Superspace unitary operator in QED with Dirac and complex scalar fields: Superfield approach

D. Shukla\textsuperscript{1(a)}, T. Bhanja\textsuperscript{1(b)} and R. P. Malik\textsuperscript{1,2(c)}

\textsuperscript{1} Physics Department, Centre of Advanced Studies, Banaras Hindu University - Varanasi - 221 005, (U.P.), India
\textsuperscript{2} DST Centre for Interdisciplinary Mathematical Sciences, Faculty of Science, Banaras Hindu University Varanasi - 221 005, India

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Abstract – We exploit the strength of the superspace (SUSP) unitary operator to obtain the results of the application of the horizontality condition (HC) within the framework of the augmented version of the superfield formalism that is applied to the interacting systems of Abelian 1-form gauge theories where the $U(1)$ Abelian 1-form gauge field couples to the Dirac and complex scalar fields in the physical four ($3 + 1$)-dimensions of spacetime. These interacting theories are generalized onto a (4, 2)-dimensional supermanifold that is parametrized by the four ($3 + 1$)-dimensional (4D) spacetime variables and a pair of Grassmannian variables. To derive the (anti-)BRST symmetries for the matter fields, we impose the gauge-invariant restrictions (GIRs) on the superfields defined on the (4, 2)-dimensional supermanifold. We discuss various outcomes that emerge out from our knowledge of the SUSP unitary operator and its Hermitian conjugate. The latter operator is derived without imposing any operation of Hermitian conjugation on the parameters and fields of our theory from outside. This is an interesting observation in our present investigation.

Introduction. – One of the most elegant and geometrically intuitive approaches to the $p$-form ($p = 1, 2, 3, \ldots$) gauge theories, described within the framework of the Becchi-Rouet-Stora-Tyutin (BRST) formalism, is the superfield approach (see, e.g., [1–5]). In particular, in [1–3], it has been shown that one can derive the proper (i.e. off-shell nilpotent and absolutely anticommuting) (anti-)BRST symmetry transformations for the non-Abelian 1-form gauge and corresponding (anti-)ghost fields by exploiting the potential and power of the horizontality condition (HC) where the supercurvature 2-form ($\tilde{F}^{(2)}$), defined on the $(D, 2)$-dimensional supermanifold, is equated with the ordinary curvature 2-form ($F^{(2)}$) defined on the $D$-dimensional Minkowskian flat spacetime manifold. However, the above superfield formalism [1–5] does not shed any light on the derivation of (anti-)BRST symmetry transformations associated with the matter fields of a given interacting non-Abelian 1-form gauge theory where there is a coupling between the gauge field and the Noether conserved current constructed with the matter fields.

In a set of papers [6–9], the above superfield formalism [1–5] has been consistently generalized so as to derive the proper (anti-)BRST symmetry transformations for the matter fields (in addition to the gauge and (anti-)ghost fields where the input from the outcomes of the HC plays an important role (see, e.g., [8,9] for details)). The generalized version of superfield formalism (where the HC and gauge-invariant restrictions (GIRs) are exploited together) has been christened as the augmented version of superfield formalism. In refs. [1–3], a superspace (SUSP) unitary operator has been intelligently chosen which provides the (anti-)BRST symmetry transformations for the matter, (anti-)ghost and gauge fields where the gauge group structure of the specific gauge theory is very elegantly maintained. However, the explicit mathematical derivation of this operator has not been provided in these seminal works [1–3]. It would be a nice idea to exploit the key concepts of the augmented superfield formalism to derive this SUSP unitary operator clearly.

\textsuperscript{(a)}E-mail: dheerajkumars.hukla@gmail.com
\textsuperscript{(b)}E-mail: tapobroto.bhanja@gmail.com
\textsuperscript{(c)}E-mail: rpmalik1995@gmail.com
The purpose of our present paper is to derive the above SUSP unitary operator elegantly and explicitly in the case of interacting Abelian 1-form gauge theories with Dirac and complex scalar fields. In this connection, first of all, we exploit the potential of the HC to derive the (anti-)BRST symmetry transformations for the Abelian 1-form gauge and corresponding (anti-)ghost fields. Subsequently, we utilize this result to derive the (anti-)BRST symmetry transformations for the matter fields (i.e., Dirac and complex scalar fields) to obtain the explicit form of the SUSP unitary operator where the SUSP unitary group structure is maintained. We exploit the explicit mathematical form of this operator to derive the results of HC and prove the reasons behind the imposition of HC in the superfield approach to BRST formalism. This is one of the highlights of our present investigation.

One of the key consequences of the SUSP unitary operator is that the matter field transforms (e.g., \( \Psi^0(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) \psi(x) \)) in such a manner that the SUSP \( U(1) \) gauge group structure is respected in the transformation space. As a result, one can define the covariant derivative which would also transform in exactly the same manner (i.e., \( D\Psi^0(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) D\psi(x) \)). This, in turn, defines the transformation of the supercurvature 2-form (i.e. \( F^2 = U(x, \theta, \bar{\theta}) F^{(2)} U^\dagger(x, \theta, \bar{\theta}) \)) which leads to the derivation of the HC (i.e. \( F^{(2)} = F^{(2)} \)) (because the SUSP unitary operator \( U(x, \theta, \bar{\theta}) \) is Abelian in nature and \( U^U = U U^U = 1 \)). Thus, we obtain an alternative to the HC in the language of the SUSP unitary operator and, in some sense, we provide the proof for the validity of the HC (i.e., \( \tilde{F}^{(2)} = F^{(2)} \)) in the context of the superfield approach to any arbitrary D-dimensional Abelian gauge theory (described within the framework of BRST formalism).

Our present endeavor is motivated by the following factors. First, the SUSP unitary operator \( U(x, \theta, \bar{\theta}) \) has been judiciously chosen in [1-3]. However, it has not been theoretically derived. We have accomplished this goal in our present endeavor. Second, the accurate derivation of this SUSP operator provides the proof behind the imposition of the HC in the context of the superfield approach to BRST formalism. Third, the \( U(1) \) group structure appears very naturally in the theory due to the transformation property (e.g., \( \Psi^0(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) \psi(x) \), etc.). Fourth, the results of HC are reproduced by using the SUSP unitary operator which provides, in some sense, an alternative to it. Finally, our present endeavor for the Abelian theory is our first modest step towards our main goal of obtaining the SUSP unitary operator for the non-Abelian theory.

Our present paper is organized as follows. In the next section, we discuss the importance of HC in the derivation of a complete set of proper (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields of this theory. The third section lays emphasis on the derivation of (anti-)BRST symmetries for the matter fields and the SUSP unitary operator (which is responsible for the shift transformations along the Grassmannian directions of the supermanifold). In the fourth section, we derive the (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields by exploiting the strength of the SUSP unitary operator (which is equivalent to the application of HC). Finally, we make some concluding remarks and point out a few future directions for further investigations in the last section.

**Preliminaries: HC and (anti-)BRST symmetries.** – We start off with the following (anti-)BRST invariant Lagrangian density \( L_B^{(2)} \) for the interacting 4D \( U(1) \) gauge theory with Dirac fields \( \psi \) and \( \bar{\psi} \) (with mass \( m \) and electric charge \( e \)) as

\[
L_B^{(2)} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + B(\partial \cdot A) + \frac{B^2}{2} - i \partial_\mu \bar{C} \partial^\mu C, \tag{1}
\]

where the covariant derivative \( D_\mu \psi = \partial_\mu \psi + i e A_\mu \psi \) and the 2-form \( F^{\mu\nu} = dA^{(1)} \) defines the curvature tensor \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) for the 1-form \( A^{(1)} = dx^\mu A_\mu \) connection \( A_\mu \), where \( d = dx^\mu \partial_\mu \) (with \( d^2 = 0 \)) is the exterior derivative. In the above, \( B \) field is the Nakanishi-Lautrup auxiliary field which is used for the linearization of the gauge-fixing term: \( -\frac{1}{4} (\partial A)^2 + (\bar{C} C) \) are the fermionic \( (C^2 = \bar{C}^2 = 0, \bar{C} C + C \bar{C} = 0) \) (anti-)ghost fields. The above Lagrangian density respects the following (anti-)BRST symmetry transformations\(^3\) [8],

\[
\begin{align*}
 s_b A_\mu &= \partial_\mu C, & s_b C &= i B, & s_b \bar{C} &= i B, & s_b B &= 0, & s_b \bar{\psi} &= -i e C \psi, & s_b \bar{\bar{\psi}} &= i e \bar{C} \psi, & s_b \bar{\phi}_{A_\mu} &= \partial_\mu \bar{C}, & \bar{\bar{s}}_b \bar{C} &= 0, & \bar{\bar{s}}_b C &= -i B, & \bar{\bar{s}}_b B &= 0, & \bar{\bar{s}}_b \bar{\psi} &= -i e C \psi, & \bar{\bar{s}}_b \bar{\bar{\psi}} &= -i e \bar{C} \psi.
\end{align*}
\]

It can be shown that the above transformations are off-shell nilpotent (\( \bar{s}_b s_b = 0 \)) of order two and absolutely anti-commuting (\( s_b \bar{s}_b + \bar{s}_b s_b = 0 \)) in nature.

The 4D (anti-)BRST invariant Lagrangian density \( L_B^{(C)} \) for the complex scalar fields \( \varphi(x) \) and \( \varphi^*(x) \) (with mass \( m \) and electric charge \( e \)) is (see, e.g., [9])

\[
L_B^{(C)} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D_\mu \varphi)^* (D^\mu \varphi) - m^2 \varphi^* \varphi + B(\partial \cdot A) + \frac{B^2}{2} - i \partial_\mu \bar{C} \partial^\mu C, \tag{3}
\]

where \( D_\mu \varphi = (\partial_\mu + i e A_\mu) \varphi \) and \( (D_\mu \varphi)^* = (\partial_\mu - i e A_\mu) \varphi^* \) are the covariant derivatives on the fields \( \varphi(x) \) and \( \varphi^*(x) \) and the rest of the symbols in \( L_B^{(C)} \) have been explained.

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1We adopt here the convention and notations such that the 4D Minkowskian flat spacetime metric \( (\eta_{\mu\nu}) \) has the signatures \((+1, -1, -1, -1)\) so that \( (\partial \cdot A) = \partial_\mu A^\mu \equiv \eta_{\mu\nu} A^\nu = \partial_\mu A_\nu - \partial_\nu A_\mu \), where the Greek indices \( \mu, \nu, \lambda \ldots = 0, 1, 2, 3 \) correspond to the spacetime directions and the Latin indices \( i, j, k \ldots = 1, 2, 3 \ldots \) stand for the space directions only.

2We shall use, throughout the whole body of our text, the notations \( s_{(a)} \) for the continuous and infinitesimal (anti-)BRST symmetry transformations connected with the 4D interacting Abelian 1-form gauge theories of the Dirac and complex scalar fields. 

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after eq. (1). It can be seen that the Lagrangian density
\[ \mathcal{L}_B^{(2)} \]
respects the following off-shell nilpotent \((s^2)_{(a)b} = 0\) (anti-)BRST symmetry transformations \(s(a)b\), namely,
\[
\begin{align*}
  s_b A_{\mu} &= \partial_\mu C, \\
  s_b C &= 0, \\
  s_b \bar{C} &= \bar{1} B, \\
  s_b B &= 0, \\
  s_b \varphi &= -i e C \varphi, \\
  s_b \varphi^* &= +i e \varphi^* C, \\
  s_b \bar{F}_{\mu\nu} &= 0, \\
  s_b \bar{A} = \partial_\mu \bar{C}, \\
  s_b \bar{C} &= 0, \\
  s_b C &= -i B, \\
  s_b \varphi &= -i e \bar{C} \varphi, \\
  s_b \varphi^* &= +i e \varphi^* C, \\
  s_b \bar{F}_{\mu\nu} &= 0.
\end{align*}
\] (4)

We also note that \(s_b\) and \(s_b\) absolutely anti-commute
\((s_b, s_b) = 0\) with each other. Physically, the nilpotency property encapsulates the fermionic nature of (anti-)BRST symmetry transformations and the linear independence of (anti-)BRST symmetry transformations is encoded in the property of absolute anticommutativity. It is worthwhile to mention that, unlike in the case of fermionic Dirac fields, the complex scalar fields \(\varphi(x)\) and \(\varphi^*(x)\) commute with the (anti-)ghost fields \(C\) and \(\bar{C}\).

To derive the proper (i.e., nilpotent and absolutely anticommuting) (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields, within the framework of superfield formalism [1–3], we apply the HC on the (4, 2)-dimensional supermanifold (with the help of super exterior derivative \(\tilde{d}\)) as [1–3,8,9]
\[
\tilde{d} A^{(1)} = d A^{(1)} \iff \tilde{F}^{(2)} = F^{(2)},
\] (5)

where \(F^{(2)} = [(dx^\mu \wedge dx^n) / 2] F_{\mu\nu}\) is the curvature 2-form defined on the 4D ordinary spacetime manifold and \(\tilde{F}^{(2)} = [(dZ^M \wedge dZ^N) / 2] \tilde{F}_{MN}\) is the supercurvature 2-form defined on the (4, 2)-dimensional supermanifold. We have the following explicit generalizations, namely:
\[
\begin{align*}
  d &= dx^\mu \partial_\mu \rightarrow \tilde{d} = dZ^M \partial_M \\
  &= dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}, \\
  A^{(1)} &= dx^\mu A_{\mu} \rightarrow \tilde{A}^{(1)} = dZ^M A_M \\
  &= dx^\mu B_{\mu}(x, \theta, \bar{\theta}) + d\theta F(x, \theta, \bar{\theta}) + d\bar{\theta} \tilde{F}(x, \theta, \bar{\theta}),
\end{align*}
\] (6)

where the superspace coordinate \(Z^M = (x^\mu, \theta, \bar{\theta})\) and the superderivative \(\partial_M = (\partial_\mu, \partial_{\theta}, \partial_{\bar{\theta}})\) characterizes the (4, 2)-dimensional supermanifold and \(A_M = (B_{\mu}, F, \tilde{F})\) corresponds to a vector supermultiplet. Here the spacetime coordinates \(x^\mu\) (with \(\mu = 0, 1, 2, 3\)) are the bosonic variables and \((\theta, \bar{\theta})\) is a pair of Grassmannian variables (with \(\theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} = \theta \bar{\theta} = 0\)). The superfields \(B_{\mu}(x, \theta, \bar{\theta}), F(x, \theta, \bar{\theta})\) and \(\tilde{F}(x, \theta, \bar{\theta})\) can be expanded along the Grassmannian directions of the (4, 2)-dimensional supermanifold as [1–3,8,9]
\[
\begin{align*}
  B_{\mu}(x, \theta, \bar{\theta}) &= A_{\mu}(x) + \theta R_{\mu}(x) + \bar{\theta} R_{\bar{\mu}}(x) + i \theta \bar{\theta} S_{\mu}(x), \\
  F(x, \theta, \bar{\theta}) &= C(x) + i \theta B_1(x) + i \bar{\theta} B_\bar{1}(x) + i \theta \bar{\theta} S(x), \\
  \tilde{F}(x, \theta, \bar{\theta}) &= \tilde{C}(x) + i \theta B_2(x) + i \bar{\theta} B_\bar{2}(x) + i \theta \bar{\theta} \tilde{S}(x).
\end{align*}
\] (7)

which yield the basic fields \((A_{\mu}, C, \bar{C})\) of our starting Lagrangian densities (1) and (3) in the limit \(\theta = \bar{\theta} = 0\). In the above, the fields \((R_{\mu}, R_{\bar{\mu}}, B_1, B_\bar{1}, s, \bar{s}, B_2, B_\bar{2})\) are the secondary fields which are to be determined in terms of the basic and auxiliary fields of the Lagrangian density (1). In fact, it can be explicitly checked that we obtain (see, e.g., [8,9] for details)
\[
\begin{align*}
  R_\mu &= \partial_\mu C, \\
  S_\mu &= \partial_\mu B, \\
  \bar{R}_{\bar{\mu}} &= \partial_{\bar{\mu}} \bar{C}, \\
  B_1 &= s = \bar{\bar{s}} = B_\bar{2} = 0, \\
  \bar{B}_1 + B_2 &= 0, \quad \Rightarrow \quad \bar{B}_1 = -B = -B_2,
\end{align*}
\] (8)

when we exploit the HC (5). The substitution of (8) into (7) yields
\[
\begin{align*}
  B^{(h)}_{\mu}(x, \theta, \bar{\theta}) &= A_{\mu} + \theta (\partial_\mu \bar{C}) + \bar{\theta} (\partial_\mu C) + \theta \bar{\theta} (i \partial_{\mu} B), \\
  \equiv A_{\mu} + \theta (s_{ab} A_{\mu}) + \bar{\theta} (\bar{s}_{ab} A_{\mu}) + \theta \bar{\theta} (s_b s_b A_{\mu}), \\
  F^{(h)}(x, \theta, \bar{\theta}) &= C + \theta (-i B) \equiv C + \theta (s_{ab} C), \\
  \tilde{F}^{(h)}(x, \theta, \bar{\theta}) &= \tilde{C} + \theta (i B) \equiv \tilde{C} + \theta (s_b \bar{C}),
\end{align*}
\] (9)

where the superscript \((h)\) stands for the superfields that have been derived after the application of HC. It is clear, from the above, that we have already obtained the (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields (cf. (2) and (4)) for the interacting system of the \(U(1)\) Abelian 1-form gauge theories (with Dirac and complex scalar fields). It is to be noted that the (anti-)BRST transformations for the gauge and (anti-)ghost fields are the same for both the interacting \(U(1)\) gauge theories under consideration. Furthermore, it is interesting to point out that the (anti-)BRST symmetry transformations \((s_{\mu}a b)\) are connected with the translational generators \(\partial_\mu\) and \(\partial_{\bar{\mu}}\) along the Grassmann directions of the supermanifold by the relationships:
\[
s_b \iff \partial_\mu, \quad s_b \iff \partial_{\bar{\mu}}.
\]

Gauge-invariant restrictions and SUSY unitary operator: (anti-)BRST symmetries for matter fields. – In our previous section, we have derived the (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields but have not discussed the (anti-)BRST symmetries associated with the Dirac fields. To obtain these symmetry transformations, we impose the following gauge-invariant restriction (GIR) on the matter superfields (see, e.g., [8] for details):
\[
\Psi(x, \theta, \bar{\theta}) \tilde{D}(x, \theta, \bar{\theta}) \Psi(x, \theta, \bar{\theta}) = \tilde{\psi}(x) D \psi(x),
\] (10)

where \(D = d + i e A^{(1)}\) and \(\tilde{D} = \tilde{d} + i e \tilde{A}^{(1)}\). The super 1-form \(\tilde{A}^{(1)}\) connection (with \(\tilde{d} \tilde{A}^{(1)} = d A^{(1)}\) is defined, in
terms of the superfields (9), as follows:

\[
\begin{align*}
\tilde{A}_{(b)}(x, \theta, \bar{\theta}) &= dx^\mu B^b_\mu(x, \theta, \bar{\theta}) + d\theta F^b(x, \theta, \bar{\theta}) \\
&\quad + d\bar{\theta} F^b(x, \theta, \bar{\theta}),
\end{align*}
\]

(11)

where the explicit expansions of \(B^b_\mu(x, \theta, \bar{\theta}), F^b(x, \theta, \bar{\theta})\) and \(F^b(x, \theta, \bar{\theta})\) are given in (9). The matter superfields \(\Psi(x, \theta, \bar{\theta})\) and \(\Psi(x, \theta, \bar{\theta})\) have the following expansions along the Grassmannian directions of the (4, 2)-dimensional supermanifold, namely,

\[
\begin{align*}
\Psi(x, \theta, \bar{\theta}) &= \psi(x) + i \theta \bar{b}_1(x) + i \bar{\theta} b_1(x) + i \theta \bar{\theta} t(x), \\
\Psi(x, \theta, \bar{\theta}) &= \bar{\psi}(x) + i \theta \bar{b}_2(x) + i \bar{\theta} b_2(x) + i \theta \bar{\theta} \bar{t}(x),
\end{align*}
\]

(12)

where \((b_1, b_1, b_2, b_2, t, \bar{t})\) are the secondary fields which would be determined in terms of the basic and auxiliary fields of our present theory described by the Lagrangian density (1). In this connection, the GIR in (10) helps us to obtain the following equations (see, e.g., [8] for details):

\[
\begin{align*}
b_1 &= -e C \psi, \quad b_1 = -e C \psi, \quad t = -i e (B - e C C), \\
b_2 &= -e \bar{\psi} C, \quad b_2 = -e \bar{\psi} C, \quad \bar{t} = i e (B - e C C).
\end{align*}
\]

(13)

The substitution of these expressions into (12) yields the following explicit expansions for the matter superfields in terms of the (anti-)BRST symmetry transformations (2):

\[
\begin{align*}
\Psi^{(a)}(x, \theta, \bar{\theta}) &= \psi(x) + \theta (-i e C \psi) + \bar{\theta} (-i e C \psi) \\
&\quad + \theta e (B - e C C) \psi, \\
\Psi^{(a)}(x, \theta, \bar{\theta}) &= \bar{\psi}(x) + \theta (sab \psi) + \bar{\theta} (sab \psi) \\
&\quad + \theta \bar{\theta} (sab \psi),
\end{align*}
\]

(14)

where the superscript \((g)\) denotes the expansions of the superfields obtained after the application of the GIR (10).

We are now in the position to state the precise form of the SUSY unitary operator which transforms the ordinary Dirac matter fields \(\psi(x)\) and \(\bar{\psi}(x)\) to their counterparts \(\Psi^{(a)}(x, \theta, \bar{\theta})\) and \(\Psi^{(a)}(x, \theta, \bar{\theta})\). In fact, using the expansions (14), it is clear that\(^4\)

\[
\begin{align*}
\Psi^{(a)}(x, \theta, \bar{\theta}) &= [1 + \theta (-i e C) + \bar{\theta} (-i e C)] \psi(x), \\
&\quad + \theta e (B - e C C) \psi(x), \\
\Psi^{(a)}(x, \theta, \bar{\theta}) &= \bar{\psi}(x) [1 + \theta (i e C) + \bar{\theta} (i e C)] \\
&\quad + \theta \bar{\theta} (B - e C C),
\end{align*}
\]

(15)

where \(U(x, \theta, \bar{\theta})\) and \(U^\dagger(x, \theta, \bar{\theta})\) are the SUSY generators which, primarily, lead to the shift transformations along the Grassmannian directions (because \(\Psi^{(a)}(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) \psi(x)\) and \(\Psi^{(a)}(x, \theta, \bar{\theta}) = \psi(x) U^\dagger(x, \theta, \bar{\theta})\)). These SUSY operators \((i.e., U, U^\dagger)\) can be expressed in the mathematically precise exponential forms\(^5\)

\[
\begin{align*}
U(x, \theta, \bar{\theta}) &= \exp [\theta (-i e C) + \bar{\theta} (-i e C)] \\
&\quad + \theta \bar{\theta} (e B)], \\
U^\dagger(x, \theta, \bar{\theta}) &= \exp [\theta (i e C) + \bar{\theta} (i e C)] \\
&\quad + \theta \bar{\theta} (-e B)],
\end{align*}
\]

(16)

which directly establish that the SUSY operator \(U(x, \theta, \bar{\theta})\) is unitary \((i.e., U U^\dagger = U^\dagger U = 1)\). This statement can be proven to be true by using the explicit expressions for \(U\) and \(U^\dagger\) that are quoted in (15) (and that are equivalent to (16)). The crucial observation is that the SUSY operator \(U(x, \theta, \bar{\theta})\) forms a \(U(1)\) group in the space of transformations where the exponential form (16) of the operator \(U(x, \theta, \bar{\theta})\) plays an important role. A similar statement could be made with the operator \(U^\dagger(x, \theta, \bar{\theta})\), too.

To obtain the (anti-)BRST symmetry transformations associated with the complex scalar fields \(\varphi(x)\) and \(\varphi^*(x)\) (cf. eq. (4)), we impose the following gauge-invariant restrictions (GIRs) on the superfields defined in the (4, 2)-dimensional supermanifold [9]:

\[
\begin{align*}
\Phi^{\star}(x, \theta, \bar{\theta}) \left( \tilde{a} + i e \tilde{A}_{(b)} \right) \Phi(x, \theta, \bar{\theta}) &= \varphi^*(x), \\
\Phi(x, \theta, \bar{\theta}) \left( \tilde{a} - i e \tilde{A}_{(b)} \right) \Phi^\dagger(x, \theta, \bar{\theta}) &= \varphi(x).
\end{align*}
\]

(17)

where the superfields \(\Phi(x, \theta, \bar{\theta})\) and \(\Phi^\dagger(x, \theta, \bar{\theta})\) have the expansions along the Grassmannian directions of the (4, 2)-dimensional supermanifold as [9]

\[
\begin{align*}
\Phi(x, \theta, \bar{\theta}) &= \varphi(x) + i \theta f_1(x) + i \bar{\theta} f_2(x) + i \theta \bar{\theta} b(x), \\
\Phi^\dagger(x, \theta, \bar{\theta}) &= \varphi^*(x) + i \theta f_2(x) + i \bar{\theta} f_1(x) + i \theta \bar{\theta} b^*(x).
\end{align*}
\]

(18)

In the above, we have the secondary fields on the r.h.s. as \((f_1, f_1^*, f_2, f_2^*, b, b^*)\). These fields could be determined in terms of the basic and auxiliary fields of the Lagrangian density \(L_{\varphi}^{(G)}\) due to the GIRs in (17). It is worthwhile to mention that the r.h.s. of (17) are gauge-invariant quantities and, therefore, they are (anti-)BRST invariant, too.

In our earlier work [9], all the secondary fields of (18) have been determined in a systematic manner by exploiting the strength of GIRs in (18). The outcome is\(^5\)

\[
\begin{align*}
f_1 &= -e C \varphi, \quad f_2 = -e C \varphi, \quad b = -i e (B - e C C) \varphi, \\
f_1^* &= e C \varphi^*, \quad f_2^* = e C \varphi^*, \quad b^* = i e (B - e C C) \varphi^*.
\end{align*}
\]

(19)

\(^4\)We note that the relationship \(\Psi^{(a)}(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) \psi(x)\) is exactly of the same kind as the \(U(1)\) gauge transformation on the Dirac field: \(\psi(x) \rightarrow \psi'(x) = U(x, \theta, \bar{\theta}) \psi(x)\) where the operator \(U(x) = e^{-i e C(x)}\) (with gauge parameter \(a(x)\)) forms the \(U(1)\) group as it satisfies all the group properties under product.

\(^5\)Under the Hermitian conjugation operations \(\theta^\dagger = -\bar{\theta}, \bar{\theta}^\dagger = -\theta, C^\dagger = \pm C, C^\dagger = \pm C, B^\dagger = B, e^\dagger = e, i^\dagger = -i\), it can be readily checked that SUSY operators \(U(x, \theta, \bar{\theta})\) and \(U^\dagger(x, \theta, \bar{\theta})\) interchange with each other and the FP ghost part \((i.e., -i \partial_{\bar{\theta}} C \theta \psi C)\) of the Lagrangian densities (1) and (3) remains invariant.
The substitution of these expressions into the expansion (18) leads to the following explicit expansions in terms of the (anti-)BRST symmetry transformations (4), namely,
\[ \Phi^{(g)}(x, \theta, \bar{\theta}) = \varphi(x) + \theta (-ieC\varphi) + \bar{\theta} (-ieC\varphi) + \theta \bar{\theta} \left[ e(B - eCC)\varphi \right] \]
\[ \equiv \varphi(x) + \theta (s_{ab}\varphi) + \bar{\theta} (s_{ab}\varphi) + \theta \bar{\theta} (s_{ab}s_{ab}\varphi), \]
\[ \Phi^{(g)}(x, \theta, \bar{\theta}) = \varphi^*(x) + \theta (ie\varphi^*C) + \bar{\theta} (ie\varphi^*C) + \theta \bar{\theta} \left[ -e(B - eCC)\varphi^* \right] \]
\[ \equiv \varphi^*(x) + \theta (s_{ab}\varphi^*) + \bar{\theta} (s_{ab}\varphi^*) + \theta \bar{\theta} (s_{ab}s_{ab}\varphi^*), \quad (20) \]
where the superscript (g) denotes the expansions of the superfields after the application of G IRs in (17). It is pretty obvious that the above superfields can be expressed in terms of the SUSP unitary operators \( U(x, \theta, \bar{\theta}) \) and \( U^\dagger(x, \theta, \bar{\theta}) \) exactly like (15) and (16) where \( \Phi^{(g)}(x, \theta, \bar{\theta}) \) would be replaced by the superfields \( \Phi^{(g)}(x, \theta, \bar{\theta}) \) and \( \Phi^{(g)}(x, \theta, \bar{\theta}) \) of (20). Thus, we note that the form of the SUSP unitary operators (16) remains the same for both interacting models of QED where there is an interaction between the \( U(1) \) gauge field \( A_\mu \) and the Noether conserved current constructed by the Dirac fields as well as by the charged complex scalar fields. This happens because of the existence of the local \( U(1) \) gauge group behind the construction of both these interacting theories.

**SUSP unitary operator and HC: salient features.**

As a result of the group property in the transformation space, we can define a super covariant derivative on the Dirac superfield in the following fashion:
\[
\begin{align*}
\psi(x) &\rightarrow \Psi(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) \psi(x), \\
D \psi(x) &\rightarrow \tilde{D} \Psi(x, \theta, \bar{\theta}) = U(x, \theta, \bar{\theta}) D \psi(x),
\end{align*}
\]
where \( \tilde{D} = \tilde{d} + i e A^{(1)}(x, \theta, \bar{\theta}) \) and \( D = d + i e A(1)(x) \).

It is now crystal clear that the super 1-form connection \( \tilde{A}^{(1)}(x, \theta, \bar{\theta}) \) and the ordinary 1-form connection \( A^{(1)}(x) \) are connected by the following equation due to the relationships quoted in (21), namely,
\[
\begin{align*}
\tilde{A}^{(1)}(x, \theta, \bar{\theta}) &= U(x, \theta, \bar{\theta}) A^{(1)}(x) U^\dagger(x, \theta, \bar{\theta}) \\
&+ \frac{i}{e} \left( i U(x, \theta, \bar{\theta}) \right) U^\dagger(x, \theta, \bar{\theta}) \equiv d_{\mu} B^{(h)}_{\mu}(x, \theta, \bar{\theta}) \\
&+ d\tilde{D} F^{(h)}(x, \theta, \bar{\theta}) + d\bar{D} F^{(h)}(x, \theta, \bar{\theta}).
\end{align*}
\]

It is evident that if we use the Abelian \( U(1) \) nature of the operator \( \tilde{U}(x, \theta, \bar{\theta}) \), the first term on the r.h.s. of (22) yields the following expression:
\[
U(x, \theta, \bar{\theta}) A^{(1)}(x) U^\dagger(x, \theta, \bar{\theta}) = A^{(1)}(x) \equiv d_{\mu} A_{\mu}(x).
\]

The second term on the r.h.s. (i.e., \( + (\tilde{d} U) U^\dagger \)) leads to the following explicit expression, namely,
\[
\begin{align*}
d_\mu \left( \partial_\mu C \right) + \theta \left( \partial_\mu C \right) + \bar{\theta} \left( i \partial_\mu B \right) \\
+ d\tilde{D} \left[ C(x) + i \theta B(x) \right] + d\bar{D} \left[ C(x) - i \theta B(x) \right],
\end{align*}
\]
where we have used the expansions for \( \tilde{d}, U(x, \theta, \bar{\theta}) \) and \( U^\dagger(x, \theta, \bar{\theta}) \) from equations (6) and (15).

Now, it is obvious that the comparison of the coefficients of \( d_{\mu}A_{\mu} \) and \( d\bar{D} \) leads to the derivation of (anti-)BRST symmetry transformations \( \text{exactly} \) in the same manner as has been done in the second section where we have exploited the HC. In other words, we obtain exactly the same expressions for the \( B^{(h)}_{\mu}, F^{(h)}(h) \) and \( F^{(h)} \) as has been defined in eq. (9).

From the above discussion, we can claim that the HC used in eq. (5) is equivalent to the relationship (22) where the SUSP unitary operator plays a decisive role. To corroborate the above claim, we note that the following property of the ordinary covariant derivatives on the Dirac field is true, namely,
\[
DD \psi(x) = i e F^{(2)}(x) \psi(x),
\]
where the covariant derivative \( D = d\mu \left( \partial_\mu + i eA_\mu \right) \) and \( F^{(2)} = \left( [d\mu \wedge d\sigma]\right)/2 F_{\mu}(x) \).

This property can be expressed in terms of the SUSP unitary operator as
\[
DD \psi(x) \rightarrow \tilde{D}\tilde{D} \psi(x, \theta, \bar{\theta}) = i e\tilde{F}^{(2)}(x, \theta, \bar{\theta}) \psi(x),
\]
where \( D, \tilde{D} \) and \( \tilde{F}^{(2)} \) are defined earlier. Now, using the relationship given in (21), we obtain \( \tilde{D} = DU^\dagger \). If we substitute this value into the l.h.s. (i.e., \( \tilde{D}\tilde{D} \)) of (26), we obtain the following relationship:
\[
U\tilde{D} \psi = i e\tilde{F}^{(2)}(x) \psi \xrightarrow{U} \tilde{F}^{(2)} = U(x, \theta, \bar{\theta}) F^{(2)}(x) U^\dagger(x, \theta, \bar{\theta}).
\]

Focusing on the Abelian nature of \( U(x, \theta, \bar{\theta}) \) and \( U^\dagger(x, \theta, \bar{\theta}) \), it is crystal clear that the r.h.s of (27) would yield \( F^{(2)} \) only (because \( UU^\dagger = U^\dagger U = I \)). Thus, we have obtained the HC condition (5) for the Abelian theory (i.e., \( \tilde{F}^{(2)} = F^{(2)} \iff \tilde{A}^{(1)} = dA^{(1)} \)).

The above argument and discussion can be replicated in the context of QED with complex scalar fields where, once again, we obtain the analogue of relation (22) which provides an alternative to the HC. Furthermore, we note that the following is true in the context of QED with complex scalar fields, namely,
\[
DD \varphi = i e F^{(2)}(x) \varphi, \quad (DD\varphi)^* = -i e F^{(2)}(x) \varphi^*.
\]

The above equation can be translated into the superfield formalism as
\[
\begin{align*}
DD \varphi(x) &\rightarrow \tilde{D}\tilde{D} \Phi(x, \theta, \bar{\theta}) = i e\tilde{F}^{(2)}(x) \Phi(x, \theta, \bar{\theta}) \\
&\equiv i e \tilde{F}^{(2)}(x) \Phi(x, \theta, \bar{\theta}) \varphi(x) \\
(DD\varphi)^* &\rightarrow (\tilde{D}\tilde{D} \Phi)^*(x, \theta, \bar{\theta}) = -i e\tilde{F}^{(2)}(x) \Phi^*(x, \theta, \bar{\theta}) \\
&\equiv -i e \tilde{F}^{(2)}(x) \varphi(x)^*(x, \theta, \bar{\theta}),
\end{align*}
\]
where we have \( \tilde{D} \Phi(x, \theta, \bar{\theta}) = (\tilde{d} + i e A^{(1)}(x, \theta, \bar{\theta}) \Phi(x, \theta, \bar{\theta}) \) and \( (\tilde{D} \Phi)^*(x, \theta, \bar{\theta}) = (\tilde{d} - i e A^{(1)}(x, \theta, \bar{\theta}) \Phi^*(x, \theta, \bar{\theta}) \).

With the input \( \tilde{D} = DU^\dagger \), it can be checked that
\[
\begin{align*}
U(x, \theta, \bar{\theta}) DD \varphi = i e \tilde{F}^{(2)}(x) \Phi(x, \theta, \bar{\theta}) &\xrightarrow{U} \tilde{F}^{(2)} = U(x, \theta, \bar{\theta}) F^{(2)}(x) U^\dagger(x, \theta, \bar{\theta}),
\end{align*}
\]
The Abelian nature of $U(x, \theta, \bar{\theta})$, once again, implies that we have obtained an alternative to the HC (i.e., $F(2) = F(2)$) in the language of the SUSP unitary operator $U(x, \theta, \bar{\theta})$ because we have $U(x, \theta, \bar{\theta}) F(2) (x) U^\dagger(x, \theta, \bar{\theta}) = F(2) (x)$ on the r.h.s. of (30).

The celebrated HC can also be obtained from the relationship given in (22). This is due to the fact that when we operate by $\tilde{d}$ on this equation from the left, we obtain the following explicit equation, namely,

$$\tilde{d} \tilde{A}^{(1)}_{(b)}(x, \theta, \bar{\theta}) = \tilde{d} A^{(1)}(x) - \frac{i}{e} \tilde{d} U(x, \theta, \bar{\theta}) \tilde{d} U^\dagger(x, \theta, \bar{\theta}),$$  

(31)

where we have used (23) and the property $\tilde{d}^2 = 0$. The first term on the r.h.s. of the above equation produces: $\tilde{d} A^{(1)} = \tilde{d} F(2)$ due to the fact that the ordinary 1-form $(A^{(1)}(x))$ connection is independent of the Grassmannian variables implying that $\partial_{\theta} A^{(1)} = \partial_{\bar{\theta}} A^{(1)} = 0$ when we use the definition of $\tilde{d}$ from (6). Taking the explicit expressions for the $U(x, \theta, \bar{\theta})$ and $U^\dagger(x, \theta, \bar{\theta})$ from (15), it can be checked that $\tilde{d} \tilde{d} U \tilde{d} U^\dagger = 0$. This statement becomes very clear if we take a close look at the exponential forms of $U$ and $U^\dagger$ in (16). It is obvious that the quantity in the exponent of these operators differs only by a sign factor. Thus, it can be readily checked that $\tilde{d} \tilde{d} U \tilde{d} U^\dagger = 0$. This implies that the last term in (31) is zero (i.e., $\tilde{d} U \tilde{d} U^\dagger = 0$) which, ultimately, leads to the validity of HC (i.e., $F(2) = F(2)$).

For readers’ convenience, we have carried out the explicit computations of $\tilde{d} U$ and $\tilde{d} U^\dagger$ which are present now in the footnote6 and it can be re-checked that the second term on the r.h.s. of (31) is zero (i.e., $\tilde{d} \tilde{d} U \tilde{d} U^\dagger = 0$).

**Conclusions.**—The central objective of our present investigation has been to derive an explicit expression for the SUSP unitary operator $U(x, \theta, \bar{\theta})$ which generates the shift symmetry transformations on the gauge, (anti-)ghost and matter superfields along the Grassmannian directions of the (4, 2)-dimensional supermanifold on which the ordinary 4D interacting Abelian 1-form gauge theories (with Dirac and complex scalar fields) are generalized. In fact, the SUSP unitary operator, ultimately, leads to the derivation of proper (i.e., off-shell nilpotent and absolutely anticommuting) (anti-)BRST symmetry transformations for the above interacting Abelian 1-form gauge theories in physical four $(3 + 1)$-dimensions of spacetime.

One of the highlights of our present endeavor is the observation that the correct derivation of the SUSP unitary operator provides an alternative to the HC in addition to encompassing in its folds the sanctity of the $U(1)$ gauge group structure in the transformation space. It is the latter property which allows us to define the covariant derivative on the super matter fields (cf. (21) and (29)). This definition, in turn, leads to the derivation of a connection between the supercurvature 2-form and the ordinary curvature 2-form (cf. (26) and (29)). This is due to the fact that the commutator of two covariant derivatives defines the field strength tensor $F_{\mu\nu}$ through the relationship $[D_{\mu}, D_{\nu}] \psi = i e F_{\mu\nu} \psi$. This result has been captured in the relationships given in eqs. (25), (26) and (28). One of the key features of our present endeavor is the observation that $U(x, \theta, \bar{\theta})$ and $U^\dagger(x, \theta, \bar{\theta})$, for both the interacting theories, turn out to be the same.

It would be a nice future endeavor to extend our present idea to derive explicitly the SUSP unitary operator in the context of interacting 4D non-Abelian 1-form gauge theory with Dirac fields which has been intelligently chosen in [1–3]. We also plan to pursue this direction of investigation in the context of interacting higher $p$-form $(p = 2, 3 \ldots)$ gauge theories which are the limiting cases of (super)string theories (see, e.g., [10]). We are presently intensively involved with these issues and our future publications would resolve these in a cogent and convincing manner [11].

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6Using (15), it is obvious that $\tilde{d} U = dz^\mu[-i e \theta \partial_\mu C - i e \bar{\theta} \partial_\mu \bar{C} + e \theta \partial_\mu (B - e C C) + \partial_\mu (i e C + e \theta 
\leftarrow e (B - e C C)] + \partial_\mu (i e C - e \theta (B - e C C)] + \tilde{d} U^\dagger = dz^\mu[-i e \bar{\theta} \partial_\mu \bar{C} + i e \theta \partial_\mu C - e \theta \partial_\mu (B - e C C) + \partial_\mu (i e \bar{C} - e \theta (B - e C C) + \partial_\mu (i e \bar{C} + e \theta (B - e C C)].$ From these lucid expressions, one can also check that $\tilde{d} \tilde{d} U \tilde{d} U^\dagger = 0$. 

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