New type of beam size effect and the $W$-boson production at $\mu^+\mu^-$ colliders

K. Melnikov  
*Institut fur Physik, Universität Mainz*

and

V.G. Serbo  
†Institut fur Theoretische Physik, Universität Leipzig

November 18, 2018

Abstract

The cross section for the reactions of the type $\mu^-\mu^+ \rightarrow e\bar{\nu}_e X$ can not be calculated by the standard methods due to the $t$–channel singularity in the physical region. In this letter we show that accounting for the finite sizes of the colliding beams results in the regularization of this singularity. The finite cross section, which is obtained in this way, turns out to be linear proportional to the transverse sizes of the colliding beams. As an application of the above result, we calculate the cross section of the $W$ boson production at $\mu^+\mu^-$ colliders in the reaction $\mu^-\mu^+ \rightarrow e\bar{\nu}_e W^+$.

MZ-TH-96-02

---

*D 55099 Germany, Mainz, Johannes Gutenberg Universität, Institut für Physik, THEP, Staudinger weg 7; e-mail: melnikov@dipmza.physik.uni-mainz.de

†Permanent address: Novosibirsk State University, 630090, Novosibirsk, Russia; e-mail: serbo@phys.nsu.nsk.su
1. Introduction.— It is known since early 60’s, that some high-energy processes can have a $t$-channel singularity in the physical region \[^1\]. There is no commonly accepted solution for this problem. Typically, this situation occurs when initial particles in a given reaction are unstable and the masses of the final particles are such that the real decay of the initial particles can take place.

Recently, in the paper \[^2\] it was stressed that this problem turned out to be a practical issue for the reaction $\mu^- \mu^+ \rightarrow e\bar{\nu}_e W^+$. It is shown in \[^2\] that the standard calculations lead to the infinite cross section in this case. Indeed, if the invariant mass of the final $e\bar{\nu}_e$ system is smaller than the mass of the muon, the square of the momentum transfer in the $t$-channel $q^2$ can be both positive and negative, depending on the scattering angle. This results in the power-like singularity in the cross section

$$d\sigma \propto \frac{dq^2}{|q^2|^2}$$

and the standard calculations turn out to be impossible. It is therefore necessary to regularize this divergence in order to produce definite prediction for the measurable number of events.

In this paper we show, that accounting for the finite sizes of the colliding beams gives a finite answer for the processes with the $t$-channel singularity in the physical region. This is the main result of our paper. As an example, we consider reaction $\mu^- \mu^+ \rightarrow e\bar{\nu}_e W^+$ and show that the actual cross section of this process is approximately 1 fb for the typical transverse beam sizes of the order of $10^{-3}$ cm.

The beam size effect (BSE) at the high-energy colliders is well studied both experimentally and theoretically (for the review see Ref. \[^3\]). For the first time this effect was observed at the VEPP-4 collider (Novosibirsk) in 1980-81 during the study of a single bremsstrahlung in the electron-positron collisions \[^4\]. This year the BSE was observed at HERA in the reaction $ep \rightarrow ep\gamma$ \[^5\]. In both cases the number of observed photons was smaller than it was expected according to the standard calculations. The decreased number of photons is explained by the fact that impact parameters, which give essential contribution to the standard cross sections of these reactions, are larger by 2-3 orders of magnitude compared to the transverse beam sizes.

In all the previous cases, when these effects were studied, the results depended logarithmically on the beam sizes. Below we show, that in our case (the $t$-channel singularity in the physical region ) the cross section for a scattering process is linear proportional to the transverse sizes of the colliding beams.

2. Cross section of the reaction $\mu^- \mu^+ \rightarrow e\bar{\nu}_e X$.— We introduce the following notations: $s = (p_1 + p_2)^2 = 4E^2$ is the square of the total energy in the center of mass frame, $m$ is the muon mass, $p_1^2 = p_2^2 = m^2$, $p_3$ is the 4-momentum of the final $e\bar{\nu}_e$ system, $y = p_3^2/m^2$ is the square of the invariant mass of the $e\bar{\nu}_e$ system in units of the square of the muon mass, $q = p_1 - p_3 = (\omega, q)$ is the momentum transfer in the $t$-channel and $x = \omega/E$ is the fraction of the initial muon energy transferred to the $t$-channel.

From simple kinematics it follows that

$$q^2 = -\frac{q_\perp^2}{1-x} + t_0, \quad -s(1-x) < q^2 < t_0, \quad t_0 = m^2 \frac{x(1-x-y)}{1-x}.$$
Here $\mathbf{q}_\perp$ is the component of the momentum $\mathbf{q}$ which is transverse to the momenta of the initial muons. Note, that $t_0 > 0$ as far as $y < 1 - x$ and that $q^2 = 0$ at

$$|\mathbf{q}_\perp| = q^0_\perp = m \sqrt{x(1 - x - y)}. \quad (1)$$

Let us consider the region of $|q^2| < \Lambda^2$ where $\Lambda \ll m$. In this region the main contribution comes from the diagram with the exchange of the muonic neutrino in the $t$-channel (Fig.1). Since for such $q^2$ the exchanged neutrino is almost real, the corresponding matrix element can be considerably simplified.

We present the matrix element $M$ in the form

$$M = -M_\mu \frac{1}{q^2 + i\epsilon} M_{\nu\mu}. \quad (2)$$

Here $M_\mu$ is the matrix element for the muon decay and $M_{\nu\mu}$ is the matrix element for the $\nu_\mu \mu^+ \to X$ process. In both of these subprocesses we take $q^2$ equal to zero.

![Feynman graph](image)

Figure 1: The Feynman graph for the reaction $\mu^- \mu^+ \to e\bar{\nu}_e X$, which gives the leading contribution in the region of small $|q^2|$.

Using the matrix elements for these subprocesses we express the cross section of the reaction $\mu^- \mu^+ \to e\bar{\nu}_e X$ through the muon decay width $\Gamma$ and the cross section $\sigma_{\nu\mu}$ of the $\nu_\mu \mu^+ \to X$ process:

$$d\sigma = \frac{1}{\pi} x \ m \ d\Gamma \ \frac{dq^2}{|q^2|^2} d\sigma_{\nu\mu}, \quad d\Gamma = 2\Gamma(1 - y)(1 + 2y) dx dy. \quad (3)$$

Let us call the coefficient in front of $d\sigma_{\nu\mu}$ as the number of neutrinos $dN_\nu$

$$dN_\nu = \frac{1}{\pi} x \ m \ d\Gamma \ \frac{dq^2}{|q^2|^2}. \quad (3)$$

From the Eq.(3) it is clear that the standard calculation of this cross section turns out to be impossible due to the power-like singularity since the point $q^2 = 0$ is within the physical region for $y < 1 - x$.

The main result of our study of the BSE in the above process can be formulated as follows:

*accounting for the BSE results in the following treatment of the divergent integral in the Eq. (3):*

$$B = \int \frac{dq^2}{|q^2|^2} \to \pi \frac{a}{q^0_\perp}. \quad (4)$$
The exact expression for the quantity \( a \) will be given below (see Eqs. (10) and (12)). We just mention here that it is proportional to the transverse sizes of the colliding beams. For the identical round Gaussian beams with the mean square radii \( \sigma_{ix} = \sigma_{iy} = \sigma_\perp, \ i = 1, 2 \) this quantity is equal to
\[
a = \sqrt{\pi} \sigma_\perp.
\]
Using this result in the expression for the number of neutrinos and integrating it over \( y \), we arrive to the following spectrum of the neutrino:
\[
dN_\nu(x) = \frac{\pi a}{2ct} f(x), \quad f(x) = \frac{24}{5\pi} \sqrt{x(1-x)} \left( 1 + \frac{22}{9} x - \frac{16}{9} x^2 \right), \quad \int_0^1 f(x) dx = 1.
\]
Here \( \tau \) is the life time of the muon at rest, \( c\tau = 660 \) m.

After the number of neutrinos (Eq. (5)) is obtained, the cross section for the reaction \( \mu^- \mu^+ \rightarrow e\bar{\nu}_e X \) is given by the equation:
\[
d\sigma = dN_\nu(x) \ d\sigma_{\nu\mu}(xs).
\]
Subsequent integration over \( x \) can be performed without further difficulties.

3. Derivation of the basic formula.— Now let us prove our result presented in the previous section (see Eq. (4)).

To begin with, we note that the standard notion of the cross section is an approximation itself. As is well known, it corresponds to the plane waves approximation for the initial and final particles. In the real experiments the particles are confined to the beams of a relatively small size and it is the collision of such beams that leads to the measurable number of events.

In order to arrive to a more general formulas for the description of the scattering processes, we have to be able to describe the collisions of the wave packets instead of the plane waves. In view of the fact that the movement of the particles inside a beam is quasiclassical, a simple and efficient technique for taking into account the beam size effects in the actual calculations has been developed.

Below we present some results from the Ref. [3], which are essential for our discussion. For simplicity, we neglect the energy and angular spread of the particles in the colliding beams.

Let us remind that in the standard approach the number of events \( N \) is the product of the cross section \( \sigma \) and the luminosity \( L \):
\[
dN = d\sigma L, \quad d\sigma = \frac{(2\pi)^4 \delta(p_1 + p_2 - P_f)|M|^2}{4p_1 p_2 \prod_f (2\pi)^3 2E_f} \cdot \int d^3 p_f n_2(r,t) d^3 r dt.
\]
where \( 2 = |v_1 - v_2| \) for the head-on collision of the ultrarelativistic beams. The quantities \( n_i(r,t) \) are the particle densities of the beams.

The transformation from the plane waves to the colliding wave packets results in the following changes. The squared matrix element \( |M|^2 \) with the initial state in the form of the plane waves with the momenta \( p_1 \) and \( p_2 \) transforms to the product:
\[
|M|^2 \rightarrow M_{fi} M^*_{fi'}. \quad (7)
\]
Here the initial state $|i⟩$ is the direct product of the plane waves with the momenta $p_1 + \frac{1}{2}\kappa$ and $p_2 - \frac{1}{2}\kappa$, while the initial state $|i′⟩$ is the direct product of the plane waves with the momenta $p_1 - \frac{1}{2}\kappa$ and $p_2 + \frac{1}{2}\kappa$. Instead of the luminosity $L$ the number of events starts to depend on the quantity $L(\歷) = 2\int n_1(r,t) n_2(r + \參,t)d^3rdt$

through the following formula:

$$dN = d\sigma(\kappa)L(\參)\exp(i\kappa\參) \frac{d^3\kappa d^3\參}{(2\pi)^3};$$

$$d\sigma(\kappa) = \frac{(2\pi)^4\delta(p_1 + p_2 - P_f)}{4p_1 p_2} M_{f_1} M_{f_2}^{*} \prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}. \quad (8)$$

The characteristic values of $\kappa$ are of the order of the inverse beam sizes, i.e. $\kappa \sim 1/\sigma_\perp$. Usually this quantity is much smaller than the typical scale for the variation of the matrix element with respect to the initial momenta. In this case we can put $\kappa = 0$ in $d\sigma(\kappa)$ which results in the standard expression for the number of events Eq.(6). Otherwise, one should use complete formulas which take into account the effect of the finite beam sizes.

In view of the discussion given in the previous section, this is indeed the situation which occurs in our case. Now we want to show how the finite result for the number of events can be obtained starting from the complete formula Eq.(8).

Let us first define the “observable cross section” by the relation $d\sigma = dN/L$ where $L$ is the standard luminosity. By writing the number of events in such a way, we push the BSE to the quantity $d\sigma$.

The study of the matrix element of the discussed process (2) suggests that the only quantity sensitive to the small variation of the initial momenta is the denominator of the neutrino propagator. Henceforth, the transformation (7) reduces to:

$$\frac{1}{|q^2|^2} \rightarrow \frac{1}{t} \frac{1}{t'}.$$  

Here

$$t = q^2 - \kappa \ 垂^2 + i\epsilon, \quad t' = q^2 + \kappa \ 垂^2 - i\epsilon.$$ 

In the expression for $t$ and $t'$ we only keep the terms which are linear in $\kappa$.

As a result the quantity $B$ (cf. Eq. (4)) transforms to:

$$B = \int \frac{dq^2}{t \ t'} \frac{L(\參)}{L} \exp(i\kappa\參) \frac{d^3\kappa d^3\参}{(2\pi)^3}. \quad (9)$$

To proceed further, we extend the region of integration over $q^2$ up to $\pm\infty$ and take $q^2$ in the point where $q^2 = 0$ \footnote{More detailed discussion of this calculation will be given elsewhere \cite{6}.}. After that integrations become simple and we obtain:

$$B = \frac{\pi}{q^2_\perp} a, \quad a = \int_0^\infty dq \frac{L(\參)}{L}, \quad n = \frac{q^2_\perp}{q^2_\perp}, \quad (10)$$

\footnote{More detailed discussion of this calculation will be given elsewhere \cite{6}.}
This completes the proof of the Eq. (11).

At high-energy colliders the distribution of particles in colliding beams can be often considered as Gaussian. In this case \( L(\rho n) \) equals:

\[
L(\rho n) = L \exp \left\{ - \rho^2 \left( \frac{\cos^2 \varphi}{2a_x^2} + \frac{\sin^2 \varphi}{2a_y^2} \right) \right\}, \quad \mathbf{n} = (\cos \varphi, \sin \varphi)
\]

(11)

where \( a_x^2 = \sigma_{1x}^2 + \sigma_{2x}^2 \) and \( a_y^2 = \sigma_{1y}^2 + \sigma_{2y}^2 \).

This results in the following expression for \( a \):

\[
a = \sqrt{\frac{\pi}{2}} \frac{a_x a_y}{\sqrt{a_x^2 \cos^2 \varphi + a_y^2 \sin^2 \varphi}}.
\]

(12)

4. The cross section of the process \( \mu^- \mu^+ \rightarrow e\bar{\nu}_e W^+ \).— Now we consider a special example, taking \( X = W^+ \). The cross section of the reaction \( \nu_\mu \mu^+ \rightarrow W^+ \) is equal to

\[
\sigma(\nu_\mu \mu^+ \rightarrow W^+) = 12\pi^2 \frac{\Gamma(W \rightarrow \mu\nu)}{M} \delta(xs - M^2)
\]

(13)

where \( \Gamma(W \rightarrow \mu\nu) = 0.22 \text{ GeV} \) is the partial W decay width and \( M = 80.2 \text{ GeV} \) is the W boson mass. The integration over the fraction of the neutrino energy \( x \) becomes trivial and finally we obtain:

\[
\sigma(\mu^- \mu^+ \rightarrow e\bar{\nu}_e W^+) = \sigma_0 \frac{\pi a}{2cT} x_0 f(x_0), \quad x_0 = \frac{M^2}{s},
\]

\[
\sigma_0 = \frac{12\pi^2}{M^2} \frac{\Gamma(W \rightarrow \mu\nu)}{M} = 19.7 \text{nb}.
\]

For numerical estimates we take \( a = \sqrt{\pi} \sigma_\perp \) (which corresponds to the case of the round identical Gaussian beams) with \( \sigma_\perp = 10^{-3} \text{ cm} \) (see Ref. [7]).

This cross section reaches the maximum of 0.76 fb for \( \sqrt{s} = 93 \text{ GeV} \). For larger energies this cross section decreases as \( s^{-3/2} \).

First, let us compare this “non-standard” contribution with the “standard” one. By the “standard” contribution we mean the cross section of the same reaction calculated by the standard rules excluding the region of the final phase space where \( q^2 > -m^2 \). This contribution was calculated [8] using the CompHEP package [9]. The comparison of both contributions is shown in Fig. 2. It is seen that the non-standard piece dominates up to the energies \( \sqrt{s} \approx 105 \text{ GeV} \).

Second, we compare our “non-standard” cross section with the cross section for the single W boson production in the reaction \( \mu^- \mu^+ \rightarrow \mu^- \bar{\nu}_\mu W^+ \). The latter is completely standard process since it has no \( t \)-channel singularity. The reasonable estimate of its cross section can be quickly obtained with the help of the equivalent photon approximation. It gives the cross section \( \approx 1 \text{ fb} \) at \( \sqrt{s} \approx 95 \text{ GeV} \) (where it almost coincides with our non-standard cross section). At higher energies the process \( \mu^- \mu^+ \rightarrow \mu^- \bar{\nu}_\mu W^+ \) strongly dominates as compared with the discussed process \( \mu^- \mu^+ \rightarrow e\bar{\nu}_e W^+ \).

In summary, we have shown that the \( t \)-channel singularity in the physical region can be regularized by taking into account the finite sizes of the colliding beams. Our results, being applied to the reaction \( \mu^- \mu^+ \rightarrow e\bar{\nu}_e W^+ \), show that the non-standard contribution for this reaction dominates up to the energies \( \sim 105 \text{ GeV} \).
Figure 2: “Standard” (solid line) and “non-standard” (dashed line) contributions to the cross sections (fb) of the reactions $\mu^-\mu^+ \rightarrow e\bar{\nu}_eW^+$ as a function of the total energy $\sqrt{s}$, GeV. The standard contribution is evaluated with the cut $-q^2 > m^2$.

K.M. is grateful to Graduiertenkolleg “Teilchenphysik”, Universität Mainz for support and to E. Sherman for a number of fruitful conversations. V.G.S. acknowledges support of the Sächsisches Staatsministerium für Wissenschaft und Kunst and of the Russian Fund of Fundamental Research. We thank I.F. Ginzburg and G.L. Kotkin for valuable discussions. We are grateful to E.E. Boos and A.E. Pukhov for providing us with the results of the CompHEP calculations.

References

[1] R.F. Peirles, Phys.Rev.Lett. 6, 641 (1961)

[2] I.F. Ginzburg, Preprint DESY 95-168 (1995)

[3] G.L. Kotkin, V.G. Serbo, A.Schiller, International Journ. Modern Physics A7, 4707 (1992)

[4] A.I. Blinov et al, Phys.Lett. B113, 423 (1982)

[5] K. Piotrzkowski, Preprint DESY 95-051 (1995)

[6] K. Melnikov, V.G. Serbo, in preparation.
[7] D. Cline, Nucl. Instrum. Methods 350, 24 (1995); R.B. Palmer, Beam Dynamics Newsletters, 8(1995).

[8] E.E. Boos and A.E. Pukhov, private communication.

[9] E.E. Boos et al. Preprint SNUTP-94-116, hep-ph/9503280