A BRANCHED COVERING OF DEGREE 2 OF THE SPHERE
WITH A COMPLETELY INVARIANT INDECOMPOSABLE
CONTINUUM.

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Abstract. We construct a branched covering of degree 2 of the sphere $S^2$
having a completely invariant indecomposable continuum $K$. The existence of
such an object is not known for rational maps of the sphere.

1. Introduction

A longstanding question in holomorphic dynamics asks if the Julia set of a ra-
tional function can be an indecomposable continuum. In [MR] and [CMR] several
necessary and sufficient conditions are given for the Julia set of a polynomial to be
an indecomposable continuum, but without knowing the existence of any example.

For some time we have been interested in the following open problem: let $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be a continuous map of degree $d, |d| > 1$, and let $N_n f$ denote the number of
fixed points of $f^n$. When does the growth rate inequality $\limsup \frac{1}{n} \log N_n f \geq \log |d|
hold for $f$? (This is Problem 3 posed in [S]). There are many recent papers written
on the subject. For example the following result holds: Assume that $f$ is a degree
$d (|d| > 1)$ branched covering of the sphere with a completely invariant simply
connected region $R$ whose boundary component is locally connected. If the set of
critical points in the boundary of $R$ is not reduced to a fixed point with multiplicity
$d − 1$, then $f$ satisfies the growth rate inequality.

This result appears in [PRX], where many other references on this problem can
be found. An example to have in mind is when $f$ is a complex polynomial with
connected and locally connected Julia set. Then $f$ has a superattracting fixed point
at infinity and the region $R$ is its basin of attraction, which is the complement of
the filled Julia set. So, in particular, the theorem mentioned above is a topological
version of the fact that complex polynomials (with connected and locally connected
Julia set) satisfy the growth rate inequality.

However, it is not known if the hypothesis on local connectivity is necessary. It
is a technical hypothesis that allowed to find the periodic points using prime end
theory. It is then natural to ask what happens if this hypothesis is dropped. Using
the ideas in [MR] and [CMR] one is quickly led to consider the case where $\partial R$ is an
indecomposable continuum. As the mere existence of such an example is unknown,
the next natural question is the topic of the present paper. We prove the following:

Theorem 1. There exists a degree 2 branched covering $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ supporting a
completely invariant indecomposable continuum.

The construction is based on the derived of pseudo-Anosov map on $S^2$, i.e.,
the Plykin repeller. In this example, the stable foliation of a generalized pseudo-
Anosov of $S^2$ is split open along a single leaf to create a 1-dimensional repeller. This
repeller has a very regular structure: it is an indecomposable continuum which looks locally like the product of the Cantor set and an open interval and which is the common boundary of its complementary domains. Naturally, these examples are homeomorphisms, but we manage to adapt these constructions for higher degree.

In the middle sections the main properties of indecomposable continua are analyzed and some primary questions about the existence and properties of indecomposable continua supporting noninvertible maps are studied.

2. INDECOMPOSABLE CONTINUA.

In this section we present the definition and properties of indecomposable continua.

Definition 1. A continuum is a compact connected set. The composant (briefly called compos) $C_p$ of a point $p$ is the union of all proper continua containing $p$. A continuum is decomposable if it is the union of two proper subcontinua. Otherwise, it is indecomposable.

Some properties follow, a good reference is [K]. From now on $X$ is a continuum.

1. If $K$ is a proper subcontinuum and $X \setminus K$ is disconnected, then $X$ is decomposable.
2. If a proper subcontinuum has interior, then $X$ is decomposable.
3. A continuum is decomposable if and only if there exists a nondense open set which disconnects it.
4. Every compos is dense. $X$ is decomposable iff every compos is $X$.

Example 1. Examples of indecomposable continua are the closure of the stable manifold of a horseshoe, the solenoid attractor, or the attractor of a derived from Anosov homeomorphism of the torus.

It is assumed from now on in this section that $X$ is a continuum and every $x \in X$ has a neighborhood $U$ which is homeomorphic to the product of the usual Cantor set $C$ and an open interval $I$. To avoid unnecessary notations we will drop the homeomorphism between the neighborhood and $C \times I$.

The local leaf of a point $\bar{x} = (x, a)$ relative to its neighborhood $C \times I$ is defined as the set $\{x\} \times I$. A segment of $X$ is a connected union of a finite number of local leaves. The leaf at $\bar{x}$, denoted $H_{\bar{x}}$, is the union of the segments containing $x$.

The following theorem will imply the indecomposability of the example we construct.

Theorem 2. If $X$ is locally the product of a Cantor set times an interval, and there exist a dense leaf, then $X$ is indecomposable.

Lemma 1. Two leaves are disjoint or equal. Every leaf of $X$ is an immersion of $\mathbb{R}$ in $X$.

Proof: Note that if $z$ belongs to $H_p \cap H_q$ then the local leaf of $z$ is contained in both $H_p$ and $H_q$, and this implies the first assertion.

Let $p \in X$; consider the class $\mathcal{L}$ which elements are the pairs $(\varphi, J)$ where $J$ is an interval and $\varphi$ an immersion of $J$ in $X$, and whose image is contained in $H_p$. A partial order is defined in $\mathcal{L}$ as usual, a pair is bigger than another if its associated immersion extends the other. Clearly $\mathcal{L}$ is nonempty, every chain is bounded and a maximal element $(\varphi, J)$ has its image contained in $H_p$. It remains to show that
\( \varphi(J) \) contains \( H_p \). Let \( q \in H_p \); by definition of \( H_p \) there exists a finite set of local leaves \( \ell_1, \ldots, \ell_n \) relative to neighborhoods \( U_1, \ldots, U_n \), and whose union contains \( p \) and \( q \). Note that if \( \ell_i \cap \ell_j \) is nonempty, and \( \varphi(J) \) intersects \( \ell_i \), then \( \varphi(J) \) contains both \( \ell_i \) and \( \ell_j \). It follows that \( \varphi(J) \) contains every \( \ell_i \), and so contains \( q \).

The following assertion is proved using the definition of leaf and the local structure. It will referred below as “continuity of leaves”. A section at \( p \) is a set \( \Sigma_p \) containing \( p \) and having the form \( C \times \{a\} \), where \( U = C \times I \) is a neighborhood of \( p \). Assume now that \( q \in H_p \), and that \( \Sigma_p \) and \( \Sigma_q \) are corresponding sections; let also \( K \) be a continuum contained in \( H_p \) containing \( p \) and \( q \), and \( U \) a neighborhood of \( K \). Then there exists a neighborhood \( V \) of \( p \) in \( \Sigma_p \) such that, for every \( y \in V \) the leaf \( H_y \) contains a continuum \( L \) contained in \( U \) and intersecting \( \Sigma_q \).

We turn now to the proof of Theorem 2. We will assume from now on that there is some point \( z \) such that \( H_z \) is dense. A standard argument that we repeat now implies that this is still true for a residual set of points: for each nonempty open set \( U \subset X \) define the set \( X(U) = \{x \in X : X_x \cap U \neq \emptyset\} \). By continuity of leaves, the set \( X(U) \) is open; it is also dense because \( H_z \) is dense. If \( \{U_n\}_{n \in \mathbb{N}} \) is a countable basis of open sets, then by Baire Theorem \( X' := \bigcap_n X(U_n) \) is a residual set, and we conclude that \( X' \) is a dense set of points \( z \) whose leaves \( H_z \) are dense.

Next define, for each nonempty open set \( U \), the set \( X_0(U) \) as the set of points \( x \) such that each component of \( X_x \setminus \{x\} \) is dense. It is claimed that \( X_0(U) \) is dense in \( X \) for every \( U \). It suffices to prove that \( X_0(U) \) is dense in \( X' \), which is dense in \( X \). Let \( x \in X' \setminus X_0(U) \), \( V \) an arbitrary neighborhood of \( x \), and \( \varphi : \mathbb{R} \to X_x \) an immersion. Assume that \( \varphi(\mathbb{R}^-) \) does not intersect \( U \). Let \( t_1 > 0 \) be minimum such that \( \varphi_x(t_1) \in U \). Taking a smaller \( V \) if necessary, it can be assumed that for every \( y \in V \) there exists an immersion \( \varphi_y : \mathbb{R} \to H_y \) such that \( \varphi_y(t_y) \in U \) for some positive \( t_y \) close to \( t_1 \). Using that \( \varphi_x(\mathbb{R}^-) \) is not a dense set and the continuity of leaves, it follows that there exists \( t^- < 0 \) and an open set \( W \) disjoint from \( V \) and \( U \) such that the following conditions hold:

1. \( \varphi_x(t^-) \in \partial W \).
2. \( W \cap \varphi_x(\mathbb{R}^-) = \emptyset \).
3. \( W \subset \bigcup \{\varphi_y(\mathbb{R}^-) : y \in V'\} \).

On the other hand, using that \( H_x = \varphi_x(\mathbb{R}) \) is dense and that (by condition (2)) \( \varphi_x(\mathbb{R}^-) \) does not intersect \( W \), it follows that there exists some \( t_2 > t_1 \) such that \( \varphi_x(t_2) \in W \). Taking a smaller \( W \) if necessary, and using again the continuity of leaves it follows that for every point in \( w \in W \) the the leaf \( H_w \) contains a segment close to \( \varphi_x(t_1, t_2) \), thus entering \( U \).

Besides, it follows by property (3) above that for every \( w \in W \) the map \( \varphi_w(t) = \varphi_y(t - \tau) \), where \( y \in V \) and \( w = \varphi_y(-\tau) \) is an immersion of \( \mathbb{R} \) onto \( H_w \).

Now take any point \( y \in V \) such that \( w = \varphi_y(-\tau) \in W \) for some positive \( \tau \). Then there exists some \( T < 0 \), close to \( \tau + t_2 - t_1 \) such that \( \varphi_y(-T) \in U \). This proves that \( y \in X_0(U) \). This proves that \( X_0(U) \) is open and dense and proves the claim.

Using Baire’s Theorem again, we conclude that the set of points in the intersection \( X'' \) of all the \( X_0(U) \) is dense, in particular nonempty. This is used in the proof of the Proposition below.
Proof of Theorem 2. By the assertion above, the assumption that a leaf is dense implies that $X''$ is nonempty. Let $z \in X''$.

Let $p$ be a point not belonging to the leaf $H_z$. Let $K$ be a compact proper set containing $z$ and $p$. If we show $K$ is not connected, then $X$ is indecomposable, by definition of compos and property 4 of continua. As $K$ is a proper compact, there exists $U = C \times I$ disjoint from $K$. Let $C'$ be a compact subset of the Cantor set $C$, not containing 0 nor 1. As each component of $H_z \setminus \{z\}$ is dense there exists a maximal arc $\ell$ containing $z$ and contained in $H_z \cap (X \setminus C' \times I)$.

For $i = 0, 1$ let $\bar{a}_i = (\alpha_i, i)$ be the extreme point of $\ell$ in $C \times \{i\}$.

Let $a_1$ and $a_2$ be extreme points of the Cantor set $C'$ such that:

1. $a_1 < a_0 < a_2$
2. $a_1$ is the right extreme point of a gap of $C$ and $a_2$ a left one.
3. There exist maximal arcs $\ell_1$ and $\ell_2$ contained in $X \setminus (C' \times I)$ and contained as extreme points $a_1$ and $a_2$ in $C' \times \{0\}$ and $b_1$ and $b_2$ in $C' \times \{1\}$.

It follows by the continuity of leaves that for every $\bar{x} = (x, 0)$ with $x \in [a_1, a_2]$ there exists a maximal arc $\ell_{\bar{x}}$ containing $\bar{x}$ and contained in $X \setminus (C' \times I)$. Then $a_1$ and $a_2$ can be taken as close to $\alpha_0$ as necessary to obtain that $p \notin \ell_{\bar{x}}$ for every $x$ between $a_1$ and $a_2$.

Consider the set $W = \cup\{\ell_{\bar{x}} : x \in [a_1, a_2]\}$. $W$ is closed by continuity of leaves and open by the choice of the points $a_1$ and $a_2$. As $K$ contains $z$ and $p$ we conclude that its intersections with both $W$ and its complement are non empty, implying that $K$ is not connected, as wished.

Example 2. (1) Let $\Sigma = 2^Z$ and $\sigma$ the shift. If $X$ is the suspension of $\sigma$ then it is locally Cantor times interval, there are dense leaves and closed leaves. So $X$ is indecomposable. Question: is it true that if $X$ is contained in a surface then the existence of dense leaves implies that every leaf is dense?

(2) Let $h$ be a homeomorphism of the interval $[0, 1]$ and $C$, as usual, the middle thirds Cantor set. Assume that $h$ carries $C \cap [0, 1/3]$ to $C \cap (0, 1/3]$ and $C \cap [1/3, 1]$ in $C \cap [2/3, 1]$. If $X$ is the suspension of the restriction of $h$ to $C$ then $X$ is on the hypothesis of this section and no leaf is dense. Of course, $X$ is decomposable the compos of each point strictly contains the leaf.

3. An explicit example

In this section we give an example of a 2 : 1 branched covering map of the sphere with a 2 : 1 invariant indecomposable continuum.

We consider the linear hyperbolic toral endomorphism $A$ induced by the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

The map $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a nonexpanding Anosov endomorphism of degree 2; its dynamics is well known, it has invariant stable and unstable foliations with dense leaves. It has the very special property that the unstable space does not depend on the preorbit chosen to define it, so it is not structurally stable: generic perturbations do not satisfy this property.

The equivalence relation $x \sim -x$, $x \in \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ is preserved by $A$ by linearity, so it induces a map $f : \mathbb{T}^2 / \sim \to \mathbb{T}^2 / \sim$, where $x \sim -x$, $x \in \mathbb{T}^2$. Note that $\mathbb{T}^2 / \sim$
is topologically a sphere and that the quotient map $\pi : \mathbb{T}^2 \to S^2$ is a degree 2 branched covering with exactly four critical points; namely, the classes in $\mathbb{T}^2$ of the points $(1/2, 1/2), (1, 0), (1/2, 0), (0, 1/2)$. By definition the map $f : S^2 \to S^2$ satisfies $\pi\sigma = f\pi$.

By construction $f$ is a degree two branched covering map; hence by the Riemann-Hurwitz formula it has two critical points that can be easily found by computing those points having just one preimage. The critical points are $(1/4, 0), (1/4, 1/2)$ with corresponding critical values $(0, 1/2), (1/2, 0)$ (formally, the projected images of these points in $S^2$). Moreover, the images of the critical values are the fixed points of $f$; namely, $f((0, 1/2)) = (1/2, 1/2)$ and $f((1/2, 0)) = (0, 0)$. That is, $f$ is postcritically finite (or a Thurston map); it has the property that every future critical orbit is finite. As we shall see in the next section, any example constructed in this way will have this property.

The dynamics of $f$ can also be described: for the time being, $f$ is just a $C^0$ map, but as $\pi$ is a local diffeomorphism except at the four branched points, then $f$ is everywhere differentiable but at these exceptional points. The projections of the stable and unstable invariant foliations of $\mathbb{A}$ have a local product structure in $S^2$ at every point except at the branched points, so the local product structure holds at every point with the exception of four points.

The fixed points of $f$ are projections of branched points of $\pi$. The stable set at $\pi((0, 0))$, that for $\mathbb{A}$ had two separatrices, is also fixed by $f$, but has now only one separatrix; and the same occurs with the unstable separatrix. The local behaviour of $f$ at these fixed points is a pseudo-Anosov type singularity with one separatrix. The map $f$ is not differentiable at the image of the branched points of $\pi$.

The next step is to produce a modification of $f$ at its fixed points in order to turn them into attractors. This is similar to the procedure to transform a saddle type fixed point into an attractor, creating a pair of saddles: the well known Derived from Anosov construction. Not so famous is the similar procedure to transform the map in a neighborhood of the pseudo-Anosov singularity in such a way that the modified map is differentiable, it has two hyperbolic fixed points, one of them is an attractor and the other a saddle. We refer the reader to the appendix in [BM], where this procedure is carefully explained.

This procedure is performed around both fixed points of $f$. Note that $f$ being a local homeomorphism at these points, and the perturbation being also local, the procedure carries out exactly as in [BM]. These points are turn into attracting fixed points, while a pair of points of saddle type are created, $p_1$ close to $\pi((0, 0))$ and $p_2$ close to $\pi((1/2, 1/2))$. The new map, denoted $F$, satisfies the following properties:

- The immediate basins of attraction $B_1$ of $\pi((0, 0))$ and $B_2$ of $\pi((1/2, 1/2))$ have as accessible boundary the stable manifolds of $p_1$ and $p_2$ respectively.
- $F$ is differentiable, the critical points belong to the basins of attraction.
- The restriction of $F$ to the complement $K$ of the union of the basins is a $2 : 1$ map having a hyperbolic structure. It follows that $K$ is a repellor, since it contains the stable manifolds of its points.
- The set $K$ is indecomposable by Theorem 2.

4. Characterizing this family of examples

Of course there is nothing special about the matrix $A$ we used to construct the example in the previous section, and we could have used other non-expanding Anosov
endomorphism. However, the fact that the map \( F : S^2 \to S^2 \) we constructed in the previous section lifts to the covering \((T^2, \pi)\) impose some restrictions on the kind of examples that can be created this way. We explore some of them in this final section.

We say that \( f : S^2 \to S^2 \) is a Lattès map if there exists a non-invertible covering map \( \tilde{f} : T^2 \to T^2 \) such that \( \pi \tilde{f} = f \pi \).

**Lemma 2.** Let \( f \) be a Lattès map. Then, the critical values of \( f \) are contained in the critical values of \( \pi : f(\text{Crit}(f)) \subset \pi(\text{Crit}(\pi)) \).

**Proof.** If \( x \in S^2 \setminus \pi(\text{Crit}(\pi)) \), then \( \# \pi^{-1}(x) = 2 \) and \( \# \tilde{f}^{-1}(\pi^{-1}(x)) = 2d \), where \( d > 1 \) is the degree of \( \tilde{f} \). So, \( \# \pi(\tilde{f}^{-1}(\pi^{-1}(x))) = \# f^{-1}(x) \geq d \), and therefore \( x \) is not a critical value. \( \square \)

**Lemma 3.** Let \( f \) be a Lattès map. Then, the critical values of \( \pi \) are not critical points of \( f : \text{Crit}(f) \cap \pi(\text{Crit}(\pi)) = \emptyset \).

**Proof.** If \( x \in \text{Crit}(\pi) \), then \( \pi \) is 2 : 1 at \( x \). Therefore, if \( \pi(x) \) is a critical point of \( f \), \( f\pi \) is \( k : 1 \) at \( x \), where \( k > 2 \). But \( f\pi = \pi \tilde{f} \) that is at most 2 : 1 at \( x \). \( \square \)

**Lemma 4.** Let \( f \) be a Lattès map. Then, the critical values of \( \pi \) are \( f \)-invariant: \( f(\pi(\text{Crit}(\pi))) \subset \pi(\text{Crit}(\pi)) \).

**Proof.** Let \( x \) be a critical value of \( \pi \). By the previous lemma \( x \) is not a critical point for \( f \), then \( f\pi \) is locally 2 : 1 at \( z = \pi^{-1}(x) \). As \( f\pi = \pi \tilde{f} \), \( \tilde{f}(z) \) must be a critical point of \( \pi \), that is \( \pi \tilde{f}(z) = f\pi(z) = f(x) \) is a critical value of \( \pi \). \( \square \)

As the critical values of \( \pi \) is a set of four points, these points must be \( f \)-periodic. By Lemma 2 the critical points of \( f \) are pre-periodic. This already imposes a restriction on our family of examples: they are all Thurston maps. Moreover, by Proposition C8.7 in [H] none of these examples are equivalent to a rational function.

Furthermore, the following holds:

**Lemma 5.** Let \( f \) be a Lattès map. Then, the critical points of \( f \) are not periodic.

**Proof.** This follows immediately from lemmas 3 and 4. \( \square \)

As a consequence, Lattès maps cannot be topological polynomials, that is having a point \( \infty \in S^2 \) such that \( f^{-1}(\infty) = \infty \). In particular, the question as to whether there exists a topological polynomial supporting a completely invariant indecomposable continuum remains open.

Finally, regarding the growth rate inequality, one immediately obtains:

**Lemma 6.** Let \( f \) be a Lattès map such that the induced map on homology of its lift \( \tilde{f} \) to the torus, \( \tilde{f}_* : H_1(T^2, \mathbb{Z}) \to H_1(T^2, \mathbb{Z}) \) is hyperbolic. Then, \( f \) satisfies the growth rate inequality.

**Proof.** Immediate from the fact that the lift \( \tilde{f} : T^2 \to T^2 \) satisfies the growth rate inequality (see, for example Theorem 1.2 page 618 of [BFGJ]). \( \square \)
Note that the hypothesis on the homology class of the lift is necessary, as the product of \( z^2 \) times an irrational rotation acting on \( S^1 \times S^1 \) shows.

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