Neutrino oscillations: Measuring $\theta_{13}$ including its sign

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In neutrino phenomenology, terms in the oscillation probabilities linear in $\sin \theta_{13}$ lead naturally to the question “How can one measure $\theta_{13}$ including its sign?” Here we demonstrate analytically and with a simulation of neutrino data that $P_{e\mu}$ and $P_{\mu\mu}$ at $L/E = 2\pi/\Delta_{21}$ exhibit significant linear dependence on $\theta_{13}$ in the limit of vacuum oscillations. Measurements at this particular value of $L/E$ can thus determine not only $\theta_{13}$ but also its sign, if CP violation is small.

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INTRODUCTION

To incorporate neutrino oscillations the standard model is conventionally extended by adding a mass term and a mixing matrix. This theory of three-flavor neutrino oscillations has been successful in accommodating the results of neutrino oscillation experiments, save LSND [1]. The theory contains six independent parameters: three mixing angles $\theta_{jk}$, two independent mass-squared differences $\Delta_{jk} := m_j^2 - m_k^2$, and one Dirac CP phase $\delta$. Ref. [2], for example, provides a summary of the current knowledge of the values of these parameters and describes future experiments. The long baseline (LBL) K2K experiment [3] is in the process of confirming the results of the Super-K atmospheric experiment [4] by measuring the parameters $\Delta_{32}$ and $\theta_{23}$, independent of atmospheric neutrino flux models. Future LBL experiments, MINOS [5], OPERA [6], and ICARUS [7], will improve upon the bounds for these parameters. Additionally, a global analysis of these LBL experiments could provide a lower bound on the magnitude of $\theta_{13}$ [8]. Future LBL experiments might also resolve the question of mass hierarchy and the level, if any, of CP violation in the neutrino sector [9].

We here examine the related question of how to best measure $\theta_{13}$, including its sign. In Ref. [10], we have shown that, even in the presence of matter effects [11], neutrino oscillations can be uniquely and completely parameterized with the following bounds on the angles: the CP phase $\delta$ lies in the range $[0, \pi)$; $\theta_{13}$ lies in the range $[-\pi/2, \pi/2]$; and the remaining mixing angles lie within the first quadrant. This choice of bounds has two advantages. First, present experiments limit $\theta_{13}$ to a small asymmetric region about zero [12, 13]. Other choices would break this region into two disconnected regions. Secondly, the CP violating phase is restricted to the first two quadrants; this range is sufficient to characterize all CP violating effects. Terms proportional to $\cos \delta$ are thus able to uniquely determine its value, assuming knowledge of all the other parameters.

In the next section, we analytically examine the terms of the neutrino oscillation probability formulae that are first order (linear) in $\theta_{13}$. These terms are proportional to either $\sin \delta$ or $\cos \delta$, as indicated in Refs. [2, 14]. It has been suggested [14, 15] that the presence of such terms, in part, can explain the excess of electron-like events in the Super-K atmospheric experiment [4]. In this work, we find that experiments which lie in the oscillatory region for the small (solar) mass-squared difference, an $L/E$ on the order of $10^4$ m/MeV, are sensitive to the linear $\theta_{13}$ terms. We further find that by judicious choice of the value of $L/E$ the effects of the CP violating phase $\delta$ can be suppressed if $\delta$ is near zero or $\pi$, while at the same time the effect of the linear term in $\theta_{13}$ is maximized. In the subsequent section, we utilize a simulation [13] of the existing neutrino oscillation data to further examine the ability of new data to determine $\theta_{13}$, including its sign. We assume that either CP is conserved or that our choice of the value $L/E$ has provided sufficient suppression of the CP violating terms. In the final section we summarize our conclusions and provide some thoughts on needed future theoretical work.

FORMAL ANALYSIS

In this section we provide explicit analytic expressions for three neutrino oscillations valid for the incoherent limit of the atmospheric mass-squared difference. We confine our discussion to vacuum oscillations. For LBL experiments through the earth, we indicate which values of $L$ and $E$ yield the cleanest measurement of $\theta_{13}$ by avoiding significant contributions from matter effects [11]. Additionally, we indicate qualitatively the consequences of straying outside these energies and baselines. We use these analytic expressions to examine where the effects of the linear terms can best be seen. As the magnitude of $\theta_{13}$ and the mass-scale ratio $\alpha := |\Delta_{21}|/|\Delta_{32}|$ are known to be small, one may expand the oscillation probability formulae about these parameters. In these perturbations (cf. [17]), terms which are linear in $\theta_{13}$ are suppressed by a factor of $\alpha \sim 0.03$. From this, one might conclude that effects relevant to the sign of $\theta_{13}$ are forever relegated to the realm of the unobservable.
this is not the case as we look beyond the valid region of these expansions.

We use the standard representation 17 of the three-neutrino mixing matrix with the notation $c_{jk} = \cos \theta_{jk}$, $s_{jk} = \sin \theta_{jk}$, and $\delta$ is the CP violating phase. In a three-neutrino theory, the probability that a neutrino with relativistic energy $E$ and flavor $\alpha$ will be detected a distance $L$ away as a neutrino of flavor $\beta$ is given by

$$P_{\alpha\beta}(L/E) = \delta_{\alpha\beta} - 4 \sum_{j,k=1}^{3} \text{Re}(U_{\alpha j} U_{\alpha k}^* U_{\beta j} U_{\beta k}^*) \sin^2 \varphi_{jk}$$

$$+ 2 \sum_{j,k=1}^{3} \text{Im}(U_{\alpha j} U_{\alpha k}^* U_{\beta j} U_{\beta k}^*) \sin 2\varphi_{jk},$$  

(1)

where $\varphi_{jk} := \Delta_{jk} L/E$ with $\Delta_{jk} := m_j^2 - m_k^2$.

Examining the terms which are linear in $\sin \theta_{13}$ motivates us to consider the limit in which the oscillations due to the mass-squared differences $|\Delta_{12}| \equiv |\Delta_{12}| \sim 10^{-3} eV^2$ are incoherent while the oscillations due to $|\Delta_{21}| \sim 10^{-5} eV^2$ are still relatively coherent. In this limit, we may take

$$\sin^2 \varphi_{23} = \sin^2 \varphi_{13} = \frac{1}{2}, \quad \sin 2\varphi_{23} = \sin 2\varphi_{13} = 0.$$  

(2)

The oscillation probabilities $P_{\mu e}$, $P_{\mu\mu}$, and $P_{\mu\tau}$, in the limit of incoherent atmospheric oscillations, are then given by

$$P_{\mu e} = \frac{1}{2} \sin 2\theta_{12} \cos 2\theta_{13} \sin 2\theta_{23} c_{\delta} + \sin^2 2\theta_{12} c_{13}^2 (c_{23}^2 - s_{13}^2 s_{23}^2) \sin^2 \varphi_{12} + \frac{1}{2} \sin^2 2\theta_{13} s_{23}^2 + 2J \sin 2\varphi_{12},$$

(3)

$$P_{\mu\mu} = 1 - \sin^2 2\theta_{12} \cos 2\theta_{13} \sin 2\theta_{23} c_{\delta} + (1 - \sin^2 2\theta_{12} c_{13}^2) s_{13}^2 \sin^2 2\theta_{23} + \sin^2 2\theta_{13} (c_{23}^2 - s_{13}^2 s_{23}^2) \sin^2 \varphi_{12} - 2 c_{13}^2 s_{23}^2 (1 - c_{13}^2 s_{23}^2),$$

(4)

$$P_{\mu\tau} = \sin 2\theta_{12} \cos 2\theta_{13} (1 + s_{13}^2) \sin 2\theta_{23} c_{\delta} (1 - \sin^2 2\theta_{12} c_{13}^2) s_{13}^2 \sin^2 2\theta_{23} - \frac{1}{2} \sin^2 2\theta_{12} (1 + s_{13}^2) \sin^2 2\theta_{23} + \sin^2 2\theta_{13} s_{13}^2 \sin^2 \varphi_{12} + \frac{1}{2} \sin^2 2\theta_{23} c_{13}^2 + 2J \sin 2\varphi_{12},$$

(5)

with

$$J = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{13} c_{13} \sin 2\theta_{23} c_{\delta},$$

(6)

As we are interested in the sign of $\theta_{13}$, we isolate those terms that are odd with respect to $\theta_{13}$

$$P_{\mu e}(\theta_{13}) - P_{\mu e}(-\theta_{13}) = 4J \sin 2\varphi_{12} + \sin 2\theta_{12} \cos 2\theta_{13} \sin 2\theta_{23} c_{\delta} \sin^2 \varphi_{12},$$

(7)

$$P_{\mu\mu}(\theta_{13}) - P_{\mu\mu}(-\theta_{13}) = -4(c_{13}^2 - s_{13}^2 s_{23}^2) \times \sin 2\theta_{12} \cos 2\theta_{13} \sin 2\theta_{23} c_{\delta} \sin^2 \varphi_{12},$$

(8)

$$P_{\mu\tau}(\theta_{13}) - P_{\mu\tau}(-\theta_{13}) = 4J \sin 2\varphi_{12} + 2 \sin 2\theta_{12} \cos 2\theta_{13} \sin 2\theta_{23} c_{\delta} \sin^2 \varphi_{12}.$$  

(9)

Note that the sign of $\theta_{13}$ exhibits the maximal effect whenever $\sin^2 \varphi_{12}$ is maximal for $c_{\delta} \sim 1$. This occurs whenever $\varphi_{12} = (2n + 1)\pi/2$ or, in other terms, for $L/E = 2(2n + 1)\pi/\Delta_{12}$. These oscillations will be more coherent for the smaller values of $n$.

This choice is fortuitous in that it also removes from Eqs. (4) and (8) terms dependent on $s_{13}$ as $\sin 2\varphi_{12} = 0$ whenever $\varphi_{21}$ is an odd-integer multiple of $\pi/2$. This removes the CP violating terms from consideration. The remaining terms are modulated by $c_{\delta}$. Thus for $\delta$ near zero or $\pi$, we would have a clean measurement of $\theta_{13}$. Also, should CP violation be found to be maximal, then these terms involving $c_{\delta}$ vanish.

A consequence of removing the dependence on $J$ is that CP violating effects are suppressed. At these local values of $L/E$ we have $P_{\alpha\beta} = P_{\beta\alpha}$, where $\pi$ indicates an antineutrino of flavor $\alpha$; assuming CPT is invariant, this can be expressed as $P_{\alpha\beta} = P_{\beta\alpha}$. This supports our previous statement that it is sufficient to only consider $P_{\mu e}$, $P_{\mu\mu}$, and $P_{\mu\tau}$ in regards to their dependence on the sign of $\theta_{13}$. These probabilities in addition to $P_{ee}$, which is a function of $\delta_{\alpha\beta}$, will give us all the other oscillation probabilities at this value of $L/E$. The remaining probabilities are

$$P_{ee} = 1 - P_{\mu e} - P_{\mu\mu},$$

(10)

$$P_{\tau\tau} = 1 - P_{\mu\tau} - P_{\mu e},$$

(11)

so that the dependence of $P_{e\tau}$ on the sign of $\theta_{13}$ can be surmised from the statements made concerning $P_{\mu e}$ and, likewise, the behavior of $P_{\tau\tau}$ can be surmised from $P_{\mu\tau}$ and $P_{\mu e}$. In what follows, we will assume that any CP violation is small so that we may set the phase equal to $0$.

We have previously performed 12 a simulation, assuming no CP violation, of the world’s neutrino oscillation data. An analysis which includes more recent data 13, in preparation, produces $\Delta \chi^2 := \chi^2 - \chi_{\text{min}}$ as pictured in Fig. 1. Included in the analysis are data for neutrinos from the sun 18, from reactors 14, atmospheric neutrinos 20, and beam-stop neutrinos 3. For one standard deviation, the analysis bounds $\theta_{13}$ to lie within $[-0.17, 0.22]$ with two minima located at $\theta_{13} = 0.11$ and $-0.04$. For the absolute minimum $\theta_{13} = 0.11$, we find
θ_{12} = 0.48, θ_{23} = 0.80, ∆_{21} = 7.7 \times 10^{-5} \text{ eV}^2, \text{ and } ∆_{12} = 2.6 \times 10^{-3} \text{ eV}^2.

Since P_{ee} is a function of sin^2 θ_{13}, the asymmetry seen in Fig. 1 must arise from the atmospheric and K2K data, which involve P_{eμ} and P_{μμ}. If we employ the “one mass-squared dominance” approximation, as is often done, we find the dashed curve presented in Fig. 1. This approximation gives oscillation probabilities that are a function of sin^2 θ_{13}.

In order to demonstrate the relative size of the effect of the sign of θ_{13}, we choose some realistic values for the mixing angles: θ_{12} = 0.56 and θ_{23} = 0.78. The first two peaks of sin^2 ϕ_{12} occur around L/E = 1.6 \times 10^4 \text{ m}/\text{MeV} and 4.8 \times 10^4 \text{ m}/\text{MeV}. For such values of L/E, the oscillations due to ∆_{32} and ∆_{31} would be incoherent. It is clear from Eq. (9) that the screening effect of maximal mixing for θ_{23} results in independence of the sign of θ_{13} for P_{μτ}. For the remaining two oscillation channels, we have a sizable effect. The oscillation probabilities evaluated at the one-standard-deviation points for θ_{13}, −0.17 and 0.22, are

\[ P_{eμ}(θ_{13} = −0.17) = 0.35, \quad P_{eμ}(θ_{13} = 0.22) = 0.50, \]  
\[ P_{μμ}(θ_{13} = −0.17) = 0.37, \quad P_{μμ}(θ_{13} = 0.22) = 0.22. \]  

The relative differences are more appropriate quantities to consider; for P_{μμ} we have the most significant effect

\[ \frac{P_{μμ}(θ_{13} = 0.22)− P_{μμ}(θ_{13} = −0.17)}{P_{μμ}(θ_{13} = 0.22) + P_{μμ}(θ_{13} = −0.17)} = −0.25, \]

while the effect is still large for P_{eμ},

\[ \frac{P_{eμ}(θ_{13} = 0.22) − P_{eμ}(θ_{13} = −0.17)}{P_{eμ}(θ_{13} = 0.22) + P_{eμ}(θ_{13} = −0.17)} = 0.17. \]  

We compare the one-sigma extremes of θ_{13} in order to demonstrate the potential size of the effect.

**SIMULATION**

The previous analytic work tells us where to look if we wish to observe the terms in the oscillation formulae which are linear in sin θ_{13}. We investigate this further by utilizing the analysis from Ref. [13]. We proceed by fixing all of the oscillation parameters except θ_{13} to their values that are given by an analysis in which θ_{13} is set to zero.

In Figs. 2-4 we present the oscillation probabilities P_{ee}, P_{eμ}, and P_{μμ} as a function of L/E respectively. In all cases we assume a Gaussian spread in energy of twenty percent. In all cases curves are presented for θ_{13} = 0 and θ_{13} = ±0.2. In Fig. 2 for P_{ee} there are only two curves as P_{ee} is a function of sin^2 θ_{13}. Thus the curves for θ_{13} = ±0.2 are identical. The curve is presented for completeness and to note that there is a measurable dependence of P_{ee} on the magnitude of θ_{13} near the peak at L/E = 3.2 \times 10^4 \text{ m}/\text{MeV}, near the location of KamLAND as has been indicated in Ref. [13].

In Fig. 3 we verify two facts as derived analytically in the previous section. First, there is a significant linear dependence of P_{eμ} on sin θ_{13}. Secondly, the dependence is maximal at ϕ_{12} = π/2 and 3π/2. The optimal value of L/E to measure θ_{13}, including its sign, is thus L/E = 2π/∆_{21}. The linear term in θ_{13} clearly dominates near the maximum of the oscillation. However, the constant term in Eq. (7) proportional to sin^2 θ_{13} becomes relevant whenever the probability is minimal.
the oscillations presented above remain the same. For such a situation, the qualitative features of choosing a neutrino energy twice that of the resonance can be minimized by \( \theta \)

The most notable changes for oscillations through the mantle of the curves in Figs. (3) and (4) can be salvaged. The effects become significant though the qualitative features of at least a few hundred MeV’s. Here, matter likely one’s neutrino source would produce neutrinos with energies would have to be proportionally smaller. More mixing parameters for a fit with

Finally, in Fig. 3 we present the values of \( P_{ee}, P_{em}, \) and \( P_{\mu\mu} \) as a function of \( \theta_{13} \). The other parameters are fixed at their values for a fit with \( \theta_{13} = 0 \). We use a value of \( L/E = 1.6 \times 10^4 \) m/MeV chosen so as to maximize the relative importance of the linear terms in \( \theta_{13} \). Again, \( P_{ee} \) is quadratic in \( \theta_{13} \). However, \( P_{em} \) and \( P_{\mu\mu} \) are nearly linear in \( \theta_{13} \) over this rather large range of \(-0.4 \leq \theta_{13} \leq 0.4 \). The near linearity reinforces our observation that the measurement of \( P_{em} \) and \( P_{\mu\mu} \) at \( L/E = 2\pi/\Delta_{21} \) is a way of determining \( \theta_{13} \), including its sign.

The phenomenology of neutrino oscillations, in the absence of CP violation and various exotica such as a fourth sterile neutrino, has been performed in the context of determining three mixing angles and two mass-squared differences. Historically, the results were presented in terms on \( \sin^2 \theta_{13} \), which yields an upper limit. When the bounds on the mixing angles were explicitly quoted, they were all stated to be bounded by \( \pi/2 \). In Ref. [21], it was shown that, in the physical case which necessarily includes the MSW matter effect, a second branch corresponding to \( \delta = \pi \) must also be included. In Ref. [11], we extended the derivation, again in the physical case which necessarily includes the MSW matter effect, to show that only the \( \delta = 0 \) branch is required if the mixing angle \( \theta_{13} \) is allowed to vary from \(-\pi/2\) to \(+\pi/2\). There are two advantages to this convention. First, the allowed region for \( \theta_{13} \) consists of a single region that extends on either side of 0, rather than two disjoint regions, one for \( \delta = 0 \) and one for \( \delta = \pi \). Secondly, in the presence of CP violation, the CP phase is bounded by 0 and \( \pi \). Thus a measurement that is sensitive to \( \cos \delta \) could uniquely determine the quadrant in which \( \delta \) lies.

CONCLUSION

The natural question that arises from this formal work is whether you can measure \( \theta_{13} \) including its sign. In Ref. [12], we demonstrated that for a model analysis of the world’s data, the \( \chi^2 \) space was asymmetric in \( \theta_{13} \),
thus demonstrating that such a measurement might be possible. A more recent analysis [1], which includes recent data verified this. Here we address the question directly and demonstrate that the answer to our question is yes – the mixing angle $\theta_{13}$ does appear linearly in the oscillation probabilities at a level where it dominates the quadratic term for the correctly chosen oscillation probabilities when measured at the correct value of $L/E$.

We find that this is true for measurements of $P_{ee}$ and $P_{\mu\mu}$ at $L/E = 2\pi/\Delta_{21}$. This fortuitously also corresponds to a value of $L/E$ where the contribution from the CP violating effects is minimal. A consequence of this work is that it reinforces our earlier thesis that parameterizing neutrino oscillation probabilities as a function of $\sin^2 \theta_{13}$ is inadequate.

There remains additional work to be done. We find that present data indicate a correlation between the allowed value of $\theta_{13}$ and $\theta_{23}$ when the linear terms are included in the analysis. We are investigating the implications for $\theta_{13}$ that will result from measurements of $\theta_{23}$. We have assumed no CP violation in our model analysis. Since the CP phase and $\theta_{13}$ are interrelated, this needs further clarification. For cleanliness of interpretation, we propose do experiments in a region of $E$ where the Earth MSW effect is small. We are examining in a quantitative way how the Earth MSW effect might modify an analysis should this be necessary. We, like others, have excluded the LSND [1] experiments. If MiniBoone verifies the LSND results, then a whole new physics will be needed to reach a consistent understanding of neutrino oscillations.

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