(s,p)-VALENT FUNCTIONS

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Abstract. We introduce the notion of \((\mathcal{F}, p)\)-valent functions. We concentrate in our investigation on the case, where \(\mathcal{F}\) is the class of polynomials of degree at most \(s\). These functions, which we call \((s,p)\)-valent functions, provide a natural generalization of \(p\)-valent functions (see [12]). We provide a rather accurate characterizing of \((s,p)\)-valent functions in terms of their Taylor coefficients, through “Taylor domination”, and through linear non-stationary recurrences with uniformly bounded coefficients. We prove a “distortion theorem” for such functions, comparing them with polynomials sharing their zeroes, and obtain an essentially sharp Remez-type inequality in the spirit of [18] for complex polynomials of one variable. Finally, based on these results, we present a Remez-type inequality for \((s,p)\)-valent functions.

1. Introduction

Let us introduce the notion of “\((\mathcal{F}, p)\)-valent functions”. Let \(\mathcal{F}\) be a class of functions to be precise later. A function \(f\) regular in a domain \(\Omega \subset \mathbb{C}\) is called \((\mathcal{F}, p)\)-valent in \(\Omega\) if for any \(g \in \mathcal{F}\) the number of solutions of the equation \(f(z) = g(z)\) in \(\Omega\) does not exceed \(p\).

For example, the classic \(p\)-valent functions are obtained for \(\mathcal{F}\) being the class of constants, these are functions \(f\) for which the equation \(f = c\) has at most \(p\) solutions in \(\Omega\) for any \(c\). There are many other natural classes \(\mathcal{F}\) of interest, like rational functions, exponential polynomial, quasi-polynomials, etc. In particular, for the class \(\mathcal{R}_s\) consisting of rational functions \(R(z)\) of a fixed degree \(s\), the number of zeroes of \(f(z) - R(z)\) can be explicitly bounded for \(f\) solving linear ODEs with polynomial coefficients (see, e.g. [4]). Presumably, the collection of \((\mathcal{R}_s, p)\)-valent functions with explicit bounds on \(p\) (as a function of \(s\)) is much wider, including, in particular, “monogenic” functions (or
“Wolff-Denjoy series”) of the form \( f(z) = \sum_{j=1}^{\infty} \frac{\gamma_j}{z-z_j} \) (see, e.g. [13, 15] and references therein).

However, in this note we shall concentrate on another class of functions, for which \( \mathcal{F} \) is the class of polynomials of degree at most \( s \). We denote it in short as \((s, p)\)-valent functions. For an \((s, p)\)-valent function \( f \) the equation \( f = P \) has at most \( p \) solutions in \( \Omega \) for any polynomial \( P \) of degree \( s \). We shall always assume that \( p \geq s + 1 \), as subtracting from \( f \) its Taylor polynomial of degree \( s \) we get zero of order at least \( s + 1 \). Note that this is indeed a generalization of \( p \)-valent functions, simply take \( s = 0 \), and every \((0, p)\)-valent function is \( p \)-valent.

As we shall see this class of \((s, p)\)-valent functions is indeed rich and appears naturally in many examples: algebraic functions, solutions of algebraic differential equations, monogenic functions, etc. In fact, it is fairly wide (see Section 2). It possesses many important properties: Distortion theorem, Bernstein-Markov-Remez type inequalities, etc. Moreover, this notion is applicable to any analytic function, under an appropriate choice of the domain \( \Omega \) and the parameters \( s \) and \( p \). In addition, it may provide a useful information in very general situations.

The following example shows that an \((s, p)\)-valent function may be not \((s + 1, p)\)-valent:

**Example 1.1.** Let \( f(x) = x^p + x^N \) for \( N \geq 10^p + 1 \). Then, for \( s = 0, \ldots, p - 1 \), the function \( f \) is \((s, p)\)-valent in the disk \( D_{1/3} \), but only \((p, N)\)-valent there.

Indeed, taking \( P(x) = x^p + c \) we see that the equation \( f(x) = P(x) \) takes the form \( x^N = c \). So for \( c \) small enough, it has exactly \( N \) solutions in the \( D_{1/3} \). Now, for \( s = 0, \ldots, p - 1 \), take a polynomial \( P(x) \) of degree \( s \leq p - 1 \). Then, the equation \( f(x) = P(x) \) takes the form \( x^p - P(x) + x^N = 0 \). Applying Lemma 3.3 of [17] to the polynomial \( Q(x) = x^p - P(x) \) of degree \( p \) (with leading coefficient 1) we find a circle \( S_\rho = \{ |x| = \rho \} \) with \( 1/3 \leq \rho \leq 1/2 \) such that \( |Q(x)| \geq (1/2)^{10p} \) on \( S_\rho \). On the other hand \( x^N \leq (1/2)^{10p+1} < (1/2)^{10p} \) on \( S_\rho \). Therefore, by the Rouché principle the number of zeroes of \( Q(x) + x^N \) in the disk \( D_\rho \)
is the same as for $Q(x)$, which is at most $p$. Thus, $f$ is $(s, p)$-valent in the disk $D_{1/3}$, for $s = 0, \ldots, p - 1$.

This paper is organized as follows: in Section 2 we characterize $(s, p)$-valent functions in terms of their Taylor domination and linear recurrences for their coefficients. In Section 3 we prove a Distortion theorem for $(s, p)$-valent functions. In Section 4 we make a detour and investigate Remez-type inequalities for complex polynomials, which is interesting in its own right. Finally, in Section 5 we extend the Remez-type inequality to $(s, p)$-valent functions, via the Distortion theorem.

2. Taylor domination, bounded recurrences

In this section we provide a rather accurate characterization of $(s, p)$-valent functions in a disk $D_R$ in terms of their Taylor coefficients. “Taylor domination” for an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an explicit bound of all its Taylor coefficients $a_k$ through the first few of them. This property was classically studied, in particular, in relation with the Bieberbach conjecture: for univalent $f$ we always have $|a_k| \leq k|a_1|$ (see [2, 3, 12] and references therein). To give an accurate definition, let us assume that the radius of convergence of the Taylor series for $f$ is $\hat{R}$, for $0 < \hat{R} \leq +\infty$.

**Definition 2.1** (Taylor domination). Let $0 < R < \hat{R}$, $N \in \mathbb{N}$, and $S(k)$ be a positive sequence of a subexponential growth. The function $f$ is said to possess an $(N, R, S(k))$-Taylor domination property if

$$|a_k| R^k \leq S(k) \max_{i=0, \ldots, N} |a_i| R^i, \quad k \geq N + 1.$$  

The following theorem shows that $f$ is an $(s, p)$-valent function in $D_R$, essentially, if and only if its lower $s$-truncated Taylor series possesses a $(p - s, R, S(k))$-Taylor domination.

**Theorem 2.1.** Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an $(s, p)$-valent function in $D_R$, and let $\hat{f}(z) = \sum_{k=1}^{\infty} a_{s+k} z^k$ be the lower $s$-truncation of $f$. Put $m = p - s$. Then, $\hat{f}$ possesses an $(m, R, S(k))$-Taylor domination, with $S(k) = \left(\frac{A_m}{m}\right)^{2m}$, and $A_m$ being a constant depending only on $m$. 

Conversely, if \( \hat{f} \) possesses an \((m, R, S(k))\)-Taylor domination, for a certain sequence \( S(k) \) of a subexponential growth, then for \( R' < R \) the function \( f \) is \((s, p)\)-valent in \( D_{R'} \), where \( p = p(s + m, S(k), R'/R) \) depends only on \( m + s \), the sequence \( S(k) \), and the ratio \( R'/R \). Moreover, \( p \) tends to \( \infty \) for \( R'/R \to 1 \), and it is equal to \( m + s \) for \( R'/R \) sufficiently small.

**Proof.** First observe that if \( f \) is \((s, p)\)-valent in \( D_{R} \), then \( \hat{f} \) is \( m \)-valent there, with \( m = p - s \). Indeed, put \( P(z) = \sum_{k=0}^{s} a_k z^k + cz^s \), with any \( c \in \mathbb{C} \). Then, \( f(z) - P(z) = z^s(\hat{f}(z) - c) \) may have at most \( p \) zeroes. Consequently, \( \hat{f}(z) - c \) may have at most \( m \) zeroes in \( D_{R} \), and thus \( \hat{f} \) is \( m \)-valent there. Now we apply the following classic theorem:

**Theorem 2.2** (Biernacki, 1936, [3]). If \( f \) is \( m \)-valent in the disk \( D_{R} \) of radius \( R \) centered at \( 0 \in \mathbb{C} \) then

\[
|a_k|R^k \leq \left( \frac{A_m k}{m} \right)^{2m} \max_{i=1,\ldots,m} |a_i|R^i, \quad k \geq m + 1,
\]

where \( A_m \) is a constant depending only on \( m \).

In our situation, Theorem 2.2 claims that the function \( \hat{f} \) which is \( m \)-valent in \( D_{R} \), possesses an \((m, R, (A_m k)^{2m})\)-Taylor domination property. This completes the proof in one direction.

In the opposite direction, for polynomial \( P(z) \) of degree \( s \) the function \( f - P \) has the same Taylor coefficients as \( \hat{f} \), starting with the index \( k = s + 1 \). Consequently, if \( \hat{f} \) possesses an \((m, R, S(k))\)-Taylor domination, then \( f - P \) possesses an \((s + m, R, S(k))\)-Taylor domination. Now a straightforward application of Theorem 2.3 of [1] provides the required bound on the number of zeroes of \( f - P \) in the disk \( D_{R} \).

A typical situation for natural classes of \((s, p)\)-valent functions is that they are \((s, p)\)-valent for any \( s \) with a certain \( p = p(s) \) which depends on \( s \). However, it is important to notice that essentially any analytic function possesses this property, with some \( p(s) \).

**Proposition 2.1.** Let \( f(z) \) be an analytic function in an open neighbourhood \( U \) of the closed disk \( D_{R} \). Assume that \( f \) is not a polynomial.
Then, the function $f$ is $(s, p(s))$-valent for any $s$ with a certain sequence $p(s)$.

**Proof.** Let $f$ be given by its Taylor series $f(z) = \sum_{k=0}^{\infty} a_k z^k$. By assumptions, the radius of convergence $\hat{R}$ of this series satisfies $\hat{R} > R$. Since $f$ is not a polynomial, for any given $s$ there is the index $k(s) > s$ such that $a_{k(s)} \neq 0$. We apply now Proposition 1.1 of [1] to the lower truncated series $\hat{f}(z) = \sum_{k=1}^{\infty} a_{s+k} z^k$. Thus, we obtain, an $(m, \hat{R}, S(k))$-Taylor domination for $\hat{f}$, for certain $m$ and $S(k)$. Now, the second part of Theorem 2.1 provides the required $(s, p(s))$-valency for $f$ in the smaller disk $D_R$, with $p(s) = p(s + m, S(k), R/\hat{R})$. □

More accurate estimates of $p(s)$ can be provided via the lacunary structure of the Taylor coefficients of $f$. Consequently, $(s, p)$-valency becomes really interesting only for those classes of analytic functions $f$ where we can specify the parameters in an explicit and uniform way. The following theorem provides still very general, but important such class.

**Theorem 2.3.** Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be $(s, s+m)$-valent in $D_R$ for any $s$. Then, the Taylor coefficients $a_k$ of $f$ satisfy a linear homogeneous non-stationary recurrence relation

$$a_k = \sum_{j=1}^{m} c_j(k) a_{k-j} \tag{2.1}$$

with uniformly bounded (in $k$) coefficients $c_j(k)$ satisfying $|c_j(k)| \leq C \rho^j$, with $C = e^2 A_m^2, \rho = R^{-1}$, where $A_m$ is the constant in the Biernecki’s Theorem 2.2.

Conversely, if the Taylor coefficients $a_k$ of $f$ satisfy recurrence relation (2.1), with the coefficients $c_j(k)$, bounded for certain $K, \rho > 0$ and for any $k$ as $|c_j(k)| \leq K \rho^j$, $j = 1, \ldots, m$, then for any $s$, $f$ is $(s, s+m)$-valent in a disk $D_R$, with $R = \frac{1}{2^{m+1}(2K+2)\rho}$.

**Proof.** Let us fix $s \geq 0$. As in the proof of Theorem 2.1 we notice that if $f$ is $(s, s+m)$-valent in $D_R$, then its lower $s$-truncated series $\hat{f}$ is
$m$-valent there. By Biernacki’s Theorem 2.2 we conclude that
\[ |a_{s+m+1}| R^{m+1} \leq \left( \frac{A_m(m+1)}{m} \right)^{2m} \max_{i=1,\ldots,m} |a_{s+i}| R^i \leq C \max_{i=1,\ldots,m} |a_{s+i}| R^i, \]
with $C = e^2 A_m^2$. Putting $k = s + m + 1$, and $\rho = R^{-1}$ we can rewrite this as
\[ |a_k| \leq C \max_{j=1,\ldots,m} |a_{k-j}| \rho^j. \]
Hence we can chose the coefficients $c_j(k)$, $k = s + m + 1$, in such a way that $a_k = \sum_{j=1}^m c_j(k) a_{k-j}$, and $|c_j(k)| \leq C \rho^j$. Notice that the bound on the recursion coefficients is sharp, and take $f(z) = [1 - (\frac{z}{R})^m]^{-1}$ (in this case, as well as for other lacunary series with the gap $m$, the coefficients $c_j(k)$ are defined uniquely). This completes one direction of the proof.

In the opposite direction, the result follows directly from Theorem 4.1 of [1], and Lemma 2.2.3 of [14], with $R = \frac{1}{2^{m+1}(2K+2)}$. \qed

3. Distortion theorem

In this section we prove a distortion-type theorem for $(s, p)$-valent functions which shows that the behavior of these functions is controlled by the behavior of a polynomial with the same zeroes.

First, let us recall the following theorem for $p$-valent functions, which is our main tool in proof.

**Theorem 3.1.** [12, Theorem 5.1] Let $g(z) = a_0 + a_1 z + \ldots$ be a regular non-vanishing $p$-valent function in $D_1$. Then, for any $z \in D_1$
\[ \left( \frac{1 - |z|}{1 + |z|} \right)^{2p} \leq |g(z)/a_0| \leq \left( \frac{1 + |z|}{1 - |z|} \right)^{2p}. \]

Now, we are at the point to formulate a distortion-type theorem for $(s, p)$-valent functions.

**Theorem 3.2** (Distortion theorem). Let $f$ be an $(s, p)$-valent function in $D_1$ having there exactly $s$ zeroes $x_1, \ldots, x_s$ (always assumed to be
counted according to multiplicity). Define a polynomial

\[ P(x) = A \prod_{j=1}^{s} (x - x_j), \]

where the coefficient \( A \) is chosen such that the constant term in the Taylor series for \( f(x)/P(x) \) is equal to 1. Then, for any \( x \in D_1 \)

\[ \left( \frac{1 - |x|}{1 + |x|} \right)^{2p} \leq |f(x)/P(x)| \leq \left( \frac{1 + |x|}{1 - |x|} \right)^{2p}. \]

**Proof.** The function \( g(x) = f(x)/P(x) \) is regular in \( D_1 \) and does not vanish there. Moreover, \( g \) is \( p \)-valent in \( D_1 \). Indeed, the equation \( g(x) = c \) is equivalent to \( f(x) = cP(x) \) so it has at most \( p \) solutions by the definition of \((s,p)\)-valent functions. Now, apply Theorem 3.1 to the function \( g \). \( \square \)

It is not clear whether the requirement for \( f \) to be \((s,p)\)-valent is really necessary in this theorem. The ratio \( g(x) = \frac{f(x)}{P(x)} \) certainly may be not \( p \)-valent for \( f \) being just \( p \)-valent, but not \((s,p)\)-valent. Indeed, take \( f(x) = x^p + x^N \) as in Example 1.1. By this example \( f \) is \( p \)-valent in \( D_{1/3} \) and it has a root of multiplicity \( p \) at zero. So \( g(x) = f(x)/x^p = 1 + x^{N-p} \) and the equation \( g(x) = c \) has \( N-p \) solutions in \( D_{1/3} \) for \( c \) sufficiently close to 1. So \( g \) is not \( p \)-valent there.

### 4. Complex polynomials

The distortion theorem 3.2 proved in the previous section, allows us easily to extend deep properties from polynomials to \((s,p)\)-valent functions, just by comparing them with polynomials having the same zeros. In this section we make a detour and investigate one specific problem for complex polynomials, which is interesting in its own right: a Remez-type inequality for complex polynomial (compare [16][18]).

Denote by

\[ V_\rho(g) = \{ z : |g(z)| \leq \rho \} \]

the \( \rho \) sub-level set of a function \( g \). For polynomials in one complex variable a result similar to the Remez inequality is provided by the
classic Cartan (or Cartan-Boutroux) lemma (see, for example, [11] and references therein):

**Lemma 4.1** (Cartan’s lemma [7], in form of [11]). Let \( \alpha, \varepsilon > 0 \), and let \( P(z) \) be a monic polynomial of degree \( d \). Then

\[
V_{\varepsilon^d}(P) \subset \bigcup_{j=1}^{p} D_{r_j},
\]

where \( p \leq d \), and \( D_{r_1}, \ldots, D_{r_p} \) are balls with radii \( r_j > 0 \) satisfying

\[
\sum_{j=1}^{p} r_j^\alpha \leq e(2\varepsilon)^\alpha.
\]

In [5, 6, 19, 20] some generalizations of the Cartan-Boutroux lemma to plurisubharmonic functions have been obtained, which lead, in particular, to the bounds on the size of sub-level sets. In these lines in [3] some bounds for the covering number of sublevel sets of complex analytic functions have been obtained, similar to the results of [18] in the real case. Now, we shall derive from the Cartan lemma both the definition of the invariant \( c_{d,\alpha} \) and the corresponding Remez inequality.

**Definition 4.1.** Let \( Z \subset D_1 \). The \((d, \alpha)\)-Cartan measure of \( Z \) is defined as

\[
c_{d,\alpha}(Z) = \min \left\{ \left( \sum_{j=1}^{p} r_j^\alpha \right)^{1/\alpha} : \text{there is a cover of } Z \text{ by } p \leq d \text{ balls with radii } r_j > 0 \right\}.
\]

Note that the \( \alpha \)-dimensional Hausdorff content of \( Z \) is defined in a similar way

\[
H_\alpha(Z) = \inf \left\{ \sum_{j} r_j^\alpha : \text{there is a cover of } Z \text{ by balls with radii } r_j > 0 \right\}.
\]

Thus, by the above definitions, we have \( H_\alpha^\frac{1}{\alpha}(Z) \leq c_{d,\alpha}(Z) \).

For \( \alpha = 1 \) the \((d, 1)\)-Cartan measure \( c_{d,1}(Z) \) was introduced and used, under the name “\( d \)-th diameter”, in [3, 9]. In particular, Lemma 3.3 of [8] is, essentially, equivalent to the case \( \alpha = 1 \) of our Theorem 4.1.

In Section 4.1 below we provide some initial geometric properties of \( c_{d,\alpha}(Z) \) and show that a proper choice of \( \alpha \) may improve geometric sensitivity of this invariant.
Now we can state and proof our generalized Remez inequality for complex polynomials:

**Theorem 4.1.** Let $P(z)$ be a polynomial of degree $d$. Let $Z \subset D_1$. Then, for any $\alpha > 0$

$$\max_{D_1} |P(z)| \leq \left( \frac{6e^{1/\alpha}}{c_{d,\alpha}(Z)} \right)^d \max_{Z} |P(z)| \leq \left( \frac{6e}{H_\alpha(Z)} \right)^d \max_{Z} |P(z)|.$$

**Proof.** Assume that $|P(z)| \leq 1$ on $Z$. First, we prove that the absolute value $A$ of the leading coefficient of $P$ satisfies

$$A \leq \left( \frac{2e^{1/\alpha}}{c_{d,\alpha}(Z)} \right)^d.$$

Indeed, we have $Z \subset V_1(P)$. By the definition of $c_{d,\alpha}(Z)$ for every covering of $V_1(P)$ by $p$ disks $D_{r_1}, \ldots, D_{r_p}$ of the radii $r_1, \ldots, r_d$ (which is also a covering of $Z$) we have \( \sum_{i=1}^d r_i^\alpha \geq c_{d,\alpha}(Z)^\alpha \). Denoting, as above, the absolute value of the leading coefficient of $P(z)$ by $A$ we have by the Cartan lemma that for a certain covering as above

$$c_{d,\alpha}(Z)^\alpha \leq \sum_{i=1}^d r_i^\alpha \leq e \left( \frac{2}{A^{1/d}} \right)^\alpha.$$

Now, we write $P(z) = A \prod_{j=1}^d (z - z_j)$, and consider separately two cases:

1) All $|z_j| \leq 2$. Thus, $\max_{D_1} |P(z)| \leq 4^d \leq \left( \frac{2e^{1/\alpha}}{c_{d,\alpha}(Z)} \right)^d \leq \left( \frac{6e}{H_\alpha(Z)} \right)^d \leq 1$.

2) For $j = 1, \ldots, d_1 < d$, $|z_j| \leq 2$, while $|z_j| > 2$ for $j = d_1 + 1, \ldots, d$. Denote

$$P_1(z) = A \prod_{j=1}^{d_1} (z - z_j), \quad P_2(z) = \prod_{j=d_1+1}^{d} (z - z_j),$$

and notice that for any two points $v_1, v_2 \in D_1$ we have $|P_2(v_1)/P_2(v_2)| < 3^{d-d_1}$. Consequently we get

$$\frac{\max_{D_1} |P(z)|}{\max_Z |P(z)|} < 3^{d-d_1} \frac{\max_{D_1} |P_4(x)|}{\max_{D_1} |P_4(x)|},$$
All the roots of $P_1$ are bounded in absolute value by 2, so by first part we have

$$
\frac{\max_{D_1} |P_1(z)|}{\max_{Z} |P_1(z)|} \leq \left( \frac{2e^{1/\alpha}}{c_{d,\alpha}(Z)} \right)^d 3^{d_1}.
$$

Application of the inequality $H_\alpha(Z) \leq c_{d,\alpha}(Z)^{\alpha}$ completes the proof. \hfill \square

Let us stress a possibility to chose an optimal $\alpha$ in the bound of Theorem 4.1. Let

$$
K_d(Z) = \inf_{\alpha > 0} \left( \frac{6e^{1/\alpha}}{c_{d,\alpha}(Z)} \right)^d, \quad K_d^H(Z) = \inf_{\alpha > 0} \left( \frac{6e}{H_\alpha(Z)} \right)^{\frac{d}{\alpha}}.
$$

**Corollary 4.1.** Let $P(z)$ be a polynomial of degree $d$. Let $Z \subset D_1$. Then,

$$
\max_{D_1} |P(z)| \leq K_d(Z) \max_{Z} |P(z)| \leq K_d^H(Z) \max_{Z} |P(z)|.
$$

### 4.1. Geometric and analytic properties of the invariant $c_{d,\alpha}$

Clearly, the invariant $c_{d,\alpha}(Z)$ is monotone in $Z$, that is, for $Z_1 \subset Z_2$ we have $c_{d,\alpha}(Z_1) \leq c_{d,\alpha}(Z_2)$. Also, for any $Z$, we have

**Proposition 4.1.** Let $\alpha > 0$. Then, $c_{d,\alpha}(Z) > 0$ if and only if $Z$ contains more than $d$ points. In the latter case, $c_{d,\alpha}(Z)$ is greater than or equal to one half of the minimal distance between the points of $Z$.

**Proof.** Any $d$ points can be covered by $d$ disks with arbitrarily small radii. But, the radius of at least one disk among $d$ disks covering more than $d + 1$ different points is greater than or equal to the one half of a minimal distance between these points. \hfill \square

The lower bound of Proposition 4.1 does not depend on $\alpha$. However, in general, this dependence is quite prominent.

**Example 4.1.** Let $Z = [a, b]$. Then, for $\alpha \geq 1$ we have $c_{d,\alpha}(Z) = (b - a)/2$, while for $\alpha \leq 1$ we have $c_{d,\alpha}(Z) = d^{\frac{1}{\alpha} - 1}(b - a)/2$.

Indeed, in the first case the minimum is achieved for $r_1 = (b - a)/2, r_2 = \cdots = r_d = 0$, while in the second case for $r_1 = r_2 = \cdots = r_d = (b - a)/2d$. 

Proposition 4.2. Let $\alpha > \beta > 0$. Then, for any $Z$

\[ c_{d,\alpha}(Z) \leq c_{d,\beta}(Z) \leq d^{(\frac{1}{\beta} - \frac{1}{\alpha})} c_{d,\alpha}(Z). \] (4.1)

Proof. Let $r = (r_1, \ldots, r_d)$ and $\gamma > 0$. Consider $||r||_{\gamma} = (\sum_{j=1}^{d} r_j^\gamma)^{\frac{1}{\gamma}}$. Then, by the definition, $c_{d,\gamma}(Z)$ is the minimum of $||r||_{\gamma}$ over all $r = (r_1, \ldots, r_d)$ being the radii of $d$ balls covering $Z$. Now we use the standard comparison of the norms $||r||_{\gamma}$, that is, for any $x = (x_1, \ldots, x_d)$ and for $\alpha > \beta > 0$,

\[ ||x||_{\alpha} \leq ||x||_{\beta} \leq d^{(\frac{1}{\beta} - \frac{1}{\alpha})} ||x||_{\alpha}. \]

Take $r = (r_1, \ldots, r_d)$ for which the minimum of $||r||_{\beta}$ is achieved, and we get

\[ c_{d,\alpha}(Z) \leq ||r||_{\alpha} \leq ||r||_{\beta} = c_{d,\beta}(Z). \]

Now taking $r$ for which the minimum of $||r||_{\alpha}$ is achieved, exactly in the same way we get the second inequality. \[ \Box \]

Now, we compare $c_{d,\alpha}(Z)$ with some other metric invariants which may be sometimes easier to compute. In each case we do it for the most convenient value of $\alpha$. Then, using the comparison inequalities of Proposition 4.2 we get corresponding bounds on $c_{d,\alpha}(Z)$ for any $\alpha > 0$. In particular, we can easily produce a simple lower bound for $c_{d,2}(Z)$ through the measure of $Z$:

Proposition 4.3. For any measurable $Z \subset D_1$ we have

\[ c_{d,2}(Z) \geq (\mu_2(Z)/\pi)^{1/2}. \]

Proof. For any covering of $Z$ by $d$ disks $D_1, \ldots, D_d$ of the radii $r_1, \ldots, r_d$ we have $\pi(\sum_{i=0}^{d} r_i^2) \geq \mu_2(Z)$. \[ \Box \]

However, in order to deal with discrete or finite subsets $Z \subset D_1$ we have to compare $c_{d,\alpha}(Z)$ with the covering number $M(\varepsilon, Z)$ (which is, by definition, the minimal number of $\varepsilon$-disks covering $Z$).

Definition 4.2. Let $Z \subset D_1$. Define

\[ \omega_{cd}(Z) = \sup_{\varepsilon} \varepsilon(M(\varepsilon, Z) - d)^{1/2}, \quad \rho_d(Z) = d\varepsilon_0, \]
where $\varepsilon_0$ is the minimal $\varepsilon$ for which there is a covering of $Z$ with $d$ $\varepsilon$-disks. Note that, writing $y = M(\varepsilon, Z) = \Psi(\varepsilon)$, and taking the inverse $\varepsilon = \Psi^{-1}(y)$, we have $\varepsilon_0 = \Psi^{-1}(d)$.

As it was mentioned above, a very similar invariant 

$$\omega_d(Z) = \sup_{\varepsilon} \varepsilon(M(\varepsilon, Z) - d)$$

was introduced and used in [18] in the real case. We compare $\omega_{cd}$ and $\omega_d$ below.

**Proposition 4.4.** Let $Z \subset D_1$. Then, $\omega_{cd}(Z)/2 \leq c_{d,2}(Z) \leq c_{d,1}(Z) \leq \rho_d(Z)$.

**Proof.** To prove the upper bound for $c_{d,1}(Z)$ we notice that it is the infimum of the sum of the radii in all the coverings of $Z$ with $d$ disks, while $\rho_d(Z)$ is such a sum for one specific covering.

To prove the lower bound, let us fix a covering of $Z$ by $d$ disks $D_i$ of the radii $r_i$ with $c_{d,2}(Z) = (\sum_{i=0}^{d} r_i^2)^{1/2}$. Let $\varepsilon > 0$. Now, for any disk $D_j$ with $r_j \geq \varepsilon$ we need at most $4r_j^2/\varepsilon^2$ $\varepsilon$-disks to cover it. For any disk $D_j$ with $r_j \leq \varepsilon$ we need exactly one $\varepsilon$-disk to cover it, and the number of such $D_j$ does not exceed $d$. So, we conclude that $M(\varepsilon, Z)$ is at most $d + (4/\varepsilon^2) \sum_{i=0}^{d} r_i^2$. Thus, we get $c_{d,2}(Z) = (\sum_{i=0}^{d} r_i^2)^{1/2} \geq \varepsilon/2(M(\varepsilon, Z) - d)^{1/2}$. Taking supremum with respect to $\varepsilon > 0$ we get $c_{d,2}(Z) \geq \omega_{cd}(Z)/2$. \qed

Since $M(\varepsilon, Z)$ is always an integer, we have

$$\omega_d(Z) \geq \omega_{cd}(Z).$$

For $Z \subset D_1$ of positive plane measure, $\omega_d(Z) = \infty$ while $\omega_{cd}(Z)$ remains bounded (in particular, by $\rho_d(Z)$).

Some examples of computing (or bounding) $\omega_d(Z)$ for “fractal” sets $Z$ can be found in [18]. Computations for $\omega_{cd}(Z)$ are essentially the same. In particular, in an example given in [18] in connection to [10] we have that for $Z = Z_r = \{1, 1/2^r, 1/3^r, \ldots, 1/k^r, \ldots\}$

$$\omega_d(Z_r) \asymp \frac{r^r}{(r+1)^{r+1}d^r}, \quad \omega_{cd}(Z_r) \asymp \frac{(2r+1)^r}{(2r+2)^{r+1}d^{r+1/2}}.$$
The asymptotic behavior here is for $d \to \infty$, as in [10].

4.2. An example. We conclude this section with one very specific example. Let
\[ Z = Z(d, h) = \{ z_1, z_2, \ldots, z_{2d-1}, z_{2d} \} , \quad x_i \in \mathbb{C}, \ d \geq 2. \]

We assume that $Z$ consists of $d, 2\eta$-separated couples of points, with points in each couple being in a distance $2h$. Let $2D(Z)$ be the diameter of the smallest disk containing $Z$, where $h \ll 1$, and $2\eta \gg h$.

Proposition 4.5. Let $Z$ be as above. Then,
\begin{enumerate}
  \item $\omega_d(Z) = dh$,
  \item $\omega_{cd}(Z) = \sqrt{dh}$.
  \item For $\alpha > 0$, we have $c_{d,\alpha}(Z) \leq d^{\frac{\alpha}{2}} h$.
  \item For $\alpha \gg 1$, we have $c_{d,\alpha}(Z) = d^{\frac{1}{\alpha}} h$.
  \item For $\kappa = \lfloor \log_d(\frac{D(Z)}{h}) \rfloor^{-1}$, we have $c_{d,\kappa}(Z) \geq \eta$.
\end{enumerate}

Proof. For $\varepsilon > h$, we have $M(\varepsilon, Z) \leq d$, and hence $M(\varepsilon, Z) - d$ is negative. For $\varepsilon < h$, we have $M(\varepsilon, Z) = 2d$, and $M(\varepsilon, Z) - d = d$. Thus the supremum of $\varepsilon(M(\varepsilon, Z) - d)$, or the supremum of $\varepsilon(M(\varepsilon, Z) - d)^{\frac{1}{2}}$, is achieved as $\varepsilon < h$ tends to $h$. Therefore, $\omega_d(Z) = dh$, and $\omega_{cd}(Z) = \sqrt{dh}$.

Covering each couple with a separate ball of radius $h$, we get for any $\alpha > 0$ that $c_{d,\alpha}(Z) \leq d^{\frac{\alpha}{2}} h$. For $\alpha \gg 1$ it is easy to see that this uniform covering is minimal. Thus, for such $\alpha$ we have the equality $c_{d,\alpha}(Z) = d^{\frac{1}{\alpha}} h$.

Now let us consider the case of a “small” $\alpha = \kappa$. Take a covering of $Z$ with certain disks $D_j$, $j \leq d$. If there is at least one disk $D_j$ containing three points of $Z$ or more, the radius of this disk is at least $\eta$. Thus, for this covering $(\sum_{j=1}^{d} r_j^\kappa)^{\frac{1}{\kappa}} \geq \eta$. If each disk in the covering contains at most two points, it must contain exactly two, otherwise these disks could not cover all the $2d$ points of $Z$. Hence, the radius of each disk $D_j$ in such covering is at least $h$, an their number is exactly $d$. We have, by the choice of $\kappa$, that $(\sum_{j=1}^{d} r_j^\kappa)^{\frac{1}{\kappa}} \geq d^{\frac{1}{\kappa}} h = D(Z) \geq \eta$. \qed
Proposition 4.5 shows that $c_{d,1}(Z) \leq dh$, while we have $c_{d,\kappa}(Z) \geq \eta$. So using $\alpha = 1$ and $\alpha = \kappa$ in the Remez-type inequality of Theorem 4.1 we get two bounds for the constant $K_d(Z)$:

$$K_d(Z) \leq \left( \frac{6e}{dh} \right)^d \quad \text{or} \quad K_d(Z) \leq \left( \frac{6e^{1/\kappa}}{\eta} \right)^d .$$  \hspace{1cm} (4.2)

But $e^{1/\kappa} = e^{\log_d(D(Z)/h)} = \left( \frac{D(Z)}{h} \right)^{\frac{1}{\ln d}}$. So the second bound of (4.2) takes a form

$$K_d(Z) \leq \left( \frac{6D(Z)}{\eta^{\ln d} h} \right)^{\frac{d}{\ln d}} .$$

We see that for $d \geq 3$ and for $h \to 0$ the asymptotic behavior of this last bound, corresponding to $\alpha = \kappa$, is much better than of the first bound in (4.2), corresponding to $\alpha = 1$. Notice, that $\kappa$ depends on $h$ and $D(Z)$, i.e. on the specific geometry of the set $Z$.

5. Remez inequality

Now, we present a Remez-type inequality for $(s, p)$-valent functions. We recall that by Proposition 2.1 above, any analytic function in an open neighborhood $U$ of the closed disk $D_R$ is $(s, p(s))$-valent in $D_R$ for any $s$ with a certain sequence $p(s)$. Consequently, the following theorem provides a non-trivial information for any analytic function in an open neighborhood of the unit disk $D_1$. Of course, this results becomes really interesting only in cases where we can estimate $p(s)$ explicitly.

**Theorem 5.1.** Let $f$ be an analytic function in an open neighborhood $U$ of the closed disk $D_1$. Assume that $f$ has in $D_1$ exactly $s$ zeroes, and that it is $(s, p)$-valent in $D_1$. Let $Z$ be a subset in the interior of $D_1$, and put $\rho = \rho(Z) = \min\{\eta : Z \subset D_\eta\}$. Then, for any $R < 1$ function $f$ satisfies

$$\max_{D_R} |f(z)| \leq \sigma_p(R, \rho) K_s(Z) \max_Z |f(z)| ,$$

where $\sigma_p(R, \rho) = \left( \frac{1+R}{1-R} \cdot \frac{1+\rho}{1-\rho} \right)^{2p} . $
Proof. Assume that $|f(x)|$ is bounded by 1 on $Z$. Let $x_1, \ldots, x_s$ be zeroes of $f$ in $D_1$. Consider, as in Theorem 3.2, the polynomial

$$P(x) = A \prod_{j=1}^{l} (x - x_j),$$

where the coefficient $A$ is chosen in such a way that the constant term in the Taylor series for $g(x) = f(x)/P(x)$ is equal to 1. Then by Theorem 3.2 for $g$ we have

$$\left(\frac{1 - |x|}{1 + |x|}\right)^{2p} \leq |g(x)| \leq \left(\frac{1 + |x|}{1 - |x|}\right)^{2p}. $$

We conclude that $P(x) \leq (1 + \rho)^{2p}$ on $Z$. Hence by the polynomial Remez inequality provided by Theorem 4.1 we obtain

$$|P(x)| \leq K_s(Z) \left(\frac{1 + \rho}{1 - \rho}\right)^{2p}$$

on $D_1$. Finally, we apply once more the bound of Theorem 3.2 to conclude that

$$|f(x)| \leq K_s(Z) \left(\frac{1 + R}{1 - R}\right)^{2p} \left(\frac{1 + \rho}{1 - \rho}\right)^{2p}$$

on $D_R$. \qed

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