Dimension of the space of intertwining operators from degenerate principal series representations

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Abstract

Let $X$ be a homogeneous space of a real reductive Lie group $G$. It was proved by T. Kobayashi and T. Oshima that the regular representation $C^\infty(X)$ contains each irreducible representation of $G$ at most finitely many times if a minimal parabolic subgroup $P$ of $G$ has an open orbit in $X$, or equivalently, if the number of $P$-orbits on $X$ is finite. In contrast to the minimal parabolic case, for a general parabolic subgroup $Q$ of $G$, we find a new example that the regular representation $C^\infty(X)$ contains degenerate principal series representations induced from $Q$ with infinite multiplicity even when the number of $Q$-orbits on $X$ is finite.

Keywords: degenerate principal series, multiplicity, spherical variety, intertwining operators, real spherical.

MSC2010: primary 22E46; secondary 22E45, 53C30.

1 Introduction

Let $G$ be a real reductive algebraic Lie group, and $H$ an algebraic subgroup of $G$. T. Kobayashi and T. Oshima established the criterion of finite multiplicity for regular representations on $G/H$.

Fact 1.1 ([10, Theorem A]). The following two conditions on the pair $(G, H)$ are equivalent:

(i) $\dim \text{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty$ for all $(\pi, \tau) \in \hat{G}_{\text{smooth}} \times \hat{H}_f$.

(ii) $G/H$ is real spherical.

Here $\hat{G}_{\text{smooth}}$ denotes the set of equivalence classes of irreducible smooth admissible Fréchet representations of $G$ with moderate growth, and $\hat{H}_f$ that of algebraic irreducible finite-dimensional representations of $H$. Given $\tau \in \hat{H}_f$, we write $C^\infty(G/H, \tau)$ for the Fréchet space of smooth sections of the $G$-homogeneous vector bundle over $G/H$ associated to $\tau$. The terminology real sphericity was introduced by T. Kobayashi in his search of a broader framework for global analysis on homogeneous spaces than the usual (e.g., reductive symmetric spaces).
Definition 1.2. A homogeneous space $G/H$ is *real spherical* if a minimal parabolic subgroup $P$ of $G$ has an open orbit in $G/H$.

The following equivalence is well known by the work of B. Kimelfeld [7] and the real rank one reduction of T. Matsuki [13]:

Fact 1.3 ([8 Theorem 2.2]). $G/H$ is real spherical if and only if the number of $H$-orbits on $G/P$ is finite. In other words, the condition (ii) in Fact 1.1 is equivalent to the following condition (iii):

$$(iii) \#(H\backslash G/P) < \infty.$$ 

Therefore, for a minimal parabolic $P$, the three conditions (i), (ii), and (iii) are equivalent by Fact 1.1 and Fact 1.3 (see Figure 1.1 below). Then one might ask a question what will happen to the relationship among the three conditions, if we replace $P$ by a general parabolic subgroup $Q$ of $G$. For this, we need to make a precise definition of variants of (i), (ii), and (iii) for a parabolic subgroup $Q$ of $G$.

Definition 1.4 ([9 Definition 6.6]). We say $\pi \in \hat{G}_{\text{smooth}}$ belongs to $Q$-series if $\pi$ occurs as a subquotient of the degenerate principal series representation $C^\infty(G/Q, \tau)$ for some $\tau \in \hat{Q}$. 

$P: \text{minimal parabolic}$

Figure 1.1

$Q: \text{general parabolic}$

$\pi, \tau \in \hat{G}_{\text{smooth}} \times \hat{H}_{\text{f}}$.

We set $\hat{G}^Q_{\text{smooth}} := \{ \pi \in \hat{G}_{\text{smooth}} \mid \pi \text{ belongs to } Q\text{-series} \}$. Obviously, $\hat{G}^Q_{\text{smooth}} \supset \hat{G}^{Q'}_{\text{smooth}}$ if $Q \subset Q'$. Moreover, $\hat{G}^Q_{\text{smooth}}$ is equal to $\hat{G}_{\text{smooth}}$ if $Q = P$ (minimal parabolic) by Harish-Chandra’s subquotient theorem [5] and to $\hat{G}_{\text{f}}$ if $Q = G$.

Definition 1.5. For a parabolic subgroup $Q$ of $G$, we define the three conditions $(i_Q)$, $(ii_Q)$, and $(iii_Q)$, respectively, as follows:

$(i_Q) \text{ dim Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty \text{ for all } (\pi, \tau) \in \hat{G}^Q_{\text{smooth}} \times \hat{H}_{\text{f}}.$

$(ii_Q) Q \text{ has an open orbit in } G/H.$

$(iii_Q) \#(H\backslash G/Q) < \infty.$

The conditions $(i_Q)$, $(ii_Q)$, and $(iii_Q)$ reduce to (i), (ii), and (iii), respectively, if $Q = P$ (minimal parabolic), and we know from Fact 1.1 and Fact 1.3 (see also Figure 1.1) that the following equivalences hold:

$$(i_Q) \iff (ii_Q) \iff (iii_Q) \text{ if } Q = P.$$
Further, it is obvious from the Frobenius reciprocity that the condition (i\_Q) automatically holds if Q = G; (ii\_Q) and (iii\_Q) obviously hold. Hence

\[(i\_Q) \iff (ii\_Q) \iff (iii\_Q) \text{ if } Q = G.\]

In the general setting, clearly, (iii\_Q) implies (ii\_Q), however the converse may fail if Q is not a minimal parabolic subgroup of G. On the other hand, the implication (i\_Q) \Rightarrow (ii\_Q) is true. In fact, the following stronger theorem holds:

**Fact 1.6** ([9, Corollary 6.8]). If there exists \( \tau \in \hat{H} \) such that for all \( \pi \in \hat{G}^\text{smooth} \)
\[\dim \text{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty,\]
then (ii\_Q) holds.

An open problem is whether the converse statement holds or not.

**Question.** Does the finite-multiplicity condition (i\_Q) in representation theory follows from the geometric condition (ii\_Q) (or (iii\_Q))?

We give a negative answer to this question in this paper. Explicitly, we prove the theorem below:

**Theorem 1.7.** Let Q be a maximal parabolic subgroup of \( G = SL(2n, \mathbb{R}) \) such that \( G/Q \) is isomorphic to the real projective space \( \mathbb{RP}^{2n-1} \). Then if \( n \geq 2 \), there exists an algebraic subgroup \( H \) of G satisfying the following two conditions:

1) \( \#(H \setminus G/Q) < \infty, \)
2) \( \dim \text{Hom}_G(C^\infty(G/Q, \chi), C^\infty(G/H)) = \infty \) for some one-dimensional representation \( \chi \) of Q.

Furthermore, if \( n \geq 3 \), H satisfies the following condition:

2') \( \dim \text{Hom}_G(C^\infty(G/Q, \chi), C^\infty(G/H)) = \infty \) for any one-dimensional representation \( \chi \) of Q.

We summarize the relationship among the conditions (i\_Q), (ii\_Q), and (iii\_Q) as follows: (i\_Q) \Rightarrow (ii\_Q) is true by Fact 1.6. Theorem 1.7 implies that neither (iii\_Q) \Rightarrow (i\_Q) nor (ii\_Q) \Rightarrow (i\_Q) holds, see Figure 1.2.

**Remark 1.8.** The recent paper [2, Theorem D] claimed the following: Suppose that a real algebraic group \( H \) acts on a real algebraic smooth variety \( M \) with \( \#(H \setminus M) < \infty \) and that \( E \) is an algebraic \( H \)-homogeneous vector bundle on \( M \). Then, for any \( n \in \mathbb{N} \),

\[\sup_{\tau \in \hat{H}} \dim \text{Hom}_H(\tau, S^\tau(M, E)) < \infty. \quad \text{(1.1)}\]

We note that \( S^\tau(M, E) \) can be identified with the space \( \mathcal{D}'(M) \) of distributions in the case that \( M \) is compact and \( E \) is the trivial bundle \( M \times \mathbb{C} \) [1, Chapter 1.5]. Therefore (1.1) would imply

\[\dim \text{Hom}_H(1, \mathcal{D}'(M)) = \dim \mathcal{D}'(M)^H < \infty, \quad \text{(1.2)}\]
when \( #(H\backslash M) < \infty \) and \( M \) is compact. Here \( 1 \) denotes the trivial one-dimensional representation of \( H \).

However, one sees from Fact 2.2 that (1.2) contradicts to Theorem 1.7 when applied to \( M = \mathbb{RP}^{2n-1} \). Thus Theorem 1.7 is a counterexample to [2] Theorem D]. Indeed, it seems to the author that a gap in the proof of [2] Theorem D] comes from a false statement \( #(H\backslash G/Q) < \infty \Rightarrow #(H\backslash G_c\backslash Q_c) < \infty \), see Remark 4.9 below.

The outline of this article as follows: In Section 2, we recall some general facts concerning distribution kernels, which were proved by T. Kobayashi and B. Speh [11]. In Section 3, we fix some basic notation for distributions on the complex Euclidean space. In Section 4, we construct the subgroup \( H \) of \( G \) and give a proof of Theorem 1.7.

### 2 Reduction to distribution kernels

In this section, we reformulate the condition 2) of Theorem 1.7 by means of distribution kernels using Fact 2.2 below.

**Definition 2.1.** Let \( G \) be a real Lie group and \( H \) a closed subgroup of \( G \). For \( \tau \in \mathcal{H}_1 \), we define the finite-dimensional representation of \( H \) by \( \tau_{2p}^\vee := \tau^\vee \otimes \mathbb{C}_{2p} \) where \( \tau^\vee \) is the contragredient representation of \( \tau \) and \( \mathbb{C}_{2p} \) denotes the one-dimensional representation of \( H \) given by \( h \mapsto | \det(\text{Ad}(h)) : g/h \to g/h |^{-1} \).

**Fact 2.2 ([11] Proposition 3.2).** Let \( G \) be a real Lie group. Suppose that \( G' \) and \( H \) are closed subgroups of \( G \) and that \( H' \) is a closed subgroup of \( G' \). Let \( \tau \) and \( \tau' \) be finite-dimensional representations of \( H \) and \( H' \), respectively.

1. There is a natural injective map:
\[
\text{Hom}_{\mathcal{G}'}(C^\infty(G/H, \tau), C^\infty(G'/H', \tau')) \hookrightarrow (\mathcal{D}'(G/H, \tau_{2p}^\vee) \otimes \tau')^{H'}.
\]

Here \( (\mathcal{D}'(G/H, \tau_{2p}^\vee) \otimes \tau')^{H'} \) denotes the space of \( H' \)-fixed vectors under the diagonal action.

2. If \( H \) is cocompact in \( G \) (e.g., a parabolic subgroup of \( G \) or a uniform lattice), then (2.1) is a bijection.

We apply this fact to the setting of Theorem 1.7. Recall that \( G = \text{SL}(2n, \mathbb{R}) \) and \( Q \) is a maximal parabolic subgroup of \( G \) such that \( G/Q \simeq \mathbb{RP}^{2n-1} \). For \( \lambda \in \mathbb{C} \), we define a one-dimensional representation \( \chi_\lambda : Q \to \text{GL}(1, \mathbb{C}) \) by \( g \mapsto | \det(\text{Ad}(g)) : g/q \to g/q |^{\lambda} \). We denote by \( \mathcal{D}'(\mathbb{R}^{2n}\backslash \{0\})_{\text{even}, \lambda-2n} \) the space of even homogeneous distributions of degree \( \lambda - 2n \) on \( \mathbb{R}^{2n}\backslash \{0\} \).

**Corollary 2.3.** For any closed subgroup \( H \) of \( G \), we have
\[
\text{Hom}_{\mathcal{G}'}(C^\infty(G/Q, \chi_\lambda), C^\infty(G/H)) \simeq \mathcal{D}'(\mathbb{R}^{2n}\backslash \{0\})_{\text{even}, \lambda-2n}^H.
\]

**Proof.** This follows from Fact 2.2 because \( \mathbb{C}_{2p} = \chi_{2n} \) as representations of \( Q \) and \( \mathcal{D}'(G/Q, \chi_\lambda) \simeq \mathcal{D}'(\mathbb{R}^{2n}\backslash \{0\})_{\text{even}, -\lambda} \) in the setting of Corollary 2.3. \( \square \)
3 Notation for distributions on the complex Euclidean space

In Section 4 we shall consider a linear group action on \( \mathbb{C}^n \) regarded as a real vector space. In order to avoid possible confusion, we prepare some notation for distributions on the complex Euclidean space \( \mathbb{C}^n \) regarded as a real vector space. Identifying \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \) by \( z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) \), we write \( \mathcal{D}(\mathbb{C}^n) \) and \( \mathcal{D}'(\mathbb{C}^n) \) for the spaces of \( C^\infty \) functions with compact support and distributions on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \), respectively. We define a distribution \( \delta(z_n, \overline{z}_n) \in \mathcal{D}'(\mathbb{C}^n) \cong \mathcal{D}'(\mathbb{R}^{2n}) \) by

\[
\delta(z_n, \overline{z}_n)(\phi) := \frac{1}{(2\pi)^n} \int_{\mathbb{C}^{n-1}} \phi(z_1, \ldots, z_{n-1}, 0) \, dz_1 \, d\overline{z}_1 \ldots \, dz_{n-1} \, d\overline{z}_{n-1}
\]

for every test function \( \phi \in \mathcal{D}(\mathbb{C}^n) \cong \mathcal{D}(\mathbb{R}^{2n}) \) where \( x' + iy' := (x_1 + iy_1, \ldots, x_{n-1} + iy_{n-1}) \). We write \( \delta(\cdot) \) for the usual Dirac delta function on \( \mathbb{R} \) and regard it as a distribution on \( \mathbb{R}^{2n} \) by the pull-back via the projection \( \mathbb{R}^{2n} \to \mathbb{R} \). Then we have

\[
\delta(z_n, \overline{z}_n) = (-2i)^{-1} \delta(x_n) \delta(y_n)
\]

as distributions on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \). Since the multiplication by \( x_n \) or \( y_n \) kills (3.1), so does it by \( z_n \) or \( \overline{z}_n = x_n - iy_n \), that is,

\[
z_n \delta(z_n, \overline{z}_n) = \overline{z}_n \delta(z_n, \overline{z}_n) = 0.
\]

We define differential operators on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) by

\[
\frac{\partial}{\partial z_j} := \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) \quad (1 \leq j \leq n).
\]

Multiplication of \( \frac{\partial^l}{\partial z_n} \delta(z_n, \overline{z}_n) \) by distributions of \( z_1, \overline{z}_1, \ldots, z_{n-1}, \overline{z}_{n-1} \) makes sense. We note that a finite family \( \{T_i\}_{i=1}^m \) of distributions on \( \mathbb{C}^{n-1}\{0\} \) vanish if the following equality as distributions on \( \mathbb{C}^{n-1}\{0\} \cong \mathbb{R}^{2n-1}\{0\} \) holds:

\[
\sum_{i=1}^m T_i(z_1, \ldots, z_{n-1}) \frac{\partial^l}{\partial z_n} \delta(z_n, \overline{z}_n) = 0.
\]

Suppose a group \( G \) acts linearly on \( \mathbb{C}^n \) regarded as a real vector space. In turn, \( G \) acts on the spaces of \( C^\infty \) functions \( f \), distributions \( T \), and differential operators \( D \) on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \). We shall denote these actions by

\[
(g \cdot f)(z) := f(g^{-1} \cdot z), \\
(g \cdot T)(\phi) := T(g^{-1} \cdot \phi), \\
(g \cdot D)(f) := g \cdot (D(g^{-1} \cdot f)),
\]

where \( g \in G, z \in \mathbb{C}^n \), and \( \phi \in \mathcal{D}(\mathbb{C}^n) \cong \mathcal{D}(\mathbb{R}^{2n}) \).
4 Proof of Theorem 1.7

In this section, we take \( G \) to be \( SL(2n, \mathbb{R}) \), and construct an algebraic subgroup \( H \) satisfying the two conditions 1) and 2) in Theorem 1.7. We begin with a 4-dimensional \( \mathbb{R} \)-algebra \( \mathcal{R} \) defined by

\[
\begin{align*}
\mathcal{R} := \mathbb{C} \oplus \mathbb{C} \varepsilon & \quad \text{as a vector space,} \\
(a + b\varepsilon)(c + d\varepsilon) := (ac + bd\overline{\varepsilon}) + (b\overline{c} + ad)\varepsilon & \quad \text{as a ring,}
\end{align*}
\]

with \( \varepsilon \) being just a symbol, and \( a, b, c, d \in \mathbb{C} \). Regarding \( \mathbb{C} \) as an \( \mathbb{R} \)-vector space, we let \( \mathcal{R} \) act \( \mathbb{R} \)-linearly on \( \mathbb{C} \) by

\[
(a + b\varepsilon) \cdot z := az + b\overline{z} \quad (a + b\varepsilon \in \mathcal{R}, \ z \in \mathbb{C}).
\]

(4.2)

Remark 4.1. We write \( i \) for the imaginary unit of \( \mathbb{C} \), then by (4.1) we have

\[
\varepsilon^2 = 1, \quad i^2 = -1, \quad i\varepsilon = -\varepsilon i.
\]

Therefore \( \mathcal{R} \) is isomorphic to the real Clifford algebra \( C(1, 1) \) as an \( \mathbb{R} \)-algebra. Hence we have \( \mathcal{R} \simeq C(1, 1) \simeq M_2(\mathbb{R}) \) (for example, [12, Proposition 4.4.1]).

Let \( M_n(\mathcal{R}) \) be the \( \mathbb{R} \)-algebra of all \( n \times n \) matrices over \( \mathcal{R} \). The left multiplication defines a (real) representation of \( M_n(\mathcal{R}) \) on \( \mathbb{C}^n \) regarded as a vector space over \( \mathbb{R} \). This representation induces an injective \( \mathbb{R} \)-algebra homomorphism

\[
\iota: M_n(\mathcal{R}) \hookrightarrow M_{2n}(\mathbb{R}),
\]

which is also surjective because the real dimensions of \( M_n(\mathcal{R}) \) and \( M_{2n}(\mathbb{R}) \) are the same. We define a subgroup \( H \) of \( M_n(\mathcal{R}) \) by

\[
H := \left\{ h^\theta(a) := \begin{pmatrix} e^{i\theta} & a_1\varepsilon & a_2\varepsilon^2 & \cdots & a_{n-1}\varepsilon^{n-1} \\ e^{i\theta} & a_1\varepsilon & \ddots & \cdots & \cdots \\ \vdots & \ddots & a_2\varepsilon^2 & \ddots & \ddots \\ \vdots & \cdots & \ddots & a_1\varepsilon & \varepsilon \end{pmatrix} \mid \begin{array}{c} \theta \in \mathbb{R} \\ a \in \mathbb{C}^{n-1} \end{array} \right\},
\]

(4.4)

where \( a = (a_1, \ldots, a_{n-1}) \in \mathbb{C}^{n-1} \). Then \( \iota(H) \) is a subgroup of \( GL(2n, \mathbb{R}) \).

Lemma 4.2. \( \det(\iota(H)) = \{1\} \).

Proof. For any \( a \in \mathbb{C}^{n-1} \), it is clear that \( \det(\iota(h^0(a))) = 1 \) since \( \iota(h^0(a)) \in GL(2n, \mathbb{R}) \) is a unipotent matrix. Moreover dividing \( \iota(h^0(0, \ldots, 0)) \in GL(2n, \mathbb{R}) \) into \( 2 \times 2 \) block matrices, we have \( \det(\iota(h^0(0, \ldots, 0))) = 1 \) for any \( \theta \in \mathbb{R} \) because \( e^{i\theta} \) acts on \( \mathbb{C} \simeq \mathbb{R}^2 \) as rotation. Since the group \( H \) is generated by elements of the form \( h^0(a) \) and \( h^0(0, \ldots, 0) \), the lemma is proved. \( \square \)
By Lemma 4.2, we may identify $H$ in $M_n(\mathbb{R}_e)$ with $\iota(H)$ in $G = SL(2n, \mathbb{R})$ via $\iota$.

The following proposition shows that the subgroup $H$ of $G$ satisfies the condition 1) in Theorem 1.7.

**Proposition 4.3.** For every $j \in \{1, 2, \ldots, n\}$, there exists exactly one $H$-orbit on $G/Q$ of real dimension $2j - 1$. These orbits exhaust all $H$-orbits on $G/Q$. In particular, $\#(H\backslash G/Q) = n < \infty$.

**Proof.** Let $\mathbb{R}^\times := GL(1, \mathbb{R})$ act on $\mathbb{C}^n$ by scalar multiplication and put $X := (\mathbb{C}^n \setminus \{0\})/\mathbb{R}^\times$. Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we have $X \simeq \mathbb{RP}^{2n-1} \simeq G/Q$ and these isomorphisms induce a bijection:

$$H \backslash X \simeq H \backslash G/Q. \quad (4.5)$$

For $j \in \{1, 2, \ldots, n\}$, we define a real $(2j - 1)$-dimensional submanifold of $X$ by

$$Y_{2j-1} := \{ (z_1, \ldots, z_j, 0, \ldots, 0) \in \mathbb{C}^n \mid z_j \neq 0 \}/\mathbb{R}^\times \subset X. \quad (4.6)$$

Then the group $H$ leaves $Y_{2j-1}$ invariant, and in fact it acts transitively. Thus we have an orbit decomposition

$$H \backslash X = \bigcup_{j=1}^{n} Y_{2j-1}.$$ 

Therefore $\#(H\backslash G/Q) = \#(H\backslash X) = n < \infty$. \hfill \square

Let us prove that the subgroup $H$ of $G$ satisfies the condition 2') of Theorem 1.7 in the case of $n \geq 3$. We define two real analytic vector fields $D$ and $D'$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ for $n \geq 3$ by

$$D := z_{n-2} \frac{\partial}{\partial \overline{z}_{n-1}} + z_{n-1} \frac{\partial}{\partial z_n}, \quad D' := z_{n-2} \frac{\partial}{\partial z_{n-1}} + \overline{z}_{n-1} \frac{\partial}{\partial \overline{z}_n}. \quad (4.7)$$

For $l \in \mathbb{N}$, we define nonzero two distributions $T^l_\lambda, \overline{T}^l_\lambda \in \mathcal{D}'(\mathbb{C}^n \setminus \{0\})$ with holomorphic parameter $\lambda \in \mathbb{C}$ by

$$T^l_\lambda(z) := \frac{1}{\Gamma(2 - \frac{l}{2})} D^l (|z_{n-1}|^{2-\lambda} \delta(z_n, \overline{z}_n)), \quad (4.8)$$

$$\overline{T}^l_\lambda(z) := \frac{1}{\Gamma(2 - \frac{l}{2})} D'^l (|z_{n-1}|^{2-\lambda} \delta(z_n, \overline{z}_n)), \quad (4.9)$$

where $\Gamma(\cdot)$ denotes the gamma function. We note that $|z_{n-1}|^{2-\lambda} = (x_{n-1}^2 + y_{n-1}^2)^{1-\lambda}$ has a simple pole at $\lambda \in 2\mathbb{N} + 4$ as a distribution and $\Gamma(2 - \frac{l}{2})$ has a simple pole at $\lambda \in 2\mathbb{N} + 4$. Therefore $T^l_\lambda$ and $\overline{T}^l_\lambda$ define distributions with holomorphic parameter $\lambda \in \mathbb{C}$ (for example, see [4 Appendix B1.4]). Moreover
Lemma 4.5. Let $T^l_\lambda$ and $\overline{T}^l_\lambda$ be homogeneous distributions of degree $-\lambda$ because $|z_{n-1}|^{2-\lambda}$ and $\delta(z_n, \overline{z}_n)$ are homogeneous of degree $2-\lambda$ and $-2$, respectively, and the operators $\overline{D}$ and $D$ preserve the degrees. Clearly, $T^l_\lambda$ and $\overline{T}^l_\lambda$ are even distributions, therefore $T^l_\lambda, \overline{T}^l_\lambda \in \mathcal{D}'(\mathbb{C}^n\setminus\{0\})_{even,-\lambda} \cong \mathcal{D}'(G/Q, \chi_\lambda)$.

Proposition 4.4. Suppose $n \geq 3$. Then for any $\lambda \in \mathbb{C}$ and any $l \in \mathbb{N}$, the distributions $T^l_\lambda$ and $\overline{T}^l_\lambda$ are $H$-invariant, that is, $T^l_\lambda, \overline{T}^l_\lambda \in \mathcal{D}'(\mathbb{C}^n\setminus\{0\})^H_{even,-\lambda}$.

Proof. We prove only the claim for $T^l_\lambda$ as that for $\overline{T}^l_\lambda$ can be shown similarly. We define elements of $H$ by the equality

$$h(\theta) := h^0(0, \ldots, 0), \quad h_j(a) := h^j(0, \ldots, 0, a, 0, \ldots, 0), \quad (1.10)$$

where $\theta \in \mathbb{R}$, $a \in \mathbb{C}$ and, $j \in \{1, 2, \ldots, n-1\}$ (see [4.4] for notation). Then it is sufficient to prove that $h(\theta) \cdot T^l_\lambda = T^l_\lambda$ for any $\theta \in \mathbb{R}$ and $h_j(a) \cdot T^l_\lambda = T^l_\lambda$ for any $a \in \mathbb{C}$ and $j \in \{1, 2, \ldots, n-1\}$ because the group $H$ is generated by elements of the form $h(\theta)$ and $h_j(a)$. The first claim follows easily from $h(\theta) \cdot z = e^{i\theta}z$ for $z \in \mathbb{C}^n$. For the case of $j = 1$ of the second claim, we need the following:

Lemma 4.5. Let $D$ be the vector field defined in (4.7). Then, we have

$$h_1(a) \cdot D = D + a(\overline{z}_{n-2} - \overline{z}_{n-1} + |a|^2 \overline{z}_n) \frac{\partial}{\partial z_{n-2}} - a \overline{z}_n \frac{\partial}{\partial z_n} \quad (a \in \mathbb{C}).$$

This is an easy calculation, hence we omit the proof.

By Lemma 4.3, the following equality as distributions on $\mathbb{C}^n\setminus\{0\} \cong \mathbb{R}^{2n}\setminus\{0\}$ holds:

$$(h_1(a) \cdot T^l_\lambda)(z) = \frac{1}{\Gamma(2-\frac{\lambda}{2})} (h_1(a) \cdot D)^l \left(|z_{n-1} - a \overline{z}_n|^{2-\lambda} \delta(z_n, \overline{z}_n)\right)$$

$$= \frac{1}{\Gamma(2-\frac{\lambda}{2})} \left(D - a \overline{z}_n \frac{\partial}{\partial z_n}\right)^l \left(|z_{n-1}|^{2-\lambda} \delta(z_n, \overline{z}_n)\right)$$

$$= \frac{1}{\Gamma(2-\frac{\lambda}{2})} D^l \left(|z_{n-1}|^{2-\lambda} \delta(z_n, \overline{z}_n)\right)$$

$$= T^l_\lambda(z).$$

We have used (3.2) and $\frac{\partial}{\partial z_{n-2}} \left(|z_{n-1}|^{2-\lambda} \delta(z_n, \overline{z}_n)\right) = 0$ in the second equality.

For $j \in \{2, 3, \ldots, n-1\}$, $h_j(a) \cdot T^l_\lambda = T^l_\lambda$ can be shown similarly in the case $j = 1$. Therefore $T^l_\lambda$ is $H$-invariant. Thus the proof of proposition completes. \hfill \Box

Proposition 4.6. If $n \geq 3$, for any $\lambda \in \mathbb{C}$ we have

$$\dim \mathcal{D}'(\mathbb{C}^n\setminus\{0\})^H_{even,-\lambda} = \infty.$$
Proof. We know from Proposition 4.4 that \( T^l_\lambda \in D'(\mathbb{C}^n \setminus \{0\})^H_{\text{even},-\lambda} \) for all \( l \in \mathbb{N} \). Therefore it is sufficient to prove that \( \{ T^l_\lambda \}_{l \in \mathbb{N}} \) is linearly independent. But this is a consequence of (3.3) and the following equality as distributions on \( \mathbb{C}^n \setminus \{0\} \simeq \mathbb{R}^{2n} \setminus \{0\} \):

\[
T^l_\lambda(z) = \frac{1}{\Gamma(2-\lambda/2)} \sum_{k=0}^{l} \binom{l}{k} \left( z_{n-2} \frac{\partial}{\partial z_{n-1}} + z_{n-1} \frac{\partial}{\partial z_n} \right) \left( |z_{n-1}|^{2-\lambda} \delta(z_n, \bar{z}_n) \right)^k \left( |z_n|^{2-\lambda} \delta(z_n, \bar{z}_n) \right)^{l-k} \frac{\partial^{l-k} |z_{n-1}|^{2-\lambda}}{\partial z_{n-1}^{l-k}} \delta(z_n, \bar{z}_n).
\]

We have used the binomial expansion in the second equality. \( \square \)

**Proof of Theorem 1.7 in the case \( n \geq 3 \).** We take \( H \) to be the subgroup (4.4) via the inclusion \( \iota : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \). Then \( H \) satisfies 1) by Proposition 4.3. Moreover \( H \) satisfies 2') by Corollary 2.3 and Proposition 4.6 because \( D'(\mathbb{R}^{2n} \setminus \{0\})^H_{\text{even},-\lambda} \simeq D'(\mathbb{C}^n \setminus \{0\})^H_{\text{even},-\lambda} \). We note that any one-dimensional representation \( \chi \) of \( Q \) is of the form \( \chi_\lambda \) for some \( \lambda \in \mathbb{C} \). \( \square \)

Next we discuss in the case \( n = 2 \). For \( \lambda = 2 \) in (4.8) and (4.9), the binomial expansion shows

\[
T^l_z(z) = \left( z_{n-2} \frac{\partial}{\partial z_{n-1}} + z_{n-1} \frac{\partial}{\partial z_n} \right)^l \delta(z_n, \bar{z}_n) = \left( \frac{\partial}{\partial z_n} \right)^l \delta(z_n, \bar{z}_n),
\]

\( 4.11 \)

\[
\overline{T}^l_z(z) = \left( \bar{z}_{n-1} \frac{\partial}{\partial z_{n-1}} \right)^l \delta(z_n, \bar{z}_n).
\]

\( 4.12 \)

In the second equality, we have used \( \frac{\partial}{\partial z_{n-1}} \delta(z_n, \bar{z}_n) = 0 \) because \( \delta(z_n, \bar{z}_n) \) does not depend on the variable \( z_{n-1} \). Then we define \( T^l_z \) and \( \overline{T}^l_z \) in the case of \( (n, \lambda) = (2, 2) \) by (4.11) and (4.12), respectively, in which the variables \( z_{n-2}, \bar{z}_{n-2} \) do not appear. By using these distributions, we prove the case \( n = 2 \) of Theorem 1.7.

**Proof of Theorem 1.7 in the case of \( n = 2 \).** We take \( H \) to be the subgroup (4.4) via the inclusion \( \iota : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) as in the case of \( n \geq 3 \), then \( H \) satisfies 1) by Proposition 4.3. Set \( D' := z_1 \frac{\partial}{\partial z_2} \). By (4.11) we have

\[
T^l_z(z) = (D')^l \delta(z_2, \bar{z}_2).
\]
We note that the group $H$ is generated by elements of the form $h(\theta)$ and $h_1(a)$ in the case of $n = 2$. Just like before, $h(\theta) \cdot T^l_\lambda = T^l_\lambda$ follows from $h(\theta) \cdot z = e^{i\theta} z$ for $z \in \mathbb{C}^2$. Moreover, direct computation shows

$$h_1(a) \cdot D' = D' + \pi (z_1 - a\overline{z}_2) \frac{\partial}{\partial z_1} - a\overline{z}_2 \frac{\partial}{\partial z_2} \quad (a \in \mathbb{C}).$$

Hence in the same way as in $n \geq 3$, the following equality of distributions on $\mathbb{C}^2 \setminus \{0\} \cong \mathbb{R}^4 \setminus \{0\}$ holds:

$$h_1(a) \cdot T^l_2(z) = \left( D' + \pi (z_1 - a\overline{z}_2) \frac{\partial}{\partial z_1} - a\overline{z}_2 \frac{\partial}{\partial z_2} \right)^l \delta(z_2, \overline{z}_2) = (D')^l \delta(z_n, \overline{z}_n) = T^l_2(z).$$

Therefore we have $T^l_2 \in \mathcal{D}'(\mathbb{C}^2 \setminus \{0\})^H_{\text{even}, -2}$ for any $l \in \mathbb{N}$. Furthermore, we have $\dim \mathcal{D}'(\mathbb{C}^2 \setminus \{0\})^H_{\text{even}, -2} = \infty$ because $\{T^l_2\}_{l \in \mathbb{N}}$ are linearly independent. Thus $H$ satisfies 2) by Corollary 2.3. Therefore the proof of the case of $n = 2$ completes.

**Remark 4.7.** For $n = 2$, the dimension of $\mathcal{D}'(\mathbb{C}^2 \setminus \{0\})^H_{\text{even}, -\lambda}$ is finite-dimensional for generic $\lambda \in \mathbb{C}$. Indeed one can show that

$$\dim \mathcal{D}'(\mathbb{C}^2 \setminus \{0\})^H_{\text{even}, -\lambda} \leq 2 \quad \text{for } \lambda \in \mathbb{C} \setminus \{2\}.$$

Finally, we discuss the supports of elements of $\mathcal{D}'(G/Q, \chi_\lambda)^H$. If $\lambda \notin 2\mathbb{N} + 4$, we have $\text{supp}(T^l_2) = cl(Y_{2n-3})$ by (4.8). Here $cl(Y_{2n-3})$ denotes the closure of $Y_{2n-3}$ in $X$ (See (1.12) for the definition of $Y_{2j-1} \subset X$ for $j \in \{1, 2, \ldots, n\}$ and hereafter we regard as $Y_{2j-1} \subset G/Q$ by $X \cong G/Q$ in (1.5)). We put $X_j := cl(Y_{2j-1}) \subset X$. Then we have

$$\dim \left( \mathcal{D}'_{X_{n-1}}(G/Q, \chi_\lambda)^H / \mathcal{D}'_{X_{n-2}}(G/Q, \chi_\lambda)^H \right) = \infty,$$

where $\mathcal{D}'_{X_{n-1}}(G/Q, \chi_\lambda)^H := \{ F \in \mathcal{D}'(G/Q, \chi_\lambda)^H \mid \text{supp}(F) \subset X_{j-1} \}$. Furthermore, the following statement holds more generally:

**Proposition 4.8.** Suppose $n \geq 3$. Let $G$ and $Q$ be as in Theorem 1.7. Then for any $j \in \{2, 3, \ldots, n-1\}$, we have

$$\dim \left( \mathcal{D}'_{X_j}(G/Q, \chi_\lambda)^H / \mathcal{D}'_{X_{j-1}}(G/Q, \chi_\lambda)^H \right) = \infty$$

for any $\lambda \in \mathbb{C} \setminus (2\mathbb{N} + 2 + 2n - 2j)$.

**Proof.** Let $D_j$ be a real analytic vector field on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ given by $D_j := \overline{z}_j \frac{\partial}{\partial z_j} + z_j \frac{\partial}{\partial \overline{z}_j}$. For $l \in \mathbb{N}$, we define a distribution $T^l_{\lambda,j} \in \mathcal{D}'(\mathbb{C}^n \setminus \{0\})$ with holomorphic parameter $\lambda \in \mathbb{C}$ by

$$T^l_{\lambda,j}(z) := \frac{1}{\Gamma(n - j + 1 + \frac{l}{2})} D_j^l \left( |z_j|^{2(n-j)-\lambda} \prod_{k=j+1}^{n} \delta(z_k, \overline{z}_k) \right). \quad (4.13)$$
Then we have $T_{1,j}^l \in \mathcal{D}'(\mathbb{C}^n \setminus \{0\})^H_{\text{even}, -\lambda} \simeq \mathcal{D}'(G/Q, \chi, \lambda)^H$ in the same way as the case of $T^l_1$. Moreover $\text{supp} (T_{1,j}^l) = c(Y_{2,j-1}) = X_j$ follows easily from (4.13) if $\lambda \in \mathbb{C} \setminus (2\mathbb{N} + 2n - 2j)$. This completes the proof of Proposition 4.10. \[ \square \]

**Remark 4.9.** Let $G_C, Q_C$ and $H_C$ be complexifications of $G, Q$ and $H$, respectively. Then if $\#(H_C \setminus G_C/Q_C) < \infty$, we have $\dim \mathcal{D}'(\mathbb{C}^n \setminus \{0\})^H_{\text{even}, -\lambda} < \infty$ for any $\lambda \in \mathbb{C}$ by the general theory of holonomic systems due to Sato-Kashiwara-Kawai [6, Theorems 5.1.7, and 5.1.12]. Therefore we have $\#(H_C \setminus G_C/Q_C) = \infty$ because $\dim \mathcal{D}'(\mathbb{C}^n \setminus \{0\})^H_{\text{even}, -\lambda} = \infty$ by Proposition 4.8. Alternatively we can show that $\#(H_C \setminus G_C/Q_C) = \infty$ by direct calculation as below.

**Proposition 4.10.** Suppose $G, Q$ are as in Theorem 4.7 and $H$ is the subgroup of $G$ defined in (4.4). Let $G_C, Q_C$ and $H_C$ be complexifications of $G, Q$ and $H$, respectively. Then if $n \geq 2$, we have $\#(H_C \setminus G_C/Q_C) = \infty$.

Before the proof of Proposition 4.10 we discuss the complexifications of $\mathbb{C}$ and $R_\varepsilon$ in order to make calculation clear. We write $\overline{\mathbb{C}}$ for the complex conjugate space of $\mathbb{C}$, that is, $\overline{\mathbb{C}} = \mathbb{C}$ as a set, and scalar multiplication of $c \in \mathbb{C}$ given by $c \cdot v := \overline{c} v$ for $v \in \mathbb{C}$. Then the complexification $\mathbb{C} \otimes_R \mathbb{C}$ of $\mathbb{C}$ is isomorphic to $\mathbb{C} \oplus \overline{\mathbb{C}}$ as a $\mathbb{C}$-algebra by the following map:

$$e_\pm \frac{a \otimes 1}{2} + e_+ \frac{c \otimes 1}{2} \mapsto (a, c) \quad (a, c \in \mathbb{C}), \quad (4.14)$$

where $e_\pm := 1 \otimes 1 \pm i \otimes i \in \mathbb{C} \otimes_R \mathbb{C}$. Here the multiplication of $\mathbb{C} \otimes_R \mathbb{C}$ is given by $(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$. Similarly, we define an isomorphism $R_\varepsilon \otimes_R \mathbb{C} = (\mathbb{C} \oplus \varepsilon) \otimes_R \mathbb{C} \cong (\mathbb{C} \oplus \overline{\mathbb{C}}) \oplus (\mathbb{C} \oplus \varepsilon)$ as a $\mathbb{C}$-algebra by

$$e_\varepsilon' \frac{(a + b\varepsilon) \otimes 1}{2} + e_+ \frac{(c + d\varepsilon) \otimes 1}{2} \mapsto (a, c) + (b, d)\varepsilon \quad (a, b, c, d \in \mathbb{C}), \quad (4.15)$$

where $e_\varepsilon' := 1 \otimes 1 \pm i \otimes i \in (\mathbb{C} \oplus \varepsilon) \otimes_R \mathbb{C}$. Then the multiplication on $(\mathbb{C} \oplus \overline{\mathbb{C}}) \oplus (\mathbb{C} \oplus \varepsilon)$ induced from this isomorphism is given below,

$$(a, c) + (b, d)\varepsilon \cdot (a', c') + (b', d')\varepsilon = (aa' + b\overline{c} + cc' + d\overline{b}) + (ab' + be\overline{c} + cd' + d\overline{e}) \varepsilon,$$

where $(a, c) + (b, d)\varepsilon \cdot (a', c') + (b', d')\varepsilon \in (\mathbb{C} \oplus \overline{\mathbb{C}}) \oplus (\mathbb{C} \oplus \varepsilon).$ Hereafter we identify $R_\varepsilon \otimes_R \mathbb{C}$ with $(\mathbb{C} \oplus \overline{\mathbb{C}}) \oplus (\mathbb{C} \oplus \varepsilon)$ via (4.15). For the proof of Proposition 4.10 we need:

**Lemma 4.11.** The complexification of the representation of $R_\varepsilon$ on $\mathbb{C}$ defined in (4.2) is given below under the identifications of (4.14) and (4.15),

$$(a, c) + (b, d)\varepsilon \cdot (z, w) = (az, cw) + (\overline{bw}, d\overline{e}),$$

where $(a, c) + (b, d)\varepsilon \in R_\varepsilon \otimes_R \mathbb{C}$ and $(z, w) \in \mathbb{C} \oplus \overline{\mathbb{C}} \simeq \mathbb{C} \otimes_R \mathbb{C}$.

This follows from easy calculation, hence we omit the proof.
Proof of Proposition 4.10. \( M_n(\mathbb{R} \otimes \mathbb{C}) \) acts on \((\mathbb{C} \oplus \overline{\mathbb{C}})^n \simeq \mathbb{C}^n \otimes \mathbb{C}\) by left multiplication. This action induces \( \iota_C : M_n(\mathbb{R} \otimes \mathbb{C}) \to M_{2n}(\mathbb{C}) \) in the same way as \( \iota \) in (4.3). Then the complexification of \( H \) in \( M_n(\mathbb{R} \otimes \mathbb{C}) \) is the following:

\[
H_C := \left\{ h^a(A) := \begin{pmatrix} (e^{ia}, e^{\overline{ia}}) A_1 e^{\overline{ia}} & \cdots & A_{n-1} e^{\overline{ia(n-1)}} \\ (e^{ia}, e^{\overline{ia}}) & \ddots & \vdots \\ \vdots & \ddots & (e^{ia}, e^{\overline{ia}}) \end{pmatrix} \bigg| a \in \mathbb{C}, \ A \in (\mathbb{C} \oplus \overline{\mathbb{C}})^{n-1} \right\}, \tag{4.16}
\]

where \( A = (A_1, \ldots, A_{n-1}) \in (\mathbb{C} \oplus \overline{\mathbb{C}})^{n-1} \). Similarly to the case of \( H \) in (4.3), \( \iota(H_C) \) is a subgroup of \( G_C = SL(2n, \mathbb{C}) \) and we may identify \( H_C \) in \( M_n(\mathbb{R} \otimes \mathbb{C}) \) with \( \iota_C(H_C) \) in \( G_C = SL(2n, \mathbb{C}) \). Let \( C^\times := GL(1, \mathbb{C}) \) act on \((\mathbb{C} \oplus \overline{\mathbb{C}})^n \) by scalar multiplication. Then, for \( c \in C^\times \) and \((z_1, w_1), \ldots, (z_n, w_n) \in (\mathbb{C} \oplus \overline{\mathbb{C}})^n \), we have

\[
c \cdot ((z_1, w_1), \ldots, (z_n, w_n)) = ((cz_1, \overline{cw}_1), \ldots, (cz_n, \overline{cw}_n)).
\]

We put \( X_C := \left( (\mathbb{C} \oplus \overline{\mathbb{C}})^n \setminus \{0\} \right) / C^\times \). By regarding \((\mathbb{C} \oplus \overline{\mathbb{C}})^n \) as \( \mathbb{C}^{2n} \), we have \( X_C \simeq \mathbb{C}^{2n-1} \simeq G_C/Q_C \) and these isomorphisms induce a bijection:

\[
H_C \backslash X_C \simeq H_C \backslash G_C/Q_C.
\]

On the other hand, the action of \( H_C \) on \((\mathbb{C} \oplus \overline{\mathbb{C}})^n \) is given below by Lemma 4.11 (See (4.10) for the definition of \( h^a(A) \in H_C \)).

\[
\begin{pmatrix} (z_1, w_1) \\ \vdots \\ (z_{n-1}, w_{n-1}) \\ (z_n, w_n) \end{pmatrix} \cdot \begin{pmatrix} (e^{ia} z_1, e^{\overline{ia}} w_1) + \sum_{j=1}^{n-1} (a_j, b_j) e^{\overline{ia}} \cdot (z_{j+1}, w_{j+1}) \\ \vdots \\ (e^{ia} z_{n-1} + a_1 \overline{w}_n, e^{\overline{ia}} w_{n-1} + b_1 \overline{w}_n) \\ (e^{ia} z_n, e^{\overline{ia}} w_n) \end{pmatrix},
\]

where \( a \in \mathbb{C}, \ A = (A_1, \ldots, A_{n-1}) = ((a_1, b_1), \ldots, (a_{n-1}, b_{n-1})) \in (\mathbb{C} \oplus \overline{\mathbb{C}})^{n-1} \) and \((z_1, w_1), \ldots, (z_n, w_n) \in (\mathbb{C} \oplus \overline{\mathbb{C}})^n \). For \( \zeta \in \mathbb{C} \), we define a complex \((2n-3)\)-dimensional submanifold of \( X_C \) by

\[
Y_{2n-3}^\zeta := \left\{ (z_j, w_j)_{j=1}^{n} \in (\mathbb{C} \oplus \overline{\mathbb{C}})^n \big| w_n = 0, \ z_n \neq 0, \ z_{n-1} = \zeta z_n \right\} / C^\times \subset X_C.
\]

Then for any \( \zeta \in \mathbb{C} \), the group \( H_C \) leaves \( Y_{2n-3}^\zeta \) invariant, and in fact it acts transitively. Moreover if \( \zeta \neq \mu \), \( Y_{2n-3}^\zeta \) and \( Y_{2n-3}^\mu \) have no intersection. Therefore we have \( \#(H_C \backslash G_C/Q_C) = \#(H_C \backslash X_C) = \infty \). \(\square\)

Acknowledgement

The author is grateful to Professor Toshiyuki Kobayashi for his much helpful advice and constant encouragement and thanks my parents for their support.
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