In these lectures, I cover the most elementary aspects of $N = 1$ supersymmetry, and its application to low energy phenomenology. Since there is no end to the subject, I decided not to cover supergravity, rather concentrate on the basic techniques of global supersymmetry, in the context of the $N = 1$ Standard Model. I do discuss, but only cursorily, the all important question of supersymmetry breaking. The lectures are organized as follows:

I-) Motivation

II-) Tools: The Chiral and Vector Supermultiplets

III-) The Minimal $N = 1$ Standard Model

IV-) More Tools: Supersymmetry Breaking

I-) MOTIVATION

The $N = 0$ Standard Model is a very compact model, described by three gauge groups, and nineteen parameters. The quantum numbers of the fermions strongly suggest that the three gauge groups are part of a more integrated structure, but its parameters range all over over the place, and show few if any discernable patterns.

We also know that the model becomes inconsistent at distances shorter than the Planck length because of the divergent nature of quantum gravity. Thus we should regard the model as an effective theory, valid at larger distances, and view the Planck scale as Nature’s own ultraviolet cutoff.

It is natural to ask if the standard model can be described in simpler terms at shorter distances. The only tool at our disposal is the renormalization group. If we assume no new physics at shorter distances, the renormalization group allows us to extrapolate the parameters of the standard model to the deep ultraviolet, and look for the patterns suggested by the quantum numbers.

To start, the inverse of the fine structure constants for the three gauge groups evolve
linearly with the logarithm of the scale

\[ \frac{d\alpha_i^{-1}}{dt} = \frac{1}{2\pi} b_i , \]

where \( b_i = (-\frac{41}{10}, \frac{19}{6}, 7) \) for \( U(1), SU(2) \) and \( SU(3) \), respectively, and \( t = \ln(\mu/\mu_0) \), \( \mu \) being the scale normalized to an arbitrary scale \( \mu_0 \).

Using low energy data as boundary conditions, the two weak fine structure constants meet at a scale of \( 10^{13} \) GeV, with a value \( \alpha^{-1} = 43 \). At the same scale, the QCD fine structure constant weighs in at \( \alpha_3^{-1} = 38 \). Thus the road to Grand Unified Theories (GUTs), the apparent unification suggested by the quantum numbers, is not achieved in the \( N = 0 \) Standard Model.

Only the hypercharge couplings becomes non-perturbative at shorter distances, but only well beyond the Planck scale, as does the electric coupling in QED.

The Yukawa couplings also behave supinely in the ultraviolet. Generically their renormalization group equations are ruled by two competing effects. One is the Yukawa couplings themselves, which tend to make the coupling blow up at short distances, the other from the gauge couplings, does the opposite. In the \( N = 0 \) Standard Model, the QCD gauge couplings dominate, and the Yukawa couplings gently settle to non-perturbative values. To give an example, the top Yukawa coupling varies according to

\[ \frac{dy_t}{dt} = \frac{y_t}{16\pi^2} \left( \frac{9}{2} y_t^2 - 8 g_3^2 \right) , \]

which is negative around \( M_Z \). Although both the Yukawa and gauge coupling decrease, the \( \beta \) function does not change sign at shorter distances. As for the leptons, their Yukawas are too small to dominate the electroweak couplings.

This leaves us with the Higgs self-coupling. Its \( \beta \) function is ruled by two effects which work in opposite directions. The contribution from the self coupling itself forces it to blow up, while the fermion loop correction works in the opposite direction; in addition there are contributions from the gauge couplings, but they are small at experimental energies, since the Higgs has no color. Neglecting the gauge contributions, we have

\[ \frac{d\lambda}{dt} = \frac{1}{16\pi^2} \left( 12\lambda^2 + 12y_t^2 \lambda - 12y_t^4 \right) . \]

It is convenient to discuss the behavior of this coupling in terms of the Higgs mass, which is proportional to \( \sqrt{\lambda} \). If the Higgs mass is large, then so is \( \lambda \), and the first term dominates. It quickly drives the coupling to non-perturbative values. The larger the Higgs mass, the sooner it blows up; since there is no evidence of non-perturbative behavior, this sets an upper bound on the Higgs mass. This is called the triviality bound. If the Higgs mass is found to be above 200 GeV, it is certain that non-perturbative physics is present in the TeV region. Since there is no evidence of strong coupling in the electroweak model at experimental scales, the Higgs cannot be arbitrarily massive; it must be lighter than 600 to 800 GeV.
On the other hand, if the Higgs mass is small, the last term which does not depend on $\lambda$ takes over and drives $\lambda$ towards negative values. However a negative $\lambda$ means that the potential is no longer stable, since its potential becomes unbounded below. Larger field configurations are favored, which takes us beyond perturbation theory. With the recent value of the top quark mass, if the Higgs is less than 150 GeV, the standard model description leaves perturbation theory at some scale below the Planck scale. For a Higgs mass around 100 GeV, this “instability bound” sets in around one TeV. It can be cured in a variety of ways, say by adding new degrees of freedom. Supersymmetry is one theory which obviates this bound by eliminating the $\lambda$ coupling altogether!

We conclude that in the $N = 0$ standard model, non-perturbative physics below the Planck scale is expected for a wide range of the Higgs mass. Should there be non-perturbative physics at some scale below the Planck mass, we must view the $N = 0$ standard model as an effective theory with that scale as a cut-off, rather than the Planck scale.

The dependence of the standard model parameters on the cut-off is illuminating. The fermion masses, for instance, depend on the logarithm of the cut-off, as can be seen by evaluating the one-loop correction to the mass. The reason for this mild dependence is chiral symmetry; it softens the degree of divergence of the diagram. This protective symmetry works because the theory becomes chirally invariant in the limit of massless fermions.

Another nearly massless (on the scale of Planck mass) particle is the Higgs scalar, but its dependence on the cut-off is linear! There is no symmetry to protect it. It is therefore natural to expect that the mass of the Higgs is of the order of the cut-off of the $N = 0$ standard model, times some coupling constant. Thus the expectation that the natural cut-off is of the order of TeVs. Generically, theories which detail this possibility are called technicolor theories. In such theories, the longitudinal W bosons interact strongly under the strong technicolor force, in direct analogy to the pions in the strong interactions. We note with some amusement, that historically, attempts to formulate such theories have led to string theories, which then led to superstring theories (Plus ça change,...).

One could conceivably avoid this conclusion if the parameters of the theory are so finely tuned that the cut-off dependence of the Higgs mass is relegated to higher loop effects. Since we do not know the origin of the parameters, this is a logical, albeit unfair possibility to keep in mind. It could be that the parameters are at a fixed point of a non-linear mother theory, with fractal-like relations among themselves. I thought I would mention this to open your ears to this possible application of chaotic phenomena.

One can demonstrably avoid strong coupling at low energies, by generalizing the $N = 0$ standard model to supersymmetry, which we call the $N = 1$ standard model. Supersymmetry links the Higgs field to a chiral fermion of equal mass. Then the chiral symmetry which protects the fermion protects the Higgs as well, and results in the same logarithmic dependence of the Higgs mass on the cut-off. As long as the boson-fermion supersymmetry is unbroken, that is. The lack of evidence of such symmetry in the low energy world, indicates that it must be broken, and the trick is to break it at a scale that is not too high lest it unravels its salutory effect. All that supersymmetry does is to allow all the couplings to remain perturbative all the way to at near the Planck mass. Then, the $N = 1$
standard model is predictive to the Planck scale, but none of the mysteries associated with the breaking of the electroweak symmetry have been explained; they have just been shuffled in the yet to come explanation of the breaking of supersymmetry. However, in generic situations, it turns out that supersymmetry breaking induces electroweak breaking. All that remains is to explain how supersymmetry breaking comes about, and at what scale.

II-) SUPERSYMMETRY TOOLBOX

This is the first of a series of sections about $N = 1$ supersymmetry. It is not meant to be complete, but rather helpful in presenting the relevant facts. In the absence of gravity, supersymmetry employs two collections of fields, arranged in supersymmetric multiplets. The first, called the chiral or Wess-Zumino supermultiplet, consists of one left-handed Weyl spinor and a complex scalar field, and it serves as a generalization of the fermion and Higgs fields of the Standard Model. The second, called the gauge supermultiplet, contains the gauge vector bosons, as well as their supersymmetric partners, the gauginos. The $N = 1$ Standard Model is described by these supermultiplets in interaction with one another.

The Chiral Supermultiplet

We start with a brief description of our notation and conventions. The student unfamiliar with these is encouraged to consult standard texts on Advanced Quantum Mechanics and Field Theory.

In four space-time dimensions, the algebra of the Lorentz group is isomorphic to that of $SU_2 \times SU_2$ (up to factors of $i$), the first generated by $\vec{J} + i\vec{K}$, the second by $\vec{J} - i\vec{K}$; $\vec{J}$ are the generators of angular momentum and $\vec{K}$ are the boosts. Thus these two $SU_2$ are seen to be connected by complex conjugation ($i \rightarrow -i$) and/or parity ($\vec{K} \rightarrow -\vec{K}, \vec{J} \rightarrow \vec{J}$), and are therefore left invariant by the combined operation of CP. In accordance with this algebraic structure, spinor fields appear in two varieties, left-handed spinors, transforming only under the second $SU_2$ as spin $\frac{1}{2}$ representations, and right-handed spinors transforming only under the first $SU_2$. They are represented by two-component complex spinor fields, called Weyl spinors,

$$\psi_L(x) \sim (2, 1), \quad \psi_R(x) \sim (1, 2).$$

The spinor fields must be taken to be anticommuting Grassmann variables, in accordance with the Pauli exclusion principle. Their Lorentz transformation properties can be written in terms of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
which satisfy

\[ \sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k. \]

The action of the Lorentz group on the spinor fields is

\[ \psi_{L,R} \rightarrow \Lambda_{L,R} \psi_{L,R} \equiv e^{i \vec{\omega} \cdot \vec{\sigma}} e^{i \vec{v} \cdot \vec{\sigma}} \psi_{L,R}, \]

where \( \vec{\omega} \) and \( \vec{v} \) are the real rotation and boost angles, respectively. This corresponds to the representation where

\[ \vec{J} = \frac{\vec{\sigma}}{2}; \quad \vec{K} = -i \frac{\vec{\sigma}}{2}. \]

It is possible to make left-handed spinors out of right-handed antispinors, and vice-versa. One checks that

\[ \bar{\psi}_L \equiv \sigma_2 \psi_R^* \sim (2,1), \]
\[ \bar{\psi}_R \equiv \sigma_2 \psi_L^* \sim (1,2). \]

Under charge conjugation the fields behave as

\[ C : \psi_L \rightarrow \sigma_2 \psi_R^*, \quad \psi_R \rightarrow -\sigma_2 \psi_L^*. \]

Under parity

\[ P : \psi_L \rightarrow \psi_R, \quad \psi_R \rightarrow \psi_L. \]

This purely left-handed notation is convenient to describe fermions that interact by the weak interactions which violate parity. For instance the neutrinos appear only as left-handed fields, while the antineutrinos are purely right-handed. On the other hand, fermions which interact in a parity invariant way as in QED and QCD, have both left- and right-handed parts. In that case, it is far more convenient to use the Dirac four-component notation. The fields \( \psi_L \) and \( \psi_R \) are put together into a four-component Dirac spinor (in the Weyl representation)

\[ \Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \]

on which the operation of parity is well-defined. In this representation, called the Weyl representation, the anticommuting Dirac matrices are (in \( 2 \times 2 \) block form)

\[ \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \]
\[ \gamma_5 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Since one can generate right-handed fields starting from left-handed ones, it suffices to consider only polynomials made out of left-handed fields.
We will be doing many manipulations in Weyl language, and it is useful to see how the Fierz transformations read. Let $\zeta$ and $\eta$ be complex two-component Weyl spinors, each transforming as $(2,1)$ of the Lorentz group. Thus the convenient Fierz decompositions

$$\zeta\eta^T \sigma_2 = -\frac{1}{2} \sigma_2 \eta^T \sigma_2 \zeta - \frac{1}{2} \sigma^i \eta^T \sigma_2 \sigma^i \zeta,$$

corresponding to

$$(2,1) \otimes (2,1) = (1,1) \oplus (3,1).$$

As the combinations $\sigma_2 \zeta^\ast$ and $\sigma_2 \eta^\ast$ transform according to the $(1,2)$ representation, we also have

$$\zeta\eta^\dagger = -\frac{1}{2} \eta^\dagger \zeta - \frac{1}{2} \sigma^i \eta^\dagger \sigma^i \zeta,$$

corresponding to

$$(2,1) \otimes (1,2) = (2,2).$$

The right hand side of this equation does indeed correspond to the vector representation, as we can see by introducing the matrices

$$\sigma^\mu = (\sigma^0 = 1, \sigma^i) ; \quad \bar{\sigma}^\mu = (\sigma^0 = 1, -\sigma^i),$$

in terms of which we rewrite

$$\zeta\eta^\dagger = -\frac{1}{2} \bar{\sigma}^\mu \eta^\dagger \sigma_\mu \zeta.$$

For the Pauli matrices, we have

$$\sigma_2 \sigma^i \sigma_2 = -\sigma^i T = -\sigma^i \ast,$$

so that

$$\sigma_2 \bar{\sigma}^\mu \sigma_2 = \sigma^\mu T.$$

The simplest set of fields on which $N = 1$ supersymmetry is realized is the chiral or Wess-Zumino multiplet which contains the three fields,

$$\varphi(x) , \text{ a complex scalar ,}$$

$$\psi(x) , \text{ a Weyl spinor ,}$$

$$F(x) , \text{ a complex auxiliary field .}$$

The Lagrangian density is given by

$$\mathcal{L}^{WS}_0 \equiv \partial_\mu \varphi^\ast \partial^\mu \varphi + \psi^\dagger \sigma^\mu \partial_\mu \psi + F^\ast F ;$$

it is invariant, up to a surface term, under the following transformations
\[\delta \varphi = \alpha^T \sigma_2 \psi,\]
\[\delta \psi = \alpha F - \bar{\sigma}^\mu \sigma_2 \alpha^* \partial_\mu \varphi,\]
\[\delta F = -\alpha^\dagger \sigma^\mu \partial_\mu \psi.\]

Here \(\alpha\) is the parameter of the supersymmetry transformation; it is a Weyl spinor. Note that we treat Grassmann variables as plain anticommuting numbers, so that for any two of them

\[(\zeta^T \sigma_2 \chi)^* = \zeta^\dagger \sigma^*_2 \chi^* = -\zeta^\dagger \sigma_2 \chi^*;\]

this is the reason there is no \(i\) in front of the fermion kinetic term. Take note of the following: \(\varphi\) and \(\psi\) have the canonical dimensions, \(-1\) and \(-3/2\), but \(F\) has the non-canonical dimension of \(-2\), and \(\alpha\) has dimension 1/2. Also, \(F\) transforms as a total divergence.

Under two supersymmetry transformations, labelled \(\delta_1\) and \(\delta_2\), with parameters \(\alpha_1\) and \(\alpha_2\), we find that

\[\left[\delta_1, \delta_2\right] = (\alpha_1^\dagger \sigma^\mu \alpha_2 - \alpha_2^\dagger \sigma^\mu \alpha_1) \partial_\mu \varphi,\]

where \(\ast\) stands for any of the three fields, \(\varphi\), \(\psi\), and \(F\). This equation shows that the result of two supersymmetry transformations is just a translation by the amount

\[\delta x^\mu = (\alpha_1^\dagger \sigma^\mu \alpha_2 - \alpha_2^\dagger \sigma^\mu \alpha_1),\]

recalling that \(P_\rho = -i\partial_\rho\) is the generator of translations. Thus supersymmetry transformations are the square root of translations, and we will see later how the Poincaré group is altered to accomodate these new transformations.

Let us verify this equation for one of the fields. For example

\[\delta_1 \delta_2 F = -\alpha_2^\dagger \sigma^\mu \partial_\mu \delta_1 \psi,\]

\[= -\alpha_2^\dagger \sigma^\mu \alpha_1 \partial_\mu F - \alpha_2^\dagger \sigma^\mu \sigma^\rho \alpha_2 \alpha^* \partial_\mu \partial_\rho \varphi.\]

Now because of the symmetry of \(\partial_\mu \partial_\rho \varphi\), we can set

\[\sigma^\mu \sigma^\rho = \frac{1}{2}(\sigma^\mu \sigma^\rho + \sigma^\rho \sigma^\mu) = g^{\mu \rho},\]

leading to

\[\delta_1 \delta_2 F = -\alpha_2^\dagger \sigma^\mu \alpha_1 \partial_\mu F - \alpha_2^\dagger \sigma^\mu \sigma^\rho \partial_\mu \partial_\rho \varphi.\]

Now \(\alpha_2^\dagger \sigma_2 \alpha_1^*\) is symmetric under the \((1 \leftrightarrow 2)\) interchange, and drops out from the commutator, giving the desired result

\[\left[\delta_1, \delta_2\right] = (\alpha_1^\dagger \sigma^\mu \alpha_2 - \alpha_2^\dagger \sigma^\mu \alpha_1) \partial_\mu F.\]
The other two expressions for \( \varphi \) and \( \psi \) work out in a similar way, making use of Fierz identities when applied to \( \psi \).

All these results can be neatly summarized by introducing a two-component Weyl Grassmann variable \( \theta \). We introduce the superfield \( \Phi(x, \theta) \) which depends only on \( \theta \), its most general expansion is

\[
\Phi(x, \theta) = \varphi(x) + \theta^T \sigma_2 \psi(x) + \frac{1}{2} \theta^T \sigma_2 \theta F(x) ,
\]

such that its change under supersymmetry can be obtained by acting on the fields,

\[
\delta \Phi = \delta \varphi + \theta^T \sigma_2 \delta \psi + \frac{1}{2} \theta^T \sigma_2 \theta \delta F ,
\]

or as operator acting on the coordinates

\[
\delta \Phi = \left[ \alpha^T \sigma_2 \frac{\partial}{\partial \theta} + \alpha^\dagger \sigma^\mu \theta \partial_{\mu} \right] \Phi ,
\]

where we have introduced the Grassmann derivative, defined through

\[
\frac{\partial}{\partial \theta} \theta^T \sigma_2 = 1 .
\]

Note that expressing the supersymmetry transformations as generated by derivative operators enables us to derive the commutator formula in a much more elegant way. It also enables us to see why the change in the coefficient of \( \theta^T \sigma_2 \theta \) is a total divergence: it can only come from the term linear in \( \theta \) in the generator, which contains the space-time derivative, acting on the term linear in \( \theta \) in the superfield.

We can express the effect of a supersymmetry transformation on the chiral superfield in another way, namely

\[
\Phi(x^\mu, \theta) \rightarrow \Phi(x^\mu + \alpha^\dagger \sigma^\mu \theta, \theta + \alpha) ; \quad \frac{\partial \Phi}{\partial \theta^*} = 0 .
\]

While formally pleasing, we note that the change in \( x^\mu \) is not real. If we decompose it into its real plus imaginary parts,

\[
\alpha^\dagger \sigma^\mu \theta = \frac{1}{2} (\alpha^\dagger \sigma^\mu \theta - \theta^\dagger \sigma^\mu \alpha) + \frac{1}{2} (\alpha^\dagger \sigma^\mu \theta + \theta^\dagger \sigma^\mu \alpha) ,
\]

we note that the imaginary part can itself be written as the change of half the quantity \( \theta^\dagger \sigma_\mu \theta \) under a shift of the Grassmann variables.

Such considerations lead us to construct the superfield

\[
V(x^\mu, \theta, \theta^*) = \Phi^* (x^\mu + \frac{1}{2} \theta^\dagger \sigma^\mu \theta, \theta) \Phi(x^\mu + \frac{1}{2} \theta^\dagger \sigma^\mu \theta, \theta) .
\]

It is manifestly real.
\[ V^*(x^\mu, \theta, \theta^*) = V(x^\mu, \theta, \theta^*) \]

and transforms under supersymmetry in an aesthetic way, namely the change in the coordinate \( x_\mu \) is real:

\[ V(x^\mu, \theta, \theta^*) \rightarrow V(x^\mu + \frac{1}{2}(\alpha^\dagger \sigma^\mu \theta - \theta^\dagger \sigma^\mu \alpha), \theta + \alpha, \theta^* + \alpha^*) . \]

However this superfield depends on the Grassmann variables and their conjugates, but the change in the coordinate is now real. To verify this equation, we start from the chiral superfield

\[ \Phi(x^\mu + \frac{1}{2} \theta^\dagger \sigma^\mu \theta, \theta) = \varphi(x) + \theta T \sigma_2 \psi(x) + \frac{1}{2} \theta^T \sigma_2 \theta F(x) \]

\[ + \frac{1}{2} \theta^\dagger \sigma^\mu \theta \partial_\mu \varphi(x) - \frac{1}{4} \theta^T \sigma_2 \theta \partial^\mu \partial_\mu \varphi(x) + \frac{1}{16} |\theta^T \sigma_2 \theta|^2 \partial^\mu \partial_\mu \varphi(x) , \]

where we have used some Fierzing and the identity

\[ \theta^\dagger \sigma^\mu \theta \theta^\dagger \sigma^\nu \theta = \frac{1}{2} g^\mu\nu \theta |\theta^T \sigma_2 \theta|^2 . \]

It follows from the above that the real superfield is given by

\[ V(x, \theta, \theta^*) = \varphi^*(x) \varphi(x) + [\theta^T \sigma_2 \psi \varphi^* - \theta^\dagger \sigma_2 \psi \varphi] \]

\[ + \frac{1}{2}[\theta^T \sigma_2 \theta \varphi^* F - \theta^\dagger \sigma_2 \theta \varphi F^* + \theta^\dagger \sigma^\mu \theta (\varphi^* \partial_\mu \varphi - \partial_\mu \varphi^* \varphi - \psi^\dagger \sigma_\mu \psi)] \]

\[ - \frac{1}{4} \theta^\dagger \sigma_2 \theta (2 \varphi^* \psi T \sigma_2 + \varphi \partial_\mu \psi^\dagger \sigma^\mu - \partial_\mu \varphi^* \psi \sigma_\mu \theta) \]

\[ - \frac{1}{4} \theta^T \sigma_2 \theta \partial^\mu (2 \varphi^* \psi F + \varphi^* \sigma^\mu \partial_\mu \varphi - \sigma^\mu \partial_\mu \varphi^* \psi) \]

\[ + \frac{1}{8} (\theta^T \sigma_2 \theta)^2 (2 \varphi^* F - \partial_\mu \varphi^* \partial^\mu \varphi + \frac{1}{2} (\varphi^* \partial^\mu \partial_\mu \varphi + \varphi \partial^\mu \partial_\mu \varphi^*) + \psi^\dagger \sigma^\mu \partial_\mu \psi - \partial_\mu \psi^\dagger \sigma^\mu \partial_\mu \psi) . \]

The alert student will recognize the last term as the Lagrange density, plus an overall divergence.

You can verify the transformation law. We show it to hold on a subset of terms of the form \( \partial_\mu \varphi \varphi^* \). On the one hand, we get

\[ V(x^\mu + \frac{1}{2}(\alpha^\dagger \sigma^\mu \theta - \theta^\dagger \sigma^\mu \alpha), \theta + \alpha, \theta^* + \alpha^*) \]

\[ = \cdots + \frac{1}{2} \varphi^*(\alpha^\dagger \sigma^\mu \theta - \theta^\dagger \sigma^\mu \alpha) \partial_\mu \varphi + \frac{1}{2}(\alpha^\dagger \sigma^\mu \theta + \theta^\dagger \sigma^\mu \alpha) \varphi^* \partial_\mu \varphi + \cdots \]

\[ = \alpha^\dagger \sigma^\mu \theta \varphi^* \partial_\mu \varphi . \]
On the other hand, by varying the fields directly, we obtain the very same term
\[
\theta^T \sigma_2 \delta \psi \phi^* = -\theta^T \sigma_2 \bar{\sigma}_2 \alpha^* \partial_\mu \phi \phi^* + ,
\]
\[
= \alpha^\dagger \sigma^\mu \theta \partial_\mu \phi \phi^* .
\]

We can also express the supersymmetric change on the real superfield as resulting from the action of differential operators, namely
\[
\delta V = \{ \alpha^T \sigma_2 \frac{\partial}{\partial \theta} - \alpha^\dagger \sigma_2 \left( \frac{\partial}{\partial \bar{\theta}} \right)^* + 1 \alpha^\dagger \sigma^\mu \theta - \theta^\dagger \sigma^\mu \alpha \} \partial_\mu \} V(x, \theta, \theta^*),
\]
\[
= \{ \alpha^T \sigma_2 \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \bar{\sigma}^\mu \sigma_2 \theta^* \partial_\mu \right) - \alpha^\dagger \sigma_2 \left( \left( \frac{\partial}{\partial \bar{\theta}} \right)^* - \frac{1}{2} \sigma_2 \sigma^\mu \theta \partial_\mu \right) \} V(x, \theta, \theta^*),
\]
where we have used the identity
\[
\theta^\dagger \sigma^\mu \alpha = -\alpha^T \sigma_2 \bar{\sigma}^\mu \theta^\dagger .
\]

Introduce the generators of supersymmetry
\[
Q = \frac{\partial}{\partial \theta} + \frac{1}{2} \bar{\sigma}^\mu \sigma_2 \theta^* \partial_\mu, ,
\]
\[
Q^* = \left( \frac{\partial}{\partial \bar{\theta}} \right)^* - \frac{1}{2} \sigma_2 \sigma^\mu \theta \partial_\mu, ,
\]
to write the change in the real superfield
\[
\delta V = (\alpha^T \sigma_2 Q - \alpha^\dagger \sigma_2 Q^*) V(x, \theta, \theta^*),
\]
The supersymmetry generators satisfy the anticommutation relations
\[
\{ Q, Q \} = \{ Q^*, Q^* \} = 0 ,
\]
\[
\{ Q, Q^* \} = \bar{\sigma}^\mu \partial_\mu .
\]
When added to the generators of the Poincaré group, these generators form the super-Poincaré group, and the particles described by supersymmetry must form irreducible representations of this supergroup. We first note that the supersymmetry generators commute with translations,
\[
[Q, P_\mu] = 0 ,
\]
hence with $P_\mu P^\mu$, the Casimir operator whose value is the mass squared. Since the Poincare group is a subgroup, any representation of the supergroup contains several representations of the Poincaré group of the same mass.
It is simplest to start with massless representations. The massless representations of the Poincaré group are labelled by the helicity \( \lambda \) which runs over positive and negative integer and half-integer values. In local field theory, each helicity state \(| \lambda >\) is accompanied by its CPT conjugate \(| - \lambda >\). For example, the left polarized photon \(| \lambda = +1 >\) and its CPT conjugate the right polarized photon \(| \lambda = -1 >\).

Let us go to the infinite momentum frame \( P_0 = P_3 \neq 0 \), where the supersymmetry algebra reduces to the Clifford algebra

\[
\{Q_1, Q_1^\ast\} = iP_0 ,
\]

all other anticommutators being zero. It follows that we have just one supersymmetry operator and its conjugate, acting like a raising operator. Thus starting with any state \(| \lambda >\), we generate only one other state \(Q^\ast | \lambda >\), which has helicity \( \lambda + 1/2 \). A repeated application of the raising operator yield zero since \(Q_1^2 = 0\). Hence there are no other states. This yields the only irreducible representation for supersymmetry of massless states: two states, differing by half a unit of helicity. (I first learned this elegant proof from Gell-Mann and Neéman in 1975)

The Wess-Zumino multiplet corresponds to the representation \(| 0 > \oplus | 1/2 >\), together with its CPT conjugate \(| 0 > \oplus | -1/2 >\), which describes one Weyl fermion and two scalar degrees of freedom.

In the same way we can expect the gauge multiplet which contains the states \(| 1 > \oplus | 1/2 >\), together with their conjugates: a vector particle and a Weyl fermion. There are other representations, such as the graviton-gravitino combination, made up of \(| 2 > \oplus | 3/2 >\), plus conjugate. These are all realized in local field theory. The number of bosonic and fermionic degrees of freedom match exactly. For instance, the chiral multiplet has, using the equations of motion, two fermionic degrees of freedom, exactly matched by the complex scalar field. If the equations of motion are not used, the number of fermions doubles, but the excess fermions is exactly matched by adding two boson fields, the complex auxiliary field \( F \).

Massive multiplets can always be obtained by assembling massless multiplets, à la Higgs.

The real supermultiplet is highly reducible. It can be checked that the covariant derivative operator

\[
D \equiv \frac{\partial}{\partial \theta} - \frac{1}{2} \sigma^\mu \sigma_2 \theta^* \partial_\mu ,
\]

and its complex conjugate anticommute with the generators of supersymmetry. By requiring that they vanish on the real superfield, we obtain the chiral superfield. I leave this an exercise to the hardiest among you.

This notation in terms of Grassmann variables allows us to write supersymmetric invariants in a very elegant way. We have already noted that the highest component of a superfield transforms as a four-divergence, so that its integral over space-time is supersymmetric invariant. We can define integration over the \( \theta \) Grassmann variables
\[ \int d\theta = 0, \quad \int d\theta \theta = 1; \]

note that since \( \theta \) has dimension 1/2, \( d\theta \) has the opposite dimension, -1/2. Integration enables us to rewrite the invariant in the form

\[ \int d^4x \int d^2\theta \Phi(x, \theta) = \int d^4x F. \]

However, any product of \( \Phi(x, \theta) \) is itself a chiral superfield. To see this, note that by Fierzing,

\[ \theta \theta^T \sigma_2 \theta = -\frac{1}{2} \sigma_2 \theta^T \sigma_2 \theta \sigma_2 \theta = -\frac{1}{2} \theta^T \sigma_2 \theta \theta, \]

so that

\[ \theta^T \sigma_2 \theta = 0, \]

Thus the polynomial expansion in \( \theta \), is exactly of the same form as that of \( \Phi \).

It follows that for any number of chiral superfields \( \Phi_a, a = 1, \ldots, N \), all the quantities

\[ \int d^4x \int d^2\theta \Phi_a \cdots \Phi_{an} \quad \text{for all } a_i \text{ and } n, \]

are supersymmetric invariants. In terms of components, the lowest polynomials are given by

\[ m \int d^2\theta \Phi_1 \Phi_2 = m(\varphi_1 F_2 + \varphi_2 F_1 - \psi_1^T \sigma_2 \psi_2), \]

\[ \lambda \int d^2\theta \Phi_1 \Phi_2 \Phi_3 = \lambda(\varphi_1 \varphi_2 F_3 + \varphi_1 F_2 \varphi_3 + F_1 \varphi_2 \varphi_3 \]

\[ - \varphi_1 \psi_2^T \sigma_2 \psi_3 - \varphi_2 \psi_1^T \sigma_2 \psi_3 - \varphi_3 \psi_1^T \sigma_2 \psi_2). \]

The quadratic terms in the superfields are mass terms, and the cubic contain the renormalizable Yukawa interactions. In addition, they contain interactions with the auxiliary fields, which lead to the same mass term for the bosons, and their quartic renormalizable self-interactions. Higher order polynomials yield non-renormalizable interactions.

For a real superfield, transforming under supersymmetry like \( V \), it is easy to show that its component along \( |\theta^T \sigma_2 \theta|^2 \) transforms as a four-divergence. This term is called the D-term. Thus its space-time integral is a supersymmetric invariant. By integrating over both \( \theta \) and \( \theta^* \), we can extract the D-term. Our only example so far is the kinetic part of the Action

\[ \int d^4x \int d^2\theta \int d^2\bar{\theta} \bar{\Phi}(x_{\mu} + \theta^i \bar{\sigma}_{\mu} \theta, \theta)^2. \]

It has the right dimension: the superfield has dimension one, and the four Grassmann integral bring dimension two.
The potential part of the Action is given by

\[ \int d^4x \int d^2\theta P(\Phi) + c.c. , \]

where the function \( P \) is called the superpotential; it depends only on the chiral superfields, not their conjugates. For renormalizable theories, it is at most cubic

\[ P = m_{ij} \Phi_i \Phi_j + \lambda_{ijk} \Phi_i \Phi_j \Phi_k . \]

It is straightforward to see that the physical potential is simply expressed in terms of the superpotential

\[ V(\varphi) = \sum_i F_i^* F_i = \sum_i \left| \frac{\partial P(\varphi)}{\partial \varphi_i} \right|^2 ; \]

it is obviously positive definite, which is a general feature of global supersymmetry.

It is easy to implement internal symmetries: just assume that the whole superfield transforms as some representation of some internal group. The kinetic part (only if the invariance is global) is automatically invariant by summing over all the internal degrees of freedom. The superpotential may not be invariant, which restricts its form.

The kinetic term has a special global symmetry, called R-symmetry; it is not an internal symmetry since it does not commute with supersymmetry. R-symmetry is a global phase symmetry on the Grassmann variables

\[ \theta \rightarrow e^{i\beta} \theta , \quad \theta^* \rightarrow e^{-i\beta} \theta^* . \]

This means that the Grassmann measures transform in the opposite way

\[ d\theta \rightarrow e^{-i\beta} d\theta , \quad d\theta^* \rightarrow e^{i\beta} d\theta^* . \]

The Grassmann integration measure for the kinetic term is invariant. The most general R-type transformation that leaves the kinetic integrand invariant is

\[ \Phi_i(x, \theta) \rightarrow e^{i m_i \beta} \Phi_i(x, e^{i \beta} \theta) . \]

This symmetry is not necessarily shared by the superpotential, unless it transforms under R as

\[ P \rightarrow e^{2i\beta} P , \]

to match the transformation of the Grassmann measure. This further restricts the form of the superpotential.

To see the role of the auxiliary fields, consider two chiral superfields with only the mass term in their superpotential, \( m \Phi_1 \Phi_2 \). In the Lagrangian density we find the terms

\[ F_1^* F_1 + F_2^* F_2 + \{ m(\varphi_1 F_2 + \varphi_2 F_1 - \psi_1^T \sigma_2 \psi_2) + c.c. \} . \]

The equations of motions for the auxiliary fields such as

\[ F_1^* = -m \varphi_2 , \text{ etc} , \]
allow us to simply rewrite them in terms of the physical fields, with the result

\[-m^2|\varphi_1|^2 - m^2|\varphi_2|^2 - m\psi_1^T \sigma_2 \psi_2.\]

These are the mass terms for four real scalars and one Dirac fermion of mass \(m\). This leads us to the mass sum rule

\[\sum_{J=0} m^2 = 2 \sum_{J=1/2} m^2,\]

where we count the number of Weyl fermions (1 Dirac = 2 Weyl). If we had only one superfield with interaction \(\frac{m}{2} \Phi \Phi\), the extra term in the Lagrangean would have been simply

\[m(\varphi F - \frac{1}{2} \psi^T \sigma_2 \psi),\]

which describes one complex scalar of mass \(m\) and one Weyl of mass \(m\); in this case the sum rule

\[\sum_{J=0,1/2} (2J + 1)(-1)^J m_J^2 = m^2 + m^2 - 2m^2 = 0,\]

is again satisfied.

Functions of a Chiral Superfield

In the following, we work out certain functions of superfields, which are of some interest in discussing non-renormalizable supersymmetric theories. As we have seen, products of chiral superfields are themselves chiral superfields, so that any special function of a chiral superfield is defined through its series expansion.

Logarithm

Given a chiral superfield

\[\Phi = \varphi(x) + \theta^T \sigma_2 \psi(x) + \frac{1}{2} \theta^T \sigma_2 \theta F(x),\]

we have

\[\ln \Phi = \ln \{\varphi[1 + \theta^T \sigma_2 \hat{\psi}(x) + \frac{1}{2} \theta^T \sigma_2 \theta \hat{F}(x)]\},\]

\[= \ln \varphi + \ln[1 + \theta^T \sigma_2 \hat{\psi}(x) + \frac{1}{2} \theta^T \sigma_2 \theta \hat{F}],\]

where

\[\hat{\psi} = \frac{\psi}{\varphi}, \quad \hat{F} = \frac{F}{\varphi}.\]

We then use the series expansion of the logarithm to obtain

\[\ln \Phi = \ln \varphi + (\theta^T \sigma_2 \hat{\psi} + \frac{1}{2} \theta^T \sigma_2 \theta \hat{F}) - \frac{1}{2} (\theta^T \sigma_2 \hat{\psi} + \frac{1}{2} \theta^T \sigma_2 \theta \hat{F})^2,\]

\[= \ln \varphi + \theta^T \sigma_2 \hat{\psi} + \frac{1}{2} \theta^T \sigma_2 \theta (\hat{F} + \frac{1}{2} \hat{\psi}^T \sigma_2 \hat{\psi}),\]
where we have used the Fierz identities and the fact that the expansion in $\theta$ cuts off after second order.

**Power**

The arbitrary power of a chiral superfield is given by its series expansion, since

$$
\Phi^a = \varphi^a \{ 1 + \theta^T \sigma_2 \hat{\psi} + \frac{1}{2} \theta^T \sigma_2 \theta \hat{F} \}^a,
$$

$$
= \varphi^a \{ 1 + a \theta^T \sigma_2 \hat{\psi} + \frac{1}{2} a \theta^T \sigma_2 \theta \hat{F} + \frac{1}{2} a(a - 1)(\theta^T \sigma_2 \hat{\psi})^2 \},
$$

which, after a Fierz, yields the exact result.

$$
\Phi^a = \varphi^a \{ 1 + a \theta^T \sigma_2 \hat{\psi} + \frac{1}{2} a \theta^T \sigma_2 \theta \hat{F} - \frac{a(a - 1)}{2} \theta^T \sigma_2 \hat{\psi} \}.
$$

**The Real Superfield**

We have already seen how to construct a real superfield out of a chiral superfield. In general, however, we should be able to build it in terms of the four real Grassmann variables which describe the Weyl spinor $\theta$. An elegant way to do this is to rewrite the two-component Weyl into a four component Majorana spinor. In the Majorana representation for the Dirac matrices, all four components of a Majorana spinor are real, so that we are dealing with four real anticommuting degrees of freedom. The real superfield is the most general expansion in terms of the real Majorana spinor

$$
\Theta = \begin{pmatrix} \theta \\ -\sigma_2 \theta^* \end{pmatrix},
$$

shown here in the Weyl representation.

Because they anticommute, the expansion will stop at the fourth order. Naive counting results in having 4 components to the first order, $\frac{4\cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} = 6$ components at the second, $\frac{4\cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} = 4$ at the third, and $\frac{4\cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} = 1$ component at the fourth. Hence a real superfield contains $(1, 4, 6, 4, 1)$ degrees of freedom, half commuting, half anti-commuting. We can form the six quadratic covariants

$$
\overline{\Theta} \Theta, \quad \overline{\Theta} \gamma_5 \Theta, \quad \overline{\Theta} \gamma_5 \gamma_\mu \Theta,
$$

where the bar denotes the usual Pauli adjoint

$$
\overline{\Theta} = \Theta^\dagger \gamma^0.
$$

It is easy to check the reality conditions

$$
(\overline{\Theta} \Theta)^* = -\overline{\Theta} \Theta, \quad (\overline{\Theta} \gamma_5 \Theta)^* = \overline{\Theta} \gamma_5 \Theta,
$$

$$
(\overline{\Theta} \gamma_5 \gamma_\mu \Theta)^* = -\overline{\Theta} \gamma_5 \gamma_\mu \Theta.
$$
The further identities
\[ \overline{\Theta} \gamma_5 \Theta \gamma_5 = -\overline{\Theta} \gamma_5 \Theta \gamma_5 = \frac{1}{4} \overline{\Theta} \gamma_5 \gamma_\mu \Theta \gamma_5 \gamma^\mu, \]
\[ \overline{\Theta} \gamma_5 \gamma_\mu \Theta \Theta = -\overline{\Theta} \Theta \gamma_5 \gamma_\mu, \quad \overline{\Theta} \gamma_5 \gamma^\mu \Theta \Theta = g^{\mu \nu} (\overline{\Theta} \Theta)^2, \]
are useful in arriving at the general Lorentz covariant expansion of a real superfield
\[ V(x^\mu, \Theta) = A(x) + i\overline{\Theta} \psi(x) + i\overline{\Theta} M(x) + \overline{\Theta} \gamma_5 \Theta N(x) + i\overline{\Theta} \gamma_5 \gamma^\mu \Theta A_{\mu}(x) + \overline{\Theta} \Theta \Lambda(x) + (\overline{\Theta} \Theta)^2 D(x). \]

In Weyl notation, the same real superfield reads
\[ V(x^\mu, \theta, \theta^*) = A(x) - i(\theta^T \sigma_2 \psi + \theta^T \sigma_2 \psi^*) - \theta^T \sigma_2 \theta C - i\theta^T \sigma_2 \theta^* C^* + i\theta^T \sigma_2 \theta A_{\mu} + \theta^T \sigma_2 \theta \theta^T \sigma_2 \lambda + \theta^T \sigma_2 \theta^* \sigma_2 \theta^T \sigma_2 \lambda^* + |\theta^T \sigma_2 \theta|^2 D, \]
where
\[ C(x) = M(x) - iN(x), \]
and
\[ \Psi(x) = \begin{pmatrix} \psi(x) \\ -\sigma_2 \psi^*(x) \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda \\ -\sigma_2 \lambda^* \end{pmatrix}. \]

The real superfield also contains a chiral superfield and its conjugate, made up of the non-canonical fields \( A, \psi, \) and \( C. \) We can always write it in the form
\[ V(x, \theta, \theta^*) = -i(\Phi(x, \theta) - \Phi^*(x, \theta)) + \hat{V}(x, \theta, \theta^*), \]
where
\[ \Phi(x, \theta) = \frac{1}{2}(B(x) + iA(x)) + \theta^T \sigma_2 \psi(x) + \theta^T \sigma_2 \theta C(x). \]

If the real superfield is dimensionless, the vector field \( A_{\mu} \) and the Weyl spinor \( \lambda \) have the right canonical dimension to represent a gauge field, and a spinor field. The real superfield describes the vector supermultiplet we have encountered in classifying the representations of the super Poincaré group, but with many extra degrees of freedom, which happen to fall neatly in chiral multiplets. This is no accident, since they in fact turn out to be gauge artifacts.

For future reference, let us work out some functions of a real superfield which are useful in some physical applications. Starting with the power of a real superfield, we have
\[ V^a = [A(1 + X)]^a, \]
with
\[ X = i\overline{\Theta} \Psi + i\overline{\Theta} \Theta \dot{M} + i\overline{\Theta} \gamma_5 \Theta \dot{N} + i\overline{\Theta} \gamma_5 \gamma^\mu \Theta \dot{A}_{\mu} + \overline{\Theta} \Theta \dot{\Lambda} + (\overline{\Theta} \Theta)^2 \dot{D}, \]
where the hat denotes division by $A$. Then, noting that $X^5 = 0$, a little bit of algebra gives

$$V^a = A^a[1 + aX + a(a - 1)\frac{X^2}{2!} + a(a - 1)(a - 2)\frac{X^3}{3!} + a(a - 1)(a - 2)(a - 3)\frac{X^4}{4!}].$$

The Fierz identity shown here for any two Dirac four component spinors $\Psi_\Lambda = -\frac{1}{4}\bar{\Psi}\gamma_5\Psi_{\gamma^5\gamma^\rho\Lambda} - \frac{1}{4}\gamma^\rho\bar{\Psi}\gamma_\rho\Lambda + \frac{1}{2}\sigma_{\mu\nu}\bar{\Psi}\sigma^{\mu\nu}\Lambda,$

is used repeatedly to rewrite the powers of $X$ in terms of the standard expansion for a real superfield. We leave it as an exercise in fierce Fierzing to work out the general formula. Here we just concentrate on the D-term. The contributions to the D-term are as follows:

$$X : (\bar{\Theta}\Theta)^2\hat{D};$$

$$X^2 : 2i\bar{\Psi}\vec{\Theta}\Theta\bar{\Theta}\hat{\Lambda} + (i\bar{\Theta}\Theta\hat{M} + \bar{\Theta}\gamma_5\Theta\hat{N} + i\bar{\Theta}\gamma_5\gamma^\mu\Theta\hat{A}_\mu)^2,$$

$$= (\bar{\Theta}\Theta)^2\left\{-\frac{i}{2}\hat{\Lambda}\hat{\Psi} - M^2 + N^2 - \hat{A}_\mu\hat{A}^\mu\right\};$$

$$X^3 : -3(i\bar{\Theta}\Theta\hat{M} + \bar{\Theta}\gamma_5\Theta\hat{N} + i\bar{\Theta}\gamma_5\gamma^\rho\Theta\hat{A}_\rho)(\bar{\Theta}\hat{\Psi})^2,$$

$$= \frac{3}{4}(\bar{\Theta}\Theta)^2(i\hat{M}\hat{\Psi}\bar{\Psi} - \hat{N}\hat{\Phi}_{\gamma_5}\bar{\Psi} - \hat{A}_\rho\hat{\Phi}_{\gamma_5}\gamma^\rho\bar{\Psi});$$

$$X^4 : (i\Theta\hat{\Psi})^4,$$

$$= \frac{1}{16}(\bar{\Theta}\Theta)^2(\bar{\Lambda}\Psi)^4[1 + 1 + g^\mu].$$

Putting it all together, we obtain for the D-term

$$(V^a)_{\hat{D}} = A^a[a\hat{D} + a\frac{(a - 1)}{2}(-\frac{i}{2}\hat{\Lambda}\hat{\Psi} - \hat{M}^2 + \hat{N}^2 - \hat{A}_\mu\hat{A}^\mu)$$

$$+ \frac{1}{8}a(a - 1)(a - 2)(i\hat{M}\hat{\Phi}_{\gamma_5}\bar{\Psi} - \hat{N}\hat{\Phi}_{\gamma_5}\bar{\Psi} - \hat{A}_\rho\hat{\Phi}_{\gamma_5}\gamma^\rho\bar{\Psi})].$$

You can use these formulae to show that the real superfield, expunged of its chiral components, satisfies $\hat{V}^3 = 0.$

The Vector Supermultiplet

We have seen that group theory implies the existence of a vector supermultiplet which generalizes gauge fields to supersymmetry; it contains (taking the Abelian case for simplicity)
\[ A_\mu(x) : \text{a gauge field} \]
\[ \lambda(x) : \text{a Weyl spinor (called the gaugino)}, \]
\[ D(x) : \text{an auxiliary field}. \]

The auxiliary field is here to provide the right count between bosonic and fermionic degrees of freedom. Without using the massless Dirac equation, the spinor is described by four degrees of freedom. The gauge field is described by three degrees of freedom, leaving the \( D \) to make up the balance. With the use of the equations of motion, the Weyl field has two degrees of freedom, and so does the massless gauge field, and the auxiliary field disappears.

Sometimes the gaugino is called a Majorana fermion, but there should be no confusion between a Weyl fermion and a Majorana fermion: in two-component notation they look exactly the same.

The Action
\[
S = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \lambda^\dagger \sigma^\mu \partial_\mu \lambda + \frac{1}{2} D^2 \right],
\]
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), is invariant under the following supersymmetry transformations
\[
\begin{align*}
\delta A_\mu &= -i \lambda^\dagger \sigma_\mu \alpha - i \alpha^\dagger \sigma_\mu \lambda, \\
\delta \lambda &= \frac{1}{2} \left( D + \frac{i}{2} \sigma^{\mu\nu} F_{\mu\nu} \right) \alpha, \\
\delta D &= \partial_\mu \lambda^\dagger \sigma^\mu \alpha - \alpha^\dagger \sigma^\mu \partial_\mu \lambda,
\end{align*}
\]

where
\[
\sigma^{\mu\nu} = \frac{1}{2} (\bar{\sigma}_\mu \sigma^\nu - \bar{\sigma}_\nu \sigma^\mu).
\]

Note again that \( D \) transforms as a four-divergence, so that the integral of \( D \) is a supersymmetric invariant. Let us check the commutation relations of the algebra:
\[
\begin{align*}
\delta_1 \delta_2 D &= \partial_\mu \delta_1 \lambda^\dagger \sigma^\mu \alpha_2 - \alpha^\dagger_2 \sigma^\mu \partial_\mu \delta_1 \lambda, \\
&= \frac{1}{2} (\alpha^\dagger_1 \sigma^\mu \alpha_2 - \alpha^\dagger_2 \sigma^\mu \alpha_1) \partial_\mu D - \\
&\quad - \frac{i}{4} \left( (\sigma^{\rho\sigma} \alpha_1)^\dagger \sigma^\mu \alpha_2 - \alpha^\dagger_2 \sigma^\mu \sigma^{\rho\sigma} \alpha_1 \right) \partial_\mu F_{\rho\sigma}.
\end{align*}
\]

The use of
\[
\sigma^{\rho\sigma} = -\bar{\sigma}^{\rho\sigma} = -\frac{1}{2} (\sigma^{\rho\sigma} \bar{\sigma}^\sigma - \bar{\sigma}^{\rho\sigma})
\]
and of the identity
\[
\sigma^\mu \sigma^{\rho\tau} = -i \epsilon^{\mu\rho\tau} \delta_\sigma + g^{\mu\rho} \sigma^\tau - g^{\mu\tau} \sigma^\rho,
\]

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and its conjugate, leads us to the equation
\[
[\delta_1, \delta_2] D = \alpha_1^\dagger \sigma^\mu \alpha_2 \partial_\mu D
+ \frac{i}{4} \alpha_1^\dagger (\sigma^{\rho \tau} \sigma^\mu + \sigma^\mu \sigma^{\rho \tau}) \alpha_2 \partial_\mu F_{\rho \tau} - (1 \leftrightarrow 2) ,
\]
whence
\[
[\delta_1, \delta_2] D = (\alpha_1^\dagger \sigma^\mu \alpha_2 - \alpha_2^\dagger \sigma^\mu \alpha_1) \partial_\mu D
+ \frac{1}{2} (\alpha_1^\dagger \sigma^\lambda \alpha_2 - \alpha_2^\dagger \sigma^\lambda \alpha_1) \epsilon^{\mu \rho \tau \lambda} \partial_\mu F_{\rho \tau} .
\]
The last term vanishes because of the Bianchi identity. (What if it did not? Any implications for the monopole?) Similarly, we compute
\[
[\delta_1, \delta_2] A_\mu = -i \delta_1 \lambda^\dagger \sigma_\mu \alpha_2 - i \alpha_2^\dagger \sigma_\mu \delta_1 \lambda - (1 \leftrightarrow 2),
= \frac{1}{4} \alpha_1^\dagger (\overline{\sigma}^{\rho \tau} \sigma_\mu - \sigma_\mu \sigma^{\rho \tau}) \alpha_2 F_{\rho \tau} - (1 \leftrightarrow 2) ,
= (\alpha_1^\dagger \sigma^\rho \alpha_2 - \alpha_2^\dagger \sigma^\rho \alpha_1) F_{\rho \mu} ,
\]
skipping over several algebraic steps. The right hand side contains the desired term, namely \( \partial_\rho A_\mu \), but it also contains \(-\partial_\mu A_\rho \); clearly it could not be otherwise from the transformation laws: their right-hand side is manifestly gauge invariant, which \( \delta A_\mu \) certainly is not. Indeed our result can be rewritten in the form
\[
[\delta_1, \delta_2] A_\mu = (\alpha_1^\dagger \sigma^\rho \alpha_2 - \alpha_2^\dagger \sigma^\rho \alpha_1) \partial_\rho A_\mu - \partial_\mu \Sigma ,
\]
where the field dependent gauge function is
\[
\Sigma = (\alpha_1^\dagger \sigma^\rho \alpha_2 - \alpha_2^\dagger \sigma^\rho \alpha_1) A_\rho .
\]
This shows clearly that a supersymmetry transformation (in this form) is accompanied by a gauge transformation. It also means that the description of the gauge multiplet we have just presented is not gauge invariant, but rather in a specific gauge; this gauge is called the Wess-Zumino gauge. It is possible to eliminate the gauge transformation in the commutator of two supersymmetries by introducing extra fields which are needed for a gauge invariant description. We leave it as an exercise to derive the full gauge invariant set of fields. These fields can be neatly assembled in a real superfield, which is not gauge invariant, but undergoes the transformation
\[
V \rightarrow V + i(\Xi - \Xi^*) ,
\]
where \( \Xi(x, \theta) \) is a chiral superfield. This nicely connects with the remarks of the previous section. The Wess-Zumino gauge is that for which the extraneous components of the real superfield are set to zero \( A = \psi = C = 0 \).
Verify that the third commutator yields the expected result

\[ [\delta_1, \delta_2] \lambda = (\alpha_1^\dagger \sigma^\mu \alpha_2 - \alpha_2^\dagger \sigma^\mu \alpha_1) \partial_\mu \lambda(x) . \]

Generalization to the non-Abelian case is totally straightforward. The only difference is that the gaugino and auxiliary fields \( \lambda^A(x) \) and \( D^A(x) \) now transform covariantly as members of the adjoint representation. Thus the ordinary derivative acting on \( \lambda^A(x) \) has to be replaced by the covariant derivative

\[(D_\mu \lambda)^A = \partial_\mu \lambda^A + ig(T^C)^A_B \lambda^B ,\]

where the representation matrices are expressed in terms of the structure functions of the algebra through

\[(T^C)^A_B = -i f^C_{AB} .\]

The non-Abelian Yang-Mills Lagrangean generalizes to

\[-\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} + \lambda^A \sigma^\mu (D_\mu \lambda)^A + \frac{1}{2} D^A D^A .\]

In the Wess-Zumino gauge, the fields of the vector supermultiplet can also be very neatly arranged in a chiral superfield which transforms as a Weyl spinor under the Lorentz group. It is given by

\[ W^A(x, \theta, \bar{\theta}) = \lambda^A(x) + \frac{1}{2} \left[ D^A(x) + \frac{i}{2} \sigma^{\mu\nu} G^A_{\mu\nu}(x) \right] \theta + \frac{1}{4} \theta^T \sigma_2 \theta \sigma^\mu \sigma_2 \partial_\mu \lambda^A(x) ,\]

where we have not shown the spinor index. Under a gauge transformation, this superfield transforms covariantly, as a member of the adjoint representation. One can also easily show that, under a supersymmetry transformation, \( W^A(x, \theta) \) does indeed transform as a chiral superfield, that is

\[ W^A(x^\mu, \theta) \rightarrow W^A(x^\mu + \alpha^\dagger \sigma^\mu \theta, \theta + \alpha) .\]

This reformulation allows us to easily build invariants out of products of this superfield. As for the Wess-Zumino multiplet, invariants are the F-term of the products of this superfield. This time, we must take care that Lorentz and gauge invariance is satisfied. In particular, the Yang-Mills Lagrange density is just

\[ \mathcal{L}_{SYM} = \int d^2 \theta(W^A)^T \sigma_2 W^A + c.c. \]

One can form another invariant

\[ \mathcal{L}_{SST} = i \int d^2 \theta(W^A)^T \sigma_2 W^A + c.c. , \]
which is the usual Yang-Mills surface term
\[ \mathcal{L}_{SST} = G^A_{\mu\nu} \tilde{G}^{A\mu\nu} - i\partial_{\mu}(\lambda^\dagger\sigma^\mu\lambda) . \]

We can make many other supersymmetric invariants; for \( SU(N) \) with \( N > 2 \), we can consider the gauge adjoint “anomaly” composite
\[ \int d^2\theta d^{ABC}(W^A)^T \sigma_2 W^B . \]
One can even build composites which transform as a self-dual antisymmetric second rank Lorentz tensor, and member of the adjoint representation of the gauge group, such as
\[ f^{ABC}(W^B)^T \sigma_2 \sigma^i W^C . \]
Some of these constructions prove useful in the context of supersymmetric dynamical models.

Finally, we can implement R-symmetry on the gauge supermultiplet, provided that
\[ W \to e^{i\beta} W , \]
which means that the gaugino itself carries one unit of R-symmetry, and the \( D \) and gauge fields have no R-number.

Chiral and Vector Supermultiplets in Interaction

Renormalizability restricts the spin of the fields to be no higher than one-half. For supersymmetry, it means that the only type of matter that can couple to the gauge supermultiplet is a collection of chiral Wess-Zumino multiplets.

Let us first consider the coupling of a Wess-Zumino multiplet to an Abelian gauge superfield. Consider the Action for a number of chiral superfields. It is clearly invariant under the global phase transformations
\[ \Phi_a(x, \theta) \to e^{i\eta_a} \Phi_a(x, \theta) , \]
as long as the \( \eta_a \) are constants, independent of the coordinates. Assume that the superpotential is invariant under these transformations. Then its invariance group is much larger. Indeed, the most general local phase transformation on these chiral superfields which leaves the superpotential invariant is
\[ \Phi_a(x, \theta) \to e^{i\eta_a \Xi(x, \theta)} \Phi_a(x, \theta) , \]
where \( \Xi(x, \theta) \) is a chiral superfield. The kinetic term, however, is no longer invariant, since
\[ \Phi^*_a(y, \theta) \Phi_a(y, \theta) \to e^{i\eta_a (\Xi(y, \theta) - \Xi^*(y, \theta))} \Phi^*_a(y, \theta) \Phi_a(y, \theta) , \]
where
\[ y_\mu = x_\mu + \frac{1}{2} \theta^\dagger \sigma_\mu \theta . \]

This is analogous to the situation in usual field theory. To restore invariance, the kinetic term is generalized by adding a real superfield, which transforms as
\[ V \rightarrow V - i(\Xi - \Xi^*) . \]

The change of the argument translates in a redefinition of \( \Lambda(x) \) and \( D(x) \) in the real superfield, and does not affect the counting of the number of degrees of freedom.

The new kinetic Action is just
\[
\int d^4 x \int d^2 \theta d^2 \bar{\theta} \sum_a \Phi_a^\dagger(y, \theta)e^{\eta_a}V(y, \theta, \theta^*) \Phi_a(y, \theta) .
\]

In the Wess-Zumino gauge, this expression reduces to
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \lambda^\dagger \sigma^\mu \partial_\mu \lambda + \frac{1}{2} D^2 \\
+ (D_\mu \varphi)^* (D^\mu \varphi)^* + \psi^\dagger \sigma^\mu D_\mu \psi + F^* F \\
+ gD_\varphi^* \varphi - 2g \lambda^T \sigma_2 \psi \varphi^* + 2g \lambda^\dagger \sigma_2 \psi^* \varphi ,
\]
with the gauge covariant derivatives
\[ D_\mu \varphi = (\partial_\mu + igA_\mu) \varphi ; \quad D_\mu \psi = (\partial_\mu + igA_\mu) \psi . \]

The last line gives new interactions, over the usual construction of gauge invariant theories, with derivatives replaced by covariant derivatives. The reason is that the new interaction terms created in this way, all proportional to the charge, are not supersymmetric invariants. The extra terms restore invariance under supersymmetry. However it is a bit tricky to check the invariance because we are in the Wess-Zumino gauge. This entails changes in the transformation properties of the fields of order \( g \). Let us give some examples.

Consider the variation of the interaction of the fermion current with the gauge potential. We find
\[
\delta \left( ig \psi^\dagger \sigma^\mu \psi A_\mu \right) = ig \psi^\dagger \sigma^\mu \alpha FA_\mu + g \psi^\dagger \sigma^\mu \psi (\lambda^\dagger \sigma_\mu \alpha + \alpha^\dagger \sigma_\mu \lambda) .
\]
To offset the last term we need the variation
\[-2g \lambda^T \sigma_2 \psi \delta \varphi^* = -g \alpha^\dagger \sigma_\mu \lambda \psi^\dagger \sigma^\mu \psi .
\]
By the same token, the variation
\[-2g \delta \lambda^T \sigma_2 \psi \varphi^* = -g D_\alpha^T \sigma_2 \psi \varphi^* + \cdots ,
\]
is compensated by
\[ gD_\varphi^* \delta \varphi = gD_\varphi^* \alpha^T \sigma_2 \psi . \]
This procedure goes on *ad nauseam*. The alert student may have noticed the presence of a term proportional to $F$. The only way to compensate for it is to add a term in the variation of $F$ itself. The extra variation
\[ \delta_{WZ}F^* = -ig\psi^\dagger\sigma^\mu\alpha A_\mu , \]
does the job. Its effect is to replace the derivative by the covariant derivative in the transformation law, which we do for all of them. Even then we are not finished: we still have one stray term proportional to $F$. Indeed we have
\[ -2g\lambda T^\sigma_2\delta\psi\varphi^* = -2gF\lambda T^\sigma_2\alpha\varphi^* + \cdots , \]
which can only cancelled by adding a term in the variation of $F$, yielding
\[ \delta_{WZ}F^* = -ig\psi^\dagger\sigma^\mu\alpha A_\mu - 2g\alpha^\dagger\sigma_2\lambda^*\varphi . \]
You have my word that it is the last change, but to the non-believer, I leave the full verification of the modified supersymmetric algebra in the Wess-Zumino gauge as an exercise during half-time.

This Lagrangian is of course does not lead to a satisfactory quantum theory because of the ABJ anomaly associated with the $U(1)$; it can be cancelled by introducing another chiral superfield with opposite charge. Then the extra terms beyond the covariant derivatives read
\[ gD(\varphi_1^\dagger \varphi_1 - \varphi_2^\dagger \varphi_2) - \left(2g\lambda T^\sigma_2(\psi_1^* \varphi_1^* - \psi_2^* \varphi_2^*) + \text{c.c.}\right) . \]
From the equations of motion, the value of the auxiliary field is
\[ D = -g(\varphi_1^\dagger \varphi_1 - \varphi_2^\dagger \varphi_2) , \]
yielding the potential
\[ V = \frac{g^2}{2}(\varphi_1^\dagger \varphi_1 - \varphi_2^\dagger \varphi_2)^2 . \]

Generalization to the non-Abelian case is straightforward. We merely quote the results for a chiral matter superfield transforming as a representation $r$ of the gauge group. The derivatives on the matter fields $\psi_a$ and $\varphi_a$ are replaced by the covariant derivatives
\[ D_\mu = \partial_\mu + igT^B A^B_\mu , \]
where $T^B$ represent the gauge algebra in the representation of the chiral superfield. The auxiliary fields $D^A(x)$ now couple through the term
\[ gD^A \varphi^\dagger_a (T^A)_a^b \varphi_b , \]
and the gauginos by the terms
\[ -2g\varphi^\dagger a (T^A)_a^b \psi_b^T \sigma_2 \lambda^A + 2g\lambda^A\sigma_2 \psi^*_a (T^A)_a^b \varphi^a , \]
where we have shown the internal group indices (but not the spinor indices).

To conclude this section, we note that the gauge coupling preserves R-symmetry, irrespective of the R-value of the chiral superfield. Having assembled all the pieces necessary for the generalization of the $N=0$ standard model to $N=1$, we are ready for its description.

III-) THE SUPERSYMMETRIC STANDARD MODEL

The $N=0$ standard model is easily made supersymmetric. Its Weyl fermions are put in chiral Wess-Zumino multiplets, its gauge bosons now form vector supermultiplets, and the Higgs boson is part of a chiral multiplet. We note that an odd number of Weyl fermions cannot be implemented if there is more than one supersymmetry, which is the reason we focus on $N=1$.

The alert among you has noticed that the left-handed lepton doublets and the Higgs doublet have the same gauged electroweak quantum numbers, although the lepton doublets have one unit of lepton number while the Higgs has none. Also that there is only one Higgs doublet and three lepton doublets. Can we build a model where the Higgs is the superpartner of a lepton doublet, using R symmetry as a compensator for lepton number? The student is encouraged to try models in this direction.

The most “economical” way of introducing N=1 supersymmetry is to associate with every spin 1/2 particle a chiral superfield,

$$ L \to \Phi_L, \quad Q \to \Phi_Q, \quad \bar{u} \to \Phi_{\bar{u}}, \quad \bar{d} \to \Phi_{\bar{d}}, \quad \bar{e} \to \Phi_{\bar{e}}; $$

this procedure associates to each fermion a scalar partner of sfermions with the same electroweak quantum number

$$ \Phi_L : \left( \begin{array}{c} \nu_L \\ e_L \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} \tilde{\nu}_L \\ \tilde{e}_L \end{array} \right) \quad \text{(slepton)}, $$

$$ \Phi_{\bar{e}} : \bar{e}_L \quad \text{and} \quad \tilde{e}_R \quad \text{(antislepton)}, $$

$$ \Phi_Q : \left( \begin{array}{c} u_L \\ d_L \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} \tilde{u}_L \\ \tilde{d}_L \end{array} \right) \quad \text{(squark)}, $$

$$ \Phi_{\bar{u}} : \bar{u}_L \quad \text{and} \quad \tilde{u}_R \quad \text{(antisquark)}, $$

$$ \Phi_{\bar{d}} : \bar{d}_L \quad \text{and} \quad \tilde{d}_R \quad \text{(antisquark)}. $$

The Higgs doublet of the Standard Model is interpreted as the scalar component of a new chiral superfield. This introduces a left-handed doublet of Weyl fermions, the Higgsinos. However, we cannot stop here, because this Higgsino doublet makes the hypercharge anomalous in two different ways. One type of anomaly is the triangle anomaly; the second is Witten’s global anomaly, which says that any theory with an odd number of half-integer spin representations of $SU(2)$, path-integrates to zero. Thus we had better do something about it. Both problems are solved by postulating the existence of another doublet of Higgsinos, which is the vector-like completion of the first with opposite hypercharge (There
are ways to chirally cancel anomalies, but they lead to much more complicated theories, which do not concern us here). We thus have two chiral superfields in the \( N = 1 \) model:

\[
\Phi_{H_d} : \left( \begin{array}{c} \varphi^0 \\ \varphi^- \\ \varphi_L \\ \varphi_L^0 \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} \varphi^0 \\ \varphi^+ \\ \varphi^+ \\ \varphi^+ \\ \varphi^+ \end{array} \right) \quad \text{(Higgsino)} ,
\]

\[
\Phi_{H_u} : \left( \begin{array}{c} \varphi^0 \\ \varphi^+ \\ \varphi^+ \\ \varphi^0 \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} \varphi^0 \\ \varphi^+ \\ \varphi^+ \\ \varphi^0 \end{array} \right) \quad \text{(Higgsino)} .
\]

It is amusing that we come to the same conclusion from phenomenology: with only one Higgs superfield, we cannot give masses to both charge 2/3 and charge -1/3, -1 fermions. Recall that this is possible in the \( N = 0 \) standard model by using the conjugate of the Higgs field in the coupling. We have seen that supersymmetry-invariant couplings in the superpotential are analytic functions of the superfields, and do not involve their conjugates. Hence we need another Higgs doublet of opposite hyper charge, which is exactly the conclusion reached from anomaly considerations.

The Yukawa interactions of the standard model are extracted from the superpotential

\[
Y_{ij}^u \Phi_{Q_i}^T \Phi_{\tau_2} \Phi_{H_u}^j + Y_{ij}^d \Phi_{Q_i}^T \Phi_{\tau_2} \Phi_{H_d}^j + Y_{ij}^\ell \Phi_{Q_i}^T \Phi_{\tau_2} \Phi_{H_d}^j ,
\]

The indices \( i, j = 1, 2, 3 \) label the three chiral families. This cubic superpotential contains no mass term for the Higgs fields, and has also many global symmetries, some nefarious to phenomenology.

The reduction of the flavor Yukawa matrices \( Y_{ij} \) proceeds as in the \( N = 0 \) model. Without loss of generality, we can bring the lepton Yukawa to diagonal form,

\[
Y_{ij}^\ell \rightarrow Y_{ii}^\ell .
\]

We diagonalize the down Yukawa, setting

\[
Y^d = U^T_d M_d V_d , \quad (M_d \text{ diagonal}) ,
\]

and rewriting the Lagrangean in terms of the superfields

\[
\Phi_{\tau_2}^i = (V_d \Phi_{\tau_2})^i , \quad \Phi_{Q}^i = (U_d \Phi_{Q})^i .
\]

The same reduction of the up quarks Yukawa matrix yields

\[
Y^u = U^T_u M_u V_u , \quad (M_u \text{ diagonal}) ,
\]

while redefining

\[
\Phi_{\tau_2}^i = (V_u \Phi_{\tau_2})^i .
\]

The \( V_u \) matrix disappears from the Lagrangean, and we have no further freedom for \( \Phi_{Q}^i \). Thus the most we can do (after dropping the primes) is

\[
Y_{ii}^\ell \Phi_{L_i}^T \Phi_{\tau_2} \Phi_{H_d}^j + M_d^{ii} \Phi_{Q_i}^T \Phi_{\tau_2} \Phi_{H_d}^j + \Phi_{Q}^{Tj} (U^T)_{ij} M_u^{jj} \Phi_{\tau_2} \Phi_{H_u}^j ,
\]

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with the flavor mixing matrix

\[ \hat{U}^T = U_d^T U_u ; \]

it reduces to the CKM matrix, after Iwasawa decomposition, to expunge extraneous phases. Thus, the Yukawa couplings of the \( N = 0 \) and \( N = 1 \) models are exactly the same, if one allows for two Higgs of opposite hypercharge.

The global phase symmetries of this superpotential are easy to identify. In the lepton sector, we still have conservation of the relative lepton numbers. In the quark sector, no distinction between families is allowed, since the CKM matrix is different from one. Global transformations on the superfields appear as

\[ \Phi_f \rightarrow e^{in_f \eta} \Phi_f , \]

where \( f \) denotes the species: \( L, \overline{e}, \overline{\nu}, \overline{d}, \) or \( Q \). The transformations which preserve supersymmetry obey the relations

\[
\begin{align*}
    n_{L_i} + n_{\overline{e}_i} + n_{H_d} &= 0 , & i = e, \mu, \tau , \\
    n_Q + n_{\overline{\nu}} + n_{H_u} &= 0 , & \text{any flavor} , \\
    n_Q + n_{\overline{d}} + n_{H_d} &= 0 , & \text{any flavor} .
\end{align*}
\]

With only one family, there are seven fields, with seven independent phases, obeying three relations from the couplings, leaving four independent symmetries; these are easily identified to be

| \( n_L \) | \( n_{\overline{e}} \) | \( n_Q \) | \( n_{\overline{\nu}} \) | \( n_{\overline{d}} \) | \( n_{H_u} \) | \( n_{H_d} \) |
|---|---|---|---|---|---|---|
| L | 1 | -1 | 0 | 0 | 0 | 0 | 0 |
| B | 0 | 0 | 1/3 | -1/3 | -1/3 | 0 | 0 |
| Y | -1 | 2 | 1/3 | -4/3 | 2/3 | 1 | -1 |
| PQ | -1/2 | -1/2 | -1/2 | -1/2 | -1/2 | 1 | 1 |

They are: two global symmetries, total lepton number (L), and baryon number (B), one local symmetry, hypercharge (Y), and the chiral Peccei-Quinn (PQ) symmetry. With three families, there are also two conserved relative lepton numbers, \( L_e - L_\mu \) and \( L_\mu - L_\tau \). Of these, only the Peccei-Quinn symmetry does not occur in the standard model.

A special feature of supersymmetric theory is the global R-symmetry, under which

\[ \theta \rightarrow e^{i\eta} \theta \]
\[ \Phi \rightarrow e^{i2\eta/3} \Phi \quad \text{for all chiral matter superfields} . \]

The chiral spinor superfields that contain the gauge bosons transform as well

\[ W^a(x, \theta) \rightarrow e^{i\eta} W^a(x, \theta) , \]

(shown here for \( SU(2)_W \) only) so that the gauginos also transform under R

\[ \lambda(x) \rightarrow e^{i\eta} \lambda(x) . \]
This symmetry requires massless gauginos since their Majorana masses have two units of R. However, gauginos must be massive, if supersymmetry is to describe the real world, and R symmetry must be broken. We note in passing that R-symmetry is anomalous but this does not mean that the gauginos acquire arbitrary masses since they are tied by supersymmetry to the vector particles whose masses are set by gauge invariance.

The PQ symmetry also causes a problem since it is carried only by fields that transform as weak isospinors. It is well-known that this leads to a weakly coupled axion with large mass, a possibility that is experimentally ruled out. Hence this symmetry must be broken as well.

This embarrassment of symmetries is somewhat alleviated when it is realized that with the minimal set of fields of the $N = 1$ model, we can add to the superpotential the so-called $\mu$-term

$$\mu \Phi^T_{H_u} \tau_2 \Phi_{H_d},$$

without violating supersymmetry. It introduces in the model a mass term $\mu$, but breaks both PQ and R symmetries, leaving the linear combination

$$R' = R + \frac{1}{3} PQ,$$

invariant, since it has $\Delta PQ = 2$, and $\Delta R = -2/3$.

This symmetry still keeps the gauginos massless. It is anomalous, and explicitly broken by QCD. Both Higgs superfields and the gauginos carry one unit of $R'$, while the other chiral superfields carry half a unit. When $H_u$ and $H_d$ get their electroweak breaking values, it is spontaneously broken, leading to the unacceptable visible axion model. The terms we have introduced in the superpotential define the minimal $N = 1$ standard model.

With the chiral superfields of the standard model, we can add to the superpotential other renormalizable terms which are invariant under both supersymmetry and the gauge groups $SU(3) \times SU(2) \times U(1)$. They are, suppressing all family indices,

$$\Phi_d \Phi_d \Phi_h; \quad \Phi_Q \Phi_d \Phi_L; \quad \Phi_L \Phi_L \Phi_e; \quad \Phi_L \Phi_{H_u}.$$

The first term violates quark number by three units, and the others lepton number by one unit. These new terms are allowed by supersymmetry, but in view of the the excellent experimental limits on both baryon and lepton numbers, they should appear with tiny coefficients, if at all. All violate $R'$ symmetry (mod 4), since they have $R' = 3/2$, which does not leave any discrete remnant. In addition, baryon number is broken only (mod 3), leaving behind the discrete group $Z_3$. There is also a parity under which all weak doublets are odd, all singlets even, but this is a consequence of invariance under the weak $SU(2)$.

If $N = 1$ supersymmetric models are to describe the real world, they must include mechanisms which break supersymmetry, break the electroweak symmetry, and break $R'$ symmetry.

First we remark that breaking of supersymmetry necessarily generates a mass for the gauginos, in order to create a mass gap with the massless gauge bosons. This automatically
breaks $R'$ symmetry, but in a specific way, leaving behind a discrete symmetry. In the minimal $N = 1$ standard model this symmetry is $Z_4$, under which we have

$$\Phi_f \rightarrow i\Phi_f; \quad \Phi_{H_{u,d}} \rightarrow -\Phi_{H_{u,d}}.$$  

In addition, the gauginos are odd under this symmetry. Since it is an $R$ symmetry, particles and their superpartners do not have the same multiplicative quantum number. The Higgs scalar doublets are odd under this 4-fold symmetry. It would seem that electroweak breaking would break it down to $Z_2$, creating potential domain wall problems, but some of this symmetry can be expressed in terms of hypercharge, baryon number and lepton number, which means that the only extra symmetry is $Z_2$, which is $R$-parity. $R$ parity is an exact symmetry of the minimal $N = 1$ standard model. It is easy to see that all quarks, leptons, and Higgs bosons are even under $R$-parity; all their superpartners are odd.

It has the important consequence that superpartners can only decay into an odd number of lighter superpartners. Thus, the lightest of these, the lightest superpartner (LSP) must be stable. While two heavy to have been produced in the laboratory, many believe that the LSP pervades the universe as a stable remnant of the cosmological soup; it might just be what dark matter is made of.

In the non-minimal model, there is no $R$-parity, and thus no stable particle, although in some models the LSP could be long-lived.

As we have emphasized, like turtles which carry their own houses, supersymmetric theories contain their own potential. Thus it is natural to ask if the potential in the $N = 1$ standard model is capable of the heroic deed of breaking any of the above symmetries. For that purpose, we must first discuss supersymmetry breaking.

**IV-) SUPERSYMMETRY BREAKING**

It is time to understand how to break supersymmetry. We disregard hard breaking, since one of the rationale for supersymmetry is to tame quantum corrections. We may then consider two different types of breaking which do not alter the ultraviolet properties of the theory.

One is soft breaking, with the symmetry broken by adding to the theory terms of dimension 2 and 3. Intuitively, they do not affect the theory in the limit where all masses are taken to zero, relative to the scale of interest. The most direct way is to give the superpartners of the massless chiral fermions a mass. This can be done without breaking electroweak symmetry. Also we can give each Higgs doublet a supersymmetry-breaking mass, and finally we can put in gaugino mass terms. This clearly splits the mass degeneracy between the particles within a supermultiplet. This is exactly like adding the effect of the quark masses in the chiral Lagrangian. Finally, there are additional possible terms of dimension three between sparticles and Higgs particles as well. Of these terms, the gaugino Majorana masses,

$$M_i \lambda_i^T \sigma_2 \lambda_i,$$
break $R'$ by two units, leaving the minimal model with an unbroken discrete subgroup, R-parity. Soft breaking is not fundamental, rather an effective manifestation of symmetry breaking.

Another way is spontaneous breaking of supersymmetry, which we now discuss. A symmetry is spontaneously broken if the field configuration which yields minimum energy no longer sustains the transformation under that symmetry. Let us remind ourselves how it works for a garden variety symmetry. The simplest is when the order parameter is a complex field $\varphi(x)$ with dynamics invariant under the following transformation

$$
\delta \varphi(x) = e^{i\beta} \varphi(x) .
$$

Now suppose that in the lowest energy configuration, this field has a constant value

$$
< \varphi(x) >_0 = v .
$$

Expanding $\varphi(x)$ away from this vacuum configuration, setting

$$
\varphi(x) = e^{i\eta(x)}(v + \rho(x)) ,
$$

we find that under the transformation, the angle $\eta(x)$ undergoes a simple shift

$$
\eta(x) \rightarrow \eta(x) + \delta ,
$$

meaning that the dynamics is invariant under that shift. Geometrically, this variable is the angle which parametrizes the closed line of minima. The dynamical variable associated with this angle is identified with the massless Nambu-Goldstone boson, $\zeta(x)$, divided by the vacuum value. It couples to the rest of the physical system as

$$
L_{NG} = \frac{1}{v} \zeta(x) \partial_{\mu} J^{\mu} ,
$$

where $J^{\mu}(x)$ is the Noether current of the broken symmetry. Clearly, a constant shift in $\zeta$ generates a surface term and leaves the Action invariant.

Let us apply this acquired wisdom to the supersymmetric case, starting with the chiral superfield. In a constant field configuration, the supersymmetry algebra reads

$$
\delta \varphi_0 = \alpha^T \sigma_2 \psi_0 ,
\delta \psi_0 = \alpha F_0 ,
\delta F_0 = 0 .
$$

Any non-zero value of $\psi_0$ breaks both supersymmetry and Lorentz invariance. Since we are only interested in Lorentz-invariant vacua, we set $\psi_0 = 0$, obtaining the only Lorentz invariant possibility

$$
\delta \varphi_0 = 0, \ \delta \psi_0 = \alpha F_0, \ \delta F_0 = 0 ;
$$
with $\varphi_0 \neq 0$ and $F_0 \neq 0$. The only way this configuration can break the supersymmetry is to require that

$$F_0 \neq 0 : \text{ broken supersymmetry }.$$ 

Since $F$ is a function of the scalar fields, it means that some $\varphi_0 \neq 0$. It must be noted that when $F_0 = 0$, and $\varphi_0 \neq 0$, any internal symmetry carried by $\varphi_0$ is broken. This fits nicely with our earlier remarks because a non-zero value for $F$ gives the potential a positive minimum.

When $F_0 \neq 0$, the chiral fermion shifts under supersymmetry: it is the Nambu-Goldstone fermion associated with the breakdown of supersymmetry, as expected, since the broken symmetric is fermionic. It often goes under the name Goldstino, although I would prefer, for historical reasons, to call it Nambino.

A similar analysis carries to the vector multiplet. There, the only vacuum configuration which does not break Lorentz invariance, is that where $A_\mu$ and $\lambda$ vanish in the vacuum, for which we have

$$\delta A_\mu^0 = 0 \, , \, \delta \lambda_0 = \alpha D_0 \, , \, \delta D_0 = 0 .$$

It is clear that the only way to break supersymmetry is to give $D_0$ a vacuum value, and in this case, it is the gaugino $\lambda$ that plays the role of the Nambino (Goldstino).

Thus, as long as we have chiral and vector superfields, the spontaneous breakdown of supersymmetry comes about when the dynamics is such that either $F$ or $D$ is non-zero in the vacuum. Another way of arriving at the same conclusion is to note that the potential from these theories is given by

$$V = F_i^* F_i + \frac{1}{2} D^2 ,$$

when $F_i$ and $D$ take on their values obtained from the equations of motion. Since $V$ is the sum of positive definite quantities it never becomes negative and if supersymmetry is spontaneously broken, its value at minimum is non-zero.

It is possible to formulate a general argument based on the fundamental anticommutation relations. In theories with exact supersymmetry, the vacuum state is annihilated by the generators of supersymmetry. However, the square of the same supersymmetry generators is nothing but the energy: the energy of the supersymmetric ground state is necessarily zero. Since it is also the state of lowest energy, it follows that the potential is necessarily positive definite. This is what we have just seen above.

Now suppose that supersymmetry is spontaneously broken. This requires that the action of supersymmetry on the vacuum is not zero, and therefore that the vacuum energy be positive. Comparing with the form of the potential, this can happen only if $F$ and/or $D$ is non-zero.

We can now examine the potential of the minimal $N = 1$ standard model. The $F$ terms couple in the following way
The potential coming from these terms is just the sum of the absolute values squared of the terms which multiply each $F$. The expression that results is pretty complicated, but it is not over, as we still have to get the contribution from the $D$-terms. There are three types of $D$-terms, corresponding to each of the gauge groups

$$U(1): D = \frac{1}{2}g_1[-\tilde{L}^\dagger_{Li} \tilde{L}_{Li} + 2\tilde{e}^*_{Ri} \tilde{e}_{Ri} + \frac{1}{3} \tilde{Q}^\dagger_{Li} \tilde{Q}_{Li} + \frac{4}{3} \tilde{u}^\dagger_{Ri} \tilde{u}_{Ri} - \frac{2}{3} \tilde{d}^\dagger_{Ri} \tilde{d}_{Ri} + H_{u}^{\dagger}H_{u} - H_{d}^{\dagger}H_{d}]$$

$$SU(2): D^a = g_2[\tilde{L}^\dagger_{Li} \tau^a_{Li} \tilde{L}_{Li} + \tilde{Q}^\dagger_{Li} \tau^a_{Li} \tilde{Q}_{Li} + H_{u}^{\dagger}\tau^a_{2}H_{u} + H_{d}^{\dagger}\tau^a_{2}H_{d}]$$

$$SU(3): D^A = g_3[\tilde{Q}^\dagger_{Li} \lambda^A_{Li} \tilde{Q}_{Li} + \tilde{u}^\dagger_{Ri} \lambda^A_{Ri} \tilde{u}_{Ri} + \tilde{d}^\dagger_{Ri} \lambda^A_{Ri} \tilde{d}_{Ri}]$$

giving to the potential the contribution

$$(D^2 + D^a D^a + D^A D^A) .$$

We parenthetically remark that without the $\mu$ term, this potential is purely quartic. The $\mu$ term gives an equal mass to the Higgs and the Higgsinos, and also creates cubic couplings among the Higgs and sleptons and squarks.

It would be too much to hope for this potential to break both electroweak and supersymmetry. Since it is the sum of squares, its minimum, if allowed, occurs when all the auxiliary fields are set to zero. Clearly, there is one solution when all the fields are set to zero. This solution breaks no symmetry, and thus has the lowest energy. So, the best we get from this potential is to ask if there are other degenerate vacua with desirable electroweak breaking features.

To see if it is possible, we work out the value of the potential at the required electroweak breaking field configuration. We set

$$H_u = \frac{v_u}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad H_d = \frac{v_d}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

and evaluate the $F$ and $D$ terms, all other fields being zero (unless we want $L$-violation as the only charge conserving possibility is $\tilde{\nu}_L \neq 0$ which violates $L$ spontaneously). From
we find that, in the electroweak vacuum, there are non vanishing D and F fields, namely

\[ D = \frac{1}{4} g_1 (|v_u|^2 - |v_d|^2) , \]
\[ D^3 = -\frac{1}{4} g_2 (|v_u|^2 - |v_d|^2) , \]
\[ F_{H_u} = \mu \tau_2 \frac{v_d}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \]
\[ F_{H_d} = -\mu \tau_2 \frac{v_u}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \]

We conclude that the electroweak vacuum configuration is not a minimum of this potential. Thus more physics has to be added to this minimal model, in the form of mechanisms for both supersymmetry and electroweak symmetry breakings.

There are many models of spontaneous supersymmetry breaking in the context of renormalizable theories. Models with F breaking were first investigated by O’Raifeartaigh. Their general feature is that they have a global R-like symmetry, and they preserve the sum rule

\[ \text{Str} M^2 \equiv \sum_{J=0,1/2} (-1)^{2J} (2J + 1) m_J^2 = 0 . \]

The other type of breaking is D-breaking, or Fayet-Iliopoulos breaking. It requires a local \( U(1) \) symmetry, to allow for a gauge singlet term linear in the D associated with the gauge supermultiplet to the Lagrangian. The sum rule is modified to read

\[ \text{Str} M^2 = g D \text{Tr} \sum_i q_i , \]

where \( q_i \) are the charges of the Weyl fermions. If the anomaly of this \( U(1) \) is cancelled in a vector-like way, the right-hand side is zero.

In both cases, these sum rules cause phenomenological problems, although they can be modified by quantum corrections. For that reason, such models have not proved easy to implement. It is a good thing that these sum rules are modified when supersymmetry is extended to supergravity. Supergravity, of which I say little, is the theory of local supersymmetry. It generalizes gravity, and must be present in a supersymmetric world. It also has the advantage of eating the massless Nambino, when the spin 3/2 gravitino gets a mass from supersymmetry breaking.

Let us conclude this short survey with a discussion of dynamical supersymmetry breaking. Explicit renormalizable models of F- and D-type breakings of global supersymmetry
have appeared in the literature. It has proven much more difficult to produce models of dynamical breaking of global supersymmetry.

By multiplying two chiral superfields, we obtain a composite superfield with components

$$\varphi_1 \varphi_2, \psi_1 \varphi_2 + \psi_2 \varphi_1, \varphi_1 F_2 + \varphi_2 F_1 - \psi_1^T \sigma_2 \psi_2.$$ 

It would appear that its F-term could acquire a non-zero vacuum value, if the fermions were subject to a strong force, which, in analogy with chiral symmetry, would cause a condensate like $\psi_1^T \sigma_2 \psi_2$ to form. This would break supersymmetry. In the Standard Model, such condensates occur as a result of QCD. Hence if these fields were like quarks and antiquarks, supersymmetry could be broken dynamically when quarks condense.

On the other hand, we do not expect a gaugino condensate to break global supersymmetry, since it is not part of an F component of a chiral composite. Indeed the gaugino condensate appears as the scalar term of $(W^A)^T \sigma_2 W^A$.

Thus we are led to consider a theory with a Non-Abelian gauge supermultiplet, in interaction with a number of chiral superfields. Naive expectations is that the strong force will cause both gauginos and matter fermions to form chiral condensates, and the matter condensates will dynamically break supersymmetry. However the situation is not at all that simple.

First of all, in the absence of matter, gaugino condensation occurs, and, as expected, supersymmetry is not broken. Secondly, with chiral matter, supersymmetry is not necessarily broken dynamically. If the matter chiral multiplets have a common mass, supersymmetry is not broken dynamically, even with strong coupling. Only when the matter is massless can supersymmetry be broken dynamically, but then the lowest energy configuration usually corresponds to infinite field values, except in some very special, and more complicated models.

A very important tool in the study of dynamical breaking of global supersymmetry is the Witten index. We have seen, that because of the supersymmetry algebra, it is easy to determine the breaking of supersymmetry: supersymmetry is broken if and only if the state of lowest energy is positive.

In supersymmetry theories, the potential is the sum of squares of F and D terms, and it is positive definite. If the potential at minimum is positive, it means that either the F and/or D terms have non-zero values, and supersymmetry is broken.

Witten considers a supersymmetric theory in a finite volume V. To preserve the translation symmetry he imposes periodic boundary conditions, and examines the vacuum energy $E(V)$. He argues that if $E(V) = 0$ for finite $V$, it will remain so as the infinite volume limit is taken.

The Hamiltonian will have discrete eigenvalues. Its spectrum is made up of two types of states, boson and fermions. They are distinguished by the value of the operator $e^{i\pi J_z}$. 

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which has value 1 on bosons, and $-1$ of fermions. He observes that states of finite energy always come in degenerate boson fermion pairs. This is a result of the algebra, which states that the supersymmetry generator is the square root of the Hamiltonian.

The situation is entirely different for the zero energy states, since the application of the supersymmetry generator to any zero energy state does not produce another state, since the energy is zero. There need not be the same number of bosonic and fermionic states of zero energy. Let there be $n_b$ bosonic and $n_f$ fermionic states of zero energy.

Now let us assume that the system undergoes adiabatic changes, such as changes of couplings, and other parameters. The occupation number of states at a given energy level will change, and so will the energy eigenvalues. However, to preserve supersymmetry, the states will migrate in pairs from one level to the next. For instance, one positive energy pair may migrate into the zero energy state. In that case, both $n_b$ and $n_f$ increase by one. Alternatively, two states in the zero energy state may migrate into a state of positive energy. In this case, both $n_b$ and $n_f$ decrease by one unit. However in both cases, the difference

$$\Delta \equiv n_b - n_f$$

is left unchanged. It is very insensitive to most changes in the system, and thus can be computed more easily, for instance in the perturbative regime, where it is easier to calculate.

What use is this difference? Suppose it is different from zero. Then, necessarily $n_b$ and/or $n_f$ is itself different from zero, indicating that the zero energy state is occupied: supersymmetry is unbroken.

On the other hand, if $\Delta = 0$, it may mean one of two thing: either $n_b = n_f \neq 0$, in which case supersymmetry is unbroken, or $n_b = n_f = 0$, which indicates that there are no states of zero energy and the breaking of supersymmetry.

The idea is to compute $\Delta$ for a value of parameters which lends itself to calculability. If it is not zero, supersymmetry is not broken. If it is zero, one cannot say anything.

The Witten index can be computed for a super pure Yang-Mills theory, where it is found to be equal to the rank of the group plus one. Thus, as expected, supersymmetry is not broken.

It can also be computed when massive chiral matter is added. Again, it is found to be non-zero. However, when the mass of the chiral superfields is taken to zero, computation of the Witten index ceases to be trustworthy, because the potential no longer favors small field values.

Before leaving this topic, we should mention that the situation is thought to be quite different in the case of local supersymmetry. There, the gaugino condensate is capable of breaking supersymmetry. It is a favorite scenario of superenthusiasts to believe that this is what happens in nature: a strong QCD-like force causes gauginos to condense. This breaks supersymmetry. This strong force operates in the hidden sector, a sector of the theory that is connected with ours only by the universal force of gravity. This is technicolor in the hidden sector! Thus supersymmetry breaking appears in our phenomenological theories through a universal mechanism, given in terms of soft breaking parameters. No model in
which this actually happens has been formulated, but it is sociologically true. Next, we discuss the effective soft breaking of supersymmetry this phenomenon is believed to cause in our sector.

Thus supersymmetry breaking is added to the $N = 1$ standard model in the form of soft terms. The incredible thing is that this simple hypothesis triggers spontaneous breaking of the electroweak symmetry!

In order to appreciate the rationale behind such a picture, it is useful to reason by analogy with low energy chiral symmetry, which although an approximate symmetry of nature, has proved to be very important in the analysis of low energy strong interactions.

Assume for a moment that the energy available to your machines is below that of a pion (yes there was such a time!). Perhaps someone had postulated back then that the chiral symmetry limit is an interesting limit in which to study the Strong Interactions. In this picture, the chiral symmetry is spontaneously broken, and the nucleons are supplemented by massless pions. It also predicts, in the form of low energy theorems, the couplings of pion to matter. But massless pions have not been seen, and this sounds like a pretty weird thing to do. Clearly chiral symmetry must be broken, but how?

Contrast the situation to the point of view I have conveyed in these lectures: it makes sense to generalize the $N = 0$ Standard Model to $N = 1$ supersymmetry. This means the invention of squarks, sleptons, gauginos, etc..., together with predictions of their couplings with ordinary none of which have been seen. Clearly supersymmetry must be broken, but how?

Chiral symmetry: The pion is found, and it is light in terms of strong interactions; it means that the picture of approximate chiral symmetry makes sense.

Supersymmetry: a gluino is found, and the whole thing makes sense.

Chiral symmetry: there is more than one pion, and theorists postulate that chiral symmetry breaking appears in the form of soft terms, with definite symmetry characteristics. These are in turns used to derive sum rules relating the masses of the pseudoscalars.

Supersymmetry: Theorists assume soft supersymmetry breaking with definite symmetry characteristics. A simple assumption is universality of the breaking; it happens to be one of the most economical ways to introduce the soft breaking. Theorists deduce sum rules. This means that the form of the supersymmetry breaking may some day be determined by experiment, but only after as many measurements as the number of soft breaking parameters.

Chiral symmetry: The meaning of the soft chiral symmetry breaking is now easily understood in terms of the underlying theory: QCD; it has the same quantum numbers as the quark mass terms in the Lagrangian.

Supersymmetry: one expects that it is experiment which will eventually determine the form of the soft supersymmetry breaking parameters. If the breaking is found to be universal, this will be a strong indication for the hidden sector scenario. Need we say that it occurs most naturally in the Heterotic String theory?

This analogy is not perfect. For one chiral symmetry is a global symmetry, while we expect supersymmetry to be a local symmetry. This is a crucial difference, but the role of
effective soft breaking terms is similar, in the sense that they are manifestations of deeper theory in both cases, and are used for phenomenology in the same way.

In the universal breaking picture, all the squarks, sleptons, and Higgs are given a common supersymmetry breaking mass $m_0$. The three gauginos are also given masses $M_i$. They need not all be the same, without extra assumptions. If the mother theory is grand unified, then it is natural to take all three masses to be the same. This is also true of some string theories. One also implements the soft breaking mass term of the Higgs with a parameter $B$. Finally, terms of dimension three appear as sparticle-sparticle-Higgs interactions of the same symmetry character as the Yukawa couplings. All these parameters are supposed to have values in the hundreds of GeV range, reflecting the strength of supersymmetry breaking.

The soft breaking parameters appear as boundary conditions in the renormalization group equations that govern the running of the same parameters. The scale at which they are specified is assumed to be in the deep ultraviolet, near or at Planck scale.

Several remarkable things are seen to happen. First of all, the square of the mass of the Higgs that couples to the up quarks, starting from its ultraviolet value, is seen to become negative in the infrared. Amazingly, the evolution equations are such that it is the Higgs that becomes tachionic, indicating the spontaneous breaking of electroweak symmetry. This is possible only because of the large value of the top quark mass. I have no time to cover this beautiful development in these introductory lectures, but you should be left with the appropriate sense of awe.

Fortunately for you, there is a lot of work yet to be done. I believe that the most important question of deep theoretical interest is dynamical supersymmetry breaking. I look forward someday to hear that one of you has actually solved its mechanism.

I wish to thank Professors J. Donoghue and K.T. Mahanthappa for their kind hospitality during my stay at the 1994 TASI. I also wish to thank the students for their keen interest and challenging questions. To some I apologize for restoring the $\sigma_2$, but I could not cope with that many indices. This work was supported in part by the United States Department of Energy under Contract No. DEFG05-86-ER-40272.

I have not included references in the text, since the lectures are quite introductory. Rather I draw your attention to several excellent elementary books on the subject, as well as to reviews and reviews of reviews. These are

J. Bagger and J. Wess, *Supersymmetry and Supergravity*, Princeton University Press, second edition (1993).

P. West *Introduction to Supersymmetry and Supergravity*, World Scientific, Singapore (1990).

*Supersymmetry and Supergravity*, a collection of Physics Reports, edited by M. Jacob (1986).