AN ANALOGUE OF LIOUVILLE’S THEOREM AND AN APPLICATION TO CUBIC SURFACES

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ABSTRACT. We prove a strong analogue of Liouville’s Theorem in Diophantine approximation for points on arbitrary algebraic varieties. We use this theorem to prove a conjecture of the first author for cubic surfaces in \( \mathbb{P}^3 \).

1. Introduction

The famous theorem of K.F. Roth (see for example [4, Part D]) gives a sharp upper bound on how well an irrational algebraic number can be approximated by rational numbers. In [10], the authors prove an analogue of Roth’s Theorem for algebraic points on arbitrary algebraic varieties.

In this paper we generalize, in the sense of [10], Liouville’s approximation theorem to arbitrary varieties, as well as giving an extension involving the asymptotic base locus. On \( \mathbb{P}^1 \), except for the case that \( x \in \mathbb{P}^1 \) is a rational point of the base number field, Liouville’s theorem is weaker than Roth’s. On arbitrary varieties the extension involving the asymptotic base locus makes it slightly more useful and we use this to verify a conjecture of the first author for cubic surfaces in \( \mathbb{P}^3 \).

The point of view of [10] is that the Roth and Liouville theorems are examples of “local Bombieri-Lang phenomena” whereby local positivity of a line bundle influences local accumulation of rational points. Specifically, given a variety \( X \), an algebraic point \( x \in X \), and an ample line bundle \( L \) on \( X \), these theorems are expressed as inequalities between \( \epsilon_x(L) \), the Seshadri constant, measuring local positivity of \( L \) near \( x \), and \( \alpha_x(L) \), an invariant measuring how well we can approximate \( x \) by rational points.

In §2 we review the definitions and elementary properties of \( \alpha_x \) and \( \epsilon_x \). In §3 we prove the generalized Liouville theorem (Theorem 3.3). We close the paper in §4 by computing \( \alpha_x \) and \( \epsilon_x \) for an arbitrary nef line bundle and rational point, not on a line, on a smooth cubic surface (where the lines are also rational); we then use this to verify Conjecture 3.2 from [9].

2. Elementary properties of \( \alpha \) and \( \epsilon \)

In this section, we give a brief overview of the properties of \( \alpha \) and \( \epsilon \) used in this paper. For a more detailed discussion of \( \alpha \), see [10]. For a more detailed discussion of \( \epsilon \), there are many good references – see for example [6, chap. 5]. Proofs of all of the facts listed below can be found in [10].
The constant $\alpha_x$. In order to motivate the definition of $\alpha_x$ it is helpful to recall the classical case of approximation on the line. For a point $x \in \mathbb{R}$ the approximation exponent $\tau_x$ of $x$ is the unique extended real number $\tau_x \in (0, \infty)$ such that the inequality

$$\left| x - \frac{a}{b} \right| \leq \frac{1}{b^{\tau_x + \delta}}$$

has only finitely many solutions $a/b \in \mathbb{Q}$ whenever $\delta > 0$ (respectively has infinitely solutions $a/b \in \mathbb{Q}$ whenever $\delta < 0$). The approximation exponent measures a certain tension between our ability to closely approximate $x$ by rational numbers (the distance term $|x - a/b|$) and the complexity (the $1/b$ term) of the number required to make this approximation. In this notation the 1844 theorem of Liouville [7] is that $\tau_x \leq \delta$ for $x \in \mathbb{R}$ algebraic of degree $d$ over $\mathbb{Q}$.

To generalize $\tau_x$ to arbitrary projective varieties defined over a number field $k$ we replace the function $|x - a/b|$ by a distance function $d_v(x, \cdot)$ depending on a place $v$ of $k$, and measure the complexity of a point via a height function $H_v(\cdot)$ depending on an ample line bundle $L$. For an introduction to the theory of heights the reader is referred to any one of [1, Chap. 2], [4, Part B], [5, Chap. III], or [13, Chap. 2]. Unless otherwise specified all height functions in this paper are multiplicative, relative to $k$, and come from line bundles on $X$ defined over $k$. In this paper we normalize our height functions as follows. The absolute values are normalized with respect to $k$: if $v$ is a finite place of $k$, $\pi$ a uniformizer of the corresponding maximal ideal, and $k$ the residue field then $|\pi|_v = 1/\#k$; if $v$ is an infinite place corresponding to an embedding $i:k \hookrightarrow \mathbb{C}$ then $|\pi|_v = |i(\pi)|^m_v$ for all $x \in k$, where $m_v = 1$ or 2 depending on whether $v$ is real or complex. The heights are then normalized so that for a point $x = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$, the height with respect to $\mathcal{O}_{\mathbb{P}^n}(1)$ is

$$H(x) = \prod_v \max(|x_0|_v, \ldots, |x_n|_v)$$

where the product ranges over all the places $v$ of $k$.

In order to define a distance function we fix a place $v$ of $k$ and extension (which we also call $v$) to $\overline{k}$.

If $v$ is archimedean: We choose a distance function on $X(\overline{k})$ by choosing an embedding $X \hookrightarrow \mathbb{P}^n_k$ defined over $k$, and pulling back (via $v$) the distance function on $\mathbb{P}^n(\mathbb{C})$ given by the Fubini-Study metric on $\mathbb{P}^n$. We denote this distance by $\text{dist}(\cdot, \cdot)$. We set $d_v(\cdot, \cdot) = \text{dist}(\cdot, \cdot)^m_v$ where $m_v = 1$ if $v$ is real and $m_v = 2$ if $v$ is complex. This distance function depends on the choice of embedding, but by [10, Proposition 2.1] any two embeddings give equivalent distance functions and the choice of embedding will not matter for the definition of $\alpha_x$.

If $v$ is non-archimedean: Again choose a projective embedding $X \hookrightarrow \mathbb{P}^n_k$ defined over $k$. If $x, y \in X(\overline{k})$, consider the corresponding projective coordinates $x = [x_0: \cdots : x_m], y = [y_0: \cdots : y_m]$, and set $d_v(x, y) = \frac{H_v(x, y)}{H_v(x)H_v(y)}$ where $H_v$ is the local height at the place $v$ (this is the definition given in [1, 2.8.16] although we are using a different normalization for height than [1]).

This definition is somewhat opaque on first reading but is a compact way of stating a very concrete notion of $v$-adic distance: points $x$ and $y$ are close if the corresponding curves in an integral model of $X$ have high order of contact at the place $v$ (see e.g., [10, §2]). In other words, two points $x$ and $y$ are close if they are congruent modulo a high power of the maximal ideal $\mathfrak{m}_v$ of $\mathcal{O}_{k_v}$. For any fixed $x \in X(\overline{k})$, different embeddings give equivalent functions $d_v(x, \cdot)$, see [10, Corollary 2.3].
Definition 2.1. Let $X$ be a projective variety defined over a number field $k$, $L$ an ample line bundle defined over $k$, and $x \in X(\overline{k})$. Then we define $\alpha_x = \alpha_x(L)$ to be the unique extended real number $\alpha_x \in (0, \infty]$ such that the inequality

$$d_v(x, y)^{\alpha_x + \delta} < H_L(y)^{-1}$$

has only finitely many solutions $y \in X(k)$ (respectively infinitely many solutions $y \in X(k)$) for any $\delta < 0$ (respectively any $\delta > 0$).

The one essential change in our definition of $\alpha_x$ over $\tau_x$ is that we have moved the exponent from the height term to the distance term. As a result, for $x \in \mathbb{R} = \mathbb{A}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{R})$ we have $\alpha_x(\mathcal{O}_{\mathbb{P}^1}(1)) = \frac{1}{\tau_x}$. In particular the theorem of Liouville becomes $\alpha_x(\mathcal{O}_{\mathbb{P}^1}(1)) \geq \frac{1}{\delta}$ for $x \in \mathbb{R}$ of degree $d$ over $\mathbb{Q}$, and it is this type of lower bound that we wish to generalize to arbitrary varieties. The choice of moving the exponent is justified by the resulting formal similarity with the Seshadri constant, and more natural behaviour when we vary $L$ (see, for example, Proposition 2.11).

In proving results about $\alpha_x$ it is useful to have a characterization of $\alpha_x$ in terms of “test sequences”, and to associate an approximation constant to such a sequence.

Definition 2.2. Let $X$ be a projective variety, $x \in X(\overline{k})$, $L$ a line bundle on $X$. For any sequence $\{x_i\} \subset X(k)$ of distinct points with $d_v(x, x_i) \to 0$ (which we denote by $\{x_i\} \to x$), we set

$$A(\{x_i\}, L) = \{ \gamma \in \mathbb{R} \mid d_v(x, x_i)^{\gamma}H_L(x_i) \text{ is bounded from above} \}.$$ 

Remarks. (a) It follows easily from the definition that if $A(\{x_i\}, L)$ is nonempty then it is an interval unbounded to the right, i.e., if $\gamma \in A(\{x_i\}, L)$ then $\gamma + \delta \in A(\{x_i\}, L)$ for any $\delta > 0$.

(b) If $\{x'_i\}$ is a subsequence of $\{x_i\}$ then $A(\{x_i\}, L) \subseteq A(\{x'_i\}, L)$.

Definition 2.3. If $A(\{x_i\}, L)$ is empty we set $\alpha_x(\{x_i\}, L) = \infty$. Otherwise we set $\alpha_x(\{x_i\}, L)$ to be the infimum of $A(\{x_i\}, L)$. We call $\alpha_x(\{x_i\}, L)$ the approximation constant of $\{x_i\}$ with respect to $L$.

As $i \to \infty$ we have $d_v(x, x_i) \to 0$ and $H_L(x_i) \to \infty$. We thus expect that $d_v(x, x_i)^{\gamma}H_L(x_i)$ goes to $0$ for large $\gamma$ and to $\infty$ for small $\gamma$. The number $\alpha_x(\{x_i\}, L)$ marks the transition point between these two behaviours.

By remark (b) above if $\{x'_i\}$ is a subsequence of $\{x_i\}$ then $\alpha_x(\{x'_i\}, L) \leq \alpha_x(\{x_i\}, L)$. Thus we may freely replace a sequence with a subsequence when trying to establish lower bounds.

Proposition 2.4. Let $X$ be a projective variety defined over a number field $k$, $L$ an ample line bundle defined over $k$, and $x \in X(\overline{k})$. Then $\alpha_{x,L}$ is the infimum of all approximation constants of sequences of points in $X(k)$ converging to $x$. If no such sequence exists then $\alpha_{x,L}(L) = \infty$.

Proof: This is an elementary argument using sequences and the fact that if $L$ is ample there are only finitely many rational points of bounded height. For details see [10, Proposition 2.9]. □

The following lemma gives an equivalent local expression for the distance, useful for calculating with test sequences.
Lemma 2.5. Let $x$ be a point of $X(k)$ and let $U$ be an open affine of $X$ containing $x$. Let $u_1, \ldots, u_r$ be elements of $\Gamma(U, \mathcal{O}_X)$ which generate the maximal ideal of $x$. Then there are constants $c$ and $C$ such that

$$cd_v(x, y) \leq \min \left( 1, \max \left( |u_1(y)|_v, \ldots, |u_r(y)|_v \right) \right) \leq C d_v(x, y)$$

for all $y \in U(\overline{k})$. I.e., on $U(\overline{k})$ the function $\min(1, \max(|u_1(\cdot)|_v, \ldots, |u_r(\cdot)|_v))$ is equivalent to the function $d_v(x, \cdot)$.

Proof: See [10, Lemma 2.4]. □

We need two results on $\alpha_x$ before continuing onto the Seshadri constant. First, we will need to know how to calculate $\alpha_x$ in one simple case.

Lemma 2.6. Let $x \in \mathbb{P}^n(k)$. Then $\alpha_{x, \mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(1)) = 1$.

Proof: This is Lemma 2.11 from [10]. □

Second, it will be useful to know how the approximation constant changes when we change the field $k$. We use the notation that for an extension field $K/k$, $\alpha_x(\{x_i\}, L)_K$ (respectively $\alpha_{x}(L)_K$) denotes the approximation constant of a sequence (resp. point $x$) computed with respect to $K$. This means that when computing $\alpha$, we use the height $H_L$ relative to $K$ and normalize $d_v$ relative to $K$. If $d = [K:k]$ and $m_v = [K_v:k_v]$ (where $K_v$ and $k_v$ denote the completions of $K$ and $k$ with respect to $v$) then this means simply that $H_L(x_i)_K = H_L(x_i)_k^d$ and $d_v(x, x_i)_K = d_v(x, x_i)_k^{m_v}$.

Proposition 2.7. Suppose $x \in X(\overline{k})$, $L$ a line bundle defined over $k$, and $\{x_i\} \to x$ a sequence of points in $X(k)$ approximating $x$. Let $K$ be any finite extension of $k$. Then $\{x_i\} \to x$ can also be considered to be a set of points of $X(K)$ approximating $x$. Set $m_v = [K_v:k_v]$, and let $d = [K:k]$. Then

$$\alpha_x(\{x_i\}, L)_K = \frac{d}{m_v} \alpha_x(\{x_i\}, L)_k.$$ 

In particular, we have the bound $\alpha_x(L)_K \leq \frac{d}{m_v} \alpha_x(L)_k$.

Proof: The claim that $\alpha_x(\{x_i\}, L)_K = \frac{d}{m_v} \alpha_x(\{x_i\}, L)_k$ follows immediately from the equalities $H_L(\cdot)_K = H_L(\cdot)_k^d$ and $d_v(\cdot, \cdot)_K = d_v(\cdot, \cdot)_k^{m_v}$. The inequality $\alpha_x(L)_K \leq \frac{d}{m_v} \alpha_x(L)_k$ then follows since the sequences of $k$-points approximating $x$ are a subset of the sequences of $K$-points approximating $x$. □

Remark. Let $x$ be a point of $X(\overline{k})$ and let $K$ be the field of definition of $x$. If $K \not\subset k_v$, or equivalently, $K_v \neq k_v$ then it will be impossible to find a sequence of points of $X(k)$ converging (in terms of $d_v$) to $x$. For example, when $v$ is archimedean this happens when $k_v = \mathbb{R}$ and $K_v = \mathbb{C}$. Thus, if we can approximate $x$ by points of $X(k)$ we may assume that $K_v = k_v$ and so $m_v = 1$.

The following result (appearing in [10] as Theorem 2.14, and incorrectly in [9] as Theorem 2.8) is obtained by combining the Roth and Dirichlet theorems for approximation on $\mathbb{P}^1$, as well as the local information about the singularity type, shows how to calculate $\alpha_x$ on any singular $k$-rational curve.

Theorem 2.8. Let $C$ be any singular $k$-rational curve and $\varphi: \mathbb{P}^1 \to C$ the normalization map. Then for any ample line bundle $L$ on $C$, and any $x \in C(\overline{k})$ we have the equality:

$$\alpha_{x, C}(L) = \min_{q \in \varphi^{-1}(x)} \frac{d}{r_q m_q}.$$
where \( d = \text{deg}(L) \), \( m_q \) is the multiplicity of the branch of \( C \) through \( x \) corresponding to \( q \), and
\[
    r_q = \begin{cases} 
        0 & \text{if } \kappa(q) \neq k_v \\
        1 & \text{if } \kappa(q) = k_v \\
        2 & \text{otherwise}.
    \end{cases}
\]

Here \( \kappa(q) \) means the residue field of the point \( q \), and we use \( r_q = 0 \) as a shorthand for \( d/r_qm_q = \infty \).

**The Seshadri constant.** The Seshadri constant was introduced by Demailly in \([2]\) for the purposes of measuring the local positivity of a line bundle.

**Definition 2.9.** Let \( X \) be a projective variety, \( x \) a point of \( X \), and \( L \) a nef line bundle on \( X \). The Seshadri constant, \( \epsilon_x(L) \), is defined to be
\[
    \epsilon_x(L) := \sup \{ \gamma \geq 0 \mid \pi^*L - \gamma E \text{ is nef} \}
\]
where \( \pi : \tilde{X} \rightarrow X \) is the blowup of \( X \) at \( x \), with exceptional divisor \( E \).

In the discussion of Conjecture 4.2 below we will need the following alternate characterization of the Seshadri constant:

**Proposition 2.10.** With the same setup as definition 2.9,
\[
    \epsilon_x(L) = \inf_{x \in C \subseteq X} \left\{ \frac{(L \cdot C)}{\text{mult}_x(C)} \right\}
\]
where the infimum is taken over all reduced irreducible curves \( C \) passing through \( x \).

**Proof:** This is \([6, \text{Proposition 5.15}]\). \qed

In order to indicate the parallels between \( \alpha_x \) and \( \epsilon_x \), and for use below, we list a few of their formal properties here.

**Proposition 2.11.** Let \( X \) be a projective variety defined over \( k \), \( x \in X(\overline{k}) \), and let \( L \) be any ample line bundle on \( X \) (also defined over \( k \), following our conventions above).

(a) For any positive integer \( m \), \( \alpha_x(mL) = m\alpha_x(L) \) and \( \epsilon_x(mL) = m\epsilon_x(L) \). (Thus \( \alpha \) and \( \epsilon \) also make sense for ample \( \mathbb{Q} \)-divisors.)

(b) \( \alpha_x \) and \( \epsilon_x \) are concave functions of \( L \): for any positive rational numbers \( a \) and \( b \), and any ample \( \mathbb{Q} \)-divisors \( L_1 \) and \( L_2 \) (again defined over \( k \)) we have
\[
    \alpha_x(aL_1 + bL_2) \geq a\alpha_x(L_1) + b\alpha_x(L_2) \quad \text{and} \quad \epsilon_x(aL_1 + bL_2) \geq a\epsilon_x(L_1) + b\epsilon_x(L_2)
\]

(c) If \( Z \) is a subvariety of \( X \) defined over \( k \) then for any point \( z \in Z(\overline{k}) \) we have \( \alpha_x(L|_Z) \geq \alpha_x(L) \) and \( \epsilon_x(L|_Z) \geq \epsilon_x(L) \).

(d) If \( Y \) is also a variety defined over \( k \), \( x \in X(k) \), \( y \in Y(k) \) and \( L_X \) and \( L_Y \) are nef line bundles defined on \( X \) and \( Y \) respectively then
\[
    \alpha_{x,y,x \times y}(L_X \boxtimes L_Y) = \min(\alpha_{x,X}(L_X), \alpha_{y,Y}(L_Y))
\]
and
\[
    \epsilon_{x,y,x \times y}(L_X \boxtimes L_Y) = \min(\epsilon_{x,X}(L_X), \epsilon_{y,Y}(L_Y)).
\]
Note that by $L_X \boxplus L_Y$ we mean the line bundle $pr_X^*L_1 + pr_Y^*L_2$ on $X \times Y$, where $pr_X$ and $pr_Y$ are the projections. We prefer additive notation for line bundles since this is in line with the behaviour of $\alpha_x$ and $\epsilon_x$, and hence use $L_X \boxplus L_Y$ rather than $L_1 \boxplus L_2$.

Proof: All the proofs follow from elementary arguments using the definitions. For the statements about $\alpha_x$ see [10, Proposition 2.12], and for the statements about $\epsilon_x$ see [10, Proposition 3.4]. \hfill \square

3. A Liouville lower bound for $\alpha$

In this section, as in the previous one, we fix a number field $k$ and let $X$ be a projective variety defined over $k$.

Lemma 3.1. Let $x$ be a point of $X(k)$, and $\pi: \tilde{X} \to X$ the blow up of $X$ at $x$ with exceptional divisor $E$. Choose an embedding $\varphi:X \to \mathbb{P}^n$ so that $x \mapsto [1:0: \cdots :0]$. Let $Z_0, \ldots, Z_n$ be the coordinates on $\mathbb{P}^n$ and define functions $u_i$, $i = 1, \ldots, n$ on the open subset where $Z_0 \neq 0$ by $u_i = Z_i/Z_0$.

For each place $w$ of $k$, define a function $e_w:X(k) \to \mathbb{R}_{\geq 0}$ by

$$e_w(y) = \begin{cases} 1 & \text{if } Z_0(y) = 0, \\ \min (1, \max(|u_1(y)|_w, \ldots, |u_n(y)|_w)) & \text{if } Z_0(y) \neq 0. \end{cases}$$

Then

(a) $e_w \leq 1$ for all places $w$.
(b) $e_w$ is equivalent to $d_v$.
(c) For $y \in X(k)$, $y \neq x$, we have $H_E(y) = (\prod_w e_w(y))^{-1}$.

Proof: Part (a) is clear from the definition. Part (b) is precisely Lemma 2.5. In (c) we are considering points $y \in X(k)$, $y \neq x$ also to be points of $\tilde{X}(k)$ via the birational map $\pi$. To prove (c) it suffices, by using the functoriality of heights under pullback, to consider the case that $X = \mathbb{P}^n$. Then the blow up $\mathbb{P}^n_\varphi$ of $\mathbb{P}^n$ at $x$ is a subvariety of $\mathbb{P}^n \times \mathbb{P}^{n-1}$ and $\mathcal{O}_{\mathbb{P}^n}(E)$ is the restriction of $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^{n-1}}(1, -1)$ to $\mathbb{P}^n_\varphi$. From this description of $\mathcal{O}_{\mathbb{P}^n}(E)$ we obtain the formula

$$H_E(y) = \prod_w \frac{\max(|Z_0(y)|_w, |Z_1(y)|_w, \ldots, |Z_n(y)|_w)}{\max(|Z_1(y)|_w, \ldots, |Z_n(y)|_w)}$$

from which (c) follows easily. \hfill \square

Lemma 3.2. Suppose that $x \in X(k)$ and let $\pi: \tilde{X} \to X$ be the blow up at $x$ with exceptional divisor $E$. Let $L$ be an ample line bundle on $X$ and $\gamma > 0$ a rational number such that $L_\gamma := \pi^*L - \gamma E$ is in the effective cone of $\tilde{X}$. Let $B'$ be the asymptotic base locus of $L_\gamma$ and set $B = \pi(B')$.

Then for any sequence $\{x_i\} \to x$ such that all points of $\{x_i\}$ are outside of $B$, $\alpha(\{x_i\}, L) \geq \gamma$.

Proof: Let $U = \tilde{X} \setminus B'$. Since $B'$ is the asymptotic base-locus of $L_\gamma$ there is a constant $c$ so that $H_{L_\gamma}(y) \geq c$ for all $y \in U(k)$. Applying Lemma 3.1 we then have

$$c \leq H_{L_\gamma}(x_i) = H_L(x_i)H_E(x_i)^{\gamma} \leq c H_L(x_i) \left( \prod_w e_w(x_i) \right)^{\gamma} \leq H_L(x_i) e_v(x_i)^\gamma.$$

By Lemma 3.1(b) \( d_\alpha(x, x_i) \) and \( e_\varepsilon(x_i) \) are equivalent functions on \( X(k) \) and therefore \( H_L(x_i)d_\alpha(x, x_i)^\gamma \geq c' \) for some positive constant \( c' \).

For any \( \delta > 0 \) we thus have \( H_L(x_i)d_\alpha(x, x_i)^\gamma - \delta \geq c'd_\alpha(x, x_i)^{-\delta} \) and so conclude that \( \gamma - \delta \notin A(\{x_i\}, L) \) since \( c'd_\alpha(x, x_i)^{-\delta} \to \infty \) as \( i \to \infty \). Therefore \( \gamma \leq \alpha(\{x_i\}, L) \). \( \square \)

The main result of this section is the following implication of Lemma 3.2.

**Theorem 3.3.** Let \( X \) be an algebraic variety defined over \( k \), \( x \in X(\overline{k}) \) any point, and set \( d = [K : k] \) where \( K \) is the field of definition of \( x \).

Let \( \pi : \tilde{X} \to X \) be the blowup of \( X \) at \( x \), with exceptional divisor \( E \), \( L \) an ample line bundle on \( X \), and \( \gamma > 0 \) a rational number such that \( L_\gamma := \pi^*L - \gamma E \) is in the effective cone of \( \tilde{X} \). Finally let \( B' \) be the asymptotic base locus of \( L_\gamma \) and set \( B = \pi(B') \). Then

(a) For any sequence \( \{x_i\} \to x \) of \( k \)-points approximating \( x \) if infinitely many points of \( \{x_i\} \) are outside \( B \) then \( \alpha(\{x_i\}, L) \geq \gamma/d \).

(b) If \( \alpha_x(L) < \gamma/d \) then \( x \in B \) and \( \alpha_x(L) = \alpha_x(L|_B) \).

(c) If \( x \in B \) and \( \alpha_x(L|_B) \geq \gamma/d \) then \( \alpha_x(L) \geq \gamma/d \).

Note that \( \pi, \tilde{X}, E, \) and \( B' \) are only defined over \( K \). However since \( \pi \) is a morphism of \( k \)-schemes, \( B \) is defined over \( k \).

**Proof:** Let \( \{x_i\} \) be a sequence approximating \( x \). If infinitely many \( x_i \) lie outside of \( B \) then we may pass to the subsequence of points outside of \( B \), which could only have the effect of lowering the approximation constant of the sequence. To prove part (a) we may therefore assume that all points of \( \{x_i\} \) lie outside \( B \). Applying Lemma 3.2 to estimate the approximation constant computed relative to \( K \) we conclude that \( \alpha(\{x_i\}, L)_K \geq \gamma \). Since there is a sequence of \( k \)-points approximating \( x \) we conclude by the remark on page 4 that (in the notation of Proposition 2.7) \( m_v = 1 \). Therefore by Proposition 2.7 \( \alpha(\{x_i\}, L)_K = \frac{1}{d} \alpha(\{x_i\}, L)_K \geq \gamma/d, \) proving (a).

If \( \alpha_x(L) < \gamma/d \) then there must be a sequence \( \{x_i\} \) approximating \( x \) such that \( \alpha(\{x_i\}, L) < \gamma/d \). By part (a) this implies that all but finitely many \( x_i \) lie in \( B \). Thus \( x \in B \) since \( B \) is closed. Since omitting finitely many elements of a sequence does not change the approximation constant we may assume that all \( x_i \) are contained in \( B \). Since \( \alpha_x(L) \) is the infimum of the approximation constants for sequences \( \{x_i\} \) with \( \alpha(\{x_i\}, L) < \gamma/d \) we conclude that \( \alpha_x(L) = \alpha_x(L|_B) \) proving (b).

If \( \alpha_x(L) < \gamma/d \) then part (b) along with the hypothesis for part (c) lead to an immediate contradiction. Thus, under the hypotheses of part (c), \( \alpha_x(L) \geq \gamma/d \). \( \square \)

**Remark.** Theorem 3.3 still holds if we replace \( B \) by the Zariski closure of \( B(k) \). This has the added advantage that every component of \( B \) is then absolutely irreducible (see [10, Lemma 2.15]).

**Corollary 3.4.** For all ample line bundles \( L \) on \( X \) we have \( \alpha_x(L) \geq \varepsilon_x(L)/d \).

**Proof:** Let \( \pi : \tilde{X} \to X \) be the blow up of \( X \) at \( x \). By the definition of \( \varepsilon_x(L) \) for all rational \( \gamma \) satisfying \( 0 < \gamma < \varepsilon_x(L) \) the line bundle \( \pi^*L - \gamma E \) is ample on \( \tilde{X} \) and in particular the asymptotic base locus of \( \pi^*L - \gamma E \) is empty. Thus by Theorem 3.3(a) we conclude that \( \alpha_x(L) \geq \gamma/d \) for any such \( \gamma \), and hence that \( \alpha_x(L) \geq \varepsilon_x(L)/d \). \( \square \)
Remark. If $X = \mathbb{P}^1$ then Corollary 3.4 and the fact that $\epsilon_x(\mathcal{O}_{\mathbb{P}^1}(1)) = 1$ give $\alpha_x(\mathcal{O}_{\mathbb{P}^1}(1)) \geq 1/d$. Thus on $\mathbb{P}^1$ Corollary 3.4 amounts to the classic Liouville bound $\tau_x \leq d$. For this reason we consider Theorem 3.3 and Corollary 3.4 to be “Liouville bounds” for $\alpha_x$.

The effective cone is usually larger than the ample cone, and in general the parts of Theorem 3.3 imply a much stronger lower bound for $\alpha_x(L)$ than Corollary 3.4. We will use this in the next section to compute $\alpha$ for the cubic surface, but give a brief illustration now by calculating $\alpha$ for rational points of a non-split quadric surface in $\mathbb{P}^3$. (For a split quadric surface $\alpha_x(\mathcal{O}_{\mathbb{P}^1}(a,b)) = \min(a,b)$ when $a,b > 0$, as implied by Proposition 2.11(d) and computed in both [9, Theorem 3.1] and [10, §2; Example (c)].)

Example. Let $X$ be a smooth quadric surface in $\mathbb{P}^3$ defined over $k$, and set $L = \mathcal{O}_{\mathbb{P}^3}(1)|_X$. We assume that no lines on $X$ are defined over $k$. Let $x$ be a $k$-point of $X$. By intersecting with a (rationally defined) hyperplane we may find a conic $C$ passing through $x$ such that $C$ is isomorphic to $\mathbb{P}^1$ over $k$. By Lemma 2.6 and Proposition 2.11(a,c), we therefore have $\alpha_{x,X}(L) \leq \alpha_{x,\mathbb{P}^1}(L|_C) = \alpha_{x,\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(2)) = 2$. Since $x$ lies on a line (over $\overline{k}$), we have $\epsilon_x(L) = 1$, and applying Corollary 3.4 we obtain $\alpha_x(L) \geq 1$. Thus $1 \leq \alpha_x(L) \leq 2$, i.e., Corollary 3.4 does not give enough information to determine $\alpha_x(L)$ in this case.

However, let $\pi: \widetilde{X} \rightarrow X$ the blow up of $X$ at $x$ with exceptional divisor $E$. Then $\pi^*L - 2E$ is effective with base locus the proper transform of the two lines passing through $x$. In particular the image $B$ of this base locus is the union of the two lines of ruling passing through $x$. Since (by assumption) neither of these lines is defined over $k$, $x$ is the only $k$-point of $B$. Thus by Theorem 3.3(a) if $\{x_i\}$ is any sequence of $k$-points approximating $x$ then $\alpha(\{x_i\}, L) \geq 2$, and in particular $\alpha_x(L) \geq 2$. Thus $\alpha_x(L) = 2$ for all $k$-points of $X$.

Since $X$ is non-split the Picard group of $X$ (over $k$) has rank one with generator $L$. Thus the above computation and the homogeneity in Proposition 2.11(a) determines $\alpha$ for all $x \in X(k)$ and all ample line bundles on $X$ defined over $k$.

4. The cubic surface

In this section, we will compute $\alpha_x$ and $\epsilon_x$ for all $k$-rational points $x$ on the blowup $X$ of $\mathbb{P}^2$ at six $k$-rational points in general position.

To begin, we will recall some notions from [9].

Definition 4.1. A sequence $\{x_i\} \rightarrow x$ whose approximation constant is equal to $\alpha_x(L)$ (if such a sequence exists) is called a sequence of best approximation to $x$. A curve $C$ passing through $x$ is called a curve of best approximation (with respect to $L$) if $C$ contains a sequence of best approximation to $x$.

In other words, if $C$ is a curve of best approximation to $x$ on $X$, then the rational points on $C$ approximate $x$ roughly as well as the rational points on $X$ approximate $x$.

In the example of the non-split quadric — and in many others considered in [9] — there is always a curve of best approximation to $x$. In [9, §4] it is shown that if Vojta’s main conjectures are true, then $\alpha_x(L)$ finite implies that $\alpha_x(L)$ is computed on a subvariety $V \subseteq X$ of negative Kodaira dimension (possibly $X$ itself, if $X$ has negative Kodaira dimension). Since varieties of negative Kodaira dimension are (again, conjecturally) covered by rational curves, one is led to the following further prediction ([9, Conjecture 2.7]):
Conjecture 4.2. Let $X$ be an algebraic variety defined over $k$, and $L$ any ample divisor on $X$. Let $x$ be any $k$-rational point on $X$ and assume that there is a rational curve defined over $k$ passing through $x$. Then there exists a curve $C$ (necessarily rational) of best approximation to $x$ on $X$ with respect to $L$.

In [9], the first author proves this conjecture in many cases, and shows that in many others it follows from Vojta's Conjecture. Those proofs use a slightly different definition of $\alpha$, but the proofs do not essentially change in the new setting.

The Seshadri-constant analogue of a curve of best approximation is called a Seshadri curve (cf. Proposition 2.10):

Definition 4.3. Let $L$ be a nef divisor on an algebraic variety $X$, and $x \in X$ any point. A Seshadri curve for $x$ with respect to $L$ is a curve $C$ such that $\epsilon_{x,X}(L) = (L \cdot C)/\text{mult}_x(C)$.

In all currently known examples, there exists a Seshadri curve for $x$ with respect to $L$, but it is conjectured that this is not always the case. In particular, it is possible that the Seshadri constant might sometimes be irrational (see [6, Remark 5.1.13]).

It is useful to know that for a fixed curve $C$, the set of line bundles for which $C$ is a curve of best approximation form a subcone of the Néron-Severi group, and similarly for the property of being a Seshadri curve.

Proposition 4.4. Let $X$ be a variety defined over $k$, and let $x \in X(k)$ be any $k$-rational point. Let $D_1$ and $D_2$ be nef divisors on $X$ with height functions $H_1$ and $H_2$ bounded below by a positive constant in some neighbourhood of $x$. Let $a_1$ and $a_2$ be non-negative integers, and let $D = a_1D_1 + a_2D_2$.

(a) If $C$ is a curve of best approximation for $D_1$ and $D_2$, then $C$ is also a curve of best approximation for $D$.

(b) If $C$ is a Seshadri curve for $x$ with respect to $D_1$ and $D_2$, then $C$ is also a Seshadri curve for $x$ with respect to $D$.

Proof: Part (a) appears as [9, Corollary 3.2]. To prove part (b), note that Proposition 2.11(b) implies the estimate

$$\epsilon_x(a_1D_1 + a_2D_2) \geq \epsilon_x(D_1) + \epsilon_x(D_2).$$

On the other hand, the hypotheses of part (b) give

$$\frac{C \cdot D}{\text{mult}_x C} = \frac{C \cdot (a_1D_1 + a_2D_2)}{\text{mult}_x C} = \frac{a_1(C \cdot D_1)}{\text{mult}_x C} + \frac{a_2(C \cdot D_2)}{\text{mult}_x C} = \epsilon_x(D_1) + \epsilon_x(D_2).$$

Thus, by Proposition 2.10, $\epsilon_x(D_1) + \epsilon_x(D_2)$ is an upper bound for $\epsilon_x(D)$. Therefore $\epsilon_x(a_1D_1 + a_2D_2) = \epsilon_x(D_1) + \epsilon_x(D_2)$ and $C$ is a Seshadri curve for $D$, proving (b).

We are now ready to begin the proof of the main result of this section. Before we state and prove the general result, we will illustrate the fundamental techniques in the case $L = -K$.

Theorem 4.5. Let $X$ be a smooth cubic surface in $\mathbb{P}^3$ defined over $k$, and isomorphic over $k$ to the blowup of $\mathbb{P}^2$ at six $k$-rational points in general position. Let $x \in X(k)$ be any $k$-rational point, and let $C_x$ be the curve of intersection of $X$ with the tangent plane to $X$ at $x$. Then

$$\epsilon_x(-K) = \begin{cases} 1 & \text{if } x \text{ lies on one of the 27 lines of } X \\ \frac{3}{2} & \text{otherwise} \end{cases}$$
while
\[
\alpha_x(-K) = \begin{cases} 
1 & \text{if } x \text{ lies on one of the 27 lines of } X \\
3 & \text{if } x \text{ is not on one of the 27 lines, and if either} \\
\quad \circ \ C_x \text{ is cuspidal at } x, \text{ or} \\
\quad \circ \ C_x \text{ is nodal at } x \text{ with tangent lines having slopes} \\
\quad \quad \text{in } k_v \text{ but not } k \\
2 & \text{otherwise} \\
\quad (i.e., C_x \text{ is nodal at } x, \text{ and the slopes of the tangent lines} \\
\quad \quad \text{are in } k \text{ or not in } k_v.) 
\end{cases}
\]

Proof: Set \( L = -K = \mathcal{O}_{\mathbb{P}^3}(1)|_X \), and let \( x \) be a point of \( X(k) \). If \( x \) lies on a line \( \ell \) then by Proposition 2.11(c) we have \( \epsilon_{x,\ell}(L|_\ell) \geq \epsilon_{x,X}(L) \geq \epsilon_{x,\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(1)) \). Since \( \epsilon_{x,\ell}(L|_\ell) = \epsilon_{x,\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(1)) = 1 \), we conclude that \( \epsilon_x(L) = 1 \). Similarly (again using Proposition 2.11(c)) we conclude that \( \alpha_x(L) = 1 \).

We now suppose that \( x \) does not lie on a line. Let \( \pi: Y \to X \) be the blowup of \( X \) at \( x \), with exceptional divisor \( E \). Then \( C_x \) is a Seshadri curve for \( x \) with respect to \( L \). To see this, note first that \( C_x \) satisfies \( C_x.L/\text{mult}_x(C_x) = 3/2 \), so \( \epsilon_x(L) \leq 3/2 \). Conversely, if \( a > 3/2 \), then \( \pi^*L - aE \) is not nef, because \( (\pi^*L - aE)(\pi^*L - 2E) = 3 - 2a < 0 \) and \( \pi^*L - 2E \) is the class of the proper transform of \( C_x \). Thus, \( \epsilon_x(L) \geq 3/2 \), implying \( \epsilon_x(L) = 3/2 \), and \( C_x \) is a Seshadri curve for \( x \) with respect to \( L \).

We now turn to the computation of \( \alpha \). The asymptotic base locus of \( \pi^*L - 2E \) is \( \widetilde{C}_x \), the proper transform of \( C_x \). Hence by Theorem 3.3(b) either \( \alpha_x(L) \geq 2 \) or \( \alpha_x(L) = \alpha_{x,C_x}(L|_{C_x}) \) (note that \( d = 1 \)). By intersecting \( X \) with a hyperplane containing \( x \) and one of the lines, we produce a \( k \)-rational conic passing through \( x \), and approximating on the conic gives us \( 2 \geq \alpha_x(L) \). We therefore conclude that \( \alpha_x(L) = \min(2, \alpha_{x,C_x}(L|_{C_x})) \).

The curve \( C_x \) is singular at \( x \), and since \( x \) does not lie on a line, \( C_x \) also irreducible. In particular, \( C_x \) is an irreducible curve of geometric genus zero, and since \( x \) is defined over \( k \), \( C_x \) is birational to \( \mathbb{P}^1 \) over \( k \), via projection from \( x \) in the tangent plane.

Applying Theorem 2.8 to \( C_x \), we find that
\[
\alpha_{x,C_x}(L|_{C_x}) = \begin{cases} 
2 & \text{if } C_x: \circ \text{ is cuspidal, or} \\
\quad \circ \text{ is nodal and the tangent lines have slopes in } k_v \\
\quad \text{but not in } k \\
3 & \text{if } C_x \text{ is nodal and the slopes of the tangent lines are in } k \\
\infty & \text{if } C_x \text{ is nodal and the slopes of the tangent lines are not in } k_v
\end{cases}
\]

and this implies the stated values of \( \alpha_x(L) \) above. \( \Box \)

We now treat the case of a general nef divisor \( D \). In what follows, we assume that the point \( x \) does not lie on a \((-1)\)-curve on \( X \). We begin with a calculation of the Seshadri constant \( \epsilon \). To do this, we will need some notation.

Let \( \phi: X \to \mathbb{P}^2 \) be the blowing down map, and let \( E_1, \ldots, E_6 \) be the exceptional divisors of \( \phi \). We define the following linear equivalence classes on \( X \):

\begin{itemize}
  \item \( L = \phi^*\mathcal{O}(1) \)
\end{itemize}
• \( L_i = L - E_i \), the strict transform of a line through \( P_i = \phi(E_i) \)
• \( L_{ij} = 2L - (\sum E_n) + E_i + E_j \), the strict transform of a conic through the four points \( P_i \) with \( n \neq i, j \)
• \( B_i = 3L - (\sum E_n) - E_i \), the strict transform of a cubic curve through all six points \( P_n \), with a node at \( P_i \).

Let \( h \) be the class of a hyperplane in the anticanonical embedding \( X \in \mathbb{P}^3 \). For any line \( \ell \) on \( X \), the hyperplanes containing \( \ell \) give (after removing \( \ell \)) a base-point-free pencil on \( X \). If \( x \in X \) does not lie on a line then the unique curve in this pencil through \( x \) is smooth and irreducible. The classes \( \{L_i, L_{ij}, B_i\} \) defined above are the 27 pencils coming from the lines. Recall that for any point \( x \) on \( X \) we use \( C_x \) for the intersection of \( X \) with its tangent plane at \( x \) (so \( C_x \) has class \( h \)). If \( x \) does not lie on a line, then \( C_x \) is a plane cubic curve with one double point, at \( x \).

**Theorem 4.6.** Let \( x \) be a point on \( X \) that does not lie on a \((-1)\)-curve, and let \( D \) be a nef divisor on \( X \). The Seshadri constant \( \epsilon_x(D) \) is equal to \( \min \{D.L_i, D.L_{ij}, D.B_i, (D.h)/2\} \).

**Proof:** The nef cone \( \Gamma \) of \( X \) has 99 generators, which are listed in §5, Table 1. Let \( S \) be the set of 27 divisor classes \( \{L_i, L_{ij}, B_i\} \) as \( i \) and \( j \) range over all possible values, and for each element \( C \in S \), we define the subcone \( \Gamma(C) \) by:

\[
\Gamma(C) = \left\{ D \in \Gamma | D.C = \min_{C' \in S} \{D.C'\} \text{ and } D.C \leq (D.h)/2 \right\}.
\]

Further define the subcone \( \Gamma(h) \) to be:

\[
\Gamma(h) = \left\{ D \in \Gamma | (D.h)/2 \leq \min_{C' \in S} \{D.C'\} \right\}.
\]

It is clear that \( \Gamma \) is the union of these 28 subcones. To prove Theorem 4.6, it suffices to show that for every subcone \( \Gamma(C) \), with \( C \in S \), the curve through \( x \) in the pencil corresponding to \( C \) is a Seshadri curve for \( x \) with respect to \( D \) for all \( D \in \Gamma(C) \) (respectively, in the case of the subcone \( \Gamma(h) \), that \( C_x \) is a Seshadri curve for \( x \) with respect to \( D \) for all \( D \in \Gamma(h) \)). By Proposition 4.4(b) it further suffices to prove this for \( D \) a generator of the cone \( \Gamma(C) \) (respectively \( \Gamma(h) \)).

The fundamental group of the space of all smooth cubic surfaces acts via monodromy on the Néron-Severi lattice of \( X \). This monodromy action preserves the hyperplane class \( h \) and acts transitively on the classes of the lines. Thus, up to monodromy action, there are only two of these subcones: \( \Gamma(L_1) \) and \( \Gamma(h) \). Generators for each of these subcones can also be found in §5. Let \( F = F_{x,L_1} \) be the unique curve in the pencil \( L_1 \) passing through \( x \). For each generator \( D \) of \( \Gamma(L_1) \), it is straightforward to verify that \( F \) is a Seshadri curve for \( x \) with respect to \( D \). These verifications also appear in §5. Each generator \( G \) of \( \Gamma(h) \) is also a generator of one of the other twenty-seven subcones \( \Gamma(C) \), and for each such \( G \), we have \( G.C = (G.h)/2 = (C.C_x)/\text{mult}_x C_x \). Thus, since \( C \) is a Seshadri curve for \( x \) with respect to \( G \), it follows that \( C_x \) is also a Seshadri curve for \( x \) with respect to \( G \), and so \( C_x \) is a Seshadri curve for every element of \( \Gamma(h) \). This concludes the proof. \( \square \)

The next step is to calculate \( \alpha_x \) for a point on a cubic surface.

**Theorem 4.7.** Let \( x \) be a point on \( X \) that does not lie on a \((-1)\)-curve, and let \( D \) be a nef divisor on \( X \). If the tangent curve \( C_x \) is a cuspidal cubic, or a nodal cubic whose tangent lines at \( x \) are defined over \( k \), but not defined over \( k \), then \( \alpha_x(D) = \epsilon_x(D) \). Otherwise, \( \alpha_x(D) = \min \{D.L_i, D.L_{ij}, D.B_i\} \).
we have seen that respectively. Thus is the divisor class 2 gives holds for every point. We use the notation from § way. For instance, Table with each of the 27 lines on the cubic surface. The other tables were generated in a similar

5. Appendix: Generators of nef cones and subcones for the cubic surface

A version of this appendix, with additional tables and larger font, may be found at [11]. We use the notation from §4. In each of the tables in this appendix the first column is a numerical identifier of the vector in that row. The subsequent columns represent the coefficients of the vector with respect to the basis \( \{ L, E_1, \ldots, E_6 \} \) of the Néron-Severi group of \( X \). Thus, vector number 1 in Table 1 is the divisor class 2 \( 2 \times 8 \alpha_x(D|_{F_{x,C}}) 2.11(c) \geq \alpha_x(D) \geq \epsilon_x(D) \geq D.C \),

where, reading from left to right, the equalities and inequalities are given by Theorem 2.8, Proposition 2.11(c), Corollary 3.4, and Theorem 4.5 respectively. Thus \( \alpha_x(D) = D.C \) and \( F_{x,C} \) is a curve of best approximation with respect to \( D \).

Now suppose that \( D \in \Gamma(h) \). If \( C_x \) is cuspidal, or nodal with tangent lines having slopes in \( k_v \) but not \( k \), then Theorem 2.8 gives \( \alpha_x(D|_{C_x}) = D.C_x/2 = D.C_x/\text{mult}_x C_x \). By Theorem 4.5, \( \epsilon_x(D) = D.C_x/2 \), and so as above we conclude that \( \alpha_x(D) = D.C_x = \epsilon_x(D) \), and that \( C_x \) is a curve of best approximation for \( D \).

We now assume that \( C_x \) is nodal and the slopes of the tangent lines are in \( k \) or not in \( k_v \). The codimension one faces of \( \Gamma(h) \) (i.e., the facets) occur where one of the inequalities defining \( \Gamma(h) \) becomes an equality, so that each facet is the intersection of \( \Gamma(h) \) and \( \Gamma(C) \) for some \( C \in S \). For each \( C \in S \) set \( \hat{\Gamma}(C) \) to be the cone generated by \( \Gamma(C) \) and \( -K \). Since \( -K \) is in the interior of \( \Gamma(h) \) it follows that \( \Gamma \) is the union of the \( \hat{\Gamma}(C), C \in S \).

For any \( C \in S \), let \( F_{x,C} \) be the member of the pencil corresponding to \( C \) passing through \( x \), as in the first part of the argument. In the proof of Theorem 4.5 we have seen that \( F_{x,C} \) is a curve of best approximation for \( -K \), and in the first part of the argument above that \( F_{x,C} \) is a curve of best approximation for all \( D \in \Gamma(C) \). By Proposition 4.4(a) we conclude that \( F_{x,C} \) is a curve of best approximation for all \( D \in \hat{\Gamma}(C) \). The result follows. \( \square \)

Note that as part of the proof we have shown that Conjecture 4.2 holds for every point \( x \in X \) not on a \((-1)\)-curve.
In Table 2 which follows, we use $D_n$ to refer to the divisor class represented by row $n$ of Table 2. For any point $x \in X$ not on a $(-1)$-curve, the unique curve $F = F_{x,L}$ in the pencil $L_1$ passing through $x$ is smooth and irreducible. In each line of the table “Reason” is a — very brief! — justification of why $F$ is a Seshadri curve for $x$ with respect to $D_n$.

For instance, in row 1 of Table 2, the “Reason” is $L_1.D_1 = 1$, and thus $F.D_1 = L_1.D_1 = 1$. We claim that for the divisors $D_i$, $\epsilon_x$ is always at least one if it is nonzero. To see this, notice that the generators of the nef cone (see Table 1) are all either morphisms to $\mathbb{P}^1$ corresponding to pencils of conics on the cubic surface, or else morphisms to $\mathbb{P}^2$ that are the blowing down of six pairwise disjoint $(-1)$-curves. In both cases, the Seshadri constant is easily seen to be either zero or at least one. It is straightforward to check that all the generators listed in Table 2 are non-negative integer linear combinations of the generators of the nef cone, and therefore (by Proposition 2.11(b)) enjoy the same property: for any point $x$, the Seshadri constant $\epsilon_x(D_i)$ is either zero or else is at least one.

By assumption, $x$ does not lie on any $(-1)$-curve, which are the only curves contracted by any $D_i$ (except for $D_{18} = L_1$, for which $\epsilon = 0$ for all points). Therefore, since $F$ has degree 1 with respect to $D_1$, $F$ is a Seshadri curve for $x$ with respect to $D_1$.

As a second example, in row 29 of Table 2, the comment “$L + L_{56}$” means that the divisor $D_{29}$ represented by that row is the sum of $L$ and $L_{56}$. Any curve that has nonzero
intersection with $L$ must have $L.C/\text{mult}_x(C) \geq 1$, for any $x$ not lying on a $(-1)$-curve, since $L$ is an isomorphism away from $(-1)$-curves. Similarly, any curve not contracted by $L_{56}$ must also satisfy $L_{56}.C/\text{mult}_x(C) \geq 1$, so any curve not contracted by $L_{56}$ or $L$ must satisfy $(L+L_{56}).C/\text{mult}_x(C) \geq 2$. If $C$ is contracted by $L_{56}$, then it is either a $(-1)$-curve, or else it is an element of the divisor class $L_{56}$ itself, in which case it satisfies $(L+L_{56}).C/\text{mult}_x(C) = 2$ by direct calculation. In all cases, since $x$ does not lie on a $(-1)$-curve, we see that $\epsilon_x(L+L_{56}) \geq 2$, and since $L_1.L = L_1.L_{56} = 1$, we conclude that $\epsilon_x(L+L_{56}) = 2$, and so the curve in the class $L_1$ through $x$ is a Seshadri curve for $x$ with respect to $D_{29} = L + L_{56}$. Similar arguments explain the other reasons of the form “$A + B$” or “$A + B + C$”.

In light of these arguments, for Table 2, it is useful to know that $L_1$ has intersection number one with the divisors $L, B_1, L_i$ for $i \neq 1$, and $L_{ij}$ for $i, j \neq 1$. 
Table 2: Generators of the cone $\Gamma(L_1)$

| # | $L_1$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | Reason |
|---|---|---|---|---|---|---|---|---|
| 1 | 4 | -3 | -1 | -1 | -1 | -1 | -1 | $L_1.D_1 = 1$ |
| 2 | 2 | -1 | -1 | 0 | 0 | 0 | 0 | $L_1.D_2 = 1$ |
| 3 | 2 | -1 | -1 | 0 | 0 | 0 | 0 | $L_1.D_3 = 1$ |
| 4 | 2 | -1 | 0 | -1 | 0 | 0 | 0 | $L_1.D_4 = 1$ |
| 5 | 2 | -1 | 0 | 0 | 0 | 0 | 0 | $L_1.D_5 = 1$ |
| 6 | 2 | -1 | 0 | 0 | 0 | 0 | -1 | $L_1.D_6 = 1$ |
| 7 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | $L_1.D_7 = 1$ |
| 8 | 3 | -2 | -1 | -1 | 0 | 0 | 0 | $L_1.D_8 = 1$ |
| 9 | 3 | -2 | -1 | -1 | 0 | 0 | 0 | $L_1.D_9 = 1$ |
| 10 | 3 | -2 | -1 | -1 | 0 | 0 | 0 | $L_1.D_{10} = 1$ |
| 11 | 3 | -2 | -1 | 0 | -1 | 0 | -1 | $L_1.D_{11} = 1$ |
| 12 | 3 | -2 | -1 | 0 | -1 | 0 | -1 | $L_1.D_{12} = 1$ |
| 13 | 3 | -2 | -1 | 0 | -1 | 0 | -1 | $L_1.D_{13} = 1$ |
| 14 | 3 | -2 | 0 | -1 | -1 | 0 | -1 | $L_1.D_{14} = 1$ |
| 15 | 3 | -2 | 0 | -1 | -1 | 0 | -1 | $L_1.D_{15} = 1$ |
| 16 | 3 | -2 | 0 | -1 | 0 | -1 | 0 | $L_1.D_{16} = 1$ |
| 17 | 3 | -2 | 0 | -1 | 0 | 0 | -1 | $L_1.D_{17} = 1$ |
| 18 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | $L_1.D_{18} = 0$ |
| 19 | 2 | -1 | -1 | -1 | 0 | 0 | 0 | $L_1.D_{19} = 1$ |
| 20 | 2 | -1 | -1 | 0 | -1 | 0 | 0 | $L_1.D_{20} = 1$ |
| 21 | 2 | -1 | -1 | 0 | 0 | -1 | 0 | $L_1.D_{21} = 1$ |
| 22 | 2 | -1 | -1 | 0 | 0 | -1 | 0 | $L_1.D_{22} = 1$ |
| 23 | 2 | -1 | 0 | 0 | 0 | 0 | 0 | $L_1.D_{23} = 1$ |
| 24 | 2 | -1 | 0 | 0 | 0 | 0 | 0 | $L_1.D_{24} = 1$ |
| 25 | 2 | -1 | -1 | 0 | 0 | 0 | 0 | $L_1.D_{25} = 1$ |
| 26 | 2 | -1 | 0 | -1 | 0 | 0 | 0 | $L_1.D_{26} = 1$ |
| 27 | 2 | -1 | 0 | -1 | 0 | 0 | 0 | $L_1.D_{27} = 1$ |
| 28 | 2 | -1 | 0 | -1 | 0 | 0 | 0 | $L_1.D_{28} = 1$ |
| 29 | 3 | -1 | -1 | -1 | 0 | 0 | 0 | $L + L_{56}$ |
| 30 | 3 | -1 | -1 | -1 | 0 | 0 | 0 | $L + L_{46}$ |
| 31 | 3 | -1 | -1 | -1 | 0 | 0 | 0 | $L + L_{45}$ |
| 32 | 3 | -1 | 0 | -1 | 0 | -1 | 0 | $L + L_{36}$ |
| 33 | 3 | -1 | 0 | -1 | 0 | -1 | 0 | $L + L_{35}$ |
| 34 | 3 | -1 | 0 | -1 | 0 | -1 | 0 | $L + L_{34}$ |
| 35 | 3 | -1 | 0 | -1 | 0 | -1 | 0 | $L + L_{26}$ |
| 36 | 3 | -1 | 0 | -1 | 0 | -1 | 0 | $L + L_{25}$ |
| 37 | 3 | -1 | 0 | -1 | 0 | -1 | 0 | $L + L_{24}$ |
| 38 | 3 | -1 | 0 | -1 | 0 | -1 | 0 | $L + L_{23}$ |
| 39 | 3 | -1 | 0 | -1 | 0 | -1 | 0 | $L + L_{22}$ |
| 40 | 3 | -1 | 0 | -1 | 0 | -1 | 0 | $L + L_{21}$ |
| 41 | 3 | -1 | -1 | -1 | 0 | 0 | 0 | $L + L_{24}$ |
| 42 | 3 | -1 | -1 | -1 | 0 | 0 | 0 | $L + L_{23}$ |
| 43 | 3 | -1 | -1 | 0 | -1 | 0 | 0 | $L + L_{21}$ |
| 44 | 4 | -1 | -1 | -1 | -1 | -1 | -1 | $L_{23} + L_{4} + L_{3}$ |
| 45 | 3 | -2 | -1 | -1 | -1 | -1 | -1 | $L_1.D_{45} = 1$ |
| 46 | 3 | -2 | -1 | -1 | -1 | -1 | -1 | $L_1.D_{46} = 1$ |
| 47 | 3 | -2 | -1 | -1 | 0 | -1 | 0 | $L_1.D_{47} = 1$ |
| 48 | 3 | -2 | -1 | 0 | -1 | -1 | 0 | $L_1.D_{48} = 1$ |
| 49 | 3 | -2 | 0 | -1 | -1 | -1 | 0 | $L_1.D_{49} = 1$ |
| 50 | 4 | -2 | -2 | -1 | -1 | -1 | -1 | $D_{42} + L_2$ |

In Table 3, the rightmost column of row $n$ contains a divisor class $C \in S$ such that $G_n$ (the divisor corresponding to the $n$th row of Table 3) is also a generator of the subcone $\Gamma(C)$. From the definition of the cones $\Gamma(C)$ and $\Gamma(h)$, this implies that $G_n.C = (G_n.h)/2$. As explained in the proof of Theorem 4.7, this provides a verification that $C_x$ is a Seshadri curve for $x$ with respect to $G_n$. 
Table 3: Generators of the cone $\Gamma(h)$

| #  | $L$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | Divisor Class |
|----|-----|-------|-------|-------|-------|-------|-------|---------------|
| 1  | 8   | -3    | -3    | -3    | -3    | -3    | -3    | $B_3$         |
| 2  | 4   | -1    | -1    | -1    | -1    | -1    | -1    | $L_1$         |
| 3  | 4   | -2    | -2    | -1    | -1    | -1    | -1    | $L_1$         |
| 4  | 4   | -2    | -1    | -2    | -1    | -1    | -1    | $L_1$         |
| 5  | 4   | -2    | -1    | -1    | -2    | -1    | -1    | $L_1$         |
| 6  | 4   | -2    | -1    | -1    | -2    | -1    | -1    | $L_1$         |
| 7  | 4   | -2    | -1    | -1    | -2    | -1    | -1    | $L_1$         |
| 8  | 4   | -1    | -2    | -2    | -1    | -1    | -1    | $L_2$         |
| 9  | 4   | -1    | -2    | -2    | -1    | -1    | -1    | $L_2$         |
| 10 | 4   | -1    | -2    | -1    | -2    | -1    | -1    | $L_2$         |
| 11 | 4   | -1    | -2    | -1    | -1    | -2    | -1    | $L_2$         |
| 12 | 4   | -1    | -2    | -2    | -1    | -1    | -1    | $L_2$         |
| 13 | 4   | -1    | -2    | -1    | -2    | -1    | -1    | $L_2$         |
| 14 | 4   | -1    | -1    | -2    | -1    | -2    | -1    | $L_3$         |
| 15 | 4   | -1    | -1    | -2    | -1    | -2    | -1    | $L_3$         |
| 16 | 4   | -1    | -1    | -1    | -2    | -1    | -1    | $L_3$         |
| 17 | 4   | -1    | -1    | -1    | -2    | -1    | -1    | $L_3$         |
| 18 | 5   | -3    | -2    | -1    | -1    | -2    | -1    | $L_4$         |
| 19 | 5   | -2    | -2    | -1    | -2    | -1    | -1    | $L_4$         |
| 20 | 5   | -2    | -1    | -1    | -2    | -2    | -1    | $L_4$         |
| 21 | 5   | -2    | -2    | -1    | -1    | -2    | -2    | $L_4$         |
| 22 | 5   | -2    | -1    | -2    | -1    | -2    | -2    | $L_4$         |
| 23 | 5   | -2    | -1    | -2    | -1    | -2    | -2    | $L_4$         |
| 24 | 5   | -2    | -2    | -1    | -1    | -2    | -2    | $L_4$         |
| 25 | 5   | -2    | -1    | -2    | -1    | -2    | -2    | $L_4$         |
| 26 | 5   | -2    | -1    | -2    | -1    | -2    | -2    | $L_4$         |
| 27 | 5   | -2    | -1    | -1    | -2    | -2    | -2    | $L_4$         |
| 28 | 5   | -1    | -2    | -2    | -1    | -1    | -1    | $L_5$         |
| 29 | 5   | -1    | -2    | -2    | -1    | -1    | -1    | $L_5$         |
| 30 | 5   | -1    | -2    | -1    | -1    | -1    | -1    | $L_5$         |
| 31 | 5   | -1    | -2    | -1    | -2    | -1    | -1    | $L_5$         |
| 32 | 5   | -1    | -2    | -1    | -2    | -1    | -1    | $L_5$         |
| 33 | 5   | -1    | -2    | -1    | -2    | -1    | -1    | $L_5$         |
| 34 | 5   | -1    | -2    | -1    | -2    | -1    | -1    | $L_5$         |
| 35 | 5   | -1    | -2    | -1    | -2    | -1    | -1    | $L_5$         |
| 36 | 5   | -1    | -2    | -1    | -2    | -1    | -1    | $L_5$         |
| 37 | 5   | -1    | -1    | -2    | -1    | -2    | -1    | $L_6$         |
| 38 | 5   | -1    | -1    | -2    | -1    | -2    | -1    | $L_6$         |
| 39 | 5   | -1    | -1    | -2    | -1    | -2    | -1    | $L_6$         |
| 40 | 5   | -1    | -1    | -2    | -1    | -2    | -1    | $L_6$         |
| 41 | 5   | -1    | -1    | -2    | -1    | -2    | -1    | $L_6$         |
| 42 | 5   | -1    | -1    | -2    | -1    | -2    | -1    | $L_6$         |
| 43 | 5   | -1    | -2    | -2    | -2    | -2    | -2    | $B_1$         |
| 44 | 5   | -1    | -2    | -2    | -2    | -2    | -2    | $B_1$         |
| 45 | 5   | -1    | -2    | -2    | -2    | -2    | -2    | $B_1$         |
| 46 | 5   | -1    | -2    | -2    | -2    | -2    | -2    | $B_1$         |
| 47 | 5   | -1    | -2    | -2    | -2    | -2    | -2    | $B_1$         |
| 48 | 5   | -1    | -2    | -2    | -2    | -2    | -2    | $B_1$         |
| 49 | 5   | -1    | -2    | -2    | -2    | -2    | -2    | $B_1$         |
| 50 | 5   | -1    | -2    | -2    | -2    | -2    | -2    | $B_1$         |

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