Power law fluid model on wave mitigation, 2D simulation using smoothed particle hydrodynamics

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Abstract. This article is focused on numerical modelling to describe influence of coastal vegetation in wave mitigation. The model based on Navier-Stokes equations with stress tensor written in power law model. Numerical approach used to solve the problem is SPH (Smoothed Particle Hydrodynamics). Three numerical simulation are conducted; plane Couette-Poiseuille flow, wave mitigation on flat bottom, and wave mitigation on incline bottom. The first simulation shows that our numerical results are in good agreement with analytic solution provided in [13]. The second simulation shows that existence of the coastal vegetation reduce the wave amplitude. Whereas the last simulation shows that the vegetation reduce the run up height. Further, the power law constant influences how high the run up.

1. Introduction
Mangrove has important role in protecting coastal area due to its ability in reducing amplitude and velocity of wave. Moreover, the coastal vegetation can support ecosystem at the coastal area. These advantages often lead mangrove to be as an option to prevent coastal area from devastation.

Resistance of mangrove can be modelled as porous medium as described in [1]. Further, porous medium also can be modelled by adding a certain friction term such as the linearized Dupuit-Forcheimer’s formula, see [2] for more detail. Another way to model the resistance is by inserting a friction term directly to the governing equations. Elaboration of the model can be seen in detail in [3]. In the reference, three kind of resistances are used such as Darcy, Manning, and Laminar viscosity resistance.

Formulations of wave interaction with porous medium can use Navier-Stokes type equations for instances see [4], [5], [6] and [7]. The different between those three references is in considering the force acting on the fluid flow. In [4] and [6], the stress tensor is considered whereas in [5] and [7], physical properties of the porous medium such as porosity and intrinsic permeability are considered in friction term.

In this article, the wave mitigation due to the coastal vegetation is modelled using Navier-Stokes type equations. The stress tensor is written in power law model as elaborated in detail in [6]. Whereas the friction term used is Manning resistance given in [3]. In Lagrangian form, the governing equations can be written as follows:
\[
\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u},
\]
\[
\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla P + g - c_f \frac{\mathbf{u} | \mathbf{u} |}{H^{1/3}} + \frac{1}{\rho} \nabla \cdot \tau,
\]
\[
\frac{D\mathbf{x}}{Dt} = \mathbf{u},
\]
\[(1)\]

where \(\rho\) denotes fluid density, \(\mathbf{u}\) is flow velocity, \(P\) is pressure, \(g\) is gravitational force \((0, -9.81)^T\), \(c_f\) is the Manning friction coefficient, \(H\) is water depth, \(\mathbf{x}\) is position of particles and \(\tau\) is the stress tensor defined by
\[
\tau_{ij} = 2\eta \zeta_{ij}, \quad \zeta_{ij} = \frac{1}{2} \left( \partial_i \mathbf{u}_j + \partial_j \mathbf{u}_i \right)
\]
\[
\eta = \mu \xi^{n-1}, \quad \dot{\xi} = \sqrt{2\zeta_{ij}\zeta_{ij}},
\]
\[(2)\]
where \(n\) denotes the power law constant (model parameter) and \(\mu = 10^{-3} \text{ Pa.s}\) is the molecular dynamics viscosity.

This article is devoted to develop numerical model to describe influence of coastal vegetation in wave mitigation. The model is based on the Navier-Stokes equations with stress tensor written in power law model and with Manning friction term. The governing equations will be solved numerically using Smoothed Particle Hydrodynamics (SPH) method that will be elaborated in Section 2. Three kinds of numerical experiments will be conducted to see the SPH code performance and propagation of waves on flat and incline bottom. Detail and results of the simulation will be presented in Section 3. Finally, some conclusions will be given in Section 4 to close this article.

2. Numerical method

In general numerical method can be grid based or particle based. In this article, the governing equations are solved using particle method namely Smoothed Particle Hydrodynamics (SPH). Briefly explanation about the SPH method will be given in this section, detail description of the method can be found, for instances in [8], [9], [10], and [11]. Here, the explanation of the method is taken from those references.

2.1. SPH fundamentals

Monaghan in [10] described that the SPH method is meshless method based on integral interpolation. In this method, any kind of functions is interpolated by a certain kernel function that has several properties. There are several kernel function that can be used but the common one is cubic spline kernel. Further, here the fluid in the computational domain is approximated by a finite number of particles that have several physical properties such as mass \(m\), velocity \(\mathbf{u}\), position \(\mathbf{x}\), density \(\rho\), and other related properties.

The discrete form of the integral interpolation of function \(\phi(\mathbf{x})\) of point \(i\) with smoothing length of the kernel function \(h\) can be written as
\[
\phi(\mathbf{x}_i) = \sum_j m_j \frac{\phi_j}{\rho_j} W(|\mathbf{x}_i - \mathbf{x}_j|, h),
\]

where the summation is carried out for all particle \((j)\) within compact support area with length \(2h\) centered at position of point \(i\) and \(W\) denotes the cubic spline kernel
\[
W(r, h) = \beta \begin{cases} 
1 - \frac{3}{2} q^2 + \frac{3}{4} q^3 & 0 \leq q < 1 \\
\frac{1}{4} (2 - q)^3 & 1 \leq q < 2 \\
0 & q \geq 2
\end{cases}
\]
\[(4)\]
where $\beta = \frac{10}{7\pi h^2}$ denotes the normalization parameter, $q = \frac{r}{h}$ represents the non-dimensional distance between particle $i$ and $j$ and $r$ is the distance between particles.

In the method, the fluid is treated as weakly compressible by adding an equation of state. In this article, the following equation is used to update pressure of the fluid

$$P(\rho) = \frac{c_0\rho_0}{\gamma} \left[ \left( \frac{\rho}{\rho_0} \right)^\gamma - 1 \right]$$  \hspace{1cm} (5)

where $\rho_0 = 1000 \text{ kg/m}^3$ denotes the reference density, $c_0$ is the speed sound of particle at $\rho_0$, and $\gamma = 7$.

2.2. SPH formulations of the governing equations

Implying the SPH method to the governing equations, we get the following equations.

$$\frac{D\rho_i}{Dt} = \sum_{j=1}^{N} m_j (u_i - u_j) \cdot \nabla W_{ij},$$  \hspace{1cm} (6)

$$\frac{Du_i}{Dt} = -\sum_{j=1}^{N} m_j \left( \frac{P_i + P_j}{\rho_i\rho_j} + \Pi_{ij} \right) \nabla_i W_{ij} + g - \frac{c_f |u_i|}{H_i^\Pi} + \frac{1}{\rho} (\nabla \cdot \tau)_i.$$  \hspace{1cm} (7)

The term $\Pi_{ij}$ represents the artificial viscosity added to get numerical stability and defined by

$$\Pi_{ij} = \begin{cases} \frac{-\alpha \epsilon \dot{\epsilon}_{ij} \psi_{ij}}{\bar{\rho}_{ij}} & u_{ij} \cdot x_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (8)

where $\alpha$ is a constant that should be calibrated depend on the problem [12] and

$$\dot{\epsilon}_{ij} = \frac{h_{ij} u_{ij} \cdot x_{ij}}{r_{ij}^2 + (0.1h_{ij})^2}, \quad u_{ij} = u_i - u_j, \quad x_{ij} = x_i - x_j$$

$$\bar{\rho}_{ij} = \left( \frac{\rho_i + \rho_j}{2} \right), \quad h_{ij} = \left( \frac{h_i + h_j}{2} \right).$$  \hspace{1cm} (9)

Formulations of the stress in SPH terms can be seen in detail in [6], [13], [14], and [15]. Let $a$ and $b$ denote 2D coordinate ($x$ and $y$ direction) and $\delta^{ba}$ is delta Dirac function with value 1 if $b = a$ and 0 if $b \neq a$. Following Shaimy in [13], SPH formulations of the stress tensor are given as:

$$\frac{1}{\rho} (\nabla \cdot \tau)_i^a = \sum_{j=1}^{N} m_j \left( \frac{\mu_j \dot{\xi}_{i}^{a-1} \delta^{ba} + \mu_j \dot{\xi}_{j}^{a-1} \delta^{ba}}{\rho_i \rho_j} \right) \partial W_{ij} \partial x_i^a,$$

$$\zeta_i^{ba} = \sum_{j=1}^{N} m_j u_{ji}^b \cdot \frac{\partial W_{ij}}{\partial x_i^a} + \sum_{j=1}^{N} m_j \rho_j \frac{\partial W_{ij}}{\partial x_i^a} = \left\{ \frac{2}{3} \sum_{j=1}^{N} m_j u_{ji}^b \cdot \nabla W_{ij} \right\} \delta^{ba}. \hspace{1cm} (10)$$

3. Numerical simulation

In this section, three numerical simulations are conducted. The first simulation are carried out to validate our numerical code performance. The second simulation is to see the influence of the coastal vegetation on flat seabed. Whereas the last simulation is conducted to describe propagation of wave on inlined seabed and see its run up height. In this article, the Manning coefficient is $c_f = 0.25$.

3.1. Plane Couette-Poiseuille flow

To validate our SPH code of the power-law fluid model, here we solve the Couette-Poiseuille flow problem. In this case, the fluid flows through two parallel plates. The flow is generated by the pressure difference in the gap of the plates and by the movement of one of the plates. In this article, the pressure
difference is applied by giving a body force $F$ and the movement of one of plates is generated by giving initial velocity $U$.

Following Shaimy in [13], the solution of the problem is given as the following equation

$$u(y, t) = \frac{U}{l} y + \frac{F}{2\mu} y(y - l) + \sum_{i=1}^{\infty} \frac{2U}{in} (-1)^i \sin \left( \frac{i\pi y}{l} \right) e^{-\nu \left( \frac{in}{l} \right)^2} t \right) + \sum_{i=0}^{\infty} \frac{4Fl^2}{\mu(2i+1)^3 \pi^3} \sin \left( \frac{(2i+1)\pi y}{l} \right) e^{-\nu \left( \frac{(2i+1)\pi}{l} \right)^2} t.$$  \hspace{1cm} (11)

In [13], detail of derivation of the solution is provided. Here we use the same numerical set up and boundary condition as the reference such as distance between the plates $l = 10^{-3}$ m, $F = 4 \times 10^{-4}$ kgm/s$^2$, $U = 5 \times 10^{-5}$ m/s. Result of the test case for power-law constant $n = 1$ is given in Figure 1.

![Figure 1](image1.png)

**Figure 1.** Velocity profile of Couette-Poiseuille flow at time $t = 0.01$ s (a), 0.05s (b), 0.1s (c), 1s (d).

Comparison of analytic solution provided by Shaimy in [13] and our numerical solution is presented in Figure 1. Here the velocity profile of numerical solution are plotted together with the analytic solution for several times with power-law constant $n = 1$. The pressure different and movement of the plate will push the fluid particles to move. Velocity profile of the particle in certain area is observed. The results are presented in the figure. As it is shown in the figure, the velocity profile will change until it reach the steady state solution. Moreover, it is hard to distinguish between analytic and numerical solution. The numerical solution is closest enough to the analytic solution. Therefore, it can be said that our numerical solutions are in good agreement with the analytic solution. In other words, our power-law SPH code is rational and can be used.

### 3.2. Wave mitigation on flat bottom

In this subsection, numerical experiment to see influence of the coastal vegetation is carried out. The coastal vegetation are placed at location $3.5 \leq x \leq 5$ m. Piston type wavemaker as used in [3] is adopted to generate periodic waves. At the beginning all particles are set at rest with distance 0.0278 m, smoothing length $h = 0.0256$ m, and placed in front of the wavemaker.

As the time goes higher, the wavemaker will push the water (particles) and create waves. The waves will propagate to the right and enter the vegetation area until they hit the right boundary and bounce back. Influence of the coastal vegetation will spread after the waves enter the area. The simulation is run in computational domain $\Omega = [0, 5]$ with time step $\Delta t = 10^{-4}$ s and initial water depth 0.5 m. Results of the simulation are presented in Figure 2.
It is clearly seen in the Figure 2 that the wavemaker push the water and generate waves that propagate to right side. Figure 2(b) shows that the waves enter the vegetation area. The waves continue to propagate and hit the wall boundary shown in Figure 2(e). Further, the waves bounce back and propagate to the opposite direction due to the hard wall as shown in Figure 2(f). Figure 2 describes the wave generation and propagation over flat bottom, but it is hard to tell the influence of the coastal vegetation. Therefore we conduct the next simulation to see the influence.

![Figure 2](image1)

**Figure 2.** Wave propagation over flat bottom of SPH power-law fluid model with power-law constant $n = 1$ at time $t = 0.3s$ (a), 0.35s (b), 0.4s (c), 0.45s (d), 0.5s (e), and 0.55s (f).

![Figure 3](image2)

**Figure 3.** Wave surface of SPH power-law fluid model for several power-law constant $n$ at time $t = 0.3s$ (a), 0.35s (b), 0.4s (c), 0.45s (d), 0.5s (e), and 0.55s (f) propagating over flat bottom. $x$-axis denotes position in meter and $y$-axis denotes water surface in meter.
To see effect of power law constant in wave mitigation due to the vegetation, next we simulate the same scenario using several power law constant. In order to see the mitigation, water surface of each power law constant are plotted together with water surface in case there is no vegetation. Snapshots of those scenarios are given in Figure 3. Here we focus on area near the vegetation area. Note that all water surface in current article is detected by using free surface algorithm as described in detail in [16].

Figure 3(a) and 3(b) show the wave enter the coastal vegetation area. In these figures, the reduction of wave amplitude does not happen yet. All scenarios have the same surface with no friction case, in this article the no friction case means there is no coastal vegetation. The reduction of the wave amplitude occurs in Figure 3(c) – 3(f). In these figures, water surface of all scenarios is below the no friction case. Further, in Figure 3(c) we can see that wave profile of no friction case is in front of the other wave profiles. It means that there is reduction of wave velocity due to the coastal vegetation.

### 3.3. Wave mitigation on incline bottom

In this subsection, numerical simulation is used to see wave mitigation on incline seabed. Initial set up of SPH parameters is same as initial set up in the previous subsection. Results of the simulation are plotted in Figure 4. We capture surface of water for several power-law constant near the vegetation area to see run up height and its result is plotted together with water surface in case there is no vegetation in Figure 5.

![Wave propagation](image)

**Figure 4.** Wave propagation over inclined bottom of SPH power-law fluid model with power-law constant \( n = 1 \) at time \( t = 0.3s \) (a), 0.35s (b), 0.4s (c), 0.45s (d), 0.5s (e), and 0.55s (f).

Figure 4 shows profile of wave propagations over an inclined bottom. The wavemaker generates wave. The wave propagates to the right, enters the vegetation area, and propagates over the inclined bottom. The recorded run up height is 0.518416 m at position \( x = 4.9359 \) m. To see influence of the coastal vegetation in the reduction of wave run up, we compare the results with case of no coastal vegetation. Further, to see effect of the power-law constant, we conduct several scenarios. In this simulation, we choose several values of the constant \( n = 0.5, 1.0, 1.5, 2.0 \). Results of the scenarios are plotted together in the Figure 5.

Figure 5 is presented comparison of water surface for power-law constant \( n = 0.5, 1.0, 1.5, 2.0 \) and \( n = 1 \) in case there is no vegetation. As shown in Figure 5(a) – 5(d), influence of the vegetation area is seen clearly from the shape of the surface. The reduction of the wave amplitude occurs is
shown in the figures. In terms of run up height, Figure 5(d) shows the reduction of the run up height. Here we record the run up height for each scenario and its result is given in Table 1.

![Figure 5. Wave surface of SPH power-law fluid model for several power-law constant $n$ at time $t = 0.4s$ (a), $0.45s$ (b), $0.5s$ (c), and $0.55s$ (d) propagating over incline bottom. x-axis denotes position in meter and y-axis denotes water surface in meter.]

| $n$   | $x$ (m)     | Run up (m) |
|-------|-------------|------------|
| 0.5   | 4.534083    | 0.336439   |
| 1.0   | 4.560988    | 0.329222   |
| 1.5   | 4.527846    | 0.316588   |
| 2.0   | 4.518687    | 0.308246   |

According to Table 1, the highest run up height is obtained when the power law constant is $n = 0.5$. Further the higher the constant is, the smaller run up height is. Based on position in $x$-axis, the waves get the furthest position when the power law constant is $n = 1.0$. When the constant is less than $n = 1.0$ or greater than $n = 1.0$, the waves reach closer position. In case there is no vegetation and the power law constant is $n = 1.0$, the run up height is $0.518416$ m at position $x = 4.9359$ m. Therefore existence of the vegetation reduce the run up height and the power law constant influences how high the run up could be. It is shown that all run up height of the scenarios in Table 1 are below the run up height of the case of no vegetation.

4. Conclusions

Elaboration of SPH power-law fluid model in wave mitigation due to coastal vegetation has been carried out. In this case, the power-law model is used to model the shear stress tensor whereas the coastal vegetation is modelled directly by adding Manning friction term. Numerical approach used to solve the problem is SPH (Smoothed Particle Hydrodynamics). Three numerical simulation are conducted; plane Couette-Poiseuille flow, wave mitigation on flat bottom, and wave mitigation on incline bottom. The first simulation is used to validate our numerical code by solving the Couette-Poiseuille flow problem. Its results show that our numerical code get results in good agreement with analytic solution. Influence of the coastal vegetation can be seen from the wave profile as shown in the second simulation. From the last simulation, existence of the mangrove reduce the run up height and the power law constant influences how high the run up could be. The highest run up height is obtained when the power law constant is $n = 0.5$ and when value of the constant increase, the height of run-up decrease. Moreover, the waves get the furthest position when the power law constant is $n = 1.0$. When the constant is less than $n = 1.0$ or greater than $n = 1.0$, the waves reach closer position.
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