TECHNICAL REPORT

Note on the computation of the Metropolis-Hastings ratio for Birth-or-Death moves in trans-dimensional MCMC algorithms for signal decomposition problems

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Abstract

Reversible jump MCMC (RJ-MCMC) sampling techniques, which allow to jointly tackle model selection and parameter estimation problems in a coherent Bayesian framework, have become increasingly popular in the signal processing literature since the seminal paper of Andrieu and Doucet (IEEE Trans. Signal Process., 47(10), 1999). Crucial to the implementation of any RJ-MCMC sampler is the computation of the so-called Metropolis-Hastings-Green (MHG) ratio, which determines the acceptance probability for the proposed moves.

It turns out that the expression of the MHG ratio that was given in the paper of Andrieu and Doucet for “Birth-or-Death” moves—the simplest kind of trans-dimensional move, used in virtually all applications of RJ-MCMC to signal decomposition problems—was erroneous. Unfortunately, this mistake has been reproduced in many subsequent papers dealing with RJ-MCMC sampling in the signal processing literature.

This note discusses the computation of the MHG ratio, with a focus on the case where the proposal kernel can be decomposed as a mixture of simpler kernels, for which the MHG ratio is easy to compute. We provide sufficient conditions under which the MHG ratio of the mixture can be deduced from the MHG ratios of the elementary kernels of which it is composed. As an application, we consider the case of Birth-or-Death moves, and provide a corrected expression for the erroneous ratio in the paper of Andrieu and Doucet.

1 Introduction

Model selection and parameter estimation are fundamental tasks arising in many (if not all) signal processing problems, when parametric models are employed. Let us consider a collection of models \( \{ M_k, k \in K \} \), indexed by some finite or countable set \( K \subset \mathbb{N} \), with parameter vector \( \theta_k \in \Theta_k \subset \mathbb{R}^{n_k} \) under model \( M_k \). In a Bayesian framework, model selection (or averaging) and parameter estimation can in principle be carried out jointly, using the posterior distribution of the pair \( (k, \theta_k) \).

\[
\pi (k, \theta_k) \propto p (y | k, \theta_k) \ p (k, \theta_k), \tag{1}
\]

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where $y$ is the observed data and $\propto$ indicates proportionality. Note that the distribution $\pi$ is defined on $X = \bigcup_{k \in \mathcal{K}} \{k\} \times \Theta_k$, which is a disjoint union of subspaces with differing dimensionality. Generic Markov Chain Monte Carlo (MCMC) methods for probability distributions defined on such spaces became available during the 90’s, most notably Green’s widely applicable RJ-MCMC sampler [17], making it possible to use a fully Bayesian approach for model selection (or averaging) and parameter estimation in all sorts of applications. The reader is referred to [4, 9, 18, 37, 45] for a broader view on trans-dimensional sampling techniques (including alternatives to the RJ-MCMC sampler).

Green’s RJ-MCMC sampler can be seen as a generalization of the well-known Metropolis-Hastings sampler [19, 28], which is capable of exploring not only the fixed-dimensional parameter spaces $\Theta_k$, but also the space $\mathcal{K}$ of all models under consideration. This algorithm relies on an accept/reject mechanism, with an acceptance ratio calibrated in such a way that the invariant distribution of the chain is the target distribution $\pi$. The computation of this acceptance ratio for trans-dimensional moves is in general a delicate issue\(^1\), involving measure theoretic considerations.

Andrieu and Doucet [1] pioneered the use of RJ-MCMC sampling in “signal decomposition” problems, by tackling joint model selection and parameter estimation for an unknown number of sinusoidal signals observed in white Gaussian noise. (At the same period, RJ-MCMC also became popular for image processing tasks such as segmentation and object recognition; see, e.g., [11, 21, 32, 33, 41]). This seminal papers was followed by many others in the signal processing literature [3, 5, 6, 10, 20, 25–27, 30, 31, 40, 42, 43], relying systematically on the original paper [1] for the computation of the acceptance ratio of “Birth-or-Death” moves—the most elementary type of trans-dimensional move, which either adds or removes a component from the signal decomposition. Unfortunately, the expression of the acceptance ratio for Birth-or-Death moves provided by [1, Equation (20)] turns out to be erroneous, as will be explained later. Worse, the exact same mistake has been reproduced in most of the following papers, referred to above.

The aim of this note is to provide clear statements of some mathematical results, perhaps not completely new but never stated explicitly, which can be used for a clean justification of the acceptance ratio of Birth-or-Death moves in signal decomposition (and similar) problems. Section 2 recalls, very quickly, the basics of MCMC methods, with a focus on Metropolis-Hastings algorithms on general state spaces (also known as RJ-MCMC algorithms). Section 3 discusses the computation of the acceptance ratio for mixture kernels, and provides conditions under which the ratio of the mixture can be directly derived from the ratio of the elementary kernels of which it is composed. Section 4 defines Birth-or-Death moves and provides the expression of the ratio; several distinct but related mathematical representations—“unsorted vectors”, “sorted vectors” and Point processes—are discussed. As an illustration, Section 5 returns to the problem considered in [1] and provides a corrected expression for the Birth-or-Death ratio. Section 6 concludes the paper.

### 2 Background on MCMC methods

This section recalls basic definitions and results for the MCMC method. The reader is referred to [16–18, 29, 37, 38, 48, 49] for more detailed explanations.

\(^1\)Fortunately, the simple and powerful “dimension matching” argument [17] allows to bypass this difficulty for a large class of proposal distributions.
2.1 MCMC with reversible kernels

Let \( \pi \) be a probability distribution on a measurable space \((\mathcal{X}, \mathcal{B})\), which is to be sampled from. MCMC sampling methods proceed by constructing a time-homogeneous Markov chain \((x_n)\) with invariant distribution \(\pi\), using a transition kernel \(P\) that is reversible with respect to \(\pi\), i.e., a kernel that satisfies the detailed balance condition

\[
\pi(dx) P(x, dx') = \pi(dx') P(x', dx).
\]  

(2)

For all measurable sets \(A \in \mathcal{B}\), integrating (2) on \(\mathcal{X} \times A\) yields

\[
\int_{\mathcal{X}} \pi(dx) P(x, A) = \pi(A),
\]

which means that \(\pi\) is an invariant distribution for the kernel \(P\) (it is also said that “\(P\) leaves \(\pi\) invariant”).

If the transition kernel \(P\) is \(\pi\)-irreducible and aperiodic, then [48, Theorem 1] \(\pi\) is the unique invariant distribution and the chain converges in total variation to \(\pi\) for \(\pi\)-almost all starting states \(x\). If \(P\) is also Harris recurrent, then convergence occurs for all initial distributions [37, Theorem 6.51].

**Remark** Some of the above requirements on the chain \((x_n)\) can be relaxed. Most notably, time-inhomogeneous chains are used in the context of “adaptive MCMC” algorithms; see, e.g., [2, 7] and the references therein. It is also possible to depart from the reversibility assumption, which is a sufficient but not necessary condition for \(\pi\) to be an invariant distribution (see, e.g., [13]), though the vast majority of MCMC algorithms considered in the literature are based on reversible kernels.

2.2 Metropolis-Hastings-Green kernels

The very popular Metropolis-Hastings-Green kernels, sometimes simply called Metropolis-Hastings kernels, correspond to the following two-stage sampling procedure: first, given that the current state of the Markov chain is \(x \in \mathcal{X}\), a new state \(x' \in \mathcal{X}\) is proposed from a transition kernel \(Q(x, dx')\); second, this move is accepted with probability \(\alpha(x, x')\) and rejected otherwise—in which case the new state is equal to \(x\). More formally, for all \(x \in \mathcal{X}\) and \(B \in \mathcal{B}\), the transition kernel is given by

\[
P(x, B) = \int_B Q(x, dx') \alpha(x, x') + s(x) \mathbb{1}_B(x),
\]  

(3)

where \(\mathbb{1}_B\) denotes the indicator function of \(B\), and

\[
s(x) = \int_{\mathcal{X}} Q(x, dx') \left(1 - \alpha(x, x')\right)
\]

is the probability of rejection at \(x\). It is easily seen that the detailed balance condition (2) holds if and only if [17, 48, 49]

\[
\pi(dx) Q(x, dx') \alpha(x, x') = \pi(dx') Q(x', dx) \alpha(x', x).
\]  

(4)

This is achieved, for instance, by the acceptance probability

\[
\alpha(x, x') = \min \{1, r(x, x')\},
\]

(5)

where \(r(x, x')\) denotes the Metropolis-Hastings-Green (MHG) ratio

\[
r(x, x') = \frac{\pi(dx') Q(x', dx)}{\pi(dx) Q(x, dx')}. \]

(6)
The right-hand side of (6) is the Radon-Nykodim derivative of $\pi (dx') Q (x', dx)$ with respect to $\pi (dx) Q (x, dx')$; see [49, Section 2] for technical details.

**Remark** It is proved in [49, Section 4] that the acceptance probability (5) is optimal in the sense of minimizing the asymptotic variance of sample path averages among all acceptance rates satisfying (4).

# 3 Mixture of proposal kernels

## 3.1 Metropolis-Hastings-Green ratio for mixture of proposal kernels

It is often convenient to consider a proposal kernel $Q$ built as a mixture of simpler transition kernels $Q_m$, with $m$ in some finite or countable index set $\mathbb{M}$. In this case we have

$$Q (x, dx') = \sum_{m \in \mathbb{M}} j (x, m) Q_m (x, dx'),$$

where $j (x, m)$ is the probability of choosing the move type $m$ given that the current state is $x$. Note that the actual value of $Q_m (x, dx)$ is irrelevant when $j (x, m) = 0$.

It turns out that, under some assumptions, the MHG ratio for a mixture kernel $Q$ can be conveniently deduced from the elementary ratios computed for each individual kernel $Q_m$ using the formula

$$r (x, x') = \frac{j (x', m') \pi (dx') Q_{m'} (x', dx)}{j (x, m) \pi (dx) Q_m (x, dx')},$$

where $m \in \mathbb{M}$ denotes the specific move that has been used to propose $x'$, and $m' \in \mathbb{M}$ is the corresponding "reverse move". Equation (8) is routinely used in applications of the RJ-MCMC algorithm, and is alluded to in Green’s paper [17, p. 717] in the sentence: "If [other] discrete variables are generated in making proposals, the probability functions of their realized values are multiplied into the move probabilities"—but it is wrong in general. Sufficient conditions for Equation (8) to hold are provided by the following result:

**Proposition 1.** Let

$$R_m (dx, dx') = j(x, m) \pi(dx) Q_m(x, dx').$$

Assume that there exists a family of disjoint sets $\mathbb{W}_m \in \mathcal{B} \otimes \mathcal{B}$ indexed by $\mathbb{M}$ such that:

i) For each $m \in \mathbb{M}$, $R_m$ is supported by $\mathbb{W}_m$, which means $R_m (\mathbb{X}^2 \setminus \mathbb{W}_m) = 0$.

ii) Each move $m \in \mathbb{M}$ has a unique "reverse move" $\varphi(m) \in \mathbb{M}$ in the sense that $\mathbb{W}_{\varphi(m)} = \mathbb{W}_{m}^T$, where $\mathbb{W}_{m}^T = \{(x', x) : (x, x') \in \mathbb{W}_m\}$.

Then, then MHG ratio is given by Equation (8) with $m' = \varphi(m)$.

**Proof.** For $\pi(dx) Q(x, dx')$-almost everywhere on $\mathbb{X}^2$, there is a unique $m = m_{x,x'} \in \mathbb{M}$ such that $(x, x') \in \mathbb{W}_m$. Equation (8) can be rewritten as:

$$r (x, x') = \frac{R_{\varphi(m_{x,x'})} (dx', dx)}{R_{m_{x,x'}} (dx, dx')}.$$

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Then, for all \(A \in \mathcal{B} \otimes \mathcal{B}\),

\[
\int\int_A r(x, x') R(dx, dx')
= \int\int_A \frac{R_{\varphi(m, x')}(dx', dx)}{R_{m, x'}(dx, dx')} \cdot \sum_{m_0 \in \mathcal{M}} R_{m_0}(dx, dx')
= \sum_{m_0 \in \mathcal{M}} \int\int_{A \setminus W_{m_0}} \frac{R_{\varphi(m_0)}(dx', dx)}{R_{m_0}(dx, dx')} R_{m_0}(dx, dx')
= \sum_{m_0 \in \mathcal{M}} \int\int_{A \setminus W_{m_0}} R_{\varphi(m_0)}(dx, dx')
= \int\int_A R(dx, dx') \quad \text{because } W^T_{m_0} = W_{\varphi(m_0)}
= \int\int_A R(dx', dx).
\]

\[\square\]

### 3.2 Mixture representation of trans-dimensional kernels

Consider the case of a variable-dimensional space, that can be written as \(\mathcal{X} = \bigcup_{k \in \mathcal{K}} \{k\} \times \Theta_k\), with \(\mathcal{K}\) a finite or countable set (usually \(\mathcal{K} \subset \mathbb{N}\) and \(\Theta_k \subset \mathbb{R}^{n_k}\)). A point \(x \in \mathcal{X}\) is a pair \((k, \theta)\) with \(k \in \mathcal{K}\) and \(\theta \in \Theta_k\). The problem of sampling a (posterior) distribution on such a space typically occurs in the context of Bayesian model selection or averaging.

Set \(\mathcal{X}_k = \{k\} \times \Theta_k\). Any kernel \(Q\) on \(\mathcal{X}\) admits a natural representation as a mixture of fixed-dimensional and trans-dimensional kernels:

\[
Q(x, dx') = \sum_{(k, l) \in \mathcal{K}^2} p_{k,l}(x) Q_{k,l}(x, dx'), \tag{9}
\]

where

\[
p_{k,l}(x) = 1_{\mathcal{X}_k}(x) Q(x, \mathcal{X}_l),
Q_{k,l}(x, \cdot) = \frac{1}{p_{k,l}(x)} Q(x, \cdot \cap \mathcal{X}_l).
\]

(An arbitrary value can be chosen for \(Q_{k,l}(x, \cdot)\) when \(p_{k,l}(x) = 0\) to make it a completely defined transition kernel.) The kernels \(Q_{k,k}\), \(k \in \mathcal{K}\), correspond to the “fixed-dimensional” part of the transition kernel \(Q\); while the kernels \(Q_{k,l}\), \((k, l) \in \mathcal{K}^2\), \(k \neq l\), correspond to the “trans-dimensional” part.

The mixture representation (9) satisfy the assumptions of Proposition 1 with \(\mathcal{M} = \mathcal{K}^2\), \(W_{k,l} = \mathcal{X}_k \times \mathcal{X}_l\) for all \((k, l) \in \mathcal{M}\) and \(\varphi(k, l) = (l, k)\). Therefore, if the current state \(x\) is in \(\mathcal{X}_k\) and the proposed state \(x'\) in \(\mathcal{X}_l\), the MHO ratio (8) reads

\[
r(x, x') = \frac{p_{l,k}(x')}{p_{k,l}(x)} \frac{\pi(dx') Q_{l,k}(x', dx)}{\pi(dx) Q_{k,l}(x, dx')} \tag{10}
\]

In most “tutorial” papers about the RJ-MCMC method, this expression is directly written in the special case where Green’s dimension matching argument can be applied (see, e.g., [18], Sections 2.2 and 2.3). Unfortunately, the dimension matching argument does not apply directly to the
commonly used Birth-or-Death kernels (see next section) if the mixture representation (9), which leads to (10), is used.

4 Birth-or-Death kernels

4.1 Birth-or-Death kernels on (unsorted) vectors

Let us consider the situation where a point \( x \in X \) describes a set of \( k \) objects \( s_1, \ldots, s_k \in S \), with \((S, \nu)\) an atomless\(^2\) measure space and \( k \in \mathbb{N} \). One possible—and commonly used—way of representing this is to consider pairs \( (k, s) \), where the objects \( s_i, 1 \leq i \leq k \), have been arranged in a vector \( s = (s_1, \ldots, s_k) \in S^k \). The corresponding space is \( X = \bigcup_{k \geq 0} X_k \), \( X_k = \{ k \} \times S^k \), with the convention that \( S^0 = \{ \emptyset \} \).

Remark The results that will be presented in this section are easily generalized if the model includes additional (fixed-dimensional) parameters that are left unchanged by the Birth-or-Death moves (for instance the parameters \( \Lambda \) and \( \delta^2 \) in a fully Bayes version of the model presented in Section 5).

Birth-or-death kernels are the most natural kind of trans-dimensional moves in such spaces. Given \( k \in \mathbb{N}, s = (s_1, \ldots, s_k) \in S^k \) and \( s^* \in S \), we introduce the notations

\[
\begin{align*}
  s_{-i} &= (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k) \in S^{k-1}, \\
  s \oplus_i s^* &= (s_1, \ldots, s_{i-1}, s^*, s_i, \ldots, s_k) \in S^{k+1},
\end{align*}
\]

where \( 1 \leq i \leq k \) in the first case and \( 1 \leq i \leq k + 1 \) in the second case. Starting from \( x = (k, s) \), a birth move inserts a new component \( s^* \in S \), generated according to some proposal distribution \( q(s) \nu(ds) \), at a randomly selected location:

\[
Q_b(x, \cdot) = \frac{1}{k+1} \sum_{i=1}^{k+1} \int_S \delta_{(k+1, s \oplus_i s^*)} q(s^*) \nu(ds^*).
\]  

(11)

A death move, on the contrary, removes a randomly selected component form the current state:

\[
Q_d(x, \cdot) = \frac{1}{k} \sum_{i=1}^{k} \delta_{(k-1, s_{-i})}.
\]  

(12)

Finally, the birth-or-death kernel is a mixture of the two:

\[
Q(x, \cdot) = p_b(x) Q_b(x, \cdot) + p_d(x) Q_d(x, \cdot),
\]  

(13)

with \( p_b(x) \geq 0, p_d(x) \geq 0, p_b(x) + p_d(x) = 1 \), and \( p_d((0, \emptyset)) = 0 \).

4.2 Expression of the MHG ratio

The following proposition provides the expression of the MHG ratio for the model and kernel described in Section 4.1.

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\(^2\)See, e.g., [14]. As a concrete example, think of \( S = \mathbb{R}^d \) endowed with its usual Borel \( \sigma \)-algebra and \( \nu \) equal to Lebesgue’s measure. We will use the following property in the proof of Proposition 2: if \((S, \nu)\) is atomless, then the diagonal \( \Delta = \{(s, s) : s \in S\} \) is \( \nu \otimes \nu \)-negligible in \( S \times S \).
Proposition 2. Assume that, for all \( k \geq 1 \), the target measure \( \pi \) restricted to \( \mathbb{X}_k \) admits a probability density function \( f_k \) with respect to \( \nu^{\otimes k} \). Then the MHG ratio is

\[
r(x, x') = \frac{f_{k+1}(x')}{f_k(x)} \frac{p_d(x')}{p_b(x)} \frac{q(s^*)}{1}
\]

for a birth move from \( x = (k, s) \) to \( x' = (k + 1, s \oplus_i s^*) \).

Proof. Although a direct computation of the MHG ratio would be possible based on Equations (11)–(13), we find it much more illuminating to deduce the result from Proposition 1 using kernels which are simpler than \( Q_b \) and \( Q_d \). To do so, let us consider the family of elementary kernels \( Q_m \), with \( m \) in the index set

\[
\mathbb{M} = \{(\alpha, k, i) \in \{0, 1\} \times \mathbb{N}^2 : 1 \leq i \leq k + \alpha\}
\]

where \( Q_{1,k,i} \) is the kernel from \( \mathbb{X}_k \) to \( \mathbb{X}_{k+1} \) that inserts a new component \( s^* \sim q(s)\nu(ds) \) in position \( i \), and \( Q_{0,k,i} \) is the kernel from \( \mathbb{X}_k \) to \( \mathbb{X}_{k-1} \) that removes the \( i \)th component. Then we can write

\[
Q(x, \cdot) = \sum_{m \in \mathbb{M}} j(x, m) Q_m(x, \cdot),
\]

with \( j(x, m) \) defined for all \( x = (k, s) \in \mathbb{X} \) as

\[
j(x, m) = \begin{cases} p_b(x)/(k + 1) & \text{if } m = (1, k, i), 1 \leq i \leq k + 1, \\ p_d(x)/k & \text{if } m = (0, k, i), 1 \leq i \leq k, \\ 0 & \text{otherwise.} \end{cases}
\]

Denote by \( \tilde{\mathbb{X}}_k \) the set of all \( x \in \mathbb{X}_k \) in which no two components are equal. For all \( k \), \( \pi(\mathbb{X}_k \setminus \tilde{\mathbb{X}}_k) = 0 \), since \( \pi|_{\mathbb{X}_k} \) admits a density with respect to the product measure \( \nu^{\otimes k} \). The mixture representation (15) thus satisfies the assumptions of Proposition 1 with

\[
\mathbb{W}_{(1,k,i)} = \{(x, x') \in \tilde{\mathbb{X}}_k \times \tilde{\mathbb{X}}_{k+1} : \exists s \in S^k, \exists s^* \in S, \ x = (k, s), \ x' = (k + 1, s \oplus_i s^*)\},
\]

\[
\mathbb{W}_{(0,k,i)} = \mathbb{W}_{(1,k-1,i)^T}, \ \varphi(1, k, i) = (0, k + 1, i) \text{ and } \varphi(0, k, i) = (1, k - 1, i). \]

According to Proposition 1, the MHG ratio for a birth move \( m = (1, k, i) \) is thus

\[
r(x, x') = \frac{p_d(x')}{p_b(x)} \frac{\pi(dx') Q_{0,k+1,i}(x', dx)}{\pi(dx) Q_{1,k,i}(x, dx')}.
\]

Observe that the \( 1/(k + 1) \) terms, in the move selection probabilities, cancel each other. To complete the proof, it remains to show that

\[
\frac{\pi(dx') Q_{0,k+1,i}(x', dx)}{\pi(dx) Q_{1,k,i}(x, dx')} = \frac{f_{k+1}(x')}{f_k(x)} \frac{1}{q(s^*)}.
\]

This can be obtained, in the general case\(^3\), by a direct computation of the densities with respect to the symmetric measure

\[
\xi(dx, dx') = \nu^{\otimes k}(ds) \left[ \delta_{(k-1,s \oplus_i s^*)}(dx') \\
+ \int_S \delta_{(k+1,s \oplus_i s^*)} \nu(ds^*) \right].
\]

\(^3\)In the important special case where \( S \subset \mathbb{R}^d \) and \( \nu \) is (the restriction of) the \( d \)-dimensional Lebesgue measure, (16) can be simply seen as the result of Green’s dimension matching argument [17, Section 3.3], in a very simple case where the Jacobian is equal to one.
We emphasize that (15) is not the usual mixture representation of trans-dimensional kernels introduced in Section 3.2. Indeed, starting, for example, from \( \mathcal{X}_k \), there are several elementary kernels that can propose a point in \( \mathcal{X}_{k+1} \). This shows the usefulness of Proposition 1, which provides sufficient conditions for (8) to hold beyond the case of the usual mixture representation (9).

### 4.3 Birth-or-Death kernels on sorted vectors

Let us assume now that the objects are “sorted”, in some sense, before being arranged in the vector \( s = (s_1, \ldots, s_k) \in \mathbb{S}^k \). This happens, in practice, either when there is a natural ordering on the set of objects (e.g., the jump times in signal segmentation or multiple change-point problems [17, 34]) or when artificial constraints are introduced to restore identifiability in the case of exchangeable components (see [9, 23, 35, 36, 46] for the case of mixture models).

To formalize this, let us consider the same space \( \mathcal{X} \) as in Section 4.1. Assume that \( \mathbb{S} \) is endowed with a total order and that the corresponding “sort function” \( \psi : \mathcal{X} \to \mathbb{S} \) is measurable. What we are assuming now is that the target measure, denoted by \( \tilde{\pi} \) in this section, is supported by \( \psi(\mathcal{X}) \)—in other words, the components of \( x \in \mathcal{X} \) are \( \tilde{\pi} \)-almost surely sorted.

In such a setting, the definition of the Birth-or-Death kernel has to be slightly modified in order to accommodate the sort constraint: the death kernel is unchanged, but new components are introduced in Section 4.3. Birth-or-Death moves in the point process framework, with the Poisson point process, as the jump times in signal segmentation or multiple change-point problems [12, 24, 41, 47].

**Remark** Another option, when the components of the vector \( s = (s_1, \ldots, s_k) \) are exchangeable, is to forget about the indices and consider the set \( \{s_1, \ldots, s_k\} \) instead. The object of interest is then a (random) finite set of points in \( \mathbb{S} \)—in other words, a point process on \( \mathbb{S} \). The expression of the MHG ratio for Birth-or-Death moves in the point process framework, with the Poisson point process as a reference measure, has been given in [15] (one year before the publication of Green’s paper [17]). Point processes have been widely used, since then, in image processing and object identification (see, for example, [12, 24, 41, 47]).
5 Example: joint detection and estimation of sinusoids in white Gaussian noise

The results presented in Section 4 can be used to compute the MHG ratio easily in many signal decomposition problems. Let us illustrate this with the joint Bayesian model selection and parameter estimation of sinusoids in white Gaussian noise, as first considered by [1]. As explained in the introduction, this seminal paper introduced the RJ-MCMC methodology in the signal processing community, and at the same time introduced an erroneous expression of the MHG ratio that has been, since then, reproduced in a long series of papers. We follow closely the model and notations of [1]; the reader is referred to the original paper for more details.

Let \( y = (y_1, y_2, \ldots, y_N)^T \) be a vector of \( N \) observations of an observed signal. We consider the finite family of nested models \( M_0 \subset M_1 \subset \cdots \subset M_{k_{\text{max}}} \), where \( M_k \) assumes that \( y \) is composed of \( k \) sinusoids observed in white Gaussian noise. Let \( \omega_k = (\omega_{1,k}, \ldots, \omega_{k,k}) \) and \( a_k = (a_{c_{1,k}}, a_{s_{1,k}}, \ldots, a_{c_{k,k}}, a_{s_{k,k}}) \) be the vectors of radial frequencies and cosine/sine amplitudes under model \( M_k \), respectively; moreover, let \( D_k \) be the corresponding \( N \times 2k \) design matrix. Then, the observed signal \( y \) follows under \( M_k \) a normal linear regression model:

\[
y = D_k a_k + n,
\]

where \( n \) is a white Gaussian noise with variance \( \sigma^2 \). The unknown parameters are, then, assumed to be the number of components \( k \), the component-specific parameters \( \theta_k = \{a_k, \omega_k\} \) and the noise variance \( \sigma^2 \) which is common to all models. The joint prior distribution is chosen to have the following hierarchical structure:

\[
p(k, \theta_k, \sigma^2) = p(a_k | k, \omega_k, \sigma^2) p(\omega_k | k) p(k) p(\sigma^2),
\]

where the prior over \( a_k \) is the conventional \( g \)-prior distribution [50], which is a zero mean Gaussian with \( \sigma^2 \delta^2 \left( D_k^T D_k \right)^{-1} \) as its covariance matrix. Conditional on \( k \), the radial frequencies are independent and identically distributed, with a uniform distribution on \((0, \pi)\). The noise variance \( \sigma^2 \) is endowed with Jeffreys improper prior, i.e. \( p(\sigma^2) \propto 1/\sigma^2 \). The number of components \( k \) is given a Poisson distribution with mean \( \Lambda \), truncated to \( \{0, 1, \ldots, k_{\text{max}}\} \). The parameters \( a_k \) and \( \sigma^2 \) can be integrated out analytically, and the resulting marginal posterior becomes

\[
p(k, \omega_k | y) \propto (y^T P_k y)^{-N/2} \frac{\Lambda^k \pi^{-k}}{k! (\delta^2 + 1)^k} \mathbf{1}_{(0, \pi)^k}(\omega_k),
\]

with

\[
P_k = I_N - \frac{\delta^2}{1 + \delta^2} D_k (D_k^T D_k)^{-1} D_k^T.
\]

when \( k \geq 1 \) and \( P_0 = I_N \).

Inference under this hierarchical Bayesian model is carried out in [1] using an RJ-MCMC sampler on \( \Theta = \bigcup_{k=0}^{k_{\text{max}}} \{k\} \times (0, \pi)^k \) with target density (18). We only focus here on the “between-models” moves, which are Birth-or-Death moves of the kind described in Section 4.1, with a uniform density on \((0, \pi)\) for the proposal distribution of the new frequency in the birth moves.

Let us now compute the MHG ratio for a birth move. Note that the posterior density (18) is written in the case of “unsorted” components described in Sections 4.1–4.2. We shall therefore make use of Proposition 2, which assumes that new component is inserted at a random position \( i \) (all components being selected with the same probability). The correct MHG ratio, for a birth move from \( x = (k, \omega_k) \) to \( x' = (k + 1, \omega_k \oplus_i \omega^*) \), turns out to be

\[
r(x, x') = \frac{p(k + 1, \omega_k \oplus_i \omega^* | y)}{p(k, \omega_k | y)} \times \frac{p_i(x')}{p_i(x)} \times \frac{1}{q(\omega^*)},
\]

(19)
Figure 1: The pdf’s of Poisson (gray) and accelerated Poisson (black) distributions with mean \( \Lambda = 5 \). Both distributions are truncated to the set \( \{0, \ldots , 32\} \).

where \( q \) denotes the uniform distribution of \((0, \pi)\). Using

\[
\frac{p_a(x')}{p_b(x)} = \frac{p_0(k)}{p_0(k+1)} = \frac{k + 1}{\Lambda}
\]
as in [1], with \( p_0 \) standing for the (truncated Poisson) prior distribution of \( k \), we finally find

\[
r(x, x') = \left( \frac{y^T P_{k+1} y}{y^T P_k y} \right)^{-N/2} \frac{\Lambda \pi^{-1}}{(1 + k)(1 + \delta^2)}
\times \frac{k + 1}{\Lambda} \times \frac{1}{\pi^{-1}}
\]

\[
= \left( \frac{y^T P_{k+1} y}{y^T P_k y} \right)^{-N/2} \frac{1}{1 + \delta^2}.
\]

(20)

Note that the expression of the ratio proposed in [1, Equation (20)] differs from the one we find here by a factor \( 1/(k + 1) \). A similar mistake in computing RJ-MCMC ratios has been reported in the field of genetics [22, 44].

In fact, using the expression of the birth ratio with an additional factor of \( 1/(k + 1) \), as in [1], amounts to assigning a different prior distribution over \( k \) called “accelerated Poisson distribution” [44] which reads

\[
p_2(k) \propto \frac{e^{-\Lambda} \Lambda^k}{(k!)^2} \mathbb{I}_N(k).
\]

(21)

Figure 1 illustrates the difference between both the accelerated (black) and the usual (gray) Poisson distributions when mean \( \Lambda = 5 \). It can be observed that the accelerated Poisson distribution (21) puts a stronger emphasis on “sparse” models, i.e., models with a small number of components.

Let us consider an experiment in which the observed signal of length \( N = 64 \) consists of \( k = 3 \) sinusoidal components with the radial frequencies \( \omega_k = (0.63, 0.68, 0.73)^T \) and amplitudes \( a_{ci,k}^2 + a_{si,k}^2 = (20, 6.32, 20)^T, 1 \leq i \leq k \). The signal to noise ratio, defined as \( \text{SNR} \triangleq \|D_k a_k\|^2 / (N \sigma^2) \), is set to a moderate value of 7dB. Samples from the posterior distribution of \( k \) are obtained using the RJ-MCMC sampler of [1], with an inverse Gamma prior \( IG(2, 100) \) on \( \delta^2 \) and a Gamma prior \( \mathcal{G}(1, 10^{-3}) \) on \( \Lambda \). For each observed signal in 100 replications of the experiment, the sampler was run twice: once with the correct expression of the ratio, given by (20), and once with the erroneous expression from [1]. Figure 2 shows the frequency of
selection of each model under both the Poisson and the accelerated Poisson distribution as a prior for $k$. It appears that the (unintended) use of the accelerated Poisson distribution, induced by the erroneous expression of the MHG ratio, can result in a significant shift to the left of the posterior distribution of $k$.

Remark Working with “sorted” vectors of frequencies would be quite natural in this problem, since the frequencies are exchangeable under the posterior (18). As explained in Section 4.3, the expression of the MHG ratio would be the same.

Remark The reason why the MHG ratio in [1] is wrong can be understood from a subsequent paper [4], where the same computation is explained in greater detail. There we can see that the authors, working with an “unsorted vector” representation, consider that the new component in a birth move is inserted at the end. The death move, however, is defined as in the present paper: a sinusoid to be removed is selected randomly among the existing components. Here is the mistake: if the new component is inserted at the end during a birth move, then any attempt at removing a component which is not the last one should be rejected during a death move. In other words, the acceptance probability should be zero when any component but the last one is picked to be removed during a death move.

6 Conclusion

The computation of MHG ratios is a delicate matter involving measure-theoretic considerations, for which practitioners need clear mathematical statements that can be used “out of the box”. Such a statement has been available for a long time in the classical fixed-dimensional Metropolis-Hastings sampler, and more recently provided by Green [17] for trans-dimensional moves that comply with the assumptions of his dimension matching argument.

In this note, we have provided the expression of the MHG ratio for Birth-or-Death moves, using a general result for mixtures of proposal kernels, and corrected the erroneous expression provided by [1]. A similar correction has to be applied to the ratios used in the long series of
signal processing papers [3–6, 10, 20, 25–27, 30, 31, 40, 42, 43] that have been found to contain the same mistake.

While writing this note, we discovered that a very similar mistake had been detected and corrected in the field of genetics by [22], from which we borrow our concluding words: The fact that this error has remained in the literature for over 5 years [12 years in the present case] underscores the view that while Bayesian analysis using Markov chain Monte Carlo is incredibly flexible and therefore powerful, the devil is in the details. Furthermore, incorrect analyses can give results that seem quite reasonable.

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