NON-HERMITIAN SYMMETRIC $N = 2$ COSET MODELS, POINCARÉ POLYNOMIALS, AND STRING COMPACTIFICATION

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Abstract. The field identification problem, including fixed point resolution, is solved for the non-hermitian symmetric $N = 2$ superconformal coset theories. Thereby these models are finally identified as well-defined modular invariant conformal field theories. As an application, the theories are used as subtheories in $N = 2$ tensor products with $c = 9$, which in turn are taken as the inner sector of heterotic superstring compactifications. All string theories of this type are classified, and the chiral ring as well as the number of massless generations and anti-generations are computed with the help of the extended Poincaré polynomial. Several equivalences between a priori different non-hermitian coset theories show up; in particular there is a level-rank duality for an infinite series of coset theories based on $C$ type Lie algebras. Further, some general results for generic $N = 2$ coset theories are proven: a simple formula for the number of identification currents is found, and it is shown that the set of Ramond ground states of any $N = 2$ coset model is invariant under charge conjugation.

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\section{Introduction}

While the conditions necessary for the consistency of a superstring theory seem to be too weak to pinpoint a ‘theory of everything’, string theory remains an interesting approach to unify the fundamental interactions including gravity. Furthermore, the study of strings has given new and deep insight in various topics in mathematics and physics so that there are good reasons, beyond possible direct application to phenomenology, to have a closer look at the structures arising in string theory.

A class of two-dimensional field theories for which this point of view is particularly justified are the \( N = 2 \) superconformal theories which are needed for the inner sector of (heterotic) string theories. The enlarged \( (N = 2) \) world sheet supersymmetry for the right-moving part of the theory is in this case dictated \[ \text{(1.1)} \] by the requirement of space-time supersymmetry, a property imposed for phenomenological reasons, such as to ‘solve’ the gauge hierarchy problem. In the present paper, we consider theories for which \( N = 2 \) supersymmetry is present in the left-moving part as well; just like in the generic case, these \( N = 2 \) theories are interesting in their own right, as they are singled out by the presence of new structures such as the ring of chiral primary fields and the connection with Calabi-Yau manifolds \[ \text{(1.2)}. \] Furthermore, there exist deep relations between \( N = 2 \) superconformal field theories and conformal field theories in general, including the interpretation of the fusion ring of any rational conformal field theory as a deformation of the chiral ring of some \( N = 2 \) theory \[ \text{(1.3)}. \]

There exist several approaches to construct the inner sector of a heterotic string theory: non-linear sigma models with a Calabi-Yau manifold as their target space \[ \text{(1.4)} \], the description in terms of Landau-Ginzburg potentials \[ \text{(1.5)} \], and exactly solvable models (these approaches are closely interrelated, but the question to which extent they are equivalent has not yet been resolved completely). By exactly solvable we mean that all correlation functions can (at least in principle) be calculated exactly. Among the solvable superconformal field theories there are free field constructions employing the Coulomb gas approach \[ \text{(1.6)} \], and theories constructed by algebraic methods. In the algebraic approach the \textit{coset construction} \[ \text{(1.7)} \] plays a prominent rôle, for it allows to obtain many superconformal theories within the framework of affine Kac-Moody algebras.

In \[ \text{(1.8)} \] Kazama and Suzuki considered coset models \( C \) of the form

\[ C[g \oplus \text{so}(2d)/h]_k. \]  \hspace{1cm} \text{(1.1)}

Here \( g \) stands for a semi-simple Lie algebra, and \( h \) is a reductive subalgebra of \( g \); the integer \( d \) is defined as \( 2d = \dim g - \dim h \), while the integer \( k \) denotes the level of the affinization \( g^{(1)}(g) \) of \( g \). As shown in \[ \text{(1.9)} \], the symmetry algebra of such coset models a complete list of all \( N = 2 \) coset theories of the form \[ \text{(1.10)} \] was obtained. Indeed the following conditions are necessary and sufficient for a coset theory of the form \[ \text{(1.11)} \] to have \( N = 2 \) superconformal symmetry:

1. The embedding \( h \hookrightarrow g \) has to be regular.
2. The number
\[ n := \frac{1}{2} (\text{rank } g - \text{rank } h) \quad (1.2) \]
must be an integer.

3. Denoting the simply connected compact Lie groups having \( g \) and \( h \) as their Lie algebras by \( G \) and \( H \), respectively, the coset manifold
\[ \frac{G}{H \times U(1)^{2n}} \quad (1.3) \]
has to be Kählerian.

Up to now the following theories solving these constraints have been considered in the literature:

- Tensor products of \( N = 2 \) minimal models [16], including models which employ non-diagonal and non-product modular invariants [17, 18, 19, 20, 21].

- Tensor products of the so-called projective cosets [22], corresponding to coset theories of the form
\[ C[\text{su}(n+1) \oplus \text{so}(2n)/\text{su}(n) \oplus u(1)]_k. \quad (1.4) \]

For these models non-diagonal modular invariants have been investigated, too [23].

- Tensor products of arbitrary hermitian symmetric coset theories (‘HSS-cosets’) with the diagonal modular invariant [24].

Note that \( N = 2 \) minimal models can be considered as projective cosets with \( n = 1 \), and projective cosets are a subclass of the hermitian symmetric cosets.

From the classification [14, 15] of \( N = 2 \) superconformal coset models in the Kazama-Suzuki framework it is well known that there exist even more models that possess \( N = 2 \) superconformal symmetry; the hermitian symmetric coset theories constitute only a subclass. In this paper we shall consider the general case. The paper is organized as follows. First, we recall in section 2 the classification of \( N = 2 \) superconformal coset models obtained in the Kazama-Suzuki framework. As a by-product we prove a simple characterization of hermitian symmetric spaces which differs from the one given in the standard literature. Based on the general classification, we then provide a complete list of all non-hermitian symmetric cosets that can be used in tensor products with conformal central charge \( c = 9 \).

We proceed by specifying the conformal field theories defining the cosets of our interest. This is necessary because a ‘Lie-algebraic coset’ \( C \) as it stands in (1.1) is in itself far from defining a consistent modular invariant conformal field theory. We emphasize that although the theories described in this paper have been introduced as formal cosets (1.1) already five years ago, they have previously not been shown to describe consistent conformal field theories. (By consistency of a conformal field theory we understand among other requirements
that the characters of the theory carry a (projective) unitary representation of the modular group. Note that up to now it is even unknown whether a conformal field theory can be associated with every coset, and if so, whether this theory is unique.) To define the theories, we first determine the precise form of the affinization of the subalgebras involved. In particular we identify, in section 3.1, the level of the $u(1)$-subalgebra that is present in each of the models.

Moreover, as is well known \[26,27\], in order to obtain a modular invariant partition function, ‘fields’ in the coset theory have to be ‘identified’. Problems arise when the length of the identification orbits is not constant; orbits of non-maximal length have to be ‘resolved’ \[27,28\], which in general is a rather delicate issue. It is common to assume \[27\] that the field identifications can be reduced to purely group-theoretical selection rules (see however the counter example found in \[29\]). In section 3.2 of our paper we determine these identification rules. Furthermore we derive a formula, valid for any $N = 2$ coset theory of the form (1.1), for the order of the abelian group that is generated by the identification currents; this provides a convenient check for the completeness of the identification rules. The resolution of fixed points is dealt with in section 3.3.

In section 4 we apply the results of section 3 to string compactification. We first calculate in section 4.1 the ring of chiral primary fields \[3\] of the models. To be able to read off the Poincaré polynomial and (part of) the spectra of massless particles of the models, we derive in section 4.2 a formula giving the full superconformal $u(1)$-charge of any Ramond ground state in terms of the length of an associated element of the Weyl group. (This length is conveniently calculated by means of Hasse diagrams; the diagrams corresponding to our models are described in appendix A.) After presenting the results for the Poincaré polynomials, we also show, in section 4.3, that the set of Ramond ground states of any $N = 2$ coset theory is symmetric under charge conjugation.

In section 4.4 we present the complete list of all tensor products of coset theories that involve at least one non-hermitian symmetric coset theory and have central charge $c = 9$, providing thus consistent vacua for heterotic string compactification to four space-time dimensions \[25\]. When combined with the list of tensor products involving only minimal models \[18\] and with the corresponding list for hermitian symmetric spaces \[22\], this completes the list of all tensor products of $N = 2$ coset theories that can be obtained from cosets of the type (1.1). Note that the set of all string vacua is much bigger than the set of all tensor products of coset theories, as in general by choosing different modular invariants of the $g$- and $h$-WZW theories one gets different string vacua. However, to obtain this set is, at present, beyond reach, as a complete classification of modular invariants is still lacking for WZW theories based on simple Lie algebras other than $A_1$.

The spectra of these compactifications are computed with the help of the extended Poincaré polynomial that, as described in section 4.4, can be deduced from the ordinary Poincaré polynomial and the action of the so-called spinor current. Sometimes the results obtained for the extended Poincaré polynomials of a priori different coset theories are identical, which suggests that the corresponding conformal field theories are closely related and maybe even identical.
In section 4.5 we present an infinite series of theories for which this phenomenon occurs and describe a map that provides a one to one correspondence between the primary fields of the respective theories.

Finally, in section 5 we conclude with a brief summary and an outlook on possible further work.

2 Classification

In [13] a supersymmetric extension of the coset construction [12] was used to obtain a large class of superconformal coset models. By bosonizing the fermions of the super WZW theories involved in the construction of these models, one arrives at a level one $so(2d)_{1}$ WZW theory. As a consequence, the models can be written as

$$C[g \oplus so(2d)_{1}/h]_{k}.$$ (2.1)

In the sequel we will adopt the notation of [14] and denote indices referring to generators of the algebra $g$ by capital letters $A, B, \ldots$, indices referring to the subalgebra $h$ by $a, b, \ldots$, and indices referring to the set $g \setminus h$, and hence also to $so(2d)$, by $\bar{a}, \bar{b}, \ldots$. Thus in particular the currents generating $g$ are denoted by $\hat{J}^{A}$, and the $so(2d)$ algebra is generated by $\dim g/h$ fermions $j^{\bar{a}}$.

Denoting the structure constants of $g$ by $f_{AB}^{C}$, the currents

$$\tilde{J}^{a} = \hat{J}^{a} - \frac{i}{k} f_{bc}^{\bar{a}} \hat{J}^{b} \hat{J}^{c}$$ (2.2)

then specify the embedding of $h$ in $g \oplus so(2d)$. From the embedding (2.2) we can read off the levels of the simple subalgebras $h_{i}$ of

$$h = \hat{h} \oplus u(1)m = \bigoplus_{i} h_{i} \oplus u(1)m.$$ (2.3)

Namely,

$$k(h_{i}) = I_{i}(k + g^{\vee}) - h_{i}^{\vee},$$ (2.4)

where $g^{\vee}$ and $h_{i}^{\vee}$ denote the dual Coxeter numbers of $g$ and $h_{i}$, respectively, and where $I_{i}$ is the Dynkin index of the embedding $h_{i} \hookrightarrow g$, i.e. the relative length squared of the highest roots $\theta_{g}$ of $g$ and $\theta_{i}$ of $h_{i}$,

$$I_{i} := \frac{\langle \theta_{g}, \theta_{g} \rangle}{\langle \theta_{i}, \theta_{i} \rangle}.$$ (2.5)

(Here and below we often follow the habit of referring to an untwisted affine Kac–Moody algebra $f^{(1)}$ by its horizontal subalgebra $f$, and to the Heisenberg algebra $\hat{u}(1)$ by its horizontal subalgebra $u(1)$, whenever no confusion can arise. In particular, the reductive subalgebra $h$ will sometimes stand for its affinization $\bigoplus_{i} h_{i}^{(1)} \oplus \hat{u}(1)m$. Also, we use the short hand notation $f_{k}$ if $f^{(1)}$ is at level $k$.)

With (2.4), the conformal central charge of the coset theory becomes

$$c = \frac{3}{2} (\dim g - \dim h) - \frac{\langle \theta_{g}, \theta_{g} \rangle}{\langle \theta_{g}, \theta_{g} \rangle} \frac{g^{\vee} \dim g - \sum_{i} \langle \theta_{i}, \theta_{i} \rangle h_{i}^{\vee} \dim h_{i}}{\langle \theta_{g}, \theta_{g} \rangle (k + g^{\vee})}.$$ (2.6)
The symmetry algebra of the models (2.1) always contains the $N = 1$ supersymmetry algebra. To find a second supercurrent $G^2$, one starts with the most general ansatz expressing a spin 3/2 current of the coset theory (1.1) in terms of the currents $J^A$ and the fermions $j^a$ [14],

$$G^2(z) = \frac{2}{k} (K_{\hat{a}\hat{b} : j^{\hat{a}} \cdot j^{\hat{b}} :} - \frac{i}{3k} S_{\hat{a}\hat{b}\hat{c}} : j^{\hat{a}} j^{\hat{b}} j^{\hat{c}} :).$$  \hspace{1cm} (2.7)

Here the colons denote normal ordering, and $S$ is a totally antisymmetrical tensor. This ansatz mimics the structure of the first supercurrent $G^1$ for which $K_{\hat{a}\hat{b}}$ and $S_{\hat{a}\hat{b}\hat{c}}$ are given by the Killing form $\kappa_{\hat{a}\hat{b}}$ and by the structure constants $f_{\hat{a}\hat{b}\hat{c}}$, respectively.

The calculation of the relevant operator products that involve $G^2(z)$ shows that the following set of equations for $K$ and $S$ is necessary and sufficient for enlarged supersymmetry:

$$K_{\hat{a}\hat{b}} = - K_{\hat{b}\hat{a}} , \quad K_{\hat{a}\hat{b}} K_{\hat{b}\hat{c}} = - \delta_{\hat{a}\hat{c}} ,$$ \hspace{1cm} (2.8)

$$f_{\hat{a}\hat{b} : f_{\hat{c}\hat{d}} :} = \frac{1}{2} K_{\hat{a}\hat{b} : f_{\hat{c}\hat{d}} :} ,$$ \hspace{1cm} (2.9)

$$f_{\hat{a}\hat{b} : f_{\hat{c}\hat{d}} :} = K_{\hat{a}\hat{p}} K_{\hat{q}\hat{d}} f_{\hat{p}\hat{q}\hat{c}} + \text{cyclic permutations in } \hat{a}, \hat{b} \text{ and } \hat{c} , \hspace{1cm} (2.10)$$

$$S_{\hat{a}\hat{b}\hat{c}} = K_{\hat{a}\hat{p}} K_{\hat{q}\hat{d}} K_{\hat{r}\hat{e}} f_{\hat{p}\hat{q}\hat{r}\hat{e}} .$$ \hspace{1cm} (2.11)

The condition (2.8) means that $K$ is a complex structure on $G/H$, which is $h$-invariant by (2.9). (2.10) is a consistency condition, while (2.11) can be used to eliminate $S$ from the problem.

This set of equations can also be understood in more geometrical terms. Namely, let $t$ denote the orthogonal complement of $h$ with respect to the Killing form $\kappa$ of $g$ (this is well defined since, $g$ being semi-simple, $\kappa$ is non-degenerate). Then the model $C[g \oplus so(2d)/h]_k$ is $N = 2$ supersymmetric if and only if there exists a direct sum decomposition of vector spaces,

$$t = t_+ \oplus t_- ,$$ \hspace{1cm} (2.12)

which obeys the conditions that $\dim t_+ = \dim t_-$, that $t_+$ and $t_-$ separately form closed Lie algebras, and that the restriction of the Killing form to $t_+$ and to $t_-$ vanishes,

$$\kappa_{|t\pm} = 0 .$$ \hspace{1cm} (2.13)

This geometric characterization is in fact rather easy to prove [14]. Suppose first that the theory $C[g \oplus so(2d)/h]_k$ is $N = 2$ supersymmetric. Define $t_\pm$ to be the eigenspaces corresponding to the eigenvalues $\pm i$ of the complex structure $K$. Then the relations $t = t_+ \oplus t_-$ and $\dim t_+ = \dim t_-$ are immediate. Using (2.8) to (2.11), it is also easy to show that

$$[t_{\pm ;} , t_{\pm ;} ] = \frac{1}{2} (i f_{\hat{a}\hat{b} : f_{\hat{c}\hat{d}} :} \pm S_{\hat{a}\hat{b}\hat{c}} ) t_{\pm ;} ,$$ \hspace{1cm} (2.14)

where $t_{\pm ;}$ denotes the component of $t_{\hat{a}}$ in $t_\pm$. Thus the elements of $t_\pm$ close under the Lie bracket. Finally, for arbitrary $r_\pm , s_\pm \in t_\pm$ the antisymmetry (2.8) of $K$ implies $\kappa (r_\pm , s_\pm ) = \mp i \kappa (K r_\pm , s_\pm ) = \pm i \kappa (r_\pm , K s_\pm ) = - \kappa (r_\pm , s_\pm ) = 0$, so
that (2.13) holds. Conversely, given a decomposition like (2.12), define $K$ by requiring $t_\pm$ to be the eigenspaces of $K$ corresponding to the eigenvalues $\pm i$, assuring that the second equation of (2.3) is fulfilled. Then (2.9), (2.10) can be shown to follow from the fact that $t_\pm$ are subalgebras, while (2.13) implies the first part of (2.8). Namely, for arbitrary $r, s \in t$ one has $r = r_+ + r_-$ and $s = s_+ + s_-$ with $r_\pm, s_\pm \in t_\pm$, and therefore $\kappa(Kr, s) = \kappa(ir_+ - ir_-, s_+ + s_-) = i\kappa(r_+, s_-) - i\kappa(r_-, s_+) = -\kappa(r_+ + r_-, is_+ - is_-) = -\kappa(r, Ks)$.

Our task is now to classify embeddings satisfying (2.8) to (2.11), or, equivalently, (2.12) and (2.13). As the following remarks show, we can assume that $g$ and $h$ are of equal rank. In [14] a sequential method has been introduced which allows us to reduce $N = 2$ coset theories with rank $h < \text{rank } g$ to the equal rank case. (It is worthwhile mentioning that the validity of this sequential algorithm has been proven in [14] only as far as the symmetry algebras of the models are concerned. As for the field contents, the general belief is that for a chain of embeddings $f \hookrightarrow h \hookrightarrow g$ the coset theory $C[g/f]$ carries the structure of the tensor product of the theories $C[g/h]$ and $C[h/f]$, albeit a non-product modular invariant must be used. This is easy to see if no field identification is necessary, and should also hold in the case when the identification currents do not have fixed points.) To apply the sequential method, one needs an intermediate subalgebra satisfying (2.15) (direct sum of Lie algebras). Such an intermediate algebra exists only [15] for the so-called regular subalgebras. A regular subalgebra $h \hookrightarrow g$ is by definition (see e.g. [30]) a subalgebra for which every generator associated to a root of the subalgebra $h$ is also associated to a root of the overlying algebra $g$; all other subalgebras are called special. In [15] it was shown that the cosets derived from special subalgebras never have enlarged supersymmetry; correspondingly we can restrict ourselves in the sequel to regular subalgebras, and hence the sequential algorithm is applicable. Regular subalgebras have been classified by Dynkin [31]; their Dynkin diagram must be a subdiagram of the extended Dynkin diagram of the overlying algebra (the extended Dynkin diagram of a simple Lie algebra $g$ coincides with the Dynkin diagram of its affinization $g^{(1)}$).

In short, we can restrict our attention to regular embeddings satisfying rank $g = \text{rank } h$. We now turn to the classification of such embeddings generating $N = 2$ superconformal coset theories. From the $N = 2$ conditions (2.8) to (2.11), one easily deduces that

$$f^{cde} K_{\hat{a} \hat{b}} f^{\hat{a} \hat{b} \hat{c}} = 0$$

(2.16)

for all $c, d$. We will denote by $\Delta_+, \Delta_-$, and $\Delta_h$ the sets of roots of $t_+, t_-$, and $h$, respectively, and define

$$\tilde{\nu}_0 := \sum_{\tilde{\alpha} \in \Delta_+} \tilde{\alpha}.$$  (2.17)

Writing (2.16) in a Cartan-Weyl basis and comparing prefactors, we find

$$\langle \tilde{\nu}_0, \gamma \rangle = 0 \quad \text{iff} \quad \gamma \in \Delta_h.$$  (2.18)
This relation implies that

\[
\left[ \sum_{\bar{\alpha} \in \Delta_+} \bar{\alpha}_i H^i, T^a \right] = 0 \quad \text{for all} \quad T^a \in h, \quad (2.19)
\]

where by \( H^i \) we denote the generators of the Cartan subalgebra, i.e. that \( h \) contains a \( u(1) \) ideal with generator \( \sum_{\bar{\alpha} \in \Delta_+} \bar{\alpha}_i H^i \). Thus the embedding \( h \hookrightarrow g \) is such that the Dynkin diagram of \( h \) is obtained from the extended Dynkin diagram of \( g \) by removing at least two nodes. One can also show [14] that

\[
(\tilde{v}_0, \bar{\beta}) \geq (\bar{\beta}, \bar{\beta}) > 0 \quad (2.20)
\]

for all \( \bar{\beta} \in \Delta_+ \).

We claim that the subalgebras yielding \( N = 2 \) superconformal cosets are precisely diagram subalgebras, i.e. subalgebras whose Dynkin diagram is contained in the non-extended Dynkin diagram of \( g \). Moreover, if the Dynkin diagram of \( h \) is obtained from that of \( g \) by removing more than one node, then the sequential method alluded to above can be applied to reduce the theory to a tensor product; hence we can assume that only a single node is deleted. We will denote by \( i_o \) the label of this distinguished node of the Dynkin diagram of \( g \); thus, for example, \( \alpha^{(i_o)} \) is the corresponding simple \( g \)-root that is not a root of \( h \). Note that the notation \( \tilde{v}_0 \) introduced in (2.17) was chosen with foresight; for instance, denoting the fundamental \( g \)-weights by \( \Lambda_{(i)} \), the relation (2.18) can be rephrased as

\[
\tilde{v}_0 \propto \Lambda_{(i_o)} \quad (2.21)
\]

(the constant of proportionality, obtainable with the help of the strange formula, reads

\[
\frac{(\theta_g, \theta_g) g^\vee \dim g - \sum_i (\theta_i, \theta_i) h^\vee_i \dim h_i}{12 \sum_{ij} G_{ij}} \quad (2.22)
\]

where \( G_{ij} = (\Lambda_{(i)}, \Lambda_{(j)}) \) denotes the metric on the weight space of \( g \), i.e. the inverse of the symmetrized Cartan matrix).

To prove the above claim, we have to show that the highest root \( \theta_g \) of \( g \) is not a root of \( h \). If \( \theta_g \) were a root of \( h \), then according to (2.18) it would satisfy \( (\tilde{v}_0, \theta_g) = 0 \). But this is not allowed, as can be seen with the help of the decomposition of \( \theta_g \) in terms of the simple \( g \)-roots \( \alpha^{(i)} \),

\[
\theta_g = \sum_{i=1}^{\text{rank } g} a_i \alpha^{(i)}. \quad (2.23)
\]

Namely, the coefficients \( a_i \) on the right hand side of (2.23), known as the Coxeter labels of \( g \), are positive integers, and hence the inequality (2.20) implies \( (\tilde{v}_0, \theta_g) = \sum_i a_i (\tilde{v}_0, \alpha^{(i)}) \geq \sum_{\bar{\alpha} \in \Delta_+} a_i (\bar{\alpha}, \alpha) > 0 \). Thus \( \theta_g \) is not a root of \( h \), so that \( h \) is a diagram subalgebra of \( g \).

The converse is seen as follows. Given a diagram subalgebra \( h \) of \( g \), assign the root \( \bar{\alpha} \) of \( t \) to belong to \( \Delta_+ \) and \( \Delta_- \), respectively, iff it is a positive respectively a negative root of \( g \). Since we assumed that \( g \) and \( h \) have equal rank, this prescription yields a decomposition of \( t \) of the form (2.12). It is now
straightforward to check that the vector spaces generated by the elements corresponding to $\Delta_{\pm}$ satisfy the geometrical formulation of the $N = 2$ conditions. Namely, nilpotency (2.13) is immediate from the well-known properties of the Killing form in a Cartan-Weyl basis; the dimensions of $t_+$ and $t_-$ coincide because positive and negative roots of $g \setminus h$ come in pairs; and the assertion that $t_\pm$ close under the Lie bracket can be verified by using the fact that $(\bar{v}_0, \bar{a}) > 0$ iff $\bar{a} \in \Delta_+$. 

Clearly, the $N = 2$ conditions (2.8) to (2.11) are particularly simple if the structure constants $f_{\bar{a} \bar{b} \bar{c}}$ vanish. As we will see shortly, the corresponding coset manifold is then a hermitian symmetric space. In this case we automatically have rank $h = \text{rank } g$. Moreover using the Jacobi identity together with the relation

$$2f^{\bar{a}\bar{c}\bar{d}}f_{\bar{b}\bar{c}\bar{d}} = f^{\bar{a}C\bar{D}}f_{\bar{b}C\bar{D}} = g^\vee \delta_{\bar{a}\bar{b}},$$

it is easy to show that

$$f_{\bar{c}\bar{d}} K_{\bar{a}\bar{b}} f^{\bar{a}\bar{b}\bar{c}} = g^\vee K_{\bar{c}\bar{d}}.$$  

Similarly as with (2.18), another useful relation is obtained by writing (2.25) in a Cartan-Weyl basis; comparing prefactors one finds

$$(\bar{v}_0, \bar{\gamma}) \equiv \sum_{\bar{a} \in \Delta_+} (\bar{a}, \bar{\gamma}) = g^\vee \iff \bar{\gamma} \in \Delta_+.$$  

With these results, we are in a position to classify all subalgebras yielding hermitian symmetric spaces. Let us first sketch the way these spaces are usually described in the mathematical literature (see e.g. [32]). Given $f_{\bar{a} \bar{b} \bar{c}} = 0$, it is possible to define an involutive automorphism $\sigma$ of the Lie algebra $g$ such that the subalgebra left invariant by $\sigma$ is equal to $h$, namely $\sigma(T^\bar{a}) := T^\bar{a}$, $\sigma(T^\bar{a}) := -T^\bar{a}$. Lie algebras admitting such an automorphism are called orthogonal involutive Lie algebras and have been classified by Cartan; a complete list can be found e.g. in [32, p. 354]. Because of (2.19), among the orthogonal involutive Lie algebras one only has to consider those whose fixed algebra contains a $u(1)$ ideal. Finally, one verifies by inspection that for all such Lie algebras the $N = 2$ conditions are fulfilled.

(The nomenclature used above arises from the following geometrical setting. The fact that $g$ and $h$ form an orthogonal involutive Lie algebra can be shown to be equivalent to the property that the homogeneous space $G/H$, with $G$ and $H$ the compact simply connected Lie groups corresponding to $g$ and $h$, is a $\text{Riemannian globally symmetric space}$. These spaces are defined as follows. For a Riemannian manifold, a neighbourhood of any point $p$ of the manifold can be described by mapping a sphere in the tangent space at $p$ on the neighbourhood; via this map the reflection about the origin of the tangent space (the pre-image of $p$) induces a mapping $\tau$ of this neighbourhood. If $\tau$ is an isometry, the manifold is called a locally symmetric space; if in addition $\tau$ can be extended to a global isometry, the manifold is called a globally symmetric space. It can be shown that all globally symmetric spaces are homogeneous spaces, i.e. isomorphic to the quotient of a simply connected Lie group by a closed subgroup. In this geometrical context the condition (2.19) means that $G/H$ carries in addition an almost complex structure $J$ which is hermitian, i.e.
Table 1: Hermitian symmetric coset theories (HSS) and their Virasoro charges

| $C[g/h]_k$ | $c$ | name               |
|-------------------|-----------------|----------------------|
| $C[A_{m+n-1}/A_{m-1} \oplus A_{n-1} \oplus u(1)]_k$ | $3kmn/(k+m+n)$ | $(A, m, n, k)$       |
| $C[B_{n+1}/B_n \oplus u(1)]_k$ | $3k(2n+1)/(k+2n+1)$ | $(B, 2n+1, k)$      |
| $C[D_{n+1}/D_n \oplus u(1)]_k$ | $6kn/(k+2n)$ | $(B, 2n, k)$        |
| $C[C_n/A_{n-1} \oplus u(1)]_k$ | $3kn(n+1)/2(k+n+1)$ | $(C, n, k)$        |
| $C[D_n/A_{n-1} \oplus u(1)]_k$ | $3kn(n-1)/2(k+n-2)$ | $(D, n, k)$        |
| $C[E_6/D_5 \oplus u(1)]_k$ | $48k/(k+12)$ | $(E_6, k)$          |
| $C[E_7/E_6 \oplus u(1)]_k$ | $81k/(k+18)$ | $(E_7, k)$          |

the metric $g$ satisfies $g(JX, JY) = g(X, Y)$ for all elements $X, Y$ of the tangent space. It can be shown that for homogeneous spaces this automatically implies that $J$ is Kählerian, i.e. covariantly constant. In the general case where $f_{\bar{a}\bar{b}\bar{c}}$ is non-vanishing (which is the situation in which we are interested in the present paper), the homogeneous space $G/H$ is no longer a Riemannian globally symmetric space, but as was shown in [14], it is nonetheless still a Kählerian space iff the $N=2$ conditions are fulfilled. We remark that for our purposes these geometric characterizations are of little use. In fact, one of the main achievements of the theory of homogeneous spaces was precisely to recast the problems in purely Lie algebraic terms, which finally provided a powerful handle on the geometric objects.

Alternatively, the classification of hermitian symmetric spaces can be found by the following simple prescription [33]: the hermitian symmetric spaces are obtained by deleting a node of the Dynkin diagram of $g$ that corresponds to a so-called [34] cominimal fundamental weight, i.e. a fundamental $g$-weight $\Lambda_{(i)}$ such that $a_i = 1$ in the decomposition (2.23) of the highest $g$-root $\theta_g$. To prove this characterization, we proceed as follows. Multiplying both sides of (2.23) with $\tilde{v}_o$ as defined in (2.17), one obtains

$$ (\tilde{v}_o, \theta_g) = \sum_{i=1}^{\text{rank } g} \sum_{\bar{\alpha} \in \Delta_+} a_i (\bar{\alpha}, \alpha^{(i)}). \quad (2.27) $$

Now suppose that $\theta_g$ is a root of $h$. Then according to (2.18) one has $\sum_{\bar{\alpha} \in \Delta_+} (\bar{\alpha}, \theta_g) = 0$. Given the fact that the Coxeter labels $a_i$ are positive, we thus learn from (2.27) that $(\tilde{v}_o, \alpha^{(i)}) = 0$ for all simple roots. But then (2.18) and (2.26) imply that all simple roots of $g$ are contained in $h$, and hence $g = h$, showing that the coset would be trivial in this case. In conclusion, $\theta_g$ cannot be a root of $h$. From (2.26) we then learn that the left hand side of (2.27) equals $g^\vee$. The right hand side can take this value only in the case when exactly one simple root of $g$ with Coxeter label equal to 1 is not contained in $h$. Now using the
Table 2: Non-hermitian symmetric coset theories relevant for $c = 9$ tensor products

| $C[g/h]_k$                                                                 | $c$                                                                 | name         |
|---------------------------------------------------------------------------|----------------------------------------------------------------------|--------------|
| $C[B_n/A_{n-1} \oplus u(1)]_k$                                            | $\frac{3}{2} n(n + 1) - \frac{3n^3}{k + 2n - 1}$                     | $(BA, n, k)$ |
| $C[B_n/B_{n-2} \oplus A_1 \oplus u(1)]_k$                                | $12n - 15 - \frac{24(n - 1)^2}{k + 2n - 1}$                          | $(BB, n, k)$ |
| $C[C_n/C_{n-1} \oplus u(1)]_k$                                           | $6n - 3 - \frac{6n^2}{k + n + 1}$                                   | $(CC, n, k)$ |
| $C[C_3/A_1 \oplus A_1 \oplus u(1)]_k$                                    | $21 - \frac{75}{k + 4}$                                             | $(C3, k)$    |
| $C[C_4/A_2 \oplus A_1 \oplus u(1)]_k$                                    | $36 - \frac{162}{k + 5}$                                            | $(C4, k)$    |
| $C[D_4/A_1 \oplus A_1 \oplus A_1 \oplus u(1)]_k$                         | $27 - \frac{150}{k + 6}$                                            | $(D4, k)$    |
| $C[D_5/A_2 \oplus A_1 \oplus A_1 \oplus u(1)]_k$                         | $45 - \frac{324}{k + 8}$                                            | $(D51, k)$   |
| $C[D_5/A_3 \oplus A_1 \oplus u(1)]_k$                                    | $39 - \frac{294}{k + 8}$                                            | $(D52, k)$   |
| $C[F_4/C_3 \oplus u(1)]_k$                                               | $45 - \frac{384}{k + 9}$                                            | $(F4, k)$    |
| $C[G_2/A_1^+ \oplus u(1)]_k$                                             | $15 - \frac{50}{k + 4}$                                             | $(G2_1, k)$  |
| $C[G_2/A_1^- \oplus u(1)]_k$                                             | $15 - \frac{54}{k + 4}$                                             | $(G2_2, k)$  |

classification of regular subalgebras [31], it is straightforward to check that one obtains in this way exactly the same list as before.

In table 1 we recall the list of all HSS models and their Virasoro charges (the short-hand notation displayed in column 3 is taken from [22]). We now return to the general case. Let us stress that we are in a position to give a complete list of all $N = 2$ coset models. However, even when grouping these theories (of which there are infinitely many) into a finite number of series, this list still remains rather long, and we will not present it here in full detail. Rather, we list only those models that can be used as factor theories in tensor products with conformal central charge $c = 9$ (as well as some other models which fall into infinite series that contain models relevant for $c = 9$). The interest in these models comes from superstring theory where they can be used for the inner sector of heterotic string vacua [25], and from the possible relation with Calabi-Yau manifolds and with Landau-Ginzburg theories.

The result of our classification is presented in table 2, where we supply the
coset theories together with their conformal central charge (as calculated according to (2.6)) and with a short-hand name that derives from the Lie algebras involved. From the classification of regular subalgebras described above, the relevant embedding \( h \hookrightarrow g \) is determined uniquely by the pair \( g, h \) of Lie algebras for all entries in table 2 except for the two models with \( g = G_2 \). In the latter cases we use the superscripts \( ' < ' \) and \( ' > ' \) to indicate that the \( A_1 \)-subalgebra corresponds to the short and long simple root of \( G_2 \), respectively.

For convenience we have grouped some models in the table in three series. From the above remarks it should be clear that there is no physical distinction between the models within these series and the other models. The different appearance is a mere artefact of our string theory-oriented condition on the central charges. We also emphasize that the list in table 2 does not contain all \( N = 2 \) coset theories with central charge \( c \leq 9 \). Their number is much larger, but most of them cannot be combined with other known \( N = 2 \) theories to obtain \( c = 9 \) tensor product theories. For instance, we have not included the model \( C[D_6/D_4 \oplus A_1 \oplus u(1)]_k \), which has \( c = 51 - \frac{486}{k+10} \). For level \( k = 1 \) the conformal central charge is \( c = \frac{75}{11} < 9 \), but there does not exist any \( N = 2 \) model with \( c = \frac{75}{11} \) which could be tensored with this theory to arrive at a \( c = 9 \) conformal field theory.

Note that the number of the models so obtained is relatively small. This can be traced back to two simple facts. First, if \( g \) is a Lie algebra of \( A \) type, all subalgebras lead to coset theories of the HSS type. Second, for any fixed Lie algebra \( g \), the central charge of the coset theory grows rather fast when one moves the node with label \( i \) away from the 'margin' of the Dynkin diagram of \( g \) towards the inner part (note that except for \( A_r \) all cominimal fundamental weights, i.e. those leading to hermitian symmetric cosets, correspond to marginal nodes).

3 Specification of the coset theories

As already emphasized, the ‘Lie-algebraic coset’ as it stands in (1.1) is in itself far from defining a consistent modular invariant conformal field theory. In this section we will provide a detailed specification of the conformal field theory.

In fact, the first step to do so was already taken in the previous section when we computed the levels (2.4) of the semi-simple part of the subalgebra \( h \), i.e. of the simple ideals in the decomposition (2.3), which in the case of our interest reads

\[
h = \hat{h} \oplus u(1) = \bigoplus_i h_i \oplus u(1).
\] (3.1)

But the abelian ideal of \( h \) must be specified as well.

3.1 The \( u(1) \) subalgebra

The conformal field theory corresponding to a \( u(1) \) algebra has Virasoro charge \( c = 1 \). As all \( c = 1 \) conformal field theories have been classified [35, 36] and their field contents is known, it is sufficient to have a look at the conformal dimensions occurring in the conformal field theory we are after, which, as we shall show now, in turn are fixed by the embedding.
The direction of the $u(1)$ in root space is given by $\tilde{v}_0$. From the embedding (2.2) we read off the precise form of the $u(1)$-generator $Q$; it is proportional to

$$Q(z) := (\tilde{v}_0, H(z)) + \sum_{\bar{\alpha} \in \Delta^+} (\tilde{v}_0, \bar{\alpha}) :\Psi^{\bar{\alpha}}\Psi^{-\bar{\alpha}}(z).$$

(3.2)

Here $:\Psi^{\bar{\alpha}}\Psi^{-\bar{\alpha}}:$ denotes the fermion number operator for the complex fermion that is associated to the root $\bar{\alpha}$; it takes integer values in the Neveu-Schwarz sector and half-integer values in the Ramond sector; $H$ stands for the Cartan subalgebra currents of $g$.

By replacing $\tilde{v}_0$ in (3.2) by an appropriate multiple $v$ of $\tilde{v}_0$, all eigenvalues of $Q$ can be taken to be integers. We will assume that we have chosen the smallest multiple fulfilling this requirement (otherwise we would be forced later on to introduce additional identification currents that have a non-trivial component only in the $u(1)$ part), and write

$$\tilde{v}_o \equiv \sum_{\bar{\alpha} \in \Delta^+} \bar{\alpha} = \xi_o v_o;$$

(3.3)

the number $\xi_o$ turns out to be an integer or half integer in all cases except for the model of type $G2$; for which $\xi_o = 5/3$. The operator product of $Q$ with itself then reads

$$Q(z) \cdot Q(w) \sim \frac{N}{(z-w)^2},$$

(3.4)

with

$$N = (v_0, v_0) k + \sum_{\bar{\alpha} \in \Delta^+} (v_0, \bar{\alpha})^2 = (v_0, v_0) (k + g^\vee).$$

(3.5)

Denote by $\varphi$ a canonically normalized free boson, satisfying $i \partial \varphi(z) i \partial \varphi(w) \sim (z-w)^{-2}$. Expressing $Q$ in terms of $\varphi$, i.e. $Q = \sqrt{N} i \partial \varphi$, we obtain the energy-momentum tensor

$$T = \frac{1}{2} : i \partial \varphi i \partial \varphi : = \frac{1}{2N} : QQ :.$$

(3.6)

Thus the conformal dimension $\Delta$ of a primary field is

$$\Delta = \frac{Q^2}{2N},$$

(3.7)

with $Q$ the $u(1)$-charge of the field, i.e. the eigenvalue of $Q$.

Thus the $u(1)$ theory in question is the conformal field theory of a free boson compactified on a torus whose radius is adjusted (or, in other words, the chiral algebra is enlarged) precisely in such a manner that the charges are identified modulo $N$. In the sequel we will denote this theory by $u(1)_N$. The relevant values of the integer $N$ (as well as the explicit values of the levels of the simple ideals $h_{\bar{\alpha}}$ computed according to (2.4) for the cases of our interest are provided in table 3.

\[\text{Thus e.g. } u(1)_2 \text{ is the theory for which the extended algebra is the level one } A_1^{(1)} \text{ Kac-Moody algebra, and } u(1)_4 \cong so(2)_1.\]
part one could use (as has been done in \[24\]) hermitian symmetric cosets, however, we encounter two cases, namely (one correspondence with the

For any coset theory, a plausible guess would seem to be that they are in one to one correspondence with the \(N\) for non-hermitian symmetric coset theories.

| name               | \(C[g_k \oplus \text{so}(2d)_1 / \bigoplus_i (h_i)_{k_i} \oplus \text{u}(1)_{N^i}]\) |
|-------------------|-----------------------------------------------------------------|
| \((BA, n, k)\), \(n\) even | \(C[(B_n)_{k} \oplus \text{so}(n^2 + n)_1 / (A_{n-1})_{k+n-1} \oplus \text{u}(1)_{n(k+2n-1)}]\) |
| \((BA, n, k)\), \(n\) odd  | \(C[(B_n)_{k} \oplus \text{so}(n^2 + n)_1 / (A_{n-1})_{k+n-1} \oplus \text{u}(1)_{4n(k+2n-1)}]\) |
| \((BB, 3, k)\)      | \(C[(B_3)_{k} \oplus \text{so}(14)_1 / (A_1)_{2k+8} \oplus (A_1)_{k+3} \oplus \text{u}(1)_{2(k+5)}]\) |
| \((BB, n, k)\), \(n > 3\) | \(C[(B_n)_{k} \oplus \text{so}(8n - 10)_1 / (B_{n-2})_{k+4} \oplus (A_1)_{k+2n-3} \oplus \text{u}(1)_{2(k+2n-1)}]\) |
| \((CC, n, k)\)      | \(C[(C_n)_{k} \oplus \text{so}(4n - 2)_1 / (C_{n-1})_{k+1} \oplus \text{u}(1)_{2(k+n+1)}]\) |
| \((C3, k)\)        | \(C[(C_3)_{k} \oplus \text{so}(14)_1 / (A_1)_{k+2} \oplus (A_1)_{2k+6} \oplus \text{u}(1)_{4(k+4)}]\) |
| \((C4, k)\)        | \(C[(C_4)_{k} \oplus \text{so}(24)_1 / (A_2)_{2k+7} \oplus (A_1)_{k+3} \oplus \text{u}(1)_{6(k+5)}]\) |
| \((D4, k)\)        | \(C[(D_4)_{k} \oplus \text{so}(18)_1 / (A_1)_{k+4} \oplus (A_1)_{k+4} \oplus \text{u}(1)_{2(k+6)}]\) |
| \((D5_1, k)\)      | \(C[(D_5)_{k} \oplus \text{so}(30)_1 / (A_2)_{k+5} \oplus (A_1)_{k+6} \oplus (A_1)_{k+6} \oplus \text{u}(1)_{12(k+8)}]\) |
| \((D5_2, k)\)      | \(C[(D_5)_{k} \oplus \text{so}(26)_1 / (A_3)_{k+4} \oplus (A_1)_{k+6} \oplus \text{u}(1)_{2(k+8)}]\) |
| \((F4, k)\)        | \(C[(F_4)_{k} \oplus \text{so}(30)_1 / (C_3)_{k+5} \oplus \text{u}(1)_{2(k+9)}]\) |
| \((G2_1, k)\)      | \(C[(G_2)_{k} \oplus \text{so}(10)_1 / (A_1)_{k+2} \oplus \text{u}(1)_{6(k+4)}]\) |
| \((G2_2, k)\)      | \(C[(G_2)_{k} \oplus \text{so}(10)_1 / (A_1)_{3k+10} \oplus \text{u}(1)_{2(k+4)}]\) |

For hermitian symmetric cosets it was noticed \[24\] that \(N\) is always a divisor of \(N_0(g, h)\), where

\[
N_0(g, h) = I_c(g) \cdot I_c(h) \cdot (k + g').
\] (3.8)

Here \(I_c\) stands for the index of connection (i.e. the number of conjugacy classes, which is equal to the order of the center \(Z\) of the corresponding universal covering Lie group) of a Lie algebra, and \(I_c(h_i) \equiv \prod_i I_c(h_i)\), where \(h_i\) are the simple algebras which appear in the decomposition \(h = \bigoplus_i h_i\) of \(h\) into simple ideals. In fact, in most cases one even has \(N = N_0(g, h)\); also, by introducing additional identification currents with a non-trivial component only in the \(u(1)\) part one could use (as has been done in \[24\]) \(N_0(g, h)\) in place of \(N\). For non-hermitian symmetric cosets, however, we encounter two cases, namely \((G2_1, k)\) and the models \((BA, n, k)\) with \(n\) odd, where the value of \(N\) is larger than \(N_0(g, h)\).

### 3.2 Selection rules and field identification

Our next task is to identify the physical fields of the theories of our interest. For any coset theory, a plausible guess would seem to be that they are in one to one correspondence with the branching functions \(b_{\Lambda, Q}^{k+3}\) of the embedding. These
objects are the coefficient functions in the decomposition

\[ \chi_{\Lambda,\lambda}(\tau) = \sum_{\lambda} b_{\Lambda,\lambda}^{\Lambda,\lambda}(\tau) \chi_{\lambda,Q}(\tau) \]  

(3.9)
of the product of the characters of \( g \) and \( \text{so}(2d) \) with respect to the characters of \( h \). Here \( \Lambda \) and \( \lambda \) stand for integrable highest weights of \( g \) and \( \hat{h} \), respectively, and \( Q \) for an allowed \( u(1) \)-charge, while \( x \) denotes an integrable highest weight of \( \text{so}(2d) \) at level one, i.e. the singlet \((0)\), vector \((v)\), spinor \((s)\), or conjugate spinor \((c)\) highest weight.

However, this naive ansatz is usually in conflict with the requirement that the characters of the coset theory carry a (projective) unitary representation of the modular group \( \text{PSL}(2,\mathbb{Z}) \). Namely, on one hand, generically some branching functions turn out to vanish identically, while the matrix \( S_g S_h^* \) that describes the transformation of the branching functions under the modular transformation \( \tau \mapsto -\frac{1}{\tau} \) has also non-zero elements between vanishing and non-vanishing branching functions (this is an immediate consequence of the fact that in any conformal field theory the modular matrix \( S \) obeys \( S_{0i} \geq S_{00} > 0 \), and that for any embedding of affine Lie algebras the module with highest \( h \)-weight zero always appears in the decomposition of the \( g \)-module with highest weight zero). On the other hand, typically branching functions for distinct combinations of highest weights coincide, in such a way that the matrix \( S_g S_h^* \) has identical rows and columns and hence cannot be unitary.

One may imagine three distinct sources for the vanishing of a branching function. First, usually the matching of conjugacy classes of \( g \)- and \( h \)-modules provides selection rules. Second, a state that naively is expected to be a highest weight state may turn out to be a null state of a Verma module of the affinization of \( h \). And third, it might happen that a given highest weight module \( L \) of the reductive subalgebra \( h \) does appear in some module of the affinization of \( g \), but that it gets always combined with other \( h \)-modules to modules of the affinization \( h^{(1)} \) of \( h \) that carry a different highest \( h \)-weight, so that (the affine extension of) the highest weight of \( L \) never occurs as a highest weight of a module of \( h^{(1)} \). Although no general arguments excluding the latter possibility are known, no example where it is realized has been found so far. It is therefore common to assume that this last mentioned possibility never arises in coset theories. Moreover, one usually also assumes that null states must only be taken into account for \( c = 0 \) coset theories.\(^3\)

The correct way to arrive at a modular invariant theory is to interpret the primary fields of the coset theory in terms of equivalence classes of branching functions [24, 27, 37]; by a slight abuse of terminology, this prescription is usually referred to as field identification. Under the assumptions just mentioned, the equivalence relation is uniquely determined by the conjugacy class selection rules. If all equivalence classes have the same number of elements, one can simply define a primary field as an equivalence class of branching functions.

\(^3\) As already mentioned in the introduction, there exists one counter example to this assumption, namely [29] the coset theory \( \mathcal{C}[(A_2)_2/(A_1)_4] \). Note that the similar coset theory \( \mathcal{C}[(A_2)_1/(A_1)_4] \) has \( c = 0 \), i.e. the underlying embedding is a conformal embedding.
Its character is then just any of the (identical) branching functions of its representatives, and accordingly the primary field can be denoted as \( \Phi_{\lambda,Q}^{\Lambda,x} \), where \( (\Lambda, x, \lambda, Q) \) is a representative combination of the relevant highest weights \( \Lambda \) of \( g \), \( x \) of \( \text{so}(2d) \), \( \lambda \) of \( \hat{h} \), and \( Q \) of \( u(1) \). If, on the other hand, several distinct sizes of equivalence classes are present, one has to be more inventive; the additional manipulations, known as the resolution of fixed points, will be addressed in the next subsection.

Our task is thus to find the relevant selection rules and deduce the identifications implied by them. This is a straightforward exercise in group theory, but is still somewhat involved owing to the non-trivial embedding of \( \hat{h} \) in \( \text{so}(2d) \).

A convenient way to state these selection rules is by means of simple currents. A simple current \( \phi_J \) of a conformal field theory is by definition a primary field whose fusion product with any primary field \( \phi_i \) consists of a single primary field, \( \phi_J \star \phi_i = \phi_{J \star i} \); the combination

\[
Q_m(\phi_i) := \Delta(\phi_i) + \Delta(\phi_J) - \Delta(\phi_{J \star i}) \tag{3.10}
\]

of conformal dimensions is known as the monodromy charge of \( \phi_i \) with respect to \( \phi_J \). The simple currents of WZW theories are all known [38], and for the \( u(1) \) theories a primary field of arbitrary charge \( Q \) is a simple current. For any conformal field theory, the subring of the fusion ring that is generated by the simple currents of the theory is isomorphic to the group ring of an abelian group \( \mathcal{G} \) whose group operation is the one implied by the fusion product. Now denote by \( \mathcal{G}_a \) the direct product of these groups obtained from the simple currents of the \( g-, h-i- \) and \( u(1)- \) parts of the coset theory. It is possible [39, 28] to characterize the non-vanishing branching functions by the fact that their monodromy charge with respect to some subgroup of \( \mathcal{G}_a \) vanishes. We will refer to this subgroup as the identification group \( \mathcal{G}_{id} \) of the coset theory and denote its order by \( |\mathcal{G}_{id}| \). The elements of \( \mathcal{G}_{id} \) are usually called identification currents. Their orbits on the branching functions are just the equivalence classes we are looking for. To qualify as an identification current, a simple current must have integer conformal weight [28] (this allows for a simple check of our results for the identification currents); this condition must be met because any identification current is a representative of the equivalence class describing the identity primary field, and conformal weights are constant modulo integers on each identification orbit.

To begin the description of identification currents for the theories of our interest, we derive a formula for the order \( |\mathcal{G}_{id}| \) of the identification group of any \( N = 2 \) coset theory of the form \(|\Box|\). This provides an important check for the completeness of the selection rules that will be listed below. Our starting point is the formula [3]

\[
|\mathcal{G}_{id}| = \left| \frac{L^*_g}{L^*_h} \right| \tag{3.11}
\]

Here \( L \) denotes the root lattice of a reductive algebra, and \( L^\vee \) the corresponding coroot lattice. The symbol ‘*’ is used to indicate the dual lattice; in particular \( (L^\vee)^* = L^W \), where \( L^W \) the weight lattice. Writing the relation (3.11) in terms
of the dual lattices and denoting the volume of the unit cell by ‘vol’, we see that
\[ |G_{id}| = \left| \frac{(L_h^W)^*}{(L_g^*)} \right| = \left| \frac{L_h^W}{L_g} \right| = \frac{\text{vol}(L_g)}{\text{vol}(L_h^W)}. \] (3.12)

Since the direction of the u(1) is orthogonal to \( \hat{h} \) in weight space, it follows that
\[ \text{vol}(L_h^W) = \text{vol}(L_h^W) \cdot 1 = \text{vol}(L_h^W) \] (3.13)
and
\[ \text{vol}(L_g) = \text{vol}(L_h) \cdot Q_{i_o}, \] (3.14)
where \( Q_{i_o} \) is the u(1)-charge of the simple root \( \alpha^{(i_o)} \). Thus
\[ |G_{id}| = Q_{i_o} \frac{\text{vol}(L_h)}{\text{vol}(L_h^W)} = Q_{i_o} \left| \frac{L_h^W}{L_h} \right| \equiv Q_{i_o} I_c(\hat{h}). \] (3.15)

Here \( I_c(\hat{h}) = \prod I_c(h_i) \) as in \[3.8\], and we made use of the fact that \( I_c(h) = |L_h^W/L_h| \) for any simple Lie algebra \( h \).

While the result \[3.15\] is completely general, the precise form of the group theoretical selection rules must be determined in a case by case study. To do so, a rather tedious investigation of the way \( \hat{h} \) is embedded in \( g \oplus \text{so}(2d) \) is necessary. In particular a careful handling of the embedding of \( h \) in \( \text{so}(2d) \) (best to be described in an orthogonal basis which corresponds to the free fermion realization of \( \text{so}(2d) \)) is required. We list in table \[3\] our results for the identification currents \( \phi_J \) of all non-hermitian symmetric \( N = 2 \) coset theories that can be used in \( c = 9 \) tensor products. We use the notation \( \phi_J \equiv (j^{(g)}, j^{(\text{so}(d))}, j^{(h_1)}, j^{(h_2)}, \ldots, j^{(u(1))}) \). In the individual entries, we write \( J_v \) for the vector simple current, and \( J_h \) and \( J_c \) for the spinor and conjugate spinor simple currents, respectively, of \( B \) and \( D \) type algebras, while for \( A \) type algebras, \( J \) stands for the simple current that acts as \( \mu^i \mapsto \mu^{i+1 \mod (r+1)} \) on the Dynkin labels of an \( A_r \)-weight (this current is associated with a marginal node of the Dynkin diagram; it has maximal order, and hence generates all simple currents of the theory); finally, for the u(1) part a field is simply denoted by its u(1)-charge \( Q \). Notice that in table \[3\] we only give a set of generators of the group \( G_{id} \) rather than all of its elements.

The way in which we arrived at these results is best described by giving an example. Thus let us have a look at the coset theory denoted by \((C4, k)\). We denote the Dynkin labels of weights of \( g = C_4 \) by \( \Lambda^i \), \( i = 1, 2, 3, 4 \), of weights of \( h_1 = A_2 \) by \( \lambda^1 \) and \( \lambda^2 \), of weights of \( h_2 = A_1 \) by \( \lambda^3 \), and the u(1)-charge by \( Q \). By analysing the embedding, we find that these numbers must be related by
\[ 3\Lambda^1 + 3\Lambda^3 - 6N + 2\lambda^1 + 4\lambda^2 + 3\lambda^4 + Q \equiv 0 \mod 6, \] (3.16)
where \( 6N \) stands for the sum of six different eigenvalues of the Cartan generators of \( \text{so}(24) \), which have integer values in the Neveu-Schwarz sectors and half

\[ \text{mod 6}. \] (3.16)

\[ 6 \] This is not in conflict with the previously mentioned result \[3\] that \( N = 2 \) symmetry requires regular embeddings. The part of the embedding that must be regular is \( h \leftrightarrow g \) rather than \( h \leftrightarrow \text{so}(2d) \).
Table 4: The identification groups for non-hermitian symmetric coset theories

| name          | $|G_{id}|$ | generators of $G_{id}$                  | fixed p. |
|---------------|---------|-----------------------------------------|----------|
| $(BA, n, k)$, n even | $n$     | $(J, 1 / J, k + 2n - 1)$                | −        |
| $(BA, n, k)$, n odd   | $2n$    | $(J, J_v / J, 2(k + 2n - 1))$           | −        |
| $(BB, n, k)$       | $4$     | $(J, 1 / J, 1, 0)$                      | ++       |
|                  |         | $(J, 1 / J, 1, 1, 0, ±(k + 2n - 1))$     | −        |
| $(CC, n, k)$       | $2$     | $(J, (J_v)^n / J, ±(k + n - 1))$         | −        |
| $(C3, k)$         | $4$     | $(J, 1 / J, 1, 0)$                      | +        |
|                  |         | $(J, 1 / J, 1, ±2(k + 4))$               | −        |
| $(C4, k)$         | $6$     | $(J, J_v / J, J, -(k + 5))$              | −        |
| $(D4, k)$         | $8$     | $(J, 1 / J, J, 1, 0)$                    | +        |
|                  |         | $(J_s, 1 / J, J, 1, 0)$                  | +        |
|                  |         | $(1, J_v / J, J, J, ±(k + 6))$           | −        |
| $(D5_1, k)$       | $24$    | $(J, J_v / J, J, 1, -(k + 8))$           | −        |
|                  |         | $(J_s, 1 / J, J, 1, 0)$                  | −        |
| $(D5_2, k)$       | $8$     | $(J_v, 1 / J, J, 1, 0)$                  | +        |
|                  |         | $(J_v, J_v / J, J, 1, ±(k + 8))$         | −        |
|                  |         | $(J_v, 1 / J, J, 1, ±(k + 8))$           | −        |
| $(F4, k)$         | $2$     | $(1, 1 / J, ±(k + 9))$                   | −        |
| $(G2_1, k)$       | $2$     | $(1, J_v / J, ±3(k + 4))$                | −        |
| $(G2_2, k)$       | $2$     | $(1, J_v / J, ±(k + 4))$                 | −        |

integer values in the Ramond sector. We want to interpret this result as a relation for monodromy charges, namely

$$Q_m[C_4] + Q_m[so(24)] + Q_m[A_2] + Q_m[A_1] + Q_m[u(1)] \equiv 0 \mod 1. \quad (3.17)$$

It is easily checked that $N \mod \mathbb{Z}$ is the monodromy charge with respect to the vector current $J_v$ of $so(2d)$, and that $Q/p$ is the monodromy charge with respect to the current with $u(1)$-charge $-N/p$ of the $u(1)$ theory. For the $g$ and $h_i$ parts, the identification currents can also be fixed uniquely, simply because all simple currents, as well as the associated monodromy charges, of the corresponding WZW theories are known. We then arrive at the combination

$$\phi_J = (J, J_v / J, J, -(k + 5)) \quad (3.18)$$

of simple currents that has (3.17) as its monodromy charge. This current has order 6. This coincides with the result of formula (3.13) for the order of the identification group, and hence we have already found all identification currents.
3.3 Fixed points

If the equivalence classes described in the previous subsection have different sizes $N$, the identification procedure becomes more complicated. Note that the maximal size of a class is equal to the size $N_0 = |G_{id}|$ of the equivalence class of the identity primary field, and that any other allowed size is a divisor of $N_0$. The equivalence classes of size $N < N_0$ should correspond to $N_0/N$ distinct physical fields [27, 28]. The required resolution of classes of non-maximal size into primary fields is problematic because not all necessary pieces of information are directly supplied by the embedding; in other words, the resolution potentially introduces some arbitrariness in the description of primary fields. In particular we do not know the characters of the individual primary fields into which such a class $f$ is resolved. We do know, however, their sum, since modular invariance imposes the constraint

$$\sum_i \mathcal{X}_{fi} = \frac{N_0}{N_f} \mathcal{X}_f,$$  \hspace{1cm} (3.19)

where $\mathcal{X}_f$ denotes the original branching function of the class $f$.

In addition, fortunately certain sum rules for the modular transformation matrix $S$ of the full theory can be derived [27]. For brevity, we will refer to a class of non-maximal size as a fixed point of the field identification. Now given the naive $S$-matrix element $S_{fg}$ between two fixed points $f$ and $g$, one can make the ansatz

$$\tilde{S}_{fi, gj} = \frac{N_f N_g}{N_0} S_{fg} + \Gamma_{fg}^{ij}$$  \hspace{1cm} (3.20)

for the full $S$-matrix between different fields $f_i, g_i$ into which the fixed points are to be resolved. The matrix $\Gamma$ introduced here must be symmetric (with respect to the double index $(f, i)$), but a priori is otherwise arbitrary. Modular invariance can be shown to imply the sum rules

$$\sum_i \Gamma_{fi, gj} = 0 = \sum_j \Gamma_{fi, gj}.$$  \hspace{1cm} (3.21)

To find a solution for $\Gamma$ we assume that with respect to the individual entries of the multi-index $(f, i) \equiv (\Lambda, x, \lambda, Q, i)$ it factorizes as

$$\Gamma_{\Lambda x, \lambda Q; \Lambda' x'; \lambda' Q'} = \Gamma_{\Lambda \lambda; \Lambda' \lambda'}^{\Lambda x \lambda Q; \Lambda' x' \lambda' Q'} = \Gamma_{(g)}^{\Lambda \Lambda'} \Gamma_{(so(d))}^{x x'} \Gamma_{(\hat{h})}^{\lambda \lambda'} \Gamma_{(u(1))}^{Q Q'} P_{ij},$$  \hspace{1cm} (3.22)

where

$$P_{ij} = \delta_{ij} - \frac{N_f}{N_0}.$$  \hspace{1cm} (3.23)

Since in all cases of our interest the fixed points $f$ have order $N_0/N_f = 2$ and must therefore be resolved into two fields, the fact that (3.23) can be factored out is an immediate consequence of the sum rules (3.21). Following [28], with the factorization assumption (3.22) we can identify in all cases a so-called fixed point conformal field theory, whose characters can be added to the branching functions to get the full collection of primary fields; these characters are nothing but the summands $\mathcal{X}_f(\tau)$ in the decomposition (3.19) above.
This procedure of fixed point resolution is certainly quite important, because it is only after having accomplished this task that we really deal with a well-defined conformal field theory (it is even unknown whether the prescription works for an arbitrary coset theory, and whether the conformal field theory it provides is unique). However, it is not difficult to see that some important quantities we will be interested in, namely the number of generations and anti-generations in a four-dimensional string compactification, can be obtained in our case without a detailed knowledge of the resolution procedure (see also the comments in the following section).

In the third column of table 4 we marked whether identification fixed points occur in the theories in question. The following notation is used: ‘–’ indicates that fixed points never occur in the corresponding theory; ‘+’ means that fixed points can occur, but not at any of the levels that are relevant for $c = 9$ tensor products (this typically happens when we are only interested in low levels where the associated outer automorphisms of $g$ act freely on the integrable representations of $g$); finally ‘++’ is used to indicate that fixed points occur and have to be resolved. Note that an identification current can possess a fixed point only if it has vanishing $u(1)$-charge.

### 3.4 Modular invariants

It should be noted that the discussion of field identification in the previous subsections refers only to one chiral half of the conformal field theory. For the full theory, one has to use all fields as identification currents, i.e. as representatives of the identity primary field, that have non-vanishing branching functions and are identification currents with respect to both the holomorphic and the anti-holomorphic part. For example, for the $N = 2$ minimal models this prescription implies the presence of left-right asymmetric identification currents if the $D_{\text{even}}$, $E_6$, or $E_8$ type invariants of the associated $A_1$ WZW theory are chosen.

For the $N = 2$ theories of our present interest, we will confine ourselves to analyse only the situation where the diagonal modular invariants of $g$, $h$ and $\text{so}(2d)$ are used. As a consequence, the identification currents are just the left-right symmetric version of the chiral currents listed in table 3. The extension to any known non-diagonal modular invariant is immediate; note however that the classification of modular invariants of simple Lie algebras other than $A_1$ (and of their tensor products) is far from being complete.

### 4 Chiral ring and Poincaré polynomials

In this section we present some results for quantities relevant to string compactification. We have computed the quantities which are the most relevant ones for the phenomenological aspects, namely the number of (anti-)generations for a compactification of the heterotic string to four space-time dimensions. To obtain these numbers, we need some information on the collection of chiral primary fields of the theory; these fields are by definition those primary fields which satisfy $q_{\text{suco}} = \Delta/2$. They generate the chiral ring of the theory; this
is a finite-dimensional nilpotent ring $R$ whose product is the naive operator product $\lim_{z \to w} \phi(z)\phi'(w)$.

For the models under consideration, it is in fact easier to work with the ground states of the Ramond sector, which owing to spectral flow\footnote{In the literature sometimes a normalization is chosen where $\frac{1}{2} q_{\text{suco}}$ is the superconformal charge.} provide equivalent information on the theory. Namely, the chiral primary fields (with superconformal charge $\frac{1}{2} q_{\text{suco}}$) are via spectral flow in one to one correspondence with Ramond ground states (with superconformal charge $q_{\text{suco}} - c/6$). In all $N = 2$ coset models of the form (1.1) we can identify the simple current in the Ramond sector which generates the flow; it is the unique Ramond ground state with highest superconformal charge, which has been termed spinor current in \cite{spinor_current}. It is easily seen that one representative of the spinor current is the field

$$S = \Phi_{0,Q_s}^0,$$

with

$$Q_s = (v_g, \rho_g - \rho_h).$$

Here $\rho_g = \sum_i \Lambda_{(i)}$ and $\rho_h$ are the Weyl vectors, i.e. half the sum of positive roots, of $g$ and of $\hat{h}$, respectively.

The information on the multiplicities of chiral states with a given superconformal charge is encoded in the Poincaré polynomial\footnote{\footnotesize Compare the remark about $E_6$ singlets in section 4.4 below.}, which can be defined as a trace over the chiral ring $R$, \[ P(t, \bar{t}) := \text{Tr}_{R} t^{J_0} \bar{t}^{\bar{J}_0}. \] Here $J_0$ denotes the generator of the superconformal $u(1)$, and the barred quantities refer to the second chiral half of the theory. In the sequel we will only consider the left-right symmetric diagonal modular invariant; correspondingly we can restrict ourselves to one chiral half and replace $t\bar{t}$ for the sake of simplicity by $t$.

4.1 Ramond ground states

To determine the ground states of the Ramond sector one can use a simple formula for the $g$- and $h$-weights of these states which can be derived\footnote{Compare the remark about $E_6$ singlets in section 4.4 below.} by means of an index argument. The advantage of this formula is twofold. First, in coset models it is usually difficult to calculate the integer part of the conformal weight $\Delta$ of a primary field; for Ramond ground states (which all have $\Delta = c/24$), however, the index argument makes it possible to identify the state without having to evaluate a formula for $\Delta$. Second, the formula automatically takes care of possibly arising null states; again, this is a rather delicate issue in the general case.\footnote{Denote by $W_g$ the Weyl group of $g$, by $|W_g|$ its order, and by $W_h$ and $|W_h|$ the analogous quantities for $\hat{h}$. For any integrable $g$-weight $\Lambda$, the recipe of [3]...}

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provides \(|W_g|/|W_h|\) Ramond ground states. The \(h\)-weight \(\tilde{\lambda}\) of each of these Ramond ground states is related to its \(g\)-weight by

\[
\tilde{\lambda} = w(\Lambda + \rho_g) - \rho_h. \tag{4.4}
\]

Here the weight \(\tilde{\lambda} \equiv (\lambda, Q)\) incorporates both the weight \(\lambda\) of the semi-simple part \(\hat{h}\) of \(h\) and the \(u(1)\)-charge \(Q\). Also, the map \(w\) in (4.4) is the representative of any class of the coset \(W_g/W_h\) possessing the property that \(\tilde{\lambda}\) is a dominant integral highest weight of \(h\); each class of \(W_g/W_h\) contains a unique representative \(w\) satisfying this requirement [40]. If \(\text{sign}(w) = 1\), then the highest weight of \(so(2d)\) that is associated to \(\tilde{\lambda}\) is the spinor (s), while for \(\text{sign}(w) = -1\), it is the conjugate spinor (c). Note that \(\rho_g - \rho_h \propto \Lambda\left(\varepsilon\right)\), the constant of proportionality being \(\sum_j (\Lambda_{(j)}, \Lambda_{(\varepsilon)}) / (\Lambda_{(\varepsilon)}, \Lambda_{(\varepsilon)})\) as can be deduced from \((\rho_h, \rho_g - \rho_h) = 0\).

To implement the formula (4.4) on a computer, it is convenient not to start with the weights of \(g\), but rather to scan all dominant weights of \(h\) that are allowed by the selection rules. For each such weight \(\tilde{\lambda}\) one determines the unique dominant integral \(g\)-weight which lies on the same \(W_g\)-orbit as \(\tilde{\lambda} + \rho_h\) (if this \(g\)-weight is not integrable at the relevant level of the affine algebra \(g^{(1)}\), then the corresponding state has to be rejected). To do so, one only has to know the action of the fundamental reflections \(w_i \in W_g\) (see e.g. [41]). This method has the advantage that one needs not know the whole \(W_g\)-orbit of a highest \(g\)-weight which, especially for large rank algebras, would require a lot of memory.

### 4.2 Poincaré polynomials

Having found the Ramond ground states, we can proceed to compute the Poincaré polynomial of an \(N = 2\) coset theory. To do so, we also need the superconformal charge of the Ramond ground states. This charge is given by

\[
q_{\text{suco}} = \sum_{\tilde{\lambda} \in \Delta^+} \tilde{\lambda}^\alpha - \frac{\xi_o Q}{k + g^\alpha}. \tag{4.5}
\]

Here \(\xi_o = \sqrt{(\bar{v}_o, \bar{v}_o)/(v_o, v_o)} = \sqrt{(k + g^\alpha)(\bar{v}_o, \bar{v}_o)/N}\) is the number defined by (3.3), \(Q\) is the \(u(1)\)-charge of the Ramond ground state, and \(\tilde{\Lambda}^\alpha \in \{\frac{1}{2}, -\frac{1}{2}\}\) are the components of its \(so(2d)\)-weight in the orthogonal basis. Unfortunately the index argument \([3]\) leading to (4.4) does not provide the full weight \(\hat{\Lambda}\), but only yields the information whether it is a weight of the the spinor or of the conjugate spinor module of \(so(2d)\), or in other words, only the value of \(\sum_{\alpha \in \Delta^+} \tilde{\Lambda}^\alpha\) modulo 2. To translate (4.5) into a more convenient formula, we proceed as follows.

Denote by \(\Delta^+_g, \Delta^-_g, \) and \(\Delta^g\) the sets of positive roots, of negative roots, and of all roots, respectively, of the Lie algebra \(g\), and by \(\Delta^+_h, \Delta^-_h\) the corresponding quantities for \(\hat{h}\). For an arbitrary element \(w\) of the Weyl group \(W_g\) define

\[
\Delta^w_\pm := \{\alpha \in \Delta^g \mid w^{-1}(\alpha) \in \Delta^w_\pm\}. \tag{4.6}
\]

\(^7\) An analogous result has been obtained in [6] for simply laced hermitian symmetric cosets at level one, and in [3] for all hermitian symmetric cosets in their free field realization.
For any \( w \in W_g \), \( \Delta^g \) is the disjoint union of \( \Delta^g_+ \) and \( \Delta^g_- \). We can express the image of the Weyl vector \( \rho_g \) under \( w \) as

\[
\rho_g(w) = \frac{1}{2} \left[ \sum_{\alpha \in \Delta^g_+} \alpha - \sum_{\alpha \in \Delta^g_-} \alpha \right],
\]

as is easily verified by applying \( w^{-1} \) to both sides of the equation.

Given a subalgebra \( h \) of \( g \), we call \( w \in W_g \) \( h \)-positive \( \text{[6]} \), iff

\[
\Delta^h_+ \subset \Delta^g_+ \cap \Delta^w_+.
\]

We claim that in order to compute \( \sum_{\bar{\alpha} \in \Delta^g_+} \hat{\Lambda} \bar{\alpha} \), we only need to identify the \( h \)-positive representative \( w \) of the coset \( W_g / W_h \) that appears in (4.4), and that the components \( \hat{\Lambda} \bar{\alpha} \) of the \( \text{so}(2d) \)-weight \( \hat{\Lambda} \) are given by

\[
\hat{\Lambda} \bar{\alpha} = \hat{\Lambda}^w_{\bar{\alpha}} := \begin{cases} \frac{1}{2} & \text{if } \bar{\alpha} \in \Delta^g_+, \\ -\frac{1}{2} & \text{if } \bar{\alpha} \in \Delta^g_- \end{cases}.
\]

This can be seen as follows. Let \( \alpha \) be an arbitrary element of \( \Delta^h_+ \). For any highest \( h \)-weight \( \hat{\lambda} \) we have \( (\hat{\lambda} + \rho_h, \alpha) > 0 \); as a consequence,

\[
0 < (\hat{\lambda} + \rho_h, \alpha) = (w(\Lambda + \rho_g), \alpha) = (\Lambda + \rho_g, w^{-1}(\alpha)).
\]

This shows that \( w^{-1}(\alpha) \in \Delta^g_+ \), or in other words, that \( \Delta^h_+ \subset \Delta^g_+ \). Now the general form of the Cartan currents of \( \hat{h} \) reads

\[
H^i_\hat{h} = H^i_g + \sum_{\bar{\alpha} \in \Delta^g_+} \bar{\alpha}^i :\Psi^{\bar{\alpha}} \Psi^{-\bar{\alpha}}:.
\]

As a consequence, under the embedding \( h \hookrightarrow g \oplus \text{so}(2d) \) the state with weight \( (w(\Lambda), \hat{\Lambda}^w_{\bar{\alpha}}) \) branches to

\[
\hat{\lambda} = w(\Lambda) + \frac{1}{2} \sum_{\bar{\alpha} \in \Delta^g_+} \bar{\alpha} - \frac{1}{2} \sum_{\bar{\alpha} \in \Delta^g_-} \bar{\alpha}
= w(\Lambda) + w(\rho_g) - \frac{1}{2} \sum_{\alpha \in \Delta^g_+ \cap \Delta^g_-} \alpha + \frac{1}{2} \sum_{\alpha \in \Delta^g_- \cap \Delta^h_+} \alpha.
\]

This reduces to \( w(\Lambda + \rho_g) - \rho_h \), i.e. yields the correct result (4.4), iff \( w \) is \( h \)-positive. Note that the weight \( (w(\Lambda), \hat{\Lambda}^w_{\bar{\alpha}}) \) is always present in the weight system of the \( g \oplus \text{so}(2d) \)-module with highest weight \( (\Lambda, s) \) or \( (\Lambda, c) \), because the Weyl group orbit of any weight of a highest weight module with dominant integral highest weight is contained in the weight system of the module.

Inserting our result (4.9) into the formula (4.5) for the superconformal charge \( q_{\text{suco}} \), we obtain

\[
q_{\text{suco}} = \frac{1}{2} \left( |\Delta^g_+ \cap \Delta^h_+| - |\Delta^g_- \cap \Delta^g_+| \right) - \frac{\xi_k Q}{k + g_y}.
\]
To simplify this formula further, we recall that the length \( l(w) \) of a Weyl group element \( w \), which is defined as the minimal number of fundamental reflections needed to obtain \( w \), obeys [42, sect. 1.7]

\[
l(w) = |\Delta^w_+ \cap \Delta_+|.
\]  

(4.14)

Using the identity

\[
|\Delta^w_+ \cap \Delta_+| + |\Delta^-_w \cap \Delta_+| = d,
\]

(4.15)

we finally obtain

\[
q_{\text{suco}} = \frac{1}{2}d - l(w) - \frac{\xi_c Q}{k + g^\vee}. 
\]

(4.16)

The length of the relevant elements of \( W_g/W_h \) can be obtained conveniently via the so-called Hasse diagram of the embedding \( h \hookrightarrow g \) (for some details, see the Appendix), and hence the formula (4.16) is easily implemented in a computer program. For the spinor current (4.1), one has \( w = \text{id} \) so that (4.16) reduces to

\[
q_{\text{suco}}(S) = \frac{1}{2}d - \frac{(v_c, \rho_g - \rho_h)}{k + g^\vee} = \frac{c}{6},
\]

(4.17)

where the last equality follows with (2.6) and the strange formula.

We are now in a position to compute the Poincaré polynomials of the theories listed in section 2. For notational simplicity, we will present the Poincaré polynomials in the form \( P(t^\ell) \), with \( \ell \) the smallest positive integer for which all values of \( \ell q_{\text{suco}} \) of chiral primary fields are integers. We find that for the three series \((BB, m + 2, 1)\), \((CC, 2, 2m + 1)\), and \((CC, 2m + 2, 1)\) with \( m \in \mathbb{Z}_{\geq 0} \), the Poincaré polynomials are given by a common formula, namely \( \ell = m + 2 \) and

\[
P(t^{m+2}) = \sum_{j=0}^{m} (j + 1) (t^{j} + t^{3m+2-j}) + (3m + 4) \sum_{j=m+1}^{2m+1} t^{j}.
\]

(4.18)

The Poincaré polynomials of the remaining models are listed in table 5.

To conclude this subsection, we remark that the resolution of fixed points does not alter the number of Ramond ground states. In other words, independently of its length each identification orbit that contains a representative satisfying (4.4) provides exactly one Ramond ground state [24].
Table 5: Poincaré polynomials for non-hermitian symmetric coset theories

| name | $\ell$ | $P(t^\ell)$ |
|------|--------|-------------|
| $(BA, 3, 1)$ | 4 | $1 + 3 t^2 + 4 t^3 + 3 t^4 + t^6$ |
| $(BA, 3, 2)$ | 14 | $1 + t^6 + t^8 + 2 t^{10} + t^{11} + 3 t^{12} + t^{13} + 2 (t^{14} + t^{15} + t^{16}) + t^{17} + 3 t^{18} + t^{19} + 2 t^{20} + t^{21} + t^{22} + t^{24} + t^{30}$ |
| $(BA, 3, 4)$ | 6 | $1 + (t^2 + t^3) + 3 t^4 + 2 t^5 + 9 t^6 + 7 t^7 + 14 t^8 + 12 t^9 + 14 t^{10} + 7 t^{11} + 9 t^{12} + 2 t^{13} + 3 t^{14} + t^{15} + t^{16} + t^{18}$ |
| $(BA, 4, 1), (C3, 1), (G2, 2)$ | 2 | $1 + 4 t + 14 t^2 + 4 t^3 + t^4$ |
| $(BA, 5, 1)$ | 4 | $1 + 5 t^2 + 10 t^4 + 16 t^5 + 16 t^6 + 5 t^8 + t^{10}$ |
| $(BA, 6, 1)$ | 2 | $1 + 6 t + 15 t^2 + 25 t^3 + 35 t^4 + 6 t^5 + t^6$ |
| $(BB, 3, 3)$ | 2 | $1 + 4 t + 17 t^2 + 40 t^3 + 17 t^4 + 4 t^5 + t^6$ |
| $(BB, 4, 2)$ | 3 | $1 + 2 t + 8 t^2 + 14 t^3 + 35 t^4 + 35 t^5 + 14 t^6 + 8 t^7 + 2 t^8 + t^9$ |
| $(CC, 2, 2), (CC, 3, 1), (G2, 1)$ | 5 | $1 + 2 t^2 + 3 (t^3 + t^4) + 2 t^5 + t^7$ |
| $(CC, 2, 4), (CC, 5, 1)$ | 7 | $1 + 2 t^2 + 3 t^4 + 5 t^5 + 4 (t^6 + t^7) + 5 t^8 + 3 t^9 + 2 t^{11} + t^{13}$ |
| $(CC, 2, 6), (CC, 7, 1)$ | 9 | $1 + 2 t^2 + 3 t^4 + 4 t^6 + 7 t^7 + 5 t^8 + 6 (t^9 + t^{10}) + 5 t^{11} + 7 t^{12} + 4 t^{13} + 3 t^{15} + 2 t^{17} + t^{19}$ |
| $(CC, 3, 2), (CC, 3, 5), (CC, 6, 2)$ | 2 | $1 + 6 t + 16 t^2 + 6 t^3 + t^4$ |
| $(CC, 4, 3)$ | 3 | $1 + 3 t + 12 t^2 + 20 t^3 + 48 (t^4 + t^5) + 20 t^6 + 12 t^7 + 3 t^8 + t^9$ |
| $(C4, 1)$ | 2 | $1 + 8 t + 29 t^2 + 64 t^3 + 29 t^4 + 8 t^5 + t^6$ |
| $(D4, 1)$ | 2 | $1 + 4 t + 15 t^2 + 40 t^3 + 15 t^4 + 4 t^5 + t^6$ |
| $(D5, 1)$ | 7 | $1 + t^2 + 3 t^4 + 4 t^5 + 3 (t^6 + t^7) + 4 t^8 + 3 t^9 + t^{11} + t^{13}$ |
| $(D5, 2)$ | 3 | $1 + t + 4 t^2 + 12 t^3 + 22 (t^4 + t^5) + 12 t^6 + 4 t^7 + t^8 + t^9$ |
| $(F4, 1)$ | 5 | $1 + t + 2 (t^2 + t^3) + 9 (t^4 + t^5 + t^6 + t^7) + 2 (t^8 + t^9) + t^{10} + t^{11}$ |
| $(G2, 1)$ | 3 | $1 + 5 (t^2 + t^3) + t^5$ |
| $(G2, 2)$ | 18 | $1 + t^{10} + t^{12} + t^{13} + t^{14} + t^{15} + 2 t^{16} + t^{17} + t^{18} + t^{19} + 2 t^{20} + t^{21} + t^{22} + t^{23} + 2 t^{24} + 2 t^{25} + t^{26} + t^{27} + t^{28} + t^{30} + t^{40}$ |
| $(G2, 5)$ | 3 | $1 + t + 4 t^2 + 8 t^3 + 22 (t^4 + t^5) + 8 t^6 + 4 t^7 + t^8 + t^9$ |
4.3 Charge conjugation

From (4.18) and the results in table 5 one can read off that the superconformal charges of chiral primary fields lie between zero and $c/3$, as it must be. One also notes that according to the results the Poincaré polynomials obey

$$ P(t) = t^{c/3} P(t^{-1}). \quad (4.19) $$

In terms of the Ramond sector this means that the collection of Ramond ground states is symmetric with respect to the charge conjugation $q_{\text{suco}} \mapsto -q_{\text{suco}}$.

In fact, using the formulæ (4.4) and (4.16) it is possible to show that this is a generic feature of all $N = 2$ coset theories of the form (1.1). To show this, consider along with an arbitrary Ramond ground state $\Phi_{\Lambda, x}$, also the field represented by $\Phi_{\Lambda^+, x'} \equiv \Phi_{\Lambda, x} \, \tilde{\lambda}^+ \equiv \Phi_{\Lambda^+, \tilde{x'}}$ defined as follows. As before, $x$ stands for either the spinor or conjugate spinor, and we define $x'$ to be equal to $x$ if $d$ is even, and to belong to the opposite conjugacy class if $d$ is odd.

Moreover,

$$ \Lambda^+ := -w^g_{\text{max}}(\Lambda), \quad (4.20) $$

$$ \tilde{\lambda}^+ := -w^h_{\text{max}}(\tilde{\lambda}) \quad (4.21) $$

(recall that $w_{\text{max}}$, denoting the longest element of a Weyl group $W$, acts as the negative of the conjugation in the representation ring of a Lie algebra). In the definition (4.22), $w^h_{\text{max}}$ is to be considered as an element of the Weyl group $W_h$. As a consequence, $w^h_{\text{max}}$ acts on the $h$-weights like the usual conjugation of weights and maps $Q$ to $-Q$. Namely, by virtue of (2.18) each fundamental Weyl reflection of $W_h$, and thus any element of $W_h$, acts on $v_0$ as the identity.

Using the identities $\rho_g = -w^g_{\text{max}}(\rho_g)$ and $\rho_h = -w^h_{\text{max}}(\rho_h)$, we see that the highest weight $\tilde{\lambda}^+ + \rho_h$ of $h$ can be written as

$$ \tilde{\lambda}^+ + \rho_h = -w^h_{\text{max}}(\tilde{\lambda} + \rho_h) = -w^h_{\text{max}}w(\Lambda + \rho_g) = w^+(\Lambda^+ + \rho_g), \quad (4.22) $$

where

$$ w^+ := w^h_{\text{max}}w^g_{\text{max}}, \quad (4.23) $$

and where $w$ is the Weyl group element introduced in (4.4). To calculate the sign of $w^+$, which determines the $\text{so}(2d)$ conjugacy class, we observe (by inspection) that for all simple Lie algebras $g$ the relation

$$ \text{sign} (w^g_{\text{max}}) = (-1)^{n_+} \quad (4.24) $$

is satisfied, where $n_+ = |\Delta^g_+|$ is the number of positive $g$-roots. Therefore

$$ \text{sign} (w^+) = \text{sign} (w^h_{\text{max}}) \, \text{sign} (w^g_{\text{max}}) \, \text{sign} (w) = (-1)^{(\dim g - \dim h)/2} \, \text{sign} (w), \quad (4.25) $$

and (4.4) now clearly implies that the state $\Phi_{\Lambda^+, \tilde{x'}}$ is again a Ramond ground state. Also note that as a by-product we proved that along with $w$ also $w^+$ is $h$-positive.
So far we have seen that the set of Ramond ground states is symmetric in the \( u(1) \)-charge. The symmetry in the superconformal charge then follows from (4.16) together with the identity

\[
I(w) + I(w^+) = d. 
\]  

(4.26)

This relation arises as follows. Let \( \bar{\alpha} \) be an arbitrary root in \( \Delta^+ \). Then either \( w^{-1}(\bar{\alpha}) \in \Delta^- \) or \( (w^+)^{-1}(\bar{\alpha}) \in \Delta^- \), because the map \( w \mapsto w^+ \) swaps exactly from negative to positive roots of \( g \setminus h \). Thus \( \Delta^+ \) is the disjoint union of \( \Delta^+_+ \) and \( \Delta^-_+ \), which by (4.14) and (4.15) proves the assertion.

Let us also note that the unique Ramond ground state with minimal superconformal charge \( q_{\text{suco}} = -c/6 \) (which via spectral flow corresponds to the identity primary field) is obtained by applying the above prescription to the spinor current (4.1), and hence is given by \( \Phi_{0,-Q^s} \). For this field the relevant \( h \)-positive Weyl group element is \( w = w^{h}_{\text{max}}w^{g}_{\text{max}} \) so that (4.27) implies

\[
I(w^{h}_{\text{max}}w^{g}_{\text{max}}) = d, 
\]  

(4.27)

while by setting \( \Lambda = \lambda = 0 \) in (4.3), one obtains \( \rho_h = \frac{1}{2} (\rho_g + w^{h}_{\text{max}}w^{g}_{\text{max}}(\rho_g)) = \frac{1}{2} (\rho_g - w^{h}_{\text{max}}(\rho_g)) \). Since the \( h \)-positive representative \( w^{g/h}_{\text{max}} \) of \( W_g/W_h \) with largest length \( d \) is unique \([42, 4.27]\) shows that this representative is given by

\[
w^{g/h}_{\text{max}} = w^{h}_{\text{max}}w^{g}_{\text{max}}. 
\]  

(4.28)

4.4 Extended Poincaré polynomials and string theory spectra

Knowing the exact form of the Ramond ground states, we are in a position to calculate the massless spectrum of the string theory that employs an \( N = 2 \) coset model as its inner part, or more precisely, the numbers \( N^2_7 \) of ‘generations’ and \( N^2_{77} \) of ‘anti-generations’ which carry the two inequivalent 27-dimensional representations of the \( E_6 \) part of the space-time gauge group of the string theory. One possibility to find these numbers (and, in addition, the number \( N^1_1 \) of \( E_6 \) singlets) is the ‘method of beta vectors’ that was introduced \([16]\) in the context of \( N = 2 \) minimal models. In practise, this is not the most convenient approach, as the dimensionality and structure of the lattice spanned by the beta vectors depends strongly on the algebras involved, so that one would be forced into a lengthy case by case analysis. (However, for the calculation of the number of massless states carrying the singlet representation of the space-time gauge group \( E_6 \), the method of beta vectors is still the only known algorithm.

Unfortunately the knowledge of the Ramond ground states is not sufficient to get the singlets. Now while for Ramond ground states the correct treatment of null states is already implemented through (4.4), for general \( N = 2 \) coset theories the presence of null states makes the determination of the singlets a hard problem. In fact, the singlet numbers have so far not been determined for (tensor products of) \( N = 2 \) coset models other than the minimal ones. For the latter theories, the representation theory of the \( N = 2 \) algebra gives a good handle on null states.)
An alternative algorithm is provided by the extended Poincaré polynomial \( \mathcal{P} \) that was introduced in [24]. This polynomial depends on the variable \( t \) of the ordinary Poincaré polynomial and on an additional variable \( x \) which keeps track of the intersections of the orbits of the spinor current \( S \) with the set of Ramond ground states. To determine the exact form of the extended Poincaré polynomial is a somewhat tricky issue, as in fact we only know some specific representatives of the fields which are Ramond ground states (the formula (4.4) does not provide all members of an equivalence class), while for the calculation of \( \mathcal{P}(t, x) \) in principle all representatives are required. Fortunately, one can show that the following procedure yields the full result. Take a single representative for each Ramond ground state, and act on it with all even powers of all representatives of \( S \) that have \( g \)-weight \( \Lambda = 0 \). This is sufficient because of the fact, proven in the appendix of [3], that for any representative \( R \) of a Ramond ground state there exists at least one representative \( R' \) that belongs to the set obtained via (4.4) and that has the same \( g \)-weight as \( R \). If any of the states so obtained is either a Ramond ground state or a superpartner of a Ramond ground state, there is a corresponding contribution

\[
\pm t^{\text{sup}(R) + c/6} x^m
\]  

(4.29)

to the extended Poincaré polynomial, with \( m \) the power of \( S \) that has been applied, and with the sign being positive for the case of a Ramond ground state and negative for the case of the superpartner of a Ramond ground state, respectively. Here by ‘superpartner’ of a state we mean the state that is obtained by taking the fusion product with the simple current \( \Phi^0_{0,0} \) which is the generator of the world-sheet supersymmetry.\(^8\) Note that for large enough power \( M \) the Ramond ground state itself or its superpartner is reproduced, so that in fact one has an infinite power series in \( x \), which however is periodic such that it can be factored into a polynomial times \((1 \mp x^M)^{-1}\).

If in \( \mathcal{P}(t, x) \) the highest (and, due to charge conjugation invariance proven in section 4.3, also the lowest) power in \( t \) gets multiplied with more than two distinct powers of \( x \), then additional gravitinos that lead to extended space-time supersymmetry (respectively, additional gauge bosons, yielding an extension of the space-time gauge group \( E_6 \) to \( E_7 \) or \( E_8 \)) are present [24]. In the tables below we have marked all models where this happens by an asterisk on the net generation number. Note that in the tables we display the number of \( E_6 \) multiplets even if the gauge group gets extended. (All models of this type that appear in our list describe in fact string propagation on the manifold \( K3 \times T^2 \), and hence have \( N_{27} = N_{27} = 21 \). The number \( N_{56} \) of the associated \( E_7 \) multiplets is in these cases \( N_{56} = N_{27} - 1 = 20 \), as one generation-antigeneration pair becomes part of the gauge boson multiplet.)

If the gauge symmetry is not extended, then as argued in [24] it is straightforward to read off the numbers \( N_{27} \) and \( N_{27} \) from the extended Poincaré poly-

---

\(^8\) Note that a primary field and the field related to it by world-sheet supersymmetry are to be treated as distinct primary fields. In particular \( \Phi^0_{0,0} \) is itself a primary field.
nomial of a $c = 9$ theory. Namely, if $P$ is written as

\[ P(t, x) = \sum_i \sum_m a_i^{(m)} t^i x^{2m}, \]  

(4.30)

then

\[ N_{27} + N_{\overline{27}} = \sum_{m=0}^{M_s/2-1} |a_m^{(1)}|, \quad N_{27} - N_{\overline{27}} = \sum_{m=0}^{M_s/2-1} (-1)^m a_m^{(1)}. \]  

(4.31)

Here $M_s$ denotes the smallest positive integer such that the $(2M_s + 1)$st power of the spinor current is either equal to the spinor current itself or to its superpartner.

As an illustration, we present one example of an extended Poincaré polynomial, namely for the theory $(G_{21}, 2)$. This has the (somewhat atypical) property that to some powers of $t$ other than the highest and the lowest ones there are associated more than two different powers of $x$. The ‘polynomial’ reads

\[ P(t^{18}, x) = \{ (1 + x^2) + t^{10} (1 + x^{10}) + t^{12} (1 + x^8 + x^{18} + x^{26}) + t^{13} (1 - x^{16}) + t^{14} (1 + x^6 - x^{12} - x^{30}) + t^{15} (1 - x^{32}) + t^{16} (2 + x^4 + x^{18} + 2x^{22}) + t^{17} (1 - x^{12}) + t^{18} (1 + x^2 + x^{18} + x^{20}) + t^{19} (1 - x^{28}) + t^{20} (2 - x^6 - x^{12} + 2x^{18} - x^{24} - x^{30}) + t^{21} (1 - x^{8}) + t^{22} (1 + x^6 + x^{18} + x^{34}) + t^{23} (1 - x^{24}) + t^{24} (2 + 2x^{14} + x^{18} + x^{32}) + t^{25} (1 - x^{4}) + t^{26} (1 - x^6 - x^{24} + x^{30}) + t^{27} (1 - x^{20}) + t^{28} (1 + x^{10} + x^{18} + x^{28}) + t^{30} (1 + x^{26}) + t^{40} (1 + x^{34}) \} (1 - x^{36})^{-1}. \]  

(4.32)

In the presence of fixed points the above prescription for obtaining the extended Poincaré polynomial is not yet quite complete, since from the quantum numbers of a fixed point alone it cannot be decided whether a field into which the fixed point is resolved and which appears in the orbit of another Ramond ground state is a Ramond ground state (or the superpartner of a Ramond ground state) or not. In principle one could resolve this ambiguity by using the full $S$-matrix of the theory to calculate the fusion rules which, in turn, determine the orbits of the spinor current. But again, there is a way to avoid this involved calculation, which has the additional benefit of showing that the results for the extended Poincaré polynomial do not depend as strongly on the details of the resolution as one might imagine. To this end we note that an important check of the spectra obtained via the extended Poincaré polynomial is provided by the results of [43], where an independent way to calculate the net generation number $\delta N$ by means of the ordinary Poincaré polynomial $P(t)$ was found. Namely,

\[ \delta N \equiv N_{27} - N_{\overline{27}} = \frac{1}{M_s} \sum_{r,s=0}^{M_s/2-1} P(e^{2\pi i d(r,s)/M_s}), \]  

(4.33)
where \( d(r, s) \) stands for the largest common divisor of the integers \( r \) and \( s \).

Now since (4.33) determines the net generation number \( \delta N \) from the ordinary Poincaré polynomial alone, \( \delta N \) cannot depend on the resolution procedure [24]. To determine the correct extended Poincaré polynomial, we thus simply have to start with the most general ansatz compatible with the prescriptions given above and calculate, for all possible values of the unknown parameters that arise from the orbits containing resolved fixed points, the net generation number for string vacua that involve the model under investigation as one factor theory. If the net generation number generated this way does not fit the value prescribed by (4.33), we can exclude the corresponding set of values for the unknown parameters. (To resolve all ambiguities uniquely, it is sometimes necessary to take into account that we can apply all formulas not only to tensor products with \( c = 9 \), but to tensor products with \( c = 3 + 6n \) for any positive integer \( n \) as well.)

As an example, let us have a look at the theory \((BB, 4, 2)\) which has \( c = 9 \). The analysis of the Ramond ground states shows that the coefficient of \( t \) in the extended Poincaré polynomial is the polynomial

\[
14 + a_1 x^2 - a_2 x^4 \tag{4.34}
\]

(multiplied with the irrelevant factor \((1+x^6)^{-1}\)). Here \( a_1 \) and \( a_2 \) are parameters arising from the fixed point ambiguities just described; they must be integers between 4 and 7. Now (4.33) shows that \( \delta N = 0 \), so that (4.31) yields \( a_1 + a_2 = 14 \), which in the given range has the unique solution \( a_1 = a_2 = 7 \). Once the exact form of the extended Poincaré polynomial is known, we can read off the number of generations and antigenerations separately, namely \( N_{27} = N_{27'} = 14 \).

In fact, in most cases one deals with a tensor product of \( N = 2 \) coset theories rather than with a single theory. The ordinary Poincaré polynomial \( P_{\text{tot}}(t) \) of a tensor product is just the product of the ordinary Poincaré polynomials \( P_i(t) \) of the factor theories. Concerning the extended Poincaré polynomial, it must be noticed that for a tensor product the considerations above are to be applied to the total spinor current \( S_{\text{tot}} = S^{(0)} S^{(1)} \ldots S^{(n)} \); \( S_{\text{tot}} \) is by definition the product of the spinor currents \( S^{(i)}, i = 1, 2, \ldots, n \), (given by (4.1)) of the \( n \) factor theories, and of the spinor \( S^{(0)} \) of the \( D_5 \) WZW theory at level one that describes either part the gauge sector of the string theory or bosonized fermions and superconformal ghosts. (The simple current \( S_{\text{tot}} \) is just the space-time supersymmetry generator, and hence for space-time supersymmetric theories it should be part of the chiral algebra of the theory, i.e. the modular invariant to be chosen for the tensor product is just the corresponding simple current invariant. This corresponds to the generalized GSO projection, and in fact it is possible to recover the usual projection condition, namely odd integer superconformal charges, from the projection to integer monodromy charges with respect to \( S_{\text{tot}} \).

To obtain the extended Poincaré polynomial \( P_{\text{tot}}(t, x) \) of the total theory from the extended Poincaré polynomials \( P_i(t, x_i) \) of all factor theories, one has to implement the following prescription [24] that is consistent with the multiplication of ordinary Poincaré polynomials. Multiply all \( P_i(t, x_i) \), and
remove afterwards all terms in which the powers of the variables $x_i$ do not coincide. Finally, replace the product $x_1 \cdot x_2 \cdot ... \cdot x_n$ by $x$. This procedure implements the fact that some power $(S_{\text{tot}})^n$ of the spinor current maps a Ramond ground state on the (superpartner of) another Ramond ground state iff the components of $(S_{\text{tot}})^n$ do so in any factor of the theory.

We present the results of our calculations in tables 6 to 8. In table 6 we list all tensor products that can be written as the tensor product of a $c = 6$ and of a $c = 3$ theory and in which at least one factor is neither a hermitian symmetric coset nor the model $(CC, 2, 1)$ that will be dealt with separately. The un-numbered lines contain the relevant non-hermitian symmetric theories, while the numbered lines provide the spectra for those $c = 9$ theories that are obtained by tensoring the $c = 6$ part with the following $c = 3$ models, respectively:

$1 - 1 - 1$,

$1 - 4$ or $(A, 1, 2, 3),$ \hspace{1cm} (4.35)

$2 - 2$,

$(CC, 2, 1)$

(the theories $1 - 4$ and $(A, 1, 2, 3)$ possess the same extended Poincaré polynomial and therefore yield the same spectrum). Here and below, the symbol ‘$-$’ is used to indicate the tensor product, and a single integer $k$ stands for the $N = 2$ minimal model at level $k$.

Next we display, in table 7, all tensor products that contain the model $(CC, 2, 1)$ which has $c = 3$, but do not contain any other non-hermitian symmetric coset theory. We can tensor this model twice and use the five $c = 3$ models listed in (4.35); as the model $(CC, 2, 1)$ itself occurs in that list, this includes tensoring three copies of the model. We can also tensor it with 17 different combinations of minimal models and 27 other combinations of hermitian symmetric cosets with $c = 6$. Altogether, this yields $15 \times 5 + 4 + 17 + 27 = 123$ models with $c = 9$ that involve non-hermitian symmetric cosets and contain a $c = 3$ part.

Finally, in table 8 we list all tensor products having $c = 9$ in which at least one factor is not a hermitian symmetric coset and which do not contain a tensor product with $c = 3$. We find 75 models of this type. The number of theories that we count as different gets reduced by various identifications, to be discussed below, among the total of 198 theories. We have taken care of these identifications, thereby reducing the number of entries in the tables 6 to 8 to 112.

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\textsuperscript{9} The list in \cite{22}, containing 28 hermitian symmetric cosets with $c = 6$, is incomplete in several respects. First, rather than $(D, 5, 2) - 16$, one must use the combinations $(D, 5, 2)$ and $(D, 5, 1) - 16$. Further, it was not realized that the coset theory $(A, 1, 2, 2)$ (appearing in three of the 28 theories) coincides with the minimal model at level 8. Finally, the theories $(B, 3, 6)$ and $(B, 6, 3)$ which in \cite{22} were supposed to be identical, are in fact \cite{24} distinct conformal field theories. Implementing these corrections, the number of the models gets reduced by one, leading to the correct number of 27 models.
Table 6: $c = 9$ tensor product theories that contain a $c = 6$ part combined with a non-hermitian symmetric factor (different from $(CC,2,1)$), and the associated generation and anti-generation numbers

| #  | Model | $N_{27}$ | $N_{27}^{-}$ | $\delta N$ |
|----|-------|----------|--------------|------------|
| 1  | $(BA,3,1)$ | 21 | 21 | * 0 |
| 2  | $-2 - 1 - 1 - 1$ | 31 | 7 | 24 |
| 3  | $-2 - 2 - 2$ | 39 | 3 | 36 |
| 4  | $-2 - (CC,2,1)$ | 31 | 7 | 24 |
| 5  | $(BA,4,1)$ or $(C3,1)$ or $(G22,2)$ | 21 | 21 | * 0 |
| 6  | $-1 - 1 - 1$ | 31 | 7 | 24 |
| 7  | $-1 - 4$ | 31 | 7 | 24 |
| 8  | $-2 - 2$ | 31 | 7 | 24 |
| 9  | $(BB,3,1)$ or $(CC,2,3)$ or $(CC,4,1)$ | 51 | 3 | 48 |
| 10 | $-1 - 1 - 1 - 1$ | 51 | 3 | 48 |
| 11 | $-1 - 1 - 4$ | 21 | 21 | * 0 |
| 12 | $-1 - 2 - 2$ | 21 | 21 | * 0 |
| 13 | $(BB,4,1)$ or $(CC,2,5)$ or $(CC,6,1)$ | 21 | 21 | * 0 |
| 14 | $-1 - 1 - 1$ | 21 | 21 | * 0 |
| 15 | $-1 - 4$ | 23 | 23 | 0 |
| 16 | $-2 - 2$ | 44 | 8 | 36 |
| 17 | $(CC,2,2)$ or $(CC,3,1)$ or $(G22,1)$ | 21 | 21 | * 0 |
| 18 | $-3 - 1 - 1 - 1$ | 21 | 21 | * 0 |
| 19 | $-3 - 1 - 4$ | 21 | 21 | * 0 |
| 20 | $-3 - 2 - 2$ | 21 | 21 | * 0 |
| 21 | $(CC,3,2)$ | 21 | 21 | * 0 |
| 22 | $-1 - 1 - 1$ | 21 | 21 | * 0 |
| 23 | $-1 - 4$ | 41 | 5 | 36 |
| 24 | $-2 - 2$ | 41 | 5 | 36 |
| 25 | $(G21,1)$ | 21 | 21 | * 0 |
| 26 | $-1 - 1 - 1 - 1$ | 29 | 5 | 24 |
| 27 | $-1 - 1 - 4$ | 29 | 5 | 24 |
| 28 | $-1 - 2 - 2$ | 21 | 21 | * 0 |
| 29 | $-1 - (CC,2,1)$ | 21 | 21 | * 0 |
Table 7: $c = 6$ tensor products, and the net generation number $\delta N$ for the $c = 9$ models obtained by tensoring in addition with $(CC,2,1)$

| #  | Model (c = 6 part) | $N_{27}$ | $N_{27}'$ | $\delta N$ |
|----|-------------------|----------|-----------|------------|
| 1  | 1 – 1 – 1 – 1 – 1 – 1  | 21       | 21        | 0          |
| 2  | 1 – 1 – 1 – 1 – 4    | 35       | 11        | 24         |
| 3  | 1 – 1 – 1 – 2 – 2    | 21       | 21        | 0          |
| 4  | 1 – 1 – 2 – 10       | 35       | 11        | 24         |
| 5  | 1 – 1 – 4 – 4        | 51       | 3         | 48         |
| 6  | 1 – 2 – 2 – 4        | 51       | 3         | 48         |
| 7  | 2 – 2 – 2 – 2        | 61       | 1         | 60         |
| 8  | 1 – 5 – 40           | 35       | 35        | 0          |
| 9  | 1 – 6 – 22           | 43       | 19        | 24         |
| 10 | 1 – 7 – 16           | 43       | 19        | 24         |
| 11 | 1 – 8 – 13           | 27       | 27        | 0          |
| 12 | 1 – 10 – 10          | 59       | 11        | 48         |
| 13 | 2 – 3 – 18           | 39       | 15        | 24         |
| 14 | 2 – 4 – 10           | 45       | 9         | 36         |
| 15 | 2 – 6 – 6            | 55       | 7         | 48         |
| 16 | 3 – 3 – 8            | 39       | 15        | 24         |
| 17 | 4 – 4 – 4            | 60       | 6         | 54         |
| 18 | (A,1,2,4) – 12       | 38       | 20        | 18         |
| 19 | (A,1,2,5) – 6        | 55       | 7         | 48         |
| 20 | (A,1,2,6) – 1 – 1    | 21       | 21        | 0          |
| 21 | (A,1,2,6) – 4        | 23       | 23        | 0          |
| 22 | (A,1,2,7) – 3        | 39       | 15        | 24         |
| 23 | (A,1,2,9) – 2        | 45       | 9         | 36         |
| 24 | (A,1,2,15) – 1       | 43       | 19        | 24         |
| 25 | (A,1,3,3) – 5        | 21       | 21        | 0          |
| 26 | (A,1,3,4) – 2        | 51       | 3         | 48         |
| 27 | (A,1,3,5) – 1        | 21       | 21        | 0          |
| 28 | (A,1,3,8)           | 45        | 9       | 36        |
| 29 | (A,1,4,5)           | 41        | 5       | 36        |
| 30 | (A,2,2,4)           | 51        | 3       | 48        |
| 31 | (B,6,3)             | 21        | 21       | 0          |
| 32 | (C,2,3) – 2         | 51        | 3       | 48        |
| 33 | (C,2,6)             | 23        | 23       | 0          |
| 34 | (C,3,2)             | 21        | 21       | 0          |
| 35 | (C,4,1) – 1         | 35        | 11       | 24        |
| 36 | (D,5,2)             | 21        | 21       | 0          |
| 37 | (CC,2,1) – 1 – 1 – 1 | 21       | 21       | 0          |
| 38 | (CC,2,1) – 1 – 4    | 51        | 3       | 48        |
| 39 | (CC,2,1) – 2 – 2    | 51        | 3       | 48        |
| 40 | (CC,2,1) – (CC,2,1) | 51        | 3       | 48        |
Table 8: $c = 9$ tensor products that contain a non-hermitian symmetric coset and cannot be decomposed in the tensor product of a $c = 3$ and a $c = 6$ theory

| #  | Model                                      | $N_{27}$ | $N_{37}$ | $\delta N$ |
|----|--------------------------------------------|----------|----------|------------|
| 1  | $(BA, 3, 1)$                               | 19       | 19       | 0          |
| 2  | $-1 - 1 - 10$                              | 23       | 23       | 0          |
| 3  | $-3 - 18$                                  | 27       | 15       | 12         |
| 4  | $-4 - 10$                                  | 35       | 11       | 24         |
| 5  | $(A, 1, 2, 9)$                             | 27       | 15       | 12         |
| 6  | $(A, 1, 3, 4)$                             | 31       | 7        | 24         |
| 7  | $(B, 6, 2)$                                | 35       | 11       | 24         |
| 8  | $(C, 2, 3)$                                | 31       | 7        | 24         |
| 9  | $(BA, 3, 1)$                               | 15       | 15       | 0          |
| 10 | $(BA, 3, 2)$                               | 12       | 8        | 4          |
| 11 | $(BA, 3, 4)$                               | 14       | 2        | 12         |
| 12 | $(BA, 5, 1)$                               | 15       | 15       | 0          |
| 13 | $(BA, 6, 1)$                               | 15       | 15       | 0          |
| 14 | $(BB, 3, 1)$ or $(CC, 2, 3)$ or $(CC, 4, 1)$| 35       | 11       | 24         |
| 15 | $-2 - 10$                                  | 43       | 7        | 36         |
| 16 | $(A, 1, 2, 6)$                             | 35       | 11       | 24         |
| 17 | $(A, 2, 2, 2)$                             | 51       | 3        | 48         |
| 18 | $(BB, 3, 3)$                               | 17       | 5        | 12         |
| 19 | $(BB, 4, 2)$                               | 14       | 14       | 0          |
| 20 | $(BB, 5, 1)$ or $(CC, 2, 7)$ or $(CC, 8, 1)$| 39       | 15       | 24         |
| 21 | $(BB, 6, 1)$ or $(CC, 2, 9)$ or $(CC, 10, 1)$| 29       | 29       | 0          |
| 22 | $-1 - 1$                                  | 44       | 14       | 30         |
| 23 | $(BB, 8, 1)$ or $(CC, 2, 13)$ or $(CC, 14, 1)$| 34       | 34       | 0          |
| 24 | $(BB, 12, 1)$ or $(CC, 2, 21)$ or $(CC, 22, 1)$| 43       | 43       | 0          |
| 25 | $(CC, 2, 2)$ or $(CC, 3, 1)$ or $(G2, 1)$ | 23       | 23       | 0          |
| 26 | $-1 - 1 - 28$                              | 29       | 29       | 0          |
| 27 | $-4 - 28$                                  | 47       | 11       | 36         |
| 28 | $(A, 1, 2, 12)$                           | 35       | 17       | 18         |
| 29 | $(B, 8, 2)$                                | 41       | 5        | 36         |
| 30 | $(CC, 2, 4)$ or $(CC, 5, 1)$              | 29       | 14       | 15         |
Table 8: continued.

| #  | Model                                      | $N_{27}$ | $N_{27}$ | $\delta N$ |
|----|--------------------------------------------|----------|----------|------------|
| 31 | (CC, 2, 6) or (CC, 7, 1)                   | 27       | 27       | 0          |
| 32 | (CC, 3, 5) or (CC, 6, 2)                   | 20       | 20       | 0          |
| 33 | (CC, 4, 3)                                 | 29       | 9        | 20         |
| 34 | (C4, 1)                                    | 15       | 7        | 8          |
| 35 | (D4, 1) – (A, 1, 2, 4)                     | 23       | 11       | 12         |
| 36 | (D5, 1)                                    | 12       | 0        | 12         |
| 37 | (D5, 1), 16                                | 19       | 19       | 0          |
| 38 | (F4, 1) – 8                               | 25       | 13       | 12         |
| 39 | (G2, 1)                                    | 17       | 17       | 0          |
| 40 | – 4 – 4                                   | 23       | 11       | 12         |
| 41 | – (A, 1, 2, 6)                             | 17       | 17       | 0          |
| 42 | – (A, 2, 2, 2)                             | 29       | 5        | 24         |
| 43 | (G2, 2) – 7                               | 9        | 9        | 0          |
| 44 | (G2, 5)                                    | 20       | 20       | 0          |

### 4.5 Level-rank duality

As it turns out, the extended Poincaré polynomials for several theories that are defined as distinct naive coset theories coincide. The cases where this happens can be easily read off the tables as follows. If the extended Poincaré polynomials of some theories are identical, these theories are listed together in an un-numbered line; the numbered line(s) following this line then contain the theories with which each of them can be tensored to obtain a $c = 9$ theory. For instance, the line preceding the lines numbered from 25 to 29 in table 8 shows that the theories (CC, 2, 2), (CC, 3, 1) and (G2, 1) have identical extended Poincaré polynomials. We have also taken into account the known fact that the extended Poincaré polynomials of the hermitian symmetric cosets (A, 1, 2, 3), (A, 2, 2, 2), (C, 3, 1), and (D, 5, 1) coincide with those of the tensor products 1 – 4, 1 – 1 – 1 – 1, 3 – 3, and 1 – 7 of minimal models, respectively.

From the experience with coset constructions, the observation that there exist a priori distinct coset theories with coinciding extended Poincaré polynomials is not very spectacular. What is surprising, however, is that in fact for all non-hermitian $N = 2$ coset theories for which the ordinary Poincaré polynomials are identical (compare table 5 above), the same is true for the extended Poincaré polynomials.

In particular, the extended Poincaré polynomials of the two theories (CC, r, k) and (CC, k + 1, r – 1) for $r \geq 2$ are identical for all values of $r$ and $k$ for which $r \geq 2$

\[10\] However, in the table we have nevertheless kept the entries # 17 and # 42 containing (A, 2, 2, 2), because after identification with 1 – 1 – 1 – 1, they would correspond to entries in a different table, namely table 6.
we calculated them. Since the extended Poincaré polynomial describes explicitly also part of the structure of the chiral ring (whereas the ordinary Poincaré polynomial essentially counts multiplicities), this is in itself already a rather strong hint that these theories should be closely related, if not be identical as conformal field theories.

When looking at the conformal field theory beyond the chiral ring one realizes that the number of primary fields in both theories is identical. But we can do even better and construct a map between primary fields of these theories which can be shown to preserve most, if not all, of the conformal field theory properties. Namely, the \( \hat{h}' = (C_k)_{r'} \)-weight \( \lambda' \) of a (representative of a) primary field of \((CC, k+1, r-1)\) is related to the \( g = (C_r)_k \)-weight \( \Lambda \) of the associated field of \((CC, r, k)\) by

\[
\Lambda \mapsto \lambda' : \quad Y_{\lambda'} = (Y_{\Lambda})^c, \tag{4.36}
\]

where \( Y_{\Lambda} \) denotes the Young tableau associated to the highest weight \( \Lambda \), and the symbol ‘\(^c\)’ stands for the operation of first taking the complement of \( Y_{\Lambda} \) with respect to the rectangular Young tableau \( Y_{k \Lambda(c)} \) and then reflecting the tableau so obtained along an axis perpendicular to the main diagonal in such a way as to obtain a standard Young tableau of \((C_k)_r\). This map is well-known from the so-called ‘level-rank duality’ of the \((C_r)_k\) and \((C_k)_r\) WZW theories. To obtain the duality map for the coset theories, it has to be supplemented by an analogous relation between the \( g' \)- and \( \hat{h} \)-weights, and by prescriptions for the \( \mathfrak{so}(2d) \) and \( \mathfrak{u}(1) \) part which include in particular

\[
Q' = \begin{cases} 
-Q & \text{for } x \in \{s, c\}, \\
 k + r + 1 - Q & \text{for } x \in \{v, 0\}. 
\end{cases} \tag{4.37}
\]

In fact we expect that the coset theories \((CC, r, k)\) and \((CC, k+1, r-1)\) are merely two different descriptions of one and the same conformal field theory, so that there is even no need for a marginal flow to interpolate between them. We will come back to this issue in a forthcoming note.

5 Conclusions

In this paper we have presented a detailed analysis of non-hermitian symmetric \( N = 2 \) superconformal coset theories and of compactifications of the heterotic string that contain such coset theories in their inner sector. In addition, we have proven some general statements on the structure of any \( N = 2 \) coset theory. Concerning the non-hermitian symmetric coset theories themselves, we have shown that they indeed allow for an interpretation as a consistent conformal field theory; this lends further support to the expectation that any coset theory, naively ‘defined’ as \( \mathcal{C}[g/h]_k \), possesses such an interpretation. In particular, it was shown that the fixed points that arise in the process of field identification can be resolved by the methods of [27].

The spectra of string compactifications that we obtained are certainly not spectacular, but rather similar to those obtained for previously analyzed classes of compactifications. This confirms the by now common lore that extending the
set of string compactifications does not have a very large impact on the set of known spectra. The results also confirm the experience that when employing more complicated conformal field theories, the numbers of generations and anti-generations tend to be smaller than in the case of simpler (say, $N = 2$ minimal) theories.

There still remain several directions for further work on the subject. First, one may consider modular invariant combinations of characters of the $g$- and $h$-WZW theories other than the diagonal one, in particular non-diagonal invariants of tensor product theories that are not obtained from products of the invariants of the affine Lie algebras associated to the individual factor theories. One may also investigate whether the coset theories, or at least their tensor products with $c = 3n$, might have a description in terms of Landau-Ginzburg potentials or Calabi-Yau manifolds, or of orbifolds thereof (while it is generally assumed that such a connection should exist, the arguments supporting this expectation are far from being rigorous). To identify these different descriptions it would be very useful to have a more detailed knowledge of the discrete symmetries of the models. One of these discrete symmetries is obvious, namely the symmetry of the operator products induced by conservation of the superconformal $u(1)$-charge; but generically there may be further symmetries, and it is not clear how one could find all of them. Of course, once discrete symmetries are known, one can divide out some of them so as to obtain orbifolds of our models.

We also mention that a complete computation of massless string spectra, i.e. including the fields that are singlets under $E_6$, would clearly be welcome. To this end one would have to compute the character decompositions by means of the Kac-Weyl character formula (in order to identify null states and to obtain the integer part of the conformal weight of a field), and implement the beta vector method known from tensor products of minimal models. It is evident that this is a laborious procedure, and any alternative method would be of great interest.

Another interesting aspect of the string spectra obtained in the paper is that the extended Poincaré polynomials $P(t, x)$, and hence the generation numbers $N_{27}$ and $N_{277}$ of the associated string compactifications, of two theories are identical whenever the ordinary Poincaré polynomials $P(t) = P(t, 0)$ are. This indicates that the structure of the extended Poincaré polynomial is to a large extent already dictated by the information contained in the ordinary Poincaré polynomial; in particular (compare [24]), in the presence of fixed points the numbers of massless generations and anti-generations do not depend at all on the details of the resolution procedure. A general proof of this observation is however still lacking.

Let us finally come back to the hypothesis that, given a chain of subalgebras $h_1 \hookrightarrow h_2 \hookrightarrow g$, the coset theory $C[g/h_1]_k$ should correspond to the tensor product of the two cosets $C[g/h_2]_k$ and $C[h_2/h_1]_{k'}$, with a suitably chosen non-product modular invariant. We emphasize that in the presence of fixed points this hypothesis is far from being proven. With the methods employed in the present paper it should be straightforward to examine the structure of both $C[g/h_1]_k$ and the tensor product of $C[g/h_2]_k$ and $C[h_2/h_1]_{k'}$ in detail, and thereby
test the hypothesis for any given chain of embeddings. To prove the equivalence in full generality, however, still a deeper understanding of the structure of coset conformal field theories seems to be necessary.
A Appendix: Hasse diagramms

The Hasse diagram [45] for an embedding $h \hookrightarrow g$ of a reductive Lie algebra in a simple Lie algebra is the graph of the coset $W_g/W_h$, interpreted as a subgraph of the graph of $W_g$, with the edges as prescribed by the Bruhat ordering of $W_g$. (Hasse diagrams also arise in the description of the topological structure of generalized flag manifolds and of the structure of the Bernstein–Gelfand–Gelfand-resolution of Verma modules.) The nodes of the Hasse diagram correspond to those representatives of elements of $W_g/W_h$ that send a dominant $g$-weight $\Lambda$ to a dominant $\hat{h}$-weight $\lambda$, i.e. to $h$-positive elements of $W_g$, and the integer $i$ attached to an edge indicates that the two nodes connected by the edge correspond to Weyl group elements $w$ and $w'$ related by $w' = w(i)w$, with $w(i)$ the $i$th fundamental reflection. For an embedding $h \hookrightarrow g$ for which the Dynkin diagram of $h$ is obtained by deleting the node with label $i_0$ from the Dynkin diagram of $g$, the Hasse diagram is isomorphic to the $W_g$-orbit of $\Lambda(i_0)$, i.e. to the ‘restricted weight diagram’ that one obtains when acting successively on the weight $\Lambda(i_0)$ with the fundamental reflections.

The Hasse diagrams for the embeddings relevant to hermitian symmetric cosets have been described in [33]. Below we present the Hasse diagrams for some of the non-hermitian symmetric cases which appear in table 2 (the diagrams for the remaining cases, i.e. the $BA$ and $BB$ series and the two $D_5$ theories look more complicated, and we refrain from drawing them here).

11 Hasse diagram of $W(C_n)/W(C_{n-1})$:

Hasse diagram of $W(C_3)/W(A_1 \oplus A_1)$:

Hasse diagram of $W(C_4)/W(A_2 \oplus A_1)$:

---

11 The Hasse diagram of $W(F_4)/W(C_3)$ can also be found in [48, p. 86].
Hasse diagram of $W(D_4)/W(A_1 \oplus A_1 \oplus A_1)$:

Hasse diagram of $W(F_4)/W(C_3)$:
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