Algebra of local symmetric operators and braided fusion \( n \)-category
– symmetry is a shadow of topological order

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Symmetry is usually defined via transformations described by a (higher) group. But a symmetry really corresponds to an algebra of local symmetric operators, which directly constrains the properties of the system. In particular, isomorphic operator algebras correspond to equivalent symmetries. In this paper, we pointed out that the algebra of local symmetry operators actually contains extended string-like, membrane-like, etc operators. The algebra of those extended operators in \( n \)-dimensional space gives rise to a non-degenerate braided fusion \( n \)-category, which happens to describe a topological order in one higher dimension (for finite symmetry). This allows us to show that the equivalent classes of finite symmetries actually correspond to topological orders in one higher dimension. Such a holographic theory not only describes (higher) symmetries, it also describes anomalous (higher) symmetries, non-invertible (higher) symmetries (also known as algebraic higher symmetries), and (invertible and non-invertible) gravitational anomalies, in a unified and systematic way. We demonstrate this unified holographic framework via some simple examples of (higher and/or anomalous) symmetries, as well as a symmetry that is neither anomalous nor anomaly-free. We also show the equivalence between \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry with mixed anomaly and \( \mathbb{Z}_4 \) symmetry, as well as between many other symmetries, in 1-dimensional space.

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A comprehensive theory was developed along this line. A holographic point view of symmetry. To place the above holographic point view on a firmer foundation, we note that even though we use transformations described by groups or higher groups to define symmetries, in fact, a symmetry is not about transformations. What a symmetry really does is to select a set of local symmetric operators which form an algebra. The algebra of all local symmetric operators determines the possible quantum phases and phase transitions, as well as all other properties allowed by the symmetry. Thus, isomorphic algebras give rise to the same physical properties, and should be regarded as equivalent symmetries. Those isomorphic classes of algebra were referred to as categorical symmetries in Ref. 47 and 48, which, by definition, describe all known and unknown types of symmetries. However, the term “categorical symmetry”, later, was also used to refer to algebraic higher symmetry (i.e. non-invertible symmetry) by some people. So here, we use categorical symmetry to stress that the term is used in the sense of Ref. 47 and 48.

In this paper, we show that an algebra of local symmetric operators not only involve point-like local symmetric operators, it must also involve extended operators, such as string-like operators, membrane-like operators, etc. We find that, for a finite symmetry in n-dimensional space, such an algebra of extended symmetric operators determines a braided fusion n-category. If the algebra include all local symmetric operators, the braided fusion n-category will be non-degenerate. Furthermore, isomorphic algebras give rise to the same non-degenerate braided fusion n-category. Thus categorical symmetries are described by non-degenerate braided fusion n-categories, which happen to correspond to topological orders in one higher dimension. In other words, we suggest that group is not a proper description of symmetry, since (higher) symmetries and anomalous (higher) symmetries described by different (higher) groups can be equivalent. Finite symmetries are really described by non-degenerate braided fusion n-categories (i.e. topological orders in one higher dimension).

The calculation in this paper is based on operator algebra. A similar picture was obtained in Ref. 50 based on ground state and their excitations. The operator algebra discussed in this paper may be related to the nets of local observable algebras in Ref. 51 and topological net of extended defects in Ref. 52.

The holographic theory of symmetry allows us to identify equivalent (higher and/or anomalous) symmetries, that can look quite different. For example, two (higher and/or anomalous) symmetries can be realized at boundaries of two symmetry protected topological (SPT) states with those symmetries in one higher dimension. If after gauging the respected symmetries in the SPT states, we obtain the same topological order, then the two corresponding symmetries have the the same categorical symmetry and are equivalent. This is a systematic way to identify equivalent symmetries and their categorical symmetry.

In Ref. 48 it was conjectured that if two anomaly-free (invertible or non-invertible) symmetries described by local fusion higher categories, $\mathcal{R}$ and $\mathcal{R}'$, are equivalent (i.e. have the equivalent monoidal center $\mathcal{Z}(\mathcal{R}) \simeq \mathcal{Z}(\mathcal{R}')$), then the two symmetries provide the same constraint on the physical properties. In other words, for any pair of equivalent symmetries, there is a lattice duality map, that maps a lattice model with one symmetry $\mathcal{R}$ to a lattice model with another symmetry $\mathcal{R}'$. More specifically, the sets of local symmetric operators selected by the two symmetries, $\{O_{\mathcal{R}}\}$ and $\{O_{\mathcal{R}'}\}$, have an one-to-one correspondence and generate the same algebra, under such a duality map. The duality map also maps the lattice Hamiltonians (as sums of local symmetric operators) of the two lattice models into each other. The two lattice models have identical dynamical properties, such as have identical energy spectrum in symmetric sub Hilbert space. This can be viewed as the physical meaning of
“equivalent symmetry”.

This conjecture is motivated and supported by the studies of some explicit examples of well known and new dualities. Ref. 47 used Kramers–Wannier duality and its generalization to study the equivalence and its holographic understanding of 1d $\text{Rep}_{\mathbb{Z}_2}$-symmetry (the $\mathbb{Z}_2$ 0-symmetry) and $\text{Vec}_{\mathbb{Z}_2}$-symmetry (the dual $\mathbb{Z}_2$ 0-symmetry), as well as 2d $2\text{Rep}_{\mathbb{Z}_2}$-symmetry (the $\mathbb{Z}_2$ 0-symmetry) and $2\text{Vec}_{\mathbb{Z}_2}$-symmetry (the $\mathbb{Z}_2^{(1)}$ 1-symmetry). Ref. 48 used a lattice duality map to study the equivalence and its holographic understanding of 1d $n\text{Rep}_{G}$-symmetry (the 0-symmetry described by a finite group $G$) and $n\text{Vec}_{G}$-symmetry (the dual non-invertible $(n-1)$-symmetry). Ref. 50 studied the duality maps and holographic equivalence of 1d $\text{Rep}_{\mathbb{Z}_2}$-symmetry, $\text{Vec}_{\mathbb{Z}_2}$-symmetry, and $s\text{Rep}_{\mathbb{Z}_2}$-symmetry (the 1d $\mathbb{Z}_2^f$ fermionic symmetry). In the above examples, the duality map can be viewed as gauging process. In Ref. 53, a more general duality map between 1d lattice systems is discussed via fusion category and tensor network operators.

In this paper, we studied a duality between anomaly-free symmetry and anomalous symmetry. We obtain a new duality map between two equivalent symmetries: 1d $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with the mixed anomaly and anomaly-free $\mathbb{Z}_4$ symmetry.

Viewing symmetry as topological order in one higher dimension generalizes the fundamental concept of symmetry. It allows us to discover new type of non-invertible symmetries (also called algebraic (higher) symmetries)[48, 54–57] that are beyond group and higher group, as well as new type of symmetries that are neither anomalous nor anomaly-free. But why do we want a more general notion of symmetry?

We know that symmetry can emerge at low energies. So we hope our notion of symmetry can include all the possible emergent symmetry. It turns out that the low energy emergent symmetries can be the usual higher and/or anomalous symmetries. They can also be non-invertible symmetries. They can even be symmetries that are neither anomalous nor anomaly-free. Therefore, we need a most general and unified view of higher and/or anomalous symmetries and beyond, if we want to use emergent symmetry as a guide to systematically understand or even classify gapless states of matter.

For example, using this generalized notion of symmetry, we gain a deeper understanding of quantum critical points. We find that the symmetry breaking quantum critical point for a symmetry described by a finite group $G$ in $n$-dimensional space is the same as the symmetry breaking quantum critical point for an algebraic higher symmetry described by fusion $n$-category $n\text{Vec}_{G}$.[47, 48] In fact, both the ordinary symmetry described by group $G$ and the algebraic higher symmetry (a non-invertible symmetry) described by fusion $n$-category $n\text{Vec}_{G}$ are present and are not spontaneously broken at this critical point. The $G$-symmetry and the algebraic higher symmetry $n\text{Vec}_{G}$ may give us a more comprehensive understanding of the symmetry breaking quantum critical point.

Symmetry can constrain the properties of a physical system. On the other hand, when certain excitations in a system have a large energy gap, below that energy gap, the system can have emergent symmetry, which can be anomalous and/or non-invertible.[47, 48, 58] In this case, we can use the emergent symmetry to reflect and to characterize the special low energy properties of the system below the gap. Here we make a preparation to go one step further. We prepare to propose and explore the possibility that the low energy property is the emergent symmetry and the emergent symmetry is the low energy property. In other words, we propose that the full emergent symmetry may fully characterize the low energy effective theory. We may be able to study and to classify all possible low energy effective theories by studying and classifying all possible emergent symmetries.

Such an idea cannot be correct if the above symmetries are still considered as being described by groups and higher groups. This is because the symmetries described by groups and higher groups are quite limited, and they cannot capture the much richer varieties of possible low energy effective theories. However, after we greatly generalize the notion of symmetry to algebraic higher symmetry, and even further to topological order in one higher dimensions – which includes (anomalous and/or higher) symmetries, (invertible and non-invertible) gravitational anomalies, and beyond – then it may be possible that those generalized symmetries can largely capture the low energy properties of quantum many-body systems. This may be a promising new direction to study low energy properties of quantum many-body systems.

The above proposal is supported by the recent study of 1d gapless conformal field theory where a topological skeleton was identified for each conformal field theory.[59–61] Such a topological skeleton is a non-degenerate braided fusion category corresponding to a 2d topological order, where the involved conformal field theory is one of the gapless boundary.

The low energy properties of quantum many-body systems are described by quantum field theories. A systematic understanding and classification of low energy properties is equivalent to a systematic understanding and classification of quantum field theories. Thus the holographic view of symmetry can have an impact on our general understanding of quantum field theories. Using this holographic point of view of symmetry, one can also obtain a classification of topological order and symmetry protected topological orders, with those generalized symmetry, for bosonic and fermionic systems, and in any dimensions.[48, 49]

II. NOTATIONS

In this paper, we will use $n+1D$ to represent spacetime dimensions, and $nd$ to represent spatial dimensions. We will use mathematical font $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ to describe fusion categories,
III. BOSONIC QUANTUM SYSTEM AND ITS ALGEBRA OF LOCAL OPERATORS

A. Total Hilbert space, local operators algebra, and local Hamiltonian

A lattice bosonic quantum system is defined by four components:

1. A triangulation of space (see Fig. 1).
2. A total Hilbert space
   \[ \mathcal{V} = \bigotimes_i \mathcal{V}_i, \]
   where \( \mathcal{V}_i = \text{span}\{ |g\rangle \mid g \in G \} \) is the local Hilbert space on vertex-\( i \). The basis vectors of \( \mathcal{V}_i \) are labeled by the elements in a finite set \( G \).
3. An algebra of local operators \( \mathcal{A} \) formed by all the local operators, \( \mathcal{A} = \{ O_i \} \). Here local operator is defined as an operator \( O_i \) that acts within the tensor product of a few nearby local Hilbert spaces, say near a vertex-\( i \).
4. A local Hamiltonian \( H = -\sum_i O_i \) which is a sum of hermitian local operators.

B. Transparent patch operators

The algebra of local operators will play a central role in this paper. One may naively expect the algebra of local operators is formed by local operators. However, beyond 0-dimensional space, the products of local operators can generate extended operators that can be string-like, membrane-like, etc. Thus the closure of the algebra of local operators must contain those extended operators. An algebra of local operators may have many different categories. For each vertex-\( i \), there is a patch operator \( O_{\text{patch}, i} \) which is a sum of all operator patterns \( \Phi(\{ a_i \}) \prod_{i \in \text{patch}} O_i^{a_i} \) where \( \Phi \) is a function that gives rise to different operator patterns. This is just one representation of tensor network operator.

Definitions:

Definition 1. A patch operator is a tensor network operator (see Fig. 2). It also has the following form

\[ O_{\text{patch}} = \sum_{\{ a_i \}} \Phi(\{ a_i \}) \prod_{i \in \text{patch}} O_i^{a_i} \]

where \( \Phi \) has a topology of \( n \)-dimensional disk, \( n = 0, 1, 2, \ldots \). A transparent patch (t-patch) operator is a patch operator that satisfies the following transparent condition (or invisible-bulk condition):

\[ O_{\text{patch}} O_{\text{patch}'} = O_{\text{patch}'} O_{\text{patch}}, \]

if the boundaries of two patches, \( \partial \text{patch} \) and \( \partial \text{patch}' \), are not linked and are far away from each other.

In the above definition, \( O_i^{a_i} \)'s are local operators acting near vertex-\( i \). For each vertex, there can be several different local operators (including the trivial identity operator) which are labeled by \( a_i \). \( \prod_{i \in \text{patch}} O_i^{a_i} \) is a product of those local operators over all the vertices \( i \) in the patch. Different choices of \( \{ a_i \} \) give rise to different operator patterns. \( \sum_{\{ a_i \}} \Phi(\{ a_i \}) \) is the sum of all operator patterns with complex weight \( \Phi(\{ a_i \}) \). One may think \( O_i^{a_i} \) 's create different types of particles labeled by \( a_i \) at vertex-\( i \). Then \( O_{\text{patch}} = \sum_{\{ a_i \}} \Phi(\{ a_i \}) \prod_{i \in \text{patch}} O_i^{a_i} \) creates a quantum liquid state of those particles on the patch. The quantum liquid state is described by the many-body wave function \( \Phi(\{ a_i \}) \).

In the above definition, we also used a notion of far way which is not rigorously defined. To define such a notion, we first introduce a notion of small local operators as operators acting on vertices whose separations are less than a number \( L_{\text{op}} \). (The separations between two vertices is defined as the minimal number of links connecting the two vertices.) In the rest of this paper, the terms “local operator” and “0-dimensional patch operator” will refer to this kind of small local operators.

![FIG. 1. A 2d lattice bosonic model, whose degrees of freedom live on the vertices and are labeled by the elements in a set: \( g_i \in G \).](image1)

![FIG. 2. The matrix elements of a string-like tensor network operator, \( O_{m_1,m_2,\ldots,n_1,n_2,\ldots} \), can be given by a contraction of rank-4 tensors \( T_{n_2,m_2,a,b} \), etc. Each tensor is represented by a vertex, where the legs of the vertex correspond to the indices of the tensor. The connected legs have the same index and is summed over (which correspond to the tensor contraction). This is just one representation of tensor network operator.](image2)
However, the algebra of small local operators contains big local operators, acting on vertices whose separations are larger than the number $L_{\text{op}}$. “$n$-dimensional patch operator” for $n > 0$ refer to those big local operators. The notion of far away means further than the distance $L_{\text{op}}$. When we take the large system size limit: $L_{\text{sys}} \to \infty$, we also assume $L_{\text{op}} \to \infty$ and $L_{\text{op}}/L_{\text{sys}} \to 0$. We will see in this paper that it is this particular way to take the large system size limit that ensures the algebra of small local operators to contain large extended operators. Such an algebra of small local operators and large extended operators in $n$-dimensional space have a structure of non-degenerate braided fusion $n$-category. This emergent phenomenon is the key point of this paper.

There is another important motivation to introduce transparent patch operators. The bulk of transparent patch operators is invisible. Thus a transparent open string operator can be viewed as two point-like particles, one for each string end. A transparent disk operator can be viewed as a closed string at the boundary of the disk. In general, a transparent patch operator gives rise to an extended excitation in one lower dimension, corresponding to the boundary of the patch. Later we will see that those point-like, string-like, etc excitations can fuse and braid, forming a braided fusion category that describe the operator algebra.

C. Patch symmetry and patch charge operators

Symmetry transformation operators and symmetry-charge creation operators play important roles in our theory about symmetry (including higher symmetry and algebraic higher symmetry). Those operators also appear in our setup of local operator algebra after we include the extended operators.

Definition 2. Roughly speaking, a t-patch operator is said to have an empty bulk if $O_{\text{symm}}^i = \text{id}_i$ for all $i$‘s far away from the boundary of the patch. A t-patch operator with an empty bulk is also referred to as a patch charge operator. A t-patch operator with non-empty bulk is referred to as a patch symmetry operator (see Section IVB for a more precise definition).

We like to remark that due to the transparency condition eqn. (3), a charge patch operator always commutes with symmetry transformation operator (acting on the whole space for 0-symmetry, or closed sub-manifold for higher symmetries). Thus the patch charge operator always carry zero total charge. So the patch charge operators are not charged operators, since charged operators do not commute with symmetry transformations. The patch charge operators defined above are something like operators that create a pair of charge and anti-charge, which correspond to a charge fluctuations with vanishing total net charge.

We want to point out that the definition 2 is not that important physically, since the notations of charge and symmetry transformation are not the notions of algebra of local operators. They are the notions of a representation of an operator algebra. For different representations of the same operator algebra, the same operator in the algebra can some times be patch charge operator and other times be patch symmetry operator.

In next section, we will discuss a concrete simple example: a bosonic system in 1-dimensional space with $Z_2$ symmetry, to illustrate the above abstract definition. We will give the explicit form of t-patch operators, to show how they reveal a braided fusion category in the algebra of local operators. In Appendix A, we will discuss an example of bosonic system in 3-dimensional space without symmetry. We will illustrate how they give rise to a non-degenerate braided fusion 3-category 3Vec.

IV. A 1d BOSONIC QUANTUM SYSTEM WITH $Z_2$ SYMMETRY

In this section, we consider the simplest symmetry – $Z_2$ symmetry in one spatial dimension. A bosonic system with $Z_2$ symmetry is obtained by modifying the algebra of the local operators. For convenience, let we assume the degrees of freedom live on vertices, which are labeled by elements in the $Z_2$ group +1 and −1.

A. $Z_2$ symmetry and its algebra of local symmetric operators

In the standard approach, a symmetry is described by a symmetry transformation, which has the following form for our example:

$$ W = \bigotimes_{i \in \text{whole space}} X_i, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4) $$

Since $W^2 = 1$ which generates a $Z_2$ group, we call the symmetry a $Z_2$ symmetry. We can use the $Z_2$ transformation $W$ to define an algebra of local operators:

$$ \mathcal{A} = \{ O_{\text{symm}}^i W O_{\text{symm}}^i \} \quad (5) $$

The local operator $O_{\text{symm}}^i$, satisfying $O_{\text{symm}}^i W = WO_{\text{symm}}^i$, is called local symmetric operator.

The symmetric Hamiltonian is a sum of local symmetric operator $H = \sum_i O_{\text{symm}}^i$. If our measurement equipments do not break the symmetry, then the measurement results are correlations of local symmetric operators. We see that a symmetry is actually described (or defined) by the algebra of local symmetric operators, rather than by the symmetry transformations. In particular

Isomorphic operator algebra $\leftrightarrow$ Equivalent symmetry.

$$ (6) $$

In this paper, we will view symmetry from this operator algebra point of view:
Definition 3. A categorical symmetry is an equivalence class of isomorphic local operator algebra.

We remark that categorical symmetry is different from the usual symmetry defined via the symmetry transformations. Two symmetries defined by different symmetry transformations may have isomorphic algebra of local symmetric operators. In that case, the two symmetries correspond to the same categorical symmetry, and are said to be equivalent.

To see the connections between operator algebra and braided fusion category, we use the t-patch operators introduced in last section to organize the local symmetric operators:

1. 0-dimensional t-patch operators: \( X_i, \ Z_iZ_{i+1}, \ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

2. 1-dimensional t-patch operators – string operators: for \( i < j \)

\[
Z_{str,ij} = Z_iZ_j, \quad Z_{str,ji} = Z_{str,ij}^\dagger,
\]

where the string_{ij} connects the vertex-\( i \) and vertex-\( j \). The above string operator has an empty bulk and is called as patch charge operator. We have another string operator: for \( i < j \)

\[
X_{str,ij} = X_{i+1}X_{i+2}\cdots X_{j}, \quad X_{str,ji} \equiv X_{str,ij}^\dagger.
\]

Note that the boundaries of X-strings actually live on the links \((i,i+1)\) and \((j,j+1)\). We labeled those links by \(i,j\). This leads to the special choice of the boundary of the string operator. The second string operator has a non-trivial bulk, which generates our \( Z_2 \) symmetry.

We remark that, in general, the operators in the string may not commute and the order of the operator product will be important in that case. Here we adopted a convention that in string operator \( O_{str,ij} \), the operators near \( i \) appear on the left side of the operators near \( j \).

In terms of t-patch operators, algebra of local symmetric operators takes the following form (only important operator relations are listed)

\[
\begin{align*}
Z_{str,ij}Z_{str,kj} &= Z_{str,ik}, \\
X_{str,ij}X_{str,kj} &= X_{str,ik} \\
Z_{str,ij}X_{str,kj} &= -X_{str,ik}Z_{str,ij}, \\
Z_{str,ij}X_{str,kj} &= +X_{str,ik}Z_{str,ij},
\end{align*}
\]

Eqn. (9) and (10) describe the fusion of string operators (see Fig. 3a). The commutator between the two kinds of string operators depends on their relative positions. If one string straddles the boundary of the other string, such as \( i < k < j < l \) as in Fig. 3b, commutator has a non-trivial phase. Otherwise (see Fig. 3c), the string operators commute, which ensure the string operators to be t-patch operators. In Section IV F, we will discuss the full algebra of t-patch operators in more detail.

B. Patch symmetry transformation

We note that \( Z_i \) operator transforms as the non-trivial representation of \( Z_2 \) group:

\[
WZ_iW^{-1} = -Z_i.
\]

Thus we say \( Z_i \) carries a non-trivial representation, or more commonly, a non-trivial \( Z_2 \) charge. The string operator \( Z_{str} \) is formed by two \( Z_2 \) charges and carry a trivial total \( Z_2 \) charge. In fact, by definition, all local symmetric operators carry trivial \( Z_2 \) charge (see later discussion).

We have stressed that a symmetry is fully characterized by its algebra of local symmetric operators. But all those local symmetric operators carry no symmetry charge. It appears that a key component of symmetry, the symmetry charge (i.e., the symmetry representation) is missing in our description.

In fact, the symmetry representation can be recovered. As pointed out in Ref. 47, there is a better way to describe symmetry transformations using t-patch operators. We notice that the only use of the symmetry transformations is to select local symmetric operators. After that we no longer need the symmetry transformations. Since local symmetric operators are local, we do not need the symmetry transformations that act on the whole space. We only need symmetry transformations that act on patches to select local symmetric operators. This motivates us to introduce patch symmetry transformation

\[
W_{\text{patch}} = \bigotimes_{i \in \text{patch}} X_i.
\]

We can use the patch symmetry transformation \( W_{\text{patch}} \) to define the local symmetric operators:

\[
\mathcal{A} = \{ O_i^{\text{symm}} \mid O_i^{\text{symm}}W_{\text{patch}} = W_{\text{patch}}O_i^{\text{symm}} \}.
\]

So a symmetry can also be defined via the patch symmetry transformations.

For the \( Z_2 \) symmetry in 1-dimensional space, the patch symmetry transformations happen to be generated by one of the string operators with non-empty bulk, \( X_{str} \), and this is why we call them patch symmetry operators. In this example, we also see that the string operator \( Z_{str,ij} \) with empty bulk corresponds to a charge-anti-charge pair
operator. This is the why we call t-patch operators with empty bulk as patch charge operators.

The patch symmetry transformations have an advantage that they can detect the symmetry charge hidden in the patch charge operators (which have zero total charge): when the patch charge operator \( Z_{\text{str}} \) straddle the boundary of the patch symmetry operator, the two operators have a non-trivial commutation relation:

\[
Z_{\text{str}} W_{\text{patch}} = -W_{\text{patch}} Z_{\text{str}}. \tag{16}
\]

This non-trivial commutation relation measures the charge carried by one end of the string operator.

If we view the order of the operator product as the order in time, and view the string as world line of a particle in spacetime (see Fig. 4), then the commutation relation eqn. (16) can be viewed as a braiding of the charged particle around the boundary of the patch symmetry operator. The boundary of the patch symmetry operator can be viewed as a “symmetry twist flux”. The charge is measured by a braiding of symmetry charge around symmetry twist flux. This is why we refer to eqn. (11) and eqn. (12) as “braiding” relations in Fig. 3.

C. The algebra of patch charge operators and its braided fusion category

Let us concentrate on patch charge operators. The properties of the charges of a symmetry can be systematically and fully described by a braided fusion category. To connect the \( \mathbb{Z}_2 \) symmetry charges to fusion category, we view the local symmetric operators \( O_{\text{symm}} \) as the morphisms, and the ends of string operator \( Z_{\text{str}}(i,j) \) (i.e. the point-like \( \mathbb{Z}_2 \)-charge) as objects \( e_i \) and \( \bar{e}_j \) in a fusion category. In other words, we write the string operator as

\[
Z_{\text{str}} = T_e(i \rightarrow j). \tag{17}
\]

The notation \( T_e(i \rightarrow j) \) is more precise and carries several meanings. (1) We view \( T_e(i \rightarrow j) \) as a world-line of a particle labeled by \( e \) that travels from \( i \) to \( j \). \( T_e(i \rightarrow j) \) can also be viewed as a hopping operator of \( e \) from \( i \) to \( j \). Here, we have adopted a convention that the arrow indicate the direction of the hopping. (2) The notation of string operator \( T_e(i \rightarrow j) \) also specify the ordering of operators: the operators near left index \( i \) appears to the left of the operators near the right index \( j \).

Since the local symmetric operators \( Z_{\text{str}}(i,j) \) (the morphisms) can move the string ends (the \( \mathbb{Z}_2 \)-charges):

\[
e_i \otimes^{O_{\text{symm}}} e_i', \quad e_i' \otimes^{O_{\text{symm}}} e_i, \tag{18}
\]

the \( \mathbb{Z}_2 \)-charges (at the string ends) at different places are isomorphic \( e_i \cong e_i' \), i.e. they belong to the same type of excitations. More generally, two excitations that can be connected by local symmetric operators are regarded as the same type of excitations.

We note that \( T_e(i \rightarrow j) \) describes the hopping of \( e \) particle from \( i \) to \( j \). What operator describes the hopping of the anti-particle \( \bar{e} \)? Here we define the hopping operator of the anti-particle \( \bar{e} \) as

\[
T_{\bar{e}}(i \leftrightarrow j) \equiv T_e(i \rightarrow j). \tag{19}
\]

This is a general way to define hopping operators of anti-particle. For our case, we find that \( T_{\bar{e}}(j \leftarrow i) = T_e(i \rightarrow j) \), and thus \( e = \bar{e} \).

From the above expression of t-patch operators, we can compute the fusion ring

\[
a \otimes b = \bigoplus_c N_{ab}^{bc} c. \tag{20}\]

of the braided fusion category. Notice that \( T_e(-\infty \rightarrow i) \) creates an \( e \) particle at \( i \) (and creates another particle at \( -\infty \) which we ignore). Creating two \( e \) particles, we obtain

\[
T_e(-\infty \rightarrow i) T_e(-\infty \rightarrow i) = \text{id}. \tag{21}\]

In other words, we get a trivial particle \( 1 \). This allows us to obtain the fusion rule

\[
e_i \otimes e_i = 1. \tag{22}\]

The isomorphic relation is an equivalence relation. After quotienting out the equivalence relation, \( e_i \cong e_j \), we find that the fusion category has only two objects: \( 1 \), i.e. the point-like \( \mathbb{Z}_2 \)-charge. The morphism of the fusion category is given by local symmetric operators \( O_{\text{symm}} \) near a vertex-\( i \).

However, the fusion rule \( N_{ab}^{bc} \) fails to completely determines the fusion category, i.e. some times, two different fusion categories can have the same fusion rule. To complete the description of the fusion category, we also need to compute the F-symbol, which is defined as the relative phases of different ways to fuse three particles \( a, b, c \) together, \( a \otimes b \otimes c \rightarrow (ab) \otimes c \rightarrow (ab)c \) and \( a \otimes b \otimes c \rightarrow a \otimes (bc) \rightarrow a(bc) \) (see Fig. 5), if we treat the result of fusion, as quantum state or as operator:

\[
O((ab)c) = F(a, b, c)(a(bc)).
\]

Following Ref. 62, the F-symbol is computed from the relative phase of the two way to compute operator prod-
FIG. 5. Two ways to fuse three particles \(a, b, c\) into \(abc\), as operator product. The phase difference of the two resulting operators is \(F(a, b, c)\). The horizontal lines and the corresponding \(45^\circ\) lines correspond to hopping operators. For example \(1 \xrightarrow{b} 2 \sim T_b(1 \rightarrow 2)\). The hopping operators with higher location are applied first. Thus we have a relation \(T_b(1 \rightarrow 2)T_a(0 \rightarrow 1)T_b(2 \rightarrow 1)T_c(3 \rightarrow 1) = F(a, b, c)T_c(3 \rightarrow 1)T_a(0 \rightarrow 1)T_b(1 \rightarrow 2)T_a(0 \rightarrow 1)T_b(2 \rightarrow 1)\).

FIG. 6. The two ways of \(a, b\) particle hopping give rise to two configurations which exchange their positions. When \(a = b\), the phase difference of the two resulting operators is \(\theta_a\), which is the self statistics of the \(a\)-particle. Thus we have a relation \(T_b(3 \rightarrow 1)T_a(1 \rightarrow 2)T_a(0 \rightarrow 1) = e^{i\theta_a}T_a(0 \rightarrow 1)T_b(1 \rightarrow 2)T_a(3 \rightarrow 1)\).

FIG. 7. The two ways of \(a, b\) particles hopping give rise to the same final configuration but via different braiding paths. The phase difference of two hopping processes is \(e^{i\theta_{ab}}\), which is the mutual statistics of the \(a\)- and \(b\)-particles. Thus we have a relation \(T_b(3 \rightarrow 1)T_a(0 \rightarrow 2) = e^{i\theta_{ab}}T_a(0 \rightarrow 2)T_b(3 \rightarrow 1)\).

self statistics of \(e\) particle using the statistical hopping algebra prescription introduced in Ref. 63 and depicted in Fig 6.

\[ T_e(0 \rightarrow 1)T_e(1 \rightarrow 2)T_e(3 \rightarrow 1) = e^{i\theta_e}T_e(3 \rightarrow 1)T_e(1 \rightarrow 2)T_e(0 \rightarrow 1) \] (25)

from which we can read off the self-statistical angle \(e^{i\theta_e} = 1\). This shows that \(e\) particles have bosonic self-statistics. We can also use Fig 7 to compute mutual statistics of \(1, e\) particles. We find that \(1, e\) particles are bosons with trivial mutual statistics. This implies that the category formed by \(1, e\) is a braided fusion category \(\text{Rep}_Z^2\). It is actually a special braided fusion category called symmetric fusion category, since all the mutual statistics are trivial.

According Tannaka duality, the symmetric fusion category \(\text{Rep}_Z^2\) can fully describe the symmetry group \(G = Z_2\). So, instead of using a group \(G\) (formed by the transformations), we can also use a symmetric fusion category of patch charge operators (i.e. formed by charged objects or the representations) to fully describe a symmetry.

D. Representation category and symmetry

Above picture also works for generic finite group \(G\): a symmetry \(G\) can also be described by a symmetric fusion category \(\text{Rep}_G\) (formed by the representations of \(G\)). This is the categorical point of view of symmetry, which was used in Ref. 49 and 64 and will be used in this paper. The symmetric fusion category generated by patch charge operators is nothing but the mathematical framework that describes the properties of symmetry charges (such as their fusion and braiding).

Definition 4. We will call the symmetric fusion category \(\mathcal{R}\) formed by patch charge operators as representation category [31]
In fact, there is another definition of representation category. We may ignore the braiding structure and consider the fusion category $\mathcal{R}$ formed by patch charge operators. Instead of the braiding structure, we consider a symmetry-breaking structure, i.e. a faithful functor $\beta : \mathcal{R} \to \text{Vec}$ (which is also called fiber functor) that describe the process of ignoring the symmetry:

**Definition 5.** If a fusion category $\mathcal{R}$ has a fiber functor $\beta$, then the pair $(\mathcal{R}, \beta)$ will be called a local fusion category. Such a local fusion category can also be viewed as the representation category of the symmetry. \cite{48}.

In higher dimension, the notion of symmetric fusion higher category used in Ref. 49 may be hard to define. In this case, the second Definition 5 is can be used as in Ref. 48.

Thus we can say that an anomaly-free symmetry in 1-dimensional space described by a finite group $G$ is fully described by its representation category, a symmetric fusion category $\text{Rep}_G$ or a local fusion category $(\text{Rep}_G, \beta)$. This point of view can be generalized to described anomaly-free symmetries beyond group and higher group. In Ref. 48, it is proposed that the most general anomaly-free symmetries in $n$-dimensional space are fully described by local fusion $n$-categories $(\mathcal{R}, \beta)$. Such a description includes non-invertible symmetries (i.e. algebraic higher symmetries).

In the above we have used a notion of anomaly-free symmetry. For symmetry described by group and/or higher group, an anomaly-free symmetry is defined as a symmetry that can be gauge. But such a definition does not apply to non-invertible symmetries, for which we do not how to gauge them. To solve this problem, Ref. 48 proposed the following macroscopic definition without using gauging

**Definition 6.** Anomaly-free symmetry is the symmetry that allows non-degenerate symmetric gapped states for any closed space manifolds.

A microscopic definition was also proposed

**Definition 7.** Anomaly-free symmetry is the symmetry that allows symmetric state of form $\vert \Psi \rangle = \bigotimes_i \vert \psi_i \rangle$, where $\vert \psi_i \rangle$ is a symmetric state on site-$i$.

We like to remark that representation categories (i.e. symmetric fusion $n$-categories or local fusion $n$-categories) only fully describe anomaly-free symmetries, but fail to fully describe anomalous symmetries. This is because different anomalous symmetries can have the same representation category. In fact, an anomalous symmetry $G$ can be described by symmetry transformations $W_g, g \in G$: $W_g W_h = W_{gh}$ that may not be on-site. The non-invariant local operators that form representations of of the symmetry group $G$. Thus

**Proposition 1.** all the different anomalous symmetries of the same group $G$ have the same representation category $\text{Rep}_G$.

Later in Section VII, we will give a 1d example of emergent $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, whose representation category formed by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ charge is not a local fusion category. This implies that the emergent $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is not an anomaly-free symmetry, since the representation categories of all anomaly-free symmetries are local fusion categories. This also implies that the emergent $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is not an anomalous symmetry (in the usual sense), since the representation categories of all anomalous symmetries are also described by local fusion categories. Here we have an example of an emergent symmetry that is neither anomalous nor anomaly-free.

### E. The algebra of patch symmetry operators and its braided fusion category – transformation category

In the above, we show that the operator algebra of a class of string operators, the patch charge operators $Z_{\text{str}}$, gives rise to a symmetric fusion category $\text{Rep}_{\mathbb{Z}_2}$. In this section, we are going to consider the operator algebra of another class of $t$-patch operators $X_{\text{str}}$, the patch symmetry operators, and show that they give rise to a symmetric fusion category $\text{Vec}_{\mathbb{Z}_2}$ which happens to be isomorphic to $\text{Rep}_{\mathbb{Z}_2}$.

Patch symmetry operators are defined by restricting the global symmetry to finite patches as discussed in Sec IV B (assume $i < j$)

$$W_{\text{patch}_{ij}} = X_{\text{str}_{ij}} = X_{i+1} \cdots X_{j-1} X_j. \quad (26)$$

Since the bulk of a patch symmetry operator is invisible, it is completely legitimate to think of the boundary of the 1d patch symmetry operator as particles. We can define a fusion operation of these particles, which is called $m$.\footnote{The reason for this name will become clear in next subsection.} Analogous to the discussion in the previous subsection, we may construct a braided fusion category corresponding to these $m$ particles.

To do so, we can think of these patch symmetry operators as operators that transport $m$ particles from one point to another on the one-dimensional space

$$W_{\text{patch}_{ij}} = T_m(i \rightarrow j), \quad (27)$$

The above can also be viewed as a world-line of $m$ particle from $i$ to $j$. In fact, the $m$ particle live on the link, such as $(i, i + 1)$. In the above, we view such a $m$ particle as located at $i$.

The hopping operators of $\bar{m}$ particle is given by

$$T_{\bar{m}}(i \leftarrow j) = T_m(i \rightarrow j). \quad (28)$$

We see that $T_m(i \leftarrow j) = T_m(j \rightarrow i)$. Thus $m = \bar{m}$. We can also work out the fusion of the $m$ particles as we did for the $e$ particles in Sec IV C. From

$$T_m(-\infty \rightarrow i)T_m(-\infty \rightarrow i) = \text{id} \quad (29)$$
we find the fusion rule \( m \otimes m = 1 \).

Next, we work out \( F \)-symbol from Fig. 5. We find that \( F(a, b, c) = 1 \) for \( a, b, c = 1, m \). This is not surprising because the patch operators all commute with each other since they are just products of Pauli \( X \) operators and identity operators. Thus \( 1, m \) form a fusion category \( \text{Vec}_{Z_2} \), which is isomorphic to \( \text{Rep}_{Z_2} \).

\( 1, m \) also have a braiding structure and form a braided fusion category. Using Fig. 6, we find that \( m \) particles have trivial self-statistics. Using Fig. 7, we find that \( 1 \) and \( m \) particles have trivial mutual statistics. This allows us to show that \( 1, m \) form a symmetric fusion category \( \text{Vec}_{Z_2} \).

**Definition 8.** We will call the symmetric fusion category \( \mathcal{T} \) formed by patch symmetry operators as transformation category.[34]

Similar to representation category, we believe that the transformation category in \( n \)-dimension space can also be described by local fusion \( n \)-categories. We ignore the braiding structure and consider the fusion category \( \mathcal{T} \) formed by patch symmetry operators. We replace braiding structure with a faithful functor \( \beta : \mathcal{T} \to \text{Vec} \).[48] The local fusion category \((\mathcal{T}, \beta)\) can also be viewed as the transformation category of the symmetry.

**F. The algebra of all string operators and its non-degenerate braided fusion category**

In this subsection, we are going to consider the operator algebra of all string operators, i.e. the patch charge operators \( Z_{\text{str}} \) and the patch symmetry operators \( X_{\text{str}} \). The isomorphic class of such a complete operator algebra is called a categorical symmetry.

We have seen that the algebra of \( Z_{\text{str}} \) corresponds to a symmetric fusion category \( \text{Rep}_{Z_2} \), and the algebra of \( X_{\text{str}} \) corresponds to a symmetric fusion category \( \text{Vec}_{Z_2} \). The algebra of \( Z_{\text{str}} \) and \( X_{\text{str}} \) corresponds to a braided fusion category formed by \( \text{Rep}_{Z_2} \) and \( \text{Vec}_{Z_2} \). Here, we like to show that such a braided fusion category describes the topological excitations in \( Z_2 \)-topological order with topological excitations \( 1, e, m, f \) in \( 2 \)-dimensional space. We will denote such a braided fusion category as \( \text{Rep}_2 \).

The algebra of \( Z_{\text{str}} \) and \( X_{\text{str}} \) also contain their product

\[
X_{\text{str},ij}Z_{\text{str},ij} = T_f(i \to j) = T_f(j \to i).
\]

\( T_f(i \to j) \) is the world-line of a new particle \( f \). We see that a \( f \) particle at \( i \) is the bound state of an \( e \) particle at \( i \) and an \( m \) particle on the link \( \langle i, i+1 \rangle \). \( T_f(i \to j) \) satisfies the following algebra

\[
T_f(i \to j)T_f(j \to k) = X_{i+1} \cdots X_jZ_iZ_jX_{j+1} \cdots X_kZ_jZ_k = X_{i+1} \cdots X_kZ_iZ_k = T_f(i \to k)
\]

From

\[
T_f(-\infty \to i)T_f(-\infty \to i) = (-\text{id})O_{-\infty},
\]

where \( O_{-\infty} \) is a local symmetric operator at \(-\infty\), we find that \( f \otimes f = 1 \).

We can calculate the self-statistics of the \( f \) particle using the hopping algebra method used in previous subsections,

\[
T_f(3 \to 1)T_f(1 \to 2)T_f(0 \to 1) = (X_2X_3Z_1Z_3)(X_2Z_1Z_2)(X_1Z_0Z_1) = -(X_1Z_0Z_1)(X_2Z_1Z_2)(X_2X_3Z_1Z_3) = -T_f(0 \to 1)T_f(1 \to 2)T_f(3 \to 1),
\]

from which we find that \( f \) particles have fermionic self-statistics.

Mutual statistics of \( e, m \), and \( f \) particles can be obtained by the use of the patch operators. For example, when \( i < k < j < l \), we have

\[
Z_{\text{str},ij}X_{\text{str},kl} = -X_{\text{str},ij}Z_{\text{str},kl}.
\]

Thus the \( e \) and \( m \) particles have \( \pi \) mutual statistics. In fact, the \( e, m, f \) particles all have \( \pi \) mutual statistics respect to each other.

Since every non-trivial topological excitations (i.e. \( e, m, f \)) can be detected remotely via mutual statistics, the particles \( 1, e, m, f \) form a non-degenerate braided fusion category \( \text{Rep}_{Z_2} \). We believe that such a non-degenerate braided fusion category fully characterized the isomorphic class of the algebra of local symmetric operators. Thus categorical symmetry is fully characterized by non-degenerate braided fusion category. Since the non-degenerate braided fusion category describes a topological order in 2-dimensional space, we can also say that categorical symmetry is fully characterized by topological order in one higher dimension. This connection between algebra of local symmetric operators and non-degenerate braided fusion category, as well as topological order in one higher dimension is the key result of this paper.

**G. A holographic way to compute categorical symmetry**

In the above, we have computed the categorical symmetry of \( Z_2 \) symmetry directly from the definition of categorical symmetry, i.e. from the algebra of local symmetry operators and their string extensions. We find that the categorical symmetry of \( Z_2 \) symmetry is a topological order in one higher dimension. In fact, we can compute this topological order in one higher dimension directly.

We know that a system with \( Z_2 \) symmetry can be realized as a boundary of a trivial product state with \( Z_2 \) symmetry in one higher dimension. If we gauge the bulk symmetric product state, we will obtain a \( Z_2 \) topological order \( \text{Rep}_2 \) described by \( Z_2 \) gauge theory. Such a \( Z_2 \) topological order in one higher dimension happen to be the categorical symmetry of \( Z_2 \) symmetry.
This result can be generalized. An anomaly-free (higher) symmetry described by (higher) group $G$ can be realized as a boundary of a trivial product state with $G$ (higher) symmetry in one higher dimension. If we gauge the bulk symmetric product state, we will obtain a topological order described by $G$ (higher) gauge theory. Such a topological order in one higher dimension is the categorical symmetry of the $G$ (higher) symmetry.

V. A 1d BOSONIC QUANTUM SYSTEM WITH ANOMALOUS $\mathbb{Z}_2$ SYMMETRY

Now we discuss the next simplest example: a bosonic system

$$H_{\mathbb{Z}_2} = -B \sum_{i=1}^{L} Z_i Z_{i+1} + J_1 \sum_{i=1}^{L} (X_i - Z_{i-1} X_i Z_{i+1})$$

$$+ J_2 \sum_{i=1}^{L} Z_{i-1} (X_i + Z_{i-1} X_i Z_{i+1}),$$

(35)

in 1-dimensional space with an anomalous $\mathbb{Z}_2$ symmetry (i.e. a non-on-site symmetry). Our discussions here follow closely the discussions in the last section.

The non-on-site $\mathbb{Z}_2$ symmetry [47, 65–67] is described by the symmetry operator

$$W = \prod_{i} X_i \prod_{s_{i+1}} s_{i+1} = \prod_{i} X_i \prod_{1} \frac{Z_{i+1} Z_{i+1} + Z_{i+1} Z_{i+1} - 1}{2}$$

(36)

which we represent pictorially as

```
\[ \begin{array}{cccccc}
  X & X & X & X & X \\
  \vdots & s & s & s & s & s \\
\end{array} \]
```

where the operators are ordered top to bottom. The phase factor $s_{i+1}$ is real despite appearances as can be checked by substituting $\{+1, -1\}$ for $Z_i$ and $Z_{i+1}$ (i.e. we work in the Z basis). It’s easy to see that it evaluates to +1 when there is no domain wall between $i$ and $i + 1$. Moreover it evaluates to −1 for only one kind of domain walls, the +1 $\rightarrow$ −1 kind; it evaluates to +1 for the −1 $\rightarrow$ +1 kind.

A. Braided fusion category of patch symmetry transformation

In order to identify the braided fusion category (i.e. the categorical symmetry) corresponding to this anomalous symmetry, we will work out the patch symmetry operators corresponding to the above global symmetry operation. Legitimate patch symmetry operators must satisfy the transparent and fusion properties i.e.

1. $W_{\text{patch}_{ij}} W_{\text{patch}_{jk}} = W_{\text{patch}_{ik}}$ for $i < j < k$

2. $W_{\text{patch}_{ij}} W_{\text{patch}_{jk}} W_{\text{patch}_{ik}} = W_{\text{patch}_{ki}}$ for $i < k < j$

In order to ensure these properties are satisfied, we need to choose appropriate boundary operators for the $W_{\text{patch}_{ij}}$. To that end, we propose the following definition:

$$W_{\text{patch}_{ij}} = O_{1}^{j} \prod_{k=i+1}^{j} X_{k} (Z_{j} O_{l}) \prod_{k=i+1}^{j} s_{k,k+1}$$

(37)

where $O_{l} = (1 - iZ_{j})/\sqrt{2}$. We may write this operator pictorially as

```
\[ \begin{array}{cccccc}
  i & j \\
  O & X & X & X & ZX \\
  \vdots & s & s & s & s & s \\
\end{array} \]
```

It is straightforward to check that this satisfies the properties mentioned above. Let us label the particles at the boundaries of this patch operator as $s$. The patch symmetry operator $W_{\text{patch}_{ij}}$ can also be understood as an operator transporting an $s$ particle from $i$ to $j$, i.e.

$$T_{s}(i \rightarrow j) = W_{\text{patch}_{ij}}$$

(38)

The fusion of $s$ particles turns out to be identical to that of the $e$ particles discussed above: they are their own antiparticles so that $s \otimes s = 1$. Here 1 is the trivial particle, an end of trivial string formed by product of identity operators. To see this fact, we consider the product of two semi-infinite strings as in eqn. (21).

$$T_{s}(-\infty \rightarrow i) T_{s}(-\infty \rightarrow i) \stackrel{\text{mod phase}}{=} O_{-\infty} Z_{1} Z_{i+1} = (1d) O_{-\infty} O_{i}$$

(39)

where we use $O_{-\infty}$ and $O_{i}$ to indicate local symmetric operators at $-\infty$ and near site $i$, respectively. Note that a local symmetric operator represents a trivial particle 1. A graphical proof of this is shown below.

```
\[ \begin{array}{cccccc}
  -\infty & & & & \leftarrow \\
  X & X & X & X & X \\
  \vdots & s & s & s & s & s \\
\end{array} \]
```

\[ \begin{array}{cccccc}
  Z & Z \\
  \vdots & s & s & s & s & s \\
\end{array} \]

$\stackrel{\text{mod phase}}{=} Z O$

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Thus complete fusion ring is given by
\[ s \otimes s = 1, \quad s \otimes 1 = s, \quad 1 \otimes 1 = 1. \] (40)

or
\[ N^{11}_1 = N^{ss}_1 = N^s1 = N^1s = 1, \quad \text{others} = 0. \] (41)

Fusion ring \( N^{ab}_s \) is only a part of data that describe the braided fusion category. We need to supply the \( F \)-symbol, \( F(a, b, c) \), to promote the fusion ring to a fusion category. Similar to the \( e \) particles, we have \( F(1, s, s) = F(1, 1, 1) = 1 \). However, the \( F \)-symbol \( F(s, s, s) \) is different (again, referring to Fig 5):
\[ W_{\text{patch}}^W_{\text{patch}}^W_{\text{patch}} W_{\text{patch}}^W_{\text{patch}} W_{\text{patch}}^W_{\text{patch}} = F(s, s, s) W_{\text{patch}}^W_{\text{patch}} W_{\text{patch}}^W_{\text{patch}} W_{\text{patch}}^W_{\text{patch}} \] \( W_{\text{patch}}^W_{\text{patch}} W_{\text{patch}}^W_{\text{patch}} W_{\text{patch}}^W_{\text{patch}} W_{\text{patch}}^W_{\text{patch}} = F(s, s, s) \) \( W_{\text{patch}}^W_{\text{patch}} W_{\text{patch}}^W_{\text{patch}} W_{\text{patch}}^W_{\text{patch}} W_{\text{patch}}^W_{\text{patch}} \)

Working out the algebra (see Appendix B 1) gives us \( F(s, s, s) = -1 \). This distinguishes the fusion category generated by the anomalous \( Z_2 \) patch symmetry operators from that of the anomaly-free \( Z_2 \) symmetry without even considering the braiding structures.

Similarly, the fusion category data, \( (N^{ab}_c, F(a, b, c)) \), is only a part of data to describe a braided fusion category. To obtain the full data to describe a braided fusion category, we need to supply the data that describes mutual and self statistics. The mutual statistics between \( s \) and \( 1 \) is trivial \( \theta_{s1} = 0 \). We can calculate the self statistics of \( s \) by calculating the statistical hopping algebra of the particle-like endpoints of the patch symmetry operator, as outlined above in Fig. 6. In this case, we find (see Appendix B 2) \( e^{i\theta_s} = i \), i.e. a statistical phase of \( \theta_s = \pi/2 \). This shows that the endpoints are semions. Thus unlike the anomaly-free \( Z_2 \) symmetry, the transformation category of anomalous \( Z_2 \) symmetry is not a symmetric fusion category. The transformation category happens to be non-degenerate, and correspond to the single-semion topological order in 2d, which will be denoted as \( N_{\text{single-semion}} \). (Usually, a transformation category is degenerate, and does not correspond to a topological order in one higher dimension.)

\[ Z_{\text{string},ij} = Z_i Z_j \] (43)

Let us label the particles at the ends of this operator as \( b \). This operator is identical to the patch charge operator in the case of anomaly-free \( Z_2 \) symmetry discussed in the previous section. All the results discussed there carry forward to this case. In particular, these patch charge operators produce the representation category, which is a symmetric fusion category \( \text{Rep} \mathbb{Z}_2 \). We see that the representation category cannot distinguish anomalous and anomaly-free symmetries, but the transformation category can.

### C. Braided fusion category of all t-patch operators

To consider all t-patch operators, we must consider fusion of the semion and the boson. The \( b \) particles fuse with \( s \) to give another semion, let’s call it \( \tilde{s} \). Along with the trivial one, we thus end up with four particles. We can easily check that \( s \) and \( b \) have \( \pi \) mutual statistics,
\[ Z_{\text{string},ij} W_{\text{patch}}^W_{\text{patch}} = -W_{\text{patch}}^W_{\text{patch}} Z_{\text{string},ij} \] (44)

Combining this with the fact that \( s \) has semionic self-statistics, we see that \( s \) and \( \tilde{s} \equiv s \otimes b \) have trivial mutual statistics.

Putting the transformation category \( N_{\text{single-semion}} \) and the representation category \( \text{Rep} \mathbb{Z}_2 \), together, the above set of anyons and their braiding and fusion data corresponds to the double-semion topological order \( N_{\text{double-semion}} \). Double-semion is an Abelian topological order which are classified \( K \)-matrix.\[^{68, 69}\] The \( K \)-matrix for the double-semion topological order is given by
\[ K_{\text{DS}} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \] (45)

The topological quasiparticles are described by integer vectors \( l \), and there \( \det(K) = 16 \) is them. The trivial particle \( 1 \) is described by \( 1 \sim (0, 0)^T \), semion \( s \sim (0, 1)^T \), semion \( \tilde{s} \sim (1, 0)^T \), and boson \( b \sim (1, 1)^T \). The self statistics of anyon \( l \) is given by \( \theta_l = \pi l^T K_{\text{DS}}^{-1} l \), the mutual statistics between anyon \( l_1 \) and \( l_2 \) is given by \( \theta_{l_1 l_2} = 2\pi l_1^T K_{\text{DS}}^{-1} l_2 \). The above \( K \)-matrix reproduces the self/mutual statistics of \( s, \tilde{s}, b \). Thus, the categorical symmetry for the anomalous \( Z_2 \) symmetry in 1-dimensional space is the double-semion topological order \( N_{\text{double-semion}} \) in 2-dimensional space.

### D. A holographic way to compute categorical symmetry

We can also compute categorical symmetry of anomalous symmetry directly by computing the corresponding topological order in one higher dimension. We know that a system with (certain) anomalous \( G \) (higher) symmetry can be realized as a boundary of a \( G \)-symmetry protected topological (SPT) state in one higher dimension. If we gauge the \( G \)-symmetric in the bulk SPT state, we will obtain a topological order described by a twisted \( G \) (higher) gauge theory. Such a topological order in one higher dimension is the categorical symmetry of the \( G \) (higher) symmetry.
Applying this method to 1d anomalous $\mathbb{Z}_2$ symmetry, we find the corresponding categorical symmetry to be the 2d double-semion topological order. The connection between 1d anomalous $\mathbb{Z}_2$ symmetry and 2d double-semion topological order was first observed in Ref. 70.

VI. A 1d BOSONIC QUANTUM SYSTEM WITH $\mathbb{Z}_2 \times \mathbb{Z}_2$ SYMMETRY WITH A MIXED ANOMALY

In this section, we calculate categorical symmetry (i.e. the non-degenerate braided fusion category) for bosonic $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with the mixed anomaly in 1-dimensional space. Following Ref. 67, (see Appendix C for details) we have two qubits on each site and two symmetry generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$,

\[ W = \prod_i X_i \]
\[ \tilde{W} = \prod_i \tilde{X}_i \prod_i s_{i,i+1} \]

where $s_{i,i+1} = i^{\frac{1}{2}(Z_{i+1}-Z_i)(\tilde{Z}_{i+1}-\tilde{Z}_i)+1}$ is the non-on-site phase factor that encodes the mixed anomaly. $X_i, \tilde{Z}_i$ act on one qubit and $\tilde{X}_i, Z_i$ on the other qubit.

A. Braided fusion category of patch operators

The operators $W$ and $\tilde{W}$ above are global symmetry transformations, which have corresponding t-patch symmetry operators as discussed in the previous sections.

\[ W_{\text{patch}}_{ij} = \tilde{O}_i \left( \prod_{k=i}^{j-1} X_k \right) \tilde{O}_j^\dagger \]  \hspace{1cm} (48)
\[ \tilde{W}_{\text{patch}}_{ij} = \prod_{k=i+1}^{j} \tilde{X}_k \prod_{k=i}^{j-1} s_{k,k+1} \]  \hspace{1cm} (49)

To satisfy the transparency condition and the composition algebra of the t-patch operators (see Fig. 3), $\tilde{O}_j$ in eqn. (48) needs to be chosen carefully: $\tilde{O}_j = (1 - i\tilde{Z}_j)/\sqrt{2}$. Pictorially, we can represent $W_{\text{patch}}_{ij}$ as

\[ \tilde{O} \]
\[ \tilde{X} \quad X \quad X \quad X \]

and $\tilde{W}_{\text{patch}}_{ij}$ as

\[ \tilde{O} \quad \tilde{X} \quad \tilde{X} \quad \tilde{X} \quad \tilde{X} \quad s \quad s \quad s \]

We label the endpoints of these patch operators $m$ and $\tilde{m}$, respectively. More carefully, we should choose one end of the string to be $m$ (or $\tilde{m}$) while the other end is its antiparticle $\tilde{m}$ ($m$ respectively). The patch charge operators are generated by

\[ Z_{\text{string},ij} = Z_i Z_j \]  \hspace{1cm} (50)
\[ \tilde{Z}_{\text{string},ij} = \tilde{Z}_i \tilde{Z}_j \]  \hspace{1cm} (51)

Let us name the charge operators at the ends of these as $e$ and $\tilde{e}$. We note here that $m, \tilde{m}$ are order 2 whereas $e, \tilde{e}$ are order 2. We can see this from the fact that $W_{\text{patch}}^{ij} = 1 = \tilde{W}_{\text{patch}}^{ij}$ while $W_{\text{patch}}^{2ij} \neq 1, \tilde{W}_{\text{patch}}^{2ij} \neq 1$. On the other hand, $Z_{\text{string}}^{ij} = 1 = \tilde{Z}_{\text{string}}^{ij}$. The fusion of $m$ and $\tilde{m}$ gives $s$ (say). The self-statistics of $e$ and $\tilde{e}$ are trivial, by the same logic as in the anomaly-free $\mathbb{Z}_2$ symmetry discussed in Sec IV C. We can also check that $m$ and $\tilde{m}$ have trivial self-statistics. However, $s$ particles have semionic self-statistics, as can be seen from the hopping algebra calculation. This is closely related to the fact that $m$ and $\tilde{m}$ have $\pi/2$ mutual statistics; we find (cf. Fig 7)

\[ W_{\text{patch}2} \tilde{W}_{\text{patch}3} = i\tilde{W}_{\text{patch}4} W_{\text{patch}02} \]  \hspace{1cm} (52)

Further details may be found in Appendix B 3. We also note that the $m$ and $e$ particles have $\pi$ mutual statistics, and so do $\tilde{m}$ and $\tilde{e}$.

The particles $m, \tilde{m}, e, \tilde{e}$ generate a non-degenerate braided fusion category that correspond to a 2d Abelian topological order. By comparing the self/mutual statistics of those topological excitations, we find that the 2d Abelian topological order is described by the $K$-matrix

\[ K = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \]  \hspace{1cm} (53)

This 2d topological order is the categorical symmetry for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with the mixed anomaly in 1-dimensional space. The topological excitations in such an Abelian topological order are labeled by integer vectors $l$. The $m, \tilde{m}, e, \tilde{e}$ correspond to the following integer vectors:

\[ e \sim (1, 0, 0, 0)^\top, \quad m \sim (0, 1, 0, 0)^\top, \quad \tilde{e} \sim (0, 0, 1, 0)^\top, \quad \tilde{m} \sim (0, 0, 0, 1)^\top. \]  \hspace{1cm} (54)

The self-statistics of particle $l$ and mutual statistics between particles $l_1$ and $l_2$ can be calculated via

\[ \theta_l = \pi l^\top K^{-1} l, \quad \theta_{l_1,l_2} = 2\pi l_1^\top K^{-1} l_2, \]  \hspace{1cm} (55)

where

\[ K^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}. \]  \hspace{1cm} (56)
The entry $\frac{1}{2}$ in $K^{-1}$ gives rise to the $\pi/2$ mutual statistics between $m$ and $\bar{m}$.

B. A holographic calculation of categorical symmetry

The above 2d Abelian topological order (i.e. the categorical symmetry) can be obtained via another approach. We know that the $Z_2 \times Z_2$ symmetry with the mixed anomaly is realized by the boundary of a 2d $Z_2 \times Z_2$ SPT state. If we gauge the 2d $Z_2 \times Z_2$ symmetry, we will turn the 2d $Z_2 \times Z_2$ SPT state into a 2d topological order. Such a 2d topological order is the Abelian topological order described above. Such an Abelian topological order was given by the $K$-matrix in equations (64) and (67) in Ref. 71. For our case, we need to substitute the values $n_1 = n_2 = 2$, and $m_0 = m_3 = 0, m_2 = 1$, which gives us the $K$-matrix in eqn. (53). This Abelian topological order is the categorical symmetry for the 1d $Z_2 \times Z_2$ symmetry with the mixed anomaly. The holographic calculation gives rise to the same result as the operator algebra calculation.

C. The equivalence between 1d $Z_2 \times Z_2$ symmetry with mixed anomaly and 1d $Z_4$ symmetry

Generalizing our $Z_2$ result, we know that the categorical symmetry of 1d anomaly-free $Z_4$ symmetry is the 2d $Z_4$ topological order ($Z_4$ gauge theory), denoted as $\text{Sau}_{Z_4}$ and described by the $K$-matrix,

$$K_{Z_4} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$$

The set of topological quasiparticle is described by integer vectors $\{(a, b)^T | a, b \in Z_4\}$, and there also $\det K_{Z_4} = 16$ of them. Their self and mutual statistics can be read off from the inverse of the $2 \times 2$ $K$-matrix, which are the same as those for the $4 \times 4$ $K$-matrix in eqn. (53). This allows us to make the following identification,

$$(0, 1)^T \leftrightarrow m, \quad (1, 0)^T \leftrightarrow \bar{m}, \quad (1, 1)^T \leftrightarrow s
\quad (2, 0)^T \leftrightarrow e, \quad (0, 2)^T \leftrightarrow \bar{e}\tag{58}$$

For example, note that $(0, 1)^T$ and $(1, 0)^T$ have trivial self statistics,

$$\pi \cdot (0, 1)^T \cdot K^{-1} \cdot (0, 1)^T = 0 \quad (59)$$

$$\pi \cdot (1, 0)^T \cdot K^{-1} \cdot (1, 0)^T = 0 \quad (60)$$

but have $\frac{\pi}{2}$ mutual statistics,

$$2\pi \cdot (0, 1)^T \cdot K^{-1} \cdot (1, 0)^T = \frac{\pi}{2} \quad (61)$$

so these must correspond to the $m, \bar{m}$ particles. Note also that these are order 4 quasiparticle vectors, i.e. 4 of them will fuse to a trivial quasiparticle. On the other hand, the quasiparticle vectors $(2, 0)^T$ and $(0, 2)^T$ correspond to $e, \bar{e}$ particles because not only do they have trivial self statistics,

$$\pi \cdot (0, 2)^T \cdot K^{-1} \cdot (0, 2)^T = 0 \quad (62)$$

$$\pi \cdot (2, 0)^T \cdot K^{-1} \cdot (2, 0)^T = 0 \quad (63)$$

but they also have trivial mutual statistics,

$$2\pi \cdot (0, 2)^T \cdot K^{-1} \cdot (2, 0)^T = 2\pi \quad (64)$$

Similar calculations show that $(0, 2)^T$ and $(1, 0)^T$ have $\pi$ mutual statistics, and so do $(2, 0)^T$ and $(0, 1)^T$.

In fact, 2d Abelian topological orders described by (53) and (57) are actually the same topological order [72]. It turns out, this K-matrix in (53) can be transformed $K \rightarrow WKW^T$ by an integer matrix $W$ with $\det(W) = \pm 1$ into a $Z_4$ K-matrix, direct summed with a trivial block.

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad WKW^T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix} \tag{65}$$

To summarize, 1d $Z_2 \times Z_2$ symmetry with the mixed anomaly is realized by the boundary of a 2d $Z_2 \times Z_2$ SPT state. 1d anomaly-free $Z_4$ symmetry is realized by the boundary of a 2d trivial $Z_4$ SPT state. The categorical symmetry of the 1d mixed-anomalous $Z_2 \times Z_2$ symmetry is the $Z_2 \times Z_2$ gauging of the 2d $Z_2 \times Z_3$ SPT state. The categorical symmetry of the 1d $Z_4$ symmetry is the $Z_4$ gauging of the 2d trivial $Z_4$ SPT state. The two symmetries give rise to the same 2d topological order. Thus 1d $Z_2 \times Z_2$ symmetry with the mixed anomaly and 1d anomaly-free $Z_4$ symmetry have the same categorical symmetry and are equivalent.

D. A new duality mapping

By comparing with the corresponding table for $Z_4$ in the additive presentation $\{0, 1, 2, 3\}$, we can make the following (non-unique) one-to-one mapping between these

\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
$Z_2 \cdot \lambda_{e_2} \cdot Z_2$ & (0,0) & (0,1) & (1,0) & (1,1) \\
\hline
(0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\
(0,1) & (0,1) & (0,0) & (0,1) & (0,1) \\
(1,0) & (1,0) & (1,1) & (0,1) & (0,1) \\
(1,1) & (1,1) & (1,0) & (0,0) & (0,1) \\
\hline
\end{tabular}
\end{center}
two representations of $Z_4$.

\[(0, 0) \leftrightarrow 0 \quad (0, 1) \leftrightarrow 2 \quad (66)\]
\[(1, 0) \leftrightarrow 3 \quad (1, 1) \leftrightarrow 1 \quad (67)\]

The above holographic equivalence of 1d mixed-anomalous $Z_2 \times Z_2$ symmetry and 1d anomaly-free $Z_4$ symmetry suggests the existence of a new duality mapping, between a model with $Z_2 \times Z_2$ non-on-site symmetry and model with $Z_4$ on-site symmetry. Such an exact duality maps between the $Z_4$ patch symmetry/charge operators and the patch symmetry/charge operators of the mixed-anomalous $Z_2 \times Z_2$ symmetry we have been outlining in this section. This duality is a Kramers-Wannier-like transformation that transforms one set of $Z_2$ variables from order to disorder (or site to link) variables, followed by an on-site (local) unitary transformation. To state the duality mapping, we first re-write the group $Z_4$ as a cocycle-twisted product of two $Z_2$ groups, as described in Appendix N of Ref. 73. With $G = Z_4$, and $A = Z_2 \leq G$, we extend $A$ by $H = Z_2$ with $\alpha = 1d$ and $e_2(h_1, h_2) = [h_1, h_2]_{\text{mod} 2}$. The group operation with these choices can be expressed as

\[(h_1, x_1) \ast (h_2, x_2) = (h_1 + h_2, x_1 + x_2 + e_2(h_1, h_2)) \quad (68)\]

where the additions are to be understood modulo 2. Using this, we may write elements of $Z_4$ using two $Z_2$ labels as $g \equiv (h, x)$ where $x \in A$ and $h \in H$. There are four $Z_4$ symmetry transformations: one trivial and three nontrivial. Taking $Z_4$ to be represented as $\{0, 1, 2, 3\}$, with the group operation being addition modulo 4, we have two generators $L_{+1}$ and $L_{+3}$ of the symmetry group,

\[
L_{+1} |g\rangle = |g + 1 \text{ mod } 4\rangle
L_{+3} |g\rangle = |g + 3 \text{ mod } 4\rangle
\quad (69)
\]

In the $(h, x)$ representation, what do these generators look like? We can work this out by looking at the group “multiplication” table of $Z_4$ in this representation: see Table I.

Using this mapping, we re-write eqn. (69) as follows.

\[
L_{+1} |(h, x)\rangle = |(h, x) \ast (1, 1)\rangle
L_{+3} |(h, x)\rangle = |(h, x) \ast (1, 0)\rangle
\quad (70)
\]

Inspecting this case-by-case, one observes that the generator $L_{+3}$ is nothing but the operator $X_1 C X_1 0$, acting on kets $|(h, x)\rangle$. Here $h$ and $x$ are labeled as qubits 1 and 0 respectively, and $C X_1 0$ denotes the controlled NOT gate with qubit 1 as the control.

Now we apply a duality transformation on the t-patch operators of the $Z_2 \times Z_2$ symmetry with mixed anomaly in order to show that we recover the t-patch operators of anomaly-free $Z_4$ symmetry. The reader who is interested in the explicit form of the duality instead of the steps leading up to it is invited to skip to the end of this subsection.

On the $Z_2 \times Z_2$ side, our states are defined by a pair of $Z_2$ variables on each site $i$, denoted $(g_i, \bar{g}_i)$. The definitions $g_i = \bar{Z}_2^{-1}, \bar{g}_i = \bar{Z}_2^{3}$ map the Z-basis $\{1\}$ to the additive $Z_2$ basis $\{0, 1\}$.

Step 1 of duality transformation $D$: We transform $(g_i, \bar{g}_i)$ to $(g_i', \bar{g}_i-1/2)$ by defining $\bar{g}_i - 1/2 = g_i - g_i - 1 \text{ mod } 2$ and $g_i' = \bar{g}_i$. The Pauli operators transform as

\[
X_i \rightarrow \bar{X}_i + \bar{X}_i, \quad Z_i Z_{i+1} \rightarrow \bar{Z}_i Z_{i+1} \quad (71)
\]

The new degrees of freedom may be shown pictorially as

\[\bar{g} \quad g' \quad \bar{g} \quad g' \quad \bar{g} \quad g' \quad \bar{g} \quad \bar{g} \quad \cdots \quad i - 1 \quad i \quad i + 1 \quad \cdots \]

For each site $i$, let us define $g''_i = \bar{g}_i - 1/2$. Then we have a two-qubit Hilbert space labeled as $(g''_i, g''_i)$ associated with site $i$. Let us choose $g''_i$ as qubit-1 and $g''_i$ as qubit-2.

Step 2 of duality transformation $D$: Now we perform a Hadamard transformation on qubit-2 of each site. The states transform as

\[|g'_i \otimes g''_i \rangle \rightarrow |g'_i \rangle \otimes (H |g''_i \rangle) \quad (72)\]

where $H$ is the Hadamard operator. We will instead work in the Heisenberg picture, where the Hadamard transformation acts on the operators and interchanges $\hat{X}$ and $\hat{Z}$. Then the states on which these transformed operators act are labeled by $Z_4$ elements in the $(h, x)$ representation with $h_i = g'_i$ and $x_i = g''_i = \bar{g}_i - 1/2 = g_i - g_i - 1 \text{ mod } 2$.

Summarizing the mapping of the basis states,

\[(g_i, \bar{g}_i) \rightarrow (g'_i = \bar{g}_i, g''_i = g_i - g_i - 1 \text{ mod } 2) \quad (73)\]

with $(g_i, \bar{g}_i) \in Z_2 \times \bar{Z}_2$ and $(g''_i, g''_i) \in Z_2 \wedge e_2 Z_2 \cong Z_4$. On the other hand, under the combined effect of steps 1 and 2 of $D$, we have the operator maps.

\[X_i \rightarrow Z''_i Z_{i+1}'', \quad Z_i Z_{i+1} \rightarrow X''_{i+1} \quad (74)\]

Using this, one finds that the operator $s_{i-1,i}$ becomes $CX(g'_i, g''_i)$. We can also denote this as $CX_{1,0} |i\rangle$ with the qubit labels 1 and 0 as described above. In fact, we can check that the patch operators in the left column of Table II are transformed to those in the right column, under the transformation $D$.

In particular, we find the dual of $\tilde{W}_{\text{patch}, i}$ to be the patch symmetry operator corresponding to the $L_{+3}$ transformation discussed above. This operator then generates all the $Z_4$ patch symmetry operators in the $Z_2 \wedge e_2 Z_2$ representation. On the other hand, the dual of $W_{\text{patch}, i}$ is a t-patch operator with empty bulk that has order 4. This operator may be identified with one of

---

2 The multiplication of elements of $H$ in the definition of $e_2$ is understood to be done in $Z$ and then mapped back to $Z_2$.
the charge patch operators of anomaly-free $Z_4$ symmetry. This completes the mapping between patch operators on both sides of our duality $D : (Z_2 \times Z_2)_{\text{mix}} \leftrightarrow Z_4$. This exact duality mapping allows us to show that the 1d $Z_2 \times Z_2$ symmetry with mixed anomaly and anomaly-free $Z_4$ symmetry have isomorphic local operator algebra i.e. they have the same categorical symmetry.

A comment on gauging: The duality we described above can also be understood as coupling the degrees of freedom of $Z_4$ symmetric system to a $Z_2$ gauge field. The Kramers-Wannier-like transformation in the first step of $D$ essentially amounts to such a gauging procedure. In the case of $Z_2$ symmetry in 1d, the Kramers-Wannier duality transformation allows one to relate $Z_2$ order and disorder operators, where the latter can be obtained from the former by gauging the local $Z_2$ symmetry and then restricting to the $Z_2$ charge even sector of the Ising gauge theory. Our duality transformation above involves an on-site unitary (Hadamard) transformation in addition to this gauging procedure.

**VII. A 1d BOSONIC QUANTUM SYSTEM WITH A $Z_2 \times Z_2$ SYMMETRY “BEYOND ANOMALY”**

In this section, we are going to study a case of emergent symmetry. We find that the emergent symmetry is neither anomaly-free nor anomalous. It illustrates that categorical symmetry (i.e. topological order in one higher dimension) is a better way to view symmetry. We get a simpler, more uniform, and more systematic picture.

Let us briefly recall the model from section II.C of Ref. 47. This model describes a 1D bosonic quantum system with spin-1/2 degrees of freedom on each site and each link. The Hamiltonian describing the model is:

$$H = - \sum_i \left( B \bar{X}_{i-\frac{1}{2}} X_i \bar{X}_{i+\frac{1}{2}} + J \bar{Z}_{i+\frac{1}{2}} \right) + U \sum_i \left( 1 - Z_i \bar{Z}_{i+\frac{1}{2}} Z_{i+1} \right) \quad (75)$$

This Hamiltonian has two on-site (i.e. anomaly-free) $Z_2$ symmetries, generated by

$$W = \prod_k X_k, \quad \bar{W} = \prod_k \bar{Z}_{k+\frac{1}{2}} \quad (76)$$

Let us denote the corresponding symmetries as $Z_2$ and $\bar{Z}_2$. The algebra of local operators is constrained by these symmetries. We add an additional constraint on this algebra: the low-energy constraint. This constraint is imposed by taking the limit of $U \to \infty$. Low energy sector of the Hilbert space must then satisfy

$$1 - Z_i \bar{Z}_{i+\frac{1}{2}} Z_{i+1} = 0, \quad \forall i \quad (77)$$

In operator language, we demand that the allowed local operators commute with the operator appearing in eqn. (77). The algebra of the allowed local operators will give rise to emergent low energy symmetry.

The question is then, how does this additional constraint change the algebra of local operators? It turns out that this modified local operator algebra involves a non-trivial relationship between the $Z_2$ and $\bar{Z}_2$ symmetries. To be clear, this is not a case of mixed anomaly of two $Z_2$ symmetries like the case discussed in the previous section. Nor is this a case of an anomaly-free symmetry: the patch symmetry operators form a non-symmetric fusion category. This is thus an example of a symmetry that is, in some sense, beyond the usual notion of “anomalous symmetry”.

The categorical symmetry of the low-energy sector of this model is not $\text{Su}_2(Z_2 \times Z_2)$, as would be the case for an anomaly-free global $Z_2 \times Z_2$ symmetry. Instead it has the categorical symmetry $\text{Su}_2$, same as that of anomaly-free global $Z_2$ symmetry. Let us now expand on this using the language we have been developing in the previous sections.

The local operator algebra is generated by t-patch operators:

1. 0-dimensional t-patch operators are the local symmetric operators that act within the low-energy sector:

$$\bar{X}_{i-\frac{1}{2}} X_i \bar{X}_{i+\frac{1}{2}}, \quad \bar{Z}_{i+\frac{1}{2}} \quad (78)$$

---

3 The experienced reader may note that this is sometimes colloquially referred to as “gauging” in the literature. We are particular about not calling it by this name since we don’t introduce any extra unphysical, or gauge, degrees of freedom in this discussion. Instead we are restricting to a subspace of the full Hilbert space to focus on the effective theory.
2. 1-dimensional t-patch operators – string operators:

\[ Z_{str_{ij}} = \prod_{k=i}^{j-1} \hat{Z}_{k+\frac{1}{2}} = Z_i Z_j, \]

\[ X_{str_{ij}} = \hat{X}_{i-\frac{1}{2}} \prod_{k=i}^{j} X_k \hat{X}_{j+\frac{1}{2}}. \] (79)

One may note that the new constraint, eqn. (77) has the effect of restricting the t-set of allowed t-patch operators. For example, the two string operators \( \prod_{k=i}^{j-1} \hat{Z}_{k+\frac{1}{2}} \) and \( Z_i Z_j \) become identical within the low energy subspace. Also two string operators \( \prod_{k=i}^{j} X_k \) and \( \hat{X}_{i+1/2} \hat{X}_{j+1/2} \) must appear together. Without this constraint, the list of t-patch operators above would be a bigger one – one that would encode anomaly-free \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry.

The algebra of the above t-patch operators takes the form:

\[ Z_{str_{ij}} X_{str_{ij}} = \pm X_{str_{ij}} Z_{str_{ij}} \] (80)
\[ Z_{str_{ij}} Z_{str_{jk}} = Z_{str_{ik}} \] (81)
\[ X_{str_{ij}} X_{str_{jk}} = X_{str_{ik}} \] (82)

where the sign in eqn. (80) is \( - \) if \( i < k < j < l \), and \( + \) otherwise. We see here that the algebra of the patch operators above mirrors that of anomaly-free \( \mathbb{Z}_2 \) symmetry, as discussed in Section IV F. Specifically, note that eqn. (81) and eqn. (82) are identical to eqn. (9) and eqn. (10) respectively. These represent the fusion of the endpoints of these t-patch operators. The mutual statistics of these endpoints are also identical in the two cases as can be seen by comparing eqn. (80) with eqn. (11) and eqn. (12).

Therefore, the exact 1d \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) on-site symmetry in the model (75) becomes a different \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry at low energies. The new \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry has the categorical symmetry \( \mathfrak{sa}u_{\mathbb{Z}_2} \), while the original \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) on-site symmetry has the categorical symmetry \( \mathfrak{sa}u_{\mathbb{Z}_2 \times \mathbb{Z}_2} \). The new \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry has a special property: a gapped state must spontaneously break one of the \( \mathbb{Z}_2 \) symmetries. A state with both \( \mathbb{Z}_2 \) symmetry must be gapless. There is no state that can spontaneously break both the \( \mathbb{Z}_2 \) symmetries.[47, 74] Those properties have some similarities to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry with the mixed anomaly.

VIII. 2d \( \mathbb{Z}_2 \) SYMMETRY AND ITS DUAL

In the above, we have discussed symmetries and categorical symmetries in 1-dimensional space. In this section, we will start to consider symmetries in higher dimensions, where we will encounter higher symmetries.

First, we consider the simplest symmetry – \( \mathbb{Z}_2 \) symmetry, in 2-dimensional space. For convenience, let us assume the degrees of freedom on each vertex (labeled by \( i \)) are labeled by elements in the \( \mathbb{Z}_2 \) group.
boundary of the disk operator $W_{\text{patch}}$, the two operators have a non-trivial commutation relation:

$$Z_{\text{str}} X_{\text{disk}} = -X_{\text{disk}} Z_{\text{str}}.$$  \hfill (91)

This non-trivial commutation relation measures the charge carried by one end of the string operator. If we view the order of the operator product as the order in time, and view the string as world line of a particle in spacetime (see Fig. 13), then the commutation relation eqn. (91) can be viewed as a braiding of the particle around the boundary of the disk operator. The charge is measured by such a braiding process.

### B. Algebra of patch charge operators and braided fusion higher category of charge objects

The properties of the charges of an anomaly-free symmetry in n-dimensional space can be systematically and fully described by a braided fusion n-category or a local n-fusion category.[48] Let us first give a brief physical introduction of fusion n-category (see Fig. 8). A fusion n-category can be used to describe extended physical objects in nd space. For example, in 3d space, 2-dimensional membranes (co-dimension-1) correspond to the objects in the fusion 3-category. 1-dimensional strings (co-dimension-2) correspond 1-morphisms, and 0-dimensional particles (co-dimension-3) correspond 2-morphisms. The above are physical excitations. Instantons or local operators (0-dimensional in spacetime) correspond 3-morphisms, which are top morphisms. The physical excitations and local operators form the fusion n-category.

To connect the $Z_2$ symmetry in 2-dimensional space to a braided fusion 2-category, we view the local symmetric operators $O^{\text{symm}}_s$ as the 2-morphisms, and the end of string operator $Z_{\text{str}}$ (i.e. $Z_2$-charge) as a 1-morphism $e$ in a fusion 2-category. Operator product of string operator can be viewed as fusion of string ends, which gives rise to the fusion rule of the 1-morphisms $e_i$:

$$e \otimes e = 1, \quad 1 \otimes e = e \otimes 1 = e.$$ \hfill (92)

$e$’s are the point-like $Z_2$-charges for $Z_2$ symmetry. Those $Z_2$-charges can form a 1d quantum liquid state, which correspond to a string excitation [39]. Let $s_{Z_2}$ be a string excitation that corresponds to the 1d spontaneous symmetry breaking state formed by the $Z_2$-charges (which is a state with a non-zero energy gap). (Note that the $Z_2$-charges have a $Z_2$ conservation as implies by the $Z_2$ fusion $e \otimes e = 1$. So they can form a non-trivial 1d gapped quantum liquid state – a spontaneous symmetry breaking state.) We have another string excitation $1_{\text{str}}$ which is formed by $Z_2$ charges along the string in a gapped symmetric state. Note that a string with no $Z_2$ charge is also a symmetric gapped state. So $1_{\text{str}}$ may mean null string, a string that does not have any thing. The string formed by $Z_2$ charges in gapped symmetric state and the string formed by nothing are equivalent (i.e. they can deform into each other without closing the energy gap), and both are denoted as $1_{\text{str}}$.

In addition to the point-like excitation $e$, we have another point-like excitation, denoted as $b_s$, which is the domain wall that connects the string-$s_{Z_2}$ and string-$1_{\text{str}}$. Since, string-$1_{\text{str}}$ is trivial (i.e. can be nothing), $b_s$ can also be viewed as a boundary of string $s_{Z_2}$. The fusion of $e$ and $b_s$ gives us the third point-like excitation $e \otimes b_s$.

The above excitations, plus the $Z_2$ symmetric local operators form a symmetric fusion 2-category denoted as $2\text{Rep}_{Z_2}$.[48]

1. The string-like excitations $1_{\text{str}}$ and $s_{Z_2}$ are objects in $2\text{Rep}_{Z_2}$.

2. The point-like excitations $1$, $e$, and $b_s$ are 1-morphisms:

$$1_{\text{str}} \xrightarrow{b_s} s_{Z_2}, \quad s_{Z_2} \xrightarrow{b_s} 1_{\text{str}}, \quad 1_{\text{str}} \otimes b_s \xrightarrow{e} s_{Z_2}, \quad s_{Z_2} \otimes b_s \xrightarrow{e} 1_{\text{str}}.$$

3. The symmetric operators $O^{\text{symm}}_s$ are 2-morphisms:

$$1 \xrightarrow{O^{\text{symm}}} 1, \quad e \xrightarrow{O^{\text{symm}}} e, \quad e \otimes b_s \xrightarrow{O^{\text{symm}}} e \otimes b_s,$$

$$b_s \xrightarrow{O^{\text{symm}}} b_s, \quad b_s \otimes b_s \xrightarrow{O^{\text{symm}}} e \otimes b_s, \quad e \otimes b_s \xrightarrow{O^{\text{symm}}} b_s.$$ \hfill (94)
$e \otimes b_s$ and $b_s$ are connected by 2-morphisms. Physically, it means that the $\mathbb{Z}_2$ charge $e$ can disappear or appear by itself near $b_s$, by processes induced by symmetric operators. This is expected $b_s$ is connected to a spontaneous symmetry breaking state. We also note that $e$ is the $\mathbb{Z}_2$ charge which is not connected to the trivial excitation $1$ by any 2-morphisms.

Here we like to introduce a notion elementary-type:[39, 48]

**Definition 9.** Two morphisms (or objects which can be viewed as 0-morphisms) connected by higher morphisms are said to have the same elementary-type.

We see that $2\text{Rep}_{\mathbb{Z}_2}$ has only one elementary-types of objects, which is the trivial elementary-type, i.e. both string-$1_{\text{str}}$ and string-$s_{\mathbb{Z}_2}$ belong to trivial elementary-type. $2\text{Rep}_{\mathbb{Z}_2}$ has only three elementary-types of 1-morphisms (particles), $1$, $e$, and $b_s \equiv e \otimes b_s$. $e$ is an excitation in the usual physics sense since it connect string-$1_{\text{str}}$ to string-$1_{\text{str}}$. $e$ is not connected to trivial excitation $1$ by 2-morphisms, and thus is a non-trivial elementary excitation.

In the above, we describe the symmetric fusion 2-category $2\text{Rep}_{\mathbb{Z}_2}$ from the point of view of excitations. We can also describe the symmetric fusion 2-category $2\text{Rep}_{\mathbb{Z}_2}$ from the point of view of patch charge operators, generated by $Z_{\text{str}}, ij$. Note that patch charge operators from a sub-algebra of the algebra of all t-patch operators.

To switch from the excitation point of view to operator point of view, we replace the excitations by the patch charge operators, that create the corresponding excitations from $\mathbb{Z}_2$ symmetric product state. The $\mathbb{Z}_2$ symmetric product state is given by

$$|\Psi_{\text{symm}}\rangle = \bigotimes_i |0\rangle_i, \quad |0\rangle = |+1\rangle + |-1\rangle \frac{1}{\sqrt{2}}.$$  

This gives rise to a description of symmetric fusion 2-category $2\text{Rep}_{\mathbb{Z}_2}$ in terms of patch charge operators (i.e. t-patch operator with empty bulk)

1. The object $1_{\text{str}}$ in $2\text{Rep}_{\mathbb{Z}_2}$ corresponds to a disk-operator (a patch-operator with 2-dimensional patch) with empty bulk

$$\hat{1}_{\text{str}}(\text{loop}) = \prod_{i' \in \text{loop} = \partial \text{disk}} \text{id}_{i'},$$  

where $\text{id}_i$ is the identity operator. Here loop is a closed string, corresponding to the boundary of the disk. The algebra of the operator $1_{\text{str}}$

$$\hat{1}_{\text{str}}(\text{loop}) \hat{1}_{\text{str}}(\text{loop}) = \hat{1}_{\text{str}}(\text{loop}).$$  

is consistent with the fusion of the object

$$1_{\text{str}} \otimes 1_{\text{str}} = 1_{\text{str}}.$$  

The object $s_{\mathbb{Z}_2}$ corresponds to a different disk-operator with empty bulk

$$\hat{s}_{\mathbb{Z}_2}(\text{loop}) = \prod_{i' \in \text{loop} = \partial \text{disk}} P_{+,i'} + \prod_{i' \in \text{loop} = \partial \text{disk}} P_{-,i'},$$  

$$P_{\pm} = \frac{\text{id} \pm Z}{2}.$$  

(Here $P_{\pm}$ can be any local operators that satisfy $P_{\mp} \neq P_\pm$ and $WP_{\pm} = P_{W}$. Again, $s_{\mathbb{Z}_2}$ is a closed string, corresponding to the boundary of the disk. We note that string $s_{\mathbb{Z}_2}$ correspond to a spontaneous symmetry breaking state that has a 2-fold degenerate ground states, $|\otimes_i |+1\rangle_i$ and $|\otimes_i |-1\rangle_i$. The operator $\prod_{i' \in \text{loop} = \partial \text{disk}} P_{+,i'}$ creates the state $|\otimes_i |+1\rangle_i$ from $|\Psi_{\text{symm}}\rangle$, while the operator $\prod_{i' \in \text{loop} = \partial \text{disk}} P_{-,i'}$ creates the state $|\otimes_i |-1\rangle_i$. A particular superposition of the two states $\prod_{i' \in \text{loop} = \partial \text{disk}} P_{+,i'} + \prod_{i' \in \text{loop} = \partial \text{disk}} P_{-,i'}$ is invariant under the $\mathbb{Z}_2$ symmetry transformation $W$. The operator $s_{\mathbb{Z}_2}(\text{loop})$ creates such $\mathbb{Z}_2$ symmetric state, and satisfies

$$\hat{s}_{\mathbb{Z}_2}(\text{loop})X_{\text{disk}} = X_{\text{disk}}\hat{s}_{\mathbb{Z}_2}(\text{loop}),$$  

as long as the string is far away from the boundary of patch symmetry operator $X_{\text{disk}}$.

The operator algebra

$$\hat{s}_{\mathbb{Z}_2}(\text{loop})\hat{s}_{\mathbb{Z}_2}(\text{loop}') = \left( \prod_{i' \in \text{loop}} P_{+,i'} \prod_{i' \in \text{loop}'} P_{+,i'} + \prod_{i' \in \text{loop}} P_{-,i'} \prod_{i' \in \text{loop}'} P_{-,i'} \right) + \left( \prod_{i' \in \text{loop}} P_{+,i'} \prod_{i' \in \text{loop}'} P_{-,i'} + \prod_{i' \in \text{loop}} P_{-,i'} \prod_{i' \in \text{loop}'} P_{+,i'} \right)$$

$$\equiv \hat{s}_{\mathbb{Z}_2}(\text{loop}') + \hat{s}_{\mathbb{Z}_2}(\text{loop}''),$$  

implies the following fusion rule for the loop-like object $s_{\mathbb{Z}_2}$:

$$s_{\mathbb{Z}_2} \otimes s_{\mathbb{Z}_2} = s_{\mathbb{Z}_2} \otimes s_{\mathbb{Z}_2} = 2s_{\mathbb{Z}_2},$$  

which is non-trivial. Here, we have assumed that the two strings, loop and loop', are not on top of each other, but are just nearby. Also

$$\hat{s}_{\mathbb{Z}_2}(\text{loop}'') = \prod_{i' \in \text{loop}} P_{+,i'} \prod_{i' \in \text{loop}'} P_{-,i'} + \prod_{i' \in \text{loop}} P_{-,i'} \prod_{i' \in \text{loop}'} P_{+,i'},$$

$$\hat{s}_{\mathbb{Z}_2}(\text{loop}''') = \prod_{i' \in \text{loop}} P_{+,i'} \prod_{i' \in \text{loop}'} P_{-,i'} + \prod_{i' \in \text{loop}} P_{-,i'} \prod_{i' \in \text{loop}'} P_{+,i'},$$  

(103)

and they both create spontaneous symmetry breaking states.

2. The 1-morphisms $1$, $e$, and $b_s$ (or more precisely, pairs of 1-morphisms) correspond to boundary of
open-string operators:
\[ \hat{1}_i \hat{1}_j = \prod_{i' \in \partial \text{str}_{ij}} \text{id}_{i'} = \text{id}_i \text{id}_j, \]
\[ \hat{e}_i \hat{e}_j = \prod_{i' \in \partial \text{str}_{ij}} Z_{i'} = Z_i Z_j, \]
\[ \hat{b}_{s,i} \otimes \hat{b}_{s,j} = \prod_{i' \in \partial \text{str}_{ij}} P_{+,i'} + \prod_{i' \in \partial \text{str}_{ij}} P_{-,i'} \]  
(104)

They are consistent with eqn. (93), which describes how objects are connected by the 1-morphisms. We like to remark that \( \hat{1}_i \hat{1}_j \) and \( \hat{e}_i \hat{e}_j \) are t-patch operators with an 1-dimensional patch, while \( \hat{b}_{s,i} \otimes \hat{b}_{s,j} \) is a t-patch operators with a 2-dimensional patch (i.e. a disk). The string \( s_{Z_2} \) form a part of the boundary of the disk, and the string \( 1_{\text{str}} \) form the other part of the boundary. The two types of boundaries are connected by the 1-morphism \( b_s \).

3. The symmetric operators \( O_{\text{symm}} \) are 2-morphisms. From the operator algebra
\[ \hat{e}_i \hat{e}_j (\hat{b}_{s,i} \otimes \hat{b}_{s,j}) = Z_i Z_j \prod_{i' \in \partial \text{str}_{ij}} P_{+,i'} + Z_i Z_j \prod_{i' \in \partial \text{str}_{ij}} P_{-,i'} \]
\[ = \prod_{i' \in \partial \text{str}_{ij}} P_{+,i'} + \prod_{i' \in \partial \text{str}_{ij}} P_{-,i'} \]
\[ = \hat{b}_{s,i} \otimes \hat{b}_{s,j} \]  
(105)

we see that we cannot distinguish \( b_{s,i} \) from \( e_i \otimes b_{s,i} \), i.e. they are connected by identity operator. This implies the relations \( b_{s,i} \sim_{O_{\text{symm}}} e_i \otimes b_{s,i} \) and \( e_i \otimes b_{s,i} \sim_{O_{\text{symm}}} b_{s,i} \), proposed in eqn. (94). We also note that operator \( \hat{1}_i \hat{1}_j = \text{id}_i \text{id}_j \) cannot be connected to operator \( \hat{e}_i \hat{e}_j = Z_i Z_j \) via local symmetric operators near vertex-\( i \) and vertex-\( j \). This implies that there is no 2-morphisms connecting \( 1 \) and \( e \).

In our above description of symmetric fusion 2-category \( 2\text{Rep}_{Z_2} \), we include descendant excitations[39, 48, 75, 76] \( s_{Z_2} \) formed by elementary excitations \( e \). Such a descendant string excitation \( s_{Z_2} \) is a spontaneous \( Z_2 \) symmetry breaking state formed by 1d \( e \) gas.

In the above description of operator algebra, we construct the string operators (or the disk operator with empty bulk) via operators \( P^x \) on the string. In general, the disk operator with empty bulk is given by a tensor network operator, whose structure is given in Fig. 9.

Since descendant excitations are formed by lower dimensional excitations, their existence and properties can be derived. Thus, we may drop all the descendant excitations and use only elementary excitations[39, 40] to obtain a simpler description of the symmetric fusion 2-category:

1. The string-like excitations \( 1 \sim_{O_{\text{symm}}} \) is the only elementary object in \( 2\text{Rep}_{Z_2} \).
2. The point-like excitations \( 1 \) and \( e \) are the only elementary 1-morphisms:
\[ 1 \sim_{O_{\text{symm}}} \rightarrow 1, \quad e \sim_{O_{\text{symm}}} \rightarrow e. \]  
(107)

Note that the elementary morphisms (or objects) \( 1 \) and \( e \) are not connected to any other elementary morphisms (except themselves) by higher morphisms. This defines the elementary morphisms or objects[39, 40].

Through the above example, we see that the algebra of the patch charge operators generated by \( Z_{\text{str}_{ij}} \) from a symmetric fusion 2-category \( 2\text{Rep}_{Z_2} \). Such a symmetric fusion 2-category \( 2\text{Rep}_{Z_2} \) fully characterize \( Z_2 \) symmetry in 2-dimensional space, which is called the representation category of the symmetry.

Similarly, we can use a fusion 2-category to describe the symmetry transformations of the \( Z_2 \) 0-symmetry, i.e. to describe the operator algebra generated by the patch symmetry operators \( X_{\text{disk}} \). The boundary of the disk operators \( X_{\text{disk}} \) are labeled by the group elements in \( G = Z_2 \), and correspond to the objects in the fusion 2-category. Adding the trivial 1-morphisms and the top 2-morphisms formed by the local operators \( X_i \) (i.e. the small disk operators), we get a fusion 2-category \( 2\text{Vec}_{Z_2} \). The fusion 2-category \( 2\text{Vec}_{Z_2} \) fully describes the \( Z_2 \) 0-symmetry in 2d space, which is the transformation category of the symmetry.

FIG. 9. The structure of a disk-like operator with empty bulk in term of tensor network. The short detached vertical lines represent identity operators on different sites, which given rise to the empty bulk of the disk-like operator. The non-trivial string operator on the boundary of the disk may have a Wess-Zumino form, i.e. may be given by a tensor network on the disk bounded by the string.
The key relations of the operator algebra are given by the following open string operators and disk operators: like symmetry [1] or higher form symmetry [4]. Such kind of symmetry was called

\[ \langle X \rangle \in \text{str} \]

where \( \langle \rangle \) is the Pauli X-operators acting in the link \((ij)\). Local symmetric operators satisfy

\[ W(S^1)O_i^{\text{symm}} = O_i^{\text{symm}}W(S^1), \quad \forall \text{ loops } S^1. \] (109)

Such kind of symmetry was called \(d\)-dimensional gauge-like symmetry [1] or higher form symmetry [4].

The algebra of local symmetric operators is generated by the following open string operators and disk operators:

\[ \tilde{X}_{ij} = \bigotimes_{(ij) \in S^1} \tilde{X}_{ij}, \quad \tilde{Z}_{\text{disk}} = \bigotimes_{(ij) \in \partial \text{disk}} \tilde{Z}_{ij} \] (110)

The key relations of the operator algebra are given by

\[ \tilde{X}_{ij} = \tilde{X}_{ij}, \quad \tilde{Z}_{\text{disk}} = \tilde{Z}_{\text{disk}}, \quad \tilde{X}_{ij} \tilde{X}_{ij} \tilde{X}_{ij} = \tilde{X}_{ij} \tilde{X}_{ij} \tilde{X}_{ij} \] (111)

where the \(\pm\) signs depend on the relation between the string and the disk (see Fig. 12(c,d)). Here and later in this paper, we will ignore the operators associated with the descendant excitations. All those descendant operators are generated by the elementary operators (associated with the elementary excitations) listed above.

We can also use the patch operators \(\tilde{X}_{\text{str}}\) on open strings to define the 1-symmetry (see Fig. 13):

\[ \tilde{X}_{\text{str}}O_i^{\text{symm}} = O_i^{\text{symm}}\tilde{X}_{\text{str}}, \] (112)

where \(i\) is far away from string ends. Using such patch symmetry operators, we can measure the \(Z^{(1)}_2\) 1-charge on the boundary of the disk operator \(\tilde{Z}_{\text{disk}}:\)

\[ \tilde{Z}_{\text{disk}}\tilde{X}_{\text{str}} = -\tilde{X}_{\text{str}}\tilde{Z}_{\text{disk}} \] (113)

when the string straddle across the boundary of the disk. We see that a \(Z^{(1)}_2\) 1-charge in 2-dimensional space is a 1-dimensional extended object. In general, an \(n\)-dimensional charge object correspond to \(n\)-symmetry, in any space dimensions.

We can use a fusion 2-category to describe the charges of the \(Z^{(1)}_2\) 1-symmetry, \(i.e.\) to described the operator algebra of the patch charge operators \(\tilde{Z}_{\text{disk}}\). The 1-dimensional (co-dimension-1) extended charge objects (the boundary of the disk operators \(\tilde{Z}_{\text{disk}}\)) are labeled by the group elements in \(G = Z_2\), and correspond to the objects in the fusion 2-category. Adding the trivial 1-morphisms and the top 2-morphisms formed by the local operators \(\prod_{i \in \text{small loop}} \tilde{Z}_i\) (\(i.e.\) the small disk operators), we get a fusion 2-category 2Vec\(_{Z_2}\). The fusion
2-category $\text{Vec}_{Z_2}$ fully describes the $Z_2$ 1-symmetry in 2d space. Such a fusion 2-category $\text{Vec}_{Z_2}$ is the representation category of the symmetry.

We can also use a fusion 2-category to describe the symmetry transformations of the $Z_2^{(1)}$ 1-symmetry, i.e. to describe the operator algebra the patch symmetry operators $\tilde{X}_{\text{str}}$. The boundary of the string operators $\tilde{X}_{\text{str}}$ are labeled by the representations in $G = Z_2$, and correspond to the 1-morphisms in the fusion 2-category. Adding the trivial objects and the top 2-morphisms formed by the local operators $\tilde{X}_i$ (i.e. the small string operators), we get a fusion 2-category $\text{Rep}_{Z_2}$. The fusion 2-category $\text{Rep}_{Z_2}$ fully describes the $Z_2$ 1-symmetry in 2d space. Such a fusion 2-category $\text{Rep}_{Z_2}$ is the transformation category of the $Z_2^{(1)}$ 1-symmetry in 2-dimensional space.

D. The equivalence between $Z_2$ 0-symmetry and $Z_2^{(1)}$ 1-symmetry in 2d space

We have seen that a $Z_2$ 0-symmetry can be fully described by a representations category $\text{Rep}_{Z_2}$ or by a transformation category $\text{Vec}_{Z_2}$. We also see that a $Z_2^{(1)}$ 1-symmetry can be fully described by a representations category $\text{Vec}_{Z_2}$ or by a transformation category $\text{Rep}_{Z_2}$. Now it is clear that the two very different looking symmetries, $Z_2$ and $Z_2^{(1)}$, are closely related, i.e. they become identical if we exchange what we call patch charge operators and what we call patch symmetry operators.

In fact, the two symmetries, $Z_2$ and $Z_2^{(1)}$, are indeed equivalent, if we consider the operator algebras of all local symmetric operators, i.e. the operator algebras generated by both patch charge operators and patch symmetry operators. The full operator algebra of $Z_2$ symmetry is defined via the following operator relations

$$Z_{str,i}Z_{str,j} = Z_{str,i+j}, \quad X_{\text{disk},i}X_{\text{disk},j} = X_{\text{disk},i+j}$$

$$Z_{str}X_{\text{disk}} = \pm X_{\text{disk}}Z_{str}, \quad (114)$$

The full operator algebra of $Z_2^{(1)}$ symmetry is defined via the following operators relations

$$\tilde{X}_{str,i}\tilde{X}_{str,j} = \tilde{X}_{str,i+j}, \quad \tilde{Z}_{\text{disk},i}\tilde{Z}_{\text{disk},j} = \tilde{Z}_{\text{disk},i+j}$$

$$\tilde{X}_{str}\tilde{Z}_{\text{disk}} = \pm \tilde{Z}_{\text{disk}}\tilde{X}_{str}, \quad (115)$$

We see that the two operator algebras are isomorphic. Thus the $Z_2$ and $Z_2^{(1)}$ symmetries have the same categorical symmetry, which implies that they are equivalent.

In fact, the categorical symmetry from the full operator algebra corresponds to a non-generate braided fusion 2-category $\text{2Gau}_{Z_2}$ (which describes the excitations in a 2d $Z_2$-gauge theory). The boundary of the disk operators are labeled by the group elements of $Z_2$, and correspond to the object in the braided fusion 2-category $\text{2Gau}_{Z_2}$. The ends of the string operators are labeled by the group representations, and correspond to the 1-morphisms in $\text{2Gau}_{Z_2}$. The local symmetric operators (i.e. the small string and small disk operators) correspond to the 2-morphisms in $\text{2Gau}_{Z_2}$. The string-like elementary excitations (the objects) and the point-like elementary excitations (the 1-morphisms) can fully detect each others, due to their non-trivial mutual statistics, as implied by the operator relation

$$Z_{\text{str}}X_{\text{disk}} = \pm X_{\text{disk}}Z_{\text{str}}. \quad (116)$$

Thus the braided fusion 2-category for the full operator algebra is non-degenerate. The “full” in operator algebra implies “non-degeneracy” in braided fusion category.

IX. 2d NON-ABELIAN SYMMETRY AND ITS DUAL

In the last section, we see that a 2d $Z_2$ 0-symmetry is equivalent to its dual, a 2d $Z_2^{(1)}$ 1-symmetry. The dual symmetry is obtained by exchanging patch charge operators and patch symmetry operators. We note that for a symmetry described by a non-Abelian finite group $G$, it also has patch charge operators and patch symmetry operators. Naturally, one may ask what is the dual of the $G$ symmetry? Are symmetry and dual symmetry equivalent? In this section, we will discuss the algebra of local symmetric operators for a non-Abelian symmetry, and the dual of a non-Abelian symmetry.

A. The $G$ 0-symmetry in 2d space

Let us consider a bosonic quantum system, whose degrees of freedoms live on the vertices and are labeled by a non-Abelian group $G$. In other words, the total Hilbert space is given by $\mathcal{V} = \bigotimes_i \mathcal{V}_i \ (\mathcal{V}_i = \text{span}\{|g_i\} \ g_i \in G\} )$.

The $G$ 0-symmetry is defined by the transformations on the whole 2d space

$$T_h = \prod_i T_i(h), \quad h \in G, \quad (117)$$

where $T_i(h)$ acts on $\mathcal{V}_i$:

$$T_i(h)|g_i\rangle = |hg_i\rangle. \quad (118)$$

The associated t-patch symmetry operator is given by

$$\hat{t}_{\text{disk}} = \prod_{i \in \text{disk}} T_i(h), \quad \tilde{t}_{\text{disk}} = \sum_{h \in \chi} \hat{t}_{\text{disk}}, \quad (119)$$

where $\chi$ is a conjugacy class of $G$. Note that here we need to sum over conjugacy class as required by the transparency condition (i.e. $T_{\text{disk}}(\chi)$ must carry vanishing total charge):

$$T_{\text{disk}}(\chi)T_{\text{disk}}(\chi') = T_{\text{disk}}(\chi')T_{\text{disk}}(\chi), \quad (120)$$

where the boundaries, $\partial_{\text{disk}}$ and $\partial_{\text{disk}}'$, are far away (i.e. do not intersect). Thus for non-Abelian symmetry,
the patch symmetry transformations are not labeled by group elements, but by conjugacy classes.

Local symmetric operators satisfy
\[
\hat{\chi}_{\text{disk}} O_i^{\text{symm}} = O_i^{\text{symm}} \hat{\chi}_{\text{disk}}, \quad \forall \chi,
\]
where \(i\) is far away from \(\partial\text{disk}\). The patch charge operators, with empty bulk, are given by
\[
\hat{R}_{\text{str}_ij} = \text{Tr}[R(\hat{g}_i) R(\hat{g}_j^{-1})], \quad \hat{g}_i | g_i \rangle = g_i | g_i \rangle,
\]
where \(R\) is an irreducible matrix representation of \(G\). The transparency condition requires us to take the trace:
\[
\hat{\chi}_{\text{disk}} \hat{R}_{\text{str}_ij} = \hat{R}_{\text{str}_ij} \hat{\chi}_{\text{disk}},
\]
where \(\text{str}_{ij}\) is far away from \(\partial\text{disk}\). But the one end of string operator carries a non-zero charge, which can be seen by trying to calculate the commutation between \(\hat{\chi}_{\text{disk}}\) and \(\hat{R}_{\text{str}_ij}\), with one end of string, \(i\) inside the disk and the other end of string, \(j\) outside the disk:
\[
\hat{\chi}_{\text{disk}} \hat{R}_{\text{str}_ij} - \hat{R}_{\text{str}_ij} \hat{\chi}_{\text{disk}} = \left( \sum_{h \in \chi} \prod_{i \in \text{disk}} T_i(h) \right) \text{Tr}(R(\hat{g}_i) R(\hat{g}_j^{-1}))
\]
}\[
= \left( \sum_{h \in \chi} \text{Tr}(R(h) R(\hat{g}_i) R(\hat{g}_j^{-1})) \right) \prod_{i \in \text{disk}} T_i(h)
\]
We see that commutator is complicated. In fact, they do not even form a proper commutator. The non-trivial relation indicates that the ends of string carries non-trivial charge. But for a non-Abelian group \(G\), the charge is not described by a simple phase factor.

The algebra of local symmetric operators is generated by the patch charge operators \(\hat{R}_{\text{str}_ij}\) and patch symmetry operators \(\hat{\chi}_{\text{disk}}\). After we sum over the conjugacy class, the symmetry transformations \(\hat{\chi}_{\text{disk}}\) are no longer invertible. They form a more general algebra
\[
\hat{\chi}_{1,\text{sphere}} \hat{\chi}_{2,\text{sphere}} = \sum_{\chi_3} M_{\chi_1, \chi_2}^{\chi_3} \hat{\chi}_{3,\text{sphere}}
\]
where \(M_{\chi_1, \chi_2}^{\chi_3}\) is the fusion coefficients of the conjugacy classes of \(G\):
\[
\chi_1 \otimes \chi_2 = \bigoplus_{\chi_3} M_{\chi_1, \chi_2}^{\chi_3} \chi_3
\]
\[
\chi = \bigoplus_{g \in \chi} g, \quad \chi = \text{a conjugacy class of } G,
\]
\[
g_1 \otimes g_2 = (g_1 g_2), \quad g_1, g_2 \in G.
\]
Here we consider symmetry transformations in a sphere to avoid the possible boundary terms. We see that, from the point of view of \(t\)-patch operators, a non-Abelian symmetry is already similar to a non-invertible symmetry.

This non-invertible feature is expected and reasonable. We know that the algebra generated by the patch charge operators \(\hat{R}_{\text{str}_ij}\) and patch symmetry operators \(\hat{\chi}_{\text{disk}}\) should correspond to a 3d topological order. For the present case, such a 3d topological order should be the one described by the \(G\)-gauge theory. The 1d boundary of the disk operator \(\hat{\chi}_{\text{disk}}\) corresponds to the flux loop in the \(G\)-gauge theory. When \(G\) is non-Abelian, the flux loop in \(G\)-gauge theory is not labeled by a group element \(g \in G\), but rather by a conjugacy class \(\chi\). The fusion of the flux loops is given by eqn. (126). This is why the patch symmetry operators are not invertible when \(G\) is non-Abelian.

The patch charge operator \(\hat{R}_{\text{str}_ij}\) corresponds to the charge excitations in the 3d \(G\)-gauge theory on \(S^3\), i.e., on two points with one carries charge \(\hat{R}\) and the other charge \(R\). Here \(R\) is a representation of \(G\) and \(\hat{R}\) is its charge conjugate. The fusion of the charges is given by the fusion of \(G\)-representations
\[
R_1 \otimes R_2 = \bigoplus_{R_3} N_{R_1, R_2}^{R_3} R_3.
\]

To measure the charge in the \(G\)-gauge theory, we need to braid a charge \(R\) around a flux \(\chi\). When \(G\) is non-Abelian, both the charge \(R\) and the flux \(\chi\) can be degenerate. The degeneracy of the charge \(R\) is \(\dim(R)\). The degeneracy of the flux \(\chi\) is the number of group elements in conjugacy class \(\chi, |\chi|\). With those degeneracies, the braiding of a charge around a flux loop is not simply a phase factor. This is why the commutation eqn. (124) is complicated.

The above correspondence suggests that the categorical symmetry of a 2d \(G\)-0-symmetry is a 3d topological order described by a \(G\)-gauge theory, which will be denoted as \(2\text{gaug}\). \(2\text{gaug}\) can also be viewed as a non-degenerate braided fusion 2-category describing the point-like excitations (the \(G\)-gauge charge) and string-like excitations (the \(G\)-flux) in the 3d \(G\)-gauge theory. Thus the categorical symmetry of a 2d \(G\)-0-symmetry is a non-degenerate braided fusion 2-category \(2\text{gaug}\).

We like to mention that the patch charge operators \(\hat{R}_{\text{str}}\) should generate a symmetric fusion 2-category \(2\text{Rep}_G\). The patch symmetry operators \(\hat{\chi}_{\text{disk}}\) should generate a braided fusion 2-category \(2\text{Vec}_G\). We like to remark that the simple objects in \(2\text{Vec}_G\) are labeled by the elements \(g\) of the group \(G\). The boundary of the patch symmetry operator \(\hat{\chi}_{\text{disk}}\) correspond to a composite object \(\chi = \bigoplus_{g \in \chi} g\), where \(\chi\) is a conjugacy class of \(G\). Since both 0-symmetry \(G\) and the algebraic 1-symmetry \(G_{\text{rep}}^{(1)}\) have the same categorical symmetry, they are equivalent symmetries. The class of quantum systems

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5 We may also say that the patch symmetry operators \(\hat{\chi}_{\text{disk}}\) generate a braided multi-fusion 2-category \(2\text{Ve}c\), which is the “symmetrized” \(2\text{Vec}_G\). The simple objects in \(2\text{Ve}c\) are given by \(\chi = \bigoplus_{g \in \chi} g\) in \(2\text{Vec}_G\). \(2\text{Ve}c\) is a multi-fusion category since \(\chi \otimes \check{\chi} = |\chi| 1 \oplus \cdots\). In this paper, we will promote the multi-fusion category to fusion category, for example, promote \(2\text{Ve}c^{\text{inv}}\) to \(2\text{Vec}_G\) by splitting \(\chi\) into \(g\).
with 0-symmetry $G$ and the class of quantum systems with with algebraic 1-symmetry $G_{\text{rep}}^{(1)}$ will have a 1-to-1 correspondence, so that the corresponding quantum systems have identical properties.

**B. The $G_{\text{rep}}^{(1)}$ 1-symmetry in 2d space**

The $Z_2^{(1)}$ 1-symmetry discussed before is described by a higher group. In this section, we are going to study a symmetry that is beyond higher group since the symmetry transformation is not invertible. Such a symmetry is called algebraic higher symmetry in Ref. 48.

Let us consider a bosonic quantum system, whose degrees of freedoms live on the links and are labeled by an non-Abelian group $G$. In other words, the total Hilbert space is given by $V = \bigotimes_{(ij)} V_{ij} \left( V_{ij} = \text{span}\{ |g_{ij} \rangle \mid g_{ij} \in G \} \right)$.

The symmetry is defined by the transformations on all the loops $S^1$:

$$W_R(S^1) = \text{Tr} \prod_{(ij) \in S^1} R(\hat{g}_{ij}), \quad \hat{g}_{ij}|g_{ij}\rangle = g_{ij}|g_{ij}\rangle, \quad (128)$$

for all matrix representation $R$ of $G$. Local symmetric operators satisfy

$$W_R(S^1)O_i^{\text{symm}} = O_i^{\text{symm}}W_R(S^1), \quad \forall \ S^1, R. \quad (129)$$

We will call such a symmetry as $G_{\text{rep}}^{(1)}$ 1-symmetry.

The algebra of local symmetric operators is generated by the following two kinds of operators:

$$\hat{R}_{\text{str},ij} = \text{Tr}(\hat{g}_{ik}R(\hat{g}_{kl})\cdots R(\hat{g}_{mj})), \quad \hat{\chi}_{\text{disk}} = \sum_{h \in \chi} \hat{h}_{\text{disk}}, \quad (130)$$

where $\chi$ is a conjugacy class of $G$, $\hat{h}_{\text{disk}} = \prod_{i \in \text{disk}} T_i(h)$, and the $T_i(h)$ operator (for $h \in G$) is defined as

$$T_i(h)|\cdots, g_{ki}, g_{ij} \cdots \rangle = |\cdots, g_{ki}h^{-1}, h_gg_{ij} \cdots \rangle. \quad (131)$$

One can check that the above patch operators are t-patch operators, satisfying the transparency condition eqn. (3). The trace in the definition of $\hat{R}_{\text{str},ij}$ and the sum over conjugacy class in the definition of $\hat{\chi}_{\text{disk}}$ are important to ensure the transparency property.

The symmetry transformations $W_R(S^1) = \text{Tr} \prod_{(ij) \in S^1} R(g_{ij})$ are not invertible. They form a more general algebra

$$W_R_1(S^1)W_R_2(S^1) = \sum_{R_3} N_{R_1,R_2}^{R_3} W_{R_3}(S^1), \quad (132)$$

where $N_{R_1,R_2}^{R_3}$ is the fusion coefficients of the irreducible representations, $R_1, R_2, R_3$, of $G$ (see eqn. (127)). Thus the symmetry generated by $W_R(S^1)$‘s is a new kind of symmetry.

We like to remark that non-invertible symmetry also exist in 1-dimensional space, which can be constructed in a very similar way. In 1d, the non-invertible symmetry is still described by the transformation $W_R(S^1) = \text{Tr} \prod_{(ij) \in S^1} R(g_{ij})$, which correspond to an non-invertible 0-symmetry denoted as $G_{\text{rep}}$. Those 1d beyond-group symmetries have been studied under the name (1) topological defect-lines/twisted-boundary-conditions in 1+1D (spacetime dimension) CFT [56, 77–79]; (2) fusion category symmetry [57, 80]; (3) quantum group symmetry [81]; etc.

Now let us go back to 2-dimensional space. We can use the $t$-patch operators $R_{\text{str}}$ on open strings to define the 1-symmetry, i.e. to select the local symmetric operators:

$$\hat{R}_{\text{str}} O_i^{\text{symm}} = O_i^{\text{symm}} \hat{R}_{\text{str}}, \quad i \text{ far away from string ends} \quad (133)$$

The patch charge operator $\hat{\chi}_{\text{disk}}$ carry vanishing 1-charge since

$$\hat{R}_{\text{str}} \hat{\chi}_{\text{disk}} = \hat{\chi}_{\text{disk}} \hat{R}_{\text{str}} \quad (134)$$

if the disk of $\hat{\chi}_{\text{disk}}$ is far away from the string ends of $\hat{R}_{\text{str}}$. However, a segment of the boundary of the disk operator $\hat{\chi}_{\text{disk}}$ can carry a non zero 1-charge. To measure such a 1-charge, we try to compute the commutator

$$\hat{\chi}_{\text{disk}} \hat{R}_{\text{str},ij} = \hat{\chi}_{\text{disk}} \hat{R}_{\text{str},ij} \quad (135)$$

assuming one end of string, $i$, is inside the disk and the other end of string, $j$, is outside the disk. We see that commutator is complicated. The non-trivial relation at least indicates that the boundary of the disk carries non-trivial 1-charge. But for a non-Abelian group $G$, the 1-charge is not described by a simple phase factor. This, in fact, is an expected result.

The above discussion suggests the algebra of local symmetric operator for 2d algebraic 1-symmetry $G_{\text{rep}}^{(1)}$ is isomorphic to the algebra of local symmetric operator from 2d 0-symmetry $G$. To see this more clearly, we remove the trace and the sum over conjugacy class in equations (124) and (135), and rewrite them as

$$\hat{h}_{\text{disk}} \hat{R}_{\text{str},ij} = \left( \prod_{i \in \text{disk}} T_i(h) \right) \left( R(g_{i})R(g_{j}^{-1}) \right)^{\alpha\beta} \quad (136)$$

$$= \left( \sum_{\gamma} R(h)^{\alpha\gamma} R(g_{j}) \right)^{\gamma\beta} \prod_{i \in \text{disk}} T_i(h) \quad (136)$$
and
\[
\hat{h}_{\text{disk}} R_{\text{str},i}^{\alpha \beta} = \left( \prod_{i \in \text{disk}} T_i(h) \right) (R(g_{ik}) R(g_{ki}) \cdots R(g_{mj}))^{\alpha \beta} \\
= \left( \sum_{\gamma} R(h)^{\alpha \gamma} (R(g_{ik}) R(g_{ki}) \cdots R(g_{mj}))^{\gamma \beta} \prod_{i \in \text{disk}} T_i(h) \right) \\
= \sum_{\gamma} R(h)^{\alpha \gamma} \hat{R}_{\text{str},i}^{\gamma \beta} \hat{h}_{\text{disk}}
\]
(137)

The above two equations have the same form, indicating that the two operator algebras are isomorphic. In this case, the 2d algebraic 1-symmetry \( G_{\text{rep}} \) also has the categorical symmetry \( 2\text{Rep}_G \), the 3d \( G \)-gauge theory. The only difference is that, for 2d algebraic 1-symmetry \( G_{\text{rep}} \), the patch symmetry operators generate a symmetric fusion 2-category \( 2\text{Rep}_G \), while the patch charge operators generate a braided fusion 2-category \( 2\text{Vec}_G \). So compare to 2d 0-symmetry \( G \), the patch symmetry operators and the patch charge operators are switched.

X. A REVIEW OF HOLOGRAPHIC THEORY OF (ALGEBRAIC HIGHER) SYMMETRY

In the previous sections, we studied many simple examples, trying to demonstrate a holographic theory of (algebraic higher) symmetry via algebras of local symmetric operator. In this section, we are going to present the holographic theory for generic cases. Such a holographic theory was developed in Ref. 48 via excitations above the symmetric ground state. Here we will present a simplified version, ignoring some subtleties.

A. Representation category

We know that symmetries are classified by groups and higher symmetries are classified by higher groups. As demonstrated in the last section, it turns out that algebraic higher symmetries (i.e. non-invertible symmetries) are described by fusion higher categories,\[48\] which is the representation category\[31\] generated by patch charge operators that we introduced in Section IV D.

However, not all fusion higher categories can be representation categories that describe algebraic higher symmetries. To identify which fusion higher category can describe a symmetry, we note that a symmetry is breakable. The symmetry breaking will change the fusion higher category into a trivial fusion higher category \( n\text{Vec} \). This motivate Ref. 48 to conjecture that local fusion higher categories \( R \) (i.e. representation categories generated by patch charge operators) describe and classify algebraic higher symmetries:

**Definition 10.** A fusion \( n \)-category \( R \) equipped with a top-faithful surjective monoidal functor \( \beta \) from \( R \) to the trivial fusion \( n \)-category, \( \beta_{\text{rep}} : n\text{Vec} \), is called a local fusion \( n \)-category. Here, top-faithful means that the functor \( \beta \) is injective when acting on the top morphisms (i.e. the \( n \)-morphism in this case). The pair \( (R, \beta) \) classify anomaly-free algebraic higher symmetries in \( n \)-dimensional space (which include anomaly-free symmetries, higher symmetries, and non-invertible symmetries).

To be brief, we usually drop \( \beta \) in the pair. This generalizes the discussion in Section IV D. Physically, the functor \( \beta \) means “ignore the symmetry” or “explicitly break the symmetry by small perturbations”. Thus at the top-morphism level, \( \beta \) maps local symmetric operators to local operators, which is a injective map. At lower-morphism/object level, the charged excitations in \( R \) are mapped to the excitations in \( n\text{Vec} \). This implies that all the objects and morphisms in a local fusion higher category \( R \) have integral quantum dimensions.

For example, if we have an \( SU(2) \) symmetry, then there is a “charged” excitation, spin-1/2 excitation (carrying the 2-dim representation of \( SU(2) \)). If we ignore the \( SU(2) \) symmetry, such a spin-1/2 excitation can be viewed as an accidental degeneracy of two trivial excitations:

\[
\underbrace{\text{spin-1/2}}_{\in R} \xrightarrow{\beta} \underbrace{1 \oplus 1}_{\in n\text{Vec}}
\]
(138)

As we have mentioned above, \( \beta \) is a symmetry breaking process. We can also view \( R \) as the fusion higher category describing the (extended) excitations in a symmetric product state with the symmetry. From this angle, we can view \( \beta \) as a domain wall between \( R \) and \( n\text{Vec} \). The domain wall is transparent to all the top morphisms in \( R \) (see Fig. 14).
The holographic principle of topological order: boundary $C^n$ uniquely determines bulk $M^{n+1}$.

Two symmetries described by fusion $n$-categories $\mathcal{R}$ and $\mathcal{R}'$ are equivalent (i.e., have the same categorical symmetry) if they have the same bulk topological order in one higher dimension: $Z(\mathcal{R}) \cong Z(\mathcal{R}')$.

### B. A holographic point view of symmetry

Consider two nd (algebraic higher) symmetries described by two local fusion $n$-categories, $\mathcal{R}$ and $\mathcal{R}'$. We know that the two symmetries are equivalent if their algebras of local symmetric operators are isomorphic. We have demonstrated that an isomorphic class of operator algebras is described by a braided fusion $n$-category, and called such an isomorphic class as a categorical symmetry. So what is the categorical symmetry for a symmetry described by local fusion higher categories, $\mathcal{R}$?

To answer this question, let us first review the holographic principle of topological order: boundary uniquely determines bulk. In physics, topological orders (i.e., gapped quantum liquids) in $n + 1$-dimensional space are characterized by their co-dimension-1, co-dimension-2, ... excitations. In other words, such a topological orders are characterized by fusion $n + 1$-category $M^{n+1}$.

On a $n$-dimensional gapped boundary of the $n + 1$-dimensional topological order, the excitations are described by a fusion $n$-category $C^n$. The holographic principle of topological order state that the boundary $C^n$ uniquely determines the bulk $M^{n+1}$. Such a boundary-bulk relation is given by the center map $Z$ in mathematics (see Fig. 15):[39, 40, 46]

$$Z(C^n) = M^{n+1}.$$ (139)

We see that the physical meaning of “center” is “bulk”. The center map (or the bulk map) $Z$ has a property that the center of a center (or the bulk of a bulk) is trivial

$$Z(Z(C^n)) = (n + 2)\text{Vec}.$$ (140)

This is dual to the well known fact: the boundary of a boundary is trivial.

It was conjectured that,[48] in $n$-dimensional space, the relation between the representation category $\mathcal{R}$ generated by all the patch charge operators and the braided fusion $n$-category $M$ (i.e., the categorical symmetry) generated by all the patch charge operators and the patch symmetry operators is given by another center map, denoted as $\Omega Z$,[39, 40, 46] that maps a fusion $n$-category $\mathcal{R}$ into a braided fusion $n$-category $M$. The new center map $\Omega Z$ is closely related to the previous center map $Z$ that maps a fusion $n$-category $\mathcal{R}$ into a braided fusion $n$-category $M$. This is because both fusion $(n + 1)$-category $M$ and braided fusion $n$-category $M$ can be used to fully describe an anomaly-free topological order in $n + 1$-dimensional space.

We note that in an anomaly-free topological order in $(n + 1)$-dimensional space, all the co-dimension-1 excitations are descendant (i.e., formed by lower dimension excitations). Dropping the co-dimension-1 excitations (called looping $\Omega$) maps a fusion $(n + 1)$-category $M$ into a braided fusion $n$-category $M$: $\mathcal{M} = \Omega M$. Adding back the descendant co-dimension-1 excitations is called de-looping: $\Sigma M = \mathcal{M}$.[39, 76] Thus the anomaly-free topological order can be described either by the braided fusion $n$-category $\mathcal{M}$, or by fusion $(n + 1)$-category $\mathcal{M}$. The anomaly-free condition of topological order corresponds to the non-degeneracy condition for the braided fusion $n$-category $\mathcal{M}$, which becomes the trivial center condition for the fusion $(n + 1)$-category $\mathcal{M}$: $Z(\mathcal{M}) = (n + 2)\text{Vec}$. The two kinds of center maps are related by

$$\Sigma Z = Z, \quad \Omega Z = Z.$$ (141)

This mathematical result provides a macroscopic way to compute the holographic equivalence classes of symmetries (i.e., the topological order in one higher dimension). In particular, the two symmetries, described by two representations categories $\mathcal{R}$ and $\mathcal{R}'$, are equivalent, if they have equivalent centers (i.e., have the same bulk topological order, or have the same categorical symmetry, see Fig. 16)[48]

$$Z(\mathcal{R}) \cong Z(\mathcal{R}').$$ (142)

Not every braided fusion higher category describes a categorical symmetry. The operator algebra is formed by all the local symmetric operators. The condition of all, is translated into a condition on the braided fusion higher category $\mathcal{M}$: $\mathcal{M}$ must be non-degenerate, i.e., satisfying $Z(\mathcal{M}) = (n + 1)\text{Vec}$. Therefore, categorical symmetries (i.e., the isomorphic classes of algebras of symmetric local operators) in $n$-dimensional space are classified by non-degenerate braided fusion $n$-categories $\mathcal{M}$.

### C. Transformation category – dual of the representation category

Instead of representation category generated by patch charge operators, we can also use transformation category generated by patch symmetry operators to fully describe an (algebraic higher) symmetry. We believe both characterizations are complete characterizations. This belief is supported by the following result [48]:
FIG. 17. $\mathcal{R} \boxtimes_{\mathcal{M}} \mathcal{R}^{\text{rev}}$ is a fusion $n$-category that describes the excitations in a slab of topological order in $(n+1)$-dimensional space. One boundary of the slab has excitations described by fusion $n$-category $\mathcal{R}$. The other boundary of the slab has excitations described by fusion $n$-category $\mathcal{R}$. The condition $\mathcal{R} \boxtimes_{\mathcal{M}} \mathcal{R}^{\text{rev}} = n\mathcal{Vec}$ ensure that all the excitations on the boundary $\mathcal{R}$ and $\mathcal{R}$ comes from symmetry described by the bulk $\mathcal{M}$. In other words, all the excitations on the boundary are symmetry charges. There is no topological excitations. FIG. 17, then both $\mathcal{R}$ and $\mathcal{R}$ are local fusion categories. Furthermore, for each $\mathcal{R}$, $\mathcal{R}$ is unique. We say that $\mathcal{R}$ is the dual of $\mathcal{R}$.

Proposition 2. Consider two fusion $n$-category $\mathcal{R}$ and $\mathcal{R}$, such that $\mathcal{M} = \mathcal{Z}(\mathcal{R}) = \mathcal{Z}(\mathcal{R})$. If $n\mathcal{Vec} = \mathcal{R} \boxtimes_{\mathcal{M}} \mathcal{R}^{\text{rev}}$ (see Fig. 17), then both $\mathcal{R}$ and $\mathcal{R}$ are local fusion $n$-categories. Furthermore, for each $\mathcal{R}$, $\mathcal{R}$ is unique. We say that $\mathcal{R}$ is the dual of $\mathcal{R}$.

For example, an $nd$ bosonic lattice model with a finite symmetry $G$ has a representations category $n\text{Rep}_G$ and a transformation category is $n\mathcal{Vec}_G$. $n\mathcal{Vec}_G$ happens to be the dual of $n\text{Rep}_G$. Such a bosonic model has a dual lattice model with a dual symmetry $G^{(n-1)}$ (see Ref. 48 for an explicit construction). The representations category of the dual symmetry $G^{(n-1)}$ is $n\mathcal{Vec}_G$, and the transformation category of the dual symmetry is $n\text{Rep}_G$. This example illustrates the dual relation between the representations category and the transformation category.

Putting the representations category and the transformation category together corresponds to putting the algebras of patch charge operators and patch symmetry operators together, which give us the full algebra of local symmetric operators. This gives us the categorical symmetry, which represents the essence of symmetry. Symmetry and dual symmetry have the same categorical symmetry and are equivalent. They only differ by swapping the names for patch charge operators and patch symmetry operators.

D. A simple example

In this subsection, we are going to discuss a simple example, to illustrate the above abstract discussions.

1. Holographic view of 2d $\mathbb{Z}_2$ 0-symmetry

As we have discussed in Section VIII A, the representation category of 2d $\mathbb{Z}_2$ 0-symmetry is a fusion 2-category $\mathcal{R} = 2\text{Rep}_{\mathbb{Z}_2}$. The transformation category of 2d $\mathbb{Z}_2$ 0-symmetry is a fusion 2-category $\mathcal{R} = 2\mathcal{Vec}_{\mathbb{Z}_2}$. It has the categorical symmetry $2\text{Aut}_{\mathbb{Z}_2} = \mathcal{Z}(2\text{Rep}_{\mathbb{Z}_2}) = \mathcal{Z}(2\mathcal{Vec}_{\mathbb{Z}_2})$, which is the 3d topological order described by $\mathbb{Z}_2$ gauge theory. In the following, we will use the holographic picture to understand the above results.

The elementary excitations in 3d $\mathbb{Z}_2$-gauge theory include point-like excitations $e$ (the bosonic $\mathbb{Z}_2$ charge) and string-like excitations $s$ (the bosonic $\mathbb{Z}_2$-flux string), as well as the trivial excitations $1$ and $1_{\text{str}}$. They satisfy the fusion rule:

$$e \otimes e = 1 \quad s \otimes s = 1_{\text{str}}$$

The string-like excitation $s$ corresponds to the flux line in the 3d $\mathbb{Z}_2$-gauge theory, which is an elementary excitation. The 3d $\mathbb{Z}_2$-gauge theory also has a non-elementary excitation (i.e. descendant) string-like excitation, $s_{\mathbb{Z}_2}$, which is a $\mathbb{Z}_2$ spontaneous-symmetry-break state formed by the $e$-particles. Here we ignore all the descendant excitations.

$\mathcal{R} = 2\text{Rep}_{\mathbb{Z}_2}$ is a boundary of $2\text{Aut}_{\mathbb{Z}_2}$, induced by the $\mathbb{Z}_2$-flux loop condensation, so on the boundary $s \sim 1_{\text{str}}$. The boundary excitations then are described by $\{1, e\}$ are $2\text{Rep}_{\mathbb{Z}_2}$. Fig. 18 represents the picture that a symmetry characterized by representation category $\mathcal{R} = 2\text{Rep}_{\mathbb{Z}_2}$ has the categorical symmetry $2\text{Aut}_{\mathbb{Z}_2}$. The Lagrangian condensible algebra is generated by $s$, which corresponds to the transformation category $\mathcal{R} = 2\mathcal{Vec}_{\mathbb{Z}_2}$. Thus Fig. 18 also represents the picture that a symmetry characterized by transformation category $\mathcal{R} = 2\mathcal{Vec}_{\mathbb{Z}_2}$ has the categorical symmetry $2\text{Aut}_{\mathbb{Z}_2}$.

2. Holographic view of 2d $\mathbb{Z}_2$ 1-symmetry

As we have discussed in Section VIII C, the representation category of 2d $\mathbb{Z}_2$ 0-symmetry is a fusion 2-category

FIG. 18. A boundary of 3d $\mathbb{Z}_2$ topological order $\mathcal{M} = 2\text{Aut}_{\mathbb{Z}_2}$ induced by $s$-string condensation. The boundary excitations is described by fusion 2-category $\mathcal{R} = 2\text{Rep}_{\mathbb{Z}_2}$.
XI. EQUIVALENT SYMMETRIES

One application of the holographic theory of symmetry is to identify equivalence between symmetries, higher symmetries, anomalous (higher) symmetries, algebraic (higher) symmetries, and gravitational anomalies. All those (anomalous and/or higher) symmetries and gravitational anomalies impose constraint on the low energy dynamics of the system. They are equivalent if they impose the identical constraint.

As we have discussed in this paper, two symmetries (described by representation categories $\mathcal{R}$ and $\mathcal{R}'$) are equivalent if they have the same categorical symmetry, i.e. have the same bulk topological order:

$$\mathcal{Z}(\mathcal{R}) \cong \mathcal{Z}(\mathcal{R}').$$

(144)

In practice, if we know a (higher) symmetry $\mathcal{R}$ is realized as a boundary of a SPT state or a symmetric product state, then the categorical symmetry is simply the bulk topological order obtained by gauging the (higher) symmetry in the bulk SPT state or the symmetric product state. We can identify many equivalent symmetries this way.

A. Some known examples

First, let us list some known examples. In $nd$ space, $Z_{N}^{(m)}$ $m$-symmetry can be realized by a boundary of $(n+1)d$ product state with $Z_{N}^{(m)}$ $m$-symmetry. Thus the categorical symmetry of nd $Z_{N}^{(m)}$ $m$-symmetry is the $(n+1)d$ $Z_{N}$ $(m+1)$-gauge theory. In $(n+1)$-dimensional space, $Z_{N}$ $(m+1)$-gauge theory and $Z_{N}$ $(n-m)$-gauge theory correspond to the same topological order. Therefore, in nd space, $Z_{N}^{(m)}$ $m$-symmetry is equivalent to $Z_{N}^{(n-m-1)} (n-m-1)$-symmetry:

$$Z_{N}^{(m)} \sim Z_{N}^{(n-m-1)}. \quad (145)$$

Furthermore, the two symmetries are dual to each other. Using the similar argument, we can obtain the following results

- In 2d, $Z_{3} \times Z_{2} \sim Z_{3}^{(1)} \times Z_{2}$. This is actually a direct application of eqn. (145).

- In 2d, $S_{3} = Z_{3} \times Z_{2} \sim Z_{3}^{(1)} \times Z_{2}$.[47] This is the twisted version of the above. $Z_{3}^{(1)} \times Z_{2}$ is a non-trivial mix of $Z_{3}$ 1-symmetry and $Z_{2}$ 0-symmetry. The charge objects of $Z_{3}^{(1)}$ are strings labeled by $s, \bar{s}$. The $Z_{2}$ 0-symmetry exchange $s$ and $\bar{s}$.

- In 1d, an anomalous $Z_{2} \times Z_{2} \times Z_{2}$ symmetry is equivalent to $D_{4}$ symmetry, for a very different reason than the above examples.[82, 83]
B. Equivalence between anomalous and anomaly-free $\mathbb{Z}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetries in 1-dimensional space

In Section VI C, we find an equivalence between 1d $\mathbb{Z}_4$ symmetry and $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with the mixed anomaly. In this section, we like to generalize that result. An 1d anomalous $\mathbb{Z}_n$ symmetry is realized by a boundary of 2d $\mathbb{Z}_n$ SPT state. After gauging the $\mathbb{Z}_n$ symmetry in the 2d SPT state, we obtain a 2d Abelian bosonic topological order, which is classified by even $K$-matrices.[68] In the present case, the corresponding topological order is given by

$$K = \begin{pmatrix} -2m & n & 0 \\ n_1 & 0 & 0 \\ -m_2 & 0 & -2m_1 & n_2 \\ 0 & 0 & 0 \\ 0 & n_2 & 0 \end{pmatrix}$$

(146)

where $m \in H^3(\mathbb{Z}_n; \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ charactering the $\mathbb{Z}_n$ anomaly ($m = 0$ for anomaly-free). We will label the anomalous $\mathbb{Z}_n$ symmetry by $(n; m)$.

Similarly, the anomalous 1d $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetry is realized by a boundary of 2d $\mathbb{Z}_n \times \mathbb{Z}_n$ SPT state. After gauging the $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetry, we obtain a 2d Abelian topological order characterized by

$$K = \begin{pmatrix} -2m_2 & n_1 & -m_{12} & 0 \\ n_1 & 0 & 0 & 0 \\ -m_{12} & 0 & -2m_1 & n_2 \\ 0 & 0 & 0 & n_2 \end{pmatrix}$$

(147)

where $m_1 \in \mathbb{Z}_{n_1}$ describing the anomaly of the $\mathbb{Z}_{n_1}$ symmetry, $m_2 \in \mathbb{Z}_{n_2}$ describing the anomaly of the $\mathbb{Z}_{n_2}$ symmetry, and $m_{12} \in \mathbb{Z}_{\gcd(n_1, n_2)}$ describing the mixed anomaly of the $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ symmetry. We will label the anomalous $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetry by $(n_1, n_2; m_1, m_{12}, m_2)$.

By computing the $S,T$ matrices of the 2d topological orders[44, 84] described by $K$-matrices, we can identify a set of $K$-matrices that give rise to the same 2d topological order, and hence correspond to equivalent symmetries. This allows us to find the following sets of equivalent symmetries:

- $(2,2;0,0,1)$, $(2,2;1,0,0)$, $(2,2;1,0,1)$
- $(4;0)$, $(2,2;0,1,0)$, $(2,2;0,1,1)$, $(2,2;1,1,0)$
- $(5;2)$, $(5;3)$
- $(5;1)$, $(5;4)$
- $(6;1)$, $(2,3;1,0,1)$
- $(6;5)$, $(2,3;1,0,2)$
- $(6;3)$, $(2,3;1,0,0)$
- $(6;4)$, $(2,3;0,0,1)$
- $(6;2)$, $(2,3;0,0,2)$
- $(6;0)$, $(2,3;0,0,0)$
- $(7;3)$, $(7;5)$, $(7;6)$
- $(7;1)$, $(7;2)$, $(7;4)$
- $(2,4;0,0,1)$, $(2,4;1,0,1)$
- $(2,4;0,0,3)$, $(2,4;1,0,3)$
- $(2,4;1,1,1)$, $(2,4;1,1,3)$
- $(8;0)$, $(2,4;0,1,0)$, $(2,4;0,1,2)$, $(2,4;1,1,0)$, $(2,4;1,1,2)$
- $(8;4)$, $(2,4;0,1,1)$, $(2,4;0,1,3)$
- $(3,3;0,0,1)$, $(3,3;1,0,0)$, $(3,3;1,1,1)$, $(3,3;1,2,1)$
- $(9;1)$, $(9;4)$, $(9;7)$
- $(9;2)$, $(9;5)$, $(9;8)$
- $(3,3;0,0,2)$, $(3,3;2,0,0)$, $(3,3;2,1,2)$, $(3,3;2,2,2)$
- $(9;0)$, $(3,3;0,1,0)$, $(3,3;0,2,0)$, $(3,3;0,1,1)$, $(3,3;0,2,1)$, $(3,3;0,1,2)$, $(3,3;0,2,2)$, $(3,3;1,1,0)$, $(3,3;1,2,0)$, $(3,3;1,0,2)$, $(3,3;2,1,0)$, $(3,3;2,2,0)$, $(3,3;2,0,1)$
- $(10;3)$, $(10;7)$
- $(10;1)$, $(10;9)$
- $(10;2)$, $(10;8)$
- $(10;4)$, $(10;6)$
- $(11;2)$, $(11;6)$, $(11;7)$, $(11;8)$, $(11;10)$
- $(11;1)$, $(11;3)$, $(11;4)$, $(11;5)$, $(11;9)$
- $(12;1)$, $(3,4;1,0,1)$
- $(12;7)$, $(3,4;1,0,3)$
- $(12;5)$, $(3,4;2,0,1)$
- $(12;11)$, $(3,4;2,0,3)$
- $(12;9)$, $(3,4;0,0,1)$
- $(12;3)$, $(3,4;0,0,3)$
- $(12;10)$, $(3,4;1,0,2)$
- $(12;4)$, $(3,4;1,0,0)$
- $(12;2)$, $(3,4;2,0,2)$
- $(12;8)$, $(3,4;2,0,0)$
- $(12;6)$, $(3,4;0,0,2)$
- $(12;0)$, $(3,4;0,0,0)$
- $(13;2)$, $(13;5)$, $(13;6)$, $(13;7)$, $(13;8)$, $(13;11)$
We see that the two symmetries (4; 0) and (2; 0, 1, 1) are equivalent. Thus the $\mathbb{Z}_4$ symmetry is also equivalent to $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with the mixed anomaly and an anomaly in one of the $\mathbb{Z}_2$ symmetry. More generally, it appears that $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetry with a particular mixed anomaly is equivalent to $\mathbb{Z}_{n^2}$ symmetry. It is also interesting to note that, for $\mathbb{Z}_p$ group ($2 < p = \text{prime}$), its $p - 1$ anomalous symmetries form just two equivalent classes, and its anomaly-free symmetry form its own equivalent class.

We also see that (4; 0) and (2; 0, 1, 1) are equivalent. Thus the $\mathbb{Z}_4$ symmetry is also equivalent to $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with the mixed anomaly and an anomaly in one of the $\mathbb{Z}_2$ symmetry. More generally, it appears that $\mathbb{Z}_n \times \mathbb{Z}_n$ symmetry with a particular mixed anomaly is equivalent to $\mathbb{Z}_{n^2}$ symmetry. It is also interesting to note that, for $\mathbb{Z}_p$ group ($2 < p = \text{prime}$), its $p - 1$ anomalous symmetries form just two equivalent classes, and its anomaly-free symmetry form its own equivalent class.

**XII. SUMMARY – THE ESSENCE OF A SYMMETRY**

With so many equivalences between symmetries labeled by (higher) groups and anomalies, it is clear that group, higher group, anomalies, local fusion higher categories, etc are not the best notions to describe a symmetry. From the very beginning, it is clear that symmetries, as well as (invertible and non-invertible) anomalies are actually described by algebra of local symmetric operators. In this paper, we show that an algebra of local symmetric operators is actually described by a non-degenerate braided fusion $n$-category, which happens to correspond to a topological order in one higher dimension. This leads to a holographic theory of symmetries and anomalies.

Symmetry (and anomaly) is actually the shadow of topological order in one higher dimension (see Fig. 21). The topological order in one higher dimension – the categorical symmetry – is the essence of symmetry.

We end the paper by listing different aspects of categorical symmetry:

A categorical symmetry is

- a symmetry plus its dual symmetry [47, 48].
- a non-invertible gravitational anomaly [37, 39–42, 40].
• a class of isomorphic algebras of local symmetric operators.
• a non-degenerate braided fusion higher category.
• a topological order in one higher dimension [47, 48].

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Appendix A: Local operator algebra and non-degenerate braided fusion 3-category – a 3-dimensional example without symmetry

Let us discuss an example to illustrate the definitions in Section III, for the case without any symmetry. We assume the space to be 3-dimensional. On each vertex-\(i\), we have two degrees of freedom labeled by elements in \(Z_2 \equiv \{+1, -1\}\), i.e. the local Hilbert space \(\mathcal{V}_i\) on a vertex 2-dimensional. The algebra of local operators is then generated by \(X_i, Z_i\) acting on \(\mathcal{V}_i\):

\[
\mathcal{A} = \{X_i, Z_i, X_iZ_i, X_iX_j, Z_iZ_j, \ldots \} \quad (A1)
\]

where \(i, j\) are near each other, and the Pauli-\(X, Z\) operators are defined by

\[
X| \pm 1\rangle = | \mp 1\rangle, \quad Z| \pm 1\rangle = \pm | \pm 1\rangle. \quad (A2)
\]

Our local operator algebra (after the closure by the extended operators) is generated by the following \(t\)-\textit{patch} operators:

1. 0-dimensional \(t\)-patch operators, \(X_i, Z_i\),

2. 1-dimensional \(t\)-patch operators – string operators,

\[
X_{\text{str}_{ij}} = X_iX_j, \quad Z_{\text{str}_{ij}} = Z_iZ_j, \quad (A3)
\]

where the string \(s_{ij}\) connects the vertex-\(i\) and vertex-\(j\). The string operators must have an empty bulk to commute with the 0-dimensional \(t\)-patch operators, when they are far away from the ends of the strings.

3. 2-dimensional \(t\)-patch operators – disk operators,

\[
X_{\text{disk}} = \prod_{i \in \partial \text{disk}} X_i, \quad Z_{\text{disk}} = \prod_{i \in \partial \text{disk}} Z_i,
\]

\[
O_{\text{disk}} = \prod_{i \in \partial \text{disk}} O_i, \quad (A4)
\]

where \(O_i\) can be any local operators.

4. 3-dimensional \(t\)-patch operators – ball operators,

\[
X_{\text{ball}} = \prod_{i \in \partial \text{ball}} X_i, \quad Z_{\text{ball}} = \prod_{i \in \partial \text{ball}} Z_i,
\]

\[
O_{\text{ball}} = \sum_{\{m_i\}} \Psi(\{m_i\}) \prod_{i \in \partial \text{ball}} O_i(m_i). \quad (A5)
\]

where \(O_i(m_i)\) can be any local operators. For example, \(O_i(0) = \text{id}\) and \(O_i(1) = X_i\). (More precisely, \(O_{\text{ball}}\) is a tensor network operator on the boundary of the ball, \(\partial \text{ball}\).)

We see that the \(t\)-patch operators all have empty bulk, \(i.e.\) are patch charge operators. There is no patch symmetry operators. This implies that our bosonic system has no symmetry.

If some \(t\)-patch operators have non-trivial bulk, then our system will have non-trivial symmetry, as we see in the examples in Section IV and beyond of the main text. In fact, the non-trivial bulk of the \(t\)-patch operators will generate the corresponding symmetries, higher symmetries, and/or non-invertible higher symmetries.

We believe that the above algebra of \(t\)-patch operators is closely related to a braided fusion 3-category \(3\text{Vec}\). At moment, we can only give a very rough description of this connection. A 3-category is formed by 0-morphisms (also called objects), 1-morphisms, 2-morphisms, and 3-morphisms (also called top morphisms). All those morphisms have relations between them. In fact, the collection of all relations between \(n\)-morphisms is the collection of all \((n + 1)\)-morphisms. The ball operators correspond to the objects, the disk operators the 1-morphisms, the string operators the 2-morphisms, and the local operators the top 3-morphisms. The difference of two ball operators are given by the disk operators, the difference of two disk operators are given by the string operators, etc.

For example, if two string operators \(O_{\text{str}_{ij}}\) and \(O'_{\text{str}_{ij}}\) are related by local operators \(O_i\) and \(O_j\):

\[
O'_{\text{str}_{ij}} = O_iO_jO_{\text{str}_{ij}}, \quad (A6)
\]

we say the 2-morphism \(O_{\text{str}_{ij}}\) connects to the 2-morphism \(O'_{\text{str}_{ij}}\) via the 3-morphism \(O_iO_j\) on the left:

\[
O_{\text{str}_{ij}} \xrightarrow{L: O_iO_j} O'_{\text{str}_{ij}}, \quad (A7)
\]

Similarly, if \(O_{\text{str}_{ij}}\) and \(O'_{\text{str}_{ij}}\) are related by local operators \(O_i\) and \(O_j\) on the right:

\[
O'_{\text{str}_{ij}} = O_{\text{str}_{ij}}O_iO_j, \quad (A8)
\]

we also say the 2-morphism \(O_{\text{str}_{ij}}\) connects to the 2-morphism \(O'_{\text{str}_{ij}}\) via the 3-morphism \(O_iO_j\):

\[
O_{\text{str}_{ij}} \xrightarrow{R: O_iO_j} O'_{\text{str}_{ij}}. \quad (A9)
\]

The 3-morphisms connecting 2-morphisms allow us to defined the notion of \textit{simple} 2-morphisms. A 2-morphism \(O_{\text{str}_{ij}}\) is simple if an existence of 3-morphism
Always indicates the existence of $3$-morphism $O'_{str,j}$ in the opposite direction. It turns out that $X_{str,j}$ and $Z_{str,j}$ introduced above are not simple. The following string operators are simple

$$P_{str,j}^\pm = P_i^\pm P_j^\pm, \quad P_i^\pm = \frac{1 \pm Z_i}{2}, \quad P_j^\pm = \frac{1 \pm Z_j}{2}. \tag{A10}$$

Certainly, the notion of simpleness applies to all morphisms.

If two 2-morphisms, $O_{str,j}$ and $O'_{str,j}$, satisfy

$$O_{str,j} \circ f = O_{str,j}', \quad O'_{str,j} \circ g = O_{str,j}', \quad O_{str,j} \circ f = O_{str,j}, \quad O'_{str,j} \circ g = O'_{str,j}, \tag{A11}$$

then we say the two 2-morphisms are isomorphic. In the above example, $O_{str,j} \circ f = O_{str,j}'$ if $O_i$ and $O_j$ are invertible, then the 2-morphism $O'_{str,j}$ connects to the 2-morphism $O_{str,j}$ via the 3-morphism $O_i^{-1}O_j^{-1}:

$$O_{str,j} = O_i^{-1}O_j^{-1}O_{str,j}',$$

or

$$O'_{str,j} = O_i^{-1}O_j^{-1}O_{str,j}'. \tag{A12}$$

In this case, the two 2-morphisms $O_{str,j}$ and $O'_{str,j}$ are isomorphic.

The isomorphic relations between two 2-morphisms is an equivalent relation. For example $P_{str,j}^{-\pm} \cong P_{str,j}^{\pm \pm}$. Although there are infinite many simple 2-morphisms in our example, there is only one type of simple 2-morphisms. A representative in this equivalence class is given by $P_{str,j}^\pm = P_i^\pm P_j^\pm$.

In this paper, when we refer to objects and morphisms, we usually refer to the equivalence classes of objects and morphisms, under the isomorphisms discussed above. Combining the definition of simpleness and isomorphism, we see that two simple morphisms cannot be connected by a higher morphism if they are not isomorphic. In other words, different types of morphisms (i.e., different equivalence classes of morphisms) cannot be connected by a higher morphism.

We like to stress that although the t-patch operator considered above all have an empty bulk, the tensor network operator on the boundary can have a Wess-Zumino form. For example, $O_{ball}$ is a tensor network operator on the boundary of the ball, but it can be defined i.e. defined by a tensor network on an extension of $\partial ball$ in one higher dimension. Such a tensor network can be viewed as a spacetime path integral on the ball, which can give rise to a topologically ordered state on $\partial ball$ described by wave function $\Psi$ (m_\{i\}). We see that we can have infinitely many types of ball operators, each type corresponds to a topological order in 2-dimensional space. Since there is no non-trivial topological order in 0- and 1-dimensional space, thus we have only one type of string-like t-patch operators and one type of membrane-like t-patch operators. Such a structure matches the structure of braided fusion 3-category 3Vec.$[48, 76]$

### Appendix B: Detailed calculations

#### 1. Calculation of $F(s, s, s)$

To compute the F-symbol $F(s, s, s)$, described in eqn. (42), we refer to Fig. 5 and substitute $a = b = c = s$. Using the definitions in eqn. (37) and eqn. (38), this picture translates to the following calculation:

$$
\begin{array}{c|c|c|c|c|c|c|c|c}
0 & 1 & 2 & 3 \\
\hline
\end{array}
\begin{array}{c}
O^\dagger & X & X & Z & X \\
\hline
- & - & - & - & -
\end{array}
\begin{array}{c}
O & O^\dagger & - & - & -
\end{array}
\begin{array}{c}
- & - & - & -
\end{array}
$$

This tells us that $F(s, s, s) = -1$. Note that our operator ordering convention is top-to-bottom and left-to-right (when in the same row).

#### 2. Self-statistics of $s$ particles

We express Fig 6 in equations as

$$
T_s(0 \rightarrow 1)T_s(1 \rightarrow 2)T_s(3 \rightarrow 1) = W_{\text{patch}_{01}}W_{\text{patch}_{12}}W_{\text{patch}_{13}}^\dagger = e^{i\theta_s}W_{\text{patch}_{13}}^\dagger W_{\text{patch}_{12}}W_{\text{patch}_{01}} = e^{i\theta_s}T_s(3 \rightarrow 1)T_s(1 \rightarrow 2)T_s(0 \rightarrow 1) \tag{B1}
$$
The l.h.s. can be simplified as

\[ \begin{array}{l}
0 & 1 & 2 & 3 \\
\hline
\end{array} \]

\[ O \uparrow \begin{array}{cccc}
X & X & ZX & X \\
\end{array} \]

\[ O \uparrow \begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array} \]

\[ O \uparrow \begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array} \]

while the r.h.s. can be simplified as

\[ \begin{array}{l}
0 & 1 & 2 & 3 \\
\hline
\end{array} \]

\[ O \uparrow \begin{array}{cccc}
X & X & ZX & X \\
\end{array} \]

\[ O \uparrow \begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array} \]

\[ O \uparrow \begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array} \]

3. Mutual and self statistics of \( m, \tilde{m}, s \) particles in \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) with mixed anomaly

First we calculate the mutual statistics of \( m \) and \( \tilde{m} \), as discussed in eqn. (52). Representing it pictorially, we find

\[ \begin{array}{l}
0 & 1 & 2 & 3 \\
\hline
\end{array} \]

\[ \tilde{O} \begin{array}{cccc}
X & X & X & X \end{array} \]

\[ \tilde{O} \begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array} \]

\[ \tilde{O} \begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array} \]

This proves eqn. (52). Now, recall that \( s \) is a bound state of \( m \) and \( \tilde{m} \). In other words,

\[ W^s_{\text{patch}_{ij}} \overset{\text{def}}{=} W_{\text{patch}_{ij}} \cdot \tilde{W}_{\text{patch}_{ij}} \tag{B2} \]

Then the self-statistics calculation shown in Fig 6 corresponds to the computation of the phase in the following sequence of operations:

\[ \begin{array}{l}
0 & 1 & 2 & 3 \\
\hline
\end{array} \]

\[ \tilde{O} \begin{array}{cccc}
X & X & X & X \end{array} \]

\[ \tilde{O} \begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array} \]

\[ \tilde{O} \begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array} \]

Comparing the two, we can see that the self-statistics phase \( e^{i\theta_s} \) equals 1, i.e. \( \theta_s = \pi/2 \). Thus, the \( s \) particles have semionic self-statistics.
From the above picture, it is clear that the computation of self-statistics of \( s \) particles is equivalent to the computation of mutual statistics of \( m \) and \( \bar{m} \) particles.

**Appendix C: Global action of 1+1D \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry with mixed anomaly**

Symmetry protected topological (SPT) states in \( d \) space dimensions are associated with anomalous symmetry actions on their \((d-1)\)-dimensional boundary. Such non-onsite action of the symmetry encodes a ‘t Hooft anomaly of the symmetry, when considered exclusively on the boundary. In Ref. 67 (Section 4), the authors wrote down an exactly soluble path integral model (also known as cocycle model[85]) to realize SPT states in general \( d \) space dimensions. These were then used to construct the corresponding anomalous symmetry action for the boundary effective theory. This framework then provides us with a recipe to write down a representative symmetry action for any anomalous symmetry in any number of dimensions. In particular, we can use this recipe to write down the anomalous (non-onsite) symmetry action for the 1+1D bosonic theory having a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry with a mixed anomaly. For this we must consider an SPT state in 2+1D that is protected by \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry.\(^6\) The path integral is defined on a 3-manifold \( M^3 \) with boundary \( M^2 = \partial M^3 \), and involves a 3-cocycle \( \nu_3 \). In Euclidean signature, the integrand of the path integral reads

\[
e^{-\int_{M^3} \mathcal{L}_{Bulk} d^3 x} = \prod_{(i,j)} e^{\pi i \nu_3(g_i, g_j, g_0 g^*)} \langle \{ g_i \} \rangle \tag{C1}
\]

where \((i,j)\) are nearest neighbors on the 1d spatial boundary, \(-g\) denotes the inverse of the group element \( g \in \mathbb{Z}_2 \times \mathbb{Z}_2 \), and \( g^* \) is an arbitrary reference group element, which can be taken to be the identity element of the symmetry group without any loss of generality. One choice of \( \nu_3 \) that encodes the mixed anomaly of two \( \mathbb{Z}_2 \) symmetries is

\[
\nu_3 = a_1 \sim a_2 \sim a_2 \tag{C3}
\]

with \( a = dg \) taking values on links, and the subscripts on \( a \) labeling the two \( \mathbb{Z}_2 \) groups. Using equations C2 and C3 allows us to write down the global symmetry generators in equations 46 and 47.

---

\( ^6 \) We use the additive presentation of the \( \mathbb{Z}_2 \) group in this appendix.
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