Ab Initio Approach to the Non-Perturbative Scalar Yukawa Model

Yang Li\textsuperscript{a,}\textsuperscript{*}, V. A. Karmanov\textsuperscript{b}, P. Maris\textsuperscript{a}, J. P. Vary\textsuperscript{a}

\textsuperscript{a}Department of Physics and Astronomy, Iowa State University, Ames, Iowa, USA. 50011
\textsuperscript{b}Lebedev Physical Institute, Leninsky Prospekt 53, 119991 Moscow, Russia

Abstract

We report on the first non-perturbative calculation of the quenched scalar Yukawa model in the four-body Fock sector truncation. The light-front Hamiltonian approach with a Fock sector dependent renormalization is applied. We study the Fock sector contribution and the electromagnetic form factor in the non-perturbative region. We find that the one- and two-body contributions dominate the Fock space up to coupling $\alpha \approx 1.7$. By comparing with lower Fock sector truncations, we show that the form factor converges with respect to the Fock sector expansion. As we approach the coupling $\alpha \approx 2.2$, we discover that the four-body contribution rises rapidly and overtakes the two- and three-body contributions.

Keywords: Light Front Hamiltonian, Fock Sector Dependent Renormalization, Scalar Yukawa Model

1. Introduction

Solving quantum field theories in the non-perturbative regime is not only a theoretical challenge but also essential to understand the structure of hadrons from first principles. The light-front (LF) Hamiltonian quantum field theory approach provides a natural framework to tackle this issue \cite{1,2}. A great advantage of this approach is that it provides direct access to the hadronic observables. In the LF dynamics, the system is defined at a fixed LF time $x^+ \equiv t + z$. The physical states are obtained by diagonalizing the LF Hamiltonian operator. The vacuum in LF quantization is trivial. As a result, it is particularly convenient to expand the physical states in the Fock space. For example, a physical pion state can be written in terms of quarks ($q$), antiquarks ($\bar{q}$) and gluons ($g$) as $|\pi\rangle = |qq\rangle + |qqq\rangle + |qqgg\rangle + \cdots$.

In order to do practical calculations, the Fock space has to be truncated. A natural choice, taking advantage of the LF dynamics, is the Fock sector truncation, also known as the light-front Tamm-Dancoff (LFTD) \cite{2}. A number of non-perturbative renormalization schemes have been developed based on the LFTD \cite{3,4,5,6}. Thus we arrive at a few-body problem and predictions can be systematically improved by including more Fock sectors. The LFTD method is a non-perturbative approach in Minkowski space, which can be compared with other non-perturbative methods, e.g., Lattice quantum field theory in Euclidean space. Of course, this approach only works if the Fock sector expansion converges in the non-perturbative region. In practice, one can compare successive Fock sector truncations and check numerically whether the relevant physical observables converge. We will see that good convergence is achieved for the scalar Yukawa model in a non-perturbative regime with a four-body Fock sector truncation. Similar results, though by a different method, were found in Refs. \cite{7,8} for the Wick-Cutkosky model \cite{9}.

We apply this approach to a scalar version of the Yukawa model that describes the pion-mediated nucleon-nucleon interaction. The Lagrangian density of the model reads

$$\mathcal{L} = \partial_{\mu} N^\dagger \partial^{\mu} N - m^2 |N|^2 + \frac{1}{2} \partial_{\mu} \pi \partial^{\mu} \pi - \frac{1}{2} \mu_0^2 \pi^2 + g_0 |N|^2 \pi + \delta m^2 |N|^2, \quad (1)$$

where $g_0$ is the bare coupling and $\delta m^2$ is the mass counterterm of the field $N(x)$. It is convenient to introduce a dimensionless coupling constant $\alpha = $
For the sake of brevity, we refer to the fundamental degrees-of-freedoms (d.o.f.’s) \(N(x)\) and \(\pi(x)\) as nucleon and pion field respectively. We also introduce a Pauli-Villars (PV) pion (with mass \(\mu_1\)) to regularize the ultraviolet (UV) divergence \[10\]. Then, a sector dependent method known as the Fock sector dependent renormalization (FSDR) developed in \[6\] is used to renormalize the theory. FSDR is a systematic non-perturbative renormalization scheme based on the covariant light-front dynamics (CLFD, see Ref. \[11\] for a review) and Fock sector expansion. It has shown great promise in the application to the Yukawa model and QED \[12\], \[13\].

The scalar Yukawa model is known to exhibit a vacuum instability \[14\]. It can be stabilized through the antinucleon d.o.f. The nucleon and pion pair \(N^+\) will generate non-perturbative dynamics at large coupling sufficient for our purposes.

Previously, this model has been solved in the same approach up to three-body truncation (one nucleon, two pions) \[8\]. The results from the two- and three-body truncations agree at small couplings; yet they deviate in the large coupling region. Therefore, it is crucial to extend the non-perturbative calculation to higher Fock sectors. In this paper, we present the calculation of the four-body truncation (one nucleon, three pions). By comparing successive truncations, we can examine the convergence of the Fock sector expansion. We presented a preliminary version of this work here. It should be emphasized, though, that our formalism is capable of dealing with the antinucleon d.o.f. The nucleon and pion d.o.f.’s will generate non-perturbative dynamics at large coupling sufficient for our purposes.

We first introduce our formalism in the next section. The LF Hamiltonian field theory will be briefly mentioned and the non-perturbative renormalization procedure will be explained. Then a set of coupled integral equations will be derived for the four-body truncation. In Sec. \[3\] we present the numerical results, including the calculation of the electromagnetic form factor. We conclude in Sec. \[4\].

\[2. \text{Light-Front Hamiltonian Field Theory}\]

The LF Hamiltonian for the scalar Yukawa model is

\[\hat{P}^- = \int d^3x \left[ \partial^+ N^+ N + m^2 |N|^2 + \frac{1}{2} \partial^+ \pi \partial^+ \pi \right.\]

\[+ \frac{1}{2} \mu_1^2 \pi^2 - g_0 |N|^2 \pi - \delta m^2 |N|^2 \big] \big|_{x^+ = 0}. \] \hspace{2cm} (2)

The physical states can be obtained by solving the time-independent Schrödinger equation

\[\hat{P}^- |p\rangle = \frac{p_\perp^2 + M^2}{p^+} |p\rangle, \] \hspace{2cm} (3)

where \(p_\perp\) and \(p^+\) are the transverse and longitudinal momentum, respectively. Thanks to boost invariance in the LF dynamics, we can take \(p_\perp = 0\) without loss of generality.

The physical state admits a Fock space expansion,

\[|p\rangle = \sum_n \int D_n \psi_n(k_{1\perp}, x_1, \cdots k_{n\perp}, x_n; p^2)\]

\[\times |k_{1\perp}, x_1, \cdots k_{n\perp}, x_n\rangle, \] \hspace{2cm} (4)

where \(x_i \equiv \frac{k_i^+}{p^+}\).

\[D_n = 2(2\pi)^3 \delta^{(2)}(k_{1\perp} + \cdots k_{n\perp})\delta(x_1 + \cdots x_n - 1)\]

\[\times \prod_{i=1}^n \frac{d^2k_{i\perp}}{(2\pi)^2} dx_i. \]

The \(n\)-body Fock state \(|k_{1\perp}, x_1, \cdots, k_{n\perp}, x_n\rangle\) consists \((n - 1)\) pions and 1 nucleon. We use the last pair \(|k_{n\perp}, x_n\rangle\) to denote the momentum of the constituent nucleon. \(\psi_n\), known as the LF wave function (LFWF), is a boost invariant. The LFWFs are normalized to unity, \(\sum_n I_n = 1\), where

\[I_n = \frac{1}{(n-1)!} \int D_n \left| \psi_n(k_{1\perp}, x_1, \cdots, k_{n\perp}, x_n; p^2) \right|^2 \] \hspace{2cm} (5)

is the probability that the system appears in the \(n\)-body Fock sector. In the scalar Yukawa model, these quantities are regulator independent, in contrast to more realistic theories such as Yukawa and QED. Note that \(\psi_1 = \sqrt{I_1}\) is a constant.

It is convenient to introduce the \(n\)-body vertex functions,

\[\Gamma_n(k_{1\perp}, x_1, \cdots, k_{n-1\perp}, x_{n-1}; p^2) = (s_1, \cdots, n-1 - p^2) \psi_n(k_{1\perp}, x_1, \cdots, k_{n\perp}, x_n; p^2) \] \hspace{2cm} (6)
for $n > 1$ and $\Gamma_1 = (m^2 - \rho^2)\psi_1$, where

$$s_1, \ldots, s_{n-1} = (k_1 + \cdots + k_n)^2 = \sum_{i=1}^{n-1} \frac{k_{\perp i}^2 + \mu_i^2}{x_i} + \frac{k_{\perp n}^2 + m^2}{x_n}$$

is the invariant mass squared of the Fock state. We have suppressed $k_{\perp n}$ and $x_n$ in $\Gamma_n$, by virtue of the momentum conservations $k_{1\perp} + k_{2\perp} + \cdots + k_{n\perp} = 0$, $x_1 + x_2 + \cdots + x_n = 1$. For simplicity we will also omit the dependence on $p^2$ in $\Gamma_n$ for the ground state $p^2 = m^2$.

Written in terms of the vertex functions $\Gamma$, Eq. (3) can be represented diagrammatically using the LF graphical rules [17] [18] (see Ref. [11] for a review). Figure 1 shows the diagrams for the four-body truncation.

The two-body vertex function $\Gamma_2$ plays a particular role in renormalization. It comprises all the radiative corrections allowed by the Fock sector truncation, which include both the amputated vertex $V_2(k_1, k_2, p)$ and the self-energy correction $\Sigma((p - k_1)^2)$ (see Fig. 2):

$$\Gamma_2(k_{1\perp}, x_1; p^2) = Z((p - k_1)^2)V_2(k_1, k_2, p)\sqrt{t_1}. \quad (7)$$

Here the function $Z(q^2) = (1 - \Sigma(q^2) - \Sigma(m^2))^{-1}$ is a generalization of the field strength renormalization constant $Z = (1 - \frac{\partial}{\partial q^2}\Sigma(q^2))^{-1}_{q^2 = m^2} = I_1$. Note the presence of the pion spectator, which means that in the $n$-body truncation, the self-energy correction in the expression for $\Gamma_2$ is the $(n-1)$-body self-energy.

The dependence of renormalization constants on the Fock sector is a general feature of the Fock sector expansion. We use $g_{0n}$ and $\delta m_n^2$ to denote the bare coupling and the mass counterterm from the $n$-body truncation, respectively. According to the LSZ reduction formula, the physical coupling $g = T_{f_1} = \sqrt{Z}V_3(k_1, k_2, p)\sqrt{T_1}$. Here “$*$” means that $V_3$ is evaluated at the renormalization point, the physical mass shell $s_1 = m^2 \Rightarrow k_{1\perp}^2 = -(1 - x_1)\mu_0^2 - x_1^2m^2 \equiv k_{1\perp}^2$. These relations provide the on-shell renormalization condition [5] [6] [12],

$$\Gamma_2(n)(k_{1\perp}, x_1; p^2 = m^2) = g\sqrt{Z(n-1)}. \quad (8)$$

Here the Fock sector dependence is shown explicitly. For example, $\Gamma_2(3)$ represents the two-body vertex function found in the $n$-body truncation. Note that $k_{1\perp}^2$ is negative, which means Eq. (5) has to be imposed through analytical continuation.

The two-body vertex function $\Gamma_2$ also provides a
Figure 3: The self-energy correction, loop correction $\Sigma$ plus mass counterterm $\delta m^2$, expressed in terms of the two-body vertex function $\Gamma_2$. The mass renormalization condition in the on-shell scheme implies $\delta m^2 = \Sigma(m^2)$.

As mentioned, the system of equations for $\Gamma_{2-4}$ resulted from truncating Eq. (3) to at most four-body (one nucleon and three pions) are shown in Fig. 4. After substituting $\Gamma_4$ into the second equation and applying the renormalization condition Eq. (8), the system of equations becomes

$$\Gamma_3^{j_2}(k_{1\perp}, s_1) = g/\sqrt{f_4^{(j_2)}} + \delta m^2 \Gamma_3^{j_2}(k_{1\perp}, s_1) \left( \frac{1}{s_1-m^2} \right)$$

Then the mass renormalization condition in the on-shell scheme implies $\delta m^2 = \Sigma(m^2)$.

We solved the system at $m = 0.94$ GeV, $\mu_0 = 0.14$ GeV. The numerical results are obtained using Cray XE6 Hopper at NERSC.
Hamiltonian in the three-body truncation becomes
\[ \alpha g \]

\[ \text{singularity in truncation to the four-body truncation via} \]

\[ \alpha \approx \alpha_f \]

\[ \text{similar case in QED) propagates from the two-body} \]

\[ \text{bare coupling, } \alpha \]

\[ \text{suffices for our purposes here.} \]

\[ \text{larger } \mu_1 \text{ requires more coverage in the UV} \]

\[ \text{hence larger grid size. A PV mass } \mu \]

\[ \text{PN mass (nucleon mass unit)} \]

\[ \alpha = 2.0 \]

\[ \text{PV mass (nucleon mass unit)} \]

\[ \alpha = 1.0 \]

\[ \begin{align*}
1 & \\
2 & \\
4 & \\
8 & \\
16 & \\
32 & \\
64 & \end{align*} \]

\[ \begin{align*}
0.0 & \\
0.2 & \\
0.4 & \\
0.6 & \\
0.8 & \\
1.0 & \end{align*} \]

\[ \text{Figure 4: The Fock sector norms } I_{1,4} \text{ as a function of the} \]

\[ \text{PV mass } \mu_1 \text{ for } \alpha = 1.0 \text{ (top) } \alpha = 2.0 \text{ (bottom). Results} \]

\[ \text{evaluated on different grids are shown.} \]

\[ \text{increases. However, for a fixed grid, increasing } \mu_1 \]

\[ \text{would increase the numerical error while decreasing} \]

\[ \text{the systematic error introduced by the finite regulator,} \]

\[ \text{as larger } \mu_1 \text{ requires more coverage in the UV} \]

\[ \text{hence larger grid size. A PV mass } \mu_1 = 15 \text{ GeV} \]

\[ \text{suffices for our purposes here.} \]

\[ \text{There exist two critical couplings at } \alpha_c \approx 2.6 \text{ and} \]

\[ \alpha_c' \approx 2.2. \text{ In the two-body truncation, one finds the} \]

\[ \frac{1}{g^2} - \frac{1}{g_{02}^2} = \frac{1}{16\pi^2m^2} \left[ f\left( \frac{\mu_0}{m} \right) - f\left( \frac{\mu_1}{m} \right) \right], \]

\[ \text{where } f(\lambda) = \int_0^1 dx (1-x)/(1-x)^2 + x^2) \text{ and} \]

\[ f(\lambda \to \infty) = 0. \text{ If the physical coupling constant } \alpha > \alpha_c \equiv \pi/f(\mu_0/m), \]

\[ \text{the two-body bare coupling } g_{02} \text{ diverges at some finite PV mass.} \]

\[ \text{Such a singularity in } g_{02} \text{ (known as the “Landau pole” in a} \]

\[ \text{similar case in QED) propagates from the two-body truncation to the} \]

\[ \text{four-body truncation via } g_{02} \text{ used in the FSDR. At } \alpha = \alpha_c', \text{ the determinant of the} \]

\[ \text{Hamiltonian in the three-body truncation becomes} \]

\[ \begin{align*}
\langle p + q | J^+ (0) | p \rangle = 2 p^+ F(Q^2), \quad (13) \end{align*} \]

\[ \text{where } q^+ = 0, Q^2 = -q^2 = q_{\perp}^2 > 0. \text{ In LF dynamics, the form factor admits a probabilistic formula-} \]

\[ \text{Figure 5: The Fock sector norms } I_{1,4} \text{ as a function of the} \]

\[ \text{coupling constant } \alpha. \text{ Results are evaluated on the grid} \]

\[ N_{\text{f.N}} = 41, N_{\text{rad}} = N_{\text{ang}} = 20, \text{ with a PV mass } \mu_1 = 15 \text{ GeV.} \]
Figure 6: Comparison of the Fock sector norms $I_1$ (top) and $I_2$ (bottom) from successive two-, three- and four-body truncations.

$$F(Q^2) = \sum_n \frac{1}{(n-1)!} \int D_n \psi^*_n(k_{i\perp}, x_1, \cdots, k_{n\perp}, x_n) \times \psi_n(k_{1\perp}, x_1, \cdots, k_{n\perp}, x_n)$$

where $k_{i\perp} = k_{i\perp} - x_i q_{\perp}$, ($i = 1, 2, \cdots, n -1$), for the spectators and $k_{n\perp} = k_{n\perp} + (1 - x_n) q_{\perp}$ for the struck parton.

Figure 7 shows the form factor for some selected couplings. In the limit of $Q^2 \rightarrow 0$, $F(0) = 1$, consistent with the charge conservation; in the limit of $Q^2 \rightarrow \infty$, $F(\infty) = I_1$. The form factors can be approximated by

$$F(Q^2) \approx \frac{1 + c I_1 Q^2}{1 + c Q^2}.$$

Figure 8 compares the form factors obtained from the two-, three- and four-body truncations for two selected couplings. The three- and four-body truncation results show good agreement even at the non-perturbative couplings, suggesting a reasonable convergence with respect to the Fock sector expansion.

4. Discussion and Conclusions

We solve the quenched scalar Yukawa model in light-front dynamics within a four-body Fock sector truncation. Fock sector dependent renormalization is implemented. The coupled system of linear integral equations is derived and solved numerically. The numerical study of the Fock sector norms suggests that up to $\alpha \approx 1.7$ the system is dominated by the lowest Fock sectors. By comparing the form factors from successive Fock sector truncations (two-, three- and four-body), we find that the Fock space expansion of the form factor converges as the number of pions increases even in the non-perturbative region.

Solving the one-nucleon sector is also the first step for the study of the two-nucleon sector – a bound-state problem, which has been extensively studied in various approaches [20]. However not all these approaches are from first principles. In our approach, the two-nucleon sector obeys similar integral equations. The bare couplings and the mass counterterms, according to FSDR, are already provided by the one-nucleon sector (up to three dressing pions). Therefore, our approach allows a systematic study of the theory with a non-perturbative renormalization.

This calculation demonstrates that the light-front Tamm-Dancoff, equipped with the Fock sector dependent renormalization, is a general \textit{ab initio}
non-perturbative approach to quantum field theories. While the solution of the scalar Yukawa model may be useful for, e.g., chiral effective field theory studies, this approach has also been applied to more realistic field theories, including the Yukawa model (truncation up to one spinor and two scalars) [13] and QED (truncation up to one electron and two photons) [5]. The study of the higher Fock sector expansion in these models is in principle similar to the current one, which indicates the potential of this approach as an alternative to other first-principle methods, e.g. the lattice gauge theory, especially in the study of hadron structures.

Acknowledgements

We are indebted to A. V. Smirnov for kindly providing us some numerical benchmark results for the three-body truncation. We wish to thank J. Carbonell, J.-F. Mathiot and X. Zhao for valuable discussions. One of us (V.A.K.) is sincerely grateful to the Nuclear Theory Group at Iowa State University for kind hospitality during his visits. This work was supported in part by the Department of Energy under Grant Nos. DE-FG02-87ER40371 and DESC0008485 (SciDAC-3/NUCLEI) and by the National Science Foundation under Grant No. PHY-0904782. Computational resources were provided by the National Energy Research Supercomputer Center (NERSC), which is supported by the Office of Science of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

References

[1] B. L. G. Bakker et al., Nucl. Phys. Proc. Suppl. 251-252, 165 (2014); [arXiv:1309.6333 [hep-ph]].
[2] R. J. Perry, A. Harindranath, and K. G. Wilson, Phys. Rev. Lett. 65, 2959 (1990).
[3] R. J. Perry and A. Harindranath, Phys. Rev. D 43, 4051 (1991).
[4] S. D. Glazek and K. G. Wilson, Phys. Rev. D 48, 5863 (1993).
[5] J. R. Hiller and S. J. Brodsky, Phys. Rev. D 59, 016006 (1998); [arXiv:hep-ph/9806541].
[6] V. A. Karmanov, J.-F Mathiot, A. V. Smirnov, Phys. Rev. D 77, 085028 (2008); [arXiv:0801.4507 [hep-th]].
[7] D. S. Hwang and V. A. Karmanov, Nucl. Phys. B 696, 413 (2004); [arXiv:hep-th/0405035].
[8] S. J. Brodsky, J. R. Hiller, G. McCartor, Ann. Phys. 321, 1240 (2006).
[9] G. C. Wick, Phys. Rev. 96, 1124 (1954);
R. E. Cutkosky, ibid. 96, 1135 (1954).
[10] S. J. Brodsky, J. R. Hiller and G. McCartor, Phys. Rev. D 64, 114023 (2001).
[11] J. Carbonell, B. Desplanques, V. A. Karmanov, and J.-F. Mathiot, Phys. Rep. 300, 215 (1998); [arXiv:nucl-th/9804029].
[12] V. A. Karmanov, J.-F Mathiot, A. V. Smirnov, Phys. Rev. D 82, 056010 (2010).
[13] V. A. Karmanov, J.-F Mathiot and A. V. Smirnov, Phys. Rev. D 86, 085006 (2012).
[14] Gordon Baym, Phys. Rev. 117, 886 (1960).
[15] F. Gross, C. Șavklı, and J. Tjon, Phys. Rev. D 64, 076008 (2001).
[16] Y. Li, V. A. Karmanov, P. Maris and J. P. Vary, published online in Few-Body Syst. (2015); [arXiv:1411.1707 [nucl-th]].
[17] S. Weinberg, Phys. Rev. 150, 1313 (1966).
[18] V. A. Karmanov, Sov. Phys. JETP 44, 210 (1976).
[19] S. D. Drell and T. M. Yan, Phys. Rev. Lett. 24, 181 (1970); West G. B., ibid. 24, 1206 (1970).
[20] See, e.g., J. J. Vivoda and J. R. Hiller, Phys. Rev. D 47, 4647 (1993); C.-R. Ji, Phys. Lett. B 322, 389 (1994); C. Șavklı, J. Tjon and F. Gross, Phys. Rev. C 60, 055210 (1999); J. Carbonell and V. A. Karmanov, Eur. Phys. J. A 27, 11 (2006); Chueung-Ryong Ji and Yukihisa Tokunaga, Phys. Rev. D 86, 054011 (2012), and the references therein.