Parallel Stochastic Gradient Descent with Sound Combiners

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Abstract
Stochastic gradient descent (SGD) is a well-known method for regression and classification tasks. However, it is an inherently sequential algorithm—at each step, the processing of the current example depends on the parameters learned from the previous examples. Prior approaches to parallelizing linear learners using SGD, such as HOGWILD! and ALLREDUCE, do not honor these dependencies across threads and thus can potentially suffer poor convergence rates and/or poor scalability. This paper proposes SYMSGD, a parallel SGD algorithm that, to a first-order approximation, retains the sequential semantics of SGD. Each thread learns a local model in addition to a model combiner, which allows local models to be combined to produce the same result as what a sequential SGD would have produced. This paper evaluates SYMSGD’s accuracy and performance on 6 datasets on a shared-memory machine shows up to 11× speedup over our heavily optimized sequential baseline on 16 cores and 2.2×, on average, faster than HOGWILD!

1. Introduction
Stochastic Gradient Descent (SGD) is an effective method for many machine learning problems. It is a simple algorithm with few hyper-parameters and its convergence rates are well understood both theoretically and empirically. However, its performance scalability is severely limited by its inherently sequential computation. SGD iteratively processes its input dataset where the computation at each iteration depends on the parameters learned from the previous iteration.

Current approaches for parallelizing SGD learn local models per thread and combine these models in ways that do not honor this inter-step dependence. For instance, threads in HOGWILD! (Recht et al., 2011) racily update a shared global model without holding any locks. In parameter-server (Li et al., 2014a), each thread (or machine) periodically sends its model deltas to a server that applies them to a global model, even though the deltas were computed on a stale model from a few updates ago.

While these algorithms are guaranteed to eventually converge, they need to carefully manage the communication-staleness trade-off. On the one hand, HOGWILD! communicates after processing every input example to achieve bounded staleness but the resulting communication cost limits scalability even in a single machine on sparse datasets — as we show in our experiments, even sparse datasets have frequent features that produce significant cache traffic. On the other hand, techniques such as ALLREDUCE (Agarwal et al., 2014) the staleness causes a drop in accuracy on the same number of examples with respect to a sequential baseline.

This paper presents SYMSGD, a parallel SGD algorithm that allows the threads to communicate less frequently but achieve a high-fidelity approximation to what the threads would have produced had they run sequentially. The key idea is for each thread to generate a sound model combiner that precisely captures the first-order effects of a SGD computation starting from an arbitrary model. Periodically, the threads update a global model with their local model while using the model combiner to account for changes in the global model that occurred in the interim.

While the algorithm can be generalized to different machine learning problems and on different parallel settings, such as distributed clusters and GPUs, we focus our evaluation on linear learners on multithread machines. This is primarily motivated by the fact that this setting forms the core of machine learning today. At Microsoft, developers trained over 1 million models per month in 2016 on single-node installations. Likewise, Databrick’s 2015 survey showed almost 50% of Spark installations are single-node (Databricks). As machines with terabytes of memory become a commonplace (As of February 2017, one can rent an X1 instance from AWS with 2 TB memory and 128 cores for less than $4 per hour (AWS-X1)), machine learning tasks on large datasets can be done efficiently on single machines without paying the inherent cost of distribution (McSherry et al., 2015).

Our evaluation shows that SYMSGD is fast, scales well on multiple cores, and achieves the same accuracy as se-
2. Parallel SymSGD Algorithm

Given a set of \( N \) input examples \( z_i = (x_i, y_i) \), where \( x_i \) is a vector of \( f \) feature values and \( y_i \) is the label to learn, let \( C(w) = \frac{1}{N} \sum_i C_z(w, x_i, y_i) \) be the convex cost function to minimize. That is, we seek to find

\[
\hat{w} = \arg\min_{w \in \mathbb{R}^f} \sum_{i=0}^n C_z(w, x_i, y_i)
\]

The cost function can optionally include a regularization term. We define \( G \triangleq \frac{\partial C}{\partial w} \) and \( G_z \triangleq \frac{\partial C_z}{\partial w} \) for the gradients, and \( H \triangleq \frac{\partial G}{\partial w} \) and \( H_z \triangleq \frac{\partial G_z}{\partial w} \) for the Hessian of the cost function.

At each step \( t \), SGD picks \( z_t = (x_t, y_t) \) uniformly randomly from the input dataset and updates the current model \( w_t \) along the gradient \( G_z \cdot \hat{w} \):

\[
w_{t+1} = w_t - \alpha_t G_z(w_t, x_t, y_t)
\]

Here, \( \alpha_t \) is the learning rate that determines the magnitude of the update along the gradient. As this equation shows, \( w_{t+1} \) is dependent on \( w_t \) and this dependence makes parallelization of SGD across iterations difficult.

Figure 1 demonstrates this difficulty. Say, a processor performs SGD on a sequence of examples \( D_1 \) from a global model \( w_g \) to reach \( w_1 \). When processing a subsequent sequence \( D_2 \), a sequential SGD algorithm would have started from \( w_1 \) to reach \( w_h \). Now, we desire to process \( D_1 \) and \( D_2 \) in parallel, but the computation on \( D_2 \) cannot start on \( w_1 \), which is known only after the computation on \( D_1 \) has finished.

State of the art parallelization techniques such as HOGWILD! and ALLREDUCE approach this problem by processing \( D_1 \) and \( D_2 \) starting from the same model \( w_g \) and respectively reaching their local models \( w_1 \) and \( w_2 \). Then, they combine their local models into a global model, but do so in an ad-hoc manner. For instance, ALLREDUCE computes a weighted average of \( w_1 \) and \( w_2 \), where the per-feature weights are chosen so as to prefer the processor that has larger update for that feature. This weighted average is depicted pictorially as \( w_a \) in Figure 1. But doing so does not necessarily reach \( w_h \), the model that a sequential SGD would have produced. HOGWILD! attempts to get around this staleness problem by communicating frequently after every input example (that is, the size of \( D_1 \) and \( D_2 \) is 1). But the resulting communication cost hurts scalability particularly across multiple sockets. This is true even for sparse datasets due to the presence of frequently-occurring features.

2.1. Symbolic SGD

The goal of this paper is to soundly combine local models with the hope of producing the same model as what a sequential SGD would have produced. In Figure 1, we seek a method to combine \( w_1 \) and \( w_2 \) into the global model \( w_h \). This requires “adjusting” the computation of \( D_2 \) for the staleness \( w_1 - w_g \) in the starting model.

To do so, the second processor performs its computation from \( w_g \) and \( \Delta w \), where \( \Delta w \) is an unknown symbolic vector. This allows the second processor to both compute a local model (resulting from the concrete part) and a model combiner (resulting from the symbolic part) that accounts for changes in the initial state. Once both processors are done learning, second processor finds \( w_h \) by setting \( \Delta w \) to \( w_1 - w_g \) where \( w_1 \) is computed by the first processor. This parallelization approach of SGD can be extended to multiple processors where all processor produce a local model and a combiner (except for the first processor) and the local models are combined sequentially using the combiners.

2.2. Model Combiners

Let \( S_D(w) \) represent the SGD computation of dataset \( D \) starting from \( w \). For example, \( w_1 = S_{D_1}(w_g) \) in Figure 1. To generate the model combiner, we need to reason about \( S_{D_1}(w + \Delta w) \). Assuming that \( S_D \) is differentiable, we have the following Taylor series expansion:

\[
S_D(w + \Delta w) = \underbrace{S_D(w)}_{\text{local model}} + \underbrace{\frac{\partial S_D}{\partial w}(w)}_{\text{model combiner}} \cdot \Delta w + O(|\Delta w|^2)
\]

We define \( M_D \triangleq \frac{\partial S_D}{\partial w} \) as the model combiner. In the equation above, the model combiner captures the first-order effect of how a \( \Delta w \) change in \( w_g \) will affect the SGD computation. For instance, by using \( \Delta w = w_1 - w_g \) in this equation, one can combine the local models in Figure 1 to
generate \( w_i \).

When \( \Delta w \) is sufficiently small, one can neglect the second order term and use the model combiner to combine local models with sufficient fidelity. Section 8.5 in the Appendix shows that convergence is guaranteed when neglecting the higher order terms under certain general assumptions of the cost function, provided \( ||\Delta w||_2 \) is bounded.

The following lemma shows how to generate a model combiner.

**Lemma 2.1.** Let \( D = (z_1, z_2, \ldots, z_n) \) be a sequence of input examples and \( D_i \) represent the subsequence \( (z_1, \ldots, z_i) \). The model combiner is given by

\[
M_D(w) = \prod_{i=n}^{1} (I - \alpha_i \cdot H_{z_i}(S_{D_{i-1}}(w), x_i, y_i))
\]

with \( S_{D_i}(w) = w \)

**Proof.** We have

\[
S_D(w) = S_{z_n}(S_{z_{n-1}}(\ldots(S_{z_1}(w))))
\]

The proof follows from Equation 1 and the chain rule. □

### 2.3. The Parallel SGD Algorithm

Model combiners provide a lot of flexibility to design parallel SGD algorithms. Section 5 explores both a map-reduce version and an asynchronous version. We describe the former here for completeness.

In the map phase, each processor \( i \in [1, N] \) starts from the same global model \( w_g \) and computes its local model \( S_{D_i}(w_g) \) and the model combiner \( M_D(w_g) \) in parallel. A subsequent reduction phase combines the local models by adjusting the input of processor \( i \) by \( w_{i-1} - w_g \).

\[
w_i = S_{D_i}(w_g) + M_{D_i}(w_g) \cdot (w_{i-1} - w_g)
\]

### 2.4. Examples

Many interesting machine learning algorithms, such as linear regression, linear regression with L2 regularization, and polynomial regression have a linear update to the model parameters (but not necessarily linear on the input example). In such cases, the higher order terms in Equation 2 vanish. For such learners, model combiners generate exactly the same model as a sequential SGD.

Specifically, considering standard linear regression with square loss, the combiner matrix is given by

\[
M_D(w) = \prod_{i=n}^{1} (I - \alpha_i \cdot x_i \cdot x_i^T)
\]

when computing on \( D = (x_1, y_1) \ldots (x_n, y_n) \). Since the model combiner is independent of \( w \), this can be computed once and reused in subsequent phases provided the learning rates do not change.

For logistic regression, which has the update rule

\[
w_i = w_{i-1} - \alpha \cdot (\sigma(x_i \cdot w_{i-1}) - y) \cdot x_i
\]

where \( \sigma \) is the sigmoid function, the model combiner is given by

\[
M_D(w) = \prod_{i=n}^{1} (I - \alpha \cdot \sigma'(x_i \cdot w_{i-1}) \cdot x_i \cdot x_i^T)
\]

where \( w_0 = w \). The model combiner for logistic regression is the model combiner generated for linear regression but with \( \alpha \) scaled by \( \sigma'(x_i \cdot w_{i-1}) \).

Table 1 provides the model combiners for a few linear learners. When the SGD update function is not differentiable, using the Taylor expansion in Equation 2 can result in errors at points of discontinuity. However, assuming bounded gradients, these errors do not affect the convergence of SYMSGD (Section 8.5).

### 3. Dimensionality Reduction of a Model Combiner

One key challenge in using model combiners as described above is that they are large \( f \times f \) matrices. Machine learning problems typically involve learning over tens of thousands to billions of features. Thus, it is impossible to represent the model combiner explicitly. This section describes mechanisms to address this problem.

The basic idea is to project the combiner matrix into a smaller dimension while maintaining its fidelity. This projection is inspired by the Johnson-Lindenstrauss (JL) lemma (Johnson & Lindenstrauss, 1984) and follows the treatment of Achlioptas (Achlioptas, 2001). While this projection generates an unbiased estimate of the combiner, its variance could potentially affect convergence. Our convergence proof in Section 8.5 show that with appropriate bounds on this variance, convergence is guaranteed.

#### 3.1. Random Projection

We observe that the only use of a combiner matrix \( M_D \) in SYMSGD is to multiply it with a \( \Delta w \). To avoid representing \( M_D \) explicitly, we instead maintain \( M_D \cdot A \) for a randomly generated \( f \times k \) matrix \( A \) with \( k \ll f \). Then we estimate \( M_D \cdot \Delta w \) with \( M_D \cdot A \cdot A^T \cdot \Delta w \). The following lemma describes when this estimation is unbiased.

Let \( [m_{ij}]_{ij} \) represents a matrix with \( m_{ij} \) as the element in the \( i \)th row and \( j \)th column.
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| Algorithm | SGD Update for $z = (x, y)$ | Model Combiner $M_z(w, x, y, \alpha)$ |
|-----------|-----------------|----------------------------------|
| OLS       | $w - \alpha(x \cdot w - y)$ | $I - \alpha \cdot x \cdot x^T$ |
| Logistic  | $w - \alpha(\sigma(x \cdot w) - y)$ | $I - \alpha \sigma'(x \cdot w) \cdot x \cdot x^T$ |
| Perceptron| $w + \alpha(y \cdot x - \delta_{y \cdot x \cdot w < 0})$ | $I$ |
| SVM       | $w - \alpha(\lambda w - y \cdot x \cdot \delta_{x \cdot w > 1})$ | $(1 - \alpha \lambda)I$ |
| Lasso     | $\max(0, u_i \triangleq w_i - \alpha(\lambda + s(i)(y - w \cdot x))x_i) \cdot [\delta_{u_i > 0}(\delta_{i=j} - \alpha s(i)x_i x_j)]$ |

Table 1. Model combiners for various linear learners

Lemma 3.1. Let $A = [a_{ij}]_{i,j}$ be a random $f \times k$ matrix with

$$a_{ij} = d_{ij}/\sqrt{k}$$

where $d_{ij}$ is independently sampled from a random distribution $D$ with $E[D] = 0$ and $\text{Var}[D] = 1$. Then

$$E[A \cdot A^T] = I_{f \times f}$$

Proof. If $B = [b_{ij}]_{i,j} \triangleq A \cdot A^T$, we have $E[b_{ij}] = \frac{1}{k} \sum_{a_{ik}a_{jk}} E[a_{ik}a_{jk}]$. When $i \neq j$, $E[b_{ij}] = 0$ as $a_{ik}$ and $a_{jk}$ are independent random variables with mean 0. $E[b_{ii}] = 1$ as the variance of $a_{ii}$ is 1.

With this lemma, the model combination with Equation 4 becomes

$$w_i \approx S_{D_k}(w_y) + M_{D_k}(w_y) \cdot A \cdot A^T(w_{i-1} - w_y)$$  \hspace{1cm} (6)

This allows an efficient algorithm that only computes the projected version of the combiner matrix while still producing the same answer as the sequential algorithm in expectation. This projection incurs a space and time overhead of $O(z \times k)$ where $z$ is the number of non-zeros in an example, $x_i$. This overhead is acceptable for small $k$ and in fact in our experiments in Section 5, $k$ is between 7 to 15 across all benchmarks. Most of the overhead for such a small $k$ is hidden by utilizing SIMD hardware within a processor (SymSGD with one thread is only half as slow as the sequential SGD as discussed in Section 5). After learning a local model and a projected model combiner in each processor, SymSGD combines the resulting local models using the combiners, but additionally employs the optimizations discussed in Section 3.2.

Note that a subset of the data, $D_k$, often contains a subset of total number of features. Our implementation takes advantage of this property and allocates and initializes $\hat{k}$ for only these observed features.

3.2. The Variance of Projection

The unbiased estimation above is useful only if the variance of the approximation is acceptably small. The following lemma describes the variance of the random projection described above.

The trace of a matrix $M$, $\text{tr}(M)$ is the sum of the diagonal elements. Let $\lambda_i(M)$ by the $i$th eigenvalue of $M$ and $\sigma_i(M) = \sqrt{\lambda_i(M^T \cdot M)}$ the $i$th singular value of $M$. Let $\sigma_{\max}(M)$ be the maximum singular value of $M$.

Lemma 3.2. Let $v = M \cdot A \cdot A^T \cdot \Delta w$. Then the trace of the covariance matrix $\text{tr}(C(v))$ is bounded by

$$\text{tr}(C(v)) \geq \frac{\|\Delta w\|^2}{k} \sum_i \sigma_i^2(M)$$

$$\text{tr}(C(v)) \leq \frac{\|\Delta w\|^2}{k} \left(\sum_i \sigma_i^2(M) + \sigma_{\max}^2(M)\right)$$

Proof. See Section 8.3.

The covariance is small if $k$, the dimension of the projected space, is large. But increasing $k$ proportionally increases the overhead of the parallel algorithm. Similarly, covariance is small if the projection happens on small $\Delta w$. Looking at Equation 6, this means that $w_{i-1}$ should be as close to $w_s$ as possible, implying that processors should communicate frequently enough such that their models are roughly in sync. Finally, the singular values of $M$ should be as small as possible. The next section describes a crucial optimization that achieves this.

3.3. Reducing the Variance

Equation 3 suggests that when $\alpha_i$ is small, the model combiner $M_{D_k}(w)$ is dominated by the $I$ term. From Lemma 8.1 in Section 8.4 shows that the combiner matrix $M_{D_k}(w)$ generated from $n$ examples, $M_{D_k}(w) - I$ has at most $n$ non-zero singular values. Because each processor operates on a subset of the data it is likely that $n$ examples $\ll f$ features. We
use these observations to lower the variance of dimensionality reduction by projecting the matrix $N_D$ instead of $M_D$. This optimization is crucial for the scalability of SYMSGD.

With this optimization the model combiner update becomes

$$w_i \approx S_D(w_g) + w_{i-1} - w_g + N_D(w_g) \cdot A \cdot A^T \cdot (w_{i-1} - w_s)$$

(7)

Lemma 3.1 guarantees that the approximation above is unbiased.

An important factor in controlling the singular values of $N_D(w_g)$ is the frequency of model combinations which is a tunable parameter in SYMSGD. As it is shown in Appendix 7, with more communication, the smaller the singular values of $N_D(w_g)$ and the less variance (error) in Equation 7.

3.4. Empirical Evaluating Singular Values of $M_D(w)$

Figure 7 empirically demonstrates the benefit of taking identity off. This figure plots the singular values of $M_D(w)$ for RCV1 (described in Section 5) after processing 64, 128, 256, 512 examples for logistic and linear regression. As it can be seen, the singular values are close to 1. However, the singular values of $N_D(w) = M_D(w) - I$ are roughly the same as those of $M_D(w)$ minus 1 and consequently, are small. Finally, the smaller $\alpha$ (not shown), the closer the singular values of $M_D(w)$ are to 1 and the singular values of $N_D(w)$ are close to 0. Also, note that the singular values of $M_D(w)$ decrease as the numbers of examples increase and therefore, the singular values of $N_D(w)$ increase. As a result, the more frequent the models are combined, the less variance (and error) is introduced.

4. Parallel SYMSGD Implementation

This section discusses the SYMSGD implementations. Section 2.3 gives a general specification of a parallel SGD algorithm where Section 2.2 describes how to build model combiners. There are many ways to implement these general specifications and in this section we discuss some of our implementation strategies for shared-memory machine.

Section 2.3 describes a map-reduce style version of SYMSGD which we call MR-SYMSGD. In contrast to our algorithm, HogWild! asynchronously updates the model parameters. Because MR-SYMSGD requires computing model combiners, it does strictly more work than HogWild! and is thus a constant factor slower, theoretically. However, even sparse datasets have a frequently used subset of features which are likely to show up in many input examples and as we show in Section 5, this frequent subset causes scalability issues for HogWild!. When cache-lines are invalidated across sockets, which happens often for these frequently accessed subset, HogWild! incurs large overheads which limit its scalability.

Async-SYMSGD is a hybrid implementation of SYMSGD which blends asynchronous updates of infrequent model parameters with MR-SYMSGD style updates for the frequent ones. Because the frequently accessed subset of features is often much smaller than the infrequently accessed ones, Async-SYMSGD has low-overhead, like HogWild!. However, because cache-lines are not invalidated as often, it scales to multiple sockets.

The 5th column in Table 3 (Average NFNZ Ratio) shows the average number of frequent features in each input example divided by the number of non-zero features in that input example. A value of 0 means all features are infrequent and 1 means all features are frequent. We define a frequent feature as to whether a particular feature shows up in at least 10% of the input examples. At runtime, Async-SYMSGD samples 1000 input examples to find frequent features and builds a model combiner for that subset and asynchronously updates those features not in that subset.

Frequency of Model Combination Equation 2 shows that the error in SYMSGD is dependent on the norm of $\Delta w$; the smaller the norm of $\Delta w$, the less the error. The way that we control the norm of $\Delta w$ is by limiting the number of examples that each processor sees before it combines its local model with the global model. We call this parameter the block size. The trade-offs of high and low values of block size are clear: large block size allows the SYMSGD communicate less often and improve overall running time but can potentially suffer in accuracy due to size of $\Delta w$. On the other hand, low values for block size enjoys better convergence but the overhead of model combination may affect the performance.

Block size is set to a constant value per benchmark (part of a parameter sweep discussed in Section 4) throughout the execution of SYMSGD. In future work we expect to dynamically adjust when to communicate by measuring the norm of $\Delta w$.

Details While, in theory, the computational complexity of computing a model combiner $O(k)$ (where $k < 15$ in all experiments), we do not see a $k \times$ slowdown. Each processor consecutively stores each of the $k$ vectors in $A$ so SYMSGD can exploit good cache locality in addition to SIMD units. This is apparent in our experiments: Figure 3 shows that the difference between Async-SYMSGD and HogWild! at 1 processor is almost 0 even though the former does $k$ times more work than the latter.

Lastly, SYMSGD uses a sparse projection (Achlioptas, 2001) to further reduce the overhead of computing $A$. Each element of $A$ is independently chosen from $\{1, -\frac{1}{3}, 0\}$ with probability $\{\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\}$, respectively. This approach
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Figure 2. Distribution of singular values of model combiners for RCV1 dataset for logistic regression with \( \alpha = 0.01 \) and for linear regression with \( \alpha = 0.001 \). Different lines correspond to different block sizes.

Table 3. Datasets characteristics.

| Dataset | #Feat | #Examples | Average NNZ | Average NFNZ Ratio | AUC | SymSGD speedup over Hogwild |
|---------|-------|-----------|-------------|--------------------|-----|-----------------------------|
| Logistic | Linear | Logistic | Linear |
| RCV1    | 47153 | 781265    | 74.71       | 0.219              | 0.958 | 0.959 | 2.60 | 2.60 |
| AdClick | 3076  | 499980    | 969.38      | 0.947              | 0.744 | 0.7654 | 2.99 | 2.94 |
| Epsilon | 2000  | 400000    | 2000        | 1.00               | 0.958 | 0.959 | 2.55 | 2.45 |
| URL     | 3231961 | 1677282 | 111.62      | 0.765              | 0.999 | 0.9986 | 1.90 | 1.04 |
| Criteo  | 1703961 | 1000000  | 33.32       | 0.530              | 0.7627 | 0.7633 | 2.05 | 1.91 |
| Webspam | 16609143 | 2799999 | 3727.75     | 0.693              | 0.9992 | 0.9909 | 1.43 | 1.78 |

**5. Evaluation**

All experiments described in this section were performed on an Intel Xeon E5-2630 v3 machine clocked at 2.4 GHz with 256 GB of RAM. The machine has two sockets with 8 cores each, allowing us to study the scalability of the algorithms across sockets. We disabled hyper-threading and turbo boost. We also explicitly pinned threads to cores in a compact way which means that thread \( i + 1 \) was placed as close as possible to thread \( i \). The machine runs Windows 10. All of our implementations were compiled with Intel C/C++ compiler 16.0 and relied heavily on OpenMP primitives for parallelization and MKL for efficient linear algebra computations. And, finally, to measure runtime, we use the average of five independent runs on an otherwise idle machine.

**Algorithms** Section 4 discusses how we implement three SGD algorithms: Async-SYM SGD, MR-SYM SGD, and HogWild!. For each, we experimented with ordinary least squares (OLS) regression and logistic regression (See Table 1 for the model combiners). This section presents results for logistic regression. The results for OLS are similar so we present them in Appendix 8.1.

When studying the scalability of a parallel algorithm, it is important to compare the algorithms against an efficient baseline (Bailey, 1991; McSherry et al., 2015). Otherwise, it is empirically not possible to differentiate between the scalability achieved from the parallelization of the inefficiencies and the scalability inherent in the algorithm. We spent a significant effort to implement a well-tuned version of all algorithms. For example, Async-SYM SGD, MR-SYM SGD, and HogWild! with 1 thread are between 1.5× to 4.2× faster than Vowpal Wabbit (Langford et al., 2007), a widely used public library.

**Datasets** Table 3 describes various statistics of each benchmark. They are all freely available, with the exception of AdClick, which is an internal Ad dataset. For each algorithm and benchmark, we did parameter sweep over the learning rate, \( \alpha \), and picked that \( \alpha \) which gave the best AUC after 10 passes over the data. For Async-SYM SGD and MR-SYM SGD, we then fixed \( \alpha \) and swept over block size and \( k \) and picked the configuration which maintained sequential accuracy up to the fourth digit.

**Results** The last two columns of Table 3 summarize the speedup of Async-SYM SGD over HogWild! for both logistic and linear regression. Async-SYM SGD is, on average, 2.25X faster than HogWild!. The reason is that Async-SYM SGD is able to scale to multiple sockets: cache-traffic from the frequent subset of each example causes HogWild! to suffer scalability when moving from 8 to 16 cores. FIG-
Figure 3 shows this phenomenon in greater detail. A point on this graph (x-axis, y-axis) shows the speedup of Async-SYM\textsuperscript{SGD}, MR-SYM\textsuperscript{SGD}, and HogWild!, respectively (y-axis) as a function of the number of threads (x-axis). In all benchmarks, HogWild! is slower on 16 threads than 8 (with the exception of Webspam wherein performance stays roughly constant). In contrast both Async-SYM\textsuperscript{SGD} and MR-SYM\textsuperscript{SGD} scale across sockets roughly linearly. Because Async-SYM\textsuperscript{SGD} uses a model combiner only for those frequently accessed subset of features, its overhead is lower than MR-SYM\textsuperscript{SGD} and is thus consistently faster. The results are similar for linear regression with the exception of URL: Async-SYM\textsuperscript{SGD} stops scaling at 8 threads, like HogWild!.

### 6. Related Work

Most schemes for parallelizing SGD learn local models independently and communicate to update the global model. The algorithms differ in how and how often the update is performed. These choices determine the applicability of the algorithm to shared-memory or distributed systems.

To the best of our knowledge, our approach is the only one that seeks to retain the semantics of the sequential SGD algorithm. Given a tight coupling of the processing units, Langford et al. (Langford et al., 2009) suggest on a round-robin scheme to update the global model allowing for some staleness. However, as the SGD computation per example is usually much smaller when compared to the locking overhead, HOGWILD! (Recht et al., 2011) improves on this approach to perform the update in a “racy” manner. While HOGWILD! is theoretically proven to achieve good convergence rates provided the dataset is sparse enough and the processors update the global model fast enough, our experiments show that the generated cache-coherence traffic limits its scalability particularly across multiple sockets. Lastly, unlike SYMSGD, which works for both sparse and dense datasets, HOGWILD! is explicitly designed for sparse data. Recently, (Sallinen et al., 2016) proposed applying lock-free HOGWILD! approach to mini-batch. However, mini-batch converges slower than SGD and also they did not study multi-socket scaling.

Zinkevich et al. (Zinkevich et al., 2010) propose a MapReduce-friendly framework for SGD. The basic idea is for each machine/thread to run a sequential SGD on its local data. At the end, the global model is obtained by averaging these local models. Our experiments with this approach show it converges very slow in comparison to a sequential algorithm because the model parameters derived from sparse features are penalized by that average at ev-
ery step. Alekh et al. (Agarwal et al., 2014) extend this approach by using MPI_AllReduce operation. Additionally, they use the adagrad (Duchi et al., 2011) approach for the learning rates at each node and use weighted averaging to combine local models with processors that processed a feature more frequently having a larger weight. Our experiments on our datasets and implementation shows that it does not achieve the sequential accuracy for similar reasons as Zinkevich et al.

Several distributed frameworks for machine learning are based on parameter server (Li et al., 2014b;a) where clients perform local learning and periodically send the changes to a central parameter server that applies the changes. For additional parallelism, the models themselves can be split across multiple servers and clients only contact a subset of the servers to perform their updates.

Lastly, there is a significant body of work in the high-performance computing literature on linear solvers. For example, MKL has optimized routines for dense linear least squares problems (Intel). We found these routines to be significantly slower than even our sequential baseline running OLS on dense datasets and MKL does not deal with non-linear terms nor sparse data. Likewise, randomized numerical linear algebra methods, like RandNLA, use random projections to solve linear least squares problems quickly (Drineas & Mahoney, 2016). While both our technique and RandNLA use randomized projections, our insight of taking $I$ off of the matrix we project is a critical step to controlling the accuracy of our approach. Further, RandNLA is specific to linear least squares.

7. Conclusion

With terabytes of memory available on multicore machines today, our current implementation has the capability of learning from large datasets without incurring the communication overheads of a distributed system. That said, we believe the ideas in this paper apply to distributed SGD algorithms we plan to pursue in future work.

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8. Appendix

8.1. Ordinary Least Squares Regression Results

![Graphs showing speedup](image)

*Figure 4.* Speedup of Least Squares regression training when using Async-SymSGD, MR-SymSGD, and HogWild! on a 16 core machine.

8.2. Variance and Covariance of \( \frac{1}{r} M \cdot A \cdot A^T \cdot \Delta w \)

In here, for the sake of simplicity, we use \( w \) instead of \( \Delta w \) and instead of \( k \) for the size of the projected space, we use \( r \) since \( k \) is used for summation indices in here, heavily. We want to estimate \( v = M \cdot w \) with \( \frac{1}{r} M \cdot A \cdot A^T \cdot w \), where \( A \) is a \( f \times r \) matrix, where \( a_{ij} \) is a random variable with the following properties: \( \mathbb{E}(a_{ij}) = 0 \), \( \mathbb{E}(a_{ij}^2) = 1 \), and \( \mathbb{E}(a_{ij}^4) = \rho = 3 \).

Let \( m^T \) be some row of \( M \). Its estimation in \( M \cdot w \) is \( v_s = \frac{1}{r} \cdot m^T \cdot A \cdot A^T \cdot w \). From Lemma 3.1 \( \mathbb{E}(v_s) = m^T \cdot w \).

We will use the notation \( ij = kl \) to mean \( i = k \wedge j = l \), and \( ij \neq kl \) to mean its negation. Let \( m_s, m_t \) be two rows of \( M \). We want to find the covariance of the resulting \( v_s \) and \( v_t \).
In other words

\[ r^2 \cdot \mathbb{E}(v_s, v_t) \]

\[ = r^2 \cdot \mathbb{E}\left( \frac{1}{r^2} \sum_{i,j,k} m_s m_{t;i} a_{ij} a_{k,j} w_k \right) \cdot \sum_{i',j',k'} m_{t';i'} a_{i'j'} a_{k',j'} w_{k'} \]

\[ = \sum_{i,j,k,i',j',k'} m_s m_{t;i} w_k w_{k'} \mathbb{E}(a_{ij} a_{k,j} a_{i'j'} a_{k',j'}) \]

\[ + \sum_{i,j,k,i',j',k'} m_s m_{t;i} w_k w_{k'} \mathbb{E}(a_{ij} a_{k,j} a_{i'j'} a_{k',j'}) \]

\[ + \sum_{i,j,k,i',j',k'} m_s m_{t;i} w_k w_{k'} \mathbb{E}(a_{ij} a_{k,j} a_{i'j'} a_{k',j'}) \]

\[ + \sum_{i,j,k,i',j',k'} m_s m_{t;i} w_k w_{k'} \mathbb{E}(a_{ij} a_{k,j} a_{i'j'} a_{k',j'}) \]

\[ = \sum_{i,j} m_s m_{t;i} w_i w_j \rho + \sum_{i,j,i',j':i \neq i'} m_s m_{t;i} w_i w_{i'} \]

\[ + \sum_{i,j,k: i \neq k} m_s m_{t;i} w_k w_k + \sum_{i,j,k: i \neq k} m_s m_{t;k} w_k w_i \]

\[ = \rho \sum_{i,j} m_s m_{t;i} w_i^2 \]

\[ + \sum_{i,j,i',j'} m_s m_{t;i} w_i w_{i'} - \sum_{i,j,i':i \neq i'} m_s m_{t;i} w_i w_{i'} \]

\[ + \sum_{i,j,k} m_s m_{t;i} w_k^2 \]

\[ + \sum_{i,j,k} m_s m_{t;k} w_k w_i \]

\[ = (\rho - 3) \sum_{i,j} m_s m_{t;i} w_i^2 + \sum_{i,j,i':i \neq i'} m_s m_{t;i} w_i w_{i'} \]

\[ + \sum_{i,j,k} m_s m_{t;i} w_k^2 \]

\[ + \sum_{i,j,k} m_s m_{t;k} w_i w_k \]

\[ = r^2 \sum_{i,i'} m_s m_{t;i} w_i w_{i'} + r \sum_{i,k} m_s m_{t;i} w_k^2 + r \sum_{i,k} m_s m_{t;k} w_i w_k \]

\[ = (r^2 + r) \sum_{i,i'} m_s m_{t;i} w_i w_{i'} + r \cdot m_s^T \cdot m_t \sum_k w_k^2 \]

as \( \rho = 3 \) and \( j \in [1 \ldots k] \)

In other words

\[ \mathbb{E}(v_s v_t) = (1 + \frac{1}{r}) \sum_{i,i'} m_s m_{t;i} w_i w_{i'} + \frac{1}{r} \cdot m_s^T \cdot m_t \sum_k w_k^2 \]

The covariance \( \text{Cov}(a, b) = \mathbb{E}(a \cdot b) - \mathbb{E}(a)\mathbb{E}(b) \). Using this we have
\[
\text{Cov}(v_s, v_t) = (1 + \frac{1}{r}) \sum_{i, i'} m_{si} m_{ti'} w_i w_{i'} + \frac{1}{r} m_s^T m_t \sum_k w_k^2 - \mathbb{E}(v_s)\mathbb{E}(v_t)
\]

\[
= (1 + \frac{1}{r}) \sum_{i, i'} m_{si} m_{ti'} w_i w_{i'} + \frac{1}{r} m_s^T m_t \sum_k w_k^2 - \mathbb{E}(v_s)\mathbb{E}(v_t)
\]

\[
= (1 + \frac{1}{r}) \mathbb{E}(v_s)\mathbb{E}(v_t) + \frac{1}{r} m_s^T m_t \sum_k w_k^2 - \mathbb{E}(v_s)\mathbb{E}(v_t)
\]

\[
= \frac{1}{r} \mathbb{E}(v_s)\mathbb{E}(v_t) + \frac{1}{r} m_s^T m_t \sum_k w_k^2
\]

Let \( C(v) \) be the covariance matrix of \( v \). That is, \( C(v)_{ij} = \text{Cov}(v_i, v_j) \). So, we have

\[
C(v) = \frac{1}{r} (M \cdot w) \cdot (M \cdot w)^T + \frac{1}{r} (M \cdot M^T) \|w\|_2^2
\]

Note that we can use this computation for matrix \( N = M - I \) as well since we did not assume anything about the matrix \( M \) from the beginning. Therefore, for \( v' = w + \frac{1}{r} N \cdot A \cdot A^T \cdot w \), \( C(v') = \frac{1}{r} (N \cdot w) \cdot (N \cdot w)^T + \frac{1}{r} (N \cdot N^T) \|w\|_2^2 \) since \( w \) is a constant in \( v' \) and \( C(a + x) = C(x) \) for any constant vector \( a \) and any probabilistic vector \( x \). Next we try to bound \( C(v) \).

### 8.3. Proof of Lemma 3.2

We can bound \( C(v) \) by computing its trace since \( \text{tr}(C(v)) = \sum_i \text{var}(v_i) \), the summation of the variance of elements of \( v \).

\[
\text{tr}(C(v)) = \frac{1}{r} \text{tr}((M \cdot w) \cdot (M \cdot w)^T) + \frac{1}{r} \|w\|_2^2 \text{tr}(MM^T)
\]

\[
= \frac{1}{r} \|M \cdot w\|_2^2 + \frac{1}{r} \|w\|_2^2 \left( \sum_i \lambda_i(M \cdot M^T) \right)
\]

\[
= \frac{1}{r} \|M \cdot w\|_2^2 + \frac{1}{r} \|w\|_2^2 \left( \sum_i \sigma_i(M)^2 \right)
\]

where \( \lambda_i M \cdot M^T \) is the \( i^{th} \) largest eigenvalue of \( M \cdot M^T \) which is the square of \( i^{th} \) largest singular value of \( M, \sigma_i(M)^2 \).

Since \( \|M \cdot w\|_2^2 \leq \|w\|_2^2 \|M\|_2^2 = \|w\|_2^2 \sigma_{\text{max}}(M)^2 \), we can bound \( \text{tr}(C(v)) \) as follows:

\[
\text{tr}(C(v)) \leq \frac{1}{r} (\sigma_{\text{max}}(M)^2) + \frac{1}{r} \|w\|_2^2 \left( \sum_i \sigma_i(M)^2 \right)
\]

It is trivial to see that:

\[
\frac{1}{r} \|w\|_2^2 \left( \sum_i \sigma_i(M)^2 \right) \leq tr(C(v))
\]

Combining the two inequalities, we have:

\[
\frac{1}{r} \|w\|_2^2 \left( \sum_i \sigma_i(M)^2 \right) \leq tr(C(v)) \leq \frac{1}{r} (\sigma_{\text{max}}(M)^2) + \frac{1}{r} \|w\|_2^2 \left( \sum_i \sigma_i(M)^2 \right)
\]
The same bounds can be derived when $N = M - I$ is used.

8.4. Rank of Matrix $N = M - I$

Now we show that subtracting $I$ from a model combiner results in a matrix with small rank. Thus, most of its singular values are zero. We assume that the model combiner is generated for a linear learner and thus it is of the form $\prod_{t} (I - \alpha x_t x_t^T)$ where any nonlinear scalar terms from the Hessian are factored into $\alpha$.

Lemma 8.1. For the matrix $M_{a\rightarrow b} = \prod_{i=b}^{a} (I - \alpha x_i \cdot x_i^T)$, $\text{rank}(M_{a\rightarrow b} - I) \leq b - a$.

Proof. The proof is by induction. The base case is when $a = b$ and $M_{a\rightarrow b} = I$. It is clear that $I - I = 0$ which is of rank zero. For the inductive step, assume that $\text{rank}(M_{a\rightarrow b-1} - I) \leq b - a - 1$. We have

$$M_{a\rightarrow b} - I = (I - \alpha x_b \cdot x_b^T) M_{a\rightarrow b-1} - I$$

$$= (M_{a\rightarrow b-1} - I) - \alpha x_b \cdot (x_b^T \cdot M_{a\rightarrow b-1})$$

Term $\alpha x_b \cdot (x_b^T \cdot M_{a\rightarrow b-1})$ is a rank-1 matrix and term $(M_{a\rightarrow b-1} - I)$ is of rank $b - a - 1$ by induction hypothesis. Since for any two matrices $A$ and $B$, $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$, $\text{rank}(M_{a\rightarrow b} - I) \leq \text{rank}(M_{a\rightarrow b-1}) + \text{rank}(-\alpha x_b \cdot (x_b^T \cdot M_{a\rightarrow b-1})) \leq b - a - 1 + 1 = b - a$. \hfill \Box

8.5. Convergence Proof

Let the sequence $w_0, w_1, \ldots, w_t$ represent the sequence of weight vectors produced by a sequential SGD run. We know that this sequence converges to the desired minimum $w^*$. Our goal is to show that SYMSGD also converges to $w^*$. Consider a process processing example sequences $D$ starting with model $w_t - \Delta w$ that is $\Delta w$ different from the “true” model $w_t$ that a sequential SGD would have started with. The output of this processor is

$$w_{t+1} = S_D(w_t - \Delta w) + M_D \Delta w$$

(8)

where the model combiner after the projection by taking the I off is given by

$$M_D = I + (S_D'(w_t - \Delta w) - I) A A^T$$

Applying Taylor’s theorem, we have for some $0 \leq \mu \leq 1$

$$S_D(w_t) = S_D(w_t - \Delta w) + S_D'(w_t - \Delta w) \Delta w + \frac{1}{2} \Delta w^T S_D''(w_t - \mu \Delta w) \Delta w$$

(9)

Comparing Equation 9 with Equation 8, we see that SYMSGD introduces two error terms to a sequential SGD

$$w_{t+1} = S_D(w_t) + FR_D(w_t, \Delta w) + SR_D(w_t, \Delta w)$$

where the first-order error term $FR$ comes due to the projection approximation

$$FR_D(w_t, \Delta w) = (I - S_D'(w_t - \Delta w))(I - A A^T)$$

and the second-order error term $SR$ comes due to neglecting the higher-order terms in the Taylor expansion.

$$SR_D(w_t, \Delta w) = \frac{1}{2} \Delta w^T S_D''(w_t - \mu \Delta w) \Delta w$$

To prove convergence SYMSGD, we show that SGD is “robust” with respect to adding these error terms. The proof follows along the same lines as the convergence proof of SGD by Bottou (Bottou, 2012) and uses similar notations. We state below the assumptions and Lemmas required for the main proof. The proof of these Lemmas is shown later.
**Assumption 1.** Convexity of the cost function

\[(w - w^*), G(w) > 0\]

for \(w \neq w^*\).

**Assumption 2.** Bounded gradients. For any input \(z = (X, y)\)

\[\|G_z(w)\|_2 \leq b_G \|w - w^*\|_2\]

for some \(b_G \geq 0\).

**Lemma 8.2.** Bounds on the mean and second moment of \(FR\)

\[E_A(FR_D(w_t, \Delta w)) = 0\]
\[E_A(||FR_D(w_t, \Delta w)||_2^2) \leq b_{FR} ||w_t - w^*||_2^2\]

for some \(b_{FR} \geq 0\)

**Lemma 8.3.** Bounds on \(SR\)

\[||SR_D(w_t, \Delta w)||_2 \leq b_{SR} ||w_t - w^*||_2\]

for some \(b_{SR} \geq 0\)

Convergence of SYMSGD follows if the following sequence converges almost surely to 0.

\[h_t = ||w_t - w^*||_2^2\]

We assume the worst case where the error terms are added every step of the SGD. This make the proof much simpler and more along the lines of the proof in Bottou (Bottou, 2012). Note that this is indeed the worst case as the error bounds from Lemma 8.2 and Lemma 8.3 are for arbitrary steps.

**Theorem 8.4.** The sequence \(h_t\) converges to 0 almost surely.

**Proof.** As in Bottou (Bottou, 2012), we denote \(P_t\) denote all the random choices made by the algorithm at time \(t\). For terseness, we use the following notation for the conditional expectation with respect to \(P_t\):

\[CE(x) = E(x|P_t)\]

The key technical challenge is in showing that the infinite sum of the positive expected variations in \(h_t\) is bounded, which we show below. Let \(z = (X, y)\) be the example processed at time \(t\). We use following short hand.

\[R_z(w_t, \Delta w) = FR_z(w_t, \Delta w) + SR_z(w_t, \Delta w)\]
\[ CE(h_{t+1} - h_t) \]

\[ = -2\gamma_t (w_t - w^*) CEG(z(w_t) + R_z(w_t, \Delta w)) + \gamma_t^2 CE(\|z(w_t) + R_z(w_t, \Delta w)\|_2^2) \]

\[ = -2\gamma_t (w_t - w^*) (G(w_t) + CE(R_z(w_t, \Delta w))) + \gamma_t^2 CE(\|z(w_t) + R_z(w_t, \Delta w)\|_2^2) \]

\[ \leq -2\gamma_t (w_t - w^*) CE(R_z(w_t, \Delta w)) + \gamma_t^2 CE(\|z(w_t) + R_z(w_t, \Delta w)\|_2^2) \quad \text{(from Assumption 1)} \]

\[ \leq -2\gamma b_{SR} \|w_t - w^*\|_2^2 + \gamma_t^2 CE(\|z(w_t) + R_z(w_t, \Delta w)\|_2^2) \quad \text{(from Lemmas 8.2 and 8.3)} \]

\[ \leq \gamma_t^2 CE(\|z(w_t) + R_z(w_t, \Delta w)\|_2^2) + 2CE(G_z(w_t)) + 2CE(G_z(w_t)R_z(w_t, \Delta w)) \]

\[ \leq \gamma_t^2 ((b_{FR} + b_{SR}) \|w_t - w^*\|_2^2 + 2CE(G_z(w_t)R_z(w_t, \Delta w))) \quad \text{(as FR has a zero mean (Lemma 8.2) and G, SR do not depend on A)} \]

\[ \leq \gamma_t^2 ((b_{FR} + b_{SR} + 2b_{FR}b_{SR}) \|w_t - w^*\|_2^2) \quad \text{(from Assumption 2 and Lemma 8.3)} \]

In other words, for \( B = b_{FR} + b_{SR} + 2b_{FR}b_{SR} \), we have

\[ CE(h_{t+1} - (1 + \gamma_t^2 B)h_t) \leq 0 \quad \text{(10)} \]

From here on, the proof proceeds exactly as in Bottou (Bottou, 2012). Define auxiliary sequences \( \mu_t = \Pi_{i=1}^t \frac{1}{1 + \gamma_t B} \) and \( h'_t = \mu_t h_t \). Assuming \( \Sigma_t \gamma_t^2 < \infty \), \( \mu_t \) converges to a nonzero value. Since Equation (10) implies \( CE(h'_{t+1} - h'_t) \leq 0 \), from quasi-martingale convergence theorem, \( h'_t \) and thus \( h_t \) converges almost surely. Under the additional assumption that \( \Sigma_t \gamma_t = \infty \), we can show that this convergence is to 0.

The proof above crucially relies on lemmas 8.2 and 8.3 that we now prove. But first we make some assumptions and prove supplementary lemmas. We restrict the discussion, as in Lemma 8.1, to linear learners and that the model combiners are of the form \( M_z(w) = (I - \alpha H_z(w)x_i x_i^T) \) for a scalar Hessian \( H_z(w) \)

**Assumption 3.** SYMSGD synchronizes sufficiently enough so that \( \Delta w \) does not grow too large.

\[ \|\Delta w\|_2 \leq \min(1, b_{\Delta w} \|w_t - w^*\|_2) \]

for some \( b_{\Delta w} > 0 \)

**Assumption 4.** Bounded Hessian.

\[ |H_z(w)| \leq b_H \]

for some \( b_H > 0 \)

**Lemma 8.5.** The model combiner \( M_D(w) = \Pi_i (I - \alpha H_z(w)x_i x_i^T) \) has bounded eigenvalues

**Proof.** The proof follows from induction on \( i \) using Assumption 4.

---

**8.6. Proof of Lemma 8.2**

The mean is a simple restatement of Lemma 3.1. The second moment follows from Assumption 3, Lemma 8.5 applied to \( M \) and \( M^T \), and from Lemma 3.2.
8.7. Proof of Lemma 8.3

For linear learners, we have

\[
S_z(w) = w - \gamma G_z(x \cdot w, y) \cdot x
\]
\[
\frac{\partial S_z(w)}{\partial w} = I - \gamma H_z(x \cdot w, y) xx^T
\]
\[
\frac{\partial^2 S_z(w)}{\partial w^2} = H'_z(x \cdot w, y) x \otimes x \otimes x
\]

where \( H \) is the second derivative of the cost with respect to \( x \cdot w \), and \( \otimes \) is the tensor outer product.

In the last equation above, if the input is composed with a previous SGD phase we have

\[
\frac{\partial^2 S_z(S_D(w))}{\partial w^2} = H'_z(S_D(x) \cdot w, y)x \otimes x \otimes (\frac{\partial S_D(w)}{\partial w})^T x
\]

For notational convenience, let \( M_{b \rightarrow a} \triangleq \prod_{i=b}^{a} (I - \alpha x_i x_i^T) \). Explicitly differentiating \( S'_n(w) \), we can show that

\[
\frac{\partial S'_n(w)}{\partial w} = \sum_j \left( -\alpha_j H'_j(s_j(w)) M_{n \rightarrow j+1} x_j x_j^T M_{j \rightarrow 1} \right) \otimes (s'_j(w))^T x
\]

Each element of \( SR_z \) is obtained by \( \Delta w^T P \Delta w \) where \( P \) is an outer product of a row from the first term above and \( s'_j(w)^T x \). Using Lemma 8.5 twice we can show that each of these vectors are bounded. This proves the lemma.