TOWARDS A COMBINATORIAL UNDERSTANDING OF
LATTICE PATH ASYMPTOTICS

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Abstract. We provide a new strategy to compute the exponential growth constant of enumeration sequences counting walks in lattice path models restricted to the quarter plane. The bounds arise by comparison with half-planes models. In many cases the bounds are provably tight, and provide a combinatorial interpretation of recent formulas of Fayolle and Raschel (2012) and Bostan, Raschel and Salvy (2013). We discuss how to generalize to higher dimensions.

Introduction

A lattice path model is a set of walks defined by the types of steps that comprise each walk, and the region to which the walks are confined. Lattice path models offer a robust framework well suited to the study of many physical and chemical phenomena, in addition to purely combinatorial objects. The basic enumerative question is to determine the number of walks of a given length from a given model. A first approximation to this value is the exponential growth constant, which itself carries combinatorial and probabilistic information. For example, it is directly related to the limiting free energy in statistical mechanical models.

A model in which the set of allowable steps is contained in \( \{0, \pm 1\}^2 \), is said to have small steps. The enumeration of small step models restricted to the first quadrant has been a subject of several recent works \[7, 5\]. Fayolle and Raschel \[7\] have determined expressions for the growth constant for these models using boundary value problem techniques. In the case of excursions, that is, walks that return to the origin, explicit asymptotic expressions are known. Denisov and Wachtel \[6, Section 1.5\] consider related models in probability, and have provided results for models in arbitrary dimension. Bostan, Raschel and Salvy \[4\] made these results explicit in the enumeration context. In all of these cases the results are obtained with sophisticated machinery which does not maintain a clear underlying combinatorial picture.

This paper is based on the following simple observation: any half plane including the first quadrant can be used to give a bound for the growth constant of the quarter plane model walks. Such a bound is readily computable using the results of Banderier and Flajolet \[1\]. Taking the best such half plane, the bound is tight for walks with small steps, in view of the results of Fayolle and Raschel. Furthermore, it is insightfully tight in that it gives a simple combinatorial interpretation of the cases of Fayolle and Raschel. This one idea unifies their six cases, which depend on

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various parameters of the model. We use only the elementary calculus observation that a minimum of a continuous function must occur either at an end point or at a critical point. Our approach is combinatorial and readily adaptable to models with larger steps, weighted steps, and to models in higher dimensions.

**Conventions and notation.** A lattice path model is a combinatorial class $R(S)$ defined by a multiset $S \subset \mathbb{R}^2$ of vectors, called the step set; and a region $R \subset \mathbb{R}^2$. A walk $w = w_1, \ldots, w_n$ is a sequence of vectors from $S$, and it remains in region $R$ if the vector sum $\sum_{i=1}^\ell w_i \in R$ for $\ell = 1 \ldots n$. The class $R(S)$ is the set of all walks which stay in the region $R$. We denote the set of all walks of length $n$ by

$$R(S)_n = \{w_1 w_2 \ldots w_n : w_i \in S, w_1 + \cdots + w_j \in R \text{ for } 1 \leq j \leq n\}.$$ 

The central quantity we investigate is the number of walks with $n$ steps in a given model, $|R(S)_n|$. We write $H = \mathbb{R} \times [0, \infty)$ for the upper half plane and $Q = \mathbb{R} \times [0, \infty)$ for the first quadrant, and abbreviate $h_n = |H(S)_n|$ and $q_n = |Q(S)_n|$ when $S$ is clear. In this work our goal is new results for the first quadrant $Q$.

A step set is said to be made of small steps if $S \subseteq \{0, \pm 1\}^2$ and in this case we use the compass abbreviations $NW \equiv (-1,1), N \equiv (0,1), NE \equiv (1,1), \text{etc.}$ We also consider larger regions, and more general step sets. We say a model is non-trivial if it contains at least one walk of positive length, and if for every boundary of the region, there exists a walk which touches that boundary. The excursions are the sub-class consisting of walks which also end at the origin.

The present work determines bounds for the growth constant, $K_S$, of the sequence $\{q_n\}$ which is defined by

$$K_S = \limsup_{n \to \infty} q_n^{1/n}.$$ 

This quantity is the multiplicative inverse of the dominant singularity of the generating function $Q(z) = \sum q_n z^n$ [3, Theorem IV.7].

Our strategy uses the simple relation that if $Q \subset R$, then $Q(S)_n \subset R(S)_n$ and hence $q_n \leq |R(S)_n|$. This is true for all $n$, hence it is also true for their growth factors. Thus, by considering well chosen regions, we are able to bound $K_S$. The key bounding regions for us are the half planes $H_\theta = \{(x,y) : x \sin \theta + y \cos \theta \geq 0\}$ where $\theta \in [0, \pi/2]$. We denote by $K_S(\theta)$ the growth constant of the sequence of the number of walks of length $n$ in this region:

$$K_S(\theta) = \limsup_{n \to \infty} |H_\theta(S)_n|^{1/n}.$$ 

**The main result and the plan of the paper.** To begin, in Lemma[7] we adapt the formulas of Banderier and Flajolet to compute $K_S(\theta)$. We show that $K_S(\theta)$ defines a continuous function in $\theta$. Since each $H_\theta$ contains $Q$, $K_S(\theta)$ is an upper bound on $K_S$. Finally, we determine the location of the minimum upper bound in Theorem[9].

We prove in Theorem[15] that in the case of small steps, these minimum bounds are precisely the values found by Fayolle and Raschel, hence the bounds are tight. This vindicates the description of this work as a combinatorial interpretation of the formulas provided by Fayolle and Raschel.

Our strategy applies to more general classes of models where at present there are no formulas with which to compare. For example, multiple steps in the same direction, longer steps, and higher dimensional models. The quantities we recover in these classes are, transparently, upper bounds and they can be compared against
1. Walks in the Half-Plane

Models restricted to a half plane are well understood. The set \( H(S) \) of walks restricted to the upper half plane with steps from the finite multiset \( S \) are in bijection with unidimensional walks with steps from the multiset \( A = \{ j : (i, j) \in S \} \) because horizontal movement does not lead to any interaction with the boundary of \( H \). We denote the class of unidimensional walks restricted to \( H \) by \( H(A) \). Here a multiset \( A \) is non-trivial if it contains at least one positive and one negative value. Define the drift of \( A \) to be \( \delta(A) = \sum_{a \in A} a \) and the inventory of \( A \) to be \( \chi(u) = \sum_{a \in A} u^a \); notice that these are related by \( \delta(A) = \chi'(1) \).

In this section we examine the growth constant for the counting sequence of walks restricted to a half-plane.

**Theorem 1.** [1, Extracted from Theorem 4] Let \( A \) be a finite multiset of integers and let \( \chi(u) = \sum_{a \in A} u^a \). The number \( h_n \) of walks of length \( n \) in \( H(A) \) depends on the sign of the drift \( \delta(A) = \chi'(1) \) as follows:

- \( \delta(A) < 0 \): \( h_n \sim \nu_0 \chi(\tau)^n (2\pi n^3)^{-1/2} + O(1/n) \);
- \( \delta(A) = 0 \): \( h_n \sim \nu_1 \chi(1)^n (\pi n)^{-1/2} + O(1/n) \);
- \( \delta(A) > 0 \): \( h_n \sim \nu_2 \chi(1)^n + \nu_0 \chi(\tau)^n (2\pi n^3)^{-1/2} + O(1/n) \).

Here \( \tau \) is the unique positive real number which satisfies \( \chi'(\tau) = 0 \), and \( \nu_0, \nu_1, \) and \( \nu_2 \) are known, real constants.

The proof in [1] of the above formulas applies transfer theorems to an explicit generating function, which they also derive. This construction requires integer step lengths. As we focus only on the exponential growth, we are able to generalize this result to a wider class of step sets.

**Theorem 2.** Let \( A \) be a finite, non-trivial multiset of real numbers, and let \( \chi(u) = \sum_{a \in A} u^a \). The growth constant \( K_A = \limsup_{n \to \infty} h_n^{1/n} \) of the sequence \( h_n = |H(A)_n| \) depends on the sign of the drift \( \delta(A) \) as follows:

\[
K_A = \begin{cases} 
|A| & \text{if } \delta(A) \geq 0 \\
\chi(\tau) & \text{otherwise.} 
\end{cases}
\]

Here \( \tau \) is the unique positive real number which satisfies \( \chi'(\tau) = 0 \).

Theorem 1 establishes this formula for \( A \subset \mathbb{Z} \) and we proceed from this case in three steps:

1. If Equation (1.1) holds for the multiset \( A \), then it is also true for the multiset \( rA = \{ ra : a \in A \} \) when \( r \neq 0 \) (Lemma 3);
2. We deduce that the formula holds for multisets of rationals (Remark 5);
3. We prove that the formula holds for multisets of reals.

We remark that the growth constant of the sequence counting the number of excursions of length \( n \) in \( H \) to the \( x \)-axis can be shown to be \( \chi(\tau) \) using a similar strategy.
1.1. Some facts about $\chi(u)$. We begin with some essential properties of $\chi(u)$.

**Lemma 3 (Scaling Lemma).** Let $A$ be a finite, non-trivial multiset of real numbers with inventory polynomial $\chi_A(u)$. Suppose further that Equation (1.1) holds for $K_A$. For any $r > 0$, define $B = rA = \{ra : a \in A\}$. Then the growth constant $K_B$ of the sequence $b_n = |H(B)_n|$ also satisfies

$$K_B = \begin{cases} |B| & \text{if } \delta(B) \geq 0 \\ \chi_B(\tau_B) & \text{otherwise} \end{cases}.$$  

Here $\chi_B(u) = \sum_{b \in B} u^b$ and $\tau_B > 0$ satisfies $\chi_B'(\tau_B) = 0$.

**Proof.** $H(B)$ is in bijection with $H(A)$, so their growth constants are the same. The formula follows since their drifts have the same sign, and because $\chi_B(\tau_B) = \chi_A(\tau_A)$. □

**Lemma 4 ($\chi(u)$ is convex).** Given a finite, non-trivial multiset $A$ of real numbers, the real valued function $\chi_A(u) = \sum_{a \in A} u^a$ is convex for $u > 0$ and has a unique positive critical point. The function is minimized at this point. Furthermore, if $\delta(A) = 0$, then the unique critical point occurs at $u = 1$.

**Proof.** If $\min_{a \in A} |a| \geq 1$, then the function is convex on the positive reals, since term by term the second derivative is positive when evaluated at a positive number $u$.

If this is not the case, we consider a scaled version of $A$, call it $B$, where each element is of modulus greater than one and deduce the convexity of $\chi_A(u)$ from the convexity of the scaled set and the composition $\chi_B(u) = \chi_A(u^r)$. The uniqueness of the fixed point follows from the relation $\tau_A = \tau_B$.

Since $A$ is non-trivial, $\chi(0) = \lim_{u \to \infty} \chi(u) = +\infty$ and so the positive critical point $\tau$ is such that $\chi(\tau)$ is a global minimum for the domain $(0,\infty)$. Finally note that as $\chi'(1) = \sum_{a \in A} a$, if $\delta(A) = 0$ then the unique critical point occurs at $u = 1$. □

**Remark 5.** To prove Theorem 2 in the case of rational steps, simply scale by the lcm of the denominators of the step lengths. Theorem 1 applies to the scaled steps and so by the scaling lemma, $K_A$ also satisfies Equation (1.1).

1.2. The continuity of $\chi(\tau)$ as a function of $A$. In order to prove Theorem 2 we consider the function $\chi(u)$, evaluated at its critical point, and how it behaves when viewed as a function of the step lengths.

**Lemma 6.** Let $F : \mathbb{R}^\ell \times \mathbb{R}_{>0} \to \mathbb{R}$ be the function defined by

$$F((x_1, x_2, \ldots, x_\ell), u) = F(x,u) = \sum_{j=1}^\ell u^{x_j}.$$  

For any $a \in \mathbb{R}^\ell$ with at least one positive component and one negative component, the function $F(a,u)$ is a convex function of $u$ with a unique positive critical point $\tau(a)$. Furthermore, there is a neighbourhood $U$ containing $a$ such that the function

$$\kappa(x) = F(x, \tau(x)),$$

is smooth on the neighbourhood $U$.  

Proof: In view of Lemma 4, the result is a simple consequence of the Implicit Function Theorem.

1.3. Proof of Theorem 2.

Proof of Theorem 2. Let \( A \subset \mathbb{R} \) be as in the hypotheses. To prove that \( K_A \) satisfies the formula of Equation (1.1) we build two sequences of rational steps which converge to \( A \) in order to squeeze the growth constant \( K_A \) of \( h_n = |H(A)_{n}| \) into the desired form.

For each \( a \in A \), let \( \{ a^+_i \} \) and \( \{ a^-_i \} \) be rational sequences satisfying

\[
0 \leq a^+_i - a \leq \frac{1}{2^i} \quad \text{and} \quad 0 \leq a - a^-_i \leq \frac{1}{2^i}.
\]

We define two multisets

\[
A^+_i = \{ a^+_i : a \in A \} \quad \text{and} \quad A^-_i = \{ a^-_i : a \in A \}.
\]

The drift is additive, thus for each \( i \),

\[
\delta(A^-_i) = \sum_{a \in A} a^-_i \leq \sum_{a \in A} a \leq \sum_{a \in A} a^+_i = \delta(A^+_i).
\]

Note that Remark 5 applies to both \( A^-_i \), and \( A^+_i \) because they are multisets of rational numbers and hence Equation (1.1) is valid for their growth factors, \( K_{A^-_i} \) and \( K_{A^+_i} \) respectively.

Now, let \( \iota \) be the bijective map \( \iota : A^-_i \to A \) sending \( a^-_i \mapsto a \) for \( a^-_i \in A^-_i \). Since \( a^-_i \leq \iota(a^-_i) \), for \( w = w_1 \ldots w_n \in H(A^-_i) \) and any \( k \leq n \),

\[
0 \leq \sum_{j=1}^{k} w_j \leq \sum_{j=1}^{k} \iota(w_j).
\]

For any \( i \), the map \( \iota \) induces an injection of the set of walks \( H(A^-_i) \) into \( H(A) \). Consequently, \( h^-_{i,n} \leq h_n \). Similarly, \( h_n \leq h^+_{i,n} \). These inequalities extends to the growth factors:

\[
K_{A^-_i} \leq K_A \leq K_{A^+_i}.
\]

We claim

\[
\lim_{i \to \infty} K_{A^-_i} = K_A = \lim_{i \to \infty} K_{A^+_i},
\]

and hence \( K_A \) is given by the formula of Theorem 1. To prove this claim we split into three cases, based on the drift of \( A \).

Case 1: \( \delta(A) > 0 \). First note that \( \delta(A^+_i) \geq \delta(A) > 0 \), hence \( K_{A^+_i} = |A^+_i| = |A| \). Next consider that since \( A^-_i \to A \), as \( i \) tends to infinity, \( \delta(A^-_i) \to \delta(A) \). Consequently, for sufficiently large \( i \), \( \delta(A^-_i) > 0 \), and hence \( K_{A^-_i} = |A^-_i| = |A| \) as well. Thus, \( K_A = |A| \) by Equation (1.2).
Case 2: $\delta(A) < 0$. By a similar argument to Case 1, when $i$ is sufficiently large, $\delta(A^+_i) < 0$. In this case, we have $K_{A^+_i} = \chi_{A^+_i}(\tau_{A^+_i})$ and $K_{A^-_i} = \chi_{A^-_i}(\tau_{A^-_i})$. To prove that these values converge as $i \to \infty$ we appeal to Lemma [3].

Relabel the elements of $\mathcal{A} = \{a_1, \ldots, a_\ell\}$ such that $a_1 \leq a_2 \leq \cdots \leq a_\ell$ and define $\mathbf{a} = (a_1, \ldots, a_\ell)$. Define vectors $\mathbf{a}^+_i, \mathbf{a}^-_i$ similarly from $A^+_i$ and $A^-_i$. Consequently,

\begin{equation}
\lim_{i \to \infty} \mathbf{a}^+_i = \mathbf{a} \quad \text{and} \quad \lim_{i \to \infty} \mathbf{a}^-_i = \mathbf{a}.
\end{equation}

Recall the function $\kappa(x) = F(x, \tau(x))$ from Lemma [3] and in particular the fact that $\kappa(x)$ is smooth on a neighbourhood $\mathcal{V}$ containing $\mathbf{a}$.

Since $\mathbf{a}^+_i \to \mathbf{a}$, there is some $M \in \mathbb{N}$ such that for $i > M$, the set $\mathcal{V}$ contains $\mathbf{a}$, $\mathbf{a}^+_i$ and $\mathbf{a}^-_i$. Then the smoothness and uniqueness of $\kappa$ on $\mathcal{V}$ give

$$\lim_{i \to \infty} K_{A^+_i} = \lim_{i \to \infty} \chi_{A^+_i}(\tau_{A^+_i}) = \lim_{i \to \infty} \kappa(\mathbf{a}^+_i) = \kappa(\mathbf{a})$$

and similarly $\lim_{i \to \infty} K_{A^-_i} = \kappa(\mathbf{a})$.

Thus, by Lemma [3] when $\delta(A) < 0$, $K_A = \kappa(\mathbf{a}) = \chi_A(\tau_A)$, as required.

Case 3: $\delta(A) = 0$. This is a mix of the previous two cases. First, as $\delta(A^+_i) > 0$, then $K_{A^+_i} = |A|$. Next, since $\delta(A^-_i) < 0$, then $K_{A^-_i} = \chi_{A^-_i}(\tau_{A^-_i})$. Recall from the proof of Lemma [4] that when $\delta(A) = 0$, $\tau_A = 1$ and thus $\kappa(\mathbf{a}) = |A|$. As before, by continuity,

$$\lim_{i \to \infty} K_{A^-_i} = \lim_{i \to \infty} \kappa(\mathbf{a}^-_i) = \kappa(\mathbf{a}) = |A|.$$

It follows that when $\delta(A) = 0$, $K_A = |A|$, as required.

Thus, in all cases we have that the growth constant $K_A$ satisfies Equation (1.1). □

1.4. Other half-planes. Next, we extend to other half-planes, each defined by an angle: $H_\theta = \{ (x, y) : x \sin \theta + y \cos \theta \geq 0 \}$. Note that for $\theta \in [0, \pi/2)$ this region is equal to $\{ (x, y) : y \geq -mx \}$ where $m = \tan \theta$. The upper half plane is given by $H_\theta$ and the right half plane is $H_{\pi/2}$. In this latter case, we use the extended reals, and write $m = \infty$. The enumeration of lattice path models in $H_\theta$ emulates the enumeration of models in $H$.

Lemma 7. Let $S \subset \mathbb{Z}^2$ be a finite, non-trivial multiset and let $H_\theta = \{ (x, y) : x \sin \theta + y \cos \theta \geq 0 \}$ and let $A(\theta) = \{ i \sin \theta + j \cos \theta : (i, j) \in S \}$. There is a bijective equivalence

$$H_\theta(S) \equiv H(A(\theta)).$$

Consequently, the growth constant $K_S(\theta)$ for the sequence $|H_\theta(S)_n|$ is determined by Theorem [1].

Proof. Here it suffices to consider the displacement of each step in the step set in the direction orthogonal to the boundary. The steps $(0, 1)$ and $(1, 0)$ respectively have displacement $\cos \theta$ and $\sin \theta$ in this direction, the other steps follow by linearity. This gives rise to a unidimensional half-plane model to which Theorem [1] applies. □

Example 8 ($S = \{ N, E, SW \} = \mathcal{F})$. For any $\theta$, there is a bijective equivalence to the unidimensional model $H_\theta(S) \equiv H(\{ \cos \theta, \sin \theta, -\cos \theta - \sin \theta \})$. When $\theta \neq \pi/2$, we can scale the model by $\cos \theta^{-1}$. Let $m = \tan \theta$, then $H_\theta(S) \equiv H(\{ 1, m, -m-1 \})$. 
If \( \theta = \pi/2 \), we substitute \( \pi/2 \) into the parametrisation by \( \theta \) and avoid creating a model with steps of infinite length. In this case, \( H_{\pi/2}(S) \equiv H(\{1,0,-1\}) \).

2. Bounds for lattice path models in the quarter plane

As we have already noted in the introduction, the exact enumeration of quarter plane models has been well explored recently. In the case of small steps, Bousquet-Méloû and Mishna identified 79 non-isomorphic, non-trivial models \( [5] \). Furthermore, they provided expressions for the generating functions as diagonals of rational functions. Bostan and Kauers \( [3] \) provided asymptotic expressions in the cases when the model satisfies a linear differential equation.

Most recently, Fayolle and Raschel have determined expressions for the dominant singularities of the generating functions of 74 models. We summarize their formulas in Section 2.3 The remaining five models are treated using the iterative kernel method \( [10] \).

Define the inventory of the model to be the Laurent polynomial

\[
P(x, y) = \sum_{(i,j) \in S} x^i y^j.
\]

This is the two dimensional analog to \( \chi \). Similarly, the drift is a straightforward generalization

\[
\delta(S) = \sum_{s \in S} s = (\delta_x, \delta_y),
\]

where

\[
\delta_x = \frac{\partial}{\partial x} P(x, 1) \bigg|_{x=1} = P_x(1, 1), \quad \delta_y = \frac{\partial}{\partial y} P(1, y) \bigg|_{y=1} = P_y(1, 1).
\]

Note that \( P(1, 1) = |S| \).

2.1. Bounds from 1/2-plane models. An upper bound on the growth constant of a quarter plane model can always be determined by appealing to a half plane model on the same steps restricted to lie in a region containing the first quadrant. In this section we describe how to determine the half plane which gives the best bound. The main result is Theorem 9. It is followed by examples of its application, its proof, and then in Section 2.3 a proof that, in the case of small steps, the bound is the same as the formula of Fayolle and Raschel.

**Theorem 9.** Let \( S \subset \mathbb{Z}^2 \) be a finite multiset that defines a non-trivial quarter plane model \( Q(S) \) with inventory \( P(x, y) \). Suppose that the pair \( (\alpha, \beta) \in \mathbb{R}^2_{\geq 0} \) satisfies

\[
P_x(\alpha, \beta) = P_y(\alpha, \beta) = 0.
\]

Then the growth constant \( K_S = \lim_{n \to \infty} q_n^{1/n} \) satisfies

\[
K_S \leq K_S(\theta) \quad \text{for all} \quad 0 \leq \theta \leq \pi/2,
\]

where \( K_S(\theta) \) is as defined in Lemma 7. Furthermore, \( K_S(\theta) \) is a continuous function of \( \theta \). If \( \log \alpha / \log \beta \geq 0 \), then the minimum value is attained at \( \theta^* = \arctan \left( \frac{\log \alpha}{\log \beta} \right) \). Otherwise, the minimum is attained at one of the endpoints of the range: either \( \theta \) or \( \pi/2 \).
Note that the minimum being obtained as described does not preclude it also being attained elsewhere, as occurs for instance when $K_S(\theta)$ is constant on all or part of its domain.

Before we prove this result, we consider two examples to develop some intuition on the behaviour.

**Example 10** ($S = \mathbb{N}$). The inventory $P(x, y) = y + \frac{1}{y} + \frac{x}{y} + \frac{1}{x}$ has a unique positive critical point at $(1, \sqrt{3})$, and so $\theta^* = \arctan(\log(1)/\log(\sqrt{3})) = 0$. We use $H(S) \equiv H(A(0)) = H(\{-1, -1, -1, -1\})$ to compute the growth constant $K_S(0) = \chi(\sqrt{3}) = 2\sqrt{3}$ and deduce the bound $K_S \leq 2\sqrt{3}$. In fact, $K_S(0) = K_S$, as computed by Fayolle and Raschel. Perhaps unsurprisingly, both formulas arise from the same computations. We formalize this connection in Lemma 14.

**Example 11** ($S = \mathbb{N}$). We numerically compute the critical point $(\alpha, \beta)$ of the inventory $P(x, y) = y + \frac{x}{y} + \frac{1}{y} + \frac{1}{x} + \frac{1}{x}$: $(\alpha, \beta) \approx (1.6760, 1.8091)$. Consequently, the minimising angle is $\theta^* \approx 0.2281\pi \approx \arctan(0.8712)$. The growth factor is computed $K_S(\theta^*) \approx 4.2148$. Lemma 14 shows that this is the correct value.

This example disproves the natural conjecture that the best half plane is defined by the perpendicular to the drift vector, which in this case is $\delta(S) = (-1, -2)$. Rather, the slope is connected to the Cramer transformation in probability. In this context, the transformation assigns the probability $P(\alpha, \beta)$ to the step $(i, j)$ so that the weighted drift is $(0, 0)$, and hence tools for walks with no drift may apply. In particular, this is used by Denisov and Wachtel in their work on excursions. Bostan, Rachel and Salvy in [4] relate the transformation to the inventory. More generally, our work determines a combinatorial interpretation of $P(\alpha, \beta)$, which is the growth factor for excursions in the quarter plane.

2.2. **Proof of Theorem 9**. First we require the following result.

**Lemma 12.** Let $S \subseteq \mathbb{Z}^2$. Suppose that $(\alpha, \beta)$ is the unique critical point in $(0, \infty)^2$ of $P(x, y) = \sum_{(i,j) \in S} x^i y^j$. If $\beta \neq 1$, then set $\theta^* = \arctan(\log \alpha / \log \beta)$, otherwise set $\theta^* = \pi/2$. Then $P(\alpha, \beta) = \chi^*(\theta^*)$.

**Proof of Lemma 12.** Consider the function $G(u, \theta) = \sum_{(i,j) \in S} u^i \sin \theta^* \cos \theta$. Remark $G(u, \theta) = P(u \sin \theta, u \cos \theta)$. Straightforward analysis of the system $G_\alpha(u, \theta) = 0, G_\beta(u, \theta) = 0$ using the fact that $(\alpha, \beta)$ the unique positive fixed point of $P(x, y)$, we derive the critical point $(u^*, \theta^*) \in (0, \infty) \times [0, 2\pi]$ of $G(u, \theta)$ satisfies $u^* = \beta^{1/\cos \theta^*}$ and $\theta^* = \arctan(\ln \alpha / \ln \beta)$. Finally, we derive the equivalence

$$
\chi^*(u^*) = P\left(\left(\frac{1}{\beta^{1+\frac{1}{\cos \theta^*}}}\right)^{\sin \theta^*}, \left(\frac{1}{\beta^{1+\frac{1}{\cos \theta^*}}}\right)^{\cos \theta^*}\right) = P(\beta^{\tan \theta^*}, \beta) = \sum_{(i,j) \in S} \beta^i \log \alpha / \log \beta^j = \sum_{(i,j) \in S} \alpha^i \beta^j = P(\alpha, \beta).
$$

**Proof of Theorem 9.** As previously noted, for any $\theta \in [0, \pi/2]$, $|q_n| \leq |H(A(\theta))_n|$, and their growth constants are similarly related: $K_S \leq K_S(\theta)$. This proves the first statement.
Establishing continuity. We divide the proof of continuity into three cases which depend of the drift $\delta(S)$, as in the proof of Lemma 6. First we remark that for any $\theta$,

$$\delta(A(\theta)) = \sum_{(i,j) \in S} (i \sin \theta + j \cos \theta) = \delta_x \sin \theta + \delta_y \cos \theta.$$ 

In the interval $[0, \pi/2]$, $\cos \theta$ and $\sin \theta$ are both non-negative.

**Case 1:** $\delta_x \geq 0, \delta_y \geq 0$: In this case, for any $\theta$, $\delta(A(\theta)) \geq 0$. Thus $K_S(\theta) = |S|$ is constant, therefore continuous, and the minimum is attained at any $\theta$ in the range.

**Case 2:** $\delta_x < 0, \delta_y < 0$: Similarly, since for all $\theta \in [0, \pi/2]$, $\delta(A(\theta)) < 0$, it follows that $K_S(\theta) = \chi_0(\tau_0)$.

The continuity of $K_S(\theta)$ is a direct consequence of Lemma 6. Recall it states some conditions under which the function $\kappa(x) = \sum_{j=1}^4 \tau(x)^{\tau_j}$ is a smooth function in a neighbourhood of a given vector $a$.

Consider the map $x$ which takes $\theta \in [0, \pi/2]$ and $T \subset \mathbb{Z}^2$ with its contents labelled in lexicographic order $\{ (i_1, j_1), \ldots, (i_\ell, j_\ell) \}$ to $x(T, \theta) = (i_1 \sin \theta + j_1 \cos \theta, \ldots, i_\ell \sin \theta + j_\ell \cos \theta)$. Then,

$$K_S(\theta) = \chi_0(\tau_0) = \kappa(x(S, \theta)).$$

The substitution $x_k \mapsto i_k \sin \theta + j_k \cos \theta$ is a smooth change of variables. Lemma 6 then implies that for each $\theta$ the composition $K_S(\theta)$ is smooth in some neighbourhood of $\theta$. As our domain is a closed interval, the continuity of $K_S$ follows by compactness.

**Case 3:** $\delta_x \cdot \delta_y \leq 0$: Without loss of generality we may assume $\delta_x < 0, \delta_y \geq 0$. There is a smallest point $\overline{\theta}$ where the drift of $A(\overline{\theta})$ is zero since $\delta_x \sin \theta + \delta_y \cos \theta$ is weakly monotone in this range. Thus if $\delta_x < 0$ and $\delta_y \geq 0$

$$K_S(\theta) = \begin{cases} |S| & \theta \geq \overline{\theta} \\ \chi_0(\tau_0) & \theta < \overline{\theta}. \end{cases}$$

This establishes continuity in the interior. To establish continuity on the whole domain it suffices to consider the value of $K_S$ at the point $\overline{\theta}$. In particular, consider the limit $\lim_{\theta \to \overline{\theta}} \chi_0(\tau_0)$.

Since $\delta(A(\overline{\theta})) = 0$, $\chi_0(\tau_0) = |S|$ by Lemma 11. As $\theta \to \overline{\theta}$, for each $k$ the component $i_k \sin \theta + j_k \cos \theta \to i_k \sin (\delta_x \sin \theta + \delta_y \cos \theta)$. Therefore $\chi_0 \to \chi_{\delta_y} \chi_0(\tau_0)$ and

$$\kappa((i_1 \sin \theta + j_1 \cos \theta, \ldots, i_\ell \sin \theta + j_\ell \cos \theta)) \to \kappa(x_{\delta_y}) = |S|.$$ 

**Location of the minimum.** If the value $\log \alpha/\log \beta$ is non-negative, then $\theta^* = \arctan \frac{\log \alpha}{\log \beta}$ is in the domain. In this case, $K_S(\theta^*) = \chi_0(\tau_{\theta^*}) = P(\alpha, \beta)$, by Lemma 12. This is the minimal value of $P(x, y)$ for any real $x$ and $y$, and since the critical points coincide, it is also the minimal value of $\chi_0(u) = P(u \sin \theta, u \cos \theta)$. The value of the minimum is $\chi_0(\tau_{\theta^*}) = K_S(\theta^*)$.

On the other hand, if $\log \alpha/\log \beta < 0$ then as per Lemma 12 there are no critical points of the function $G(u, \theta) = \chi_0(u)$ with $\theta$ inside the interval $[0, \pi/2]$ and any other critical point satisfying $G_\theta(u, \theta) = 0$ in $(0, \infty) \times [0, \pi/2]$ must be a local maximum. Thus, the only other candidates for local minima with $\theta \in [0, \pi/2]$ occur at the endpoints. If $\delta(S) = (0/+/-)$, then the minimum occurs at $\theta^* = 0$, since the $K_S(\pi/2) = |S|$. Similarly, if $\delta(S) = (-/-/+)$, then $\theta^* = \pi/2$. □

---

1 Either $i_k < i_{k+1}$, or $i_k = i_{k+1}$ and $j_k \leq j_{k+1}$ for $k = 1, \ldots, \ell - 1$. 
The best bound of this form for \( K_S \) is then the minimum value of \( K_S(\theta) \) over \([0, \pi/2] \). This bound is tight in all known cases.

### 2.3. The case of small steps: confirmed tight bounds

Fayolle and Raschel \cite{FayolleRaschel} describe the location of the dominant singularity in the generating function for most quarter plane models with small steps. Their formula depends on the drift \( \delta(S) = (\delta_x, \delta_y) \), along with another parameter of the model called the covariance. The covariance is defined to be

\[
\gamma(S) = \frac{\partial^2}{\partial x \partial y} P(x, y) \bigg|_{(x,y)=(1,1)} - \delta_x \delta_y.
\]

In the case of small steps the inventory always has the form

\[
P(x, y) = a(x)y + b(x) + c(x)y^{-1} = \tilde{a}(y)x + \tilde{b}(y) + \tilde{c}(y)x^{-1}.
\]

They prove that there are four possibilities for \( K_S \):

\[
\begin{align*}
\delta_x &> 0 & \delta_y &> 0 & \rho_0 &> 0 & \rho_X &< 0 & \rho_Y &< 0 & 1/|S| &< 0 \\
\delta_x &< 0 & \delta_y &> 0 & \rho_0 &< 0 & \rho_X &> 0 & \rho_Y &< 0 & 1/|S| &> 0 \\
\delta_x &> 0 & \delta_y &< 0 & \rho_0 &< 0 & \rho_X &< 0 & \rho_Y &> 0 & 1/|S| &> 0 \\
\delta_x &< 0 & \delta_y &< 0 & \rho_0 &> 0 & \rho_X &> 0 & \rho_Y &> 0 & 1/|S| &< 0
\end{align*}
\]

Remark 13 (Fayolle and Raschel, Remark 4.9\cite{FayolleRaschel}). For any non-trivial quarter-plane model \( Q(S) \) with \( S \subseteq \{\pm 1, 0\}^2 \), the dominant singularity \( \rho \) of the generating function \( \sum_n q_n z^n \) is determined by \( \delta(S) \) and \( \gamma \) as summarized in the following table:

| Drift | Possible Values for \( \rho \) | \( 1/|S| \) |
|-------|----------------|--------|
| \( +/0 \) | \( +/0 \) | \( \times \) |
| \( + - \) | \( + - \) | \( \times \) |
| \( 0 - \) | \( \gamma \leq 0 \) | \( \gamma \geq 0 \) |
| \( - + \) | \( \gamma \leq 0 \) | \( \gamma \geq 0 \) |
| \( - - \) | \( \times \) |        |

These results have some natural interpretations. If \( \delta(S) \) is non-negative in both components then the growth constant is as for unrestricted walks. If \( \delta(S) \) is positive in one component, and negative in the other, then growth constant is given by the walks that return to the relevant axis. The value \( P(\alpha, \beta) \) is the growth factor for excursions \cite{DenisovWachtel}, hence, analogously to the half plane case, if \( \delta(S) \) is negative in both components then the growth constant is the same as the growth constant for excursions in the region. The remaining cases are more difficult to describe on account of the influence of the covariance factor.

Remarkably, we are able unify the six cases, and deliver a single interpretation of the formulas. The remainder of this section is a proof that \( K_S = \min_{\theta \in [0, \pi/2]} K_S(\theta) \) for all small step quarter plane models. Thus, for this family of lattice path models, all growth constants arise as the growth constants the same step set of some containing half plane. This connects us back to the work of Denisov and Wachtel. Even though they consider excursions, they do show that once the steps are suitably weighted, in the enumeration, the choice of cone affects the sub-exponential growth, but not exponential growth. We are seeing a similar version of this in the
general walks, once the “correct” family of cones is considered by choosing the best half plane.

We now prove that the bounds are tight for small steps by showing that the values of $K_S(\theta^*)$ agree with the values from Remark 13.

**Lemma 14.** For any non-trivial quarter plane model $Q(S)$ with $S \subset \{0, \pm 1\}^2$, the following equalities hold:

\[
\rho_X^{-1} = \chi_0(\tau_0) \quad \rho_Y^{-1} = \chi_{\pi/2}(\tau_{\pi/2}).
\]

**Proof.** This is proved by simply unraveling the notation:

\[
\chi_0(u) = \sum_{(i,j) \in S} u^j = P(1,u)
\]

and so

\[
\chi_0(u) = [u]P(1,u) - \frac{1}{u^2}P(1,u) = \tau_0 = \sqrt{\frac{[u^{-1}]P(1,u)}{[u]P(1,u)}}.
\]

Consequently,

\[
\chi_0(\tau_0) = [y^0]P(1,y) + 2\sqrt{[y]P(1,y) \cdot [y^{-1}]P(1,y)} = \rho_X^{-1},
\]

which is precisely the formula for $\rho_X^{-1}$ given in Equation 2.1. The $\rho_Y$ case is similar since $\chi_{\pi/2}(u) = \sum_{(i,j) \in S} u^i$. 

**Theorem 15.** Let $S \subseteq \{0, \pm 1\}^2$ be a finite set defining a non-trivial quarter-plane lattice path model. The growth constant $K_S$ for the number $q_n$ of walks of length $n$ in $Q(S)$ satisfies

\[
K_S = \min_{\theta \in [0, \pi/2]} K_S(\theta).
\]

The location of the minimum is given in Theorem 9.

**Proof.** To prove the equivalence, we demonstrate that our formula agrees with theirs in each case. To do so we need to understand when $\log \alpha / \log \beta$ is non-negative. The symmetries inherent in the problem allow us to reduce the cases we consider. Specifically, the map $x \mapsto 1/x$ moves the critical point from $(\alpha, \beta)$ to $(1/\alpha, \beta)$, and it also changes the sign on $\delta_x$, and the problem is symmetric in $x$ and $y$. Hence it suffices to consider $\delta_x \geq \delta_y \geq 0$. Under this assumption, the five relevant cases are summarized in Table 1.

| Case | $\delta_x$ | $\delta_y$ | $\gamma$ | $\alpha$ | $\beta$ |
|------|------------|------------|----------|---------|---------|
| 1    | 0          | 0          | 1        | 1       | 1       |
| 2    | +          | +          | < 1      | < 1     | < 1     |
| 3    | +          | 0          | > 0      | < 1     | > 1     |
| 4    | +          | 0          | < 1      | 1       | < 1     |
| 5    | +          | 0          | < 0      | < 1     | > 1     |

**Table 1.** Location of critical point by drift and covariance value

The fact that $P_x(1,1) = \delta_x = 0$ and $P_y(1,1) = \delta_y = 0$, and hence $\alpha = \beta = 1$. 

Next, assume \( \delta_x > 0 \). Since we are in the case of small steps, the inventory is of the form

\[
P(x, y) = a x y + b x + c x/y + d y/x + e/x + f/y x + g y + h/y,
\]

with \( a, b, c, d, e, g, f, h \in \{0, 1\} \). The proof makes use of simple arithmetic consequences of \( \delta_x \geq \delta_y \geq 0 \), \( P_x(\alpha, \beta) = P_y(\alpha, \beta) = 0 \), and the sign of \( \gamma \). The fact that the step sets are not degenerate also intervenes. We prove Case 2 explicitly, and the remaining three cases are similar, given the additional constraint added by the covariance. The condition \( \delta_x \geq \delta_y > 0 \) implies

\[
\delta_x - \delta_y > 0 \implies b + 2c + h > 2d + e + g.
\]

Similarly, \( \alpha P_x(\alpha, \beta) - \beta P_y(\alpha, \beta) = 0 \) implies

\[
\alpha P_x(\alpha, \beta) + \beta P_y(\alpha, \beta) = 0 \implies b\alpha + 2c\alpha/\beta + h/\beta = 2d\beta/\alpha + e/\alpha + g\beta.
\]

If \( \alpha \geq 1 \) and \( \beta < 1 \), then

\[
b + 2c + h < b\alpha + 2c\alpha/\beta + h/\beta = 2d\beta/\alpha + e/\alpha + g\beta
\]

and

\[
b + 2c + h \geq 2d + e + g > 2d\beta/\alpha + e/\alpha + g\beta.
\]

This is a contradiction. Hence at least one of \( \alpha < 1 \) or \( \beta \geq 1 \) is true. Suppose that \( \alpha > 1 \), and \( \beta \geq 1 \), and consider sums:

\[
\delta_x + \delta_y > 0 \implies 2a + b + g > e + h + 2f
\]

\[
\alpha P_x(\alpha, \beta) + \beta P_y(\alpha, \beta) = 0 \implies 2\alpha\alpha + b\alpha + g\beta = e/\alpha + h/\beta + 2f/(\alpha\beta).
\]

These two statements are contradictory, hence \( \alpha < 1 \) and \( \beta \leq 1 \). To refine the bound on \( \beta \), consider two cases. If \( \delta_x = \delta_y \) then by swapping the roles of \( x \) and \( y \) we get that \( \beta < 1 \) as desired. Otherwise, by nontriviality, \( \delta_x = 2 \) and \( \delta_y = 1 \), so \( a = b = c = 1 \), \( \{g, d\} = \{0, 1\} \) and all other coefficients are 0, but this gives a trivial walk and thus is outside of our consideration. Thus, in the case \( \delta_x \geq \delta_y > 0 \), \( \alpha < 1 \) and \( \beta < 1 \), and Case 2 is shown.

The arguments for the remaining cases are quite similar, using the additional conditions from the covariance \( \gamma = a + f - c - d \).

Tracing back through the initial transformations, Table 1 establishes the locations of the minima depending on the value of the drift and the covariance and hence determines the sign of \( \log \alpha/\log \beta \) in all cases. Therefore, by Theorem 9, our bounds match the results of Fayolle and Raschel. This proves the theorem.  

\[\square\]

2.4. Elementary methods for lower bounds. Ideally, we would like to pair our upper bound strategy with combinatorial methods to provide lower bounds in order to describe exact formulas. In the absence of a general mechanism, however, the methods of [9], while elementary and somewhat ad-hoc, are surprisingly powerful. In the case of the quarter plane, using only two combinatorial lemmas, in combination with a handful of well known cases, one can deduce the growth factors of 73 of the 79 small step models.

2.4.1. The shuffle. The shuffle of two lattice paths \( w = w_1 \ldots w_m \) and \( v = v_1 \ldots v_m \), denoted \( v \shuffle w \), is the set of all paths created by all possible interlacing \( w \) and \( v \).
maintaining the relative order of steps from \(w\) and \(v\). This is extended to define the shuffle of two sets of walks:

\[
R(S_1) \shuffle R(S_2) = \bigcup_{w \in R(S_1), v \in R(S_2)} v \shuffle w.
\]

**Lemma 16** (Proposition 8.3.12, [9]). Let \(R(S_1)\) and \(R(S_2)\) be two lattice models and set \(C = R(S_1) \shuffle R(S_2)\). Then,

\[
C \subseteq R(S_1 \cup S_2) \quad \text{and} \quad \lim_{n \to \infty} |C_n|^{1/n} = K_{S_1} + K_{S_2}.
\]

Consequently, \(K_{S_1} + K_{S_2} \leq K_{S_1 \cup S_2}\).

The proof uses the asymptotic implications on a shuffle as developed in [9]. To create a bound, a set is divided into two subsets, and the lemma is applied. For example, let \(S = \{N, NE, S, SW, W\} = \mathcal{H}\). To determine a lower bound on \(K_{S}\), we apply the lemma to \(S_1 = \{NE, SW, W\} = \mathcal{H}\) and \(S_2 = \{N, S\} = 1\). This results in the bound \(2 + 2\sqrt{2} \leq K_{S}\), which matches our upper bound so it is tight. For negative drift walks, a shuffle bound will typically be tight if one of the subwalks is drift 0.

2.4.2. **Rotating a step.** We define an operator which modifies a single step in a step set. In the case of small steps it is equivalent to a rotation towards the diagonal. Both rotation operators act on a single step:

\[
r_x : (a, b) \mapsto (a + 1, b) \quad \text{and} \quad r_y : (a, b) \mapsto (a, b + 1).
\]

The effect on the enumeration is as follows.

**Lemma 17** (Proposition 8.3.3, [9]). Let \(S\) and \(T\) be subsets of \(Z\) defining quarter plane models such that they differ precisely by a single step: \(S \setminus T = \{(i, j)\}\), \(r_x(i, j) \in T\) but \(r_x(i, j) \notin S\). Then, \(K_S \leq K_T\). The result is also true when \(r_x\) is replaced by \(r_y\).

The proof is a direct consequence of the injection that the rotation of a single step induces. A complete proof is given in [9].

**Example 18.** The pair \(S = \Upsilon\) and \(T = \Upsilon\) satisfy the hypotheses of the Lemma since \(r_y(E) = NE\). Thus, \(K_S \leq K_T\). In this case, as \(K_S = 3 = K_T\), the lower bound is tight.

These two lemmas are sufficient to determine lower bounds for most small step models, starting from a small selection of models which were known before. For example, in [9] Johnson uses this strategy to prove tight lower bounds for the 23 \(D\)-finite models and here we use these results to find bounds for 31 of the remaining cases invoking only one new base case. We write \(S_1 \rightarrow_n S_2\) to indicate that step set \(S_1\) shuffled with an easily enumerated step set with exponential growth \(n\), for example Dyck prefixes for \(n = 2\), gives a lower bound on \(S_2\). Write \(S_1 \rightarrow_r S_2\) to mean that \(S_1\) gives a lower bound on \(S_2\) by rotating a step of \(S_1\). Then the following table indicates how to select step sets so that the lower bounds on the exponential growth generated by Lemma [10] and Lemma [17] are tight:
In this table a lower bound coming from a half plane model (where walks interact with at most one boundary) is represented by $H$.

2.4.3. **Excursions.** For any given model $S$, the asymptotic growth formula for the subset of excursions always acts as a lower bound. These are known, and agree with the upper bound that we find when the drift is negative or zero in both components. The formulas for the asymptotic growth of excursions are general, and will always form a lower bound for the asymptotic growth of the complete class of walks. Thus, in the case of negative drift, we can apply the bound from the excursions, and this will be tight. This is true for larger step sets as well.

Combining our upper and lower bounds, which are derived from only simple combinatorial arguments, we obtain $K_S$ for all but 6 of the small step models. The goal, rather than to recompute known values, is to develop a toolbox to compute $K_S$ for other families of models for which there are no known formulas, for example, the case of 3 dimensions, or in the case of longer steps. Additionally, we envision combining our strategy, plus guesses resulting from the method of Bostan and Kauers for D-finite models to build lower bounds for many models. Higher dimensional walks do not need to be shuffle-reducible. For example, shuffle $\downarrow$ in the $xy$-plane with $\uparrow$ in the $yz$-plane. One further rotation gives $2\sqrt{2} + 4 \leq K_S$ for $S = \{(1,1,1), (0,1,1), (0,1,-1), (-1,0,0), (-1,1,0), (-1,-1,0)\}$, which is tight.

3. **Extensions**

Our half-plane bounding strategy does not rely on the size of the steps nor the region in which the paths are restricted. Naive numerical calculations on examples of quarter plane walks with larger steps and of three dimensional models suggest the bounds remain tight.

Furthermore, in a recent study of three dimensional walks [2], Bostan, Bousquet-Mélou, Kauers, and Melczer guessed differential equations satisfied by the generating functions of some small step models, and using this they were able to conjecture the exact growth factors. We verified that, in each of the 28 cases for which we had data from [2], the growth factor for the model $O(S)$, where $O = \mathbb{R}^3_{\geq 0}$, is equal to the minimal bound.

These observations lead us to the following conjecture.

**Conjecture 1.** Let $S \subset \mathbb{Z}^d$ be a finite multiset of steps. Let $K_S$ be the growth constant for the enumerative sequence counting the number of such walks restricted to the first orthant. Let $P$ be the set of hyperplanes through the origin in $\mathbb{R}^d$ which...
do not meet the interior of the first orthant. Given \( p \in \mathcal{P} \) let \( K_S(p) \) be the growth constant of the walks on \( S \) which are restricted to the side of \( p \) which includes the first orthant. Then

\[
K_S = \min_{p \in \mathcal{P}} K_S(p).
\]

Note that in all dimensions, \( K_S(p) \) can be computed by projecting the steps onto the normal to \( p \), and enumerating the resulting unidimensional model. Thus, if true, our conjecture would give an elementary way to understand and compute the growth constant of any class of first orthant restricted walks.

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