Polarization coherent states and geometric phases in quantum optics

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Abstract

Polarization coherent states (PCS) are considered as generalized coherent states of $SU(2)_p$ group of the polarization invariance of the light fields. The geometric phases of PCS are introduced in a way, analogous to that used in the classical polarization optics.
I. INTRODUCTION

For several recent decades the polarization properties of light have been widely investigated both in theoretical and in applied aspects (see, e.g., [1–8] and references therein). In particular, some fundamental problems of quantum mechanics, related to the “hidden” variables, Bell’s inequalities and Einstein-Podolsky-Rosen paradox, quantum chaos, Berry’s and other geometric phases, etc., are successfully studied by means of quantum polarization optics.

It is well known [3,9,10] that the generalized coherent states (GCS), generated by the action of the displacement operators $D(g) = \exp(\sum d_i F_i)$ of the groups $G^{DS}$ on certain fixed reference vectors $|\psi_0\rangle$ in the given space $L^D$ of the representation $D(G^{DS}) = \{D(g), g \in G^{DS}\}$ of the groups $G^{DS}$, present an effective tool for the study of quantum systems having the dynamic symmetry (DS) groups $G^{DS}$. In particular, the average values $\langle\{\alpha_i\}; \psi_0| f(\{F_i\})|\{\alpha_i\}; \psi_0\rangle$ of the arbitrary functions $F(\{F_i\})$, corresponding to the observables and depending on the generators $F_i$ of $G^{DS}$, as well as the quasiprobability distribution functions $Q(\{\alpha_i\}; \psi_0) = \langle\{\alpha_i\}; \psi_0|\rho|\{\alpha_i\}; \psi_0\rangle$, $\rho$ being the density matrix, are widely used for the description of quasiclassical properties of the appropriate quantum systems near the “classical limit” [2,10]. For example, in quantum optics similar quantities, defined using the conventional Glauber’s coherent states and associated with Weyl-Heisenberg group $W(m)$, are widely used for the description of $m$-mode electromagnetic fields [2]. The GCS associated with $SU(m)$ groups play the same role for the systems of $n$-level emitters of radiation [3].

Recently it was shown [1,3,7] that the DS group adequate to the polarization properties of quantum light is the $SU(2)_p$ group of the polarization invariance of the free light fields. It’s generators $P_\alpha$, $\alpha = 0, 1, 2$ (or $\alpha = 0, \pm$) are the components of the polarization ($P$) quasispin, which corresponds to the Stokes vector $\vec{\Sigma} = (\Sigma_\alpha)$ [6], parameterized on the so-called Poincare sphere $S_\rho^2$ [4] in the classical statistical optics.

The aim of the present paper is to investigate the GCS of the $SU(2)_p$ group (see
also \((\ddot{E}, \ddot{\Pi})\) in the \(2m\)-mode Fock space \(L_F(2m) = \text{Span}\{\prod_{j=1}^{m}(a_+^{(j)})^{n_+}(a_-^{(j)})^{n_-}|0\}\) with two polarization and \(m\) spatiotemporal (ST) modes in the helicity \((\pm)\) polarization basis in both mathematical and physical aspects. An attention is paid to differences between pictures of independent (uncorrelated) and correlated ST modes which correspond, respectively, to one-mode and broad-band measuring devices (cf. \([12]\)); from the mathematical viewpoint these cases differ by using collective \((P_\alpha = \sum_{j=1}^{m} P_\alpha(j), S_P^2)\) or ”individual” \((P_\alpha(j), S_P^2(j))\) \(P\)-quasispin components and Poincaré spheres. Then Pancharatnam-type geometric phase acquired by these states during the cyclic evolution on the Poincaré sphere is derived and compared with the classical results \([8]\). We also briefly discuss some other applications of polarization GCS to the quasiclassical description of the polarization properties of quantum light.

II. POLARIZATION COHERENT STATES

The general definition of GCS of the \(SU(2)\) group is well known \([1,10]\)

\[
|\xi; \psi_0\rangle \equiv |\theta, \varphi; \psi_0\rangle = \exp(\xi J_+ - \xi^* J_-)|\psi_0\rangle,
\]  

(1)

where \(\xi = -\theta/2 \exp(-i\varphi), \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi\) are the angular coordinates of the “classical” quasispin \(\vec{J} = (J_\alpha)\) in its “phase” space, i.e. the Poincaré sphere \(S_P^2(\theta, \varphi); |\psi_0\rangle\) is a certain reference vector in the space \(L\) of the states of the system.

From the physical viewpoint, the states \(|\xi; \psi_0\rangle\) describe output light beams obtained by means of action of quantum ”\(SU(2)\)-rotators” with Hamiltonians

\[
H_{SU(2)} = gJ_+ + g^*J_-
\]  

(2)

on the input beams in the quantum state \(|\psi_0\rangle\) (see, e.g., \([3]\) and references therein for possible realizations of such rotators in experimental devices for \(m = 1\)).

For spin systems having a fixed spin value \(j\) one of the basis vectors \(|jm\rangle\) of the irrep \(D_j(SU(2))\) is used as \(|\psi_0\rangle\), and the values of \(m = \pm j\) correspond to the “squeezed” GCS most near to classical states \([10]\). A peculiarity of the polarization
spin \((J_\alpha = P_\alpha)\) of the light fields is that the Fock spaces \(L_F(2m)\) of the field states may be viewed as direct sums of the specific \(SU(2)\) fiber bundles and contain the subspaces \(L^j_\sigma\) of the irrep \(D^j(SU(2))\) with \(j = p = 0, 1/2, 1, \ldots, \) generally (in the picture of correlated ST modes) with a certain multiplicity, where the index \(\sigma\) labels the \(SU(2)\)-equivalent subspaces \(L^j_\sigma\) and corresponds to the additional (non-polarizational) degrees of freedom [1][7]. Hence to get the “complete phase portrait“ of the quantum light field in all \(L_F(2m)\) one should have complete sets of the GCS of \(SU(2)\) similar to Eq. (1) with a set of reference vectors \(|\psi_0\rangle = |\psi_0^{p,\sigma}\rangle \in L^{p,\sigma}, \ p = 0, 1/2, \ldots, \) or with a set of vectors \(|\psi_0\rangle = |\psi_0^{p,\gamma}\rangle\), having nonzero projections on each of the subspaces \(L^{p,\sigma}\), \(p = 0, 1/2, \ldots\).

In the first case, basing on the maximal classicality criterion in the polarization degree of freedom it seems natural to choose (in the picture of correlated ST modes) for \(|\psi_0^{p,\sigma}\rangle\) the vectors

\[
|p, \pi = \pm p; n, \lambda\rangle = \sum_{\alpha_j^{\pm}=2p, \sum_{ij} \gamma_{ij}=-p} C(\{\alpha_j^{\pm}; \gamma_{ij}\}) \prod_{j=1}^m (a_j^{\pm}(j))^\alpha_j^{\pm} \prod_{i<j} (X_{ij}^{\pm})^{\gamma_{ij}}|0\rangle,
\]

where \(X_{ij}^{\pm} \equiv a_i^{\pm}(j)a_{i^{\pm}}(j) - a^\dagger_i^{\pm}(i)a^\dagger_i^{\pm}(j)\) are the creation operators of the \(SU(2)\)-invariant two-photon clusters. These vectors belong to a polarization-invariance adapted orthonormalized basis in \(L_F(2m)\) \(\{|p, \pi; n, \lambda\}\), which is defined by the following equations [1][7][12]

\[
P^2|p, \pi; n, \lambda\rangle = p(p + 1)|p, \pi; n, \lambda\rangle,
\]

\[
P_0|p, \pi; n, \lambda\rangle = \pi|p, \pi; n, \lambda\rangle,
\]

\[
N|p, \pi; n, \lambda\rangle = n|p, \pi; n, \lambda\rangle,
\]

where \(P^2 = \frac{1}{2}(P_+P_- + P_-P_+)\) is the \(SU(2)\) Casimir operator, \(P_\pm = \sum_{j=1}^m P_\pm(j) = \sum_{j=1}^m a^\pm_{\pm}(j)a^{\pm}_{\pm}(j), P_0 = \sum_{j=1}^m P_0(j) = \frac{1}{2} \sum_{j=1}^m [a^+_\pm(j)a_\pm(j) - a^\pm_\pm(j)a^-_\pm(j)],\)
\(N = \sum_{j=1}^m [a_\pm^+(j)a^-_\pm(j) + a^\pm_\pm(j)a^-_\pm(j)]\) is the total photon number, and \(\lambda\) is the extra quantum label. In particular cases \(m = 1, 2\) the vectors given by Eq. (3) take the form [7][11]:
\[ |p\rangle_\pm \equiv |p, \pi = \pm p; n = 2p\rangle = [(2p)!]^{-\frac{1}{2}}(a_{\pm}^+(1))^{2p}|0\rangle, \quad 2p = 0, 1, \ldots, \quad (5) \]

and
\[
|p, n, t\rangle_\pm \equiv |p, \pi = \pm p; n, t\rangle = \frac{1}{(2p+1)!}
\left[ (n/2+p+1)!(n/2-p)!(p+t)!(p-t)! \right]^{-\frac{1}{2}}
(a_{\pm}^+(1))^{p+t}(a_{\pm}^+(2))^{p-t}(X_{12}^+)^{n/2-p}|0\rangle, \quad (6)
\]

where \(2t\) is the difference \(N(1) - N(2) = N_+(1) + N_-(1) - N_+(2) - N_-(2)\) of the photon numbers in the first and second ST modes.

Now making use of the definition given by Eq. (1) and of the transformation properties of the operators \(a_{\pm}^+(j), X_{ij}^+\) with respect to the group SU(2)_p, \([1,7]\) we get the sets of the polarization GCS generated by the reference vectors (3) (compare with \([7]\)):
\[
|\theta, \varphi; p, n, \lambda\rangle_\pm \equiv \exp(\xi P_+ - \xi^* P_-)|p, \pm p; n\rangle
= \sum C(\{\alpha_j^\pm, \gamma_{ij}\}) \prod_{j=1}^m (a_{\pm}^+(j; \theta, \varphi))^{\alpha_j^\pm} \prod_{i<j} (X_{ij}^+)^{\gamma_{ij}}|0\rangle, \quad (7)
\]

where the operator \(a_{\pm}^+(j; \theta, \varphi) \equiv a_{\pm}^+(j) \cos \frac{\theta}{2} \pm a_{\pm}^+(j) \exp(\pm i\varphi) \sin \frac{\theta}{2}\) may be interpreted as the creation operator of the elliptically polarized photon in the \(j\)-th ST mode having the ellipticity parameters defined by the angles \(\theta, \varphi\) \([8]\).

Since the quasispin \(\vec{P}\) is the vector of SU(2) space one can see from the analogy with the theory of transformation of angular momentum that from the mathematical point of view the action of polarization rotator is equivalent to the multiplication of the initial state vector by the spherical function of the finite rotation of the first order.

Changing the direction averaged over the quasispin state will be analogous to turning the coordinate system by \(\theta\) and \(\varphi\) angles and can be expressed in the following way
\[
\langle \xi, \psi_0 | P_{\alpha} | \xi, \psi_0 \rangle = \sum_{\beta} D_{\alpha\beta}^1(\theta, \varphi) \langle \psi | P_{\beta} | \psi \rangle, \quad (8)
\]

where \(D_{\alpha\beta}^1(\theta, \varphi)\) is the Wigner function.
As follows from this expression, according to the group theory, the sum of the squares of the mean values of the polarization quasispin components will be \( SU(2) \)-invariant:

\[
\sum_{\alpha} \langle \xi, \psi_0 | P_\alpha | \xi, \psi_0 \rangle^2 = C = R^2, \tag{9}
\]

where the constant \( R = \sqrt{\sum_{\alpha} \langle \psi_0 | P_\alpha | \psi_0 \rangle^2} \) is the radius of the polarization sphere (Poincaré).

Indeed, averaging the polarization quasispin components over states \( \rho \) leads to the conventional parameterization of Poincaré sphere (when the poles correspond to the states with circular polarization):

\[
\begin{align*}
\langle \xi; \psi_0 | P_0 | \xi; \psi_0 \rangle &= R \cos \theta, \\
\langle \xi; \psi_0 | P_1 | \xi; \psi_0 \rangle &= R \sin \theta \cos \phi, \\
\langle \xi; \psi_0 | P_2 | \xi; \psi_0 \rangle &= R \sin \theta \sin \phi;
\end{align*} \tag{10}
\]

where the parameters \( \theta \) and \( \varphi \) are the angular coordinates of the point on the sphere with the radius \( R = \langle \psi_0 | P_0 | \psi_0 \rangle \). This point defines the direction of vector of "classical" quasispin after the single ST mode with minimum value of polarization uncertainty have passed through the polarization rotator.

The action of this rotator on the state of this type with \( m = 2 \) may be described in the follows way

\[
| \theta, \varphi; p, n, t \rangle \pm \equiv | \theta, \varphi; p, \pi = \pm p; n, t \rangle = \frac{1}{(2p+1)!} \left[ \frac{(n/2 + p + 1)!(n/2 - p)!(p+t)!(p-t)!}{(2p + 1)!} \right]^{-1/2} \times (a_{\pm}(1; \theta, \varphi))^{p+t}(a_{\pm}(2; \theta, \varphi))^{p-t}(X_{12}^+)^{n/2-p} | 0 \rangle, \tag{11}
\]

The sets of the polarization GCS given by Eq. (6) belong , from the mathematical point of view, to the class of semi-coherent ones (which are coherent (quasiclassical) in polarization degrees of freedom and orthonormalized (strongly quantum) in other ones) \( \rho \), and yield the following decomposition of the identity operator \( \hat{I} \) in \( L_F(2m) \) \( 5,7 \), which is an expression of the basis set completeness:
\[ \hat{i} = \sum_{n,p,\lambda} \int_0^\pi \int_0^{2\pi} \frac{(2p+1)}{4\pi} \sin \theta d\theta d\varphi |\theta, \varphi; p, n, \lambda \rangle (\pm \langle \theta, \varphi; p, n, \lambda | \pm) \]  

Choosing for \(|\psi_0^{p,\sigma}\rangle\) the reference vectors different from those given by Eq. (3), one may construct, by means of Eq. (1), other sets of GCS of the \(SU(2)_p\) group, which in the mathematical aspect are equivalent to Eq. (7) \[7,11\].

The construction of Eq. (7) is simplified in the picture of independent ST modes, when the group \(SU(2)_p\) acts in the space \(L_F(2)\) of each \(j\)-th ST mode independently, and its action is determined by the angles \((\theta_j, \varphi_j)\):

\[
|\{\theta_j, \varphi_j\}; \{n_j\}\rangle = \prod_{j=1}^m \exp((\xi_j P_+(j) - \xi_j^* P_-(j))(a_\pm^+(j))^{n_j}|0\rangle
= \prod_{j=1}^m \frac{(a_\pm^+(j); \theta_j, \varphi_j)^{n_j}}{(n_j!)^{1/2}}|0\rangle. 
\]  

(13)

The set of GCS (13) is complete (an analog of Eq. (12) is valid for it) and yields the “polarization phase portrait of the field” adequate to independent measurements for each ST mode. The connection between the sets (7) and (13) is realized via the generalized Clebsh-Gordan coefficients of \(SU(2)_p\) \[9\].

We note that states (13) are the Fock states in terms of “rotated” photon operators \(a_\pm^+(j, \theta, \varphi)\) and are unitarily equivalent to initial states (3); hence there are some difficulties to produce them (as well as states (7) in physical experiments. Therefore, from the physical viewpoint it is of interest to consider alternate types of GCS of \(SU(2)_p\), which do not contain the discrete parameters \(n, \lambda\). For example, if instead of Eq. (3) one takes the vectors \(|\psi_0^{p,\{z\}}\rangle = \exp(\sum_i z_i F_i)(a_\pm^+(1)2^{p})(2p)!|0\rangle\), \(F_i\) being the generators of the \(SO^*(2m)\) group, complementary to \(SU(2)_p\), we obtain states which are GCS with respect to both \(SU(2)_p\) and \(SO^*(2m)\) groups and display a specific squeezing in both polarization and biphoton degrees of freedom\[12\].

Another type of such “physical” polarization GCS may be obtained if instead of Eq. (3) one takes the sets of reference vectors \(|\psi_0^{z}\rangle\), having the nonzero projections on all \(L^{p,\sigma}\). An example of such a set is the familiar set of Glauber’s coherent states:

\[
|\{\alpha_j^+, \alpha_j^-\}\rangle = \prod_{j=1}^m \exp[\alpha_j^+ a_\pm^+(j) + \alpha_j^- a_\pm^-(j) - (\alpha_j^+)^* a_\pm^+(j) - (\alpha_j^-)^* a_\pm^-(j)]|0\rangle. 
\]  

(14)
Then using the definition (1) and taking the $SU(2)_p$ transformation properties of $a_\pm(j)$ into account, one gets from Eq. (14) the set of GCS

$$|\theta,\varphi;\{\alpha^+_j,\alpha^-_j\}\rangle \equiv \exp(\xi P_+ - \xi^* P_-) |\{\alpha^+_j,\alpha^-_j\}\rangle = |\{\tilde{\alpha}^+_j(\theta,\varphi),\tilde{\alpha}^-_j(\theta,\varphi)\}\rangle,$$  

(15)

$$\tilde{\alpha}^\pm_j(\theta,\varphi) = \alpha^\pm_j \cos \frac{\theta}{2} \mp \exp(\mp i\varphi) \alpha^\mp_j \sin \frac{\theta}{2},$$

which can be obtained experimentally by the action of quantum polarization "rotators" described by Hamiltonians of the form

$$H_{SU(2)_p} = \sum_{\alpha,\beta=\pm} \zeta_{\alpha,\beta} \sum_{j=1}^m a_\alpha^+(j) a_\beta(j)$$

(16)
on the initial states (14).

The states (13) are analogous to the initial set (14), but with $SU(2)$-rotated parameters $\alpha_j^\pm$ involving two additional (redundant from the mathematical viewpoint) parameters $\theta$ and $\varphi$ that is of no importance from the physical viewpoint. We also note that states (13) can be represented in the form:

$$|\{\alpha^+_j,\alpha^-_j\}\rangle = \prod_{j=1}^m |\theta_j,\varphi_j;\alpha_j\rangle_+,$$

$$|\theta_j,\varphi_j;\alpha_j\rangle_+ \equiv |\theta_j,\varphi_j;\alpha_j^+\rangle = |\alpha_j^+ \cos \frac{\theta_j}{2},\alpha_j^+ \exp(i\varphi_j) \sin \frac{\theta_j}{2}\rangle,$$

(17)

where the polarization coordinates are picked out explicitly (unlike in the form (15)).

### III. GEOMETRIC PHASES OF POLARIZATION COHERENT STATES

In classical polarization optics it is well known [8,13], that during the cyclic evolution of its polarization state the classical plane wave acquires an additional phase shift equal to half the solid angle subtended by the trajectory of the tip of the Stokes vector on the Poincaré sphere. This additional phase is shown to be a particular case of the Pancharatnam’s phase [14] associated with the $SU(2)$ symmetry of the polarization states. It is invariant with respect to deformations of the trajectory leaving the solid angle unchanged and, therefore, is of purely geometric nature. Pancharatnam’s ideas
have been used \cite{15} to set a generalized definition of the geometric phase, valid for a
wide class of quantum evolutions, generally, non-cyclic.

It is natural to pose a question, what happens to the states of quantum light in
the similar situation. The considerations presented above make it possible to apply
the general definitions \cite{15,16} of the geometric phase

$$\gamma = - \oint_{C} A_s ds,$$

where the gauge potential $A_s$ is expressed as

$$A_s = \text{Im} \langle \xi(\theta, \varphi), \psi_0 | \frac{d}{ds} \xi(\theta, \varphi), \psi_0 \rangle$$

$$= \text{Im} \langle \xi(\theta, \varphi), \psi_0 | \nabla_{\Omega} \xi(\theta, \varphi), \psi_0 \rangle \frac{d\Omega}{ds},$$

$s$ is an evolution variable which determines the motion of the system along the evo-
lution trajectory $C$. The states $|\xi(\theta, \varphi), \psi_0 \rangle$ are supposed to be the normalized polar-
ization GCS defined by Eq. (1) with a certain particular choice of the reference state
vectors mentioned above. The explicit form of the derivatives on the unit sphere may
be written as

$$\frac{d\Omega}{ds} = \tilde{e}_\theta \frac{d\theta}{ds} + \tilde{e}_\varphi \sin \theta \frac{d\varphi}{ds},$$

and

$$\langle \xi(\theta, \varphi), \psi_0 | \nabla_{\Omega} \xi(\theta, \varphi), \psi_0 \rangle = \left[ \tilde{e}_\theta \frac{\partial}{\partial u} + \tilde{e}_\varphi \frac{\partial}{\sin \theta \varphi \partial v} \right] \langle \xi(\theta, \varphi), \psi_0 | \xi(u \theta, v \varphi), \psi_0 \rangle_{u=v=1}. \quad (21)$$

As the first example let us consider the polarization GCS \cite{7} with $m = 1, 2$ and
the reference state vectors given by Eqs. \cite{14}, \cite{8}. The overlap integral in Eq. (21)
then takes the form

$$\langle \xi(\theta, \varphi), \psi_0 | \xi(u \theta, v \varphi), \psi_0 \rangle \equiv \langle \theta, \varphi; p, 2p | u \theta, v \varphi; p, 2p \rangle \pm$$

$$= \left[ \cos \frac{\theta}{2} \cos \frac{u \theta}{2} + \sin \frac{\theta}{2} \sin \frac{u \theta}{2} \exp(\mp \varphi(v - 1)) \right]^{2p}. \quad (22)$$
where \( p = n/2 \) for \( m = 1 \) and \( 0 \leq p \leq n/2 \) for \( m = 2 \). The derivatives are easily calculated explicitly for any \( p \), and from Eqs. (18)-(22) we obtain

\[
\gamma = \pm 2p \int_C \sin^2 \frac{\theta}{2} d\varphi,
\]

(23)

which is the \( 2p \) multiple to half the solid angle subtended by \( C \) on the Poincare sphere. In particular, for \( p = 1/2 \) this result coincides with that for classical plane waves.

For the GCS given by Eq. (15) with the reference state vectors (14) one gets

\[
\langle \xi(\theta,\varphi), \psi_0 | \xi(u\theta, v\varphi), \psi_0 \rangle \equiv \langle \theta, \varphi; \{\alpha_j^+, \alpha_j^-\} | u\theta, v\varphi; \{\alpha_j^+, \alpha_j^-\} \rangle
\]

\[
= \exp \left[ -\frac{1}{2} \sum_{j=1}^{m} \left( |\tilde{\alpha}_j^+(\theta, \varphi)|^2 + |\tilde{\alpha}_j^-(\theta, \varphi)|^2 + |\tilde{\alpha}_j^+(u\theta, v\varphi)|^2 + |\tilde{\alpha}_j^-(u\theta, v\varphi)|^2 \right) \right]
\]

\[
\times \exp \left[ \sum_{j=1}^{m} \{ \tilde{\alpha}_j^+(\theta, \varphi)(\tilde{\alpha}_j^+(u\theta, v\varphi))^* + \tilde{\alpha}_j^-(\theta, \varphi)(\tilde{\alpha}_j^-(u\theta, v\varphi))^* \} \right],
\]

(24)

where the notation \( \tilde{\alpha}_j^\pm(\theta, \varphi) \) is clear from Eq. (15). Making use of Eqs. (19)-(21), (24) we get the explicit expression of the gauge potential \( A_s \)

\[
A_s = A_s^{(1)} + A_s^{(2)};
\]

(25)

\[
A_s^{(1)} = -\frac{d\varphi}{ds} \sin^2 \frac{\theta}{2} \sum_{j=1}^{m} \left( |\alpha_j^+|^2 - |\alpha_j^-|^2 \right);
\]

(26)

\[
A_s^{(2)} = -Re \left[ \left( \sin \theta \frac{d\varphi}{ds} + i \frac{d\theta}{ds} \right) \exp(-i\varphi) \sum_{j=1}^{m} \alpha_j^- (\alpha_j^+)^* \right].
\]

(27)

When going from presentation in circular basis to Cartesian basis for polarization quasispin component the final expression for geometric phase takes form

\[
\gamma = \gamma^{(0)} + \gamma^{(1)} + \gamma^{(2)}
\]

(28)

\[
\gamma^{(0)} = 2\langle P_0 \rangle \int_C \sin^2 \frac{\theta}{2} d\varphi, \quad 2\langle P_0 \rangle = \sum_{j=1}^{m} \left( |\alpha_j^+|^2 - |\alpha_j^-|^2 \right),
\]

(29)

\[
\gamma^{(1)} = -\langle P_1 \rangle \int_C [\sin \theta \cos \varphi d\varphi + \sin \varphi d\theta], \quad \langle P_1 \rangle = Re \left[ \sum_{j=1}^{m} \alpha_j^- (\alpha_j^+)^* \right],
\]

(30)

\[
\gamma^{(2)} = \langle P_2 \rangle \int_C [\sin \theta \sin \varphi d\varphi - \cos \varphi d\theta], \quad \langle P_2 \rangle = -Im \left[ \sum_{j=1}^{m} \alpha_j^- (\alpha_j^+)^* \right]
\]

(31)

Therefore the geometric phase is expressed as a scalar product of two vectors. The first one is the vector of polarization quasispin averaged over initial Glauber
state (14). This vector corresponds to the quantum polarization state of the input beam and represents the possible nonclassical properties of light source. The second vector consists of contour integrals on Poincaré sphere, does not depend on the state of the light beam and characterizes transformation of polarization on the track.

In particular, for PGCS (3) with $p = n/2$ and $m = 1, 2$ we have $\langle P_0 \rangle = p$, $\langle P_1 \rangle = 0$, $\langle P_2 \rangle = 0$, that leads to (23).

It may be seen that the contribution of $A_s^{(1)}$ to the geometric phase (18) is just the classical half the solid angle subtended by the cyclic evolution loop $C$ on the Poincaré sphere, multiplied by a factor depending on mode structure of the field. If for each $j$ either $\alpha_j^+$ or $\alpha_j^-$ equals zero then $A_s^{(2)}$ vanishes, and Eq. (29) represents the total geometric phase. This is valid, in particular, for the single-mode states (17) associated to elliptically polarized waves obtained after transmissions of coherent light beams with a definite circular polarization ($< |P_0| > \neq 0$, $< |P_1| >= 0$, $< |P_2| >= 0$) through polarization rotators. At the same time it is not the case for general GCS (15) with initial (reference) vectors (14); specifically, even in the case of one ST mode $A_s^{(2)}$ does not vanish that reflects a specific correlation of polarization modes after a transmission of beams (14) with $\alpha_j^\pm$ being purely real through “polarization rotators” (10). A similar situation (when $A_s^{(2)}$ does not vanish) also occurs in the case $|\alpha_j^+| = |\alpha_j^-|$ in Eqs. (12),(13) corresponding to a transmission of ”helicityless” ( $< |P_0| >= 0$) coherent light beams [1,12] through polarization rotators. In general, Eqs. (24-27) describe a structure (nature) of influences of polarization rotators on initial Glauber’s CS in dependence on their polarization properties since ”energetic” multipliers in these equations are related to expectation values of different components of polarization quasispin $P_\alpha$. 

IV. DISCUSSIONS AND IMPLICATIONS

The sets of GCS obtained above may be also used for the quasiclassical analysis of the polarization properties of quantum light fields. In particular, following the general rules \[2] one can use the definition \[11\] to introduce the complete polarization distribution functions of the quasiprobability \[11\] as

\[
Q(\theta, \varphi; \psi_0)_{\rho} \equiv \langle \theta, \varphi; \psi_0 | \rho | \theta, \varphi; \psi_0 \rangle \equiv \text{Tr}[\rho | \theta, \varphi; \psi_0 \rangle \langle \theta, \varphi; \psi_0 |],
\]

(32)

where $\rho$ is the complete density matrix for the state of the field, $| \theta, \varphi; \psi_0 \rangle$ being defined by Eq. \(1\). Then, substituting the specifications \(7\), \(13\), \(15\), \(17\) for $| \theta, \varphi; \psi_0 \rangle$ found above into Eq. \(32\), we get the appropriate concrete types of the complete polarization quasiprobability functions. Note, however, that such functions, besides the dependence on the polarization parameters $\theta, \varphi$, involve the additional quantum numbers $n, p, \lambda, \{ \alpha_j^\pm \}$, etc., which characterize the non-polarization degrees of freedom of the field. Therefore, to obtain its “polarization quasiclassical portrait” in the $\rho$-state one may make use of the reduced polarization quasiprobability functions $Q(\varphi, \theta; \psi_0)_{\rho}$, resulting from Eq. \(14\) after the summation (or integration) over the non-polarization variables. Such functions may be used to analyse the “polarization squeezing” \[12\] in analogy with the familiar $Q$-functions in case of the standard quadrature squeezing \[17\].

So, we have demonstrated the presence of geometric phase in the polarization generalized coherent states. This phase is due to the cyclic evolution in the space of the parameters $\theta, \varphi$, which are the angular coordinates of the “classical” quasispin on the Poincare sphere. The explicit expression of the geometric phase is shown to depend on the polarization quasispin $p$ and the choice of the reference state vectors, and only in the particular case of $p = 1/2$ , the GCS generated by the set of essentially non-classical reference vectors \(3\) demonstrate exactly the same geometric phase as in the classical case studied by other authors \[13\]. For particular sets of GCS generated by Glauber’s CS the geometric phase is shown to be a product of the classical expression
by a factor depending on the mode structure in the picture of independent ST modes, whereas the result is different for GCS generated by multimode Glauber’s CS in the picture of correlated ST modes.

Thus we have presented a straightforward generalization of the classical theory of the geometric phase induced by the evolution of the polarization onto a new class of quantum light states. The expressions derived are to be used in further investigation of the geometric phases in quantum optics. Specifically, it is of interest to calculate geometric phases for different types of non-classical states of unpolarized and partially polarized light displaying polarization squeezing [12] that may prove to be useful for a practical identification of these states.

We also note that the difference of Eqs. (25)-(27) from the classical results and Eq. (28) reflects the fact the states (14) and (15) are essentially less quasiclassical as compared to GCS (7) with respect to polarization degrees of freedom. Therefore Glauber’s CS do not simulate completely classical light waves.

To compare our results correctly with those known in classical polarization optics [13] it is necessary to emphasize that the phase measured in usual interference experiments is not the phase of the field state vector but the phase acquired by the field amplitude operator of the field state vector [18]. As follows from numerous examples from [18], the latter may be identified with the classical Hannay angle [19]. The relation between the Hannay angle and the Berry phase of the corresponding quantum states has been considered in several papers [20–22]. The rigorous relation follows from the decomposition of the Berry’s phase in powers of $\hbar$ in the quasiclassical limit. If one omits the terms of the order $\hbar^2$ and higher, the Hannay angle $h_j$, associated with the $j$-th classical degree of freedom, appears to be equal to

$$h_j = -\frac{\partial \gamma(n)}{\partial n_j},$$

where $\gamma(n_j)$ is the geometric phase of the state having the set of quantum numbers $\{n_j\}$ which are coupled to the corresponding classical action variables via the Bohr-Zommerfeld condition $I_j = (n_j + \mu_j)\hbar$ [22].
In our case $I_j$ is a classic integral of motion in the space defined by the averaged components of polarization quasispin $\langle P_\alpha \rangle (\alpha = 0, 1, 2)$, i.e. $I_p = hR$, where $R = \sqrt{\langle P_0 \rangle^2 + \langle P_1 \rangle^2 + \langle P_2 \rangle^2}$ is the radius of Poincaré sphere. Correspondingly, the expression for polarization Hannay angle, according with (10), (29)-(31) and (33), have the form

$$h_p = 2 \cos \theta_0 \oint_C \sin^2 \frac{\theta}{2} d\varphi - \sin \theta_0 \cos \varphi_0 \oint_C (\sin \theta \cos \varphi d\varphi + \sin \varphi d\theta)$$
$$+ \sin \theta_0 \sin \varphi_0 \oint_C (\sin \theta \sin \varphi d\varphi - \cos \varphi d\theta),$$

(34)

where $(\theta_0, \varphi_0)$ are the angular coordinates of the initial position of the classical polarization quasispin vector on the Poincaré sphere.

Now it becomes clear that the geometric phase of the "polarizationally most classical" PGCS (23) completely agrees with the classical results [13] since after the calculation of the derivative (33) in accordance with [18] it follows from (23) that the phase shift observed in a usual interferometer (Hannay angle) is just half the solid angle on the Poincaré sphere.

For the PGCS obtained from the Glauber states the connection between the geometric phase (28)-(31) and the phase observed in a usual interferometer is less evident, since the state-dependent factors multiplied by the contour integrals on the Poincaré sphere are no longer "good" quantum numbers, and the states themselves are strongly nonclassical in the polarization degrees of freedom. However, it may be shown that the phase acquired by the field amplitude (Hannay angle) will be half the solid angle subtended by $C$ again.

Therefore, the quantum nature of light doesn’t manifest itself in the experiments like [8] where the total intensity of the light is measured at the output of the interferometer even if one proceeds to the photon-counting technique. This result agrees with the conventional point of view [5].

However, new phase information may be obtained if at the output of the interferometer one measures physical observables other than the total intensity, for example, the components of the P-quasispin or the polarization noise. Even in the
simplest case of Fock states the result is different from the usual one. Indeed, according to [18] during the cyclic evolution the field operators acquire the phase factors $a^+_+ \rightarrow \tilde{a}^+_+ = e^{ih_+}a^+_+$, $a^+_- \rightarrow \tilde{a}^+_- = e^{ih_-}a^+_-$ with different Hannay angles for the right- and left-hand-polarized components. At the output the superposition of the fields is described by the operators $\tilde{a}^+_+ = ra^+_+ + t\tilde{a}^+_+$. Then both $h_\pm$ themselves and their combinations $h_+ \pm h_-$ appear in the mean values of $P^k_{1,2}$ composed of these operators using the general definition.

The search for the ways to measure the GPs of the state vectors of quantum light, particularly, PGCS, as well as for most simple and convenient technologies of the experimental realization of multimode quantum polarization rotators is a subject of our further studies. One of the promising approaches is suggested in [23]. This approach involves a combination of experimental two-photon interferometer setups, aimed at the study of entangled states, with the ideas of the geometric phases defined according to [15].

In conclusion we note that the polarization GCS of $SU(2)_p$ group obtained may be applied also to the analysis of other aspects of the polarization quasiclassical description. Among them one should mention the polarization uncertainty relations and so-called intelligent states [24], as well as the general problem of the description of the quantum light field phase [25].

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