MEAN CURVATURE TYPE FLOW AND SHARP MICHEAL-SIMON INEQUALITIES

JINGSHI CUI AND PEIBIAO ZHAO

Abstract. In this paper, we first investigate a new locally constrained mean curvature flow (1.5) and prove that if the initial hypersurface $M_0$ is of smoothly compact starshaped, then the solution $M_t$ of the flow (1.5) exists for all time and converges to a sphere in $C^\infty$-topology. Following this flow argument, not only do we achieve a new proof of the celebrated sharp Michael-Simon inequality for mean curvature in Euclidean space $\mathbb{R}^{n+1}$, but we also get the necessary and sufficient condition for the establishment of the equality.

In the second part of this paper, we study a mean curvature type flow (1.7) of static convex hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$, and prove that the flow (1.7) has a unique smooth solution $M_t$ for all time $t \in [0, +\infty)$, and the static convexity of the hypersurface is preserved along the flow (1.7). Moreover, $M_t$ converges exponentially to a sphere of radius $R$ in $C^\infty$-topology as $t \to +\infty$. By exploiting the properties of this flow, we develop and present a new sharp Michael-Simon inequality for $k$th mean curvature.

Keywords: Locally constrained mean curvature flow; Mean curvature type flow; Michael-Simon inequality; $k$th mean curvature.

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1. Introduction

It is well known that the classical Michael-Simon inequality as a Sobolev inequality on submanifolds of Euclidean space is not sharp. The classical Michael-Simon inequality says as follows

**Theorem 1.1.** ([23]) Let $i : M^n \to \mathbb{R}^N$ be an isometric immersion ($N > n$). Let $U$ be an open subset of $M^n$. For a function $\phi \in C^\infty_c(U)$, there exists a constant $C$, such that

$$\left( \int_M |\phi|^{\frac{n}{n-1}} d\mu_M \right)^{\frac{n-1}{n}} \leq C \int_M (|H| \cdot |\phi| + |\nabla \phi|) d\mu_M$$

where $H$ is the mean curvature of $M^n$.

If $\phi \equiv 1$, the Michael-Simon inequality gives the relationship between the area and the integral of mean curvature. The best constant problem in (1.1) is still an open issue, even in the case of minimal surfaces. In a very recent paper, Brendle [5] had confirmed a sharp version of the Michael-Simon inequality as below.

**Theorem 1.2.** ([5]) Let $M$ be a compact hypersurface in $\mathbb{R}^{n+1}$ (possibly with boundary $\partial M$), and let $f$ be a positive smooth function on $M$. Then

$$\int_M \sqrt{|\nabla f|^2 + f^2 H^2} + \int_{\partial M} f \geq n |B^n|^\frac{1}{n} \left( \int_M f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}$$

where $H$ is the mean curvature of $M$ and $|B^n|$ is the area of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Moreover, if equality holds, then $f$ is constant and $M$ is a flat disk.

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The method of proving the inequality (1.2) in [5] is inspired by the Alexandrov-Bakelman-Pucci maximum principle (see e.g. [7], [27]) and the key step in the proof is to simplify the original inequality (1.2), which can be divided into the following three folds.

The first fold is to derive that there holds
\[
\int_M \sqrt{|\nabla^M f|^2 + f^2 H^2} d\mu + \int_{\partial M} f d\mu = n \int_M f \frac{\mu}{|B^n|} d\mu;
\]

The second fold is to prove that there is a function \( \psi : M \to \mathbb{R} \) which solves the PDE
\[
\text{div}_M (f \nabla^M \psi) = nf \frac{\mu}{|B^n|} - \sqrt{|\nabla^M f|^2 + f^2 H^2}
\]
on \( M \) with Neumann boundary condition \( \langle \nabla^M \psi, \vec{n} \rangle = 1 \) on \( \partial M \). Here, \( \vec{n} \) denotes the co-normal to \( M \). Notice that according to the standard theory of elliptic regularity, \( \psi \) belongs to the class \( C^{2,\beta} \) for each \( 0 < \beta < 1 \);

The third fold is to derive the following inequality
(1.3) \[
\int_M f \frac{\mu}{|B^n|} \geq |B^n|
\]
which is reduced from the original inequality (1.2).

In addition to the Michael-Simon inequality for mean curvature, it is certainly meaningful to establish the Michael-Simon type inequality for \( k \)th mean curvature \( \sigma_k(\kappa) \). In [9], Sun-Yung Alice Chang and Yi Wang presented the following inequality in \( \mathbb{R}^{n+1} \).

**Theorem 1.3.** ([9]) Let \( i : M^m \to \mathbb{R}^{n+1} \) be an isometric immersion. Let \( U \) be an open subset of \( M \) and \( u \in C^\infty_c(U) \) be a nonnegative function. For \( m = 2, \ldots, n-1 \), if \( M \) is \((m+1)\)-convex, then there exists a constant \( C \) depending only on \( n \) and \( m \), such that for \( 1 \leq k \leq m \)
\[
\left( \int_M \sigma_{k-1}(\kappa) u \frac{\mu}{\kappa} d\mu_M \right)^{\frac{n-k}{n-k+1}} \leq C \int_M \left( \sigma_k(\kappa) u + \sigma_{k-1}(\kappa) |\nabla u| + \cdots + |\nabla^k u| \right) d\mu_M
\]
If \( m = n \), then the inequality holds when \( M \) is \( n \)-convex. If \( m = 1 \), then the inequality holds when \( M \) is 1-convex. \((m = 1 \) case is a corollary of the Michael-Simon inequality).

In [9], the method used to prove the above result is optimal transport theory. M. Gromov first proposed the idea of proving geometric inequalities by means of mappings between domains and spheres which are optimal transport maps for special cases (see e.g. [10, 32]). But what is the case in the equality holds? And what is the sharp constant?

Motivated by Theorem 1.2 and Theorem 1.3, we propose the following conjecture about the sharp Michael-Simon inequality for \( k \)th mean curvature in Euclidean space \( \mathbb{R}^{n+1} \).

**Conjecture 1.4.** Let \( M \) be a compact hypersurface in \( \mathbb{R}^{n+1} \) (possibly with boundary \( \partial M \)) and let \( f \) be a positive smooth function on \( M \). For \( 1 \leq k \leq n-1 \), there holds
(1.4) \[
\int_M \sqrt{\sigma_k^2 f^2 + \sigma_{k-1}^2 |\nabla^M f|^2} + \int_{\partial M} \sigma_{k-1} f \geq n |B^n| \frac{1}{\kappa^{n-1}} \left( \int_M \sigma_{k-1} f \frac{\mu}{\kappa^{n-1}} \right)^{\frac{n-k}{n-k+1}}
\]
where \( \sigma_k := \sigma_k(\kappa) \) is the \( k \)th mean curvature. Equality holds in (1.4) if and only if \( M \) is a sphere and \( f \) is constant.

We know that the constrained curvature flow is a very effective tool for proving new and old geometric inequalities that do not generate singularities during the flow. The first example of such flow was proposed by Huisken [18] in Euclidean space \( \mathbb{R}^{n+1} \) and it is called volume preserving mean curvature flow. Huisken proved that if the initial hypersurface is uniformly convex, then the flow exists for all time and converges to a sphere in \( C^\infty \)-topology as \( t \to +\infty \).
Notice that the velocity function of this flow contains a global constrained term so that the volume of the closed domain enclosed by the hypersurface is unchanging and the area of the hypersurface is monotonically decreasing. This property makes the evolving hypersurface converge to the solution of the isoperimetric type problem in $\mathbb{R}^{n+1}$. For similar curvature flows, one can refer to [2, 8, 24, 30] and the references therein for details.

The globally constrained curvature flow is very difficult to compute a priori estimates, mainly due to the global constraint term. In [15], Guan and Li defined a locally constrained curvature flow in the space form, which is based on Minkowski identity. This type of flow only requires the initial hypersurface to be a starshaped to obtain the longtime existence and convergence. Subsequently, locally constrained curvature flow attracted a lot of attention, such as in [14, 19, 20, 26, 31]. However, its applications are mainly focused on proving the isoperimetric type inequalities.

Mean curvature flow also has promising applications in physics. It can not only describe the evolution of interfaces, such as propagation at material interfaces, fluid free boundary motion, and crystal growth (see e.g. [22, 25]), but is also widely developed in fields such as image processing, computer-aided design, and algorithms.

In this paper, we will introduce the new mean curvature type flow and apply it to prove some remarkable inequalities such as Michael-Simon inequality.

In the first part of the present paper, we introduce a new locally constrained mean curvature flow $X : M \times [0, T) \to \mathbb{R}^{n+1}$, $n \geq 2$ of starshaped hypersurfaces in $\mathbb{R}^{n+1}$ which satisfies

$$
\begin{cases}
\frac{\partial}{\partial t} X(x, t) = - \left( fH + \frac{n}{n-1} \frac{\partial f}{\partial x} \right) \nu(x, t) \\
X(\cdot, 0) = X_0(\cdot)
\end{cases}
$$

(1.5)

where $\nu(x, t)$ and $H$ are the unit outer normal and the mean curvature of $M_t = X(M, t)$, respectively, and $f : M \times [0, T, \infty) \to \mathbb{R}$ is a positive smooth function.

Suppose $M_t$ is starshaped with respect to a point $p$, given by a smooth embedding $X(\cdot, t) : S^n \to M_t \subset \mathbb{R}^{n+1}$. The radial function $r : S^n \times [0, T) \to \mathbb{R}_+$ represents the distance from $X(\cdot, t)$ to $p$, then we have $X(\xi, t) = r(\xi, t) \xi, \xi \in S^n$ and $f$ can be expressed as $f[r(\xi, t)]$. Therefore, the evolution problem (1.5) can be expressed, in terms of $r$, as a scalar PDE

$$
\begin{cases}
\frac{\partial}{\partial t} r = - \left( fH + \frac{n}{n-1} \frac{\partial f}{\partial r} \sqrt{1 + r^{-2} |\nabla r|^2} \right) \sqrt{1 + r^{-2} |\nabla r|^2} \\
r(\cdot, 0) = r_0
\end{cases}
$$

(1.6)

we will prove that the flow (1.5) has longtime existence and convergence provided that $f$ satisfies the following assumption.

**Assumption 1.5.** $\tilde{f}(r) := (n-1) \frac{1}{r^2} f(r) + \frac{1}{r} \frac{\partial f}{\partial r}(r)$ is monotonically increasing with respect to $r$, where $r > 0$, and there exists a zero point for $\tilde{f}(r)$.

Now we state the main properties of flow (1.5).

**Theorem 1.6.** Let $X_0 : M \to \mathbb{R}^{n+1}(n \geq 2)$ be a smooth embedding of a compact hypersurface $M$ in $\mathbb{R}^{n+1}$ such that $M_0 = X_0(M)$ is starshaped, and assume that $f$ satisfies Assumption 1.5. Then the flow (1.5) has a unique smooth solution $M_t = X_t(M)$ for all time $t \in [0, +\infty)$. Moreover, $M_t$ converges to a sphere as $t \to +\infty$ in $C^\infty$-topology.

In addition, we can apply the convergence result of flow (1.5) to prove the inequality (1.2) for starshaped hypersurfaces.
Theorem 1.7. Let $M$ be a smooth, compact, starshaped hypersurface in $\mathbb{R}^{n+1}$ (possibly with boundary $\partial M$). For a positive function $f \in C^\infty(M)$, there holds

$$\int_M \sqrt{|\nabla f|^2 + f^2 H^2} \, d\mu + \int_{\partial M} f \, d\mu \geq n|B^n|^\frac{1}{n} \left( \int_M f \frac{n}{n-1} \, d\mu \right)^{\frac{n-1}{n}}$$

Equality holds if and only if $M$ is a sphere and $f$ is constant.

Further, we briefly explain the proof of Theorem 1.7 in three steps.

Step 1. We apply the divergence theorem and the standard elliptic regular theory to simplify the original inequality (1.2) by scaling;

Step 2. We will show that $\int_M f \frac{n}{n-1} \, d\mu$ decreases monotonically along the flow (1.5), this property makes the inequality (1.2) hold for any evolving hypersurfaces;

Step 3. Following the convergence result of flow (1.5), we obtain the sharp constant in the inequality (1.2) and the necessary and sufficient condition for the establishment of the equality.

In the second part of this paper, we will introduce a new mean curvature type flow and use it to solve Conjecture 1.4. Let $X_0 : M \to \mathbb{R}^{n+1}$ be a smooth embedding such that $M$ is a closed, convex hypersurface in Euclidean space $\mathbb{R}^{n+1}$. We consider a smooth family of embeddings $X : M \times [0, T) \to \mathbb{R}^{n+1}$ satisfying

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} X(x, t) = \left( 1 - h \frac{E_k(\nu)}{E_{k-1}(\nu)} \right) \nu(x, t), \quad k = 1, \ldots, n \\
X(\cdot, 0) = X_0(\cdot)
\end{array} \right.$$ (1.7)

where $\nu(x, t)$ and $\kappa = (\kappa_1, \ldots, \kappa_n)$ are the unit outer normal and the principal curvatures of $M_t = X(M, t)$ respectively, $E_k(\kappa) = \binom{n}{k}^{-1} \sigma_k(\kappa)$ is the normalized $k$th mean curvature and $h(\nu) = \langle X, \nu \rangle$ is the support function of $M_t$. When $k = 1$, by scaling, the flow (1.7) is equivalent to the mean curvature type flow $\frac{\partial}{\partial t} X = (n - Hh) \nu$ introduced in [15].

Our construction of the flow (1.7) is inspired by the fully nonlinear flow defined by Guan and Li in [16], which takes the following form

$$\frac{\partial}{\partial t} X(x, t) = \left( c_{k-1} - h \frac{\sigma_k(\nu)}{\sigma_{k-1}(\nu)} \right) \nu(x, t) = c_{k-1} \left( 1 - h \frac{E_k(\nu)}{E_{k-1}(\nu)} \right) \nu(x, t)$$ (1.8)

where $c_{k-1} = \frac{\sigma_k(l)}{\sigma_{k-1}(l)} = \frac{n+1-k}{k}$. They show that starting from a smooth, closed, convex hypersurface in $\mathbb{R}^{n+1}$ ($n \geq 2$), the solution of the flow (1.8) exists for all positive time and converges smoothly and exponentially to a sphere. A nice characteristic of flow (1.8) is that the $(k-1)$th quermassintergral is unchanging and the $k$th quermassintergral is monotonically decreasing, so it can be used to prove the Alexandrov-Fenchel inequality.

Definition 1.8. For a bounded domain $\Omega$ in $\mathbb{R}^{n+1}$ with smooth boundary $M = \partial \Omega$, it is called static convex if its second fundamental form satisfies

$$h_{ij} \geq h^{-1} g_{ij} > 0$$ (1.9)

everywhere on $M$, where $h^{-1}$ is the inverse of the support function.

Concerning the flow (1.7), we obtain a new property that if the initial hypersurface is static convex, then the solution of the flow (1.7) remains static convex for all $t > 0$. It is worth noting that this property is crucial in proving Conjecture 1.4. From the inequality (1.9), we get the static convexity implies the strict convexity. Further, it can be found that the velocity functions of flow (1.7) and flow (1.8) differ only by a constant multiple.
Therefore, the longtime existence and convergence of flow (1.7) can be obtained when the initial hypersurface is static convex.

**Theorem 1.9.** Let \( X_0 : M \rightarrow \mathbb{R}^{n+1} (n \geq 2) \) be a smooth embedding of a closed, static convex hypersurface \( M_0 = X_0(M) \) in \( \mathbb{R}^{n+1} \). Then the flow (1.7) has a unique smooth solution \( M_t = X_t(M) \) for all time \( t \in [0, +\infty) \). Moreover, \( M_t \) is static convex for each \( t \geq 0 \) and it converges exponentially to a sphere of radius \( R \) in \( C^\infty \)-topology as \( t \rightarrow +\infty \), where the radius \( R \) determined by \( V_{k-1}(\Omega_0) = V_{k-1}(B_{R}^{n+1}) \).

Inspired by the idea of the proof of the Michael-Simon inequality for mean curvature, We can use the properties of flow (1.7) to prove a new sharp Michael-Simon inequality for \( k \)th mean curvature. To facilitate our discussion, let \( M_t \) be parametrized by the inverse Gauss map \( X_t : S^n \rightarrow M_t \subset \mathbb{R}^{n+1} \), then the positive function \( f \in C^\infty (M_t) \) can be expressed as \( f [h(\nu)] \). Before presenting the detailed results, we need to make the following assumption about \( f \).

**Assumption 1.10.** \( g(h) := f \frac{(n-k+1)}{(n-k)} (h) \) is convex and monotonically increasing with respect to \( h \), where \( h > 0 \).

**Theorem 1.11.** Let \( M \) be a smooth, compact and static convex hypersurface in \( \mathbb{R}^{n+1} \) (possibly with boundary \( \partial M \)), and \( \Omega \) be the domain enclosed by \( M \). Assume that \( f \) satisfies Assumption 1.10. Then for any \( 1 \leq k \leq n-1 \), there holds

\[
\int_M \sqrt{\sigma_k^2 f^2 + \sigma_{k-1}^2 |\nabla f|^2} d\mu + \int_{\partial M} \sigma_{k-1} f d\mu \\
\geq n \left( y_k \circ z_k^{-1} (V_{k-1}(\Omega)) \right)^{\frac{n-k}{n+1-k}} \left( \int_M \sigma_{k-1} f \frac{(n-k+1)}{(n-k)} d\mu \right)^{\frac{n-k}{n+1-k}}
\]

(1.10)

where \( y_k(r) = \binom{n}{k} f \frac{(n-k)}{(n-k+1-k)} z_k(r), z_k(r) = V_k(B_{r}^{n+1}) \), the \( k \)th quermassintegral for the sphere of radius \( r \), and \( z_k^{-1} \) is the inverse function of \( z_k \). Equality holds in (1.10) if and only if \( M \) is a sphere and \( f \) is constant.

In particular, if \( M \) is a sphere with radius \( R \) determined by \( V_{k-1}(\Omega) = V_{k-1}(B_{R}^{n+1}) \) and we consider the special case that \( f = R^{-(n-k)} \) in Theorem 1.11 and derive the following conclusion.

**Corollary 1.12.** Let \( M \) and \( f \) be given as above, then the inequality (1.4) for static convex hypersurface is established. Equality holds in (1.4) if and only if \( M \) is a sphere and \( f \) is constant.

**Remark 1.13.** (1) Corollary 1.12 is the weaker form of Conjecture 1.4.

(2) If \( k = 1 \), the inequality (1.4) is the sharp Michael-Simon inequality for mean curvature.
2. Preliminaries

In this section, we first collect three parametrizations of (locally) hypersurfaces embedded in $\mathbb{R}^{n+1}$. These are parametrization by radial function, by a graph of a function, by support function. The first two parameterizations are used for the flow (1.5), where the second parameterization is used in estimating the principal curvatures. the last one is used for the flow (1.7). Last, we review the properties of normalized elementary symmetric functions.

2.1 Parametrization by radial graph

As $M \subset \mathbb{R}^{n+1}$ is a smooth, compact, starshaped hypersurface with respect to a point, we may suppose to be the origin of $\mathbb{R}^{n+1}$. $M$ can be represented as

$$M = \{ r(\xi) \xi : \xi \in \mathbb{S}^n \}$$

where $r = |X|$ is the radial function and $\xi = X/|X|$. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal coordinate system of $\mathbb{S}^n$, and denote $\nabla$ be the gradient on $\mathbb{S}^n$. The induced metric on $M$ now has the form (see e.g. [12])

$$g_{ij} = \langle \nabla_i X, \nabla_j X \rangle = r^2 e_{ij} + \nabla_i r \nabla_j r$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^{n+1}$ and $e_{ij}$ is the metric of $\mathbb{S}^n$. The inverse metric $(g_{ij})^{-1}$ is

$$g^{ij} = r^{-2}(e^{ij} - \frac{\nabla^i r \nabla^j r}{r^2 + |\nabla r|^2})$$

where $\nabla^i = e^{ik} \nabla_k$. The outer unit normal is given by

$$\nu = (r \xi - \nabla r) \left( r^2 + |\nabla r|^2 \right)^{-\frac{1}{2}}$$

The second fundamental form and the mean curvature of $M$ are as follows

$$h_{ij} = -\langle \nabla_i X, \nu \rangle = (r^2 + |\nabla r|^2)^{\frac{1}{2}} \left( r^2 e_{ij} + 2 \nabla_i r \nabla_j r - r \nabla_i r \right)$$

$$H = \sum g^{ij} h_{ij} = \frac{1}{r \sqrt{1 + r^{-2} |\nabla r|^2}} \left[ n - \left( e^{ij} - \frac{\nabla^i \omega \nabla^j \omega}{1 + |\nabla \omega|^2} \right) \nabla_{ij} \omega \right]$$

where $\nabla_{ij} = \nabla_i \nabla_j$ and $\omega = \ln r$. The principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ of $M$ are the eigenvalues of the second fundamental form relative to the metric, or equivalently, they are the solutions of the equation

$$\det (\lambda_{ij} - \kappa \delta_{ij}) = 0$$

where the symmetric matrix $[\lambda_{ij}]$ is given by (see e.g. [28])

$$[\lambda_{ij}] = \left[ g^{ik} \right]^{\frac{1}{2}} [h_{kl}] \left[ g^{lj} \right]^{\frac{1}{2}}$$

$[g^{ij}]^{\frac{1}{2}}$ denotes the positive square root of $[g^{ij}]$ and can be expressed as

$$[g^{ij}]^{\frac{1}{2}} = r^{-1} \left[ e^{ij} - \frac{\nabla^i r \nabla^j r}{\sqrt{r^2 + |\nabla r|^2} \left( r^2 + \sqrt{r^2 + |\nabla r|^2} \right)} \right]$$

Note that, according to [28], if the flow equation is

$$\frac{\partial}{\partial t} X(x, t) = \Phi(x, t) \nu(x, t)$$
where \( \Phi(x, t) \) is a smooth function on \( M_t \), then the radial function \( r \) satisfies the following equation

\[
\begin{aligned}
\frac{\partial}{\partial t} r &= \sqrt{1 + r^{-2} |\nabla r|^2} \Phi(x, t) \quad \text{on} \quad S^n \times \mathbb{R}_+
\end{aligned}
\]

(2.7)

Thus the flow (1.5) can be converted into (1.6).

2.2 Parametrization by local graph

It is known that \( M \) is smooth, and by the implicit function theorem, we may assume that \( M \) can be locally represented by a graph of a function on hyperplane \( \Sigma \subset \mathbb{R}^{n+1} \). By rotating coordinates, \( \Sigma \) can become \( \mathbb{R}^n \), then there is a \( C^\infty \) function \( u : \Sigma \to \mathbb{R} \) defined in a neighborhood \( U \) of a point \( P \in M \) that satisfies

\[
M \cap U = \{(\eta, u(\eta)); \eta \in U \subset \Sigma\}
\]

The metric, the inverse metric, the second fundamental form and the principal curvatures of \( M \) are given by (see e.g. [28])

\[
\begin{aligned}
g_{ij} &= \delta_{ij} + D_i u D_j u; \quad g^{ij} = \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2}; \quad h_{ij} = \frac{D_i u D_j u}{1 + |Du|^2}^\frac{1}{2} \\
\end{aligned}
\]

and

\[
\lambda_{ij} = \frac{1}{v} \left\{ D_{ij} u - \frac{D_i u D_j u}{v(1 + v)} - \frac{D_j u D_i u}{v(1 + v)} + \frac{D_i u D_j u}{v^2(1 + v)^2} \right\}
\]

(2.8)

where \( D \) is the usual gradient in \( \mathbb{R}^n \), and \( v = (1 + |Du|^2)^\frac{1}{2} \) (see [6]). Together with (2.6), we can obtain that \( u(\cdot, t) : \Sigma \times [0, T) \to \mathbb{R} \) is the solution of the following equation

\[
\begin{aligned}
\frac{\partial}{\partial t} u &= -\sqrt{(1 + |Du|^2)} \Phi(x, t) \quad \text{on} \quad \Sigma \times \mathbb{R}_+
\end{aligned}
\]

(2.9)

2.3 Parametrization by support function

Let \( M \) be a smooth, compact, strictly convex hypersurface in \( \mathbb{R}^{n+1} \), one may suppose that the origin in its interior, then \( M \) can be parametrized by the inverse Gauss map \( X : S^n \to \mathbb{R}^n \subset \mathbb{R}^{n+1} \), defined as

\[
X(\nu) = h(\nu) \nu + \nabla h(\nu)
\]

where \( \nu \) is the unit outer normal and \( h(\nu) \) is the support function. The second fundamental form of \( M \) is given by (see e.g. [29])

\[
\begin{aligned}
h_{ij} &= \nabla_j \nabla_i h + e_{ij}
\end{aligned}
\]

(2.10)

From the Guass-Weingarten formula \( \nabla_i \nu = h_{ik} g^{kl} \nabla_l X \), we have

\[
\begin{aligned}
e_{ij} &= \langle \nabla_i \nu, \nabla_j \nu \rangle = h_{ik} g^{kl} h_{jl}
\end{aligned}
\]

and

\[
\begin{aligned}
g_{ij} &= h_{ik} e^{kl} h_{jl}
\end{aligned}
\]

According to (2.6), the general evolution equation of \( h \) is

\[
\begin{aligned}
\frac{\partial}{\partial t} h &= \Phi(x, t) \quad \text{on} \quad S^n \times \mathbb{R}_+
\end{aligned}
\]

\[
\begin{aligned}
h(\cdot, 0) = h_0
\end{aligned}
\]
Thus, the flow equation (1.7) can be expressed in terms of $h$ as the following equation

$$\begin{align*}
\frac{\partial}{\partial t} h &= 1 - h \frac{E_k(\kappa)}{E_{k-1}(\kappa)} \\
h(\cdot, 0) &= h_0
\end{align*}$$

(2.12)

**Lemma 2.1.** Let $M$ be a smooth, strictly convex hypersurface in $\mathbb{R}^{n+1}$, and the position vector $X : S^n \to \mathbb{R}^{n+1}$. Then the support function $h$ satisfies

$$\begin{align*}
(1) \nabla_i h &= <X, x_k> h_k^i \\
(2) \nabla_j \nabla_i h &= h_{ij} - h(h^2)_{ij}
\end{align*}$$

(2.13) \hspace{1cm} (2.14)

where $\{x_1, \cdots, x_n\}$ is a coordinate system in the tangent space of $M$.

**Proof.** (1) Differentiating $h = \langle X, \nu \rangle$ gives

$$\nabla_i h = \nabla_i \langle X, \nu \rangle = \langle \nabla_i X, \nu \rangle + \langle X, \nabla_i \nu \rangle = \langle X, \nabla_i \nu \rangle$$

and combining the Guass-Weingarten formula, we have

$$\nabla_i h = h^i_l \langle X, \nabla_l X \rangle = h^i_l \langle X, x_l \rangle$$

(2) The third equation can be directly obtained by (2.10). \hfill \Box

2.4 Normalized elementary symmetric functions

For each $k = 1, \ldots, n$, the normalized $k$th elementary symmetric functions for $\kappa = (\kappa_1, \ldots, \kappa_n)$ are

$$E_k(\kappa) = \binom{n}{k}^{-1} \sigma_k(\kappa) = \binom{n}{k}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}$$

and we can set $E_0(\kappa) = 1$ and $E_k(\kappa) = 0$ for $k > n$. If $A = [A_{ij}]$ is an $n \times n$ symmetric matrix and $\kappa = \kappa(A) = (\kappa_1, \cdots, \kappa_n)$ are the eigenvalues of $A$, then $E_k(A) = E_k(\kappa(A))$ can be expressed as

$$E_k(A) = \frac{(n - k)!}{n!} \delta_{i_1 \cdots j_k} A_{i_1,j_1} \cdots A_{i_k,j_k}, \quad k = 1, \ldots, n$$

Now, let’s review some of the properties of the normalized $k$th elementary symmetric functions (see e.g.\([13]\)).

**Lemma 2.2.** Let $\hat{E}^{ij}_k = \frac{\partial E_k}{\partial A_{ij}}$, then we have

$$\begin{align*}
\sum_{i,j} \hat{E}^{ij}_k g_{ij} &= k E_{k-1} \\
\sum_{i,j} \hat{E}^{ij}_k A_{ij} &= k E_k \\
\sum_{i,j} \hat{E}^{ij}_k (A^2)_{ij} &= n E_k E_k - (n - k) E_{k+1}
\end{align*}$$

(2.15) \hspace{1cm} (2.16) \hspace{1cm} (2.17)

where $(A^2)_{ij} = \sum_{k=1}^n A_{ik} A_{kj}$

Next, we recall the well-know Minkowski identity (see e.g.\([15]\)).

**Lemma 2.3.** Let $M$ be a smooth closed hypersurface in $\mathbb{R}^{n+1}$. Then

$$\int_M E_{k-1}(\kappa)d\mu = \int_M h E_k(\kappa)d\mu$$

(2.18)
Lemma 2.4. If $\kappa \in \Gamma_k^+$, the following inequality is called the Newton-MacLaurin inequality

\[(2.19)\]
\[E_{m+1}(\kappa)E_{k-1}(\kappa) \leq E_k(\kappa)E_m(\kappa), \quad 1 \leq k \leq m\]

where $\Gamma_k^+ = \{ x \in \mathbb{R}^n : E_i(x) > 0, i = 1, \ldots, k \}$. Equality holds if and only if $\kappa_1 = \cdots = \kappa_n$.

Let $F(\kappa) = \frac{E_k(\kappa)}{E_{k-1}(\kappa)}$, $F(\kappa)$ is a smooth symmetric function on $\mathbb{R}^n$. $\bar{F}^p$ and $\bar{F}^{pq}$ are the components of the first and second derivatives of $F$ with respect to its argument, i.e.

\[\bar{F}^p(\kappa) = \frac{\partial F(\kappa)}{\partial \kappa_p}, \quad \bar{F}^{pq}(\kappa) = \frac{\partial^2 F(\kappa)}{\partial \kappa_p \partial \kappa_q}\]

For a diagonal matrix $A$ with eigenvalues $\kappa = \kappa(A)$, similarly, we can view $F(\kappa)$ as a smooth symmetric function $F(A) = F(\kappa(A))$, where $F(A) = \frac{E_k(A)}{E_{k-1}(A)}$. In a local orthonormal frame, the first and second derivatives of $F(A)$ satisfy (see [1], [3])

\[\bar{F}^{pq}(A) = \bar{F}^p(\kappa)\delta_{pq}\]

and

\[\bar{F}^{pq,ij}(A)B_{pq}B_{ij} = \sum_{p,q} \bar{F}(\kappa)B_{pp}B_{qq} + 2 \sum_{p<q} \frac{\bar{F}^p(\kappa) - \bar{F}^q(\kappa)}{\kappa_p - \kappa_q} (B_{pq})^2\]

where $B \in \text{Sym}(n)$. The later formula makes sense as a limit in the case of $\kappa_p = \kappa_q$. Using (2.15), (2.17) and (2.19), we have the following corollary.

Corollary 2.5. Let $F(A) = \frac{E_k(A)}{E_{k-1}(A)}$ and $\kappa(A) \in \Gamma_k^+$, then

\[(2.20)\]
\[1 \leq \sum_{i,j} \bar{F}^{ij}g_{ij} \leq k\]

\[(2.21)\]
\[F^2 \leq \sum_{i,j} \bar{F}^{ij} (A^2)_{ij} \leq (n - k + 1)F^2\]

Lemma 2.6. See[3], $F(\kappa) = \frac{E_k(\kappa)}{E_{k-1}(\kappa)}$ satisfies the following, where $\kappa \in \Gamma_k^+$

1) $F(\kappa)$ is strictly increasing, i.e. $\bar{F}^p = \frac{\partial F}{\partial \kappa_p} > 0$ on $\Gamma_k^+$, $\forall p = 1, \ldots, n$;

2) $F(\kappa)$ is homogeneous of degree 1, i.e. $F(ak) = aF(\kappa)$ for any $a > 0$;

3) $F(\kappa)$ is strictly positive on $\Gamma_k^+$ and is normalized such that $F(1) = 1$;

4) $F(\kappa)$ is concave;

5) $F(\kappa)$ is inverse concave, i.e. the function

\[F_*(\kappa_1, \cdots, \kappa_n) = F^{(k_n^{-1}, \cdots, k_1^{-1})^{-1}}\]

is concave.

3. Evolution equation

Along the general flow (2.6) that

\[\frac{\partial}{\partial t}X(x,t) = \Phi(x,t)\nu(x,t)\]

in Euclidean space $\mathbb{R}^{n+1}$, we have the following evolution equations (see [14]).
Lemma 3.1.

(3.1) \[ \frac{\partial}{\partial t} g_{ij} = 2\Phi h_{ij} \]

(3.2) \[ \frac{\partial}{\partial t} d\mu_t = nE_1 \Phi d\mu_t \]

(3.3) \[ \frac{\partial}{\partial t} h_{ij} = -\nabla_j \nabla_i \Phi + \Phi(h^2)_{ij} \]

(3.4) \[ \frac{\partial}{\partial t} h_i^j = -\nabla^j \nabla_i \Phi - \Phi(h^2)_i^j \]

(3.5) \[ \frac{\partial}{\partial t} E_{k-1} = \frac{\partial E_{k-1}}{\partial h_i^j} \frac{\partial h_i^j}{\partial t} = \dot{E}_{k-1}^i ( -\nabla_j \nabla_i \Phi - \Phi(h^2)_{ij} ) \]

Lemma 3.2. Along the flow (1.7), we have the following evolution equations.

1. The second fundamental form evolves

\[ \frac{\partial}{\partial t} h_{ij} = h\dot{F}^{kl} \nabla_k \nabla_i \nabla_{ij} + h\dot{F}^{kl, pq} \nabla_i \nabla_{kl} \nabla_j h_{pq} + \nabla_i F \nabla_j h + \nabla_i h \nabla_j F \]

\[ \frac{\partial}{\partial t} h_{ij} = (F + h\dot{F}^{kl}(h^2)_{kl}) h_{ij} + (1 - 3hF)(h^2)_{ij} \]

2. Let \( S_{ij} = h_{ij} - h^{-1} g_{ij} \), it evolves as

\[ \frac{\partial}{\partial t} S_{ij} = h\dot{F}^{kl} \nabla_k \nabla_i S_{ij} + h^{-2} \dot{F}^{kl} \nabla_k h \nabla_i h_{ij} + h \dot{F}^{kl, pq} \nabla_i \nabla_{kl} h_{pq} + \nabla_i h \nabla_j F + \nabla_j h \nabla_j F \]

\[ \frac{\partial}{\partial t} S_{ij} = (1 - 3hF)(S^2)_{ij} + (h\dot{F}^{kl}(h^2)_{kl} - 3F) S_{ij} + 2\dot{F}^{kl}(h^2)_{kl} g_{ij} - 2h^{-1} F g_{ij} \]

Proof. (1) We have know that \( \frac{\partial}{\partial t} h_{ij} = -\nabla_i \nabla_j \Phi + \Phi(h^2)_{ij} \) and \( \Phi = (1 - hF) \), then

\[ \frac{\partial}{\partial t} h_{ij} = -\nabla_i \nabla_j (1 - hF) + (1 - hF)(h^2)_{ij} \]

\[ \frac{\partial}{\partial t} h_{ij} = F \nabla_i \nabla_j h + h \nabla_i \nabla_j F + \nabla_i F \nabla_j h + \nabla_i h \nabla_j F + (1 - hF)(h^2)_{ij} \]

where \((h^2)_{ij} = h_i^k h_{kj}\). By the Simons’s identity \( \nabla_k \nabla_i \nabla_{ij} = \nabla_i \nabla_j h_{kl} - (h^2)_{kl} h_{ij} + (h^2)_{ij} h_{kl} \), we have

\[ \nabla_i \nabla_j F = \nabla_i \left( \dot{F}^{kl} \nabla_j h_{kl} \right) = \dot{F}^{kl, pq} \nabla_i h_{pq} \nabla_j h_{kl} + \dot{F}^{kl} \nabla_i \nabla_j h_{kl} \]

\[ \nabla_i \nabla_j F = \dot{F}^{kl} \nabla_i \nabla_j h_{kl} + \dot{F}^{kl, pq} \nabla_i \nabla_j h_{kl} + \dot{F}^{kl}(h^2)_{kl} h_{ij} - \dot{F}^{kl} h_{kl}(h^2)_{ij} \]

Since \( F \) is homogeneous of 1, then \( F = \dot{F}^{kl} h_{kl} \). Substituting (2.14) and (3.9) into (3.8), we get

\[ \frac{\partial}{\partial t} S_{ij} = h\dot{F}^{kl} \nabla_i \nabla_k \nabla_j h_{ij} + h\dot{F}^{kl, pq} \nabla_i \nabla_k h_{kl} \nabla_j h_{pq} + \nabla_i F \nabla_j h + \nabla_i h \nabla_j F \]

(2) Differentiating \( S_{ij} = h_{ij} - h^{-1} g_{ij} \) with respect to \( t \) and substituting (2.12), (3.1) and (3.6)
yields
\[
\frac{\partial}{\partial t}S_{ij} = h\tilde{F}^{kl}\nabla_k\nabla_i h_{ij} + h\tilde{F}^{kl,pq}\nabla_i h_{kl}\nabla_j h_{pq} + \nabla_i F\nabla_j h + \nabla_i h\nabla_j F
\]
+ \left(F + h\tilde{F}^{kl}(h^2)_{kl}\right) h_{ij} + (1 - 3hF)(h^2)_{ij}
+ h^{-2}(1 - hF)g_{ij} - 2h^{-1}(1 - hF)h_{ij}
\]
and by direct calculation, we get the following formula
\begin{align}
\nabla_k\nabla_i S_{ij} &= \nabla_k\nabla_i h_{ij} + g_{ij}h^{-2}\nabla_k\nabla_i h - 2h^{-3}g_{ij}\nabla_kh\nabla_i h \\
(S^2)_{ij} &= S^k_kS_{kj} = (h^2)_{ij} - 2h^{-1}h_{ij} + h^{-2}g_{ij}
\end{align}

Thus, we have
\[
\frac{\partial}{\partial t}S_{ij} = h\tilde{F}^{kl}\nabla_k\nabla_i S_{ij} + 2h^{-2}g_{ij}\tilde{F}^{kl}\nabla_k h \nabla_i h + h\tilde{F}^{kl,pq}\nabla_i h_{kl}\nabla_j h_{pq} + \nabla_i h\nabla_j F + \nabla_j h\nabla_j F
+ (1 - 3hF)(S^2)_{ij} + \left(h\tilde{F}^{kl}(h^2)_{kl} - 3F\right) S_{ij} + 2\tilde{F}^{kl}(h^2)_{kl}g_{ij} - 2h^{-1}Fg_{ij}
\]
\[\square\]

4. Long time existence and smooth convergence of flow 1.5

To obtain the longtime existence of the flow (1.5), we need to derive a priori estimates, thus using the standard theory of parabolic partial differential equations. We already know that the flow (1.5) can be converted into scaler PDE (1.6), or equivalently,
\begin{align}
\begin{cases}
\frac{\partial r}{\partial t} = -\sqrt{1 + r^{-2}|\nabla r|^2}fH - \frac{n-1}{n-1}f(1 + r^{-2}|\nabla r|^2) \\
r(\cdot, 0) = r_0
\end{cases}
\end{align}

Firstly, we perform $C^0$ estimate of $r$.

**Lemma 4.1.** Let $r \in C^\infty(S^n \times [0, T))$ be a smooth solution to the initial value problem (4.1), then there are positive constants $C_1 = C_1(\delta, r_{\text{min}}(0))$ and $C_2 = C_2(\varepsilon, r_{\text{max}}(0))$, such that
\begin{align}
C_1 \leq r(\cdot, t) \leq C_2
\end{align}

**Proof.** Set $r_{\text{min}}(t) := \min_{x \in S^n} r(x, t)$, we have $\nabla r_{\text{min}} = 0$ and $\nabla^2 r_{\text{min}} \geq 0$. In view of (2.3), we get
\[
H(r_{\text{min}}) = \frac{1}{r_{\text{min}}} \left[ n - \frac{1}{r_{\text{min}}} \nabla^2 r_{\text{min}} \right]
\]
Thus
\[
\frac{\partial}{\partial t} r_{\text{min}} = -nf r_{\text{min}}^{-1} + f r_{\text{min}}^{-2} \nabla^2 r_{\text{min}} - \frac{n}{n-1} \frac{\partial f}{\partial r}(r_{\text{min}})
\]
\[
\geq -nf r_{\text{min}}^{-1} - \frac{n}{n-1} \frac{\partial f}{\partial r}(r_{\text{min}})
\]
If $-nf r_{\text{min}}^{-1} - \frac{n}{n-1} \frac{\partial f}{\partial r}(r_{\text{min}}) \geq 0$, then
\[
\frac{\partial}{\partial t} r_{\text{min}} \geq 0
\]
and by the above differential inequality, we infer that
\[ r_{\min}(t) \leq \left( \frac{f(r_{\min}(0))}{f(r_{\min}(t))} \right)^{\frac{1}{n-1}} r_{\min}(0) \]
and since \( \frac{\partial f}{\partial r}(r_{\min}) \leq -\frac{1}{n-1} f^{-1}_r \leq 0 \), we have
\[ \frac{f(r_{\min}(0))}{f(r_{\min}(t))} \geq 1 \]
Then
\[ r_{\min}(t) \geq r_{\min}(0) \]
If \( -nf^{-1}_r - \frac{n}{n-1} \frac{\partial f}{\partial r}(r_{\min}) < 0 \), then \( \hat{f}(r_{\min}) > 0 \). Since \( \hat{f} \) has a zero point, then there are constants \( 0 < \varepsilon < \delta \) such that
\[ \hat{f}(\varepsilon) < 0; \quad \hat{f}(\delta) > 0 \]
Thereby
\[ r_{\min}(t) > \delta \]
Combining the above two scenarios, we get
\[ r_{\min}(t) \geq C_1 := \max\{\delta, r_{\min}(0)\} \]
In the same way, Set \( r_{\max}(t) := \max_{x \in S^n} r(\cdot, t) \), we have \( \nabla r_{\max} = 0 \) and \( \nabla^2 r_{\max} \leq 0 \). Similarly, we obtain
\[ \frac{\partial r_{\max}}{\partial t} = -nfr^{-1}_r + fr^{-2}_{r_{\max}} \nabla^2 r_{\max} - \frac{n}{n-1} \frac{\partial f}{\partial r}(r_{\max}) \]
\[ \leq -nfr^{-1}_r - \frac{n}{n-1} \frac{\partial f}{\partial r}(r_{\max}) \]
If \( -nfr^{-1}_r - \frac{n}{n-1} \frac{\partial f}{\partial r}(r_{\max}) > 0 \), we have \( \hat{f}(r_{\max}) < 0 \), thus
\[ r_{\max}(t) < \varepsilon \]
If \( -nfr^{-1}_r - \frac{n}{n-1} \frac{\partial f}{\partial r}(r_{\max}) \leq 0 \), then
\[ r_{\max}(t) \leq \min \left\{ r_{\max}(0), \left( \frac{f(r_{\max}(0))}{f(r_{\max}(t))} \right)^{\frac{1}{n}} r_{\max}(0) \right\} \]
When \( \frac{\partial f}{\partial r}(r_{\max}) < 0 \), we have \( \frac{f(r_{\max}(0))}{f(r_{\max}(t))} < 1 \), then
\[ r_{\max}(t) \leq \left( \frac{f(r_{\max}(0))}{f(r_{\max}(t))} \right)^{\frac{1}{n}} r_{\max}(0) := \theta r_{\max}(0), \quad \theta < 1 \]
When \( \frac{\partial f}{\partial r}(r_{\max}) \geq 0 \), we have \( \frac{f(r_{\max}(0))}{f(r_{\max}(t))} \geq 1 \), then
\[ r_{\max}(t) \leq r_{\max}(0) \]
Hence
\[ r_{\max}(t) \leq \min\{r_{\max}(0), \theta r_{\max}(0)\} = \theta r_{\max}(0) \]
Combining the above two situations, we obtain
\[ r_{\max}(t) \leq C_2 := \min\{\varepsilon, \theta r_{\max}(0)\} \]
Next, we estimate the gradient of \( r \) which is not very precise but is sufficient.

**Lemma 4.2.** Let \( r \in C^\infty(S^n \times [0, T)) \) be a smooth solution to the initial value problem (4.1). For any time \( t \in [0, T) \), there is a positive constant \( C \) depending on 1 and \( M_0 \), such that

\[
\max_{x \in S^n} |\nabla r(\cdot, t)| \leq C
\]  

**Proof.** Let \( \omega = \ln r \) and \( \varphi = \frac{1}{2} |\nabla \omega|^2 \). We re-express geometric quantities in terms of \( \omega \).

\[
g_{ij} = e^{2\omega} (e_{ij} + \nabla_i \omega \nabla_j \omega) \\
g^{ij} = e^{-2\omega} (e_{ij} - \frac{\nabla_i \omega \nabla_j \omega}{1 + |\nabla \omega|^2}) \\
h_{ij} = e^{\omega} \left(1 + |\nabla \omega|^2 \right)^{-\frac{1}{2}} (e_{ij} + \nabla_i \omega \nabla_j \omega - \nabla_{ij} \omega)
\]

and

\[
H = \sum g^{ij} h_{ij} = e^{-\omega} \left(1 + |\nabla \omega|^2 \right)^{-\frac{1}{2}} \left[ n - (e_{ij} - \frac{\nabla_i \omega \nabla_j \omega}{1 + |\nabla \omega|^2}) \nabla_{ij} \omega \right]
\]

From (4.1), \( \omega \) is the solution of the initial value problem

\[
\begin{aligned}
\frac{\partial}{\partial t} \omega &= e^{-\omega} \sqrt{1 + |\nabla \omega|^2} \left(-fH - \frac{n}{n-1} e^{-\omega} \frac{\partial f}{\partial \omega} \nabla \omega \nabla \omega \right) \\
\omega(\cdot, 0) &= \omega_0 = \ln r_0
\end{aligned}
\]

Combining \( \frac{\partial}{\partial t} \varphi = \nabla^k \omega \nabla_k \left( \frac{\partial}{\partial \omega} \omega \right) \), we obtain the evolution equation of \( \varphi \)

\[
\begin{aligned}
\frac{\partial}{\partial t} \varphi &= - \nabla^k \omega \nabla_k (e^{-\omega} \sqrt{1 + |\nabla \omega|^2} fH) - e^{-\omega} \nabla^k \omega \nabla_k \left( \sqrt{1 + |\nabla \omega|^2} \right) fH \\
&\quad - e^{-\omega} \sqrt{1 + |\nabla \omega|^2} \nabla^k \omega \nabla_k fH - e^{-\omega} \sqrt{1 + |\nabla \omega|^2} f \nabla^k \omega \nabla_k H \\
&\quad - \frac{n}{n-1} \nabla^k \omega \nabla_k (e^{-2\omega}) \frac{\partial f}{\partial \omega} (1 + |\nabla \omega|^2) - \frac{n}{n-1} e^{-2\omega} \nabla^k \omega \nabla_k \left( \frac{\partial f}{\partial \omega} \right) (1 + |\nabla \omega|^2) \\
&\quad - \frac{n}{n-1} e^{-2\omega} \frac{\partial f}{\partial \omega} \nabla^k \omega \nabla_k (1 + |\nabla \omega|^2)
\end{aligned}
\]

where

\[
\nabla_k f = \frac{\partial f}{\partial \omega} \nabla_k \omega; \quad \nabla_k \left( \frac{\partial f}{\partial \omega} \right) = \frac{\partial^2 f}{\partial \omega^2} \nabla_k \omega
\]

Suppose \( \varphi \) attains the spatial maximum at point \((x_t, t) \in (S^n \times [0, T))\), we have

\[
\begin{aligned}
\nabla \varphi &= \nabla^m \omega \nabla_{km} \omega = 0 \\
\nabla^2 \varphi &= \nabla^k \omega \nabla_{km} \omega + \nabla^k \omega \nabla_{klm} \omega \leq 0 \quad k = 1, \ldots, n
\end{aligned}
\]
and also
\[
\nabla_k \left( \sqrt{1 + |\nabla \omega|^2} \right) = (1 + |\nabla \omega|^2)^{-\frac{1}{2}} \nabla^m \omega \nabla_k \omega = 0
\]

\[
H \nabla^k \omega = n e^{-\omega} (1 + |\nabla \omega|^2)^{-\frac{3}{2}} \nabla^k \omega
\]

\[
\nabla_k H = - e^\omega \nabla_k \omega (1 + |\nabla \omega|^2)^{-\frac{1}{2}} \left[ n - \left( e^{ij} - \frac{\nabla_i \omega \nabla_j \omega}{1 + |\nabla \omega|^2} \right) \right] \nabla_{ij} \omega
\]

\[
- e^\omega \left( e^{ij} - \frac{\nabla_i \omega \nabla_j \omega}{1 + |\nabla \omega|^2} \right) \nabla_{kij} \omega
\]

\[
\nabla^k \omega \nabla_k H \geq - n e^{-\omega} |\nabla \omega|^2 (1 + |\nabla \omega|^2)^{-\frac{1}{2}}
\]

Bring (4.7), (4.8) and the above equations into (4.6), we obtain
\[
\frac{\partial}{\partial t} \varphi \leq e^{-2\omega} |\nabla \omega|^2 \left[ 2nf - n \frac{\partial f}{\partial \omega} + \frac{2n}{n - 1} \frac{\partial f}{\partial \omega} (1 + |\nabla \omega|^2) - \frac{n}{n - 1} \frac{\partial^2 f}{\partial \omega^2} (1 + |\nabla \omega|^2) \right]
\]

Since \( \tilde{f} \) is monotonically increasing with respect to \( r \), we get
\[
2nf - n \frac{\partial f}{\partial r} r \leq \frac{n}{n - 1} \frac{\partial^2 f}{\partial r^2} r^2 - \frac{n}{n - 1} \frac{\partial f}{\partial r}
\]

Since
\[
\frac{\partial f}{\partial r} = r^{-1} \frac{\partial f}{\partial \omega}, \quad \frac{\partial^2 f}{\partial r^2} = r^{-2} \left( \frac{\partial^2 f}{\partial \omega^2} - \frac{\partial f}{\partial \omega} \right)
\]
then the inequality (4.10) has the following form
\[
2nf - n \frac{\partial f}{\partial \omega} \leq \frac{n}{n - 1} \frac{\partial^2 f}{\partial \omega^2} - \frac{2n}{n - 1} \frac{\partial f}{\partial \omega}
\]

If \( \frac{n}{n - 1} \frac{\partial^2 f}{\partial \omega^2} - \frac{2n}{n - 1} \frac{\partial f}{\partial \omega} \geq 0 \), we have
\[
2nf - n \frac{\partial f}{\partial \omega} \leq \frac{n}{n - 1} \frac{\partial^2 f}{\partial \omega^2} - \frac{2n}{n - 1} \frac{\partial f}{\partial \omega} \leq \left( \frac{n}{n - 1} \frac{\partial^2 f}{\partial \omega^2} - \frac{2n}{n - 1} \frac{\partial f}{\partial \omega} \right) (1 + |\nabla \omega|^2)
\]
Then
\[
\frac{\partial \varphi}{\partial t} \leq 0
\]

If \( \frac{n}{n - 1} \frac{\partial^2 f}{\partial \omega^2} - \frac{2n}{n - 1} \frac{\partial f}{\partial \omega} < 0 \), we have
\[
\frac{\partial \varphi}{\partial t} \leq 2\varphi e^{-2\omega} \left( 2nf - n \frac{\partial f}{\partial \omega} \right) - 2\varphi (1 + 2\varphi) e^{-2\omega} \left( \frac{n}{n - 1} \frac{\partial^2 f}{\partial \omega^2} - \frac{2n}{n - 1} \frac{\partial f}{\partial \omega} \right)
\]
\[
\leq 2 \left( 2nf + \frac{3n - n^2}{n - 1} \frac{\partial f}{\partial \omega} - \frac{n}{n - 1} \frac{\partial^2 f}{\partial \omega^2} \right) \varphi e^{-2\omega} - 4 \left( \frac{n}{n - 1} \frac{\partial^2 f}{\partial \omega^2} - \frac{2n}{n - 1} \frac{\partial f}{\partial \omega} \right) \varphi e^{-2\omega}
\]

Let
\[
C_3 := 2e^{-2\omega} \left( 2nf + \frac{3n - n^2}{n - 1} \frac{\partial f}{\partial \omega} - \frac{n}{n - 1} \frac{\partial^2 f}{\partial \omega^2} \right) \leq 0
\]
\[
C_4 := -4e^{-2\omega} \left( \frac{n}{n - 1} \frac{\partial^2 f}{\partial \omega^2} - \frac{2n}{n - 1} \frac{\partial f}{\partial \omega} \right) \geq 0
\]
then the inequality (4.9) is expressed as

\[ \frac{\partial \varphi}{\partial t} \leq C_3 \varphi + C_4 \varphi^2 \]  

Assume that \( C_3 + C_4 \varphi(t) \geq 0 \), otherwise we have \( \varphi(t) < -\frac{C_3}{C_4} \) and \( \frac{\partial \varphi}{\partial t} < 0 \). Since

\[ -\frac{C_3}{C_4} = -\frac{2e^{-2\omega}}{C_4} \left( \frac{2nf - n \frac{\partial f}{\partial \omega}}{n-1} \right) \leq 1 \]

Hence

\[ \varphi < C := \min\{1, \varphi(0)\} \]

We know that \( \varphi(t) \geq -\frac{C_3}{C_4} \). This implies there exists \( t_1 \in [0, T) \) such that \( \varphi_1 := \varphi(t_1) = -\frac{C_3}{C_4} \), then

\[ \frac{\partial \varphi}{\partial t} < C_3 \varphi + \left( C_4 - \frac{C_4}{C_3} \right) \varphi^2 \]

where \( -\frac{C_3}{C_4} \geq 1 \). Solving the above inequality yields

\[ \varphi < \frac{-C_3}{C_4 - \frac{C_4}{C_3}} \leq -\frac{C_3}{C_4} \]

This contradicts the hypothesis, so \( \varphi < C \) holds. \( \Box \)

Finally, we establish the uniform positive upper bound of the principal curvatures.

**Lemma 4.3.** Along the flow (1.5), the principal curvatures of \( M_t \) satisfy

\[ \kappa_i \leq \tilde{C}, \quad i = 1, \ldots, n \]

where \( \tilde{C} \) is a positive constant and depends only on \( M_0 \).

**Proof.** Suppose that at time \( t_0 \in [0, T) \), \( \kappa \) attains its maximum at point \( x_0 \) in the direction \( \zeta_0 \in T_{X(x_0,t_0)}M_{t_0} \). After rotating coordinate, we may assume that \( x_0 \in S^n \) in the north pole and \( M_t \) can be represented as the graph of \( u \) in the neighborhood of \( (X_0, t_0) \subset \Sigma \times [0, T) \), where \( X_0 = X(x_0, t_0) \subset M_{t_0}, \Sigma \subset T_{X_0}M_{t_0} \) is the tangent hyperplane, and \( f \) can be expressed as \( f(u) \). We know that

\[ \frac{\partial f}{\partial t} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial t} = \frac{\partial f}{\partial u} D_t u \]

Bring (2.7) and (2.9) into (4.14) respectively, we can get

\[ \frac{\partial f}{\partial t} = \frac{\partial f}{\partial r} \left( \sqrt{1 + r^2} |\nabla r|^2 \right) \Phi(x,t) \]

and

\[ \frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \left( -\sqrt{1 + |Du|^2} \right) \Phi(x,t) \]
Then, $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial u}$ are related as follows

\[
\frac{\partial f}{\partial r} = -\frac{\partial f}{\partial u} \sqrt{1 + |Du|^2} \sqrt{1 + r^{-2} |\nabla r|^2}
\]

(4.15)

Therefore, $u$ is the solution of the following equation

\[
\begin{cases}
D_t u = \sqrt{1 + |Du|^2} \left( f H - \frac{n}{n-1} \frac{\partial f}{\partial u} \sqrt{1 + |Du|^2} \right) \\
u(\cdot, 0) = u_0
\end{cases}
\quad \text{on } \Sigma \times [0, T)
\]

(4.16)

For convenience, we choose an orthogonal coordinate system $\{\eta_1, \ldots, \eta_n\}$ in $\Sigma$ with the origin at $X_0$. Then in coordinates centered at the origin and parallel to $\{\eta_1, \ldots, \eta_n\}$ we have

\[
X_0 = (a_1, \ldots, a_n, a)
\]

for appropriate constants $a_1, \ldots, a_n, a$ and $a > 0$. Thus

\[
X_t = (a_1, \ldots, a_n, a) + (\eta, u(\eta)),
\]

\[
\nu = \frac{(Du, -1)}{|v|}
\]

where $v = \left(1 + |Du|^2\right)^{\frac{1}{2}}$. Without loss of generality, we can assume that the maximum principal curvature of the graph $u(\cdot, t)$ is obtained at $(0, t_0)$ in the direction $\eta_1$. In the vicinity of $(0, t_0)$, we have

\[
\kappa_1 = \frac{D_{11} u}{v \left(1 + |D_1 u|^2\right)}.
\]

(4.17)

and

\[
u = \frac{(Du, -1)}{|v|}
\]

(4.18)

\[
u = \frac{(Du, -1)}{|v|}
\]

(4.19)

We can rotate $\{\eta_2, \ldots, \eta_n\}$ such that $D^2 u(0, t_0)$ is diagonal and $D_{11} u(0, t_0) > 0$. In the equations above and below, the convention of summing over Latin indices will be used instead of Greek indices. By calculating

\[
D_t \kappa_1 = \frac{D_{11} u}{v \left(1 + |D_1 u|^2\right)} - \frac{2D_1 u D_{11} u D_{11} u}{v \left(1 + |D_1 u|^2\right)^2} - \frac{v^{-1} D_k u D_{k1} u}{v^2 \left(1 + |D_1 u|^2\right)}
\]

(4.20)

Then, at $(0, t_0)$

\[
D_t \kappa_1 = D_{11} u
\]

We also have

\[
D_\alpha \kappa_1 = \frac{D_{11} \alpha u}{v \left(1 + |D_1 u|^2\right)} - \frac{2D_1 u D_{1\alpha} u D_{11} u}{v \left(1 + |D_1 u|^2\right)^2} - \frac{v^{-1} D_k u D_{k\alpha} u}{v^2 \left(1 + |D_1 u|^2\right)}
\]
and

\[
D_{\alpha\alpha\kappa_1} = \frac{D_{11\alpha\alpha}u}{v(1 + |D_1u|^2)} - \frac{D_{11\alpha}uD_\alpha v}{v^2(1 + |D_1u|^2)} - \frac{2D_{11\alpha}uD_1^1uD_1\alpha u}{v(1 + |D_1u|^2)^2} - \frac{D_{11\alpha}uD_\alpha v + D_{11\alpha}uD_{\alpha\alpha}v}{v^2(1 + |D_1u|^2)} + \frac{2D_{11\alpha}uD_1^1uD_{1\alpha}u}{v^2(1 + |D_1u|^2)^{3/2}} \left[ (1 + |D_1u|^2)D_\alpha v + 4v(1 + |D_1u|^2)D_1uD_1\alpha u \right] - \frac{2}{v(1 + |D_1u|^2)^2} \left[ D_{11\alpha}uD_1^1uD_{1\alpha}u + D_{11\alpha}uD_1^1uD_{1\alpha\alpha}u + D_{11\alpha}uD_1^1uD_{1\alpha\alpha}u \right]
\]

where

\[
D_\alpha v = v^{-1}D^k u D_{k\alpha} u \\
D_{\alpha\alpha} v = -v^{-3} \left( D^k u D_{k\alpha} u \right)^2 + v^{-1} (D_{\alpha\alpha} u)^2 + v^{-1} D^k u D_{k\alpha\alpha} u
\]

Thus, at \((0, t_0)\)

\[
D_\alpha v = 0 \\
D_{\alpha\alpha} v = (D_{\alpha\alpha} u)^2
\]

and

\[
D_\alpha \kappa_1 = D_{11\alpha} u = 0 \\
D_{\alpha\alpha} \kappa_1 = D_{\alpha\alpha}(D_{11} u) - D_{11} u(D_{\alpha\alpha} v) - 2D_{11} u(D_1 u)^2 \\
= D_{11\alpha\alpha} u - D_{11} u(D_{\alpha\alpha} u)^2 - 2(D_{11} u)^3 \leq 0
\]

Next, we compute \(D_{11\mu} u\). Differentiating (4.17) twice in \(\eta_1\) direction, we get

\[
D_{11\mu} u = D_{11} u \left( f H - \frac{n}{n - 1} \frac{\partial f}{\partial u} \right) + 2D_1 u \left( H D_1 f + f D_1 H - \frac{n}{n - 1} \frac{\partial f}{\partial u} D_1 u \right) + v \left[ H D_{11} f + f D_{11} H + 2D_1 f D_1 H - \frac{n}{n - 1} v D_{11} \left( \frac{\partial f}{\partial u} \right) - \frac{n}{n - 1} \frac{\partial f}{\partial u} D_{11} u \right] - 4 \frac{n}{n - 1} v D_1 \left( \frac{\partial f}{\partial u} \right)
\]

and at \((0, t_0)\)

\[
D_{11\mu} u = (D_{11} u)^2 \left( f H - \frac{n}{n - 1} \frac{\partial f}{\partial u} \right) - \frac{n}{n - 1} D_{11} \left( \frac{\partial f}{\partial u} \right) - \frac{n}{n - 1} \frac{\partial f}{\partial u} (D_{11} u)^2 + HD_{11} f + f D_{11} H + 2D_1 f D_1 H
\]
From $H = S_1(\kappa) = \kappa_1 + \cdots + \kappa_n$, we compute that
\[D_1 H = S_1^{ij} D_1 \lambda_{ij} = D_1 \lambda_{ij}\]
\[D_{11} H = S_1^{ij,pq} D_1 \lambda_{ij} D_1 \lambda_{pq} + S_1^{ij} D_{11} \lambda_{ij} = S_1^{ij} D_{11} \lambda_{ij}\]

at $(0, t_0)$. By direct calculation,
\[D_1 f = \frac{\partial f}{\partial u} D_1 u = 0 \quad D_{11} f = \frac{\partial^2 f}{\partial u^2} (D_1 u)^2 + \frac{\partial f}{\partial u} D_{11} u = \frac{\partial f}{\partial u} D_{11} u\]
and
\[D_1 \left( \frac{\partial f}{\partial u} \right) = \frac{\partial^2 f}{\partial u^2} D_1 u = 0 \quad D_{11} \left( \frac{\partial f}{\partial u} \right) = \frac{\partial^3 f}{\partial u^3} (D_1 u)^2 + \frac{\partial^2 f}{\partial u^2} D_{11} u = \frac{\partial^2 f}{\partial u^2} D_{11} u\]

at $(0, t_0)$. Bring the above formulas into (4.25), we get
\[D_{11} u = (D_{11} u)^3 \left( f H - \frac{n}{n-1} \frac{\partial f}{\partial u} \right) + \frac{\partial f}{\partial u} H D_{11} u + f S_1^{ij} D_{11} \lambda_{ij}\]
(4.26)
\[= - \frac{n}{n-1} \frac{\partial^2 f}{\partial u^2} (D_{11} u) - \frac{n}{n-1} \frac{\partial f}{\partial u} (D_{11} u)^2\]

and the equation (2.8) implies that
\[D_{11} \lambda_{ij} = D_{11} u u_{ij} - D_{1k} u D_{1k} u_{ij} - 2 D_{1i} u D_{1k} u D_{jk} u.\]
(4.27)

at $(0, t_0)$. From $D^2 u$ and $\left[ S_1^{ij} \right]$ are diagonal at $(0, t_0)$, we have
\[D_{11} \lambda_{11} = D_{11} u u_{11} - 3 (D_{11} u)^3\]
\[D_{11} \lambda_{ii} = D_{11} u u_{ii} - D_{ii} u (D_{11} u)^2\]
\[S_1^{ij} D_{11} \lambda_{ii} = D_{11} \lambda_{11} + \cdots + D_{nn} \lambda_{nn}\]

So by (4.24),
\[D_{11} u u_{ii} \leq D_{11} u (D_{ii} u)^2 + 2 (D_{11} u)^3\]
(4.28)
Taking (4.28) into (4.27), we get
\[D_{11} \lambda_{ii} \leq D_{11} u u_{ii} + 2 (D_{11} u)^3 - D_{ii} u (D_{11} u)^2\]
Then
\[S_1^{ij} D_{11} \lambda_{ii} \leq D_{11} u S_1^{ij} ((D_{ii} u)^2 + 2 (D_{11} u)^2 - D_{ii} u D_{11} u)\]
(4.29)
\[\leq (n + 2) (D_{11} u)^3 - H (D_{11} u)^2\]

Using (4.26) and (4.29) in (4.20), we obtain, at $(0, t_0)$
\[D_t \kappa_1 \leq (n + 2) f (D_{11} u)^3 + \frac{\partial f}{\partial u} H D_{11} u - \frac{n}{n-1} \frac{\partial^2 f}{\partial u^2} D_{11} u - \frac{2n}{n-1} \frac{\partial f}{\partial u} (D_{11} u)^2\]
(4.30)

Now extended the condition (4.10) to $f[u(\eta, t)]$. According to the different representations of $f$
\[f := f(r) = f \circ r \circ \xi : M_t \to \mathbb{S}^n \to \mathbb{R}; \quad \xi : M_t \to \mathbb{S}^n; \quad X_t(x) = r(\xi, t) \xi\]
\[f := f(u) = f \circ u \circ \eta : M_t \to \Sigma \to \mathbb{R}; \quad \eta : M_t \to \Sigma; \quad X_t(x) = (\eta, u(\eta, t))\]
we have
\[\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial \xi}{\partial x} = \frac{\partial f}{\partial r} \nabla r \frac{\partial \xi}{\partial x}\]
and

\[ \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial x}{\partial u} \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial x}{\partial u} \]

By (4.18) we have \( |\frac{\partial f}{\partial x}| = 0 \), i.e. \( |\frac{\partial f}{\partial r} \nabla r \frac{\partial r}{\partial x}| = 0 \) at \((0, t_0)\). There is either \( \frac{\partial f}{\partial r} = 0 \) or \( \nabla r \frac{\partial r}{\partial x} = 0 \).

Case 1: If \( \frac{\partial f}{\partial r} = 0 \), from (4.17), we obtain \( \frac{\partial f}{\partial u} = 0 \). Then, at \((0, t_0)\),

\[ D_t \kappa_1 \leq (n + 2)f(D_{11}u)^3 - \frac{n}{n-1} \frac{\partial^2 f}{\partial u^2} D_{11}u \]

and the inequality (4.10) becomes

\[ 0 \leq 2nf \leq \frac{n}{n-1} r_0^2 \frac{\partial^2 f}{\partial r^2} \]

where \( r_0 := r^2(0, t_0) = a_1^2 + \cdots + a_n^2 + a^2 \). Similarly, we have

\[ \frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial r^2} \left( \frac{\partial r}{\partial t} \right)^2 + \frac{\partial f}{\partial r} \frac{\partial^2 r}{\partial t^2} = \frac{\partial^2 f}{\partial u^2} \left( \frac{\partial u}{\partial t} \right)^2 + \frac{\partial f}{\partial u} \frac{\partial^2 u}{\partial t^2} \]

and

\[ \frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial r^2} \left( \frac{\partial r}{\partial t} \right)^2 + \frac{\partial f}{\partial r} \frac{\partial^2 r}{\partial t^2} = \frac{\partial^2 f}{\partial u^2} \left( \frac{\partial u}{\partial t} \right)^2 \]

at \((0, t_0)\). Using (4.14) and (4.15), we get

\[ \frac{\partial u}{\partial t} = - \left( \frac{\sqrt{1 + |Du|^2}}{\sqrt{1 + r^{-2} |\nabla r|^2}} \right) \frac{\partial r}{\partial t} \]

Thus, at \((0, t_0)\),

\[ \frac{\partial^2 f}{\partial r^2} = \frac{\partial^2 f}{\partial u^2} \frac{1}{1 + r^{-2} |\nabla r|^2} \leq \frac{\partial^2 f}{\partial u^2} \]

and \( \frac{\partial^2 f}{\partial u^2} \geq 0 \) can be obtained from (4.32). Hence, \( f(u) \) satisfies the following condition

\[ 0 \leq 2nf \leq \frac{n}{n-1} r_0^2 \frac{\partial^2 f}{\partial u^2} \]

at \((0, t_0)\). Bring (4.35) into (4.31), we have, at \((0, t_0)\),

\[ \frac{\partial \kappa_1}{\partial t} \leq \frac{1}{2} \frac{n + 2}{n-1} \frac{\partial^2 f}{\partial u^2} r_0^2 (D_{11}u)^3 - \frac{n}{n-1} \frac{\partial^2 f}{\partial u^2} D_{11}u \]

\[ = D_{11}u \frac{\partial^2 f}{\partial u^2} \left( \frac{1}{2} \frac{n + 2}{n-1} r_0^2 (D_{11}u)^2 - \frac{n}{n-1} \right) \]

We can choose \( X_0 \) to be close enough to the original origin such that \( r_0 \) is small enough and \( r_0^2 (D_{11}u)^2 \leq \epsilon \), where \( \frac{1}{4} < \epsilon < \frac{2n}{n+2} \). Thus, we get

\[ D_t \kappa_1 \leq 0 \]

Case 2: If \( \nabla r \frac{\partial r}{\partial x} = 0 \), we obtain, at \((0, t_0)\)

\[ \nabla \frac{\partial X}{\partial x} = \frac{\partial r}{\partial x} = |X|^{-1} \left( \frac{\partial X}{\partial x} \cdot X \right) = \left( \frac{\partial X}{\partial x} \cdot \xi \right) = 0 \]
From the above equation, we deduce that the unit normal vector is proportional to $\xi$, so we have

\begin{equation}
\nabla r = 0
\end{equation}

at $(0, t_0)$. Combining with (4.15) and (4.34), we also have

\begin{equation}
\frac{\partial f}{\partial u} = \frac{-\partial f}{\partial r}; \quad \frac{\partial u}{\partial t} = \frac{-\partial r}{\partial t}
\end{equation}

Through direct calculation, we can get

\[
\frac{\partial^2 r}{\partial t^2} = \frac{\partial}{\partial t} \left( -\frac{\sqrt{1 + r^{-2} |\nabla r|^2}}{\sqrt{1 + |Du|^2}} \frac{\partial u}{\partial t} \right)
\]

\[
= -\left(1 + |Du|^2\right)^{-\frac{1}{2}} \sqrt{1 + |Du|^2} \frac{\partial}{\partial t} \left( \frac{\sqrt{1 + r^{-2} |\nabla r|^2}}{\sqrt{1 + |Du|^2}} \right) \frac{\partial u}{\partial t}
\]

\[
- \sqrt{1 + r^{-2} |\nabla r|^2} \frac{\partial}{\partial t} \left( \frac{\sqrt{1 + |Du|^2}}{\sqrt{1 + |Du|^2}} \frac{\partial u}{\partial t} \right)
\]

\[
- \left(\frac{\sqrt{1 + r^{-2} |\nabla r|^2}}{\sqrt{1 + |Du|^2}}\right) \frac{\partial^2 u}{\partial t^2}
\]

where

\[
\frac{\partial}{\partial t} \left( \sqrt{1 + |Du|^2} \right) = (1 + |Du|^2)^{-\frac{1}{2}} Du D_t(Du)
\]

\[
\frac{\partial}{\partial t} \left( \sqrt{1 + r^{-2} |\nabla r|^2} \right) = (1 + r^{-2} |\nabla r|^2)^{-\frac{1}{2}} r^{-4} \left( r^2 \nabla r \frac{\partial}{\partial t} \left( \nabla r - r |\nabla r|^2 \frac{\partial}{\partial t} r \right) \right)
\]

and

\[
\frac{\partial}{\partial t} \left( \sqrt{1 + |Du|^2} \right) = 0
\]

\[
\frac{\partial}{\partial t} \left( \sqrt{1 + r^{-2} |\nabla r|^2} \right) = 0
\]

at $(0, t_0)$. Then

\begin{equation}
\frac{\partial^2 r}{\partial t^2} = -\frac{\sqrt{1 + r^{-2} |\nabla r|^2}}{\sqrt{1 + |Du|^2}} \frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 u}{\partial t^2}
\end{equation}

Taking (4.37) and (4.38) into (4.33), we get, at $(0, t_0)$,

\begin{equation}
\frac{\partial^2 f}{\partial r^2} = \frac{\partial^2 f}{\partial u^2}
\end{equation}

Therefore, combining (4.10) and the above relation equations, we have

\begin{equation}
2nf + nr_0 \frac{\partial f}{\partial u} \leq \frac{n}{n-1} r_0^2 \frac{\partial^2 f}{\partial u^2} + \frac{n}{n-1} r_0 \frac{\partial f}{\partial u}
\end{equation}

at $(0, t_0)$. Bring (4.40) into (4.30) we get

\[
D_t \kappa_1 \leq \left( \frac{1}{n-1} r_0^2 \frac{\partial^2 f}{\partial u^2} + \frac{1}{n-1} (2-n) r_0 \frac{\partial f}{\partial u} \right) (D_{11} u)^3
\]

\[
+ HD_{11} u \frac{\partial f}{\partial u} - \frac{2n}{n-1} \frac{\partial f}{\partial u} D_{11} u - \frac{2n}{n-1} \frac{\partial f}{\partial u} (D_{11} u)^2
\]
It is known that the following formula is valid
\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2} \left( \frac{\partial u}{\partial \eta} \right)^2 + \frac{\partial f}{\partial u} \frac{\partial^2 u}{\partial \eta^2} \left( \frac{\partial \eta}{\partial \xi} \right)^2 + \frac{\partial f}{\partial u} \frac{\partial u}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2}
\]
and the equation (4.18) and (4.36) implies
\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial r} \frac{\partial^2 r}{\partial \xi^2} \left( \frac{\partial \xi}{\partial x} \right)^2 + \frac{\partial f}{\partial r} \frac{\partial^2 r}{\partial \eta^2} \left( \frac{\partial \eta}{\partial \xi} \right)^2 + \frac{\partial f}{\partial r} \frac{\partial r}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2}
\]
and
\[
\Delta r D^2 u = \frac{\partial^2 f}{\partial u^2} \left( \frac{\partial u}{\partial \eta} \right)^2 + \frac{\partial f}{\partial u} \frac{\partial^2 u}{\partial \eta^2} \left( \frac{\partial \eta}{\partial \xi} \right)^2 + \frac{\partial f}{\partial u} \frac{\partial u}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2}
\]
From the definition of \( \xi \), it follows that \( \xi = \frac{X}{|X|} = \frac{(\eta, u)}{\sqrt{\eta^2 + u^2}} \). Differentiating \( \xi \) with respect to \( \eta \)
\[
\frac{\partial \xi}{\partial \eta} = \frac{1}{\sqrt{\eta^2 + u^2}} (\eta, u) - \frac{(\eta, u)}{(\eta^2 + u^2)^{3/2}} (\eta + uD)u)
\]
and
\[
\frac{\partial \xi}{\partial \eta} = a \frac{r_0}{r_0^4} (a, -a_1, \ldots, -a_n)
\]
at \((0, t_0)\). Therefore
\[
\frac{\Delta r}{D^2 u} = \frac{\partial^2 r}{\partial \xi^2} \left( \frac{\partial \xi}{\partial \eta} \right)^2 = -\left( \frac{\partial \eta}{\partial \xi} \right)^2 = -\left( \frac{\partial \xi}{\partial \eta} \right)^2 = -\frac{r_0^4}{a^2}
\]
Using \( r = |X| \), we calculate that
\[
\Delta r = -|X|^2 \langle \nabla X, X \rangle \langle \nabla X, X \rangle + |X|^{-1} \langle \nabla^2 X, X \rangle + |X|^{-1} \langle \nabla X, \nabla X \rangle
\]
\[
= -\langle \nabla r \rangle^2 + |X|^{-1} \langle \nabla^2 X, X \rangle + |X|^{-1} \langle \nabla X, \nabla X \rangle
\]
Thus
\[
\Delta r = |X|^{-1} \langle \nabla^2 X, X \rangle + |X|^{-1} \langle \nabla X, \nabla X \rangle
\]
at \((0, t_0)\). From \( X = (\eta, u(\eta)) \), we have
\[
DX = \frac{\partial X}{\partial \eta} = \nabla X \frac{\partial \xi}{\partial \eta}
\]
\[
D^2 X = \nabla^2 X \left( \frac{\partial \xi}{\partial \eta} \right)^2 + \nabla X \frac{\partial^2 \xi}{\partial \eta^2}
\]
\[
\langle DX, DX \rangle = \langle \nabla X, \nabla X \rangle \left( \frac{\partial \xi}{\partial \eta} \right)^2
\]
\[
\langle D^2 X, X \rangle = \langle \nabla^2 X, X \rangle \left( \frac{\partial \xi}{\partial \eta} \right)^2 + \langle \nabla X, X \rangle \frac{\partial^2 \xi}{\partial \eta^2}
\]
and
\[ \langle D^2 X, X \rangle = \langle \nabla^2 X, X \rangle \left( \frac{\partial \xi}{\partial \eta} \right)^2 \]
at \((0, t_0)\). Hence
\[ \Delta r = r_0^{-1} \left( \frac{\langle D^2 X, X \rangle}{r_0^2} + \frac{\langle DX, DX \rangle}{r_0^2} \right) = r_0^{-1} \left( \frac{\partial \eta}{\partial \xi} \right)^2 (-aD^2 u + 1) \]

By (4.41) and (4.42), we get
\[ \Delta r = \frac{r_0^4}{a^2(r_0 + a)} \]
at \((0, t_0)\). Thus
\[ H(0, t_0) = \frac{1}{r_0} \left( \frac{n - \frac{r_0^3}{a^2(r_0 + a)}}{n - 1} \right) \]
We may assume that \( a^2 = \frac{1}{c^2}r_0^2 \) where \( c \in (1, \frac{4}{3}) \), then
\[ 0 < (n - 1)r_0^{-1} < H(0, t_0) < nr_0^{-1} \]
By selecting a suitable \( X_0 \) can make \( \frac{1}{4} < (r_0D_{11}u)^2 < \epsilon \), then
\[ D_t \kappa_1 \leq \frac{1}{2} \frac{n + 2}{n - 1} \left( \frac{\partial^2 f}{\partial u^2} D_{11}u - (n - 2) \frac{\partial f}{\partial u} r_0^{-1} D_{11}u \right) r_0^2(D_{11}u)^2 + \frac{\partial f}{\partial u} D_{11}u H \]
\[ \leq \frac{n}{n - 1} \frac{\partial f}{\partial u} r_0^{-1} D_{11}u - \frac{n}{n - 1} \frac{\partial f}{\partial u} (D_{11}u)^2 \]
\[ \leq \frac{n}{n - 1} \frac{\partial f}{\partial u} D_{11}u \left( \frac{n}{n - 1} - \frac{2n - 1}{n - 1} r_0 D_{11}u \right) \leq 0 \]
Furthermore, we can apply the maximum principle to conclude that \( \kappa \) is bounded from above.
\[ \Box \]
Combining the above Lemmas we can get the longtime existence of flow (1.5).

**Proposition 4.4.** Let \( T^* \) be the maximal existence time of the flow (1.5). Then \( T^* = \infty \).

**Proof.** From (4.1), we know that
\[ \frac{\partial}{\partial t} r = -\sqrt{1 + r^{-2}|\nabla r|^2} fH - \frac{n}{n - 1} \frac{\partial f}{\partial r} (1 + r^{-2}|\nabla r|^2) := \tilde{\Phi} \]
where \( \tilde{\Phi} := \tilde{\Phi}(\nabla r, \nabla^2 r) \), and \( Q^{ij} = \frac{\partial \tilde{\Phi}}{\partial r_{ij}} \) is positive. Hence, (1.6) is a parabolic equation on \( S^n \times \mathbb{R}_+ \), the short time existence of the flow (1.5) can be derived from the theory of parabolic equation. By Proposition 4.1 and Proposition 4.2, we have the \( C^0 \) and \( C^1 \) estimate for the flow (1.5). In addition, from Proposition 4.3 we get the upper bounded of the principal curvature, thus we obtain the uniform \( C^2 \) estimate of the flow (1.5). After that, applying Krylov’s [21] theory and standard parabolic Schauder estimate to derive \( C^{2, \alpha} \) estimate, and higher order regular estimates respectively. Therefore, we get the longtime existence of the flow (1.5).
Finally, we prove the flow (1.5) converges to the unique sphere by making an exact estimate of the gradient of \( r \).

**Proposition 4.5.** Let \( r \in C^\infty(S^n \times [0, \infty)) \) be a smooth solution to the initial value problem (4.1). For any time \( t \in [0, \infty) \), there is a positive constant \( C \) depending only on \( M_0 \), such that

\[
\max_{\xi \in S^n} |\nabla r(\cdot, t)| \leq C e^{-\gamma t}
\]

**Proof.** From Lemma 4.2

\[
\frac{\partial \varphi}{\partial t} \leq e^{-2\omega |\nabla \omega|^2} \left[ 2nf - n \frac{\partial f}{\partial \omega} + \frac{2n}{n-1} \frac{\partial f}{\partial \omega} (1 + |\nabla \omega|^2) - n \frac{\partial^2 f}{\partial \omega^2} \right]
\]

If \( \frac{n}{n-1} \frac{\partial^2 f}{\partial \omega^2} - \frac{2n}{n-1} \frac{\partial f}{\partial \omega} \geq 0 \), we have

\[
\frac{\partial \varphi}{\partial t} \leq e^{-2\omega |\nabla \omega|^2} \left( 2nf + \frac{3n - n^2 \partial f}{n-1} - \frac{n}{n-1} \frac{\partial^2 f}{\partial \omega^2} \right)
\]

Let

\[
C_6 : = e^{-2\omega} \left( 2nf + \frac{3n - n^2 \partial f}{n-1} - \frac{n}{n-1} \frac{\partial^2 f}{\partial \omega^2} \right)
\]

and since \( f \in C^\infty(M) \), so \( \frac{\partial f}{\partial r} \) and \( \frac{\partial^2 f}{\partial r^2} \) are continuous functions. From Proposition 4.1, we know that \( r \) is bounded, then \( \frac{\partial f}{\partial r} \) and \( \frac{\partial^2 f}{\partial r^2} \) are bounded. Therefore, \( C_6 \) is bounded.

For some positive constant \( \gamma \geq -\frac{1}{2} \max C_6 \), we obtain

\[
\frac{\partial \varphi}{\partial t} \leq -\gamma \varphi
\]

and this implies that the gradient of the radial function decreases exponentially.

If \( \frac{n}{n-1} \frac{\partial^2 f}{\partial \omega^2} - \frac{2n}{n-1} \frac{\partial f}{\partial \omega} < 0 \), we have

\[
\frac{\partial \varphi}{\partial t} \leq C_3 \varphi + C_4 \varphi^2
\]

where \( C_3 + C_4 \varphi < 0 \). There exists a constant \( \gamma_1 > 0 \) such that \( C_3 + C_4 \varphi \leq -\gamma_1 \), then

\[
\frac{\partial \varphi}{\partial t} \leq -\gamma_1 \varphi
\]

So far the proof of Proposition 4.5 is complete. \( \square \)

It can be inferred from Proposition 4.5 that \( r_\infty := \lim_{t \to \infty} r(\cdot, t) \) exists and

\[
\max_{\xi \in S^n} |r(\cdot, t) - r_\infty| \leq C e^{-\gamma t}
\]

Now we can apply the interpolation inequality and Sobolev embedding theorem on \( S^n \) to derive the convergence of \( r(\cdot, t) \) to \( r_\infty \) is smooth. From the comparison principle, the uniqueness of \( r_\infty \) follows in a standard way. This completes the proof of Theorem 1.6.
5. New proof of sharp Michael-Simon inequality

In this section, we will apply the smooth convergence result in Theorem 1.6 to give a new proof of the sharp Michael-Simon inequality (1.2) and find the necessary and sufficient condition for the establishment of the equality. We know that the inequality (1.2) can be simplified to (1.3), so the monotonicity of $\int_{M} f^{\frac{n}{n-1}} d\mu_t$ is the key point.

**Lemma 5.1.** Along flow (1.5), we have

$$\frac{\partial}{\partial t} \int_{M_t} f^{\frac{n}{n-1}} d\mu_t \leq 0$$

**Proof.**

$$\frac{\partial}{\partial t} \int_{M_t} f^{\frac{n}{n-1}} d\mu_t = \int_{M_t} \left( \frac{n}{n-1} f^{\frac{1}{n-1}} \frac{\partial f}{\partial t} + f^{\frac{n}{n-1}} \Phi H \right) d\mu_t$$

$$= \int_{M_t} f^{\frac{1}{n-1}} \left( \frac{n}{n-1} \frac{\partial f}{\partial r} \sqrt{1 + r^{-2} |\nabla r|^2 + fH} \right) \Phi d\mu_t$$

and $\Phi = - \left( \frac{n}{n-1} \frac{\partial f}{\partial r} \sqrt{1 + r^{-2} |\nabla r|^2 + fH} \right)^{\frac{1}{n-1}}$, thus

$$\frac{\partial}{\partial t} \int_{M_t} f^{\frac{n}{n-1}} d\mu_t = - \int_{M_t} \left( \frac{n}{n-1} \frac{\partial f}{\partial r} \sqrt{1 + r^{-2} |\nabla r|^2 + fH} \right)^2 \leq 0$$

The monotonicity of $\int_{M_t} f^{\frac{n}{n-1}} d\mu_t$ yields

$$\int_{M_0} f^{\frac{n}{n-1}} d\mu \geq \int_{M_t} f^{\frac{n}{n-1}} d\mu_t \geq \int_{M_\infty} f^{\frac{n}{n-1}} d\mu_\infty$$

and from the smooth convergence of the flow (1.5), $M_\infty = B_{r_\infty}$, then

$$H = nr_\infty^{-1}, \quad \nabla r_\infty = 0$$

and

$$\nabla f(r_\infty) = \frac{\partial f}{\partial r_\infty} \nabla r_\infty = 0$$

Therefore, $f(r_\infty)$ is constant and

$$\int_{M_\infty} f^{\frac{n}{n-1}} d\mu_\infty = \int_{B_{r_\infty}} f^{\frac{n}{n-1}} d\mu_\infty = f^{\frac{n}{n-1}} (r_\infty) r_\infty^{n} |B^n|$$

The same, we have $\frac{\partial}{\partial t} \int_{M} f^{\frac{n}{n-1}} d\mu_t = 0$ when $M_t = B_{r_\infty}$, then

$$\frac{n}{n-1} \frac{\partial f}{\partial r} \sqrt{1 + r^{-2} |\nabla r|^2 + fH} = 0$$

i.e.

$$nf(r_\infty) r_\infty^{-1} + \frac{n}{n-1} \frac{\partial f}{\partial r_\infty} (r_\infty) = 0$$

Solving the above ODE, we infer that

$$f(r_\infty) = r_\infty^{\frac{n}{n-1}}$$
and adding (5.4) into (5.2), we get
\[
\int_{M_\infty} f^{\frac{n}{n-1}} d\mu_\infty = |B^n|
\]
hence
\[
\int_M f^{\frac{n}{n-1}} d\mu \geq |B^n|
\]
which means that the starshaped hypersurface satisfies the inequality (1.2).

It is obvious that equality holds for the sphere, we just need to prove the converse. Suppose that the smooth starshaped hypersurface \(M_t\) makes the equality hold
\[
(5.5) \quad \int_{M_t} f^{\frac{n}{n-1}} d\mu = f^{\frac{n}{n-1}}(r_\infty) r_\infty^n |B^n| = |B^n|
\]
Then, along the flow (1.5), the integral \(\int_{M_t} f^{\frac{n}{n-1}} d\mu_t\) remains to be a constant and there is (5.3) that
\[
\frac{n}{n-1} \frac{\partial f}{\partial r} \sqrt{1 + r^{-2} \vert \nabla r \vert^2} + fH = 0
\]
Thereby, we find
\[
\frac{\partial r}{\partial t} = -\sqrt{1 + r^{-2} \vert \nabla r \vert^2} \left( \frac{n}{n-1} \frac{\partial f}{\partial r} \sqrt{1 + r^{-2} \vert \nabla r \vert^2} + fH \right) = 0
\]
and
\[
\frac{\partial}{\partial t} \left( f^{\frac{n}{n-1}} r^n \right) = f^{\frac{n}{n-1}} \left( \frac{n}{n-1} \frac{\partial f}{\partial r} r + nf \right) r^{n-1} \frac{\partial r}{\partial t} = 0
\]
Thus
\[
r^n f^{\frac{n}{n-1}}(r) = r_\infty^n f^{\frac{n}{n-1}}(r_\infty)
\]
The equation (5.5) is equivalent to
\[
(5.6) \quad \int_{M_t} f^{\frac{n}{n-1}} d\mu = f^{\frac{n}{n-1}}(r_\infty) r_\infty^n |B^n| = f^{\frac{n}{n-1}}(r) r^n |B^n|
\]
and we can deduce that
\[
(5.7) \quad r^n = \frac{\int_{M_t} f^{\frac{n}{n-1}} d\mu_t}{f^{\frac{n}{n-1}}(r) |B^n|}
\]
Differentiating (5.7) and combining with (5.6), we get
\[
\nabla(r^n) = \nabla \left( \frac{\int_{M_t} f^{\frac{n}{n-1}} d\mu_t}{f^{\frac{n}{n-1}}(r) |B^n|} \right) = -\frac{n}{n-1} \frac{\nabla f}{f} r^n
\]
Thus
\[
\nabla(r^n) + \frac{n}{n-1} \frac{\nabla f}{f} r^n = \nabla r \left( f + \frac{n}{n-1} \frac{\partial f}{\partial r} r \right) = 0
\]
It can be inferred that either \(\nabla r = 0\) or \(f + \frac{n}{n-1} \frac{\partial f}{\partial r} r = 0\).

Case1: If \(\nabla r = 0\), \(r\) is constant, then \(M_t = B_r\) and \(\nabla f = \frac{\partial f}{\partial r} \nabla r = 0\) i.e. \(f\) is constant.

Case2: If \(f + \frac{1}{n-1} \frac{\partial f}{\partial r} r = 0\), then
\[
(5.8) \quad f(r) = r^{-(n-1)}
\]
Taking (5.8) into (5.3), we have
\[
\left( -\frac{1}{n-1}rH + \frac{n}{n-1}\sqrt{1 + r^{-2}|\nabla r|^2} \right) \frac{\partial f}{\partial r} = 0
\]
i.e.
\[
-\frac{1}{n-1}rH + \frac{n}{n-1}\sqrt{1 + r^{-2}|\nabla r|^2} = 0
\]
Furthermore, we can get
\[
H = \frac{n\sqrt{r^2 + |\nabla r|^2}}{r^2}
\]
Owing to
\[
H = g^{ij}h_{ij} = \frac{n(r^2 + 2|\nabla r|^2 - r\Delta r)}{(r^2 + 2|\nabla r|^2)^{\frac{3}{2}}}
\]
Combining (5.9) and (5.10), we have
\[
r\Delta r = -r^{-2}|\nabla r|^4
\]
and the equation (2.2) implies that
\[
h_{ij} = \frac{e_{ij}}{\sqrt{r^2 + |\nabla r|^2}} (r + r^{-1}|\nabla r|^2)^2
\]
Thus
\[
|A|^2 = g^{ij}g^{kl}h_{ik}h_{jl} = \frac{n^2}{(r^2 + |\nabla r|^2)^3} (r + r^{-1}|\nabla r|^2)^4
\]
Note that
\[
\frac{|A|^2}{H^2} = 1
\]
Therefore, \(M_t\) is sphere and it can be inferred that \(f\) is constant.

6. Preserving of static convexity

In this section, we will show that static convexity is preserved along the flow (1.7). The main tool we use in our proof is the tensor maximum principle shown below, which was proved by Andrews.

**Theorem 6.1.** ([3]) \ Let \(S_{ij}\) be a smooth time-varying symmetric tensor field on a compact manifold \(M\) satisfying
\[
\frac{\partial}{\partial t} S_{ij} = a^{kl} \nabla_k \nabla_l S_{ij} + b^k \nabla_k S_{ij} + N_{ij}
\]
where \(a^{kl}\) and \(b\) are smooth, \(\nabla\) is a (possibly time-dependent) smooth symmetric connection, and \(a^{kl}\) is positive definite everywhere. Suppose that
\[
N_{ij}v^i v^j + \sup_{\Lambda_k^p} 2a^{kl} (2\Lambda_k^p \nabla_i S_{ip} v^j - \Lambda_k^p \Lambda_l^q S_{pq}) \geq 0
\]
whenever \(S_{ij} \geq 0\) and \(S_{ij} v^j = 0\), where the supremum is taken over all \(n \times n\) matrix \(\Lambda_k^p\). If \(S_{ij} \geq 0\) everywhere on \(M\) at \(t = 0\) and on \(\partial M\) for \(0 \leq t \leq T\), then \(S_{ij} \geq 0\) holds on \(M\) for \(0 \leq t \leq T\).

Now, we state the main conclusions of this section.
Theorem 6.2. If the initial hypersurface $M_0$ is static convex, then along the flow (1.7), the evolving hypersurfaces $M_t$ is static convex for $t > 0$.

Proof. Let $S_{ij} = h_{ij} - h^{-1}g_{ij}$, then static convexity is equivalent to the positive of the tensor $S_{ij}$. By (3.7), we have

$$
\frac{\partial}{\partial t} S_{ij} = h\dot{F}^{kl}\nabla_k\nabla_l S_{ij} + h^{-2}\dot{F}^{kl}\nabla_k h\nabla_l h g_{ij} + h\dot{F}^{kl, pq}\nabla_i h_{kl} \nabla_j h_{pq} + \nabla_i h \nabla_j F + \nabla_j h \nabla_i F + \nabla_i h \nabla_j F + \nabla_j h \nabla_i F
$$

$$
+ (1 - 3hF)(S^2)_{ij} + \left( h\dot{F}^{kl}(h^2)_{kl} - 3F \right) S_{ij} + 2\dot{F}^{kl}(h^2)_{kl} g_{ij} - 2h^{-1} F g_{ij}
$$

From Theorem 6.1, we need to prove the inequality (6.1) whenever $S_{ij} \geq 0$ and $S_{ij} v^i = 0$ ($v$ is a null vector of $S_{ij}$). Suppose that at $(x, t)$, $S_{ij}$ has a null vector $v$ and the principal curvatures is strictly monotonically increasing, i.e. $\kappa_1 < \kappa_2 < \cdots < \kappa_n$. $S_{ij} v^i = 0$ implies that $v = e_1$ and $S_{11} = \kappa_1 - h^{-1} = 0$ at $(x, t)$. Since $S_{ij}$ and $(S^2)_{ij}$ satisfy the null vector condition, it remains to show that

$$Q_1 := 2h^{-2}\dot{F}^{kl}\nabla_k h \nabla_l h g_{ij} + h\dot{F}^{kl, pq}\nabla_i h_{kl} \nabla_j h_{pq} + \nabla_i h \nabla_j F + \nabla_j h \nabla_i F + \nabla_i h \nabla_j F + \nabla_j h \nabla_i F$$

$$+ 2\dot{F}^{kl}(h^2)_{kl} g_{ij} - 2h^{-1} F g_{ij} + 2h \sup_A \dot{F}^{kl} (2\Lambda^p_q \nabla_l S_{1p} - \Lambda^p_k \Lambda^q_l S_{pq})$$

$$\geq 0$$

at $(x, t)$ for all matrix $(\Lambda^p_k)$. We know that $S_{11} = 0$ and $\nabla_k S_{11} = 0$ at $(x, t)$, then

$$\dot{F}^{kl} (2\Lambda^p_k \nabla_l S_{1p} - \Lambda^p_k \Lambda^q_l S_{pq}) = \sum_{k=1}^n \sum_{p=2}^n \dot{F}^k \left( 2\Lambda^p_k \nabla_k S_{1p} - (\Lambda^p_k)^2 S_{pp} \right)$$

$$= \sum_{k=1}^n \sum_{p=2}^n \dot{F}^k \left( \frac{(\nabla_k S_{1p})^2}{S_{pp}} - \left( \frac{\nabla_k S_{1p}}{S_{pp}} \right)^2 S_{pp} \right)$$

It follows that the supremum is obtained by choosing $\Lambda^p_k = \frac{\nabla_k S_{1p}}{S_{pp}}$ for $p \geq 2, k \geq 1$ and $\Lambda^1_k = 0$ for all $k$. Thus, $Q_1$ becomes:

$$Q_1 := 2h^{-2}\dot{F}^{kl}\nabla_k h \nabla_l h g_{ij} + h\dot{F}^{kl, pq}\nabla_i h_{kl} \nabla_j h_{pq} + \nabla_i h \nabla_j F + \nabla_j h \nabla_i F$$

$$+ 2\dot{F}^{kl}(h^2)_{kl} g_{ij} - 2h^{-1} F g_{ij} + 2h \sum_{k=1}^n \sum_{p=2}^n \dot{F}^k \left( \frac{(\nabla_k S_{1p})^2}{S_{pp}} \right)$$

By Lemma 2.6, $F(\kappa)$ is inverse concave. Through direct calculation, such as in [4], we have

$$\dot{F}^{kl, pq}\nabla_1 h_{kl} \nabla_1 h_{pq} + 2h \sum_{k=1}^n \sum_{p=2}^n \dot{F}^k \left( \frac{(\nabla_k S_{1p})^2}{S_{pp}} \right)$$

$$\geq 2h F^{-1} |\nabla_1 F|^2 + 2 \sum_{k=1, p=1}^n \dot{F}^k \left( \frac{1}{\kappa_{kp} - h^{-1}} - \frac{1}{\kappa_p} \right) (\nabla_1 h_{kp})$$

$$\geq 2h F^{-1} |\nabla_1 F|^2 + 2 \sum_{k=1}^n \dot{F}^k \left( \frac{h^{-1}}{(\kappa_k - h^{-1})\kappa_k} \right) (\nabla_1 h_{kk})$$

In addition, using the Cauchy-Schwarz inequality, we have

$$\sum_{k=2}^n \frac{\dot{F}^k}{\kappa_k (\kappa_k - h^{-1})} (\nabla_1 h_{kk})^2 \cdot \sum_{k=2}^n \dot{F}^k \kappa_k (\kappa_k - h^{-1}) \geq \left( \sum_{k=2}^n \dot{F}^k |\nabla_1 h_{kk}| \right)^2 \geq |\nabla_1 F|^2$$
where we use $\nabla_{1} S_{11} = \nabla_{1} \kappa_{1} = \nabla_{1} h^{-1} = 0$ and the inequalities (2.21) that $\sum_{k=1}^{n} \dot{F}^{k} \kappa_{k}^{2} - F \geq F^{2} - F > 0$. Since $h^{-1} < \kappa_{2} < \cdots < \kappa_{n}$, we get

$$\sum_{k=2}^{n} \frac{\dot{F}^{k} \kappa_{k}}{(\kappa_{k} - h^{-1})} (\nabla_{1} h_{kk})^{2} \geq \frac{|\nabla_{1} F|^{2}}{\sum_{k=1}^{n} F^{k} \kappa_{k}^{2} - h^{-1} F} \geq 0$$

Thereby

$$Q_{1} \geq 2h F^{-1} |\nabla_{1} F|^{2} + 2h \sum_{k=1}^{n} \frac{\dot{F}^{k} \kappa_{k}}{(\kappa_{k} - h^{-1})} (\nabla_{1} h_{kk})^{2}$$

$$+ 2h^{-2} \dot{F}^{kl} \nabla_{k} h \nabla_{l} h + 2 \dot{F}^{kl}(h^{2})_{kl} - 2h^{-1} F$$

$$\geq 2h F^{-1} |\nabla_{1} F|^{2} + \sum_{k=1}^{n} \frac{|\nabla_{1} F|^{2}}{F^{k} \kappa_{k}^{2} - h^{-1} F} + 2 \dot{F}^{kl}(h^{2})_{kl} - 2h^{-1} F$$

$$+ 2h^{-2} \dot{F}^{kl} \langle X, x_{i} \rangle \langle X, x_{j} \rangle h_{k}^{i} h_{l}^{j}$$

$$\geq 2h^{-2} F^{2} + 2F^{2} - 2h^{-1} F$$

$$\geq 2h^{-1} F (h^{-1} F - 1) + 2F^{2} \geq 0$$

\[Q_{1} \geq 2h F^{-1} |\nabla_{1} F|^{2} + 2h \sum_{k=1}^{n} \frac{\dot{F}^{k} \kappa_{k}}{(\kappa_{k} - h^{-1})} (\nabla_{1} h_{kk})^{2} + 2h^{-2} \dot{F}^{kl} \nabla_{k} h \nabla_{l} h + 2 \dot{F}^{kl}(h^{2})_{kl} - 2h^{-1} F \geq 2h^{-1} F (h^{-1} F - 1) + 2F^{2} \geq 0\]

**Proof of Theorem 1.9:** We proved that static convexity is preserved along the flow (1.7), then $M_{t}$ is strictly convex for all $t > 0$ and can be represented $M_{t}$ in terms of the support function $h$. First of all, directly applying the maximum principle we can obtain $C^{0}$ estimate of $h$, which is equivalent to $C^{0}$ estimate of the solution $M_{t}$. $C^{1}$ estimate and $C^{2}$ estimate follow from the same steps in [17]. Secondly, since the initial hypersurface $M_{0}$ is static convex, which implies that $M_{0}$ is strictly convex, then one can obtain that flow (1.7) is uniformly parabolic. Through Krylov’s theory, we get $C^{2,\alpha}$ estimate. The longtime existence of the flow (1.7) can be obtained from standard theory for parabolic equations. Finally, as in [16], By estimating the gradient of the radial function of $M_{t}$, we can derive that $M_{t}$ is exponentially converging to a sphere.

### 7. Sharp Michael-Simon inequality for $kh$ mean curvature

In this section, we use the convergence result of the flow (1.7) to prove the inequality (1.10) for static convex hypersurface. First of all, by scaling, we may assume that

$$\int_{M} \sqrt{\sigma_{k}^{2} f^{2} + \sigma_{k-1}^{2} |\nabla M f|^{2}} + \int_{\partial M} \sigma_{k-1} f = n \int_{M} \sigma_{k-1} f \frac{n-k+1}{n-k}$$

This normalization ensures that we can find a function $\vartheta : M \to \mathbb{R}$ which solves the PDE

$$\text{div}_{M} (\sigma_{k-1} f \nabla M \vartheta) = n \sigma_{k-1} f \frac{n-k+1}{n-k} - \sqrt{\sigma_{k}^{2} f^{2} + \sigma_{k-1}^{2} |\nabla M f|^{2}}$$

on $M$ with Neumann boundary condition $\langle \nabla M \vartheta, \vec{n} \rangle = 1$ on $\partial M$. Here, $\vec{n}$ denotes the co-normal to $M$. Note that $\vartheta$ is of class $C^{2,\beta}$ for each $0 < \beta < 1$ by standard elliptic regularity theory. Now we only need to prove the following inequality

$$\int_{M} \sigma_{k-1} f \frac{n-k+1}{n-k} d\mu \geq y_{k} \circ z_{k-1}^{-1}(V_{k-1}(\Omega))$$

\[(7.1) \int_{M} \sigma_{k-1} f \frac{n-k+1}{n-k} d\mu \geq y_{k} \circ z_{k-1}^{-1}(V_{k-1}(\Omega))\]
Secondly, we prove that the flow (1.7) preserves the \((k-1)\)th quermassintergral \(V_{k-1}(\Omega_t)\) where \(\Omega_t\) is the domain enclosed by \(M_t\), as following

\[
\frac{\partial}{\partial t} V_{k-1}(\Omega_t) = (n + 2 - k) \int_{M_t} \left( 1 - h \frac{E_k}{E_{k-1}} \right) E_{k-1} d\mu_t = \int_{M_t} (E_{k-1} - hE_k) d\mu_t = 0
\]

where the last equality we use Minkowski identity \((2.18)\).

Last but not least, we need to establish the monotonicity of \(\int_{M_t} \sigma_{k-1}(\kappa) \int n-k-1 d\mu_t\) along the flow \((1.7)\). For computational convenience, using \(g = \int n-k-1\)

\[
\frac{\partial}{\partial t} \int_{M_t} \sigma_{k-1} g d\mu_t = \int_{M_t} \left( g \frac{\partial \sigma_{k-1}}{\partial t} + \sigma_{k-1} \frac{\partial g}{\partial f} \frac{\partial h}{\partial t} + \sigma_1 \sigma_{k-1} g \Phi \right) d\mu_t
\]

\[
= \int_{M_t} g \binom{n}{k-1} \dot{E}^{ij}_{k-1} (\nabla_j \nabla_i \Phi - \Phi (h^2)_{ij}) d\mu_t
\]

\[
+ \int_{M_t} \left( \sigma_{k-1} \frac{\partial g}{\partial f} \frac{\partial h}{\partial t} + g \sigma_{k-1} \sigma_1 \right) \Phi d\mu_t
\]

\[
= \int_{M_t} - \binom{n}{k-1} \left( \dot{E}^{ij}_{k-1} \nabla_j \nabla_i g \right) \Phi d\mu_t + \int_{M_t} \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} \sigma_{k-1} \Phi d\mu_t
\]

where the third equality we use integration by parts. Combining \((2.13)\) and \((2.14)\), we have

\[
\frac{\partial}{\partial t} \int_{M_t} \sigma_{k-1} g d\mu_t = \int_{M_t} - \binom{n}{k-1} \left( \frac{\partial^2 g}{\partial f^2} \left( \frac{\partial f}{\partial h} \right)^2 + \frac{\partial g}{\partial f} \frac{\partial^2 f}{\partial h^2} \right) \dot{E}^{ij}_{k-1} (h^2)_{ij} \Phi d\mu_t
\]

\[
- \int_{M_t} \left( \binom{n}{k-1} \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} \dot{E}^{ij}_{k-1} (h_{ij} - h (h^2)_{ij}) \Phi d\mu_t
\]

\[
+ \int_{M_t} k \binom{n}{k} gE_k \Phi d\mu_t + \int_{M_t} \left( \binom{n}{k-1} \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} E_{k-1} \Phi d\mu_t
\]

\[
= \int_{M_t} - \binom{n}{k-1} \left( \frac{\partial^2 g}{\partial f^2} \left( \frac{\partial f}{\partial h} \right)^2 + \frac{\partial g}{\partial f} \frac{\partial^2 f}{\partial h^2} \right) \dot{E}^{ij}_{k-1} (h^2)_{ij} \Phi d\mu_t
\]

\[
+ \int_{M_t} \left( \binom{n}{k-1} \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} h \dot{E}^{ij}_{k-1} (h^2)_{ij} \Phi d\mu_t
\]

\[
+ \int_{M_t} k \binom{n}{k} gE_k \Phi d\mu_t + \int_{M_t} \left( \binom{n}{k-1} \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} E_{k-1} \Phi d\mu_t
\]

\[
(7.2)
\]

\[
- \int_{M_t} \left( \binom{n}{k-1} \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} E_{k-1} \Phi d\mu_t
\]

\[
\]
where we use \((2.16)\) in the last equality. Adding \(\Phi = 1 - h \frac{E_k}{E_{k-1}}\) in \((7.2)\), we get
\[
\frac{\partial}{\partial t} \int_{M_t} \sigma_{k-1} g d\mu_t
\]
\[
= \int_{M_t} - \binom{n}{k}(k-1) \left( \frac{\partial^2 g}{\partial f^2} \left( \frac{\partial f}{\partial h} \right)^2 + \frac{\partial g}{\partial f} \frac{\partial^2 f}{\partial h^2} \right) \left( nE_1 - (n-k+1) \frac{E_k}{E_{k-1}} \right) (E_{k-1} - hE_k) d\mu_t
\]
\[
- \int_{M_t} \binom{n}{k-1} (k-2) \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} (E_{k-1} - hE_k) d\mu_t + \int_{M_t} \binom{n}{k} g \frac{E_k}{E_{k-1}} (E_{k-1} - hE_k) d\mu_t
\]
\[
+ \int_{M_t} n \binom{n}{k-1} \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} hE_1 (E_{k-1} - hE_k) d\mu_t
\]
\[
- \int_{M_t} (n-k+1) \binom{n}{k-1} \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} E_{k-1} (E_{k-1} - hE_k) d\mu_t
\]
where we use \((2.16)\). From Assumption 1.10, we can get the following inequalities
\[
\frac{\partial f}{\partial h} \geq 0; \quad \frac{\partial^2 g}{\partial f^2} \left( \frac{\partial f}{\partial h} \right)^2 + \frac{\partial g}{\partial f} \frac{\partial^2 f}{\partial h^2} \leq 0
\]
and using \((1.9)\) and \((2.19)\), we also have
\[
h_{ij} \geq h^{-1} g_{ij} \quad \text{(i.e. } E_1 \geq h^{-1}) \quad \text{;} \quad (E_{k-1} - hE_k) = \frac{1}{k-1} \dot{E}_{k-1}^{ij} \left( g_{ij} - hh_{ij} \right) \leq 0; \quad E_k \leq E_1 E_{k-1}
\]
Thus, combining the above several inequalities, we get
\[
\frac{\partial}{\partial t} \int_{M_t} \sigma_{k-1} g d\mu_t \leq \int_{M_t} - \binom{n}{k-1} (k-2) \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} (E_{k-1} - hE_k) d\mu_t
\]
\[
- \int_{M_t} \binom{n}{k-1} (k-2) \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} (E_{k-1} - hE_k) d\mu_t + \int_{M_t} n \binom{n}{k-1} \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} hE_1 (E_{k-1} - hE_k) d\mu_t
\]
\[
\leq \int_{M_t} - \binom{n}{k-1} (k-2) \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} (E_{k-1} - hE_k) d\mu_t + \int_{M_t} \frac{n}{k-1} \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} hE_1 (E_{k-1} - hE_k) d\mu_t
\]
\[
\leq \int_{M_t} \frac{n}{k-1} \frac{\partial g}{\partial f} \frac{\partial f}{\partial h} (E_{k-1} - hE_k) d\mu_t \leq 0
\]
(7.3)

Therefore
\[
\int_{M_t} \sigma_{k-1} g d\mu_t \geq \int_{M_\infty} \sigma_{k-1} g d\mu_\infty = \int_{B_R} \sigma_{k-1} g d\mu
\]
The equality in the above formula is obtained from the convergence result of the flow \((1.7)\) and \(B_R = \partial B_{R}^{n+1}\). Also \(\nabla f = \frac{\partial f}{\partial h} \nabla h = 0\) on \(B_R\) i.e. \(f\) is constant on \(B_R\).

We already know the flow \((1.7)\) preserved \((k-1)\)th quermassintergral, i.e.
\[
V_{k-1}(\Omega_0) = V_{k-1}(\Omega_t) = V_{k-1}(B_{R}^{n+1})
\]
and
\[
\int_{B_R} \sigma_{k-1} g d\mu = \binom{n}{k-1} \int_{B_R} E_{k-1}(\kappa) d\mu = \binom{n}{k-1} \int_{B_R^+} V_k(B_R^{n+1}) \\
= y_k \circ z_{k-1}^{-1}(V_{k-1}(B_R^{n+1})) = y_k \circ z_{k-1}^{-1}(V_{k-1}(\Omega_t)) = y_k \circ z_{k-1}^{-1}(V_{k-1}(\Omega_0))
\]

Then
\begin{equation}
\int_{M_t} \sigma_{k-1} g d\mu_t \geq y_k \circ z_{k-1}^{-1}(V_{k-1}(\Omega_t)) \tag{7.4}
\end{equation}

Now we just need to show that \( M \) is a sphere and \( f \) is constant when the equality holds in (1.10). If the smooth static convex hypersurface \( M_t \) attains the equality
\[
\int_{M_t} \sigma_{k-1} f \frac{n-k+1}{n-k} d\mu_t = y_k \circ z_{k-1}^{-1}(V_{k-1}(\Omega_t))
\]

Then the equality in (7.3) holds, which implies that the principal curvature of \( M_t \) satisfies \( \kappa_1 = \cdots = \kappa_n = h^{-1} \) i.e. \( M \) is the sphere. In the same way, we can infer that \( f \) is constant. This completes the proof of Theorem 1.11.

Without loss of generality, if \( M \) is a sphere \( B_R \) and we take \( f = R^{-(n-k)} \), then
\[
\int_{B_R} \sigma_{k-1} f \frac{n-k+1}{n-k} d\mu = |B^n|
\]

Therefore, we obtain the inequality (1.4) for the static convex hypersurface.

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