The Thermodynamical Behaviors of Kerr–Newman AdS Black Holes

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We reconsider the study of critical behaviors of Kerr–Newman Anti-de Sitter (AdS) black holes in four dimensions. The study is made in terms of the moduli space parameterized by the charge $Q$ and the rotation parameter $a$, relating the mass $M$ of the black hole and its angular momentum $J$ via the relation $a = J/M$. Specifically, we discuss such thermodynamical behaviors in the presence of a positive cosmological constant considered as a thermodynamic pressure and its conjugate quantity as a thermodynamic volume. The equation of state for a charged Reissner–Nordstrom AdS black hole predicts a critical universal number depending on the $(Q, a)$ moduli space. In the vanishing limit of the $a$ parameter, this prediction recovers the usual universal number in four dimensions. Then, we find the bounded region of the moduli space allowing the consistency of the model with real thermodynamical variables.

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Four-dimensional black holes have received increasing attention in the context of supergravity theories embedded either in superstrings or in M-theory compactified on internal spaces.1,2 This involves Reissner-Nordstrom (RN) black holes being static, spherically symmetric configurations minimizing the Maxwell–Einstein action. These solutions are completely defined by giving two parameters: the charge of the black hole $Q$ and the mass $M$. Such parameters have been explored to find some stringy moduli using the attractor mechanism.3,4

Recently, intensive efforts have been devoted to investigations of the thermodynamic behaviors of the black hole backgrounds including Reissner–Nordstrom Anti-de Sitter (RN-AdS) black solutions in four and in various dimensions.5,6,8–14 More precisely, the equation of state for certain black holes has been worked out and it has been realized that this analysis shares a similar feature with the van der Waals $P-V$ diagram.14,6

The $P-V$ criticality of RN-AdS black holes with spherical configurations has been extensively investigated using the same method. In this way, the behavior of the Gibbs free energy in the fixed charge ensemble has been dealt with. In particular, the phase transition in the $(P, T)$-plane has been studied in Ref. [6]. In fact, a nice connection between the behavior of the RN-AdS black hole system and the van der Waals fluid has been shown. Moreover, it has been realized that $P-V$ criticality, Gibbs free energy, first-order phase transition, and the behavior near the critical points can be associated with the liquid-gas system.

Following the strategy of Ref. [6] to describe the four-dimensional case, we have discussed the critical behavior of charged RN-AdS black holes in arbitrary dimensions of spacetime.5 Considering the cosmological constant $\Lambda$ as a thermodynamic pressure and its conjugate quantity as a thermodynamic volume, we have given a comparative study in terms of the dimension and the displacement of the critical points. These parameters can be used to control the transition between small and large black holes. More precisely, it has been shown that such behaviors vary nicely in terms of the dimension of the spacetime in which the black holes reside. In an arbitrary dimension $d$, we have obtained an universal number given by

$$\chi = \frac{P_v v_c}{T_c} = \frac{2d - 5}{4d - 8},$$

(1)

connecting the space-time dimension $d$ with the universal number $\chi$. This equation recovers the four-dimensional value

$$\chi = \frac{P_v v_c}{T_c} = \frac{3}{8}.$$  

(2)

More recently, the critical behaviors of the charged and the rotating AdS black holes in the presence of the electrodynamic effects have been investigated in Ref. [15]. In four dimensions, it has been found that neutral slowly rotating black holes involve the same critical behavior. Among others, the relation $\chi = \frac{P_v v_c}{T_c} = \frac{5}{12}$ has been obtained. However, this number, in fact, should depend on the extra rotating parameter and should recover the usual value (2) in the vanishing limit of such a parameter.

The aim of this work is to contribute to these topics by reconsidering the study of the critical behaviors of the Kerr–Newman AdS black holes. The present study is made in terms of the moduli space parameterized by the charge $Q$ and the rotation parameter $a$. The latter is given in terms of the mass $M$ of the black hole and its angular momentum $J$ via the expression $a = J/M$. In this way, these two parameters
together with the displacement of the critical points can be used to control the transition between small and large black holes. Among others, we find that Eq. (2) has been modified. It will be given in terms of the $Q$ and $a$ parameters. Then, we present numerically the bounded region of the moduli space allowing the consistency of the model with real thermodynamical quantities.

To proceed, we consider the Einstein–Maxwell AdS action in four dimensions given by

$$\mathcal{I} = -\frac{1}{16\pi G} \int_M dx^4 \sqrt{-g} [R - F^2 + 2\Lambda],$$

(3)

where $F = dA$ is the field strength with $A$ being the potential 1-form. Here $\Lambda$ can be identified with $-\frac{\hbar}{4G}$ defining the cosmological constant associated with the characteristic length scale $\ell$. The variation of the above action with respect to the metric tensor provides the Kerr–Newman AdS solution, given in the Boyer–Lindquist coordinate by\(^{(7)}\)

$$ds^2 = -\frac{1}{\Sigma} [\Delta_t - \Delta_\theta a^2 \sin^2 \theta] dt^2 - \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Sigma^2}{\Delta_r^2} [\Delta_\theta (r^2 + a^2)]^2 - \Delta_r a^2 \sin^2 \theta \sin^2 \theta d\phi^2 - \frac{2\alpha}{\Sigma} [\Delta_\theta (r^2 + a^2) - \Delta_r] \sin^2 \theta dtd\phi,$$

(4)

where

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Xi = 1 + \frac{1}{3} 4a^2,$$

(5)

$$\Delta_\theta = 1 + \frac{1}{3} 4a^2 \cos^2 \theta,$$

(6)

$$\Delta_r = (r^2 + a^2) (1 - \frac{1}{3} 4a^2) - 2Mr + Q^2.$$

(7)

In the negative values of $\Lambda$, the horizons of the metric (4) can be derived from the vanishing condition of the following quantity

$$\Delta_r = (r^2 + a^2) (1 - \frac{1}{3} 4a^2) - 2Mr + Q^2$$

$$= -\frac{1}{3} A [r^4 - \left(\frac{3}{4} - a^2\right) r^2 + \frac{6M}{A} r - \frac{3}{A} (a^2 + Q^2)]$$

$$= -\frac{1}{3} A (r - r_+)(r - r_-)(r - r_+)(r - r_-).$$

(8)

In fact the equation $\Delta_r = 0$ has four different roots, as shown in Ref.\(^{(10)}\): $r_+$ and $r_-$ are a pair of complex conjugate roots, while $r_+$ and $r_-$ are two real positive roots. Assuming that $r_+ > r_-$, the condition $r = r_+$ describes the event horizon.

Using the thermodynamical calculation techniques\(^{(17)}\) the black hole temperature reads

$$T = \frac{1}{\beta} = \frac{3r_+^4 + (a^2 + \ell^2)r_+^2 - \ell^2(a^2 + Q^2)}{4\pi \ell^2 r_+ (r_+^2 + a^2)}.$$

(9)

Following the analysis in Ref.\(^{(6)}\), we can obtain the equation of state for a Kerr–Newman AdS black hole $P = P(V,T)$ in four dimensions. For a generic point in the $(Q,a)$ moduli space, the calculation leads to

$$P = \frac{3(r_+(4\pi a^2 T + r_+(4\pi r_+ - 1) + a^2 + Q^2))}{8\pi r_+^2 (a^2 + 3r_+^2)}.$$  

(10)

In this expression, the event horizon radius $r_+$ is given by

$$r_+ = \left(\frac{3V}{4}\right)^{1/3},$$

(11)

where $V$ is the thermodynamic volume of the black hole, being the volume of the four-dimensional sphere. It is worth noting that Eq. (11) can be identified with the equation given in Ref.\(^{(15)}\) by considering a particular normalization for the mass and charge parameters.

As given in Ref.\(^{(6)}\), the physical pressure and temperature take the following forms:

$$\text{Press} = \frac{\hbar c}{\ell_P} P, \quad \text{Temp} = \frac{\hbar c}{k} T,$$

(12)

where the Planck length is given by $\ell_P^2 = \hbar G_4/c^3$. Multiplying Eq. (11) by $\frac{\hbar c}{kT}$, one can obtain a compact expression for the thermodynamical variable Press. Indeed, the calculations give the form

$$\text{Press} = \frac{3kr_+ \text{Temp}}{2(a^2 + 3r_+^2)\ell_P^2} + \cdots.$$

(13)

A close inspection around the van der Waals equation given by $(P + \frac{\beta}{V})(V - b) = kT$ has shown that we can
identify the specific volume \( v \) with
\[
v = 2\ell_p^2 r_+.
\]
(14)

In this way, the equation of state (11) takes the form
\[
P = \frac{3\left(a^2(8\pi T v + 4) + 4Q^2 + v^2(2\pi T v - 1)\right)}{8\pi a^2 v^2 + 6\pi v^4}.
\]
(15)

Having derived such quantities, we present a numerical discussion on the obtained results. These calculations allow one to discuss the corresponding \( P-V \) diagram, as plotted in Fig. 1.

It is seen from Fig. 1 that for \( Q \neq 0 \) and for \( T < T_c \), the behavior looks like an extended van der Waals gas and the corresponding system involves inflection points defining critical points. These points satisfy the conditions
\[
\frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0.
\]
(16)

After calculations, we obtain the following expressions
\[
v_c = 2\left(3a^2 + 2Q^2 + \frac{32a^4 + 41a^2Q^2 + 12Q^4}{\sqrt{3}X} + \frac{X}{32/3}\right)^{1/2},
\]
(17)

\[
T_c = 3\sqrt{\frac{Y}{332\pi a^6}}\left[8Y(311a^2 + 207Q^2)
+ 5424a^2(a^2 + Q^2) - 69Y^2\right],
\]
(18)

while the critical pressure takes the form
\[
P_c = \frac{X}{6656\pi a^6 Z(a^2 X + Z)} - \frac{2496a^6 Z}{3\sqrt{3}X}
+ 3\sqrt{3}XZ'\left(\frac{Z}{X} + 7488a^6 X(a^2 + Q^2)\right)
+ \frac{\sqrt{3}ZZ'}{\left(\frac{Z}{X}\right)}.
\]
(19)

The quantities \( X, Y, Z \) and \( Z' \), appearing in the above equations, are given, respectively, by
\[
X = \left\{312a^6 + 600a^4Q^2 + 369a^2Q^4 + \sqrt{3} - 320a^{12}
- 1152a^{10}Q^2 - 1488a^8Q^4 - 809a^6Q^6
- 153a^4Q^8\right\}^{1/3},
\]
(20)

\[
Y = \frac{128a^4 + 16a^2Q^2}{\sqrt{3}X} + \frac{16a^2Q^2}{\sqrt{3}X} + 12a^2 + 16Z^{3/2}Q^4
+ 8Q^2 + \frac{4X}{3\sqrt{3}},
\]
(21)

\[
Z = 323^{2/3}a^4 + a^2(41 \times 3^{2/3}Q^2 + 9X) + 123^{2/3}Q^4
+ 6Q^2X + \sqrt{3}X^2,
\]
(22)

\[
Z' = 16272a^4\sqrt{Y} + 16272a^2Q^4\sqrt{Y} + 7464a^2Y^{3/2}
+ 4968Q^2Y^{3/2} - 207Y^{5/2}.
\]
(23)

It is worth noting that the vanishing condition of the \( a \) parameter recovers the result of Ref. [6],
\[
T_{a0} = \frac{1}{3\sqrt{6}\pi Q}, \quad v_{a0} = 2\sqrt{6}Q,
\]
\[
P_{a0} = \frac{1}{96\pi Q}, \quad V_{a0} = 8\sqrt{6}\pi Q^3.
\]
(24)

In what follows, we discuss the effect of parameter \( a \) on the locality of the critical points. We plot in Fig. 2 the case of some particular points living in the vertical line \( Q = 1 \) of the \((Q, a)\) moduli space. General study could be carried out by varying also the charge parameter.

It is found from Fig. 2 that parameter \( a \) controls the position of the critical point change. For a large value of \( a \), the van der Waals behavior tends to the ideal-gas case.

After an examination of the generic regions of the \((Q, a)\) moduli space, we observe that the critical temperature is not defined for all values of \( a \) and \( Q \). At the vicinity of the origin of the \((Q, a)\) moduli space, the temperature becomes maximal.

In the small values of \( a \), the critical coordinates can be reduced to
\[
T_c = \frac{1}{3\sqrt{6}\pi Q} - \frac{23a^2}{72(\sqrt{6}\pi Q^3)} + O(a^4),
\]
(25)

\[
v_c = 2\sqrt{6}Q + \frac{59a^2}{6\sqrt{6}Q} + O(a^4),
\]
(26)

\[
P_c = \frac{1}{96\pi Q^2} - \frac{11a^2}{576(\pi Q^4)} + O(a^4).
\]
(27)

This limit produces a critical universal number
\[
\chi = \frac{3}{8} - \frac{a^2}{48Q^2} + O(a^4).
\]
(28)

This expression has the following nice features: (i) It depends on the \((Q, a)\) moduli space. (ii) It is valid only for charged black holes. (iii) For a vanishing value of the \( a \) parameter, it can be reduced to the usual value \( 3/8 \) given in Eq. (2).

Moreover, we note that the line \((0, a)\) of the moduli space \((Q, a)\) is singular corresponding to an indefinite critical point. In this singular line, the equation of state (11) reduces to
\[
P|Q=0 = \frac{3(4\pi a^2 r_+ T + a^2 + 4\pi a^2 T - r_+^2)}{8\pi r_+^2 (a^2 + 3r_+^2)},
\]

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representing an ideal gas behavior in the absence of the critical points. This singularity can be removed by taking a charged black hole.

![Fig. 3. The critical temperature and pressure in the \((Q, a)\) moduli space.](image)

At this level, we note that Eq. (28) recovers exactly the result of Ref. [6] by setting \(a = 0\). It also produces the result in Ref. [7] by considering a four-dimensional black hole. It should be interesting to see the connection with the result of Ref. [15]. We believe that this link deserves more study. We decide to explore this issue for future work.

Extra information on the \((Q, a)\) moduli space can be derived by studying the critical thermodynamical variables. In what follows, we will be interested in such quantities. The latter will shed light on the physical regions of the \((Q, a)\) moduli space in which the thermodynamical variables have real values. In particular, we give a numerical study. Indeed, the numerical result of the critical temperature and the pressure are plotted in Fig. 3.

From Fig. 3, we observe that the critical pressure is not well defined in all regions of the \((Q, a)\) moduli space. For this reason, we plot the regions in which the coordinate of the critical point is allowed physically. These regions are plotted in Fig. 4.

According to the region of the \((Q, a)\) moduli space, we distinguish several cases. More precisely, non-black-hole transition has been observed in the region defined by \(Q > 1\) and \(a < 1\). However, in the region defined by \(0 < Q < 1\) and \(0 < a < 2\), a black hole transition can be produced for very small \(T_c\) and \(P_c\). Moreover, it has been shown that the black hole transition can be easily produced for non vanishing values of the charge \(Q\) and the parameter \(a\).

At the end of this work, it is interesting to note that the extra constraints on the \((Q, a)\) moduli space can be fixed by calculating the entropy function. The physical conditions, derived from the thermodynamical principals, on such a function raise new constraints on the \((Q, a)\) moduli space. To see that, we consider the following action [17]

\[
I = \frac{\beta}{4G\Xi} \left[ -r_+^3 + \Xi r_+^2 + \frac{\ell^2(a^2 + Q^2)}{r_+} + 2\frac{\ell^2 Q r_+}{a^2 + r_+^2} \right].
\]

It is known that this action corresponds to the Gibbs free energy

\[
G = G(P, T) = \frac{1}{4} r_+ \left( 1 - \frac{6Q^2}{8\pi a^2P - 3(a^2 + r_+^2)} \right)
\]

\[
= \frac{3(a^2 + Q^2)}{4r_+(8\pi a^2P - 3)} + \frac{2\pi P r_+^3}{8\pi a^2P - 3},
\]

where \(r_+\) can be understood as a function of the pressure and the temperature, \(r_+ = r_+(P, T)\), via the state equation (11). Roughly speaking, the free energy reads

\[
F(T, V) = G - PV = \frac{r_+^2}{4r_+} \left( 1 - \frac{2Q^2}{(a^2 + r_+^2)(a^2X - 1)} \right)
\]

\[
+ \frac{a^2 + Q^2}{1 - a^2X} + r_+^4 X \left( \frac{1}{a^2X - 1} - 2 \right),
\]

where

\[
X = \frac{4\pi P r_+^3 + 4\pi r_+^4 + (4\pi r_+^4 + 4\pi r_+(4\pi r_+T - 1) + a^2 + Q^2)}{r_+^4(a^2 + 3a^2)}.
\]

Based on these quantities, we get the entropy function

\[
S(T, V) = \{ \pi r_+^2(2a^2X(a^2X - 2) + 3)
\]

\[
+ a^2r_+^4(2a^2X(a^2X - 2) + 3)
\]

\[
- r_+^2(a^4 + 3a^2Q^2) - a^4(a^2 + Q^2) \}
\]

\[
\cdot \{ r_+^2(a^2 + 3a^2)(a^2X - 1)^2 \}^{-1}
\]

\[
\text{(33)}
\]

In the vanishing limit of the \(a\) parameter, the entropy function becomes

\[
S(T, V)|_{a=0} = \pi r_+^2.
\]

\[
\text{(34)}
\]
Based on the above formula of the entropy, we can find the following constraint, for \( \ell = \sqrt{3} \) and \( r_+ = 1 \),

\[
Q \in \mathcal{R}, \quad a < \frac{1}{\sqrt{14}} \sqrt{-9Q^2 + \sqrt{81Q^4 - 36Q^2 + 256} + 2}.
\]

(35)

The constraints derived from the entropy agree with the previous ones obtained from the critical values of \( T_c \) and \( P_c \).

In summary, we have studied the critical behavior of Kerr–Newmann AdS black holes in four dimensions and showed similarities with the van der Waals gas in the asymptotic limit. In particular, we have derived the coordinates of the critical points and then discussed the effect of the \((Q, a)\) space moduli on such points. As a by-product, we have worked out a universal number depending on the \((Q, a)\) moduli space and recovered the usual value \(3/8\) for a Reissner Nordstrom AdS black hole.

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