Fractionally charged impurity states of a fractional quantum Hall system

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Abstract
The single-particle spectral function for an incompressible fractional quantum Hall state in the presence of a scalar short-ranged attractive impurity potential is calculated via exact diagonalization within the spherical geometry. In contrast to the noninteracting case, where only a single bound state below the lowest Landau level forms, electron–electron interactions strongly renormalize the impurity potential, effectively giving it a finite range, which can support many quasi-bound states (long-lived resonances). Averaging the spectral weights of the quasi-bound states and extrapolating to the thermodynamic limit, for filling factor \(\nu = 1/3\) we find evidence consistent with localized fractionally charged \(e/3\) quasi-particles. For \(\nu = 2/5\), the results are slightly more ambiguous, due to finite size effects and possible bunching of Laughlin-quasi-particles.

1. Introduction
The ubiquitous presence of impurities in condensed matter systems is often a contentious problem. But they are also responsible for such intriguing phenomena as Anderson localization and the Kondo effect. While impurities and their effects are often a concern or subject of experiments, they can also be used as an experimental tool. For instance, by probing the scattering or bound states induced by a single localized impurity, information about the underlying microscopic—as opposed to thermodynamic—properties of the impurity-free bulk...
can be obtained. A striking example of this is in high-temperature superconducting systems, where tunneling experiments have been able to measure the local density of states near an impurity atom, in the superconducting state \cite{1, 2}. The spatial structure of which is related to the symmetry of the superconducting order parameter, or Cooper-pair wave function of the bulk, e.g. s- or d-wave. Similarly, a fractional quantum Hall (FQH) system can support exotic quasi-particles that carry a fractional charge. These quasi-particles, predicted by Laughlin \cite{3}, were first observed in the nonequilibrium shot-noise of the current carrying FQH edge states \cite{4}. More recently, experimental evidence of fractionally charged quasi-particles occurring in the bulk \cite{5} has been found, as well as through numerical simulations \cite{6–8}.

On the theoretical side, the effects of impurities on FQH systems has been a topic of great interest. Prior works have mostly focused on the effects of a single localized scalar or magnetic potential on the local density \cite{6, 9–11}. Here, we report on another interesting aspect, which involves probing the impurity-induced bound states in a FQH system to test for the existence of localized fractionally charged Laughlin-quasi-particles. To this end, we numerically calculated the spectral function of a FQH droplet by exact diagonalization in the presence of an attractive impurity potential and extracted the spectral weights of the resulting bound states. In principle the bound-state spectral weight(s) correspond to the fraction of a bare electron in the bound state, i.e. the fractional charge.

In a noninteracting system the spectral weight of a bound state $Z_b$ is unity. For a Landau–Fermi liquid $Z_b \simeq Z$, where $Z$ is the so-called wave-function renormalization, or quasi-particle amplitude, which roughly corresponds to the fraction of a bare electron that remains in a quasi-particle. Because an incompressible ground state of a FQH system is believed to support fractionally charged Laughlin quasi-particles, the presence of an attractive impurity potential could bind one or more of these quasi-particles into a localized state.

Furthermore, Jain’s highly successful composite fermion theory \cite{12, 13} and the elastic model of Conti and Vignale \cite{14, 15} predict a Fermi-liquid-like spectral peak at the chemical potential that corresponds to a single-particle excitation of a bound complex of multiple composite fermions, which re-forms an electron–quasi-particle. The electron–quasi-particle peak has not been observed in tunneling experiments \cite{16–19} or in exact diagonalization studies of finite-sized systems \cite{20}, including the present one. These non-Laughlin-quasi-particles, if present, could also be bound to an impurity potential. A signature of this would be a bound-state spectral weight that approaches unity. For the largest system sizes studied in the present work, we do find such a state, but, owing to finite size effects, its identification as an electron–quasi-particle remains tenuous.

In the following section, we outline how impurity bound states and energies can be identified within standard diagrammatic many-body theory for the single-particle Green’s function, with and without electron–electron interactions, using the $T$-matrix formalism. After which, details are given in sections 3 and 4 for the exact numerical calculations of the Green’s function and relevant spectral function for a finite-sized FQH system, in the spherical geometry. Finally, results and conclusions are discussed in section 5.

2. Determination of impurity bound states

The location of the possibly complex poles, in frequency space, of the single-particle Green’s function determines the energies and lifetimes of the single-particle excitations of the
system [21]. For example, in a noninteracting impurity-free system, the Fourier transformed zero-temperature (retarded) Green’s function is ($\hbar = 1$)

$$G_0(r, \omega) = \lim_{\eta \to 0^+} \frac{1}{V} \sum_k \frac{e^{i k \cdot r}}{\omega - (\epsilon_k - \mu) + i \eta},$$

(1)

where $V$ is the system volume, $\mu$ the chemical potential and $\epsilon_k$ is the dispersion. As one expects, the poles of $G_0(\omega)$ occur when $\omega = \epsilon_k - \mu$. In the presence of an attractive impurity potential, additional poles can form. These indicate the impurity-induced bound states. For a static impurity potential $V(r)$, and neglecting electron–electron interactions for the moment, the Dyson equation for the time-ordered Green’s function is

$$G(r, r', \omega) = G_0(r, r', \omega) + \int dr_1 dr_2 G_0(r, r_1, \omega) \Sigma(r_1, r_2, \omega) G(r_2, r', \omega),$$

(2)

where the self-energy $\Sigma(r, r', \omega) = V(r) \delta(r - r')$. Any additional poles of $G(\omega)$ that are related to the impurity are more easily determined by rewriting Dyson’s equation (2) into a $T$-matrix equation. By iterating (2) the Green’s function can also be expressed as

$$G(r, r', \omega) = G_0(r, r', \omega) + \int dr_1 dr_2 G_0(r, r_1, \omega) T(r_1, r_2, \omega) G_0(r_2, r', \omega),$$

(3)

where the so-called $T$-matrix is given by

$$T(r, r', \omega) = \Sigma(r, r', \omega) + \int dr_1 dr_2 \Sigma(r, r_1, \omega) G_0(r_1, r_2, \omega) T(r_2, r', \omega).$$

(4)

As the poles of the clean system are given by $G_0(\omega)$, from equation (3) any additional ones, related to the impurity, must be determined by poles of the $T$-matrix.

In one-dimension and for a delta-function impurity potential, the $T$-matrix equation can be straightforwardly solved. Its single pole $\omega_b$, below the bottom of the band, corresponds to the well-known solution of Schrödinger’s equation for the bound-state energy of an attractive delta-function potential. Near the bound-state energy, the local density of states $N(r, \omega) = -\frac{1}{\pi} \text{Im} G(r, r, \omega)$ can be written as

$$N(r, \omega \to \omega_b) = |\psi_b(r)|^2 \delta(\omega - \omega_b),$$

(5)

where $\psi_b(r)$ is the normalized bound-state wave function. The spectral weight $Z_b$ of the bound state is then

$$Z_b = \int dr \int_{\omega_b-0^+}^{\omega_b+0^+} d\omega N(r, \omega) = 1.$$

(6)

Although in two-dimensions (2D) and three-dimensions a delta-impurity potential must be treated more carefully, bound states in noninteracting quantum Hall systems have been extensively studied [22–26].

As electron–electron interactions dominate in the FQH regime, they must also be included in the calculation of the electron Green’s function, along with the impurity potential. The $T$-matrix formulation of the impurity problem can be generalized to a interacting system [27].
In general, interactions lead to a finite lifetime of bound states producing a resonance, or quasi-bound state. Additionally, as shown in the following, interactions renormalize the impurity potential. This renormalized or effective potential can support additional quasi-bound states.

For a system with electron–electron interactions, for example Coulomb, Dyson’s equation (2) retains the same form, but now the self-energy \( \Sigma(\omega) \) contains both electron–electron correlations and impurity scattering diagrams. These two contributions can be formally separated by writing \( \Sigma(\omega) = \Sigma_U(\omega) + \Sigma_{UV}(\omega) \), where \( \Sigma_U(\omega) \) contains all irreducible diagrams that involve only the Coulomb interaction between electrons and \( \Sigma_{UV}(\omega) \) contains all diagrams with at least one impurity potential. \( \Sigma_{UV}(\omega) \) accounts for the bare impurity potential and terms involving combinations of both the impurity and Coulomb interaction to all orders. Dyson’s equation can then be formally rewritten as

\[
G(r, r', \omega) = G_U(r, r', \omega) + \int \, dr_1 \, dr_2 \, G_U(r, r_1, \omega) \Sigma_{UV}(r_1, r_2, \omega) G(r_2, r', \omega),
\]

where \( G_U(\omega) \) is the exact interacting Green’s function in the absence of the impurity. In analogy with the noninteracting case, equation (2), the diagonal matrix elements (in position space) of \( \Sigma_{UV}(r, r', \omega) \) can then be identified as an effective or renormalized impurity potential \( V_{\text{eff}}(r, \omega) \).

In general even if \( V(r) \) is short-ranged, \( V_{\text{eff}}(r, \omega) \) is not. Furthermore, a generalized T-matrix equation can also be found from (7) and can be formally expressed by

\[
T(r, r', \omega) = \Sigma_{UV}(r, r', \omega) + \int \, dr_1 \, dr_2 \, \Sigma_{UV}(r, r_1, \omega) G_U(r_1, r_2, \omega) T(r_2, r', \omega).
\]

The poles of which determine the quasi-bound states of an interacting system.

### 3. Fractional quantum Hall effect on a Haldane sphere

Haldane [28] was the first to introduce the spherical geometry to study boundaryless finite-sized fractional quantum Hall (FQH) systems. The electrons are confined to the surface of a 2D sphere of radius \( R \). The quantizing magnetic field \( B \) is produced by a Dirac magnetic monopole located at the center of the sphere

\[
B(r) = \frac{2Q\Phi_0}{4\pi r^2} \hat{r},
\]

where \( Q \) is the so-called monopole strength, which can be a positive or negative integer or half-integer and \( \Phi_0 = \hbar c/e \) is the flux quantum. The total magnetic flux through the surface of the sphere is then \( 2|Q|\Phi_0 \), i.e. \( 2|Q|\Phi_0 = 4\pi R^2|B| \); thus, in units of \( h = 1 \) and \( \ell = \sqrt{c/|e|B|} \) the sphere’s radius is \( R = \ell \sqrt{|Q|} \). Various vector potentials \( A \) such that \( \nabla \times A = B \) which differ by the location and number of Dirac strings can be commonly found in the literature. Here, we use

\[
A(r) = -\frac{c}{er} Q \cot(\theta) \hat{\phi}.
\]

The noninteracting first quantized Hamiltonian for an electron with charge \(-e\) confined to the surface of the sphere is then taken to be

\[
H_0 = \frac{1}{2m} \left[ -\frac{i}{R} \nabla_\Omega + \frac{e}{c} A \right]^2,
\]
where $\nabla_\Omega = \hat{\theta} \partial_\theta + \frac{1}{\sin(\theta)} \hat{\phi} \partial_\phi$. Defining the gauge-invariant orbital angular momentum

$$\Lambda = R\hat{r} \times \left[ -\frac{1}{R} \nabla_\Omega + \frac{e}{c} A \right]$$

the Hamiltonian (11) can then be expressed as

$$H_0 = \frac{\Lambda^2}{2mR^2}.$$  \hspace{1cm} (13)

The one electron normalized eigenfunctions of (13) are given by the so-called monopole, or spin-weighted, spherical harmonics

$$\varphi Y_{lm}(\Omega) = N_{Qlm} 2^{-m} (1-x)^{\frac{m-Q}{2}} (1+x)^{\frac{m+Q}{2}} \times P_{l-m}^{m-Q+Q}(x) e^{imQ}$$  \hspace{1cm} (14)

with $x = \cos(\theta)$,

$$N_{Qlm} = \left( \frac{2l + 1}{4\pi} \frac{(l-m)!}{(l+Q)!(l-Q)!} \right)^{1/2}$$  \hspace{1cm} (15)

and $P_{\alpha,\beta}^n(x)$ are the Jacobi polynomials defined by

$$P_{\alpha,\beta}^n(x) = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} (x-1)^{n-j}(x+1)^j.$$  \hspace{1cm} (16)

The allowed values of $l$ and $m$ are

$$l = |Q|, |Q|+1, \ldots,$$

$$m = -l, -l+1, \ldots, l.$$  \hspace{1cm} (17)

The associated energy eigenvalues are

$$\epsilon_l = \frac{l(l+1) - Q^2}{2|Q|} \omega_c,$$  \hspace{1cm} (18)

where $\omega_c = e|B|/(mc)$. In the lowest Landau level (LLL) $l = |Q|$, the eigenfunctions (14) simplify to

$$\varphi Y_{Qm}(\Omega) = (-1)^{Q-m} \left[ \frac{2Q+1}{4\pi} \frac{2Q}{Q-m} \right]^{1/2} \cos^{Q+m}(\theta/2) \sin^{Q-m}(\theta/2) e^{imQ}.$$  \hspace{1cm} (19)

Because of the compact geometry of the sphere, the degeneracy of the LLL is finite. The single-particle Hilbert space is span by only $2|Q|+1$ states.

In the LLL approximation the kinetic energy can be neglected; thus, for $N$ particles interacting through the Coulomb potential $U$ the LLL Hamiltonian is simply

$$H_{\text{LLL}} = \frac{1}{2} \sum_{i \neq j}^{N} U(\Omega_i - \Omega_j) = \frac{e^2}{2R} \sum_{i \neq j}^{N} \frac{1}{|\Omega_i - \Omega_j|},$$  \hspace{1cm} (20)

See [12] and references therein for further details.
where the chord distance between the particles on the sphere has been used. The two-body matrix elements of the Coulomb interaction in the LLL Hilbert space are given by

$$\langle Qm_1, Qm_2|U|Qm'_1, Qm'_2\rangle = \frac{e^2}{R} \sum_{L=0}^{2Q} \sum_{M=-L}^{L} U^{(Q)}_L \langle Qm_1, Qm_2|LM\rangle \langle Qm'_1, Qm'_2|LM\rangle,$$  \hspace{1cm} (21)

where \(\langle Qm_1, Qm_2|LM\rangle\) are Clebsch–Gordan coefficients and

$$U^{(Q)}_L = 2 \left( \begin{array}{c} 2Q-2L \\ 2Q-L \end{array} \right) \left( \begin{array}{c} 4Q+2L+2 \\ 2Q+L \end{array} \right) \left( \begin{array}{c} 4Q+2 \\ 2Q+1 \end{array} \right).$$  \hspace{1cm} (22)

A short-ranged attractive impurity potential with strength \(g < 0\) located at the north pole of the sphere can be written as

$$V(\Omega) = \frac{g \delta(\theta)}{2\pi \sin(\theta)}.$$  \hspace{1cm} (23)

The impurity matrix elements in the LLL Hilbert space are then

$$\langle Qm|V|Qm'\rangle = \frac{g(2Q+1)}{4\pi} \delta_{Q,m} \delta_{Q,m'} \equiv \lambda \delta_{Q,m} \delta_{Q,m'}.$$  \hspace{1cm} (24)

In general the relationship between the number of particles \(N\), the total number of magnetic flux quanta \(N_{\phi}\) and the filling factor \(\nu\) of the FQH state depends on the topology of the system, and is given by

$$N_{\phi} = N\nu^{-1} - S,$$  \hspace{1cm} (25)

where \(S\) is a topological quantum number commonly called the shift \([30–33]\). For a sphere \(N_{\phi} = 2Q\), and for filling factors \(\nu = p/(2p+1)\) with \(p \in \mathbb{N}\), the shift is \(S = 2 + p\).

For a given filling factor and number of electrons, the corresponding monopole strength can be determined and the \(N\)-particle LLL Hilbert space basis constructed using the single-particle states, equation (19). The matrix elements of the \(N\)-particle Hamiltonian (20), with or without an impurity (23), in the \(N\)-particle basis can be obtained using Slater–Condon rules \([34]\) along with the one- and two-body matrix elements (24) and (21). Numerical diagonalization of the resulting Hamiltonian then allows one to calculate the Green’s function, as shown in the following section. An exact full diagonalization calculation, as opposed to iterative methods, such as Lanczos, for the single-particle Green’s function is preformed. This restricts our calculations to smaller systems sizes, but provides highly accurate spectral weights for all states, which is required.

4. Spectral function

At zero temperature the retarded single-particle Green’s function for the angular coordinates \(\Omega = (\theta, \phi)\) is defined as

$$G^r(\Omega, \Omega', t) = -i \Theta(t) \langle \psi_N^\dagger \hat{H}(\Omega, t), \hat{H}^\dagger(\Omega', 0) \rangle \psi_N^\dagger.$$  \hspace{1cm} (26)
where \(|\psi_N^g\rangle\) is the exact ground state of an interacting \(N\)-particle system, and \(\hat{\Psi}^{(t)}(\Omega)\) are the second quantized fermionic annihilation (creation) operators. Even in the presence of the impurity potential (23), \(L_z\) remains a conserved quantity. Thus, in the LLL approximation

\[
G^r(\Omega, \Omega', t) \approx -i\Theta(t) \sum_{m=-Q}^{Q} Q_{Qm}(\Omega) Q_{Qm}^*(\Omega') \langle \psi_N^g | [\hat{a}_m(t), \hat{a}_m^\dagger(0)] | \psi_N^g \rangle
\]

\[
= \sum_{m=-Q}^{Q} Q_{Qm}(\Omega) Q_{Qm}^*(\Omega') G_{m}^r(t). \tag{27}
\]

The \(L_z\)-resolved Green's function \(G_{m}^r(t)\) can be expressed in terms of the so-called lesser and greater correlations functions [35]

\[
G_{m}^<(t) = \langle \psi_N^g | \hat{a}_m^\dagger(0) \hat{a}_m(t) | \psi_N^g \rangle \tag{28}
\]

and

\[
G_{m}^>(t) = \langle \psi_N^g | \hat{a}_m(t) \hat{a}_m^\dagger(0) | \psi_N^g \rangle \tag{29}
\]

as

\[
G_{m}^r(t) = -i\Theta(t) [G_{m}^>(t) + G_{m}^<(t)]. \tag{30}
\]

The lesser and greater correlations functions, equations (28) and (29), can be evaluated as follows. Explicitly writing out the time dependence of the operators

\[
G_{m}^<(t) = \langle \psi_N^g | \hat{a}_m^\dagger(0) e^{i(\hat{H} - \mu \hat{N})t} \hat{a}_m e^{-i(\hat{H} - \mu \hat{N})t} | \psi_N^g \rangle
\]

\[
= \langle \psi_N^g | \hat{a}_m^\dagger(0) e^{i(\hat{H} - \mu \hat{N})t} \hat{a}_m | \psi_N^g \rangle e^{-i(E_{\mu} - \mu) t}. \tag{31}
\]

Next, one inserts a resolution of the identity operator expressed as a complete set of energy eigenstates of a system containing \(N-1\) particles

\[
G_{m}^<(t) = \sum_{\alpha} e^{i(E_{\alpha} - \mu(N-1))t} e^{-i(E_{\mu} - \mu) t} \langle \psi_N^g | \hat{a}_m^\dagger(0) | \alpha, N-1 \rangle \langle \alpha, N-1 | \alpha \hat{a}_m | \psi_N^g \rangle
\]

\[
= \sum_{\alpha} e^{-i(E_{\mu} - E_{N-1,\alpha} - \mu) t} | \langle N-1, \alpha | \hat{a}_m | \psi_N^g \rangle |^2. \tag{32}
\]

For the \(N-1\) particle system, the monopole strength is held fixed, i.e. it corresponds to the value used to define the ground state \(|\psi_N^g\rangle\). Similarly for \(G_{m}^>(t)\), equation (29)

\[
G_{m}^>(t) = \sum_{\alpha} e^{-i(E_{\mu} - E_{N-1,\alpha} - \mu) t} | \langle N+1, \alpha | \hat{a}_m^\dagger | \psi_N^g \rangle |^2. \tag{33}
\]

The Fourier transform of \(G_{m}^r(t)\), equation (30), is then

\[
G_{m}^r(\omega) = \int_{-\infty}^{\infty} dt \; G_{m}^r(t) e^{i\omega t}
\]

\[
= -i \lim_{\eta \to 0} \int \frac{d\omega'}{2\pi} \frac{G_{m}^>(\omega') + G_{m}^<(\omega')}{\omega - \omega' + i\eta}. \tag{34}
\]
Figure 1. The top panel shows all spectral weights \( Z^\geq_m,\alpha \) of the spectral function, equation (37), in absence of an impurity, for \( N = 7 \) particles at filling factor \( \nu = 1/3 \), approximately 58,000 states. The bottom (note scale) shows the same for a system in the presence of a delta-impurity potential having strength \( \lambda = -10 e^2/\ell \).

where

\[
G^>_m(\omega) = 2\pi \sum_\alpha |\langle N+1, \alpha | \hat{a}^+_m | \psi^g_N \rangle|^2 \delta(\omega + \mu - E_{N+1,\alpha} + E^g_N)
\]

\[
\equiv 2\pi \sum_\alpha Z^>_m,\alpha \delta(\omega + \mu - E_{N+1,\alpha} + E^g_N)
\]

and

\[
G^<_m(\omega) = 2\pi \sum_\alpha |\langle N-1, \alpha | \hat{a}_m | \psi^g_N \rangle|^2 \delta(\omega + \mu + E_{N-1,\alpha} - E^g_N)
\]

\[
\equiv 2\pi \sum_\alpha Z^<_m,\alpha \delta(\omega + \mu + E_{N-1,\alpha} - E^g_N).
\]

The \( L_z \)-resolved spectral function is then defined as

\[
A_m(\omega) = -\frac{1}{\pi} \text{Im} G^>_m(\omega)
\]

\[
= \sum_\alpha \left[ Z^>_m,\alpha \delta(\omega + \mu - E_{N+1,\alpha} + E^g_N) + Z^<_m,\alpha \delta(\omega + \mu + E_{N-1,\alpha} - E^g_N) \right]
\]

which satisfies the sum rule

\[
\sum_{m=-Q}^Q \int_{-\infty}^\mu d\omega A_m(\omega) = N.
\]

Figures 1 and 2 show the calculated spectral weights \( Z^\geq_m,\alpha \) and location of the single-particle excitation energies for \( \nu = 1/3 \) and \( 2/5 \) respectively, with (bottoms panels) and without
Figure 2. The top panel shows all spectral weights $Z_{m,\alpha}$ of the spectral function, equation (37), in absence of an impurity, for $N = 8$ particles at filling factor $\nu = 2/5$, approximately 28,000 states. The bottom shows the same for a system in the presence of a delta-impurity potential having strength $\lambda = -10 \ hbar^2/\ell$.

(top panels) an impurity. For numerical purposes the chemical potential is taken to be $\mu = (\mu_+ + \mu_-)/2$, where $\mu_+ = E_{N+1}^G - E_N^G$ and $\mu_- = E_N^G - E_{N-1}^G$. For a noninteracting system in the presence of an attractive delta-impurity potential, of the form given by equations (23) and (24), a single bound state below the LLL forms at $\omega_b \approx \omega_c/2 - \mu + \lambda \simeq \lambda$. As seen in the bottom panels of figures 1 and 2, for an interacting system this remains qualitatively the same. Although instead of a single bound state, many additional resonances, with an energy width of approximately $hbar^2/\ell$, appear. As discussed in section 2, these new quasi-bound states appear because of the interaction renormalization of the bare impurity potential, effectively giving a finite width to the delta function.

The total $L_z$-resolved spectral weights of the quasi-bound states are obtained by integrating the spectral functions over the energy width of the impurity resonances

$$Z_{b,m} = \int_{\text{impurity states}} d\omega A_m(\omega).$$

(39)

The strength of the impurity is chosen large enough to well-separate the quasi-bound states from the rest of the spectrum and thus facilitate their identification and the integration region for (39). In principle they could be identified, for an arbitrary impurity strength, by the poles of the generalized $T$-matrix (8), but, for finite-sized systems, this is numerically difficult.

Technically, as was pointed out in [23], to obtain a truly localized and mathematically well-defined bound state higher Landau levels as well as a renormalization or regularization of the delta impurity have to be included. However in the strong field limit $\lambda/\omega_c \ll 1$ the LLL approximate bound state energy and wave function are logarithmically accurate.
For the data shown in figures 1 and 2, the integrated spectral weights of the impurity states in each angular momentum channel are shown in the top panels of figures 3 and 4. The averaged integrated spectral weights $\bar{Z}_b = (2|Q| + 1)^{-1} \sum_m Z_{b,m}$ of the quasi-bound states as a function of $1/N$ are shown in the bottom panels of figures 3 and 4 for filling factors $\nu = 1/3$ and $2/5$ respectively and listed in table 1. The variance $\sigma$ of the $L_z$-resolved weights is also listed in

Figure 3. The top panel shows the integrated spectral weights of impurity states per angular momentum channel $m$, for $N = 7$ particles at filling factor $\nu = 1/3$. The bottom panel shows the averaged integrated impurity spectral weights versus total particle number, along with a simple linear fit to extrapolate to the thermodynamic limit.

Figure 4. The top panel shows the integrated spectral weights of impurity states per angular momentum channel $m$, for $N = 8$ particles at filling factor $\nu = 2/5$. The bottom panel shows the averaged integrated impurity spectral weights versus total particle number, along with a simple linear fit to extrapolate to the thermodynamic limit.

5. Results and discussion

For the data shown in figures 1 and 2, the integrated spectral weights of the impurity states in each angular momentum channel are shown in the top panels of figures 3 and 4. The averaged integrated spectral weights $\bar{Z}_b = (2|Q| + 1)^{-1} \sum_m Z_{b,m}$ of the quasi-bound states as a function of $1/N$ are shown in the bottom panels of figures 3 and 4 for filling factors $\nu = 1/3$ and $2/5$ respectively and listed in table 1. The variance $\sigma$ of the $L_z$-resolved weights is also listed in

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Here, the data shown in the bottom panels of figures 3 and 4 is given: the averaged integrated impurity spectral weights $\overline{Z}_b = (2|Q|+1)^{-1} \sum_m Z_{b,m}$ for each particle number $N$ and filling factors $\nu$. The variance $\sigma$ of the integrated impurity spectral weights is also stated.

| $N$  | $\overline{Z}_b$ | $\sigma$ |
|------|------------------|----------|
| $\nu = 1/3$ |
| 4    | 0.253            | 0.265    |
| 5    | 0.258            | 0.198    |
| 6    | 0.287            | 0.152    |
| 7    | 0.269            | 0.117    |
| $\nu = 2/5$ |
| 4    | 0.265            | 0.305    |
| 6    | 0.257            | 0.197    |
| 8    | 0.258            | 0.133    |

While the variance is quite large, it is monotonically decreasing as the system size increases. As can be seen from figure 3, for $\nu = 1/3$ the extrapolation to the thermodynamic limit gives a bound-state weight, or fractional charge, of approximately $e^* \approx 0.31e$, which is consistent with the value $e^* = e/3$. For $\nu = 2/5$, see figure 4, we find $e^* \approx 0.25e$, which, although reasonable, is further from the predicted $e^* = e/5$ value. The accuracy or lack of compared to the $\nu = 1/3$ case could simply be a finite-size effect, as the Hilbert space for the $\nu = 1/3$ system is approximately twice as large, even through the $\nu = 2/5$ system has a greater number of electrons. This result could also be due to bunching of Laughlin-quasi-particles, where two $e^* = e/5$ charged particles form a single quasi-bound state. This bunching effect has been recently observed in shot-noise experiments at low temperature [36].

Finally, we discuss what appear to be outliers, both large and small, in the data for the $L_z$-resolved weights. These can be most easily seen in the top panels of figures 3 and 4 for the smallest $m$-values. These states are localized in space on the opposite side of the Haldane sphere than that of the impurity potential. For several values of angular momentum the bound-state spectral weights appear to vanish or become very small. These correspond to extremely poor quasi-bound states with short lifetimes. But with no a priori reason for exclusion they are included in the average for $\overline{Z}_b$. In contrast there is a single weight, at the smallest $m$-value for each filling factor, with a comparatively large integrated spectral weight, which seemingly approaches unity. Although this is also included in the average, as mentioned in section 1, these states could be an indication of a bound electron–quasi-particle. Notwithstanding no other indication of such single-particle excitations are seen in the spectral function of the impurity-free systems, e.g. an electron–quasi-particle peak at the chemical potential.

6. Conclusions

In conclusion, we have calculated the fractional charge of impurity bound states in FQH systems by an exact diagonalization calculation for the electron Green’s function and extracting the spectral weights of the impurity bound states; these correspond to the fraction of a bare electron that remains in each single-particle state. We find evidence consistent with the theoretically predicted fractional charge carried by Laughlin-quasi-particles: $e^* \approx 0.31e$ for filling factor $\nu = 1/3$ and $e^* \approx 0.25e$ for $\nu = 2/5$. 
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