On a Ricci Quarter-Symmetric Metric Recurrent Connection and a Projective Ricci Quarter-Symmetric Metric Recurrent Connection in a Riemannian Manifold

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Abstract. Two new types of connections, Ricci quarter-symmetric metric recurrent connection and projective Ricci quarter-symmetric metric recurrent connection, were introduced and some interesting geometrical and physical characteristics were achieved.

1. Introduction

The concept of the semi-symmetric connection was introduced by Friedman and Schouten in [6] for the first time, Hayden in [11] introduced the metric connection with torsion, and Yano in [21] defined a semi-symmetric metric connection and studied its geometric properties. N. Agache and M. Chafle [1] investigated the semi-symmetric non-metric connection. Recently, De, Han and Zhao in [2] studied the semi-symmetric non-metric connection. On the other hand, the Schur’s theorem of a semi-symmetric non-metric connection is well known ([12, 13]) based only on the second Bianchi identity. A semi-symmetric metric connection that is a geometrical model for scalar-tensor theories of gravitation was studied ([3]) and a conjugate symmetry condition of the Amari-Chentsov connection with metric recurrent was also studied. Recently in [9] the similar topics were further studied in sub-Riemannian manifolds. A quarter-symmetric connection in [8] was defined and studied. Afterwards, several types of a quarter-symmetric metric connection were studied ([4, 10, 19, 22]). In [7, 14, 20, 23, 24], the geometric and physic properties of conformal and projective the semi-symmetric metric recurrent connections were studied. And in [17, 18] a projective conformal quarter-symmetric metric connection and a generalized quarter-symmetric metric recurrent connection were studied. In [5] a curvature copy problem of the symmetric connection was studied. And in [18] the mutual connection of a semi-symmetric connection was studied.

Motivated by the previous researches we define newly in this note the Ricci quarter-symmetric metric recurrent connection and the projective Ricci quarter-symmetric metric recurrent connection and study

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their properties. And the Schur’s theorem of the Ricci quarter-symmetric metric recurrent connection and the projective Ricci quarter-symmetric metric recurrent connection and several types of these connections with constant curvature are discovered.

2. A Ricci Quarter Symmetric Metric Recurrent Connection

Let \((M, g)\) be a Riemannian manifold \((\text{dim} M \geq 2)\), \(g\) be the Riemannian metric on \(M\), and \(\nabla\) be the Levi-Civita connection with respect to \(g\). Let \(X(M)\) denote the collection of all vector fields on \(M\).

**Definition 2.1.** A connection \(\nabla\) is called a Ricci quarter-symmetric metric recurrent connection, if it satisfies

\[
\nabla_Z g(X, Y) = 2\omega(Z)g(X, Y), T(X, Y) = \pi(Y)UX - \pi(X)UY
\]

where \(U\) is a Ricci operator, \(\omega\) and \(\pi\) are 1-form respectively. If \(U(X) = X\), then \(\nabla\) is a semi-symmetric metric recurrent connection studied in [24].

Let \((x^i)\) be the local coordinate, then \(g, \nabla, \omega, \pi, U\) and \(T\) have the local expressions \(g_{ij}, \Gamma^k_{ij}, \Gamma^k_{ij}, \omega_i, \pi_i, U^i_j\) and \(T^k_{ij}\) respectively. At the same time the expression (1) can be rewritten as

\[
\nabla_k g_{ij} = 2\omega_k g_{ij}, T^k_{ij} = \pi_i U^k_j - \pi_j U^k_i
\]

The coefficient of \(\nabla\) is given as

\[
\Gamma^{k}_{ij} = \{^k_i_j \} - \omega_j \delta^k_i - \omega_i \delta^k_j + g_{ij} \omega^k + \pi_i U^k_j - U^k_i \pi^k
\]

where \(U_{ij}\) is a Ricci tensor of the Levi-Civita connection \(\nabla\). From (3), the curvature tensor of \(\nabla\), by a direct computation, is

\[
\nabla^j_\ell \Gamma^i_k = \nabla^j_\ell K^i_{jk} + \delta^j_\ell \nabla^i_k - \delta^j_\ell \nabla^i_k + g_{jk} \nabla^i_\ell - g_{ik} \nabla^j_\ell + U^k_{ij} b^k_j - U^k_{jk} b^k_i
\]

\[
+ U_{ik} b^l_j - U_{jk} b^l_i + c^i_l \pi_k - c^i_k \pi_l + c^j_l \pi_k - c^j_k \pi_l - \delta^j_\ell (\omega_{ij} - \omega_{ji})
\]

where \(K^i_{jk}\) is the curvature tensor of the Levi-Civita connection \(\nabla\) and other notations are given as

\[
a^k_{ij} = \nabla_{i} \omega_{jk} + \omega_{i} \omega_{jk} + U^\rho_{jk} \omega_{\rho} \pi_{\kappa} - U^\rho_{ij} \omega_{\rho} \pi_{\kappa} - \frac{1}{2} g_{k\rho} \omega_{\rho} \omega_{\kappa}
\]

\[
b^k_{ij} = \nabla_{i} \pi_{jk} + \pi_{i} \omega_{jk} - U^\rho_{jk} \pi_{\rho} \pi_{\kappa} - \frac{1}{2} U_{ik} \pi_{\rho} \pi_{\rho}
\]

\[
c^k_{ij} = \nabla_{i} U_{jk}
\]

\[
\omega_{ij} = \nabla_{i} \omega_{j}
\]

Let

\[
A^i_{jk} = \delta^i_\ell \nabla^j_\ell + a^i_{jk} g_{jk} - U^l_{ij} b^l_k - b^l_{ik} U^l_j + c^l_j \pi_k - c^l_k \pi_j - \delta^i_\ell \omega_{ij}
\]

Then, we get

\[
K^i_{jk} = A^i_{jk} + A^i_{jk} - A^i_{jk}
\]

So there exists the following.

**Theorem 2.2.** When \(A^i_{jk} = A^i_{jk}\), then the curvature tensor will keep unchanged under the connection transformation \(\nabla \rightarrow \nabla\).
From (3), the coefficient of dual connection $\hat{\nabla}$ of the Ricci quarter-symmetric metric recurrent connection $\nabla$ is

$$\hat{\Gamma}_{ij}^k = \frac{1}{n} \hat{\omega}^k_{ji} + \hat{\omega}^k_{ij} + g_{ij} \hat{\omega}^k - \pi_i \hat{\omega}^k = \frac{1}{n} \hat{\omega}^k_{ji} + \hat{\omega}^k_{ij} + g_{ij} \hat{\omega}^k - \pi_i \hat{\omega}^k$$

(6)

By using the expression (6), the curvature tensor of dual connection $\hat{\nabla}$ is

$$\hat{R}_{ijk}^l = R_{ijk}^l + \hat{\delta}^l_{jk} - \hat{\delta}^l_{ik} + g_{ik} \hat{\delta}^l_{jk} - g_{jk} \hat{\delta}^l_{ik} + \hat{\Omega}^l_{ijk} - \hat{\Omega}^l_{jik}$$

$$+ \hat{U}^l_{ijk} - \hat{U}^l_{jik} + c^l_{ij} \pi_k - c^l_{jk} \pi_i + c^l_{ki} \pi_j + \hat{\delta}^l_{ij} (\omega_{ij} - \omega_{ji})$$

(7)

In the Riemannian manifold $(M, g)$ if $R_{ijk}^l = \hat{R}_{ijk}^l$, then the connection $\nabla$ is called a conjugate symmetry and if $R_{ijk} = \hat{R}_{ijk}$, then the connection $\nabla$ is called a conjugate Ricci symmetry, and if $P_{ij} = \hat{P}_{ij}$, then the connection $\nabla$ is called a conjugate quasi-Ricci (or Volume) symmetry, where $P_{ij} = g^{lk} R_{ijkl}$.

**Theorem 2.3.** In a Riemannian manifold $(M, g)$ with a Ricci quarter-symmetric metric recurrent connection $\nabla$ if a 1-form $\omega$ is a closed form, then the Riemannian manifold $(M, g, \nabla)$ is a quasi-Ricci flat and the Ricci quarter-symmetric metric recurrent connection is a conjugate symmetric.

*Proof.* By using the contraction of the indices $k$ and $l$ in the (4) we have

$$P_{ji} = \hat{P}_{ji} - n(\omega_{ji} - \omega_{ij})$$

where $\hat{P}_{ij} = \hat{K}_{ijk}^k = 0$. If a 1-form $\omega$ is a closed form, then $\omega_{ij} = \omega_{ji}$. Hence $P_{ji} = 0$. Consequently the Riemannian manifold $(M, g, \nabla)$ is a quasi-Ricci flat. On the other hand, from the expressions (4) and (7), we obtain

$$\hat{R}_{ijk}^l = R_{ijk}^l + 2 \hat{\delta}^l_{jk} (\omega_{ij} - \omega_{ji})$$

(8)

If a 1-form $\omega$ is a closed form, then $\omega_{ij} = \omega_{ji}$. Hence from the expression (8), we have $\hat{R}_{ijk}^l = R_{ijk}^l$. Consequently, the Ricci quarter-symmetric metric recurrent connection $\nabla$ is a conjugate symmetry. □

**Theorem 2.4.** The Ricci quarter-symmetric metric recurrent connection $\nabla$ on a Riemannian manifold $(M, g)$ is a conjugate symmetry if and only if it is a Ricci symmetry or a conjugate volume symmetry.

*Proof.* By using the contraction of the indices $i$ and $l$ in (8) we have

$$\hat{R}_{jk} = R_{jk} - 2 (\omega_{jk} - \omega_{kj}).$$

From this expression, we arrive at

$$\omega_{jk} - \omega_{kj} = \frac{1}{2} (R_{jk} - \hat{R}_{jk}).$$

Substituting this expression into (8), we have

$$\hat{R}_{ijk}^l + \hat{\delta}^l_{jk} R_{ij} = R_{ijk}^l + \hat{\delta}^l_{jk} R_{ij}$$

(9)

From the equation (9) it is easy to show that $R_{ijk}^l = \hat{R}_{ijk}^l$ if and only if $R_{jk} = \hat{R}_{jk}$. On the other hand, by using the contraction of the indices $k$ and $l$ in (8), we have

$$\hat{P}_{ij} = P_{ij} + 2 n (\omega_{ij} - \omega_{ji})$$

From this expression, we arrive at

$$\omega_{jk} - \omega_{kj} = \frac{1}{2n} (R_{jk} - \hat{R}_{jk}).$$
Similarly, the formula (3) shows
\[ R_{ijlk} - \frac{1}{n} \delta^p_l R_{ijkl} = R_{ijlk} - \frac{1}{n} \delta^p_l R_{ijkl} \] (10)
From the equation (10), it is easy to show that \( R_{ijlk} = \hat{R}_{ijlk} \) if and only if \( P_{ij} = \hat{P}_{ij} \). \( \square \)

It is well known that a sectional curvature at a point \( p \) in a Riemannian manifold is independent of \( \Pi \) (a 2-dimensional subspace of \( T_p(M) \)), the curvature tensor is
\[ R_{iklj} = k(p)(\delta^i_l g_{jk} - \delta^i_j g_{lk}) \] (11)
In this case, if \( k(p) = \text{const} \), then the Riemannian manifold is a constant curvature manifold.

**Theorem 2.5.** Suppose that \( (M,g)(\text{dim} M \geq 3) \) is a connected Riemannian manifold associated with an isotropic Ricci quarter-symmetric metric recurrent connection \( \nabla \). If there holds
\[ \omega_n = -s_h \] (12)
then \( (M,g,\nabla) \) is a constant curvature manifold, where \( s_h = \frac{1}{n-1} T^p_{hp} \) (Schur’s theorem for the Ricci quarter-symmetric metric recurrent connection)

**Proof.** Substituting the expression (11) and using the expression (2) into the second Bianchi identity of the curvature tensor of the Ricci quarter-symmetric metric recurrent connection \( \nabla \), we get
\[ \nabla_i R_{jk} + \nabla_j R_{hi} + \nabla_h R_{ij} = T^m_{ij} R_{jk} + T^m_{ij} R_{hi} + T^m_{ij} R_{hk} \]
then we have
\[ \begin{align*}
(V_h k(p) + 2 \omega_h k(p))(\delta^i_l g_{jk} - \delta^i_j g_{lk}) + (V_j k(p) + 2 \omega_j k(p))(\delta^i_l g_{hk} - \delta^i_h g_{jk}) \\
+ (V_i k(p) + 2 \omega_i k(p))(\delta^i_l g_{hk} - \delta^i_h g_{jk})
\end{align*} 
= k(p)[(\delta^i_l U_{jk} - \delta^i_j U_{lk}) + \pi_i(\delta^i_l g_{hk} - \delta^i_h g_{jk})] \\
+ \pi_i(\delta^i_l U_{hk} - \delta^i_h U_{jk})]
\]
Contracting the indices \( i \) and \( l \), then we obtain
\[ \begin{align*}
(n - 2)(V_h k(p) + 2 \omega_h k(p))g_{jk} - (n - 2)(V_j k(p) + 2 \omega_j k(p))g_{hk}
\end{align*} 
= k(p)[(n - 3)(\pi_h U_{jk} - \pi_j U_{hk}) + \pi_i(\delta^i_l g_{hk} - \delta^i_h g_{jk})] \\
+ \pi_i(\delta^i_l U_{hk} - \delta^i_h U_{jk})]
\]
Multiplying both sides of this expression by \( g^{jk} \), then we have
\[ \begin{align*}
(n - 1)(n - 2)(V_h k(p) + 2 \omega_h k(p)) = 2(n - 2)k(p)(\pi_i U^p_{jk} - \pi_i U^p_{hk})
\end{align*} 
\]
From this equation above we obtain
\[ V_h k(p) = -2(\omega_h + s_h)k(p). \]
Consequently, from that we know \( k(p) = \text{const} \) if and only if \( \omega_n = -s_h \). \( \square \)

By Theorem 2.5, the expression (2) for the Ricci quarter-symmetric metric connection with a constant curvature satisfies
\[ \nabla_i g_{jk} = -2s_h g_{jk}, T^k_{ij} = \pi_i U^k_{ij} - \pi_j U^k_{ij} \] (13)
Similarly, the formula (3) shows
\[ \Gamma^k_{ij} = [ij]^k + s_i \delta^j_k + s_j \delta^i_k - g_{ij}s^k + \pi_i U^k_{ij} - U^k_{ij}\pi^k \] (14)
If the Riemannian manifold is an Einstein manifold, then we obtain
\[ U_{jk} = \frac{k}{n} g_{jk} \]  
(15)

From the expression (15), we have
\[ s_k = -\frac{k}{n} \pi_k. \]

Hence, for an Einstein manifold, the expression (13) shows
\[ \nabla_k 1_{ij} = 2 \frac{k}{n} \pi_k 1_{ij} \]
\[ T_k 1_{ij} = \frac{k}{n} (\pi_j 1^k - \pi_i 1^j) \]  
(16)

Similarly, the formula (14) shows
\[ \Gamma_k 1_{ij} = \{ k 1_{ij} \} - \frac{k}{n} \pi_k 1^j \]  
(17)

This connection was studied in [3].

From the expression (3), the coefficient of mutual connection of the Ricci quarter-symmetric metric recurrent connection is
\[ m^k 1_{ij} = \{ k 1_{ij} \} - \omega_i 1^k - \omega_j 1^k + \frac{1}{k} 1_{ij} \omega_k - \pi_i 1^k + \pi_j 1^k \]  
(18)

This connection satisfies the relation
\[ m^k 1_{ij} = 2 \omega 1_{ij} + 2 \pi_k 1_{ij} + \pi_k 1_{ij} + \pi_k 1_{ij} = \pi_i 1^k - \pi_j 1^k. \]  
(19)

From the expressions (18) and (19), the coefficient of dual connection of the mutual connection is
\[ \hat{m}^k 1_{ij} = \{ k 1_{ij} \} - \omega_i 1^k - \omega_j 1^k + \frac{1}{k} 1_{ij} \omega_k - \pi_i 1^k + \pi_j 1^k. \]  
(20)

On the other hand, in a Riemannian manifold the Weyl connection satisfies the relation
\[ \nabla_k 1_{ij} = 2 \omega 1_{ij} = 2 \pi_k 1_{ij} + \pi_k 1_{ij} = \pi_i 1^k - \pi_j 1^k. \]  
(21)

and the coefficient of \( \hat{V} \) is
\[ \hat{m}^k 1_{ij} = \{ k 1_{ij} \} - \omega_i 1^k - \omega_j 1^k + \frac{1}{k} 1_{ij} \omega_k. \]  
(22)

From the expressions (21) and (22), the coefficient of a dual connection of the Weyl connection is
\[ \hat{w}^k 1_{ij} = \{ k 1_{ij} \} + \omega_i 1^k - \omega_j 1^k + \frac{1}{k} 1_{ij} \omega_k. \]  
(23)

**Theorem 2.6.** In a Riemannian manifold \((M, g)\) the dual connection of the mutual connection of a Ricci quarter-symmetric metric recurrent connection is projective equivalent to dual connection of the Weyl connection.

**Proof.** From the expressions (20) and (23), we have
\[ \hat{m}^k 1_{ij} = \hat{w}^k 1_{ij}, \]
where \((ij)\) expresses the symmetry of the indices. Hence the connection \( \hat{V} \) has the same geodesic as \( \hat{V} \). Thus the connection \( \hat{V} \) is projective equivalent to the connection \( \hat{V} \). \(\square\)
3. A Projective Ricci Quarter-Symmetric Metric Recurrent Connection

**Definition 3.1.** In a Riemannian manifold \((M,g)\), a connection \(\tilde{\nabla}\) is called a projective Ricci quarter-symmetric metric recurrent connection, if the \(\tilde{\nabla}\) is projective equivalent to a Ricci quarter-symmetric metric recurrent connection \(\nabla\).

In a Riemannian manifold \((M,g)\), a projective Ricci quarter-symmetric metric recurrent connection \(\tilde{\nabla}\) satisfies the relation

\[
\tilde{\nabla}_Z g(X,Y) = -2[\Psi(Z) - \omega(Z)]g(X,Y) - \Psi(Y)g(Y,Z) - \Psi(Y)g(X,Z),
\]

\[
\tilde{\nabla}^p T(X,Y) = \pi(Y)UY - \pi(X)UY.
\]

The local expression of this relation is

\[
\begin{align*}
\tilde{\nabla}_k g_{ij} &= -2(\Psi_k - \omega_k)g_{ij} - \Psi_j g_{ik} - \Psi_i g_{jk}, \\
\tilde{\nabla}^p \Gamma^k_{ij} &= \pi j U^k_i - \pi i U^k_j
\end{align*}
\]

and the coefficient of \(\tilde{\nabla}\) is

\[
\Gamma^k_{ij} = (\Psi_i - \omega_i)\delta^k_j + (\Psi_j - \omega_j)\delta^k_i + g_{ij}\omega^k + \pi_i U^k_j - U^k_i\pi^k_j.
\]

where \(\Psi_i\) is a projective component.

From (25), we find that the curvature tensor of \(\tilde{\nabla}\) is

\[
\tilde{R}^{\varphi i}_{jk} = K^{\varphi i}_{jk} + \delta^i_{jk} - \delta^i_{j} \delta^k_{j} - g_{jk} \omega_{i} + \pi_{ij} U_{jk}^{\varphi} - U_{jk}^{\varphi} \pi_{ij} - \delta^i_{j} \delta^k_{j} \omega_{i} + \delta^i_{j} \delta^k_{j} \pi_{ij} - \delta^i_{j} \delta^k_{j} \pi_{ij} - U_{jk}^{\varphi} \pi_{ij} + \frac{1}{2} U_{jk}^{\varphi} \pi_{ij} - \delta^i_{j} \delta^k_{j} \omega_{i} + \delta^i_{j} \delta^k_{j} \pi_{ij} - \delta^i_{j} \delta^k_{j} \pi_{ij} - U_{jk}^{\varphi} \pi_{ij} + \frac{1}{2} U_{jk}^{\varphi} \pi_{ij}
\]

where \(K^{\varphi i}_{jk}\) is the curvature tensor of the Levi-Civita connection \(\tilde{\nabla}\), and the other notations are given as

\[
\begin{align*}
\tilde{a}^{\varphi}_{jk} &= \tilde{\nabla}_{k}(\Psi_{j} - \omega_{j}) - (\Psi_{j} - \omega_{j})(\Psi_{k} - \omega_{k}) + U_{jk}^{\varphi}(\Psi_{p} - \omega_{p})\pi^{p} - U_{jk}^{\varphi} \pi^{p} + \frac{1}{2} U_{jk}^{\varphi} \pi^{p} + \frac{1}{2} U_{jk}^{\varphi} \pi^{p} \\
\tilde{b}^{\varphi}_{jk} &= \tilde{\nabla}_{k}(\Psi_{j} - \omega_{j}) + \omega_{j}\omega_{k} + U_{jk}^{\varphi} \pi^{p} - U_{jk}^{\varphi} \omega_{p} \pi^{p} + \frac{1}{2} U_{jk}^{\varphi} \pi^{p} + \frac{1}{2} U_{jk}^{\varphi} \pi^{p} \\
\tilde{c}^{\varphi}_{jk} &= \tilde{\nabla}_{k} \pi_{j} - \pi_{j}(\Psi_{k} - \omega_{k}) - U_{jk}^{\varphi} \pi^{p} + \frac{1}{2} U_{jk}^{\varphi} \pi^{p} + \frac{1}{2} U_{jk}^{\varphi} \pi^{p} \\
\tilde{\psi}_{ij} &= \tilde{\nabla}_{j} \omega_{i} + \omega_{j}\omega_{k} + U_{jk}^{\varphi} \pi^{p} - U_{jk}^{\varphi} \omega_{p} \pi^{p} + \frac{1}{2} U_{jk}^{\varphi} \pi^{p} + \frac{1}{2} U_{jk}^{\varphi} \pi^{p} \\
\tilde{\mu}_{ij} &= \tilde{\nabla}_{j} \pi_{i} - \pi_{i}(\Psi_{j} - \omega_{j}) - U_{jk}^{\varphi} \pi^{p} + \frac{1}{2} U_{jk}^{\varphi} \pi^{p} + \frac{1}{2} U_{jk}^{\varphi} \pi^{p}
\end{align*}
\]

Let

\[
B^{\varphi i}_{jk} = \delta^i_{jk} - g_{jk} \omega_{i} + \pi_{ij} U_{jk}^{\varphi} - U_{jk}^{\varphi} \pi_{ij} - \delta^i_{jk} \omega_{i} + \delta^i_{jk} \pi_{ij} - \delta^i_{jk} \pi_{ij} - U_{jk}^{\varphi} \pi_{ij} + \frac{1}{2} U_{jk}^{\varphi} \pi_{ij}
\]

Then we get

\[
\tilde{R}^{\varphi i}_{jk} = K^{\varphi i}_{jk} + B^{\varphi i}_{jk}.
\]

So there exists the following.
Theorem 3.2. When \( B_{jk}^l = B_{jk}^l \), then the curvature tensor will keep unchanged under the connection transformation \( \nabla \rightarrow \nabla' \).

From (25) and (26), the coefficient of dual connection \( \nabla \) of the projective Ricci quarter-symmetric metric recurrent connection \( \nabla' \) is

\[
\hat{\nabla}^k_{ij} = \{\hat{\nabla}^k_{ij} - (\Psi_l - \omega_l)\hat{\nabla}^k_{ij} - (\Psi^k - \omega^k)g_{ij} - \omega_j \delta^k_i + \pi_j\dot{\Gamma}_ij - U_{ij}^k - U_{ij}^kT^k \}.
\]

By using the expression (28), the curvature tensor of dual connection \( \nabla' \) is

\[
\hat{\nabla}_{jk}^l = \hat{\nabla}_{jk}^l + \delta^p_k\hat{\nabla}_{jk}^p - \delta^p_k\hat{\nabla}_{jk}^p + g_{jk}\dot{\Gamma}_ik + U_{jk}^l - U_{jk}^l\dot{\Gamma}_ik
\]

From the expressions (26) and (29), we have

\[
\hat{\Gamma}_{ijk} = \hat{\Gamma}_{ijk} + \delta^p_i\hat{\Gamma}_{jk}^p + g_{jk}\delta^p_i + U_{jk}^l - U_{jk}^l\delta^p_i
\]

Let

\[
D_{ijk}^l = \delta^p_i\hat{\Gamma}_{jk}^p + g_{jk}\delta^p_i + U_{jk}^l - U_{jk}^l\delta^p_i + U_{jk}^l - U_{jk}^l\delta^p_i + U_{jk}^l - U_{jk}^l\delta^p_i + 2\delta^l_i(\Psi_{ij} - \Psi_{ji})
\]

Then we get

\[
\hat{\nabla}_{jk}^l = \hat{\nabla}_{jk}^l + D_{ijk}^l - D_{ijk}^l.
\]

So there exists the following.

Theorem 3.3. In the Riemannian manifold \( (M, g, \nabla) \), if 1-form \( \Psi \) and \( \omega \) are of closed forms, then the Riemannian manifold is a quasi-Ricci(or volume) flat and if \( D_{jk}^l = D_{jk}^l \), then the projective Ricci quarter-symmetric recurrent connection \( \nabla' \) is a conjugate symmetry.

Proof. By using the contraction of the indices \( k \) and \( l \) in the expression (26) we have

\[
\hat{\nabla}_{ij}^k = \hat{\nabla}_{ij}^k + \hat{\nabla}_{ij}^k - \hat{\nabla}_{ij}^k - \hat{\nabla}_{ij}^k + U_{ij}^l - U_{ij}^l\delta^k_i + U_{ij}^l - U_{ij}^l\delta^k_i
\]

where \( P_{ij} = R_{ijkl}g^{kl}, \hat{P}_{ij} = K_{ijkl}g^{kl} = 0 \), and \( \epsilon_{ij}^k\pi_k = c_{ij}^k\pi^k \).
Using the expression (28), there holds the following
\[
\begin{align*}
\vec{a}_{ij} - a_{ij} &= (\Psi_{ij} - \Psi_{ji}) - (\omega_{ij} - \omega_{ji}) - U^p_i (\Psi_p - \omega_p) \pi_j + U^p_j (\Psi_p - \omega_p) \pi_i, \\
\vec{b}_{ij} - b_{ij} &= \omega_{ij} - \omega_{ji} - U^p_i \omega_p \pi_j + U^p_j \omega_p \pi_i, \\
U^p_{ik} \tilde{\omega}^j_{jk} - U^p_j \tilde{\omega}^i_{jk} &= U^p_j \tilde{\omega}^i_{jk} - U^p_j \tilde{\omega}^i_{jk} - U^p_i (\Psi_k - \omega_k) \pi_j + U^p_i (\Psi_k - \omega_k) \pi_j, \\
U_{ik} d^k - U_{jk} d^k &= U_{ik} \tilde{\omega}^j_{jk} - U_{jk} \tilde{\omega}^i_{jk} + U_{ik} \omega^j \pi_j - U_{jk} \omega^j \pi_j, \\
e^i_{jk} \pi_k - e^j_{jk} \pi_k &= 0, \\
e^i_{jk} \pi_k - e^j_{jk} \pi_k &= 0.
\end{align*}
\]

Substituting these expressions into the expression (32) and using 1-form \(\Psi\) and \(\omega\) are of closed 1-forms, then \(\vec{P}_{ij} = 0\). Hence the Riemannian manifold \((M, g, \tilde{V})\) is a quasi-Ricci(or volume) flat.

On the other hand, from the expression (31) if \(D_{j}^p = D_{jk}^l\), then \(\tilde{R}_{jk}^l = \tilde{R}_{jk}^l\). Hence the projective Ricci quarter-symmetric recurrent connection is of conjugate symmetry. \(\square\)

**Theorem 3.4.** Suppose that \((M, g)(\dim M \geq 3)\) is a connected Riemannian manifold associated with an isotropic Ricci quarter-symmetric metric recurrent projective connection. If there holds
\[
\Psi_h = 2(\omega_h + s_h)
\]
then \((M, g, \tilde{V})\) is a constant curvature manifold, where \(s_h = \frac{1}{n-1} T^p_{hp} \) (the Schur’s theorem for the Ricci quarter-symmetric metric recurrent projective connection).

**Proof.** Substituting the expression (11) into the second Bianchi identity of the curvature tensor of the projective Ricci quarter-symmetric metric recurrent connection, we get
\[
\begin{align*}
\vec{V}_{h} R_{jk}^l + \vec{V}_{i} R_{jhk}^l + V_{j} \tilde{R}_{hjk}^l &= T_{hi} R_{jk}^l + T_{ij} \tilde{R}_{hjk}^l + T_{jk} \tilde{R}_{hij}^l
\end{align*}
\]

then by using the expression (24) we have
\[
\begin{align*}
&\left[\vec{V}_{h} k(p) + (2\omega_h - \Psi_h) k(p)\right] \left[\delta^i_{j} g_{jk} - \delta^j_{i} g_{jk}\right] + \left[\vec{V}_{i} k(p) + (2\omega_i - \Psi_i) k(p)\right] \left[\delta^j_{i} g_{jk} - \delta^i_{j} g_{jk}\right] \\
&+ \left[\vec{V}_{h} k(p) + (2\omega_h - \Psi_h) k(p)\right] \left[\delta^j_{i} g_{jk} - \delta^i_{j} g_{jk}\right] \\
&= k(p) \left[\pi_h (\delta^i_{j} U_{jk} - \delta^j_{i} U_{jk}) + U^l_{i} g_{jk} - U^l_{j} g_{jk}\right] + \pi_i (\delta^j_{i} U_{jk} - \delta^i_{j} U_{jk} + U^l_{i} g_{jk} - U^l_{j} g_{jk}) \\
&+ \pi_j (\delta^i_{j} U_{jk} - \delta^j_{i} U_{jk} + U^l_{i} g_{jk} - U^l_{j} g_{jk})
\end{align*}
\]

Contracting the indices \(i\) and \(l\), we obtain
\[
\begin{align*}
(n-1) \left[\vec{V}_{h} k(p) + (2\omega_h - \Psi_h) k(p)\right] g_{jk} - (n-1) \left[\vec{V}_{i} k(p) + (2\omega_i - \Psi_i) k(p)\right] g_{jk} \\
+ \left[\vec{V}_{h} k(p) + (2\omega_h - \Psi_h) k(p)\right] g_{jk} - \left[\vec{V}_{h} k(p) + (2\omega_h - \Psi_h) k(p)\right] g_{jk} \\
= k(p) \left[\pi_h [(n-2) U_{jk} + g_{jk} U^l_{i}] - \pi_j [(n-2) U_{jk} + g_{jk} U^l_{i}] + \pi_k U_{jk} - \pi_k U_{jk}\right] \\
+ g_{jk} \pi_k \nabla_v g_{jk} - g_{jk} \pi_k \nabla^v g_{jk}
\end{align*}
\]

Multiplying both sides of this expression by \(g^k\), then we have
\[
(n-1)(n-2) \left[\vec{V}_{h} k(p) + (2\omega_h - \Psi_h) k(p)\right] = 2(n-2) k(p) \left[\pi_h U^v - \pi_k U^v\right]
\]
From this equation above we obtain
\[ \tilde{\nabla}_h k(p) = [\Psi_h - 2(\omega_h + s_h)]k(p) \]
Consequently from that we know \( k(p) = \text{const} \) if and only if \( \Psi_h = 2(\omega_h + s_h) \).

**Theorem 3.5.** If an Einstein manifold \((M, g)(\dim M \geq 3)\) associated with a projective Ricci quarter-symmetric metric recurrent connection \( \tilde{\nabla} \) has a constant curvature, then the Riemannian manifold \((M, g, \tilde{\nabla})\) is conformal flat.

**Proof.** Adding the expressions (26) and (29), we obtain
\[ \tilde{R}_{ijk} + R_{ijk} = 2K_{ijk} + \delta^l_j(a_{ik} - b_{ik}) - \delta^l_i(a_{jk} - b_{jk}) + g_{il}(a_{lj} - b_{lj}) \]
Substituting this expression into (36) and putting
\[ \alpha \]
Multiplying both sides of (37) by \( n \),
\[ \tilde{U}_{ik} = \frac{k}{n} g_{ik} \]
Using this expression, from (27) we obtain
\[ \tilde{\nabla}^p g_{ijk} = 0. \]
Using these expressions, from the expression (34), we have
\[ \tilde{R}_{ik} + R_{ik} = 2K_{ik} - (n - 2)a_{ik} - g_{ik} \alpha_i \]
where \( a_{ik} = a_{ik} - b_{ik} + \frac{1}{2} (c_{ik} + d_{ik}) \). Contracting the indices \( i \) and \( l \) of (36), we get
\[ \tilde{R}_{ik} + R_{ik} = 2K_{ik} - (n - 2)a_{ik} - g_{ik} \alpha_i \]
Multiplying both sides of (37) by \( g^{ik} \), then we arrive at
\[ \tilde{R} + R = 2K - 2(n - 1) \alpha_i \]
From this expression above we have
\[ \alpha_i = \frac{1}{2(n - 1)} [2K - \tilde{R} + R] \]
Using the expression from (37), we have
\[ \alpha_i = \frac{1}{n - 2} \left( \tilde{R}_{ik} + R_{ik} - \frac{1}{2(n - 1)} g_{ik} [2K - (\tilde{R} + R)] \right) \]
Substituting this expression into (36) and putting
\[ C_{ijk} = R_{ijk} - \frac{1}{n - 2} \left( \delta^l_j R_{ik} - \delta^l_i R_{jk} + g_{il} R_{kj} - g_{lk} R_{ij} \right) + \frac{\tilde{R}}{(n - 1)(n - 2)} (\delta^l_i g_{jk} - \delta^l_j g_{ik}) \]
Using this expression, from (40) we have

\[
\bar{C}_{ijk} = R_{ijk} - \frac{1}{n-2} \left( \delta^i_j \bar{R}_k - \delta^i_k \bar{R}_j + \delta^i_l \bar{R}_j + g_{ik} \bar{R}_j - g_{jk} \bar{R}_i \right) + \frac{\bar{R}}{(n-1)(n-2)} (\delta^i_j g_{lk} - \delta^i_l g_{jk})
\]

which means that the Riemannian manifold \((\mathcal{M}, g, \bar{V})\) is of conformal flat.

\[
\bar{C}_{ijk} = K_{ijk} - \frac{1}{n-2} \left( \delta^i_j K_{lk} - \delta^i_k K_{lj} + \delta^i_l K_{ij} + g_{ik} K_{lj} - g_{jk} K_{li} \right) + \frac{K}{(n-1)(n-2)} (\delta^i_j g_{lk} - \delta^i_l g_{jk})
\]

then by a direct computation, we obtain

\[
\bar{P}^i_{jkl} = \bar{C}_{jkl} + \bar{C}_{ikl} = 2\bar{C}_{ijk}
\] (38)

By using the fact that \(\bar{V}\) has a constant curvature, thus we have \(\bar{C}_{ijk} = \bar{C}_{jik} = 0\). Hence, one gets

\[
\bar{C}_{ijk} = 0.
\]

This means that the Riemannian manifold \((\mathcal{M}, g, \bar{V})\) is of conformal flat.

**Theorem 3.6.** The projective Ricci quarter-symmetric metric recurrent connection \(\bar{\nabla}\) on an Einstein manifold \((\mathcal{M}, g, \bar{V})(\dim\mathcal{M} \geq 3)\) is a conjugate symmetry if and only if it is a conjugate Ricci symmetry and conjugate volume symmetry.

**Proof.** From (26) and (29), we get

\[
\bar{R}_{ijk} = R_{ijk} + \delta^i_k \beta_{jk} - \delta^i_j \beta_{ik} + g_{ik} \beta_j - g_{jk} \beta_i + 2\delta^i_k \gamma_{ij}
\]

(39)

where \(\beta_{jk} = a_{jk} + b_{jk} + \frac{\bar{R}}{n} (c_{jk} + \bar{\delta}_{jk})\), \(\gamma_{ij} = (\omega_{ij} - \omega_{ji}) - (\Psi_{ij} - \Psi_{ji})\). By using contraction of indices \(i\) and \(l\) of (39), we obtain

\[
\bar{P}_{jk} = R_{jk} + n\beta_{jk} - g_{jk} \beta_i + 2\gamma_{jk}.
\]

(40)

Alternating the indices \(k\) and \(j\) of this expression, we obtain

\[
\bar{P}_{jk} - \bar{P}_{kj} = \bar{R}_{jk} - \bar{R}_{kj} + n(\beta_{jk} - \beta_{kj}) - 2\gamma_{jk}
\]

On one hand, contracting the indices \(k\) and \(l\) of (39) and changing index \(i\) for \(j\), index \(j\) for \(k\), we get

\[
\bar{P}_{jk} = \bar{P}_{kj} + 2(\beta_{jk} - \beta_{kj}) - 2\gamma_{jk}
\]

From these expressions above we have

\[
\gamma_{jk} = \frac{1}{2(n^2 - 4)} \left[ 2 \left( \bar{P}_{jk} - \bar{P}_{kj} \right) - (\bar{R}_{jk} - \bar{R}_{kj}) + n(\bar{R}_{jk} - \bar{R}_{kj}) \right]
\]

Using this expression, from (40) we have

\[
\beta_{jk} = \frac{1}{n} \left( \bar{R}_{jk} - \bar{R}_{kj} + g_{jk} \beta_i + \frac{1}{n^2 - 4} \left[ 2 \left( \bar{P}_{jk} - \bar{P}_{kj} \right) - (\bar{R}_{jk} - \bar{R}_{kj}) + n(\bar{R}_{jk} - \bar{R}_{kj}) \right] \right)
\]
Substituting the above two expressions into (39), we obtain
\[
\begin{align*}
R_{ijk}^l - &\frac{1}{n}(\delta_i^l R_{jk}^p - \delta_j^l R_{ik}^p + g_{ik} R_{jp}^l - g_{jk} R_{ip}^l) - \frac{2}{n(n^2 - 4)}[\delta_i^l (R_{jk}^p - R_{kj}^p) \\
- &\delta_j^l (R_{ik}^p - R_{ki}^p) + g_{ik} (R_{jp}^l - R_{pj}^l) - g_{jk} (R_{ip}^l - R_{pi}^l) + n\delta_i^l (R_{kj}^p - R_{jk}^p) + n\delta_j^l (R_{ki}^p - R_{ik}^p) \\
- &\frac{1}{n^2 - 4}(\delta_i^l p_{jk}^p - \delta_j^l p_{ik}^p + g_{ik} p_{jl}^l - g_{jk} p_{il}^l + n\delta_i^l p_{kj}^p + n\delta_j^l p_{ki}^p)
\end{align*}
\]

From this expression we arrive at \( R_{ijk}^l = \tilde{R}_{ijk}^l \) if and only if \( R_{jk}^p = \tilde{R}_{jk}^p, P_{jk} = \tilde{P}_{jk} \). Where \( \tilde{R}_{ij}^l = R_{jil} g^i, \tilde{R}_{ij}^l = R_{jil}^l g^i \). This ends the proof of Theorem 3.6.

From the expression (25), the coefficient of mutual connection \( \tilde{\nabla}^p \) of the projective Ricci quarter-symmetric metric recurrent connection \( \nabla^p \) is
\[
\Gamma_{ij}^k = \frac{1}{n} (t_{ij}^k) - (\Psi_l - \omega_i) \delta_j^k + (\Psi_j - \omega_j) \delta_i^k + g_{ij} \omega^k + \pi_i \omega_U^k - U_{ij} \pi^k.
\] (41)

This connection satisfies the relation
\[
\begin{align*}
\nabla^m_{k} g_{ij} &= -2(\Psi_k - \omega_k) g_{ij} - \Psi_j g_{ik} - \Psi_i g_{jk} - 2\pi_k U_{ij} + U_{ik} \pi_j + U_{ij} \pi_k \\
T_{ij}^k &= \pi_i \omega_U^k - \pi_j \omega_U^k
\end{align*}
\] (42) (43)

From the expressions (41) and (42), the coefficient of dual connection \( \tilde{\nabla}^m \) of the mutual connection \( \nabla^m \) is
\[
\tilde{\Gamma}_{ij}^k = \frac{1}{n} (t_{ij}^k) - (\Psi_i - \omega_i) \delta_j^k + (\Psi_j - \omega_j) \delta_i^k - (\Psi^k - \omega^k) g_{ij} - \omega_j \delta_i^k - \pi_i \omega_U^k + U_{ij} \pi^k.
\] (44)

On the other hand, the coefficient of a dual connection \( \tilde{\nabla}^w \) of the Weyl projective connection \( \nabla^w \) is given as
\[
\tilde{\Gamma}_{ij}^k = \frac{1}{n} (t_{ij}^k) - (\Psi_i - \omega_i) \delta_j^k - (\Psi^k - \omega^k) g_{ij} - \omega_j \delta_i^k.
\] (45)

**Theorem 3.7.** In a Riemannian manifold the dual connection \( \tilde{\nabla}^m \) of the mutual connection \( \nabla^m \) of the projective Ricci quarter-symmetric metric recurrent connection \( \nabla^p \) is projective equivalent to dual connection \( \tilde{\nabla}^w \) of the Weyl projective connection \( \nabla^w \).

**Proof.** From the expressions (44) and (45), we have
\[
\tilde{\Gamma}_{ij}^k = \tilde{\Gamma}_{ij}^k
\]

Hence, the connection \( \tilde{\nabla}^m \) has the same geodesic as \( \tilde{\nabla}^w \). Thus the connection \( \tilde{\nabla}^m \) is projective equivalent to the connection \( \tilde{\nabla}^w \).
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