Adiabatic invariants for the regular region of the Dicke model

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Abstract
Adiabatic invariants for the non-integrable Dicke model are introduced. They are shown to provide approximate second integrals of motion in the energy region where the system exhibits a regular dynamics. This low-energy region, present for any set of values of the Hamiltonian parameters is described both with a semiclassical and a full quantum analysis in a broad region of the parameter space. Peres lattices in this region exhibit that many observables vary smoothly with energy, along distinct lines which beg for a formal description. It is demonstrated how the adiabatic invariants provide a rationale to their presence in many cases. They are built employing the Born–Oppenheimer approximation, valid when a fast system is coupled to a much slower one. As the Dicke model has one bosonic and one fermionic degree of freedom, two versions of the approximation are used, depending on which one is the faster. In both cases a noticeably accord with exact numerical results is obtained. The employment of the adiabatic invariants provides a simple and clear theoretical framework to study the physical phenomenology associated to these regimes, far beyond the energies where a quadratic approximation around the minimal energy configuration can be used.

Keywords: Dicke model, Born–Oppenheimer approximation, integrability

(Some figures may appear in colour only in the online journal)
1. Introduction

Eighteen years after Rabi proposed his model for the description of a two-level atom interacting with a single mode of radiation field inside a QED cavity [1], Dicke made the generalization for $N$ two-level atoms [2]. Shortly after their proposals, both models were approximated by simpler (and integrable) Hamiltonians by means of the rotating-wave approximation (RWA), i.e. the Jaynes-Cummings (JC) and the Tavis-Cummings (TC) models [3, 4]. In this work we focus on the Dicke model, which has been a topic of discussion for several years thanks to its most important trait: the prediction of the superradiant phase transition at finite temperature, which in the limit of zero temperature leads to a quantum phase transition (QPT) [5–8]. Along many years several authors have pointed out that this phase transition cannot be achieved in a QED cavity systems due to a no-go theorem [9], opening an active discussion, which is far from closed [10–15]. Despite objections to the QPT, the Dicke model remains of great interest in our days.

The Dicke Hamiltonian is composed of three terms, one describing a monochromatic quantized radiation field inside the cavity, a second one related with the collective atomic system, and a third one which describes the interaction between them. It reads (from now on we set $\hbar = 1$)

$$\hat{H}_D = \omega \hat{a}^{\dagger} \hat{a} + \omega_0 \hat{J}_z + \frac{2 \gamma}{\sqrt{N}} (\hat{a} + \hat{a}^{\dagger})\hat{J}_x .$$  \hspace{1cm} (1)

Here, the frequency of the radiation mode is $\omega$, associated with the number operator $\hat{a}^{\dagger} \hat{a}$. For the atomic part, $\omega_0$ is the excitation energy of the single two-level system, while $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ are collective atomic pseudospin operators obeying the $\text{su}(2)$ algebra. The $\hat{J}_z$ operator quantifies the relative atomic population. Besides, it holds that if $j(j + 1)$ is the eigenvalue of the $\text{su}(2)$ Casimir operator $\hat{J}_+^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$, then $j = N/2$ (the pseudospin length) defines the symmetric atomic subspace which includes the ground-state. Finally, in the case of cavity QED systems, the interaction parameter $\gamma$ depends principally on the atomic dipolar moment.

As mentioned before, the most representative feature of the model is its second-order QPT in the thermodynamic limit, a paradigmatic example of quantum collective behavior. When the atom-field interaction reaches the critical value $\gamma_c = \sqrt{\omega_0 \omega}/2$, its ground state goes from a normal phase ($\gamma < \gamma_c$), with no photons and no excited atoms, to a superradiant phase ($\gamma > \gamma_c$), where the number of photons and excited atoms becomes comparable to the total number of atoms in the system, i.e. a macroscopic population of the upper atomic level.

Being the simplest non-integrable atom-field system exhibiting quantum chaos [16, 17], the Dicke model became a paradigm in quantum optics first, and later in the quantum information community, as it describes more generally the interaction of a set of $N$ two-level systems (qubits) with a bosonic field, like quantum dots, Bose–Einstein condensates, and polaritons in circuit QED systems [18–22]. Thanks to the advance in sophisticated experimental techniques to control quantum systems, the superradiant QPT was observed experimentally in several systems during the last years. For the Dicke model it was first simulated by means of a Bose–Einstein condensate in an optical cavity [23, 24]. Also, an open version of the model was realized employing Raman transitions [25], and a dynamical non-equilibrium superradiant phase transition has been also observed [26]. Moreover, by means of superconducting QED, the Rabi model [27, 28] and few-atoms Dicke-like models [29, 30] have been likewise explored.

More than ten years ago, Emary and Brandes built approximate solutions of the Dicke model by means of the Holstein–Primakoff realization of the $\text{su}(2)$ algebra in the thermodynamic limit [17]. This approach describes the behavior of the model for the ground- and low
energy-states, exhibiting the superradiant QPT easily and showing the presence of chaos in the spectrum of the Hamiltonian. Besides, the truncated Holstein–Primakoff approach makes it possible to extract the critical exponents for the ground-state energy per particle, the fraction of excited atoms, the number of photons per atom, their fluctuations, and the concurrence [31, 32].

Enhanced by these experimental and theoretical results, and thanks to its algebraic properties, the Dicke model has become an excellent tool from the theoretical point of view for exploring several features of quantum many-body systems with collective degrees of freedom, e.g. the QPT, the excited-state quantum phase transitions (ESQPT) [33–36] and their relation with the thermal phase transition [37, 38], the onset of quantum chaos and its correspondence to the classical limit [39–42], quantum quenches [43, 44] and the problem of equilibrium and thermalization in isolated many-body quantum systems [45, 46], to name a few.

This is the motivation for looking for as many analytical or semi-analytical descriptions of the spectra and observables of the model as it is possible. The truncated Holstein–Primakoff approximation decouples the low energy modes into two independent harmonic oscillators. It is valid when the number of excitations is small compared with the number of atoms, allowing for the description of a few low-energy states far from the QPT [47, 48]. The lack of a general analytical description is rooted in the fact that the Dicke model does not have as many integrals of motion as degrees of freedom, being in this sense non-integrable, unlike the TC model. Numerical calculations, which take advantage of the Hamiltonian’s integrable limit $\omega_0 \to 0$, provide useful results in truncated subspaces [49–51].

On the other hand, it has been recently shown that, thanks to being restricted to the single qubit case, the Rabi model can be considered as integrable [52–54], though the controversy about this fact remains [55, 56]. A similar method has been recently applied to the Dicke model with three [53] and two atoms [57, 58]. However, a concise analytic solution of the Dicke model is far to be obtained, and in the general case is quite probably impossible.

In this work we construct second integrals of motion in the Dicke model employing the Born–Oppenheimer approximation (BOA) [59]. They are useful and predictive within the non-chaotic energy regime and within a wide range of values of the external parameters, coupling and frequencies, both in the normal and superradiant phases. The BOA validity relies on the fact that part of the system oscillates much faster than the other one, causing an effective decoupling where the slow (adiabatically changing) variables enters in the fast dynamics as simple parameters. As the Dicke model, restricted to a $j$-subspace, has only two degrees of freedom, one bosonic and one fermionic, two versions of the approximation can be used, depending on which one is the faster. Similar approaches have already been reported in the Rabi model, the Dicke model, as well as for the JC and TC modes. In the Rabi model the fast atomic BOA has been employed to determine the entanglement of a single atom with the bosonic field [60], and to unveil previously unnoticed aspects of the RWA in the JC model [61]. In the Dicke model the fast atomic BOA was applied to study its ground-state properties, like the finite size scaling of the entanglement between the components of the system and other physical observables [62]. Also, it has been used to study the finite size dependence of the tunneling driven ground-state energy splitting in the superradiant phase [63]. The fast boson BOA has been applied to the Rabi model in [64], and then extended to develop the so called generalized RWA [65]. Likewise, in [66, 67] the fast boson BOA has been employed to describe the different phases that can be found in the Dicke model realization employing Bose–Einstein condensates in optical cavities [23].

In this contribution, we extend the use of the BOA to higher energies in the Dicke Hamiltonian, and show that it can describe the regular, non-chaotic energy regime of the model, which extends from the ground-state up to an upper energy (above or below the
ESQPT critical energy) which depends strongly on the level splitting and the frequency of the field [68]. For both cases, the fast boson and fast atomic BOA, we explicitly derive the approximate second integral of motion that makes regular the low energy regime of the Dicke model, and show its range of applicability in the model parameter space. The BOA extends the theoretical tools to describe the, so far unexplained, whole non-chaotic or regular energy region of the Dicke model, far beyond the region of validity of the quadratic approximation of the Holstein–Primakoff description. We present solid numerical evidence showing that the BOA, and the second integrals of motion coming from it, shed new light in the study of the quantum dynamics of the model. This contribution extends the study presented in [69] by some of us, where the basic idea was introduced and complementary results were given for the fast pseudospin BOA.

The paper is organized as follows, in section 2 the fast pseudospin and the fast boson approximations are presented, and their low-energy frequencies are calculated. These frequencies are compared in section 3 with the exact ones, in order to establish the region of validity of the approximations in the Dicke model parameter space. In section 4 semi-analytical expressions are deduced for the expectation values of observables through a semiclassical formalism associated with the slow variables. These expressions are extensively compared in section 5 with exact results coming from a numerical diagonalization of the Dicke Hamiltonian. In their region of applicability, the approximations reproduce nicely the exact numerical results, giving a simple framework to understand them. In section 6 the adiabatic invariants associated with the two fast-slow approximations are unveiled and numerical tests are presented, showing them as approximated integrals of motion, both in the semiclassical and quantum versions of the Dicke model, until the appearance of chaos in the energy space. In this same section, the quantization of the semiclassical slow variable is presented and the resulting spectra are compared with the exact ones which come from numerical diagonalizations, showing again a remarkable accord. Finally in section 7 we present our conclusions.

2. Adiabatic approximation

The Dicke Hamiltonian commutes with the $\hat{J}^2$ operator, defining different subspaces with fixed $j$. As mentioned before, the maximum value $j = N/2$ designates the symmetric subspace where the ground-state lies. Inside this subspace, the Hamiltonian only has two degrees of freedom, one for the collective atomic excitations (pseudospin) and one for the bosons. An effective decoupling between them can occur as a consequence of the different temporal scales in their respective intrinsic dynamics, in two different ways. One when the pseudospin dynamics is much faster than the bosonic one, and the second in the opposite case. In the following two sections we present the two approaches separately. To derive in a simple way the approximations, we start with a quantum-classical approach, quantum for the fast variables and classical for the slow ones. In the following sections full quantum and full classical results are presented in the framework provided by the fast-slow approximations.

2.1. Fast pseudospin approximation

For the sake of clarity, let us consider the Dicke Hamiltonian in terms of classical boson variables, i.e. by substituting the annihilation (creation) operators according to $\hat{a} \to (1/\sqrt{2})(q + ip)$ and $\hat{a}^\dagger \to (1/\sqrt{2})(q - ip)$. 


\[ \hat{H}_{PS} = \frac{\omega}{2}(p^2 + q^2) + \omega_\gamma \hat{J}_z + \frac{2\gamma}{\sqrt{j}} q \hat{J}_c. \]  

(2)

If we freeze the bosonic slow variables, a simple Hamiltonian for the pseudospin variables is obtained, which can be easily diagonalized by considering a rotation (\(\beta\)) around the \(y\)-axis

\[ \hat{H}_{PS} = \frac{\omega}{2}(p^2 + q^2) + \omega_\rho(q) \hat{J}_{z'} \]  

with

\[ \hat{J}_{z'} = \cos \beta \hat{J}_z + \sin \beta \hat{J}_c, \quad \cos \beta = \frac{\omega_\rho}{\omega_\rho(q)}, \quad \sin \beta = \frac{2\gamma}{\sqrt{j}} \frac{q}{\omega_\rho(q)}, \quad \text{and} \]

\[ \omega(p) = \sqrt{\omega_\rho^2 + \frac{2\gamma^2}{j}} = \omega_\rho \sqrt{1 + \frac{\omega_\rho}{\omega_\rho} \left( \frac{f q}{\sqrt{j}} \right)^2}, \]  

(4)

where \(f \equiv \gamma/\gamma_c\) with \(\gamma_c \equiv \sqrt{\omega_\rho^2/2}\) the critical value separating the normal and superradiant phases. The rotation angle \(\beta\) can be positive or negative depending on the sign of \(q\). The above Hamiltonian describes the precession of the pseudospin around the rotated \(z'\) axis with angular velocity \(\omega(p)\), and is diagonalized by considering eigenstates of the rotated operator \(\hat{H}_{PS}(j m') = m' | j m'\rangle\)

\[ \hat{H}_{PS}(j m') = H_m(p, q) | j m'\rangle = \left[ \frac{\omega}{2}(p^2 + q^2) + \omega(p) m' \right] | j m'\rangle, \]  

(5)

with \(m' = -j, -j+1, \ldots, j-1, j\). The eigenvalues, \(H_m(p, q)\), depend on the, up to here, frozen bosonic variables. If we let these variables evolve, these eigenvalues define, for each \(m'\), an effective Hamiltonian for the slow bosonic variables, which has the standard form of a classical particle moving in a conservative so-called adiabatic potential

\[ V_m(q) = \frac{\omega}{2} q^2 + \omega_\rho \sqrt{1 + \frac{\omega_\rho}{\omega_\rho} \left( \frac{f q}{\sqrt{j}} \right)^2} m'. \]  

(6)

We have manipulated the Dicke Hamiltonian to obtain two effective dynamics for the pseudospin and boson variables respectively, where the other variables enter as simple parameters. This approach will be self consistently valid only if the frequency of the precessing pseudospin \(\omega(p) = \omega(p)\) is much larger than the frequency of the bosonic variables, \(\omega_\rho\), which evolve according to the effective Hamiltonian \(H_m(p, q) = \frac{\omega}{2} p^2 + V_m(q)\). A detailed analysis of the validity of this approach is presented below.

An important consequence of this approach is that the dynamics of the semiclassical model and the spectrum in the quantized version, are organized in a finite number of bands, characterized each by the quantum number \(m'\). Each band is associated to a different effective Hamiltonian, \(H_m(q, p)\), for the bosonic variables, and since the energy of the effective Hamiltonian is only lower bounded, each band extends infinitely in energy. This band structure provides an useful description of the regular part of the energy spectra, up to an energy where the approximation breaks down.

To determine the values of the fast and slow frequencies (\(\omega_F\) and \(\omega_B\)) as a function of coupling and energy, and to be able to establish the range of validity of the previous approach, we analyse the dynamics of the slow bosonic variables and focus our attention on the effective potential \(V_m(p, q)\).
The minimum of the potential \( V_{qm} \), for \( m \) integer or half integer according to \( j \), is given by

\[
\omega = -\frac{\omega_0}{2} \left( \left| \frac{1}{j} \right|^2 + \left| \frac{m'}{j} \right|^2 f^2 \right) \quad \text{for} \quad -j \leq m' \leq -j + f^2 \quad \text{and} \quad f \geq 1
\]

(7)

The value of \( q \) which minimize the potential is

\[
q_{\text{min}}' = \frac{\omega_0 \sqrt{j}}{2\gamma} \sqrt{\frac{m'}{j} f^4 - 1} \quad \text{for} \quad -j \leq m' \leq -j + f^2 \quad \text{and} \quad f \geq 1
\]

(8)

The form of the potentials \( V_{qm}(q) \) is illustrated in figure 1 for two representative cases in the normal \((\gamma < \gamma_c, f < 1)\) and superradiant \((\gamma > \gamma_c, f > 1)\) phases. In both cases the smaller the value of \( m' \), the lower the associated potential. In the normal phase, figure 1(a), the potentials for all \( m' \) have their minimum at \( q = 0 \), whereas in the superradiant phase, figure 1(b), the potentials for the values of \( m' \) in the interval \(-j \leq m' \leq -j + f^2\) present a spontaneous breaking of the parity symmetry \((q \rightarrow -q)\), and have two degenerate minima. The right-hand side well, with \( q > 0 \), corresponds to a positive rotation angle \( \beta \), whereas the left-hand side, with \( q < 0 \), is associated to a negative \( \beta \) angle. For \(-j + f^2 < m' < j\) the parity symmetry of the potentials is spontaneously restored and the minima are located again at \( q = 0 \).

2.1.1 Fast pseudospin approximation: boson and pseudospin frequencies. For each band \( m' \) associated to the potential \( V_{qm}(q) \), it is easy to estimate the frequencies of its lowest energy \( E_{m_{\text{min}}} \). For the slow bosonic variables, the frequency is estimated by expanding the potential around the minimum \( q_{\text{min}} \); the quadratic term of this expansion gives the frequency (see appendix A); the result is

\[
\omega_{\text{B}}(E_{m_{\text{min}}}) = \begin{cases} 
\omega \sqrt{\frac{f^4 - \left( \frac{1}{j} \right)^2}{f^4}} = \omega \sqrt{1 + \frac{1}{j^2}} = \omega \sqrt{1 + \frac{m'}{j^2}} & \text{for} \quad -j \leq m' \leq -j + f^2 \quad \text{and} \quad f \geq 1 \\
\omega \sqrt{\frac{1 + \frac{m'}{j^2}}{f^2}} = \omega \sqrt{1 + \frac{m'}{j^2}} & \text{otherwise}
\end{cases}
\]

(9)
The expressions in terms of \( \omega \equiv \prime E_j m o_{\text{min}}/() \varepsilon \) were obtained by substituting \( \prime m \) as a function of \( \prime E_{\text{min}} \) using equation (7). The slow boson frequency for the lowest energy band (\( \prime = \prime j m \)) reduces to

\[
\omega_{\text{qB}} \equiv \omega_{\text{qB}}(E_{\text{min}}') = \left\{ \begin{array}{ll}
\omega \sqrt{1 - f^2} & \text{for } f \geq 1 \\
\omega \sqrt{1 - f^{-2}} & \text{for } f < 1
\end{array} \right.
\]

(10)

The fast frequencies of the precessing pseudospin are obtained by evaluating \( \omega_{\text{qP}}(q) \), equation (4), at \( q_{\text{min}}' \) using equation (8), \( \omega_{\text{qP}}(E_{\text{min}}') = \omega_{\text{qP}}(q_{\text{min}}') \), the result is

\[
\omega_{\text{qP}}(E_{\text{min}}') = \left\{ \begin{array}{ll}
\omega_{\text{p}} \left| \frac{m'}{f} \right| f^2 = \omega_{\text{p}} \sqrt{1 + 2 \epsilon f^2} & \text{for } -j \leq m' \leq -j f^2 \text{ and } f \geq 1 \\
\omega_{\text{p}} & \text{otherwise}
\end{array} \right.
\]

(11)

The fast pseudospin frequency for the lowest energy band (\( m' = -j \)) reduces to

\[
\omega_{\text{qP}} \equiv \omega_{\text{qP}}(E_{\text{min}}') = \left\{ \begin{array}{ll}
\omega_{\text{p}} f^2 & \text{for } f \geq 1 \\
\omega_{\text{p}} & \text{for } f < 1
\end{array} \right.
\]

(12)

2.1.2. Fast pseudospin approximation: application range. As mentioned above, the fast pseudospin approximation is valid if \( \omega_{\text{qP}}(E_{\text{min}}') \gg \omega_{\text{qB}}(E_{\text{min}}') \). In figure 2 we illustrate the behavior of the lowest energy frequencies of each band \( m' \) and their respective ratio \( \omega_{\text{qP}}(E_{\text{min}}')/\omega_{\text{qB}}(E_{\text{min}}') \) as a function of \( m' \). We have chosen one case in the normal (\( f = 0.8, \omega/\omega_{\text{p}} = 0.2 \), figures 2(a) and (d)), and two cases in the superradiant phase (\( f = 2, \omega = \omega_{\text{p}} \), figures 2(b) and (e) and \( f = 3, \omega/\omega_{\text{p}} = 0.2 \), figures 2(c) and (f)).
In the case of the normal phase, figures 2(a) and (d), the pseudospin frequency is equal to \( \omega_p \) for all \( m' \), and \( \omega_{pB}(E_{mm}^{\prime}) \) increases monotonically with \( m' \), consequently the ratio \( \omega_p(E_{mm}^{\prime})/\omega_{pB}(E_{mm}^{\prime}) \) decreases as \( m' \) increases. In the superradiant phase, for \( m' \) below \(-jff^2\) the pseudospin frequency descends linearly with \( m' \), whereas the boson frequency decreases much more slowly reaching a zero value at \( m' = -jff^2 \). A local maximal ratio \( \omega_p(E_{mm}^{\prime})/\omega_{pB}(E_{mm}^{\prime}) \) is obtained in the minimal value \( m' = -j \). For \( m' \) above \(-jff^2\) the pseudospin frequency becomes constant and equal to \( \omega_p \), but the boson frequency increases with \( m' \) making that the ratio \( \omega_p(E_{mm}^{\prime})/\omega_{pB}(E_{mm}^{\prime}) \) decreases as a function of \( m' \).

A divergence in the ratio \( \omega_p(E_{mm}^{\prime})/\omega_{pB}(E_{mm}^{\prime}) \) is observed for \( m' \) around \(-jff^2\) (vertical dashed lines in figures 2(d) and (e)), which is the value where the parity symmetry is spontaneously restored and the effective potential passes from a double well to a single well. This passage is accompanied by a flattening of the minimum of the potential, which causes that the boson frequency goes to zero inducing a divergent ratio. This behavior resembles the second order quantum phase transition which takes place in the ground-state at the critical value \( \gamma_{c} \), but here the parameter is the excitation energy instead of the coupling constant. It is worth noting that this divergence does not imply that the approximation becomes invalid (see discussion in [69]), but it is just a consequence of the critical behavior in the bosonic frequency.

Observe that in the \( f = 3 \) out of resonance case, figures 2(c) and (f), the ratio \( \omega_p/\omega_B \) is one order of magnitude greater than in the resonant \( f = 2 \) case, which indicates that the approximation must work much better in the former case, as confirmed numerically below.

As the lower energies are associated with the more negatives values of \( m' \), in the three cases the approximation works better in the low-energy region, where the ratio \( \omega_p(E_{mm}^{\prime})/\omega_{pB}(E_{mm}^{\prime}) \) is larger, and ceases to work as the energy increases. The \( m' \) value where this happens must be lower for values of \( f \) closer to 1 and/or for smaller values of the ratio \( \omega_p/\omega_B \).

Since the ratio \( \omega_p(E_{mm}^{\prime})/\omega_{pB}(E_{mm}^{\prime}) \) (except for the divergence in the narrow region around \(-jff^2\) in the superradiant phase) decreases as a function of \( m' \), a simple and necessary condition for the validity of the fast pseudospin approximation can be established by looking at the frequencies in the lowest \( m' = -j \) band

\[
\frac{\omega_p}{\omega_B} = \frac{\omega_p(E_{mm}^{\prime})}{\omega_{pB}(E_{mm}^{\prime})} = \begin{cases} \frac{\omega_p}{\omega_B} = \frac{f^4}{\sqrt{f^4 - 1}} & \text{for } f \geq 1 \\ \frac{\omega_p}{\omega_B} = \frac{1}{\sqrt{1 - f^2}} & \text{for } f < 1 \end{cases}
\]

(13)

the condition \( \omega_p/\omega_B \gg 1 \) entails the following condition on the model parameters

\[
\frac{\omega_p}{\omega_B} \ll \frac{f^4}{\sqrt{f^4 - 1}} \quad \text{for } f \geq 1 \quad \text{and} \quad \frac{\omega_p}{\omega_B} \ll \frac{1}{\sqrt{1 - f^2}} \quad \text{for } f < 1.
\]

(14)

The level curves of the ratio \( \omega_p/\omega_B \) in the parameter space of the Dicke model (\( \omega/\omega_p \) versus \( \gamma/\gamma_c \)) are shown in the figure 3(a). The gray area indicates the region where the ratio \( \omega_p/\omega_B < 1 \) and the fast pseudospin is not valid. The solid thick line indicates a ratio \( \omega_p/\omega_B = 1 \). Observe that the condition \( \omega_p/\omega_B \ll 1 \) is fulfilled even in the resonant case \( \omega/\omega_p = 1 \) for couplings large enough in the superradiant phase, a result contrary to the naive expectation that the adiabatic approximation with fast pseudospin and slow bosons is only valid if \( \omega/\omega_p \ll 1 \). In the normal phase, the condition \( \omega_p/\omega_B \ll 1 \) likewise holds for small enough \( \omega/\omega_p \).
However, the previous condition is a necessary but not sufficient condition to establish the validity of the fast pseudospin approximation. As discussed in section 3, a more restrictive and accurate condition can be established by comparing the frequencies $\omega_{FB}$ with those obtained from a Holstein–Primakoff or quadratic approximation around the minimal energy configuration of the Dicke model.

### 2.2. Fast boson approximation

Now we turn our attention to the opposite limit, i.e. when the slow variables are the pseudospin ones and the boson variables are fast. To derive in a simple way the approximation, let us consider this time the Dicke Hamiltonian with the pseudospin variables classical and frozen ($\hat{J}_j \rightarrow j$)

$$\hat{H}_B = \omega \hat{a}^\dagger \hat{a} + \omega_{\text{d}} \hat{a}^\dagger + \frac{\sqrt{2} \gamma}{\sqrt{J}} (\hat{a}^\dagger \hat{a}) \hat{h}_i.$$ 

This Hamiltonian can be easily diagonalized by considering a shift transformation $\hat{a} = \hat{b} - \frac{\sqrt{2} \gamma}{\omega_{\text{d}}} \hat{h}_i$, the resulting Hamiltonian is

$$\hat{H}_B = \omega \hat{b}^\dagger \hat{b} + \omega_{\text{d}} \hat{b}^\dagger - \frac{2 \gamma^2}{\omega_{\text{d}}^2} \hat{h}_i^2.$$ 

By considering eigenstates of the number operator $\hat{b}^\dagger \hat{b} |n'\rangle = n'|n\rangle$, we obtain, for each $n'$, an effective Hamiltonian, $H_n(j)$, for the slow classical pseudospin variables

$$\hat{H}_B |n'\rangle = H_n(j) |n'\rangle \equiv \left( \omega n' + \omega_{\text{d}} + \frac{2 \gamma^2}{\omega_{\text{d}}^2} \hat{h}_i^2 \right) |n'\rangle.$$ 

The resulting Hamiltonian for the pseudospin variables, $H_n(j)$, is a Lipkin–Meshkov–Glick one (as already noticed in [66, 67]), whose energy minimum is...
with the variables minimizing the energy given by

\[
E_{\text{min}}^{n'} = \begin{cases} 
-\omega_j j + \omega n' & \text{for } f < 1 \\
\frac{\omega_j j}{2} \left( f^2 + \left( \frac{1}{f} \right)^2 \right) + \omega n' & \text{for } f \geq 1 
\end{cases}
\]  
(16)

The phase space of the effective Hamiltonian for the pseudospin slow variables, \( H_{jn}(\vec{f}) \), is the surface of a sphere of radius \( J \). To illustrate its behavior, we show in figure 4 contour plots of the equipotential lines of \( H_{jn}(\vec{f}) \) (without the additive term \( \omega n' \)), using the canonical variables

\[
(j_{x,\text{min}}, j_{y,\text{min}}, j_{z,\text{min}}) = \begin{cases} 
(0, 0, -j) & \text{for } f < 1 \\
(\pm j \sqrt{1 - f^4}, 0, -\frac{j}{f^2}) & \text{for } f \geq 1 
\end{cases}
\]  
(17)

The spontaneous breaking of the parity symmetry is seen by the appearance of double degenerated minima for couplings above the critical value. It is worth mentioning that, contrary to the fast pseudospin approximation, in this case the spontaneous breakings of the parity symmetry and the accompanied quantum phase transitions occur simultaneously for all the effective Hamiltonians \( H_{jn}(\vec{f}) \) when the coupling reaches the critical value \( \gamma_c \). The term in the effective Hamiltonians depending on the fast boson quantum number \( \omega n' \) is a simple additive constant without effect in the quantum phase transition, which is in contrast with the fast pseudospin approximation where the quantum number of the fast variable \( n' \) determines, for
the superradiant phase, if the associated effective potential for the slow variables presents or not a spontaneous breaking of the parity symmetry (see panel (b) of figure 1).

Note that the fast boson approximation implies the existence of an infinite number of bands in the corresponding quantum energy spectrum. Each band, labelled by the quantum number \( n' \), is associated to an effective Hamiltonian for the pseudospin variables \( \hat{H}_p(j) \). Since this effective Hamiltonian is lower and upper bounded in energy, every band has a finite extension in energy. Additionally, observe that the pseudospin dynamics is independent of the quantum number \( n' \), whose only effect is an energy shift by an amount \( \omega n' \).

2.2.1. Fast boson approximation: boson and pseudospin frequencies. In order to determine the region of applicability of the previous fast boson approximation, the frequencies of the boson and pseudospin variables have to be evaluated. A necessary condition of applicability of the fast boson approximation is \( \omega_B \gg \omega_F \).

For the bosonic variables the frequency is simply the frequency of the non-interacting case \( \omega_B = \omega \), regardless of the value of the coupling. For the slow pseudospin variables, we calculate the frequency for the minimal energy of the effective Hamiltonian \( \hat{H}_p \) by expanding it up to second order around the minimal energy configuration, equation (17). The result (see appendix A for details) is independent of the quantum number \( n' \) and is given by

\[
\omega_f(n'_\text{min}) = \begin{cases} 
\omega \sqrt{1 - f^2} & \text{for } f < 1 \\
\omega \sqrt{f^4 - 1} & \text{for } f \geq 1 
\end{cases}
\]  

(19)

2.2.2. Fast boson approximation: application range. With the previous expression for the frequencies, the ratio \( \omega_B/\omega_F \) is

\[
\frac{\omega_B}{\omega_F} = \begin{cases} 
\frac{\omega}{\omega_0} \sqrt{1 - f^2} & \text{for } f < 1 \\
\frac{\omega}{\omega_0} \sqrt{f^4 - 1} & \text{for } f \geq 1 
\end{cases}
\]  

(20)

therefore the condition \( \omega_B/\omega_F \gg 1 \) entails

\[
\frac{\omega}{\omega_0} \gg \sqrt{1 - f^2} \quad \text{for } f < 1 \quad \text{and} \quad \frac{\omega}{\omega_0} \gg \sqrt{f^4 - 1} \quad \text{for } f \geq 1.
\]  

(21)

The level curves of the ratio \( \omega_B/\omega_F \) in the relevant parameter space of the Dicke model are shown in panel (b) of figure 3. Similar to the fast pseudospin approximation, the region where the fast boson approximation is not valid (\( \omega_B/\omega_F < 1 \)) is indicated by the gray area.

Similarly to the condition derived for the fast pseudospin approximation, equation (14), the previous condition for the validity of the fast boson approximation is necessary but not sufficient. When they are combined, one realizes that there exists a region in the parameter space, the region between the dashed black and thick black lines in both panels of figure 3, where none of the fast-slow approximations is ruled out by the simple conditions \( \omega_F/\omega_B > 1 \) or \( \omega_B/\omega_F > 1 \). This would imply that in this region both approximations would be, to a certain extent, valid, which is clearly impossible. In order to overcome this contradiction more
restrictive conditions for the validity of the fast-slow approximations have to be established. These more restrictive conditions can be obtained by comparing the above derived frequencies, \( \omega_{BF} \), with those obtained from a Holstein–Primakoff approximation [17] as is discussed in the following section.

3. Tighter applicability limits of the fast-slow approximations

While in the previous section simple conditions on the parameters of the model were established to determine the regions of validity of the two fast-slow approximations, tighter and consistent conditions can be derived by comparing the frequencies coming from the fast-slow approximations with those \( \omega_{H} \) obtained from a Holstein–Primakoff approximation to the Dicke model [17], which are the same that are obtained by considering a quadratic approximation around the minimal energy configuration in the exact Dicke model.

The Holstein–Primakoff approximation [70] is constructed in two steps. First, an exact mapping of the quasi-spin operators is made in terms of one boson, as an infinite series. A coherent state is built with these bosons and its parameter determined to provide a mean field description of the ground state which becomes exact in the thermodynamic limit [48]. Second, a truncated version of the Hamiltonian, quadratic in the boson operators, provides a description of the Dicke Hamiltonian as two decoupled harmonic oscillators [17]. The Holstein–Primakoff mapping and its truncation reads

\[
\begin{align*}
J_+ &= 2j \sqrt{1 - \frac{\hat{b} \hat{b}^\dagger}{2j}} \hat{b} \approx 2j \hat{b} \\
J_- &= 2j \hat{b}^\dagger \sqrt{1 - \frac{\hat{b} \hat{b}^\dagger}{2j}} \approx 2j \hat{b}^\dagger \\
\hat{J}_z &= \hat{b}^\dagger \hat{b} - j.
\end{align*}
\]

They provide a good approximation for the first excited states, except in the vicinity of the quantum phase transition where the mean field approximation breaks down. They can be employed up to slightly higher energies, but when these states start having a significant number of bosons the quadratic approximation is no longer useful.

After substituting the Holstein–Primakoff truncated mapping and by employing a Bogoliubov transformation in the resulting Hamiltonian, we obtain the frequencies of the two harmonic oscillators, which are given by

\[
2\omega_\pm^2 = \omega_0^2 + \omega_0^2 \pm \sqrt{(\omega_0^2 - \omega^2)^2 + 16\gamma^2\omega_0^4},
\]

for \( f < 1 \) (\( \gamma < \gamma_c \)) and by

\[
2\gamma^4 \omega_\pm^2 = \omega_0^2 \gamma_4^4 + \omega_0^2 \gamma_\gamma^4 \pm \sqrt{(\omega_0^2 \gamma_4^4 - \omega_0^2 \gamma_\gamma^4)^2 + 16\omega_0^8 \gamma_\gamma^4 \gamma_c^8},
\]

for the superradiant phase \( f \geq 1 \) (\( \gamma \geq \gamma_c \)).

As these frequencies \( \omega_\pm \) provide the exact low energy frequencies in the thermodynamic limit, except for the critical coupling (\( \gamma = \gamma_c \)) where one of the quadratic terms is exactly zero and the lowest order approximation involves a quartic term, we naturally demand that the frequencies obtained from our fast-slow approximations reproduce this low energy limit.

In figure 5 we compare the frequencies \( \omega_{BF} \) (dashed) with \( \omega_F \) and \( \omega_B \) coming from the fast pseudospin (red) and fast boson (blue) approximation. The curves show the frequencies
(in units of $\omega_o$) as a function of the ratio $\omega/\omega_o$ for fixed values of the ratio $f = \gamma/\gamma_c$: (a) $f = 0.15$, (b) $f = 0.8$, (c) $f = 2$ and (d) $f = 3$. In the interval $\omega/\omega_o$ between the vertical orange lines, the fast-slow approximation frequencies differ for more than 5% from the quadratic ones. The vertical black dashed line is the $\omega/\omega_o$ value where the fast pseudospin frequencies intersect the fast boson ones, this value is 1 for the normal phase and $f^2$ in the superradiant one.

The regions between the vertical dashed orange lines correspond to the ratios $\omega/\omega_o$ where the $\omega_{B,F}$ frequencies differ from the $\omega_{\pm}$ ones for more than 5%. Therefore, these regions are excluded from the region of validity of any fast-slow approximation. Observe that these excluded regions are equally those where the ratio $\omega_{B,F}/\omega_0 \approx 1$. Summarizing, we have identified regions in the parameter space of the Dicke model, where the conditions $\omega_B/\omega_F \gg 1$ (fast boson approximation) or $\omega_F/\omega_B \gg 1$ (fast pseudospin approximation) are fulfilled and, additionally, the frequencies $\omega_{B,F}$ reproduce very well the Holstein–Primakoff ones $\omega_{\pm}$.

In figure 6, the validity analysis described above is extended to all the couplings in the interval $f = \gamma/\gamma_c \in [0,4]$. The region of validity of the fast pseudospin approximation is indicated in light red, whereas the region where the fast boson is valid is indicated in light blue. The white region indicates the region in the parameter space where none of the two approximations is valid, in this region the frequencies $\omega_{F,B}$ differ from the normal frequencies $\omega_{\pm}$ in more than 5% as discussed above. The figure 6 is a conclusive and very concrete result that gives a valuable guide to determine if one, the other or neither fast-slow approximation can be used for a given set of values of the Hamiltonian parameters. In the same figure different
level curves for the ratio of boson and pseudospin frequencies are drawn. In the upper light blue region the curves correspond to $\omega/\omega_o = 2$ (dotted), $\omega/\omega_o = 5$ (dashed) and $\omega/\omega_o = 10$ (solid) respectively. In the bottom light red region similar curves are shown but for the ratio $\omega_p/\omega_o$. The larger these frequencies ratios, the better the respective approximation is expected to work. The dashed black lines in the white region corresponds to 1 for $f = \gamma/\gamma_c < 1$ and $f^2$ for $f \geq 1$. The brown points coordinates, indicating the cases shown in figures 7–9, are: $(\gamma/\gamma_c, \omega/\omega_o) = (0.8, 0.1), (0.8, 1.5), (0.8, 10), (2, 1), (2, 4), (2, 10), (3, 0.2), (3, 9)$.

Some important observations,

(i) In the normal phase, the resonant case $\omega/\omega_o = 1$ is excluded from the region of validity of any fast-slow approximation, but in the superradiant phase it can be described with the fast pseudospin approximation and the greater the coupling, the better the approximation is expected to work.

(ii) Although we have imposed the condition $\omega_p, \omega_f \sim \omega_o$ for the lowest energy excitations, this does not mean that both approximations are equivalent. As we show numerically below, the fast-slow approximations give a better approximation to the exact Dicke solution and extends to higher energies, well above the energies where the Holstein–Primakoff approximation breaks down.

(iii) The present analysis for the validities of the fast-slow approximations is based on the frequencies coming from the respective approximations calculated in the lowest (ground-state) energy region. For given values of the Hamiltonian parameters, the validities of the
approximations as a function of the energy for a given band ($m'$ in the fast pseudospin approximation or $n'$ in the fast boson one) are determined numerically in the next sections. There it will be shown that, as the ratio $\omega FB/\omega FB$ increases, the fast pseudospin (fast boson) approximation is able to provide a very good description of the system up to higher energies.

(iv) A very simple criterion to establish which of the two approximations is valid can be obtained; in the normal phase ($f < 1$) the criterion is $\omega \omega 1o$ for the fast boson BOA whereas for the fast pseudospin BOA the criterion is $\omega \omega 1o$. In the superradiant phase the criterion changes as a consequence of the different nature of the fundamental effective excitations, the simple criterion is now $\omega \omega 2o$ for the fast boson and fast pseudospin BOA, respectively. The line 1 for $f < 1$ and $\hat{f}$ for $f > 1$, marking roughly the border between one and the other BOA is indicated in figure 6 by dashed black lines.

4. Semiclassical expectation values

In the previous sections we introduced two approximated, semiclassical Hamiltonians which allow to decouple the fast and slow modes and determined the characteristic frequencies associated with these modes and their range of validity. In this section we present the calculation of expectation values of selected observables ($O$) in stationary states. In general, for a fully ergodic $K$-degrees of freedom Hamiltonian $[H_{q,p}, i i]$, with canonical variables $q_{i}, p_{i}, i = 1, 2, ..., K$, the semiclassical approximation to the expectation value of a given observable with semiclassical limit $O_{q,p}$, in a stationary state with energy $E$ is [71]

$$
\langle O \rangle_E = \frac{\int \prod^{K} dq dp O(q,p) \delta(E - H_{q,p})}{\int \prod^{K} dq dp \delta(E - H_{q,p})}.
$$

If the ergodicity is prevented by a conserved quantity $I$, the integration domain must be restricted to the subspace associated to a fixed value of $I$. In the case of our fast-slow approximations, the subspaces are determined by the quantum numbers $m'$ and $n'$ for the fast pseudospin and fast boson approximation respectively, and the integration domain is reduced to that of the respective slow variables.

For the fast pseudospin approximation the expectation value of any observable $O_{q,p}$ in a stationary state of energy $E$ and quantum number $m'$ can be semiclassically evaluated by

$$
\langle O \rangle = \frac{\int dq dp O_{m'}(p,q) \delta [E - H_{m'}(p,q)]}{\int dq dp \delta [E - H_{m'}(p,q)]}, \quad (23)
$$

where $O_{m'}(p,q) = \langle j, m' | O(a^{\dagger}, a^{\dagger}; a, a; j, j, j) | j, m' \rangle$ with $| j, m' \rangle$ eigenvectors of the rotated operator $J_z, H_{m'}(p,q)$ is the effective Hamiltonian for the slow boson variables, equation (5), and the bosonic operators ($a^{\dagger}, a$) have to be written in terms of their classical limit variables ($q, p$):

$$
\hat{a} \rightarrow \frac{1}{\sqrt{2}} (q + ip) \quad \text{and} \quad \hat{a}^{\dagger} \rightarrow \frac{1}{\sqrt{2}} (q - ip).
$$

For the fast boson approximation, the semiclassical approximation to the expectation values in stationary states of energy $E$ and quantum number $n'$ is

$$
\langle O \rangle = \frac{\int dq dp O_{n'}(\phi, \phi) \delta [E - H_{n'}(\phi^{\dagger})]}{\int dq dp \delta [E - H_{n'}(\phi^{\dagger})]}, \quad (24)
$$
where $O_{\alpha'}(j, \phi) = \langle n'|O(\alpha', a; J, J_0)|n'\rangle$ with $|n'\rangle$ eigenvectors of the shifted boson number operator $b^\dagger b$, $H_{\alpha'}(j)$ is the effective Hamiltonian for the slow pseudospin variables, equation (15), and the pseudospin operators are written in terms of classical canonical variables $(j, \phi)$: $(J, J_0, J_0) \rightarrow (j \sqrt{1 - (j_0 j)^2} \cos \phi, j \sqrt{1 - (j_0 j)^2} \sin \phi, j_0)$.

We particularize the previous general expressions for two observables, the number of photons $\hat{a}^\dagger \hat{a}$ and the population difference between the two level systems $J_z$. The details of the calculations yielding to the given expressions can be found in appendix B. The behavior of the expectation values of these observables restricted to the subspaces of the adiabatic invariants of each BOA, as the quasi-conserved quantities $I$, will be analysed employing Peres lattices in the next section.

4.1. Density of States and semiclassical expectation values: fast pseudospin

First, we calculate the expression appearing in the denominator of equation (23), which is proportional to the semiclassical approximation to the density of states $\nu(E, m')$ or Weyl’s formula

$$2\pi \nu(E, m') = \int dq \int dp \delta [E - H_m(q, p)] = \frac{2}{\omega} \int_{q \in I_{E,m'}} dq \sqrt{\frac{E - V_m(q)}{V_m(q)}},$$

(25)

where $I_{E,m'}$ is the classical allowed region in $q [E - V_m(q) \geq 0]$ and $V_m(q)$ is defined in equation (6). For energies inside the double well potential this region is formed by two disconnected intervals, whereas for any other case is given by a sole interval (see figure 1)

$$I_{E,m'} = \begin{cases} [-q_-(E, m'), -q_+(E, m')] \cup [q_-(E, m'), q_+(E, m')] & \text{for } -j \leq m' \leq -j f^2, f \geq 1 \\ [-q_+(E, m'), q_-(E, m')] & \text{otherwise} \end{cases}$$

where $q_{\pm}$ are the returning points ($V_m(q) = E$) given by

$$q_{\pm}(E, m') = \sqrt{2j} \sqrt{\frac{\omega \omega_{jd}}{\omega_{jd}}} \left( \frac{m'}{j} \right)^{1/2} \pm \frac{m'}{j} \sqrt{1 + \frac{2E}{\omega_{jd}} - \left( \frac{m'}{j} \right)^2}.$$  

This structure is directly linked to the presence of an excited-state quantum phase transition (ESQPT) [33–36], defined as a singularity in the Density of States and the expectation values of some observables as a function of energy for given Hamiltonian parameters, as can be seen in the numerical results displayed in figure 9(b). The change from two disconnected regions to one sole interval entails that the phase-space level curve $H_m(q, p) = \omega_m m'$, for $-j \leq m' \leq -j f^2$, crosses itself at a singular point. This fact produces a density of states with a logarithmic singularity at the critical energy $E_c = \omega_m m'$. The same phenomenon happens in all the one-dimensional collective quantum systems displaying ESQPTs [72].

For the expectation value of the $J_z$ operator we obtain

$$\langle \hat{J}_z \rangle_{E, m'} = \frac{1}{2\pi \nu(E, m')} \int dq \int dp \langle jm'|J_z|jm'\rangle \delta [E - H_m(p, q)]$$

$$= \frac{1}{2\pi \nu(E, m')} \frac{2}{\omega} \int_{q \in I_{E,m'}} dq \sqrt{\frac{m'}{\omega}} \frac{\sqrt{\frac{E - V_m(q)}{V_m(q)}}}{\sqrt{1 + \frac{2E}{\omega_{jd}} - \left( \frac{m'}{j} \right)^2}}.$$  

(26)
whereas for the number of bosons, the results is

$$\langle \hat{\mathcal{N}} \rangle_{E,m'} = \frac{1}{2\pi \nu'(E,m')} \int dp \, dq \, \frac{1}{2} (p^2 + q^2) \delta [E - H_m(p,q)]$$

$$= \frac{1}{2\pi \nu'(E,m')} \frac{1}{\omega} \int \frac{1}{\omega} \int I_{E,m'} dq \, \frac{E - \omega, \sqrt{1 + \frac{\omega'(L_0')^2}{\omega} \frac{I_{E,m'}}{\sqrt{E - V_m(q)}}}}{\sqrt{E - V_m(q)}}. \tag{27}$$

Both observables behave in the same way than the density of states, showing a logarithmic singularity at the critical energy at which $I_{E,m'}$ changes from having two disconnected regions to a sole interval.

### 4.2. Density of States and semiclassical expectation values: fast boson

In a similar way as above, we evaluate first the denominator of the general formula (24), which is proportional to the semiclassical approximation to the energy Density of States $[\nu''(E,n')]$

$$2\pi \nu''(E,n') = \int \delta j \, \delta [E - H''(j)] = \begin{cases}  \frac{1}{\omega_0} \int_0^{\phi_0} \frac{d\phi}{\sqrt{F(\cos \phi)}}, & \text{for } \epsilon'' \leq -1 \\ \frac{1}{\omega_0} \int_0^{2\pi} \frac{d\phi}{\sqrt{F(\cos \phi)}}, & \text{for } \epsilon'' > -1 \end{cases} \tag{28}$$

where we have defined $F(x) = 1 + 2f^2 \epsilon_0 x^2 + f^4 x^4$ with $\epsilon_0 \equiv (E - \omega')(\omega_0')$, and $\phi_0$ is given by $\cos^2(\phi_0) = \frac{1}{r^2} \left[ -\epsilon'' + \sqrt{\epsilon''^2 - 1} \right]$.

The properties of this density of states are the same than the corresponding to the fast-pseudospin approximation. If $\epsilon'' \leq -1$, the allowed region for the angle $\phi$ is also composed by two disconnected regions, but the integral can be simplified to the expression displayed in equation (28) (see appendix B for a detailed discussion); if $\epsilon'' > -1$, $\phi$ covers the whole interval $[0, 2\pi)$. Thus the level curve $H_{E''}(\phi, j; j') = -\omega'' + \omega n'$ also crosses itself at a singular point, and therefore $\epsilon'' = -1$ is the critical energy of the ESQPT, as it happens in the Lipkin–Meshkov–Glick model [73]. Both the density of states itself, and the expectation values of $\hat{J}_z$ and $\hat{\mathcal{N}}$ show a logarithmic singularity at this energy (see figures 8(a) and 9(a)).

The expectation value of $\hat{J}_z$:

$$\langle \hat{J}_z \rangle'_{E,n'} = \frac{1}{2\pi \nu''(E,n')} \int \delta j \, \delta [E - H''(j)],$$

is given by

$$\langle \hat{J}_z \rangle'_{E,n'} = \begin{cases} -\frac{8j}{f^2 \omega_0 2\pi \nu''(E,n')} \int_0^{\phi_0} \frac{\phi \, d\phi}{\cos^2(\phi) \sqrt{F(\cos \phi)}}, & \text{for } \epsilon'' \leq -1 \\ -\frac{j}{f^2 \omega_0 2\pi \nu''(E,n')} \int_0^{2\pi} \frac{d\phi}{\cos^2(\phi) \sqrt{F(\cos \phi)}}, & \text{for } \epsilon'' > -1 \end{cases} \tag{29}$$
Finally, for the expectation value of the number of bosons,

\[
\langle \hat{a}^\dagger \hat{a} \rangle_{E,n} = \frac{1}{2\pi \nu(E,n)} \int d\phi \, d\phi' |\langle n'|\hat{a}^\dagger \hat{a} |n'\rangle| \delta [E - H_\phi(j)]
\]

the result is

\[
\langle \hat{a}^\dagger \hat{a} \rangle_{E,n} = \begin{cases} 
  n - \frac{8j}{f^2 2\pi \nu(E,n')} \int_0^{\phi'} d\phi' \frac{1 + \cos^2 \phi \epsilon_{\text{eff}}^2}{\cos^2 \phi \sqrt{\mathcal{F}(\cos \phi)}} & \text{for } \epsilon_{\text{eff}} \leq -1 \\
  n - \frac{j}{f^2 2\pi \nu(E,n')} \int_0^{2\pi} d\phi \frac{1 + \epsilon_{\text{eff}}^2 \cos^2 \phi - \sqrt{\mathcal{F}(\cos \phi)}}{\cos^2 \phi \sqrt{\mathcal{F}(\cos \phi)}} & \text{for } \epsilon_{\text{eff}} > 1
\end{cases}
\]

The above expressions, equations (26), (27), (29) and (30), which look involved in their forms, follow from the detailed and lengthy analysis presented in appendix B. The possibility of obtaining these semi-analytical expressions for the most relevant observables, the expected number of photons and of excited atoms, in the regular regions of the Dicke model, is one the main contributions of this work.

5. Comparison with the exact numerical results

In this section we compare the results for the observables obtained employing the fast-slow approximations with those evaluated by numerical diagonalization of the exact Dicke Hamiltonian.

5.1. Peres lattices and fast-slow approximations

5.1.1. Peres lattices. Peres lattices [74, 75] are useful tools to study the regular or chaotic behavior in quantum systems with two degrees of freedom. The main idea is that, when this kind of systems have an integrable Hamiltonian, plotting the eigenvalues of a constant of motion (Peres operator) against the Hamiltonian eigenenergies a regular lattice is obtained, where each Hamiltonian eigenstate contributes with one point. The conserved quantities provide a natural way to label the eigenstates. An extra term can be added to the Hamiltonian, making it non-integrable. In this case the Peres operator will no longer be a conserved quantity, and its expectation value can be plotted as a function of the eigenenergies. Such plots are called Peres lattices. A small perturbation introduces localized distortions in the lattice without destroying it. As the perturbation increases irregular regions appear in the lattice, which are associated with chaotic dynamics. Peres lattices provide a qualitative way to identify regions of regularity or chaos, being the counterpart of the Poincaré sections employed to identify regions exhibiting classical chaos in phase space [41]. They are quite useful to visualize the competition between regular and chaotic behavior in the quantum spectrum of a system [74, 75]. By choosing different Peres operator, various properties of individual states can be followed, being some of them more suitable to exhibit the route to chaos.

5.1.2. Numerical analysis. The exact numerical full quantum Dicke results are presented using Peres lattices. In this case we have selected as Peres operator the (displaced) fraction of excited atoms \( \langle \hat{J}_z \rangle / \mu \), plotted against the corresponding energy eigenvalue. To make the comparisons between the exact and approximate expectation values, we have selected nine different sets of parameters. Three different ratios \( f = \gamma / \gamma_c \) are chosen, one in the normal \( f = 0.8 \).
and two in the superradiant phase ($f = 2$ and $f = 3$). For each value $f$, three different ratios $\omega / \omega_p$ are used. One where the fast pseudospin approximation is expected to work, other where the valid approximation is the fast boson one, and a third ratio where neither the fast pseudospin nor the fast boson approximation is valid. Concretely, for $f = 0.8$ the ratios $\omega / \omega_p = 0.1, 1.5$ and 10 are used. For $f = 2$ we use the ratios $\omega / \omega_p = 1, 4$ and 10, and finally, for $f = 3$ we use $\omega / \omega_p = 0.2, 9$ and 20. All these points (except the last one that is out of the plot range) are indicated in figure 6 by brown dots.

In figure 7 the three cases in the normal phase are shown. In panel (a) a ratio $\omega / \omega_p = 10$ is used, where the fast boson approximation is expected to be valid. The exact results are shown as black dots, and the approximated ones by solid lines. Since $\omega_p / \omega_j = 16.67$, the interval of validity of the approximation extends to high energies, as can be seen in the figure, where the agreement is excellent. According to the fast boson approximation, the spectrum must be organized in finite bands labelled by the quantum number $n'$, extending from $E(\omega_j) = \omega_j n' - 1$ until $E(\omega_j) = \omega_j n' + 1$, exactly what is observed in the numerical results. The end of the lowest band at $E(\omega_j) = 1$ signals an ESQPT, where the whole pseudospin Bloch sphere becomes available.

In figure 7(b) exact numerical and fast pseudospin approximation results are shown for a smaller ratio $\omega / \omega_p = 0.1$, where the pseudospin to boson frequency ratio is $\omega_p / \omega_B = 16.67$, and consequently the fast pseudospin approximation must provide a good description of the

Figure 7. Peres lattices ($J_j / j$ versus $E(\omega_j)$) (black dots) and results from the two fast-slow approximations (solid lines) for $f = 0.8$ and three different ratios $\omega / \omega_p$. (a) $\omega / \omega_p = 10$ where the fast boson approximation (blue lines) is valid. (b) $\omega / \omega_p = 0.1$ where the fast pseudospin approximation (red lines) is the valid one. (c) and (d) correspond to the same ratio $\omega / \omega_p = 1.5$, where, according to figure 6, neither the fast boson approximation (blue lines in panel (c)) nor the fast pseudospin approximation (red lines in panel (d)) is valid. Insets show zooms to the low energy regions. A value $j = 60$ was used.
exact results. This expectation is confirmed by the results shown in the figure, where the Peres lattice is very well described by the approximated results. Observe that the energy scale is much smaller than the one used in figure 7(a), because the density of states is much larger in this case, even so the number of lowest energy quantum states well described by the approximation is similar. According to the fast pseudospin approximation, the spectrum has to be organized in a finite number of infinite bands, the figure 7(b) shows the first twelve of these bands, both in the approximated and numerical exact results.

Finally in figures 7(c) and (d) the same Peres lattice is shown but compared with results of the fast boson and fast pseudospin approximations, respectively. The panels correspond to a ratio $\omega/\omega_0 = 1.5$ where, according to our previous analysis, neither one nor the other approximation is expected to give a good description of the exact results, which is confirmed by the figures. Observe that in the case of the panel (c) a very small number of states in the lowest energy band are described by the fast boson approximation. In the case of the fast pseudospin approximation not even this small number exists. This fact can be understood by looking at the pseudospin to boson frequency ratios, according to the boson approximation this ratio is $\omega_B/\omega_P = 2.5$, whereas from the fast pseudospin one the ratio is very close to one $\omega_P/\omega_B = 1.1$. However, as discussed in section 3, in both cases the $\omega_{B,P}$ differ (for more than 5%) from the $\omega_{B,P}$ frequencies, and consequently both approximations have, as confirmed, to provide a very poor description of the exact results.

In figure 8 three cases in the superradiant phase with $f = 2$ are shown. In panel (a) a ratio $\omega/\omega_0 = 10$ is employed, which corresponds to $\omega_B/\omega_P = 5.77$. In accord to the analysis of section 3, the fast boson approximation (blue lines) describes very well the numerical exact results (dots) until very large excitation energies. Even if some numerical discrepancies can be seen (attributable to the fact that the ratio $\omega_B/\omega_P = 5.77$ is not much larger than 1), the overall structure
of the Peres lattice is completely reproduced by the fast boson approximation. The finite bands (labelled by the quantum number $n^\prime$) are clearly distinguishable in the Peres lattice, and the two lowest bands can be seen completely (more complete bands are hidden because of the plot range used). According to the approximation, the lowest band ($n^\prime = 0$) must begin at energy $E(\omega_0, j) = -\frac{1}{2}(f^2 + f^{-2})$ and end at $E(\omega_0, j) = 1$, the energy of the ESQPT where the whole Bloch sphere becomes available. These features are clearly seen in the exact numerical results.

It is known [33–35] that in the superradiant phase, a second ESQPT appears at energy $E(\omega_0, j) = -1$. A singular behavior of the lowest band is observed exactly at this energy in figure 8(a). What is more, every band presents a similar singular behavior at energies $E(\omega_0, j) = -1 + \frac{1}{2}j$, which is associated to a logarithmic divergence in the Density of States $\nu(E, n^\prime)$ of the one degree of freedom effective Hamiltonian $H_n(j)$, as discussed in section 4 and in agreement with the general theory of ESQPTs [76]. It is worth remarking that the ESQPT at $E(\omega_0, j) = -1$ reported in [33–35] for the full Dicke model does not show a divergence in the Density of States, but in its first derivative. This is because in these Refs. the ESQPTs were studied without considering the invariant subspaces coming from the BOA. In the parameter region where the BOA is valid until the energy of the ESQPT (as the case shown in figure 8(a)), our results show divergent Densities of States for each invariant band in this energy region. This situation changes dramatically if the ESQPT takes place within the chaotic region, where the second adiabatic invariant does not exist any more and the dynamics spreads over the two degrees of freedom of the Hamiltonian, exhibiting a continuous Density of States (with divergent derivative).

The case shown in figure 8(b) for a ratio $\omega/\omega_0 = 1$, corresponds to $\omega_F/\omega_0 = 4.13$, where the fast pseudospin approximation is valid, at least in a certain energy interval above the ground-state energy. Because the ratio $\omega_F/\omega_0 = 4.13$ is not much larger than 1, it is expected that this energy interval does not extend too far from the ground-state. According to the fast pseudospin approximation, the spectrum has to be organized in a finite number ($2j + 1$) of bands labelled by the quantum number $n^\prime$. The Peres lattice shows only a small fraction of them (about 16), and fewer are correctly reproduced by the fast pseudospin approximation (around the first 6). The number of lowest states correctly reproduced by the fast pseudospin approximation decreases as we look at more excited bands. The fast pseudospin approximation provides a qualitative, and sometimes quantitative, description of the regular part of the Peres lattice, until the onset of chaos in the model. It is worth emphasizing that the fast pseudospin approximation gives a more accurate and extended description of the exact spectrum than the Holstein–Primakoff approach [69], the only analytic approximation to the low energy region of the Dicke model known so far.

In figures 8(c) and (d), corresponding to $\omega/\omega_0 = 4$ and $\omega_F/\omega_0 \sim 1$, the numerical exact results confirm what is expected from the analysis of section 3. None of the fast-slow approximations gives a good description of the Peres lattice, not even in the low energy part. It interesting to note that the Peres lattice of this case is completely regular in the low energy sector, remaining as a challenge to obtain an analytical approximation to the Dicke model able to describe it.

In figure 9(a) large coupling $f = 3$ in the superradiant phase is considered. In panel (a) a ratio $\omega/\omega_0 = 20$ is used, in the region of applicability of the fast boson approximation. Even if $\omega_F/\omega_0$ is much larger than 1, the boson to pseudospin ratio is not so large $\omega_F/\omega_F = 2.24$. The exact Peres lattice is globally very well described by the approximation, though numerical discrepancies can be observed in the low-energy region. The expected finite bands of the approximation are clearly seen in the numerical results. Every band exhibits its own ESQPT at $E(\omega_0, j) = -1 + \omega F(\omega_0, j)n$ and ends at energies close to $E(\omega_0, j) = 1 + \omega F(\omega_0, j)n$.

In figure 9(b), the results of the fast pseudospin approximation are shown together with the exact numerical results for $\omega/\omega_0 = 0.2$, which gives $\omega_F/\omega_0 = 45.28$. Since this particular case is numerically more challenging, a more modest system size was employed ($j = 15$), while the
The energy range considered includes the energy of the ESQPT \( \omega_j = -E_j \), and almost all the exact states below this energy are very well reproduced by the approximation. The approximation seems to work very well until the onset of chaos in the system, which appears at large excitation energies, larger for more excited bands (larger \( m' \)). This result confirms that the ESQPT does not imply that the fast-pseudospin approximation becomes invalid around the critical energy. On the contrary, this approximation provides a valuable tool to study the properties of the ESQPT in this model.

In figures 9(c) and (d), the same Peres lattice for \( \omega_\omega = 9 \) is shown. In panel (c) is compared with the results of the fast boson approximation and in panel (d) with those of the fast pseudospin one. Except for the ground-state, as expected from the analysis of section 3, the exact Peres lattice is not even closely reproduced by the results of any approximation. Although in this case no approximation gives a close description of the exact results, as in the previous figures, the low energy part of the Peres lattice is regular. As mentioned, it would be interesting to find an approximation able to describe this regular region.

6. Conserved quantities

Associated to the fast boson and to the fast pseudospin approximations, adiabatic invariants can be identified. These adiabatic invariants are, to the extent that one or the other approximation is valid, approximate integrals of motion. In this section, these adiabatic invariants
are given and discussed. Numerical results are presented to illustrate that these invariants are effectively approximated integrals of motion. Numerical results are presented both in the quantum and classical version of the Dicke model.

6.1. Adiabatic invariant in the fast pseudospin approximation

In the case of the fast pseudospin approximation, the adiabatic invariant is the projection of the pseudospin along the precession axis, which, in turn, is a function of the \( q \) coordinate of the slow bosonic variables

\[
\hat{J}_c = \frac{1}{\sqrt{1 + \frac{\omega_j^2}{\omega_0^2}q^2}} \hat{J}_z + \frac{f}{\sqrt{1 + \frac{\omega_j^2}{\omega_0^2}q^2}} \hat{J}_x, \tag{31}
\]

where we express the adiabatic invariant in terms of \( f = \gamma/\gamma_c \) and \( \omega/\omega_0 \). In the classical version of the model, the previous expression gives the approximate integral of motion by considering the pseudospin components as classical variables, \( \hat{J}_i \to j_i \), of a vector of magnitude \( \| \mathbf{j} \| = \sqrt{j(j+1)} \). In the quantum case, the corresponding quantum observable is obtained by substituting the variable \( q \) by creation and annihilation operators \( \{ q \to (\hat{a} + \hat{a}^\dagger)/\sqrt{2} \} \)

\[
\hat{J}_c = \frac{1}{\sqrt{1 + \frac{\omega_j^2}{\omega_0^2}(\hat{a} + \hat{a}^\dagger)^2}} \hat{J}_z + \frac{f}{\sqrt{1 + \frac{\omega_j^2}{\omega_0^2}(\hat{a} + \hat{a}^\dagger)^2}} \hat{J}_x. \tag{32}
\]

In figure 10 full quantum results are presented. The Peres lattice of \( \hat{J}_c \) is shown for \( f = \gamma/\gamma_c = 3 \) and a ratio \( \omega/\omega_0 = 0.2 \) (the same parameters used in panel (b) of figure 9). The results show that, in the low-energy region where the approximation is valid, the Hamiltonian eigenstates are organized in horizontal bands associated each with the quantum number \( m' \).
The calculation of the uncertainty $\Delta J_z$ gives values very close to zero for the lowest energy eigenstates of each band, showing that these Hamiltonian eigenstates are simultaneously approximated eigenstates of operator $\hat{J}_z$.

In figure 11, similar results are presented, but for the classical version of the model (see appendix C for the definition of the classical Hamiltonian employed). In panels (a) and (b) the variance of temporal averages of the classical variable $\hat{J}_z$ is calculated integrating numerically the classical trajectories between the times 0 to $T$, for a wide sample of initial conditions in a surface of constant energy $[E(\omega_0 f) = -1.4]$ with $p = 0$.

Figure 11. Temporal variances $\Delta \hat{J}_z$ of the classical Dicke model for a wide sample of initial conditions in the energy surface $E(\omega_0 f) = -1.4$ with $p = 0$. Panel (a) is for $f = 2$ with $\omega_0 = 1$ and panel (b) for $f = 3$ with $\omega_0 = 0.2$. Panels (c) and (d) show, respectively, Poincaré sections for the same parameters and energy. (horizontal dashed lines).
The same sets of parameters as figures 8(b) and 9(b) respectively, were employed: $f = 2$ and $\omega / \omega_0 = 1$ in figure 11(a) and $f = 3$ and $\omega / \omega_0 = 0.2$ in figure 11(b). The variance of $j_z$ takes values very close to zero (indicating to what extent $j_z$ is an approximated constant of motion) for initial conditions in the regular part of the phase space. Figures 11(c) and (d) display the corresponding Poincaré sections, allowing to identify regular and chaotic regions. For initial conditions in the chaotic regions of the phase space, the variance of $j_z$ takes large values. These results suggest that the onset of chaos in the model is intimately related with the breaking of the fast pseudospin approximation, where the dynamical variable $j_z$ ceases to be an approximated integral of motion.

A complementary result showing the ability of the fast pseudospin approximation to explain and reproduce the regular energy regime of the classical Dicke model, is shown in figure 12 for the case $\omega / \omega_0 = 0.2$ with $f = 3$. We select as initial conditions those corresponding to the minimal energy configurations of the effective potentials $V_{\mu}(q)$, for different values of $m' \in [-j, j]$ (in the classical version $m'$ is a continuous variable). We integrate the classical equations of motion, obtaining the classical trajectories for these initial conditions. A Fourier analysis of a given dynamical variable (in this case $q$) allows to extract its fundamental frequencies. The most relevant classical frequencies are compared with those obtained from the fast pseudospin approximation given in equations (9) and (11). The agreement between the set of classical frequencies and their analytic approximated estimation is very good, until the onset of chaos in the system, which occurs at energy $E(\omega_j, j) \sim 0$ for the band head states (lowest energy states of each $m'$), as can be seen in figure 9(b).

### 6.2. Adiabatic invariant in the fast boson approximation

In the case of the fast boson approximation, the adiabatic invariant is the number of quanta of the shifted operator $\hat{b}^\dagger \hat{b}$, which written in terms of the boson and pseudospin variables is

$$\hat{b}^\dagger \hat{b} = \hat{a}^\dagger \hat{a} + \sqrt{\frac{\omega_0}{\omega}} \frac{f}{\sqrt{2j}} (\hat{a}^\dagger \hat{a} + \hat{a}^\dagger) \hat{j}^\dagger + \frac{\omega_0}{\omega} \frac{f^2}{2j} \hat{j}^2,$$

Figure 12. Light dots (green) depict the principal frequencies of the exact dynamical classical variable $q$, as a function of energy, for initial conditions corresponding to the minimal energy configurations of the effective potentials $V_{\mu}(q)$. Solid black lines are the analytic pseudospin and bosonic frequencies obtained from the fast pseudospin approximation, equations (9) and (11). Insets show a closer view of the fast and slow frequencies. The system considered was $\omega / \omega_0 = 0.2$ with $f = 3$. 

The same sets of parameters as figures 8(b) and 9(b) respectively, were employed: $f = 2$ and $\omega / \omega_0 = 1$ in figure 11(a) and $f = 3$ and $\omega / \omega_0 = 0.2$ in figure 11(b). The variance of $j_z$ takes values very close to zero (indicating to what extent $j_z$ is an approximated constant of motion) for initial conditions in the regular part of the phase space. Figures 11(c) and (d) display the corresponding Poincaré sections, allowing to identify regular and chaotic regions. For initial conditions in the chaotic regions of the phase space, the variance of $j_z$ takes large values. These results suggest that the onset of chaos in the model is intimately related with the breaking of the fast pseudospin approximation, where the dynamical variable $j_z$ ceases to be an approximated integral of motion.
Again, we express it in terms of \( \gamma / \gamma \) and \( \omega / \omega \) parameters.

In figure 13, the Peres lattice of operator \( \hat{b}^\dagger \hat{b} \) is shown for the case \( f = 2, \omega / \omega = 10 \) (the same parameters used in panel (a) of figure 8). It can be seen that the finite bands are associated to the quantum number \( n' = 0, 1, ... \) (dashed lines). The colors of the points in the Peres lattice indicate the uncertainty \( \Delta \hat{b}^\dagger \hat{b} \). Observe that the lowest energy states of every band and all the members of the lowest band (\( n' = 0 \)) have very low uncertainty, which indicates that these Hamiltonian eigenstates are, to some extent, also simultaneous eigenstates of operator \( \hat{b}^\dagger \hat{b} \).

6.3. Quantization

Finally, we discuss the full quantum treatment that can be performed from the fast-slow approximations. In the case of the fast pseudospin approximation, once the pseudospin part has been diagonalized by considering a rotation around the \( y \) axis, we are left with the Hamiltonian

\[
H_{m'}(q, p) = \frac{\omega}{2} \hat{p}^2 + V_{m'}(q),
\]

for each \( m' \). This Hamiltonian can be quantized in the standard way to obtain a Schrödinger equation

\[
\left( -\frac{\omega}{2} \partial_q^2 + V_{m'}(q) \right) \Psi_{E}(q) = E \Psi_{E}(q).
\]

(34)

Figure 14 shows the energy spectrum as a function of the coupling, obtained from solving this equation. The approximated spectra are compared with numerical exact results for couplings ranging from 0 until the superradiant phase, in the case \( \omega / \omega = 0.1 \) with \( j = 10 \). Only the lowest 10 exact energy states of each parity sector are plotted. The approximate results reproduce remarkably well the exact spectrum. The approximation allows to classify the states according to the quantum number \( m' \). In the figure, states associated to the two lowest \( m' \) values, \( m' = -j \) and \( m' = -j + 1 \), are indicated by black and red lines respectively. In the approximation, crossings between same parity states associated to different \( m' \)-sectors are allowed.

Figure 13. Peres lattice of operator \( \hat{b}^\dagger \hat{b} \) for \( f = 2 \) and \( \omega / \omega = 10 \). The point colors indicate the uncertainty of the respective Hamiltonian eigenstate. Horizontal dashed lines indicate the values \( n' = 0, 1, ... \) the lines begin and finish at the energies predicted by the fast boson approximation.
A more detailed study is necessary to determine if these crossings become avoided ones in the exact spectrum. Preliminary results indicate that a numerical precision beyond the standard double precision, which has been employed in our numerical codes, is needed in order to be conclusive about this issue.

For the fast-boson approach, the procedure is similar. After diagonalizing the fast boson variable by using the shift transformation, we are left, for every quantum number $n'$, with an effective Hamiltonian for the pseudospin variables, equation (15). By considering the pseudo-spin components as $su(2)$ quantum operators, the resulting Hamiltonian is a particular version of the Lipkin–Meshkov–Glick model that can be numerically diagonalized, giving the spectrum associated to band $n'$. It is interesting to note that the same LMG model is obtained for each band $n'$, and the only difference between the bands’ spectra is an additive simple constant $\omega n'$.

In figure 15 we compare the approximate spectra obtained from the previous approach with the exact ones obtained numerically. We use a ratio $\omega / \omega_0 = 10$ and couplings from 0 until $\gamma = 2 \gamma_c$. Similar to the fast pseudospin case, the agreement is very good.
A complete survey of the Born–Oppenheimer approximation (BOA) applied to the Dicke model has been presented. The study includes both the case when the fast variable is the pseudospin one and the case when the fast variable is the bosonic one. The ranges of validity of both versions of the BOA were unveiled. It was found that in the normal phase a simple criterion is \( \omega / \omega_0 \gg 1 \) for the fast boson BOA and \( \omega / \omega_0 \ll 1 \) for the fast pseudospin one, but in the superradiant phase, as a consequence of the change in the fundamental effective excitations, the criterion changes to \( \omega / \omega_{0f} \gg f^2 \) and \( \omega / \omega_{0f} \ll f^2 \) for the fast boson and fast pseudospin BOAs respectively (\( \gamma = f_\gamma = \sqrt{\omega_{0f} \omega_f / 2} \)).

More restrictive and detailed criteria were obtained by comparing the low energy frequencies coming from the BOAs (\( \omega_B \) and \( \omega_F \) for the boson and pseudospin respectively) with those obtained from a quadratic approximation around the minimal energy configuration. Ample numerical tests were presented to show the effectiveness of these criteria and the ability of the BOAs to reproduce the exact results of the Dicke model, both in its classical and quantum versions. It was found that the BOAs reproduce very well the exact results in the normal and superradiant phase, from the ground state until the energy where chaos appears in the model. The approximations explain in a clear way the appearance of independent energy bands in the regular part of the spectrum in a wide region of the model parameter space. Quantitative discrepancies are found, but they are reduced for larger ratios \( \omega_B / \omega_F \) in the case of the fast boson BOA and \( \omega_F / \omega_B \) in the case of the fast pseudospin one.

The adiabatic invariants associated to the two different BOAs were explicitly identified. They make possible to label single states with two quantum numbers within the energy regions where the BOAs are valid. We have found that they are quasi commuting operators with the quantum Hamiltonian and very approximately conserved quantities in the classical case. Our results show that the BOA is very efficient to explore and explain the phenomenology observed in the regular regime of the Dicke model, up to energies where chaos appears and no known approximation is valid.

Figure 15. Similar to figure 14 but for \( \omega / \omega_0 = 10 \) with \( j = 60 \), and the approximate results are obtained from the fast boson approximation. Black lines correspond to the quantum number \( n' = 0 \), whereas the upper red lines are associated to \( n' = 1 \).
Besides the understanding of the spectrum in a wide region above the ground-state, these results provide a valuable tool to study dynamic phenomena in the Dicke model. In particular, we expect that the second adiabatic invariant determines the non-equilibrium dynamics and the further thermalization after a quench, leading the system to a generalized Gibbs ensemble, instead of a standard Gibbs ensemble, within the region in which either the fast boson or the fast pseudospin approximation is valid. Also, as we have foreseen in this work, the properties of the ESQPTs in this model are delimited by the existence (or not) of the second adiabatic invariant. Within the region in which the BOA is valid, the critical behavior is the corresponding to a system of just one degree of freedom, showing a logarithmic divergence in the density of states and expectation values of observables. It is worth noting that the same kind of critical behavior has been recently reported in the Rabi model, in the limit in which the atomic frequency is much larger than the frequency of the field [78], that is, in a region in which the fast pseudospin approximation is expected to hold.

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Appendix A. Low energy frequencies of slow variables

In this appendix we derive the lowest energy frequencies of the slow variables for the two BOAs, by expanding the effective Hamiltonians around the minimal energy configurations until quadratic terms, \( H = \frac{1}{2}Ap^2 + \frac{1}{2}Bq^2 \), and using the very well known result for the frequency of quadratic Hamiltonians \( \omega = \sqrt{AB} \).

A.1. Fast pseudospin: low energy boson frequencies for each \( m' \)

We expand the potential \( V_m(q) \) appearing in the effective Hamiltonian (5), around the minimal energy configuration given in equation (8). When the potential is a double well \((-j \leq m' \leq -jf^2)\) this expansion gives

\[
H_m(q, p) = E_{\text{min}}^m + \frac{\omega}{2} p^2 + \frac{\omega}{2} \left( 1 - \left( \frac{j}{m'} \right)^2 \frac{1}{f^4} \right) (q - q_{\text{min}}^m)^2 + \ldots,
\]

therefore the boson frequency is

\[
\omega_B(E_{\text{min}}^m) = \sqrt{\omega^2 \left( 1 - \left( \frac{j}{m'} \right)^2 \frac{1}{f^4} \right) + \frac{f^4 - \left( \frac{j}{m'} \right)^2}{f^4}}.
\]

When the potential has a single minimum at \( q = 0 \), the expansion of the potential around it, yields

\[
H_m(q, p) = \omega m' + \frac{\omega}{2} p^2 + \frac{\omega}{2} \left( 1 + \frac{m'}{j} f^2 \right) q^2 + \ldots.
\]
from where the boson frequency can be easily obtained

\[ \omega_B(F'_{\min}) = \sqrt{\omega^2 \left(1 + \frac{m' f^2}{j^2}\right)} = \omega \sqrt{1 + \frac{m' f^2}{j^2}}. \]

### A.2. Fast boson: low energy pseudospin frequency

To find the slow pseudospin frequencies, we express the effective Hamiltonian (15) in terms of the canonical variables shown in equation (18)

\[ H_n(Q, P) = \omega n' - \omega_0 j + \frac{\omega_0 f}{2} \left[ \frac{P^2}{j^2} + Q^2 - \frac{f^2 Q^2}{4} \left(4 - \frac{P^2}{j^2} - Q^2\right)\right]. \]

Then, we expand this Hamiltonian around the minimal energy configuration given in equation (17). For \( \gamma < \gamma_c \), the minimal energy configuration in terms of the canonical variables is \((Q_{\min}, P_{\min}) = (0, 0)\). The expansion of the Hamiltonian until quadratic terms is

\[ H_n(Q, P) = \omega n' - \omega_0 j + \frac{1}{2} \omega_0 (1 - f^2) Q^2 + \ldots \]

From this expression we obtain the pseudospin frequency

\[ \omega_f(F'_{\min}) = \sqrt{(\omega_0 f) \omega_0 (1 - f^2)} = \omega_0 \sqrt{1 - f^2}. \]

For \( \gamma > \gamma_c \), the minimal energy configuration in terms of variables \( Q \) and \( P \) reads \((Q_{\min}, P_{\min}) = (\pm \sqrt{2(1 - f^{-2})}, 0)\). Expanding the Hamiltonian around this point gives

\[ H_n = E_{\min} + \frac{1}{2} \omega_0 (1 + f^2) P^2 + \frac{1}{2} \omega_0 2j(f^2 - 1)Q^2 + \ldots, \]

therefore the pseudospin frequency is

\[ \omega_f(F'_{\min}) = \sqrt{\omega_0 2j(f^2 - 1) \frac{\omega_0 (1 + f^2)}{2j}} = \omega_0 \sqrt{(f^2 - 1)(1 + f^2)} = \omega_0 \sqrt{f^4 - 1}. \]

### Appendix B. Details of the semiclassical calculations of observables

In this appendix we present in detail the calculations yielding to the expressions for the expectation values of selected observables shown in section 4.

#### B.1. Fast pseudospin calculation

First, we calculate the denominator appearing in the expressions for the expectation values, which is proportional to the semiclassical approximation to the Density of States \([\nu^f(E, m')]\) or Weyl’s formula

\[ 2\pi \nu^f(E, m') = \int dq \, dp \, \delta [E - H_{\min}(q, p)]. \] (B.1)
The integration over $p$ can be performed, by expressing the delta function in terms of the zeros of the equation $E - H_{m'}(q, p) = 0$,

$$p_{\pm} = \pm \frac{1}{\omega} \sqrt{\frac{2}{\omega} (E - V_{m'}(q))}. \quad (B.2)$$

From where, we obtain

$$2\pi\nu/(E, m') = \int dq \int dp \left\{ \frac{\delta(p - p_{+})}{|\partial H_{m'}/\partial p|_{p_{+}}} + \frac{\delta(p - p_{-})}{|\partial H_{m'}/\partial p|_{p_{-}}} \right\}. \quad (B.3)$$

Since $|\partial H_{m'}/\partial p|_{p_{+}} = |p_{+}| = \sqrt{2\omega(E - V_{m'}(q))}$, the integration over variable $p$ gives

$$2\pi\nu/(E, m') = \frac{2}{\sqrt{E - V_{m'}(q)}} \int_{q \in \mathcal{I}_{E, m'}} dq,$$ \quad (B.4)

provided that $p_{\pm}$ are real numbers. $V_{m'}(q)$ is defined in equation (6). This latter condition is guaranteed by the classically allowed integration region $q \in \mathcal{I}_{E, m'}$, defined by $E \geq V_{m'}(q)$. The boundaries of this integration region are obtained by solving $E = V_{m'}(q)$, which implies the following quadratic equation for the variable $q^2$

$$E^2 - \omega Eq^2 + \frac{\omega^2}{4} q^4 = m'^2 \left( \frac{\omega^2}{4} + \frac{4\gamma^2}{j} q^2 \right),$$

whose solutions are

$$q^2 = \left( \frac{2j\omega}{\omega^j} \right) \left( \frac{E}{\omega^j} + \left( \frac{m'}{j} \right)^2 f^2 + \frac{m'^2}{j^2} \right),$$

where we have used the definition $f = \gamma/j\omega$ with $\gamma = \sqrt{\omega^2 - 1}$. For energies inside the double well potential, $-j m' \leq -j f^2$ and $E_{\text{min}} < E \leq \omega m'$ (see panel (b) of figure 1), the classically allowed region is formed by two disconnected intervals

$$\mathcal{I}_{E, m'} = [-q_{+}(E, m'), -q_{-}(E, m')] \cup [q_{+}(E, m'), q_{+}(E, m')] .$$

For any other case the classical allowed region is given by the interval

$$\mathcal{I}_{E, m'} = [-q_{+}(E, m'), q_{+}(E, m')] .$$

To calculate the expectation value of the bosonic number operator, we substitute $\hat{a}^\dagger \hat{a} \to \frac{1}{2}(p^2 + q^2)$ to obtain

$$\langle \hat{a}^\dagger \hat{a} \rangle_{E, m'} = \frac{1}{2\pi\nu/(E, m')} \int dp dq \frac{1}{2}(p^2 + q^2) \delta[E - H_{m'}(p, q)].$$

As before, we express the delta function in terms of the zeros of its argument (B.2), to obtain

$$\langle \hat{a}^\dagger \hat{a} \rangle_{E, m'} = \frac{1}{2\pi\nu/(E, m')} \int dq \int dp \frac{1}{2}(p^2 + q^2) \left\{ \frac{\delta(p - p_{+})}{|\partial H_{m'}/\partial p|_{p_{+}}} + \frac{\delta(p - p_{-})}{|\partial H_{m'}/\partial p|_{p_{-}}} \right\}.$$
\[ \int_{q \in \mathcal{I}_{E,m'}} \frac{1}{2} \left( \frac{p_x^2 + q^2 + q_x^2 + q_y^2}{\sqrt{2\omega(E-V_m(q))}} \right) dq = \frac{1}{2\sqrt{2\omega}} \int_{q \in \mathcal{I}_{E,m'}} \left( \frac{2q^2 + \frac{2}{\omega}(E-V_m(q))}{\sqrt{E-V_m(q)}} \right) dq. \]

where we have used, the interval defined above \( q \in \mathcal{I}_{E,m'} \) to guarantee that \( p_\perp \) are real numbers. By using, in the numerator of the integrand, the explicit expression of the potential \( V_m(q) \) (6), we obtain after some direct simplifications the result

\[ \langle \hat{a}^\dagger \hat{a} \rangle_{E,m'} = \frac{1}{2\pi \nu'(E,m')} \frac{1}{\omega} \sqrt{\frac{2}{\omega}} \int_{\mathcal{I}_{E,m'}} \frac{E - \sqrt{\omega_0^2 + \frac{2q^2}{\sqrt{\rho}}}}{\sqrt{E-V_m(q)}} dq. \]

To calculate the expectation value of \( J_z \), we take advantage of the inverse rotation \( \beta = \cos \beta' m \) to evaluate

\[ \langle jm' \mid \hat{J}_z \mid j, m' \rangle = \cos \beta \frac{m'}{\sqrt{\omega_0^2 + \frac{2q^2}{\sqrt{\rho}}}} m', \]

where we have used \( \langle jm' \mid \hat{J}_z \mid jm' \rangle = m' \), \( \langle jm' \mid \hat{J}_z \mid jm' \rangle = 0 \), and the definition of \( \cos \beta \) in equation (3). With this result the expectation value of \( \hat{J}_z \) for each \( m' \) and given energy \( E \) is

\[ \langle \hat{J}_z \rangle_{E,m'} = \frac{1}{2\pi \nu'(E,m')} \int dp dq \frac{\omega_0 m'}{\sqrt{\omega_0^2 + \frac{2q^2}{\sqrt{\rho}}}} \delta[E - H_m(p,q)]. \]

The integral appearing in the previous expression

\[ \int dq dp \frac{\omega_0 m'}{\sqrt{\omega_0^2 + \frac{2q^2}{\sqrt{\rho}}}} \delta[E - H_m(p,q)] = \int dq \frac{\omega_0 m'}{\sqrt{\omega_0^2 + \frac{2q^2}{\sqrt{\rho}}}} \int dp \delta[E - H_m(p,q)], \]

has the same integral over the variable \( p \) that we have already calculated before (B.1). Therefore, using the result of equation (B.4), we obtain

\[ \langle \hat{J}_z \rangle_{E,m'} = \frac{1}{2\pi \nu'(E,m')} \frac{\omega_0}{\sqrt{\omega_0^2 + \frac{2q^2}{\sqrt{\rho}}}} \sqrt{\int_{\mathcal{I}_{E,m'}} \frac{\omega_0 m' dq}{\sqrt{\omega_0^2 + \frac{2q^2}{\sqrt{\rho}}}}}. \]

\[ B.2. Fast boson calculations \]

As in the previous case, first we calculate the denominator appearing in the expression for the expectation values, which is proportional to the semiclassical approximation to the Density of States \( \nu'(E,m') \)

\[ 2\pi \nu'(E,m') = \int d\phi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \delta[E - H_m(j)], \]

(B.5)
\[ \int d\phi \int_{-j}^j d\phi \left( \frac{\delta(j_k - j_{k+})}{\partial H_{\phi}/\partial j_k} \right) \frac{\delta(j_k - j_{k-})}{\partial H_{\phi}/\partial j_k} \]  

where \( j_{k\pm} \) are the solutions of the equation \( E - H_{\phi}(j_k, \phi) = 0 \), given by

\[ \frac{\dot{j}_{k\pm}}{f} \cos^2 \phi + 1 = \pm \sqrt{F(\cos \theta)} \]

where we have used the definition \( F(x) \equiv 1 + 2f^2 \epsilon \rho x^2 + f^4 x^4 \), with \( \epsilon_n \equiv (E - \omega_n')/(\omega_n) \). Calculating the derivatives

\[ \partial H_{\phi}/\partial j_k = \omega_0 \left( \frac{\dot{j}_k}{f} \cos^2 \phi + 1 \right) \]

and evaluating them in \( j_{k\pm} \), we obtain

\[ \left| \partial H_{\phi}/\partial j_k \right|_{j_{k\pm}} = \omega_0 \sqrt{1 + 2 \cos^2 \phi \epsilon_n f^2 + \cos^4 \phi f^4} \equiv \omega_0 \sqrt{\mathcal{F}(\cos \phi)} \]

Therefore the integral reads

\[ 2\pi \nu^2(\epsilon, n') = \frac{1}{\omega_0} \int d\phi \int_{-j}^j d\phi \frac{\delta(j_k - j_{k+}) + \delta(j_k - j_{k-})}{\sqrt{\mathcal{F}(\cos \phi)}} \]

To perform the integration over the variable \( j_k \), we have to investigate when the solutions \( j_{k\pm} \) are in the interval \( [j, j] \). We can distinguish two cases depending on the value of \( \epsilon_n \). If \( \epsilon_n < -1 \), the two solutions \( j_{k\pm} \) are in the interval \( [j, j] \) if and only if the square root argument appearing in the denominator is greater or equal than 0:

\[ \mathcal{F}(\cos \phi) = 1 + 2 \cos^2 \phi \epsilon_n f^2 + \cos^4 \phi f^4 \geq 0 \]

The previous condition is satisfied in two intervals around \( \phi = 0 \) and \( \phi = \pi \)

\[ \phi \in [-\phi_0, \phi_0] \cup [\pi - \phi_0, \pi + \phi_0] \]

where \( \phi_0 \) is the solution of the equation \( 1 + 2 \cos^2 \phi \epsilon_n f^2 + \cos^4 \phi f^4 = 0 \), given by

\[ \cos^2(\phi_0) = \frac{1}{f^2} \left[ -\epsilon_n + \sqrt{\epsilon_n^2 + 1} \right] \]

Therefore in this case, \( \epsilon_n < -1 \), the integral over \( j_k \) gives

\[ 2\pi \nu^2(\epsilon, n') = \frac{1}{\omega_0} \int_{\phi_0}^{\pi - \phi_0} \frac{d\phi}{\sqrt{\mathcal{F}(\cos \phi)}} \frac{2}{\sqrt{\mathcal{F}(\cos \phi)}} \]

Now, since the dependence on \( \phi \) in the integrand enters through \( \cos^2 \phi \), the integral over \( \phi \in [-\phi_0, \phi_0] \cup [\pi - \phi_0, \pi + \phi_0] \) can be expressed as four times the integral in the interval \( \phi \in [0, \phi_0] \). In this way the final form for the integral is

\[ 2\pi \nu^2(\epsilon, n') = \frac{8}{\omega_0} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\mathcal{F}(\cos \phi)}} \]

For \( -1 < \epsilon_n \leq 1 \), only the solution \( j_{k+} \) is in the interval \( [-j, -j] \), and that for any value of \( \phi \in [0, 2\pi] \). Therefore in this case the integral takes the form
Gathering the previous results, we obtain the equation (28).

The expectation value of \( \hat{J} \),

\[
\langle \hat{J} \rangle_{\nu,E} = \frac{1}{2 \pi \nu^\alpha(E,n')} \int_{\phi_0}^{\phi_{\nu}} d\phi \int_{j_-}^{j_+} d\phi' \delta(E - H_{\nu}(\phi')) ,
\]

can be calculated similarly. By expressing the Dirac delta in terms of the roots of its argument, \( j_{\pm} \), we obtain

\[
\frac{1}{2 \pi \nu^\alpha(E,n')} \int d\phi \int_{j_-}^{j_+} d\phi' \frac{\delta(j_+ - \phi') + \delta(j_- - \phi')}{\omega_0 \sqrt{f(\cos \phi)}} .
\]

As before, in the case \( \epsilon_n \lesssim -1 \) the two roots are in \([-j, j]\) for \( \phi \) in the region defined above, therefore the integral gives

\[
\frac{4}{2 \pi \nu^\alpha(E,n') \omega_0} \int_{\phi_0}^{\phi_{\nu}} d\phi \sqrt{\cos \phi}.
\]

The sum of the roots is \( j_+ + j_- = -2j(j^2 \cos \phi) \), yielding to the expression

\[
\langle \hat{J} \rangle_{\nu,E} = -\frac{8j}{j^2 \omega_0 2 \pi \nu^\alpha(E,n')} \int_{\phi_0}^{\phi_{\nu}} d\phi \frac{\cos^2 \phi \sqrt{f(\cos \phi)}}{\cos \phi}.
\]

For \(-1 < \epsilon_n \lesssim 1\), since only the root \( j_+ \) is in the interval \([-j, j]\) and \( \phi \) is not restricted, the expectation value is

\[
\langle \hat{J} \rangle_{\nu,E} = \frac{1}{2 \pi \nu^\alpha(E,n') \omega_0} \int_{\phi_0}^{\phi_{\nu}} d\phi \frac{j_+}{\sqrt{f(\cos \phi)}} = \frac{j}{j^2 \omega_0 2 \pi \nu^\alpha(E,n')} \int_{\phi_0}^{\phi_{\nu}} d\phi \frac{1 - \sqrt{f(\cos \phi)}}{\cos^2(\phi) \sqrt{f(\cos \phi)}}.
\]

The summary of these results is shown in equation (29).

To calculate the expectation value of the boson number operator, \( \hat{a}^\dagger \hat{a} \), we write it in terms of the \( \hat{b}^\dagger \) and \( \hat{b} \) operators

\[
\hat{a}^\dagger \hat{a} = \hat{b}^\dagger \hat{b} + \frac{2 \gamma^2}{j \omega} j_+ - \left| \frac{2}{j} \right| \gamma \hat{\mu} (\hat{b}^\dagger + \hat{b}) ,
\]

(B.7)

to calculate \( \langle n'| \hat{a}^\dagger \hat{a} | n' \rangle = n' + \frac{2 \gamma^2}{j \omega} j_+ \), where we have used that \( \langle n'| (\hat{b}^\dagger + \hat{b}) | n' \rangle = 0 \) and \( \langle n'| \hat{b} \hat{b} | n' \rangle = n' \). With the previous result, the expectation value of the number of photons in the fast boson approximation reads

\[
\langle \hat{a}^\dagger \hat{a} \rangle_{\nu,E} = \frac{1}{2 \pi \nu^\alpha(E,n')} \int d\phi \delta(n' + \frac{2 \gamma^2}{j \omega} j_+) \delta(E - H_{\nu}(\phi'))
\]

\[
= n' + \frac{2 \gamma^2}{2 \pi \nu^\alpha(E,n') j \omega} \int d\phi (j^2 - j^2_+) \cos^2 \phi \delta(E - H_{\nu}(\phi')) ,
\]

(B.8)
where we have expressed the \( j_x \) component in terms of the canonical variables \( \phi = -j_{j} j \cos x z^2 \).

The two cases previously identified have to be considered separately to evaluate the integral of the second term. In the case \( \epsilon_a' \leq 1 \) the integral is

\[
\int \frac{2\gamma^2}{2\pi \nu'(E, n', j) \omega} \int d\phi d\phi' (j_x^2 - j_{x'}^2) \cos^2 \phi \delta \left[ (E - H_a(\vec{j}) \right] \]

\[
= \int \frac{2\gamma^2}{2\pi \nu'(E, n', j) \omega} \left[ j_x^2 - j_{x'}^2 \right] \cos^2 \phi d\phi.
\]

Now, by using the expressions for the roots \( j_{k+} \) and the definition of \( F(\cos \phi) \), we obtain

\[
j^2 - j_{x'}^2 + j^2 - j_{x}^2 = -4j^2 \frac{1 + f^2 \epsilon_a' \cos^2 \phi}{f^4 \cos^2 \phi}.
\]

By substituting this result in the integral, we obtain

\[
-\int \frac{2\gamma^2}{2\pi \nu'(E, n', j) \omega} \left( \frac{16j^2}{f^4 \omega} \right) \int \frac{1 + f^2 \epsilon_a' \cos^2 \phi}{\cos^2 \phi \sqrt{F(\cos \phi)}} d\phi.
\]

Recalling that \( f = \gamma/\gamma_c \) with \( \gamma_c = \sqrt{\omega \omega_c}/2 \), we simplify the factor in front of the previous integral to obtain the following expression for the expectation value of \( a'a \), valid in the case \( \epsilon_a' \leq 1 \)

\[
\langle \hat{a}a' \rangle_{E, n'} = n' - \frac{8j}{f^2 2\pi \nu'(E, n')} \int \phi d\phi \frac{1 + f^2 \epsilon_a' \cos^2 \phi}{\cos^2 \phi \sqrt{F(\cos \phi)}}.
\]

For \( -1 < \epsilon_a' \leq 1 \), the integral is

\[
\int \frac{2\gamma^2}{2\pi \nu'(E, n', j) \omega} \int d\phi d\phi' (j_x^2 - j_{x'}^2) \cos^2 \phi \delta \left[ (E - H_a(\vec{j}) \right] \]

\[
= \int \frac{2\gamma^2}{2\pi \nu'(E, n', j) \omega} \int d\phi \frac{j^2 - j_{x'}^2}{\omega \sqrt{F(\cos \phi)}} \cos^2 \phi.
\]

By using the expression for the root \( j_{x'} \) and the definitions of \( F \) and \( f \), the previous integral simplifies to yield the following expression

\[
\langle \hat{a}a' \rangle_{E, n'} = n' \int \frac{j}{f^2 2\pi \nu'(E, n')} \int \frac{2\pi}{2\pi \nu'(E, n')} \phi d\phi \frac{1 + \epsilon_a f^2 \cos^2 \phi - \sqrt{F(\cos \phi)}}{\cos^2 \phi \sqrt{F(\cos \phi)}}.
\]

The previous results are gathered in equation (30).

**Appendix C. Classical equations of motion of the Dicke model**

The full classical Dicke Hamiltonian that we use in this contribution is the one obtained from a semiclassical approximation to the quantum propagator written in terms of coherent states [79]

\[
H_\Delta = \langle \alpha; z | \hat{H}_D | \alpha, z \rangle.
\]
where $|\alpha; z\rangle = |\alpha\rangle \otimes |z\rangle$ with $|z\rangle$ and $|\alpha\rangle$ Bloch (for the pseudospin) and Glauber (for the bosons) coherent states respectively

\[
|z\rangle = \frac{1}{(1+|z|^2)^j} e^{izj}|j, -j\rangle
\]
\[
|\alpha\rangle = e^{-|\alpha|^2/2} e^{i\alpha'j}|0\rangle.
\]

The expectation value of the Dicke Hamiltonian in the coherent states reads (we take $\mathcal{N} = 2j$)

\[
H_{\text{cl}} = \omega' \langle \alpha \rangle^2 - \omega_j \left( \frac{1 - |z|^2}{1 + |z|^2} \right) + \frac{2\gamma}{\sqrt{2j}} (\alpha + \alpha^*) z^* + z^*.
\]

We consider canonical variables $(q, p)$ and $(\phi, j_z)$, related to the complex coherent parameters through

\[
\alpha = \frac{1}{\sqrt{2}} (q + ip) \quad z = \tan(\theta/2)e^{-i\phi} = \frac{1 + (j_z/\sqrt{2j})}{1 - (j_z/\sqrt{2j})} e^{-i\phi}.
\]

The expectation value of the Hamiltonian, written in terms of the previous variables define the classical Hamiltonian

\[
H_{\text{cl}}(q, p, \phi, j_z) = \frac{\omega'}{2}(p^2 + q^2) + \omega_j j_z + \frac{2\gamma}{\sqrt{2j}} q \sqrt{j_z^2 - j_z^2} \cos \phi,
\]

with classical Hamilton equations given by

\[
\frac{dq}{dt} = \partial_p H_{\text{cl}} \quad \frac{dp}{dt} = -\partial_q H_{\text{cl}}
\]
\[
\frac{d\phi}{dt} = \partial_{j_z} H_{\text{cl}} \quad \frac{dj_z}{dt} = -\partial_{\phi} H_{\text{cl}}.
\]

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