Fixed-Parameter Algorithms for Graph Constraint Logic

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Abstract

Non-deterministic constraint logic (NCL) is a simple model of computation based on orientations of a constraint graph with edge weights and vertex demands. NCL captures PSPACE and has been a useful tool for proving algorithmic hardness of many puzzles, games, and reconfiguration problems. In particular, its usefulness stems from the fact that it remains PSPACE-complete even under severe restrictions of the weights (e.g., only edge-weights one and two are needed) and the structure of the constraint graph (e.g., planar and/or graphs of bounded bandwidth). While such restrictions on the structure of constraint graphs do not seem to limit the expressiveness of NCL, the building blocks of the constraint graphs cannot be limited without losing expressiveness: We consider as parameters the number of weight-one edges and the number of weight-two edges of a constraint graph, as well as the number of and/or vertices of an and/or constraint graph. We show that NCL is fixed-parameter tractable (FPT) for any of these parameters. In particular, for NCL parameterized by the number of weight-one edges or the number of and/or vertices, we obtain a linear kernel. It follows that, in a sense, NCL as introduced by Hearn and Demaine is defined in the most economical way for the purpose of capturing PSPACE.

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1 Introduction

Non-deterministic constraint logic (NCL) has been introduced by Hearn and Demaine [7] as a model of computation in order to show that many puzzles and games are complete in their natural complexity classes. For instance, they showed that the 1-player games Sokoban and Rush Hour are PSPACE-complete [7] and there are many follow-up results showing hardness of a large number of puzzles, games, and reconfiguration problems. An NCL constraint graph is a graph with edge-weights one and two and a configuration is given by an orientation of the constraint graph, such that the in-weight at each vertex is at least two. Two configurations are adjacent if they differ with respect to the orientation of a single edge. The question whether two given configurations are connected by a path, i.e., a sequence of adjacent configurations, is known to be PSPACE-complete, even if the constraint graph is a planar graph of maximum degree three (in fact, a planar AND/OR graph, to be defined shortly) [7]. Similar hardness results are known for the question whether it is possible to reverse a single given edge, or whether there is a transformation between two configurations, such that each edge is reversed at most once.

One of the main advantages of NCL, apart from its simplicity, is its hardness on constraint graphs with a severely restricted structure, which entails strong hardness results for other problems. In particular, NCL is PSPACE-complete on AND/OR graphs, which are cubic graphs, where each vertex is either incident to three weight-two edges (“AND vertex”) or exactly one weight-two edge (“OR vertex”), see Figure 1. It remains PSPACE-complete if in addition we assume that the constraint graphs are planar [7] and have bounded bandwidth [15]. We investigate the possibility of obtaining a further strengthening by restricting the composition of the constraint graph. In particular we consider constraint graphs with a bounded number of weight-one or weight-two edges, and AND/OR graphs with a bounded number of AND or OR vertices. Our main result is that NCL parameterized by any of the four quantities admits an FPT algorithm. That is, for the purpose of capturing PSPACE, the definition of NCL given by Hearn and Demaine is as economical as possible. We furthermore hope that based on our results, NCL may become of interest for investigating the parameterized complexity of puzzles, games, and reconfiguration problems.

In the following we adhere to the historical convention that an edge of weight one (resp., weight two) of a constraint graph is called red (resp., blue). We refer to the question whether a given configuration of a constraint graph is reachable from another given configuration as configuration-to-configuration (C2C). Furthermore, by configuration-to-edge (C2E) we refer to the question whether, we can reach from a given configuration another one such that a given edge is reversed.

Figure 1: The two types of vertices that occur in AND/OR constraint graphs. Edges must be oriented such that the in-weight at each vertex is at least two. By convention, weight-one edges are red and weight-two edges are blue.

Our Contribution We consider four natural parameterizations of NCL and show that the corresponding parameterized problems admit FPT algorithms. In particular we consider as parameters

1. the number of AND vertices of an AND/OR graph,
2. the number of OR vertices of an AND/OR graph,
3. the number of red edges of a constraint graph, and
4. the number of blue edges of a constraint graph.

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2. the number of OR vertices of an AND/OR graph,
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4. the number of blue edges of a constraint graph.
Table 1: Parameterized Complexity of NCL. For entries marked with † we obtain a linear kernel.

| Parameter(s)                           | C2C      | C2E      |
|----------------------------------------|----------|----------|
| treewidth and max. degree              | PSPACE-c | PSPACE-c |
| transformation length                  | W[1]-hard| W[1]-hard|
| transformation length and max. degree   | FPT      | FPT      |
| # of AND vertices (AND/OR graphs)      | FPT† (Cor. 1) | FPT† (Cor. 1) |
| # of OR vertices (AND/OR graphs)       | FPT† (Thm. 1) | FPT† (Thm. 1) |
| # of red edges                         | FPT† (Thm. 3) | FPT† (Cor. 7) |
| # of blue edges                        | FPT† (Cor. 18) | FPT† (Cor. 18) |

Note that none of these parameterizations trivially leads to an XP algorithm that just enumerates all orientations for the constant number of red/blue edges according to the parameter. For an overview of the parameterized complexity results on NCL, including our results, please refer to Table 1.

NCL is known to be PSPACE-complete on AND/OR constraint graphs, which are undirected edge-weighted graphs where each vertex is either an AND vertex or an OR vertex as shown in Figure 1. We show that C2C and C2E parameterized by the number of AND vertices or the number of OR vertices admits an FPT algorithm. The algorithm first performs a preprocessing step followed by a reduction to the problem Binary Constraint Satisfiability Reconfiguration (BCSR for short). Hatanaka et al. have shown that BCSR can be solved in time \( O(d^{O(p)}) \), where \( d \) and \( p \) are the maximum size of a domain and the number of non-Boolean variables, respectively.

On general constraint graphs we obtain a linear kernel for C2C parameterized by the number of red edges. For this purpose we introduce three reduction rules, which, when applied exhaustively, yield a kernel of linear size. To the best of our knowledge, this is the first polynomial kernel for a parameterization of NCL. The first rule states that each component containing at least two blue cycles can be replaced by a gadget of constant size for each red edge that is attached to the component. The second rule states that vertices incident to a blue edge only can be deleted, since the orientation of this edge is the same for every orientation. The third rule is inverse to subdividing a blue edge: any vertex incident to precisely two blue edges can be deleted and replaced by a single blue edge connecting its former neighbors. Note that the number of red edges in an AND/OR graph is precisely the number of AND vertices in an AND/OR graph. Hence, a linear kernel for NCL parameterized by the number of red edges implies a linear kernel for NCL parameterized by the number of AND vertices of an AND/OR graph. Furthermore, we show that slightly modified reduction rules can be applied in order to obtain a linear kernel for C2E.

Finally, we consider C2C and C2E parameterized by the number \( k \) of blue edges and show that it admits an FPT algorithm. Our key idea is to partition the set of feasible orientations of the constraint graph into \( 2^{O(k)} \) classes, such that in each class, all blue edges are oriented in the same way and the red edges have the same indegree sequence. Denote the set of these classes by \( \mathcal{F} \), and define a mapping \( \phi \) from the set of orientations of the constraint graph to \( \mathcal{F} \) (see Section 5.1 for the details). Then, in Section 5.2 we define an adjacency relation between elements in \( \mathcal{F} \), which is consistent with the reachability of configurations of the constraint graph in some sense. In our algorithm, instead of the original reconfiguration problem, we first solve the reconfiguration problem in \( \mathcal{F} \), which can be done in \( 2^{O(k)} \cdot \text{poly}(|V|) \) time, where \( V \) is the set of vertices of the constraint graph. If it is impossible to reach the target configuration in \( \mathcal{F} \), then we can conclude that it is also impossible with respect to the original constraint graph. Otherwise, we can reduce the original problem to the case that the blue edges agree in the initial and target configuration and the set of red edges in the initial configurations whose orientation differs from the target configuration consists of arc-disjoint dicycles (see Section 5.3). Finally, in Section 5.4 we test whether the direction of each dicycle can be reversed or not.

Related Work A large number of puzzles, games, and reconfiguration problems have been shown to be hard using reductions from NCL and its variants. Examples include motion planning problems, where rectangular pieces have to be moved to certain final positions and sliding block puzzles such as Rush Hour [9, 7], Sokoban [7], Snowman [4] and other puzzle games such as Bloxors [16]. In the bounded length version of NCL, the orientation of each edge may be reversed at most once. This variant has been used to show NP-completeness of the games Klondike, Mahjong Solitaire and Nonogram [8]. Note that NCL gives a uniform view on games as computation and often allows for simpler proofs and strengthenings of known complexity results in this area. Furthermore, deciding proof equivalence in multiplicative linear
logic has been shown to be \textsc{PSPACE}-complete by a reduction from NCL \cite{9}.

NCL is also very useful for showing hardness of reconfiguration problems. In a reconfiguration problem we are given two configurations and agree on some simple “move” that produces a new configuration from a given one. The question is whether we can reach the second configuration from the first by a sequence of moves. For surveys on reconfiguration problems, please refer to \cite{12,14}. For many reconfiguration problems, such as token sliding on graphs \cite{7}, a variant of independent set reconfiguration \cite{11}, as well as vertex cover reconfiguration \cite{10}, dominating set reconfiguration \cite{4}, reconfiguration of paths \cite{2}, and deciding Kempe-equivalence of 3-colorings \cite{1}, reductions from NCL establish \textsc{PSPACE}-hardness even on planar graphs of low maximum degree. Van der Zanden showed that there is some constant \(c\), such that NCL is \textsc{PSPACE}-complete on planar subcubic graphs of bandwidth at most \(c\) \cite{15}. Note that this property is often maintained in the reductions \cite{1,2,4,7,10} and it implies that NCL remains hard on graphs of bounded treewidth.

Tractable special cases of NCL have received much less attention. Concerning parameterized complexity, NCL remains \textsc{PSPACE}-complete when parameterized by treewidth and maximum degree of the constraint graph. On the other hand, NCL parameterized by the length of the transformation is \(W[1]\)-hard and it becomes FPT when parameterized by the length of the transformation and the maximum degree \cite{15}. If additionally each edge may be reversed at most once in a transformation, NCL is FPT when parameterized by treewidth and the maximum degree, or by the length of the transformation \cite{15}.

\textbf{Organization} The paper is organized as follows. In the next section we give some preliminaries about NCL and introduce notation used throughout the paper. Section 3 contains our FPT algorithm for NCL parameterized by the number of vertices. The linear kernel for NCL parameterized by the number of red edges, which also implies the result for \textsc{and} vertices, can be found in Section 4. Finally, in Section 5 we give an FPT algorithm for NCL parameterized by the number of blue edges. Section 6 concludes the paper and gives some open problems.

\section{Preliminaries}

Let \(G = (V, E)\) be an undirected graph, which may have multiple edges and (self) loops. We denote by \(V(G)\) (resp., \(E(G)\)) the set of vertices (resp., set of edges) \(G\). Each edge in an undirected graph which joins two vertices \(x\) and \(y\) is represented as an unordered pair \(xy\) (or equivalently \(yx\)). On the other hand, each arc in a digraph which leaves \(x\) and enters \(y\) is written as an ordered pair \((x, y)\). Let \((V, E, w)\) be a constraint graph, that is, an undirected graph \((V, E)\) with edge weights \(w : E \to \{1, 2\}\). We denote by \(E^\text{red}\) and \(E^\text{blue}\) the sets of red (weight one) and blue (weight two) edges in \(E\), respectively, and have that \(E = E^\text{red} \cup E^\text{blue}\). We denote by \(V_{\text{AND}}(G)\) and \(V_{\text{OR}}(G)\) the sets of \textsc{and} and \textsc{or} vertices in a graph \(G\), respectively; we sometimes drop \(G\), and simply write \(V_{\text{AND}}\) and \(V_{\text{OR}}\) if it is clear from the context. A constraint graph is called \textsc{and/or} graph if each vertex is an \textsc{and} or \textsc{or} vertex; thus, an \textsc{and/or} graph is 3-regular.

An \textit{orientation} \(A\) of \(E\) is a multi-set of arcs obtained by replacing each edge in \(E\) with a single arc having the same end vertices. We refer to \(G\) as the underlying graph of the digraph \((V, A)\). For an orientation \(A\) of \(E\), we always denote by \(A^\text{red}\) and \(A^\text{blue}\) the subsets of \(A\) corresponding to \(E^\text{red}\) and \(E^\text{blue}\), respectively. For any arc subset \(B \subseteq A\) and a vertex \(v \in V\), let \(\rho_B(v)\) denote the number of arcs in \(B\) that enter \(v\). Then, \(\rho_B\) can be regarded as a vector in \(\mathbb{Z}_{\geq 0}^{|V|}\), where \(\mathbb{Z}_{\geq 0}\) is the set of all nonnegative integers. An orientation \(A\) of \(E\) is \textit{feasible} if \(\rho_{A^\text{red}}(v) + 2 \cdot \rho_{A^\text{blue}}(v) \geq 2\) for every \(v \in V\); a feasible orientation is synonymously referred to as \textit{configuration}.

For two orientations \(B\) and \(B'\) of an edge subset \(F \subseteq E\), we write \(B \leftrightarrow B'\) if \(B = B'\) or there exists an arc \((x, y) \in B\) such that \(B' = (B \setminus \{(x, y)\}) \cup \{(y, x)\}\). For notational convenience, we simply write \(B' = B - (x, y) + (y, x)\) in the latter case. For an orientation \(B\) of \(F\), \textit{reversing} the direction of an edge \(xy \in F\) is the operation which yields from \(B\) an orientation \(B'\) of \(F\), such that \(B' = B - (x, y) + (y, x)\) if \((x, y) \in B\) and \(B - (y, x) + (x, y)\) otherwise. For two feasible orientations \(A\) and \(A'\) of \(E\), a sequence \((A_0, A_1, \ldots, A_{\ell})\) of feasible orientations of \(E\) is called a \textit{reconfiguration sequence} between \(A\) and \(A'\) if \(A_0 = A\), \(A_{\ell} = A'\), and \(A_{k-1} \leftrightarrow A_k\) for all \(k \in \{1, 2, \ldots, \ell\}\). We write \(A \leftrightarrow A'\) if there exists a reconfiguration sequence between \(A\) and \(A'\) (or \(A' \leftrightarrow A\) if not). Given a constraint graph \(G\) and two feasible orientations \(A_{\text{ini}}\) and \(A_{\text{tar}}\) of \(E(G)\), the problem \textsc{C2C} asks whether \(A_{\text{ini}} \leftrightarrow A_{\text{tar}}\) or not. Similarly, given a constraint graph \((G, w)\), a feasible orientation \(A_{\text{ini}}\) of \(E(G)\), and an edge \(e \in E(G)\), the problem
C2E asks whether there is a feasible orientation \(A_{\text{tar}}\), such that \(A_{\text{ini}} \leadsto A_{\text{tar}}\) and the direction of \(e\) is different in \(A_{\text{ini}}\) and \(A_{\text{tar}}\). We denote by a triple \((G, A_{\text{ini}}, A_{\text{tar}})\) an instance of C2C and by a triple \((G, A_{\text{ini}}, v, w)\) an instance of C2E.

3 NCL for AND/OR graphs

In this section, we consider NCL when restricted to AND/OR constraint graphs. Recall that NCL remains PSPACE-complete on AND/OR graphs [7]. We thus prove that C2C and C2E on AND/OR constraint graphs is fixed-parameter tractable when parameterized by the number of OR vertices. An analogous result for C2C and C2E parameterized by the number of AND vertices follows from our FPT result for NCL parameterized by the number of red edges in the next section (see Theorem 3). Therefore, the main result here is the following theorem.

**Theorem 1.** C2C and C2E on AND/OR constraint graphs with \(n\) vertices parameterized by the number \(k\) of OR vertices admits a \(2^{\O(k)} \cdot \text{poly}(n)\)-time algorithm.

In the reminder of this section, we give an overview of the proof of Theorem 1. Our strategy is to give an FPT-reduction from C2C on AND/OR constraint graphs to the binary constraint satisfaction reconfiguration problem (BCSR, for short) [5], which will be defined in Section 3.2. To do so, we first apply some preprocessing to a given instance of C2C on an AND/OR graph (in Section 3.1), and then give our FPT-reduction to BCSR (in Section 3.2). By similar arguments we obtain the result for C2E.

### 3.1 Preprocessing

The preprocessing subdivides each blue edge that is not a loop into two blue edges. It is not hard to see that a single subdivision yields an equivalent instance. Let \(uv\) be a blue edge of a constraint graph \(\hat{G}\) and consider the constraint graph \(G\) obtained by subdividing \(uv\) into two blue edges \(uz\) and \(zv\), where \(z\) is a new vertex we call middle vertex. Let \(G\) be the resulting constraint graph and observe that from any feasible orientation \(A\) of \(\hat{G}\) we may obtain a feasible orientation \(A\) of \(G\) by letting \(A = \hat{A} - (u, v) + (u, z) + (z, v)\) if \((u, v) \in \hat{A}\) and \(A = \hat{A} - (v, u) + (v, z) + (z, u)\) otherwise.

Furthermore, in any feasible orientation of \(\hat{G}\), we can transfer in-weight from, say, \(u\) to \(v\) by reversing the arc \((u, v)\) iff the in-weight at \(u\) is at least four. Furthermore, due to the orientation of \(uv\), the corresponding arc contributes to the in-weight of precisely one of \(u\) and \(v\). Conversely, in an orientation of \(G\), we can transfer in-weight from, say, \(u\) to \(v\) by reversing the directions of the arcs corresponding to \(uz\) and \(zv\) iff the in-weight at \(u\) is at least four. Furthermore, in any orientation of \(G\), the arcs corresponding to \(uz\) and \(zv\) contribute in-weight to at most one of \(u\) and \(v\). Hence, by subdividing a blue edge of \(\hat{G}\) from an instance \((\hat{G}, A_{\text{ini}}, A_{\text{tar}})\) of C2C, we obtain an equivalent instance. Let \((G, A_{\text{ini}}, A_{\text{tar}})\) be the instance of C2C obtained from \((\hat{G}, A_{\text{ini}}, A_{\text{tar}})\) by subdividing each blue edge of \(\hat{G}\) that is not a loop. By repetition of the above argument we obtain the following result.

**Lemma 2.** \((G, A_{\text{ini}}, A_{\text{tar}})\) is a yes-instance if and only if \((\hat{G}, A_{\text{ini}}, A_{\text{tar}})\) is.

### 3.2 FPT-reduction to BCSR

In this subsection, we sketch our FPT-reduction to BCSR. We start by formally defining the problem BCSR. Let \(H = (X, F)\) be an undirected graph. We call each vertex \(x \in X\) a variable. Each \(x \in X\) has a finite set \(D(x)\), called a domain of \(x\). A variable \(x\) is called a Boolean variable if \(|D(x)| \leq 2\), and otherwise called a non-Boolean variable. Each edge \(xy \in F\) has a subset \(C(xy) \subseteq D(x) \times D(y)\), called a (binary) constraint of \(xy\). A mapping \(\Gamma: X \rightarrow \bigcup_{x \in X} D(x)\) is a solution of \(H\) if \(\Gamma(x) \in D(x)\) for every \(x \in X\). In addition, a solution \(\Gamma\) of \(H\) is proper if \(\{x \in X : \Gamma(x) \neq \Gamma'(x)\}\) = 1. Given an undirected graph \(H\), a domain \(D(x)\) for each \(x \in X\), a constraint \(C(xy)\) for each \(xy \in F\), and two proper solutions \(\Gamma_{\text{ini}}\) and \(\Gamma_{\text{tar}}\) of \(H\), the binary constraint satisfaction reconfiguration problem (BCSR) asks whether there exists a sequence \(\{\Gamma_0, \Gamma_1, \ldots, \Gamma_\ell\}\) of proper solutions of \(H\) such that \(\Gamma_0 = \Gamma_{\text{ini}}\), \(\Gamma_\ell = \Gamma_{\text{tar}}\), and \(\Gamma_{i-1} \leftrightarrow \Gamma_i\) for each \(i \in \{1, 2, \ldots, \ell\}\). Let \((H, D, C, \Gamma_{\text{ini}}, \Gamma_{\text{tar}})\) an instance of BCSR.
It is known that BCSR can be solved in time $O^*(d^{O(p)})$, where $d := \max_{x \in X} |D(x)|$ and $p$ is the number of non-Boolean variables in $X$ [5, Theorem 18]. To prove Theorem 4 given an instance $(\hat{G}, \hat{A}_{ini}, \hat{A}_{tar})$ of C2C on an AND/OR constraint graph with at most $k$ AND/OR vertices, we first perform the preprocessing from Section 4.1 to obtain an instance $(G, A_{ini}, A_{tar})$ of C2C. Note that $G$ is not an AND/OR graph, and $V$ can be partitioned into $V_{AND}(G)$, $V_{OR}(G)$ and $V_{UND}(G)$, where $V_{OR}(G)$ (or simply $V_{UND}$) is the set of middle vertices in $G$. By Lemma 2 we have that $(G, A_{ini}, A_{tar})$ is a yes-instance if and only if $(\hat{G}, \hat{A}_{ini}, \hat{A}_{tar})$. Hence, to conclude the proof of Theorem 4 we provide an FPT-reduction from a preprocessed instance $(G, A_{ini}, A_{tar})$ of C2C with the parameter $|V_{OR}(G)| = |V_{OR}(G)| \leq k$ to an instance $(H, D, C, \Gamma_{ini}, \Gamma_{tar})$ of BCSR such that both $d$ and $p$ are bounded by some computable functions depending only on $k$.

Due to the preprocessing, observe that the constraint graph $G$ has no two parallel blue edges. In addition, no edge in $G$ joins an AND vertex and an OR vertex, and hence we can partition $E$ into two sets $E_{AND}$ and $E_{OR}$, defined as follows: $E_{AND}$ is the set of edges of $G$ that are incident to an AND vertex; $E_{OR}$ is the set of edges of $G$ that are incident to an OR vertex. The high-level idea of the reduction to BCSR is the following. For each OR vertex $v$, we create an OR variable $x_v$. Observe that the in-weight requirement at $v$ is violated only if each arc is pointing away from $v$. We forbid such orientations by giving each OR variable $x_v$ a domain of size seven corresponding to the seven legal orientations of the incident edges of $v$.

The remaining in-weight requirements and consistency requirements are modelled by adding constraints, which also define the set of edges in $H$. For each edge $e$ of $G$, we create a Boolean edge-variable $x_e$, whose domain represents the two possible orientations of an edge. The construction of domains above ensures that in-weight requirement is satisfied for each AND vertex. To ensure the same property for all other vertices, we add three types of constraints for middle vertices and AND vertices, to enforce the following constraints:

**Type 1:** Constraints for middle vertices.

Let $v$ be a middle vertex between two vertices $v_1$ and $v_2$. Since both $v_1v$ and $vv_2$ are blue edges, the in-weight requirement at $v$ is satisfied if and only if $v_1v$ or $vv_2$ points towards $v$.

**Type 2-1:** Constraints for AND vertices having loops.

Let $v$ be an AND vertex having a loop $vv$. So $vv$ must be red and the remaining edge $vv_3 \in E_{AND}$ is blue where $v_3$ is a middle vertex. Then, the in-weight requirement at $v$ is satisfied if and only if $vv_3$ is oriented towards $v$.

**Type 2-2:** Constraints for AND vertices without loops.

Let $v$ be an AND vertex, and let $vv_1$, $vv_2$, $vv_3$ be three (distinct) edges incident to $v$ such that $vv_1$ and $vv_2$ are red, and $vv_3$ is blue; it may hold that $v_1 = v_2$. Then, the in-degree requirement at $v$ is satisfied if and only if i) $vv_1$ or $vv_3$ are oriented towards $v$ and ii) $vv_2$ or $vv_3$ are oriented towards $v$.

By the construction of constraints above, we know that a solution $\Gamma$ of $H$ is proper if and only if the corresponding orientation $A_{\Gamma}$ of $E$ is feasible. Therefore, we can define proper solutions $\Gamma_{ini}$ and $\Gamma_{tar}$ of $H$ which correspond to feasible orientations $A_{ini}$ and $A_{tar}$ of $E$, respectively. In this way, from a preprocessed instance $(G, A_{ini}, A_{tar})$ of C2C with the parameter $|V_{OR}(G)| \leq k$, we have constructed in polynomial time a corresponding equivalent instance $(H, D, C, \Gamma_{ini}, \Gamma_{tar})$ of BCSR such that $d = \max_{x \in X} |D(x)| = 7$ and $p \leq |V_{OR}(G)| \leq k$.

**4 NCL parameterized by the number of red edges**

Our main result in this section is a linear kernel for C2C parameterized by the number of red edges of the constraint graph.

**Theorem 3.** There is a polynomial-time algorithm that, given an instance of C2C on a constraint graph with $k$ red edges, outputs an equivalent instance of C2C of size $O(k)$.

In particular, Theorem 3 implies that C2C parameterized by the number of red edges admits a $O^*(2^O(k))$-time algorithm. By observing that in any AND/OR constraint graph, the number of red edges is equal to the number of AND vertices, we immediately obtain the following result.
Corollary 4. C2C on AND/or graphs parameterized by the number $k'$ of AND vertices admits a kernel of size $O(k')$.

It can be shown by similar arguments that there is also a linear kernel for C2E parameterized by the number of red edges of the constraint graph. In the remainder of this section, we prove Theorem 3. Let $I = (G, A_{ini}, A_{tar})$ be an instance of C2C, where $G$ is any constraint graph with $k$ red edges. We give four reduction rules, and show that applying them repeatedly preserves the answer. Furthermore, we show that applying them exhaustively yields an instance of size $O(k)$, where $k = |E^{red}|$. To conclude the proof, we show that the reduction can be applied in polynomial time.

We say that a vertex is blue if all its incident edges are blue. Otherwise, if at least one incident edge is red, we call the vertex red. A subset $V' \subseteq V$ is called a blue component if it is a connected component in the graph $(V, E^{blue})$. Note that a blue component may contain red vertices of $G$. The first reduction rule removes blue components of $G$ that are directed cycles. Observe that no arc in such a component can be reversed. The second reduction rule removes blue components that contain at least two cycles and attaches to each red vertex $v$ of the component a copy of the gadget shown in Figure 2. The gadget consists of a cycle on five new vertices $\{v_0, v_1, v_2, v_3, v_4\}$ with two chords $\{v_1, v_3\}$ and $\{v_2, v_4\}$. Additionally we add an edge joining $v$ and $v_0$. All edges of the gadget have weight two. The third reduction rule removes blue vertices of degree one and the last rule removes the center vertex of a blue path on three vertices.

While modifying $G$ we also modify $A_{ini}$ and $A_{tar}$ accordingly. That is, if we delete edges of $G$, these edges are also deleted in $A_{ini}$ and $A_{tar}$. If we add a gadget to $G$, then the arcs in $A_{ini}$ and $A_{tar}$ have the same orientation on the gadget. Note that the number $k$ of red vertices is not altered by an application of any of the rules. Here is a more formal description of the four rules:

Reduction rule 1. Let $C$ be a component of $G$ that is a blue chordless cycle. If the orientations $A_{ini}$ and $A_{tar}$ agree on $C$, then we remove $C$ from the graph and adjust $A_{ini}$ and $A_{tar}$ accordingly. Otherwise we output a no-instance.

Reduction rule 2. Let $C$ be a blue component that contains at least two cycles. Then we remove from $G$ every blue vertex in $C$ and attach to each red vertex in $C$ a copy of the gadget in Figure 2. Additionally we modify $A_{ini}$ and $A_{tar}$ accordingly such that both agree on each copy of the gadget.

Reduction rule 3. If $G$ has a blue vertex $v$ of degree one, delete $v$ and its incident edge from $G$ and remove the corresponding arc(s) from $A_{ini}$ and $A_{tar}$.

Reduction rule 4. Suppose $G$ has a blue vertex $v$ of degree 2, such that the two neighbors $u$ and $w$ of $v$ are non-adjacent in $G$. Then delete $v$ and its incident edges from $G$ and add the blue edge $uv$. Remove any arcs incident to $v$ from $A_{ini}$ and $A_{tar}$. Finally, add $(u, w)$ to $A_{ini}$ (resp. $A_{tar}$) if $(u, v) \in A_{ini}$ (resp., $A_{tar}$) and $(w, u)$ otherwise.

We show that applying any of the four rules is safe, that is any application results in a yes-instance if and only if $I$ is a yes-instance.

Proposition 5. Reduction rules 1–4 are safe for C2C.

By applying a depth-first-search we can check if any of the rules can be applied. Thus we have the following.

Proposition 6. Reduction rules 1–4 can be applied exhaustively in time $O(|V| \cdot (|V| + |E|))$.

Theorem 3 now follows by the previous propositions and a simple counting argument. Using similar arguments we show that there is also a linear kernel for C2E parameterized by the number $k$ of red edges. The main difference is that in reduction rule 2 we only add the gadget to each red vertex that is...
part of a cycle or connected to two distinct cycles by two disjoint paths. Furthermore, if the edge \( e \) that we wish to reverse is part of of a component containing two cycles, we add a gadget to the tail of \( e \).

**Corollary 7.** C2E parameterized by the number \( k \) of red edges admits a kernel of size \( O(k) \). Furthermore, C2E on AND/OR graphs parameterized by the number \( k' \) of AND vertices admits a kernel of size \( O(k') \).

5 NCL parameterized by the number of blue edges

The objective of this section is to show that C2C parameterized by the number \( k \) of blue edges is fixed parameter tractable.

**Theorem 8.** C2C parameterized by the number \( k \) of blue edges can be solved in time \( 2^{O(k)} \cdot \text{poly}(|V|) \).

In the remainder of this section, we prove Theorem 8. Let \( I = (G, A_{\text{ini}}, A_{\text{tar}}) \) be an instance of C2C, where \( G \) is any constraint graph and denote by \( V \) and \( E \) the set of vertices and edges of \( G \), respectively.

Let \( A \) denote the set of all feasible orientations of \( E \). Our key idea is to classify the feasible orientations into \( 2^{O(k)} \) classes, where each class is determined by the orientation \( A_{\text{blue}} \) of \( E_{\text{blue}} \) and the indegree sequence of \( A_{\text{red}} \). Denote the set of these classes by \( \mathcal{F} \), and define a mapping \( \phi \) from \( A \) to \( \mathcal{F} \) (see Section 5.1 for details). Then, in Section 5.2, we define a reconfiguration relation \( \leftrightarrow \) between elements in \( \mathcal{F} \), which is consistent with \( \rightarrow \) in some sense. In our algorithm, instead of the original reconfiguration problem in \( A \), we first solve the reconfiguration problem in \( \mathcal{F} \), which can be done in \( 2^{O(k)} \cdot \text{poly}(|V|) \) time. If it has no reconfiguration sequence, then we can conclude that there is no reconfiguration sequence in the original problem. Otherwise, we can reduce the original problem to the case when \( A_{\text{ini}}^{\text{blue}} = A_{\text{tar}}^{\text{blue}} \) and \( A_{\text{ini}} \setminus A_{\text{tar}}^{\text{red}} \) consists of arc-disjoint dicycles (see Section 5.3). Finally, in Section 5.4, we test whether the direction of each cycle can be reversed or not.

Before starting the main part of the proof of Theorem 8, we show the following lemma that plays an important role in our argument. Roughly, it says that we can change the orientation of \( E_{\text{red}} \) keeping a fixed indegree constraint.

**Lemma 9.** Let \( A_{\text{ini}}^{\text{red}} \) and \( A_{\text{tar}}^{\text{red}} \) be orientations of \( E_{\text{red}} \). Then, there exists a sequence \( A_0^{\text{red}}, A_1^{\text{red}}, \ldots, A_l^{\text{red}} \) of orientations of \( E_{\text{red}} \) such that \( A_0^{\text{red}} = A_{\text{ini}}^{\text{red}}, \rho_{A_i^{\text{red}}} = \rho_{A_{\text{tar}}^{\text{red}}}, A_i^{\text{red}} \leftrightarrow A_{i+1}^{\text{red}} \) for \( i = 1, \ldots, l \), and \( \rho_{A_i^{\text{red}}}(v) \geq \min\{\rho_{A_{\text{ini}}^{\text{red}}}(v), \rho_{A_{\text{tar}}^{\text{red}}}(v)\} \) for any \( v \in V \) and any \( i \in \{0, 1, \ldots, l\} \).

**Proof.** We prove the lemma by induction on \( |A_{\text{ini}}^{\text{red}} \setminus A_{\text{tar}}^{\text{red}}| \). If \( \rho_{A_{\text{ini}}^{\text{red}}} = \rho_{A_{\text{tar}}^{\text{red}}} \), then the claim is obvious, because the sequence consisting of only one orientation \( A_0^{\text{red}} = A_{\text{ini}}^{\text{red}} \) satisfies the conditions. Thus, it suffices to consider the case when \( \rho_{A_{\text{ini}}^{\text{red}}} \neq \rho_{A_{\text{tar}}^{\text{red}}} \). In this case, there exists a vertex \( u \in V \) such that \( \rho_{A_{\text{ini}}^{\text{red}}}(u) > \rho_{A_{\text{tar}}^{\text{red}}}(u) \), because \( \sum_{v \in V} \rho_{A_{\text{ini}}^{\text{red}}}(v) = \sum_{v \in V} \rho_{A_{\text{tar}}^{\text{red}}}(v) \). Then, there exists an arc \( a \in A_{\text{ini}}^{\text{red}} \setminus A_{\text{tar}}^{\text{red}} \) that enters \( u \). Let \( A_1^{\text{red}} \) be the orientation of \( E_{\text{red}} \) obtained from \( A_{\text{ini}}^{\text{red}} \) by reversing the direction of \( a \). Since \( |A_{\text{ini}}^{\text{red}} \setminus A_{\text{tar}}^{\text{red}}| < |A_{\text{ini}}^{\text{red}} \setminus A_{\text{tar}}^{\text{red}}| \), by induction hypothesis, there exists a sequence \( A_1^{\text{red}}, \ldots, A_l^{\text{red}} \) of orientations of \( E_{\text{red}} \) such that \( \rho_{A_{\text{ini}}^{\text{red}}} = \rho_{A_{\text{tar}}^{\text{red}}}, A_0^{\text{red}} \leftrightarrow A_i^{\text{red}} \) for \( i = 2, \ldots, l \), and \( \rho_{A_i^{\text{red}}}(v) \geq \min\{\rho_{A_{\text{ini}}^{\text{red}}}(v), \rho_{A_{\text{tar}}^{\text{red}}}(v)\} \) for any \( v \in V \) and any \( i \in \{1, \ldots, l\} \). By letting \( A_0^{\text{red}} = A_{\text{ini}}^{\text{red}} \), the sequence \( A_0^{\text{red}}, A_1^{\text{red}}, \ldots, A_l^{\text{red}} \) satisfies the conditions, because \( A_0^{\text{red}} = A_{\text{ini}}^{\text{red}} \), \( \rho_{A_i^{\text{red}}}(v) \geq \rho_{A_{\text{ini}}^{\text{red}}}(v) \) for each \( v \in V \setminus \{u\} \), and \( \min\{\rho_{A_i^{\text{red}}}(u), \rho_{A_{\text{ini}}^{\text{red}}}(u)\} = \rho_{A_{\text{ini}}^{\text{red}}}(u) = \rho_{A_{\text{tar}}^{\text{red}}}(u) \).

The proof of Lemma 9 is constructive, and hence we can find such a sequence efficiently.

5.1 Classification of \( A \)

In this subsection, we classify the feasible orientations into \( 2^{O(k)} \) classes. Let \( X \subseteq V \) be the set of all vertices to which edges in \( E_{\text{blue}} \) are incident. Define \( \mathcal{F} \) as the set of all pairs \((A_{\text{blue}}, d)\) where \( A_{\text{blue}} \) is an orientation of \( E_{\text{blue}} \) and \( d \) is a vector in \( \{0, 1, 2\}^{\lambda} \) satisfying the following conditions:

1. \( 2\rho_{A_{\text{blue}}}(v) + d(v) \geq 2 \) for any \( v \in X \).
(2) There exists an orientation $A^{\text{red}}$ of $E^{\text{red}}$ such that for any $v \in V$,

$$\rho_{A^{\text{red}}}(v) \begin{cases} 
0 & \text{if } v \in X \text{ and } d(v) = 0, \\
1 & \text{if } v \in X \text{ and } d(v) = 1, \text{ and} \\
\geq 2 & \text{otherwise.}
\end{cases}$$

We note that $|F| \leq 2|E^{\text{blue}}| \cdot 3^{|X|} = 2^{|E^{\text{blue}}|}$, because $|X| \leq 2|E^{\text{blue}}|$. For a vector $d \in \{0,1,2\}^X$, we say that an orientation $A^{\text{red}}$ of $E^{\text{red}}$ realizes $d$ if $A^{\text{red}}$ satisfies the condition (2) above. We can easily see that if $(A^{\text{blue}}, d) \in F$ holds and $A^{\text{red}}$ realizes $d$, then $A := A^{\text{blue}} \cup A^{\text{red}}$ is a feasible orientation of $E$. Conversely, if $A = A^{\text{blue}} \cup A^{\text{red}}$ is a feasible orientation of $E$ (i.e., $A \in A$), then the vector $d \in \{0,1,2\}^X$ defined by $d(v) = \min\{\rho_{A^{\text{red}}}(v), 2\}$ for each $v \in X$ satisfies that $(A^{\text{blue}}, d) \in F$. This defines a mapping $\phi$ from $A$ to $F$.

We can also see that the membership problem of $F$ can be decided in polynomial time.

**Lemma 10.** For an orientation $A^{\text{blue}}$ of $E^{\text{blue}}$ and a vector $d \in \{0,1,2\}^X$, we can test whether $(A^{\text{blue}}, d) \in F$ or not in polynomial time.

**Proof.** We can easily check the condition (1). To check the condition (2), we construct a digraph $\hat{G} = (\hat{V}, \hat{A})$ and consider a network flow problem in it. Introduce a new vertex $w_e$ for each $e \in E^{\text{red}}$ and two new vertices $s$ and $t$, and define $\hat{V} := V \cup \{w_e | e \in E^{\text{red}}\} \cup \{s, t\}$. Define the arc set $\hat{A} := \hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_3$ by

$$\hat{A}_1 := \{(s, w_e) | e \in E^{\text{red}}\},$$

$$\hat{A}_2 := \{(w_e, v) | e \in E^{\text{red}}, v \in V, e \text{ is incident to } v \text{ in } G\},$$

$$\hat{A}_3 := \{(v, t) | v \in V\}.$$

For each $a \in \hat{A}$, define the lower bound $l(a)$ and the upper bound $u(a)$ of the amount of flow through $a$ as follows.

- For each $(s, w_e) \in \hat{A}_1$, define $l(s, w_e) := u(s, w_e) := 1$.
- For each $(w_e, v) \in \hat{A}_2$, define $l(w_e, v) := 0$ and $u(w_e, v) := 1$.
- For each $(v, t) \in \hat{A}_3$, define $l(v, t) := u(v, t) := d(v)$ if $v \in X$ and $d(v) \in \{0,1\}$, and define $l(v, t) := 2$ and $u(v, t) := +\infty$ otherwise.

Then, the condition (2) holds if and only if $\hat{G}$ has an integral $s$-$t$ flow satisfying the above constraint. This can be tested in polynomial time by a standard maximum flow algorithm (see e.g. [13] Corollary 11.3a)).

### 5.2 Reconfiguration in $F$

In this subsection, we consider a reconfiguration between elements in $F$. For $(A_1^{\text{blue}}, d_1), (A_2^{\text{blue}}, d_2) \in F$, we denote $(A_1^{\text{blue}}, d_1) \not\leftrightarrow F (A_2^{\text{blue}}, d_2)$ if

- $d_1 = d_2$ and $A_1^{\text{blue}} \not\leftrightarrow F A_2^{\text{blue}}$, or
- $A_1^{\text{blue}} = A_2^{\text{blue}}$.

If there exists a sequence $(A_0^{\text{blue}}, d_0), (A_1^{\text{blue}}, d_1), \ldots, (A_l^{\text{blue}}, d_l) \in F$ such that $(A_i^{\text{blue}}, d_{i-1}) \not\leftrightarrow F (A_{i+1}^{\text{blue}}, d_i)$ for $i = 1, \ldots, l$, then we denote $(A_0^{\text{blue}}, d_0) \not\leftrightarrow F (A_l^{\text{blue}}, d_l)$. Then, we can easily see the following.

**Lemma 11.** Let $A_{\text{ini}}, A_{\text{tar}} \in A$. If $A_{\text{ini}} \not\leftrightarrow F A_{\text{tar}}$, then $\phi(A_{\text{ini}}) \not\leftrightarrow F \phi(A_{\text{tar}})$.

**Proof.** If $A_{\text{ini}} \leftrightarrow F A_{\text{tar}}$, then $\phi(A_{\text{ini}}) \leftrightarrow F \phi(A_{\text{tar}})$ by definition. By using this relationship repeatedly, we obtain the claim. \hfill \square
Although the opposite implication is not true, we show the following statement.

**Lemma 12.** Let $A_{ini}, A_{tar} \in \mathcal{A}$. If $\phi(A_{ini}) \not\sim \phi(A_{tar})$, then there exists $A_{tar}^{\circ} \in \mathcal{A}$ such that $\phi(A_{tar}^{\circ}) = \phi(A_{ini})$ and $A_{ini} \sim \sim A_{tar}^{\circ}$.

**Proof.** It suffices to consider the case when $\phi(A_{ini}) \not\sim \phi(A_{tar})$. Denote $\phi(A_{ini}) = (A_{ini}^{blue}, d_{ini})$ and $\phi(A_{tar}) = (A_{tar}^{blue}, d_{tar})$. By definition, we have either $d_{ini} = d_{tar}$ and $A_{ini}^{blue} \leftrightarrow A_{tar}^{blue}$, or $A_{ini}^{blue} = A_{tar}^{blue}$.

If $d_{ini} = d_{tar}$ and $A_{ini}^{blue} \leftrightarrow A_{tar}^{blue}$, then $A_{ini} \leftrightarrow A_{ini}^{blue} \cup A_{tar}^{red}$ and $\phi(A_{tar}^{blue} \cup A_{tar}^{red}) = (A_{tar}^{blue}, d_{tar}) = (A_{tar}^{blue}, d_{ini}) = (A_{tar}^{blue}, d_{tar}) = (A_{tar}^{blue} \cup A_{tar}^{red})$, which means that $A_{tar}^{\circ} = A_{tar}^{blue} \cup A_{ini}^{red}$ satisfies the conditions.

Otherwise, let $A_{ini}^{blue} := A_{ini}^{blue} = A_{tar}^{blue}$. By Lemma 9, we obtain a sequence $A_{0}^{red}, A_{1}^{red}, \ldots, A_{l}^{red}$ of orientations of $E^{red}$ such that $\phi(A_{ini}^{red}) = \phi(A_{ini}^{red})$, $\rho_{A_{i}^{red}} = \rho_{A_{i}^{red}}$, $A_{1}^{red} \leftrightarrow A_{l}^{red}$ for $i = 1, \ldots, l$, and $\rho_{A_{i}^{red}}(v) \geq \min\{\rho_{A_{i}^{red}}(v), \rho_{A_{i}^{red}}(v)\}$ for any $v \in V$ and any $i \in \{0, 1, \ldots, l\}$. Then, for any $i \in \{0, 1, \ldots, l\}$, we have

$$2\rho_{A_{i}^{red}}(v) + \rho_{A_{i}^{red}}(v) \geq \min\{2\rho_{A_{i}^{red}}(v) + \rho_{A_{i}^{red}}(v), 2\rho_{A_{i}^{red}}(v) + \rho_{A_{i}^{red}}(v)\} \geq 2$$

for any $v \in V$, and hence $A_{i}^{blue} \cup A_{i}^{red}$ is feasible. Since $A_{i}^{blue} \cup A_{i-1}^{red} \leftrightarrow A_{i}^{blue} \cup A_{i}^{red}$ for $i = 1, \ldots, l$, we have

$$(A_{ini} =) A_{i}^{blue} \cup A_{i}^{red} \sim A_{i}^{blue} \cup A_{i}^{red}.$$  
Furthermore, since $\rho_{A_{i}^{red}} = \rho_{A_{i}^{red}}$, we have $\phi(A_{i}^{blue} \cup A_{i}^{red}) = \phi(A_{i}^{red})$. Therefore, $A_{tar}^{\circ} := A_{i}^{blue} \cup A_{i}^{red}$ satisfies the conditions in the lemma.

Note that we can construct $A_{tar}^{\circ}$ and a reconfiguration sequence in Lemma 12 efficiently.

### 5.3 Algorithm

Let $I = (G, A_{ini}, A_{tar})$ be an instance of C2C. We first compute $\phi(A_{ini})$ and $\phi(A_{tar})$, and test whether $\phi(A_{ini}) \not\sim \phi(A_{tar})$ or not. If $\phi(A_{ini}) \not\sim \phi(A_{tar})$, then we can immediately conclude that $A_{ini} \not\sim \sim A_{tar}$ by Lemma 13.

Thus, in what follows, suppose that $\phi(A_{ini}) \sim \phi(A_{tar})$. In this case, by applying Lemma 12, we can construct $A_{tar}^{\circ} \in \mathcal{A}$ with $\phi(A_{tar}^{\circ}) = \phi(A_{tar})$ such that $A_{ini} \sim \sim A_{tar}^{\circ}$. This shows that $A_{ini} \sim \sim A_{tar}$ is equivalent to $A_{tar}^{\circ} \sim \sim A_{tar}$, which means that we can regard $A_{tar}^{\circ}$ as a new initial configuration instead of $A_{ini}$. Thus, the problem is reduced to the case when $\phi(A_{ini}) = \phi(A_{tar})$. In particular, we have $A_{ini}^{blue} = A_{tar}^{blue}$.

Suppose that $A_{ini}^{blue} = A_{tar}^{blue}$ = $A_{tar}^{blue}$ and $\rho_{A_{tar}^{red}} \neq \rho_{A_{tar}^{red}}$. Then, by applying Lemma 9, we obtain an orientation $A_{i}^{red}$ of $E^{red}$ such that $\rho_{A_{i}^{red}} = \rho_{A_{i}^{red}}$ and $A_{ini} \sim A_{i}^{blue} \cup A_{i}^{red}$. This shows that $A_{ini} \sim A_{tar}$ is equivalent to $A_{i}^{blue} \cup A_{i}^{red} \sim A_{tar}$, which means that we can regard $A_{i}^{blue} \cup A_{i}^{red}$ as a new initial configuration instead of $A_{ini}$. Thus, the problem is reduced to the case when $\rho_{A_{i}^{red}} = \rho_{A_{i}^{red}}$. If $A_{ini} = A_{tar}$, we conclude that $A_{ini} \sim A_{tar}$. Otherwise, since $\rho_{A_{i}^{red}} = \rho_{A_{i}^{red}}$, the set $A_{i}^{red} \setminus A_{i}^{red}$ can be decomposed into arc-disjoint cycles.

Note that all of the above procedures can be executed in $O(k) \cdot \text{poly}(|V|)$ time, since $|\mathcal{F}| = 2^O(k)$. In what follows, we give an algorithm for testing whether the direction of each cycle can be reversed or not. For this purpose, we show the following lemma.

**Lemma 13.** Let $A_{ini} \in \mathcal{A}$ and let $C$ be a dicycle with all the arcs in $A_{i}^{red}$. Then, the followings are equivalent.

1. $A_{ini} \sim (A_{ini} \setminus C) \cup \overline{C}$, where $\overline{C}$ is the reverse dicycle of $C$.
2. For any arc $a$ in $C$, there exists an orientation $A \in \mathcal{A}$ such that $A_{ini} \sim A$ and $a \notin A$.
3. For any $u \in V(C)$, there exists an orientation $A \in \mathcal{A}$ such that $A_{ini} \sim A$ and $2\rho_{A_{ini}}(u) + \rho_{A_{ini}}(u) \geq 3$.

**Proof.** We prove (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i), respectively.

[(i) $\Rightarrow$ (ii)] If (i) holds, then $A := (A_{ini} \setminus C) \cup \overline{C}$ satisfies the conditions in (ii), since it contains no arc in $C$. 

10
Algorithm 1: Algorithm for the Reconfiguration Problem

\textbf{Input}: a graph $G = (V, E)$ and orientations $A_{\text{ini}}, A_{\text{tar}} \in \mathcal{A}$.
\textbf{Output}: “yes” if $A_{\text{ini}} \leftrightarrow A_{\text{tar}}$, and “no” otherwise.

1. Compute $\mathcal{F}, \phi(A_{\text{ini}})$, and $\phi(A_{\text{tar}})$;
2. if $\phi(A_{\text{ini}}) \not\leq \phi(A_{\text{tar}})$ then return “no”;
3. if $\phi(A_{\text{ini}}) \neq \phi(A_{\text{tar}})$ or $\rho_{A_{\text{ini}}} \neq \rho_{A_{\text{tar}}}$ then
4. Compute $A \in \mathcal{A}$ such that $\phi(A) = \phi(A_{\text{tar}})$, $\rho_{A_{\text{ini}}} = \rho_{A_{\text{tar}}}$, and $A_{\text{ini}} \leftrightarrow A$;
5. $A_{\text{ini}} \leftarrow A$; // See Section 5.2
6. while $A_{\text{ini}}^{\text{red}} \setminus A_{\text{tar}}^{\text{red}}$ contains a dicycle $C$ do
7. Take $u \in V(C)$ and solve PROBLEM A;
8. if PROBLEM A has no feasible solution then
9. Return “no”;
10. else
11. $A_{\text{ini}} \leftarrow (A_{\text{ini}} \setminus C) \cup \bar{C}^{-}$; // See Section 5.3
12. return “yes”;

\[ [(\text{ii}) \Rightarrow (\text{iii})] \text{ We prove the contraposition. Assume that (iii) does not hold, that is, there exists a vertex } u \in V(C) \text{ such that } 2\rho_{A_{\text{ini}}} + \rho_{A_{\text{tar}}} = 2 \text{ for any } A \in \mathcal{A} \text{ with } A_{\text{ini}} \leftrightarrow A. \text{ Let } a \text{ be the arc in } C \text{ that enters } u. \text{ Since we cannot reverse the direction of } a \text{ without violating the feasibility, } a \text{ is contained in any orientation } A \in \mathcal{A} \text{ with } A_{\text{ini}} \leftrightarrow A. \]

\[ [(\text{iii}) \Rightarrow (\text{i})] \text{ Suppose that (iii) holds. We take a sequence } A_0, A_1, \ldots, A_l \text{ of feasible orientations of } E \text{ such that } A_0 = A_{\text{ini}}, A_l \text{ is obtained from } A_{l-1} \text{ by reversing an arc } a_i \in A_{l-1} \text{ for } i \in \{1, 2, \ldots, l\}, \text{ and there exists } u \in V(C) \text{ such that } 2\rho_{A_{\text{ini}}} + \rho_{A_{\text{tar}}} \geq 3. \text{ By taking a minimal sequence with these conditions, we may assume that } a_i \text{ is not contained in } C \text{ for } i \in \{1, 2, \ldots, l\}. \text{ Since } 2\rho_{A_{\text{ini}}} + \rho_{A_{\text{tar}}} \geq 3, \text{ starting from } A_1, \text{ we can change the direction of each arc in } C \text{ one by one without violating the feasibility, which shows that } A_{\text{ini}} \leftrightarrow (A_l \setminus C) \cup \bar{C}. \text{ On the other hand, since } (A_l \setminus C) \cup \bar{C} \text{ is obtained from } (A_{l-1} \setminus C) \cup \bar{C} \text{ by reversing } a_i \text{ for } i \in \{1, 2, \ldots, l\}, \text{ we obtain } (A_{\text{ini}} \setminus C) \cup \bar{C} \leftrightarrow (A_l \setminus C) \cup \bar{C}. \text{ Thus, it holds that } A_{\text{ini}} \leftrightarrow (A_l \setminus C) \cup \bar{C} \leftrightarrow (A_{\text{ini}} \setminus C) \cup \bar{C}. \]

Let $C$ be a dicycle in $A_{\text{ini}}^{\text{red}} \setminus A_{\text{tar}}^{\text{red}}$. Fix a vertex $u \in V(C)$ and consider the following problem, for which an algorithm is presented later in Section 5.4

\textbf{PROBLEM A}

\textbf{Input}: A constraint graph $G$, an orientation $A_{\text{ini}} \in \mathcal{A}$, and a vertex $u \in V(G)$.
\textbf{Task}: Find an orientation $A \in \mathcal{A}$ s.t. $2\rho_{A_{\text{ini}}} + \rho_{A_{\text{tar}}} \geq 3$ and $A_{\text{ini}} \leftrightarrow A$ (if exists).

If \textbf{PROBLEM A} has no solution, then condition (iii) in Lemma 13 does not hold. This shows that the condition (ii) in Lemma 13 does not hold, that is, there exists an arc $a$ in $C$ that is contained in any orientation $A \in \mathcal{A}$ with $A_{\text{ini}} \leftrightarrow A$. In this case, since $a \in A_{\text{ini}} \setminus A_{\text{tar}}$, we conclude that $A_{\text{ini}} \not\leftrightarrow A_{\text{tar}}$.

Otherwise, \textbf{PROBLEM A} has a solution, and hence the condition (iii) in Lemma 13 holds. Since it is equivalent to the condition (i) in Lemma 13, we have that $A_{\text{ini}} \leftrightarrow (A_{\text{ini}} \setminus C) \cup \bar{C}$. Therefore, $A_{\text{ini}} \leftrightarrow A_{\text{tar}}$ is equivalent to $(A_{\text{ini}} \setminus C) \cup \bar{C} \leftrightarrow A_{\text{tar}}$, which means that we can regard $(A_{\text{ini}} \setminus C) \cup \bar{C}$ as a new initial configuration instead of $A_{\text{ini}}$. Then, the problem is reduced to the case with smaller $|A_{\text{ini}} \setminus A_{\text{tar}}|$. By applying this procedure at most $O(|E|)$ times repeatedly, we can solve the original reconfiguration problem. The entire algorithm is shown in Algorithm 1

5.4 Algorithm for \textbf{PROBLEM A}

The remaining task is to give a polynomial time algorithm for \textbf{PROBLEM A}. For this purpose, we use a similar argument to Section 5.2. Suppose we are given a graph $G = (V, E)$ and a vertex $u \in V$. Recall that $X \subseteq V$ is the set of all vertices to which edges in $E_{\text{blue}}$ are incident. Define $\mathcal{F}_u$ as the set of all pairs $(A_{\text{blue}}, d)$, where $A_{\text{blue}}$ is an orientation of $E_{\text{blue}}$ and $d$ is a vector in $\{0, 1, 2, 3\}^{X \cup \{u\}}$ satisfying the following conditions:
Lemma 16. Proposition 17. φ
\[ F \]

We note that C2C of a general constraint graph. We give FPT algorithms for the
Proof.
\[ ( \text{Lemma 15.} \]

\[ ( \]

\[ A \]

\[ \text{Conversely, assume that there exists a pair (} A^{\text{blue}}, d) \in F_u, \text{ such that} \]

\[ 2 \rho_{A^{\text{blue}}}(u) + d(u) = 3, \text{ and} \]

\[ \phi_u(A_{\text{ini}}) \iff \phi_u(A_{\text{tar}}). \]

\[ \text{Problem A has a solution if and only if there exists a pair (} A^{\text{blue}}, d) \in F_u \text{ such that} \]

\[ 2 \rho_{A^{\text{blue}}}(u) + d(u) \geq 3 \text{ and} \phi_u(A_{\text{ini}}) \iff (A^{\text{blue}}, d). \]

\[ \text{Proof. If A is a solution of Problem A, then} \phi_u(A) = (A^{\text{blue}}, d) \text{ satisfies the conditions by Lemma 15. Conversely, assume that there exists a pair (} A^{\text{blue}}, d) \in F_u \text{ such that} \]

\[ 2 \rho_{A^{\text{blue}}}(u) + d(u) \geq 3 \text{ and} \phi_u(A_{\text{ini}}) \iff (A^{\text{blue}}, d). \]

\[ \text{By Lemma 16 there exists an orientation} A \in \mathcal{A} \text{ with} \phi_u(A) = (A^{\text{blue}}, d) \text{ such that} A_{\text{ini}} \iff A. \text{ Since} 2 \rho_{A^{\text{blue}}}(u) + \rho_{A^{\text{red}}}(u) \geq 2 \rho_{A^{\text{blue}}}(u) + d(u) \geq 3, A \text{ is a solution of Problem A.} \]

By this proposition, in order to solve Problem A, it suffices to test whether there exists a pair (A^{\text{blue}}, d) such that 2 \rho_{A^{\text{blue}}}(u) + d(u) \geq 3 and \phi_u(A_{\text{ini}}) \iff (A^{\text{blue}}, d). Since |F_u| = 2^{O(k)}, it can be checked in \(2^{O(k)} \cdot \text{poly}(|V|)\) time. Note that the elements of F_u can be computed in \(2^{O(k)} \cdot \text{poly}(|V|)\) time by Lemma 17. Thus, Algorithm 7 solves the problem C2C in \(2^{O(k)} \cdot \text{poly}(|V|)\) time.

Using similar arguments as in Theorem 8 we can also solve the C2E version.

Corollary 18. C2E parameterized by the number k of blue edges can be solved in time \(2^{O(k)} \cdot \text{poly}(|V|)\).

6 Conclusion

We investigated the parameterized complexity of NCL for four natural parameters related to the constraint graph: The number of AND/OR vertices of an AND/OR graph and the number of red/blue edges of a general constraint graph. We give FPT algorithms for the C2C and C2E version of NCL for each parameter and in particular a linear kernel for NCL parameterized by he number of red edges. An interesting question for future work is whether there is a polynomial kernel for NCL parameterized by the number of OR vertices or the number of blue edges.

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A Proofs Omitted from Section 3

In this section we prove Theorem 1. To do so, we first formally define and prove the correctness of the preprocessing step. We then proceed to give an FPT-reduction from preprocessed instances of C2C parameterized by the number of or vertices to BCSR.

A.1 Correctness of the Preprocessing

Let $\hat{G}, \hat{A}_{ini}, \hat{A}_{tar}$ be a given instance of NCL such that $\hat{G} = (\hat{V}, \hat{E})$ is an AND/OR graph. For each $xy \in \hat{E}^{\text{blue}}$ which is not a loop, we subdivide it by adding a new vertex $z$; and let $w(xz) = w(zy) = 2$. We call the newly inserted vertex $z$ a middle vertex between $x$ and $y$. Let $G = (V, E)$ be the resulting graph.

Note that $G$ is not an AND/OR graph, and $V$ can be partitioned into $V_{\text{red}}(G)$, $V_{\text{blue}}(G)$ and $V_{\text{sm}}(G)$, where $V_{\text{sm}}(G)$ (or simply $V_{\text{sm}}$) is the set of middle vertices in $G$. Since $V_{\text{red}}(G) = V_{\text{blue}}(G)$ holds, we will show in this subsection that solving $(\hat{G}, \hat{A}_{ini}, \hat{A}_{tar})$ with the parameter $|V_{\text{red}}(G)| \leq k$ is equivalent to solving $(G, A_{ini}, A_{tar})$ with the parameter $|V_{\text{red}}(G)| = |V_{\text{red}}(G)| \leq k$.

Consider any orientation $\hat{A}$ of the original edge set $\hat{E}$. Then, we define an orientation $A$ of $E$, as follows: for each $(x, y) \in E^{\text{blue}}$ such that $x \neq y$, we delete $(x, y)$ from $A$, and add two arcs $(x, z)$ and $(z, y)$; let $A$ be the resulting orientation of $E$. We observe that the following lemma holds.

In particular, we construct two (feasible) orientations $A_{ini}$ and $A_{tar}$ of $E$ which correspond to the original (feasible) orientations $\hat{A}_{ini}$ and $\hat{A}_{tar}$ of $\hat{E}$, respectively. In this way, we obtain the instance $(G, A_{ini}, A_{tar})$ of NCL as the result of the preprocessing to $(\hat{G}, \hat{A}_{ini}, \hat{A}_{tar})$.

Lemma 19. $A$ is a feasible orientation of $E$ if and only if $\hat{A}$ is a feasible orientation of $\hat{E}$.

Proof. For each middle vertex $z \in V_{\text{sm}}$ between $x$ and $y$, it holds that $\rho_{\text{red}}(z) + 2 \cdot \rho_{\text{blue}}(z) = 2$ because $(x, z) \in A$ and $(y, z) \notin A$. Thus, $A$ is feasible if and only if $\rho_{\text{red}}(v) + 2 \cdot \rho_{\text{blue}}(v) \geq 2$ for every $v \in V(G) \setminus V_{\text{sm}} = V(\hat{G})$. Therefore, the lemma holds.

Let $(G, A_{ini}, A_{tar})$ be a given instance of NCL such that $\hat{G} = (V, E)$ is an AND/OR graph. For each $xy \in \hat{E}^{\text{blue}}$ which is not a loop, we subdivide it by adding a new vertex $z$; and let $w(xz) = w(zy) = 2$. We call the newly inserted vertex $z$ a middle vertex between $x$ and $y$. Let $G = (V, E)$ be the resulting graph.

Note that $G$ is not an AND/OR graph, and $V$ can be partitioned into $V_{\text{red}}(G)$, $V_{\text{blue}}(G)$ and $V_{\text{sm}}(G)$, where $V_{\text{sm}}(G)$ (or simply $V_{\text{sm}}$) is the set of middle vertices in $G$. Since $V_{\text{red}}(G) = V_{\text{blue}}(G)$ holds, we will show in this subsection that solving $(G, A_{ini}, A_{tar})$ with the parameter $|V_{\text{red}}(G)| \leq k$ is equivalent to solving $(G, A_{ini}, A_{tar})$ with the parameter $|V_{\text{red}}(G)| = |V_{\text{red}}(G)| \leq k$.

Consider any orientation $\hat{A}$ of the original edge set $\hat{E}$. Then, we define an orientation $A$ of $E$, as follows: for each $(x, y) \in \hat{E}^{\text{blue}}$ such that $x \neq y$, we delete $(x, y)$ from $A$, and add two arcs $(x, z)$ and $(z, y)$; let $A$ be the resulting orientation of $E$. We observe that the following lemma holds.

In particular, we construct two (feasible) orientations $A_{ini}$ and $A_{tar}$ of $E$ which correspond to the original (feasible) orientations $\hat{A}_{ini}$ and $\hat{A}_{tar}$ of $\hat{E}$, respectively. In this way, we obtain the instance $(G, A_{ini}, A_{tar})$ of NCL as the result of the preprocessing to $(\hat{G}, \hat{A}_{ini}, \hat{A}_{tar})$. Lemma 2 ensures that the preprocessing preserves the reconfigurability.

Proof of Lemma 2. We first prove the if direction. Assume that $(\hat{G}, \hat{A}_{ini}, \hat{A}_{tar})$ is a yes-instance, and hence there exists a reconfiguration sequence $(A_{0}, A_{1}, \ldots, A_{\ell})$ between $A_{0} = A_{ini}$ and $A_{\ell} = A_{tar}$. For each $i \in \{0, 1, \ldots, \ell\}$, we define the arc set $A_{i}$ by replacing $(x, y) \in A_{i}$ with two arcs $(x, z), (z, y)$ for all middle vertices $z \in V_{\text{sm}}$ that subdivide $xy \in E^{\text{blue}}$. Since each $A_{i}$ is a feasible orientation of $E$, Lemma 19 says that $A_{i}$ is a feasible orientation of $E$. In addition, by the definition, we know that $A_{0} = A_{ini}$ and $A_{\ell} = A_{tar}$. Then, the following claim proves that $(G, A_{ini}, A_{tar})$ is a yes-instance.

Claim 1. $A_{i-1} \leftrightarrow A_{i}$ holds for each $i \in \{1, 2, \ldots, \ell\}$.

Proof of Claim 1. If $\hat{A}_{i-1} = \hat{A}_{i}$ holds, then we have $A_{i-1} = A_{i}$ and hence the claim trivially holds. We thus assume that $\hat{A}_{i-1} \neq \hat{A}_{i}$. Since $\hat{A}_{i-1} \leftrightarrow \hat{A}_{i}$, there exists an arc $(x, y) \in \hat{A}_{i-1}$ such that $\hat{A}_{i-1} = A_{i-1} - (x, y) + (y, x)$ where $x \neq y$. If the arc $(x, y) \in A_{i-1}$ is red, then $(x, y) \in A_{i-1}$ and we have $A_{i-1} - (x, y) + (y, x) = A_{i}$. Therefore, $A_{i-1} \leftrightarrow A_{i}$ holds, and hence we have $A_{i-1} \leftrightarrow A_{i}$.

We thus consider the remaining case, that is, the arc $(x, y) \in \hat{A}_{i-1}$ is blue and $x \neq y$. Let $z$ be the middle vertex between $x$ and $y$. Then, we know that $A_{i-1} \setminus A_{i} = \{(x, z), (z, y)\}$ and $A_{i} \setminus A_{i-1} = \{(z, x), (y, z)\}$. Let $A := A_{i-1} - (z, y) + (y, z)$, then $A_{i-1} \leftrightarrow A$ holds. We now prove that $(A_{i-1}, A, A_{i})$ is
a reconfiguration sequence between \( A_{i-1} \) and \( A_i \). Note that \( A \leftrightarrow A_i \) holds since \( A_i = A - (x, z) + (z, x) \). Therefore, it suffices to show that \( A \) is a feasible orientation of \( E \). To see this, we observe that for each vertex \( v \in V(G) \),

\[
\rho_{A_{red}}(v) + 2 \cdot \rho_{A_{blue}}(v) = \begin{cases} 
4 & \text{if } v = z; \\
\rho_{A_{red}}(v) + 2 \cdot \rho_{A_{blue}}(v) & \text{if } v = y; \\
\rho_{A_{red}}(v) + 2 \cdot \rho_{A_{blue}}(v) & \text{otherwise.}
\end{cases}
\]

Since both \( A_{i-1} \) and \( A_i \) are feasible orientations of \( E \), it thus holds that \( \rho_{A_{red}}(v) + 2 \cdot \rho_{A_{blue}}(v) \geq 2 \) for all \( v \in V(G) \). Therefore, \( A \) is also feasible, and hence we have \( A_{i-1} \leftrightarrow A_i \).

This completes the proof of the if direction.

We then prove the only-if direction. Assume that \((G, A_{ini}, A_{tar})\) is a yes-instance, and hence there exists a reconfiguration sequence \((A_0, A_1, \ldots, A_\ell)\) between \( A_0 = A_{ini} \) and \( A_\ell = A_{tar} \). For each \( i \in \{0, 1, \ldots, \ell\} \), we define an orientation \( A_i \) of \( E \) from \( A_i \), as follows: for each middle vertex \( z \) in \( G \) which subdivides \( xy \in E \), we introduce a variable 

\[
\rho_{A_{red}}(v) + 2 \cdot \rho_{A_{blue}}(v) = \begin{cases} 
4 & \text{if } v = z; \\
\rho_{A_{red}}(v) + 2 \cdot \rho_{A_{blue}}(v) & \text{if } v = y; \\
\rho_{A_{red}}(v) + 2 \cdot \rho_{A_{blue}}(v) & \text{otherwise.}
\end{cases}
\]

for each \( v \in V(G) \). Then, we have the following claim.

\textbf{Claim 2.} For each \( i \in \{1, 2, \ldots, \ell\} \), it holds that \( A_{i-1} \leftrightarrow A_i \).

\textbf{Proof of Claim} \( \clubsuit \) If \( A_{i-1} = A_i \) holds, then we have \( A_{i-1} = A_i \) and hence the claim holds. We thus assume that \( A_{i-1} \neq A_i \). Since \( A_{i-1} \leftrightarrow A_i \), there exists an arc \((u, v) \in A_{i-1}\) such that \( A_i = A_{i-1} - (u, v) + (u, v) \) where \( u \neq v \). Then, we know that either \( u \) or \( v \) is a middle vertex \( z \) which was inserted to a blue edge \( xy \in E_{blue} \). If \( (u, v) = (z, y) \in A_{i-1} \), then \( (y, z) \in A_i \) and hence we have \( (x, y) \in A_{i-1} \) and \( A_i - (u, v) + (v, u) = A_i \). Therefore, \( A_{i-1} \leftrightarrow A_i \) holds.

We thus consider the remaining case, that is, the arc \((u, v) \in A_{i-1}\) is blue and \( u \neq v \). Then, we know that either \( u \) or \( v \) is a middle vertex \( z \) which was inserted to a blue edge \( xy \in E_{blue} \). If \( (u, v) = (z, y) \in A_{i-1} \), then \( (y, z) \in A_i \) and hence we have \((x, y) \in A_{i-1} \) and \( A_i - (x, y) + (y, x) = A_i \); it thus holds that \( A_{i-1} \leftrightarrow A_i \). If \((u, v) \neq (z, y)\), then we have \( A_{i-1} = A_i \) and hence \( A_{i-1} \leftrightarrow A_i \). \( \square \)

By Claim \( \clubsuit \) we can obtain a reconfiguration sequence between \( A_{ini} = A_0 \) and \( A_{tar} = A_\ell \) as a subsequence of \((A_0, A_1, \ldots, A_\ell)\) by ignoring repetitions of the same orientations. Thus, \((G, A_{ini}, A_{tar})\) is a yes-instance. This completes the proof of the only-if direction.

In this way, to prove Theorem \( \heartsuit \) it suffices to solve the instance \((G, A_{ini}, A_{tar})\) with the parameter \(|V_{ori}(G)| \leq k\). Recall that \( V(G) \) can be partitioned into three subsets \( V_{AND}(G) \), \( V_{OR}(G) \) and \( V_{OR}(G) \). By the construction of \( G \), observe that \( G \) has no multiple blue edges. In addition, no edge in \( G \) joins an AND vertex and an OR vertex, and hence we can partition \( E \) into two subsets \( E_{AND} \) and \( E_{OR} \), defined as follows: \( E_{AND} \) is the set of edges in \( G \) that are incident to AND vertices in \( V(G) \); and \( E_{OR} \) is the set of edges in \( G \) that are incident to OR vertices in \( V(G) \).

\subsection*{A.2 FPT-reduction to BCSR}

In this subsection, we construct an FPT-reduction to BCSR. Recall that BCSR can be solved in time \( O^*(d^{O(p)}) \), where \( d := \max_{x \in X} |D(x)| \) and \( p \) is the number of non-Boolean variables in \( X \) \( \clubsuit \) Theorem 18\( \). Therefore, as a proof of Theorem \( \heartsuit \) we construct an FPT-reduction from a preprocessed instance \((G, A_{ini}, A_{tar})\) of NCL with the parameter \(|V_{ori}(G)| \leq k\) to an instance \((H, D, C, \Gamma_{ini}, \Gamma_{tar})\) of BCSR such that both \( d \) and \( p \) are bounded by some computable functions depending only on \( k \).

We first construct the set \( X \) of variables, as follows:

- for each \( v \in V_{ori}(G) \), we introduce a variable \( x_v \), called an OR variable; and
• for each \( e \in E_{\text{AND}} \), we introduce a variable \( x_e \), called an edge variable.

We then construct the domain \( D(x) \) for each \( x \in X \), as follows:

• If \( x \) is an or variable \( x_v \) for \( v \in V_{\text{OR}}(G) \), then we consider the following two cases:
  
  – Consider the case where there is a loop \( vv \in E_{\text{OR}} \). Since \( v \) is an or vertex, it has exactly one blue edge \( vv' \in E_{\text{OR}} \) such that \( v' \) is a middle vertex. In this case, let \( D(x_v) := \{ \emptyset, \{ v' \} \} \).

    Consider the other case, that is, \( vv \not\in E_{\text{OR}} \). Since \( G \) has no multiple blue edges, \( v \) has three distinct neighbors (middle vertices), say \( v_1, v_2 \) and \( v_3 \), in \( G \). In this case, let \( D(x_v) := \{ \emptyset, \{ v_1 \}, \{ v_2 \}, \{ v_3 \}, \{ v_1, v_2 \}, \{ v_2, v_3 \}, \{ v_3, v_1 \} \} \). We regard that assigning \( \{ v' \} \in D(x_v) \) to \( x_v \) corresponds to directing the edge \( vv' \) as \( (v, v') \), while \( \emptyset \in D(x_v) \) to \( (v', v) \). Note that the loop \( vv \) has only one possible direction, and any orientation of \( E \) contains the arc \( (v, v') \).

• If \( x \) is an edge variable \( x_e \) for \( e = uv \in E_{\text{AND}} \), then let \( D(x_e) := \{ \{ u \}, \{ v \} \} \). We regard that assigning \( \{ u \} \in D(x_e) \) to \( x_e \) corresponds to directing \( uv \) as \( (u, v) \), while \( \{ v \} \in D(x_e) \) to \( (v, u) \).

Since \( E \) can be partitioned into \( E_{\text{OR}} \) and \( E_{\text{AND}} \), any solution \( \Gamma \) of \( G \) defines an orientation of \( E \). Conversely, any orientation of \( E \) defines a solution \( \Gamma \) of \( G \). We note that \( d = \max_{x \in X} |D(x)| = 7 \). Furthermore, notice that only or variables \( x_v \) without loops are non-Boolean variables, and the other variables are Boolean variables. Therefore, \( p \leq |V_{\text{OR}}(G)| \leq k \), where \( p \) is the number of non-Boolean variables in \( X \).

We finally construct the set of constraints, which also defines the set of edges in \( H \). Our aim here is to ensure that a solution \( \Gamma \) of \( G \) is proper if and only if the corresponding orientation \( A_{\Gamma} \) of \( E \) is feasible. By the construction of domains above, we know that \( \rho_{A_{\Gamma}^\text{red}}(v) + 2 \cdot \rho_{A_{\Gamma}^\text{blue}}(v) \geq 2 \) holds for each \( v \in V_{\text{OR}}(G) \). Therefore, we construct three types of constraints for middle vertices and AND vertices, as follows:

**Type 1:** Constraints for middle vertices.

Let \( v \) be a middle vertex between two vertices \( v_1 \) and \( v_2 \). Since both \( v_1v \) and \( v_2v \) are blue edges, \( \rho_{A_{\Gamma}^\text{red}}(v) + 2 \cdot \rho_{A_{\Gamma}^\text{blue}}(v) \geq 2 \) holds if and only if \( (v_1, v) \in A_{\Gamma} \) or \( (v_2, v) \in A_{\Gamma} \) hold. For each \( i \in \{1, 2\} \), let

\[
\begin{align*}
  x_i = \begin{cases}
    x_{v_i} & \text{if } v_i \text{ is an OR vertex;} \\
    x_{vv_i} & \text{otherwise.}
  \end{cases}
\end{align*}
\]

Then, we let \( C(x_1x_2) := \{ S_1S_2 \in D(x_1) \times D(x_2) : v \in S_1 \text{ or } v \in S_2 \} \).

**Type 2-1:** Constraints for AND vertices having loops.

Let \( v \) be an AND vertex having a loop \( vv \). Since \( v \) is an AND vertex, we know that \( vv \) must be red, and the remaining edge \( v_3v \in E_{\text{AND}} \) is blue where \( v_3 \) is a middle vertex. Then, \( \rho_{A_{\Gamma}^\text{red}}(v) + 2 \cdot \rho_{A_{\Gamma}^\text{blue}}(v) \geq 2 \) holds if and only if \( (v_1, v) \in A_{\Gamma} \) or \( (v_2, v) \in A_{\Gamma} \) hold. Since \( vv \) and \( v_3v \) are in \( E_{\text{AND}} \), there are corresponding edge variables \( x_{vv} \) and \( x_{v_3v} \). Then, we let \( C(x_{vv}x_{v_3v}) := \{ SS_3 \in D(x_{vv}) \times D(x_{v_3v}) : v \in S_3 \} \).

**Type 2-2:** Constraints for AND vertices having no loop.

Let \( v \) be an AND vertex, and let \( v_1, v_2, v_3 \) be three (distinct) edges incident to \( v \) such that \( vv_1 \) and \( vv_2 \) are red, and \( vv_3 \) is blue; it may hold that \( v_1 = v_2 \). Then, \( \rho_{A_{\Gamma}^\text{red}}(v) + 2 \cdot \rho_{A_{\Gamma}^\text{blue}}(v) \geq 2 \) holds if and only if \( A_{\Gamma} \) satisfies both the following two conditions:

1. it holds that \( (v_1, v) \in A_{\Gamma} \) or \( (v_3, v) \in A_{\Gamma} \); and
2. it holds that \( (v_2, v) \in A_{\Gamma} \) or \( (v_3, v) \in A_{\Gamma} \).

Since \( vv_1, vv_2 \) and \( vv_3 \) are in \( E_{\text{AND}} \), there are corresponding edge variables \( x_{vv_1}, x_{vv_2} \) and \( x_{vv_3} \). Then, we let \( C(x_{vv_1}x_{vv_2}x_{vv_3}) := \{ S_1S_2S_3 \in D(x_{vv_1}) \times D(x_{vv_2}) \times D(x_{vv_3}) : v \in S_1 \text{ or } v \in S_2 \text{ or } v \in S_3 \} \), and \( C(x_{vv_1}x_{vv_3}) := \{ S_2S_3 \in D(x_{vv_2}) \times D(x_{vv_3}) : v \in S_2 \text{ or } v \in S_3 \} \).
By the construction of constraints above, we know that a solution $\Gamma$ of $G$ is proper if and only if the corresponding orientation $A_{\Gamma}$ of $E$ is feasible. Therefore, we can define proper solutions $\Gamma_{ini}$ and $\Gamma_{tar}$ of $H$ which correspond to feasible orientations $A_{ini}$ and $A_{tar}$ of $E$, respectively.

In this way, from a preprocessed instance $(G, A_{ini}, A_{tar})$ of NCL with the parameter $|V_{or}(G)| \leq k$, we have constructed the corresponding instance $(H, D, C, \Gamma_{ini}, \Gamma_{tar})$ of BCSR such that $d = \max_{x \in X} |D(x)| = 7$ and $p \leq |V_{or}(G)| \leq k$. In addition, we have shown that there is a one-to-one correspondence between proper solutions of $H$ and feasible orientations of $E$. Since BCSR can be solved in time $O^*(d^{O(p)})$ [5], the following lemma completes the proof of Theorem [1] for C2C.

**Lemma 20.** $(G, A_{ini}, A_{tar})$ is a yes-instance of NCL if and only if $(H, D, C, \Gamma_{ini}, \Gamma_{tar})$ is a yes-instance of BCSR.

**Proof.** We first prove the only-if direction. Assume that $(G, A_{ini}, A_{tar})$ is a yes-instance, and hence there exists a reconfiguration sequence $(A_0, A_1, \ldots, A_\ell)$ of feasible orientations of $E$ between $A_{ini}$ and $A_{\ell} = A_{tar}$. For each $i \in \{0, 1, \ldots, \ell\}$, let $\Gamma_i$ be the proper solution of $H$ defined by $A_i$. Then, we know that $\Gamma_0 = \Gamma_{ini}$ and $\Gamma_\ell = \Gamma_{tar}$ hold. To show that $(H, D, C, \Gamma_{ini}, \Gamma_{tar})$ is a yes-instance, it thus suffices to prove that $\Gamma_{i+1} \leftrightarrow \Gamma_i$ holds for each $i \in \{1, 2, \ldots, \ell\}$. Since $A_{i+1} \leftrightarrow A_i$, there exists an arc $(u, v)$ in $A_{i+1}$ such that $A_i = A_{i+1} \sim (u, v) + (v, u)$. We know that the edge $(u, v)$ in $G$ is either in $E_{AND}$ or in $E_{or}$. If $(u, v) \in E_{AND}$, then we have $|\{x \in X : \Gamma_{i+1}(x) \neq \Gamma_i(x)\}| = |\{x_{uv}\}| = 1$. Otherwise (i.e., if $(u, v) \in E_{or}$), then we have $|\{x \in X : \Gamma_{i+1}(x) \neq \Gamma_i(x)\}| = |\{x_{uv}\}| = 1$ where we assume without loss of generality that $v$ is an $OR$ vertex and $u$ is a middle vertex. Therefore, $\Gamma_{i+1} \leftrightarrow \Gamma_i$ holds for both cases, as claimed. This completes the proof of the only-if direction.

We then prove the if direction. Assume that $(H, D, C, \Gamma_{ini}, \Gamma_{tar})$ is a yes-instance, and hence there exists a sequence $(\Gamma_0, \Gamma_1, \ldots, \Gamma_\ell)$ of proper solutions of $H$ such that $\Gamma_0 = \Gamma_{ini}$, $\Gamma_\ell = \Gamma_{tar}$, and $\Gamma_{i+1} \leftrightarrow \Gamma_i$ holds for each $i \in \{1, 2, \ldots, \ell\}$. For each $i \in \{0, 1, \ldots, \ell\}$, let $A_i$ be the feasible orientation of $E$ defined by $\Gamma_i$. Then, we know that $A_0 = A_{ini}$ and $A_\ell = A_{tar}$ hold. To show that $(G, A_{ini}, A_{tar})$ is a yes-instance, we thus prove that $A_{i+1} \sim A_i$ holds for each $i \in \{1, 2, \ldots, \ell\}$. Since $\Gamma_{i+1} \leftrightarrow \Gamma_i$, there exists exactly one variable $x \in X$ such that $\Gamma_{i+1}(x) \neq \Gamma_i(x)$. Then, we consider the following three cases.

**Case 1.** $x$ is an edge variable $x_e$ for $e \in E_{AND}$.

In this case, we know that the difference between $A_{i+1}$ and $A_i$ is only the direction of $e$. Therefore, $A_i$ can be obtained from $A_{i+1}$ by reversing the direction of $e$, and hence we have $A_{i+1} \leftrightarrow A_i$. Thus, $A_{i+1} \sim A_i$ holds.

**Case 2.** $x$ is an OR variable $x_v$ for $v \in V_{or}(G)$ having a loop $vv \in E_{or}$.

In this case, we know that the difference between $A_{i+1}$ and $A_i$ is only the direction of $vv' \in E_{or}$, where $v'$ is the (unique) middle vertex adjacent to $v$. Therefore, we have $A_{i+1} \leftrightarrow A_i$, and hence $A_{i+1} \sim A_i$ holds.

**Case 2.** $x$ is an OR variable $x_v$ for $v \in V_{or}(G)$ having no loop.

Let $v_1, v_2, v_3$ be three (distinct) middle vertices adjacent to $v$. Recall that a solution $\Gamma$ of $H$ defines the directions of three edges $v_1v, v_2v$ and $v_3v$ in the corresponding orientation of $E$, as follows: $(v, v')$ if $v' \in \Gamma(v)$, and $(v', v)$ if $v' \in \{v_1, v_2, v_3\} \setminus \Gamma(v)$. Then, we construct a sequence of orientations of $E$ between $A_{i+1}$ and $A_i$, as follows:

1. for each $v' \in \Gamma_{i+1}(v) \setminus \Gamma_i(v)$, reverse the direction of $vv'$ from $(v, v')$ to $(v', v)$ one by one; and
2. for each $v' \in \Gamma_i(v) \setminus \Gamma_{i+1}(v)$, reverse the direction of $vv'$ from $(v', v)$ to $(v, v')$ one by one.

Let $(A^0, A^1, \ldots, A^\ell)$ be the sequence of orientations of $G$ defined as above, where $A^0 := A_{i+1}$ and $q := |\Gamma_{i+1}(v) \Delta \Gamma_i(v)|$. By the construction of the sequence, we know that $A^q = A_i$ and $A^j \leftrightarrow A^{j+1}$ for each $j \in \{1, 2, \ldots, q\}$. Furthermore, all orientations $A^j$ are feasible. Thus, $(A^0, A^1, \ldots, A^\ell)$ is a reconfiguration sequence between $A^q = A_{i+1}$ and $A^0 = A_i$, and hence we have $A_{i+1} \sim A_i$.

This completes the proof of the if direction.

It remains to prove the statement for C2E. For the C2E case, we give a reduction from NCL to BCSR. We use the same reduction for C2E case. While an FPT-algorithm for C2C of BCSR is given in [5], that for C2E is not. However, we can simply improve the FPT-algorithm for C2C to C2E as
follows: In the FPT-algorithm of [5], the authors first construct a contracted solution graph (CSG). Then they determine whether there is a path on CSG between two nodes corresponding to the initial and the target solutions. Since the size of CSG is FPT-size, the algorithm takes only FPT-time. Then for the C2E case, it is enough to determine whether there is a path on CSG between two nodes corresponding to the initial solution and any solution we wish.

In the remaining part of this proof, we show that the edge we wish to reverse in NCL is which variable in BCSR. In the preprocessing of our reduction, we subdivide each blue edge that is not a loop into two blue edges. Let \( G \) be the graph before the preprocessing and \( \hat{G} \) be a graph after preprocessing. Let \((u^*, v^*)\) be the orientation in \( A_{ini} \) of \( G \) that we wish to reverse. Note that the edge \( u^*v^* \) may be subdivided into \( u^*z^* \) and \( z^*v^* \) in the preprocessing. In other words, reversing \((u^*, v^*)\) in \( G \) corresponds to reversing both of \((u^*, z^*)\) and \((z^*, v^*)\) in \( \hat{G} \). However, since \( z^*v^* \) is a blue edge, after \((z^*, v^*)\) is reversed, \( z^* \) has enough in-weight and we can always reverse \((u^*, z^*)\). Therefore, reversing \((u^*, v^*)\) in \( G \) corresponds to reversing only \((z^*, v^*)\) in \( \hat{G} \).

Let \( e^* \) be the edge of \( \hat{G} \) that we wish to reverse. If \( e^* \) is incident to an OR vertex, the three edges including \( e^* \) in NCL correspond to one non-Boolean variable in BCSR, say \( v \). While \( v \) can be assigned seven values, four of them correspond to incoming direction of \( e^* \), and three of them correspond to outgoing direction of \( e^* \). Therefore, we need to search for a path between a node corresponding to the initial orientation and any node such that the value of \( v \) corresponds to a different direction of \( e^* \) from the initial one.

If \( e^* \) is incident to an AND vertex, \( e^* \) in NCL corresponds to a Boolean variable in BCSR. Therefore, reversing the direction of \( e^* \) corresponds to changing a value of this variable. In this case, a node corresponds to the initial orientation and a node corresponds to an orientation we wish to obtain. However, some nodes might be contracted in CSG. Before some nodes are contracted, some variables of which we cannot change the value at all are deleted. Therefore, if \( e^* \) corresponds to such a variable, we answer NO. Otherwise, we only check whether there exists at least one feasible solution in BCSR such that the variable has a different value from the initial one. Since contracted nodes in CSG are always connected, if there exists such a feasible solution, we answer YES, and otherwise NO.

By above discussion, our reduction for C2C also works for C2E.

### B Proofs Omitted from Section 4

**Proof of Proposition 3.** We prove that the four reduction rules are safe one by one. We first show that Reduction Rule 1 is safe.

Let \( C \) be a component of \( G \) that is a blue chordless cycle. Observe that no arc on \( C \) can be reversed. Therefore, if \( A_{ini} \) and \( A_{tar} \) disagree on \( C \), then we have a no instance. On the other hand, if \( A_{ini} \) and \( A_{tar} \) agree on \( C \) then we may remove the component \( C \) from \( G \) and continue.

We now show that Reduction Rule 2 is safe. Let \( C \) be a component of \((V, E^{blue})\) containing at least two cycles \( K_1 \) and \( K_2 \) and let \( kc \) be the number of red vertices of \( C \). Without loss of generality we assume that no proper subset of \( E(K_1) \cup E(K_2) \) contains two distinct cycles. Let \( G_C := (V(C), E^{blue} \cap E(C)) \) be the graph induced by the vertices of the component \( C \) of the blue subgraph of \( G \) and let \( A_{ini} \) and \( A_{tar} \) be the start and target orientations of \( E \). We first prove the following two claims.

**Claim 1.** There is a feasible orientation \( A_{ini}^c \) of the constraint graph \( G \) with the following properties:

1. The orientations \( A_{ini} \) and \( A_{ini}^c \) agree on \( E - E(C) \).
2. \( K_1 \) and \( K_2 \) are oriented cycles with respect to \( A_{ini}^c \).
3. each vertex \( v \) of \( C \) has at least one edge in \( E(C) \) oriented towards \( v \) by \( A_{ini}^c \), and
4. there is a transformation from \( A_{ini} \) to \( A_{ini}^c \).

**Proof of Claim 1.** If \( K_1 \) or \( K_2 \) is a directed cycle with respect to the orientation \( A_{ini} \), we leave the orientation of the cycle as it is. Otherwise, assume without loss of generality that \( K_1 \) is not a directed cycle with respect to \( A_{ini} \). Then there is at least one vertex \( v \) of \( K_1 \) having two blue edges of \( K_1 \) oriented towards \( v \) by \( A_{ini} \). Reversing one of the two arcs yields a feasible orientation. After this step one of the neighbors of \( v \), say \( w \), has one additional incoming edge of \( K_1 \). If the remaining edge of \( w \) on \( K_1 \) is an incoming edge, we reverse this edge. Otherwise we leave it as it is. By performing these steps in a
consistent manner we obtain an orientation such that $K_1$ is an oriented cycle. If $K_1$ and $K_2$ intersect, due to the minimality of $E(K_1) \cup E(K_2)$, they intersect in a path $P$. In a similar way as above, we turn $K_2$ into an oriented cycle. In the case that $K_1$ and $K_2$ intersect we orient the edges of $K_2$, such that the orientation is consistent with that of $P$. Let $A_{ini}$ be the resulting feasible orientation. Consider a spanning tree $T$ of $G_C$. Since each vertex of $K_1$ and $K_2$ has in-degree at least one with respect to $A_{ini}$, we may (iteratively) direct each edge in $E(T) - (E(K) \cup E(K'))$ away from $K_1$. Let $A_{ini}'$ be the resulting feasible orientation. Observe that each vertex in $G_C$ has in-degree at least one with respect to $A_{ini}$ and that only the orientation of edges in $G_C$ were changed. Hence the three claimed properties are satisfied by $A_{ini}'$. Furthermore, there is a transformation from $A_{ini}$ to $A_{ini}'$. 

Claim 2. There is a feasible orientation $A_{tar}^o$ of the constraint graph $G$ with the following properties:

1. The orientations $A_{tar}$ and $A_{tar}^o$ agree on $E - E(C)$,
2. $A_{ini}$ and $A_{tar}^o$ agree on $E(C)$, and
3. there is a transformation from $A_{tar}$ to $A_{tar}^o$.

Proof of Claim 2. We distinguish the two cases that $K_1$ and $K_2$ are disjoint or not. Let us first assume that $K_1$ and $K_2$ are disjoint. We first show that we can transform $A_{tar}$ into a feasible orientation that agrees with $A_{ini}$ on $K_1$ and $K_2$. If $K_1$ has a vertex that has at least two incoming edges, we apply the same procedure as in the proof of Claim 1. Else, $K_1$ is a directed cycle (but possibly not directed as in $A_{ini}$). Since $C$ is connected, $K_1$ and $K_2$ are connected by some path $P$. Since $K_1$ is a directed cycle, we can direct $P$ away from $K_1$ towards $K_2$. Hence, there is at least one vertex in $V(K_2)$ that has at least two incoming edges. By the same steps of the proof of Claim 1 we may obtain a feasible orientation that agrees with $A_{ini}^o$ on $E(K_2)$. By reversing $P$ and applying the same steps for $K_1$, we obtain an orientation $A_{tar}$ such that $A_{ini}$ and $A_{tar}$ agree on $K_1$ and $K_2$. We consider the same spanning tree $T$ as in the proof of Claim 1 and (iteratively) direct all edges of $T$ away from $K_1$. Let $A_{tar}'$ be the resulting feasible orientation. Since each vertex of $G_C$ has at least one incoming arc with respect to $A_{tar}'$, we can direct the remaining edges $E(G_C) - (E(K_1) \cup E(K_2) \cup E(T))$ as in $A_{ini}^o$. Let the resulting feasible orientation be $A_{tar}^o$. Observe that in the steps above only the orientation of edges of $G_C$ were changed, thus the first property of Claim 2 holds. Also observe that $A_{tar}$ and $A_{ini}^o$ agree on $E(G_C)$. Thus, Property 2 also holds.

It remains to consider the case that $K_1$ and $K_2$ are not disjoint. To obtain a feasible orientation $A_{tar}^o$ with the desired properties, simply apply the steps in the proof of Claim 1 to $A_{tar}$.

It follows that there is a transformation from $A_{ini}$ to $A_{tar}$ if and only if there is a transformation from $A_{ini}^o$ to $A_{tar}^o$. Let $G'$ be the constraint graph obtained from $G$ by deleting the blue vertices of $C$ and connecting each red vertex of $C$ with a copy of the gadget shown in Figure 2. We refer to the copies of the gadget as $G_1, G_2, \ldots, G_{kC}$. Let $A_{ini}^{new}$ (resp. $A_{tar}^{new}$) be an orientation of $E(G')$ such that $A_{ini}^o$ and $A_{tar}^o$ agree on $E(C)$, and $A_{ini}^{new}$ (resp. $A_{tar}^{new}$) agree on $E(G') - \bigcup_{1 \leq i \leq kC} E(G_i)$. Furthermore, at each gadget $G_i$, $1 \leq i \leq kC$, the orientations $A_{ini}^{new}$ and $A_{tar}^{new}$ are as shown in Figure 2. Observe that by construction each red vertex of $G'$ has at least one incoming blue arc from the gadget shown in Figure 2, so the orientations $A_{ini}^{new}$ and $A_{tar}^{new}$ are feasible.

In order to show that Rule 2 is safe it remains to prove that there is a transformation from $A_{ini}$ to $A_{tar}$ if and only if there is a transformation from $A_{ini}^o$ to $A_{tar}^o$. So first suppose that there is a transformation from $A_{ini}$ to $A_{tar}$. Then we obtain a transformation from $A_{ini}^o$ to $A_{tar}^o$ by skipping all the moves that change the orientation of an edge in $E(C)$. On the other hand, from a transformation from $A_{ini}^o$ to $A_{tar}^o$ we obtain a transformation from $A_{ini}$ to $A_{tar}$ by ignoring all the moves that change the orientation of an edge of one of the gadgets $G_i$, $1 \leq i \leq kC$. Therefore, Rule 2 is safe.

We now prove that Reduction Rule 3 is safe. Consider a blue vertex $v$ of degree one in $G$. Observe that Rule 3 is safe since in any feasible orientation of $E(G)$, the blue edge incident to $v$ is oriented towards $v$.

It remains to prove the safeness of Reduction Rule 3. Let $v$ be a blue vertex of degree 2 in $G$ and let $u$ and $w$ be the neighbors of $v$, such that $uw \notin E(G)$ and let $A$ be a feasible orientation of $E$. Since $v$ is a blue vertex of degree 2, there are at most three possible orientation of the edges $uw$ and $vw$, $uw$ and $uw$, and $uw$ and $uw$. Suppose we obtain the graph $G'$ by replacing the vertex $v$ by a new blue edge $uw$. We obtain from a feasible orientation of $G$ a feasible orientation of $G'$ by orienting
in the proof of Claim 1 we showed that there is an orientation \( A \) of the old instance is a
\( C \) gadget to a pseudo-forest in which each blue component has at most one cycle.

Thus if \( v \) is an edge satisfying a) or b), the answer to the decision problem is yes. Else, \( e^* \) is an edge
of type c). If the target configuration of \( e^* \) is oriented away from the cycles, then we can output yes.
Otherwise we will now work with \( A^e_\text{ini} \). Let \( e^* = (uv) \) be oriented from \( v \) to \( u \) in \( A^e_\text{ini} \). Let \( P \) be the
shortest path (neglecting orientations) in \( C \) from \( v \) to a vertex \( u \) of a cycle in \( C \).
Note that \( P \) and \( u \) are unique. Let \( L \subseteq V \) be all vertices that can be reached from \( w \) by the arcs in \( A^e_\text{ini} \).

Rule 2 is now modified in the following way. Instead adding a gadget to every red vertex of \( C \) as before, we only add a vertex to each red vertex of \( C \) that is not contained in \( L \). Furthermore, we add a
gadget to \( v \) (even though it might be a blue vertex). We then delete all edges and blue vertices of \( C \)
that are not contained in \( L \).

Observe that, similar to Rule 2, we have that the modified instance is a yes instance if and only if
the old instance is a yes instance. After applying the modified Rule 2 until no longer possible, we obtain
a pseudo-forest in which each blue component has at most one cycle.

Next, we consider Rule 3. If \( e^* \) is not affected by the Rule, it can be applied safely. Otherwise, if \( e^* \)
is the only edge adjacent to some vertex, then \( e^* \) can not be reversed and we output no. Hence Rule 3
is safe.

Finally, we consider Rule 4. Again, if \( e^* \) is not affected by the rule then applying the rule is safe.
Otherwise we can assume that \( w \) is adjacent to precisely two blue edges, say \( e^* \) and \( e \). Similar to
the original proof of Rule 2 we can show that contracting \( e \) is safe. Hence applying any of the three (modified)
rules is safe. After applying the Rules 2, 3 and 4 until no longer possible, we have that the resulting instance
has at most \( O(k) \) vertices and edges and thus we have a kernel of size \( O(k) \). We then have to check
whether the modified start configuration is connected to a configuration in which \( e^* \) is reversed.

Proof of Theorem 3 Let \( (G, A^\text{ini}, A^\text{tar}) \) be the instance of NCL that we obtained by applying the reduction
rules 1, 2, 3 until no longer possible. We show that \( G = (V, E) \) has at most \( 8k \) vertices and 11k edges,
where \( k = |E^\text{red}| \). Let \( G_B = (V, E^\text{blue}) \) be the blue subgraph containing blue edges only. Furthermore, let \( V_B \)
be the set of blue vertices of the copies of the gadget shown in Figure 2 present in \( G \).

We first bound the number of vertices and edges in \( G_B - V_B \). Note that each component in \( G_B - V_B \)
is a pseudo-forest, since otherwise Rule 2 is applicable. Also note that there are at most \( k \) blue vertices.
of degree 2 that are not contained in a blue cycle in $G_B - V_B$. This is due to the fact that a vertex of degree 2 in $G_B - V_B$ is one of the exceptions of Rule 3. All remaining vertices of $G_B - V_B$ have degree at least three. Let $G_B'$ be the graph obtained from $G_B$ by contracting each vertex of degree 2 in $G_B - V_B$.

We argue that $|V(G_B' - V_B)| \leq 2k$ and $|E(G_B' - V_B)| \leq 2k$. Since $G_B - V_B$ is a pseudo-forest, we have that $|E(G_B' - V_B)| \leq |V(G_B' - V_B)|$. For now let us assume that $k \leq |V(G_B' - V_B)|/2 - 1$. This implies that there are at least $|V(G_B' - V_B)|/2 + 1$ blue vertices in $G_B - V_B$. Also note that each red vertex is incident to at least one blue edge in $G_B' - V_B$, as otherwise it is an isolated vertex in $G_B' - V_B$. But since each blue vertex has at least 3 incident edges, we have that $|E(G_B' - V_B)| \geq (3(|V(G_B' - V_B)|)/2 + 1) + |V(G_B' - V_B)|/2 - 1)/2 > |V(G_B' - V_B)|$, a contradiction. Hence we have $k \geq |V(G_B' - V_B)|/2$ and therefore $|E(G_B' - V_B)| \leq 2k$. This also implies $|E(G_B' - V_B)| \leq 2k$, as claimed. Since there are up to $k$ vertices of degree 2 in $G_B - V_B$, we get $|V(G_B - V_B)| \leq 3k$ and $|E(G_B - V_B)| \leq 3k$.

For each gadget $G$, we have that $|V(G_i)| = 5$ and $|E(G_i)| = 8$. Since $G$ contains at most $k$ gadgets we have $|V(G)\leq 8k$ and $|E(G)| \leq 11k$.

Hence NCL for $G$ admits a kernel of size $O(k)$; and, in particular, NCL for $G$ can be solved in time $O^*(2^{O(k)})$.

\[ \square \]

C Proofs Omitted from Section 5

Proof of Lemma 14. We prove the lemma by induction on $|A^\text{red}_\text{ini} \setminus A^\text{tar}_\text{ini}|$. If $\rho^\text{ini} = \rho^\text{tar}_\text{ini}$, then the claim is obvious, because the sequence consisting of only one orientation $A^\text{ini}_1 = A^\text{ini}_0$ satisfies the conditions. Thus, it suffices to consider the case when $\rho^\text{ini} \neq \rho^\text{tar}_\text{ini}$. In this case, there exists a vertex $u$ in $V$ such that $\rho^\text{ini}_u > \rho^\text{tar}_u(u)$, because $\sum_{v \in V} \rho^\text{ini}_v(v) = \sum_{v \in V} \rho^\text{tar}_v(v)$. Then, there exists an arc $a \in A^\text{ini}_0 \setminus A^\text{ini}_1$ that enters $u$. Let $A^\text{ini}_1$ be the orientation of $E^\text{ini}$ obtained from $A^\text{ini}_0$ by reversing the direction of $a$. Since $|A^\text{ini}_1 \setminus A^\text{tar}_1| < |A^\text{ini}_0 \setminus A^\text{tar}_1|$, by induction hypothesis, there exists a sequence $A^\text{ini}_1, \ldots, A^\text{ini}_i$ of orientations of $E^\text{ini}$ such that $\rho^\text{ini}_i = \rho^\text{ini}_i = A^\text{ini}_i \rightarrow A^\text{ini}_{i+1}$ for $i = 2, \ldots, l$, and $\rho^\text{ini}_{l}\rho^\text{ini}_{l+1}(v) \geq \min\{\rho^\text{ini}_{l}(v), \rho^\text{ini}_{l+1}(v)\}$ for any $v \in V$ and any $i \in \{1, \ldots, l\}$. By letting $A^\text{ini}_0 = A^\text{ini}_1$, the sequence $A^\text{ini}_1, \ldots, A^\text{ini}_l$ satisfies the conditions, because $A^\text{ini}_0 \rightarrow A^\text{ini}_1 \rightarrow A^\text{ini}_2(v) \geq \rho^\text{ini}_0(v)$ for each $v \in V \setminus \{u\}$, and $\min\{\rho^\text{ini}_0(u), \rho^\text{ini}_1(u)\} = \rho^\text{ini}_0(u)$. \[ \square \]

Proof of Lemma 15. We can easily check the condition (1). To check the condition (2), we construct a digraph $\bar{G} = (V, \bar{A})$ and consider a network flow problem in it. Introduce a new vertex $w_e$ for each $e \in E^\text{red}$ and two new vertices $s$ and $t$, and define $V := V \cup \{w_e \mid e \in E^\text{red}\} \cup \{s, t\}$. Define the arc set $\bar{A} := \bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3$ by

- $\bar{A}_1 := \{(s, w_e) \mid e \in E^\text{red}\}$,
- $\bar{A}_2 := \{(w_e, v) \mid e \in E^\text{red}, v \in V, v \text{ is incident to } v \text{ in } G\}$,
- $\bar{A}_3 := \{(v, t) \mid v \in V\}$.

For each $e \in \bar{A}$, define the lower bound $l(e)$ and the upper bound $u(e)$ of the amount of flow through $e$ as follows.

- For each $(s, w_e) \in \bar{A}_1$, define $l(s, w_e) := u(s, w_e) := 1$.
- For each $(w_e, v) \in \bar{A}_2$, define $l(w_e, v) := 0$ and $u(w_e, v) := 1$.
- For each $(v, t) \in \bar{A}_3$, define $l(v, t) := 0$ and $u(v, t) := +\infty$ otherwise.

Then, the condition (2) holds if and only if $\bar{G}$ has an integral $s$-$t$ flow satisfying the above constraint. This can be tested in polynomial time by a standard maximum flow algorithm (see e.g. 13 Corollary 11.3(a)). \[ \square \]

Proof of Lemma 16. It suffices to consider the case when $\phi(A^\text{ini}_i) \leftrightarrow \phi(A^\text{tar}_i)$. Denote $\phi(A^\text{ini}_i) = (A^\text{blue}_i, d^\text{ini}_i)$ and $\phi(A^\text{tar}_i) = (A^\text{blue}_i, d^\text{tar}_i)$. By definition, we have either $d^\text{ini}_i = d^\text{tar}_i$ and $A^\text{blue}_i \leftrightarrow A^\text{blue}_i$, or $d^\text{ini}_i = d^\text{tar}_i$ and $A^\text{blue}_i \leftrightarrow A^\text{blue}_i$, or $d^\text{ini}_i = d^\text{tar}_i$.

If $d^\text{ini}_i = d^\text{tar}_i$ and $A^\text{blue}_i \leftrightarrow A^\text{blue}_i$, then $A^\text{ini}_i \leftrightarrow A^\text{blue}_i \cup A^\text{red}_i$ and $\phi(A^\text{blue}_i \cup A^\text{red}_i) = (A^\text{blue}_i, d^\text{ini}_i) = (A^\text{blue}_i, d^\text{tar}_i) = \phi(A^\text{tar}_i)$, which means that $A^\text{ini}_i := A^\text{blue}_i \cup A^\text{red}_i$ satisfies the conditions.

21
Otherwise, let $A^\text{blue} := A^\text{blue}_\text{ini} = A^\text{blue}_\text{tar}$. By Lemma 8 we obtain a sequence $A_0^\text{red}, A_1^\text{red}, \ldots, A_l^\text{red}$ of orientations of $E^\text{red}$ such that $A_0^\text{red} = A^\text{red}_\text{ini}, \rho_{A_i^\text{red}} = \rho_{A_{i-1}^\text{red}}, A_i^\text{red} \leftrightarrow A_{i+1}^\text{red}$ for $i = 1, \ldots, l$, and $\rho_{A_i^\text{red}}(v) \geq \min\{\rho_{A_i^\text{red}}(v), \rho_{A_{i-1}^\text{red}}(v)\}$ for any $v \in V$ and any $i \in \{0, 1, \ldots, l\}$. Then, for any $i \in \{0, 1, \ldots, l\}$, we have
\[
2\rho_{A^\text{blue}}(v) + \rho_{A^\text{red}}(v) \geq \min\{2\rho_{A^\text{blue}}(v) + \rho_{A^\text{red}}(v), 2\rho_{A^\text{blue}}(v) + \rho_{A^\text{red}}(v)\} \geq 2
\]
for any $v \in V$, and hence $A^\text{blue} \cup A_i^\text{red}$ is feasible. Since $A^\text{blue} \cup A_i^\text{red} \leftrightarrow A^\text{blue} \cup A_i^\text{red}$ for $i = 1, \ldots, l$, we have
\[
(A^\text{ini}) = A^\text{blue} \cup A_{i-1}^\text{red} \leftrightarrow A^\text{blue} \cup A_i^\text{red}.
\]
Furthermore, since $\rho_{A_i^\text{red}} = \rho_{A_i^\text{red}}$, we have $\phi(A^\text{blue} \cup A_i^\text{red}) = \phi(A_i^\text{red})$. Therefore, $A_{i+1}^\text{red} := A^\text{blue} \cup A_i^\text{red}$ satisfies the conditions in the lemma.

Proof of Lemma 8. We prove (i)⇒(ii), (ii)⇒(iii), and (iii)⇒(i), respectively.

[(i)⇒(ii)] If (i) holds, then $A := (A^\text{ini} \setminus C) \cup \overline{C}$ satisfies the conditions in (ii), since it contains no arc in $C$.

[(ii)⇒(iii)] We prove the contraposition. Assume that (iii) does not hold, that is, there exists a vertex $u \in V(C)$ such that $2\rho_{A^\text{blue}}(u) + \rho_{A^\text{red}}(u) = 2$ for any $A \in A$ with $A^\text{ini} \leftrightarrow A$. Let $a$ be the arc in $C$ that enters $u$. Since we cannot reverse the direction of $a$ without violating the feasibility, $a$ is contained in any orientation $A \in A$ with $A^\text{ini} \leftrightarrow A$.

[(iii)⇒(i)] Suppose that (iii) holds. We take a sequence $A_0, A_1, \ldots, A_l$ of feasible orientations of $E$ such that $A_0 = A^\text{ini}, A_i$ is obtained from $A_{i-1}$ by reversing an arc $a_i \in A_{i-1}$ for $i \in \{1, 2, \ldots, l\}$, and there exists $u \in V(C)$ such that $2\rho_{A_i^\text{blue}}(u) + \rho_{A_i^\text{red}}(u) \geq 3$. By taking a minimal sequence with these conditions, we may assume that $a_i$ is not contained in $C$ for $i \in \{1, 2, \ldots, l\}$. Since $2\rho_{A_i^\text{blue}}(u) + \rho_{A_i^\text{red}}(u) \geq 3$, starting from $A_l$, we can change the direction of each arc in $C$ one by one without violating the feasibility, which shows that $A_l \leftrightarrow (A_l \setminus C) \cup \overline{C}$. On the other hand, since $(A_1 \setminus C) \cup \overline{C}$ is obtained from $(A_{i-1} \setminus C) \cup \overline{C}$ by reversing $a_i$, for $i \in \{1, 2, \ldots, l\}$, we obtain $(A_{i-1} \setminus C) \cup \overline{C} \leftrightarrow (A_i \setminus C) \cup \overline{C}$. Thus, it holds that $A^\text{ini} \leftrightarrow A_l \leftrightarrow (A_l \setminus C) \cup \overline{C} \leftrightarrow (A_i \setminus C) \cup \overline{C}$.

Proof of Proposition 17. If $A$ is a solution of PROBLEM A, then $\phi_A(A) = (A^\text{blue}, d)$ satisfies the conditions by Lemma 10. Conversely, assume that there exists a pair $(A^\text{blue}, d) \in F_u$ such that $2\rho_{A^\text{blue}}(u) + d(u) \geq 3$ and $\phi_A(A^\text{ini}) \leftrightarrow (A^\text{blue}, d)$. By Lemma 10, there exists an orientation $A \in A$ with $\phi_A(A) = (A^\text{blue}, d)$ such that $A^\text{ini} \leftrightarrow A$. Since $2\rho_{A^\text{blue}}(u) + \rho_{A^\text{red}}(u) \geq 2\rho_{A^\text{blue}}(u) + d(u) \geq 3$, $A$ is a solution of PROBLEM A.

Proof of Corollary 18. Let $e^*$ be the edge of the orientation $A^\text{ini}$ that we wish to reverse. If $e^*$ is a blue edge, we simply solve the reconfiguration problem in $F$ (see Section 5.2). Since $|F| = 2^{O(k)}$, this can be done in FPT time. If $e^*$ is a red edge, we solve PROBLEM A with $u$ being the head of $e^*$ (see Section 5.3). This can also be done in FPT time.