Third-Order Differential Subordination Results for Analytic Functions Associated with a Certain Differential Operator

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Abstract: In this research, we study suitable classes of admissible functions and establish the properties of third-order differential subordination by making use a certain differential operator of analytic functions in \( U \) and have the normalized Taylor-Maclaurin series of the form: 
\[
 f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U).
\]
Some new results on differential subordination with some corollaries are obtained. These properties and results are symmetry to the properties of the differential superordination to form the sandwich theorems.

Keywords: analytic function; univalent function; differential operator; third-order; admissible functions; dominant; differential subordination

1. Introduction

Let \( S \) be the class of functions which are analytic in the open unit disk \( U = \{z \in \mathbb{C} : |z| < 1\} \). Also let \( S_\tau[a,n] \ (n \in \mathbb{N} = \{1,2,3, \cdots\}; \ a \in \mathbb{C}\) be the subclass of \( S \) in which the functions satisfy the following form:
\[
 f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \quad (z \in U).
\]

Let \( G \) be a subclass of \( S \) which are analytic in \( U \) and have the normalized Taylor-Maclaurin series of the form:
\[
 f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U). \quad (1)
\]

Suppose that \( f \) and \( g \) are analytic functions in \( S \). We say that \( f \) is subordinate to \( g \), written as follows:
\[
 f \prec g \quad \text{in} \quad U \quad \text{or} \quad f(z) \prec g(z), \quad (z \in U)
\]
if there exists a Schwarz function \( \omega \in S \), which is analytic in \( U \), with \( \omega(0) = 0 \ and \ |\omega(z)| < 1 \ (z \in U) \), such that \( f(z) = g(\omega(z)), \ (z \in U) \).

Furthermore, if \( g \) is univalent in \( U \), we have [1]:
\[
 g(z) \prec f(z) \Leftrightarrow g(0) = f(0) \quad \text{and} \quad g(U) \subset f(U), \quad (z \in U).
\]

For a function \( f(z) \in G \) given by (1) and \( g(z) \in G \), defined by:
\[
 g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (2)
\]
the Hadamard product (or convolution) of $f(z)$, $g(z)$ denoted by $f \ast g$ is defined by
\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g \ast f)(z).
\] (3)

We’ll go over some additional terms and concepts from the differential subordination theory here.

**Definition 1.** [2] Let $\Pi : C^4 \times U \to C$ and suppose that the function $h(z)$ is univalent in $U$. If the function $p(z)$ is analytic in $U$ and satisfies the following third-order differential subordination:
\[
\Pi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right) \prec h(z)
\] (4) then $p(z)$ is called a solution of the differential subordination (4). Furthermore, a given univalent function $q(z)$ is called a dominant of the solutions of (4) or more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (4). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (4) is said to be the best dominant.

**Definition 2.** [2] Let $Q$ be the set of all functions $q$ that are analytic and univalent on $U \setminus E(q)$, where
\[
E(q) = \left\{ \xi : \xi \in \partial U : \lim_{z \to \xi} q(z) = \infty \right\}
\] (5) and $\min |q'(z)| = p > 0$ for $\xi \in \partial U \setminus E(q)$. Further, let the subclass of $Q$ for which $q(0) = a$, be denoted by $Q(a)$ with
\[
Q(0) = Q_0 \text{ and } Q(1) = Q_1.
\] (6)

The subordination methodology is applied to appropriate classes of admissible functions. The following class of admissible functions was given by Antonino and Miller [2].

**Definition 3.** [2] Let $\Omega$ be a set in $C$. Also $q \in Q$ and $n \in \mathbb{N} \setminus \{1\}$. The class $\Psi_n[\Omega, q]$ of admissible functions consists of those functions $\Pi : C^4 \times U \to C$, which satisfy the following admissibility conditions:
\[
\Pi(r,s,t,u; z) \notin \Omega
\]
whenever
\[
r = q(\xi), \ s = k\xi q'(\xi), \ \Re \left( \frac{t}{s} + 1 \right) \geq k \Re \left( \frac{\xi q''(\xi)}{q'(\xi)} + 1 \right)
\]
and
\[
\Re \left( \frac{u}{s} \right) \geq k^2 \Re \left( \frac{\xi^2 q'''(\xi)}{q''(\xi)} \right),
\]
where $z \in U$, $\xi \in \partial U \setminus E(q)$ and $k \geq n$.

**Lemma 1.** [2]: Let $p \in S[a, n]$ with $n \geq 2$ and $q \in Q(a)$ satisfying the following conditions:
\[
\Re \left( \frac{\xi^2 q'''(\xi)}{q''(\xi)} \right) \geq 0 \text{ and } \left| \frac{z p'(z)}{q'(\xi)} \right| \leq k,
\]
where $z \in U$, $\xi \in \partial U \setminus E(q)$ and $k \geq n$. If $\Omega$ is a set in $C$, $\Pi \in \Psi_n[\Omega, q]$, and
\[
\Pi \left( p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z \right) \in \Omega,
\]
then
p(z) \sim q(z) \quad (z \in \mathbb{U}).

The geometric function theory relies heavily on the study of operators. Convolution of certain analytic functions may be used to express several differential and integral operators. This formalism, it is noticed, facilitates further mathematical research and also aids in a better understanding of the geometric aspects of such operators. The importance of convolution in the theory of operators may be understood by the following set of examples given by Barnard and Kellogg [3] and Carlson–Shaffer [4], etc.

Now, we introduce new operator by using the convolution in our this study.

**Definition 4.** Let \( f, g \in \mathcal{G}, \, \sigma \in \mathbb{N}_0 \) and \( \delta, \lambda \in \mathbb{N} \), we define the operator:

\[
\mathcal{D}_{\sigma,\delta,\lambda}^1 : \mathcal{G} \rightarrow \mathcal{G}
\]

by

\[
\mathcal{D}_{\sigma,\delta,\lambda}^1(f \ast g)(z) = \frac{\sigma}{\sigma + \delta + \lambda} \mathcal{D}_{\sigma,\delta,\lambda}^0(f \ast g)(z) + \frac{\delta + \lambda}{\sigma + \delta + \lambda} z \left( \mathcal{D}_{\sigma,\delta,\lambda}^1(f \ast g)(z) \right)'
\]

\[
= z + \sum_{n=2}^{\infty} \frac{\delta + \lambda}{\sigma + \delta + \lambda} \left( \frac{\sigma + (\delta + \lambda)n}{\sigma + \delta + \lambda} \right) a_n b_n z^n,
\]

where

\[
\mathcal{D}_{\sigma,\delta,\lambda}^0(f \ast g)(z) = (f \ast g)(z).
\]

Let \( f, g \in \mathcal{G}, \, \sigma \in \mathbb{N}_0 \) and \( \delta, \lambda \in \mathbb{N} \). Then

\[
\mathcal{D}_{\sigma,\delta,\lambda}^1(f \ast g)(z) = \frac{\sigma}{\sigma + \delta + \lambda} \mathcal{D}_{\sigma,\delta,\lambda}^0(f \ast g)(z) + \frac{\delta + \lambda}{\sigma + \delta + \lambda} z \left( \mathcal{D}_{\sigma,\delta,\lambda}^1(f \ast g)(z) \right)'
\]

\[
= \frac{\sigma}{\sigma + \delta + \lambda} \left[ z + \sum_{n=2}^{\infty} a_n b_n z^n \right] + \frac{\delta + \lambda}{\sigma + \delta + \lambda} z \left[ z + \sum_{n=2}^{\infty} a_n b_n z^n \right]'
\]

\[
= \frac{\sigma}{\sigma + \delta + \lambda} z + \sum_{n=2}^{\infty} \frac{\sigma}{\sigma + \delta + \lambda} a_n b_n z^n + \frac{\delta + \lambda}{\sigma + \delta + \lambda} z \left[ 1 + \sum_{n=2}^{\infty} n a_n b_n z^{n-1} \right]
\]

\[
= z + \sum_{n=2}^{\infty} \frac{\sigma + (\delta + \lambda)n}{\sigma + \delta + \lambda} a_n b_n z^n.
\]

In general

\[
\mathcal{D}_{\sigma,\delta,\lambda}^m(f \ast g)(z) = \frac{\sigma}{\sigma + \delta + \lambda} \mathcal{D}_{\sigma,\delta,\lambda}^{m+1}(f \ast g)(z) + \frac{\delta + \lambda}{\sigma + \delta + \lambda} z \left( \mathcal{D}_{\sigma,\delta,\lambda}^m(f \ast g)(z) \right)'
\]

\[
= z + \sum_{n=2}^{\infty} \left( \frac{\sigma + (\delta + \lambda)n}{\sigma + \delta + \lambda} \right)^m a_n b_n z^n.
\]

By simple calculation, we obtain

\[
(\delta + \lambda)z \left( \mathcal{D}_{\sigma,\delta,\lambda}^m(f \ast g)(z) \right)' = (\sigma + \delta + \lambda) \left( \mathcal{D}_{\sigma,\delta,\lambda}^{m+1}(f \ast g)(z) \right) - \sigma \left( \mathcal{D}_{\sigma,\delta,\lambda}^m(f \ast g)(z) \right) \mathcal{D}_{\sigma,\delta,\lambda}^m(f \ast g)(z).
\]

The notion of the third-order differential subordination can be found in the work of Ponnusamy and Juneja [5]. The recent work by several authors (see for example, [6,7]; see also [8,9]) on the differential subordination attracted many researchers in this field. For example, see [8,10–20].

In this research, we investigate suitable classes of admissible functions associated with the new differential operator \( \mathcal{D}_{\sigma,\delta,\lambda}^m(f \ast g)(z) \) and establish the properties of third-order differential subordination by making use a certain new differential operator of analytic functions in \( \mathbb{U} \) and have the normalized Taylor–Maclaurin series of the form: \( f(z) = \)
z + \sum_{n=2}^{\infty} a_n z^n, \ (z \in \mathbb{U}). \) Some new results on differential subordinations with some corollaries are obtained. Here, we obtain the symmetry of the differential superordination results.

2. Results Related to the Third-Order Differential Subordination

We start with a given set \( \Omega \) and a function \( q \) in this section, and we establish a set of acceptable functions so that (4) holds true. We construct the following new class of admissible functions for this purpose, which will be needed to establish the key third-order differential subordination theorems for the operator \( D_{\sigma,\delta}^m(q,\alpha) \) described by (7).

**Definition 5.** Let \( \Omega \) be a set in \( C \) and \( q \in Q_0 \cap S_0 \). The class \( \mathcal{G}_{\Omega, q} \) of admissible functions consists of those functions \( \phi : C^4 \times U \to C, \) that satisfy the following admissibility conditions:

\[
\phi(\alpha, \beta, \gamma, \nu; z) \notin \Omega,
\]

whenever

\[
\alpha = q(\xi), \ \beta = \frac{k(\delta + \lambda)q'\xi(\xi) + \sigma q(\xi)}{(\sigma + \delta + \lambda)} \geq kR (q''(\xi) + 1)
\]

and

\[
R\left(\frac{(\sigma + \delta + \lambda)^2(3\gamma) + (\sigma + \delta + \lambda)(2(2\sigma + \delta + \lambda)) + \sigma(3\sigma - 2(2\sigma + \delta + \lambda))}{(\delta + \lambda)^2(2\sigma + \delta + \lambda) - \sigma(\delta + \lambda)^2\beta^2\alpha} + \frac{2q''(\xi)}{q'q''(\xi)}\right) \geq k^2 R \left(\frac{q''(\xi)}{q'(\xi)}\right),
\]

where \( z \in U, \ \xi \in \partial U \setminus E(q) \) and \( k \geq 2 \).

**Theorem 1.** Let \( \phi \in \mathcal{G}_{\Omega, q} \). If the functions \( f, g \in \mathcal{G} \) and \( q \in Q_0 \), satisfy the following conditions:

\[
R\left(\frac{q''(\xi)}{q'(\xi)}\right) \geq 0, \quad \left| \frac{D_{\sigma,\delta}^m(f \ast g)(z)}{q'(\xi)} \right| \leq k \quad (9)
\]

and

\[
\{ \phi \left( D_{\sigma,\delta}^m(f \ast g)(z), D_{\sigma,\delta}^{m+1}(f \ast g)(z), D_{\sigma,\delta}^{m+2}(f \ast g)(z), D_{\sigma,\delta}^{m+3}(f \ast g)(z); z \right) : z \in U \} \subset \Omega,
\]

then

\[
D_{\sigma,\delta}^m(f \ast g)(z) \prec q(z), \quad (z \in U).
\]

**Proof.** Define the analytic function \( p(z) \) in \( U \) by

\[
p(z) = D_{\sigma,\delta}^m(f \ast g)(z).
\]

Form Equations (8)–(11), we have

\[
D_{\sigma,\delta}^{m+1}(f \ast g)(z) = \frac{(\delta + \lambda)zp'(z) + \sigma p(z)}{(\sigma + \delta + \lambda)}. \quad (12)
\]

By a similar argument, we get

\[
D_{\sigma,\delta}^{m+2}(f \ast g)(z) = \frac{(\delta + \lambda)^2z^2p''(z) + (\delta + \lambda)(2\sigma + \delta + \lambda)zp'(z) + \sigma^2p(z)}{(\sigma + \delta + \lambda)^2}. \quad (13)
\]
and
\[
\mathcal{D}^{m+3}_{v,\delta,\lambda}(f * g)(z) = \frac{(\delta + \lambda)^3 \sigma^3 (\sigma + \delta + \lambda)^2 z^2 p''(z) + (\delta + \lambda)^2 [\sigma^2 + (\sigma + \delta + \lambda)(2\sigma + \delta + \lambda) + \sigma^2] z p(z)}{(\sigma + \delta + \lambda)^3}
\]
(14)
Define the transformation from \( \mathbb{C}^4 \) to \( \mathbb{C} \) by
\[
\alpha(r, s, t, u) = r, \quad \beta(r, s, t, u) = \frac{(\delta + \lambda)s + \sigma r}{(\sigma + \delta + \lambda)}, \quad \gamma(r, s, t, u) = \frac{(\delta + \lambda)^2 t + (\delta + \lambda)(2\sigma + \delta + \lambda)s + \sigma^2 t}{(\sigma + \delta + \lambda)^2}
\]
(15)
\[
v(r, s, t, u) = \frac{(\delta + \lambda)^3 u + (\delta + \lambda)^2 3(\sigma + \delta + \lambda)t + (\delta + \lambda) [(\sigma + \delta + \lambda)(2\sigma + \delta + \lambda) + \sigma^2] s + \sigma^2 t}{(\sigma + \delta + \lambda)^3}.
\]
(16)
Let
\[
\Pi(r, s, t, u) = \phi(\alpha, \beta, \gamma, v; z)
\]
(17)
Using the Equations (11)–(14), and from the Equation (17), we have
\[
\Pi(p(z), zq'(z), z^2 q''(z), z^3 q'''(z); z) = \phi\left(\mathcal{D}^m_{v,\delta,\lambda}(f * g)(z), \mathcal{D}^{m+1}_{v,\delta,\lambda}(f * g)(z), \mathcal{D}^{m+2}_{v,\delta,\lambda}(f * g)(z), \mathcal{D}^{m+3}_{v,\delta,\lambda}(f * g)(z); z\right).
\]
(18)
Hence, clearly (10) becomes
\[
\Pi\left(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z\right) \in \Omega,
\]
we note that
\[
\frac{t}{s} + 1 = \frac{(\sigma + \delta + \lambda)[(\sigma + \delta + \lambda)\gamma - 2\sigma \beta + \sigma^2 \alpha]}{(\delta + \lambda) [(\sigma + \delta + \lambda)\beta - \sigma \alpha]}
\]
and
\[
\frac{u}{s} = \frac{(\sigma + \delta + \lambda)^3 (v - 3\gamma) + 2(\sigma + \delta + \lambda)(2\sigma + \delta + \lambda)\beta + \sigma(3\sigma - 2(2\sigma + \delta + \lambda)\alpha)}{(\delta + \lambda)^2 [(\sigma + \delta + \lambda)\beta - \sigma \alpha]}
\]
Thus clearly, the admissibility condition for \( \phi \in \mathcal{A}_2[\Omega, q] \) in Definition 4, is equivalent to admissibility condition \( \Pi \in \Psi_2[\Omega, q] \) as given in Definition 3 with \( n = 2 \).
Therefore, by using (9) and Lemma 1, we have
\[
\mathcal{D}^m_{v,\delta,\lambda}(f * g)(z) \prec q(z).
\]
This completes the proof. \( \square \)

Our next result is consequence of Theorem 1, when the behavior of \( q(z) \) on \( \partial \Omega \) is not known.
Corollary 1. Let $\Omega \subset C$ and let the function $q$ be univalent in $U$ with $q(0) = 1$. Let $\phi \in \mathcal{H}[\Omega, q_p]$ for some $p \in (0, 1)$, where $q_p(z) = q(pz)$. If the function $f, g \in \mathcal{G}$ and $q_p$ satisfies the following conditions:

$$\Re\left(\frac{\xi q_p''(\xi)}{q_p'(\xi)}\right) \geq 0, \quad \left|\frac{D_{\sigma, \delta, \lambda}(f \ast g)(z)}{q_p''(\xi)}\right| \leq k, \quad (z \in \Omega; k \geq 2; \xi \in \partial U \setminus E(q_p))$$

then

$$D_{\sigma, \delta, \lambda}(f \ast g)(z) < q(z), \quad (z \in U).$$

Proof. By applying Theorem 1, we get

$$D_{\sigma, \delta, \lambda}(f \ast g)(z) < q_p(z), \quad (z \in U).$$

The result asserted by Corollary 1 is now deduced from following subordination property

$$q_p(z) < q(z) \quad (z \in U).$$

This completes the proof of Corollary 1. □

If $\Omega \neq C$, is a simply connected domain, the $\Omega = h(U)$ for some conformal mapping $h(z)$ of $U$ on to $\Omega$. In this case the class $\mathcal{H}[h(U), q]$ is written as $\mathcal{H}[h, q]$. This leads to the following immediate consequence of Theorem 1.

Theorem 2. Let $\phi \in \mathcal{H}[h, q]$. If the function $f, g \in \mathcal{G}$ and $q \in \mathcal{Q}_0$, satisfy the following conditions:

$$\Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \quad \left|\frac{D_{\sigma, \delta, \lambda}(f \ast g)(z)}{q''(\xi)}\right| \leq k$$

and

$$\phi\left(D_{\sigma, \delta, \lambda}^m(f \ast g)(z), D_{\sigma, \delta, \lambda}^{m+1}(f \ast g)(z), D_{\sigma, \delta, \lambda}^{m+2}(f \ast g)(z), D_{\sigma, \delta, \lambda}^{m+3}(f \ast g)(z); z\right) \prec h(z),$$

then

$$D_{\sigma, \delta, \lambda}^m(f \ast g)(z) < q(z), \quad (z \in U).$$

The next result is an immediate consequence of Corollary 1.

Corollary 2. Let $\Omega \subset C$ and let the function $q$ be univalent in $U$ with $q(0) = 1$. Also let $\phi \in \mathcal{H}[\Omega, q_p]$ for some $p \in (0, 1)$, where $q_p(z) = q(pz)$. If the function $f, g \in \mathcal{G}$ and $q_p$ satisfies the following conditions:

$$\Re\left(\frac{\xi q_p''(\xi)}{q_p'(\xi)}\right) \geq 0, \quad \left|\frac{D_{\sigma, \delta, \lambda}^m(f \ast g)(z)}{q_p''(\xi)}\right| \leq k, \quad (z \in \Omega; k \geq 2; \xi \in \partial U \setminus E(q_p))$$

and

$$\phi\left(D_{\sigma, \delta, \lambda}^m(f \ast g)(z), D_{\sigma, \delta, \lambda}^{m+1}(f \ast g)(z), D_{\sigma, \delta, \lambda}^{m+2}(f \ast g)(z), D_{\sigma, \delta, \lambda}^{m+3}(f \ast g)(z); z\right) \prec h(z),$$

then

$$D_{\sigma, \delta, \lambda}^m(f \ast g)(z) < q(z), \quad (z \in U).$$
Theorem 3. Let the function $h$ be univalent in $U$. Also let $\phi : C^4 \times U \to C$ and $\Pi$ given by (17). Suppose that following differential equation

$$\Pi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z),$$

(21)

then

$$D_{c,d,\lambda}^m(f \ast g)(z) \prec (f \ast g)(z), \ (z \in U).$$

The following result yield the best dominant of differential subordination (20).

**Theorem 3.** Let the function $h$ be univalent in $U$. Also let $\phi : C^4 \times U \to C$ and $\Pi$ given by (17). Suppose that following differential equation

$$\Pi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z),$$

(21)

then

$$D_{c,d,\lambda}^m(f \ast g)(z) \prec q(z), \ (z \in U)$$

and $q(z)$ is the best dominant.

**Proof from Theorem 1.** We see that $q$ is a dominant of (19) since $q$ satisfies (20), it is also a solution of (19). Therefore, $q$ will be dominated by all dominants. Hence, $q$ is the best dominant. This completes the proof of Theorem 3. $\Box$

In view of Definition 5, and in special case when $q(z) = Mz \ (M > 0)$, the class $\mathfrak{J}_J[\Omega, q]$, of admissible functions, denoted by $\mathfrak{J}_J[\Omega, M]$ is expressed follows.

**Definition 6.** Let $\Omega$ be set in $C$ and $M > 0$. The class $\mathfrak{J}_J[\Omega, M]$ of admissible functions consists of those functions $\phi : C^4 \times U \to C$, such that

$$\phi\left(Me^{it}, \left(\frac{k(d\lambda+\gamma)}{(2r+\alpha+\lambda)}\right)Me^{it}, \frac{(d\lambda+\gamma)\lambda+[(d\lambda+\gamma)(2r+\delta+\lambda)K+\sigma^2]Me^{it}}{(r\lambda+\alpha+\delta)^t} \right) \not\in \Omega,$$

(22)

where $z \in U$,

$$R\left(Le^{-it}\right) \geq (K - 1)KM$$

and

$$R\left(Ne^{-it}\right) \geq 0, \ \forall \theta \in \mathbb{R}; k \geq 0.$$

**Corollary 3.** Let $\phi \in \mathfrak{J}_J[\Omega, M]$. If the function $f, g \in \mathcal{G}$ satisfies the following conditions

$$\left|D_{c,d,\lambda}^{m+1}(f \ast g)(z)\right| \leq kM, \ (z \in U; k \geq 2; M > 0)$$

and

$$\left(D_{c,d,\lambda}^m(f \ast g)(z), D_{c,d,\lambda}^{m+1}(f \ast g)(z), D_{c,d,\lambda}^{m+2}(f \ast g)(z), D_{c,d,\lambda}^{m+3}(f \ast g)(z); z\right) \in \Omega,$$

then

$$\left|D_{c,d,\lambda}^m(f \ast g)(z)\right| < M.$$

In special case, when $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\mathfrak{J}_J[\Omega, M]$ is simple denoted by $\mathfrak{J}_J[M]$. Corollary 3 can now be rewritten in the following from.

**Corollary 4.** Let $\phi \in \mathfrak{J}_J[M]$. If the function $f, g \in \mathcal{G}$ satisfies the following conditions

$$\left|D_{c,d,\lambda}^{m+1}(f \ast g)(z)\right| \leq kM, \ (z \in U; k \geq 2; M > 0)$$
and
\[ |D_{\sigma,\delta,\lambda}^m(f \ast g)(z), D_{\sigma,\delta,\lambda}^{m+1}(f \ast g)(z), D_{\sigma,\delta,\lambda}^{m+2}(f \ast g)(z), D_{\sigma,\delta,\lambda}^{m+3}(f \ast g)(z); z| < M, \]
then
\[ |D_{\sigma,\delta,\lambda}^m(f \ast g)(z)| < M. \]

**Corollary 5.** Let \( k \geq 2, \ 0 \neq q \in C \) and \( \mathcal{M} > 0 \). If the function \( f, g \in \mathcal{G} \) satisfies the following conditions:
\[ |D_{\sigma,\delta,\lambda}^{m+1}(f \ast g)(z)| \leq k\mathcal{M} \]
and
\[ |D_{\sigma,\delta,\lambda}^m(f \ast g)(z) - D_{\sigma,\delta,\lambda}^{m+1}(f \ast g)(z)| < \frac{Mz}{|\sigma + \delta + \lambda|}, \]
then
\[ |D_{\sigma,\delta,\lambda}^m(f \ast g)(z)| < M. \]

**Proof.** Let \( \phi(a, \beta, \gamma, \nu; z) = \beta - a \) and \( \Omega = \mathcal{h}(\mathbb{U}), \) where
\[ \mathcal{h}(z) = \frac{\mathcal{M}z}{|\sigma + \delta + \lambda|}, \ (\mathcal{M} > 0) \]
use Corollary 3, we need to show that \( \phi \in \mathcal{J}_j[\Omega, \mathcal{M}] \), that is the admissibility condition (22), is satisfied. This follows readily, since it is seen that
\[ |\phi(a, \beta, \gamma, \nu; z)| = \left| \frac{(k - 1)\mathcal{M}e^{i\theta}}{\sigma + \delta + \lambda} \right| \geq \frac{\mathcal{M}}{|\sigma + \delta + \lambda|}, \]
where \( z \in \mathbb{U}, \ \theta \in \mathbb{R} \) and \( k \geq 2 \). The required result now follows from Corollary 3. This completes the proof. \( \square \)

**Definition 7.** Let \( \Omega \) be a set in \( C \) and \( q \in Q_1 \cap S_1 \), the class \( \mathcal{J}_j[\Omega, q] \) of admissible functions consists of those functions \( \phi : C^4 \times \mathbb{U} \rightarrow C \), which satisfy the following admissibility conditions
\[ \phi(a, \beta, \gamma, \nu; z) \notin \Omega, \]
whenever
\[ a = q(\xi), \ \beta = \frac{k(\delta + \lambda)\xi q'(\xi) + q\xi}{\sigma + \delta + \lambda}, \]
\[ \mathfrak{R}\left( \frac{(\sigma + \delta + \lambda)^2 - (\sigma + \delta + \lambda)(2\sigma + 3(\delta + \lambda))(\beta - a)}{(\sigma + \delta + \lambda)(\sigma + \delta + \lambda)(\beta - a)} \right) \geq k^2 \mathfrak{R}\left( \frac{\xi q''(\xi)}{q'(\xi)} + 1 \right), \]
and
\[ \mathfrak{R}\left( \frac{(\nu + \delta - 2\alpha)(\sigma + \delta + \lambda)^2 - (\gamma - \alpha)(\sigma + \delta + \lambda)(\beta - a)(2\sigma + 3(\delta + \lambda))(\beta - a)}{(\sigma + \delta + \lambda)(\sigma + \delta + \lambda)(\beta - a)} \right) \]
\[ \geq k^2 \mathfrak{R}\left( \frac{\xi q'''(\xi)}{q'(\xi)} \right), \]
where \( z \in \mathbb{U}, \ \xi \in \partial \mathbb{U}/E(q) \) and \( k \geq n \).

**Theorem 4.** Let \( \phi \in \mathcal{J}_j[\Omega, q] \). If the function \( f, g \in \mathcal{G} \) and \( q \in Q_1 \), satisfy the following conditions:
\[ R\left( \frac{\tilde{q}'(\xi)}{q'(\xi)} \right) \geq 0, \quad \left| \mathfrak{D}^{m+1}_{\sigma, \delta, \lambda}(f * g)(z) \right| \leq k \]  

(23)

and

\[ \left\{ \phi\left( \mathfrak{D}^m_{\sigma, \delta, \lambda}(f * g)(z) \right), \frac{\mathfrak{D}^{m+1}_{\sigma, \delta, \lambda}(f * g)(z)}{z}, \frac{\mathfrak{D}^{m+2}_{\sigma, \delta, \lambda}(f * g)(z)}{z}, \frac{\mathfrak{D}^{m+3}_{\sigma, \delta, \lambda}(f * g)(z)}{z}; \ z \in \mathbb{U} \right\} \subset \mathcal{Q}, \]  

(24)

then

\[ \frac{\mathfrak{D}^m_{\sigma, \delta, \lambda}(f * g)(z)}{z} < q(z) \ (z \in \mathbb{U}). \]  

Proof. Define \( p(z) \) in \( \mathbb{U} \), by

\[ p(z) = \frac{\mathfrak{D}^m_{\sigma, \delta, \lambda}(f * g)(z)}{z} \]  

(25)

from Equations (8) and (11), we have

\[ \frac{\mathfrak{D}^{m+1}_{\sigma, \delta, \lambda}(f * g)(z)}{z} = \frac{(\delta + \lambda)zp'(z) + (\sigma + \delta + \lambda)p(z)}{(\sigma + \delta + \lambda)^2} \]  

(26)

By a similar argument, we get

\[ \frac{\mathfrak{D}^{m+2}_{\sigma, \delta, \lambda}(f * g)(z)}{z} = \frac{(\delta + \lambda)^2z^2p''(z) + (\delta + \lambda)(2\sigma + 3(\delta + \lambda))zp'(z) + (\sigma + \delta + \lambda)^2zp(z)}{(\sigma + \delta + \lambda)^3} \]  

(27)

and

\[ \frac{\mathfrak{D}^{m+3}_{\sigma, \delta, \lambda}(f * g)(z)}{z} = \frac{(\delta + \lambda)^3z^3p'''(z) + (\delta + \lambda)^2[6(\delta + \lambda) + 3\sigma]z^2p''(z) + (\delta + \lambda)[(2\sigma + 3(\delta + \lambda))z^2p'(z) + (\sigma + \delta + \lambda)^2zp(z)]}{(\sigma + \delta + \lambda)^4} \]  

(28)

Define the transformation from \( C^4 \) to \( C \) by

\[ \alpha(t, s, t, u) = t, \quad \beta(t, s, t, u) = \frac{(\delta + \lambda)s + (\sigma + \delta + \lambda)t}{(\sigma + \delta + \lambda)}, \]  

(29)

\[ \gamma(t, s, t, u) = \frac{(\delta + \lambda)^2t + (\delta + \lambda)[2\sigma + 3(\delta + \lambda)]s + (\sigma + \delta + \lambda)^2t}{(\sigma + \delta + \lambda)^2}, \]  

and

\[ v(t, s, t, u) = \frac{(\delta + \lambda)^3u + (\delta + \lambda)^2[6(\delta + \lambda) + 3\sigma]t + (\delta + \lambda)[(2\sigma + 3(\delta + \lambda))s + (\sigma + \delta + \lambda)^3t]}{(\sigma + \delta + \lambda)^3}. \]  

(30)

Let

\[ \Pi(t, s, t, u) = \phi(\alpha, \beta, \gamma, v; z) = \phi\left( \frac{(\delta + \lambda)s + (\sigma + \delta + \lambda)t}{(\sigma + \delta + \lambda)}; \ \frac{(\delta + \lambda)^2t + (\delta + \lambda)[2\sigma + 3(\delta + \lambda)]s + (\sigma + \delta + \lambda)^2t}{(\sigma + \delta + \lambda)^2}, \ \frac{(\delta + \lambda)^3u + (\delta + \lambda)^2[6(\delta + \lambda) + 3\sigma]t + (\delta + \lambda)[(2\sigma + 3(\delta + \lambda))s + (\sigma + \delta + \lambda)^3t]}{(\sigma + \delta + \lambda)^3} \right) \]  

(31)
The proof will make use of lemma 1. Using the Equations (25)–(27) and from the Equation (31), we have

\[
\Pi(p(z), zp'(z), z^2p'(z), z^3p'''(z); z) = \\
\phi \left( \frac{\mathcal{D}^{m}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+2}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+3}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}; z \right). \tag{32}
\]

Hence, clearly (24) becomes

\[
\Pi(p(z), zp'(z), z^2p'(z), z^3p'''(z); z) \in \Omega
\]

We note that

\[
\frac{1}{s} + 1 = \frac{(\sigma + \delta + \lambda)(\gamma - \alpha) - 2(\sigma + \delta + \lambda)(\beta - \alpha)}{(\delta + \lambda)(\beta - \alpha)}
\]

and

\[
\frac{u}{s} = \frac{(\sigma + \delta + \lambda)^2(\nu + \beta - 2\alpha) - (\sigma + \delta + \lambda)(\gamma - \alpha) + (2\sigma + 3(\delta + \lambda))(\sigma + 4(\delta + \lambda))(\beta - \alpha)}{(\delta + \lambda)((\sigma + \delta + \lambda)(\gamma - \alpha) - (2\sigma + 3(\delta + \lambda))(\beta - \alpha))}. \tag{33}
\]

Thus, clearly, the admissibility condition for \( \phi \in J_{\alpha,1}[\Omega, q] \) in Definition 7 is equivalent to admissibility condition for \( \Pi \in \Psi_2[\Omega, q] \), as given in Definition 3 with \( n \geq 2 \).

Therefore, by using (23) and Lemma 1, we have

\[
\text{Re} \left( \frac{\xi q''(\xi)}{q'(\xi)} + k \right) \geq 0, \quad \left\vert \frac{\mathcal{D}^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{zq'(\xi)} \right\vert \leq k
\]

\[
\phi \left( \frac{\mathcal{D}^{m}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+2}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+3}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}; z \right) < h(z), \tag{34}
\]

then

\[
\frac{\mathcal{D}^{m}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z} < q(z) \quad (z \in \mathbb{U}).
\]

If \( \Omega \neq \mathbb{C} \), is a simply connected domain, the \( \Omega = h(U) \) for some conformal mapping \( h(z) \) of \( U \) on to \( \Omega \). In this case the class \( J_{\alpha,1}[h(U), q] \) is written as \( J_{\alpha,1}[h, q] \). This leads to the following immediate consequence of Theorem 4 is stated below.

**Theorem 5.** Let \( \phi \in J_{\alpha,1}[\Omega, q] \). If \( f, g \in \mathcal{G} \) and \( q \in Q_1 \), satisfy the following conditions:

\[
\text{Re} \left( \frac{\xi q''(\xi)}{q'(\xi)} \right) \geq 0, \quad \left\vert \frac{\mathcal{D}^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{zq'(\xi)} \right\vert \leq k \tag{33}
\]

and

\[
\phi \left( \frac{\mathcal{D}^{m}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+2}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+3}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z}; z \right) < h(z),
\]

then

\[
\frac{\mathcal{D}^{m}_{\sigma,\delta,\lambda}(f \ast g)(z)}{z} < q(z) \quad (z \in \mathbb{U}).
\]

In view of Definition 7, and in special case when \( q(z) = Mz \) (\( M > 0 \)), the class \( J_{\alpha,1}[\Omega, q] \), of admissible functions, denoted by \( J_{\alpha,1}[\Omega, M] \) is expressed follows.

**Definition 8.** Let \( \Omega \) be set in \( \mathbb{C} \) and \( M > 0 \). The class \( J_{\alpha,1}[\Omega, M] \) of admissible functions consists of those function \( \phi : \mathbb{C} \times \mathbb{U} \to \mathbb{C} \), such that
\[\phi \left( \frac{\mathcal{M}e^{i\theta}}{L} \left[ \frac{(\delta+\lambda)^2L + |(\delta+\lambda)(2\alpha+3(\delta+\lambda))k + (\sigma+\delta+\lambda)^2|\mathcal{M}e^{i\theta}}{(\sigma+\delta+\lambda)^2} \right] \right) \notin \Omega, \quad (35)\]

whenever, \(z \in U,\)
\[\Re\left(Le^{-i\theta}\right) \geq (K-1)K\]

and
\[\Re\left(Ne^{-i\theta}\right) \geq 0, \forall \theta \in \mathbb{R}; k \geq 0.\]

**Corollary 6.** Let \(\phi \in \mathfrak{J}_{1,1}[\Omega, \mathcal{M}].\) If the function \(f, g \in \mathcal{G}\) satisfies the following conditions
\[
\left| \frac{\mathcal{D}^{m+1}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z} \right| \leq k\mathcal{M}, \quad (z \in U; k \geq 2; \mathcal{M} > 0)
\]

and
\[\phi \left( \frac{\mathcal{D}^{m}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+1}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+2}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+3}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z} ; z \right) \in \Omega,
\]

then
\[
\left| \frac{\mathcal{D}^{m}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z} \right| < \mathcal{M}.
\]

When \(\Omega = q(U) = \{\omega : |\omega| < \mathcal{M}\},\) the class \(\mathfrak{J}_{1,1}[\Omega, \mathcal{M}]\) is simply denoted by \(\mathfrak{J}_{1,1}[\mathcal{M}].\)

**Corollary 7.** Let \(\phi \in \mathfrak{J}_{1,1}[\Omega, \mathcal{M}].\) If the function \(f, g \in \mathcal{G}\) satisfies the following conditions
\[
\left| \frac{\mathcal{D}^{m+1}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z} \right| \leq k\mathcal{M}, \quad (z \in U; k \geq 2; \mathcal{M} > 0)
\]

and
\[\phi \left( \frac{\mathcal{D}^{m}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+1}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+2}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z}, \frac{\mathcal{D}^{m+3}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z} ; z \right) < \mathcal{M},
\]

then
\[
\left| \frac{\mathcal{D}^{m}_{\psi,\alpha,\lambda}(f \ast g)(z)}{z} \right| < \mathcal{M}.
\]

**Definition 9.** Let \(\Omega\) be a set in \(C.\) Also \(q \in Q_1 \cap S_1,\) the class \(\mathfrak{J}_{1,2}[\Omega, q]\) of admissible functions consists of those functions \(\phi : C^4 \times U \rightarrow C,\) which satisfy the following admissibility conditions
\[
\phi(\alpha, \beta, \gamma, \nu; z) \notin \Omega,
\]
whenever
\[ \alpha = q(\xi), \beta = \frac{(\delta + \lambda)\alpha}{\tau} + (\sigma + \delta + \lambda)\gamma, \]
\[ \Re \left( (\sigma + \delta + \lambda)^2 \gamma (\beta - \alpha) + (\sigma + \delta + \lambda) \gamma [(\sigma + \delta + \lambda) \gamma - (\beta - \alpha)(\alpha + 2)] - (\sigma + \delta + \lambda) \alpha (\beta - \alpha) \right), \]
\[ \geq k \Re \left( \frac{q''(\xi)}{q'(\xi)} + 1 \right) \]
and
\[ \Re \left( \frac{\beta^2}{(\beta + \lambda)^2} (\alpha - \gamma)(\sigma + \delta + \lambda)^2 - \left( \frac{(\sigma + \delta + \lambda)}{(\sigma + \lambda)} \right)^2 (\beta(\gamma - \beta)(1 - \beta - 3\alpha) + \alpha^2 (\beta - \alpha)) \right) \]
\[ - \frac{3(\sigma + \delta + \lambda)}{(\sigma + \lambda)} (\beta(\gamma - \beta) - \alpha(\beta - \alpha)) + \left( \frac{(\sigma + \delta + \lambda)}{(\sigma + \lambda)} \right)^2 (\beta - \alpha)^2 ((\beta - 5\alpha) - 3) \]
\[ + 2(\beta - \alpha)(\beta - \alpha)^{-1} \geq k^2 \Re \left( \frac{q''(\xi)}{q'(\xi)} \right), \]
where \( z \in \mathbb{U}, \xi \in \partial \mathbb{U}/E(q) \) and \( k \geq n. \)

**Theorem 6.** Let \( \phi \in \mathcal{D}_{\mathbb{U},\mathbb{Q}}[\Omega, \alpha]. \) If the function \( f, g \in \mathcal{G} \) and \( q \in \mathcal{Q}_1, \) satisfy the following conditions:
\[ \Re \left( \frac{\xi q''(\xi)}{q'(\xi)} \right) \geq 0, \quad \left| \frac{\mathcal{D}^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{\mathcal{D}^m_{\sigma,\delta,\lambda}(f \ast g)(z)} \right| \leq k \quad (36) \]
and
\[ \left\{ \begin{array}{l} \phi \left( \frac{\mathcal{D}^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{\mathcal{D}^m_{\sigma,\delta,\lambda}(f \ast g)(z)} \right), \quad \frac{\mathcal{D}^{m+2}_{\sigma,\delta,\lambda}(f \ast g)(z)}{\mathcal{D}^m_{\sigma,\delta,\lambda}(f \ast g)(z)}, \quad \frac{\mathcal{D}^{m+3}_{\sigma,\delta,\lambda}(f \ast g)(z)}{\mathcal{D}^m_{\sigma,\delta,\lambda}(f \ast g)(z)}, \quad \frac{\mathcal{D}^{m+4}_{\sigma,\delta,\lambda}(f \ast g)(z)}{\mathcal{D}^m_{\sigma,\delta,\lambda}(f \ast g)(z)}; \ z \in \mathbb{U} \end{array} \right\} \subset \Omega, \quad (37) \]
then
\[ \frac{\mathcal{D}^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{\mathcal{D}^m_{\sigma,\delta,\lambda}(f \ast g)(z)} \prec q(z), \quad (z \in \mathbb{U}). \]

**Proof.** Define the analytic function \( p(z) \) in \( \mathbb{U} \) by
\[ p(z) = \frac{\mathcal{D}^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{\mathcal{D}^m_{\sigma,\delta,\lambda}(f \ast g)(z)}. \quad (38) \]
From Equations (8) and (38), we have
\[ \frac{\mathcal{D}^{m+2}_{\sigma,\delta,\lambda}(f \ast g)(z)}{\mathcal{D}^m_{\sigma,\delta,\lambda}(f \ast g)(z)} = \frac{1}{(\sigma + \delta + \lambda)} \left[ (\delta + \lambda) z p'(z) + (\sigma + \delta + \lambda)(p(z))^2 \right] = \frac{A}{(\sigma + \delta + \lambda)}. \quad (39) \]
By a similar argument, we get
\[ \frac{\mathcal{D}^{m+3}_{\sigma,\delta,\lambda}(f \ast g)(z)}{\mathcal{D}^m_{\sigma,\delta,\lambda}(f \ast g)(z)} = \frac{B}{(\sigma + \delta + \lambda)} \quad (40) \]
and
\[ \frac{\mathcal{D}^{m+4}_{\sigma,\delta,\lambda}(f \ast g)(z)}{\mathcal{D}^m_{\sigma,\delta,\lambda}(f \ast g)(z)} = \frac{(\delta + \lambda)}{(\sigma + \delta + \lambda)} \left[ B + B^{-1}(C + A^{-1}D - A^{-2}C^2) \right], \quad (41) \]
where

\[
B = \frac{1}{(\sigma + \delta + \lambda)} \left[ (\sigma + \rho(z)) + \frac{(\delta + \lambda) \rho''(z)}{\rho(z)} - \frac{(\delta + \lambda) \rho''(z)}{\rho(z)} - 2(\delta + \lambda) \rho'(z) \right],
\]

\[
C = \left( \frac{(\delta + \lambda)^2}{(\sigma + \delta + \lambda)^2} \right) \left[ (\sigma + \rho(z)) + (\delta + \lambda)^2(2\sigma + \delta + \lambda) \frac{\rho''(z)}{\rho(z)} - \frac{(\delta + \lambda)^2}{(\sigma + \delta + \lambda)^2} \frac{\rho''(z)}{\rho(z)} \right]^2
\]

and

\[
D = \frac{3(\delta + \lambda)^3}{(\sigma + \delta + \lambda)^2} \frac{\rho''(z)}{\rho(z)} + \frac{(\delta + \lambda)^3}{(\sigma + \delta + \lambda)^3} \frac{\rho'''(z)}{\rho(z)} - \frac{(\delta + \lambda)^2}{(\sigma + \delta + \lambda)^2} \frac{\rho''(z)}{\rho(z)} + 2 \left( \frac{(\delta + \lambda)^2}{(\sigma + \delta + \lambda)^2} \frac{\rho''(z)}{\rho(z)} \right)^3
\]

\[
+ (\delta + \lambda)^2(\sigma + \delta + \lambda) \rho'(z) + (\delta + \lambda)^2(\sigma + \delta + \lambda)^2 \rho''(z).
\]

We now define the transformation from \( C^4 \) to \( C \) by

\[
a(r, s, t, u) = \frac{1}{(\sigma + \delta + \lambda)} \left[ (\delta + \lambda) \frac{\rho(z)}{\rho(z)} + (\sigma + \delta + \lambda) r \right] = \frac{\mathcal{E}}{(\sigma + \delta + \lambda)},
\]

\[
\gamma(t, s, u) = \frac{1}{(\sigma + \delta + \lambda)} \left[ (\sigma + \rho(z)) + (\delta + \lambda)^2 \frac{\rho''(z)}{\rho(z)} - \frac{(\delta + \lambda)^2}{(\sigma + \delta + \lambda)^2} \frac{\rho''(z)}{\rho(z)} \right] = \frac{\mathcal{F}}{(\sigma + \delta + \lambda)}
\]

and

\[
\nu(t, s, u) = \frac{1}{(\sigma + \delta + \lambda)} \left[ \mathcal{F} + \mathcal{F}^{-1} \left( \mathcal{L} + \mathcal{E}^{-1} \mathcal{H} - \mathcal{E}^{-2} \mathcal{L}^{-2} \right) \right],
\]

where

\[
\mathcal{L} = \left( \frac{(\delta + \lambda)}{(\sigma + \delta + \lambda)} \right)^2 \frac{\sigma}{r} + (\delta + \lambda)^2(2\sigma + \delta + \lambda) \frac{s}{r} - \frac{(\delta + \lambda)^2}{(\sigma + \delta + \lambda)^2} \frac{\rho''(z)}{\rho(z)}(\sigma + \delta + \lambda) + (\delta + \lambda)(2\sigma + \delta + \lambda)^2
\]

and

\[
\mathcal{H} = \frac{3(\delta + \lambda)^3}{(\sigma + \delta + \lambda)^2} \frac{\rho''(z)}{\rho(z)} + \frac{(\delta + \lambda)^3}{(\sigma + \delta + \lambda)^3} \frac{\rho'''(z)}{\rho(z)} - 3(\delta + \lambda)^3 \frac{\rho''(z)}{\rho(z)} + 2 \left( \frac{(\delta + \lambda)^2}{(\sigma + \delta + \lambda)^2} \frac{\rho''(z)}{\rho(z)} \right)^3
\]

\[
+ (\delta + \lambda)^2(\sigma + \delta + \lambda)^2 \rho'(z) + (\delta + \lambda)^2(\sigma + \delta + \lambda)^2 \rho''(z).
\]

Let

\[
\Pi(r, s, t, u) = \phi(\alpha, \beta, \gamma, \nu; z) = \phi \left( \frac{\mathcal{E}}{(\sigma + \delta + \lambda)} \frac{\mathcal{F}}{(\sigma + \delta + \lambda)} \right) \frac{1}{(\sigma + \delta + \lambda)} \left[ \mathcal{F} + \mathcal{F}^{-1} \left( \mathcal{L} + \mathcal{E}^{-1} \mathcal{H} - \mathcal{E}^{-2} \mathcal{L}^{-2} \right) \right].
\]

The proof will make use of lemma 1. Using the Equations (38)–(41), and from the Equation (44), we have

\[
\Pi(p(z), z^2 \rho'(z), z^2 \rho''(z), z^3 \rho'''(z); z)
\]

\[
= \phi \left( \frac{\mathcal{D}^{\rho'''}_{\rho'}(f \ast g)(z)}{\mathcal{D}^{\rho'''}_{\rho''}(f \ast g)(z)} \frac{\mathcal{D}^{\rho''''}_{\rho''}(f \ast g)(z)}{\mathcal{D}^{\rho''''}_{\rho'''}(f \ast g)(z)} \right).
\]

Hence, clearly (37) becomes

\[
\Pi(p(z), z^2 \rho'(z), z^2 \rho''(z), z^3 \rho'''(z); z) \in \Omega.
\]
We note that
\[
\frac{1}{g} + 1 = 
\gamma(\beta-\alpha)(\sigma+\delta+\lambda)^2 + \gamma(\sigma+\delta+\lambda)(\sigma+\delta+\lambda)\gamma-(\beta-\alpha)(\alpha+2) - a(\sigma+\delta+\lambda)((\beta-\alpha)^2 - \delta + (\delta+\lambda)(\beta-\alpha) \\
(\delta+\lambda)/(\beta-\alpha)
\]
and
\[
\frac{u}{g} = \left[ \frac{(\sigma+\delta+\lambda)^2(\beta-\gamma)}{(\beta+\lambda)^2} (\beta+\beta(1-\beta-3\alpha)) - \frac{3(\sigma+\delta+\lambda)}{(\beta+\lambda)^2} (\beta(\gamma-\beta) - a(\beta-\alpha)) \right] \\
+ \frac{(\beta-\alpha)^2(\sigma+\delta+\lambda)}{(\beta+\lambda)^2} ((\beta-5\alpha) - 3) + \frac{3(\beta-\alpha)^2(\sigma+\delta+\lambda)^2}{(\beta+\lambda)^2} + 2(\beta-\alpha)^{-1}.
\]

Thus clearly, the admissibility condition for \( \phi \in \mathcal{J}_{12}[\Omega, q] \), in Definition 9 is equivalent to admissibility condition for \( \Pi \in \Psi_2[\Omega, q] \), as given in Definition 3 with \( n = 2 \). Therefore, by using (36) and Lemma 1, we have
\[
\frac{D^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{D^m_{\sigma,\delta,\lambda}(f \ast g)(z)} < q(z).
\]
(46)

This completes the Proof of Theorem 6. □

If \( \Omega \neq \mathbb{C} \), is simply-connected domain, then \( \Omega = \mathbb{h}(U) \) for some conformal mapping \( \mathbb{h}(z) \) of \( U \) on to \( \Omega \). In this case, the class \( \mathcal{J}_{12}[\mathbb{h}[U], q] \) is written as \( \mathcal{J}_{12}[h, q] \). An immediate consequence of Theorem (4) is now stated below without proof.

Theorem 7. Let \( \phi \in \mathcal{J}_{12}[\Omega, q] \). If \( f, g \in \mathcal{G} \) and \( q \in \mathcal{Q}_1 \), satisfy the following conditions (37) and
\[
\phi\left( \frac{D^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{D^m_{\sigma,\delta,\lambda}(f \ast g)(z)}, \frac{D^{m+2}_{\sigma,\delta,\lambda}(f \ast g)(z)}{D^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}, \frac{D^{m+3}_{\sigma,\delta,\lambda}(f \ast g)(z)}{D^{m+2}_{\sigma,\delta,\lambda}(f \ast g)(z)}, \frac{D^{m+4}_{\sigma,\delta,\lambda}(f \ast g)(z)}{D^{m+3}_{\sigma,\delta,\lambda}(f \ast g)(z)} \right) < h(z), \tag{47}
\]
then
\[
\frac{D^{m+1}_{\sigma,\delta,\lambda}(f \ast g)(z)}{D^m_{\sigma,\delta,\lambda}(f \ast g)(z)} < q(z), \quad (z \in U).
\]

3. Discussion

We study classes of admissible functions and establish the properties of third-order differential subordination using certain differential operator of analytic functions in \( U \) and have the normalized Taylor–Maclaurin series of the form: \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), \( (z \in U) \). Some new results on differential subordination with some corollaries are obtained. These properties and results are symmetry to the properties of the differential superordination to form the sandwich theorems. Our results are different from the previous results for the other authors. We opened some windows for authors to generalize our new subclasses to obtain some new results in univalent and multivalent function theory using the results in the paper.

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