Optimal market making with persistent order flow

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Abstract
We address the issue of market making on electronic markets when taking into account the self exciting property of market order flow. We consider a market with order flows driven by Hawkes processes where one market maker operates, aiming at optimizing its profit. We characterize an optimal control solving this problem by proving existence and uniqueness of a viscosity solution to the associated Hamilton Jacobi Bellman equation. Finally we propose a methodology to approximate the optimal strategy.

Keywords: Hawkes processes, market making, high frequency trading, stochastic control, viscosity solutions.

1 Introduction

Most electronic exchanges are organized as an anonymous continuous double auction system. Market participants can send limit orders to a central limit order book (LOB for short) displaying a volume of shares and a price at which they are ready to buy or sell. Limit orders in the LOB can be canceled (cancellation order). Market participants can also use market orders specifying a volume in order to buy or sell instantaneously at the best available price. In a very stylized view we can consider that there are two types of market participants: market takers seeking to buy or sell shares for strategic purposes using market orders and market makers filling the LOB with limit orders so that they play the role of intermediaries between buyer and seller market takers.

In practice one of the main risk faced by a market maker is the inventory risk. For example if he has a large positive inventory, price may decrease to his disadvantage. Market makers thus adapt their strategies to mitigate this risk. Basically we expect a market maker with a large positive inventory to set attractive ask prices and less competitive bid prices, in order to attract more buy than sell market orders. To do so in a relevant way he must therefore adapt his strategy to the main statistical features of the order flow.

Some key aspects that market makers should incorporate in their trading strategies are the clustering and long memory properties of order flow. The clustering property refers to the fact that buy and sell market orders are not distributed homogeneously in time but tend to be clustered, see [15]. In practice it means that after a buy (for say) market order it is likely that a new one is going to be sent shortly. Long memory of order flow designates the fact that the autocorrelation function of trade sign (+1 for a buy order and −1 for a sell order) exhibits a power-law tail, see [20]. These two properties imply that market order flow is very persistent. Our goal in this paper is to propose a method to design market making strategies that take into account those two features. For this purpose we consider a market with one market maker controlling the best bid and ask prices and with market takers using only market orders (of unit volume).

The issue of market making while managing an inventory risk has been notably addressed in [3,13] where market order flow is modeled using Poisson processes, see also the books [6,12]. However these
processes neither reproduce the clustering nor the long memory property of order flow. To take them into account, the authors of \cite{3, 7} use a refined model based on Hawkes processes with exponential kernels. Such modeling is also used in \cite{2, 15} to design optimal liquidation strategies. In this work we consider generalized Hawkes processes. More precisely \( N \) is a generalized Hawkes process with intensity \( \lambda_i \) if
\[
\lambda_i = \Phi\left( \int_0^t K(t-s)dN_s \right),
\]
where \( \Phi \) is a continuous function and \( K \) a completely monotone \( L^1 \) function\(^1\). In \cite{2, 7} the authors consider exponential functions for \( K \). For such kernel Hawkes processes manage to reproduce the clustering property of the order flow but not its long memory. However when the kernel \( K \) has a power law tail: \( K(t) \sim t^{-\beta} \) for some \( \beta > 1 \), both properties are reproduced, see \cite{5, 19}. So in this paper we extend the works \cite{3, 7, 13} to market order flows driven by Hawkes processes with general kernels.

We denote by \( N^b_t \) (resp. \( N^a_t \)) the total number of buy (resp. sell) market orders sent between time \( 0 \) and time \( t \) and write \( i_t := N^b_t - N^a_t \) for the market maker’s inventory, which is null at time \( 0 \). As in \cite{3} the market maker controls the bid and ask spreads, denoted by \( \delta^a \) and \( \delta^b \). The corresponding best ask and bid prices are \( P + \delta^a \) and \( P - \delta^b \), where \( P \) is the fundamental price of the underlying asset. The set of admissible controls is then
\[
A = \{ \delta = (\delta^a, \delta^b) \in \mathbb{R}_+^2, \text{ s.t. } \delta \text{ is predictable} \},
\]
where predictability is relative to the natural filtration generated by \((P, N^a, N^b)\), see Section 5.1 for more details. Since market takers are seeking for low transaction costs, their trading intensity is decreasing with the spreads. More precisely we know from classical financial economics results, see \cite{10, 21, 26}, that the average number of trades per unit of time is a decreasing function of the ratio between spread and volatility. To model this we consider that when the spreads are \( \delta^a \) and \( \delta^b \) market order intensities are given by
\[
\lambda^{a,\delta}_t = e^{-\frac{k}{\sigma} \delta^a} \lambda^{a,0}_t \quad \text{and} \quad \lambda^{b,\delta}_t = e^{-\frac{k}{\sigma} \delta^b} \lambda^{b,0}_t,
\]
where \( k \) is a positive constant and \( \sigma \) is the volatility of price and
\[
\lambda^{a,0}_t = \Phi\left( \int_0^t K(t-s)dN^a_s \right), \quad \lambda^{b,0}_t = \Phi\left( \int_0^t K(t-s)dN^b_s \right).
\]
Regarding the dynamic of \( P \) we assume it is given by
\[
dP_t = d(t,P_t)dt + \sigma dW_t
\]
(1)
where \( d \) is a Lipschitz function. Finally we formalize the market maker problem as a general stochastic control problem
\[
\sup_{\delta \in A} \mathbb{E}^\delta \left[ G(i_T, P_T)e^{-rT} + \int_0^T e^{-rs}(g(i_s, P_s)ds + \delta^a_s dN^a_s + \delta^b_s dN^b_s) \right],
\]
(2)
where \( r \) is a positive constant and \( g \) and \( G \) are two continuous functions with at most quadratic growth. The former represents a continuous reward received by the market maker (besides its P&L) and the latter is a final lump sum payment paid to the market maker at the end of its trading. Typical choices would be \( G(x, y) = xy \) and \( g(x, y) = x^2 \). The notation \( \mathbb{E}^\delta \) denotes the expectation under the law corresponding to the control \( \delta \) (see Section 5.1 for details).

In \cite{3}, to solve the market maker’s optimization problem in a Poisson context, the authors study the associated Hamilton Jacobi Bellman equation (HJB for short). They can use this method since \((P, i, N^a, N^b)\) is Markovian in this case. However when \( N^a \) and \( N^b \) are Hawkes processes, notably when the kernel is not exponential, \((P, i, N^a, N^b)\) is not Markovian\(^2\). In order to circumvent this difficulty

\(^1\)In this paper we consider complete monotony on \( \mathbb{R}_+ \).

\(^2\)In the exponential case the process \((P, i, N^a, N^b, \lambda^a, \lambda^b)\) is actually Markovian.
we consider auxiliary state variables enabling us to work in a Markovian setting. More precisely we consider the process $X = (P, i, \theta^a, \theta^b)$ where
\[ \theta^a_t = \int_0^t K(\cdot - s)dN^a_s \quad \text{and} \quad \theta^b_t = \int_0^t K(\cdot - s)dN^b_s. \]

Note that here $\theta^a_t$ and $\theta^b_t$ are random functions from $\mathbb{R}_+$ into $\mathbb{R}_+^3$. The process $(t, X_t)_{t \geq 0}$ is Markovian. Moreover, studying the HJB equation associated with this representation, we prove in Section 2 that the optimal control problem (2) admits a solution of the form
\[ \delta^*_t = \delta^K(t, X_t), \]
where $\delta^K$ is a feedback control function. In our approach the HJB equation is defined on a subset of an infinite dimensional vector space. So in general we cannot rely on classical numerical methods to approximate $\delta^K$. To tackle this issue we propose the following strategy.

- We show that if $(K_n)_{n \geq 0}$ converges towards $K$ in $L^1$ and uniformly on $[0, T]$ then $(\delta^{K_n})_{n \geq 0}$ converges point-wise towards $\delta^K$.
- We show that when $K(t) = \sum_{i=1}^n \alpha_i e^{-\gamma_i t}$ then there exists a Markovian representation of the model in dimension $2n + 2$. Therefore in this case the optimal control $\delta^K$ can be approximated numerically.
- Inspired by [11], we prove that for any completely monotone kernel $K$ in $L^1$, we can find a sequence $(K_n)_{n \geq 0}$, converging towards $K$ in $L^1$ and uniformly on $[0, T]$ and such that for any $n$, $K_n$ is a linear combination of $n$ decreasing exponential functions.

Those three points give a simple methodology to approximate $\delta^K$. However when $n$ is large we cannot rely on finite differences methods to compute $\delta^{K_n}$ since the dimension is too large. So for numerical experiments we use the probabilistic representation of semi-linear partial differential equations (PDEs for short) introduced in [14].

The paper is organized as follows. In Section 2 we prove existence of a solution to Problem (2) based on the study of its associated HJB equation. In Section 3 we explain how to approximate the optimal control obtained in Section 2. Finally in Section 4 we present some numerical experiments. The proofs are relegated to the Appendix.

2 Solving the market maker problem using viscosity solutions

In this section we prove existence of a solution to Problem (2). First we define an appropriate set for the process $X$. Then we show that the associated HJB equation has a unique viscosity solution with polynomial growth and prove the existence of an optimal control solving (2).

2.1 Appropriate domain for the process $X$

To study the uniqueness of solution to a PDE in the sense of viscosity, it is convenient to deal with locally compact domain. We have $X = (P, i, \theta^a, \theta^b) \in \mathbb{R} \times \mathbb{Z} \times L^1 \times L^1$, but since $L^1$ is not locally compact we need to specify more precisely the set in which the processes $\theta^a$ and $\theta^b$ belong. Obviously we have for $j = a$ or $b$
\[ \theta^j_t \in \Theta^K_t = \{ \sum_{i=1}^n K(\cdot - T_i), n \in \mathbb{N}, T_1 \leq \cdots \leq T_n \leq t \} \subset \Theta^K_T. \]

We naturally endow $\Theta^K_T$ with the topology of $L^1$ and prove in Appendix A that it enjoys the following topological properties.

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3To define $\theta^a_t$ and $\theta^b_t$ we consider that $K$ is extended to $\mathbb{R}$ with value 0 on $\mathbb{R}_-^*$.
Lemma 2.1.

(i) The set $\Theta^K_T$ is a locally compact closed subset of $L^1$.

(ii) For any sequence $(s_n, \theta_n)_{n \geq 0}$ with values in $[0, T] \times \Theta^K_T$ such that for any $n$, $\theta_n \in \Theta^K_T$, if $(s_n, \theta_n)_{n \geq 0}$ converges towards $(s, \theta)$ then we have $\theta \in \Theta^K_T$ and $\theta_n \to \theta(s)$ when $n \to +\infty$.

(iii) Moreover if $K$ is a sum of exponential functions then we have for any $k \geq 0$,

$$\theta_n^{(k)}(T) \to \theta^{(k)}(T), \text{ when } n \to +\infty.$$ 

From point (i) in Lemma 2.1 the set $\Theta^K_T$ is adapted to our purpose. Points (ii) and (iii) are purely technical and are used in Section 3. Based on the sets $(\Theta^K_T)_{t \in [0, T]}$ we define a locally compact domain for $X$. More precisely for any $t \in [0, T]$ we consider

$$Z^K_t = \{(i, \theta^a, \theta^b) \in \mathbb{Z} \times \Theta^K_t \times \Theta^K_t \} \text{ and } \mathcal{X}^K_t = \{(p, i, \theta^a, \theta^b) \text{ s.t. } p \in \mathbb{R} \text{ and } (i, \theta^a, \theta^b) \in Z^K_t \}.$$ 

The set $Z^K_t$ (resp. $\mathcal{X}^K_t$) is a locally compact closed subset of $\mathbb{Z} \times L^1 \times L^1$ (resp. $\mathbb{R} \times \mathbb{Z} \times L^1 \times L^1$). We finally define

$$\mathcal{E}^K = \{(t, x) \in [0, T] \times \mathcal{X}^K_T \text{ s.t. } x \in \mathcal{X}^K_t \}$$

which is a locally compact closed subset of $[0, T] \times \mathbb{R} \times \mathbb{Z} \times L^1 \times L^1$. Obviously we have $(t, X_t) \in \mathcal{E}^K$ for any $t \in [0, T]$. To lighten the notations when we consider $x \in \mathcal{E}^K$ (resp. $x \in \mathcal{X}^K_t$, $z \in Z^K_t$) we implicitly assume that $x = (t, p, i, \theta^a, \theta^b)$ (resp. $x = (p, i, \theta^a, \theta^b)$, $z = (i, \theta^a, \theta^b)$). We also define for any $x = (p, i, \theta^a, \theta^b) \in \mathbb{R} \times L^1 \times L^1 \times \mathbb{Z}$ the norm $\|x\| = \sqrt{p^2 + i^2 + \|\theta^a\|^2 + \|\theta^b\|^2}$ and for any non-negative $R$ the set,

$$\mathcal{E}^K_R = \{(t, x) \in \mathcal{E}^K, \text{ s.t. } \|x\| \leq R \},$$

which is a compact subset of $\mathcal{E}^K$ as consequence of Lemma 2.1 (i).

Now that we have defined a set adapted to PDE analysis we derive in the next section the HJB equation related to the stochastic control problem [2].

2.2 Hamilton-Jacobi-Bellman equation associated to the control problem

The stochastic control problem [2] is written in an unconventional way because of the integrals

$$\int_0^T \delta^a_s dN^a_s \text{ and } \int_0^T \delta^b_s dN^b_s.$$ 

However up to a $P^\delta$-local martingale those terms are respectively equal to $\int_0^T \delta^a_s \lambda^a_s \delta^s ds$ and $\int_0^T \delta^b_s \lambda^b_s \delta^s ds$. So as consequence of Appendix [2.3] for any $\delta \in \mathcal{A}$ we have

$$\mathbb{E}^\delta[G(iT, P_T)e^{-rT} + \int_0^T e^{-rs} \left(g(i_s, P_s)ds + \delta^a_s dN^a_s + \delta^b_s dN^b_s \right)]$$

$$= \mathbb{E}^\delta[G(iT, P_T)e^{-rT} + \int_0^T e^{-rs} \left(g(i_s, P_s) + \delta^a_s \lambda^a_s \delta^s + \delta^b_s \lambda^b_s \delta^s \right)ds].$$

Thus [2] is equivalent to the stochastic control problem

$$\sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta[G(iT, P_T)e^{-rT} + \int_0^T e^{-rs} \left(g(i_s, P_s) + \delta^a_s \lambda^a_s \delta^s + \delta^b_s \lambda^b_s \delta^s \right)ds].$$

In order to give intuition on the HJB equation related to this stochastic control problem we write the Ito formula related to $X$. We consider a function $\varphi$ defined on $[0, T] \times \mathbb{R} \times \mathbb{Z} \times L^1 \times L^1$ that is $C^{2, 2, 0, 0, 0}$. We call any function with such regularity a test function. For any $s < t \in [0, T]$ we have

$$\varphi(t, X_t) - \varphi(s, X_s) = \int_s^t \left(\delta^a \varphi(u, X_{u-}) + L^P \varphi(u, X_{u-}) + \sum_{j=a, b} D^K_j \varphi(u, X_{u-}) e^{-\frac{1}{2} \delta^j_s} \Phi(\theta^{(j)}_u(u)) \right)du$$

$$+ \delta^a \varphi(u, X_{u-}) \sigma dW_u + D^K_a \varphi(u, X_{u-}) dM^a_u + D^K_b \varphi(u, X_{u-}) dM^b_u,$$
where

\[ M_t^{a,\delta} = N_t^a - \int_0^t \lambda_s^{a,\delta} \, ds \quad \text{and} \quad M_t^{b,\delta} = N_t^b - \int_0^t \lambda_s^{b,\delta} \, ds \]

are \( \mathbb{P}^\delta \)-uniformly integrable martingales, see Appendix B.1 for details. The operator \( \mathcal{L}^p \) is the infinitesimal generator related to the diffusion of \( P \) and is defined for any test function \( \varphi \) and \( (t,x) \in \mathcal{E}^K \) by

\[ \mathcal{L}^p \varphi(t,x) = d(t,p)\partial_p \varphi(t,x) + \frac{1}{2} \sigma^2 \partial_{pp} \varphi(t,x). \]

The operators \( D_a^K \) and \( D_b^K \) correspond to the infinitesimal generators related to the diffusion of \( N^a \) and \( N^b \). They are defined for \( (t,x) \in \mathcal{E}^K \) by

\[
D_a^K \varphi(t,x) = \varphi(t,p,i - 1, \theta^a + K(\cdot - t), \theta^b) - \varphi(t,p,i, \theta^a, \theta^b), \\
D_b^K \varphi(t,x) = \varphi(t,p,i + 1, \theta^a, \theta^b + K(\cdot - t)) - \varphi(t,p,i, \theta^a, \theta^b).
\]

Hence the HJB equation associated to the control problem (3) is

\[
(HJB)_K : \left\{ \begin{array}{l}
F(x,U(x),\nabla U(x),\partial_{pp} U(x),D^K_U(x)) = 0 \quad \text{for } x \in \mathcal{E}^K, \\
U(T,y) = G(i,p) \quad \text{for } y \in \mathcal{X}_T^K
\end{array} \right.
\]

with \( \nabla U = (\partial_i U, \partial_p U) \), \( D^K_U = (D_a^K U, D_b^K U) \) and where the function \( F \) is defined for \( (x,u,q,A,I) \in \mathcal{E}_t^K \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \) by

\[
F(x,u,q,A,I) = ru - q_1 - d(t,p)q_2 - \frac{1}{2} \sigma^2 A - g(i,p) - \sup_{\delta \in \mathbb{R}_+} \Phi(b^a(t)) e^{-\frac{1}{2} \delta} (\delta + I_1) - \sup_{\delta \in \mathbb{R}_+} \Phi(b^b(t)) e^{-\frac{1}{2} \delta} (\delta + I_2).
\]

A simple computation gives the maximizers

\[
\delta^a = \left( \sigma/k - I_1 \right)_+ \quad \text{and} \quad \delta^b = \left( \sigma/k - I_2 \right)_+.
\] (4)

Note that the dependence in \( K \) of \( (HJB)_K \) lies in the operator \( D^K \). It seems hard to prove existence of a smooth solution to the integro-partial differential equation (IPDE for short) \( (HJB)_K \). Therefore in the next section we look for viscosity solutions.

### 2.3 Viscosity solutions: some definitions

Since we are dealing with an IPDE defined on a non usual set and in order to make things precise we define the notion of viscosity solution in our framework. First we give the classical definition and then its counterparts based on semi jets.

**Definition 2.1.**

- A locally bounded function \( U \in USC(\mathcal{E}^K) \) (the set of upper semi-continuous function on \( \mathcal{E}^K \)) is a viscosity sub-solution of \( (HJB)_K \) if for all \( x \in \mathcal{E}^K \) and test function \( \phi \) such that \( x \) is a maximum on \( \mathcal{E}^K \) of \( U - \phi \) we have

\[
F(x,\phi(x),\nabla \phi(x),\partial_{pp} \phi(x),D^K_U(x)) \leq 0.
\]

- A locally bounded function \( U \in LSC(\mathcal{E}^K) \) (the set of lower semi-continuous function on \( \mathcal{E}^K \)) is a viscosity super-solution of \( (HJB)_K \) if for all \( x \in \mathcal{E}^K \) and test function \( \phi \) such that \( x \) is a minimum on \( \mathcal{E}^K \) of \( U - \phi \) we have

\[
F(x,\phi(x),\nabla \phi(x),\partial_{pp} \phi(x),D^K_U(x)) \geq 0.
\]

- A continuous function \( U \) defined on \( \mathcal{E}^K \) is a viscosity solution of \( (HJB)_K \) if it is a viscosity super-solution and a viscosity sub-solution.
In the above definition it is equivalent to consider local (or local strict) extrema. We have not replaced \( U \) by \( \phi \) for the last operator \( D^K \). This is because \( D^K U \) do not require regularity assumption on the function \( U \) to be defined. However it is equivalent to replace \( D^K \) by \( D^K \phi \) in Definition 2.1. Indeed in the case of sub-solution for say, we can always build a sequence of test functions \((\phi_n)_{n \geq 0}\) satisfying \( U \leq \phi_n \) with equality at point \( x \) and such that

\[
(\nabla \phi_n(x), \partial_{pp}^2 \phi_n(x)) = (\nabla \phi(x), \partial_{pp}^2 \phi(x)) \text{ with } D^K \phi_n(x) \xrightarrow{n \to +\infty} D^K U(x).
\]

By continuity of \( F \) we get the equivalence. This also holds for super-solution.

We now introduce the notions of semi super and sub-jets in our framework. For \( U \) a USC function on \( \mathcal{E}^K \) and \( x = (t, p, z) \in \mathcal{E}^K \), the super-jet of \( U \) at point \( x \) is the set

\[
\mathcal{J}^+ U(x) = \{(g, A, h) \in \mathbb{R}^2 \times \mathbb{R} \times C^0(\mathcal{Z}_T^K) \text{ s.t. for any } y = (s, q, v) \in \mathcal{E}^K \text{ we have} \}
\]

\[
U(s, y) \leq U(t, x) + g_1(t - s) + g_2(p - q) + \frac{1}{2}A(p - q)^2 + h(z - v) + o(|t - s| + |p - q|)
\]

and \( h(0) = 0 \}

and the semi super-jet of \( U \) at point \( x \) is

\[
\overline{\mathcal{J}}^+ U(x) = \{(g, A, h) \in \mathbb{R}^2 \times \mathbb{R} \times C^0(\mathcal{Z}_T^K) \text{ s.t. there exists a sequence} \}
\]

\[
(x_n, g_n, A_n, h_n)_{n \geq 0} \text{ with for any } n \geq 0 \ (g_n, A_n, h_n) \in \mathcal{J}^+ U(x_n)
\]

\[
\text{and such that } (x_n, U(x_n), g_n, A_n, h_n) \xrightarrow{n \to +\infty} (x, U(x), g, A, h) \}
\]

In the above definition the convergence of \( h_n \) is taken in the sense of locally uniform convergence around 0. By analogy for a LSC function \( U \) we define the sub-jet \( \mathcal{J}^- U(x) \) and the semi sub-jet \( \overline{\mathcal{J}}^- U(x) \). We can now give another characterization of viscosity sub and super-solutions relying on the notions of semi jets.

**Definition 2.2.**

- A locally bounded function \( U \in USC(\mathcal{E}^K) \) is a viscosity sub-solution of \((\text{HJB})_K\) if for all \( x \in \mathcal{E}^K \), and \((g, A, h) \in \mathcal{J}^+ U(x)\) we have

\[
F(x, U(x), g, A, D^K U(x)) \leq 0.
\]

- A locally bounded function \( U \in LSC(\mathcal{E}^K) \) is a viscosity super-solution of \((\text{HJB})_K\) if for all \( x \in \mathcal{E}^K \), and \((g, A, h) \in \overline{\mathcal{J}}^- U(x)\) we have

\[
F(x, U(x), g, A, D^K U(x)) \geq 0.
\]

We show that Definition 2.1 and 2.2 are equivalent in Appendix C.

In the next section based on the study of \((\text{HJB})_K\) we prove that the control problem \((3)\) admits a solution.

### 2.4 Existence of an optimal control

In this section we prove existence of a solution to Problem \((3)\). Before stating the result we present a sketch of the proof.

We start by proving uniqueness of a viscosity solution with polynomial growth to \((\text{HJB})_K\) using a comparison result. The main difficulty is to adapt the Crandall-Ishi’s lemma to our framework, which is done in Appendix D. Using a verification argument we then check that the continuation utility function \( U^K \) associated to the problem \((3)\) is actually this unique solution. The maximizers of the Hamiltonian given in Equation \((4)\) then naturally provide a control solving Problem \((3)\). The full proof is given in Section 5.2.
Theorem 2.1.

(i) There exists a unique viscosity solution $U^K$ with polynomial growth to $(HJB)_K$.

(ii) This solution satisfies

$$U^K(0) = \sup_{\delta \in A} \mathbb{E}^\delta [G(i_T, P_T) e^{-rT} + \int_0^T e^{-rs} \left( g(i_s, P_s) + \delta_s^a \lambda_s^a + \delta_s^b \lambda_s^b \right) ds].$$

(iii) The problem (3) admits a solution given by

$$\delta^K(t) = \delta^K(t, X_t), \quad \text{with} \quad \delta^K = (\delta^K_a, \delta^K_b),$$

where

$$\delta^K_a = (\sigma/k - D^K_b U^K)_+ \quad \text{and} \quad \delta^K_b = (\sigma/k - D^K_a U^K)_+.$$  (5)

It is important to remark that to obtain existence of an admissible optimal control we have benefited from the fact that we are controlling counting processes, whose infinitesimal generators are defined for any finite functions. From a practical point of view Theorem 2.1 implies that if we manage to compute $U^K$ it is possible to implement the optimal control by monitoring the processes $\theta^K_a$ and $\theta^K_b$. Note that to do this it is sufficient to monitor the arrival times of buy and sell market orders. However since $\mathcal{E}_K$ is a subset of an infinite dimensional vector space, we cannot approximate $U^K$ using classic numerical methods. Therefore we need to find another way to approach the control $\delta^K$. We tackle this problem in the next section.

3 How to approach the optimal control

In this section we explain how to approach numerically the feedback control $\delta^K$. We proceed in three steps:

- We show that if $(K_n)_{n \geq 0}$ converges towards $K$ in $L^1$ and uniformly on $[0, T]$ then $(\delta^K_n)_{n \geq 0}$ converges point-wise towards $\delta^K$.

- We prove that when $K(t) = \sum_{i=1}^n \alpha_i e^{-\gamma_i t}$ there exists a Markovian representation of the model in dimension $2n + 2$.

- Inspired by [1] we show that for any completely monotone function $K$ in $L^1$ we can find a sequence $(K_n)_{n \geq 0}$ converging towards $K$ in $L^1$ and uniformly on $[0, T]$ such that for any $n$, $K_n$ is a linear combination of $n$ decreasing exponential functions.

Those three points give a simple method to implement an approximate version of the control $\delta^K$: choose $\tilde{K}$, written as sum of positive decreasing exponential functions, close enough to $K$. Use the finite dimensional representation to compute $U^K$ and implement $\tilde{\delta}$ instead of $\delta^K$. We make precise this method in the last part of this section.

3.1 Convergence of solutions and optimal controls

Consider a completely monotone function $K$ in $L^1$. We show that if a sequence of continuous $L^1$ functions $(K_n)_{n \geq 0}$ converges towards $K$ in $L^1$ and uniformly on $[0, T]$ then the sequence $(\delta^K_n)_{n \geq 0}$ converges point-wise towards $\delta^K$. With respect to Equation (5) it is sufficient to prove that the sequence $(U^K_n)_{n \geq 0}$ converges point-wise towards $U^K$.

From Theorem 5.8 in [25] we observe that the notion of viscosity solution is perfectly adapted to prove the convergence of solutions to a sequence of PDEs. Hence we prove in Section 5.3 the following result which is an adaptation of Theorem 5.8 in [25] to our framework.
Proposition 3.1. Consider a sequence \((K_n)_{n \geq 0}\) of continuous \(L^1\) functions converging towards a completely monotone function \(K\) in \(L^1\) and uniformly on \([0, T]\), then for any \(x \in \mathcal{E}^K\) we have
\[
U^K(x) = \lim_{(y, n) \in \bar{\mathcal{E}} \rightarrow (x, +\infty)} U^{K_n}(y) \tag{6}
\]
where
\[
\bar{\mathcal{E}} = \mathcal{E}^K \times \{\infty\} \cup \left( \bigcup_{n \geq 0} \mathcal{E}^{K_n} \times \{n\} \right).
\]

The main technical difficulty in the proof of Proposition 3.1 compared to Theorem 5.8 in [25], is that the functions \((U^{K_n})_{n \geq 0}\) are defined on different domains. From now on, when we consider a limit as in Equation (6) we forget to write \(\bar{\mathcal{E}}\) to lighten notations. Proposition 3.1 perfectly fits our purpose of approaching \(U^K\) for any \(K\). Indeed suppose we manage to find a dense subset of the completely monotone \(L^1\) functions such that for any \(K\) in this subset, the function \(U^K\) can be approximated numerically. Then Proposition 3.1 guarantees that for any completely monotone function \(K\) in \(L^1\) we can approach numerically \(U^K\). We show in the next two sections that the set
\[
\mathcal{SE} = \bigcup_{n \geq 0} \{ \sum_{i=1}^{n} \alpha_i e^{-\gamma_i} 1_{\mathbb{R}_+} \text{ s.t. } \alpha \in \mathbb{R}_+^n \text{ and } \gamma \in \mathbb{R}_+^n \}
\]
satisfies those two conditions. Note that \(\mathcal{SE}\) is simply the set of positive linear combination of decreasing exponential functions.

We start by studying Problem (3) when the function \(K\) is in \(\mathcal{SE}\). Then we show that \(\mathcal{SE}\) is dense in the set of completely monotone functions in \(L^1\).

3.2 Solving the optimal control for \(K \in \mathcal{SE}\)

We explain in this section how to solve the stochastic control problem (3) when the function \(K\) belongs in \(\mathcal{SE}\).

We consider that the kernel of the Hawkes processes \(N^a\) and \(N^b\) is given by \(K_{a, \gamma}(t) = \sum_{i=1}^{n} \alpha_i e^{-\gamma_i} 1_{\mathbb{R}_+}(t)\), where \(n\) is a positive integer, \(\alpha \in \mathbb{R}_+^n\) and \(\gamma \in \mathbb{R}_+^n\). For \(i \in \{1, \ldots, n\}\) and \(j = a\) or \(b\) we define the process
\[
\lambda_{t}^{j, i} = \int_{0}^{t} \alpha_i e^{-\gamma_i(t-s)} dN_s^j.
\]
Then \(Y_t^{a, \gamma} = (t, P_t, \nu, (e_t^{a, i})_{1 \leq i \leq n}, (e_t^{b, i})_{1 \leq i \leq n})\) is a Markovian process since
\[
\lambda_{t}^{j, 0} = \Phi\left( \sum_{i=1}^{n} \lambda_{t}^{j, i} \right) \text{ and } \Phi_{t}^{j, i} = -\gamma_i \lambda_{t}^{j, i} dt + \alpha_i dN_t^j \text{ for } j = a \text{ or } b.
\]

The domain associated to this representation is \(\mathcal{E}^n = [0, T] \times \mathbb{R} \times \mathbb{Z} \times \mathbb{R}_+^n \times \mathbb{R}_+^n\), which is locally compact. As for \(\mathcal{E}^K\), when we have \((t, x) \in \mathcal{E}^K\) we implicitly consider that \(x = (p, i, e^a, e^b)\). We can naturally go from the first representation to the second one. More precisely we prove in Appendix B\dagger\footnote{\text{Here dense is intended in the sense of convergence in } L^1 \text{ together with uniform convergence on } [0, T].} that there exists a continuous function \(R^{a, \gamma}\) from \(\mathcal{E}^{K_{a, \gamma}}\) into \(\mathcal{E}^n\) such that for any \(t > 0\) we have \(R^{a, \gamma}(t, X_t) = (t, Y_t^{a, \gamma})\). However notice that the second representation is somehow larger than the first one: if we consider \(y = (t, p, i, e^a, e^b) \in \mathcal{E}^n\) there is a priori no \(x \in \mathcal{E}^{K_{a, \gamma}}\) such that
\[
y = R^{a, \gamma}(x).
\]

This is because in general there does not exist \(m \geq 0\) and \((T_i)_{1 \leq i \leq m}\) in \([0, T]\) such that for any \(i \in \{1, \ldots, n\}\)
\[
e^{a, i} = \sum_{j=1}^{m} e^{-\gamma_i(t-T_j)}.
\]
The infinitesimal generators associated to the processes $N^a$ and $N^b$ for the new representation are denoted by $D^a_\alpha$ and $D^b_\alpha$. They are defined for any function $U$ on $E^n$ by
\[
D^a_\alpha U(x) = U(t, p, i - 1, c^a + \alpha, c^b) - U(t, p, i, c^a, c^b), \quad D^b_\alpha U(x) = U(t, p, i + 1, c^a, c^b + \alpha) - U(t, p, i, c^a, c^b).
\]

The HJB equation related to Problem (3) in this new representation is therefore
\[
(HJB)_{\alpha,\gamma} : \left\{ \begin{array}{l}
G_{\alpha,\gamma}(x, U(x), \nabla^c U(x), \nabla U(x), \partial^2_{p}\partial^c U(x), D^a_\alpha U(x)) = 0, \text{ for } x \in E^n, \\
U(T, y) = G(i, p) \text{ for } (T, y) \in E^n
\end{array} \right.
\]
with $\nabla^c U = (\nabla^c_a U, \nabla^c_b U)$ where for $j = a$ or $b$, $\nabla^c_j U = (\partial_{c^j}\partial p U)_{1 \leq i \leq n}$, $\nabla U(t, x) = (\partial_t U(t, x), \partial_p U(t, x))$,
\[
D^a_\alpha U(t, x) = (D^a_\alpha U(t, x), D^b_\alpha U(t, x))
\]
and where the function $G_{\alpha,\gamma}$ is defined for $(x, u, h, q, A, I) \in E^n \times \mathbb{R} \times (\mathbb{R}^n)^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ by
\[
G_{\alpha,\gamma}(x, u, h, q, A, I) = ru - h_1 - d(t, p)h_2 - \frac{1}{2}a^2A - \langle \gamma, q_1 \rangle - \langle \gamma, q_2 \rangle - g(i, p)
\]
\[- \sup_{\delta \in \mathbb{R}^+} \Phi(\sum_{i=1}^n c^{a,i}e^{-\frac{1}{2}\delta}(\delta + I_1)) - \sup_{\delta \in \mathbb{R}^+} \Phi(\sum_{i=1}^n c^{b,i}e^{-\frac{1}{2}\delta}(\delta + I_2)).
\]

We can easily adapt the proof of Theorem 2.1 to $(HJB)_{\alpha,\gamma}$ and prove the following result.

**Theorem 3.1.**

(i) There exists a unique continuous viscosity solution with polynomial growth $U^{\alpha,\gamma}$ to $(HJB)_{\alpha,\gamma}$.

(ii) The solution $U^{\alpha,\gamma}$ satisfies
\[
U^{\alpha,\gamma}(0) = \mathbb{E}^\mathbb{A}[G(i_T, P_T)e^{-rT} + \int_0^T e^{-rs}(g(s, P_s) + \delta^a_\alpha \lambda^a_\alpha s + \delta^b_\alpha \lambda^b_\alpha s)ds].
\]

(iii) The stochastic control problem (3) admits a solution that is written
\[
\delta^{\alpha,\gamma}(t, Y^{t,\alpha,\gamma}), \text{ with } \delta^{\alpha,\gamma} = (\delta^{a,\gamma}_\alpha, \delta^{b,\gamma}_\alpha)
\]
where
\[
\delta^{a,\gamma}_\alpha = (\sigma/k - D^a_\alpha U^{\alpha,\gamma})_+ \text{ and } \delta^{b,\gamma}_\alpha = (\sigma/k - D^b_\alpha U^{\alpha,\gamma})_+.
\]

(iv) We have $U^{K,\alpha,\gamma} = U^{\alpha,\gamma} \circ R^{\alpha,\gamma}$.

The proof of the three first points is exactly the same as the proof of Theorem 2.1. We deal with point (iv) in Section 5.4. Points (iii) and (iv) of Theorem 3.1 imply that for any $\alpha$ and $\gamma$ in $\mathbb{R}^n_+$ we can approach numerically $\delta^{K,\alpha,\gamma}$. We just need to approximate $U^{\alpha,\gamma}$ using any numerical method, which is possible because the domain of $(HJB)_{\alpha,\gamma}$ is a subset of a finite dimensional vector space. Then using the change of variable $R^{\alpha,\gamma}$ one gets
\[
\delta^{K,\alpha,\gamma} = \delta^{\alpha,\gamma} \circ R^{\alpha,\gamma}.
\]

This shows that the optimal control processes given in Theorem 2.1 (iii) and Theorem 3.1 (iii) are actually the same.
3.3 Density of $\mathcal{SE}$ in the set of completely monotone function

In this section we show that $\mathcal{SE}$ is dense in the set of completely monotone functions in $L^1$. Before giving the result we present a sketch of the proof. The main point is that any completely monotone function can be written as the Laplace transform of a positive measure $m$, see Lemma 2.3 in [22].

$$K(x) = \int_0^{+\infty} e^{-ux} m(du).$$

(7)

Moreover if $K(0) < +\infty$ then $m$ is $L^1$ and if $K$ is in $L^1$ then $\int_0^{+\infty} \frac{m(du)}{u} < \infty$. Hence using a Riemann sum to approach the integral in (7) we get a natural approximation of $K$ by a function in $\mathcal{SE}$. Based on this idea we prove the following result in Appendix F.

**Lemma 3.1.** For any completely monotone function $K$ in $L^1$ we can find a sequence $(\alpha_n, \gamma_n)_{n \geq 0}$, where for any $n$ $(\alpha_n, \gamma_n) \in \mathbb{R}_+ \times \mathbb{R}_+$, such that the sequence $(K_{\alpha_n, \gamma_n})_{n \geq 0}$ converges towards $K$ in $L^1$ and uniformly on every compact set of $\mathbb{R}_+$. Moreover we may choose $(\alpha_n, \gamma_n)_{n \geq 0}$ such that

$$\|K_{\alpha_n, \gamma_n}\|_1 = \|K\|_1 \text{ and } K_{\alpha_n, \gamma_n}(0) = K(0).$$

Lemma 3.1 concludes on the existence of a procedure to approach $\delta^K$. In the next section we sum up our results and explain how one may implement in practice an approximation of the optimal control.

3.4 Conclusion on approaching the optimal control

We fix a completely monotone function $K$ in $L^1$ and a sequence $(\alpha_n, \gamma_n)_{n \geq 0}$ such that $(K_{\alpha_n, \gamma_n})_{n \geq 0}$ converges towards $K$ in $L^1$ and uniformly on $[0, T]$. We write $K_n$ instead of $K_{\alpha_n, \gamma_n}$ to lighten notations. The existence of such sequence is given by Lemma 3.1. Moreover from Proposition 3.1, for any $x \in \mathcal{E}^K$ we have

$$U^K(x) = \lim_{(y, n) \to (x, +\infty)} U^{K_n}(y).$$

Therefore using point $(iii)$ of Theorems 2.1 and 3.1 we get

$$\delta^K(x) = \lim_{(y, n) \to (x, +\infty)} \delta^{K_n}(y),$$

and Theorem 3.1 $(iv)$ gives that for any $x \in \mathcal{E}^K$

$$\delta^K(x) = \lim_{(y, n) \to (x, +\infty)} \delta^{\alpha_n, \gamma_n} \circ R^{\alpha_n, \gamma_n}(y).$$

For a given $x = (t, p, i, \theta^a, \theta^b) \in \mathcal{E}^K$ we now explicit a sequence $(x_n)_{n \geq 0}$ converging towards $x$ and such that $x_n \in \mathcal{E}^{K_n}$ for any $n$. By definition of $\Theta^K$ there exists $m_a$ and $m_b$ two non-negative integers and two sequences $(T_i^a)_{1 \leq i \leq m_a}$ and $(T_i^b)_{1 \leq i \leq m_b}$ in $[0, t]$ such that

$$\theta^a = \sum_{i=1}^{m_a} K(\cdot - T_i^a) \text{ and } \theta^b = \sum_{i=1}^{m_b} K(\cdot - T_i^b).$$

Consequently for any $n > 0$ we naturally define :

$$\theta^{n,a} = \sum_{i=1}^{m_a} K_n(\cdot - T_i^a) \text{ and } \theta^{n,b} = \sum_{i=1}^{m_b} K_n(\cdot - T_i^b) \in \Theta_t^{K_n},$$

and $x_n = (t, p, i, \theta^{n,a}, \theta^{n,b})$ which obviously belongs in $\mathcal{E}^{K_n}$. Because of Lemma 3.1 the sequence $(\theta^{n,a})_{n \geq 0}$ (resp. $(\theta^{n,b})_{n \geq 0}$) converges in $L^1$ towards $\theta^a$ (resp. $\theta^b$). Therefore we get $(x_n, n)$ converges towards $(x, +\infty)$ as $n$ goes to infinity, consequently

$$\delta^K(x) = \lim_{n \to +\infty} \delta^{\alpha_n, \gamma_n} \circ R^{\alpha_n, \gamma_n}(x_n).$$

Hence for $n$ large enough we can consider that for any $t \in [0, T]$

$$\delta^t_t = \delta^K(X_t) \approx \delta^{\alpha_n, \gamma_n}(Y_t^{\alpha_n, \gamma_n}).$$
In conclusion to implement an approached version of the optimal control \( \delta^K \) one must:

1. Fix \( n \) positive and find \( \alpha, \gamma \in \mathbb{R}^n_+ \) such that \( K_{\alpha, \gamma} \) is close to \( K \). See Appendix F for a method to build \( K_{\alpha, \gamma} \).

2. Approach numerically \( U^{\alpha, \gamma} \), the solution of \( (HJB)_{\alpha, \gamma} \) which is equivalent to approach numerically the feedback \( \delta^{\alpha, \gamma} \).

3. Monitor \( Y^{\alpha, \gamma} \) and apply the control \( \delta^{\alpha, \gamma}(Y^{\alpha, \gamma}) \).

The only flaw of this method is that the set \( \mathcal{E}^n \) is a subset of a vector space of dimension \( 2n + 2 \). Hence when \( n \) is larger than 2 it is very unlikely that simple finite differences methods can be used to solve numerically \( (HJB)_{\alpha, \gamma} \). To tackle this issue one has to use other numerical methods such as neural networks, see [4, 16] for example, or probabilistic method, see [14]. In this article we propose to use the later method for numerical applications.

## 4 Numerical applications

In this section we present some numerical experiments illustrating our results.

We consider a simplified version of the market maker’s problem:

\[
(N) : \sup_{\delta \in \mathcal{A}} \mathbb{E}\left[ \int_0^T \delta_s^a dN_s^a + \delta_s^b dN_s^b - \mu_i^2 \right].
\]

This corresponds to \( G = 0 \) and \( g(i, p) = -\mu^2 \). We take \( k/\sigma = 50 \) and \( \mu = 0.02 \). We note \( U_K \) the unique viscosity solution (with polynomial growth) of the HJB equation associated to \( (N) \) when the Hawkes processes’ kernel is \( K \). In all this section we discard the price variable from the IPDEs since it does not appear in the optimization problem.

We first consider in Section 4.1 the cases of kernels in \( \mathcal{SE} \) with \( n = 2 \). We use a finite differences method to solve the IPDEs. Then in Section 4.2 we deal with more complex functions \( K \). To solve the IPDEs we use the probabilistic representation introduced in [14] which is described in Appendix G.

### 4.1 The small dimension case

We consider three control functions \( \delta^0, \delta^1 \) and \( \delta^2 \) computed in the following way:

- The control \( \delta^0 \) is computed by a market maker that believes buy and sell order flows are Poisson processes with intensity \( \mu_0 \).

- The control \( \delta^1 \) is computed by a market maker that believes order flows are driven by Hawkes processes with intensity \( \mu_1 \) and kernel \( K_1(t) = \alpha_1 e^{-\gamma_1 t} \).

- The control \( \delta^2 \) is computed by a market maker that believes order flows are driven by Hawkes processes with intensity \( \mu_2 \) and kernel \( K_2(t) = \alpha_2 e^{-\gamma_2 t} + \alpha_2 e^{-\gamma_2 t} \).

We use the following parameters settings:

- \( \mu_0 = 0.01 \)
- \( \mu_1 = 0.001, \gamma_1 = 1 \) and \( \alpha^1 = 0.9 \)
- \( \mu_2 = 0.001, \gamma_2 = (1, 1) \) and \( \alpha^2 = (0.45, 0.45) \).

These parameters are consistent with respect to the average intensity of market orders (in a stationary version):

\[
\mu_0 = \frac{\mu_1}{1 - \|K_1\|} = \frac{\mu_2}{1 - \|K_2\|}.
\]
In order to estimate the gain made by market makers using refined strategies we compute the value function associated to each control when the order flows actually follows the modeling of the third market maker, see Figures 1, 2 and 3. As expected the control $\delta^2$ is optimal and $\delta^0$ is sub-optimal compared $\delta^1$. We observe in Figure 1 that considering a one factor model for the order flows leads to a 10% gain compared with a strategy considering that market order flows is a Poisson process. Using two factors leads to another 10% gain compared to the one factor case.

![Figure 1: Value function along the time for controls $\delta^0$, $\delta^1$ and $\delta^2$ with initial condition $c^a = (0, 10)$, $c^b = (0, 10)$ and $i = -10$.](image)

### 4.2 The large dimension case

In this section we apply the method presented in Section 3.4 to estimate $U_K$ at several points when the function $K$ is a positive linear combination of $n$ decreasing exponential functions and $n$ is large.

More precisely for $n \in \{1, \ldots, 200\}$ we consider the kernel $K_n$ given by $K_n(t) = \sum_{i=1}^{n} \alpha_i e^{-\gamma_i t}$. We write $K = K_{200}$ and for any $n \leq 200$ set $\alpha^n = (\alpha_i)_{1 \leq i \leq n}$. The parameters $(\alpha_i)_{1 \leq i \leq 200}$ and $(\gamma_i)_{1 \leq i \leq 200}$ are given in Figure 4.2. For any $n$ using the probabilistic representation of $[14]$, see Appendix G for more details, we estimate $U_{K,n}$ at the points

$$x^n_0 = (0, 0, K^n, 0), \quad x^n_1 = (0, 10, K^n, 0) \quad \text{and} \quad x^n_2 = (0, -10, K^n, 0) \quad \text{in} \quad E^{K_n}.$$ 

We consider $x_0 = (0, 0, K, 0)$, $x_1 = (0, 10, K, 0)$ and $x_2 = (0, -10, K, 0)$ in $E^K$. According to Proposition 3.1 we have for any $i \in \{1, \ldots, 2\}$

$$U_{K,n}(x^n_i) \xrightarrow{K \rightarrow +\infty} U_K(x_i).$$

This convergence is clearly illustrated in Figure 4.2. This prove the tractability of our approach to take into account the self exciting properties of market order flow into market making strategies.

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Figure 2: Difference between the value function associated to control $\delta^2$ and $\delta^1$ for $c^a = (10, 0)$, $c^{b,1} = 10$.

Figure 3: Difference between the value function associated to control $\delta^2$ and $\delta^0$ for $c^a = (10, 0)$, $c^{b,1} = 10$.  

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Figure 4: Estimation of the value function $U_{K_n}$ at points $x_i^n$ for $n \in \{1, \ldots, 200\}$ and $i \in \{0, 1, 2\}$.

Figure 5: Parameters $(\alpha_i)_{1 \leq i \leq 200}$ (in blue) and $(\gamma_i)_{1 \leq i \leq 200}$ (in red).
5 Proofs

5.1 Formal definition of the probability space

In this section we make precise the probability space we are working on. In particular we give a proper definition to $\mathbb{E}^t$. First we define the canonical process and the probability space associated to our stochastic control problem.

- Consider $\Omega_d$ the set of increasing piecewise constant càdlàg functions from $[0, T]$ into $\mathbb{N}$ with jumps equal to 1 and $\Omega_p$ the set of continuous functions from $[0, T]$ into $\mathbb{R}$. We define $\Omega = \Omega_p \times \Omega_d^2$.

- We let $(W_t, N_t^a, N_t^b)_{t \in [0, T]}$ be the canonical process on $\Omega$.

- The associated filtration is $\mathbb{F} = (\mathcal{F}_t^a \otimes \mathcal{F}_t^b \otimes \mathcal{F}_t^d)_{t \in [0, T]}$ where $(\mathcal{F}_t^d)_{t \in [0, T]}$ (resp. $(\mathcal{F}_t^p)_{t \in [0, T]}$) is the right continuous completed filtration associated with $N^a$ (or $N^b$) (resp. $W$).

- We denote by $\mathbb{P}_0$ the probability measure on $(\Omega, \mathbb{F})$ such that $(M^a_s = N^a_s - s\lambda_0, M^b_s = N^b_s - s\lambda_0)_{s \in [0, T]}$ for $\lambda_0 > 0$, are local martingales and $(W_s)_{s \in [0, T]}$ is a Brownian motion.

We now introduce some process related to our model. For a fixed $(t, x) \in \mathcal{E}^K$ we define $X^{t,x} = (P^t_{s,x}, i^{t,x}, \theta^{t,x,a}, \theta^{t,x,b})$ that is the state of the system after time $t$ when starting from point $(t, x)$. The dynamic of $X^{t,x}$ is given on $[t, T]$ by

$$
\begin{align*}
\frac{dP^{t,x}_s}{d\mathbb{P}_0} &= L^{t,x;\delta} \\
\mathbb{E}^{t,x;\delta}_t &= \int_0^t \frac{\lambda^a(s, X^{t,x}_s, \delta_s) - \lambda_0}{\lambda_0} 1_{s \geq t} dM^a_s + \frac{\lambda^b(s, X^{t,x}_s, \delta_s) - \lambda_0}{\lambda_0} 1_{s \geq t} dM^b_s.
\end{align*}
$$

Since $\lambda^a(t, x, \delta) \leq C(1 + \|x\|)$ and $\lambda^b(t, x, \delta) \leq C(1 + \|x\|)$, by the Corrolary 2.6 in [23], for any $(t, x) \in \mathcal{E}^K$, $(L^{t,x;\delta}_s)_{s \in [t, T]}$ is a true $\mathbb{P}_0$ martingale. Moreover by Theorem III-3.11 in [18] the processes

$$M^{t,x,a;\delta} = N^a - \int_t^\cdot \lambda^a(u, \delta_u, X^{t,x}_u) du \quad \text{and} \quad M^{t,x,b;\delta} = N^b - \int_t^\cdot \lambda^b(u, \delta_u, X^{t,x}_u) du$$

are $\mathbb{E}^{t,x;\delta}$-local martingales on $[t, T]$. Actually they are true martingales, see Appendix [B.1]

For $(t, x) \in \mathcal{E}^K$ and $\delta \in \mathcal{A}$ we note $\mathbb{E}^{t,x;\delta}_t$ the expectation under the law $\mathbb{P}^{t,x;\delta}$ and note $\mathbb{E}_t$ instead of $\mathbb{E}^{t,x;\delta}_t$.

Finally, for any $F$ bounded continuous function, $\delta \in \mathcal{A}$ and $\theta$ stopping time with values in $[t, T]$ we have:

$$\mathbb{E}^{t,x}_t[F(X^{t,x}_T)|\mathcal{F}_\theta] = \mathbb{E}^{t,x}_t[F(X^{t,x}_\theta)|\mathcal{F}_\theta]$$

(8)

where, $\delta^\theta$ is the restriction to $[\theta, T]$ of $\delta$. This prove that for any $(t, x) \in \mathcal{E}^K$ the process $(s, X^{t,x}_s)_{s \geq t}$ is Markovian.
5.2 Proof of Theorem 2.1

We proceed in 5 steps.

1. Section 5.2.1 Using a comparison result we show that (HJB)$_K$ admits a unique viscosity solution with polynomial growth.

2. Section 5.2.2 For any $K$ we define $U^K$ the continuation utility function associated to (3).

3. Section 5.2.3 We prove a dynamic programming principle for $U^K$.

4. Section 5.2.4 Using a verification argument we show that $U^K$ is the unique viscosity solution (with polynomial growth) of (HJB)$_K$.

5. Section 5.2.5 We show that the control given in Equation (5) solves the control problem (3).

5.2.1 Comparison result for (HJB)$_K$

We start by proving a comparison result for bounded solutions, then we extend it to functions with polynomial growth.

**Proposition 5.1.** Let $U \in USC(\mathcal{E}^K)$ be a bounded from above viscosity sub-solution of (HJB)$_K$ and $V \in LSC(\mathcal{E}^K)$ be a bounded from below viscosity super-solution of (HJB)$_K$ such that $U(T, \cdot) \leq V(T, \cdot)$ then

$U \leq V$ on $\mathcal{E}^K$.

**Proof.** We suppose that there exists some $(t_0, x_0) \in \mathcal{E}^K$ such that

$U(t_0, x_0) - V(t_0, x_0) = \delta > 0$.

By hypothesis necessarily $t_0 \in [0, T)$. We show that this implies a contradiction. We consider the following quantities

$N_{\varepsilon} = \sup_{(t,x) \in \mathcal{E}^K} U(t,x) - V(t,x) - 2\varepsilon \|x\|^2$

and

$N_{\varepsilon}^\alpha = \sup_{(t,x),(t,y) \in \mathcal{E}^K} U(t,x) - V(t,y) - \varepsilon (\|x\|^2 + \|y\|^2) - \alpha \|x - y\|^2.$

The function $U$ and $-V$ being bounded from above we have

$\lim_{\|x\|+\|y\| \to +\infty} U(t,x) - V(t,y) - \alpha \|x - y\|^2 - \varepsilon \|x\|^2 - \varepsilon \|y\|^2 = -\infty$

uniformly in $t$. Thus we can restrict the supremums to bounded sets that depends only on $\varepsilon$. More precisely

$N_{\varepsilon} = \sup_{(t,x) \in \mathcal{E}^K} U(t,x) - V(t,x) - 2\varepsilon \|x\|^2$  \hspace{1cm} (9)

$N_{\varepsilon}^\alpha = \sup_{(t,x),(t,y) \in \mathcal{E}^K} U(t,x) - V(t,y) - \varepsilon (\|x\|^2 + \|y\|^2) - \alpha \|x - y\|^2$  \hspace{1cm} (10)

where $R$ only depends on $\varepsilon$. We remind that the set $\mathcal{E}^K$ is compact. Hence the supremum $N_{\varepsilon}^\alpha$ is achieved at some $(t_{\varepsilon}^\alpha, x_{\varepsilon}^\alpha, y_{\varepsilon}^\alpha)$. We show at the end of the proof that when $\alpha \to +\infty$, up to a subsequence, we have

$\lim_{\alpha \to +\infty} (t_{\varepsilon}^\alpha, x_{\varepsilon}^\alpha, y_{\varepsilon}^\alpha) = (t_{\varepsilon}, x_{\varepsilon}, x_{\varepsilon})$  \hspace{1cm} (11)

where $(t_{\varepsilon}, x_{\varepsilon})$ achieves the supremum $N_{\varepsilon}$. We also prove that

$\lim_{\alpha \to +\infty} \alpha \|x_{\varepsilon}^\alpha - y_{\varepsilon}^\alpha\|^2 = 0$, $\lim_{\alpha \to 0} N_{\varepsilon}^\alpha = N_{\varepsilon}$, $\lim_{\varepsilon \to 0} \varepsilon \|x_{\varepsilon}\|^2 = 0$  \hspace{1cm} (12)

and that

$\lim_{\varepsilon \to 0} N_{\varepsilon} = N = \sup_{(t,x) \in \mathcal{E}^K} U(t,x) - V(t,x).$  \hspace{1cm} (13)
A consequence of Equation (13) is that
\[
\lim_{\alpha \to +\infty} \left( U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}), V(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}) \right) = (U(t_{\varepsilon}, x_{\varepsilon}), V(t_{\varepsilon}, x_{\varepsilon})).
\] (14)

We use the notations \( x^\alpha_{\varepsilon} = (P^\alpha_{\varepsilon}, i^\alpha_{\varepsilon}, \theta^\alpha_{\varepsilon}) \) and \( y^\varepsilon_{\alpha} = (Q^\varepsilon_{\alpha}, j^\varepsilon_{\alpha}, \beta^\varepsilon_{\alpha}, \gamma^\varepsilon_{\alpha}) \).

With respect to Lemma D.1, which is an adaptation of the Crandall-Ishi’s lemma to our framework, for any \( \beta > 0 \) there exists \((\lambda^\alpha_{\varepsilon}, p^\alpha_{\varepsilon}), A^\beta_{\varepsilon} \), \( h \in \mathcal{F}^+ U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}) \) and \((\hat{\lambda}^\alpha_{\varepsilon}, q^\alpha_{\varepsilon}), B^\beta_{\varepsilon} \), \( g \in \mathcal{F}^- V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}) \) such that
\[
-(\beta^{-1} + 2\varepsilon + 4\alpha)I_2 \leq \begin{pmatrix} A^\beta_{\varepsilon} & 0 \\ 0 & -B^\beta_{\varepsilon} \end{pmatrix} \leq (2\varepsilon + 3\alpha^2)I_2 + (2\alpha + 8\beta)(\alpha\varepsilon + \alpha^2) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\]

with \( p^\alpha_{\varepsilon} = 2\varepsilon P^\alpha_{\varepsilon} + 2\alpha(P^\alpha_{\varepsilon} - Q^\alpha_{\varepsilon}), q^\alpha_{\varepsilon} = -2\varepsilon Q^\alpha_{\varepsilon} - 2\alpha(Q^\alpha_{\varepsilon} - P^\alpha_{\varepsilon}), \lambda^\alpha_{\varepsilon} = 0 \) and \( \hat{\lambda}^\alpha_{\varepsilon} = 0 \).

Remark that for \( \varepsilon \) small enough
\[
U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}) - V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}) \geq \delta - \varepsilon \| x_0 \|^2 > \frac{\delta}{2}
\]

We now walk towards a contradiction by showing that
\[
\limsup_{\varepsilon \to 0} \limsup_{\alpha \to +\infty} U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}) - V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}) \leq 0.
\]

According to the definition of sub-solution and super-solution we have
\[
F(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}, U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}), (\lambda^\alpha_{\varepsilon}, p^\alpha_{\varepsilon}), A^\beta_{\varepsilon} U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon})) \leq 0
\]
and
\[
F(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}, V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}), (\hat{\lambda}^\alpha_{\varepsilon}, q^\alpha_{\varepsilon}), B^\beta_{\varepsilon} V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon})) \geq 0.
\]

By definition of \( F \):
\[
r(U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}) - V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon})) \leq F(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}, U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}), (\lambda^\alpha_{\varepsilon}, p^\alpha_{\varepsilon}), A^\beta_{\varepsilon} U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}))
\]
\[
- F(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}, V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}), (\lambda^\alpha_{\varepsilon}, p^\alpha_{\varepsilon}), A^\beta_{\varepsilon} U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}))
\]
thus
\[
r(U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}) - V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon})) \leq F(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}, V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}), \hat{\lambda}^\alpha_{\varepsilon}, q^\alpha_{\varepsilon}, B^\beta_{\varepsilon} V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}))
\]
\[
- F(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}, V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}), \lambda^\alpha_{\varepsilon}, p^\alpha_{\varepsilon}, A^\beta_{\varepsilon} U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}))
\]
\[
\leq d(t^\alpha_{\varepsilon}, P^\alpha_{\varepsilon}) p^\alpha_{\varepsilon} - d(t^\alpha_{\varepsilon}, Q^\alpha_{\varepsilon}) q^\alpha_{\varepsilon} + \frac{1}{2} \sigma^2 A^\beta_{\varepsilon} - \frac{1}{2} \sigma^2 B^\beta_{\varepsilon}
\]
\[
+ H(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}, D^\alpha_{\varepsilon} V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon})) - H(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}, D^\alpha_{\varepsilon} U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}))
\]
where
\[
H(t, x, I) = - \sup_{\delta \in \mathbb{R}_+} \Phi(\theta^\alpha(t)) e^{-\frac{\varepsilon}{2}(\delta + I_1)} - \sup_{\delta \in \mathbb{R}_+} \Phi(\theta^\beta(t)) e^{-\frac{\varepsilon}{2}(\delta + I_2)} + g(i, p).
\]

Note that the function \( H \) is Lipschitz continuous. Taking \( \beta = \alpha^{-1} \) we get
\[
\sigma^2 A^\beta_{\varepsilon} - \sigma^2 B^\beta_{\varepsilon} \leq 2(2\varepsilon + \alpha^{-1}4\varepsilon^2)\sigma^2.
\]

The RHS can be taken arbitrarily small when \( \alpha \to +\infty \) and \( \varepsilon \to 0 \) by Equation (12). Using the Lipschitz property of \( d \) we have
\[
d(t^\alpha_{\varepsilon}, P^\alpha_{\varepsilon}) p^\alpha_{\varepsilon} - d(t^\alpha_{\varepsilon}, Q^\alpha_{\varepsilon}) q^\alpha_{\varepsilon} = 2\varepsilon (d(t^\alpha_{\varepsilon}, P^\alpha_{\varepsilon}) P^\alpha_{\varepsilon} + d(t^\alpha_{\varepsilon}, Q^\alpha_{\varepsilon}) Q^\alpha_{\varepsilon} + 2\alpha(P^\alpha_{\varepsilon} - Q^\alpha_{\varepsilon})(d(t^\alpha_{\varepsilon}, P^\alpha_{\varepsilon}) - d(t^\alpha_{\varepsilon}, q^\alpha_{\varepsilon}))
\]
\[
\leq 2\varepsilon C(1 + \| y^\alpha_0 \|^2 + \| x^\alpha_0 \|^2) + Cl\alpha\| x^\alpha_0 - y^\alpha_0 \|^2.
\]
Here again the RHS goes to zero when $\alpha \to +\infty$ and then $\varepsilon \to 0$ because of Equation 12. Finally by Equation 14 and since $U$ (resp. $V$) is a USC (resp. LSC) function and $H$ is continuous and decreasing with respect to its last variable we have

$$\limsup_{\alpha \to +\infty} H(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}, D^K V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon})) - H(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}, D^K U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon})) \leq H(t_{\varepsilon}, x_{\varepsilon}, D^K V(t_{\varepsilon}, x_{\varepsilon})) - H(t_{\varepsilon}, x_{\varepsilon}, D^K U(t_{\varepsilon}, x_{\varepsilon})).$$

Remark that for any $z$ such that $(t_{\varepsilon}, x_{\varepsilon} + z) \in \mathcal{E}^K$ we have by definition of $(t_{\varepsilon}, x_{\varepsilon})$

$$U(t_{\varepsilon}, x_{\varepsilon}) - V(t_{\varepsilon}, x_{\varepsilon}) - 2\varepsilon\|x_{\varepsilon}\|^2 \geq U(t_{\varepsilon}, x_{\varepsilon} + z) - V(t_{\varepsilon}, x_{\varepsilon} + z) - 2\varepsilon\|x_{\varepsilon} + z\|^2.$$ 

Consequently we have

$$V(t_{\varepsilon}, x_{\varepsilon} + z) - V(t_{\varepsilon}, x_{\varepsilon}) \geq U(t_{\varepsilon}, x_{\varepsilon} + z) - U(t_{\varepsilon}, x_{\varepsilon}) - 2\varepsilon(\|x_{\varepsilon} + z\|^2 - \|x_{\varepsilon}\|^2)$$

and so

$$D^K V(t_{\varepsilon}, x_{\varepsilon}) \leq D^K (U - 2\varepsilon\|\cdot\|)(t_{\varepsilon}, x_{\varepsilon}).$$

The monotony and Lipschitz regularity of $H$ implies

$$\limsup_{\alpha \to +\infty} H(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}, D^K V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon})) - H(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}, D^K U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}))$$

$$\leq H(x_{\varepsilon}, D^K U(t_{\varepsilon}, x_{\varepsilon})) - H(x_{\varepsilon}, D^K U(t_{\varepsilon}, x_{\varepsilon}))$$

$$\leq C\varepsilon\|x_{\varepsilon}\|\|D^K\|\|2(t_{\varepsilon}, x_{\varepsilon})\|.$$ 

Notice that for any $x \in \mathcal{E}^K$

$$D^K\|\|2(x) = \left(\|\theta^0 + K(\cdot - t)\|^2 - \|\theta^0\|^2 + |i + 1|^2 - |i|^2\right)^2$$

$$+ \left(\|K(\cdot - t)\|^2 - \|\theta^0\|^2 + |i - 1|^2 - |i|^2\right)^2$$

$$\leq \left(\|K(\cdot - t)\|^2 + 2\|K(\cdot - t)\|\|\theta^0\| + 1 + 2i\right)^2$$

thus there exists $C > 0$ such that $\|D^K\|\|2(t, x)\| \leq C(1 + \|x\|)$. Consequently we get

$$\limsup_{\alpha \to +\infty} H(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}, D^K V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon})) - H(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}, D^K U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon})) \leq C\varepsilon(1 + \|x_{\varepsilon}\|^2)$$

that goes to zero when taking the limit $\varepsilon \to 0$. Finally we have shown that

$$\limsup_{\varepsilon \to 0} \limsup_{\alpha \to +\infty} U(t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}) - V(t^\alpha_{\varepsilon}, y^\alpha_{\varepsilon}) \leq 0.$$ 

We get a contradiction.

We finally prove the statements 11, 12 and 13. We consider $(t_{\varepsilon}, x_{\varepsilon}, y_{\varepsilon}) \in (t^\alpha_{\varepsilon}, x^\alpha_{\varepsilon}, y^\alpha_{\varepsilon})_{\alpha \geq 0}$ that exists since $\mathcal{E}^K$ is compact. Since $N^\alpha_{\varepsilon} \geq N_{\varepsilon}$ then necessarily $x_{\varepsilon} = y_{\varepsilon}$. We now prove the first limit of 12 and that $(t_{\varepsilon}, x_{\varepsilon})$ corresponds to a point where the supremum $N_{\varepsilon}$ is achieved. Passing to the lower limit we get

$$U(t_{\varepsilon}, x_{\varepsilon}) - V(t_{\varepsilon}, x_{\varepsilon}) - 2\varepsilon\|x_{\varepsilon}\|^2 \geq \limsup_{\alpha \to +\infty} \alpha\|x^\alpha_{\varepsilon} - y^\alpha_{\varepsilon}\|^2 \geq N_{\varepsilon}.$$

Hence by definition of $N_{\varepsilon}$ we necessarily have that

$$\lim_{\alpha \to +\infty} \alpha\|x^\alpha_{\varepsilon} - y^\alpha_{\varepsilon}\|^2 = 0$$

and that

$$N_{\varepsilon} = U(t_{\varepsilon}, x_{\varepsilon}) - V(t_{\varepsilon}, x_{\varepsilon}) - 2\varepsilon\|x_{\varepsilon}\|^2.$$

To conclude we show that $N_{\varepsilon} \to N$ and that $\varepsilon\|x_{\varepsilon}\|^2 \to 0$. For $\xi > 0$ consider $(t, x)$ that is $\xi$-optimal in the definition of $N$:

$$U(t, x) - V(t, x) \geq N - \xi.$$ 

For $\varepsilon$ small enough $2\varepsilon\|x\|^2$ is lower than $\xi$, and we get

$$N \geq N_{\varepsilon} \geq U(t, x) - V(t, x) - 2\varepsilon\|x\|^2 \geq N - 2\xi.$$
Therefore we get convergence of \( N_\varepsilon \) towards \( N \) and as consequence
\[
U(t_\varepsilon, x_\varepsilon) - V(t_\varepsilon, x_\varepsilon) - 2\varepsilon\|x_\varepsilon\|^2 \to N.
\]
Since for any \( \varepsilon \) we have
\[
N \geq U(t_\varepsilon, x_\varepsilon) - V(t_\varepsilon, x_\varepsilon) \geq U(t_\varepsilon, x_\varepsilon) - V(t_\varepsilon, x_\varepsilon) - 2\varepsilon\|x_\varepsilon\|^2 = N_\varepsilon,
\]
we get that \( \varepsilon\|x_\varepsilon\|^2 \to 0 \). This concludes the proof.

Now we extend Proposition 5.1 to the case of functions with polynomial growth.

**Proposition 5.2.** Let \( U \in USC(\mathcal{E}^K) \) with polynomial growth be a viscosity sub-solution of Equation (HJB)_K and \( V \in LSC(\mathcal{E}^K) \) with polynomial growth be a viscosity super-solution of Equation (HJB)_K such that \( U(T, \cdot) \leq V(T, \cdot) \). Then
\[
U \leq V \text{ on } \mathcal{E}^K.
\]

**Proof.** There exists \( k > 0 \) such that
\[
\lim_{\|x\| \to +\infty} \frac{|U(t, x)| + |V(t, x)|}{1 + \|x\|^k} = 0.
\]
We introduce the following function
\[
w(t, x) = e^{K(T-t)}(1 + \|x\|^{2k}).
\]
We have
\[
D^K w(t, x) = e^{K(T-t)} \left( P^{11}_{2k-1}(\|\phi\|_1^1) P^{12}_{2k-1}(i) P^{12}_{2k-1}(i) \right)
\]
with \( (P^{ij}_{2k-1})_{i, j \in \{1, 2\}} \) polynomials with degree \( 2k - 1 \). Consequently for some \( C > 0 \)
\[
\|x\||D^K w(t, x)|| \leq Cw(t, x).
\]
We have
\[
\sigma^2 \partial^2_p w(t, x) \leq C(1 + \|x\|^2) e^{K(T-t)} Q_{2k-2}(\|x\|) \leq Cw(t, x)
\]
and
\[
d(t, x) \partial_p w(t, x) \leq e^{K(T-t)} C(1 + \|x\|) Q_{2k-1}(\|x\|) \leq Cw(t, x)
\]
where \( Q_{2k-2} \) and \( Q_{2k-1} \) are two polynomials with respective degree \( 2k - 2 \) and \( 2k - 1 \). Consequently for any constant \( B \)
\[
-\partial_t w(t, x) - d(t, x) \partial_p w(t, x) - \frac{1}{2} \sigma^2 \partial^2_p w(t, x) - B\|x\||D^K w(t, x)|| \geq w(t, x)(K - C)
\]
which is positive for \( K \) large enough. Hence for any \( \varepsilon > 0 \) the function \( U - \varepsilon w \) is a bounded from above viscosity sub-solution of Equation (HJB)_K. Indeed if \( U - \varepsilon w \leq \phi \) then \( U \leq \phi + \varepsilon w \) consequently
\[
F(t, x, U(t, x), \nabla(\phi + \varepsilon w)(t, x), \partial^2_{pp}(\phi + \varepsilon w)(t, x), D^K U(t, x)) \leq 0.
\]
We have for \( K \) large enough
\[
F(t, x, U(t, x) - \varepsilon w(t, x), \nabla \phi(t, x), \partial^2_{pp}(\phi)(t, x), D^K(U - \varepsilon w)(t, x))
- F(t, x, U(t, x), \nabla(\phi + \varepsilon w)(t, x), \partial^2_{pp}(\phi + \varepsilon w)(t, x), D^K U(t, x))
\leq \varepsilon(-r w(t, x) + \partial_t + d \partial_p + \frac{1}{2} \sigma^2 \partial^2_{pp} w(t, x) + C\|x\||D^K w(t, x)||
\]
\[
< 0.
\]
It implies that
\[
F(t, x, U(t, x) - \varepsilon w(t, x), (\partial_t \phi(t, x), \partial_p \phi(t, x)), \partial^2_{pp} \phi(t, x), D^K(U - \varepsilon w)(t, x)) \leq 0.
\]
We show in the same way that \( V + \varepsilon w \) is a bounded from below viscosity super-solution. Then from Proposition 5.1 we have
\[
U - \varepsilon w \leq V + \varepsilon w
\]
and taking \( \varepsilon \) to 0 we get the stated result. \( \Box \)
An immediate consequence from Proposition 5.2 is that there exists a unique viscosity solution with polynomial growth to \((\text{HJB})_K\). We now prove the existence of such solution using a verification argument.

### 5.2.2 Definition of the continuation utility function

For \((t,x) \in \mathcal{E}^K\) and \(\delta \in \mathcal{A}\) we define

\[
J^K(t,x;\delta) = \mathbb{E}^\delta_t[x \left[ G(t,x,T) e^{-r(T-t)} + \int_t^T e^{-r(s-t)} \tilde{g}(s,X_{s}^{t,x},\delta_s) ds \right] ]
\]

where

\[
\tilde{g}(s,x,\delta) = g(s,P) + \delta^a \lambda^a(s,x,\delta) + \delta^b \lambda^b(s,x,\delta).
\]

We also define

\[
U^K(t,x) = \sup_{\delta \in \mathcal{A}} J^K(t,x;\delta)
\]

that is the maximal utility than can expect a market maker starting its trading from time \(t\) with initial market condition given by \(x\). By Lemma B.1 we get that \(U^K\) has polynomial growth. More precisely there exists a positive constant \(\kappa\) such that

\[
U^K(t,x) \leq \kappa(1 + \|x\|^2).
\]

We also define

\[
\mathcal{A}_t = \{ \delta \in \mathcal{A} \text{ s.t. } \delta \text{ is independent of } \mathcal{F}_t \},
\]

the set of controls starting from \(t\) and independent from the past. Since under \(P_0\) the processes \(N^a\) and \(N^b\) have independent increments, using the same arguments than in Remark 2.2-(iv) in \([25]\) we get

\[
U^K(t,x) = \sup_{\delta \in \mathcal{A}_t} J^K(t,x;\delta).
\]

In the next sections we show that the function \(U^K\) is the unique viscosity solution with polynomial growth to \((\text{HJB})_K\). For this we prove a dynamic programming principle for \(U^K\) and then conclude using a verification argument.

### 5.2.3 Dynamic programming principle

Consider the lower and upper semi-continuous version of \(U^K\):

\[
U^K_*(x) = \liminf_{y \to x} U^K(y) \quad \text{and} \quad U^K^*(x) = \limsup_{y \to x} U^K(y).
\]

Inspired by \([25]\) we prove the following dynamic programming principle.

**Theorem 5.1.** Let \((t,x) \in \mathcal{E}^K\) be fixed and \(\{\theta^\delta, \delta \in \mathcal{A}_t\}\) be a family of finite stopping times with values in \([t,T]\). Assume that for any \(\delta\), \((X^{t,x}_s 1_{s \in [t,\theta^\delta]}), s \in [0,T]\) is \(L^\infty\)-bounded. Then we have

\[
U^K(t,x) \geq \sup_{\delta \in \mathcal{A}_t} \mathbb{E}^\delta_t[e^{-r(\theta^\delta-t)}U^K_*(\theta^\delta,X^{\theta^\delta,t,x}_\theta) + \int_t^{\theta^\delta} e^{-r(s-t)} \tilde{g}(s,X^{\theta^\delta,s,x}_s,\delta_s) ds]
\]

and

\[
U^K(t,x) \leq \sup_{\delta \in \mathcal{A}_t} \mathbb{E}^\delta_t[e^{-r(\theta^\delta-t)}U^K_*(\theta^\delta,X^{\theta^\delta,t,x}_\theta) + \int_t^{\theta^\delta} e^{-r(s-t)} \tilde{g}(s,X^{\theta^\delta,s,x}_s,\delta_s) ds].
\]

The proof of Theorem 5.1 is the same as the one of Theorem 2.3 in \([25]\). However since we are working on non-usual domains we write the proof for the sake of completeness.
Proof. We first show the first inequality. We consider a continuous function \( \psi \) such that \( U^K \geq \psi \). By definition of \( U^K \) for any \((t,x) \in E^K\) there is an admissible control \( \delta^{t,x} \in A_t \) that is \( \varepsilon \) optimal:

\[
J^K(t,x;\delta^{t,x}) \geq U^K(t,x) - \varepsilon.
\]

The function \( G \) and \( \tilde{g} \) being lower semi-continuous, the function \( J^K(\cdot;\delta^{t,x}) \) is also lower semi-continuous by Fatou’s lemma. Then \( \psi \) being upper semi-continuous we can find a family of positive real \((r_{t,x})_{t,x \in E^K}\) such that for any \((t,x) \in E^K\) we have

\[
\psi(t,x) - \psi(s,y) \geq -\varepsilon \text{ and } J^K(t,x;\delta^{t,x}) - J^K(s,y;\delta^{t,x}) \leq \varepsilon, \text{ for } (s,y) \in B(t,x;r_{t,x})
\]

where

\[
B(t,x;r) = \{(s,y) \in E^K \text{ s.t. } s \in (t-r,t), \|x-y\| < r\}.
\]

The system \((B(t,x;r_{t,x}))_{t,x \in E}\) forms an open covering of \( E^K \). With the topology it is endowed \( E^K \) is second countable since \([0,T] \times \mathbb{R} \times \mathbb{Z} \times \mathbb{L}_1 \times \mathbb{L}_1\) is second countable. So by the Lindelöf covering Theorem we can extract from \((B(t,x;r(t,x)))_{(t,x) \in E^K}\) a countable subfamily that covers \( E^K \). Thus we have \((t_i,x_i,r_i)_{i \in \mathbb{N}}\) such that

\[
E^K \subseteq \bigcup_{i \in \mathbb{N}} B(t_i,x_i;r_i).
\]

Set \( A^n = \bigcup_{0 \leq i \leq n} A_i \). Consider \( A_0 = \{T\} \times X^K_T \), \( C_{-1} = \emptyset \) and define the sequence

\[
A_{i+1} = B(t_{i+1},x_{i+1};r_{i+1}) \setminus C_i, \text{ where } C_i = C_{i-1} \cup A_i, \ i \geq 0.
\]

Now fix \( \delta \in A_i \). With the above construction, we have \((\theta^\delta,X^{t,\delta}) \in \bigcup_{i \geq 0} A_i \) and for \( i \geq 1 \), we have

\[
J^K(\cdot;\delta^{t,\delta}) \geq \psi - 3\varepsilon \text{ on } A_i.
\]

We define the control process \( \delta^{t,\varepsilon} \) by

\[
\delta^{t,\varepsilon}_s = 1_{[t,\theta^\delta]}(s)\delta_s + 1_{[\theta^\delta,T]}(s)(1_{A^n}(\theta^\delta,X^{t,\delta})\delta_s + \sum_{i=1}^n 1_{A_i}(\theta^\delta,X^{t,\delta})\delta^{t,\delta} s^{t,\delta}).
\]

The control \( \delta^{t,\varepsilon} \) is in \( A_i \). By Equation (8) we have

\[
\begin{align*}
\mathbb{E}^{\delta^{t,\varepsilon},n}_{t,x}[G(t^{t,\delta},P^{t,\delta})e^{-r(T-t)} & + \int_t^T e^{-r(s-t)}\tilde{g}(s,X^{t,\delta},\delta_s)ds|F_{\theta^\delta}]1_{A^n}(\theta^\delta,X^{t,\delta}) \\
= (U^K(T,X^{t,\delta})e^{-r(T-t)} & + \int_t^T e^{-r(s-t)}\tilde{g}(s,X^{t,\delta},\delta_s)ds)1_{A_0}(\theta^\delta,X^{t,\delta}) \\
+ \sum_{i=1}^n (e^{-r(\theta^\delta)}J^K(\theta^\delta,X^{t,\delta};\delta^{t,\delta}) & + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s,X^{t,\delta},\delta_s)ds)1_{A_i}(\theta^\delta,X^{t,\delta}) \\
\geq \sum_{i=0}^n (e^{-r(\theta^\delta)}\psi(\theta^\delta,X^{t,\delta}) & - 3\varepsilon + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s,X^{t,\delta},\delta_s)ds)1_{A_i}(\theta^\delta,X^{t,\delta}) \\
\geq (e^{-r(\theta^\delta)}\psi(\theta^\delta,X^{t,\delta}) & - 3\varepsilon + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s,X^{t,\delta},\delta_s)ds)1_{A^n}(\theta^\delta,X^{t,\delta}).
\end{align*}
\]

Thus we get

\[
U^K(t,x) \geq J^K(t,x;\delta^{t,\varepsilon}) \geq\mathbb{E}^{\delta^{t,\varepsilon},n}_{t,x}[G(t^{t,\delta},P^{t,\delta})e^{-r(T-t)} + \int_t^T e^{-r(s-t)}\tilde{g}(s,X^{t,\delta},\delta^{t,\varepsilon})ds|F_{\theta^\delta}]
\]

\[
\geq\mathbb{E}^{\delta^{t,\varepsilon},n}_{t,x}[(e^{-r(\theta^\delta)}\psi(\theta^\delta,X^{t,\delta}) - 3\varepsilon + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s,X^{t,\delta},\delta_s)ds)1_{A^n}(\theta^\delta,X^{t,\delta})]
\]

\[
+\mathbb{E}^{\delta^{t,\varepsilon},n}_{t,x}[(G(t^{t,\delta},P^{t,\delta})e^{-r(T-t)} + \int_t^T e^{-r(s-t)}\tilde{g}(s,X^{t,\delta},\delta_s)ds)1_{A^n}(\theta^\delta,X^{t,\delta})].
\]
Since $L^{t,x;\delta_n}$ is a true martingale and $L_s^{t,x;\delta_n} = L_s^{t,x;\delta}$ for $s \in [t, \theta^\delta]$ we have

$$U^K(t, x) \geq \mathbb{E}_{t, x}^{\delta}(e^{-r(\theta^\delta - t)}\psi(\theta^\delta, X_{\theta^\delta}^{t,x}) - 3\varepsilon + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X_s^{t,x}, \delta_s)ds) 1_{A_\varepsilon}(\theta^\delta, X_{\theta^\delta}^{t,x})$$

$$+ \mathbb{E}_{t, x}^{\delta}(G(t^\delta_T, P^\delta_T) e^{-r(T-t)} + \int_t^T e^{-r(s-t)}\tilde{g}(s, X_s^{t,x}, \delta_s)ds) 1_{A_\varepsilon}(\theta^\delta, X_{\theta^\delta}^{t,x}).$$

By dominated convergence letting $n \to +\infty$ we get

$$U^K(t, x) \geq -3\varepsilon + \mathbb{E}_{t, x}^{\delta}(e^{-r(\theta^\delta - t)}\psi(\theta^\delta, X_{\theta^\delta}^{t,x}) + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X_s^{t,x}, \delta_s)ds).$$

Since $\varepsilon$ is any positive real we have

$$U^K(t, x) \geq \mathbb{E}_{t, x}^{\delta}(e^{-r(\theta^\delta - t)}\psi(\theta^\delta, X_{\theta^\delta}^{t,x}) + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X_s^{t,x}, \delta_s)ds).$$

We now explain how to pass from $\psi$ dominated by $U^K$ to $U^K$. By hypothesis for any $\delta$ we can find $r$ such that almost surely $\|X_{t,x}^{\delta}\| \leq r$ for any $s \in [t, \theta^\delta]$. Then we can find an increasing sequence of continuous functions on $\mathcal{E}_K^\delta$, $(\Phi_n)_{n \geq 0}$ such that $\Phi_n \leq U^K \leq U^K$ and such that $\Phi_n$ converges pointwise towards $U^K$ on $[0, T] \times B_r(x)$ and such that $\Phi_n (t, x)$ is a viscosity super (resp. sub)-solution of $(\text{HJB})_K$ (see Lemma 3.5. in [23]), where $B_r(x) = \{ y \in \mathbb{R} \times \mathbb{Z} \times L^1 \times L^1 \text{ s.t. } ||y - x|| \leq r \}$. Consequently from monotone convergence Theorem we have

$$U^K(t, x) \geq \mathbb{E}_{t, x}^{\delta}(e^{-r(\theta^\delta - t)}U^K(\theta^\delta, X_{\theta^\delta}^{t,x}) + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X_s^{t,x}, \delta_s)ds).$$

Then we can pass to the supremum in $\delta \in A_t$ to get the result.

Now we show the first inequality. Take $\delta \in A_t$ and consider $\tilde{\theta}$ the controlled process obtained after freezing the trajectory of $\delta$ up to time $\theta^\delta$. By definition of $U^K$ we have

$$U^K(\theta^\delta, X_{\theta^\delta}^{t,x}) \geq \mathbb{E}_{\theta^\delta, X_{\theta^\delta}^{t,x}}^{\theta^\delta}[e^{-r(T-\theta^\delta)}G(\theta^\delta, X_{\theta^\delta}^{t,x}, P_{\theta^\delta}^{t,x}) \int_{\theta^\delta}^T e^{-r(s-\theta^\delta)}\tilde{g}(s, X_s^{t,x}, \delta_s)ds].$$

Using Equation (8) this gives

$$U^K(\theta^\delta, X_{\theta^\delta}^{t,x})e^{-r(\theta^\delta - t)} + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X_s^{t,x}, \delta_s)ds$$

$$\geq \mathbb{E}_{t, x}^{\delta}[e^{-r(T-t)}G(t^\delta_T, P^\delta_T) + \int_t^T e^{-r(s-t)}\tilde{g}(s, X_s^{t,x}, \delta_s)ds].$$

Now taking the average, by arbitrariness of $\delta$ we get the second inequality

$$\sup_{\delta \in A_t} \mathbb{E}_{t, x}^{\delta}[U^K(\theta^\delta, X_{\theta^\delta}^{t,x})e^{-r(\theta^\delta - t)} + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X_s^{t,x}, \delta_s)ds] \geq U^K(t, x).$$

In the next section we show that $U^K$ is a viscosity solution of $(\text{HJB})_K$ using a verification argument based on Theorem 5.1.

### 5.2.4 Verification

In this section using the dynamic programming principle proved previously we prove that $U^K$ (resp. $U^K$) is a viscosity super (resp. sub)-solution of $(\text{HJB})_K$. The proof is inspired from the proof of Propositions 6.2 and 6.3 in [25].
Proposition 5.3. The function $U^K_\ast$ (resp. $U^{K \ast}$) is a viscosity sub (resp. super)-solution of (HJB)$_K$.

Proof. We first show that $U^K_\ast$ is a viscosity super-solution and then that $U^{K \ast}$ is a viscosity sub-solution.

Let $(t, x) \in \mathcal{E}$ and $\phi$ be a test function such that

$$(U^K_\ast - \phi)(t, x) = \min_{\mathcal{E}^K} U^K_\ast - \phi = 0$$

and $(t_n, x_n)$ a sequence in $\mathcal{E}^K$ such that

$$(t_n, x_n) \to (t, x)$$

and $U^K(t_n, x_n) \to U^K_\ast(t, x)$.

Since $\phi$ is continuous we have

$$\eta_n = U^K(t_n, x_n) - \phi(t_n, x_n) \to 0.$$ 

Let $\delta \in \mathbb{R}_+^2$ and consider the constant control process equal to $\delta$. We use the notation $X^n = X^{t_n, x_n}$ and $E^\delta_n = E^\delta_{t_n, x_n}$. Finally, for all $n > 0$ we define the stopping time:

$$\tau_n = \inf\{s > t_n \text{ s.t. } (s - t_n, X^n_s - x_n) \notin [0, h_n) \times B_\alpha\},$$

where $B_\alpha$ the ball for $\| \cdot \|$, centered in 0 with radius $\alpha$ positive and small enough such that if a jump occurs then the stopping time $\tau_n$ is immediately reached. We take

$$h_n = \sqrt{\eta_n}1_{\eta_n \neq 0} + n^{-1}1_{\eta_n = 0}.$$ 

Notice that $\tau_n \to t$ almost surely.

From the first inequality in the dynamic programming principle, we have

$$0 \leq E^\delta_n [U^K(t_n, x_n) - e^{-r(\tau_n - t_n)}U^K_\ast(\tau_n, X^n_{\tau_n}) - \int_{t_n}^{\tau_n} e^{-r(s - t_n)}\tilde{g}(s, X^n_{\tau_n}, \delta)ds].$$

Now using that $U^K_\ast \geq \phi$ we get

$$0 \leq \eta_n + E^\delta_n [\phi(t_n, x_n) - e^{-r(\tau_n - t_n)}\phi(\tau_n, X^n_{\tau_n}) - \int_{t_n}^{\tau_n} e^{-r(s - t_n)}\tilde{g}(s, X^n_{\tau_n}, \delta)ds].$$

We can use the Ito formula since $\phi$ is smooth. Thus we get

$$0 \leq \eta_n - E^\delta_n \left[ \int_{t_n}^{\tau_n} e^{-r(s - t_n)}((-r\phi + \partial_t \phi + \mathcal{L}^\delta \phi)(s, X^n_s) + \tilde{g}(s, X^n_s, \delta))ds \right] - E^\delta_n [M^n_s],$$

where

$$\mathcal{L}^\delta \phi(s, x) = \mathcal{L}^\delta \phi(s, x) + \sum_{j=a,b} D_{j}^2 \phi(s, x)e^{-\frac{s}{2}\delta^2} \Phi(\theta^j(s))$$

and with

$$M^n_s = \int_{t_n}^{s} e^{-r(s - t_n)}(D_K^n \phi(s, X^n_s)dM^a_s + D_K^n \phi(s, X^n_s)dM^b_s + \sigma \partial_p \phi(s, X^n_s)dW_s)$$

The function $\phi$ being continuous, the integrands in the term $M^n_s$ are all bounded so the expectation of $M^n_s$ under $\mathbb{F}_n^\delta$ is 0. Consequently we have

$$0 \leq \eta_n - E^\delta_n \left[ \frac{1}{h_n} \int_{t_n}^{\tau_n} e^{-r(s - t_n)}((-r\phi + \partial_t \phi + \mathcal{L}^\delta \phi)(s, X^n_s) + \tilde{g}(s, X^n_s, \delta))ds \right] .$$

Taking $n \to +\infty$ using dominated convergence and arbitrariness of $\delta$ we get

$$0 \leq (r\phi - \partial_t \phi - \mathcal{L}^\delta \phi)(t, x) - \tilde{g}(t, x, \delta).$$
The control $\delta$ being arbitrary we finally have that
\[ F(t, x, \phi(t, x), \nabla \phi(t, x), \partial_{pp}^2 \phi(t, x), D^K \phi(t, x)) \geq 0. \]
Thus $U^K_*$ is a viscosity supersolution of $(\text{HJB})_K$.

Now we suppose that $U^{K*}$ is not a viscosity subsolution of $(\text{HJB})_K$ and exhibit a contradiction. According to the definition of viscosity subsolution we can find $\phi$ a test function and $(t_0, x_0)$ such that
\[ 0 = (U^{K*} - \phi)(t_0, x_0) > (U^{K*} - \phi)(t, x), \; \forall \; (t, x) \in \mathcal{E}_K \setminus \{(t_0, x_0)\} \]
and that
\[ F(t_0, x_0, \phi(t_0, x_0), \nabla \phi(t_0, x_0), \partial_{pp}^2 \phi(t_0, x_0), D^K \phi(t_0, x_0)) > 0. \] (16)
By continuity of $\phi$ and $F$ we have existence of a $r > 0$ small enough such that on $B_r(t_0, x_0)\setminus \{(t_0, x_0)\}$ we have
\[ h = -F(., \phi, \nabla \phi, \partial_{pp}^2 \phi, D^K \phi) < 0. \]
Moreover we can find some $\eta > 0$ (up to a change of $r$), such that
\[ \sup_{\partial B_r(t_0, x_0) \cap \mathcal{J}(t_0, x_0)} U^{K*} - \phi = -2\eta e^{rT} \]
where $\mathcal{J}(t_0, x_0)$ is the set of all values that can be reached if a jump occurs inside $B_r(t_0, x_0)$. Note that it is a compact set. We consider a sequence $(t_n, x_n)_{n \geq 0} \in \mathcal{E}_K$ such that
\[ \lim_{n \to +\infty} (t_n, x_n) = (t_0, x_0) \text{ and } \lim_{n \to +\infty} U^K(t_n, x_n) = U^{K*}(t_0, x_0). \]
Since $U^K(t_n, x_n) - \phi(t_n, x_n) \to 0$ we can assume that
\[ |U^K(t_n, x_n) - \phi(t_n, x_n)| \leq \eta \text{ for any } n \geq 1. \]
For a fixed control $\delta \in \mathcal{A}_{t_n}$ we define the stopping time
\[ \tau_n = \inf\{t > t_n \text{ s.t. } X_n^{t_n, t, \delta} \notin B_r(t_0, x_0)\}. \]
At the stopping time, either the process $X_n^{t_n, t, \delta}$ has not jumped and so is on $\partial B_r(t_0, x_0)$ or has jumped and is in $\mathcal{J}(t_0, x_0)$. Thus
\[ e^{-r(\tau_n - t_n)} \phi(\tau_n, X_n^{t_n, t, \delta}) \geq 2\eta + e^{-r(\tau_n - t_n)} U^K(\tau_n, X_n^{t_n, t, \delta}). \]
We derive from the Ito formula
\[ U^K(t_n, x_n) \geq -\eta + \phi(t_n, x_n) \]
\[ = -\eta + \mathbb{E}^\delta_n[e^{-r(\tau_n - t_n)} \phi(\tau_n, X_n^{t_n, t, \delta}) - \int_{t_n}^{\tau_n} e^{-r(s - t_n)}(-r + \partial_t + \mathcal{L}^\delta)\phi(s, X^s_n)ds]. \]
So by to Equation (16) we have
\[ U^K(t_n, x_n) \geq -\eta + \mathbb{E}^\delta_n[e^{-r(\tau_n - t_n)} \phi(\tau_n, X_n^{t_n, t, \delta}) + \int_{t_n}^{\tau_n} e^{-r(s - t_n)}g(s, X^s_n, \delta)ds] \]
\[ \geq \eta + \mathbb{E}^\delta_n[e^{-r(\tau_n - t_n)} U^{K*}(\tau_n, X_n^{t_n, t, \delta}) + \int_{t_n}^{\tau_n} e^{-r(s - t_n)}g(s, X^s_n, \delta)ds]. \]
Since $\delta$ is any control and $\eta$ is positive this contradict the second equation of Theorem 5.1. Thus $U^{K*}$ is a viscosity sub-solution of $(\text{HJB})_K$.

\[ \square \]

A direct consequence of Proposition 5.3 together with Proposition 5.2 is that
\[ U^K_* \geq U^{K*}. \]
But obviously we have $U^K \leq U^{K*}$, therefore $U^K = U^{K*} = U^K$. In particular $U^K$ is continuous and therefore is the unique continuous viscosity solution with polynomial growth to $(\text{HJB})_K$.
5.2.5 Proof of Theorem 2.1 (iii)
To prove Theorem 2.1 (iii) we must show that $J^K(\cdot ; \delta^*) = U^K$.

As we did previously we can show that $J^K(\cdot ; \delta^*)$ is the unique viscosity solution with polynomial growth of

$$ (LHJB)_K : \begin{cases} \ rU - \partial U - \mathcal{L}^K U - g - \sum_{j=a,b} e^{-\frac{\|\sigma_j^*\|}{\delta^*}}\Phi(\theta_j^*(u))(D_j^K U + \delta_j^K) = 0, \text{ on } \mathcal{E}^K. \end{cases} $$

But since $U^K$ is a viscosity solution of $(HJB)_K$ and by definition of $\delta_K$, $U^K$ is also a viscosity solution with polynomial growth of $(LHJB)_K$. So we get the result.

5.3 Proof of Proposition 3.1
We define the following functions on $\mathcal{E}^K$:
$$ U(x) = \limsup_{(y,n) \rightarrow (x, +\infty)} U^K_n(y) \text{ and } \overline{U}(x) = \liminf_{(y,n) \rightarrow (x, +\infty)} U^K_n(y), $$

We show that $U$ and $\overline{U}$ are respectively a viscosity super-solution and a viscosity sub-solution of $(HJB)_K$.

Consider $\phi$ a test function and $x \in \mathcal{E}^K$ a strict minimizer of $U - \phi$. We have existence of a sequence $(x_n, \sigma_n)_{n \in \mathbb{N}}$ in $\mathcal{E}$ such that
$$ (x_n, \sigma_n) \rightarrow (+\infty, x) \text{ and } U^K_n(x_n) \rightarrow U(x). $$

Consider $B_r(x)$ the closed ball of $[0, T] \times \mathbb{R} \times \mathbb{Z} \times \mathcal{L}_1^1 \times \mathcal{L}^1$ with radius $r > 0$ centered in $x$. Then we can always suppose that $x_n \in B_r(x)$, $\forall n \geq 0$. Let $\underline{x}_n$ be a minimizer of the difference $U^K_n - \phi$ on $\mathcal{E}^K \cap B_r(x)$ (exists because $\mathcal{E}^K_n$ is locally compact). We note $\underline{x}_n = (t_n, p_n, \theta^{n,a}_j, \theta^{n,b}_j)$. We show at the end of the proof that there exists $x \in \mathcal{E}^K$ such that $(x, +\infty)$ is the limit of a subsequence of $(\underline{x}_n, \sigma_n)_{n \geq 0}$ and that $\theta^{n,j}(t_n) \rightarrow \theta^j(t)$ for $j = a$ and $b$. So we can write
$$ U(x) - \phi(x) = \lim_{n \rightarrow +\infty} U^K_n(x_n) - \phi(x_n)
\geq \liminf_{n \rightarrow +\infty} U^K_n(\underline{x}_n) - \phi(\underline{x}_n)
\geq U(x) - \phi(x). $$

Thus by definition of $x$ we get that $(\underline{x}_n)_{n \geq 0}$ converges towards $x$ and that
$$ U^K_n(\underline{x}_n) \rightarrow U(x). $$

As a consequence when $n$ is large enough $\underline{x}_n$ is a local minimizer of $U^K_n - \phi$ (because it is in the interior of $\overline{B}$) hence by definition of viscosity solutions
$$ F(\underline{x}_n, U^K_n(\underline{x}_n), \nabla \phi(\underline{x}_n), \partial_{pp} \phi(\underline{x}_n), D_K U^K_n(\underline{x}_n)) \geq 0. $$

Then by definition of $U$ and since $U^K_n(\underline{x}_n) \rightarrow U(x)$:
$$ \liminf_{n \rightarrow +\infty} D_K U^K_n(x_n) \geq D_K U(x). $$

Finally since $F$ is decreasing with respect to the last variable and since $\theta^{n,j}(t_n)$ converges towards $\theta^j(t)$ for $j = a$ and $b$ we have
$$ F(x, U(x), \nabla \phi(x), \partial_{pp} \phi(x), D_K U(x)) \geq \limsup_{n \rightarrow +} F(\underline{x}_n, U^K_n(\underline{x}_n), \nabla \phi(\underline{x}_n), \partial_{pp} \phi(\underline{x}_n), D_K U^K_n(\underline{x}_n)) \geq 0. $$

So by Definition 2.1 $U$ is a viscosity super-solution of $(HJB)_K$. In the same way we can show that $\overline{U}$ is a viscosity sub-solution of $(HJB)_K$. Moreover since the a priori inequalities on $U^K_n$ can be chosen
uniform in $n$ (because $\|K_n\|_1 \to \|K\|_1$) they are true for $\underline{U}$ and $\overline{U}$. So Proposition 5.2 implies that $\underline{U} \geq \overline{U}$. Because we have the other inequality by definition we get $\overline{U} = \underline{U} = U^K$, the unique viscosity solution with polynomial growth of $(HJB)_K$.

To complete the proof we show that $(x_n)_{n \geq 0}$ admits a subsequence converging towards some $x \in \mathcal{E}^K$ and that for $j = a$ and $b$, $\theta^{n,a}(t_n)$ converges $\theta^a(t)$.

We have $x_n = (t_n, p_n, i_n, \theta^{n,a}, \theta^{n,b})$ with

$$\theta^{n,a} = \sum_{j=1}^{m_{n,a}} K_n(-T^a_j) \quad \text{and} \quad \theta^{n,b} = \sum_{j=1}^{m_{n,b}} K_n(-T^b_j)$$

where $m_{n,a}$ and $m_{n,b}$ are non-negative integers, $(T^a_j)_{1 \leq j \leq m_{n,a}}$ and $(T^b_j)_{1 \leq j \leq m_{n,b}}$ are in $[0, t_n]$. We recall that $(\|x_n\|_{n \geq 0})$ is bounded. Hence up to a subsequence $(t_n, p_n, i_n, \|\theta^{n,a}\|_1, \|\theta^{n,b}\|_1)_{n \geq 0}$ converges towards some $(t, p, i, \theta^a, \theta^b)$. Since we have assumed that $\|K\|_1$ is positive the convergence of $(\|\theta^{n,a}\|_1)_{n \geq 0}$ and $(\|\theta^{n,b}\|_1)_{n \geq 0}$ imply those of $(m_{n,a})_{n \geq 0}$ and $(m_{n,b})_{n \geq 0}$. Consequently those sequences are eventually constant equal to $m^a$ and $m^b$ for $n$ large enough. Then up to a subsequence we have convergence of $(T^a_j)_{1 \leq j \leq m^a}$ and $(T^b_j)_{1 \leq j \leq m^b}$ which are compact sets. We consider $(T^a_j)_{1 \leq j \leq m^a}$ and $(T^b_j)_{1 \leq j \leq m^b}$ their limits. We now show that $(\theta^{n,a})_{n \geq 0}$ converges in $L^1$ towards

$$\theta^a = \sum_{j=1}^{m^a} K(\cdot - T^a_j) \in \Theta^K_t.$$ 

Since by comparison theorem $T^a_j \leq t$ it is enough to show that $K_n(\cdot - T^a_0)$ converges in $L^1$ towards $K(\cdot - T^a_0)$ to conclude. We write

$$\|K(\cdot - T^a_n) - K_n(\cdot - T^a_0)\|_1 \leq \|K_n(\cdot - T^a_0) - K(\cdot - T^a_0)\|_1 + \|K(\cdot - T^a_0) - K(\cdot - T^a_n)\|_1 \leq \|K - K\|_1 + \|K(\cdot - T^a_0) - K(\cdot - T^a_n)\|_1.$$ 

The first term goes to 0 by hypothesis, the second by dominated convergence. Same results holds for $(\theta^{n,b})_{n \geq 0}$ and $\theta^b$. Consequently we have proved the convergence of $(x_n)_{n \geq 0}$ towards

$$x = (t, p, i, \theta^a, \theta^b) \in \mathcal{E}^K.$$ 

We finally show that $\theta^{n,a}(t_n)$ converges towards $\theta^a(t)$, the same methodology holds for $b$. We have for $n$ large enough

$$|\theta^{n,a}(t_n) - \theta^a(t)| \leq \sum_{j=1}^{m^a} |K_n(t_n - T^a_j) - K(t - T^a_j)|.$$ 

The uniform convergence of $K_n$ towards $K$ implies that $K_n(t_n - T^a_0)$ converges towards $K(t - T^a_0)$. This concludes the proof.

### 5.4 Proof of point (iv) of Theorem 3.1

We recall that the proof of Theorem 3.1 is exactly the same of Theorem 2.1. So for any $(t, y) \in \mathcal{E}^n$ we define for $(t, y) \in \mathcal{E}^n$ the process $Y^{t, y} = (t^y, P^{t, y}, e^{t, y, a}, e^{t, y, b}) \in \mathcal{E}^n$ by analogy with the process $X^{t, x}$ defined in Section 5.2.2. Note that by construction for any $(t, x) \in \mathcal{E}^{b, a, \gamma}$ and for any $s \in [t, T]$ we have for $(t, y) = \mathcal{R}^{a, \gamma}(t, x)$ 

$$\langle s, Y^{t, y} \rangle = \mathcal{R}^{a, \gamma}(s, X^{t, x}) .$$ (17)
is the unique viscosity solution with polynomial growth of \(\text{(HJB)}_{\alpha,\gamma}\). Moreover for any \((t,x) \in \mathcal{E}^{\alpha,\gamma}\) and \((t,y) = R_{R}^{\alpha,\gamma}(t,x)\) by Equation (17) we have:

\[
U^{\alpha,\gamma}(t,y) = \sup_{\delta \in \mathcal{A}} \mathbb{E}^{\delta}[G(i^{t,x,x}_{T},P^{T,x}_{T}) + \int_{t}^{T} (g(i^{t,x,s}_{s},P^{s,x}_{s}) + \delta^{a} \lambda_{a}^{s} + \delta^{b} \lambda_{b}^{s}) ds] = U^{K_{\alpha,\gamma}}(t,x).
\]

Therefore for any \((t,x) \in \mathcal{E}^{K_{\alpha,\gamma}}\) we have \(U^{K_{\alpha,\gamma}}(t,x) = U^{\alpha,\gamma} \circ R^{\alpha,\gamma}(t,x)\). This concludes on the proof of point \((iv)\) of Theorem 3.1.

A Proof of Lemma 2.1

We first prove (i). Consider \((\theta_{k})_{k \geq 0}\) a sequence with values in \(\Theta_{t}^{K}\) that converges towards some \(\theta\) in \(L^{1}\). We have

\[
\theta_{k} = \sum_{j=1}^{N_{k}} K(\cdot - T^{k}_{j}).
\]

The convergence of \(\|\theta_{k}\|_{1}\) towards \(\|\theta\|_{1}\) gives that \(N_{k}\) is constant equal to some \(N\) up to a certain rank. Finally for any subsequence \(((T^{\sigma(k)}_{j})_{1 \leq j \leq N})_{k \geq 0}\) converging to some \((T_{j})_{1 \leq j \leq N}\) we have:

\[
\theta_{\sigma(k)} \to \sum_{j=1}^{N} K(\cdot - T_{j}) = \theta, \text{ in } L^{1}.
\]

so \(\Theta_{t}^{K}\) is closed. Now with the same notation we consider a bounded sequence \((\theta_{k})_{k \geq 0}\). We can find an extraction \(\sigma\) such that \(N_{\sigma(k)}\) is constant equal to some \(N\) and such that for \(j = 1 \ldots N\), \(T^{\sigma(k)}_{j} \to T_{j}\). This implies that

\[
\theta_{\sigma(k)} \to \sum_{j=1}^{N} K(\cdot - T_{j}).
\]

This show that that \(\Theta_{t}^{K}\) is locally compact.

Now we prove (ii). Consider a converging sequence \((s_{k},\theta_{k})_{k \geq 0}\) such that \(\theta_{k} \in \Theta_{t}^{K}\) for any \(k\) and let \(\theta = \sum_{i} N_{k} K(\cdot - T_{j})\) be the limit of \((\theta_{k})_{k \geq 0}\). Then necessarily \(((T^{k}_{j})_{1 \leq j \leq N})_{k \geq 0}\) converges towards \((T_{j})_{1 \leq j \leq N}\). Moreover by comparison we have \(T_{j} \leq s\) and by continuity of \(K\) that

\[
\theta_{k}(s_{k}) \to \sum_{j=1}^{N} K(s - T_{j}).
\]

Finally consider now that \(K(t) = \alpha e^{-\gamma t}\), for \(l \in \mathbb{N}\) we have

\[
(\theta^{(l)}_{k})(T) = \sum_{i=1}^{N} \alpha(-\gamma)^{l} e^{-\gamma(T - T^{k}_{i})}.
\]

The convergence of \((T^{k}_{j})_{k \geq 0}\) thus imply that \(\theta^{(l)}_{k}(T) \to \theta^{(l)}(T)\).

B A priori inequalities

In this section we prove some a priori inequalities.

B.1 Hawkes processes

Consider a Hawkes process \(N\) with kernel \(K = c1_{\mathbb{R}_{+}}\) and exogenous intensity \(\mu\). The intensity of \(N\) is given by

\[
\lambda_{t} = \mu + N_{t}c.
\]
The existence of such process is proved in [17]. Consider \( T_p = \inf\{ s \text{ s.t. } N_s > p \} \), by to [17], \( T_\infty = \lim_{n \to +\infty} T_n = +\infty \). Then let \( N^p = N^{T_p} \). We have for any \( t \in [0, T] \)

\[
E[N^p_t] = E[\int_0^{t\wedge T_p} \lambda_s ds] \leq E[\int_0^t C(1 + N^p_s) ds].
\]

thus using a Gronwall lemma we get \( E[N^p_T] \leq CT e^{CT} \). The RHS being independent of \( p \) and using monotonous convergence we get

\[
E[N_T] < +\infty.
\]

We also have

\[
E[(N^p)^2] = E[\int_0^{t\wedge T_p} (2N^p_s + 1)dN_s] = E[\int_0^{t\wedge T_p} (2N^p_s + 1)\lambda_s ds] \leq E[\int_0^{t\wedge T_p} C(2N^p_s + 1)(N^p_s + 1) ds] \leq E[\int_0^{t\wedge T_p} (2N^p_s + 1)(N^p_s + 1) ds] \leq C T e^{CT}.
\]

Using again a Gronwall lemma we deduce that \( E[(N^p_T)^2] \leq CT^2 e^{CT} \) with \( C \) independent of \( p \), so

\[
E[(N_T)^2] < +\infty.
\]

Now consider a Hawkes process \( N \) with kernel \( K \) bounded and intensity given by

\[
\lambda_t = \Phi\left( \int_0^t K(t - s) dN_s \right)
\]

with \( \Phi \) non decreasing in its last variable and such that \( |\Phi(x)| \leq C(1 + |x|) \) for some \( C > 0 \). By thining we can see \( N \) as dominated by some Hawkes process \( \tilde{N} \) with kernel \( C1_{R_+} \) and exogenous intensity \( C \).

Remark that as a consequence \( \tilde{\lambda} \) dominates \( \lambda \). So we get

\[
E[\tilde{N}_T + \int_0^T \tilde{\lambda}_s] < +\infty
\]

then consequently

\[
E[N_T] < +\infty, \ E[N^2_T] < +\infty \text{ and } E[\int_0^T \delta e^{-k\delta} \lambda_s ds] < +\infty.
\]

This ensures that the function \( U^K \) defined in Equation (15) is well defined. This also implies that the martingales \( M^{t,x,a,\delta} \) and \( M^{t,x,b,\delta} \) are uniformly integrable martingales.

### B.2 A priori estimates on \( X \)

We prove here that the value function \( U^K \) defined in Equation (15) has polynomial growth in \( x \). For this we show some inequalities on the norm of \( (X^{t,x})_{(t,x) \in E^K} \). More precisely we prove the following result:

**Lemma B.1.** There exists some positive constant \( C \) depending only on \( T \) and on the regularity constants of \( G \) and \( g \) such that for any \( (t, x) \in E \)

\[
E^{t,x}[\sup_{s \in [t,T]} \|X^s_{t,x}\|^2] \leq C(1 + \|x\|^2)
\]

and

\[
|U^K(t, x)| \leq C(1 + \|x\|^2).
\]

To prove Lemma B.1 consider \( (t, x) \in E^K \) and \( \delta \in \mathcal{A}_t \) with \( x = (P, t, \theta^a, \theta^b) \). We show different a priori estimates on the subprocesses composing \( X^{t,x} = (P^{t,x}, t^{t,x}, \theta^{t,x,a}, \theta^{t,x,b}) \) under the probability \( P^{t,x,\delta} \).
A priori estimates on $\theta^{t,x;a}$ and $\theta^{t,x;b}$: We have
\[ N_s^a - N_t^a = M_s^{t,x;a,\delta} + \int_t^s \lambda^a(u, X(u, X, \delta_u)) du \]
since $\lambda^a(t, x, \delta) \leq C(1 + \|\theta^a\|_1)$ and $\|\theta^a_u\|_1 = \|\theta^a\|_1 + (N_u^a - N_t^a)\|K\|_1$ we have
\[ N_s^a - N_t^a \leq M_s^{t,x;a,\delta} + \int_t^s C(1 + \|\theta^a_u\|_1 + \|K\|_1(N_u^a - N_t^a)) du. \]
Passing to the average under $\mathbb{P}^{t,x,\delta}$ using the fact that $M^{t,x;d,\delta}$ is a true martingale and a Grönwall lemma we get
\[ \mathbb{E}^{t,x}[N_s^a - N_t^a] \leq C(1 + \|\theta^a\|_1) \] (18)
where $C$ only depends only on the Lipshitz constant $\|K\|$ and on the model constants. Consequently we have for some positive constant $\lambda$
\[ \mathbb{E}^{t,x}[\|\theta^{t,x,a}\|_1] \leq C(1 + \|\theta^a\|_1). \]
We now give an a priori estimate for the second order moment.
\[ (N_s^a - N_t^a)^2 = \int_t^s (2(N_u^a - N_t^a) + 1) \lambda_u du + \int_t^s (2(N_u^a - N_t^a) + 1)dM_s^{t,x,a,\delta} \]
\[ \leq \int_t^s (2(N_u^a - N_t^a) + 1)C(1 + \|\theta^{t,x,a}\|_1 + N_u^a - N_t^a) du + \int_t^s (2(N_u^a - N_t^a) + 1)dM_s^{t,x,a,\delta} \]
\[ \leq \int_t^s C(N_u^a - N_t^a)^2 + N_u^a - N_t^a(1 + \|\theta^{t,x,a}\|_1) du + \int_t^s (2(N_u^a - N_t^a) + 1)dM_s^{t,x,a,\delta}. \]
The average of the last term of the right hand side is 0 as consequence of Appendix B.1. Thus taking the average and using a Grönwall lemma we get
\[ \mathbb{E}^{t,x}_t[(N_s^a - N_t^a)^2] \leq C(1 + \|\theta^a\|^2). \] (19)
A priori estimates on $P^{t,x}$: We have
\[ dP_s^{t,x} = d(s, P_s^{t,x})ds + \sigma dW_s, \text{ with } P_t^{t,x} = P. \]
By assumptions there exists $k > 0$ such that $|d(t, p) - d(t, q)| \leq k|p - q|$. We have the classic apriori estimates (see for exemple Theorem 1.2 in [25]).
\[ \mathbb{E}^{t,x}_t[\sup_{s \leq t} P_s^2] \leq C(1 + P^2). \] (20)
Where $C$ only depends only on the Lipshitz constant $k$ and on $T$.
A priori estimates on $X^{t,x}$: We have
\[ i_s = i + N_s^a - N_t^a + N_s^b - N_t^b \]
\[ \|\theta^{i}_s\|_1 = \|\theta^i\|_1 + \|K\|_1(N_s^i - N_t^i), \text{ for } j = a, b. \]
Thus we have
\[ \|X_s^{t,x}\|^2 \leq C(1 + i^2 + \|\theta^a\|^2_1 + \|\theta^b\|^2_1) + (N_s^a - N_t^a)^2 + (N_s^b - N_t^b)^2 + P_s^2). \]
Taking the average and using Equations (18), (19) and (20) we get
\[ \mathbb{E}^{t,x}_t[\sup_{s \leq t} X_s^{t,x,\delta}^2] \leq C(1 + i^2 + \|\theta^a\|^2_1 + \|\theta^b\|^2_1 + P^2) = C(1 + |x|^2) \] (21)
where $C$ is independent of $\delta$. 

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A priori estimates on the value function: By the quadratic growth of $G$ and $\tilde{y}$ we get

$$|J^K(t, x; \delta)| \leq E^t_{t, x}[e^{-r(T-t)}C(1 + \|X^t_x\|^2) + \int_t^T e^{-r(s-t)}C(1 + \|X^s_x\|^2)].$$

Because of the a priori estimates \[18\], \[19\], \[20\] and \[21\] we have

$$|J^K(t, x; \delta)| \leq C(1 + t^2 + \|\theta^a\|^2 + \|\theta^b\|^2 + P^2) \leq C(1 + \|x\|^2)$$

where $C$ only depends on $T$ and the regularity constants. We conclude by arbitrariness of $\delta$.

B.3 Rewriting of the utility

We show that for any $\delta \in A$ we have

$$E^\delta[\int_0^T e^{-rs}\delta_a^s dN^a_s] = E^\delta[\int_0^T e^{-rs}\delta_a^s \lambda^a_{s, \delta} ds].$$

The same result for $b$ holds by the same arguments. To conclude it is enough to show that

$$\overline{M}_t = \int_0^t e^{-rs}\delta_a^s dM^a_s$$

is a true martingale. We have

$$|\overline{M}|_t = \int_0^t e^{-2rs}(\delta_a^s)^2 dN^a_s$$

and since $(\delta_a^s)^2 \lambda^a_{s, \delta} \leq C(1 + \|X_s\|_1)$ we get

$$\langle \overline{M}\rangle_T \leq \int_0^T e^{-2rs}C(1 + \|X_s\|_1) ds \leq TC(1 + \sup_{s \in [0, T]} \|X_s\|_1).$$

The last term of the RHS is integrable by Lemma B.1 So by the monotone convergence $|\overline{M}|_T$ is also integrable so $\overline{M}$ is a uniformly integrable martingale. As consequence we get

$$E^\delta[G(i_T, P_T)e^{-rT} + \int_0^T e^{-rs}\left(g(i_s, P_s) ds + \delta_a^s dN^a_s + \delta_b^s dN^b_s\right)]$$

$$= E^\delta[G(i_T, P_T)e^{-rT} + \int_0^T e^{-rs}\left(g(i_s, P_s) + \delta_a^s \lambda^a_{s, \delta} + \delta_b^s \lambda^b_{s, \delta}\right) ds].$$

C Equivalence between the two definitions of viscosity solutions

Lemma C.1. Definition 2.1 and Definition 2.2 are equivalent.

Proof. We show it for sub-solutions, the demonstration is the same for super-solutions.

Consider $U$ a USC function sub-solution of (HJB)$_K$ in the sense of Definition 2.2 Now consider $\phi$ a test function such that $0 = U(t_0, x_0) - \phi(t_0, x_0) = \sup_{V} U - \phi$ for $V$ a neighborhood of $(t_0, x_0)$ in $\mathcal{E}^K$.

We show that

$$F(t_0, x_0, U(t_0, x_0), \nabla \phi(t_0, x_0), \partial^2_{pp} \phi(t_0, x_0), D^K \phi(t_0, x_0)) \leq 0.$$

Writing $x = (p, z) \in \mathbb{R} \times Z^K_T$ we have $\phi(t, p, z) = \phi(t_0, p_0, z_0) + \partial_t \phi(t_0, x_0)(t - t_0) + \partial_p \phi(t_0, x_0)(p - p_0) + \partial^2_{pp} \phi(t_0, x_0) \frac{(p - p_0)^2}{2} + o(|p - p_0|^2 + |t - t_0|^2) + h(z - z_0)$, where $h$ is a modulus of continuity of $\phi$. Thus we have

$$(\nabla \phi(t_0, x_0), \partial^2_{pp} \phi(t_0, x_0), h) \in J^+ u(t_0, x_0).$$
Consequently
\[ F(t_0, x_0, U(t_0, x_0), \nabla \phi(t_0, x_0), \partial^2_{pp} \phi(t_0, x_0), DKU(t_0, x_0)) \geq 0. \]
So \( U \) is a viscosity sub-solution of \((HJB)_K\) in the sense of Definition 2.1.

Now we show the opposite implication. Consider \( U \) a USC function sub-solution of \((HJB)_K\) in the sense of Definition 2.1. Consider \((d, A, h) \in J^+U(t_0, x_0)\), we built a test function \( \phi \) dominating \( U \) with equality at point \((t_0, x_0)\) and such that
\[
(\nabla \phi(t_0, x_0), \partial^2_{pp} \phi(t_0, x_0)) = (d, A).
\]
We will then get the expected inequality that will extend directly to \( \mathcal{J}^+U(t_0, x_0) \) by continuity of \( F \).

Using the notation \((t, x) = (t, p, z) \in [0, T] \times \mathbb{R} \times Z^K_t\) we have
\[
U(t, x) \leq U(t_0, x_0) + d_1(t - t_0) + d_2(p - p_0) + \frac{1}{2} A(p - p_0)^2 + h(z - z_0) + o(|p - p_0|^2) + o(|t - t_0|).
\]
hence
\[
U(t, p, z) - h(z - z_0) \leq U(t_0, x_0) + d_1(t - t_0) + d_2(p - p_0) + \frac{1}{2} A(p - p_0)^2 + o(|p - p_0|^2) + o(|t - t_0|).
\]
We take the supremum on \( z \) over a compact neighborhood of \( z_0 \), and consider
\[
\hat{U}(t, p) = \sup_{z \in B_r(z_0)} U(t, p, z) - h(z - z_0).
\]
Since \( \hat{U}(t_0, p_0) = U(t_0, x_0) \) we get
\[
\hat{U}(t, p) \leq \hat{U}(t_0, p_0) + d_1(t - t_0) + d_2(p - p_0) + \frac{1}{2} A(p - p_0)^2 + o(|p - p_0|^2) + o(|t - t_0|).
\]
We prove at the end that \( \hat{U} \) is a USC function and assume this is true. The last equation means that \((d, A) \in J^+\hat{U}(t_0, p_0)\). Then by an argument developed for the analysis of viscosity solutions on \( \mathbb{R}^d \) (see for example [11] Lemma 4.1.) we have existence of a function \( \phi \in C^{1,2} \) such that
\[
\hat{U}(t, p) - \phi(t, p) \leq \hat{U}(t_0, p_0) - \phi(t_0, p_0) \text{ with } (\nabla \phi(t_0, p_0), \partial^2_{pp} \phi(t_0, p_0)) = (d, A).
\]
So finally we have on a compact neighborhood of \( x_0 \):
\[
U(t, p, e) - \phi(t, p) - h(e - e_0) \leq U(t_0, p_0, e_0) - \phi(t_0, p_0) - h(e_0 - e_0).
\]
This local domination can then be extended to the whole domain \( \mathcal{E}^K \).

Finally we show that \( \hat{U} \) is a USC function. Fix \( \varepsilon > 0 \) and \((t, p)\). Since \( U \) is USC and \( h \) continuous, for any \( e \in B_r(e_0) \) we can find \( r_e \) such that on \( B_{r_e}(t, p, e) \) we have
\[
U + h(\cdot - e_0) \leq U(t, p, e) + h(e - e_0) + \varepsilon.
\]
The collection \( \{B_{\varepsilon_i}(t, p, e)\}_{e \in B_r(e_0)} \) forms an open covering of \( \{t\} \times \{p\} \times B_r(e_0) \) which is a compact set by Lemma 2.1. Thus we may find a finite sequence \( \{B_{\varepsilon_i}(t, p, e_i)\}_{1 \leq i \leq N} \) that covers \( \{t\} \times \{p\} \times B_r(e_0) \). Consider \( r_* = \min_{1 \leq i \leq N} \frac{\varepsilon_i}{2} \). Now take any \((s, q) \in B_{r_*}(t, p)\), then for any \( e \in B_r(e_0) \) there is some \( i \in \{1, \ldots, N\} \) such that \((t, p, e) \in B_{r_e/2}(t, p, e_i)\). Hence we get
\[
||(s, q, e) - (t, p, e_i)|| \leq \frac{r_e}{2} + r_* \leq r_e,
\]
so \((s, q, e) \in B_{r_e}(t, p, e_i)\) and consequently
\[
U(s, q, e) - h(e - e_0) \leq U(t, p, e_i) + h(e_i - e_0) + \varepsilon \leq \hat{U}(t, p) + \varepsilon.
\]
Passing to the supremum in \( e \in B_r(e_0) \) in the LHS we get that \( \hat{U} \) is USC. \( \square \)
D Crandall Ishi’s lemma

The most crucial point to prove comparison result for viscosity solutions is the Crandall-Ishi’s lemma that allows to deal with the second order terms. In the general case the Crandall Ishi’s lemma is proved for subset of $\mathbb{R}^n$, see [9]. Hence our particular domain requires an adaptation of the classic version of the Lemma.

**Lemma D.1.** Let $\phi_1 \in C^2((0,T]^2)$, $\phi_2 \in C^1(\mathbb{R}^2)$ and $\phi_3 \in C^0((Z_F^2)^2)$, $u \in USC(\mathcal{E}^K)$ and $v \in LSC(\mathcal{E}^K)$. Suppose we have $(t_0, p_0, z_0) \in \mathbb{R}^2 \times \mathbb{R}^2 \times (Z_F^2)^2$ such that

$$u(t_0^1, p_0^1, z_0^1) - v(t_0^2, p_0^2, z_0^2) - \phi_1(t_0) - \phi_2(p_0) - \phi_3(z_0)
= \sup_{t, p, z \in \mathbb{R}^2 \times [0, T]^2 \times (Z_F^2)^2} u(t^1, p^1, z^1) - v(t^2, p^2, z^2) - \phi_1(t) - \phi_2(p) - \phi_3(z). \quad (22)$$

Then for any $\varepsilon$ there is $(A_\varepsilon, h)$ and $(B_\varepsilon, h)$ in $\mathbb{R} \times C^0(Z_F^2)$ such that

$$((\nabla_1 \phi_1(t_0), \nabla_1 \phi_2(p_0)), A_\varepsilon, h) \in \mathcal{J}^+ u(t_0^1, p_0^1, z_0^1), \quad ((-\nabla_2 \phi_1(t_0), -\nabla_2 \phi_2(p_0)), B_\varepsilon, h) \in \mathcal{J}^- v(t_0^2, p_0^2, z_0^2)$$

and that

$$- (\varepsilon^{-1} + |H \phi_2(p_0)|) I_2 \leq \begin{pmatrix} A_\varepsilon & 0 \\ 0 & -B_\varepsilon \end{pmatrix} \leq H \phi_2(p_0) + \varepsilon H \phi_2(p_0)^2 \quad (23)$$

where $H$ is the Hessian operator and $|A|$ denotes the spectral radius of the matrix $A$.

Note that even though this extension is not straightforward we benefit from the fact that in $(HJB)_K$ the second order derivative is related to a real variable. Therefore the strategy of the proof is to bring back the problem in the classic framework.

**Proof.** We can consider that there exists $V$ a compact neighborhood of $(t_0, p_0, z_0)$ in $\mathcal{E}^K$ such that on $V \setminus (t_0, p_0, z_0)$ we have

$$(u - v)(t_0, p_0, z_0) \geq (u - v)(t, p, z) - \phi_2(p) + \phi_2(p_0) - \phi_1(t) + \phi_1(t_0) - \phi_3(z) + \phi_3(z_0)
\geq (u - v)(t, p, z) - \phi_2(p) + \phi_2(p_0) + \nabla \phi_1(t_0)(t_0 - t) + O(||t - t_0||^2) - \phi_3(z) + \phi_3(z_0)
\geq (u - v)(x, y, z) - \phi_2(p) + \phi_2(p_0) + \nabla \phi_1(t_0)(t_0 - t) - C ||t - t_0||^2 - H(z_0^1 - z_1^1) + H(z_0^2 - z_2^2)$$

where $h$ is any modulus of continuity of the function $\phi_3$ and $C$ a positive constant. For $x \in \mathbb{R}$ consider $g_j(x) = \partial_j \phi_1(t_0)x - Cx^2$ for $j = 1$ or 2. So on $V \setminus (t_0, p_0, z_0)$ we have:

$$(u - v)(t_0, p_0, z_0) \geq u(t^1, p^1, z^1) - v(t^2, p^2, z^2) - \phi_2(p) + \phi_2(p_0)
- h(z_0^1 - z_1^1) - h(z_0^2 - z_2^2) + g_1(t_0^1 - t^1) + g_2(t_0^2 - t^2) \quad (24)$$

with equality at $(t_0, p_0, z_0)$ and with $h(0) = g_i(0) = 0$, $g_j'(0) = \partial_j \phi_1(t_0)$.

We can always assume that there exists $r > 0$ so that $V$ is of the form

$$V = (B_r(t_0) \times B_r(p_0) \times B_r(z_0)) \cap \mathcal{E}^K$$

where $B_r(x)$ denotes the closed ball with center $x$ and radius $r$. We define the following functions

$$\tilde{u}(p^1) = \sup_{t, x \in B_r(t_0) \times B_r(z_0)} u(t^1, p^1, z^1) - h(z_0^1 - z_1^1) + g_1(t_0^1 - t^1)
\text{ and } \tilde{v}(p^2) = \inf_{t, x \in B_r(t_0) \times B_r(z_0)} v(t^2, p^2, z^2) - h(z_0^2 - z_2^2) + g_2(t_0^2 - t^2)$$

where the supremums above are taken for $(t, z)$ such that $(t, p, z) \in \mathcal{E}^K$. The functions $\tilde{u}$ and $\tilde{v}$ are respectively USC and LSC functions since the supremums are taken over compact subsets (see the proof of Lemma 4.1). And we have

$$\tilde{u}(p^1) - \tilde{v}(p^2) - \phi_2(p) \leq \tilde{u}(p_0^1) - \tilde{v}(p_0^2) - \phi_2(p_0).$$

Thus by the Crandall Ishi’s lemma (see for example Theorem 6.1. in [10]) there exists $(A_\varepsilon, B_\varepsilon)$ satisfying (23) such that

$$(\partial_1 \phi_2(p_0), A_\varepsilon) \in \mathcal{J}^+ \tilde{u}(p_0^1) \text{ and } (-\partial_2 \phi_2(p_0), B_\varepsilon) \in \mathcal{J}^- \tilde{v}(p_0^2).$$
Consequently there exist a sequence \((q_n, A_n, p_n^1, u(p_n^1))_{n \in \mathbb{N}}\) such that
\[
\lim_{n \to +\infty} (q_n, A_n, p_n^1, u(p_n^1)) = (\partial \phi_2(p_0), A_\varepsilon, p_0^1, \tilde{u}(p_0)), \quad \text{and } \forall \ n \geq 0, \ (q_n, A_n, p_n^1, u(p_n^1)) \in \mathcal{J}^+ \tilde{u}(p_n^1).
\]
So for any \(n\) we have
\[
\tilde{u}(p_1^1) \leq \tilde{u}(p_n^1) + q_n(p^1 - p_n^1) + \frac{1}{2} A_n(p^1 - p_n^1)^2 + o(|p^1 - p_n^1|^2).
\]
Consider \(t_n^1\) and \(z_n^1\) such that
\[
\tilde{u}(p_n^1) = u(t_n^1, p_n^1, z_n^1) - h(z_0^1 - z_n^1) + g_1(t_0^1 - t_n^1)
\]
such maximizers exist by compactness. We show that \((t_n^1, p_n^1, z_n^1)\) converges towards \((t_0^1, p_0^1, z_0^1)\), we assume it for now. Equation \([24]\) implies that for any \((t, p, z)\) we have
\[
u(t^1, p^1, z^1) \leq u(t_n^1, p_n^1, z_n^1) + q_n(p^1 - p_n^1) + \frac{1}{2} A_n(p^1 - p_n^1)^2 + o(|p^1 - p_n^1|^2) \]
\[- h(z_0^1 - z_n^1) + h(z_0^1 - z^1) + g_1(t_0^1 - t_n^1) - g_1(t_0^1 - t^1).
\]
Consider the function \(h_n(z^1) = -h(z_0^1 - z_n^1) + h(z_0^1 - z_n^1 - z^1)\) such that \(h_n(0) = 0\) and
\[
\frac{h_n(z^1 - z_n^1)}{h_n(z_0^1 - z_n^1) + h(z_0^1 - z^1)}
\]
Since \(z_n^1\) converges towards \(z^1\) the sequence \((h_n)_{n \geq 0}\) converges uniformly towards \(h\) because \(h\) is continuous and because we are working on compact neighborhood. Consider \(q_n^1 = \partial_1 \phi_1(t_0) - 2C(t_0^1 - t_n^1)\) that converges towards \(\partial_1 \phi_1(t_0)\)
\[
g_1(t_0^1 - t_n^1) - g_1(t_0^1 - t^1) = q_n^1(t^1 - t_n^1) + C(t_n^1 - t^1)^2.
\]
Thus we have
\[
u(t^1, p^1, z^1) \leq u(t^1, p^1, z^1) + q_n(p^1 - p_n^1) + \frac{1}{2} A_n(p^1 - p_n^1)^2 + o(|p^1 - p_n^1|^2) \]
\[+ \frac{1}{2} A_n(p^1 - p_n^1)^2 + o(|p^1 - p_n^1|^2) \]
hence \(((q_n^1, q_n), A_n, h_n) \in \mathcal{J}^+ u(t_n^1, p_n^1, z_n^1)\) and
\[
((q_n^1, q_n), A_n, h_n) \rightarrow ((\partial_1 \phi_1(t_0), \partial_1 \phi_2(p_0), A_\varepsilon, h)\]
Finally we show that \((t_n^1, p_n^1, z_n^1) \rightarrow (t_0^1, p_0^1, z_0^1)\) which will imply the conclusion that
\[
((\partial_1 \phi_1(t_0), \partial_2 \phi_2(p_0)), A_\varepsilon, h) \in \mathcal{J}^+ u(t_0^1, p_0^1, z_0^1).
\]
We have for any \(n \geq 0:\)
\[
\tilde{u}(p_n^1) = u(t_n^1, p_n^1, z_n^1) - h(z_0^1 - z_n^1) - g_1(t_0^1 - t_n^1).
\]
Consider any \((t^1, z^1) \in (t_n^1, z_n^1)_{n \geq 0}\). Since \(\tilde{u}(p_n^1) \rightarrow \tilde{u}(p_0^1)\), by upper semi-continuity of \(u\) and by the definition of \(\tilde{u}\) we get
\[
u(t^1, p_0^1, z^1) - h(z_0^1 - z^1) + g_1(t_0^1 - t^1) \geq u(t_0^1, p_0^1, z_0^1).
\]
Which implies that \((t^1, z^1) = (t_0^1, z_0^1)\) since everywhere else the above inequality is false because of Equation \([24]\). So we get
\[
((q_n^1, q_n), A_n, h_n, (x_n^1, y_n^1, z_n^1)_{n \geq 0}) \rightarrow ((\partial_1 \phi_1(t_0), \partial_1 \phi_2(p_0)), A_\varepsilon, h, (t_0^1, p_0^1, z_0^1))
\]
and so
\[
((\partial_1 \phi_1(t_0), \partial_1 \phi_2(p_0)), A_\varepsilon, h) \in \mathcal{J}^+ u(t_0^1, p_0^1, z_0^1).
\]
Similarly we get
\[
((\partial_2 \phi_1(t_0), -\partial_2 \phi_2(p_0)), B_\varepsilon, h) \in \mathcal{J}^- v(t_0^2, p_0^2, z_0^1).
\]
This concludes the proof. \(\square\)
E  Existence of $R^{\alpha,\gamma}$

Consider $t \geq 0$ we have for any $j \geq 0$

$$\theta_t^{(j)}(T) = \sum_{i=1}^{n} c_t^{a,i}(-\gamma_i)^je^{-\gamma_i(T-t)} \text{ and } \theta_t^{b,(j)}(T) = \sum_{i=1}^{n} c_t^{b,i}(-\gamma_i)^je^{-\gamma_i(T-t)}. \quad (25)$$

So let $A$ be the matrix with coefficient $A_{ij} = (-\gamma_i)^j$. This is a Vandermonde matrix which is invertible. By Equation (25) we have

$$c_t^{a,i} = e^{\gamma_i(T-t)} \sum_{j=1}^{n} (A^{-1})_{ij} \theta_t^{a,(j)}(T) \text{ and } c_t^{b,i} = e^{\gamma_i(T-t)} \sum_{j=1}^{n} (A^{-1})_{ij} \theta_t^{b,(j)}(T).$$

So we define $R^{\alpha,\gamma}$ for $(t, x) \in \mathcal{E}^{K,\gamma}$ by

$$R^{\alpha,\gamma}(t, x) = (t, p, i, c^a(t, x), c^b(t, x))$$

where

$$c^a(t, x) = (e^{\gamma_i(T-t)} \sum_{j=1}^{n} (A^{-1})_{ij} \theta_t^{a,(j)}(T))_{1 \leq j \leq n} \text{ and } c^b(t, x) = (e^{\gamma_i(T-t)} \sum_{j=1}^{n} (A^{-1})_{ij} \theta_t^{b,(j)}(T))_{1 \leq j \leq n}.$$ 

By Lemma 2.1, the map $R^{\alpha,\gamma}$ is continuous and by construction we have for any $t \geq 0$

$$R^{\alpha,\gamma}(t, X_t) = (t, Y_t^{\alpha,\gamma}).$$

F  Proof of Lemma 3.1

We are going to approximate the integral in Equation (7) by Riemann sum. We take $A_n = \sqrt{n}$ and $(a_i)_{0 \leq i \leq n-1}$ a regular grid of $[0, A_n]$ with mesh $\frac{1}{\sqrt{n}}$. We set

$$K_n(t) = \sum_{i=0}^{n-1} e^{-a_i + t} \int_{a_i}^{a_{i+1}} m(du) \leq K(t).$$

For $t \in \mathbb{R}_+$, we have

$$K(t) - K_n(t) = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} m(du) t e^{-tu} dv - \int_{A_n}^{+\infty} e^{-tuv} m(du).$$

Hence for any $T$ and $t \leq T$:

$$|K_n(t) - K(t)| \leq \sum_{i=0}^{n-1} T \int_{a_i}^{a_{i+1}} m(du) (a_{i+1} - a_i) + \int_{A_n}^{+\infty} m(du)$$

which goes to 0 when $n$ goes to infinity, uniformly on $t \in [0, T]$. Hence the sequence $K_n$ converges almost surely towards $K$ and is dominated by $K$ so $K_n$ converges in $L_1$ towards $K$.

Set $\alpha_n = K(0) - K_n(0)$ and $\beta_n = \frac{\alpha_n}{\|K\|_1}$ and consider $\tilde{K}_n = K_n + \alpha_n e^{-\beta_n \cdot}$, we have for any $n$

$$\tilde{K}_n(0) = K(0) \text{ and } \|\tilde{K}_n\|_1 = \|K\|_1$$

and $\tilde{K}_n \rightarrow K$ in $L_1$. Thus the sequence $(\tilde{K}_n)_{n \geq 0}$ gives the result.
G Probabilistic representation of IPDE in high dimension

We are going to use a probabilistic representation based on branching processes. This method is insensitive to the dimension of the domain of the IPDE. Theoretically the method works for any semilinear IPDE admitting a strong solution and with a generator that can be written as a power serie. Thought this is not the case for \((HJB)_{\alpha,\gamma}\), in order to implement this method we approximate the generator of the IPDE by a second order polynomial and assume that the approached IPDE have a strong solution. Thus we are left with an IPDE of the form

\[
(HJB)_{\alpha,\gamma}': \quad -\partial_t U - \mathcal{L} U - f(U, D^{\alpha} U) = 0, \quad u(T, \cdot) = 0 \text{ on } \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n
\]

where

\[
f(U, D^{\alpha} U)(t, x) = f_0(t, x) + f_1(t, x) U(t, x)
+ f_{2,1}(t, x) D_0^\alpha U(t, x) + \frac{f_{2,1}^b(t, x)}{2} (D_0^\alpha U(t, x))^2
+ f_{2,2}(t, x) D_0^2 U(t, x) + \frac{f_{2,2}^b(t, x)}{2} (D_0^2 U(t, x))^2.
\]

The operator \(\mathcal{L}\) is defined by \(\mathcal{L} U(t, x) = -\langle \Gamma, \nabla \xi U(t, x) \rangle - \langle \Gamma, \nabla \xi U(t, x) \rangle\).

Consider a process \(\tilde{X}^{t,x}\) starting at time \(t\) with initial state \(x\) such that \((t, x) \in \mathcal{E}^n\) and which dynamic is driven by the infinitesimal generator \(\mathcal{L}\). The Feynman-Kac formula gives

\[
U(t, x) = \mathbb{E}\left[\frac{f(U, D^{\alpha} U)(\tau, \tilde{X}_{\tau}^{t,x})}{\rho(\tau)} \mathbb{1}_{t+\tau < T} ds\right]
\]

where \(\tau\) is a positive random variable with density \(\rho\).

We show in Appendix \([G.1]\) that there exists an appropriate probability measure \(\mathbb{P}_\tau\) on the set

\[\mathcal{T} = \{0, 1, (2, j, d, \varepsilon), \text{ with } d \in \{0, 1, 2\}, \text{ and } \varepsilon \in \{0, 1\}^d\}\]

and a set of functions \((g_r)_{\tau \in \mathcal{T}}\) from \([0, T] \times \mathbb{Z} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+\) such that for any random variable \(\xi\) with law \(\mathbb{P}_\tau\) we have

\[
f(U, D^{\alpha} U)(t, x) = \mathbb{E}[g_\xi(U, D^{\alpha} U)(t, x)].
\]

The set \((g_r)_{\tau \in \mathcal{T}}\) is defined by

\[
g_0(U, D^{\alpha} U)(t, x) = f_0(t, x) \mathbb{P}(l = 0)^{-1}, \quad g_1(U, D^{\alpha} U)(t, x) = f_1(t, x) \mathbb{P}(l = 1)^{-1} U(t, x)
\]

and

\[
g_{(2, j, d, \varepsilon)}(U, D^{\alpha} U)(t, x) = f_{2,j}^d(t, x) \mathbb{P}(l = 2, j, d, \varepsilon)^{-1} \prod_{k=1}^d U(t, x + \Delta^k \varepsilon_k)(-1)^{1-\varepsilon_k},
\]

where \(\Delta^a\) (resp. \(\Delta^b\)) is the jump corresponding to a ask (resp. bid) market order, namely \(\Delta^a = (-1, 0, \alpha)\) and \(\Delta^b = (1, 0, \alpha)\) (We recall that the price variable is no longer part of the domain).

We now define a branching process in the following way: any particle is noted by \((t, x, l_0, l_1, \ldots, l_n)\) where \((x, t) \in \mathcal{E}^n\) and the \(l_i\)’s belong in \(\mathcal{T}\). The variable \(x\) denotes the initial position of the particle and \(t\) its birth time, \(l_n\) is the label of the particle, \(l_{n-1}\) the label of its parent, and so on. The lifetime of the particles are i.i.d random variables with density \(\rho\).

We now describe the evolution of the particle. Consider a particle born at time \(s\) at the state \(x\) with lifetime \(\tau\). During its lifetime the particle state is described by its position: \((l^{s,x}_i, c^{s,x,a}_i, c^{s,x,b}_i)\) \(s \leq t \leq s + \tau\) in \(\mathbb{Z} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+\). The dynamic of the particle position is given by

\[
dc^{i}_t = -\gamma_i c^{i}_t dt, \text{ for } i \in \{1, \ldots, n\} \text{ and } j = a \text{ or } b.
\]

The other components are constants. Note that this dynamic corresponds to the infinitesimal generator \(\mathcal{L}\). When the particle dies it gave birth to independent particles. The number and type of child particles depend on the label \(l_n\) of the particle:
• if $l_n = 0$: 0 child
• if $l = 1$: 1 child
• if $l_n = (2, d, j, \varepsilon)$: $d$ children
  
  – if $j = a$ the initial state of the $i$–th child particle is $X^a_{s,t} + \Delta^a \varepsilon_i$
  
  – if $j = b$ the initial state of the $i$–th child particle is $X^b_{s,t} + \Delta^b \varepsilon_i$

The labels of the child particles are i.i.d. random variables with law $\mathbb{P}_\tau$. We note $\mathcal{C}_p$ the set of the child particles.

Considering a particle starting at point $(t, x)$, Equations (27) and (26) give

$$U(t, x) = \mathbb{E}\left[\frac{a(l, t + \tau, X^l_{t,\tau})}{\rho(\tau)} \prod_{c \in \mathcal{C}_p} U(t + \tau, X_c) 1_{t + \tau < T}\right]$$

where $X_c$ denotes the initial position of the child particle $c$ and where $a$ is defined by

$$a(i, t, x) = f_0(t, x) \mathbb{P}(l = i)^{-1}, \text{ for } i = 1\text{ or } 2$$

$$a((2, j, d, \varepsilon), t, x) = f^1_{j, d}(t, x) \mathbb{P}(l = (2, j, d, \varepsilon))^{-1} \prod_{k=1}^{d} (-1)^{1-\varepsilon_k}$$

By iterating the above equality to the descendents of the particle and assuming that the number of descendent particles born before the time horizon $T$ is almost surely finite we can evaluate $U(t, x)$ using Monte Carlo simulation. For more details on this method we refer to [14].

**G.1 Existence of a measure for the particle method**

We have

$$f(u, Du)(t, x) = \mathbb{E}[f_I(u, Du)(t, x)]$$

where $I$ is a random variable with values in $\{0, 1, 2\}$, and

$$f_0(u, Du)(x) = f_0(t, x) \mathbb{P}(I = 0)^{-1}$$

$$f_1(u, Du)(x) = f_1(t, x) u(t, x) \mathbb{P}(I = 1)^{-1}$$

$$f_2(u, Du)(x) = \mathbb{E}[f_1(u, Du)(t, x)] \mathbb{P}(I = 2)^{-1}$$

where $l$ is a random variable with values in $\{(a, 1), (b, 1), (a, 2), (b, 2)\}$ and

$$f_{(j, d)}(u, Du)(t, x) = f^j_{2, d}(t, x) D^j u(t, x)^d \mathbb{P}(l = (j, d))^{-1}.$$

Finally we have

$$D^j u(t, x)^d = 2^d \mathbb{E} \prod_{k=1}^{d} u(t, x + \Delta^j \varepsilon_k) (-1)^{1-\varepsilon_k}$$

with $(\varepsilon_i)_{1 \leq i \leq d}$ i.i.d. random variables with law $\text{Ber}(\frac{1}{2})$. Thus finally

$$f(u, Du)(t, x) = \mathbb{E}[g_I(u, Du)(t, x)]$$

with $l$ is a random variable whose law is the uniform probability on the set $\mathcal{L} = \{0, 1, (2, j, d, \varepsilon)\}$, with $d \in \{0, 1, 2\}, \ j \in \{a,b\}, \ \varepsilon \in \{0, 1\}^d$ and where

$$g_0(u, Du)(t, x) = f_0(t, x) \ P(l = 0)^{-1}$$

$$g_1(u, Du)(t, x) = f_1(t, x) u(t, x) \ P(l = 1)^{-1}$$

$$g_{(2, j, d, \varepsilon)}(u, Du)(t, x) = f^j_{2, d}(t, x) \mathbb{P}(l = (2, j, d, \varepsilon))^{-1} \prod_{k=1}^{d} u(t, x + \Delta^j \varepsilon_k) (-1)^{1-\varepsilon_k}.$$
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