A variational formulation for constitutive laws described by bipotentials

Marius Buliga
Simion Stoilow Institute of Mathematics of the Romanian Academy, Romania

Géry de Saxcé
Laboratoire de Mécanique de Lille, Université des Sciences et Technologies de Lille, France

Claude Vallée
Institut Pprime, France

Abstract
We propose a rigorous variational formulation and an algorithm for solving the discretisation in time of the evolution problem for an implicit standard material, in terms of bipotentials.

Keywords
bipotentials theory, non-associated constitutive laws, variational principles

1. Introduction
Berga and de Saxcé [1] proposed a bipotential for the constitutive law of a soil and further they gave a variational formulation of this model.

We are interested in the precise formulation of the model, and we would especially like to understand from a mathematical viewpoint the recipe proposed by them for using bipotentials in order to get a variational formulation of their model. We regard this as the first step towards the establishment of a general variational theory of bipotentials.

The following paragraph, extracted from Berga and de Saxcé [1] page 414, is revealing for two reasons: (a) for understanding the motivation for introducing the bifunctional in order to adapt the Uzawa algorithm for implicit constitutive laws; (b) the imprecision concerning the understanding of the proposed new algorithm, because, as we shall see, the simultaneous minimisation of the bifunctional is not in fact how the algorithm works.

‘One of the advantages of the new formulation is to extend the classical Calculus of Variations to non-associated constitutive laws. In the theoretical frame of the Implicit Standard Materials, a new functional, called bifunctional, is introduced, depending on both the displacement and stress field. The exact solution of the Boundary Value Problem corresponds to the simultaneous minimisation of the bifunctional, firstly with respect to kinematically admissible displacement fields, when the stress field is equal with the exact one, and
secondly with respect to statically admissible stress fields, when the displacement field is the exact one. The two minimisation problems are the direct extension of the dual variational principles of displacements and stresses.’

The bipotential (see Definition 3.1) was introduced in Saxcé and Feng [2], in order to formulate a large family of non-associated constitutive laws in terms of convex analysis. The basic idea is explained further in a few words. In mechanics, associate constitutive laws are simply relations \( y \in \partial \phi(x) \), with \( \phi : X \to \mathbb{R} \cup \{ +\infty \} \), a convex and lower semicontinuous (lsc) function. According to the Fenchel inequality such a relation is equivalent to \( \phi(x) + \phi^*(y) = \langle x, y \rangle \), where \( \phi^* \) is the Fenchel conjugate of \( \phi \). It can be noticed that often in the mathematical study of problems related to the associated constitutive laws, that instead of the function \( \phi \), we find the expression

\[
b(x, y) = \phi(x) + \phi^*(y)
\]

which we call the ‘separable bipotential’. The idea is then to use as a basic notion the bipotential \( b : X \times Y \to \mathbb{R} \cup \{ +\infty \} \), which is convex and lsc in each argument and satisfies a generalisation of the Fenchel inequality. Thus, for non-associated constitutive laws there correspond bipotentials, which are not separable.

There are many such laws that can be studied with the help of bipotentials, as witnessed by the papers listed below. In many of these papers bipotentials are used for numerical purposes and several ad hoc algorithms have been suggested and exploited for applications. Here is a partial list of constitutive laws that have been suggested and exploited for applications. Here is a partial list of constitutive laws that have been suggested and exploited for applications.

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The authors started a mathematical study of bipotentials and their relation to convex analysis [17–19]. This paper is another contribution to this subject, concerning mathematically sound variational formulations and algorithms for numerically solving the quasistatic evolution problem for constitutive laws of implicit standard materials. For another paper that contains a variational formulation via bipotentials for the particular case of separated bipotentials, see Matei and Niculescu [20].

2. Notation and prerequisites from convex analysis

\( X \) and \( Y \) are topological, locally convex, real vector spaces of dual variables \( x \in X \) and \( y \in Y \), with the duality product \( \langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R} \). We shall suppose that \( X, Y \) have topologies compatible with the duality product, that is: any continuous linear functional on \( X \) (respectively \( Y \)) has the form \( x \mapsto \langle x, y \rangle \), for some \( y \in Y \) (respectively \( y \mapsto \langle x, y \rangle \), for some \( x \in X \)). We use the following notation:

1. \( \overline{\mathbb{R}} = \mathbb{R} \cup \{ +\infty \} \).
2. The domain of a function \( \phi : X \to \overline{\mathbb{R}} \) is dom \( \phi = \{ x \in X : \phi(x) \in \mathbb{R} \} \).
3. \( \Gamma_0(X) = \{ \phi : X \to \overline{\mathbb{R}} : \phi \text{ is lsc and dom } \phi \neq \emptyset \} \).
4. For any convex and closed set \( A \subset X \), its indicator function, \( \Psi_A \), is defined by

\[
\Psi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise.} \end{cases}
\]

5. The subgradient of a function \( \phi : X \to \overline{\mathbb{R}} \) at a point \( x \in X \) is the (possibly empty) set

\[
\partial \phi(x) = \{ u \in Y \mid \forall z \in X (z - x, u) \leq \phi(z) - \phi(x) \}
\]

6. The inf-convolution of two functions \( \phi, \psi \in \Gamma_0(X) \) is the function \( \phi \Box \psi \in \Gamma_0(X) \) defined by: for any \( x \in X \)

\[
\phi \Box \psi(x) = \inf \{ \phi(u) + \psi(v) : u + v = x \}
\]

3. Bipotentials and syncs

Definition 3.1. A bipotential is a function \( b : X \times Y \to \overline{\mathbb{R}} \), with the properties:

(a) for any \( x \in X \), if dom \( b(x, \cdot) \neq \emptyset \) then \( b(x, \cdot) \in \Gamma_0(X) \); for any \( y \in Y \), if dom \( b(\cdot, y) \neq \emptyset \) then \( b(\cdot, y) \in \Gamma_0(Y) \);
Definition 3.2. A sync (synchronised convex function) is a function $c : X \times Y \to [0, +\infty]$ with the properties:

(a) for any $x \in X$, if $\text{dom} \ c(x, \cdot) \neq \emptyset$ then $c(x, \cdot) \in \Gamma_0(X)$; for any $y \in Y$, if $\text{dom} \ c(\cdot, y) \neq \emptyset$ then $c(\cdot, y) \in \Gamma_0(Y)$;
(b) for any $x \in X$, if $\text{dom} \ c(x, \cdot) \neq \emptyset$ and the minimum $\min \{c(x, y) : y \in Y\}$ exists then this minimum equals $0$; for any $y \in Y$, if $\text{dom} \ c(\cdot, y) \neq \emptyset$ and the minimum $\min \{c(x, y) : x \in X\}$ exists then this minimum equals $0$.

Remark 1. The string of equivalences (1) justifies the name ‘synchronised convex function’, as it expresses the fact that critical points of functions $c(x, \cdot)$ are related to critical points of functions $c(\cdot, y)$.

With the notation from Proposition 1, we have $M(b) = c^{-1}(0)$. Also, for any $x \in X$ and $y \in Y$, property (a) of Definition 3.2 for syncs is equivalent to

$$\text{epi}(c) \cap \{x\} \times Y \times \mathbb{R} \text{ and } \text{epi}(c) \cap X \times \{y\} \times \mathbb{R}$$

are closed convex sets, where $\text{epi}(c)$ is the epigraph of $c$

$$\text{epi}(c) = \{(x, y, r) \in X \times Y \times \mathbb{R} : c(x, y) \leq r\}$$

An interesting fact is that there are no duality products in the definition of syncs. As an application, let $(X, Y)$ be a pair of spaces, $(\cdot, \cdot)$ and $(\cdot, \cdot)'$ be two duality products, defined on $X \times Y$, and $c : X \times Y \to [0, +\infty]$ be a sync. We define the applications

$$b, b' : X \times Y \to \mathbb{R} \cup \{+\infty\}$$

$$b(x, y) = c(x, y) + \langle x, y \rangle$$

$$b'(x, y) = c(x, y) + \langle x, y \rangle'$$

Then $b$ is a bipotential with respect to $(\cdot, \cdot)$ and $b'$ is a bipotential with respect to $(\cdot, \cdot)'$. As a corollary, if we have a bipotential $b$ with respect to the duality product $(\cdot, \cdot)$ and $(\cdot, \cdot)'$ is another duality product, then the application $b'$ defined by $b'(x, y) = b(x, y) - \langle x, y \rangle + \langle x, y \rangle'$ is a bipotential with respect to the duality product $(\cdot, \cdot)'$ and $M(b) = M(b')$ (they describe the same law). More generally, we have the following proposition concerning transformations of syncs.

Proposition 2. Let $(X, Y, (\cdot, \cdot)), (X', Y', (\cdot, \cdot)')$ be two pairs of spaces with their respective duality products, $T : X \to X'$ and $L : Y \to Y'$ be two linear, bijective, continuous transformations, $\alpha > 0$ and $c' : X' \times Y' \to [0, +\infty]$ be a sync. Then the function

$$c : X \times Y \to [0, +\infty], \quad c(x, y) = \alpha c'(Tx, Ly)$$

is a sync and $c^{-1}(0) = c'^{-1}(0)$. 
Proof. The application $c'$ is a sync, therefore it satisfies conditions (a) and (b) in Definition 3.2. It is straightforward to see that $c$ is convex and lsc in each argument, therefore condition (a) in Definition 3.2 is a consequence of the same condition for $c'$. Also, because $T$ and $L$ are bijective, condition (b) for the application $c$ follows from the same condition for $c'$. □

The following is Definition 3.1 from Buliga et al. [18].

**Definition 3.3.** A non-empty set $M \subset X \times Y$ is a BB graph (bi-convex and bi-closed) if for any $x \in X$ and $y \in Y$ the sections

$$M(x) = \{ y \in Y \mid (x, y) \in M \}$$

$$M^*(y) = \{ x \in X \mid (x, y) \in M \}$$

are convex and closed.

For any BB graph $M$ the indicator function $\Psi_M$ is obviously a sync. For this sync, there is a corresponding bipotential

$$b_\infty(x, y) = \langle x, y \rangle + \Psi_M(x, y)$$

In particular, this shows that we may associate more than one bipotential with a BB graph. Indeed, if $M$ is maximal cyclically monotone then it is the graph of a separable bipotential, but also the graph of the bipotential associated with the sync $\Psi_M$ (that is a bipotential of the form $b_\infty$). Therefore maximal cyclically monotone graphs have at least two distinct bipotentials.

4. **Implicit standard materials described by bipotentials**

In the mechanics of standard materials, the evolution problem is generally given by a set of equations, inequalities, boundary conditions and initial conditions. They can be structured as three groups: kinematical equations, equilibrium equations and the constitutive law modelling the material’s behaviour.

4.1. **Notation**

The configuration of a body is represented by $\Omega$, an open, bounded set with piecewise smooth boundary $\partial \Omega$.

We denote by $n$ the dimension of the configuration space ($n = 1, 2$ or $3$), thus $\Omega \subset \mathbb{R}^n$.

The boundary decomposes into two disjoint parts: on $\partial_0 \Omega$ displacements are imposed, while given surface forces act on the remaining part of the boundary denoted by $\partial_1 \Omega$. The closure of $\Omega$ is denoted by $\overline{\Omega}$.

The following quantities are required:

1. $u$ is the displacement of the body with respect to the configuration $\Omega$.
2. $\varepsilon = D(u) = 1/2 \left( \nabla u + \nabla u^T \right)$ is the associated strain. The trace of the strain is denoted by $\varepsilon_m = 1/n \text{tr} \varepsilon$ and the strain deviator is

$$\varepsilon = \varepsilon^e + \varepsilon^p$$

3. The strain $\varepsilon$ decomposes additively into elastic and plastic strains

4. The stress field is denoted by $\sigma$, its trace is the hydrostatic pressure $s_m = \text{tr} \sigma$ and $s$ denotes the stress deviator.
5. $S$ is the elasticity tensor modulus.
6. The density of volumic forces is $f_v$; the surface forces act with density $f_s$ on $\partial_1 \Omega$. The class of stress fields $\sigma$ that satisfy the equilibrium equations

$$\text{div} \sigma + f_v = 0 \text{ in } \Omega, \quad \sigma \cdot n = f_s \text{ on } \partial_1 \Omega$$

is denoted by $SA(f_v, f_s)$. 
7. The imposed boundary displacements on $\partial_0 \Omega$ are denoted by $\tilde{u}$. In fact, it is useful for further computation to consider the imposed boundary displacement $\tilde{u}$ to be defined over all $\Omega$. The class of displacements $u$, such that $u - \tilde{u} = 0$ on $\partial_0 \Omega$ (possibly in the sense of trace) is denoted by $CA(\tilde{u})$ and called the class of displacements that are kinematically admissible with respect to $\tilde{u}$.

Let $\text{Sym}(n)$ be the space of $n \times n$ real symmetric matrices and $\text{Sym}_0(n) \subset \text{Sym}(n)$ be the subspace of real symmetric matrices with null trace. The decomposition of a real symmetric matrix into hydrostatic and deviatoric parts can be expressed by the linear bijective transformations

$$T_1 : \text{Sym}(n) \to \mathbb{R} \times \text{Sym}_0(n), \quad T_1(\varepsilon) = \left( \frac{1}{n} \text{tr} \varepsilon, \varepsilon - \frac{1}{n} (\text{tr} \varepsilon) I_n \right)$$

$$T_2 : \text{Sym}(n) \to \mathbb{R} \times \text{Sym}_0(n), \quad T_2(\sigma) = \left( \text{tr} \sigma, \sigma - \frac{1}{n} (\text{tr} \sigma) I_n \right)$$

With the defined notation, for any strain value $\varepsilon \in \text{Sym}(n)$, or for any stress value $\sigma \in \text{Sym}(n)$, the decomposition into the hydrostatic and deviatoric parts is

$$T_1(\varepsilon) = (e_m, e), \quad T_2(\sigma) = (s_m, s)$$

(In order to keep track of physical dimensions, we should introduce two spaces $\text{Sym}(n)$, one for strains and the other for stresses, or introduce units of measure, but we feel that such notation only makes the presentation unnecessarily complicated.)

We shall consider the following duality products

$$\langle \cdot, \cdot \rangle : \text{Sym}(n) \times \text{Sym}(n) \to \mathbb{R}, \quad \langle \varepsilon, \sigma \rangle = \text{tr} (\varepsilon \sigma)$$

$$\langle \cdot, \cdot \rangle' : (\mathbb{R} \times \text{Sym}_0(n)) \times (\mathbb{R} \times \text{Sym}_0(n)) \to \mathbb{R}, \quad (\langle e_m, e \rangle, (s_m, s))' = e_m s_m + \langle e, s \rangle$$

Notice that the first duality product is the one in the formulation of the dissipation (as an integral over the body configuration $\Omega$ of $\langle \dot{\varepsilon}^p, \sigma \rangle$). The second duality product will be used for the plastic bipotential; see later for the example of the Berga and de Saxcé bipotential for the non-associative Drücker–Prager law. The relation between these dualities is

$$\langle \varepsilon, \sigma \rangle = \langle T_1(\varepsilon), T_2(\sigma) \rangle'$$

therefore (by passing to associated syncing and back) we can easily transform bipotentials expressed in coordinates $(\varepsilon, \sigma)$ into bipotentials expressed in coordinates $((e_m, e), (s_m, s))$.

The kinematical equations are

$$\varepsilon = \frac{1}{2} (\nabla u + \nabla u^T), \quad u \in CA(\tilde{u}) \quad (3)$$

The equilibrium equations are

$$\sigma \in SA(f_v, f_s) \quad (4)$$

The constitutive equations (besides the additive decomposition of the strain into elastic and plastic parts) are expressed with two bipotentials: the elastic and the plastic bipotentials, respectively.

The elastic bipotential is defined by the elasticity tensor modulus and it has the form

$$b_e(\varepsilon^e, \sigma) = \frac{1}{2} (\varepsilon^e, S \varepsilon^e) + \frac{1}{2} (S^{-1} \sigma, \sigma) \quad (5)$$

The elastic bipotential is defined over pairs of dual variables (elastic strain, stress). It is a separable bipotential, expressed as the sum of the (density of) the elastic energy and its dual. Moreover, this bipotential is quadratic in each variable.

The plastic bipotential

$$b_p = b_p(\dot{\varepsilon}^p, \sigma) \quad (6)$$
is defined over another pair of dual variables, namely (plastic strain rate, stress). In the case of standard materials, the plastic bipotential is separated (expressed as the sum of the plastic potential and its dual). For implicit standard constitutive laws that can be expressed by a bipotential (for example the non-associative Drucker–Prager law), the bipotential is not separated.

The constitutive equations are

\[ \varepsilon = \varepsilon^e + \varepsilon^p \]  
\[ \varepsilon^e \in \partial b_e(\varepsilon^e, \cdot)(\sigma) \]  
\[ \dot{\varepsilon}^p \in \partial b_p(\dot{\varepsilon}^p, \cdot)(\sigma) \]

The constitutive equation (8) is equivalent to \( \varepsilon^e = S^{-1}\sigma \), which is a linear equation. In order to enhance the resemblance between (8) and (9), we differentiate with respect to time the constitutive equation for \( \varepsilon^e \) and then express the result with the help of the elastic bipotential

\[ \dot{\varepsilon}^e \in \partial b_e(\dot{\varepsilon}^e, \cdot)(\sigma) \]

5. Non-associated Drucker–Prager elastoplasticity

An important example of an implicit standard material is provided by the non-associated Drucker–Prager constitutive law. Here we follow Berga and de Saxcé [1].

5.1. Plastically admissible stresses

The model is characterised by a Drucker–Prager plastic yielding surface. The set of plastically admissible stresses is the following cone

\[ K_{\text{stress}} = \left\{ \sigma = \frac{1}{3}s_mI + s \text{ such that } \frac{1}{k_d}||s|| + s_m \tan \phi \leq c \right\} \]

Here \( c \) is the cohesion, \( \phi \) is the friction angle and \( k_d \) is a constant whose significance is explained in Berga and de Saxcé [1] Section 3, relations (3.1) and (3.2).

We denote by \( K'_{\text{stress}} = T(K_{\text{stress}}) \) the same cone in coordinates \((s_m, s)\) of the stresses.

5.2. Plastically admissible strain rates

Let \( \theta \in [0, \phi] \) be the dilatancy angle (if \( \theta = \phi \) then we have the associated Drucker–Prager elastoplasticity). The set of admissible plastic strain rates is the cone

\[ K_{\text{strain}} = \left\{ \dot{\varepsilon}^p = \dot{\varepsilon}_m^p I + \dot{\varepsilon}^p \text{ such that } k_d \tan \theta ||\dot{\varepsilon}^p|| \leq \dot{\varepsilon}_m^p \right\} \]

We denote by \( K'_{\text{strain}} = T(K_{\text{strain}}) \) the same cone in the representation \((\varepsilon_m, \varepsilon)\) of the strains.

5.3. The flow rule

The constitutive equation for the evolution of the plastic strain has the following expression

\[ \left( (\dot{\varepsilon}_m^p + k_d(tg \phi - tg \theta) ||\dot{\varepsilon}^p||), \dot{\varepsilon}^p \right) \in \partial \Psi_{K'_{\text{stress}}}(s_m, s) \]

Theorems 4.1 and 4.2 from Berga and de Saxcé [1] are collected into the following.
Theorem 5.1. Let \( b'_p : (\mathbb{R} \times \text{Sym}(n)) \times (\mathbb{R} \times \text{Sym}(n)) \to \mathbb{R} \cup \{+\infty\} \) be the function

\[
b'_p((e_m, e), (s_m, s)) = \begin{cases} 
C_1 e_m + C_2(s_m - \frac{e}{tg \phi}) & \text{if } (s_m, s) \in \bar{K}'_{\text{stress}} \text{ and } (e_m, e) \in \bar{K}'_{\text{strain}} \\
+\infty & \text{otherwise}
\end{cases}
\]

where \( \|e\| \) is the norm defined by \( \langle e, e \rangle = \frac{e^\top e}{2} \) and the constants

\[
C_1 = \frac{c}{tg \phi}, \quad C_2 = k_d(tg \theta - tg \phi)
\]

are from the flow rule (11). Then:

(a) \( b'_p \) is a bipotential with respect to the duality product \( \langle \cdot, \cdot \rangle' \),

(b) the non-associated Drücker–Prager constitutive equation for the evolution of the plastic strain (11) can be expressed with the help of the bipotential \( b'_p \) as

\[
b'_p((\dot{e}'_m, \dot{e}'_e), (s_m, s)) = \langle (\dot{e}'_m, \dot{e}'_e), (s_m, s) \rangle'
\]

As an application of Proposition 2, we obtain the following characterisation of the Drücker–Prager constitutive law.

Corollary 1. In the coordinates \( (\varepsilon, \sigma) \in \text{Sym}(n) \times \text{Sym}(n) \), with the duality product \( \langle \varepsilon, \sigma \rangle = \text{tr}(\varepsilon \sigma) \), the non-associated Drücker–Prager constitutive law (11) can be expressed with the help of the bipotential \( b_p : \text{Sym}(n) \times \text{Sym}(n) \to \mathbb{R} \cup \{+\infty\} \) defined by

\[
b_p(\varepsilon, \sigma) = \Psi_{\text{Kstress}}(\sigma) + \Psi_{\text{Kstrain}}(\varepsilon) + \frac{1}{n} C_1 \text{tr} \varepsilon + C_2 \left( \text{tr} \sigma - \frac{c}{tg \phi} \right) \left\| \varepsilon - \frac{1}{n} (\text{tr} \varepsilon) I \right\|
\]

Remark 2. The term containing \( C_2 \) represents a coupling between the hydrostatic part of the stress and the deviatoric part of the strain (rate). If \( C_2 = 0 \) then we get the associated Drücker–Prager constitutive law. In this case the last term from the right-hand side of expression (13) can be eliminated by modifying the cones \( \bar{K}_{\text{stress}} \) and \( \bar{K}_{\text{strain}} \). But if \( C_2 \neq 0 \) such a modification cannot be made because of the coupling between the deviatoric and hydrostatic parts, so this last term in the expression of \( b_p \) cannot disappear by a modification of the cones \( \bar{K}_{\text{stress}} \) and \( \bar{K}_{\text{strain}} \).

6. Time discretisation of the evolution problem

Given initial data, the displacement \( u_0 \) and the initial plastic strain \( \varepsilon^p_0 \), the boundary data \( \bar{u} = \bar{u}(t), f_s = f_s(t) \) and the volume forces \( f_v = f_v(t) \), for \( t \in [0, T] \), a solution of the evolution problem is a collection \((u, \varepsilon^p, \varepsilon^e, \sigma)\) of fields dependent on \( t \), which satisfy the kinematical, equilibrium and constitutive equations, as well as the initial and boundary conditions.

We want to give a variational formulation of the time discretisation of the evolution problem. For this we consider a discretisation

\[
\{t_0 = 0, t_1, \ldots, t_N = T\}
\]

of the time interval \([0, T]\). For each \( k = 0, \ldots, N \) we denote by \((u_k, \varepsilon^p_0, \varepsilon^e_k, \sigma_k)\) the unknowns at the moment \( t_k \). We shall use also the notation: for any \( k = 0, \ldots, N \), let \( \Delta t_k = t_{k+1} - t_k, \Delta u_k = u_{k+1} - u_k \), and so on, for all fields, known or unknown.

Later, we will replace the time derivatives from the evolution equation by finite differences with respect to the considered time discretisation. The problem which we want to solve is the following:

6.1. Problem (Pdisc)

Given \((u_k, \varepsilon^p_k, \varepsilon^e_k, \sigma_k)\), find \((\Delta u, \Delta \varepsilon^p, \Delta \varepsilon^e, \Delta \sigma)\), the solution of

\[
\Delta \varepsilon^e + \Delta \varepsilon^p = D(\Delta u)
\]
\[ \Delta \varepsilon^e = S \Delta \sigma \] (15)

\[ \frac{1}{\Delta t_k} \Delta \varepsilon^p \in \partial b_p \left( \frac{1}{\Delta t_k} \Delta \varepsilon^p, \cdot \right) (\sigma_k + \Delta \sigma) \] (16)

\[ \Delta \sigma \in SA(\Delta f_{s,k}, \Delta f_{v,k}) \] (17)

\[ \Delta u \in_CA(\Delta \tilde{u}_k) \] (18)

The unknowns \((u_{k+1}, \varepsilon_{k+1}^p, \varepsilon_{k+1}^e, \sigma_{k+1})\) are obtained as

\[ u_{k+1} = u_k + \Delta u, \quad \varepsilon_{k+1}^p = \varepsilon_k^p + \Delta \varepsilon^p, \ldots \]

Our first concern is to express \((P_{\text{disc}})\) using bipotentials.

**Lemma 6.1.** For any \(k = 0, \ldots, N - 1\), the function

\[ b_{p,k}(\Delta \varepsilon^p, \Delta \sigma) = \Delta t_k b_p \left( \frac{1}{\Delta t_k} \Delta \varepsilon^p, \sigma_k + \Delta \sigma \right) - \langle \Delta \varepsilon^p, \sigma_k \rangle \] (19)

is a bipotential and equation (16) is equivalent to

\[ \Delta \varepsilon^p \in \partial b_{p,k}(\Delta \varepsilon^p, \cdot)(\Delta \sigma) \] (20)

**Proof.** We will show that

\[ c_{p,k}(\Delta \varepsilon^p, \Delta \sigma) = b_{p,k}(\Delta \varepsilon^p, \Delta \sigma) - \langle \Delta \varepsilon^p, \Delta \sigma \rangle \]

is a sync. For this we introduce the sync associated with the bipotential \(b_p\), namely

\[ c_p(\Delta \varepsilon^p, \Delta \sigma) = b_p(\Delta \varepsilon^p, \Delta \sigma) - \langle \Delta \varepsilon^p, \Delta \sigma \rangle \]

Observe that

\[ c_{p,k}(\Delta \varepsilon^p, \Delta \sigma) = \Delta t_k c_p \left( \frac{1}{\Delta t_k} \Delta \varepsilon^p, \sigma_k + \Delta \sigma \right) \]

We apply Proposition 2 and get the result. As a consequence, the function \(b_{p,k}\) defined by (19) is a bipotential. From here, the second part of the proposition is a straightforward computation, left for the interested reader. \(\Box\)

### 6.2. Simplification of the boundary conditions and volume forces

It is not a restriction of generality to suppose that the boundary conditions and volume forces are trivial, that is to suppose that equations (17) and (18) have the following form

\[ \Delta \sigma \in SA(0, 0) \] (21)

\[ \Delta u \in CA(0) \] (22)

Indeed, let us choose a field \(\Delta \tilde{\sigma} \in SA(\Delta f_{v,k}, \Delta f_{v,k})\). If we define the new unknowns

\[ \Delta u' = \Delta u - \Delta \tilde{u}, \quad \Delta \sigma' = \Delta \sigma - \Delta \tilde{\sigma} \]

then we could use Proposition 2 again in order to prove that the constitutive equations, in the new unknowns, can be expressed by bipotentials.

In order not to use excessive notation, we shall assume equations (21) and (22) and we shall neglect the change of unknowns (thus maintaining the notation \(\Delta u, \Delta \sigma\)).
6.3. Elimination of several unknowns

We can simplify the problem (Pdisc) by a standard argument involving the elimination of the unknowns $\Delta \varepsilon^e, \Delta \varepsilon^p$, by using an inf-convolution.

Indeed, let us denote $\Delta \varepsilon = \Delta \varepsilon^e + \Delta \varepsilon^p$. By equation (14), $\Delta \varepsilon$ can be deduced from $\Delta u$.

For any $\Delta \sigma$, the functions $b_c(\cdot, \Delta \sigma)$ and $b_{p,k}(\cdot, \Delta \sigma)$ are not everywhere infinite, are convex and lower semicontinuous, therefore we can define the inf-convolution of them

$$\Delta b_k(\Delta \varepsilon, \Delta \sigma) = \left( b_c(\cdot, \Delta \sigma) \Box b_{p,k}(\cdot, \Delta \sigma) \right)(\Delta \varepsilon)$$  \hfill (23)

**Lemma 6.2.**

$$\Delta \sigma \in \partial \Delta b_k(\cdot, \Delta \sigma)(\Delta \varepsilon)$$  \hfill (24)

is equivalent to: there are $\Delta \varepsilon^e, \Delta \varepsilon^p$, such that $\Delta \varepsilon = \Delta \varepsilon^e + \Delta \varepsilon^p$, which satisfy, together with $\Delta \sigma$, equations (15) and (16).

**Proof.** By a well-known property of inf-convolutions, equation (24) is equivalent to: there are $\Delta \varepsilon^e, \Delta \varepsilon^p$, such that $\Delta \varepsilon = \Delta \varepsilon^e + \Delta \varepsilon^p$, which satisfy

$$\Delta \sigma \in \partial b_c(\cdot, \Delta \sigma)(\Delta \varepsilon^e)$$  \hfill (25)

$$\Delta \sigma \in \partial b_{p,k}(\cdot, \Delta \sigma)(\Delta \varepsilon^p)$$  \hfill (26)

But both $b_c$ and $b_{p,k}$ are bipotentials, therefore (25) is equivalent to (15) and (26) is equivalent to (20), which is equivalent to (16) by Lemma 6.1. \hfill \Box

**Remark 3.** Because of the particular form of $b_c$ (a quadratic function), the inf-convolution $\Delta b_k(\cdot, \Delta \sigma)$ is differentiable, with Lipschitz gradient, as a kind of Moreau–Yosida regularisation. Therefore the inclusion (24) is equivalent to a standard equality, because the set from the right-hand side contains only one element. This is a well-known advantage of this elimination of unknowns in associated plasticity.

Let us list the properties of the function $\Delta b_k$:

1. It is lower semicontinuous (even differentiable, with Lipschitz gradient in the first argument).
2. $\Delta b_k$ is defined via an inf-convolution of a bipotential of type (13) with the elastic bipotential $b_c$, therefore it satisfies the same growth inequality as $b_c$, namely, there is a constant $C > 0$ such that for any $\Delta \varepsilon \in \text{Sym}(n)$ and $\Delta \sigma \in \text{Sym}(n)$, if $\Delta b_k(\Delta \varepsilon, \Delta \sigma) < +\infty$ then

$$\Delta b_k(\Delta \varepsilon, \Delta \sigma) \leq C \left( \| \Delta \varepsilon \|^2 + \| \Delta \sigma \|^2 \right)$$

where $\| \cdot \|$ is an arbitrary Euclidean norm on the space $\text{Sym}(n)$.
3. It satisfies a weak form of the Fenchel inequality

$$\Delta b_k(\Delta \varepsilon, \Delta \sigma) \geq \langle \Delta \varepsilon, \Delta \sigma \rangle$$

4. It is convex in the first argument, but not in the second, therefore it is not a bipotential, as stated in Theorem 6.1 of Berga and de Saxcé [1]. Notice, however, that the proof of Lemma 6.2 uses the fact that the function $b_{p,k}$ is a bipotential.

We collect the partial results obtained so far into the following theorem, which provides a simplified form of the problem (Pdisc).

**Theorem 6.3.** The problem (Pdisc) is equivalent to the following one: find $(\Delta u, \Delta \sigma) \in CA(0) \times SA(0, 0)$ that satisfy (24).
7. Variational formulation of the problem (Pdisc)

We give further a variational formulation à la Nayroles [21] of the following general problem, which contains (Pdisc) as a particular case.

We consider a first pair of spaces in duality:

1. \( X = L^2(\Omega, \text{Sym}(n)) \) is the space of the deformation fields \( \epsilon \).
2. \( Y = L^2(\Omega, \text{Sym}(n)) \) is the space of stress fields \( \sigma \).

Instead of equalities \( X, Y = L^2(\Omega, \text{Sym}(n)) \), we may consider that \( X \) and \( Y \) are topological, locally convex, real vector spaces of dual variables \( \epsilon \in X \) and \( \sigma \in Y \), with the duality product \( \langle \cdot, \cdot \rangle_1 : X \times Y \to \mathbb{R} \), endowed with two injective continuous linear transformations \( A : X \to L^2(\Omega, \text{Sym}(n)) \) and \( B : X \to L^2(\Omega, \text{Sym}(n)) \) such that

\[
\langle \epsilon, \sigma \rangle_1 = \int_\Omega \langle A(\epsilon)(x), B(\sigma)(x) \rangle \, dx = \langle A(\epsilon), B(\sigma) \rangle
\]

In the integral we see the duality product (scalar product) on the space \( \text{Sym}(n) \) of \( n \times n \) symmetric real matrices. In the right-hand side we see the duality product (scalar product) of \( L^2 \) with itself.

The space \( X, Y \) may be finite dimensional (for example associated with a discretisation in space by finite elements) or infinite dimensional. In the following we shall omit to mention the injections \( A, B \) or any other similar transformations which may appear. As an exception, in the following Theorem 7.1, part (I), we need the two injective continuous linear transformations \( A : X \to L^2(\Omega, \text{Sym}(n)) \) and \( B : X \to L^2(\Omega, \text{Sym}(n)) \) such that

\[
\langle \epsilon, \sigma \rangle_1 = \int_\Omega \langle A(\epsilon)(x), B(\sigma)(x) \rangle \, dx = \langle A(\epsilon), B(\sigma) \rangle
\]

We consider a function \( b : \text{Sym}(n) \times K \to \mathbb{R} \) with the following properties:

1. \( K \subseteq \text{Sym}(n) \) is a closed convex set of the form \( K = a + K_0 \), with \( a \in \text{Sym}(n) \) and \( K_0 \subseteq \text{Sym}(n) \) a closed convex cone, such that \( 0 \in K \) (this is the set of plastically admissible stresses, as in the definition of the Drücker–Prager plasticity). Let \( \pi_K : \text{Sym}(n) \to K \) be the projection on this cone.
2. \( b \) is lower semicontinuous in both arguments, differentiable with Lipschitz gradient and convex in the first argument; moreover we suppose that the Lipschitz constant of the gradient of \( b \) in the first argument is continuous with respect to the second variable.
3. \( b \) satisfies, for any \( \epsilon, \sigma \in \text{Sym}(n) \), the inequality: \( b(\epsilon, \sigma) \geq \langle \epsilon, \sigma \rangle \).
4. There is a constant \( C > 0 \) such that for any \( \epsilon \in \text{Sym}(n) \) and \( \sigma \in K \) we have

\[
b(\epsilon, \sigma) \leq C \left( \|\epsilon\|^2 + \|\sigma\|^2 \right)
\]

Associated with the function \( b \) is the ‘bifunctional’ of Berga and de Saxcé

\[
B(\epsilon, \sigma) = \int_\Omega b(\epsilon(x), \sigma(x)) \, dx
\]

Our main theorem is the following:

**Theorem 7.1.** Suppose that the function \( b \) takes only finite values, that is for any \( (\epsilon, \sigma) \in \text{Sym}(n) \times \text{Sym}(n) \) we have \( b(\epsilon, \sigma) < +\infty \).

(I) Let \( u \in U \) and \( \sigma \in Y_0 \). The pair \( (u, \sigma) \) satisfies almost everywhere in \( \Omega \)

\[
\sigma \in \partial b(\cdot, \cdot)(D(u))
\]

if and only if for any \( \epsilon \in X \) we have

\[
B(D(u), \sigma) \leq B(\epsilon, \sigma) - \langle \epsilon, \sigma \rangle_1
\]

(II) For any \( u^0 \in U \), \( \sigma^0 \in Y_0 \) there is a sequence \( (u^k, \sigma^k)_k \) in \( U \times Y_0 \), such that for any \( k \in \mathbb{N} \)
a. (global condition) for all $v \in U$ the displacement $u^{k+1} \in U$ satisfies

$$B(D(u^{k+1}), \sigma^k) \leq B(D(v), \sigma^k)$$

b. (local condition) the stress $\sigma^{k+1}$ satisfies almost everywhere in $\Omega$ the relation

$$\sigma^{k+1} \in \partial b(\cdot, \sigma^k)(D(u^{k+1}))$$

(III) If a sequence $(u^k, \sigma^k)_k$ from (II) has a subsequence (denoted by the same symbols) such that $u^k$ converges weakly in $W^{1,2}$ to $u$ and $\sigma^k$ converges weakly in $L^2$ to $\sigma$, then $(u, \sigma)$ is a solution of the problem (28).

Proof. (I) We follow the convention: we will identify an element of $g \in L^2(\Omega, \text{Sym}(n))$ (which is an equivalence class of functions) with its representant, defined almost everywhere in $\Omega$ by the Lebesgue theorem.

Let $u \in U$ and $\sigma \in Y_0$, such that we have $\sigma \in \partial b(\cdot, \sigma)(D(u))$ almost everywhere in $\Omega$. Let us take $\varepsilon \in X$. Then, by integration of the constitutive relation (and by the definition of $Y_0$), we have

$$\int_{\Omega} b(\varepsilon(x), \sigma(x)) \, dx - \int_{\Omega} \langle \varepsilon(x), \sigma(x) \rangle \, dx \geq \int_{\Omega} b(D(u)(x), \sigma(x)) \, dx$$

which is exactly the relation (28).

Conversely, let us start from the last integral inequality, supposed to be true for any $\varepsilon \in X$. Further we suppose also that $X = L^2(\Omega, \text{Sym}(n))$. Let us pick an arbitrary $x_0$ in the intersection of the Lebesgue sets of $D(u)$ and $\sigma$. For any open ball $B(x_0, r) \subset \Omega$ centered in $x_0$ we define $\varepsilon_r \in X$ such that $\varepsilon_r = D(u)$ almost everywhere outside $B$. We obviously get that

$$\frac{1}{|B(x_0, r)|} \left( \int_{B(x_0, r)} b(\varepsilon_r(x), \sigma(x)) \, dx - \int_{B(x_0, r)} \langle \varepsilon_r(x) - D(u)(x), \sigma(x) \rangle \, dx \right) \geq \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} b(D(u)(x), \sigma(x)) \, dx$$

For any $\bar{\varepsilon} \in \text{Sym}(n)$ we can choose for any $r > 0$ (but sufficiently small) an $\varepsilon_r$, such that

$$\lim_{r \to 0} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} \varepsilon_r(x) \, dx = \bar{\varepsilon}$$

and such that we can pass to the limit with $r$ to 0, to obtain

$$b(\bar{\varepsilon}, \sigma(x_0)) - \langle \bar{\varepsilon} - D(u)(x_0), \sigma(x_0) \rangle \geq b(D(u)(x_0), \sigma(x_0))$$

This is equivalent to the satisfaction of the constitutive relation almost everywhere.

(II) Suppose that for $k \in \mathbb{N}$ we have an element $(u^k, \sigma^k)$ of the sequence. We want to prove the existence of $(u^{k+1}, \sigma^{k+1})$ that satisfy the global condition (a), the local condition (b) and $\sigma^{k+1} \in Y_0$.

By the convexity, growth and continuity conditions on $b$, we easily obtain the existence of a minimiser of the functional $u \in U \mapsto B(D(u), \sigma^k)$. This proves the existence of $u^{k+1}$ that satisfy the global condition (a). The local condition (b) is in fact the definition of $\sigma^{k+1}$. Because of the differentiability and continuity conditions on $b$, it easily follows from $\sigma^k \in Y$ that $\sigma^{k+1} \in Y$. We have to prove that $\sigma^{k+1} \in Y_0$. For this, we choose an arbitrary $v \in U$ and we integrate the local condition (b). We obtain that

$$B(D(v), \sigma^k) - B(D(u^{k+1}), \sigma^k) \geq \langle D(v) - D(u^{k+1}), \sigma^{k+1} \rangle$$

The left-hand side of this inequality is non-negative (by the global condition) and it can be made arbitrarily small, for example by choosing $v = u^{k+1} + \lambda w$, for a given, but arbitrary $w \in U$ and $\lambda > 0$ smaller and smaller. In conclusion we obtain that for any $w \in U$ we have
\[ \langle D(w), \sigma^{k+1} \rangle \leq 0 \]

which implies that \( \sigma^{k+1} \in Y_0 \).

(III) Suppose that \((u^k, \sigma^k)\) converges, in the given sense, to \((u, \sigma)\). The sequence of functionals \( v \mapsto B(D(v), \sigma^k) \) converges in the variational sense to the functional \( v \mapsto B(D(v), \sigma) \), so, up to the choice of a subsequence, the minimisers of these respective functionals (namely the \( u^{k+1} \)) converge to a minimiser of the latter functional. Therefore \((u, \sigma)\) satisfy the condition

\[ B(D(v), \sigma) \geq B(D(u), \sigma) \]

for any \( v \in U \).

The limit \( \sigma \) is in \( Y_0 \) by construction. We can also pass to the limit in the integral form of the local condition, which is: for any \( \varepsilon \in X \)

\[ B(\varepsilon, \sigma^k) - \langle \varepsilon - D(u^{k+1}), \sigma^{k+1} \rangle \geq B(D(u^{k+1}), \sigma^k) \]

and we get the relation (28). \( \square \)

The previous theorem contains at part (II) an algorithm for finding a solution of the problem (Pdisc). This algorithm is the rigorous reformulation of an algorithm proposed in Berga and de Saxcè [1], Section 8.

However, this theorem can be improved (and will be in further research) in several respects. Firstly, for Drucker–Prager plasticity, the function \( \Delta b_k \) takes infinite values. In this case the algorithm for solving the problem (Pdisc) should take the following form. Let \( K \) denote the set of plastically admissible stresses. Then:

0. initialise \((u^0, \sigma^0)\) (for example take them equal to \((0, 0))\).
1. repeat: given \((u^k, \sigma^k) \in U \times Y\),
   a. (global condition) find \( u^{k+1} \) such that for all \( v \in U \)

\[ B(D(u^{k+1}), \sigma^k) - \langle D(u^{k+1}), \sigma^{k+1} \rangle \leq B(D(v), \sigma^k) - \langle D(v), \sigma^k \rangle \]

b. (local condition) define the stress \( \sigma^{k+1} \) almost everywhere in \( \Omega \) by the relation

\[ \sigma^{k+1} \in \pi_K \left( \partial b(\cdot, \sigma^k)(D(u^{k+1})) \right) \]

We don’t know yet how to prove that such a sequence converges to a solution of the problem (28), which is the weak form of the problem (Pdisc).

Secondly, by exploiting the particular expression of the functions \( b \) that appear in real plasticity problems, we may be able to prove that sequences \((u^k, \sigma^k)\) have convergent subsequences, for example by a boundedness argument.

Another potentially very interesting subject concerns Coulomb friction. This law can be expressed by a bipotential [2, 17]. It should be interesting to explore the corresponding variational formulation, where the bifunctional will contain volume integrals as well as surface integrals. Related to this see also Laborde and Renard [16].

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**Conflict of interest**

None declared.

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