OPTIMAL SPINOR SELECTIVITY FOR QUATERNION ORDERS

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Abstract. Let \( D \) be a quaternion algebra over a number field \( F \), and \( \mathscr{G} \) be an arbitrary genus of \( O_F \)-orders of full rank in \( D \). Let \( K \) be a quadratic field extension of \( F \) that embeds into \( D \), and \( B \) be an \( O_F \)-order in \( K \) that can be optimally embedded into some member of \( \mathscr{G} \). We provide a necessary and sufficient condition for \( B \) to be optimally spinor selective for the genus \( \mathscr{G} \), which generalizes previous existing optimal selectivity criterions for Eichler orders as given by Arenas, Arenas-Carmona and Contreras, and by Voight independently. This allows us to obtain a refinement of the classical trace formula for optimal embeddings, which will be called the spinor trace formula. When \( \mathscr{G} \) is a genus of Eichler orders, we extend Maclachlan’s relative conductor formula for optimal selectivity from Eichler orders of square-free levels to all Eichler orders.

1. Introduction

Let \( F \) be a number field, \( K \) be a quadratic field extension of \( F \), and \( D \) be a quaternion \( F \)-algebra. According to the Hasse-Brauer-Noether-Albert Theorem [28, Theorem III.3.8], \( K \) embeds into \( D \) over \( F \) if and only if there is no place of \( F \) that is simultaneously ramified in \( D \) and split in \( K \). The optimal (spinor) selectivity question studies an integral refinement of this theorem. To state the question and the answers precisely, we set up some notations and definitions.

Throughout this paper, we assume that \( K \) embeds into \( D \) over \( F \). Orders in \( D \) (resp. \( K \)) refer exclusively to \( O_F \)-orders of full rank in \( D \) (resp. \( K \)), where \( O_F \) is the ring of integers of \( F \) as usual. Two orders \( \mathcal{O} \) and \( \mathcal{O}' \) in \( D \) are said to be locally isomorphic if their \( p \)-adic completions \( \mathcal{O}_p \) and \( \mathcal{O}'_p \) are isomorphic at every finite prime \( p \) of \( F \). This defines an equivalence relation on the set of orders in \( D \), and an equivalence class is called a genus of orders in \( D \). For example, all maximal orders of \( D \) form a single genus. The genus represented by an (arbitrary) order \( \mathcal{O} \) will be denoted by \( \mathscr{G}(\mathcal{O}) \). Similarly, two orders \( \mathcal{O} \) and \( \mathcal{O}' \) in \( D \) are said to be of the same type if they are \( O_F \)-isomorphic, or equivalently, if there exists \( x \in D^\times \) such that \( \mathcal{O}' = x\mathcal{O}x^{-1} \). The type of \( \mathcal{O} \) will be denoted by \( [\mathcal{O}] \), i.e. \( [\mathcal{O}] := \{ x\mathcal{O}x^{-1} \mid x \in D^\times \} \). Each genus \( \mathscr{G} \) of orders in \( D \) is subdivided into finitely many types [29, §17.1], and we write \( Tp(\mathscr{G}) \) for the finite set of types in \( \mathscr{G} \). If \( \mathscr{G} = \mathscr{G}(\mathcal{O}) \), then we put \( Tp(\mathcal{O}) := Tp(\mathscr{G}(\mathcal{O})) \) and regard it as a pointed set with \([\mathcal{O}]\) as the base point.

Given an order \( B \) in \( K \) and an order \( \mathcal{O} \) in \( D \), we write \( \text{Emb}(B, \mathcal{O}) \) for the set of optimal embeddings of \( B \) into \( \mathcal{O} \), that is

\[
\text{Emb}(B, \mathcal{O}) := \{ \varphi \in \text{Hom}_F(K, D) \mid \varphi(K) \cap \mathcal{O} = \varphi(B) \}.
\]
The unit group $O^\times$ acts on $\text{Emb}(B, O)$ from the right by conjugation: $\varphi \mapsto u^{-1}\varphi u$ for every $u \in O^\times$. It is well known \cite[§30.3–30.5]{voight} that the number of orbits is finite both in the global and local cases, so we put

$$m(B, O, O^\times) := |\text{Emb}(B, O)/O^\times|, \quad m(B_p, O_p, O_p^\times) := |\text{Emb}(B_p, O_p)/O_p^\times|.$$ 

Clearly, whether $\text{Emb}(B, O) = \emptyset$ or not depends only on the type $[O]$. Moreover, if $\text{Emb}(B, O) \neq \emptyset$, then

$$(*) \quad \text{Emb}(B_p, O_p) \neq \emptyset \quad \text{for every finite prime } p \text{ of } F.$$ 

We aim to study the converse of this implication. For the rest of this paper, we assume that $\mathcal{G}$ is a genus of orders in $D$ such that condition \footnote{The “not all” part is automatic since we assume condition \footnote{1} throughout.} holds for $B$ and for every $O \in \mathcal{G}$. This guarantees the existence of $[O_0] \in \text{Tp}(\mathcal{G})$ such that $\text{Emb}(B, O_0) \neq \emptyset$ by \cite[Corollary 30.4.18]{voight}. The optimal selectivity question further asks whether assumption \footnote{1} implies that $\text{Emb}(B, O) \neq \emptyset$ for every type $[O] \in \text{Tp}(\mathcal{G})$.

The optimal selectivity question is best studied when $D$ satisfies the Eichler condition (i.e. there exists an archimedean place of $F$ that does not ramify in $D$), for otherwise one can easily construct examples that answer negatively to this question (See Example \footnote{2.1}). However, the answer can still be negative even if one assumes the Eichler condition, as discovered by Chinburg and Friedman \cite{chinburg-friedman}. Indeed, it was them who first introduced the notion of “selectivity”.

**Definition 1.1** (\cite[Definition 31.1.5]{voight}). Suppose that $D$ satisfies the Eichler condition. If $\text{Emb}(B, O') = \emptyset$ for some but not all orders $O' \in \mathcal{G}$, then we say that $B$ is optimally selective for $\mathcal{G}$.

Naturally, one further asks exactly when optimally selectivity happens, and how to characterize those types of orders $[O] \in \text{Tp}(\mathcal{G})$ that do admit optimal embeddings from $B$. A lot research has been carried out on these topics following Chinburg and Friedman’s pioneering work. Maclachlan \cite{maclachlan} first obtained an optimal selectivity theorem for Eichler orders of square-free levels. Independently, Arenas et al. \cite{arenas} and Voight \cite[Chapter 31]{voight} removed the square-free condition and obtained theorems for Eichler orders of arbitrary levels. See \cite[§31.7.7]{voight} for a brief historical note on the contributions of Chan-Xu \cite{chan-xu}, Guo-Qin \cite{guo-qin}, Linowitz \cite{linowitz}, Arenas-Carmona \cite{arenas-carmona1,arenas-carmona2,arenas-carmona3,arenas-carmona4}, and many others.

Central to the theory of selectivity is a class field $\Sigma_{\mathcal{G}}$ attached to the genus $\mathcal{G}$ (See Definition \footnote{2.3}). It is an abelian extension of $F$ of exponent 2 satisfying $[\Sigma_{\mathcal{G}} : F] = [\text{Tp}(\mathcal{G})]$ (still assuming that $D$ satisfies the Eichler condition, cf. \footnote{2.3} and Remark \footnote{2.3}). For the reader’s convenience, we reproduce Voight’s formulation of optimal selectivity theorem for Eichler orders.

**Theorem 1.2** (\cite[Theorem 31.1.7]{voight}). Keep the assumption that $D$ satisfies the Eichler condition. Let $\mathcal{G}$ be a genus of Eichler orders. Then the following statements hold.

(i) $B$ is optimally selective for the genus $\mathcal{G}$ if and only if $K \subseteq \Sigma_{\mathcal{G}}$.

(ii) If $B$ is optimally selective for the genus $\mathcal{G}$, then $\text{Emb}(B, O) \neq \emptyset$ for exactly half of the types $[O] \in \text{Tp}(\mathcal{G})$.

(iii) In all cases, $m(B, O, O^\times) = m(B, O', O'^\times)$ for $O, O' \in \mathcal{G}$ whenever both sides are nonzero.
In this paper, we obtain a number of generalizations of the above theorem. Our main theorem (Theorem 2.15) gives the necessary and sufficient conditions for optimal selectivity that applies to any arbitrary genus of quaternion orders. The global condition $K \subseteq \Sigma_G$ is still necessary, but the sufficiency condition requires additional local considerations at finitely many places where $\mathcal{O}$ has Eichler invariant zero (See Definition 2.13). On one hand, this criterion of optimal selectivity shows that Theorem 1.2 extends verbatim to orders in $D$ that have nonzero Eichler invariants at all finite places of $F$ (which include the Eichler orders as a proper subclass). On the other hand, it paves the way for the discovery in [23] of pairs $(B, \mathcal{G})$ for which the condition $K \subseteq \Sigma_\mathcal{G}$ is insufficient for optimal selectivity.

When $B$ is optimally selective for the genus $G$, there is a method determining all the types $[\mathcal{O}] \in \text{Tp}(\mathcal{G})$ with $\text{Emb}(B, \mathcal{O}) \neq \emptyset$ using the Artin symbol, provided one of such type is known prior. See [1 Lemma 2.2], [29 §31.1.9], or Theorem 2.15 of the present paper. However, in practice it might not be easy to construct explicitly an order $\mathcal{O}_0 \in \mathcal{G}$ with $\text{Emb}(B, \mathcal{O}_0) \neq \emptyset$. In the study of optimal selectivity for Eichler orders of square-free levels, Maclachlan [22] had the novel idea of relating the optimal selectivity of two distinct orders $B, B' \subset K$ via their relative conductor (See (3.1)). For example, to know whether $\text{Emb}(B, \mathcal{O}) = \emptyset$ or not for an Eichler order $\mathcal{O}$ of square-free level, we can choose a suitable $F$-embedding $\varphi : K \rightarrow D$ and put $B' := \varphi^{-1}(\mathcal{O})$. Then the relative conductor $f(B'/B)$ encodes the desired information for $\text{Emb}(B, \mathcal{O})$. In Theorem 3.2, we shall remove the square-free condition and extend Maclachlan’s result to all Eichler orders.

Built on the previous work of Vignéras [28 §III.5], Voight gives an explicit formula for $m(B, \mathcal{O}, \mathcal{O}^\times)$ in [29 Corollary 31.1.10] (cf. (4.2)) by combing part (iii) of Theorem 1.2 together with the trace formula [28 Theorem III.5.11] [29 Theorem 30.4.7]. Here the Eichler condition is essential as remarked by him in [29 Remark 31.6.2]. On the other hand, when studying certain class number formulas attached to orders in totally definite quaternion algebras (i.e. quaternion algebras that do not satisfy the Eichler condition), the present authors are confronted with the task of generalizing the Vignéras-Voight formula to the totally definite case. As a prerequisite, we need a version of the optimal selectivity theorem that applies to the totally definite case. This in itself is not new. In [1 Remark, p. 99], M. Arenas et al. remark that the first two parts of Theorem 1.2 still hold as soon as the words “conjugate class” and “conjugate” are replaced by “spinor genera” and by “spinor equivalence” respectively. We shall take this approach and formulate all our results in terms of optimal spinor selectivity (see Definition 2.4). Such a formulation allows us to get a similar result as part (iii) of Theorem 1.2 except that both sides of the equality are of the form $\sum_i m(B, \mathcal{O}_i, \mathcal{O}_i^\times)$, where the $\mathcal{O}_i$’s range over the left orders of right $\mathcal{O}$-ideal classes within one spinor class (See (4.14)). This leads to a refinement of the classical trace formula, which will be called the spinor trace formula in Proposition 4.3. If $D$ satisfies the Eichler condition, then the summation consists of only one term, and the spinor trace formula reduces back to the Vignéras-Voight formula.

This paper is organized as follows. In Section 2 we prove the optimal spinor selectivity theorem for an arbitrary genus of quaternion orders, thus generalizing the first two parts of Theorem 1.2. Section 3 focuses on extending Maclachlan’s relative conductor formula to all Eichler orders. In Section 4 we prove the spinor trace formula, which generalizes the last part of Theorem 1.2.
Notation. Throughout this paper, \( p \) denotes a finite prime of \( F \). If \( M \) is a finite dimensional \( F \)-vector space or a finite \( \hat{O}_F \)-module, then we write \( M_p \) for the \( p \)-adic completion of \( M \). Let \( \hat{\mathbb{Z}} = \lim \mathbb{Z} / n\mathbb{Z} = \prod_p \mathbb{Z}_p \) be the profinite completion of \( \mathbb{Z} \). If \( X \) is a finitely generated \( \mathbb{Z}\)-module or a finite dimensional \( \mathbb{Q}\)-vector space, we set \( \hat{X} := X \otimes \mathbb{Z} \). For example, \( \hat{F} \) is the ring of finite adeles of \( F \), and \( \hat{O}_F = \prod_p \hat{O}_{F_p} \). Here \( \hat{O}_{F_p} \) denotes the ring of integers of \( F_p \), and the product runs over all finite primes of \( F \). The reduced norm map of \( D \) is denoted as \( \text{Nr} : D \to \hat{F} \). Given a set \( Y \subseteq \hat{D} \), we write \( Y^1 \) for the subset of elements of reduced norm 1, that is, \( Y^1 := \{ y \in Y \mid \text{Nr}(y) = 1 \} \). In particular, \( \hat{D}^1 = \text{ker}(\hat{D}^\times \to \hat{F}^\times) \).

2. The optimal spinor selectivity theorem

In this section, we derive the aforementioned theorem (Theorem 2.15) of optimal spinor selectivity that applies to any arbitrary genus of quaternion orders. Our new contribution to this already highly developed theory mostly includes verifying certain local condition for optimal embeddings in Lemma 2.14 at the places of \( \mathbb{F} \) where the Eichler invariant of the quaternion order is \(-1\). This provides simplifications to the existing theory and makes generalizations possible.

Different from Theorem 1.2, the quaternion algebra \( D \) in this section is allowed to be arbitrary (i.e. not necessarily satisfying the Eichler condition). By definition, \( D \) is totally definite if and only if \( F \) is a totally real field and \( D \otimes_{\mathbb{F},\sigma} \mathbb{R} \) is isomorphic to the Hamilton quaternion algebra \( \mathbb{H} \) for every embedding \( \sigma : F \to \mathbb{R} \). In the totally definite case, Theorem 1.2 does not always hold without modifications, as shown by the following simple example.

Example 2.1. Let \( p \in \mathbb{N} \) be a prime, and \( D_{p,\infty} \) be the unique quaternion algebra over \( \mathbb{Q} \) ramified precisely at \( p \) and \( \infty \). Let \( \mathcal{O}_{p,\infty} \) be the genus of maximal \( \mathbb{Z}\)-orders in \( D_{p,\infty} \). From [25 Proposition V.3.2], if \( p \equiv 3 \pmod{4} \), then there is a unique member \( [\mathcal{O}_0] \in \text{Tp}(\mathcal{O}_{p,\infty}) \) such that \( \text{Emb}(\mathbb{Z}[\sqrt{-1}], \mathcal{O}_0) \neq \emptyset \), while \( |\text{Tp}(\mathcal{O}_{p,\infty})| \) tends to infinity as \( p \) goes to infinity. This already violates part (ii) of Theorem 1.2.

As remarked by M. Arenas et al. [1, Remark, p. 99], one needs the notion of spinor genus to extend the optimal selectivity theorem to the totally definite case. This notion was previously studied by Brzezinski [7 §1].

Definition 2.2. (1) Let \( \mathcal{G} \) be an arbitrary genus of orders in \( D \). Two orders \( \mathcal{O}, \mathcal{O}' \in \mathcal{G} \) are said to be in the same spinor genus (and denoted by \( \mathcal{O} \sim \mathcal{O}' \)) if there exists \( x \in D^\times \hat{D}^1 \) such that \( \mathcal{O}' = x\mathcal{O}x^{-1} \).

(2) The spinor genus of \( \mathcal{O} \) is the set \( [\mathcal{O}]_{\text{sg}} \) consisting of all orders \( \mathcal{O}' \) with \( \mathcal{O}' \sim \mathcal{O} \). The set of spinor genera within \( \mathcal{G} \) is denoted by \( \text{SG}(\mathcal{G}) \), that is, \( \text{SG}(\mathcal{G}) := \{ [\mathcal{O}]_{\text{sg}} \mid \mathcal{O} \in \mathcal{G} \} \). Often we write \( \text{SG}(\mathcal{O}) \) for \( \text{SG}(\mathcal{G}) \) and regard it as a pointed set with base point \( [\mathcal{O}]_{\text{sg}} \).

(3) Given orders \( B \subseteq K \) and \( \mathcal{O} \in \mathcal{G} \), we define the optimal spinor selectivity symbol as follows

\[
\Delta(B, \mathcal{O}) = \begin{cases} 
1 & \text{if } \exists \mathcal{O}' \in [\mathcal{O}]_{\text{sg}} \text{ such that } \text{Emb}(B, \mathcal{O}') \neq \emptyset, \\
0 & \text{otherwise}.
\end{cases}
\]

Remark 2.3. If \( D \) satisfies the Eichler condition, Brzezinski [7 Proposition 1.1] shows that each spinor genus consists of exactly one type. Thus in this case, \( \Delta(B, \mathcal{O}) = 1 \) if and only if \( \text{Emb}(B, \mathcal{O}) \neq \emptyset \).
Definition 2.4. We say $B$ is optimally spinor selective (selective for short) for the genus $\mathcal{G}$ if $\Delta(B, \mathcal{O}') = 0$ for some but not all $[\mathcal{O}']_{\text{sg}} \in \text{SG}(\mathcal{G})$. If $B$ is selective for $\mathcal{G}$, then a spinor genus $[\mathcal{O}]_{\text{sg}}$ with $\Delta(B, \mathcal{O}) = 1$ is said to be selected by $B$.

To give a preliminary characterization of the spinor genera selected by $B$, we take the “two class field” approach as mapped out by Arenas-Carmona [2 §3] and combine it with inputs from Voight’s work [29 §31].

First, let us introduce the class field $\Sigma_{\mathcal{G}}/F$ attached to the genus $\mathcal{G}$ as mentioned in the introduction. Following [28, §III.4], we write $F_{\mathcal{D}}^\times$ for the subgroup of $F^\times$ consisting of the elements that are positive at each infinite place of $F$. Let $\text{Nr} : D^\times \to F^\times$ be the reduced norm map. The Hasse-Schilling-Maass theorem [25 Theorem 33.15] [28 Theorem III.4.1] implies that $\text{Nr}(D^\times) = F_{\mathcal{D}}^\times$. Moreover, we have $\text{Nr}(\hat{D}^\times) = \hat{F}^\times$, the finite idele group of $F$. Let $\mathcal{N}(\hat{O})$ be the normalizer of $\hat{O}$ in $\hat{D}^\times$. There is an adelic description of $\text{SG}(\mathcal{O})$ as follows [7 Propositions 1.2 and 1.8]

$$\text{SG}(\mathcal{O}) \simeq (D^\times \hat{D}^\dagger) \setminus \hat{D}^\times / \mathcal{N}(\hat{O}) \xrightarrow{\text{Nr}} F_{\mathcal{D}}^\times \setminus \hat{F}^\times / \text{Nr}(\mathcal{N}(\hat{O})), \tag{2.2}$$

where the two double coset spaces are canonically bijective via the reduced norm map. Clearly, the group $\text{Nr}(\mathcal{N}(\hat{O}))$ depends only on the genus $\mathcal{G}$ and not on the choice of $\mathcal{O}$.

Definition 2.5 ([2 §2], [21 §3]). The spinor genus field\footnote{This field is often called the spinor class field in the literature [12 §9]. However, we are going to introduce a concept called spinor class following [4 §1]. To avoid confusion, the field $\Sigma_{\mathcal{G}}$ will be called the spinor genus field since it is uniquely determined by the genus $\mathcal{G}$. Moreover, if $F$ is a quadratic field, and $\mathcal{O}$ is an Eichler order, then $\Sigma_{\mathcal{G}}$ is a subfield of the classical (strict) genus field in [13 Definition 15.29] or [14 §6], so the terminology is consistent in that sense.} of $\mathcal{G}$ is the abelian field extension $\Sigma_{\mathcal{G}}/F$ corresponding to the open subgroup $F_{\mathcal{D}}^\times \text{Nr}(\mathcal{N}(\hat{O})) \subseteq \hat{F}^\times$ via the class field theory [19 Theorem X.5].

Since $\text{Nr}(\mathcal{N}(\hat{O}))$ is an open subgroup of $\hat{F}^\times$ containing $(\hat{F}^\times)^2$, the Galois group $\text{Gal}(\Sigma_{\mathcal{G}}/F)$ is a finite elementary 2-group [21 Proposition 3.5]. We have a canonical identification of the pointed sets

$$\text{SG}(\mathcal{O}) \simeq F_{\mathcal{D}}^\times \setminus \hat{F}^\times / \text{Nr}(\mathcal{N}(\hat{O})) \simeq \text{Gal}(\Sigma_{\mathcal{G}}/F), \tag{2.3}$$

where the base point $[\mathcal{O}]_{\text{sg}}$ is identified with the identity element of $\text{Gal}(\Sigma_{\mathcal{G}}/F)$. This equips $\text{SG}(\mathcal{O})$ with an abelian group structure. Given another order $\mathcal{O}' \in \mathcal{G}$, we define $\rho(\mathcal{O}, \mathcal{O}')$ to be the element of $\text{Gal}(\Sigma_{\mathcal{G}}/F)$ identified with $[\mathcal{O}']_{\text{sg}} \in \text{SG}(\mathcal{O})$ via (2.3). More canonically, we regard the base point free set $\text{SG}(\mathcal{G})$ as a principal homogeneous space over $\text{Gal}(\Sigma_{\mathcal{G}}/F)$ via (2.3). Then $\rho(\mathcal{O}, \mathcal{O}')$ is the unique element of $\text{Gal}(\Sigma_{\mathcal{G}}/F)$ that sends $[\mathcal{O}]_{\text{sg}}$ to $[\mathcal{O}']_{\text{sg}}$. Since $\text{Gal}(\Sigma_{\mathcal{G}}/F)$ is an elementary 2-group, $\rho$ defines a map $\mathcal{G} \times \mathcal{G} \to \text{Gal}(\Sigma_{\mathcal{G}}/F)$ that is symmetric in its two variables. By definition, $\rho(\mathcal{O}, \mathcal{O}')$ depends only on the spinor genera of $\mathcal{O}$ and $\mathcal{O}'$. Moreover, $\rho(\mathcal{O}, \mathcal{O}') = 1$ if and only if $\mathcal{O} \sim \mathcal{O}'$. If we write the group law of $\text{Gal}(\Sigma_{\mathcal{G}}/F)$ additively, then

$$\rho(\mathcal{O}, \mathcal{O}'') = \rho(\mathcal{O}, \mathcal{O}') + \rho(\mathcal{O}', \mathcal{O}''), \quad \forall \mathcal{O}, \mathcal{O}', \mathcal{O}'' \in \mathcal{G}. \tag{2.4}$$

Example 2.6. Suppose for the moment that $\text{Nr}(\mathcal{O}_{\mathcal{F}}^\times) = O_{\mathcal{F}}^\times$ for every prime $p \subseteq \mathcal{O}_{\mathcal{F}}$ so that $\Sigma_{\mathcal{G}}/F$ is unramified at all the finite places. For instance, this assumption holds for all Eichler orders in $D$. Let $\mathcal{O}$ and $\mathcal{O}'$ be two members of $\mathcal{G}$. There exists...
an $O_F$-lattice $I$ in $D$ linking $O$ and $O'$ as in [28 §1.4], that is, $O' = O_1(I)$ and $O = O_r(I)$, where
\begin{equation}
O_1(I) := \{ \alpha \in D \mid \alpha I \subseteq I \}, \quad O_r(I) := \{ \alpha \in D \mid I\alpha \subseteq I \}.
\end{equation}
Such a lattice $I$ is locally principal both as a fractional right $O$-ideal and as a fractional left $O'$-ideal. Then $\rho(O, O') \in \text{Gal}(\Sigma_{\mathfrak{g}}/F)$ is given by the Artin symbol $(\text{Nr}(I), \Sigma_{\mathfrak{g}}/F)$. See [1 §2] and [29 §31.1.9].

Next, we introduce another class field $E_{op}/F$ closely related to the optimal embeddings. For each $F$-embedding $\varphi : K \hookrightarrow D$, consider the following sets
\begin{equation}
\mathcal{E}(\varphi, B, O) := \{ \alpha \in D^\times \mid \varphi(K) \cap \alpha O \alpha^{-1} = \varphi(B) \},
\end{equation}
\begin{equation}
\mathcal{E}_p(\varphi, B, O) := \{ g_p \in D_p^\times \mid \varphi(K_p) \cap g_p O_p g_p^{-1} = \varphi(B_p) \}, \quad \forall \text{ prime } p \subset O_F,
\end{equation}
\begin{equation}
\widehat{\mathcal{E}}(\varphi, B, O) := \{ g = (g_p) \in \hat{D}^\times \mid \varphi(K) \cap g \hat{O} g^{-1} = \varphi(B) \}.
\end{equation}
Given any other $F$-embedding $\varphi' : K \hookrightarrow D$ and another order $O' \in \mathcal{G}$, we pick $\alpha \in D^\times$ and $x \in \hat{D}^\times$ such that $\varphi' = \alpha^{-1} \varphi \alpha$ and $\hat{O} = x \hat{O} x^{-1}$. Then
\begin{equation}
\widehat{\mathcal{E}}(\varphi', B, O') = \alpha^{-1} \widehat{\mathcal{E}}(\varphi, B, O) x^{-1}.
\end{equation}

Since condition (4) is assumed to hold throughout this paper, there is always an order $O \in \mathcal{G}$ such that $\text{Emb}(B, O) \neq \emptyset$. Keep such an $O$ and an optimal embedding $\varphi \in \text{Emb}(B, O)$ fixed. For simplicity, we identify $K$ with its image $\varphi(K) \subset D$. Clearly, $\widehat{\mathcal{E}}(\varphi, B, O)$ is left translation invariant by $\hat{K}^\times$ and right translation invariant by $\mathcal{N}(\hat{O})$. Moreover, it contains 1 by the choice of $O$ and $\varphi$, so we have
\begin{equation}
\widehat{\mathcal{E}} := \widehat{\mathcal{E}}(\varphi, B, O) \supseteq \hat{K}^\times \mathcal{N}(\hat{O}).
\end{equation}

**Definition 2.7 ([2 §2], [21 §3]).** The **optimal representation field** attached to the pair $(B, \mathcal{G})$ is the abelian field extension $E_{op}/F$ corresponding to $F_K^\times \text{Nr}(\widehat{\mathcal{E}}) \subseteq \hat{F}^\times$ via the class field theory.

A priori, for this definition to make sense, one needs to know that $F_K^\times \text{Nr}(\widehat{\mathcal{E}})$ is an open subgroup of $\hat{F}^\times$. This has already been verified by Voight [29 §31.3.14], who also shows that $[\hat{F}^\times : F_K^\times \text{Nr}(\widehat{\mathcal{E}})] \leq 2$. His proof makes use of the following chain of inclusions
\begin{equation}
F_K^\times \text{Nr}(\hat{K}^\times) \subseteq F_K^\times \text{Nr}(\hat{K}^\times) \text{Nr}(\mathcal{N}(\hat{O})) \subseteq F_K^\times \text{Nr}(\hat{E}) \subseteq \hat{F}^\times.
\end{equation}
Here $F_K^\times$ is the subgroup of $F^\times$ consisting of the elements that are positive at each infinite place of $F$ that is ramified in $K/F$. The assumption that $K$-embeddable into $D$ implies that $F_K^\times \subseteq F_D^\times$. Observe that $F_K^\times \text{Nr}(\hat{K}^\times)$ is an open subgroup of $\hat{F}^\times$ of index 2 by the class field theory. In fact, this is the key idea behind Voight’s arguments.

We claim that the group $F_K^\times \text{Nr}(\hat{E})$ is uniquely determined by the pair $(B, \mathcal{G})$. In other words, if $O' \in \mathcal{G}$ is another order with $\varphi' \in \text{Emb}(B, O')$, then $F_K^\times \text{Nr}(\widehat{\mathcal{E}}') = F_K^\times \text{Nr}(\widehat{\mathcal{E}})$, where $\widehat{\mathcal{E}}' := \widehat{\mathcal{E}}(\varphi', B, O')$. Indeed, from (2.9), we have $F_K^\times \text{Nr}(\widehat{\mathcal{E}}') = F_K^\times \text{Nr}(\widehat{\mathcal{E}}) \text{Nr}(\alpha x)\alpha^{-1}$ for suitable $\alpha \in D^\times$ and $x \in \hat{D}^\times$. On the other hand, $\alpha x \in \widehat{\mathcal{E}}$ by definition (2.3). The claim is verified since $F_K^\times \text{Nr}(\widehat{\mathcal{E}})$ is a subgroup of $\hat{F}^\times$.

At each finite prime $p$ of $F$, put $\mathcal{E}_p := \mathcal{E}_p(\varphi, B, O)$ for simplicity. There is a similar chain of inclusions as the one in (2.11):
\begin{equation}
\text{Nr}(K_p^\times) \subseteq \text{Nr}(K_p^\times) \text{Nr}(\mathcal{N}(O_p)) \subseteq \text{Nr}(\mathcal{E}_p) \subseteq F_p^\times.
\end{equation}
The same argument as those in [29, §31.3.14] shows that \( \text{Nr}(\mathcal{E}_p) \) is a subgroup of \( F'_p \) of index at most 2. Moreover, \( \text{Nr}(\mathcal{E}_p) \) depends only on the isomorphism classes of the local orders \( B_p \) and \( \mathcal{O}_p \) by a local version of the calculation above.

Note that the first two terms of (2.11) correspond via the class field theory to \( K \) and \( K \cap \Sigma_\mathcal{G} \) respectively. It follows that

\[
(2.13) \quad K \supseteq K \cap \Sigma_\mathcal{G} \supseteq E_\text{op} \supseteq F.
\]

The following lemma is adapted from [1, Lemma 2.2] (see also [22, Theorem 3.2] and [29, Proposition 31.4.4]), so we omit its proof.

**Lemma 2.8.** Let \( \mathcal{O} \in \mathcal{G} \) be an order such that there exists \( \varphi \in \text{Emb}(B, \mathcal{O}) \). Given another order \( \mathcal{O}' \in \mathcal{G} \), we have \( \Delta(B, \mathcal{O}') = 1 \) if and only if \( \rho(\mathcal{O}, \mathcal{O}') \in \text{Gal}(\Sigma_\mathcal{G}/F) \) restricts to identity on \( E_\text{op} \).

Since \( [K : F] = 2 \), the field \( E_\text{op} \) coincides with either \( F \) or \( K \) by (2.13). In light of the identification \( \text{SG}(\mathcal{O}) \simeq \text{Gal}(\Sigma_\mathcal{G}/F) \) in (2.10), the corollary below follows directly from the above lemma.

**Corollary 2.9.** The order \( B \subseteq K \) is selective for the genus \( \mathcal{G} \) if and only if \( E_\text{op} = K \). More explicitly,

(i) if \( E_\text{op} = F \), then \( \Delta(B, \mathcal{O}') = 1 \) for every spinor genus \( [\mathcal{O}']_{sg} \in \text{SG}(\mathcal{G}) \), and \( B \) is non-selective for the genus \( \mathcal{G} \);

(ii) if \( E_\text{op} = K \), then \( K \subseteq \Sigma_\mathcal{G} \), and exactly half of the spinor genera in \( \text{SG}(\mathcal{G}) \) are selected by \( B \).

In particular, if \( K \cap \Sigma_\mathcal{G} = F \), then \( B \) is non-selective for the genus \( \mathcal{G} \).

So far everything has been pretty standard: almost all the ideas have appeared in the literature somewhere. Now we employ the local-global compatibility of class field theory [26, §6] to reduce it to purely local considerations. This will eventually allow us to insert our own input into the theory.

**Lemma 2.10.** We have \( K \subseteq \Sigma_\mathcal{G} \) if and only if both of the following conditions hold:

(i) \( F'_K = F'_D \), or equivalently by weak approximation, \( K \) and \( D \) are ramified at exactly the same (possibly empty) set of real places of \( F \);

(ii) \( \text{Nr}(\mathcal{N}(\mathcal{O}_p)) \subseteq \text{Nr}(K_p^\times) \) for every finite prime \( p \) of \( F \).

**Proof.** Recall that \( F'_K \subseteq F'_D \) since \( K \) embeds into \( D \) over \( F \) by assumption. Clearly, (i) and (ii) guarantee that \( F'_K \text{Nr}(\hat{K}^\times) \supseteq F'_D \text{Nr}(\hat{K}^\times) \), and hence \( K \subseteq \Sigma_\mathcal{G} \).

Conversely, suppose that \( K \subseteq \Sigma_\mathcal{G} \) so that \( F'_K \text{Nr}(\hat{K}^\times) \supseteq F'_D \text{Nr}(\hat{K}^\times) \). By definition, \( \Sigma_\mathcal{G} \) splits completely at all the real places of \( F \) that are unramified in \( D \), and hence (i) necessarily holds. It follows from \( F'_K = F'_D \) that

\[
(2.14) \quad \text{Nr}(\mathcal{N}(\hat{K})) \subseteq F'_K \text{Nr}(\hat{K}^\times).
\]

If \( p \) splits in \( K \), then \( \text{Nr}(K_p^\times) = F_p^\times \), so \( \text{Nr}(\mathcal{N}(\mathcal{O}_p)) \subseteq \text{Nr}(K_p^\times) \) automatically. On the other hand, according to the local-global compatibility of class field theory [26, §6], for each \( p \) non-split in \( K \), there is a commutative diagram

\[
\begin{array}{ccc}
F_p^\times / \text{Nr}(K_p^\times) & \cong & \text{Gal}(K_p/F_p) \\
\text{Gal}(K_p/F_p) & \supseteq & \text{Gal}(K/F).
\end{array}
\]
Given \(a_p \in F_p^\times\), we have \(a_p \in \text{Nr}(K_p^\times)\) if and only if \(a := (\ldots, 1, a_p, 1, \ldots)\) lies in \(F_K^\times \text{Nr}(K^\times)\). Therefore, condition (ii) is necessary as well.

According to Corollary 2.9, for \(B\) to be selective for the genus \(\mathcal{G}\), it is necessary that \(K \subseteq \Sigma_{\mathcal{G}}\). Suppose that this is the case for the moment. Then \(F_K^\times = F_B^\times\) by this assumption. The same proof as Lemma 2.10 shows that (2.15) \(E_{op} = K\) if and only if \(\text{Nr}(K_p^\times) = \text{Nr}(\mathcal{E}_p), \forall \text{ finite prime } p \subseteq O_F.\)

Note that \(\text{Nr}(K_p^\times) = \text{Nr}(\mathcal{E}_p)\) holds automatically at every prime \(p\) split in \(K\) since \(\text{Nr}(K_p^\times) = F_p^\times\) for such \(p\). At each remaining \(p\) we already have \(\text{Nr}(K_p^\times) = \text{Nr}(K_p^\times) \text{Nr}(\mathcal{N}(O_p))\) thanks to the assumption \(K \subseteq \Sigma_{\mathcal{G}}\). In light of the chain of inclusions (2.12), one further asks whether the following equality holds:

(2.16) \(\text{Nr}(K_p^\times) \text{Nr}(\mathcal{N}(O_p)) = \text{Nr}(\mathcal{E}_p)\).

It is often convenient to treat this purely as a local question, discussed separately from the assumption \(K \subseteq \Sigma_{\mathcal{G}}\).

**Example 2.11.** Let \(p\) be a finite prime of \(F\) that is unramified in \(K\) and coprime to the reduced discriminant \(\delta(O)\) of \(O\). The latter condition implies that \(O_p \simeq M_2(O_{F_p})\), the ring of \(2 \times 2\) matrices with entries in \(O_{F_p}\). According to [28, Theorem II.3.2], all optimal embeddings of \(B_p\) into \(O_p\) are \(O_p^\times\)-conjugate, so \(\mathcal{E}_p = K_p^\times O_p^\times\). On the other hand, \(\text{Nr}(K_p^\times) \supseteq O_{F_p}^\times\), and \(\mathcal{N}(O_p) = F_p^\times O_p^\times\). Therefore, (2.17) \(\text{Nr}(K_p^\times) = \text{Nr}(K_p^\times) \text{Nr}(\mathcal{N}(O_p)) = \text{Nr}(\mathcal{E}_p)\).

This already shows that (2.16) holds for almost all finite primes of \(F\).

A key to Voight’s optimal selectivity theorem as quoted in Theorem 1.2 is the following equality result by himself [29, Proposition 31.5.7].

**Proposition 2.12.** If \(O_p\) is an Eichler order, then equality (2.16) holds at \(p\).

To go beyond the Eichler orders, we recall the notion of Eichler invariants from [6, Definition 1.8].

**Definition 2.13.** Let \(p\) be a finite prime of \(F\), \(\xi_p := O_{F_p}/p\) be the finite residue field of \(p\), and \(\xi'_p/\xi_p\) be the unique quadratic field extension. When \(O_p \not\simeq M_2(O_{F_p})\), the quotient of \(O_p\) by its Jacobson radical \(\mathfrak{J}(O_p)\) falls into the following three cases:

\(O_p/\mathfrak{J}(O_p) \simeq \xi_p \times \xi_p, \quad \xi_p, \quad \text{or} \quad \xi'_p,\)

and the **Eichler invariant** \(e_p(O)\) of \(O\) at \(p\) is defined to be \(1, 0, -1\) accordingly. As a convention, if \(O_p \simeq M_2(O_{F_p})\), then \(e_p(O)\) is defined to be \(2\).

Similarly, we write \((K/p)\) for the symbol\(^3\) that takes value \(1, 0, -1\) according to whether \(p\) is split, ramified or inert in \(K/F\).

For example, if \(D\) is ramified at \(p\) and \(O_p\) is maximal, then \(e_p(O) = -1\). It is shown in [6, Proposition 2.1] that \(e_p(O) = 1\) if and only if \(O_p\) is a non-maximal Eichler order (particularly, \(D\) is split at \(p\)). As a result, if \(O\) is an Eichler order, then \(e_p(O) \neq 0\) for every finite prime \(p\).

We extend Voight’s equality result above to local quaternion orders with Eichler invariant \(-1\).

\(^3\)This is usually called the Artin symbol, but we want to distinguish it from the Artin symbol \((a, K/F) \in \text{Gal}(K/F)\) to be used later where we identify \(\text{Gal}(K/F)\) with \(\mathbb{Z}/2\mathbb{Z}\).
Lemma 2.14. If $e_p(O) = -1$, then $\text{Nr}(K_p^\times) \text{Nr}(N(O_p)) = \text{Nr}(E_p)$.

Proof. Without lose of generality, we assume that $p$ is non-split in $K$. By [6 Proposition 3.1], $O_p$ is a Bass order (see [14 §37], [8 §1], [11] and [33 §3.1]). Let $L_p/F_p$ be the unique unramified quadratic field extension. It is shown in [8 Proposition 1.12] that $\text{Emb}(L_p, O_p) \neq \emptyset$, which implies that $\text{Nr}(O_p^\times) = O_{F_p}^\times$ since

\begin{equation}
O_{F_p}^\times \supseteq \text{Nr}(O_p^\times) \supseteq N_{L_p/F_p}(O_{L_p}^\times) = O_{F_p}^\times.
\end{equation}

Thus if $K_p/F_p$ is ramified, then $\text{Nr}(K_p^\times) \text{Nr}(N(O_p)) = F_p^\times$ and the equality is trivial again.

Let $\delta(O_p)$ be the reduced discriminant of $O_p$, and $\nu_p : F_p^\times \to \mathbb{Z}$ be the normalized discrete valuation of $F_p$. From [6 Corollary 3.2], $\nu_p(\delta(O_p))$ is odd if and only if $D_p$ is division. When $D_p$ is division, there exists $u \in N(O_p)$ such that $\nu_p(Nr(u))$ is odd by [8 Theorem 2.2]. It follows that $\text{Nr}(N(O_p)) = F_p^\times$ in this case and equality is trivial once more.

Lastly, assume that $K_p = L_p$ and $D_p \cong M_2(F_p)$. In this case, $\text{Nr}(N(O_p)) = \text{Nr}(K_p^\times = F_p^\times$ by [8 Theorem 2.2]. If $B_p = O_{K_p}$, then $\text{Nr}(O_p)$ acts transitively on $\text{Emb}(B_p, O_p)$ as shown in the start of the proof of [8 Theorem 3.3, p. 178]. In other words, $E_p = K_p^\times N(O_p)$, and the equality (2.10) holds. Now suppose that $B_p$ is non-maximal, and $B'_p \supseteq B_p$ is the unique $O_{F_p}$-order in $K_p$ such that $B_p = O_{F_p} + pB'_p$. Let $O'_p \supseteq O_p$ be the unique minimal overorder of $O_p$. We claim that $\text{Nr}(N(O'_p)) = \text{Nr}(K_p^\times)$ holds for $O'_p$ as well. Indeed, if $O'_p \neq M_2(O_{F_p})$, then it has Eichler invariant $-1$ again by [6 Corollary 3.2], so the equality holds as shown above. If $O'_p \cong M_2(O_{F_p})$, then the equality holds by (2.11). The claim is verified.

Let $\varphi_p : B_p \to O_p$, and put $E'_p := E_p(\varphi_p, B'_p, O'_p)$. According to [8 Lemma 3.7], every optimal embedding $B_p \to O_p$ extends to an optimal embedding $B'_p \to O'_p$, which implies that $E_p \subseteq E'_p$. Thus to show that $\text{Nr}(K_p^\times) = \text{Nr}(E'_p)$, it is enough to show that $\text{Nr}(K_p^\times) = \text{Nr}(E'_p)$ for the pair $(B'_p, O'_p)$. Iterating the above argument using the ascending chain of orders in [6 Corollary 3.2], we eventually arrive at a pair $(B''_p, O''_p)$ where either $B''_p = O_{K_p}$ or $O''_p \cong M_2(O_{F_p})$. It has already been shown that $\text{Nr}(K_p^\times) = \text{Nr}(E''_p)$, and the lemma is proved.

In the next theorem, we no longer keep $O \in \mathcal{G}$ fixed with $\text{Emb}(B, O) \neq \emptyset$. Rather, $O$ is allows to be any arbitrary member of $\mathcal{G}$. This does not affect the subgroup $\text{Nr}(E_p) \subseteq F_p^\times$, which depends only on $B$ and the genus $\mathcal{G}$. Since $e_p(O)$ is independent of the choice of $O \in \mathcal{G}$ as well, it makes sense to put $e_p(\mathcal{G}) := e_p(O)$ for any $O \in \mathcal{G}$.

Theorem 2.15. Let $\mathcal{G}$ be an arbitrary genus of orders in $D$. Let $K/F$ be a quadratic field extension that is $F$-embeddable into $D$, and $B$ be an order in $K$. Suppose that condition (5) holds for $B$ and $\mathcal{G}$. Let $S$ be the following finite (possibly empty) set of primes of $F$:

\begin{equation}
S := \{ p \mid (K/p) \neq 1 \quad \text{and} \quad e_p(\mathcal{G}) = 0 \},
\end{equation}

Then $B$ is (optimally spinor) selective for $\mathcal{G}$ if and only if

\begin{equation}
K \subseteq \Sigma_{\mathcal{G}}, \quad \text{and} \quad N_{K/F}(K_p^\times) = \text{Nr}(E_p) \quad \text{for every} \quad p \in S.
\end{equation}

If $B$ is selective, then
(1) for any two orders \( \mathcal{O}, \mathcal{O}' \in \mathcal{G} \),
\[
\Delta(B, \mathcal{O}) = \rho(\mathcal{O}, \mathcal{O}')|_K + \Delta(B, \mathcal{O}'),
\]
where \( \rho(\mathcal{O}, \mathcal{O}')|_K \) is the restriction of \( \rho(\mathcal{O}, \mathcal{O}') \in \text{Gal}(\Sigma \mathcal{O}/F) \) to \( K \), and the summation on the right is taken inside \( \mathbb{Z}/2\mathbb{Z} \) with the canonical identification \( \text{Gal}(K/F) \simeq \mathbb{Z}/2\mathbb{Z} \);

(2) exactly half of the spinor genera in \( \text{SG}(\mathcal{G}) \) are selected by \( B \).

**Proof.** The first part of the theorem follows from combining Corollary 2.9 with Proposition 2.12 and Lemma 2.14. Equation (2.21) is a reformulation of Lemma 2.8, and the last statement is already contained in Corollary 2.9. □

When combined with the criterion for \( K \subseteq \Sigma \mathcal{O} \) in Lemma 2.10, Theorem 2.15 reduces the selectivity problem to purely local studies of quaternion orders over complete discrete valuation rings. It also singles out the finitely many places \( p \) of \( F \) with \( e_p(\mathcal{O}) = 0 \) as the possible places of obstruction for the condition \( K \subseteq \Sigma \mathcal{O} \) to be sufficient for selectivity. This point of view has already borne fruit in [23], where Deke Peng and the first named author applied Theorem 2.15 to obtain a selectivity theorem for quaternion Bass orders that are well-behaved at the dyadic primes of \( F \). This provides the first known (as far as we are aware) concrete examples for which the condition \( K \subseteq \Sigma \mathcal{O} \) is insufficient for selectivity. Nevertheless, if we impose the condition that \( e_p(\mathcal{O}) \neq 0 \) for every finite prime \( p \), then we get results that look exactly like the ones in [11] Theorem 1.1 and [29] Theorem 31.1.7, which are stated for Eichler orders.

**Corollary 2.16.** Keep the assumption of Theorem 2.15 and assume further that \( e_p(\mathcal{G}) \neq 0 \) for every finite prime \( p \) of \( F \). Then \( B \) is optimally spinor selective for \( \mathcal{G} \) if and only if \( K \subseteq \Sigma \mathcal{O} \).

Similarly, the following lemma is a generalization of [29] Proposition 31.2.1 (see also [11] Theorem 1.1(3) and [21] Proposition 5.11).

**Lemma 2.17.** Suppose that \( e_p(\mathcal{G}) \neq 0 \) for every finite prime \( p \) of \( F \). Let \( \mathfrak{d}(\mathcal{O}) \) be the reduced discriminant of an order \( \mathcal{O} \in \mathcal{G} \). Then \( K \subseteq \Sigma \mathcal{O} \) if and only if both of the following conditions hold:

(a) the extension \( K/F \) and the \( F \)-algebra \( D \) are unramified at every finite prime \( p \) of \( F \) and ramify at exactly the same (possibly empty) set of infinite places;

(b) if \( p \) is a finite prime of \( F \) with \( \nu_p(\mathfrak{d}(\mathcal{O})) \equiv 1 \pmod{2} \), then \( p \) splits in \( K \).

**Proof.** First note that \( \text{Nr}(\mathcal{O}_p^\times) = O_{F_p}^\times \) for every finite prime \( p \) of \( F \). Indeed, if \( e_p(\mathcal{O}) = -1 \), then this has been proved in (2.18); if \( e_p(\mathcal{O}) \in \{1, 2\} \), then \( \mathcal{O}_p \) is an Eichler order, so this is obvious. Next it follows from [8] Theorem 2.2 that
\[
\text{Nr}(\mathcal{N}(\mathcal{O}_p)) = \begin{cases} 
F_p^{x^2}O_{F_p}^\times & \text{if } \nu_p(\mathfrak{d}(\mathcal{O})) \equiv 0 \pmod{2}, \\
F_p^{x^2} & \text{if } \nu_p(\mathfrak{d}(\mathcal{O})) \equiv 1 \pmod{2}.
\end{cases}
\]

From Lemma 2.10, we find that \( K \subseteq \Sigma \mathcal{O} \) if and only if
- \( K \) and \( D \) are ramified at exactly the same (possibly empty) set of real places of \( F \);
- \( K \) is unramified at every prime \( p \subset O_F \) with \( \nu_p(\mathfrak{d}(\mathcal{O})) \equiv 0 \pmod{2} \);
- \( K \) splits at every prime \( p \subset O_F \) with \( \nu_p(\mathfrak{d}(\mathcal{O})) \equiv 1 \pmod{2} \).
Thus conditions (a) and (b) are clearly sufficient. For the necessity, recall that if \( e_p(\mathcal{O}) \in \{1, 2\} \), then \( D_p \simeq M_2(F_p) \). Thus non-zero-ness assumption on the Eichler invariants implies that \( e_p(\mathcal{O}) = -1 \) at every finite prime \( p \) ramified in \( D \). On the other hand, if \( D_p \) is division and \( e_p(\mathcal{O}) = -1 \), then \( \nu_p(\mathfrak{d}(\mathcal{O})) \) is odd by [8 Corollary 3.2]. Thus if \( K \subseteq \Sigma_{\mathcal{O}} \), then \( K \) splits at every \( p \) ramified in \( D \). This forces \( D \) to be unramified at all finite places of \( F \) since \( K \) embeds into \( D \) by our assumption. The conditions are necessary as well. \( \square \)

If \( e_p(\mathcal{O}) \neq 0 \) for every finite prime \( p \) of \( F \) and \( K \subseteq \Sigma_{\mathcal{O}} \), then for any two orders \( \mathcal{O}, \mathcal{O'} \in \mathcal{G} \), the element \( \rho(\mathcal{O}, \mathcal{O'})|_K \in \text{Gal}(K/F) \) can be computed using an ideal \( I \) linking \( \mathcal{O} \) and \( \mathcal{O'} \) as in Example 2.6 since we do have \( \text{Nr}(\mathcal{O}_{K'}^e) = O_{K_p}^e \) for every finite prime \( p \).

3. Extending Maclachlan’s relative conductor formula

Let \( (B, \mathcal{G}) \) be as in the previous section, particularly the order \( B \subset K \) satisfies condition [23]. Throughout this section, we let \( \mathcal{G} = \mathcal{G}_n \) be the genus of Eichler orders in \( D \) of level \( \mathfrak{n} \subseteq O_F \). For simplicity, put \( \Sigma_n := \Sigma_{\mathcal{G}_n} \), and assume that \( K \subseteq \Sigma_n \) so that every order in \( K \) satisfying condition [24] is selective for \( \mathcal{G}_n \). Let \( \mathfrak{f}(B) \) be the conductor of \( B \), i.e. \( \mathfrak{f}(B) \) is the unique ideal of \( O_K \) such that \( B = O_K + \mathfrak{f}(B)O_K \). If \( B' \subseteq O_K \) is another \( O_K \)-order in \( K \), we put \( \mathfrak{f}(B'/B) := \mathfrak{f}(B')^{-1}\mathfrak{f}(B) \) and call it the relative conductor of \( B \) with respect to \( B' \). This is a fractional ideal of \( O_K \). Assume that \( B' \) satisfies condition [25] as well. If \( n \) is square-free, Maclachlan shows in [22] Theorem 3.3] that for any two orders \( \mathcal{O}, \mathcal{O'} \in \mathcal{G}_n \), the selectivity symbol formula [22] may be enhanced to

\[
\Delta(B, \mathcal{O}) = (\mathfrak{f}(B'/B), K/F) + \rho(\mathcal{O}, \mathcal{O'})|_K + \Delta(B', \mathcal{O'}). 
\]

Here \( (\mathfrak{f}(B'/B), K/F) \in \text{Gal}(K/F) \) is the Artin symbol [19 §X.1], which is well-defined since the assumption \( K \subseteq \Sigma_n \) forces \( K/F \) to be unramified at all the finite places of \( F \) by Lemma 2.17. The summation on the right hand side is taken inside \( \mathbb{Z}/2\mathbb{Z} \) with the canonical identification \( \text{Gal}(K/F) = \mathbb{Z}/2\mathbb{Z} \) as before. In this section, we show that Maclachlan’s formula holds without the square-free assumption on \( n \).

First, let us workout condition [24] more concretely in terms of \( n \) and \( \mathfrak{f}(B) \). Since \( K \subseteq \Sigma_n \) by our assumption, \( D \) and \( K \) are unramified at all the finite places of \( F \) according to Lemma 2.17. Particularly, the reduced discriminant \( \mathfrak{d}(\mathcal{O}) \) of \( \mathcal{O} \) coincides with the level \( n \). For each finite prime \( p \) of \( F \), let \( \nu_p : F_p^* \rightarrow \mathbb{Z} \) be the normalized discrete valuation. Define numerical invariants of \( \mathcal{O} \) and \( B \) at \( p \) as follows:

\[
n_p := \nu_p(\mathfrak{d}(\mathcal{O})) = \nu_p(\mathfrak{n}), \quad i_p(B) := \nu_p(\mathfrak{f}(B)).
\]

We quote the relevant part of the criterion for the existence of local optimal embeddings given by Guo and Qin in [17 Lemma 2.2], which in turn is based on a theorem of Brzezinski [9 Theorem 1.8].

**Lemma 3.1.** Keep the assumption that \( p \) is unramified in \( K \). Then \( \text{Emb}(B_p, \mathcal{O}_p) \neq \emptyset \) if and only if one of the following conditions holds

1. \( (K/p) = 1; \)
2. \( (K/p) = -1 \) and \( n_p \leq 2i_p(B) \).

In particular, if \( \text{Emb}(B_p, \mathcal{O}_p) \neq \emptyset \), then \( \text{Emb}(B'_p, \mathcal{O}_p) \neq \emptyset \) for any order \( B'_p \subseteq B_p \).
Theorem 3.2. Let \( n \) be an arbitrary ideal of \( O_F \). Then formula (3.1) holds for any orders \( \mathcal{O}, \mathcal{O}' \in \mathcal{G}_n \) and for any orders \( B, B' \) in \( K \) both satisfying condition (3).

In particular, we can take \( \mathcal{O}' = \mathcal{O} \) and pick a suitable \( B' \) to compute \( \Delta(B, \mathcal{O}) \).

Corollary 3.3. Keep \( n, B, \mathcal{O} \) be as in Theorem 3.2. Let \( B' := \varphi^{-1}(\mathcal{O}) \) for an \( F \)-embedding \( \varphi : K \to D \). Then

\[
\Delta(B, \mathcal{O}) = \left( \mathfrak{j}(B'/B), K/F \right) + 1.
\]

Proof of Theorem 3.2. As usual, all the orders in \( K \) (such as \( B, B', B_0 \) etc.) considered below are assumed to satisfy condition (3). We make a couple of simplifications.

(a) If (3.1) holds for a fixed pair of orders \( \mathcal{O}_0, \mathcal{O}'_0 \in \mathcal{G} \), then it holds for every pair \( \mathcal{O}, \mathcal{O}' \in \mathcal{G} \). This follows directly from Lemma 2.17 and the additive property of \( \mathcal{G}(\mathcal{O}, \mathcal{O}') \) in (2.4).

(b) If (3.1) holds for \( B = B_0 \) for a fixed order \( B_0 \) (with \( B' \) still being arbitrary), then it holds for every \( B \). This again follows from Lemma 2.17 and the multiplicative property of the relative conductors:

\[
\mathfrak{j}(B'/B) = \mathfrak{j}(B'/B_0)\mathfrak{j}(B_0/B).
\]

Now pick \( B_0 \) to be the unique order in \( K \) satisfying

\[
i_p(B_0) = \begin{cases} 0 & \text{if } n_p \text{ is odd}, \\ n_p & \text{if } n_p \text{ is even}. \end{cases}
\]

Since \( n_p = 0 \) for almost all \( p \), such an order does exist in \( K \). Moreover, it satisfies condition (3) by Lemma 2.17 and Lemma 3.1. Thus there exists an order \( \mathcal{O}_1 \in \mathcal{G}_n \) that admits an optimal embedding \( \varphi \in \text{Emb}(B_0, \mathcal{O}_1) \). We shall modify \( \mathcal{O}_1 \) locally at finite many places to obtain another order \( \mathcal{O}_0 \in \mathcal{G}_n \) that is more conductive to our purpose. For each \( \varepsilon \in \{ \pm 1 \} \), let \( T_\varepsilon \) be the following finite set of primes \( p \subset O_F \):

\[
T_\varepsilon := \{ p \mid \left( K/p \right) = \varepsilon \text{ and } i_p(B') \neq i_p(B_0) \}.
\]

It follows from Lemma 2.17 and the assumption \( K \subseteq \Sigma_n \) that \( n_p = 0 \) is even for every \( p \in T_{-1} \). In particular, \( i_p(B_0) = n_p \) for every \( p \in T_{-1} \) by definition.

At each prime \( p \subset O_F \) inert in \( K \), there exists an element \( \omega_p \in O_{K_p}^\times \) satisfying a quadratic equation

\[
\omega_p^2 - t_p \omega_p + u_p = 0, \quad \text{with } t_p \in O_{F_p}, \ u_p \in O_{F_p}^\times,
\]

such that \( O_{K_p} = O_{F_p} + \omega_p O_{F_p} \). For each \( p \in T_{-1} \), we choose a suitable identification \( D_p = M_2(F_p) \) such that \( \varphi : K_p \to D_p \) sends \( \omega_p \) to the element \( \begin{bmatrix} 0 & -u_p \\ t_p & 1 \end{bmatrix} \), where \( \varphi \in \text{Emb}(B_0, \mathcal{O}_1) \) is the optimal embedding as above. Let \( \pi_p \) be a uniformizer in \( F_p \). There is a unique order \( \mathcal{O}_0 \in \mathcal{G}_n \) satisfying the following local properties:

\[
(O_0)_p = \begin{bmatrix} O_{F_p} & O_{F_p} \\ \pi_p O_{F_p} & \pi_p O_{F_p} \end{bmatrix}, \quad \forall p \in T_{-1}, \quad \text{and} \quad (O_0)_p = (O_1)_p, \quad \forall p \notin T_{-1}.
\]

By our construction, for every \( p \in T_{-1} \) we have

\[
\varphi(K_p) \cap (O_0)_p = \varphi(O_{F_p} + \pi_p n_p \omega O_{F_p}) = \varphi((B_0)_p),
\]

so \( \varphi \) defines an optimal embedding of \( B_0 \) into \( \mathcal{O}_0 \). In particular, \( \Delta(B_0, \mathcal{O}_0) = 1 \).

Next, we produce another order \( \mathcal{O}_0' \) such that \( \varphi \) defines an optimal embedding of \( B' \) into \( \mathcal{O}_0' \). For each \( p \) in \( T_1 \), we pick an element \( x_p \in D_p^\times \) such that \( \varphi \in \text{Emb}(B_0, \mathcal{O}_0) \).
\[ \text{Emb}(B'_p, x_p(O_0)_{p}^{-1}). \] Such an \( x_p \) exists by part (1) of Lemma 3.1. For each \( p \in T_{-1} \), we put \( x_p = \begin{bmatrix} 0 & 1 \\ \pi_p^{-i'_p} & 0 \end{bmatrix} \), where \( i'_p := i_p(B') \). One easily computes that
\[ x_p(O_0)_{p}x_p^{-1} = \begin{bmatrix} O_{F_p} & \pi_p^{n_p-i'_p} O_{F_p} \\ \pi_p^{i'_p} O_{F_p} & O_{F_p} \end{bmatrix}. \]

We have \( n_p - i'_p \leq i'_p \) by part (2) of Lemma 3.1. Since \( u_p \in O_{F_p}^\times \) by construction, it follows that
\[ \varphi(K_p) \cap x_p(O_0)_{p}x_p^{-1} = \varphi(O_{F_p} + \pi_p^{i'_p} \omega_p O_{F_p}) = \varphi(B'_p), \quad \forall p \in T_{-1}. \]

Lastly, we define \( O'_0 \) to be the unique order such that \( \hat{O}'_0 = x \hat{O}_0 x^{-1} \), where \( x \in \hat{D}^\times \) is the element such that \( x_p \) is defined above if \( p \in T_1 \cup T_{-1} \), and \( x_p = 1 \) at every other \( p \). The construction above guarantees that \( \varphi \in \text{Emb}(B', O'_0) \), so \( \Delta(B', O'_0) = 1 \).

Now to show that formula (3.1) holds for \( B = B_0, O = O_0 \) and \( O' = O'_0 \), it is enough to check that
\[ (f(B'/B_0), K/F) + \rho(O_0, O'_0)|_K = 0. \]
From (3.3), we have \( f(B'/B_0) = \prod_{p \in T_{-1}} p^{n_p-i'_p} \prod_{p \in T_1} p^{i_p(B_0)-i'_p} \). Since \( (p, K/F) \) vanishes if \( p \in T_1 \) and \( n_p \) is even at every \( p \in T_{-1} \), we obtain
\[ (f(B'/B_0), K/F) = \sum_{p \in T_{-1}} (p^{-i'_p}, K/F), \]
where \( \text{Gal}(K/F) \) is identified with \( \mathbb{Z}/2\mathbb{Z} \) as usual. On the other hand,
\[ \rho(O_0, O'_0)|_K = (\text{Nr}(x), K/F) = \sum_{p \in T_{-1}} (p^{i'_p}, K/F), \]
where the last equality holds because those \( x_p \) for \( p \in T_1 \) make no contribution to the Artin symbol. Now (3.6) follows from combining (3.7) and (3.8), and the Theorem is proved by the observations at the start of the proof. \( \square \)

4. The spinor trace formula

In this section, we assume that \( G \) is an arbitrary genus of orders in \( D \), and \( K \) is not necessarily contained in \( \Sigma_g \). Let \( O \) be an order in \( G \), and \( B \) be an order in \( K \) satisfying condition (4.1). Recall from (4.1) that \( m(B, O, O^\times) \) denotes the number of optimal embeddings of \( B \) into \( O \) up to \( O^\times \)-conjugacy. Our goal of this section is to generalize the Vignéras-Voight formula (39) Corollary 31.1.10] for \( m(B, O, O^\times) \) to the totally definite case. Let us first set up some notations and reproduce this formula for the convenience of the reader.

By definition, the class number of \( O \) (denoted as \( h(O) \)) is the cardinality of the finite set \( \text{Cl}(O) \) of locally principal right \( O \)-ideal classes in \( D \). It is well known that \( h(O) \) depends only on the genus \( G \) and not on the choice of \( O \in G \). Similarly, let \( h(B) \) denote the class number of \( B \). We define the symbol
\[ s(K, G) = \begin{cases} 1 & \text{if } K \subseteq \Sigma_g \\ 0 & \text{otherwise.} \end{cases} \]
Definition 4.2. the trace formula.

The Eichler condition.
in (2.5). The trace formula holds for any quaternion algebras, i.e. it does not require
we group the right
O
longer viable to produce a formula for
m
Here \[ I \]
denotes the right
O
-I-
ideals into
O
\] denotes the right
O
-I-
ideals. Instead, we group the right
O
-I-
ideals classes into spinor classes and produce a refinement of the trace formula.

Definition 4.2. Two locally principal right \( \mathcal{O} \)-ideals \( I \) and \( I' \) are in the same spinor class if there exists \( x \in D^x \sim \tilde{D}^1 \) such that \( \tilde{P} = x \tilde{I} \).

Note that if \( I \) and \( I' \) belong to the same spinor class, then their left orders \( \mathcal{O}(I) \) and \( \mathcal{O}(I') \) are in the same spinor genus. The spinor class of \( I \) is denoted by \( [I]_{\text{sc}} \). For a fixed spinor class \( [I]_{\text{sc}} \), the set of ideal classes in \( [I]_{\text{sc}} \) is denoted by \( \text{Cl}(\mathcal{O}, [I]_{\text{sc}}) \). In other words,

\[
\text{Cl}(\mathcal{O}, [I]_{\text{sc}}) := \{ [I'] \in \text{Cl}(\mathcal{O}) | [I'] \subseteq [I]_{\text{sc}} \}.
\]

For simplicity, we put \( \text{Cl}_{\text{sc}}(\mathcal{O}) := \text{Cl}(\mathcal{O}, [\mathcal{O}]_{\text{sc}}) \).

Let \( \text{SCI}(\mathcal{O}) \) be the finite set of spinor classes of locally principal right \( \mathcal{O} \)-ideals. It can be described adelically as follows

\[
\text{SCI}(\mathcal{O}) \simeq (D^x \sim \tilde{D}^1 \backslash \tilde{D}^x / \tilde{\mathcal{O}}^x) \cong \mathcal{F}_D^x / \mathcal{F}_D^x / \mathcal{N}_r(\tilde{\mathcal{O}}^x),
\]

where the two double coset spaces are canonically bijective via the reduced norm map. This equips \( \text{SCI}(\mathcal{O}) \) with an abelian group structure (whose identity element
is \([\mathcal{O}_{\text{sc}}]\), so we call it the spinor class group of \(\mathcal{O}\). In light of the adelic description of \(\text{SG}(\mathcal{O})\) in (2.2), there is a canonical surjective group homomorphism
\[(4.6) \quad \text{SCl}(\mathcal{O}) \rightarrow \text{SG}(\mathcal{O}), \quad [I]_{\text{sc}} \mapsto [\mathcal{O}_l(I)]_{\text{sg}}.\]

Our spinor trace formula can be stated as follows.

**Proposition 4.3** (Spinor trace formula). Let \(\mathcal{G}\) be an arbitrary genus of orders in \(D\). Suppose that either \(K \cap \Sigma_{\mathcal{G}} = F\) or \(B\) is selective for \(\mathcal{G}\). Then for each \([J]_{\text{sc}} \in \text{SCl}(\mathcal{O})\), we have
\[(4.7) \quad \sum_{[I] \in \text{Cl}(\mathcal{O}, [J]_{\text{sc}})} m(B, \mathcal{O}_l(I), \mathcal{O}_l(I)^{\times}) = \frac{2^\sigma(K, \mathcal{G}) \Delta(B, \mathcal{O}_l(J)) h(B)}{[\text{SCl}(\mathcal{O})]} \prod_p m_p(B).\]

**Remark 4.4.** We make a few observations.

(a) If \(D\) is further assumed to satisfy the Eichler condition, then \(\text{Cl}(\mathcal{O}, [J]_{\text{sc}})\) is a singleton with the unique member \([J]\) by [7, Proposition 1.1], so we recover a slightly more general form of the Vignéras-Voight formula (4.2).

(b) From Corollary 2.16 if \(e_p(\mathcal{O}) \neq 0\) for every finite prime \(p\) of \(F\) (e.g. if \(\mathcal{O}\) is an Eichler order), then the assumption of the proposition holds automatically. Moreover, in this case \([\text{SCl}(\mathcal{O})]\) is equal to the restricted class number \(h_D(F)\) of \(F\) with respect to \(D\) since \(\text{Nr}(\mathcal{O}_F^{\times}) = \mathcal{O}_F^{\times}\), for every prime \(p \subset O_F\).

(c) Form Corollary 2.18 \(B\) is selective for the genus \(\mathcal{G}\) if and only if \(E_{\text{op}} = K\), where \(E_{\text{op}}\) is the optimal representation field in Definition 2.7. Recall from (2.13) that we have the following chain of inclusions:
\[F \subseteq E_{\text{op}} \subseteq K \cap \Sigma_{\mathcal{G}} \subseteq K.\]

The assumption of the proposition covers the two cases below:
\[F = E_{\text{op}} = K \cap \Sigma_{\mathcal{G}} \subseteq K, \quad F \subsetneq E_{\text{op}} = K \cap \Sigma_{\mathcal{G}} = K.\]

In the last remaining case where \(F = E_{\text{op}} \subsetneq K \cap \Sigma_{\mathcal{G}} = K\), it is not known whether (4.7) still holds true or not. Examples of \((B, \mathcal{O})\) falling into this third case do exist, as shown by Peng and the first named author in [23, §5].

To prove Proposition 4.3 we reformulate the summation in (4.7) adelicly. If \(\tilde{I} = x \tilde{\mathcal{O}}\) for some \(x \in \tilde{D}^{\times}\), then we put
\[(4.8) \quad \tilde{[I]}_{\text{sc}} := D^{\times} \tilde{D}^{1} x \tilde{\mathcal{O}}^{\times} = D^{\times} \tilde{D}^{1} \tilde{\mathcal{O}}^{\times} x,\]
where the last equality follows from the fact that
\[(4.9) \quad \tilde{D}^{1} \tilde{\mathcal{O}}^{\times} = \tilde{D}^{1} x \tilde{\mathcal{O}}^{\times} x^{-1}, \quad \forall x \in \tilde{D}^{\times}.\]

Clearly, \(\tilde{[I]}_{\text{sc}}\) depends only on the spinor class \([I]_{\text{sc}}\). The set \(\text{Cl}(\mathcal{O}, [I]_{\text{sc}})\) may be described adelically as
\[(4.10) \quad \text{Cl}(\mathcal{O}, [I]_{\text{sc}}) \simeq D^{\times} \backslash \tilde{[I]}_{\text{sc}}/\tilde{\mathcal{O}}^{\times} \simeq D^{\times} \backslash (D^{\times} \tilde{D}^{1} x \tilde{\mathcal{O}}^{\times} x^{-1})/(x \tilde{\mathcal{O}}^{\times} x^{-1}),\]
where the last isomorphism is induced from the right multiplication of \(\tilde{[I]}_{\text{sc}}\) by \(x^{-1}\). If we set \(\mathcal{O}' := \mathcal{O}_l(I)\) so that \(\tilde{\mathcal{O}}' = x \tilde{\mathcal{O}} x^{-1}\), then there is a bijection
\[(4.11) \quad \text{Cl}(\mathcal{O}, [I]_{\text{sc}}) \simeq \text{Cl}(\mathcal{O}', [\mathcal{O}']_{\text{sc}}) =: \text{Cl}_{\text{sc}}(\mathcal{O}').\]

Fix an embedding \(\varphi : K \rightarrow D\) and identify \(K\) with its image in \(D\) as before. At the moment, we do not require \(\varphi \in \text{Emb}(B, \mathcal{O})\) yet. Let \(\tilde{E} := \tilde{E}(\varphi, B, \mathcal{O})\) be as
defined in \cite{28}. Using the same method as in the proofs of \cite{28} Theorem III.5.11 and \cite{29} Theorem 30.4.7, we immediately obtain the following lemma.

**Lemma 4.5.** We have
\begin{equation}
\sum_{[I] \in \Cl(O,I)} m(B, O_I(I), O_I(I)^\times) = |K^\times \cap [\hat{J}]_{sc}/\hat{O}^\times|.
\end{equation}

We leave it as an exercise using Lemma 4.5 to show that if $O$ and $O'$ belong to the same spinor genus, then
\begin{equation}
\sum_{[I] \in \Cl(O)} m(B, O_I(I), O_I(I)^\times) = \sum_{[I'] \in \Cl(O')} m(B, O_I(I'), O_I(I')^\times).
\end{equation}

Now we are ready to prove the spinor trace formula.

**Proof of Proposition 4.3.** Recall that the canonical map $\Cl(O) \to \SG(O)$ in \eqref{4.6} is a group homomorphism. From part (ii) of Corollary 2.9, if $B$ is selective for $\mathcal{G}$, then exactly half of the spinor classes $[I]_{sc} \in \Cl(O)$ satisfy that $\Delta(B, O_I(I)) = 1$.

In light of the trace formula \eqref{4.3}, the proposition reduces to a statement reminiscent of part (iii) of Theorem 1.2: if $\Delta(B, O_I(J)) = \Delta(B, O_I(J')) = 1$ for two spinor classes $[J]_{sc}$ and $[J']_{sc}$ in $\Cl(O)$, then
\begin{equation}
\sum_{[I] \in \Cl(O,J)_{sc}} m(B, O_I(I), O_I(I)^\times) = \sum_{[I'] \in \Cl(O,J')_{sc}} m(B, O_I(I'), O_I(I')^\times).
\end{equation}

Now let $O' \in \mathcal{G}$ be another member of $\mathcal{G}$, and $M'$ be a locally principal right $O'$-ideal with $O_I(M') = O$. The map $I \mapsto IM'$ induces a bijection between locally principal right ideals of $O$ and those of $O'$, and it preserves ideal classes, spinor classes, and associated left orders. Therefore,
\begin{equation}
\sum_{[I] \in \Cl(O,J)_{sc}} m(B, O_I(I), O_I(I)^\times) = \sum_{[I'] \in \Cl(O,J'M')_{sc}} m(B, O_I(I'), O_I(I')^\times).
\end{equation}

Replacing $O$ by a suitable $O'$ if necessary, we shall assume that there exists an optimal embedding $\varphi \in \Emb(B, O)$. Clearly, if \eqref{4.11} holds for $J' = O$, then it holds for all $J'$ with $\Delta(B, O_I(J')) = 1$, so we further take $J' = O$.

Keep the optimal embedding $\varphi \in \Emb(B, O)$ fixed and identify $K$ with $\varphi(K) \subseteq D$ as before. By the assumption, either $K \cap \Sigma_{\mathcal{G}} = F$ or $B$ is selective for $\mathcal{G}$. This guarantees that
\begin{equation}
F^x_B \Nr(\hat{K}^\times) \Nr(\mathcal{N}(\hat{O})) = F^x_B \Nr(\hat{\mathcal{E}}).
\end{equation}

Indeed, from \eqref{2.11} and \eqref{2.13}, if $K \cap \Sigma_{\mathcal{G}} = F$, then both sides are equal to $\hat{F}^\times$; if $B$ is selective for $\mathcal{G}$, then both sides are equal to $F^x_K \Nr(\hat{K}^\times)$.

Write $\hat{J} = z\hat{O}$ for some $z \in \hat{D}^\times$. Then $O_I(\hat{J}) = z\hat{O}z^{-1}$. By Lemma 2.8, the assumption $\Delta(B, O_I(J)) = 1$ implies that $\Nr(z) \in F^x_B \Nr(\hat{\mathcal{E}})$. Thanks to \eqref{4.16}, there exist
\begin{equation}
y \in D^\times \hat{D}^1, \quad k \in \hat{K}^\times \quad \text{and} \quad u \in \mathcal{N}(\hat{O})
\end{equation}
such that $yz = ku$. Thus we have
\begin{equation}
[\hat{J}]_{sc} = D^\times \hat{D}^1 yz\hat{O}^\times = D^\times \hat{D}^1 ku\hat{O}^\times = kD^\times \hat{D}^1 \hat{O}^\times u.
\end{equation}
Since \( \tilde{\mathcal{E}} \) is left invariant by \( \tilde{K}^\times \) and right invariant by \( \mathcal{N}(\tilde{\mathcal{O}}) \), we get
\[
(4.19) \quad \tilde{\mathcal{E}} \cap [\tilde{J}]_{sc} = \tilde{\mathcal{E}} \cap (kD^\times \tilde{D}^1 \tilde{\mathcal{O}}^\times) = k(\tilde{\mathcal{E}} \cap \tilde{D}^\times \tilde{\mathcal{O}}^\times) u = k(\tilde{\mathcal{E}} \cap [\tilde{\mathcal{O}}]_{sc}) u.
\]
Lastly, observe that the map \( x \mapsto kxu \) for \( x \in \tilde{\mathcal{E}} \cap [\tilde{\mathcal{O}}]_{sc} \) induces a bijection
\[
K^\times \backslash (\tilde{\mathcal{E}} \cap [\tilde{\mathcal{O}}]_{sc}) / \tilde{\mathcal{O}}^\times \to K^\times \backslash (\tilde{\mathcal{E}} \cap [\tilde{J}]_{sc}) / \tilde{\mathcal{O}}^\times.
\]
Therefore, (4.14) holds for \( J' = \mathcal{O} \) by Lemma 4.5. The proposition is proved. \( \square \)

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