There has been much interest in so-called SIC-POVMs: rank 1 symmetric informationally complete positive operator valued measures. In this paper we discuss the larger class of POVMs which are symmetric and informationally complete but not necessarily rank 1. This class of POVMs is of some independent interest. In particular it includes a POVM which is closely related to the discrete Wigner function. However, it is interesting mainly because of the light it casts on the problem of constructing rank 1 symmetric informationally complete POVMs. In this connection we derive an extremal condition alternative to the one derived by Renes et al.
1. Introduction

There has been much interest in rank 1 symmetric, informationally complete positive operator valued measures [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13]: SIC-POVMs, as they are often called. In \( d \)-dimensional Hilbert space these are systems of \( d^2 \) operators \( \hat{E}_r = (1/d) \hat{P}_r \) such that each \( \hat{P}_r \) is a rank 1 projector and

\[
\text{Tr}(\hat{P}_r \hat{P}_s) = \frac{1}{(d+1)}(1 + \delta_{rs})
\]

for all \( r, s \). In that case it can be shown that \( \sum_{r=1}^{d^2} \hat{E}_r = 1 \), so the operators \( \hat{E}_r \) constitute a POVM. Moreover the POVM is informationally complete [14] [15] [16] [17] [18] [19] [20] (meaning that an arbitrary density matrix \( \hat{\rho} \) is completely specified by the probabilities \( \text{Tr}(\hat{E}_r \hat{\rho}) \)). POVMs of this kind have been constructed [1] [5] [7] [8] [21] [22] (analytically and/or numerically) for every dimension \( d \leq 45 \). It is still an open question whether they exist in dimensions > 45.

The purpose of this paper is to discuss POVMs which are still symmetric, in the sense that

\[
\text{Tr}(\hat{E}_r \hat{E}_s) = \alpha + \beta \delta_{rs}
\]

for fixed numbers \( \alpha, \beta \), and informationally complete, but which are not assumed to be rank 1. We will refer to such POVMs as SI-POVMs (“S” for “symmetric”, “I” for “informationally complete”). SI-POVMs which are also rank 1 we will refer to as SI(1)-POVMs (so an SI(1)-POVM is what in the literature is often called a SIC-POVM).

SI-POVMs are of some independent interest. In particular, we will show in Section 7 that the discrete Wigner function is closely related to a POVM of this type. However, our main reason for studying them is to gain additional insight into the problem of constructing SI(1)-POVMs. To that end we derive an extremal condition alternative to the one used by Renes et al in their numerical work.

The plan of the paper is as follows. In Section 2 we discuss some geometrical features of quantum state space which will be needed in the sequel. In Section 3 we relate this discussion to the problem of devising a tomographical procedure which is, in some suitably defined sense, optimal. In Section 4 we prove a theorem characterising the structure of an arbitrary SI-POVM. In Section 5 we specialise to the case of SI-POVMs covariant under the Weyl-Heisenberg group (or generalized Pauli group as it is often called). We show that such POVMs have a very simple representation in terms of the Weyl-Heisenberg displacement operators. In Section 6 we turn to the problem of constructing SI(1)-POVMs, and derive an extremal condition alternative to the one derived by Renes et al [1]. Finally, in Section 7 we construct an SI-POVM which is closely related to the discrete Wigner function.

2. The Bloch Body

Let \( \mathcal{H} \) be a \( d \) dimensional Hilbert space, and let \( \mathcal{D} \) be the space of density matrices defined on \( \mathcal{H} \). If \( d = 2 \) it is well known that \( \mathcal{D} \) can be identified with the Bloch sphere. To be specific: let \( \mathcal{B} \) be the unit ball in \( \mathbb{R}^3 \) (i.e. the set of vectors \( \in \mathbb{R}^3 \) having length \( \leq 1 \)). Then a \( 2 \times 2 \) complex matrix \( \hat{\rho} \) is a density matrix if and only if it can be written in the form

\[
\hat{\rho} = \frac{1}{2} (1 + b \cdot \hat{\sigma})
\]

where \( b \in \mathcal{B} \) (the Bloch ball) and \( \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3 \) are the Pauli matrices.

With the appropriate modifications this construction can be generalized to higher dimensions [4] [5] [23] [24] [25] [26] [27] [28] [29] [30]. Let \( \text{su}(d) \) be the \( d^2 - 1 \) dimensional
real vector space consisting of all trace zero Hermitian $d \times d$ complex matrices.\footnote{\textit{su}(d) is the Lie algebra for the special unitary group \textit{SU}(d). This group theoretical fact is highly relevant to the problem of characterizing the geometry of quantum state space \cite{24,25,26,27,28,29,30}. However, it will play no part in the considerations of this paper.}

Let $\mathcal{B}$ be the convex subset consisting of all $\hat{B} \in \text{su}(d)$ for which $\hat{B} \geq -1$. Then a $d \times d$ matrix $\hat{\rho}$ is a density matrix if and only if

$$\hat{\rho} = \frac{1}{d}(1 + \hat{B})$$

(4)

for some $\hat{B} \in \mathcal{B}$. We refer to $\mathcal{B}$ as the Bloch body, and to its elements as Bloch vectors.\footnote{What we are calling Bloch vectors are of course matrices. Some authors introduce a standard basis for $\text{su}(d)$ at this point and reserve the term “Bloch vector” for the components of $\hat{B}$ in that basis (as has been the long-standing practice in the 2 dimensional case—see Eq. (3) above). However, it appears to us that this makes the notation needlessly complicated.}

It is convenient to define an inner product on $\text{su}(d)$ by

$$\langle \hat{B}_1, \hat{B}_2 \rangle = \frac{1}{d(d-1)} \text{Tr}(\hat{B}_1 \hat{B}_2)$$

(5)

for all $\hat{B}_1, \hat{B}_2 \in \text{su}(d)$ (so $\langle \hat{B}_1, \hat{B}_2 \rangle$ is just the Hilbert-Schmidt inner product rescaled by the factor $\frac{1}{d(d-1)}$). Let

$$\| \hat{B} \| = \sqrt{\langle \hat{B}, \hat{B} \rangle}$$

(6)

be the corresponding norm.

If $d = 2$ a vector $\hat{B} \in \text{su}(d)$ is a Bloch vector if and only if $\| \hat{B} \| \leq 1$. Moreover, the corresponding density matrix is a pure state if and only if $\| \hat{B} \| = 1$. For $d > 2$ the situation is more complicated. Let $\mathcal{B}_i$ and $\mathcal{B}_o$ be the balls

$$\mathcal{B}_i = \{ \hat{B} \in \text{su}(d): \| \hat{B} \| \leq \frac{1}{d-1} \}$$

(7)

$$\mathcal{B}_o = \{ \hat{B} \in \text{su}(d): \| \hat{B} \| \leq 1 \}$$

(8)

and let

$$\mathcal{S}_i = \{ \hat{B} \in \text{su}(d): \| \hat{B} \| = \frac{1}{d-1} \}$$

(9)

$$\mathcal{S}_o = \{ \hat{B} \in \text{su}(d): \| \hat{B} \| = 1 \}$$

(10)

be the bounding spheres. Then \cite{23,26,29}

$$\mathcal{B}_i \subseteq \mathcal{B} \subseteq \mathcal{B}_o$$

(11)

It can further be shown \cite{23,26,29} that $\mathcal{B}_i$ and $\mathcal{B}_o$ are respectively the largest and smallest balls centred on the origin for which this is true. Specifically:

1. If $r > 1/(d-1)$ there exists $\hat{B} \in \text{su}(d)$ such that $\| \hat{B} \| = r$ and $\hat{B} \notin \mathcal{B}$.

2. If $0 \leq r \leq 1$ there exists $\hat{B} \in \mathcal{B}$ such that $\| \hat{B} \| = r$.

Moreover a Bloch vector $\hat{B} \in \mathcal{B}$ corresponds to a pure state if and only if it has norm $= 1$ (i.e. if and only if it is in $\mathcal{B} \cap \mathcal{S}_o$).

It is worth noting that Bengtsson and Ericsson \cite{6} have proved a stronger result: in any dimension for which either a full set of MUBs (mutually unbiased bases) or an SU(1)-POVM exist $\mathcal{B}_i$ is the largest \textit{ellipsoid} which can be inscribed in $\mathcal{B}$.

If $d = 2$ we have $\mathcal{B}_i = \mathcal{B} = \mathcal{B}_o$ and $\mathcal{B} \cap \mathcal{S}_o = \mathcal{S}_o$, so the Bloch body has a very simple geometrical structure (it is just a ball of radius 1 centred on the origin, with the pure states comprising the boundary). For $d > 2$ these relations no longer hold, and the geometry is much harder to appreciate intuitively. One gets some additional intuitive feeling for the geometry, at least in low dimension, by looking...
at the 2-dimensional sections of $\mathcal{B}$ which have been calculated \[24, 25, 26, 29\] for $d = 3$ and 4.

Let $\hat{B}$ be any vector $\in S_\circ$ (not necessarily a Bloch vector). An immediate consequence of Eq. (11) is that $x\hat{B} \in \mathcal{B}$ whenever $|x| \leq 1/(d-1)$. Kimura and Kossakowski \[29\] have proved some much stronger results. In the first place they have shown

**Theorem 1.** Let $\hat{B}$ be any vector $\in S_\circ$ (not necessarily a Bloch vector). Let $-m_-$ be the smallest eigenvalue of $\hat{B}$ and let $m_+$ be the largest (so $-m_- \leq \hat{B} \leq m_+$). Then

1. The quantities $m_\pm$ satisfy the inequalities
   \[1 \leq m_- \leq d-1\] (12)
   and
   \[1 \leq m_+ \leq d-1\] (13)
   Moreover $m_- = 1$ if and only if $m_+ = d-1$, and $m_+ = 1$ if and only if $m_- = d-1$
2. $\hat{B}$ is a Bloch vector (in fact the Bloch vector corresponding to a pure state) if and only if $m_- = 1$. Similarly $-\hat{B}$ is a Bloch vector (in fact the Bloch vector corresponding to a pure state) if and only if $m_+ = 1$.

**Proof.** See Kimura and Kossakowski \[29\]. □

Theorem 1 characterizes the vectors $\in \mathcal{B} \cap S_\circ$ (i.e. the Bloch vectors corresponding to pure states) in terms of their eigenvalues. The next theorem relates the diameter of the Bloch body in the direction $\hat{B}$ to the eigenvalues of $\hat{B}$.

**Theorem 2.** Let $\hat{B}$ and $m_\pm$ be as in the statement of Theorem 1 and let $x \in \mathbb{R}$. Then $x\hat{B} \in \mathcal{B}$ if and only if

\[-1/m_+ \leq x \leq 1/m_-\] (14)

**Proof.** See Kimura and Kossakowski \[29\]. □

**Remark.** As Kimura and Kossakowski point out, it follows from Theorems 1 and 2 that a point where the boundary of $\mathcal{B}$ touches the outer sphere $S_\circ$ is always diametrically opposite to a point where the boundary of $\mathcal{B}$ touches the inner sphere $S_i$ (and conversely).

We conclude this section by proving a theorem which shows that, instead of considering the eigenvalues (as in Theorem 1), one can use the quantity $\text{Tr}(\hat{B}^3)$ to tell whether a vector $\hat{B} \in S_\circ$ is the Bloch vector corresponding to a pure state. We first need to prove

**Lemma 3.** Let $\hat{P}$ be any $d \times d$ Hermitian matrix (not necessarily a positive matrix). Suppose
\[\text{Tr}(\hat{P}^2) = 1\] (15)
Then
\[\text{Tr}(\hat{P}^3) \leq 1\] (16)
with equality if and only if $\hat{P}$ is a one dimensional projector.

**Remark.** It is not assumed that $\text{Tr}(\hat{P}) = 1$. 

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_d$ be the eigenvalues of $\hat{P}$ (not necessarily distinct). In view of Eq. (15)

$$\sum_{r=1}^{d} \lambda_r^2 = 1$$  \hspace{1cm} (17)

Define

$$\kappa = \sum_{r=1}^{d} |\lambda_r|^3$$  \hspace{1cm} (18)

It follows from Eq. (17) that $|\lambda_r| \leq 1$ for all $r$, and consequently that $1 - |\lambda_r| \geq 0$ for all $r$. So

$$1 - \kappa = \sum_{r=1}^{d} (\lambda_r^2 - |\lambda_r|^3)$$
$$= \sum_{r=1}^{d} \lambda_r^2 (1 - |\lambda_r|)$$
$$\geq 0$$  \hspace{1cm} (19)

with equality if and only if $\lambda_r^2 (1 - |\lambda_r|) = 0$ for all $r$. Consequently

$$\kappa \leq 1$$  \hspace{1cm} (20)

for all $r$.

It is now immediate that

$$\text{Tr}(\hat{P}^3) \leq |\text{Tr}(\hat{P}^3)| \leq \kappa \leq 1$$  \hspace{1cm} (22)

Suppose

$$\text{Tr}(\hat{P}^3) = 1$$  \hspace{1cm} (23)

Then it follows from Eq. (22) that $\kappa = 1$ which means, in view of Eqs. (20) and (21), that $\lambda_r^2 (1 - |\lambda_r|) = 0$ for all $r$. Consequently, for each $r$, $|\lambda_r| = 0$ or $1$. The fact that $\sum_r \lambda_r^2 = 1$ then implies that $|\lambda_r| = 1$ for exactly one value of $r$ and $= 0$ for all the others. Since, by assumption, $\sum_r \lambda_r^2 = 1$ we must actually have $\lambda_r = 1$ for exactly one value of $r$ and $= 0$ for all the others—implying that $\hat{P}$ is a one dimensional projector.

If, on the other hand, $\hat{P}$ is a one dimensional projector it is immediate that $\text{Tr}(\hat{P}^3) = 1$.

We are now in a position to prove our main result:

**Theorem 4.** Let $\hat{B}$ be any vector $\in S_o$ (not necessarily a Bloch vector). Then

1. The quantity $\text{Tr}(\hat{B}^3)$ satisfies the inequalities

$$-d(d-1)(d-2) \leq \text{Tr}(\hat{B}^3) \leq d(d-1)(d-2)$$  \hspace{1cm} (24)

2. The upper bound in Inequalities (24) is achieved if and only if $\hat{B} \in \mathcal{B} \cap S_o$ (and is therefore the Bloch vector corresponding to a pure state).

3. The lower bound in Inequalities (24) is achieved if and only if $-\hat{B} \in \mathcal{B} \cap S_o$ (and is therefore the Bloch vector corresponding to a pure state).

**Proof.** The fact that $\hat{B} \in S_o$ means

$$\text{Tr}(\hat{B}^2) = d(d-1)$$  \hspace{1cm} (25)

Define

$$\tilde{P}_\pm = \frac{1}{d} (1 \pm \hat{B})$$  \hspace{1cm} (26)
Then Eq. (25) implies
\[ \text{Tr}(\hat{P}_2^2) = \frac{1}{d^2}(d + \text{Tr}(\hat{B}^2)) = 1 \] (27)
We may therefore use Lemma 3 to deduce
\[ \frac{1}{d^2}(d + 3 \text{Tr}(\hat{B}^2) \pm \text{Tr}(\hat{B}^3)) = \text{Tr}(\hat{P}_3^3) \leq 1 \] (28)
with equality if and only if \( \hat{P}_3 \) is a one dimensional projector. In view of Eq. (25) this means
\[ \text{Tr}(\hat{B}^3) \leq d(d-1)(d-2) \] (29)
with equality if and only if \( \hat{B} \) is a one dimensional projector, and
\[ \text{Tr}(\hat{B}^3) \geq -d(d-1)(d-2) \] (30)
with equality if and only if \( -\hat{B} \) is a one dimensional projector. But \( \hat{P}_3 \) is a one dimensional projector if and only if \( \hat{B} \in \mathcal{B} \cap \mathcal{S}_0 \), and \( \hat{P}_- \) is a one dimensional projector if and only if \( -\hat{B} \in \mathcal{B} \cap \mathcal{S}_0 \). The claim is now immediate. \( \square \)

3. BLOCH GEOMETRY AND TOMOGRAPHY

The geometry of the Bloch body is intimately related to the problem of devising measurement schemes which are, in some suitably defined sense, tomographically optimal. The connection works both ways. On the one hand knowledge of the geometry tells us what measurement schemes are possible. On the other hand a knowledge of possible measurement schemes provides important insight into the geometry. In this section we summarize the Bloch geometrical aspects of two such measurement schemes: namely, schemes based on a full set of mutually unbiased bases or MUBs [3, 4, 5, 6, 22, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40] and schemes based on SI(1)-POVMs [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] (or SIC-POVMs as they are often called). Much of the material in this section amounts to a review of the relevant parts of Bengtsson [4] and Bengtsson and Ericsson [6], but using a slightly different terminology and notation.

We begin with the case of a full set of MUBs. Suppose one has a large number of copies of a \( d \)-dimensional quantum system, all presumed to be in the same quantum state. Suppose one takes a fixed von Neumann measurement having \( d \) distinct outcomes, and performs it on many copies of the system. Suppose one then identifies the relative frequencies obtained with the corresponding probabilities. This will give one \( d \) probabilities \( p_1, p_2, \ldots, p_d \). Taking into account the normalisation condition \( \sum_{r=1}^d p_r = 1 \) this means one has \( d-1 \) independent numbers. On the other hand a full specification of the quantum state requires \( d^2-1 \) independent numbers. It follows that if one wants to perform tomography using only von Neumann measurements one needs to divide the set of copies of the system into a minimum of \( d+1 \) subsets, and to perform different von Neumann measurements on the copies belonging to different subsets. We will refer to a measurement scheme based on the minimum number of \( d+1 \) different von Neumann measurements, each having \( d \) distinct outcomes, as a \textit{minimal Von Neumann scheme}.

The question now arises: what is the best way of choosing the \( d+1 \) different measurements in a minimal von Neumann scheme? Let \( \hat{P}_1^r, \hat{P}_2^r, \ldots, \hat{P}_d^r \) be the \( d \) orthogonal, one dimensional projectors describing the \( r \)th measurement and let \( \hat{B}_1^r, \hat{B}_2^r, \ldots, \hat{B}_d^r \) be the corresponding Bloch vectors. So
\[ \hat{P}_a^r = \frac{1}{d}(1 + \hat{B}_a^r) \] (31)
for all \( a, r \). Notice that, whereas in Section 2 we used Bloch vectors to describe quantum states, now we are using them to describe quantum measurements.
also that the fact that the $\hat{P}_r^a$ are all one dimensional projectors means that the vectors $\hat{B}_r^a$ all lie on $B \cap S_\circ$.

The orthonormality condition
\[ \text{Tr}(\hat{P}_r^a \hat{P}_r^b) = \delta_{ab} \] (32)

together with Eq. (5) implies
\[ \langle \hat{B}_r^a, \hat{B}_r^b \rangle = \begin{cases} 1 & a = b \\ \frac{1}{d-1} & a \neq b \end{cases} \] (33)

from which one sees that for each $r$ the Bloch vectors $\hat{B}_1^r, \hat{B}_2^r, \ldots, \hat{B}_d^r$ are the vertices of a regular $d-1$ dimensional simplex. So the $d+1$ families of orthogonal projectors define $d+1$ regular simplices, each having its vertices in $B \cap S_\circ$. One might guess, and detailed calculation confirms \([33, 41]\), that the optimal choice from a tomographic point of view is to choose the projectors in such a way that the polytope formed by all $d^2 + d$ Bloch vectors has maximal volume. This is achieved if the $d+1$ different simplices are mutually orthogonal:
\[ \langle \hat{B}_a^r, \hat{B}_s^b \rangle = 0 \] (34)

for all $a$, $b$ and $r \neq s$.

This condition is often stated in a slightly different form. Suppose we choose vectors $|\psi_r^a\rangle \in H$ such that $\hat{P}_r^a = |\psi_r^a\rangle\langle \psi_r^a|$ (so for each $r$ the set $|\psi_1^r\rangle, |\psi_2^r\rangle, \ldots, |\psi_d^r\rangle$ is an orthonormal basis for $H$). Then the requirement that the simplices corresponding to different bases be mutually orthogonal is equivalent to the requirement that
\[ |\langle \psi_a^r | \psi_s^b \rangle| = \frac{1}{\sqrt{d}} \] (35)

for all $a$, $b$ and $r \neq s$. A family of orthonormal bases for which this condition is satisfied is said to be mutually unbiased.

The question now arises: do families of $d+1$ mutually unbiased bases (MUBs) actually exist? This is a difficult geometrical problem. As Bengtsson and Ericsson \([4, 6]\) have noted, what makes it hard is, in essence, the fact that $B \cap S_\circ$ has a much lower dimension than $S_\circ$. Consider, for instance, the case $d = 3$. In that case the problem is to orientate a set of 4 mutually orthogonal equilateral triangles in such a way that all 12 vertices lie in $B \cap S_\circ$. It is very easy, almost trivial, to construct a family of 4 mutually orthogonal equilateral triangles with vertices on the 7 dimensional sphere $S_\circ$. The difficult part is then to rotate them so that every vertex lies on the 4 dimensional subspace $B \cap S_\circ$. As it happens the problem has been solved for $d = 3$, and also for every other dimension which is the power of a prime number \([31, 33]\). But for values of $d$ which are not prime powers the question is still open \([3, 4, 5, 6, 22, 31, 32, 34, 35, 36, 37, 38, 39, 40]\). So we have here an important physical problem the solution to which depends on gaining a better understanding of the geometry of the Bloch body.

Let us now turn to a different measurement scheme. Suppose that, instead of using $d+1$ different von Neumann measurements, we wanted to use a single POVM measurement. The POVM would obviously need to have the property that specifying the probability of each of the distinct outcomes fixes the quantum state. Such a POVM is said to be \textit{informationally complete} \([14, 15, 16, 17, 18, 19, 20]\). As we remarked earlier, a complete specification of the quantum state requires the specification of $d^2 - 1$ independent numbers. Taking into account the normalisation condition (the fact that the probabilities must sum to unity) this means that an informationally complete POVM must have at least $d^2$ distinct outcomes. We will say that a POVM is minimal informationally complete if it has precisely this minimum number of $d^2$ distinct outcomes. The question we have then to answer is:
which minimal informationally complete POVMs are tomographically optimal? As with the MUB problem, the answer to this question depends on achieving a better understanding of the geometry of the Bloch body.

Let \( \hat{E}_1, \hat{E}_2, \ldots, \hat{E}_{d^2} \) be an arbitrary POVM having \( d^2 \) distinct elements. Define

\[
t_r = \text{Tr}(\hat{E}_r)
\]

(36)

We may assume that \( \hat{E}_r \neq 0 \) and consequently \( t_r \neq 0 \) for all \( r \) (otherwise the POVM would effectively reduce to one having fewer than \( d^2 \) elements). It follows that for each \( r \) the operator \((1/t_r)\hat{E}_r\) is a density matrix. We may therefore write, for all \( r \),

\[
\hat{E}_r = \frac{t_r}{d^2}(1 + \hat{B}_r)
\]

(37)

where \( \hat{B}_r \in \mathcal{B} \). The fact that \( \sum_{r=1}^{d^2} \hat{E}_r = 1 \) implies

\[
\sum_{r=1}^{d^2} t_r = d
\]

(38)

and

\[
\sum_{r=1}^{d^2} t_r \hat{B}_r = 0
\]

(39)

It is easily seen that the POVM is informationally complete if and only if the Bloch vectors \( \hat{B}_r \) span \( \text{su}(d) \). This in turn will be true if and only if the vectors \( \hat{B}_r \) are the vertices of a \( d^2 - 1 \) dimensional simplex (typically an irregular simplex) having non-zero volume. One might guess, and detailed calculation confirms \([11]\), that the POVM would be optimal from a tomographic point of view if we could arrange that (a) the simplex is regular and (b) the vertices all lie on \( \mathcal{B} \cap \mathcal{S}_0 \) (because the volume of the simplex would then be maximal). In other words we would like to arrange that

\[
\langle \hat{B}_r, \hat{B}_s \rangle = \begin{cases} 1 & r = s \\ -\frac{1}{d-1} & r \neq s \end{cases}
\]

(40)

In that case

\[
\sum_{r=1}^{d^2} \langle \hat{B}_r, \hat{B}_s \rangle = 0
\]

(41)

for all \( s \). Since the vectors \( \hat{B}_s \) span \( \text{su}(d) \) this means

\[
\sum_{r=1}^{d^2} \hat{B}_r = 0
\]

(42)

Eqs. (38), (39) and (42), taken in conjunction with the fact that the \( d^2 \) vectors \( \hat{B}_r \) span the \( d^2 - 1 \) dimensional space \( \text{su}(d) \), then imply

\[
t_1 = t_2 = \cdots = t_{d^2} = \frac{1}{d}
\]

(43)

so that the POVM elements take the form

\[
\hat{E}_r = \frac{1}{d^2}(1 + \hat{B}_r)
\]

(44)

Since the \( \hat{B}_r \) all belong to \( \mathcal{B} \cap \mathcal{S}_0 \) we may alternatively write

\[
\hat{E}_r = \frac{1}{d} \hat{P}_r
\]

(45)
where the $\hat{P}_r$ are a family of one dimensional projectors satisfying

$$\text{Tr}(\hat{P}_r\hat{P}_s) = \begin{cases} 1 & r = s \\ \frac{1}{d+1} & r \neq s \end{cases}$$

(46)

The converse is also true: if $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_d$ is any family of one dimensional projectors satisfying Eq. (46) then $\frac{1}{d}\hat{P}_1, \frac{1}{d}\hat{P}_2, \ldots, \frac{1}{d}\hat{P}_d$ is an informationally complete POVM.

A POVM which satisfies the defining Eq. (40) (equivalently: a POVM which is rank 1 and which satisfies Eqs. (45) and (46)) is usually referred to as a SIC-POVM (symmetric informationally complete POVM). It appears to us that this terminology is unsatisfactory as, besides being symmetric and informationally complete, POVMs of the type in question are also rank 1. We therefore suggest that POVMs of the type in question would be better described as SI(1)-POVMs (“S” for symmetric, “I” for informationally complete, “1” for rank 1). The larger class of POVMs, which are symmetric and informationally complete but not necessarily rank 1, we will refer to as SI-POVMs.

Do SI(1)-POVMs exist? This is a difficult geometrical problem. Moreover, it is difficult for essentially the same reason that the MUB problem is difficult: namely, the submanifold $\mathcal{B} \cap S_0$ has much lower dimension than the sphere $S_0$ if $d > 2$. It is easy to construct a regular $d^2 - 1$ dimensional simplex with vertices in the $d^2 - 2$ dimensional sphere $S_0$, but very hard then to rotate the simplex so that every vertex lies in the $2(d-1)$ dimensional subspace $\mathcal{B} \cap S_0$ (except, of course, when $d = 2$). Moreover the difficulty increases with increasing $d$ (because $2(d-1)/(d^2 - 2) \to 0$ as $d \to \infty$). SI(1)-POVMs have been constructed analytically and numerically in dimensions 2 to 10 inclusive, and in dimensions 12, 13 and 19. They have been constructed numerically in dimensions 5 to 45 inclusive. It is an open question whether they exist in dimensions $> 45$.

Most (not all) of the SI(1)-POVMs which have been constructed to date are covariant under the action of the Weyl-Heisenberg group (or generalized Pauli group, as it is sometimes called). For a summary of the pertinent facts concerning this group see Appendix A.

Let $\mathbb{Z}_d^2$ be the set of integer pairs $p = (p_1, p_2)$ such that $0 \leq p_1, p_2 \leq d - 1$, and for each $p \in \mathbb{Z}_d^2$ let $\hat{D}_p$ be the corresponding Weyl-Heisenberg displacement operator, as defined by Eq. (93). Let $B$ be any Bloch vector in $\mathcal{B} \cap S_0$. Then the fact that the $\hat{D}_p$ are unitary means that for each $p$ $\hat{D}_p \hat{B} \hat{D}_p^\dagger$ also belongs to $\mathcal{B} \cap S_0$. Suppose that the $\hat{D}_p$ constitute a regular simplex:

$$\langle \hat{D}_p, \hat{D}_q \rangle = \begin{cases} 1 & p = q \\ -\frac{1}{d-1} & p \neq q \end{cases}$$

(48)

Then the corresponding SI(1)-POVM is said to be Weyl-Heisenberg covariant.

\(^3\)To see this note that Eq. (49) implies that the corresponding Bloch vectors satisfy Eq. (40). It follows that the Bloch vectors span $\text{su}(d)$ (because if $M$ is the $(d^2 - 1) \times (d^2 - 1)$ matrix with elements $M_{rs} = \langle \hat{B}_r, \hat{B}_s \rangle$ for $r, s = 1, 2, \ldots, d^2 - 1$ then $\text{Det} M = d^{2(d-2)}/(d^2 - 1)^{d^2 - 1} \neq 0$) and consequently that $\sum_{r=1}^{d^2} \hat{B}_r = 0$ (by the same argument that led to Eq. (42)). The claim is now immediate.
4. SI-POVMs in General

In the last section we discussed SI(1)-POVMs: POVMs which are not only symmetric and informationally complete but also rank-1 (so that each element of the POVM is proportional to a one dimensional projector). We now want to broaden the discussion, and consider POVMs which, though symmetric and informationally complete, are not necessarily rank 1.

Consider an arbitrary POVM \( \hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n \) defined on a \( d \)-dimensional Hilbert space. Without loss of generality it may be assumed that \( \hat{E}_r \neq 0 \) for all \( r \). We saw in the last section that we can write

\[
\hat{E}_r = \frac{t_r}{d} (1 + \hat{B}_r)
\]  

(49)

where \( \hat{B}_r \in \mathcal{B} \) for all \( r \), where \( t_r > 0 \) for all \( r \), and where

\[
\sum_r t_r = d \quad \text{and} \quad \sum_r t_r \hat{B}_r = 0
\]  

(50) and (51)

Conversely, if we have a set of Bloch vectors \( \hat{B}_r \) and positive numbers \( t_r \) satisfying these conditions then Eq. (49) defines a POVM.

We say that the POVM is informationally complete if the probabilities \( \text{Tr}(\hat{\rho} \hat{E}_r) \) completely specify an arbitrary density matrix \( \hat{\rho} \). We say that it is symmetric if

\[
\text{Tr}(\hat{E}_r \hat{E}_s) = \alpha + \beta \delta_{rs}
\]  

(52)

for all \( r, s \) and fixed numbers \( \alpha, \beta \).

We then have the following theorem:

**Theorem 5.** Let \( \hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n \) be a POVM having \( n \) elements (all non-zero) defined on a \( d \)-dimensional Hilbert space. The POVM is symmetric and informationally complete if and only if

1. \( n = d^2 \).
2. The POVM elements are of the form

\[
\hat{E}_r = \frac{1}{d^2} (1 + \hat{B}_r)
\]  

(53)

where the Bloch vectors \( \hat{B}_r \) satisfy

\[
\langle \hat{B}_r, \hat{B}_s \rangle = \begin{cases} 
\kappa^2 & r = s \\
-\frac{\kappa^2}{d^2-1} & r \neq s 
\end{cases}
\]  

(54)

with \( 0 < \kappa \leq 1 \).

**Remark.** We will refer to \( \kappa \) as the efficiency parameter as it determines the volume of the regular simplex spanned by the Bloch vectors \( \hat{B}_r \), and consequently the efficiency of the POVM for tomographic purposes [41]. The POVM is maximally efficient if and only if \( \kappa = 1 \) in which case it is rank one (an SI(1)-POVM in the terminology explained in the last section).

**Proof.** We first prove necessity. Suppose the POVM is symmetric and informationally complete. We can write it in the form

\[
\hat{E}_r = \frac{t_r}{d} (1 + \hat{B}_r)
\]  

(55)
for Bloch vectors $\hat{B}_r$ and positive numbers $t_r$ satisfying Eqs. (50) and (51). The symmetry condition Eq. (52) then implies

$$t_r = \text{Tr}(\hat{E}_r) = \sum_{s=1}^{n} \text{Tr}(\hat{E}_r \hat{E}_s) = n\alpha + \beta$$

for all $r$. In view of Eq. (50) this means

$$t_r = \frac{d}{n}$$

for all $r$, and consequently

$$\alpha = \frac{d - n\beta}{n^2}$$

Using these results, Eq. (55) and the symmetry condition Eq. (52) we deduce

$$\langle \hat{B}_r, \hat{B}_s \rangle = -\frac{n\beta}{d(d-1)} + \frac{n^2\beta}{d(d-1)} \delta_{rs}$$

(59)

The fact that the $\hat{B}_r$ are Bloch vectors means $\langle \hat{B}_r, \hat{B}_r \rangle \leq 1$. We must also have $\langle \hat{B}_r, \hat{B}_r \rangle > 0$ (because otherwise $\hat{E}_r = \frac{1}{n}$ for all $r$, in which case the POVM would not be informationally complete). Consequently

$$0 < \beta \leq \frac{d(d-1)}{n(n-1)}$$

(60)

Let $\hat{M}$ be the $n \times n$ matrix with elements $\hat{M}_{rs} = \langle \hat{B}_r, \hat{B}_s \rangle$. Since the POVM is informationally complete the Bloch vectors $\hat{B}_r$ must span the $d^2 - 1$ dimensional space su($d$). So $\hat{M}$ must have rank $d^2 - 1$. On the other hand

$$\text{Det}(\hat{M} - \lambda) = -\lambda \left( \frac{n^2\beta}{d(d-1)} - \lambda \right)^{n-1}$$

(61)

It follows from this that $\hat{M}$ has $n - 1$ non-zero eigenvalues (since we have shown that $\beta > 0$). However, the fact that $\hat{M}$ is rank $d^2 - 1$ means that it must have $d^2 - 1$ non-zero eigenvalues. We conclude that $n = d^2$. Making the substitutions $n = d^2$ and $\beta = \frac{n^2\beta}{d(d-1)}$ in Eq. (59) we obtain Eq. (54). Moreover, it follows from Eq. (60) that $0 < \kappa \leq 1$.

Having proved necessity, it remains to prove sufficiency. Suppose $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_{d^2}$ are Bloch vectors satisfying Eq. (54). Let $\hat{M}$ be the $d^2 \times d^2$ matrix with elements $\hat{M}_{rs} = \langle \hat{B}_r, \hat{B}_s \rangle$. Then

$$\text{Det}(\hat{M} - \lambda) = -\lambda \left( \frac{\kappa^2d^2}{d^2 - 1} - \lambda \right)^{d^2-1}$$

(62)

Since, by assumption, $\kappa > 0$ it follows that $\hat{M}$ has $d^2 - 1$ non-zero eigenvalues, and is therefore rank $d^2 - 1$. Consequently the Bloch vectors span the $d^2 - 1$ dimensional vector space su($d$).

Eq. (54) also implies

$$\left\langle \left( \sum_{a=1}^{d^2} \hat{B}_a \right), \hat{B}_r \right\rangle = 0$$

(63)

for all $r$. Since the $\hat{B}_r$ span su($d$) we deduce

$$\sum_{a=1}^{d^2} \hat{B}_a = 0$$

(64)
It follows from this that if we define
\[ \hat{E}_r = \frac{1}{d^2} (1 + \hat{B}_r) \] (65)
the operators \( \hat{E}_1, \hat{E}_2, \ldots, \hat{E}_{d^2} \) constitute a POVM. The fact that the \( \hat{B}_r \) span su(d) means the POVM is informationally complete. The fact that the POVM is symmetric is immediate. □

We noted in the last section that the existence problem for SI(1)-POVMs is hard, and still unsolved for dimensions > 45. But if one relaxes the demand that the POVM be rank 1, and simply looks for an SI-POVM of arbitrary rank, the problem becomes much easier.

To construct an SI-POVM of arbitrary rank all we have to do is construct a regular simplex in su(d) with its vertices all on \( S_0 \) (since \( S_0 \) is a sphere such simplices are guaranteed to exist). Let \( \hat{B}_1, \hat{B}_2, \ldots, \hat{B}_{d^2} \) be the vertices. Then
\[ \langle \hat{B}_r, \hat{B}_s \rangle = \begin{cases} 1 & \text{if } r = s \\ -\frac{1}{d^2-1} & \text{otherwise} \end{cases} \] (66)
If the \( \hat{B}_r \) were Bloch vectors this would give us an SI(1)-POVM. However, if the simplex is chosen at random they are very unlikely to be Bloch vectors (because the manifold \( B \cap S_0 \) has much lower dimension than \( S_0 \)). Nevertheless, we can still use them to construct an SI-POVM by shrinking the simplex until the vertices are all in \( B \). In fact, let \( -m_r \) be the smallest eigenvalue of \( \hat{B}_r \). It follows from Theorem 1 that \( 1 \leq m_r \leq d - 1 \) for all \( r \). Now define
\[ \kappa = \min_{1 \leq r \leq d^2} \left( \frac{1}{m_r} \right) \] (67)
We have \( \frac{1}{d^2} \leq \kappa \leq 1 \). Moreover, it follows from Theorem 2 that \( \hat{B}_r' = \kappa \hat{B}_r \in B \) for all \( r \). By construction
\[ \langle \hat{B}_r', \hat{B}_s' \rangle = \begin{cases} \kappa^2 & \text{if } r = s \\ -\frac{\kappa^2}{d^2-1} & \text{otherwise} \end{cases} \] (68)
So we can use Theorem 5 to deduce that the POVM with elements
\[ \hat{E}_r = \frac{1}{d^2} (1 + \hat{B}_r') \] (69)
is symmetric, informationally complete with efficiency parameter = \( \kappa \).

The argument just given shows that in every dimension \( d \) there exists an SI-POVM with efficiency parameter \( \geq \frac{1}{d^2-1} \). We will see in Section 7 that at least when \( d \) is odd it is possible to considerably improve on that.

5. SI-POVMs which are Weyl-Heisenberg Covariant

In Section 6 we will discuss the bearing of the above results on the really difficult problem: i.e. the problem of constructing POVMs which are, not merely symmetric and informationally complete, but also rank 1 (have efficiency parameter = 1). In preparation for that we first need to prove a result concerning SI-POVMs (with efficiency parameter not necessarily = 1) which are covariant under the Weyl-Heisenberg group.

We begin with a definition. Let \( \hat{B} \in S_0 \) (we do not assume that \( \hat{B} \) is a Bloch vector), and for each \( p \in \mathbb{Z}_d^2 \) let \( \hat{B}_p = \hat{D}_p \hat{B} \hat{D}_p^\dagger \) (where \( \mathbb{Z}_d^2 \) and \( \hat{D}_p \) are as defined
in Appendix A. We say that $\hat{B}$ is the generating vector for a Weyl-Heisenberg covariant regular simplex if

$$\langle \hat{B}, \hat{B}_p \rangle = \begin{cases} 1 & \text{if } p = (0,0) \\ -\frac{1}{d^2-1} & \text{otherwise} \end{cases}$$

(70)

It is easily seen that if that is the case

$$\langle \hat{B}_p, \hat{B}_q \rangle = \begin{cases} 1 & \text{if } p = q \\ -\frac{1}{d^2-1} & \text{otherwise} \end{cases}$$

(71)

meaning that the vectors $B_p$ are the vertices of a regular simplex.

We now have the following lemma:

**Lemma 6.** A vector $\hat{B} \in S_o$ is the generating vector for a Weyl-Heisenberg covariant regular simplex if and only if

$$\hat{B} = \frac{1}{\sqrt{d+1}} \sum_{q \in (\mathbb{Z}_2^d)^*} e^{i\theta_q} \hat{D}_q$$

(72)

for any set of real numbers $\theta_q$ satisfying the condition $e^{i\theta_q} = s_{-q} e^{-i\theta_q}$ (where $(\mathbb{Z}_2^d)^*$, $s_{-q}$ and $q$ are as defined in Appendix A).

**Proof.** We know from Eq. (112) that any vector $\hat{B} \in S_o$ can be written

$$\hat{B} = \sum_{q \in (\mathbb{Z}_2^d)^*} c_q \hat{D}_q$$

(73)

where the expansion coefficients $c_q = (1/d) \text{Tr}(\hat{D}_q^\dagger \hat{B})$ satisfy the condition $c_q = s_{-q} c_q^*$. By a straightforward application of Eq. (95) we find

$$\hat{B}_p = \sum_{q \in (\mathbb{Z}_2^d)^*} \tau^2(p,q) c_q \hat{D}_q$$

(74)

In view of Lemma 7 in the Appendix it follows

$$\langle \hat{B}, \hat{B}_p \rangle = \frac{1}{(d-1)} \sum_{q \in (\mathbb{Z}_2^d)^*} |c_q|^2 \tau^2(p,q)$$

(75)

Suppose now that $|c_q| = 1/\sqrt{d+1}$ for all non-zero $q$. Then Eq. (75) implies

$$\langle \hat{B}, \hat{B}_p \rangle = \frac{1}{d^2-1} \left(-1 + \sum_{q \in \mathbb{Z}_2^d} \tau^2(p,q)\right) = \frac{1}{d^2-1} \left(-1 + d^2 \delta_{00}\right)$$

(76)

So $\hat{B}$ is the generating vector for a Weyl-Heisenberg covariant regular simplex.

To prove necessity, suppose that Eq. (76) is satisfied. Using the fact that

$$\sum_{p \in \mathbb{Z}_2^d} \tau^2(p,q-r) = d^2 \delta_{qr}$$

(77)

for all $q, r \in \mathbb{Z}_2^d$ to invert Eq. (75) one finds $|c_q| = 1/\sqrt{d+1}$ for all non-zero $q$. □

This lemma gives us an easy way to construct SI-POVMs. Simply choose an arbitrary set of phases $e^{i\theta_q}$ satisfying the condition $e^{i\theta_q} = s_{-q} e^{-i\theta_q}$ and construct the vector $\hat{B}$ specified by Eq. (72). Let $-1/\kappa$ be the minimum eigenvalue of $\hat{B}$. It follows from Theorem 2 that $1/(d-1) \leq \kappa \leq 1$. Moreover $-1/\kappa$ is also the minimum eigenvalue of $\hat{B}_p$ for all $p$. So it follows from Theorem 2 that the operators

$$\hat{E}_p = \frac{1}{d^2}(1 + \kappa \hat{B}_p)$$

(78)
constitute a POVM. By construction the POVM is SI, Weyl-Heisenberg covariant, and has efficiency parameter $\kappa \geq 1/(d-1)$.

6. Construction of SI(1)-POVMs

Of course, what we would really like to do is to construct a POVM which is, not merely symmetric and informationally complete, but also rank 1. The POVM defined by Eq. (78) will be rank 1, with efficiency parameter $\kappa = 1$, if and only if $\hat{B}$ is a Bloch vector. The question therefore arises: how do we choose the phases in Eq. (72) so as to ensure that that is the case?

We can answer that question by appealing to Theorem 4. The vector $\hat{B}$ in Eq. (72) is on the sphere $S_0$. So Theorem 4 tells us that

$$\text{Tr}(\hat{B}^3) \leq d(d-1)(d-2)$$

(79)

with equality if and only if $\hat{B}$ is a Bloch vector. In terms of the phases on the right hand side of Eq. (72) the condition reads (using Lemma 7 in the Appendix)

$$\sum_{p, q, p \oplus q \in (\mathbb{Z}_2)^*} s_{p+q} e^{i (\theta_p + \theta_q - \theta_{p \oplus q})} \leq (d-1)(d-2)(d+1)^{3/2}$$

(80)

with equality if and only if $\hat{B}$ is a Bloch vector.

This gives us an extremal condition alternative to the one used by Renes et al [1]. Renes et al [1] base their numerical construction of Weyl-Heisenberg covariant SI(1)-POVMs on the fact that, if $\hat{P}$ is an arbitrary rank 1 projector and $\hat{P}_p = \hat{D}_p \hat{P} \hat{D}_p^\dagger$, then

$$\left(\text{Tr}(\hat{P}_p)\right)^2 \geq \frac{2d}{d+1}$$

(81)

with equality if and only if $\frac{1}{d} \hat{P}_p$ constitute an SI(1)-POVM. The inequality we have derived provides us with an alternative procedure: instead of looking for a projector $\hat{P}$ which minimizes the expression on the left hand side of Eq. (81), one can look for a set of phases which maximize the expression on the left hand side of Eq. (80).

It should be said that if one is specifically looking for a method of constructing SI(1)-POVMs numerically a procedure based on Eq. (81) is likely to be more efficient than one based on Eq. (80). This is because the expression on the left hand side of Eq. (81) is a function of $2(d-1)$ real parameters (i.e. the number of parameters needed to specify the projector $\hat{P}$), whereas the one on the left hand side of Eq. (80) is a function of $(d^2 + 1)/2$ real parameters if $d$ is odd and $(d^2 + 4)/2$ real parameters if $d$ is even (i.e. the number of independent phase angles).

However, although the extremal condition represented by Eq. (80) would appear not to have any advantages from a concrete numerical point of view, it may perhaps be interesting from a more abstract mathematical point of view, as providing additional insight into the problem. In particular, the fact that the phase angles appear in combinations of the form $\theta_p + \theta_q + \theta_r$ with $p + q + r = 0 \pmod{d}$ may possibly provide some clues as to the origin of the order 3 symmetry found in every Weyl-Heisenberg covariant SI(1)-POVM constructed to date.

\footnote{Note that Grassl [5] has constructed a counter-example in dimension 12 to conjecture C of ref. [2]. However, his example is still invariant under a canonical order 3 unitary. Specifically his matrix $T_{12}$ is a representative of the Clifford operation $\begin{pmatrix} 4 & 3 \\ 9 & 7 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -6 \end{pmatrix}$, which it will be seen has Clifford trace $= -1$ (notation and terminology as in ref. [7]). His example is therefore consistent with conjecture A of ref. [7].}
7. The Wigner POVM

Suppose that \( d \) is odd. In that case we can set the phase angles on the right hand side of Eq. (72) equal to zero, giving

\[
\hat{B} = \frac{1}{\sqrt{d+1}} \sum_{q \in \mathbb{Z}_d^*} \hat{D}_q
\]

(82)

(notice that if \( d \) was even this choice of phases would not be permissible because when \( d \) is even the signs \( s_{-q} \) are not all positive). For reasons explained below we will refer to the SI-POVM corresponding to this choice of \( \hat{B} \) as the Wigner POVM.

We wish to determine the efficiency parameter of the Wigner POVM. For that purpose it is convenient to consider the operator

\[
\hat{U} = \frac{1}{d} (1 + \sqrt{d+1} \hat{B}) = \frac{1}{d} \sum_{q \in \mathbb{Z}_d^*} \hat{D}_q
\]

(83)

\( \hat{U} \), like \( \hat{B} \), is an Hermitian operator. Moreover

\[
\hat{U}^2 = \frac{1}{d^2} \sum_{q, r \in \mathbb{Z}_d^*} \tau(q,r) \hat{D}_{q+r}
\]

\[
= \frac{1}{d^2} \sum_{q, r \in \mathbb{Z}_d^*} \tau(q,r) \hat{D}_q
\]

\[
= 1
\]

(84)

where we used the fact that \( \sum_{r \in \mathbb{Z}_d^*} \tau(q,r) = d^2 \delta_{q0} \) (note that this depends on the fact that \( d \) is odd). It follows that the eigenvalues of \( \hat{U} \) all \( \pm 1 \). Taking into account the fact that \( \text{Tr}(\hat{U}) = 1 \) we deduce that \( \hat{U} \) must have \( (d+1)/2 \) eigenvalues \( = 1 \) and \( (d-1)/2 \) eigenvalues \( = -1 \). Consequently the smallest eigenvalue of \( \hat{B} \) is \(-\sqrt{d+1}\). In view of Theorem 2 it follows that \( (1/\sqrt{d+1}) \hat{B} \) is a Bloch vector. Hence the \( d^2 \) operators

\[
\hat{E}_p = \frac{1}{d^2} \left( 1 + \frac{1}{\sqrt{d+1}} \hat{B}_p \right)
\]

(85)

constitute an SI-POVM of rank \((d+1)/2\). We will refer to this as the Wigner POVM.

It has efficiency parameter \( 1/\sqrt{d+1} \)—which is a considerable improvement on the worst case value \( 1/(d-1) \) calculated in Section 5, although still greatly inferior to the best case value \( \kappa = 1 \).

Let us now explain the connection between the Wigner POVM and the Wigner function. Let \( \hat{\rho} \) be an arbitrary density matrix, and let

\[
\rho_p = \frac{1}{d} \text{Tr}(\hat{D}_p \hat{\rho})
\]

(86)

We define the Wigner function \( W_p \) to be the discrete Fourier transform of the coefficients \( \rho_p \):

\[
W_p = \frac{1}{d} \sum_{q \in \mathbb{Z}_d^*} \tau^{-2}(p,q) \rho_p
\]

(87)

This definition agrees with that of Wootters \[32\] in the case when \( d \) is prime. If \( d \) is non-prime the Wigner function as defined by this formula loses some of the properties which Wootters considers desirable. However, it appears to us that it

5In the notation of ref. \[2\] \( \hat{U} \) is a representative of the Clifford operation \[ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} \] . Its action on the standard basis used to define the operators \( \hat{D}_p \) (see Eqs. 91 and 92) is \( \hat{U} | r \rangle = | \bar{r} \rangle \). So \( \hat{U} \) can be thought of as a discrete parity operator.
retains sufficiently many of these properties for it still to be considered a reasonable way of defining the Wigner function. For further discussion of the discrete Wigner function see refs. [32, 39, 42, 43, 44, 45] and references cited therein.

The Wigner function can be expressed in terms of the operators $\hat{U}_p = \hat{D}_p \hat{U} \hat{D}_p^\dagger$ (where $\hat{U}$ is the operator defined in Eq. (83)). In fact

$$\hat{U}_p = \frac{1}{d} \sum_{q \in \mathbb{Z}_d^2} \tau^{2(p \cdot q)} \hat{D}_q = \frac{1}{d} \sum_{q \in \mathbb{Z}_d^2} \tau^{-2(p \cdot q)} \hat{D}_q^\dagger$$  \hspace{1cm} (88)$$

Eqs. (88) and (87) then imply

$$W_p = \frac{1}{d} \text{Tr}(\hat{U}_p \hat{\rho})$$ \hspace{1cm} (89)$$

Taking into account Eqs. (83) and (85) we deduce

$$W_p = (d + 1) \text{Tr}(\hat{E}_p \hat{\rho}) - \frac{1}{d}$$ \hspace{1cm} (90)$$

Of course, the fact that the Wigner function is a linear function of the probabilities $\text{Tr}(\hat{E}_p \hat{\rho})$ is an automatic consequence of the fact that the POVM is informationally complete. However, in the case of the Wigner POVM the relationship is particularly simple: to obtain the Wigner function one merely has to rescale the probabilities by a constant amount and then shift them by a constant amount.

Eq. (89) is conceptually interesting because it establishes a connection between SI-POVMs and the Wigner function. At first sight it may appear that it also has a more concrete, pragmatic significance, as providing a good way to determine the Wigner function tomographically. However, a little reflection will dispel that impression. The trouble is that the Wigner POVM has efficiency parameter $= 1/\sqrt{d} + 1$, which is $< 1$ (and $\ll 1$ if $d$ is large). So if one wants to determine the numbers $W_p$ it would be much more efficient (would give much less statistical uncertainty for a given number of measurements) to use a tomographic scheme based on an SI(1)-POVM, or a full set of MUBs (in dimensions where such exist), and then to perform the appropriate linear transformation on the relative frequencies obtained $^\text{[1]}'$.

Finally, let us note that Miquel et al $^\text{[42]}$ have described a scheme for “directly measuring” the individual numbers $W_p$. This scheme might, perhaps, have some advantages over a scheme based on an SI(1)-POVM or a full set of MUBs in a case where one was only interested in some of the numbers $W_p$.

8. Conclusion

We originally undertook the investigations reported here in the hope that they might lead to a solution of the really challenging problem, which is to demonstrate the existence (or, as it may be, the non-existence) of SI(1)-POVMs in every finite dimension. We did not succeed in that primary aim. Nevertheless, we derive some consolation from the fact that the class of SI-POVMs is of some intrinsic interest. Also, it is not impossible that the results reported here contain clues that may help us to solve the main problem.

Appendix A. Weyl-Heisenberg Group

In this appendix we summarise those facts concerning the Weyl-Heisenberg group (or generalized Pauli group as it is sometimes called) which are needed in the main text. Our definitions are those of ref. $^\text{[7]}$, and may differ slightly from the ones used by other authors. Let $|0\rangle, |1\rangle, \ldots, |d-1\rangle$ be an orthonormal basis for $\mathcal{H}$, and let the reason for defining $\tau = -e^{\pi i/d}$ rather than $\tau = e^{\pi i/d}$ is that it means $\tau^{d^2} = 1$ for all $d$.
\( \tau = -e^{\pi i/d} \). Define operators \( \hat{T} \) and \( \hat{S} \) by

\[
\hat{T}|r\rangle = \tau^{2r}|r\rangle \quad \hat{S}|r\rangle = \begin{cases} |r + 1\rangle & r = 0, 1, \ldots, d - 2 \\ |0\rangle & r = d - 1 \end{cases}
\]

(91) (92)

Then define, for each pair of integers \( \mathbf{p} = (p_1, p_2) \in \mathbb{Z}^2 \),

\[
\hat{D}_p = \tau^{p_1 p_2} \hat{S}^{p_1} \hat{T}^{p_2}
\]

(93)

The operators \( \hat{D}_p \) are the displacement operators of the Weyl-Heisenberg group. The reason for including the factor \( \tau \) is that it means that the operators have the following nice properties:

\[
\hat{D}^\dagger_p = \hat{D}_{-p} \quad \hat{D}_p \hat{D}_q = \tau^{\langle p, q \rangle} \hat{D}_{p+q}
\]

(94) (95)

where

\[
\langle p, q \rangle = p_2 q_1 - p_1 q_2
\]

(96)

The fact that \( \langle p, p \rangle = 0 \) means

\[
(\hat{D}_p)^n = \hat{D}_{np}
\]

(97)

for all \( p \in \mathbb{Z}^2 \) and \( n \in \mathbb{Z} \). In particular the operators \( \hat{D}_p \) are unitary:

\[
\hat{D}^\dagger_p \hat{D}_p = 1
\]

(98)

for all \( p \). It is also worth noting that

\[
(\hat{D}_p)^d = 1
\]

(99)

for all \( p \) (this is one of the reasons for setting \( \tau = -e^{\pi i/d} \)). If, instead, one set \( \tau = e^{\pi i/d} \) it would sometimes happen that \( (\hat{D}_p)^d = -1 \).

The presence of the factor \( \tau^{\langle p, q \rangle} \) on the right hand side of Eq. (95) means that the operators \( \hat{D}_p \) do not constitute a group. However, one obtains a group (the Weyl Heisenberg group) if one takes the set of all operators of the form \( e^{i\alpha} \hat{D}_p \), where \( e^{i\alpha} \) is an arbitrary phase (alternatively, one can define the Weyl-Heisenberg group to be the set of all operators of the form \( \tau^n \hat{D}_p \), where \( n \) is an arbitrary integer).

If \( p = q \) (mod \( d \)) then \( \hat{D}_p = \hat{D}_q \) up to a sign. Specifically:

\[
\hat{D}_p = \begin{cases} \hat{D}_q & \text{if } d \text{ is odd} \\ (-1)^{\frac{1}{2}\langle p, q \rangle} \hat{D}_q & \text{if } d \text{ is even} \end{cases}
\]

(100)

(to prove this formula write \( p = q + du \) and then use Eq. (93)). It is therefore often convenient to restrict ourselves to values of \( p \) lying in the set \( \mathbb{Z}^2_d = \{ (p_1, p_2) : p_1, p_2 = 0, 1, 2, \ldots, d - 1 \} \). Given arbitrary \( p \in \mathbb{Z}^2 \) let \( [p] \) be the unique element of \( \mathbb{Z}^2_d \) such that \( [p] = p \) mod \( d \). It is also convenient to define

\[
\mathbf{p} \oplus \mathbf{q} = [\mathbf{p} + \mathbf{q}]
\]

(101)

\[
\mathbf{p} \ominus \mathbf{q} = [\mathbf{p} - \mathbf{q}]
\]

(102)

\[
\mathbf{p} = [-\mathbf{p}]
\]

(103)

and

\[
s_p = \begin{cases} 1 & \text{if } d \text{ is odd} \\ (-1)^{\frac{1}{2}\langle p, [p] \rangle} & \text{if } d \text{ is even} \end{cases}
\]

(104)
We then have, for all \( p, q \in \mathbb{Z}_d^2 \),

\[
\hat{D}_p^\dagger = s_p \hat{D}_p \quad (105)
\]

\[
\hat{D}_p \hat{D}_q = s_{p+q} \tau^{(p,q)} \hat{D}_{p \oplus q} \quad (106)
\]

It is also easily verified that

\[
\text{Tr}(\hat{D}_p^\dagger \hat{D}_q) = d \delta_{pq} \quad (107)
\]

for all \( p, q \in \mathbb{Z}_d^2 \). This means that, relative to the Hilbert-Schmidt inner product, the operators \( \frac{1}{\sqrt{d}} \hat{D}_p \) are an orthonormal basis for the \( d^2 \) complex dimensional space \( \mathcal{L}(\mathcal{H}) \) consisting of all \( d \times d \) complex matrices. So an arbitrary matrix \( \hat{A} \in \mathcal{L}(\mathcal{H}) \) can be expanded

\[
\hat{A} = \sum_{p \in \mathbb{Z}_d^2} A_p \hat{D}_p \quad (108)
\]

where the expansion coefficients are given by

\[
A_p = \frac{1}{d} \text{Tr}(\hat{D}_p^\dagger \hat{A}) \quad (109)
\]

It follows from Eqs. (105), (108) and (109) that \( \hat{A} \) is Hermitian if and only if

\[
A_p = s_p A_p^* \quad (110)
\]

for all \( p \in \mathbb{Z}_d^2 \).

Let \( (\mathbb{Z}_d^2)^* = \{ p \in \mathbb{Z}_d^2 : p \neq (0,0) \} \). The fact that

\[
\text{Tr}(\hat{D}_p) = \begin{cases} d & \text{if } p = 0 \text{ (mod } d) \\ 0 & \text{otherwise} \end{cases} \quad (111)
\]

means that \( \hat{A} \in \text{su}(d) \) if and only if it has an expansion

\[
\hat{A} = \sum_{p \in (\mathbb{Z}_d^2)^*} A_p \hat{D}_p \quad (112)
\]

where the coefficients satisfy Eq. (110).

The following lemma tells us how to calculate the expansion coefficients and traces of double and triple products:

**Lemma 7.** Let \( \hat{A}, \hat{B}, \hat{C} \in \mathcal{L}(\mathcal{H}) \). Then

\[
(\hat{A} \hat{B})_p = \sum_{q \in \mathbb{Z}_d^2} s_{p-q} \tau^{(q,p)} A_q B_p \quad (113)
\]

\[
(\hat{A} \hat{B} \hat{C})_p = \sum_{q, r \in \mathbb{Z}_d^2} s_{p-q-r} \tau^{(q+r,p+q)} A_q B_r C_p \quad (114)
\]

where \( (\hat{A} \hat{B})_p = (1/d) \text{Tr}(\hat{D}_p^\dagger \hat{A} \hat{B}) \) and \( (\hat{A} \hat{B} \hat{C})_p = (1/d) \text{Tr}(\hat{D}_p^\dagger \hat{A} \hat{B} \hat{C}) \) are the expansion coefficients as given by Eq. (109). Traces are given by

\[
\text{Tr}(\hat{A} \hat{B}) = d \sum_{q \in \mathbb{Z}_d^2} s_{-q} A_q B_q \quad (115)
\]

\[
\text{Tr}(\hat{A} \hat{B} \hat{C}) = d \sum_{q, r \in \mathbb{Z}_d^2} s_{-q-r} \tau^{(q,r)} A_q B_r C_q \quad (116)
\]
If $\hat{A}, \hat{B}, \hat{C}$ are Hermitian we can alternatively write

$$\text{Tr}(\hat{A}\hat{B}) = d \sum_{q \in \mathbb{Z}_d^2} A_q B_q^*$$

and

$$\text{Tr}(\hat{A}\hat{B}\hat{C}) = d \sum_{q, r \in \mathbb{Z}_d^2} s_{q+r} \tau(q, r) A_q B_r C_{q+r}^*$$

(117

(118)

Proof. Let $\hat{A}, \hat{B} \in \mathcal{L}(\mathcal{H})$. It follows from Eq. (108) that

$$\hat{A}\hat{B} = \sum_{q, r \in \mathbb{Z}_d^2} A_q B_r \hat{D}_q \hat{D}_r$$

(119)

where in the last line we used the fact that $s_{q+r} \tau(q, r) = s_{q+r} \tau(q, r)$. Eq. (113) is now immediate.

To prove Eq. (114) we apply Eq. (113) twice:

$$(\hat{A}\hat{B}\hat{C})_p = \sum_{q \in \mathbb{Z}_d^2} s_{p-q} \tau(q, p) A_q (\hat{B}\hat{C})_p$$

(120)

where in the last line we used the identity $s_{p-q} s_{r} \tau(r, p) = s_{p-r} \tau(r, p)\tau(q, r)$.

To prove Eqs. (115) and (116) set $p = 0$ in Eqs. (113) and (114) and use the fact that $\text{Tr}(M) = dM_0$ for all $M \in \mathcal{L}(\mathcal{H})$. Eq. (117) follows from Eq. (115). Eq. (118) follows from Eq. (116) and the identity $s_{p+q} s_{-p} = s_{-p} s_{p}$.

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