The Chern Character in the Simplicial de Rham Complex

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Abstract

On the basis of Dupont’s work, we exhibit a cocycle in the simplicial de Rham complex which represents the Chern character.

1 Introduction

In [5], Dupont introduced the differential forms on a simplicial manifold and showed how to use them to construct a homomorphism from $I^*(G)$, the $G$-invariant polynomial ring over Lie algebra $G$, to $H^*(BG)$, the cohomology ring of the classifying space $BG$. This map is called the Bott-Shulman map. There he used the property of $H^*(BG)$ that it is isomorphic to the total cohomology of a complex $\Omega^*(\mathcal{N}G(\ast))$ which is associated to a simplicial manifold $\{\mathcal{N}G(\ast)\}$. In brief, $\{\mathcal{N}G(\ast)\}$ is a sequence of manifolds $\{\mathcal{N}G(p) = G^p\}_{p=0,1,\ldots}$ together with face operators $\varepsilon_i : \mathcal{N}G(p) \to \mathcal{N}G(p-1)$ for $i = 0, \ldots, p$ satisfying relations $\varepsilon_i \varepsilon_j = \varepsilon_{j-1} \varepsilon_i$ for $i < j$ (The standard definition also involves degeneracy operators but we do not need them here). The image of the Bott-Shulman map in $\Omega^*(\mathcal{N}G(\ast))$ is called the Bott-Shulman-Stasheff form. In [11], Shulman gave the shape of it.

Theorem 1.1 (Shulman). In $\Omega^*(\mathcal{N}G(\ast))$, the characteristic classes $\Phi$ which are given as the image of Chern-Weil homomorphism is of the following form:

$$\Phi = \Phi_{q-1} + \Phi_{q-2} + \cdots + \Phi_0 \quad \Phi_i \in \Omega^{q+i}(\mathcal{N}G(q-i)).$$

This theorem was also showed by Kamber-Tondeur [8], and Bott-Hochschild [1]. In [3], Brylinski described the value of the Bott-Shulman-Stasheff form.
In this paper we will exhibit the cocycles more precisely when they represent the Chern characters.

The outline is as follows. In section 2, we briefly recall the universal Chern-Weil theory due to Dupont. In section 3, we obtain the Bott-Shulman-Stasheff form in $\Omega^*(NG(\ast))$ which represents the Chern character $ch_p$. In section 4, we introduce some result about the Chern-Simons forms.

2 Review of the universal Chern-Weil Theory

In this section we recall the universal Chern-Weil theory following [6]. For any Lie group $G$, we have simplicial manifolds $NG$, $\overline{NG}$ and simplicial $G$-bundle $\gamma: \overline{NG} \to NG$ as follows:

\[
NG(q) = \underbrace{G \times \cdots \times G}_{q \text{-times}} \ni (h_1, \cdots, h_q):
\]

face operators $\varepsilon_i : NG(q) \to NG(q-1)$

\[
\varepsilon_i(h_1, \cdots, h_q) = \begin{cases} (h_2, \cdots, h_q) & i = 0 \\ (h_1, \cdots, h_i, h_{i+1}, \cdots, h_q) & i = 1, \cdots, q-1 \\ (h_1, \cdots, h_{q-1}) & i = q \end{cases}
\]

\[
\overline{NG}(q) = \underbrace{G \times \cdots \times G}_{q+1 \text{-times}} \ni (g_1, \cdots, g_{q+1}):
\]

face operators $\overline{\varepsilon}_i : \overline{NG}(q) \to \overline{NG}(q-1)$

\[
\overline{\varepsilon}_i(g_1, \cdots, g_{q+1}) = (g_1, \cdots, g_i, g_{i+2}, \cdots, g_{q+1}) & \quad i = 0, 1, \cdots, q
\]

We define $\gamma : \overline{NG} \to NG$ as $\gamma(g_1, \cdots, g_{q+1}) = (g_1g_2^{-1}, \cdots, g_qg_{q+1}^{-1})$.

For any simplicial manifold $X = \{X_\ast\}$, we can associate a topological space $\|X\|$ called the fat realization. Since any $G$-bundle $\pi : E \to M$ can be realized as a pull-back of the fat realization of $\gamma$, $\|\gamma\|$ is the universal bundle $EG \to BG$ [10].

Now we construct a double complex associated to a simplicial manifold.

**Definition 2.1.** For any simplicial manifold $\{X_\ast\}$ with face operators $\{\varepsilon_\ast\}$, we define double complex as follows:

\[
\Omega^{p,q}(X) \overset{\text{def}}{=} \Omega^q(X_p)
\]
Derivatives are:

\[ d' := \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*, \quad d'' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p). \]

For NG and N\(\bar{G}\) the following holds \[2\] \[6\] \[9\].

**Theorem 2.1.** There exist ring isomorphisms

\[ H(\Omega^*(NG)) \cong H^*(BG), \quad H(\Omega^*(N\bar{G})) \cong H^*(EG). \]

Here \(\Omega^*(NG)\) and \(\Omega^*(N\bar{G})\) mean the total complexes.

There is another double complex associated to a simplicial manifold.

**Definition 2.2.** A simplicial \(n\)-form on a simplicial manifold \(\{X_p\}\) is a sequence \(\{\varphi^{(p)}\}\) of \(n\)-forms \(\varphi^{(p)}\) on \(\Delta^p \times X_p\) such that

\[ (\varepsilon^i \times id)^* \varphi^{(p)} = (id \times \varepsilon_i)^* \varphi^{(p-1)}. \]

Here \(\varepsilon^i\) is the canonical \(i\)-th face operator of \(\Delta^p\).

Let \(A^{k,l}(X)\) denote the set of simplicial \((k + l)\)-forms on \(\Delta^p \times X_p\) which are expressed locally of the form

\[ \sum a_{i_1 \cdots i_k, j_1 \cdots j_l} (dt_{i_1} \wedge \cdots \wedge dt_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}) \]

where \(t_i\) are the barycentric coordinates in \(\Delta^p\) and \(x_j\) are the local coordinates in \(X_p\). We call these forms \((k, l)\)-form on \(\Delta^p \times X_p\) and define derivatives as:

\[ d' := \text{the exterior differential on } \Delta^p \]

\[ d'' := (-1)^p \times \text{the exterior differential on } X_p. \]

Then \((A^{k,l}(X), d', d'')\) is a double complex. We denote the total complex by \(A^*(X)\).
Now we define a map $I_\Delta : A^*(X) \to \Omega^*(X)$ as follows:

$$I_\Delta(\alpha) \overset{\text{def}}{=} \int_{\Delta^p} (\alpha|_{\Delta^p \times X_p}).$$

Then the following theorem holds [5].

**Theorem 2.2.** $I_\Delta$ induces a natural ring isomorphism

$$I_\Delta^* : H(A^*(X)) \cong H(\Omega^*(X)).$$

Let denote $\mathcal{G}$ the Lie algebra of $G$. A connection on a simplicial $G$-bundle $\pi : \{E_p\} \to \{M_p\}$ is a sequence of 1-forms $\{\theta\}$ on $\{E_p\}$ with coefficients $\mathcal{G}$ such that $\theta$ restricted to $\Delta^p \times E_p$ is a usual connection form on a principal $G$-bundle $\Delta^p \times E_p \to \Delta^p \times M_p$.

Let $\theta$ be the Maurer-Cartan form of $G$. Dupont constructed a canonical connection $\theta \in A^1(\mathcal{G})$ on $\gamma : N\mathcal{G} \to NG$ in the following way:

$$\theta|_{\Delta^p \times N\mathcal{G}(p)} \overset{\text{def}}{=} t_1 \theta_1 + \cdots + t_{p+1} \theta_{p+1}.$$

Here $\theta_i$ is defined by $\theta_i = \text{pr}_i^* \theta$ where $\text{pr}_i : \Delta^p \times N\mathcal{G}(p) \to G$ is the projection into the $i$-th factor of $N\mathcal{G}(p)$. We also obtain the curvature $\Omega \in A^2(\mathcal{G})$ on $\gamma$ as:

$$\Omega|_{\Delta^p \times N\mathcal{G}(p)} = d\theta|_{\Delta^p \times N\mathcal{G}(p)} + (-1)^p \frac{1}{2} [\theta|_{\Delta^p \times N\mathcal{G}(p)}, \theta|_{\Delta^p \times N\mathcal{G}(p)}].$$

Now let denote $I^*(G)$ the ring of $G$-invariant polynomials on $\mathcal{G}$. For $P \in I^*(G)$, we restrict $P(\Omega) \in A^*(\mathcal{G})$ to each $\Delta^p \times N\mathcal{G}(p) \to \Delta^p \times NG(p)$ and apply the usual Chern-Weil theory then we have the simplicial $2k$-form $P(\Omega)$ on $NG$. We summarize:

**Theorem 2.3** ([5] [11]). There is a canonical homomorphism

$$\omega : I^*(G) \to H(\Omega^*(NG))$$

which maps $P \in I^*(G)$ to $\omega(P) = [I_\Delta(P(\Omega))]$. 

□
3 The Chern character in the double complex

In this section we exhibit the cocycle in $\Omega^{*\ast}(NG)$ which represents the Chern character. Throughout this section, $G = GL(n; \mathbb{C})$ and $ch_p$ means both the $p$-th Chern character and the polynomial to define it.

Note that the diagram below is commutative, since $I_\Delta$ acts on only the differential forms on $\Delta^*$, and so does $\gamma^\ast$ on differential forms on each $NG(\ast)$.

$$A^{\ast\ast}(NG) \xrightarrow{I_\Delta} \Omega^{*\ast}(NG)$$

$$\gamma^\ast$$

$$A^{\ast\ast}(NG) \xrightarrow{I_\Delta} \Omega^{*\ast}(NG)$$

We first give the cocycle in $\Omega^{p+q}(NG(p−q))(0 \leq q \leq p−1)$ which corresponds the $p$-th Chern character by restricting $(1/p)! \text{tr} ((−\Omega/2\pi i)^p) \in A^{2p}(NG)$ to $A^{p−q,p+q}(\Delta^{p−q} \times NG(p−q))$ and integrating it along $\Delta^{p−q}$. Then we give the cocycle in $\Omega^{p+q}(NG(p−q))$ which hits to $ch_p(\Omega) \in \Omega^{p+q}(NG(p−q))$ by $\gamma^\ast$.

Since $[\theta_i, \theta_j] = \theta_i \wedge \theta_j + \theta_j \wedge \theta_i$ for any $i, j$,

$$\Omega|_{\Delta^{p−q} \times NG(p−q)} = \sum_{i=1}^{p−q} dt_i \wedge (\theta_i - \theta_{i+1}) - (-1)^{p−q} \sum_{1 \leq i < j \leq p−q+1} t_i t_j (\theta_i - \theta_j)^2.$$  

Now

$$dt_i \wedge (\theta_i - \theta_{i+1}) = dt_i \wedge (\theta_i - \theta_{i+1}) + (\theta_{i+1} - \theta_{i+2}) + \cdots + (\theta_{p−q} - \theta_{p−q+1})$$

and for any $G$-valued differential forms $\alpha, \beta, \gamma$ and any integer $1 \leq \forall x \leq p−q$, the equation $\alpha \wedge (dt_i \wedge (\theta_x - \theta_{x+1})) \wedge \beta \wedge (dt_j \wedge (\theta_x - \theta_{x+1})) \wedge \gamma = -\alpha \wedge (dt_j \wedge (\theta_x - \theta_{x+1})) \wedge \beta \wedge (dt_i \wedge (\theta_x - \theta_{x+1})) \wedge \gamma$ holds, so the terms of the forms above cancel each other in $(\Omega|_{\Delta^{p−q} \times NG(p−q)})^p$. Then we obtain:

$$(\Omega|_{\Delta^{p−q} \times NG(p−q)})^p = \left(\sum_{i=1}^{p−q} dt_i \wedge (\theta_i - \theta_{i+1}) - (-1)^{p−q} \sum_{1 \leq i < j \leq p−q+1} t_i t_j (\theta_i - \theta_j)^2\right)^p.$$

For convenience we define $\text{Det}$ for the matrices which consists of $G$-valued differential forms $\mu_{st}$ $(1 \leq s, t \leq n)$ as follows:

$$\text{Det} \left( \begin{array}{ccc} \mu_{11} & \cdots & \mu_{1n} \\ \vdots & \ddots & \vdots \\ \mu_{n1} & \cdots & \mu_{nn} \end{array} \right) \overset{\text{def}}{=} \sum_{\sigma \in S_n} (\text{sgn}(\sigma)) \mu_{1\sigma(1)} \cdots \mu_{n\sigma(n)}.$$
Now we obtain the following theorem.

**Theorem 3.1.** We set:

\[
\bar{S} = \begin{pmatrix}
\theta_1 - \theta_2 & \theta_2 - \theta_3 & \cdots & \theta_{p-q} - \theta_{p-q+1} \\
\vdots & \vdots & & \vdots \\
\theta_1 - \theta_2 & \cdots & \cdots & \theta_{p-q} - \theta_{p-q+1}
\end{pmatrix}
\]

Then the cocycle in \(\Omega^{p+q}(N\bar{G}(p-q))\) \((0 \leq q \leq p-1)\) which corresponds the \(p\)-th Chern character \(\text{ch}_p\) is

\[
\frac{1}{p!}(-1)^p(-1)^{p-p-q-1/2} \times 
\sum \left( \text{Tr}(\text{Det}(\bar{S})) \times (-1)^q(p-q+1) \int_{\Delta_{p-q}} \prod_{i<j} (t_i t_j)^{a_{ij}(\text{Tr}(\text{Det}(\bar{S})))} dt_1 \wedge \cdots \wedge dt_{p-q} \right).
\]

Here \(\text{Tr}(\text{Det}(\bar{S}))\) means the terms that \((\theta_i - \theta_j)^2\) \((1 \leq i < j \leq p-q+1)\) are put \(q\) -times between \((\theta_{k-1} - \theta_k)\) and \((\theta_k - \theta_{k+1})\) in \(\text{tr}(\text{Det}(\bar{S}))\) permitting overlaps; \(a_{ij}(\text{Tr}(\text{Det}(\bar{S})))\) means the number of \((\theta_i - \theta_j)^2\) in it. \(\sum\) means the sum of all such terms. \(\square\)

\(\theta_i\) on the \(\Omega^*(N\bar{G}(n))\) corresponds \((\text{pr}_2 \circ \phi_{a_i})^*(g^{-1}dg)\) on the Čech-de Rham complex \(\Omega^*(\pi^{-1}(U_{a_1}) \cap \cdots \cap \pi^{-1}(U_{a_{n+1}}))\) for the total space of the induced bundle. So from this theorem we can check that for any flat \(G\)-bundle the image of Chern-Weil homomorphism is 0, because in the case that the induced bundle is flat, \((\text{pr}_2 \circ \phi_{a_i})^*(g^{-1}dg) = (\text{pr}_2 \circ \phi_{a_j})^*(g^{-1}dg)\) for any \(i\) and \(j\).

In purpose of getting the differential forms in \(\Omega^{*\ast}(NG)\) which hit the cocycles in Theorem 3.1 by \(\gamma^\ast\), we introduce the following corresponding \(\psi\).

**Definition 3.1.** The value of \(\psi\) is decided when it is located between \(dh_i\) and \(dh_j\) as follows:

\[
dh_i \psi dh_j = \begin{cases} 
\begin{aligned}
&dh_i dh_j & j = i + 1 \\
&dh_i(h_{i+1} \cdots h_{j-1})dh_j & j > i + 1 \\
&dh_i(h_j h_{j+1} \cdots h_{i-1} h_i)^{-1} dh_j & j \leq i.
\end{aligned}
\end{cases}
\]

Here \(h_i\) is the \(i\)-th factor of \(NG(\ast)\). \(\square\)
A straightforward calculation shows that
\[
\gamma^* \text{tr}(dh_{i_1}\psi dh_{i_2}\psi \cdots dh_{i_{p-1}}\psi dh_{i_p}\psi) = \text{tr}(\theta_{i_1} - \theta_{i_1+1})(\theta_{i_2} - \theta_{i_2+1}) \cdots (\theta_{i_p} - \theta_{i_p+1}).
\]

From the above, we conclude:

**Theorem 3.2.** We set:

\[ R_{ij} = (dh_i\psi + dh_{i+1}\psi + \cdots + dh_{j-1}\psi)^2 \quad (1 \leq i < j \leq p - q + 1) \]

\[ S = \begin{pmatrix}
  dh_1\psi & dh_2\psi & \cdots & dh_{p-q}\psi \\
  \vdots & \vdots & & \vdots \\
  dh_1\psi & \cdots & \cdots & dh_{p-q}\psi 
\end{pmatrix} \]

Then the cocycle in \( \Omega^{p+q}(NG(p-q)) \) (0 \( \leq q \leq p - 1 \) which represents the \( p \)-th Chern character \( \text{ch}_p \) is

\[
\frac{1}{p!} \left( \frac{-1}{2\pi i} \right)^p (-1)^{(p-q)(p-q-1)/2} \times \\
\sum \left( \text{Tr}(\text{Det}(S)) \times (-1)^{q(p-q+1)} \int_{\Delta_{p-q}} \prod_{i<j} (t_i t_j)^{a_{ij}(\text{Tr}(\text{Det}(S)))} \, dt_1 \wedge \cdots \wedge dt_{p-q} \right). 
\]

Here \( \text{Tr}(\text{Det}(S)) \) means the term that \( R_{ij} \) \( (1 \leq i < j \leq p - q + 1) \) are put \( q \) -times between \( dh_i\psi \) and \( dh_{i+1}\psi \) in \( \text{tr}(\text{Det}(S)) \) permitting overlaps; \( a_{ij}(\text{Tr}(\text{Det}(S))) \) means the number of \( R_{ij} \) in it. \( \sum \) means the sum of all such terms. \( \square \)

**Example 3.1.** The cocycle which represents the second Chern character \( \text{ch}_2 \) in \( \Omega^4(NG) \) is the sum of the following \( C_{1,3}, C_{2,2} \):

\[
0 \quad \overset{d''}{\rightarrow} \quad C_{1,3} \in \Omega^3(G) \overset{d'}{\rightarrow} \Omega^3(NG(2)) \quad \overset{d''}{\rightarrow} \quad C_{2,2} \in \Omega^2(NG(2)) \overset{d'}{\rightarrow} 0
\]

\[
C_{1,3} = \left( \frac{-1}{2\pi i} \right)^2 \frac{1}{6} \text{tr}(h^{-1} dh)^3, \quad C_{2,2} = \left( \frac{-1}{2\pi i} \right)^2 \frac{-1}{2} \text{tr}(h_2^{-1} h_1^{-1} dh_1 dh_2).
\]
Corollary 3.1. The cocycle which represents the second Chern class $c_2$ in \( \Omega^4(NG) \) is the sum of the following $c_{1,3}, c_{2,2}$:

\[
\begin{align*}
0 \\
\uparrow d'' \\
c_{1,3} \in \Omega^3(G) \rightarrow d' \cdots \Omega^3(NG(2)) \\
\uparrow d'' \\
c_{2,2} \in \Omega^2(NG(2)) \rightarrow d' \cdots 0
\end{align*}
\]

\[
c_{1,3} = \left(\frac{-1}{2\pi i}\right)^2 \frac{-1}{6} \text{tr}(h^{-1}dh)^3
\]

\[
c_{2,2} = \left(\frac{-1}{2\pi i}\right)^2 \frac{1}{2} \text{tr}(h_2^{-1}h_1^{-1}dh_1dh_2) - \left(\frac{-1}{2\pi i}\right)^2 \frac{1}{2} \text{tr}(h_1^{-1}dh_1)\text{tr}(h_2^{-1}dh_2).
\]

Example 3.2. The cocycle which represents the 3rd Chern character $ch_3$ in \( \Omega^6(NG) \) is the sum of the following $C_{1,5}, C_{2,4}, C_{3,3}$:

\[
\begin{align*}
0 \\
\uparrow d'' \\
C_{1,5} \in \Omega^5(G) \rightarrow d' \cdots \Omega^5(NG(2)) \\
\uparrow d'' \\
C_{2,4} \in \Omega^4(NG(2)) \rightarrow d' \cdots \Omega^4(NG(3)) \\
\uparrow d'' \\
C_{3,3} \in \Omega^3(NG(3)) \rightarrow d' \cdots 0
\end{align*}
\]

\[
C_{1,5} = \frac{1}{3!} \left(\frac{-1}{2\pi i}\right)^3 \frac{-1}{10} \text{tr}(h^{-1}dh)^5
\]

\[
C_{2,4} = \frac{1}{3!} \left(\frac{-1}{2\pi i}\right)^3 \left(\frac{1}{2} \text{tr}((dh_1h_1^{-1}dh_1h_1^{-1}dh_1dh_2h_2^{-1}h_1^{-1}))
\]

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\[ C_{3,3} = \frac{1}{3!} \left( \frac{-1}{2\pi i} \right)^3 \left( \frac{1}{2} \text{tr}(dh_1dh_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}) ight) \\
- \frac{1}{2} \text{tr}(dh_2h_1^{-1}dh_1h_2dh_3h_3^{-1}h_2^{-1}). \]

4 The Chern-Simons Form

We briefly recall the notion of the Chern-Simons form in [4].

Let \( \pi : E \to M \) be any principal \( G \)-bundle and denote \( \theta, \Omega \) its connection form and the curvature. For any \( P \in \mathbb{I}^k(G) \), we define the \((2k-1)\)-form \( TP(\theta) \) on \( E \) as:

\[
TP(\theta) \overset{\text{def}}{=} \int_0^1 P(\theta \wedge \phi_t^{k-1})dt
\]

Here \( \phi_t \overset{\text{def}}{=} t\Omega + t(t-1)[\theta,\theta] \). Then the equation \( d(TP(\theta)) = P(\Omega^k) \) holds and \( TP(\theta) \) is called the Chern-Simons form of \( P(\Omega^k) \). When the bundle is flat, then its curvature vanishes hence \( d(TP(\theta)) = P(\Omega^k) = 0 \).

Now we put the simplicial connection into \( TP \) and using the same argument in section 3, then we get the Chern-Simons form in \( \Omega^{2p-1}(NG) \).

**Proposition 4.1.** The Chern-Simons form in \( \Omega^3(NU(n)) \) which corresponds the second Chern class \( c_2 \) is the sum of the following \( Tc_{0,3}, Tc_{1,2} \):

\[
\begin{array}{ccc}
0 & \overset{d''}{\longrightarrow} & \Omega^3(U(n)) \\
\uparrow{d''} & & \Omega^3(NU(n))(1) \\
Tc_{0,3} \in \Omega^3(U(n)) & \overset{d}{\longrightarrow} & \Omega^3(NU(n))(1) \\
\uparrow{d''} & & \Omega^2(NU(n))(2) \\
Tc_{1,2} \in \Omega^2(NU(n))(1) & \overset{d'}{\longrightarrow} & \Omega^2(NU(n))(2)
\end{array}
\]
\[ T_{c_{0,3}} = \left( \frac{-1}{2\pi i} \right)^2 \frac{1}{6} \text{tr}(g^{-1}dg)^3 \]

\[ T_{c_{1,2}} = \left( \frac{-1}{2\pi i} \right)^2 \left( \frac{1}{2} \text{tr}(g_1^{-1}dg_1g_2^{-1}dg_2) - \frac{1}{2} \text{tr}(g_1^{-1}dg_1)\text{tr}(g_2^{-1}dg_2) \right). \]

□

Remark 4.1. The term \[-\left( \frac{-1}{2\pi i} \right)^2 \frac{1}{2} \text{tr}(g_1^{-1}dg_1)\text{tr}(g_2^{-1}dg_2)\] vanishes when we restrict \( U(n) \) to \( SU(n) \).

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