ON STRICTLY WEAK MIXING $C^*$-DYNAMICAL SYSTEMS AND A WEIGHTED ERGODIC THEOREM

FARRUKH MUKHAMEDOV

Abstract. We prove that unique ergodicity of tensor product of $C^*$-dynamical system implies its strictly weak mixing. By means of this result a uniform weighted ergodic theorem with respect to $S$-Besicovitch sequences for strictly weak mixing dynamical systems is proved. Moreover, we provide certain examples of strictly weak mixing dynamical systems.

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1. Introduction

Recently, the investigation of the ergodic properties of quantum dynamical systems had a considerable growth. Since the theory of quantum dynamical systems provides convenient mathematical description of the irreversible dynamics of an open quantum system (see [9], sec.4.3, [31],[27]). In this setting, the matter is more complicated than in the classical case. Some differences between classical and quantum situations are pointed out in [4],[26]. This motivates an interest to study of dynamics of quantum systems (see [4],[13],[17]). Therefore, it is then natural to address the study of the possible generalizations to quantum case of the various ergodic properties known for classical dynamical systems. A lot of papers (see, [12], [14],[23],[24],[32]) were devoted to the investigations of mixing properties of dynamical systems.

It is known [21] that a strong ergodic property for a classical system is the unique ergodicity. Namely, a classical dynamical system $(\Omega, T)$ consisting of a compact Hausdorff space $\Omega$ and a homeomorphism $T$ is said to be uniquely ergodic if there exists a unique invariant Borel measure $\mu$ for $T$. It is seen that the ergodic average $\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k$ converges uniformly to the constant function $\int f \, d\mu$ in this case. A pivotal example of classical uniquely ergodic dynamical system is given by an irrational rotation on the unit circle, see e.g. [21]. In quantum setting, the last property is formulated as follows (see also Sec. 2). Let $(\mathcal{A}, T)$ be a $C^*$–dynamical system based on the $C^*$-algebra $\mathcal{A}$ and a unital completely positive (ucp) map $T$ on $\mathcal{A}$. The unique ergodicity or equivalently strict ergodicity for $(\mathcal{A}, \alpha)$ is equivalent (cf. [2, 25]) to the norm convergence

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^n(a) = E(a), \quad a \in \mathcal{A},$$

where $E$ is a conditional expectation, given by $E = \varphi(\cdot)\mathbb{1}$, onto the fixed–point subspace of $T$, consisting of the constant multiples of the identity. Here, $\varphi \in \mathcal{S}(\mathcal{A})$ is the unique
invariant state for \( T \). Some generalizations of unique ergodicity have been investigated in [2, 3], where the conditional expectation \( E \) in (1.1) (necessarily unique) is taken as a projection onto the fixed-point subspace of \( T \), which, in general, is supposed to be nontrivial.

In [25] we have introduced a property stronger than the unique ergodicity, called \textit{strict weak mixing}. This property for \( (A, T) -C^*\)-dynamical system requires the existence of a state \( \varphi \in \mathcal{S}(A) \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \psi(T^k(a)) - \varphi(a) \right| = 0, \quad a \in A,
\]

for each \( \psi \in \mathcal{S}(A) \). It can be shown (see below) that \( \varphi \) is the unique invariant state for \( T \). If \( (A, T) \) is strictly weak mixing, then it is uniquely ergodic (see Proposition 2.2). Conversely, the irrational rotations on the unit circle provide examples of uniquely ergodic dynamical systems which are not strictly weak mixing, see [25], Example 2.

In this paper we are going to prove for the strict mixing an analog of the well-known classical result stating that a transformation is weakly mixing if and only if its Cartesian product of dynamical systems. In the final Section 5, by means of the main result we prove one uniform weighted ergodic theorem for strictly weak mixing systems.

2. Preliminaries

In this section we recall some preliminaries concerning \( C^*\)-dynamical systems.

Let \( A \) be a \( C^*\)-algebra with unit \( 1 \). An element \( x \in A \) is called \textit{self-adjoint} (resp. \textit{positive}) if \( x = x^* \) (resp. there is an element \( y \in A \) such that \( x = y^*y \)). The set of all self-adjoint (resp. positive) elements will be denoted by \( A_{sa} \) (resp. \( A_+ \)). By \( A^* \) we denote the conjugate space to \( A \). A linear functional \( \varphi \in A^* \) is called \textit{Hermitian} if \( \varphi(x^*) = \varphi(x) \) for every \( x \in A \). A Hermitian functional \( \varphi \) is called \textit{positive} if \( \varphi(x^*x) \geq 0 \) for every \( x \in A \). A positive functional \( \varphi \) is said to be a \textit{state} if \( \varphi(1) = 1 \). By \( \mathcal{S}(A) \) (resp. \( \mathcal{S}_h(A) \)) we denote the set of all states (resp. Hermitian functionals) on \( A \). Let \( B \) be another \( C^*\)-algebra with unit. By \( A \otimes B \) we denote the algebraic tensor product of \( A \) and \( B \). A completion of \( A \otimes B \) with respect to the minimal \( C^*\)-tensor norm on \( A \otimes B \) is denoted by \( A \otimes B \), and it would be also a \( C^*\)-algebra with a unit (see, [29]). A linear operator \( T : A \to A \) is called \textit{positive} if \( Tx \geq 0 \) whenever \( x \geq 0 \). By \( M_n(A) \) we denote the set of all \( n \times n \)-matrices \( a = (a_{ij}) \) with entries \( a_{ij} \) in \( A \). A linear mapping \( T : A \to A \) is called \textit{completely positive} if the linear operator \( T_n : M_n(A) \to M_n(A) \) given by \( T_n(a_{ij}) = (T(a_{ij})) \) is positive for all \( n \in \mathbb{N} \). A completely positive map \( T : A \to A \) with \( T1 = 1 \) is called a \textit{unital completely positive (ucp) map}. A pair \( (A, T) \) consisting of a \( C^*\)-algebra \( A \) and a ucp map \( T : A \to A \) is called a \textit{\( C^*\)-dynamical system}. In the sequel, we will call any triplet \( \mathcal{A}, \varphi, T \) consisting of a \( C^*\)-algebra \( A \), a state \( \varphi \) on \( A \) and a ucp map \( T : A \to A \) with \( \varphi \circ T = \varphi \), that is a dynamical system with an invariant state, a state preserving \( C^*\)-dynamical system. A state preserving \( C^*\)-dynamical system is a non-commutative \( C^*\)-probability space \((\mathcal{A}, \varphi)\) (see [10]) together with a ucp map \( T \) on \( \mathcal{A} \) preserving the non-commutative probability \( \varphi \). It is known [29] that if \( (\mathcal{A}, T) \) and \( (\mathcal{B}, H) \) are two \( C^*\)-dynamical systems, then \( (\mathcal{A} \otimes \mathcal{B}, T \otimes H) \) is also \( C^*\)-dynamical system. Since a mapping \( T \otimes H : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A} \otimes \mathcal{B} \) given by \( (T \otimes H)(x \otimes y) = Tx \otimes Hy \) is a ucp map.
We say that the state preserving $C^*$-dynamical system $(\mathcal{A}, \varphi, T)$ is er
dodic (respectively, weakly mixing, strictly weak mixing) with respect to $\varphi$ if
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (\varphi(yT^k(x)) - \varphi(y)\varphi(x)) = 0, \text{ for all } x, y \in \mathcal{A}. \] (2.1)
(respectively,
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\varphi(yT^k(x)) - \varphi(y)\varphi(x)| = 0, \text{ for all } x, y \in \mathcal{A}, \] (2.2)
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x)) - \psi(1)\varphi(x)| = 0, \text{ for all } x \in \mathcal{A}, \psi \in \mathcal{A}^*. \] (2.3)

The state preserving $C^*$-dynamical system $(\mathcal{A}, \varphi, T)$ is called uniquely ergodic with respect to $\varphi$ if $\varphi$ is the unique invariant state under $T$.

Remark 2.1. If we take a functional $\varphi(xy)$ instead of $\psi(x)$ in (2.3), then one can see that strict weak mixing implies weak mixing. Converse, is not true. A related example was provided in [25], Example 3.

In [25] (see also [2]) we have proved the following characterization of unique ergodicity of dynamical systems.

Theorem 2.1. Let $(\mathcal{A}, \varphi, T)$ be a state preserving $C^*$-dynamical system. The following conditions are equivalent
(i) $(\mathcal{A}, \varphi, T)$ is uniquely ergodic ;
(ii) For every $x \in \mathcal{A}$ the following equality holds
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k(x) = \varphi(x)1, \]
where convergence in norm of $\mathcal{A}$;
(iii) For every $x \in \mathcal{A}$ and $\psi \in \mathcal{A}^*$ the following equality holds
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(T^k(x)) = \psi(1)\varphi(x). \]

Remark 2.2. From this Theorem we immediately infer that unique ergodicity implies ergodicity of $C^*$-dynamical system.

Proposition 2.2. If the $C^*$-dynamical system $(\mathcal{A}, \varphi, T)$ is strictly weak mixing, then it is uniquely ergodic.

Proof. Let $\psi \in \mathcal{A}^*$, then one gets
\[ \left| \frac{1}{n} \sum_{k=0}^{n-1} (\psi(T^k(x)) - \psi(1)\varphi(x)) \right| \leq \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x)) - \psi(1)\varphi(x)| \to 0 \]
whenever $n \to \infty$, as $(\mathcal{A}, T)$ is strictly weak mixing. By using the Jordan decomposition of bounded linear functionals (cf. [29]), we conclude that (iii) of Theorem 2.1 is satisfied. \qed

In many interesting situations, the ergodic behavior of dynamical systems is connected with some spectral properties, see e.g. [11, 22, 26, 32]. It is not possible to extend such results to our situation. However, a strictly weak mixing map $T$ cannot have eigenvalues on the unit circle $\mathbb{T}$ except $z = 1$. 
Let $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit disk in the complex plane, and $\mathring{\mathbb{D}} = \{z \in \mathbb{C} : |z| < 1\}$ its interior. If $T$ has norm one, we have $\sigma(T) \subset \mathbb{D}$, $\sigma(T)$ being the spectrum of $T$.

Let $T : \mathfrak{A} \to \mathfrak{A}$ be a linear map. Denote

$$\mathfrak{A}_z = \{x \in \mathfrak{A} : T(x) = zx\},$$
$$\mathfrak{A}_z^* = \{f \in \mathfrak{A}^* : f \circ T = zf\},$$

where $z \in \mathbb{C}$. Furthermore,

**Proposition 2.3.** Let $(\mathfrak{A}, T)$ be a strictly weak mixing $C^*$-dynamical system. Then $z \in \mathbb{T}\setminus\{1\}$ implies $\mathfrak{A}_z = \{0\}$ and $\mathfrak{A}_z^* = \{0\}$

**Proof.** Assume that $T(x_0) = zx_0$ for some $z \neq 1$. Then $\varphi(x_0) = \varphi(T(x_0)) = z\varphi(x_0)$ which means $\varphi(x_0) = 0$. In addition, the strict weak mixing implies

$$0 = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x_0)) - \psi(1)\varphi(x_0)| = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |z^k \psi(x_0)|$$
$$= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(x_0)| = |\psi(x_0)|.$$

Namely, $\psi(x_0) = 0$ for every $\psi \in \mathfrak{A}^*$, hence $x_0 = 0$. The second part can be proceeded similarly. $\square$

**Remark 2.3.** For any linear map $T$ of $\mathfrak{A}$, it is obvious that if $z \in \mathring{\mathbb{D}}$ and $x \in \mathfrak{A}_z$, then $\lim_{k} T^k(x) = 0$.

3. **Tensor product of strictly weak mixing dynamical systems**

This section is devoted to tensor product of uniquely ergodic and strictly weak mixing dynamical systems. Here we prove the main result of the paper.

Set

$$\mathfrak{A}_1^* = \{g \in \mathfrak{A}^* : \|g\|_1 \leq 1\}, \quad \mathfrak{A}_{1,h}^* = \mathfrak{A}_1^* \cap \mathfrak{A}_h^*.$$ 

Now we are going to prove an analogous result of [1, 31] for the strictly weak mixing dynamical systems.

**Theorem 3.1.** Let $(\mathfrak{A}, \varphi, T)$, $(\mathfrak{B}, \varphi_1, H)$ be two state preserving $C^*$-dynamical systems. For the following assertions

(i) The state preserving $C^*$-dynamical system $(\mathfrak{A} \otimes \mathfrak{B}, \varphi \otimes \varphi_1, T \otimes H)$ is strictly weak mixing;

(ii) $(\mathfrak{A}, \varphi, T)$ and $(\mathfrak{B}, \varphi_1, H)$ are strictly weak mixing;

the implication (i)$\Rightarrow$(ii) holds.

If in addition one has $(\mathfrak{A} \otimes \mathfrak{B})^* = \mathfrak{A}^* \otimes \mathfrak{B}^*$, then (ii)$\Rightarrow$(i) also holds.

**Proof.** The implication (i)$\Rightarrow$(ii) immediately follows from the definition.

Now assume that $(\mathfrak{A} \otimes \mathfrak{B})^* = \mathfrak{A}^* \otimes \mathfrak{B}^*$ holds. Let us consider the implication (ii)$\Rightarrow$(i). It is clear that the state $\varphi \otimes \varphi_1$ is invariant with respect to $T \otimes H$.

Let $\psi \in \mathfrak{A}^*$ and $\phi \in \mathfrak{B}^*$ be arbitrary functionals and $x \in \ker \varphi$, $y \in \ker \varphi_1$. Then according to (ii) we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))| = 0. \quad (3.1)$$
The Schwartz inequality implies that
\[
\frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))\phi(H^k(y))| \leq \frac{1}{n} \left( \sum_{k=0}^{n-1} |\psi(T^k(x))|^2 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{n-1} |\phi(H^k(y))|^2 \right)^{\frac{1}{2}}
\]
\[
= \sqrt{\frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))|^2} \sqrt{\frac{1}{n} \sum_{k=0}^{n-1} |\phi(H^k(y))|^2}
\]
\[
\leq \||\phi||, y\| \sqrt{\frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))|^2}
\]  \quad (3.2)

Moreover, the relations
\[
\frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))|^2 \leq \sup_{0 \leq k \leq n-1} |\psi(T^k(x))| \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))|
\]
\[
\leq \||\psi||, x\| \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))|
\]
with (3.1) yield that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))\phi(H^k(y))| = 0.
\]

Thus,
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi \circ \phi(T^k \otimes H^k(x \otimes y))| = 0, \quad (3.3)
\]
for \( x \in \ker \varphi, y \in \ker \varphi_1, \psi \in \mathfrak{A}^*, \phi \in \mathfrak{B}^* \).

Let \( \mathfrak{A}^* \otimes \mathfrak{B}^* \) be the algebraic tensor product of \( \mathfrak{A}^* \) and \( \mathfrak{B}^* \). Thanks to our assumption one can see that the \( \|\cdot\|_1 \)-closure of \( \mathfrak{A}^* \circ \otimes \mathfrak{B}^* \) is \( (\mathfrak{A} \otimes \mathfrak{B})^* \). So, using the norm-denseness of the elements \( \sum_{i=1}^{m} \psi_i \otimes \phi_i \) in \( (\mathfrak{A} \otimes \mathfrak{B})^* \) from (3.3) one gets
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\omega(T^k \otimes H^k(x \otimes y))| = 0, \quad (3.4)
\]
for \( \omega \in (\mathfrak{A} \otimes \mathfrak{B})^* \).

Let \( x \in \mathfrak{A} \) and \( y \in \mathfrak{B} \). Denoting \( x^0 = x - \varphi(x) \mathbf{1}, y^0 = y - \varphi_1(y) \mathbf{1} \) we have \( x^0 \in \ker \varphi, y^0 \in \ker \varphi_1 \), so for them (3.4) holds.

Denote \( \omega_1(x) = \omega(x \otimes \mathbf{1}), x \in \mathfrak{A} \) and \( \omega_2(y) = \omega(\mathbf{1} \otimes y), y \in \mathfrak{B} \). Then according to condition (ii) we find
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\omega_1(T^k(x)) - \omega(\mathbf{1} \otimes \mathbf{1})\varphi(x)| = 0, \quad (3.5)
\]
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\omega_2(H^k(y)) - \omega(\mathbf{1} \otimes \mathbf{1})\varphi_1(y)| = 0. \quad (3.6)
\]
Now from
\[
\frac{1}{n} \sum_{k=0}^{n-1} |\omega(T^k \otimes H^k (x \otimes y)) - \omega(1 \otimes 1)\varphi(x)\varphi_1(y)|
\]
\[
\leq |\varphi_1(y)| \left( \frac{1}{n} \sum_{k=0}^{n-1} |\omega_1(T^k (x)) - \omega(1 \otimes 1)\varphi(x)| \right)
\]
\[
+ |\varphi(x)| \left( \frac{1}{n} \sum_{k=0}^{n-1} |\omega_2(H^k (y)) - \omega(1 \otimes 1)\varphi_1(y)| \right)
\]
\[
+ \frac{1}{n} \sum_{k=0}^{n-1} |\omega(T^k \otimes H^k (x^0 \otimes y^0))|
\]
and (3.4)-(3.6) we get
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\omega(T^k \otimes H^k (x \otimes y)) - \omega(1 \otimes 1)\varphi(x)\varphi_1(y)| = 0. \tag{3.7}
\]

The norm-denseness of the elements \( \sum_{i=1}^m x_i \otimes y_i \) in \( \mathfrak{A} \otimes \mathfrak{B} \) with (3.7) yields
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\omega(T^k \otimes H^k (z)) - \omega(1 \otimes 1)\varphi \otimes \varphi_1(z)| = 0.
\]

for arbitrary \( z \in \mathfrak{A} \otimes \mathfrak{B} \). So, \( (\mathfrak{A} \otimes \mathfrak{B}, \varphi \otimes \varphi_1, T \otimes H) \) is strictly weak mixing. \( \square \)

Remark 3.1. Note that analogous results for weak mixing dynamical system defined on von Neumann algebras were proved in [22],[32].

From the proved theorem we get the following

**Corollary 3.2.** Let \( (\mathfrak{A}, \varphi, T) \) be a state preserving \( C^* \)-dynamical systems. For the following assertions

(i) The state preserving \( C^* \)-dynamical system \( (\mathfrak{A} \otimes \mathfrak{A}, \varphi \otimes \varphi, T \otimes T) \) is uniquely ergodic;

(ii) \( (\mathfrak{A}, \varphi, T) \) is strictly weak mixing;

the implication \( (i) \Rightarrow (ii) \) holds.

If, in addition, \( \mathfrak{A}^* \otimes \mathfrak{A}^* = (\mathfrak{A} \otimes \mathfrak{A})^* \) is satisfied then both \( (i), (ii) \) assertions are equivalent to

(iii) The state preserving \( C^* \)-dynamical system \( (\mathfrak{A} \otimes \mathfrak{A}, \varphi \otimes \varphi, T \otimes T) \) is strictly weak mixing;

Proof. \( (i) \Rightarrow (ii) \). Let \( (\mathfrak{A} \otimes \mathfrak{A}, \varphi \otimes \varphi, T \otimes T) \) be uniquely ergodic. Let \( x \in \ker \varphi, x = x^* \). The unique ergodicity of the dynamical system (see Theorem 2.1) implies
\[
\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k \otimes T^k (x \otimes x) \right\| = 0.
\]

Hence,
\[
\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \psi \otimes \psi(T^k \otimes T^k (x \otimes x)) \right\| = 0, \quad \text{for all } \psi \in \mathfrak{A}_{1,h}^*.
\]

Self-adjointness of \( x \) yields
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k (x))|^2 = 0 \quad \text{for all } \psi \in \mathfrak{A}_{1,h}^*.
\]
By the Schwartz inequality one finds
\[
\frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))| \leq \frac{1}{n} \left[ \sum_{k=0}^{n-1} 1 \right] \left[ \sum_{k=0}^{n-1} |\psi(T^k(x))|^2 \right]^{1/2} = \left[ \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))|^2 \right]^{1/2},
\]
which with (3.8) implies
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))| = 0 \quad \text{for all } \forall \psi \in \mathfrak{A}^*_i,h.
\]

Now let \( x \in \ker \varphi \) and \( \psi \in \mathfrak{A}^*_i \) be arbitrary. Then they can be represented as \( x = x_1 + ix_2, \psi = \psi_1 + i\psi_2, \) where \( x_1, x_2 \in \ker \varphi, x_j^* = x_j, \psi_j \in \mathfrak{A}^*_i,h, j = 1, 2. \) From
\[
\frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))| \leq \frac{1}{n} \sum_{i,j=1}^2 \sum_{k=0}^{n-1} |\psi_i(T^k(x_j))|
\]
and (3.9) it follows that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))| = 0,
\]
for \( x \in \ker \varphi, \psi \in \mathfrak{A}^*_i. \)

Finally let \( x \in \mathfrak{A}. \) Then the last relation (3.10) for the element \( x^0 = x - \varphi(x)I \) implies the assertion.

Let \( \mathfrak{A}^* \otimes \mathfrak{A}^* = (\mathfrak{A} \otimes \mathfrak{A})^* \) be satisfied, then the implication (ii)⇒(iii) is a direct consequence of Theorem 3.1. The implication (iii)⇒(i) immediately follows from Proposition 2.2. □

An idea of the proof of Theorem 3.1 allows us to get some adaptation of a result of [1] for strictly weak mixing dynamical systems. Namely we have the following

**Theorem 3.3.** Let \( (\mathfrak{A}, \varphi, T) \) be a state preserving \( C^* \)-dynamical systems. For the following assertions

(i) For every state preserving uniquely ergodic \( C^* \)-dynamical system \( (\mathfrak{B}, \varphi_1, H) \) the state preserving \( C^* \)-dynamical system \( (\mathfrak{A} \otimes \mathfrak{B}, \varphi \otimes \varphi_1, T \otimes H) \) is uniquely ergodic;

(ii) \( (\mathfrak{A}, \varphi, T) \) is strictly weak mixing,

(iii) For every state preserving uniquely ergodic \( C^* \)-dynamical system \( (\mathfrak{B}, \varphi_1, H) \) such that \( \mathfrak{A}^* \otimes \mathfrak{B}^* = (\mathfrak{A} \otimes \mathfrak{B})^* \) the state preserving \( C^* \)-dynamical system \( (\mathfrak{A} \otimes \mathfrak{B}, \varphi \otimes \varphi_1, T \otimes H) \) is uniquely ergodic;

the following implications hold \( (i) \Rightarrow (ii) \Rightarrow (iii). \)

**Proof.** (i)⇒(ii). According to the condition \( (\mathfrak{A} \otimes \mathfrak{B}, \varphi \otimes \varphi_1, T \otimes H) \) is uniquely ergodic, this means that the state \( \varphi \otimes \varphi_1 \) is a unique for it. Take arbitrary functional \( \psi \in \mathfrak{A}^* \) and \( \phi \in \mathcal{S}(\mathfrak{B}) \), then the unique ergodicity due to Theorem 2.1 implies that
\[
0 = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (\psi \otimes \phi(T^k \otimes H^k(x \otimes 1)) - \psi(1)\varphi(x))
\]
\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (\psi(T^k(x)) - \psi(1)\varphi(x)).
\]
This shows unique ergodicity of \((A, \varphi, T)\). Then condition (i) implies that \(T \otimes T\) is also uniquely ergodic, therefore Corollary 3.2 yields that \(T\) is strictly weak mixing.

(ii)\(\Rightarrow\)(iii). Let \((B, \varphi_1, H)\) be a completely positive, uniquely ergodic dynamical system such that \(A^* \otimes B^* = (A \otimes B)^*\). Then it is clear that the state \(\varphi \otimes \varphi_1\) is invariant with respect to \(T \otimes H\). The argument used in the proof of Theorem 3.1 implies that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \omega(T^k \otimes H^k(x \otimes y)) = 0.
\]

(3.11)

for every \(x \in \ker \varphi, y \in B\) and \(\omega \in (A \otimes B)^*\).

Let \(x \in A\). Then (3.11) holds for \(x^0 = x - \varphi(x)I\). The unique ergodicity of \((B, \varphi_1, H)\) implies

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (\omega_2(H^k(y)) - \omega(I \otimes I)\varphi_1(y)) = 0,
\]

(3.12)

here as before \(\omega_2(x) = \omega(I \otimes x)\).

Now from

\[
\left| \frac{1}{n} \sum_{k=0}^{n-1} \omega(T^k \otimes H^k(x \otimes y)) - \omega(I \otimes I)\varphi_1(y) \right|
\]

\[
\leq \left| \frac{1}{n} \sum_{k=0}^{n-1} (\omega(T^k \otimes H^k)(x^0 \otimes y)) \right|
\]

\[
+ |\varphi(x)| \left| \frac{1}{n} \sum_{k=0}^{n-1} (\omega_2(H^k(y)) - \omega(I \otimes I)\varphi_1(y)) \right|
\]

and (3.11), (3.12) it follows that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \omega(T^k \otimes H^k(x \otimes y)) - \omega(I \otimes I)\varphi_1(y) = 0
\]

(3.13)

The density argument used in the proof of Theorem 3.1 and Theorem 2.1 yield the required assertion.

\(\square\)

Remark 3.2. If in condition (i) of Theorem 3.3 we take not all state preserving uniquely ergodic \(C^*\)-dynamical systems, then the assertion of the theorem fails. Indeed, let us consider the following example. Let \(S^1 = \{z \in \mathbb{C} : |z| = 1\}\) and \(\lambda\) be the Lebesgue measure on \(S^1\) such that \(\lambda(S^1) = 1\). The measure induces a positive linear functional \(\varphi_\lambda(f) = \int f(z) d\lambda(z)\) such that \(\varphi_\lambda(1) = 1\). Consider a \(C^*\)-algebra \(A = C(S^1)\), where \(C(S^1)\) is the space of all continuous functions on \(S^1\). Fix an element \(a = \exp(2\pi i \alpha)\), where \(\alpha \in [0, 1)\) is an irrational number. Define a mapping \(T_\alpha : C(S^1) \mapsto C(S^1)\) by \((T_\alpha(f))(z) = f(az)\) for all \(f \in C(S^1)\). It is clear that \((C(S^1), \varphi_\lambda, T_\alpha)\) is a state preserving \(C^*\)-dynamical system. Since \(\alpha\) is irrational, then Theorem 2 of Chapter 3 of [21] implies that the defined dynamical system is uniquely ergodic. According to that theorem the tensor product \(T_\alpha \otimes T_\beta\), acting on \(C(S^1) \otimes C(S^1)\), is also uniquely ergodic for every \(\beta\) which is rationally independent with \(\alpha\).

But \(T_\alpha\) is not strictly weak mixing. Indeed, take a linear functional \(h \in C(S^1)^*\) defined by \(h(f) = \int z f(z) d\lambda(z), f \in C(S^1)\). Then we have \(h(T_\alpha(f)) = a^{-2} h(f)\) for all \(f \in C(S^1)\). Thus, Proposition 2.3 implies that \(T_\alpha\) is not strictly weak mixing. According to Corollary 3.2 the tensor product \(T_\alpha \otimes T_\alpha\), acting on \(C(S^1) \otimes C(S^1)\), is not uniquely ergodic. Moreover,
\( T_\alpha \otimes T_\alpha \) is not ergodic. Indeed, using the well known equality \( C(S^1 \times S^1) = C(S^1) \otimes C(S^1) \) we see that \( T_\alpha \otimes T_\alpha \) acts as follows
\[
(T_\alpha \otimes T_\alpha)(f(x, y)) = f(ax, ay), \quad x, y \in S^1,
\]
where \( f \in C(S^1 \times S^1) \). For the element \( g \) of \( C(S^1 \times S^1) \) defined by \( g(x, y) = x/y \), we have \((T_\alpha \otimes T_\alpha)(g)(x, y) = g(x, y)\) which means that \( T_\alpha \otimes T_\alpha \) is not ergodic.

### 4. Examples

In this section we are going to provide certain examples of strictly weak mixing ucp maps.

1. Let \( \mathfrak{A} = M_2(\mathbb{C}) \) and \( \tau \) be the normalized trace on \( \mathfrak{A} \). By \( e_{ij}, i, j = 1, 2 \) we denote the matrix units (in the standard basis of \( \mathbb{C} \)) of \( \mathfrak{A} \). Consider \( \mathcal{E} : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A} \) - the canonical conditional expectation, i.e. \( \mathcal{E}(x \otimes y) = \tau(y)x \) (see [29]). Take \( V \in \mathfrak{A} \otimes \mathfrak{A} \) such that \( \mathcal{E}(VV^*) = \mathbb{I} \). Define \( T_V : \mathfrak{A} \to \mathfrak{A} \) by \( T_V(x) = \mathcal{E}(V(1 \otimes x)V^*) \), \( x \in \mathfrak{A} \). Then it is clear that \( T_V \) is a ucp map with \( \tau(T_Vx) = \tau(x) \) for all \( x \in \mathfrak{A} \). If its peripheral spectrum is \{1\}, then \( T_V^n \to \tau \mathbb{I} \) as \( n \to \infty \). In this case \((\mathfrak{A}, \tau, T_V)\) would be strictly weak mixing. In particular, if we choose \( V \) as follows
\[
V_\beta = \sqrt{\frac{2}{1 + \cosh(2\beta)}} \exp\{\beta(e_{12} \otimes e_{21} + e_{21} \otimes e_{12})\}, \quad \beta \in \mathbb{R}
\]
then all the required conditions are satisfied.

2. Let \((C(K), \nu, T)\) be a commutative strictly weak mixing dynamical system. Now with the aid of above Example 1 and Theorem 3.1 one finds that \((C(K) \otimes M_2(\mathbb{C}), \nu \otimes \tau, T \otimes T_V^\beta)\) is a non-commutative strictly weak mixing dynamical system.

3. First we formulate a result relating to adaptation of the Blum-Hanson theorem (see [5, 7, 18, 30]) for strictly weak mixing dynamical systems, which will be used below.

**Theorem 4.1.** A state preserving \( C^*\)-dynamical system \((\mathfrak{A}, \varphi, T)\) is strictly weak mixing if and only if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} T^{k_m}(x) = \varphi(x) \mathbb{I} \quad (4.1)
\]
for every \( x \in \mathfrak{A} \) and increasing sequence of positive numbers \( \{k_n\} \) such that \( \sup_n k_n/n < \infty \). Here the convergence is meant with respect to the uniform norm.

Now let \( \mathbb{F}_\infty \) be the free group on infinitely many generators \( \{g_i\}_{i \in \mathbb{Z}} \). Let \( \lambda \) be the regular representation of \( \mathbb{F}_\infty \) on \( \ell^2(\mathbb{F}_\infty) \). If \( \delta_t, t \in \mathbb{F}_\infty \) denotes the unit vectors
\[
\delta_t(s) = \begin{cases} 1, & s = t \\ 0, & s \neq t \end{cases}
\]
in \( \ell^2(\mathbb{F}_\infty) \), then one has \( \lambda(s)\delta_t = \delta_{st} \) for every \( s, t \in \mathbb{F}_\infty \). The \( C^*\)-algebra \( C^*_\lambda(\mathbb{F}_\infty) \) associated with the regular representation of \( \mathbb{F}_\infty \) is the norm-closure in \( B(\ell^2(\mathbb{F}_\infty)) \) of \( \text{span}\{\lambda(s) : s \in \mathbb{F}_\infty\} \). Note that any element \( s \in \mathbb{F}_\infty \) has a unique expression as a finite product of \( g_i \) (\( i \in \mathbb{Z} \)). This expression is called the **word** for \( s \). The number of factors in the word is called the **length** of the word. Let \( \beta : \mathbb{F}_\infty \to \mathbb{F}_\infty \) be the shift-automorphism, i.e. \( \beta(g_i) = g_{i+1} \) for all \( i \in \mathbb{Z} \). The induced by \( \beta \) free-shift automorphism of \( C^*_\lambda(\mathbb{F}_\infty) \) is denoted by \( \alpha_\beta \). In [2] it has been proved that \( \alpha_\beta \) is uniformly ergodic. Now we are going to show that it is strictly weak mixing.
By a standard density argument, it is enough to show that the sequence \( \{ \alpha_{j}^{n}(\lambda(s)) \}_{n \geq 1} \) is weakly mixing to zero whenever \( \beta(s) \neq s \), that is
\[
\frac{1}{n} \sum_{k=1}^{n} |f(\alpha_{j}^{k}(\lambda(s)))| \to 0 \quad (4.2)
\]
for each \( f \in C_\lambda^*(F_\infty)^* \).

Let \( s \) be a nontrivial element of word length \( p \), then by Haagerup's inequality (cf. [16]), for each sequence \( \{ k_j \} \) of natural numbers, one has
\[
\frac{1}{n} \sum_{j=1}^{n} \left\| \frac{1}{n} \sum_{j=1}^{n} \beta_{k_j}^{j}(s) \right\|_{\ell^2(F_\infty)} = \frac{p+1}{\sqrt{n}}.
\]

Now according to Theorem 4.1 we get (4.2).

**Remark 4.1.** Note that in [15] some examples of strictly weak mixing dynamical systems, related to free shift of the reduced \( C^* \)-algebras of RG-groups and amalgamated free product \( C^* \)-algebras have been provided.

## 5. Uniform weighted ergodic theorem

From Theorem 4.1 we know that subsequential ergodic theorem holds for strictly weak mixing dynamical system. But it would be interesting to obtain some weighted uniform ergodic theorems. Note that similar problem has been investigated in [5] for Hilbert spaces. Namely, they found the necessary and sufficient conditions for the convergence of
\[
\frac{1}{n} \sum_{k=0}^{n-1} a_k T^k x \quad (5.1)
\]
for every contraction \( T \) on a Hilbert space \( H \) and every \( x \in H \). In our case, a situation is different, since we are dealing with \( C^* \)-algebras, which are not Hilbert spaces. In this section we are going to give a sufficient condition for the uniform convergence of weighted averages (5.1) for strictly weak mixing \( C^* \)-dynamical systems.

By analogy of a Besicovitch sequences (see [19]) we introduce a notion of \( S \)-Besicovitch sequences as follows: we say that a bounded sequence \( \{ b_n \} \subset \mathbb{C} \) is a \( S \)-Besicovitch if for any \( \epsilon > 0 \) there exists a uniquely ergodic dynamical system \( (C(K), \nu, T_1) \), a function \( f_0 \in C(K) \) and \( \omega_0 \in K \) such that
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| b_k - (T_1^k f_0)(\omega_0) \right| < \epsilon. \quad (5.2)
\]

Now we are ready to formulate the result.

**Theorem 5.1.** Let \( (\mathfrak{A}, \varphi, T) \) be a strictly weak mixing \( C^* \)-dynamical system. Then for every \( x \in \mathfrak{A} \) and \( S \)-Besicovitch sequence \( \{ b_n \} \) the averages
\[
\frac{1}{n} \sum_{k=0}^{n-1} b_k T^k(x) \quad (5.3)
\]
converge uniformly in \( \mathfrak{A} \).
Proof. Let \( \epsilon > 0 \) be an arbitrary number. Assume that \((C(K), \nu, T_1)\), \( f_0, \omega_0 \) is a generating system for the sequence \( \{b_n\} \). Due to commutativity of \( C(K) \) one has \((C(K) \otimes \mathcal{A})^* = C(K)^* \otimes \mathcal{A}^*\), therefore, Theorem 3.3 implies that a dynamical system \((C(K) \otimes \mathcal{A}, \nu \otimes \varphi, T_1 \otimes T)\) is uniquely ergodic, i.e. for every \( \mathbf{x} \in C(K) \otimes \mathcal{A} \) the following holds

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (T_k^1 \otimes T^k) (\mathbf{x}) = (\nu \otimes \varphi)(\mathbf{x}) \mathbf{1}.
\]

In particular, for \( f_0 \otimes x \in C(K) \otimes \mathcal{A} \) we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (T_k^1 f_0)(\omega_0) T^k (x) = \nu(f_0) \varphi(x) \mathbf{1}.
\]

This means that there is \( N_0 \in \mathbb{N} \) such that

\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} (T_k^1 f_0)(\omega_0) T^k (x) - \frac{1}{m} \sum_{l=0}^{m-1} (T_l^1 f_0)(\omega_0) T^l (x) \right\| < \epsilon \quad (5.4)
\]

for all \( n, m \geq N_0 \).

Now from (5.2) and (5.4) we find

\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} b_k T^k (x) - \frac{1}{m} \sum_{l=0}^{m-1} b_l T^l (x) \right\| \leq \left\| \frac{1}{n} \sum_{k=0}^{n-1} b_k T^k (x) - \frac{1}{n} \sum_{k=0}^{n-1} (T_k^1 f_0)(\omega_0) T^k (x) \right\|
\]

\[
+ \left\| \frac{1}{m} \sum_{k=0}^{m-1} b_l T^l (x) - \frac{1}{m} \sum_{l=0}^{m-1} (T_l^1 f_0)(\omega_0) T^l (x) \right\|
\]

\[
+ \left\| \frac{1}{n} \sum_{k=0}^{n-1} (T_k^1 f_0)(\omega_0) T^k (x) - \frac{1}{n} \sum_{k=0}^{n-1} (T_k^1 f_0)(\omega_0) T^k (x) \right\|
\]

\[
\leq \frac{1}{n} \sum_{k=0}^{n-1} |b_k - (T_k^1 f_0)(\omega_0)|||x||
\]

\[
+ \frac{1}{m} \sum_{l=0}^{m-1} |b_l - (T_l^1 f_0)(\omega_0)|||x|| + \epsilon
\]

\[
\leq \epsilon(2||x|| + 1)
\]

for all \( n, m \geq N_0 \). This completes the proof. \( \square \)

**Example.** Consider the uniquely ergodic \( C^* \)-dynamical system \((C(S^1), T_0)\) defined in Remark 3.2. For fixed \( m \in \mathbb{N} \) take \( f_{0,m}(z) = z^m \) and \( \omega_0 = 1 \). Then one can see that a sequence \( \{b_n^{(m)}\}_{n \in \mathbb{N}} \) given by \( b_n^{(m)} = a^{nm} \), here as before \( a = \exp\{2\pi i \alpha\} \), is \( S \)-Besicovitch. From the just proved theorem due to \( \varphi_\lambda (f_{0,m}) = 0 \) one concludes that for every \( \lambda \in \mathfrak{A} \)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} a^{km} T^k (x) = 0,
\]

for every strictly weak mixing \( C^* \)-dynamical system \((\mathfrak{A}, \varphi, T)\).

**Remark 5.1.** We note that Besicovitch weighted ergodic type theorems were studied in ([5],[19],[28]).
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Farrukh Mukhamedov, Department of Computational & Theoretical Sciences, Faculty of Sciences, International Islamic University Malaysia, P.O. Box, 141, 25710, Kuantan, Pahang, Malaysia

E-mail address: far75m@yandex.ru