Lifting U-Dualities

K. Koepsell and F. Roose

Department of Mathematics
King’s College London
Strand, London WC2R 2LS, U.K.

Abstract

We present a novel global $E_{7(7)}$ symmetry in five-dimensional maximal supergravity as well as an $E_{8(8)}$ symmetry in $d=4$. These symmetry groups which are known to be present after reduction to $d=4$ and $d=3$, respectively, appear as conformal extensions of the respective well-known hidden-symmetry groups. A global scaling symmetry of the Lagrangian is the key to enhancement of $E_{6(6)}$ to $E_{7(7)}$ in $d=5$ and $E_{7(7)}$ to $E_{8(8)}$ in $d=4$. The group action on the physical fields is induced by conformal transformations in auxiliary spaces $M$ of dimensions 27 and 56, respectively. The construction is analogous to the one where the conformal group of Minkowski space acts on the boundary of AdS$_5$ space. A geometrical picture underlying the action of these “conformal duality groups” is given.
1 Introduction

In the late 70s, extended supergravity theories in $d$ dimensions, ($d \leq 9$) were demonstrated to exhibit certain continuous global symmetries, termed ‘hidden symmetries’ at the time [1]. More specifically, they were found as invariances of the equations of motion supplemented with the Bianchi identities. It was not until ’94 that Hull and Townsend conjectured a discrete version of these symmetry groups to persist as exact symmetries of (compactified) type II string theories [2]. These discrete groups have become known since as ‘U-duality groups’.

Compactification on a $d$-torus relates the maximally-extended supergravity in $11-d$ dimensions to the unique eleven-dimensional one [3]. The hidden-symmetry group in $1 \leq 11-d \leq 9$ dimensions is observed to be $E_{d(d)}^2$.

Despite this observation, it is a non-trivial question whether global symmetries that become manifest only in lower-dimensional theories might be secretly present nonetheless in higher dimensions. Indeed, in Refs. [5, 6], eleven-dimensional supergravity was reformulated such that the space-time tangent-space $SO(1,10)$ symmetry was being replaced by the local symmetries that otherwise would become manifest only upon reduction to four and three dimensions, respectively. In these constructions the dependence on all 11 space-time coordinates is retained; hence, the symmetries are already present before dimensional reduction. Thus, part of the hidden symmetries of dimensionally reduced supergravity can be “lifted” to eleven dimensions. Moreover, some evidence has been given that the full $E_{8(8)}$ duality group [7, 8] and even the Kac-Moody algebra $E_{11}$ [9] can be realised in the 11-dimensional theory hinting at the existence of an “exceptional geometry” [7] in eleven dimensions.

In the present paper, we describe a procedure to lift manifest global symmetries up by one dimension. Rather than restricting ourselves to the maximal compact subgroup, we do not meet any obstacles to perform the lift for the entire group $G$. The key object in the construction is an auxiliary space, $\mathcal{M}$, on which $G$ has a natural nonlinear action. This nonlinear action is most easily derived by viewing $\mathcal{M}$ as the ‘boundary’ of a submanifold that is appropriately embedded in a larger space $\mathcal{M}^\#$, on which $G$ acts linearly. As will be clarified, the steps involved are, in spirit, reminiscent of those leading to the standard nonlinear action of the conformal group in Minkowski space, whereby the latter is identified with the boundary of AdS.

Further, the idea of this additional (besides the space-time dimensions, that is) space was already present in Ref. [10]. Here, however, the new

\footnote{For $d < 6$ this is a classical group (see e.g. [4]).}
ingredient is that the supergravity fields will now be allowed to live on $\mathcal{M}$, even though they are propagating only in space-time. Therefore, these extra dimensions are not to be considered truly physical. Rather, the underlying geometry of $\mathcal{M}$ provides a means to make an otherwise hidden symmetry manifest, via the induced action on the fields in the theory.

The idea of making seemingly nontrivial dualities manifest via some underlying geometry can hardly be called new: F-theory [11], where the $SL(2,\mathbb{Z})$ symmetry of a class of compactified IIB models is geometrically understood as the modular group acting on an auxiliary two-torus (the elliptic fibre), provides a prototypical example. Note that also there, no physical significance is assigned to the added two torus directions.

The structure of the paper is as follows: in section 2 we outline the concept of conformal realisations. Starting with the familiar example of the conformal group in Minkowski space in section 2.1, we apply the same construction to $E_7(7)$ acting conformally on a 27-dimensional space $\mathcal{M}$ in section 2.2; next, in section 2.3 we put the relation between the conformal $E_7(7)$ duality in $d=5$ and the linearly realised $E_7(7)$ in four dimensions in a geometrical perspective. In section 3, we exploit the construction to find a realisation of $E_7(7)$ in five-, and $E_8(8)$ in four-dimensional maximal supergravity theories. First, we review the known structure of five-dimensional supergravity in section 3.1, and point out the minimal consistency conditions that a true “lifted” symmetry must satisfy. Next, section 3.2 introduces the 27-dimensional auxiliary space $\mathcal{M}$. An analysis of the consistency requirements fixes a peculiar dependence of all fields on $\mathcal{M}$ only via the cubic $E_6(6)$-norm $N$. Moreover, an $E_7(7)$ embedding in the diffeomorphism group of $\mathcal{M}$, namely, as the conformal group w.r.t. $N$, is demonstrated to be consistent. Some additional remarks are collected in section 3.3; section 3.4, finally, contains a similar construction resulting in an $E_8(8)$ conformal duality symmetry in $d=4$.

2 Conformal realisations

As will become clear shortly, the nonlinear realisation of $E_7(7)$ in a 27-dimensional flat space bears striking similarities to that of the conformal group $Conf(M^{3,1})$ in four-dimensional Minkowski space, $M^{3,1}$. Since the latter has a clearer geometrical picture, it will serve as a model throughout the paper.
2.1 The conformal group in $M^{3,1}$

Minkowski four-space, $M \equiv M^{3,1}$, is the (flat) vector space $\mathbb{R}^4$ endowed with the indefinite metric $\eta$. On $M$, the conformal group $Conf(M^{3,1})$ consists of elements $g$ with an action on $x \in M$ that preserves the lengths of vectors up to a scale factor:

$$\eta(gx, gx) = \lambda_g(x) \eta(x, x).$$

(1)

As is well-known, this condition singles out the transformations whose infinitesimal form is given by:

$$\delta_e x^\mu = e^\mu,$$

(2)

$$\delta_\Lambda x^\mu = \Lambda^\mu_\nu x^\nu + h x^\mu,$$

(3)

$$\delta_f x^\mu = 2(x^\nu f_\nu) x^\mu - (x^\nu x_\nu) f^\mu,$$

(4)

A word about notation: $e^\mu, f^\mu \in M^{3,1}$ parametrise translations and so-called special conformal transformations$^3$, respectively, while Eq. (3) contains the Lorentz transformations and dilatations, parametrised by (antisymmetric) $\Lambda$ and $h \in \mathbb{R}$. The given transformations enlarge the Poincaré algebra to the conformal algebra $so(4,2)$, which has a three-grading

$$so(4,2) = g^{-1} \oplus g^0 \oplus g^+\oplus (e^\mu) \oplus (\Lambda^\mu_\nu, h) \oplus (f^\mu)$$

(5)

i.e., the degree of the commutator of two elements equals the sum of their degrees. For example, Lorentz rotation generators have degree 0. Note further that the infinitesimal transformations in grade 0 act linearly, while grade +1 elements have a quadratic action on the coordinates.

2.2 Conformal extension of $E_{6(6)}$ to $E_{7(7)}$

In order to extend the global symmetry group $E_{6(6)}$ in five dimensions to $E_{7(7)}$ we will introduce a 27-dimensional auxiliary space $\mathcal{M}$ which admits $E_{6(6)}$ as “generalised Lorentz group”. Like in Minkowski space, we will extend this group to a “generalised conformal group”. In fact, to introduce such generalised space-times and the action of the corresponding conformal group is an old idea [12, 13]. More recently, it was already suggested to extend space-time by extra dimensions [10].

$^3$The integrated version of these are best thought of as a succession of inversion ($x^\mu \rightarrow \frac{x^\mu}{\eta(x,x)}$), translation, and inversion again.
We shall make use of the following branching rule for the adjoint representation of $E_{7(7)}$ with respect to $E_{6(6)}$:

$$
\begin{align*}
133 \rightarrow & \ 27 \oplus [78 \oplus 1] \oplus 27 \\
& \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1}.
\end{align*}
$$

(6)

The $E_{6(6)}$ adjoint 78 acts linearly on $\mathcal{M}$, the carrier space of the fundamental 27 representation; $\mathcal{M}$ can be endowed with a triple norm $N_3$ that is preserved under $E_{6(6)}$, analogously to $\eta$ being Lorentz invariant in Minkowski space. Moreover, Eq. (6) displays the three-graded structure of $E_{7(7)}$, similar to that of $Conf(M^{3,1})$ in Minkowski space (see Eq. (5)), and thus hints towards viewing $E_{7(7)}$ as a conformal extension of $E_{6(6)}$. Indeed, following the construction of Ref. [14], one may define the following operations on $\mathcal{M}$ with coordinates $Y^m$:

$$
\begin{align*}
\delta_{27} Y^m &= E^m, \\
\delta_{78} Y^m &= \Lambda^m_n Y^n, \\
\delta_1 Y^m &= H Y^m, \\
\delta_{27} Y^m &= \frac{1}{2} F_{n K^{mn}} Y^n Y^r, \\
\delta_{78} Y^m &= \frac{1}{2} F_{n K^{mn}} Y^n Y^r,
\end{align*}
$$

(7)

These transformations define a nonlinear realisation of $E_{7(7)}$ on $\mathcal{M}$, provided that the coefficients $K^{mn}_{\ pq}$ are identified with the structure constants of the $E_{6(6)}$-invariant triple product in the 27 representation of $E_{6(6)}$ (see Appendix A). These are a generalisation of the transformations in Eq. (2) - Eq. (4). Moreover, the first two types of transformations leave $N_3(Y - Y')$ invariant, while this quantity gets rescaled only under transformations of the latter two types. As such, $E_{6(6)}$ acts on $\mathcal{M}$ as generalised Lorentz rotations, and the $-1$ (+1) subspace as translations (special conformal transformations). It is thus fair to say that Eq. (7) define a conformal realisation of $E_{7(7)}$ on a 27-dimensional space.

### 2.3 Hidden exceptional geometry

In $d = 4, N = 8$ supergravity, the gauge vectors and their electric-magnetic duals combine into the 56 linear $E_{7(7)}$ representation. To see the relation with the $d = 5$ situation, where $E_{7(7)}$ will be realised on 27, the geometric perspective adopted in this section may give additional insight.

For simplicity, we start by reviewing the extension of the Lorentz $so(3,1)$ to the conformal $so(4,2)$ algebra in Minkowski four-space. First, $so(4,2)$

\[\text{The explicit form of the coefficients } K^{mn}_{\ pq} \text{ can be found in Ref. [14].}\]
is linearly realised on a six-dimensional vectorspace $M^{4,2}$, endowed with a metric $\hat{\eta}$ of signature $(4,2)$. In coordinates $X$, the following equation defines a codimension-1 subspace:

$$\hat{\eta}(X,X) = R^2,$$

for arbitrary fixed $R \in \mathbb{R}$. The thus-defined five-manifold with the induced metric $g$ is identified as AdS$_5$. Since the linear action on $M^{4,2}$ leaves the defining equation Eq. (8) invariant, so$(4,2)$ descends to the isometry group of $g$. Alternatively, the fact that AdS$_5 \simeq SO(4,2)/ISO(3,1)$ makes this property manifest.

A coordinate system on AdS$_5$ that will prove particularly useful, is that of the so-called Poincaré coordinates $(x^\mu, u), \mu = 0\ldots3$. They enjoy the following properties:

1. the subalgebra iso$(3,1)$ of isometries has a linear realisation on $(x^\mu)$;
2. the dilatation descends to a rescaling $(\lambda x^\mu, \lambda^{-1} u)$.

In these coordinates, AdS$_5$ is a foliation parametrised by $u$. Moreover, Minkowski space arises in this picture as the boundary of AdS$_5$: $\{u = \infty\}$, and special conformal transformations are nonlinearly realised on $(x^\mu)$ (as given in Eq. (4)). In summary, the picture outlined picture yields an understanding how transformations that initially act linearly get translated into
nonlinear ones, now realised on the boundary of an invariantly embedded submanifold, though.

Next, when $E_{7(7)}$ is viewed as a conformal extension of $E_{6(6)}$, a closely parallel geometric reasoning will lead to the desired relation between the 56 and 27 realisations. The branching rule

$$56 \xrightarrow{E_{6(6)}} 1 \oplus 27 \oplus \overline{27} \oplus \overline{1}. \quad (9)$$

suggests a choice of coordinates on the carrier space $\mathcal{M}_{56}^\#$ of 56 that reflect this decomposition: $\hat{Z} := (z, Z^m, Z_m, \tilde{z})$. Now, define the following quantities:

$$\begin{bmatrix} z \\ Z^m \\ Z_m \\ \tilde{z} \end{bmatrix} := \tilde{Z} = \begin{bmatrix} U \\ Y^m \\ Y_m \\ 1 \end{bmatrix} \quad (10)$$

After tedious though straightforward algebra, one finds that the system of equations

$$\begin{align*}
U &= N(Y^m), \\
Y^m &= (Y_m)^\#
\end{align*} \quad (11)$$

is invariant under the induced $E_{7(7)}$. These equations define a 28-dimensional curved submanifold $\mathcal{M}$ in 56. Furthermore, it can be shown that $(Y^m, \tilde{z})$ form a system of Poincaré coordinates on $\mathcal{M}$. The 27-dimensional boundary is recovered as the set $\{\tilde{z} = \infty\}$, thus making the parallel with the AdS story complete.

### 3 Conformal duality symmetries

The maximal supergravity theories in dimensions $2 \leq d \leq 10$ can be obtained by toroidal dimensional reduction of the $N = 1$ supergravity [3] in eleven dimensions. After dualisation of some of the $n$-form fields they all obey a global $E_{11-d}$ symmetry. This symmetry was first discovered by Cremmer and Julia in the four-dimensional theory [1] and was named “hidden symmetry”. An exhaustive treatment of all dimensions $3 \leq d \leq 10$ can be found in Ref. [4]. In the following, we will apply our construction to the five- and four-dimensional maximal supergravity theories. However, we believe that similar constructions exist in all other dimensions.
3.1 Global symmetries of $d=5$, $N=8$ supergravity

$N=8$ supergravity in 5 dimensions [15] is described by the fields $e_{\mu}^{\alpha}$, $\psi_{\mu}^{a}$, $A_{\mu}^{ab}$, $\chi^{abc}$, $\phi^{abcd}$ which transform under the $USp(8)$ $R$-symmetry and are antisymmetric and traceless in their indices $a,b,\ldots = 1,\ldots,8$.

It is well-known that this theory admits a global $E_{6}(6)$ symmetry which leaves the Lagrangian invariant. $E_{6(6)}$ acts on the vector fields $A_{\mu}^{m}(m=1,\ldots,27)$ in the $27$ representation and on the scalars which can be grouped together into a 27-bein $V_{m}^{ab} \in E_{6(6)}/USp(8)$ in the $\overline{27}$ representation. The fields are summarised in Table 3.1.

The bosonic part of the Lagrangian is given by

$$
\mathcal{L} = -\frac{1}{4} e R + \frac{1}{24} e \partial_{\mu}G_{mn}\partial^{\mu}(G^{-1})^{mn} - \frac{1}{8} e g^{\mu\rho}g^{\nu\sigma}G_{mn}F_{\mu\nu}^{m}F_{\rho\sigma}^{m} \\
+ \frac{1}{12} \epsilon^{\mu\rho\sigma\lambda} F_{\mu\nu}^{m}F_{\rho\sigma}^{n}A_{\lambda}^{p}C_{mnp},
$$

(13)

where $G_{mn}$ is the metric derived from the vielbein $V_{m}^{ab}$ and $C_{mnp}$ are the coefficients of the norm $N_{3}$ (see Appendix A).

In this $E_{6(6)}$-covariant formulation, the hidden global $E_{6(6)}$ symmetry is manifest. There is an analogous form of the Lagrangian in all dimensions $3 \leq d \leq 10$ [4]. However, it is only in odd dimensions $d$ that the global symmetry group $E_{11-d}$ be effectively realised on the Lagrangian; in even dimensions the hidden symmetry is only a full symmetry of the equations of motion.
the graviton: \( e_\mu^\alpha \) 2 1 1
8 gravitini: \( \psi_\mu^a \) 3/2 8 1
27 vector fields: \( A_{\mu}^{\alpha} \) 1 1 27
48 spin-1/2 fermions: \( \chi^{abc} \) 1/2 48 1
42 scalars: \( V_{mab} \) 0 27 27

Table 1: Field content of \( d=5, N=8 \) supergravity

Besides this global \( E_{6(6)} \), an additional scaling symmetry \( D \) was discovered to exist in all maximal supergravities. This so-called “trombone symmetry” [16] finds its origin in the scaling symmetry of the eleven-dimensional supergravity theory. In the dimensionally-reduced theories, the vielbein scales linearly and all \( n \)-index potential forms have scaling weight \( n \); scalar fields are left invariant:

\[
e_\mu^\alpha \rightarrow \lambda e_\mu^\alpha, \quad A_\mu \rightarrow \lambda A_{\mu\nu\rho}, \quad (\mu, \alpha = 0, \ldots, 4)
\]

Under \( D \), the Lagrangian scales as \( \mathcal{L} \rightarrow \lambda^3 \mathcal{L} \).

In Ref. [16], the authors pointed out the following problem: in the five-dimensional quantum theory, the symmetries are expected to be broken to a discrete subgroup by the Dirac quantisation condition. As to \( E_{6(6)} \times D \), one could, in principle, restrict both factor separately to discrete subgroups. Since the only discrete subgroup in the second factor is \( \{ \pm 1 \} \), the scaling would not survive quantisation.

In view of this problem, it looks desirable to find a larger group \( \mathcal{G} \supset E_{6(6)} \times D \) such that both factors are no longer independent but rather, are united in a nontrivial way dictated by the group structure of \( \mathcal{G} \): upon restriction to \( \mathcal{G}(\mathbb{Z}) \), the initially independent quantisations of \( E_{6(6)} \) and \( D \) would be naturally related. A glance at Eq. (6) suggests an immediate (minimal) candidate for \( \mathcal{G} \), namely \( E_{7(7)} \). This is one piece of motivation for our quest for \( E_{7(7)} \) in \( d=5 \) maximal supergravity.

What are the minimal requirements that such an \( E_{7(7)} \) realisation must meet?

1. To begin with, \( E_{6(6)} \times D \) should be embedded in such a way that the standard known action is retrieved upon restriction from \( E_{7(7)} \).
2. Secondly, upon dimensional reduction, the action must be such that it reduces to the standard linear $E_{7(7)}$ action on 56 upon dimensional reduction to $d = 4$. If met, this condition would guarantee a natural identification of the $E_{7(7)}$ in five-dimensions with the four-dimensional (familiar) $E_{7(7)}$. In that case, the duality symmetry would be lifted, indeed, up by one dimension.

3.2 Extending the global symmetries beyond $E_{6(6)}$

At this stage, we introduce an auxiliary 27-dimensional space $\mathcal{M}$, with co-ordinates $Y^m$. Let us temporarily assume that the fields listed in Table 3.1 depend on $Y^m$. In other words, the fields now live in the auxiliary 27-dimensional space $\mathcal{M}$. Bold a step as this may seem at first sight, it will be motivated by verifying that the minimal requirements of the previous section can effectively be fulfilled. For now, this hypothesis allows us to derive the required transformation properties of the supergravity fields in $d = 5$. From the last column in Table 3.1, the scalar coset 27-bein $V_m^{ab}$ may be viewed as a one-form on $\mathcal{M}$, while the 27 vector fields $A_\mu^m$ transform as the components of a vector field on $\mathcal{M}$. The remaining supergravity fields are scalars on $\mathcal{M}$.

From tensor calculus, the following transformation rules for arbitrary tensor fields $T(Y)^m$ and $T(Y)_m$ are induced by the $E_{6(6)}$ action on $\mathcal{M}$, as given in Eq. (7):

\[
\delta_{E_{6(6)}} T(Y)^m = \Lambda^p_n Y^p \partial_n T(Y)^m + \Lambda^m_n T(Y)^m,
\]

\[
\delta_{E_{6(6)}} T(Y)_m = \Lambda^p_n Y^p \partial_n T(Y)_m + \Lambda^m_n T(Y)_m;
\]

the first terms on the r.h.s. are the transport terms and the parameters of $E_{6(6)}$ are related as $\Lambda^m_n = -\Lambda^m_n$. For mixed-type and higher-order tensors, one derives similar transformation rules.

The second terms in the r.h.s. of Eq. (15) characterise $T(Y)^m$ and $T(Y)_m$ as 27 and 27 linear representations of $E_{6(6)}$. This is the actual behaviour that we wish to recover. For this to occur, the transport terms in Eq. (15) must vanish. Therefore, in order to satisfy the first condition on p. 9, we conclude that the fields can at most depend on $Y$ via the $E_{6(6)}$-invariant norm $N_3(Y)$:

\[
e_\mu^\alpha(Y) = e_\mu^\alpha (N_3(Y)),
\]

\[
A_\mu^m(Y) = A_\mu^m (N_3(Y)),
\]

\[
V_m^{ab}(Y) = V_m^{ab} (N_3(Y)).
\]

How about the scaling behaviour under $D$? As tensors, $T(Y)^m$ and $T(Y)_m$ behave as follows under scale transformations:

\[
\delta_1 T(Y)^m = H Y^n \partial_n T(Y)^m + H T(Y)^m,
\]

\[
\delta_1 T(Y)_m = H Y^n \partial_n T(Y)_m + H T(Y)_m;
\]

\[
\delta_1 Y^m = H Y^n \partial_n Y^m + H Y^m.
\]
\[ \delta_1 T_m(Y) = H Y^n \partial_n T_m - H T_m. \]  

(17)

Matching this with the scaling laws in Eq. (14), we are to conclude that, initially,

\[ e_\mu^\alpha(Y) = N_3(Y) e_\mu^{0\alpha} \]
\[ A_\mu^m(Y) = N_3(Y) A_\mu^{0m} \]

(18)

where the fields \( e_\mu^{0\alpha} \), \( A_\mu^{0m} \), and also the scalars \( \nu_m^{ab} \) do not depend on \( Y \).

The resulting Lagrangian is

\[ \mathcal{L}(Y) = N(Y)^3 \mathcal{L}_0 \]

(19)

where \( \mathcal{L}_0 \) is independent of \( Y \) and only the spacetime-global factor in front of \( \mathcal{L}_0 \) will change under reparametrisations in the \( Y \)-space. In particular, the group \( E_{7(7)} \) which is realised as a conformal group on the 27 coordinates \( Y^m \) only affects the global prefactor \( N(Y)^3 \) and thus becomes a genuine symmetry at the level of the equations of motion.

Finally, the conformal action of \( E_{7(7)} \) on \( \mathcal{M} \) given in Eq. (7), is found to induce the transformation rules below on tensors:

\[ \delta_\mathcal{L} T^m(Y) = \frac{1}{2} F_n K^{pq} r Y^q Y^r \partial_p T^m + F_n K^{pq r} Y^q T^m, \]

\[ \delta_\mathcal{L} T_m(Y) = \frac{1}{2} F_n K^{pq r} Y^q \partial_p T_m + F_n K^{pm q} Y^q T_p. \]

(20)

(21)

with parameters related by \( K^{pq r} Y^q \partial_p T_m + F_n K^{pm q} Y^q T_p \).

3.3 Some remarks on conformal dualities

The proposed realisation of \( E_{7(7)} \) deserves some further comments.

First, the form of the Lagrangian, Eq. (19), is a product of two factors: a constant \( \mathcal{L}_0 \) (that is, as far as the \( Y \)-dependence is concerned), and a prefactor \( N^3(Y) \). Under the action of \( \text{Diff}(\mathcal{M}) \), this Lagrangian would generically transform into \( f[N(Y)]\mathcal{L}_0 \); since this may be viewed as a global rescaling of the initial Lagrangian, one might be tempted to infer that the equations of motion remain invariant under arbitrary reparametrisations \( \text{Diff}(\mathcal{M}) \). However, there is a catch: the factorised form of \( \mathcal{L} \) resulted from the particular \( Y \)-dependence of all tensor fields via \( N(Y) \) solely. For

\[ N(Y)^3 \mathcal{L}_0 \rightarrow f[N(Y)]\mathcal{L}_0 \]

(22)

to be induced from the transformations of individual tensor fields, one must be careful not to destroy the specific dependence of the fields via \( N(Y) \). General diffeomorphisms on the space \( \mathcal{M} \) would generically introduce an
explicit field dependence on all 27 $Y^m$-coordinates; as such, factoring out $N(Y)$ (like in Eq. (18)) would no longer be possible. The only admissible coordinate transformations on $\mathcal{M}$ therefore, are those which leave the norm $N(Y)$ conformally invariant. This singles out $E_{7(7)}$, embedded in Diff($\mathcal{M}$) as proposed in Eq. (7).

Next, it is noteworthy that our proposed realisation on $\mathcal{V}_{m}^{ab}(Y), A_{\mu}^{m}(Y)$ is linear in the fields. This feature has two important implications:

1. the space-time gauge invariance of the vector fields is manifestly preserved; nonlinear transformations not spoiling gauge invariance are not evident.

2. upon dimensional reduction to $d = 4$, the vector fields in $27$ combine with the Kaluza–Klein vector from the space-time metric (and their duals) into the linear $56$ of $E_{7(7)}$; had our construction been nonlinear in the fields, demonstrating that things do work out consistently would be far harder a task.

Thirdly, inspection of Eq. (20) and Eq. (21) reveals that the space on which the symmetry is realised is $\mathcal{W} = \Phi \otimes \mathbb{C}[Y^m]$, $\Phi$ being the space of ($Y$-independent) supergravity fields and their space-time derivatives; further, $\mathbb{C}[Y^m]$ is the polynomial ring in 27 variables. The latter has the structure of a graded space,

$$\mathbb{C}[Y^m] \simeq \bigoplus_{n=0}^{\infty} \text{Sym}^n(V) ,$$

where $\mathcal{M}$ is a 27-dimensional abstract vector space. In other words, the complete polynomial ring is graded by the degree of homogeneous polynomials. Accordingly, we decompose $\mathcal{W} = \oplus_n \mathcal{W}_n$ in an obvious notation. The nice feature of this fact is that the transformation rules for $Y$-dependent fields show that the grading is respected by the realisation, i.e.,

$$0 \leftarrow^\Phi \mathcal{W}_1 \rightarrow^\mathcal{W}_2 \rightarrow^\mathcal{W}_3 \rightarrow^\mathcal{W}_4 \cdots .$$

In particular, the dilatation (1) and $E_{6(6)}$ (78) do not shift degrees, hence map $\Phi$ into itself, while the 27 translations annihilate the degree-0 space. $\Phi$ implements $27 = g^{-1}$ trivially. Put equivalently, only the $g^0 \oplus g^{+1}$ subalgebra of $E_{7(7)}$ is seen to act nontrivially on the supergravity fields.
Finally, let us comment on the physical (in)significance of introducing $Y$-dependence in the supergravity fields. As explained in section 3.2, the particular way in which quantities are made $Y$-dependent effects in an initial $N(Y)^3$ prefactor in front of the lagrangian. Acting with $E_{7(7)}$ on $N(Y)^3\mathcal{L}_0$ could rescale $N(Y)$, at worst. If so, however, the scale factors could be absorbed into the coupling constant in front of the action:

$$S = \frac{1}{\ell_p^3} \int e\mathcal{L}$$

(25)

(where $\ell_p$ is the Planck length in five dimensions). Namely, $\ell_p \rightarrow \ell_p(N(Y))$ would result in a $Y$-dependent coupling, and the conformal duality transformations, Eq. (20), would map a theory with a given $\ell_p$ to one with a possibly different $\ell_p'$. That is, after fixing an arbitrary choice of a point $Y$. The picture suggested is thus: there is a one-dimensional space $\mathcal{T}$ of $N(8)$ theories, distinguished by different values of $\ell_p$. Rather than by $\ell_p$, one may choose to parametrise this line by $N(Y)$; the 27-dimensional space of $Y^m$ is thus viewed as a highly redundant description of $\mathcal{T}$: the one-parameter family of 26-dimensional subspaces $\{N(Y) = N_0\}$ in the (carrier space of) $\mathcal{T}$ reflect this redundancy.

### 3.4 Duality symmetries of $d = 4, N = 8$ supergravity

We want to discuss how to extend the global $E_{7(7)}$ symmetry in four dimensions to $E_{8(8)}$. For this purpose we need a realisation of $E_{8(8)}$ on a 56-dimensional vector space.

The problem in this case is the fact that $E_{8(8)}$ does not admit a three-grading Eq. (5). However, it admits a 5-graded decomposition w.r.t. its subgroup $E_{7(7)} \times SL(2,\mathbb{R})$. Denoting its Lie algebra by $\mathfrak{e}_8$ we have

$$\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2} \oplus 1 \oplus 56 \oplus (133 \oplus 1) \oplus 56 \oplus 1$$

(26)

An important property of this decomposition is the fact that the subspaces of grade $-1$ and $-2$ together form a maximal Heisenberg subalgebra. The corresponding generators $X^{ij}, X_{ij} \in \mathfrak{g}^{-1}$ and $x \in \mathfrak{g}^{-2}$ obey the commutation relations

$$[X^{ij}, X_{kl}] = -2 \delta^{ij}_{kl} x.$$  \hspace{1cm} (27)

In Ref. [14] a realisation of $E_{8(8)}$ on the $(56 + 1)$-dimensional Heisenberg subalgebra $\mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$ has been constructed. This so-called quasiconformal realisation is similar to the conformal realisation of $E_{7(7)}$ in the sense that an
$E_7(7)$ invariant norm $N_4(X^{ij}, X_{ij}, x)$ on that 57-dimensional space is conformally invariant under the $E_8(8)$ action. The norm is given by

$$N_4(X^{ij}, X_{ij}, x) = I_4(X^{ij}, X_{ij}) - x^2$$

(28)

where $I_4$ is the quartic invariant of $E_7(7)$ in the 56 representation. Enforcing the $E_8(8)$-invariant condition $N_4 = 0$, or equivalently

$$I_4(X^{ij}, X_{ij}) = x^2,$$

(29)

eliminates the 57th variable $x$ and yields a realisation of $E_8(8)$ on $\mathbb{R}^{56}$. Following the same procedure as in section 3.2, we can realise $E_8(8)$ on the fields of maximal supergravity in $d = 4$ and the Lagrangian will by construction again only change by a global factor.

4 Conclusions

In this paper, we have shown how to lift the continuous hidden-symmetry group $E_7(7)$ from $d = 4$ to $d = 5$. In five dimensions, the symmetries are conformally realised in the sense of section 3.3, where the idea of “trombone symmetry” was borrowed from Ref.[16]: rather than $E_6(6) \times D$ there, in our approach the entire $E_7(7)$ emerges.

The presented construction hinges largely on the fact that $E_7(7)$ may be viewed as a conformal extension of $E_6(6)$. As explained in section 3.3, a “natural” way to lift the $E_7(7)$, whilst preserving the known $E_6(6)$, is to make the supergravity fields live, but not propagate, in a 27-dimensional space $\mathcal{M}$, on which $E_7(7)$ acts by (generalised) conformal transformations. In the context of hidden symmetries, this auxiliary space $\mathcal{M}$ had already appeared in the literature in Ref. [10], although in a different guise.

A geometrical picture of the situation illustrated how $E_7(7)$, acting in the 56 linear representation, ends up realised as nonlinear coordinate transformations on $\mathcal{M} \hookrightarrow 56$. Rather than $E_7(7)$, the continuous group of hidden symmetries, it is believed that only the U-duality discrete subgroup $E_7(7)(\mathbb{Z}) \subset E_7(7)$ survives quantisation [2]. In principle, using this embedding, the U-dualities of $d=4, N=8$ supergravity can now be lifted to the five dimensional theory.

In summary, the results presented in this paper may be said to give additional insight in the origin of four-dimensional symmetries from a five-dimensional perspective. As to higher dimensions, we believe that our construction can be generalised with minor modifications to account for the different structure of the U-duality groups.
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A Properties of $E_{6(6)}$

The Lie algebra of $E_{6(6)}$ has maximal compact subalgebra $USp(8)$. We denote tensors transforming in the fundamental $27$ transformation by $T^m$ $(m = 1, \ldots, 27)$ and tensors transforming in $\overline{27}$ by $T_m$. An $E_{6(6)}$ transformation in the $27$ representation is given by

$$\delta T^m = \Lambda^m{}_n T^n \quad \text{and} \quad \delta T_m = \Lambda_m{}^n T_n$$

with a traceless matrix $\Lambda^m{}_n = -\Lambda_m{}^n$.

The tensor product

$$27 \times 27 = 1 + 78 + 650$$

contains a singlet $\delta^m_n$, whereas

$$27 \times 27 = \overline{27} + 351 + 351'$$

$$\overline{27} \times \overline{27} = 27 + 351 + 351'$$

(32)

does not. This is reflected by the fact that $E_{6(6)}$ does not possess a quadratic invariant. However, there is an invariant $N_3$ of third order:

$$N_3(T) = T^m T^n T^p C_{mnp}$$

(33)

The $27$ can be associated with the $27$-dimensional exceptional Jordan algebra $J_{3OS}$ where $\mathbb{O}_S$ denotes the split real form of the octonions $\mathbb{O}$. This Jordan algebra possesses a symmetric Jordan product $X \circ Y$ with invariance group $F_4(4)$ and a triple product\(^5\)

$$\{X, Y, Z\} = 16(X^{ca} Z_{cd} Y^{db} + Z^{ca} X_{cd} Y^{db}) + 4X^{ab} Y^{cd} Z_{cd} + 4Y^{ab} X^{cd} Z_{cd} + 4Z^{ab} X^{cd} Y_{cd} + 4\Omega^{ab}(X_{cd} Y_{de} Z_{ef} \Omega_{ef})$$

(34)

which is invariant under $E_{6(6)}$. This triple product is crucial in the construction of $E_{7(7)}$ as the conformal extension of $E_{6(6)}$. We denote its structure constants by $K^{mn}_{pq}$:

$$\{X, Y, Z\}^m = K^{mn}_{pq} X^p Y_n Z^q$$

(35)

Using Eq. (32) one can also define a “conjugations” $\#$ which maps $27$ into $\overline{27}$ and vice versa. It is quadratic in the sense that $(\lambda T)^\# = \lambda^2 T^\#$ and obeys the relation $T^{\#\#} = N(T) T$.

\(^5\)The corresponding formula for the triple product in [14] should read correctly

$$\{X, Y, Z\}^{ab} = 16(X^{ac} Z_{cd} Y^{db} + Z^{ac} X_{cd} Y^{db}) + 4X^{ab} Y^{cd} Z_{cd} + 4Y^{ab} X^{cd} Z_{cd} + 4Z^{ab} X^{cd} Y_{cd} + 4\Omega^{ab}(X_{cd} Y_{de} Z_{ef} \Omega_{ef})$$
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