \in U_{\xi,m}} Z_d^{d,\alpha_m}(p^m) \right\}_{m \geq 1} \). By Corollary 5.14, the stabilizer of \( \{ Z_{\xi,m}^d \}_{m \geq 1} \) in \( G_0^d \) equals \( G_\xi^0 \). Since the quotient \( G_d^0/G_\xi^0 \) is cyclic, we have \( G_d^0 \)-equivariant isomorphisms

\[
\mathcal{H}_c^d \simeq \lim_{\to} H_{c}^{d-1}(Z^d(p^m)_C, \overline{\mathbb{Q}_\ell}) \left( \frac{d-1}{2} \right)
\]

\[
\simeq \text{Ind}_{G_\xi^0}^{G_d^0} \left( \lim_{\to} H_{c}^{d-1}(Z_{\xi,m}^d, \overline{\mathbb{Q}_\ell}) \left( \frac{d-1}{2} \right) \right)
\]

by (5.23). Hence, we have an isomorphism

\[
\mathcal{H}_c^d \otimes \xi' \simeq \mathcal{H}_c^d
\]

as \( G_d^0 \)-representations. Therefore, we have isomorphisms

\[
\mathcal{U}_c^d \otimes (\xi \circ \delta_1^\ell) \simeq \text{c-Ind}_{G_\xi^0}^{G_d^0} (\mathcal{H}_c^d \otimes \xi') \simeq \text{c-Ind}_{G_\xi^0}^{G_d^0} \mathcal{H}_c^d = \mathcal{U}_c^d
\]

as \( G_d \)-representations. Hence, the required assertion follows.

\[\square\]

5.2.3 Main results

In the following, we always assume that \( d = 2 \). We omit the indices \( d \) in the notations in §5.2.1 and §5.2.2.

We have an isomorphism

\[
\lim_{\to} H^1_c(\text{LT}(p^n)_C, \overline{\mathbb{Q}_\ell}) \left( \frac{1}{2} \right) \cong \bigoplus_{h \in \mathbb{Z}} \lim_{\to} H^1_c(X^{(h)}(p^n)_C, \overline{\mathbb{Q}_\ell}) \left( \frac{1}{2} \right) \cong \mathcal{U}_c \tag{5.24}
\]

as \( G \)-representations (cf. [Da] (3.5.2) or [Fa2] §4.5]). Let \( \text{LT}(p^n)/\varpi^\mathbb{Z} \) denote the quotient of \( \text{LT}(p^n) \) by the action of \( \varpi \in D^\times \). Note that the subgroup \( \{(x,x) \in G_D \mid x \in F^\times \} \subset G_D \) acts on (5.24) trivially. We set

\[
\mathcal{U}_c = \lim_{\to} H^1_c(\text{LT}(p^m)/\varpi^\mathbb{Z})_C, \overline{\mathbb{Q}_\ell}) \left( \frac{1}{2} \right), \tag{5.25}
\]

which is regarded as a \( \text{GL}_2(F)/\varpi^\mathbb{Z} \times D^\times / \varpi^\mathbb{Z} \times W_F \) representation. As a \( \text{GL}_2(F)/\varpi^\mathbb{Z} \times D^\times / \varpi^\mathbb{Z} \)-representation, this is smooth (cf. [St, Lemma 2.5.1]). By (5.24), the representation (5.25) is regarded as a \( G \)-subrepresentation of \( \mathcal{U}_c \).

We will prove the following proposition in a purely local manner in §6.
Proposition 5.16. Let char $F$ denote the characteristic of $F$. Assume that char $F = p \neq 2$. For any admissible pair $(L/F, \chi)$, let $\pi_\chi$, $\tau_\chi$ and $\rho_\chi$ be as in (5.7). We choose a uniformizer $\varpi$ of $F$. Assume that $(L/F, \chi)$ is minimal and $\chi(\varpi) = 1$. Then, there exists a $G$-equivariant injection
\[ \pi_\chi \otimes \rho_\chi^\vee \otimes \tau_\chi \hookrightarrow \overline{\mathcal{U}}_c. \]

By admitting Proposition 5.16 and using Corollary 5.15 we obtain our main theorem in this paper.

Theorem 5.17. Let $(L/F, \chi)$ be an admissible pair.
1. We have a $G$-equivariant injection
\[ \pi_\chi \otimes \rho_\chi^\vee \otimes \tau_\chi \hookrightarrow \mathcal{U}_c. \] (5.26)
2. The injection (5.26) induces the isomorphism
\[ \text{Hom}_{\GL_2(F)}(\mathcal{U}_c, \pi_\chi) \simeq \rho_\chi \otimes \tau_\chi \]
as $D^\times \times W_F$-representations.

Proof. We prove the first assertion. Let $(L/F, \chi)$ be an admissible pair. We take a minimal admissible pair $(L/F, \chi')$ and a character $\xi$ of $F^\times$ such that $\chi = \chi' \otimes \xi$. We take a second root $c_0 \in \overline{\mathbb{Q}}_\ell'$ of $\chi'(\varpi)$. Let $\lambda$ be the unramified character of $F^\times$ which sends $\varpi$ to $c_0$. Then, $(L/F, \chi \otimes (\xi \lambda)_L^{-1})$ is a minimal admissible pair such that $(\chi \otimes (\xi \lambda)_L^{-1})(\varpi) = 1$. Let $\phi$ be the character of $G$ which is the composite of $\delta^3$ and the character $\xi \lambda$ of $F^\times$. By Lemma 5.4 and Proposition 5.16 we have a $G$-equivariant injection
\[ (\pi_\chi \otimes \rho_\chi^\vee \otimes \tau_\chi) \otimes \phi^{-1} \simeq \pi_{\chi \otimes (\xi \lambda)_L^{-1}} \otimes \rho_{\chi \otimes (\xi \lambda)_L^{-1}} \otimes \tau_{\chi \otimes (\xi \lambda)_L^{-1}} \hookrightarrow \overline{\mathcal{U}}_c \subset \mathcal{U}_c. \] (5.27)
By twisting this by $\phi$ and using Corollary 5.15 we obtain the $G$-equivariant injection
\[ \pi_\chi \otimes \rho_\chi^\vee \otimes \tau_\chi \hookrightarrow \mathcal{U}_c \otimes \phi \simeq \mathcal{U}_c. \] (5.28)
Hence, we obtain the claim.

We prove the second assertion. We simply write $\pi$, $\rho$ and $\tau$ for $\pi_\chi$, $\rho_\chi$ and $\tau_\chi$ respectively. We simply write $\xi_1$, $\xi_2$ and $\xi_3$ for $\phi|_{\GL_2(F) \times \{1\} \times \{1\}}$, $\phi|_{\{1\} \times D^\times \times \{1\}}$ and $\phi|_{\{1\} \times \{1\} \times W_F}$ respectively. Clearly, we have $\phi = \xi_1 \otimes \xi_2 \otimes \xi_3$ as $G$-representations. We have $D^\times \times W_F$-equivariant homomorphisms
\[ \text{Hom}_{\GL_2(F)}(\mathcal{U}_c, \pi) \simeq \text{Hom}_{\GL_2(F)}(\mathcal{U}_c \otimes \xi_1^{-1}, \pi \otimes \xi_1^{-1}) \]
\[ \simeq \text{Hom}_{\GL_2(F)}(\mathcal{U}_c, \pi \otimes \xi_1^{-1}) \otimes (\xi_2 \otimes \xi_3) \]
\[ \simeq \text{Hom}_{\GL_2(F)/\varpi^2}(\overline{\mathcal{U}}_c, \pi \otimes \xi_1^{-1}) \otimes (\xi_2 \otimes \xi_3) \]
\[ \to \rho \otimes \tau, \] (5.29)
where we use $\mathcal{U}_c \simeq \mathcal{U}_c \otimes \phi$ by Corollary 5.15 at the second isomorphism, the third isomorphism follows from the same argument as the proof of [ST, Theorem 2.5.2], and the fourth homomorphism is induced by (5.27). The fourth map in (5.29) is surjective, because $\overline{\mathcal{U}}_c$ is a smooth $\GL_2(F)/\varpi^2$-representation, and $\pi \otimes \xi_1^{-1}$ is an injective object in the category of smooth $\GL_2(F)/\varpi^2$-representations by [CS, Theorem 5.4.1]. Hence, (5.29) gives the $D^\times \times W_F$-equivariant surjection
\[ \text{Hom}_{\GL_2(F)}(\mathcal{U}_c, \pi) \to \rho \otimes \tau, \] (5.30)
which corresponds to the one induced by (5.28). By [Mi, Theorem 3.7] and [St, Theorem 2.5.2 (ii)] (cf. [Mi2] and [IT2, Proposition 1.1]), it is shown that

$$\dim_{\mathbb{Q}_l} \text{Hom}_{GL_2(F)}(\mathcal{U}_c, \pi) = 2 \dim_{\mathbb{Q}_l} JL(\pi)$$

in a purely geometric manner. Hence, it suffices to show

$$\dim_{\mathbb{Q}_l} JL(\pi) = \dim_{\mathbb{Q}_l} \rho$$

(5.31) to prove that (5.30) is a bijection. By Lemmas 5.5 and 5.6, we have $c(\rho) = c(JL(\pi))$. Since the JL preserves character twists, to prove (5.31), it suffices to show it in the case where $\pi$ is minimal by (5.9). By the construction of the middle bijection in (5.7) in the minimal case, the dimension of a minimal irreducible smooth representation of $D^\times$ depends only on its conductor and the number $q$. Hence, we obtain the claim.

We recall the non-abelian Lubin-Tate theory for GL(2). This is proved by Deligne if $F = \mathbb{Q}_p$ and $p \neq 2$ in a letter to Piatetskii-Shapiro, and by Carayol in general (cf. [Ca] and [Ca2]).

**Theorem 5.18. (NALT for GL(2))**

1. For any irreducible cuspidal representation $\pi$ of $GL_2(F)$, there exists a $G$-equivariant injection

$$\pi \otimes JL(\pi)^\vee \otimes LL(\pi)^\vee \hookrightarrow \mathcal{U}_c.$$  

(5.32)

2. Let the notation be as in 1. The inclusion (5.32) induces the isomorphism

$$\text{Hom}_{GL_2(F)}(\mathcal{U}_c, \pi) \simto JL(\pi) \otimes LL(\pi)$$

as $D^\times \times W_F$-representations.

In the following, we introduce a consequence of Theorem 5.17.

**Theorem 5.19.** We assume that $\text{char } F = p \neq 2$. Then, Theorem 5.8 is equivalent to Theorem 5.18.

**Proof.** By using (5.7) and Theorem 5.17, we immediately obtain the required assertion.

**Remark 5.20.** Let $(F^\times)^\vee$ denote the set of all smooth characters of $F^\times$. We have an isomorphism

$$\lim_{m} H^2_c(LT(p^m), \mathbb{Q}_l) \simeq \bigoplus_{\chi \in (F^\times)^\vee} \chi \circ \delta^1$$

(5.33)

as $G$-representations by Poincaré duality and results in §5.2.2. Let $St$ denote the Steinberg representation of $GL_2(F)$. For a character $\chi$ of $F^\times$, let $St_\chi$ denote the twist of $St$ by $\chi \circ \det$. By using (5.33), we can prove that

$$\text{Hom}_{GL_2(F)}(\mathcal{U}_c, St_\chi) \simeq \chi \circ \text{Nrd}_{D/F} \otimes \chi$$

as $D^\times \times W_F$-representations by applying [Fa2, Corollary 4.8] to the case where $n = 2$, $q = 1$ and $i(\mathfrak{s}) = 1$ (cf. [IT, Proposition 2.1]).

6 Proof of Proposition 5.16

In this section, on the basis of the analysis given in §2, §3 and §4, we will give a proof of Proposition 5.16 case by case.
6.0.4 Unramified case of level zero

Let \( \varpi_1 \in \mathcal{F}[p_F]_{\text{prim}} \). We recall the group action on \( \overline{X}_{1,1} \). By Lemma 2.9 the reduction \( \overline{X}_{1,1} \) is isomorphic to the curve \( X_{\text{DL}} \) in \( \S 1.3 \).

**Lemma 6.1.** Let \((g,d) \in \text{GL}_2(O_F) \times O_D^\times \). Then, the induced action of \((g,d)\) on \( \overline{X}_{1,1} \simeq X_{\text{DL}} \) equals the action of \((\bar{g}, \bar{d}) \in \text{GL}_2(F_\ell) \times F_\ell^\times \) given in \( \S 4.3 \).

**Proof.** It is well-known (cf. \[Yo\]). We omit its proof. \( \square \)

Let \( \chi \in C \). Let \( \pi_0^\chi \) be the inflation of the irreducible cuspidal \( \text{GL}_2(F_\ell) \)-representation \( H^1_c(X_{\text{DL}, F}, \mathbb{Q}_\ell)_\chi \) in \( \S 4.3 \) by the reduction map \( \text{GL}_2(O_F) \to \text{GL}_2(F_\ell) \). We regard \( \chi \) as a character of \( O_D^\times \) via the canonical map \( O_D^\times \to F_\ell^\times \), for which we write \( \chi_D \).

Let \( W'_{F_2} = \{ (1, \varpi^{-n_z}, \sigma) \in G \mid \sigma \in W_{F_2} \} \).

Let \( \Delta_0^\chi' \) denote the character of \( W'_{F_2} \) defined by

\[
\Delta_0^\chi'(1, \varpi^{-n_z}, \sigma) = \Delta_0(\sigma) \chi\left( \frac{a_{F_2, \varpi}(\sigma)}{w_{F_2}} \right)
\]

for \( \sigma \in W_{F_2} \).

**Lemma 6.2.** We have an isomorphism

\[
H^1_c(\overline{X}_{1,1}, \mathbb{Q}_\ell)_{\text{cusp}} \left( \frac{1}{2} \right) \simeq \bigoplus_{\chi \in C} (\pi_0^\chi \otimes \chi_D \otimes \Delta_0^\chi')
\]

as \( \text{GL}_2(O_F) \times O_D^\times \times W'_{F_2} \)-representations.

**Proof.** The required assertion follows from Lemma 3.1 \( \S 4.34 \), Lemmas 4.18.2 and 6.1 \( \square \)

The subgroup \( F^\times \subset G_D \) acts on the Lubin-Tate tower trivially (cf. \[Ca2\] p. 20 in \( \S 1.3 \)). We regard the both sides in \( (6.1) \) as \( (F^\times(\text{GL}_2(O_F) \times O_D^\times)) \times W'_{F_2} \)-representations with trivial \( F^\times \)-action. We set

\[
\mathcal{G}_1 = (F^\times(\text{GL}_2(O_F) \times O_D^\times))W'_{F_2} \simeq (F^\times(\text{GL}_2(O_F) \times O_D^\times)) \times W'_{F_2},
\]

\[
\mathcal{J}_1 = (1, \varpi^{2, 1})\mathcal{G}_1 = J_{1,1} \times J_{2,1} \times W_{F_2}.
\]

We consider the affinoid \( X_{1,1} \subset X^{(0)}(p) \subset \text{LT}(p) \), for which we write \( X^{(0)}_{1,1} \). Let \( \varpi : \text{LT}(p) \to \text{LT}(p) \) be the automorphism induced by the action of \( \varpi \in F^\times \subset D^\times \) (cf. \[SL\] \S 2.2.2). Then, we consider \( \prod_{i \in \mathbb{Z}} \varpi^i X^{(0)}_{1,1} \), on which \( (1, \varpi^{2, 1})\mathcal{G}_1 \) acts. Recall that \( \text{GL}_2(O_F) \times D^\times \times W_{F_2} \) acts on \( \text{LT}(p) \) as in \( \S 6.2 \). Then, we have a \((1, \varpi^{2, 1})\mathcal{G}_1 \)-equivariant injection

\[
X_{1,1,C} \simeq \left( \prod_{i \in \mathbb{Z}} \varpi^i X^{(0)}_{1,1,C} \right)/\varpi^{2} \hookrightarrow (\text{LT}(p)/\varpi^{2})_C,
\]

where \( X_{1,1,C} \) admits a trivial action of the element \( \varpi \in F^\times \subset D^\times \). The right hand side of \( (6.2) \) is isomorphic to a disjoint union of two copies of \( X(p) \).

**Proposition 6.3.** Let \( (F_2/F, \chi) \) be a minimal admissible pair such that \( l(\chi) = 0 \) and \( \chi(\varpi) = 1 \). We have a \( \mathcal{J}_1 \)-equivariant injection

\[
\pi_{\chi^\vee} \otimes \rho_{\chi} \otimes \tau_{\chi} \to \mathcal{U}_c.
\]
Proof. By applying Lemma 4.5.2 with $W = H^1_c(\mathbb{X}_{1,1}, \overline{\mathbb{Q}}_\ell)$ by Lemma 4.1.1, (3.24) and (6.2), we have $J_1$-equivariant injections

$$H^1_c(\mathbb{X}_{1,1}, \overline{\mathbb{Q}}_\ell)_{\text{cusp}} \hookrightarrow H^1_c((\mathbb{LT}(p)/\omega^2)_C, \overline{\mathbb{Q}}_\ell) \left( \frac{1}{2} \right) \subset \mathcal{U}_c.$$ (6.3)

We set $\chi_0 = \chi_{[U_{2}/F_2]}$, and $\Delta_0 \chi'_0$ as a $J_1$-representation with trivial $(1, \omega, 1)$-action. Then, this is clearly isomorphic to $\Lambda_{\chi'} \otimes \Lambda'_\chi \otimes \Delta_0 \chi$ in (5.1.3) as $J_1$-representations. Hence, by Lemma 6.2 and (6.3), we have a $J_1$-equivariant injection

$$\Lambda_{\chi'} \otimes \Lambda'_\chi \otimes \Delta_0 \chi \hookrightarrow \mathcal{U}_c.$$ (6.4)

We consider the usual topology on $G_D$, and the discrete topology on $W_F$. Then, we regard $\mathcal{U}_c$ as a smooth representation of $G$. Hence, the required assertion follows from Frobenius reciprocity and irreducibility of

$$\pi_{\chi'} \otimes \rho_\chi \otimes \tau_\chi = c\text{-Ind}_{J_1}^G(\Lambda_{\chi'} \otimes \Lambda'_\chi \otimes \Delta_0 \chi).$$

\[\square\]

Remark 6.4. In [Yo], he constructs a semi-stable model of $X(p)$ in any dimensional case. As a result, restricted to the height two case, he proves that

$$H^1_c(X_{\text{DL,F}}, \overline{\mathbb{Q}}_\ell) \simeq H^1_c(X(p), \overline{\mathbb{Q}}_\ell).$$

Hence, the kernel $H$ of the canonical map $H^1_c(X_{\text{DL,F}}, \overline{\mathbb{Q}}_\ell) \to H^1_c(X_{\text{DL,F}}, \overline{\mathbb{Q}}_\ell)$ also contributes to the cohomology of the generic fiber $X(p)$. For $\chi \in (F_q^\times)^\vee$, let $\text{St}_\chi$ denote the twist by $\chi \circ \det$ of the Steinberg representation $\text{St}$ of $\text{GL}_2(F_q)$. Then, by (4.3.3), the kernel $H$ is isomorphic to $\bigoplus_{\chi \in (F_q^\times)^\vee} (\text{St}_\chi \otimes \chi \circ \text{Nrf}_{F_q^2/F_q})$ as $\text{GL}_2(F_q) \times F_q^\times$-representations. Since we focus on cuspidal representations, the analysis in this subsection is enough for our purpose.

6.0.5 Unramified case of positive level

Let $n \geq 2$ be a positive integer. We choose an element $\zeta \in F_{n,q} \setminus F_q$. We consider the $F$-embeddings $F_2 \hookrightarrow M_2(F)$ and $F_2 \hookrightarrow D$ as in (3.1) and (3.13) respectively. Let $H_n^\zeta$ and $H_{\zeta,D}^n$ be as in (3.8) and (3.15) respectively. We put

$$V^v_\zeta = p^{|\mathfrak{m}|} \mathcal{C}_1 \times p^{|\mathfrak{m}-1|} \mathcal{C}_2 \subset M_2(\mathcal{O}_F) \times \mathcal{O}_D,$$

$$K^\zeta = 1 + p_{F_2}^{2n-1} + p^{n+1}\mathcal{C}_1 \subset H^\zeta_1,$$

$$K_{\zeta,D}^n = 1 + p_{F_2}^{2n-1} + p^{|\mathfrak{m}^\alpha|} \mathcal{C}_2 \subset H_{\zeta,D}^n.$$ (6.4)

Then $F_2^\times \times F_2^\times$ normalizes $K^n_\zeta \times K^n_{\zeta,D}$ and $H^n_\zeta \times H^n_{\zeta,D}$ in $G_D$. We set

$$\mathcal{L}_{n,\zeta} = F^\times U_{F_2}(H^n_{\zeta,D} \times H^n_{\zeta,D}) \subset \mathcal{P}_{n,\zeta} = F^\times U_{F_2}(K^n_\zeta \times K^n_{\zeta,D}) \subset \mathcal{K}_{n,\zeta} = F^\times (H^n_\zeta \times H^n_{\zeta,D}) \subset G_D.$$ (6.4)

These subgroups are studied in [We, §4.3] and [We3, §3.6]. It is not difficult to check that $\mathcal{L}_{n,\zeta}$ is a normal subgroup of $\mathcal{K}_{n,\zeta}$. Any element of $\mathcal{K}_{n,\zeta}$ has the form $\pi^k(x, x + \omega^{n-1}y) + v$ with $k \in \mathbb{Z}$, $x \in \mathcal{O}_F^\times$, $y \in \mathcal{O}_{F_2}$ and $v \in V_\zeta^n$.

In the sequel, we define an isomorphism $\phi_n : V_\zeta^n = V_\zeta^n / V_\zeta^{n+1} \to F_{n,q}$ such that

$$\phi_n(xvy) = (xy^q)^{q^{n-1}} \phi_n(v) \quad \text{for} \quad x, y \in F_{n,q} \quad \text{and} \quad v \in V_\zeta^n.$$ (6.4)
Assume that \( n \) is odd. We set \( n = 2m - 1 \). Let \( v_0 = \begin{pmatrix} 1 & \zeta + \zeta^q \\ 0 & -1 \end{pmatrix} \). We can easily check that

\[
v_0\zeta = \zeta^qv_0, \quad g(a, b) = (a + b\zeta) v_0 \quad \text{for any } g(a, b) \in C_1,
\]

\[
(c + d\zeta)^{-1}(g(a, b)^{(\nu+1)} g(ac + ad(\zeta + \zeta^q) + bd\zeta^{q+1}, bc - ad) \quad \text{for } c + d\zeta \in \mathbb{F}_q^\times.
\]

(6.5)

We have \( V^n_\zeta \leftarrow p^{m-1}c_1/p^m c_1 \cong \mathbb{F}_q^{\omega^{m-1}v_0} \). We define \( \phi_n \) by

\[
\phi_n \left( x^{\omega^{m-1}v_0} \right) = x
\]

for \( x \in \mathbb{F}_q^{\omega} \). We can easily check (6.4) by (6.5).

Assume that \( n \) is even. We set \( n = 2m \). We have \( V^n_\zeta \leftarrow p^{m-1}c_2/p^m c_2 \cong \mathbb{F}_q^{\omega^{m-1}\varphi} \). We define \( \phi_n \) by

\[
\phi_n \left( x^{\omega^{m-1}\varphi x} \right) = x
\]

(6.7)

for \( x \in \mathbb{F}_q^{\omega} \). Then we have (6.4).

We have \( V^n_\zeta V^n_\zeta \subset p^{n-1}_F \times p^{n-1}_F \). We consider the map \( p^{n-1}_F \times p^{n-1}_F \to \mathbb{F}_q^n \) defined by \( (x_1, x_2) \mapsto (x_1 - x_2)/\omega^{n-1} \). For \( v, w \in V^n_\zeta \), we write \( v \cdot w \) for the image of \( vw \) by this map. Then, we can check that \( v \cdot w = (\phi_n(v)\phi_n(w))^q^{n-1} \) for \( v, w \in V^n_\zeta \). Hence, we obtain the isomorphism

\[
K_{n, \zeta}/L_{n, \zeta} \cong Q; \quad (x, x + \omega^{n-1}y + v)^{(-1)^{n-1}} \mapsto g(\bar{x}, \phi_n(v), (-1)^{n}y)
\]

(6.8)

(cf. [We] the proof of Proposition 4.3.4). Let \( \nu: K_{n, \zeta} \to Q \) denote the composite of \( K_{n, \zeta} \to K_{n, \zeta}/L_{n, \zeta} \) and the isomorphism (6.8). The image of the subgroup \( P_{n, \zeta} \) by the map \( \nu \) equals the center \( Z \) of \( Q \). Let \( s_1: M_2(F) \to F_2 \) be the projection and \( s_2: D \to F_2; a + \varphi b \mapsto a \) for \( a, b \in F_2 \) (cf. (3.13) and (3.14)). We set \( s: M_2(F) \times D \to F_2; (x, y) \mapsto s_1(x) - s_2(y) \). Let \( \psi \in \mathcal{C} \).

We define a character \( \bar{\psi}_\zeta \) of the subgroup \( P_{n, \zeta} \subset K_{n, \zeta} \) by

\[
\bar{\psi}_\zeta(x(1 + w)) = \psi\left(\omega^{-(n-1)}s(w)\right) \quad \text{for } x \in F^xU^1_F \text{ and } 1 + w \in K^n_\zeta \times K^n_{\zeta,D}.
\]

Let \( \tau_{\zeta, \psi} \) denote the inflation of the irreducible \( Q \)-representation \( \tau_0^\psi \) in Lemma 4.1 by \( \nu \). Then, \( \tau_{\zeta, \psi} \) is the \( \varphi \)-dimensional irreducible representation of \( K_{n, \zeta} \) satisfying

\[
\tau_{\zeta, \psi}|_{F^x} = 1^\varphi, \quad \tau_{\zeta, \psi}|_{P_{n, \zeta}} = \bar{\psi}_\zeta^\varphi, \quad \text{Tr} \tau_{\zeta, \psi}(x) = -1 \quad \text{for } x \in \mu_{q^2-1}(F_2) \setminus \mu_{q-1}(F),
\]

(6.9)

where \( 1 \) is the trivial character of \( K_{n, \zeta} \).

**Remark 6.5.** The representation \( \tau_{\zeta, \psi} \) appears in [We] Theorem 5.0.3 and [We3] Proposition 3.8.

We consider the subgroup of \( G \):

\[
W^{(n)}_{F_2} = \left\{ \{(1, \omega^{-n_\sigma}, \sigma) \in G \mid \sigma \in W_{F_2} \} \right\} \quad \text{if } n \text{ is odd},
\]

\[
\left\{ \{(\omega^{n_\sigma}, 1, \sigma) \in G \mid \sigma \in W_{F_2} \} \right\} \quad \text{if } n \text{ is even}.
\]

We have a natural isomorphism \( a^{(n)}: W_{F_2} \cong W^{(n)}_{F_2} \). Let \( I^{(n)}_{F_2} \) be as in (3.19). The subgroup \( W^{(n)}_{F_2} \) normalizes \( I^{(n)}_{F_2} \). We consider the subgroups \( W^{(n)}_{F_2, n-1} = W^{(n)}_{F_2, n-1}I^{(n)}_{F_2} \subset W^{(n)}_{F_2} = W^{(n)}_{F_2}I^{(n)}_{F_2} \) in \( G \). We have

\[
W^{(n)}_{F_2} \cap I^{(n)}_{F_2} = \{(1, 1, \sigma) \mid \sigma \in I_{F_2}, a^{(n)}_{F_2}(\sigma) = 1\}.
\]

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We define a character $\psi'_\zeta$ of $\tilde{W}_{F_2,n-1}^{(n)}$ by
\[
\psi'_\zeta \left( a^{(n)}(\sigma)y \right) = (-1)^n \psi \left( \pi_{n-1}(\sigma) \right) \quad \text{for} \quad \sigma \in W_{F_2,n-1} \quad \text{and} \quad y \in I_{F_2}^{(n)}.
\]
In the following, we often regard a subgroup $H \subset G_D$ as a subgroup of $G$ by $G_D \hookrightarrow G$; $(g, d) \mapsto (g, d, 1)$. Then, $\tilde{W}_{F_2}^{(n)}$ normalizes $K_{n,\zeta}$. We set $\mathcal{H}_{n,\zeta} = K_{n,\zeta} \tilde{W}_{F_2,n-1}^{(n)}$. Note that
\[
K_{n,\zeta} \cap \tilde{W}_{F_2,n-1}^{(n)} = \begin{cases} 
\{ (1, d, 1) \mid d \in U_{F_2}^{n-1} \} & \text{if } n \text{ is odd}, \\
\{ (g, 1, 1) \mid g \in U_{F_2}^{n-1} \} & \text{if } n \text{ is even}.
\end{cases}
\]
Then, the representation $\tau_{\zeta,\psi}$ of $K_{n,\zeta}$ and the character $\psi'_\zeta$ of $\tilde{W}_{F_2,n-1}^{(n)}$ are consistent. Further $\tilde{W}_{F_2,n-1}^{(n)}$ normalizes $\tau_{\zeta,\psi}$, because we have $\nu(w^{-1}hw) = \nu(h)$ for any $w \in \tilde{W}_{F_2,n-1}^{(n)}$ and $h \in K_{n,\zeta}$.

We set \[
\tau_{\zeta,\psi}(hw) = \psi'_\zeta(w)\tau_{\zeta,\psi}(h) \quad \text{for} \quad h \in K_{n,\zeta} \quad \text{and} \quad w \in \tilde{W}_{F_2,n-1}^{(n)},
\]
which is a representation of $\mathcal{H}_{n,\zeta}$.

Let $\varpi_n \in \mathcal{F}[p^n_{F_2}]_{\text{prim}}$. Then, $\mathcal{H}_{n,\zeta}$ stabilizes the affinoid $X_{n,n,\zeta,\varpi_n}$, and the action on the reduction $\overline{\mathbf{X}}_{n,n,\zeta,\varpi_n}$ is described in §3.

**Proposition 6.6.** We have an isomorphism
\[
H^1_c(\overline{\mathbf{X}}_{n,n,\zeta,\varpi_n}, \mathcal{O}_\ell) \left( \frac{1}{2} \right) \simeq \bigoplus_{\psi \in \mathcal{C}} \tau_{\zeta,\psi}'
\]
as $\mathcal{H}_{n,\zeta}$-representations.

**Proof.** By using (6.5), in the notation of Proposition 3.3 we have
\[
\beta \left( 1 + \varpi^{[\frac{1}{2}]}(c+d\zeta)^{-1}g(a,b) \right) = \frac{(a+bc\zeta)/(c+d\zeta)}{c+d\zeta} \quad \text{for} \quad c+d\zeta \in F_{\varpi}^\times \quad \text{and} \quad g(a,b) \in C_1.
\]

Let $X_0$ be the curve in 4.12. Let $K_{n,\zeta}$ act on $X_0$ through the isomorphism (6.8). Then, by Propositions 2.10, 3.3, 3.4, Lemma 3.5, 4.25, (6.5), (6.6) and (6.7), we have a $K_{n,\zeta}$-equivariant purely inseparable map
\[
\overline{\mathbf{X}}_{n,n,\zeta,\varpi_n} \to X_0; \quad (X,Y) \mapsto \left( X, Y^{q^{n-1}} \right).
\]
Hence, by Lemma 1.16.1, we have isomorphisms
\[
H^1_c(\overline{\mathbf{X}}_{n,n,\zeta,\varpi_n}, \mathcal{O}_\ell) \xrightarrow{\sim} H^1_c(X_0, \mathcal{O}_\ell) \simeq \bigoplus_{\psi \in \mathcal{C}} \tau_{\zeta,\psi}'
\]
as $K_{n,\zeta}$-representations. The claim on the action of Weil group follows from Lemmas 3.1, 3.6 and 4.16.2.

The group $U_{F_2}^0 \times U_{F_2}^0$ normalizes $K_{n,\zeta}$ and $\tilde{W}_{F_2}^{(n)}$ respectively. We set
\[
\mathcal{G}_{n,\zeta} = (U_{F_2}^0 \times U_{F_2}^0) K_{n,\zeta} \tilde{W}_{F_2}^{(n)} \subset G.
\]
Then, we have
\[
\mathcal{G}_{n,\zeta} = \Delta_{\zeta}(\varpi) \left[ (U_{F_2}^0 H_{\zeta}^n \times U_{F_2}^0 H_{\zeta,D}^n) W_{F_2}^{(n)} \right].
\]
Note that
\[
(U_{F_2}^0 H_{\zeta}^n \times U_{F_2}^0 H_{\zeta,D}^n) W_{F_2}^{(n)} \simeq U_{F_2}^0 H_{\zeta}^n \times U_{F_2}^0 H_{\zeta,D}^n \times W_{F_2}^{(n)}.
\]
The group $\mathcal{G}_{n,\zeta}$ acts on $\overline{\mathbf{X}}_{n,\zeta}$ by §3.
Lemma 6.7. We have an isomorphism

$$H^1_c(\mathbf{X}_{n,\zeta}, \overline{Q}_\ell) \simeq \text{Ind}_{\mathcal{H}_{n,\zeta}}^{G_{n,\zeta}} H^1_c(\mathbf{X}_{n,n,\zeta,\pi_n}, \overline{Q}_\ell)$$

as $G_{n,\zeta}$-representations.

Proof. We can check that

$$[G_{n,\zeta} : \mathcal{H}_{n,\zeta}] = |U_{F_2}^n/U_{F_2}^{n-1}|.$$  \hspace{1cm} (6.11)

By Frobenius reciprocity, we have a $G_{n,\zeta}$-equivariant injection

$$H^1_c(\mathbf{X}_{n,\zeta}, \overline{Q}_\ell) \hookrightarrow \text{Ind}_{\mathcal{H}_{n,\zeta}}^{G_{n,\zeta}} H^1_c(\mathbf{X}_{n,n,\zeta,\pi_n}, \overline{Q}_\ell).$$  \hspace{1cm} (6.12)

By (3.3) and (6.11), the both sides of (6.12) have the same dimension. Hence, the required assertion follows. \hfill \Box

We identify $U_{F_2}^{n-1}/U_{F_2}^n$ with $\mathbb{F}_q$ by $1 + \varpi^{n-1} x \mapsto \bar{x}$ for $x \in \mathcal{O}_{F_2}$. Let $C$ be as in (4.2). We set

$$I = \left\{ \chi_0 \in (U_{F_2}^0/U_{F_2}^n) \mid \chi_0|_{U_{F_2}^{n-1}/U_{F_2}^n} \in C \right\}.$$  \hspace{2cm} \hspace{1cm}

Let $\chi_0 \in I$. We set $\psi = \chi_0|_{U_{F_2}^{n-1}/U_{F_2}^n} \in C$. Let $(F_2/F, \chi)$ be the minimal admissible pair such that

$$l(\chi) = n - 1, \quad \chi|_{U_{F_2}^0/U_{F_2}^n} = \chi_0, \quad \chi(\varpi) = 1.$$  \hspace{1cm}

Let $\Lambda^{\psi}$ and $\Lambda_\chi$ denote the irreducible representations of $J_{1,n}$ and $J_{2,n}$ in (5.1.3) respectively. Note that

$$U_{F_2}^0 H_{\zeta}^n = U_{F_2}^0 U_{2n}^{[\frac{2}{k}]} \subset F_2^x U_{2n}^{[\frac{2}{k}]} = J_{1,n},$$

$$U_{F_2}^0 H_{\zeta,D}^n = U_{F_2}^0 U_{D}^{n-1} \subset F_2^x U_{D}^{n-1} = J_{2,n},$$

because $U_{F_2}^0 U_{D}^{2k+1} = U_{F_2}^0 U_{D}^{2k}$ for any integer $k \geq 0$. We set

$$\tau_{\zeta,\chi_0} = \Lambda^{\psi}|_{U_{F_2}^0 H_{\zeta}^n}, \quad \tau_{\zeta,\chi_0}^D = \Lambda_\chi|_{U_{F_2}^0 H_{\zeta,D}^n}. \hspace{2cm} (6.13)$$

Note that one of (6.13) is one-dimensional, and the other is $q$-dimensional. Let $\xi'_{\zeta,\chi_0}$ denote the character of $W_{F_2}^{(n)}$ defined by

$$\xi'_{\zeta,\chi_0}(a^{(n)}(\sigma)) = (-1)^{n^*} \chi_0(a_{F_2,\varpi}(\sigma)) \quad \text{for} \quad \sigma \in W_{F_2}.$$  \hspace{2cm}

We put

$$\pi_{\zeta,\chi_0} = \tau_{\zeta,\chi_0} \otimes \tau_{\zeta,\chi_0}^D \otimes \xi'_{\zeta,\chi_0},$$

which can be regarded as a $G_{n,\zeta}$-representation with trivial action of $\Delta_\chi(\varpi)$ (cf. (6.10)).

Remark 6.8. Let $\chi$ be a character of $F_2^x$ such that $l(\chi) = n - 1$ and $\chi|_{U_{F_2}^0/U_{F_2}^n} \in I$. Then, $(F_2/F, \chi)$ is a minimal admissible pair by [Se, Proposition 2 ii) in V §2].

Proposition 6.9. We have an isomorphism

$$H^1_c(\mathbf{X}_{n,\zeta}, \overline{Q}_\ell) \left( \frac{1}{2} \right) \simeq \bigoplus_{\chi_0 \in I} \pi_{\zeta,\chi_0} \hspace{2cm} (6.14)$$

as $G_{n,\zeta}$-representations.
Proof. We set $\psi = \chi_0|_{U_{p_2}^1/U_{p_2}^0} \in C$. We will check $\pi_{\zeta,\chi_0}|_{U_{n,\zeta}^+} \simeq \tau_{\zeta,\psi}$. As mentioned in $\text{(5.1.3)}$, the restrictions $\Lambda_\chi|_{U_{p_2}^1/U_{p_2}^0(m+1)}$ and $\Lambda_\chi|_{U_{p_2}^1/U_{p_2}^0}$ are multiples of characters. Note that

$$H^\zeta_{n+1} \subset U_{p_2}^{[n+1]}; \quad H^\zeta_{n,D} \subset U_{p_2}^n.$$ By $\text{(5.1.3)}$, we can check that the restriction $(\tau_{\zeta,\chi_0} \otimes \tau_{\zeta,\chi_0}^D)|_{K_{n,\zeta}}$ is trivial, and the representation $(\tau_{\zeta,\chi_0} \otimes \tau_{\zeta,\chi_0}^D)|_{K_{n,\zeta}}$ satisfies $(6.9)$. Hence, by Lemma $\text{4.14}$, we have an isomorphism $\tau_{\zeta,\psi} \simeq (\tau_{\zeta,\chi_0} \otimes \tau_{\zeta,\chi_0}^D)|_{K_{n,\zeta}}$ as $K_{n,\zeta}$-representations. The restriction $\pi_{\zeta,\chi_0}|_{U_{p_2}^1}$ is trivial. Clearly, we have $\xi_{\zeta,\chi_0}|_{U_{p_2}^1} = \psi'_{\zeta,\psi}$. Hence, we have $\pi_{\zeta,\chi_0}|_{U_{n,\zeta}^+} \simeq \tau_{\zeta,\psi}_{\zeta,\chi_0}$.

The representations $\{\pi_{\zeta,\chi_0}\}_{\chi_0 \in I}$ are different from each other. Hence, by Proposition $\text{6.6}$, Lemma $\text{6.4}$ and Frobenius reciprocity, we have a $G_{n,\zeta}$-equivariant injection

$$\bigoplus_{\chi_0 \in I} \pi_{\zeta,\chi_0} \hookrightarrow H^1_c(X_{n,\zeta}, \overline{Q}_\ell) \left(\frac{1}{2}\right). \quad (6.15)$$ We have $|I| = q^{2n-3}(q-1)(q^2-1)$ and $\dim \pi_{\zeta,\chi_0} = q$. Since the both sides of $(6.15)$ are $q^{2n-1}(q-1)(q^2-1)$-dimensional by Proposition $\text{2.10}$ $(5.3)$ and Lemma $\text{4.16}$, the required assertion follows.

We consider the affinoid $X_{n,\zeta} \subset X(0)(p^n) \subset \text{LT}(p^n)$, for which we write $X_{n,\zeta}^{(0)}$. Let $\varpi: \text{LT}(p^n) \rightarrow \text{LT}(p^n)$ be the automorphism induced by the action of $\varpi \in F^\times \subset D^\times$. We set

$$J_n = (1, \varpi^Z, 1)G_{n,\zeta} = J_{1,n} \times J_{2,n} \times W_{F_2} \subset (F^\times \text{GL}_2(\mathcal{O}_F)) \times D^\times \times W_F.$$ Then, we consider $\coprod_{i \in \mathbb{Z}} \varpi^i X^{(0)}_{n,\zeta,\mathbb{C}}$, on which $J_n$ acts. Then, we have a $J_n$-equivariant injection

$$X_{n,\zeta,\mathbb{C}} \simeq \left(\prod_{i \in \mathbb{Z}} \varpi^i X^{(0)}_{n,\zeta,\mathbb{C}}\right) / \varpi^Z \hookrightarrow \left(\text{LT}(p^n)/\varpi^Z\right)_\mathbb{C}, \quad (6.16)$$ where $X_{n,\zeta,\mathbb{C}}$ admits the trivial action of the element $\varpi \in F^\times \subset D^\times$. The right hand side of $(6.16)$ is non-canonically isomorphic to a disjoint union of two copies of $X(p^n)$.

**Proposition 6.10.** Let $(F_2/F, \chi)$ be a minimal admissible pair such that $l(\chi) \geq 1$ and $\chi(\varpi) = 1$. Then we have a $G$-equivariant injection

$$\pi_{\chi^\psi} \otimes \rho_\chi \otimes \tau_\chi \hookrightarrow \mathcal{U}_c.$$ Proof. Let $l(\chi) = n - 1$ with $n \geq 2$. By Proposition $\text{2.10}$ Lemmas $\text{4.5}$ $2$, $(4.16)$, $(5.24)$ and $(6.16)$, we have a $J_n$-equivariant injection

$$H^1_c(X_{n,\zeta}, \overline{Q}_\ell) - \left(\frac{1}{2}\right) \hookrightarrow H^1_c((\text{LT}(p^n)/\varpi^Z)_\mathbb{C}, \overline{Q}_\ell) - \left(\frac{1}{2}\right) \subset \mathcal{U}_c. \quad (6.17)$$ We put $\chi_0 = \chi|_{U_{p_2}^0/U_{p_2}^0}$. We have $\chi_0 \in I$. We regard $\pi_{\zeta,\chi_0}$ as a $J_n$-representation with trivial $(1, \varpi, 1)$-action. Let

$$\pi_{\zeta,\chi_0} = \Lambda_\chi^{\psi} \otimes \Lambda_0^\chi \otimes \Delta_0 \chi,$$ be as in $\text{(5.1.3)}$, for which we write $\Lambda_\chi$. It is clear that

$$\Lambda_\chi(1, \varpi, 1) = 1, \quad \Lambda_\chi|_{U_{p_2}^0 H_\zeta^{x_0} \times U_{p_2}^0 H_\zeta^{D}} \simeq \tau_{\zeta,\chi_0} \otimes \tau_{\zeta,\chi_0}^D. \quad (6.18)$$
For $\sigma \in W_{E_2}$, we have
\[
\Lambda_x (a^{(n)}(\sigma)) = \Delta_0(\sigma) \chi(\sigma) = (-1)^{n_\sigma} \chi(\sigma) = (-1)^{n_\sigma} \chi_0 (a_{F_2,\infty}^0(\sigma)) = \xi c_{\chi_0} (a^{(n)}(\sigma)). \quad (6.19)
\]
By (6.18) and (6.19), we have an isomorphism $\pi_{\xi,\chi_0} \simeq \Lambda_x$ as $\mathbf{J}_n$-representations. Hence, by Proposition 6.9 and (6.17), we obtain a $\mathbf{J}_n$-equivariant injection $\Lambda_x \hookrightarrow \mathcal{U}_c$. The required assertion follows from Frobenius reciprocity.

\[\square\]

6.0.6 Ramified case

Let $E$ be a totally ramified quadratic extension of $F$. We take a uniformizer $\varpi_E$ of $E$ such that $\varpi_E^2 \in F$. We write $\varpi$ for $\varpi_E^2$. Let $n$ be an odd positive integer. Let $H_{E}^n$ and $H_{E,D}^n$ be as in (3.25) and (3.33) respectively. Let $E^\times$ as a subgroup of $G_D$ via $\Delta_E$ in (3.35). We set
\[
\mathcal{P}_{E,n} = F^\times U_E^1 \left( H_{E}^n \times H_{E,D}^n \right) \subset \mathcal{K}_{E,n} = E^\times \mathcal{P}_{E,n} \subset G_D.
\]
We consider the projections $s_{E,1} : M_2(F) \rightarrow E$ and $s_{E,2} : D \rightarrow E$ which are induced by (3.24) and (3.29) respectively. We set $s_{E} : M_2(F) \times D \rightarrow E$; $(x,y) \mapsto s_{E,2}(y) - s_{E,1}(x)$. Let $\psi \in \mathbb{F}_q' \setminus \{1\}$. We set
\[
\tilde{\psi} (x(1 + y)) = \psi \left( 2 \varpi^n s_{E}(y) \right) \text{ for } x \in F^\times U_E^1 \text{ and } 1 + y \in H_{E}^n \times H_{E,D}^n. \quad (6.20)
\]
This is well-defined and determines a character of $\mathcal{P}_{E,n}$. Furthermore, $E^\times$ normalizes the character $\tilde{\psi}$. This character extends uniquely to a character $\tilde{\psi}'$ of $\mathcal{K}_{E,n}$ such that $\tilde{\psi}'(\varpi_E) = -1$.

Let $W_{E,n}^' = \{(1, \varphi^{-n_\sigma}, \sigma) \in G \mid \sigma \in W_{E,n} \} \subset W_{E}^' = \{(1, \varphi^{-n_\sigma}, \sigma) \in G \mid \sigma \in W_{E} \}.$

Let $I_{E}^n$ be as in (3.38). Note that $W_{E}^'$ normalizes $I_{E}^n$. We set $\bar{W}_{E,n} = W_{E,n} I_{E}^n \subset \bar{W}_{E} = W_{E} I_{E}^n$. Let $\kappa : \{\pm 1\} \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be the non-trivial character. Then, we have $G_{1,2,1}(\kappa, \psi) = -\tau(\kappa_{E/F}; \psi_F)$ in the notation of (4.11). We define a character $\psi'$ of $\bar{W}_{E,n}$ by
\[
\bullet \; \psi'(1, \varphi^{-n_\sigma}, \sigma) = \left\{ - \left( \frac{1}{q} \right)^{m-1} \lambda_{E/F}(\psi_F) \right\}^{n_\sigma} \psi(2 \pi_{E,n}(\sigma)) \text{ for } \sigma \in W_{E,n}, \text{ and}
\]
\[
\bullet \; \psi'(1, d, \sigma) = \left( \frac{d}{q} \right)^{\kappa_{E,n}} \text{ for } (1, d, \sigma) \in I_{E}^n.
\]
We set $\mathcal{H}_{E,n} = \mathcal{K}_{E,n} \bar{W}_{E,n}$. Note that
\[
\mathcal{K}_{E,n} \cap \bar{W}_{E,n} = \{(1, d, 1) \mid d \in U_{E}^n \},
\]
\[
\tilde{\psi}'|_{\mathcal{K}_{E,n} \cap \bar{W}_{E,n}} = \psi'|_{\mathcal{K}_{E,n} \cap \bar{W}_{E,n}}.
\]
Hence, $\tilde{\psi}'$ and $\psi'$ determines the character $\Phi_{\psi}$ of $\mathcal{H}_{E,n}$.

Let $\varpi_{E,n+1} \in \mathcal{P}^{[n+1]}_{\text{prim}}$. Then, $\mathcal{H}_{E,n}$ stabilizes the affinoid $Z_{n,n,\varpi_{E,n+1}}$, and the action on the reduction $\bar{Z}_{n,n,\varpi_{E,n+1}}$ is described in 33

**Proposition 6.11.** We have an isomorphism
\[
H_{c}^1(\overline{Z_{n,n,\varpi_{E,n+1}}}, \mathbb{Q}_{\ell}) \left( \frac{1}{2} \right) \simeq \bigoplus_{\psi \in \mathbb{F}_q' \setminus \{1\}} \Phi_{\psi}
\]
as $\mathcal{H}_{E,n}$-representations.
Proof. By Proposition \[2.12\] Lemma \[3.2\] all the results in \[3.5\] and \[3.6\] Lemma \[4.10\] 1, Corollary \[4.11\] 1 and Lemma \[4.12\] we have the claim. \[\square\]

We set
\[G_{E,n} = \left( (U^0_E \times U^0_E) \mathcal{K}_{E,n} \right) \tilde{W}_E = \left( E^\times (U^0_E H^0_E \times U^0_E H^0_{E,D}) \right) W'_E \subset G.\]

**Lemma 6.12.** Let \(n\) be an odd positive integer. Then, we have an isomorphism
\[H^1_c(\mathbb{Z}_{\varpi E,n}, \overline{Q}_l) \simeq \text{Ind}_{H^0_{E,n}}^{G_{E,n}} H^1_c(\mathbb{Z}_{n,n,\varpi E,n+1}, \overline{Q}_l)\]
as \(G_{E,n}\)-representations.

**Proof.** We can check that
\[\left[ G_{E,n} : H_{E,n} \right] = \left| U^0_E / U^0_E \right|. \tag{6.21}\]
We have a \(G_{E,n}\)-equivariant injection
\[H^1_c(\mathbb{Z}_{\varpi E,n}, \overline{Q}_l) \hookrightarrow \text{Ind}_{H^0_{E,n}}^{G_{E,n}} H^1_c(\mathbb{Z}_{n,n,\varpi E,n+1}, \overline{Q}_l). \tag{6.22}\]
Since the both sides of \((6.22)\) have the same dimension by \((3.5)\) and \((6.21)\), the claim follows. \[\square\]

We identify \(U^0_E / U^0_{E+1}^{n+1}\) with \(\mathbb{F}_q\) by \(1 + \varpi_n x \mapsto \bar{x}\) for \(x \in \mathcal{O}_E\). We set
\[I^E = \left\{ x_0 \in \left( U^0_E / U^0_{E+1}^{n+1} \right)^\vee \mid x_0|_{U^0_E / U^0_{E+1}^{n+1}} \in \mathbb{F}_q \setminus \{1\} \right\}. \tag{6.23}\]

Let \(\chi_0 \in I^E\). For a character \(\phi \in \mathbb{F}_q^\times \setminus \{1\}\), let \(\phi_2\) denote the character defined by \(x \mapsto \phi(2x)\) for \(x \in \mathbb{F}_q\). Let \(\psi \in \mathbb{F}_q^\times \setminus \{1\}\) be the character such that \(\psi_2 = \chi_0|_{U^0_E / U^0_{E+1}^{n+1}}\). Recall that the character \(\tilde{\psi}\) of \(\mathcal{P}_{E,n}\) is normalized by \(E^\times\). We define characters \(\psi_{\chi_0}\) of \(U^0_E H^0_E\) and \(\psi_{\chi_0}^D\) of \(U^0_E H^0_{E,D}\) by
\[
\psi_{\chi_0}(gx) = \chi_0(g)\tilde{\psi}(g,1) \quad \text{for} \quad x \in U^0_E \quad \text{and} \quad g \in H^0_E,
\]
\[
\psi_{\chi_0}^D(xd) = \chi_0(x)\tilde{\psi}(1,d) \quad \text{for} \quad x \in U^0_E \quad \text{and} \quad d \in H^0_{E,D}
\]
respectively (cf. \((6.20)\)). Note that
\[(E^\times (U^0_E H^0_E \times U^0_E H^0_{E,D})) \cap W'_E = \{1\}.\]

We define a character \(\Phi_{E,\chi_0}\) of \(G_{E,n}\) by
\[
\bullet \quad \Phi_{E,\chi_0}|_{U^0_E H^0_E \times U^0_E H^0_{E,D}} = \psi_{\chi_0} \otimes \psi_{\chi_0}^D,
\]
\[
\bullet \quad \Phi_{E,\chi_0}(\varpi_E) = -1, \quad \text{and}
\]
\[
\bullet \quad \Phi_{E,\chi_0}(1, \varphi^{-m}, \sigma) = \left\{ -\left( \frac{1}{\varpi_q} \right)^{m-1} \lambda_{E/F}(\psi_F) \right\}^{n_o} \left( \frac{\chi_0(a_{E,\varpi E}(\sigma))}{\varpi_q} \right) \chi_0(a_{E,\varpi E}(\sigma)) \quad \text{for} \quad \sigma \in W_E.
\]

We can directly check that
\[
\Phi_{E,\chi_0}|_{\mathcal{K}_{E,n}} = \tilde{\psi}', \quad \Phi_{E,\chi_0}|_{\overline{W}_E} = \psi'.
\]
Hence, we have
\[\Phi_{E,\chi_0}|_{H_{E,n}} = \Phi_{\psi}. \tag{6.23}\]
Remark 6.13. Let \( \chi \) be a character of \( E^\times \) such that \( l(\chi) = n \) and \( \chi|_{U_E^n U_E^{n+1}} \in I^E \). Then, \((E/F, \chi)\) is a minimal admissible pair, because any character of \( E^\times \), which factors through the norm map \( \text{Nr}_{E/F} \), has an even level by \([\text{Se}, \text{Corollary 3 in V} \S 3]\).

Proposition 6.14. We have an isomorphism

\[
H_1^c(\mathcal{Z}_{\varphi E,n}, \mathfrak{g}_\ell) \left( \frac{1}{2} \right) \cong \bigoplus_{\chi_0 \in I^E} \Phi_{E,\chi_0}
\]

as \( G_{E,n} \)-representations.

Proof. Let \( \chi_0 \in I^E \). Let \( \psi \in F_q^\times \setminus \{1\} \) be the character such that \( \psi^2 = \chi_0|_{U_E^n U_E^{n+1}} \). By Proposition 6.11, Lemma 6.12, (6.23) and Frobenius reciprocity, we have a \( G_{E,n} \)-equivariant injection

\[
\bigoplus_{\chi_0 \in I^E} \Phi_{E,\chi_0} \hookrightarrow H_1^c(\mathcal{Z}_{\varphi E,n}, \mathfrak{g}_\ell) \left( \frac{1}{2} \right).
\]

Since the both sides of (6.25) are \( q^n - 1 \) \( q^n - 1 \)-dimensional by Proposition 2.12, (3.5) and Lemma 4.11.1, we obtain the claim. \( \square \)

We set

\[
\text{LT}'(p^n) = \text{LT}(p^{n+1})/U_3^{n+1}, \quad Y^{(h)}(p^n) = X^{(h)}(p^n)/U_3^{n+1}.
\]

Then, we have

\[
\text{LT}'(p^n) = \prod_{h \in \mathbb{Z}} Y^{(h)}(p^n).
\]

Note that \( U_3^{n+1} \) is a normal subgroup of the standard Iwahori subgroup \( I^\times \), and \( \varphi_E^{-1} U_3^{n+1} \varphi_E = U_3^{n+1} \). Thereby, the product group \((\varphi_E^{-1} \varphi_E) \times D^\times \times W_F \) acts on \( \text{LT}'(p^n)_C \) (cf. \( \S 3.1 \)). Let \( \varphi : \text{LT}'(p^n) \to \text{LT}'(p^n) \) be the automorphism induced by the action of \( \varphi \in D^\times \). We consider the affinoid \( L_{\varphi E,n} \subset Y^{(0)}(p^n) \subset \text{LT}'(p^n) \), for which we write \( Z_{\varphi E,n} \). We have

\[
(1, \varphi_Z, 1)G_{E,n} \subset (\varphi_E^{-1} \varphi_E) \times D^\times \times W_F.
\]

Hence, we have a \( (1, \varphi_Z, 1)G_{E,n} \)-equivariant injection

\[
Z_{\varphi E,n}' = \left( \bigotimes_{i \in \mathbb{Z}} \phi_i^* Z_{\varphi E,n, C}^{(0)} \right)/\varphi_Z \hookrightarrow (\text{LT}'(p^n)/\varphi_Z)_C.
\]

We set

\[
J_{E,n}' = (1, \varphi_Z, 1)G_{E,n} \subset J_{E,n} = (1, \varphi_Z, 1)G_{E,n} = J_{E,1,n} \times J_{E,2,n} \times W_E.
\]

We have \([J_{E,n} : J_{E,n}'] = 2\).

Proposition 6.15. Let \((E/F, \chi)\) be a minimal admissible pair such that \( \chi(\varphi) = 1 \). Then, we have a \( G \)-equivariant injection

\[
\pi_{\chi^\vee} \otimes \rho_{\chi} \otimes \tau_{\chi} \hookrightarrow \mathcal{U}_c.
\]
Proof. Let \( n \) be the level of \( \chi \). Note that \( \mathbb{Z}_E^{n} \) is isomorphic to a disjoint union of two copies of \( \mathbb{Z}_{E,n,C} \). By Proposition 2.12, Lemma 4.5.2, Corollary 4.11.2, (5.24) and (6.17), we have an injective \( J_{E,n} \)-equivariant homomorphism

\[
H_1^c(\mathbb{Z}_{E,n}, \mathbb{Q}_\ell) \left( \frac{1}{2} \right) \hookrightarrow H_1^c((\mathcal{L}^{n}((\mathcal{L}^{n}(p_n)/\mathbb{Z})_{C}, \mathbb{Q}_\ell)) \left( \frac{1}{2} \right) \subset \mathcal{U}_c.
\]

(6.27)

We have an isomorphism

\[
H_1^c(\mathbb{Z}_{E,n}, \mathbb{Q}_\ell) \left( \frac{1}{2} \right) \simeq \text{Ind}_{J_{E,n}}^{J_{E,n}} H_1^c(\mathbb{Z}_{E,n}, \mathbb{Q}_\ell) \left( \frac{1}{2} \right)
\]

(6.28)

as \( J_{E,n} \)-representations. We set \( \chi_0 = \chi \mid U_0 \). We have \( \chi_0 \in I_{E,n} \). For \( \iota \in \{\pm 1\} \), let \( \Phi_{E,\chi_0}^{\iota} \) be the character of \( J_{E,n} \) such that \( \Phi_{E,\chi_0}^{\iota}(1, \varphi, 1) = \iota \). Then, by Proposition 6.14, (6.27) and (6.28), we have

\[
\Phi_{E,\chi_0}^{\iota} \subset H_1^c(\mathbb{Z}_{E,n}, \mathbb{Q}_\ell) \left( \frac{1}{2} \right) \subset \mathcal{U}_c.
\]

(6.29)

Assume that \( \chi_0(\varpi_E) = \iota \). Let

\[
\Lambda_{E,\chi_0} \otimes \Lambda_{E,\chi_0} \otimes \Delta_{E,\chi_0, \iota}
\]

be the \( J_{E,n} \)-representation defined in §5.1.4. We simply write \( \Lambda_{\chi_0} \) for it. Then, we easily check that

\[
\Lambda_{\chi_0}(1, \varphi, 1) = -\iota, \quad \Lambda_{\chi_0}(\psi^{H}_E \times \psi^{H}_E) = \psi_{\chi_0} \otimes \psi_{\chi_0}^D, \quad \Lambda_{\chi_0}(\varpi_E, \varpi_E, 1) = -1.
\]

(6.30)

The element \( \zeta(\alpha, \chi_0) \) in the notation of §5.1.4 equals 1. Hence, for \( \sigma \in W_E \), we have

\[
\Lambda_{\chi_0}(1, \varphi^{-\sigma}, \sigma) = (-\iota)^{\nu_{E}} \Delta_{E,\chi_0}(\sigma) \chi(\sigma)
\]

\[
= (-\lambda_{E,F}(\psi_F)_{\sigma})^{\nu_{E}} \Delta_{E,\chi_0}(a_{E,\varpi_E}^{0}(\sigma)) \chi_0(a_{E,\varpi_E}^{0}(\sigma))
\]

(6.31)

where we use the definition of \( \Delta_{E,\chi_0} \) and (5.5) at the second and the last equalities. By (6.30) and (6.31), as \( J_{E,n} \)-representations, \( \Phi_{E,\chi_0}^{\iota} \) is isomorphic to \( \Lambda_{\chi_0} \). Therefore, by (6.29) and Frobenius reciprocity, we obtain the claim.

\[\square\]

6.0.7 Conclusion

As a result of the analysis in this section, we obtain Proposition 5.16.

Corollary 6.16. Proposition 5.16 holds.

Proof. The required assertion follows from Propositions 6.3, 6.10 and 6.15.

\[\square\]

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