Bohr-Sommerfeld Lagrangian submanifolds
as minima of convex functions

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We prove more convexity properties for Lagrangian submanifolds in symplectic and Kähler manifolds. Namely, every closed Bohr-Sommerfeld Lagrangian submanifold $Q$ of a symplectic/Kähler manifold $X$ can be realised as a Morse-Bott minimum for some ‘convex’ exhausting function defined in the complement of a symplectic/complex hyperplane section $Y$. In the Kähler case, ‘convex’ means strictly plurisubharmonic while, in the symplectic case, it refers to the existence of a Liouville pseudogradient. In particular, $Q \subset X \setminus Y$ is a regular Lagrangian submanifold in the sense of Eliashberg-Ganatra-Lazarev.

1. Introduction

Rational convexity properties of Lagrangian submanifolds were first discovered in $\mathbb{C}^2$ by Duval and then investigated further by Duval-Sibony, Gayet and Guedj. In particular, generalising a result established by Duval-Sibony [DS95] in $\mathbb{C}^n$, Guedj [Gue99] obtained the following theorem: in a complex projective manifold $X$, every closed Lagrangian submanifold $Q$ is rationally convex, which means that $X \setminus Q$ is filled up with smooth complex hypersurfaces. More precisely, these complex hypersurfaces $Y$ are very ample divisors of arbitrarily large degrees, so their complements are affine manifolds and possess exhausting $\mathbb{C}$-convex functions $f : X \setminus Y \to \mathbb{R}$. In this work, which was motivated by the study of vanishing cycles in global Picard-Lefschetz theory, we give a necessary and sufficient condition for the existence of such a function $f$ admitting $Q$ as a Morse-Bott (i.e. transversally non-degenerate) minimum. This condition refers to a Kähler class and can be more generally stated as follows in the symplectic setting:

**Definition 1.** Let $(X, \omega)$ be an integral symplectic manifold, meaning that $X$ is a closed manifold and $\omega$ a symplectic form with integral periods. We say that a Lagrangian submanifold $Q$ satisfies the Bohr-Sommerfeld condition
— or simply is Bohr-Sommerfeld — if the homomorphism $H_2(X, Q, \mathbb{Z}) \to \mathbb{R}$ defined by integration of $\omega$ takes its values in $\mathbb{Z}$.

In the Kähler setting, our main result is:

**Theorem 2.** Let $(X, \omega)$ be a closed integral Kähler manifold and $Q$ a closed Lagrangian submanifold satisfying the Bohr-Sommerfeld condition. Then, for every sufficiently large integer $k$, there exist a complex hyperplane section $Y$ of degree $k$ in $X$ avoiding $Q$ and an exhausting $C$-convex function $f : X \setminus Y \to \mathbb{R}$ that has a Morse-Bott minimum at $Q$ and is Morse away from $Q$ with finitely many critical points.

To be more explicit, there exists a holomorphic line bundle $L \to X$ with first Chern class $\omega$ such that the complex hypersurface $Y$ is the zero-set of a holomorphic section of some large tensor power of $L$.

In [AGM01], Auroux-Gayet-Mohsen reproved Guedj’s above theorem and extended it to the symplectic setting using the ideas and techniques developed by Donaldson in [Don96]. Theorem 2 also has a symplectic version, whose statement below appeals to the following terminology:

- A *symplectic hyperplane section of degree $k$* in a closed integral symplectic manifold $(X, \omega)$ is a symplectic submanifold $Y$ of codimension 2 that is Poincaré dual to $k\omega$.

- A function $f : X \setminus Y \to \mathbb{R}$ is *$\omega$-convex* if it admits a pseudogradient that is a Liouville (i.e. $\omega$-dual to some primitive of $\omega$) vector field.

With this wording, Donaldson’s main theorem in [Don96] is that every closed integral symplectic manifold contains symplectic hyperplane sections of all sufficiently large degrees. Furthermore, according to Auroux-Gayet-Mohsen [AGM01], such symplectic hyperplane sections can be constructed away from any given closed Lagrangian submanifold. On the other hand, Giroux showed in [Gir18] that, for all sufficiently large degrees, the complements of Donaldson’s symplectic hyperplane sections admit exhausting $\omega$-convex functions (and hence are Weinstein manifolds). Mixing these ingredients, we obtain:

**Theorem 3.** Let $(X, \omega)$ be a closed integral symplectic manifold and $Q$ a closed Bohr-Sommerfeld Lagrangian submanifold of $X$. Then, for every sufficiently large integer $k$, there exist a symplectic hyperplane section $Y$ of degree $k$ in $X$ avoiding $Q$ and an exhausting $\omega$-convex function $f : X \setminus Y \to$
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R that has a Morse-Bott minimum at \( Q \) and is Morse away from \( Q \) with finitely many critical points.

In \[\text{EGL15}\], Eliashberg-Ganatra-Lazarev introduced the following definition: a Lagrangian submanifold \( Q \) in a Weinstein manifold \((W, \omega)\) is ‘regular’ if there exists a Liouville pseudogradient on \( W \) that is tangent to \( Q \) (or equivalently there exists a primitive of \( \omega \) vanishing on \( Q \)). This property, which implies that \( Q \) is an exact Lagrangian submanifold, is known for quite a long time to be a strong constraint. For instance, it is elementary to see (without any holomorphic curve theory) that a closed Lagrangian submanifold in \( \mathbb{C}^n \) cannot be regular. In the same time, though we do not have any example of a non-regular closed exact Lagrangian submanifold in a Weinstein manifold, we do not know any general method to prove that exact Lagrangian submanifolds should \textit{a priori} be regular. Theorems \( 2 \) and \( 3 \) show that, in the complement of the complex and symplectic hyperplane sections constructed, the Bohr-Sommerfeld Lagrangian submanifold \( Q \) is included in the zero-set of a Liouville pseudogradient and is therefore regular.

In Section \( 2 \) we explain why the Bohr-Sommerfeld condition is necessary for our purposes and describe some of properties of Bohr-Sommerfeld Lagrangians. In Section \( 3 \) we prove Theorem \( 3 \) applying the main technical result from \[\text{Gir18}\]. In Section \( 4 \) we prove Theorem \( 2 \) and a complex-geometric analogue, using techniques that go back to \[\text{DS95}\].

Acknowledgements. This work is part of my Ph.D. prepared at ÉNS de Lyon under the supervision of Emmanuel Giroux. I warmly thank him for his help and support and Jean-Paul Mohsen for his comments on a draft of this paper. This work was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program “Investissements d’Avenir” (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR), and by the UMI 3457 of CNRS-CRM.

2. Bohr-Sommerfeld Lagrangian submanifolds

Let us first remark that Cieliebak-Mohnke proved, in \[\text{CM17}, \text{Thm. 8.3}\], a version of the main theorem of \[\text{AGM01}\] that is specific to Bohr-Sommerfeld Lagrangian submanifolds.

The Bohr-Sommerfeld condition in Theorems \( 2 \) and \( 3 \) is necessary, indeed:
Lemma 4. Let \((X, \omega)\) be a closed symplectic manifold and \(Q\) a Lagrangian submanifold. Suppose that there exist a closed submanifold \(Y \subset X\) Poincaré-dual to \(\omega\) avoiding \(Q\) and \(\lambda\) a primitive of \(\omega\) over \(X \setminus Y\) such that \(\lambda|_Q\) is exact. Then \(Q\) is a Bohr-Sommerfeld Lagrangian submanifold of \((X, \omega)\).

Proof. It suffices to prove the following (well-known) claim: Let \(X\) be a closed connected oriented manifold, \(Y \subset X\) a closed codimension 2 submanifold and \(\omega\) a non-exact closed 2-form on \(X\) that is Poincaré-dual to \(Y\). Then, for every compact surface \(\Sigma \subset X\) with boundary disjoint from \(Y\) and primitive \(\lambda\) of \(\omega\) on \(X \setminus Y\) whose restriction to the submanifold \(Q\) is exact,

\[
\int_{\Sigma} \omega = \Sigma.Y. \tag{1}
\]

We first suppose that \(Y\) is connected. For any embedded 2-disc \(D\) intersecting \(Y\) transversely at one point, with sign \(\epsilon(D) = \pm 1\), set \(r := \epsilon(D)(\int_D \omega - \int_{\partial D} \lambda)\). The ‘residue’ \(r\) does not depend on the disc \(D\). To see this we will prove that, for two such discs \(D\) and \(D'\),

\[
\epsilon(D') \int_{\partial D'} \lambda - \epsilon(D) \int_{\partial D} \lambda = \int_C \omega = \epsilon(D') \int_{D'} \omega - \epsilon(D) \int_D \omega.
\]

Connectedness of \(Y\) gives an oriented cylinder \(C\) in \(X \setminus Y\) bounding \(-\epsilon(D')\partial D'\) and \(\epsilon(D)\partial D\). On the one hand, by Stokes theorem,

\[
\int_C \omega = \epsilon(D') \int_{\partial D'} \lambda - \epsilon(D) \int_{\partial D} \lambda.
\]

On the other hand, the capped cylinder \(C + \epsilon(D)D - \epsilon(D')D'\) is a boundary in \(X\) and \(\omega\) is closed so

\[
\int_C \omega + \epsilon(D) \int_D \omega - \epsilon(D') \int_{D'} \omega = 0.
\]

Finally, the ‘residue’ \(r\) is independent of \(D\).

Let \(\Sigma \subset X\) be a compact surface intersecting \(Y\) away from \(\partial \Sigma\). By a general position argument we may suppose the intersection is transverse. For each point \(p_i \in \Sigma \cap Y\), take a disc \(D_i \subset \Sigma\) that intersects \(Y\) only at \(p_i\). Stokes theorem gives \(\int_{\Sigma \cup D_i} \omega = -\sum_i \int_{\partial D_i} \lambda\), then:

\[
\int_{\Sigma} \omega = \Sigma.Y \ r. \tag{1}
\]
Since $\omega$ is not exact, we can apply (1) to some closed surface $\Sigma_0$ with
$\Sigma_0.Y = \int_{\Sigma_0} \omega \neq 0$. This gives $r = 1$; so (1) proves the claim.
Suppose $Y$ is not connected. If $\dim X \geq 3$, the cycle $[Y]$ may be repre-
sented by a closed connected submanifold, namely an embedded (away from
$\partial \Sigma_0$) connected sum of the connected components of $Y$. If $\dim X = 2$, we
may represent $[Y]$ by some integral multiple of any point. Consequently, we
reduce to the previous case. □

Meanwhile, the Bohr-Sommerfeld condition can be easily obtained after
a modification of the symplectic form:

Lemma 5 (Approximation and rescaling). Let $(X, \omega)$ be a closed sym-
plectic manifold and $Q$ a closed Lagrangian submanifold. Then there exists
a small closed 2-form $\epsilon$ and an integer $k$ such that $Q$ is a Bohr-Sommerfeld
Lagrangian submanifold of $(X, k(\omega + \epsilon))$.

Proof. We argue as in [AGM01]: the 2-form $\omega$ vanishes on $Q$ so, in view
of the exact sequence $\cdots \to H^2(X, \mathbb{Q}) \to H^2(X, \mathbb{R}) \to H^2(Q, \mathbb{R}) \to \cdots$, it is the image of a relative class $c \in H^2(X, Q; \mathbb{R})$. We approximate $c$ by
some $r \in H^2(X, Q; \mathbb{Q})$ and take a small closed form $\epsilon$ vanishing on $Q$ that
represents $c - r$. Then the closed form $\omega - \epsilon$ is symplectic, vanishes on $Q
$ and its relative periods — given by evaluation of $r$ — are rational. □

We now give the characterisation of Bohr-Sommerfeld Lagrangian sub-
manifolds that we will use to prove Theorems 2 and 3.

Lemma 6 (Hermitian flat line bundles). Let $(X, \omega)$ be an integral sym-
plectic manifold and $Q$ a submanifold. Then $Q$ is a Bohr-Sommerfeld
Lagrangian submanifold if and only if there exist a Hermitian line bundle
$L \to X$ and a unitary connection $\nabla$ of curvature $-2i\pi \omega$ such that $(L, \nabla)|_Q$ is
a trivial flat bundle. If $Q$ is a Bohr-Sommerfeld Lagrangian and, in addition,
$(X, \omega)$ is Kähler, then one can take for $(L, \nabla)$ a holomorphic Hermitian line
bundle with its Chern connection.

Proof. Suppose that $Q$ is a Bohr-Sommerfeld Lagrangian submanifold. Since
$\omega$ has integral periods, we may fix a lift of its cohomology class to $H^2(X, \mathbb{Z})$.
We take a Hermitian line bundle $L_0 \to X$ with first Chern class $c$ and a unitary connection $\nabla_0$ of curvature $-2i\pi \omega$. The submanifold $Q$ is Lagrangian so the restriction $(L_0, \nabla_0)|_Q$ is a flat Hermitian bundle.

We will construct a flat Hermitian line bundle $(L_1, \nabla_1) \to X$ whose re-
striction to $Q$ is isomorphic to $(L_0, \nabla_0)|_Q$. Then the desired line bundle will be
$L_0 \otimes L_1^{-1}$. 
Recall that flat Hermitian line bundles over a manifold \( Y \) are classified up to isomorphism by their holonomy representation \( H_1(Y, \mathbb{Z}) \to U(1) \) (cf. proposition 3.6.15 in [Thu97]). To construct the flat bundle \( (L_1, \nabla_1) \) it suffices to extend the holonomy representation \( \rho : H_1(Q, \mathbb{Z}) \to U(1) \) of the flat bundle \( (L_0, \nabla_0)|_Q \) to a homomorphism \( H_1(X, \mathbb{Z}) \to U(1) \).

We first show that \( \rho \) is trivial on the kernel of the group homomorphism \( i : H_1(Q, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \) induced by inclusion. Consider the exact sequence of the pair \( (X, Q) \):

\[
\cdots \to H_2(X, Q; \mathbb{Z}) \xrightarrow{\partial} H_1(Q, \mathbb{Z}) \xrightarrow{i} H_1(X, \mathbb{Z}) \to \cdots
\]

where \( \partial \) is the homomorphism given by the boundary of chains. It suffices to show that \( \rho \circ \partial = 0 \). Every \( a \in H_2(X, Q; \mathbb{Z}) \) can be represented by an embedded surface \( \Sigma \subset X \) whose (possibly empty) boundary is included in \( Q \). It then follows from (well-known) lemma 7 that:

\[
\rho(\partial a) = \exp \left( 2i\pi \int_a \omega \right).
\]

Since the Lagrangian submanifold \( Q \) is Bohr-Sommerfeld, \( \rho(\partial a) = 1 \).

Thus \( \rho \) factors through a homomorphism \( \tilde{\rho} : H_1(Q, \mathbb{Z})/\ker i \to U(1) \) where \( H_1(Q, \mathbb{Z})/\ker i \) injects into \( H_1(X, \mathbb{Z}) \). Now \( U(1) \) is a divisible abelian group so it is an injective \( \mathbb{Z} \)-module (see for instance [Wei95, Corollary 2.3.2]). Hence \( \tilde{\rho} \) extends to \( H_1(X, \mathbb{Z}) \).

In the case where \( (X, \omega) \) is Kähler, the above Hermitian line bundle \( (L_0, \nabla_0) \) can be chosen holomorphic with its Chern connection (see, e.g., [Dem12, Theorem 13.9.b]). On the other hand the flat line bundle \( (L_1, \nabla_1) \) is isomorphic to the quotient of the trivial flat bundle \( \tilde{X} \times \mathbb{C} \) by the diagonal action of the fundamental group, acting on its universal cover \( \tilde{X} \) by deck transformations and on \( \mathbb{C} \) by the holonomy representation \( H_1(X, \mathbb{Z}) \to U(1) \) (cf. proposition 3.6.15 in [Thu97]). Therefore the trivial holomorphic structure and the trivial connection on \( \tilde{X} \times \mathbb{C} \) respectively induce a holomorphic structure and the Chern connection on \( L_1 \). Consequently, the bundle \( L_0 \otimes L_1^{-1} \) has the desired properties.

Conversely, let \( (X, \omega) \) be a symplectic manifold and a Hermitian line bundle \( L \to X \) with a unitary connexion of curvature \(-2i\pi\omega\) such that \( (L, \nabla)|_Q \) is a trivial flat bundle. Then the (trivial) holonomy representation \( \rho \) of \( (L, \nabla)|_Q \) satisfies \([2]\), so \( Q \) is a Bohr-Sommerfeld Lagrangian. □

**Lemma 7 (Gauss-Bonnet).** Let \( X \) be a manifold and \( L \to X \) a Hermitian line bundle with a unitary connection \( \nabla \) whose curvature 2-form
is written $-2i\pi \omega$. Let $\Sigma$ be a connected oriented surface with non-empty boundary and $f : \Sigma \to X$ a map. The holonomy of $\nabla$ along the loop $f|_{\partial \Sigma}$ is $\exp(2i\pi \int_{\Sigma} f^* \omega) \in U(1)$.

**Proof.** We may assume $X = \Sigma$ and $f = \text{Id}_\Sigma$ by pulling back the line bundle $L$ by $f$. There is a unit section $s : \Sigma \to L$ in the trivialisation of $L$ given by $s$. In the trivialisation of $L$, there is a primitive $\alpha$ of $\omega$ such that the connection $\nabla$ reads $d - 2i\pi \alpha$. By Stokes theorem

$$\int_{\Sigma} \omega = \int_{\partial \Sigma} \alpha.$$ 

We may assume that $\partial \Sigma$ is connected. Take $\beta : [0, 1] \to \partial \Sigma$ a parametrisation of $\partial \Sigma$. For every unit parallel lift $\gamma : [0, 1] \to L$ of $\beta$ and for all $t \in [0, 1]$, $\gamma'(t) = 2i\pi \gamma(t) (\beta^* \alpha)(\partial_t)$ hence

$$2i\pi \int_{\partial \Sigma} \alpha = \int_{[0,1]} \frac{\gamma'(t)}{\gamma(t)} \ dt = \log \frac{\gamma(1)}{\gamma(0)}.$$ 

An exponentiation gives the result. $\square$

### 3. As minima of $\omega$-convex functions

In this section we prove Theorem 3 so $Q$ is a closed Bohr-Sommerfeld Lagrangian submanifold in a closed integral symplectic manifold $(X, \omega)$. We will adopt the following notation:

- $J$: a fixed $\omega$-compatible almost complex structure on $X$;
- $g = \omega(\cdot, J \cdot)$: the corresponding Riemannian metric;
- $\lambda_0$: the Liouville form on $T^*Q$;
- $f_0 : p \in T^*Q \mapsto \pi |p|^2 \in \mathbb{R}$ where $| \cdot |$ is the norm on each fibre of $T^*Q \to Q$ induced by the restriction of the metric $g$ to $Q$.

Using Weinstein’s normal form theorem, we identify a neighbourhood $N$ of $Q \subset (X, \omega)$ with a tube $\{ f_0 < c \}$ around the zero section $Q$ in $(T^*Q, d\lambda_0)$ in such a way that, for all $q \in Q$, the subspaces $T_qQ, T_q^*Q \subset T_q(T^*Q)$ are $g$-orthogonal.

Using Lemma 6, we fix a Hermitian line bundle $L \to X$ with a unitary connection $\nabla$ of curvature $-2i\pi \omega$ and a unit parallel section $s_0$ of the flat bundle $(L, \nabla)|_Q$. 
The main characters of the next lemmata are the two following sections of $L|_N$:

1. $s : N \to L|_N$: the extension of the section $s_0$ by parallel transport by $\nabla$ along the rays in the fibres of $T^* Q$;
2. $s_0 := e^{-f_0} s : N \to L|_N$ (which is well-defined since $f_0|_Q = 0$).

For any positive integer $k$, we denote by $L^k$ the $k$-th tensor power of the line bundle $L$, whose induced connection has curvature $\nabla - 2k \pi i \omega$, and we set $g_k := kg$ the rescaled metric. For any integer $r \geq 0$, we endow the vector bundle $\bigotimes^r T^* X \otimes L^k$ with the connection induced by the Levi-Civita connection for the metric $g_k$ and our connection on $L^k$; we still write this connection $\nabla$. We define the $C^r$ norm of a section $u : X \to L^k$ by $\|u\|_{C^r,g_k} := \sup |u| + \sup |\nabla u|_{g_k} + \cdots + \sup |\nabla^r u|_{g_k}$. The $J$-linear and $-J$-linear parts of the connection $\nabla$ are written $\nabla^J$ and $\nabla^J$. For any 1-form $\lambda$ on $X$, we will denote by $\vec{\lambda}$ the vector field that is $\omega$-dual to $\lambda$.

**Lemma 8.** There exists a constant $C > 0$ such that, for every integer $k \geq 1$, the function $f_0$ and the section $s_0^k$ satisfy the following bounds on $N$:

$$\vec{\lambda}_0 \cdot (kf_0) \geq C^{-1} (|\vec{\lambda}_0|^2_{g_k} + |d(kf_0)|^2_{g_k}), \quad C^{-1} (kf_0)^{1/2} \leq |d(kf_0)|_{g_k} \leq C (kf_0)^{1/2},$$

$$|\nabla s_0^k|_{g_k} \leq C (kf_0)^{1/2} e^{-kf_0}, \quad \|\nabla^2 s_0^k\|_{C^0,g_k} \leq C \quad \text{and} \quad \|\nabla^r s_0^k\|_{C^r,g_k} \leq C k^{-1/2}.$$

**Proof.** By rescaling, it suffices to establish the first two bounds of the statement for $k = 1$. The function $f_0$ is Lyapounov for the vector field $\vec{\lambda}_0$. This implies the first bound. The submanifold $Q$ is a Morse-Bott minimum for $f_0$, hence the second bound.

Since $s_0 = e^{-f_0} s$ with $s$ parallel,

$$\nabla s_0 = -d f_0 e^{-f_0} s + e^{-f_0} \nabla s = -(d f_0 + 2\pi i \lambda_0) s_0.$$

Therefore, $\nabla s_0$ vanishes identically on the zero section. Hence, there exists a constant $C > 0$ such that $|\nabla s_0|_g \leq Cf_0^{1/2}$. Moreover, the 1-jet of $\nabla^r s_0$ vanishes at each point of $Q$. Indeed, by the identity $\vec{\lambda}_0 = -\omega(\cdot, \vec{\lambda}_0)$ (here $k = 1$) and by $J$-linearity of the 1-form $g(\cdot, \vec{\lambda}_0) - i \omega(\cdot, \vec{\lambda}_0)$,

$$\nabla^r s_0 = -2\pi \left( \frac{df_0}{2\pi} + i \lambda_0 \right)^r s_0 = -2\pi \left( \frac{df_0}{2\pi} - g(\cdot, \vec{\lambda}_0) \right)^r s_0,$$

so it suffices to show that the 1-jet of the 1-form $\frac{df_0}{2\pi} - g(\cdot, \vec{\lambda}_0)$ vanishes identically along $Q$. Its 0-jet clearly vanishes, and, for each vector $v = (v_1, v_2)$
in the $g$-orthogonal sum $T(T^*Q)|_Q = TQ \oplus T^*Q,$
\[d(g(\cdot, \gamma_0))(v, v) = g(v, v, \gamma_0) = g(v, v_2) = g(v_2, v_2) = (d^2 f_0)(v, v)/(2\pi),\]
hence its 1-jet vanishes too. Consequently, there exists a constant $C > 0$ such that $|\nabla^n s_0|_g \leq C f_0^{1/2}$ and $|\nabla^n s_0|_g \leq C f_0$. Therefore, by the Leibniz rule, we obtain the desired bounds on $\nabla^n s_0^k$ and $\nabla^2 s_0^k$, and the two bounds $|\nabla^n s_0^k|_{g_k} \leq C k^{1/2} f_0 e^{-k f_0}$, $|\nabla^n s_0^k|_{g_k} \leq (k f_0^{1/2} + f_0^{1/2}) C e^{-k f_0}$. The two latter real-valued Gaussian functions of $f_0$ both reach their global maximum at $Constant \times k^{-1}$ so we obtain the last bound of the statement. \[\Box\]

In particular, our sections $s_0^k$ are asymptotically holomorphic in the following sense:

**Definition 9 (Donaldson, Auroux).** Sections $s_k : X \to L^k$ are called **asymptotically holomorphic** if there exists a constant $C > 0$ such that for every positive integer $k$, $\|\nabla^n s_k\|_{c^1, g_k} \leq C k^{-1/2}$ and $\|s_k\|_{c^2, g_k} \leq C$.

The following result was already observed in Auroux-Gayet-Mohsen [AGM01] Remark p.746]. Recall that our neighbourhood $N$ of $Q$ is identified with the cotangent tube $\{f_0 < c\}$.

**Lemma 10.** Let $\beta : N \to [0, 1]$ be a compactly supported function (independent of $k$) with $\beta = 1$ on a tube $\{f_0 < b\}$. Then, the sections $s_0, k := \beta s_0^k : X \to L^k$ are asymptotically holomorphic.

**Proof of lemma 10** The sections $s_0^k$ satisfy the estimates of lemma 8 on $N$. Then, there exists a constant $C > 0$ such that:
\[\|\nabla^n s_0, k\|_{c^0, g_k} \leq \|d\beta\|_{c^0, g_k} \sup_{\{f_0 > b\}} |s_0^k| + \|\nabla^n s_0^k\|_{c^0, g_k} \leq C k^{-1/2} e^{-bk} + C k^{-1/2}.
\]

Similarly:
\[\|\nabla^n s_0, k\|_{c^0, g_k} \leq \|d^2 \beta\|_{c^0, g_k} \sup_{\{f_0 > b\}} |s_0^k| + 2\|d\beta\|_{c^0, g_k} \sup_{\{f_0 > b\}} |\nabla s_0^k|_{g_k} + \|\nabla^n s_0^k\|_{c^0, g_k} \leq C k^{-1} e^{-bk} + 2C k^{-1/2} (C k^{1/2} e^{-bk}) + C k^{-1/2}.
\]

Hence, there exists a constant $C > 0$ such that, for all $k$, $\|\nabla^n s_0, k\|_{c^1, g_k} \leq C k^{-1/2}$. In the same way, we obtain the bound $\|s_0, k\|_{c^2, g_k} \leq C$. \[\Box\]
Giroux’s theorem below provides transverse perturbations of our sections $s_{0,k}$ with the following property.

**Definition 11 (Giroux).** Let $\kappa \in (0,1)$. A section $s : X \to L^k$ is called $\kappa$-quasiholomorphic if $|\nabla'' s| \leq \kappa |\nabla' s|$ at each point of $X$.

**Theorem 12 ([Gir18, Proposition 13]).** Let $\epsilon > 0$, $\kappa \in (0,1)$ and $s_{0,k} : X \to L^k$ asymptotically holomorphic sections. Then, for any sufficiently large integer $k$, there exists a section $s_{1,k} : X \to L^k$ with the following properties:

- $s_{1,k}$ vanishes transversally;
- $s_{1,k}$ is $\kappa$-quasiholomorphic;
- $\|s_{1,k} - s_{0,k}\|_{C^1,g_k} < \epsilon$;
- $- \log |s_{1,k}| : \{p \in X, s_{1,k}(p) \neq 0\} \to \mathbb{R}$ is a Morse function.

Let us now bring the previous facts together to prove Theorem 3.

**Proof of Theorem 3.** Using lemma 10, we fix sections $s_{0,k} : X \to L^k$ with $s_{0,k} = s_k^0$ on a tube $\{f_0 < b\}$. We then fix $\epsilon \in (0,1)$ and take sections $s_{1,k} : X \to L^k$ provided by Theorem 12. The subset $Y := \{s_{1,k} = 0\} \subset (X,\omega)$ is a symplectic hyperplane section of degree $k$ (because of the first two properties of Theorem 12, see for instance proposition 3 in [Don96]) avoiding the submanifold $Q$ (because $|s_0| = 1$ on $Q$ and by the third property of Theorem 12).

It remains to construct an $\omega$-convex exhaustion $f : X \setminus Y \to \mathbb{R}$ that has a Morse-Bott minimum at $Q$ and is Morse away from $Q$ with finitely many critical points. In order to do so, we will glue the function $f_{0,k} := kf_0 : N \to \mathbb{R}$, which clearly has a Morse-Bott minimum at $Q$, with the exhaustive function $f_{1,k} := - \log |s_{1,k}|$, which is Morse (by the last property of Theorem 12) and has finitely many critical points (because $s_{1,k}$ vanishes transversally).

Before gluing, let us note that, by lemma 8, $f_{0,k}$ is Lyapounov for the Liouville vector field $\lambda_0^\omega$ with Lyapounov constant in the metric $g_k$ that is independent of $k$. On the other hand, a Liouville pseudogradient for $f_{1,k}$ is provided by Giroux’s following lemma. In order to state it, we set $\lambda_{1,k}$ the real 1-form such that, in the unitary trivialisation of $L^k|_{X \setminus Y}$ given by $s_{1,k}/|s_{1,k}|$, the connection $\nabla$ reads $d - 2k\pi i \lambda_{1,k}$. We also recall that the notation $\lambda^\omega$ stands for the $\omega$-dual vector field to a given 1-form $\lambda$. 


Lemma 13 ([Gir18, Lemma 12]). Let $\kappa \in (0, 1)$ and $s_{1,k} : X \to L^k$ a $\kappa$-quasiholomorphic section. Then

$$\overrightarrow{\lambda_{1,k}} f_{1,k} \geq \frac{1 - \kappa^2}{1 + \kappa^2} \left( |df_{1,k}|^2_{g_k} + |\overrightarrow{\lambda_{1,k}}|^2_{g_k} \right).$$

Hence the function $f_{1,k}$ is Lyapounov for the Liouville vector field $\overrightarrow{\lambda_{1,k}}$, with a uniform Lyapounov constant in the metric $g_k$.

Finally, the desired function $f$ is constructed in the following lemma, by gluing, on an annular region $\{a < f_{0,k} < b\}$ about $Q$, the standard (Morse-Bott) Weinstein structure $(\overrightarrow{\lambda}_0, f_{0,k})$ on $T^*Q$ with the Weinstein structure $(\overrightarrow{\lambda}_{1,k}, f_{1,k})$ given by Giroux’s above theorem and lemma. $\square$

In the following lemma, the number $c$ still refers to the size of our cotangent tube $\{f_0 < c\}$ about $Q$.

Lemma 14. Let $\kappa \in (0, 1)$ and $a, b \in (0, c)$ with $a < b$. Then, for every sufficiently small $\epsilon \in (0, 1)$ and for every $k \geq k_0(\epsilon)$ sufficiently large, there exist a Liouville vector field $\overrightarrow{\lambda}$ on $X \setminus Y$ and a Lyapounov function $f : X \setminus Y \to \mathbb{R}$ for $\overrightarrow{\lambda}$ such that $(\overrightarrow{\lambda}, f) = (\overrightarrow{\lambda}_0, f_{0,k})$ on $\{f_{0,k} \leq a\}$, $(\overrightarrow{\lambda}, f) = (\overrightarrow{\lambda}_{1,k}, f_{1,k})$ away from $\{f_{0,k} < b\}$ and $f$ has no critical point on $\{a \leq f_{0,k} \leq b\}$.

Proof. We will omit the indices $k$ in the proof.

For now, we admit the following two facts: there exists a constant $C > 0$ (independent of $k, \epsilon$) such that

$$\|f_0 - f_1\|_{C^1(N), g_k} \leq C \epsilon$$

and, for sufficiently small $\epsilon > 0$, the form $\lambda_1 - \lambda_0$ is exact on $N$.

We will glue the Weinstein structures in two steps. Let us fix two numbers $a < a_+ < b_- < b$. For $\epsilon < \min\left(\frac{a_+ - a}{2}, \frac{b - b_-}{2}\right)$, the annular region $\{a < f_0 < b\}$ contains the level sets $\{f_1 = a_-\}$ and $\{f_1 = b_+\}$ (by the bound (3)).

First, let us glue the functions in the inner collar $\{f_0 \geq a\} \cap \{f_1 \leq a_-\}$; more precisely, let us construct a Lyapounov function $f : X \to \mathbb{R}$ for the vector field $\overrightarrow{\lambda}_0$ with $f = f_0$ on $\{f_0 \leq a\}$ and $f = f_1$ away from $\{f_1 < a_+\}$. It suffices to show that $\overrightarrow{\lambda}_0$ is transverse to the level sets of $f_0$ and $f_1$ in this inner collar; indeed, increasing from $a$ to $b$ along each trajectory of $\overrightarrow{\lambda}_0$ gives a function $f$ transverse to the level sets of $f_0$ and $f_1$. There exists a constant $C' > 0$ such that $\|d f_0\|_{g_k} \geq C'$ (by lemma 8). By the latter bound and (3), $\|d f_1\|_{g_k} \geq C' - C \epsilon$. So, by the Lyapounov conditions, there exists a constant $C' > 0$ such that $\overrightarrow{\lambda}_0 \cdot f_0 \geq C'$ (in particular $\overrightarrow{\lambda}_0$ is transverse to the
level sets of $f_0$) and $\lambda_1 f_1 \geq C'$. By the latter bound and again [3], $\lambda_0$ is transverse to the level sets of $f_1$.

Second, we glue the Liouville vector fields in the outer collar $\{a_- < f_1 < b_+\}$ (where $f = f_1$); more precisely we construct a Liouville vector field $\hat{\lambda}$ which is transverse to the level sets of the function $f$ in the outer collar and coincides with $\lambda_0$ on $\{f_1 < a_-\}$ and with $\lambda_1$ outside $\{f_1 < b_+\}$. The 1-form $\lambda_1 - \lambda_0$ is exact (by our initial claim) so we have a function $H$ such that $\lambda_1 - \lambda_0 = dH$. Let us fix a cutoff function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta = 0$ near $\mathbb{R}_{< (b-a)/2}$ and $\beta = 1$ near $\mathbb{R}_{\geq b}$ and set $\beta_1 := \beta \circ f_1$. Then the vector field $\lambda := \lambda_0 + d(\beta_1 H)$ is Liouville and satisfies the desired boundary conditions. Moreover, $\beta_1$ is tangent to the level sets of $f_1$ and, by the above paragraph, $\lambda_0$ and $\lambda_1$ are positively transverse to these, so $\lambda = (1 - \beta_1)\lambda_0 + \beta_1 \lambda_1 + \beta_1 H$ is transverse to these too.

It remains to prove the two initial claims. On the one hand, $f_0 - f_1 = \text{Re} \log(s_1 s_0^{-1})$ and, since $u_j := s_j/|s_j|$ satisfies $\nabla u_j = -2k \pi i \lambda_j u_j$,

$$\lambda_1 - \lambda_0 = \frac{1}{2k \pi i} (u_0^{-1} \nabla u_0 - u_1^{-1} \nabla u_1) = \frac{1}{2k \pi i} \nabla \log(u_0 u_1^{-1}) = \frac{1}{2k \pi i} \nabla \log(s_1 s_0^{-1}).$$

On the other hand, $\|\log(s_1 s_0^{-1})\|_{C^1, g_k} \leq C$; this is a consequence of the three bounds $\|s_1 - s_0\|_{C^1, g_k} < \epsilon$, $\inf |s_0| > e^{-\epsilon}$, and $\|\nabla s_0\|_{C^0, g_k} \leq \text{Constant}$ (from lemma 3). In particular we obtain the bound 3 and for $\epsilon$ sufficiently small, $\|\log(s_1 s_0^{-1})\|_{C^0} < \pi/3$ so $\lambda_1 - \lambda_0$ is exact. □

**Remark 15 (An alternative proof of the regularity of $Q \subset X \setminus Y$).**

For sufficiently large $k$, it is possible to choose our $\kappa$-quasiholomorphic perturbation $s_{1,k} : X \to L^k$ (vanishing transversally and away from $Q$) of $s_{0,k}$ in such a way that the quotient function $(s_{1,k}/s_{0,k})|_Q$ is real-valued. The latter property, which can be achieved by implementing techniques from Auroux-Munoz-Presas’ [AMP05] in the proof of [Gir18, Proposition 13], implies that the Liouville pseudogradients $\lambda_{1,k}$ of the function $-\log|s_{1,k}|$ is tangent to $Q$.

### 4. As minima of C-convex functions

This section deals with the proof of Theorem 3 so $Q$ is a closed Bohr-Sommerfeld Lagrangian submanifold in a closed integral Kähler manifold $(X, \omega)$. Using Lemma 6 we fix a holomorphic Hermitian line bundle $L \to X$ with Chern curvature $-2\pi i \omega$ and a parallel unit section $s_0 : Q \to L|_Q$. We denote by $\nabla$ the Chern connection. Let $k$ be a positive integer. We will use the following notation:
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\[ g_k = \kappa \omega(\cdot, i \cdot) \text{; the rescaled metric ;} \]
\[ d_k : \text{the distance function to } Q \text{ in the metric } g_k ; \]
\[ B_k(Q, c) = \{ d_k < c \}. \]

We recall that we endow the vector bundle \( \bigotimes^r T^* X \otimes L^k \) with the connection induced by the Levi-Civita connection for the metric \( g_k \) and the connection on \( L^k \) — we still write this connection \( \nabla \). We define the \( C^r \) norm of a section \( u : X \to L^k \): \( \| u \|_{C^r, g_k} = \sup |u| + \sum_{j=1}^r \sup |\nabla^j u|_{g_k} \).

Since \( Q \) is a totally real submanifold, there are many \( C^r \)-convex functions on a neighbourhood of \( Q \) having a Morse-Bott minimum at \( Q \) (namely the squared distance function \( d_2^2 \)). In the next two lemmas construct such a function under the form \( f_0 = - \log |s_0| \), where \( s_0 \) is a section of \( L \) over a neighbourhood of \( Q \) whose powers are asymptotically holomorphic (in a sense made precise in lemma 18) and will be later modified into genuine global holomorphic sections of \( L^k \) (see proposition 19).

**Lemma 16.** There exists a number \( c > 0 \) such that the restriction of the line bundle \( L \) to \( N := B_1(Q, c) \) admits a non-vanishing holomorphic section \( s : N \to L|_N \). Furthermore, given any integer \( r \geq 1 \), the complex-valued function \( s_0/(s|_Q) \) extends to a smooth function \( F : N \to \mathbb{C} \) such that the form \( d''F \) vanishes identically along \( Q \) together with its \( r \)-jet.

We will eventually choose \( r = n \), the complex dimension of the manifold \( X \).

**Proof.** Since \( Q \) is a totally real submanifold of \( X \), it has a neighbourhood on which the squared distance function \( d_2^2 \) is \( C^r \)-convex (see for instance Proposition 2.15 in [CE12]), for sufficiently small \( c > 0 \), the neighbourhood \( N \) is a Stein manifold. So the first assertion follows from results of Oka [Oka39] and Grauert [Gra58].

For any positive integer \( r \), [CE12, Proposition 5.55] shows that the complex-valued function \( s_0/(s|_Q) \) extends to a smooth function \( F : N \to \mathbb{C} \) such that, at each point of \( Q \), \( d''F \) vanishes together with its \( r \)-jet. \( \square \)

The desired local section and local function are respectively:

\[ \bullet \ s_0 := Fs : N \to L|_N, \text{ extending the section } s_0 : Q \to L|_Q; \]
\[ \bullet \ f_0 := - \log |s_0| : N \to \mathbb{R}. \]

**Remark 17 (The real-analytic case).** If the submanifold \( Q \) is real-analytic, then one can take for \( s_0 : N \to L|_N \) a holomorphic section. Indeed,
one may ensure that the connection \( \nabla \) on the bundle \( L \) provided by Lemma \( 6 \) is real-analytic. In that case, the section \( s_0 : Q \to L|_Q \) is real-analytic and can be complexified.

**Lemma 18.** Suppose that \( r \geq 1 \). Then, the submanifold \( Q \) is a Morse-Bott minimum for the function \( f_0 \). Besides, there exists a constant \( C > 0 \) such that, for every integer \( k \geq 1 \), the section \( s_0^k : N \to L^k|_N \) satisfies the following bounds on \( N \):

\[
|2\pi k\omega - dd^c \log |s_0^k|_{g_k}| \leq Ck^{-1/2}d_k, \\
e^{-Cd_k^2} \leq |s_0^k| \leq e^{-d_k^2/C}, \\
|\nabla s_0^k|_{g_k} \leq Cd_k e^{-d_k^2/C}, \| \nabla'' s_0^k \|_{C^1,g_k} \leq Ck^{-r/2}.
\]

**Proof.** We first observe that \( \nabla s_0 \) vanishes at every point \( p \in Q \). Indeed, \( T_pX = T_pQ \oplus i T_pQ \) (because \( Q \) is totally real of middle dimension), \( \nabla s_0(p) = \nabla'' s_0(p) \) (because \( \nabla'' s_0(p) \) vanishes) and \( \nabla s_0(p) \) vanishes on \( T_pQ \) (because \( s_0|_Q \) is parallel). Thus, there exists a constant \( C > 0 \) such that \( |\nabla s_0| \leq C d_1 \).

Similarly, since the \( r \)-jet of \( \nabla'' s_0 \) vanishes identically on \( Q \), there exists a constant \( C > 0 \) such that \( |\nabla'' s_0|_{g_1} \leq C d_1^{r+1} \) and \( |\nabla'' s_0|_{g_1} \leq C d_1^r \).

The function \( f_0 = -\log |s_0| \) vanishes together with its 1-jet at \( p \); indeed, \( f_0(p) = 0 \) and

\[
df_0(p) = \frac{1}{2} d\log(|s_0|^2) = \frac{1}{2} |s_0|^{-2} \log(|s_0|^2) = |s_0|^{-2} \Re \langle \nabla s_0, s_0 \rangle = 0.
\]

Moreover,

\[
2\pi \omega_p + (dd^c f_0)_p = dd^c \log \frac{|s_0|}{s_0} = -i d'd'' \log \frac{|s_0|}{s_0} = -(i d'd'' \log |F|^2)_p = 0
\]

because the 1-jet of the form \( d''F \) vanishes at \( p \). Therefore, there exists a constant \( C > 0 \) such that \( |2\pi \omega + dd^c f_0|_g \leq C d_1 \). Multiplying this by \( k \) gives the first bound of the statement. On the other hand, the Hessian quadratic form \( (d^2 f_0)_p : T_pX \to \mathbb{R} \) vanishes on \( T_pQ \) and satisfies, for every vector \( v \in T_pX \),

\[
(d^2 f_0)(v,v) + (d^2 f_0)(iv,iv) = -(dd^c f_0)(v,iv) = 2\pi \omega(v,iv) = 2\pi g(v,v).
\]

Hence, \( (d^2 f_0)_p \) is positive definite on \( iT_pQ \) and \( Q \) is a Morse-Bott minimum for \( f_0 \). Since \( Q \) is compact, one can find a constant \( C > 0 \) such that, on
some neighbourhood of $Q$ for the metric $g_1$:

$$C^{-1}d_1^2 \leq f_0 \leq Cd_1^2.$$ 

In other words, $e^{-Cd_1^2} \leq |s| \leq e^{-d_1^2/C}$. We obtain the second bound of the statement by taking the $k$-th power. The third bound and the bounds

$$|\nabla'' s^k_0|_g \leq C[kd^1_1 + 1]e^{-kd_1^2/C}, \quad |\nabla\nabla'' s^k_0|_g \leq Ckd^1_1(1 + kd_1^2)e^{-kd_1^2/C}$$

follow from this bound and the bounds on $\nabla s_0$, $\nabla'' s_0$ and $\nabla\nabla'' s_0$ by the Leibniz rule applied to $s^k_0$. The two latter real-valued Gaussian functions of $d_1$ both reach their global maximum at $\text{Constant} \times k^{-1/2}$. By expressing these bounds in the rescaled metric $g_k$, we obtain the last bound of the statement. □

The following is the main result of this section. Recall that the number $c$ is the size of the tube $N = B_1(Q, C)$.

**Proposition 19.** Let $\rho \in (0, c)$. There exist holomorphic sections $s^k : X \to L^k$ such that, for every $\epsilon > 0$ and for $k \geq k_0(\epsilon)$ sufficiently large, $s_k$ vanishes transversally and $\|s_k - s^k_0\|_{C^1, g_k} < \epsilon$ on $B_1(Q, \rho)$, the $\rho$-neighbourhood of $Q$ in the metric $g$.

We postpone the proof of proposition [19] and first explain how it implies Theorem [2].

**Proof of Theorem 2.** We fix a radius $\rho \in (0, c)$ and, by proposition [19], holomorphic sections $s^k : X \to L^k$: for every $\epsilon > 0$ and for $k \geq k_1(\epsilon)$ sufficiently large, the zero-set $Y := s^{-1}_k(0)$ is a (smooth) complex hyperplane section and $\|s_k - s^k_0\|_{C^1, g_k} < \epsilon$ on $B_1(Q, \rho)$. By the second and third inequalities in lemma [18], there exists a constant $C > 0$ (independent of $k$ and $\epsilon$) such that, for $\epsilon > 0$ sufficiently small, on $B_k(Q, \rho)$, the functions $f_1 := -\log |s_k|$ and $f_0 = -\log |s^k_0|$ satisfy

$$\|f_1 - f_0\|_{C^1, g_k} < C\epsilon.$$ 

Take a cutoff function $\beta_k : X \to [0, 1]$ supported in $B_k(Q, \rho)$, with $\beta_k = 1$ on $B_k(Q, \rho/2)$ and $\|\beta_k\|_{C^2, g_k} \leq C'$ for some constant $C' > 0$ (independent of $k$). The function $f := \beta_k f_0 + (1 - \beta_k)f_1 : X \setminus Y \to \mathbb{R}$ is exhausting, reaches a Morse-Bott minimum at $Q$ and its critical points remain in a compact subset. (We remark that, for sufficiently small $\epsilon$, this minimum is global. Indeed,
on \( \{ \beta_k = 1 \} \), \( f = f_0 \), and on \( \{ \beta_k < 1 \} \), \( f_1 \geq - \log(|s_0| + \epsilon) \geq - \log(e^{-p^2/C} + \epsilon) > 0 \). Let us show that \( f \) is \( C \)-convex. First, since \( s_k \) is holomorphic, \(-dd^c f_1 = 2k \pi \omega \). Then, by the first bound of lemma \[ \ref{lemma:bound} \] there exists a constant \( C'' > 0 \) such that \( \|dd^c(f_0 - f_1)\|_{C^0,g_k} \leq C'' k^{-1/2} \). Hence,

\[
\|2k \pi \omega + dd^c f\|_{C^0,g_k} = \|dd^c(\beta_k(f_0 - f_1))\|_{C^0,g_k} \\
\leq \|\beta_k\|_{C''} k^{-1/2} + \|(f_0 - f_1)dd^c \beta_k\| \\
+ \|\text{d}(f_0 - f_1) \wedge d^c \beta_k\| + \|d^c(f_1 - f_0) \wedge d \beta_k\| \\
\leq C'' k^{-1/2} + 3(C \epsilon) C'.
\]

Consequently, for every \( \epsilon > 0 \) sufficiently small and for every \( k \geq k_0(\epsilon) \) sufficiently large, \( \|2k \pi \omega + dd^c f\|_{C^0,g_k} < 2\pi \). This inequality ensures that the function \( f \) is \( C \)-convex. A \( C^2 \)-small perturbation of the function \( f \) with support in a compact subset of \( Y \setminus Q \) is Morse away from \( Q \) and satisfies the properties of Theorem \[ \ref{thm:main} \].

Our next aim is to prove proposition \[ \ref{prop:holomorphic} \]. The following lemma defines global smooth sections of \( L^k \) which will be later modified into genuine holomorphic sections. The \( L^2 \)-norm of a section \( s : X \to \bigotimes^r T^* X \otimes L^k \) for the rescaled metric \( g_k \) is defined by

\[
\|s\|_{L^2,g_k} := \left( \int_X |s|_{g_k}^2 \frac{(k \omega)^n}{n!} \right)^{1/2}.
\]

**Lemma 20.** Let \( \beta : X \to [0,1] \) a function supported in \( N \) with \( \beta = 1 \) on a tube \( B(Q, \rho) \). There exists a constant \( C > 0 \) such that the sections \( s_0,k := \beta s_0^k : X \to L^k \) satisfy the following bounds:

\[
\|\nabla'' s_{0,k}\|_{C^0,g_k} \leq C k^{-r/2}, \|\nabla'' s_{0,k}\|_{L^2,g_k} \leq C k^{(n-r)/2}
\]

**Proof.** The sections \( s^k \) satisfy the bounds of Lemma \[ \ref{lemma:bound} \] on \( N \). Then, there exists a constant \( C > 0 \) such that:

\[
\|\nabla'' s_{0,k}\|_{C^0,g} \leq \|\text{d}\beta\|_{C^0,g} \sup_{\{d_i > \rho\}} |s_0^k| + \sup_{B(Q,2\rho)} |\nabla'' s_0^k|_g \\
\leq C(e^{-k/C} + k^{-(r-1)/2}).
\]
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In the same way:

\[ \| \nabla^n s_{0,k} \|_{C^0,g_k} \leq \| d^2 \beta \|_{C^0,g_k} \sup_{\{d_i > \rho\}} |s_0^k| + 2 \| \beta \|_{C^0,g_k} \sup_{\{d_i > \rho\}} |\nabla s_0^k|_g + \sup_{\alpha(B,Q_2 \rho)} |\nabla s_0^k|_g \]

\leq Ce^{-k/C} + Ce^{-k/C} + Ck^{-(r-2)/2}.

Since

\[ \| \nabla^n s_{0,k} \|_{L^2,g_k} \leq Ck^{n/2} \| \nabla^n s_{0,k} \|_{C^0,g_k}, \]

the \(C^1\) and the \(L^2\) norms, in the metric \(g_k\), satisfy the bounds of the statement. \(\square\)

We now use the following version of Hörmander’s \(L^2\)-estimates:

**Theorem 21 (cf. [Dem12, Theorem VIII.6.5] and the discussion thereafter).** Let \((X, \omega)\) be a closed integral Kähler manifold and \(L \to X\) a holomorphic Hermitian line bundle with Chern curvature \(-2\pi i \omega\). Set \(C := \sup \left| \frac{\text{Ric}([\omega])}{2\pi} \right|_g\). Then, for every \(k > C\) and for every smooth section \(u : X \to \bigwedge^1 T^* X \otimes L^k\) such that \(\nabla^n u = 0\), there exists a smooth section \(t : X \to L^k\) satisfying:

\[ \nabla^n t = u \text{ and } \| t \|_{L^2} \leq \frac{1}{n(k-C)} \| u \|_{L^2}^2. \]

Applying this theorem to the sections \(s_{0,k}\) of lemma 20, we obtain smooth sections \(t_k : X \to L^k\) satisfying \(\| t_k \|_{L^2,g_k} \leq Ck^{(n-r-1)/2}\), and, for \(k\) sufficiently large, \(\nabla^n (s_{0,k} - t_k) = 0\). The following lemma converts our \(L^2\)-estimates to \(C^1\)-estimates.

**Lemma 22.** Let \((X, \omega)\) be a closed integral Kähler manifold, \(L \to X\) a holomorphic Hermitian line bundle with Chern curvature \(-2\pi i \omega\). There exists a constant \(C > 0\) such that for every integer \(k\) and for every section \(t : X \to L^k\):

\[ \| t \|_{C^1,g_k} \leq C(\| \nabla^n t \|_{C^1,g_k} + \| t \|_{L^2,g_k}). \]

**Proof.** The desired bound is local. At a given point \(p \in X\), we will obtain it on a \(g_k\)-ball of uniform radius about \(p\) — where, for sufficiently large \(k\), the geometry of \(L^k\) compares with the trivial line bundle over the unit ball of
euclidean space \((\mathbb{C}^n, g_0)\). There exist constants \(R, C > 0\) and a family (indexed by \(p \in X, k \geq 1\)) of holomorphic charts \(z^k_p : B_k(p, R) \to \mathbb{C}^n\) centered at \(p\) such that,

\[
\|(z^k_p)^* g_k - g_0\|_{C^1, g_0} \leq Ck^{-1/2} \text{ over } (z^k_p)(B_k(p, R)).
\]

We first explain this when \(k = 1\). There exist constants \(R, C_0 > 0\) and a family of holomorphic charts \(z_p : B_1(p, R) \to \mathbb{C}^n\) centered at \(p\) with \(\|\nabla (z^*_p g)||_{g_0} < C_0\), where the covariant derivative and the norm are taken for the flat metric. Furthermore, after post-composing each chart by an element of \(\text{GL}(n, \mathbb{C})\), we may assume that \((z^*_p g)(p) = g_0\). Then, the family \(z_p\) satisfies the bound \(5\) with \(C = C_0(1 + R)\). In the general case \(k \geq 1\), to get the desired charts \(z^k_p\), it suffices to post-compose each chart \(z_p\) by the centered dilation \(\mathbb{C}^n \to \mathbb{C}^n\) of ratio \(k^{1/2}\).

Let us take a Hörmander holomorphic peak section at \(p\) (see for instance [Don96, Proposition 34]): for sufficiently large \(k\), there exists a holomorphic section \(s_p : X \to L^k\) satisfying the bounds:

\[|s_p(p)| = 1, \inf_{B_k(p, R)} |s_p| \geq C^{-1} \text{ and } \|s_p\|_{C^1, g_k} \leq C,\]

for some constant \(C > 0\) independent of \(p\) and \(k\).

Let \(t\) be a section of \(L^k\) and \(p \in X\). We set \(f := \frac{1}{t} s_p\). In view of the identities \(\nabla t = df \, s_p + f \nabla s_p\), \(\nabla \nabla t = d^2 f \, s_p + 2df \otimes \nabla s_p + f \nabla \nabla s_p\), and the bounds on the peak sections, it suffices to show that for sufficiently large \(k\),

\[\|f\|_{C^1(B_k(p, R/5)), g_k} \leq C\|d^0 f\|_{C^1(B_k(p, R)), g_k} + C\|f\|_{L^2(B_k(p, R)), g_k}.\]

In the following, we will identify the domain of the chart \(z^k_p\) with its image in \(\mathbb{C}^n\). We denote by \(B_0(q, R)\) the ball of radius \(R\) at a point \(q\) in \(\mathbb{C}^n\) and by \(\mu\) the Euclidean volume form on \(\mathbb{C}^n\). Let us prove the (standard) following bound:

\[\|f\|_{C^1(B_0(0, R/5)), g_0} \leq C\|d^0 f\|_{C^1(B_0(0, R/2)), g_0} + C\|f\|_{L^2(B_0(0, R/2)), g_0}.\]

This will end the proof because, in view of the comparison \(5\) of the rescaled metric \(g_k\) with the flat metric \(g_0\), for sufficiently large \(k\), we have the inclusions \(B_k(p, R/6) \subset B_0(0, R/5)\) and \(B_0(0, R/2) \subset B_k(p, R)\), and there exists
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a constant $C > 0$ (independent on $k$ and $p$) such that, over $B_0(0, R/2)$,

$$\mu \leq (1 + C k^{-n/2}) \frac{(k\omega)^n}{n!} \text{ and } (1 - C k^{-1/2}) | \cdot |_{g_0} \leq | \cdot |_{g_k} \leq (1 + C k^{-1/2}) | \cdot |_{g_0}.$$ 

On the one hand, [HW68, Lemma 4.4] gives:

$$\|f\|_{C^0(B_0(0, R/4))} \leq C \|d''f\|_{C^0(B_0(0, R/2))} + C \|f\|_{L^2(B_0(0, R/2)), g_0}.$$

On the other hand, we have the following standard bound (cf. [CE12, Lemma 8.37] for instance):

$$\|f\|_{C^1(B_0(0, R/5)), g_0} \leq C \|d''f\|_{C^1(B_0(0, R/4)), g_0} + C \|f\|_{C^0(B_0(0, R/4))}.$$ 

In the two above estimates the constants depend only on $R$ and $n$. Therefore we obtain the desired bound. □

By lemmas \[22\] and \[20\] and using $r = n$, we obtain the following estimate: for every $\epsilon > 0$, for $k \geq k_1(\epsilon)$ sufficiently large,

$$\|t_k\|_{C^1, g_k} \leq C(\|\nabla'' s_{0,k}\|_{C^1, g_k} + k^{-1/2} \|s_{0,k}\|_{L^2, g_k}) \leq C k^{(n-r-1)/2} < \epsilon/2.$$ 

On the other hand, by Bertini theorem, for sufficiently large $k$ there exists a holomorphic section $s_k : X \to L^k$ vanishing transversally with

$$\|s_k - (s_{0,k} - t_k)\|_{C^1, g_k} < \epsilon/2.$$

Therefore the sections $s_k$ satisfy the conclusions of proposition \[19\]. This ends the proof of Theorem \[2\].

Let us finish with a complex-geometric variant of Theorem \[2\].

**Theorem 23.** Let $X$ be a closed complex manifold, $a$ a Kähler class and $Q$ a closed submanifold. Suppose that $Q$ is a Bohr-Sommerfeld Lagrangian submanifold for some Kähler form in $a$. Then, there exists a holomorphic line bundle $L \to X$ with first Chern class $a$, and, for every sufficiently large $k$, there exist a Hermitian metric $h_k$ on $L^k$ with positive Chern curvature and a holomorphic section $s_k : X \to L^k$ vanishing transversally such that the function $-\log |s_k|_{h_k} : X \setminus s_k^{-1}(0) \to \mathbb{R}$ has a Morse-Bott minimum at $Q$ and is Morse elsewhere.

**Proof of Theorem 23.** We fix a Kähler form $\omega \in \mathcal{A}$ with $\omega|_Q = 0$ as well as a Hermitian holomorphic line bundle $L \to X$ with Chern curvature $-2i\pi \omega$ whose restriction to $Q$ is a trivial flat bundle (by lemma \[6\]). We fix $\epsilon, \rho > 0$
and repeat the construction of section 4 to obtain sections $s_k^0$, $s_k : X \to L^k$ with the properties stated in lemma 18 and proposition 19. We keep the notation $f_0 = - \log |s_k^0|$ and $f_1 = - \log |s_k|$.

To construct the desired Hermitian metric on $L^k$, we will proceed as in the final step of the proof of Theorem 2 but we will modify the initial Hermitian metric $h_k$ of $L^k$ instead of the function $f_1$. Take a cutoff function $\beta_k : X \to [0,1]$ with support in $B_k(Q, \rho)$ with $\beta_k = 1$ on $B_k(Q, \rho/2)$ and such that $\|d\beta_k\|_{C^1, g_k} < C' $, for some constant $C' > 0$ independent of $k$. We define a new Hermitian metric on $L^k$ by:

$$h'_k = e^{2\beta_k(f_1 - f_0)} h_k.$$ 

The exhaustion function $- \log |s_k|_{h'_k} : \{s_k \neq 0\} \to \mathbb{R}$ equals $f_0$ on $B_k(Q, \rho/2)$ hence has a Morse-Bott local minimum at $Q$. Furthermore,

$$2k\pi \omega - \text{dd}^c \log |s_k|_{h'_k} = -\text{dd}^c(\beta_k(f_1 - f_0)).$$

Therefore, by repeating the estimation (4), for every $\epsilon < \epsilon_0$ sufficiently small and for $k \geq k_0(\epsilon)$ sufficiently large, $\|2k\pi \omega - \text{dd}^c \log |s_k|_{h'_k}\|_{C^0, g_k} < 2\pi$. This inequality ensures that the function $- \log |s_k|_{h'_k}$ is $C$-convex. Finally, there exists a $C^2$-small function $\eta_k : X \setminus Y \to \mathbb{R}$ with compact support away from $Q$ such that, setting the Hermitian metric $h''_k := e^{-2\eta_k} h'_k$, the function $- \log |s_k|_{h''_k} = - \log |s_k|_{h'_k} + \eta_k$ is Morse away from $Q$.

In conclusion, the Hermitian metric $h''_k$ and the sections $s_k : X \to L^k$ have the desired properties. 

□

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Received July 12, 2018

Accepted October 17, 2018
