VARIATIONAL PRINCIPLES FOR THE TOPOLOGICAL PRESSURE OF MEASURABLE POTENTIALS

MARC RAUCH∗

Mathematical Institute, University of Jena
Ernst-Abbe-Platz 2, 07745 Jena, Germany

Abstract. We introduce notions of topological pressure for measurable potentials and prove corresponding variational principles. The formalism is then used to establish a Bowen formula for the Hausdorff dimension of cookie-cutters with discontinuous geometric potentials.

1. Introduction. Let $(X,d)$ be a compact metric space and $T : X \to X$ be a continuous transformation. Throughout this paper we consider $(X,T)$ to be a time-discrete dynamical system. An important notion in the field of dynamical systems and its associated thermodynamic formalism is the topological pressure. For a function $\varphi : X \to \mathbb{R}$, the topological pressure with respect to $(X,T)$ on a given subset $Z \subseteq X$ is defined to be

$$P_Z(T, \varphi) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{E} \exp \sum_{i=0}^{n-1} \varphi(T^i x),$$

where the supremum is taken over all $(\epsilon,n)$-separated sets $E$ in $Z$. Above definition was introduced and discussed for $Z = X$ and $\varphi \in C(X, \mathbb{R})$ in [18]. The variational principle was also proven there: One has

$$P_X(T, \varphi) = \sup \left\{ h_\mu(T) + \int_X \varphi \, d\mu \right\},$$

where the supremum is taken over all ergodic $T$-invariant Borel probability measures $\mu$ on $X$.

The aim of this paper is to extend the definition of pressure to not necessarily continuous functions $\varphi$, and to prove a corresponding variational principle. Up to now there seem to be at least two systematic attempts to treat this task. In [15] dynamical systems were considered, where the invariant set under study can be exhausted by an increasing sequence of subsets, such that $\varphi$ is continuous on the closure of each subset. Consequently, the topological pressure is then defined to be the supremum of the topological pressures of $\varphi$ on those sets. A corresponding variational principle holds under some integrability assumptions. A variational principle for sub-additive, upper semi-continuous sequences of functions was established in [4] and [2], as well as in [13] for $\mathbb{Z}_+^d$-actions. A generalization was recently given in [11].

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∗ Corresponding author: Marc Rauch.
for weighted topological pressure on systems with upper semi-continuous entropy mapping. We note, that in [15] and [11] Carathéodory dimension type definitions of pressure (as introduced and discussed in [17] and [16]) were used, whereas [4] and [13] extended the classical topological pressure, defined via separated sets.

In this paper we also stick to the original pressure definitions given in [18]. More precisely, we extend those definitions of pressure to discontinuous \( \varphi \), and compare them to the classical ones. Furthermore we determine several classes of functions, which admit variational inequalities and principles. Various examples are given. In particular we construct an example for a potential, which is not upper semi-continuous and for which the formalism of [15] cannot be applied, but which admits a variational principle for the pressure considered here.

As an application, we establish a Bowen formula for the Hausdorff dimension of attractors of cookie-cutters with discontinuous geometric potentials. This is done by connecting Hofbauer’s Bowen formula [12] to the pressure defined in [1] (see Theorem 7.4 and Remark 21 (a)). We then use this relation to show a continuity property of the Hausdorff dimensions of a sequence of cookie-cutters (see Theorem 8.2 and Remark 23).

2. Main results. Let \((X, T)\) be a dynamical system and \(\varphi : X \to \mathbb{R}\) be a measurable function. For every subset \(Z \subseteq X\), define \(P_Z(T, \varphi)\) as in (1).

**Theorem A** (Mass Distribution Principle, Theorem 5.2). Let \(Z \subseteq X\) be a Borel set. If \(\mu\) is a Borel probability measure on \(X\) satisfying \(\mu(Z) > 0\), one has

\[
P_Z(T, \varphi) \geq \inf_{z \in Z} P_{\mu}(T, \varphi, z),
\]

where

\[
P_{\mu}(T, \varphi, z) := \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \left( -\log \mu(B_{\delta_n}(x, \delta)) + \sum_{i=0}^{n-1} \varphi(T^ix) \right).
\]

Above result is well known for \(\varphi \in \mathcal{C}(X, \mathbb{R})\), and we show in the present paper, that the method of proof works well in the more general setting of measurable functions. Using Brin-Katok’s theorem (see [3]), we can derive the following variational inequality:

**Theorem B** (Variational Inequality, Theorem 5.3). Let \(h_{\text{top}}(T) < \infty\) and \(\mu\) be a \(T\)-invariant ergodic Borel probability measure. If \(\varphi : X \to \mathbb{R}\) is quasi-integrable with respect to \(\mu\), then there exists a Borel set \(G \subseteq X\) such that \(\mu(G) = 1\) and

\[
P_G(T, \varphi) \geq h_{\mu}(T) + \int_X \varphi \, d\mu. \tag{3}
\]

In particular, if \(\varphi : X \to \mathbb{R}\) is quasi-integrable with respect to \(T\) (see Definition 5.4), then

\[
P_X(T, \varphi) \geq \sup \left\{ h_{\mu}(T) + \int_X \varphi \, d\mu \right\}, \tag{4}
\]

where the supremum is taken over all \(T\)-invariant Borel probability measures \(\mu\) on \(X\).

Inequality (4) was already proven for upper semi-continuous functions in [4] and [13] (see Remark 15). In both proofs it is used, that \(\varphi\) is bounded from above.
Inequality (3) is well known for Carathéodory dimension types of topological pressure of continuous $\varphi$ (see [17]). Our formulations extend those results to potentials, which are quasi-integrable with respect to $T$-invariant measures on $(X,T)$.

Since in the proof of the upper bound of the classical variational principle only a weak version of upper semi-continuity is used, we are able to state the following generalized variational principle:

**Theorem C** (Variational Principle, Theorem 6.4). Let $h_{\text{top}}(T) < \infty$. If $\varphi : X \to \mathbb{R}$ is upper semi-continuous with respect to $T$ (see Definition 6.3), then the variational principle (2) holds for $\varphi$.

This extends the statements given in [4] and [13].

3. Preliminaries. We first recall and restate some basic notions.

**Definition 3.1.** Let $(X,d)$ be a non-empty, compact metric space and $T : X \to X$ be a continuous transformation. For each $n \in \mathbb{N}$ define for $x,y \in X$ the metric

$$d_n(x,y) := \max \left\{ d(T^ix,T^iy) : 0 \leq i < n \right\}.$$

Given some $\epsilon > 0$, a subset $\emptyset \neq E \subseteq Z \subseteq X$ is called $(\epsilon,n)$-separated in $Z$, if

$$\inf \{ d_n(x,y) : x \neq y \in E \} \geq \epsilon.$$

In addition, $E \subseteq Z$ is called maximal $(\epsilon,n)$-separated in $Z$, if for all $z \in Z$ the set $E \cup \{ z \}$ is not $(\epsilon,n)$-separated anymore. One can easily verify the following:

**Lemma 3.2.** Let $E \subseteq Z$ be $(\epsilon,n)$-separated. Then there exists some $E \subseteq E' \subseteq Z$ such that $E'$ is maximal $(\epsilon,n)$-separated. In particular, for every non-empty subset $Z \subseteq X$ there exists a maximal $(\epsilon,n)$-separated set $E \subseteq Z$.

**Definition 3.3.** Let $\emptyset \neq Z \subseteq X$, $\delta > 0$ and $n \in \mathbb{N}$. A finite subset $\emptyset \neq F \subseteq Z$ is called $(\delta,n)$-cover of $Z$, if

$$Z \subseteq \bigcup_{z \in F} B_{d_n}(z,\delta),$$

where $B_{\rho}(z,\epsilon) := \{ x \in X : \rho(x,z) < \epsilon \}$ denotes the open $\epsilon$-ball with respect to some given metric $\rho$ on $X$.

If $E$ is a maximal $(\epsilon,n)$-separated set for $Z$, then one has by definition $Z \subseteq \bigcup_{z \in E} B_{d_n}(z,\epsilon)$. Hence by Lemma 3.2 and compactness of $X$ we have the following:

**Lemma 3.4.** Every $(\epsilon,n)$-separated set in $X$ is finite, and there exists a $(\delta,n)$-cover for every $Z \neq \emptyset$.

**Definition 3.5.** Let $(\Omega,\mathcal{A})$ be a measurable space and $T : \Omega \to \Omega$ be measurable. The set of all $T$-invariant probability measures on $\Omega$ is denoted by $\mathcal{M}_T(\Omega)$, and the set of all $T$-invariant, ergodic ones is denoted by $\mathcal{E}_T(\Omega) \subseteq \mathcal{M}_T(\Omega)$. Recall for $\mu \in \mathcal{M}_T(\Omega)$ the quantity $h_{\mu}(T)$ to be the measure-theoretic entropy of $\mu$ with respect to $T$. For dynamical systems $(X,T)$, the quantity $h_{\text{top}}(T)$ denotes the topological entropy of $(X,T)$.

Usually ergodic theory is formulated for integrable functions. In this paper we are going to use the notion of quasi-integrable functions:
Definition 3.6. Let $(\Omega, A, \mu)$ be a probability space and $\varphi : \Omega \to \mathbb{R}$ be a measurable function. We call $\varphi$ to be quasi-integrable with respect to $\mu$, if either $\int_\Omega \varphi^+ \, d\mu < \infty$ or $\int_\Omega \varphi^- \, d\mu < \infty$. In case $\varphi$ is quasi-integrable, we define its integral to be

$$\int_\Omega \varphi \, d\mu := \int_\Omega \varphi^+ \, d\mu - \int_\Omega \varphi^- \, d\mu.$$ 

Quasi-integrable functions share some important properties with integrable functions. Some of them are recalled in the next two lemmas:

Lemma 3.7. Let $f : \Omega \to \mathbb{R}$ be a quasi-integrable function with respect to $\mu$. Then one has

a) $-f$ is quasi-integrable with respect to $\mu$ satisfying $\int_\Omega -f \, d\mu = -\int_\Omega f \, d\mu$;

b) If $(\Gamma, B, \nu)$ is another probability space such that $T : \Gamma \to \Omega$ is measurable and $\mu = \nu \circ T^{-1}$, one has

$$\int_\Omega f \, d\mu = \int_\Gamma f \circ T \, d\nu;
$$

c) If $g : \Omega \to \mathbb{R}$ is quasi-integrable with respect to $\mu$ such that $f \leq g$ $\mu$-almost everywhere, one has

$$\int_\Omega f \, d\mu \leq \int_\Omega g \, d\mu.$$

Lemma 3.8. Let $f, g : \Omega \to \mathbb{R}$ be quasi-integrable functions with respect to $\mu$, such that $\int_\Omega f \, d\mu + \int_\Omega g \, d\mu$ is well-defined. Then $f + g$ is well-defined $\mu$-almost everywhere and quasi-integrable with respect to $\mu$, and

$$\int_\Omega (f + g) \, d\mu = \int_\Omega f \, d\mu + \int_\Omega g \, d\mu.$$

The ergodic theorem of Birkhoff and the ergodic decomposition theorem can be restated for quasi-integrable functions. Assume $(X, T)$ to be a dynamical system for the rest of this section.

Theorem 3.9. Let $\mu \in \mathcal{M}_T(X)$ and $\varphi : X \to \mathbb{R}$ be quasi-integrable with respect to $\mu$. Then there exists some quasi-integrable function $\psi : X \to \mathbb{R}$ with respect to $\mu$, such that $\psi \circ T = \psi$ $\mu$-almost everywhere and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) = \psi(x)$$

$\mu$-almost everywhere. Furthermore one has

$$\int_X \psi \, d\mu = \int_X \varphi \, d\mu,$$

and in particular, if $\mu$ is ergodic, one has

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) = \int_X \varphi \, d\mu$$

for $\mu$-almost every $x \in X$. 
Theorem 3.10. Fix some \( \mu \in \mathcal{M}_T(X) \) and denote by \( m_\mu \) the ergodic decomposition of \( \mu \), that is \( \mu = \int_{\mathcal{E}_T(X)} \nu \, dm_\mu(\nu) \). If \( \varphi : X \to \mathbb{R} \) is quasi-integrable with respect to \( \mu \), then one has
\[
\int_X \varphi \, d\mu = \int_{\mathcal{E}_T(X)} \left( \int_X \varphi \, d\nu \right) \, dm_\mu(\nu).
\]
In particular, \( \mathcal{E}_T(X) \ni \nu \mapsto \int_X \varphi \, d\nu \) is a quasi-integrable with respect to \( m_\mu \).

4. Topological pressure for arbitrary potentials. In this section we introduce three notions of topological pressure for not necessarily continuous potentials. Let \( (X,T) \) be a dynamical system.

Definition 4.1. An function \( \varphi : X \to \mathbb{R} \) is called potential. For given potential \( \varphi \), \( \emptyset \neq Z \subseteq X \), \( \epsilon > 0 \) and \( n \in \mathbb{N} \) define
\[
M_Z(T,\varphi,\epsilon,n) := \sup_E \sum_{x \in E} \exp \left( n^{-1} \sum_{i=0}^{n-1} \varphi(T^i x) \right),
\]
where the supremum is taken over all \((\epsilon,n)\)-separated sets \( E \) in \( Z \). Likewise, define
\[
M_Z(T,\varphi,\delta,n) := \inf_F \sum_{x \in F} \exp \left( n^{-1} \sum_{i=0}^{n-1} \varphi(T^i x) \right),
\]
where the infimum is taken over all \((\delta,n)\)-covers \( F \) of \( Z \). Both definitions make sense: The set \( \{ z \} \) is \((\epsilon,n)\)-separated for all \( z \in Z \), \( \epsilon > 0 \) and \( n \in \mathbb{N} \), and by Lemma 3.4 there exists always a \((\delta,n)\)-cover for \( Z \). We set \( M_Z(T,\varphi,\epsilon,n) := 0 \).

Next, define
\[
P_Z(T,\varphi,\epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log M_Z(T,\varphi,\epsilon,n),
\]
\[
Q_Z(T,\varphi,\delta) := \limsup_{n \to \infty} \frac{1}{n} \log M_Z(T,\varphi,\delta,n),
\]
and
\[
P_Z(T,\varphi) := \lim_{\epsilon \to 0} P_Z(T,\varphi,\epsilon),
\]
\[
Q_Z(T,\varphi) := \lim_{\delta \to 0} Q_Z(T,\varphi,\delta).
\]
Both limits exist, as every \((\epsilon,n)\)-separating set is also \((\epsilon',n)\)-separating for \( 0 < \epsilon' < \epsilon \), and every \((\delta',n)\)-cover is a \((\delta,n)\)-cover for \( 0 < \delta' < \delta \). The quantity \( P_Z(T,\varphi) \) is called upper topological pressure of \( \varphi \) on \( Z \) with respect to \( T \), and \( Q_Z(T,\varphi) \) is called lower topological pressure of \( \varphi \) on \( Z \) with respect to \( T \).

Remark 1. In case \( \varphi : X \to \mathbb{R} \) is continuous, the definitions of \( Q_X(T,\varphi) \) and \( P_X(T,\varphi) \) coincide with the classical definitions given in [19]. In particular, one has then by [19] Theorem 9.1
\[
Q_X(T,\varphi) = P_X(T,\varphi).
\]
However, as we shall see in Remark 5 above equality does not hold in general for discontinuous \( \varphi \).

Remark 2. By definition, the quantity \( M_Z(T,\varphi,\epsilon,n) \) is always finite. In contrast, \( \overline{M}_Z(T,\varphi,\epsilon,n) \) may take values in \([0,\infty]\). We also have the estimate
\[
\overline{M}_Z(T,\varphi,\epsilon,n) \leq \sup_E \# E \cdot \exp \left( n \cdot \sup_{x \in X} \varphi(x) \right).
\]
So if \( \varphi \) is bounded from above, \( \overline{M}_Z(T, \varphi, \epsilon, n) < \infty \) follows from Remark \( 7.2 \) (5). We also note that \( Q_Z(T, \varphi, \delta), P_Z(T, \varphi, \epsilon) \in [\infty, +\infty] \) for all \( Z \subseteq X \) and \( Q_\varphi(T, \varphi) = P_\varphi(T, \varphi) = -\infty \). If \( \varphi \) is not bounded from above, but from below on one trajectory, we have the following:

**Proposition 1.** Suppose there is a \( 0 \leq C \in \mathbb{R} \) and an \( x_0 \in X \) such that \( \varphi(T^i x_0) \geq -C \) for all \( i \geq 0 \). If \( \sup_{i \geq 0} \varphi(T^i x_0) = \infty \), then one has \( P_X(T, \varphi, \epsilon) = \infty \) for every \( \epsilon > 0 \).

**Proof.** Choose for every \( n \geq 1 \) some \( k_n \geq 0 \) such that \( \varphi(T^{k_n} x_0) \geq 2^n + (n + 1) \cdot C \).

Set \( x_n := T^{k_n} x_0 \).

Thus, as \( \{x_n\} \) is a \((\epsilon, n)\)-separated set in \( X \) for every \( \epsilon > 0 \),

\[
\frac{1}{n} \log \overline{M}_X(T, \varphi, \epsilon, n) \geq \frac{1}{n} \log \exp \sum_{i=0}^{n-1} \varphi(T^i x_n) \geq \frac{2^n}{n} \to \infty
\]

as \( n \to \infty \). \( \square \)

The next lemma follows readily from the definitions of the pressure:

**Lemma 4.2.** Let \( Y \subseteq Z \subseteq X \), then one has \( P_Y(T, \varphi) \leq P_Z(T, \varphi) \). For every \( x \in X \) one has in addition

\[
Q_{\{x\}}(T, \varphi) = P_{\{x\}}(T, \varphi) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x).
\]

**Remark 3.** In general, \( Q_Y(T, \varphi) \leq Q_Z(T, \varphi) \) does not hold for \( Y \subseteq Z \). Consider for example \( \text{Id} \) to be the identity on \([0, 1]\), and \( \varphi(0) := 1, \varphi(x) := 0 \) for \( x \neq 0 \). Then \( 0 = Q_{[0,1]}(\text{Id}, \varphi) < Q_{\{0\}}(\text{Id}, \varphi) = 1 \).

**Definition 4.3.** Let \( \delta > 0, \alpha \in \mathbb{R} \) and \( n \in \mathbb{N} \). For \( \emptyset \neq Z \subseteq X \) we define

\[
M_Z(T, \varphi, \delta, \alpha, n) := \inf_F \sum_{x \in F} \exp \left( -\alpha \cdot n + \sum_{i=0}^{n-1} \varphi(T^i x) \right),
\]

where the infimum is taken over all \((\delta, n)\)-covers \( F \) of \( Z \). By Lemma 3.4 above quantity is well-defined and finite for all \( \emptyset \neq Z \), and we set

\[
M_\emptyset(T, \varphi, \delta, \alpha, n) := 0
\]

Next define for all \( Z \subseteq X \), \( \delta > 0 \) and \( \alpha \in \mathbb{R} \)

\[
M_Z(T, \varphi, \delta, \alpha) := \limsup_{n \to \infty} M_Z(T, \varphi, \delta, \alpha, n).
\]

Note that \( M_\emptyset(T, \varphi, \delta, \alpha) = 0 \) for all \( \alpha \in \mathbb{R} \).

**Lemma 4.4.** Let \( \emptyset \neq Z \subseteq X \). If there is an \( \alpha \in \mathbb{R} \) such that \( M_Z(T, \varphi, \delta, \alpha) < \infty \), then \( M_Z(T, \varphi, \delta, \beta) = 0 \) for all \( \beta > \alpha \).

**Proof.** Denote \( \varphi_n(x) := \sum_{i=0}^{n-1} \varphi(T^i x) \). Let \( C \in \mathbb{R} \) and \( F_n \) a \((\delta, n)\)-cover of \( Z \) such that

\[
\sum_{x \in F_n} \exp \left( -\alpha \cdot n + \varphi_n(x) \right) < M_Z(T, \varphi, \delta, \alpha) + 1 < C
\]
for all $n > N_0$, $N_0$ large enough. Then
\[
0 \leq M_Z(T, \varphi, \delta, \beta, n) \leq \sum_{x \in F_n} \exp\left(-\beta \cdot n + \varphi_n(x)\right)
= \sum_{x \in F_n} \frac{\exp\left(-\alpha \cdot n + \varphi_n(x)\right)}{\exp((\beta - \alpha) \cdot n)} \leq \frac{C}{\exp((\beta - \alpha) \cdot n)} \to 0
\]
as $n \to \infty$. Thus $M_Z(T, \varphi, \delta, \beta) = 0$. \hfill \Box

The last lemma justifies the next definition:

**Definition 4.5.** Let $Z \subseteq X$ and $\delta > 0$. Then
\[
CP_Z(T, \varphi, \delta) := \inf \{ \alpha \in \mathbb{R} : M_Z(T, \varphi, \delta, \alpha) = 0 \}
\]
is well-defined. As every $(\delta', n)$-cover is a $(\delta, n)$-cover for $0 < \delta' < \delta$, the limit
\[
CP_Z(T, \varphi) := \lim_{\delta \to 0} CP_Z(T, \varphi, \delta)
\]
exists and is called **topological capacity pressure of $\varphi$ on $Z$ with respect to $T$**.

**Remark 4.** Although the quantity $CP_Z(T, \varphi)$ is called capacity pressure and its definition looks like a lower Carathéodory capacity (see \[16\] for a detailed introduction and discussion of this subject), it is important to emphasize, that it is not a proper Carathéodory construction for general $\varphi$. In particular, one cannot hope monotonicity of $Z \mapsto CP_Z(T, \varphi)$, if $\varphi$ is discontinuous. However, the next theorem shows that $CP_Z(T, \varphi)$ recovers $Q_Z(T, \varphi)$, and gives a monotonicity relation between the pressures defined so far. This is of importance in the proof of Theorem 5.2.

**Theorem 4.6.** One has $CP_Z(T, \varphi) = Q_Z(T, \varphi) \leq P_Z(T, \varphi)$ for all $Z \subseteq X$.

**Proof.** We may assume $CP_Z(T, \varphi) > -\infty$, which implies $Z \neq \emptyset$. Denote $\varphi_n(x) := \sum_{i=0}^{n-1} \varphi(T^i x)$. Fix some $\delta_0 > 0$ such that $CP_Z(T, \varphi, \delta) > -\infty$ for all $0 < \delta < \delta_0$. Fix furthermore an $-\infty < \alpha < CP_Z(T, \varphi, \delta)$. Then there exists some sequence $\{n_l\}_{l \in \mathbb{N}}$, which depends on $\delta$ and $\alpha$, such that
\[
\infty = M_Z(T, \varphi, \delta, \alpha) = \lim_{l \to \infty} \inf_{F} \sum_{x \in F} \left( \exp \varphi_{n_l}(x) \cdot \exp(-\alpha \cdot n_l) \right),
\]
where the infimum is taken over all $(\delta, n_l)$-covers $F$ of $Z$. Hence there has to be a $l_0 \in \mathbb{N}$ large enough, such that
\[
\inf_{F} \sum_{x \in F} \exp \varphi_{n_l}(x) = \frac{M_Z(T, \varphi, \delta, n_l)}{\exp(\alpha \cdot n_l)} \geq \exp(\alpha \cdot n_l)
\]
and
\[
\alpha \leq \frac{1}{n_l} \log \frac{M_Z(T, \varphi, \delta, n_l)}{\exp(\alpha \cdot n_l)} \leq \limsup_{n \to \infty} \frac{1}{n} \log \frac{M_Z(T, \varphi, \delta, n)}{\exp(\alpha \cdot n)} + \delta
\]
for all $l > l_0$. As $\alpha < CP_Z(T, \varphi, \delta)$ was arbitrarily chosen, letting $\alpha \to CP_Z(T, \varphi, \delta)$ yields
\[
CP_Z(T, \varphi, \delta) \leq \limsup_{n \to \infty} \frac{1}{n} \log M_Z(T, \varphi, \delta, n) + \delta = Q_Z(T, \varphi, \delta) + \delta.
\]
Repeating above argument for $CP_Z(T, \varphi, \delta) < \infty$, one obtains in addition
\[
Q_Z(T, \varphi, \delta) \leq CP_Z(T, \varphi, \delta)
\]
for all \( \delta > 0 \). Next we pick by \( Z \neq \emptyset \) and Lemma 3.2 some maximal \((\delta, n)\)-separated set \( E_n \subseteq Z \). That means \( Z \subseteq \bigcup_{x \in E_n} B_{d_n}(x, \delta) \), hence
\[
M_Z(T, \varphi, \delta, n) \leq \sum_{x \in E_n} \exp \varphi_n(x) \leq M_Z(T, \varphi, \delta, n)
\]
for all \( n \in \mathbb{N} \). Thus, for all \( \delta < \delta_0 \),
\[
Q_Z(T, \varphi, \delta) \leq CP_Z(T, \varphi, \delta) \leq Q_Z(T, \varphi, \delta) + \delta \leq P_Z(T, \varphi, \delta) + \delta,
\]
and letting \( \delta \to 0 \) we finally obtain
\[
CP_Z(T, \varphi) = Q_Z(T, \varphi) \leq P_Z(T, \varphi).
\]

Remark 5. In general, it can happen that \( Q_X(T, \varphi) < P_X(T, \varphi) \).
To see this, recall the example of Remark 3. It yields
\[
Q_{[0,1]}(\text{Id}, \varphi) < Q_{\{0\}}(\text{Id}, \varphi) \leq P_{\{0\}}(\text{Id}, \varphi) \leq P_{[0,1]}(\text{Id}, \varphi).
\]
Other examples for differing lower and upper pressures were given in [7].

5. Mass distribution principle and variational pressure. Let \((X, T)\) be a dynamical system and \( \varphi : X \to \mathbb{R} \) be a potential. We introduce various measure-theoretic notions of pressure for discontinuous potentials. The main goal of this section is then to establish analogs of the classical mass distribution principle (see [8]) for those pressures.

Definition 5.1. Given some Borel probability measure \( \mu \) on \( X \), define for \( x \in X \) and \( \delta > 0 \)
\[
P_{\mu}(T, \varphi, x, \delta) := \liminf_{n \to \infty} \frac{1}{n} \left( -\log \mu(B_{d_n}(x, \delta)) + \sum_{i=0}^{n-1} \varphi(T^i x) \right).
\]
As \( \mu(B_{d_n}(x, \delta')) \leq \mu(B_{d_n}(x, \delta)) \) for \( 0 < \delta' < \delta \), the limit
\[
P_{\mu}(T, \varphi, x) = \lim_{\delta \to 0} P_{\mu}(T, \varphi, x, \delta)
\]
extists. It is called measure-theoretic pressure of \( \varphi \) on \( x \) with respect to \( \mu \) and \( T \).

In case \( \varphi : X \to \mathbb{R} \) is measurable and bounded from below, the function
\[
x \mapsto \frac{1}{n} \left( -\log \mu(B_{d_n}(x, \delta)) + \sum_{i=0}^{n-1} \varphi(T^i x) \right)
\]
is also measurable and bounded from below. Thus \( x \mapsto P_{\mu}(T, \varphi, x, \delta) \) is quasi-integrable with respect to \( \mu \) for every \( \delta > 0 \). Denote by \( \mathcal{M}(X) \) the set of all Borel probability measures on \( X \), and by \( \mathcal{B}(X) \) the set of all measurable functions \( \varphi : X \to \mathbb{R} \) bounded from below. For \( \mu \in \mathcal{M}(X) \) and \( \varphi \in \mathcal{B}(X) \) the quantity
\[
P_{\mu}(T, \varphi) := \lim_{\delta \to 0} \int_X P_{\mu}(T, \varphi, x, \delta) \, d\mu(x)
\]
is called mean measure-theoretic pressure of \( \varphi \) with respect to \( T \) and \( \mu \).
Immediately by monotone convergence
\[
P_{\mu}(T, \varphi) = \int_X P_{\mu}(T, \varphi, x) \, d\mu(x)
\]
follows.

We state now three versions of the so-called mass distribution principle, beginning with the most important:

**Theorem 5.2.** Let \( \mu \in \mathcal{M}(X) \). Suppose \( Z \) to be a Borel set satisfying \( \mu(Z) > 0 \). Then one has

\[
P_Z(T, \varphi) \geq \inf_{z \in Z} P_{\mu}(T, \varphi, z)
\]

for every measurable \( \varphi : X \to \mathbb{R} \).

*Proof.* Denote \( \varphi_n(x) := \sum_{i=0}^{n-1} \varphi(T^i x) \) and \( L := \inf_{z \in Z} P_{\mu}(T, \varphi, z) \). In case \( L = -\infty \) nothing is to be shown. In case \( -\infty < L < +\infty \) fix \( \epsilon > 0 \). Define for \( \delta > 0 \) and \( N \in \mathbb{N} \)

\[
Z_{\delta,N} := \{ z \in Z : \mu(B_{d_e}(z, \delta)) \leq \exp\left( -n \cdot (L - \epsilon) + \varphi_n(z) \right) \text{ for all } n \geq N \}.
\]

Assume that \( \mu(Z_{\delta,N}) = 0 \) for all \( \delta > 0 \), \( N \in \mathbb{N} \). By definition of \( L \), for every \( z \in Z \) there exists a \( 0 < \delta_z^* \) and an \( N_z^* \in \mathbb{N} \) such that

\[
L - \epsilon \leq \frac{1}{n}\left( -\log \mu(B_{d_e}(z, \delta)) + \varphi_n(z) \right)
\]

for all \( 0 < \delta < \delta_z^* \) and \( n \geq N_z^* \). This shows \( Z = \bigcup_{n \geq 1} \bigcup_{N \geq 1} Z_{1,n,N} \) and \( \mu(Z) = 0 \), which is a contradiction. Hence we can choose \( \delta^* > 0 \), \( N^* \in \mathbb{N} \) such that \( \mu(Z_{\delta^*,N^*}) > 0 \). Next let \( F \) be a \((\delta,n)\)-cover of \( Z_{\delta^*,N^*} \), for \( 0 < \delta < \delta^* \), \( n \geq N^* \). We obtain then by \( F \subseteq Z_{\delta^*,N^*} \subseteq Z_{\delta,N} \) the estimate

\[
\sum_{z \in F} \exp\left( -n \cdot (L - \epsilon) + \varphi_n(z) \right) \geq \sum_{z \in F} \mu(B_{d_e}(z, \delta)) \geq \mu(Z_{\delta^*,N^*}) > 0.
\]

As the cover \( F \) was arbitrarily chosen, letting \( n \to \infty \) results in

\[
M_{Z_{\delta,N}}(T, \varphi, \delta, L - \epsilon) \geq \mu(Z_{\delta^*,N^*}) > 0.
\]

Hence

\[
CP_{Z_{\delta,N}}(T, \varphi, \delta) \geq L - \epsilon
\]

for all \( 0 < \delta < \delta^* \). Letting \( \delta \to 0 \) and using Theorem 4.6 yields

\[
P_Z(T, \varphi) \geq P_{Z_{\delta^*,N^*}}(T, \varphi) \geq CP_{Z_{\delta^*,N^*}}(T, \varphi) \geq L - \epsilon.
\]

Now letting \( \epsilon \to 0 \) gives us \( P_Z(T, \varphi) \geq L \). The case \( L = \infty \) can be proven in the same way as above by considering the sets

\[
Z_{\delta,N}^C := \{ z \in Z : \mu(B_{d_e}(z, \delta)) \leq \exp\left( -n \cdot C + \varphi_n(z) \right) \text{ for all } n \geq N \}
\]

for \( C \to \infty \). \( \square \)

If we assume \( \varphi \) to be measurable and bounded from below, we immediately obtain the second version of the mass distribution principle:

**Corollary 1.** Let \( \mu \in \mathcal{M}(X) \) and \( \varphi \in \mathcal{B}(X) \). Suppose \( Z \) to be a Borel set satisfying \( \mu(Z) = 1 \). Then one has

\[
P_Z(T, \varphi) \geq P_{\mu}(T, \varphi).
\]
Proof. The proof works in a similar way like the proof of Theorem 1.2 (i) in [10]. Assume $-\infty < P_\mu(T, \varphi) < \infty$ and fix $\epsilon > 0$. Clearly

$$Z_\epsilon := \{ z \in Z : P_\mu(T, \varphi, z) \geq P_\mu(T, \varphi) - \epsilon \}$$

is a Borel set such that $\mu(Z_\epsilon) > 0$. Hence by Theorem 5.2

$$P_Z(T, \varphi) \geq P_{Z_\epsilon}(T, \varphi) \geq \inf_{z \in Z_\epsilon} P_\mu(T, \varphi, z) \geq P_\mu(T, \varphi) - \epsilon.$$ 

Taking the limit $\epsilon \to 0$ yields $P_Z(T, \varphi) \geq P_\mu(T, \varphi)$. The case $P_\mu(T, \varphi) = \infty$ works in same way. \hfill \Box

Remark 6. In the proof of Theorem 5.2, $\varphi(x) \in \mathbb{R}$ for all $x \in X$ was the only property we used, whereas Corollary 1 needed more assumptions. Actually, Theorem 5.2 gives us the third important version of the mass distribution principle, if we assume finite topological entropy and quasi-integrable $\varphi$:

Theorem 5.3. Let $h_{\text{top}}(T) < \infty$ and $\mu \in \mathcal{E}_T(X)$. If $\varphi : X \to \mathbb{R}$ is quasi-integrable with respect to $\mu$, then there exists a Borel set $G \subseteq X$ such that $\mu(G) = 1$ and

$$P_G(T, \varphi) \geq h_\mu(T) + \int_X \varphi \, d\mu.$$ 

In particular one has $P_X(T, \varphi) \geq h_\mu(T) + \int_X \varphi \, d\mu$.

Proof. Note that by $h_{\text{top}}(T) < \infty$ also $h_\mu(T) < \infty$ follows, thus the term $h_\mu(T) + \int_X \varphi \, d\mu$ is well-defined. By Theorem 3.9 there exists some Borel set $G_1 \subseteq X$ such that $\mu(G_1) = 1$ and

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) = \int_X \varphi \, d\mu$$

for every $x \in G$. By Brin-Katok’s theorem [3] there exists another Borel set $G_2 \subseteq X$ such that $\mu(G_2) = 1$ and

$$\lim_{\delta \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu(B_{d_n}(x, \delta)) = h_\mu(T).$$

Combining both yields

$$P_\mu(T, \varphi, x) = h_\mu(T) + \int_X \varphi \, d\mu \quad (5)$$

for all $x \in G := G_1 \cap G_2$. Here we used $\lim \inf_{n \to \infty} (a_n + b_n) = \lim \inf_{n \to \infty} a_n + \lim \inf_{n \to \infty} b_n$, if $b_n$ converges in $\mathbb{R}$ and the sum of both limits is well-defined. Hence applying Lemma 4.2 and Theorem 5.2 we obtain $P_X(T, \varphi) \geq P_G(T, \varphi) \geq h_\mu(T) + \int_X \varphi \, d\mu$. \hfill \Box

Remark 7. Note that the statement of Theorem 5.3 also holds, if $h_{\text{top}}(T) = \infty$ and $\int_X \varphi \, d\mu > -\infty$.

Corollary 2. Assume $\varphi : X \to \mathbb{R}$ to be measurable such that for every $N > 0$ there exists a $\mu_N \in \mathcal{M}_T(X)$ with the properties:

(i) $\varphi$ is quasi-integrable with respect to $\mu_N$ for all $N > 0$;
(ii) $\int_X \varphi \, d\mu_N > N$ for all $N > 0$.

Then one has $P_X(T, \varphi) = \infty$. In particular above statement holds, if there is a $\mu_\infty \in \mathcal{M}_T(X)$ satisfying $\int_X \varphi \, d\mu_\infty = \infty$. 

Proof. By Theorem 3.10 we have
\[ N < \int_X \varphi \, d\mu_N = \int_{E_T(X)} \left( \int_X \varphi \, d\nu \right) \, d\mu_N(\nu), \]
where \( \mu_N \) denotes the ergodic decomposition of \( \mu_N \). Thus there must exist some \( \nu_N \in E_T(X) \) such that \( \int_X \varphi \, d\nu_N \geq N \). That means by Remark 7 \( P_X(T, \varphi) \geq N \to \infty \) as \( N \to \infty \). 

We now assume that \( \varphi : X \to \mathbb{R} \) is quasi-integrable for all invariant measures, which allows us to introduce the variational pressure.

**Definition 5.4.** The set of all measurable \( \varphi : X \to \mathbb{R} \), which are quasi-integrable for all \( \mu \in \mathcal{M}_T(X) \), is denoted by \( Q_T(X) \). We call \( \varphi \in Q_T(X) \) quasi-integrable with respect to \( T \). Note that \( B(X) \subseteq Q_T(X) \). If the expression \( h_\mu(T) + \int_X \varphi \, d\mu \) is well-defined for all \( \mu \in \mathcal{M}_T(X) \), we can introduce the quantity
\[
S(T, \varphi) := \sup \left\{ h_\mu(T) + \int_X \varphi \, d\mu : \mu \in \mathcal{M}_T(X) \right\},
\]
which is called variational pressure of \( \varphi \) with respect to \( T \). In particular \( S(T, \varphi) \) is well-defined for all \( \varphi \in Q_T(X) \) in case of finite topological entropy.

As \( \varphi \) is a real-valued function, one sees immediately the following:

**Proposition 2.** Let \( \varphi \in Q_T(X) \) and \( h_{top}(T) < \infty \). If there exists a periodic orbit with respect to \( T \), then \( S(T, \varphi) > -\infty \).

**Remark 8.** The variational pressure can be \( \pm \infty \). Consider for example the unit circle \( S^1 \) with irrational rotation \( R \) on \( S^1 \). The unique \( R \)-invariant measure on \( S^1 \) is the normalized Hausdorff measure \( \mathcal{H}^1 \), which satisfies \( h_{\mathcal{H}^1}(R) = 0 \). One can choose now a Borel measurable partition \( \{A_n\}_{n=1}^\infty \) of \( X \), such that \( \mathcal{H}^1(A_n) = 1/2^n \). Define \( \psi(x) = -2^n/x \) if \( x \in A_n \). Then \( \psi \in Q_R(S^1) \) and
\[
S(R, \varphi) = \int_{S^1} \psi \, d\mathcal{H}^1 = -\sum_{n=1}^\infty \frac{1}{n} = -\infty.
\]

It turns out that by ergodic decomposition, the variational pressure can be computed as the supremum over all ergodic measures (as in the classical case):

**Lemma 5.5.** Let \( h_{top}(T) < \infty \) and \( \varphi \in Q_T(X) \). Then one has
\[
S(T, \varphi) = \sup \left\{ h_\mu(T) + \int_X \varphi \, d\mu : \mu \in \mathcal{E}_T(X) \right\}.
\]

**Proof.** Denote \( s := \sup \left\{ h_\mu(T) + \int_X \varphi \, d\mu : \mu \in \mathcal{E}_T(X) \right\} \). We may assume \( S(T, \varphi) > -\infty \), as \( S(T, \varphi) \geq s \), let \( \mu \in \mathcal{M}_T(X) \) and \( m_\mu \) be its ergodic decomposition. As \( \mathcal{E}_T(X) \ni \nu \mapsto h_{\nu}(T) \geq 0 \) and \( \mathcal{E}_T(X) \ni \nu \mapsto \int_X \varphi \, d\nu \) are quasi-integrable functions with respect to \( m_\mu \), one has by Lemma 3.8 and Theorem 3.10
\[
h_\mu(T) + \int_X \varphi \, d\mu = \int_{E_T(X)} \left( h_{\nu}(T) + \int_X \varphi \, d\nu \right) \, d\mu_N(\nu). \quad (6)
\]
Next assume \( \mu_n \in \mathcal{M}_T(X) \) to be an sequence such that \( \lim_{n \to \infty} (h_{\mu_n}(T) + \int_X \varphi \, d\mu_n) = S(T, \varphi) \). In case \( S(T, \varphi) < \infty \) we can choose by \( (6) \) some ergodic measure \( \nu_n \in \mathcal{E}_T(X) \) such that
\[
h_{\mu_n}(T) + \int_X \varphi \, d\mu_n - \frac{1}{n} \leq h_{\nu_n}(T) + \int_X \varphi \, d\nu_n,
\]
Remark 10. For Theorem 6.1 it is not enough to suppose the condition which shows by the statement. In case $S(T, \varphi) = \infty$ we can choose in a similar way ergodic measures $\nu_n \in \mathcal{E}_T(X)$ such that
\[ n \leq h_{\nu_n}(T) + \int_X \varphi \, d\nu_n \]
for all $n \in \mathbb{N}$. \hfill \qed

Remark 9. The condition $h_{\text{top}}(T) < \infty$ in above lemma can be dropped, if $\varphi \in Q_T(X)$ satisfies $\int_X \varphi \, d\mu > -\infty$ for all $\mu \in \mathcal{M}_T(X)$.

6. Variational principles for measurable potentials. Let $(X, T)$ be a dynamical system. We give a first version of the variational principle for quasi-integrable functions:

Theorem 6.1. Let $h_{\text{top}}(T) < \infty$ and $\varphi \in Q_T(X)$. In case $S(T, \varphi) = \infty$ one has $P_X(T, \varphi) = S(T, \varphi)$.

Proof. As $h_{\text{top}}(T) < \infty$ and $S(T, \varphi) = \infty$, by Lemma 5 there must be a sequence of ergodic measures $\nu_N \in \mathcal{E}_T(X)$ such that $\int_X \varphi \, d\nu_N > N$. But then by Corollary 2 $P_X(T, \varphi) > N \to \infty$ as $N \to \infty$ follows. \hfill \qed

Remark 10. For Theorem 6.1 it is not enough to suppose $P_X(T, \varphi) = \infty$. Consider the unit circle $S^1$ with irrational rotation $R$ on $S^1$. One can now choose a dense orbit $O = \{ x_0, T^1x_0, T^2x_0, \ldots \} \subseteq S^1$. Define $\psi(T^n x_0) := n$ for $n \geq 0$ and $\psi(x) := 0$ for $x \in S^1 \setminus O$. Then by Proposition 1 we obtain $0 = S(R, \psi) < P_{S^1}(R, \psi) = \infty$.

One might suspect that if $S(T, \varphi) = -\infty$, similarly to Theorem 6.1 one has $Q_X(T, \varphi) = -\infty$. It is not clear whether this holds in general, but we have the following positive result:

Proposition 3. Let $T$ be an isometry, that is $d(T^ix, T^iy) = d(x, y)$ for all $x, y \in X$ and $i \geq 1$. Suppose there exists an ergodic measure $\nu$ such that $\nu(U) > 0$ for all open sets $U \subseteq X$. Fix a $\varphi \in Q_T(X)$. If $S(T, \varphi) = -\infty$, one has $Q_X(T, \varphi) = -\infty$.

Proof. First note that $B_d(x, \epsilon) = B_d(x, \epsilon)$ for all $x \in X$ and $\epsilon > 0$. Therefore $h_{\text{top}}(T) = 0$, and $S(T, \varphi)$ is well-defined. Furthermore $\int_X \varphi \, d\mu = -\infty$ for all $\mu \in \mathcal{M}_T(X)$. By Theorem 1.7 we can choose now a dense orbit $O = \{ x_0, T^1x_0, T^2x_0, \ldots \}$ satisfying $\lim_{n \to \infty} \frac{1}{n} \sum_{i<n} \varphi(T^ix) = -\infty$. That means
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^iy) = -\infty
\]
for all $y \in O$. Given an $\epsilon > 0$, pick a finite subset $F_{\epsilon} \subseteq O$ such that $F_{\epsilon}$ is an $(\epsilon, n)$-cover of $X$. Then
\[
\frac{1}{n} \log \sum_{y \in F_{\epsilon}} \exp \sum_{i=0}^{n-1} \varphi(T^iy) \leq \frac{1}{n} \log \#F_{\epsilon} + \max_{y \in F_{\epsilon}} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^iy) \to -\infty
\]
as $n \to \infty$. Hence $Q_X(T, \varphi, \epsilon) = -\infty$ for all $\epsilon > 0$, and $Q_X(T, \varphi) = -\infty$ follows. \hfill \qed

The next theorem gives us one half of the variational principle:
Theorem 6.2. In case \( \varphi \in B(X) \) one has
\[
P_X(T, \varphi) \geq \sup \{ P_\mu(T, \varphi) : \mu \in \mathcal{M}(X) \}.
\]
In case \( h_{\text{top}}(T) < \infty \) and \( \varphi \in Q_T(X) \), one has
\[
P_X(T, \varphi) \geq S(T, \varphi).
\]
In case \( h_{\text{top}}(T) < \infty \) and \( \varphi \in B(X) \), one has
\[
P_X(T, \varphi) \geq \sup \{ P_\mu(T, \varphi) : \mu \in \mathcal{M}(X) \} \geq S(T, \varphi).
\]

Proof. The first two statements are consequences of Lemma 5.5, Theorem 5.3 and Corollary 1. By (5), for every ergodic \( \nu \in E_T(X) \) we have
\[
P_\nu(T, \varphi) = \int_X P_\nu(T, \varphi, x) \, d\nu = \int_X \left( h_\nu(T) + \int_X \varphi \, d\nu \right) \, d\nu = h_\nu(T) + \int_X \varphi \, d\nu.
\]
That means by Lemma 5.5
\[
\sup \{ P_\mu(T, \varphi) : \mu \in \mathcal{M}(X) \} \geq \sup \{ P_\nu(T, \varphi) : \nu \in E_T(X) \} = S(T, \varphi).
\]

In view of Proposition 1, the following second version of the variational principle holds:

Corollary 3. Let \( \varphi \in B(X) \). If there is an \( x_0 \in X \) such that
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x_0) = \infty,
\]
then
\[
P_X(T, \varphi) = \sup \{ P_\mu(T, \varphi) : \mu \in \mathcal{M}(X) \} = \infty.
\]

Proof. We show that \( \delta_{x_0} \) is an equilibrium measure. By \(- \log \delta_{x_0}(B_{d_n}(x_0, \delta)) = 0 \) we have for all \( \delta > 0 \)
\[
P_{\delta_{x_0}}(T, \varphi) = P_{\delta_{x_0}}(T, \varphi, x_0, \delta) = \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x_0) = \infty
\]
\[
\Box
\]

Remark 11. Above proof combined with Lemma 4.2 shows that for all \( \varphi \in B(X) \) and \( x \in X \)
\[
P_{\delta_{x}}(T, \varphi) \leq P_{\{x\}}(T, \varphi).
\]

To prove upper estimates for the variational principle, we first introduce a new class of functions:

Definition 6.3. Let \( \varphi \in Q_T(X) \). We call \( \varphi \)\textbf{ upper semi-continuous with respect to} \( T \), if the following holds: If \( \{ \mu_n \}_{n \in \mathbb{N}} \) is a sequence of atomic probability measures \( \mu_n = \sum_{i=1}^{k_n} \lambda_n^i \delta_{x_i} \), where \( \{ \lambda_n^i \}_{i=1}^{k_n} \) are some probability vectors and \( \{ x_i \}_{i=1}^{k_n} \subseteq X \) for \( n \in \mathbb{N} \), such that there exists a \( \mu \in \mathcal{M}_T(X) \) satisfying \( \mu_n \to \mu \) in the weak* topology, then
\[
\limsup_{n \to \infty} \int_X \varphi \, d\mu_n \leq \int_X \varphi \, d\mu.
\]

The set of all upper semi-continuous functions with respect to \( T \) is denoted by \( U_T(X) \subseteq Q_T(X) \).
Remark 12. The example of Remark 3 is a system, where each function in $U_T(X)$ needs to be bounded from above. Assume there is a $\varphi \in U_{id}([0, 1])$, which is not bounded from above. Pick a sequence $x_n \in [0, 1]$, such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} f(x_n) = \infty$. Then one has

$$\limsup_{n \to \infty} \int_{[0, 1]} \varphi \, d\delta_n = \infty > \varphi(x) = \int_{[0, 1]} \varphi \, d\delta_x,$$

which is a contradiction.

It turns out that many systems exhibit the same behaviour as the above example:

**Proposition 4.** If $(X, T)$ has a periodic orbit, than each $\varphi \in U_T(X)$ is bounded from above.

**Proof.** Let $\{x_0, Tx_0, \ldots, T^{p-1}x_0\} \subseteq X$ be a periodic orbit with minimal period $p \geq 1$. Then $\mu = \frac{1}{p} \sum_{0 \leq i < p} \delta_{T^i x_0} \in \mathcal{E}_T(X)$. Assume there is a measurable function $\varphi : X \to \mathbb{R}$ such that for each $n \geq 1$ there is an $x_n \in X$ satisfying $\varphi(x_n) \geq n$.

Define

$$\mu_n := \frac{1}{n} \delta_{x_n} + \left(1 - \frac{1}{n}\right) \mu.$$

Then it is easy to see that $\mu_n \to \mu$ as $n \to \infty$. On the other hand one has

$$\limsup_{n \to \infty} \int_X \varphi \, d\mu_n = \limsup_{n \to \infty} \left(\frac{1}{n} \varphi(x_n) + \left(1 - \frac{1}{n}\right) \int_X \varphi \, d\mu\right) \geq 1 + \int_X \varphi \, d\mu.$$

This shows $\varphi \notin U_T(X)$. \qed

We do not know whether the last statement also holds for systems where each ergodic measure has no atoms. Hence, to cover the full generality, we have to deal with cases where the pressure is infinite:

**Proposition 5.** Suppose $\varphi \in U_T(X)$ and $P_X(T, \varphi, \epsilon) = \infty$ for some $\epsilon > 0$. Then there exists a $\mu \in \mathcal{M}_T(X)$ such that $\int_X \varphi \, d\mu = \infty$.

**Proof.** Let $\xi = \{A_1, \ldots, A_k\}$ be a measurable partition of $X$ such that $\text{diam}(A_i) < \epsilon$ for all $i = 1, \ldots, k$. Denote by

$$\bigvee_{i=0}^{n-1} T^{-i} \xi := \left\{ \bigcap_{i=0}^{n-1} B_i : B_i \in T^{-i} \xi \right\}.$$

If $\sigma$ is some probability measure on $X$, define

$$H_\sigma \left( \bigvee_{i=0}^{n-1} T^{-i} \xi \right) := -\sum_{B \in \bigvee_{i=0}^{n-1} T^{-i} \xi} \sigma(B) \log \sigma(B).$$

Note that for all $n \geq 1$

$$H_\sigma \left( \bigvee_{i=0}^{n-1} T^{-i} \xi \right) \leq n \cdot \log k. \tag{7}$$

We first consider subsequences $\{n_j\}_{j \in \mathbb{N}}$ such that

$$\lim_{j \to \infty} \frac{1}{n_j} \log M_X(T, \varphi, n_j) = \infty.$$
and $-\infty < \overline{M}_X(T, \varphi, \epsilon, n_j) < \infty$ for all $j \in \mathbb{N}$. Set $\varphi_n(x) := \sum_{i=0}^{n-1} \varphi(T^ix)$ and choose $(\epsilon, n_j)$-separated sets $E_{n_j}$ satisfying

$$\log \sum_{x \in E_{n_j}} \exp \varphi_{n_j}(x) \geq \log \overline{M}_X(T, \varphi, \epsilon, n_j) - 1. \quad (8)$$

Next define the probability measure

$$\sigma_{n_j} := \frac{\sum_{x \in E_{n_j}} \exp \varphi_{n_j}(x) \cdot \delta_x}{\sum_{x \in E_{n_j}} \exp \varphi_{n_j}(x)}$$

for every $j \in \mathbb{N}$. Then by the definition of $\sigma_{n_j}$ and Lemma 9.9

$$H_{\sigma_{n_j}} \left( \bigvee_{i=0}^{n_j-1} T^{-i} \xi \right) + \int_X \varphi_{n_j} \, d\sigma_{n_j} = \log \sum_{x \in E_{n_j}} \exp \varphi_{n_j}(x). \quad (9)$$

Next define

$$\mu_{n_j} := \frac{1}{n_j} \sum_{i=0}^{n_j-1} \sigma_{n_j} \circ T^{-i}.$$ 

Combining (7), (8) and (9) yields

$$\log k + \int_X \varphi \, d\mu_{n_j} \geq \frac{1}{n_j} \left( \log \overline{M}_X(T, \varphi, \epsilon, n_j) - 1 \right).$$

Furthermore there exists a subsequence $\{ n_{j_l} \}$ and a $\mu \in \mathcal{M}_T(X)$ such that $\lim_{l \to \infty} \mu_{n_{j_l}} = \mu$ in the weak* topology. Thus using $\varphi \in U_T(X)$

$$\log k + \int_X \varphi \, d\mu \geq \log k + \limsup_{l \to \infty} \int_X \varphi \, d\mu_{n_{j_l}} \geq \lim_{l \to \infty} \frac{1}{n_{j_l}} \log \overline{M}_X(T, \varphi, \epsilon, n_{j_l}) = \infty.$$ 

This implies $\int_X \varphi \, d\mu = \infty$.

Now let $\{ n_j \}_{j \in \mathbb{N}}$ be a subsequence such that $\overline{M}_X(T, \varphi, \epsilon, n_j) = \infty$ for all $j \in \mathbb{N}$. We can then choose $(\epsilon, n_j)$-separated sets $E_{n_j}$ satisfying

$$\log \sum_{x \in E_{n_j}} \exp \varphi_{n_j}(x) \geq 2^{n_j},$$

and the statement follows in the same way as above.

Proposition 5 gives us a third version of the variational principle, which is in some sense the reverse direction of Theorem 6.1:

**Corollary 4.** Let $h_{\text{top}}(T) < \infty$ and $\varphi \in U_T(X)$. If there is an $\epsilon > 0$ such that $P_X(T, \varphi, \epsilon) = \infty$, then one has $S(T, \varphi) = P_X(T, \varphi)$.

**Proof.** By the last proposition and $P_X(T, \varphi, \epsilon) \leq P_X(T, \varphi)$ we have $S(T, \varphi) = \infty = P_X(T, \varphi, \epsilon) = P_X(T, \varphi)$. 

We are now able to state the variational principle for all $\varphi \in U_T(X)$:

**Theorem 6.4.** Let $h_{\text{top}}(T) < \infty$ and $\varphi \in U_T(X)$. Then one has

$$P_X(T, \varphi) = S(T, \varphi).$$
Proof. By Theorem \ref{theorem:proof} it remains to show that \( P_X(T, \varphi) \leq S(T, \varphi) \). Furthermore, by Corollary \ref{corollary:corollary} we may assume \( F_X(T, \varphi, \epsilon) < \infty \) for all \( \epsilon > 0 \). In this situation, the proof follows the conventional proof of the classical variational principle as given in \cite{19}. We are only outlining it here. Denote \( \varphi_n(x) := \sum_{i=0}^{n-1} \varphi(T^i x) \). Fix \( \epsilon > 0 \) and choose for all \( n \in \mathbb{N} \) a \((\epsilon, n)\)-separated set \( E_n \) such that \( \log \sum_{x \in E_n} \exp \varphi_n(x) \geq \log M_X(T, \varphi, \epsilon, n) - 1 \). Define

\[
\sigma_n := \frac{\sum_{x \in E_n} \exp \varphi_n(x) \cdot \delta_x}{\sum_{x \in E_n} \exp \varphi_n(x)}
\]

and \( \mu_n := \frac{1}{n} \sum_{i=0}^{n-1} \sigma_n \circ T^{-i} \). Next pick some subsequence \( \{ n_j \}_{j \in \mathbb{N}} \) such that \( \lim_{j \to \infty} \frac{1}{n_j} \log M_X(T, \varphi, \epsilon, n_j) = P_X(T, \varphi, \epsilon) \) and \( \{ \mu_{n_j} \}_{j \in \mathbb{N}} \) converges to some \( \mu \in \mathcal{M}_T(X) \) in the weak* topology. Furthermore we can choose a measurable partition \( \xi = \{ A_1, \ldots, A_k \} \) satisfying \( \text{diam}(A_i) < \epsilon \) and \( \mu(\partial A_i) = 0 \) for all \( i = 1, \ldots, k \). This yields the choice of \( E_n \) and \( \xi \)

\[
H_{\sigma_n} \left( \bigvee_{i=0}^{n_j-1} T^{-i} \xi \right) + \int_X \varphi_n \, d\sigma_n = \log \sum_{x \in E_n} \exp \varphi_n(x).
\]

Given \( 1 \leq q < n_j \) and \( 0 \leq m \leq q - 1 \), define \( a(m) := \lfloor (n - m)/q \rfloor \). One can now decompose \( \bigvee_{i=0}^{n_j-1} T^{-i} \xi \) as

\[
\bigvee_{i=0}^{n_j-1} T^{-i} \xi = \bigvee_{r=0}^{a(m)-1} T^{-(rq+m)} \left( \bigvee_{i=0}^{q-1} T^{-i} \xi \right) \vee \bigvee_{l \in S} T^{-l} \xi,
\]

where \( S \) is a set with cardinality at most \( 2q \). Hence

\[
\log \sum_{x \in E_{n_j}} \exp \varphi_{n_j}(x) = H_{\sigma_{n_j}} \left( \bigvee_{i=0}^{n_j-1} T^{-i} \xi \right) + \int_X \varphi_{n_j} \, d\sigma_{n_j}
\]

\[
\leq \sum_{r=0}^{a(m)-1} H_{\sigma_{n_j} \circ T^{-rq+m}} \left( \bigvee_{i=0}^{q-1} T^{-i} \xi \right) + 2q \log k + \int_X \varphi_{n_j} \, d\sigma_{n_j}.
\]

Summing over all \( m = 0, \ldots, q-1 \) gives

\[
q \cdot \log \sum_{x \in E_{n_j}} \exp \varphi_{n_j}(x) \leq \sum_{p=0}^{n_j-1} H_{\sigma_{n_j} \circ T^{-p}} \left( \bigvee_{i=0}^{q-1} T^{-i} \xi \right) + 2q^2 \log k + q \cdot \int_X \varphi_{n_j} \, d\sigma_{n_j},
\]

and dividing by \( n_j \) yields

\[
\frac{q}{n_j} \left( \log M_X(T, \varphi, \epsilon, n_j) - 1 \right) \leq \frac{q}{n_j} \cdot \log \sum_{x \in E_{n_j}} \exp \varphi_{n_j}(x)
\]

\[
\leq H_{\mu_{n_j}} \left( \bigvee_{i=0}^{q-1} T^{-i} \xi \right) + \frac{2q^2}{n_j} \log k + q \cdot \int_X \varphi \, d\mu_{n_j}.
\]

Using \( \varphi \in U_T(X) \) and \( \mu(\partial A_i) = 0 \) for all \( i = 1, \ldots, k \) we obtain by \( j \to \infty \)

\[
q \cdot P_X(T, \varphi, \epsilon) \leq H_{\mu} \left( \bigvee_{i=0}^{q-1} T^{-i} \xi \right) + q \cdot \int_X \varphi \, d\mu.
\]
Finally dividing by \( q \) and letting \( q \to \infty \) gives
\[
P_X(T, \varphi, \epsilon) \leq h_\mu(T) + \int_X \varphi \, d\mu,
\]
that is \( P_X(T, \varphi, \epsilon) \leq S(T, \varphi) \) for every \( \epsilon > 0 \). This shows the statement. \( \square \)

**Remark 13.** The condition \( h_{\text{top}}(T) < \infty \) in Theorem 6.4 can be dropped, if \( \varphi \in U_T(X) \) satisfies \( \int_X \varphi \, d\mu > -\infty \) for all \( \mu \in \mathcal{M}_T(X) \).

Next we introduce two non-trivial classes of (dis-)continuous functions, which satisfy Theorem 6.4.

**Definition 6.5.** Let \((Y, \rho)\) an arbitrary metric space and \( \varphi : Y \to \mathbb{R} \) measurable. The set
\[
D_\varphi := \{ x \in Y : \varphi \text{ is not continuous in } x \}
\]
is called **set of discontinuity points of** \( \varphi \). One can show that \( D_\varphi \) is Borel measurable. Denote by \( C_T(X) \) the set of all bounded, Borel measurable functions \( \varphi : X \to \mathbb{R} \), such that \( \mu(D_\varphi) = 0 \) for all \( \mu \in \mathcal{M}_T(X) \).

**Proposition 6.** Let \( \{ \mu_n \}_{n \in \mathbb{N}} \) be a sequence of Borel probability measures with limit measure \( \mu \) in the weak* topology. Then one has for all \( \varphi \in C_T(X) \)
\[
\lim_{n \to \infty} \int_X \varphi \, d\mu_n = \int_X \varphi \, d\mu.
\]
In particular one has \( C_T(X) \subseteq U_T(X) \).

**Proof.** The first statement is part of the Portmanteau theorem, see for example [14], Theorem 13.16. The second statement follows immediately. \( \square \)

**Corollary 5.** One has \( P_X(T, \varphi) = S(T, \varphi) \) for all \( \varphi \in C_T(X) \).

**Proof.** As \( \varphi \) is bounded, the statement follows from Proposition 6, Theorem 6.4 and Remark 13. \( \square \)

**Remark 14.** We give an illustration of above corollary. Suppose \((X, d)\) to be a non-empty, compact space and \( T : X \to X \) to be a contraction. By the Banach fixed-point theorem there exists a unique fixed-point \( x_0 \in X \). It is then easy to see that for every continuous \( \psi : X \to \mathbb{R} \) and every \( x \in X \)
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \psi(T^i x) = \psi(x_0),
\]
which means \( \mathcal{E}_T(X) = \{ \delta_{\{x_0\}} \} \). Thus for every bounded, measurable \( \varphi : X \to \mathbb{R} \) satisfying \( x_0 \notin D_\varphi \) one has
\[
P_X(T, \varphi) = \int_X \varphi \, d\delta_{\{x_0\}} = \varphi(x_0).
\]
In particular by Lemma 4.2 one has \( P_X(T, \varphi) = P_{\{x\}}(T, \varphi) \) for every \( x \in X \).

**Definition 6.6.** A function \( \varphi : X \to \mathbb{R} \) is called **upper semi-continuous** on \( X \), if \( \{ x \in X : f(x) < c \} \) is an open set for every \( c \in \mathbb{R} \). We denote the set of all upper semi-continuous functions \( \varphi : X \to \mathbb{R} \) by \( U(X) \). As \( X \) is compact, every \( \varphi \in U(X) \) is bounded from above (see for example [1] Theorem 2.43). This immediately yields \( U(X) \subseteq Q_T(X) \). In addition, the following holds:
Proposition 7. Let \( \{ \mu_n \}_{n \in \mathbb{N}} \) be a sequence of Borel probability measures with limit measure \( \mu \) in the weak\(^*\) topology. Then one has for \( \varphi \in \mathcal{U}(X) \)

\[
\limsup_{n \to \infty} \int_X \varphi \, d\mu_n \leq \int_X \varphi \, d\mu.
\]
In particular one has \( \mathcal{U}(X) \subseteq \mathcal{U}_T(X) \) for every continuous mapping \( T : X \to X \).

Corollary 6. Let \( h_{\text{top}}(T) < \infty \) and \( \varphi \in \mathcal{U}(X) \). Then one has \( P_X(T, \varphi) = S(T, \varphi) \).

Proof. The statement follows from Proposition 7 and Theorem 6.4.

Remark 15. Corollary 6 was already proven (as a special case) in [4] Theorem 1.1 together with [2] Remark 2.4 (1). However, the proof in [4] for the lower bound of the variational principle requires the functions \( f_n := \exp \left( \sum_{i<n} \varphi(T^i) \right) \) to attain a maximum on every compact subset of \( X \). This is clearly the case if \( \varphi \) is upper semi-continuous, as this implies the upper semi-continuity of \( f_n \). Another proof for Corollary 6 can be found in [13] (see Theorem 4.4.11). Here for the lower bound of the variational principle, the functions \( f_n \) need to be bounded from above on \( X \).

In the method of proof used in the present paper, above properties are not needed. Instead, the lower estimate is first proven for ergodic measures with the help of an ergodic theorem. After that it can be extended to all invariant measures via ergodic decomposition.

Remark 16. Recall the example given in Remark 14 and consider the indicator function

\[
\chi(x) := \begin{cases} 
0, & x \neq x_0, \\
1, & x = x_0.
\end{cases}
\]
As one has \( D_{\chi} = \{ x_0 \} \) in general, we can not apply Corollary 5. One the other hand the function \( \chi \) is upper semi-continuous, as \( \{ x_0 \} \) is closed. Thus again by Corollary 6

\[
P_X(T, \chi) = \chi(x_0) = 1.
\]

Remark 14 motivates the following example, which shows that the pressure defined in this paper might be applied to systems, which are unavailable to the pressures and its variational principle derived in [15].

Proposition 8. Let \( \alpha \in (0,1) \cap \mathbb{Q} \) and \( E := [0,1] \cap \mathbb{Q} \). Then there exists a measurable function \( \varphi_\alpha : [0,1] \to \mathbb{R} \), which is not upper semi-continuous, such that \( D_{\varphi_\alpha} = [0,1] \setminus \{ \alpha \} \). In particular for every increasing sequence \( \Lambda_l \subseteq \Lambda_{l+1} \subseteq [0,1] \) of subsets satisfying \( [0,1] = \bigcup_{l=1}^\infty \Lambda_l \) there exists a \( k \geq 1 \) such that \( \varphi_\alpha \) is not continuous on the closure of \( \Lambda_k \).

Proof. Define

\[
\varphi_\alpha(x) := \begin{cases} 
-\alpha^{-1} \cdot x + 1, & x \in [0,\alpha] \setminus E, \\
(1-\alpha)^{-1} \cdot (x-\alpha), & x \in [\alpha,1] \setminus E, \\
0, & x \in E.
\end{cases}
\]
Clearly \( \varphi_\alpha \) is measurable and continuous in \( \alpha \) only. Also one has \( \limsup_{\alpha \to x} \varphi_\alpha(x) > 0 = \varphi_\alpha(x) \) for every \( x \in E \setminus \{ \alpha \} \), which proves \( \varphi_\alpha \) not to be upper semi-continuous. Now by \( [0,1] = \bigcup_{l=1}^\infty \Lambda_l \) we also have \( [0,1] = \bigcup_{l=1}^\infty \Lambda_l \). According to the Baire category theorem there exists some \( k \geq 1 \) and open interval \( (a,b) \) such that \( (a,b) \subseteq \overline{\Lambda}_k \). But that means \( \varphi_\alpha \) is not continuous on \( \overline{\Lambda}_k \).
Corollary 7. Let \( X := [0,1] \), \( \alpha \in (0,1) \cap \mathbb{Q} \) and \( T : X \to X \) contracting such that \( T(\alpha) = \alpha \). If \( \varphi_\alpha \) is the function constructed in Proposition 8, then one has

\[
P_{[0,1]}(T, \varphi_\alpha) = \sup \left\{ h_\mu(T) + \int_{[0,1]} \varphi_\alpha \, d\mu : \mu \in \mathcal{M}_T([0,1]) \right\}.
\]

Proof. This follows from Remark 14 and Proposition 8.

Remark 17. Clearly both sets \( U(X) \) and \( C_T(X) \) contain all continuous functions \( \varphi : X \to \mathbb{R} \). Moreover, Corollary 7 can be seen as variational principle for potentials which are continuous from a measure theoretical point of view. We want to emphasize that the set \( C_T(X) \) heavily depends on the mapping \( T : X \to X \). The set \( U(X) \) on the other hand only depends on the metric \( d \) on \( X \). Note that in general it may happen that \( C_T(X) \lessdot U(X) \neq \emptyset \), which implies \( U(X) \lessdot \emptyset \). This can be seen from the following statement:

Proposition 9. Assume that \( (X,d) \) has no isolated points, and there exists a non-atomic \( \mu \in \mathcal{E}_T(X) \). Then there is a function \( \varphi \in C_T(X) \), which is neither upper nor lower semi-continuous on \( X \).

Proof. As \( \mu \) is non-atomic, there are two distinct points \( x_1 \neq x_2 \in X \) such that \( \lim_{n \to \infty} \sum_{i<n} \delta_{T^i x_1} = \lim_{n \to \infty} \sum_{i<n} \delta_{T^i x_2} = \mu \). Next define \( \varphi := 1_{\{x_1\}} - 1_{\{x_2\}} \). As \( X \) has no isolated points, \( \varphi \) is not lower semi-continuous in \( x_1 \), and not upper semi-continuous in \( x_2 \). That means \( D_\varphi = \{x_1,x_2\} \), and as \( \mu \) has no atoms, \( \varphi \in C_T(X) \) follows.

7. Cookie-cutters with discontinuous geometric potentials. In this section we introduce cookie-cutter systems with discontinuous geometric potentials and their corresponding attractors. Following this, we use the topological pressure and its variational principle for discontinuous functions to compute the Hausdorff dimension of those attractors. The definitions and notations are based on the classical treatment of cookie-cutters in [9].

Definition 7.1. Let \( N \geq 2 \) and \( J_i = [x_i,y_i] \subseteq [0,1] \) be pairwise disjoint closed subintervals for \( i = 1, \ldots, N \) and set \( J := \bigcup_{i=1}^N J_i \) as well as \( I := \bigcup_{i=1}^N (x_i,y_i) \). Now let \( T : J \to [0,1] \) be a continuous function and \( D \subseteq I \) a finite set such that

(I) \( T : I \setminus D \to [0,1] \) is continuously differentiable such that the left and right derivatives on the interval endpoints \( x_i, y_i \) for \( i = 1, \ldots, N \) exist;

(II) For each point \( x \in D \) the one-sided limits \( \lim_{y \to x\pm} T'(y) \) exist and are different;

(III) \( \inf_{\xi \in I \setminus D} |T'(\xi)| > 1 \);

(IV) \( T|_{J_i} : J_i \to [0,1] \) is bijective for all \( i = 1, \ldots, N \).

A mapping \( T : J \to [0,1] \) satisfying the above properties is called cookie-cutter. We define the derivative of \( T \) on the interval endpoints \( x_i, y_i, i = 1, \ldots, N \) to be the left or right derivatives respectively. By definition, \( T' \) is not well-defined in the set \( D \), and there is also no way to extend \( T' \) continuously to \( D \).

By (III) there exists for every \( i = 1, \ldots, N \) an inverse branch \( \varphi_i := (T|_{J_i})^{-1} : [0,1] \to J_i \). The existence of a global attractor is ensured by the following proposition:

Proposition 10. There exists a number \( c < 1 \) such that \( |\varphi_i(x) - \varphi_i(y)| \leq c \cdot |x-y| \) for all \( x,y \in [0,1] \). In particular there is a unique compact set \( \emptyset \neq X \subseteq [0,1] \).
In Case such that \( X = \bigcup_{i=1}^{N} \varphi_{i}(X) \). In addition the set \( X \) can be expressed as \( X = \{ x \in [0, 1] : T^{k}(x) \text{ is defined and in } J \text{ for all } k \geq 0 \} \).

Proof. Fix \( i \in \{ 1, \ldots, N \} \). Clearly \( \varphi_{i} : I_{i} := (0, 1) \setminus T(D \cap J_{i}) \to J_{i} \) is continuously differentiable satisfying \( c_{i} := \sup_{\xi \in I_{i}} |\varphi'_{i}(\xi)| < 1 \). Fix \( x < y \in [0, 1] \) and denote by \( z_{1} < \cdots < z_{m}, m \geq 0 \), the set of points \( [x, y] \cap T(D \cap J_{i}) \). Set \( z_{0} := x \) and \( z_{m+1} := y \). As \( \varphi_{i} \) is strict monotonic on \( [0, 1] \), we have

\[
\begin{align*}
|\varphi_{i}(x) - \varphi_{i}(y)| & = \sum_{k=0}^{m} |\varphi_{i}(z_{k}) - \varphi_{i}(z_{k+1})| \\
& \leq \sum_{k=0}^{m} \sup_{\xi \in (z_{k}, z_{k+1})} |\varphi'_{i}(\xi)||\varphi_{i}(z_{k}) - \varphi_{i}(z_{k+1})| \\
& \leq \sup_{\xi \in I_{i}} |\varphi'_{i}(\xi)| \sum_{k=0}^{m} |z_{k} - z_{k+1}| = \sup_{\xi \in I_{i}} |\varphi'_{i}(\xi)| |x - y|.
\end{align*}
\]

Therefore the number \( c := \max_{i=1, \ldots, N} c_{i} \) has the desired property. The second part follows for example from [8] Theorem 9.1.

**Definition 7.2.** The compact set \( X \) satisfying \( X = \bigcup_{i=1}^{N} \varphi_{i}(X) \) is called **attractor of the cookie-cutter** \( T : J \to [0, 1] \). As \( T|X \) is continuous and \( X = T(X) \), the tuple \( (X, T|X) \) is a dynamical system. Furthermore, \( T'|X \setminus D \) is continuous. Thus, if \( \varphi : X \to \mathbb{R} \) is some function satisfying \( \varphi|X \setminus D = T'|X \setminus D \), one has \( D \cap X = D_{\varphi} \) (see Definition 6.5). We call such a function \( \varphi \) to be an **extension of \( T' \) to \( X \)**. Clearly an extension \( \varphi \) of \( T' \) to \( X \) is continuous if and only if one has \( D \cap X = \emptyset \).

In that case \( \varphi := T'|X \) by definition is the only possible extension.

If \( \varphi : X \to \mathbb{R} \) is an extension from \( T' \) to \( X \), the function \( \log|\varphi| : X \to \mathbb{R} \) is called **geometric potential** of \( T \). In case \( D \cap X = \emptyset \) the system is called **cookie-cutter with discontinuous geometric potentials**, and every possible extension \( \varphi \) defines a corresponding geometric potential.

**Remark 18.** We shall give some examples to illustrate the notion of cookie-cutters. Let \( J := [0, \frac{1}{2}] \cup [\frac{3}{4}, 1] \). Define

\[
T_{1}(x) := \begin{cases} 
3x, & 0 \leq x \leq \frac{1}{4}, \\
-3x + 3, & \frac{3}{4} \leq x \leq 1.
\end{cases}
\]

In this case the corresponding attractor \( X_{1} \) is the middle-third Cantor set. The derivative \( T'_{1} \) is well-defined everywhere. If we change \( T_{1} \) to

\[
T_{2}(x) := \begin{cases} 
3x, & 0 \leq x \leq \frac{1}{5}, \\
-\frac{12}{5}x + \frac{13}{5}, & \frac{3}{5} \leq x \leq \frac{2}{5}, \\
-\frac{18}{5}x + \frac{18}{5}, & \frac{5}{6} \leq x \leq 1,
\end{cases}
\]

we see that \( T'_{2} \) does not exist in \( \frac{4}{5} \). One the other hand one has \( T(\frac{5}{6}) = \frac{3}{5} \notin J \), which means \( \frac{5}{6} \notin X_{2} \) and \( \varphi := T'_{2}|X_{2} \) is the continuous extension. To obtain a cookie-cutter satisfying \( D_{\varphi} = \emptyset \) for every extension \( \varphi \) of \( T' \) to \( X \), one can easily modify the second example in a way that the point of discontinuity is a fixed point (see Figure 1).

The goal is to compute the Hausdorff dimension \( \dim_{H}X \) of an attractor \( X \). In case \( D \cap X = \emptyset \) it is determined by the zero of a certain pressure function, which
involves the (continuous) geometric potential. We restate the well-known, classical result for the convenience of the reader:

**Theorem 7.3.** Let \( T : J \to [0, 1] \) be a cookie-cutter, \( X \) its attractor and \( D \cap X = \emptyset \). Denote \( \psi := T'|X \). Then \( P_X(T|X, -s \cdot \log |\psi|) = 0 \) if and only if \( s = \dim_H X \).

**Proof.** As \( T' \) is continuously differentiable on \( J \setminus D \),
\[
a(x) := \lim_{y \to x} \frac{|(T|X)(y) - (T|X)(x)|}{|y - x|} = |\psi(x)|
\]
holds for all \( x \in X \), thus \( a : X \to \mathbb{R} \) is continuous. In addition by the third property of Definition 7.1 one has \( 1 < \inf_{x \in X} a(x) \), which implies
\[
0 < \inf_{x \in X} \liminf_{n \to \infty} \frac{1}{n} \sum_{k < n} \log a(x).
\]
Then by \([6]\) Theorem 2.4 it follows that \( \tilde{P}_X(T|X, -s \cdot \log a) = 0 \) if and only if \( s = \dim_H X \). Here \( \tilde{P}_Z \) denotes a Carathéodory dimension type definition of topological pressure, which was first given in \([17]\). Furthermore, it is well-known that for all continuous \( \varphi : X \to \mathbb{R} \) and all non-empty compact \( T \)-invariant subsets \( Z \subseteq X \) one has
\[
\tilde{P}_Z(T|X, \varphi) = P_Z(T|X, \varphi).
\]
Hence the statement follows.

**Remark 19.** Actually Theorem 2.4 in \([6]\) is much more powerful than above proof suggests. It basically states, that the Hausdorff dimension of every subset of an attractor is the zero of the pressure function \( s \mapsto \tilde{P}_Z(T|X, -s \cdot \log a) \), provided the system \( (X, T|X) \) is conformal and reasonable expanding. Conformal in this context means that the expression (10) is well-defined and continuous on \( X \). However the theorem cannot be applied anymore, if the limit \( a(x) \) in (10) does not exist for even one \( x \in X \). In higher dimensions this can happen, if the derivative in a point exists, but has distinct eigenvalues. For a survey of recent research on the topic of non-conformal repellers, see \([5]\).

As indicated in Remark 19, the classical thermodynamic formalism and its celebrated Bowen formula cannot be applied directly to cookie-cutters \( T : J \to [0, 1] \), as
the derivative might not exist in finitely many points on the attractor. Nevertheless its still possible to establish an analogous formula for them:

**Theorem 7.4.** Let $T : J \to [0, 1]$ be a cookie-cutter and $X$ its attractor. Then there exists a geometric potential $\log |\psi| : X \to \mathbb{R}$ such that $P_X(T|X, -s \cdot \log |\psi|) = 0$ if and only if $s = \dim_H X$.

**Remark 20.** As we shall see, the geometric potential can be constructed with the lower semi-continuous extension of $T'$. It might be considered as the natural choice among all possible geometric potentials.

To prove above theorem, we collect some preparatory results first:

**Lemma 7.5.** There exists a function $f : J \to \mathbb{R}$ such that

(i) $f|J \setminus D = T'|J \setminus D$, that is, $f|X$ is an extension of $T'$ to $X$;

(ii) $x \mapsto -s \cdot \log |f(x)|$ is upper semi-continuous for every $s \geq 0$;

(iii) The one-sided limits $\lim_{y \to x-} f(y)$ and $\lim_{y \to x+} f(y)$ exist for all $x \in (x_i, y_i)$ respectively $x \in [x_i, y_i], i = 1, \ldots, N$.

(iv) One has $\int_a^b f(x) \, dx = T(b) - T(a)$ for all $[a,b] \subseteq J \setminus D$.

**Proof.** First choose some $\delta > 0$ such that $T$ restricted to $B_x := (I \setminus D) \cap B_{\delta}(x)$ is strict monotonic for all $x \in J$. By definition of $T$, each $x \in D$ is a jump discontinuity of $T'$. We therefore define $f : J \to \mathbb{R}$ by

$$ f(x) := \begin{cases} T'(x), & x \in J \setminus D, \\ \lim \inf_{y \to x} T'(y), & x \in D, T'|B_x > 1, \\ \lim \sup_{y \to x} T'(y), & x \in D, T'|B_x < -1. \end{cases} $$

Thus $x \mapsto |f(x)|$ is lower semi-continuous. As $\log(\cdot)$ is strictly monotone increasing, the function $x \mapsto \log |f(x)|$ remains lower semi-continuous, whereas $x \mapsto -s \cdot \log |f(x)|$ is upper semi-continuous for all $s \geq 0$. The remaining parts easily follow. \hfill \Box

**Lemma 7.6.** The dynamical system $(X, T|X)$ is topological transitive and satisfies $h_{top}(T|X) = \log N$.

**Proof.** Denote by $(\Sigma^+_N, \sigma_N)$ the full one-sided shift on $N$ symbols, that is $\Sigma^+_N := \{1, \ldots, N\}^\mathbb{N}$ and $\sigma_N : \Sigma^+_N \to \Sigma^+_N, (\omega_1, \omega_2, \ldots) \mapsto (\omega_2, \omega_3, \ldots)$. It is well known that $(X, T|X)$ and $(\Sigma^+_N, \sigma_N)$ are topological conjugated via $\pi : \Sigma^+_N \to X, (\omega_1, \omega_2, \ldots) \mapsto \bigcap_{n=1}^\infty \varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_n}(X)$ (see for example [8]). This shows the statement. \hfill \Box

**Proof of Theorem 7.4.** The idea of the proof is to embed $(X, T|X)$ in a piecewise monotonic system. We refer for detailed definitions and discussion of the following notions to [12].

We call a map $\tau : [0, 1] \to [0, 1]$ expanding piecewise monotonic or EPM, if there exist

$$ 0 = c_0 < c_1 < \cdots < c_M = 1 $$

such that $\tau|[c_{i-1}, c_i)$ is monotonic and continuously differentiable for $i = 1, \ldots, M$ and $\inf_{\xi \in [0, 1] \setminus \{c_1, \ldots, c_M\}} \tau'(\xi) > 1$. Therefore it suffices to extend $T$ from $J$ to the whole interval $[0, 1]$ in a proper way.

Consider the set $[0, 1] \setminus J = [0, x_1) \cup (y_1, x_2) \cup \cdots \cup (y_{N-1}, x_N) \cup (y_N, 1]$. We assume without restriction $[0, x_1)$ and $(y_N, 1]$ to be empty, that is $x_1 = 0$ and
$y_N = 1$, as one can perform the following construction process on those half-open intervals too.

Fix $0 < \epsilon < \frac{1}{4}$. Define

$$g(x) := \begin{cases} 
(2 - 4\epsilon)x + \epsilon, & x \in (0, \frac{1}{2}), \\
(2 - 4\epsilon)(x - \frac{1}{2}) + \epsilon, & x \in (\frac{1}{2}, 1).
\end{cases}$$

Clearly $|g(x)| = (2 - 4\epsilon) > 1$ and $0 < g(x) < 1$ for all $x \in (0, 1)$. Fix an $i \in \{1, \ldots, N\}$. Define an affine scaling $\Phi : (0, 1) \to (y_i - 1, x_i)$ by

$$\Phi_i(x) := (x_i - y_i - 1)x + y_i - 1$$

and $T_i : (y_i - 1, x_i) \to (y_i - 1, x_i)$ by

$$T_i := \Phi_i \circ g \circ \Phi_i^{-1}.$$

Finally define $\tau : [0, 1] \to [0, 1]$ by

$$\tau(x) := \begin{cases} 
T(x), & x \in J, \\
T_i(x), & x \in (y_i - 1, x_i).
\end{cases}$$

It is easy to see that $\tau$ satisfies the properties of an EPM map, if we linearly order all points of $D \cup \{x_i, y_i \mid i = 1, \ldots, N\} \cup \{0, 1\}$ as demanded in (11). In addition one has $\tau(x) \notin X$ for every $x \notin X$. This follows from the construction of $\tau$: For every $x \in [y_i - 1, x_i]$ one has $T_i(x) \in (y_i - 1, x_i)$. Furthermore every $x \in J \setminus X$ will eventually be mapped by $T$ into one of the intervals $(y_i - 1, x_i)$, where it cannot escape into $X$. Hence $X$ is a closed completely invariant subset of the EPM mapping $\tau : [0, 1] \to [0, 1]$ (see [12]). Clearly $\tau | X = T | X$. Next take the function $f$ constructed in Lemma 7.5 and define

$$\gamma(x) := \begin{cases} 
f(x), & x \in J, \\
(2 - 4\epsilon), & x \in [0, 1] \setminus J.
\end{cases}$$

We then observe:
We conclude this section with some additional remarks.

(i) The one-sided limits \( \lim_{y \to x^-} \gamma(y) \) and \( \lim_{y \to x^+} \gamma(y) \) exist for all \( x \in (0, 1) \);
(ii) One has \( \int_a^b \gamma(x) \, dx = \tau(b) - \tau(a) \) for all \( [a, b] \subseteq J \setminus D \cup \{ x_i, y_i \mid i = 1, \ldots, N \} \cup \{ 0, 1 \} \).

This makes \( \gamma \) to a regular derivative of \( \tau \) (see [12]). Define the pressure function

\[
p(s) := \sup \left\{ h_\mu(\tau|X) + \int_X -s \cdot \log |\gamma|X| \, d\mu \right\},
\]

where the supremum is taken over all \( \tau|X \)-invariant Borel probability measures \( \mu \) on \( X \). By [12] the function \( s \mapsto p(s) \) has a unique zero \( s_0 \in (0, 1] \) such that \( s_0 = \dim_H X \), if \( p(0) > 0 \). By \( \tau|X = T|X \) clearly \( p(s) = S(T|X, -s \cdot \log |\gamma|X|) \).

Thus using Lemma 7.6 and the variational principle for topological pressure

\[
p(0) = S(T|X, 0) = h_{\text{top}}(T|X) = \log N > 0.
\]

This means the unique zero \( s_0 \in (0, 1] \) such that \( s_0 = \dim_H X \) actually exists. On the other hand by Lemma 7.5 and construction of \( \gamma \) we know that

\[
-s \cdot \log |\gamma|X|(x) = -s \cdot \log |f|X|(x) |f|X|(x)
\]

is upper semi-continuous in all \( x \in X \) and for every \( s \geq 0 \). Hence by Corollary 6 and \( \psi := f|X \),

\[
p(s) = P_X(T|X, -s \cdot \log |\psi|)
\]

for all \( s \geq 0 \). This shows the statement. \( \square \)

**Remark 21.** We conclude this section with some additional remarks.

(a) We want to emphasize, that if one wants to compute the dimension of a cookie cutter, one can readily apply the Bowen formula which was established in [12]. That is, using the variational pressure \( p(s) \) defined in [12], the function \( s \mapsto p(s) \) has a unique zero, which coincides with the Hausdorff and box counting dimension. In the present paper we establish in addition a Bowen formula for the topological pressure, by using Corollary 6. Both types of pressure use a discontinuous potential, if \( X \cap D \neq \emptyset \). On the other hand, as we are dealing with \( \mathbb{R}^1 \), the underlying EPM map can be made continuous (see [13] A.5), and this fact is implicitly used in the proof of the Bowen formula in [12]. Thus, by changing the topology, one can always relate \( p(s) \) to the classical topological pressure of a continuous potential on the new system. However, as we will see in Remark 23 for our application it is more convenient to use the pressure for discontinuous potentials, instead of changing the whole system to allow one continuous potential.

(b) Denote by \( \psi \) the constructed proper extension in Theorem 7.4. It is easy to see that

\[
\dim_H X = \sup \left\{ \frac{h_\mu(T|X)}{\int_X \log |\psi| \, d\mu} : \mu \in \mathcal{M} \right\},
\]

where \( \mathcal{M} \) can be the set \( \mathcal{M}_{T|X}(X) \), the set \( \mathcal{E}_{T|X}(X) \), or the set \( \mathcal{E}^+_{T|X}(X) := \{ \nu \in \mathcal{E}_{T|X}(X) : h_\nu(T|X) > 0 \} \). As no ergodic measure \( \nu \in \mathcal{E}^+_{T|X}(X) \) has any atoms, we can change \( f \) on countable many points to some new function \( \tilde{\psi} : X \to \mathbb{R} \setminus \{ 0 \} \) and again obtain

\[
\dim_H X = \sup \left\{ \frac{h_\mu(T|X)}{\int_X \log |\tilde{\psi}| \, d\mu} : \mu \in \mathcal{E}^+_{T|X}(X) \right\}.
\]
On the other hand the corresponding pressure functions for \( \log|\psi| \) might not have a zero, nor a variational principle for the topological pressure has to exist.

(c) By Lemma 7.6 the entropy mapping \( h : \mathcal{M}_{T|X}(X) \to [0, \infty], \mu \mapsto h_\mu(T|X) \), is upper semi-continuous. Similarly, by Lemma 7.5 and Proposition 7 the mapping \( \mu \mapsto \int_X -s \cdot \log|\psi| \, d\mu \) is upper semi-continuous too. Hence there exists for every \( s \geq 0 \) an equilibrium state \( \mu_s \in \mathcal{M}_{T|X}(X) \) such that

\[
\begin{align*}
  h_\mu_s(T|X) + \int_X -s \cdot \log|\psi| \, d\mu_s &= P_X(T|X, -s \cdot \log |\psi|).
\end{align*}
\]

In particular there is a \( \mu_{s_0} \in \mathcal{M}_{T|X}(X) \) such that

\[
\dim_H X = \frac{h_{\mu_{s_0}}(T|X)}{\int_X \log|\psi| \, d\mu_{s_0}}.
\]

8. **Continuity of pressure and continuity of the Hausdorff dimension of cookie-cutters.** The aim of the last section is to prove a continuity property of the Hausdorff dimension of cookie-cutters. This can be easily done with the help of the topological pressure, as it defines a Lipschitz continuous operator on the space of bounded measurable functions.

**Definition 8.1.** Let \((X, d)\) be a metric space. For every measurable and bounded \( \varphi : X \to \mathbb{R} \), we define the norm of \( \varphi \) on \( X \) to be

\[ ||\varphi||_X := \sup_{x \in X} |\varphi(x)|. \]

Thus \( ||\cdot||_X \) induces a norm on the \( \mathbb{R} \)-vector space \( I(X) \), which is defined as the set of all measurable, bounded functions \( \varphi : X \to \mathbb{R} \).

**Proposition 11.** Let \((X, T)\) be a dynamical system such that \( h_{\text{top}}(T) < \infty \). Then one has for all \( \varphi \in I(X) \)

\[ h_{\text{top}}(T) - ||\varphi||_X \leq P_X(T, \varphi) \leq h_{\text{top}}(T) + ||\varphi||_X, \]

hence \( P_X(T, \varphi) \) is finite. In particular one has

\[ |P_X(T, \varphi_1) - P_X(T, \varphi_2)| \leq ||\varphi_1 - \varphi_2||_X \]

for all \( \varphi_1, \varphi_2 \in I(X) \), that is \( P_X(T, \cdot) : I(X) \to \mathbb{R}, \varphi \mapsto P_X(T, \varphi) \) is Lipschitz continuous.

**Proof.** The first statement follows from the definition of pressure. By Remark 2 we have \( \overline{M}_X(T, \varphi, \epsilon, n) \) to be finite for all \( n \in \mathbb{N}, \epsilon > 0 \). Thus the second statement can be similarly proven like [19] Theorem 9.7 (iv).

**Theorem 8.2.** Let \( T_n : J \to [0, 1] \) be cookie-cutters with corresponding attractors \( X_n \) for \( n \in \mathbb{N} \). Denote by \( f_n : J \to \mathbb{R} \) the lower semi-continuous extensions of \( T_n \) as constructed in Lemma 7.3. Suppose there is a cookie-cutter \( T_\infty : J \to [0, 1] \) with attractor \( X_\infty \) and its lower semi-continuous extension \( f_\infty : J \to \mathbb{R} \) such that

\[ \lim_{n \to \infty} ||f_n - f_\infty||_J = 0. \]

Then one has

\[ \lim_{n \to \infty} \dim_H X_n = \dim_H X_\infty. \]
Remark 22. Above theorem basically states, that if one has a classical, smooth cookie-cutter $T : J \to [0,1]$, the Hausdorff dimension of its attractor changes only slightly, if one adds some tiny corners to $T$. Another way to view this theorem is that in terms of the dimension, a smooth cookie-cutter can be approximated by cookie-cutters with discontinuous geometric potentials. Note that the theorem cannot be used the other way around, i.e. to approximate a cookie-cutter with discontinuous geometric potentials by smooth cookie-cutters, as $\lim_{n \to \infty} \|f_n - f_\infty\|_J = 0$ cannot hold in this case.

Figure 3. The cookie-cutters $T_n$ approaching the limit cookie-cutter $T_\infty$.

Before we prove the theorem, we recall the following lemma:

Lemma 8.3. Let $(X,d)$, $(Y,\rho)$ be metric spaces, $S : X \to Y$ be continuous and $f : Y \to \mathbb{R}$ be upper semi-continuous. Then $f \circ S : X \to \mathbb{R}$ is upper semi-continuous.

Proof of Theorem 8.2. Clearly one has $f_n \in I(J)$ for all $n \in \mathbb{N} \cup \{\infty\}$. Assume $\sup_{n \in \mathbb{N}} \|f_n\|_J = \infty$, then there exist some $n_k \in \mathbb{N}$, $x_k \in J$ such that $\lim_{k \to \infty} |f_{n_k}(x_k)| = \infty$. Thus $\lim_{k \to \infty} |f_{n_k}(x_k) - f_\infty(x_k)| = \infty$, which is a contradiction. Hence there is a constant $C > 1$ such that

$$\|f_n(x)\| \in [1, C] \quad (13)$$

for all $x \in J$, $n \in \mathbb{N} \cup \{\infty\}$. As $x \mapsto \log(x)$ is Lipschitz continuous on $[1, C]$ with Lipschitz constant $1$, we in addition have

$$\lim_{n \to \infty} \left\| - \log |f_\infty| + \log |f_n| \right\|_J \leq \lim_{n \to \infty} \left\| |f_n| - |f_\infty| \right\|_J = 0. \quad (14)$$

Next denote for each $n \in \mathbb{N} \cup \{\infty\}$ by $\pi_n : \Sigma_N^+ \to X_n$ the topological conjugation introduced in the proof of Lemma 7.6. Define

$$\lambda_n^s := -s \cdot \log |f_n \circ \pi_n| : \Sigma_N^+ \to \mathbb{R}$$

for all $n \in \mathbb{N} \cup \{\infty\}$, $s \geq 0$. All $\lambda_n^s$ are bounded, and by Lemma 8.3 upper semi-continuous. Thus using the variational principle one has

$$P_{\Sigma_N^+}(\sigma_N, \lambda_n^s) \supset \left\{ h_\mu(\sigma_N) + \int_{\Sigma_N^+} \lambda_n^s \, d\mu : \mu \in \mathcal{M}(\Sigma_N^+) \right\}$$
for all \( n \in \mathbb{N} \cup \{ \infty \} \), \( s \geq 0 \). The topological conjugations \( \pi_n \) induce a one-to-one correspondence between \( \mathcal{M}_{\sigma_N}(\Sigma_N) \) and \( \mathcal{M}_{T_n}(X_n) \) by \( \mu \mapsto \mu \circ \pi_n^{-1} \). In particular one has \( h_{\mu}(\sigma_N) = h_{\mu \circ \pi_n^{-1}}(T_n|X_n) \) and
\[
\int_{\Sigma_N^+} \lambda_n^* \, d\mu = \int_{X_n} -s \cdot \log |f_n| \cdot d(\mu \circ \pi_n^{-1}).
\]
This means
\[
P_{\Sigma_N^+}(\sigma_N, \lambda_n^*) = \sup \left\{ h_{\mu}(T_n|X_n) + \int_{X_n} -s \cdot \log |f_n| \cdot d\mu : \mu \in \mathcal{M}_{T_n|X_n}(X_n) \right\},
\]
hence, using the variational principle again,
\[
P_{\Sigma_N^+}(\sigma_N, \lambda_n^*) = P_{X_n}(T_n|X_n, -s \cdot \log |f_n|)
\]
for all \( n \in \mathbb{N} \cup \{ \infty \} \), \( s \geq 0 \).

Now denote \( s_n := \dim_X X_n \) for all \( n \in \mathbb{N} \cup \{ \infty \} \). Let \( s_{n_k} \) be some convergent subsequence with limit \( s^* \). Recall that by (15) and Theorem 7.4 each \( s_{n_k} \) is the unique zero of \( s \mapsto P_{\Sigma_N^+}(\sigma_N, \lambda_{n_k}^*) \). Thus one has by Proposition 11 (13) and (14)
\[
\left| P_{\Sigma_N^+}(\sigma_N, \lambda_{\infty}^*) \right| \\
\leq \left| P_{\Sigma_N^+}(\sigma_N, \lambda_{n_k}^*) - P_{\Sigma_N^+}(\sigma_N, \lambda_{n_k}^*) \right| + \left| P_{\Sigma_N^+}(\sigma_N, \lambda_{n_k}^*) - P_{\Sigma_N^+}(\sigma_N, \lambda_{n_k}^*) \right| \\
\leq s^* \cdot \| -\log |f_{\infty} \circ \pi_{\infty}| + \log |f_{n_k} \circ \pi_{n_k}| \|_{\Sigma_N^+} + \| s^* - s_{n_k} \| \cdot \| \log |f_{n_k} \circ \pi_{n_k}| \|_{\Sigma_N^+} \\
\leq s^* \cdot \| -\log |f_{\infty}| + \log |f_{n_k}| \|_{f} + \| s^* - s_{n_k} \| \cdot \| \log |f_{n_k}| \|_{f} \to 0
\]
as \( k \to \infty \). This means \( P_{\Sigma_N^+}(\sigma_N, \lambda_{\infty}^*) = 0 \), hence
\[
s^* = \dim_H X_\infty = \lim_{k \to \infty} \dim_H X_{n_k}.
\]
As \( s_n \in [0, 1] \) for all \( n \in \mathbb{N} \), there exists at least one convergent subsequence. This shows by (16) that the limit of \( s_n \) exists and one has
\[
\lim_{n \to \infty} \dim_H X_n = \dim_H X_\infty.
\]

\[\square\]

Remark 23. The two key observations for the proof of Theorem 8.2 are:

(a) The operator \( \varphi \mapsto P_X(T, \varphi) \) is Lipschitz continuous.

(b) For each \( n \in \mathbb{N} \cup \{ \infty \} \) there is a topological conjugation \( \Sigma_N^+ \to X_n \).

To apply (a), one has to relate the variational pressure and the topological pressure via a variational principle. For this, we used Corollary 8. As mentioned in Remark 21 there is also the option to change the underlying EPM system of each \( X_n \) into a new system, where dynamics and the potential are continuous again. Then one would be able to use the classical variational principle. However, by changing the system, one has to introduce new symbolic spaces \( \mathcal{F}_n \subset \Sigma_{N_n}^+ \), where the numbers \( N_n \) depend on the discontinuities of the functions \( f_n \), and each \( \mathcal{F}_n \) is a subshift of the full shift on \( \Sigma_{N_n}^+ \). Using this approach together with our method of proof, only a weaker version of Theorem 8.2 can be proven: To satisfy (b), one has to assume in addition that all \( \mathcal{F}_n \) are pairwise topological conjugated for \( n \in \mathbb{N} \cup \{ \infty \} \).
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Received October 2015; revised November 2016.
E-mail address: marc.rauch@uni-jena.de