Little-Parks Oscillation of Superconducting Möbius Strip

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Little-Parks oscillation of a Möbius strip (or ring equivalently) made of a superconductor is studied based on Ginzburg-Landau theory. It is shown that, if the strip is wide enough, a novel state appears when the number of magnetic flux quanta threading the ring is close to a half odd integer. As a result the shape of Little-Parks oscillation of critical temperature is modified. We estimate the free energy of this state and give the phase diagram of the superconducting Möbius strip in a magnetic field.

KEYWORDS: Superconductivity, Little-Parks oscillation, Topological Matter, Möbius strip

Recently novel type of crystals have been grown by the group in Hokkaido University,1) which are referred to as “topological matter”. These crystals are ring, disk or cylinder shaped with one of their crystal axis along the azimuthal direction. Surprisingly, even a ring of a quasi-one-dimensional charge-density-wave system, e.g., NbSe3, was created and the formation of the charge-density-wave was experimentally confirmed,2) which indicates that these samples are good crystals and relatively free from the expected disorder originating from the bending of the crystal axis.

Among these crystals we concentrate on one of the most subtle structure, the Möbius strip. Möbius strip is a representative object which has non-trivial topology. Therefore it is interesting to see how its topological nature affects the physical properties of the system. Experimentally, the Möbius strip of NbSe3 has been created.3) We especially pay attention to the case where NbSe3 behaves as a superconductor. In this sample, the most conducting axis is along the azimuthal direction. As we see later this is actually the best configuration to observe non-trivial topological effects in the superconducting state.

The Möbius geometry was previously investigated concerning the correlated electrons in mesoscopic systems. Persistent current through the so-called “Möbius ladder” was theoretically studied by Mila, Stafford and Coppeni.4) The Möbius ladder is a simplified version of Möbius strip which consists of only one closed electronic channel winding twice along a circle. In case of a Möbius strip, however, the situation is more complicated, since there are many channels along the strip. The aim of this letter is to clarify how the persistent current (=supercurrent) through the superconducting Möbius strip is affected by its topological structure.

To answer this question we study Little-Parks oscillation, which is the oscillation of superconducting critical temperature as a function of the magnetic flux threading the ring.5, 6) We study this phenomenon based on the Ginzburg-Landau (GL) theory. Bearing the experimental situation in mind, we assume that the Möbius ring is formed by an array of coupled one-dimensional chains, although the continuum limit is taken in the end of the calculation.

Throughout this letter, the thickness of the strip is disregarded and the strip is assumed to be two-dimensional.

Our model is described by the following GL free energy,

$$F = \sum_{i=K+1}^{K} \int_{0}^{L} dx \left\{ \frac{1}{2m^*} \left| \frac{\hbar}{i} \partial_x + \frac{e^*}{c} A_x \right| \psi_i \right|^2 + \alpha |\psi_i|^2 + \frac{b}{2} |\psi_i|^4 \right\} \right.$$  

$$+ \sum_{i=-K+1}^{K-1} \int_{0}^{L} dx \left| \psi_{i+1} - \psi_i \right|^2. \tag{1}$$

Here the $x$-coordinate ($0 < x \leq L$) is taken along the ring. The geometry of the system is shown in fig. 1. $c$, $m^*$ and $-e^*$ are the speed of light, and the mass and charge of a Cooper pair, respectively. $v$ is a parameter of interchain Josephson coupling, and $a$ and $b$ are GL parameters. The order parameter of the $i$-th chain is denoted by $\psi_i(x)$. Boundary condition for the order parameter is given by,

$$\psi_i(0) = \psi_{i+1}(L), \quad \psi_i(L) = \psi_{i-1}(0). \tag{2}$$

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**Fig. 1.** (a) Structure of the Möbius ring threaded by a magnetic flux $\Phi$. (b) Chains consisting the ring. Both ends are identified.
This comes from the fact that the \(i\)-th chain and the \((-i+1)\)-th chain are connected, making a closed chain of total length \(2L\), which is a direct conclusion of the Möbius strip (see fig. 1). Number of chains in the cross section of the strip is assumed to be even (= 2\(K\)) for simplicity. The \(x\)-component of vector potential \(A_x\) is set to be a constant \(A_x = \Phi/L\), where \(\Phi\) is magnetic flux threading the ring. In this letter we consider only the Aharonov-Bohm flux, neglecting the effects of magnetic field applied on the strip.

We write the order parameter as \(\psi_l(x) = \chi_i e^{i\theta_l(x)}\) and assume the amplitude \(\chi_i\) to be a constant. The following choice of \(\theta_l(x)\) may be natural,
\[
\theta_l(x) = \begin{cases} 
\frac{\pi l}{L} + \pi l_{i-1} & (i \leq 0) \\
\frac{\pi l}{L} & (i \geq 1)
\end{cases},
\]
where \(l_i\)'s are real numbers such that \((l_i + l_{i+1})/2\) is an integer. This makes the order parameter single-valued. Since the \(i\)-th chain and the \((-i+1)\)-th chain are essentially the same chain, we assume that they have same amplitude and wave number: \(\chi_i = \chi_{-i+1}\) and \(l_i = l_{i+1}\). Then \(l_i\)'s are further limited to integers. By substituting these forms into Eq. (1) and performing the integration over \(0 < x \leq L\), we obtain the following expression,
\[
F = L \sum_{i = -K+1}^{K} \left[ \frac{\hbar^2}{2m^*} \left( \frac{2\pi}{L} \right)^2 \left( \frac{l_i}{2} - f \right)^2 \chi_i^2 
+ \alpha \chi_i^2 + \frac{b}{2} \chi_i^4 \right] 
+ L \sum_{i = -K+1}^{K-1} v \left( \chi_{i+1}^2 + \chi_i^2 - 2 \chi_{i+1} \chi_i \delta_{i+1,i} \right) 
+ 2Lv(1 - \cos \pi l_1)\chi_0 \chi_1,
\]
where \(f = \Phi/\phi_0\) with \(\phi_0 = hc/e^*\). Next we minimize the free energy with respect to \(\chi_i\)'s and \(l_i\)'s. In order to minimize the free energy in the bulk, namely the third term of the second line, we set \(l_i = l\) for all \(i\). This, however, gives rise to an increase of the free energy arising from the last term of Eq. (4), if \(l\) is an odd number.

Here we transfer to the continuum description of the free energy Eq. (4) assuming that the characteristic length scale in the transverse direction, namely the transverse coherence length, is much larger than interchain spacing which we denote by \(a\). We define the continuous variable by \(y = a\). After scaling the order parameter and the constants appropriately,
\[
\frac{\chi_i}{\sqrt{a}} \rightarrow \Psi(y), \quad v \rightarrow \frac{v}{a^2}, \quad b \rightarrow \frac{b'}{a},
\]
we obtain the continuum free energy as,
\[
F = L \int_{-W/2}^{W/2} dy \left[ \frac{\hbar^2}{2m^*} \left( \frac{2\pi}{L} \right)^2 \left( \frac{\bar{v}}{2} - f \right)^2 \Psi^2 
+ \alpha \Psi^2 + \frac{b'}{2} \Psi^4 \right] 
+ 2L\bar{v}(1 - \cos \pi l)\Psi(0)^2.
\]
(6)

The width of the strip is given by \(W = 2Ka\) and \(\bar{v}\) is given by \(\alpha + \frac{\hbar^2}{2m^*} \left( \frac{2\pi}{L} \right)^2 \left( \frac{1}{2} - f \right)^2\). By setting \(\alpha \equiv \alpha(0)(\frac{L}{\pi l - 1})\) and introducing zero-temperature longitudi-
\[ t - 1 + 12(\xi_\perp(0)/W)^2 = 0 \] determines the critical temperature. By comparing this with the critical temperature of even-\( l \) state, we can determine the phase diagram near \( T_c \), which is schematically shown in fig. 2. As one can see from this figure, the odd-\( l \) state is realized only when \( t_1 > t_2 \) is satisfied, where \( t_1 \) and \( t_2 \) are the maximum of the critical temperature of odd-\( l \) state and the minimum of that of even-\( l \) state, respectively, as shown in fig. 2. Here we introduce new parameters, \( r_\perp = \xi_\perp(0)/W \) and \( r_\parallel = \xi_\parallel(0)/L \). Since \( t_1 = 1 - 12r_\perp^2 \) and \( t_2 = 1 - \pi^2r_\parallel^2 \) according to the present calculation, we obtain the criterion for the odd-\( l \) state as \( r_\perp < \pi r_\parallel/(2\sqrt{3}) \).

![Fig. 2. Critical temperature of each state labelled by \( l \). The maximum of the critical temperature of the odd-\( l \) state is indicated by \( t_1 \), whereas the minimum of that of even-\( l \) state is indicated by \( t_2 \).](image)

Next we study the low temperature region \( t \ll 1 \) of odd-\( l \) state. As a solution of Eq. (8), we adopt the following formula, which satisfies \( \lim_{y \to \pm \infty} \Psi(y) = \text{const.} \)

\[
\Psi(y) = \sqrt{\frac{1 - t}{b}} \tanh \left( \sqrt{\frac{1 - t}{\xi_\perp(0)}} |y| + \frac{d}{2} \xi_\perp(0) \right),
\]

where \( d \) is determined so that Eq.(9) is satisfied. Here we have neglected the effect of the boundaries \( y = \pm W/2 \) assuming \( \xi_\perp(0) \ll W \). The condition for \( d \) reduces to,

\[
\sqrt{\frac{1 - t}{2} \xi_\perp(0)} = \sinh \left( \sqrt{\frac{1 - t}{2} \frac{2d}{\xi_\perp(0)}} \right),
\]

which is fulfilled by setting \( d \approx a/2 \) because \( a \ll \xi_\perp(0) \) is supposed. The free energy is calculated as follows. By substituting Eq. (10) into Eq. (7) we obtain the following expression,

\[
F \simeq -LW \frac{\alpha(0)(1 - t)^2}{2b} \left[ 1 - \frac{8\sqrt{2}}{3} \frac{r_\perp}{\sqrt{1 - t}} \right], \quad (r_\perp \ll (12))
\]

Again we have taken \( a \to 0 \) limit here and the last term of Eq. (7) vanished.

Next we determine the phase boundary between even-\( l \) and odd-\( l \) state. The free energy of even-\( l \) and odd-\( l \) state, normalized by \( LW\alpha(0)/(2b) \), are given respectively by

\[
f_l^{(o)} = -(1 - t)^2 + 2(1 - t)(2\pi r_\parallel)^2 \left( \frac{l}{2} - f \right)^2 + O(r_\parallel^4),
\]

and odd-\( l \) state is supposed. The free energy is calculated as follows.

\[
\begin{align}
\bar{f}_l^{(o)} &= -(1 - t)^2 + 2(1 - t)(2\pi r_\parallel)^2 \left( \frac{l}{2} - f \right)^2 + \frac{8\sqrt{2}}{3} (1 - t)^{3/2} r_\perp + O(r_\parallel^4, r_\perp^2 r_\parallel).
\end{align}
\]

By equating \( f_2^{(o)} \) and \( f_3^{(o)} \) (\( k \) is an integer) we obtain the boundary between even-\( l \) and odd-\( l \) state. For example, the boundary between \( l = 0 \) and \( l = 1 \) state is given by

\[
t_1 + \frac{1 + \sqrt{1 - t}}{\sqrt{1 - t}} = f. \quad (12)
\]

From this, we can see that, at \( T = 0 \), the odd-\( l \) state can exist only when \( r_\perp < 3\pi^2 r_\parallel^2/(4\sqrt{2}) \) is satisfied, since otherwise the boundary between \( l = 0 \) and \( l = 1 \) state crosses the boundary between \( l = 1 \) and \( l = 2 \) state at nonzero temperature, giving rise to direct transition between \( l = 0 \) and \( l = 2 \) state below the crossing temperature.

The obtained phase diagram is given in fig. 3. Solid lines show the phase boundaries obtained from the above argument. The broken lines show guess of the phase boundaries in the intermediate temperature region, where the present treatment is not applicable. As we see from the figure, there can be three types of behaviors depending on the geometry of the system. As is pointed out above, the odd-\( l \) state are not at all stable when \( \frac{2\sqrt{3}}{3} r_\parallel < r_\perp \). In this case we obtain the situation depicted in fig. 3 (a). If \( \frac{3\pi^2}{4\sqrt{2}} r_\parallel^2 < r_\perp < \frac{2\sqrt{3}}{3} r_\parallel \) is satisfied, the odd-\( l \) state appears though it is not yet stable at zero temperature and we obtain a phase diagram like fig. 3 (b). If \( r_\perp < \frac{3\pi^2}{4\sqrt{2}} r_\parallel^2 \) is satisfied, the odd-\( l \) state survives until zero temperature is reached, as depicted in fig. 3 (c). In this letter we have assumed that \( r_\parallel \) is small so that \( \frac{3\pi^2}{4\sqrt{2}} r_\parallel^2 < \frac{2\pi}{3} r_\parallel \) is satisfied. In the actual Möbius crystal fabricated by Tanda and coworkers,\(^3\) \( \xi_\parallel(0) \) is much larger than \( \xi_\perp(0) \) because of the sample anisotropy, and basically the situation (b) or (c) is rather easily realized even if the strip width is not so large.

As is clarified in the above argument, the odd-\( l \) state includes a \( \pi \)-phase shift line in the midst of the strip. This line can be viewed as a vortex line confined in the strip, which we call in-plane vortex line. In fig. 4 (a) - (d), we have schematically depicted how an in-plane vortex line is formed. It is interesting to see that the in-plane vortex line is formed from the ordinary vortex penetrating the strip (fig. 4 (a)), which appears in the intermediate state of the transition between two even-\( l \) states, a phenomenon usually called the “phase slip”. Then the state (a) must have larger free energy than the two even-\( l \) states before and after the phase slip. From this we can conclude that when the odd-\( l \) state (d) is more stable than even-\( l \) states, it must be more stable than the state (a). Although it is not clear whether or not the odd-\( l \) state can be, for a certain \( f \), the most stable configuration among all the possible configurations of the order parameter, we consider that the odd-\( l \) state is a strong candidate because of its uniformity along the strip. Further investigation of this point, however, is left for the future studies.

Our results are experimentally accessible using topo-
Fig. 3. Phase diagram of the Möbius ring for the case of (a) \( r_{\parallel} < r_{\perp} \), (b) \( \frac{3\pi^2}{4\sqrt{2}} r_{\parallel}^2 < r_{\perp} < \frac{\pi^2}{2\sqrt{2}} r_{\parallel} \) and (c) \( r_{\perp} < \frac{3\pi^2}{4\sqrt{2}} r_{\parallel} \). Odd-\( l \) states appears in the hatched region.

Fig. 4. Formation process of the odd-\( l \) state with a \( \pi \)-phase shift line in the midst of the Möbius strip. The strip is shown in the expanded form. The line AB should be identified with A'B'. The bold solid line indicates a vortex line. The vortex line is drawn also in the outside of the strip for the sake of clarity. (a) A vortex is penetrating the strip. (b) The vortex is stretched in the strip to form an in-plane vortex segment. (c) The segment is almost encircling the ring. (d) By annihilating the out-of-plane vortex segments, the in-plane segment forms a closed loop encircling the Möbius ring.

However, one should note that the Little-Parks oscillation is actually observable only for the samples with a diameter less than 1 \( \mu m \). In such systems, effect of thermal fluctuation is not negligible. This effect is most significant when the number of the magnetic flux quanta in the ring is close to a half odd integer and, therefore, it is serious for the odd-\( l \) states. Because of this, the condition for the observation of odd-\( l \) state can be severer in actual systems. To overcome this point, further development of fabrication and manipulation technique of the topological matter is required.

In summary, we have investigated the superconducting Möbius strip under a magnetic field. We have pointed out a possibility of a new state which appears when the number of magnetic flux quanta in the ring is close to a half odd integer and clarified its structure. The magnetic phase diagram is obtained for systems with various radius and strip width.

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