R-triviality of groups of type $F_4$ arising from the first Tits construction

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Abstract

Any group of type $F_4$ is obtained as the automorphism group of an Albert algebra. We prove that such a group is R-trivial whenever the Albert algebra is obtained from the first Tits construction. Our proof uses cohomological techniques and the corresponding result on the structure group of such Albert algebras.

1. Introduction

The notion of path connectedness for topological spaces is natural and well known. In algebraic geometry, this notion corresponds to that of R-triviality. Recall that the notion of R-equivalence, introduced by Manin in [13], is an important birational invariant of an algebraic variety defined over an arbitrary field and is defined as follows: if $X$ is a variety over a field $K$ with $X(K) \neq \emptyset$, two $K$-points $x, y \in X(K)$ are called elementarily R-equivalent if there is a path from $x$ to $y$, that is, if there exists a rational map $f : \mathbb{A}^1 \to X$ defined at 0, 1 and mapping 0 to $x$ and 1 to $y$. This generates an equivalence relation $R$ on $X(K)$ and one can consider the set $X(K)/R$.

If $X$ has in addition a group structure, that is, $X = G$ is an algebraic group over a field $K$, then the set $G(K)/R$ has a natural structure of an (abstract) group and this gives rise to a functor

$$G/R : \text{Fields}/K \to \text{Groups},$$

where $\text{Fields}/K$ is the category of field extensions of $K$, and $G/R$ is given by $L/K \to G(L)/R$. One says that $G$ is R-trivial if $G(L)/R = 1$ for all field extensions $L/K$.

The computation of the group $G(K)/R$ is known only in a few cases. Here are two examples.

Example 1. Let $A$ be a central simple algebra over a field $K$ and $G = \text{GL}(1, A)$ the group of ‘invertible elements of $A$’. Denote by $H$ the kernel $\text{SL}(1, A)$ of the reduced norm homomorphism $\text{Nrd} : G \to G_\text{m}$, and let $\text{RSL}(1, A)$ be the (abstract) subgroup of $H(K)$ consisting of R-trivial elements. Then by [26], $G(K)/\text{RSL}(1, A)$ equals $A^\times/[A^\times, A^\times]$, which is naturally isomorphic to the Whitehead group $K_1(A)$ in algebraic K-theory (cf. [15, §3]). The group $H(K)/R$ is known as the reduced Whitehead group $SK_1(A)$. The groups $K_1(A)$ and $SK_1(A)$ have been extensively studied in the framework of algebraic K-Theory.

Example 2 (cf. [5]). Let $f$ be a quadratic form over a field $F$ of characteristic not 2 and let $G = \text{Spin}(f)$. Consider the complex

$$\cdots \to \prod_{x \in X(1)} K_2F(x) \xrightarrow{\partial} \prod_{x \in X(0)} K_1F(x) \xrightarrow{N} K_1F = F^\times.$$
where $\mathbf{X}$ is the projective quadric corresponding to $f$. Let $A_0(\mathbf{X}, K_1)$ be the last homology group of this complex. Then $\mathbf{G}(F)/R \simeq A_0(\mathbf{X}, K_1)$. This result allows one to use the machinery of algebraic $K$-theory when dealing with groups of $R$-equivalence classes and is useful in both directions. For instance, if $f$ is a Pfister form, then $\mathbf{G}$ is a rational variety, hence $\mathbf{G}(F)/R = 1$ and therefore $A_0(\mathbf{X}, K_1) = 0$. This triviality result is originally due to Rost and was used in the proof of the Milnor conjecture. Conversely, if $f$ is a generic quadratic form over the field $F = k(t_1, \ldots, t_m)$, where $t_1, \ldots, t_n$ are independent variables over the field $k$, then using techniques of Chow groups one shows that $A_0(\mathbf{X}, K_1) = 0$, implying $\mathbf{G}(F)/R = 1$.

These two examples show that the group $\mathbf{G}(K)/R$ encodes deep arithmetic properties of $\mathbf{G}$, and that its computation is of great importance in the theory of algebraic groups. It is in general a difficult question to determine whether $\mathbf{G}$ is $R$-trivial. For algebraic groups, this is the case if $\mathbf{G}$ is rational, that is, birationally equivalent to affine space (cf. [7]). Also, it is known that semisimple groups of rank at most 2 are rational, hence $R$-trivial (cf. [26]).

The group $\mathbf{G}(K)/R$ for algebraic tori was computed in [7] and for adjoint classical semisimple groups in [14]. The latter computation enables one to construct first examples of non-rational adjoint groups. For simply connected semisimple groups, the computation of the group of $R$-equivalence classes is known only for type $\text{A}_n$ (cf. [4]), for spinor groups $\text{Spin}(f)$ (see the above Example 2) and in some easy cases when $\mathbf{G}/R$ is trivial because the corresponding algebraic group is rational (types $\text{C}_n$ and $\text{G}_2$).

The case of exceptional groups is wide open. Beyond $\text{G}_2$, which is rational by the above, not much is known. In particular, it is not known whether groups of type $\text{F}_4$ are $R$-trivial (the only exception is the isotropic case, where $R$-triviality was established in [6]). These groups arise as automorphism groups of Albert algebras (that is, simple exceptional Jordan algebras, defined below). Recall that any Albert algebra can be obtained through the first or the second Tits construction. The following theorem is the main result of our paper.

**Theorem 1.1.** Any algebraic group of type $\text{F}_4$ over a field $K$ of characteristic not 2 or 3 arising from the first Tits construction is $R$-trivial.

We conclude this introduction by mentioning three standard conjectures about the functor $\mathbf{G}/R$: (i) it takes values in the category of abelian groups, (ii) $\mathbf{G}(K)/R$ is a finite group if $K$ is finitely generated over its prime subfield, and (iii) the functor $\mathbf{G}/R$ has transfers.

If $\mathbf{G}$ is an algebraic torus, then the first and third conjectures trivially hold and the second was proved by Colliot-Thélène and Sansuc [7]. For semisimple groups, not much is known. Our main result provides an indication that all three conjectures are true for reductive algebraic groups.

## 2. Preliminaries on algebraic groups

In this section, we collect some facts about algebraic groups for later use. We start with an observation on subgroups of type $\text{D}_4$ inside the split group of type $\text{F}_4$.

**Proposition 2.1.** Let $\mathbf{H}$ be a split group of type $\text{F}_4$ defined over an arbitrary field $K$. Then any $K$-subgroup $\mathbf{M} \subset \mathbf{H}$ of type $\text{D}_4$ is quasi-split.

**Proof.** Consider a Borel $K$-subgroup $\mathbf{B}$ of $\mathbf{H}$. Since the dimension of $\mathbf{B}$ is 28, it intersects the 28-dimensional subgroup $\mathbf{M}$ non-trivially. Thus $\mathbf{M}$ contains a split unipotent subgroup or a split torus $\mathbf{G}_m$. In both cases, $\mathbf{M}$ is $K$-isotropic.
Let $S \subset M$ be a maximal $K$-split torus. If $\dim(S) \geq 3$, then $M$ is necessarily quasi-split. It remains to check the cases where the dimension of $S$ is 1 or 2. Note that if $M$ is a trialitarian group of type $3, 6 D_4$ and if $\dim(S) = 2$, then $M$ is quasi-split. The remaining cases correspond to the following Tits $K$-indices (cf. [25]).

2.1. Type $^1D_4$

2.2. Type $^2D_4$

2.3. Type $^3, 6D_4$

We will show that neither case can occur.

Case (1): Here $\dim(S) = 1$. The semisimple anisotropic kernel $M_{ss}$ of $M$ is a group of type $^1A_3$, and is thus of dimension 15. By [10, 26.2 Corollary A], the centralizer $C_H(S)$ in $H$ is a $K$-split† reductive group of rank 4 and by construction contains $M_{ss}$. Hence the semisimple part

$C_H(S)_{ss} = [C_H(S), C_H(S)]$

of $C_H(S)$ is a semisimple $K$-split group of rank 3 containing $M_{ss}$. Since $C_H(S)_{ss}$ is split, $C_H(S)_{ss} \neq M_{ss}$. Therefore the type of $C_H(S)_{ss}$ is $B_3$ or $C_3$, and so its dimension is 21. In both cases, any Borel $K$-subgroup of $C_H(S)_{ss}$ has dimension 12, and thus for dimension reasons intersects $M_{ss}$ non-trivially, contradicting the assumption that $M_{ss}$ is $K$-anisotropic.

Case (2): Here $\dim(S) = 2$, and the semisimple anisotropic kernel $M_{ss}$ of $M$ is a group of type $A_1 \times A_1$, hence of dimension 6. Again, we consider the centralizer $C_H(S)$; its semisimple

†It is $K$-split because every split subtorus in $H$, in particular $S$, is contained in a maximal split torus of $H$. 
part $C_H(S)_{ss} = [C_H(S), C_H(S)]$ is a $K$-split semisimple group of rank 2 containing $M_{ss}$. Therefore it is either of type $A_2$ or $B_2 = C_2$. A dimension count shows that the intersection of a Borel $K$-subgroup of $C_H(S)_{ss}$ and $M_{ss}$ is non-trivial, contradicting anisotropy.

Case (3): Here, up to isogeny, $M$ is of the form $O^+(D, f)$ where $D$ is a quaternion algebra and $f$ is a skew-Hermitian form over $D$ of dimension 4 with Witt index 1. From [18], we know that after passing to the function field of the Severi–Brauer variety of $D$, the anisotropic part of $f$ remains anisotropic, hence the Tits index of $M$ is of the form (2). But this case is impossible by the above consideration.

Cases (4)–(6) are treated in the same way as cases (1)–(3), respectively, since the dimensions of the corresponding subgroups are the same. In case (7), the uncircled vertices correspond by the above consideration.

2.4. The norm principle

Let $G$ be a semisimple group defined over a field $K$ and $Z \subset G$ a central subgroup. Let $c = [\xi]$ be an element in

$$\ker [H^1(K, Z) \to H^1(K, G)].$$

**Definition.** Let $K(t)$ be the field of rational functions over $K$. We say that $[\xi]$ is $R$-trivial if there exists

$$[\xi(t)] \in \ker [H^1(K(t), Z) \to H^1(K(t), G)]$$

such that $\xi(t)$ is defined at $t = 0$ and $t = 1$, and satisfies $[\xi(0)] = 1$ and $[\xi(1)] = [\xi]$.

**Remark 2.2.** Here, if $J$ is an algebraic group over $K$, we say that a cocycle in $Z^1(K(t), J)$ is defined at 0 and 1 if it lies in the image of $Z^1(O, J)$, where $O$ is the intersection of the localizations $K[t]_{(t)}$ and $K[t]_{(t-1)}$ in $K(t)$. Using the evaluation maps $\varepsilon_0, \varepsilon_1 : O \to K$, such a cocycle can be evaluated at 0 and 1.

**Remark 2.3.** The short exact sequence

$$1 \to Z \to G \to G/Z \to 1$$

induces an exact sequence

$$1 \to Z(K) \to G(K) \to (G/Z)(K) \to H^1(K, Z) \to H^1(K, G)$$

and the map $f$ induces a bijection between the orbits of $G(K)$ in $(G/Z)(K)$ and the kernel in (8).

**Example 2.4.** Let $G$ be a quasi-split (absolutely) simple simply connected group. Then

$$H^1(K, Z) = \ker [H^1(K, Z) \to H^1(K, G)]$$

and every $[\xi] \in H^1(K, Z)$ is $R$-trivial. Indeed, $Z$ is contained in a maximal $K$-quasi-split torus $T$ and $H^1(F, T) = 1$ for any field extension $F/K$. Hence the required equality holds. Furthermore, the long exact sequence in Remark 2.3 shows that $f : G/Z(K) \to H^1(K, Z)$ is surjective. To show that $H^1(K, Z)$ is $R$-trivial it remains to note that the variety $G/Z$ is $K$-rational (because $G/Z$ is quasi-split and absolutely simple), and this implies that $G/Z$ is $R$-trivial.
Example 2.5. Let $G = \text{SL}(1, D)$ where $D$ is a central simple algebra of arbitrary degree $n$ over $K$, and let $Z$ be the centre of $G$. Since $Z \cong \mu_n$, we have $H^1(K, Z) \cong K^\times / K^\times n$. Also, $H^1(K, G) \cong K^\times / \text{Nrd}(D^\times)$. Therefore,

$$\text{Ker } [H^1(K, Z) \to H^1(K, G)] = \text{Nrd}(D^\times) / K^\times n.$$ 

Since $D$ is an affine space, any $[\xi]$ in the above kernel is $R$-trivial.

For any finite extension $L/K$, we have the restriction map

$$\text{res}_K^L : H^1(K, Z) \to H^1(L, Z)$$

and the corestriction map

$$\text{cor}_K^L : H^1(L, Z) \to H^1(K, Z).$$

**Definition.** Let $L/K$ be a finite field extension. We say that the norm principle holds for a cohomology class

$$[\eta] \in \text{Ker } [H^1(L, Z) \to H^1(L, G)]$$

if

$$\text{cor}_K^L([\eta]) \in \text{Ker } [H^1(K, Z) \to H^1(K, G)].$$

We also say that the norm principle holds for the pair $(Z, G)$ if it holds for all classes

$$[\eta] \in \text{Ker } [H^1(L, Z) \to H^1(L, G)]$$

for each finite field extension $L/K$.

**Theorem 2.6** [9]. Let $L/K$ be a finite field extension and assume that $[\eta] \in \text{Ker } [H^1(L, Z) \to H^1(L, G)]$ is $R$-trivial. Then the norm principle holds for $[\eta]$.

3. Preliminaries on Albert algebras

3.1. Albert Algebras

Let $K$ be a field of characteristic different from 2 and 3. A Jordan algebra over $K$ is a unital, commutative $K$-algebra$^1$ $A$ in which the Jordan identity

$$(xy)(xx) = x(y(xx))$$

holds for all $x, y \in A$ (whence $A$ is power associative).

If $B$ is an associative algebra with multiplication denoted by $\cdot$, then the anti-commutator $\frac{1}{2}(x \cdot y + y \cdot x)$ endows $B$ with a Jordan algebra structure, which we denote by $B^+$. A Jordan algebra $A$ is called special if it is isomorphic to a Jordan subalgebra of $B^+$ for some associative algebra $B$, and exceptional otherwise. An Albert algebra is then defined as a simple, exceptional Jordan algebra. The dimension of any Albert algebra is 27, and, over separably closed fields, all Albert algebras are isomorphic [12, 37.11]. From this it follows that all Albert algebras over $K$ are twisted forms of each other.

Over a general field $K$, any Albert algebra arises from one of the two Tits constructions [12, 39.19].$^2$ In this note, we will mainly be concerned with those arising from the first Tits construction. Recall that in this case, $A = D \oplus D \oplus D$ as a vector space, where $D$ is a central

$^1$We do not assume the algebra to be associative.

$^2$A historical account of this result of Tits' is provided in the notes at the end of Chapter IX of [12].
simple algebra over $K$ of degree 3. The multiplication of $A$ is determined by a scalar $\mu \in K^\times$. More precisely, define the cross product on $D$ by
\[ u \times v = uv + vu - T_D(u)v - T_D(v)u + T_D(uv) - T_D(v)u, \]
where juxtaposition denotes the associative multiplication of $D$, and $T_D$ denotes the reduced trace, and write
\[ \tilde{u} = T_D(u) - u \]
for $u \in D$. For any $\mu \in K^\times$, the product $(x, y, z)(x', y', z')$ in $A = D \oplus D \oplus D$ is then given by
\[ \frac{1}{2}(xx' + y\tilde{y}' + \tilde{y}y' + \tilde{z} \tilde{x} + \tilde{x} \tilde{z} + zy'). \]
One then writes $A = J(D, \mu)$ and says that $A$ arises from $D$ and $\mu$ via the first Tits construction.

In some arguments, we will also need to consider reduced Albert algebras, and we briefly recall their construction. Given an octonion algebra $C$ over $K$ and $\Gamma = (\gamma_1, \gamma_2, \gamma_3) \in (K^\times)^3$, the set $H_3(C, \Gamma)$ of $\Gamma$-Hermitian $3 \times 3$-matrices over $C$, that is, those of the form
\[ \begin{pmatrix} \xi_1 & c_3 & \gamma_1^{-1} \gamma_2 \gamma_3 \xi_2 \\
\gamma_2^{-1} \gamma_1 \xi_3 & \xi_2 & c_1 \\
c_2 & \gamma_3^{-1} \gamma_2 c_1 \xi_3 & \xi_3 \end{pmatrix}, \]
with $\xi_i \in K$ and $c_i \in C$ for $i = 1, 2, 3$, and where $-$ denotes the involution on $C$, is an Albert algebra under the anti-commutator of the matrix product. From [20], we know that an Albert algebra over $K$ is a division algebra if and only if it is not of this form.

### 3.2. Automorphisms and similarities

If $A$ is an Albert algebra over $K$, then $H = \text{Aut}(A)$ is a simple algebraic group over $K$ of type $F_4$. In fact, the assignment $A \rightarrow \text{Aut}(A)$ establishes an equivalence between the category of Albert algebras over $K$ and the category of simple $K$-groups of type $F_4$ [12, 26.18]. Moreover, $A$ is endowed with a cubic form $N : A \rightarrow K$ known as the norm of $A$. The structure group $\text{Str}(A)$ of $A$ is the affine group scheme whose $R$-points are defined by
\[ \text{Str}(A)(R) = \{ x \in \text{GL}(A)(R) \mid x \text{ is an isotopy of } A \otimes_K R \} \]
for every $K$-ring $R$. By an isotopy of an Albert algebra $B$ over a ring $R$, one understands an isomorphism of Jordan algebras from $B$ to the isotope $B^{(p)}$ for some $p \in B^\times$ with $N(p) \in R^\times$, the algebra $B^{(p)}$ having the same underlying module as $B$, with multiplication
\[ x \cdot y = (xy) + (yx) - (xy)p. \]

It is clear that every isotopy is a norm similarity of $A$. This holds for Albert algebras over commutative rings (see [16] for a discussion of such algebras) and implies that the structure group is a subgroup of the group of norm similarities. By [11, VI Theorem 7], if $F$ is a field with more than three elements, then every norm similarity is an isotopy. Since these groups are smooth, it follows that they coincide, that is, for any $K$-ring $R$,
\[ \text{Str}(A)(R) = \{ x \in \text{GL}(A)(R) \mid N_R(x(a)) = \nu(x)N_R(a) \ \forall a \in A \otimes_K R \}, \]
where $N_R$ is the scalar extension of $N$ to $A \otimes_K R$, and $\nu(x)$ is a scalar in $R^\times$ (called a multiplier) depending on $x$ only. Note that $\text{Str}(A)$ contains a (central) split torus $G_m$ consisting of all homotheties of $A$.

It is well known that the derived subgroup
\[ G = [\text{Str}(A), \text{Str}(A)] \]
of \( \text{Str}(A) \) is a strongly inner form of a split simple simply connected algebraic group of type \( E_6 \) and that \( \text{Str}(A) \) is an almost direct product of \( G_m \) and \( G \) (their intersection is the centre of \( G \)) (see, for example, [22, Theorem 7.3.2]). Thus,

\[
\mathcal{G} = \text{Str}(A)/G_m
\]

is an adjoint group of type \( E_6 \). Since our main result is known for split or isotropic \( G \) (see [6]) we may assume without loss of generality that \( G \) is \( K \)-anisotropic, which is equivalent to saying that \( A \) is a division algebra or, equivalently, that the norm map \( N \) is anisotropic, that is, the equation \( N(a) = 0 \) has no non-zero solutions over \( K \).

To each Albert algebra over \( K \), one associates the cohomological invariants \( f_3 \in H^3(K, \mu_2) \), \( f_5 \in H^5(K, \mu_2) \), and \( g_3 \in H^3(K, \mathbb{Z}/3\mathbb{Z}) \). The invariants \( g_3 \) and \( f_5 \) are due to Serre, and are defined in [8], while the invariant mod 3 is due to Rost, and is defined in [19] (recall that we are working over a field of characteristic different from 2 and 3). By the aforementioned equivalence of categories, one can view these as invariants of groups of type \( F_4 \) as well.

### 3.3. Action of the structure group \( \text{Str}(A) \) on \( A \)

In what follows, we denote the group of \( K \)-points of \( \text{Str}(A) \) by \( \text{Str}(A) \).

Let \( W(A) \) be the additive \( K \)-group corresponding to \( A \), that is \( W(A)(R) = A \otimes_K R \) for all \( K \)-ring \( R \). Under the natural action of \( \text{Str}(A) \) on \( W(A) \), the group \( H \) coincides with the stabilizer of \( 1 \in A \), as follows from [22, 5.9.4]. This action of \( \text{Str}(A) \) on \( W(A) \) induces an action on the open \( K \)-subvariety \( U \) of norm-units defined by

\[
U(R) = \{ a \in A \otimes_K R \mid N_R(a) \in R^\times \}.
\]

**Lemma 3.1.** Let \( A \) be split over \( K \). Set \( U = U(K) \). Then the action \( \text{Str}(A) \times U \to U \) is transitive.

The transitivity of the action over a separable closure of \( K \) follows from the fact [12, 37.11] that over a separably closed field, all Albert algebras, and thus in particular all isotopes of a given Albert algebra, are isomorphic. Therefore, to prove the lemma, it suffices to show that the natural map \( H^1(K, H) \to H^1(K, \text{Str}(A)) \) has trivial kernel. Furthermore, since \( A \) is split so is \( \text{Str}(A) \). Therefore the canonical map

\[
\text{Str}(A)(K) = \text{Str}(A) \to (\text{Str}(A)/G)(K) = G_m(K) = K^\times
\]

is surjective. This in turn implies that

\[
\text{Ker}[H^1(K, G) \to H^1(K, \text{Str}(A))] = 1.
\]

Thus we are reduced to establishing that

\[
\text{Ker}[H^1(K, H) \to H^1(K, G)] = 1.
\]

We will give two proofs of this statement, one using the structure theory of algebraic groups, and the other more Jordan theoretic in nature, to illuminate both sides of the statement.

**First proof.** Let \( [\xi] \) be in the kernel. Consider the twisted group \( ^\xi H \). It is a \( K \)-subgroup of the split group \( ^\xi G \simeq G \). Let \( B \subseteq G \) be a Borel \( K \)-subgroup. A dimension count shows that \( ^\xi H \cap B \) is non-trivial. Hence \( ^\xi H \) is isotropic, thus it is split by a quadratic extension, which implies that the invariant \( g_3 \) of \( ^\xi H \) is trivial, and thus by [21], that the invariant \( f_5 \) of \( ^\xi H \) is trivial as well. Thus, up to equivalence in \( H^1(K, H) \), we may assume that \( \xi \) takes values in a split subgroup of \( H \) of type \( G_2 \) which is in turn a subgroup of the split simple simply connected group \( M \) of type \( D_4 \) generated by the roots \( \alpha_2, \ldots, \alpha_5 \). The centralizer of \( M \) in \( G \) is
a split 2-dimensional torus, say \( S \), and \( C_G(S) = S \cdot M \). Since \( \xi \) takes values in \( M \), and \( S \) and \( M \) commute, it follows that

\[
\xi(S \cdot M) = S \cdot \xi M \subset \xi G \simeq G.
\]

Since \( S \subset \xi G \) and \( \xi G \) is split, there is a 6-dimensional split torus \( T \subset \xi G \) containing \( S \). Thus

\[
T \subset C_{\xi G}(S) = S \cdot \xi M.
\]

Therefore \( \xi M \) is split and this implies \( [\xi] = 1 \). \( \square \)

**Second proof.** By [2, Theorem 2.5], this kernel parametrizes the isomorphism classes of isotopes of \( A \). But from [16, Corollary 60], it follows that the invariants mod 2 and 3 of any isotope of the split Albert algebra \( A \) are trivial. Thus all such isotopes are isomorphic, and the above kernel is trivial, as desired.

### 3.4. Subgroups and a cohomology class attached to a point in \( A \)

Let \( a \in A \). We associate to it a subgroup of type \( D_4 \) in \( G \) and a 2-dimensional torus as follows. Let \( L \subset A \) be an étale subalgebra containing \( a \). Then \( L \) is a subfield, since \( A \) is a division algebra. Define the group \( G^L \) by

\[
G^L(R) = \{ x \in G(R) \mid x(l) = l \ \forall l \in L \otimes_K R \}
\]

for any \( K \)-ring \( R \). Since \( G^L \) stabilizes \( 1 \in L \subset A \), one gets \( G^L \subset H \subset G \). By [1, 4.4], over a separable closure of \( K \) the group \( G^L \) is isomorphic to \( \text{Spin}_4 \), hence it has type \( D_4 \). The inclusions \( \text{Spin}_4 \subset H \subset G \) and the fact that all root subsystems of type \( D_4 \) inside \( E_6 \) are conjugate imply that over a separable closure of \( K \) the group \( G^L \) is conjugate to the standard subgroup in \( G \) of type \( D_4 \) generated by the roots \( \alpha_2, \alpha_3, \alpha_4, \alpha_5 \). Therefore \( S^L = C_G(G^L) \subset G \) is a 2-dimensional torus over \( K \). Clearly, we have \( S^L(L) = L \). Using an explicit model of \( A \) over a separable closure of \( K \) or Steinberg generators and relations in a split simple simply connected algebraic group, one can easily verify that

\[
Z^L := S^L \cap H = S^L \cap G^L
\]

is the centre of \( G^L \).

Let \( S^L \subset \text{Str}(A) \) be a 3-dimensional torus generated by \( S^L \) and \( G_m \). Let

\[
N^L(R) = \text{Str}(A, L)(R) = \{ x \in \text{Str}(A)(R) \mid x(L \otimes_K R) = L \otimes_K R \}
\]

for each \( K \)-ring \( R \).

**Lemma 3.2.** \( S^L \cdot G^L \) is the connected component of \( N^L \) and over the separable closure of \( K \), one has \( N^L/(S^L \cdot G^L) \simeq S_3 \).

**Proof.** Note that since for each \( K \)-ring \( R \), every element of \( G^L(R) \) fixes the identity of \( A \), the group \( G^L \) is contained in \( \text{Aut}(A) \). Let \( N^L = \text{Aut}(A, L) \) be the intersection of \( N^L \) and \( \text{Aut}(A) \) in \( \text{Str}(A) \). To conclude, we may assume that \( K \) is separably closed. By [12, 39.13] combined with [1, 4.4], we then have \( N^L_0 / G^L \simeq S_3 \). The lemma follows from this result, provided that we find, for each \( f \in N^L_0(K) \), an element \( g \in S^L(K) \) such that \( gf \in N^L_0(K) \). Given such an element \( f \), we have \( f(1) \in L \simeq K \times K \times K \). Since \( K \) is assumed separably closed, there exists \( x \in L \) such that \( x^2 = y \). Then the \( U \)-operator \( U_x \) belongs to \( S^L(K) \) and satisfies \( U_x(1) = y \). Setting \( g = U_x^{-1} \), we have \( gf(1) = 1 \), whence \( gf \in \text{Aut}(A)(K) \), implying \( gf \in N^L_0(K) \) as desired. \( \square \)

Since by the above proof the representatives of \( N^L/(S^L \cdot G^L) \) can be chosen in \( H \), we have a natural identification \( N^L/(S^L \cdot G^L) \simeq \text{Aut}(L/K) \).
Our next goal is to describe a structure of $S^L$. Before doing so, we recall a standard fact about actions of algebraic groups on varieties and the corresponding transporters. Let $L_1, L_2 \subset A$ be two cubic étale (commutative) subalgebras. Consider the transporter defined by

$$\text{Transp}(L_1, L_2)(R) = \{ x \in \text{Str}(A)(R) \mid x(L_1 \otimes_K R) = L_2 \otimes_K R \}.$$ 

for each $K$-ring $R$. It is an $N^{L_1}$-torsor, hence it is represented by a cohomology class

$$[\xi] \in \text{Ker}[H^1(K, N^{L_1}) \to H^1(K, \text{Str}(A))].$$

By construction, this class is trivial if and only if $L_1$ and $L_2$ are conjugate over $K$. Furthermore, let $[\tilde{\xi}]$ be the image of $[\xi]$ in

$$H^1(K, N^{L_1}/(S^{L_1} \cdot G^{L_1})) = H^1(K, \text{Aut}(L_1/K)).$$

Then we have $\tilde{\xi}L_1 \simeq L_2$. In particular, if $L_1 \simeq L_2$, then $[\tilde{\xi}]$ is trivial, hence up to equivalence in $H^1(K, N^{L_1})$ we may assume that $\xi$ takes values in $S^{L_1} \cdot G^{L_1}$. This observation leads us to the following.

**Lemma 3.3.** Let $A = H_3(C, \Gamma)$ be a reduced Albert algebra, where $C$ is an octonion algebra over $K$ and $\Gamma \in (K^\times)^3$. Let $L_1 \subset A$ be the diagonal subalgebra and let $L_2 \subset A$ be an arbitrary split cubic subalgebra. Then $L_1$ and $L_2$ are conjugate in $\text{Str}(A)$.

**Proof.** By the discussion preceding the lemma, the obstacle for conjugacy is a cohomology class $[\xi] \in H^1(K, (S^{L_1} \cdot G^{L_1}))$ which maps to the trivial class in $H^1(K, \text{Str}(A))$. Since $L_1$ consists of diagonal matrices in $H_3(C, \Gamma)$, it readily follows that $S^{L_1}$ is a split $K$-torus, hence $S^{L_1} \cdot G^{L_1}$ is a Levi subgroup of a parabolic subgroup $P \subset \text{Str}(A)$. It is well known that both arrows

$$H^1(K, (S^{L_1} \cdot G^{L_1})) \to H^1(K, P) \to H^1(K, \text{Str}(A))$$

have trivial kernel. It follows that $[\xi] = 1$, as desired. \hfill \Box

We come back to an arbitrary Albert algebra $A$ over $K$, and a cubic subfield $K \subset L \subset A$.

**Proposition 3.4.** One has $S^L \simeq R_{L/K}(G_m)$ and $S'^L \simeq R^{(1)}_{L/K}(G_m)$.

**Proof.** Let $F/K$ (respectively, $F'/K$) be the minimal splitting field of $R^{(1)}_{L/K}(G_m)$ (respectively, $S'^L$). Let $\Gamma = \text{Gal}(F/K)$ and $\Gamma' = \text{Gal}(F'/K)$. These two Galois groups act naturally on the corresponding character lattices of the tori $R^{(1)}_{L/K}(G_m)$ and $S'^L$, respectively, and hence both embed naturally into $\text{GL}_2(\mathbb{Z})$. Recall that an arbitrary $n$-dimensional torus is determined uniquely by its minimal splitting field and the conjugacy class of the image of the corresponding Galois group in $\text{GL}_n(\mathbb{Z})$.

**Step 1:** $F' \subset F$. The base extension $F/K$ completely splits $L$. Hence the $g_3$-invariant of $A$ becomes trivial over $F$, implying that $A_F \simeq H_3(C, \Gamma)$ where $C$ is the octonion algebra over $F$ corresponding to $f_3(A_F)$, and $\Gamma \in (F^\times)^3$. By the above lemma, up to conjugacy we may assume that $L \otimes_K F \subset A_F$ coincides with the diagonal subalgebra. Explicit computations show that $(S^L)_F$ is a split 3-dimensional torus, and that so is $(S'^L)_F$.

**Step 2:** $F' \subset F'$. Since the field extension $F'/K$ splits $S^L$ it follows from Tits’ classification of isotropic simple algebraic groups of type $E_6$ that the semisimple part $G^L$ of $C_{\text{Str}(A)}(S^L)$ is isomorphic to $\text{Spin}(f)$, where $f$ is the 3-fold Pfister form as above, which is either split over $F'$ or splits over a quadratic field extension $E/F'$. Therefore, the same holds for $\text{Str}(A)$ and for $A$. In both cases, the $g_3$-invariant of $A_{F'}$, being an invariant mod 3, is trivial, implying
that $A_{F'} \simeq H_3(C, \Gamma)$ for an octonion $F'$-algebra $C$ and $\Gamma \in (F' \times)^3$. Since all maximal split tori in $\text{Str}(A_{F'})$ are conjugate over $F'$, we may assume that, up to conjugacy, the split torus $(S^L)_{F'}$ preserves the diagonal subalgebra of $H_3(C, \Gamma)$; then so does its centralizer in $\text{Str}(A_{F'})$. Furthermore, an explicit computation show that 

$$(G^L)_{F'} = [C_{\text{Str}(A_{F'})}(S^L)_{F'}, C_{\text{Str}(A_{F'})}(S^L)_{F'}]$$

acts trivially on the diagonal. Since $L$ consists of precisely those points of $A$ fixed pointwise under the action of $G^L$, it follows that $L \otimes_K F' \subset A_{F'}$ coincides with the diagonal. Thus $L$ is split over $F'$.

**Step 3:** The images of $\Gamma$ and $\Gamma'$ in $\text{GL}_2(Z)$ are conjugate. Recall that up to conjugacy the group $\text{GL}_2(Z)$ has two maximal finite subgroups, isomorphic to the dihedral groups $D_4$ and $D_6 \simeq D_3 \times Z/2$. Since the orders of $\Gamma$ and $\Gamma'$ are divisible by 3 we may, up to conjugacy, assume that $\Gamma, \Gamma' \subset D_6 \subset \text{GL}_2(Z)$. If $|\Gamma| = |\Gamma'| = 3$, then these groups coincide with the unique 3-Sylow subgroup of $D_6$ and we are done. Otherwise $\Gamma$ and $\Gamma'$ have order 6, hence $\Gamma \simeq \Gamma' \simeq S_3$. The group $D_6 \simeq D_3 \times Z/2 \simeq S_3 \times Z/2$ has two subgroups isomorphic to $S_3$. They are not conjugate in $D_6$, but explicit computations shows that they are conjugate in $\text{GL}_2(Z)$. This completes the proof. 

\[ \square \]

In the proof, we used the fact that the inclusion 

$L \subset \{ a \in A \mid g(a) = a \ \forall g \in G^L(K) \}$

is an equality. Hence the restriction of the action of $\text{Str}(A)$ (on $A$) to $S^L$ defines an action $S^L \times L^\times \rightarrow L^\times$. Since $S^L \cap H = S^L \cap H = Z^L$, the stabilizer of $1 \in L^\times$ in $S^L$ is equal to $Z^L$. It follows that over a separable closure of $K$ the action $S^L \times L^\times \rightarrow L^\times$ is transitive. Thus we may identify the varieties $S^L/Z^L \simeq L^\times$. The exact sequence 

$1 \rightarrow Z^L \rightarrow S^L \xrightarrow{\phi} L^\times \rightarrow 1$

and the fact that $H^1(K, S^L) = 1$ imply that we have a canonical surjective map $L^\times \rightarrow H^1(K, Z^L)$. Thus to each point $a \in L^\times$ we can attach its image $[\xi_a] \in H^1(K, Z^L)$.

**Remark 3.5.** By construction, the map $\phi$ is the restriction of the map $\text{Str}(A) \rightarrow U$ sending $g$ to $g(1)$.

4. $R$-triviality

We keep the notation introduced in Section 3; in particular $U \subset A$ is the open subset consisting of the elements $a$ in $A$ whose norm $N(a)$ is invertible.

**Proposition 4.1.** Let $A$ be an Albert algebra arising from the first Tits construction. Then the group $\text{Str}(A)$ acts transitively on $U$.

**Remark 4.2.** Proposition 4.1 is equivalent to Corollary 4.9 of [17]. The methods, however, are entirely different. Most importantly, the cohomological considerations used in our proof are essential for establishing the main result of [3].

**Proof.** Recall that the stabilizer of $1 \in A$ is $H$. Thus we may identify the variety $\text{Str}(A)/H$ with $U$. It follows from the exact sequence 

$$1 \rightarrow H \rightarrow \text{Str}(A) \rightarrow U \rightarrow 1$$
that there is a canonical one-to-one correspondence between the set of Str$(A)$-orbits in $U$ and the set
\[ \text{Ker} \left[ H^1(K, H) \rightarrow H^1(K, \text{Str}(A)) \right]. \]
Therefore our assertion amounts to proving the triviality of the above kernel.

Let $a \in U$ and let 
\[ [\eta] \in \text{Ker} \left[ H^1(K, H) \rightarrow H^1(K, \text{Str}(A)) \right] \]
be the corresponding cohomology class. We take a cubic subfield $L$ containing $a$, construct the class $[\xi_a] \in H^1(K, Z^L)$ as in the end of the previous section, and write $[\xi]$ for its image under the composition
\[ H^1(K, Z^L) \rightarrow H^1(K, G^L) \rightarrow H^1(K, H). \]
By Remark 3.5, $[\eta] = [\xi]_H$. The following claim completes the proof of the proposition.

**Claim.** The map $H^1(K, Z^L) \rightarrow H^1(K, H)$ is trivial.

Indeed, let $[\xi] \in H^1(K, Z^L)$. Take its restriction $[\xi]_L \in H^1(L, (Z^L)_L)$. Upon extending scalars to $L$, the $g_3$-invariant of $H$ becomes trivial. Since $A$ is obtained from the first Tits construction, its invariants $f_3$ and $f_5$ are trivial, and thus the group $H_L$ is split. Then by Proposition 2.1, the group $G^L$ is quasi-split over $L$. Choose a maximal quasi-split torus $T^L_L$ in $G^L \times_K L$. It contains the centre $(Z^L)_L$ of $(G^L)_L$ and has trivial cohomology in dimension 1. This implies that
\[ [\xi]_L \in \text{Ker} \left[ H^1(L, (Z^L)_L) \rightarrow H^1(L, (G^L)_L) \right]. \]
By Example 2.4, every element in $H^1(L, (Z^L)_L)$ is $R$-trivial and therefore the norm principle holds for each such element. Thus
\[ \text{cor}_K^L(\text{res}_L([\xi])) \in \text{Ker} \left[ H^1(K, Z^L) \rightarrow H^1(K, G^L) \right]. \]
The claim follows upon noting that
\[ \text{cor}_K^L(\text{res}_L([\xi])) = [\xi]^3 = [\xi], \]
because $H^1(K, Z^L)$ is a group of exponent 2. \hfill \Box

In the course of the proof of the proposition, we established the following.

**Corollary 4.3.** Let $F/K$ be an arbitrary field extension.

(a) We have $\text{Ker} \left[ H^1(F, H) \rightarrow H^1(F, \text{Str}(A)) \right] = 1$.

(b) The variety $H \times U$ is birationally isomorphic to $\text{Str}(A)$; in particular, $H$ is $R$-trivial if and only if so is $\text{Str}(A)$.

**Proof.** (a) was already established and (b) follows immediately from (a).

**Corollary 4.4.** Let $K$ be a field of characteristic $\neq 2, 3$. Let $H$ be an arbitrary algebraic group of type $F_4$ over $K$ arising from the first Tits construction. Then $H$ is $R$-trivial.

**Proof.** Let $A$ be the corresponding Albert algebra. By [23, 24], the structure group $\text{Str}(A)$ is $R$-trivial. Hence the assertion follows from the above corollary. \hfill \Box

**Remark 4.5.** In the forthcoming paper [3], we will deal with automorphism and structure groups of arbitrary Albert algebras. \hfill \Box
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