Anomalous Zero Sound

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Abstract

We show that the anomalous term in the current, recently suggested by Son and Yamamoto, modifies the structure of the zero sound mode in the Fermi liquid in a magnetic field.
1 Waves in Fermi Liquids

The standard theory of a Fermi liquid \cite{1} allows us to predict a lot of interesting collective phenomena in quantum liquids, in particular, the existence of different modes of density fluctuations.

The main tool of the analysis of Fermi liquids is the kinetic equation

\[
\frac{\partial n(p, x)}{\partial t} + \frac{\partial \epsilon(p, x)}{\partial n(p, x)} \frac{\partial n(p, x)}{\partial x} - \frac{\partial \epsilon(p, x)}{\partial x} \frac{\partial n(p, x)}{\partial p} = 0
\]  

(1)

Here and below \( n(p, x) \) is the density of particles in the phase space, and \( \epsilon(p, x) \) is the energy density. It is important to understand that the energy density in a specified mode is not equivalent to the dispersion law for the free mode, since we deal with interacting systems, wherein the whole system “contributes energy” into the given mode with momentum \( p \) and position \( x \). The other crucial thing for understanding Fermi liquids is that the natural basis for the Hilbert space of excited states is the quasiparticle basis. The quasiparticle vacuum is the state with all energy levels below the Fermi surface are filled with particles. Thus while we deal with small excitations, the quasiparticles have the energy \( \epsilon = \epsilon_F \) and momentum \( p \), confined to the Fermi sphere.

The kinetic equation can be understood either as the semiclassical limit of the evolution equation for the density operator \( n \) by means of the Hamiltonian \( H \)

\[
\frac{\partial n}{\partial t} = \frac{i}{\hbar} [n, H],
\]

(2)

with the classical Poisson bracket of \( \{\epsilon, n\} \) emerging from the commutator \( [n, H] \), or as the classical continuity equation for the current

\[
\frac{\partial n(x)}{\partial t} + \frac{\partial j(x)}{\partial x} = 0,
\]

(3)

where the current is normally given by

\[
j(x) = -\int \frac{d^3p}{(2\pi)^3} \epsilon(p, x) \frac{\partial n(p, x)}{\partial p} = \int \frac{d^3p}{(2\pi)^3} n(p, x) \frac{\partial \epsilon(p, x)}{\partial p} = n(x)v,
\]

(4)

where we dropped the boundary terms under the integral sign.

The basic assumption for the standard Fermi liquid theory is that the energy in the \( p \) mode “feels” the fluctuations of the rest of the liquid. Thus we can say that this model starts from a self-consistent density distribution of interacting quasiparticles, the interaction strength not necessarily being small. While the “equilibrium” energy density \( \epsilon_0(p) \) remains function of momentum only, the energy fluctuation \( \epsilon(p, x) \) is a function of both momenta and coordinates

\[
\epsilon(p, x) = \epsilon_0(p) + \delta \epsilon(p, x),
\]

(5)

and is related to the density fluctuations \( \delta n \), which, in turn, are defined as

\[
n(p, x) = n_0(p) + \delta n(p, x),
\]

(6)
by means of a convolution with the “interaction kernel" $f(p, p')$

$$\delta \epsilon(p, x) = \int \frac{d^3 p'}{(2\pi \hbar)^3} f(p, p') \delta n(p, x), \quad (7)$$

which, assuming the quasiparticle density localized on the Fermi sphere, becomes

$$\delta \epsilon(p, x) = \int d^2 \Omega F(\theta, \theta') \delta n(\theta', x), \quad (8)$$

where the dimensionless formfactor function $F(\theta, \theta') \left( F = \frac{\rho_{\text{Fermi}}}{2\pi^2 \hbar^3} f \right)$ can be represented as a series in the spherical harmonics. Actually the interaction kernel $f$ is a matrix that includes spin-spin interaction formfactor $\sim \sigma \sigma'$; the integral sign in that case must be understood as a trace over the spin indices as well. We omit the spin indices from our analysis, since we limit ourselves to the class of interactions with $f \sim \hat{1}$; The equilibrium density and energy distributions $n_0, \epsilon_0$ are time- and coordinate-independent. If $\delta n$ describes a space-dependent fluctuation of particle density, the kinetic equation (1) will become

$$\frac{\partial \delta n}{\partial t} + \frac{\partial \epsilon_0}{\partial \mathbf{p}} \frac{\partial \delta n}{\partial \mathbf{x}} - \frac{\partial \delta \epsilon}{\partial x} \frac{\partial n_0}{\partial \mathbf{p}} = 0. \quad (9)$$

Now, consider a flat-wave fluctuation of the particle density

$$\delta n = \delta(\epsilon - \epsilon_F) \nu(\theta, \phi) e^{i(\omega t - \mathbf{k} \mathbf{r})}. \quad (10)$$

Taking into account that

$$\frac{\partial n_0(p, x)}{\partial \mathbf{p}} = -v_F \delta(\epsilon - \epsilon_F), \quad (11)$$

we get the following integral equation from (9)

$$(\omega - kv) \nu(\theta) = kv \int \frac{d^2 \Omega'}{4\pi} F(\theta, \theta', \phi, \phi') \nu(\theta'), \quad (12)$$

where $d^2 \Omega = \sin \theta d\theta d\phi$. Restricting our analysis to $\phi$-independent modes solely, choosing $k$ as the $Oz$ direction on the Fermi sphere, noticing that $kv = kv_F \cos \theta$, introducing a convenient parameter $s \equiv \frac{\omega}{kv_F}$, we get

$$(s - \cos \theta) \nu(\theta) = \cos \theta \int \frac{\sin \theta' d\theta'}{2} \nu(\theta') F(\theta, \theta'). \quad (13)$$

Let us limit ourselves with the the zeroth and the first harmonics of $F$

$$F = F_0 + F_1 \left( \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi') \right). \quad (14)$$

This will mean that a solution to (13) must be sought in the form

$$\nu(\theta) = (C_0 + C_1 \cos \theta) \frac{\cos \theta}{s - \cos \theta}, \quad (15)$$

$2$
where $C_0, C_1$ are some constants which are subject to conditions

$$C_0 = F_0 \int \frac{\sin \theta d\theta}{2} (C_0 + C_1 \cos \theta) \frac{\cos \theta}{s - \cos \theta},$$

$$C_1 = F_1 \int \frac{\sin \theta d\theta}{2} \cos \theta (C_0 + C_1 \cos \theta) \frac{\cos \theta}{s - \cos \theta}.$$  \hspace{1cm} (16)

This system of linear equations upon $C_0, C_1$ must have a zero determinant to be solvable, which imposes a condition upon $s$ thus providing us with a dispersion relation following from the equation

$$\left( \frac{1}{2} s \log \left( \frac{s + 1}{s - 1} \right) - 1 \right) (F_0(F_1 + 3) + 3F_1 s^2) = F_1 + 3.$$  \hspace{1cm} (17)

The case of small $F_0, F_1$ (both are dimensionless formfactors, describing interaction of quasi-particles; they can be directly inferred e.g. from the four-particle interaction Hamiltonian) may be called weakly interacting phase, whereas the case of large formfactors – strongly interacting. In the weakly interacting phase we expand the equation (17) around $s = 1$ and get the dispersion law

$$s = 1 + 2e^{-\frac{2(F_0(F_1 + 3) + 3F_1 s^2)}{F_0(F_1 + 3) + 3F_1}},$$

whereas for the strongly-interacting case the expansion takes place around $s = \infty$ and we get

$$s = \frac{1}{\sqrt{3}} \sqrt{F_0 + \frac{3F_1}{5} + \frac{F_0 F_1}{3}}.$$  \hspace{1cm} (19)

The oscillation mode we have described so far is known as the zero sound mode. It is important for us that this phenomenon is equally successfully realized both in the framework of the field theory (Landau Fermi liquid theory, explicated above) and in holography [2]. Of importance is also the fact that holography, advertised as the appropriate language for the strongly-coupled theory, indeed reproduces the dispersion law (19) for the large $F_0, F_1$, rather than the weakly-coupled (18) regime. In particular, even the quantum attenuation contribution (the absorptive part in the dispersion law not shown here) is reproduced form holography as well [2]. Let us emphasize that zero sound is not the single quasiparticle excitation but the collective excitation of the Fermi surface.

Now we proceed to an interesting modification of zero sound due to an anomaly in the magnetic field.

2 Anomaly and Chirality

Recently Son and Yamamoto [3] have suggested that in theories with anomalies the expression for the current (24) be supplied with two extra terms

$$j(x) = \int \frac{d^3 p}{(2\pi)^3} \left[ -\epsilon(p, x) \frac{\partial n(p, x)}{\partial p} - \left( \Omega \frac{\partial n(p, x)}{\partial p} \right) \epsilon(p, x) B - \epsilon(p, x) \left( \Omega \times \frac{\partial n(p, x)}{\partial x} \right) \right].$$  \hspace{1cm} (20)
Here the crucial element of the construction is the “dual” magnetic field strength $\Omega$ in the momentum space.

This extra current leads to the existence of several interesting effects in the presence of an external magnetic field. One of them is the chiral magnetic effect [4], which amounts to generation of an electric current along a magnetic field in presence of an axial chemical potential $\mu_A$

$$ j = \frac{\mu_A B}{2\pi^2}. \tag{21} $$

This phenomenon was realized in the holographic approach to QCD in several different ways [5, 6, 7] and observed on lattice [8]. Holography models the chiral magnetic effect via a Chern-Simons term

$$ S_{CS} = - \frac{N_c}{24\pi^2} \int \epsilon^{\mu\nu\lambda\rho\sigma} A_\mu F_{\nu\lambda} F_{\rho\sigma}, \tag{22} $$

which generates the anomalous current. Recently the same term in the holographic action has been shown to be responsible for another very interesting effect, the chiral magnetic wave [9]. The essence of this effect is propagation of a chirality wave through a medium in a magnetic field $B = (0, 0, B)$

$$ n_A \sim e^{i(\omega t - kz)}. \tag{23} $$

The anomalous term in the current in the magnetic case has the immediate counterpart for the rotating Fermi fluid. This is the generic situation when the following substitution $B \to 2m\omega$ yields the correct anomaly for the rotating matter. In particular the chiral magnetic effect has the parallel chiral vortical effect. The presence of the proper term in the current in the rotating case has been remarked in [10]. In what follows we have in mind that the anomalous zero sound splitting can be equally considered for the magnetic or rotating cases.

## 3 Anomalous Current

The modification of the current (24) allows us to include the anomaly-driven dynamics into the kinetic equation. The realization of the anomaly as a contribution to the current via Berry phase in [3] has paved the way to understanding how the modes of density oscillations in Fermi liquids are modified due to anomalies in a magnetic field. A geometrical argument for the appearance of the Wess-Zumino term for a Fermi surface due to a Berry phase was suggested recently by Zahed [11]. Consider a Fermi liquid with right and left quasiparticles, described by their densities $n_R$ and $n_L$. We have now two currents

$$ j_{R,L}(x) = \int \frac{d^3 p}{(2\pi)^3} \left[ -\epsilon_{R,L}(p, x) \frac{\partial n_{R,L}(p, x)}{\partial p} - \left( \Omega_{R,L} \frac{\partial n_{R,L}(p, x)}{\partial p} \right) \epsilon_{R,L}(p, x)B - \epsilon_{R,L}(p, x) \left( \Omega_{R,L} \times \frac{\partial n_{R,L}(p, x)}{\partial x} \right) \right], \tag{24} $$
The oscillations of the left and right densities, that to be identical and free one from each other become now coupled via the anomaly term. This will lift the degeneracy in the dispersion law, yielding two different types of zero sound. This phenomenon will be called by us “anomalous zero sound”.

One might question the validity of such an approach given that we combine the non-relativistic physics of quasiparticles living on a Fermi surface, and an explicitly relativistic extra current by Son-Yamamoto. Indeed, since the extra current terms are derived from the Berry phase, they require the presence of a monopole at the origin of the momentum space. Son and Yamamoto have clearly related the extra term in the current with the anomaly. Since the anomalous contribution exists both for massive and massless particles we should keep the monopole at the origin of the momentum space for the massive particles as well as Son and Yamamoto did.

Alternatively, one could appeal to the relativistic formulation of the Fermi liquid theory by Baym and Chin [12], which has been shown to be much similar to the original Landau non-relativistic formalism. Then the argumentation by Son-Yamamoto based on Berry’s phase would be directly applicable.

The “dual” magnetic field $\Omega$ is given by the condition that its flux through the Fermi surface is the topological charge

$$\frac{1}{2\pi} \int dS \Omega = \pm 1,$$

whence we can choose for $\Omega$ a hedgehog Ansatz

$$\Omega_{R,L} = \pm \frac{n}{2p_F},$$

where $p_F$ is the Fermi momentum, $n$ is the unit normal vector on the Fermi surface, $dS$ is the integration measure on the Fermi surface.

The extra contributions in the kinetic equation $\frac{\partial n(x)}{\partial t} + \frac{\partial j(x)}{\partial x} = 0$ that come from $\left(\Omega \frac{\partial n(p,x)}{\partial p} \right) \epsilon(p,x) B$ are the following terms

$$(\nabla j)_1 = \nabla_i \left( \Omega^j \frac{\partial \hat{n}}{\partial p^j} (p,x) \right) \epsilon(p,x) B^i,$$

$$(\nabla j)_2 = \nabla_i \left( \Omega^j \frac{\partial n(p,x)}{\partial p^j} \right) \hat{k} (p,x) B^i,$$

here the vertical arrows point out where the differential operator acts. It is clear from elementary vector algebra identity $\vec{k} (\vec{k} \times \hat{n}) = 0$ that the contribution of $\nabla \left[ \epsilon(p,x) \left( \Omega \times \frac{\partial n(p,x)}{\partial x} \right) \right]$ is zero since the wave of density as well as the wave of energy fluctuations propagate in the same direction of the wave-vector $\vec{k}$. The density waves will be sought in the same form as prescribed by Landau theory for zero sound. We notice that such fluctuations, assuming
are precisely of the form that the chiral spiral wave predicted from holography
\[
\delta n_{R,L}(\theta, \phi) = \delta(\epsilon - \epsilon_F)\nu_{R,L}(\theta, \phi)e^{i(\omega t - kr)}.
\]
\[(28)\]

For simplicity we stay in the axially symmetric sector \(\nu_{R,L} = \nu_{R,L}(\theta)\); certainly there will exist other zero sound modes corresponding to decomposition in higher harmonics according to the azimuthal angle \(\phi\). Since we deal with a system “left fermions + right fermions + other degrees of freedom”, we cannot make a precise statement on how the fluctuations of the densities correspond to the energy fluctuations. Thus we model it as
\[
\delta\epsilon_R = \int (F_{SV_R}(\theta') + F_{AV_L}(\theta')) \frac{\sin \theta' d\theta'}{2},
\]
\[
\delta\epsilon_L = \int (F_{AV_R}(\theta') + F_{SV_L}(\theta')) \frac{\sin \theta' d\theta'}{2}
\]
\[(29)\]

The “interaction form factors” \(F_S, F_A\) contain essentially all the interesting dynamics of the problem. Here we chose the simplest case with \(F_{S,A} = \text{const}\), corresponding to the lowest Legendre polynomials in the spherical functions expansion. Defining them dimensionless as above, we allow ourselves to speak about a “weakly coupled system” where \(F_{S,A} \to 0\), and a “strongly coupled system” where \(F_{S,A} \to \infty\). Anticipating the next section of this work we should warn the reader that it is not at all evident that the “strongly coupled” Landau model of the Fermi liquid based on the nonrelativistic Fermi sphere picture and the collisionless kinetic equation is not a priori deemed to be equivalent to the holographic model of the same object.

Under the definitions given above, the two contributions in questions become
\[
(\nabla j_R(\theta))_1 = \frac{ik\nu_R(\theta)\delta(\epsilon - \epsilon_F)Bv_F}{4p_F^2},
\]
\[
(\nabla j_R(\theta))_2 = \frac{ik\delta(\epsilon - \epsilon_F)Bv_F}{2p_F^2} \int (F_{SV_R}(\theta') + F_{AV_L}(\theta')) \frac{\sin \theta' d\theta'}{2}.
\]
\[(30)\]

and analogously we can write down the contributions for the left current \(j_L\). In the equation for \((\nabla j)_1\) we have used an integration by parts trick in order to avoid nasty terms like \(\nu_{R,L}(\theta)\).

Thus the equation \[(13)\] is transformed into
\[
(s - \cos \theta - b)\nu_R(\theta) = (b + \cos \theta)(F_S C_R + F_A C_L),
\]
\[
(s - \cos \theta - b)\nu_L(\theta) = (b + \cos \theta)(F_A C_R + F_S C_L),
\]
\[(31)\]

where a convenient dimensionless parameter
\[
b = \frac{Bv_F}{2p_F^2}
\]
\[(32)\]

has been introduced. Solving this system with regard to \(\nu_{R,L}\) and imposing the conditions
\[
\int \sin \theta' d\theta' \nu_R(\theta'; C_R, C_L) = C_R
\]
\[
\int \sin \theta' d\theta' \nu_L(\theta'; C_R, C_L) = C_L
\]
\[(33)\]
we get thus a homogeneous system upon the coefficients \( C_{R,L} \). Requiring its determinant to be zero we obtain the dispersion laws. They can conveniently be written down for the case of small and large coupling. For small coupling we see that the degeneracy has indeed been lifted

\[
s = 1 \pm b + e^{-\frac{F_{S}}{2F_{S}}(1 \mp b)}.
\]

(34)

At zero magnetic field the Landau result is restored (obtainable from (18) taking \( F_{1} = 0, F_{S} = F_{A} = F_{0} \)). If we look at the eigenfunctions it can be shown that the axial zero sound is proportional to the magnetic field while the vector zero sound acquires the correction to the usual velocity.

At large coupling we get

\[
s = \sqrt{\frac{F_{S}}{3} \left( 1 - \frac{F_{A}^2 - F_{S}^2}{2F_{S}} \right)}
\]

(35)

Again, we comply with the Landau result (19) taking \( F_{0} = F_{S} \gg 1, F_{S} = F_{A} \). Notice that in both cases the field \( B \) has been held arbitrary; the only perturbative expansion used was the expansion in small or large \( F_{S,A} \). We did not consider here the non-anomalous effects of the magnetic field on the structure of the modes, since in the linear order in the magnetic field the non-anomalous effect will be upon spin waves rather than zero sound waves.

These dispersion laws are the main result of the field-theoretical part of this work. We interpret this situation in terms of two types of zero sound, axial and vector one propagating through a medium of left and right fermions with zero net chirality, which interact due to the presence of the \( B \) field. Below we analyze the two zero sounds (or the anomalous zero sound) from the holographic point of view. As we have remarked above the same is true for the rotating case.

4 Anomalous zero sound in holography

Anomalous splitting

Recently a great deal of activity has taken place, advocating understanding quantum liquids (both of Fermi and non-Fermi type) from holography; for some of the results see \([13, 14, 15, 16, 17, 18, 19]\); a full review of the AdS/condensed state correspondence would certainly go beyond the scope of the present paper; for a pedagogic introduction see e.g. \([20]\).

One can try to elucidate the nature of the mode we have observed from the holographic point of view. The normal zero sound (i.e. without the anomaly) in holography has been discovered by Karch, Son and Starinets \([2]\). It can be easily modified to account for the anomalous effects, resident in the Chern-Simons term. Consider a holographic model with the action

\[
S = S_{DBI}^{L} + S_{DBI}^{R} + S_{CS}^{L} - S_{CS}^{R}
\]

(36)
where the (Dirac–Born–Infeld-motivated) action for either left or right modes is

$$S_{DBI}^{L,R} = -T \int d^8 \xi \sqrt{-\det (g_{ij} + 2\pi \alpha' F_{ij}^{L,R})}$$  \hspace{1cm} (37)$$

which in the metric

$$ds^2 = \frac{r^2}{R^2} dx_{1,3}^2 + \frac{R^2}{r^2} (dr^2 + r^2 d\Omega_5^2)$$  \hspace{1cm} (38)$$
in presence of the vector-potentials $A_0^{L,R}(r)$, and a constant field $F_{12}^R = F_{12}^L = B$ becomes

$$S = -NV_0 \int dr \sqrt{1 - \left( \frac{\partial A_0^{L,R}}{\partial r} \right)^2} (r^4 + B^2)$$  \hspace{1cm} (39)$$

(here we have included the $2\pi \alpha'$ into the definitions of the fields). The Chern–Simons part is

$$S_{CS} = -\frac{N_c}{24\pi^2} \int dr d^4 x \epsilon^{\mu\nu\lambda\rho\sigma} A_\mu F_{\nu\lambda} F_{\rho\sigma}. \hspace{1cm} (40)$$

Let us consider a system analogous to our model in the four-dimensional part of this work. We have already mentioned two types of holographically represented matter; now we must say something about the chemical potential. To comply with the previous sections we choose $\mu_L = \mu_R \equiv \mu$. Thus we can solve the classical equations of motion for the $L$ and $R$ modes independently by the Ansatz

$$A_{0,cl}^{L,R}(r) = \frac{d}{\sqrt{d^2 + r^6 + r^2 B^2}}$$  \hspace{1cm} (41)$$

where the integration constant $d \sim \mu^{\frac{1}{3}}$. When fluctuations

$$A_0^{L,R}(r) = A_{0,cl}^{L,R}(r) + a_0^{L,R}(r, x, t)$$
$$A_3^{L,R}(r) = a_3^{L,R}(r, x, t)$$  \hspace{1cm} (42)$$

are taken into account, the Chern–Simons term would not have contributed to the fluctuation equations had there been a single type of fermions solely. Yet due to having both left and right modes, we can extract the VVA “anomalous vertex” structure and obtain the following modification of the equations of motion (25)a-c in [2]

$$\partial_z (f^3 g a V) - \frac{q f}{z} (\omega a V + qa V) + i b a V = 0,$$
$$\partial_z (f g a V) + \frac{\omega}{z} (\omega a V + qa V) - i b a V = 0,$$
$$\partial_z (f^3 g a V') - \frac{q f}{z} (\omega a V + qa V') - i b a V' = 0,$$
$$\partial_z (f g a V') + \frac{\omega}{z} (\omega a V + qa V') + i b a V' = 0,$$  \hspace{1cm} (43)$$

where

$$a_{0,3}^{L,R}(r, x) = \int \frac{dq d\omega}{(2\pi)^2} a_0^{L,R}(r, \omega, q) e^{-i\omega t + iq x}, \hspace{1cm} (44)$$
the remnant gauge fixing condition being

\[ f^2 \omega a_0 V^A + q a_3 V^A = 0, \]  

(45)

where we have switched to the variable \( z \equiv \frac{1}{r} \) and introduced the functions

\[ f = \sqrt{1 + \frac{d^2 z^6}{1 + z^4 B^2}}, \]  

(46)

and

\[ g = \sqrt{1 + B^2 z^4} \]  

(47)

and absorbed the normalization factor into \( b \sim B \). Then following [2] we introduce two gauge-invariant field strengths

\[ E^V = \omega a_3^V + qa_0^V \]

\[ E^A = \omega a_3^A + qa_0^A \]  

(48)

and construct a complex variable out of them

\[ E = E^V + iE^A, \]  

(49)

now representing the dynamics in the chiral plane, thus we can finally write down the equation upon the complex variable \( E \)

\[ E''(z) + E'(z) \left( \frac{f'(z) \left(3q^2 - \omega^2 f(z)^2\right)}{f(z) \left(q^2 - \omega^2 f(z)^2\right)} + \frac{g'(z)}{g(z)} \right) + \frac{E(z) \left(\omega^2 f(z)^2 - q^2\right)}{f(z)^2} + \frac{ibq\omega f(z) E'(z)}{f(z) (q^2 - \omega^2 f(z)^2)} = 0 \]  

(50)

This equation is then studied numerically. It is subject to the boundary equations

\[
\begin{aligned}
E(z)|_{z \to 0} &= 0 \\
\partial_z (z E(z))|_{z \to \infty} &= i\omega z E(z)|_{z \to \infty}
\end{aligned}
\]  

(51)

The first of these conditions is the normal Dirichlet boundary condition on the boundary; the second one is the infalling wave condition on the horizon. These two conditions fix eigenvalues of this equation

\[ \omega = \omega(k) = \alpha(B)k - i\beta(B)k^2. \]  

(52)

We chose the Ansatz for \( \omega(k) \) containing a linear and a quadratic term only, which is confirmed by numeric calculations. It can be also verified by numerical analysis that the term linear in \( k \) is indeed purely real whereas the term quadratic in \( k \) is purely imaginary. The numerical dependencies of the phase velocity of the wave \( \alpha \) and the diffusion coefficient \( \beta \) on the magnetic field are given in Fig. [1] and Fig. [2] respectively for both modes of the anomalous zero sound. One can readily see that the \( \alpha \) dependence starts with a correct \( 1/\sqrt{3} \) value, and then changes linearly for small values of the field. This is surely the Chern-Simons contributions, since the DBI could have contributed only in the order of \( \sim B^2 \). The field lifts the degeneracy,
and the splitting between the modes for small values of the field is linear for small fields; at large fields the DBI terms quadratic in $B$ start to contribute.

Thus we have produced in this section the dispersion relation for the chiral anomalous zero sound in holography generalizing analysis of [2]. Notice that the fluctuation effect that we have described exists independently from the static anomalous axial current that will be present in our setting as well [21] but has not been analyzed here as in the leading order the two effects are essentially decoupled.

5 Discussion

In this note we have discussed the modification of the zero sound excitation of the Fermi surface due to the anomaly in the external magnetic field. Since the vector and axial modes are coupled in the magnetic field we get two independent zero sound modes instead of the
single one. Using the nontrivial Berry phase in the momentum space we get the dispersion relations for both zero sound modes in the framework of the kinetic equations.

Our study was focused on the peculiar longitudinal modes and a detailed analysis for the transverse modes can be taken into account as well. We have considered only the fluctuations of the components of the vector and axial currents ignoring the spin effects due to the fluctuations of the tensor currents. There should be interesting effect on the zero spin sound due to the mixing of scalar and tensor modes in the external magnetic field discussed in [24]. Another study case for us would be the involvement of the spin-spin interaction and influence of the magnetic field on interaction between the spin waves and sound waves. It would be also useful to discuss separately the effects of the possible Fermi points in the spin sound case.

Our consideration has a lot in common with the chiral spiral wave [25] however in that case the modes in plasma have been investigated implying the high temperature while in our case we discuss the dense matter at very small or vanishing temperature. At the holographic side we have generalized the consideration of [23] taking into account the Chern-Simons term and have found the clear-cut manifestation of the anomalous contributions at the strong coupling. It would be interesting to investigate the possibility of the experimental observation of the magnetic or rotational anomalous zero sound.

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References

[1] L. Landau, “The theory of a Fermi liquid,” *Zh. Eksp. Teor. Fiz.* 30 (1956) 1058.

[2] A. Karch, D. Son, and A. Starinets, “Zero Sound from Holography,” arXiv:0806.3796 [hep-th].

[3] D. T. Son and N. Yamamoto, “Berry Curvature, Triangle Anomalies, and Chiral Magnetic Effect in Fermi Liquids,” arXiv:1203.2697 [cond-mat.mes-hall].
[4] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, “The Chiral Magnetic Effect,”
Phys.Rev. **D78** (2008) 074033, arXiv:0808.3382 [hep-ph]

[5] H.-U. Yee, “Holographic Chiral Magnetic Conductivity,” JHEP **0911** (2009) 085
arXiv:0908.4189 [hep-th]

[6] A. Rebhan, A. Schmitt, and S. A. Stricker, “Anomalies and the chiral magnetic effect in
the Sakai-Sugimoto model,” JHEP **1001** (2010) 026, arXiv:0909.4782 [hep-th]

[7] A. Gorsky, P. Kopnin, and A. Zayakin, “On the Chiral Magnetic Effect in Soft-Wall
AdS/QCD,” Phys.Rev. **D83** (2011) 014023, arXiv:1003.2293 [hep-ph].

[8] P. Buividovich, M. Chernodub, E. Luschevskaya, and M. Polikarpov, “Numerical
evidence of chiral magnetic effect in lattice gauge theory,” Phys.Rev. **D80** (2009)
054503, arXiv:0907.0494 [hep-lat].

[9] D. E. Kharzeev and H.-U. Yee, “Chiral Magnetic Wave,” Phys.Rev. **D83** (2011) 085007,
arXiv:1012.6026 [hep-th]

[10] V. Kirilin, Z. Khaidukov, and A. Sadofyev, “Chiral Vortical Effect in Fermi Liquid,”
arXiv:1203.6612 [cond-mat.mes-hall]

[11] I. Zahed, “Anomalous Chiral Fermi Surface,” arXiv:1204.1955 [hep-th]

[12] G. Baym and S. A. Chin, “Landau Theory of Relativistic Fermi Liquids,” Nucl.Phys.
**A262** (1976) 527

[13] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, “Building a Holographic
Superconductor,” Phys.Rev.Lett. **101** (2008) 031601, arXiv:0803.3295 [hep-th]

[14] K. Balasubramanian and J. McGreevy, “Gravity duals for non-relativistic CFTs,”
Phys.Rev.Lett. **101** (2008) 061601, arXiv:0804.4053 [hep-th]

[15] M. Cubrovic, J. Zaanen, and K. Schalm, “String Theory, Quantum Phase Transitions
and the Emergent Fermi-Liquid,” Science **325** (2009) 439–444, arXiv:0904.1993
[hep-th]

[16] T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, “Emergent quantum criticality, Fermi
surfaces, and AdS(2),” Phys.Rev. **D83** (2011) 125002, arXiv:0907.2694 [hep-th]

[17] M. Ammon, J. Erdmenger, S. Lin, S. Muller, A. O’Bannon, et al., “On Stability and
Transport of Cold Holographic Matter,” JHEP **1109** (2011) 030, arXiv:1108.1798
[hep-th]
[18] T. Faulkner and J. Polchinski, “Semi-Holographic Fermi Liquids,” *JHEP* **1106** (2011) 012, arXiv:1001.5049 [hep-th].

[19] S. A. Hartnoll, J. Polchinski, E. Silverstein, and D. Tong, “Towards strange metallic holography,” *JHEP* **1004** (2010) 120, arXiv:0912.1061 [hep-th]. 71 pages, 8 figures.

[20] S. A. Hartnoll, “Lectures on holographic methods for condensed matter physics,” *Class. Quant. Grav.* **26** (2009) 224002, arXiv:0903.3246 [hep-th].

[21] A. Rebhan, A. Schmitt, and S. Stricker, “Holographic chiral currents in a magnetic field,” *Prog. Theor. Phys. Suppl.* **186** (2010) 463–470, arXiv:1007.2494 [hep-th].

[22] N. Jokela, G. Lifschytz, and M. Lippert, “Magnetic effects in a holographic Fermi-like liquid,” *JHEP* **1205** (2012) 105, arXiv:1204.3914 [hep-th].

[23] M. Goykhman, A. Parnachev, and J. Zaanen, “Fluctuations in finite density holographic quantum liquids,” arXiv:1204.6232 [hep-th].

[24] A. Gorsky, P. Kopnin, A. Krikun, and A. Vainshtein, “More on the Tensor Response of the QCD Vacuum to an External Magnetic Field,” *Phys. Rev.* **D85** (2012) 086006, arXiv:1201.2039 [hep-ph].

[25] D. E. Kharzeev and H.-U. Yee, “Chiral helix in AdS/CFT with flavor,” *Phys. Rev.* **D84** (2011) 125011, arXiv:1109.0533 [hep-th].