PRINCIPAL SUBSPACES FOR THE QUANTUM AFFINE VERTEX
ALGEBRA IN TYPE $A^{(1)}_1$ 

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Abstract. By using the ideas of Feigin and Stoyanovsky and Calinescu, Lepowsky and Milas we introduce and study the principal subspaces associated with the Etingof–Kazhdan quantum affine vertex algebra of integer level $k \geq 1$ and type $A^{(1)}_1$. We show that the principal subspaces possess the quantum vertex algebra structure, which turns to the usual vertex algebra structure of the principal subspaces of generalized Verma and standard modules at the classical limit. Moreover, we find their topological quasi-particle bases which correspond to the sum sides of certain Rogers–Ramanujan-type identities.

1. Introduction

Feigin and Stoyanovsky [9] introduced the notion of principal subspace which can be associated with generalized Verma modules and integrable highest weight modules of affine Kac–Moody Lie algebras. These rather remarkable objects provide a connection between representation theory of affine Lie algebras and combinatorial identities via their quasi-particle bases; see, e.g., [2, 3, 9, 12]. Many other interesting aspects of principal subspaces were also extensively studied; see, e.g., [6, 19, 25, 26] and references therein.

In this paper we consider the Etingof–Kazhdan quantum affine vertex algebra $\mathcal{V}_c(\mathfrak{sl}_2)$ associated with the rational $R$-matrix, as defined in [8]. Motivated by the aforementioned results on the principal subspaces for affine Lie algebras we investigate their quantum counterparts using the structure of $\mathcal{V}_c(\mathfrak{sl}_2)$. By generalizing the definition of Feigin and Stoyanovsky [9] we introduce the principal subspace $W_{\mathcal{V}_c(\mathfrak{sl}_2)}$ for $\mathcal{V}_c(\mathfrak{sl}_2)$ and show that it is a quantum vertex subalgebra of $\mathcal{V}_c(\mathfrak{sl}_2)$ for all $c \in \mathbb{C}$. Although it turns to the commutative vertex algebra at the classical limit, $W_{\mathcal{V}_c(\mathfrak{sl}_2)}$ by itself is not commutative and, furthermore, its vertex operator map $Y(z)$ possesses poles at $z = 0$ of infinite order, i.e. $Y(v, z)w$ can have infinitely many negative powers of the variable $z$ for $v, w \in W_{\mathcal{V}_c(\mathfrak{sl}_2)}$.

Motivated by the presentations of principal subspaces found by Calinescu, Lepowsky and Milas [4, 5], we introduce and study a family of ideals $\mathcal{I}_{\mathcal{V}_k(\mathfrak{sl}_2)}^t$, $t \in \mathbb{C}$, of the quantum vertex algebra $W_{\mathcal{V}_k(\mathfrak{sl}_2)}$, where $k > 0$ is an integer. The form of the ideal generators for $t = 0$ comes from the integrability relation, $x^+_k(z)^{k+1} = 0$ on the level $k$ standard $\hat{\mathfrak{sl}}_2$-module, found by Lepowsky and Primc [22]. On the other hand, the generators for

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\[ \mathcal{T}_{\mathfrak{h}(\mathfrak{sl}_2)}^t \] with \( t \neq 0 \) are inspired by the \( h \)-adic integrability relation of vertex operators in Iohara’s bosonic realization of level 1 modules for the double Yangian for \( \mathfrak{sl}_2 \) [14], \( x^+(z)x^+(z + h) = 0 \). We show that the classical limit of the quotient quantum vertex algebra \( W_{\mathcal{L}_k(\mathfrak{sl}_2)} = W_{\mathfrak{h}(\mathfrak{sl}_2)}/\mathcal{T}_{\mathcal{V}_{\mathfrak{h}(\mathfrak{sl}_2)}}^t \) coincides with the vertex algebra over the principal subspace of the level \( k \) integrable highest weight \( \mathfrak{sl}_2 \)-module \( L(k\Lambda_0) \). In particular, using the underlying quantum vertex algebra structure, we define quantum quasi-particles, certain operators on the underlying quantum vertex algebra structure, we define quantum quasi-particles, are subject to the relations of \( \mathcal{U}_{\mathfrak{aff}} \) the affine Kac–Moody Lie algebra. Its generators are equipped with the structure of vertex algebra; see [13] for more details. Let \( \mathfrak{sl}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t^\pm] \) be the corresponding affine Kac–Moody Lie algebra. Its generators \( K \) and \( x(r) = x \otimes t^r \) with \( x \in \mathfrak{sl}_2 \) and \( r \in \mathbb{Z} \) are subject to the relations

\[ [a(m), b(n)] = [a, b](m + n) + m \langle a, b \rangle \delta_{m+n,0} K \quad \text{and} \quad [K, a(m)] = 0. \]

Fix \( c \in \mathbb{C} \). The generalized Verma module \( N(c\Lambda_0) \) for \( \mathfrak{sl}_2 \) is defined as the induced module

\[ N(c\Lambda_0) = U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{sl}_2 \otimes \mathbb{C}[t^\pm \mathbb{C}K])} \mathbb{C}_c, \]

where \( U(\mathfrak{g}) \) is the universal enveloping algebra of the Lie algebra \( \mathfrak{g} \) and the structure of \( U(\mathfrak{sl}_2 \otimes \mathbb{C}[t] \otimes \mathbb{C}K) \)-module on the one-dimensional space \( \mathbb{C}_c = \mathbb{C} \) is defined so that \( \mathfrak{sl}_2 \otimes \mathbb{C}[t] \) acts trivially and \( K \) acts as the scalar \( c \). We refer to \( v_{N(c\Lambda_0)} = 1 \otimes 1 \in N(c\Lambda_0) \) as the highest weight vector. By the Poincaré–Birkhoff–Witt theorem the generalized Verma module \( N(c\Lambda_0) \) is isomorphic to \( U(\mathfrak{sl}_2 \otimes t^{-1}\mathbb{C}[t^{-1}]) \) as a vector space. For any integer \( k > 0 \) let \( L(k\Lambda_0) \) be the level \( k \) standard module, i.e. the integrable highest weight \( \mathfrak{sl}_2 \)-module which equals the unique simple quotient of \( N(k\Lambda_0) \). Let \( v_{L(k\Lambda_0)} \) be its highest weight vector, i.e. the image of the \( v_{N(k\Lambda_0)} \) in \( L(k\Lambda_0) \); see [17] for more details. It is well-known that the generalized Verma module \( N(c\Lambda_0) \) and the standard module \( L(k\Lambda_0) \) are naturally equipped with the structure of vertex algebra; see [1,10,11].

The principal subspaces were introduced in [9]. For a generalized Verma module \( V = N(c\Lambda_0) \) and standard module \( V = L(k\Lambda_0) \) define its principal subspace \( W_V \) as

\[ W_V = U(\mathbb{C}x_\alpha \otimes \mathbb{C}[t^\pm]) \cdot v_V \subset V. \] (2.1)
They can be described in terms of their presentations or in terms of their quasi-particle bases. We now recall Calinescu–Lepowsky–Milas’ presentations [4, 5]. Set

\[ I_{N(k\Lambda_0)} = \sum_{p \geq k+1} U^- R_{N(k\Lambda_0)}(p) \text{ for } R_{N(k\Lambda_0)}(p) = \sum_{r_1, \ldots, r_{k+1} \geq 1} \alpha(r_1) \ldots \alpha(r_{k+1}), \quad (2.2) \]

where \( U^- = U(\mathbb{C}x_\alpha \otimes \mathbb{C}[t^{-1}]) \) and \( k > 0 \) is an integer. Note that \( R_{N(k\Lambda_0)}(p) \cdot v_{N(k\Lambda_0)} \) equals the coefficient of \( u^{p-k-1} \) in the series \( \alpha(u)^{k+1}v_{N(k\Lambda_0)} = \alpha(u)^{k+1}v_{N(k\Lambda_0)} \), where

\[ x_\alpha^+(u) = \sum_{r=1}^\infty x_\alpha(-r)u^{r-1} \quad \text{and} \quad x_\alpha(u) = \sum_{r \in \mathbb{Z}} x_\alpha(-r)u^{r-1}. \]

For any positive integer \( k \) let \( \mathcal{I}_{N(k\Lambda_0)} = I_{N(k\Lambda_0)} \cdot v_{N(k\Lambda_0)} \). Recall that the principal subspace \( W_{N(c\Lambda_0)} \) with \( c \in \mathbb{C} \) is a vertex subalgebra of \( N(c\Lambda_0) \) with the vertex operator map

\[ Y(x_\alpha^+(u_1) \ldots x_\alpha^+(u_n)v_{N(c\Lambda_0)}, z) x_\alpha^+(v_1) \ldots x_\alpha^+(v_m)v_{N(c\Lambda_0)} = x_\alpha^+(z+u_1) \ldots x_\alpha^+(z+u_n)x_\alpha^+(v_1) \ldots x_\alpha^+(v_m)v_{N(c\Lambda_0)}. \quad (2.3) \]

**Theorem 2.1** ([4, 5]). \( \mathcal{I}_{N(k\Lambda_0)} \) is the ideal of the vertex algebra \( W_{N(k\Lambda_0)} \) generated by the element \( x_\alpha(-1)^{k+1} \cdot v_{N(k\Lambda_0)} \). Moreover, the quotient vertex algebra \( W_{N(k\Lambda_0)}/\mathcal{I}_{N(k\Lambda_0)} \) coincides with the vertex algebra \( W_{L(k\Lambda_0)} \).

We now recall the construction of quasi-particle bases for principal subspaces; see [2, 9, 12]. For any integer \( m > 0 \) set

\[ x_{m\alpha}^+(u) = \sum_{r \geq m} x_{m\alpha}(-r)u^{r-m} = x^+_\alpha(u)^m. \quad (2.4) \]

The coefficients \( x_{m\alpha}(-r) \) are called quasi-particles of charge \( m \). Consider the monomials

\[ x_{m_1\alpha}(n_1) \ldots x_{m_r\alpha}(n_r) \quad \text{with} \quad r \geq 0, \quad m_1 \geq \ldots \geq m_r \geq 1, \quad n_1, \ldots, n_r \leq -1. \quad (2.5) \]

Denote by \( B_{N(c\Lambda_0)} \) the set of all quasi-particle monomials (2.5) which satisfy

\[ n_{s+1} \leq n_s - 2m_s \quad \text{if} \quad m_{s+1} = m_s \quad \text{and} \quad n_s \leq -m_s - 2(s-1)m_s \quad (2.6) \]

for all \( s = 1, \ldots, r - 1 \). The set

\[ B_{N(c\Lambda_0)} = \{ b \cdot v_{N(c\Lambda_0)} : b \in B_{N(c\Lambda_0)} \} \subset W_{N(c\Lambda_0)} \]

forms a basis for \( W_{N(c\Lambda_0)} \); see [2, 12]. As for the standard modules, by [9, 12] the set

\[ B_{L(k\Lambda_0)} = \{ b \cdot v_{L(k\Lambda_0)} : b \in B_{L(k\Lambda_0)} \} \subset W_{L(k\Lambda_0)}, \quad \text{where} \]

\( B_{L(k\Lambda_0)} = \{ b \in B_{N(k\Lambda_0)} : b \text{ consists of quasi-particles of charges less than or equal to } k \} \), forms a basis for the principal subspace \( W_{L(k\Lambda_0)} \) for all integers \( k > 0 \). Note that in the definition of the set \( B_{L(k\Lambda_0)} \) the quasi-particle monomials are regarded as operators on \( L(k\Lambda_0) \). The bases \( B_{L(k\Lambda_0)} \) provide an interpretation of certain Rogers–Ramanujan-type identities, while \( B_{N(c\Lambda_0)} \) can be combined with the Poincaré–Birkhoff–Witt bases to derive new combinatorial identities; see [3, Sect. 7] for more details and references.
3. Quantum affine vertex algebra in type $A_1^{(1)}$

Let $h$ be a formal parameter. In this paper we employ the usual expansion convention where the negative powers of the expressions of the form $z_1 + \ldots + z_n$, with each $z_i$ denoting a variable or a parameter, are expanded in the negative powers of the variable on the left. For example, we have

$$(z_1 + z_2)^{-r} = \sum_{l \geq 0} \left( \frac{-r}{l} \right) z_1^{-l-1} z_2^l \neq \sum_{l \geq 0} \left( \frac{-r}{l} \right) z_2^{-l-1} z_1^l = (z_2 + z_1)^{-r} \quad \text{for } r > 0.$$  

Define the Yang $R$-matrix in the variable $u$ by

$$R(u) = 1 - hu^{-1} \in \text{End} \mathbb{C}^2 \otimes \text{End} \mathbb{C}^2[h/u],$$  

where $1$ is the identity and $P$ the permutation operator. There exists a unique series $g(u) \in 1 + \frac{h}{u} \mathbb{C}[h/u]$ satisfying

$$g(u + 2h) = g(u)(1 - h^2 u^{-2}).$$  

The normalized $R$-matrix $\overline{R}(u) = g(u)R(u)$ possesses the unitarity property,

$$\overline{R}(u)\overline{R}(-u) = 1$$  

and the crossing symmetry properties,

$$\overline{R}(-u)_{RL} \overline{R}(u + 2h) = 1 \quad \text{and} \quad \overline{R}(-u)_{LR} \overline{R}(u + 2h) = 1,$$

where the subscript RL (LR) indicates that the first tensor factor of $\overline{R}(-u) = \overline{R}_{12}(-u)$ is applied from the right (left) while its second tensor factor is applied from the left (right); see, e.g., [16, Sect. 2.2] for more details.

The dual Yangian $Y^+(\mathfrak{sl}_2)$ is the associative algebra over the commutative ring $\mathbb{C}[[h]]$ with generators $t_{ij}^{(-r)}$, where $i,j = 1,2$ and $r = 1,2,\ldots$, and the defining relations

$$R(u-v)T_1^+(u)T_2^+(v) = T_2^+(v)T_1^+(u)R(u-v),$$  

$$\text{qdet} T^+(u) = 1.$$  

Regarding the relations (3.4), the element $T^+(u) \in \text{End} \mathbb{C}^2 \otimes Y^+(\mathfrak{sl}_2)[[u]]$ is defined by

$$T^+(u) = \sum_{i,j=1,2} e_{ij} \otimes t_{ij}^+(u) \quad \text{for} \quad t_{ij}^+(u) = \delta_{ij} - h \sum_{r=1}^\infty t_{ij}^{(-r)} u^{r-1},$$

where $e_{ij}$ are the matrix units. Throughout the paper, we indicate a copy of the matrix in the tensor product algebra $(\text{End} \mathbb{C}^2)^{\otimes m} \otimes Y^+(\mathfrak{sl}_2)$ by subscripts, so that, e.g., we have

$$T_r^+(u) = \sum_{i,j=1,2} 1^{\otimes (r-1)} \otimes e_{ij} \otimes 1^{\otimes (m-r)} \otimes t_{ij}^+(u).$$

In particular, we have $m = 2$ and $r = 1,2$ in the defining relation (3.4). As for the relation (3.5), for more information on the quantum determinant see [7, Sect. 3].

Let us recall the Etingof–Kazhdan construction [8, Sect. 2] of the quantum vertex algebra structure over the $h$-adically completed dual Yangian; see [8, Sect. 1.4] for the precise definition of the notion of quantum vertex algebra (quantum VOA). From now
on the tensor products are understood as $h$-adically completed. For families of variables $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$ and a single variable $z$ define

$$T_{[n]}^+(u|z) = \prod_{i=1,\ldots,n} T_i^+(z + u_i) \quad \text{and} \quad \overline{T}_{nm}^{12}(u|v|z) = \prod_{i=1,\ldots,n} \prod_{j=n+1,\ldots,n+m} \overline{R}_{ij}(z + u_i - v_{j-n}),$$

where the arrows indicate the order of the factors. Relation (3.4) implies the identity

$$\overline{T}_{nm}^{12}(u|v|z - w)\overline{T}_{m}^{13+13}(u|z)\overline{R}_{nm}^{13}(u|w) = \overline{T}_{m}^{13}(u|w)\overline{T}_{m}^{13}(u|z)\overline{T}_{nm}^{12}(u|v|z - w)$$

(3.6) for the operators on

$$((\text{End } \mathbb{C}^2)^\otimes_n \otimes (\text{End } \mathbb{C}^2)^\otimes_m \otimes Y^+(\mathfrak{sl}_2))[[h]],$$

(3.7) where the superscripts in (3.6) indicate the tensor factors, as given by (3.7). Omitting the variable $z$ we get the operators,

$$T_{[n]}^+(u) = \prod_{i=1,\ldots,n} T_i^+(u_i) \quad \text{and} \quad \overline{T}_{nm}^{12}(u|v) = \prod_{i=1,\ldots,n} \prod_{j=n+1,\ldots,n+m} \overline{R}_{ij}(u_i - v_{j-n})$$

which satisfy the identity

$$\overline{T}_{nm}^{12}(u|v)\overline{T}_{m}^{13+13}(u)\overline{T}_{m}^{13}(u)\overline{T}_{nm}^{12}(u|v)$$

on (3.7). The next lemma is a quantum current reformulation of [8, Lemma 2.1]; cf. [21].

**Lemma 3.1.** Let $c \in \mathbb{C}$ and set $\mathcal{V}_c(\mathfrak{sl}_2) = Y^+(\mathfrak{sl}_2)[[h]]$. For any integer $n \geq 1$ and the family of variables $u = (u_1, \ldots, u_n)$ there exists a unique operator

$$\overline{\mathcal{T}}_{[n]}(u) = \overline{\mathcal{T}}_{[n]}(u_1, \ldots, u_n) \in (\text{End } \mathbb{C}^2)^\otimes_n \otimes \text{Hom}(\mathcal{V}_c(\mathfrak{sl}_2), \mathcal{V}_c(\mathfrak{sl}_2)((u_1, \ldots, u_n))[[h]])$$

such that for all $m \geq 0$ and the family of variables $v = (v_1, \ldots, v_m)$ we have

$$\overline{\mathcal{T}}_{[n]}^{13}(u)\overline{T}_{m}^{13+13}(v)1 = \overline{T}_{nm}^{12}(u)\overline{T}_{m}^{13}(u)\overline{T}_{nm}^{12}(u|v)$$

(3.8) on (3.7), where $1$ is the unit in the algebra $Y^+(\mathfrak{sl}_2)$ and $u \pm ha = (u_1 \pm ha, \ldots, u_n \pm ha)$ for $a \in \mathbb{C}$. In particular, we have $\overline{\mathcal{T}}_{[n]}(u)1 = \overline{T}_{[n]}^+(u)1$.

The next theorem is a particular case of Etingof–Kazhdan’s construction [8, Thm. 2.3].

**Theorem 3.2.** For any $c \in \mathbb{C}$ there exists a unique structure of quantum vertex algebra on $\mathcal{V}_c(\mathfrak{sl}_2)$ such that the vacuum vector is $1$, the vertex operator map is defined by

$$Y(T_{[n]}^+(u), 1, z) = \overline{\mathcal{T}}_{[n]}(u|z),$$

(3.9)

where $\overline{\mathcal{T}}_{[n]}(u|z) = \overline{\mathcal{T}}_{[n]}(z + u_1, \ldots, z + u_n)$, and the braiding map $S(z)$ is defined by

$$S(z)(T_{[n]}^{13+13}(u)T_{m}^{13+24}(v)(1 \otimes 1)) = \overline{T}_{nm}^{12}(u|v|z - h(c + 2))^{-1} \cdot \overline{T}_{nm}^{13}(u|w|z) \cdot \overline{T}_{nm}^{12}(u|v|z + h(c + 2) T_{m}^{13}(u) T_{m}^{13}(v) T_{nm}^{12}(u|v|z)(1 \otimes 1))$$

(3.10)

for operators on $(\text{End } \mathbb{C}^2)^\otimes_n \otimes (\text{End } \mathbb{C}^2)^\otimes_m \otimes \mathcal{V}_c(\mathfrak{sl}_2) \otimes \mathcal{V}_c(\mathfrak{sl}_2)$.

\(^1\)The original expression for the braiding map in [8] differs from (3.10). However, by using the crossing symmetry property (3.3) one easily checks that both definitions coincide.
4. Principal subspaces for the quantum vertex algebra $\mathcal{V}_c(\mathfrak{sl}_2)$

4.1. Principal subspaces of vacuum modules. Let $c \in \mathbb{C}$. From now on we write $x(-r) = t^{(r-t)}_{12} \in Y^+(\mathfrak{sl}_2)$ for $r = 1, 2, \ldots$ and

$$x^+(u) = \sum_{r=1}^{\infty} x(-r) u^{-r-1} \in Y^+(\mathfrak{sl}_2)[[u]].$$  \hspace{1cm} (4.1)

Defining relations (3.4) for the dual Yangian imply

$$[x^+(u), x^+(v)] = 0.$$  

Let $W_{\mathcal{V}_c(\mathfrak{sl}_2)}$ be the $h$-adic completion of the unital subalgebra of $Y^+(\mathfrak{sl}_2)$ generated by all $x(-r)$ with $r = 1, 2, \ldots$. We adopt the terminology coming from the representation theory of affine Lie algebras and refer to $W_{\mathcal{V}_c(\mathfrak{sl}_2)}$ as the principal subspace. However, we should mention that, unlike the principal subspaces for affine Lie algebras, $W_{\mathcal{V}_c(\mathfrak{sl}_2)}$ is not a complex vector space but rather a topologically free $\mathbb{C}[[h]]$-module. By the Poincaré–Birkhoff–Witt theorem for the dual Yangian, see [7, Sect. 3.4], the set

$$\mathcal{B}_{\mathcal{V}_c(\mathfrak{sl}_2)} = \{x(r_m) \ldots x(r_1) 1 : r_m \leq \ldots \leq r_1 \leq -1, m \geq 0\}$$

forms a topological basis for $W_{\mathcal{V}_c(\mathfrak{sl}_2)}$. By this we mean that $\mathcal{B}_{\mathcal{V}_c(\mathfrak{sl}_2)}$ is linearly independent over $\mathbb{C}[[h]]$ and that for every $n \geq 1$ and $v \in W_{\mathcal{V}_c(\mathfrak{sl}_2)}$ there exists an element $w$ in the $\mathbb{C}[[h]]$-span of $\mathcal{B}_{\mathcal{V}_c(\mathfrak{sl}_2)}$ such that $v \equiv w \mod h^n$. The classical limit of $W_{\mathcal{V}_c(\mathfrak{sl}_2)}$, i.e. the complex algebra $W_{\mathcal{V}_c(\mathfrak{sl}_2)}/hW_{\mathcal{V}_c(\mathfrak{sl}_2)}$ is the universal enveloping algebra $U(\mathbb{C}x_\alpha \otimes t^{-1}\mathbb{C}[t^{-1}])$. Moreover, the images of the generators $x(-r)$ with $r \geq 1$ in the quotient $W_{\mathcal{V}_c(\mathfrak{sl}_2)}/hW_{\mathcal{V}_c(\mathfrak{sl}_2)}$ coincide with the elements $x_\alpha(-r) = x_\alpha \otimes t^{-r} \in U(\mathbb{C}x_\alpha \otimes t^{-1}\mathbb{C}[t^{-1}])$.

Remark 4.1. As discussed above, the universal enveloping algebra $U(\mathbb{C}x_\alpha \otimes t^{-1}\mathbb{C}[t^{-1}])$ is the classical limit of the algebra $W_{\mathcal{V}_c(\mathfrak{sl}_2)}$. More generally, by taking the classical limit $h = 0$ of the dual Yangian for $\mathfrak{sl}_2$ one obtains the algebra $U(\mathfrak{sl}_2 \otimes t^{-1}\mathbb{C}[t^{-1}])$. This correspondence, as well as its generalization to the $\mathfrak{sl}_N$ case, extends to the underlying (quantum) vertex algebra structures; see [8, Sect. 2.2]. Another way to go from the quantum to the classical setting can be found in, e.g., [16, Sect. 2.2] and [24, Sect. 15] in terms of the dual (and also double) Yangian for $\mathfrak{gl}_N$ defined over the complex field $\mathbb{C}$. The classical limit $U(\mathfrak{gl}_N \otimes t^{-1}\mathbb{C}[t^{-1}])$ is there obtained as the corresponding graded algebra of the dual Yangian with respect to a certain ascending filtration.

Motivated by the quasi-particles for affine Lie algebras, see [2,9,12], we define the series

$$x^{+,t}_{(m)}(u) = \sum_{r \geq m} x^t_{(m)}(-r) u^{-r-m} = x^+(u) x^+(u + th) \ldots x^+(u + (m-1)th)$$  \hspace{1cm} (4.2)

for any integer $m > 0$ and $t \in \mathbb{C}$. In particular, we have $x^{+,0}_{(m)}(u) = x^+(u)^m$. The coefficients $x^t_{(m)}(-r)$ for $m > 1$ and $t \neq 0$ are no longer elements of the dual Yangian $Y^+(\mathfrak{sl}_2)$ because of the $h$-shifted arguments. However, they do belong to its $h$-adic completion and, more specifically, to $W_{\mathcal{V}_c(\mathfrak{sl}_2)}$. Observe that $x^{+,t}_{(1)}(u) = x^+(u)$ for all $t \in \mathbb{C}$. Following [2,9,12] we refer to the coefficients $x^t_{(m)}(-r)$ as quasi-particles of charge $m$. Consider the set

$$\mathcal{B}^t_{\mathcal{V}_c(\mathfrak{sl}_2)} = \{b \cdot 1 : b \in \mathcal{B}^t_{\mathcal{V}_c(\mathfrak{sl}_2)}\},$$
where \( B^t_{\mathcal{V}_c(\mathfrak{sl}_2)} \) is the set of all monomials
\[
x^t_{(m_1)}(n_r) \cdots x^t_{(m_1)}(n_1) \quad \text{with} \quad r \geq 0, \ m_1 \geq \ldots \geq m_r \geq 1, \ n_1, \ldots, n_r \leq -1.
\]
which satisfy difference conditions (2.6) for all \( s = 1, \ldots, r - 1 \).

**Proposition 4.2.** For all \( c, t \in \mathbb{C} \) the set \( B^t_{\mathcal{V}_c(\mathfrak{sl}_2)} \) forms a topological basis for \( W_{\mathcal{V}_c(\mathfrak{sl}_2)} \).

**Proof.** By comparing (2.4) and (4.2) we see that the classical limit of the quasi-particle \( x^t_{(m_1)}(-r) \) coincides with the quasi-particle \( x_{m_1}(-r) \). Hence the linear independence of the set \( B^t_{\mathcal{V}_c(\mathfrak{sl}_2)} \) is clear as its classical limit is the basis \( B_{N(c\Lambda_0)} \) of the principal subspace \( W_{N(c\Lambda_0)} \). Choose any nonzero \( v \in W_{\mathcal{V}_c(\mathfrak{sl}_2)} \) and positive integer \( n \). We will prove that there exists an element \( w \in \mathbb{C}[[h]] \)-span of \( B^t_{\mathcal{V}_c(\mathfrak{sl}_2)} \) such that \( v = \bar{w} \mod h^n \). As \( W_{\mathcal{V}_c(\mathfrak{sl}_2)} \) is \( h \)-adically complete, this verifies the proposition. Let \( \bar{v} \in W_{\mathcal{V}_c(\mathfrak{sl}_2)}/hW_{\mathcal{V}_c(\mathfrak{sl}_2)} = W_{N(c\Lambda_0)} \) be the image of \( v \) with respect to the classical limit. The \( \mathbb{C}[[h]] \)-module \( W_{\mathcal{V}_c(\mathfrak{sl}_2)} \) is topologically free so we can assume that \( \bar{v} \) is nonzero. Indeed, otherwise we consider the element \( v' = h^{-m}v \) instead of \( v \), where the integer \( m > 0 \) is chosen so that \( v' \) belongs to \( W_{\mathcal{V}_c(\mathfrak{sl}_2)} \) and possesses nonzero classical limit. There exist basis elements \( \bar{b}_{1,0}, \ldots, \bar{b}_{r,0} \in B_{N(c\Lambda_0)} \) and \( \beta_{1,0}, \ldots, \beta_{r,0} \in \mathbb{C} \) such that \( \bar{v} = \sum_i \beta_{i,0} \bar{b}_{i,0} \). Choose \( b_{1,0}, \ldots, b_{r,0} \in B^t_{\mathcal{V}_c(\mathfrak{sl}_2)} \) such that their classical limits equal \( \bar{b}_{1,0}, \ldots, \bar{b}_{r,0} \) respectively. This implies \( v = \sum_i \beta_{i,0} b_{i,0} \mod h \).

Next, we express the classical limit \( \bar{v}_1 \) of \( v_1 = h^{-1}(v - \sum \beta_{i,0} b_{i,0}) \) as a linear combination \( \bar{v}_1 = \sum_i \beta_{i,1} \bar{b}_{i,1} \) for some \( \bar{b}_{1,1}, \ldots, \bar{b}_{r,1} \in B_{N(c\Lambda_0)} \) and \( \beta_{1,1}, \ldots, \beta_{r,1} \in \mathbb{C} \). This produces the equality \( v_1 = \sum_i \beta_{i,1} b_{i,1} \mod h \), where \( b_{i,1} \in B^t_{\mathcal{V}_c(\mathfrak{sl}_2)} \) are chosen so that their classical limits coincide with the corresponding elements \( \bar{b}_{i,1} \). Therefore, we can express \( v \) as
\[
v = \sum_i \beta_{i,0} b_{i,0} + h \sum_i \beta_{i,1} b_{i,1} \mod h^2.
\]
By continuing this procedure, now starting with \( v_2 = h^{-1}(v_1 - \sum \beta_{i,1} b_{i,1}) \), after \( n - 2 \) more steps we obtain a \( \mathbb{C}[h] \)-linear combination of some elements of \( B^t_{\mathcal{V}_c(\mathfrak{sl}_2)} \) which coincides with \( v \) modulo \( h^n \), as required.

We now show that \( W_{\mathcal{V}_c(\mathfrak{sl}_2)} \) inherits the quantum vertex algebra structure from \( \mathcal{V}_c(\mathfrak{sl}_2) \).

Let \( z, u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_m) \) be the variables. Introduce the functions
\[
g_{nm}(u|v|z) = \prod_{i=1,\ldots,n} g(z + u_i - v_j) \quad \text{and} \quad p_{nm}(u|v|z) = \prod_{j=1,\ldots,m} \left( 1 - \frac{h}{z + u_i - v_j} \right). \tag{4.3}
\]
Furthermore, we write \( x^+_{(m)}(u) = x^+(u_1) \cdots x^+(u_n) \) and \( z + u = (z + u_1, \ldots, z + u_n) \).

**Theorem 4.3.** \( W_{\mathcal{V}_c(\mathfrak{sl}_2)} \) is a quantum vertex subalgebra of \( \mathcal{V}_c(\mathfrak{sl}_2) \). Moreover, for all integers \( m, n \geq 1 \) and the variables \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_m) \) we have
\[
Y(x^+_{(m)}(u), z) x^+_{(m)}(v) 1 = p_{nm}(-u|v|z) g_{nm}(u + h(c + 2)|v|z) x^+_{(m+n)}(z + u, v) 1, \tag{4.4}
\]
\[
S(z)(x^+_{(m)}(u) 1 \otimes x^+_{(m)}(v) 1) = p_{nm}(u|v|z)^2 g_{nm}(u|v|z)^2 g_{nm}(-u|v|z)(z + (c + 2)h)(x^+_{(m)}(u) 1 \otimes x^+_{(m)}(v) 1) . \tag{4.5}
\]
Proof. As the vacuum vector $\mathbf{1}$ belongs to $W_{\mathcal{V}(\mathfrak{sl}_2)}$ it is sufficient to prove that

$$Y(a, z)b \in W_{\mathcal{V}(\mathfrak{sl}_2)}[[z^\pm 1]] \quad \text{and} \quad \mathcal{S}(z)a \otimes b \in W_{\mathcal{V}(\mathfrak{sl}_2)} \otimes W_{\mathcal{V}(\mathfrak{sl}_2)}[[z^\pm 1]]$$

(4.6)

for all $a, b \in W_{\mathcal{V}(\mathfrak{sl}_2)}$. In fact, it is sufficient to verify (4.6) for the basis elements $a, b \in B_{\mathcal{V}(\mathfrak{sl}_2)}$ only. However, this follows directly from formulae (4.4) and (4.5). As for the aforementioned formulae, by combining (3.8) and (3.9) we find

$$Y(T_{[n]}^{+13}(u) \mathbf{1}, z)T_{[m]}^{-23}(v) \mathbf{1} = \mathcal{T}_{nm}^{\mathfrak{sl}}(u + h(c + 2)|v|z) \cdot \mathcal{T}_{nm}^{\mathfrak{sl}}(z + u, v) \mathbf{1} \mathcal{T}_{nm}^{\mathfrak{sl}}(u|v|z)^{-1}.$$  

(4.7)

By using the unitarity property (3.2) and taking the matrix entries of $e_{12}^{\otimes(n+m)}$ in (4.7) and (3.10) we obtain the identities (4.4) and (4.5) respectively, as required. □

Remark 4.4. Due to the form of the functions in (4.3), by taking the classical limit $h = 0$ of (4.4) we obtain the vertex operator map (2.3), while the classical limit of the braiding map (4.5) is the identity. Although the vertex operator map for $\mathcal{N}(c\Lambda_0)$ is commutative, this is no longer true for the quantum vertex algebra structure on $\mathcal{V}_c(\mathfrak{sl}_2)$. Its vertex operator map possesses the $\mathcal{S}$-locality property, as defined in [8, Sect. 1.3]. For example, the particular case of the $\mathcal{S}$-locality for the vertex operator $Y(x(-1) \mathbf{1}, z)$ takes the form

$$g(z_1 - z_2)g(-z_1 + z_2 - hc) \left( 1 - \frac{h}{z_1 - z_2} \right) Y(x(-1) \mathbf{1}, z_1)Y(x(-1) \mathbf{1}, z_2)$$

$$= g(z_2 - z_1)g(-z_2 + z_1 - hc) \left( 1 - \frac{h}{z_2 - z_1} \right) Y(x(-1) \mathbf{1}, z_2)Y(x(-1) \mathbf{1}, z_1).$$

(4.8)

In addition, by (4.4) the vertex operator map possesses poles of infinite order at $z = 0$. It is worth noting that, due to the identity $g(z)g(-z + 2h) = 1$ (see [16, Sect. 2.2]), equality (4.8) at the critical level $c = -2$ produces one of the defining relations for the double Yangian for $\mathfrak{sl}_2$ (see [14, Cor. 3.4]),

$$(z_1 - z_2 - h)E_1(z_1)E_1(z_2) = (z_1 - z_2 + h)E_1(z_2)E_1(z_1) \quad \text{with} \quad E_1(z) = Y(x(-1) \mathbf{1}, z).$$

4.2. Ideals of principal subspaces. Let $V$ be a quantum vertex algebra with the vertex operator map $Y(z)$ and the braiding map $\mathcal{S}(z)$. One can introduce the notion of ideal of $V$ in parallel with vertex algebra theory. First, as in [23, Def. 3.4], for any $I \subseteq V$ define

$$[I] = \{ v \in V : h^nv \in I \text{ for some } n \geq 0 \}.$$ 

A topologically free $\mathbb{C}[[h]]$-submodule $I$ of $V$ is said to be an ideal of $V$ if $I = [I]$ and

$$Y(z)(V \otimes I), Y(z)(I \otimes V) \subseteq I((z))[[h]],$$

$$\mathcal{S}(z)(V \otimes I) \subseteq V \otimes I((z))[[h]] \quad \text{and} \quad \mathcal{S}(z)(I \otimes V) \subseteq I \otimes V((z))[[h]].$$

(4.9)

(4.10)

As with vertex algebra theory, one easily verifies that the quotient of a quantum vertex algebra over its ideal is naturally equipped with the quantum vertex algebra structure. In particular, the assumption that $I$ is a topologically free $\mathbb{C}[[h]]$-module satisfying $I = [I]$ ensures that the quotient $V/I$ is topologically free.

---

2The same conclusion follows from the more general results in [8]; also recall Remark 4.1.
Let $k > 0$ be an integer and $t \in \mathbb{C}$. Write $R^i_{V_k(s_lz)}(p) = x^i_{(k+1)}(-p)$ for $p > k$ so that $R^i_{V_k(s_lz)}(p) \cdot 1$ is the coefficient of $u^{p-k-1}$ in $x^+_{(k+1)}(u) \cdot 1 \in V_k(s_lz)$. Motivated by the Calinescu–Lepowsky–Milas presentations of principal subspaces from Theorem 2.1, also recall (2.2), we define

$$T^i_{V_k(s_lz)} = [T^i_{V_k(s_lz)} \cdot 1] [[h]] \quad \text{for} \quad T^i_{V_k(s_lz)} = \sum_{p > k+1} W_{V_k(s_lz)} R^i_{V_k(s_lz)} (p),$$

where the action of $W_{V_k(s_lz)}$ on $R^i_{V_k(s_lz)}(p)$ is given by the algebra multiplication.

**Proposition 4.5.** $T^i_{V_k(s_lz)}$ is the ideal of the quantum vertex algebra $W_{V_k(s_lz)}$.

**Proof.** First, we note that $T^i_{V_k(s_lz)}$ is topologically free by construction. Next, the constraint $T^i_{V_k(s_lz)} = [T^i_{V_k(s_lz)}]$ follows by [23, Prop. 3.7]. Hence it remains to verify the requirements imposed by (4.9) and (4.10). Choose any two basis monomials $a, b \in B^i_{V_k(s_lz)}$ and $p \geq k+1$. Let us prove that the images of $a R^i_{V_k(s_lz)}(p) \otimes b$ and $b \otimes a R^i_{V_k(s_lz)}(p)$ with respect to the vertex operator map $Y(z)$ and braiding $S(z)$ belong to $T^i_{V_k(s_lz)}((z))[[h]]$ and $T^i_{V_k(s_lz)} \otimes V_k(s_lz)((z))[[h]]$ respectively. Apply the substitution

$$(u_{n-k}, \ldots, u_n) = (w, w+th, \ldots, w+kth) \quad \text{or} \quad (v_{n-k}, \ldots, v_n) = (w, w+th, \ldots, w+kth),$$

where $w$ is a single variable and $n > k$ or $m > k$ arbitrary integers, to formulae (4.4) and (4.5). This turns the expressions $x^+_{[n+m]}(z+u, v) 1$ and $x^+_{[m]}(v) 1 \otimes x^+_{[n]}(u) 1$, which appear on the right hand sides of (4.4) and (4.5), into power series with coefficients in $T^i_{V_k(s_lz)}$ and in $T^i_{V_k(s_lz)} \otimes V_k(s_lz)$ or $V_k(s_lz) \otimes T^i_{V_k(s_lz)}$ respectively. Hence the same happens with the entire right hand sides of (4.4) and (4.5) as well. Therefore, by extracting the coefficients for suitably chosen $n$ and $m$ we conclude that the images of $a R^i_{V_k(s_lz)}(p) \otimes b$ and $b \otimes a R^i_{V_k(s_lz)}(p)$ with respect to $Y(z)$ and $S(z)$ belong to (4.12) respectively, as required. Finally, as $Y(z)$ and $S(z)$ are $\mathbb{C}[[h]]$-module maps, this conclusion extends from the elements of the form $a R^i_{V_k(s_lz)}(p) \otimes b$ and $b \otimes a R^i_{V_k(s_lz)}(p)$ to all elements of the $h$-adically completed tensor products $T^i_{V_k(s_lz)} \otimes V_k(s_lz)$ and $V_k(s_lz) \otimes T^i_{V_k(s_lz)}$, so the proposition follows.

Denote by $B^i_{V_k(s_lz)}$ the set of all monomials in $B^i_{V_k(s_lz)}$ which contain at least one quasi-particle of charge greater than or equal to $k+1$. We write $B^i_{V_k(s_lz)} = \{ b \cdot 1 : b \in B^i_{V_k(s_lz)} \}$.

By Proposition 4.2 the set $B^i_{V_k(s_lz)}$ is linearly independent. Consider the identities

$$x^+_{(k+1)}(z) = x^+_{(l)}(z+(k+1)th) x^+_{(k+1)}(z) \quad \text{for} \quad l > 0, \ t \in \mathbb{C}.$$  

They imply that for any integer $n \geq 1$ each element of $B^i_{V_k(s_lz)}$ can be expressed as a $\mathbb{C}[[h]]$-linear combination of the elements of $R^i_{V_k(s_lz)} \cdot 1$ modulo $h^n$. Therefore, as the ideal $T^i_{V_k(s_lz)}$ is $h$-adically complete, it contains the set $B^i_{V_k(s_lz)}$. From now on, we consider the $t = 0$ case. As for the $t \neq 0$ case, see Section 4.3 and, in particular, Remark 4.11 below.

**Proposition 4.6.** The set $B^0_{V_k(s_lz)}$ forms a topological basis of $T^0_{V_k(s_lz)}$.

**Proof.** By the discussion above the given set is linearly independent. The proof that $B^0_{V_k(s_lz)}$ spans an $h$-adically dense $\mathbb{C}[[h]]$-submodule of $T^0_{V_k(s_lz)}$ goes by repeating the argument of
Jerković and Primc [15, Sect. 4.4]. It relies on the fact that quasi-particles of charges \( p \) and \( q \) with \( p \leq q \) satisfy \( 2p \) independent relations of the form

\[
\left( \frac{d}{dz} x_+^{0, 0}(z) \right) x_+^{0, 0}(z) = A_l(z), \quad l = 0, \ldots, 2p - 1.
\] (4.13)

Here \( A_0(z), \ldots, A_{2p-1}(z) \) are certain formal power series with coefficients in the set of quasi-particle polynomials such that each coefficient contains at least one quasi-particle of charge greater than or equal to \( q + 1 \). By arguing as in the affine Lie algebra setting, one checks that for any integer \( n \geq 1 \) relations (4.13) can be used to express any element of \( T^0_{\bar{V}_k}\) as a \( \mathbb{C}[h] \)-linear combination of the elements of \( \bar{B}^0_{\bar{V}_k} \) modulo \( h^n \). \( \square \)

Due to Proposition 4.5 we can introduce the quotient quantum vertex algebra

\[ W_{\mathcal{L}_k} = W_{\bar{V}_k} / T^0_{\bar{V}_k}. \]

We now regard the monomials in \( B^0_{\bar{V}_k} \) as operators on \( W_{\mathcal{L}_k} \). Define

\[ B_{\mathcal{L}_k} = \{ b \cdot 1 : b \in B^0_{\bar{V}_k} \setminus \bar{B}^0_{\bar{V}_k} \} \subset W_{\mathcal{L}_k}. \]

Clearly, the set \( B^0_{\bar{V}_k} \setminus \bar{B}^0_{\bar{V}_k} \) consists of all quasi-particle monomials in \( B^0_{\bar{V}_k} \) whose quasi-particle charges are less than or equal to \( k \).

**Theorem 4.7.** The classical limit of the quantum vertex algebra \( W_{\mathcal{L}_k} \) coincides with the vertex algebra \( W_{L(k,\Lambda_0)} \). Moreover, the set \( B_{\mathcal{L}_k} \) forms a topological basis of \( W_{\mathcal{L}_k} \).

**Proof.** The fact that \( B_{\mathcal{L}_k} \) spans an \( h \)-adically dense \( \mathbb{C}[h] \)-submodule of \( W_{\mathcal{L}_k} \) goes by a usual argument which relies on the quasi-particle relations (4.13) and closely follows Jerković and Primc [15, Sect. 4.4]; also recall the proof of Proposition 4.6. Moreover, the set \( B_{\mathcal{L}_k} \) is linearly independent as it is complementary to the topological basis \( \bar{B}^0_{\bar{V}_k} \) of the ideal \( T^0_{\bar{V}_k} \) established in Proposition 4.6. Finally, as the quantum vertex algebra structure on \( W_{\mathcal{L}_k} \) comes from (4.4) and (4.5), we conclude by Remark 4.4 that its classical limit produces the vertex algebra \( W_{L(k,\Lambda_0)} \), as required. \( \square \)

**Remark 4.8.** The quantum analogue of the defining identity for quasi-particles,

\[ x_{ma}(z) = x_{a}(z)^m = Y(x_{a}(-1)^m u_{L(k,\Lambda_0)}, z), \] (4.14)

see [12, Sect. 3], takes the form

\[ x_{(m)}(z) := Y(x_{m,t}(-1) 1, z), \text{ where } x_{m,t}(-1) = x(-1)x^+(th)\ldots x^+((m-1)th), \] (4.15)

\( x^+(ath) \) denote the evaluations of the series (4.1) at \( u = ath \) with \( t \in \mathbb{C} \) and \( x(-1) \) is the constant term of (4.1). In particular, by setting \( t = 0 \) in (4.15) we get

\[ x_{(m)}^0(1) = Y(x_{m,0}(-1) 1, z) = Y(x(-1)^m 1, z). \] (4.16)

However, in contrast with (4.14), the expression in (4.16) is not the \( m \)-th power of the series \( x_{(1)}(z) \). In fact, the \( m \)-th power of \( x_{(1)}^0(z) \) is not even well defined. Also note that \( x_{(m)}^t(1) = x_{(m)}^{0,t}(1) \) for all \( t \in \mathbb{C} \) and, furthermore, that the classical limits of \( x_{(m)}^t(z) \) and \( x_{(m)}^{0,t}(z) \) coincide. On the other hand, the actions of the operators \( x_{(m)}^t(z) \) and \( x_{(m)}^{0,t}(z) \) on the quotient \( W_{\bar{V}_k} / T^t_{\bar{V}_k} \) differ when \( k \geq m \) and are both trivial for \( k < m \). The
Let $r \in \mathfrak{sl}_2$. Quasi-particle relations at $t \neq 0$. In contrast with \cite{2,9,12}, where the construction of quasi-particle bases of principal subspaces relies on finding a suitable family of quasi-particle relations, in the proof of Proposition 4.2 we bypassed this step using the classical limit. However, the quantum quasi-particles, as defined by (4.17), still satisfy certain relations which can be employed instead of classical limit to directly verify that the set $B^0_{V_k(sl_2)}$ spans $h$-adically dense $\mathbb{C}[[[h]]]$-submodule of the corresponding principal subspace. For $t = 0$ these are given by (4.13). Let us derive such relations for $t \neq 0$. First, for integers $q \geq p > 0$ we have the following $2p$ identities:

\begin{align}
&x_{(k+1)}^+(z + c_k h) x_{(k)}^+(z + pth) = x_{(k-1)}^+(z + pth) x_{(p+q-k+1)}^+(z + c_k h), \quad k = 1, \ldots, p, \quad (4.17) \\
&x_{(k)}^+(z + c_{p+k} h) x_{(q)}^+(z + pth) = x_{(p-k)}^+(z + c_{p+k} h) x_{(q+k)}^+(z + pth), \quad k = 1, \ldots, p, \quad (4.18)
\end{align}

where we set $x_{(0)}^+(z) = 1$ and

\[(c_1, \ldots, c_p; c_{p+1}, \ldots, c_{2p}) = (0, t, \ldots, t(p-1); t(q+1), t(q+2), \ldots, t(q+p)).\]

Remark 4.10. Relations (4.17) and (4.18) present a rational $R$-matrix counterpart of the quasi-particle relations for the Ding–Feigin operators associated with the quantum affine algebra in type $A$; cf. \cite[Sect. 3.1]{20}.
For \( l = 1, \ldots, 2p \) let \((\alpha_{1,l}, \ldots, \alpha_{l,l})\) be the solution of the system of \( l \) linear equations

\[
c^k_l \alpha_{1,l} + c^k_2 \alpha_{2,l} + \ldots + c^k_l \alpha_{l,l} = \delta_{k,l-1}(l-1)!, \quad k = 0, \ldots, l-1,
\]

(4.19)

where \( c^k_l = \delta_{k,0} \). Clearly, each of these \( 2p \) systems possesses a unique solution as its coefficients form a Vandermonde matrix. Indeed, we have \( c_i \neq c_j \) for \( i \neq j \) because of \( q \gg p \). Denote the expressions on the right hand sides of equalities (4.17) and (4.18) obtained for \( k = 1, \ldots, p \) by \( R^l_k \) and \( R^l_{p+k} \) respectively. We have \( 2p \) relations,

\[
(th)^{1-l} \sum_{k=1}^l \alpha_{k,l} x^+_{(p)}(z + c_k h) x^+_{(q)}(z + ph) = (th)^{1-l} \sum_{k=1}^l \alpha_{k,l} R^l_k \quad \text{for} \quad l = 1, \ldots, 2p.
\]

(4.20)

By using (4.19) and the formal Taylor theorem, \( a(z + z_0) = e^{z_0 \frac{\partial}{\partial z}} a(z) \) one checks that the sum on the left hand side of (4.20) possesses a zero of order greater than or equal to \( l - 1 \) at \( h = 0 \) so that the expressions in (4.20) are well defined. Relations (4.20) are linear combinations of equalities (4.17) and (4.18) and each coefficient on the right hand side of (4.20) contains one quasi-particle of charge strictly greater than \( q \). Furthermore, by (4.19) we have

\[
(th)^{1-l} \sum_{k=1}^l \alpha_{k,l} x^+_{(p)}(z + c_k h) x^+_{(q)}(z + ph) = \left( \frac{d^{l-1}}{dz^{l-1}} x^+_{(p)}(z) \right) x^+_{(q)}(z) \mod h
\]

(4.21)

for \( l = 1, \ldots, 2p \). As the classical limit of the right hand side of (4.21) is equal to

\[
\left( \frac{d^{l-1}}{dz^{l-1}} x^+_{pa}(z) \right) x^+_{qa}(z),
\]

it follows by [15, Sect. 4.4], see also [2, Lemma 2.2.1], that the quasi-particle relations obtained by equating the right hand sides of (4.20) and (4.21),

\[
\left( \frac{d^{l-1}}{dz^{l-1}} x^+_{(p)}(z) \right) x^+_{(q)}(z) = \left( th \right)^{1-l} \sum_{k=1}^l \alpha_{k,l} R^l_k \mod h \quad \text{for} \quad l = 1, \ldots, 2p
\]

(4.22)

are independent. More specifically, they can be used in parallel with the corresponding quasi-particle relations in the affine Lie algebra setting [2, 9, 12] to prove directly that the set \( \mathcal{B}^h_{\mathcal{V}_c(sl_2)} \) spans \( h \)-adically dense \( \mathbb{C}[\hbar] \)-submodule of the corresponding principal subspace. However, as relations (4.22) are given modulo \( h \), the classical argument has to be applied \( n \) times to express a given element of \( W_{\mathcal{V}_c(sl_2)} \) as a \( \mathbb{C}[\hbar] \)-linear combination of elements of \( \mathcal{B}^h_{\mathcal{V}_c(sl_2)} \) modulo \( h^n \). Consequently, the argument relies on the fact that the principal subspace \( V = W_{\mathcal{V}_c(sl_2)} \) is topologically free and possesses the following property:

\[
a = b \mod h \quad \text{for} \quad a, b \in V \quad \text{implies} \quad h^{-1}(a - b) \in V.
\]

(4.23)

**Remark 4.11.** It is not clear whether the constraint in (4.23) holds for \( V = \mathcal{I}^l_{\mathcal{V}_c(sl_2)} \cdot 1 \).

Hence the aforementioned classical argument does not necessarily lead to the construction of topological basis of \( \mathcal{I}^l_{\mathcal{V}_c(sl_2)} \) with \( t \neq 0 \). A major difference between the ideal \( \mathcal{I}^l_{\mathcal{V}_c(sl_2)} \) and the ideals \( \mathcal{I}^l_{\mathcal{V}_c(sl_2)} \) with \( t \neq 0 \) appears to be the fact that \( \mathcal{I}^l_{\mathcal{V}_c(sl_2)} \) is of the form \( \mathcal{I}^l_{\mathcal{N}(k\Lambda_0)}[\hbar] \).

On the other hand, consider the following simple example. Let \( t \neq 0 \) and \( k = 1 \). By
regarding $x_{(2)}^+(z) 1 \in \mathcal{I}_{V_1(\mathfrak{sl}_2)}$ as a power series with respect to the parameter $\hbar$ and then taking the classical limit of each coefficient we get

$$x_{(2)}^+(z) 1 \equiv x_{2\alpha}(z)v_{N(\Lambda_0)} + \hbar t \frac{d}{dz} x_{2\alpha}(z)v_{N(\Lambda_0)} + \hbar^2 t^2 x_{\alpha}(z) \frac{d^2}{dz^2} x_{\alpha}(z)v_{N(\Lambda_0)} + \ldots .$$

The coefficients with respect to the variable $z$ of the first two summands on the right hand side belong to $\mathcal{I}_{N(\Lambda_0)}[\hbar]$, which is no longer true for the third summand.

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