SHARP SPECTRAL MULTIPLIERS FOR HARDY SPACES ASSOCIATED TO NON-NEGATIVE SELF-ADJOINT OPERATORS SATISFYING DAVIES-GAFFNEY ESTIMATES

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Abstract. We consider the abstract non-negative self-adjoint operator $L$ acting on $L^2(X)$ which satisfies Davies-Gaffney estimates and the corresponding Hardy spaces $H^p_L(X)$. We assume that doubling condition holds for the metric measure space $X$. We show that a sharp Hörmander-type spectral multiplier theorem on $H^p_L(X)$ follows from restriction type estimates and the Davies-Gaffney estimates. We also describe the sharp result for the boundedness of Bochner-Riesz means on $H^p_L(X)$.

1. Introduction

Suppose that $L$ is a non-negative self-adjoint operator acting on $L^2(X,\mu)$, where $X$ is a measure space with measure $\mu$. Then $L$ admits a spectral resolution $E(\lambda)$ and for any bounded Borel function $F : [0, \infty) \to \mathbb{C}$, one can define the operator

$$F(L) = \int_0^\infty F(\lambda)dE(\lambda).$$

(1.1)

By the spectral theorem, this operator is bounded on $L^2(X)$. Spectral multiplier theorems give sufficient conditions on $F$ and $L$ which imply the boundedness of $F(L)$ on various functional spaces defined on $X$. This is one of active topics in harmonic analysis and has been studied extensively. We refer the reader to [1, 5, 7, 8, 10, 13, 15, 16, 19, 21, 29, 33] and the references therein.

Before we state our main result we describe our basic assumptions. We throughout assume that the considered metric measure space $(X,d,\mu)$ with a distance $d$ and a non-negative Borel measure $\mu$ satisfies the volume doubling condition: there exists a constant $C > 0$ such that for all $x \in X$ and for all $r > 0$,

$$V(x,2r) \leq CV(x,r) < \infty,$$

(1.2)

where $V(x,r)$ is the volume of the ball $B(x,r)$ centered at $x$ of radius $r$. In particular, $X$ is a space of homogeneous type. See for example [11].

Note that the doubling condition (1.2) implies that there exist some constants $C,n > 0$ such that

$$V(x,\lambda r) \leq C\lambda^nV(x,r)$$

(1.3)

uniformly for all $\lambda \geq 1$ and $x \in X$. In the sequel, we shall consider $n$ as small as possible. In the Euclidean space with Lebesgue measure, the parameter $n$ is the dimension of the space.

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In this paper, we consider the following two conditions corresponding to the operator $L$.

First, the operator $L$ is a non-negative self-adjoint operator acting on $L^2(X)$ and the semigroup $\{e^{-tL}\}_{t>0}$ generated by $L$ satisfies the Davies-Gaffney condition (See for example [12]). That is, there exist constants $C, c > 0$ such that for any open subsets $U_1, U_2 \subset X$,

$$\langle e^{-tL} f_1, f_2 \rangle = C \exp \left(-\frac{\text{dist}(U_1, U_2)^2}{ct} \right) \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)}, \quad \forall t > 0,$$

for every $f_i \in L^2(X)$ with $\text{supp} f_i \subset U_i, \ i = 1, 2$, where $\text{dist}(U_1, U_2) := \inf_{x \in U_1, y \in U_2} d(x, y)$.

Second, the operator $L$ satisfies restriction type estimates. Given a subset $E \subseteq X$, we define the projection operator $P_E$ by multiplying by the characteristic function of $E$, that is,

$$P_E f(x) = \chi_E(x) f(x).$$

For a function $F : \mathbb{R} \to \mathbb{C}$ and $R > 0$, we denote by $\delta_R F : \mathbb{R} \to \mathbb{C}$ the function $x \to F(Rx)$. Following [8], we say that a non-negative self-adjoint operator $L$ satisfies restriction type estimates if for any $R > 0$ and all Borel functions $F$ such that $\text{supp} \ F \subset [0, R]$, there exist some $p_0$ and $q$ satisfying $1 \leq p_0 < 2$ and $1 \leq q \leq \infty$ such that

$$\|F(\sqrt{L}) P_{B(x,r)}\|_{p_0 \to 2} \leq CV(x,r)^{\frac{1}{2} - \frac{1}{p_0}} (Rr)^{n\left(\frac{1}{q_0} - \frac{1}{2}\right)} \|\delta_R F\|_{L^q}$$

for all $x \in X$ and $r \geq 1/R$, where $n$ is the dimension entering doubling volume condition (1.3). When $L = -\Delta$ on $\mathbb{R}^n$, this estimate is equivalent to the classic $(p,2)$ restriction estimate of Stein-Tomas. See [8] or Proposition 2.4 below.

The aim of this paper is to obtain a sharp Hörmander-type spectral multiplier theorem for abstract operators which generate semigroups satisfying Davies-Gaffney condition. More precisely, our result shows that restriction type estimates imply sharp spectral multipliers on Hardy spaces $H^p_L(X)$ for $p > 0$, where $H^p_L(X)$ is a new class of Hardy spaces associated to $L$ ( [2, 3, 14, 17, 18, 19, 22, 23, 24, 26], see Section 2 below). The theorem is valid for abstract self-adjoint operators. However, before the result can be applied one has to verify conditions (GE) and (1.4). Usually proving restriction type condition (1.4) is difficult. See discussions in [8]. We discuss several examples of operators which satisfy required restriction type estimates in Section 4. On the other hand, condition (1.4) with $p_0 = 1$ and $q = \infty$ follows from Gaussian estimates (1.8) for the heat kernel corresponding to the operator. See discussions in [8] and [14].

Let $\phi$ be a nontrivial compact supported smooth function and define the Sobolev norm

$$\|F\|_{W^{s,q}} = \|(I - d^2/dx^2)^{s/2} F\|_{L^2}.$$ 

The following theorem is the main result of the paper.

**Theorem 1.1.** Consider the doubling metric measure space $(X, d, \mu)$ which satisfies (1.5) with dimension $n$. Assume that the operator $L$ satisfies Davies-Gaffney estimate (GE) and the restriction type condition (1.4) for some $p_0, q$ satisfying $1 \leq p_0 < 2$ and $1 \leq q \leq \infty$. Suppose that $0 < p \leq 1$ and for a bounded Borel function $F$, there exists some constant $s > n(1/p - 1/2)$ such that

$$\sup_{t > 0} \|\phi \delta_t F\|_{W^{s,q}} < \infty.$$ 

Then the operator $F(\sqrt{L})$ is bounded on $H^p_L(X)$, i.e., there exists a constant $C > 0$ such that

$$\|F(\sqrt{L}) f\|_{H^p_L(X)} \leq C \|f\|_{H^p_L(X)}.$$
Remarks:

i) Theorem 1.1 is sharp when $q = 2$ by considering Bochner-Riesz means on the spaces $H^p_L(X)$. See Corollary 1.2 below. When $L = -\Delta$ on $\mathbb{R}^n$, it satisfies (1.4) with $q = 2$ for all $1 \leq p_0 \leq (2n + 2)/(n + 3)$ and from this theorem we can obtain sharp results for the boundedness of classic Bochner-Riesz means on Hardy spaces $H^p(\mathbb{R}^n)$.

ii) In [19], Theorem 1.1 was obtained under the condition (1.5) with the norm $W^{s,\infty}$. Note that for fixed $p_0$ if condition (1.4) holds for some $q \in [1, \infty)$, then (1.4) holds for all $q' \geq q$ including the case $q' = \infty$ and also note that the smaller $q$ is, the weaker condition (1.5) is. Although $q = 2$ leads to the sharp result, we have examples that the operator satisfies (1.4) with some $q > 2$ but it does not satisfy (1.4) with $q = 2$. For example, harmonic oscillator $L = -d^2/dx^2 + x^2$ acting on $L^2(\mathbb{R})$ satisfies (1.4) with $p_0 = 1$ and $q = 4$ other than $q = 2$. See [15, Section 7.5] for more discussion.

iii) We do not expect $q < 2$ for condition (1.4). If (1.4) holds for some $q < 2$, by Theorem 1.1, the classic Bochner-Riesz mean operator would be bounded for some $\delta > n(1/p - 1/2) - 1/2$ and this contradicts the well known result that $\delta > n(1/p - 1/2) - 1/2$ is necessary for the classic Bochner-Riesz summability. However, if we consider $p_0 \to r$ norm in (1.4) with some $r > 2$ instead of $p_0 \to 2$ norm, using analogous argument, we can get similar results for some $1 \leq q < 2$.

A standard application of spectral multiplier theorems is to consider the boundedness of Bochner-Riesz means. Let us recall that Bochner-Riesz means of order $\delta$ for a non-negative self-adjoint operator $L$ are defined by the formula

\begin{equation}
S^\delta_R(L) = \left(I - \frac{L}{R^2}\right)^\delta, \quad R > 0.
\end{equation}

In Theorem 1.1 if one chooses $F(\lambda) = (1 - \lambda^2)^\delta_\lambda$ then $F \in W^{\beta,q}$ if and only if $\delta > \beta - 1/q$. We then have the following corollary, which generalizes the classical result due to Sjölin [32] and Stein-Tableleson-Weiss [34] on the Bochner-Riesz means, and this result is sharp for Laplacian on $\mathbb{R}^n$ (see [32]).

**Corollary 1.2.** Assume that the operator $L$ satisfies Davies-Gaffney estimate (GE) and the restriction type condition (1.4) for some $p_0, q$ satisfying $1 \leq p_0 < 2$ and $1 \leq q \leq \infty$. Suppose that $0 < p \leq 1$. Then for all $\delta > n(1/p - 1/2) - 1/q$, we have

\begin{equation}
\left\| \left(I - \frac{L}{R^2}\right)^\delta \right\|_{H^p_{p_0} \to H^p_L} \leq C
\end{equation}

uniformly in $R > 0$.

Note that when the semigroup $e^{-tL}$ generated by $L$ has a kernel $p_t(x,y)$ satisfying a Gaussian upper bound, that is

\begin{equation}
|p_t(x,y)| \leq \frac{C}{V(x,\sqrt{t})} \exp \left( - \frac{d^2(x,y)}{ct} \right)
\end{equation}

for all $t > 0$, and $x, y \in X$, then the Hardy space $H^p_L(X)$ coincides with $L^p(X)$ for every $1 < p < \infty$ (see [2] [22]). Hence the following corollary is a consequence of Theorem 1.1.
Corollary 1.3. Assume that the heat kernel corresponding to the operator $L$ satisfies (1.8) and the operator $L$ satisfies the restriction type condition (1.4) for some $p_0, q$ satisfying $1 < p_0 < 2$ and $1 \leq q \leq \infty$. Then for any even bounded Borel function $F$ such that $\sup_{t>0} \|\phi t F\|_{W^{s,q}} < \infty$ for some $s > n(1/p_1 - 1/2)$ and $1 \leq p_1 \leq p_0$, the operator $F(\sqrt{L})$ is bounded on $L^p(X)$ for $p_1 < p < p_1'$, i.e., there exists a constant $C > 0$ such that

$$\|F(\sqrt{L})f\|_{L^p(X)} \leq C\|f\|_{L^{p_1}(X)}.$$  

The paper is organized as follows. In Section 2, we recall some preliminary results about finite speed propagation property, restriction type estimates and Hardy space $H^p_L(X)$ associated to an operator $L$, and state a criterion for boundedness of spectral multipliers on $H^p_L(X)$. In Section 3, we will prove our main result, Theorem 1.1, by using some estimates for the operator $F(\sqrt{L})$ away from the diagonal and the restriction type estimates.

Throughout, the letter “$C$” and “$c$” will denote (possibly different) constants that are independent of the essential variables.

2. Preliminaries

To simplifying the notation, we shall often just use $B$ instead of $B(x,r)$. Given $\lambda > 0$, we will write $\lambda B$ for the $\lambda$-dilated ball which is the ball with the same center as $B$ and with radius $\lambda r$. For $1 \leq p \leq \infty$, we denote the norm of a function $f \in L^p(X, d\mu)$ by $\|f\|_p$. If $T$ is a bounded linear operator from $L^p(X, d\mu)$ to $L^q(X, d\mu)$, $1 \leq p, q \leq \infty$, we write $\|T\|_{p \rightarrow q}$ for the operator norm of $T$. Let $\phi$ be a non-negative $C_c^\infty$-function such that

$$(1.1) \quad \text{supp}\phi \subseteq \left(\frac{1}{4}, 1\right) \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \phi(2^{-k}\lambda) = 1 \quad \text{for all } \lambda > 0.$$  

2.1. Finite speed propagation for the wave equation. Following [9], we set

$$D_\rho = \{(x, y) \in X \times X : d(x, y) \leq \rho\}.$$  

Given an operator $T$ from $L^p(X)$ to $L^q(X)$, we write

$$\text{(2.1)} \quad \text{supp} K_T \subseteq D_\rho$$

if $\langle Tf_1, f_2 \rangle = 0$ whenever $f_k$ is in $C(X)$ and $\text{supp} f_k \subseteq B(x_k, \rho_k)$ when $k = 1, 2$, and $\rho_1 + \rho_2 + \rho < d(x_1, x_2)$. One says that $\cos(t\sqrt{L})$ satisfies finite speed propagation property if there holds

$$(\text{FS}) \quad \text{supp} K_{\cos(t\sqrt{L})} \subseteq D_t \quad \forall t \geq 0.$$  

More precisely, we have the following result.

**Proposition 2.1.** Let $L$ be a non-negative self-adjoint operator acting on $L^2(X)$. Then the finite speed propagation property (FS) and Davies-Gaffney estimate (GE) are equivalent.

**Proof.** For the proof, we refer the reader to Theorem 2 in [30] and Theorem 3.4 in [9]. See also [6].

The following lemma is a straightforward consequence of (FS).

**Lemma 2.2.** Assume that $L$ satisfies (FS) and that $F$ is an even bounded Borel function with Fourier transform $\hat{F}$ satisfying $\text{supp} \hat{F} \subseteq [-\rho, \rho]$. Then

$$\text{supp} K_{F(\sqrt{L})} \subseteq D_\rho.$$
Proof. If $F$ is an even function, then by the Fourier inversion formula,

$$F(\sqrt{L}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{F}(t) \cos(t \sqrt{L}) \, dt.$$ 

But $\text{supp} \, \hat{F} \subset [-\rho, \rho]$ and Lemma 2.2 follows from (FS).

2.2. Restriction type estimates. The following result was obtained in [8, Proposition 2.3].

Proposition 2.3. Suppose that $(X, d, \mu)$ satisfies properties (1.3). Let $1 \leq p_0 < 2$ and $N > n(1/p - 1/2)$. Then condition (1.4) with $q = \infty$ is equivalent with each of the following conditions:

(a) For all $x > 0$ and $r \geq t > 0$ we have

$$(G_{p_0,2}) \quad \|e^{-t^2L} P_{B(x,r)}\|_{p_0 \to 2} \leq CV(x, r)^{\frac{1}{2} - \frac{1}{p_0}} \left(\frac{r}{t}\right)^{n(\frac{1}{p_0} - \frac{1}{2})}.$$ 

(b) For all $x \in X$ and $r \geq t > 0$ we have

$$(E_{p_0,2}) \quad \|(1 + t\sqrt{L})^{-N} P_{B(x,r)}\|_{p_0 \to 2} \leq CV(x, r)^{\frac{1}{2} - \frac{1}{p_0}} \left(\frac{r}{t}\right)^{n(\frac{1}{p_0} - \frac{1}{2})}.$$ 

Following [21], one says that $L$ satisfies $L^{p_0}$ to $L^{p_0'}$ restriction estimates if the spectral measure $dE_{\sqrt{T}}(\lambda)$ maps $L^{p_0}(X)$ to $L^{p_0'}(X)$ for some $p_0$ satisfying $1 \leq p_0 \leq 2n/(n + 1)$, with an operator norm estimate

$$(R_{p_0}) \quad \|dE_{\sqrt{T}}(\lambda)\|_{p_0 \to p_0'} \leq C\lambda^{n(\frac{1}{p_0} - \frac{1}{p_0'})^{-1}}$$ 

for all $\lambda > 0$.

Proposition 2.4. Suppose that there exist positive constants $0 < C_1 \leq C_2 < \infty$ such that $C_1 r^n \leq V(x, r) \leq C_2 r^n$ for every $x \in X$ and $r > 0$. Then conditions (R_{p_0}) and (1.4) with $q = 2$ are equivalent.

Proof. For the proof, we refer the reader to [8, Proposition 2.4].

2.3. Hardy spaces $H^p_L(X)$. The following definition of Hardy spaces $H^p_L(X)$ comes from [19] (see also [22]). Following [3], one can define the $L^2$ adapted Hardy space

(2.3)

$$H^2(X) := \overline{R(L)},$$

that is, the closure of the range of $L$ in $L^2(X)$. Then $L^2(X)$ is the orthogonal sum of $H^2(X)$ and the null space $N(L)$.

Consider the following quadratic operators associated to $L$

(2.4)

$$S_{h,K} f(x) = \left( \int_0^\infty \int_{d(x,y) < t} |(t^2 L)^K e^{-t^2 L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}, \quad x \in X$$

where $f \in L^2(X)$. For each $K \geq 1$ and $1 \leq p < \infty$, we now define

$$D_{K,p} = \left\{ f \in H^2(X) : S_{h,K} f \in L^p(X) \right\}, \quad 0 < p < \infty.$$
Definition 2.5. Let \( L \) be a non-negative self-adjoint operator on \( L^2(X) \) satisfying the Davies-Gaffney condition \((GE)\).

(i) For each \( 0 < p \leq 2 \), the Hardy space \( H^p_L(X) \) associated to \( L \) is the completion of the space \( D_{\lambda, p} \) in the norm
\[
\|f\|_{H^p_L(X)} = \|S_{\lambda, 1}f\|_{L^p(X)}.
\]

(ii) For each \( 2 < p < \infty \), the Hardy space \( H^p_L(X) \) associated to \( L \) is the completion of the space \( D_{K_0, p} \) in the norm
\[
\|f\|_{H^p_L(X)} = \|S_{h, K_0}f\|_{L^p(X)}, \quad K_0 = \left[ \frac{n}{4} \right] + 1.
\]

Under an assumption of Gaussian upper bounds \((1.8)\), it was proved in \([2]\) that \( H^p_L(X) = L^p(X) \) for all \( 1 < p < \infty \). Note that, in this paper, we only assume the Davies-Gaffney estimates on the heat kernel of \( L \), and hence for \( 1 < p < \infty \), \( p \neq 2 \), \( H^p_L(X) \) may or may not coincide with the space \( L^p(X) \). However, it can be verified that \( H^2_L(X) = H^2(X) \) and the dual of \( H^p_L(X) \) is \( H^{p'}_L(X) \), with \( 1/p + 1/p' = 1 \) (see Proposition 9.4 of \([22]\)). We also recall that the \( H^p_L(X) \) spaces \( (1 \leq p < +\infty) \) are a family of interpolation spaces for the complex interpolation method. See \([22\), Proposition 9.5]).

2.4. A criterion for boundedness of spectral multipliers on \( H^p_L(X) \). We now state a criterion from \([19]\) that allows us to derive estimates on Hardy spaces \( H^p_L(X) \). This generalizes the classical Calderón-Zygmund theory and we would like to emphasize that the conditions imposed involve the multiplier operator and its action on functions but not its kernel.

Lemma 2.6. Let \( L \) be a non-negative self-adjoint operator acting on \( L^2(X) \) satisfying the Davies-Gaffney estimate \((GE)\). Let \( m \) be a bounded Borel function. Suppose that \( 0 < p \leq 1 \) and \( M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2}) \). Assume that there exist constants \( s > n(\frac{1}{p} - \frac{1}{2}) \) and \( C > 0 \) such that for every \( j = 2, 3, \ldots \),
\[
\|F(L)(I - e^{-r_B^2L})^M f\|_{L^2(2^j B, 2^{j-1} B)} \leq C 2^{-js} \|f\|_{L^2(B)}
\]
for any ball \( B \) with radius \( r_B \) and for all \( f \in L^2(X) \) with supp \( f \subset B \). Then the operator \( F(L) \) extends to a bounded operator on \( H^p_L(X) \). More precisely, there exists a constant \( C > 0 \) such that for all \( f \in H^p_L(X) \)
\[
\|F(L)f\|_{H^p_L(X)} \leq C \|f\|_{H^p_L(X)}.
\]

Proof. For the proof, we refer the reader to \([19, Theorem 3.1]\).

3. Proof of Theorem 1.1

For \( s \in \mathbb{R} \) and \( p, q \) in \([1, \infty]\), we denote by \( B^p_q \) the usual Besov space (see, e.g., \([4]\)). In order to prove Theorem 1.1, let us first show the following useful auxiliary lemma.

Lemma 3.1. Assume that operator \( L \) satisfies the finite speed propagation property \((FS)\) and condition \((1.4)\) for some \( p_0, q \) satisfying \( 1 \leq p_0 < 2 \) and \( 1 \leq q \leq \infty \). Next assume that
function $F$ is even and supported on $[-R, R]$. Then for any $s > \max\{n(1/p_0 - 1/2) - 1, 0\}$, there exists a constant $C_s$ such that for any ball $B = B(x, r)$ and for every $j = 1, 2, \ldots$

$$\|P_{B(x, 2^jr)}F(\sqrt{L})P_{B(x, r)}\|_{p_0 \to 2} \leq \begin{cases} C_s V(x, r)^{\frac{3}{2} - \frac{1}{p_0}} (Rr)^{n(\frac{1}{p_0} - \frac{1}{2})} (2^j r R)^{-s} \|\delta_R F\|_{B_2^{1, 1}} & r R \geq 1; \\
C_s V(x, R^{-1})^{\frac{3}{2} - \frac{1}{p_0}} (2^j r R)^{-s} \|\delta_R F\|_{B_2^{1, 1}} & r R < 1.
\end{cases}$$

**Proof.** Our approach is inspired by the proof of Lemma 3.4 in [9]. Fix $r, j$ and $R$ such that $2^{j-5} r R > 1$. Otherwise, by condition (1.4) the proof of (3.1) is trivial. Note that $\phi_0$ and $\phi_k$ are smooth even functions supported in $[-4, 4]$ and $[2^k, 2^{k+2}] \cup [-2^{k+2}, -2^k]$ respectively. Further, $\phi_0(\lambda) + \sum_{k \geq 1} \phi_k(\lambda) = 1$ for all $\lambda$ and $\phi_0 = 1$ on $[-2, 2]$. We set $\psi(\lambda) = \phi_0(\lambda/(2^{j-3} r))$ and $\tilde{\psi}(\lambda) = \phi_0(\lambda/(2^{j-3} r))$. Define $T_\psi$ as $T_\psi F = \phi \tilde{F}$. Since $\text{supp } \psi \subset [-2^{j-1} r, 2^{j-1} r]$, it follows by Lemma 2.2

$$\text{supp } K_{T_\psi F(\sqrt{L})} \subset \{(z, y) \in X \times X : d(z, y) \leq 2^{j-1} r\}.$$ 

Hence,

$$K_{F(\sqrt{L})}(z, y) = K_{[F - T_\psi F(\sqrt{L})]}(z, y)$$

for all $z, y$ such that $d(z, y) > 2^{j-1} r$. We obtain

$$\|P_{B(x, 2^jr)}F(\sqrt{L})P_{B(x, r)}\|_{p_0 \to 2} \leq \|[F - T_\psi F(\sqrt{L})]P_{B(x, r)}\|_{p_0 \to 2}.$$ 

Now,

$$F - T_\psi F = \sum_{k \geq 0} \delta_{R^{-1}}(\phi_k)[F - T_\psi F] = \delta_{R^{-1}}(\phi_0)[F - T_\psi F] - \sum_{k \geq 1} \delta_{R^{-1}}(\phi_k)T_\psi F = \delta_{R^{-1}}(\phi_0)[F - T_\psi F] - (1 - \delta_{R^{-1}}(\phi_0))T_\psi F,$$

since $k \geq 1$, supp $\delta_{R^{-1}}(\phi_k) \subset [-2^{k+2} R, -2^k R] \cup [2^k R, 2^{k+2} R]$, and supp $F \subset [-R, R]$. It follows that

$$\|P_{B(x, 2^jr)}F(\sqrt{L})P_{B(x, r)}\|_{p_0 \to 2} \leq \|\delta_{R^{-1}}(\phi_0)[F - T_\psi F](\sqrt{L})P_{B(x, r)}\|_{p_0 \to 2} + \|[1 - \delta_{R^{-1}}(\phi_0)]T_\psi F(\sqrt{L})P_{B(x, r)}\|_{p_0 \to 2}.$$ 

**(3.3)**

**Case 1: $r R \geq 1$.**

Note that supp $\delta_{R^{-1}}(\phi_0) \subset [-4R, 4R]$. By condition (1.4),

$$\|\delta_{R^{-1}}(\phi_0)[F - T_\psi F](\sqrt{L})P_{B(x, r)}\|_{p_0 \to 2} \leq CV(x, r)^{\frac{3}{2} - \frac{1}{p_0}} (Rr)^{n(\frac{1}{p_0} - \frac{1}{2})} \|\phi_0 \delta_R[F - T_\psi F]\|_{L^q} \leq CV(x, r)^{\frac{3}{2} - \frac{1}{p_0}} (Rr)^{n(\frac{1}{p_0} - \frac{1}{2})} \|\delta_R F - T_\psi_0(\delta_R F)\|_{L^q} = CV(x, r)^{\frac{3}{2} - \frac{1}{p_0}} (Rr)^{n(\frac{1}{p_0} - \frac{1}{2})} \sum_{i \geq 0} T_{\phi_i}[I - T_\psi]\delta_R F\|_{L^q}.$$
Note that $\phi_\lambda(1-\psi_\lambda) = \phi_\lambda(1-\psi_\lambda)$ for all $\lambda \in \mathbb{R}$ unless $\lambda \geq 2^i \geq 2^{-4}rR$. Consequently, $T_{\phi_\lambda}[I - T_{\psi_\lambda}]\delta R F = 0$ unless $i \geq i_0$, where $i_0 = \log_2(2^{-4}rR)$, and
\[
\|\delta R^{-1}(\phi_\lambda)[F-T_{\psi} F](\sqrt{L})P_{B(x,r)}\|_{p_0 \to 2} \leq CV(x,r)^{\frac{1}{2} - \frac{1}{p_0}}(Rr)^{\frac{1}{2} - \frac{1}{p_0}} \sum_{i \geq i_0} \|T_{\phi_\lambda}[I - T_{\psi_\lambda}]\delta R F\|_{L^q}
\leq CV(x,r)^{\frac{1}{2} - \frac{1}{p_0}}(Rr)^{\frac{1}{2} - \frac{1}{p_0}} \sum_{i \geq i_0} \|T_{\phi_\lambda}\delta R F\|_{L^q}
\leq CV(x,r)^{\frac{1}{2} - \frac{1}{p_0}}(Rr)^{\frac{1}{2} - \frac{1}{p_0}}2^{-i_0} \sum_{i \geq i_0} 2^i \|T_{\phi_\lambda}\delta R F\|_{L^q}
\leq CV(x,r)^{\frac{1}{2} - \frac{1}{p_0}}(Rr)^{\frac{1}{2} - \frac{1}{p_0}}(2^j r R)^{-s} \|\delta R F\|_{L^q}.
\]
(3.4)

We now treat the remain term in formula (3.3). We claim that for any $s > 0$,
\[
\sup_\lambda (1 - \delta R^{-1}(\phi_\lambda))(\lambda)T_{\psi} F(\lambda)(1 + R^{-1} |\lambda|)^{s+1} \leq C(2^j r R)^{-s} \|\delta R F\|_{L^q}.
\]
(3.5)

Let $\hat{f}$ denotes the inverse Fourier transform of function $f$. We observe that $|\lambda - y| \approx |\lambda|$ if $|\lambda| \geq 2R$ and $|y| \leq R$, and hence
\[
\sup_\lambda (1 - \delta R^{-1}(\phi_\lambda))(\lambda)T_{\psi} F(\lambda)(1 + R^{-1} |\lambda|)^{s+1}
\leq \sup_\lambda (1 - \phi_\lambda(\lambda/R))(\lambda)(1 + |\lambda|/R)^{s+1}
\leq \sup_\lambda (1 - \phi_\lambda(\lambda/R)) \left| \int_{-R}^R F(y)\hat{\psi}(\lambda - y)dy \right| (1 + |\lambda|/R)^{s+1}
\leq \sup_\lambda (1 - \phi_\lambda(\lambda/R))2^{j-3}r \int_{-R}^R |F(y)|(1 + 2^{j-3}r |\lambda - y|)^{-s-1}dy(1 + |\lambda|/R)^{s+1}
\leq C \sup_\lambda (1 - \phi_\lambda(\lambda/R))2^{j-3}r R(1 + 2^{j-3}r |\lambda|)^{-s-1}(1 + |\lambda|/R)^{s+1} \|\delta R F\|_{L^q}
\leq C(2^j r R)^{-s} \|\delta R F\|_{L^q}.
\]
From (3.3) and Proposition 2.3 it follows that for any $s > \max\{n(1/p_0 - 1/2) - 1, 0\}$,
\[
\|(1 - \delta R^{-1}(\phi_\lambda))T_{\psi} F(\sqrt{L})P_{B(x,r)}\|_{p_0 \to 2}
\leq \sup_\lambda \left| (1 - \delta R^{-1}(\phi_\lambda))(\lambda)T_{\psi} F(\lambda)(1 + R^{-1} |\lambda|)^{s+1} \right| \|(I + R^{-1}\sqrt{L})^{-s-1}P_{B(x,r)}\|_{p_0 \to 2}
\leq CV(x,r)^{\frac{1}{2} - \frac{1}{p_0}}(Rr)^{\frac{1}{2} - \frac{1}{p_0}}(2^j r R)^{-s} \|\delta R F\|_{L^q}.
\]
(3.6)

Then estimates of (3.3), (3.4) and (3.6) imply estimate (3.1) for any $s > \max\{n(1/p_0 - 1/2) - 1, 0\}$.

Case 2: $r R < 1$.

It follows from (3.3) that
\[
\|P_{B(x,2^j r R)} F(\sqrt{L})P_{B(x,r)}\|_{p_0 \to 2}
\leq \|\delta R^{-1}(\phi_\lambda)[F-T_{\psi} F](\sqrt{L})P_{B(x,R^{-1})}\|_{p_0 \to 2} + \|(1 - \delta R^{-1}(\phi_\lambda))T_{\psi} F(\sqrt{L})P_{B(x,R^{-1})}\|_{p_0 \to 2}.
\]
(3.7)

Replacing $B(x,r)$ by $B(x,R^{-1})$ in (3.4) and (3.6), a similar argument as in Case 1 shows (2.6), and we skip it here. The proof of Lemma 3.1 is complete. □
Proof of Theorem 1.1} To prove Theorem 1.1 by Lemma 2.6 it suffices to verify condition (2.3). Recall that \( \phi \) is a non-negative \( C_0^\infty \) function such that

\[
\text{supp } \phi \subseteq \left( \frac{1}{4}, 1 \right) \quad \text{and} \quad \sum_{\ell \in \mathbb{Z}} \phi(2^{-\ell} \lambda) = 1 \quad \text{for all } \lambda > 0.
\]

Then

\[
F(\lambda) = \sum_{\ell \in \mathbb{Z}} \phi(2^{-\ell} \lambda) F(\lambda) = \sum_{\ell \in \mathbb{Z}} F_\ell(\lambda) \quad \text{for all } \lambda > 0.
\]

For every \( \ell \in \mathbb{Z} \) and \( r > 0 \), set \( F^\ell_{r,M} = F_\ell(\lambda)(1 - e^{-r^2 \lambda^2})^M \). So for any ball \( B = B(x, r) \),

\[
\|F(\sqrt{L})(I - e^{-r^2 L})^M f\|_{L^2(2^{j-1} B, 2^j B)} \leq \sum_{\ell \in \mathbb{Z}} \|F^\ell_{r,M}(\sqrt{L}) f\|_{L^2(2^{j-1} B, 2^j B)}.
\]

Fix \( f \in L^2(X) \) with \( \text{supp } f \subseteq B \) and \( j \geq 2 \). Note that \( \text{supp } F^\ell_{r,M} \subseteq [-2^\ell, 2^\ell] \). So if \( r2^\ell < 1 \), it follows Lemma 3.1 that for any \( s > \max\{n(1/p_0 - 1/2) - 1, 0\} \)

\[
\|F^\ell_{r,M}(\sqrt{L}) f\|_{L^2(2^{j-1} B, 2^j B)} \leq \|P_{2^{j-1} B, 2^j B} F^\ell_{r,M}(\sqrt{L}) P_B\|_{p_0 \rightarrow 2} \|f\|_{L^{p_0}} \\
\leq CV(x, 2^{-\ell} \frac{1}{p_0} (2^j r^2)^{-s}) \|\delta_{2^\ell} F^\ell_{r,M}\|_{B^{s,1}} \|f\|_{L^{p_0}} \\
\leq CV(x, 2^{-\ell} \frac{1}{p_0} (2^j r^2)^{-s}) \|\delta_{2^\ell} F^\ell_{r,M}\|_{B^{0,1}} V(x, r) \|f\|_{L^2} \\
\leq C 2^{-js}(2^j r)^{2M-s} \|\phi\|_{B^{s,1}} \|f\|_{L^2}.
\]

(3.9)

Similarly if \( r2^\ell \geq 1 \), then

\[
\|F^\ell_{r,M}(\sqrt{L}) f\|_{L^2(2^{j-1} B, 2^j B)} \leq \|P_{2^{j-1} B, 2^j B} F^\ell_{r,M}(\sqrt{L}) P_B\|_{p_0 \rightarrow 2} \|f\|_{L^{p_0}} \\
\leq CV(x, r) \|\delta_{2^\ell} F^\ell_{r,M}\|_{B^{0,1}} \|f\|_{L^{p_0}} \\
\leq C(2^\ell)^{n(\frac{1}{p_0} - \frac{1}{2})} (2^j r^2)^{-s} \|\delta_{2^\ell} F^\ell_{r,M}\|_{B^{0,1}} \|f\|_{L^2} \\
\leq C 2^{-js}(2^j r)^{2M-s} \|\phi\|_{B^{0,1}} \|f\|_{L^2}.
\]

(3.10)

Note that for any \( \varepsilon > 0 \) \( \|f\|_{B^{s,1}} \leq C \|f\|_{W^{s,\varepsilon}} \) (see, e.g. [4]). Choosing \( s \) such that \( M > s > n(1/p_0 - 1/2) \), it follows from (1.5), (3.3), (3.9) and (3.10) that

\[
\|F(\sqrt{L})(I - e^{-r^2 L})^M f\|_{L^2(2^{j-1} B, 2^j B)} \leq C 2^{-js} \|f\|_{L^2}.
\]

This proves (2.5). Hence, by Lemma 2.6 \( F(\sqrt{L}) \) can be extended to be a bounded operator on \( H^p_{L_s}(X) \). The proof of Theorem 1.1 is complete. \( \square \)

Proof of Corollary 1.3} Firstly, we note that when \( p_1 = p_0 \), Corollary 1.3 follows from Theorem 4.1. Secondly, it follows by Theorem 1.1 that Corollary 1.3 holds for \( p_1 = 1 \). Now we follow an idea as in [29] to construct a family of spectral multipliers \( \{F_z : z \in \mathbb{C}, 0 \leq Re z \leq 1\} \) as follows:

\[
F_z(\lambda) = \sum_{j=-\infty}^{\infty} \eta(2^{-j} \lambda) \left( 1 - 2^{2j} \frac{d}{d\lambda^2} \right)^{\frac{z-\theta}{2}\frac{n(1/p_0 - 1/2)}} \|F(\lambda)\phi(2^{-j} \lambda)\|
\]
where \( \theta = (1 - 1/p_1)/(1 - 1/p_0) \) and \( \eta \in C_c^\infty([1/4, 4]), \phi \in C_c^\infty([1/2, 2]), \eta = 1 \) on \([1/2, 2]\) and \( \sum_j \eta(2^{-j} \lambda) = \sum_j \phi(2^{-j} \lambda) = 1 \) for all \( \lambda > 0 \). Observe that if \( z = 1 + iy \), then
\[
\sup_{t>0} \| \phi \delta_tF_{1+iy} \|_{W^{s_1,q}} \leq C \sup_{t>0} \| \phi \delta_tF \|_{W^{s_0,q}} (1 + |y|)^{n/2}
\]
for some \( s_1 > n(1/p_0 - 1/2) \). On the other hand, if \( z = iy \), then
\[
\sup_{t>0} \| \phi \delta_tF_{iy} \|_{W^{s_2,q}} \leq C \sup_{t>0} \| \phi \delta_tF \|_{W^{s_0,q}} (1 + |y|)^{n/2}
\]
for some \( s_2 > n/2 \). It follows by [8, Theorem 4.1] that \( F_{1+iy}(\sqrt{L}) \) is bounded on \( H^p_L(X) \) for \( p_0 < p < p'_0 \) and by Theorem 1.1 that \( F_{iy}(\sqrt{L}) \) is bounded on \( H^1_L(X) \). Applying the three line theorem, we get \( F_0(\sqrt{L}) = F(\sqrt{L}) \) is bounded on \( H^p_L(X) \), that is, \( F(\sqrt{L}) \) is bounded on \( L^p(X) \) for \( p_1 < p < p'_1 \).

4. Applications

Theorem 1.1 is valid for abstract self-adjoint operators. However, before the result can be applied one has to verify conditions (GE) and (1.4). Usually proving restriction type condition (1.4) is difficult. See discussions in [8]. In this section, we discuss several examples of operators which satisfy required restriction type estimates and apply our main results to these operators.

4.1. Sub-Laplacians on homogeneous groups. Let \( G \) be a homogeneous Lie group of polynomial growth with homogeneous dimension \( n \) (see for examples, [7, 13, 20, 25]) and let \( X_1, \ldots, X_k \) be a system of left-invariant vector fields on \( G \) satisfying the Hörmander condition. We define the sub-Laplace operator \( L \) acting on \( L^2(G) \) by the formula
\[
L = -\sum_{i=1}^k X_i^2.
\]

**Proposition 4.1.** Let \( L \) be the homogeneous sub-Laplacian defined by the formula (4.1) acting on a homogeneous group \( G \). Then condition (1.4) holds for \( p_0 = 1 \) and \( q = 2 \), and hence results of Theorem 1.1 and Corollary 1.2 hold for \( q = 2 \).

**Proof.** It is well known that the heat kernel corresponding to the operator \( L \) satisfies Davies-Gaffney estimate (GE). It is also not difficult to check that for some constant \( C > 0 \)
\[
\|F(\sqrt{L})\|_{L^2(L^2(\mathbb{R}))} = C \int_0^\infty |F(t)|^2 t^{n-1} dt.
\]
See for example equation (7.1) of [15] or [7, Proposition 10]. It was proved that the above equality implies condition (1.4) with \( p_0 = 1 \) and \( q = 2 \) (see [8, Section 12]). Then Theorem 1.1 and Corollary 1.2 imply Proposition 4.1. \( \square \)

Proposition 4.1 can be extended to “quasi-homogeneous” operators acting on homogeneous groups, see [31] and [15].
4.2. Schrödinger operators on asymptotically conic manifolds. Asymptotically conic manifolds (see [28]) are defined as the interior of a compact manifold $M$ with boundary, such that the metric $g$ is smooth on the interior and in a collar neighbourhood of the boundary has the form

$$
g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2}\$$

where $x$ is a smooth boundary defining function and $h(x)$ is a smooth family of metrics on the boundary.

**Proposition 4.2.** Let $(M, g)$ be a nontrapping asymptotically conic manifold of dimension $n \geq 3$, and let $x$ be a smooth boundary defining function of $\partial M$. Let $L := -\Delta + V$ be a Schrödinger operator with $V \in \mathfrak{c}^\infty(M)$ and assume that $L$ is a positive operator and 0 is neither an eigenvalue nor a resonance. Then condition (1.4) is true with $q = 2$ for all $1 \leq p_0 \leq (2n + 2)/(n + 3)$, and hence results of Theorem 1.1 and Corollary 1.2 hold for $q = 2$.

**Proof.** It was proved in [21, Theorem 1.3] that condition $(R_{p_0})$ is satisfied for $L$ when $1 \leq p_0 \leq (2n + 2)/(n + 3)$. By Proposition 2.4 Theorem 1.1 and Corollary 1.2 Proposition 4.2 is proved. □

4.3. Schrödinger operators with the inverse-square potential. In this subsection, we consider Schrödinger operators $L = -\Delta + V$ on $L^2(\mathbb{R}^n, dx)$, where $V(x) = \frac{c}{|x|^2}$. We assume that $n > 2$ and $c > -(n - 2)^2/4$. The classical Hardy inequality

$$(4.2) \quad -\Delta \geq \frac{(n - 2)^2}{4}|x|^{-2},$$

shows that the self-adjoint operator $L$ is non-negative if $c > -(n - 2)^2/4$. Set $p_c^* = n/\sigma$, $\sigma = \max\{(n - 2)/2 - \sqrt{(n - 2)^2/4 + c, 0}\}$. If $c \geq 0$ then the semigroup $\exp(-tL)$ is pointwise bounded by the Gaussian upper bound (1.8) and hence acts on all $L^p$ spaces with $1 \leq p \leq \infty$. If $c < 0$, then $\exp(-tL)$ acts as a uniformly bounded semi-group on $L^p(\mathbb{R}^n)$ for $p \in ((p_c^*)', p_c^*)$ and the range $((p_c^*)', p_c^*)$ is optimal (see for example [27]).

For these Schrödinger operators, we have the following proposition.

**Proposition 4.3.** Assume that $n > 2$ and let $L = -\Delta + V$ be a Schrödinger operator on $L^2(\mathbb{R}^n, dx)$ where $V(x) = \frac{c}{|x|^2}$ and $c > -(n - 2)^2/4$. Suppose that $p_0 \in ((p_c^*)', 2n/(n + 2)]$ where $p_c^* = n/\sigma$ and $\sigma = \max\{(n - 2)/2 - \sqrt{(n - 2)^2/4 + c, 0}\}$ and $(p_c^*)' \sigma$ its dual exponent. Then condition (1.4) is true with $q = 2$, and hence results of Theorem 1.1 and Corollary 1.2 hold for $q = 2$.

**Proof.** It was proved in [8, Section 10] that $L$ satisfies restriction estimate $(R_{p_0})$ for all $p_0 \in ((p_c^*)', \frac{2n}{n+2}]$. If $c \geq 0$, then $p = (p_c^*)' = 1$ is included. By Proposition 2.4 $(R_{p_0})$ and (1.3) with $q = 2$ are equivalent. Now Proposition 4.3 follows from Theorem 1.1 and Corollary 1.2. □

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References

[1] G. Alexopoulos, Spectral multipliers on Lie groups of polynomial growth. *Proc. A.M.S.*, 46 (1994), 457-468.
[2] P. Auscher, X.T. Duong and A. McIntosh, Boundedness of Banach space valued singular integral operators and Hardy spaces. Unpublished preprint (2005).
[3] P. Auscher, A. McIntosh and E. Russ, Hardy spaces of differential forms on Riemannian manifolds. *J. Geom. Anal.*, 18 (2008), 192-248.
[4] J. Bergh and J. Löfström, *Interpolation spaces*. Springer-Verlag, Berlin-New York, 1976.
[5] S. Blunck, A Hörmander-type spectral multiplier theorem for operators without heat kernel. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 2 (2003), 449-459.
[6] J. Cheeger, M. Gromov and M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplacian and the geometry of complete Riemannian manifolds. *J. Differential Geom.*, 17 (1982), 15-53.
[7] M. Christ, $L^p$ bounds for spectral multipliers on nilpotent groups. *Trans. Amer. Math. Soc.*, 328 (1991), 73-81.
[8] P. Chen, E.M. Ouhabaz, A. Sikora and L.X. Yan, Endpoint estimates for Bochner-Riesz means and sharp spectral multipliers. Submitted (2011), arXiv:1202.4052v1.
[9] T. Coulhon and A. Sikora, Gaussian heat kernel upper bounds via Phragmén-Lindelöf theorem. *Proc. Lond. Math.*, 96 (2008), 507-544.
[10] M. Cowling and A. Sikora, A spectral multiplier theorem for a sublaplacian on SU(2). *Math. Z.*, 238 (2001), 1-36.
[11] R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*. Lecture Notes in Mathematics, 242. Springer, Berlin-New York, 1971.
[12] E.B. Davies, Heat kernel bounds, conservation of probability and the Feller property. *J. Anal. Math.*, 58 (1992), 99C119. Festschrift on the occasion of the 70th birthday of Shmuel Agmon.
[13] L. De Michele and G. Mauceri, $H^p$ multipliers on stratified groups. *Ann. Mat. Pura Appl.*, 148 (1987), 353-366.
[14] X.T. Duong and J. Li, Hardy spaces associated to operators satisfying bounded $H^\infty$ functional calculus and Davies-Gaffney estimates. Preprint (2009).
[15] X.T. Duong, E.M. Ouhabaz and A. Sikora, Plancherel-type estimates and sharp spectral multipliers. *J. Funct. Anal.*, 196 (2002), 443-485.
[16] J. Dziubański and M. Preisner, Remarks on spectral multiplier theorems on Hardy spaces associated with semigroups of operators. *Rev. Un. Mat. Argentina* 50 (2009), 201-215.
[17] J. Dziubański and M. Preisner, On Riesz transforms characterization of $H^1$ spaces associated with some Schrödinger operators. *Potential Anal.* 35 (2011), 39-50.
[18] X.T. Duong and L.X. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds. *J. Amer. Math. Soc.*, 18 (2005), 943-973.
[19] X.T. Duong, L.X. Yan, Spectral multipliers for Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates. *J. Math. Soc. Japan* 63 (2011), 295-319.
[20] G. Folland and E.M. Stein, *Hardy spaces on Homogeneous Groups*, Princeton Univ. Press, 1982.
[21] C. Guillarmou, A. Hassell and A. Sikora, Restriction and spectral multiplier theorems on asymptotically conic manifolds. Preprint (2010).
[22] S. Hofmann, G.Z. Lu, D. Mitrea, M. Mitrea and L.X. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates. *Memoirs of the Amer. Math. Soc.* 214 (2011), no. 1007.
[23] S. Hofmann and S. Mayboroda, Hardy and BMO spaces associated to divergence form elliptic operators, *Math. Ann.*, 344 (2009), 37-116.
[24] S. Hofmann, S. Mayboroda and A. McIntosh, Second order elliptic operators with complex bounded measurable coefficients in $L^p$, Sobolev and Hardy spaces, to appear in *Ann. Sci. École Norm. Sup.* (2011).
[25] A. Hulanicki and E.M. Stein, Marcinkiewicz multiplier theorem for stratified groups, unpublished manuscript.
[26] R. Jiang and D. Yang, Orlicz-Hardy spaces associated with operators satisfying Davies-Gaffney estimates. *Communications in Contemporary Mathematics* **13** (2011), no. 2, 331-373.

[27] V. Liskevich, Z. Sobol and H. Vogt, On the $L^p$ theory of $C^0$-semigroups associated with second-order elliptic operators II. *J. Funct. Anal.* **193** (2002), no. 1, 55–76.

[28] R.B. Melrose, Spectral and scattering theory for Laplacian on asymptotically Euclidian spaces, in *Spectral and scattering theory*, M. Ikawa, ed., Marcel Dekker, (1994).

[29] A. Miyachi, On some singular Fourier multipliers. *J. Fac. Sci. Univ. Tokyo*, **28** (1981), 267-315.

[30] A. Sikora, Riesz transform, Gaussian bounds and the method of wave equation. *Math. Z.*, **247** (2004), 643-662.

[31] A. Sikora, On the $L^2 \to L^\infty$ norms of spectral multipliers of “quasi-homogeneous” operators on homogeneous groups. *Trans. Amer. Math. Soc.*, **351** (9) (1999), 3743–3755.

[32] P. Sjölin, Convolution with oscillating kernels in $H^p$ spaces, *J. London Math. Soc.*, **23** (1981), 442-454.

[33] E.M. Stein, *Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, (1993).

[34] E.M. Stein, M.H. Taibleson and G. Weiss, Weak type estimates for maximal operators on certain $H^p$ spaces, *Rend. Circ. mat. Palermo Suppl.* **1** (1981), 81-97.

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