QUANTIZING SL(N) SOLITONS AND THE HECKE ALGEBRA

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ABSTRACT

The problem of quantizing a class of two-dimensional integrable quantum field theories is considered. The classical equations of the theory are the complex $sl(n)$ affine Toda equations which admit soliton solutions with real masses. The classical scattering theory of the solitons is developed using Hirota’s solution techniques. A form for the soliton $S$-matrix is proposed based on the constraints of $S$-matrix theory, integrability and the requirement that the semi-classical limit is consistent with the semi-classical WKB quantization of the classical scattering theory. The proposed $S$-matrix is an intertwiner of the quantum group associated to $sl(n)$, where the deformation parameter is a function of the coupling constant. It is further shown that the $S$-matrix describes a non-unitary theory, which reflects the fact that the classical Hamiltonian is complex. The spectrum of the theory is found to consist of the basic solitons, scalar states (or breathers) and excited (or ‘breathing’) solitons. It is also noted that the construction of the $S$-matrix is valid for any representation of the Hecke algebra, allowing the definition of restricted $S$-matrices, in which case the theory is unitary.

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Introduction

It is an outstanding problem to understand non-perturbative effects in quantum field theory. In two-dimensions the situation promises to be more tractable. Massless Euclidean quantum field theories, which describe the critical scaling behaviour of two-dimensional statistical systems, exhibit an infinite dimensional symmetry, described by the group of, possibly extended, conformal transformations. In a sense, the infinite problem of quantum field theory is reduced to a finite problem involving the representation theory of an infinite algebra [1]. In the space of massive two-dimensional field theories, there is a class of theories that shares the property of integrability with the conformal field theories. In a massive integrable theory there exists an infinite number of commuting conserved charges, and so there exists some transformation to action-angle variables, and the theory is separable (at least classically). For the relativistically invariant classical integrable theories, it is interesting to speculate as to the nature of the associated relativistic quantum field theories. It is thought that the property of integrability will generally survive quantization and the resulting theory will be particularly simple because its $S$-matrix will factorize [2,3].

For all their simplicity, it has still proved to be very difficult to quantize the classical integrable theories. There are a number of approaches, for instance the quantum inverse scattering formalism [4] or the formalism of ref. [5]. Another approach, which has proved particularly useful, might be called the ‘direct $S$-matrix approach’ [3]. The success of this latter approach, rests on the fact that the constraints of $S$-matrix theory, along with the hitherto mentioned property of factorizability, strongly constrain the allowed form of the $S$-matrix, in fact to such an extent that it becomes possible, with a bit of foresight, to conjecture a form for the $S$-matrix, up to possible ambiguities of CDD type.

The $S$-matrix approach is particularly well adapted to quantizing classical soliton theories because in such theories there is a direct relationship between the semi-classical limit of the $S$-matrix and the classical soliton scattering solutions [6]. In a classical soliton theory the scattering of solitons, as described by multi-soliton solutions to classical equations, can be described in a very simple way. In the distant past, an $N$-soliton solution has the form of $N$ well separated solitons of given velocity. In the far future, the solution also consists of $N$ solitons with exactly the same distribution of velocities; this is due to the
existence of conserved higher spin integrals of motion. If the scattering solution is analysed in detail, then one finds that only the centres-of-mass of the solitons change (consistent with the overall conservation of momentum). A shift in the centre-of-mass of a soliton can alternatively be thought of as a time-delay or advance. The important point is that the scattering of the $N$ solitons can be thought of in terms of $\frac{1}{2}N(N-1)$ elementary pairwise scatterings, each of which contributes to the shift in the centres-of-mass of the participating solitons.

The picture in the quantum theory is thought to be very similar in the sense that the scattering of $N$ solitons can also be thought of in terms of pairwise scattering, with the property that individual velocities are conserved (this is the factorizability assumption [2,3]). In the quantum theory there can be non-trivial mixings due to mass degeneracies (such processes are seen in the classical theory as solutions involving a complex time trajectory [7]) and the analogue of the time-delay is now played by a non-trivial (momentum dependent) phase factor. The semi-classical limit of the phase factor is directly related to the classical time-delay. So there is a very concrete connexion between the $S$-matrix and classical scattering theory. The idea is to use the axioms of $S$-matrix theory and the semi-classical limit in order to propose a form for the $S$-matrix of a classical soliton scattering theory.

The known class of relativistic integrable field theories includes the sine and sinh-Gordon theories, the Toda field theories, the chiral models and various fermion models. In this work, we will consider the affine Toda theories associated to the algebra $sl(n)$, which include the sine and sinh-Gordon theories as special cases when $n = 2$.

The sinh and sine-Gordon theories are almost completely understood, in the sense that their complete spectra and $S$-matrices are known [3,8]. Since we shall argue that the more general Toda theories share many features of the sine/sinh-Gordon theory it is worth discussing these theories in some detail. The equation of motion for both theories may be written

$$\Box \phi = -\frac{2m^2}{\beta} \sin(\sqrt{2}\beta \phi). \quad (1)$$

The field $\phi(x, t)$ is a scalar, whilst $m$ and $\beta$ are coupling constants. The sinh-Gordon theory differs only in that $\beta$ is taken to be purely imaginary.

The spectrum of the sinh-Gordon theory is particularly simple, there is only a single

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1 The notation used is not conventional, usually $\beta \to \beta/\sqrt{2}$, however, it will prove more useful for comparing with the $sl(n)$ generalization.
scalar particle, which is identified with $\phi$ the fundamental field of the theory. The situation for the sine-Gordon theory is, in comparison, much more complicated. Classically the sine-Gordon theory admits soliton, or kink, solutions of the form
\[
\phi(x, t) = \pm \frac{2\sqrt{2}}{\beta} \tan^{-1} \left( e^{\sigma (x-\nu t-\xi)} \right),
\]
where $\nu$, $\sigma$ and $\xi$ are constants, and $\sigma^2(1-\nu^2) = 4m^2$. The two solutions in (2) represent the soliton and anti-soliton. Multiple soliton solutions also exist which describe the scattering of solitons. There are other classical solutions, known as the breather or doublet solutions, which have the interpretation of a soliton and anti-soliton oscillating about a fixed centre with some period. In a remarkable series of papers by various authors [8,9], the complete spectrum in the quantum theory has been determined. One has to distinguish two regions; the attractive regime which, in our conventions, is $0 \leq \beta^2 \leq 2\pi$ and the repulsive regime, $2\pi \leq \beta^2 \leq 4\pi$ (the language refers to the remarkable connexion of the sine-Gordon theory with the massive Thirring model [10]). The theory is not well-defined when $\beta^2 > 4\pi$. In the attractive regime, the spectrum consists of a soliton and anti-soliton of mass
\[
\hat{M} = \frac{8m}{\beta^2} - \frac{2m}{\pi}.
\]
The first term is the classical soliton mass, whilst the second is the first quantum correction. There are arguments to suggest that (3) is actually exact to all order in $\beta^2$ [9]. In addition to the solitons there is series of soliton anti-soliton bound states that arise from quantizing the breather solutions, yielding a discrete mass spectrum
\[
\hat{m}(k) = 2\hat{M} \sin \left( \frac{k\gamma}{8} \right), \quad k = 1, 2, \ldots, < \frac{4\pi}{\gamma},
\]
where
\[
\gamma = \frac{\beta^2}{1 - \beta^2/4\pi}.
\]
Remarkably, the ground state of the breather system is identified with the fundamental particle of the sine-Gordon Lagrangian. Indeed, $\hat{m}(1) = 2m + O(\beta^2)$, where the terms $O(\beta^2)$ agree, order by order, with the loop corrections to the mass of the fundamental particle. This theory exhibits the phenomenon of nuclear democracy; the solitons can be though of as coherent excitations of the fundamental particle, which itself can be though of as the bound state of a soliton and anti-soliton. The full $S$-matrix for the combined breather/soliton system was found in [3].
In the repulsive regime the situation is not so straightforward [8]. The vacuum becomes unstable and the spectrum is changed. New scalar states appear which cannot be thought of as bound states of the solitons.

The sine-Gordon theory is also remarkable in that it can be reformulated as the theory of a massive interacting fermion, namely the massive Thirring model. In this picture the solitons are represented by a Dirac spinor field and the breather states are then fermion bound states [10].

We now turn our attention to the $sl(n)$ generalizations of the sine/sinh Gordon theories. These are the affine $sl(n)$ Toda field theories. The equation of motion of the theory is

$$
\Box \phi = -\frac{m^2}{\dot{\beta}} \sum_{j=1}^{n} \alpha_j e^{\dot{\beta} \alpha_j \cdot \phi}.
$$

The field $\phi(x,t)$ is an $n-1$ (= rank $sl(n)$)-dimensional vector. The inner products are taken with respect to the Killing form of $sl(n)$ restricted to the Cartan subalgebra. The $\alpha_j$'s, for $j = 1,\ldots,n-1$ are the simple roots of $sl(n)$; $\alpha_n$ is the extended root (minus the highest root)$^2$. The fact that the extended root is included in the sum, distinguishes the affine theories from the non-affine ones. In the affine theories the $\alpha_j$'s are linearly dependent:

$$
\sum_{j=1}^{n} \alpha_j = 0.
$$

Notice that the sinh-Gordon theory is recovered by choosing the algebra to be $sl(2)$.

For the moment we consider the theories which generalize the sinh-Gordon theory, so $\dot{\beta}$ is real. In the weak coupling limit we expect the theory to include $n-1$ scalars with masses

$$
\dot{m}_a = m_a \left( 1 + \frac{\dot{\beta}^2}{4n} \cot \frac{\pi}{n} + \cdots \right), \quad a = 1,2,\ldots,n-1,
$$

where the first term is the ‘classical mass’:

$$
m_a = 2m \sin \left( \frac{\pi a}{n} \right), \quad j = 1,2,\ldots,n-1.
$$

An $S$-matrix for the scalars was conjectured in [11], which is consistent with the Feynman diagram expansion in the limit of weak coupling. The $S$-matrix only has poles on the

$^2$ For convenience we shall think of the labels on the $\alpha_j$'s as being defined modulo $n$, so that $\alpha_j \equiv \alpha_{j+n}$
‘physical strip’ corresponding to the exchange of the scalar particles, and so the spectrum is complete. This just generalizes the situation for the sinh-Gordon theory.

Our interest is in soliton theories for which we need to generalize the sine-Gordon theory. To do this we can consider the Toda equations when $\tilde{\beta}$ is purely imaginary, so $\tilde{\beta} = i\beta$. For $n > 2$ this means that the equations are now complex. Classically, there exists a well-defined set of classical soliton solutions with real masses. Rather than take the usual route to quantization we shall proceed directly to an $S$ matrix, showing that our proposal is consistent, in the semi-classical limit, with the classical scattering theory. Since the Hamiltonian of the theory is a complex quantity (for $n > 2$), we expect that the theory is non-unitary; indeed the issue of unitarity can be addressed directly at the level of the $S$-matrix.

Our proposal for the $S$-matrix involves the intertwiner of the $sl(n)$ quantum group, and hence the Hecke algebra. In fact the deformation parameter of the quantum group is the coupling constant for the solitons, such that in the weak coupling limit of the solitons the quantum group reduces the universal enveloping algebra of the Lie algebra $sl(n)$ and the Hecke algebra reduces to the symmetric group, which in physical terms means that the $S$-matrix is just a permutation and hence describes a non-interacting theory. In crude terms the $S$-matrix has a quantum group symmetry, as well as certain momentum dependent symmetries which lead to new integrals of motion. We also point out that the $S$-matrix can be constructed for any representation of the Hecke algebra.

The plan of the paper is as follows. In §1, we review the construction of the classical soliton theory [12]. §2 provides an introduction to the $sl(n)$ quantum group and the Hecke algebras, as well as proving a few results which are needed in the following sections. A factorizable $S$-matrix is constructed in §3, which in §4 is proposed as the soliton $S$-matrix of the complex $sl(n)$ Toda theories. Various properties of the $S$-matrix are investigated, for instance we show that the spectrum consists of additional scalar states as well as excited solitons. In addition we show, as expected, that the $S$-matrix describes a non-unitary quantum field theory. In §5 we show that the $S$-matrix is consistent with the semi-classical quantization of the classical scattering theory, via a WKB approximation. Finally, some comments are made in §6.
1. The Classical Solitons

The soliton solutions of the complex \( sl(n) \) Toda theory were constructed in [12]. Here, we recall some of the details in order to motivate the construction of the \( S \)-matrix in following sections.

The equation of motion of the complex \( sl(n) \) affine Toda field theory is

\[
\Box \phi = -\frac{m^2}{i\beta} \sum_{j=1}^{n} \alpha_j e^{i\beta \alpha_j \cdot \phi}.
\]  

(1.1)

This equation is integrable in the sense that there exists an infinite number of Poisson-commuting conserved charges (see for example [13]), one of which is interpreted as the energy. This is the integral of the density

\[
H = \frac{1}{2} \left( (\partial_t \phi)^2 + (\partial_x \phi)^2 \right) - \frac{m^2}{\beta^2} \sum_{j=1}^{n} \left( e^{i\beta \alpha_j \cdot \phi} - 1 \right).
\]  

(1.2)

Notice that the field is periodic with respect to the weight lattice, \( \Lambda^* \), of \( sl(n) \). More precisely

\[
\phi \sim \phi + \frac{2\pi}{\beta} w, \quad \forall w \in \Lambda^*.
\]  

(1.3)

The constant field configurations \( \phi = (2\pi/\beta)w, \forall w \in \Lambda^* \), have zero energy. There exist kink, or soliton, solutions which interpolate between these configurations as \( x \) goes from \(-\infty\) and \( \infty \). One can associate a topological charge

\[
t = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \phi \in \Lambda^*, \quad \forall \phi \in \Lambda^*,
\]  

(1.4)

to each solution.

Explicit expressions for the soliton solutions were found in [12] using the Hirota formalism [14]. The idea is as follows: first one performs a change of variables

\[
\phi = -\frac{1}{i\beta} \sum_{j=1}^{n} \alpha_j \log \tau_j.
\]  

(1.5)

The equation of motion (1.1) is equivalent to

\[
t_j \tau_j - \tau_j^2 - \tau_j'' \tau_j + \tau_j'^2 = m^2 (\tau_{j-1} \tau_{j+1} - \tau_j^2), \quad j = 1, \ldots, n,
\]  

(1.6)
where, as for the roots, $\tau_j \equiv \tau_{j+n}$. (1.6) is an equation of ‘Hirota bilinear type’ [14]. The characteristic polynomial in this case is

$$F(\sigma, \lambda, a) = \sigma^2 - \lambda^2 - 4m^2 \sin^2 \left( \frac{\pi a}{n} \right),$$

(1.7)

where $\sigma$ and $\lambda$ are continuous variables and $a \in \{1, 2, \ldots, n-1\}$. It is useful to introduce $v = \lambda/\sigma$, which is the velocity of a soliton. The $N$-soliton solution is then written in terms of functions $\Phi^{(p)}(p, t)$, one for each soliton, and $\gamma^{(pq)}$ associated to each soliton pair, $1 \leq p < q \leq N$, where

$$\Phi^{(p)}_j(x, t) = \sigma_px - \lambda_pt + \frac{2\pi i}{n}a_pj + \xi_p, \quad j = 1, \ldots, n,$$

(1.8)

subject to the constraint

$$F(\sigma_p, \lambda_p, a_p) = 0.$$  

(1.9)

In the above $\xi_p$ is a constant, whose real part represents an arbitrary shift in the centre-of-mass, and whose imaginary part will determine the topological charge of the soliton. The ‘interaction function’ is given by

$$\exp \gamma^{(pq)} = -\frac{F(\sigma_p - \sigma_q, \lambda_p - \lambda_q, a_p - a_q)}{F(\sigma_p + \sigma_q, \lambda_p + \lambda_q, a_p + a_q)},$$

(1.10)

where $F$ is the characteristic polynomial in (1.7). Another useful way to write the interaction function is

$$\exp \gamma^{(pq)}(\theta) = \frac{\sin \left( \frac{\theta}{2i} + \frac{\pi(a_p-a_q)}{2n} \right) \sin \left( \frac{\theta}{2i} - \frac{\pi(a_p-a_q)}{2n} \right)}{\sin \left( \frac{\theta}{2i} + \frac{\pi(a_p+a_q)}{2n} \right) \sin \left( \frac{\theta}{2i} - \frac{\pi(a_p+a_q)}{2n} \right)},$$

(1.11)

where $\theta = \theta_p - \theta_q$ is the rapidity difference of the two solitons. The general $N$ soliton solution is then

$$\tau_j(x, t) = \sum_{\mu_1=0}^{1} \cdots \sum_{\mu_N=0}^{1} \exp \left( \sum_{p=1}^{N} \mu_p \Phi^{(p)}_j + \sum_{1 \leq p < q \leq N} \mu_p \mu_q \gamma^{(pq)} \right).$$

(1.12)

Let us now analyse the one-soliton solution in more detail. The explicit expression for the solution is

$$\phi(x, t) = -\frac{1}{i\beta} \sum_{j=1}^{n} \alpha_j \log \left\{ 1 + \exp \left( \sigma(x - vt) + \xi + \frac{2\pi i a}{n}j \right) \right\},$$

(1.13)

$\beta$ The rapidity is related to the velocity by $v = \tanh \theta$, i.e. $\tanh \theta = \lambda/\sigma$. 

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with
\[ \sigma^2(1 - v^2) = 4m^2 \sin^2 \left( \frac{\pi a}{n} \right). \] (1.14)

Clearly the solution represents a kink, whose centre is at \( vt - \sigma^{-1}\text{Re}(\xi) \), moving with velocity \( v \) and with characteristic size \( \sigma^{-1} \). (1.14) simply expresses the relativistic invariance of the theory: the faster a soliton goes the narrower it becomes.

We now find the topological charge of the *elementary* soliton, that is the soliton with \( a = 1 \). Assuming \( \sigma > 0 \), as \( x \to -\infty \phi \to 0 \). The limit as \( x \to \infty \) is more subtle and depends upon the choice for the imaginary part of the constant \( \xi \). One obtains \( n \) different values, for instance with the choice

\[ \text{Im}(\xi) = \begin{cases} 
\frac{2\pi(m-1)}{\pi(2m-1)} n & \text{if } m = 1, 2, \ldots, n \text{ odd} \\
\frac{n}{m} n & \text{if } m = 1, 2, \ldots, n \text{ even}. 
\end{cases} \] (1.15)

At this point it is convenient to introduce the weights of the \( n \) dimensional representation of \( sl(n) \), \( e_j \) (again \( e_{j+n} \equiv e_j \)). These vectors have the property that \( \sum_{j=1}^{n} e_j = 0 \), and the simple roots, along with \( \alpha_n \), are given by \( \alpha_j = e_j - e_{j+1} \). By using the \( n \) possibilities for \( \xi \) one finds that the topological charges of the elementary soliton fill out the set of weights of the \( n \) dimensional representation.

Similarly one can show that the solitons for other values of \( a \) have topological charges in the set of weights of the \( a \)th *fundamental* representation of \( sl(n) \). In addition to the soliton solutions one can also construct breather solutions. They are obtained by considering the two soliton solution, for solitons of equal mass, *i.e.* either two solitons of type \( a \) or one of type \( a \) and one of type \( n - a \). In the centre-of-mass frame the breather solution is obtained by taking \( \lambda_1 = -\lambda_2 = i\omega \), for \( \omega \in \mathbb{R} \). The solution is interpreted as the two solitons oscillating about a common centre. The new feature of the more general theories is that breather solutions can have non-zero topological charge, and so there are ‘breathing soliton’ solutions.

One can easily calculate the masses of the soliton solutions from the Hamiltonian (1.2). The resulting masses only depend on the particular fundamental representation that the topological charge of the soliton lies in, one finds [12]

\[ M_a = \frac{4mn}{\beta^2} \sin \left( \frac{\pi a}{n} \right), \quad a = 1, \ldots, n - 1. \] (1.16)

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4 The fundamental representations are the irreducible representations whose highest weights are dual to the simple roots.
Notice that these masses are proportional to the masses of the Toda particles of the real coupling constant theory (9). It seems rather miraculous that the resulting expressions for the masses are real considering that the Hamiltonian is in general complex. However, this can be traced to the fact that the soliton solutions satisfy a reality condition of the form $\phi^* = -M\phi$, where $M$ is an orthogonal transformation with $M^2 = 1$, which acts as a $\mathbb{Z}_2$ permutation on the roots $\alpha_j$, $j = 1, \ldots, n$. The presence of the involution $M$ implies that energy of a soliton is real, since $H(\phi, \beta)^* = H(-M\phi, -\beta) = H(\phi, \beta)$ due to invariance of the Hamiltonian under permutation of the roots. Some issues concerning the reality of the complex Toda field theories are addressed in ref. [15]. The fact that the masses are proportional to the masses of the fundamental Toda particles is explained in ref. [12], via a relationship between the bootstrap equations of the scattering theory of the fundamental particles and a purely classical analogue of the bootstrap equations for the soliton theory.

For the case when $n = 2$ the doublet of soliton solutions of (1.13) reduce to the soliton and anti-soliton solutions of the sine-Gordon theory in (2).

The multi-soliton solutions describe the scattering of the single solitons. For example, consider an $N$ soliton solution with $v_1 > v_2 \cdots > v_N$. Initially, i.e. as $t \to -\infty$, the solution is approaches a set of isolated solitons, in the order $1, 2, \ldots, N$ as $x$ goes from $-\infty$ to $\infty$; the position of the $p^{th}$ soliton at time $t$ being

$$v_p t - \frac{1}{\sigma_p} \left( \sum_{q=1}^{p-1} \gamma^{(pq)} + \text{Re}(\xi_p) \right). \tag{1.17}$$

Finally, i.e. as $t \to \infty$, the solution also approaches that of $N$ isolated solitons moving with the same set of velocities as those in the initial state, but now with order $N, N-1, \ldots, 1$ as $x$ goes from $-\infty$ to $\infty$, the position of the $p^{th}$ soliton being

$$v_p t - \frac{1}{\sigma_p} \left( \sum_{q=p+1}^{N} \gamma^{(pq)} + \text{Re}(\xi_p) \right). \tag{1.18}$$

If the solution is analysed in detail the following picture emerges. Each soliton retains its integrity except when near another soliton, within a distance $\sim m^{-1}$. In this interaction region the solution is complicated, however, the solitons emerge with the same velocities and the only effect of the interaction is to shift the centre-of-mass of each soliton (with the overall centre-of-mass being preserved). So an $N$ soliton scattering solution can be analysed in terms of $\frac{1}{2}N(N-1)$ two-soliton scatterings. This is illustrated in figure 1,
Figure 1. A multi-soliton scattering process.

where the circles indicate the regions where the solution is not approximated by a set of isolated solitons; these are the ‘interaction’ regions.

The shift in the centre-of-mass of $q^{th}$ soliton as it ‘interacts’ with the $p^{th}$ soliton is $-\gamma^{(pq)}/\sigma_q$. Another way to interpret this is to say that the $q^{th}$ soliton on interacting with the $p^{th}$ soliton will experience at time-delay given by $-\gamma^{(pq)}/(\sigma_q v_q)$. This shift in the center-of-mass is illustrated in figure 2.

Figure 2. Two soliton scattering

The question that we now address is: is there a factorizable quantum soliton $S$-matrix whose semi-classical limit yields the time-delays of the classical scattering theory described above? In our search for the $S$-matrix we are guided by the situation for the sine-Gordon soliton $S$-matrix. It is known that the $S$-matrix in this case is, up to a scalar factor, the intertwiner of the $sl(2)$ quantum group. So it appears that we should consider the $sl(n)$ quantum groups in order to construct the more general $S$-matrices. In the next section we consider some relevant facts about the $sl(n)$ quantum group and its commutant, the Hecke algebra. In §3 and §4 we go on to propose a form for the factorizable $S$-matrix describing the quantum solitons.

2. The $sl(n)$-Quantum Group and Hecke Algebra

We have seen that classically each soliton mass eigenstate is associated with a fundamental representation of $sl(n)$, in the sense that the allowed topological charge is a weight of such a representation. In the quantum theory we expect that the asymptotic state representing a soliton to carry two quantum numbers, the velocity (or rapidity) and the topological charge. Therefore associated to each external state is a vector in one of the fundamental modules of $sl(n)$. We denote the module corresponding to the fundamental representation having Dynkin labels ‘one’ over the $a^{th}$ simple root, and zero elsewhere, as $V_a$. The soliton with mass $M_a$ is then associated to the module $V_a$.

Since the theory is integrable we assume that the $S$-matrix is factorizable. This means that the individual momenta of each external state is separately conserved, so there is no particle creation and all the elements of the $S$-matrix can be deduced from the two-body...
process [3]. Notice that this is analogous to the classical scattering theory of the solitons discussed in the last section. In particular, a quantum scattering process may be analysed in terms of pairwise scattering, so the multi-soliton $S$-matrix can be constructed in terms of the two-body $S$-matrix. From a Lie algebraic point of view, the two-body $S$-matrix must act as the intertwiner:

$$ S^{a,b} : V_a \otimes V_b \mapsto V_b \otimes V_a. \quad (2.1) $$

Notice that if there were no interaction then the $S$-matrix would simply permute the vector spaces. This gives a clue about how to construct a non-trivial $S$-matrix, since there exist natural generalizations of the permutation groups known as the Hecke algebras, which reduce to the former in some limit.

In the rest of this section we introduce the Hecke algebras and the associated quantum groups. Our approach follows that of M. Jimbo in refs. [16].

Central to the subject is the Yang-Baxter Equation (YBE) which can be written as an identity in $\text{End}(V \otimes V \otimes V)$, for some vector space $V$:

$$ (\tilde{R}(x) \otimes I)(I \otimes \tilde{R}(xy))(\tilde{R}(y) \otimes I) = (I \otimes \tilde{R}(y))(\tilde{R}(xy) \otimes I)(I \otimes \tilde{R}(x)), \quad (2.2) $$

where $\tilde{R}(x) \in \text{End}(V \otimes V)$, $I$ is the identity in $\text{End}(V)$, and $x \in \mathbb{C}^*$ is the spectral parameter. We are interested in the solutions for which $\tilde{R}(x)$ is trignometric in $u = \log x$, and when $V \simeq \mathbb{C}^n$ is the $n$ dimensional module of the Lie algebra $sl(n)$. The solution depends upon an additional parameter $q$, the deformation parameter, which for the rest of this section we take to be generic (not equal to a root of unity).

Trignometric solutions of the YBE are naturally described by a quantum group; in this case the $q$-deformation of the universal enveloping algebra of $g = sl(n)$, denoted $U_q(sl(n))$. The parameter $q$ sets the degree of deformation and as $q \to 1$, $U_q(g) \to U(g)$. $U_q(g)$ is endowed with the structure of a Hopf algebra. That is an algebra homomorphism

$$ \Delta^{(m)} : U_q \longrightarrow U_q^{\otimes m}. \quad (2.3) $$

$\Delta^{(m)}$ defines the action of $U_q(g)$ on tensor products of the $n$ dimensional representation $\varrho : U_q(g) \to \text{End}(V)$. For generic $q$ (not equal to a root of unity) the irreducible representations of $U_q(g)$ are in one-to-one correspondence with those of $g$. The trignometric solution of the YBE commutes with $(\varrho \times \varrho)(\Delta^{(2)}(x))$, $x \in U_q(g)$, and so $\tilde{R}(x)$ lies in the commutant of $\Delta^{(2)}(U_q(g))$. 

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In general, the commutant of $\Delta^{(m)}(U_q(g))$ is equal to the Hecke algebra $\mathcal{H}_m$, which can be thought of as a deformation of the symmetric group on $m$ objects, $S_m$. A representation of $\mathcal{H}_m$ is generated by $\{T_a, \ a = 1, \ldots, m-1\}$, with $T_a \in \text{End}(V^{\otimes m})$, subject to the following relations

$$(T_a - q^{-1})(T_a + q) = 0$$

$$T_a T_{a+1} T_a = T_{a+1} T_a T_{a+1}$$

$$[T_a, T_b] = 0 \quad |a - b| \geq 2.$$  

(2.4)

In the limit $q \to 1$ we recover the relations for the symmetric group $S_m$, with $T_a \to \sigma_a$ being the generator which permutes the $a$th and $(a+1)$th space in the tensor product. In general, we can label elements $T_w \in \mathcal{H}_m$ with elements $w \in S_m$, such that $T_{ww'} = T_w T_{w'}$ (if $l(ww') = l(w) + l(w')$, where $l(w)$ is the length of $w \in S_m$).

It can be readily verified that the basic $\tilde{R}$ matrix can be expressed as

$$\tilde{R}(x) = xT^{-1} - x^{-1}T,$$  

(2.5)

which satisfies the YBE by virtue of the relations of the Hecke algebra $\mathcal{H}_3$. To make the above discussion more explicit we introduce a basis $\{e_i, i = 1, \ldots, n\}$ for $V$. From ref. [16] we have

$$T e_i \otimes e_j = \begin{cases} 
q^{-1} e_i \otimes e_i & i = j \\
(q^{-1} - q)e_i \otimes e_j + e_j \otimes e_i & i > j \\
e_j \otimes e_i & i < j.
\end{cases}$$  

(2.6)

The representation constructed above actually satisfies an additional constraint, over and above the Hecke algebra relations, this is the generalized Temperley-Lieb condition, which is the vanishing of the deformed full anti-symmetrizer $s^{-}_{n+1}$ defined below.

Just as one can generate higher irreducible representations of $sl(n)$ by considering the action of projection operators, formed from elements of the symmetric group $S_m$, encoded by the Young Tableaux, on the $m$-fold tensor product of the basic representation, so one can form higher irreducible representations of $U_q(sl(n))$ by considering the same process with $S_m$ replaced by $\mathcal{H}_m$. In particular, we will be interested in the analogues of the fundamental representations $\varrho_a$ of $U_q(sl(n))$, $a = 1, \ldots, n-1$, with $\varrho_a : U_q(sl(n)) \to \text{End}(V_a)$ (where $\varrho_1 \equiv \varrho$ and $V_1 \equiv V$). $V_a$ can be projected out of the $a$-fold tensor product $V_1^{\otimes a}$ with the Hecke algebra analogue of the full anti-symmetrizer:

$$V_a \simeq s_{a}^- (V^{\otimes a}).$$  

(2.7)

\footnote{For generic $q$, not equal to a root of unity.}
An expression for the Hecke algebra analogue of the full symmetrizer $s_m^+$ and anti-symmetrizer $s_m^-$ has been given in ref. [16]

$$s_m^\pm = \frac{1}{[m]!} \sum_{w \in S_m} (\pm)^{l(w)} q^{\pm (m(m-1)/2 - l(w))} T_w,$$  \hspace{0.5cm} (2.8)

where $[m]! = [m][m-1] \cdots [1]$ and $[m] = (q^m - q^{-m})/(q - q^{-1})$. $s_m^\pm$ are projection operators, so that $(s_m^\pm)^2 = s_m^\pm$. For example

$$s_2^+ = \frac{1}{[2]} (q + T), \quad s_2^- = \frac{1}{[2]} (q^{-1} - T),$$

and so in terms of these projection operators

$$\tilde{R}(x) = (xq - x^{-1}q^{-1})s_2^+ + (x^{-1}q - xq^{-1})s_2^-.$$  \hspace{0.5cm} (2.9)

Up till now, we have only considered solutions of the YBE for $\tilde{R}(x) \in \text{End}(V_1 \otimes V_1)$. One can also look for solutions to the more general YBE

$$(\tilde{R}^{U_2,U_3}(x) \otimes I_{U_1})(I_{U_2} \otimes \tilde{R}^{U_1,U_3}(xy))(\tilde{R}^{U_1,U_2}(y) \otimes I_{U_3})$$

$$= (I_{U_3} \otimes \tilde{R}^{U_1,U_2}(y))(\tilde{R}^{U_1,U_3}(xy) \otimes I_{U_2})(I_{U_1} \otimes \tilde{R}^{U_2,U_3}(x)),$$  \hspace{0.5cm} (2.10)

where $\tilde{R}^{U_i,U_j}(x) = \sigma \cdot R^{U_i,U_j}(x)$, with

$$R^{U_i,U_j}(x) \in \text{End}(U_i \otimes U_j).$$  \hspace{0.5cm} (2.11)

Here, $\sigma$ is the permutation $\sigma(w \otimes u) = u \otimes w$ for $w \in U_i$, $u \in U_j$ and the $U_i$ are arbitrary representations of $U_q(g)$. Notice that both the left and right-hand sides of (2.10) map $U_1 \otimes U_2 \otimes U_3$ to $U_3 \otimes U_2 \otimes U_1$.

Since higher representations can be formed by taking tensor products of the basic representation, the higher $\tilde{R}^{W,U}$-matrices can be built out of the basic $\tilde{R}$-matrix, via the fusion procedure. We briefly describe this process following refs. [16].

If we know $\tilde{R}^{U'',U}(x)$ and $\tilde{R}^{U'',U'}(x)$ then we can write down a new solution of the YBE:

$$\tilde{R}^{U'',U' \otimes U}(x) = (I \otimes \tilde{R}^{U'',U}(xy_1))(\tilde{R}^{U'',U'}(xy_2) \otimes I).$$  \hspace{0.5cm} (2.12)

With an appropriate choice of $y_1, y_2$ we can restrict the above to give a new solution to the YBE $\tilde{R}^{U'',W}(x)$, where $W$ is in the decomposition of the tensor product $U' \otimes U$. The choice of $y_1, y_2$ is determined by the requirement that

$$W \simeq \tilde{R}^{U',U}(y_2/y_1)(U' \otimes U) \subset U \otimes U',$$  \hspace{0.5cm} (2.13)
A simple application of the YBE then shows that $\hat{R}^{U'', W}(x)$ so defined, is indeed a homomorphism from $U'' \otimes W$ to $W \otimes U''$. So for example, notice that since $\hat{R}((-q)^{-1}) \propto s_{2}^{-}$ we have

$$V_{2} \simeq \hat{R}((-q)^{-1})(V_{1} \otimes V_{1}),$$

so we can find $\hat{R}^{1,2}(x)$ acting on $V_{1} \otimes V_{2} \simeq (I \otimes s_{2}^{-})(V_{1} \otimes V_{1} \otimes V_{1})$:

$$\hat{R}^{1,2}(x) \equiv \hat{R}^{V_{1}, V_{2}}(x) = (I \otimes \hat{R}(xy_{1}))((\hat{R}(xy_{1})(-q)^{-1}) \otimes I), \quad (2.14)$$

where $y_{1}$ is arbitrary. In a similar way one finds

$$\hat{R}^{2,1}(x) \equiv \hat{R}^{V_{2}, V_{1}}(x) = (I \otimes \hat{R}(xy_{1}))((I \otimes \hat{R}(xy_{1})(-q)^{-1})\), \quad (2.15)$$

acting on $V_{2} \otimes V_{1} \simeq (s_{2}^{-} \otimes I)(V_{1} \otimes V_{1} \otimes V_{1})$. By repeating the fusion procedure we can find $\hat{R}^{a,b}(x)$, for $a, b = 1, \ldots, n - 1$, acting in $V_{a} \otimes V_{b} \simeq (s_{a}^{-} \otimes s_{b}^{-})(V_{1} \otimes V_{1})$, a subspace of the $a + b$-fold tensor product of $V_{1}$.

In general [16], $\hat{R}^{a,b}(x)$ has the following spectral decomposition

$$\hat{R}^{a,b}(x) \equiv \hat{R}^{V_{a}, V_{b}}(x) = \sum_{U} \rho_{U}(x)P_{U}, \quad (2.16)$$

where $P_{U}$ is the orthogonal projector, relative to a $U_{q}(sl(n))$ invariant scalar product, onto the irreducible representation $U$ in the tensor product $V_{a} \otimes V_{b} \subset V_{1} \otimes V_{1} \otimes V_{1}$, and $\rho_{U}(x)$ is a scalar function of $x$. Notice that (2.9) is of the form (2.16), because $V_{1} \otimes V_{1} = V_{2} \oplus W$, since $s_{2}^{-}$ is the projector onto the fundamental representation $V_{2}$, and $W \simeq s_{2}^{+}(V_{1} \otimes V_{1})$ is the analogue of the symmetric tensor. For generic $q$, this implies that between fundamental representations:

$$\hat{R}^{a,b}(x) \equiv \hat{R}^{V_{a}, V_{b}}(x) = \rho_{a,b}(x)s_{a+b}^{-} + \cdots,$$

where the dots represent non-fundamental representations which appear in the tensor product $V_{a} \otimes V_{b}$, and the full anti-symmetrizer $s_{a+b}^{-}$ is the projector onto the unique fundamental representation $V_{a+b}$, which appears in the tensor product, (and $a + b$ is taken modulo $n$).

Notice that at each stage of the fusion procedure we are free to choose an overall shift in $x$, as indicated by the presence of $y_{1}$ in (2.14) and (2.15). We claim that the following application of the fusion procedure defines a solution to the YBE acting in the reducible
module $V_1 \oplus V_2 \oplus \cdots \oplus V_{n-1}$, the sum of the fundamental representations, by recursion from the basic solution in equation (2.5):

$$
\tilde{R}^{a,b+c}(x) = (I \otimes \tilde{R}^{a,b}(x(-q)^{c/2}))(\tilde{R}^{a,c}(x(-q)^{-b/2}) \otimes I)
$$

$$
\tilde{R}^{b+c,a}(x) = (\tilde{R}^{b,a}(x(-q)^{c/2}) \otimes I)(I \otimes \tilde{R}^{c,a}(x(-q)^{-b/2})).
$$

(2.17)

At each stage we have set the overall multiplicative shift in $x$. The first equation in (2.17) is to be understood as being restricted to $V_a \otimes V_{b+c} \simeq (s_a^- \otimes s_{b+c}^-)V_1^{\otimes (a+b+c)}$ and the second to $V_{b+c} \otimes V_a \simeq (s_{b+c}^- \otimes s_a^-)V_1^{\otimes (a+b+c)}$. In the above we assume that $a, b, c < n$ and $b+c < n$. The YBE will be satisfied, by virtue of the fusion procedure, if (2.13) is true, i.e.

$$
\tilde{R}^{c,b}((-q)^{-(c+b)/2}) \propto s_{c+b}^-,
$$

(2.18)

since $s_{b+c}^-$ is the appropriate projection operator. Since we have not managed to find an economical proof of (2.18), the details have been relegated to appendix A.

In the following section we will need to know the positions of any zeros that $\tilde{R}^{a,b}(x)$ might have. Using the fusion equations we have

$$
\tilde{R}^{2,1}((-q)^{1/2}) = (\tilde{R}(-q) \otimes I)(I \otimes \tilde{R}(1))(s_2^- \otimes I).
$$

However, $\tilde{R}(1) = (q - q^{-1})I \otimes I$ and $\tilde{R}(-q) = -(q^2 - q^{-2})s_2^+$, therefore

$$
\tilde{R}^{2,1}((-q)^{1/2}) = -(q^2 - q^{-2})(q - q^{-1})(s_2^+ \otimes I)(s_2^- \otimes I) = 0.
$$

More generally, by using the fusion equations, one can easily show that $\tilde{R}^{a,b}(x)$, for $a \geq b$, has zeros at $x = (-q)^{-(a+b-2i-2j)/2}$, for $i = 1, 2, \ldots, a - 1$ and $j = 1, 2, \ldots, b$.

Finally we note that using properties of the Hecke algebra alone one can show

$$
\tilde{R}(x)\tilde{R}(x^{-1}) = (q^{-1}x - qx^{-1})(q^{-1}x^{-1} - qx)I \otimes I,
$$

and so by using (2.17) we have

$$
\tilde{R}^{a,b}(x)\tilde{R}^{b,a}(x^{-1}) = \prod_{i,j=1}^{a,b} g(x(-q)^{-(a+b-2i-2j+2)/2})I \otimes I,
$$

(2.19)

where

$$
g(x) = (q^{-1}x - qx^{-1})(q^{-1}x^{-1} - qx).
$$

This concludes our discussion of the quantum group $U_q(sl(n))$ and the Hecke algebra. The central result that we will use in the next section is the statement that (2.17) generate a consistent solution of the YBE on the reducible module $\oplus_{a=1}^{n-1} V_a$. 

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3. A Quantum Group S-matrix

An S-matrix constructed from the quantum group intertwiner discussed in the last section will form the conjectured S-matrix describing the solitons of the complex Toda field theory. The deformation parameter $q$, in this context, is a coupling constant. The ansatz, which was first written down in ref. [17], is a generalization of the soliton S-matrix for the sine-Gordon theory, where the appropriate quantum group in that case is the deformation of $U(sl(2))$. For the moment we consider the general problem of constructing an S-matrix out of the $sl(n)$ quantum group intertwiner; the connexion with the Toda theories will be addressed in §4.

In general, the axioms of S-matrix theory do not strongly constrain the form of the S-matrix. The situation in two dimensions for integrable theories is exceptional [3]. To start with, integrability implies that there can be no particle creation or destruction, since the individual momenta of the incoming particles must be conserved. In addition an $N$-particle S-matrix can be factorized as a product of $\frac{1}{2}N(N-1)$ 2-particle S-matrices. Suppose that the particle states form a number of degenerate multiplets labelled by a set of finite dimensional vector spaces $\{V_a, a = 1, 2, \ldots, n-1\}$, with masses $m_a$. Since the masses of the particles must be preserved on scattering the only possible processes involve ‘flavour changing’, i.e. the S–matrix is determined by a set of maps

$$S^{a,b}(\theta): V_a \otimes V_b \longrightarrow V_b \otimes V_a,$$

where $\theta = \theta_a - \theta_b$ is the rapidity difference of the incoming particles. Consistency with factorization implies that the two-body S-matrix must satisfy the Yang Baxter equation

$$(I \otimes S^{a,b}(\theta_1))(S^{a,c}(\theta_1 + \theta_2) \otimes I)(I \otimes S^{b,c}(\theta_2)) = (S^{c,b}(\theta_2) \otimes I)(I \otimes S^{a,c}(\theta_1 + \theta_2))(S^{a,b}(\theta_1) \otimes I),$$

where $I$ is the identity operator on the appropriate vector space. Further conditions come from the axioms of S–matrix theory and integrability:

(i) Unitarity\(^6\).

$$S^{a,b}(\theta)S^{b,a}(-\theta) = I_b \otimes I_a,$$

\(^6\) Unitarity in this context does not imply that the underlying quantum field theory itself is unitary; that issue depends on the signs of the residues of the S-matrix at particle poles, an issue that we address in §4.
where $I_b \otimes I_a$ is the identity in $\text{End}(V_b \otimes V_a)$.

(ii) Crossing symmetry.

$$S^{\bar{a},b}(\theta) = (I \otimes C_a) \cdot (\sigma \cdot S^{b,a}(i\pi - \theta))^{t_1} \cdot \sigma \cdot (C_{\bar{a}} \otimes I), \quad (3.4)$$

where $C_a : V_a \rightarrow V_{\bar{a}}$, $\bar{a} \in \{1, 2, ..., n - 1\}$, is the charge conjugation operator, and $C_{\bar{a}}C_a = I_a$ is the identity operator in $V_a$. As before, $\sigma$ denotes the permutation, $\sigma(u \otimes v) = v \otimes u$, and $t_1$ means ‘transpose’ in the second space which is well defined because $\sigma \cdot S^{b,a}(\theta) \in \text{End}(V_b \otimes V_a)$.

(iii) Analyticity. $S(\theta)$ is a meromorphic function of $\theta$. The only singularities on the physical strip, $0 \leq \text{Im} \theta \leq \pi$, are along $\text{Re} \theta = 0$ (since there can be no particle production for physical values of the rapidity), and the simple poles correspond to direct or cross-channel resonances. If $S^{a,b}(\theta)$ has a simple pole at $\theta = iu_{ab}^c$ in the direct channel we say that a particle of mass

$$m_c^2 = m_a^2 + m_b^2 + 2m_am_b \cos u_{ab}^c, \quad (3.5)$$

is a bound state of $a$ and $b$. The new particle must itself be included in the particle spectrum. If $ab \rightarrow c$ can occur then so can $a\bar{c} \rightarrow \bar{b}$ and $b\bar{c} \rightarrow \bar{a}$, where bar denotes charge conjugation. From (3.5) we deduce the following identity

$$u_{ab}^c + u_{a\bar{c}}^\bar{b} + u_{b\bar{c}}^\bar{a} = 2\pi. \quad (3.6)$$

(iv) The Bootstrap equations. The bootstrap equations give a non–linear relation between $S$–matrix elements. If $S^{a,b}(\theta)$ has a direct channel pole at $\theta = iu_{ab}^c$, corresponding to a particle in $V_c = P_{ab}^c(V_a \otimes V_b)$, where $P_{ab}^c$ is a projection operator, then

$$S^{d,c}(\theta) = (I \otimes S^{d,a}(\theta - iu_{a\bar{c}}^\bar{b}))(S^{d,b}(\theta + i\bar{u}_{b\bar{c}}^\bar{a}) \otimes I), \quad (3.7)$$

restricted to $V_d \otimes V_c \subset V_d \otimes V_b \otimes V_a$, and similarly

$$S^{c,d}(\theta) = (S^{b,d}(\theta - i\bar{u}_{b\bar{c}}^\bar{a}) \otimes I))(I \otimes S^{a,d}(\theta + i\bar{u}_{a\bar{c}}^\bar{b})), \quad (3.8)$$

restricted to $V_c \otimes V_d \subset V_b \otimes V_a \otimes V_d$. In the above $\bar{u}_{ab}^c = \pi - u_{ab}^c$, etc.

The bootstrap constraints are very powerful because they must be consistent with the integrability of the theory. What we mean by this is that the spectrum of possible spins of the conserved charges is tightly constrained and this in turn highly constrains the possible
masses of the physical states. There is a class of minimal solutions to the above axioms for which each particle state is non-degenerate, i.e. $V_a \simeq 1$, and the $S$-matrix has the minimum number of poles and zeros needed to satisfy the axioms. Each minimal solution is related to a simply-laced Lie algebra. The minimal $S$-matrix (when multiplied by a function of the coupling constant which introduces no additional poles onto the physical strip) is then conjectured to be the $S$-matrix of the particles of the affine Toda field theories for real coupling constant discussed in the introduction. The spins of the conserved charges of a theory described by the minimal $S$-matrix are equal to the exponents of the finite Lie algebra $g$ modulo its Coxeter number, and the number of particle states is equal to the rank of $g$. So, for example, the particle spectrum of the $sl(n)$ theory consists of $n-1$ particles with masses

$$m_a = m_0 \sin \left( \frac{\pi a}{n} \right), \quad a = 1, \ldots, n - 1.$$  \hspace{1cm} (3.9)$$

Notice that these masses are, up to an overall scale factor, just the classical masses of the fundamental Toda particles for real coupling (9), but also the classical soliton masses in the complex Toda theories (1.16). The possible fusions are $ab \rightarrow (a + b) \mod n$, which occur at the rapidity values $\theta = iu_{ab}$:

$$u_{ab} = \begin{cases} \frac{a+b}{n} \pi & a+b < n \\ \left(2 - \frac{a+b}{n}\right) \pi & a+b \geq n. \end{cases}$$ \hspace{1cm} (3.10)$$

The charge conjugation operator maps $a \rightarrow \bar{a} = n - a$. The explicit form for $S_{\text{min}}^{a,b}(\theta)$, from ref. [11], is

$$S_{\text{min}}^{a,b}(\theta) = (a + b)(a + b - 2)^2(a + b - 4)^2 \cdots (|a - b|),$$ \hspace{1cm} (3.11)$$

where the following notation has been borrowed from ref. [18]:

$$(x) = \frac{\sin \left( \frac{\theta}{2} + \frac{\pi x}{2n} \right)}{\sin \left( \frac{\theta}{2} - \frac{\pi x}{2n} \right)}.$$ 

The $S$-matrix element $S_{\text{min}}^{a,b}(\theta)$ has one direct channel pole at $\theta = iu_{ab}$ corresponding to the exchange of the particle $a + b \mod n$, and a cross-channel pole at $\theta = iu_{a\bar{b}}$ corresponding to the exchange of particle $a - b \mod n$. Notice that since there is only one possible pole in $S_{\text{min}}^{a,b}(\theta)$ corresponding to a bound state in the direct channel, we can unambiguously write $u_{ab} \equiv u_{ab}^c$. 

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We now construct a series of factorizable $S$--matrices, with degenerate particle states, based on the ansatz

$$S^{a,b}(\theta) = X^{a,b}(\theta)\tilde{R}^{a,b}(x(\theta)), \quad (3.12)$$

where $X^{a,b}(\theta)$ is a scalar function of $\theta$, and the $\tilde{R}^{a,b}(x)$ are the solutions to the YBE constructed in the last section. It is convenient to split the scalar prefactor $X^{a,b}(\theta)$ into two pieces:

$$X^{a,b}(\theta) = S^{a,b}_{\text{min}}(\theta)f^{a,b}(x(\theta)), \quad (3.13)$$

where $S^{a,b}_{\text{min}}(\theta)$ is the minimal $S$-matrix written down in (3.11).

The form of the ansatz identifies a particle of mass proportional to $m_a$ with the $a$th fundamental representation of $U_q(sl(n))$. We now show that the above ansatz is consistent with the axioms of $S$-matrix theory. One should bear in mind that $S^{a,b}_{\text{min}}(\theta)$, by itself, satisfies all the axioms of $S$-matrix theory.

The YBE equation for $S(\theta)$ is satisfied because $\tilde{R}(x)$ itself satisfies the YBE. However, we deduce that the spectral parameter $x$ and the rapidity $\theta$ must be related by $x = \exp(c\theta + d)$, for constants $c$ and $d$.

Unitarity can be satisfied by an appropriate choice of $f^{a,b}(x)$. Using (2.19) we deduce that $d = 0$, i.e. $x = \exp c\theta$ and

$$f^{a,b}(x) = \prod_{i,j=1}^{a,b} f(x(-q)^{-(a+b-2i-2j+2)/2}), \quad (3.14)$$

where $f(x)$ is a scalar function satisfying

$$f(x)f(x^{-1}) = \frac{1}{(q^{-1}x^{-1} - qx)(q^{-1}x - qx^{-1})}. \quad (3.15)$$

The $\tilde{R}^{a,b}(x)$ matrix must be consistent with the bootstrap equations (3.7) and (3.8). In addition, the residue of $S^{a,b}(\theta)$ at the pole $\theta = i u_{ab}$, corresponding to the direct channel bound state $c = a + b$, must be proportional to the projection operator $s_{a+b}^-$. For example, $S^{1,1}(\theta)$ should be proportional to $s_2^-$ when $\theta = i u_{11} = 2\pi i/n$. By using equation (2.9), we deduce that

$$x = (-q)^{-n\theta/2\pi i}. \quad (3.16)$$

Now we know how $x$ is related to $\theta$ we can rewrite the bootstrap equations in terms of the multiplicative variable $x$. It is easy to show that the two bootstrap equations (3.7) and

\footnote{We ignore any ambiguities of the CDD type.}
(3.8) then are identical to the two fusion equations in (2.17). In addition, equation (2.18) says that the residue of $S^{a,b}(\theta)$ at the pole $\theta = iu_{ab}$ is proportional to $s^{-a+b}$, as required.

To exhibit crossing symmetry, we have to specify the charge conjugation operator $C_a : V_a \to V_{n-a}$. In fact, $V_a$ is naturally dual to $V_{n-a}$ via the action of the Hecke algebra analogue of the $\epsilon$-tensor, namely $s_n$. This is because $\dim(s_n(V_a \otimes V_{n-a})) = 1$, where $V_a \otimes V_{n-a} \subset V_1 \otimes V_n$. From the action $T(e_i \otimes e_j) = e_j \otimes e_i$, for $i < j$ and equation (2.8) one finds

$$s_n^{-1}(e_1 \otimes e_2 \otimes \ldots \otimes e_n) = \frac{q^{-n(n-1)/2}}{[n]!} \sum_{\{ij\} \in P_n} (-q)^{l(\{ij\})} e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n},$$

(3.17)

where $P_n$ is the set of permutations of $\{1,2,\ldots,n\}$, and $l(w)$ is the length of the permutation $w \in P_n$, with respect to simple transpositions. If we define a set of dual vectors $\{e_i^*, i = 1,\ldots,n\}$, with $e_i^* \cdot e_j = \delta_{ij}$, then the explicit expression for the charge conjugation operator is

$$C_a = \lambda_a \sum_{\{ij\} \in P_n} (-q)^{l(\{ij\})} (e_{i_{a+1}} \otimes \ldots \otimes e_{i_n}) (e_{i_1}^* \otimes \ldots \otimes e_{i_a}^*).$$

(3.18)

In the above, $\lambda_a$ is a normalization constant determined by $C_a C_a = I_a$. This gives

$$\lambda_a \lambda_a = \frac{(-a\bar{a}q^{-a(a-1)/2-a(\bar{a}-1)/2-a\bar{a}}}{[a]![\bar{a}]!}.$$ 

For the case $n = 2$, corresponding to the sine-Gordon theory, the action of the charge conjugation operator is unconventional. In the usual formulation of the soliton $S$-matrix in sine-Gordon theory, the ansatz of (3.12) needs to be conjugated with momentum dependent factors in order to ensure crossing symmetry [19]. However, in the present formulation this is not required because the charge conjugation operator acts in an unconventional way:

$$Ce_1 = (-q)^{1/2} e_2, \quad Ce_2 = (-q)^{-1/2} e_1,$$

(3.19)

to compare with the usual action $Ce_1 = e_2$ and $Ce_2 = e_1$. The unconventional action (3.19) makes the introduction of momentum dependent factors in the definition of the $S$-matrix unnecessary, but is obviously equivalent to the usual formulation by a redefinition of the in and out states.

To determine the constraint implied by crossing symmetry, it is sufficient to consider the relationship between $S^{1,1}(\theta)$ and $S^{n-1,1}(\theta)$. It is possible to show directly that the
S-matrix elements are related by crossing symmetry if the following equation is satisfied by the function $f(x)$:

$$
\prod_{i=1}^{n-1} f(x^{-1}(-q)^{-i}) \prod_{i=1}^{n-2} (x^{-1}(-q)^{-i-1} - x(-q)^{i+1}) = f(x). \tag{3.20}
$$

To complete the construction of a consistent $S$-matrix, we must find a function $f(x)$, which satisfies (3.15) and (3.20). Introducing the notation $q = -\exp -i\pi \lambda$, the explicit expression for $f(x)$, derived in appendix B, is

$$
f(x) = \frac{\Gamma\left(\frac{in\lambda \theta}{2\pi} + \lambda\right) \Gamma\left(1 - \frac{in\lambda \theta}{2\pi} - \lambda\right)}{2\pi i} \prod_{j=1}^{\infty} \frac{\Gamma\left(1 + \frac{in\lambda \theta}{2\pi} + (j - 1)n\lambda\right)}{\Gamma\left(1 - \frac{in\lambda \theta}{2\pi} + (j - 1)n\lambda\right)} \times \frac{\Gamma\left(\frac{in\lambda \theta}{2\pi} + jn\lambda\right) \Gamma\left(-\frac{in\lambda \theta}{2\pi} + (j - 1)n + 1\lambda\right) \Gamma\left(1 - \frac{in\lambda \theta}{2\pi} + (jn - 1)\lambda\right)}{\Gamma\left(-\frac{in\lambda \theta}{2\pi} + jn\lambda\right) \Gamma\left(\frac{in\lambda \theta}{2\pi} + (j - 1)n + 1\lambda\right) \Gamma\left(1 + \frac{in\lambda \theta}{2\pi} + (jn - 1)\lambda\right)} \tag{3.21}
$$

To summarize: the fusion procedure for the $R$-matrix provides a solution to the bootstrap equations if we identify the particle of mass proportional to $m_a$ with the fundamental representation $\rho_a$ of $U_q(sl(n))$.

### 4. The Soliton $S$-Matrix

The quantum group $S$-matrix in equation (3.12) represents our ansatz for the soliton-soliton $S$-matrix. We now consider some of the implications of this proposal. First of all, the fact that the poles of the minimal part of the $S$-matrix encode the fusing between the solitons implies the quantum soliton masses are proportional to (3.9). This means that, up to an overall multiplicative renormalization, the quantum soliton masses $\hat{M}_a$ are proportional to the classical soliton masses $M_a$ in (1.16), or the masses of the fundamental Toda particles $m_a$.

The topological charge operator is proportional to the Cartan subalgebra generator $h$ of $U_q(sl(n))$, which has the following action on $V_1$:

$$
he_i = e_i e_i, \tag{4.1}
$$

where $e_i$ is one of the weights of the $n$ dimensional representation. It is immediately apparent that

$$
[h \otimes I + I \otimes h, \tilde{R}(x)] = 0, \tag{4.2}
$$
and so the proposed $S$-matrix conserves topological charge. In the quantum theory there is a soliton state for every weight of a fundamental representation. In the classical scattering theory, not all the weights of the fundamental representations are obtained (except for the $n$ and $\bar{n}$ representations).

The fact that the fusing rules of the solitons (3.10) are exactly the same as the those of the fundamental Toda particles in the real coupling constant theories, implies that the spectrum of conserved charges is the same. All these conserved charges are scalars with respect to the quantum group. However, there exist symmetries which are non-trivial with respect to the quantum group. These ‘residual quantum symmetries’ generalize the situation for the sine-Gordon theory \[19,20\]. To see this in more detail we have to introduce the generators of $U_q(sl(n))$, \{$e_i, f_i, h_i, i = 1, \ldots, n-1$\}. (The $e_i$'s should not be confused with the weights of the $n$ dimensional representation.) The algebra of the generators may be found in ref. [16]. Recall that $\hat{R}(x)$ is invariant under the action of the quantum group whose action is defined by the comultiplication $\Delta^{(2)}$:

\[
\Delta^{(2)}(e_i) = q^{h_i/2} \otimes e_i + e_i \otimes q^{-h_i/2} \\
\Delta^{(2)}(f_i) = q^{h_i/2} \otimes f_i + f_i \otimes q^{-h_i/2} \\
\Delta^{(2)}(h_i) = h_i \otimes I + I \otimes h_i.
\] (4.3)

So crudely speaking the $S$-matrix is quantum group invariant. However, the $\hat{R}(x)$ matrix also enjoys a momentum dependent symmetry. To see this we note that the $n$ dimensional representation of $U_q(sl(n))$ is identical to that of $sl(n)$. Let $(e_0, f_0)$ be the raising and lowering operators associated to the highest root. $\hat{R}(x)$ satisfies

\[
\hat{R}(x_1/x_2)(x_1^2 e_0 \otimes q^{-h_0/2} + x_2^2 q^{h_0/2} \otimes e_0) = (x_2^2 e_0 \otimes q^{-h_0/2} + x_1^2 q^{h_0/2} \otimes e_0) \hat{R}(x_1/x_2),
\] (4.4)

where $h_0 = -\sum_{i=1}^{n-1} h_i$. A similar equation holds with $e_0$ replaced by $f_0$ and $x_1$ and $x_2$ interchanged. We interpret the above as a momentum dependent symmetry of the $S$-matrix. The spin of the generator of such a symmetry follows from the relation between $\theta$ and $x$ in equation (3.16). With $q = \exp -\pi i \lambda$, the spin of the generator is $n\lambda$.

Since we have the full $S$-matrix of solitons we can discover whether there are any additional states in the theory, which will manifest themselves as simple poles on the physical strip. These new states must then be added into the list of states of the theory. The

\[8\] Here, $h_i = \alpha_i \cdot h$, where $h$ is the Cartan subalgebra generator introduced above.
bootstrap equations can then be used to extract the $S$-matrix elements of these additional states. For instance, in the case of $sl(2)$, the sine-Gordon theory, one finds a set of poles, whose position depends on the coupling constant, corresponding to the exchange of scalar states. These are the ‘breathers’ or ‘doublets’. The masses of the breathers, as written down in (4), follow from the positions of the poles. So we must search for simple poles on the physical strip, over and above the simple poles already interpreted as being due to the exchange of solitons.

To begin with, we consider just the scattering of elementary solitons. There are three possible classes of process to consider. (i) The transmission process\(^9\), $ik \rightarrow ki$, for $i \neq k$. (ii) The identical particle process, $ii \rightarrow ii$. (iii) The reflection process, $ik \rightarrow ik$. These processes are illustrated in figure 3.

\[ S_{ik\rightarrow ki}(\theta) = \prod_{j=1}^{\infty} \frac{\Gamma\left(\frac{i\theta}{2\pi} + 1 + \frac{j}{n\lambda}\right) \Gamma\left(-\frac{i\theta}{2\pi} + \frac{j}{n\lambda}\right)}{\Gamma\left(-\frac{i\theta}{2\pi} + 1 + \frac{j}{n\lambda}\right) \Gamma\left(-\frac{i\theta}{2\pi} + \frac{j-1}{n\lambda}\right)} \] (4.5)

and

\[ S_{ii\rightarrow ii}(\theta) = \frac{\sin\left(\pi\lambda - \frac{in\lambda\theta}{2}\right)}{\sin\left(\frac{in\lambda\theta}{2}\right)} S_{ik\rightarrow ki}(\theta). \] (4.6)

Consider the $S$-matrix element $S_{ik\rightarrow ki}(\theta)$. It has simple poles on the physical strip ($0 \leq \text{Im}(\theta) \leq \pi$) at

\[ \theta = -\frac{2\pi i}{n\lambda} p + \frac{2\pi i}{n}, \quad p = 0, 1, \ldots, [\lambda]. \] (4.7)

In the above, the notation $[\lambda]$ means the largest integer less than $\lambda$. The simple pole at $\theta = 2\pi i/n$, corresponds to the exchange of a soliton with topological charge $e_i + e_k$ and mass $\hat{M}_2$ in the direct channel. The residues of the poles in (4.7) are proportional to $s_2^-$, and so they are interpreted as being due to the exchange of excited solitons, or ‘breathing solitons’, in the direct channel with topological charge $e_i + e_k$. The soliton corresponding

\[ \text{The notation here is short for } (e_i, e_k) \rightarrow (e_k, e_i). \]
to the simple pole at $\theta = 2\pi i/n$, is just the ground state of this system. The masses of the excited solitons are

$$\hat{M}_2(p) = 2\hat{M}_1 \cos\left(\frac{\pi}{n} \left(1 - \frac{p}{\lambda}\right)\right), \quad p = 0, 1, \ldots, [\lambda],$$

(4.8)

where $\hat{M}_2 = \hat{M}_2(0)$. For the sine-Gordon theory the poles (4.7) and the masses have a different interpretation. In this case the states being exchanged are the breathers; their topological charge is zero and the masses in (4.8) are equal to the breather masses (4) (with the identification $\gamma = 4\pi/\lambda$).

Similarly the element $S_{ii\rightarrow ii}(\theta)$ has simple poles on the physical strip at

$$\theta = \frac{2\pi i}{n\lambda} p, \quad p = 1, 2, \ldots, \left[\frac{1}{2} n\lambda\right].$$

(4.9)

These poles are at $x = e^{-\pi ip}$ for which

$$\tilde{R}(x = e^{-i\pi p}) = (-)^p (q - q^{-1}) I \otimes I,$$

(4.10)

and so it is natural to interpret these poles as being due to the exchange of scalar states (i.e. with zero topological charge) in the cross channel of $ii \rightarrow ii$: these scalar states are the analogues of the sine-Gordon breathers. The masses of the states are

$$\hat{m}_1(p) = 2\hat{M}_1 \sin\left(\frac{\pi p}{n\lambda}\right), \quad p = 1, 2, \ldots, \left[\frac{1}{2} n\lambda\right],$$

(4.11)

which generalizes the sine-Gordon result (4).

We now consider the other $S$-matrix elements $S^{a,b}(\theta)$, where we take $a + b \leq n$ and $a \geq b$, without loss of generality. Rather than give a full discussion, we only present the results; a complete analysis will appear elsewhere. There are two types of simple pole, those whose positions do not depend on the coupling constant and those whose positions do. In the first set are

$$\theta = \frac{\pi i}{n} (a + b), \quad \frac{\pi i}{n} (a - b),$$

(4.12)

which correspond to the soliton $a + b$, in the direct channel, and $\bar{a} + b$, in the crossed channel. In the second set are

$$\theta = -\frac{2\pi i}{n\lambda} p + \frac{\pi i}{n} (a + b - 2j + 2), \quad p = 1, 2, \ldots, \left[\frac{1}{2} \lambda (a + b - 2j + 2)\right],$$

(4.13)

with $j = 1, 2, \ldots, b$, and

$$\theta = \frac{2\pi i}{n\lambda} p + \frac{\pi i}{n} (a + b - 2j), \quad p = 1, 2, \ldots, \left[\frac{1}{2} \lambda (2j - a - b + n)\right],$$

(4.14)
also with $j = 1, 2, \ldots, b$. The poles in (4.13) are direct channel poles and those in (4.14) are crossed channel poles. The poles correspond to solitons transforming in non-fundamental representations, excited solitons in both fundamental and non-fundamental representations, and scalar states. The excited solitons corresponding to the fundamental representation $q_{a+b}$ have masses given by the square root of

$$
\sqrt{\hat{M}_a^2 + \hat{M}_b^2 + 2\hat{M}_a\hat{M}_b \cos \left( \frac{\pi}{n} \left( a + b - \frac{2p}{\lambda} \right) \right)}, \quad p = 1, 2, \ldots, \left[ \frac{\lambda}{2}(a + b) \right], \quad (4.15)
$$

which generalizes (4.8). The scalar states correspond to simple poles in the direct channel of $S^{a,a}(\theta)$, and have masses

$$
m_a(p, j) = 2\hat{M}_a \sin \left( \frac{\pi}{n} \left( \frac{p}{\lambda} + j - 1 \right) \right), \quad j = 1, 2, \ldots, a, \quad p = 1, 2, \ldots, \left[ \frac{\lambda}{2}(n - 2j + 2) \right], \quad (4.16)
$$

which generalizes (4.11).

Another issue concerns the question as to whether the $S$-matrix describes a unitary quantum field theory. As we have already pointed out, this is a separate issue from whether the $S$-matrix is unitary as a matrix. A discussion of this point may be found in ref. [21] which discusses the non-unitary field theory describing the Lee-Yang edge singularity.

Consider the $S$-matrix element describing the scattering of two equal mass particles. If $S$ has a pole at $\theta = iu$ corresponding to the exchange of particle in the direct channel, then for a unitary theory one has

$$
S(\theta) \sim \frac{i\rho^2}{\theta - iu}, \quad (4.17)
$$

for some $\rho \in \mathbb{R}$. For a non-unitary theory the residue might have a different sign. For example, consider the excited soliton poles of equation (4.7) for the process $ik \rightarrow ki$. The explicit expression for the $S$-matrix element for this process is equation (4.5). The pole corresponding to the $p^{th}$ excited soliton comes from the factor

$$
\Gamma \left( -\frac{i\theta}{2\pi} - \frac{1}{n} + \frac{p}{n\lambda} \right).
$$

The only ‘dangerous’ terms as regards the sign of the residue are

$$
\Gamma \left( -\frac{p}{n\lambda} \right) \Gamma \left( -\frac{p-1}{n\lambda} \right) \cdots \Gamma \left( -\frac{1}{n\lambda} \right),
$$

which contributes an overall sign of $(-)^p$ to the residue. So the first excited soliton corresponds to a non-unitary coupling, the second a unitary coupling $etc$. The message of this
result is that the underlying quantum field theory is, in general, non-unitary. This was only to be expected, considering the complex form for the Hamiltonian (1.2). However, as we have seen for the elementary solitons, there exists a regime for which all the non-unitary states decouple. In the next section we make some comments about this for the full $S$-matrix.

The $S$-matrix reduces to a very simple expression when $\lambda = 1$. In this case $q = 1$ and so the Hecke algebra reduces to the symmetric group and the $S$-matrix becomes trivial in the space of topological charges. One can easily verify from the explicit expressions that

$$S^{a,b}(\theta)\big|_{\lambda=1} = S_{\min}^{a,b}(\theta).$$

Finally let us consider the case for $n = 2$, where the $S$-matrix is the soliton $S$-matrix of the sine-Gordon theory. In this case $S_{\min}(\theta)$ has no poles on the physical strip; it is a CDD type ambiguity and so for this case alone we drop this part of the $S$-matrix without affecting any properties of the ansatz. The resulting $S$-matrix is exactly that of ref. [3], up to the unconventional action of the charge conjugation operator (3.19).

5. The Semi-Classical Limit

In this section we will verify that some particular elements of the conjectured $S$-matrix are consistent with the classical scattering theory. The idea is to use the relation between the semi-classical limit of the $S$-matrix (the limit as $\hbar \to 0$)\(^{10}\) and the time-delays of the classical scattering theory. We will only consider the scattering of the elementary solitons (the $n$-dimensional representation) for simplicity.

As we discussed in §3 there are only three possible classes of process involving elementary solitons. The transmission, identical particle and reflection processes. It turns out that only the semi-classical limit of the first two processes can be connected in a simple way with the classical scattering theory. A discussion of the reflection amplitude requires an analysis of a complex time trajectory in the classical theory [7], which we will postpone for a future publication.

For the transmission and identical particle processes, there is a very simple relation between the leading term of the semi-classical limit of the $S$-matrix element and the corresponding time-delay of the classical theory. We simply quote the result which follows from

\(^{10}\) In this section we shall re-introduce $\hbar$ into our formulas.
the WKB analysis of ref. [6]. If $E$ is the energy in the channel in question and $\Delta t(E)$ is the classical time-delay, defining

$$\delta(E) = \frac{1}{2}n_B\pi + \frac{1}{2}\int_{E_{\text{th}}}^{E} dE' \Delta t(E'),$$

(5.1)

where the number of bound states, or resonances, in the channel is the largest integer less than $n_B$ (which is denoted $[n_B]$), and $E_{\text{th}}$ is the threshold energy where the resonances are just unbound, then the leading behaviour of the $S$-matrix is

$$S(\theta) = \exp\left(\frac{2i}{\hbar}(\delta(\theta) + O(\hbar))\right).$$

(5.2)

Let us apply (5.1) to the scattering of elementary solitons. In the centre-of-mass the total energy is $E = 2M_1\cosh(\theta/2)$, where $\theta$ is the relative rapidity. The threshold energy is $2M_1$. The classical time-delay is from §1

$$\Delta t(\theta) = -\frac{2n\gamma^{(12)}(\theta)}{M_1\beta^2\sinh(\theta/2)}.$$  

(5.3)

Writing the integral (5.1) in terms of the rapidity we have

$$\delta(\theta) = \frac{1}{2}n_B\pi - \frac{n}{\beta^2}\int_0^\theta d\theta' \gamma^{(12)}(\theta').$$

(5.4)

In (5.4), $\gamma^{(12)}(\theta)$ is the ‘interaction function’ (1.11) for the classical soliton scattering. For two elementary solitons we have

$$\gamma^{(12)}(\theta) = \log\left(\frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2} + \frac{\pi}{n}\right) \sin\left(\frac{\theta}{2} - \frac{\pi}{n}\right)}\right).$$

(5.5)

The two relevant $S$-matrix elements are written down in (4.5) and (4.6). In order to implement the semi-classical limit we have to know how the coupling constant $\lambda$ (or $q$) depends on $\hbar$. We have already seen that the limit $q \to 1$ can be thought of as the weak coupling limit for the solitons, since the $S$-matrix becomes trivial, however, this cannot correspond to the limit $\beta \to 0$ because the soliton masses are proportional to $\beta^{-2}$. This situation is familiar from the sine-Gordon theory, the point being that the solitons are weakly coupled when the coupling constant $\beta$ is large. Mirroring the situation of the sine-Gordon theory, we shall find that a behaviour of the form

$$\lambda = \frac{1}{\hbar\beta^2}(\lambda_0 + O(\hbar\beta^2)),$$

(5.6)
will be necessary. So as $\hbar \to 0 \lambda \to \infty$. In this limit the product of the gamma functions in (4.5) may be approximated by an integral in the exponent, so the leading order behaviour is

$$S_{ik \to ki}(\theta) \to \exp \left\{ \frac{n\lambda_0}{\hbar \beta^2} \int_0^{\infty} dx \log \left( \frac{\Gamma \left( \frac{i\theta}{2\pi} + 1 + x \right) \Gamma \left( \frac{i\theta}{2\pi} + 1 + x \right)}{\Gamma \left( \frac{-i\theta}{2\pi} + \frac{1}{n} + x \right) \Gamma \left( \frac{-i\theta}{2\pi} - \frac{1}{n} + x \right)} \right) \right\}. \quad (5.7)$$

But this is equal to

$$\exp \left\{ \frac{n\lambda_0}{\hbar \beta^2} \int_0^{\frac{1}{n}} dx \log \left( \frac{\sin \left( -\pi x - \frac{i\theta}{2\pi} \right)}{\sin \left( \pi x - \frac{i\theta}{2\pi} \right)} \right) \right\} = \exp \left\{ i\pi \frac{\lambda_0}{\hbar \beta^2} + \frac{i n \lambda_0}{2\pi \hbar \beta^2} \int_0^\theta d\theta' \log \left( \frac{\sin \left( \frac{\theta'}{2\pi} + \frac{\pi}{n} \right) \sin \left( \frac{\theta'}{2\pi} - \frac{\pi}{n} \right)}{\sin^2 \left( \frac{\theta'}{2\pi} \right)} \right) \right\}. \quad (5.8)$$

From this we can extract the phase shift

$$\delta_{ik \to ki}(\theta) = \frac{\lambda_0 \pi}{2\beta^2} + \frac{n \lambda_0}{4\pi \beta^2} \int_0^\theta d\theta' \log \left( \frac{\sin \left( \frac{\theta'}{2\pi} + \frac{\pi}{n} \right) \sin \left( \frac{\theta'}{2\pi} - \frac{\pi}{n} \right)}{\sin^2 \left( \frac{\theta'}{2\pi} \right)} \right). \quad (5.9)$$

Now compare the above with (5.4) and the expression for $\gamma^{(12)}(\theta)$ in (5.5). The two expressions agree if

$$\lambda_0 = 4\pi, \quad (5.10)$$

and $n_B$, the parameter which relates to the number of bound states in the channel $ik \to ki$, is equal to $\lambda$ (at this order in $\beta^2$). This is consistent with the number of bound states found in the direct channel of this process ($\sim \lambda$) found in §4.

In a similar way we can repeat the analysis for the process $ii \to ii$. However, it is more straightforward to take the semi-classical limit of (4.6)

$$S_{ii \to ii}(\theta) = S_{ik \to ki}(\theta) \exp \left( -i\pi \frac{\lambda_0}{\hbar \beta^2} \right), \quad (5.11)$$

from which we deduce

$$\delta_{ii \to ii}(\theta) = \frac{n \lambda_0}{4\pi \beta^2} \int_0^\theta d\theta' \log \left( \frac{\sin \left( \frac{\theta'}{2\pi} + \frac{\pi}{n} \right) \sin \left( \frac{\theta'}{2\pi} - \frac{\pi}{n} \right)}{\sin^2 \left( \frac{\theta'}{2\pi} \right)} \right). \quad (5.12)$$
Again this is the correct semi-classical limit if \( n_B \), in this case, is zero. This is consistent with the findings of §4, where we found no bound states in the direct channel of the process \( ii \to ii \) (however, there are bound states in the crossed channel).

So we have shown, at least for some of the processes, that the conjectured \( S \)-matrix with the following functional form for \( q \)

\[
q = \exp - \frac{1}{\hbar \beta^2} \left( 4\pi^2 i + O(\hbar \beta^2) \right), \tag{5.13}
\]

does indeed describe the quantization of the classical scattering theory. In order to find the higher order terms in (5.13) it would be necessary to go beyond the WKB approximation. Such an analysis has been done for the sine-Gordon theory in refs. [9,22] yielding the result

\[
q = \exp \left( -\frac{4\pi^2 i}{\hbar \beta^2} + \pi i \right), \tag{5.14}
\]

or equivalently \( \gamma = 4\pi/\lambda \), where \( \gamma \) is defined in (5).

6. Discussion

We have constructed the classical scattering theory of the solitons of the complex affine \( sl(n) \) Toda equations. The solitons have topological charges which are weights of the fundamental representations of the Lie algebra \( sl(n) \), and masses whose ratios are equal to those of the conventional Toda particles. Using the sine-Gordon theory as a paradigm, we proposed a form for the \( S \)-matrix of the solitons by generalizing the \( sl(2) \) quantum group to the \( sl(n) \) quantum group. The quantum group acts in space of topological charge of the quantum soliton theory and the resulting \( S \)-matrix commutes with its action, as well as with a momentum dependent operator leading to non-trivial conserved quantities over and above the ones associated to a real Toda theory. The ansatz related the coupling constant of the Toda theory to the deformation parameter of the quantum group (5.13). The quantum masses of the solitons are proportional to their classical masses, up to an overall renormalization. This is consistent with the lowest order quantum corrections to the soliton masses, which can be computed in the following way. One looks at the linearized equation around the soliton solutions. The equation is a multi-component Schrödinger equation. Surprisingly, given the fact that the potential is complex, the frequencies of the modes are real, and hence the solitons are stable to small perturbations. Furthermore, the
zero-point energies of the modes may be summed to give the lowest order correction to the soliton masses. Details of this calculation will be presented elsewhere.

The spectrum of the $sl(n)$ theories, for $n > 2$, is a good deal more complicated than the sine-Gordon theory. In addition to the scalar breather states, there are also excited soliton states. Ideally, one would like to construct the $S$-matrix elements of these new states by applying the bootstrap equations. For the sine-Gordon theory this procedure terminates, no new states are then produced, and the complete spectrum just consists of the soliton anti-soliton and breathers. The problem of finding the complete spectrum for the general soliton theories looks rather formidable. The $S$-matrix describes a non-unitary theory, as expected from the complex form for the Hamiltonian. For the sine-Gordon theory the ground state of the breather is identified with the ‘elementary particle’ of the theory. We now show that such an identification can be made for the $sl(n)$ theories. If we expand the equations of motion (1.1) in powers of $\phi$ then in the linear approximation we expect to see modes corresponding the ‘classical’ masses of (9), as in the real Toda theories. On quantizing we might expect these modes to appear as quanta with mass $m_a + O(\beta^2)$, where the corrections arise from loops. Indeed, we did find scalar particles with masses given by (4.16). For each $a = 1, 2, \ldots, n - 1$ there is a discrete spectrum of particles. The ground states of these spectra have mass

$$\hat{m}_a(1, 1) = 2\hat{M}_a \sin \left(\frac{\pi}{n\lambda}\right), \quad (6.1)$$

In the limit of weak coupling, or $\hbar \to 0$, using the expression for the classical soliton masses and the expression for $\lambda$ we deduce

$$\hat{m}_a(1, 1) = 2m \sin \left(\frac{\pi a}{n}\right) + O(\beta^2), \quad (6.2)$$

which are the masses of the elementary particles in the weak coupling limit.

From the point of view of the $S$-matrix $S^{a,b}(\theta)$, there are two types of particle appearing as bound states, depending on whether the position of the associated pole depends on the coupling constant (4.12), or not, (4.13) and (4.14). In the former set, there are only the original solitons themselves, associated to the fundamental representations. The latter set contains the excited solitons, the solitons corresponding to non-fundamental representations and the scalar particles. There is a region for the coupling constant, namely

$$\lambda < \frac{2}{n}, \quad (6.3)$$
for which all the poles corresponding to the latter states are no longer on the physical strip. Remarkably the resulting S-matrix now describes a unitary quantum field theory, since all the non-unitary couplings are associated with the second set of states. Hence, for (6.3) the soliton S-matrix is complete and describes a unitary theory. Full details of this will be presented elsewhere.

For the sine-Gordon theory, \( \lambda < 1 \) is equivalent to \( \gamma > 4\pi \), or \( \beta^2 > 2\pi \), which is the repulsive regime, for which the S-matrix, as presented, is no longer valid. It would clearly be of interest to discover whether the general Toda theories have an analogue of the repulsive regime.

Returning to the S-matrix constructed in \( \S 2 \), it is easy to see that the axioms of S-matrix theory are satisfied algebraically (from the point of view of the Hecke algebra). This means that they will hold for any representation of the Hecke algebra. In particular, it means that we can formulate the theory in the Interaction Round a Face (IRF), or Solid-On-Solid (SOS), picture. From a soliton point of view this corresponds to labelling the processes by specifying the vacua between the solitons, rather than the topological charges of the solitons. Obviously this is completely equivalent to the ‘vertex’ point of view adopted in \( \S 2 \). However, this equivalence is only true for generic values of \( q \) assumed in \( \S 2 \). When \( q \) is a root of unity, say \( \lambda = 1/p \), where \( p \in \mathbb{Z} > n \), the ‘vertex’ description is no longer appropriate since some of the elements of the R-matrix become singular. In the ‘face’ picture, however, there is a way of restricting the allowed variables to a finite set for which the R-matrix is well-defined and the Hecke algebra relations are still satisfied. In terms of the solitons this would correspond to restricting the allowed set of vacua to some finite set (recall that the allowed set of vacua is isomorphic to the weight lattice of \( sl(n) \)). For these restricted models the spin of the symmetry generator in (4.4) is equal to \( n/p \), hence these theories have fractional symmetries generalizing the situation for the restricted sine-Gordon theories in refs. [20]. Notice that the restricted S-matrices would lie in the regime (6.3), so they would describe unitary quantum field theories.

In fact there are more general representations of the Hecke algebra associated to certain graphs [23] to which one could also associate a factorizable soliton S-matrix. In addition, one could also consider quantum groups related to other Lie algebras.
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Appendix A

In this appendix we prove the result that the $\check{R}^{a,b}(x)$ matrices defined in equation (2.17) satisfy the YBE. As we have already stated in the text the fusion procedure will guarantee that the YBE if (2.18) is satisfied. For reasons of space, the proof that we present is not self-contained, since we shall assume two lemmas which may be found in the second reference of [16].

In the following we use the notation $\check{R}_a(x) = I \otimes \cdots \otimes I \otimes \check{R}(x) \otimes I \otimes \cdots \otimes I$ to denote the basic $\check{R}(x)$ acting between the $a^{th}$ and $(a + 1)^{th}$ space in the tensor product $V_1 \otimes \cdots \otimes V_1$, and $s^-_{a,b}$ to denote the Hecke algebra analogue of the full anti-symmetrizer $s^-_{b-a+1}$ acting between the $a^{th}$ space and the $b^{th}$ space, inclusive, in the tensor product $V_1 \otimes \cdots \otimes V_1$.

**Lemma 1.** Equation (9) of the second reference of [16].

$$\check{R}_a(x)\check{R}_{a-1}(x(-q)^{-1})\cdots \check{R}_1(x(-q)^{-a+1})s^-_{2,a+1}$$

$$= s^-_{1,a} \check{R}_a(x(-q)^{-a+1})\check{R}_{a-1}(x(-q)^{-a+2})\cdots \check{R}_1(x),$$

and similarly

$$\check{R}_1(x)\check{R}_2(x(-q)^{-1})\cdots \check{R}_a(x(-q)^{-a+1})s^-_{1,a}$$

$$= s^-_{2,a+1} \check{R}_1(x(-q)^{-a+1})\check{R}_2(x(-q)^{-a+2})\cdots \check{R}_a(x).$$

**Lemma 2.**

$$s^-_{1,a+1} = \frac{1}{(-q)^{-a-1} - (-q)^{a+1}} s^-_{1,a} \check{R}_a((-q)^{-a}) s^-_{1,a}$$

$$= \frac{1}{(-q)^{-a-1} - (-q)^{a+1}} s^-_{2,a+1} \check{R}_1((-q)^{-a}) s^-_{2,a+1}.$$ 

By applying Lemma 2 $b$ times one can easily show

$$s^-_{1,a+1} \propto s^-_{1,a} \check{R}_a((-q)^{-a})\check{R}_{a-1}((-q)^{-a+1})\cdots \check{R}_{a-b+1}((-q)^{-a+b-1})s^-_{1,a-b+1}, \quad (A.1)$$
and
\[ s_{1,a+1}^- \propto s_{2,a+1}^- R_1((-q)^{-a}) R_2((-q)^{-a+1}) \cdots R_b((-q)^{-a+b-1}) s_{b+1,a+1}^- . \] (A.2)

We now turn to the main proof. \( \tilde{R}^{a,b}(x) \) is defined recursively in terms of \( \tilde{R}(x) \) by using the fusion equations (2.17). What we need to show is that the \( \tilde{R}^{a,b}(x) \) so defined satisfies (2.18):
\[ \tilde{R}^{a,b}((-q)^{-a+b}/2) \propto s_{a+b}^- . \] (A.3)

We proceed by induction. Firstly, equation (2.9) implies that (A.3) is satisfied for \( a = b = 1 \):
\[ \tilde{R}^{1,1}((-q)^{-1}) \equiv \tilde{R}((-q)^{-1}) \propto s_2^- . \]

Now consider
\[ \tilde{R}^{a+1,b}((-q)^{-(a+b+1)/2}) = (\tilde{R}^{a,b}((-q)^{-a+b}/2) \otimes I)(I \otimes \tilde{R}^{1,b}((-q)^{-a-(b+1)/2}))(s_{a+1}^- \otimes s_b^-) , \] (A.4)

where we have introduced the projection operator \( (s_{a+1}^- \otimes s_b^-) \) to implement the restriction \( V_{a+1} \otimes V_b \subset V_1^{\otimes(a+b+1)} \), explicitly. Assuming that (A.3) is true for \( \tilde{R}^{a,b}(x) \) we can write the right-hand side of (A.4) as
\[ (s_{a+b}^- \otimes I)(I \otimes \tilde{R}^{1,b}((-q)^{-a-(b+1)/2}))(s_{a+1}^- \otimes s_b^-) . \] (A.5)

By applying (2.17) \( b - 1 \) times we can express \( \tilde{R}^{1,b}(x) \) in terms of the basic \( \tilde{R}(x) \)
\[ \tilde{R}^{1,b}(x) = \tilde{R}_b(x(-q)^{(b-1)/2}) \tilde{R}_{b-1}(x(-q)^{(b-3)/2}) \cdots \tilde{R}_1(x(-q)^{-(b-1)/2}) s_{2,b+1}^- . \] (A.6)

Using Lemma 1 we can rewrite the right-hand side of (A.6) as
\[ s_{1,b}^- \tilde{R}_b(x(-q)^{-(b-1)/2}) \tilde{R}_{b-1}(x(-q)^{-(b-3)/2}) \cdots \tilde{R}_1(x(-q)^{(b-1)/2}) . \]

Substituting this into (A.5) we find
\[ s_{1,a+b}^- \tilde{R}_{b+a}((-q)^{-a-b}) \tilde{R}_{b+a-1}((-q)^{-a-b+1}) \cdots \tilde{R}_{a+1}((-q)^{-a-1}) s_{1,a+1}^- , \]
where we have using the fact that \( s_{a,b}^- s_{c,d}^- = s_{a,b}^- \) for \( a \leq c < d \leq b \). But by (A.1), the corollary of Lemma 2, this is proportional to \( s_{a+b+1}^- \), as required.
To complete the proof we must consider
\[
\tilde{R}^{a,b+1} \left( (-q)^{-(a+b+1)/2} \right)
= (I \otimes \tilde{R}^{a,b}((-q)^{-(a+b)/2})) (\tilde{R}^{a,1}((-q)^{-(b-a+1)/2}) \otimes I) (s_a^- \otimes s_{b+1}^-).
\] (A.7)

The discussion proceeds in the same way. Assuming (A.3) for \( \tilde{R}^{a,b}(x) \) the right–hand side of (A.7) is
\[
(I \otimes s_{a+b}^-)(\tilde{R}^{a,1}((-q)^{-(b-a+1)/2}) \otimes I) (s_a^- \otimes s_{b+1}^-).
\] (A.8)

By applying (2.17) \( a - 1 \) times we have
\[
\tilde{R}^{a,1}(x) = \tilde{R}_1(x(-q)^{(a-1)/2}) \tilde{R}_2(x(-q)^{(a-3)/2}) \cdots \tilde{R}_a(x(-q)^{(a-1)/2}) s_{1,a}^-.
\]

Using Lemma 1 we can rewrite this as
\[
s_{2,a+1}^- \tilde{R}_1(x(-q)^{(a-1)/2}) \tilde{R}_2(x(-q)^{(a-3)/2}) \cdots \tilde{R}_a(x(-q)^{(a-1)/2}).
\]

Plugging this into (A.8), we have
\[
s_{2,a+b+1}^- \tilde{R}_1((-q)^{-a-b}) \tilde{R}_2((-q)^{-a-b+1}) \cdots \tilde{R}_a((-q)^{-b-1}) s_{a+1,a+b+1}^-.
\]

But by (A.2) this is proportional to \( s_{a+b+1}^- \), as required. This completes the proof.

**Appendix B**

In this appendix we find the function \( f(x) \) that satisfies equations (3.15) and (3.20). By using (3.15) we may rewrite (3.20) as
\[
\prod_{i=0}^{n-1} f(x^{-1}(-q)^{-i}) = ((-q)^{-1} x - (-q)x^{-1})^{-1} \prod_{i=1}^{n-1} (x^{-1}(-q)^{-i} - x(-q)^{i})^{-1}.
\]

Now we introduce \( u(x) = ((-q)^{-1} x^{-1} - (-q)x)f(x) \), where (3.15) implies \( u(x)u(x^{-1}) = 1 \). \( u(x) \) satisfies
\[
\prod_{i=0}^{n-1} u(x^{-1}(-q)^{-i}) = \prod_{i=1}^{n-1} \frac{x(-q)^{i-1} - x^{-1}(-q)^{-i+1}}{x^{-1}(-q)^{-i} - x(-q)^{i}}.
\]

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In order to solve this equation for $u(x)$ we introduce the Gamma function representation of the sine function. Defining $q = -\exp -i\pi \lambda$ and $x = \exp -i\pi \mu$, so $\mu = -n\lambda \theta / 2\pi i$, we have

$$
x^{-1}(-q)^{-a} - x(-q)^{a} = \frac{2\pi i}{\Gamma(\mu + a\lambda)\Gamma(1 - \mu - a\lambda)}.
$$
in terms of this

$$
\prod_{i=0}^{n-1} u(x^{-1}(-q)^{-i}) = \prod_{i=1}^{n-1} \frac{\Gamma(\mu + i\lambda)\Gamma(1 - \mu - i\lambda)}{\Gamma(-\mu + (1 - i)\lambda)\Gamma(1 + \mu - (1 - i)\lambda)}.
$$

To solve this equation we introduce the notation

$$
w(x) = \prod_{i=0}^{n-1} \Gamma(x + i\lambda),
$$
in terms of which

$$
\prod_{i=1}^{n-1} \Gamma(\mu + i\lambda) = \prod_{j=1}^{\infty} \frac{w(\mu + ((j - 1)n + 1)\lambda)}{w(\mu + jn\lambda)},
$$
and similarly for the other terms in (B.1). Using these facts we can write down an expression for $u(x)$ as an infinite product of Gamma functions. In fact because

$$( -q)^{-1}x^{-1} - (q)x^{-1} = \Gamma(\mu + \lambda)\Gamma(1 - \mu - \lambda)/2\pi i$$

the explicit expression for $f(x)$ is

$$
f(x) = \frac{\Gamma(\mu + \lambda)\Gamma(1 - \mu - \lambda)}{2\pi i} \prod_{j=1}^{\infty} \frac{\Gamma(1 + \mu + (j - 1)n\lambda)}{\Gamma(1 - \mu + (j - 1)n\lambda)}
\times \frac{\Gamma(\mu + jn\lambda)\Gamma(-\mu + ((j - 1)n + 1)\lambda)\Gamma(1 - \mu + (jn - 1)\lambda)}{\Gamma(-\mu + jn\lambda)\Gamma(\mu + ((j - 1)n + 1)\lambda)\Gamma(1 + \mu + (jn - 1)\lambda)}.
$$

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