Integrable Mappings Related to the Extended Discrete KP Hierarchy

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Abstract. We investigate self-similar solutions of the extended discrete KP hierarchy. It is shown that corresponding ansätze lead to purely discrete equations with dependence on some number of parameters together with equations governing deformations with respect to these parameters. Some examples are provided. In particular it is shown that well known discrete first Painlevé equation (dP1) and its hierarchy arises as self-similar reduction of Volterra lattice hierarchy which in turn can be treated as reduction of the extended discrete KP hierarchy. It is written down equations which naturally generalize dP1. It is shown that these discrete systems describe Bäcklund transformations of Noumi-Yamada systems of type $A_{2(n-1)}^{(1)}$. We also consider Miura transformations relating different infinite- and finite-field integrable mappings. Simplest example of this kind of Miura transformations is given.

Keywords: extended discrete KP hierarchy, self-similar solutions, discrete Painlevé equations

1. Introduction

In Refs. [14], [15] we exhibit and investigate two-parameter class of invariant submanifolds of Darboux-KP (DKP) chain which is in fact the chain of copies of (differential) KP hierarchy glued together by Darboux map. The DKP chain is formulated in terms of formal Laurent series

\[ h(i) = z + \sum_{k=2}^{\infty} h_k(i) z^{-k+1}, \ a(i) = z + \sum_{k=1}^{\infty} a_k(i) z^{-k+1} : i \in \mathbb{Z} \]

and reads as [7]

\[ \partial_p h(i) = \partial H^{(p)}(i), \]

\[ \partial_p a(i) = a(i)(H^{(p)}(i+1) - H^{(p)}(i)) \]  

with \( H^{(p)}(i) : p \geq 1 \) (with fixed value of \( i \)) being the currents calculated at the point \( h(i) \) [2]. Restriction on invariant submanifold \( S^n \) is defined by condition \( z^{l-n+1} a^{[n]}(i) \in H_{+}(i), \ \forall i \in \mathbb{Z} \) where \( H_{+}(i) = \langle 1, H^{(1)}(i), H^{(2)}(i), ... > \) and \( a^{[k]}(i) \)'s are Faà di Bruno discrete iterates defined by recurrence relation \( a^{[k+1]}(i) = a(i)a^{[k]}(i) \) with \( a^{[0]}(i) \equiv 1 \). As was shown in Refs. [14], [15], the restriction of DKP

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chain on $S^n_0$ leads to linear discrete systems (see also [12])

$$Q_{(n)}^r \Psi = z^r \Psi$$  \hspace{1cm} (2)

and

$$z^{p(n-1)} \partial_p^{(n)} \Psi = (Q_{(n)}^m)^+ \Psi,$$  \hspace{1cm} (3)

with the pair of discrete operators

$$Q_{(n)}^r \equiv \Lambda^r + \sum_{k \geq 1} q_k^{(n,r)} z^{k(n-1)} \Lambda^{r-kn}$$

and

$$(Q_{(n)}^m)^+ \equiv \Lambda^m + \sum_{k=1}^p q_k^{(n,m)} z^{k(n-1)} \Lambda^{(p-k)m}$$

whose coefficients $q_k^{(n,r)} = q_k^{(n,r)}(i)$ uniquely defined as polynomials in coordinates $a_k(i)$ with the help of the relation

$$z^r = a^{(r)(i)} + z^{n-1} q_1^{(n,r)}(i) a^{(r-n)(i)} + z^{2(n-1)} q_2^{(n,r)}(i) a^{(r-2n)(i)} + \ldots, \ r \in \mathbb{Z}.$$  

The consistency condition of (2) and (3) reads as Lax equation

$$z^{p(n-1)} \partial_p^{(n)} Q_{(n)}^r = [(Q_{(n)}^m)^+, Q_{(n)}^r]$$  \hspace{1cm} (4)

and can be rewritten in explicit form

$$\partial_p^{(n)} q_k^{(n,r)}(i) = Q_{k,p}^{(n,r)}(i) = q_k^{(n,r)}(i + pn) - q_k^{(n,r)}(i)$$

$$+ \sum_{s=1}^p q_s^{(n,m)}(i) \cdot q_{k-s+p}^{(n,r)}(i + (p-s)n)$$

$$- \sum_{s=1}^p q_s^{(n,m)}(i + r - (k-s+p)n) \cdot q_{k-s+p}^{(n,r)}(i).$$  \hspace{1cm} (5)

Lax equations (4) naturally contains, for $n = 1$, equations of the discrete KP hierarchy and we call them the extended discrete KP hierarchy [12], [14], [15].

The formula (5) is proved to be a container for many differential-difference systems [15] (see, also [11], [13]) but to derive them from (5) one needs to take into account purely algebraic relations

$$q_k^{(n,r_1+r_2)}(i) = q_k^{(n,r_1)}(i) + \sum_{s=1}^{k-1} q_s^{(n,r_1)}(i) \cdot q_{k-s}^{(n,r_2)}(i + r_1 - sn) + q_k^{(n,r_2)}(i + r_1).$$
\[ q_k^{(n,r_2)}(i) + \sum_{s=1}^{k-1} q_s^{(n,r_2)}(i) \cdot q_k^{(n,r_1)}(i + r_2 - sn) + q_k^{(n,r_1)}(i + r_2) \]  

which are coded in permutability operator relation

\[ Q_{n}^{r_1 + r_2} = Q_{n}^{r_1} Q_{n}^{r_2} = Q_{n}^{r_2} Q_{n}^{r_1}. \]

It is quite obvious that equations (5) admit reductions with the help of simple conditions

\[ q_k^{(n,r)}(i) \equiv 0, \quad k \geq l + 1, \quad l \geq 1. \]

which are also consistent with the algebraic relations (6).

As was shown in [14], [15], this reductions can be properly described in geometric setting as double intersections of invariant manifolds of DKP chain: \( S_{n,r,l} = S_{0}^{n} \cap S_{l-1}^{n-r} \). For example, the restriction of the DKP chain on \( S_{2,1,1} \) yields Volterra lattice hierarchy.

Our principal goal in this Letter is to investigate self-similar solutions of equations (5) supplemented by (6). In Section 2, we perform auxiliary calculations aiming to select some quantities which do not depend on evolution parameters if solution \( \{ q_k^{(n,r)}(i) \} \) of the extended discrete KP hierarchy, or more exactly its subhierarchy corresponding to multi-time \( t^{(n)} \), is invariant with respect to suitable scaling transformation. In Section 3, we show that self-similar ansatzes yield a large class of purely discrete systems supplemented by equations governing deformations of parameters entering these systems. In Section 4, we consider the simplest example corresponding to Volterra lattice hierarchy and deduce dP\(_I\) and some discrete equations which can be treated as higher members in dP\(_I\) hierarchy. We recover there known relationship between dP\(_I\) and fourth Painlevé equation (P\(_{IV}\)). In Section 5, we show one-component discrete equations naturally generalizing dP\(_I\) in a sense that they govern Bäcklund transformation for higher order generalizations of P\(_{IV}\). Section 6 is devoted to constructing some class of Miura transformations relating different discrete equations. We provide the reader by simple examples of such transformations.

2. Auxiliary calculations

To prepare a ground for further investigations, we need in following technical statement

**Proposition 2.1.** By virtue of relations (6) with \( r_1 = pn, \ r_2 = sn \) and \( k = p + s \), the relation

\[ Q_{p,s}^{(n,pm)}(i) = Q_{s,p}^{(n,sn)}(i). \]
with arbitrary \( p, s \in \mathbb{N} \) is identity.

**Proof.** By direct calculations.

The formula (7), by definition of \( Q_{k,p}^{(n,r)} \), can be rewritten in the form

\[
\partial_s^{(n)} q_p^{(n,m)} (i) = \partial_p^{(n)} q_s^{(n,m)} (i), \quad \forall p, s \in \mathbb{N}.
\]

In particular, we have

\[
\partial_s^{(n)} q_p^{(n,m)} (i) = \partial_p^{(n)} q_1^{(n,m)} (i) = \partial_p^{(n)} \left( \sum_{s=1}^{n} q_1^{(n,1)} (i + s - 1) \right)
= \sum_{s=1}^{n} Q_{1,p}^{(n,1)} (i + s - 1).
\]

Each \( Q_{1,p}^{(n,1)} (i) \) is uniquely expressed as a polynomial of \( q_k (i) \equiv q_k^{(n,1)} (i) \).

So, we can suppose that there exists the polynomials \( \xi_p^{(n)} (i) \) in \( q_k (i) \) such that

\[
q_p^{(n,m)} (i) = \sum_{s=1}^{n} \xi_p^{(n)} (i + s - 1), \quad \forall p \geq 1
\]

and consequently

\[
\partial_s^{(n)} \xi_p^{(n)} (i) = \partial_p^{(n)} q_1 (i) = Q_{1,p}^{(n,1)} (i).
\]

For example, we can write down the following:

\[
\xi_1^{(n)} (i) = q_1 (i), \quad \xi_2^{(n)} (i) = q_2 (i) + q_2 (i + n) + q_1 (i) \cdot \sum_{s=1}^{n-1} q_1 (i + s).
\]

Unfortunately we cannot by now to write down the polynomials \( \xi_3^{(n)} (i) \), \( \xi_4^{(n)} (i) \), ... in explicit form for arbitrary \( n \), but we believe that one can do it for arbitrary concrete values of \( n \). From (8) we have the following

**Proposition 2.2.** By virtue of equations of motion of the extended discrete KP hierarchy,

\[
\partial_s^{(n)} \xi_p^{(n)} (i) = \partial_p^{(n)} \xi_s^{(n)} (i), \quad \forall s, p \geq 1.
\]

Fixing any integer \( p \geq 2 \), let us define

\[
\alpha_i^{(n)} = \sum_{k=1}^{p} k t_k^{(n)} \xi_k^{(n)} (i).
\]
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Taking into account the above conjecture, we obtain

\[
\partial_s^{(n)} \alpha_i^{(n)} = s \xi_s^{(n)}(i) + \sum_{k=1}^p k t_k^{(n)} \partial_s^{(n)} \xi_k^{(n)}(i)
\]

\[
= s \xi_s^{(n)}(i) + \sum_{k=1}^p k t_k^{(n)} \partial_s^{(n)} \xi_s^{(n)}(i)
\]

\[
= \left\{ s + \sum_{k=1}^p k t_k^{(n)} \partial_s^{(n)} \right\} \xi_s^{(n)}(i), \quad s = 1, \ldots, p. \quad (11)
\]

In what follows, we are going to investigate some class of self-similar solutions of extended discrete KP hierarchy and the latter auxiliary formula is of great importance from the point of view of constructing purely discrete systems to which lead corresponding ansatzes.

3. Self-similar solutions and integrable mappings

It is evident, that linear systems (2) and (3) and its consistency relations (5) and (6) are invariant under group of dilations

\[
q_k^{(n,r)}(i) \to \epsilon^k q_k^{(n,r)}(i), \quad t_l \to \epsilon^{-l} t_l, \quad z \to \epsilon z, \quad \Psi_i \to \epsilon^i \Psi_i.
\]

In what follows we consider dependence only on finite number of evolution parameters \( t_1^{(n)}, \ldots, t_p^{(n)} \). Invariants of this group are

\[
T_l = \frac{t_l^{(n)}}{(p t_p^{(n)})^{l/p}}, \quad l = 1, \ldots, p - 1, \quad \xi = (p t_p^{(n)})^{1/p} z,
\]

\[
\psi_i = z^i \Psi_i, \quad x_k^{(n,r)}(i) = (p t_p^{(n)})^{k/p} q_k^{(n,r)}(i).
\]

From this one gets the ansatzes for self-similar solutions

\[
q_k^{(n,r)}(i) = \frac{1}{(p t_p^{(n)})^{k/p}} x_k^{(n,r)}(i), \quad \Psi_i = z^i \psi_i(\xi; T_1, \ldots, T_{p-1}). \quad (12)
\]

Here \( x_k^{(n,r)}(i) \)'s are supposed to be unknown functions of \( T_1, \ldots, T_{p-1} \). Direct substitution of (12) into (5) gives

\[
\partial_{T_l} x_k^{(n,r)}(i) = X_{k,l}^{(n,r)}(i), \quad l = 1, \ldots, p - 1
\]

and

\[
Y_{k,p}^{(n,r)}(i) = k x_k^{(n,r)}(i) + T_1 X_{k,1}^{(n,r)}(i) + 2T_2 X_{k,2}^{(n,r)}(i) + \ldots
\]
Here $X^{(n,r)}_{k,l}(i)$'s are rhs's of (5) where $q^{(n,r)}_{k}(i)$'s are replaced by $x^{(n,r)}_{k}(i)$'s. Moreover, we must take into account algebraic relations

$$x^{(n,r_1+r_2)}_k(i) = x^{(n,r_1)}_k(i) + \sum_{s=1}^{k-1} x^{(n,r_1)}_s(i) \cdot x^{(n,r_2)}_{k-s}(i+r_1-sn) + x^{(n,r_2)}_k(i+r_1)$$

$$= x^{(n,r_2)}_k(i) + \sum_{s=1}^{k-1} x^{(n,r_2)}_s(i) \cdot x^{(n,r_1)}_{k-s}(i+r_2-sn) + x^{(n,r_1)}_k(i+r_2)$$

which follows from (6). Corresponding auxiliary linear equations are transformed to the following form:

$$\partial_l T_l \psi = (X^{ln}_{(n)}(i)_+)_l \psi, \ l = 1, ..., p-1,$$

$$\xi \psi \xi = \left\{ T_1(X^{n}_{(n)}(i))_++ 2T_2(X^{2n}_{(n)}(i))_++ \ldots 
+(p-1)T_{p-1}(X^{(p-1)n}_{(n)}(i))_+ + (X^{pn}_{(n)}(i))_+ \right\} \psi,$$

$$X^{r}_{(n)} \psi = \xi \psi,$$

where

$$X^{r}_{(n)} \equiv \xi A^r + \sum_{k \geq 1} \xi^{1-k} x_k^{(n,r)} A^{r-kn}, \ r \in \mathbb{Z}.$$ 

Let us now observe that if $\{q^{(n,r)}_{k}(i)\}$ represent self-similar solution than by virtue of (11) the quantities $\alpha^{(n)}_i$ do not depend on $t^{(n)}_1, ..., t^{(n)}_p$. Moreover, one can write

$$\alpha^{(n)}_i = \sum_{k=1}^{p-1} kT_k \zeta^{(n)}_k(i) + \zeta^{(n)}_p(i),$$

with $\zeta^{(n)}_k(i) = (p^{(n)}_k)^{k/n} \zeta^{(n)}_k(i)$.

So, one can state that $\alpha^{(n)}_i$ in these circumstances do not depend on parameters $T_1, ..., T_{p-1}$.

Simplest situation in which one can easily to derive pure discrete equations from (14) supplemented by deformation equations in a nice form is when $r = 1$ and $p = 2$. Equations (14) are specified in this case as

$$Y^{(n,1)}_{k,2}(i) = kx_k(i) + T_1 X^{(n,1)}_{k,1}(i) + X^{(n,1)}_{k,2}(i) = 0.$$  

\footnote{In what follows, $x_k(i) \equiv x^{(n,1)}_k(i)$}
In addition, one must take into account deformation equations
\[ \partial T x_k(i) = X_k^{(n,1)}(i) = x_{k+1}(i+n) - x_k(i) \]
\[ + x_k(i) \left( \sum_{s=1}^{n} x_1(i+s-1) - \sum_{s=1}^{n} x_1(i+s-kn) \right). \]  

(17)

**Proposition 3.1.** The relations (16) read as infinite-field mapping
\[ \alpha_i^{(n)} = T_1 x_1(i) + x_1(i) \cdot \sum_{s=1-n}^{n-1} x_1(i+s) + x_2(i) + x_2(i+n), \]  

(18)

\[ X_k(i) = x_k(i) \left( k + \sum_{s=1}^{n} \alpha_i^{(n)} + n \sum_{s=1}^{n} \alpha_i^{(n)} - k \right) \]
\[ + x_{k+1}(i+n) \left( T_1 + \sum_{s=1}^{2n} x_1(i+s-1) \right) \]
\[ - x_{k+1}(i) \left( T_1 + \sum_{s=1}^{2n} x_1(i+s-(k+1)n) \right) \]
\[ + x_{k+2}(i+2n) - x_{k+2}(i) = 0, \quad k \geq 1 \]  

(19)

with \( \alpha_i^{(n)} \)'s being arbitrary constants.

**Proof.** The proposition is proved by straightforward calculations. In what follows we use the formula (9) with \( p = 1, 2 \), where \( q_k(i) \)'s are replaced by \( x_k(i) \)'s. We have
\[ X_k^{(n,1)}(i) = x_{k+1}(i+n) - x_{k+1}(i) + x_k(i) \left( x_1^{(n,n)}(i) - x_1^{(n,n)}(i+1-kn) \right) \]
\[ = x_{k+1}(i+n) - x_{k+1}(i) + x_k(i) \left( \sum_{s=1}^{n} \xi_1^{(n)}(i+s-1) \right. \]
\[ \left. - \sum_{s=1}^{n} \xi_1^{(n)}(i+s-kn) \right), \]  

(20)

\[ X_k^{(n,1)}(i) = x_{k+2}(i+2n) + x_1^{(n,2n)}(i) x_{k+1}(i+n) + x_2^{(n,2n)}(i) x_k(i) \]
\[ - x_{k+2}(i) - x_1^{(n,2n)}(i+1-(k+1)n) x_{k+1}(i) - x_2^{(n,2n)}(i+1-kn) x_k(i) \]
\[ x_{k+2}(i + 2n) + x_{k+1}(i + n) \cdot \sum_{s=1}^{2n} x_1(i + s - 1) + x_k(i) \cdot \sum_{s=1}^{n} \xi_2(n)(i + s) \]
\[-x_{k+2}(i) - x_{k+1}(i) \cdot \sum_{s=1}^{2n} x_1(i + s - (k + 1)n) + x_k(i) \cdot \sum_{s=1}^{n} \xi_2(n)(i + s - kn). \]

One can easily check that substituting (20) and (21) into (16) gives (19) with \( \alpha_i(n) \) given by (18). Moreover, it is already proved that \( \alpha_i(n) \)'s do not depend on \( t_1^{(n)}, ..., t_p^{(n)} \) and therefore on \( T_1, ..., T_{p-1} \).

It is obvious that the system (18) and (19) admits as well as equations (17) reduction with the help of simple condition
\[ x_k(i) \equiv 0, \quad k > l, \quad l \geq 1. \]
Then \( l \)-th equation in (19) is specified as
\[ x_l(i) \cdot \left\{ l + \sum_{s=1}^{n} \alpha_i^{(n)}_{i+s-1} - \sum_{s=1}^{n} \alpha_i^{(n)}_{i+s-ln} \right\} = 0. \]
Since it is supposed that \( x_l(i) \neq 0 \) then the constants \( \alpha_i^{(n)} \) are forced to be subjects of constraint
\[ l + \sum_{s=1}^{n} \alpha_i^{(n)}_{i+s-1} - \sum_{s=1}^{n} \alpha_i^{(n)}_{i+s-ln} = 0. \]

4. \( dP I \) and its hierarchy

Let us consider, as a simplest example, the case corresponding to Volterra lattice hierarchy, that is \( n = 2, r = 1, l = 1 \). Take \( p = 2 \). The equations (16) and (17) are written down as follows:
\[ x'_i = x_i(x_{i+1} - x_{i-1}), \quad ' \equiv \partial / \partial T_1, \]
\[ x_i + T_1 x_i \left\{ x_1^{(2,2)}(i) - x_1^{(2,2)}(i - 1) \right\} + x_i \left\{ x_2^{(2,4)}(i) - x_2^{(2,4)}(i - 1) \right\} = 0. \]

Here we denote \( x_i = x_1^{(2,1)}(i) \). Using (7) one calculates
\[ x_1^{(2,2)}(i) = x_i + x_{i+1}, \]
\[ x_2^{(2,4)}(i) = x_i(x_{i-1} + x_i + x_{i+1}) + x_{i+1}(x_i + x_{i+1} + x_{i+2}). \]
Taking into account these relations, the equation (23) turns into
\[ T_1 x_i + x_i(x_{i-1} + x_i + x_{i+1}) = \alpha_i^{(2)}, \]
\[ 1 + \alpha_i^{(2)} - \alpha_{i-1}^{(2)} = 0. \] (24)

One can rewrite (24) as
\[ x_{i-1} + x_i + x_{i+1} = -T_1 + \frac{\alpha_i^{(2)}}{x_i}, \] (25)
where \( \alpha_i^{(2)} \)'s are some constants forced to be subjects of the quasi-periodicity constraint: \( \alpha_{i+2}^{(2)} = \alpha_i^{(2)} - 1 \). One can immediately to write down the solution of this equation: \( \alpha_i^{(2)} = \alpha - \frac{1}{2} + \beta(-1)^i \), where \( \alpha \) and \( \beta \) are some complex constants. Provided these conditions, (25) is dP\( L \) [4].

Observe that evolution equation (22) with (25) turns into
\[ x_i' = 2x_i x_{i+1} + x_i^2 + T_1 x_i - \alpha_i^{(2)}. \]
It can be easily checked that together with (25) this lattice is equivalent to the pair of ordinary first-order differential equations
\[ w_1' = 2w_1 w_2 + w_1^2 + T_1 w_1 + a, \]
\[ w_2' = -2w_1 w_2 - w_2^2 - T_1 w_2 - b \] (26)
with discrete symmetry transformation (cf. [6])
\[ \overline{w}_1 = w_2, \quad \overline{w}_2 = -w_1 - w_2 - T_1 - \frac{b}{w_2}, \quad \overline{a} = b, \quad \overline{b} = a + 1, \]
where \( w_1 \equiv x_i, \quad w_2 = x_{i+1}, \quad a \equiv -\alpha_i^{(2)}, \quad b \equiv -\alpha_{i+1}^{(2)} \) for some fixed (but arbitrary) value \( i = i_0 \). In turn the system (26) is equivalent to second-order equation
\[ w'' = \frac{(w')^2}{2w} + \frac{3}{2} w^3 + 2T_1 w^2 + \left( \frac{T_1^2}{2} + a - 2b + 1 \right) w - \frac{a^2}{2w}, \quad w \equiv w_1 \] (27)
with corresponding symmetry transformation
\[ \overline{w} = \frac{w' - w^2 - T_1 w - a}{2w}, \quad \overline{a} = b, \quad \overline{b} = a + 1. \]
In fact this is P\( IV \) with B"acklund transformation [6]. By rescaling \( T_1 \to \sqrt{2} T_1, \quad w \to w/\sqrt{2} \) it can be turned to following canonical form:
\[ w'' = \frac{(w')^2}{2w} + \frac{3}{2} w^3 + 4T_1 w^2 + 2(T_1^2 + a - 2b + 1) w - \frac{2a^2}{w} \] (27)
$$w = w' - w^2 - 2T_1 w - 2a + 2T_1 w - 2a$$, \( \sigma = b, \ b = a + 1 \).

As a result, we obtain the well-known relationship between \( d\Pi\) (25) and \( P_{IV}\) (27) (see, for example [5]).

Remark 4.1. The equations (26) can be interpreted as self-similar reduction of Levi system [8]

$$v_{1t2} = (-v_1' + v_1^2 + 2v_1v_2)',$$
$$v_{2t2} = (v_2' + v_2^2 + 2v_1v_2)'$$

with the help of the ansatz

$$v_k = \left(\frac{1}{2t_2}\right)^{1/2}w_k(T_1), \ k = 1, 2.$$

Let us discuss now the higher members in \( d\Pi\) hierarchy. To construct them, one needs to consider the cases \( p = 3, 4\) and so on, that is

$$x_i + T_1 x_i \left\{ x_1^{(2,2)}(i) - x_1^{(2,2)}(i - 1) \right\} + 2T_2 x_i \left\{ x_2^{(2,4)}(i) - x_2^{(2,4)}(i - 1) \right\}$$
$$+ x_i \left\{ x_3^{(2,6)}(i) - x_2^{(2,6)}(i - 1) \right\} = 0,$$

$$x_i + T_1 x_i \left\{ x_1^{(2,2)}(i) - x_1^{(2,2)}(i - 1) \right\} + 2T_2 x_i \left\{ x_2^{(2,4)}(i) - x_2^{(2,4)}(i - 1) \right\}$$
$$+ 3T_3 x_i \left\{ x_3^{(2,6)}(i) - x_2^{(2,6)}(i - 1) \right\} + \left\{ x_4^{(2,8)}(i) - x_4^{(2,8)}(i - 1) \right\} = 0.$$

We have calculated, to wit the following

$$x_3^{(2,6)}(i) = \zeta_3^{(2)}(i) + \zeta_3^{(2)}(i + 1)$$

with

$$\zeta_3^{(2)}(i) = x_i \left\{ \zeta_2^{(2)}(i - 1) + \zeta_2^{(2)}(i) + \zeta_2^{(2)}(i + 1) + x_{i-1}x_{i+1} \right\}$$

and

$$x_4^{(2,8)}(i) = \zeta_4^{(2)}(i) + \zeta_4^{(2)}(i + 1)$$

with

$$\zeta_4^{(2)}(i) = x_i \left\{ \zeta_3^{(2)}(i - 1) + \zeta_3^{(2)}(i + 1) + x_{i-1}x_{i+1}(x_{i-2} + x_{i-1} + x_i + x_{i+1} + x_{i+2}) \right\}.$$
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\[ T_1 x_i + 2T_2 \xi_2^{(2)}(i) + 3T_3 \xi_3^{(2)}(i) + \xi_4^{(2)}(i) = \alpha_i^{(2)}, \]

with corresponding constants \( \alpha_i^{(2)} \) being subjected to the constraint
\[ 1 + \alpha_{i+1}^{(2)} - \alpha_{i-1}^{(2)} = 0. \]
These results entirely correspond to that of the work [3].

5. Generalizations of \( dP_I \)

Let us consider one-field reductions of the system given by equations (18) and (19) when \( n \) is arbitrary. The analogues of the equations (22) and (23) in this case are

\[ x_i' = x_i \left( \sum_{s=1}^{n-1} x_{i+s} - \sum_{s=1}^{n-1} x_{i-s} \right), \tag{30} \]

\[ x_i + T_1 x_i \left\{ x_1^{(n,n)}(i) - x_1^{(n,n)}(i+1-n) \right\} \\
+ x_i \left\{ x_2^{(n,2n)}(i) - x_2^{(n,2n)}(i+1-n) \right\} = 0. \tag{31} \]

We take into account that
\[ x_1^{(n,n)}(i) = \sum_{s=1}^{n} x_{i+s-1}, \quad x_2^{(n,2n)}(i) = \sum_{s=1}^{n} x_{i+s-1} \left( \sum_{s_1=1}^{2n-1} x_{i+s_1+s_1-n-1} \right). \]

Then the equation (31) can be cast into the form
\[ T_1 x_i + x_i (x_{i+1-n} + \ldots + x_{i+n-1}) = \alpha_i^{(n)}, \]
\[ 1 + \sum_{s=1}^{n-1} \alpha_{i+s}^{(n)} - \sum_{s=1}^{n-1} \alpha_{i-s}^{(n)} = 0. \tag{32} \]

One can write the solution of (32) as
\[ \alpha_i^{(n)} = \alpha - \frac{1}{(n-1)n} i + \beta \omega^i \tag{33} \]

with arbitrary constants \( \alpha, \beta \in \mathbb{C} \) and \( \omega = e^{\frac{2\pi i}{n}} = \exp(2\pi\sqrt{-1}/n) \).

So, one concludes that in this case self-similar ansatz leads to equation
\[ x_{i+1-n} + \ldots + x_{i+n-1} = -T_1 + \frac{\alpha_i^{(n)}}{x_i}, \tag{34} \]
where the constants \( \alpha_i^{(n)} \)'s are given by (33).
Standard analysis of singularity confinement shows that this property for (34) is valid provided that
\[ \alpha^{(n)}_{i+n} - \alpha^{(n)}_i = \alpha^{(n)}_{i+2n-1} - \alpha^{(n)}_{i+n-1}. \] (35)
but one can show that this equation does not contradict to (32), but it is more general. Indeed, it follows from (32) that
\[ -1 = \sum_{s=1}^{n-1} \alpha^{(n)}_{i+s+n-1} - \sum_{s=1}^{n-1} \alpha^{(n)}_{i+s} = \sum_{s=1}^{n-2} \alpha^{(n)}_{i+s+2n-2} - \sum_{s=1}^{n-2} \alpha^{(n)}_{i+s-1}. \] (36)
Second equality in (36) can be rewritten in the form
\[ \sum_{s=2}^{n-1} \alpha^{(n)}_{i+s+n-1} + \alpha^{(n)}_{i+2n-1} - \sum_{s=1}^{n-2} \alpha^{(n)}_{i+s} - \alpha^{(n)}_{i+n-1} = \alpha^{(n)}_{i+n} + \sum_{s=3}^{n} \alpha^{(n)}_{i+s+n-2} - \alpha^{(n)}_i - \sum_{s=2}^{n-1} \alpha^{(n)}_{i+s-1}. \]
From the latter one obtains (35).
Finally, let us show that equations (30), (32) and (34) are equivalent to higher order Painlevé equation of type $A_{l}^{(1)}$ [10] with $l = 2(n-1)$, i.e. higher order generalization of $P_{IV}$. Following the line of previous section, one can observe that equations (30), (32) and (34) are equivalent to the system
\[ w'_k = w_k \left( 2 \sum_{s=1}^{n-1} w_{k+s} + w_{k} + T_1 \right) + a_k, \]
\[ w'_{k+n-1} = -w_{k+n-1} \left( 2 \sum_{s=1}^{n-1} w_{k+s-1} + w_{k+n-1} + T_1 \right) - a_{k+n-1} \] (37)
\[ (k = 1, \ldots, n - 1) \]
supplemented by Bäcklund transformation
\[ \overline{w}_k = w_{k+1}, \ k = 1, \ldots, 2n - 3, \ \overline{w}_{2(n-1)} = -\sum_{s=1}^{2(n-1)} w_s - \frac{a_n}{w_n} - T_1, \]
\[ \overline{a}_k = a_{k+1}, \ k = 1, \ldots, 2n - 3, \ \overline{a}_{2(n-1)} = \sum_{s=1}^{n-1} a_s - \sum_{s=1}^{n-2} a_{n+s} + 1. \] (38)
Here we identify $w_k = x_{i+k-1}$ and $a_k = -\alpha^{(n)}_{i+k-1}$. To write down the equations (37) in symmetric form one need to introduce new variables $\{ f_0, f_1, \ldots, f_{2(n-1)} \}$ by
\[ f_{2k} = -w_k, \ f_{2k-1} = -w_{k+n-1} \ (k = 1, \ldots, n - 1), \ f_0 = \sum_{s=1}^{2(n-1)} w_s + T_1. \]
It is evident that $\sum_{s=0}^{2(n-1)} f_s = T_1$. By straightforward calculations one can check that the system (37) cast into following symmetric form

$$f'_k = f_k \left( \sum_{r=1}^{n-1} f_{k+2r-1} - \sum_{r=1}^{n-1} f_{k+2r} \right) + c_k \quad (k = 0, \ldots, 2(n-1)) \quad (39)$$

where subscripts are supposed to be an elements of $\mathbb{Z}/(2n-1)\mathbb{Z}$. The constants $c_k$’s are related with $a_k$’s by the relations

$$c_{2k} = -a_k, \quad c_{2k-1} = a_{k+n-1}, \quad k = 1, \ldots, n-1,$$

$$c_0 = \sum_{s=1}^{n-1} a_s - \sum_{s=1}^{n-1} a_{s+n-1} + 1, \quad \sum_{s=0}^{2(n-1)} c_s = 1.$$

The system (39) is nothing but Noumi-Yamada system of $A_l^{(1)}$ type with $l = 2(n-1)$. As for Bäcklund transformation (38), one can verify that it coincides with an element $T = s_1 \pi^2$ of the extended affine Weil group $\tilde{W} = \langle \pi, s_0, \ldots, s_{2(n-1)} \rangle$ [10].

For the sake of completeness, let us provide the reader by suitable information on some representation of affine Weil group of type $A_l^{(1)}$ which is useful for constructing of Bäcklund transformations of P IV and P V and its generalizations. The action of automorphisms $\{s_0, \ldots, s_l\}$ on the field of rational functions in variables $\{c_0, \ldots, c_l\}$ and $\{f_0, \ldots, f_l\}$ can be defined as follows [10]:

$$s_k(c_k) = -c_k, \quad s_k(c_l) = c_l + c_k \quad (l = k \pm 1),$$

$$s_k(f_k) = f_k, \quad s_k(f_l) = f_l + \frac{c_k}{f_k} \quad (l = k \pm 1).$$

One also defines an automorphism $\pi$ by the rules $\pi(c_k) = c_{k+1}$ and $\pi(f_k) = f_{k+1}$. It is known by [9] that this set of automorphisms define a representation of the extended affine Weil group $\tilde{W}$ and represents a collection of Bäcklund transformations of the system (39).

6. Miura transformations

We showed in Ref. [15] that symmetry transformation

$$g_k : \begin{cases} a(i) \to z^{-k} a^{[k]}(ki) \\ h(i) \to h(ki) \end{cases}$$

on the space of DKP chain solutions is a suitable basis for constructing of some class of lattice Miura transformations. In the language
of invariant submanifolds we have \(g_k(S_{kn}^0) \subset S_{kn}^0\) and \(g_k(S_{kn,rl}) \subset S_{n,rl}^0\). In terms of coordinate \(q_{s}^{(n,r)}(i)\) one can use simple relation \(\tilde{q}_{s}^{(n,r)}(i) = q_{s}^{(kn,kr)}(ki)\) but to get lattice Miura transformation in the form \(\tilde{q}_{k}^{(n,r)}(i) = F(\{q_{s}^{(kn,kr)}(ki)\})\) one needs to make use algebraic relations (6). For example, Miura transformation \(g_2(S_{0}^0) \subset S_{0}^0\) reads

\[
\tilde{q}_{k}^{(1,1)}(i) = q_{k}^{(2,2)}(2i)|_{(6)}
\]

\[
= q_{k}^{(2,1)}(2i) + \sum_{s=1}^{k-1} q_{s}^{(2,1)}(2i) \cdot q_{k-s}^{(2,1)}(2i - 2s + 1) + q_{k}^{(2,1)}(2i + 1). \quad (40)
\]

We believe that this approach can be applied for constructing Miura transformations relating different integrable mappings. Let us show the simplest example. Denote \(y_k(i) = x_k^{(1,1)}(i)\) and \(\beta_i = \alpha_i^{(1)}\). The equations (18) and (19) defining integrable mapping in this case are specified as

\[
\beta_i = T_1 y_1(i) + y_1^2(i) + y_2(i) + y_2(i + 1), \quad (41)
\]

\[
Y_k(i) = y_k(i) (k + \beta_i - \beta_{i-k+1}) + y_{k+1}(i + 1) (T_1 + y_1(i) + y_1(i + 1))
- y_{k+1}(i) (T_1 + y_1(i - k) + y_1(i - k + 1))
+ y_{k+2}(i + 2) - y_{k+2}(i) = 0, \quad k \geq 1. \quad (42)
\]

Denote \(x_k(i) = x_k^{(2,1)}(i)\) and \(\alpha_i = \alpha_i^{(2)}\). Infinite-field discrete system in the case \(n = 2\) is

\[
\alpha_i = T_1 x_1(i) + x_1(i)(x_1(i - 1) + x_1(i) + x_1(i + 1)) + x_2(i) + x_2(i + 2), \quad (43)
\]

\[
X_k(i) = x_k(i) (k + \alpha_i + \alpha_{i+1} - \alpha_{i-2k+1} - \alpha_{i-2k+2})
+ x_{k+1}(i + 2) (T_1 + x_1(i) + x_1(i + 1) + x_1(i + 2) + x_1(i + 3))
- x_{k+1}(i) (T_1 + x_1(i - 2k - 1) + x_1(i - 2k))
+ x_1(i - 2k + 1) + x_1(i - 2k + 2))
+ x_{k+2}(i + 4) - x_{k+2}(i) = 0, \quad k \geq 1. \quad (44)
\]

Restriction of the latter system on \(S_{2,1,1}\) gives \(dP_I\) (25), while the restriction of the system (41) and (42) on \(S_{1,1,2}\) yields two-field system

\[
\beta_i = T_1 y_1(i) + y_1^2(i) + y_2(i) + y_2(i + 1),
\]
$$y_1(i) + y_2(i+1)(T_1 + y_1(i) + y_1(i+1)) - y_2(i)(T_1 + y_1(i-1) + y_1(i)) = 0,$$

$$2 + \beta_i - \beta_{i-1} = 0. \tag{45}$$

One can easily write down Miura transformation relating infinite-field mappings given by pairs of equations (41), (42) and (43), (44), respectively, by replacing $q_1^{(1,1)}(i) \to y_k(i)$ and $q_1^{(2,1)}(i) \to x_k(i)$ in (40), to wit

$$y_k(i) = x_k(2i) + \sum_{s=1}^{k-1} x_s(2i) \cdot x_{k-s}(2i - 2s + 1) + x_k(2i + 1). \tag{46}$$

Moreover we have $\beta_i = \alpha_{2i} + \alpha_{2i+1}$. By straightforward but tedious calculations one can check that substituting (46) into (42) gives

$$Y_k(i)|_{(46)} = X_k(2i) + \sum_{s=1}^{k-1} X_s(2i) \cdot X_{k-s}(2i - 2s + 1) + X_k(2i + 1) = 0.$$

As for transformation $g_2(S_{2,1,1}) \subset S_{1,1,2}$ relating $dP_I$ with the system (45), we have it in the form

$$y_1(i) = x(2i), \quad y_2(i) = x(2i + 1), \quad \beta_i = \alpha_{2i} + \alpha_{2i+1}.$$

### 7. Concluding remarks

We showed that in the simplest case corresponding to the restriction of dynamics on phase-space of the DKP chain on invariant submanifold $S_{n,1,1}$ one derives one-field discrete dynamical system governing Bäcklund transformation of Noumi-Yamada system of $A_{2(n-1)}^{(1)}$ type which is a natural higher-order generalization of $P_{IV}$. As is known higher-order generalizations of $P_{V}$ correspond to affine Weil groups of type $A_{l}^{(1)}$ with odd $l$.

On the other hand, these generalizations are also described by Veselov-Shabat periodic dressing chains

$$r'_i + r'_{i+1} = r_i^2 - r_{i+1}^2 + \alpha_i, \quad r_{i+N} = r_i, \quad \alpha_{i+N} = \alpha_i$$

with odd $N \geq 3$ for $P_{IV}$ and with even $N \geq 4$ for $P_{V}$, respectively [16], [1]. So, in a sense $P_{IV}$ and $P_{V}$ go in parallel in these two settings. We can expect that our approach also covers $P_{V}$ and its higher-order generalizations, but to save the space we leave this question for subsequent publications.
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