Hochschild cohomology of quantized symplectic orbifolds and the Chen-Ruan cohomology

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Abstract

We prove the additive version of the conjecture proposed by Ginzburg and Kaledin in [23]. This conjecture states that if \(X/G\) is an orbifold modeled on a quotient of a smooth affine symplectic variety \(X\) (over \(\mathbb{C}\)) by a finite group \(G \subset \text{Aut}(X)\) and \(A\) is a \(G\)-stable quantum algebra of functions on \(X\) then the graded vector space \(HH^\bullet(A^G)\) of the Hochschild cohomology of the algebra \(A^G\) of invariants is isomorphic to the graded vector space \(H^\bullet_{CR}(X/G)\) of the Chen-Ruan (stringy) cohomology of the orbifold \(X/G\).

MSC-class: 16E40, 53D55.

1 Introduction

A topological space which is obtained as a quotient of another space with respect to an action of a group is still a challenging object in modern geometry. Orbifolds provide us with an important class of examples of such spaces.

In this paper we consider a smooth affine algebraic variety \(X\) (over \(\mathbb{C}\)) endowed with an algebraic symplectic structure and with an action \(\pi\) of a finite group \(G\) which preserves the symplectic structure. Given these data it is not hard to construct a \(G\)-invariant (associative) star-product \(\star\) in the vector space \(A_+ = \mathcal{O}(X)[[\hbar]]\) of formal power series over regular functions on \(X\), which deforms the commutative multiplication on \(\mathcal{O}(X)\) in the sense of the deformation quantization [2], [3]. If \(A = A_+[[\hbar]]\) denotes the localization of \(A_+\) then it is natural to refer to the algebra \(A^G\) of invariants as the algebra of quantum functions of the orbifold modeled on \(X/G\).

The main result of this paper is
Theorem 1 If the group $G$ acts faithfully on $X$ then there exists an isomorphism

$$HH^\bullet(A^G) \cong H^\bullet_{CR}(X/G, \mathbb{C}((\hbar)))$$

between the graded vector space $HH^\bullet(A^G)$ of Hochschild cohomology of the quantum algebra $A^G$ of functions on $X/G$ and the graded vector space $H^\bullet_{CR}(X/G, \mathbb{C}((\hbar)))$ of Chen-Ruan (=stringy) cohomology [7] of the orbifold $X/G$ with coefficients in the field $\mathbb{C}((\hbar))$.

This statement is an additive version of the conjecture stated in paper [23]. More precisely, in [23] V. Ginzburg and D. Kaledin conjectured the existence of an isomorphism (1) which also respects the corresponding multiplicative structures.

We have to mention that the algebras we deal with are defined over the field $\mathbb{C}((\hbar))$ of formal Laurent power series in an auxiliary variable $\hbar$. Therefore our algebras are endowed with the natural $\hbar$-adic topology and, in the definition of Hochschild homological complex, we have to replace the algebraic tensor product by the one completed in this topology.

We would like to mention recent results of N. Ganter who proposed in her PhD thesis [20] a homotopic refinement of the theory of the orbifold elliptic genus and the Dijkgraaf-Moore-Verlinde-Verlinde (DMVV) product formulas [12]. In her thesis she also defines higher chromatic versions of the orbifold elliptic genus and proves the DMVV-type formula for these versions of the orbifold elliptic genus. In recent paper [19] E. Frenkel and M. Szczesny proved that the graded vector space of the Chen-Ruan orbifold cohomology is isomorphic to the graded vector space of the cohomology of the chiral De Rham complex, which can be naturally associated with an orbifold modeled on a quotient of a smooth complex variety. In [19] the authors also proposed a formula which expresses the orbifold elliptic genus in terms of a partition function of the corresponding chiral vertex algebra. It would be interesting to generalize this formula of E. Frenkel and M. Szczesny to the higher chromatic versions of the orbifold elliptic genus proposed in [20] by N. Ganter.

The authors of this paper were unaware about the results of X. Tang who computed in his thesis [29] Hochschild and periodic cyclic homology of the quantum algebra of functions on a proper étale groupoid equipped with an invariant symplectic structure. Combining the result of X. Tang (see theorem 4.3.2) together with the Van Den Bergh duality [30] one can easily deduce theorem 1. However, we have to mention that, in order to compute Hochschild homology of the above quantum algebra, X. Tang used the formality theorem for Hochschild chains [10] and we do use any version of the formality theorem in our proof.

The paper is organized as follows. In the following section we outline a sketch of the proof of theorem 1. This proof is based on three propositions 2, 3, and 4. The first proposition is the Van Den Bergh duality [30] between Hochschild cohomology and Hochschild homology. The second proposition is the decomposition theorem of the Hochschild (co)homology of the twisted group algebra, and the third proposition says that one can express the Hochschild homology of the quantum algebra $A$ with values in the bimodule $Ag$ twisted by the action of an element $g \in G$ in terms of De Rham cohomology of the subvarieties of fixed points of $X$ under the action of $g$. The third section of the paper is devoted to the proof of proposition 3 and the forth most technical section is devoted to the proof of proposition 4. We conclude our paper by giving an application of our result to deformation theory, namely to the definition of a “global” analog of symplectic reflection algebras from [14].

Notation. Throughout the paper summation over repeated indices is assumed. $X$ is a smooth affine algebraic variety (over $\mathbb{C}$) equipped with an algebraic symplectic form. The
dimension of \( X \) is \( 2n \). We omit symbol \( \wedge \) referring to a local basis of exterior forms as if we thought of \( dx^i \)'s as anti-commuting variables. By a nilpotent linear operator we always mean one whose second power vanishes. If an associative algebra \( A \) is endowed with an action of a finite group \( G \) then by \( a^g \) we denote the left action of an element \( g \in G \) on an element \( a \in A \) and by \( A[G] = \mathbb{C}[G] \triangleright A \) we denote the corresponding twisted group algebra. If \( B \) is an associative algebra then \( B^{op} \) denotes the algebra \( B \) with the opposite multiplication. For a vector bundle \( V \) we denote by \( S^m V \) the \( m \)-th symmetric tensor power of \( V \).

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2 Sketch of the proof.

We start with an observation that \( A \) is a simple algebra. Indeed, if \( I \) is a non-zero two-sided ideal in \( A \) then \( I_+ = I \cap A_+ \) is a saturated ideal in \( A_+ \). This implies that \( I_0 = I_+ / h I_+ \) is a non-zero Poisson ideal in \( A_0 = \mathcal{O}(X) \). But \( X \) is symplectic and therefore \( I_0 = A_0 \). Since \( I_+ \) is saturated, \( I_+ = A_+ \) and hence \( I = A \).

Next, we would like to mention the following

**Proposition 1** ([27]) *If \( B \) is a simple algebra and \( G \) is a finite group which acts on \( B \) by outer automorphisms then the twisted group algebra \( B[G] \) is also simple.*

It is not hard to see that the algebra \( A \) does not have inner automorphisms of finite order. Therefore, since \( G \) acts on \( A \) faithfully, we conclude that the twisted group algebra \( A[G] \) is simple. The latter implies that the \((A^G, A[G])\)-bimodule

\[ P_e = e A[G] \]

and the \((A[G], A^G)\) -bimodule

\[ Q_e = A[G] e, \quad e = \frac{1}{|G|} \sum_{g \in G} g \]

establish a Morita equivalence between \( A[G] \) and \( A^G \). Hence, we can replace \( HH^\bullet(A^G) \) in (1) by the graded vector space \( HH^\bullet(A[G]) \) of Hochschild cohomology of the twisted group algebra \( A[G] \).

Let \( g \) be an element of the group \( G \). The subvariety \( X_g \) of fixed points in \( X \) under the action of \( g \) may, in principle, be reducible and have components of different dimensions. However, it is not hard to show that \( X_g \) is smooth\(^1\). Therefore, if \( \{X_Q^g\}, Q = 1, 2, \ldots \) is the set of irreducible components of \( X_g \) then two different components do not intersect each other. Thus each irreducible (and also connected) component \( X_Q^g \) of \( X_g \) is a smooth affine subvariety of \( X \).

For the orbifold modeled on the quotient \( X/G \) of a smooth affine symplectic variety \( X \) (over \( \mathbb{C} \)) the definition of the Chen-Ruan cohomology [7] simplifies drastically. Namely, we

\(^1\)To prove this assertion one can use normal coordinates associated with a \( G \)-invariant connection on \( X \).
have [23] that

$$H_{\text{CR}}^\bullet(X/G) = \left( \bigoplus_{g \in G, Q} H_{\text{DR}}^{\text{codim}X^g_Q}(X^g_Q) \right)^G,$$  \hspace{1cm} (2)

where the notation $H_{\text{DR}}$ stands for algebraic De Rham cohomology. Notice that our coefficient field is $\mathbb{C}((h))$. Hence, by the algebraic De Rham cohomology of a variety $Y$ we mean the cohomology of the De Rham complex of the formal Laurent power series in $h$ with values in algebraic exterior forms on $Y$. By Grothendieck’s theorem this cohomology is isomorphic to the singular cohomology of $Y$ with the coefficients in the field $\mathbb{C}((h))$.

The proof of theorem 1 is essentially based on the three propositions given below.

**Proposition 2 (M. Van Den Bergh,[30])** Suppose that an algebra $B$ over a field $k$ has a finite Hochschild dimension and

$$\text{Ext}_{B \otimes B^{op}}^i(B, B \otimes B^{op}) = \begin{cases} U & \text{if } i = d, \\ 0 & \text{otherwise} \end{cases}$$

as $B$-bimodules, where $U$ is an invertible $B$-module. Then for any $B$-bimodule $M$ the graded vector spaces

$$HH^\bullet(B, M) \quad \text{and} \quad HH_{d-\bullet}(B, U \otimes_B M)$$

are naturally isomorphic.

Following [15] we denote by $VB(d)$ the class of associative algebras $B$ satisfying the conditions of proposition 2 with $U = B$.

Notice that if $M$ is a bimodule over the twisted group algebra $A[G]$ then $M$ is also a $G$-bimodule and an $A$-bimodule. Therefore, to any $A[G]$-bimodule $M$ we assign the Hochschild complexes $C_\bullet(A, M)$ and $C^\bullet(A, M)$ endowed with the following action of $G$:

$$\alpha(g)(m, a_1, a_2, \ldots, a_q) = (gm^{-1}, (a_1)^g, (a_2)^g, \ldots, (a_q)^g),$$ \hspace{1cm} (3)

$$\alpha(g)(\Psi)(a_1, a_2, \ldots, a_q) = g\Psi((a_1)^{g^{-1}}, (a_2)^{g^{-1}}, \ldots, (a_q)^{g^{-1}})g^{-1},$$ \hspace{1cm} (4)

g $\in G$, $(m, a_1, \ldots, a_q) \in C_q(A, M)$, $\Psi \in C^q(A, M)$.

**Proposition 3** If $A$ is an associative algebra (over $\mathbb{C}$) acted on by a finite group $G$, and $M$ is a bimodule over the twisted group algebra $A[G]$, then

$$HH_\bullet(A[G], M) = \left( HH_\bullet(A, M) \right)^G,$$ \hspace{1cm} (5)

and

$$HH^\bullet(A[G], M) = \left( HH^\bullet(A, M) \right)^G,$$ \hspace{1cm} (6)

where the action of $G$ on $HH_\bullet(A, M)$ (resp. on $HH^\bullet(A, M)$) is induced by (3) (resp. by (4)).
Proposition 4  The graded $G$-modules

$$HH_{\bullet}(A, A[G])$$

and

$$\bigoplus_{g \in G, Q} H^\dim_{DR} X_Q^{-\bullet}(X_Q^g)$$

are isomorphic.

Remark. We would like to mention that in the case of the trivial group $G = \{e\}$ the above proposition reduces to the algebraic analogue of the well-known theorem of R. Nest and B. Tsygan (theorem A2.1 in [28]) which is proved for the quantum algebra of compactly supported functions of a smooth symplectic manifold.

Theorem 1 will follow immediately from the above propositions if we prove that $A[G]$ belongs to the class $VB_{2n}$, where $2n$ is the dimension of $X$.

For this purpose we first prove that

Proposition 5  If an associative algebra $A$ (over $\mathbb{C}$) satisfies the conditions of proposition 2 and

$$\text{Ext}^d_{A \otimes A^{op}}(A \otimes A^{op}) = A$$

both as an $A$-bimodule and a $G$-module, then the twisted group algebra $A[G]$ also belongs to the class $VB(d)$.

Proof. By the assumption $A$ has a finite Hochschild dimension. Hence, due to proposition 3, so does the algebra $A[G]$.

Using proposition 3 once again we get

$$\text{Ext}^i_{A[G] \otimes A[G]^{op}}(A[G], A[G] \otimes A[G]^{op}) = HH^i(A[G], A[G] \otimes A[G]^{op}) = \left( \bigoplus_{g, h \in G} HH^i(A, A \otimes h A^{op}) \right)^G.$$  

For any pair $g, h \in G$ the $A$-bimodule $Ag \otimes h A^{op}$ is isomorphic to $A \otimes A^{op}$ and hence

$$\text{Ext}^i_{A[G] \otimes A[G]^{op}}(A[G], A[G] \otimes A[G]^{op}) = 0$$

for any $i \neq d$. Furthermore, using condition (7) we get

$$\text{Ext}^d_{A[G] \otimes A[G]^{op}}(A[G], A[G] \otimes A[G]^{op}) = \left( \bigoplus_{h \in G} A[G] \otimes h \right)^G,$$

where $A[G] \otimes h$ denotes the $A[G]$-bimodule $A[G]$ twisted from the right by $h$, and $g \in G$ acts on the direct sum $\bigoplus_{h \in G} A[G] \otimes h$ multiplying the first component by $g^{-1}$ from the right and the second component by $g$ from the left.

The direct sum $\bigoplus_{h \in G} A[G] \otimes h$ is free as a $G$-module. Hence,

$$\text{Ext}^d_{A[G] \otimes A[G]^{op}}(A[G], A[G] \otimes A[G]^{op}) = A[G]$$

as an $A[G]$-bimodule and the desired statement follows. $\square$
Proposition 6 ([15]) The algebra $A$ of quantum functions on $X$ has a finite Hochschild dimension,

$$\text{Ext}^i_{A \otimes A^{\text{op}}}(A, A \otimes A^{\text{op}}) = 0$$

for any $i \neq 2n = \dim X$ and

$$\text{Ext}^{2n}_{A \otimes A^{\text{op}}}(A, A \otimes A^{\text{op}}) = A$$

both as an $A$-bimodule and as a $G$-module.

The proof of this proposition given in [15] misses some details. For this reason we decided to repeat the proof of [15] below and spend more time on the details omitted in [15].

**Proof.** Let us suppose for a moment that

$$A_0 = \mathcal{O}(X) \in VB(2n)$$

and

$$\text{Ext}^{2n}_{A_0 \otimes A_0}(A_0, A_0 \otimes A_0) = A_0$$

as $G$-modules.

These assumptions have a pure algebraic geometric meaning and have nothing to do with the quantization of $X$. For this reason we postpone the proofs of the assumptions until the very end of the section.

Since $A$ is a quantization of the commutative algebra $A_0$, for any $A$-module $M$ the Hochschild complex $C_\bullet(A, M)$ is endowed with the filtration induced by degrees in $\hbar$. For the corresponding spectral sequence

$$E^1_{p,q}(C_\bullet(A, M)) = HH_{p+q}(A_0, M).$$

Hence due to our assumption (9) $A$ has a finite Hochschild dimension.

Let us now consider the Hochschild complex $C_\bullet(A, A \otimes A^{\text{op}})$ of $A$-bimodules and $G$-modules. The natural filtration induced by degrees in $\hbar$ gives us the spectral sequence converging to $\text{Ext}^i_{A \otimes A^{\text{op}}}(A, A \otimes A^{\text{op}})$ and having

$$E^1_{p,q}(C_\bullet(A, A \otimes A^{\text{op}})) = HH_{p+q}(A_0, A_0 \otimes A_0).$$

Due to our assumptions (9) and (10) the right hand side of (11) is isomorphic to $A_0$ as an $A_0$-bimodule and as a $G$-module if $p + q = 2n$, and vanishes if $p + q \neq 2n$. The latter implies that all the differentials $d_1, d_2, d_3, \ldots$ starting from $d_1$ in $E^1_\bullet, \bullet$ vanish. Thus,

$$E^1_{p,q}(C_\bullet(A, A \otimes A^{\text{op}})) = E^\infty_{p,q}(C_\bullet(A, A \otimes A^{\text{op}})),$$

$$\text{Ext}^i_{A \otimes A^{\text{op}}}(A, A \otimes A^{\text{op}}) = 0$$

for any $i \neq 2n$ and the associated graded $A_0$-bimodule and $G$-module of the filtered $A$-bimodule and $G$-module

$$U_A = \text{Ext}^{2n}_{A \otimes A^{\text{op}}}(A, A \otimes A^{\text{op}})$$

is naturally isomorphic to $A_0((\hbar))$. This implies that there exists a $G$-invariant automorphism

$$\nu = Id + \hbar \nu_1 + \ldots$$
of $A_+$ such that $U_A$ is isomorphic to $A\nu$ both as an $A$-bimodule and as a $G$-module.

In particular, we see that $A$ satisfies the conditions of proposition 3. Hence,

$$HH^0(A, \nu^{-1}A) = HH_{2n}(A).$$

Using proposition 4 for the trivial group $G = e$ we get that $HH_{2n}(A) = H^0(X)((h))$. Hence, $HH^0(A, \nu^{-1}A) = \mathbb{C}((h))$ and therefore there is a non-zero element $b \in A, b \neq 0$ such that for any $a \in A_+$

$$\nu^{-1}(a) \ast b = b \ast a$$

or equivalently,

$$a \ast b = b \ast \nu(a), \quad \forall \ a \in A_+. \quad (13)$$

We can safely assume that $b \in A_+$.

Denoting by $a_0$ and $b_0$ the zero-th terms of the elements $a$ and $b$, respectively, and comparing the terms of the first degree in $h$ in (13), we get

$$\nu_1(a_0)b_0 = \{a_0, b_0\}.$$ \quad (14)

If $b_0$ vanishes at some point $p \in X$ then one can pick a function $a_0 \in A_0 = \mathcal{O}(X)$ such that the left hand side of (13) has a bigger order of vanishing than the right hand side. From this, we conclude that $b_0$ is a nowhere vanishing function and therefore $b$ is an invertible element of $A_+$. Hence equation (13) implies that the automorphism $\nu$ is inner and $A$ indeed belongs to the class $VB(2n)$.

To prove that (8) is an isomorphism of $G$-modules it suffices to show that the element $b$ in (13) is $G$-invariant. Since the automorphism $\nu$ is $G$-invariant we have that for any $a \in A_+$ and $g \in G$

$$b^{-1} \ast a \ast b = (b^g)^{-1} \ast a \ast b^g$$

or equivalently,

$$(b^g \ast b^{-1}) \ast a = a \ast (b^g \ast b^{-1}).$$

But the algebra $A_+$ has a one-dimensional center $\mathbb{C}[[h]] \subset A_+$. Hence, $b$ generates a one-dimensional representation of the group $G$ over the ring $\mathbb{C}[[h]]$:

$$b^g = \alpha_g b, \quad \alpha_g \in \mathbb{C}[[h]].$$

Since the associated graded module of the filtered $G$-module (12) is naturally isomorphic to $A_0((h))$ as a $G$-module, for any $g \in G$

$$\alpha_g \bigg|_{h=0} = 1.$$\quad (14)

But we also have that $(\alpha_g)^{N_g} = 1$, where $N_g$ is the order of the element $g$. Hence, $\alpha_g = 1$ for any $g \in G$. \quad \Box

We are now left with our assumptions (9) and (10) about the algebra $A_0 = \mathcal{O}(X)$ of regular functions on the affine variety $X$. It is not hard to see that these assumptions are consequences of the following proposition and the fact that $X$ is endowed with a $G$-invariant algebraic symplectic form.
Proposition 7 If $A_0$ is the algebra of regular functions on a smooth affine algebraic variety $X$ (over a field of characteristic zero) of dimension $d$, then $A_0$ has a finite Hochschild dimension,

$$Ext^i_{A_0 \otimes A_0}(A_0, A_0 \otimes A_0) = 0,$$

if $i \neq d$, and the $A_0$-bimodule

$$Ext^d_{A_0 \otimes A_0}(A_0, A_0 \otimes A_0)$$

is naturally isomorphic to

$$Hom_{A_0}(\Omega^d(A_0), A_0),$$

where $\Omega^d(A_0)$ is the module of top degree exterior forms on $X$.

Proof is parallel to the proof of the Hochschild-Kostant-Rosenberg theorem [24]. It is based on the standard lemma of commutative algebra which says that if a morphism

$$\mu : M_1 \rightarrowtail M_2$$

of modules over a commutative algebra $B$ becomes an isomorphism after the localization with respect to any maximal ideal $m$ of $B$ then $\mu$ is an isomorphism of modules.

First, since $X$ is smooth, the kernel of the multiplication map

$$\mu_0 : A_0 \otimes A_0 \rightarrow A_0$$

is a locally complete intersection. In other words, for any maximal ideal $m \subset A_0$, the kernel of the localized map

$$(\mu_0)_m : (A_0 \otimes A_0)_{\mu^{-1}(m)} \rightarrow (A_0)_m$$

is generated by a regular sequence of length $d$.

Thus, applying to $(A_0)_m$ the Koszul resolution, we get that for any maximal ideal $m \subset A_0$ and for any $i \neq d$

$$(Ext^i_{A_0 \otimes A_0}(A_0, A_0 \otimes A_0))_m = 0.$$ Hence,

$$Ext^i_{A_0 \otimes A_0}(A_0, A_0 \otimes A_0) = 0$$

for any $i \neq d$.

If $i = d$ then we have a pair of morphisms of modules

$$Ext^d_{A_0 \otimes A_0}(A_0, A_0 \otimes A_0) \twoheadrightarrow Ext^d_{A_0 \otimes A_0}(A_0, A_0),$$

$$Hom_{A_0}(\Omega^d(A_0), A_0) \twoheadrightarrow Ext^d_{A_0 \otimes A_0}(A_0, A_0).$$

The first morphism is induced by multiplication $\mu_0$ and the second one is the so-called anti-symmetrization morphism given by the formula

$$\varepsilon(\gamma)(a_1, \ldots, a_d) = \gamma(da_1 \wedge \ldots \wedge da_d), \quad \gamma \in Hom_{A_0}(\Omega^d(A_0), A_0).$$

Using the Koszul resolution once again, one can show that for any maximal ideal $m \subset A_0$ the localizations of the morphisms (16) are isomorphisms. Thus the above lemma from commutative algebra completes the proof of the proposition. \(\square\)
3 The decomposition of the (co)homology of the twisted group algebras.

Proof of proposition 3. The idea of the proof is to guess a free left resolution of $A[G]$ as an $A[G] \otimes A[G]^{op}$-module which allows us to get the desired decompositions (5) and (6) for free. Namely, we consider the following complex of free $A[G] \otimes A[G]^{op}$-modules

$$\begin{array}{cccccc}
\downarrow^{\beta'} & & & & & \\
\beta & C_{m,q} & \beta' & C_{m-1,q} & \beta & \ldots \\
\downarrow^{\beta'} & & \downarrow^{\beta'} & & & \\
\beta & C_{m,q-1} & \beta' & C_{m-1,q-1} & \beta & \ldots \\
\downarrow^{\beta'} & & \downarrow^{\beta'} & & & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}$$

... $\xrightarrow{\beta} A[G] \otimes A \otimes A[G] \otimes k[G] \xrightarrow{\beta} A[G] \otimes A[G] \otimes k[G] \xrightarrow{\beta} 0$

... $\xrightarrow{\beta} A[G] \otimes A \otimes A[G] \xrightarrow{\beta} A[G] \otimes A[G] \xrightarrow{\beta} A[G]$, where

$$C_{m,q} = A[G] \otimes A^{\otimes m} \otimes A[G] \otimes k[G]^{\otimes q}. \quad (18)$$

The horizontal differential $\beta$ is defined by

$$\beta (ag \otimes a_1 \otimes \ldots \otimes a_m \otimes bh \otimes g_1 \otimes \ldots \otimes g_q) =$$

$$(-)^q (aa^q g \otimes a_2 \ldots \otimes a_m \otimes bh \otimes g_1 \otimes \ldots \otimes g_q - ag \otimes a_1 a_2 \otimes a_3 \ldots \otimes a_m \otimes bh \otimes g_1 \otimes \ldots \otimes g_q) + \ldots + (-)^m ag \otimes a_1 \otimes \ldots \otimes a_m bh \otimes g_1 \otimes \ldots \otimes g_q \quad (19)$$

where $a, a_1, \ldots, a_m, b \in A$ and $g, h, g_1, \ldots, g_q \in G$. If $m = 0$ and $q \neq 0$ then $\beta$ vanishes and if both $m = 0$ and $q = 0$ then $\beta$ is the ordinary multiplication in $A[G]$.

The vertical differential $\beta'$ is just the boundary operator of the group Eilenberg-Mac Lane complex:

$$\beta' (ag \otimes a_1 \otimes \ldots \otimes a_m \otimes bh \otimes g_1 \otimes \ldots \otimes g_q) =$$

$$agg_1 \otimes a_1^{g_1^{-1}} \otimes a_2^{g_1^{-1}} \ldots \otimes a_m^{g_1^{-1}} \otimes b^{g_1^{-1}} g_1^{-1} h \otimes g_2 \otimes \ldots \otimes g_q$$

$$-ag \otimes a_1 \otimes \ldots \otimes a_m \otimes bh \otimes g_1 g_2 \otimes g_3 \otimes \ldots \otimes g_q + \ldots + (-)^q - 1 ag \otimes a_1 \otimes \ldots \otimes a_m \otimes bh \otimes g_1 \otimes \ldots \otimes g_{q-2} \otimes g_{q-1} g_q +$$

$$+(-)^q ag \otimes a_1 \otimes \ldots \otimes a_m \otimes bh \otimes g_1 \otimes \ldots \otimes g_{q-2} \otimes g_{q-1}$$

(20)
where \(a, a_1, \ldots, a_m, b \in A\) and \(g, h, g_1, \ldots, g_q \in G\). It is not hard to see that both differentials respect the natural \(A[G] \otimes A[G]^\text{op}\)-module structure:

\[
a_0 g_0 (ag \otimes a_1 \otimes \ldots \otimes a_m \otimes bh \otimes g_1 \otimes \ldots \otimes g_q) b_0 h_0 = a_0 a_0^g g_0 a_1 \otimes \ldots \otimes a_m \otimes bh_0 h_0 \otimes g_1 \otimes \ldots \otimes g_q,
\]

and anti-commute with each other:

\[
\beta \beta' + \beta' \beta = 0.
\]

It turns out that the total complex of (18) is acyclic in all terms. We prove this by defining explicitly the homotopy operator \(\chi\) which acts on chains as follows. If \(m \neq 0\) then

\[
\chi (ag \otimes a_1 \otimes \ldots \otimes a_m \otimes bh \otimes g_1 \otimes \ldots \otimes g_q) = (-)^q g \otimes a_{g^{-1}} \otimes a_1 \otimes \ldots \otimes a_m \otimes bh \otimes g_1 \otimes \ldots \otimes g_q,
\]

and if \(m = 0\) then

\[
\chi (ag \otimes bh \otimes g_1 \otimes \ldots \otimes g_q) = (-)^q g \otimes a_{g^{-1}} \otimes bh \otimes g_1 \otimes \ldots \otimes g_q + 1 \otimes ab \otimes g \otimes g_1 \ldots g_q.
\]

Finally, the action of \(\chi\) on \(A[G]\) is defined by

\[
\chi (ag) = 1 \otimes ag.
\]

The desired property of \(\chi\)

\[
\chi (\beta + \beta') + (\beta + \beta') \chi = Id
\]

can be proved by the direct computation.

Since our base field \(\mathbb{C}\) obviously contains the inverse \(|G|^{-1}\) of the order of the group \(G\), the rest is evident from the definition of Hochschild (co)homology, the isomorphism between the module of invariants and the module of coinvariants, and the acyclicity of the (co)homological group Eilenberg-Mac Lane complex in positive dimensions.

\[\square\]

**Remark.** Different statements closely related to the assertion of proposition 3 were obtained by various authors. Thus, the decomposition (5) was given in preprint [4] of J.-L. Brylinski. In the same year J.-L. Brylinski [5] considered cyclic homology of a natural generalization of the twisted group algebra to the case when the initial algebra is the algebra of functions on a smooth manifold and the manifold is acted by a Lie group. J.-L. Brylinski suggested in [5] that the cyclic homology of this algebra is a good replacement of the equivariant \(K\)-theory in the case when the Lie group is noncompact. Simultaneously with J.-L. Brylinski, B. Feigin and B. Tsygan proposed in [18] the corresponding decomposition for the cyclic homology of an arbitrary twisted group algebra. Another derivation of this result of B. Feigin and B. Tsygan was obtained in paper [21] by E. Getzler and J.D.S. Jones. The proof given in [21] involves a notion of the paracyclic module which is a natural generalization of the cyclic module. Using this notion in [22], J. Block, E. Getzler and J. D. S. Jones considered the case of the topological algebra acted by a Lie group. Finally, in [25] M. Lorenz proved the decomposition (5) of the Hochschild homology for any \(G\)-graded associative algebra which is a natural generalization of the twisted group algebra.
4 Hochschild homology $HH_\bullet(A, Ag)$.

4.1 The case of the symplectic affine space.

Before speaking about the general symplectic affine variety, we will mention the necessary results for the case of the symplectic affine space acted on by the cyclic group $\{1, g, g^2, \ldots, g^{N-1}\}$ of order $N$. These results we need are essentially contained in paper [1]. The only difference is that unlike the authors of [1], we speak about the formal Weyl algebra, rather than the ordinary one.

Let $V$ be a $2n$-dimensional vector space (over $\mathbb{C}$) endowed with a symplectic form $B$. Let $\{y^1, \ldots, y^{2n}\}$ be a basis in $V$. We denote by $B_{ij}$, $(i, j = 1, \ldots, 2n)$ the matrix $B_{ij} = B(y^i, y^j)$ of the form $B$ in this basis.

**Definition 1** The Weyl algebra $W$ associated with the symplectic vector space $V$ is the vector space $\mathbb{C}[\mathbb{C}[[V]][(\hbar)]]$ of the formally completed symmetric algebra of $V$ equipped with the following (associative) multiplication

$$(a \circ b)(y, \hbar) = \exp \left( \frac{\hbar}{2} B_{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right) a(y, \hbar) b(z, \hbar) \bigg|_{y=z}. \quad (26)$$

One can easily see that the multiplication defined by (26) does not depend on the choice of a basis in $V$. We view $W$ as an algebra over the field $\mathbb{C}((\hbar))$.

The Weyl algebra $W$ is naturally filtered with respect to the degree of monomials $2[\hbar] + [y]$ where $[\hbar]$ is the degree in $\hbar$ and $[y]$ is the degree in $y$:

$$\ldots \subset W^1 \subset W^0 \subset W^{-1} \ldots \subset W,$$

$$W^m = \{ a = \sum_{2k+p \geq m} \hbar^k a_{k; i_1 \ldots i_p} y^{i_1} \ldots y^{i_p} \} \quad (27)$$

This filtration defines the $2[\hbar] + [y]$-adic topology in $W$.

Let us assume that a symplectic vector space $(V, B)$ is acted on by a cyclic group $\mathbb{Z}_N = \{1, g, g^2, \ldots, g^{N-1}\}$ of order $N$, and $g$ preserves the symplectic form $B$. It is not hard to show that in this situation $V$ becomes a direct sum of two symplectic vector spaces:

$$V = \text{Ker} (\pi(g) - Id) \oplus \text{Im} (\pi(g) - Id), \quad (28)$$

where $\pi(g)$ is the action of $g$ on $V$. We denote the subspace $\text{Ker} (\pi(g) - Id)$ by $V_g$, the dimension of $V_g$ by $2m_g$, and the dual form to the corresponding symplectic form on $V_g$ by $B^o$

$$B^o = \frac{1}{2} B_{ab}^g y^a \wedge y^b,$$

where $\{y^1, \ldots, y^{2m_g}\}$ is a basis in $V_g$.

The group $\mathbb{Z}_N$ naturally acts on the Weyl algebra $W$. In particular, we can define a $W$-bimodule $Wg$ twisted from the right by the action of $g$. For this bimodule we have the following

$^2$More precisely, we should call it the *formal Weyl algebra*. 


Proposition 8. With the above notations,

\[ HH_q(W, Wg) = \begin{cases} 
\mathbb{C}((\hbar)), & \text{if } q = 2m_g, \\
0, & \text{otherwise}.
\end{cases} \] (29)

If \( \varepsilon_{a_1 \ldots a_{2m_g}} \) are components of the Liouville volume form

\[ \varepsilon = \wedge^{m_g} B^g \] (30)
in the basis \( \{y^1, \ldots, y^{2m_g}\} \) then the element

\[ \psi = \varepsilon_{a_1 \ldots a_{2m_g}} 1 \otimes y^{a_1} \otimes \ldots \otimes y^{a_{2m_g}} \] (31)
is a non-trivial cycle of the normalized Hochschild complex \( \overline{\mathcal{C}}(W, Wg) \).

Proof. The first claim was essentially proved in [1]. The only subtlety we have to mention is that unlike the authors of [1] we consider the formal rather than the ordinary Weyl algebra. However, it is not hard to verify that the arguments of [1] can be extended to the formal Weyl algebra as well\(^3\).

Let us mention that formula (31) does not depend on the choice of the local basis in \( V_g \) and therefore the chain \( \psi \in \overline{\mathcal{C}}_{2m_g}(W, Wg) \) is well-defined. A simple computation shows that (31) is indeed a cycle in the normalized Hochschild complex \( \overline{\mathcal{C}}_\bullet(W, Wg) \).

To prove that the cycle (31) is non-trivial we consider the spectral sequence associated with the filtration induced by degrees in \( \hbar \). It is obvious that

\[ E^1_{p,q}(W, Wg) = HH_{p+q}(\mathbb{C}[[V]], \mathbb{C}[[V]]g) \],

where \( \mathbb{C}[[V]] \) is viewed as a commutative algebra.

The splitting (28) gives us the natural projection \( pr \) from the ring \( \mathbb{C}[[V]] \) to the ring \( \mathbb{C}[[V_g]] \). Using this projection we define the following analogue of the antisymmetrization map

\[ \mu(a_0, a_1, \ldots, a_q) = pr(a_0)d(pr(a_1)) \wedge \ldots \wedge d(pr(a_q)) \], (32)

from the normalized Hochschild complex \( \overline{\mathcal{C}}_\bullet(\mathbb{C}[[V]], \mathbb{C}[[V]]g) \otimes \mathbb{C}((\hbar)) \) to the graded vector space

\[ \mathbb{C}[[V_g]]((\hbar)) \otimes \wedge^\bullet(V_g) \).

With the help of the standard Koszul resolution of \( \mathbb{C}[[V]] \) one can show that (32) is a quasi-isomorphism from \( \overline{\mathcal{C}}_\bullet(\mathbb{C}[[V]], \mathbb{C}[[V]]g) \otimes \mathbb{C}((\hbar)) \) to the complex

\[ (\mathbb{C}[[V_g]]((\hbar)) \otimes \wedge^\bullet(V_g), 0) \] (33)

with the vanishing differential.

Thus,

\[ E^1_{p,q} = \mathbb{C}[[V_g]] \otimes \wedge^{p+q}(V_g) \].

It is not hard to show that the differential

\[ d_1 : E^1_{p,q} \mapsto E^1_{p+1,q-2} \]

\(^3\)See, for example, appendix B of paper [11] in which this issue is examined for the cohomological complex in the case of the trivial group action.
is the De Rham differential in (33) twisted by the symplectic Hodge operator [6]. Hence,
\begin{equation}
E^2_{p,q} = \begin{cases} 
\mathbb{C}, & \text{if } p + q = 2m_g, \\
0, & \text{otherwise.}
\end{cases}
\end{equation}

It is easy to see that the cycle \( \psi \) (31) projects onto a generator of \( E^2 \). Therefore, \( \psi \) is nontrivial and the proposition follows. □

4.2 Fedosov’s construction.

In this subsection we briefly recall Fedosov’s construction [16] of the star-product on the symplectic manifold \( X \) focusing our attention on the equivariant aspects of the construction.

The Weyl algebra bundle \( \mathcal{W} \) is a bundle over \( X \) whose sections are the following formal power series
\begin{equation}
a = a(x, y, \hbar) = \sum_{k,l} \hbar^k a_{k; i_1i_2...i_l}(x)y^{i_1}...y^{i_l},
\end{equation}
where \( y = (y^1...y^{2n}) \) are fiber coordinates of the tangent bundle \( TX \), \( a_{k; i_1i_2...i_l}(x) \) are symmetric covariant tensors, and the summation over \( k \) is bounded below.

Multiplication of two sections of \( \mathcal{W} \) is given by the Weyl formula
\begin{equation}
a \circ b(x, y, \hbar) = \exp \left( \frac{\hbar}{2} \omega^{ij}(x) \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right) a(x, y, \hbar)b(x, z, \hbar) |_{y = z},
\end{equation}
where \( \omega^{ij}(x) \) are components of the corresponding Poisson tensor
\begin{equation}
\overline{\omega} = \omega^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \quad \omega^{ik}\omega_{kj} = \delta^i_j.
\end{equation}

For any point \( p \in X \) the fiber \( \mathcal{W}_p \) of the Weyl algebra bundle at \( p \) is isomorphic to the Weyl algebra \( W \) associated with the cotangent space \( T^*_p(X) \) at the point \( p \) with the symplectic form \( \omega_p \).

The filtration (27) of the Weyl algebra gives us a natural filtration of the bundle \( \mathcal{W} \)
\begin{equation}
\cdots \subset \mathcal{W}^1 \subset \mathcal{W}^0 \subset \mathcal{W}^{-1} \subset \cdots \subset \mathcal{W},
\end{equation}
\begin{equation}
\Gamma(\mathcal{W}^m) = \{ a = \sum_{2k+p \geq m} \hbar^k a_{k;i_1i_2...i_p}(x)y^{i_1}...y^{i_p} \}.
\end{equation}
This filtration defines a \( 2[\hbar] + [y] \)-adic topology in the algebra \( \Gamma(\mathcal{W}) \) of sections of \( \mathcal{W} \).

The vector space \( \Omega^\bullet(\mathcal{W}) \) of smooth exterior forms with values in \( \mathcal{W} \) is naturally a graded associative algebra with the product induced by (36) and the following graded commutator
\begin{equation}
[a, b] = a \circ b - (-)^{q_aq_b} b \circ a,
\end{equation}
where \( q_a \) and \( q_b \) are exterior degrees of \( a \) and \( b \), respectively. The filtration of \( \mathcal{W} \) (37) gives us a filtration of the algebra \( \Omega^\bullet(\mathcal{W}) \)
\begin{equation}
\cdots \subset \Omega^\bullet(\mathcal{W}^1) \subset \Omega^\bullet(\mathcal{W}^0) \subset \Omega^\bullet(\mathcal{W}^{-1}) \subset \cdots \subset \Omega^\bullet(\mathcal{W}).
\end{equation}
(In the definition of this filtration, the degree of $dx^i$ is put to be zero). Notice that the algebra $\Omega(W)$ is endowed with the natural action of the group $G$.

It is not hard to show that on $TX$ there exists a torsion free connection $\partial^s$ compatible with the symplectic structure $\omega$ and invariant under the action of $G$. Using this connection we define the following linear operator

$$\nabla: \Omega^*(W) \mapsto \Omega^{*+1}(W),$$

$$\nabla = dx^i \frac{\partial}{\partial x^i} - dx^i \Gamma^k_{ij}(x)y^j \frac{\partial}{\partial y^k},$$

where $\Gamma^k_{ij}(x)$ are the Christoffel symbols of $\partial^s$.

Thanks to the compatibility of $\partial^s$ with the symplectic structure $\omega$ and with the action of $G$ the operator (38) is a $G$-equivariant derivation of the graded algebra $\Omega^*(W)$. A simple computation shows that

$$\nabla^2 a = \frac{1}{2} [R, a], \quad \forall a \in \Omega(W),$$

where

$$R = \frac{1}{2\hbar} \omega_{km}(R_{ij})^m_l(x)y^k y^l dx^i dx^j,$$

and $(R_{ij})^m_l(x)$ is the Riemann curvature tensor of $\partial^s$.

**Definition 2** The Fedosov connection is a nilpotent derivation of the graded algebra $\Omega^*(W)$ of the following form

$$D = \nabla + \frac{1}{\hbar} [A, \cdot], \quad A = -dx^i \omega_{ij}(x)y^j + r,$$

where $r$ is an element in $\Omega^1(W)$

The flatness of $D$ is equivalent to the fact that the Fedosov-Weyl curvature

$$C^W = \hbar R + 2\nabla A + \frac{1}{\hbar} [A, A]$$

of $D$ belongs to the subspace $\Omega^2(X)((\hbar)) \subset \Omega^2(W)$. A simple analysis of degrees in $\hbar$ and $y$ shows that $C^W$ is of the form

$$C^W = -\omega + \Omega_h, \quad \Omega_h \in h\Omega^2(X)[[\hbar]]$$

whereas the Bianchi identity $D(C^W) = 0$ implies that $\Omega_h$ is a series of two-forms closed with respect to the De Rham differential.

Let us remark that the Fedosov connection (39) can be rewritten as

$$D = \nabla - \delta + \frac{1}{\hbar} [r, \cdot],$$

where

$$\delta = \frac{1}{\hbar} [dx^i \omega_{ij}(x)y^j, \cdot] = dx^i \frac{\partial}{\partial y^i}$$

is the Koszul derivation of the algebra $\Omega^*(W)$. 14
For our purposes we will need the homotopy operator for the Koszul differential $\delta$

$$\delta^{-1}a = y^k i \left( \frac{\partial}{\partial x^k} \right) \int_0^1 a(x, h, ty, tdx) \frac{dt}{t},$$

(44)

where $i(\partial/\partial x^k)$ denotes the contraction of an exterior form with the vector field $\partial/\partial x^k$, and $\delta^{-1}$ is extended to $\Gamma(W)$ by zero.

Simple calculations show that $\delta^{-1}$ is indeed the homotopy operator for $\delta$, namely

$$a = \sigma(a) + \delta\delta^{-1}a + \delta^{-1}\delta a, \quad \forall a \in \Omega(W)$$

(45)

where $\sigma$ is the natural projection

$$\sigma(a) = a \bigg|_{y=0, \ dx=0}, \quad a \in \Omega^*(W)$$

(46)

from $\Omega^*(W)$ onto the vector space $A_0((\hbar))$, where $A_0 = \mathcal{O}(X)$.

The proof of the following theorem is contained in section 5.3 of [17]. More precisely, see theorem 5.3.3 and remarks at the end of section 5.3.

**Theorem 2 (Fedosov, [17])** If $\partial^s$ is a symplectic connection on $X$ and $\Omega_h$ is a series of closed two-forms in $\hbar\Omega^2(M)[[\hbar]]$ then

1. Iterating the equation

$$r = \delta^{-1}(R - \Omega_h) + \delta^{-1}(\nabla r + \frac{1}{\hbar}r \circ r)$$

(47)

in degrees in $y$’s we get an element $r \in \Omega^1(W^g)$ such that the derivation

$$D = \nabla - \delta + \frac{1}{\hbar}[r, \cdot]$$

(48)

is nilpotent and has the Fedosov-Weyl curvature (40) $C^W = -\omega + \Omega_h$.

2. Given a Fedosov connection (39) one can construct a vector space isomorphism $\lambda$

$$\lambda : A_0((\hbar)) \rightarrow \Gamma_D(W)$$

(49)

from $A_0((\hbar))$ to the algebra $\Gamma_D(W)$ of flat sections of $W$ with respect to $D$. The inverse isomorphism to $\lambda$ is the projection $\sigma$ (46) and the product in $A_0((\hbar))$

$$a \ast b = \sigma(\lambda(a) \circ \lambda(b)), \quad a, b \in A_0((\hbar))$$

(50)

induced via the isomorphism $\lambda$ is the desired star-product in $A$.

**Remark 1.** The cohomology class of the Fedosov-Weyl curvature (40) is a well-defined characteristic class of a star-product in $A_0((\hbar))$. This characteristic class is referred to as the Deligne-Fedosov class.

**Remark 2.** Due to the naturality of the Fedosov construction the Fedosov differential (48) corresponding to a $G$-invariant series $\Omega_h$ of closed 2-forms is $G$-equivariant and the
corresponding star-product (50) is $G$-invariant.

**Remark 3.** For any function $a \in A_0 = \mathcal{O}(X)$

$$\left( \frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} \lambda(a) \right) \bigg|_{y=0} = \partial_{x^{i_1}} \cdots \partial_{x^{i_p}} a(x) + \text{lower order derivatives of } a. \quad (51)$$

Due to the algebraic geometric version [11] of the theorem of P. Xu [31] the star-product $\star$ in $A$ is equivalent to some Fedosov star-product (50) corresponding to the Deligne-Fedosov class $[-\omega + \Omega_\hbar]$. We claim that

**Proposition 9** The star-product $\star$ in $A$ is equivalent to the Fedosov star-product (50) corresponding to the formal series $\Omega_\hbar$ of $G$-invariant closed 2-forms and the equivalence between $\star$ and (50) can be established by a $G$-invariant operator.

**Proof.** Due to theorem 2 in [11] the $\star$ in $A$ is equivalent to the Fedosov star-product corresponding to the formal series $\Omega_\hbar$ such that for any $g \in G$

$$\pi(g)^\star(\Omega_\hbar) - \Omega_\hbar$$

is a series of exact 2-forms. Therefore, the form

$$\frac{1}{|G|} \sum_{g \in G} \pi(g)^\star(\Omega_\hbar)$$

is $G$-invariant, represents the same class as $\Omega_\hbar$, and gives the Fedosov star-product (50) which is equivalent to $\star$ in $A$.

The second claim is proved in [11] (see corollary 1). $\square$

### 4.3 The proof of proposition 4.

The idea of the proof is based on the construction of a complex of $G$-modules $K^\bullet$ and a pair of $G$-equivariant quasi-isomorphisms

$$\bigoplus_{g \in G} C^\bullet(A, Ag) \xrightarrow{\gamma} K^\bullet \xleftarrow{\gamma} \bigoplus_{g, Q} (\Omega^{2m_Q^g} \ast (X_Q^g)((\hbar)), d),$$

where $C^\bullet(A, Ag)$ is the Hochschild complex of $A$ with values in the twisted bimodule $Ag$, $X_Q^g$ as above denotes the $Q$-th connected (and irreducible) component of the subvariety $X_g$ of the points fixed under the action of $g \in G$, and $2m_Q^g = \dim X_Q^g$.

Due to proposition 9 and remark 2 after theorem 2 we may safely assume that the product in $A$ is the Fedosov star-product (50) corresponding to a $G$-invariant series $\Omega_\hbar$ of closed 2-forms and the Fedosov differential (48) is $G$-equivariant.

Let us denote by $E^{Q,g}$ the restriction of the tangent bundle $TX$ to $X_Q^g$

$$TX\bigg|_{X_Q^g} = E^{Q,g}.$$ 

Similarly we restrict the Weyl algebra bundle $\mathcal{W}$ over $X$ to $X_Q^g$ and denote this new bundle of algebras by $\mathcal{W}^{Q,g}$. The sections of $\mathcal{W}^{Q,g}$ are the following formal power series in fiber coordinates $y^i$ of the bundle $E^{Q,g}$

$$a(x, y, \hbar) = \sum_{k, p} h^k a_{k:i_1\ldots i_p}(x) y^{i_1} \cdots y^{i_p},$$

(53)
where \( a_{k;i_1...i_p}(x) \) are components of sections of \( S^p(E^{Q,g})^* \) and the summation in \( h \) is bounded below.

Since any element \( h \in G \) maps the subvariety \( X^Q_g \) onto the subvariety \( X^Q_{gh^{-1}} \), the sets of bundles \( \{E^{Q,g} \rightarrow X^Q_g\} \) and \( \{W^{Q,g} \rightarrow X^Q_g\} \) are equipped with an action of \( G \). In particular, the bundle \( W^{Q,g} \) is acted on by the cyclic group \( Z_{N_g} \) generated by \( g \in G \).

To the \( Z_{N_g} \)-bundle \( \mathcal{W}^{Q,g} \) of algebras we attach the bundle of fiberwise Hochschild chains \( \mathcal{C}^{Q,g} \) over \( X^Q_g \)

\[
\mathcal{C}^{Q,g} = \bigoplus_{m=0}^{\infty} \mathcal{C}^{Q,g}_m, \tag{54}
\]

where

\[
\mathcal{C}^{Q,g}_m = (W^{Q,g}) \otimes (m+1),
\]

and \( \otimes \) stands for the tensor product over \( \mathcal{O}(X^Q_g)((h)) \) completed in the adic topology (37).

The sections of \( \mathcal{C}^{Q,g}_m \) are the following formal power series in \( m+1 \) collections of fiber coordinates \( y^i_0,...,y^i_m \) of \( E^{Q,g} \)

\[
a(x,h,y_0,...,y_m) = \sum_k \sum_{\alpha_0...\alpha_m} h^k a_{k;\alpha_0...\alpha_m}(x)y^\alpha_0...y^\alpha_m, \tag{55}
\]

where the summation in \( k \) is bounded below \( \alpha \)'s are multi-indices \( \alpha = j_1...j_l \),

\[
y^\alpha = y^{j_1}y^{j_2}...y^{j_l},
\]

\( a_{\alpha_0...\alpha_m}(x) \) are sections of the bundle \( S^{[\alpha_0]}E^{Q,g} \otimes \cdots \otimes S^{[\alpha_m]}E^{Q,g} \), and \( |\alpha| \) is the length of the multi-index \( \alpha \).

The differential in \( \Gamma(C^{Q,g}) \) is given by the formula

\[
(ba)(x,h,y_0,...,y_{m-1}) = S_0a(x,h,z,\pi(g)(y_0),...,y_{m-1})|_{z=y_0} - S_1a(x,h,y_0,z,y_1,...,y_{m-1})|_{z=y_1} + \cdots \tag{56}
\]

\[
+(-)^{m-1}S_{m-1}a(x,h,y_0,y_1,...,z,y_{m-1})|_{z=y_{m-1}} + (-)^mS_0(x,h,y_0,y_1,...,y_{m-1},z)|_{z=y_0}
\]

where \( S_a \ (a = 0,...,m-1) \) denotes the following operator:

\[
S_a = \exp \left( \frac{h}{2\omega^{ij}(x)} \frac{\partial}{\partial z^i} \frac{\partial}{\partial y^a_j} \right),
\]

and \( \pi(g) \) stands for the action of \( g \) on the fibers of \( E^{Q,g} \).

Notice that, since \( W^{Q,g} \) is the restriction of the Weyl algebra bundle \( \mathcal{W} \) to \( X^Q_g \), we can restrict the flat Fedosov connection (48) on \( \mathcal{W} \) to the flat connection on \( W^{Q,g} \):

\[
D^{Q,g} = \nabla^{Q,g} + \frac{1}{h}[\mathcal{A}^{Q,g},\cdot], \tag{57}
\]

where \( \nabla^{Q,g} \) and \( \mathcal{A}^{Q,g} \) are the restrictions of the connection (38) and form \( \mathcal{A} \) in (39) to \( X^Q_g \), respectively.
Extending the connection $D^{Q,g}$ to the bundle $C^{Q,g}_m = \mathcal{W} \otimes (m+1)$ we get the nilpotent differential on the graded vector space $\Omega^\bullet(C^{Q,g})$ of exterior forms on $X^Q_g$ with values in the sheaf $C^{Q,g}_m = W^{\tilde{\otimes}} \otimes (m+1)$.

For the sake of simplicity, we use the same notation $D^{Q,g}$ for this differential.

Since the Fedosov differential $D$ (48) is a $G$-equivariant derivation of the product (26), the differential $D^{Q,g}$ anticommutes with the fiberwise Hochschild differential $b$

$$D^{Q,g}b + bD^{Q,g} = 0$$

if we use the convention

$$b dx^i = -dx^i b.$$ 

For our purposes we will also need the bundle $\mathcal{C}^{Q,g}$ of normalized fiberwise Hochschild chains over each subvariety $X^Q_g$. Namely,

$$\mathcal{C}^{Q,g} = \bigoplus_{m=0}^\infty \mathcal{C}^{Q,g}_m,$$ 

where

$$\mathcal{C}^{Q,g}_m = \mathcal{W}^{Q,g} \otimes (\mathcal{W}^{Q,g}) \otimes m,$$

$\otimes$ stands for the tensor product over $O(X^Q_g)((h))$ completed in the adic topology (37), and $W^{Q,g} = \mathcal{W}^{Q,g}/\mathbb{C}((h))1$.

It is not hard to see that the differentials (56) and (58) still make sense for the normalized chains (59). Furthermore, since the Fedosov differential (48) is $G$-equivariant, the direct sum of double complexes

$$\mathcal{D}_{p,q} = \bigoplus_{Q,g} (\Omega^p(\mathcal{C}^{Q,g}_q), D^{Q,g} + b)$$

form a double complex of $G$-modules.

A standard lemma of homological algebra (see, for example, proposition 1.6.5 in [26]) says that the natural projection

$$\Pi : \bigoplus_{Q,g} (\Omega^p(\mathcal{C}^{Q,g}_q), D^{Q,g} + b) \mapsto \mathcal{D}_{p,q}$$

is a quasi-isomorphism of complexes. We also mention that the projection (61) is $G$-equivariant.

We want to show that the total complex

$$\mathcal{K}^\bullet = \bigoplus_{q-p=\bullet} \mathcal{D}_{p,q}$$

of the double complex $\mathcal{D}_{p,q}$ is the desired middle term in (52).

For this purpose we mention that the complex $\bigoplus_{g \in G} C_\bullet(A, A_g)$ of $G$-modules can be naturally embedded into a bigger graded $G$-module $J$

$$J = \bigoplus_m J_m, \quad J_m = \bigoplus_{g \in G, Q} \Gamma(X^Q_g, Jets_{X^Q_g}(X^m))$$

(62)
where $\text{Jets}_{X^Q_g}(X^m)$ denotes the sheaf of jets of regular functions on the product $X^m$ of $m$ copies of $X$ near the subvariety $X^Q_g$ embedded into the main diagonal $\Delta X^m \cong X$ of $X^m$. It is not hard to see the Hochschild differential of $C_\bullet(A, Ag)$ still makes sense for the graded module (62) and is compatible the action of $G$.

Although the complex (62) is bigger than the complex $\bigoplus_{g \in G} C_\bullet(A, Ag)$, we have

**Proposition 10** The natural embedding

$$\bigoplus_{g \in G} C_\bullet(A, Ag) \hookrightarrow J_\bullet$$

is a quasi-isomorphism of complexes.

**Proof.** It is clear from the construction that (63) is a morphism of complexes of $G$-modules. Furthermore, the map (63) induces a quasi-isomorphism of $E_1$-terms of the spectral sequences associated with the filtration induced by degrees in $\hbar$. Hence, due to the standard snake-lemma argument of homological algebra, the map (63) induces an isomorphism of homology as graded $G$-modules. ☐

Notice that, thanks to remark 3 after theorem 2, the map $\lambda$ (49) gives us the natural embedding of complexes of $G$-modules

$$\lambda^J : J_\bullet \hookrightarrow \bigoplus_{Q, g} \Omega^0(C^Q_{Q, g}) .$$

We claim that

**Proposition 11** For any $q$ the complex $(\Omega^\bullet(C^Q_{Q, g}), D^Q_{Q, g})$ is acyclic in positive dimension and the morphism (64) restricted to $\Gamma(X^Q_g, Jets_{X^Q_g}(X^q))$ gives us the isomorphism

$$\lambda^{Q, g} : \Gamma(X^Q_g, Jets_{X^Q_g}(X^q)) \xrightarrow{\sim} \ker D^Q_{Q, g} \cap \Gamma(C^Q_{Q, g}) .$$

**Proof.** To prove the first claim we introduce, on the complex $(\Omega^\bullet(C^Q_{Q, g}), D^Q_{Q, g})$, the filtration induced by degrees in $y^a_0$ of the sections (55), where $y^a_0$ are fiber coordinates of the tangent bundle $TX^Q_g \subset E^Q_{Q, g}$. The zero-th differential $d_0$ of the corresponding spectral sequence has the form of the Koszul differential

$$\delta^Q_{Q, g} = dx^a \frac{\partial}{\partial y^a_0} ,$$

where $x^a$ are local coordinates on $X^Q_g$. Hence, the $E^Q$-term of the spectral sequence is acyclic in positive dimension and the first claim follows.

To prove the second claim we mention that the morphism (65) is obviously injective. Thus we are left with a proof of surjectivity. In fact it suffices to prove the surjectivity locally on $X^Q_g$.

Indeed, if $a$ is a $D^Q_{Q, g}$-flat section of $C^Q_{Q, g}$ and for each local chart $V \subset X^Q_g$ we have $j_V \in \Gamma(V, Jets_{X^Q_g}(X^q))$ such that $a|_V = \lambda^{Q, g}(j_V)$. Then, by the injectivity of $\lambda^{Q, g}$, $j_V = j'_V$ on any intersection $V \cap V'$. Hence we have $j \in Jets_{X^Q_g}(X^q)$ such that $\lambda^{Q, g}(j) = a$. 

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An obvious analogue of theorem 5.5.1 in [17] says that the differential $D^{Q,g}$ on $W^{Q,g}$ can be locally conjugated to

$$D_0^{Q,g} = dx^a \frac{\partial}{\partial x^a} - dx^a \frac{\partial}{\partial y^a},$$

where $x^a$ are local coordinates on $X^Q_g$ and $y^a$ are fiber coordinates of the tangent bundle $TX^Q_g \subset E^{Q,g}$. Thus the question of surjectivity of $\lambda^{Q,g}$ reduces to a simple exercise of the theory of partial differential equations. □

Next, we claim that

Proposition 12 For any connected component $X^Q_g$ one can construct an element

$$\kappa^{Q,g} \in \Omega^0(\overline{C}^{Q,g}_{2m_g}),$$

such that

$$D^{Q,g}\kappa^{Q,g} = 0, \quad b\kappa^{Q,g} = 0,$$

the evaluation $(\kappa^{Q,g})_p$ at any point $p \in X^Q_g$ is a non-trivial $b$-cycle: in the fiber $(\overline{C}^{Q,g})_p$, and for any $h \in G$

$$\pi(h)^*\kappa^{Q',gh^{-1}} = \kappa^{Q,g}.$$  \hspace{1cm} (68)

Let us first mention that if we have the elements $\kappa^{Q,g}$ satisfying properties (67) and (68) the proof of proposition 4 is completed.

Indeed, using $\kappa^{Q,g}$ we can construct a collection of morphisms

$$\psi^{Q,g} : (\Omega^\bullet(X^Q_g)((h)),d) \mapsto (\Omega^\bullet(\overline{C}^{Q,g}_{2m_g}),D^{Q,g} + b),$$

$$\psi^{Q,g}(\eta) = \eta \kappa^{Q,g}, \quad \eta \in \Omega^\bullet(X^Q_g)((h))$$

of complexes such that for any $h \in G$ the diagram

$$\begin{array}{ccc}
\Omega^\bullet(X^Q_g)(\overline{h}) & \xrightarrow{\psi^{Q,g}} & \Omega^\bullet(\overline{C}^{Q,g}_{2m_g}) \\
\downarrow{\pi(h)^*} & & \downarrow{\pi(h)^*} \\
\Omega^\bullet(X^Q_g)(\overline{h}) & \xrightarrow{\psi^{Q,g}} & \Omega^\bullet(\overline{C}^{Q,g}_{2m_g})
\end{array}$$

(70)

commutes.

Due to (70) the maps (69) form a morphism of complexes of $G$-modules

$$\psi^{tot} : \bigoplus_{g,Q}\Omega^{2m_g-\bullet}(X^Q_g)((h)) \mapsto \bigoplus_{q-p=\bullet} \bigoplus_{Q,g}(\Omega^p(\overline{C}^{Q,g}_q), D^{Q,g} + b).$$

(71)

The fibers of the complex of bundles $\overline{C}^{Q,g}_\bullet$ is isomorphic to the normalized Hochschild complex $\overline{C}(W,W_g)$ of the Weyl algebra with values in the twisted module $W_g$. Thus, due to proposition 8 and the fact that the evaluations of the elements $\kappa^{Q,g}$ are non-trivial cycles in fibers of $\overline{C}^{Q,g}$, the map (71) induces an isomorphism between the $G$-modules

$$\bigoplus_{Q,g} \Omega^\bullet(X^Q_g)((h)) \xrightarrow{\sim} \bigoplus_{Q,g} H_b(\Omega^\bullet(\overline{C}^{Q,g})).$$

(72)
where $H_b$ denotes cohomology of the double complex $\Omega^p(\mathcal{C}_q^{Q,g})$ with respect to the differential $b$.

Furthermore, proposition 11 implies that the map (64) composed with the projection (61) induces an isomorphism of the $G$-modules

$$J_* \sim \bigoplus_{Q,g} H_{DQ,g}(\Omega(\mathcal{C}_*^{Q,g})),$$

where $H_{DQ,g}$ denotes cohomology of the double complex $\Omega^p(\mathcal{C}_q^{Q,g})$ with respect to the differential $D_{Q,g}$.

Proposition 11 also implies that $\Omega^p(X_g^{Q},\mathcal{C}_q^{Q,g})$ is acyclic along all the rows outside the intersection with the zeroth column and proposition 8 implies that $\Omega^p(X_g^{Q},\mathcal{C}_q^{Q,g})$ is acyclic along all the columns outside the intersection with the $2m_g^Q$-th row, where $2m_g^Q$ is the dimension of $X_g^{Q}$. Thus, since the subcomplexes $\Omega^p(X_g^{Q},\mathcal{C}_q^{Q,g})$ of the double complex $D_{p,q}^{Q,g}$ (60) are placed in the strips $0 \leq p \leq 2m_g^Q$, we can apply to $D_{p,q}^{Q,g}$ the standard argument of the spectral sequence and conclude that the composition of the morphism of (63), (64), and (61) is the first desired quasi-isomorphism in (52) and the map (71) is the second desired quasi-isomorphism in (52).

**Proof of proposition 12** We set

$$\kappa_0^{Q,g} = \varepsilon_{a_1...a_{2m_g^Q}} \Pi(1 \otimes y^{a_1} \otimes \ldots \otimes y^{a_{2m_g^Q}}) \in \Omega^0(\mathcal{C}_{2m_g^Q})^Q,$$

where $\varepsilon_{a_1...a_{2m_g^Q}}$ are the components of the Liouville volume form associated with the symplectic form on $X_g^{Q}$. $y^a$ are fiber coordinates in $TX_g^{Q}$ and $\Pi$ is the projection (61).

By proposition 8

$$b \kappa_0^{Q,g} = 0$$

and the evaluations of $\kappa_0^{Q,g}$ represent non-trivial cycles in fibers of $\mathcal{C}^{Q,g}$.

Unfortunately, $D_{Q,g}^{Q,g} \kappa_0^{Q,g}$ could be non-zero. However, due to (75) the element $\nu_1 = D_{Q,g}^{Q,g} \kappa_0^{Q,g}$ is a $b$-cycle $b \nu_1 = 0$. Let us show that $\nu_1 = D_{Q,g}^{Q,g} \kappa_0^{Q,g}$ is a $b$-boundary.

By definition (57)

$$\nu_1 = \nabla^{Q,g} \kappa_0^{Q,g} + \frac{1}{\hbar} [A^{Q,g}, \kappa_0^{Q,g}],$$

where the commutator is defined componentwise.

The first term in (76) vanishes since the symplectic form and hence the Liouville tensor on $X_g^{Q}$ are preserved by the connection $\nabla^{Q,g}$. Therefore,

$$\nu_1 = \frac{1}{\hbar} \varepsilon_{a_1...a_{2m_g^Q}} \Pi(1 \otimes y^{a_1} \otimes \ldots \otimes y^{a_{k-1}} \otimes [A^{Q,g}, y^{a_k}] \otimes y^{a_{k+1}} \ldots \otimes y^{a_{2m_g^Q}}).$$

But the right hand side of (77) is the $b$-boundary of the following element:

$$\nu_2 = \frac{1}{\hbar} \varepsilon_{a_1...a_{2m_g^Q}} \Pi(1 \otimes A^{Q,g} \otimes y^{a_1} \otimes \ldots \otimes y^{a_{2m_g^Q}})$$

4This fact has a simple explanation in terms of the action of Hochschild cochains on Hochschild chains of any associative algebra [9].
Thus, \[ D^{Q,g} \kappa_0^{Q,g} = b \nu_2. \]

Due to this observation and the fact that the double complex \( \Omega^p(\overline{C}_{Q,g}^Q) \) is acyclic along all the columns outside the intersection with the \( 2m_g^Q \)-th row we can extend the element (74) to the sum
\[
\kappa^{Q,g} = \kappa_0^{Q,g} + \sum_{i=1}^{2m_g^Q} \kappa_i^{Q,g}, \quad \kappa_i^{Q,g} \in \Omega^i(\overline{C}_{Q,g}^Q),
\]

such that
\[
(D^{Q,g} + b)\kappa^{Q,g} = 0.
\]

Due to acyclicity of the double complex \( \Omega^p(\overline{C}_{Q,g}^Q) \) along all the rows outside the intersection with the zeroth column we can construct a cycle \( \kappa^{Q,g} \) which is placed in the subgroup \( \Omega^0(\overline{C}_{Q,g}^Q) \) and is homologically equivalent to (78).

It is clear from the construction that the difference between \( \kappa^{Q,g} - \kappa_0^{Q,g} \) is a \( b \)-boundary. Hence the evaluation \( (\kappa^{Q,g})_{\mid p} \) at any point \( p \in X^Q_g \) is a non-trivial \( b \)-cycle in the fiber \( (\overline{C}_{Q,g})_p \).

It is also obvious that for any \( h \in G \)
\[
\pi(h)^* \kappa_0^{Q,'hgh^{-1}} = \kappa_0^{Q,g}.
\]

Therefore, since we use homotopy operators which are compatible with the action the general linear group in fiber coordinates \( y^a \), the constructed elements \( \kappa^{Q,g} \) satisfy equation (68) for any \( h \in G \).

We finish the proof of proposition 12 and the proof of theorem 1. \( \Box \)

5 Applications to deformation theory

Let \( A \) and \( G \) be as in Theorem 1.

Conjecture 1 The deformations of the algebras \( A[G] \) and \( A^G \) are unobstructed. Thus there exists universal formal deformations \( H_c, H'_c \) of these algebras parametrized by \( c \in H^2(A[G]) = H^2(A^G) \).

Note that according to Theorem 1,
\[
H^2(A^G) = H^2_{CR}(X/G) = (H^2(X) \bigoplus_{g \in G} \bigoplus_{Q: \text{codim} X^Q_g = 2} H^0(X^Q_g))^G.
\]

(where the coefficients on the RHS are \( \mathbb{C}((\hbar)) \)). Thus the conjecture implies that if \( G \) contains symplectic reflections (i.e., elements \( g \) for which \( X^Q_g \) contains components of codimension 2 in \( X \)), then there exist “interesting” deformations of \( A^G \), i.e., ones not coming from \( G \)-invariant deformations of \( A \).

Let us give a few examples motivating this conjecture.
Example 1 If $G$ is trivial, the conjecture is true, as follows e.g. from theorem 5.3.3 in [17].

Example 2 Let $X = T^*Y$, where $Y$ is a smooth affine variety, and suppose $G$ acts on $Y$. In this case the conjecture is true as follows from Theorem 2.13 of [13], and $H_c$ is called the Cherednik algebra of $X, G$. If $X$ is a vector space and $G$ acts linearly, then $H_c$ is the rational Cherednik algebra from [14].

Example 3 If $X = V$ is a symplectic vector space and $G$ acts linearly then the conjecture holds by Theorem 2.16 of [14], and $H_c$ is called the Cherednik algebra of $X, G$. If $X$ is a vector space and $G$ acts linearly, then $H_c$ is the rational Cherednik algebra from [14].

Example 4 Let $X = V/L$, where $V$ is a symplectic vector space and $L$ a lattice in $V$ (i.e., $L$ is the abelian group generated by a basis of $V$). Thus $X$ is an algebraic torus with a symplectic form. We assume that the symplectic form is integral and unimodular on $L$. Let $G \subset Sp(L)$ be a finite subgroup; then $G$ acts naturally on $X$. In this case we expect that $H_c$ is an “orbifold Hecke algebra” defined in [13], Example 3.11.

More precisely, let $V_\mathbb{R}$ be the tensor product $L \otimes_\mathbb{Z} \mathbb{R}$. This space has a real symplectic structure $\omega$ obtained by restricting the symplectic structure on $V$. Pick a $G$-invariant Kähler structure on $V_\mathbb{R}$ whose imaginary part is $\omega$. This makes $V_\mathbb{R}$ into a complex vector space, and thus $X_\mathbb{R} = V_\mathbb{R}/L$ is a complex torus (analytic abelian variety). The construction of paper [13] attaches to $X_\mathbb{R}, G$ the Hecke algebra $H_r(X_\mathbb{R}, G)$, parametrized exactly by the space $H^2(A[|G|])$. We expect this algebra to coincide with $H_c$.

Note that this class of examples includes Cherednik’s double affine Hecke algebras which are attached to affine Weyl groups [8].

Motivated by these examples, we propose to call $H_c$ the symplectic reflection algebra of $X, G$.

Theorem 3 Assume that $H^3(X, \mathbb{C})^G = 0$, and for any $g, Q$ such that $X^Q_g$ has codimension 2, one has $H^1(X^Q_g, \mathbb{C})^{Z(g)} = 0$, where $Z(g)$ is the centralizer of $g$ in $G$. Then conjecture 1 is true.

Proof. According to Theorem 1, the assumptions imply that $H^3(A[|G|]) = H^3(A^G) = 0$, so the statement follows from classical deformation theory. □

Note that in the case when $X = Y^n$, where $Y$ is a smooth affine symplectic variety, and $G = S_n$, this theorem was proved in [15]. In this case, interesting deformations of $A^G$ arise if and only if $Y$ is an algebraic surface.

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