1

Discovering the Kerr and Kerr–Schild metrics

Roy Patrick Kerr
ICRANet, Piazzale della Repubblica 10, I-65122 Pescara, Italy,
and University of Canterbury, Christchurch, New Zealand.

1.1 Introduction

The story of this metric begins with a paper by Alexei Zinovievich Petrov (1954) where the simultaneous invariants and canonical forms for the metric and conformal tensor are calculated at a general point in an Einstein space. This paper took a while to be appreciated in the West, probably because the Kazan State University journal was not readily available, but Felix Pirani (1957) used it as the foundation of an article on gravitational radiation theory. He analyzed gravitational shock waves, calculated the possible jumps in the Riemann tensor across the wave fronts, and related these to the Petrov types.

I was a graduate student at Cambridge, from 1955 to 1958. In my last year I was invited to attend the relativity seminars at Kings College in London, including one by Felix Pirani on his 1957 paper. At the time I thought that he was stretching when he proposed that radiation was type N, and I said so, a rather stupid thing for a graduate student with no real supervisor to do†. It seemed obvious that a superposition of type N solutions would not itself be type N, and that gravitational waves near a macroscopic body would be of general type, not Type N.

Perhaps I did Felix an injustice. His conclusions may have been oversimplified but his paper had some very positive consequences. Andrzej Trautman computed the asymptotic properties of the Weyl tensor for outgoing radiation by generalizing Sommerfeld’s work on electromagnetic radiation, confirming that the far field is Type N. Bondi, M.G.J. van der Burg and Metzner (1962) then introduced appropriate null coor-

† My nominal supervisor was a particle physicist who had no interest in general relativity.
ordinates to study gravitational radiation in the far zone and related this to the results of Petrov and Pirani.

In 1958 I went to Syracuse University as a research associate of Peter Bergman. While there I was invited to join Joshua Goldberg at the Aeronautical Research Laboratory in Dayton Ohio. There was another relativist at the lab, Dr Joseph Schell, who had studied Einstein’s unified field theory under Vaclav Hlavaty. Josh was about to go on study leave to Europe for a few months, and did not want to leave Joe by himself.

Before he left, Josh and I became interested in the new methods that were entering general relativity from differential geometry at that time. We did not have a copy of Petrov’s paper so our first project was to re-derive his classification using projective geometry, something which was being done by many other people throughout the world at that time. In each empty Einstein space, \(E\), the conformal tensor determines four null “eigenvectors” at each point. The metric is called algebraically special (AS) if two of these eigenvectors coincide. This vector is then called a principal null vector (PNV) and the field of these is called a principal null “congruence”.

After this, we used a tetrad formulation to study vacuum Einstein spaces with degenerate holonomy groups (Goldberg and Kerr (1961), Kerr and Goldberg (1961)). The tetrad used consisted of two null vectors and two real orthogonal space-like vectors,

\[ ds^2 = (\omega^1)^2 + (\omega^2)^2 + 2\omega^3\omega^4. \]

We proved that the holonomy group must be an even dimensional subgroup of the Lorentz group at each point, and that if its dimension is less than six then coordinates can be chosen so that the metric has the following form:

\[ ds^2 = dx^2 + dy^2 + 2du(dv + \rho dx + \frac{1}{2}(\omega - \rho \cdot v)du), \]

† There is a claim spread on internet that we were employed to develop an antigravity engine to power spaceships. This is rubbish! The main reason why the The US Air Force had created a General Relativity section was probably to show the navy that they could also do pure research. The only real use that the USAF made of us was when some crackpot sent them a proposal for antigravity or for converting rotary motion inside a spaceship to a translational driving system. These proposals typically used Newton’s equations to prove non-conservation of momentum for some classical system.

‡ The term algebraically degenerate is sometimes used instead.
where both $\rho$ and $\omega$ are independent of $v$ and

$$
\rho_{,xx} + \rho_{,yy} = 0
$$

$$
\omega_{,xx} + \omega_{,yy} = 2\rho_{,ux} - 2\rho\rho_{,xx} - (\rho_x)^2 + (\rho_y)^2
$$

This coordinate system was not quite uniquely defined. If $\rho$ is bilinear in $x$ and $y$ then it can be transformed to zero, giving the well-known plane-fronted wave solutions. These are type N, and have a two-dimensional holonomy groups. The more general metrics are type III with four-dimensional holonomy groups.

In September 1961 Joshua joined Hermann Bondi, Andrzej Trautman, Ray Sachs and others at King’s College in London. By this time it was well known that all such AS spaces possess a null congruence whose vectors are both geodesic and shearfree. These are the degenerate “eigenvectors” of the conformal tensor at each point. Andrzej suggested to Josh and Ray how they might prove the converse. This led to the celebrated Goldberg-Sachs theorem (see Goldberg and Sachs (1962)):

**Theorem 1** A vacuum metric is algebraically special if and only if it contains a geodesic and shearfree null congruence.

Either properties of the congruence, being geodesic and shear-free, or a property of the conformal tensor, algebraically degeneracy, could be considered fundamental with the others following from the Goldberg-Sachs theorem. It is likely that most thought that the algebra was fundamental, but I believe that Ivor Robinson and Andrzej Trautman (1962) were correct when they emphasized the properties of the congruence instead. They showed that for any Einstein space with a shear-free null congruence which is also hypersurface orthogonal there are coordinates for which

$$
ds^2 = 2r^2 P^{-2} d\zeta d\bar{\zeta} - 2du dr - (\Delta \ln P - 2r(\ln P)_{,u} - 2m(u)/r)du^2,
$$

where $\zeta$ is a complex coordinate, $= (x + iy)/\sqrt{2}$, say, so that

$$
2d\zeta d\bar{\zeta} = dx^2 + dy^2.
$$

The one remaining field equation is,

$$
\Delta \Delta (\ln P) + 12m(\ln P),_u - 4m, = 0, \quad \Delta = 2P^2 \partial_\zeta \partial_{\bar{\zeta}}.
$$

The PNV $h$ is $k = k_\mu \partial_\mu = \partial_r$, where $r$ is an affine parameter along

$\S$ The simple way that the coordinate $v$ appears was to prove typical of these algebraically special metrics.

$\dagger$ The letter $k$ will be used throughout this article to denote the PNV.
Fig. 1.1. Ivor Robinson and Andrzej Trautman constructed all Einstein spaces possessing a hypersurface orthogonal shearfree congruence. Whereas Bondi and his colleagues were looking at spaces with these properties asymptotically, far from any sources, Robinson and Trautman went a step further, constructing exact solutions. (Images courtesy of Andrzej Trautman and the photographer, Marek Holzman)

the rays. The corresponding differential form is \( k = k_\mu dx^\mu = du \), so that \( k \) is the normal to the surfaces of constant \( u \). The coordinate \( u \) is a retarded time, the surfaces of constant \( r, u \) are distorted spheres with metric \( ds^2 = 2r^2 P^{-2}d\zeta d\bar{\zeta} \) and the parameter \( m(u) \) is loosely connected with the system’s mass. This gives the complete solution to the Robinson–Trautman problem‡.

In 1962 Goldberg and myself attended a month-long meeting in Santa Barbara. It was designed to get mathematicians and relativists talking to each other. Perhaps the physicists learned a lot about more modern mathematical techniques, but I doubt that the geometers learned much from the relativists. All that aside, I met Alfred Schild at this

‡ In the study of exact solutions, “solving” a problem usually means introducing a useful coordinate system, solving the easier Einstein equations and replacing the ten components of the metric tensor with a smaller number of functions, preferably of less than four variables. These will then have to satisfy a residual set of differential equations, the harder ones, which usually have no known complete solution. For example, the remaining field equation for the Robinson–Trautman metrics is highly nonlinear and has no general solution.
The Kerr and Kerr–Schild metrics

Fig. 1.2. Ezra T Newman, with T. Unti and L.A. Tambourino, studied the field equations for diverging and rotating algebraically special Einstein spaces.

conference. He had just persuaded the Texas state legislators to finance a Center for Relativity at the University of Texas, and had arranged for an outstanding group of relativists to join. These included Roger Penrose and Ray Sachs, but neither could come immediately and so I was invited to visit for the 62-63 academic year.

After Santa Barbara, Goldberg and myself flew to a conference held at Jablonna near Warsaw. This was the third precursor to the triennial meetings of the GRG society and therefore it could be called GR3. Robinson and Trautman (1964) presented a paper on “Exact Degenerate Solutions” at this conference. They spoke about their well-known solution and also showed that when the rays are not hypersurface orthogonal coordinates can be chosen so that

\[ ds^2 = -P^2[(d\xi - ak)^2 + (d\eta - bk)^2] + 2dpk + ck^2, \]

where, as usual, \( k \) is the PNV. Its components, \( k_\alpha \), are independent of \( \rho \), but \( a, b, c \) and \( P \) may be functions of all four coordinates.

I was playing around with the structure of the Einstein equations during 1962, using the new (to physicists) methods of tetrads and differential forms. I had written out the equations for the curvature using a complex null tetrad and self-dual bivectors, and then studied their integrability conditions. In particular, I was interested in the same problem that Robinson and Trautman were investigating where \( k \) was not a gra-
dient, i.e. twisting, but there was a major road block in my way. Alan Thompson had also come to Austin that year and was also interested in these methods. Although there seemed to be no reason why there should not be many algebraically special spaces, Alan kept quoting a result from a preprint of a paper by Newman, Tambourino and Unti (1963) in which they had “proved” that the only possible space with a diverging and rotating PNV is NUT space, a one parameter generalization of the Schwarzschild metric. They derived this result using the new Newman–Penrose spinor formalism (N–P). Their equations were essentially the same as those obtained by people such as myself using self-dual bivectors: only the names are different. I could not understand how the equations that I was studying could possibly lead to their claimed result, but could only presume it must be so since I did not have a copy of their paper.

In the spring of 1963 Alan obtained a preprint of this paper and loaned it to me. I thumbed through it quickly, trying to see where their hunt for solutions had died. The N–P formalism assigns a different Greek letter to each component of the connection, so I did not try to read it carefully, just rushed ahead until I found what appeared to be the key equation,

$$\frac{1}{3}(n_1 + n_2 + n_3)a^2 = 0,$$

where the $n_i$ were all small integers, between -4 and +4. Their sum was not zero so this gave $a = 0$. I had no idea what $a$ represented, but its vanishing seemed to be disastrous and so I looked more carefully to see where this equation was coming from. Three of the previous equations, each involving first derivatives of some of the connection components, had been differentiated and then added together. All the second derivatives cancelled identically and most of the other terms were eliminated using other N–P equations, leaving

$$n_1 + n_2 + n_3 \equiv 0.$$

In effect, they rediscovered one component of the identities, but with
The Kerr and Kerr–Schild metrics

numerical errors. The real fault was the way the N–P formalism confuses the Bianchi identities with the derived equations involving derivatives of the $\Psi_i$ variables.

Alan Thompson and myself were living in adjoining apartments, so I dashed next door and told him that their result was incorrect. Although it was unnecessary, we recalculated the first of the three terms, $n_1$, obtained a different result to the one in the preprint, and verified that (1.2) was now satisfied. Once this blockage was out of the way, I was then able to continue with what I had been doing and derive the metric and field equations for twisting algebraically special spaces. The coordinates I constructed turned out to be essentially the same as the ones given in Robinson and Trautman (1962). This shows that they are the “natural” coordinates for this problem since the methods used by them were very different to those used by me. Ivor loathed the use of such things as N–P or rotation coefficients, and Andrzej and he had a nice way of proving the existence of their canonical complex coordinates $\zeta$ and $\bar{\zeta}$. I found this same result from one of the Cartan equations, as will be shown in the next section, but I have no doubt that their method is more elegant. Ivor explained it to me on more than one occasion, but unfortunately I never understood what he was saying†.

At this point I presented the results at a monthly Relativity conference held at the Steven’s Institute in Hoboken, N.J. When I told Ted Newman that (1.1) should have been identically zero, he said that they knew that $n_1$ was incorrect, but that the value for $n_2$ given in the preprint was a misprint and so (1.2) was still not satisfied. I replied that since the sum had to be zero the final term, $n_3$ must also be incorrect. Alan and I recalculated it that evening, confirming that (1.2) was satisfied‡.

1.2 Discovery of the Kerr metric

When I realized that the attempt by Newman et al. to find all rotating AS spaces had foundered and that Robinson and Trautman appeared to have stopped with the static ones, I rushed headlong into the search for these metrics.

Why was the problem so interesting to me? Schwarzschild, by far the most significant physical solution known at that time, has an event

† While writing this article I read their 1962 paper and finally understood how they derived their coordinates. It only took me 45 years.
‡ Robinson and Trautman (1962) also doubted the original claim by Newman et al. since they knew that the linearized equations had many solutions.
horizon. A spherically symmetric star that collapses inside this is forever lost to us, but it was not known whether angular momentum could stop this collapse to a black hole. Unfortunately, there was no known metric for a rotating star. Schwarzschild was an example of the Robinson–Trautman metrics, none of which could contain a rotating source as they were all hypersurface orthogonal. Many had tried to solve the Einstein equations assuming a stationary and axially symmetric metric, but none had succeeded in finding any physically significant rotating solutions. The equations for such metrics are complicated nonlinear PDEs in two variables. What was needed was some extra condition that would reduce these to ODEs, and this might be the assumption that the metric is AS.

The notation used in the rest of this paper is fairly standard. There were two competing formalisms being used around 1960, complex tetrads and spinors. I used the former, Newman et al. the latter. The derived equations are essentially identical, but each approach has some advantages. Spinors make the the Petrov classification trivial once it has been shown that a tensor with the symmetries of the conformal tensor corresponds to a completely symmetric spinor, $\Psi_{ABCD}$. The standard notation for the components of this tensor is

$$\Psi_0 = \Psi_{0000}, \quad \Psi_1 = \Psi_{0001}, \ldots \quad \Psi_4 = \Psi_{1111}.$$ 

If $\zeta^A$ is an arbitrary spinor then the equation

$$\Psi_{ABCD}\zeta^A\zeta^B\zeta^C\zeta^D = 0$$

is a homogeneous quartic equation with four complex roots, $\zeta_i^A$. The related real null vectors, $Z_i^{\alpha\dot{\alpha}} = \zeta_i^\alpha \zeta_i^{\dot{\alpha}}$, are the four PNVs of Petrov. The spinor $\zeta^\alpha = \delta^\alpha_0$ gives a PNV if $\Psi_0 = 0$. It is a repeated root and therefore it is the principal null vector of an AS spacetime if $\Psi_1 = 0$ as well.

The Kerr (1963) letter presented the main results of my calculations but gave few details. The methods that I used to solve the equations for AS spaces are essentially those used in “Exact Solutions” by Stephani et al. (2003), culminating in their equation (27.27). I will try to use the same notation as in that book since it is almost identical to the one I used in 1963, but I may get some of the signs wrong. Beware!

† Robinson and Trautman also had a fairly natural complex tetrad approach.
‡ I spent many years trying to write up this research but, unfortunately, I could never decide whether to use spinors or a complex tetrad, and thus it did not get written up until Kerr and Debney (1969). George Debney also collaborated with Alfred Schild and myself on the Kerr–Schild metrics in Debney et al. (1970).
Suppose that \((e_a) = (m, \bar{m}, l, k)\) is a null tetrad, i.e., a set of four null vectors where the last two are real and the first two are complex conjugates. The corresponding dual forms are \((\omega^a) = (\bar{m}, m, -k, -l)\) and the metric is
\[
 ds^2 = 2(m\bar{m} - kl) = 2(\omega^1\omega^2 - \omega^3\omega^4) \tag{1.3}
\]

The vector \(k\) is a PNV and so its direction is uniquely defined, but the other directions are not. The form of the metric tensor in (1.3) is invariant under a combination of a null rotation \((B)\) about \(k\), a rotation \((C)\) in the \(m \wedge \bar{m}\) plane and a Lorentz transformation \((A)\) in the \(l \wedge k\) plane,
\[
 k' = k, \quad m' = m + Bk, \quad l' = l + B\bar{m} + \bar{B}m + B\bar{B}k, \tag{1.4a}
 k' = k, \quad m' = e^{iC}m, \quad l' = l, \tag{1.4b}
 k' = Ak, \quad m' = m, \quad l' = A^{-1}l. \tag{1.4c}
\]

The most important connection form is
\[
 \Gamma_{41} = \Gamma_{41a}\omega^a = m^\alpha k_\alpha dx_\beta
\]

The optical scalars of Ray Sachs for \(k\) are just the components of this form with respect to the \(\omega^a\).
\[
 \sigma = \Gamma_{411} = \text{shear},
 \rho = \Gamma_{412} = \text{complex divergence},
 \kappa = \Gamma_{414} = \text{geodesy},
\]

The fourth component, \(\Gamma_{413}\), is not invariant under a null rotation about \(k\),
\[
 \Gamma'_{413} = \Gamma_{413} + B\rho,
\]
and has no real geometric significance. It can be set to zero using an appropriate null rotation. Also, since \(k\) is geodesic and shearfree for AS metrics, both \(\kappa\) and \(\sigma\) are zero and therefore
\[
 \Gamma_{41} = \rho\omega^2. \tag{1.5}
\]

If we use the simplest field equations,
\[
 R_{44} = 2R_{4142} = 0, \quad R_{41} = R_{4112} - R_{4134} = 0, \quad R_{11} = 2R_{4113} = 0,
\]
† I personally hate the minus sign in this expression and did not use it in 1963, but it seems to have become standard. By the time I finish this article I am sure that I will wish I had stuck with all positive signs!
a total of 5 real equations, and the fact that the metric is AS,

\[ \Psi_0 = -2R_{4141} = 0, \quad 2\Psi_1 = -R_{4112} - R_{4134} = 0, \]

then the most important of the second Cartan equations simplifies to

\[ d\Gamma_{41} - \Gamma_{41} \wedge (\Gamma_{12} + \Gamma_{34}) = R_{41 ab} \omega^a \wedge \omega^b = R_{4123} \omega^2 \wedge \omega^3. \] (1.6)

Taking the wedge product of (1.6) with \( \Gamma_{41} \) and using (1.5),

\[ \Gamma_{41} \wedge d\Gamma_{41} = 0. \] (1.7)

This was the key step in my study of these metrics but this result was not found in quite such a simple way. At first, I stumbled around using individual component equations rather than differential forms to look for a useful coordinate system. It was only after I had found this that I realized that using differential forms from the start would have short-circuited several days analysis.

Equation (1.7) is the integrability condition for the existence of complex functions, \( \zeta \) and \( \Pi \), such that

\[ \Gamma_{41} = d\zeta/\Pi, \quad \Gamma_{42} = d\bar{\zeta}/\bar{\Pi}. \]

The two functions \( \zeta \) and its complex conjugate, \( \bar{\zeta} \), will be used as (complex) coordinates. They are not quite unique since \( \zeta \) can always be replaced by an arbitrary analytic function \( \Phi(\zeta) \).

Using the transformations in (1.4b) and (1.4c),

\[ \Gamma'_{41} = Ae^{iC} \Gamma_{41} = Ae^{iC} d\zeta/\Pi, \quad \Rightarrow \quad \Pi' = A^{-1} e^{-iC} \Pi. \]

\( \Pi' \) can be set to 1 by choosing \( Ae^{iC} = \Pi \), and that is what I did in 1963, but it is also common to just use the \( C \)-transformation to convert it to a real function \( P \),

\[ \Gamma_{41} = \rho \omega^2 = d\zeta/P. \] (1.8)

This is the derivation for two of the coordinates used in 1963. Since \( \omega^1 k^a = 0 \rightarrow k(\zeta) = 0 \), these functions, \( \zeta, \bar{\zeta} \), are constant along the PNV.

The other two coordinates were very standard and were used by most people considering similar problems at that time. The simplest field equation is

\[ R_{44} = 0 \quad \Rightarrow \quad k\rho = \rho_{|4} = \rho^2, \]

so that the real part of \(-\rho^{-1}\) is an affine parameter along the rays. This
was the obvious choice for the third coordinate, \( r \),

\[
\rho^{-1} = -(r + i\Sigma).
\]

There was no clear choice for the fourth coordinate, so \( u \) was chosen so that \( P^u u,\alpha = 1, \) \( k^\alpha u,\alpha = 0 \), a pair of consistent equations.

Given these four coordinates, the basis forms are

\[
\begin{align*}
\omega_1 &= m_\alpha dx^\alpha = -d\zeta/P = (r - i\Sigma)d\zeta/P, \\
\omega_2 &= \bar{m}_\alpha dx^\alpha = -d\bar{\zeta}/P \rho = (r + i\Sigma)d\bar{\zeta}/P, \\
\omega_3 &= k_\alpha dx^\alpha = du + Ld\zeta + \bar{L}d\bar{\zeta}, \\
\omega_4 &= l_\alpha dx^\alpha = dr + Wd\zeta + \bar{W}d\bar{\zeta} + H\omega_3.
\end{align*}
\]

where \( L \) is independent of \( R \), and the coefficients \( \Sigma, W \) and \( H \) have still to be determined.

When all this was substituted into the first Cartan equation, (1.42), and (1.6), the simplest component of the second Cartan equation, (1.43), \( \Sigma \) and \( W \) were calculated as functions of \( L \) and its derivatives

\[
\begin{align*}
2i\Sigma &= P^2(\partial \bar{L} - \partial \bar{\zeta}), & \partial = \partial \zeta - L\partial u, \\
W &= -(r + i\Sigma)L_u + i\partial \Sigma.
\end{align*}
\]

The remaining field equations, the “hard” ones, were more complicated, but still fairly straightforward to calculate. Two gave \( H \) as a function of a real “mass” function \( m(u, \zeta, \bar{\zeta}) \) and certain functions of the higher derivatives of \( P \) and \( L \).

\[
\begin{align*}
H &= \frac{1}{2}K - r(\ln P)_u - \frac{mr + M\Sigma}{r^2 + \Sigma^2}, \\
M &= \Sigma K + P^2 \text{Re}[\partial \bar{\zeta} - 2L_u \partial \Sigma - \Sigma \partial u \partial \bar{\zeta}], \\
K &= 2P^{-2} \text{Re}[\partial(\partial \ln P - \bar{L}, u)]
\end{align*}
\]

Finally, the first derivatives of the mass function, \( m \), are given by the rest of the field equations, \( R_{31} = 0 \) and \( R_{33} = 0 \),

\[
\begin{align*}
\partial(m + iM) &= 3(m + iM)L_u, & (1.9a) \\
\bar{\partial}(m - iM) &= 3(m - iM)\bar{L}_u, & (1.9b) \\
[P^{-3}(m + iM)]_u &= P[\partial + 2(\partial \ln P - L_u)\partial I], & (1.9c)
\end{align*}
\]

† In Kerr (1963) \( \Omega, D \) and \( \Delta \) were used instead of \( L, \partial \) and \( \Sigma \), but the results were the same, mutatis mutandis.

‡ This expression for \( M \) was first published by Robinson et al. (1969). The corresponding expression in Kerr (1963) is for the gauge when \( P = 1 \). The same is true for equation (1.33).
As was said in Kerr (1963), there are two natural choices that can be made to restrict the coordinates and simplify the final results. One is to rescale $r$ so that $P = 1$ and $L$ is complex, the other is to take $L$ to be pure imaginary with $P \neq 1$. I chose to do the first since this gives the most concise form for $M$ and the remaining field equations. It also gives the smallest group of permissible coordinate transformations, simplifying the task of finding all possible Killing vectors. The results for this gauge are

\[ M = \text{Im}(\bar{\partial}\partial L), \]

\[ \partial(m + iM) = 3(m + iM)L_{,u}, \]

\[ \bar{\partial}(m - iM) = 3(m - iM)\bar{L}_{,u}, \]

\[ \partial_u[m - \text{Re}(\bar{\partial}\partial L)] = |\partial_u\partial L|^2. \]

Since all derivatives of $m$ are given, the commutators were calculated to see whether the field equations were completely integrable. This gives $m$ as a function of the higher derivatives of $L$ unless both $\Sigma_{,u}$ and $L_{,uu}$ are zero. As stated in Kerr (1963), if these are both zero then there is a coordinate system in which $P$ and $L$ are independent of $u$ and $m = cu + A(\zeta, \bar{\zeta})$, where $c$ is a real constant. If it is zero then the metric is independent of $u$ and therefore stationary. The field equations are

\[ \nabla[\nabla(\ln P)] = c, \]

\[ \nabla = P^2\partial^2/\partial\zeta\partial\bar{\zeta}, \]

\[ M = 2\Sigma\nabla(\ln P) + \nabla\Sigma, \]

\[ m = cu + A(\zeta, \bar{\zeta}), \]

\[ cL = (A + iM)\zeta, \quad \Rightarrow \quad \nabla M = c\Sigma. \]

We shall call these metrics quasi-stationary.

In Kerr (1963) I stated that the solutions of these equations include the Kerr metric (for which $c = 0$). This is true but it is not how this solution was found. Furthermore, in spite of what many believe, its construction did not use the Kerr–Schild ansatz.

### 1.3 Symmetries in Algebraically Special Spaces

The field equations, (1.9) or (1.11), are so complicated that some extra assumptions were needed to reduce them to a more manageable form. My next step in the hunt for physically interesting solutions was to assume that the metric is stationary. Fortunately, I had some tricks that
allowed me to find all possible Killing vectors without actually solving Killing’s equation.

The key observation is that any Killing vector generates a 1-parameter group which must be a subgroup of the group $C$ of coordinate transformations that preserve all imposed coordinate conditions.

Suppose that $\{x^*, \omega^*_\}$ is another set of coordinates and tetrad vectors that satisfy the conditions imposed in the previous section. If we restrict our coordinates to those that satisfy $P = 1$ then $C$ is the group of transformations $x \to x^*$ for which

$$
\begin{align*}
\zeta^* &= \Phi(\zeta), \\
u^* &= |\Phi_\zeta|(u + S(\zeta, \bar{\zeta})), \\
r^* &= |\Phi_\zeta|^{-1}r,
\end{align*}
$$

and the basic metric functions, $L^*$ and $m^*$, are given by

$$
\begin{align*}
L^* &= (|\Phi_\zeta|/\Phi_\zeta)[L - S_\zeta - \frac{1}{2}(\Phi_\zeta/\Phi_\zeta)(u + S(\zeta, \bar{\zeta}))], \\
m^* &= |\Phi_\zeta|^{-3}m.
\end{align*}
$$

Let $S$ be the identity component of the group of symmetries of a space. If we interpret these as coordinate transformations, rather than point transformations, then it is the set of transformations $x \to x^*$ for which $g_{\alpha\beta}(x^*) = g_{\alpha\beta}(x^*)$.

It can be shown that $S$ is precisely the subgroup of $C$ for which

$$
\begin{align*}
m^*(x^*) &= m(x^*), \\
L^*(x^*) &= L(x^*).
\end{align*}
$$

Suppose that $x \to x^*(x, t)$ is a 1-parameter group of motions,

$$
\begin{align*}
\zeta^* &= \psi(\zeta; t), \\
u^* &= |\psi_\zeta|(u + T(\zeta, \bar{\zeta}; t)), \\
r^* &= |\psi_\zeta|^{-1}r.
\end{align*}
$$

Since $x^*(x; 0) = x$, the initial values of $\psi$ and $T$ are

$$
\psi(\zeta; 0) = \zeta, \quad T(\zeta, \bar{\zeta}; 0) = 0.
$$

The corresponding infinitesimal transformation, $K = \partial x^*/\partial \xi$ is

$$
K^\mu = \left[\frac{\partial x^*}{\partial \xi}\right]_{t=0}.
$$

† Note that this implies that all derivatives of these functions are also invariant, and so $g_{\alpha\beta}$ itself is invariant.
If we define

\[ \alpha(\zeta) = \left[ \frac{\partial \psi}{\partial t} \right]_{t=0}, \quad V(\zeta, \bar{\zeta}) = \left[ \frac{\partial T}{\partial t} \right]_{t=0}, \]

then the infinitesimal transformation is

\[ K = \alpha \partial \zeta + \bar{\alpha} \partial \bar{\zeta} + \text{Re}(\bar{\alpha} \zeta)(u \partial_u - r \partial_r) + V \partial_u. \]  

Replacing \( \Phi(\zeta) \) with \( \psi(\zeta; t) \) in (1.13), differentiating this w.r.t. \( t \), and using the initial values for \( \psi \) and \( T \), \( K \) is a Killing vector if and only if

\[ V_\zeta + \frac{1}{2} \alpha \zeta \rho + KL + \frac{1}{2} (\alpha \bar{\zeta} - \bar{\alpha} \zeta)L = 0, \]

\[ Km + 3 \text{Re}(\alpha \zeta)m = 0. \]

The transformation rules for \( K \) under an element \((\Phi, S)\) of \( \mathcal{C} \) are

\[ \alpha^* = \Phi_\zeta \alpha, \quad V^* = \left| \Phi_\zeta \right|[V - \text{Re}(\alpha \zeta)S + KS]. \]

Since \( \alpha \) is itself analytic, if \( \alpha \neq 0 \) for a particular Killing vector then, \( \Phi \) can be chosen so that \( \alpha^* = \left[ \right] \) and then \( S \) so that \( V^* = 0 \). If \( \alpha = 0 \) then so is \( \alpha^* \), and \( K \) is already simple without the \((\Phi, S)\) transformation being used. There are therefore two canonical types for \( K \),

\[ \text{Type 1 : } K_1 = V \partial_u, \quad \text{or Type 2 : } K_2 = \partial \zeta + \partial \bar{\zeta}. \]  

These are asymptotically timelike and spacelike, respectively.

### 1.4 Stationary solutions

The obvious and easiest way to simplify the field equations was to assume that the metric was stationary. The Type 2 Killing vectors are asymptotically spacelike and so I assumed that \( \mathcal{E} \) had a Type 1 Killing vector \( K = V \partial_u \). The coordinates used in the last section assumed that \( P = 1 \). If we transform to the more general coordinates where \( P \neq 1 \), using an \( A \)-transformation with associated change in the \((r, u)\) variables, we get

\[ k' = Ak, \quad l' = A^{-1}l, r' = A^{-1}r, \quad u' = Au, \]

\[ K_1 = V \partial_u = V A \partial_{u'} = \partial_{u'} \quad \text{if } VA = 1. \]

The metric can therefore be assumed independent of \( u \), but \( P \) may not be constant. The basic functions, \( L, P \) and \( m \) are functions of \((\zeta, \bar{\zeta})\)

† Or any other analytic function of \( \zeta \) that one chooses
The Kerr and Kerr–Schild metrics

alone, and the metric simplifies to

\[ ds^2 = ds_0^2 + 2mr/(r^2 + \Sigma^2)k^2, \tag{1.16} \]

where the “base” metric, is

\[ (ds_0)^2 = 2(r^2 + \Sigma^2)P^{-2}d\zeta d\bar{\zeta} - 2l_0 k, \tag{1.17a} \]

\[ l_0 = dr + i(\Sigma,\zeta d\zeta - \Sigma,\bar{\zeta} d\bar{\zeta}) + \left[ \frac{1}{2}K - \frac{M\Sigma}{(r^2 + \Sigma^2)} \right] k. \tag{1.17b} \]

Although the base metric is flat for Schwarzschild it is not so in general. Σ, K and M are all functions of the derivatives of L and P,

\[ \Sigma = P^2 \text{Im}(L\zeta), \quad K = 2\nabla^2 \ln P, \]
\[ M = \Sigma K + \nabla^2 \Sigma, \quad \nabla^2 = P^2 \partial_\zeta \partial_{\bar{\zeta}}, \tag{1.18} \]

The mass function, m, and M are conjugate harmonic functions,

\[ m_\zeta = -iM_\zeta, \quad m_{\bar{\zeta}} = +iM_{\bar{\zeta}}, \tag{1.19} \]

and the remaining field equations are

\[ \nabla^2 K = 0, \quad \nabla^2 M = 0. \tag{1.20} \]

If m is a particular solution of these equations then so is \( m + m_0 \) where \( m_0 \) is an arbitrary constant. The most general situation where the metric splits in this way is when \( P, L \) and M are all independent of \( u \) but \( m = cu + A(\zeta, \bar{\zeta}) \). The field equations for these are given in (1.12) and in Kerr (1963). We can state this as a theorem:

**Theorem 2** If \( ds_0^2 \) is any stationary (diverging) algebraically special metric, or more generally a solution of (1.12), then so is

\[ ds_0^2 + \frac{2m_0 r}{r^2 + \Sigma^2}k^2, \]

where \( m_0 \) is an arbitrary constant. These are the most general diverging algebraically special spaces that split in this way.

These are “generalized Kerr–Schild” metrics with base spaces \( ds_0^2 \) that are not necessarily flat.

The field equations for stationary AS metrics are certainly simpler than the original ones, (1.9), but they are still PDEs, not ODEs, and their complete solution is unknown.
1.5 Axial symmetry

We are getting close to Kerr. At this point I assumed that the metric was axially symmetric as well as stationary. I should have revisited the Killing equations to look for any Killing vector (KV) that commutes with the stationary one, $\partial_u$. However, I knew that it could not also be Type 1† and therefore it must be Type 2. It seemed fairly clear that it could be transformed to the canonical form $i(\partial_\zeta - \partial_{\bar{\zeta}})$ ($= \partial_y$ where $\zeta = x + iy$) or equivalently $i(\zeta \partial_\zeta - \bar{\zeta} \partial_{\bar{\zeta}})$ ($= \partial_\phi$ in polar coordinates where $\zeta = R e^{i\phi}$). I was getting quite eager at this point so I decided to just assume such a KV and see what turned up‡.

From the first equation in (1.20), the curvature $\nabla^2 (\ln P)$ of the 2-metric $P^{-2}d\zeta d\bar{\zeta}$ is a harmonic function,

$$\nabla^2 \ln P = P^2 (\ln P)_{,\zeta\zeta} = F(\zeta) + \bar{F}(\bar{\zeta}),$$

where $F$ is analytic. There is only one known solution of this equation for $F$ not a constant,

$$P^2 = P_0 (\zeta + \bar{\zeta})^3, \quad \nabla^2 \ln P = -\frac{3}{2} P_0 (\zeta + \bar{\zeta}), \quad (1.21)$$

where $P_0$ is an arbitrary constant. The mass function $m$ is then constant and the last field equation, $\nabla^2 M = 0$, can be solved for $L$. The final metric is given in Kerr and Debney (1969), equation (6.14), but it is not worth writing out here since it is not asymptotically flat.

If $E$ is to be the metric for a localized physical source then the null congruence should be asymptotically the same as Schwarzschild. $F(\zeta)$ must be regular everywhere, including at infinity, and must therefore be constant,

$$\frac{1}{2} K = PP_{,\zeta\zeta} - P_{,\zeta} P_{,\bar{\zeta}} = R_0 = \pm P_0^2, \quad (say) \quad (1.22)$$

As was shown in Kerr and Debney (1969), the appropriate Killing equations for a $K_2$ that commutes with $K_1$ are

$$K = \alpha \partial_\zeta + \bar{\alpha} \partial_{\bar{\zeta}}, \quad \alpha = \alpha(\zeta),$$

$$K_L = -\alpha_L, \quad K_{\Sigma} = 0, \quad (1.23)$$

$$K_P = \text{Re}(\alpha_P) P, \quad K_m = 0.$$

† If it were it would be parallel to the stationary KV and therefore a constant multiple of it.
‡ All possible symmetry groups were found for diverging AS spaces in George C. Debney’s Ph.D. thesis. My 1963 expectations were confirmed there.
The Kerr and Kerr–Schild metrics

I do not remember the choice I made for the canonical form for $K_2$ in 1963, but it was probably $\partial_y$. The choice in Kerr and Debney (1969) was

\[ \alpha = i\zeta, \quad \Rightarrow \quad K_2 = i(\zeta \partial_\zeta - \bar{\zeta} \partial_{\bar{\zeta}}), \]

and that will be assumed here. For any function $f(\zeta, \bar{\zeta})$,

\[ K_2 f = 0 \quad \Rightarrow \quad f(\zeta, \bar{\zeta}) = g(Z), \quad \text{where} \quad Z = \zeta \bar{\zeta}. \]

Now $\text{Re}(\alpha, \zeta) = 0$, and therefore

\[ K_2 P = 0, \quad \Rightarrow \quad P = P(Z), \]

and therefore $K$ is given by

\[ \frac{1}{2} K = P^2(\ln P)_{\zeta \bar{\zeta}} = PP_{\zeta \bar{\zeta}} - P_\zeta P_{\bar{\zeta}} = Z_0 \quad \Rightarrow \quad P = Z + Z_0, \]

after a $\Phi(\zeta)$-coordinate transformation. Note that the form of the metric is invariant under the transformation

\[ r = A_0 r^*, \quad u = A_0^{-1} u^*, \quad \zeta = A_0 \zeta^*, \]

\[ Z_0 = A_0^{-2} Z_0^*, \quad m_0 = A_0^{-3} m_0^*, \quad (1.24) \]

where $A_0$ is a constant, and therefore $Z_0$ is a disposable constant. We will choose it later.

The general solution of (1.23) for $L$ and $\Sigma$ is

\[ L = i \bar{\zeta} P^{-2} B(Z), \quad \Sigma = ZB' - (1 - Z_0 P^{-1}) B, \]

where $B' = dB/dZ$. The complex “mass”, $m + iM$, is an analytic function of $\zeta$ from (1.19), and is also a function of $Z$ from (1.23). It must therefore be a constant,

\[ m + iM = \mu_0 = m_0 + iM_0. \]

Substituting this into (1.18), the equation for $\Sigma$,

\[ \Sigma K + \nabla^2 \Sigma = M = M_0 \quad \Rightarrow \quad P^2 \left[ Z \Sigma'' + \Sigma' \right] + 2Z_0 \Sigma = M_0. \]

The complete solution to this is

\[ \Sigma = C_0 + \frac{Z - Z_0}{Z + Z_0} [-a + C_2 \ln Z], \]

where $\{C_0, a, C_2\}$ are arbitrary constants. This gave a four-parameter metric when these known functions are substituted into (1.16) and (1.17). However, if $C_2$ is nonzero then the final metric is singular at $r = 0$. It
was therefore omitted in Kerr (1963). The “imaginary mass” is then $M = 2Z_0C_0$ and so $C_0$ is a multiple of the NUT parameter. It was known in 1963 that the metric cannot be asymptotically flat if this is nonzero and so it was also omitted. The only constants retained were $m_0, a$ and $Z_0$. When $a$ is zero and $Z_0$ is positive the metric is that of Schwarzschild. It was not clear that the metric would be physically interesting when $a \neq 0$, but if it had not been so then this whole exercise would have been futile.

The curvature of the 2-metric, $2P^{-2}d\zeta d\bar{\zeta}$, had to have the same sign as Schwarzschild if the metric was to be asymptotically flat, and so $Z_0 = +P^2_0$. The basic functions in the metric then became

$$Z_0 = P^2_0, \quad P = \zeta \bar{\zeta} + P^2_0, \quad m = m_0, \quad M = 0,$$

$$L = ia\bar{\zeta}P^{-2}, \quad \Sigma = -\frac{a\zeta}{\zeta + Z_0}. $$

The metric was originally published in spherical polar coordinates. The relationship between these and the $(\zeta, \bar{\zeta})$ coordinates is

$$\zeta = P_0\cot\frac{\theta}{2}e^{i\phi}. $$

At this point we choose $\mathcal{A}_0$ in the transformation (1.24) so that

$$2P^2_0 = 1, \quad \Rightarrow \quad k = du + a\sin^2\theta d\phi $$

Recalling the split of (1.16) and (1.17),

$$ds^2 = ds^2_0 + 2mr/(r^2 + a^2\cos^2\theta)k^2 $$

where $m = m_0$, a constant, and,

$$ds^2 = (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$- (2dr + du - a\sin^2\theta d\phi)(du + a\sin^2\theta d\phi). $$

This is the original form of Kerr (1963), except that $u$ has been replaced by $-u$ to agree with current conventions, and $a$ has been replaced with its negative.$^\dagger$

Having found this fairly simple metric, I was desperate to see whether it was rotating. Fortunately, I knew that the curvature of the base metric, $ds^2_0$, was zero, and so it was only necessary to find coordinates where this was manifestly Minkowskian. These were

$$(r + ia)e^{i\phi}\sin\theta = x + iy, \quad r \cos\theta = z, \quad r + u = -t. $$

$\dagger$ We will see why later.
The Kerr and Kerr–Schild metrics

This gives the Kerr–Schild form of the metric,

\[
ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m r^3}{r^4 + a^2 z^2}[dt + \frac{z}{r}dz] + \frac{r}{r^2 + a^2}(xdx + ydy) - \frac{a}{r^2 + a^2}(x dy - ydx)^2.
\]

where the surfaces of constant \( r \) are confocal ellipsoids of revolution about the z-axis,

\[
\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1.
\]

Asymptotically, of course, \( r \) is just the distance from the origin in the Minkowskian coordinates, and the metric is clearly asymptotically flat.

**Angular momentum**

After the metric had been put into its Kerr–Schild form I went to Alfred Schild and told him I was about to calculate the angular momentum of the central body. He was just as excited as I was and so he joined me in my office while I computed. We were both heavy smokers at that time, so you can imagine what the atmosphere was like, Alfred puffing away at his pipe in an old arm chair, and myself chain-smoking cigarettes at my desk.

The Kerr–Schild form is particularly suitable for calculating the physical parameters of the solution. My PhD thesis at Cambridge was entitled “Equations of Motion in General Relativity”. It had been claimed previously in the literature that it was only necessary to satisfy the momentum equations for singular particles to be able to integrate the EIH quasi-static approximation equations at each order. One thing shown in my thesis was the physically obvious fact that the angular momentum equations were equally important. Some of this was published in Kerr (1958) and (1960). Because of this previous work I was well aware how to calculate the angular momentum in this new metric.

It was first expanded in powers of \( R^{-1} \), where \( R = x^2 + y^2 + z^2 \) is the usual Euclidean distance,

\[
ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m}{R}(dt + dR)^2 - \frac{4ma}{R^3}(xyd - ydx)(dt + dR) + O(R^{-3})
\]

Now, if \( x^\mu \rightarrow x^\mu + a^\mu \) is an infinitesimal coordinate transformation, then
\[ ds^2 \rightarrow ds^2 + 2da_{\mu}dx^\mu. \]

If we choose
\[ a_{\mu}dx^\mu = -\frac{am}{R^2}(xdy - ydx) \]
\[ 2da_{\mu}dx^\mu = -4m\frac{am}{R^3}(xdy - ydx)dR, \]
then the approximation in (1.29) simplifies to
\[
2da_{\mu}dx^\mu = -\frac{4ma}{R^3}(xdy - ydx)dR, \]

\[ ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m}{R}(dt + dR)^2 \]

\[ -\frac{4ma}{R^3}(xdy - ydx)dt + O(R^{-3}). \]

The leading terms in the linear approximation for the gravitational field around a rotating body were well known (for instance, see Papapetrou (1974) or Kerr (1960)). The contribution from the angular momentum vector, \( \mathbf{J} \), is
\[ 4R^{-3}\epsilon_{ijk}x^iJ^jdx^kdt. \]

A comparison of the last two equations showed that the physical parameters were
\[ \text{Mass} = m, \quad \mathbf{J} = (0, 0, ma). \]

When I turned to Alfred Schild, who was still sitting in the armchair smoking away, and said “It’s rotating!” he was even more excited than I was. I do not remember how we celebrated, but celebrate we did!

Robert Boyer subsequently calculated the angular momentum by comparing the known Lense-Thirring results for frame dragging around a rotating object in linearized relativity with the frame dragging for a circular orbit in a Kerr metric. This was a very obtuse way of calculating the angular momentum since the approximation (1.30) was the basis for the calculations by Lense and Thirring, but it did show that the sign was wrong in the original paper!

**1.6 Singularities and Topology**

The first Texas Symposium on Relativistic Astrophysics was held in Dallas December 16-18, 1963, just a few months after the discovery of the rotating solution. It was organized by a combined group of Relativists

\[ \text{† Unfortunately, I was rather hurried when performing this calculation and got the sign wrong. This is why the sign of the parameter } a \text{ in Kerr(1963) is different to that in all other publications, including this one. This way of calculating } \mathbf{J} \text{ was explained at the First Texas Symposium at the end of 1963, see Kerr(1965), but I did not check the sign at that time.} \]
and Astrophysicists and its purpose was to try to find an explanation for the newly discovered quasars. The source 3C273B had been observed in March and was thought to be about a million million times brighter than the sun.

It had been long known that a spherically symmetric body could collapse inside an event horizon to become what was to be later called a black hole by John Wheeler. However, the Schwarzschild solution was non-rotating and it was not known what would happen if rotation was present. I presented a paper called “Gravitational collapse and rotation” in which I outlined the Kerr solution and said that the topological and physical properties of the event horizon may change radically when rotation is taken into account. It was not known at that time that Kerr was the only possible stationary solution for such a rotating black hole and so I discussed it as an example of such an object.

Although this was not pointed out in the original letter, Kerr (1963), the geometry of Kerr is even more complicated than the Kruskal extension of Schwarzschild. The metric is everywhere nonsingular, except on the ring

$$z = 0, \quad x^2 + y^2 = a^2.$$  

The Weyl scalar, $R_{abcd} R^{abcd} \to \infty$ near these points and so they are true singularities, not just coordinate ones. Furthermore, this ring behaves like a branch point in the complex plane. If one travels on a closed curve that threads the ring the initial and final metrics are different: $r$ changes sign. Equation (1.28) has one nonnegative root for $r^2$, and therefore two real roots, $r_{\pm}$, for $r$. These coincide where $r^2 = 0$, i.e., on the disc $D$ bounded by the ring singularity

$$D: \quad z = 0, \quad x^2 + y^2 \leq a^2.$$  

The disc can be taken as a branch cut for the analytic function $r$. We have to take two spaces, $E_1$ and $E_2$ with the topology of $R^4$ less the disc $D$. The points above $D$ in $E_1$ are joined to the points below $D$ in $E_2$ and vice versa. In $E_1$, $r > 0$ and the mass is positive at infinity; in $E_2$, $r < 0$ and the mass is negative. The metric is then everywhere analytic except on the ring.

It was trivially obvious to everyone that if the parameter $a$ is very much less than $m$ then the Schwarzschild event horizon at $r = 2m$ will be modified slightly but cannot disappear. For instance, the light cones at $r = m$ in Kerr all point inwards for small $a$. Before I went to the meeting I had calculated the behaviour of the time like geodesics up and
down the axis of rotation and found that horizons occurred at the points on the axis in $E_1$ where

$$r^2 - 2mr + a^2 = 0, \ r = |z|.$$  

but that there are no horizons in $E_2$ where the mass is negative. In effect, the ring singularity is "naked" in that sheet. I made a rather hurried calculation of the two event horizons in $E_1$ before I went to the Dallas Symposium and claimed incorrectly there, Kerr (1964), that the equations for them were the two roots of

$$r^4 - 2mr^3 + a^2 z^2 = 0,$$

whereas $z^2$ should be replaced by $r^2$ in this and the true equation is

$$r^2 - 2mr + a^2 = 0.$$  

I attempted to calculate this using inappropriate coordinates and assuming that the equation would be: "$\psi(r, z)$ is null for some function of both $r$ and $z$". I did not realize that this function depended only on $r$. The Kerr–Schild coordinates are a generalization of the Eddington–Finkelstein coordinates for Schwarzschild. For the latter future–pointing radial geodesics are well behaved but not those traveling to the past. Kruskal coordinates were designed to handle both. Similarly for Kerr, the coordinates given here only handle ingoing curves. This metric is known to be Type D and therefore it has another set of Debever–Penrose vectors and an associated coordinate system for which the outgoing geodesics are well behaved, but not the ingoing ones.

This metric consists of three blocks, outside the outer event horizon, between the two horizons and within the inner horizon (at least for $m < a$, which is probably true for all existing black holes). Just as Kruskal extends Schwarzschild by adding extra blocks, Boyer and Lindquist (1967) and Carter (1968) independently showed that the maximal extension of Kerr has a similar proliferation of blocks. However, the Kruskal extension has no application to a real black hole formed by the collapse of a spherically symmetric body and the same is true for Kerr. In fact, even what I call $E_2$, the sheet where the mass is negative, is probably irrelevant for the final state of a collapsing rotating object.

Ever since this metric was first discovered people have tried to fit an interior solution. One morning during the summer of 1964 Ray Sachs and myself decided that we would try to do so. Since the original form is useless and the Kerr–Schild form was clearly inappropriate we started by
The Kerr and Kerr–Schild metrics

transforming to the canonical coordinates for stationary axisymmetric solutions.

In Papapetrou (1966) there is a very elegant treatment of stationary axisymmetric Einstein spaces. He shows that if there is a real non-singular axis of rotation then the coordinates can be chosen so that there is only one off-diagonal component of the metric. We call such a metric quasi-diagonalize. All cross terms between \{dr, d\theta\} and \{dt, d\phi\} can be eliminated by transformations of the type

\[
\begin{align*}
&dt' = dt + Adr + Bd\theta, \quad d\phi' = d\phi + Cdr + Dd\theta,
\end{align*}
\]

where the coefficients can be found algebraically. Papapetrou proved that \(dt'\) and \(d\phi'\) are perfect differentials if the axis is regular\(^\dagger\).

Ray and I calculated the coefficients \(A \ldots D\) and transformed the metric to the Boyer-Lindquist form,

\[
\begin{align*}
dt &\rightarrow dt + \frac{2mr}{\Delta} dr \\
d\phi &\rightarrow -d\phi + \frac{a}{\Delta} dr \\
\Delta &\equiv r^2 - 2mr + a^2, \\
\Sigma &\equiv r^2 + a^2 \cos^2 \theta,
\end{align*}
\]

where, as before, \(u = -(t+r)\). The right hand sides of the first two equations are clearly perfect differentials as the Papapetrou analysis showed.

The full Boyer-Lindquist form of the metric is

\[
ds^2 = \frac{\Sigma}{\Delta} dr^2 - \frac{\Delta}{\Sigma} [dt - a \sin^2 \theta d\phi]^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} [(r^2 + a^2) d\phi - a dt]^2,
\]

after some tedious analysis that used to be easy but now seems to require an algebraic package such as Maple.

Having derived this canonical form, we studied the metric for at least ten minutes and then decided that we had no idea how to introduce a reasonable source into a metric of this form, and probably would never have. Presumably those who have tried to solve this problem in the last 43 years have had similar reactions. Soon after this failed attempt Robert Boyer came to Austin. He said to me that he had found a new quasi–diagonalized form of the metric. I said “Yes. It is the one with

\(^\dagger\) It is shown in Kerr and Weir (1976) that if the metric is also algebraically special then it is quasi-diagonalize precisely when it is Type D. These metrics are the NUT parameter generalization of Kerr.
the polynomial $r^2 - 2mr + a^2$ but for some reason he refused to believe that we had also found this form. Since it did not seem a "big deal" at that time I did not pursue the matter further, but our relations were hardly cordial after that.

One of the main advantages of this form is that the event horizons can be easily calculated since the inverse metric is simple. If $f(r, \theta) = 0$ is a null surface then

$$\Delta(r)f_{,rr} + f_{,\theta\theta} = 0,$$

and therefore $\Delta \leq 0$. The two event horizons are the surfaces $r = r_{\pm}$ where the parameters $r_{\pm}$ are the roots of $\Delta = 0$,

$$\Delta = r^2 - 2mr + a^2 = (r - r_+)(r - r_-).$$

If $a < m$ there are two distinct horizons between which all time-like lines point inwards; if $a = m$ there is only one event horizon; and for larger $a$ the singularity is bare! Presumably, any collapsing star can only form a black hole if the angular momentum is small enough: $a < m$. This seems to be saying that the body cannot rotate faster than light, if the final picture is that the mass is located on the ring radius $a$. However, it should be remembered that this radius is purely a coordinate radius, and that there is no way that the final stage of such a collapse is that all the mass is located at the singularity.

The reason for the last statement is that if the mass where to end on the ring then there would be no way to avoid the second asymptotically flat sheet where the mass appears negative. I do not believe that the star opens up like this along the axis of rotation. If we remember that the metric is discontinuous across the disc bounded by the singular ring then it is quite possible that a well-behaved finite source could be put between $z = 0_{\pm}, |R| < a$. This would correspond to the surface of the final body being $r = 0$ in Boyer-Lindquist coordinates, say, but where the interior corresponds to $r < 0$. The actual surface may be more complicated than this but I am quite sure that this is closer to the final situation than that the matter all collapses onto the ring.

What I believe to be more likely is that the inner event horizon never actually forms. As the body continues to collapse inside its event horizon it spins faster and faster so that the geometry in the region between its outer surface and the outer event horizon approaches that between the two event horizons for Kerr. The surface of the body surface will appear

\[\dagger\] This has been done using $\delta$-functions, but I am thinking more of a nonsingular source where the distance between the two sides of the disk is nonzero.
to be asymptotically null. The full metric may not be geodesically complete. Many theorems have been claimed stating that a singularity must exist if certain conditions are satisfied, but they all make assumptions that may not be true for collapse to a black hole. Furthermore, these assumptions are often (usually?) unstated or unrecognised, and the proofs are dependent on other claims/theorems that may not be correct.

These are only two of a very large range of possibilities for the interior. What happens after the outer horizon forms is still a mystery after more than four decades. It is also the main reason why I said at the end of Kerr (1963) that "It would be desirable to calculate an interior solution...". This statement has been taken by some to mean that I thought the metric only represented a real rotating star. This is untrue and is an insult to all those relativists of that era who had been looking for such a metric to see whether the event horizon of Schwarzschild would generalise to rotating singularities.

The metric was known to be Type D with two distinct geodesic and shearfree congruences from the moment it was discovered. This means that if the other is used instead of \( k \) then the metric must have the same form, i.e., it is invariant under a finite transformation that reverses "time" and possibly the axis of rotation in the appropriate coordinates. This also meant that there is an extension that is similar to the Kruskal–Szekeres extension of Schwarzschild. Both Boyer and Lindquist (1967) and a fellow Australasian, Brandon Carter (1968), solved the problem of constructing the maximal extensions of Kerr, and even that for charged Kerr. These are mathematically fascinating and the latter paper is a beautiful analysis of the problem, but the final result is of limited physical significance.

Brandon Carter's (1968) paper was one of the most significant papers on the Kerr metric during the mid-sixties for another reason. He showed that there is an extra invariant for geodesic motion which is quadratic in the momentum components: \( J = X_{ab}v^av^b \) where \( X_{ab} \) is a Killing tensor, \( X_{(abc)} = 0 \). This gave a total of four invariants with the two Killing vector invariants and \(|v|^2\) itself, enough to generate a complete first integral of the geodesic equations.

Another significant development was the "proof" that this is the only stationary metric with a simply connected bounded event horizon, i.e., the only possible black hole. Many contributed to this, including Steven Hawking (1972), Brandon Carter (1971) and another New Zealander, David Robinson (1975). The subsequent work in this area is discussed
by David in an excellent article in this book and so I will not pursue this any further here.

1.7 Kerr–Schild metrics

One morning during the fall semester 1963, sometime before the First Texas Symposium, I tried generalizing the way that the field equations split for the Kerr metric by setting

$$ds^2 = ds_0^2 + \frac{2mr}{r^2 + \Sigma^2}k^2.$$  

The base metric $ds_0^2$ was to be an algebraically special metric with $m = 0$. From an initial rough calculation this had to be flat. Also, it seemed that the coordinates could be manipulated so that $\partial L - L\zeta = 0$ and that the final metric depended on an arbitrary analytic function of the complex variable $\zeta$. At this point I lost interest since the metric had to be singular at the poles of the analytic function unless this function was quadratic and the metric was therefore Kerr.

Sometime after the Texas Symposium Jerzy Plebanski visited Austin. Alfred Schild gave one of his excellent parties for Jerzy during which I heard them talking about solutions of the Kerr–Schild type $ds_0^2 + hk^2$, where the first term is flat and $k$ is any null vector. I commented that there might be some algebraically special spaces with this structure depending on an arbitrary function of a complex variable but that this had not been checked.

At this point Alfred and I retired to his home office and calculated the simplest field equation, $R_{ab}k^a k^b = 0$. To our surprise this showed that the null vector had to be geodesic. We then calculated $k_k k_a R_{abpq}(k_k k_b k_p k_q)$, found it to be zero and deduced that all metrics of this type had to be algebraically special. This meant that all such spaces with a diverging congruence might already be known. We checked my original calculations next day and found them to be correct.

As was stated in Theorem 2 $m$ is a unique function of $P$ and $L$ unless there is a canonical coordinate system where $m$ is linear in $u$ and $\{L, P\}$ are functions of $\{\zeta, \bar{\zeta}\}$ alone. If the base space is flat then $m_u = c = 0$ and the metric is stationary. The way these Kerr–Schild metrics were found originally was by showing that in a coordinate system where $P = 1$ the canonical coordinates could be chosen so that $\partial L = 0$. Transforming

† This name came later, of course.
from these coordinates to ones where $P \neq 1$ and $\partial_u$ is a Killing vector,

$$P \zeta \zeta = 0, \quad L = P^{-2}\phi(\zeta),$$  \hspace{1cm} (1.32)

where $\phi(\zeta)$ is analytic. From the first of these $P$ is a real bilinear function of $\zeta$ and therefore of $\bar{\zeta}$,

$$P = p\zeta \bar{\zeta} + q\zeta + \bar{q}\bar{\zeta} + c.$$ 

This can be simplified to one of three canonical forms, $P = 1, 1 \pm \zeta \bar{\zeta}$ by a linear transformation on $\zeta$. We will assume henceforth that

$$P = 1 + \zeta \bar{\zeta}.$$ 

The only problem was that this analysis depended on results for algebraically special metrics and these had not been published and would not be for several years. We had to derive the same results by a more direct method. The metric was written as

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + h k^2,$$  \hspace{1cm} (1.33a)

$$k = (du + Y dv + Y d\zeta + Y \bar{\zeta} dv)/(1 + Y \bar{Y}),$$  \hspace{1cm} (1.33b)

where $Y$ is the old coordinate $\zeta$ used in (1.32)† and

$$u = z + t, \quad v = t - z, \quad \zeta = x + iy.$$  ‡

The tetrad used to calculate the field equations was defined naturally from the identity

$$ds^2 = (d\zeta + Y dv)(d\bar{\zeta} + \bar{Y} dv) - (dv + h k)k$$

Each of these spaces has a symmetry which is also a translational symmetry for the base Minkowski space, $ds_0^2$. The most interesting situation is when this is time-like and so it will be assumed that the metric is independent of $t = \frac{1}{2}(u + v)$.

If $\phi(Y)$ is the same analytic function as in (1.32) then $Y$ is determined as a function of the coordinates by

$$Y^2\bar{\zeta} + 2z Y - \zeta + \phi(Y) = 0.$$ \hspace{1cm} (1.34)

and the coefficient of $k^2$ in (1.33) is

$$h = 2m \text{Re}(2Y \zeta),$$ \hspace{1cm} (1.35)

† $Y$ is the ratio of the two components of the spinor corresponding to $k$.

‡ Note that there certain factors of $\sqrt{2}$ have been omitted to simplify the results. This does lead to a factor 2 appearing in (1.35).
where $m$ is a real constant. Differentiating (1.34) with respect to $\zeta$ gives
\[ Y_\zeta = (2Y\zeta + 2z + \phi')^{-1}. \tag{1.36} \]

The Weyl spinor invariant is given by
\[ \Psi_2 = c_0 mY_\zeta^3, \]
where $c_0$ is some power of 2, and the metric is therefore singular precisely where $Y$ is a repeated root of its defining equation, (1.34).

If the k-lines are projected onto the Euclidean 3-space $t = 0$ with \( \{x, y, z\} \) as coordinates so that $ds_E^2 = dx^2 + dy^2 + dz^2$, then the perpendicular from the origin meets the projected k-line at the point
\[ F_0 : \zeta = \frac{\phi - Y^2\bar{\phi}}{P^2}, \quad z = -\frac{\bar{\phi} - Y\phi}{P^2}, \]
and the distance of the line from the origin is
\[ D = \frac{|\phi|}{1 + YY}. \]
a remarkably simple result. This was used by Kerr and Wilson (1978) to prove that unless $\phi$ is quadratic the singularities are unbounded and the spaces are not asymptotically flat. The reason why I did not initially take the general Kerr–Schild metric seriously was that this was what I expected.

Another point that is easily calculated is $Z_0$ where the line meets the plane $z = 0$,
\[ Z_0 : \zeta = \frac{\phi + Y^2\bar{\phi}}{1 - (YY)^2}, \quad z = 0, \]

The original metric of this type is Kerr where
\[ \phi(Y) = -2iaY, \quad D = \frac{2|a||Y|}{1 + |Y|^2} \leq |a|. \]

If $\phi(Y)$ is any other quadratic function then it can be transformed to the same value by using an appropriate Euclidean rotation and translation about the $t$-axis. The points $F_0$ and $Z_0$ are the same for Kerr, so that $F_0$ lies in the $z$-plane and the line cuts this plane at a point inside the singular ring provided $|Y| \neq 1$. The lines where $|Y| = 1$ are the tangents to the singular ring lying entirely in the plane $z = 0$ outside the ring. When $a \to 0$ the metric becomes Schwarzschild and all the $Y$-lines pass through the origin.
When \( \phi(Y) = -2iaY \) becomes
\[
Y^2 \zeta + 2(z - ia)Y - \zeta = 0.
\]
There are two roots, \( Y_1 \) and \( Y_2 \) of this equation,
\[
Y_1 = \frac{r \zeta}{(z + r)(r - ia)}, \quad 2Y_1,\zeta = \frac{r^3 + iarz}{r^4 + a^2z^2}
\]
\[
Y_2 = \frac{r \zeta}{(z - r)(r + ia)}, \quad 2Y_2,\zeta = -\frac{r^3 + iarz}{r^4 + a^2z^2}
\]
where \( r \) is a real root of \( \text{(1.28)} \). This is a quadratic equation for \( r^2 \) with only one nonnegative root and therefore two real roots differing only by sign, \( \pm r \). When these are interchanged, \( r \leftrightarrow -r \), the corresponding values for \( Y \) are also swapped, \( Y_1 \leftrightarrow Y_2 \).

When \( Y_2 \) is substituted into the metric then the same solution is returned except that the mass has changed sign. This is the other sheet where \( r \) has become negative. It is usually assumed that \( Y \) is the first of these roots, \( Y_1 \). The coefficient \( h \) of \( k^2 \) in the metric, \( \text{(1.33)} \), is then
\[
h = 2m \text{Re}(2Y_1\zeta) = \frac{2mr^3}{r^4 + a^2z^2}.
\]
This gives the metric in its KS form, \( \text{(1.27)} \).

The results were published in two places, Kerr and Schild (1965a,b). The first of these was a talk that Alfred gave at the Galileo Centennial in Italy, the second was an invited talk that I gave, but Alfred wrote, at the Symposium on Applied Mathematics of the American Mathematical Society, April 25 1964. The manuscript had to be presented before the conference so that the participants had some chance of understanding results from distant fields. We stated on page 205 that

"Together with their graduate student, Mr. George Debney, the authors have examined solutions of the nonvacuum Einstein-Maxwell equations where the metric has the form (2.1). Most of the results mentioned above apply to this more general case. This work is continuing."

### 1.8 Charged Kerr

What was this quote referring to? When we had finished with the Kerr–Schild metrics, we looked at the same problem with a nonzero electromagnetic field. The first stumbling block was that \( R_{ab}k^a k^b = 0 \) no
longer implied that the k-lines are geodesic. The equations were quite intractable without this and so it had to be added as an additional assumption. It then followed that the principal null vectors were shearfree, so that the metrics had to be algebraically special. The general forms of the gravitational and electromagnetic fields were calculated from the easier field equations. The E–M field proved to depend on two functions called $A$ and $\gamma$ in Debney, Kerr and Schild (1969).

When $\gamma = 0$ the “difficult” equations are linear and similar to those for the purely gravitational case. They were readily solved giving a charged generalization of the original Kerr–Schild metrics. The congruences are the same as for the uncharged metrics, but the coefficient of $k^2$ is

$$h = 2m\text{Re}(2Y_{,\xi}) - |\psi|^2|2Y_{,\xi}|^2.$$  \hspace{1cm} (1.37)

where $\psi(Y)$ is an extra analytic function generating the electromagnetic field. This is best expressed through a potential,

$$f = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu = -d\alpha,$$$$
\alpha = -P(\psi Z + \bar{\psi} \bar{Z})k - \frac{1}{2}(\chi d\bar{Y} + \bar{\chi} dY),$$

where

$$\chi = \int P^{-2}\psi(Y) dY,$$

$\bar{Y}$ being kept constant in this integration.

The most important member of this class is charged Kerr. For this,

$$h = \frac{2m r^3 - |\psi(Y)|^2 r^2}{r^4 + a^2 z^2}.$$  \hspace{1cm} (1.38)

Asymptotically, $r = R$, $k = dt - dR$, a radial null-vector and $Y = \tan\left(\frac{1}{2}\theta\right)e^{i\phi}$. If the analytic function $\psi(Y)$ is nonconstant then it must be singular somewhere on the unit sphere and so the gravitational and electromagnetic fields will be also. The only physically significant charged Kerr–Schild is therefore when $\psi$ is a complex constant, $e + ib$. The imaginary part, $b$, can be ignored as it gives a magnetic monopole, and so we are left with $\psi = e$, the electric charge,

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2mr^3 - e^2 r^2}{r^4 + a^2 z^2} [dt + \frac{z}{r} dz,$$$$
+ \frac{r}{r^2 + a^2}(xdx + ydy) - \frac{a}{r^2 + a^2}(xdx - ydy)]^2,$$  \hspace{1cm} (1.39)

The electromagnetic potential is

$$\alpha = \frac{er^3}{r^4 + a^2 z^2} \left[ dt - \frac{a(xdy - ydx)}{r^2 + a^2} \right],$$
The Kerr and Kerr–Schild metrics

where a pure gradient has been dropped. The electromagnetic field is

\[ (F_{xt} - iF_{yz}, F_{yt} - iF_{zx}, F_{zt} - iF_{xy}) = \frac{er^3}{(r^2 + ia)^3}(x, y, z + ia). \]

In the asymptotic region this field reduces to an electric field,

\[ E = \frac{e}{R^3}(x, y, z), \]

and a magnetic field,

\[ H = \frac{ea}{R^5}(3xz, 3yz, 3z^2 - R^2). \]

This is the electromagnetic field of a body with charge \( e \) and magnetic moment \((0, 0, ea)\). The gyromagnetic ratio is therefore \( ma/ea = m/e \), the same as that for the Dirac electron. This was first noticed by Brandon Carter and was something that fascinated Alfred Schild.

This was the stage we had got to before March 1964. We were unable to solve the equations where the function \( \gamma \) was nonzero so we enlisted the help of our graduate student, George Debney. Eventually we realized that we were unable to solve the more general equations and so we suggested to George that he drop this investigation. He then tackled the problem of finding all possible groups of symmetries in diverging algebraically special spaces. He succeeded very well with this, solving many of the ensuing field equations for the associated metrics. This work formed the basis for his PhD thesis and was eventually published in Kerr and Debney (1970).

In Janis and Newman (1965) and Newman and Janis (1965) the authors defined and calculated multipole moments for the Kerr metric, using the Kerr–Schild coordinates as given in Kerr (1963). They then claimed that this metric is that of a ring of mass rotating about its axis of symmetry. Unfortunately, this cannot be so because the metric is multivalued on its symmetry axis and is consequently discontinuous there. The only way that this can be avoided is by assuming that the space contains matter on the axis near the centre. As was acknowledged in a footnote to the second paper, this was pointed out to the authors by the referee and myself before the paper was published, but they still persisted with their claim.
1.9 Newman’s construction of the Kerr–Newman metric

Newman knew that the Schwarzschild, Reissner–Nordström and Kerr metrics all have the same simple form in Eddington–Finkelstein or Kerr–Schild coordinates (see eq. 1.33). Schwarzschild and its charged generalisation have the same null congruence, \( k = dr - dt \); only the coefficient \( h \) is different,

\[
h = \frac{2m}{r} \quad \longrightarrow \quad h = \frac{2m}{r} - \frac{e^2}{r^2}.
\]

For these the complex divergence of the underlying null congruence is

\[
\rho = \bar{\rho} = \frac{1}{r}.
\]

Newman hoped to find a charged metric with the same congruence as Kerr but with \( h \) generalised to something like the Reissner–Nordström form with \( e^2/r^2 \) replaced by \( e^2\rho^2 \). This does not quite work since \( \rho \) is complex for Kerr so he had to replace \( \rho^2 \) with something real.

There are many real rational functions of \( \rho \) and \( \bar{\rho} \) that reduce to \( \rho^2 \) when \( \rho \) is real, so he wrote down several possibilities and distributed them to his graduate students. Each was checked to see whether it was a solution of the Einstein–Maxwell equations. The simplest, \( \rho^2 \rightarrow \rho\bar{\rho} \), worked! The appropriate electromagnetic field was then calculated, a non-trivial problem.

The reason that this approach was successful has nothing to do with “complexifying the Schwarzschild and Reissner–Nordström metrics” by some complex coordinate transformation, as stated in the original papers. It works because all these metrics are of Kerr–Schild form and the general Kerr–Schild metric can be charged by replacing the uncharged \( h \) with its appropriate charged version,

\[
h = 2m\text{Re}(\bar{\zeta}) - \text{Re}(\psi|\zeta|^2) \quad \longrightarrow \quad 2m\text{Re}(\rho) - e^2\rho\bar{\rho},
\]

without changing the congruence.

The charged solution was given in Newman et al. (1965). They claimed that the metric can be generated by a classical charged rotating ring. As in the previous paper Newman and Janis (1965), it was then admitted in a footnote that the reason why this cannot be true had already been explained to them.

1.10 Appendix: Standard Notation

Let \( \{e_a\} \) and \( \{\omega^a\} \) be dual bases for tangent vectors and linear 1-forms, respectively, i.e., \( \omega^a(e_b) = \delta^a_b \). Also let \( g_{ab} \) be the components of the
metric tensor,
\[ ds^2 = g_{ab} \omega^a \omega^b, \quad g_{ab} = e_a \cdot e_b. \]

The components of the connection in this frame are the Ricci rotation coefficients,
\[ \Gamma^a_{bc} = -\omega^a_{\mu \nu} e_b^\mu e_c^\nu, \quad \Gamma_{abc} = g_{ak} \Gamma^k_{bc}. \]

The commutator coefficients \( D^a_{bc} = -D^a_{cb} \) are defined by
\[ [e_b, e_c] = D^a_{bc} e_a, \quad \text{where} \quad [u, v](f) = u(v(f)) - v(u(f)). \]

or equivalently by
\[ d\omega^a = D^a_{bc} \omega^b \wedge \omega^c. \quad (1.40) \]

Since the connection is symmetric, \( D^a_{bc} = -2\Gamma^a_{[bc]} \), and since it is metrical
\[ \Gamma_{abc} = \frac{1}{2}(D_{bac} + D_{cab} - D_{abc}), \quad \Gamma_{abc} = g_{am} \Gamma^m_{bc}, \quad D_{abc} = g_{am} D^m_{bc}. \]

If it is assumed that the \( g_{ab} \) are constant, then the connection components are determined solely by the commutator coefficients and therefore by the exterior derivatives of the tetrad vectors,
\[ \Gamma_{abc} = \frac{1}{2}(D_{bac} + D_{cab} - D_{abc}). \]

The components of the curvature tensor are
\[ \Theta^a_{bcd} \equiv \Gamma^a_{bd[c} - \Gamma^a_{bc]d} + \Gamma^e_{bd} \Gamma^a_{ec} - \Gamma^e_{bc} \Gamma^a_{ed} - D^e_{ed} \Gamma^a_{bc}. \quad (1.41) \]

We must distinguish between the expressions on the right, the \( \Theta^a_{bcd} \), and the curvature components, \( R^a_{bcd} \), which the N–P formalism treat as extra variables, their \( (\Psi_i) \).

A crucial factor in the discovery of the spinning black hole solutions was the use of differential forms and the Cartan equations. The connection 1-forms \( \Gamma^a_b \) are defined as
\[ \Gamma^a_b = \Gamma^a_{bc} \omega^c. \]

These are skew-symmetric when \( g_{ab[c} = 0 \),
\[ \Gamma_{ba} = -\Gamma_{ab}, \quad \Gamma_{ab} = g_{ac} \Gamma^c_a. \]

The first Cartan equation follows from \( (1.40) \),
\[ d\omega^a + \Gamma^a_{bc} \omega^b = 0. \quad (1.42) \]
The curvature 2-forms are defined from the second Cartan equations,
\[ \Theta^{a b} \equiv d \Gamma^{a b} + \Gamma^{a c} \wedge \Gamma^{c b} = \frac{1}{2} R^{a b c d} \omega^c \omega^d. \] (1.43)

The exterior derivative of (1.42) gives
\[ \Theta^{a b} \wedge \omega^b = 0 \Rightarrow \Theta^{a [bcd]} = 0, \]
which is just the triple identity for the Riemann tensor,
\[ R^{a [bcd]} = 0. \] (1.44)

Similarly, from the exterior derivative of (1.43),
\[ d \Theta^{a b} - \Theta^{a f} \wedge \Gamma^f_b + \Gamma^a_f \wedge \Theta^f_b = 0, \]
that is
\[ \Theta^{a [bcd]} = 0, \Rightarrow R^{a b[cde]} = 0. \]

This equation says nothing about the Riemann tensor, \( R^{a b c d} \) directly. It says that certain combinations of the derivatives of the expressions on the right hand side of ((1.41)) are linear combinations of these same expressions.
\[ \Theta_{ab[cd|e]} + D_{[cd} \Theta_{e]ab} - 
\frac{1}{2} \delta_{a[c} \Theta_{de]b} - 
\frac{1}{2} \delta_{b[c} \Theta_{de]a} \equiv 0. \] (1.45)

These are the true Bianchi identities. A consequence of this is that if the components of the Riemann tensor are thought of as variables, along with the components of the metric and the base forms, then these variables have to satisfy
\[ R_{ab[cd|e]} = -2 R_{[be|c} \Gamma^f_{df],} \] (1.46)

Bibliography
Bondi, H., M.G.J. van der Burg and A.W.K. Metzner (1962). Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems. Proc. Roy. Soc. Lond. A 269, 21-52.
Boyer, R.H. and Lindquist, R.W. (1967). Maximal Analytic Extension of the Kerr Metric, J. Math. Phys. 8, 265-280.
Carter, B. (1968), Global Structure of the Kerr Family of Gravitational Fields, Phys. Rev. 174, 1559-1571.
Carter, B. (1971). Axisymmetric Black Hole Has Only Two Degrees of Freedom, Phys. Rev. Lett. 26, 331.
Debney, G.C., Kerr, R.P., and Schild, A. (1969). Solutions of the Einstein and Einstein-Maxwell equations, J. Math. Phys. 10, 1842.
Goldberg, J.N. and Kerr, R.P. (1961). Some applications of the infinitesimal-holonomy group to the Petrov classification of Einstein spaces J. Math. Phys. 2, 327.
The Kerr and Kerr–Schild metrics

Goldberg, J.N. and Sachs, R.K. (1962) A theorem on Petrov Types, *Acta Phys. Polon.*, Suppl. 22, 13.
Hawking, S.W., (1972). Black holes in general relativity, *Commun. Math. Phys.* 25, 152-166.
Janis, A.I., and Newman, E.T. (1965) Structure of Gravitational Sources. *J. Math. Phys.* 6, 902-914.
Kerr, R.P. (1958). The Lorentz-covariant Approximation Method in General Relativity I. *Il Nuovo Cim.* X13, 469.
Kerr, R.P. (1960). On the Quasi-static Approximation Method in General Relativity, *Il Nuovo Cim.* X16, 26.
Kerr, R.P. (1963). Gravitational field of a spinning mass as an example of algebraically special metrics, *Phys. Rev. Lett.* 11, 26.
Kerr, R.P. (1965). Gravitational collapse and rotation, *Quasi-stellar Sources and Gravitational Collapse, including the Proceedings of the First Texas Symposium on Relativistic Astrophysics*, Edited by I. Robinson, A. Schild and E.L. Schuckling. The University of Chicago Press, Chicago and London, 99-109.
Kerr, R.P. and Debney, G.C. (1970). Einstein spaces with symmetry groups, *J. Math. Phys.* 11, 2807.
Kerr, R.P. and Goldberg, J.N. (1961). Einstein spaces with four-parameter holonomy groups, *J. Math. Phys.* 2, 332
Kerr, R.P. and Schild, A. (1965a). A new class of vacuum solutions of the Einstein field equations, *Atti del Congregno Sulla Relativita Generale: Galileo Centenario*.
Kerr, R.P. and Schild, A. (1965b). Some algebraically degenerate solutions of Einstein’s gravitational field equations, *Proc. Sym. in Applied Math.* XVII, Edited by R. Finn. American Math. Soc, Providence, Rhode Island, 199-209.
Kerr, R.P. and Wilson, W.B. (1978). Singularities in the Kerr–Schild Metrics, *Gen. Rel. Grav.* 10, 273.
Newman, E.T., Tambourino, L.A., and Unti, T. (1963). Empty-space generalization of the Schwarzschild metric. *J. Math. Phys.* 4, p. 915.
Newman, E.T., and Janis, A.I. (1965) Note on the Kerr Spinning-Particle metric. *J. Math. Phys.* 6, 915-917.
Newman, E.T., Couch, E., Chinnapared, K., Exton, A., Prakash, A., and Torrence, R. (1965) Metric of a Rotating, Charged Mass. *J. Math. Phys.* 6, 918-919.
Papapetrou, A. (1974) *Lectures on General Relativity*, D.Reidel, Dordrecht and Boston, 116.
Penrose, R. (1960). A spinor approach to general relativity, *Ann. Phys. (NY)* 10, 171.
Petrov, A.Z. (1954). The classification of spaces defining gravitational fields, *Scientific Proceedings of Kazan State University* 114, 55, Translation by J. Jezierski and M.A.H. MacCallum, with introduction, by M.A.H. MacCallum, (2000) *Gen. Rel. Grav.* 32, 1661.
Pirani, F.A.E. (1957). Invariant formulation of gravitational radiation theory, *Phys. Rev.* 105, 1089.
Robinson, D.C. (1975). Uniqueness of the Kerr Black Hole, *Phys. Rev. Lett.* 34, 905-6.
Robinson, I. and Trautman, A. (1962). Some spherical gravitational waves in general relativity, *Proc. Roy. Soc. Lond.* A 265, 463.
Robinson, I. and Trautman, A. (1964). Exact Degenerate Solutions of Einstein’s Equations; in Proceedings on Theory of Gravitation, Conference in Warszawa and Jablonna, 1962 ed. L. Infeld (Gauthier-Villars, Paris and PWN, Warszawa), 107–114
Sachs, R.K. (1961). Gravitational waves in general relativity. VI. Waves in asymptotically flat space–time, Proc. Roy. Soc. Lond. A 264, 309.
Sachs, R.K. (1962). Gravitational waves in general relativity. VIII. Waves in asymptotically flat space–time, Proc. Roy. Soc. Lond. A 270, 103.
Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C. and Herlt, E. (2003). Exact Solutions to Einstein’s Field Equations, Second Edition Cambridge University Press, Cambridge UK.