Numerical Approximation of Integrals in Presence of Nearby Singularities

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Abstract: The quadrature formulas meant for the numerical approximation of integrals of one-dimensional real variables need to be modified for the sake of accuracy when a singularity is present in the proximity of the path of integration. The required corrective factor has been constructed and some existing quadrature rule has been applied with the corrective factor to obtain better accuracy.

Keywords: Quadrature rules, Singularity, Principal part.

AMS classification: 65 D 30

1. Introduction

Integrals with and without singularities occur quite frequently in engineering, different branches of science and applied mathematics. Many problems of practical nature in economics, statistics and atmospheric sciences involve singular integrals. It is well known that the quadrature rules even possessing a higher degree of precisions fail to produce the required degree of accuracy if singularities of the integrand are in the proximity of the path of integration. However, Atkinson (1987) defines singular integral in a general term as ‘an integral is a singular integral if the standard methods of integration either do not apply or lead to slow convergence’. We consider the integrand having singularities nearer to the path of integration. The integral considered in this paper is given by

\[ I(f) = \int_{-1}^{1} f(x)dx \]  

(1)

where f(x) is a real-valued function of one dimension.

Recently, Mohanty, Acharya (2016) (2020) have constructed different mixed quadrature rules for the numerical evaluation of the integral I(f) given in equation (1) with their error analysis. In the present work, we have considered the six-point mixed quadrature rule as follows:

\[ R_6(f) = A_{11}(f(x_{11}) + f(-x_{11})) + A_{12}(f(x_{12}) + f(-x_{12})) + A_{13}x_{13}(f'(x_{13}) - f'(-x_{13})) \]  

(2)

where the coefficients and nodes are presented in table-1.

Table-1: coefficients and nodes of six point mixed rule

| Coefficient | Node    |
|-------------|---------|
| A_{11} = 0.508701586231572 | x_{11} = 0.715671062685404 |
| A_{12} = 0.491298413768429  | x_{12} = 0.249812196512064  |
| A_{13} = 0.027452120847800  | x_{13} = 0.875914167779317  |
Even though the rule $R_1(f)$ has a degree of precision eleven which is the same with the Gauss-Legendre six-point rule. It has been observed that the accuracy of the rule is adversely affected if the integrand is having singularities nearer to the path of integration. Therefore, the quadrature rule is needed to be modified to get the desired accuracy.

2. Review of Related Studies

P.M. Mohanty and M. Acharya (2016) have derived some six point eleventh degree quadrature rules involving derivatives of the integrand. S.B. Sahoo, B.P. Acharya and M. Acharya (2015) have worked on numerical approximation of contour integrals in presence of nearby singularities of the integrand. B.P. Acharya, M. Acharya and S. Mohapatra (2012) have formulated a class of eight point rules of degree five for the said integral and the error has been determined for obtaining optimal value. They (2011) have also formulated interpolatory rules for the numerical approximation of complex Cauchy principal value integrals in two dimensions and observed the maximum accuracy of computed values which is the close proximity of the point. K. Diethelm (2000) has investigated two possible approaches to two dimensional CPV problems, corresponding to generalizations of two approaches known in the 1-D case. In principle, both methods can be applied to integration domains of arbitrary shape, although he has found that certain combinations of algorithms and domains are more useful than others. In particular, he has discussed error estimates and has shown that the methods are highly competitive. Moreover, in contrast to most of the previously discussed methods, the approaches are very efficient when integrals have to be calculated for various locations of the singularity. He (1998) has derived the expressions for error bounds for spline-based quadrature methods for strongly singular integrals. G. Monegato (1982) has examined the numerical integration (in the Cauchy principal value sense) of functions having (several) first order real poles and surveyed of results concerning some quadrature formulas of interpolatory type proposed by Delves, Hunter, Elliott and Paget and several other authors; along with the description some minor generalizations and make comments on the computational aspects are presented. B.P. Acharya, R.N. Das (1981) a quadrature rule for numerical evaluation of Cauchy principal value integrals of the type

$$\int_{-1}^{1} \frac{f(x)}{(x-a)}dx$$

where $-1 < a < 1$ and $f(x)$ possesses complex singularities near to the path of integration has been formulated. An analysis of the error has been obtained. F. Lether (1977) has derived a method for subtracting out singularities in numerical integration of the function whose singularities are present close to the path of integration.

3. Objectives Of The Study

- To construct the corrective factor for the rule given in equation (2).
- To check whether the accuracy of the rule has any influence when the singularities present in the close proximity of the path of integration.
- To find out the significant difference between the accuracy of the rule without using corrective factor and using the corrective factor.

4. Construction Of The Corrective Factor

Let the integrand $G(x)$ is having singularities close to the path of integration.

$$I(G) = \int_{-1}^{1} G(x)dx = \int_{-1}^{1} \frac{f(x)}{x^2 + a^2} dx$$  \hspace{1cm} (3)

Applying partial fraction to the above equation (3) we have,

$$I(G) = \frac{1}{2ai} \left[ \int_{-1}^{1} \frac{f(x)}{x - ai} dx - \int_{-1}^{1} \frac{f(x)}{x + ai} dx \right]$$
\[
I(G_1(x)) = I\left(G_1(x) - \phi_1(x)\right) + \int_{-1}^{1} \phi_1(x) \, dx,
\]
\[
I(G_2(x)) = I\left(G_2(x) - \phi_2(x)\right) + \int_{-1}^{1} \phi_2(x) \, dx.
\]

These two integrals \(G_1(x) - \phi_1(x)\) and \(G_2(x) - \phi_2(x)\) can be evaluated using the quadrature rule \(R_1(f)\). These integrands are denoted by
\[
\psi_1(x) = \int_{-1}^{1} (G_1(x) - \phi_1(x)) \, dx,
\]
\[
\psi_2(x) = \int_{-1}^{1} (G_2(x) - \phi_2(x)) \, dx.
\]

The second integrals are given in equation (7) i.e., \(\int_{-1}^{1} \phi_1(x) \, dx\) and \(\int_{-1}^{1} \phi_2(x) \, dx\) are obtained by direct integration.

Hence the corrective factor \(C_r\) for the integrand \(G(x)\) is
\[
C_r = \frac{\int_{-1}^{1} \phi_1(x) \, dx - \int_{-1}^{1} \phi_2(x) \, dx}{2ai} = \frac{\left\{ f(ai) + f(-ai) \right\} \log \frac{ai -1}{ai +1} + 2\left( f'(ai) - f'(-ai)\right) - ai\left( f''(ai) + f''(-ai)\right)}{2ai}
\]

Finally, the approximation of the integral \(G(x)\) applying the quadrature rule is given by
\[
\int_{-1}^{1} G(x) \, dx = \int_{-1}^{1} \frac{G_1(x) - G_2(x)}{2ai} \, dx = \frac{\psi_1(x) - \psi_2(x)}{2ai} + C_r \quad (10)
\]

5. Conclusions and Numerical Verifications

The degree of precision of each of the rule \( R_1(f) \) is eleven. However, from the angle of a degree of precision, the rule is at par with the Gauss-Legendre six-point rule. The integral \( I(G) \) given by equation (3) has been computed for the function \( f(x) = \cos x \) is given as follows:

\[
J_1(x) = \int_{-1}^{1} \frac{\cos x}{1+a^2} \, dx \quad (11)
\]

The integral \( J_1 \) has been computed for different values of ‘\( a \)’ using the rule \( R_1(f) \) and also evaluated using corrective factor \( C_r \). The exact values, corrective factors, computed values without corrective factor \( C_r \), and with corrective factor \( C_r \) are presented in Table -2. Comparing the error magnitudes of the rule for the evaluation of the integral \( J_1 \) given in the following table and from which it can be observed that errors without a corrective factor are very high, not accurate to even one decimal place. But errors using corrective factor are very less and highly accurate. The magnitude of an error associated with any standard integration rule depends upon the shortest distance of the singularity from the path of integration. Less the shortest distance, more is the magnitude of error and this leads to the failure of the standard method of integration. Fig 1 plots the graph between the value of ‘\( a \)’ and logarithm of the logarithm of error for the integral \( J_1 \) using the rule \( R_1(f) \). It is observed that when the value of ‘\( a \)’ increases the \( \log(\log(\text{error})) \) increases. It is concluded that if the integrand has nearby singularities, then the corrective factor has a significant role in restoring the accuracy of the rule.

| a     | Exact value  | Corrective factor Cr | The Magnitude of Error without Cr | The Magnitude of Error with Cr |
|-------|--------------|----------------------|----------------------------------|-------------------------------|
| 0.01  | 311.2021611240909 | 311.1749567852309 | 294.6111324281873                | 5.68e-14                     |
| 0.02  | 154.1381125760985 | 154.1109829665775 | 137.6206280539343                | 8.53e-14                     |
| 0.03  | 101.7937374922099 | 101.7667324426927 | 85.3972921393337                 | 4.26e-14                     |
| 0.04  | 75.629220226823833 | 75.602389584634125 | 59.3990827864406                 | 2.84e-14                     |
| 0.05  | 59.93659816999513 | 59.909991805785978 | 43.9150647954162                 | 1.42e-14                     |
| 0.06  | 49.479884187954902 | 49.453552002360446 | 33.7056615814113                 | 1.42e-14                     |
| 0.07  | 42.015084114924534 | 41.989076045258344 | 26.5228203599915                 | 1.42e-14                     |
| 0.08  | 36.420201126502491 | 36.394567153416403 | 21.2401717500782                 | 2.13e-14                     |
| 0.09  | 32.071903864482628 | 32.046694018652346 | 17.2298505970511                 | 1.42e-14                     |
| 0.10  | 28.596193687196820 | 28.571458056008794 | 14.1133048535637                 | 3.55e-15                     |

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Fig-1: The graph between the value of ‘a’ and logarithm of the logarithm of error for the integral $\int_1$ using the rule $R_1(f)$.

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