Uniqueness of the Fréchet algebra topology on certain Fréchet algebras

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Abstract. In 1978, Dales posed a question about the uniqueness of the ($F$)-algebra topology for ($F$)-algebras of power series in $k$ indeterminates. We settle this in the affirmative for Fréchet algebras of power series in $k$ indeterminates. The proof goes via first completely characterizing these algebras; in particular, it is shown that the Beurling-Fréchet algebras of semiweight type do not satisfy a certain equicontinuity condition due to Loy. Some applications to the theory of automatic continuity are also given, in particular the case of Fréchet algebras of power series in infinitely many indeterminates.

Key words: Fréchet algebra of power series in $k$ indeterminates, Arens-Michael representation, Loy’s condition (E), automatic continuity.
1 Introduction.

Throughout the paper, “algebra” will mean a complex, commutative algebra with identity unless otherwise specified. A Fréchet algebra is a complete, metrizable locally convex algebra $A$ whose topology $\tau$ may be defined by an increasing sequence $(p_m)_{m \geq 1}$ of submultiplicative seminorms. We may refer to $\tau$ as “the Fréchet topology of $A$” in the following. The principal tool for studying Fréchet algebras is the Arens-Michael representation, in which $A$ is given by an inverse limit of Banach algebras $A_m$ (see [11 §5] or [12 §2]).

Let $k \in \mathbb{N}$ be fixed. We write $\mathcal{F}_k$ for the algebra $\mathcal{C}[[X_1, X_2, \ldots, X_k]]$ of all formal power series in $k$ commuting indeterminates $X_1, X_2, \ldots, X_k$, with complex coefficients. A fuller description of this algebra is given in [3 §1.6]; we briefly recall some notation, which will be used throughout the paper. Let $k \in \mathbb{N}$, and let $J = (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^{\times k}$. Set

$$|J| = j_1 + j_2 + \cdots + j_k;$$

ordering and addition in $\mathbb{Z}^{\times k}$ will always be component-wise. A generic element of $\mathcal{F}_k$ is denoted by

$$\sum_{J \in \mathbb{Z}^{\times k}} \lambda_J X^J = \sum \{ \lambda_{(j_1, j_2, \ldots, j_k)} X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} : (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^{\times k} \}.$$
The algebra $\mathcal{F}_k$ is a Fréchet algebra when endowed with the weak topology $\tau_c$ defined by the coordinate projections

$$\pi_I : \sum_{J \in \mathbb{Z}^+} \lambda_J X^J \mapsto \lambda_I, \quad \mathcal{F}_k \to \mathcal{A},$$

for each $I \in \mathbb{Z}^k$. A defining sequence of seminorms for $\mathcal{F}_k$ is $(p'_m)$, where

$$p'_m(\sum_{J \in \mathbb{Z}^k} \lambda_J X^J) = \sum_{|J| \leq m} |\lambda_J| \quad (m \in \mathbb{N}).$$

A Fréchet algebra of power series in $k$ variables (shortly: FrAPS in $\mathcal{F}_k$) is a subalgebra $A$ of $\mathcal{F}_k$ such that $A$ is a Fréchet algebra containing the indeterminates $X_1, X_2, \ldots, X_k$ and such that the inclusion map $A \hookrightarrow \mathcal{F}_k$ is continuous (equivalently, the projections $\pi_I, I \in \mathbb{Z}^k$, are continuous linear functionals on $A$). It is worthwhile mentioning that in [4, Corollaries 11.3 and 11.4], we show that the time-honoured definitions of Banach and Fréchet (and, more generally, ($F$)-) algebras of power series in $\mathcal{F}_1$ contain a redundant clause of the continuity of coordinate projections; this is not possible in the several-variable case by [4, Theorem 12.3].

Though Fréchet algebras of power series in $k$ indeterminates have been considered by Loy [9], recently these algebras – and more generally, the power series ideas in general Fréchet algebras – have acquired significance in understanding the structure of Fréchet algebras [1, 3, 4, 12, 13, 14, 15].
Thus it is of interest to investigate the following:

(1) whether one can completely characterize these algebras,

(2) whether such algebras have a unique topology as Fréchet algebras.

In this paper we shall be concerned with the solution to the above problems; our argument here is kept short because it uses key ideas involved in the solution to these problems for the case $k = 1$ [12]. (See Theorem 3.1 and Corollary 4.3 below.) In Section 3, we obtain several results of independent interest. Precisely, we shall classify FrAPS in $\mathcal{F}_k$ which do not satisfy an equicontinuity condition (E): there is a sequence $(\gamma_k)_{K \in \mathbb{N}}$ of positive reals such that $(\gamma_k^{-1} \pi_k)$ is equicontinuous [9]. (See Theorem 3.10 below.)

We remark that the uniqueness of the Fréchet topology of $\mathcal{F}_k$ for each $k \in \mathbb{N}$ is established in [3], and the general case has been open since 1978 [2, Question 11]. We use the structure of the closed ideals and their powers to establish the uniqueness of the Fréchet topology of FrAPS in $\mathcal{F}_k$, this is not known for the larger algebra $\mathcal{F}_\infty = \mathcal{C}[[X_1, X_2, \ldots]]$ [14] and FrAPS in $\mathcal{F}_\infty$, and so we cannot apply our approach to establish the uniqueness of the Fréchet topology of FrAPS in $\mathcal{F}_\infty$. Not only this, but Read also showed in [14] that in the absence of the uniqueness of the Fréchet topology, the Singer-
Wermer conjecture cannot be established in the case of Fréchet algebras and thus, the situation on Fréchet algebras is markedly different from that on Banach algebras. However, we shall give some remarks, establishing the uniqueness of the Fréchet topology of FrAPS in \( \mathcal{F}_\infty \) admitting a continuous norm. The solution of Question 11 of [2] does include the uniqueness of the Fréchet topology of FrAPS in \( \mathcal{F}_1 \), established in [12], as a special case.

A Fréchet algebra \((A, (p_m))\) is said to be a Fréchet algebra with power series generators \(x_1, x_2, \ldots, x_k\) if each \(y \in A\) is of the form

\[
y = \sum_{J \in \mathbb{Z}^{+k}} \lambda_J x^J = \sum \{ \lambda_{(j_1, j_2, \ldots, j_k)} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} : (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^{+k} \},
\]

for \(\lambda_J\) complex scalars such that \(\sum_{J \in \mathbb{Z}^{+k}} |\lambda_J| \ p_m(x^J) < \infty\) for all \(m\).

Thus if \(A\) is a Fréchet algebra with finitely many power series generators \(x_1, x_2, \ldots, x_k\), then \(A\) is a commutative, separable, finitely generated Fréchet algebra generated by \(x_1, x_2, \ldots, x_k\). A semiweight function on \(\mathbb{Z}^{+k}\) is a function \(\omega : \mathbb{Z}^{+k} \to \mathbb{R}\) such that

\[
\omega(M + N) \leq \omega(M)\omega(N), \ \omega(0) = 1 \ \text{and} \ \omega(N) \geq 0 \ (M, N \in \mathbb{Z}^{+k});
\]

a semiweight function is a weight function if for all \(N \in \mathbb{Z}^{+k}, \ \omega(N) > 0\). To answer (2) above, we will investigate the following two questions on FrAPS \(A\) in \(\mathcal{F}_k\) as an intermediate step.
(A) When are $X_1, X_2, \ldots, X_k$ power series generators for $A$?

(B) When is $A$ isomorphic to an inverse limit of Banach algebras of power series in $k$ variables?

The solution to the above problems are given in Theorems 3.4 and 3.10, respectively. In Section 4, we also pose some interesting questions in automatic continuity theory.

2 Fréchet algebras.

Let $M$ be a closed maximal ideal of a Fréchet algebra $A$. We shall suppose from now on that $\dim(M/M^2) = k$ is finite (it is easy to see that for finitely generated Fréchet algebras this condition is automatically satisfied; see [15, Proposition 2.2] for the Banach case). Then, by the remark following Theorem 2.3 of [15], for each $n \in \mathbb{N}$, the homogeneous monomials of degree $n$ in $t_1, t_2, \ldots, t_k \in M$ are representatives of a basis for $M^n/M^{n+1}$ if and only if $\dim(M^n/M^{n+1}) = C_{n+k-1,n} = \frac{(n+k-1)!}{n!(k-1)!}$ for all $n$, and thus $M$ is not nilpotent. Thus, in a special case, we have the following, with an eye on [12, Lemma 2.1].
Proposition 2.1 Let \((A, (p_m))\) be a commutative, unital Fréchet algebra with the Arens-Michael isomorphism \(A \cong \lim_{\leftarrow} (A_m; d_m)\). Suppose that there exists a fixed \(k \in \mathbb{N}\) such that \(M\) is a closed maximal ideal of \(A\) such that:

(i) \(\bigcap_{n \geq 1} M^n = \{0\}\) and (ii) \(\dim(M^n/M^{n+1}) = C_{n+k-1,n}\) for all \(n\). Then there exist \(t_1, t_2, \ldots, t_k \in M\) such that \(M^n = M^{n+1} \oplus \text{sp}\{t^I : |I| = n\}\) for each \(n \geq 1\). Assume further that each \(p_m\) is a norm. Then, for each sufficiently large \(m\), \(M_m\) is a non-nilpotent maximal ideal of \(A_m\) such that:

(a) \(\bigcap_{n \geq 1} M^n_m = \{0\}\) and (b) \(\dim(M^n_m/M^{n+1}_m) = C_{n+k-1,n}\) for all \(n\).

Proof. The first half of the proof has already been discussed above. For the second half of the proof, follow \[13\], Proposition 2.3.

Concerning Proposition 2.1, the counter-examples (see \[13\]; for the one-variable case) show that the assumption that each \(p_m\) is a norm on \(A\) cannot be dropped. The algebra \(\mathcal{F}_k\) is a trivial counter-example in the several-variable case. We also remark that, in the case where \(\dim(M/M^2) = 1\), one deduces \(\dim(M^n/M^{n+1}) = 1\) for all \(n\) in \[12\], Proposition 2.3], and so we do not require \(\dim(M^n/M^{n+1}) = 1\) for all \(n\) as a stronger hypothesis, but then we do require \(M\) to be non-nilpotent there. Below, we exhibit an easy counter-example to show that the hypothesis that \(\dim(M^n/M^{n+1}) = C_{n+k-1,n}\) for all \(n \in \mathbb{N}\) is not redundant in the proposition above (many
thanks to Professor H. G. Dales for calling my attention to this counter-
example).

Let

$$B = \mathcal{F}_2 = \mathcal{O}[[X, Y]],$$

with the usual Fréchet algebra topology $\tau_c$ and let $J$ be the ideal gen-
erated by the element $X^2 - Y^3$. Since $B$ is noetherian \[17\] VII, Corollary
p. 139 and Theorem 4', all ideals in $B$ are closed by \[19\] Theorem 5],
so $J$ is closed. Hence the quotient $A = B/J$ is a noetherian Fréchet al-
gebra, with all the ideals in $A$ closed. Clearly, $M/J$ is the unique max-
imal ideal in $A$, where $M = \ker \pi_0 \ (0 = \{0, 0\})$, in $B$. The two el-
ements $X + J$ and $Y + J$ are linearly independent modulo $(M/J)^2$, and
so $\dim((M/J)/(M/J)^2) = 2$ since $M/M^2 \cong (M/J)/(M/J)^2$. However
$(X + J)^2 \in (M/J)^3$, so $\dim((M/J)^2/(M/J)^3) = 2$ since $XY + J$ and
$Y^2 + J$ are linearly independent modulo $(M/J)^3$.

Next, to see that this is a counter-example, we show that $\bigcap_{n \geq 1}(M/J)^n \neq
J$, the zero element of $A$. To see this, let us start with an element $g$ of $B$
such that

$$g \in pX^2 + qXY + rY^2 + M^3 + J,$$
for some $p, q, r \in \mathcal{C}$; that is,

$$g \in \mathcal{C}X^2 + \mathcal{C}XY + \mathcal{C}Y^2 + M^3 + J$$

and suppose that $g \in M^4 + J$. Then

$$g \in (X^2 - Y^3)(a + bX + cY) + M^4,$$

for some $a, b, c \in \mathcal{C}$ because all other terms in $J = (X^2 - Y^3)B$ are in $M^4$.

Thus we can see that there exist $a_1, b_1, c_1 \in \mathcal{C}$ such that

$$pX^2 + qXY + rY^2 = (X^2 - Y^3)(a_1 + b_1X + c_1Y) + M^3.$$

Now, equating the coefficients of $XY$ and $Y^2$, we see that $q = r = 0$, and, equating the coefficients of $X^2$, we see that $p = a_1$, and then, equate the coefficients of $Y^3$ to see that $0 = a_1$. Thus $p = 0$. We conclude that $M^3 + J = M^4 + J$. One can generalize this idea to see that $M^n + J = M^{n+1} + J$ for each $n \geq 3$, the only element of $B$ that belongs to $M^n + J$ is actually in $M^{n+1} + J$. Thus we conclude that $M^3 + J = M^n + J$ for each $n \geq 3$, and so $\bigcap_{n \geq 1}(M/J)^n = (M/J)^3 \neq J.$
3 Fréchet algebras of power series in $\mathcal{F}_k$.

We now turn to the problem of describing all those commutative Fréchet algebras which may be continuously embedded in $\mathcal{F}_k$ in such a way that they contain the polynomials in $X_1, X_2, \ldots, X_k$. The following theorem completely characterizes separable FrAPS in $\mathcal{F}_k$. The method of proof will be used again in the proof of Theorem 3.10.

**Theorem 3.1** Let $A$ be a commutative, unital Fréchet algebra. Suppose that there exists a fixed $k \in \mathbb{N}$ such that $A$ contains a closed maximal ideal $M$ such that: (i) $\cap_{n \geq 1} M^n = \{0\}$; and (ii) $\dim(M^n/M^{n+1}) = C_{n+k-1,n}$ for all $n$. Then $A$ is a Fréchet algebra of power series in $\mathcal{F}_k$. The converse holds if the polynomials in $X_1, X_2, \ldots, X_k$ are dense in $A$.

**Proof.** The proof is similar to that of [12, Theorem 3.1], and will be outlined only. Supposing $A$ satisfies the stated conditions, there exist $t_1, t_2, \ldots, t_k \in M$ such that

$$M^n = M^{n+1} \oplus \text{sp}\{t^I : |I| = n\},$$

for each $n \geq 1$, by Proposition 2.1. Let $x \in A$. Then a simple induction on $n$ shows that for $n \geq 1$, $x = \sum_{|I| \leq n} \lambda_I t^I + y_n$, where $y_n \in M^{n+1}$ and the $(\lambda_I)$ are uniquely determined. Hence the functionals $\pi_J : x \mapsto \lambda_J$ are
uniquely defined, and linear for all $J \in \mathbb{N}^k$. If $x \in \ker \pi_J$ for all $J \in \mathbb{N}^k$, then $x \in \cap_{n \geq 1} M^n = \{0\}$. Thus the mapping

$$x \mapsto \sum_{I \in \mathbb{Z}^{I \in \mathbb{N}^k}} \pi_I(x) t^I$$

is an isomorphism of $A$ onto an algebra of formal power series in $\mathcal{F}_k$.

Carrying over the topology via this isomorphism, the result will follow once we show that the functionals $\pi_J$ are continuous for each $J$. Clearly $\pi_{0,\ldots,0}$ is continuous since $M = \ker \pi_{0,\ldots,0}$ is a closed maximal ideal of $A$. Let $J \in \mathbb{N}^k$, and assume that $\pi_I$ is continuous for each $|I| < |J|$, and take $(x_n)$ in $A$ with $x_n \to 0$ as $n \to \infty$. Then

$$x_n = \sum_{|I| \leq |J|} \pi_I(x_n) t^I + y_{n,|J|},$$

for some $y_{n,|J|} \in \overline{M^{[J]+1}}$. It follows that

$$\sum_{|I| = |J|} \pi_I(x_n) t^I + y_{n,|J|} \to 0,$$

so if $\pi_I(x_n)$ does not converge to 0 for at least one $I$ ($|I| = |J|$) we deduce that some non-zero linear combination of $t^I$, $|I| = |J|$, lies in $\overline{M^{[J]+1}}$, a contradiction. Thus each $\pi_J$ is continuous.

Conversely, let $A$ be a Fréchet algebra of power series in $\mathcal{F}_k$ such that the polynomials in $X_1, X_2, \ldots, X_k$ are dense in $A$. Setting $M = \ker \pi_{0,\ldots,0}$, we
have $\overline{M^n} \subset \ker \pi_N$ for each $N \in \mathcal{Z}^+^k$ with $|N| = n-1 \ (n \in \mathcal{N})$, so that $\bigcap_{n \geq 1} \overline{M^n} = \{0\}$. Clearly

$$\overline{M^{n+1}} \oplus \text{sp}\{t^I : |I| = n\} \subseteq \overline{M^n}.$$ 

Let

$$M_{n+1} = \{a \in A : o(a) \geq n+1\}.$$ 

Since $M_{n+1}$ is closed by continuity of the functionals $\pi_J$, we have $\overline{M^{n+1}} \subseteq M_{n+1}$. Given that the polynomials are dense in $A$, the series actually converge in $A$, and so $\overline{M^{n+1}} = M_{n+1}$. Hence $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for all $n$, and the theorem follows.

We note that, in all FrAPS $A$ in $\mathcal{F}_k$, we have $M = \ker \pi_{0,...,0}$ is a non-nilpotent, closed maximal ideal such that $\bigcap_{n \geq 1} \overline{M^n} = \{0\}$. Two counter-examples in the one-variable case (see [12, Remarks 1 (b)]) show that the assumption that the polynomials are dense in $A$ cannot be dropped in the above theorem.

We now turn to answer (A) above. The following lemma, whose proof we omit, is a several-variable-analogue of [12, Lemma 3.2] (for proof, see [1, Lemma 2.2]. Define a subset $A_1$ of $A$ by

$$A_1 := \{y \in A : \sum_{J \in \mathcal{Z}^+^k} |\lambda_J| p_m(X^J) < \infty \text{ for all } m\}$$
in the lemma.

**Lemma 3.2** Let $A$ be a Fréchet algebra of power series in $\mathcal{F}_k$. Then:

1. $A_1$ is continuously embedded in $A$;
2. $A_1$ is a Fréchet algebra having power series generators $X_1, X_2, \ldots, X_k$;
3. $A_1$ is a Banach algebra provided that $A$ is a Banach algebra.

We recall that elements $x_1, x_2, \ldots, x_k$ in a Fréchet algebra $A$ generate a multi-cyclic basis if each $y \in A$ can be uniquely expressed as

$$y = \sum_{J \in \mathbb{Z}^{+k}} \lambda_J x^J = \sum \{\lambda_{(j_1, j_2, \ldots, j_k)} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} : (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^{+k}\},$$

where $\lambda_J$ are complex scalars. A seminorm $p$ on a Fréchet algebra $A$, having power series generators $x_1, x_2, \ldots, x_k$ generating a multi-cyclic basis for $A$, is a power series seminorm if

$$p(y = \sum_{J \in \mathbb{Z}^{+k}} \lambda_J x^J) = \sum_{J \in \mathbb{Z}^{+k}} |\lambda_J| p(x^J) \quad (y \in A).$$

**Corollary 3.3** Let $A$ be a Fréchet algebra $A$ having power series generators $x_1, x_2, \ldots, x_k$. Then $x_1, x_2, \ldots, x_k$ generate a multi-cyclic basis for $A$ if and only if the topology of $A$ is defined by a sequence of power series seminorms.
Proof. For the proof of “only if” part, follow [1, Lemma 2.2].

Next, we define Beurling-Fréchet algebras $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type, and list some of their useful properties.

First, we recall from Introduction that $\omega$ is a proper semiweight if $\omega(N_0) = 0$ for some $N_0 \in \mathbb{N}^k$. Let $k \in \mathbb{N}$, and let $(A, (p_m))$ be the Beurling-Fréchet algebra

$$
\ell^1(\mathbb{Z}^+, \Omega) := \{ f = \sum_{J \in \mathbb{Z}^+} \lambda_J X^J \in \mathcal{F}_k : \sum_{J \in \mathbb{Z}^+} |\lambda_J| \omega_m(J) < \infty \text{ for all } m \},
$$

where $\Omega = (\omega_m)$ is a separating and increasing sequence of semiweight functions on $\mathbb{Z}^+$ defined by $\omega_m(J) = p_m(X^J)$. If $\Omega$ is an increasing sequence of weight functions on $\mathbb{Z}^+$, then we define

$$
\rho = \sup_m \rho_m, \text{ where } \rho_m = \inf_{N \in \mathbb{Z}^+} \omega_m(N)^{1/|N|}.
$$

Thus, $\rho = 0$ if and only if $\rho_m = 0$ for each $m$, if and only if for each $m \ell^1(\mathbb{Z}^+, \omega_m)$ is a local Banach algebra in the Arens-Michael representation of $\ell^1(\mathbb{Z}^+, \Omega)$ if and only if $\ell^1(\mathbb{Z}^+, \Omega)$ is a local Fréchet algebra, and $\rho > 0$ if and only if $\rho_m > 0$ for some $m$ if and only if for each $l \geq m, \ell^1(\mathbb{Z}^+, \omega_l)$ is a semisimple Banach algebra in the Arens-Michael representation of $\ell^1(\mathbb{Z}^+, \Omega)$ if and only if $\ell^1(\mathbb{Z}^+, \Omega)$ is a semisimple Fréchet algebra.
Suppose that $\Omega$ is a separating and increasing sequence of proper semi-weights on $\mathbb{Z}^+$. Then $\rho = 0$ if and only if $\ell^1(\mathbb{Z}^+, \Omega)$ is a local Fréchet algebra if and only if the completion of $\ell^1(\mathbb{Z}^+, \omega_m)/\ker p_m$ under the induced norm $p_m$ is a local Banach algebra for all $m$. In this case, $\ell^1(\mathbb{Z}^+, \Omega)$ is either $\mathcal{F}_k$ or a local FrAPS in $\mathcal{F}_k$. We call such a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ an algebra of semiweight type. We note that the unique maximal ideal of $\ell^1(\mathbb{Z}^+, \Omega)$ is

$$\{ f = \sum_{j \in \mathbb{Z}^+} \lambda_j X^j \in \ell^1(\mathbb{Z}^+, \Omega) : \lambda_0 = 0 \}.$$ 

For example, if $k = 1$, then, by [1, Theorem 2.1], $\ell^1(\mathbb{Z}^+, \Omega) = \mathcal{F}$, which is an inverse-limit of finite-dimensional algebras, and is also a local algebra. In this case,

$$\omega_m : \mathbb{Z}^+ \to [0, \infty), \omega_m(n) = p_m(X^n),$$

is 1, if $n \leq m$ and is 0, if $n > m$ ([1, p. 131]). If $k = 2$, then, by Theorem 3.4 below, $\ell^1(\mathbb{Z}^{2}, \Omega)$ is either $\mathcal{F}_2$ or $A_X$ or $A_Y$ (all the three algebras are local), where

$$A_X := \{ f = \sum_{i,j} \lambda_{i,j} X^i Y^j \in \mathcal{F}_2 : p_m(f) := \sum_{j=0}^\infty \sum_{i=0}^m | \lambda_{i,j} | < \infty \text{ for all } m \}$$

in which case

$$\omega_m : \mathbb{Z}^+ \to [0, \infty), \omega_m(i,j) = p_m(X^i Y^j),$$

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is 1, if \( i \leq m, j \in \mathbb{Z}^+ \) and is 0, if \( i > m, j \in \mathbb{Z}^+ \), and where

\[
A_Y := \{ f = \sum_{i,j} \lambda_{i,j} X^i Y^j \in \mathcal{F}_2 : p_m(f) := \sum_{i=0}^{\infty} \sum_{j=0}^{m} |\lambda_{i,j}| < \infty \text{ for all } m \}
\]

in which case

\[
\omega_m : \mathbb{Z}^+ \to [0, \infty), \quad \omega_m(i, j) = p_m(X^i Y^j),
\]

is 1, if \( j \leq m, i \in \mathbb{Z}^+ \) and is 0, if \( j > m, i \in \mathbb{Z}^+ \). For \( \ell^1(\mathbb{Z}^+, \Omega) = \mathcal{F}_2 \), with

\[
p_m(f) := \sum_{0 \leq i + j \leq m} |\lambda_{i,j}|,
\]

we define \( \Omega = (\omega_m) \), where

\[
\omega_m : \mathbb{Z}^+ \to [0, \infty), \quad \omega_m(i, j) = p_m(X^i Y^j),
\]

is 1, if \( 0 \leq i + j \leq m \) and is 0, if \( i + j > m \) (\cite[p. 131]{Ref}). Clearly, \( A_X \cong A_Y \) under the interchange of variables \( X \) and \( Y \).

If \( k = 3 \), then, again by Theorem 3.4 below, \( \ell^1(\mathbb{Z}^+, \Omega) \) is either \( \mathcal{F}_3 \) or \( A_X \) or \( A_Y \) or \( A_Z \) or \( A_{X,Y} \) or \( A_{X,Z} \) or \( A_{Y,Z} \) defined analogously (in fact, \( A_X \cong A_Y \cong A_Z \) and \( A_{X,Y} \cong A_{X,Z} \cong A_{Y,Z} \)). We can extend above arguments for \( k \geq 4 \). Thus, for \( k \in \mathbb{Z}^+ \), we have Beurling-Fréchet algebras

\( \ell^1(\mathbb{Z}^+, \Omega) \) of semiweight type, with the following properties:
(1) $F_k$ is the only Fréchet algebra of finite type among FrAPS in $F_k$, by Corollary 3.8 below; rest of $\ell^1(\mathbb{Z}^+, \Omega)$ are not Fréchet algebras of finite type (see [7]).

(2) The Arens-Michael representations of $\ell^1(\mathbb{Z}^+, \Omega)$ do not contain BAPS in $F_k$ (for the time being, we assume that such algebras have unique Fréchet topology, which we shall prove later); for proof, see Remark preceding to Corollary 3.6 below.

(3) The polynomials in $k$ variables are dense in $\ell^1(\mathbb{Z}^+, \Omega)$.

(4) The Fréchet topology $\tau$ of $\ell^1(\mathbb{Z}^+, \Omega) (\neq F_k)$, defined by a sequence $(p_m)$, is finer than $\tau_c$ of $F_k$, but surely not equivalent otherwise the $\tau$-closure of the algebra of polynomials in $k$ variables (which is $\ell^1(\mathbb{Z}^+, \Omega)$) is equal to $F_k$, a contradiction. One can also deduce a contradiction from the statement (1) above. Hence the rest of $\ell^1(\mathbb{Z}^+, \Omega)$ differ from $F_k$ (also, from the statement (1) point of view as well).

(5) $F_r$, $1 \leq r \leq k-1$, can be regarded as closed subalgebras of $\ell^1(\mathbb{Z}^+, \Omega)$ via the obvious quotient maps (e.g., if $k = 2$, then $C[[X]] = F_1$ can be regarded as a closed subalgebra of $A_X$; the quotient map from $A_X$ obtained by setting $Y = 0$ is denoted by

$$\pi : \sum_{i,j} \lambda_{i,j} X^i Y^j \mapsto \sum_{i=0}^{\infty} \lambda_{i,0} X^i, \ A_X \rightarrow F_1).$$
Hence all Beurling-Fréchet algebras $\ell^1(\mathbb{Z}^+^k, \Omega)$ of semiweight type are local Fréchet algebras since the closed subalgebras $\mathcal{F}_r$, $1 \leq r \leq k$, are local Fréchet algebras, and the unique maximal ideal $M$ is

$$\{ f = \sum_{J \in \mathbb{Z}^+^k} \lambda_J X^J \in \ell^1(\mathbb{Z}^+^k, \Omega) : \lambda_0 = 0 \}.$$ 

Also, for a fixed $k \in \mathbb{N}$, there are finitely many Beurling-Fréchet algebras $\ell^1(\mathbb{Z}^+^k, \Omega)$ of semiweight type, and these algebras can be properly nested (for example, if $k = 3$, then $A_X \subset A_{X,Y} \subset \mathcal{F}_3$). Further, if $(A, (q_m))$ is a FrAPS in $\mathcal{F}_k$ such that the $q_m$ are proper seminorms on $A$, then $A$ is continuously embedded in the “least” Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+^k, \Omega)$ of semiweight type (note that there might be several such Beurling-Fréchet algebras of semiweight type containing $A$). Moreover it is clear that if such $A$ contains a Beurling-Fréchet algebra of semiweight type such that

$$\ell^1(\mathbb{Z}^+^k, \Omega_1) \hookrightarrow A \hookrightarrow \ell^1(\mathbb{Z}^+^k, \Omega_2)$$

continuously, then, depending on the $q_m$, $A$ is either $\ell^1(\mathbb{Z}^+^k, \Omega_1)$ or $\ell^1(\mathbb{Z}^+^k, \Omega_2)$ or none of these in which case both the inclusions are proper (for example, $A_X \hookrightarrow A_{X,Y} \hookrightarrow \mathcal{F}_3$). We use these facts in the proof of Theorem 3.7 below.

We call $\ell^1(\mathbb{Z}^+^k, \Omega)$ a Beurling-Fréchet algebra of weight type if $\Omega$ is an increasing sequence of weight functions $\omega_m$ on $\mathbb{Z}^+^k$. In this case, the topology
Theorem 3.4 Let $A$ be a Fréchet algebra of power series in $\mathcal{F}_k$. Suppose that $X_1, X_2, \ldots, X_k$ are power series generators for $A$. Then $A$ is the Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+^k, \Omega)$ for an increasing sequence $\Omega$ of semiweight functions on $\mathbb{Z}^+^k$. 

Proof. Let $A$ satisfy the stated conditions. By the uniqueness of the formal power series expression and the fact that $X_1, X_2, \ldots, X_k$ are power series generators for $A$, it follows that $\{X^J : J \in \mathbb{Z}^+^k\}$ is a multi-cyclic basis for $A$. By Corollary 3.3 the Fréchet topology $\tau$ of $A$ is defined by an increasing sequence $(p_m)$ of power series seminorms; and for each $y = \sum_{J \in \mathbb{Z}^+^k} \lambda_J X^J$ in $A$, we have $y = \lim_n \sum_{|J| \leq n} \lambda_J X^J$ in the topology $\tau$. For each $m$, 

$$\omega_m : \mathbb{Z}^+^k \rightarrow [0, \infty), \quad \omega_m(N) = p_m(X^N) \quad \text{for each } m,$$
define a separating sequence \( \Omega \) of semiweight functions. Let \( \ell^1(\mathbb{Z}^+, \Omega) \) be as defined following Corollary 3.3. Since each \( p_m \) is a power series seminorm, \( A \subset \ell^1(\mathbb{Z}^+, \Omega) \). In fact, \( A = \ell^1(\mathbb{Z}^+, \Omega) \). For let

\[
f = \sum_{J \in \mathbb{Z}^+} \lambda_J X^J \in \ell^1(\mathbb{Z}^+, \Omega).
\]

Let \( f_n = \sum_{|J| \leq n} \lambda_J X^J \). Since \( X_i \in A \) for \( i = 1, 2, \ldots, k \), each \( f_n \in A \); and \( (f_n) \) is a Cauchy sequence in \( A \). Thus \( f \in A \).

Now we have the following possibilities:

(a) all \( \omega_m \) are weights;

(b) no \( \omega_m \) is a weight;

(c) at least one \( \omega_m \) fails to be a weight.

In the case (c), let \( G = \{ \omega_m : \omega_m \text{ is not a weight} \} \). If \( G \) is finite the corresponding \( p_m \) may be deleted, if \( G \) is infinite the corresponding \( p_m \) can be taken to define the topology so reducing consideration to the case (b).

Assume (b). Then for each \( m \), there is \( N \in \mathbb{Z}^+ \) such that \( \omega_m(N) = 0 \). Then

\[
\omega_m(N + M) \leq \omega(N)\omega(M) = 0 \quad (M \in \mathbb{Z}^+).
\]

Now, depending on \( \Omega = (\omega_m) \), we have \( A = \ell^1(\mathbb{Z}^+, \Omega) \) a Beurling-Fréchet algebra of semiweight type. In the case (a), we have \( A = \ell^1(\mathbb{Z}^+, \Omega) \) a
Beurling-Fréchet algebra of weight type. The theorem follows.

Next, let $A$ be a FrAPS in $\mathcal{F}_k$. We call a seminorm $p$ on $A$ closable if for any $p$-Cauchy sequence $(f_l)$ in $A$, $f_l \to 0$ in $\tau_c$ implies that $p(f_l) \to 0$. We define $p$ to be of type (E) if given $M \in \mathbb{Z}^+$, there exists $c_M > 0$ such that

$$|\pi_M(f)| \leq c_M p(f),$$

for all $f \in A$. A seminorm of type (E) is a norm. Also, closability of a norm on a normed algebra of power series in $k$ indeterminates is a necessary and sufficient condition for the completion to be a BAPS in $\mathcal{F}_k$ (see [1, Lemma 3.5].

We now answer (B) stated in the introduction. The following proposition, whose proof we omit, is a several-variable-analogue of [1, Proposition 3.1].

**Proposition 3.5** Let $A$ be a Fréchet algebra of power series in $\mathcal{F}_k$. Let $p$ be a continuous submultiplicative seminorm on $A$. Let $\ker p = \{f \in A : p(f) = 0\}$. Let $A_p$ be the completion of $A/\ker p$ in the norm $\|f + \ker p\|_p = p(f)$. Then the following are equivalent.

(i) $p$ is a norm and $A_p$ is a Banach algebra of power series in $\mathcal{F}_k$.

(ii) $p$ is closable and of type (E).
Remark. The above proposition can be used to prove the property (2) as follows: consider $A_Y \subset \mathcal{F}_2$. The topology $\tau$ of $A_Y$ is given by a sequence $(p_m)$ of proper power series seminorms, hence they are not of type (E). So, $(A_Y)_m$ cannot be a BAPS in $\mathcal{F}_2$.

**Corollary 3.6** Let $A = \lim_{\leftarrow} A_m$ be the Arens-Michael representation of a Fréchet algebra of power series in $\mathcal{F}_k$. Assume that each $p_m$ is a norm. Then each $A_m$ is a Banach algebra of power series in $\mathcal{F}_k$ if and only if each $p_m$ is a closable norm of type (E).

It is readily seen that a Fréchet algebra of power series in $\mathcal{F}_k$ satisfies Loy’s condition (E) in [9] if and only if $A$ admits a continuous norm of type (E) if and only if the topology of $A$ is defined by a sequence of norms of type (E).

Next, we give characterizations of a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type. We have the following elementary, but crucial, theorem. By identifying the series expansion in $x_1, x_2, \ldots, x_k$ with the series expansion in $X_1, X_2, \ldots, X_k$, Fréchet algebras with a multi-cyclic basis are realized as Fréchet algebras of power series in $\mathcal{F}_k$, the projections $\pi_J$ being continuous. Note that by a proper seminorm we mean a seminorm that is not a norm.

**Theorem 3.7** Let $A$ be a Fréchet algebra of power series in $\mathcal{F}_k$. Then $A$
is either a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type or the Fréchet topology $\tau$ of $A$ is defined by a sequence $(p_m)$ of norms.

**Proof.** If $A$ is Banach, then certainly the topology $\tau$ of $A$ is defined by a norm, and so $A$ is not equal to a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type. Now suppose that $A$ is a non-Banach FrAPS in $\mathcal{F}_k$. Let $(p_m)$ be an increasing sequence of seminorms defining the Fréchet topology $\tau$ of $A$, and set

$$G = \{ l \in \mathbb{N} : p_l \text{ is a proper seminorm on } A \}.$$  

If $G$ is finite the corresponding $p_l$ may be deleted and we have a new sequence of norms, defining the same Fréchet topology $\tau$ of $A$. Otherwise, $G$ is infinite the corresponding $p_l$ can be taken to define the Fréchet topology $\tau$ of $A$. Then, by Lemma 3.2, there exists a Fréchet subalgebra $(A_1, (q_m))$ of $(A, (p_m))$ continuously embedded in $A$; $A_1$ is a Fréchet algebra with power series generators $X_1, X_2, \ldots, X_k$. By Theorem 3.4, $A_1$ is a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of (semi)weight type. To rule out a possibility of $A_1$ being a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of weight type, fix $m \geq 1$. Then there is a non-zero element $f \in A$ that belongs to $\ker p_m$. Let $f_i$ be the initial form of $f$ \cite[p. 130]{17}. Thus $f_i$ is a homogeneous polynomial of degree $n(m)$ (which is minimal as $f$ varies in $\ker p_m$). Since $\ker p_m$ is a closed
ideal in $A$ and the inclusion map $A \hookrightarrow \mathcal{F}_k$ is continuous with $A$ a dense subalgebra of $(\mathcal{F}_k, \tau_c)$, by [11, Lemma B.10], the closure of $\ker p_m$ (say $I$) in the topology $\tau_c$ is a closed ideal in $\mathcal{F}_k$ such that $\ker p_m = I \cap A$ and that $\ker p_m \subseteq I \subseteq \ker p'_l(m)$. Now, by the lemma from [17, p. 136], there exists an automorphism $\psi$ of $\mathcal{F}_k$ such that $\psi(f) = f_l$. Hence $f_l \in \ker p'_l(m)$ (note that this is true since we have replaced $\mathcal{F}_k$ by $\psi(\mathcal{F}_k) = \mathcal{F}_k$, which is a FrAPS). Since $\ker p'_l(m)$ is finitely generated by the monomials and $f_l$ is a homogeneous polynomial of degree $n(m)$ in $\ker p'_l(m)$, so it is, indeed, finitely generated by the monomials of degree $n(m)$. Since $I$ is a finitely generated ideal in $\mathcal{F}_k$ ($\mathcal{F}_k$ being noetherian) and $I \subseteq \ker p'_l(m)$, $I$ is finitely generated by the monomials of degree $n(m)$. So $f_l \in I$. Evidently, $f_l \in A$, being a homogeneous polynomial of degree $n(m)$. So $f_l \in \ker p_m$.

Hence

$$p_m(X^{n(m)}) = q_m(X^{n(m)}) = 0 \ (m \in \mathbb{N}),$$

and so, $A_1$ is, indeed, a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type. Hence $A_1$ is a local algebra, and therefore $A$ is also a local algebra.

Now there are several cases; we consider these cases for $k = 2$ (one can modify the following arguments for any $k$). Thus $f_l$ is a monomial of degree $n(m)$ in $X$ and $Y$. We have the following three cases.
Case 1. For each $m$, $\ker p_m$ is finitely generated by the monomials of degree $n(m)$ in $X$ and $Y$ in which case $I$ is also finitely generated by the monomials of degree $n(m)$ in $X$ and $Y$ as discussed earlier. Since $I \subseteq \ker p_{n(m)-1} = M_{n(m)}$, and $I$ and $M_{n(m)}$ are generated by the same generators, we have $I = M_{n(m)}$, that is, $\ker p_m = M_{n(m)}(A)$, where

$$M_{n(m)}(A) := \{ f \in A : o(f) \geq n(m) \}.$$ 

But then

$$\ker q_m = M_{n(m)}(A_1) := \{ f \in A_1 \subset A : o(f) \geq n(m) \},$$

where $A_1$ is as in Lemma 3.2. By Theorem 3.4, $A_1 = F_2$. It follows that $A = F_2$ topologically in view of the open mapping theorem.

Case 2. for each $m$, $\ker p_m$ is singly generated by the monomial $X^{n(m)}$.

In this case

$$A_1 = \{ f \in F_2 : q_m(f) = \sum_{j=0}^{\infty} \sum_{i=0}^{m} |\lambda_{i,j}| < \infty \text{ for all } m \} = A_X,$$

by Theorem 3.4, since $X^{n(m)} \in \ker q_m$. So, $A_X = A_1 \subset A \subset F_2$, by Lemma 3.2. But, as we discussed in the property (5), $A_X \subset A \subset A_X \subset F_2$ since $p_m$ is a proper seminorm on $A$ for each $m$ such that $X^{n(m)} \in \ker p_m$.

Thus we have $A = A_X$ topologically in view of the open mapping theorem.
Case 3. for each $m$, $\ker p_m$ is singly generated by the monomial $Y^{n(m)}$.

Follow the argument of Case 2, and the proof is complete.

As corollaries, we have the following characterizations of a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type as a Fréchet algebra.

**Corollary 3.8** Let $A$ be a Fréchet algebra of power series in $\mathcal{F}_k$. Then $A$ is equal to a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type if and only if the Fréchet topology of $A$ is defined by a sequence $(p_m)$ of proper seminorms. In particular, $A = \mathcal{F}_k$ if and only if the Fréchet topology of $A$ is defined by a sequence $(p_m)$ of proper seminorms with finite-dimensional cokernels.

In fact, we have the following result on an Arens-Michael representation of $A$.

**Corollary 3.9** Let $A$ be a Fréchet algebra of power series in $\mathcal{F}_k$ such that the polynomials are dense in $A$. Then $A$ is not equal to a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type if and only if $A = \varprojlim A_m$, where each $A_m$ is a Banach algebra of power series in $\mathcal{F}_k$.

*Proof.* Suppose that $A$ is not equal to a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type. Evidently, by Corollary 3.8 we may sup-
pose that each $p_m$ is a norm on $A$. Now, by Theorem 3.1, $A$ contains a closed maximal ideal $M = \ker \pi_0$ such that

$$\bigcap_{n \geq 1} M^n = \{0\} \text{ and } \dim(M^n/M^{n+1}) = C_{n+k-1,n} (n \in \mathbb{N}).$$

By Proposition 2.1, for each sufficiently large $l$, $M_l$ is a non-nilpotent maximal ideal of $A_l$ such that

$$\bigcap_{n \geq 1} M_l^n = \{0\} \text{ and } \dim(M_l^n/M_l^{n+1}) = C_{n+k-1,n} (n \in \mathbb{N}).$$

Again, by Theorem 3.1, $A_l$ is a BAPS in $\mathcal{F}_k$ for each sufficiently large $l$.

Hence, by passing to a suitable subsequence of $(p_m)$ defining the same Fréchet topology of $A$, we conclude that each $A_l$ is a BAPS in $\mathcal{F}_k$.

The converse has already been discussed in the property (2) above.

The immediate consequence of Corollary 3.9 is: if $A$ is a FrAPS in $\mathcal{F}_k$ such that the polynomials are dense in $A$ and such that it is not a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type, then $A$ satisfies Loy’s condition (E) by Corollary 3.6. A somewhat more elaborate version of the same idea enables us to drop the condition on the polynomials in order to get a more general result, given below.

We recall that if the Fréchet topology $\tau$ of $A$ is given by a sequence $(p_m)$, then each $p_m$ is of type (E) if and only if $A$ satisfies Loy’s condition (E). Also,
by a several-variable-analogue of [8, Theorem 2], $A$ satisfies Loy’s condition (E) if and only if $A$ admits a growth sequence; i.e. there exists a sequence $(\sigma_K)_{K \in \mathbb{N}^+}$ of positive reals such that $\sigma_K \pi_K(x) \to 0$ for each $x \in A$. This characterization of Loy’s condition (E) tells us that a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type does not satisfy Loy’s condition (E) since $\mathcal{F}_r, 1 \leq r \leq k - 1$, does not satisfy Loy’s condition (E), which is a closed subalgebra by the property (5).

**Theorem 3.10** Let $A$ be a Fréchet algebra of power series in $\mathcal{F}_k$. Then $A$ is not equal to the Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type if and only if $A = \lim \leftarrow A_m$, where each $A_m$ is a Banach algebra of power series in $\mathcal{F}_k$. In particular, $A$ satisfies Loy’s condition (E) in this case.

**Proof.** The proof is similar to that of [12, Theorem 3.6], and so we will merely sketch it only. Supposing $A$ is not equal to a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, \Omega)$ of semiweight type, we may suppose that each $p_m$ is a norm on $A$, by Corollary 3.8. We first show that the projections $\pi_J$ are continuous on $(A, p_m)$ for all $J$ and $m$; i.e. the inclusion maps $(A, p_m) \hookrightarrow \mathcal{F}_k$ is continuous.

Since $M = \ker \pi_0, \ldots, 0$ is a non-nilpotent, closed maximal ideal of $A$ such that $\cap_{n \geq 1} M^n = \{0\}$, $M$ is a non-nilpotent, maximal ideal of $(A, p_m)$ for each $m$; also, $M^{n+1} \neq M^n \neq \{0\}$ (closure in $(A, \tau)$) in $(A, p_m)$ for all $m$.
and $n$. In fact, we may suppose that for each sufficiently large $m$, $M$ is also closed in $(A, p_m)$ and hence that $\pi_0, \ldots, 0$ is $p_m$-continuous for each sufficiently large $m$. Also, by the argument in the proof of [12, Proposition 2.3], we may suppose that $M_m$ is a non-nilpotent, maximal ideal of $A_m$ and that $\cap_{n \geq 1} M_m^n = \{0\}$ for each $m$. Assume inductively that $\pi_I$ is continuous for each $|I| < |J|$, and take $(x_n)$ in $(A, p_m)$ with $p_m(x_n) \to 0$. Then, following the argument given in Theorem 3.1, we deduce that some non-zero linear combination of $X^I$, $|I| = |J|$, lies in $A \cap \overline{M_m^{J+1}}$ (which is, in fact, $\overline{M_m^{[J]+1}}$ in $(A, p_m)$ for each $m$), a contradiction of the fact that $\overline{M_m^{[J]+1}} \neq \overline{M_m^{[J]}}$.

Next, for each $m \in \mathbb{N}$, one shows that $p_m$ is closable on $A$ by noticing that the inclusion map $(A, p_m) \hookrightarrow \mathcal{F}_k$ can be extended to a continuous homomorphism $\phi : A_m \to \mathcal{F}_k$. So $x_n \to x$ in $(\mathcal{F}_k, \tau_c)$, and hence $p_m(x_n) \to 0$.

Then one shows that $\phi$ is, indeed, injective, and so $A_m$ is a Banach algebra of power series in $\mathcal{F}_k$ for each $m$. In particular, $A$ satisfies Loy’s condition (E), by the remark following Corollary 3.6.
4 Automatic continuity and uniqueness of topology, and open questions.

We now turn to establish that every FrAPS in $\mathcal{F}_k$ has a unique Fréchet topology. By [9, Theorem 1], it is clear that every FrAPS $A$ in $\mathcal{F}_k$ satisfying Loy’s condition (E) has a unique Fréchet topology. Since a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^k, \Omega)$ of semiweight type does not satisfy the condition (E), Theorem 3.10 gives the following result from [10] by noticing that [10, Theorem 10] holds true if $A$ is considered to be a FrAPS in $\mathcal{F}_k$ in the codomain.

**Theorem 4.1** Let $A$ be a Fréchet algebra of power series in $\mathcal{F}_k$ such that $A$ is not equal to a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^k, \Omega)$ of semiweight type. Then a homomorphism $\theta : B \to A$ from a Fréchet algebra $B$ is continuous provided that the range of $\theta$ is not one-dimensional.

Now, we prove the uniqueness of the Fréchet topology for Beurling-Fréchet algebras $\ell^1(\mathbb{Z}^k, \Omega)$ of semiweight type.

**Theorem 4.2** Let $A$ be a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^k, \Omega)$ of semiweight type. Then $A$ has a unique Fréchet topology.

**Proof.** Let $\tau'$ be another topology such that $(A, \tau')$ is a Fréchet algebra. We recall that the original Fréchet topology $\tau$ of $A$ is defined by a sequence $(p_m)$
of proper power series seminorms. First we note that $A$ is a local Fréchet algebra, and hence it has a unique maximal ideal $M = \ker \pi_{0,\ldots,0}$ (this is clearly an algebraic property). Since $(A, \tau')$ is local, $A$ is a $Q$-algebra by Corollary 3. Hence $M$ is $\tau'$-closed. For $n \in \mathbb{Z}^+$, let

$$M_n(A) := \{ f \in A : o(f) \geq n \}.$$ 

Since $(A, \tau)$ is a FrAPS in $\mathcal{F}_k$ such that the polynomials are dense in $A$, by argument of Theorem 3.1, we have

$$\bigcap_{n \geq 1} M_n^\tau = \{0\} \text{ and } M_n(A) = M_n^\tau,$$

for each $n$, hence $\bigcap_{n \geq 1} M_n(A) = \{0\}$. Now let $B$ be the closure of the polynomials in $k$ variables in the topology $\tau'$. Hence $B$ is a closed subalgebra of $(A, \tau')$. Then $M \cap B$ is a non-nilpotent, closed maximal ideal of $B$ such that $\bigcap_{n \geq 1} (M \cap B)^n = \{0\}$; the latter holds because

$$\bigcap_{n \geq 1} M_n(B) = \{0\} \text{ and } (M \cap B)^n \subset (M \cap B)^n' \subset M_n(B),$$

for each $n$. Also, since the polynomials in $k$ variables are dense in $B$, $(M \cap B)^n' = M_n(B)$, and so

$$\dim((M \cap B)^n' / (M \cap B)^{n+1}') = C_{n+k-1,n},$$

for all $n$. Now, by Theorem 3.1, $(B, \tau')$ is a FrAPS in $\mathcal{F}_k$. 32
Since the inclusion map \((B, \tau') \hookrightarrow (A, \tau')\) is continuous, by the open mapping theorem for Fréchet spaces, the inclusion map is a linear homeomorphism. Hence both \((A, \tau)\) and \((A, \tau')\) are FrAPS in \(\mathcal{F}_k\). Let \((a_n)\) be a sequence in \(A\) such that \(a_n \to 0\) in \((A, \tau)\) and \(a_n \to a\) in \((A, \tau')\). For each \(J \in \mathbb{N}^k\), the functional \(\pi_J\) is continuous on both \((A, \tau)\) and \((A, \tau')\), and so \(\pi_J(a) = 0\), whence \(a = 0\). By the closed graph theorem for Fréchet spaces, the identity map \((A, \tau) \to (A, \tau')\) is a linear homeomorphism, and so \(\tau = \tau'\) on \(A\). The proof is complete.

As a corollary of the above two results, we have the following result (and Questions) in the theory of automatic continuity. We recall that the continuity of automorphism of \(\mathcal{F}_k\) is proved in [17, p. 136], and the uniqueness of the Fréchet topology of \(\mathcal{F}_k\) is proved in [3, Theorem 4.6.1]. The following gives an answer to [2, Question 11] for FrAPS in \(\mathcal{F}_k\).

**Corollary 4.3** Let \(A\) be a Fréchet algebra of power series in \(\mathcal{F}_k\). Then \(A\) has a unique Fréchet topology.

We now list the following questions, which may have some interest in the theory of automatic continuity.

**Question 1.** Let \(A\) be a Beurling-Fréchet algebra \(\ell^1(\mathbb{Z}^+, \Omega) (\neq \mathcal{F}_k)\) of semiweight type. Is every automorphism of \(A\) continuous?
The author conjecture that the answer of Question 1 is in the affirmative; also, this question has been studied for special cases (see \cite{5, 6}). More generally, we have the following

**Question 2.** Is every homomorphism \( \theta : B \to \mathcal{F}_k \) \((k > 1)\) from a Fréchet algebra \( B \) continuous?

The above question has recently been settled partially in \cite{4}: (i) we remark that if \( \theta : B \to \mathcal{F}_k \) is a discontinuous homomorphism from an \((F)\)-algebra \( B \) (i.e., a complete, metrizable topological algebra) into \( \mathcal{F}_k \) such that \( \theta(B) \) is dense in \((\mathcal{F}_k, \tau_c)\) and such that the separating ideal of \( \theta \) has finite codimension in \( \mathcal{F}_k \), then \( \theta \) is an epimorphism (see \cite{4, Theorem 12.1}), and (ii) there exists a Banach algebra \((A, \| \cdot \|)\) such that \( \mathcal{O}[X_1, X_2] \subset A \subset \mathcal{F}_2 \), but such that the embedding \((A, \| \cdot \|) \hookrightarrow \mathcal{F}_2, \tau_c)\) is not continuous (see \cite{4, Theorem 12.3]); surprisingly, \( A \) is (isometrically isomorphic to) a Banach algebra of power series in \( \mathcal{F}_1 \) (see \cite{4, Theorem 10.1 (i))].

As shown in \cite{14}, \( \mathcal{F}_\infty \) admits two inequivalent Fréchet algebra topologies. It is clear that the unique maximal ideal \( M = \ker \pi_0 \) \((0 = (0, 0, \ldots))\), is closed under both topologies, since \( \mathcal{F}_\infty \) is a local Fréchet \( Q \)-algebra. Thus it is of interest to know whether every FrAPS in \( \mathcal{F}_\infty \) (except \( \mathcal{F}_\infty \) itself) has a unique Fréchet topology. In other words, we have the following natural
question.

**Question 3.** Is there any other proper, unital subalgebra of $\mathcal{F}_\infty$, with two inequivalent Fréchet topologies? In particular, is there any other proper subalgebra of $\mathcal{F}_\infty$ which is closed under the topology imposed by Charles Read on $\mathcal{F}_\infty$ and which is also FrAPS in $\mathcal{F}_\infty$ in its “usual” topology?

To answer the latter part of the above question, the “natural” extension of Beurling-Fréchet algebras of semiweight type (i.e., $\ell^1((\mathbb{Z}^+)^{<\omega}, \Omega)$) would be an easy target. Also, we have the following curious question.

**Question 4.** Does there exist a Fréchet algebra with infinitely many inequivalent Fréchet algebra topologies?

For $m \in \mathbb{N}$, set

$$U_m = \{ f = \sum \{ \alpha_r X^r : r \in (\mathbb{Z}^+)^{<\omega} \} \in \mathcal{F}_\infty : p_m(f) := \sum |\alpha_r| m^{|r|} < \infty \},$$

and then set

$$\mathcal{U} = \bigcap \{ U_m : m \in \mathbb{N} \}.$$

It is clear that each $(U_m, p_m)$ is a unital Banach subalgebra of $\mathcal{F}_\infty$ and that $(\mathcal{U}, (p_m))$ is a unital Fréchet subalgebra of $\mathcal{F}_\infty$. Being semisimple Fréchet algebras, the test algebra $\mathcal{U}$ for the still unsolved “Michael problem” (and $U_m$ for each $m$ appearing in the Arens-Michael representation of $\mathcal{U}$), $\ell^1((\mathbb{Z}^+)^{<\omega})$
and the algebra $\ell^1(S_c)$, where $S_c$ denotes the free semigroup on $c$ generators, have unique Fréchet topologies (the result also follows from [4, Theorems 10.1 and 10.5] and [12, Corollary 4.2]). In this regard, we first give the following result (for the proof, see either [8, Theorem 1] or [9, Theorem 2]) by noticing that FrAPS $A$ in $\mathcal{F}_\infty$ satisfies Loy’s condition (E) if there is a sequence $(\gamma_K : K \in \mathbb{N}^k, k \in \mathbb{N})$ of positive reals such that $(\gamma_K^{-1} \pi_K)$ is equicontinuous.

**Theorem 4.4** Let $A$ be a Fréchet algebra of power series in $\mathcal{F}_\infty$ satisfying Loy’s condition (E), above, and let $\phi : B \rightarrow A$ be a homomorphism from a Fréchet algebra $B$ into $A$ such that $X_1 \in \phi(B)$. Then $\phi$ is continuous. In particular, every automorphism of $A$ is continuous, and $A$ has a unique Fréchet algebra topology.

We remark that the Fréchet topology of $A$ of the above theorem is defined by a sequence of norms. We have the following “natural” generalization of [4, Theorem 10.1].

**Theorem 4.5** Let $A$ be a Fréchet algebra of power series in $\mathcal{F}_\infty$, with its topology defined by a sequence $(p_m)$ of norms. Suppose that $A$ is a graded subalgebra of $\mathcal{F}_\infty$. Then there is a continuous embedding $\theta$ of $A$ into $(\mathcal{F}, \tau_c)$. 36
such that $\theta(X_1) = X$, and so $A$ is (isometrically isomorphic to) a Fréchet algebra of power series in $F_1$. In particular, $A$ has a unique Fréchet topology.

We note that we cannot drop the assumption on $(p_m)$ in the above theorem; $(F_\infty, \tau_c)$ is a counterexample, since it can continuously be embedded in $(F_2, \tau_c)$, by [4, Theorem 9.1], but, by Theorem 2.6 (or Theorem 11.8) of [4], there is no embedding of $F_2$ into $F$.

Finally, we remark that we have recently established the uniqueness of (F)-algebra topology for (F)-algebra of power series in the indeterminate $X$ (see [4, Corollary 11.7]). Then the natural question is to extend this result for the several-variable case, in order to settle the Dales question completely. But our approach fails, because the clause of the continuity of coordinate projections cannot be dropped from the definitions of Banach and Fréchet algebras of power series in $k$ indeterminates (see [4, Theorem 12.3]). The approach here is based on a sequence $(p_m)$ of submultiplicative seminorms, and so these results cannot be extended to all (F)-algebra of power series in $k$ indeterminates.

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References

[1] S. J. Bhatt and S. R. Patel, *On Fréchet algebras of power series*, Bull. Austral. Math. Soc. 66 (2002), 135-148.

[2] H. G. Dales, *Automatic continuity: a survey*, Bull. London Math. Soc. 10 (1978), 129-183.

[3] H. G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monogr. 24, Clarendon Press, 2000.

[4] H. G. Dales, S. R. Patel and C. J. Read, *Fréchet algebras of power series*, Banach Center Publ. 91 (2010), 123-158.

[5] F. Ghahramani and J. P. McClure, *Automorphisms and derivations of a Fréchet algebra of locally integrable functions*, Studia Math. 103 (1992), 51-69.

[6] S. Grabiner, *Derivations and automorphisms of Banach algebras of power series*, Mem. Amer. Math. Soc. 146 (1974).
[7] M. K. Kopp, *Fréchet algebras of finite type*, Arch. Math. (Basel) 83 (2004), 217-228.

[8] R. J. Loy, *Uniqueness of the Fréchet space topology on certain topological algebras*, Bull. Austral. Math. Soc. 4 (1971), 1-7.

[9] R. J. Loy, *Local analytic structure in certain commutative topological algebras*, Bull. Austral. Math. Soc. 6 (1972), 161-167.

[10] R. J. Loy, *Banach algebras of power series*, J. Austral. Math. Soc. 17 (1974), 263-273.

[11] E. A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. 11 (1952).

[12] S. R. Patel, *Fréchet algebras, formal power series, and automatic continuity*, Studia Math. 187 (2008), 125-136.

[13] S. R. Patel, *Fréchet algebras, formal power series, and analytic structure*, J. Math. Anal. Appl. 394 (2012), 468-474.

[14] C. J. Read, *Derivations with large separating subspace*, Proc. Amer. Math. Soc. 130 (2002), 3671-3677.
[15] T. T. Read, *The powers of maximal ideals in a Banach algebra and analytic structure*, Trans. Amer. Math. Soc. 161 (1971), 235-248.

[16] M. P. Thomas, *Local power series quotients of commutative Banach and Fréchet algebras*, Trans. Amer. Math. Soc. 355 (2003), 2139-2160.

[17] O. Zariski and P. Samuel, *Commutative algebra. Vol. 2*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960.

[18] W. Żelazko, *On maximal ideals in commutative m-convex algebras*, Studia Math. 58 (1976), 291-298.

[19] W. Żelazko, *A characterization of commutative Fréchet algebras with all ideals closed*, Studia Math. 138 (2000), 293-300.